Terminal Coalgebras and Non-wellfounded Sets in Homotopy Type Theory

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Abstract

Non-wellfounded material sets have previously been modeled in Martin-Löf type theory by Lindström using setoids (Lindström, 1989). In this paper we construct models of non-wellfounded material sets in Homotopy Type Theory (HoTT) where equality is interpreted as the identity type. The first model satisfies Scott’s Anti-Foundation Axiom (SAFA) and dualises the construction of iterative sets. The second model satisfies Aczel’s Anti-Foundation Axiom (AFA), and is constructed by adaption of Aczel–Mendler’s terminal coalgebra theorem to type theory, which requires propositional resizing.

In an bid to extend coalgebraic theory and anti-foundation axioms to higher type levels, we formulate generalisations of AFA and SAFA, and construct a hierarchy of models which satisfies the SAFA generalisations. These generalisations build on the framework of Univalent Material Set Theory, previously developed by two of the authors (Gylterud and Stenholm, 2023).

Our results are formalised in the proof-assistant Agda.

1 Introduction

In non-wellfounded set theory, the concept of a material set is expanded beyond the cumulative hierarchy. The allowance for non-wellfounded sets, such as the Quine atom \( q := \{ q \} \), makes it easier to study circular phenomena and structures such as transitions systems and streams. In what follows, we seek to integrate non-wellfounded set theory into Homotopy Type Theory (HoTT)—a relatively new framework for mathematics, which supports higher dimensional structures as first-class citizens with the powerful Univalence Axiom and higher inductive types. Our aim is to take classical notions from universal coalgebra and non-wellfounded set theory and extend them to higher-dimensional structures.

Wellfounded material set theory has been studied in Martin-Löf type theory since 1978 with the introduction of Aczel’s setoid model of Constructive Zermelo–Fraenkel set theory (CZF) (Aczel, 1978). Non-wellfounded set theory in Martin-Löf type theory was studied already in 1989 by Lindström, when she constructed a setoid based model of constructive \( \text{ZF}^- \) (ZF without the axiom of foundation) + Aczel’s anti-foundation axiom (AFA) (Lindström, 1989).

These two models of material set theory were, as mentioned, setoid based, meaning that equality was interpreted as a binary relation distinct from Martin-Löf’s identity type. This was rectified in the model presented in the HoTT Book (Univalent Foundations Program, 2013), which constructed a model of wellfounded set theory using a higher inductive type, in which equality was interpreted as the identity type.

Gylterud (2018) then constructed a model, \((V^0,\in)\), equivalent to the HoTT Book model, but which did not require higher inductive types for the construction. This construction and its properties have been further explored in Gratzer et al. (2024). One important aspect of \( V^0 \) is its role as the initial algebra of the \( U \)-restricted powerset functor \( P_U^0 : \text{Type} \to \text{Type} \), which maps \( X \mapsto \sum_{A \subseteq U} A \mapsto X \). One of the ideas we explore here is to construct the terminal coalgebra for \( P_U^0 \) to use as a model of non-wellfounded sets, filling out the question mark in the table below.

| Setoid      | Identity type |
|-------------|---------------|
| Foundation  | Aczel 1978    | Gylterud 2018 |
| Anti-foundation | Lindström 1989 | ? |


We show that the terminal coalgebra for $P^0_U$ would indeed yield a model of Aczel’s anti-foundation axiom (AFA):

**AFA:** Any (directed) graph can be uniquely decorated with sets such that elementhood between the sets coincides with edges in the graph.

As we shall see, the classical Aczel–Mendler construction (Aczel and Mendler, 1989) can be adapted to the HoTT setting and constructs a terminal coalgebra for $P^0_U$, but it requires propositional resizing—an impredicative axiom.

In addition to the Aczel–Mendler construction, we provide a new construction, $V^\infty_0$, of non-wellfounded sets in HoTT which dualises the construction of $V^0_0$, but which surprisingly does not yield a terminal coalgebra for $P^0_U$. It is a third fixed point—neither initial nor terminal. This type is a model of Scott’s anti-foundation axiom (SAFA), an alternative anti-foundation axiom to AFA. SAFA is based on the concept of *Scott extensionality*. A graph is Scott extensional if equality of nodes in the graph coincides with isomorphism of unfolding trees (more on that later).

**SAFA:** Every Scott extensional graph can be injectively decorated with sets, and the graph of all sets with edges symbolising elementhood is Scott extensional.

We also explore possible extensions of anti-foundation axioms to higher types. In HoTT, there is a fundamental notion of $n$-type arising from the iterative application of identity types. The 0-types are the sets, where much of classical mathematics takes place. But even for down-to-earth mathematics such as combinatorics, higher types can play a role. Groupoids, that is 1-types, show up for instance in Joyal’s theory of combinatory species. We therefore propose generalisations of both AFA and SAFA to $n$-types, and our model construction $V^\infty_0$ is presented as a general construction, $V^\infty_n$, which then satisfies $k$-SAFA for each $k \leq n$.

The construction of $V^\infty_\infty$ is based on M-types. These types were constructed in HoTT in “Non-wellfounded trees in Homotopy Type Theory” (Ahrens, Capriotti, and Spadotti, 2015). We provide some further general results about M-types. In particular, we fully characterise the identity types of M-types as indexed M-types.

1.1 Contributions

The main contributions of this paper are the following:

- Construction of a fixed point for each of the non-polynomial functors $X \mapsto (\sum_{A:U} A \leftrightarrow_n X)$, which is distinct from both the initial algebra and the terminal coalgebra.
- Adapting Aczel–Mendler’s construction to type theory, assuming propositional resizing.
- Applying the HoTT version of Aczel–Mendler to construct a terminal coalgebra for the $U$-restricted powerset functor.
- A demonstration that this terminal coalgebra yields a model of set theory incorporating Aczel’s anti-foundation axiom, with the identity type serving as equality.
- Showing that Scott’s anti-foundation axiom has a constructive model in HoTT, with the identity type as equality.
- A characterisation of the identity types of M-types as indexed M-types.

1.2 Formalisation

The results in this paper has been formalised in the Agda proof assistant (The Agda development team, 2024a). Our formalisation builds on the agda-unimath library (Rijke et al., 2024), which is an extensive library of formalised mathematics from the univalent point of view. The results in Section 7 are formalised using Cubical Agda—an extension of Agda with features from cubical type theory (The Agda development team, 2024b). But as the proofs in this article demonstrate, they can be carried out in the same framework as the rest of the article.

The formalisation of Sections 1–5 in this paper has been included in a larger library on material set theory in HoTT, which can be found here: https://git.app.uib.no/hott/hott-set-theory. As the formalisation is structured slightly differently than the outline of this paper, there are a few results which do not have an exact counterpart in the code base. All these results are simple corollaries or variations of results which have been formalised. Importantly, all the main results are fully formalised.

The formalisation of Section 7 can be found at: https://github.com/niccoloveltri/aczel-mendler. Throughout the paper there will also be clickable links to specific lines of Agda code corresponding to a given result. These will be shown as the Agda logo 🍀.
1.3 Notation and conventions

The notation throughout the paper will follow common practice in HoTT. We use some categorical notations, including coercion from categories to their types of objects: We take \( x : C \) to mean \( x : \text{Obj}_C \).

The ambient type theory is assumed to contain \( \text{M-types} \). This is not a very restrictive assumption as it has been shown by Ahrens, Capriotti, and Spadotti (2015) that \( \text{M-types} \) can be constructed from inductive types in HoTT.

**Convention** Throughout the paper we will take the type of truncation levels to be the type \( \mathbb{N}_{\leq 2} \), i.e. the usual truncation levels, but with a supremum, \( \infty \), such that \( \| P \|_{\infty} \equiv P \). Moreover, for computations we have \( \infty - 1 = \infty = \infty + 1 \). We will also use \( \mathbb{N}_{\leq 1} \) for the subset of truncation levels excluding \(-2\), and \( \mathbb{N}_{-2} \) and \( \mathbb{N}_{-1} \) for the ones further excluding \( \infty \).

We will also take liberties with coercions of subtypes into their ambient type to enhance the readability of theorems and proofs. Since the results are all formalised in Agda, we allow ourselves this simplification without worry of any loss of rigour. The same goes for using some essentially small types in some places instead of their small replacements.

2 Coalgebras on Type

The notion of an \( F \)-coalgebra is usually formulated for functors on categories. In HoTT, there is a whole spectrum of notions of categories depending on how much saturation (or univalence) one wants to require and whether one wants to restrict the type level of homomorphisms or objects or both. At one end of this spectrum we find the wild categories, where objects and homomorphisms can be of any type level and no saturation is required.

In this setting we will be interested in wild functors \( F : \text{Type} \rightarrow \text{Type} \), which is an operation on types with an action \( (X \rightarrow Y) \rightarrow (F X \rightarrow F Y) \), which we denote by juxtaposition \( F f \), which preserves composition and the identity function.

An \( F \)-coalgebra is a pair \( (A, \alpha) \), where \( A : \text{Type} \) and \( \alpha : A \rightarrow F A \). As is usual in universal coalgebra, we require no comonadicity of \( F \) nor coassociativity of \( \alpha \) (i.e. \( \alpha \) being algebra for \( F \) as a comonad). We will also here settle on some notation for standard notions of universal coalgebra, adapted to the HoTT setting.

**Definition 1** (The wild category of \( F \)-coalgebras). Let \( F : \text{Type} \rightarrow \text{Type} \) be a wild endofunctor on the wild category of types and functions. The wild category of \( F \)-coalgebras, denoted \( \text{F-Coalg} \), is the wild category for which

- The type of objects is the type of \( F \)-coalgebras:
  \[
  \sum_{A : \text{Type}} A \rightarrow F A.
  \]

- Given two coalgebras \( (A, \alpha) \) and \( (B, \beta) \), the type of \( F \)-coalgebra homomorphisms from \( (A, \alpha) \) to \( (B, \beta) \) is the type
  \[
  \sum_{f : A \rightarrow B} \beta \circ f \sim F f \circ \alpha.
  \]

- The underlying map of the identity homomorphism on \( (A, \alpha) \) is \( \text{id}_A \) and the homotopy is constructed as usual by the functoriality of \( F \) on the identity homomorphism.

- Given a homomorphism \( (f, H) \) from \( (A, \alpha) \) to \( (B, \beta) \) and \( (g, K) \) from \( (B, \beta) \) to \( (C, \gamma) \), the underlying map of their composition is given by \( g \circ f \), and the homotopy is the usual composition of squares together with the functoriality of \( F \) on composition.

It is important to note that since the carrier of the codomain, \( B \), can be of any type level, the second component of \( \text{Hom}_{\text{F-Coalg}} (A, \alpha) (B, \beta) \), namely \( \beta \circ f \sim F f \circ \alpha \), is a structure, not just a property.

We will also use coalgebras for (wild) functors on indexed types. These are functorial operations \( F : (I \rightarrow \text{Type}) \rightarrow (I \rightarrow \text{Type}) \) for some \( I : \text{Type} \). We will call these indexed functors and indexed coalgebras.

**Definition 2** (The wild category of indexed \( F \)-coalgebras). Given an index \( I : \text{Type} \), let \( F : (I \rightarrow \text{Type}) \rightarrow (I \rightarrow \text{Type}) \) be a wild endofunctor on the wild category of \( I \)-indexed type families and fiberwise maps. The wild category of \( I \)-indexed \( F \)-coalgebras, \( \text{F-Coalg} \), is the wild category for which
• The type of objects is the type of \( I \)-indexed \( F \)-coalgebras:
\[
\sum_{A : \text{Type}} \prod_{i : I} A_i \to F A_i.
\]

• Given two coalgebras \((A, \alpha), (B, \beta)\), the type of \( F \)-coalgebra homomorphisms from \((A, \alpha)\) to \((B, \beta)\) is the type
\[
\sum_{f : \prod_{i : I} A_i \to B_i} \prod_{i : I} \beta_i \circ f_i \sim F f_i \circ \alpha_i.
\]

• The underlying map of the identity homomorphism on \((A, \alpha)\) is \(\lambda_i. \text{id}_{A_i}\) and the homotopy is constructed as usual by the functoriality of \(F\) on the identity homomorphism.

• Given a homomorphism \((f, H)\) from \((A, \alpha)\) to \((B, \beta)\) and \((g, K)\) from \((B, \beta)\) to \((C, \gamma)\), the underlying map of their composition is given by \(\lambda_i. g_i \circ f_i\), and the homotopy is the usual composition of squares together with the functoriality of \(F\) on composition.

**Definition 3.** An \( F \)-coalgebra \((A, \alpha)\) is **extensional** if \(\alpha : A \to F A\) is an embedding.

Through the lens of type levels, we can also see a close connection between two important properties of coalgebras, being **terminal** and being **simple**:

**Definition 4.** An \( F \)-coalgebra \((A, \alpha)\) is **terminal** if for every other \( F \)-coalgebra, \((B, \beta)\), the type of homomorphisms into \((A, \alpha)\), namely \(\text{Hom}_{F\text{-Coalg}}(A, \alpha)(B, \beta)\), is contractible.

**Definition 5.** An \( F \)-coalgebra \((A, \alpha)\) is **simple** if for every other \( F \)-coalgebra, \((B, \beta)\), the type of homomorphism into \((A, \alpha)\), namely \(\text{Hom}_{F\text{-Coalg}}(A, \alpha)(B, \beta)\), is a proposition.

The following is immediate from the definitions:

**Lemma 1.** A terminal \( F \)-coalgebra is simple.

### 2.1 Bisimulation

Bisimulation is another central notion of coalgebra theory. In short, a bisimulation is just a span in the category of \( F \)-coalgebras, or a relation on the coalgebra that relates elements in a way compatible with the coalgebra structure. We can arrange the bisimulations on a particular \( F \)-coalgebra into a (wild) category.

Although bisimulations are essentially spans, when working with dependent types, it is also useful to think of the bisimulation as stemming from a relation \( R : X \to X \to \text{Type} \). Thus, the carrier of the bisimulation is (without loss of generality) the \(\Sigma\)-type: \(|R| := \sum_{(x, x') : X \times X} R x x'\). From this carrier, we have projections \(\pi_0 \circ \pi_0 : |R| \to X\) and \(\pi_1 \circ \pi_0 : |R| \to X\), which should be \( F \)-coalgebra homomorphisms.

\[
\begin{array}{ccc}
X & \xleftarrow{\pi_0 \circ \pi_0} & |R| & \xrightarrow{\pi_1 \circ \pi_0} & X \\
\downarrow{\mu} & & \downarrow{\mu} & & \downarrow{\mu} \\
F X & \xleftarrow{F(\pi_0 \circ \pi_0)} & F |R| & \xrightarrow{F(\pi_1 \circ \pi_0)} & F X
\end{array}
\]

A morphism of bisimulations can be thought of as an \( F \)-coalgebra homomorphism between the bisimulations as \( F \)-coalgebras, along with a filling of the left and right triangular prisms of the following diagram:
Definition 6 (The wild category of F-bisimulations on an F-coalgebra). Let $F : \text{Type} \to \text{Type}$ be a wild endofunctor on the wild category of types and functions, and let $(X, m)$ be an F-coalgebra. The **wild category of F-bisimulations on** $(X, m)$, denoted $\text{F-Bisim}_{(X, m)}$, is the wild category for which

- The type of objects is the type of spans:

$$\sum_{R : X \times X \to \text{Type}} \sum_{\alpha : [R] \to F [R]} (m \circ \pi_0 \circ \pi_0 \sim F (\pi_0 \circ \pi_0) \circ \alpha) \times (m \circ \pi_1 \circ \pi_0 \sim F (\pi_1 \circ \pi_0) \circ \alpha)$$

- Given two F-bisimulations $(R, (\alpha, (H_0, H_1)))$ and $(R', (\alpha', (H'_0, H'_1)))$, the type of F-bisimulation homomorphisms from the first to the second is

$$\sum_{(f, K) : \text{Hom}_{F - \text{Coalg}} ([R], [\alpha]) ([R'], [\alpha'])} ((\pi_0 \circ \pi_0, H'_0) \circ (f, K) = (\pi_0 \circ \pi_0, H_0)) \times ((\pi_1 \circ \pi_0, H'_1) \circ (f, K) = (\pi_1 \circ \pi_0, H_1))$$

- The first component of the identity homomorphism on $(R, (\alpha, (H_0, H_1)))$ is the identity

$$\text{id} : \text{Hom}_{F - \text{Coalg}} ([R], [\alpha]) ([R], [\alpha]).$$

The higher homotopies follow from the functoriality of $F$.

- Given a homomorphism $((f, K), (p_0, p_1))$ from $(R, (\alpha, (H_0, H_1)))$ to $(R', (\alpha', (H'_0, H'_1)))$ and a homomorphism $((g, J), (q_0, q_1))$ from $(R', (\alpha', (H'_0, H'_1)))$ to $(R'', (\alpha'', (H''_0, H''_1)))$, the underlying homomorphism of their composition is given by

$$(g, J) \circ (f, K) : \text{Hom}_{F - \text{Coalg}} ([R'], [\alpha']) ([R'', [\alpha'']).$$

The higher homotopies follow from the functoriality of $F$.

When doing set level mathematics, a bisimulation homomorphism from $(R, \alpha)$ to $(R', \alpha')$ (the homotopies being propositions) would simply be an F-coalgebra homomorphism from the total space of the first relation to the total space of the second. But since we have no restrictions on the type levels of the carrier types, we also need coherence on the homotopies involved in the bisimulations.

In universal coalgebra, there are many equivalent formulations of being a simple F-coalgebra (Rutten, 2000). One of the equivalent formulations is that the identity bisimulation is the terminal bisimulation. The definition below is a strengthening of the classical definitions, allowing proof relevant bisimulations and coalgebras with higher homotopies.

Definition 7. Let $(X, m)$ be an F-coalgebra. We define the **identity bisimulation**, $(\equiv, (\alpha, (H_0, H_1)))$, on $(X, m)$, by noting that $\pi_0 \circ \pi_0 : | = | \to X$ is an equivalence, and letting $\alpha ((x, x), \text{rell}) = F (\pi_0 \circ \pi_0)^{-1}(m \ x)$. Likewise, $H_0$ and $H_1$ are defined by path induction.

Definition 8. Let $(X, m)$ be an F-coalgebra. We say that $(X, m)$ is **bisimulation simple** if $(\equiv, (\alpha, (H_0, H_1)))$ is terminal. That is: for every other bisimulation $(R, (\alpha', (H'_0, H'_1)))$ on $(X, m)$ the type of F-bisimulation homomorphisms from $(R, (\alpha', (H'_0, H'_1)))$ to $(\equiv, (\alpha, (H_0, H_1)))$ is contractible.

A type family is a pair $(A, P)$ where $A : \text{Type}$ and $P : A \to \text{Type}$. An equivalence of families between two families $(A, P)$ and $(B, Q)$ is a pair $(\alpha, \sigma)$ where $\alpha : A \simeq B$ and $\sigma : \prod_{x: A} Q(\alpha \ a) \simeq P \ a$. By univalence, we can transfer results about one family along such an equivalence to a result about the other family. We will now use this to relate equality of homomorphisms with bisimulation homomorphisms into the identity bisimulation.

Lemma 2 (**$\equiv$**). Let $(X, m)$ be an F-coalgebra. There is an equivalence of type families between bisimulations with homomorphisms into $(\equiv, (\alpha, (H_0, H_1)))$ and equality of pairs of homomorphisms into $(X, m)$. That is, there is an equivalence of families between $\text{F-Bisim}_{(X, m)}$ and $\text{F-Coalg} (\text{Hom}_{F - \text{Coalg}} (Y, n) (X, m))$ and $\lambda (\nu (f, g), f = g)$

Proof. Let $(R, (\alpha', (H'_0, H'_1)))$ be an F-coalgebra bisimulation on $(X, m)$. By definition $(\equiv, \alpha')$ is an F-coalgebra, and $(\pi_0 \circ \pi_0, H'_0)$ and $(\pi_1 \circ \pi_0, H'_1)$ are homomorphisms. The type of bisimulation homomorphisms from $(R, (\alpha', (H'_0, H'_1)))$ to $(\equiv, (\alpha, (H_0, H_1)))$ is the type of fillings of the following diagram:
Note that the projection $\pi_0 \circ \pi_0 : | = | \to X$ is an equivalence. Applying this equivalence and using the fact that $F(\pi_0 \circ \pi_0) (\alpha ((x,x), \text{refl})) = m x$, we see that this is equivalent to having a filling of the following diagram:

Such a filling is equivalent to $(\pi_0 \circ \pi_0, H') = (\pi_1 \circ \pi_0, H_1')$.

Likewise, any pair of parallel coalgebra homomorphisms $(f, H)$ and $(g, I)$ from $(Y, n)$ to $(X, m)$ gives rise to a bisimulation by letting $R x x' = \sum_{y : Y} (f y = x) \times (g y = x')$. One can check that going back and forth yields equivalent results. Thus, by univalence we have an equivalence of type families.

**Lemma 3 (\(\square\)).** An $F$-coalgebra is bisimulation simple if and only if it is simple.

**Proof.** By Lemma 2, equality between homomorphisms into $(A, m)$ is equivalent to bisimulation homomorphisms into $(=, (\alpha, (H_0, H_1)))$. Thus, if equality between homomorphisms is contractible (since $\text{Hom}_{F\text{-Coalg}} (B, n) (A, m)$ is a proposition), then $(=, (\alpha, (H_0, H_1)))$ terminal, and vice versa. \(\square\)

**Corollary 1.** Let $(X, m)$ be a terminal $F$-coalgebra. Then $(X, m)$ is bisimulation simple, i.e. the identity bisimulation, $(=, (\alpha, (H_0, H_1)))$, is the terminal bisimulation.

### 2.2 Coalgebraic view of set theory

There is a coalgebraic viewpoint of material set theory, where one replaces the usual $\in$-relation on $V$ with a coalgebra structure $V \to P(V)$ in the category of classes and class functors. The functor $P$ is the powerset functor on classes which assigns to each class the class of subsets of the class. The axiom of foundation says that $V$ is the initial $P$-algebra, while Aczel’s anti-foundation axiom says that $V$ is the terminal coalgebra. Other $P$-coalgebras are what is known in set theory as set-like models of set theory, and the Mostowski collapsing theorem can be framed in these terms. See for instance Paul Taylor’s work on these topics (Taylor, 2023).

In *Univalent Material Set Theory* (Gylterud and Stenholm, 2023), two of the authors of the current paper developed this coalgebraic viewpoint of material set theory inside HoTT, generalising it from sets to types of arbitrary type levels. Since the models developed later use this framework, we will quickly revisit the central definitions here.

The powerset functor on classes has a close correspondent in HoTT, namely the $U$-restricted powerset functor:

$$P^U_0 : \text{Type} \to \text{Type}$$

$$P^U_0 \ X := \sum_{A : U} A \to X.$$
The functorial action of $P_U^0$ is taking the forward image along the function:

$$P_U^0 f (A, v) = (\text{image}(f \circ v), \text{incl}(f \circ v)).$$

By applying the type theoretic replacement principle (Rijke, 2017), the image lands in $U$ (and thus the functorial action is well-defined) if the codomain of $f$ is locally $U$-small. We will therefore restrict the application of this function to locally small types.

This notion of powerset is different from the one attained by regarding subtypes as maps into the type of $U$-small propositions. The two notions coincide on types in $U$, but differ on large types. In particular, $X \mapsto (X \rightarrow \text{hProp}_U)$ cannot have a fixed point, due to Cantor’s paradox. There is however no such obstacle for $P_U^0$, which is already known to have an initial algebra. As we shall see later in this article, it also has a terminal coalgebra, assuming propositional resizing, and a third fixed point (without assuming any resizing). All fixed points are extensional coalgebras, which means that they model the set theoretic extensionality axiom.

In univalent material set theory, one lifts the requirement of having to deal only with subtypes, and generalises to coalgebras for the polynomial functor $P_U^n$:

$$P_U^n : \text{Type} \rightarrow \text{Type}$$

$$P_U^n X := \sum_{A \in U} A \rightarrow X.$$ 

The functorial action for $P_U^n$ is simply postcomposition:

$$P_U^n f (A, v) = (A, f \circ v).$$

Extensional coalgebras for this functor correspond to what are called $\epsilon$-structures in univalent material set theory. There is also a hierarchy of functors between $P_U^0$ and $P_U^\infty$, where we restrict to $n$-truncated maps:

$$P_U^{n+1} : \text{Type} \rightarrow \text{Type}$$

$$P_U^{n+1} X := \sum_{A \in U} A \leftrightarrow_n X.$$ 

The subscripted hooked arrow, $A \leftrightarrow_n X$, denotes an $n$-truncated function $A \rightarrow X$. The $n$ here ranges from $-1$ to $\infty$, so that $P_U^n$ is defined for all $n$ from $0$ to $\infty$. The type $P_U^1 X$, for instance, is the type of coverings of $X$.

The functorial action on $P_U^n$ is taking $n$-images of the composition:

$$P_U^n f (A, v) = (\text{image}_n(f \circ v), \text{incl}_n(f \circ v)).$$

Just as for $P_U^0$, unless $n = \infty$, this is only well-defined on locally small types.

We will almost exclusively focus on the anti-foundation axioms in this paper, but at times we will see some examples where we will use things like the empty set, $\emptyset$, and paring/finite unordered tupling. In univalent material set theory unordered tuples must be subscripted with their type level. We will only use type level 0 and type level 1 in the examples, so it is sufficient here to note that $\{a_0, \cdots, a_{n-1}\}_0$ is the usual set theoretic tupling where repetition is ignored, while $\{a_0, \cdots, a_{n-1}\}_1$ is multiset tupling where for instance $\emptyset \in \{\emptyset, \emptyset\}$ becomes a type with two elements. There is also the notion of ordered pairing, but it is uniform in type level and consists of a choice of embedding $\langle -, - \rangle : V \times V \leftrightarrow V$. See Univalent Material Set Theory (Gylterud and Stenholm, 2023) for details.

**Notation:** As we do not work with several universes in this article, we will often suppress mention of $U$ in $P_U^n$ and simply write $P^n$.

Since, we will use it already in the definition of the anti-foundation axioms, we will now take the opportunity to introduce the terminal coalgebra of $P^\infty$ which we will call $V^\infty$:

$$V^\infty := \mathcal{M}_{A : U} A.$$ 

This M-type type comes equipped with a coalgebra structure $\text{desup}^\infty : V^\infty \rightarrow P^\infty V^\infty$, which is an equivalence. Let $\text{sup}^\infty : P^\infty V^\infty \rightarrow V^\infty$ denote the inverse of $\text{desup}^\infty$. For any other $P^\infty$-coalgebra, $(X, m)$ there is a unique coalgebra homomorphism $\text{corec}^\infty (X, m) : (X, m) \rightarrow (V^\infty, \text{desup}^\infty)$. We will sometimes suppress the coalgebra $(X, m)$ and only write $\text{corec}^\infty$, when the coalgebra is clear from the context.

7
3 The identity type of an M-type

The M-types are a class of coinductive types, dual to the inductive W-types. Intuitively, while the elements of W-types are well-founded trees with specified branching types, the M-types are the types of all trees with that branching type. Formally, each M-type is the terminal coalgebra of a polynomial functor which specifies the branching type. A polynomial functor is one which is induced by a container (Abbot, Altenkirch, and Ghani, 2005; Altenkirch et al., 2015). Put simply, a polynomial functor $\Delta \rightarrow Type$ is of the form $\sum_{a:A} B a \rightarrow X$, for some $A : Type$ and $B : A \rightarrow Type$. The data $A,B$ is called a container and denoted $A \triangleleft B$. The functor $X \mapsto \sum_{a:A} B a \rightarrow X$, as induced by the container $A \triangleleft B$ is denoted by $[A \triangleleft B] : Type \rightarrow Type$. The M-type $\Lambda_{A,B} B a : Type$ is the underlying type of the terminal coalgebra of $[A \triangleleft B]$ and its coalgebra map is denoted:

$$\text{desup}_{A,B} : \Lambda_{A,B} B a \rightarrow [[A \triangleleft B](\Lambda_{A,B} B a)].$$

There is also an indexed version of polynomial functors, containers and M-types. The indexed versions generalise from functors $Type \rightarrow Type$ to functors $(I \rightarrow Type) \rightarrow (J \rightarrow Type)$. An indexed polynomial functor maps $X \mapsto \lambda j. \sum_{a:A_j} \prod_{b:B_j a} X(w j b)$, for some $A : J \rightarrow Type$ and $B : \prod_{j,J} A_j \rightarrow Type$ and $w : \prod_{j,J} \Lambda_{A_j} B_j a \rightarrow I$. The data $A,B,w$ is called an indexed container* and is denoted $A \triangleleft (B, w)$. The induced polynomial functor is denoted $[A \triangleleft (B, w)] : (I \rightarrow Type) \rightarrow (J \rightarrow Type)$. The indexed M-types are the terminal coalgebras for indexed polynomial endofunctors, i.e. when $I = J$.

Throughout the rest of this section, let $A \triangleleft B$ be a container. For convenience, we introduce some notation for $[A \triangleleft B]$-coalgebras. This notation goes back to Aczel (1978), where it was applied to its prototypical W-type, but we will use it for coalgebras in general.

**Notation:** Given $m : X \rightarrow [A \triangleleft B] X$, and $x : X$ we will denote by $\overline{x} : A$ and $\tilde{x} : B \overline{x} \rightarrow X$ the unique elements defined by $m x = (\overline{x}, \tilde{x})$, that is $\overline{x} := \pi_0 (m x)$ and $\tilde{x} := \pi_1 (m x)$. This notation suppresses the map $m$, but it should be clear from the context which map the notation refers to, whenever it is used. This notation will also be used for large Type coalgebras $m : X \rightarrow \sum_{I:Type} I \rightarrow X$.

The identity type of a W-type can be characterised inductively (Gylterud, 2019). For elements $x,y : W_{a,A} B a$ there is an equivalence:

$$(x = y) \simeq \sum_{p : \overline{x} = \overline{y}} \prod_{b : B \overline{p}} \tilde{x} b = \tilde{y} (\text{tr}_p^B b).$$

The goal of this section is to give a similar characterisation of the identity type of M-types: The identity type between two elements of an M-type is an indexed M-type (Theorem 1). This characterisation is slightly more involved than the one for W-types, which was straightforward induction, and goes through some results of bisimulation.

3.1 Characterisation of bisimulations of polynomial functors and the identity type of M-types

We will now characterise the identity type of an M-type as an indexed M-type. This result is not surprising, but is very useful for working with M-types in HoTT. When we later construct a model of Scott’s non-wellfounded sets, this characterisation is critical to prove local smallness of the model. Furthermore, the characterisation of the identity type follows from a characterisation of bisimulations of polynomial functors as coalgebras for a related indexed polynomial functor.

**Definition 9 (\text{\overline{C}‘}).** Given an $[A \triangleleft B]$-coalgebra $(X,m)$, we define the $(X \times X)$-indexed polynomial functor

$$E_{(X,m)} : (X \times X \rightarrow Type) \rightarrow (X \times X \rightarrow Type)$$

$$E_{(X,m)} R (x,y) := \sum_{p : \overline{x} = \overline{y}} \prod_{b : B \overline{p}} R (\tilde{x} b, \tilde{y} (\text{tr}_p^B b)).$$

The functorial action is postcomposition on the second component.

*Note that what we here call indexed container is what Altenkirch et al. (2015) call a doubly indexed container, which is not the same as what they call indexed containers.*
Note that the identity type is an $E_{(X,m)}$-coalgebra, for any pair $(X,m)$. Specifically, define the following map by path induction:

$$
\gamma : \prod_{(x,y) : X \times X} x = y \to E_{(X,m)} (=) (x,y)
$$

$$
\gamma (x,x) \text{ refl} := (\text{refl}, \text{refl-htpy}).
$$

As an intermediate step towards showing equivalence of $E$-coalgebras and $[A \triangleleft B]$-bisimulations, it will also be helpful to define the following $(X \times X)$-indexed polynomial functor on $(X,m)$:

**Definition 10.** Given an $[A \triangleleft B]$-coalgebra $(X,m)$, we define the $(X \times X)$-indexed polynomial functor

$$
D_{(X,m)} : (X \times X \to \text{Type}) \to (X \times X \to \text{Type})
$$

$$
D_{(X,m)} R (x,y) := \sum_{(a,\phi) : [A \triangleleft B] | R} (m x = \pi_0 \circ \pi_0) \times (m y = \pi_1 \circ \pi_0) (a,\phi).
$$

The functorial action sends a fiberwise map $g : \prod_{(x,y) : X \times X} R (x,y) \to R' (x,y)$ to the map which acts on the first component by $[A \triangleleft B] (\text{tot } g)$.

Intuitively, the operations $E_{(X,m)}$ and $D_{(X,m)}$ both unfold a relation one step as though it was a bisimulation. The difference is that $D$ uses the realisation of the polynomial, while $E$ uses the polynomial directly. But they are in fact equivalent:

**Lemma 4 (\text{\&\&}).** Given an $[A \triangleleft B]$-coalgebra $(X,m)$ and a relation $R : X \times X \to \text{Type}$, for any pair $(x,y) : X \times X$ we have a natural equivalence

$$
E_{(X,m)} R (x,y) \simeq D_{(X,m)} R (x,y),
$$

which maps $(p,\sigma) : E_{(X,m)} R (x,y)$ to the element

$$
(\pi, \lambda b. ((\tilde{x} b, \tilde{y} (\tau B b)), \sigma b)) : [A \triangleleft B] | R.
$$

**Proof.** We have the following chain of equivalences:

$$
E_{(X,m)} R (x,y) \simeq \sum_{p : \pi} \sum_{\phi_1 : B \pi \to X} (\tilde{y} \circ \tau B p = \phi_1) \times \left( \prod_{b : B \pi} R (\tilde{x} b, \phi_1 b) \right)
$$

$$
\simeq \sum_{\phi_1 : B \pi \to X} (m y = (\pi, \phi_1)) \times \left( \prod_{b : B \pi} R (\tilde{x} b, \phi_1 b) \right)
$$

$$
\simeq \sum_{a : A} \sum_{\phi_0,\phi_1 : B a \to X} (m x = (a,\phi_0)) \times (m y = (a,\phi_1)) \times \left( \prod_{b : B a} R (\phi_0 b, \phi_1 b) \right)
$$

$$
\simeq D_{(X,m)} R (x,y).
$$

By chasing $(p,\sigma) : E_{(X,m)} R (x,y)$ along the equivalences one sees that it is mapped as stated. \hfill \square

**Proposition 1 (\text{\&\&}).** For any $[A \triangleleft B]$-coalgebra $(X,m)$ there is an equivalence of types

$$
E_{(X,m)} \text{-Coalg} \simeq [A \triangleleft B] \text{-Bisim}_{(X,m)},
$$

which sends a map $f : \prod_{(x,y) : X \times X} R (x,y) \to E_{(X,m)} R (x,y)$ to the map

$$
\lambda ((x,y), r. (\pi, \lambda b. ((\tilde{x} b, \tilde{y} (\tau B (f r) b)), \pi_1 (f r) b)) : [R] \to [A \triangleleft B] | R|.
$$

**Proof.** Given a relation $R : X \times X \to \text{Type}$, by Lemma 4 there is an equivalence

$$
\prod_{(x,y) : X \times X} R (x,y) \to E_{(X,m)} R (x,y)
$$

$$
\simeq \prod_{(x,y) : X \times X} R (x,y) \to D_{(X,m)} R (x,y)
$$

$$
\simeq \sum_{a : [R] \to [A \triangleleft B] | R]} (m \circ \pi_0 \circ \pi_0 \sim [A \triangleleft B] (\pi_0 \circ \pi_0) \circ a)
$$

$$
\times (m \circ \pi_1 \circ \pi_0 \sim [A \triangleleft B] (\pi_1 \circ \pi_0) \circ a).
$$
The desired equivalence then follows by applying the equivalence above to the second component of the type $E_{(X,m)}\text{-}\text{Coalg}$. By chasing $f : \prod_{(x,y) : X \times X} R(x,y) \to E_{(X,m)} R(x,y)$ along the equivalence we see that it is mapped as stated.

**Proposition 2** (\textit{\& \&}). Given a $[A \triangleleft B]\text{-}\text{coalgebra} (X,m)$, let $e$ be the equivalence given by Proposition 1. Then for any $E_{(X,m)}\text{-}\text{coalgebras} (R,f)$ and $(R',f')$ there is an equivalence of types

$$\text{Hom}_{E_{(X,m)}\text{-}\text{Coalg}}(R,f)(R',f') \simeq \text{Hom}_{[A\triangleleft B]\text{-}\text{Bisim}_{(X,m)}}(e(R,f))(e(R',f')).$$  

**Proof.** Applying the equivalence $e$ given by Proposition 1 on $(R,f)$ and $(R',f')$, denote the components of the result as:

- $\alpha : |R| \to [A \triangleleft B]|R|,$
- $\alpha' : |R'| \to [A \triangleleft B]|R'|,$
- $H_0 : m \circ \pi_0 \circ \pi_0 \sim [A \triangleleft B](\pi_0 \circ \pi_0) \circ \alpha,$
- $H_0' : m \circ \pi_0 \circ \pi_0 \sim [A \triangleleft B](\pi_0 \circ \pi_0) \circ \alpha',$
- $H_1 : m \circ \pi_1 \circ \pi_0 \sim [A \triangleleft B](\pi_1 \circ \pi_0) \circ \alpha$
- $H_1' : m \circ \pi_1 \circ \pi_0 \sim [A \triangleleft B](\pi_1 \circ \pi_0) \circ \alpha'.$

Let $e'$ denote the equivalence given by Lemma 4. We have a chain of equivalences

$$\text{Hom}_{E_{(X,m)}\text{-}\text{Coalg}}(R,f)(R',f') \simeq \text{Hom}_{[A\triangleleft B]\text{-}\text{Coalg}}(R,e' \circ f)(R',e' \circ f')$$ (7)

$$\simeq \sum g : \prod_{(x,y) : X \times X} R(x,y) \to R'(x,y) K \circ \alpha' \circ \alpha \circ \left( H_0' \cdot K = H_0 \right) \circ \left( H_1' \cdot K = H_1 \right)$$ (8)

$$\simeq \sum g : \prod_{(x,y) : X \times X} R(x,y) \to R'(x,y) K \circ \alpha' \circ \alpha \circ \left( H_0' \cdot K = \text{tr}_{h.\text{refl}}^{\lambda h. h \sim [A\triangleleft B] h \circ \alpha} H_0 \right) \circ \left( H_1' \cdot K = \text{tr}_{h.\text{refl}}^{\lambda h. h \sim [A\triangleleft B] h \circ \alpha} H_1 \right)$$ (9)

$$\simeq \sum g : |R| \to |R'| : \sum p : \pi_0 \circ \pi_0 \circ \pi_0 = \pi_0 \circ \pi_0 \circ \pi_0 \circ \pi_0 \circ \pi_0 q : \pi_1 \circ \pi_0 \circ \pi_0 = \pi_1 \circ \pi_0 \circ \pi_0 \circ \pi_0 \circ \pi_0 \text{tot} [A\triangleleft B](\text{tot} g) \circ \alpha \circ \left( H_0' \cdot K = \text{tr}_{p.\text{refl}}^{\lambda h. h \sim [A\triangleleft B] h \circ \alpha} H_0 \right) \circ \left( H_1' \cdot K = \text{tr}_{q.\text{refl}}^{\lambda h. h \sim [A\triangleleft B] h \circ \alpha} H_1 \right)$$ (10)

$$\simeq \text{Hom}_{[A\triangleleft B]\text{-}\text{Bisim}_{(X,m)}}(e(R,f))(e(R',f'))$$ (11)

as desired.

Now we are ready to characterise the identity type on $M_{a,A}B a$ as an indexed M-type.

**Theorem 1** (\textit{\& \&}). The pair $(=,\gamma)$ is the terminal $E_{(M_{a,A}B a,\text{desup}_{A,B})}\text{-}\text{coalgebra}.$

**Proof.** Let $e$ be the equivalence given by Proposition 1. Then $e(=,f)$ is the terminal $[A \triangleleft B]\text{-}\text{coalgebra} \text{bisimulation}$ on $(M_{a,A}B a,\text{desup}_{A,B})$, by Corollary 1. The terminality of $(=,f)$ then follows by Proposition 2. 

**4 AFA and SAFA in \(\varepsilon\)-structures**

Most axioms of set theory, such as paring, union, separation and even infinity, replacement or powerset, are set existence axioms — they inform a student which sets they can construct within the theory. All the sets the student can construct from these axioms alone are wellfounded. Classically, wellfounded sets are those without an infinite membership chain:

$$a_0 \ni a_1 \ni a_2 \ni \cdots$$

Constructively, well-foundedness is instead formulated as an induction principle for $\varepsilon$ or using an accessibility predicate. In both constructive and classical traditions, the most prominent theories include an axiom which states that, all sets are wellfounded. This axiom is called regularity or foundation.
It’s a standard, classical result that the axiom of foundation is independent of the rest. What is more, under certain assumptions\(^\text{1}\) any structure defined by sets can be defined by well-founded sets.

When one removes the requirement that every material set must be well-founded, two questions arise:

1. Which non-wellfounded sets exist?
2. When are two non-wellfounded sets equal?

Anti-foundation axioms are properties of \(\in\)-structures which give answers to these two questions. In this text we consider two such axioms. The first is Aczel’s Anti-Foundation Axiom (AFA), and the second is Scott’s Anti-Foundation Axiom (SAFA). These answer the question slightly differently, and in this section we will try to capture the formulation of these in a way which generalises to \(\in\)-structures to higher type levels.

The second question arises because extensionality does not fully determine the equality between non-wellfounded sets. For instance, if two sets satisfy the equations \(x = \{x, y\}_0\) and \(y = \{x\}_0\), both \(x = y\) and \(x \neq y\) are possible – of course not in the same \(\in\)-structure. The 0-subscript on the pairing is crucial, because if we used multiset pairing, and let \(x = \{x, y\}_1\), it follows that \(x \neq y\), since a pair is never a singleton. This foreshadows the main thesis of this section, that the difference between Aczel’s and Scott’s conceptions of non-wellfounded sets is a matter of truncation level, from the perspective of HoTT.

In elementary terms, AFA states that given any graph there is a unique assignment of sets to the nodes of the graph, such that the elementhood relations between the assigned sets coincides with the edges of the graph. This both gives a way of constructing non-wellfounded sets (by giving a graph) and a way of proving equalities between non-wellfounded sets (showing that they can decorate the same node in a graph).

SAFA states that every graph where nodes have unique unfolding trees can be decorated with sets (in the same sense as in AFA) and that for sets isomorphism of unfolding trees determines equality. Additionally, the decoration is injective (since equality of nodes is determined by their unfolding trees) and is unique among such decorations. This may at the moment sound baroque and even ad hoc, but we will attempt to shed light on this.

Why all these graphs? An answer to this question comes from universal coalgebra. An \(\in\)-structure being, in general a coalgebra for the functor \(P^\infty\), and specifically a \(P^n\)-coalgebra in the case of \(n\)-level structures (cf. Gylterud and Stenholm, 2023, Theorem 3), the non-well founded sets come from coalgebra maps into the structures. In ordinary mathematics, a graph is exactly a coalgebra \(X \rightarrow P^0 X\). This emphasises looking at the out-edges from a node, and a coalgebra map into an \(\in\)-structure translates out-edges to elements. So, what we will call a decoration of a graph is precisely a coalgebra homomorphism from the induced coalgebra of the graph into the \(\in\)-structure the graph lives in.

\subsection*{4.1 Graphs and decorations}

Usually in mathematics, we think of graphs as structures consisting of nodes and edges. However, in the formulation of the anti-foundation axioms we will work with a slightly different notion of graph, as simply a set of pairs. This leaves the domain of nodes implicit, which simplifies the definition of a decoration.

\begin{definition} \((\subset\!\!\!\!\subset)\). In an \(\in\)-structure \((V, \in)\) with ordered pairing structure \((-,-)\), an element \(g : V\) is a graph if all its elements are pairs. That is, there is a map

\[
\prod_{e \in V} e \in g \rightarrow \sum_{(x,y) \in V \times V} e = \langle x, y \rangle,
\]

or equivalently, for every \(e : V\) such that \(e \in g\) there are source \(e : V\) and target \(e : V\) such that \(e = \langle \text{source } e, \text{target } e \rangle\).
\end{definition}

\begin{remark}
The notation “source \(e\)” and “target \(e\)” suppresses mention of the specific proof element of \(e \in V\) which is used to construct source \(e\) and target \(e\). However, this is justified since \(\sum_{(x,y) \in V \times V} e = \langle x, y \rangle\) is a proposition, and thus any choice of such proof object yields equal results.
\end{remark}

\begin{definition} \((\subset\!\!\!\!\subset)\). Given a graph \(g : V\) in a \(\in\)-structure \((V, \in)\) with ordered pairing structure \((-,-)\), define the type \(\text{Target } g\), the subtype of \(V\) consisting of targets of edges in \(g\), by \(\text{Target } g := \sum_{y \in V} \exists x : V . \langle x, y \rangle \in g\).
\end{definition}

\(^1\text{AC is more than sufficient, but the much milder axiom of well-founded materialisation is enough (cf. discussion in Shulman, 2010, after Lemma 6.46).}\)
Since the domain of nodes in the graph is left implicit, a decoration will be a universally defined function \( d : V \to V \), where the convention is that \( dx \) is empty if there are no edges \( (x, y) \in g \). When there is an edge \( (x, y) \) this edge should give rise to an elementhood relation \( dy \in dx \). In fact, there should for every \( z : V \) be an equivalence between \( z \in dx \) and the edges in \( (x, y) \in g \) for which \( z = dy \):

**Definition 13 (\( \mathcal{C} \mathcal{F} \).)** For \( n : \mathbb{N}_{\geq 1} \), an \((n+1)\)-decoration of a graph \( g : V \in \mathcal{C} \)-structure \((V, \varepsilon)\), with an ordered pairing structure \((\cdot, \cdot)\), is a map \( d : V \to V \) together with an element of the type

\[
\prod_{x, z : V} z \in dx \simeq \left\| \sum_{y : V} (x, y) \in g \times dy = z \right\|_n.
\]

The truncation level restricts the level of \( dx \), so that, for instance, in 0-level \( \varepsilon \)-structures \( dx \) will be a set. The notion of 0-decoration is equivalent to the classical notion of decoration as a function satisfying the equation \( d(x) = \{ (y) \mid (x, y) \in g \}_0 \) (cf. Aczel, 1988, Chapter 1). And, in terms of univalent material set theory\(^4\), an \( n \)-decoration is a function satisfying the equation \( d(x) = \{ (y) \mid (x, y) \in g \}_n \).

The notion of \( \infty \)-decoration is one where there is no truncation yielding simply:

\[
z \in dx \simeq \left( \sum_{y : V} (x, y) \in g \times dy = z \right).
\]

Intuitively it says that \( dy \) occurs in \( dx \) precisely as many times as \( (x, y) \) occurs in \( g \) (and that all elements of \( dx \) are of the form \( dy \)).

There are two simple observations we can make if we know the level of the \( \varepsilon \)-structure.

- In an \( n \)-level \( \varepsilon \)-structure, an \((n+1)\)-decoration is also an \( \infty \)-decoration since the type \( \sum_{y : V} (x, y) \in g \times dy = z \) has type level \( n \).
- In an \( n \)-level \( \varepsilon \)-structure, an \( \infty \)-decoration is also \( n \)-decoration, but the opposite is not always the case. For instance, in level 0, if \( d : V \to V \) is an \( \varepsilon \)-decoration, we know that \( \sum_{y : V} (x, y) \in g \times dy = z \) is a proposition since it is equivalent to \( z \in dx \) which is a proposition. Hence, the propositional truncation in the requirement for a 0-truncation is superfluous and \( d \) is also a 0-decoration. However, the graph \( g = \{ \langle a, b \rangle, \langle a, c \rangle \}_0 \) cannot have a \( \infty \)-decoration in any 0-level structure, if \( a, b \) and \( c \) are distinct, since \( db = dc = \emptyset \) and thus \( \emptyset \in da \simeq \{ \sum_{y : V} (a, y) \in g \times dy = \emptyset \} \simeq 2 \) which is not a proposition. But, being wellfounded, \( g \) has a 0-decoration, namely the one which assigns \( dx = \{ \emptyset \mid x = a \}_0 \).

At level 0, the \( \infty \)-decorations are the injective 0-decorations. This does not mean that \( d \) is injective on all of \( V \)—that would yield a contradiction—but rather that it becomes injective when restricted to the sets which are nodes in the graph (i.e. occurs in an edge). Classically, Scott’s axiom is formulated in terms of injective decorations, but we will instead use \( \infty \)-decorations as this generalises to higher type levels.

### 4.2 Coalgebraic characterisation of \( n \)-decorations

The \( \varepsilon \)-structures are the same as \( P^\infty \)-coalgebras, and the usual characterisation of decorations as coalgebra maps into \( V \) extends in our settings to coalgebra maps into \( P^\infty \). This is essentially what is proved in Proposition 4 below. However, to make characterisation convenient, either the functorial action must be adjusted for each \( n \), or the underlying structure must be of level \( n \). We opt to adjust the functorial action.

**Definition 14.** Let \( n : \mathbb{N}_{\geq 1} \), and define a wild functor \( P_n^\infty : \text{Type} \to \text{Type} \) by \( P_n^\infty X := \sum_{A : U} A \to X \) on types and on functions by \( P_n^\infty f(A, v) := (\text{image}_n(f \circ v), \text{incl}_n(f \circ v)) \).

**Remark:** Notice that \( P_n^\infty \) is like a hybrid between \( P^\infty \) and \( P^n \): Since \( P_n^\infty \) and \( P^\infty \) have the same action on types, a coalgebra for one is automatically a coalgebra for the other. On the other hand, if two \( P_n^\infty \)-coalgebras factor into \( P^n \)-coalgebras, the type of \( P_n^\infty \)-coalgebra homomorphisms is equivalent to the type of \( P^n \)-coalgebra homomorphisms. The following commutative diagram summarises the relationship between \( P^n \) and \( P_n^\infty \). The unnamed arrows are the \( n \)-image map and the inclusion of \( n \)-truncated functions into functions.

\(^4\)Gylterud and Stenholm (For discussion on \( n \)-truncated set comprehension and replacement, see 2023, Definitions 7 and 8).
Given Proposition 4.

For each graph \( g : V \rightarrow P_n^\infty V \) and \( d : V \rightarrow V \) there is an equivalence of types between \( d \) being an \( n \)-decoration of \( g \) and being a \( P_n^\infty \)-coalgebra homomorphism from \( m_g : V \rightarrow P_n^\infty V \) to \( m_e : V \rightarrow P_n^\infty V \). Hence, there is an equivalence of types between \( n \)-decorations of \( g \) and \( P_n^\infty \)-coalgebra homomorphisms from \( m_g \) to \( m_e \).

Let us for the rest of the subsection fix \( n : N^\infty \) and a \( U \)-like \( \in \)-structure \((V, \in)\) and its associated \( P_n^\infty \)-coalgebra structure \( m_{\in} : V \rightarrow P_n^\infty V \). Assume also that \( V \) is locally small and let \( x \approx y \) denote the small type equivalent to the identity type for \( x, y : V \).

If we have a graph in \( V \), there are several ways of constructing a coalgebra from it. Below, we define two closely related \( P_n^\infty \)-coalgebra structures: \( m_g : V \rightarrow P_n^\infty V \) and \( m_\varepsilon : \text{Target } g \rightarrow P_n^\infty (\text{Target } g) \), which will help characterise decorations and define Scott’s anti-foundation axiom.

**Proposition 3 \((\varepsilon g)\).** For each graph \( g : V \rightarrow P_n^\infty V \) such that \( \pi_0(m_g x) \simeq \sum_{y : V} \langle x, y \rangle \in g \) and \( \pi_1(m_g x) : \pi_0(m_g x) \rightarrow V \) becomes \( \pi_0 : \left( \sum_{y : V} \langle x, y \rangle \in g \right) \rightarrow V \) when transported along this equivalence.

**Proof.** Given \( x : V \) let \( m_g x := \left( \sum_{e : g} \text{source } (\tilde{g} e) \approx x, \text{target } (\tilde{g} e) \equiv y \right) \), and observe that:

\[
\sum_{e : \overline{g}} \text{source } (\tilde{g} e) \approx x \simeq \sum_{y : V} \sum_{e : \overline{g}} \left( \text{source } (\tilde{g} e) = x \right) \times \left( \text{target } (\tilde{g} e) = y \right) \\
\simeq \sum_{y : V} \sum_{e : \overline{g}} \left( \text{source } (\tilde{g} e) \right) \times \left( \text{target } (\tilde{g} e) \right) = \langle x, y \rangle \\
\simeq \sum_{y : V} \text{fiber } \tilde{g} \langle x, y \rangle \\
\equiv \sum_{y : V} \langle x, y \rangle \
\]

Note that the following diagram commutes:

\[
\begin{array}{ccc}
\sum_{e : \overline{g}} \text{source } (\tilde{g} e) \approx x & \xrightarrow{\simeq} & \sum_{y : V} \langle x, y \rangle \in g \\
\text{target } \circ \tilde{g} \circ \pi_0 & \downarrow & \pi_n \\
V & \rightarrow & V
\end{array}
\]

up to definitional equality. \( \square \)

**Remark:** Ignoring size issues, justified by Proposition 3, we will simply write:

\[ m_g x = \left( \sum_{y : V} \langle x, y \rangle \in g, \pi_0 \right). \]

This is clearer to read than coercing along an equivalence. A more careful treatment, without notational abuse, is found in the formalisation.

**Lemma 5 \((\varepsilon g)\).** If a graph \( g : V \) is an \( n \)-type in \((V, \in)\) \( (\text{i.e. } e \in g \text{ is an } n-1 \text{ type}) \) then \( \pi_1(m_g x) : \pi_0(m_g x) \rightarrow V \) is \((n-1)\)-truncated, and hence \( m_g \) factors into a \( P_n^\infty \)-coalgebra \( m_{g, \varepsilon} : V \rightarrow P_n^\infty V \).

**Proof.** The map \( \text{target } \circ \tilde{g} \circ \pi_0 \) is \((n-1)\)-truncated since, for any \( y : V \), we have an equivalence

\[
\text{fiber } (\text{target } \circ \tilde{g} \circ \pi_0) y \simeq \text{fiber } \pi_0 y \\
\simeq (x, y) \in g,
\]

and the last type is \((n-1)\)-truncated. \( \square \)

**Proposition 4.** For each graph \( g : V \rightarrow V \) and map \( d : V \rightarrow V \) there is an equivalence of types between \( d \) being an \( n \)-decoration of \( g \) and being a \( P_n^\infty \)-coalgebra homomorphism from \( m_g : V \rightarrow P_n^\infty V \) to \( m_e : V \rightarrow P_n^\infty V \). Hence, there is an equivalence of types between \( n \)-decorations of \( g \) and \( P_n^\infty \)-coalgebra homomorphisms from \( m_g \) to \( m_e \).
Proof. Given a graph \( g : V \) and a map \( d : V \to V \) we have the following chain of equivalences:

\[
(m \circ d \sim P^n_d \circ m_g) \simeq \prod_{x \in V} \text{fiber } (d x) \simeq \text{fiber } (\text{incl}_{n-1} (d \circ \text{target } g \circ \pi_0)) z
\]  

(18)

\[
\simeq \prod_{x \in V} \prod_{z \in d x} \sum_{(g, p) : \text{fiber } d z} \langle x, y \rangle \in g \bigg|_{n-1}
\]  

(19)

\[
\simeq \prod_{x \in V} \prod_{z \in d x} \sum_{y \in V} \langle x, y \rangle \in g \times dy = z \bigg|_{n-1}
\]  

(20)

\[\blacksquare\]

**Proposition 5 (\( \square \)).** For each graph \( g : V \), the coalgebra \( m_g \) restricts to \( \text{Target } g \). We will call this coalgebra structure \( n_g : \text{Target } g \to P^n (\text{Target } g) \) and the subtype inclusion \( \pi_0 : \text{Target } g \to V \) is a \( P^n \)-coalgebra homomorphism.

Proof. First, note that for any \( e : \mathcal{E} \), target \( (g e) \) lies in \( \text{Target } g \) as it is the child of source \( (g e) \). Thus let \( n_g(x, .) = \left( \sum_{x \in \mathcal{E}} \text{source } (g e) \approx x, (\lambda(e, _). (\text{target } (g e), .)) \right) \), for which we can check that \( \pi_0 \) is a \( P^n \)-coalgebra homomorphism:

\[
P^n \pi_0 (n_g (x, .))
\]

\[
= \left( \sum_{x \in \mathcal{E}} \text{source } (g e) \approx x, \pi_0 \circ (\lambda(e, _). (\text{target } (g e), .)) \right)
\]

(21)

\[
= \left( \sum_{x \in \mathcal{E}} \text{source } (g e) \approx x, (\lambda(e, _). \text{target } (g e)) \right)
\]

(22)

\[
= \left( \sum_{x \in \mathcal{E}} \langle x, y \rangle \in g, \pi_0 \right)
\]

(23)

\[
= m_g x
\]

(24)

\[
m_g (\pi_0 (x, .))
\]

(25)

\[\blacksquare\]

Remark: For \( n_g \), just as for \( m_g \), we will slightly abuse notation, justified by Proposition 5, and write:

\[
n_g (x, .) = \left( \sum_{y \in V} \langle x, y \rangle \in g, \lambda(y, e) \cdot (y, |(x, e)|) \right).
\]

Again, a more careful treatment is found in the formalisation.

### 4.3 Aczel’s anti-foundation axiom

Aczel’s anti-foundation axiom can now be formulated as generalised properties for any truncation level. We will demonstrate that if one could construct terminal coalgebras for the \( P^n \) functors, the resulting \( \varepsilon \)-structures would satisfy the generalised properties.

**Definition 15 (\( \square \)).** An \( \varepsilon \)-structure \( (V, \varepsilon) \), with an ordered pairing structure, has **Aczel n-anti-foundation** (n-AFA), for \( n : \mathbb{N}_0^\varepsilon \), if for every graph \( g : V \) the type of \( n \)-decorations of \( g \) is contractible. Equivalently, this can be split into two parts:

- **n-AFA1:** For every graph \( g : V \) the type of \( n \)-decorations of \( g \) is inhabited
- **n-AFA2:** For every graph \( g : V \) the type of \( n \)-decorations of \( g \) is a proposition.

The classical AFA axiom is equivalent to Aczel 0-anti-foundation, since 0-decorations are the usual decorations, and contractible is the HoTT way of saying “exists unique”.

As decorations are \( P^n \)-coalgebras, one type that would model AFA is the terminal \( P^n \)-coalgebra.

**Theorem 2 (\( \square \)).** Suppose \( (V, m) \) is the terminal \( P^n \)-coalgebra and that \( V \) is locally \( U \)-small. Then the induced \( \varepsilon \)-structure has Aczel n-anti-foundation.
Proof. It was shown in Gylterud and Stenholm (2023) that \((V, m)\) has an ordered pairing structure. Let \(g : V\) be a graph. By Proposition 4 we need to show that the type of \(P^n\)-coalgebra homomorphisms from the corresponding graph coalgebra, given by Proposition 3, into \((V, m)\) is contractible. But this follows from terminality of \((V, m)\).

4.4 Scott’s anti-foundation axiom

Recall that, classically, SAFA is the statement that every Scott extensional graph has a unique injective decoration and \(V\) itself is Scott extensional. A graph is defined as being Scott extensional if equality on the nodes is tree isomorphism of the corresponding unfolding trees. Note that two trees are isomorphic if there is an isomorphism between the children of the roots, such that the subtrees of each related pair of children are tree isomorphic. We can see this as the unfolding step in a \(P^\infty\)-bismulation.

The terminal \(P^\infty\)-coalgebra, \(V^\infty\), can be thought of as the type of trees, and the map induced by its terminality, \(\text{corec}^\infty(A, m) : A \to V^\infty\), is the unfolding of a coalgebra or graph into a tree (starting in a given node). Because of univalence, the identity type in \(V^\infty\) is precisely tree isomorphism. This means that we can express Scott extensionality for a graph as saying that \(\text{corec}^\infty(\text{Target} g, n_g)\) is an embedding. Every function in HoTT has an associated action on paths, which becomes an equivalence for an embedding. So, if \(\text{corec}^\infty(\text{Target} g, n_g)\) is an embedding, its action on paths of the graph provides an equivalence between equality in the graph and isomorphism of its unfolding trees.

On higher type levels, it is a bit strong to require an embedding. For instance, in multisets (which are the material set theory equivalent of groupoids), we would like to consider a graph like \(\{\emptyset, \emptyset\}\) as a Scott extensional representation of the complete binary tree. However, this tree has many non-trivial automorphisms in \(V^\infty\), which our single node, \(\emptyset\), does not have. An embedding would require nodes in the graph to come prefilled with these automorphisms, but in our models this is not required. We therefore define the notion of a graph being Scott \(n\)-extensional.

Definition 16 (\(\Downarrow^\infty\)). Given a graph \(g : V\) and \(n : N^\infty\), we say that \(g\) is Scott \((n + 1)\)-extensional if the tree unfolding map \(\text{corec}^\infty(\text{Target} g, n_g)\) is \(n\)-truncated.

Clearly, being Scott \(n\)-extensional implies being Scott \((n + 1)\)-extensional, and by the reasoning above, Scott \(0\)-extensional is the usual notion of Scott extensional in level 0 \(\in\)-structures. Furthermore, if the graph is a set level graph (meaning that \(\text{Target} g\) is a set and \(n_g\) factors through \(P^1\)), then it is automatically Scott 1-extensional.

We can now define Scott’s anti-foundation axiom for \(\in\)-structures of any type level.

Definition 17 (\(\Downarrow^\infty\)). A \(U\)-like \(\in\)-structure \((V, \in)\), with an ordered pairing structure, has Scott \(n\)-anti-foundation \((n\text{-SAFA})\), for \(n : N^\infty\), if the two properties \(n\text{-SAFA}_1\) and \(\text{SAFA}_2\) hold:

- \(n\text{-SAFA}_1\): Any Scott \(n\)-extensional graph \(g : V\) has an \(\in\)-decoration.
- \(\text{SAFA}_2\): For any graph \(g\) the type of \(\in\)-decorations is a proposition.

The classical notion of SAFA then corresponds to what is defined above as Scott 0-anti-foundation. \(\text{SAFA}_2\) is the same as \(\in\)-\(\text{AFA}_2\), and since being Scott \(\in\)-extensional is a vacuous requirement, we get that \(\in\)-SAFA is equivalent to \(\in\)-\(\text{AFA}\).

5 The coiterative hierarchy

The coiterative hierarchy is a dualisation of a specific construction of the iterative hierarchy (Gylterud, 2018). That construction starts with the type of all wellfounded trees and picks out the subset of those which are hereditarily sets (i.e. in each node each immediate subtree is unique). The coiterative hierarchy is constructed dually, starting from the type of all (possibly non-wellfounded) trees, and picking out those which are co-hereditarily sets. That is, no matter how far we go into the tree, in each node the immediate subtrees are always distinct.

In Univalent Material Set Theory (Gylterud and Stenholm, 2023), the construction of an iterative hierarchy of sets was extended to a hierarchy of \(n\)-types, \(V^n\). When dualising to coiterative sets we will keep this level of generality and construct a coiterative hierarchy of \(n\)-types, \(V_n^\infty\). The first level, \(V_0^\infty\) is then the coiterative sets.

The iterative hierarchy was carved out from the \(W\)-type \(V^\infty := W_{A_0} A\), as a subtype, using an inductive predicate is-\(n\)-type : \(V^n \to \text{Type}\). The coiterative hierarchy will, dually, be carved out as a subtype from the \(M\)-type, \(V_0^\infty := M_{A_0} A\), and a coinductive predicate is-coi\(n\)-type : \(V_0^\infty \to \text{Type}\).
Figure 1: This tree represents an iterative set: \(\{\{\emptyset, \emptyset\}_0, \emptyset\}_0\).

Figure 2: This tree does not represent an iterative set because the left child of the root has two equal children. It does however represent the iterative multiset: \(\{\{\emptyset, \emptyset\}_1, \emptyset\}_1\).

Figure 3: The full binary tree is not a coiterative set. But rather a multiset \(b = \{b, b\}_1\).

Figure 4: This infinite binary tree represents the set \(x\) which is part of the solution to the equations \(x = \{x, y\}_0\) and \(y = \{x\}_0\).

**Definition 18**. For \(n : \mathbb{N} - 2\), define the predicate:

\[
\begin{align*}
\text{is-coit-(n+1)-type} & : \mathbb{N} \to V_\infty^\infty \to \text{Type} \\
\text{is-coit-(n+1)-type}_0 x & := \text{is-n-trunc-map } \tilde{x} \\
\text{is-coit-(n+1)-type}_{\text{succ } k} x & := \prod_{a : \mathbb{N}} \text{is-coit-(n+1)-type}_k (\tilde{x} a).
\end{align*}
\]

**Proposition 6**. The type \(\text{is-coit-n-type}_k x\) is a proposition for any \(n, k\) and \(x : V_\infty^\infty\).

**Proof**. This follows by induction on \(k\) and the fact that being an \((n-1)\)-truncated map is a proposition.

A coiterative \(n\)-type is then a tree which is a coiterative \(n\)-type at every level.

**Definition 19**. For \(n : \mathbb{N} - 1\), define the predicate:

\[
\begin{align*}
\text{is-coit-n-type} & : V_\infty^\infty \to \text{Type} \\
\text{is-coit-n-type} x & := \prod_{k : \mathbb{N}} \text{is-coit-n-type}_k x
\end{align*}
\]

**Proposition 7**. The type \(\text{is-coit-n-type} x\) is a proposition for any \(x : V_\infty^\infty\).

**Proof**. A family of propositions is again a proposition.

Now we can define the type of coiterative \(n\)-types.

**Definition 20** (The coiterative hierarchy). For \(n : \mathbb{N} - 1\), let \(V_\infty^n\) denote the type of coiterative \(n\)-types:

\[
V_\infty^n := \sum_{x : V_\infty^\infty} \text{is-coit-n-type}\_x.
\]

**Proposition 8**. \(V_\infty^n\) is a subtype of \(V_\infty^\infty\), i.e. there is an embedding \(V_\infty^n \hookrightarrow V_\infty^\infty\).

In particular, the identity type on \(V_\infty^n\) is the same as the identity type on \(V_\infty^\infty\).
5.1 $V^n_\infty$ is a fixed point for $P^n$

The elements in $V^n_\infty$ are non-wellfounded trees where all branchings are $(n-1)$-truncated maps. So when one removes the root from a tree, one gets a small type and an $(n-1)$-truncated map from that type into $V^n_\infty$. Similarly, if one has a small type and an $(n-1)$-truncated map from that type into $V^n_\infty$ then one can construct a tree in $V^n_\infty$. Hence, we will show that $V^n_\infty$ is a fixed point to $P^n$.

Lemma 6 (§$\text{V}'$). For any $x : V^n_\infty$, there is an equivalence

$$\text{is-coit-}n\text{-type } x \simeq \text{is-n-trunc-map } \bar{x} \times \prod_{a \in \mathbb{F}} \text{is-coit-}n\text{-type } (\bar{x} a).$$

Proof. Follows by induction over $\mathbb{N}$.

Theorem 3 (§$\text{V}'$). $V^n_\infty$ is a fixed point for $P^n$.

Proof. Since $V^n_\infty$ is the terminal $P^\infty$-coalgebra, it is in particular a fixed point for $P^\infty$.

Let $x : V^n_\infty$, then by Lemma 6, the element $(\bar{x}, \bar{x})$ lies in $P^n V^n_\infty$. By the same token, given $A : U$ and $f : A \hookrightarrow_{n-1} V^n_\infty$, the element $\text{sup}^\infty(A, f)$ is a coiterative $n$-type.

5.2 Non-terminality of $V^0_\infty$ as a $P^0$-coalgebra

Even though $V^n_\infty$ is a fixed point for $P^n$ and is a subtype of the terminal $P^\infty$-coalgebra, it turns out to not be the terminal $P^n$-coalgebra. At least $V^0_\infty$ is not the terminal $P^0$-coalgebra. But we conjecture this result to hold for all $n$. This is surprising since the dual construction gives the initial algebra of $P^n$ (Theorem 15 of Gylterud and Stenholm (2023)). Intuitively, the reason is that in the wellfounded setting tree isomorphism coincides with bisimulation, while in the non-well-founded setting it does not.

For $V^0_\infty$ to be terminal, any graph (considered as a $P^0$-coalgebra) should have a unique representative in $V^0_\infty$. But $V^0_\infty$ contains more than one representative of some graphs, i.e. we can construct a $P^0$-coalgebra for which there are two distinct $P^0$-coalgebra homomorphisms into $V^0_\infty$. One of the maps sends each node to its unfolding tree. Because the functorial action of $P^n$ takes the $(n-1)$-image of the composite map, i.e. it collapses some structure, there is also a $P^n$-coalgebra homomorphism which maps the nodes to another tree.

Theorem 4. $V^0_\infty$ is not the terminal $P^0$-coalgebra.

Proof. Consider the following $P^0$-coalgebra $(X, m)$, represented as a graph:

```
\begin{center}
\begin{tikzpicture}
\node (x) at (0,0) {$x$};
\node (y) at (1,0) {$y$};
\draw (x) -- (y);
\end{tikzpicture}
\end{center}
```

The unfolding trees of the two nodes as given by $\text{corec}^\infty(X, m) : X \to V^\infty_\infty$ are distinct, so $\text{corec}^\infty(X, m)$ factors as a $P^0$-coalgebra homomorphism, $f : X \to V^0_\infty$, from $(X, m)$ to $(V^0_\infty, \text{desup}_0)$, such that $fx \neq fy$.

On the other hand, let $g$ be the map that sends both nodes to the infinite unary tree, which we will denote $g : V^0_\infty$. 

```
\begin{center}
\begin{tikzpicture}
\node (x) at (0,0) {$\cdot$};
\node (y) at (1,0) {$\cdot$};
\node (z) at (2,0) {$\cdot$};
\node (w) at (3,0) {$\cdot$};
\draw (x) -- (y);
\draw (y) -- (z);
\draw (z) -- (w);
\end{tikzpicture}
\end{center}
```
Clearly, \( g \) is also a \( P^0 \)-coalgebra homomorphism:
\[
P^0 g (m x) = (\text{image} (g \circ \bar{\pi}), \text{incl}) = (1, \lambda \bar{x} q) = (\bar{\eta}, \bar{q}) = \text{desup}_0 (g x)
\]
and likewise for \( y \):
\[
P^0 g (m y) = (\text{image} (g \circ \bar{\pi}), \text{incl}) = (1, \lambda \bar{x} q) = (\bar{\eta}, \bar{q}) = \text{desup}_0 (g y).
\]

However, since \( f x \neq f y \) and \( g x = g y \), we get that \( f \) and \( g \) are two distinct \( P^0 \)-coalgebra homomorphisms from \((X, m)\) to \((V^0_\infty, \text{desup}_0)\). \( \square \)

5.3 Local smallness of \( V^n \)

The functorial action of \( P^n \) takes the \( n \)-image of a map. In order for this to be small, the domain must be small and the codomain appropriately locally small. In particular, when we are considering maps into \( V^n_\infty \), we use the fact that this type is locally small, as we will show in this section. This result uses univalence and follows from the characterisation of the identity on an \( M \)-type as an indexed \( M \)-type.

The idea is that, by univalence, the indexed functor \( E_{(V^n_\infty, \text{desup})} \) is equivalent to the indexed functor \( E'_{(V^n_\infty, \text{desup})} \), for which the corresponding indexed \( M \)-type is small.

**Definition 21 (\( \mathcal{C}f \)).** Given \( X : \text{Type} \) and \( m : X \rightarrow \left( \sum_{A : \text{Type}} A \rightarrow X \right) \), define the following functors
\[
E'_{(X, m)} : (X \times X \rightarrow \text{Type}) \rightarrow (X \times X \rightarrow \text{Type})
\]
\[
E'_{(X, m)} R (x, y) := \sum_{e : \bar{x} \in \bar{x}} \prod_{a, \bar{a}} R (\bar{x} a, \bar{y} (e a)).
\]

**Proposition 10 (\( \mathcal{C}f \)).** Given \( X : \text{Type} \) and \( m : X \rightarrow \left( \sum_{A : \text{Type}} A \rightarrow X \right) \), there is a natural family of equivalences
\[
E_{(X, m)} R (x, y) \simeq E'_{(X, m)} R (x, y).
\]

**Proof.** Follows by univalence. \( \square \)

This gives us an alternative characterisation of the identity type on \( V^n_\infty \).

**Theorem 5 (\( \mathcal{C}f \)).** The identity type on \( V^n_\infty \) is the terminal \( E' \)-coalgebra.

**Proof.** By Theorem 1, the identity type on \( V^n_\infty \) is the terminal \( E \)-coalgebra. Since the functors \( E \) and \( E' \) are naturally equivalent by Proposition 10, the identity type is also the terminal coalgebra for \( E' \). \( \square \)

Note that by the theorem above, there is for any \( x, y : V^n_\infty \) an equivalence
\[
(x = y) \simeq \sum_{\bar{x} \sim \bar{y}} \prod_{a, \bar{a}} \bar{x} a = \bar{y} (e a).
\]

**Theorem 6 (\( \mathcal{C}f \)).** \( V^n_\infty \) is locally \( U \)-small.

**Proof.** Since \( E'_{(V^n_\infty, \text{desup}^\infty)} \) is an indexed polynomial functor, it has a corresponding indexed \( M \)-type which is the terminal \( E'_{(V^n_\infty, \text{desup}^\infty)} \)-coalgebra. In their paper on non-wellfounded trees in HoTT, Ahrens, Caprriott, and Spadotti (2015) constructed indexed \( M \)-types from inductive types. From this construction one can observe that the universe level of the constructed indexed \( M \)-type does not depend on the indexing type. In our case, the universe level of the indexed \( M \)-type corresponding to \( E'_{(V^n_\infty, \text{desup}^\infty)} \) is the least upper bound of the universe levels of \( \underline{\Pi} \simeq \underline{\alpha} \) and \( \underline{\Pi} \), which is \( U \).

Since (the carrier of) any two terminal \( E'_{(V^n_\infty, \text{desup}^\infty)} \)-coalgebras are equivalent, it follows that \( V^n_\infty \) is locally \( U \)-small. \( \square \)

**Corollary 2 (\( \mathcal{C}f \)).** \( V^n_\infty \) is locally \( U \)-small.

**Proof.** By Proposition 8, \( V^n_\infty \) is a subtype of \( V^n_\infty \) and thus has the same identity type. The result then follows from the fact that \( V^n_\infty \) is locally \( U \)-small, by Theorem 6. \( \square \)
5.4 \( V_n^\infty \) is a simple \( P^\infty \)-coalgebra

The first requirement to satisfy SAFRA is that the type of \( \infty \)-decorations is a proposition. By the characterisation of \( \infty \)-decorations as \( P^\infty \)-coalgebra homomorphisms, it is sufficient for the model to be a simple \( P^\infty \)-coalgebra. Thus, we show this for \( V_n^\infty \).

Note that we do not prove that \( V_n^\infty \) is simple as a \( P^n \) coalgebra. In fact, the proof of non-terminality of \( V_n^\infty \) demonstrates that it is not simple as a \( P^n \) coalgebra.

**Definition 22.** Let \( X \) and \( Y \) be types, and let \( f : X \to Y \). Given a binary relation \( R : Y \times Y \to \text{Type} \)

we define a binary relation on \( X \):

\[
Rf : X \times X \to \text{Type} \\
Rf(x, x') := R(f x, f x').
\]

Moreover, given a fiberwise map \( g : \prod_{(y, y')} Y \times Y R(y, y') \to R'(y, y') \) between relations \( R \) and \( R' \), define the fiberwise map:

\[
gf : \prod_{(x, x')} X \times X Rf(x, x') \to R'f(x, x')
\]

\[
gf(x, x') := g(f x, f x').
\]

**Proposition 11 (\( \text{\text{□}} \)).** Let \((X, m)\) and \((Y, n)\) be \( P^\infty \)-coalgebras and let \( (f, \alpha) \) be a \( P^\infty \)-coalgebra homomorphism from \((X, m)\) to \((Y, n)\). For any binary relation \( R : Y \times Y \to \text{Type} \) and every pair of elements \( x, x' : X \), there is an equivalence

\[
e_R : E_{(Y, n)}^f R(f x, f x') \simeq E_{(X, m)}^\alpha (Rf)(x, x').
\]

This family of equivalences is natural, i.e., for every fiberwise map \( g : \prod_{(y, y')} Y \times Y R(y, y') \to R'(y, y') \) the following diagram commutes:

\[
\begin{array}{ccc}
E_{(Y, n)}^f R(f x, f x') & \xrightarrow{e_R} & E_{(Y, n)}^{g f} R'(f x, f x') \\
\downarrow & & \downarrow \\uparrow{e_R'} \\
E_{(X, m)}^\alpha (Rf)(x, x') & \xrightarrow{E_{(X, m)}^{g f}(x, x')} & E_{(X, m)}^{g f} R'(f x, f x')
\end{array}
\]

Moreover, for equality we have

\[
e_{\equiv} = (\text{id-equiv, refl-htpy}) = (\text{id-equiv, refl-htpy}).
\]

**Proof.** For \( x, x' : X \), the two types are

\[
E_{(Y, n)}^f R(f x, f x') \equiv \sum_{e : \exists \bar{x} : T} \prod_{x : \bar{x} \mapsto T} R((\bar{f} x) \alpha, (\bar{f} x') \alpha(e a)), \quad (26)
\]

\[
E_{(X, m)}^\alpha (Rf)(x, x') \equiv \sum_{e : \exists \bar{x} : T} \prod_{x : \bar{x} \mapsto T} R(f (\bar{x} a), f (\bar{x}' \alpha(e a))), \quad (27)
\]

Note that we have paths

\[
\alpha x : (\bar{f} x, \bar{f} x) = (\bar{x}, f \circ \bar{x}), \quad (28)
\]

\[
\alpha x' : (\bar{f} x', \bar{f} x') = (\bar{x}', f \circ \bar{x}'). \quad (29)
\]

The desired equivalence is thus given by transporting along these paths. Naturality follows from the fact that transport preserves families. The action of \( e_{\equiv} \) on \( (\text{id-equiv, refl-htpy}) \) follows by path induction. \( \square \)

**Proposition 12 (\( \text{\text{□}} \)).** Let \((X, m)\) and \((Y, n)\) be \( P^\infty \)-coalgebras and let \( (f, \alpha) \) be a \( P^\infty \)-coalgebra homomorphism from \((X, m)\) to \((Y, n)\). Given an \( E_{(X, m)}^\alpha \)-coalgebra \((R, \sigma)\), we can define an \( E_{(Y, n)}^f \)-coalgebra \((f R, f \sigma)\).
Proof. Intuitively, \( f R \) relates any two elements which are the images of two related elements in \( X \). Formally, define the relation \( f R : Y \times Y \to \text{Type} \) by

\[
f R(y, y') := \sum_{(x, \cdot): \text{fiber } y} \sum_{(x', \cdot): \text{fiber } y'} R(x, x').
\]

For \( f \sigma \), it is enough by path induction to construct an element

\[
f \sigma (f x, f x')((x, \text{refl}), (x', \text{refl}), r) := \sum_{e: f x = f x'} \prod_{a: f x} f R((\tilde{x} a), (\tilde{x'} a) (e a)).
\]

Using the (inverse of the) equivalence \( e_{f R} \) in Proposition 11, it is enough to construct an element of the type

\[
\sum_{e: f x = f x'} \prod_{a: f x} f R(f (\tilde{x} a), f (\tilde{x'} (e a))).
\]

For this we take

\[
(\pi_0 (\sigma r), \lambda a.((\tilde{x} a, \text{refl}), (\tilde{x'} (e a), \text{refl}), \pi_1 (\sigma r))).
\]

Proposition 13 \( (\bigodot') \). Let \( (X, m) \) and \( (Y, n) \) be \( P^\infty \)-coalgebras and let \( (f, \alpha) \) be a \( P^\infty \)-coalgebra homomorphism from \( (X, m) \) to \( (Y, n) \). Let \( (R, \sigma) \) be an \( E(X, m) \)-coalgebra and let \( (S, \phi) \) be an \( E(Y, n) \)-coalgebra. There is an equivalence

\[
\hom((R, \sigma), (S f, e_S \circ \phi)) \simeq \hom((f R, f \sigma), (S, \phi)),
\]

where \( e_S \) is the equivalence given in Proposition 11.

Proof. Define the map

\[
h : \prod_{(x, x') : X \times X} R(x, x') \to f R(f x, f x')
\]

\[
h(x, x') r := ((x, \text{refl}), (x', \text{refl}), r).
\]

We have the following chain of equivalences

\[
\hom((R, \sigma), (S f, e_S \circ \phi))
\]

\[
\equiv \sum_{g: \prod_{(x, x') : X \times X} R(x, x') \to S(f x, f x')} \prod_{(x, x') : X \times X} e_S(x, x') \circ \phi(f x, f x') \circ g(x, x') \sim E'(X, m) \circ g(x, x') \circ \sigma(x, x')
\]

\[
\equiv \sum_{g: \prod_{(x, x') : X \times X} R(x, x') \to S(f x, f x')} \prod_{(x, x') : X \times X} \phi(f x, f x') \circ g(x, x') \sim e_S^{-1}(x, x') \circ E'_{(X, m)} \circ g(x, x') \circ \sigma(x, x')
\]

\[
\equiv \sum_{g: \prod_{(y, y') : Y \times Y} R(y, y') \to S(y, y')} \prod_{(y, y') : Y \times Y} \phi(f x, f x') \circ g f(x, x') \circ h(x, x') \sim e_S^{-1}(x, x') \circ E'_{(X, m)}(\lambda(x, x'), g f(x, x') \circ h(x, x'))(x, x') \circ \sigma(x, x')
\]

\[
\equiv \sum_{g: \prod_{(y, y') : Y \times Y} R(y, y') \to S(y, y')} \prod_{(y, y') : Y \times Y} \phi(f x, f x') \circ g f(x, x') \circ h(x, x') \sim e_S^{-1}(x, x') \circ E'_{(X, m)}(g f(x, x') \circ E'_{(X, m)} h(x, x') \circ \sigma(x, x'))
\]

\[\square\]
\[ \sum_{g \Pi(y, y') \times Y \times Y} f R(y, y') \rightarrow S(y, y') \]
\[ \prod_{(x, x') \in X \times X} \phi (f(x, x') \circ g(f(x, x') \circ h(x, x')) \]
\[ \sim E_{(Y, n)} g(f(x, x') \circ f\lambda (x, x') \circ h(x, x')) \]
\[ \equiv \sum_{g \Pi(y, y') \times Y \times Y} f R(y, y') \rightarrow S(y, y') \]
\[ \prod_{(x, x') \in X \times X} \phi (f(x, x') \circ g(f(x, x') \circ h(x, x')) \]
\[ \sim E_{(Y, n)} g(f(x, x') \circ f\sigma (f(x, x') \circ h(x, x')) \]
\[ \equiv \hom((f R, f \sigma), (S, \phi)) \]

Step (34) uses the naturality of the equivalence in Proposition 11.

**Corollary 3** \((\mathcal{C}^\alpha)^\ast\). Let \((X, m)\) and \((Y, n)\) be \(P^\infty\)-coalgebras and let \((f, \alpha)\) be a \(P^\infty\)-coalgebra homomorphism from \((X, m)\) to \((Y, n)\). Suppose that equality on \(Y\) is the terminal \(E_{(Y, n)}\)-coalgebra. Then equality on \(X\) is the terminal \(E_{(X, m)}\)-coalgebra if and only if \(f\) is an embedding.

**Proof.** Let \(m\) be the \(E_{(X, m)}\)-coalgebra map for equality on \(X\) and let \(m'\) be the \(E_{(Y, n)}\)-coalgebra map for equality on \(Y\). Let \(e\) be the equivalence given in Proposition 11, for equality on \(Y\). Note that since \(e((\text{id-equiv, refl-htpy}) = (\text{id-equiv, refl-htpy})\), \(ap_{f}\) is an \(E_{(X, m)}\)-coalgebra homomorphism from \((=_{X}, m)\) to \((=_{Y}, m)\).

For any \(E_{(X, m)}\)-coalgebra \((R, \sigma)\), we have an equivalence
\[ \hom((R, \sigma), (\lambda(x, x'), f x = f x', m) \cong \hom((f R, f \sigma), (=_{Y}, m')) \]

given by Proposition 13. By the terminality of \((=_{Y}, m')\) it therefore follows that \((\lambda(x, x'), f x = Y, f x', m)\) is the terminal \(E_{(X, m)}\)-coalgebra.

Since \(ap_{f}\) is an \(E_{(X, m)}\)-coalgebra homomorphism from \((=_{X}, m)\) to \((=_{Y}, m)\) and the terminal \(E_{(X, m)}\)-coalgebra is unique up to unique isomorphism, it follows that \(ap_{f}\) is a family of equivalences if and only if \((=_{X}, m)\) is the terminal \(E_{(X, m)}\)-coalgebra.

**Corollary 4.** \((V_{\infty}^{n}, \text{desup}^{n})\) is a simple \(P^\infty\)-coalgebra.

**Proof.** \(V_{\infty}^{n}\) embeds into \(V_{\infty}^{\infty}\), hence it satisfies the conditions of Corollary 3. \(E_{(V_{\infty}^{n}, \text{desup}^{n})}\)-coalgebras are equivalent to \(P^\infty\)-bisimulations on \((V_{\infty}^{n}, \text{desup}^{n})\), so it follows that \((V_{\infty}^{n}, \text{desup}^{n})\) is a bisimulation simple \(P^\infty\)-coalgebra, hence simple.

One can also use Corollary 3 to replace the condition in 0-SAFA\(_1\) with \(g\) being simple:

**Corollary 5.** Given an \(\varepsilon\)-structure \((V, \varepsilon)\), a graph \(g : V\) being Scott 0-extensional is equivalent to \((\text{Target}_{g}, n_{g})\) being a simple \(P^\infty\)-coalgebra.

### 5.5 Coalgebra homomorphisms into \(V_{\infty}^{n}\)

How do we construct a map from a \(P^\infty\)-coalgebra, say \((X, m)\), into \(V_{\infty}^{n}\)? An obvious approach is to view \((X, m)\) as a \(P^\infty\)-coalgebra and show that \(\text{corec}^{\infty} : X \rightarrow V_{\infty}^{\infty}\) lands in \(V_{\infty}^{n}\), where \(\text{corec}^{\infty}\) is the underlying map of the unique \(P^\infty\)-coalgebra homomorphism from \((X, m)\) to \((V_{\infty}^{\infty}, \text{desup}^{\infty})\). Unfortunately, this is not always the case.

Viewing \((X, m)\) as a graph, \(\text{corec}^{\infty}\) maps each node to its unfolding tree. Consider now the \(P^{0}\)-coalgebra represented by the following graph:
The topmost node is mapped by corec∞ to the tree

which is not an element of V0 as the branching at the root is not an embedding.

However, if corec∞ is an (n − 1)-truncated map, then it lands in V∞.

**Proposition 14**. Given a P^n-coalgebra (X, m), if corec∞ : X → V∞ is an (n − 1)-truncated map, then for all x : X, corec∞ x is a coiterative n-type.

**Proof.** For x : X we need to show that

\[ \prod_{k \in \mathbb{N}} \text{is-coit-n-type}_k (\text{corec}^\infty x) . \]

Proceed by induction on k.

For the base case, note that since corec∞ is a P∞-coalgebra homomorphism, we have

\[ (\text{corec}^\infty x) = \text{corec}^\infty \circ \tilde{x} . \]

Both these maps are (n − 1)-truncated, and therefore the composition is (n − 1)-truncated.

Similarly, for the induction step, since corec∞ is a homomorphism, it is enough to show that

\[ \prod_{a \in \mathcal{T}} \text{is-coit-n-type}_k (\text{corec}^\infty (\tilde{x} a)). \]

But this follows from the induction hypothesis. □

**Definition 23**. Given a P^n-coalgebra (X, m) for which corec∞ is an (n − 1)-truncated map, let

\[ \text{corec}^n : X \to V_n^\infty \]

denote the restriction of corec∞ into V^n by Proposition 14.

The map corec^n is a P^n-coalgebra homomorphism. This is an instance of a useful lemma about which maps into V^n are P^n-coalgebra homomorphisms.

**Lemma 7**. Let (X, m) be a P^n-coalgebra and let f : X → V_n^\infty. Then there is an equivalence of types between f being a P^n-coalgebra homomorphism and \( \pi_0 \circ f \) being a P^∞-coalgebra homomorphism.

**Proof.** Consider the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & V_n^\infty \\
m \downarrow & & \downarrow \pi_0 \\
P^n X & \xrightarrow{P^n f} & P^n V_n^\infty \\
\downarrow & & \downarrow \text{desup}^n \\
P^\infty X & \xrightarrow{P^\infty f} & P^\infty V_\infty \\
\end{array}
\]

\[
\begin{array}{ccc}
& & V^\infty \\
\pi_0 \downarrow & & \downarrow \text{desup}^\infty \\
P^n V_n^\infty & \xrightarrow{\text{desup}^n} & P^\infty V_\infty \\
\end{array}
\]

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The map from $P^n V^n_\infty$ to $P^\infty V^n_\infty$ is an embedding as it simply forgets that the map in the second coordinate is $(n-1)$-truncated. Additionally, $P^\infty \pi_0$ is an embedding since $\pi_0$ is an embedding. Thus, it is equivalent to show that the upper left square commutes, and showing that the two maps are equal when postcomposed with the forgetful map and $P^\infty \pi_0$.

The square on the right commutes as the inclusion $V^n_\infty$ into $V^\infty_\infty$ is a $P^\infty$-coalgebra homomorphism (Proposition 9), and since $f$ is an $(n-1)$-truncated map, the lower left square also commutes. It therefore follows that there is an equivalence between the upper left square commuting and the outer square commuting.

**Proposition 15** (\(\mathfrak{C}/\)). Let $(X, m)$ be a $P^n$-coalgebra for which $\text{corec}^{\infty}$ is an $(n-1)$-truncated map, then $\text{corec}^n$ is an $(n-1)$-truncated map, and it is a $P^n$-coalgebra homomorphism into $(V^n_\infty, \text{desup}^n)$.

Proof. Since $\text{corec}^{\infty}$ is an $(n-1)$-truncated map and $\pi_0 : V^n_\infty \to V^\infty_\infty$ is an embedding, it follows that $\text{corec}^n$ is an $(n-1)$-truncated map. By Lemma 7, since $\text{corec}^{\infty} \equiv \pi_0 \circ \text{corec}^n$ is a $P^\infty$-coalgebra homomorphism, it follows that $\text{corec}^n$ is a $P^n$-coalgebra homomorphism.

Even though $(V^n_\infty, \text{desup}^n)$ is not the terminal $P^n$-coalgebra, it is *almost* terminal — it is terminal with respect to truncated maps.

**Theorem 7** (\(\mathfrak{C}/\)). Let $(X, m)$ be a $P^n$-coalgebra for which $\text{corec}^{\infty}$ is an $(n-1)$-truncated map. Then the following type is contractible:

\[
\sum_{(f, \alpha) : \text{Hom}_{P^n -\text{coalg}}(X, m) (V^n_\infty, \text{desup}^n)} \text{is-(n-1)-trunc-map } f
\]

Proof. First we note that by Lemma 7, the type of $P^n$-coalgebra homomorphisms from $(X, m)$ to $(V^n_\infty, \text{desup}^n)$ for which the underlying map is $(n-1)$-truncated, is a subtype of the type of $P^\infty$-coalgebra homomorphisms from $(X, m)$ to $(V^\infty_\infty, \text{desup}^\infty)$. Specifically, we have the following chain of equivalences and embeddings:

\[
\sum_{f : X \to V^n_\infty} \text{desup}^n \circ f \sim P^n f \circ m \simeq \sum_{f : X \to V^n_\infty} \text{desup}^\infty \circ \pi_0 \circ f \sim P^\infty (\pi_0 \circ f) \circ m \quad (38)
\]

\[
\iff \sum_{f : X \to V^n_\infty} \text{desup}^\infty \circ \pi_0 \circ f \sim P^\infty (\pi_0 \circ f) \circ m \quad (39)
\]

\[
\iff \sum_{f : X \to V^\infty_\infty} \text{desup}^\infty \circ f \sim P^\infty f \circ m \quad (40)
\]

The last step is an instance of the fact that embeddings are monomorphisms.

By Proposition 15, the first type in the chain above is inhabited. Since any inhabited type which embeds into a proposition is contractible, it follows that the first type is contractible.

Note that this does not contradict the counter example to terminality above since the second map in that case is not an embedding.

## 6 The coiterative hierarchy as a model of set theory

A result that dates back to the fifties is that any fixed point of the powerset functor is a model of ZFC\(^-\) (ZFC without foundation/regularity) (Rieger, 1957). In a previous paper by two of the authors (Gylterud and Stenholm, 2023) a corresponding result was shown for models of set theory in HoTT — the powerset functor in this case being $P^0$. Specifically, a fixed point of $P^0$ in HoTT is a model of

- empty set,
- unordered pairing,
- restricted separation,
- replacement,
- union,
- exponentiation,
- infinity/natural numbers.
In fact, natural higher type level generalisations of these axioms were defined, and it was shown that fixed points of $P^n$ satisfy the axioms at level $n$ or less\(^\text{§}\) (Gylterud and Stenholm, 2023, Section 5). Moreover, the type $V^n$ was shown to be the initial algebra of the functor $P^n$ and as such was shown to model the axiom of foundation, in addition to the ones above.

Since $V^n_\infty$ is a fixed point of $P^n$ it is also a model of the axioms above. However, since it is not the initial algebra, it is not a model of foundation. Neither is it the terminal coalgebra, and thus not a model of Aczel’s anti-foundation axiom. In this section we will show that it is instead a model of Scott’s anti-foundation axiom.

The definition of the elementhood relation on $V^n_\infty$ is the one which is induced by its coalgebra structure.

**Definition 24.** For $x, y : V^n_\infty$, define the elementhood relation between them as

$$x \in_n y := \text{fiber} \tilde{y} x.$$  

The relation $\in_n$ is extensional: the canonical map

$$x = y \rightarrow \prod_{z : V^n_\infty} z \in_n x \simeq z \in_n y$$

is an equivalence. A type with an extensional binary relation in this sense is what is called an $\in$-structure in Gylterud and Stenholm (2023). We will use the definitions of the properties corresponding to the set theoretic axioms defined there.

The following result is an instance of the results in Section 5 of Gylterud and Stenholm (2023), which shows that a (locally small) fixed point of $P^n$ models all defined properties except foundation.

**Theorem 8.** For each $n : \mathbb{N}$, $(V^n_n, \in_n)$ satisfies the following properties, as defined in Gylterud and Stenholm (2023):

- empty set,
- $U$-restricted $n$-separation,
- $\omega$-unordered $I$-tupling, for all $k : \mathbb{N} - 1$ such that $k < n$ and $k$-types $I : U$,
- $k$-unordered $I$-tupling, for all $k : \mathbb{N} - 1$ such that $k \leq n$ and $I : U$,
- $k$-replacement, for all $k : \mathbb{N} - 1$ such that $k \leq n$,
- $k$-union, for all $k : \mathbb{N} - 1$ such that $k \leq n$,
- exponentiation, for any ordered pairing structure,
- natural numbers for any $(n - 1)$-truncated representation.

### 6.1 $V^n_\infty$ models Scott’s anti-foundation axiom

As $V^n_\infty$ is not the initial $P^n$-algebra, $(V^n_\infty, \in_n)$ is not a model of foundation. Indeed, $V^n_\infty$ contains anti-wellfounded sets, the simplest one being the infinite unary tree:

```
       /
      / \n     .   .
    /     \
   .       .
  /         \
 .           .
```

So $(V^n_\infty, \in_n)$ is a model of non-wellfounded sets. However, as discussed at the start of this paper, there are several anti-foundation axioms in material set theory, so we need to state specifically which anti-foundation axiom $(V^n_\infty, \in_n)$ is a model of. In this section we will show that $(V^n_\infty, \in_n)$ has Scott $n$-anti-foundation.

By Theorem 1 in Gylterud and Stenholm (2023) and Theorem 8, $(V^n_\infty, \in_n)$ has an ordered pairing structure. Let $(-, -) : V^n_\infty \times V^n_\infty \rightarrow V^n_\infty$ denote this structure.

**Theorem 9.** For each $n : \mathbb{N}^\omega$ the $\in$-structure $(V^n_\infty, \in_n)$ has the Scott $k$-anti-foundation property ($k$-SAFA) for any $k \leq n$.\(^\text{§}\)

\(^\text{§}\)There is also a requirement about the fixed point being appropriately locally small.
Proof. \text{SAFA}_2 \text{ is immediate from } V^n_\infty \text{ being a simple } P^\infty \text{-coalgebra by Corollary 4 and Proposition 4.}

For \text{n-SAFA}_1, \text{let } g : V^n_\infty \text{ be a Scott } n\text{-extensional graph. The } P^\infty \text{-coalgebra homomorphism}
\[
\text{corec}^\infty (\text{Target } g, n_g) : \text{Target } g \to V^\infty_\infty
\]
factors through the embedding \( V^n_\infty \hookrightarrow V^\infty_\infty \), since \( g : V^n_\infty \text{ is Scott } n\text{-extensional.} \) Denote this map \( d' : \text{Target } g \to V^\infty_\infty \).

To obtain from this a \( P^\infty \text{-coalgebra homomorphism from } (V^n, m_g) \), and thus an \( \infty \)-decoration by Proposition 4, let \( dx = \sup^n (\sum_{y, V^n_\infty} (x, y) \in g, \lambda(y, e).d'(y, [(x, e)])). \) This is a valid application of \( \sup^n \) since \( \sum_{y, V^n_\infty} (x, y) \in g \) is essentially small and \( d' \) is \( (k - 1)\)-truncated and thus its composition with the map \( (\sum_{y, V^n_\infty} (x, y) \in g) \to \text{Target } g \) sending \((y, e)\) to \( (y, [(x, e)]) \) is \( (n - 1)\)-truncated. It remains to check that the coalgebra homomorphism square commutes, i.e. \( \text{desup} (dx) = P^\infty d (m_g x) \). Note that the first component of both \( \text{desup} (dx) \) and \( P^\infty d (m_g x) \) is \( \sum_{y, V^n_\infty} (x, y) \in g \). For the second component we have the following chain of equalities:

\[
\pi_1 (P^\infty d (m_g x)) = d \circ \pi_0 = \lambda(y, e).d y = \lambda(y, e).\sup^n \left( \sum_{z, V^n_\infty} \langle y, z \rangle \in g, \lambda(z, e').d' (z, [(y, e']) \right) = \lambda(y, e).\sup^n \left( P^\infty d' \left( \sum_{z, V^n_\infty} \langle y, z \rangle \in g, \lambda(z, e').(z, [(y, e']) \right) \right) = \lambda(y, e).\sup^n \left( P^\infty d' (n_g (y, [(x, e)])) \right) = \lambda(y, e).d' (y, [(x, e)]) = \pi_1 (\text{desup} (dx)) \]
\[
\square
\]

7 The terminal \( P^0 \)-coalgebra

In this section we describe a general construction of terminal coalgebras for functors satisfying a certain accessibility condition. This is a formalization in type theory of a theorem due to Aczel and Mendler (1989), which dates back to the late 80s and states that every \textit{set-based} endofunctor on the category of proper classes has a terminal coalgebra. We describe how to translate the original proof of Aczel and Mendler, written in the language of set theory with reasoning based on classical logic, into the constructive setting of HoTT. In the type theoretic statement of the theorem, proper classes are replaced by large types, and sets are replaced by small types. The notion of set-based functor is replaced by a certain accessibility condition with respect to small types. We were able to remove all invocations of choice principles from the original proof, but not all impredicativity. In fact, the existence of terminal coalgebras is guaranteed only under the assumption of \textit{propositional resizing}, a form of impredicativity for propositions. Here we recall the principle in a formulation given by Jong and Escardo (2023).

\textbf{Definition 25 (\( \text{Prop}^f \)).} The \textbf{principle of propositional resizing} states that every proposition \( P : \text{Type} \) is essentially small, i.e. it is equivalent to a small proposition \( Q : U \).

We do not assume propositional resizing globally, but we precisely mark all theorems that require its assumption.

Remember that \( P^0 \) does not have a functorial action on all functions, only on ones with locally small codomain. In the presence of propositional resizing, these can also be functions with set-valued codomain. This means that the Aczel–Mendler theorem does not immediately apply to \( P^0 \). Nevertheless, in the last part of this section we will show how to appropriately adjust the statement and proof of the theorem in order to construct terminal coalgebras also for “functors” such as \( P^0 \).

7.1 \( U \)-based functors

Aczel and Mendler’s theorem applies to set-based endofunctors on proper classes, where, intuitively, a functor is set-based when its value on a proper class \( X \) is the colimit of values on small subsets of \( X \).
Before reformulating this accessibility condition in our type theoretic setting, we recall some definitions and establish some notation.

In this section, we globally assume functors to be set-valued, i.e $F X$ is a set, independently of the type level of $X$.

**Definition 26** ($\precong$. Let $\alpha : A \to FA$ be a coalgebra. We say that $\alpha$ is

- **$U$-simple** if, for all $B : U$ and coalgebras $\beta : B \to FB$, the type of coalgebra homomorphisms from $\beta$ to $\alpha$ is a proposition;
- **$U$-terminal** if, for all $B : U$ and coalgebras $\beta : B \to FB$, the type of coalgebra homomorphisms from $\beta$ to $\alpha$ is contractible;

Aczel and Mendler write “strongly extensional” instead of “$U$-simple”. Assuming propositional resizing, the Aczel–Mendler theorem guarantees the existence of a $U$-terminal coalgebra for every functor $F$. But the existence of a terminal coalgebra is guaranteed only in case $F$ satisfies an accessibility condition. This condition is a type-theoretic reformulation (and slight generalization) of Aczel and Mendler’s notion of set-based functor.

**Definition 27** ($\precong$. A functor $F$ is **$U$-based** if, for any large type $X : Type$ and $x : FX$, there is a small type $Y : U$, a function $i : Y \to X$ and element $y : FY$ such that $Fi y = x$.

The existential quantification in the above statement is strong, i.e. it is a $\Sigma$-type without propositional truncation around it. Intuitively, $F$ is $U$-based when $FX$ is the colimit of $FY$, where $Y$ ranges over small generalized elements of $X$. Notice that the definition is slightly different from the one of Aczel and Mendler, as the they require $Y$ to be a subset of $X$, i.e. $i$ is an embedding in their definition. This restriction is not crucial in the construction of the terminal coalgebra, so we remove it from the definition.

Notice that Definition 27 admits a slight reformulation, that will become useful later on: a functor is $U$-based whenever for all $X : Type$ the function

$$ (\lambda(A, f, a). F f a) : \left( \sum_{(A, f) : P \to X} FA \right) \to FX $$

has a section $\text{base}_F : FX \to \sum_{(A, f) : P \to X} FA$.

### 7.2 Relation lifting and precongruences

There are many ways to lift a (possibly proof-relevant) relation on a type $X$ to a relation on $FX$ (Staton, 2011). Many of these liftings are well-behaved only when the functor $F$ preserves weak pullbacks. This restriction can be avoided by employing Aczel and Mendler’s notion of relation lifting.

**Definition 28** ($\precong$. Given $X : Type$, the **relation lifting** $E_F$ takes a relation $R : X \times X \to Type$ and produces a relation $E_F R : FX \times FX \to Type$ as follows:

$$ E_F R(x, y) := (F[-]_R x = F[-]_R y) $$

where $[-]_R$ is the point constructor of the set quotient $X/R$.

Notice that $E_F R$ is always propositionally-valued since $F(X/R)$ is always a set. If $R$ is valued in $U$ instead of Type, there is no guarantee that $E_F R$ is also valued in $U$, as $F(X/R)$ may not be locally $U$-small. But this is true under the assumption of propositional resizing.

**Definition 29** ($\precong$. Given a coalgebra $\alpha : X \to FX$, a relation $R : X \times X \to Type$ is called a **precongruence** if the following type is inhabited:

$$ \text{is-precong}_{\alpha} R := \prod_{x, y : X} R(x, y) \to E_F R(\alpha x, \alpha y) $$

The type of propositionally-valued precongruences on the coalgebra $\alpha$ is denoted $\text{Precong}^U_{\alpha}$, and we write $\text{Precong}^U$ for the type of propositionally-valued small precongruences.

**Definition 30** ($\precong$. A coalgebra $\alpha : X \to FX$ is called **precongruence simple** if, for all $x, y : X$ such that $R(x, y)$ for some $R : \text{Precong}_{\alpha}$, then also $x = y$. We call it **$U$-precongruence simple** if the latter holds for $R : \text{Precong}^U_{\alpha}$.
Aczel and Mendler require the precongruence in the definition of simple coalgebra (which they call “s-extensional”) to be a congruence, i.e. an equivalence relation on X. We do not require symmetry and transitivity, as reflexivity is sufficient for our purposes. The terminology “simple” comes from Rutten (2000), denoting coalgebras for which bisimulation implies equality. We generalize the notion from bisimulation to reflexive precongruence.

Every precongruence simple coalgebra is also U-precongruence simple, but the opposite implication is not necessarily true. It becomes true if we assume the principle of propositional resizing.

The maximal precongruence on a coalgebra α is the propositional truncation of the disjoint union of all its small precongruences:

\[ x \sim_\alpha y := \prod_{R: \text{Precong}_{\alpha}^U} R(x, y) \]

It is possible to show that \((\sim_\alpha) : \text{Precong}_{\alpha}\) and, assuming propositional resizing, also \((\sim_\alpha) : \text{Precong}_{\alpha}^U\). We can form the set quotient \(X/\sim_\alpha\), which satisfies a number of important properties. First, \(X/\sim_\alpha\) has an F-coalgebra structure \(\alpha^q : X/\sim_\alpha \to F(X/\sim_\alpha)\) defined by structural recursion. The case of the point constructor is given as follows: \(\alpha^q [x]_{\sim_\alpha} := F [-]_{\sim_\alpha} (\alpha x)\). The constructor \([-]_{\sim_\alpha}\) is a coalgebra homomorphism between α and \(\alpha^q\). Moreover, the coalgebra \(\alpha^q\) is U-precongruence simple.

**Proposition 16 (\(\Box\)).** The coalgebra \(\alpha^q : X/\sim_\alpha \to F(X/\sim_\alpha)\) is U-precongruence simple.

*Proof.* Applying the elimination principle of set quotients, it is sufficient to show that given \(x, y : X\), a propositionally-valued reflexive precongruence \(R : X/\sim_\alpha \times X/\sim_\alpha \to U\) and a proof of \(R([x]_{\sim_\alpha}, [y]_{\sim_\alpha})\), then \(x \sim_\alpha y\). In other words, we need to find a small propositionally-valued reflexive precongruence \(S : X \times X \to U\) such that \(S(x, y)\). Take \(S(a, b) := R([a]_{\sim_\alpha}, [b]_{\sim_\alpha})\). Notice that the types \((X/\sim_\alpha)/R\) and \(X/S\) are isomorphic, and the underlying function \(c : (X/\sim_\alpha)/R \to X/S\) makes the following square commute:

\[
\begin{array}{c}
X \xrightarrow{[-]_S} X/S \\
\downarrow\quad\downarrow c \\
X/\sim_\alpha \xrightarrow{[-]_R} (X/\sim_\alpha)/R
\end{array}
\]

Let \(a, b : X\) and suppose \(S(a, b)\). The following sequence of equalities proves that \(S\) is a precongruence:

\[
F [-]_S (\alpha a) = F (c \circ [-]_R \circ [-]_{\sim_\alpha}) (\alpha a) = F c (F [-]_R (\alpha^q [a]_{\sim_\alpha})) = F c (F [-]_R (\alpha^q [b]_{\sim_\alpha})) = F (c \circ [-]_R \circ [-]_{\sim_\alpha}) (\alpha b) = F [-]_S (\alpha b)
\]

Step (42) follows by (41) and step (44) is the fact that \(R\) is a precongruence. Finally, in step (46) we use (41) again.

**Proposition 17 (\(\Box\)).** Every U-precongruence simple coalgebra is U-simple.

*Proof.* Let \(\alpha : X \to FX\) be a precongruence simple coalgebra and let \(f, g\) be two coalgebra homomorphisms from another coalgebra \(\beta : Y \to FY\) to \(\alpha\). For all \(y : Y\) we need to show that \(f(y) = g(y)\). From the precongruence simplicity of \(\alpha\), it is sufficient to find a reflexive precongruence relating \(f(y)\) and \(g(y)\).

Consider the relation:

\[
R' x x' := \sum_{y : Y} (x = f(y) \times (x' = g(y))
\]

and its propositional reflexive closure \(R x x' := R'(x x' + (x = x'))\). It is not hard to show that \(R\) is a precongruence on \(\alpha\), which moreover relates \(f\) and \(g\) as \([\text{inl}(y, \text{refl, refl})]\) : \(R f(y)\) \((g y)\).

**Corollary 6.** Assuming propositional resizing, the coalgebra \(\alpha^q : X/\sim_\alpha \to F(X/\sim_\alpha)\) is U-simple.
7.3 The \( U \)-terminal coalgebra

The \( U \)-terminal coalgebra of a functor \( F \) is built in two steps. First, define the weakly \( U \)-terminal coalgebra as the disjoint union of all small coalgebras:

\[
\nu\nu F_U := \sum_{X \in \mathcal{U}} \sum_{\alpha : X \to FX} X.
\]

(47)

Every small coalgebra \( \alpha : X \to FX \) clearly injects in the union \( \alpha^* : X \to \nu\nu F_U \), \( \alpha^* x := (X, \alpha, x) \). The coalgebra structure \( \zeta : \nu\nu F_U \to F(\nu\nu F_U) \) is given by \( \zeta (X, \alpha, x) := F \alpha^* (x) \). It easy to prove that \( \alpha^* \) is a coalgebra homomorphism between \( \alpha \) and \( \zeta \).

In order to turn the weakly \( U \)-terminal coalgebra into a strong \( U \)-terminal coalgebra, we quotient its carrier \( \nu\nu F_U \) by the maximal precongruence on \( \zeta \): \( \nu F_U := \nu\nu F_U / \sim_{\zeta} \). We know this has a coalgebra structure \( \zeta^q \). Moreover, given a small coalgebra \( \alpha : X \to FX \), there is a coalgebra homomorphism from it to \( \zeta^q \) given by the composition of \( \alpha^* \) and \( [\cdot]_{\sim_{\zeta}} \). Invoking Corollary 6, which assumes propositional resizing, we know that this is the only such coalgebra homomorphism.

**Theorem 10 (\( \zeta^q \)).** Assuming propositional resizing, the coalgebra \( \zeta^q : \nu F_U \to F(\nu F_U) \) is \( U \)-terminal.

7.4 The Aczel–Mendler theorem

We finally show how the \( U \)-terminal coalgebra \( \zeta^q \) is also terminal with respect to large coalgebras, provided the functor \( F \) is \( U \)-based.

First, notice that \( \mathcal{P}^\infty \) is not only a polynomial functor, but a polynomial monad. Its unit \( \eta : X \to \mathcal{P}^\infty X \) is \( \eta x := (1, \lambda * x) \). The Kleisli extension \( \text{bind} : \mathcal{P}^\infty X \to \mathcal{P}^\infty Y \) of a function \( g : X \to \mathcal{P}^\infty Y \) is obtained by forming the disjoint union of all indexing types:

\[
\text{bind} g (A, f) := \left( \sum_{a : A} \pi_0 (g(fa)), \lambda (a, y), \pi_1 (g(fa)y) \right)
\]

Given \( g : X \to \mathcal{P}^\infty X \), its Kleisli extension can be iterated a finite number of times:

\[
\begin{align*}
\text{bind}^0 g z & := z \\
\text{bind}^{n+1} g z & := \text{bind} g (\text{bind}^n g z).
\end{align*}
\]

(48)

It can also be iterated an infinite number of times, by collecting all the finite approximations:

\[
\begin{align*}
\text{bind}^\infty g & : \mathcal{P}^\infty X \to \mathcal{P}^\infty X \\
\text{bind}^\infty g z & := \left( \sum_{n \in \mathbb{N}} \pi_0 (\text{bind}^n g z), \lambda (n, x), \pi_1 (\text{bind}^n g x) \right)
\end{align*}
\]

Given a large coalgebra \( \alpha : X \to FX \) for a \( U \)-based functor \( F \), one can construct a \( \mathcal{P}^\infty \)-coalgebra structure on \( X \) as follows: \( \hat{\alpha} x := \pi_0 (\text{base}_F (\alpha (x))) \).

**Proposition 18 (\( \zeta^q \)).** Let \( F \) be a \( U \)-based functor and \( \alpha : X \to FX \) a large coalgebra. For all \( z : \mathcal{P}^\infty X \), there is a function \( \alpha_z : \pi_0 z \to F(\pi_0 (\text{bind} \hat{\alpha} z)) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\pi_0 z & \xrightarrow{\pi_1 z} & X \\
\alpha_z & \downarrow \cong & \alpha \\
F(\pi_0 (\text{bind} \hat{\alpha} z)) & \xrightarrow{F (\pi_1 (\text{bind} \hat{\alpha} z))} & FX
\end{array}
\]

Proof. Let \( a : \pi_0 z \). Since \( F \) is \( U \)-based, there exist \( A : U, \iota : A \to X \) and \( y : FA \) such that \( F \iota y = \alpha (\pi_1 z a) \). In other words \( y \equiv \pi_2 (\text{base}_F (\alpha (\pi_1 z a))) \). Take \( \alpha_z a := F (\lambda x. (a, x)) y \).

\( \square \)

The construction of Proposition 18 can be iterated, producing a family of functions \( \alpha^n_z : \pi_0 (\text{bind}^n \hat{\alpha} z) \to F(\pi_0 (\text{bind}^{n+1} \hat{\alpha} z)) \) indexed by a natural number \( n \), which makes the following family of diagrams commute:

\[
\begin{align*}
\pi_0 (\text{bind}^n \hat{\alpha} z) & \xrightarrow{\pi_1 (\text{bind}^n \hat{\alpha} z)} X \\
\alpha^n_z & \downarrow \cong \alpha \\
F(\pi_0 (\text{bind}^{n+1} \hat{\alpha} z)) & \xrightarrow{F (\pi_1 (\text{bind}^{n+1} \hat{\alpha} z))} FX
\end{align*}
\]

(49)
Proposition 19. Let $F$ be a $U$-based functor and $\alpha : X \to FX$ a large coalgebra. Then each $z : P^\infty X$ determines a small coalgebra $\alpha_z : X \to F(x)$ and a coalgebra homomorphism $k_z$ from $\alpha_z$ to $\alpha$.

Proof. Define the carrier $X_z$ as $\pi_0 (\text{bind}_x \hat{\alpha}_z)$ and its coalgebra structure as

$$\alpha_z^\infty (n, x) := F (\lambda y. n + 1, y) (\alpha_z^n x).$$

There is a function $k_z (n, x) := \pi_1 (\text{bind}_x \hat{\alpha}_z) x$ between $X_z$ and $X$. The fact that this is a coalgebra homomorphism between $\alpha_z^\infty$ and $\alpha$ follows from the commutativity of the family of diagrams in (49).

Notice also the existence of a function $u_z : \pi_0 z \to X_z$ sending $x$ to the pair $(0, x)$, which makes the triangle below commute. Since $k_z$ is a coalgebra homomorphism, the square below also commutes:

$$\begin{array}{ccc}
X_z & \xrightarrow{k_z} & X \\
\downarrow{\alpha_z^\infty} & & \downarrow{\alpha} \\
F k_z & \xrightarrow{\alpha} & F X
\end{array}$$

(50)

Given $z : P^\infty X$, a multiset of elements in $X$, and $w : P^\infty X_z$ a multiset of $X_z$, the latter determines also a multiset $w'$ of $X$ as follows: $w' := P^\infty k_z w$. The small coalgebras associated to $z$ and $w'$ by Proposition 19 are in a strong relationship with each other. We refer the interested reader to the formalization for a proof of this technical lemma.

Lemma 8. Let $F$ be a $U$-based functor and $\alpha : X \to FX$ a large coalgebra. For all $z : P^\infty X$ and $w : P^\infty X_z$, there is a coalgebra homomorphism $l_z,w$ between $\alpha_w^\infty$ and $\alpha_z^\infty$ that makes the following diagram commute:

$$\begin{array}{ccc}
\pi_0 w' & \xrightarrow{u_z} & \pi_0 w \\
\downarrow{\pi_0 w'} & & \downarrow{\pi_1 w} \\
X_w & \xrightarrow{l_z,w} & X_z
\end{array}$$

(51)

We are now ready to prove the main result of Aczel and Mendler (1989).

Theorem 11. Let $F$ be a $U$-based functor. If a coalgebra is $U$-terminal then it is also terminal.

Proof. Let $\beta : Y \to FY$ be a $U$-terminal coalgebra and let $\alpha : X \to FX$ be a large coalgebra. We construct a coalgebra homomorphism from $\alpha$ to $\beta$. Given $x : X$, we get $\eta x : P^\infty X$ and therefore, by Proposition 19, a small coalgebra $\alpha_{\eta x} : X_{\eta x} \to F(X_{\eta x})$. From $U$-terminality, there exists a unique coalgebra homomorphism $h_x$ between $\alpha_{\eta x}$ and $\beta$.

We now show how this homomorphism can be lifted to one initiating from the large coalgebra $\alpha$. First, a function $h : X \to Y$ can be defined as $h x := h_x (\eta x \ast)$, which is a coalgebra homomorphism:

$$F h (\alpha x) = F h (F k_{\eta x} (\alpha_{\eta x}^\infty (u_{\eta x} \ast)))$$

(52)

$$= F (h \circ k_{\eta x}) (\alpha_{\eta x}^\infty (u_{\eta x} \ast))$$

(53)

$$= F h_x (\alpha_{\eta x}^\infty (u_{\eta x} \ast))$$

(54)

$$= \beta (h x (\alpha_{\eta x}^\infty (u_{\eta x} \ast)))$$

(55)

$$\equiv \beta (h x)$$

(56)

Step (52) follows from (50) and step (55) is the fact that $h_x$ is a coalgebra homomorphism. The validity of step (54), i.e. the equation $h \circ k_{\eta x} = h_x$, can be justified as follows. Let $a : X_{\eta x}$ and define $a' : X$ as $a' := k_{\eta x} a$. We have the following sequence of equalities:

$$h (k_{\eta x} a) \equiv h_{a'} (u_{\eta a'} \ast) = h_x (l_{\eta x, \eta a} (u_{\eta a'} \ast)) = h_x (\pi_1 (\eta a) \ast) \equiv h_x a$$

The second equality holds since $h_{a'}$ is the unique coalgebra homomorphism from $\alpha_{\eta a'}^\infty$ to $\beta$, and the fact that $h_x$ and $l_{\eta x, \eta a}$ (which was introduced in Lemma 8) are both coalgebra homomorphisms. The third equality is an instance of (51).
The coalgebra homomorphism \( h \) is unique. Given another one \( h' \) and an element \( x : X \), we have the following sequence of equalities:

\[
  h x \equiv h_x (u_{\eta x} *) = h' (k_{\eta x} (u_{\eta x} *)) = h' (\pi_1 (\eta x) *) \equiv h' x
\]

The second equality holds since \( h_x \) is the unique coalgebra homomorphism from \( \alpha^\infty_{\eta x} \) to \( \beta \), and the fact that \( h' \) and \( k_{\eta x} \) are both coalgebra homomorphisms. The third equality is an instance of the triangle in (50).

Plugging together Theorems 10 and 11, we obtain the general terminal coalgebra theorem of Aczel and Mendler. Assuming propositional resizing, there is a \( U \)-terminal coalgebra \( \zeta^U : \nu F_U \rightarrow F(\nu F_U) \) for any functor \( F \). If the latter happens to be \( U \)-based, then this coalgebra is also terminal with respect to large coalgebras.

**Theorem 12** (\( \zeta^U \)). Let \( F \) be a \( U \)-based functor. Assuming propositional resizing, the coalgebra \( \zeta^U : \nu F_U \rightarrow F(\nu F_U) \) is terminal.

### 7.5 Adjusting the theorem for \( P^0 \)

The powerset construction \( P^0 \) is not a functor, as it only acts on functions \( f : X \rightarrow Y \) with locally small codomain. \( Y \) can also be restricted to be a set if one assumes propositional resizing. Crucially this means that the Aczel–Mendler theorem described so far does not apply to it. Luckily, this can be remedied with a few small modifications.

First, let us call \( F \) a **set-valued functor** if \( FX \) is a set and \( F \) acts exclusively on set-valued functions, i.e., its action on functions is typed \( \prod_{X : \text{Type}, Y : \text{Set}} (X \rightarrow Y) \rightarrow FX \rightarrow FY \). Clearly \( P^0 \) is a set-valued functor in this sense, assuming propositional resizing.

The notion of \( U \)-basedness in Definition 27 also needs to be adjusted. Let \( \text{Set}_U \) be the type of sets in \( U \). We now say that a set-valued functor is \( \text{Set}_U \)-**based** if, for any large set \( X : \text{Set} \) and \( x : FX \), there is a small set \( Y : \text{Set}_U \), a function \( \iota : Y \rightarrow X \) and element \( y : FY \) such that \( F \iota y = x \). In other words, both \( X \) and \( Y \) in the definition are required to be sets. This is important for the results of Section 7.4 to go through when functors only act on set-valued functions. For example, the bottom functions in (49) and (50) are well-defined only if \( X \) is a set. Similarly, the functions \( l_z \) in Lemma 8 can only be a coalgebra morphism in case \( X_z \) is a set.

**Proposition 20** (\( \zeta^U \)). \( P^0 \) is \( \text{Set}_U \)-based.

**Proof.** Let \( X : \text{Set} \) and \( x : P^0 X \). Notice that \( \pi_0 x : U \) is a set, since \( \pi_1 x : \pi_0 x \rightarrow X \) is an embedding and \( X \) is a set. Therefore we can return the triple consisting of the small set \( \pi_0 x \), the function \( \pi_1 x : \pi_0 x \rightarrow X \) and the element \( (\pi_0 x, \text{id}) : P^0(\pi_0 x) \).

The weakly \( U \)-terminal coalgebra in (47) also needs to be modified. This is because \( \nu F_U \) is not a set, so there cannot be any coalgebra homomorphism targeting it. The solution is to take its set truncation \( ||\nu F_U||_0 \) instead. It is straightforward to define a coalgebra structure on it using the elimination principle of set truncation.

Finally, assuming that \( X \) is a set in the definition of \( \text{Set}_U \)-basedness restricts the notion of terminal coalgebra in Definition 26 to work only for coalgebras with a set carrier. We say that a coalgebra \( \alpha : A \rightarrow FA \) is **terminal with respect to sets** if, for all \( B : \text{Set} \) and coalgebras \( \beta : B \rightarrow FB \), the type of coalgebra homomorphisms from \( \beta \) to \( \alpha \) is contractible.

With all these restrictions in place, the Aczel–Mendler Theorem 12 still works.

**Theorem 13** (\( \zeta^U \)). Let \( F \) be a \( \text{Set}_U \)-based set-valued functor. Assuming propositional resizing, the coalgebra \( \zeta^U : \nu F_U \rightarrow F(\nu F_U) \) is terminal with respect to sets.

As a corollary, we obtain a terminal coalgebra with respect to sets for the powerset functor \( P^0 \).

**Corollary 7** (\( \zeta^U \)). Assuming propositional resizing, \( P^0 \) admits a terminal coalgebra with respect to sets.

### 8 Conclusion

In this paper we constructed a non-initial and non-terminal fixed point of the (restricted) powerset functor and showed that it is a model of material set theory with Scott’s anti-foundation axiom. Moreover, we constructed the terminal coalgebra of the same functor, assuming propositional resizing. This is then a model of material set theory with Aczel’s anti-foundation axiom.
8.1 Related work

The result that the subtype of coiterative sets, of the type of non-wellfounded trees, is a model of Scott's anti-foundation axiom can be found in the classical literature in the paper of D'Agostino and Visser (2002). They consider universes of multisets and define two functors, $\Delta$ and $\Gamma$, from sets to multisets for which the terminal coalgebras exist classically. They then show that the subclass of unisets (i.e. multisets with, coiteratively, only one occurrence of each element) for $\Delta$ and $\Gamma$ are models of AFA and SAFA respectively. Their functor $\Gamma$ corresponds to our functor $P^\infty$, and their model of SAFA corresponds to our model, except, of course, that they work within the framework of classical set theory. They also prove in the same paper that the relation between nodes in graphs of having isomorphic unfolding trees is precisely the relation of $\Gamma$-bisimulation, which corresponds to our Theorem 1.

8.2 Future work

There are still some questions that remain unanswered, especially the initial motivation of this paper: to construct the terminal coalgebra of the powerset functor. The construction in the last section relies in a crucial way on propositional resizing. Is there a way to construct the terminal coalgebra, without any constructively questionable assumptions? Is it possible to show that assuming the existence of the terminal coalgebra implies some classical principle? Or is it independent altogether?

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