ON UNIFORM DECAY OF THE MAXWELL FIELDS ON BLACK HOLE SPACE-TIMES

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Abstract. This is the second in a series of papers in which we take a systematic study of gauge field theories such as the Maxwell equations and the Yang-Mills equations, on curved space-times. In this paper, we study the Maxwell equations in the domain of outer-communication of the Schwarzschild black hole. We show that if we assume that the middle components of the non-stationary solutions of the Maxwell equations verify a Morawetz type estimate supported around the trapped surface, then we can prove uniform decay properties for components of the Maxwell fields in the entire exterior of the Schwarzschild black hole, including the event horizon, by making only use of Sobolev inequalities combined with energy estimates using the Maxwell equations directly. This proof is entirely gauge independent, and does not pass through the scalar wave equation on the Schwarzschild black hole, and does not need to separate the middle components for the Maxwell fields. However, proving a Morawetz estimate directly using the Maxwell equations, without referring to the scalar wave equation, seems to be out of reach of the mathematical community as of today; which I was not able to solve yet in this work. If one is able to prove the Morawetz estimate directly using the Maxwell equations, this combined with the present work would give full conceptual proof of decay of the Maxwell fields on the Schwarzschild black hole, and would then be in particular useful for the non-abelian case of the Yang-Mills equations where the separation of the middle components cannot occur. The whole manuscript is written in an expository way where we detail all the calculations.

1. Introduction

In this paper, we study the Maxwell equations on the Schwarzschild black hole. In a recent paper, [DR1]-[DR2], Dafermos and Rodnianski proved decay for solutions of the free scalar wave equation $\Box_g \phi = 0$ in the exterior of the Schwarzschild black hole, up to points on the event horizon. We do not know how to make these methods work for $\Box_g \phi = \phi$ or for $\Box_g \phi = \phi^2$. Thus, this rules out the possibility of using these methods for the Maxwell equations in a hyperbolic formulation where the source terms would be $\Box_g F_{\mu\nu}$, where $F_{\mu\nu}$ is the Maxwell field. In a recent paper, [Bl], Blue proved decay for the Maxwell fields on the exterior of the Schwarzschild background. The proof of Blue required a study of a wave equation on the Schwarzschild space-time for the middle components, which can be separated from the other components in the abelian case of the Maxwell equations. This was later extended by Andersson and Blue to Kerr metrics, [AB]. However, in the non-linear case of the Yang-Mills equations, one cannot decouple the middle components from the others. Yet, it seems difficult to generalize the results of Dafermos and
Rodnianski for the free scalar wave equation to the Maxwell equations using the Maxwell energy-momentum tensor directly, without referring to the scalar wave equation, combined with suitable Sobolev inequalities. A key step to achieve this would be to bound the conformal energy without separating the so-called middle components of the Maxwell fields. This would provide a new independent proof and improves the result of Blue, and would be in particular useful for the non-abelian case of the Yang-Mills equations where such separation cannot occur. I tried to do this in the goal of proving uniform boundedness for Yang-Mills fields on the exterior of the Schwarzschild black hole and the Kerr metric. However, as Klainerman pointed out to me later, many people tried to get a more conceptual proof of decay for the Maxwell equations, without passing through the scalar wave equation, and achieving this would be very significant. While I was not able to solve this yet, in this paper I write a proof of decay for the Maxwell fields on the exterior of the Schwarzschild space-time, directly without separating the middle components or referring to the wave equation, assuming that we have a Morawetz estimate at the 0-derivative level. However, as Andersson and Blue wrote in [AB], proving a Morawetz estimate for the Maxwell field directly would be an important advance in the field, which I did not achieve yet in the present work. To explain, let us recall (see Appendix) that in the exterior, the Schwarzschild metric can be written as,

\[ ds^2 = -(1 - \frac{2m}{r})dt^2 + \frac{1}{(1 - \frac{2m}{r})}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \]

If we define,

\[ r^* = r + 2m \log(r - 2m) \]
\[ v = t + r^* \]
\[ w = t - r^* \]

then, we have,

\[ ds^2 = -(1 - \frac{2m}{r})dv dw + r^2d\sigma^2 \]
\[ = -(1 - \frac{2m}{r})\frac{1}{2}dv \otimes dw - (1 - \frac{2m}{r})\frac{1}{2}dw \otimes dv + r^2d\sigma^2 \]

Let,

\[ \frac{\partial}{\partial w} = \frac{1}{(1 - \frac{2m}{r})} \frac{\partial}{\partial w} \]
\[ \frac{\partial}{\partial v} = \frac{\partial}{\partial v} \]
\[ \frac{\partial}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta} \]
\[ \frac{\partial}{\partial \phi} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \]
and at a point in the Schwarzschild space-time, let $e_1, e_2$ be a normalized basis of $S^2$, which verifies for all $A, B \in \{1, 2\}$,

\[
\begin{align*}
\mathbf{g}(e_A, \frac{\partial}{\partial w}) &= \mathbf{g}(e_A, \frac{\partial}{\partial v}) = 0 \\
\mathbf{g}(e_A, e_B) &= \delta_{AB}
\end{align*}
\]

(5)

(6)

In this context, by "assuming a Morawetz estimate", we mean exactly that for $r_0, R_0$ as in the proof of (55), and for all $t_i = (1.1^i t_0$, where $t_0$ is any real positive number and $i$ is any natural number, if we define:

\[
\begin{align*}
J^{(G)}_{F}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0) &= \int_{t=t_i}^{t=t_{i+1}} \int_{r=r_0}^{r=R_0} \int_{S^2} \left[ |F_{r\theta}|^2 + \frac{1}{4} |F_{\phi\theta}|^2 \right] r^* \, d\sigma \, d\tau \, dt \\
\end{align*}
\]

(7)

then, we assume that the non-stationary solutions verify

\[
\begin{align*}
J^{(G)}_{F}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0) &\lesssim |\hat{E}_{F}^{(\phi)}(t_i)| + |\hat{E}_{F}^{(\phi)}(t_{i+1})| + \sum_{j=1}^{3} (|\hat{E}_{\mathcal{L}_{\Omega_j} F}(t_i)| + |\hat{E}_{\mathcal{L}_{\Omega_j} F}(t_{i+1})|)
\end{align*}
\]

(8)

where $\Omega_j, j \in \{1, 2, 3\}$, is a basis of angular momentum operators, and where

\[
\begin{align*}
\hat{E}_{F}^{(\phi)}(t) &= \int_{r=\infty}^{r=-\infty} \int_{S^2} \left[ \frac{1}{r^2(1 - \frac{2m}{r})} |F_{\theta \phi}|^2 + \frac{1}{r^2(1 - \frac{2m}{r}) \sin^2 \theta} |F_{\phi \theta}|^2 \\
&\quad + \frac{1}{r^2(1 - \frac{2m}{r})} |F_{\theta \phi}|^2 + \frac{1}{r^2(1 - \frac{2m}{r}) \sin^2 \theta} |F_{\phi \theta}|^2 \right] r^2 (1 - \frac{2m}{r}) \, d\sigma \, d\tau
\end{align*}
\]

(9)

that is the negative of the energy without the middle components $F_{t\theta}, F_{\theta \phi}$. The only stationary solutions of the Maxwell equations on the exterior of the Schwarzschild black hole, are the so-called Coulomb solutions. Hence, the assumption above is assumed for the non-Coulomb solutions. Our proof would still work with any product of Lie derivatives on the right hand side of (8), with adjusting accordingly the quantities in the theorem that depend on the initial data.

We will prove that if the middle components of the non-stationary solutions verify a Morawetz type estimate at the zero-derivative level, (8), then we can prove uniform decay properties of solutions to the Maxwell equations in the domain of outer-communication of the Schwarzschild black hole space-time, including the event horizon, by making use of suitable Sobolev inequalities combined with energy estimates using the energy momentum tensor of the Maxwell fields. We do not make any use of decomposition into spherical harmonics. We start with a Cauchy hypersurface prescribed by $t = t_0$ where the initial data has to verify certain regularity conditions (there is no vanishing condition on the bifurcate sphere for $F$). Away
from the horizon (in the region \( r \geq R > 2m \), for a fixed \( R \)), we will prove that
\[
|F_{\hat{\mu}\hat{\nu}}(w, v, \omega)| \leq \frac{C}{(1 + |v|)}
\]
for all \( \hat{\mu}, \hat{\nu} \in \{ \partial_{w}, \partial_{v}, \partial_{\theta}, \partial_{\phi} \} \), where \( F_{\mu\nu} \) is the Maxwell field. Near the horizon, and in the entire exterior region \( r \geq 2m \), up to points on the event horizon, we will prove that
\[
|F_{\hat{v}\hat{w}}(v, w, \omega)| \leq \frac{C}{\max\{1, v\}}, \quad |F_{e_{1}e_{2}}(v, w, \omega)| \leq \frac{C}{\max\{1, v\}}
\]
\[
|F_{\hat{v}e_{a}}(v, w, \omega)| \leq \frac{C}{\max\{1, v\}}, \quad |\sqrt{1 - \frac{2m}{r}} F_{\hat{w}e_{a}}(v, w, \omega)| \leq \frac{C}{\max\{1, v\}}
\]

To explain really thoroughly:

If one tries to generalize the proof of Dafermos and Rodnianski for the free scalar wave equation to the Maxwell equations, on the Schwarzschild space-time, using the Maxwell energy-momentum tensor instead, some of the difficulties that appear are:

(1) In order to bound the conformal energy of the Maxwell fields and their derivatives in the direction of Killing vector fields, one needs to control a space-time integral near the trapped surface \( r = 3m \), that involves the so-called middle components of the Maxwell field (see (44)). In fact, the terms in the space-time integral \( J^{(K)} \), obtained by applying the divergence theorem on the Morawetz vector field \( K = -w^{2} \frac{\partial}{\partial w} - v^{2} \frac{\partial}{\partial v} \), contracted with the energy momentum tensor of the Maxwell fields, are non-negative in a region \( r_{0} \leq r \leq R_{0} \) that contains \( r = 3m \). It seems that this cannot be controlled using these methods due to the presence of the other components with the "wrong" sign in the space-time integral generated from a space type vector field of control (see (49)). Indeed, using the divergence theorem with the vector field \( G = f(r^{*}) \frac{\partial}{\partial r^{*}} \), where \( f \) is a bounded function, we will obtain a space-time integral and boundary terms. However, the space-time integral obtained from \( G \) has independent terms that do not appear in \( J^{(K)} \) that enter with the wrong sign, and hence it cannot be made positive. We will prove that if we get past this, see assumption (8), we can then write a gauge independent proof of uniform decay of the Maxwell fields in the exterior of the Schwarzschild black hole up to points on the horizon, using the Maxwell equations directly.

(2) One needs to construct a new field which verifies the Maxwell equations and the Bianchi identities, that coincides with the original field in some region and vanishes identically outside another specific region (see the proof of (58)). In the case of the wave equation \( \Box_{g} \phi = 0 \), one can multiply the initial data in the Cauchy problem by a cut-off function, and consider the evolution of such data to obtain a solution that verifies the wave equation and the properties stated previously. In the case of the Maxwell and the Yang-Mills
equations, if one multiplies the initial data by a cut-off function then the constraint equations would not be satisfied anymore. It seems at first sight that one cannot get a new field that verifies the needed properties. While this is true if one wanted to do this for all components, nevertheless, one can do this for all the components except to the $F_{rt}$ and $F_{\theta\phi}$ components, where the multiplication should be at the level of the space derivative of the components. Hence, one can construct a new field that can be made to coincide with $F$ in a certain region, and vanish outside another except to these last components. The somewhat good news is that the calculations show that these "bad" terms do not appear in the boundary terms generated from the divergence theorem applied to a space-like vector field contracted with the energy-momentum tensor (see (52)). Hence, in our assumption (8), we suppose that the space-time integral near the the trapped surface $r = 3m$ of the middle components, can be bounded by the energy without the middle components, which would be the case if this estimate was obtained by controlling the space-time integral by boundary terms generated from space type vector fields multiplied by a bounded function, as shown in estimate (53). This is crucial to establish (58) that is the main estimate to bound the conformal energy in (74).

(3) In order to prove decay for a generalized energy that would control the $L^2$ norm of the fields near the horizon, one is confronted to a situation where it seems crucial to control a space-time integral supported on a bounded region in space near the event horizon, that contains all the components. We overcome this by our assumption (8); it can also be used to bound the space-time integral containing the other components as in (104).

Also, in addition to the above:

(4) As opposed to the case of the wave equation, the flux of a generalized energy that controls the $L^2$ norm of the fields near the horizon do not contain all of their components. On $v = constant$ hypersurfaces it contains only $F_{w\theta}$, $F_{\hat{w}\hat{\theta}}$, $F_{\hat{w}\hat{\theta}}$, and on $w = constant$ hypersurfaces, it contains only the components $F_{v\hat{w}}$, $F_{\hat{v}\hat{w}}$, $F_{\hat{t}\hat{\phi}}$ and $F_{\hat{v}\hat{\phi}}$. In addition, while using Sobolev inequalities near the horizon, since $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial w}$ are not Killing this would add an additional difficulty, while in the case of the wave equation, the squares of these derivatives appear in the fluxes which are easily controlled. We get around these problems by using suitable Sobolev type inequalities for each component, combined with the Bianchi identities and the field equations, in a way that the derivatives in the direction of $\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial w}$ can be controlled by Killing derivatives, $\partial_i$ and $\partial_{1i}$, of the components that appear in the flux to which we would have proved decay.

In the case of the Yang-Mills equations there is the additional impediment that is the equations are non-linear. If one gets an energy identity for the Yang-Mills fields, one cannot write directly the same energy identity for the derivatives of the field in the direction of Killing vector fields, as opposed to the Maxwell fields, due to the non-linearity of the equations.

More precisely, we will prove the following theorem,
1.1. The statement.

**Theorem 1.** Let \((M, g)\) be a maximally extended Schwarzschild space-time. We know by then that the exterior of the black hole space-time, \((\bar{M}, \bar{g})\), is isometric to \((M, g)\) where,

\[
g = -(1 - \frac{2m}{r})dt^2 + \frac{1}{(1 - \frac{2m}{r})}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)
\]

Let, \(\Sigma_{t=t_0}\) be a Cauchy hypersurface prescribed by \(t = t_0\). In a system of coordinates, let \(F_{\mu\nu}(w, v, \omega)\) be the components of the Maxwell field defined as the solution of the Cauchy problem of the Maxwell equations:

\[
\nabla^a F_{ab} = 0 \quad (10)
\]

\[
\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \quad (11)
\]

where the initial data prescribed on the Cauchy hypersurface \(\Sigma_{t=t_0}\) verifies the Maxwell constraint equations:

\[
\nabla_\nu F_{t\nu}(t_0, r, \omega) = 0 \\
|\nabla_r F_{\theta\phi} + \nabla_\theta F_{r\phi} + \nabla_\phi F_{r\theta}|(t_0, r, \omega) = 0
\]

We assume that for \(r_0\) and \(R_0\) as in the proof of \((55)\), \((r_0 \leq 3m \leq R_0)\), and for all \(t_i = (1.1)^i t_0\), where \(t_0\) is any real positive number and \(i\) is any natural number, the non-stationary solutions verify

\[
\int_{t_i}^{t_{i+1}} \int_{r^* = r_0}^{r^* = R_0} \int_{S^2} |l F_{w\theta}|^2 + \frac{1}{4} |l F_{\phi\phi}|^2 dr^* d\sigma^2 dt \lesssim |\hat{E}_F^{(\Omega)}(t_i)| + |\hat{E}_F^{(\Omega)}(t_{i+1})| + \sum_{j=1}^{3} (|\hat{E}_{\Omega_{\alpha_j}}^{(\Omega)}(t_i)| + |\hat{E}_{\Omega_{\alpha_j}}^{(\Omega)}(t_{i+1})|)
\]

where \(\Omega_j, j \in \{1, 2, 3\}\), is a basis of angular momentum operators, and where

\[
\hat{E}_F^{(\Omega)}(t) = \int_{r^* = -\infty}^{r^* = \infty} \int_{\phi = 0}^{2\pi} \int_{\theta = 0}^{\pi} 2(1 - \frac{2m}{r})^2(|F_{w\theta}|^2 + |F_{w\phi}|^2 + |F_{\phi\phi}|^2 + |F_{\phi\theta}|^2) r^2 \sin(\theta) d\theta d\phi dr^*
\]

that is the energy without the middle components \(F_{w\theta}\) and \(F_{\phi\phi}\).

**Remark 1.2.** Our proof would work with any arbitrary product of Lie derivatives of \(F\) on the right hand side of assumption (12), with an adjustment on the initial data accordingly.
Then, we have,
\[
|F_{\hat{\nu}\hat{\psi}}(v, w, \omega)| \leq \frac{C}{\max\{1, v\}}, \quad |F_{e_1 e_2}(v, w, \omega)| \leq \frac{C}{\max\{1, v\}}
\]
\[
|F_{\hat{e}_0}(v, w, \omega)| \leq \frac{C}{\max\{1, v\}} \quad \sqrt{1 - \frac{2m}{r}} F_{\hat{e}_0}(v, w, \omega) \leq \frac{C}{\max\{1, v\}}
\]
in the entire exterior region, up to points on the horizon, under certain regularity conditions on the initial data prescribed in what follows, to which we also add the assumption that their limit goes to zero at spatial infinity on the initial slice \(\Sigma_{t=t_0}\).

More precisely, away from the horizon (in the region \(r \geq R > 2m\)), we have,
\[
|F_{\hat{\mu}\hat{\nu}}(w, v, \omega)| \approx \frac{E_F}{1 + |v|}
\]
\[
|F_{\hat{\mu}\hat{\nu}}(w, v, \omega)| \approx \frac{E_F}{1 + |w|}
\]
for all normalized components \(\hat{\mu}, \hat{\nu} \in \{\frac{\partial}{\partial w}, \frac{\partial}{\partial v}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\}\), and where \(E_F\) is defined by,
\[
E_F = \left( \sum_{i=0}^{1} \sum_{j=0}^{5} E^{(K)}_{r, i} (\mathcal{L}_i) (L_{i}) F (t = t_0) + E^{(K)}_{r, 0} (L_{0}) F (t = t_0) \right)^{\frac{1}{2}}
\]
where \(L\) is the Lie derivative restricted on the 2-spheres, and where,
\[
E^{(K)}_{F}(t_i) = \int_{r^* = \infty}^{r^* = -\infty} \int_{\phi = 0}^{2\pi} d\phi \int_{\theta = 0}^{\pi} d\theta \left( w^2 (1 - \frac{2m}{r})^2 [F_{\hat{\omega}\hat{\psi}}]^2 + v^2 [F_{\hat{\psi}\hat{\phi}}]^2 + F_{\hat{\phi}\hat{\phi}}^2 \right) + \omega^2 (1 - \frac{2m}{r}) [F_{\hat{\psi}\hat{\phi}}^2 + F_{\hat{\phi}\hat{\phi}}^2] \right)^{\frac{1}{2}}
\]
and where,
We expect the angular momentum derivatives, or any other Killing derivatives, in the assumption (8) to come out due to the presence of the trapped surface \( r = 3m \).

Hence, we also assume - although we can make the proof without the following assumption by using only (8) with the price of losing more derivatives on the initial data - that the solutions we are looking at verify

\[
\int_{t = t_0}^{t = t_{i+1}} \int_{r^* = r_0^*}^{r^* = R_0^*} \int_{S^2} \left[ \frac{3}{2} \left( \frac{1}{r} - \frac{2m}{r} \right) \left( |\hat{F}_{\nu\theta}|^2 + |\hat{F}_{\nu\phi}|^2 \right) + \left( \frac{1}{r} - \frac{2m}{r} \right) \left( |\hat{F}_{\nu\theta}|^2 + |\hat{F}_{\nu\phi}|^2 \right) \right] \cdot r^2 \sin(\theta) d\theta d\phi dr^* \approx \frac{E_1}{\max\{1,v\}}.
\]

Then, near the horizon (in the region \( 2m \leq r \leq R \)), we have,

\[
|\hat{F}_{\nu\theta}(v, w, \omega)| \lesssim \frac{E_1}{\max\{1,v\}}, \quad |\hat{F}_{e_1\nu\theta}(v, w, \omega)| \lesssim \frac{E_1}{\max\{1,v\}},
\]

\[
|\hat{F}_{e_2\nu}(v, w, \omega)| \lesssim \frac{E_2}{\max\{1,v\}}, \quad \sqrt{1 - \frac{2m}{r}} F_{\mu\nu}(v, w, \omega) \lesssim \frac{E_2}{\max\{1,v\}}
\]

for \( a \in \{1, 2\} \), and where,

\[
E_1 = \sum_{j=0}^{6} E^{(\nu)}_{\nu^i(\mathcal{L})^j_F} (t = t_0) + \sum_{j=0}^{5} E^{(K)}_{\nu^i(\mathcal{L})^j_F} (t = t_0) + \sum_{j=1}^{3} E^{(\hat{\nu})}_{\nu^i(\mathcal{L})^j_F} (t = t_0)^{\frac{1}{2}}
\]

where,

\[
E^{(\hat{\nu})}_{\nu^i(\mathcal{L})^j_F} (t = t_0) = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \left[ (1 - \frac{2m}{r}) (|\hat{F}_{\nu\theta}|^2 + |\hat{F}_{\nu\phi}|^2) + (|\hat{F}_{\nu\theta}|^2 + |\hat{F}_{\nu\phi}|^2) \right] \cdot r^2 \sin(\theta) d\theta d\phi dr^* (t = t_0)
\]

and where,
\[ E_2 = \left[ E_F^2 + \sum_{i=0}^{1} \sum_{j=1}^{2} E\left( \frac{\partial}{\partial t} \right)^{j} r^{(i)(L_i)} F(t = t_0) + E\left( \frac{\partial}{\partial t} \right)^{j} r^{(K)} F(t = t_0) \right]^2 \]

\[ = \left[ \sum_{i=0}^{1} \sum_{j=0}^{5} E\left( \frac{\partial}{\partial t} \right)^{j} r^{(i)(L_i)} F(t = t_0) + E\left( \frac{\partial}{\partial t} \right)^{j} r^{(K)} F(t = t_0) \right] \]

\[ + \sum_{i=0}^{1} \sum_{j=0}^{4} E\left( \frac{\partial}{\partial t} \right)^{j} r^{(K)} F(t = t_0) + E\left( \frac{\partial}{\partial t} \right)^{j} r^{(K)} F(t = t_0) \]

\[ + \sum_{i=0}^{1} \sum_{j=1}^{2} E\left( \frac{\partial}{\partial t} \right)^{j} r^{(i)(L_i)} F(t = t_0) + E\left( \frac{\partial}{\partial t} \right)^{j} r^{(K)} F(t = t_0) \right]^2 \]

1.3. **Strategy of the proof.**

We decompose our proof of decay for the Maxwell fields, on the Schwarzschild black hole, into two parts. The first proves decay away from the horizon (in the region \( r \geq R \) where \( R > 2m \), arbitrarily fixed), and the second part deals with the region near the horizon (\( 2m \leq r \leq R \)).

1.3.1. **In the first part.** All integrations will be done on spacelike hypersurfaces prescribed by \( t = \text{constant} \), and space-time integrals will be understood as integrals on a region bounded by those hypersurfaces. The starting point of our proof is a suitable use of Sobolev inequalities. Sobolev inequalities permit one to bound the \( L^\infty \) norm of the square of the Maxwell field (the square is taken with respect to the scalar product on the Lie algebra \( \langle , \rangle \)), by the \( L^2 \) norm of the Maxwell field and its derivatives up to some order. And yet, the \( L^2 \) norm of the Maxwell fields and their derivatives in the direction of Killing vector fields can be controlled away from the horizon by the energy \( E(t) \) of those (energy obtained from the vector field \( \frac{\partial}{\partial t} \)).

We lose control on the \( L^2 \) norm of the components \( F_{\hat{v}\hat{w}}, F_{\theta\hat{\phi}}, F_{\hat{w}\theta}, F_{\hat{\theta}\phi} \) of those fields near the horizon due to the presence of the \((1-\frac{2m}{r})\) term (that vanishes at the horizon) that appears in the expression of the energy. And so, since the covariant derivatives of the Maxwell fields in the direction of non-Killing vector fields can be transformed into covariant derivatives in the direction of Killing vector fields by using the field equations and the Bianchi identities, one can bound the \( L^2 \) norm of those fields by their energy which is conserved, because we have conservation of energy for the Maxwell equations and therefore for their Killing derivatives because the Maxwell equations are linear. This way, we can bound the Maxwell fields away from horizon.

However, if we prove decay of the local \( L^2 \) norms of the Maxwell fields and their derivatives in the direction of Killing vector fields, we can prove decay of the Maxwell fields. The key point is that the \( L^2 \) norms here can be taken to be space integrals on only a bounded region. This way, away from the horizon, they can be controlled by a piece of the energy integral, that is the energy as a space integral without integrating on the whole space, but only on the bounded region (this is
because the terms that appear in the space integral of the energy are exactly the squares of the Maxwell fields, multiplied by the term \((1 - \frac{m^2}{r^2})\) for some components. These energies taken on a bounded region of space, can be bounded by the conformal energy (obtained by taking the Morawetz vector field \(K = -\omega^2 \frac{\partial}{\partial \omega} - v^2 \frac{\partial}{\partial v}\)) divided by the minimum on that region of \(u^2\) and \(w^2\) (see (72)). Consequently, if we bound the conformal energy, we have shown so far how one can possibly obtain decay from this of solutions to the Maxwell equations away from the horizon.

To bound the conformal energy of the Maxwell fields and their derivatives in the direction of Killing vector fields, we proceed as follows:

1. We will use the space-time integral \(J_F^{(G)}\) in our assumption (8), of which the terms are positive, to control \(J_F^{(K)}\) (the space-time integral obtained by using the divergence theorem on the Morawetz vector field \(K\) contracted with the energy momentum tensor of the Maxwell fields) in the following sense, see estimate (55):

\[
J_F^{(K)}(t_i \leq t \leq t_{i+1}) \lesssim t_{i+1} J_F^{(G)}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0)
\]

and from our assumption (8), we have

\[
J_F^{(G)}(t_i \leq t \leq t_{i+1}) \lesssim |\hat{E}_F(t_i)| + |\hat{E}_F(t_{i+1})| + \sum_{j=1}^{3} (|\hat{E}_{\Omega_j} F(t_i)| + |\hat{E}_{\Omega_j} F(t_{i+1})|)
\]

where \(\Omega_j, j \in \{1, 2, 3\}\), is a basis of angular momentum derivatives, and \(\hat{E}_F(t)\) is the energy without the middle components, see (9).

2. We construct a new field, \(\hat{F}\), such that it verifies the Maxwell equations and the Bianchi identities, and it coincides with the original field in some region and vanishes identically outside another specific region for the components which appear in the boundary terms \(\hat{E}_F(t)\), so that we could write:

\[
|\hat{E}_F(t)| \lesssim \frac{E_F(t)}{t^2}
\]

and consequently (see (58)):

\[
J_F^{(G)}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0) \lesssim \frac{1}{t_i} E_F(t = t_i) + \frac{1}{t_i} \sum_j E_{\Omega_j}(t = t_i)
\]

\[
+ \frac{1}{t_{i+1}} E_F(t = t_{i+1}) + \frac{1}{t_{i+1}} \sum_j E_{\Omega_j}(t = t_{i+1})
\]

For \(t_i\) such that \(t_{i+1} = (1.1)t_i\) (where \(i\) is an integer), and \(|r^*(r_0)| + |r^*(R)| \leq 0.4t_i\), we can make use of the divergence theorem to properly commutate these inequalities, and use the fact that the series \(\sum \frac{1}{t^2}\) converges, to establish a uniform bound on the conformal energy that depends...
on the initial data and its Killing Lie derivatives derivatives. We obtain
(74):
\[
E_{F}^{(K)}(t) \lesssim E_{F,r}^{(K)}(t) + E_{F,r}^{(K)}(t = 0) = E_{F}^{M}
\]

(17)

1.3.2. In the second part. To obtain decay near the horizon, we are going to integrate in rectangles in the Penrose diagram representing the exterior of the Schwarzschild black hole, of which one side is included in the horizon. We will apply the divergence theorem with the vector field
\[
H = -\left(\frac{h(r^*)}{1 - \frac{r^*}{r}}\right) \frac{\partial}{\partial v} + h(r^*) \frac{\partial}{\partial \hat{\theta}}
\]
contracted with the energy momentum tensor of the Maxwell fields, where \( h \geq 0 \) is supported in the region \( 2m \leq r \leq (1.2)r_1 \) for \( r_1 \) chosen such that, \( 2m < r_0 \leq r_1 < (1.2)r_1 < 3m \), and where \( h \) is such that \( \lim_{r^* \rightarrow -\infty} h(r^*) = 1 \), and for \( r \leq r_1 \), we have \( h > 0 \), \( h' \geq 0 \), \( h' \leq \frac{2m}{r} h \). (1 - \frac{2m}{r}) \frac{1}{2} h \leq h' \), and \( (1 + \frac{6m}{r}) h \leq \frac{2m}{r} h \). By applying the divergence theorem in the rectangles described previously with \( H \) defined as such, we get that the flux through the hypersurfaces prescribed by \( v = \text{constant} \) is roughly speaking the \( L^2 \) norms of \( \sqrt{1 - \frac{2m}{r} F_{\hat{\theta} \hat{\varphi}}} \) and \( \sqrt{1 - \frac{2m}{r} F_{\hat{\varphi} \hat{\varphi}}} \), and the \( L^2 \) norms of \( \sqrt{\frac{1 - 2m}{r} F_{\hat{\varphi} \hat{\varphi}}} \) and \( \sqrt{\frac{1 - 2m}{r} F_{\hat{\theta} \hat{\varphi}}} \). In addition, we get a space-time integral supported near the event horizon \(-I_{F}^{(H)}\), of which the terms are roughly speaking the squares of \( F_{\hat{\varphi} \hat{\varphi}}, \) \( F_{\hat{\varphi} \hat{\theta}}, \) \( F_{\hat{\theta} \hat{\theta}} \), and roughly a factor that goes to zero when \( r \) goes to \( 2m \) multiplied by the squares of \( F_{\hat{\varphi} \hat{\varphi}}, \) \( F_{\hat{\varphi} \hat{\theta}}, \) \( F_{\hat{\theta} \hat{\theta}} \), and \( F_{\hat{\theta} \hat{\varphi}} \).

(3) In order to prove decay for the flux of \( H \), that is a generalized energy that would control the \( L^2 \) norm of some components of the fields near the horizon, one is confronted to a situation where it seems crucial to control a space-time integral supported on a bounded region in space near the event horizon, that contains the non-middle components. This could be overcome by the assumed Morawetz estimate, (8), that could be used to bound the space-time integral of the non-middle components as well. Just to simplify the calculations we assume that we have (13), since we expect the derivatives in assumption (8) to come out due to the trapped surface \( r = 3m \), i.e.
\[
\int_{t = t_i}^{t = t_{i+1}} \int_{r^* = r_0}^{r^* = R_0} \int_{S^2} \left[ |F_{\hat{\varphi} \hat{\varphi}}|^2 + \frac{1}{4} |F_{\hat{\varphi} \hat{\theta}}|^2 \right] r^* \, dr^* \, d\sigma^2 \, dt 
\lesssim |\hat{E}_{F}^{(K)}(t_i)| + |\hat{E}_{F}^{(K)}(t_{i+1})|
\]

(18)

Hence, away from the horizon, in \( r \geq r_1 \), the space-time integral, \( |I_{F}^{(H)}| \), can be bounded by the standard energy supported on a bounded region in space, to which one can prove decay due to the boundedness of the conformal energy that we would have already established in the first part.

Near the horizon, in \( r \leq r_1 \), the choices \( h' \geq 0 \), \( h' \leq \frac{2m}{r} h \), \( (1 - \frac{2m}{r}) \frac{1}{2} h \leq h' \) were constructed on purpose to obtain \( 0 \leq -I_{F}^{(H)}(r \leq r_1) \). This last fact
will lead to an inequality on $-I_F^{(H)}$, that involve the flux of the standard energy that one can bound, and that from the vector field $H$ on $v = \text{constant}$ hypersurface, (107), for $v_i = t_i + r_i^*$, and $w_i = t_i - r_i^*$, where $t_i$ is defined as in the first part, $t_i = (1.1)t_0$:

$$
- I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1)
$$

$$
- F_F^{(H)}(v = v_{i+1})(w_i \leq w \leq \infty) - F_F^{(H)}(w = \infty)(v_i \leq v \leq v_{i+1})
$$

$$
\lesssim F_F^{(\mathbf{\hat{H}})}(w = w_i)(v_i \leq v \leq v_{i+1}) - F_F^{(H)}(v = v_i)(w_i \leq w \leq \infty)
$$

$$
+ |E_F^{(\mathbf{\hat{H}})}(-0.85)t_i \leq r^* \leq (0.85)t_i)(t = t_i)|
$$

(19)

Using the Cauchy stability one can bound the flux from $H$ by the initial data prescribed on the initial Cauchy hypersurface, and hence bound the space-time integral $-I_F^{(H)}(r \leq r_1)$. In addition, we can prove inequality (111):

$$
\inf_{v_i \leq v \leq v_{i+1}} - F_F^{(H)}(v)(w_i \leq w \leq \infty)
$$

$$
\lesssim - I_F^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1)
$$

$$
+ \sup_{v_i \leq v \leq v_{i+1}} F_F^{(\mathbf{\hat{H}})}(v)(w_i \leq w \leq \infty)(r \geq r_1)
$$

(20)

To prove decay to the flux from $H$ on $v = \text{constant}$ hypersurfaces near the horizon and on $w = \text{constant}$ hypersurfaces on a segment of fixed length in $v$, we apply the last two inequalities above in the rectangle prescribed by $[v_i, v_{i+1}],[w_i, \infty]$, and we commutate them properly. This will lead to decay in $v_i = \max\{1, v\}$ of the flux from $H$ on $v = \text{constant}$ near the horizon, and on $w = \text{constant}$ hypersurfaces with a fixed length in $v$, as shown in estimates (119) and (120):

$$
- F_F^{(H)}(v)(2m \leq r \leq R) \lesssim \frac{||E_F^{(\mathbf{\hat{H}}})| + E_F^{\#}(\mathbf{\hat{H}})(t = t_0) + E_F^{M}|}{v_+^2}
$$

(21)

and,

$$
- F_F^{(H)}(w)(v - 1 \leq \mathbf{\eta} \leq v) \lesssim \frac{||E_F^{(\mathbf{\hat{H}}})| + E_F^{\#}(\mathbf{\hat{H}})(t = t_0) + E_F^{M}|}{v_+^2}
$$

(22)

Finally: (4) To prove decay for the normalized components $F_{\mathbf{\eta}\mathbf{\hat{e}}}$ and $F_{\mathbf{\eta}^1 \mathbf{\hat{e}}}$, we make use of a Sobolev inequality restricted on $w = \text{constant}$ hypersurfaces with a fixed length in $v$. Using the field equations, the $L^2$ norms of derivatives in the direction of $\mathbf{\hat{n}}$ can be controlled by the $L^2$ norms of $F_{\mathbf{\eta}\mathbf{\hat{e}}}$, $F_{\mathbf{\hat{e}}\mathbf{\hat{e}}}$ and of their angular derivatives, and of $F_{\mathbf{\eta}\mathbf{\hat{e}}}$ and $F_{\mathbf{\eta}^1 \mathbf{\hat{e}}}$. This leads to a bound by the flux obtained from $H$ on $w = \text{constant}$ hypersurfaces, of $F$ and its angular momentum derivatives. Since, those are Killing derivatives, and since the Maxwell equations are linear, and we proved decay of for the flux, this leads to the desired result.
We do the same to prove decay for the components $F_{\hat{\nu}e_1}$ and $F_{\hat{\nu}e_2}$ except that this time, the $L^2$ norms of the derivatives in the direction of $\frac{\partial}{\partial v}$ can be bounded by the $L^2$ norms of the time derivatives of $F_{\hat{\nu}e_1}$, $F_{\hat{\nu}e_2}$, and of the angular derivatives of $F_{\hat{\nu}\hat{\mu}}$, $F_{e_1e_2}$, and of $F_{\hat{\nu}e_1}$, $F_{\hat{\nu}e_2}$, using the field equations and the Bianchi identities.

The components $\sqrt{1 - \frac{2m}{r}} F_{\hat{\nu}e_1}$ and $\sqrt{1 - \frac{2m}{r}} F_{\hat{\nu}e_2}$ can be controlled by using a Sobolev inequality where we integrate on the hypersurfaces $v = \text{constant}$. As a result of direct computation using the field equations and the Bianchi identities, the $L^2$ norms of derivatives in the direction of $\frac{\partial}{\partial w}$ can be controlled by the $L^2$ norms of time derivatives of $\sqrt{1 - \frac{2m}{r}} F_{\hat{\nu}e_1}$, $\sqrt{1 - \frac{2m}{r}} F_{\hat{\nu}e_2}$, of angular derivatives of $\sqrt{1 - \frac{2m}{r}} F_{\hat{\mu}\hat{\nu}}$, $\sqrt{1 - \frac{2m}{r}} F_{e_1e_2}$, and of $\sqrt{1 - \frac{2m}{r}} F_{\hat{\nu}e_1}$, $\sqrt{1 - \frac{2m}{r}} F_{\hat{\nu}e_2}$. Making use of the decay of the flux from the vector field $H$ on $v = \text{constant}$, we obtain decay of these local $L^2$ norms, and hence we prove pointwise decay for $\sqrt{1 - \frac{2m}{r}} F_{\hat{\nu}e_a}$, $a \in \{1, 2\}$.

In order to cover the whole exterior region in theorem (1), we chose in the first part $R = r_1$, and we can choose $r_1 = r_0$.

**Remark 1.4.** The whole manuscript is written in an expository way, where we detail all the calculations, and we show standard material in the Appendix.

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## 2. Conservation Laws

Let $\Psi_{\mu\nu}$ be a an anti-symmetric two tensor valued in the Lie algebra. Consider the energy-momentum tensor

$$T_{\mu\nu}(\Psi) = \langle \Psi_{\mu\beta}, \Psi_{\nu}^{\beta} \rangle - \frac{1}{4} g_{\mu\nu} \langle \Psi_{\alpha\beta}, \Psi^{\alpha\beta} \rangle$$

(23)
Considering the Schwarzschild time \( t' \) (see (141) in Appendix), and considering two spacelike hypersurfaces \( \Sigma_{t'1}, \Sigma_{t'2} \), \( t'2 > t'1 \). We consider the region \( B = J^+(\Sigma_{t'1}) \cap J^-(\Sigma_{t'2}) \cap D \), where \( D \) is the closure of the exterior region of the black hole, known as the domain of outer-communication of the black hole.

Considering a vector field \( V^\nu \) we let

\[
J_\mu(V) = V^\nu T_{\mu\nu} = T_\mu V
\]

We have,

\[
\nabla^\mu J_\mu(V) = \partial^\mu T_{\mu V} - T(\nabla^\mu e_\mu, V) = \nabla^\mu T_{\mu V} + T(e_\mu, \nabla^\mu V)
\]

Hence,

\[
\nabla^\mu J_\mu(V) = \nabla^\mu T_{\mu V} + T(e_\mu, \nabla^\mu V)
\]

\[
= V^\nu (\nabla^\mu T_{\mu\nu}) + \tau^\nu_\mu T_{\mu\nu}
\]

(by symmetry of \( T \))

Applying the divergence theorem on \( J_\mu(V) \) in the region \( B \) bounded to the past by \( \Sigma_{t'1} \) and to the future by \( \Sigma_{t'2} \), and by a null hypersurface \( N \), we obtain:

\[
\int_B V^\nu (\nabla^\mu T_{\mu\nu})dV_B + \int_B \tau^\nu_\mu (V)T_{\mu\nu}dV_B = \int_{\Sigma_{t'1}} J_\mu(V)n^\mu dV_{\Sigma_{t'1}} - \int_{\Sigma_{t'2}} J_\mu(V)n^\mu dV_{\Sigma_{t'2}}
\]

\[
- \int_N J_\mu(V)n^\mu dV_N
\]

(24)

where \( n^\mu \) are the unit normal to the hypersurfaces \( \Sigma \), \( n^\mu_N \) is any null generator of \( N \), \( dV_\Sigma \) are the induced volume forms and \( dV_N \) is defined such that the divergence theorem applies.

Considering the Maxwell field \( F \), as it verifies (10) and (11), we have

\[
\nabla^\nu T_{\mu\nu}(F) = 0
\]

(25)

Also, considering the spherical symmetry of the Schwarzschild black hole, the angular momentum operators are Killing and therefore \( L_{\Omega_j} F \) verifies the Maxwell equations(10) and (11), and thus

\[
\nabla^\nu T_{\mu\nu}(L_{\Omega_j} F) = 0
\]

where \( \Omega_j \) is a basis of angular momentum operators, \( j \in \{1, 2, 3\} \).
And given that $\frac{\partial}{\partial \tau}$ is Killing, we have

$$\nabla^\nu T_{\mu\nu}(\mathcal{L}_t F) = 0$$

Taking $\Psi$ any product of $\mathcal{L}_t$, $\mathcal{L}_{\Omega_i}$, and $F$, where $\Omega_i$, $i \in \{1, 2, 3\}$, is a basis of angular momentum operators, we get that $\Psi$ satisfies the Maxwell equations (10) and (11) since $\frac{\partial}{\partial \tau}$ and $\Omega_j$, $j \in \{1, 2, 3\}$ are Killing vector fields and therefore. Hence,

$$\nabla^\nu T_{\mu\nu}(\Psi) = 0 \quad (26)$$

Let,

$$\mu = \frac{2m}{r} \quad (27)$$

Now, let’s compute,

$$< \Psi_{\alpha\beta}, \Psi^{\alpha\beta} >$$

$$= < \Psi_{w\alpha}, \Psi^{w\alpha} > + < \Psi_{v\alpha}, \Psi^{v\alpha} > + < \Psi_{\theta\alpha}, \Psi^{\theta\alpha} > + < \Psi_{\phi\alpha}, \Psi^{\phi\alpha} >$$

$$= \frac{-2}{(1 - \mu)} < \Psi_{w\alpha}, \Psi^{w\alpha} > + \frac{-2}{(1 - \mu)} < \Psi_{v\alpha}, \Psi^{v\alpha} > + \frac{1}{r^2} < \Psi_{\theta\alpha}, \Psi^{\theta\alpha} > + \frac{1}{r^2 \sin^2 \theta} < \Psi_{\phi\alpha}, \Psi^{\phi\alpha} >$$

$$= \frac{-2}{(1 - \mu)} < \Psi_{w\theta}, \Psi^{w\theta} > - \frac{2}{(1 - \mu)} < \Psi_{v\theta}, \Psi^{v\theta} > - \frac{2}{(1 - \mu)} < \Psi_{w\phi}, \Psi^{w\phi} > - \frac{2}{(1 - \mu)} < \Psi_{v\phi}, \Psi^{v\phi} >$$

$$+ \frac{1}{r^2} < \Psi_{\theta\theta}, \Psi^{\theta\theta} > + \frac{1}{r^2} < \Psi_{\theta\phi}, \Psi^{\theta\phi} > + \frac{1}{r^2 \sin^2 \theta} < \Psi_{\phi\theta}, \Psi^{\phi\theta} >$$

$$= \frac{-2}{r^2(1 - \mu)} < \Psi_{w\theta}, \Psi^{v\theta} > - \frac{2}{r^2 \sin^2 \theta (1 - \mu)} < \Psi_{w\phi}, \Psi^{v\phi} > - \frac{4}{(1 - \mu)} < \Psi_{w\theta}, \Psi^{w\theta} >$$

$$- \frac{2}{r^2(1 - \mu)} < \Psi_{v\theta}, \Psi^{w\theta} > - \frac{2}{r^2 \sin^2 \theta (1 - \mu)} < \Psi_{v\phi}, \Psi^{w\phi} > - \frac{4}{(1 - \mu)} < \Psi_{v\theta}, \Psi^{v\theta} >$$

$$- \frac{2}{r^2 \sin^2 \theta (1 - \mu)} < \Psi_{\phi\theta}, \Psi^{w\phi} > - \frac{2}{r^2 \sin^2 \theta (1 - \mu)} < \Psi_{\phi\phi}, \Psi^{w\phi} > - \frac{1}{r^2 \sin^2 \theta} < \Psi_{\phi\theta}, \Psi^{\phi\theta} >$$

$$= \frac{-8}{r^2(1 - \mu)} < \Psi_{w\theta}, \Psi^{v\theta} > - \frac{8}{r^2 \sin^2 \theta (1 - \mu)} < \Psi_{w\phi}, \Psi^{v\phi} > - \frac{8}{(1 - \mu)} < \Psi_{w\theta}, \Psi^{w\theta} > + \frac{2}{r^4 \sin^2 \theta} < \Psi_{\phi\theta}, \Psi^{\phi\theta} >$$

Computing,

$$T_{ww} = < \Psi_{w\beta}, \Psi^{w\beta} > - \frac{1}{2} g_{ww} < \Psi_{\alpha\beta}, \Psi^{\alpha\beta} >$$

$$= < \Psi_{w\theta}, \Psi^{w\theta} > + < \Psi_{w\theta}, \Psi^{w\phi} > + < \Psi_{w\phi}, \Psi^{w\phi} >$$

$$= \frac{1}{r^2} < \Psi_{w\theta}, \Psi^{w\theta} > + \frac{1}{r^2 \sin^2 \theta} < \Psi_{w\phi}, \Psi^{w\phi} >$$
\[
T_{\nu\nu} = <\Psi_{\nu\alpha}, \Psi_{\nu}^\alpha > \\
= <\Psi_{\nu\nu}, \Psi_{\nu}^w > + <\Psi_{\nu\theta}, \Psi_{\nu}^\theta > + <\Psi_{\nu\phi}, \Psi_{\nu}^\phi > \\
= \frac{1}{r^2} |\Psi_{\nu\phi}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\nu\phi}|^2
\]

\[
T_{vw} = <\Psi_{vw}, \Psi_{w}^\alpha > - \frac{1}{4} g_{vw} <\Psi_{\alpha\beta}, \Psi^{\alpha\beta} > \\
= <\Psi_{vw}, \Psi_{w}^w > + <\Psi_{vw}, \Psi_{w}^\theta > + <\Psi_{vw}, \Psi_{w}^\phi > - \frac{1}{4} g_{vw} <\Psi_{\alpha\beta}, \Psi^{\alpha\beta} > \\
= \frac{-2}{(1 - \mu)} <\Psi_{vw}, \Psi_{vw} > + \frac{1}{r^2} <\Psi_{vw}, \Psi_{w\theta} > + \frac{1}{r^2 \sin^2 \theta} <\Psi_{vw}, \Psi_{w\phi} > \\
\quad + \frac{(1 - \mu)}{8} \frac{-8}{r^2 (1 - \mu)} <\Psi_{w\theta}, \Psi_{w\theta} > - \frac{8}{r^2 \sin^2 \theta (1 - \mu)} <\Psi_{w\phi}, \Psi_{w\phi} > \\
\quad - \frac{8}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{2}{r^4 \sin^2 \theta} |\Psi_{\phi\phi}|^2 \\
= \frac{2}{(1 - \mu)} |\Psi_{vw}|^2 + \frac{1}{r^2} <\Psi_{vw}, \Psi_{w\theta} > + \frac{1}{r^2 \sin^2 \theta} <\Psi_{vw}, \Psi_{w\phi} > \\
\quad - \frac{1}{r^2} <\Psi_{w\theta}, \Psi_{w\theta} > - \frac{1}{r^2 \sin^2 \theta} <\Psi_{w\phi}, \Psi_{w\phi} > \\
\quad - \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{(1 - \mu)}{4r^4 \sin^2 \theta} |\Psi_{\phi\phi}|^2 \\
= \frac{1}{(1 - \mu)} |\Psi_{vw}|^2 + \frac{(1 - \mu)}{4r^4 \sin^2 \theta} |\Psi_{\phi\phi}|^2
\]

\[
T_{\theta\theta} = <\Psi_{\theta\alpha}, \Psi_{\theta}^\alpha > - \frac{1}{4} g_{\theta\theta} <\Psi_{\alpha\beta}, \Psi^{\alpha\beta} > \\
= <\Psi_{\theta\nu}, \Psi_{\nu}^w > + <\Psi_{\theta\nu}, \Psi_{\nu}^\theta > + <\Psi_{\theta\nu}, \Psi_{\nu}^\phi > \\
\quad - \frac{r^2}{4} \frac{-8}{r^2 (1 - \mu)} <\Psi_{\theta\nu}, \Psi_{\theta\nu} > - \frac{8}{r^2 \sin^2 \theta (1 - \mu)} <\Psi_{\theta\phi}, \Psi_{\theta\phi} > \\
\quad - \frac{8}{(1 - \mu)^2} |\Psi_{\nu\nu}|^2 + \frac{2}{r^4 \sin^2 \theta} |\Psi_{\phi\phi}|^2 \\
= \frac{-2}{(1 - \mu)} <\Psi_{\theta\nu}, \Psi_{\theta\nu} > - \frac{2}{(1 - \mu)} <\Psi_{\theta\nu}, \Psi_{\theta\nu} > + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\theta\phi}|^2 \\
\quad + \frac{2}{(1 - \mu)} <\Psi_{\theta\nu}, \Psi_{\theta\nu} > + \frac{2}{r^2 \sin^2 \theta} <\Psi_{w\phi}, \Psi_{w\phi} > \\
\quad + \frac{2r^2}{(1 - \mu)^2} |\Psi_{\nu\nu}|^2 - \frac{1}{2r^2 \sin^2 \theta} |\Psi_{\phi\phi}|^2 \\
= \frac{-2}{(1 - \mu)} <\Psi_{\theta\nu}, \Psi_{\theta\nu} > + \frac{2}{r^2 \sin^2 \theta} <\Psi_{w\phi}, \Psi_{w\phi} > \\
\quad + \frac{2r^2}{(1 - \mu)^2} |\Psi_{\nu\nu}|^2 + \frac{1}{2r^2 \sin^2 \theta} |\Psi_{\phi\phi}|^2
\]
This gives,

\[ T_{\phi \phi} = < \Psi_{\phi \alpha}, \Psi_{\phi}^{\alpha} > - \frac{1}{4} g_{\phi \phi} < \Psi_{\alpha \beta}, \Psi^{\alpha \beta} > \]
\[ = < \Psi_{\phi \nu}, \Psi_{\nu}^{\phi} > + < \Psi_{\phi \omega}, \Psi_{\omega}^{\phi} > + < \Psi_{\phi \theta}, \Psi_{\phi}^{\theta} > \]
\[ - \frac{r^2 \sin^2 \theta}{4} \left[ -8 \frac{1}{r^2(1 - \mu)} < \Psi_{w \theta}, \Psi_{v \theta} > - \frac{8}{r^2 \sin^2 \theta(1 - \mu)} < \Psi_{w \phi}, \Psi_{v \phi} > \right] \]
\[ - \frac{8}{(1 - \mu)^2} |\Psi_{v \omega}|^2 + \frac{2}{r^4 \sin^2 \theta} |\Psi_{\phi \theta}|^2 \]
\[ = - \frac{2}{2r^2} |\Psi_{v \omega}|^2 - \frac{1}{2r^2} |\Psi_{\phi \theta}|^2 \]

\[ \pi^{\alpha \beta} T_{\alpha \beta} \]
\[ = T_{\omega \nu} \pi^{\omega \nu} + T_{v \nu} \pi^{v \nu} + 2T_{w \nu} \pi^{w \nu} + T_{\theta \theta} \pi^{\theta \theta} + T_{\phi \phi} \pi^{\phi \phi} \]
\[ = \frac{1}{r^2} |\Psi_{w \theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w \phi}|^2 \left( -\frac{2}{1 - \mu} \partial_v V^w \right) \]
\[ + \frac{1}{r^2} |\Psi_{v \phi}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\phi \theta}|^2 \left( -\frac{2}{1 - \mu} \partial_w V^v \right) \]
\[ + \frac{2}{1 - \mu} |\Psi_{v \omega}|^2 + \frac{(1 - \mu)}{4r^4 \sin^2 \theta} |\Psi_{\phi \theta}|^2 \left( -\frac{1}{1 - \mu} \left[ \partial_v V^v + \partial_w V^w + \frac{m}{r^2} (V^v - V^w) \right] \right) \]
\[ + \frac{2}{1 - \mu} |\Psi_{v \omega}|^2 + \frac{1}{2r^2 \sin^2 \theta} |\Psi_{\phi \theta}|^2 \left( \frac{(1 - \mu)^2}{2r^2} (V^v - V^w) \right) \]
\[ + \frac{2}{1 - \mu} |\Psi_{v \omega}|^2 + \frac{2}{2r^2} |\Psi_{\phi \theta}|^2 \left( \frac{(1 - \mu)^2}{2r^2 \sin^2 \theta} (V^v - V^w) \right) \]
\[ + \frac{2}{1 - \mu} |\Psi_{v \omega}|^2 + \frac{1}{2r^2} |\Psi_{\phi \theta}|^2 \left( \frac{(1 - \mu)^2}{2r^2 \sin^2 \theta} (V^v - V^w) \right) \]
Thus,

\[
\pi^{\alpha \beta}(V)T_{\alpha \beta}(\Psi) = \left[ \frac{1}{r^2} \Psi_{w \theta} \right]^2 + \frac{1}{r^2 \sin^2 \theta} \left| \Psi_{w \phi} \right|^2 \left( \frac{-2}{1 - \mu} \partial_v V^w \right)
\]

(28)

\[
+ \frac{1}{r^2} \left| \Psi_{v \theta} \right|^2 + \frac{1}{r^2 \sin^2 \theta} \left| \Psi_{v \phi} \right|^2 \left( \frac{-2}{1 - \mu} \partial_w V^v \right)
\]

\[
+ \frac{4r^2}{(1 - \mu)^2} \left| \Psi_{vw} \right|^2 + \frac{1}{4r^4 \sin^2 \theta} \left| \Psi_{\phi \theta} \right|^2 \left( \frac{(1 - \mu)}{2r^3} (V^v - V^w) \right)
\]

2.1. The vector field \( \frac{\partial}{\partial t} \).

Let,

\[ t^\gamma = \left( \frac{\partial}{\partial t} \right)^\gamma \]

We have,

\[
\frac{\partial}{\partial v} = \frac{\partial t}{\partial v} \frac{\partial}{\partial t} + \frac{\partial r^*}{\partial v} \frac{\partial}{\partial r^*}
\]

\[
= \frac{1}{2} \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial}{\partial r^*}
\]

(29)

\[
\frac{\partial}{\partial w} = \frac{\partial t}{\partial w} \frac{\partial}{\partial t} + \frac{\partial r^*}{\partial w} \frac{\partial}{\partial r^*}
\]

\[
= \frac{1}{2} \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial}{\partial r^*}
\]

(30)

Hence,

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial v} + \frac{\partial}{\partial w}
\]

\[
= t^v \frac{\partial}{\partial v} + t^w \frac{\partial}{\partial w}
\]

where \( t^v = 1 \) and \( t^w = 1 \). Thus,
\[ \pi^{\alpha\beta}(\frac{\partial}{\partial t})T_{\alpha\beta}(\Psi) \]

\[= \left[ \frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 \right] (\frac{-2}{1 - \mu}) \partial_v t^w \]

\[+ \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \left[ \frac{-2}{1 - \mu} \partial_w t^v \right] \]

\[+ \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \left[ -2[\partial_v t^v + \partial_w t^w + \frac{(3\mu - 2)}{2r}(t^v - t^w)] \right] \]

\[= 0 \]

In other words, since \( \frac{\partial}{\partial t} \) is Killing, its deformation tensor vanishes, i.e. \( \pi^{\alpha\beta}(\frac{\partial}{\partial t}) = 0 \), therefore

\[ \pi^{\alpha\beta}(\frac{\partial}{\partial t})T_{\alpha\beta}(\Psi) = 0 \] (31)

Let,

\[ \hat{\partial} \frac{\partial}{\partial t} = \frac{1}{\sqrt{1 - \mu}} \frac{\partial}{\partial t} \]

(32)

\[ \hat{\partial} \frac{\partial}{\partial r^*} = \frac{1}{\sqrt{1 - \mu}} \frac{\partial}{\partial r^*} \]

(33)

We also assume that,

\[ \lim_{r \to \infty} \Psi_{\hat{\mu}\hat{\nu}}(t = t_0) = 0 \] (34)

From a local existence result that ensures that a certain regularity will be conserved, one can prove that the condition above will be satisfied for all time, i.e.

\[ \lim_{r \to \infty} \Psi_{\hat{\mu}\hat{\nu}}(t) = 0 \] (35)

Thus, we will have no integrals on spatial infinity.

Applying the divergence theorem to the vector \( t^\mu T_{\mu\nu} \) in the region \( B \) bounded by two hypersurfaces \( \Sigma_1 \) and \( \Sigma_2 \) defined by \( t = constant \), where \( t_2 \geq t_1 \), we get,
We have,
\[ dV_{\Sigma_t} = (\sqrt{1 - \mu})r^2d\sigma^2 \]
(we have \( \sin(\theta) \geq 0 \) because \( 0 \leq \theta \leq \pi \). Thus,
\[ dV_{\Sigma_t} = (\sqrt{1 - \mu})r^2\sigma^2 \]
On the other hand,
\[ T_{\hat{t}\hat{t}}(\Psi) = \frac{1}{4} g_{\hat{t}\hat{t}} < \Psi_{\hat{t}\hat{t}} , \Psi_{\hat{t}\hat{t}} > < \Psi_{\hat{r}\hat{r}} , \Psi_{\hat{r}\hat{r}} > + \frac{1}{2} |\Psi_{\hat{w}\hat{w}}|^2 + \frac{1}{2} |\Psi_{\hat{v}\hat{v}}|^2 \]
\[ < \Psi_{\hat{t}\hat{t}}, \Psi_{\hat{t}} > = - < \Psi_{\hat{t}} , \Psi_{\hat{t}} > + < \Psi_{\hat{r}\hat{r}}, \Psi_{\hat{r}\hat{r}} > + < \Psi_{\hat{t}\hat{r}}, \Psi_{\hat{t}\hat{r}} > + < \Psi_{\hat{t}\hat{\theta}}, \Psi_{\hat{t}\hat{\theta}} > \\
= |\Psi_{\hat{t}\hat{r}}|^2 + |\Psi_{\hat{t}\hat{\theta}}|^2 + |\Psi_{\hat{t}\hat{\phi}}|^2 \]
From (29) and (30), we get,
\[ \Psi_{v\theta} = \Psi_{\mu\nu}(\partial_{\theta})^\mu(\partial_{\theta})^\nu = \Psi_{\mu\nu}(\frac{1}{2} \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial}{\partial r^*})^\mu(\frac{\partial}{\partial \theta})^\nu \]
\[ = \frac{1}{2} \Psi_{\mu\nu}(\frac{\partial}{\partial t})^\mu(\frac{\partial}{\partial \theta})^\nu + \frac{1}{2} \Psi_{\mu\nu}(\frac{\partial}{\partial r^*})^\mu(\frac{\partial}{\partial \theta})^\nu \]
\[ = \frac{1}{2} \Psi_{t\theta} + \frac{1}{2} \Psi_{r^*\theta} \tag{36} \]

\[ \Psi_{w\theta} = \Psi_{\mu\nu}(\partial_{\theta})^\mu(\partial_{\theta})^\nu = \Psi_{\mu\nu}(\frac{1}{2} \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial}{\partial r^*})^\mu(\frac{\partial}{\partial \theta})^\nu \]
\[ = \frac{1}{2} \Psi_{\mu\nu}(\frac{\partial}{\partial t})^\mu(\frac{\partial}{\partial \theta})^\nu - \frac{1}{2} \Psi_{\mu\nu}(\frac{\partial}{\partial r^*})^\mu(\frac{\partial}{\partial \theta})^\nu \]
\[ = \frac{1}{2} \Psi_{t\theta} - \frac{1}{2} \Psi_{r^*\theta} \tag{37} \]

and similarly,

\[ \Psi_{v\phi} = \frac{1}{2} \Psi_{t\phi} + \frac{1}{2} \Psi_{r^*\phi} \tag{38} \]

\[ \Psi_{w\phi} = \frac{1}{2} \Psi_{t\phi} - \frac{1}{2} \Psi_{r^*\phi} \tag{39} \]

Thus,

\[ \langle \Psi_{\tilde{t}\tilde{\theta}}, \Psi_{\tilde{t}\tilde{\theta}} \rangle = \langle \frac{\Psi_{\tilde{t}\tilde{t}} - \Psi_{\tilde{r}^*\tilde{\theta}}}{2}, \frac{\Psi_{\tilde{t}\tilde{t}} + \Psi_{\tilde{r}^*\tilde{\theta}}}{2} \rangle \]
\[ = \frac{1}{4} |\Psi_{\tilde{t}\tilde{t}}|^2 - \frac{|\Psi_{\tilde{r}^*\tilde{\theta}}|^2}{2} \]

and,

\[ \langle \Psi_{\tilde{w}\tilde{\phi}}, \Psi_{\tilde{w}\tilde{\phi}} \rangle = \frac{1}{4} |\Psi_{\tilde{w}\tilde{\phi}}|^2 - \frac{|\Psi_{\tilde{r}^*\tilde{\phi}}|^2}{2} \]

We have,
\[
\Psi_{\mu\nu} = \Psi_{\mu\nu}(0) = \Psi_{\mu\nu}(T) = \Psi_{\mu\nu}(T) = \Psi_{\mu\nu}(T)
\]

Therefore,

\[
T_{ij}(\Psi) = 2 \Psi_{ij} = \frac{1}{2} \Psi_{ij}\]

Thus,

\[
\int_B T_{ij}(\Psi) dV_B = \int_{\Sigma_{t_2}} J_{ij}(T)(\sqrt{1-\mu})dV_{\Sigma_{t_2}} - \int_{\Sigma_{t_1}} J_{ij}(T)(\sqrt{1-\mu})dV_{\Sigma_{t_1}}
\]

\[
= \int_{\Sigma_{t_1}} \frac{1}{2} (\Psi_t)^2 + (\Psi_{t\hat{t}})^2 + (\Psi_{\hat{t}\hat{t}})^2 + (\Psi_{\hat{t}\hat{\theta}})^2 + (\Psi_{\hat{\theta}\hat{\theta}})^2 (1-\mu)(t_2-t_1) dr^2 d\sigma^2
\]

\[
E_{\Psi}(t) = E_{\Psi}(t_{t_2}) - E_{\Psi}(t_{t_1})
\]

where,

\[
E_{\Psi}(t_{t_2}) = \int_{\Sigma_{t_2}} \frac{1}{2} (\Psi_t)^2 + (\Psi_{t\hat{t}})^2 + (\Psi_{\hat{t}\hat{t}})^2 + (\Psi_{\hat{t}\hat{\theta}})^2 + (\Psi_{\hat{\theta}\hat{\theta}})^2 (1-\mu)(t_2-t_1) dr^2 d\sigma^2
\]

Since \(\Psi_{\mu\nu}\) is anti-symmetric two tensor, we get,

\[
\Psi_{\mu\nu} = \frac{1}{2} \Psi_{\mu\nu}(\frac{\partial}{\partial r})^\mu (\frac{\partial}{\partial t})^\nu
\]

\[
= \frac{1}{2} \Psi_{\mu\nu}(\frac{\partial}{\partial r})^\mu (\frac{\partial}{\partial t})^\nu
\]
Taking $\Psi$ any product of $L_t, L_{\Omega_i},$ and $F,$ where $\Omega_i, i \in \{1, 2, 3\},$ is a basis of angular momentum operators, since $\int_B T^\nu (\nabla^\mu T_{\mu\nu}(\Psi)) dV_B = 0,$ we have conservation of the energy generated from the vector field $\frac{\partial}{\partial t}.$

2.2. The vector field $K.$

Let

$$K = -\omega^2 \frac{\partial}{\partial \omega} - v^2 \frac{\partial}{\partial v}$$

$$= K^\omega \frac{\partial}{\partial \omega} + K^v \frac{\partial}{\partial v}$$

We have,

$$\partial_v K^\omega = -\partial_v \omega^2 = 0$$

$$\partial_\omega K^v = -\partial_\omega v^2 = 0$$

Computing,

$$\pi^{\alpha\beta}(K)T_{\alpha\beta}(\Psi) = (v + w)[4 + \frac{(3\mu - 2)}{r}(v - w)], \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2$$

Thus,

$$\pi^{\alpha\beta}(K)T_{\alpha\beta}(\Psi) = (v + w)[4 + \frac{(3\mu - 2)}{r}(v - w)], \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2$$

Recall that $v$ and $w$ are defined as in (136) and (137), thus,

$$v + \omega = 2t$$

$$v - \omega = 2r^*$$

Therefore, we also have,

$$\pi^{\alpha\beta}(K)T_{\alpha\beta}(\Psi) = 4t[2 + \frac{(3\mu - 2)r^*}{r}], \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2$$

(44)
We define,
\[ J^{(K)}_{\psi}(t_i \leq t \leq t_{i+1}) = \int_{t=t_i}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{S^2} \pi^{\alpha\beta}(K) T_{\alpha\beta}(\Psi) dVol \]
(45)

Computing,
\[ E^{(K)}(t_i) = \int_{r^*=-\infty}^{r^*=\infty} \int_{S^2} J_\alpha(K) n^\alpha dVol_{t=t_i}(t = t_i) \]
where
\[ n^\alpha = -\frac{\partial}{\sqrt{(1 - \mu)}} \]
and
\[ dVol_{t=t_i} = r^2 \sqrt{(1 - \mu)} d\sigma dr^* \]

\[ E^{(K)}_{\psi}(t_i) \]
\[ = \int_{r^*=-\infty}^{r^*=\infty} \int_{S^2} -\frac{1}{\sqrt{(1 - \mu)}} \left( [\frac{\partial}{\partial v}] J_\alpha(K) r^2 \sqrt{(1 - \mu)} d\sigma^2 dr^* \right) \]
\[ = \int_{r^*=-\infty}^{r^*=\infty} \int_{S^2} \left( v^2 [\frac{1}{r^2(1 - \mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta(1 - \mu)} |\Psi_{w\phi}|^2] \right) \]
\[ + \omega \left( \frac{1}{(1 - \mu)^2} |\Psi_{w\theta}|^2 + \frac{1}{4r^2 \sin^2 \theta(1 - \mu)} |\Psi_{w\phi}|^2 \right) \]
\[ \right) r^2 (1 - \mu) d\sigma^2 dr^* \right) \]
(47)

2.3. The vector field \( G \).

Let
\[ G = -f(r^*) \frac{\partial}{\partial \omega} + f(r^*) \frac{\partial}{\partial v} \]
(48)
where \( f(r^*) \) depends only on \( r^* \).

Computing,
\[ \frac{\partial}{\partial \omega} G^\omega = \frac{\partial r^*}{\partial \omega} \frac{\partial}{\partial r^*} G^\omega + \frac{\partial t}{\partial \omega} \frac{\partial}{\partial t} G^\omega \]
\[ = -\frac{1}{2} \frac{\partial}{\partial r^*} G^\omega + 0 \]
\[ = \frac{1}{2} f' \]
where \( f' = \frac{\partial}{\partial v}f \)

Similarly,

\[
\begin{align*}
\frac{\partial}{\partial v} G^v &= \frac{\partial r^*}{\partial v} \frac{\partial}{\partial r^*} G^v + \frac{\partial t}{\partial v} \frac{\partial}{\partial t} G^v \\
&= \frac{1}{2} \frac{\partial}{\partial r^*} G^v + 0 \\
&= \frac{1}{2} f' \\
\frac{\partial}{\partial v} G^\omega &= \frac{\partial r^*}{\partial v} \frac{\partial}{\partial r^*} G^\omega + \frac{\partial t}{\partial v} \frac{\partial}{\partial t} G^\omega \\
&= \frac{1}{2} \frac{\partial}{\partial r^*} G^\omega + 0 \\
&= -\frac{1}{2} f' \\
\frac{\partial}{\partial \omega} G^v &= \frac{\partial r^*}{\partial \omega} \frac{\partial}{\partial r^*} G^v + \frac{\partial t}{\partial \omega} \frac{\partial}{\partial t} G^v \\
&= -\frac{1}{2} \frac{\partial}{\partial r^*} G^v + 0 \\
&= -\frac{1}{2} f'.
\end{align*}
\]

\[
\pi^{\alpha\beta}(G) T_{\alpha\beta}(\Psi) = \left[ \frac{1}{r^2} |\Psi_{\omega\theta}|^2 + \frac{1}{4r^2 \sin^2 \theta} |\Psi_{\omega\phi}|^2 \left( \frac{1}{2 f'} \right) \right] \\
+ \left[ \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{4r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \left( \frac{1}{2 f'} \right) \right] \\
+ \left[ \frac{1}{(1-\mu)^2} |\Psi_{w\theta}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{w\phi}|^2 \left( \frac{1}{2 f'} \right) \right].
\]
Finally, we obtain,

\[
T^{\alpha\beta}(\Psi_{\mu\nu})\pi_{\alpha\beta}(G) = \left[ \frac{1}{r^2} |\Psi_{\psi\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\psi\phi}|^2 + \frac{1}{r^2} |\Psi_{\nu\psi}|^2 \right] \frac{f'}{1 - \mu} \nonumber \\
-2\left[ \frac{1}{(1 - \mu)^2} |\Psi_{\psi\nu}|^2 + \frac{1}{4r^2 \sin^2 \theta} |\Psi_{\psi\phi}|^2 \right] (f' + \frac{f}{r}(3\mu - 2)) \quad (49)
\]

Computing,

\[
E_{\Psi}^{(G)}(t_i) = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} J_{\alpha}(G)(\Psi_{\mu\nu})n^{\alpha} dVol_{t = t_i}(t = t_i) \quad (50)
\]

where \(n^{\alpha} = -\frac{\partial}{\sqrt{(1 - \mu)}}\) and \(dVol_{t = t_i} = r^2 \sqrt{(1 - \mu)} d\sigma^2 dr^*\). Thus,

\[
E_{\Psi}^{(G)}(t_i) = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \frac{1}{\sqrt{(1 - \mu)}} \left( \frac{\partial}{\partial r} \right)^{\alpha} J_{\alpha}(G)(\Psi_{\mu\nu})r^2 \sqrt{(1 - \mu)} d\sigma^2 dr^*
\]

Recall (29) and (30), thus,

\[
G = f(r^*) \frac{\partial}{\partial r^*}
\]

Therefore,

\[
E_{\Psi}^{(G)}(t_i) = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} -fT_{r^*}(\Psi_{\mu\nu})r^2 d\sigma^2 dr^*
\]

\[
T_{r^*} = <\Psi_{t^\alpha}, \Psi_{r^*}^\alpha>
\]
\[
= <\Psi_{t^\alpha}, \Psi_{r^*}^\alpha> + <\Psi_{t^\theta}, \Psi_{r^*}^\theta> + <\Psi_{t^\phi}, \Psi_{r^*}^\phi>
\]
\[
= \frac{1}{r^2} <\Psi_{t^\theta}, \Psi_{r^*}^\theta> + \frac{1}{r^2 \sin^2 \theta} <\Psi_{t^\phi}, \Psi_{r^*}^\phi>
\]

Thus,

\[
E_{\Psi}^{(G)}(t_i) = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} -f \left[ \frac{1}{r^2} <\Psi_{t^\theta}, \Psi_{r^*}^\theta> + \frac{1}{r^2 \sin^2 \theta} <\Psi_{t^\phi}, \Psi_{r^*}^\phi> \right] r^2 d\sigma^2 dr^*
\]

Thus,

\[
E_{\Psi}^{(G)}(t_i)
\]
3. Bounding the Conformal Energy on $t = \text{constant}$ Hypersurfaces

Let $\Psi$ be any product of $L_t$, $L_{\Omega_i}$, and $F$, where $\Omega_i$, $i \in \{1, 2, 3\}$, is a basis of angular momentum operators. We know by then that we have (26).

3.1. Estimate for $E^{(G)}_{\Psi}$.

Let $f$ in (51) be a bounded function of $r^*$. Then, we have

$$|E^{(G)}_{\Psi}(t = t_i)| \lesssim |\hat{E}^{(G)}_{\Psi}(t = t_i)| \lesssim |\hat{E}^{(G)}_{\Psi}(t = t_i)|$$  \hspace{1cm} (53)

where,

$$\hat{E}^{(G)}_{\Psi}(t) = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \left[ \frac{1}{r^2} < \Psi_{t\theta}, \Psi_{r\theta} > + \frac{1}{r^2 \sin^2 \theta} < \Psi_{t\phi}, \Psi_{r\phi} > + \frac{1}{r^2 (1 - \mu)} |\Psi_{r\theta}|^2 + \frac{1}{r^2 (1 - \mu)} |\Psi_{r\phi}|^2 \right] r^2 (1 - \mu) ds^2 dr^*$$  \hspace{1cm} (54)

Proof

We have,

$$|E^{(G)}_{\Psi}(t_i)| = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} |f| \left[ \frac{1}{r^2} |\Psi_{t\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{t\phi}|^2 + \frac{1}{r^2 (1 - \mu)} |\Psi_{r\theta}|^2 + \frac{1}{r^2 (1 - \mu)} |\Psi_{r\phi}|^2 \right] r^2 ds^2 dr^*$$

(by using $a.b \lesssim a^2 + b^2$)

$$\lesssim \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} |f| \left[ \frac{1}{r^2} |\Psi_{t\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{t\phi}|^2 + \frac{1}{r^2 (1 - \mu)} |\Psi_{r\theta}|^2 + \frac{1}{r^2 (1 - \mu)} |\Psi_{r\phi}|^2 \right] r^2 ds^2 dr^*$$

(because $f$ is bounded).

Then, we have,
\[ E_\Psi(t = t_i) = -\int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \left( \frac{1}{2} |\Psi_{t, r^*}|^2 + \frac{1}{2} |\Psi_{\theta, \phi}|^2 + \frac{1}{2} |\Psi_{r, \theta}|^2 + \frac{1}{2} |\Psi_{r, \phi}|^2 \right) r^2 (1 - \mu) d\sigma d r^* \]

and therefore,

\[ |E_\Psi(t = t_i)| = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \left[ |\Psi_{t, \theta}|^2 + |\Psi_{t, \phi}|^2 + |\Psi_{r, \theta}|^2 + |\Psi_{r, \phi}|^2 \right] r^2 (1 - \mu) d\sigma d r^* \]

\[ \lesssim |E_\Psi(t = t_i)| \]

3.2. Controlling \( J^{(K)}_\Psi \) in terms of \( J^{(G)}_\Psi \).

\[ J^{(K)}_\Psi(t_i \leq t \leq t_{i+1}) \lesssim t_{i+1} J^{(G)}_\Psi(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \quad (55) \]

where \( r_0 \leq 3m \leq R_0 \). And we have,

\[ E^{(K)}_\Psi(t = t_{i+1}) \leq J^{(K)}_\Psi(t_i \leq t \leq t_{i+1}) + E^{(K)}_\Psi(t = t_i) \quad (56) \]

**Proof**

We have,

\[ J^{(K)}_\Psi(t_i \leq t \leq t_{i+1}) = \int_{t_{i+1}}^{t_{i+1}} \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} 4r^2 \left[ 2 + \frac{(3\mu - 2)r^*}{r} \right] \left| \frac{1}{(1 - \mu)^2} |\Psi_{v, \nu}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi, \theta}|^2 \right|^2 (1 - \mu) d\sigma d r^* \]

We remark that \( 2 + \frac{(3\mu - 2)r^*}{r} \) is positive only in a bounded interval \([r_0, R_0]\) where \( r_0 > 2m \) and \( R_0 > 3m \). To see this, notice that,

\[ \lim_{r^* \to -\infty} \left[ 2 + \frac{(3\mu - 2)r^*}{r} \right] = -\infty \]

and,
\[ \lim_{r \to \infty} \left[ 2 + \frac{(3\mu - 2)r^*}{r} \right] = 2 - 2.(1) = 0 \]

and

\[ \lim_{r \to 3m} \left[ 2 + \frac{(3\mu - 2)r^*}{r} \right] = 2 + 0 = 2 > 0 \]

More precisely, let’s look for the region where \( J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1}) \) is negative:

\[ 2 + \frac{r^*}{r}(-2 + 3\mu) \leq 0 \]

when

\[ r^*(3\mu - 2) \leq -2r \]

Choosing \( r_0 \) small enough such that \( (3\mu_0 - 2) \geq 0 \), then we need \( r_0 \) such that for \( r \leq r_0 \),

\[ r^* \leq -\frac{2r}{(3\mu - 2)} \]

so choose \( r_0 \) such that

\[ r_0^* \leq -\frac{2r_0}{(3\mu_0 - 2)} < 0 \]

or choose \( \hat{R}_0 \) large such that \( (3\mu(\hat{R}_0) - 2) \leq 0 \), and such that for \( r \geq \hat{R}_0 \), we get,

\[ r^* \geq -\frac{2r}{(3\mu - 2)} > 0 \]

then choose \( \hat{R} \) such that

\[ \hat{R}_0^* \geq -\frac{2\hat{R}_0}{(3\mu(\hat{R}_0) - 2)} > 0 \]

In conclusion choose \( r_0 \) such that

\[ r_0 < -\frac{2r_0}{(3\mu_0 - 2)} < 0 \]

and choose \( R_0 \) as the infimum of all \( \hat{R}_0 \) such that

\[ \hat{R}_0^* \geq -\frac{2\hat{R}_0}{(3\mu(\hat{R}_0) - 2)} > 0 \] (57)

Then in the region \( r \leq r_0 \) or \( r \geq R_0 \) we know that the integrand in \( J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1}) \) is negative.

Thus,
\[ J^{(K)}_{\Psi}(t_i \leq t \leq t_{i+1}) \]
\[ \leq \int_{t_i}^{t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} 4\{2 + \frac{(3\mu - 2)r^*}{r} \} [\frac{1}{(1 - \mu)^2}] |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi \theta}|^2 |r^2(1 - \mu)|d\sigma^2 dr^* dt \]
\[ \leq t_{i+1} \int_{t_i}^{t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} 4\{2 + \frac{(3\mu - 2)r^*}{r} \} [\frac{1}{(1 - \mu)^2}] |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi \theta}|^2 |r^2(1 - \mu)|d\sigma^2 dr^* dt \]
\[ \leq t_{i+1} \int_{t_i}^{t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{S^2} \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi \theta}|^2 |d\sigma^2 dr^* dt \]
\[ \leq t_{i+1} \int_{t_i}^{t_{i+1}} \int_{r^*=r_0^*}^{r^*=R_0^*} \int_{S^2} \int_{r^*=r_0^*}^{\infty} \int_{r^*=r_0^*}^{R_0^*} \int_{S^2} \chi_{[r_0^*,R_0^*]} |d\sigma^2| [\frac{1}{4} |\Psi_{vw}|^2] + \frac{1}{4} |\Psi_{\phi \theta}|^2 |.6\mu r(1 - \mu)^2 dr^* d\sigma^2 dt \]
\[ \leq t_{i+1} J^{(G)}_{\Psi}(t_i \leq t \leq t_{i+1})(r_0^* \leq r^* \leq R_0^*) \]

3.3. Estimate for $J^{(K)}_{\Psi}$ in terms of $E^{(K)}_{\Psi}$.

For,
\[ t_{i+1} = t_i + (0.1)t_i \]
and
\[ |r^*(r_0^*)| = |r^*(R)| \leq 0.4t_i \]

We have:

\[ J^{(K)}_{\Psi}(t_i \leq t \leq t_{i+1}) \leq t_{i+1} J^{(G)}_{\Psi}(t_i \leq t \leq t_{i+1})(r_0^* < r < R_0^*) \]
\[ \leq t_{i+1} \frac{1}{t_i} E^{(K)}_{\Psi}(t = t_i) + \frac{1}{t_i} \sum_{j=1}^{3} E^{(K)}_{\mathcal{L}_{\Omega_j} \Psi}(t = t_i) \]
\[ + \frac{1}{t_{i+1}} E^{(K)}_{\Psi}(t = t_{i+1}) + \frac{1}{t_{i+1}} \sum_{j=1}^{3} E^{(K)}_{\mathcal{L}_{\Omega_j} \Psi}(t = t_{i+1}) \]

\[ (58) \]

Proof

We have, from assumption (8),

\[ J^{(G)}_{\Psi}(t_i \leq t \leq t_{i+1}) \leq |E^{(G)}_{\psi}(t_{i+1})| + |\hat{E}^{(G)}_{\Psi}(t_i)| + |E^{(G)}_{\mathcal{L}_{\Omega_j} \Psi}(t_{i+1})| + |\hat{E}^{(G)}_{\mathcal{L}_{\Omega_j} \Psi}(t_i)| \]
Let \( \hat{\Psi} \) be an anti-symmetric 2-tensor, defined by the following,

\[
\hat{\Psi}_{\hat{t}\hat{r}}(t = t_i, r^*, \theta, \phi) = \hat{\chi} \left( \frac{2r^*}{t_i} \right) \Psi_{\hat{t}\hat{r}}(t = t_i, r^*, \theta, \phi) \quad (59)
\]

\[
\hat{\Psi}_{\hat{t}\hat{\theta}}(t = t_i, r^*, \theta, \phi) = \hat{\chi} \left( \frac{2r^*}{t_i} \right) \Psi_{\hat{t}\hat{\theta}}(t = t_i, r^*, \theta, \phi) \quad (60)
\]

\[
\hat{\Psi}_{\hat{r}\hat{\theta}}(t = t_i, r^*, \theta, \phi) = \hat{\chi} \left( \frac{2r^*}{t_i} \right) \Psi_{\hat{r}\hat{\theta}}(t = t_i, r^*, \theta, \phi) \quad (61)
\]

\[
\hat{\Psi}_{\hat{r}\hat{\phi}}(t = t_i, r^*, \theta, \phi) = \hat{\chi} \left( \frac{2r^*}{t_i} \right) \Psi_{\hat{r}\hat{\phi}}(t = t_i, r^*, \theta, \phi) \quad (62)
\]

\[
\nabla_{\hat{r}} \hat{\Psi}_{\hat{t}\hat{r}}(t = t_i, r^*, \theta, \phi) = \hat{\chi} \left( \frac{2r^*}{t_i} \right) \nabla_{\hat{r}} \Psi_{\hat{t}\hat{r}}(t = t_i, r^*, \theta, \phi) \quad (63)
\]

\[
\nabla_{\hat{r}} \hat{\Psi}_{\hat{t}\hat{\theta}}(t = t_i, r^*, \theta, \phi) = \hat{\chi} \left( \frac{2r^*}{t_i} \right) \nabla_{\hat{r}} \Psi_{\hat{t}\hat{\theta}}(t = t_i, r^*, \theta, \phi) \quad (64)
\]

where \( \hat{\chi} \) is a smooth cut-off function equal to one on \([-1, 1]\) and zero outside \([-\frac{3}{2}, \frac{3}{2}]\).

And for,

\[
-\frac{t_i}{2} \leq r^* \leq \frac{t_i}{2} \quad (65)
\]

\[
\hat{\Psi}_{\hat{r}\hat{t}}(t = t_i, r^*, \theta, \phi) = \Psi_{\hat{r}\hat{t}}(t = t_i, r^*, \theta, \phi) \quad (66)
\]

\[
\hat{\Psi}_{\hat{\theta}\hat{t}}(t = t_i, r^*, \theta, \phi) = \Psi_{\hat{\theta}\hat{t}}(t = t_i, r^*, \theta, \phi) \quad (67)
\]

And for,

\[
t_i \leq t \leq t_{i+1} \quad (68)
\]

\[
\nabla_{\mu} \hat{\Psi}_{\mu \nu} = 0 \quad (69)
\]

\[
\nabla_{\alpha} \hat{\Psi}_{\mu \nu} + \nabla_{\nu} \hat{\Psi}_{\nu \alpha} + \nabla_{\mu} \hat{\Psi}_{\alpha \mu} = 0 \quad (70)
\]

Then, we have that for,

\[
-\frac{t_i}{2} \leq r^* \leq \frac{t_i}{2}
\]

\[
\hat{\Psi}_{\mu \nu}(t = t_i, r^*, \theta, \phi) = \Psi_{\mu \nu}(t = t_i, r^*, \theta, \phi)
\]

And for

\[
r^* \leq \frac{3t_i}{4} \quad \text{and} \quad r^* \geq \frac{3t_i}{4}
\]

for,

\[
(k, l) \in \{(r^*, \theta), (r^*, \phi), (t, \theta), (t, \phi)\}
\]
we have,

\[ \hat{\Psi}_{kl}(t = t_i, r^*, \theta, \phi) = 0 \]

Now, considering the region

\[ t_i \leq t \leq t_{i+1} \text{, and } r_0 \leq r \leq R_0 \]

where

\[ t_{i+1} \leq t_i + (0.1)t_i, \text{ and } |r^*(r_0)| + |r^*(R_0)| \leq 0.4t_i \]

clearly, on \( t = t_i \), we have \( \hat{\Psi}_{\mu\nu}(t = t_i) = \Psi_{\mu\nu}(t = t_i) \), in the specified region. However, since the information from the initial data propagates no faster than the speed of light, i.e. along the null cones \( t = r^* \), and \( t = -r^* \), then in the region \( t_i \leq t \leq t_{i+1} \) we have \( \hat{\Psi}_{\mu\nu} = \Psi_{\mu\nu} \) if \( r_0^* \geq -\frac{1}{2} + (0.1)t_i = -(0.4)t_i \) and \( R_0^* \leq \frac{1}{2} - (0.1)t_i = (0.4)t_i \) which is satisfied in the specified region because of the condition that \( |r^*(r_0)| + |r^*(R_0)| \leq 0.4t_i \). Thus,

\[ \Psi_{\mu\nu} = \hat{\Psi}_{\mu\nu} \]

in the specified region. Therefore, we have,

\[ J_\Psi^{(G)}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \lesssim J_\Psi^{(G)}(t_i \leq t \leq t_{i+1}) \quad (71) \]

And for the same reason, on \( t = t_{i+1} \), we have \( \hat{\Psi}_{\mu\nu}(t = t_{i+1}) = 0 \) if \( r^* \geq -\frac{34}{15} + (0.1)t_i = -(0.85)t_i \) and \( r^* \leq \frac{34}{15} + (0.1)t_i = (0.85)t_i \).

\[
\hat{E}_\Psi^{(G)}(t = t_{i+1}) \\
= \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} (|\tilde{\Psi}_{\bar{t}\theta}|^2 + |\tilde{\Psi}_{\bar{t}\phi}|^2 + |\tilde{\Psi}_{r^*\theta}|^2 + |\tilde{\Psi}_{r^*\phi}|^2) \cdot r^2 (1 - \mu) d\sigma^2 dr^* \\
\lesssim \int_{r^* = -(0.85)t_i}^{r^* = (0.85)t_i} \int_{S^2} (|\tilde{\Psi}_{\bar{t}\theta}|^2 + |\tilde{\Psi}_{\bar{t}\phi}|^2 + |\tilde{\Psi}_{r^*\theta}|^2 + |\tilde{\Psi}_{r^*\phi}|^2) \cdot r^2 (1 - \mu) d\sigma^2 dr^* \\
\text{(where we used the boundedness of } \chi) \\
\lesssim \int_{r^* = -(0.85)t_i}^{r^* = (0.85)t_i} \int_{S^2} (|\tilde{\Psi}_{\bar{t}\theta}|^2 + |\tilde{\Psi}_{\bar{t}\phi}|^2 + |\tilde{\Psi}_{r^*\theta}|^2 + |\tilde{\Psi}_{r^*\phi}|^2) \cdot r^2 (1 - \mu) d\sigma^2 dr^* 
\]

We also have,
Lemma 3.4.

We proved that,
\[
\hat{E}_\Psi(t = t_i^{+}) = \int_{r^* = r_i'}^{r^* = \infty} \int_{S^2} \left( |\hat{\Psi}_{\hat{t}\hat{\theta}}|^2 + |\hat{\Psi}_{\hat{t}\hat{\phi}}|^2 + |\hat{\Psi}_{\hat{r}\hat{\theta}}|^2 + |\hat{\Psi}_{\hat{r}\hat{\phi}}|^2 \right) r^2 (1 - \mu) d\sigma^2 dr^*
\]
\[
\lesssim \int_{r^* = -\infty}^{r^* = r_i'} \int_{S^2} \left( |\hat{\Psi}_{\hat{t}\hat{\theta}}|^2 + |\hat{\Psi}_{\hat{t}\hat{\phi}}|^2 + |\hat{\Psi}_{\hat{r}\hat{\theta}}|^2 + |\hat{\Psi}_{\hat{r}\hat{\phi}}|^2 \right) r^2 (1 - \mu) d\sigma^2 dr^*
\]

(using the boundedness of \(\chi\))
\[
\lesssim \int_{r^* = -(0.85) t_i}^{r^* = (0.85) t_i} \int_{S^2} \left( |\Psi_{\hat{t}\hat{\theta}}|^2 + |\Psi_{\hat{t}\hat{\phi}}|^2 + |\Psi_{\hat{r}\hat{\theta}}|^2 + |\Psi_{\hat{r}\hat{\phi}}|^2 \right) r^2 (1 - \mu) d\sigma^2 dr^*
\]

Lemma 3.4.

\[
\int_{r^* = r_i'}^{r^* = \infty} \int_{S^2} \left( \frac{1}{r^2 (1 - \mu)} |\Psi_{\hat{t}\hat{\theta}}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{\hat{t}\hat{\phi}}|^2 + \frac{1}{r^2 (1 - \mu)} |\Psi_{\hat{r}\hat{\theta}}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{\hat{r}\hat{\phi}}|^2 \right)
\]
\[
+ \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{\hat{t}\hat{\theta}}|^2 + \frac{1}{(1 - \mu)^2} |\Psi_{\hat{t}\hat{\phi}}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\hat{r}\hat{\theta}}|^2 \right) (1 - \mu) r^2 d\sigma^2 dt^*
\]
\[
\lesssim \frac{E^{(K)}_\Psi(t)}{\min_{w \in (t) \cap \{r_i' \leq r^* \leq r_i^+\}} w^2} + \frac{E^{(K)}_\Psi(t)}{\min_{v \in (t) \cap \{r_i' \leq r^* \leq r_i^+\}} v^2}
\]

(72)

**Proof**

We proved that,
\[
E^{(K)}_\Psi(t_i) = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \left( \frac{1}{r^2 (1 - \mu)} |\Psi_{\hat{t}\hat{\theta}}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{\hat{t}\hat{\phi}}|^2 + \frac{1}{r^2 (1 - \mu)} |\Psi_{\hat{r}\hat{\theta}}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{\hat{r}\hat{\phi}}|^2 \right)
\]
\[
+ (1 - \mu) (w^2 + v^2) \left( \frac{1}{(1 - \mu)^2} |\Psi_{\hat{t}\hat{\phi}}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\hat{r}\hat{\theta}}|^2 \right) r^2 d\sigma^2 dt^*
\]

Because of this we have,
\[
\int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \left( (1 - \mu) w^2 \left( \frac{1}{r^2 (1 - \mu)} |\Psi_{\hat{t}\hat{\theta}}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{\hat{t}\hat{\phi}}|^2 \right) \right)
\]
\[
+ (1 - \mu) w^2 \left( \frac{1}{(1 - \mu)^2} |\Psi_{\hat{t}\hat{\phi}}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\hat{r}\hat{\theta}}|^2 \right) r^2 d\sigma^2 dt^*(t_i)
\]
\[
\lesssim E^{(K)}_\Psi(t_i)
\]

and,
\[
\int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \left( (1 - \mu) v^2 \left( \frac{1}{r^2 (1 - \mu)} |\Psi_{\hat{t}\hat{\theta}}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{\hat{t}\hat{\phi}}|^2 \right) \right)
\]
\[
+ (1 - \mu) v^2 \left( \frac{1}{(1 - \mu)^2} |\Psi_{\hat{t}\hat{\phi}}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\hat{r}\hat{\theta}}|^2 \right) r^2 d\sigma^2 dt^*(t_i)
\]
\[
\lesssim E^{(K)}_\Psi(t_i)
\]
and thus,
\[
\int_{r^* = r_1^*}^{r^* = r_2^*} \int_{S^2} \left( (1 - \mu) \frac{1}{r^2(1 - \mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{w\phi}|^2 \right) + (1 - \mu) \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\phi}|^2 \right) r^2 d\sigma^2 dr^*(t) \lessgtr \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t \cap (r_1^* \leq r^* \leq r_2^*)} w^2}
\]

and,
\[
\int_{r^* = r_1^*}^{r^* = r_2^*} \int_{S^2} \left( (1 - \mu) \frac{1}{r^2(1 - \mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{w\phi}|^2 \right) + (1 - \mu) \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\phi}|^2 \right) r^2 d\sigma^2 dr^*(t) \lessgtr \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t \cap (r_1^* \leq r^* \leq r_2^*)} w^2} + \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t \cap (r_1^* \leq r^* \leq r_2^*)} w^2} v^2
\]

Summing, we obtain,
\[
\int_{r^* = r_1^*}^{r^* = r_2^*} \int_{S^2} \left( (1 - \mu) \frac{1}{r^2(1 - \mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{w\phi}|^2 \right) + (1 - \mu) \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\phi}|^2 \right) r^2 d\sigma^2 dr^*(t) \lessgtr \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t \cap (r_1^* \leq r^* \leq r_2^*)} w^2} + \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t \cap (r_1^* \leq r^* \leq r_2^*)} w^2} v^2
\]

Inequality (72) gives,
\[
\int_{r^* = -(0.85)t_1}^{r^* = (0.85)t_1} \int_{S^2} \left( (1 - \mu) \frac{1}{r^2(1 - \mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{w\phi}|^2 + \frac{1}{r^2 (1 - \mu)} |\Psi_{w\theta}|^2 \right) + (1 - \mu) \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\phi}|^2 \right) r^2 d\sigma^2 dr^*(t) \lessgtr \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t \cap (-0.85)t_1 \leq r^* \leq (0.85)t_1} w^2} + \frac{E_{\Psi}^{(K)}(t)}{\min_{w \in \{t \cap (-0.85)t_1 \leq r^* \leq (0.85)t_1} w^2} v^2
\]

Examining now the term,
\[ |\hat{E}(\frac{\partial}{\partial t})_{\mathcal{L}_{\Omega_i} \Psi}(t = t_i)| \lesssim \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} |(\mathcal{L}_{\Omega_i} \hat{\Psi}_{\tilde{\omega}\theta})^2 + |\mathcal{L}_{\Omega_i} \hat{\Psi}_{\tilde{\omega}\phi}|^2 \]
\[+ |\mathcal{L}_{\Omega_i} \hat{\Psi}_{\tilde{\varphi} \phi}|^2\cdot (1 - \mu) r^2 d\sigma^* dr^*(t = t_i) \]

For,
\[(k, l) \in \{(r^*, \theta), (r^*, \phi), (t, \theta), (t, \phi)\} \]

\[\mathcal{L}_{\Omega_i} \hat{\Psi}_{kl}(t = t_i, r^*, \theta, \phi) = \mathcal{L}_{\Omega_i} \chi(\frac{2r^*}{t_i}) \Psi_{kl}(t = t_i, r^*, \theta, \phi) \]
\[= \chi(\frac{2r^*}{t_i}) \mathcal{L}_{\Omega_i} \Psi_{kl}(t = t_i, r^*, \theta, \phi) \]

Thus,
\[|\hat{E}(\frac{\partial}{\partial t})_{\mathcal{L}_{\Omega_i} \Psi}(t = t_i)| \lesssim E_{\mathcal{L}_{\Omega_i}$ \Psi}(t = t_i) \lesssim E_{\mathcal{L}_{\Omega_i}$ \Psi}(t = t_i) \]

\[E_{\mathcal{L}_{\Omega_i} \Psi}(t) \lesssim \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{j=1}^{3} E_{\mathcal{L}_{\Omega_i} \Psi, \mathcal{L}_{\Omega_j} \Psi, \mathcal{L}_{\Omega_k} \Psi}(t = t_0) + \sum_{l=1}^{3} \sum_{j=1}^{3} E_{\mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_j} \Psi}(t = t_0) \]

\[E_{\mathcal{L}_{\Omega_i} \Psi}(t) \lesssim E_{\mathcal{L}_{\Omega_i} \Psi}(t = t_0) \]

where,
\[E_{\mathcal{L}_{\Omega_i} \Psi} = \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{j=1}^{3} E_{\mathcal{L}_{\Omega_i} \Psi, \mathcal{L}_{\Omega_k} \Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_j} \Psi}(t = t_0) + \sum_{l=1}^{3} \sum_{j=1}^{3} E_{\mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_j} \Psi}(t = t_0) \]

\[E_{\mathcal{L}_{\Omega_i} \Psi} = \sum_{i=0}^{3} E_{r^*(\mathcal{L}) \Psi}(t = t_0) + \sum_{i=0}^{2} E_{r^*(\mathcal{L}) \Psi}(t = t_0) \]

3.5. Estimate for \( E_{\mathcal{L}_{\Omega_i} \Psi} \).

\[E_{\mathcal{L}_{\Omega_i} \Psi}(t) \lesssim \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{j=1}^{3} E_{\mathcal{L}_{\Omega_i} \Psi, \mathcal{L}_{\Omega_k} \Psi, \mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_j} \Psi}(t = t_0) + \sum_{l=1}^{3} \sum_{j=1}^{3} E_{\mathcal{L}_{\Omega_l} \Psi, \mathcal{L}_{\Omega_j} \Psi}(t = t_0) \]

\[E_{\mathcal{L}_{\Omega_i} \Psi}(t) \lesssim E_{\mathcal{L}_{\Omega_i} \Psi}(t = t_0) \]

Proof

Let,
\[ t_{i+1} = t_i + (0.1)t_i = (1.1)t_i \]  

(76)

For \( t_0 \) big enough, we will have

\[ |r^*(r_0)| + |r^*(R_0)| \leq 0.4t_0 \]

and therefore will be able to apply (58).

In view of (24) and (26) applied to the vector field \( K \) in the region \( t \in [t_i, t_{i+1}] \), we get

\[
E^{(K)}_\Psi(t = t_{i+1}) \leq J^{(K)}_\Psi(t_0 \leq t \leq t_{i+1}) + E^{(K)}_\Psi(t = t_0)
\]

\[
\leq t_{i+1}J^{(G)}_\Psi(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E^{(K)}_\Psi(t = t_0)
\]

(from (55))

and,

\[
E^{(K)}_{\mathcal{L}_{\alpha\beta}}\Psi(t = t_{i+1}) \leq t_{i+1}J^{(G)}_{\mathcal{L}_{\alpha\beta}}\Psi(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E^{(K)}_{\mathcal{L}_{\alpha\beta}}\Psi(t = t_0)
\]

(from (55))

Since \( \hat{\Psi} \) verifies the Maxwell equations, we have

\[
\nabla^\alpha T_{\alpha\beta}(\hat{\Psi}) = 0
\]

(77)

and thus, we can estimate,

\[
J^{(G)}_\Psi(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \lesssim \frac{1}{t_i}E^{(K)}_\Psi(t = t_i) + \frac{1}{t_i} \sum_{j=1}^{3} E^{(K)}_{\mathcal{L}_{\alpha\beta}}\Psi(t = t_i) + \frac{1}{t_{i+1}} \sum_{j=1}^{3} E^{(K)}_{\mathcal{L}_{\alpha\beta}}\Psi(t = t_{i+1})
\]

(from (58))
\[ J^{(G)}_{\psi}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \]
\[ \lesssim \frac{t_i}{t_i^2} J^{(G)}_{\psi}(t_0 \leq t \leq t_i)(r_0 \leq r \leq R_0) + \frac{1}{t_i^2} E^{(K)}_{\psi}(t = t_0) \]
\[ + \frac{t_i}{t_i^2} \sum_{j=1}^{3} J^{(G)}_{\Psi,\Omega_j \psi}(X)(t_0 \leq t \leq t_i)(r_0 \leq r \leq R_0) + \frac{1}{t_i^2} \sum_{j=1}^{3} E^{(K)}_{\Psi,\Omega_j}(t = t_0) \]
\[ + \frac{t_{i+1}}{t_{i+1}^2} J^{(G)}_{\psi}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + \frac{1}{t_{i+1}^2} \sum_{j=1}^{3} E^{(K)}_{\Psi,\Omega_j}(t = t_0) \]

We will use the notation \( J^{(G)}_{\Psi,\Omega_j \psi} = J^{(G)}_{\psi} + J^{(G)}_{\Psi,\Omega_j \psi} \), for all letters such as \( J \), and for different summations, so as to lighten the notation and be able to write:

\[ J^{(G)}_{\psi}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \]
\[ \lesssim \frac{1}{t_i^2} \sum_{j=1}^{3} J^{(G)}_{\Psi,\Omega_j \psi}(t_0 \leq t \leq t_i)(r_0 \leq r \leq R_0) + \frac{1}{t_i^2} \sum_{j=1}^{3} E^{(K)}_{\Psi,\Omega_j}(t = t_0) \]
\[ + \frac{1}{t_{i+1}^2} \sum_{j=1}^{3} J^{(G)}_{\Psi,\Omega_j \psi}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + \frac{1}{t_{i+1}^2} \sum_{j=1}^{3} E^{(K)}_{\Psi,\Omega_j}(t = t_0) \]

From this we can deduce the following,

\[ J^{(K)}_{\psi}(t_i \leq t \leq t_{i+1}) \]
\[ \lesssim t_{i+1} J^{(G)}_{\psi}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \]
\[ \quad \text{(from (55))} \]
\[ \lesssim \sum_{j=1}^{3} J^{(G)}_{\Psi,\Omega_j \psi}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + \frac{1}{t_{i+1}^2} \sum_{j=1}^{3} E^{(K)}_{\Psi,\Omega_j}(t = t_0) \]

(from the estimate above, and using the positivity of \( J^{(G)}_{\Psi,\Omega_j \psi} \))

Since,
\[ t_{i+1} = (1.1)t_i \]

we have,
\[ t_{i+1} = (1.1)^{i+1} t_0 \]
and thus,

$$\sum_{i} \frac{1}{t_{i+1}} = \sum_{i} \frac{1}{(1.1)^{i+1}t_{0}} \lesssim 1$$

Therefore,

$$J^{(K)}_{\Psi}(t_{0} \leq t \leq t_{i+1}) = \sum_{i=0}^{i+1} J^{(K)}_{\Psi}(t_{i} \leq t \leq t_{i+1}) \lesssim (i+1) \sum_{j=1}^{3} J^{(G)}_{\psi,\mathcal{L}_{\Omega_{j}}} \psi(t_{0} \leq t \leq t_{i+1}) \lesssim (i+1) \sum_{j=1}^{3} E^{(K)}_{\psi,\mathcal{L}_{\Omega_{j}}} \psi(t = t_{0})$$

(from the above)

and thus,

$$E^{(K)}_{\psi}(t = t_{i+1}) \lesssim J^{(K)}_{\psi}(t_{0} \leq t \leq t_{i+1}) + E^{(K)}_{\psi}(t = t_{0}) \lesssim (i+1) \sum_{j=1}^{3} J^{(G)}_{\psi,\mathcal{L}_{\Omega_{j}}} \psi(t_{0} \leq t \leq t_{i+1}) \lesssim (i+1) \sum_{j=1}^{3} E^{(K)}_{\psi,\mathcal{L}_{\Omega_{j}}} \psi(t = t_{0})$$

In the same manner, this leads to,

$$E^{(K)}_{\mathcal{L}_{\Omega_{i}} \psi}(t = t_{i+1}) \lesssim J^{(K)}_{\mathcal{L}_{\Omega_{i}} \psi}(t_{0} \leq t \leq t_{i+1}) + E^{(K)}_{\mathcal{L}_{\Omega_{i}} \psi}(t = t_{0}) \lesssim (i+1) \sum_{j=1}^{3} J^{(G)}_{\mathcal{L}_{\Omega_{i}} \psi,\mathcal{L}_{\Omega_{j}}} \psi(t_{0} \leq t \leq t_{i+1}) \lesssim (i+1) \sum_{j=1}^{3} E^{(K)}_{\mathcal{L}_{\Omega_{i}} \psi,\mathcal{L}_{\Omega_{j}}} \psi(t = t_{0})$$

(since $\Omega_{i}$, $i \in \{1,2,3\}$ are Killing, and therefore $\mathcal{L}_{\Omega_{i}} \Psi$ verifies the Maxwell equations).

Repeating the procedure again, we get,
Thus,

\[ J^{(G)}_{\Psi}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \]

\[ \lesssim \frac{1}{t_i} E^{(K)}_{\Psi}(t = t_i) + \frac{1}{t_i^2} \sum_{l=1}^{3} E^{(K)}_{L_{\alpha_l} \Psi}(t = t_i) + \frac{1}{t_{i+1}^2} \sum_{l=1}^{3} E^{(K)}_{L_{\alpha_l} \Psi}(t = t_{i+1}) \]

(from (58))

\[ \lesssim \frac{i}{t_i} \sum_{l=1}^{3} \sum_{j=1}^{3} J^{(G)}_{\Psi, L_{\alpha_l} \Psi, L_{\alpha_j} \Psi}(t_0 \leq t \leq t_i)(r_0 \leq r \leq R_0) + \frac{1}{t_i} \sum_{l=1}^{3} \sum_{j=1}^{3} E^{(K)}_{\Psi, L_{\alpha_l} \Psi, L_{\alpha_j} \Psi}(t = t_i) \]

\[ + \frac{1}{t_{i+1}} \sum_{l=1}^{3} \sum_{j=1}^{3} E^{(K)}_{\Psi, L_{\alpha_l} \Psi, L_{\alpha_j} \Psi}(t = t_{i+1}) \]

\[ \lesssim \frac{(i+1)}{t_{i+1}^2} \sum_{l=1}^{3} \sum_{j=1}^{3} J^{(G)}_{\Psi, L_{\alpha_l} \Psi, L_{\alpha_j} \Psi}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \]

\[ + \frac{1}{t_{i+1}} \sum_{l=1}^{3} \sum_{j=1}^{3} E^{(K)}_{\Psi, L_{\alpha_l} \Psi, L_{\alpha_j} \Psi}(t = t_{i+1}) \]

\[ \lesssim \frac{(i+1)}{t_{i+1}^2} \sum_{l=1}^{3} \sum_{j=1}^{3} J^{(G)}_{\Psi, L_{\alpha_l} \Psi, L_{\alpha_j} \Psi}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) \]

\[ + \frac{1}{t_{i+1}} \sum_{l=1}^{3} \sum_{j=1}^{3} E^{(K)}_{\Psi, L_{\alpha_l} \Psi, L_{\alpha_j} \Psi}(t = t_{i+1}) \]

(using the above).

We have,
\[
\frac{(i+1)}{t_{i+1}} = \frac{(i+1)}{(t_{i+1})^\frac{1}{2}} \frac{1}{(t_{i+1})^\frac{1}{2}} \leq C \frac{1}{(t_{i+1})^\frac{1}{2}}
\]

(where we used the fact that \( \frac{(i+1)}{(t_{i+1})^\frac{1}{2}} \leq C \))

\[
\lesssim \frac{1}{(1.1)^{\frac{1}{2}}}
\]

\[
\lesssim (\sqrt{\frac{1}{1.1}})^i
\]

\[\sum_i (\sqrt{\frac{1}{1.1}})^i\] is a geometric series with \( (\sqrt{\frac{1}{1.1}}) < 1 \), and therefore,

\[\sum_{i=0}^{\infty} (\sqrt{\frac{1}{1.1}})^i \lesssim 1\]

Finally, we have,

\[
J_{\Psi}^{(K)}(t_0 \leq t \leq t_{i+1})(-\infty \leq r^* \leq \infty)
= \sum_{i=0}^{i+1} J_{\Psi}^{(K)}(t_i \leq t \leq t_{i+1})(-\infty \leq r^* \leq \infty)
\lesssim \sum_{i=0}^{i+1} (\sqrt{\frac{1}{1.1}})^i \sum_{l=1}^{3} \sum_{j=1}^{3} J_{\Psi,\zeta_{\lambda_l} \Psi,\zeta_{\lambda_j} \Psi}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0)
+ \sum_{i=0}^{i+1} \frac{1}{t_{i+1}} \sum_{l=1}^{3} \sum_{j=1}^{3} E_{\Psi,\zeta_{\lambda_l} \Psi,\zeta_{\lambda_j} \Psi}(t = t_0)
\lesssim \sum_{l=1}^{3} \sum_{j=1}^{3} \left[ J_{\Psi,\zeta_{\lambda_l} \Psi,\zeta_{\lambda_j} \Psi}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E_{\Psi,\zeta_{\lambda_l} \Psi,\zeta_{\lambda_j} \Psi}(t = t_0) \right]
\]

(from the above)

which gives,

\[
E_{\Psi}^{(K)}(t = t_{i+1}) \lesssim J_{\Psi}^{(K)}(t_0 \leq t \leq t_{i+1})(-\infty \leq r^* \leq \infty) + E_{\Psi}^{(K)}(t = t_0)
\]

\[
\lesssim \sum_{l=1}^{3} \sum_{j=1}^{3} \left[ J_{\Psi,\zeta_{\lambda_l} \Psi,\zeta_{\lambda_j} \Psi}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E_{\Psi,\zeta_{\lambda_l} \Psi,\zeta_{\lambda_j} \Psi}(t = t_0) \right]
\]
From assumption (8),

\[ J^{(G)}_{\psi}(t_0 \leq t \leq t_{i+1})(r_0 \leq r \leq R) \lesssim \sum_{j=1}^{3} E^{\text{\#}}_{\psi, \tilde{\mathcal{L}}_{\Omega_j}} \psi \]

Thus,

\[ E^{(K)}_{\psi}(t = t_{i+1}) \lesssim \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{j=1}^{3} E^{\text{\#}}_{\psi, \tilde{\mathcal{L}}_{\Omega_k}, \tilde{\mathcal{L}}_{\Omega_l}, \mathcal{L}_{\Omega_j}} \psi (t = t_0) + \sum_{l=1}^{3} \sum_{j=1}^{3} E^{(K)}_{\psi, \tilde{\mathcal{L}}_{\Omega_j}} \psi (t = t_0) \lesssim E^{M}_{\psi} \]

because,

\[ \sum_{j=1}^{3} |\mathcal{L}_{\Omega_j} \psi|^2 = r^2 |\mathcal{L} \psi|^2 = |r \mathcal{L} \psi|^2 \quad (78) \]

We have \( t_{i+1} = (1.1)^{i+1} t_0 \), however, our proofs work with any \( a \) such that \( 1 < a < 2 \), and \( t_{i+1} = a^i t_0 \). Since for all \( t > t_0 \) there exist \( i \) and \( a \) such that \( t = a^i t_0, t_0 > 1 \), we get that,

\[ E^{(K)}_{\psi}(t) \lesssim E^{M}_{\psi} \]

for all \( t > t_0 \), and similarly for all \( t < -t_0 \). And since the region \(-t_0 \leq t \leq t_0\) is a bounded region, the above inequality holds in this region. Thus, for all \( t \), we have,

\[ E^{(K)}_{\psi}(t) \lesssim E^{M}_{\psi} \]
4. The Proof of Decay Away from the Horizon

We will prove that for all $\hat{\mu}, \hat{\nu} \in \{\frac{\partial}{\partial w}, \frac{\partial}{\partial v}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\}$, all the components of the Maxwell fields satisfy

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{E_F}{(1 + |v|)}$$

and,

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{E_F}{(1 + |w|)}$$

in $r \geq R > 2m$, for an arbitrarily fixed $R$, and where $E_F$ is defined by

$$E_F = \left[ \sum_{i=0}^{1} \sum_{j=0}^{5} E_{r_i(L^j)} F(t = t_0) + \sum_{i=0}^{1} \sum_{j=0}^{4} E_{r_i(L^j)} F(t = t_0) + \sum_{i=0}^{2} \sum_{j=0}^{4} E_{r_i(L^j)} (t = t_0) \right]^{1/2}$$

Proof

We will prove that,

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{[E^{(K)}_{F_L F_F r^2 F_F r^2 F_F r^2 F_F r^2 F_F r^3 F_F}]^{1/2}}{(1 + |v|)}$$

and,

$$|F_{\hat{\mu}\hat{\nu}}|(w, v, \omega) \lesssim \frac{[E^{(K)}_{F_L F_F r^2 F_F r^2 F_F r^2 F_F r^2 F_F r^3 F_F}]^{1/2}}{(1 + |w|)}$$

Recall that,

$$E^{(K)}_{\Psi} \lesssim \sum_{j=0}^{3} E_{r^j(L^j)} \Psi(t = t_0) + \sum_{j=0}^{2} E_{r^j(L^j)} \Psi(t = t_0)$$
Thus,

\[ E^{(K)}_{F, L, r, \mathcal{L}^r F, F, L, r^2 \mathcal{L}^r, F, r^2 (\mathcal{L}^r)^2 F, L, r^2 (\mathcal{L}^r)^2, F, r^3 (\mathcal{L}^r)^3 F} \lesssim \sum_{i=0}^{1} \sum_{j=0}^{5} E^{(K)}_{r^i (\mathcal{L}^r)^i F, F, L, r^2 (\mathcal{L}^r)^2 F, L, r^2 (\mathcal{L}^r)^2, F, r^3 (\mathcal{L}^r)^3 F} (t = t_0) + \sum_{i=0}^{4} \sum_{j=0}^{5} E^{(K)}_{r^i (\mathcal{L}^r)^i F, F, L, r^2 (\mathcal{L}^r)^2 F, L, r^2 (\mathcal{L}^r)^2, F, r^3 (\mathcal{L}^r)^3 F} (t = t_0) \]

\[ + E^{(K)}_{r^5 (\mathcal{L}^r)^5 F, F, L, r^2 (\mathcal{L}^r)^2 F, L, r^2 (\mathcal{L}^r)^2, F, r^3 (\mathcal{L}^r)^3 F} (t = t_0) \]

\[ \lesssim E_F^2 \]

**Definition 4.1.** We define positive definite Riemannian metric in the following manner:

\[ h(e_\alpha, e_\beta) = g(e_\alpha, e_\beta) + 2g(e_\alpha, \frac{\partial}{\partial t})g(e_\beta, \frac{\partial}{\partial t}) \]  \hspace{1cm} (79)

where

\[ \frac{\hat{\partial}}{\partial t} = (-g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}))^{-\frac{1}{2}} \frac{\partial}{\partial t} \]  \hspace{1cm} (80)

**Definition 4.2.** For any \( G \)-valued 2-tensor \( K \), we let

\[ |K|^2 = h_{\alpha\nu}h_{\beta\nu}|K^{\mu\nu}|.|K^{\alpha\beta}| \]  \hspace{1cm} (81)

**Lemma 4.3.** We have,

\[ \nabla_\sigma h(e_\alpha, e_\beta) = 2g(e_\alpha, \nabla_\sigma \frac{\hat{\partial}}{\partial t})g(e_\beta, \frac{\hat{\partial}}{\partial t}) + 2g(e_\alpha, \frac{\hat{\partial}}{\partial t})g(e_\beta, \nabla_\sigma \frac{\hat{\partial}}{\partial t}) \]  \hspace{1cm} (82)

**Proof**

\[ \nabla_\sigma h(e_\alpha, e_\beta) = \partial_\sigma h(e_\alpha, e_\beta) - h(\nabla_\sigma e_\alpha, e_\beta) - h(e_\alpha, \nabla_\sigma e_\beta) \]

\[ = \nabla_\sigma g(e_\alpha, e_\beta) + 2\nabla_\sigma [g(e_\alpha, \frac{\hat{\partial}}{\partial t})g(e_\beta, \frac{\hat{\partial}}{\partial t})] \]

\[ = 2\partial_\sigma [g(e_\alpha, \frac{\hat{\partial}}{\partial t})g(e_\beta, \frac{\hat{\partial}}{\partial t})] - 2g(\nabla_\sigma e_\alpha, \frac{\hat{\partial}}{\partial t})g(e_\beta, \frac{\hat{\partial}}{\partial t}) \]

\[ - 2g(e_\alpha, \frac{\hat{\partial}}{\partial t})g(\nabla_\sigma e_\beta, \frac{\hat{\partial}}{\partial t}) \]
(since the metric \( g \) is Killing)

\[
\begin{align*}
2\partial_\sigma g(e_\alpha, \frac{\partial}{\partial t} e_\beta) g(e_\beta, \frac{\partial}{\partial t} e_\alpha) + 2g(e_\alpha, \frac{\partial}{\partial t} e_\beta) g(e_\beta, \frac{\partial}{\partial t} e_\alpha) \\
-2g(\nabla_\sigma e_\alpha, \frac{\partial}{\partial t} e_\beta) g(e_\beta, \frac{\partial}{\partial t} e_\alpha) - 2g(e_\alpha, \frac{\partial}{\partial t} e_\beta) g(\nabla_\sigma e_\beta, \frac{\partial}{\partial t} e_\alpha) \\
= 2[\partial_\sigma g(e_\alpha, \frac{\partial}{\partial t}) - g(\nabla_\sigma e_\alpha, \frac{\partial}{\partial t})] g(e_\beta, \frac{\partial}{\partial t}) \\
+ 2[\partial_\sigma g(e_\beta, \frac{\partial}{\partial t}) - g(\nabla_\sigma e_\beta, \frac{\partial}{\partial t})] g(e_\alpha, \frac{\partial}{\partial t})
\end{align*}
\]

Using the fact that \( \nabla g = 0 \), we get,

\[
\nabla_\sigma h(e_\alpha, e_\beta) = 2 g(e_\alpha, \nabla_\sigma \frac{\partial}{\partial t} e_\beta) g(e_\beta, \frac{\partial}{\partial t} e_\alpha) + 2 g(e_\alpha, \frac{\partial}{\partial t} e_\beta) g(e_\beta, \nabla_\sigma \frac{\partial}{\partial t} e_\alpha)
\]

Let,

\[
\hat{t}_\alpha = \left( \frac{\partial}{\partial t} \right)_\alpha = g_{\mu\alpha} \left( \frac{\partial}{\partial t} \right)^\mu
\]

Hence, we can write (79) as,

\[
h_{\alpha\beta} = g_{\alpha\beta} + 2 \left( \frac{\partial}{\partial t} \right)_\alpha \left( \frac{\partial}{\partial t} \right)_\beta
\]

and (82) as,

\[
\nabla_\sigma h_{\alpha\beta} = 2[\nabla_\sigma \hat{t}_\alpha, \hat{t}_\beta + \hat{t}_\alpha, \nabla_\sigma \hat{t}_\beta]
\]

4.4. The region \( \omega \geq 1, \ r \geq R \).

We consider the region \( w \geq 1, \ r \geq R \), where \( R \) is fixed.

Let \( \Psi \) be a tensor, and \( |\Psi_{\mu\nu}| = <\Psi_{\mu\nu}, \Psi_{\mu\nu}>^{\frac{1}{2}} \). We can compute

\[
|\nabla |\Psi_{\mu\nu}| = \frac{2 <\mathcal{L}\Psi_{\mu\nu}, \Psi_{\mu\nu}>^{\frac{1}{2}}}{2 <\Psi_{\mu\nu}, \Psi_{\mu\nu}>^{\frac{1}{2}}} \leq \frac{|\mathcal{L}\Psi_{\mu\nu}| \|\Psi_{\mu\nu}\|}{|\Psi_{\mu\nu}|}
\]

We have the Sobolev inequality,

\[
r^2 |F|^2 \lesssim \int_{S^2} r^2 |F|^2 d\sigma^2 + \int_{S^2} r^2 |\nabla|F|^2 d\sigma^2 + \int_{S^2} r^2 |\nabla \nabla|F|^2 d\sigma^2
\]
where $\nabla$ is the covariant derivative restricted on the 2-spheres. We have

\[
\left| \mathcal{L} h_{\alpha\beta} \right|^2 = \frac{1}{r^2} \sum_{j=1}^{3} \left| \Omega_j h_{\alpha\beta} \right|^2 = \frac{1}{r^2} \sum_{i=1}^{3} \left| \Omega_j g(e_\alpha, e_\beta) + 2\Omega_j g(e_\alpha, \frac{\partial}{\partial t}) g(e_\beta, \frac{\partial}{\partial t}) \right|
\]

\[= 0
\]

Since $\mathcal{L} h_{\mu\nu} = 0$, we have

\[
r^2|F|^2 \lesssim \int_{S^2} r^2 |F|^2 d\sigma^2 + \int_{S^2} r^2 |\mathcal{L} F|^2 d\sigma^2 + \int_{S^2} r^2 |\mathcal{L} \mathcal{L} F|^2 d\sigma^2
\]

Let, $r_F$ be a value of $r$ such that $R \leq r_F \leq R + 1$, and to be determined later. we have,

\[
\int_{S^2} r^2|F|^2(t, r, \omega)d\sigma^2 \lesssim \int_{S^2} r^2|F|^2(t, r_F, \omega)d\sigma^2 + \int_{S^2} \int_{\tau = r^*}^{\tau = r_F} \nabla_{\tau^*} [r^2|F|^2](t, r, \omega)d\tau^* d\sigma^2
\]

\[\lesssim \int_{S^2} r^2|F|^2(t, r_F, \omega)d\sigma^2 + \int_{S^2} \int_{\tau = r^*}^{\tau = r_F} 2r|F|^2(t, r, \omega)(1 - \mu)d\tau^* d\sigma^2 + 2 \int_{S^2} \int_{\tau = r^*}^{\tau = r_F} r^2 \nabla_{\tau^*} |F|^2(t, r, \omega)d\tau^* d\sigma^2
\]

By the same,

\[
\int_{S^2} r^2|\mathcal{L} F|^2(t, r, \omega)d\sigma^2
\]

\[\lesssim \int_{S^2} r^2|\mathcal{L} F|^2(t, r_F, \omega)d\sigma^2 + \int_{S^2} \int_{\tau = r^*}^{\tau = r_F} 2r|\mathcal{L} F|^2(t, r, \omega)(1 - \mu)d\tau^* d\sigma^2 + 2 \int_{S^2} \int_{\tau = r^*}^{\tau = r_F} r^2 \nabla_{\tau^*} |\mathcal{L} F|^2(t, r, \omega)d\tau^* d\sigma^2
\]

and,

\[
\int_{S^2} r^2|\mathcal{L} \mathcal{L} F|^2(t, r, \omega)d\sigma^2
\]

\[\lesssim \int_{S^2} r^2|\mathcal{L} \mathcal{L} F|^2(t, r_F, \omega)d\sigma^2 + \int_{S^2} \int_{\tau = r^*}^{\tau = r_F} 2r|\mathcal{L} \mathcal{L} F|^2(t, r, \omega)(1 - \mu)d\tau^* d\sigma^2 + 2 \int_{S^2} \int_{\tau = r^*}^{\tau = r_F} r^2 \nabla_{\tau^*} |\mathcal{L} \mathcal{L} F|^2(t, r, \omega)d\tau^* d\sigma^2
\]

We showed the following estimate,
\[
\int_{r^* = r_1^*}^{r^* = r_2^*} \frac{E_{\psi}^{(K)}(t)}{\min_{w \in (t) \cap \{r_1^* \leq r^* \leq r_2^*\}} w} + \frac{E_{\psi}^{(K)}(t)}{\min_{v \in (t) \cap \{r_1^* \leq r^* \leq r_2^*\}} v^2
\]

Thus,
\[
\int_{r^* = r_1^*}^{r^* = r_2^*} \frac{E_{\psi}^{(K)}(t)}{\min_{w \in (t) \cap \{r_1^* \leq r^* \leq r_2^*\}} w} + \frac{E_{\psi}^{(K)}(t)}{\min_{v \in (t) \cap \{r_1^* \leq r^* \leq r_2^*\}} v^2
\]

Therefore,
\[
\int_{r^* = R^*}^{r^* = (R+1)^*} \int_{S^2} |\Psi|^2(t, r, \omega)(1 - \mu) r^2 d\sigma^2 dr^* \lesssim \frac{E_{\psi}^{(K)}(t)}{t^2}
\]

or,
\[
\int_{r^* = R}^{r^* = (R+1)} \int_{S^2} |\Psi|^2(t, r, \omega) r^2 d\sigma^2 dr^* \lesssim \frac{E_{\psi}^{(K)}(t)}{t^2}
\]

There exists \( r_{\psi} \), such that \( R \leq r_{\psi} \leq R + 1 \) and,
\[
\int_{S^2} r_{F}^2 |F|^2(t, r_{F}, \omega) d\sigma^2 \lesssim \frac{E_{F}^{(K)}(t)}{t^2 (R + 1 - R)}
\]

which gives,
\[
\int_{S^2} r_{F}^2 |F|^2(t, r_{F}, \omega) d\sigma^2 \lesssim \frac{E_{F}^{(K)}(t)}{t^2}
\]

By same,
\[
\int_{S^2} r_{F}^2 |F|^2(t, r_{F}, \omega) d\sigma^2 = \sum_{j=1}^{3} \frac{1}{r_{F}^2} \int_{S^2} r_{F}^2 |\mathcal{L}_{\Omega_j} F|^2(t, r_{F}, \omega) d\sigma^2 \lesssim \sum_{j=1}^{3} \frac{E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2}
\]
and,
\[
\int_{S^2} r^2 |\mathcal{E}_r F|^2 (t, r_0, \omega) \, d\sigma \geq \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{E_{\mathcal{E}_i \mathcal{L}_{ij}} F(t)}{t^2}
\]

On the other hand,
\[
\int_{r^*=r_T}^{r^*} \int_{S^2} r |\mathcal{E}_F F|^2 (1 - \mu)(t, r, \omega) \, d\sigma \geq \int_{r^*=r_T}^{r^*} \frac{r}{R} \int_{S^2} r |F|^2 (1 - \mu)(t, r, \omega) \, d\sigma \geq \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{u^2}
\]

Thus,
\[
\int_{r^*=r_T}^{r^*} \int_{S^2} r |F|^2 (1 - \mu)(t, r, \omega) \, d\sigma \geq \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{u^2}
\]

and,
\[
\int_{r^*=r_T}^{r^*} \int_{S^2} r |\mathcal{E}_r F|^2 (1 - \mu)(t, r, \omega) \, d\sigma \geq \sum_{j=1}^{3} \frac{1}{R^2} \int_{r^*=r_T}^{r^*} \int_{S^2} r |\mathcal{L}_{ij} F|^2 (1 - \mu)(t, r, \omega) \, d\sigma \geq \sum_{j=1}^{3} \frac{E_{\mathcal{L}_{ij} F}(t)}{t^2} + \frac{E_{\mathcal{L}_{ij} F}(t)}{u^2}
\]

By same,
\[
\int_{r^*=r_T}^{r^*} \int_{S^2} r |\mathcal{E}_r F|^2 (1 - \mu)(t, r, \omega) \, d\sigma \geq \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{E_{\mathcal{E}_i \mathcal{L}_{ij}} F(t)}{t^2} + \frac{E_{\mathcal{E}_i \mathcal{L}_{ij}} F(t)}{u^2} + \frac{E_{\mathcal{L}_{ij} F}(t)}{u^2}
\]

Now, we want to estimate the term,
\[
\int_{S^2} \int_{r=r^*}^{r=r^*} r^2 \nabla_{r^*} |F|^2(t, r, \omega) d\mathbf{r}^* d\sigma^2 = \int_{S^2} \int_{r=r^*}^{r=r^*} r^2 \nabla_{r^*} |F|^2(t, r, \omega) \sqrt{(1-\mu)} d\mathbf{r}^* d\sigma^2 \\
= \int_{S^2} \int_{r=r^*}^{r=r^*} r^2 [\nabla_{r^*} (h^{\mu\alpha} h^{\nu\beta}) |F_{\mu\nu}| |F_{\alpha\beta}| + h^{\mu\alpha} h^{\nu\beta} \nabla_{r^*} (|F_{\mu\nu}| |F_{\alpha\beta}|)](t, r, \omega) \sqrt{(1-\mu)} d\mathbf{r}^* d\sigma^2
\]

We have,
\[
\nabla_{r^*} h(e^\alpha, e^\beta) = 2g(e^\alpha, \nabla_{\sigma} \frac{\partial}{\partial t}) g(e^\beta, \frac{\partial}{\partial t}) + 2g(e^\alpha, \frac{\partial}{\partial t}) g(e^\beta, \nabla_{\sigma} \frac{\partial}{\partial t})
\]

and
\[
\nabla_{r^*} \frac{\partial}{\partial t} = 0
\]

Hence,
\[
\nabla_{r^*} h^{\alpha\beta} = \nabla_{r^*} h(e^\alpha, e^\beta) = 2g(e^\alpha, \nabla_{r^*} \frac{\partial}{\partial t}) g(e^\beta, \frac{\partial}{\partial t}) + 2g(e^\alpha, \frac{\partial}{\partial t}) g(e^\beta, \nabla_{r^*} \frac{\partial}{\partial t}) = 0
\]

Therefore,
\[
\int_{S^2} \int_{r=r^*}^{r=r^*} r^2 \nabla_{r^*} |F|^2(t, r, \omega) d\mathbf{r}^* d\sigma^2 \\
= \int_{S^2} \int_{r=r^*}^{r=r^*} r^2 h^{\mu\alpha} h^{\nu\beta} \nabla_{r^*} (|F_{\mu\nu}| |F_{\alpha\beta}|)(t, r, \omega) \sqrt{(1-\mu)} d\mathbf{r}^* d\sigma^2 \\
\geq \int_{S^2} \int_{r=r^*}^{r=r^*} r^2 [h^{\mu\alpha} h^{\nu\beta} |\nabla_{r^*} |F_{\mu\nu}| |F_{\alpha\beta}| + h^{\mu\alpha} h^{\nu\beta} |\nabla_{r^*} |F_{\alpha\beta}|](t, r, \omega) \sqrt{(1-\mu)} d\mathbf{r}^* d\sigma^2 \\
\geq \int_{S^2} \int_{r=r^*}^{r=r^*} r^2 [h^{\mu\alpha} h^{\nu\beta} |\mathcal{L}_{r^*} F_{\mu\nu}| |F_{\alpha\beta}| + h^{\mu\alpha} h^{\nu\beta} |\mathcal{L}_{r^*} F_{\alpha\beta}|](t, r, \omega) \sqrt{(1-\mu)} d\mathbf{r}^* d\sigma^2 \\
\geq \int_{S^2} \int_{r=r^*}^{r=r^*} r^2 [\hat{h}^{\mu\alpha} \hat{h}^{\nu\beta} |\mathcal{L}_{r^*} F_{\mu\nu}| |F_{\alpha\beta}| + \hat{h}^{\mu\alpha} \hat{h}^{\nu\beta} |\mathcal{L}_{r^*} F_{\alpha\beta}|](t, r, \omega) \sqrt{(1-\mu)} d\mathbf{r}^* d\sigma^2
\]

where \(\hat{\mu}, \hat{\nu}, \hat{\alpha}, \hat{\beta} \in \{\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}, \frac{\partial}{\partial \sigma}\}\).

As we have (see (7.3) in the Appendix),
\[ \nabla_{\hat{r}^*} \frac{\dot{\hat{r}}}{\partial t} = 0 \]
\[ \nabla_{\hat{r}^*} \frac{\dot{\hat{r}}}{\partial r^*} = 0 \]
\[ \nabla_{\hat{r}^*} \frac{\partial}{\partial \theta} = 0 \]
\[ \nabla_{\hat{r}^*} \frac{\partial}{\partial \phi} = 0 \]

we get,
\[ \mathcal{L}_{\hat{r}^*} F_{\hat{\mu} \hat{\nu}} = \nabla_{\hat{r}^*} F_{\hat{\mu} \hat{\nu}} \]

Consequently, using Cauchy-Schwarz, we obtain

\[ \int_{\tau = \tau_0}^{\tau = \tau^*} r^2 \nabla_{\hat{r}^*} |F|^2 (t, r, \omega) d\tau d\sigma^2 \]
\[ \lesssim \left[ \int_{\tau = \tau_0}^{\tau = \tau^*} \int_{S^2} \tau^2 |\nabla_{\hat{r}^*} F|^2 (1 - \mu) d\sigma^2 d\tau^* \right]^{\frac{1}{2}} \]
\[ \lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2} \]

We have,
\[ \int_{\tau = \tau_0}^{\tau = \tau^*} \int_{S^2} \tau^2 |F|^2 (1 - \mu) d\sigma^2 d\tau^* \lesssim \sqrt{\frac{E_F^{(K)}(t)}{t}} + \sqrt{\frac{E_F^{(K)}(t)}{w}} \]

Now, considering the case where \( \hat{\mu} = \hat{r}^* \), (or similarly if \( \hat{\nu} = \hat{r}^* \)), we can compute,
\[ \nabla_{\hat{r}^*} F_{\hat{r}^* \hat{\nu}} = -\nabla_{\hat{\mu}} F_{\hat{\nu} \hat{\mu}} - \nabla_{\hat{\theta}} F_{\hat{\theta} \hat{\nu}} - \nabla_{\hat{\phi}} F_{\hat{\phi} \hat{\nu}} \]
(since the Maxwell fields are divergence free)

Therefore,
\[ \nabla_{\hat{r}^*} F_{\hat{r}^* \hat{\nu}} = \nabla_{\hat{\mu}} F_{\hat{\nu} \hat{\mu}} - \nabla_{\hat{\theta}} F_{\hat{\theta} \hat{\nu}} - \nabla_{\hat{\phi}} F_{\hat{\phi} \hat{\nu}} \]

And if both \( \mu \neq \hat{r} \), and \( \nu \neq \hat{r} \), we can compute,
\[ \nabla_{\hat{r}^*} F_{\hat{\mu} \hat{\nu}} = -\nabla_{\hat{\mu}} F_{\hat{r}^* \hat{\nu}} - \nabla_{\hat{\theta}} F_{\hat{\theta} \hat{\nu}} \]
(by using the Bianchi identities)
and therefore,

$$\nabla_{\hat{r}}^* F_{\hat{\mu} \hat{\nu}} = -\nabla_\mu F_{\hat{\nu} \hat{r}} - \nabla_\nu F_{\hat{r} \hat{\mu}}$$

where $\hat{\mu}, \hat{\nu} \in \{ \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \}$.

In all cases, we get,

$$|\nabla_{\hat{r}}^* F_{\hat{\mu} \hat{\nu}}|^2 (1 - \mu) \lesssim [|\nabla_t F|^2 + |\nabla F|^2] (1 - \mu)$$

(using the triangular inequality and $a.b \lesssim a^2 + b^2$)

Hence,

$$|\nabla_{\hat{r}}^* F|^2 (1 - \mu) \lesssim [|\nabla_t F|^2 + \frac{1}{r^2} \sum_{j=1}^3 |\nabla_{\Omega_j} F|^2] (1 - \mu)$$

$$\lesssim |\nabla_t F|^2 + \sum_{j=1}^3 |\nabla_{\Omega_j} F|^2$$

$$\lesssim h^{\alpha \beta} h^{\gamma \delta} |\nabla_t F_{\alpha \beta}| |\nabla_{\Omega_j} F_{\gamma \delta}| + \sum_{j=1}^3 h^{\alpha \beta} h^{\gamma \delta} |\nabla_t F_{\alpha \beta}| |\nabla_{\Omega_j} F_{\gamma \delta}|$$

We have,

$$|\nabla_t F|^2$$

$$= h^{\alpha \beta} h^{\gamma \delta} [|\nabla_t F_{\alpha \beta}| |\nabla_t F_{\gamma \delta}|]$$

$$= h^{\alpha \beta} h^{\gamma \delta} [|L_i F_{\alpha \beta}| + |\nabla_{\Omega_j} F_{\alpha \beta}| + |\nabla_t F_{\alpha \beta}| |\nabla_{\Omega_j} F_{\alpha \beta}|]$$

$$\lesssim |L_i F|^2 + h^{\alpha \beta} h^{\gamma \delta} [|F(\nabla_{\Omega_j} F_{\alpha \beta}| |\nabla_{\Omega_j} F_{\alpha \beta}| + |\nabla_t F_{\alpha \beta}| |\nabla_{\Omega_j} F_{\alpha \beta}|]$$

(using Cauchy-Schwarz inequality and $a.b \lesssim a^2 + b^2$)

Since, $\frac{\partial}{\partial \mu}$ is a smooth vector field away from the horizon, choosing a system of coordinates to compute the contractions above, we get

$$|\nabla_t F|^2 (1 - \mu) \lesssim |L_i F|^2 + |F|^2$$

Similarly, we have,

$$|\nabla_{\Omega_j} F|^2$$

$$\lesssim |L_{\Omega_j} F|^2 + h^{\alpha \beta} h^{\gamma \delta} [|F(\nabla_{\Omega_j} F_{\alpha \beta}| |\nabla_{\Omega_j} F_{\alpha \beta}| + |\nabla_t F_{\alpha \beta}| |\nabla_{\Omega_j} F_{\alpha \beta}|]$$

$$+ h^{\alpha \beta} h^{\gamma \delta} [|F(\nabla_{\Omega_j} F_{\alpha \beta}| |\nabla_{\Omega_j} F_{\alpha \beta}| + |\nabla_t F_{\alpha \beta}| |\nabla_{\Omega_j} F_{\alpha \beta}|]$$
Since \( \Omega_j, j \in \{1, 2, 3\} \) are smooth vector fields, computing the contractions above in a system of coordinates, we obtain

\[
\sum_{j=1}^{3} |\nabla_{\Omega_j} F_{\mu}\beta|^{2} \lesssim \sum_{j=1}^{3} |\mathcal{L}_{\Omega_j} F|^{2} + |F|^{2}
\]

Finally, we have

\[
|\nabla_{\rho} F|^{2} (1 - \mu) \lesssim |F|^{2} + |\mathcal{L}_{\rho} F|^{2} + \sum_{j=1}^{3} |\mathcal{L}_{\Omega_j} F|^{2}
\]

Thus,

\[
\int_{\mathcal{T} = r}^{r_{0}} \int_{\mathcal{T} = r_{0}} \int_{\mathcal{S}} |\mathcal{L}_{\mathcal{T}} F|^{2} d\sigma^{2} d\mathbf{r}^{*}
\]

\[
\lesssim \int_{\mathcal{T} = r}^{r_{0}} \int_{\mathcal{T} = r_{0}} \int_{\mathcal{S}} |\mathcal{L}_{\mathcal{T}} F|^{2} + |\mathcal{L}_{\mathcal{T}} F|^{2} + |F|^{2} d\sigma^{2} d\mathbf{r}^{*}
\]

and therefore,

\[
\left| \int_{\mathcal{T} = r}^{r_{0}} \int_{\mathcal{T} = r_{0}} \int_{\mathcal{S}} |\mathcal{L}_{\mathcal{T}} F|^{2} d\sigma^{2} d\mathbf{r}^{*} \right|^{\frac{1}{2}}
\]

\[
\lesssim \sqrt{E_{F}^{(K)}(t) + E_{\mathcal{L}_{I}, F}(t) + \sum_{j=1}^{3} E_{\mathcal{L}_{\Omega_j}, F}(t)}
\]

Thus,

\[
\int_{\mathcal{T} = r}^{r_{0}} \int_{\mathcal{T} = r_{0}} \int_{\mathcal{S}} |\mathcal{L}_{\mathcal{T}} F|^{2} d\sigma^{2} d\mathbf{r}^{*}
\]

\[
\lesssim \left( \sqrt{E_{F}^{(K)}(t) + E_{\mathcal{L}_{I}, F}(t) + \sum_{j=1}^{3} E_{\mathcal{L}_{\Omega_j}, F}(t)} \right) + \left( \sqrt{E_{F}^{(K)}(t) + E_{\mathcal{L}_{I}, F}(t) + \sum_{j=1}^{3} E_{\mathcal{L}_{\Omega_j}, F}(t)} \right)
\]

\[
\lesssim \frac{E_{F}^{(K)}(t) + E_{\mathcal{L}_{I}, F}(t) + \sum_{j=1}^{3} E_{\mathcal{L}_{\Omega_j}, F}(t)}{t^{2}} + \frac{E_{F}^{(K)}(t) + E_{\mathcal{L}_{I}, F}(t) + \sum_{j=1}^{3} E_{\mathcal{L}_{\Omega_j}, F}(t)}{w^{2}}
\]
By same,

\[
\int_{r = r^*} \int_{S^2} r^2 \langle \mathcal{L}_r \mathcal{L} \mathcal{F}, \mathcal{L} \mathcal{F} \rangle > h(t, r, \omega) d\sigma d\tau^m
\]

\[\lesssim \sum_{i=1}^{3} \sum_{j=1}^{3} \left[ \frac{E_{L_{\Omega_i} F, L_{\Omega_j} F, L_{\Omega_i} F, L_{\Omega_j} F}(t)}{t^2} + \frac{E_{L_{\Omega_i} L_{\Omega_j} F, L_{\Omega_i} L_{\Omega_j} F, L_{\Omega_i} L_{\Omega_j} F}(t)}{u^2} \right] \]

and,

\[
\int_{r = r^*} \int_{S^2} r^2 \langle \mathcal{L}_r \mathcal{L} \mathcal{F}, \mathcal{L} \mathcal{F} \rangle > h(t, r, \omega) d\sigma d\tau^m
\]

\[\lesssim \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{l=1}^{3} \left[ \frac{E_{L_{\Omega_i} L_{\Omega_j} L_{\Omega_l} F, L_{\Omega_i} L_{\Omega_j} L_{\Omega_l} F, L_{\Omega_i} L_{\Omega_j} L_{\Omega_l} F}(t)}{t^2} + \frac{E_{L_{\Omega_i} L_{\Omega_j} L_{\Omega_l} F, L_{\Omega_i} L_{\Omega_j} L_{\Omega_l} F, L_{\Omega_i} L_{\Omega_j} L_{\Omega_l} F}(t)}{u^2} \right] \]

Using the fact that,

\[
\sum_{j=1}^{3} |\mathcal{L}_{\Omega_j} \Psi|^2 = r^2 |\mathcal{L} \Psi|^2 = |r \mathcal{L} \Psi|^2
\]

We have,

\[
|r \mathcal{L} r \mathcal{L} \Psi|^2 = r^2 [r^2 |\mathcal{L} \Psi|^2] = r^2 \sum_{j=1}^{3} |\mathcal{L}_{\Omega_j} \Psi|^2
\]

\[= \sum_{j=1}^{3} r^2 |\mathcal{L}_{\Omega_j} \Psi|^2
\]

\[= \sum_{i=1}^{3} \sum_{j=1}^{3} |\mathcal{L}_{\Omega_i} L_{\Omega_j} \Psi|^2
\]

Finally, we obtain,
\[
\begin{align*}
&\leq r^2 |F|^2(t, r, \omega) \\
&\quad + \frac{E_{F, \xi, \tau, F}(t)}{t^2} + \frac{E_{F, \xi, \tau, F}(t)}{w^2} \\
&\quad + \frac{E_{F, \xi, \tau, F, F, r^2(\xi)}(t)}{t^2} + \frac{E_{F, \xi, \tau, F, F, r^2(\xi)}(t)}{w^2} \\
&\quad + \frac{E_{F, \xi, \tau, F, F, r^2(\xi)}(t)}{t^2} + \frac{E_{F, \xi, \tau, F, F, r^2(\xi)}(t)}{w^2}
\end{align*}
\]

Since what is on the left hand side of the previous inequality is a contraction, it can be computed in the basis \( \{ \frac{\partial}{\partial w}, \frac{\partial}{\partial v}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \} \). Thus,

\[
\begin{align*}
&\leq \sum_{\mu, \nu \in \{ \frac{\partial}{\partial w}, \frac{\partial}{\partial v}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \}} |F_{\mu\nu}|(w, v, \omega) \\
&\quad \leq \frac{[E_{F, \xi, \tau, F, F, \xi, \tau, F, F, r^2(\xi)}(t)]^{\frac{1}{2}}}{rt} + \frac{[E_{F, \xi, \tau, F, F, \xi, \tau, F, F, r^2(\xi)}(t)]^{\frac{1}{2}}}{rw}
\end{align*}
\]

For \( R \) fixed, consider first the region where \( t \geq 1 \), and thus we have \( r + t \lesssim rt \). Consequently, \( v = r^* + t \lesssim 2r + t \lesssim 2r + 2t \lesssim rt \), and \( v \lesssim t + r \lesssim r + t - r^* \lesssim r + w \lesssim rw \) (since \( w \geq 1 \)).

For \( t \leq 1 \), the region, \( \omega \geq 1 \), \( r \geq R \), \( t \leq 1 \) is a bounded compact region, and therefore, in this region

\[
|F_{\mu\nu}|(w, v, \omega) \lesssim E_{F}(t = t_0) + E_{\nabla F}(t = t_0)
\]

(see [G])

\[
\lesssim E_{F, \xi, \tau, F}(t = t_0)
\]

Thus, we have,

\[
|F_{\mu\nu}|(w, v, \omega) \lesssim \frac{[E_{F, \xi, \tau, F, F, \xi, \tau, F, F, r^2(\xi)}(t)]^{\frac{1}{2}}}{(1 + |v|)}
\]

\[
\lesssim \frac{E_F}{(1 + |v|)}
\]
4.5. **The region** \( w \leq -1, \ r \geq R, \ |t| \geq 1. \)

Again, we have the Sobolev inequality

\[
r^2|\mathcal{F}_{\hat{\mu}}|^2 \lesssim \int_{S^2} r^2|F|^2 d\sigma^2 + \int_{S^2} r^2|\mathcal{L} F|^2 d\sigma^2 + \int_{S^2} r^2|\mathcal{L} \mathcal{L} F|^2 d\sigma^2
\]

We have,

\[
\int_{S^2} r^2|F|^2(t, r, \omega) d\sigma^2 \lesssim \int_{S^2} r^2|F|^2(t, \infty, \omega) d\sigma^2 + \int_{S^2} \int_{r^* = r}^{r^* = \infty} \nabla_{r^*} r^2|F|^2(t, r, \omega) d\tau^* d\sigma^2
\]

\[
\lesssim \int_{S^2} \int_{r^* = r}^{r^* = \infty} 2r|F|^2(t, r, \omega)(1 - \mu) d\tau^* d\sigma^2 + 2 \int_{S^2} \int_{r^* = r}^{r^* = \infty} r^2 < \nabla_{r^*} F, F >_h (t, r, \omega) d\tau^* d\sigma^2
\]

By the same,

\[
\int_{S^2} r^2|\mathcal{L} F|^2(t, r, \omega) d\sigma^2 \lesssim \int_{S^2} \int_{r^* = r}^{r^* = \infty} 2r|\mathcal{L} F|^2(t, r, \omega)(1 - \mu) d\tau^* d\sigma^2 + 2 \int_{S^2} \int_{r^* = r}^{r^* = \infty} r^2 < \nabla_{r^*} \mathcal{L} F, \mathcal{L} F >_h (t, r, \omega) d\tau^* d\sigma^2
\]

and,

\[
\int_{S^2} r^2|\mathcal{L} \mathcal{L} F|^2(t, r, \omega) d\sigma^2 \lesssim \int_{S^2} \int_{r^* = r}^{r^* = \infty} 2r|\mathcal{L} \mathcal{L} F|^2(t, r, \omega)(1 - \mu) d\tau^* d\sigma^2 + 2 \int_{S^2} \int_{r^* = r}^{r^* = \infty} r^2 < \nabla_{r^*} \mathcal{L} \mathcal{L} F, \mathcal{L} \mathcal{L} F >_h (t, r, \omega) d\tau^* d\sigma^2
\]

We have shown,

\[
\int_{r^* = r_1^*}^{r_2^*} \int_{S^2} (|\Psi_{\phi\hat{\theta}}|^2 + |\Psi_{\phi\hat{\phi}}|^2 + |\Psi_{\hat{\theta}\phi}|^2 + |\Psi_{\hat{\phi}\phi}|^2 + |\Psi_{\hat{\phi}\hat{\phi}}|^2)(1 - \mu) r^2 d\sigma^2 dr^*(t)
\]

\[
\lesssim \frac{E^{(K)}_{\Psi}(t)}{\min_{w(t) \cap \{r_1^* \leq r^* \leq r_2^*\}} w^2} + \frac{E^{(K)}_{\Psi}(t)}{\min_{v(t) \cap \{r_1^* \leq r^* \leq r_2^*\}} v^2}
\]

Thus,
\[ \int_{r^* = r_*^1}^{r^* = r_*^2} \int_{S^2} |\Psi|^2(t, \tau, \omega)(1 - \mu) r^2 d\sigma^2 d\tau^r \lesssim \frac{E^{(K)}_\theta(t)}{\min_{\omega \in \{t \cap (r_*^1 \leq r^* \leq r_*^2)\}} \omega^2} + \frac{E^{(K)}_\psi(t)}{\min_{\omega \in \{t \cap (r_*^1 \leq r^* \leq r_*^2)\}} \omega^2} \]

We have,

\[ \int_{\tau = r_*}^{\tau = \infty} \int_{S^2} |\tau| F|^{2}(1 - \mu)(t, \tau, \omega) d\sigma^2 d\tau^r \lesssim \int_{\tau = r_*}^{\tau = \infty} \int_{S^2} |\tau| F|^{2}(1 - \mu)(t, \tau, \omega) d\sigma^2 d\tau^r \lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{u^2} \]

Thus,

\[ \int_{\tau = r_*}^{\tau = \infty} \int_{S^2} |\tau| F|^{2}(1 - \mu)(t, \tau, \omega) d\sigma^2 d\tau^r \lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{u^2} \]

and,

\[ \int_{\tau = r_*}^{\tau = \infty} \int_{S^2} |\tau| \mathcal{L} F|^{2}(1 - \mu)(t, \tau, \omega) d\sigma^2 d\tau^r = \sum_{j=1}^{3} \int_{\tau = r_*}^{\tau = \infty} \int_{S^2} |\tau| \mathcal{L}_{\Omega_j} F|^{2}(1 - \mu)(t, \tau, \omega) d\sigma^2 d\tau^r \lesssim \sum_{j=1}^{3} \frac{E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_{\mathcal{L}_{\Omega_j} F}^{(K)}(t)}{u^2} \]

By same,

\[ \int_{\tau = r_*}^{\tau = \infty} \int_{S^2} |\tau| \mathcal{L} \mathcal{L} F|^{2}(1 - \mu)(t, \tau, \omega) d\sigma^2 d\tau^r \lesssim \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{E_{\mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F}^{(K)}(t)}{t^2} + \frac{E_{\mathcal{L}_{\Omega_i} \mathcal{L}_{\Omega_j} F}^{(K)}(t)}{u^2} \]

Now, we can estimate the term,

\[ \int_{\tau = r_*}^{\tau = \infty} \int_{S^2} \nabla_{r^*} F, F >_h (t, r, \omega) d\sigma^2 d\tau^r \lesssim \left[ \int_{\tau = r_*}^{\tau = \infty} \int_{S^2} |\nabla_{r^*} F|^2 d\sigma^2 d\tau^r \right]^{\frac{1}{2}} \cdot \left[ \int_{\tau = r_*}^{\tau = \infty} \int_{S^2} |F|^2 d\sigma^2 d\tau^r \right]^{\frac{1}{2}} \]
We have,

\[ \int_{\tau = r}^{\infty} \int_{S^2} r^2 |F|^2 (1 - \mu)(t, \tau, \omega) d\sigma^2 d\tau \leq \frac{E_F^{(K)}(t)}{t^2} + \frac{E_F^{(K)}(t)}{w^2} \]

Thus,

\[ \left[ \int_{\tau = r}^{\infty} \int_{S^2} r^2 |F|^2 d\sigma^2 d\tau \right]^\frac{1}{2} \lesssim \sqrt{E_F^{(K)}(t)} + \frac{E_F^{(K)}(t)}{w} \]

We also have the estimate:

\[ \int_{\tau = r}^{\infty} \int_{S^2} r^2 |\nabla_r F|^2 d\sigma^2 d\tau \]

\[ \lesssim \int_{\tau = r}^{\infty} \int_{S^2} r^2 (|F|^2 + |L_t F|^2 + |L F|^2) d\sigma^2 d\tau \]

\[ \lesssim \int_{\tau = r}^{\infty} \int_{S^2} r^2 |F|^2 + |L_t F|^2 d\sigma^2 d\tau + \sum_{j=1}^{3} \int_{\tau = r}^{\infty} \int_{S^2} r^2 \frac{1}{R^2} |L_{\Omega_j} F|^2 d\sigma^2 d\tau \]

\[ \lesssim \frac{E_F^{(K)}(t)}{t^2} + \frac{E_{L_t F}^{(K)}(t)}{t} + \sum_{j=1}^{3} \frac{E_{L_{\Omega_j} F}^{(K)}(t)}{w^2} \]

and therefore,

\[ \left[ \int_{\tau = r}^{\infty} \int_{S^2} r^2 |\nabla_r F|^2 d\sigma^2 d\tau \right]^\frac{1}{2} \]

\[ \lesssim \sqrt{E_F^{(K)}(t)} + \frac{E_F^{(K)}(t)}{|t|} + \sqrt{E_{L_t F}^{(K)}(t) + \sum_{j=1}^{3} E_{L_{\Omega_j} F}^{(K)}(t)} \]
Thus,
\[
\int_{\mathbb{S}^2} \tau < \nabla_{\mathcal{F}} \mathcal{F} \mathcal{F} >_h (t, r, \omega) d\sigma^2 d\tau^s
\]
\[
\lesssim \sqrt{E_F} (t) + E_{\mathcal{L}_F} (t) + \sum_{j=1}^3 E_{\mathcal{L}_j F} (t)
\]
\[
\frac{t}{w} \sqrt{\frac{E_F} {E_F} (t) + \sqrt{E_F} (t)} + \frac{w}{t} \sqrt{E_F (t) + E_{\mathcal{L}_F} (t) + \sum_{j=1}^3 E_{\mathcal{L}_j F} (t)}
\]
\[
\lesssim \frac{t^2}{w^2} \frac{E_F (t) + E_{\mathcal{L}_F} (t) + E_{\mathcal{L}_F} (t)}{E_F (t) + E_{\mathcal{L}_F} (t) + E_{\mathcal{L}_F} (t)}
\]

By same,
\[
\int_{\mathbb{S}^2} \tau < \nabla_{\mathcal{F}} \mathcal{F} \mathcal{F} >_h (t, r, \omega) d\sigma^2 d\tau^s
\]
\[
\lesssim \frac{E_F (t)}{t^2} + \frac{E_{\mathcal{L}_F} (t)}{w^2}
\]

and,
\[
\int_{\mathbb{S}^2} \tau < \nabla_{\mathcal{F}} \mathcal{F} \mathcal{F} >_h (t, r, \omega) d\sigma^2 d\tau^s
\]
\[
\lesssim \frac{E_F (t)}{t^2} + \frac{E_{\mathcal{L}_F} (t)}{w^2}
\]

Finally, we obtain,
\[
\frac{r^2 | \mathcal{F}^2 (t, r, \omega)}
\]
\[
\lesssim \frac{E_F (t)}{t^2} + \frac{E_{\mathcal{L}_F} (t)}{w^2}
\]
\[
+ \frac{E_{\mathcal{L}_F} (t)}{t^2} + \frac{E_{\mathcal{L}_F} (t)}{w^2}
\]
\[
+ \frac{E_{\mathcal{L}_F} (t)}{t^2} + \frac{E_{\mathcal{L}_F} (t)}{w^2}
\]

Thus,
\[ |F|(w, v, \omega) \lesssim \frac{[E^{(K)}_{F,L,F,r,L,F,L,F,r^2}(\mathcal{L})^2F,F,L,r^2(\mathcal{L})^2F,F,r^3(\mathcal{L})^3F(t)]^{\frac{1}{2}}}{r|t|} \]

We have,
\[ r^* = \frac{v - w}{2} \]

Thus, for \( w \leq -1 \), we have, \( r^* \geq v \), and hence \( r \geq v \). Therefore,
\[ \frac{1}{r} \lesssim \frac{1}{|v|} \]

Since in this region we have \( |w| \geq 1 \), and \( |t| \geq 1 \), we get,
\[ |F_{\mu\nu}|(w, v, \omega) \lesssim \frac{[E^{(K)}_{F,L,F,r,L,F,L,F,r^2}(\mathcal{L})^2F,F,L,r^2(\mathcal{L})^2F,F,r^3(\mathcal{L})^3F(t)]^{\frac{1}{2}}}{(1 + |v|)} \]

We also have for fixed \( R \), and \( |t| \geq 1 \), \( |w| = |t - r^*| \lesssim |r^*| + |t| \lesssim 2r + t \lesssim 2r + 2|t| \lesssim r|t| \). Thus, in this region, we also have,
\[ |F_{\mu\nu}|(w, v, \omega) \lesssim \frac{E_F}{(1 + |w|)} \]

4.6. The region \( w \leq -1 \), \( r \geq R \), \( -1 \leq t \leq 1 \).

Let,
\[ t^# = t - 2 \]

When
\[ -1 \leq t \leq 1 \]
we have
\[ -3 \leq t^# \leq -1 \]
Let,
\[ w^\# = t^\# - r^* = t - r^* - 2 \]

When
\[ w \leq -1 \]

we have
\[ w^\# \leq -3 \]

Thus, the region \( w \leq -1, r \geq R, -1 \leq t \leq 1 \), is in the new system of coordinates included in the region \( w^\# \leq -1, r \geq R, t^\# \leq -1 \).

\( \frac{\partial}{\partial t} \) is a Killing vector field, therefore, the time translation will keep the metric invariant, i.e. in the new system of coordinates \( \{t^\#, r, \theta, \phi\} \) the metric is written exactly as in the former system of coordinates \( \{t, r, \theta, \phi\} \). Consequently, we will have the same results proven previously, i.e., in the region \( w^\# \leq -1, r \geq R, t^\# \leq -1 \), we have:

\[
|F_\mu^\nu|(w^\#, v^\#, \omega) \lesssim \frac{[E_M F_{,\ell_1 F_{,\ell_2 F_{,\ell_3 F}} F_{,\ell_4 F_{,\ell_5 F_{,\ell_6 F}}} F_{,r^2(\mathcal{L})^2 F_{,r^2(\mathcal{L})^2 F_{,r^3(\mathcal{L})^3 F}}}]^{1/2}}{(1 + |v^\#|)}
\]

and,

\[
|F_\mu^\nu|(w^\#, v^\#, \omega) \lesssim \frac{[E_M F_{,\ell_1 F_{,\ell_2 F_{,\ell_3 F}} F_{,\ell_4 F_{,\ell_5 F_{,\ell_6 F}}} F_{,r^2(\mathcal{L})^2 F_{,r^2(\mathcal{L})^2 F_{,r^3(\mathcal{L})^3 F}}}]^{1/2}}{(1 + |v|)}
\]

which gives,

\[
|F_{\bar{\mu}\bar{\nu}}|(w, v, \omega) \lesssim \frac{[E_M F_{,\ell_1 F_{,\ell_2 F_{,\ell_3 F}} F_{,\ell_4 F_{,\ell_5 F_{,\ell_6 F}}} F_{,r^2(\mathcal{L})^2 F_{,r^2(\mathcal{L})^2 F_{,r^3(\mathcal{L})^3 F}}}]^{1/2}}{(1 + |v|)}
\]

and,

\[
|F_{\bar{\mu}\bar{\nu}}|(w, v, \omega) \lesssim \frac{[E_M F_{,\ell_1 F_{,\ell_2 F_{,\ell_3 F}} F_{,\ell_4 F_{,\ell_5 F_{,\ell_6 F}}} F_{,r^2(\mathcal{L})^2 F_{,r^2(\mathcal{L})^2 F_{,r^3(\mathcal{L})^3 F}}}]^{1/2}}{(1 + |w|)}
\]

in the region \( w \leq -1, r \geq R, -1 \leq t \leq 1 \).
4.7. The region \(-1 \leq w \leq 1, r \geq R\).

Let,

\[ r^{*>} = r^* + 2 \]

Then, when

\[-1 \leq w \leq 1 \]

we have,

\[-3 \leq w^* \leq -1 \]

and, when

\[ r^* \geq R^* \]

then,

\[ r^{*>} \geq R^* + 2 \geq R^* \]

Thus, the region \(-1 \leq w \leq 1, r \geq R\) is included, in the new system of coordinates, in the region \( w^{*>} \leq -1, r^{*>} \geq R^* \).

Notice that \( r^* \) is defined up to a constant. With the new definition of \( r^* \) everything we have proven with \( r^* \) works with \( r^{*>} \) by replacing in (8), \( J^{(G)}_{\Psi}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0) \) by \( J^{(G)}_{\Psi}(t_i \leq t \leq t_{i+1})(r_0 < r < R_0^{*}) \) defined by

\[
J^{(G)}_{\Psi}(r_0 < r < R_0^*) (t_i \leq t \leq t_{i+1}) = \int_{t=t_i}^{t=t_{i+1}} \int_{r=r_0}^{r=R_0^*} \int_{S^2} \left[ |\Psi_\phi| \right]^2 + \frac{1}{4} |\Psi_{\phi\theta}|^2 d\sigma^2 dt
\]

However, since the length of the interval \( w \in [-1,1] \) was arbitrary; we only wanted in the previous subsections to avoid \( w = 0 \), what is actually only needed is

\[
J^{(G)}_{\Psi}(r_0 < r < R_0 + \epsilon) (t_i \leq t \leq t_{i+1}) = \int_{t=t_i}^{t=t_{i+1}} \int_{r=r_0}^{r=R_0^* + \epsilon} \int_{S^2} \left[ |\Psi_\phi| \right]^2 + \frac{1}{4} |\Psi_{\phi\theta}|^2 dr^* d\sigma^2 dt \tag{86}
\]

in assumption (8) with \( \epsilon \) arbitrary small. Since we assume (8), we would obtain the above, or take \( R_0 \) as being the infimum of all \( R_0 \) in (57) plus \( \epsilon \) fixed. Therefore, in the region \( w^{*>} \leq -1, r^{*>} \geq R^* \), we have,

\[
|F_{\mu\nu}|(w^{*>,}w^{#},\omega) \lesssim \frac{[E^{M}_{F,C,F,r\Psi,F,C,r\Psi,F,r^2(F)\gamma F,C,r^2(F)\gamma F,r^3(F)\gamma F}] }{(1 + |v^{*}|)}
\]

\[
\lesssim \frac{[E^{M}_{F,C,F,r\Psi,F,C,r\Psi,F,r^2(F)\gamma F,C,r^2(F)\gamma F,r^3(F)\gamma F}] }{(1 + |v|)}
\]
which gives,

\[ |F|(w, v, \omega) \lesssim \frac{E_F}{(1 + |v|)} \]

in the region \(-1 \leq w \leq 1, r \geq R\).

5. \textbf{Decay of the Energy to Observers Traveling to the Black Hole on} \(v = \text{constant}\) \textbf{Hypersurfaces Near the Horizon}

5.1. \textbf{The vector field} \(H\).

Let

\[ H = -\frac{h(r^*)}{(1 - \mu)} \frac{\partial}{\partial w} - h(r^*) \frac{\partial}{\partial v} \]

\[ = H^w \frac{\partial}{\partial w} + H^v \frac{\partial}{\partial v} \quad (87) \]

Computing,

\[ \frac{\partial}{\partial w} h^w = \frac{\partial r^*}{\partial w} \frac{\partial}{\partial r^*} H^w + \frac{\partial t}{\partial w} \frac{\partial}{\partial t} H^w \]

\[ = -\frac{1}{2} \frac{\partial}{\partial r^*} H^w + 0 \]

\[ = \frac{1}{2(1 - \mu)} [h' - \frac{\mu}{r} h] \]

where \( h' = \frac{\partial}{\partial r^*} h \)

And,

\[ \frac{\partial}{\partial v} H^w = \frac{\partial r^*}{\partial v} \frac{\partial}{\partial r^*} H^w + \frac{\partial t}{\partial v} \frac{\partial}{\partial t} H^w \]

\[ = \frac{1}{2} \frac{\partial}{\partial r^*} H^w + 0 \]

\[ = \frac{-1}{2(1 - \mu)} [h' - \frac{\mu}{r} h] \]
Similarly,

\[
\frac{\partial}{\partial v} H^v = \frac{\partial r^*}{\partial v} \frac{\partial}{\partial r^*} H^v + \frac{\partial t}{\partial v} \frac{\partial}{\partial t} H^v
\]

\[
= \frac{1}{2} \frac{\partial}{\partial r^*} H^v + 0
\]

\[
= -\frac{1}{2} h^v
\]

\[
\frac{\partial}{\partial w} H^v = \frac{\partial r^*}{\partial w} \frac{\partial}{\partial r^*} H^v + \frac{\partial t}{\partial w} \frac{\partial}{\partial t} H^v
\]

\[
= -\frac{1}{2} h^v
\]

\[
+ \frac{1}{4} \left[ \frac{3\mu - 2}{2} r \right] (H^v - H^w)
\]

Thus,

\[
\begin{align*}
\pi^{\alpha\beta}(H)T_{\alpha\beta}(\Psi) & = \left[ \frac{1}{r^2} |\Psi_{\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \left( \frac{-2}{1 - \mu} \right) \\
& + \left[ \frac{1}{r^2} |\Psi_{\phi\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] \left( \frac{-2}{1 - \mu} \right) \\
& + \left[ \frac{1}{(1 - \mu)^2} |\Psi_{\phi\theta}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right]
\end{align*}
\]

\[
\begin{align*}
& \cdot \left[ -2 \left( \frac{1}{2(1 - \mu)} [(h^v - (1 - \mu)h^w) - \frac{\mu}{r} h^v] \right) + \frac{(2 - 3\mu)}{(1 - \mu)R} (h - (1 - \mu)h) \right]
\end{align*}
\]

(88)

We have,

\[
F_{\Psi}(H)(w = w_i)(v_i \leq v \leq v_{i+1}) = \int_{v = v_i}^{v = v_{i+1}} \int_{S^2} J_{\alpha}(H)n^\alpha dV \omega_{w = w_i}(w = w_i)
\]

(89)
where
\[
n^\alpha = g\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial t}\right)^{-1}\left(\frac{\partial}{\partial w}\right)^\alpha = -\frac{2}{(1-\mu)}\left(\frac{\partial}{\partial w}\right)^\alpha
\]
and
\[
dVol_{w=w_i} = g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) r^2 d\sigma^2 dv
= -(1-\mu)r^2 d\sigma^2 dv
\]

We get
\[
F^{(H)}_{(w=w_i)}(v_i \leq v \leq v_{i+1})
= \int_{v=v_i}^{v=v_{i+1}} \int_{S^2} -2[h(r^*) T_{wv} + h(r^*) T_{vv}] r^2 d\sigma^2 dv
= \int_{v=v_i}^{v=v_{i+1}} \int_{S^2} -2[h(r^*)\left(\frac{1}{1-\mu}\left|\Psi_{vw}\right|^2 + \frac{1}{4r^4\sin^2\theta}\left|\Psi_{v\phi}\right|^2\right)] r^2 d\sigma^2 dv
+ h(r^*)\left(\frac{1}{r^2}\left|\Psi_{w\phi}\right|^2 + \frac{1}{r^2\sin^2\theta}\left|\Psi_{v\phi}\right|^2\right)] r^2 d\sigma^2 dv
\]
\[
= \int_{v=v_i}^{v=v_{i+1}} \int_{S^2} -2[h(r^*)\left(\frac{1}{(1-\mu)^2}\left|\Psi_{vw}\right|^2 + \frac{1}{4r^4\sin^2\theta}\left|\Psi_{v\phi}\right|^2\right)] r^2 d\sigma^2 dv
+ h(r^*)\left(\frac{1}{r^2}\left|\Psi_{w\phi}\right|^2 + \frac{1}{r^2\sin^2\theta}\left|\Psi_{v\phi}\right|^2\right)] r^2 d\sigma^2 dv
\]
\[(90)\]

and,
\[
F^{(H)}_{(v=v_i)}(w_i \leq w \leq w_{i+1}) = \int_{w=w_i}^{w=w_{i+1}} \int_{S^2} J_\alpha(H)n^\alpha dVol_{v=v_i}(v = v_i)
\]
\[(91)\]

where
\[
n^\alpha = g\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial t}\right)^{-1}\left(\frac{\partial}{\partial w}\right)^\alpha = -\frac{2}{(1-\mu)}\left(\frac{\partial}{\partial w}\right)^\alpha
\]
and
\[
dVol_{w=w_i} = g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) r^2 d\sigma^2 dw
= -(1-\mu)r^2 d\sigma^2 dw
\]

Thus,
\[ F_{\Psi}^{(H)}(v = v_i)(w_i \leq w \leq w_{i+1}) = \int_{w = w_i}^{w = w_{i+1}} \int_{S^2} -2\left( \frac{\partial}{\partial v}\right) T_{ww} + h(r^{*})T_{vw}r^{2}\sigma^2 d\omega d\sigma \]

\[ = \int_{w = w_i}^{w = w_{i+1}} \int_{S^2} -2\left( \frac{\partial}{\partial v}\right) \frac{h(r^{*})}{(1 - \mu)} + \frac{1}{r^{2}} |\Psi_{w\theta}|^2 + \frac{1}{r^{2} \sin^2 \theta} |\Psi_{w\phi}|^2 \]

\[ + h(r^{*}) \left( \frac{1}{(1 - \mu)} |\Psi_{vw}|^2 + \frac{1}{4r^{4} \sin^2 \theta} |\Psi_{\phi\phi}|^2 \right) r^{2}\sigma^2 d\omega d\sigma \]

We get

\[ F_{\Psi}^{(H)}(v = v_i)(w_i \leq w \leq w_{i+1}) = \int_{v = v_i}^{v = v_{i+1}} \int_{w = w_i}^{w = w_{i+1}} \int_{S^2} \left( \frac{1}{r^{2}(1 - \mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^{2} \sin^2 \theta(1 - \mu)} |\Psi_{w\phi}|^2 \right) \]

\[ + (1 - \mu) h(r^{*}) \left( \frac{1}{(1 - \mu)} |\Psi_{vw}|^2 + \frac{1}{4r^{4} \sin^2 \theta} |\Psi_{\phi\phi}|^2 \right) r^{2}\sigma^2 d\omega d\sigma \]

(92)

Applying the divergence theorem for \( \Psi_{\mu\nu} \) in a rectangle in the Penrose diagram representing the exterior of the Schwarzschild space-time of which one side contains the horizon, say in the region \([w_i, \infty], [v_i, v_{i+1}]\) :

\[ \int_{v = v_i}^{v = v_{i+1}} \int_{w = w_i}^{w = w_{i+1}} \int_{S^2} \left[ \frac{1}{r^{2}(1 - \mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^{2} \sin^2 \theta(1 - \mu)} |\Psi_{w\phi}|^2 \right] r^{2}\sigma^2 d\omega d\sigma \]

\[ + (1 - \mu) h(r^{*}) \left( \frac{1}{(1 - \mu)} |\Psi_{vw}|^2 + \frac{1}{4r^{4} \sin^2 \theta} |\Psi_{\phi\phi}|^2 \right) \]

\[ + \left[ \frac{1}{(1 - \mu)} \left( h^{\prime} - (1 - \mu)h^{\prime \prime} \right) - \frac{\mu}{r} h \right] r^{2}\sigma^2 d\omega d\sigma \]

\[ = -F_{\Psi}^{(H)}(w = \infty)(v_i \leq v \leq v_{i+1}) + F_{\Psi}^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1}) \]

\[ - F_{\Psi}^{(H)}(v = v_{i+1})(w_i \leq w \leq \infty) + F_{\Psi}^{(H)}(v = v_i)(w_i \leq w \leq \infty) \]

We are going to choose \( h \) such that

\[ h(r^{*}) = -\infty = 1 \]

and for all \( r > 2m \):

\[ h \geq 0 \]
Furthermore, we let \( h \) be supported in the region \( 2m \leq r \leq (1.2)r_1 \) for \( r_1 \) chosen such that, \( 2m < r_0 \leq r_1 < (1.2)r_1 < 3m \). We choose \( h \) such that, for all \( r \leq r_1 \), we have

\[
\frac{\mu}{r} h - h' \geq 0 \tag{93}
\]

\[
h > 0 \tag{94}
\]

\[
h' \geq 0 \tag{95}
\]

\[
- \frac{1}{(1-\mu)} h' + \frac{3}{r} h \leq 0 \tag{96}
\]

\[
\mu \left[ - \frac{1}{(1-\mu)} h' + \frac{3}{r} h \right] \leq -h \tag{97}
\]

Computing

\[
\int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{S_2} \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{w\theta}|^2 \,
\]

\[
\cdot \left[ - \frac{1}{(1-\mu)} [(h' - (1-\mu)h') - \frac{\mu}{r} h] + \frac{(2-3\mu)}{(1-\mu)} r (h - (1-\mu)h)] \right] r^2 ds^2 (1-\mu) dw dv
\]

\[
= \int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{S_2} \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{w\theta}|^2 \,
\]

\[
\cdot \left[ - \frac{1}{(1-\mu)} \left( \frac{\mu}{r} h' - \frac{1}{r} h \right) + \frac{(2-3\mu)}{(1-\mu)} \frac{r}{h} \right] r^2 ds^2 (1-\mu) dw dv
\]

\[
= \int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{S_2} \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{w\theta}|^2 \,
\]

\[
\cdot \mu \left[ - \frac{1}{(1-\mu)} h' + \frac{3}{r} h \right] r^2 ds^2 (1-\mu) dw dv \tag{98}
\]

Let,

\[
I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)
\]

\[
= \int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{S_2} \left( \frac{1}{r^2 (1-\mu)^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)^2} |\Psi_{w\phi}|^2 \right) \left( h' - \frac{\mu}{r} h \right)
\]

\[
+ \left( \frac{1}{r^2} |\Psi_{vw}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 \right) \left[ - \frac{h'}{(1-\mu)} \right]
\]

\[
+ \left( \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{w\theta}|^2 \right) \mu \left[ - \frac{1}{(1-\mu)} h' + \frac{3}{r} h \right] r^2 ds^2 (1-\mu) dw dv \tag{99}
\]
Then, we have,

\[-F_{\Psi}^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1}) - F_{\Psi}^{(H)}(v = v_i)(w_i \leq w \leq \infty)\]

\[= -I_{\Psi}^{(H)}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)
- F_{\Psi}^{(H)}(w = \infty)(v_i \leq v \leq v_{i+1}) - F_{\Psi}^{(H)}(v = v_{i+1})(w_i \leq w \leq \infty)\]

(100)

We choose \(r_1\) small enough such that \((1.2)r_1 < 3m\).

5.2. Estimate 1.

For \((w_i, v_i)\) such that \(r(w_i, v_i) = r_1\), where \(r_1\) is as determined in the construction of the vector field \(H\), and for \(v_{i+1} \geq v_i\), we have

\[-F_{\Psi}^{(H)}(w = w_i)(v_i \leq v \leq v_{i+1}) \lesssim F_{\Psi}^{(\frac{\partial}{\partial v})}(w = w_i)(v_i \leq v \leq v_{i+1})\]  

(101)

**Proof**

We have,

\[F_{\Psi}^{(\frac{\partial}{\partial v})}(w = w_i)(v_i \leq v \leq v_{i+1}) = \int_{v = v_i}^{v = v_{i+1}} \int_{S^2} J_{\alpha}(\frac{\partial}{\partial t})n^\alpha dV ol_{w = w_i}(w = w_i)\]

where

\[n^\alpha = g(\frac{\partial}{\partial v}, \frac{\partial}{\partial t})^{-1}(\frac{\partial}{\partial v})^\alpha\]

\[= -\frac{2}{(1 - \mu)}(\frac{\partial}{\partial v})^\alpha\]

and

\[dV ol_{w = w_i} = g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})r^2 d\sigma^2 dv = -(1 - \mu)r^2 d\sigma^2 dv\]
We get
\[ F_{\Psi}^{(H)} (w = w_i) (v_i \leq v \leq v_{i+1}) = \int_{v = v_i}^{v = v_{i+1}} \int_{S^2} 2[T_{vv} + T_{vw}] r^2 d\sigma^2 dv \]
\[ = \int_{v = v_i}^{v = v_{i+1}} \int_{S^2} 2\left( \frac{1}{(1 - \mu)} \right) |\Psi_{vw}|^2 + \frac{(1 - \mu)}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \]
\[ + \frac{1}{r^2} |\Psi_{v\phi}|^2 + \frac{1}{r^2 \sin^2 \theta |\Psi_{v\phi}|^2} r^2 d\sigma^2 dv \]
\[ = \int_{v = v_i}^{v = v_{i+1}} \int_{S^2} 2\left( \frac{1}{(1 - \mu)} \right)^2 |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \]
\[ + \frac{1}{r^2 (1 - \mu)} |\Psi_{v\phi}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{v\phi}|^2 |r^2 (1 - \mu) d\sigma^2 dv \]

We showed that,
\[ F_{\Psi}^{(H)} (w = w_i) (v_i \leq v \leq v_{i+1}) \]
\[ = \int_{v = v_i}^{v = v_{i+1}} \int_{S^2} -2[h(r^*) (\frac{1}{(1 - \mu)} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2) \]
\[ + h(r^*) (\frac{1}{r^2} |\Psi_{v\phi}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2) r^2 d\sigma^2 dv \]

The region \( w = w_i \) and \( v_i \leq v \leq v_{i+1} \) is in the region \( r \geq r_1 \) as \( r(w_i, v_i) = r_1 \), and \( v_{i+1} \geq v_i \). Thus, in this region
\[ \frac{h(r^*)}{(1 - \mu)} \lesssim 1 \]

which gives immediately (101).

5.3. Estimate 2.

Let \( t_{i+1} = (1.1) t_i \)

Define,
\[ J^{(C)}_\psi (t_i \leq t \leq t_{i+1}) (r_0 \leq r \leq R_0) = \int_{t = t_i}^{t = t_{i+1}} \int_{r = r_0}^{r = R_0} \int_{S^2} |\Psi_{\phi\psi}|^2 + \frac{1}{4} |\Psi_{\phi\psi}|^2 |r^* - (3m)^*| d\sigma^2 dt \]

From (13), we have
\[ J^{(C)}_\psi (t_i \leq t \leq t_{i+1}) (r_0 \leq r \leq R_0) \lesssim |\hat{E}^{(\hat{\psi})}_\psi (t_i)| + |\hat{E}^{(\hat{\psi})}_\psi (t_{i+1})| \]

(103)
Recall that
\[ 2m < r_0 \leq r_1 < (1.2)r_1 < 3m \]

We have,
\[
\int_{t_i}^{t_i+1} \int_{r_*=-\infty}^{r_*=\infty} \int_{S^2} \left[ |\Psi_{r*\theta}|^2 + |\Psi_{r*\phi}|^2 + |\Psi_{t*\theta}|^2 + |\Psi_{t*\phi}|^2 \right] \chi_{[r_1^*, (1.2)r_1^*]} \, r^2 (1 - \mu) d\sigma^2 dr^* dt \\
\lesssim |E_{\Psi}(\hat{\Psi})| (-(0.85)t_i \leq r_* \leq (0.85)t_i)(t = t_i) \]

(104)

Proof

Let,
\[
f(r^*) = \int_{r_*=-\infty}^{r_*=\infty} \chi_{[r_1^*, (1.2)r_1^*]} (r^*) \, dr^* \quad (105)
\]

where \( \chi \) is the sharp cut-off function, such that,
\[
f(r^*) = 1, \quad \text{for} \quad r_1^* < r^* < (1.2)r_1^*
\]

and
\[
f(r) = 0, \quad \text{for} \quad r^* \in \left[ -\infty, r_1^* \right] \cup \left[ (1.2)r_1^*, \infty \right] \]

We get from (49) applied to \( f \),
\[
T^{\alpha\beta}(\Psi_{ww}) \pi_{\alpha\beta}(G) \\
= \frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 + \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \chi_{[r_1^*, (1.2)r_1^*]} \frac{(3\mu - 2) \, \int_{r_*=-\infty}^{r_*=\infty} \chi_{[r_1^*, (1.2)r_1^*]} dr^*}{\left(1 - \mu \right)^2} \\
- \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \left( \chi_{[r_1^*, (1.2)r_1^*]} + \frac{3\mu - 2}{r^2} \int_{r_*=-\infty}^{r_*=\infty} \chi_{[r_1^*, (1.2)r_1^*]} \, dr^* \right)
\]
Applying the divergence theorem between the two hypersurfaces \( \{ t = t_i \} \) and \( \{ t = t_{i+1} \} \), we obtain

\[
\begin{align*}
&\int_{t=t_{i}}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{S^2} \left( \frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 + \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right) \frac{\chi_{[r_i^{*},(1.2)r_i^{*}]}(1 - \mu)}{(1 - \mu)} r^2 (1 - \mu) dr^* d\sigma^2 dt \\
&= \int_{t=t_{i}}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{S^2} \left( \frac{2}{r^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right) \chi_{[r_i^{*},(1.2)r_i^{*}]} r^2 (1 - \mu) dr^* d\sigma^2 dt \\
&\quad + \frac{(3\mu - 2)}{r} \int_{r^*=-\infty}^{r^*=\infty} \chi_{[r_i^{*},(1.2)r_i^{*}]} dr^* r^2 (1 - \mu) dr^* d\sigma^2 dt \\
&\quad + E^{(G)}(t_{i+1}) - E^{(G)}(t_{i}) \\
&\leq \int_{t=t_{i}}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=r_{i}} \int_{S^2} \left( \frac{1}{r^2} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2 + \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right) \frac{\chi_{[r_i^{*},(1.2)r_i^{*}]}(1 - \mu)}{(1 - \mu)} r^2 (1 - \mu) dr^* d\sigma^2 dt \\
&\quad + \frac{(3\mu - 2)}{r} \int_{r^*=-\infty}^{r^*=(3m)^*} \chi_{[r_i^{*},(1.2)r_i^{*}]} dr^* r^2 (1 - \mu) dr^* d\sigma^2 dt \\
&\quad + E^{(G)}(t_{i+1}) - E^{(G)}(t_{i}) \\
&\leq J^{(C)}_{\Psi}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E^{(G)}_{\Psi}(t_{i+1}) - E^{(G)}_{\Psi}(t_{i})
\end{align*}
\]

Instead of \( \Psi \), take \( \hat{\Psi} \) as in the proof of (58), we get,

\[
\begin{align*}
&\int_{t=t_{i}}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{S^2} \left( \frac{1}{r^2} |\hat{\Psi}_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\hat{\Psi}_{w\phi}|^2 + \frac{1}{r^2} |\hat{\Psi}_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\hat{\Psi}_{v\phi}|^2 \right) \frac{\chi_{[r_i^{*},(1.2)r_i^{*}]}(1 - \mu)}{(1 - \mu)} r^2 (1 - \mu) dr^* d\sigma^2 dt \\
&\leq \int_{t=t_{i}}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=r_{i}} \int_{S^2} \left( \frac{1}{r^2} |\hat{\Psi}_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\hat{\Psi}_{w\phi}|^2 + \frac{1}{r^2} |\hat{\Psi}_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\hat{\Psi}_{v\phi}|^2 \right) \frac{\chi_{[r_i^{*},(1.2)r_i^{*}]}(1 - \mu)}{(1 - \mu)} r^2 (1 - \mu) dr^* d\sigma^2 dt \\
&\leq J^{(C)}_{\hat{\Psi}}(t_i \leq t \leq t_{i+1})(r_0 \leq r \leq R_0) + E^{(G)}_{\hat{\Psi}}(t_{i+1}) - E^{(G)}_{\hat{\Psi}}(t_{i})
\end{align*}
\]

Recall that we have,

\[
|E^{(G)}_{\Psi}(-\infty \leq r^* \leq \infty)(t_i)| \leq |E^{(G)}_{\Psi}((0.85)t_i \leq r^* \leq (0.85)t_i)(t = t_i)|
\]

Thus,

\[
\begin{align*}
&\int_{t=t_{i}}^{t=t_{i+1}} \int_{r^*=-\infty}^{r^*=\infty} \int_{S^2} \frac{1}{2} \left( |\Psi_{\phi}|^2 + |\hat{\Psi}_{\phi}|^2 \right) \chi_{[r_i^{*},(1.2)r_i^{*}]} r^2 (1 - \mu) dr^* d\sigma^2 dt \\
&\leq J^{(C)}_{\Psi}(t_i \leq t \leq t_{i+1})(-\infty \leq r^* \leq \infty) \\
&\leq |E^{(G)}_{\Psi}(-\infty \leq r^* \leq \infty)| \\
&\leq |E^{(G)}_{\Psi}((0.85)t_i \leq r^* \leq (0.85)t_i)(t = t_i)|
\end{align*}
\]
5.4. **Estimate 3.**

For

\[
\begin{align*}
    w_i & = t_i - r_1^* \\
    v_i & = t_i + r_1^*
\end{align*}
\]

where \( r_1 \) is as determined in the construction of the vector field \( H \), we have

\[
- I^H_\psi (v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty) \\
- F^H_\psi (w = \infty)(v_i \leq v \leq v_{i+1}) - F^H_\psi (v = v_{i+1})(w_i \leq w \leq \infty) \\
\leq C F^H_\psi (w = w_i)(v_i \leq v \leq v_{i+1}) - F^H_\psi (v = v_i)(w_i \leq w \leq \infty) \\
\]

(106)

(where \( C \) is a constant)

And,

\[
- I^H_\psi (v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \\
- F^H_\psi (v = v_{i+1})(w_i \leq w \leq \infty) - F^H_\psi (v = v_{i+1})(w_i \leq w \leq \infty) \\
\leq F^H_\psi (w = w_i)(v_i \leq v \leq v_{i+1}) - F^H_\psi (v = v_i)(w_i \leq w \leq \infty) \\
+ |F^H_\psi (- (0.85)t_i \leq r^* \leq (0.85)t_i)(t = t_i)| \\
\]

(107)

**Proof**

We showed that,

\[
- I^H_\psi (v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty) \\
- F^H_\psi (w = \infty)(v_i \leq v \leq v_{i+1}) - F^H_\psi (v = v_{i+1})(w_i \leq w \leq \infty) \\
= - F^H_\psi (w = w_i)(v_i \leq v \leq v_{i+1}) - F^H_\psi (v = v_i)(w_i \leq w \leq \infty) \\
\]

From (101), we have,

\[
- F^H_\psi (w = w_i)(v_i \leq v \leq v_{i+1}) \lesssim F^H_\psi (w = w_i)(v_i \leq v \leq v_{i+1}) \\
\]

This proves (106). On the other hand, for all \( r \leq r_1 \), we have,

\[
\frac{-1}{(1 - \mu)} h' + \frac{3}{r} h \leq 0 \\
\]

(108)
Thus,

\[
- \int_{v=v_i,r \leq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{S^2} \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \mu \left[ -\frac{1}{r^2} h' + \frac{3}{r} h \right] \cdot r^2 d\sigma^2 (1-\mu) dw dv \\
\geq 0 \tag{109}
\]

whereas,

\[
|\int_{v=v_i,r \geq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{S^2} \frac{1}{(1-\mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \mu \left[ -\frac{1}{r^2} h' + \frac{3}{r} h \right] \cdot r^2 (1-\mu) d\sigma^2 dw dv|
\]

\[
\lesssim J^{(C)}_{\Psi}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r_1 \leq r \leq (1.2)r_1)
\]

\[
\lesssim |E_{\Psi}^{(\phi)}(-(0.85)t_i \leq r^* \leq (0.85)t_i)(t=t_i)| \tag{110}
\]

\[
J^{(C)}_{\Psi}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r_1 \leq r \leq 1.2r_1)
\]

\[
\lesssim |E_{\Psi}^{(\phi)}(-(0.85)t_i \leq r^* \leq (0.85)t_i)(t=t_i)|
\]

and,

\[
|\int_{v=v_i,r \geq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{S^2} \left[ \frac{1}{r^2 (1-\mu)} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \left( -h' - \mu \frac{1}{r^2} h \right) \right] \cdot r^2 d\sigma^2 (1-\mu) dw dv|
\]

\[
\lesssim \int_{v=v_i,r \geq r_1}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} \int_{S^2} \left[ \frac{1}{r^2 (1-\mu)} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \cdot \chi_{[r_1,1.2r_1]}(r^*) \cdot r^2 d\sigma^2 dw dv
\]

where \( \chi_{[r_1,1.2r_1]} \) is a smooth positive cut-off function supported on \([r_1^*,1.2r_1^*] \).
From (104),
\[
\int_{v=v_{i+1}}^{v=v_i+1} \int_{r=r_1}^{r=r_i} \int_{w=w_i}^{w=\infty} \left[ \frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right. \\
\left. + \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right] \chi_{[r_1^*,1.2r_1^*]} r^2 d\sigma^2 dw dv \\
\lesssim |E^{(\mathbf{\Pi})}_\psi ((0.85)t_i \leq r^* \leq (0.85)t_i)(t=t_i)|
\]

We get,
\[
|f^{(H)}_\psi (v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \geq r_1) \lesssim |E^{(\mathbf{\Pi})}_\psi ((0.85)t_i \leq r^* \leq (0.85)t_i)(t=t_i)|
\]

This proves (107).

5.5. Estimate 4.

Let \( v_{i+1} \geq v_i \). We have
\[
\inf_{v_i \leq v \leq v_{i+1}} -F^{(H)}_\psi (v)(w_i \leq w \leq \infty) \\
\lesssim -F^{(H)}_\psi (v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) + \sup_{v_i \leq w \leq v_{i+1}} F^{(\mathbf{\Pi})}_\psi (v)(w_i \leq w \leq \infty)(r \geq r_1)
\]

Proof

We have,
\[
-F^{(H)}_\psi (v)(w_i \leq w \leq \infty) = \int_{w=w_i}^{w=\infty} \int_{S^2} \left[ h(r^*) \left( \frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right) \\
+ (1-\mu) h(r^*) \left( \frac{1}{(1-\mu)^2} |\Psi_{v\theta}|^2 + \frac{1}{4r^4 \sin^2 \theta (1-\mu)} |\Psi_{v\phi}|^2 \right) \right] r^2 d\sigma^2 dw
\]

and we have
Given the expression of \( h, h' \), in the region \( r \leq r_1 \), we get for \( v \geq v_i \),

\[
-F^{(H)}_\Psi(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)
= \int_{v=v_i}^{v=v_{i+1}} \int_{w=w_i}^{w=\infty} 2\left[T_{ww} + T_{wv}\right] r^4 \sin^2 \theta \, r^2 \, d\varphi \, d\sigma \, dw
\]

On the other hand, we have,

\[
F^{(H)}_\Psi(v_i \leq v \leq w_{i+1})
= \int_{w=w_i}^{w=w_{i+1}} \int_{\mathbb{S}^2} 2\left[T_{ww} + T_{wv}\right] r^2 \, d\varphi \, d\sigma \, dw
\]

Thus, from the boundedness of \( h, h' \), we have in \( r \geq r_1 \),

\[
-F^{(H)}_\Psi(v_i \leq w \leq \infty)(r \geq r_1)
\]

\[
\leq \int_{w=w_i, r \geq r_1} \int_{\mathbb{S}^2} \left( \frac{1}{r^2(1-\mu)} |\Psi_w|^2 + \frac{1}{r^2 \sin^2 \theta(1-\mu)} |\Psi_w|^2 \right) + \frac{1}{(1-\mu)^2} \frac{1}{r^4 \sin^2 \theta(1-\mu)} |\Psi_{\phi w}|^2 \right) \, r^2 \, d\varphi \, d\sigma \, dw
\]
Thus,

\[
-\frac{F_{\psi}^{(H)}(v)(w_i \leq w \leq \infty)}{w \geq \infty} \leq \int_{w = w_i, r \geq r_1} \int_{S^2} \left( \frac{1}{r^2(1 - \mu)} |\Psi_w|^2 + \frac{1}{r^2 \sin^2 \theta(1 - \mu)} |\Psi_{w\phi}|^2 \right) \left( \frac{1}{(1 - \mu)} (\frac{\mu}{r} h - h') \right)
+ \frac{1}{r^2} |\Psi_{v\phi}|^2 + \frac{1}{4r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \left( \frac{h'}{1 - \mu} \right)
+ \int_{v = v_i + 1}^{v = w_i} \frac{1}{r^2} \sin^2 \theta |\Psi_{v\phi}|^2 \left| \frac{\mu}{(1 - \mu)} (\frac{3}{r^2} h) \right| r^2 \sin^2 \theta (1 - \mu) d\sigma dv
+ \int_{v = v_i}^{v = v_i + 1} F_{\psi}^{(\psi)}(v)(w_i \leq w \leq \infty)(r \geq r_1) dv
\]

\[
\leq -(112)
\]

\[
0 \leq -F_{\psi}^{(H)}(v_i \leq v \leq v_i + 1)(w_i \leq w \leq \infty)(r \leq r_1) \leq |E_{\psi}^{(\psi)}(t) + E_{\psi}^{(\psi)}(t = t_0)}
\]

5.6. **Estimate 5.**

For

\[
w_i = t_i + r_1 \]
\[
v_i = t_i + r_1
\]

where \( r_1 \) is as determined in the construction of the vector field \( H \), we have

\[
(112)
\]
where,

$$E_{\Psi}(t = t_0)$$

$$= \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \left[ \frac{1}{r^2(1 - \mu)} |\Psi_{\theta\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{\theta\phi}|^2 + \frac{1}{r^2} |\Psi_{\phi\phi}|^2 \right] r^2 \sin^2 \theta \sigma^2 dr^* (t = t_0)$$

$$+ \frac{1}{(1 - \mu)^2} |\Psi_{v\nu}|^2 + \frac{1}{4 r^4 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right] r^2 \sigma^2 dr^* (t = t_0)$$

$$(113)$$

**Proof**

Computing,

$$E^{(H)}_{\Psi}(t) = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} J_\alpha(H)(\Psi_{\mu\nu}) n^\alpha d\text{Vol}_t$$

$$(114)$$

where $n^\alpha = -\frac{(\phi^\alpha)}{\sqrt{1 - \mu}}$ and $d\text{Vol}_{t=t_i} = r^2 \sqrt{1 - \mu} \sigma^2 dr^*$, we get,

$$E^{(H)}_{\Psi}(t) = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \frac{1}{\sqrt{1 - \mu}} (\frac{\partial}{\partial t})^\alpha J_\alpha(H)(\Psi_{\mu\nu}) r^2 \sqrt{1 - \mu} \sigma^2 dr^*$$

Therefore,

$$E^{(H)}_{\Psi}(t) = \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} -(H)^\alpha T_{\alpha\beta}(\Psi_{\mu\nu}) r^2 \sigma^2 dr^*$$

$$= \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \frac{1}{(1 - \mu)} T_{tw} h T_{te} r^2 \sigma^2 dr^*$$

$$= \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \frac{1}{(1 - \mu)} T_{ww} h T_{tw} + h T_{wv} r^2 \sigma^2 dr^*$$

$$= \int_{r^* = -\infty}^{r^* = \infty} \int_{S^2} \left[ \frac{1}{(1 - \mu)} |\Psi_{\theta\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\theta\phi}|^2 \right]$$

$$+ \frac{1}{(1 - \mu)} |\Psi_{v\theta}|^2 + \frac{1}{(1 - \mu)} |\Psi_{\phi\theta}|^2$$

$$h \left( \frac{1}{r^2} |\Psi_{v\theta}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\phi\theta}|^2 \right)$$

$$+ h \left( \frac{1}{r^2} |\Psi_{v\phi}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\phi\phi}|^2 \right)$$

$$(115)$$
Hence,

\[ E^{(H)}_{\Psi}(t) = \int_{r^* = \infty}^{r^* = -\infty} \int_{S^2} [h(1 - \mu)(|\Psi_{\hat{w}\hat{\theta}}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{w\phi}|^2)] \\
+ h(|\Psi_{\hat{v}\hat{\theta}}|^2 + \frac{1}{4} |\Psi_{\hat{v}\phi}|^2) + h(1 - \mu)(|\Psi_{\hat{v}\hat{\theta}}|^2 + \frac{1}{4} |\Psi_{\hat{v}\phi}|^2) \\
+ h(|\Psi_{\hat{v}\hat{\theta}}|^2 + |\Psi_{\hat{v}\phi}|^2)] r^2 d\sigma^2 dr^* \]

By using the divergence theorem in the region \((v \leq v_0)(t_0 \leq t \leq \infty)(r \leq r_1)\), we

\[ -F^{(H)}_{\Psi}(v \leq v_0)(t_0 \leq t \leq \infty)(r \leq r_1) \]
\[ -E^{(H)}_{\Psi}(v = v_0)(w_0 \leq w \leq \infty) - F^{(H)}_{\Psi}(w = \infty)(-\infty \leq v \leq v_0) = E^{(H)}_{\Psi}(t_0) \quad (115) \]

Due to the positivity of the terms on the left hand side, we get

\[ -E^{(H)}_{\Psi}(v = v_0)(w_0 \leq w \leq \infty) \lesssim E^{(H)}_{\Psi}(t_0) \lesssim E^{\#(\hat{w})}_{\Psi}(t = t_0) \quad (116) \]

where,

\[ E^{\#(\hat{w})}_{\Psi}(t = t_0) \]
\[ = \int_{r^* = \infty}^{r^* = -\infty} \int_{S^2} \left[ \frac{1}{r^2 (1 - \mu)} |\Psi_{\hat{w}\hat{\theta}}|^2 + \frac{1}{r^2 \sin^2 \theta (1 - \mu)} |\Psi_{w\phi}|^2 + \frac{1}{r^2} |\Psi_{\hat{v}\phi}|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{\hat{v}\phi}|^2 \right] \\
+ \left[ \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\hat{v}\phi}|^2 \right] r^2 d\sigma^2 dr^* (t = t_0) \]

From the divergence theorem and the fact that \(\frac{\partial}{\partial t}\) is Killing, it is easy to see that by integrating in a suitable region and using the positivity of the energy we get,

\[ E^{(\hat{w})}_{\Psi}(w = w_i)(v_i \leq v \leq v_{i+1}) = E^{(\hat{w})}_{\Psi}(w = w_i)(v_i \leq v \leq v_{i+1})(r \geq r_1) \]
\[ \lesssim |E^{(\hat{w})}_{\Psi}(-0.85/t_i \leq r^* \leq (0.85)/t_i)(t = t_i)| \quad (117) \]
From (107), we get,
\[ -I_H^\Psi(v_0 \leq v \leq v_1)(w_0 \leq w \leq \infty)(r \leq r_1) \]
\[ -F_H^\Psi(v = v_1)(w_0 \leq w \leq \infty) - F_H^\Psi(w = \infty)(v_0 \leq v \leq v_1) \]
\[ \lesssim F_H^\Psi(w = w_0)(v_0 \leq v \leq v_1) - F_H^\Psi(v = v_0)(w_0 \leq w \leq \infty) \]
\[ + |E_H^\Psi(-(0.85)t_0 \leq r^* \leq (0.85)t_0)(t = t_0)| \]
\[ \lesssim |E_H^\Psi(-(0.85)t_0 \leq r^* \leq (0.85)t_0)(t = t_0)| + E^M_H^\Psi(t = t_0) \]
(from (116) and (117)).

By recurrence from inequality (107), and using (117), we obtain for all \( i \) integer
\[ -I_H^\Psi(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \]
\[ -F_H^\Psi(v = v_{i+1})(w_i \leq w \leq \infty) - F_H^\Psi(w = \infty)(v_i \leq v \leq v_{i+1}) \]
\[ \lesssim F_H^\Psi(w = w_i)(v_i \leq v \leq v_{i+1}) - F_H^\Psi(v = v_i)(w_i \leq w \leq \infty) \]
\[ + |E_H^\Psi(-(0.85)t_i \leq r^* \leq (0.85)t_i)(t = t_i)| \]
\[ \lesssim |E_H^\Psi(-(0.85)t_i \leq r^* \leq (0.85)t_i)(t = t_i)| + E^M_H^\Psi(t = t_0) \]

Due to sign of \( h \), and the definition of \( h \), we have that the terms in each of the integrands on the left hand side are positive, hence, we obtain (112).

\[ \square \]

5.7. Estimate 6.

For all \( v \), let \( w_0(v) = v - 2r_1^7 \)

Let
\[ v_+ = \max\{1, v\} \] (118)

We have,
\[ -F_H^\Psi(v)(w_0(v) \leq w \leq \infty) \lesssim \frac{|E_H^\Psi| + E^M_H^\Psi(t = t_0) + E^M_H^\Psi}{v_+} \] (119)

and,
\[ -F_H^\Psi(w)(v - 1 \leq w \leq v) \lesssim \frac{|E_H^\Psi| + E^M_H^\Psi(t = t_0) + E^M_H^\Psi}{v_+^2} \] (120)
Proof

Let,

\[ v_i = t_i + r_1^* \quad (121) \]
\[ w_i = t_i - r_1^* \quad (122) \]

where \( t_i \) is defined as in (76):

\[ t_i = (1.1)^i t_0 \]

We have shown that,

\[ \inf_{v_i \leq v \leq v_{i+1}} -F^{(H)}(v)(w_i \leq w \leq \infty) \]
\[ \lesssim \frac{-F^{(H)}(v_i \leq v \leq v_{i+1})(v_i \leq w \leq \infty)}{(v_{i+1} - v_i)} + \sup_{v_i \leq v \leq v_{i+1}} F^{(K)}(v)(w_i \leq w \leq \infty)(r \geq r_1) \]

Lemma 5.8. We have,

\[ \sup_{v_i \leq v \leq v_{i+1}} F^{(K)}(v)(w_i \leq w \leq \infty)(r \geq r_1) \lesssim \frac{E^{(K)}(t_i)}{r_i^2} \quad (123) \]

Proof

By integrating in a well chosen region and using the divergence theorem we get that,

\[ \sup_{v_i \leq v \leq v_{i+1}} F^{(K)}(v)(w_i \leq w \leq \infty)(r \geq r_1) \]
\[ \lesssim \int_{r^* = r_i^*}^{r^* = r_{i+1}} \int_{S^2} \left( \frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right) \]
\[ + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{v\phi}|^2 \right) + \frac{1}{(1-\mu)} \frac{1}{4 r^4 \sin^2 \theta} |\Psi_{\phi\phi}|^2 \right). (1-\mu) r^2 d\sigma d^2 \theta (124) \]

We showed that,

\[ \int_{r^* = r_i^*}^{r^* = r_{i+1}} \int_{S^2} \left( \frac{1}{r^2(1-\mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{w\phi}|^2 \right) \]
\[ + \frac{1}{r^2 \sin^2 \theta (1-\mu)} |\Psi_{v\phi}|^2 \right) + \frac{1}{(1-\mu)} \frac{1}{4 r^4 \sin^2 \theta} |\Psi_{\phi\phi}|^2 \right). (1-\mu) r^2 d\sigma d^2 \theta \]

\[ \lesssim \frac{E^{(K)}(t_i)}{\min_{w \in \{t_i \cap \{r_i^* \leq r^* \leq r_{i+1}^*\}} w^2} + \frac{E^{(K)}(t_i)}{\min_{v \in \{t_i \cap \{r_i^* \leq r^* \leq r_{i+1}^*\}}} v^2} \]
Thus, (124) gives,
\[
\sup_{v_i \leq v \leq v_{i+1}} F^H_{\psi}(v)(w_i \leq w \leq \infty)(r \geq r_1) \lesssim \frac{E^K_{\psi}(t_i)}{t_i^2} \tag{125}
\]

And we showed that,
\[
-F^H_{\psi}(v_i \leq v \leq v_{i+1})(w_i \leq w \leq \infty)(r \leq r_1) \lesssim |E^H_{\psi}(t_i)| + E^K_{\psi}(t = t_0)
\]

Thus, we obtain,
\[
\inf_{v_i \leq v \leq v_{i+1}} F^H_{\psi}(v)(w_i \leq w \leq \infty) \lesssim \frac{1}{(v_{i+1} - v_i)}[|E^H_{\psi}(t_i)| + E^K_{\psi}(t = t_0)] + \frac{E^K_{\psi}(t_i)}{t_i^2} \tag{126}
\]

and thus, there exists a $v_i^\# \in [v_i, v_{i+1}]$ where above inequality holds.

We have,
\[
v_{i+1} - v_i = t_{i+1} + r_*^i - (t_i + r_*^i) = t_{i+1} - t_i = (1.1)^i t_0 - (1.1)^{i+1} t_0 = (1.1)^i t_0 (1.1 - 1) = (0.1)(1.1)^i t_0 = 0.1 t_i \tag{127}
\]

Let,
\[
w_i^\# = v_i^\# - 2r_*^i \tag{128}
\]

Therefore,
\[
-F^H_{\psi}(v_i^\#)(w_i^\# \leq w \leq \infty) \lesssim -F^H_{\psi}(v_i^\#)(w_i \leq w \leq \infty)
\]

(due to the positivity of $-F^H_{\psi}(v_i^\#)(w_i \leq w \leq \infty)$)
\[
\lesssim \frac{1}{t_i^2}[|E^H_{\psi}(t_i)| + E^K_{\psi}(t = t_0)] + \frac{E^K_{\psi}(t_i)}{t_i^2} \tag{by (126)}.
\]
Lemma 5.9.

We proved that, by applying the divergence theorem in a well chosen region, we get,

\[ -F_{\Psi}^{(H)}(v = v_{i+1})(w_i^\#) \leq w \leq \infty \]

\[ \lesssim F_{\Psi}^{(H)}(w = w_i^\#)(v_i^\# \leq v \leq v_{i+1}) - F_{\Psi}^{(H)}(v = v_i^\#)(w_i^\# \leq w \leq \infty) \]

\[ + |E_{\Psi}^{(\partial)}(-(0.85)t_i \leq r^* \leq (0.85)t_i)(t = t_i)| \]

**Proof**

By applying the divergence theorem in a well chosen region, we get,

\[ F_{\Psi}^{(\partial)}(w = w_i^\#)(v_i^\# \leq v \leq v_{i+1}) \lesssim \frac{E_{\Psi}^{(K)}(t_i)}{t_i^2} \]

(129)

We proved that,

\[
\left| E_{\Psi}^{(\partial)}(r_i^* \leq r^* \leq (0.1)t_i + r_i^*)(t = t_i) \right| \\
= \int_{r^* = r_i^*} \int_{S^2} \left[ \frac{1}{r^2(1 - \mu)} |\Psi_{w\theta}|^2 + \frac{1}{r^2 \sin^2 \theta(1 - \mu)} |\Psi_{w\phi}|^2 \right] \left[ \frac{1}{r^2(1 - \mu)} |\Psi_{v\theta}|^2 + \frac{1}{4r^4 \sin^2 \theta |\Psi_{\phi\theta}|^2} \right] (1 - \mu)r^2 d\sigma^2 dr^*(t) \\
\lesssim \frac{E_{\Psi}^{(K)}(t_i)}{E_{\Psi}^{(K)}(t_i)} w^2 + \frac{E_{\Psi}^{(K)}(t_i)}{E_{\Psi}^{(K)}(t_i)} \left( \frac{\min_{r^* \in \{t_i \cap (r_i^* \leq r^* \leq (0.1)t_i + r_i^*)\}} |t_i - r^*|^2}{\min_{r^* \in \{r_i^* \leq r^* \leq (0.1)t_i + r_i^*\}} |t_i - r^*|^2} \right) \left( \frac{\min_{r^* \in \{r_i^* \leq r^* \leq (0.1)t_i + r_i^*\}} |t_i - r^*|^2}{\min_{r^* \in \{r_i^* \leq r^* \leq (0.1)t_i + r_i^*\}} |t_i - r^*|^2} \right) \\
\lesssim \frac{E_{\Psi}^{(K)}(t_i)}{E_{\Psi}^{(K)}(t_i)} w^2 + \frac{E_{\Psi}^{(K)}(t_i)}{E_{\Psi}^{(K)}(t_i)} \left( \frac{\min_{r^* \in \{r_i^* \leq r^* \leq (0.1)t_i + r_i^*\}} |t_i - r^*|^2}{\min_{r^* \in \{r_i^* \leq r^* \leq (0.1)t_i + r_i^*\}} |t_i - r^*|^2} \right) \left( \frac{\min_{r^* \in \{r_i^* \leq r^* \leq (0.1)t_i + r_i^*\}} |t_i - r^*|^2}{\min_{r^* \in \{r_i^* \leq r^* \leq (0.1)t_i + r_i^*\}} |t_i - r^*|^2} \right) \\
(130)
For $r^* \in [r_1^*, (0.1)t_i + r_1^*]$,

\[ t_i - r_1^* \geq t_i - r^* \geq t_i - [(0.1)t_i + r_1^*] = (0.9)t_i - r_1^* \]

We have,

\[ (0.9)t_i - r_1^* \geq 0 \]

(for $t_i$ large enough)

Thus,

\[ |(0.9)t_i - r_1^*|^2 \leq |t_i - r^*|^2 \leq |t_i - r_1^*|^2 \]

\[ \min_{r^* \in \{r_1^* \leq r^* \leq (0.1)t_i + r_1^*\}} |t_i - r^*|^2 \geq |(0.9)t_i - r_1^*|^2 \]

Therefore,

\[ \frac{E^{(K)}_{\Psi}(t_i)}{\min_{r^* \in \{r_1^* \leq r^* \leq (0.1)t_i + r_1^*\}} |t_i - r^*|^2} \lesssim \frac{E^{(K)}_{\Psi}(t_i)}{t_i^2} \]

and thus,

\[ |E^{(\mathcal{M})}_{\Psi}(r_1^* \leq r^* \leq (0.1)t_i + r_1^*)(t = t_i)| \lesssim \frac{E^{(K)}_{\Psi}(t_i)}{t_i^2} \]

Therefore,

\[ E^{(\mathcal{M})}_{\Psi}(w = w_i^#)(v_i^# \leq v \leq v_{i+1}) \lesssim \frac{E^{(K)}_{\Psi}(t_i)}{t_i^2} \]

We also have,
Thus,

\[ |E^{(\frac{\partial}{\partial t})}_{\Psi}(-(0.85)t_i \leq r^* \leq (0.85)t_i)(t = t_i)| \lesssim \frac{E^{(K)}_{\Psi}(t_i)}{t_i^2} \]

and thus,

\[ -F^{(H)}_{\Psi}(v = v_{i+1})(w^\#_i \leq w \leq \infty) \lesssim \frac{1}{t_i} \left[ |E^{(\frac{\partial}{\partial t})}_{\Psi}| + E^{(\frac{\partial}{\partial t})}_{\Psi}(t = t_0) + \frac{E^{(K)}_{\Psi}(t_i)}{t_i^2} \right] \]

\[ \lesssim \left[ |E^{(\frac{\partial}{\partial t})}_{\Psi}| + E^{(\frac{\partial}{\partial t})}_{\Psi}(t = t_0) + E^{(K)}_{\Psi}(t_i) \right] \]

(from the above)

and thus,

\[ -F^{(H)}_{\Psi}(v = v_{i+1})(w_{i+1} \leq w \leq \infty) \lesssim -F^{(H)}_{\Psi}(v = v_{i+1})(w^\#_i \leq w \leq \infty) \]

\[ \lesssim \left[ |E^{(\frac{\partial}{\partial t})}_{\Psi}| + E^{(\frac{\partial}{\partial t})}_{\Psi}(t = t_0) + E^{(K)}_{\Psi}(t_i) \right] \]

(131)

(due to the positivity of \(-F^{(H)}_{\Psi}(v = v_{i+1})(w^\#_i \leq w \leq \infty)\)).

Repeating the same procedure,

**Lemma 5.10.**

\[ -J^{(H)}_{\Psi}(v_{i+1} \leq v \leq v_{i+2})(w_{i+1} \leq w \leq \infty)(r \leq r_1) \lesssim \left[ |E^{(\frac{\partial}{\partial r})}_{\Psi}| + E^{(\frac{\partial}{\partial r})}_{\Psi}(t = t_0) + E^{(M)}_{\Psi} \right] \]

**Proof**

We get from (107),
\[-I^{(H)}(v_{i+1} \leq v \leq v_{i+2})(w_{i+1} \leq w \leq \infty)(r \leq r_1)\]
\[\lesssim F^{(H)}(w = w_{i+1})(v_{i+1} \leq v \leq v_{i+2}) - F^{(H)}(v = v_{i+1})(w_{i+1} \leq w \leq \infty)\]
\[+ |E^{(\frac{\partial}{\partial t})}(w_{i+1} \leq r^* \leq (0.85)t_{i+1})| \]

(due to the positivity of \(-F^{(H)}(w = \infty)(v_{i+1} \leq v \leq v_{i+2}), -F^{(H)}(v = v_{i+2})(w_{i+1} \leq w \leq \infty)\) and \(-I^{(H)}(v_{i+1} \leq v \leq v_{i+2})(w_{i+1} \leq w \leq \infty)(r \leq r_1)\))

We have,

\[|E^{(\frac{\partial}{\partial t})}(-(0.85)t_{i+1} \leq r^* \leq (0.85)t_{i+1})| \lesssim \frac{E^{(K)}(t_{i+1})}{t_{i+1}^2}\]

and,

\[-F^{(H)}(v = v_{i+1})(w_{i+1} \leq w \leq \infty) \lesssim \frac{|E^{(\frac{\partial}{\partial t})} + E^{(\#)}(t = t_0) + E^{(K)}(t_i)|}{t_i}\]

(from above)

And,

\[F^{(\frac{\partial}{\partial t})}(w = w_{i+1})(v_{i+1} \leq v \leq v_{i+2}) \lesssim |E^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq v_{i+2} - \frac{2t_{i+1} - v_{i+2}}{2})|\]
\[\lesssim |E^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq (0.1)t_{i+1} + r_1^*)|\]

We proved,

\[|E^{(\frac{\partial}{\partial t})}(r_1^* \leq r^* \leq (0.1)t_{i+1} + r_1^*)| \lesssim \frac{E^{(K)}(t_{i+1})}{t_{i+1}^2}\]

hence,
\[ F_{\Psi}^{(K)}(w = w_{i+1})(v_{i+1} \leq v \leq v_{i+2}) \lesssim \frac{E_{\Psi}^{(K)}(t_{i+1})}{t_{i+1}^2} \]

Therefore,

\[ -I_{\Psi}^{(H)}(v_{i+1} \leq v \leq v_{i+2})(w_{i+1} \leq w \leq \infty)(r \leq r_1) \lesssim \left[ |E_{\Psi}^{(K)}| + E_{\Psi}^{#}(\frac{\partial}{\partial t}) t = t_0 + E_{\Psi}^{M} \right] \frac{t_{i+1}^2}{t_{i+1}} \]

Using (111), we get,

\[
\inf_{v_{i+1} \leq v \leq v_{i+2}} F_{\Psi}^{(H)}(v)(w_{i+1} \leq w \leq \infty) \lesssim \left[ |E_{\Psi}^{(K)}| + E_{\Psi}^{#}(\frac{\partial}{\partial t}) t = t_0 + E_{\Psi}^{M} \right] \frac{t_{i+1}^2}{t_{i+1}} \]

and thus, there exists a \( v_{i+1}^\# \in [v_{i+1}, v_{i+2}] \) where above inequality holds. We get,

\[
F_{\Psi}^{(H)}(v_{i+1}^\#)(w_{i+1}^\# \leq w \leq \infty) \lesssim \left[ |E_{\Psi}^{(K)}| + E_{\Psi}^{#}(\frac{\partial}{\partial t}) t = t_0 + E_{\Psi}^{M} \right] \frac{t_{i+1}^2}{t_{i+1}} \]

As before, we let,

\[ w_{i+1}^\# = v_{i+1}^\# - 2r_1^* \]

From (107), applied in the region \([w_{i+1}^\#, \infty],[v_{i+1}^\#, v_{i+2}]\), we get due to the positivity of \(-I_{\Psi}^{(H)}(v_{i+1}^\# \leq v \leq v_{i+2})(w_{i+1}^\# \leq w \leq \infty)\), and \(-F_{\Psi}^{(H)}(w = \infty)(v_{i+1}^\# \leq v \leq v_{i+2})\), that,
We proved that,

\[ \mathcal{F}^{(H)} \Psi(v = v_{i+2})(w_{i+1}^\# \leq w \leq \infty) \lesssim \mathcal{F}^{(\#)} \Psi(w = w_{i+1}^\#)(v_{i+1}^\# \leq v \leq v_{i+2}) - \mathcal{F}^{(H)} \Psi(v = v_{i+1}^\#)(w_{i+1}^\# \leq w \leq \infty) \]

\[ + |E^{(\#)}_{\Psi}(-(0.85)t_{i+1} \leq r^* \leq (0.85)t_{i+1})(t = t_{i+1})| \]

We also have,

\[ |E^{(\#)}_{\Psi}(-(0.85)t_{i+1} \leq r^* \leq (0.85)t_{i+1})(t = t_{i+1})| \lesssim \frac{E^{(K)}_{\Psi}(t_{i+1})}{t_{i+1}^2} \]

Therefore,

\[ \mathcal{F}^{(H)} \Psi(v = v_{i+2})(w_{i+1}^\# \leq w \leq \infty) \lesssim \frac{[|E^{(\#)}_{\Psi}| + E^{(\#)}_{\Psi}(t = t_0) + E^M_{\Psi}]}{t_{i+1}^2} \]

and finally,

\[ \mathcal{F}^{(H)} \Psi(v = v_{i+2})(w_{i+2} \leq w \leq \infty) \lesssim \frac{[|E^{(\#)}_{\Psi}| + E^{(\#)}_{\Psi}(t = t_0) + E^M_{\Psi}]}{v_{i+1}^2} \]

(due to the positivity of \( \mathcal{F}^{(H)} \Psi(v = v_{i+2})(w_{i+1}^\# \leq w \leq \infty) \)).
6. The Proof of Decay Near the Horizon

Let \( v_{+} \) be as defined in (118), we will prove that for all \( r \leq r_{1} \),

\[
|F_{v_0}(v, w, \omega)| \lesssim \frac{E_{1}}{v_{+}}
\]
\[
|F_{v_{1}e_2}(v, w, \omega)| \lesssim \frac{E_{1}}{v_{+}}
\]

where,

\[
E_{1} = \left[ \sum_{j=0}^{6} E_{r^j(L^j)^pF}(t = t_{0}) + \sum_{j=0}^{5} E_{r^j(L^j)^pF}(t = t_{0}) + \sum_{j=1}^{3} E_{r^j(L^j)^pF}(t = t_{0}) \right]^{\frac{1}{2}}
\]

and,

\[
E_{2} = \left[ E_{F}^{2} + \sum_{i=0}^{1} \sum_{j=0}^{2} E_{r^j(L^j)^pF}(t = t_{0}) + \sum_{i=0}^{3} E_{r^j(L^j)^pF}(t = t_{0}) \right]^{\frac{1}{2}}
\]

where,

\[
|F_{\bar{v}_1}(v, w, \omega)| \lesssim \frac{E_{2}}{v_{+}}
\]
\[
|F_{\bar{v}_2}(v, w, \omega)| \lesssim \frac{E_{2}}{v_{+}}
\]
\[
|1 - \frac{2m}{r} F_{\bar{v}_1}(v, w, \omega)| \lesssim \frac{E_{2}}{v_{+}}
\]
\[
|1 - \frac{2m}{r} F_{\bar{v}_2}(v, w, \omega)| \lesssim \frac{E_{2}}{v_{+}}
\]
Proof

6.1. Decay for $F_{\hat{v}\hat{w}}$ and $F_{e_1 e_2}$.

We have the Sobolev inequality,

$$|F_{\hat{v}\hat{w}}(v, w, \omega)|^2 \lesssim \int_{\tau=v-1}^{\tau=v} \int_{S^2} \left( |F_{\hat{v}\hat{w}}|^2 + |\mathcal{L}_v F_{\hat{v}\hat{w}}|^2 + |\mathcal{L}_v F_{\hat{v}\hat{w}}|^2 \right) d\sigma^2 d\tau$$

\begin{align*}
\mathcal{L}_v \Psi_{\hat{v}\hat{w}} &= \nabla_{\hat{v}} \Psi_{\hat{v}\hat{w}} - \Psi \left( \nabla_{\hat{v}} \frac{\partial}{\partial \hat{v}} - \frac{\partial}{\partial w} \right) - \Psi \left( \nabla_{\hat{w}} \frac{\partial}{\partial \hat{w}} - \frac{\partial}{\partial v} \right) \\
&= \frac{1}{2} \nabla_{\hat{v}} \Psi_{\hat{v}\hat{w}} - \frac{\mu}{2r} \Psi_{\hat{v}\hat{w}} + \frac{\mu}{2r} \Psi_{\hat{w}\hat{v}} \\
&= \frac{1}{2} \left( \nabla_{\hat{v}} \Psi_{\hat{v}\hat{w}} + \nabla_{\hat{w}} \Psi_{\hat{w}\hat{v}} \right) \\
&= \frac{1}{2} \left( \nabla_{\hat{e}_a} \Psi_{\hat{e}_a\hat{e}_a} + \nabla_{\hat{e}_b} \Psi_{\hat{e}_b\hat{e}_b} \right) \\
&= \frac{1}{2} \left( \mathcal{L}_{\hat{e}_a} \Psi_{\hat{e}_a\hat{e}_a} - \Psi (\nabla_{\hat{v}} \hat{e}_a, \hat{e}_a) - \nabla_{\hat{v}} \Psi_{\hat{e}_a\hat{e}_a} - \nabla_{\hat{e}_a} (\nabla_{\hat{v}} \hat{e}_a, \hat{e}_a) - \Psi (\nabla_{\hat{v}} \hat{e}_a, \hat{e}_a) \right) \\
&= \frac{1}{2} \left( \mathcal{L}_{\hat{e}_a} \Psi_{\hat{e}_a\hat{e}_a} - \Psi (\nabla_{\hat{v}} \hat{e}_a, \hat{e}_a) + \mathcal{L}_{\hat{e}_b} \Psi_{\hat{e}_b\hat{e}_b} - \Psi (\nabla_{\hat{v}} \hat{e}_b, \hat{e}_b) \right)
\end{align*}

(by using the field equations)

Thus,

\begin{align*}
\int_{\tau=v-1}^{\tau=v} \int_{S^2} |\mathcal{L}_v \Psi_{\hat{v}\hat{w}}(v, w, \omega)|^2 d\sigma^2 d\tau &\lesssim \int_{\tau=v-1}^{\tau=v} \int_{S^2} \left( |\nabla_{\hat{e}_a} \Psi_{\hat{e}_a\hat{e}_a} - \nabla_{\hat{e}_b} \Psi_{\hat{e}_b\hat{e}_b}|^2 d\sigma^2 d\tau \right) \\
&\lesssim \int_{\tau=v-1}^{\tau=v} \int_{S^2} \left( |\nabla_{\hat{e}_a} \Psi_{\hat{e}_a\hat{e}_a}|^2 + |\nabla_{\hat{e}_b} \Psi_{\hat{e}_b\hat{e}_b}|^2 \right) d\sigma^2 d\tau \\
&\lesssim \int_{\tau=v-1}^{\tau=v} \int_{S^2} \left( |\nabla_{\hat{e}_a} \Psi_{\hat{e}_a\hat{e}_a}|^2 + |\nabla_{\hat{e}_b} \Psi_{\hat{e}_b\hat{e}_b}|^2 + |\nabla_{\hat{v}} \hat{e}_a|^2 + |\nabla_{\hat{v}} \hat{e}_b|^2 \right) d\sigma^2 d\tau \\
&\lesssim \int_{\tau=v-1}^{\tau=v} \int_{S^2} \left( |r \nabla_{\hat{e}_a} \Psi_{\hat{e}_a\hat{e}_a}|^2 + |r \nabla_{\hat{e}_b} \Psi_{\hat{e}_b\hat{e}_b}|^2 + |\nabla_{\hat{v}} \hat{e}_a|^2 + |\nabla_{\hat{v}} \hat{e}_b|^2 \right) r^2 d\sigma^2 d\tau
\end{align*}

(for $r \geq 2m > 0$)
Recall that,

\[-F^{(H)}_{\Psi}(w)(v - 1 \leq \Psi \leq v)\]

\[= -F^{(H)}_{\Psi}(w)(v - 1 \leq \Psi \leq v) - F^{(H_s)}_{\Psi}(w)(v - 1 \leq \Psi \leq v)\]

\[= \int_{\tau = v-1}^{\tau = v} \int_{S^2} 2h(r^*) \left( \frac{1}{r^2} |\Psi_v\theta|^2 + \frac{1}{r^2 \sin^2 \theta} |\Psi_{v\phi}|^2 \right)\]

\[+ \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4r^4 \sin^2 \theta} |\Psi_{\theta\phi}|^2 \right) r^2 d\sigma^2 d\bar{\nu}\]

\[= \int_{\tau = v-1}^{\tau = v} \int_{S^2} 2h(r^*) \left( |\Psi_{v\varepsilon_1}|^2 + |\Psi_{v\varepsilon_2}|^2 + \frac{1}{(1 - \mu)^2} |\Psi_{vw}|^2 + \frac{1}{4} |\Psi_{\varepsilon_1\varepsilon_2}|^2 \right) r^2 d\sigma^2 d\bar{\nu}\]

(by computing (90) using the orthonormal basis $e_a$, $a \in \{1, 2\}$ instead of $\frac{\partial}{\partial \phi}$ and $\frac{\partial}{\partial \phi}$ which are singular at $\theta = 0, \pi$).

Consequently,

\[\int_{\tau = v-1}^{\tau = v} \int_{S^2} |\mathcal{L}_v \Psi_{\varepsilon_1}(v, w, \omega)|^2 d\sigma^2 d\bar{\nu} \lesssim -F^{(H)}_{\Psi, r^2}(w)(v - 1 \leq \Psi \leq v)\]

Therefore, we have,

\[|F_{\varepsilon_1}(v, w, \omega)| \lesssim \int_{\tau = v-1}^{\tau = v} \int_{S^2} \left( |F_{\varepsilon_1}|^2 + |\mathcal{L}_v F_{\varepsilon_1}|^2 + |\mathcal{L}_v F_{\varepsilon_1}|^2 \right)\]

\[+ |\mathcal{L}_v F_{\varepsilon_1}|^2 + |(\mathcal{L}_v)^2 F_{\varepsilon_1}|^2 + |(\mathcal{L}_v)^2 F_{\varepsilon_1}|^2 \right) d\sigma^2 d\bar{\nu}\]

\[\lesssim -F^{(H)}_{\mathcal{L}_v F_r^2(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3) F_r^3)}(w)(v - 1 \leq \Psi \leq v)\]

From (120), we proved that,

\[-F^{(H)}_{\Psi}(w)(v - 1 \leq \Psi \leq v) \lesssim \frac{|E_{\Psi}^{(\Theta)}(t) + E_{\Psi}^{(\Theta)}(t = t_0) + E_{\Psi}^{(M)}(t = t_0)|}{v_+^2}\]

Thus,

\[-F^{(H)}_{\mathcal{L}_v F_r^2(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3) F_r^3)}(w)(v - 1 \leq \Psi \leq v)\]

\[\lesssim \frac{E_{\mathcal{L}_v F_r^2(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3) F_r^3)}(t = t_0) + E_{\mathcal{L}_v F_r^2(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3) F_r^3} (t = t_0))}{v_+^2}\]

\[+ \frac{E_{\mathcal{L}_v F_r^2(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3(\mathcal{L}_v^2 F_r^3) F_r^3} (t = t_0))}{v_+^2}\]
\[
E_{F,r/L, r^2(L^2)^2, r^3(L^3)^3}^F(t = t_0) + E_{F,r/L, r^2(L^2)^2, r^3(L^3)^3}^F(t = t_0)
+ \sum_{j=0}^3 E_{r^j(L^j)^F}^F(t = t_0) + \sum_{j=0}^3 E_{r^j(L^j)^F}^F(t = t_0) + \sum_{j=0}^3 E_{r^j(L^j)^F}^M(t = t_0)
\]

Recall that,
\[
E^M_F = \sum_{j=0}^3 E_{r^j(L^j)^F}^F(t = t_0) + \sum_{j=0}^3 E_{r^j(L^j)^F}^F(t = t_0) + \sum_{j=0}^3 E_{r^j(L^j)^F}^M(t = t_0)
\]

Thus,
\[
\sum_{j=0}^3 E_{r^j(L^j)^F}^F(t = t_0) + \sum_{j=0}^3 E_{r^j(L^j)^F}^F(t = t_0) + \sum_{j=0}^3 E_{r^j(L^j)^F}^M(t = t_0)
\]

We get,
\[
\frac{|F_{v^\omega}(v, w, \omega)|^2}{\sum_{j=0}^6 E_{r^j(L^j)^F}^F(t = t_0) + \sum_{j=0}^5 E_{r^j(L^j)^F}^F(t = t_0) + \sum_{j=0}^5 E_{r^j(L^j)^F}^M(t = t_0)}
\]

Finally we get,
\[
|F_{v^\omega}(v, w, \omega)| \lesssim \frac{E_1}{v_+}
\]

where \( E_1 \) is defined as follows,
\[
E_1 = \left[ \sum_{j=0}^6 E_{r^j(L^j)^F}^F(t = t_0) + \sum_{j=0}^5 E_{r^j(L^j)^F}^F(t = t_0) + \sum_{j=0}^5 E_{r^j(L^j)^F}^M(t = t_0) \right]^{\frac{1}{2}}
\]
Concerning the component $F_{e_1 e_2}$, similarly, we have the Sobolev inequality,

$$|F_{e_1 e_2}(v, w, \omega)|^2 \lesssim \int_{\mathbb{T} = v}^{\mathbb{T} = v} \int_{S^2} \left( |F_{e_1 e_2}|^2 + |\mathcal{L}_v F_{e_1 e_2}|^2 + |\mathcal{Q} F_{e_1 e_2}|^2 + |\mathcal{Q} L_{e_1 e_2}|^2 + |(\mathcal{Q})^2 F_{e_1 e_2}|^2 + |(\mathcal{Q})^2 L_{e_1 e_2}|^2 \right) d\sigma^2 d\varpi$$

$$\mathcal{L}_v \Psi_{e_1 e_2} = \nabla v \Psi_{e_1 e_2} - \Psi(\nabla \hat{\nu} e_1, e_2) - \Psi(e_1, \nabla \hat{\nu} e_2)$$

$$= -\nabla e_1 \Psi_{e_2 \hat{\nu}} - \nabla e_2 \Psi_{\hat{\nu} e_1}$$

$$= \nabla e_1 \Psi_{e_2 \hat{\nu}} - \nabla e_2 \Psi_{\hat{\nu} e_1}$$

$$= \mathcal{L}_{e_2} \Psi_{e_2 \hat{\nu}} - \Psi(\nabla e_2 \hat{\nu}, e_2) - \Psi(\hat{\nu}, \nabla e_1 e_2) - \mathcal{L}_{e_2} \Psi_{\hat{\nu} e_1} + \Psi(\nabla e_2 \hat{\nu}, e_1) + \Psi(\hat{\nu}, \nabla e_2 e_1)$$

Therefore,

$$\int_{\mathbb{T} = v}^{\mathbb{T} = v} \int_{S^2} |\mathcal{L}_v \Psi_{e_1 e_2} (v, w, \omega)|^2 d\sigma^2 d\varpi$$

$$\lesssim \int_{\mathbb{T} = v}^{\mathbb{T} = v} \int_{S^2} \left( |\mathcal{L}_{e_1} \Psi_{e_2 \hat{\nu}}|^2 + |\mathcal{L}_{e_2} \Psi_{\hat{\nu} e_1}|^2 + |\Psi_{e_1 e_2}|^2 + |\Psi_{\hat{\nu} e_1}|^2 \right) d\sigma^2 d\varpi$$

$$\lesssim \int_{\mathbb{T} = v}^{\mathbb{T} = v} \int_{S^2} \left( |\mathcal{Q} \Psi_{e_2 \hat{\nu}}|^2 + |\mathcal{Q} \Psi_{\hat{\nu} e_2}|^2 + |\Psi_{e_1 e_2}|^2 \right) d\sigma^2 d\varpi$$

$$\lesssim -F_{\Psi, r \mathcal{Q} \Psi}(w)(v - 1 \leq \mathbb{T} \leq v)$$

(Using what we already proved, and the fact that $r$ is bounded in the region $0 < 2m \leq r \leq R$).

Consequently, we have,

$$|F_{e_1 e_2}(v, w, \omega)|^2 \lesssim -F_{\mathcal{Q}, r \mathcal{Q} \mathcal{Q}^{2} \mathcal{Q}^{3} F}(w)(v - 1 \leq \mathbb{T} \leq v)$$

We proved that,

$$-F_{\mathcal{Q}, r \mathcal{Q} \mathcal{Q}^{2} \mathcal{Q}^{3} F}(w)(v - 1 \leq \mathbb{T} \leq v) \lesssim \frac{E_{F, r \mathcal{Q} \mathcal{Q}^{2} \mathcal{Q}^{3} F}(t = t_0) + E_{M}^{F, r \mathcal{Q} \mathcal{Q}^{2} \mathcal{Q}^{3} F}}{v^2}$$

Thus,

$$|F_{e_1 e_2}(v, w_0, \omega)| \lesssim \frac{E_1}{v^2}$$
6.2. Decay for $F_{\psi_1}$ and $F_{\psi_2}$.

We have the Sobolev inequality,

$$|F_{\psi}(v, w, \omega)|^2 \lesssim \int_{r=\rho} \int_\mathbb{S}^2 (|F_{\psi}|^2 + |\mathcal{L}_v F_{\psi}|^2 + |\mathcal{Q} F_{\psi}|^2) + \int |\mathcal{Q} \mathcal{L}_v F_{\psi}|^2 + |(\mathcal{Q})^2 F_{\psi}|^2 + \mathcal{L}_v F_{\psi}^2 \, d\sigma^2 d\Omega$$

On one hand, we can compute,

$$\mathcal{L}_v \Psi_{\psi} = \nabla_v \Psi_{\psi} + \Psi(\nabla_{\psi}, \hat{\theta}) + \Psi(\hat{\psi}, \nabla_{\hat{\theta}})$$

$$= \nabla_v \Psi_{\psi} + \frac{\mu}{2r} \Psi_{\psi}$$

\[
\begin{align*}
\nabla_\psi \Psi_{\psi} &= \frac{1}{2} \nabla_\psi (\Psi_{t\psi} + \Psi_{r\psi}) \frac{1}{2} \nabla_\psi (\frac{\partial}{r} + \frac{\partial}{\psi}) (\Psi_{t\psi} + \Psi_{r\psi}) \\
&= \frac{1}{4} \nabla_t (\Psi_{t\psi} + \Psi_{r\psi}) + \frac{1}{4} \nabla_r (\Psi_{t\psi} + \Psi_{r\psi}) \\
&= \frac{1}{4} \nabla_t (\Psi_{t\psi} + \Psi_{r\psi}) + \frac{1}{4} \nabla_r (\Psi_{t\psi} + \Psi_{r\psi}) \\
&= \frac{1}{4} \nabla_t (\Psi_{t\psi} + \Psi_{r\psi}) - \frac{1}{4} (\nabla_\theta \Psi_{r\psi} + (1 - \mu) \nabla_\phi \Psi_{\psi}) \\
&= \nabla_t \Psi_{\psi} - \frac{1}{4} (1 - \mu) \nabla_\theta \Psi_{r\psi} + \nabla_\phi \Psi_{\psi}
\end{align*}
\]

(Where we used the field equations and the Bianchi identities).

Thus,

\[
\begin{align*}
\nabla_\psi \Psi_{\psi} &= \mathcal{L}_t \Psi_{\psi} - \Psi(\nabla_i \hat{\psi}, \hat{\theta}) - \Psi(\hat{\psi}, \nabla_i \hat{\theta}) \\
&\quad + (1 - \mu) \left[ - \mathcal{L}_\theta \Psi_{r\psi} + \Psi(\nabla_\psi \hat{\theta}, \hat{\theta}) + \Psi(\hat{\theta}, \nabla_\psi \hat{\theta}) \right] \\
&= \mathcal{L}_t \Psi_{\psi} - \frac{\mu}{2r} \Psi_{\psi} \\
&\quad + \frac{1}{4} (1 - \mu) \left[ - \mathcal{L}_\theta \Psi_{r\psi} + \Psi(\nabla_\psi \hat{\theta}, \hat{\theta}) + \Psi(\hat{\theta}, \nabla_\psi \hat{\theta}) \right] \\
&= \mathcal{L}_t \Psi_{\psi} - \frac{1}{4} (1 - \mu) \mathcal{L}_\theta \Psi_{r\psi} - \frac{1}{4} (1 - \mu) \mathcal{L}_\phi \Psi_{\psi} - \frac{\sqrt{1 - \mu}}{r} \Psi_{\psi} + \frac{\sqrt{1 - \mu}}{r} \Psi_{\psi} \\
&= \mathcal{L}_t \Psi_{\psi} - \frac{1}{4} (1 - \mu) \mathcal{L}_\theta \Psi_{r\psi} - \frac{1}{4} (1 - \mu) \mathcal{L}_\phi \Psi_{\psi} \\
&\quad - \frac{1}{r} \Psi_{\psi} + \frac{(1 - \mu)}{r} \Psi_{\hat{\theta}} \\
&= \mathcal{L}_t \Psi_{\psi} - \frac{(1 - \mu)}{2r} \mathcal{L}_\theta \Psi_{\hat{\psi}} - \frac{(1 - \mu)}{4} \mathcal{L}_\phi \Psi_{\hat{\psi}} - \frac{1}{r} \Psi_{\hat{\psi}} + \frac{(1 - \mu)}{r} \Psi_{\hat{\theta}}
\end{align*}
\]
Hence,
\[ L_{\theta} \Psi_{\phi} = \nabla_{\Phi} \Psi_{\phi} + \frac{\mu}{2r} \Psi_{\phi} - \frac{(1-\mu)}{2} L_{\phi} \Psi_{\phi} - \frac{(1-\mu)}{4} L_{\Phi} \Psi_{\phi} - \frac{(1-\mu)}{2r} \Psi_{\phi} \]
and similarly, we obtain,
\[ L_{\theta} \Psi_{\phi} = \nabla_{\Phi} \Psi_{\phi} - \frac{(1-\mu)}{2} L_{\phi} \Psi_{\phi} - \frac{(1-\mu)}{4} L_{\Phi} \Psi_{\phi} - \frac{(1-\mu)}{2r} \Psi_{\phi} \]
Thus, for \( r \geq 2m > 0 \),
\[ |L_{v} \Psi_{v\phi}|^2 \leq |L_{l} \Psi_{v\phi}|^2 + |L_{\phi} \Psi_{v\phi}|^2 + |L_{\Phi} \Psi_{v\phi}|^2 + |\Psi_{v\phi}|^2 \leq |L_{l} \Psi_{v\phi}|^2 + |r L_{\phi} \Psi_{v\phi}|^2 + |r L_{\Phi} \Psi_{v\phi}|^2 + |\Psi_{v\phi}|^2 \]
(by using \( a.b \leq a^2 + b^2 \))
We get,
\[ |L_{v} \Psi_{v\phi}|^2 \leq |L_{l} \Psi_{v\phi}|^2 + |L_{\phi} \Psi_{v\phi}|^2 + |L_{\Phi} \Psi_{v\phi}|^2 + |\Psi_{v\phi}|^2 \]
\[ |L_{v} \Psi_{v\phi}|^2 \leq |L_{l} \Psi_{v\phi}|^2 + |r L_{\phi} \Psi_{v\phi}|^2 + |r L_{\Phi} \Psi_{v\phi}|^2 + |\Psi_{v\phi}|^2 \]
(in the region \( 0 < 2m \leq r \leq R \)).
Therefore,
\[ |F_{v\phi}(v, w, \omega)|^2 \]
\[ \leq \int_{\tau=v-1}^{\tau=v} \int_{S^2} (|F_{v\phi}|^2 + |\mathcal{L}_{\Phi} F_{v\phi}|^2 + |\mathcal{L}_{\phi} F_{v\phi}|^2 + |\mathcal{L}_{\theta} F_{v\phi}|^2 + |\mathcal{L}_{\Phi} F_{v\phi}|^2 + |\mathcal{L}_{\phi} F_{v\phi}|^2 + |\mathcal{L}_{\theta} F_{v\phi}|^2) d\sigma^2 d\varphi \]
\[ \leq \int_{\tau=v-1}^{\tau=v} \int_{S^2} (|F_{v\phi}|^2 + |\mathcal{L}_{\Phi} F_{v\phi}|^2 + |r \mathcal{L}_{\phi} F_{v\phi}|^2 + |r \mathcal{L}_{\theta} F_{v\phi}|^2 + |r \mathcal{L}_{\Phi} F_{v\phi}|^2 + |r \mathcal{L}_{\phi} F_{v\phi}|^2 + |r \mathcal{L}_{\theta} F_{v\phi}|^2) d\sigma^2 d\varphi \]
\[ \leq -F_{v}^{(H)}(v, w, \omega) + |F_{v}^{(H)}(v, w, \omega)|^2 + |r \mathcal{L}_{\Phi} F_{v\phi}|^2 + |r \mathcal{L}_{\phi} F_{v\phi}|^2 + |r \mathcal{L}_{\theta} F_{v\phi}|^2 + |r \mathcal{L}_{\Phi} F_{v\phi}|^2 + |r \mathcal{L}_{\phi} F_{v\phi}|^2 + |r \mathcal{L}_{\theta} F_{v\phi}|^2 \]
\[ \leq \sum_{j=0}^{2} \sum_{i=0}^{1} -F_{r}^{(H)}(v, w, \omega) + |F_{r}^{(H)}(v, w, \omega)|^2 + |r \mathcal{L}_{\Phi} F_{v\phi}|^2 + |r \mathcal{L}_{\phi} F_{v\phi}|^2 + |r \mathcal{L}_{\theta} F_{v\phi}|^2 + |r \mathcal{L}_{\Phi} F_{v\phi}|^2 + |r \mathcal{L}_{\phi} F_{v\phi}|^2 + |r \mathcal{L}_{\theta} F_{v\phi}|^2 \]
From estimate (120),

\[-F_{\Psi}^{(H)}(w)(v - 1 \leq \psi \leq v) \lesssim \frac{\left| E_{\Psi}(\psi) \right| + E_{\Psi}(\psi)(t = t_0) + E_{\Psi}^{M}}{v^2_+} \]

Recall that,

\[ E_{\Psi}^{M} = 3 \sum_{j=0}^{3} E_{r^j(L)^{\times j^n}}^j \Psi(t = t_0) + 2 \sum_{j=0}^{2} E_{r^j(L)^{\times j^n}}^j \Psi(t = t_0) \]

Therefore,

\[ |F_{\tilde{\psi}}(v, w, \omega)| \lesssim \frac{E_2}{v_+} \]

where,

\[ E_2 = \left( \sum_{j=0}^{5} \sum_{i=0}^{1} E_{r^j(L)^{\times j^n}}^i \Psi(t = t_0) + E_{r^0(L)^{\times 0^n}}^0 \Psi(t = t_0) \right) \]

\[ + \sum_{j=0}^{4} \sum_{i=0}^{1} E_{r^j(L)^{\times j^n}}^i \Psi(t = t_0) + E_{r^0(L)^{\times 0^n}}^0 \Psi(t = t_0) \]

\[ + \sum_{j=1}^{2} \sum_{i=0}^{1} E_{r^j(L)^{\times j^n}}^i \Psi(t = t_0) + E_{r^0(L)^{\times 0^n}}^0 \Psi(t = t_0) \]

\[ = \left[ E_2^p + \sum_{j=1}^{2} \sum_{i=0}^{1} E_{r^j(L)^{\times j^n}}^i \Psi(t = t_0) + E_{r^0(L)^{\times 0^n}}^0 \Psi(t = t_0) \right]^{\frac{1}{2}} \]

and similarly,

\[ |F_{\tilde{\phi}}(v, w, \omega)| \lesssim \frac{E_2}{v_+} \]

6.3. **Decay for** $\sqrt{1 - \frac{2m}{r}F_{\tilde{\omega}e_1}}$ **and** $\sqrt{1 - \frac{2m}{r}F_{\tilde{\omega}e_2}}$.

We have the Sobolev inequality,
\[
|\sqrt{(1-\mu)F_{\tilde{w}\tilde{\theta}}(v, w, \omega)}|^2
\]
\[
\lesssim \int_{r=w_0}^{r=\infty} \int_{S^2} \left( |(1-\mu)F_{\tilde{w}\tilde{\theta}}|^2 + |L_w(\sqrt{(1-\mu)F_{\tilde{w}\tilde{\theta}}})|^2 + |\Psi(\sqrt{(1-\mu)F_{\tilde{w}\tilde{\theta}}})|^2 + |(\partial_r^2 - \partial_{\theta}^2)(r^2 - \mu^2)\partial_{\theta}^2 \partial_{\theta}^2 |^2 \right)\, d\sigma^2 \, d\Omega.
\]

We have,
\[
L_w \Psi_{\tilde{w}\tilde{\theta}} = \nabla_w \Psi_{\tilde{w}\tilde{\theta}} + \Psi(\nabla_w \tilde{w}, \tilde{\theta}) + \Psi(\tilde{w}, \nabla_{\tilde{w}} \tilde{\theta})
\]
\[
= \nabla_w \Psi_{\tilde{w}\tilde{\theta}} + \frac{1}{(1-\mu)^2} \nabla_w \Psi_{\tilde{w}\tilde{\theta}}
\]

Computing,
\[
\nabla_w \Psi_{\tilde{w}\tilde{\theta}} = \frac{1}{2} \nabla_w (\Psi_{\tilde{w}\tilde{\theta}} - \Psi_{r\tilde{\theta}}) + \frac{1}{2} \nabla_t (\Psi_{\tilde{w}\tilde{\theta}} - \Psi_{r\tilde{\theta}})
\]
\[
= \frac{1}{4} \nabla_t (\Psi_{\tilde{w}\tilde{\theta}} - \Psi_{r\tilde{\theta}}) - \frac{1}{4} \nabla_r (\Psi_{\tilde{w}\tilde{\theta}} - \Psi_{r\tilde{\theta}})
\]
\[
= \frac{1}{4} \nabla_t (\Psi_{\tilde{w}\tilde{\theta}} - \Psi_{r\tilde{\theta}}) + \frac{1}{4} (-\nabla_t \Psi_{\tilde{w}\tilde{\theta}} - \nabla_r \Psi_{r\tilde{\theta}} + \nabla_t \Psi_{r\tilde{\theta}} - \nabla_r \Psi_{\tilde{w}\tilde{\theta}} + (1-\mu) \nabla_{\tilde{\theta}} \Psi_{\tilde{w}\tilde{\theta}})
\]
\[
= \frac{1}{2} \nabla_w (\Psi_{\tilde{w}\tilde{\theta}} - \Psi_{r\tilde{\theta}}) + \frac{1}{4} \nabla_t (\Psi_{\tilde{w}\tilde{\theta}} - \Psi_{r\tilde{\theta}})
\]

(Using the field equations and the Bianchi identities).

Thus,
\[
\nabla_w \Psi_{\tilde{w}\tilde{\theta}} = \frac{1}{2} \left( \nabla_t \Psi_{\tilde{w}\tilde{\theta}} - \nabla_t \Psi_{r\tilde{\theta}} \right) + \frac{1-\mu}{4} \left( \nabla_{\tilde{\theta}} \Psi + \nabla_{\tilde{\phi}} \Psi \right)
\]
\[
= \frac{1}{2} \left( \mathcal{L}_t \Psi_{\tilde{w}\tilde{\theta}} - \frac{\mu}{2r} \Psi_{r\tilde{\theta}} - \frac{\mu}{2r} \Psi_{\tilde{w}\tilde{\theta}} \right)
\]
\[
+ \frac{1-\mu}{4} \left( \mathcal{L}_{\tilde{\theta}} \Psi_{r\tilde{\theta}} - \frac{\sqrt{1-\mu}}{r} \Psi_{\tilde{w}\tilde{\theta}} - \mathcal{L}_{\tilde{\phi}} \Psi_{\tilde{w}\tilde{\theta}} - \frac{\sqrt{1-\mu}}{r} \Psi_{r\tilde{\theta}} \right)
\]
\[
= \mathcal{L}_t \Psi_{\tilde{w}\tilde{\theta}} + \frac{(1-\mu)}{4} \left( \mathcal{L}_{\tilde{\theta}} \Psi_{r\tilde{\theta}} - \mathcal{L}_{\tilde{\phi}} \Psi_{\tilde{w}\tilde{\theta}} \right) + \frac{\mu}{2r} \Psi_{\tilde{w}\tilde{\theta}} + \frac{1-\mu}{4} \Psi_{\tilde{w}\tilde{\theta}}
\]
\[
= \mathcal{L}_t \Psi_{\tilde{w}\tilde{\theta}} + \frac{(1-\mu)}{2} \mathcal{L}_{\tilde{\theta}} \Psi_{\tilde{w}\tilde{\theta}} - \frac{(1-\mu)}{4} \mathcal{L}_{\tilde{\phi}} \Psi_{\tilde{w}\tilde{\theta}} + \Psi_{\tilde{w}\tilde{\theta}}
\]

and similarly, we obtain,
\[ \nabla_w \Psi_{\hat{w}\hat{\phi}} = \mathcal{L}_t \Psi_{\hat{w}\hat{\phi}} + (1-\mu) \frac{1}{2} \mathcal{L}_\hat{\phi} \Psi_{\hat{w}\hat{\phi}} - \frac{(1-\mu)}{4} \mathcal{L}_\hat{\theta} \Psi_{\hat{w}\hat{\phi}} + \Psi_{\hat{w}\hat{\phi}} \]

Therefore,

\[ \mathcal{L}_w \Psi_{\hat{w}\hat{\phi}} = (1-\mu) \mathcal{L}_\hat{w} \Psi_{\hat{w}\hat{\phi}} = \frac{1}{(1-\mu)} \nabla_w \Psi_{\hat{w}\hat{\phi}} \]

\[ = \mathcal{L}_t \Psi_{\hat{w}\hat{\phi}} + \frac{1}{2} \mathcal{L}_\hat{\theta} \Psi_{\hat{w}\hat{\theta}} - \frac{1}{4} \mathcal{L}_\hat{\phi} \Psi_{\hat{w}\hat{\phi}} + \Psi_{\hat{w}\hat{\phi}} \]

Hence,

\[ |\mathcal{L}_w \Psi_{\hat{w}\hat{\phi}}|^2 \lesssim |\mathcal{L}_t \Psi_{\hat{w}\hat{\phi}}|^2 + |\mathcal{L}_\hat{\theta} \Psi_{\hat{w}\hat{\theta}}|^2 + |\mathcal{L}_\hat{\phi} \Psi_{\hat{w}\hat{\phi}}|^2 + |\Psi_{\hat{w}\hat{\phi}}|^2 \]

(by using \(a.b \lesssim a^2 + b^2\))

We get,

\[ |\mathcal{L}_w \Psi_{\hat{w}\hat{\phi}}|^2 \lesssim |\mathcal{L}_t \Psi_{\hat{w}\hat{\phi}}|^2 + |\mathcal{L}_\hat{\theta} \Psi_{\hat{w}\hat{\theta}}|^2 + |\mathcal{L}_\hat{\phi} \Psi_{\hat{w}\hat{\phi}}|^2 + |\Psi_{\hat{w}\hat{\phi}}|^2 \]

\[ \lesssim |\mathcal{L}_t \Psi_{\hat{w}\hat{\phi}}|^2 + |r \mathcal{L}_\hat{\theta} \Psi_{\hat{w}\hat{\theta}}|^2 + |r \mathcal{L}_\hat{\phi} \Psi_{\hat{w}\hat{\phi}}|^2 + |\Psi_{\hat{w}\hat{\phi}}|^2 \]

(in the region \(0 < 2m \leq r \leq R\)).

We also have,

\[ \frac{\partial}{\partial w} \sqrt{(1-\mu)} = \left( \frac{1}{2} \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial}{\partial r^2} \right) \sqrt{(1-\mu)} = -\frac{(1-\mu)}{2} \frac{\partial}{\partial r} \sqrt{(1-\mu)} = -\frac{(1-\mu)}{2} \frac{\mu}{2r \sqrt{(1-\mu)}} \]

\[ = -\frac{\mu}{4r} \sqrt{(1-\mu)} \]

Thus,
\[
|\sqrt{(1 - \mu)F_{\psi\theta}(v, w, \omega)}|^2 \\
\lesssim \int_{w_0}^{w=\infty} \int_{S^2} (|F_{\psi\theta}|^2 + |\mathcal{L}_w F_{\psi\theta}|^2 + |\mathcal{L}_w^2 F_{\psi\theta}|^2 + |\mathcal{L}_w^3 F_{\psi\theta}|^2) \, d\sigma \, d\omega \\
\lesssim \int_{w_0}^{w=\infty} \int_{S^2} (1 - \mu)(|F_{\psi\theta}|^2 + |\mathcal{L}_w F_{\psi\theta}|^2 + |r \mathcal{L}_w F_{\psi\theta}|^2 + |r^2 \mathcal{L}_w^2 F_{\psi\theta}|^2 + |r^3 \mathcal{L}_w^3 F_{\psi\theta}|^2) \, d\sigma \, d\omega \\
\lesssim \sum_{j=0}^{2} \sum_{i=0}^{1} -F_{r^j \mathcal{L}_w^i F_{\psi\theta}}(v)(w_0 \leq w \leq \infty) - F_{r^3 \mathcal{L}_w^3 F_{\psi\theta}}(v)(w_0 \leq w \leq \infty)
\]

From estimate 6,

\[-F_{\psi}^{(H)}(v)(w_0 \leq w \leq \infty) \lesssim \frac{|E_{\psi}(\bar{\delta})| + E_{\psi}^{(H)}(t = t_0) + E_{\psi}^{M}}{v_+^2} \]

As

\[E_{\psi}^{M} = \sum_{j=0}^{3} E_{r^j \mathcal{L}_w^i \psi}^{(K)}(t = t_0) + \sum_{j=0}^{2} E_{r^j \mathcal{L}_w^i \psi}^{(K)}(t = t_0) \]

we have,

\[|\sqrt{(1 - \mu)F_{\psi\theta}(v, w, \omega)}| \lesssim \frac{E_{2}}{v_+} \]

and similarly,

\[|\sqrt{(1 - \mu)F_{\psi\theta}(v, w, \omega)}| \lesssim \frac{E_{2}}{v_+} \]

\[\blacksquare\]
General Relativity postulates that space-time is a 4-dimensional Lorentzian manifold \((M, g)\) that satisfies the Einstein equations,

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}
\]

where \(T_{\mu\nu}\) is a symmetric 2-tensor on \(M\) that is the stress-energy-momentum tensor of matter. In vacuum \(T_{\mu\nu} = 0\), thus the Einstein vacuum equations are \(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0\). In vacuum, this yields to \(R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R\) and since \(R = R^{ii}\) by definition, we get \(R = g^{ij}R_{ij} = \frac{1}{2}g^{ii}g_{ij}R = \frac{1}{2}R = 2R\). This means that in vacuum \(R = 0\) and therefore the Einstein vacuum equations can be written as the following:

\[
R_{\mu\nu} = 0 \tag{132}
\]

The simplest solution of the Einstein vacuum equations is the 4-dimensional Minkowski metric, it represents a flat space-time. The first black hole solution of the Einstein vacuum equations was discovered by Karl Schwarzschild about a month after the publication of Einstein’s theory of General Relativity. However, it took nearly 50 years from then for it to be fully understood as a black hole space-time. When it was first discovered, it was to represent the gravitational field outside a spherical, uncharged, non-rotating star with mass \(m\) and was written under the form

\[
ds^2 = -(1 - \frac{2m}{r})dt^2 + \frac{1}{(1 - \frac{2m}{r})}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{133}
\]

It turned out that this solution could be extended, as solutions to the Einstein vacuum equations (132), to describe the gravitational field inside the star, created by the mass of the star, and thought of as being there in vacuum, i.e. without the matter inside the star. The extended Schwarzschild solution is what became to be known as a black hole space-time. It is also good to note that according to Birkhoff’s theorem, any spherically symmetric solution of the Einstein vacuum equations is locally isometric to the Schwarzschild solution. In this sense the Schwarzschild solution is unique although it can have a different form in a different system of coordinates.

7.1. The extended Schwarzschild solution.

To obtain the extended Schwarzschild solution, as explained in [HE], first let,

\[
r^* = \int \frac{dr}{(1 - \frac{2m}{r})} = r + 2m \log(r - 2m) \tag{134}
\]
We have,
\[ dr^* = dr + \frac{2m}{r-2m} \frac{dr}{dr} = (1 + \frac{1}{2m-1})dr = (\frac{r}{2m-1})dr \]
\[ = (\frac{1}{1-\frac{2m}{r}})dr \]

Thus, the Schwarzschild space-time in the exterior (133) can be also written as:
\[ ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})dr^*2 + r^2 d\sigma^2 \quad (135) \]
where \( d\sigma^2 \) is the usual metric on the unit sphere.

Let,
\[ v = t + r^* \quad (136) \]
\[ w = t - r^* \quad (137) \]

We have
\[ t = \frac{v + w}{2} \]
\[ dt = \frac{dv + dw}{2} \]
\[ (dt)^2 = \frac{dv^2}{4} + \frac{dw^2}{4} + \frac{dvdw}{2} = \frac{dv^2}{4} + \frac{dw^2}{4} + \frac{dv \otimes dw}{4} + \frac{dw \otimes dv}{4} \]

Injecting in (135), we get
\[ ds^2 = -(1 - \frac{2m}{r})(\frac{dv^2}{4} + \frac{dw^2}{4} + \frac{dvdw}{2}) + (1 - \frac{2m}{r})(dr^*)^2 + r^2 d\sigma^2 \]

We also have
\[ r^* = \frac{v - w}{2} \]
\[ (dr^*)^2 = \frac{(dv - dw)^2}{2} = \frac{dv^2}{4} + \frac{dw^2}{4} - \frac{dvdw}{2} \]

Therefore,
\[ ds^2 = -(1 - \frac{2m}{r})(\frac{dv^2}{4} + \frac{dw^2}{4} + \frac{dvdw}{2}) + (1 - \frac{2m}{r})^{-1}(1 - \frac{2m}{r})^2[\frac{dv^2}{4} + \frac{dw^2}{4} - \frac{dvdw}{2}] + r^2 d\sigma^2 \]

Thus,
\[ ds^2 = -(1 - \frac{2m}{r})dvdw + r^2 d\sigma^2 \]
\[ = -(1 - \frac{2m}{r})\frac{1}{2} dv \otimes dw - (1 - \frac{2m}{r})\frac{1}{2} dw \otimes dv + r^2 d\sigma^2 \quad (138) \]
7.1.1. Kruskal coordinates.

Let \( v' = v'(v) \), \( w' = w'(w) \) where \( v' \), \( w' \) are arbitrary \( C^1 \) functions.

\[
\begin{align*}
\frac{dv}{dv'} &= \frac{dv}{dv'} \\
\frac{d\omega}{d\omega'} &= \frac{d\omega}{d\omega'}
\end{align*}
\]

\[
ds^2 = -(1 - \frac{2m}{r}) \frac{dv}{dv'} \frac{d\omega}{d\omega'} v' d\omega' + r^2 d\sigma^2
\] (139)

Define,

\[
\begin{align*}
\dot{x'} &= \frac{v' - w'}{2} \\
\dot{t'} &= \frac{v' + w'}{2}
\end{align*}
\]

We get

\[
\begin{align*}
v' &= \dot{t'} + x' \\
w' &= \dot{t'} - x'
\end{align*}
\]

and

\[
\begin{align*}
dv' &= d\dot{t'} + dx' \\
d\omega' &= d\dot{t'} - dx'
\end{align*}
\]

Thus,

\[
\begin{align*}
dv' d\omega' &= (d\dot{t'} + dx')(d\dot{t'} - dx') = (d\dot{t'})^2 - d\dot{t'} dx' + dx' d\dot{t'} - (dx')^2 = (d\dot{t'})^2 - (dx')^2 \\
&= -(-d\dot{t'}^2 + (dx')^2)
\end{align*}
\]

Let,

\[
F^2 = -\left(1 - \frac{2m}{r}\right) \frac{dv}{dv'} \frac{dw}{dw'} = (1 - \frac{2m}{r}) \frac{dv}{dv'} \frac{dw}{dw'}
\]

From (139), we have

\[
(ds')^2 = F^2(t', x')(-(d\dot{t'})^2 + (dx')^2) + r^2 d\sigma^2
\]

Kruskal’s choice is:

\[
\begin{align*}
v' &= \exp\left(\frac{v}{4m}\right) \\
w' &= -\exp\left(-\frac{w}{4m}\right)
\end{align*}
\]

\[
\begin{align*}
\dot{t'} &= \frac{1}{2}(v' + w') \\
(\dot{t'})^2 &= \frac{1}{4}\left((v')^2 + (w')^2 + 2v' w'\right) = \frac{1}{4}\left(\exp\left(\frac{v}{2m}\right) + \exp\left(-\frac{w}{2m}\right) + 2\exp\left(\frac{v - w}{4m}\right)\right) \\
x' &= \frac{1}{2}(v' - w') \\
(x')^2 &= \frac{1}{4}\left(\exp\left(\frac{v}{2m}\right) + \exp\left(-\frac{w}{2m}\right) + 2\exp\left(\frac{v - w}{4m}\right)\right)
\end{align*}
\]
Computing,
\[(t')^2 - (x')^2 = -\frac{2}{4}(\exp(\frac{v-w}{2m}) + \exp(\frac{v-w}{2m})) = -\frac{2}{2}\exp(\frac{v-w}{4m}) = -\exp(\frac{v-w}{4m})\]

Thus,
\[(t')^2 - (x')^2 = -\exp(\frac{r}{2m})(r-2m) \quad (144)\]

Computing,
\[
\begin{align*}
\frac{dv}{dv'} &= 1 \cdot \exp\left(\frac{v}{4m}\right), \quad \frac{dw}{dw'} = 1 \cdot \exp\left(-\frac{w}{4m}\right) \\
\frac{dv}{dv'} &= 4m \cdot \exp\left(-\frac{v}{4m}\right), \quad \frac{dw}{dw'} = 4m \cdot \exp\left(\frac{w}{4m}\right) \\
\frac{dv}{dv'} \cdot \frac{dw}{dw'} &= 16m^2 \exp\left(\frac{w-v}{4m}\right)
\end{align*}
\]

Therefore,
\[
F^2 = (1 - \frac{2m}{r})16m^2 \exp\left(-\frac{2r^*}{4m}\right)
\]

\[
= \frac{(r-2m)}{r}16m^2 \exp\left(-\frac{2}{4m}(r + 2m \log(r - 2m))\right)
\]

\[
= \frac{(r-2m)}{r}16m^2 \exp\left(-\frac{2r}{4m}\right) \exp(-\log(r - 2m))
\]

\[
= \frac{1}{r}16m^2 \exp\left(-\frac{r}{2m}\right)
\]

\[
= \frac{16m^2}{r} \exp\left(-\frac{r}{2m}\right)
\]

Finally, we obtain,
\[
ds^2 = \frac{16m^2}{r} \exp\left(-\frac{r}{2m}\right)(-(dt')^2 + (dx')^2) + r^2(t', x')d\sigma^2 \quad (145)
\]

7.2. The Penrose diagram.

The Penrose diagram, see [HE], is constructed by taking,
\[
v'' = \arctan\left(\frac{v'}{2m}\right) \quad (146)
\]
\[
w'' = \arctan\left(\frac{w'}{2m}\right) \quad (147)
\]
\[-\pi < v'' + w'' < \pi \quad (148)\]
\[-\pi < v'' < \frac{\pi}{2} \quad (149)\]
\[-\pi < w'' < \frac{\pi}{2} \quad (150)\]
7.3. The compatible symmetric connection.

Computing the Christoffel symbols for the Schwarzschild metric, we have

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left( \frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

Hence,

$$\Gamma^i_{\theta\theta} = \frac{1}{2} g^{im} \left( \frac{\partial g_{m\theta}}{\partial x^\theta} + \frac{\partial g_{m\theta}}{\partial x^\theta} - \frac{\partial g_{\theta\theta}}{\partial x^m} \right)$$

We get

$$\Gamma^\phi_{\theta\theta} = 0$$

Consequently,

$$\nabla_\theta \frac{\partial}{\partial \theta} = - (1 - \mu) r \frac{\partial}{\partial r}$$

We have,

$$\Gamma^i_{\phi\phi} = \frac{1}{2} g^{im} \left( \frac{\partial g_{m\phi}}{\partial x^\phi} - \frac{\partial g_{\phi\phi}}{\partial x^m} \right)$$

therefore

$$\Gamma^r_{\phi\phi} = \frac{1}{2} g^{rr} \left( \frac{\partial g_{m\phi}}{\partial x^\phi} - \frac{\partial g_{\phi\phi}}{\partial x^m} \right)$$

$$= \frac{1}{2} g^{rr} \left( - \frac{\partial g_{\phi\phi}}{\partial x^r} \right)$$

$$= \frac{1}{2} (1 - \mu) (-2 r \sin^2(\theta))$$

$$= - r (1 - \mu) \sin^2(\theta)$$

$$\Gamma^\theta_{\phi\phi} = \frac{1}{2} g^{\theta\theta} \left( - \frac{\partial g_{\phi\phi}}{\partial x^\theta} \right)$$

$$= \frac{1}{2} \left( r^2 \frac{\partial \sin^2(\theta)}{\partial \theta} \right)$$

$$= \frac{1}{2} (-2 \sin(\theta) \cos(\theta))$$

$$= - \sin(\theta) \cos(\theta)$$
We obtain

\[ \nabla_\phi \frac{\partial}{\partial \phi} = -(1 - \mu) r \sin^2(\theta) \frac{\partial}{\partial r} - \sin(\theta) \cos(\theta) \frac{\partial}{\partial \theta} \]
\[ = -r \sin^2(\theta) \frac{\partial}{\partial r} - \sin(\theta) \cos(\theta) \frac{\partial}{\partial \theta} \]  

(152)

We also have

\[ \Gamma^i_{\theta \phi} = \frac{1}{2} g^m_{\theta \phi} \left( \frac{\partial g_{m \theta}}{\partial x^\phi} + \frac{\partial g_{m \phi}}{\partial x^\theta} - \frac{\partial g_{\theta \phi}}{\partial x^m} \right) \]
\[ = \frac{1}{2} g^m_{\theta \phi} \left( \frac{\partial g_{m \phi}}{\partial x^\theta} \right) \]

hence

\[ \Gamma^\phi_{\theta \phi} = \frac{1}{2} g^\phi_{\theta \phi} \left( \frac{\partial g_{\phi \phi}}{\partial x^\theta} \right) \]
\[ = \frac{1}{2 r^2 \sin^2(\theta)} \left( \frac{\partial r^2 \sin^2(\theta)}{\partial \theta} \right) \]
\[ = \frac{1}{2 \sin^2(\theta)} \left( 2 \sin(\theta) \cos(\theta) \right) \]
\[ = \cos(\theta) \frac{\partial}{\partial \phi} \]

We get

\[ \nabla_\theta \frac{\partial}{\partial \phi} = \frac{\cos(\theta)}{\sin(\theta)} \frac{\partial}{\partial \phi} \]  

(153)

We also compute

\[ \Gamma^i_{\theta t} = \frac{1}{2} g^m_{\theta t} \left( \frac{\partial g_{m \theta}}{\partial x^t} + \frac{\partial g_{m t}}{\partial x^\theta} - \frac{\partial g_{\theta t}}{\partial x^m} \right) \]
\[ = 0 \]

from which we derive

\[ \nabla_\theta \frac{\partial}{\partial t} = 0 \]  

(154)

Similarly,

\[ \Gamma^i_{\phi t} = 0 \]

gives

\[ \nabla_\phi \frac{\partial}{\partial t} = 0 \]  

(155)

Computing

\[ \Gamma^i_{t t} = \frac{1}{2} g^m_{t t} \left( \frac{\partial g_{t m}}{\partial x^t} + \frac{\partial g_{m t}}{\partial x^t} - \frac{\partial g_{t t}}{\partial x^m} \right) \]
\[ = \frac{1}{2} g^m_{t t} \left( -\frac{\partial g_{t t}}{\partial x^m} \right) \]

\[ = \frac{1}{2} g^m_{t t} \left( -\frac{\partial g_{t t}}{\partial x^m} \right) \]
we derive

$$\nabla_t \frac{\partial}{\partial t} = \frac{\mu(1-\mu)}{2r} \frac{\partial}{\partial r}$$

(156)

Computing

$$\Gamma^i_{tr^*} = \frac{1}{2} g^{i m}(\frac{\partial g_{m t}}{\partial x^r} + \frac{\partial g_{m r^*}}{\partial x^r} - \frac{\partial g_{t r^*}}{\partial x^m})$$

$$\Gamma^r_{tr^*} = \frac{1}{2} g^{r^* r^*}(\frac{\partial g_{r^* t}}{\partial x^r} - \frac{\partial g_{r^* r^*}}{\partial x^r})$$

we obtain

$$\nabla_{r^*} \frac{\partial}{\partial r^*} = \frac{\mu}{2r} \frac{\partial}{\partial r^*}$$

(157)

Computing

$$\Gamma^i_{t r^*} = \frac{1}{2} g^{i m}(\frac{\partial g_{m t}}{\partial x^r} + \frac{\partial g_{m r^*}}{\partial x^r} - \frac{\partial g_{t r^*}}{\partial x^m})$$

$$\Gamma^t_{t r^*} = \frac{1}{2} g^{t t}(\frac{\partial g_{t t}}{\partial x^r})$$

we get

$$\nabla^i \frac{\partial}{\partial r^*} = \frac{\mu}{2r} \frac{\partial}{\partial t}$$
We have

\[
\Gamma_{\theta r^*}^i = \frac{1}{2} g^{im} \left( \frac{\partial g_{m\theta}}{\partial x^r} + \frac{\partial g_{mr^*}}{\partial x^\theta} - \frac{\partial g_{\theta r^*}}{\partial x^m} \right)
\]
\[
= \frac{1}{2} g^{im} \left( \frac{\partial g_{m\theta}}{\partial x^r} \right)
\]

\[
\Gamma_{\phi r^*}^\theta = \frac{1}{2} g^{\theta \phi} \left( \frac{\partial g_{\phi r^*}}{\partial x^\theta} \right)
\]
\[
= \frac{1}{2r^2} \left( \frac{\partial r^2}{\partial r^*} \right)
\]
\[
= \frac{(1 - \mu)}{2r^2} \left( \frac{\partial r^2}{\partial r} \right)
\]
\[
= \frac{(1 - \mu)}{2r^2} (2r) = \frac{(1 - \mu)}{r}
\]

therefore,

\[
\nabla_{\theta} \frac{\partial}{\partial r} = \frac{(1 - \mu)}{r} \frac{\partial}{\partial \theta}
\]  

(159)

Computing

\[
\Gamma_{\phi r^*}^i = \frac{1}{2} g^{im} \left( \frac{\partial g_{m\phi}}{\partial x^r} + \frac{\partial g_{mr^*}}{\partial x^\phi} - \frac{\partial g_{\phi r^*}}{\partial x^m} \right)
\]
\[
= \frac{1}{2} g^{im} \left( \frac{\partial g_{m\phi}}{\partial x^r} \right)
\]

\[
\Gamma_{\phi r^*}^\phi = \frac{1}{2} g^{\phi \phi} \left( \frac{\partial g_{\phi r^*}}{\partial x^\phi} \right)
\]
\[
= \frac{1}{2 \sin^2(\theta)} \left( \frac{\partial r^2 \sin^2(\theta)}{\partial r^*} \right)
\]
\[
= \frac{(1 - \mu)}{2r^2} \left( \frac{\partial r^2}{\partial r} \right)
\]
\[
= \frac{(1 - \mu)}{2r^2} (2r) = \frac{(1 - \mu)}{r}
\]

we obtain

\[
\nabla_{\phi} \frac{\partial}{\partial r} = \frac{(1 - \mu)}{r} \frac{\partial}{\partial \phi}
\]  

(160)
By letting,
\[
\frac{\dot{\partial}}{\partial t} = \frac{1}{\sqrt{(1 - \mu)}} \frac{\partial}{\partial t}
\]
\[
\frac{\dot{\partial}}{\partial r^*} = \frac{1}{\sqrt{(1 - \mu)}} \frac{\partial}{\partial r^*}
\]
(161)
\[
\frac{\dot{\partial}}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial \theta}
\]
\[
\frac{\dot{\partial}}{\partial \phi} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}
\]
(162)
we can compute,
\[
\nabla_\theta \frac{\dot{\partial}}{\partial \theta} = \frac{1}{r} \nabla_\theta \frac{\dot{\partial}}{\partial \theta} = \frac{1}{r^2 \nabla_\theta} \frac{\partial}{\partial \theta} = -\frac{1}{r} \frac{\partial}{\partial r^*}
\]
(163)
\[
\nabla_\phi \frac{\dot{\partial}}{\partial \phi} = \frac{1}{r^2 \sin^2(\theta)} \nabla_\phi \frac{\partial}{\partial \phi}
\]
\[
= -\frac{1}{r} \sqrt{(1 - \mu)} \frac{\dot{\partial}}{\partial r^*} - \frac{\cos(\theta)}{r \sin(\theta)} \frac{\dot{\partial}}{\partial \theta}
\]
(164)
\[
\nabla_\theta \frac{\dot{\partial}}{\partial \phi} = \frac{1}{r} \nabla_\theta \frac{\dot{\partial}}{\partial \phi} = \frac{1}{r^2} \nabla_\theta \left( \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} \right)
\]
\[
= -\frac{\cos(\theta)}{r^2 \sin^2(\theta)} \frac{\partial}{\partial \phi} + \frac{1}{r^2 \sin(\theta)} \nabla_\theta \frac{\partial}{\partial \phi}
\]
\[
= \frac{\cos(\theta)}{r^2 \sin^2(\theta)} \frac{\partial}{\partial \phi} + \frac{\cos(\theta)}{r^2 \sin(\theta)} \frac{\partial}{\partial \phi}
\]
\[
= 0
\]
(165)
\[
\nabla_\phi \frac{\dot{\partial}}{\partial \theta} = \frac{1}{r \sin(\theta)} \nabla_\phi \frac{\dot{\partial}}{\partial \theta} = \frac{1}{r^2 \sin(\theta)} \nabla_\phi \frac{\dot{\partial}}{\partial \theta}
\]
\[
= \frac{1}{r^2 \sin(\theta) \sin(\theta)} \frac{\cos(\theta)}{\dot{\partial}} \frac{\dot{\partial}}{\partial \phi}
\]
(166)
\[ \nabla \dot{\hat{\theta}} = \frac{1}{r \sqrt{(1 - \mu)}} \nabla \hat{\theta} \frac{\partial}{\partial t} \]
\[ = 0 \] (167)

\[ \nabla \dot{\hat{\phi}} = \frac{1}{r \sin(\theta) \sqrt{(1 - \mu)}} \nabla \hat{\phi} \frac{\partial}{\partial t} \]
\[ = 0 \] (168)

\[ \nabla \dot{\hat{\tau}} = \frac{1}{(1 - \mu)} \nabla \hat{\tau} \frac{\partial}{\partial t} \]
\[ = \frac{\mu}{2r \sqrt{(1 - \mu)}} \frac{\partial}{\partial r^*} \] (169)

\[ \nabla_c \dot{\frac{\partial}{\partial r^*}} = \frac{1}{\sqrt{(1 - \mu)}} \nabla_c \frac{\partial}{\partial r^*} = \frac{1}{\sqrt{(1 - \mu)}} \nabla_c \left( \frac{1}{\sqrt{(1 - \mu)}} \frac{\partial}{\partial r^*} \right) \]
\[ = \sqrt{(1 - \mu)} \nabla_c \left( \frac{1}{\sqrt{(1 - \mu)}} \right) \frac{\partial}{\partial r^*} + \frac{1}{(1 - \mu)} \nabla_c \frac{\partial}{\partial r^*} \]
\[ = \sqrt{(1 - \mu)} \frac{-\mu}{2r(1 - \mu)^{3/2}} \frac{\partial}{\partial r^*} + \frac{1}{(1 - \mu)} \nabla_c \frac{\partial}{\partial r^*} \]
\[ = \frac{-\mu}{2r(1 - \mu)} \frac{\partial}{\partial r^*} + \frac{1}{(1 - \mu)} \frac{\mu}{2r} \frac{\partial}{\partial r^*} \]
\[ = 0 \] (170)

\[ \nabla_i \dot{\frac{\partial}{\partial r^*}} = \frac{1}{(1 - \mu)} \nabla_i \frac{\partial}{\partial r^*} \]
\[ = \frac{\mu}{2r \sqrt{(1 - \mu)}} \frac{\partial}{\partial t} \] (171)
\[ \nabla_{r^*} \frac{\partial}{\partial t} = \frac{1}{\sqrt{1-\mu}} \nabla_{r^*} \left( \frac{1}{\sqrt{1-\mu}} \frac{\partial}{\partial t} \right) \]
\[ = \frac{1}{\sqrt{1-\mu}} \nabla_{r^*} \left( \frac{1}{\sqrt{1-\mu}} \frac{\partial}{\partial t} \right) + \frac{1}{(1-\mu)} \nabla_{t} \frac{\partial}{\partial r^*} \]
\[ = \sqrt{(1-\mu)} \frac{(-\mu)}{2r(1-\mu)^2} \frac{\partial}{\partial t} + \frac{1}{(1-\mu)} \frac{\mu}{2r} \frac{\partial}{\partial t} \]
\[ = \frac{-\mu}{2r(1-\mu)} \frac{\partial}{\partial t} + \frac{1}{(1-\mu)} \frac{\mu}{2r} \frac{\partial}{\partial t} \]
\[ = 0 \]  
(172)

\[ \nabla_{\theta} \frac{\partial}{\partial r^*} = \frac{1}{r \sqrt{(1-\mu)}} \nabla_{\theta} \frac{\partial}{\partial r^*} \]
\[ = \frac{\sqrt{1-\mu}}{r} \frac{\partial}{\partial \theta} \]  
(173)

\[ \nabla_{r^*} \frac{\partial}{\partial \theta} = \frac{1}{\sqrt{1-\mu}} \nabla_{r^*} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \right) \]
\[ = \frac{1}{\sqrt{1-\mu}} \nabla_{r^*} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \right) + \frac{1}{r \sqrt{(1-\mu)}} \nabla_{\theta} \frac{\partial}{\partial r^*} \]
\[ = \sqrt{(1-\mu)} \nabla_{r^*} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \right) + \frac{\sqrt{1-\mu}}{r} \frac{\partial}{\partial \theta} \]
\[ = -\frac{\sqrt{1-\mu}}{r^2} \frac{\partial}{\partial \theta} + \frac{\sqrt{1-\mu}}{r^2} \frac{\partial}{\partial \theta} \]
\[ = 0 \]  
(174)

\[ \nabla_{\phi} \frac{\partial}{\partial r^*} = \frac{1}{r \sin(\theta) \sqrt{(1-\mu)}} \nabla_{\phi} \frac{\partial}{\partial r^*} \]
\[ = \frac{\sqrt{(1-\mu)}}{r} \frac{\partial}{\partial \phi} \]  
(175)
\[ \nabla^*_r \frac{\partial}{\partial \phi} = \frac{1}{\sqrt{1 - \mu}} \nabla^*_r \left( \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} \right) = \frac{\sqrt{1 - \mu}}{r^{\frac{1}{2}}} \nabla^*_r \left( \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} \right) + \frac{1}{r \sin(\theta) \sqrt{(1 - \mu)}} \frac{\partial}{\partial r^*} \]

\[ = \frac{-\sqrt{1 - \mu}}{r^{\frac{1}{2}} \sin(\theta)} \frac{\partial}{\partial \phi} + \frac{1}{r \sin(\theta) \sqrt{(1 - \mu)}} \frac{\partial}{\partial r^*} \]

\[ = \frac{-\sqrt{1 - \mu}}{r^{\frac{1}{2}} \sin(\theta)} \frac{\partial}{\partial \phi} + \frac{\sqrt{1 - \mu}}{r \sin(\theta) \sqrt{(1 - \mu)}} \frac{\partial}{\partial r^*} \]

\[ = 0 \quad (176) \]

### 7.4. The deformation tensor.

We start by evaluating

\[ \nabla \frac{\partial}{\partial w} = \Gamma_{vw}^i e_i \]

We have,

\[ \Gamma_{ki}^i = \frac{1}{2} g^{im} (\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}) \]

Computing,

\[ \Gamma_{vw}^i = \frac{1}{2} g^{im} (\frac{\partial g_{mw}}{\partial w} + \frac{\partial g_{mw}}{\partial v} - \frac{\partial g_{vw}}{\partial x^m}) \]

we have,

\[ \Gamma_{vw}^w = \frac{1}{2} g^{vm} (\frac{\partial g_{vw}}{\partial w} + 0 - \frac{\partial g_{vw}}{\partial w}) \]

\[ = 0 \]

\[ \Gamma_{vw}^v = \frac{1}{2} g^{vw} (\frac{\partial g_{vw}}{\partial v} + \frac{\partial g_{vw}}{\partial v} - \frac{\partial g_{vw}}{\partial v}) \]

\[ = \frac{1}{2} g^{vw} (0 + \frac{\partial g_{vw}}{\partial v} - \frac{\partial g_{vw}}{\partial v}) \]

\[ = 0 \]

\[ \Gamma_{vw}^\theta = 0 \]

\[ \Gamma_{vw}^\phi = 0 \]

Thus,

\[ \nabla_v \frac{\partial}{\partial w} = 0 \]

also means,

\[ \nabla_w \frac{\partial}{\partial w} = 0 \]

Now, we want to compute

\[ \Gamma_{vw}^i = \frac{1}{2} g^{im} (\frac{\partial g_{mw}}{\partial w} + \frac{\partial g_{mw}}{\partial v} - \frac{\partial g_{vw}}{\partial x^m}) \]

\[ = g^{im} \frac{\partial g_{mw}}{\partial w} \]

\[ \Gamma_{vw}^i = g^{im} \frac{\partial g_{mw}}{\partial v} \]
We have,
\[ \Gamma^{w}_{ww} = g^{vw} \frac{\partial g_{vw}}{\partial w} = 0 \]
and
\[ \Gamma^{w}_{ww} = g^{vw} \frac{\partial g_{vw}}{\partial w} \]
We have,
\[ g_{vw} = -\frac{(1 - 2m)}{r^2} \]
and
\[ r^{\ast} = \frac{v - w}{2} \]
Computing,
\[ \frac{\partial r}{\partial w} = \frac{\partial r}{\partial r^{\ast}} \frac{\partial r^{\ast}}{\partial w} = \frac{-1}{2} (1 - \mu) \]
where we define \( \mu = \frac{2m}{r} \).
Computing,
\[ \frac{\partial r}{\partial v} = \frac{\partial r}{\partial r^{\ast}} \frac{\partial r^{\ast}}{\partial v} = \frac{1}{2} (1 - \mu) \]
\[ \frac{\partial g_{vw}}{\partial r} = \frac{-1}{2} \frac{\partial (1 - \frac{2m}{r})}{\partial r} = \frac{-m}{r^2} \]
Thus,
\[ \Gamma^{w}_{ww} = g^{vw} \frac{\partial g_{vw}}{\partial w} = -\frac{2}{1 - \mu} \frac{\partial g_{vw}}{\partial r} \frac{\partial r}{\partial w} \]
\[ = -\frac{1}{1 - \mu} \frac{2m}{r^2} \left( -\frac{1}{2} (1 - \mu) \right) = -\frac{m(1 - \mu)}{r^2 (1 - \mu)} \]
\[ = -\frac{m}{r^2} \]
We also have
\[ \Gamma^{\phi}_{ww} = \Gamma^{\phi}_{ww} = \Gamma^{v}_{ww} = 0 \]
hence,
\[ \nabla^{\ast}_{w} \frac{\partial}{\partial w} = -\frac{m}{r^2} \frac{\partial}{\partial w} \]
On the other hand,

\[
\Gamma^v_{vv} = g^{wv} \frac{\partial g_{wv}}{\partial v} = -\frac{2}{1-\mu} \frac{\partial g_{ov}}{\partial r} \frac{\partial r}{\partial v} = \frac{2m}{r^2(1-\mu)} \frac{\partial r}{\partial v} = \frac{m(1-\mu)}{r^2(1-\mu)} = \frac{m}{r^2}
\]

We also have,

\[
\Gamma^w_{vv} = g^{wv} \frac{\partial g_{vv}}{\partial v} = 0
\]

and,

\[
\Gamma^\theta_{vv} = \Gamma^\phi_{vv} = 0
\]

from which

\[
\nabla_v \frac{\partial}{\partial v} = \frac{m}{r^2} \frac{\partial}{\partial v}
\]

Computing,

\[
\Gamma^i_{\theta v} = \frac{1}{2} g^{im} \left( \frac{\partial g_{m\theta}}{\partial x^v} + \frac{\partial g_{m\theta}}{\partial \theta} \frac{\partial x^v}{\partial x^m} \right)
\]

\[
= \frac{1}{2} g^{im} \frac{\partial g_{m\theta}}{\partial x^v}
\]

\[
\Gamma^\phi_{\theta v} = \frac{1}{2} g^{\phi \theta} \frac{\partial g_{\phi \theta}}{\partial v} = \frac{1}{2} \frac{\partial (r^2)}{\partial r} \frac{\partial r}{\partial \theta} = \frac{1}{2} \frac{r(1-\mu)}{r^2(1-\mu)} = \frac{(1-\mu)}{2r}
\]

thus,

\[
\nabla_v \frac{\partial}{\partial v} = \frac{1}{2} \frac{\partial}{\partial \theta}
\]

And,

\[
\Gamma^i_{\phi v} = \frac{1}{2} g^{im} \left( \frac{\partial g_{m\phi}}{\partial x^v} + \frac{\partial g_{m\phi}}{\partial \phi} \frac{\partial x^v}{\partial x^m} \right)
\]

\[
= \frac{1}{2} g^{im} \frac{\partial g_{m\phi}}{\partial v}
\]

\[
\Gamma^\phi_{\phi v} = \frac{1}{2} g^{\phi \phi} \frac{\partial g_{\phi \phi}}{\partial v} = \frac{1}{2} \frac{\partial (r^2 \sin^2 \theta)}{\partial r} \frac{\partial r}{\partial \phi} = \frac{1}{2} \frac{2r \sin \theta}{2r^2 \sin^2 \theta} \frac{\partial r}{\partial v} = \frac{2r}{2r^2} \frac{1}{2} (1-\mu)
\]

thus,

\[
\nabla_v \frac{\partial}{\partial v} = \frac{1}{2r} \frac{\partial}{\partial \phi}
\]
Also,

\[
\Gamma^i_{\phi w} = \frac{1}{2} g^{im} \left( \frac{\partial g_{m\phi}}{\partial x^w} + \frac{\partial g_{mw}}{\partial \phi} - \frac{\partial g_{\phi w}}{\partial x^m} \right)
\]

\[
= \frac{1}{2} g^{im} \frac{\partial g_{m\phi}}{\partial w}
\]

\[
\Gamma^\phi_{\phi w} = \frac{1}{2} g^{\phi \phi} \frac{\partial g_{\phi \phi}}{\partial w} = \frac{1}{2r^2 \sin^2 \theta} \frac{\partial (r^2 \sin^2 \theta)}{\partial r} \frac{\partial r}{\partial w} = -\frac{r}{2r^2} (1 - \mu)
\]

\[
= -\frac{(1 - \mu)}{2r}
\]

thus,

\[
\nabla_\phi \frac{\partial}{\partial w} = -\frac{(1 - \mu)}{2r} \frac{\partial}{\partial \phi}
\]

We also have,

\[
\nabla_\theta \frac{\partial}{\partial w} = -\frac{(1 - \mu)}{2r} \frac{\partial}{\partial \theta}
\]

Therefore, in conclusion, we have:

\[
\nabla_w \frac{\partial}{\partial w} = -\frac{m}{r^2} \frac{\partial}{\partial w}
\]

(177)

\[
\nabla_v \frac{\partial}{\partial v} = \frac{m}{r^2} \frac{\partial}{\partial v}
\]

(178)

\[
\nabla_v \frac{\partial}{\partial w} = \nabla_w \frac{\partial}{\partial v} = 0
\]

(179)

\[
\nabla_v \frac{\partial}{\partial v} = \frac{(1 - \mu)}{2r} \frac{\partial}{\partial \theta}
\]

(180)

\[
\nabla_w \frac{\partial}{\partial w} = \frac{-(1 - \mu)}{2r} \frac{\partial}{\partial \theta}
\]

(181)

\[
\nabla_v \frac{\partial}{\partial v} = \frac{(1 - \mu)}{2r} \frac{\partial}{\partial \phi}
\]

(182)

\[
\nabla_w \frac{\partial}{\partial w} = \frac{-(1 - \mu)}{2r} \frac{\partial}{\partial \phi}
\]

(183)

Now, let

\[
V = V^w(v, w) \frac{\partial}{\partial w} + V^v(v, w) \frac{\partial}{\partial v}
\]

(184)

We use the notation \( V^w(v, w) = V^w \), and \( V^v(v, w) = V^v \).
Computing,

\[
\begin{align*}
\nabla_v V &= (\partial_v V^w) \frac{\partial}{\partial w} + V^w (\nabla_v \frac{\partial}{\partial w}) + (\partial_v V^w) \frac{\partial}{\partial v} + V^v (\nabla_v \frac{\partial}{\partial v}) \\
&= (\partial_v V^w) \frac{\partial}{\partial w} + (\partial_v V^v) \frac{\partial}{\partial v} + V^v \frac{m}{r^2} \frac{\partial}{\partial v} \\
\nabla_w V &= (\partial_w V^w) \frac{\partial}{\partial w} + V^w (\nabla_w \frac{\partial}{\partial w}) + (\partial_w V^v) \frac{\partial}{\partial v} + V^v (\nabla_w \frac{\partial}{\partial v}) \\
&= (\partial_w V^w) \frac{\partial}{\partial w} - V^w \frac{m}{r^2} \frac{\partial}{\partial w} + (\partial_w V^v) \frac{\partial}{\partial v} \\
\nabla_\theta V &= (\partial_\theta V^w) \frac{\partial}{\partial w} + V^w (\nabla_\theta \frac{\partial}{\partial w}) + (\partial_\theta V^v) \frac{\partial}{\partial v} + V^v (\nabla_\theta \frac{\partial}{\partial v}) \\
&= V^v \frac{(1 - \mu)}{2r} \frac{\partial}{\partial \theta} - V^w \frac{(1 - \mu)}{2r} \frac{\partial}{\partial \theta} \\
&= \frac{(1 - \mu)}{2r} (V^v - V^w) \frac{\partial}{\partial \theta} \\
\nabla_\phi V &= V^w (\nabla_\phi \frac{\partial}{\partial w}) + V^v (\nabla_\phi \frac{\partial}{\partial v}) \\
&= V^v \frac{(1 - \mu)}{2r} \frac{\partial}{\partial \phi} - V^w \frac{(1 - \mu)}{2r} \frac{\partial}{\partial \phi} \\
&= \frac{(1 - \mu)}{2r} (V^v - V^w) \frac{\partial}{\partial \phi}
\end{align*}
\]

Therefore,

\[
\begin{align*}
\nabla_v V &= (\partial_v V^w) \frac{\partial}{\partial w} + (\partial_v V^v + \frac{m}{r^2} V^v) \frac{\partial}{\partial v} \\
\nabla_w V &= (\partial_w V^w) \frac{\partial}{\partial w} + (\partial_w V^v - \frac{m}{r^2} V^w) \frac{\partial}{\partial w} \\
\nabla_\theta V &= \frac{(1 - \mu)}{2r} (V^v - V^w) \frac{\partial}{\partial \theta} \\
\nabla_\phi V &= \frac{(1 - \mu)}{2r} (V^v - V^w) \frac{\partial}{\partial \phi}
\end{align*}
\]

The deformation tensor is,

\[
\pi^{\alpha\beta}(V) = \frac{1}{2} (\nabla^\alpha V^\beta + \nabla^\beta V^\alpha)
\]
Computing,

\[
\begin{align*}
\pi^{w\bar{w}}(V) &= \nabla^w V^w = g^{w\bar{w}} \nabla_w V^w \\
&= -\frac{2}{(1 - \mu)} \partial_w V^w \\
\pi^{v\bar{v}}(V) &= \nabla^v V^v = g^{v\bar{v}} \nabla_v V^v \\
&= -\frac{2}{(1 - \mu)} \partial_v V^v \\
\pi^{v\bar{w}}(V) &= \frac{1}{2} (\nabla^v V^w + \nabla^w V^v) \\
&= \frac{1}{2} (g^{v\bar{w}} \nabla_w V^v + g^{w\bar{v}} \nabla_v V^w) \\
&= \frac{-1}{(1 - \mu)} \left[ \partial_v V^v + \partial_w V^w + \frac{m}{r^2} (V^v - V^w) \right] \\
&= \pi^{w\bar{v}}(V) \\
\pi^{\theta\bar{\theta}}(V) &= \nabla^\theta V^\theta = g^{\theta\bar{\theta}} \nabla_\theta V^\theta \\
&= \frac{(1 - \mu)}{2r^3} (V^v - V^w) \\
\pi^{\phi\bar{\phi}}(V) &= \nabla^\phi V^\phi = g^{\phi\bar{\phi}} \nabla_\phi V^\phi \\
&= \frac{(1 - \mu)}{2r^3 \sin^2 \theta} (V^v - V^w) \\
\pi^{\theta\bar{\phi}}(V) &= \frac{1}{2} (\nabla^\theta V^v + \nabla^v V^\theta) \\
&= 0 = \pi^{\phi\theta} \\
\pi^{\phi\bar{v}}(V) &= \frac{1}{2} (\nabla^\phi V^v + \nabla^v V^\phi) \\
&= 0 = \pi^{\phi\bar{w}}(V) \\
\pi^{\theta\bar{w}}(V) &= 0 = \pi^{w\theta}(V) \\
\pi^{\phi\bar{w}}(V) &= 0 = \pi^{w\phi}(V)
\end{align*}
\]

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