The aim of this paper is a study of certain class of chiral (vertex) algebras which we call chiral algebras of differential operators.

I.0. To explain this notion, let us take up a "naive" approach to vertex algebras. Let us fix (in this Introduction) a ground field \( k \) of characteristic 0. Recall (cf. [K]) that a \( \mathbb{Z}_{\geq 0} \)-graded vertex algebra is a \( \mathbb{Z}_{\geq 0} \)-graded \( k \)-vector space \( V = \oplus_{i \geq 0} V_i \) equipped with a distinguished vacuum vector \( 1 \in V_0 \) and a family of bilinear operations

\[
(n) : V \otimes_k V \longrightarrow V, \quad x \otimes y \mapsto x(n)y,
\]

(\( n \in \mathbb{Z} \)), the operation \( (n) \) having the degree \(-n-1\), that is, \( V_{i(n)}V_j \subset V_{i+j-n-1} \) where by definition \( V_i = 0 \) for \( i < 0 \). These operations must satisfy a family of quadratic Borcherds identities. Throughout this paper we shall deal only with \( \mathbb{Z}_{\geq 0} \)-graded vertex algebras, so we shall omit the words "\( \mathbb{Z}_{\geq 0} \)-graded" when speaking about them.

Note that in particular the operation \( (-1) \) has degree 0. It is neither commutative nor associative in general.
We are interested in a question: what structure on the subspace \( V_{\leq 1} := V_0 \oplus V_1 \) is induced by the structure of a vertex algebra on \( V \)?

The claims in italics below are easily proved using the Borcherds identities, see the main body of the paper.

First of all, \( V_i \) is a commutative associative \( k \)-algebra with unit 1 with respect to \((-1)\). Let us denote this algebra by \( A \), and the operation \( a_{(-1)} b \) on it by \( ab \). Note that for each \( i \) the operation \((-1)\) gives a mapping

\[ A \otimes V_i \longrightarrow V_i, \quad (I1) \]

to be denoted \( ax \), but this mapping does not make \( V_i \) an \( A \)-module since the associativity \( abx = (ab)x \) is not satisfied in general.

We have a mapping \( \partial : V_i \longrightarrow V_{i+1} \) defined by \( \partial x = x_{[i-2]}1 \). Let us denote by \( \Omega \subset V_1 \) a \( k \)-subspace generated by all elements \( a\partial b \), \( a, b \in A \). The operation \((I1)\) makes \( \Omega \) an \( A \)-module (i.e. the associativity holds true on this subspace) and \( \partial : A \longrightarrow \Omega \) is a derivation.

Let \( T \) be the quotient space \( V_1/\Omega \). The operation \((I1)\) induces the structure of an \( A \)-module on \( T \) and operation \((0)\) induces a structure of a Lie algebra on \( T \) and an action of \( T \) on \( A \) by derivations. These structures are compatible: they make \( T \) a Lie algebroid (see 0.2) over \( A \).

The operation \((0)\) induces an action of \( T \) on \( \Omega \) which makes \( \Omega \) a module over the Lie algebra \( T \), and \( \partial \) is a morphism of \( T \)-modules.

The operation \((1)\) induces an \( A \)-bilinear pairing \( T \otimes \Omega \longrightarrow A \) which is a morphism of \( T \)-modules.

Let us choose a splitting

\[ s : T \longrightarrow V_1 \quad (I2) \]

of the canonical projection \( \pi : V_1 \longrightarrow T \). Define a mapping \( \gamma : A \times T \longrightarrow \Omega \) by \( \gamma(a, \tau) = s(a\tau) - as(\tau) \). Define a symmetric mapping \( \langle , \rangle : T \times T \longrightarrow A \) by \( \langle \tau, \tau' \rangle = s(\tau_{(1)})s(\tau') \). Finally, define a skew symmetric operation \( c : T \times T \longrightarrow \Omega \) by \( c(\tau, \tau') = s([\tau, \tau']) - [s(\tau), s(\tau')] \) where we set \( [x, y] := \frac{1}{2}(x_{(0)}y - y_{(0)}x) \).

These three operations, \( \gamma \), \( \langle , \rangle \) and \( c \), satisfy certain compatibilities, listed below in 1.3, (A1) — (A5).

Thus, we have assigned to a vertex algebra \( V \) (with a splitting \((I2)\)) a collection of data \( \mathcal{A} = (A, T, \Omega, \partial, \gamma, \langle , \rangle, c) \) as above satisfying properties (A1) — (A5). We call such collection of data a vertex algebroid which is a central hero of our story.

The above procedure gives a functor from the category of vertex algebras to the (appropriately defined) category of vertex algebroids. This functor admits a left adjoint \( U \), called vertex envelope. A vertex algebra of the form \( UA \) is called a chiral algebra of differential operators.

**1.2.** Let \( X \) be a smooth algebraic variety over \( k \), \( U \subset X \) an affine Zariski open subset, \( A = O(U) \). Let us consider the class of vertex algebroids of the form \( \mathcal{A} = (A, T, \Omega, \partial, \ldots) \) with \( T = \text{Der}_k(A) \), \( \Omega = \Omega^1_k(A) \), \( \partial = d_{DR} \) — the usual de Rham differential, and with the standard action of \( T \) on \( \Omega \) by the Lie derivative.
Such vertex algebroids (with suitably defined morphisms) form a groupoid (i.e. a category where all morphisms are invertible) $\mathcal{A}lg(A)$. This category is nonempty if there exists an $A$-base of $T$ consisting of commuting vector fields.

This groupoid has a remarkable additional structure. To describe this structure, let us define another groupoid $\Omega^{(2,3)}(A)$ as follows. By definition, its objects will be all closed 3-forms on $A$. If $\omega, \omega'$ are two such 3-forms, the set of morphisms between them is defined as

$$\text{Hom}_{\Omega^{(2,3)}(A)}(\omega, \omega') = \{ \eta \in \Omega^2_{A/k} \mid d_{DR} \eta = \omega' - \omega \}$$

The composition is defined in the obvious way. The addition of 3-forms induces a functor

$$\Omega^{(2,3)}(A) \times \Omega^{(2,3)}(A) \longrightarrow \Omega^{(2,3)}(A)$$

which makes $\Omega^{(2,3)}(A)$ an Abelian group in categories.

The first main result of this paper is

Theorem I1. The groupoid $\mathcal{A}lg(A)$ admits a canonical structure of a $\Omega^{(2,3)}(A)$-Torseur.

This is Theorem 7.2 of the paper. It means that one has a functor

$$\hat{+} : \mathcal{A}lg(A) \times \Omega^{(2,3)}(A) \longrightarrow \mathcal{A}lg(A), \ (A, \omega) \mapsto A \hat{+} \omega$$

which defines an Action of the Abelian group $\Omega^{(2,3)}(A)$ on $\mathcal{A}lg(A)$, such that for each $A \in \text{Ob}\mathcal{A}lg(A)$ the induced functor

$$A \hat{+} ? : \Omega^{(2,3)}(A) \longrightarrow \mathcal{A}lg(A)$$

is an equivalence of categories.

I.3. Sheafifying the previous construction one gets a sheaf of groupoids, i.e. gerbe $\mathcal{A}lg_X$ over $X$, with the lien $\Omega^{(2,3)}_X := \Omega^2_X \longrightarrow \Omega^3_{X, closed}$ (a short complex of sheaves over $X$, with $\Omega^2_X$ sitting in degree 0).

By a simple general homological construction (see 7.3), this gerbe gives rise to a characteristic class

$$c(\mathcal{A}lg_X) \in H^2(X; \Omega^{(2,3)}_X)$$

with the following property.

The groupoid of global sections $\mathcal{A}lg_X(X)$ is nonempty iff $c(\mathcal{A}lg_X) = 0$. If this is the case then its set of connected components is a nonempty $H^1(X; \Omega^{(2,3)}_X)$-torseur and the group of automorphisms of its object is isomorphic to $H^0(X; \Omega^{(2,3)}_X)$.

Our next aim is to calculate the class $c(\mathcal{A}lg_X)$. One shows (see Theorem 7.10) that it essentially coincides with the second component of the Chern character of the tangent bundle $T_X$. Let us describe an "explicit formula" for it.

Let $E$ be an arbitrary finite dimensional vector bundle over $X$, given by a Cech cocycle $(g_{ij}) \in Z^1(\mathcal{U}; GL_r(O_X))$ on a Zariski open covering $\mathcal{U} = \{ U_i \}$ of $X$. Define a Cech two-cocycle $ch_2(E) \in Z^2(\mathcal{U}; \Omega^{(2,3)}_X)$ by

$$ch_2(E) = (\frac{1}{2} \text{tr}(g_{ij}^{-1}g_{jk}^{-1}dg_{jk}dg_{ij}), \frac{1}{6} \text{tr}((g_{ij}^{-1}dg_{ij})^3))$$

(ACS)
This class may be called the "Atiyah-Chern-Simons class" of $E$. Its first component, $\alpha(E)$, is an element of $Z^2(U; \Omega^2_X)$ which is the usual "Atiyah" representative of the degree 2 part of the Chern character of $E$ "style Hodge" living in $H^2(X; \Omega^2_X)$, while the second component, $\beta(E)$ ("Chern-Simons"), is a Cech 1-cochain with coefficients in $\Omega^3_X$. The de Rham differential of $\alpha(E)$ is equal to the Cech coboundary of $\beta(E)$. One checks that the cohomology class of $(ACS)$ does not depend on the choice of trivialization of $E$.

Our second main result (Theorem 7.5) is

**Theorem I.2.** The class $c(\mathcal{A}lg)$ is equal to (the cohomology class of) $2\text{ch}_2(T_X)$.

I.4. Our third main topic is an explicit construction of the enveloping algebra $U\mathcal{A}$ of a vertex algebroid $\mathcal{A}$ and "Poincaré-Birkhoff-Witt" type theorem for it. Let us formulate the last theorem (see Theorem 9.18).

**Theorem I.3.** Each sheaf of algebras of chiral differential operators $\mathcal{D} = U\mathcal{A}$ of a vertex algebroid $\mathcal{A}$ and "Poincaré-Birkhoff-Witt" type theorem for it. Let us formulate the last theorem (see Theorem 9.18).

Each sheaf of algebras of chiral differential operators $\mathcal{D} = U\mathcal{A}$ of a vertex algebroid $\mathcal{A}$ and of fixed conformal weight, such that the corresponding graded algebra (which is a sheaf of $\mathbb{Z}_{\geq 0}$-graded commutative vertex algebras) is

$$\text{gr}_F(\mathcal{D}) = \text{Sym}_{\mathcal{O}_X} \left\{ (\oplus_{i \geq 1} T^{(i)}) \oplus (\oplus_{i \geq 1} \Omega^{(i)}) \right\}$$

where $T^{(i)}$ (resp. $\Omega^{(i)}$) is a copy of $T_X$ (resp. of $\Omega^1_X$) living in conformal weight $i$.

As a preparation to this theorem, we study in Section 8 the vertex envelope $UC$ of an arbitrary conformal algebra $C$. Although its construction is more or less contained in Kac’s book [K], we present here in a sense more direct construction of $C$, maybe of independent interest. Note in particular an amusing Lie bracket (8.21.1) which is defined on an arbitrary conformal algebra.

In the hope of possible arithmetical applications, we adopted in this paper a "characteristic free" approach to vertex algebras. Contrary to this Introduction (and to the tradition), in the main body of this work the ground ring will be an arbitrary commutative ring containing $1/2$ (with one exception: the PBW theorem 9.18). This generalization is achieved without great effort. Note that the original Borcherds’ definition of vertex algebras was characteristic-free.

I.4. This paper may be regarded as an "algebraic" version of our last note [GMS] where we worked in the analytical category. However, the approach adopted here is quite different from op. cit. (cf. also [MSV] and [MS]).

During the preparation of this work we have enourmously benefited from the discussions and correspondence with Sasha Beilinson. He was the first to suggest (by analogy with the classical picture, [BB]) that in the algebraic category the situation is more subtle than in the analytical one. He has sent us a note [BD1] where similar questions are treated from a different point of view. In a sense, a great part of this work is a result of the attempts to understand this note (which still remains mysterious for us). For the details on Beilinson-Drinfeld approach, see [BD2], 3.8.

Another special gratitude goes to Hélène Esnault. She has found a mistake in the previous formulation of our main result whose correction lead to the discovery of the class (ACS), and greatly helped in some computations. Theorem 7.10 was
obtained in collaboration with her.

This work was mostly done while V.S. visited IHES. He is grateful to this institute for the support and excellent working conditions.
Recollections and Notation

0.0. Throughout this paper, a commutative ring (algebra) will mean a commutative associative ring (algebra) with unit.

$k$ will denote a fixed ground commutative ring. We will often assume that $1/2 \in k$; we will indicate this assumption when necessary.

Algebra will mean a $k$-algebra; $\otimes$ will mean the tensor product over $k$.

In a nonassociative algebra, $abc\ldots$ will mean $a(b(c(\ldots)))\ldots$.

$\mathbb{Z}_{\geq 0}$ will denote the set of nonnegative integers. The binomial coefficients are defined by

$$\binom{a}{b} = \frac{a(a-1)\ldots(a-b+1)}{b!}, \quad a \in \mathbb{Z}, \ b \in \mathbb{Z}_{\geq 0}.$$  \hspace{1cm} (0.0.1)

and 0 if $b < 0$. We have

$$\binom{-a-1}{b} = (-1)^b \binom{a+b}{b}$$ \hspace{1cm} (0.0.2)

for all $a \geq 0$.

0.1. Let $A$ be a commutative algebra. The Lie algebra of $k$-derivations $T_A := \text{Der}_k(A,A)$ acts on the $A$-module of Kähler 1-differentials $\Omega^1_A := \Omega^1_{A/k}$ according to the usual rule

$$\tau(ab) = \tau(a)db + a\tau(b)$$ \hspace{1cm} (0.1.1)

The de Rham differential $d : A \to \Omega^1_A$ commutes with the $T_A$-action.

We have the canonical $A$-bilinear pairing

$$\langle \ , \ \rangle : T_A \times \Omega^1_A \to A, \quad \langle \tau, adb \rangle = a\tau(b)$$ \hspace{1cm} (0.1.2)

We have

$$\tau(a\omega) = \tau(a)\omega + a\tau(\omega)$$ \hspace{1cm} (0.1.3)

$$\langle a\tau(\omega) \rangle = a\tau(\omega) + \langle \tau(\omega) \rangle da$$ \hspace{1cm} (0.1.4)

$$\tau(\langle \nu, \omega \rangle) = \langle [\tau, \nu], \omega \rangle + \langle \nu, \tau(\omega) \rangle$$ \hspace{1cm} (0.1.5)

$$\langle \tau, da \rangle = \tau(a)$$ \hspace{1cm} (0.1.6)

($a \in A, \tau, \nu \in T_A, \omega \in \Omega^1_A$). (Of course the last formula is a particular case of (0.1.2).)

0.2. A Lie $A$-algebroid is a Lie algebra $T$ acting by derivations on $A$ and equipped with a structure of an $A$-module, such that

$$[\tau, a\nu] = a[\tau, \nu] + \tau(a)\nu$$ \hspace{1cm} (0.2.1)

and

$$(ar)(b) = a\tau(b)$$ \hspace{1cm} (0.2.2)

for all $\tau, \nu \in T; \ a, b \in A$. 

0.2.1. Direct image (pushout). Let $B$ be a commutative $A$-algebra, $i : A \to B$ be the structure morphism and $T$ be an $A$-algebroid Lie. Assume that $T$, as a Lie algebra, acts on $B$ by derivations in such a way that $i$ is a morphism of $T$-modules and

$$(a \tau)(b) = a \tau(b)$$

$(a \in A, \ b \in B, \tau \in T)$.

Then the $B$-module $T_B := B \otimes_A T$ admits a canonical structure of a $B$-algebroid Lie. Namely, the Lie bracket on $T_B$ is given by

$$[b_1 \otimes \tau_1, b_2 \otimes \tau_2] = b_1 b_2 \otimes [\tau_1, \tau_2] + \tau_1(b_2)b_1 \otimes \tau_2 - \tau_2(b_1)b_2 \otimes \tau_1$$  \hspace{1cm} (0.2.1.1)$$

and the action of $T_B$ on $B$ is defined by

$$(b_1 \otimes \tau)(b_2) = b_1 \tau(b_2)$$  \hspace{1cm} (0.2.1.2)$$

Vertex algebras

0.3. In his original paper, [B], Borcherds defined a vertex algebra over an arbitrary commutative ring. However, later most people preferred to work over the complex numbers. It is fairly obvious that all the general theorems of Kac’s book [K] are true over an arbitrary field of characteristic 0. A little less obvious, but true, is that with a minor modification of the definitions, they remain true over an arbitrary commutative ring. Below we recall the definitions and results to be used in the sequel, and explain these modifications.

Throughout this work, we will deal only with $\mathbb{Z}_{\geq 0}$-graded vertex algebras.

0.4. Definition. A $\mathbb{Z}_{\geq 0}$-graded conformal algebra is a $\mathbb{Z}_{\geq 0}$-graded $k$-module $C = \oplus C_i$, together with a family of endomorphisms

$$\partial^{(j)} : C \to C, \text{ of degree } j, \ j \in \mathbb{Z}_{\geq 0},$$

such that

$$\partial^{(i)} \circ \partial^{(j)} = \binom{i + j}{i} \partial^{(i+j)}; \quad \partial^{(0)} = Id,$$  \hspace{1cm} (0.4.1)$$

and a family of bilinear operations

$$(n) : C \times C \to C, \ (a, b) \mapsto a_{(n)}b, \text{ of degree } -n - 1, \ n \in \mathbb{Z}_{\geq 0},$$

such that

$$\partial^{(i)} a_{(n)} b = (-1)^i \binom{n}{i} a_{(n-i)} b \hspace{1cm} (0.4.2)$$

$$a_{(n)} b = (-1)^{n+1} \sum_{j=0}^\infty (-1)^j \partial^{(j)}(b_{(n+j)}a) \hspace{1cm} (0.4.3)$$

$$a_{(m)} b_{(n)} c = b_{(n)} a_{(m)} c + \sum_{j=0}^{m} \binom{m}{j} (a_{(j)} b)_{(m+n-j)} c \hspace{1cm} (0.4.4)$$
for all \(a, b, c \in C; \ m, n, i \in \mathbb{Z}_{\geq 0}\).

Cf. [K], Definition 2.7b. In each conformal algebra we have the following identities:

\[
(a(m)b)_{(n)}c = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left\{ a(m-j)b_{(n+j)c} - (-1)^{m} b_{(m+n-j)\alpha(j)c} \right\}
\]  

(0.4.5)

and

\[
\partial^{(j)}(a_{(n)}b) = \sum_{p=0}^{j} \partial^{(p)}a_{(n)}\partial^{(j-p)}b
\]  

(0.4.6)

We leave their direct proof to the reader. The proof of (0.4.5) uses only the axiom (0.4.4) (and (0.4.1)), and the proof of (0.4.6) uses the axioms (0.4.2) and (0.4.3).

We also have

\[
a_{(n)}\partial^{(j)}b = \sum_{p=0}^{j} \binom{n}{p} \partial^{(j-p)}(a_{(n-p)}b)
\]  

(0.4.7)

This is proven by induction on \(j\).

**0.5. First Definition of a Vertex Algebra.** A \(\mathbb{Z}_{\geq 0}\)-graded vertex algebra is a \(\mathbb{Z}_{\geq 0}\)-graded \(k\)-module \(V = \oplus V_n\) (vacuum vector) and a family of bilinear operations

\(\langle n \rangle : V \times V \to V, (a, b) \mapsto a_{(n)}b,\) of degree \(-n-1, n \in \mathbb{Z}\),

(0.5.1)

such that

\[1_{(n)}a = \delta_{n,-1}a; a_{(-1)}1 = a; a_{(n)}1 = 0 \text{ if } n \geq 0,\]

(0.5.2)

and

\[
\sum_{j=0}^{\infty} \binom{m}{j} a_{(m+j)}b_{(m+t-j)c} = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \left\{ a_{(m+n-j)}b_{(n+t-j)c} - (-1)^n b_{(n+t-j)}a_{(m+j)c} \right\}
\]  

(0.5.3)

for all \(a, b, c \in V, m, n, t \in \mathbb{Z}\).

Cf. [B], Section 4, [K], Prop. 4.8. (b). The important particular case of (0.5.3) corresponds to \(m = 0\):

\[
(a_{(n)}b)_{(j)}c = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \left\{ a_{(n-j)}b_{(j)}c - (-1)^{n} b_{(n+t-j)}a_{(m+j)c} \right\}
\]  

(0.5.4)

cf. (0.4.5).

Set

\[
\partial^{(j)}a := a_{(-1-j)}1, \ j \in \mathbb{Z}_{\geq 0}
\]  

(0.5.5)

This way we get endomorphisms \(\partial^{(j)}\) of \(V\) of degree \(j\). It follows from (0.5.2) that

\[
\partial^{(j)}1 = \delta_{j,0}1
\]  

(0.5.6)
\[
\partial^{(0)} = Id
\]
and (0.5.4) applied to \( b = c = 1 \) gives
\[
\partial^{(i)} \cdot \partial^{(j)} = \binom{i + j}{i} \partial^{(i+j)}
\]
We have the *commutativity formula*
\[
a_{(n)} b = (-1)^{n+1} \sum_{j=0}^{\infty} (-1)^{j} \partial^{(j)} (b_{(n+j)}a)
\]
for all \( a, b \in V, \; n \in \mathbb{Z} \). The proof will be given below, see the last paragraph of the next subsection.

One deduces from (0.5.4) that
\[
(\partial^{(j)} a)_{(n)} b = (-1)^{j} \binom{n}{j} a_{(n-j)} b
\]
and
\[
\partial^{(j)} (a_{(n)} b) = \sum_{p=0}^{j} (\partial^{(p)} a)_{(n)} \partial^{(j-p)} b
\]
for all \( n \in \mathbb{Z} \). We have the *Operator Product Expansion (OPE) formula*,
\[
[x_{(m)}, y_{(n)}] = \sum_{j \geq 0} \binom{m}{j} x_{(j)} y_{(m+n-j)} \; (m, n \in \mathbb{Z})
\]
cf. [K], (4.6.7) and the end of the next Subsection.

**0.6.** Following [K], let us give an equivalent definition of a vertex algebra. If \( M \) is a \( k \)-module, \( M[[z, z^{-1}]] \) will denote as usually the module of formal power series \( \sum_{n=-\infty}^{\infty} a_n z^n, \; a_n \in M \). Let us define endomorphisms \( \partial^{(j)}, \; j \in \mathbb{Z}_{\geq 0}, \) of \( M[[z, z^{-1}]] \) by
\[
\partial^{(j)} (\sum_n a_n z^n) = \sum_n \binom{n}{j} a_n z^{n-j}
\]
\[
[x_{(m)}, y_{(n)}] = \sum_{j \geq 0} \binom{m}{j} x_{(j)} y_{(m+n-j)} \; (m, n \in \mathbb{Z})
\]

**Second Definition of a Vertex Algebra.** A \( \mathbb{Z}_{\geq 0} \)-graded vertex algebra is a \( \mathbb{Z}_{\geq 0} \)-graded \( k \)-module \( V = \bigoplus_i V_i \) equipped with

— a distinguished vector \( 1 \in V_0 \) (*vacuum vector*);

— a family of endomorphisms \( \partial^{(j)} : V \to V \) of degree \( j, \; j \in \mathbb{Z}_{\geq 0}, \) such that \( \partial^{(0)} = Id \) and \( \partial^{(i)} \cdot \partial^{(j)} = (\binom{i+j}{i}) \partial^{(i+j)}; \)

— a linear mapping \( V \to \text{End}(V)[[z, z^{-1}]], \; a \mapsto a(z) = \sum_n a_n z^{-n-1} \) such that \( \deg a_{(n)} = \deg a - n - 1. \)

These data should satisfy the following axioms.
Translation invariance. For all \( j \in \mathbb{Z}_{\geq 0} \), \([\partial^{(j)}, a(z)] = \partial^{(j)}a(z)\).

Vacuum. \( \partial^{(j)}1 = \delta_{j,0}1; \ 1(z) = 1; \ a_{(n)}1 = 0 \) for \( n \geq 0 \) and \( a_{(-1)}1 = a \).

Locality. \((z-w)^N[a(z), b(w)] = 0 \) for \( N \gg 0 \).

This is a modification of the definition given in [K], 4.1, which works over an arbitrary base commutative ring. Proposition 4.1 of op. cit. remains true if we understand by \( e^{z\partial} \) the expression

\[
e^{z\partial} := \sum_{j=0}^{\infty} z^j \partial^{(j)}\tag{0.6.2}
\]

(we cannot use Lemma 4.1 of op. cit. anymore!). The argument of op. cit. 4.2 shows that the commutativity formula (0.5.9) is true for the vertex algebras in the second definition. Finally, the proof of op. cit., Proposition 4.8 (b) shows that the first and the second definitions are equivalent. The proof from [K], 4.6 works to give the proof of the OPE formula (0.5.12).

0.7. Theorem. Let \( V = \oplus V_i \) be a \( \mathbb{Z}_{\geq 0} \)-graded \( k \)-module equipped with a distinguished vector \( 1 \in V_0 \) and a family of endomorphisms \( \partial^{(j)} \) of degree \( j \), \( j \in \mathbb{Z}_{\geq 0} \) such that \( \partial^{(0)} = Id, \partial^{(j)} \partial^{(i)} = \partial^{(i+j)} \) and \( \partial^{(j)}1 = \delta_{j,0}1 \).

Assume that we are given a family of homogeneous vectors \( \{a^\alpha\} \subset V \) and a family of formal power series ("distributions")

\[
\{a^\alpha(z) = \sum_n a_{(n)}^\alpha z^{-n-1}\} \subset \text{End}(V)[[z, z^{-1}]], \deg a_{(n)}^\alpha = \deg a^\alpha - n - 1,
\]

satisfying the following conditions

(t) \([\partial^{(j)}, a^\alpha(z)] = \partial^{(j)}a^\alpha(z)\);

(v) \( a_{(n)}^\alpha 1 = 0 \) for \( n \geq 0; \ a_{-1}^\alpha 1 = a^\alpha; \)

(l) The distributions \( a^\alpha(z) \) are mutually local, i.e. for all \( \alpha, \beta \), \((z-w)^N[a^\alpha(z), a^\beta(w)] = 0 \) for sufficiently large \( N \).

Let \( S \) denote the set of all vectors of the form

\[
a_{j_1 \ldots j_N}^{\alpha_1 \ldots \alpha_N} := a_{(-1-j_1)}^{\alpha_1} \ldots a_{(-1-j_N)}^{\alpha_N}1, \ N, j_i \in \mathbb{Z}_{\geq 0}, \tag{0.7.1}
\]

Assume that we are given a map

\[
V \rightarrow \text{End}(V)[[z, z^{-1}]], \ a \mapsto a(z) = \sum a_{(n)} z^{-n-1} \tag{*}
\]

having the following property:

(P) There exists a subset \( S' \subset S \) which generates \( V \) as a \( k \)-module such that for each \( a_{j_1 \ldots j_N}^{\alpha_1 \ldots \alpha_N} \in S' \), we have

\[
a_{j_1 \ldots j_N}^{\alpha_1 \ldots \alpha_N}(z) = \partial^{j_1} a^{\alpha_1}(z) \ldots \partial^{j_N} a^{\alpha_N}(z); \tag{0.7.2}
\]
Then the mapping (*) defines the structure of a $\mathbb{Z}_{\geq 0}$-graded vertex algebra on $V$, and (0.7.2) holds true for all $a_{j_1 \cdots j_N}^{n_1 \cdots n_N} \in S$.

This is a version of "Existence Theorem", [K], Theorem 4.5 and Corollary 4.5, whose proof goes through.

0.8. Let $\mathbb{V}ert$, $Conf$ denote the categories of $\mathbb{Z}_{\geq 0}$-graded vertex and conformal algebras respectively. We have a functor

$$c : \mathbb{V}ert \rightarrow Conf$$

(0.8.1)

which assigns to a vertex algebra $V$ the same space $V$, with the operations $(n)_n$, $n < 0$, forgotten. The axioms of a conformal algebra are satisfied due to (0.5.10), (0.5.9) and (0.5.12). This functor admits a left adjoint, to be constructed in Section 8 below.

In the sequel we will call $\mathbb{Z}_{\geq 0}$-graded vertex (conformal) algebras simply vertex (conformal) algebras.

0.9. A vertex algebra $V$ is called commutative if $a(n)_b = 0$ for all $a, b \in V$, $n \geq 0$ (Kac uses the term "holomorphic").

Let $V$ be a commutative vertex algebra. Then, with respect to the operation $ab := a_{(-1)^n} b$, $V$ becomes a commutative associative algebra with the unity $1$. The operations with negative indices are recovered from the formula

$$a_{(-1^j)} b = (\partial^{(j)}) a b$$

(0.9.1)

This way we get an equivalence of categories

$$\mathbb{V}ert Ab \xrightarrow{\sim} \partial - \mathcal{A}lg$$

(0.9.2)

Here $\mathbb{V}ert Ab$ denotes the category of commutative vertex algebras and $\partial - \mathcal{A}lg$ denotes the category whose objects are $\mathbb{Z}_{\geq 0}$-graded vector spaces $V = \bigoplus V_i$ equipped with a structure of a commutative algebra such that $V_i \cdot V_j \subset V_{i+j}$ and $1 \in V_0$, and with a family of endomorphisms $\partial^{(j)}$ of degree $j$, $j \in \mathbb{Z}_{\geq 0}$, such that

$$\partial^{(i)} \cdot \partial^{(j)} = \binom{i+j}{i} \partial^{(i+j)} , \quad \partial^{(0)} = Id$$

(0.9.3)

and

$$\partial^{(j)}(ab) = \sum_{p=0}^j \partial^{(p)} a \cdot \partial^{(j-p)} b$$

(0.9.4)

Cf. [B]. Objects of $\partial - \mathcal{A}lg$ will be called $\partial$-algebras.
§ 1. Vertex Algebroids

1.1. Let us define an extended Lie algebroid to be a quintuple $T = (A, T, \Omega, \partial, \langle \, , \rangle)$ where $A$ is a commutative $k$-algebra, $T$ is a Lie $A$-algebroid, $\Omega$ is an $A$-module equipped with a structure of a module over the Lie algebra $T$, $\partial : A \to \Omega$ is an $A$-derivation and a morphism of $T$-modules, $\langle \, , \rangle : T \times \Omega \to A$ is an $A$-bilinear pairing.

These data must satisfy the following properties ($a, \tau, \nu, \omega \in A$):

\begin{align*}
\langle \tau, \partial a \rangle &= \tau(a) \quad (1.1.1) \\
\tau(a\omega) &= \tau(a)\omega + a\tau(\omega) \quad (1.1.2) \\
(\tau\partial)(\omega) &= a\tau(\omega) + \tau(\omega)\partial a \quad (1.1.3) \\
\tau(\langle \nu, \omega \rangle) &= \langle [\tau, \nu], \omega \rangle + \langle \nu, \tau(\omega) \rangle \quad (1.1.4)
\end{align*}

Cf. 0.1. We will also say that $T = (A, T, \ldots)$ is an extended Lie $A$-algebroid.

Let define a morphism between two extended Lie algebroids $T = (A, T, \ldots)$ and $T' = (A', T', \ldots)$ to be a triple $g = (g_A, g_T, g_{\Omega})$ where $g_A : A \to A'$ is a morphism of $k$-algebras, $g_T : T \to T'$ is a morphism of Lie algebras and $A$-modules, $g_{\Omega} : \Omega \to \Omega'$ is a morphism of $A$-modules. We require that the following properties should hold:

\begin{align*}
g_A(\tau(a)) &= g_T(\tau)(g_A(a)) \quad (1.1.5) \\
g_{\Omega}(\partial a) &= \partial g_A(a) \quad (1.1.6) \\
g_A(\langle \tau, \omega \rangle) &= \langle g_T(\tau), g_{\Omega}(\omega) \rangle \quad (1.1.7)
\end{align*}

Composition of morphisms is defined in the obvious way. This way we get a category $\text{LieAlg}$ of extended Lie algebroids.

1.2. Let $T$ be a Lie $A$-algebroid. Set $\Omega := \text{Hom}_A(T, A)$. Define $\partial : A \to \Omega$ by (1.1.1); let $\langle \, , \rangle$ be the evident pairing. Action of $T$ on $\Omega$ is defined by (1.1.4). This way we get an extended Lie $A$-algebroid $\mathcal{T}_T$.

Let us call an extended Lie algebroid $\mathcal{T} = (A, T, \Omega, \langle \, , \rangle)$ perfect if the pairing $\langle \, , \rangle$ induces an isomorphism $\Omega \cong \text{Hom}_A(T, A)$.

The correspondence $T \mapsto \mathcal{T}_T$ provides an equivalence of the category of Lie algebroids with the full subcategory $\text{LieAlg}^{\text{perf}} \subset \text{LieAlg}$ of perfect extended Lie algebroids.

1.3. De Rham - Chevalley complex. Let $\mathcal{T} = (A, T, \Omega, \ldots)$ be an extended Lie $A$-algebroid. Let us define $A$-modules $\Omega^i = \Omega^i(T)$, $i \in \mathbb{Z}_{\geq 0}$, as follows. Set $\Omega^0 = A$, $\Omega^1 = \Omega$. For $i \geq 2$, $\Omega^i$ is the submodule of the module of $A$-polylinear homomorphisms $h$ from $T^{i-1}$ to $\Omega$ such that the function $\langle \tau_1, h(\tau_2, \ldots, \tau_i) \rangle$ is skew symmetric with respect to all permutations of $(\tau_1, \ldots, \tau_i)$.

For example, if $\mathcal{T}$ is as in the previous example, then $\Omega^1 = \text{Hom}_A(\Lambda^1_A T, A)$. 
Let us define the maps $d_{DR} = d_{DR}^i : \Omega^i \rightarrow \Omega^{i+1}$ as follows. For $i = 0$ we set
\[ d_{DR}^0 a = -\partial a. \]
For $i \geq 1$ we set
\[ d_{DR}^i h(\tau_1, \ldots, \tau_i) = d_{Lie} h(\tau_1, \ldots, \tau_i) - \partial(\tau_1, h(\tau_2, \ldots, \tau_i)) \] (1.3.1)
where
\[ d_{Lie} h(\tau_1, \ldots, \tau_i) = \sum_{p=1}^i (-1)^{p+1} \tau_p(h(\tau_1, \ldots, \widehat{\tau}_p, \ldots, \tau_i)) + \sum_{1 \leq p < q \leq i} (-1)^{p+q} h([\tau_p, \tau_q], \tau_1, \ldots, \widehat{\tau}_p, \ldots, \widehat{\tau}_q, \ldots, \tau_i) \] (1.3.2)
For example,
\[ d_{DR}^i \omega(\tau) = \tau(\omega) - \partial(\tau, \omega), \] (1.3.3)
for $\omega \in \Omega^1 = \Omega$; and
\[ d_{DR}^i h(\tau_1, \tau_2) = -h([\tau_1, \tau_2]) + \tau_1(h(\tau_2)) - \tau_2(h(\tau_1)) - \partial(\tau_1, h(\tau_2)) \] (1.3.4)
for $h \in \Omega^2$.
Let us introduce the action of the Lie algebra $T$ on the modules $\Omega^i$ by
\[ \tau(h)(\tau_1, \ldots, \tau_{i-1}) = \tau(h(\tau_1, \ldots, \tau_{i-1})) - \sum_{p=1}^{i-1} h(\tau_1, \ldots, [\tau, \tau_p], \ldots, \tau_i) \] (1.3.5)
Let us define the convolution operators $\langle \tau, \cdot \rangle : \Omega^i \rightarrow \Omega^{i-1}$ by
\[ \langle \tau, h \rangle(\tau_1, \ldots, \tau_{i-2}) = h(\tau, \tau_1, \ldots, \tau_{i-2}) \] (1.3.6)
The maps $\{d_{DR}^i\}$ may be characterized as a unique collection of maps such that $d_{DR}^0 = -\partial$ and the Cartan formula
\[ \tau(h) = \langle \tau, d_{DR} h \rangle + d_{DR} \langle \tau, h \rangle \] (1.3.7)
holds true.

The maps $d_{DR}$ commute with the action of $T$. One checks that $d_{DR}^2 = 0$, so we get a complex $(\Omega^*(T), d_{DR})$ called the de Rham-Chevalley complex of $T$.

1.4. In the definition below we assume that $1/2 \in k$.

A vertex algebroid is a septuple $A = (A, T, \Omega, \partial, \gamma, (\cdot, \cdot), c)$ where $A$ is a commutative $k$-algebra, $T$ is a Lie $A$-algebroid, $\Omega$ is an $A$-module equipped with an action of the Lie algebra $T$, $\partial : A \rightarrow \Omega$ is a derivation commuting with the $T$-action,
\[ (\cdot, \cdot) : (T \oplus \Omega) \times (T \oplus \Omega) \rightarrow A \]
is a symmetric $k$-bilinear pairing equal to zero on $\Omega \times \Omega$ and such that $T_A = (A, T, \Omega, \partial, (\cdot, \cdot)_{|T \oplus \Omega})$ is an extended Lie $A$-algebroid; $c : T \times T \rightarrow \Omega$ is a skew symmetric $k$-bilinear pairing and $\gamma : A \times T \rightarrow \Omega$ is a $k$-bilinear map.
The following axioms must hold \(a, b \in A; \tau, \tau_i \in T\):

\[
\begin{align*}
\gamma(ab, \tau) &= \gamma(ab) - a\gamma(b, \tau) - \tau(a)b - \tau(b)a \\
\langle a\tau_1, \tau_2 \rangle &= a\langle \tau_1, \tau_2 \rangle + \langle a, \tau_1 \rangle \tau_2 - \tau_1 \tau_2(a) \tag{A1} \\
c(a\tau_1, \tau_2) &= ac(\tau_1, \tau_2) + \gamma(a, \tau_1)\tau_2 + \tau_2(\gamma(a, \tau_1)) - \frac{1}{2}(\tau_1, \tau_2)\partial a + \frac{1}{2}\partial\tau_1\tau_2(a) - \frac{1}{2}\partial\langle \tau_2, \gamma(a, \tau_1) \rangle \tag{A2} \\
\langle [\tau_1, \tau_2], \tau_3 \rangle + \langle \tau_2, [\tau_1, \tau_3] \rangle &= \tau_1(\langle \tau_2, \tau_3 \rangle) - \frac{1}{2}\tau_2(\langle \tau_1, \tau_3 \rangle) - \frac{1}{2}\tau_3(\langle \tau_1, \tau_2 \rangle) + \langle \tau_2, c(\tau_1, \tau_3) \rangle + \langle \tau_3, c(\tau_1, \tau_2) \rangle \tag{A3} \\
\left. \frac{1}{2}\partial \right\{(\langle \tau_1, \tau_2 \rangle, \tau_3) - 2\langle \tau_3, c(\tau_1, \tau_2) \rangle \} = 0 \tag{A4}
\end{align*}
\]

where \(d_{Lie} \) is defined by (1.3.2).

To stress the dependence on \(A\), we shall sometimes say that \(A\) is a vertex A-algebroid.

1.5. Multiplying the identity (A4) by \(l\) and subtracting the \(j\)-th multiple of (A4) corresponding to the triple \((\tau_2, \tau_1, \tau_3)\) we get an equivalent form of (A4):

for every \(l, j \in k\),

\[
\begin{align*}
-j\langle \tau_1, \tau_2, \tau_3 \rangle + l\langle \tau_2, \tau_1, \tau_3 \rangle + (l + j)\langle \tau_3, \tau_1, \tau_2 \rangle \\
-l + \frac{1}{2}j)\tau_1(\langle \tau_2, \tau_3 \rangle) + \frac{1}{2}l + j)\tau_2(\langle \tau_1, \tau_3 \rangle) + \frac{1}{2}(l - j)\tau_3(\langle \tau_1, \tau_2 \rangle) + j\langle \tau_1, c(\tau_2, \tau_3) \rangle - l\langle \tau_2, c(\tau_1, \tau_3) \rangle - (l + j)\langle \tau_3, c(\tau_1, \tau_2) \rangle = 0 
&\tag{A4}_{i,j}^{bis}
\end{align*}
\]

1.6. The left hand side of (A5) is skew symmetric with respect to all permutations of \((\tau_1, \tau_2, \tau_3)\). The right hand side is manifestly symmetric only with respect to the transposition of \((\tau_1, \tau_2)\). However, if we replace \(\tau_1(\langle \tau_2, \tau_3 \rangle)\) in the right hand side by its expression from (A4) we get the following equivalent form of (A5):

\[
\begin{align*}
d_{Lie}(\tau_1, \tau_2, \tau_3) &= \frac{1}{2}\partial \{(\langle \tau_1, \tau_2, \tau_3 \rangle) + \frac{1}{2}\tau_2(\langle \tau_1, \tau_3 \rangle) - \frac{1}{2}\tau_3(\langle \tau_1, \tau_2 \rangle) + \langle \tau_2, c(\tau_1, \tau_3) \rangle - \langle \tau_3, c(\tau_1, \tau_2) \rangle \} 
&\tag{A5}_{bis}
\end{align*}
\]

The rhs of (A5)\(_{bis}\) is skew symmetric with respect to the transposition of \((\tau_2, \tau_3)\). Consequently, the rhs of (A5) is completely skew symmetric since the symmetric group \(S_3\) is generated by two transpositions (12) and (23).

Let us replace in (A5)\(_{bis}\) the triple \((\tau_1, \tau_2, \tau_3)\) by the triples \((\tau_2, \tau_3, \tau_1)\) and \((\tau_3, \tau_1, \tau_2)\) and sum up the three identities. We will get another equivalent for of (A5):

\[
3d_{Lie}(\tau_1, \tau_2, \tau_3) = \partial \{(\langle \tau_1, \frac{1}{2}\tau_2, \tau_3 \rangle + c(\tau_2, \tau_3)) + 
\]
\[ +\langle \tau_2, \frac{1}{2}\langle \tau_3, \tau_1 \rangle + c(\tau_3, \tau_1) \rangle + \langle \tau_3, \frac{1}{2}\langle \tau_1, \tau_2 \rangle + c(\tau_1, \tau_2) \rangle \} \quad (A5)^{cer} \]

1.7. Example. Let \( \mathfrak{g} \) be a Lie algebra over \( k \) equipped with a symmetric invariant form \( \langle \ , \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow k \). Then \( A_{\mathfrak{g},(\cdot)} = (k, \mathfrak{g}, 0, 0, \langle \ , \rangle, 0) \) is a vertex \( k \)-algebroid.

1.8. Let \( A \) be a commutative \( k \)-algebra and \( \mathcal{T} = (A, T, \Omega, \partial, \langle \ , \rangle) \) be an extended Lie \( A \)-algebroid. Let \( T_0 \subset T \) be a \( k \)-submodule which generates \( T \) as an \( A \)-module, and assume that we are given \( k \)-bilinear mappings

\[ \gamma : A \times T_0 \rightarrow \Omega; \quad \langle \ , \rangle : T_0 \times T_0 \rightarrow A, \text{ symmetric;} \]

\[ c : T_0 \times T_0 \rightarrow \Omega, \text{ skew symmetric} \quad (1.8.1) \]

It is clear that there exists not more than one extension of the mappings to mappings

\[ \gamma : A \times T \rightarrow \Omega; \quad \langle \ , \rangle : T \times T \rightarrow A, \quad c : T \times T \rightarrow \Omega, \quad (1.8.2) \]

satisfying (A1), (A2), (A3). This extension, if it exists, must be given by the formulas \( (a, b \in A, \tau, \tau_1 \in T_0) \):

\[ \gamma(a, b \tau) = \gamma(ab, \tau) - a\gamma(b, \tau) - \tau(a)\partial b - \tau(b)\partial a \quad (1.8.3)_\gamma \]

\[ \langle a\tau_1, b\tau_2 \rangle = ab\langle \tau_1, \tau_2 \rangle + a\langle \tau_1, \gamma(b, \tau_2) \rangle + b\langle \tau_2, \gamma(a, \tau_1) \rangle - \]

\[ - a\tau_2\tau_1(b) - b\tau_1\tau_2(a) - \tau_1(b)\tau_2(a) \quad (1.8.3)_{\langle \ , \rangle} \]

\[ c(a\tau_1, b\tau_2) = ab \cdot c(\tau_1, \tau_2) + \gamma(ab, [\tau_1, \tau_2]) + \]

\[ + \gamma(a\tau_1(b), \tau_2) - \gamma(b\tau_2(a), \tau_1) - a\tau_1(\gamma(b, \tau_2)) + b\tau_2(\gamma(a, \tau_1)) + \]

\[ + \frac{1}{2}\langle \tau_1, \tau_2 \rangle \{ a\partial b - b\partial a \} - \langle \tau_1, \tau_2 \rangle (a)\partial b - \langle \tau_1, \tau_2 \rangle (b)\partial a + \frac{1}{2}\{ \tau_1(b)\partial \tau_2(a) - \tau_2(a)\partial \tau_1(b) \} + \]

\[ + \frac{1}{2}\{ - \langle \tau_1, \gamma(b, \tau_2) \rangle \partial a + \tau_2, \gamma(a, \tau_1) \rangle \partial b + a\partial \langle \tau_1, \gamma(b, \tau_2) \rangle - b\partial \langle \tau_1, \gamma(a, \tau_1) \rangle \} + \]

\[ + \frac{1}{2}\partial \{ b\tau_1\tau_2(a) - a\tau_2\tau_1(b) \} \quad (1.8.3)_c \]

When deducing the last formula, one should take into account (1.1.3).

1.9. Theorem. (Extension of Identities) Assume that (A4) and (A5) hold true for all \( \tau, \tau_1 \in T_0 \) and that the formulas (1.8.3) provide well defined mappings (1.8.2).

Then the axioms (A1) — (A5) hold true for all \( \tau, \tau_1 \in T \), i.e. \( \mathcal{A} = (A, T, \ldots) \) is a vertex \( A \)-algebroid.

Proof. Let us check the axiom (A1). It is enough to show that if (A1) holds true for some \( \tau \) and all \( a, b \) then it holds true for \( c\tau \ (c \in A) \) and all \( a, b \). Thus, we have to check the identity

\[ \gamma(ab, c\tau) = \gamma(a, b c\tau) + a\gamma(b, c\tau) + c\tau(\partial a) b + c\tau(b)\partial a \quad (?) \]

The left hand side of it equal to

\[ \gamma(abc, \tau) - ab\gamma(c, \tau) - \tau(ab)\partial c - \tau(c)\partial(ab), \]
cf. (1.8.3)\(^\gamma\). On the other hand, the first two terms in the right hand side are equal to
\[
\gamma(a,bc\tau) = \gamma(abc,\tau) - a\gamma(bc,\tau) - \tau(a)\partial(bc) - \tau(bc)\partial a
\]
and
\[
a\gamma(b,c\tau) = a\gamma(bc,\tau) - ab\gamma(c,\tau) - a\tau(b)c - a\tau(c)b,
\]
again by (1.8.3)\(^\gamma\). Comparing, we get the desired identity (?).

The other axioms are checked in a similar way. This is a tiresome but straightforward calculation. When checking (A5) one should use the identity (1.8.3).

1.10. Pushout. Let \(\mathcal{A} = (A,T,\Omega,\ldots)\) be a vertex \(A\)-algebroid. Let \(B\) be a commutative \(A\)-algebra, \(i : A \to B\) the structure morphism. Set \(\Omega_B := B \otimes_A \Omega\), \(T_B := B \otimes T\). The \(A\)-derivation \(\partial : A \to \Omega\) induces a \(B\)-derivation \(\partial_B : B \to \Omega_B\). The \(A\)-bilinear pairing \(\langle , \rangle : T \times \Omega \to A\) uniquely extends to a \(B\)-bilinear pairing \(\langle , \rangle_B : T_B \times \Omega_B \to B\).

Assume that the Lie algebra \(T\) acts on \(B\) by derivations in such a way that \(\tau(i(a)) = i(\tau(a))\) and \((a\tau)(b) = a\tau(b)\) \((a \in A, b \in B, \tau \in T)\). Then \(T_B\) acquires a canonical structure of a Lie \(B\)-algebroid, cf. 0.2.1, and \((T_B,\Omega_B,\partial_B,\langle , \rangle_B)\) becomes an extended Lie \(B\)-algebroid.

1.10.1. Theorem. Assume that we are given a \(k\)-bilinear mapping \(\gamma : B \times T \to \Omega_B\) such that \(\gamma(i(a),\tau) = 1 \otimes \gamma(a,\tau)\) and that (A1) holds true for all \(\tau \in T\), \(a \in A\), \(b \in B\).

Then there exists a unique extension of \(\gamma\) to a \(k\)-bilinear mapping \(\gamma_B : B \times T_B \to \Omega_B\) satisfying (A1) for all \(a,b \in B\), \(\tau \in T_B\);

there exists a unique extension of the pairing \(\langle , \rangle : T \times T \to A\) to a pairing \(\langle , \rangle_B : T_B \times T_B \to B\) satisfying (A2) for all \(a,b \in B\), \(\tau_1 \in T_B\);

there exists a unique extension of the pairing \(\gamma : T \times T \to \Omega\) to a pairing \(c_B : T_B \times T_B \to \Omega_B\) satisfying (A3) for all \(a \in B\), \(\tau_1 \in T_B\).

The septuple \(\mathcal{A}_B = (B,T_B,\Omega_B,\partial_B,\gamma_B,\langle , \rangle_B,c_B)\) is a vertex \(B\)-algebroid.

Proof. Apply 1.9 to \(T_0 := \text{Im}(T \to T_B)\). \(\triangle\)

1.11. Example. Let \(A\) be a commutative \(k\)-algebra, \(\Omega = \Omega_A^1\) — the \(A\)-module of Kählerian 1-differentials, \(\partial : A \to \Omega\) the canonical \(A\)-derivation. Let \(\mathfrak{g}\) be a Lie \(k\)-algebra acting on \(A\) by derivations and equipped with an invariant bilinear form \(\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to k\).

Due to the morphism \(\mathfrak{g} \to \text{Der}_k(A)\), \(\Omega\) is equipped with a canonical structure of a \(\mathfrak{g}\)-module such that \(\partial\) is a morphism of \(\mathfrak{g}\)-modules and there is a canonical pairing \(\mathfrak{g} \times \Omega \to A\), cf. 0.1.

Set \(T = A \otimes \mathfrak{g}\). Then \(T\) is canonically a Lie \(A\)-algebroid, there exists a unique extension of the zero map \(A \times \mathfrak{g} \to A\) (resp., of \(\langle , \rangle\) and of the zero map \(\mathfrak{g} \times \mathfrak{g} \to \Omega\)) to the map \(\gamma : A \times T \to \Omega\) (resp., to the pairing \(\langle , \rangle : T \times T \to A\) and to the map \(c : T \times T \to \Omega\) which satisfies (A1) (resp., (A2) and (A3)).

This way we get a vertex \(A\)-algebroid \(\mathcal{A} = (A,T,\partial,\gamma,\langle , \rangle,c)\).

For the proof, apply Theorem 1.9 to \(T_0 = \mathfrak{g} \subset T\).
1.12. Example. Let $A$ be a commutative $k$ algebra, $g \subset T = \text{Der}_k(A)$ a Lie $k$-subalgebra such that $T = A \otimes g$, $\Omega = \Omega^1_k$.

For example, $A$ is the coordinate ring of an algebraic group, $g$ is the Lie algebra of left invariant vector fields.

Assume that $g$ is equipped with an invariant bilinear form $\langle , \rangle$. Then we are in the situation of 1.11 and get a vertex $A$-algebroid $A = A_{A,A,g,\langle \rangle}$.

1.13. Example. Let $A$ be a commutative $k$-algebra equipped with an étale morphism $f = (f_1, \ldots, f_n) : A_0 = k[x_1, \ldots, x_n] \to A$. Such an $f$ is called an étale coordinate system on $A$; it exists Zariski locally for every $A$ smooth over $k$.

The commuting vector fields $\partial/\partial x_i \in \text{Der}_k(A_0)$ admit unique liftings to vector fields $\tau_i \in T = \text{Der}_k(A)$. The $k$-submodule $g \subset T$ spanned by $\tau_i$ is an abelian Lie subalgebra; let us equip it with the trivial bilinear form. Then we are in the situation of 1.11 and we get a vertex $A$-algebroid $A = A_{A,f}$.

§2. From Vertex Algebras to Vertex Algebroids

2.1. Let $V$ be a vertex algebra over $k$, cf. 0.5. In this section we assume that $1/2 \in k$; recall that we always deal with $\mathbb{Z}_{\geq 0}$-graded vertex algebras. We denote $\partial := \partial^{(1)}$.

The identity (0.5.4) at $n = 0$ gives

$$a_{(0)}b_{(n)}c = (a_{(0)}b)_{(n)}c + b_{(n)}a_{(0)}c \quad (2.1.1)$$

for all $n$, i.e. the operation $(0)$ is a derivation with respect to all operations $(n)$.

We shall denote the operation $(1)$ by $ab$ or $a \cdot b$.

Set $A = V_0$. In the sequel $a, b, c$ will denote elements of $A$. Since $a_{(n)}b = 0$ for all $n \geq 0$, (0.5.4) and (0.5.9) imply that $A$ is a commutative associative $k$-algebra with the unity $1$ with respect to the operation $ab$.

Elements of $V_1$ will be denoted $x, y, z$. It follows from (0.5.4), taking into account that $(-1) = (-1)^2$, that

$$(ab)x = abx + \partial a b_{(0)}x + \partial b a_{(0)}x \quad (2.1.2)$$

The identity (0.5.9) implies

$$a_{(0)}x = -x_{(0)}a \quad (2.1.3)$$

and

$$xa = ax + \partial (x_{(0)}a) \quad (2.1.4)$$

Let $\Omega \subset V_1$ be the $k$-submodule generated by all elements $a\partial b$; $\omega$ will denote an element of $\Omega$.

From (2.1.2) we have

$$(ab)\partial c = ab\partial c + \partial a b_{(0)}\partial c + \partial b a_{(0)}\partial c$$
Note that
\[ \partial u(0)v = 0 \quad (2.1.5) \]
for all \( u, v \in V \), due to (0.5.10). Hence \( b_{(0)} \partial c = -\partial c_{(0)} b = 0 \); therefore \( (ab) \partial c = ab \partial c \). On the other hand, \( 1 \partial c = \partial c 1 - \partial c_{(0)} 1 = \partial c 1 = \partial c \). It follows that the operation \( a \omega \) gives a structure of an \( A \)-module on \( \Omega \). (It is not true for \( V_1 \) since \( ax \) is not associative.)

Note that by (0.5.4) \( (a \partial b)_{(0)} c = a \partial b_{(0)} c = 0 \), in other words
\[ \omega_{(0)} a = 0 \quad (2.1.6) \]

It follows from (0.5.9) that
\[ a \omega = \omega a \quad (2.1.7) \]

We have the map \( \partial : A \rightarrow \Omega \). By (0.5.11) and (2.1.7) \( \partial \) is an \( A \)-derivation.

Let \( T \) denote the quotient module \( V_1 / \Omega \); let \( \pi : V_1 \rightarrow T \) be the canonical projection. Elements of \( T \) will be denoted \( \tau \). The operation \( ax \) induces an operation \( A \times T \rightarrow T \), \( (a, \tau) \mapsto a \tau \). On the other hand,
\[
1x = x1 - \partial(x_{(0)} 1) = x1 = x
\]
(2.1.8)
i.e. the operation \( a \tau \) provides \( T \) with a structure of an \( A \)-module.

By (0.5.4)
\[
(a \partial b)_{(0)} x = a \partial b_{(0)} x - a_{(-2)} \partial b_{(1)} x + \partial b(a_{(0)} x)
\]
Note that by (0.5.10) and (2.1.7)
\[
a_{(-2)} b = \partial a b = b \partial a
\]
(2.1.9)
Therefore
\[
(a \partial b)_{(0)} x = (b_{(0)} x) \partial a + (a_{(0)} x) \partial b
\]
whence
\[
\Omega_{(0)} V_1 \subset \Omega
\]
(2.1.10)
On the other hand,
\[
x_{(0)} a \partial b = (x_{(0)} a) \partial b + a x_{(0)} \partial b
\]
by (2.1.1), and
\[
x_{(0)} \partial b = -\partial b_{(0)} x + \partial(\partial b_{(1)} x) = -\partial(b_{(0)} x) = \partial(x_{(0)} b),
\]
i.e.
\[
x_{(0)} (a \partial b) = (x_{(0)} a) \partial b + a \partial(x_{(0)} b)
\]
(2.1.11)
It follows that
\[
V_1_{(0)} \Omega \subset \Omega
\]
(2.1.12)
Therefore the operation \( \partial_{(0)} \) induces an operation \( T \times T \rightarrow T \), to be denoted \( [.] \).

By (0.5.9)
\[
x_{(0)} y = -y_{(0)} x + \partial(y_{(1)} x)
\]
(2.1.13)
hence $[,]$ is skew symmetric. (2.1.1) implies that $[,]$ satisfies the Jacobi identity, hence it provides a structure of a Lie algebra (over $k$) on $T$.

By (0.5.4) and (2.1.5) \(a\partial b)(0)c = a\partial b_0c = 0\), i.e.

\[
\Omega(0)A = 0 \quad (2.1.14)
\]

It follows that the operation $x_{(0)}a$ induces a pairing $T \times A \to A$, to be denoted $\tau(a)$. By (2.1.1), $\tau_1\tau_2(a) = [\tau_1, \tau_2](a) + \tau_2\tau_1(a)$ and $\tau(ab) = \tau(a)b + a\tau(b)$. By (0.5.4), $a\tau(b) = a\tau(b)$. Thus, $T$ is equipped with a structure of a Lie $A$-algebroid.

By (2.1.11) and (2.1.14) \(\Omega(0)\Omega = 0 \quad (2.1.15)\)

It follows that the operation $x_{(0)}\omega$ induces an operation $T \times \Omega \to \Omega$, to be denoted $\tau(\omega)$, which makes $\Omega$ a module over the Lie algebra $T$, by (2.1.1). Again by (2.1.1), $\tau(a\omega) = \tau(a)\omega + a\tau(\omega)$.

It follows from (2.1.11) that

\[
\tau(a\partial b) = \tau(a)\partial b + a\partial\tau(b) \quad (2.1.16)
\]

In particular $\tau(\partial b) = \partial\tau(b)$, i.e. $\partial$ is a morphism of $T$-modules.

By (0.5.9)

\[
x_{(1)}y = y_{(1)}x \quad (2.1.17)
\]

We have

\[
(a\partial b)_{(1)}x = a\partial b_{(1)}x = -a\partial b_0x = ax_0b \quad (2.1.18)
\]

by (0.5.4), (2.1.5), (0.5.10) and (2.1.3). It follows that

\[
\Omega(1)\Omega = 0 \quad (2.1.19)
\]

by (2.1.14). Therefore the operation $x_{(1)}y$ induces an operation $T \times \Omega \to A$, to be denoted $\langle \tau, \omega \rangle$.

By (2.1.17) and (2.1.18)

\[
\langle \tau, a\partial b \rangle = a\tau(b) \quad (2.1.20)
\]

From (0.5.4), $\langle a\tau, \omega \rangle = a\langle \tau, \omega \rangle$. From (2.1.1),

\[
\tau(\langle \tau', \omega \rangle) = \langle [\tau, \tau'], \omega \rangle + \langle \tau', \tau(\omega) \rangle
\]

Finally, again from (0.5.4), \(ax_{(0)}\omega = ax_{(0)}\omega + a_{(-2)}x_{(1)}\omega\) which implies \((a\tau)(\omega) = a\tau(\omega) + \langle \tau, \omega \rangle \partial a\), by (2.1.9).

Therefore we have canonically associated with our vertex algebra an extended Lie $A$-algebroid $\Phi(V) = (T, \Omega, \partial, \langle , \rangle)$.

2.2. Let us assume that the projection $\pi : V_1 \to T$ admits a splitting, i.e. there exists a morphism of $k$-modules $s : T \to V_1$ such that $\pi \circ s = Id_T$. Let us fix such a splitting. A vertex algebra with a chosen $s$ will be called split.
Define a skew symmetric operation $[,] : V_1 \times V_1 \rightarrow V_1$ by

$$[x, y] := \frac{1}{2}(x(y) - y(x))$$  \hspace{1cm} (2.2.1)

We have $\pi([x, y]) = [\pi(x), \pi(y)]$.

Let us use the notation $\langle x, y \rangle$ for $x(y)$. It follows from (0.5.9) that

$$[x, y] = x(y) - \frac{1}{2}\partial(x, y)$$  \hspace{1cm} (2.2.2)

Set

$$\gamma(a, \tau) = s(a\tau) - as(\tau)$$ \hspace{1cm} (2.2.3)_γ

$$\langle \tau_1, \tau_2 \rangle = \langle s(\tau_1), s(\tau_2) \rangle$$ \hspace{1cm} (2.2.3)_c

$$c(\tau_1, \tau_2) = s([\tau_1, \tau_2]) - [s(\tau_1), s(\tau_2)]$$ \hspace{1cm} (2.2.3)_c

2.3. Theorem. The septuple $\mathcal{A} = (A, T, \Omega, \partial, \gamma, \langle \cdot, \cdot \rangle, c)$ is a vertex algebroid.

Proof. We have to check the axioms (A1) — (A5) from 1.4.

Check of (A1). We have $\gamma(ab, \tau) = s(ab\tau) - (ab)s(\tau) = s(ab\tau) - as(b\tau) + as(b\tau) - (ab)s(\tau)$. By (2.1.2) and (2.1.3),

$$\gamma(ab, \tau) = ab\gamma(\tau) + \partial a \gamma(b) + \partial b \gamma(a)$$

which implies (A1).

Check of (A2). We have $\langle a\tau_1, \tau_2 \rangle = \langle s(a\tau_1), s(\tau_2) \rangle = \langle as(a\tau_1), \gamma(a, \tau_1), s(\tau_2) \rangle$.

By (0.5.4),

$$\langle ax, y \rangle = a\langle x, y \rangle - x(y)x(0)a$$ \hspace{1cm} (2.3.1)

which implies (A2).

Check of (A3). We have $c(a\tau_1, \tau_2) = s([a\tau_1, \tau_2]) - [s(a\tau_1), s(\tau_2)]$;

$s([a\tau_1, \tau_2]) = s(a[\tau_1, \tau_2] - \tau_2(a)\tau_1) = as([\tau_1, \tau_2]) + \gamma(a, [\tau_1, \tau_2]) - \tau_2(a)s(\tau_1) - \gamma(\tau_2(a), \tau_1)$;

$[s(a\tau_1), s(\tau_2)] = [s(a\tau_1) + \gamma(a, \tau_1), s(\tau_2)]$.

We have $[ax, y] = (ax)(0)y - \frac{1}{2}\partial(ax, y)$ (see (2.2.2)). By (0.5.4),

$$[ax, y] = ax(y) + a(-2)x(1)y + xa(0)y =$$

$$= ax(y) + \langle x, y \rangle \partial a - \pi(y)(a)x - \partial \pi(x)\pi(y)(a)$$ \hspace{1cm} (2.3.2)

By (2.3.1)

$$\frac{1}{2}\partial(ax, y) = -\frac{1}{2}\partial\{a(x, y) - x(0)y(0)a\} = -\frac{1}{2}(x, y)\partial a - \frac{1}{2}a\partial(x, y) + \frac{1}{2}\partial \pi(x)\pi(y)(a)$$

Therefore

$$[ax, y] = a[x, y] + \frac{1}{2}(x, y)\partial a - \frac{1}{2}\partial \pi(x)\pi(y)(a) - \pi(y)(a)x$$
On the other hand,
\[ [\omega, x] = -[x, \omega] = -\pi(x)\omega + \frac{1}{2} \partial(x, \omega) \]

It follows that
\[ [s(a\tau_1), s(\tau_2)] = a[s(\tau_1), s(\tau_2)] + \frac{1}{2} \langle \tau_1, \tau_2 \rangle \partial a - \frac{1}{2} \partial \tau_1 \tau_2(a) - \tau_2(a)s(\tau_1) - \tau_2(\gamma(a, \tau_1)) + + \frac{1}{2} \partial \gamma(a, \tau_1), \tau_2 \rangle. \]

This implies (A3).

**Check of (A4).** We shall make use of the formula
\[ s(\tau_1)(o)s(\tau_2), s(\tau_3)) = (s(\tau_1)(o)s(\tau_2), s(\tau_3)) + (s(\tau_2), s(\tau_1)(o)s(\tau_3)) \quad (2.3.3) \]

which is a particular case of (2.1.1). We have \( s(\tau_1)(o)s(\tau_2), s(\tau_3)) = \tau_1(\langle \tau_2, \tau_3 \rangle) \). On the other hand,
\[ s(\tau_1)(o)s(\tau_2) = [s(\tau_1), s(\tau_2)] + \frac{1}{2} \partial \langle \tau_1, \tau_2 \rangle = \langle [\tau_1, \tau_2] \rangle = c(\tau_1, \tau_2) + \frac{1}{2} \partial \langle \tau_1, \tau_2 \rangle \quad (2.3.4) \]

Therefore
\[ (s(\tau_1)(o)s(\tau_2), s(\tau_3)) = \langle [\tau_1, \tau_2], \tau_3 \rangle - (c(\tau_1, \tau_2), \tau_3) + \frac{1}{2} \tau_3(\langle \tau_1, \tau_2 \rangle) \]

Interchanging \( \tau_2 \) with \( \tau_3 \) and plugging these expressions into (2.3.3) we get (A4).

**Check of (A5).** We shall use the formulas
\[ s(\tau_1)(o)s(\tau_2)(o)s(\tau_3) = (s(\tau_1)(o)s(\tau_2))s(\tau_3) + s(\tau_2)(o)s(\tau_1)(o)s(\tau_3) \]

and (2.3.4). We get
\[ s(\tau_1)(o)s(\tau_2)(o)s(\tau_3) = s(\tau_1)(o)\left\{ s(\tau_2, \tau_3)] - c(\tau_2, \tau_3) + \frac{1}{2} \partial \langle \tau_2, \tau_3 \rangle \right\} = \]
\[ = s(\tau_1)([\tau_2, \tau_3]) - c(\tau_1, [\tau_2, \tau_3]) + \frac{1}{2} \partial \tau_1(\tau_2, \tau_3) - \tau_1(c(\tau_2, \tau_3)) + \frac{1}{2} \partial \tau_1(\tau_2, \tau_3) \]

Interchanging \( \tau_1 \) with \( \tau_2 \) we get
\[ s(\tau_2)(o)s(\tau_1)(o)s(\tau_3) = s([\tau_2, [\tau_1, \tau_3]]) - c(\tau_2, [\tau_1, \tau_3]) + \frac{1}{2} \partial \tau_2([\tau_1, \tau_3]) - \tau_2(c(\tau_1, \tau_3)) + \frac{1}{2} \partial \tau_2([\tau_1, \tau_3]) \]

Similarly,
\[ (s(\tau_1)(o)s(\tau_2))s(\tau_3) = \left\{ s([\tau_1, \tau_2]) - c(\tau_1, \tau_2) + \frac{1}{2} \partial \tau_1(\tau_2) \right\}o s(\tau_3) = \]
\[ = s([\tau_1, \tau_2], \tau_3)] - c([\tau_1, \tau_2], \tau_3) + \frac{1}{2} \partial \tau_3([\tau_1, \tau_2], \tau_3) + \tau_3(c(\tau_1, \tau_2)) - \partial \tau_3(c(\tau_1, \tau_2)) \]
where we have used the formula
\[
\omega(0)x = -\pi(x)(\omega) + \partial\langle \pi(x), \omega \rangle
\] (2.3.5)
which is a consequence of (0.5.9). The axiom (A5) follows. This completes the proof of the theorem. \(\triangle\)

2.4. Thus, to a pair \((V, s)\), where \(V\) is a vertex algebra and \(s : T \rightarrow V_1\) is a splitting on \(V\), we have assigned a vertex algebroid \(\mathcal{A}(V, s) = (A, T, \Omega, \partial, \gamma, \langle , \rangle, c)\). Note that by definition \(\mathcal{A}(V, s)\) has the property

(Sur) The module \(\Omega\) is generated as an \(A\)-module by the subspace \(\partial A\). Equivalently, the morphism \(\Omega_{A/k} \rightarrow \Omega\) induced by \(\partial\) is epimorphic.

§3. Category of Vertex Algebroids

3.1. Let us define a 1-truncated vertex algebra to be a septuple \(v = (V_0, V_1, 1, a_{(-1)}(0), a_{(1)}(0))\) where \(V_0, V_1\) are two \(k\)-modules, \(1\) an element of \(V_0\) \((\text{vacuum vector})\), \(\partial : V_0 \rightarrow V_1\) a morphism of \(k\)-modules,

\[
\langle i \rangle : (V_0 \oplus V_1) \times (V_0 \oplus V_1) \rightarrow V_0 \oplus V_1 \ (i = -1, 0, 1)
\]

are \(k\)-bilinear operations of degree \(-i - 1\). Elements of \(V_0\) \((\text{resp., } V_1)\) will be denoted \(a, b, c\) \((\text{resp., } x, y, z)\). So, we have 7 operations: \(a_{(-1)}b, a_{(-1)}x, x_{(-1)}a, a(0)x, x(0)a, x(0)y\) and \(x(1)y\). The following axioms must be satisfied:

(Vacuum)
\[
a_{(-1)}1 = a; \ x_{(-1)}1 = x; \ a(0)1 = 0
\] (Vac)

(Derivation)
\[
(\partial a)(0)b = 0; \ (\partial a)(0)x = 0; \ (\partial a)(1)x = -a(0)x \quad \text{(Der)}_1
\]
\[
\partial(a_{(-1)}b) = (\partial a)(-1)b + a_{(-1)}\partial b; \ \partial(x(0)a) = x(0)\partial a \quad \text{(Der)}_2
\]

(Commutativity)
\[
a_{(-1)}b = b_{(-1)}a; \ a_{(-1)}x = x_{(-1)}a - \partial(x(0)a) \quad \text{(Com)}_{-1}
\]
\[
x(0)a = -a(0)x; \ x(0)y = -y(0)x + \partial(y(1)x) \quad \text{(Com)}_0
\]
\[
x(1)y = y(1)x \quad \text{(Com)}_1
\]

(Associativity)
\[
(a_{(-1)}b)(-1)c = a_{(-1)}b(-1)c \quad \text{(Ass)}_{-1}
\]

Operation \((0)\) is a derivation with respect to all operations \((i)\), i.e.
\[
\alpha(0)\beta(i)\gamma = (\alpha(0)\beta)(i)\gamma + \beta(i)\alpha(0)\gamma, \ (\alpha, \beta, \gamma \in V_0 \oplus V_1)
\] (Ass)_0
whenever the both sides are defined.

\[(a_{(-1)}x)(0)b = a_{(-1)}x(0)b\quad(Ass)_1\]
\[(a_{(-1)}b)(-1)x = a_{(-1)}b(-1)x + (\partial a)(-1)b(0)x + (\partial b)(-1)a(0)x\quad(Ass)_2\]
\[(a_{(-1)}x)(1)y = a_{(-1)}x(1)y - x(0)y(0)a\quad(Ass)_3\]

A morphism between two 1-truncated vertex algebras \(f : v = (V_0, V_1, \ldots) \rightarrow v' = (V'_0, V'_1, \ldots)\) is a pair of maps of \(k\)-modules \(f = (f_0, f_1), f_i : V_i \rightarrow V'_i\) such that \(f_0(1) = 1', f_1(\partial a) = \partial f_0(a)\) and \(f(a_{(i)}\beta) = f(a(i))f(\beta)\), whenever both sides are defined.

This way we get a category \(\mathcal{V}ert_{\leq 1}\) of 1-truncated vertex algebras. We have an obvious truncation functor

\[t : \mathcal{V}ert \rightarrow \mathcal{V}ert_{\leq 1}\]

which assigns to a vertex algebra \(V = \oplus V_i\) the truncated algebra \(tV := (V_0, V_1, \ldots)\). In Section 9 below we shall construct a left adjoint to \(t\).

3.2. Let \(v = (V_0, V_1, \ldots)\) be a 1-truncated vertex algebra. Let \(\Omega_v \subset V_1\) be the \(k\)-submodule generated by all elements \(a_{(-1)}\partial b\). Set \(T_v = V_1/\Omega_v\); let \(\pi : V_1 \rightarrow T_v\) be the canonical projection.

Let us call \(v\) splittable if \(\pi\) admits a \(k\)-linear splitting \(s : T \rightarrow V_1\), cf. 2.2. This is of course a weak condition; it holds true for example if \(T\) is a projective \(k\)-module. Let \(\mathcal{V}ert_{\leq 1}' \subset \mathcal{V}ert_{\leq 1}\) denote the full subcategory of splittable 1-truncated vertex algebras.

A 1-truncated vertex algebra with a chosen splitting \(s\) will be called split. The argument of Section 2 assigns a vertex algebroid \(\mathcal{A}(v; s)\) to every split 1-truncated vertex algebra \((v; s)\).

We have by definition

\[\mathcal{A}(V; s) = \mathcal{A}(tV; s)\]

for every split vertex algebra \((V; s)\).

3.3. Conversely, let \(\mathcal{A} = (A, T, \Omega, \ldots)\) be a vertex algebroid. We want to assign to it a split 1-truncated vertex algebra. To this end one needs simply to invert the construction of the previous Section.

Namely \(V_0 = A, V_1 = T \oplus \Omega\); let \(\partial : V_0 \rightarrow V_1\) be the composition of \(\partial : A \rightarrow \Omega\) with the obvious embedding \(\Omega \subset V_1\); let \(s : T \rightarrow V_1\) be the obvious embedding.

Let us define the operations \((i)\) as follows:

\[a_{(-1)}b = ab; \quad a_{(-1)}\omega = a\omega; \quad a_{(-1)}\tau = a\tau - \gamma(a, \tau)\quad(3.3.1)\]
\[a_{(0)}b = a_{(0)}\omega = \omega_{(0)}' = 0; \quad \tau_{(0)}a = \tau(a); \quad \tau_{(0)}\omega = \tau(\omega)\quad(3.3.2)\]
\[\tau_{(0)}\tau' = [\tau, \tau'] - c(\tau, \tau') + \frac{1}{2}\partial(\tau, \tau')\quad(3.3.3)\]
By inverting the argument of the previous Section, one sees easily that we get a split 1-truncated vertex algebra \((V_0, V_1, \ldots)\), to be denoted by \(uA\).

Let \(v \in \mathcal{V}_{\leq 1}\). For each splitting \(s\) of \(v\) we have by construction a canonical isomorphism

\[
v \sim \rightarrow uA(v; s)
\]

(3.3.5)

which is in fact an equivalence of categories, due to (3.3.5).

Let \(\mathcal{A} = (A, T, \Omega, \ldots)\) and \(\mathcal{A}' = (A', T', \Omega', \ldots)\) be two vertex algebroids. We want to describe explicitly the set of morphisms \(\text{Hom}_{\mathcal{A}lg}(\mathcal{A}, \mathcal{A}')\) which consists of all morphisms \(g : uA \rightarrow uA'\) such that \(g(\Omega) \subset \Omega'\). This way we get a category of vertex algebroids, to be denoted \(\mathcal{A}lg\). The mapping \(u\) induces a functor

\[
u : \mathcal{A}lg \rightarrow \mathcal{V}_{\leq 1}
\]

(3.3.6)

3.4. Let \(\mathcal{A} = (A, T, \Omega, \ldots)\) and \(\mathcal{A}' = (A', T', \Omega', \ldots)\) be two vertex algebroids. We want to describe explicitly the set of morphisms \(\text{Hom}_{\mathcal{Alg}}(\mathcal{A}, \mathcal{A}')\subset \text{Hom}_{\mathcal{A}lg}(uA, uA')\).

Let \(g : uA \rightarrow uA'\) be a morphism belonging to \(\text{Hom}_{\mathcal{Alg}}(\mathcal{A}, \mathcal{A}')\). Let \(g_A : A \rightarrow A'\) be its component of weight 0. Since \(g\) preserves the operation \((-1)\) and the vacuum, \(g_A\) is a morphism of commutative k-algebras. Let \(g_1 : T \otimes \Omega \rightarrow T' \otimes \Omega'\) be its component of weight 1. We have \(\partial \circ g = g \circ \partial\); therefore \(g_1(\partial A) \subset \partial A'\). By definition, \(g_1(\Omega) \subset \Omega'\). Let us denote by \(g_1 : \Omega \rightarrow \Omega'\) and \(g_T : T = UA/\Omega \rightarrow U'A'/\Omega'\) the morphisms induced by \(g_1\). Thus, in components \(g_1\) has the form

\[
g_1(\tau, \omega) = (g_T(\tau), g_1(\omega) + h(\tau))
\]

(3.4.1)

where \(h : T \rightarrow \Omega'\) is some k-linear mapping.

3.5. Theorem. The correspondence described in 3.4 provides a canonical isomorphism of the set \(\text{Hom}_{\mathcal{Alg}}(\mathcal{A}, \mathcal{A}')\) with the set of all quadruples \((g_A, g_T, g_1, h)\) where

\[
\begin{align*}
(3.5A) &\ g_A : A \rightarrow A' \text{ is a morphism of k-algebras;} \\
(3.5\Omega) &\ g_1 : \Omega \rightarrow \Omega' \text{ is a morphism of k-modules such that } g_1(\partial a) = \partial g_A(a) \\
&\ g_1(a\omega) = g_A(a)g_1(\omega); \\
(3.5T) &\ g_T : T \rightarrow T' \text{ is a morphism of Lie k-algebras such that } g_T(\tau a) = g_A(a)g_T(\tau), g_A(\tau(a)) = g_T(\tau)(g_A(a)), g_1(\tau(\omega)) = g_T(\tau)(g_1(\omega)) \text{ and } g_A(\tau, \omega) = \langle g_T(\tau), g_1(\omega) \rangle; \\
(3.5\gamma) &\ h : T \rightarrow \Omega' \text{ is a morphism of k-modules such that } h(\tau a) = g_A(a)h(\tau) - \gamma'(g_A(a), g_T(\tau)) + g_1(\gamma(\tau, \omega)); \\
\end{align*}
\]

(3.5(`)`)

\[
\begin{align*}
(3.5c) &\ g_A(c(\tau_1, \tau_2)) = c'(g_T(\tau_1), g_T(\tau_2)) + \langle g_T(\tau_1), h(\tau_2) \rangle + \langle g_T(\tau_2), h(\tau_1) \rangle; \\
(3.5\alpha) &\ g_1(\tau_1, \tau_2) = c'(g_T(\tau_1), g_T(\tau_2)) + \frac{1}{2}\partial(g_T(\tau_1), h(\tau_2)) - \frac{1}{2}\partial(g_T(\tau_2), h(\tau_1)) - g_T(\tau_1)(h(\tau_2)) + g_T(\tau_2)(h(\tau_1)) + h([\tau_1, \tau_2]).
\end{align*}
\]
3.6. Proof. By definition, a quadruple \( g = (g_A, g_T, g_{\Omega}, h) \) defines a morphism \( uA \rightarrow uA' \) if it satisfies the identities (3.6.1) through (3.6.6) below (\( a, b \in A, \ x, y \in uA_1 \)): 

\[
\begin{align*}
g_0(1) &= 1: \ g_0(a_{(-1)}b) = g_0(a)_{(-1)}g_0(b) \\
g_1(\partial a) &= \partial g_0(a) \\
g_1(a_{(-1)}x) &= g_0(a)_{(-1)}g_0(x) \\
g_1(x_{(0)}a) &= g_1(x)_{(0)}g_0(a) \\
g_1(x_{(0)}y) &= g_1(x)_{(0)}g_1(y) \\
g_0(x_{(1)}y) &= g_1(x)_{(1)}g_1(y)
\end{align*} \tag{3.6.1, 3.6.2, 3.6.3, 3.6.4, 3.6.5, 3.6.6}
\]

The condition (3.6.1) is equivalent to (3.5A); (3.6.2) and (3.6.3) for \( x = \omega \) is equivalent to (3.5Ω). Let us write down (3.6.3) for \( x = \tau \). We have

\[
a_{(-1)}\tau = (a\tau, -\gamma(a, \tau))
\]

(3.6.7)

Therefore \( g_1(a_{(-1)}\tau) = (g(a\tau), h(a\tau) - g(\gamma(a, \tau)) \) and

\[
g_1(a)_{(-1)}g_1(\tau) = g_1(a)_{(-1)}(g(\tau), h(\tau)) = (g(a)g(\tau), -\gamma(g(a), g(\tau)) + g(a)h(\tau))
\]

It follows that (3.6.3) for \( x = \tau \) is equivalent to \( g(a\tau) = g(a)g(\tau) \) and (3.6.4) for \( x = \omega \) is vacuous. (3.6.4) for \( x = \tau \) is equivalent to \( g(a\tau) = g(a)g(\tau) \).

(3.6.6) for \( x, y \in \Omega \) is vacuous. (3.6.6) for \( x = \omega, y = \tau \) (or vice versa) is equivalent to \( g((\tau, \omega)) = (g(\tau), g(\omega)) \). We have

\[
g_1(\tau_1)_{(1)}g_1(\tau_2) = (g(\tau_1, g(\tau_2))^\tau + (g(\tau_1), h(\tau_2)) + (g(\tau_2), h(\tau_1))
\]

Therefore (3.6.6) for \( x, y \in T \) is equivalent to (3.5(,,)).

(3.6.5) for \( x = \tau, y = \omega \) (or vice versa) is equivalent to \( g(\tau(\omega)) = g(\tau)(g(\omega)) \). Finally, let us write down (3.6.5) for \( x, y \in T \). We have

\[
\tau_{1(0)}\tau_2 = ([\tau_1, \tau_2], -c(\tau_1, \tau_2) + \frac{1}{2}\partial(\tau_1, \tau_2)) \tag{3.6.8}
\]

cf. (2.2.2) and (2.2.3). Therefore

\[
g_1(\tau_{1(0)}\tau_2) = (g([\tau_1, \tau_2], h([\tau_1, \tau_2])) - g(c(\tau_1, \tau_2) + \frac{1}{2}\partial(g(\tau_1), g(\tau_2)) +
\]

\[
+ \frac{1}{2}\partial(g(\tau_1), h(\tau_2)) + \frac{1}{2}\partial(g(\tau_2), h(\tau_1))
\]

where we have used (3.5(,,)). On the other hand,

\[
g_1(\tau_{1(0)}g_1(\tau_2)) = ([g(\tau_1), g(\tau_2)], -c'(g(\tau_1), g(\tau_2)) + \frac{1}{2}\partial(g(\tau_1), g(\tau_2)) +
\]

\[
+ g(\tau_1)(h(\tau_2)) - g(\tau_2)(h(\tau_1)) + \partial(h(\tau_1), g(\tau_2))
\]
where we have used the formula
\[ \omega(\omega)\tau = -\tau(\omega) + \partial(\tau, \omega) \] (3.6.9)
following from \((0.5.9)\) (cf. \((2.3.5)\)). Therefore \((3.6.5)\) for \(x, y \in T\) is equivalent to the requirement that \(g_T\) is a morphism of Lie algebras, and to \((3.5c)\). The theorem is proved. \(\triangle\)

3.7. Let \((A, T, \ldots) \xrightarrow{g} (A', T', \ldots) \xrightarrow{g'} (A'', T'', \ldots)\) be two morphisms of vertex algebroids. Then their composition is obviously equal to
\[ g' \circ g = (g'_A g_A, g'_T g_T, g'_\Omega g_\Omega, g'h + h'g_T) \] (3.7.1)
cf. \((3.4.1)\). The identity morphisms are
\[ \text{Id}_A = (\text{Id}_A, \text{Id}_T, \text{Id}_\Omega, 0) \] (3.7.2)

3.8. Theorem. (Extension of Mappings.) In the conditions of \((3.5)\), let \(T_0 \subset T\) be a \(k\)-submodule which generates \(T\) as an \(A\)-module. Let \(g = (g_A, g_T, g_\Omega, h)\) be such a quadruple that \((3.5A), (3.5\Omega)\) and \((3.5T)\) are fulfilled and such that \((3.5\gamma), (3.5(\gamma))\) and \((3.5c)\) are fulfilled for all \(\tau, \tau_i \in T_0\).

Then \((3.5\gamma), (3.5(\gamma))\) and \((3.5c)\) hold true for all \(\tau, \tau_i \in T\), i.e. \(g\) defines a morphism \(A \longrightarrow A'\).

Cf. Theorem 1.9.

Proof. Let us prove that if \((3.5\gamma)\) is true \((a, \tau)\) with a fixed \(\tau\) and all \(a\), then it is true for all couples \((a, b\tau)\). We have to prove that
\[ h(ab \tau) = g(a)h(b \tau) - \gamma(g(a), g(b \tau)) + g(\gamma(a, b \tau)) \]
(?)
The left hand side is equal to
\[ h(ab \tau) = g(ab)h(\tau) - \gamma(g(ab), g(\tau)) + g(\gamma(ab, \tau)) = \\
= g(ab)h(\tau) - g(a)\gamma(g(b), g(\tau)) - g(\tau)(g(a))\partial g(b) - g(\tau)(g(b))\partial g(a) + \\
+ g(\gamma(a, b \tau)) + g(a)\gamma(b, \tau) + g(\tau(a)\partial b) + g(\tau(b))\partial a \]
where we have used \((A1)\). On the other hand, the first summand in the rhs is equal to
\[ g(a)h(b \tau) = g(a)g(b)h(\tau) - g(a)\gamma(g(b), g(\tau)) + g(a)g(\gamma(b, \tau)), \]
and we easily see that we have the required identity indeed.

The similar claims connected with the equations \((3.5(\gamma))\) and \((3.5c)\) are proved analogously, and we leave them to the reader. \(\triangle\)
§4. Cofibered Structure

4.1. Let us introduce objects which lie "in between" extended Lie algebroids and vertex algebroids. Let us define a vertex prealgebroid to be a sextuple $B = (A, T, \Omega, \partial, \gamma, \langle \cdot, \cdot \rangle)$ where $A, T, \Omega, \partial, \gamma, \langle \cdot, \cdot \rangle$ are as in 1.4.

The data $(A, T, \Omega, \langle \cdot, \cdot \rangle|_{T \times \Omega})$ should form an extended Lie algebroids and the axioms 1.4 (A1) and (A2) must be satisfied.

Let us define a morphism between two vertex prealgebroids $B \in \mathcal{P}$ by the rules (3.7.1), (3.7.2). This way we get a category $\mathcal{B}$ with the properties (3.7.2). In other words, $\mathcal{B}$ is a groupoid. Therefore, $\alpha \in \mathcal{B}$ is invertible, namely $\alpha^{-1} = (Id_A, Id_T, Id_\Omega, -h)$, cf. (3.7.1), (3.7.2). In other words, $\mathcal{B}$ is a groupoid.

Consider the de Rham-Chevalley complex of $T$:

$$\Omega (T) : 0 \longrightarrow A \longrightarrow \Omega \frac{dA}{\partial A} \Omega^2 \longrightarrow \ldots$$

cf. 1.3. Let $A_i = (\ldots, c_i) \in \text{Ob} \overline{\text{Alg}}_B$, $i = 1, 2$. Set $\alpha = c_1 - c_2$. It follows from (A4) that

$$\langle \tau_2, \alpha (\tau_1, \tau_0) \rangle + \langle \tau_3, \alpha (\tau_1, \tau_2) \rangle = 0,$$

e i.e. $\alpha \in \Omega^3$. It follows from (A5) that $\alpha \in \Omega^{3,cl} := \text{Ker}(\Omega^3 \longrightarrow \Omega^4)$, cf. (1.3.2).

Conversely, if $A = (\ldots, c) \in \text{Ob} \overline{\text{Alg}}_B$ and $\alpha \in \Omega^{3,cl}$ then $A + \alpha := (\ldots, c + \alpha) \in \text{Ob} \overline{\text{Alg}}_B$. We have proven that

4.2.1. The set $\text{Ob} \overline{\text{Alg}}_B$ is canonically an $\Omega^{3,cl}$-torsor.

Let $A \in \text{Ob} \overline{\text{Alg}}_B$, $A' = A + \alpha$, $\alpha \in \Omega^{3,cl}$, $g = (Id_A, Id_T, Id_\Omega, h) \in \text{Hom} \overline{\text{Alg}}_B (A, A')$. Then, due to (3.5γ), $h$ must be $A$-linear; by (3.5⟨,⟩), $h \in \Omega^2$ and by (3.5c), $d_{DR} h = -\alpha$. Therefore,
4.2.2. the set $\text{Hom}_{\text{Alg}}(A, A + \alpha)$ may be canonically identified with the set of $h \in \Omega^2$ such that $d_{DB}h = -\alpha$.

Consequently, the set of isomorphism classes of objects of the groupoid $\text{Alg}_B$ is equal to the third de Rham cohomology $H^3(\Omega(T))$.

4.3. Let $T = (A, T, \ldots), T' = (A', T', \ldots) \in \text{LieAlg}$, $g = (g_A, g_T, g)$ $\in$ $\text{Hom}_{\text{LieAlg}}(T', T)$. Let $B \in \text{PreAlg}_T$, $B' \in \text{PreAlg}_{T'}$.

Let us consider the set $\text{Hom}_g(B', B) \subset \text{Hom}_{\text{PreAlg}}(B', B)$ consisting of all morphisms $\tilde{g}$ such that $Q(\tilde{g}) = g$. Let $\tilde{g} = (g_A, g_T, g, h), \tilde{g}' = (g_A, g_T, g, h')$ $\in$ $\text{Hom}_g(B', B)$. Set $\beta := h - h': T \longrightarrow \Omega'$. Due to (3.5γ) and (3.5(1)), $\beta$ will satisfy the properties

$$\beta(\alpha) = g_A(a)\beta(\tau) \quad (4.3.1)$$

and

$$(g_T(\tau_1), \beta(\tau_2)) + (g_T(\tau_2), \beta(\tau_1)) = 0 \quad (4.3.2)$$

Let us denote by $\Omega^2_3(T)$ the set of all maps $\beta : T' \longrightarrow \Omega$ satisfying (4.3.1) and (4.3.2). It is an $A$-module in the obvious way, and consequently an $A'$-module, by restriction of scalars. In particular, $\Omega^2_{id_T} = \Omega^2(T)$.

Vice versa, for each $\beta \in \Omega^2_g$, the map $\tilde{g}' + \beta := (g_A, g_T, g, h + \beta)$ belongs to $\text{Hom}_g(B, B')$. Thus we have proven that

4.3.1. the set $\text{Hom}_g(B', B)$ is canonically an $\Omega^2_g$-torseur. In particular, if $T = T'$ then $\text{Hom}_{\text{PreAlg}}(B', B)$ is an $\Omega^2(T)$-torseur.

Similarly, let $A \in \text{Alg}_T$, $A' \in \text{Alg}_{T'}$. Let $\text{Hom}_g(A', A)$ be the subset of $\text{Hom}_{\text{Alg}}(A', A)$ consisting of all morphisms $\tilde{g}$ such that $Q(\tilde{g}) = g$. Let $\Omega^2_A$ be the $k$-module of all maps $\beta : T' \longrightarrow \Omega$ satisfying (4.3.1), (4.3.2) and

$$-\beta([\tau_1, \tau_2]) + g_T(\tau_1)(h(\tau_2)) - g_T(\tau_2)(h(\tau_1)) - \partial(g_T(\tau_1), h(\tau_2)) = 0 \quad (4.3.3)$$

In particular, $\Omega^2_{id_T} = \Omega^2(T)$, cf. (1.3.4). As above, due to (3.5c), we get

4.3.2. the set $\text{Hom}_g(A', A)$ is canonically an $\Omega^2_{g, cl}$-torseur. In particular, if $T = T'$ then $\text{Hom}_{\text{Alg}}(A', A)$ is an $\Omega^2_{\text{Alg}}(T)$-torseur.

If $g' : T'' \longrightarrow T'$ is another morphism of extended Lie algebroids, we have a composition map

$$\nu_{g,g'} : \Omega^2_g \times \Omega^2_{g'} \longrightarrow \Omega^2_{gg'}, \quad \nu_{g,g'}(\beta, \beta') := g_A\beta' + \beta g_T \quad (4.3.4)$$

It is a morphism of $A'$-modules. We have $\nu_{g,g'}(\Omega^2_{g, cl} \times \Omega^2_{g', cl}) \subset \Omega^2_{gg'}$ (cf. Remark 4.4 below). The maps (4.3.4) are associative in the obvious sense (with respect to triples of composable morphisms in $\text{LieAlg}$).

On the other hand, given $B'' \in \text{PreAlg}_{T''}$, we have the composition

$$\text{Hom}_g(B', B) \times \text{Hom}_g(B'', B') \longrightarrow \text{Hom}_{gg'}(B'', B) \quad (4.3.5)$$

4.3.3. The map (4.3.5) is compatible with (4.3.4). Therefore, we have a canonical isomorphism of $\Omega^2_{gg'}$-torseurs

$$\rho_{g,g'} : \text{Hom}_{gg'}(B'', B) \xrightarrow{\sim} \nu_{g,g'}(\text{Hom}_g(B', B) \times \text{Hom}_g(B'', B)) \quad (4.3.6)$$
The isomorphisms $\rho_{g,g'}$ satisfy obvious 2-cocycle equations connected with triples of composable morphisms $(g, g', g'')$.

This follows from (3.7.1).

4.4. Remark. Generalizing 1.3, one can define the de Rham-Chevalley complexes $\Omega_g$ such that the modules $\Omega^2_g$ and $\Omega^{2,cd}_g$ from the previous no. become really the module of two-forms and closed ones respectively. These complexes come equipped with the composition maps $\Omega^2_g \times \Omega^2_{g'} \rightarrow \Omega^2_{gg'}$ which are compatible with the de Rham differentials and satisfy associativity.

We leave the necessary definitions as an exercise to the reader.

4.5. Theorem. Let $g = (g_A, g_T, g_\Omega, h) : B \rightarrow B'$ be a morphism between vertex prealgebroids such that $g_T$ is an isomorphism. Let $\Delta \in A$. There exists a unique pair $(g_A, \tilde{g})$ where $g_A, \Delta \in A$ and $\tilde{g} : A \rightarrow g_A, A$ is a morphism of vertex algebroids such that $P(\tilde{g}) = \tilde{g}$.

Proof. Uniqueness. Let $A = (A, T, \Omega, \partial, \gamma, \langle \cdot, \cdot \rangle, c)$; let $\Delta' = (A', T', \Omega', \partial', \gamma', c') \in A$. A morphism $g : A \rightarrow A'$ such that $P(g) = g$ must of course be represented by the same quadruple as $g$, i.e. $g = (g_A, g_T, g_\Omega, h)$. It is a morphism of vertex algebroids if the condition (3.5c) is fulfilled, i.e. iff

$$c'(g_T(\tau_1), g_T(\tau_2)) = g_\Omega(c(\tau_1, \tau_2)) - \frac{1}{2} \partial'(g_T(\tau_1), h(\tau_2))' + \frac{1}{2} \partial'(g_T(\tau_2), h(\tau_1))' +$$

$$+ g_T(\tau_1)(h(\tau_2)) - g_T(\tau_2)(h(\tau_1)) - h([\tau_1, \tau_2])$$

(4.5.1)

This equation defines $c'$ uniquely since $g_T$ is an isomorphism by assumption.

Existence. We have to check that the map $c'$ defined by (4.5.1) satisfies axioms (A3), (A4), (A5). Let us check (A4) for example.

To unburden the notation, let us assume that $Q(B) = Q(B')$ and $g = (Id_A, Id_T, Id_\Omega, h)$ (the general case being treated by an identical computation). We have to prove that

$$\langle \tau_2, c'(\tau_1, \tau_3) \rangle + \langle \tau_3, c'(\tau_1, \tau_2) \rangle =$$

$$= \langle [\tau_1, \tau_2], \tau_3 \rangle + \langle [\tau_1, \tau_3], \tau_2 \rangle - \tau_1(\tau_2, \tau_3) + \frac{1}{2} \tau_2(\tau_1, \tau_3) + \frac{1}{2} \tau_3(\tau_1, \tau_2)$$

(?)

We have

$$\langle \tau_2, c'(\tau_1, \tau_3) \rangle = \langle \tau_2, c(\tau_1, \tau_3) - \frac{1}{2} \partial(\tau_1, h(\tau_3)) + \frac{1}{2} \partial(\tau_3, h(\tau_1)) + \tau_1(h(\tau_3)) -$$

$$- \tau_3(h(\tau_1)) - h([\tau_1, \tau_3]) = \langle \tau_2, c(\tau_1, \tau_3) - \frac{1}{2} \tau_2([\tau_1, h(\tau_3)]) + \frac{1}{2} \tau_2([\tau_3, h(\tau_1)]) +$$

$$+ \tau_1([\tau_2, h(\tau_3)]) - [\tau_1, \tau_2], h(\tau_3)) - \tau_3([\tau_2, h(\tau_1)]) - [\tau_3, \tau_2], h(\tau_1)) - [\tau_3, \tau_2], h(\tau_1)) = \langle \tau_2, h([\tau_1, \tau_3])$$

Similarly, interchanging $\tau_2$ with $\tau_3$,

$$\langle \tau_3, c'(\tau_1, \tau_2) \rangle = \langle \tau_3, c(\tau_1, \tau_2) - \frac{1}{2} \tau_3([\tau_1, h(\tau_2)]) + \frac{1}{2} \tau_3([\tau_2, h(\tau_1)]) +$$
and then extending the operations \( \gamma, \) algebroid (A1) — (A3). By Theorem 1.9 we get a vertex algebroid, to be denoted \( \tau, \tau \). Since \\

\[
\langle \tau_1, \tau_2 \rangle = \langle \tau_1, \tau_2 \rangle - \langle \tau_1, h(\tau_2) \rangle - \langle \tau_2, h(\tau_1) \rangle - \langle \tau_3, h(\tau_1, \tau_2) \rangle
\]

By assumption,

\[
\langle \tau_2, c(\tau_1, \tau_3) \rangle + \langle \tau_3, c(\tau_1, \tau_2) \rangle =
\]

\[
= \langle [\tau_1, \tau_2], \tau_3 \rangle + \langle [\tau_1, \tau_3], \tau_2 \rangle - \tau_1(\langle \tau_2, \tau_3 \rangle) + \frac{1}{2} \tau_2(\langle \tau_1, \tau_3 \rangle) + \frac{1}{2} \tau_3(\langle \tau_1, \tau_2 \rangle)
\]

Using this and the axiom (3.5(\( \langle, \rangle \))) which takes the form

\[
\langle \tau_1, \tau_2 \rangle' = \langle \tau_1, \tau_2 \rangle - \langle \tau_1, h(\tau_2) \rangle - \langle \tau_2, h(\tau_1) \rangle
\]

we get the required identity (?).

The other two axioms, (A3) and (A5), are checked in a similar manner, and we leave them to the reader. It is convenient to check (A5) in its equivalent form (A5)\( ^{fr} \), see 1.6. \( \triangle \)

\section{5. Chern-Simons}

5.1. Let us define a frame of a Lie \( A \)-algebroid \( T \) to be a \( k \)-submodule \( g \subset T \) such that \( A \otimes_k g = T \). For example, if \( T \) is a free \( A \)-module and \( \{ \tau_i \} \) is some \( A \)-base of \( T \) then \( g(\tau_i) := \sum k \tau_i \) is a frame in \( T \).

A frame of a vertex \( A \)-algebroid (resp. prealgebroid, extended Lie algebroid) is by definition a frame of the underlying Lie algebroid \( T \). A vertex algebroid (resp. prealgebroid, ...) equipped with a frame will be called framed.

We will call an extended Lie algebroid \( T \) quasiregular if it is perfect (see 1.2) and admits a frame.

5.2. Let us consider the situation of 4.3, and assume that \( T' \) is equipped with a frame \( g' \) and \( T \) is perfect. Let us assume that \( g_T \) is an isomorphism. Then \( g := g_T(g') \) is a frame in \( T \). Let us define a \( k \)-linear map \( h_{g'} : g' \rightarrow \Omega \) by the condition

\[
\langle g_T(\tau_1), h_{g'}(\tau_2) \rangle = \frac{1}{2} \{ g_A(\langle \tau_1, \tau_2 \rangle') - \langle g_T(\tau_1), g_T(\tau_2) \rangle \} \quad (\tau_i \in g') \quad (5.2.1)
\]

Since \( g_T \) is an isomorphism, (5.2.1) defines \( h_{g'} \) uniquely. Then the condition (3.5(\( \langle, \rangle \))) is obviously satisfied for all \( \tau_i \in g' \). There is a unique extension of \( h_{g'} \) to a map \( h_{g'} : T' \rightarrow \Omega \) satisfying (3.5(\( \gamma \))) for all \( \tau \in g' \).

By Theorem 3.8, the conditions (3.5(\( \gamma \))) and (3.5(\( \langle, \rangle \))) are then fulfilled for all \( \tau, \tau_i \in T \), that is, \( g_{g'} := (g, h_{g'}) \) is a morphism of vertex prealgebroids \( B' \rightarrow B \).

In other words,

5.2.1. the \( \Omega^2(T)^{\tau}-torseur \ hom_\tau(B', B) \) is equipped with a trivialization \( g_{g'} \).

5.3. Let \( (T, g) \) be a framed extended Lie algebroid. We can define a vertex algebroid \( \mathcal{A}_{T,g} \) by setting \( \gamma(a, \tau) = 0, \) \( \langle \tau, \tau' \rangle = c(\tau, \tau') = 0 \) for all \( a \in A, \tau, \tau' \in g \) and then extending the operations \( \gamma, \langle, \rangle, c \) to the whole of \( T \) using the axioms (A1) — (A3). By Theorem 1.9 we get a vertex algebroid, to be denoted \( \mathcal{A}_{T,g} \).
If $\mathfrak{g}$ is a Lie subalgebra of $T$ then $A_{T,\mathfrak{g}}$ may be defined as follows. Take the vertex $k$-algebroid $A_{\mathfrak{g}} = (k, \mathfrak{g}, \mathfrak{g}^*, 0, 0, \langle , \rangle, 0)$ where $\mathfrak{g}$ acts on $\mathfrak{g}^*$ in the coadjoint way, $\langle \cdot , \cdot \rangle_{\mathfrak{g} \times \mathfrak{g}^*}$ is the obvious pairing, the other components of $\langle \cdot , \cdot \rangle$ being zero. Then apply to $A_{\mathfrak{g}}$ the pushout with respect to the structure morphism $k \to A$, where we set $\gamma(a, \tau) = 0$ for $\tau \in \mathfrak{g}$ and the Lie algebra acts on $A$ due to the embedding $\mathfrak{g} \subset T$, cf. 1.10. We get a vertex $A$-algebroid which is equal to $A_{T,\mathfrak{g}}$.

We set $B_{T,\mathfrak{g}} := P(A_{T,\mathfrak{g}})$.

Let us call the frame $\mathfrak{g}$ abelian if $\mathfrak{g}$ is an abelian Lie subalgebra of $T$, i.e. $[\tau, \tau'] = 0$ for all $\tau, \tau' \in \mathfrak{g}$. We call an extended Lie algebroid $T$ regular if it is perfect and admits an abelian frame.

5.4. Let $T = (A, T, \Omega, \ldots)$ be a regular extended Lie algebroid. Let $\mathfrak{g}, \mathfrak{g}'$ be two abelian frames of $T$. Let us assume that $\mathfrak{g}$ and $\mathfrak{g}'$ are free $k$-modules of finite rank $n$. In the sequel we will need some formulas pertaining to this situation.

Choose some $k$-bases $\{\tau_i\}, \{\tau'_i\}, i = 1, \ldots, n$, of $\mathfrak{g}$ and $\mathfrak{g}'$ respectively; let $\{\omega_i\} \subset \mathfrak{g}^*, \{\omega'_i\} \subset \Omega'$ be the dual bases. Note that

$$\tau_i(\omega_j) = 0 \text{ for all } i, j,$$

(5.4.1)

since for all $p$, $\langle \tau_p, \tau_i(\omega_j) \rangle = \tau_i(\langle \tau_p, \omega_j \rangle) - \langle [\tau_i, \tau_p], \omega_j \rangle = 0$ because $\mathfrak{g}$ is abelian and $\langle \tau_p, \omega_j \rangle = \delta_{pj}$.

Recall that

$$\langle \tau, a\partial b \rangle = a\tau(b)$$

(5.4.2)

and

$$(a\tau)(\omega) = a\tau(\omega) + \langle \tau, \omega \rangle \partial a$$

(5.4.3)

Since $\langle \tau_i, \partial a \rangle = \tau_i(a)$,

$$\partial a = \tau_i(a)\omega_i$$

(5.4.4)

where we always imply the summation over repeating indices.

Define the matrices $\phi = (\phi^{ij}), \rho = (\rho^{ij}) \in \text{GL}_n(A)$ by $\tau'_i = \phi^{ij} \tau_j, \omega'_i = \rho^{ij} \omega_j$. Since $\langle \tau_i, \omega_j \rangle = \langle \tau'_i, \omega'_j \rangle = \delta_{ij}$, we have

$$\phi \cdot \rho^t = 1$$

(5.4.5)

where $(\rho^t)^{ij} = \rho^{ij}$. Since $[\tau_i, \tau_j] = [\tau'_i, \tau'_j] = 0$,

$$\phi^{ip}_q \tau_p (\phi^{jq}) = \phi^{ip}_q \tau_p (\phi^{jq})$$

(5.4.6)

for all $i, j, q$. Applying $\tau_r$ to (5.4.6) we get

$$\phi^{ip}_q \tau_p (\phi^{jq}) - \phi^{jp}_q \tau_p (\phi^{iq}) = \tau_r (\phi^{ip}) \tau_p (\phi^{jq}) - \tau_r (\phi^{jp}) \tau_p (\phi^{iq})$$

(5.4.7)

for all $i, j, q, r$. Setting here $r = q$ and summing up by $q$ we get

$$\phi^{ip}_q \tau_p (\phi^{jq}) = \phi^{jq}_r \tau_q (\phi^{ip})$$

(5.4.8)
for all $i, j$. Applying $\tau_r$ we get

$$\phi^{ip} \tau_r \tau_q \tau_p (\phi^{jq}) - \phi^{jp} \tau_r \tau_q \tau_p (\phi^{iq}) = \tau_r (\phi^{iq}) \tau_p \tau_q (\phi^{jp}) - \tau_r (\phi^{ip}) \tau_q \tau_p (\phi^{jq}) \quad (5.4.9)$$

It follows from (5.4.5) that

$$\tau_r (\rho) = -\rho \tau_r (\phi^j) \rho; \quad \tau_r (\phi) = -\phi \tau_r (\rho^j) \phi \quad (5.4.10)$$

for all $r$. Multiplying (5.4.6) by $\rho^ia$ and summing up by $i$, we get

$$\tau_a (\phi^{jq}) = \phi^{jp} \rho^ia \tau_p (\phi^{iq}) \quad (5.4.11)$$

whence

$$\rho^ib \tau_a (\phi^{iq}) = \rho^ia \tau_b (\phi^{iq}) \quad (5.4.12)$$

By (5.4.10)

$$\tau_c (\rho^{uk}) = -\rho^{uk} \tau_c (\phi^{lk}) \rho^ib \quad (5.4.11)$$

$$= -\rho^{uk} \phi^{lp} \rho^ia \tau_p (\phi^{lk}) \rho^ib = -\rho^{uk} \tau_b (\phi^{lk}) \rho^ic = \tau_b (\rho^{uc})$$

Thus,

$$\tau_a (\rho^{bc}) = \tau_c (\rho^{ba}) \quad (5.4.13)$$

5.5. In the situation 5.4, consider the vertex prealgebroids $B_{T,a} = (A, T, \Omega, \partial, \gamma, \langle, \rangle)$ and $B_{T,a'} = (A, T, \Omega, \partial, \gamma', \langle, \rangle')$. According to 5.2, we have an isomorphism

$$g_{a,a'} = (Id_A, Id_T, Id_\Omega, h_{a,a'}) : B_{T,a} \simto B_{T,a} \quad (5.5.1)$$

where $h = h_{a,a'}$ is defined by

$$\langle \tau'_i, h(\tau'_j) \rangle = -\frac{1}{2} \langle \tau'_i, \tau'_j \rangle \quad (5.5.2)$$

which is (5.2.1) in our situation.

Now consider the vertex algebroids $A_{T,a} = (\ldots, c)$, $A_{T,a'} = (\ldots, c')$. We have $g_0 A_{T,a}, A_{T,a} \in Alg_{B_{T,a}}$. Therefore, by 4.2.1 $A_{T,a} = g_0 A_{T,a} + \beta$ for some closed 3-form $\beta$.

Let us define a form $\beta = \beta_{a,a'} \in \Omega^3 (T)$ by

$$\beta(\tau'_i, \tau'_j) = \beta^{ijr} \omega'_r = \frac{1}{2} \{ \tau_u (\phi^{ip}) \tau_p (\phi^{jq}) \tau_q (\phi^{ru}) - \tau_u (\phi^{ip}) \tau_p (\phi^{jq}) \tau_q (\phi^{ru}) \} \omega'_r \quad (5.5.3)$$

5.6. Magic Lemma. The form $\beta_{a,a'}$ is closed and we have $A_{T,a} = g_0 A_{T,a} + \beta_{a,a'}$.

5.7. Proof. Let us write down the things explicitly. The following formulas hold true in the algebroid $A_{T,a}$ (they are (1.8.3) in our situation):

$$\gamma(a, b \tau_i) = -\tau_i (a) \partial b - \tau_i (b) \partial a \quad (5.7.1)$$
\begin{equation}
\langle a \tau_i, b \tau_j \rangle = -b \tau_i \tau_j(a) - a \tau_j \tau_i(b) - \tau_i(b) \tau_j(a) \tag{5.7.1}_{(\ast)}
\end{equation}
\begin{equation}
c(a \tau_i, b \tau_j) = \frac{1}{2} \{ \tau_i(b) \partial \tau_j(a) - \tau_j(a) \partial \tau_i(b) \} + \frac{1}{2} \partial \{ b \tau_i \tau_j(a) - a \tau_j \tau_i(b) \} \tag{5.7.1}_c
\end{equation}

Define the matrix \((h^{ij}) \in Mat_n(A)\) by \(h(\tau^i_i) = h^{ij} \omega_j\). The left hand side of (5.5.2) is equal to
\begin{equation}
\langle \phi^{ip} \tau_p, h^{jq} \omega_q \rangle = \phi^{ip} h^{jp}
\end{equation}
By (5.7.1)_{(\ast)}, the right hand side is equal to
\begin{equation}
\frac{1}{2} \{ \phi^{jq} \tau_p \tau_q (\phi^{ip}) + \phi^{ip} \tau_p \tau_q (\phi^{jq}) + \tau_p (\phi^{jq}) \tau_q (\phi^{ip}) \} \tag{5.4.8}
\end{equation}
\begin{equation}
= \phi^{ip} \tau_q \tau_p (\phi^{jq}) + \frac{1}{2} \tau_p (\phi^{jq}) \tau_q (\phi^{ip})
\end{equation}
Thus, the equation (5.5.2) takes the form
\begin{equation}
\phi^{ip} h^{jp} = \phi^{ip} \tau_q \tau_p (\phi^{jq}) + \frac{1}{2} \tau_p (\phi^{jq}) \tau_q (\phi^{ip})
\end{equation}
wherefrom, applying (5.4.5),
\begin{equation}
h^{ij} = \tau_p \tau_j (\phi^{ip}) + \frac{1}{2} \tau_q (\phi^{jq}) \tau_p (\phi^{jq}) \rho^{ij} \tag{5.7.2}
\end{equation}
We have to prove that
\begin{equation}
c(\tau^i_i, \tau^j_j) = g_* c'(\tau^i_i, \tau^j_j) + \beta(\tau^i_i, \tau^j_j)
\end{equation}
where \(g_* c'\) is defined by
\begin{equation}
g_* c'(\tau^i_i, \tau^j_j) = c'(\tau^i_i, \tau^j_j) + \tau^i_i(h(\tau^j_j)) - \tau^j_j(h(\tau^i_i)),
\end{equation}
by (4.5.1). Thus, we have to prove that
\begin{equation}
c(\tau^i_i, \tau^j_j) - c'(\tau^i_i, \tau^j_j) - \tau^i_i(h(\tau^j_j)) + \tau^j_j(h(\tau^i_i)) = \beta(\tau^i_i, \tau^j_j) \tag{5.7.3}
\end{equation}
By definition, \(c'(\tau^i_i, \tau^j_j) = 0\). By (5.7.1)\_c and (5.4.8) we have
\begin{equation}
c(\tau^i_i, \tau^j_j) = c(\phi^{ip} \tau_p, \phi^{jq} \tau_q) = \frac{1}{2} \{ \tau_p (\phi^{jq}) \partial \tau_q (\phi^{ip}) - \tau_q (\phi^{ip}) \partial \tau_p (\phi^{jq}) \} \tag{5.7.4}
\end{equation}
On the other hand, by (4.5.1) and (4.5.3),
\begin{equation}
\tau^i_i(h(\tau^j_j)) = (\phi^{ip} \tau_p)(h^{jq} \omega_q) = \phi^{ip} \tau_p (h^{jq} \omega_q) + (\tau_p, h^{jq} \omega_q) \partial \phi^{ip} = \phi^{ip} \tau_p (h^{jq} \omega_q) + h^{jp} \partial \phi^{ip}
\end{equation}
where we have used (4.5.1). Thus, (5.7.3) takes the form
\begin{equation}
\frac{1}{2} \{ \tau_p (\phi^{jq}) \partial \tau_q (\phi^{ip}) - \tau_q (\phi^{ip}) \partial \tau_p (\phi^{jq}) \} - \phi^{ip} \tau_p (h^{jq}) \omega_q - h^{jp} \partial \phi^{ip} + \phi^{ip} \tau_p (h^{jq}) \omega_q + h^{jp} \partial \phi^{ip} = \beta^{ijr} \omega_r' \tag{5.7.5}
\end{equation}
We have to prove that the matrix \((h^{ij})\) defined by \((5.7.2)\) satisfies the differential equation \((5.7.5)\). Using \((5.4.4)\), rewrite \((5.7.5)\) in the form

\[
\frac{1}{2} \left\{ \tau_p (\phi^{jq}) \tau_q (\phi^{ip}) - \tau_q (\phi^{ip}) \tau_p (\phi^{jq}) \right\} - \\
- \phi^{ip} \tau_p (h^{ji}) - h^{jp} \tau_j (\phi^{ip}) + \phi^{jp} \tau_p (h^{ij}) + h^{ip} \tau_i (\phi^{jp}) = \beta^{ijr} \rho^r \omega_l
\]  

\((5.7.6)\)

Denote

\[
A = \frac{1}{2} \left\{ \tau_p (\phi^{jq}) \tau_q (\phi^{ip}) - \tau_q (\phi^{ip}) \tau_p (\phi^{jq}) \right\},
\]

\[
B = B^{ij} = - \phi^{ip} \tau_p (h^{ji}) - h^{jp} \tau_j (\phi^{ip}),
\]

and

\[
C = - B^{ij} = \phi^{ip} \tau_p (h^{ji}) + h^{jp} \tau_j (\phi^{ip})
\]

We have to prove that \(A + B + C = \beta^{ijr} \rho^r \omega_l\) where \(h^{ij}\) is given by \((5.7.2)\) and \(\beta^{ijr}\) is given by \((5.5.3)\). Thus, we have

\[
B = - \phi^{ip} \tau_p \left\{ \tau_q (\phi^{jq}) + \frac{1}{2} \tau_u (\phi^{jq}) \tau_q (\phi^{ru}) \rho^l \right\} - \tau_i (\phi^{jp}) \left\{ \tau_q (\phi^{jq}) + \frac{1}{2} \tau_u (\phi^{jq}) \tau_q (\phi^{ru}) \rho^p \right\}
\]

and

\[
C = \phi^{ip} \tau_p \left\{ \tau_q (\phi^{jq}) + \frac{1}{2} \tau_u (\phi^{jq}) \tau_q (\phi^{ru}) \rho^l \right\} + \tau_i (\phi^{jp}) \left\{ \tau_q (\phi^{jq}) + \frac{1}{2} \tau_u (\phi^{jq}) \tau_q (\phi^{ru}) \rho^p \right\}
\]

Let us denote the \(n\)-th summand in an expression \(X\) by \(X_n\) (where we open the brackets). We have

\[
B_1 + C_1 = - \phi^{ip} \tau_p \tau_q (\phi^{jq}) + \phi^{ip} \tau_p \tau_q (\phi^{iu}) \quad (5.4.9)
\]

\[
= - \tau_i (\phi^{jq}) \tau_p \tau_q (\phi^{ip}) + \tau_i (\phi^{jq}) \tau_p \tau_q (\phi^{ip}) = - C_3 - B_3
\]

Next,

\[
B_2 = - \frac{1}{2} \phi^{ip} \left\{ \tau_p \tau_u (\phi^{jq}) \tau_q (\phi^{ru}) \rho^l + \tau_u (\phi^{jq}) \tau_p \tau_q (\phi^{ru}) \rho^l + \tau_u (\phi^{jq}) \tau_q (\phi^{ru}) \tau_p (\rho^l) \right\}
\]

We have

\[
B_{21} = - \frac{1}{2} \phi^{ip} \tau_p \tau_u (\phi^{jq}) \tau_q (\phi^{ru}) \rho^l \quad (5.4.7)
\]

\[
= - \frac{1}{2} \left\{ \phi^{ip} \tau_p \tau_u (\phi^{jq}) + \tau_u (\phi^{jq}) \tau_p (\phi^{iu}) - \tau_u (\phi^{jq}) \tau_p (\phi^{iu}) \right\} \tau_q (\phi^{ru}) \rho^l
\]

Next,

\[
B_{22} = - \frac{1}{2} \phi^{ip} \tau_p \tau_u (\phi^{jq}) \tau_q (\phi^{ru}) \rho^l \quad (5.4.7)
\]

\[
= - \frac{1}{2} \tau_u (\phi^{jq}) \rho^l \left\{ \phi^{ip} \tau_p \tau_q (\phi^{iu}) + \tau_q (\phi^{ip}) \tau_p (\phi^{iu}) - \tau_q (\phi^{ip}) \tau_p (\phi^{iu}) \right\}
\]

\[
= - \frac{1}{2} \tau_u (\phi^{jq}) \tau_q (\phi^{iu}) - \tau_u (\phi^{jq}) \tau_q (\phi^{ip}) \rho^l + \frac{1}{2} \tau_u (\phi^{jq}) \tau_q (\phi^{iu}) \rho^l
\]
We see that $B_{221} = -A_1$, $B_{222} = -B_{13}$ and $B_{223} = -B_{212}$. Similarly, $A_2 = -C_{221}$. We compute $B_{23}$ using (5.4.10):
\[
B_{23} = \frac{1}{2} \phi^j \tau_a (\phi^j)^a \tau_q (\phi^r)^a \rho^{ra} \tau_p (\phi^b)^b \rho^{bl} \overset{(5.4.6)}{=} \\
= \frac{1}{2} \phi^{bj} \tau_p (\phi^j)^a \tau_q (\phi^r)^a \rho^{ra} \rho^{bl} = \frac{1}{2} \tau_1 (\phi^a)^a \tau_q (\phi^r)^a \rho^{ra} = -B_{44}
\]
Finally, $B_{211} + C_{211} = \beta^{ij} \rho^{ri} \omega_i$. Everything except these last terms cancels out, and this proves the Lemma. $	riangle$

\section{6. Atiyah}

6.1. Let $T = (A, T, \Omega, \partial)$ be a perfect extended Lie algebroid. Let $g, g', g''$ be three frames in $T$. According to 5.2, we have the morphisms of the corresponding vertex prealgebroids
\[
\mathcal{B}_{\mathcal{T}:g'} \xrightarrow{\mathcal{g}'} \mathcal{B}_{\mathcal{T}:g} \xrightarrow{\mathcal{g}_{g',g''}} \mathcal{B}_{\mathcal{T}:g''}
\]
(6.1.1)
as well as the morphism $g_{g',g''}: \mathcal{B}_{\mathcal{T}:g'} \rightarrow \mathcal{B}_{\mathcal{T}:g''}$, all of them over $Id_{\mathcal{T}}$. Recall that $Hom_{\mathcal{T}\text{-Alg}}(\mathcal{B}_{\mathcal{T}:g'}, \mathcal{B}_{\mathcal{T}:g''})$ is an $\Omega^2(\mathcal{T})$-torseur, cf. 4.3.1. We are aiming to compute the discrepancy
\[
\alpha_{g,g',g''} := g_{g,g'} \circ g_{g',g''} - g_{g,g''} \in \Omega^2(\mathcal{T})
\]
(6.1.2)
We have the functions $h_{g,g'}$, etc., acting from $T$ to $\Omega$ (not $A$-linear!), as in the previous Section, which define our morphisms. By (3.7.1) the composition $g_{g,g'} \circ g_{g',g''}$ is defined by the function $h_{g,g'} + h_{g',g''}$; therefore the discrepancy (6.1.2) is defined by the $A$-linear function
\[
\alpha_{g,g',g''} = h_{g,g'} + h_{g',g''} - h_{g,g''} \in \Omega^2(\mathcal{T}) \subset Hom_A(T, \Omega)
\]
(6.1.3)
which by definition coincides with (6.1.2).

Note that the functions (6.1.3) obviously satisfy the 2-cocycle condition
\[
\alpha_{g',g'',g'''} - \alpha_{g,g'',g'''} + \alpha_{g,g',g''} - \alpha_{g,g',g'''} = 0
\]
(6.1.4)

6.2. Choose some bases $\{\tau_i\}, \{\tau'_i\}$ and $\{\tau''_i\}$ of our frames $g, g', g''$; let $\{\omega_i\}, \{\omega'_i\}, \{\omega''_i\}$ be the dual bases in $\Omega$. Define two matrices $\phi, \psi \in GL_n(A)$ by $\tau'_i = \phi^i j \tau_j$, $\tau''_i = \psi^{ij} \tau'_j$ and set $\rho := \phi^{-1}$, $\sigma := \psi^{-1}$.

The maps $h_{g,g'}$, etc., are defined by the matrices $(h_{g,g'}^{ij})$, etc., where $h_{g,g'}(\tau'_i) = h_{g,g'}^{ij} \omega_j$, etc.

By (3.5) and (5.7.1) we have
\[
h_{g,g'}(a \tau'_i) = ah_{g,g'}(\tau'_i) - \gamma(a, \tau'_i) = ah_{g,g'}(\tau'_i) - \gamma(a, \phi^i j \tau_j) = ah_{g,g'}(\tau'_i) - \gamma(a, \phi^i j \tau_j) =
\]
\[ = ah_{g,g'}(\tau'_i) + \tau_p(a)\partial \phi^{jp} + \tau_p(\phi^{jp})\partial a \]

Therefore,
\[ A := h_{g,g'}(\tau''_i) = h_{g,g'}(\psi^{ij}\tau'_j) = \psi^{ij} h_{g,g'}(\tau'_j) + \tau_p(\psi^{ij})\partial \phi^{jp} + \tau_p(\phi^{jp})\partial \psi^{ij}; \]
\[ B := h_{g',g''}(\tau''_i) = h_{g',g''}^{ia}\omega'_a = h_{g',g''}^{ia}\rho^a\omega_l = \]
by (5.7.2)
\[ = \left\{ \tau'_p\tau'_a(\psi^{jp}) + \frac{1}{2}\tau_q(\psi^{jp})\tau_p(\psi_{pq})\sigma^{ra} \right\} \rho^a\omega_l \]
and
\[ C = -h_{g,g''}(\tau''_i) = -h_{g,g''}^{il}\omega_l = -\left\{ \tau_p(\psi^{ip}) + \frac{1}{2}\tau_q(\psi^{ip})\tau_p(\psi^{pq})(\sigma)_{rl} \right\} \omega_l \]

We have to calculate \( \alpha_{g,g',g''}(\tau''_i) = A + B + C \). In the computation we shall use the same convention for the notation of various terms in \( A, B, C, \ldots \) as in 5.7.

So, we have
\[ A1 = \psi^{ij} h_{g,g'}^{il}\omega_l = \psi^{ij} \left\{ \tau_p(\psi^{jp}) + \frac{1}{2}\tau_q(\psi^{ip})\tau_p(\psi^{pq})(\sigma)_{rl} \right\} \omega_l \]

By (5.4.4),
\[ A2 = \tau_p(\psi^{ij})\tau_l(\psi^{jp})\omega_l \]
and
\[ A3 = \tau_p(\phi^{jp})\tau_l(\psi^{ij})\omega_l \]

Next,
\[ B1 = \phi^{pu}\tau_u(\psi^{ip})\rho^a\omega_l = \phi^{pu}\tau_u(\psi^{ip})\omega_l + \phi^{pu}\tau_u(\psi^{ip})(\sigma)_{rl} \omega_l \]
and
\[ B2 = \frac{1}{2}\phi^{pu}\phi^{pq} \tau_u(\psi^{ip})(\psi^{pq})\sigma^{ra} \rho^a\omega_l \]

Finally,
\[ C1 = -\tau_p(\psi^{iu})\phi^{np}\omega_l = -\phi^{up}\tau_p(\psi^{iu})\omega_l - \psi^{iu}\tau_p(\phi^{np})\omega_l - \]
\[ -\tau_p(\psi^{iu})\tau_l(\phi^{np})\omega_l - \tau_l(\psi^{iu})\tau_p(\phi^{np})\omega_l \]
and
\[ C2 = -\frac{1}{2}\left\{ \phi^{up}\tau_q(\psi^{iu}) + \psi^{iu}\tau_q(\phi^{np}) \right\} \left\{ \tau^{rq}\tau_p(\psi^{rv}) + \psi^{ru}\tau_p(\phi^{qv}) \right\} \sigma^{rs} \rho^a\omega_l = \]
\[ = -\frac{1}{2}\left\{ \phi^{up}\phi^{aq}\tau_q(\psi^{iu})\tau_p(\psi^{rv})\sigma^{rs} \rho^a + \psi^{iu}\phi^{aq}\tau_q(\psi^{rv})\tau_q(\phi^{np})\sigma^{rs} \rho^a + \\
+ \phi^{up}\tau_q(\psi^{iu})\tau_p(\phi^{eq})\rho^a + \psi^{iu}\tau_q(\phi^{np})\tau_p(\phi^{eq})\rho^a \right\} \omega_l \]
We see first of all the terms of the second order cancel out, as they should: $A_{11} = -C_{12}$ and $B_{11} = -C_{11}$. Most of the other terms also cancel out, and in $A + B + C$ we are left only with

$$B_{12} + C_{23} = \frac{1}{2} \phi^{np} \tau_p (\phi^{sq}) \tau_q (\psi^{iu}) \rho^{sl} \omega_l \quad (5.4.6)$$

$$= \frac{1}{2} \phi^{sp} \tau_p (\phi^{uq}) \tau_q (\psi^{iu}) \rho^{sl} \omega_l = \frac{1}{2} \tau_l (\phi^{uq}) \tau_q (\psi^{iu}) \omega_l$$

and

$$C_{22} = -\frac{1}{2} \psi^{iu} \phi^{uq} \tau_q (\phi^{np}) \tau_p (\psi^{rv}) \sigma^{rs} \rho^{sl} \omega_l \quad (5.4.6)$$

$$= \frac{1}{2} \psi^{iu} \phi^{uq} \tau_q (\phi^{np}) \tau_p (\psi^{rv}) \sigma^{rs} \rho^{sl} \omega_l$$

Thus, we have

$$\alpha_{\theta, \theta', \theta''} (\tau_i') = \frac{1}{2} \tau_l (\phi^{uq}) \tau_q (\psi^{iu}) \omega_l - \frac{1}{2} \psi^{iu} \phi^{uq} \tau_q (\phi^{np}) \tau_p (\psi^{rv}) \sigma^{rs} \rho^{sl} \omega_l$$

Rewriting the right hand side in the base $\{\omega_i''\}$ we get

$$\alpha (\tau_i'') := \alpha_{\theta, \theta', \theta''} (\tau_i'') = \alpha^{ir} \omega_r'' = \alpha^{ir}_1 \omega_r'' + \alpha^{ir}_2 \omega_r'' \quad (6.2.1)$$

where

$$\alpha^{ir}_1 = \frac{1}{2} \psi^{ra} \phi^{ad} \tau_l (\phi^{aq}) \tau_q (\psi^{iu}) \quad (6.2.2)$$

and

$$\alpha^{ir}_2 = -\frac{1}{2} \psi^{iu} \phi^{uq} \tau_q (\phi^{np}) \tau_p (\psi^{rv}) \quad (6.2.3)$$

So we see that $\alpha^{ir}_1 = -\alpha^{ri}_2$, i.e. the matrix $\alpha^{ir}$ is skew symmetric, that is, $\alpha \in \Omega^2 (T)$ as it should be.

**6.3.** Let us rewrite the expression (6.2.3) in terms of vector fields $\tau_i'':$

$$\alpha^{ir}_2 = -\frac{1}{2} \tau_i'' (\phi^{vp}) \phi^{-1pa} \psi^{-1ab} \tau_b'' (\psi^{rv}) =$$

(using the identity $\tau (\phi) = -\phi \tau (\phi^{-1}) \phi$)

$$= -\frac{1}{2} \phi^{ys} \tau_i'' (\phi^{-1sa}) \psi^{-1ab} \psi^{ra} \tau_b'' (\psi^{-1uc}) \psi^{sv}$$

Using (5.4.13), we have

$$\psi^{-1ab} \tau_b'' (\psi^{-1uc}) = \tau_a'' (\psi^{-1uc}) = \tau_a'' (\psi^{-1ac}) = \psi^{-1ab} \tau_b'' (\psi^{-1ac})$$

whence

$$\alpha_2 (\tau_i'') := \alpha^{ir}_2 \omega_r'' = -\frac{1}{2} \phi^{ys} \tau_i'' (\phi^{-1sa}) \psi^{ru} \psi^{-1ab} \tau_b'' (\psi^{-1ac}) \psi^{rv} \omega_r'' =$$

$$= -\frac{1}{2} \phi^{ys} \tau_i'' (\phi^{-1sa}) \tau_r'' (\psi^{-1ac}) \psi^{ru} \omega_r'' = -\frac{1}{2} tr \{ \phi^{ys} \tau_i'' (\phi^{-1}) \tau_r'' (\psi^{-1}) \} \omega_r''$$
Skew symmetrizing, we arrive at the first part of

6.4. Theorem. (a) The cocycle \( \alpha_{g',g''} \), (6.1.3) is given in coordinates by the expression

\[
\alpha_{g',g''}(\tau''_i) = \alpha(\psi, \phi)(\tau''_i) = \frac{1}{2} \text{tr}\{\tau''_i(\psi^{-1})\psi\phi\tau''_i(\phi^{-1}) - \tau''_i(\psi^{-1})\psi\phi\tau''_i(\phi^{-1})\} \omega''_r
\]

(b) The 3-form \( \beta_{g',g'} = \beta(\phi) \) (5.5.3) is equal to

\[
\beta_{g',g'}(\tau'_i, \tau'_j) = \beta(\phi)(\tau'_i, \tau'_j) = \frac{1}{2} \text{tr}\{\tau'_i(\phi^{-1})\phi\tau'_j(\phi^{-1})\phi - \tau'_j(\phi^{-1})\phi\tau'_i(\phi^{-1})\phi\} \omega'_r
\]

Part (b) is proven by the same argument as in 6.3, and we leave it to the reader.

From the expression (6.4.2) we see directly that the form \( \beta(\phi) \) is closed, and from (6.4.1) one checks that the form \( \alpha(\phi, \psi) \) satisfies the group (or Cech) cocycle condition

\[
\alpha(\psi, \chi) - \alpha(\phi, \psi, \chi) + \alpha(\phi, \psi \chi) - \alpha(\phi, \psi) = 0
\]

for all \( \phi, \psi, \chi \in GL_n(A) \), which is the same as (6.1.4), and

\[
d\alpha(\psi, \phi) = \beta(\phi) + \beta(\psi) - \beta(\phi \psi)
\]

Thus, a couple \( c(T) = (\alpha, \beta) \) represents a cohomology class in \( H^2(GL_n(A), \Omega^{2,3}(T)) \) of the group \( GL_n(A) \) with coefficients in the complex

\[
\Omega^{(2,3)}(T) := \Omega^2(T) \longrightarrow \Omega^{3,cl}(T)
\]

where \( \Omega^2(T) \) sits in degree 0 (the action of \( GL_n(A) \) on \( \Omega^{(2,3)}(T) \) being trivial).

The cocycle \( \alpha \) is classical, and essentially goes back to Atiyah; it is written down explicitly by Harris, [H], p. 280. The class \( \beta \) resembles "Chern-Simons" form.

The whole cocycle \( c(T) \) may be thought of as an integration of a cocycle \( \tilde{c} \) from [MSV], (5.16), (5.17).

§7. Gerbes of Vertex Algebroids

7.1. Let us reformulate the results of the last three Sections in language of Torseurs. This reformulation was inspired by [BD1].

Let \( T = (A, T, \ldots) \) be a quasiregular extended Lie algebroid (see 5.1). Let us define a groupoid \( \Omega^{(2,3)}_T \) is follows. We set \( \text{Ob} \Omega^{(2,3)}_T = \Omega^{3,cl}(T) \); for \( \omega_1, \omega_2 \in \Omega^{3,cl}(T) \) a morphism \( \omega_1 \longrightarrow \omega_2 \) is by definition a two-form \( \eta \in \Omega^2(T) \) such that \( d_{DR}(\eta) = \)
\( \omega_2 - \omega_1 \). The composition of morphisms is defined in an obvious manner. Note that \( \Omega_T^{(2,3)} \) is in fact an abelian group in categories.

Similarly, let \( \Omega_T^{2,cl} \) denote a groupoid with the unique object and the set of morphisms equal to \( \Omega^{2,cl}(T) \). It is also an abelian group in categories. We have a fully faithful monoidal functor

\[
\Omega_T^{2,cl} \longrightarrow \Omega_T^{(2,3)}
\]

(7.1.1)

sending the unique object in \( \Omega_T^{2,cl} \) to 0 \( \in \Omega^{3,cl}(T) \).

Consider the groupoid \( \mathcal{A}lg_T \). According to 5.3, it is nonempty.

We can define an action of \( \Omega_T^{(2,3)} \) on \( \mathcal{A}lg_T \), i.e. a monoidal functor

\[
\ast : \mathcal{A}lg_T \times \Omega_T^{(2,3)} \longrightarrow \mathcal{A}lg_T
\]

(7.1.2)

as follows. For \( A \in \mathcal{A}lg_T \), \( \omega \in \Omega^{3,cl}(T) \), a couple \( (A, \omega) \) goes to the vertex algebroid \( A + \omega \) defined in 4.2. If \( \eta \in \Omega^2(T) \), \( d_{DR}(\eta) = \omega' - \omega \) then a morphism

\[
\ast (\eta) : A + \omega \longrightarrow A + \omega'
\]

is defined according to 4.2.2.

Let us fix \( A \) and consider the functor

\[
\Omega_T^{(2,3)} \longrightarrow \mathcal{A}lg_T, \quad \omega \mapsto A + \omega
\]

(7.1.3)

induced by (7.1.2). By 4.2.2 this functor is fully faithful.

Let \( A' \in \mathcal{A}lg_T \) be another object. By 5.2, there exists a morphism of vertex prealgebroids \( g : P(A) \longrightarrow P(A') \) lying over \( Id_T \); it is necessarily an isomorphism. Consider the vertex algebroid \( g_*A \) constructed in Theorem 4.5. By definition, \( g \) is lifted to an isomorphism \( \tilde{g} : A \longrightarrow g_*A \). Since \( g_*A \in \mathcal{A}lg_{P(A')} \), by 4.2.1

\[
A' = g_*A + \omega \quad \text{for some } \omega \in \Omega^{3,cl}(T)
\]

Therefore \( \tilde{g} \) induces an isomorphism \( \tilde{\ast} \) : \( A + \omega \longrightarrow g_*A + \omega = A' \).

In other words, we have checked that (7.1.3) is surjective on isomorphism classes of objects, hence it is an equivalence of categories. This proves

7.2. Theorem. Let \( T \) be a quasiregular extended Lie algebroid. Then the Action (7.1.2) makes the groupoid \( \mathcal{A}lg_T \) a nonempty \( \Omega_T^{(2,3)} \)-Torseur. \( \triangle \)

7.3. Let \((X, \mathcal{O}_X)\) be a topological space ringed by a sheaf of commutative \( k \)-algebras \( \mathcal{O}_X \). Let us call an extended Lie \( \mathcal{O}_X \) Lie algebroid \( T = (\mathcal{O}_X, T, \ldots) \) (quasi)regular if there exists an open covering \( X = \bigcup U_i \) of \( X \) such that all \( T(U_i) \) are (quasi)regular.

For example, if \( T \) is a Lie \( \mathcal{O}_X \)-algebroid which is locally free as an \( \mathcal{O}_X \)-module then the corresponding extended algebroid \( T_T \) is quasiregular. If \( X \) is a smooth \( k \)-scheme of finite type and \( T = T_{X/k} \) is the sheaf of vector fields then \( T_T \) is regular.

We can sheafify the constructions of the previous Subsections and obtain the sheaves (champs) of groupoids (i.e. gerbes) \( \mathcal{A}lg_T \), etc.
Let $\mathcal{T}$ be quasiregular. Consider the gerbe $\mathcal{A}lg_{\mathcal{T}}$. According to 7.1 and 7.2, it is locally nonempty but not locally connected in general; its sheaf of connected components is an $H^3_{dR}(\mathcal{T})$-torseur.

By the general procedure this gerbe defines a characteristic class

$$c(\mathcal{T}) := c(\mathcal{A}lg_{\mathcal{T}}) \in H^2(X; \Omega^{(2,3)}(\mathcal{T}))$$

(7.3.1)

Here in the right hand side we consider the hypercohomology with coefficients in the complex $\Omega^{(2,3)}(\mathcal{T})$.

Let us explain how to define the class (7.3.1). Choose an open covering $\mathcal{U} = \{U_i\}$ of $X$ such that all groupoids $\mathcal{A}lg_{\mathcal{T}|U_i}$ are nonempty. Choose an object $A_i$ in each $\mathcal{A}lg_{\mathcal{T}|U_i}$. Over double intersections $U_{ij} := U_i \cap U_j$ we get two objects $A_i|_{U_{ij}}, A_j|_{U_{ij}} \in \mathcal{A}lg_{\mathcal{T}|U_{ij}}$. Choose 3-forms $\omega_{ij} \in \Omega^{3,c}(\mathcal{T}(U_{ij}))$ such that there exist isomorphisms

$$h_{ij} : A_j|_{U_{ij}} \overset{\sim}{\rightarrow} A_i|_{U_{ij}} + \omega_{ij}$$

(7.3.2)

Choose some isomorphisms (7.3.2). Then on triple intersections get isomorphisms

$$A_i|_{U_{ijk}} + \omega_{ij} + \omega_{ik} \overset{\sim}{\rightarrow} A_i|_{U_{ijk}} + \omega_{ik}$$

(7.3.3)

The isomorphisms (7.5.3) must be given by the 2-forms $\eta_{ijk} \in \Omega^2(\mathcal{T})$ such that $d\Omega^2(\eta_{ijk}) = c_{ij} - c_{ik} + c_{jk}$. Then $(\omega_{ij}, \eta_{ijk})$ is a 2-cocycle in the Cech complex $C^2(\mathcal{U}; \Omega^{2,3}(\mathcal{T}))$ representing the class (7.3.1).

7.4. Now let us assume that the groupoid $\mathcal{T}$ is regular. Theorem 6.4 calculates the class (7.3.1).

Namely, define the ”Atiyah-Chern-Simons” class $\text{ch}_2(\mathcal{T}) \in H^2(X; \Omega^{(2,3)}(\mathcal{T}))$ by the following procedure. Choose some bases of local sections $\tau^{(i)} = \{\tau^{(i)}\} \subset \mathcal{T}(U_i)$ over some open covering. Let $\phi_{ij} \in GL_n(\mathcal{O}(U_{ij}))$ be the transition matrix from $\tau^{(i)}$ to $\tau^{(j)}$ over $U_{ij}$. By definition, $\text{ch}_2(\mathcal{T})$ is represented by the Cech 2-cocycle

$$2\text{ch}_2(\mathcal{T}) := (\alpha(\mathcal{T}), \beta(\mathcal{T}))$$

(7.4.1)

where

$$\alpha(\mathcal{T}) = (\alpha(\mathcal{T})_{ijk}) = \text{tr}(\phi_{ij}^{-1} \partial \phi_{ij} \wedge \partial \phi_{jk} \cdot \phi_{ij}^{-1}) \in Z^2(\mathcal{U}; \Omega^2(\mathcal{T}))$$

(7.4.2)

and

$$\beta(\mathcal{T}) = (\beta(\mathcal{T})_{ij}) = \left\{\frac{1}{3} \text{tr}(\partial \phi_{ij} \phi_{ij}^{-1} \partial \phi_{ij} \phi_{ij}^{-1} \partial \phi_{ij} \phi_{ij}^{-1})\right\} \in C^1(\mathcal{U}; \Omega^{3,c}(\mathcal{T}))$$

(7.4.3)

Theorem 6.4 implies

7.5. **Theorem.** Let $\mathcal{T}$ be a regular extended Lie $\mathcal{O}_X$-algebroid. Then $c(\mathcal{T}) = 2\text{ch}_2(\mathcal{T})$.

Therefore, the gerbe $\mathcal{A}lg_{\mathcal{T}}$ admits a global section iff $\text{ch}_2(\mathcal{T}) = 0$. If so, then the groupoid of global sections $\mathcal{A}lg_{\mathcal{T}}(X)$ is equivalent to the groupoid of $\Omega^{(2,3)}(\mathcal{T})$-torseurs, whence $\pi_0(\mathcal{A}lg_{\mathcal{T}}(X)) \cong H^1(X; \Omega^{(2,3)}(\mathcal{T}))$ and the automorphism group of an object of this groupoid is isomorphic to $H^0(X; \Omega^{(2,3)}(\mathcal{T}))$. 
△

It is instructive to compare the previous discussion with [BB], 2.1. In the chiral situation the degree of cohomology goes one step up.

7.6. Let us identify the class \( c(T) \) when \( X \) is a smooth \( k \)-scheme and \( T = T_X \) is the tangent bundle. If \( E \) is an arbitrary vector bundle over \( X \) given by a Cech 1-cocycle

\[
g = (g_{ij}) \in Z^1(\mathcal{U}; GL_r(\mathcal{O}_X))
\]

(7.6.1)
on some open covering \( \mathcal{U} \) then (7.4.1) - (7.4.3) define a cocycle

\[
c(g) = (\alpha(g), \beta(g)) \in Z^2(\mathcal{U}; \Omega^{[2,3]}_X)
\]

(7.6.2)

If \( g_{ij} = \phi_i h_{ij} \phi_j^{-1} \) for some \( \phi = (\phi_i) \in C^1(\mathcal{U}; GL(\mathcal{O}_X)) \) then one checks by a direct computation that

7.6.1. Claim. (a)

\[
\alpha(g) - \alpha(h) = d_{\text{Cech}} \eta
\]

(7.6.3)

where \( \eta = \eta(h, \phi) = (\eta_{ij}) \in C^2(\mathcal{U}; \Omega^2_X) \) is given by

\[
\eta_{ij} = tr\{h^{-1}_{ij} d(h_{ij} \phi_j^{-1} \phi_i^{-1} d\phi_i + h_{ij}^{-1} \phi_i^{-1} d\phi_i h_{ij} \phi_j^{-1} d\phi_j)\}
\]

(7.6.4)

(b)

\[
d_{\text{DR}} \eta = \beta(g) - \beta(h) - d_{\text{Cech}} \gamma
\]

(7.6.5)

where \( \gamma = \gamma(h, \phi) = (\gamma_i) \in C^0(\mathcal{U}; \Omega^{3,cl}) \) is defined by

\[
\gamma_i = \frac{1}{3} tr((\phi_i^{-1} d\phi_i)^3)
\]

(7.6.6)

△

7.6.2. Corollary.

\[
c(g) - c(h) = de
\]

(7.6.7)

where

\[
e = (\eta, \gamma) \in C^1(\mathcal{U}; \Omega^{[2,3]}_X)
\]

(7.6.8)

△

Therefore (7.6.2) gives rise to a well defined characteristic class \( c(E) \in H^2(X, \Omega^{[2,3]}_X) \).

The following Lemma is obvious.

7.7. Lemma. (a) If \( f : Y \rightarrow X \) is an arbitrary morphism from another smooth scheme then \( c(f^* E) = f^* c(E) \).

(b) If \( 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \) is a short exact sequence of vector bundles over \( X \) then \( c(E) = c(E') + c(E'') \).

(c) If \( L \) is a line bundle then \( c(L) \) is equal to the image of \( c_1(L) \otimes c_1(L) \in H^1(X; \mathcal{O}_X^*) \otimes \mathbb{R}^2 \) under the composition

\[
H^1(X; \mathcal{O}_X^*) \otimes \mathbb{R}^2 \xrightarrow{d_{\log} \otimes 2} H^1(X; \Omega^1_X \otimes \mathbb{R}^2) \otimes \mathbb{R}^2 \rightarrow H^2(X; \Omega^2_X \otimes \mathbb{R}^2) \rightarrow H^2(X; \Omega^{[2,3]}_X)
\]
7.8. Recall (cf. [S]) that for an arbitrary \( E \) we have a characteristic class

\[
2ch_2^{(K)}(E) := c_1^{(K)2}(E) - 2c_2^{(K)} \in H^2(X; K_{2,X})
\]

from which we can get a class \( 2ch_2(E) \in H^2(X; \Omega_X^{(2,3)}) \) using the \textit{dlog} map

\[
H^2(X; K_2) \rightarrow H^2(X; \Omega_X^{2,cl}) \rightarrow H^2(X; \Omega_X^{(2,3)})
\]

The class \( 2ch_2(E) \) also satisfies properties 7.7 (a) - (c).

It is obvious that the natural map

\[
H^2(X; \Omega_X^{(2,3)}) \rightarrow H^2(X; \Omega_X^{2})
\]

is injective, where \( \Omega_X^{2} := \Omega_X^{2} \rightarrow \Omega_X^{3} \rightarrow \ldots \) is the stupidly truncated (and shifted, so that \( \Omega_X^{2} \) sits in degree 0) de Rham complex.

The proof of the following lemma was provided to us by H. Esnault.

7.9. Lemma. \textit{The inverse image map}

\[
H^2(X; \Omega_X^{2}) \rightarrow H^2(\mathbb{P}(E); \Omega_{\mathbb{P}(E)}^{2})
\]

is injective. \( \triangle \)

In fact, more is true. One can consider the cohomology theory which assigns to a smooth \( X \) a collection of cohomology groups \( \{H^i(X; \Omega_X^{j})\} \). This theory has the standard Grothendieck’s properties needed to define the Chern classes, cf. [Gr]. This fact was communicated to us by A. Beilinson.

Anyway, the inverse image map \( H^2(X; \Omega_X^{(2,3)}) \rightarrow H^2(\mathbb{P}(E); \Omega_{\mathbb{P}(E)}^{(2,3)}) \) is also injective, hence by splitting principle we get

7.10. Theorem. For all vector bundles \( E \), \( c(E) = 2ch_2(E) \). \( \triangle \)

This theorem was obtained in collaboration with H. Esnault.

7.11. Corollary. \textit{In the situation of 7.6 the class described in Theorem 7.5 is equal to} \( 2ch_2(T_X) \). \( \triangle \)

§8. Vertex Envelope of a Conformal Algebra

8.1. Our aim in this Section will be to construct a left adjoint \( U \) to the forgetful functor (0.8.1) and to prove the ”Poincaré-Birkhoff-Witt” theorem for algebras \( UC, C \in Conf \).

Let \( V \) be a vertex algebra. We have the following two particular cases of the OPE formula (0.5.12). The first one corresponds to \( m = n = -1 \):

\[
[x(-1), y(-1)] = \sum_{j \geq 0} (-1)^j \partial^{(j+1)}(x^{(j)}y)(-1)
\]

(8.1.1)
where we have used (0.5.10). The second one corresponds to $m \geq 0$, $n = -1$:

$$[x(m), y(-1)] = \sum_{j=0}^{m} \binom{m}{j} (x(j)y)(m-j-1) \quad (8.1.2)$$

8.2. Let $C = \oplus C_i$ be a conformal algebra. Let $TC = \sum_{j \geq 0} T^j C$, $T^j C := C^{\otimes j}$ be the tensor algebra of $C$ over $k$. The multiplication in $TC$ will be denoted $x \cdot y$ or $xy$. The $\mathbb{Z}_{\geq 0}$-grading of $C$ induces the $\mathbb{Z}_{\geq 0}$-grading of $TC$ such that $TC$ becomes a $\mathbb{Z}_{\geq 0}$-graded associative algebra with the unit $1 \in T^0 C = k$. We have a canonical embedding of $k$-modules $C = T^1 C \subset TC$.

There is a unique extension of the operators $\partial^{(j)}$ on $C$ to the whole space $TC$ satisfying

$$\partial^{(j)}(xy) = \sum_{p=0}^{j} \partial^{(p)} x \cdot \partial^{(j-p)} y \quad (8.2.0)$$

cf. (0.4.6). These operators will satisfy (0.4.1).

Let $R \subset TC$ be a two-sided ideal generated by all elements

$$r(x, y) := xy - yx - \sum_{j \geq 0} (-1)^j \partial^{(j+1)}(x(j)y), \ x, y \in C, \quad (8.2.1)$$

cf. (8.1.1). Set $UC = TC/R$. We have a canonical morphism $i : C \to UC$ equal to the restriction of the projection $p : TC \to UC$ to $C$.

Since the relations (8.2.1) are homogeneous, the algebra $UC$ inherits a $\mathbb{Z}_{\geq 0}$-grading from $TC$. Using (0.4.6) and an obvious identity $\partial^{(i)} \partial^{(j)} = \partial^{(j)} \partial^{(i)}$ one sees easily that the operators $\partial^{(i)}$ respect the ideal $R$. Hence they induce the operators $\partial^{(i)}$ on $UC$ of degree $i$ which satisfy (0.4.1).

8.3. **Theorem.** There is a unique structure of a vertex algebra on the $k$-module $UC$ such that for all $x \in C$, $z \in TC$,

$$p(xz) = i(x)(-1)p(z) \quad (8.3.1)$$

The correspondence $C \to UC$ defines a functor $U : Conf \to Vert$ left adjoint to the forgetful functor.

The algebra $UC$ will be called a *vertex envelope* of a conformal algebra $C$.

This Theorem will be proven in 8.20, after some preparation.

8.4. Let us define $k$-linear operators $x^{(j)}$, $x \in C$, $j \in \mathbb{Z}$ of degree $-j - 1$ acting on the module $TC$. If $j < 0$, $j = -n - 1$, we set

$$x^{(-n-1)} z = (\partial^{(n)} x) z \quad (8.4.1)$$

In particular, $x^{(-1)} z = x z$.

Each element of $TC$ is a linear combination of monomials $z = z_1 z_2 \ldots z_n$, $z_i \in C$. We define $x^{(j)} z$, $j \in \mathbb{Z}_{\geq 0}$ by induction on the length $n$ of the monomial. If $n = 1$, \ldots \ldots
i.e. \( z \in C \), then we already have \( x_{(j)}z \) due to the structure of a conformal algebra on \( C \).

If \( z = yu \), \( y \in C \), we set

\[
x_{(j)}yu = yx_{(j)}u + \sum_{p=0}^{j} \binom{j}{p} (x_{(p)}y)(j-p-1)u \tag{8.4.2}
\]

cf. (8.1.2).

We leave to the reader an easy proof of the lemma below.

**8.4.1. Lemma.** For all \( i \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z} \),

\[
(\partial^{(i)}x)_{(n)} = (-1)^{i} \binom{n}{i} x_{(n-i)} \tag{8.4.3}
\]

**8.5. Lemma.** The operators \( x_{(j)} \) introduced above respect the ideal \( R \).

The proof will be given in 8.6 — 8.8 below.

**8.6.** Since \( R \) is a left ideal, it is respected by all operators \( x_{(n)} \) with \( n < 0 \).

It follows from the commutation formula (8.4.2) that if all the operators \( x_{(n)}, n \geq 0 \), respect a subset \( S \subset TC \) then they respect the left ideal generated by \( S \). Therefore we need to prove that

\[
u_{(n)}xyz - u_{(n)}yxz - \sum_{j \geq 0} (-1)^{j} u_{(n)} \partial^{(j+1)}(x_{(j)}y)z \in R \tag{8.6.1}
\]

for all \( u, x, y \in C \), \( z \in TC \), \( n \geq 0 \). Let us denote the summands in (8.6.1) by \( A, B \) and \( C \). We have

\[
A = u_{(n)}xyz = (u_{(n)}x)yz + xu_{(n)}yz + \sum_{p=0}^{n-1} \binom{n}{p} (u_{(p)}x)(n-p-1)yz \tag{8.6.2}
\]

We shall use the same agreement as in 5.7: in an expression \( X \), the \( n \)-th summand will be denoted by \( Xn \). We have

\[
A2 = x(u_{(n)}y)z + xyu_{(n)}z + \sum_{q=0}^{n-1} \binom{n}{q} x(u_{(q)}y)(n-q-1)z \tag{8.6.3}
\]

\[
A3 = \sum_{p=0}^{n-1} \binom{n}{p} \left\{ \left( (u_{(p)}x)(n-p-1)y \right)z + y(u_{(p)}x)(n-p-1)y + \right. \right.
\]

\[
+ \left. \sum_{q=0}^{n-p-2} \binom{n-p-1}{q} \left( (u_{(p)}x)(q)y \right)(n-p-q-2)z \right\} \tag{8.6.4}
\]
Similarly,

\[ B = -u(n)y x z = -(u(n)y)x z - y u(n)x z - \sum_{p=0}^{n-1} \binom{n}{p} (u(p)y)(n-p-1)x z \quad (8.6.5) \]

\[ B2 = -y(u(n)x)z - y xu(n)z - \sum_{p=0}^{n-1} \binom{n}{p} y(u(p)x)(n-p-1)z \quad (8.6.6) \]

and

\[ B3 = - \sum_{p=0}^{n-1} \binom{n}{p} \left\{ (u(p)y)(n-p-1)z + x(u(p)y)(n-p-1)z + \sum_{q=0}^{n-p-2} \binom{n-p-1}{q} (u(q)x)(n-p-q-2)z \right\} \quad (8.6.7) \]

Next,

\[ C = - \sum_{j \geq 0} (-1)^j \left\{ (u(n)\partial^{(j+1)}(x(j)y))z + \partial^{(j+1)}(x(j)y)u(n)z + \sum_{p=0}^{n-1} \binom{n}{p} (u(p)\partial^{(j+1)}(x(j)y))(n-p-1)z \right\} \quad (8.6.8) \]

Due to (0.4.7),

\[ C1 = - \sum_{j \geq 0} (-1)^j \partial^{(j+1)}(u(n)x(j)y) + \sum_{p=1}^{\min(n,j+1)} \binom{n}{p} \partial^{(j+1-p)}(u(n-p)x(j)y)z \quad (8.6.9) \]

Next,

\[ C11 = - \sum_{j \geq 0} (-1)^j \partial^{(j+1)} \left\{ (u(n)x(j)y) + x(j)u(n)y + \sum_{p=0}^{n-1} \binom{n}{p} (u(p)x)(n+j-p)y \right\} z \quad (8.6.10) \]

and

\[ C12 = - \sum_{j \geq 0} (-1)^j \sum_{p=1}^{\min(n,j+1)} \binom{n}{p} \partial^{(j+1-p)} \left\{ (u(n-p)x(j)y) + x(j)u(n-p)y + \sum_{q=0}^{n-p-1} \binom{n-p}{q} (u(q)x)(j+n-p-q)y \right\} z \quad (8.6.11) \]

We have \( A21 + B21 + C2 = r(x,y)u(n)z \in R \). Next, \( A23 = -B32 \) and \( A32 = -B23 \). Next,

\[ A21 + B1 = x(u(n)y)z - (u(n)y)xz \overset{R}{\sim} \sum_{j} (-1)^j \partial^{(j+1)}(x(j)u(n)y)z = -C112 \]
Similarly,
\[ A1 + B21 = (u_{(n)}x)yz - y(u_{(n)}x)z \sim \sum_j (-1)^j \partial^{(j+1)}((u_{(n)}x)_{(j)}y)z = -C111 \]

8.7. Below we shall often use the identity
\[ \binom{n}{p} \binom{p}{r} = \binom{n}{r} \binom{n-r}{n-p} \]  

(8.7.1)

We claim that
\[ C113 + C121 + C123 = -A31 \]  

(8.7.2)

Indeed, we have
\[ C113 = -\sum_{j \geq 0}^{n-1} \sum_{q=0}^{n-1} (-1)^j \binom{n}{q} \partial^{(j+1)}((u_{(q)}x)_{(n-j-q)})z \]  

(8.7.3)

and
\[ C121 + C123 = -\sum_{j \geq 0}^{\min(n,j+1)} \sum_{p=1}^{n-p-1} \binom{n}{p} \binom{n-p}{q} \partial^{(j+1-p)}((u_{(q)}x)_{(j+n-p-q)})z \]  

(8.7.4)

Using (8.7.1), we have
\[ \binom{n}{p} \binom{n-p}{q} = \binom{n}{n-p} \binom{n-p}{q} = \binom{n}{n-q} \binom{n-q}{p} \]

Consider the part of (8.7.4) at a fixed \( q \), \( 0 \leq q \leq n-1 \):
\[ (8.7.4)_q = -\binom{n}{q} \sum_{j \geq 0}^{\min(n-q,j+1)} \sum_{p=1}^{n-p-1} (-1)^j \binom{n-q}{p} \partial^{(j+1-p)}((u_{(q)}x)_{(j+n-p-q)})z \]

\[ = -\binom{n}{q} \sum_{p=1}^{n-q} \sum_{j \geq p-1} (-1)^j \binom{n-q}{p} \partial^{(j+1-p)}((u_{(q)}x)_{(j+n-p-q)})z \]  

(8.7.5)

It is easy to see that the part of (8.7.5) corresponding to \( j = p - 1 \) is equal to \(-A31_q\) and the part corresponding to \( j \geq p \) equals \(-C113_q\). This proves (8.7.2).

Using the commutativity formula (0.4.3), one sees that \( B31 = -C122 \).

8.8. We claim that
\[ A33 + B33 + C3 = 0 \]  

(8.8.1)
We have

\[ C^3 = - \sum_{j \geq 0} (-1)^j \sum_{p=0}^{n-1} \binom{n}{p} \sum_{q=0}^{\min(p,j+1)} \binom{p}{q} \partial^{j+1-q}(u_{(p-q)}x_j y)(n-p-1)^z = \]

\[ = - \sum_{j \geq 0} (-1)^j \sum_{p=0}^{n-1} \binom{n}{p} \sum_{q=0}^{\min(p,j+1)} \binom{p}{q} (-1)^{j+1-q} \binom{n-p-1}{j-q+1} (u_{(p-q)}x_j y)(n-j-p+q-2)^z = \]

(we set \( r := p - q \))

\[ = \sum_{j \geq 0} \sum_{p=0}^{n-1} \binom{n}{p} \sum_{r=p-j-1}^{p} (-1)^{p-r} \binom{p}{r} \binom{n-p-1}{j-p+r+1} (u_{(r)}x_j y)(n-j-r-2)^z = \]

\[ = \sum_{j \geq 0} \sum_{p=0}^{n-1} \binom{n}{p} \sum_{r=p-j-1}^{p} (-1)^{p-r} \binom{p}{r} \binom{n-p-1}{j-p+r+1} \]

\[ \{(x_j u_{(r)})^z (n-j-r-2) + \sum_{s=0}^{r} \binom{r}{s} ((u_{(s)})^z (r+j-s))^z (n-j-r-2)^z \} \]

(8.8.2)

So, we have written \( C^3 \) as a sum of two terms. Now we claim that

\[ C^{32} = -A^{33} \]  \hspace{1cm} (8.8.3)

Set \( l := r + j - s \). We have

\[ C^{32} = \sum_{l \geq 0, s \geq 0, l+s \leq n-2} \binom{n}{s} \sum_{p=0}^{l+s+1} \binom{l+s+1}{p} \min(p,l+s) \]

\[ (-1)^{p-r} \binom{n-s}{n-p} \binom{p-s}{r-s} \binom{n-p-1}{l+s-p+1} ((u_{(s)})^z (l+j y)(n-l-s-2)^z \]

where we have used that

\[ \binom{n}{p} \binom{p}{r} \binom{r}{s} = \binom{n}{r} \binom{r}{s} \binom{n-r}{n-p} = \binom{n}{s} \binom{n-r}{n-p} \binom{n-s}{p-s} \binom{n-s}{r-s} \]

Thus, at fixed \( l, s \),

\[ C^{32}_{l,s} = \binom{n}{s} \sum_{p=0}^{l+s+1} (-1)^{p} \binom{n-s}{n-p} \binom{n-p-1}{l+s-p+1} \min(p,l+s) \sum_{r=s}^{l+s+1} (-1)^{r} \binom{p-s}{r-s} ((u_{(s)})^z (l+j y)(n-l-s-2)^z \]

The last sum is non-zero in two cases:

(a) \( p = s \), so that \( r = s = p \), and we get the coefficient \( \binom{n}{s} \binom{n-s}{l+1} \),

and

(b) \( p = l + s + 1 \) where we get \( -\binom{n}{s} \binom{n-s}{l+1} \).
Using the identities

\[ \binom{n-s-1}{l+1} - \binom{n-s}{l+1} = - \binom{n-s-1}{l} \quad (8.8.4) \]

we get

\[ C32_{l,s} = - \binom{n}{s} \binom{n-s-1}{l} = -A33_{l,s} \]

which proves (8.8.3).

Next, we claim that

\[ C31 = -B33 \quad (8.8.5) \]

Indeed,

\[
C31 = \sum_{j \geq 0} \sum_{p=0}^{n-1} (-1)^{p-r} \binom{n}{p} \sum_{r=\max(0,p-j-1)}^{p} \binom{n-p-1}{j-p+r+1} \binom{p}{r} (x(j) u(r))_{(n-j-r-2)z} = 
\]

(\text{using (0.4.3)})

\[
= \sum_{j \geq 0} \sum_{p=0}^{n-1} (-1)^{j+p} \binom{n}{p} \sum_{r=\max(0,p-j-1)}^{p} \binom{n-p-1}{j-p+r+1} \binom{p}{r} (-1)^{j+1} 
\]

\[
\sum_{s \geq 0} \binom{n-j-r-2}{s} (u(r) y)_{(j+s)x} (n-j-s-r-2)z 
\]

Set \( l := j+s \). We get

\[
C31 = \sum_{l \geq 0, r \geq 0, l+r \leq n-2} \sum_{s \geq 0, p \geq r+s \leq r+l+1} (-1)^{p-r+l-s+1} \binom{n}{p} \binom{p}{r} \binom{n-p-1}{l+r-s-p+1} \binom{n-l-r+s-2}{s} (u(r) y)_{(l+s)x} (n-l-r-2)z 
\]

Using the identities \( \binom{n}{p} \binom{p}{r} = \binom{n-r}{n-p} \) and

\[
\binom{n-p-1}{l+r-s-p+1} \binom{n-l-r+s-2}{s} = \binom{n-p-1}{n-l+s-r-2} \binom{n-l-r+s-2}{s} = 
\]

\[
= \binom{n-p-1}{n-p-s-1} \binom{n-p-s-1}{n-l-r-2} = \binom{n-p-1}{n-p-s-1} \binom{n-p-s-1}{n-l-r-2} = 
\]

\[
= \binom{n-p-1}{n-l-r-2} \binom{l+r-p+1}{s} 
\]
we get at fixed \( l, r \)

\[
C31_{l,r} = (-1)^{l+r+1} \binom{n}{r}^{r+l+1} \sum_{p=r} \binom{n-r}{p} (-1)^p \binom{n-p-1}{n-l-r-2}
\]

\[
\sum_{s=0}^{\min(r+l-p+1,l)} (-1)^s \binom{l+r-p+1}{s} ((u(r)y)_j)^x_{(n-l-r-2)} z
\]

The last sum is non-zero in two cases:

(a) \( p = l + r + 1 \), so that \( s = 0 \), and we get the coefficient \( \binom{n}{r} \binom{n-r}{l+1} \),

and

(b) \( p = r \), so that we get the coefficient \( -\binom{n}{r} \binom{n-r-1}{l+1} = -\binom{n}{l+1} \).

It follows that

\[
C31_{r,l} = \binom{n}{r} \binom{n-r-1}{l} ((u(r)y)_j)^x_{(n-l-r-2)} z = -B33_{r,l}
\]

which proves (8.8.5). The identities (8.8.3) and (8.8.5) together imply (8.8.1).

After bookkeeping, the computations of 8.6 — 8.8 show that the left hand side of (8.6.1) belongs to the ideal \( R \), which finishes the proof of Lemma 8.5. \( \triangle \)

**8.9.** By Lemma 8.5, the operators \( x_{(j)} \) induce operators

\[
x_{(j)} : UC \rightarrow UC, \ x \in C, j \in \mathbb{Z}
\]  

\[ (8.9.1) \]

**8.10. Lemma.** The operators (8.9.1) satisfy the OPE formula (0.5.12).

This lemma will be proven in 8.11 — 8.19 below.

**8.11.** We have to prove the identity between the operators acting on \( UC \),

\[
[x_{(m)}, y_{(n)}] = \sum_{j \geq 0} \binom{m}{j} (x_{(j)} y)_{(m+n-j)}
\]  

\[ (8.11.1) \]

for all \( m, n \in \mathbb{Z} \).

Let us discuss separately three cases.

**8.12. Case A.** \( m < 0 \) and \( n < 0 \). First of all, note that for \( m = n = -1 \) the relation (8.11.1) is nothing but (8.2.1) and holds true by definition, since \( R \) is a left ideal.

In general, set \( m = -1 - a, n = -1 - b \) for \( a, b \geq 0 \). We have by definition (8.4.1)

\[
[x_{(-1-a)}, y_{(-1-b)}] = [\partial^{(a)} x, \partial^{(b)} y] = \sum_{j \geq 0} (-1)^j \partial^{(a+j)} ((\partial^{(a)} x)_{(j)} \partial^{(b)} y)
\]  

\[ (8.12.1) \]
The right hand side of (8.11.1) is equal to
\[
\sum_{j \geq 0} \left( -1 - \frac{a}{j} \right) (x(j) y) (-2 - a - b - j) = \sum_{j \geq 0} (-1)^j \binom{a + j}{j} \partial^{(a+b+j+1)} (x(j) y) \quad (8.12) 
\]
where we have used that
\[
\binom{-1 - a}{j} = (-1)^j \binom{a + j}{j} 
\]
for \(a, j \geq 0\).

On the other hand,
\[
(\partial^a x)_{(j)} (\partial^b y) = (-1)^a \binom{j}{a} x_{(j-a)} \partial^b y = (-1)^a \binom{j}{a} \sum_{p=0}^b \binom{j - a}{p} \partial^{(b-p)} (x_{(j-a-p)} y),
\]
by (0.4.2) and (0.4.7). Therefore, the rhs of (8.12.1) is equal to
\[
\sum_{j \geq 0} (-1)^j + a \binom{j}{a} \sum_{p=0}^b \binom{j - a}{p} \partial^{(j+1)} \partial^{(b-p)} (x_{(j-a-p)} y) = \\
= \sum_{j \geq 0} (-1)^j + a \min(b, j-a) \sum_{p=0}^b \binom{j - a}{p} \left( \binom{j + 1 + b - p}{j + 1} \partial^{(j+1+b-p)} (x_{(j-a-p)} y) \\
\text{(substituting } k := j - a - p)\\n= \sum_{k \geq 0} \sum_{p=0}^b (-1)^{p+k} \binom{k + a + p}{a} \binom{k + p}{p} \binom{k + a + b + 1}{k + a + p + 1} \partial^{(k+a+b+1)} (x_{(k)} y) = \\
\text{(using that } \binom{k+a+p}{a} \binom{k+p}{p} = \binom{k+a+p}{k+p} \binom{k+p}{p} = \binom{k+a}{k+a})\\n= \sum_{k \geq 0} (-1)^k \binom{k + a}{a} \left( \sum_{p=0}^b (-1)^p \binom{k + a + p}{p} \binom{k + a + b + 1}{k + a + p + 1} \right) \partial^{(k+a+b+1)} (x_{(k)} y) \\
\quad (8.12)\)
\]

8.13. Lemma. For all \(b, q \in \mathbb{Z}_{\geq 0}\),
\[
\sum_{p=0}^b (-1)^p \binom{p + q}{p} \binom{b + q + 1}{p + q + 1} = 1
\]
This is easily proved by induction on \(b\). \(\triangle\)

It follows from this lemma that (8.12.4) is equal to (8.12.2), which completes the check of Case A.
8.14. **Case B.** \( m \geq 0 \) and \( n < 0 \). If \( n = -1 \) then (8.11.1) is the same as the definition (8.4.2). Now let \( n = -1 - a, \ a \geq 0 \). The lhs of (8.11.1) is equal to

\[
[x(m), y(-1-a)] = [x(m), \partial^{(a)} y] = \sum_{j=0}^{m-1} \binom{m}{j} (x(j) \partial^{(a)} y)_{(m-j-1)} + x(m) \partial^{(a)} y \quad (8.14.1)
\]

The rhs of (8.11.1) is equal to

\[
\sum_{k=0}^{m} \binom{m}{k} (x(k)y)_{(m-1-a-k)} =
\]

\[
= \sum_{k=0}^{m-1-a} \binom{m}{k} (x(k)y)_{(m-1-a-k)} + \sum_{k \geq \max(m-a,0)} \binom{m}{k} (x(k)y)_{(m-1-a-k)} \quad (8.14.2)
\]

Consider the first sum in the rhs of (8.14.1). It is equal to

\[
\sum_{j=0}^{m-1} \binom{m}{j} \min(j,a) \binom{m-j}{p} \binom{j}{p} \partial^{(a-p)} (x(j-p)y)_{(m-j-1)} =
\]

\[
= \sum_{j=0}^{m-1} \binom{m}{j} \sum_{p=0}^{\min(j,a)} \binom{m-j}{a-p} \sum_{k=\max(0, a-m+j+1)} \binom{k}{p} (-1)^{a-p} (m-j-1)_{a-p} (x(j-p)y)_{(m-j-a+p-1)} =
\]

(making the substitution \( k = j - p \))

\[
= \sum_{k=0}^{m-a-1} \sum_{p=0}^{a} (-1)^{a-p} \binom{m}{k+p} \binom{m-p-k-1}{a-p} (x(k)y)_{(m-k-a-1)} =
\]

(using that \( \binom{m}{k+p} \binom{k}{p} = \binom{m}{k} \binom{m-k}{p-k} \))

\[
= \sum_{k=0}^{m-a-1} \binom{m}{k} \left\{ \sum_{p=0}^{a} (-1)^{a-p} \binom{m-k}{m-p-k} \binom{m-p-k-1}{a-p} \right\} (x(k)y)_{(m-k-a-1)} =
\]

(substituting \( r = p - a, \ s = m - a - k - 1 \geq 0 \))

\[
= \sum_{k=0}^{m-a-1} \binom{m}{k} \left\{ \sum_{r=0}^{a} (-1)^r (a+s+1) \binom{r+s}{r} \right\} (x(k)y)_{(m-k-a-1)}
\]

By Lemma 8.13, this is equal to the first sum in the rhs of (8.14.2).

The second term in the rhs of (8.14.1) is equal to

\[
x(m) \partial^{(a)} y = \sum_{p=0}^{\min(m,a)} \binom{m}{p} \partial^{(a-p)} (x(m-p)y) =
\]
(substituting $k = m - p$)

$$
\sum_{k \geq \text{max}(m-a,0)} \binom{m}{m-k} \partial^{(a+m+k)}(x(k)y) = \sum_{k \geq \text{max}(m-a,0)} \binom{m}{k} (x(k)y)^{-1+a+m-k},
$$

which is the same as the second sum in (8.14.2).

Therefore, (8.14.1)=(8.14.2), which finishes the proof of (8.11.1) in the Case B.

8.15. Case C. $m < 0$ and $n \geq 0$. First let us treat the case $m = -1$. We need to prove that

$$
[x(-1), y(n)] = \sum_{j \geq 0} (-1)^{j+1} (x(j)y)(n-j)
$$

(8.15.1)

The lhs of (8.15.1) is equal to

$$
-[y(n), x(-1)] = -\sum_{p \geq 0} \binom{n}{p} (y(p)x)(n-p-1)
$$

(8.15.2)

On the other hand, the rhs of (8.15.1) equals (we use (0.4.3))

$$
-\sum_{j \geq 0} \sum_{q \geq 0} (-1)^{j+q+1} (y(j+q)x)(n-j-q) = \sum_{j \geq 0, q \geq 0} \binom{n-1-j}{q} (y(j+q)x)(n-j-q) =
$$

(substituting $p = j + q$)

$$
= -\sum_{p \geq 0} (y(p)x)(n-1-p) \cdot \left\{ \sum_{q=0}^{p} \binom{n-1-p+q}{q} \right\}
$$

Therefore, (8.15.1) follows from the identity below, which is easily checked by induction on $n$:

$$
\sum_{q=0}^{p} \binom{n-1-p+q}{q} = \binom{n}{p}
$$

(8.15.3)

for all $n, p \geq 0$.

Now assume that $m = -1-a, \ a \geq 0$. We need to prove that

$$
[x(-1-a), y(n)] = \sum_{p \geq 0} \binom{-1-a}{p} (x(p)y)(-1-a+n-p)
$$

(8.15.4)

The lhs is equal to

$$
[\partial^{(a)} x(-1), y(n)] = \sum_{j \geq 0} (-1)^{j} \partial^{(a+j)}(x(j)y)(n-1-j) = \sum_{j \geq a} (-1)^{j+a} \binom{j}{a} (x(j-a)y)(n-1-j) =
$$

(substituting $p = j - a$)

$$
= \sum_{p \geq 0} (-1)^{p} \binom{p+a}{a} (x(p)y)(n-1-p-a)
$$
so (8.15.4) follows from (0.0.2). The completes the proof of Case C.

8.16. Case D. \( m, n \geq 0 \). We have to prove that

\[
[x^{(m)}, y^{(n)}]v = \sum_{j=0}^{m} \binom{m}{j} (x^{(j)}y)_{(m+n-j)}v
\] (8.16.1)

for all \( x, y \in C \), \( v \in UC \). In fact, we shall prove a stronger statement, which we prefer to formulate as a separate

8.17. Lemma. If \( m \) and \( n \) are nonnegative then the OPE identity (8.11.1) holds true on TC. In other words, (8.16.1) holds true for all \( m, n \geq 0 \), \( x, y \in C \) and \( v \in TC \).

We shall prove this by induction on the length of \( v \). If \( v \in C \) then the desired identity holds true by definition of a conformal algebra, (0.4.4). Now, let \( v = zu \) where \( z \in C \), \( u \in TC \). So, we have to prove that

\[
[x^{(m)}, y^{(n)}]zu = \sum_{j=0}^{m} \binom{m}{j} (x^{(j)}y)_{(m+n-j)}zu
\] (8.17.1)

Consider the lhs first. We have

\[
A := x^{(m)}y^{(n)}zu = x^{(m)} \left\{ zy^{(n)}u + (y^{(n)}z)u + \sum_{p=0}^{n-1} \binom{n}{p} (y^{(p)}z)_{(n-p-1)}u \right\} = \\
= zx^{(m)}y^{(n)}u + (x^{(m)}z)y^{(n)}u + \sum_{q=0}^{m-1} \binom{m}{q} (x^{(q)}y)_{(m-q-1)}(y^{(n)}u) +
\]

\[
= (y^{(n)}z)x^{(m)}u + (x^{(m)}y^{(n)}z)u + \sum_{q=0}^{m-1} \binom{m}{q} (x^{(q)}y^{(n)}z)_{(m-q-1)}u + \sum_{p=0}^{n-1} \binom{n}{p} x^{(m)}(y^{(p)}z)_{(n-p-1)}u
\]

Similarly,

\[
B := -y^{(n)}x^{(m)}zu = -zy^{(n)}x^{(m)}u - (y^{(n)}z)x^{(m)}u - \sum_{p=0}^{n-1} \binom{n}{p} (y^{(p)}z)_{(n-p-1)}x^{(m)}u -
\]

\[
- (x^{(m)}z)y^{(n)}u - (y^{(n)}x^{(m)}z)u - \sum_{p=0}^{n-1} \binom{n}{p} (y^{(p)}x^{(m)}z)_{(n-p-1)}u - \sum_{q=0}^{m-1} \binom{m}{q} y^{(n)}(x^{(q)}z)_{(m-q-1)}u
\]

Obviously, the lhs of (8.17.1) is equal to \( A + B \), and we should compare this with the rhs:

\[
R := \sum_{j=0}^{m} \binom{m}{j} (x^{(j)}y)_{(m+n-j)}zu = \sum_{j=0}^{m} \binom{m}{j} \left\{ z(x^{(j)}y)_{(m+n-j)}u +
\]

\[+ ((x^{(j)}y)_{(m+n-j)}z)u + \sum_{r=0}^{m+n-j-1} \binom{m+n-j}{r} ((x^{(j)}y)_{(r)}z)_{(m+n-j-r-1)}u \right\}
\]
First of all, $A_1 + B_1 = R_1$ by induction hypothesis. Next, $A_2 = -B_4$ and $A_4 = -B_2$. Further, $A_5 + B_5 = R_2$ by the axiom of a conformal algebra. Next,

$$C := A_3 + B_7 = \sum_{q=0}^{m-1} \binom{m}{q} \left\{ (x(q)z)_{m-q-1}y(n)u - y(n)(x(q)z)_{m-q-1}u \right\} =$$

(by induction hypothesis)

$$= - \sum_{q=0}^{m-1} \binom{m}{q} \sum_{s=0}^{n} \binom{n}{s} (y(s)x(q)z)_{m+n-q-s-1}u$$

Similarly,

$$D := A_7 + B_3 = \sum_{p=0}^{n-1} \binom{n}{p} \sum_{r=0}^{m} \binom{m}{r} (x(r)y(p)z)_{m+n-p-r-1}u$$

We have

$$E := A_6 + C_{s=n} = \sum_{q=0}^{m-1} \binom{m}{q} \left\{ (x(q)y(n)z)_{m-q-1} - (y(n)x(q)z)_{m-q-1} \right\} u =$$

$$= \sum_{q=0}^{m-1} \sum_{s=0}^{q} \binom{m}{q} \binom{q}{s} ((x(s)y)_{q+n-s}z)_{m-q-1}u =$$

$$= \sum_{q=0}^{m-1} \sum_{s=0}^{q} \binom{m-s}{m-q} ((x(s)y)_{q+n-s}z)_{m-q-1}u$$

Similarly,

$$F := D_{r=m} + B_6 = \sum_{p=0}^{n-1} \binom{n}{p} \left\{ (x(m)y(p)z)_{n-p-1} - (y(p)x(m)z)_{n-p-1} \right\} u =$$

$$= \sum_{p=0}^{n-1} \sum_{r=0}^{m} \binom{n}{p} \binom{m}{r} ((x(r)y)_{m+p-r}z)_{n-p-1}u$$

Finally,

$$G := C_{s<n} + D_{r<m} = \sum_{p=0}^{n-1} \sum_{q=0}^{m-1} \binom{n}{p} \binom{m}{q} \left\{ (x(q)y(p)z)_{m+n-p-q-1} - (y(p)x(q)z)_{m+n-p-q-1} \right\} u =$$

$$= \sum_{p=0}^{n-1} \sum_{q=0}^{m-1} \binom{n}{p} \binom{m}{q} \sum_{s=0}^{q} \binom{q}{s} ((x(s)y)_{p+q-s}z)_{m+n-p-q-1}u =$$

$$= \sum_{p=0}^{n-1} \sum_{q=0}^{m-1} \binom{n}{p} \sum_{s=0}^{q} \binom{m}{s} \binom{m-s}{m-q} ((x(s)y)_{p+q-s}z)_{m+n-p-q-1}u$$
A careful examination of the sums involved, together with a formula

$$\sum_{p} \left( \frac{a}{r-p} \right) \binom{n}{p} = \binom{a+n}{r}$$  \hspace{1cm} (8.17.2)

with $a = m - j$, which is obvious from the combinatorial definition of the binomial coefficients, shows that $E + F + G = R3$.

This completes the proof of Lemma. △

8.18. Remark. Lemmas 8.17 and 8.4.1 mean that $TC$ is canonically a module over conformal algebra $C$.

8.19. Lemma 8.17 implies Case D, which completes the proof of Lemma 8.10. △

8.20. Now we can finish the proof of Theorem 8.3. According to Lemmas 8.5 and 8.10, we have the collection of mutually local fields $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$, $a \in C$, acting on the space $UC$. These fields satisfy the conditions of Theorem 0.7, which gives the desired structure of a vertex algebra on $UC$.

The other claims of Theorem 8.3 are obvious. △

8.21. Let $C$ be a conformal algebra. Let us introduce an operation $[x, y]$ on the space $C$ by

$$[x, y] = \sum_{j \geq 0} (-1)^j j^{(j+1)} (x(j)y) \hspace{1cm} (8.21.1)$$

8.22. Theorem. The operation (8.21.1) is a Lie bracket on $C$.

The space $C$ with the Lie algebra structure given by (8.21.1) will be denoted $C_{\text{Lie}}$.

The proof is given in 8.23 — 8.26 below.

8.23. Skew symmetry. Using (0.4.3), we have

$$[x, y] = \sum_{j \geq 0} (-1)^j j^{(j+1)} \left\{ (-1)^{j+1} \sum_{p \geq 0} (-1)^p p^{(p)} (y(j+p)x) \right\} =$$

$$= - \sum_{j,p \geq 0} (-1)^p \binom{j+1+p}{p} (y(j+p)x) =$$

$$= - \sum_{n \geq 0} \left\{ \sum_{p=0}^{n} (-1)^p \binom{n+1}{p} p^{(n+1)} (y(n)x) = -[y, x], \text{ QED} \right. \hspace{1cm} (8.23.1)$$

8.24. Jacobi identity. We need to prove that

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

This is an obvious consequence of two lemmas below.
8.25. Lemma. For all \( x, y, z \in C \)

\[
[x, [y, z]] = \sum_{a, b \geq 0} (-1)^{a+b} \partial^{(a+b+2)} (x_{(a)} y_{(b)} z) \quad (8.25.1)
\]

Indeed, we have

\[
[x, [y, z]] = \sum_{j \geq 0} (-1)^j [x, \partial^{(j+1)} (y_{(j)} z)] = \sum_{i, j \geq 0} (-1)^{i+j} \partial^{(i+1)} (x_{(i)} \partial^{(j+1)} (y_{(j)} z)) =
\]

(using (0.4.6))

\[
= \sum_{i, j \geq 0} (-1)^{i+j} \partial^{(i+1)} \left\{ \sum_{p \geq 0} \binom{i}{p} \partial^{(j+1-p)} (x_{(i-p)} y_{(j)} z) \right\} = \]

\[
= \sum_{i, j \geq 0} (-1)^{i+j} \sum_{p \geq 0} \binom{i}{p} \partial^{(j+2-p)} (x_{(i-p)} y_{(j)} z) =
\]

(substituting \( a := i - p, \ b := j \))

\[
= \sum_{a, b \geq 0} (-1)^{a+b} \partial^{(a+b+2)} (x_{(a)} y_{(b)} z) \cdot \sum_{p \geq 0} (-1)^p \binom{a+p}{p} \binom{a+b+2}{a+p+1},
\]

and we conclude by Lemma 8.13.

8.26. Lemma. For all \( x, y, z \in C \)

\[
[[x, y], z] = \sum_{a, b \geq 0} (-1)^{a+b} \partial^{(a+b+2)} \{ x_{(a)} y_{(b)} z - y_{(b)} x_{(a)} z \} \quad (8.26.1)
\]

Indeed, we have

\[
[[x, y], z] = \sum_{j \geq 0} (-1)^j \sum_{i \geq 0} (-1)^i \partial^{(i+1)} \{ \partial^{(j+1)} (x_{(j)} y_{(i)} z) \} =
\]

\[
= \sum_{i, j \geq 0} (-1)^{i+j+1} \binom{i}{j+1} \partial^{(i+1)} ((x_{(j)} y_{(i-j-1)} z) =
\]

(by (0.4.5))

\[
= \sum_{i, j \geq 0} (-1)^{i+j+1} \sum_{q=0}^{j} (-1)^q \binom{j}{q} \partial^{(i+1)} \{ x_{(j-q)} y_{(i-j-q-1)} z - (-1)^j y_{(i-q-1)} x_{(q)} z \},
\]

and we again conclude after an easy rearrangement of the indices and using Lemma 8.13.

This completes the proof of lemma and of the theorem. \( \triangle \)
8.27. **Corollary.** The vertex envelope $UC$ of a conformal algebra $C$ is equal to the associative enveloping algebra $UC^{\text{Lie}}$ of the Lie algebra $C^{\text{Lie}}$.

If $i : C \rightarrow UC$ denotes the canonical morphism,

$$i([x, y]) = i(x)(-1)i(y) - i(y)(-1)i(x) \quad (8.27.1)$$

for all $x, y \in C$.

This follows immediately from the definition in 8.2.

Note that in an arbitrary vertex algebra, although the operation $x(\cdot -1)y$ is not associative, the commutator $x(\cdot -1)y - y(\cdot -1)x$ is a Lie bracket, according to a remark by A. Radul cited in [K], 3.1, page 82.

This corollary supports the point of view advocated by Beilinson and Drinfeld, namely, that conformal algebras (Lie*-algebras in the language of BD) are analogs of Lie algebras, while vertex algebras (chiral algebras) are analogs of associative algebras.

8.28. Let us define a canonical increasing filtration $FUC$ on $UC$ by induction:

$$F_{-1}UC = 0, \quad F_0UC = k \cdot 1, \quad F_{i+1}UC = F_iUC + C_{(-1)}F_iUC. \quad (8.28.1)$$

Obviously, the operation $x(\cdot -1)y$ induces on the associated graded space $gr_F(UC) := \oplus_{i \geq 0} gr^i_F(UC) := \oplus_i F_iUC/F_{i-1}UC$ a structure of a commutative associative unitary $k$-algebra.

We have an evident surjective morphism of commutative algebras

$$Sym_k(C) \rightarrow gr_F(UC) \quad (8.28.1)$$

8.29. **Theorem.** If $C$ is a free $k$-module then (8.28.1) is an isomorphism.

This is immediate consequence of Corollary 8.27 and the usual Poincaré-Birkhoff-Witt theorem for Lie algebras, cf. [Se], Part I, Chapter III, Theorem 4.3.

§9. **Enveloping Algebra of a Vertex Algebroid**

**Abelian case**

9.1. Let us define a $\partial$-module (over $k$) to be a $\mathbb{Z}_{\geq 0}$-graded $k$-module $C = \oplus_{i \geq 0} C_i$ equipped with a family of endomorphisms $\partial^{(j)} : C \rightarrow C, \ j \geq 0$, where $\partial^{(0)}$ has degree $j$, $\partial^{(0)} = Id$ and

$$\partial^{(i)}\partial^{(j)} = \binom{i+j}{j} \partial^{(i+j)}.$$

Obviously, a $\partial$-module is the same as an abelian conformal algebra, i.e. a conformal algebra in which all operations $(n)$ are trivial. These objects, with obvious morphisms, form a category $\partial-\text{Mod}$. 
Let us define a 1-truncated $\partial$-module to be a triple $X = (C_0, C_1, \partial)$ where $C_i$ are $k$-modules and $\partial : C_0 \to C_1$ is a map of $k$-modules. These objects form a category $\partial - \text{Mod}_{\leq 1}$. We have an obvious truncation functor

$$\partial - \text{Mod} \to \partial - \text{Mod}_{\leq 1} \quad (9.1.1)$$

It is easy to construct the left adjoint to (9.1.1). Namely, given $X = (C_0, C_1, \partial) \in \partial - \text{Mod}_{\geq 1}$, consider the direct sum

$$M = C_0 \oplus C_1 \oplus (\oplus_{i \geq 2} \partial^{(i)} C_0) \oplus (\oplus_{i \geq 1} \partial^{(i)} C_1) \quad (9.1.2)$$

where $\partial^{(i)} X$ denotes a copy of a $k$-module $X$, whose elements are denoted $\partial^{(i)} x$, $x \in X$. Define a $\partial$-module $X^\partial = \oplus_{i \geq 0} C_i$ as the quotient of $M$ modulo the submodule generated by the following elements:

$$\partial^{(i+1)} x - (i + 1) \partial^{(i)} \partial(x), \ i \geq 1, x \in C_0 \quad (9.1.3)$$

The grading and operators $\partial^{(i)} : C_i \to C_i+j$ are defined in the obvious manner.

Note that if the ground ring $k$ contains the field of rational numbers $\mathbb{Q}$ then $X$ is equal simply to

$$X^\partial = C_0 \oplus C_1 \oplus (\oplus_{i \geq 1} \partial^{(i)} C_1) \quad (9.1.4)$$

9.2. Recall that in 0.9 one introduced the category $\partial - \text{Alg}$ of $\partial$-algebras which are identified with commutative vertex algebras. A $\partial$-ideal $J$ in a $\partial$-algebra $B$ is a $\mathbb{Z}_{\geq 0}$-graded ideal stable under all endomorphisms $\partial^{(i)}$; this is the same as a vertex ideal if $B$ is understood as a vertex algebra.

We have an obvious forgetful functor

$$\partial - \text{Alg} \to \partial - \text{Mod} \quad (9.2.1)$$

This functor admits a left adjoint

$$\text{Sym} : \partial - \text{Mod} \to \partial - \text{Alg} \quad (9.2.2)$$

that assigns to a $\partial$-module $C$ the symmetric algebra (over $k$) $\text{Sym}(C)$. It is nothing but the restriction of the functor defined in 8.3 to the full subcategory of abelian conformal algebras.

9.3. Jet algebra. Let us call a vertex algebroid $A = (A, T, \Omega, \ldots)$ abelian if $T = 0$. Thus, an abelian vertex algebroid is simply a triple $A = (A, \Omega, \partial)$ where $A$ is a commutative algebra, $\Omega$ is an $A$-module and $\partial : A \to \Omega$ is a derivation. Thus, abelian vertex algebroids are the same as ”1-truncated $\partial$-algebras”. The category of abelian vertex algebroids will be denoted $\text{AlgAb}$; it is a full subcategory of $\text{Alg}$.

We have an obvious forgetful functor

$$o : \text{AlgAb} \to \partial - \text{Mod} \quad (9.3.1)$$

and the truncation functor

$$\partial - \text{Alg} \to \text{AlgAb} \quad (9.3.2)$$
Let us construct a left adjoint to (9.3.2). Given $\mathcal{A} = (A, \Omega, \partial) \in Alg_Ab$, consider the $\partial$-module $(a\mathcal{A})^\partial$ and its symmetric algebra $S(\mathcal{A}) := Sym((a\mathcal{A})^\partial) \in \partial - Alg$. Note that the grading on $S(\mathcal{A})$ is defined in such a way that $A \subset S(\mathcal{A})$ has grading 0 and $\Omega \subset S(\mathcal{A})$ has grading 1.

Let $R \subset S(\mathcal{A})$ be a $\partial$-ideal generated by all elements

$$1_A - 1_{S(\mathcal{A})}; \; ab - a \cdot b; \; a\omega - a \cdot \omega, \quad (9.3.3)$$

$a, b \in A, \; \omega \in \Omega$ (note that these elements are indeed homogeneous). We denote the quotient $\partial$-algebra $S(\mathcal{A})/R$ by $J(\mathcal{A})$, and call it the jet algebra of $\mathcal{A}$. The assignment $\mathcal{A} \mapsto J(\mathcal{A})$ defines a functor

$$J : Alg_Ab \longrightarrow \partial - Alg \quad (9.3.4)$$

left adjoint to (9.3.2).

The composition $A \hookrightarrow (a\mathcal{A})^\partial = Sym((a\mathcal{A})^\partial) \hookrightarrow S(\mathcal{A}) \longrightarrow J(\mathcal{A})$ defines a map

$$i_A : A \longrightarrow J(\mathcal{A}) \quad (9.3.5)$$

which is a map of $k$-algebras, due to the relations (9.3.3), which identifies $A$ with $J(\mathcal{A})_0$. This makes $J(\mathcal{A})$ an $A$-algebra.

A similar composition defines a map

$$i_\Omega : \Omega \longrightarrow J(\mathcal{A}) \quad (9.3.6)$$

which is a map of left $A$-modules, again due to the relations (9.3.3). We have the compatibility: $\partial(i_\Omega(\alpha)) = i_\Omega(\partial(\alpha))$, and the triple $(J(\mathcal{A}), i_A, i_\Omega)$ has a

**9.3.1. Universal Property.** Given a triple $(B, i'_A, i'_\Omega)$ where $B$ is a $\partial$-algebra, $i'_A : A \longrightarrow B$ a morphism of $k$-algebras such that $i'_A(\mathcal{A}) \subset B_0$, $i'_\Omega : \Omega \longrightarrow B$ a morphism of $A$-modules such that $i'_\Omega(\Omega) \subset B_1$ and $i'_\Omega(\partial(\alpha)) = \partial(i'_A(a))$, there exists a unique map of $\partial$-algebras $f : J(\mathcal{A}) \longrightarrow B$ such that $f \circ i_A = i'_A$ and $f \circ i_\Omega = i'_\Omega$.

**9.4.** Let $J^+ = \oplus_{i \geq 1} J(\mathcal{A})_i$ be the augmentation ideal. Consider the associated graded algebra with respect to the $J^+$-adic filtration

$$grJ(\mathcal{A}) = \oplus_{i \geq 0} J^{+i}/J^{+i+1} \quad (9.4.1)$$

It inherits a $\mathbb{Z}_{\geq 0}$-grading from $J(\mathcal{A})$. Let $\Omega^{(i)}$ denote the image of the composition

$$\partial^{(i)} A \oplus \partial^{(i-1)} \Omega \longrightarrow J^+ \longrightarrow J^+ / J^+ $$

Note that $\Omega^{(i)}$ is an $A$-submodule of $J^+ / J^+^2$, and is contained inside the homogeneous component $(J^+ / J^+^2)_i$.

Adding up, we get a map of $A$-modules

$$\oplus_{i \geq 1} \Omega^{(i)} \longrightarrow J^+ / J^+^2 \quad (9.4.2)$$

and hence a morphism of $A$-algebras

$$Sym_A(\oplus_{i \geq 1} \Omega^{(i)}) \longrightarrow Sym_A(J^+ / J^+^2) \longrightarrow grJ(\mathcal{A}) \quad (9.4.3)$$
the second arrow being the evident canonical map.

**9.5. Theorem.** Assume that $k \supset \mathbb{Q}$ and $\Omega$ is a free $A$-module. Then

(i) the maps $\Omega \to \Omega^{(i)}$ sending $\omega$ to (the image of) $\partial^{(i-1)}\omega$, $i \geq 1$, are isomorphisms of $A$-modules;

(ii) both maps in (9.4.3) are isomorphisms of $A$-algebras.

Indeed, choose an $A$-base $\{\omega_r\}$ of $\Omega$. Let $\tilde{\Omega}^{(i)}$ denote a free $A$-module with the base $\{\partial^{(i-1)}\omega_r\}$. Let $\tilde{J}$ be the symmetric algebra $\text{Sym}_A\{A \oplus (\oplus_{i \geq 1} \tilde{\Omega}^{(i)})\}$. It has an obvious structure of a $\partial$-algebra, and satisfies the universal property 9.3.1. Therefore, we have canonical isomorphism

$$J(A) \xrightarrow{\sim} \tilde{J} \tag{9.5.1}$$

We can apply the construction of 9.4 to the algebra $\tilde{J}$ as well, and for it the claims of Theorem 9.5 are clear. On the other hand, the maps $\Omega^{(i)} \to \tilde{\Omega}^{(i)}$ induced by (9.5.1) are isomorphisms. This implies the theorem. $\triangle$

**Conformal algebra associated with a vertex algebroid**

**9.6.** Following the pattern of 3.1, let us define a 1-truncated conformal algebra to be a quintuple $c = (C_0, C_1, \partial, \alpha_0, \alpha_1)$ where $C_i$ are $k$-modules, $\partial : C_0 \to C_1$ is a $k$-module map and $\alpha_i : (C_0 \oplus C_1) \otimes_2 \to (C_0 \oplus C_1)$ are bilinear operations of degree $-i - 1$.

Elements of $C_0$ (resp. of $C_1$) will be denoted $a, b, c$ (resp. $x, y, z$).

These data must satisfy the following axioms.

**(Derivation)**

$$\begin{align*}
(\partial a)_{(0)} &= 0; \\
(\partial a)_{(1)} &= -a_{(0)}; \\
\partial(x_{(0)}a) &= x_{(0)}\partial a
\end{align*} \tag{Der}$$

**(Commutativity)**

$$\begin{align*}
x_{(0)}a &= -a_{(0)}x; \\
x_{(0)}y &= -y_{(0)}x + \partial(y_{(1)}x); \\
x_{(1)}y &= y_{(1)}x
\end{align*} \tag{Com}$$

**(Associativity)** For all $\alpha, \beta, \gamma \in C_0 \oplus C_1$ and $i = 0, 1$,

$$\alpha_{(0)}\beta_{(i)}\gamma = (\alpha_{(0)}\beta)_{(i)}\gamma + \beta_{(i)}\alpha_{(0)}\gamma \tag{Ass}$$

1-truncated conformal algebras, with obvious morphisms, form a category $\text{Conf}_{\leq 1}$.

We have an evident forgetful functor

$$\text{Vert}_{\leq 1} \to \text{Conf}_{\leq 1} \tag{9.6.1}$$

**9.7.** We have an obvious truncation functor

$$\text{Conf} \to \text{Conf}_{\leq 1} \tag{9.7.1}$$
which assigns to a conformal algebra $C$ its part $C_{\leq 1}$ of degree $\leq 1$. Let us construct a left adjoint to this functor.

Given $c = (C_0, C_1, \ldots) \in Conf_{\leq 1}$, consider $c$ as a 1-truncated $\partial$-module (forgetting the operations), and consider the corresponding $\partial$-module $C := c^0$, cf. 9.1.

9.8. Theorem. There is a unique structure of a conformal algebra on $C$ such that operations $(i)$ and $\partial$ on the subspace $C_{\leq 1}$ coincide with the ones given by the structure of a 1-truncated conformal algebra on $c$.

The assignement $c \mapsto C$ gives a functor

$$Conf_{\leq 1} \rightarrow Conf$$

left adjoint to (9.7.1).

Indeed, uniqueness is clear, due to the axioms (0.4.2) and (0.4.3) of a conformal algebra. We leave the proof of existence to the reader. The claim about adjointness is evident.

9.9. Let $A = (A, T, \Omega, \ldots)$ be a vertex algebroid. We assign to $A$ a 1-truncated conformal algebra $cA = (C_0, C_1, \ldots)$ by setting $C_0 = A$, $C_1 = T \oplus \Omega$, $\partial : C_0 \rightarrow C_1$ to be the composition of $\partial : A \rightarrow \Omega$ with the embedding $\Omega \subset C_1$, and defining the operations $(i)$ by

$$a(0)b = a(0)\omega = \omega(0)\omega' = 0; \quad \tau(0)a = \tau(a); \quad \tau(0)\omega = \tau(\omega)$$

$$\tau(0)\tau' = [\tau, \tau'] - c(\tau, \tau') + \frac{1}{2} \partial(\tau, \tau')$$

$$x(1)y = \langle x, y \rangle$$

In other words, $cA$ is the result of application of the forgetful functor (9.6.1) to the 1-truncated vertex algebroid $uA$ defined in 3.3.

9.10. We may apply the functor (9.8.1) to the 1-truncated conformal algebra $cA$, and get a conformal algebra, to be denoted $CA$.

Assume for simplicity that $k \supset \mathbb{Q}$. Then, as a $k$-module,

$$CA = A \oplus (\oplus_{i \geq 0} \partial^{(i)}T) \oplus (\oplus_{i \geq 0} \partial^{(i)}\Omega)$$

Here are the explicit formulas for the operations $(n)$ (below we agree that $\partial^{(i)}\alpha = 0$ if $i < 0$). First of all, $\alpha_{(n)}\beta = 0$ for all $\alpha, \beta \in A \oplus (\oplus_{i \geq 0} \partial^{(i)}\Omega)$ and $n \geq 0$. Next,

$$a(n)\partial^{(i)}\tau = -\partial^{(i-n)}\tau(a); \quad \tau(n)\partial^{(i)}a = \partial^{(i-n)}\tau(a)$$

$$\omega(n)\partial^{(i)}\tau = -\partial^{(i-n)}\tau(\omega) + (i + 1)\partial^{(i-n+1)}\langle \omega, \tau \rangle$$

$$\tau(n)\partial^{(i)}\omega = \partial^{(i-n)}\tau(\omega) + n\partial^{(i-n+1)}\langle \tau, \omega \rangle$$

Finally,

$$\tau(n)\partial^{(i)}\tau' = \partial^{(i-n)}\{[\tau, \tau'] - c(\tau, \tau')\} + \frac{1}{2}(i + n + 1)\partial^{(i-n+1)}\langle \tau, \tau' \rangle$$
Construction of envelope

9.11. Now we can construct the vertex envelope of a vertex algebroid \( A = (A, T, \Omega, \ldots) \). Consider the conformal algebra \( CA \); our vertex envelope \( UA \) will be a quotient of the vertex envelope \( UCA \) defined in the previous Section by certain vertex ideal.

To define this ideal, let us return to the tensor algebra \( T := TC_A \), as in 8.2. Let us define a left \( C \)-ideal \( \mathcal{R} \subset T \) generated by the following elements:

\[
\mathcal{R} = \{ (1_A - 1_T)u, u \in T \}
\]  

(9.11.1)

\[
r(a, x) := a \cdot x - ax, a \in A, x \in A \oplus \Omega; r(a, \tau) = a \cdot \tau - a\tau + \gamma(a, \tau)
\]

(9.11.2)

Here by \( C \)-ideal we mean an ideal closed under all operators \( \partial^{(i)} \).

9.12. Lemma. The ideal \( \mathcal{R} \) is respected by all operations \( \gamma(n) \), \( n \in CA \), \( n \in \mathbb{Z} \).

Indeed, one has to check this for \( n \in A \oplus T \oplus \Omega, n \geq 0 \), and only on generators of \( \mathcal{R} \). This is done by an easy case-by-case computation that we will leave to the reader, restricting ourselves by an example.

Let us check that \( \tau(0) r(a, \tau') \in S \). We have

\[
\tau(0) r(a, \tau') = \tau(0) \{ a \cdot \tau' - a\tau + \gamma(a, \tau') \};
\]

\[
\tau(0) (a \cdot \tau') = \tau(0) a \cdot \tau' + a \cdot \tau(0) \tau' = \tau(a) \cdot \tau' + a \cdot ([\tau, \tau'] - c(\tau, \tau') + \frac{1}{2} \partial(\tau, \tau'));
\]

\[
-\tau(0) (a\tau') = -[\tau, a\tau'] + c(\tau, a\tau') - \frac{1}{2} \partial(\tau, a\tau')
\]

We have

\[
-\tau(0) \tau(a) = \tau(0) \{ a \cdot \tau - a\tau + \gamma(a, \tau) \};
\]

by axiom (A3) of a vertex algebroid,

\[
c(\tau, a\tau') = -c(a\tau', \tau) = -ac(\tau', \tau) - \gamma(a, [\tau', \tau]) + c(\tau(a), \tau') - \tau(\gamma(a, \tau')) + \tau(a) + \frac{1}{2} \partial(\tau, \gamma(a, \tau')) + \frac{1}{2} \partial(\tau', \gamma(a, \tau'))
\]

By axiom (A2),

\[
-\frac{1}{2} \partial(\tau, a\tau') = -\frac{1}{2} \partial \{ a(\tau', \tau) + \gamma(a, \tau') + \tau a(\tau) \} =
\]

\[
= -\frac{1}{2} (\tau', \tau) \partial a + \frac{1}{2} a \partial(\tau', \tau) - \frac{1}{2} \partial(\gamma(a, \tau') + \frac{1}{2} \partial\gamma(a, \tau')
\]

Finally,

\[
\tau(0) \gamma(a, \tau') = \tau(\gamma(a, \tau'))
\]

After bookkeeping, we see that

\[
\tau(0) r(a, \tau') = r(a, [\tau, \tau']) + r(\tau(a), \tau') - r(\tau(a, \gamma(\tau')) + \frac{1}{2} r(a, \partial(\tau, \tau')) \in S, QED
\]
9.13. Let $\bar{R} \subset UCA$ denote the image of $R$ under the canonical projection $TCA \rightarrow UCA$. The previous lemma means that $\bar{R}$ is a vertex ideal in $UCA$. Let us denote by $UA$ the vertex algebra $UCA/\bar{R}$. It is equipped with an obvious splitting $T \rightarrow T \oplus \Omega = UA$.

Conversely, given a splittable vertex algebra $V$, choose a splitting $s : T \rightarrow V$, and consider the vertex algebroid $A(V; s)$, cf. §2. Let $Vert' \subset Vert$ denote the full subcategory of splittable vertex algebras. We get a functor

$$A : Vert' \rightarrow Alg$$

which is in fact the composition of the truncation functor (3.1.1) (restricted to $Vert'$) and of the functor quasiinverse to (3.3.6).

9.14. Theorem. The assignment $A \mapsto UA$ provides a functor

$$U : Alg \rightarrow Vert'$$

which is left adjoint to (9.13.1).

Indeed, this is evident from the construction.

Poincaré-Birkhoff-Witt

9.15. Let $A = (A, T, \Omega, \ldots)$ be a vertex algebroid. The enveloping algebra $UA$ is generated as a $k$-module by the monomials of the form

$$x^p \cdot x^{p-1} \cdot \ldots \cdot x^1$$

where $x^i$ has the form $\partial^{(j)} y$, $y = a, \omega$ or $\tau$. Here we have denoted for brevity by $x \cdot x'$ the operation $x_{(-1)}x'$.

Let us introduce a canonical increasing exhaustive filtration $F_0UA \subset F_1UA \subset \ldots UA$ by setting $F_iUA$ to be equal the $k$-submodule of $UA$ generated by all monomials (9.15.1) where there are $\leq i$ letters $x_j$ of the form $\partial^{(a)} \tau$.

Obviously all submodules $F_iUA$ are stable under $\partial^{(j)}$ and $F_iUA \cdot F_jUA \subset F_{i+j}UA$. Consider the associated graded module $gr_FUA = \oplus_{i \geq 0} F_iUA/F_{i-1}UA$.

It is easy to see that the operation $x \cdot y$ induces a commutative and associative multiplication on $gr_FUA$, i.e. $gr_FUA$ becomes a $\partial$-algebra (an abelian vertex algebra).

Let $A^{ab}$ denote an abelian vertex algebroid $(A, \Omega \oplus T, \partial)$. We have

$$A(gr_FUA) = A^{ab}$$

hence by adjunction we get a canonical map of $\partial$-algebras

$$J(A^{ab}) \rightarrow gr_FUA$$

It is clear the this map is surjective.
Recall that by 9.4 we have a canonical filtration on \( J(\mathcal{A}^{ab}) \) (let us denote it by \( G \) here) and a canonical surjective map of \( \partial \)-algebras

\[
\text{Sym}_A \left\{ \left( \bigoplus_{i \geq 1} T^{(i)} \right) \oplus \left( \bigoplus_{i \geq 1} \Omega^{(i)} \right) \right\} \rightarrow \gr_G J(\mathcal{A}^{ab}) \tag{9.15.4}
\]

where \( X^{(i)} (X = T \text{ or } \Omega) \) denotes a copy of \( X \) sitting in weight \( i \).

By refining the filtration \( F \) using the filtration \( G \), we get a canonical filtration \( H \) on \( U\mathcal{A} \) together with a surjective map (of \( \partial \)-algebras over \( A \))

\[
\text{Sym}_A \left\{ \left( \bigoplus_{i \geq 1} T^{(i)} \right) \oplus \left( \bigoplus_{i \geq 1} \Omega^{(i)} \right) \right\} \rightarrow \gr_H U\mathcal{A} \tag{9.15.5}
\]

The multiplication in the right hand side is induced by the operation \( (-1) \) on \( U\mathcal{A} \).

**9.16. Theorem.** Assume that \( k \supset \mathbb{Q} \) and both \( \Omega \) and \( T \) are free \( A \)-modules. Then the maps (9.15.3) and (9.15.5) are isomorphisms.

Filtration \( H \) is compatible with the conformal grading, finite on each component with the fixed conformal weight and is canonical in following sense: for any morphism of vertex algebroids \( \mathcal{A} \rightarrow \mathcal{A}' \), the corresponding morphism of vertex envelopes \( U\mathcal{A} \rightarrow U\mathcal{A}' \) respects these filtrations.

To prove this, we use the strategy of Serre’s proof of the usual PBW theorem, cf. [Se], pp. 14-16. Let us choose well ordered \( A \)-bases \( \{\tau_s\}, \{\omega_r\} \) of \( T \) and \( \Omega \). They give rise to a well ordered \( A \)-base

\[
\{x_u\}_{u \in I} = \{\partial^{(a)} \tau_s, \partial^{(b)} \omega_r\}
\]
of \( \left( \bigoplus_{i \geq 1} T^{(i)} \right) \oplus \left( \bigoplus_{i \geq 1} \Omega^{(i)} \right) \). For a sequence \( M = (u_1, \ldots, u_m) \), \( u_j \in I \), \( u_1 \leq u_2 \leq \ldots \leq u_m \) define a monomial \( x_M \in U\mathcal{A} \) by

\[
x_M = x_{u_1} \cdots x_{u_m}
\]
Similarly, \( M \) defines monomials \( \tilde{x}_M \in \gr_H U\mathcal{A} \) and

\[
\tilde{x}_M \in S := \text{Sym}_A \left\{ \left( \bigoplus_{i \geq 1} T^{(i)} \right) \oplus \left( \bigoplus_{i \geq 1} \Omega^{(i)} \right) \right\}
\]

Obviously, the monomials \( \{\tilde{x}_M\} \) form an \( A \)-base of \( S \). On the other hand, it is easy to see that each element of \( U\mathcal{A} \) may by written as \( \sum a_M \cdot x_M \) with some \( a_M \in A \).

To prove our claim it is enough to show that a relation

\[
\sum a_M \cdot x_M = 0 \tag{9.16.1}
\]
implies that all \( a_M = 0 \) (cf. loc. cit., Lemma 4.5).

Proceeding in a manner similar to 8.4, we define an action of the Lie algebra \( CA^{\text{Lie}} \), and hence of its envelope \( UCA^{\text{Lie}} = UCA \), on \( S \). Next one checks the relations (9.11.1) and (9.11.2) and therefore gets an action of \( U\mathcal{A} \) on \( S \).

One sees immediately from the definitions that \( (a \cdot x_M) \cdot 1_S = a \tilde{x}_M \). Hence (9.16.1) implies the relation \( \sum a_M \tilde{x}_M = 0 \) in \( S \) and therefore \( a_M = 0 \), which proves the theorem. We leave the details to the reader. \( \triangle \)
9.17. Let $X$ be a smooth $k$-scheme, $U \subset X$ a Zariski open in $X$. A vertex algebra of the form $U \mathcal{A}$ where $\mathcal{A} \in Alg_{T_X}(U)$ is a section of the gerbe $Alg_{T_X}$ discussed in Section 7, is called an algebra of chiral differential operators on $U$.

These algebras form a gerbe $\mathcal{D}_{X}$, by definition isomorphic to the gerbe $Alg_{X}$.

Note that all isomorphisms of algebras of chiral differential operators respect the canonical filtrations on them, and there is no obstruction to the gluing of associated graded algebras.

In particular, we have

9.18. Theorem. Assume that $k \supset \mathbb{Q}$. Each algebra of chiral do $\mathcal{D}_X \in \mathcal{D}_{X}(X)$ admits a canonical filtration whose graded algebra is isomorphic to

$$gr(\mathcal{D}_X) = Sym_{\mathcal{O}_X} \left\{ (\oplus_{i \geq 1} T_X^{(i)}) \oplus (\oplus_{i \geq 1} \Omega_X^{1(i)}) \right\} \quad (9.18.1)$$

where $T_X^{(i)}$ (resp. $\Omega_X^{1(i)}$) denotes a copy of the tangent bundle (resp. of the bundle of 1-forms) sitting in conformal degree $i$. △
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V.G.: Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA; vgorb@ms.uky.edu

F.M.: Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA; fmalikov@math.usc.edu

V.S.: IHES, 35 Route de Chartres, 91440 Bures-sur-Yvette, France; vadik@ihes.fr