COMPOSITION SUMS RELATED TO THE HYPERGEOMETRIC FUNCTION

R. MILSON

Abstract. The present note considers a certain family of sums indexed by the set of fixed length compositions of a given number. The sums in question cannot be realized as weighted compositions. However they can be related to the hypergeometric function, thereby allowing one to factorize the corresponding generating polynomials. This factorization leads to some interesting identities.

An $l$-composition of a natural number $n$ is an ordered list of $l$ positive integers $p = (p_1,p_2,\ldots,p_l)$ such that $p_1 + \ldots + p_l = n$. The purpose of the present note is to exhibit closed form expressions for a certain family of sums indexed by the set of fixed length compositions of a given number. There are examples of such identities relating composition sums to Stirling numbers \cite{4, 3} and to Fibonacci numbers \cite{2, 4}. It should be noted that the sums introduced in the preceding references can all be regarded as enumerations of weighted compositions \cite{4}. To be more specific, a weighted composition is one where the $j$th term is given a weight $w_j$, and the enumeration of such compositions is defined as

$$S(l, n) = \sum w_{p_1} w_{p_2} \ldots w_{p_l},$$

where the sum is taken over all $l$-compositions of a fixed $n$. It therefore follows that all such sums can be realized in terms of a certain type of generating function:

$$\frac{1}{1 - tw(x)} = \sum_{l,n} S(l, n)t^lx^n,$$

where $w(x) = w_1x + w_2x^2 + \ldots$.

The same cannot be said of the sums introduced in the present note. Instead, the identities to be discussed here come about because one can relate the sums in question to the hypergeometric function, $F(\alpha, \beta, \gamma; z)$. This is accomplished by gauge-transforming a certain

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second-order differential equation into the hypergeometric equation, and then taking the residue with respect to the \( \gamma \) parameter. For a number of other combinatorial identities involving hypergeometric functions, as well as for other properties of these fascinating mathematical objects the reader is referred to [1].

For a given composition, \( p \), let \( L(p) \) and \( R(p) \) denote, respectively, the products of the left and right partial sums of \( p \). To wit

\[
L(p) = p_1(p_1 + p_2) \cdots (p_1 + \cdots + p_{l-1}) n,
\]

\[
R(p) = p_l(p_{l-1} + p_l) \cdots (p_2 + \cdots + p_1) n.
\]

Throughout let \( e_j(x_1, \ldots, x_k) \) denote the \( j \)th elementary symmetric function of the \( x \)'s, i.e. the coefficient of \( X^{k-j} \) in the expansion of \((X + x_1) \cdots (X + x_k)\).

The weighted sums that will be discussed here are indexed by three natural numbers, and are defined by

\[
S(k, l, n) = \sum e_k(p_1, \ldots, p_l) \frac{L(p)}{L(p) R(p)},
\]

where the sum is taken over all \( l \)-compositions of \( n \). For every \( n > 0 \) define a corresponding generating polynomial by

\[
P_n(u, v) = \sum_{k=0}^{l} \sum_{l=1}^{n} S(k, l, n) u^k v^{l-k}.
\]

The main result of the present note is the following factorization of this generating polynomial.

**Theorem 1.** In the even case, say \( n = 2m \), one has

\[
(n!)^2 P_n(u, v) = \prod_{i=0}^{m-1} [(u + v + q_i)(u + v + q_{i+1}) + r_i u],
\]

where \( q_i = i(n - i) \) and \( r_i = (n - 1 - 2i)^2 \). In the odd case, say \( n = 2m + 1 \) one has

\[
(n!)^2 P_n(u, v) = (u + v + q_m) \prod_{i=0}^{m-1} [(u + v + q_i)(u + v + q_{i+1}) + r_i u].
\]

The proof of the theorem will be given below. First it is worth remarking that the above theorem implies some attractive identities:

\[
\sum_{p_1 p_2 \cdots p_l} \frac{1}{n} e_{l-1} \left( \frac{1}{1 \cdot 2}, \frac{1}{2 \cdot 3}, \ldots, \frac{1}{(n-1) \cdot n} \right),
\]

(1)
\[
\sum \frac{(p_1 - 1)(p_2 - 1) \ldots (p_l - 1)}{L(p)R(p)}
\]
\[=
\frac{n-1}{n^2} e_{l-1} \left( \frac{r_1}{q_1 q_2} \frac{r_2}{q_2 q_3} \ldots \frac{r_{m-1}}{q_{m-1} q_m} \right), \quad (2)
\]
where the sums are taken over all \( l \)-compositions of \( n \), and where \( m \) in equation \((2)\) is the largest integer smaller or equal to \( n/2 \).

To prove the identity in \((1)\) note that the left hand side is just \( S(l, l, n) \), and hence can be obtained as the coefficient of \( u^l \) in the factorizations of Theorem 1. An easy calculation shows that
\[q_i + q_{i+1} + r_i = i(i+1) + (n-i-1)(n-i).
\]
Hence
\[P_n(u, 0) = \frac{n}{n} \prod_{i=1}^{n-1} \left( \frac{u}{i(i+1)} + 1 \right)
\]
and the desired identity follows immediately.

The identity in \((2)\) is proved by noting that the left hand side is given by the alternating sum
\[\sum_{k=0}^{l} (-1)^{l-k} S(k, l, n),
\]
and hence can be obtained as the coefficient of \( u^l \) in \( P_n(u, -u) \). Using Theorem 1 to evaluate the latter leads to \((3)\).

Using Theorem 1 to evaluate \( P_n(0, v) \) one also obtains the identity
\[\sum \frac{1}{L(p)R(p)} = \frac{1}{n^2} e_{l-1} \left( \frac{1}{q_1} \frac{1}{q_2} \ldots \frac{1}{q_{m-1}} \right).
\]
However, this identity is considerably less interesting, because it can be obtained by a simple rearrangement of factors in the summands of the left hand side in question.

The proof of Theorem 1 will require a number of intermediate results. Let \( a_{ij}, i, j \in \mathbb{N} \) be indeterminates. For a composition \( p_1, p_2, \ldots, p_l \) of \( n \) let \( s_j \) denote the \( j \)th left partial sum, \( p_1 + \ldots + p_j \), and set \( a(p) = a_{0s_1}a_{s_1 s_2}a_{s_2 s_3} \ldots a_{s_{l-1} n} \).

**Lemma 2.** Defining \( f_n, n \in \mathbb{N} \) recursively by
\[f_n = \sum_{j=0}^{n-1} a_{jn} f_j, \quad f_0 = 1,
\]
one has \( f_n = \sum_p a(p) \) where the sum is taken over all compositions (of all lengths) of \( n \).
For $f(\gamma)$, a rational function of an indeterminate $\gamma$, let $\text{Res}(f; \gamma = \gamma_0)$ denote the residue of this function at the value $\gamma_0$, i.e. the coefficient of $(\gamma - \gamma_0)^{-1}$ in the Laurent series expansion of $f$ about $\gamma = \gamma_0$. Let

$$f(u, v, \gamma; z) = 1 + \sum_{n>0} f_n(u, v, \gamma)z^n$$

denote the unique formal power series solution of

$$z^2f_{zz} + (1 - \gamma)f_z + \left(\frac{vz}{1-z} + \frac{uz}{(1-z)^2}\right)f = 0, \quad f(0) = 1. \quad (3)$$

**Proposition 3.** $nP_n(u, v) = \text{Res}(f_n; \gamma = n)$.

**Proof.** Rewriting $z(1-z)^{-1}$ as $\sum_{n>0} z^n$ and $z(1-z)^{-2}$ as $\sum_{n>0} nz^n$ one obtains the following recurrence relation for the coefficients of $f$:

$$n(\gamma - n)f_n = \sum_{i=0}^{n-1} ((n-i)u+v)f_i.$$

Set

$$a_{ij} = \frac{(j-i)u+v}{j(\gamma-j)}$$

and note that for a composition $p_1, \ldots, p_l$ of $n$ one has

$$a(p) = \frac{p_1u+v}{s_1(\gamma-s_1)} \times \frac{p_2u+v}{s_2(\gamma-s_2)} \times \cdots \times \frac{p_{l-1}u+v}{s_{l-1}(\gamma-s_{l-1})} \times \frac{p_lu+v}{n(\gamma-n)}$$

$$= \frac{\sum_{k=0}^l e_k(p_1, \ldots, p_l)u^k v^{l-k}}{L(p)(\gamma-s_1)(\gamma-s_2)\cdots(\gamma-n)}.$$

Taking the residue of the right hand side at $\gamma = n$ and applying Lemma 2 gives the desired conclusion. \qed

Let $\alpha, \beta, \gamma$ be indeterminates, and let $F(\alpha, \beta, \gamma; z)$ denote the usual hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} z^n,$$

where $(x)_n$ is an abbreviation for the expression $x(x+1)\ldots(x+n-1)$.

**Proposition 4.** Setting

$$\hat{u} = \frac{1}{4} (\alpha + \beta + \gamma)(2 - \alpha - \beta - \gamma), \quad (4)$$

$$\hat{v} = \frac{1}{4} (\alpha - \beta - \gamma)(\alpha - \beta + \gamma),$$

one has

$$f(\hat{u}, \hat{v}, \gamma; z) = (1 - z)^{(\alpha + \beta + \gamma)/2} F(\alpha, \beta, 1 - \gamma; z). \quad (5)$$
Composition sum identities

Proof. Substituting (5) into (3) yields the following equation for $F$

$$z^2 F_{zz} + (1 - \gamma)z F_{z} - (\alpha + \beta + \gamma) F_{z} - \alpha \beta F = 0.$$ 

Multiplying the above by $(1 - z)/z$ yields the usual hypergeometric equation with parameters $\alpha, \beta, 1 - \gamma.$

Proof of Theorem 1. Expanding $(1 - z)^{(\alpha + \beta + \gamma)/2}$ into a power series in $z$, and using Proposition 4 one obtains that

$$f_n(\hat{u}, \hat{v}, \gamma) = \frac{(\alpha)_n (\beta)_n}{n! (1 - \gamma)_n} + \ldots,$$

where the remainder is a finite sum of rational functions in $\alpha, \beta, \gamma$ which do not contain a factor of $\gamma - n$ in the denominator. Hence, by Proposition 3

$$P_n(\hat{u}, \hat{v}) = (-1)^n \frac{(\alpha)_n (\beta)_n}{(n!)^2}.$$ 

An elementary calculation shows that for $0 \leq i < n/2$

$$(\hat{u} + \hat{v} + q_i)(\hat{u} + \hat{v} + q_{i+1}) + r_i \hat{u} = (\alpha + i)(\beta + i)(\alpha + n - 1 - i)(\beta + n - 1 - i)$$

and that for $n = 2m + 1$

$$\hat{u} + \hat{v} + q_m = -(\alpha + m)(\beta + m).$$

The desired factorizations follow immediately. \qed

References

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Dept of Math., Dalhousie U., Halifax, Canada, B3J 3J5
Current address: McGill University, Montreal
E-mail address: milson@math.mcgill.ca