Knizhnik-Zamolodchikov-Bernard equations as a quantization of nonstationary Hitchin system.

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The KZB equations for conformal blocks of the WZNW theory are written on the moduli space of holomorphic principal bundles on the surface. They become the multi-time Schrödinger equation for the nonstationary Hitchin system. From the known form of the equations we learn about the covariance of quantization with respect to changes of the coordinate frame.

The Knizhnik-Zamolodchikov-Bernard (KZB) equations were first derived as the equations for conformal blocks of the Wess-Zumino-Novikov-Witten (WZNW) theory on Riemann surfaces [KZ,B,L,I]. These equations also appear in many other mathematical problems like quantization of the moduli space of flat connections [H2,AWDP] or quantization of isomonodromic deformations [R]. Many of the known integrable systems are closely related to the KZB equations (mostly at genus one) [N,O]. The general nature of these equations requires a more unified picture of the underlying structures. This paper is a small step in this direction. As a starting point we take the KZB equations in the form previously derived in the framework of the WZNW theory [I]. We observe that these equations can be pushed down from the non-invariant “twist language” of D. Bernard [B] to the moduli space of holomorphic principal bundles. The KZB operators realize a quantization of the quadratic Hamiltonians of the Hitchin system [H1], the equations being the Schrödinger equations in the multi-time of complex moduli of the surface. Classically, this system has a quadratic kinetic Hamiltonian with a “space-time” dependent mass. Under the quantization, the space dependence of mass leads to the “potential” term in the KZB equations. The multi-dimensionality of time appears to provide an additional rigidity to the quantization procedure, limiting the possible choice of the connection involved in the KZB equations. We also observe a covariance of the KZB equations with respect to changing the coordinate frame in accordance with the classical equations of motion. This covariance generalizes the usual Galilean invariance of the free particle in the flat space, but has also special features due to the space-time dependence of mass and to the multidimensionality of time.

1. Definitions and constructions.

Let \( \mathcal{C} \) denote the moduli space of Riemann surfaces of a given genus \( N \) without punctures. For a given Riemann surface \( \Sigma \in \mathcal{C} \) we consider the moduli space \( \mathcal{M}_\Sigma \) of stable holomorphic principal bundles over \( \Sigma \) with a complex Lie group \( G_C \) (\( G_C \)-bundles). The dimension of \( \mathcal{C} \) is \( 3(N-1) \) (when \( N > 1 \)), \( \dim \mathcal{M}_\Sigma = (N-1) \dim G \). We think of \( \mathcal{M}_\Sigma \) as
a fiber of a bundle $\mathcal{E}_N$ over $C$. $\mathcal{E}_N$ is the moduli space of stable $G_C$-bundles over Riemann surfaces of genus $N$ with the structure group $G_C$. We shall parametrize $G_C$-bundles by $(0, 1)$ forms $\bar{A}(z, \bar{z})$ on $\Sigma$ with values in the complex Lie algebra $G_C$. A holomorphic section $g$ is defined as a solution to the equation

$$\bar{\partial}_A g \equiv (\bar{\partial} - \bar{A})g = 0,$$  \hspace{1cm} (1.1)

where $\bar{\partial}$ is the antiholomorphic derivative on $\Sigma$. Different $(0, 1)$ forms $\bar{A}$ and $\bar{A}'$ define the same holomorphic principal bundle if the corresponding solutions of (1.1) differ by a global shift $h(z, \bar{z})$: $g' = hg$. In other words, the forms $\bar{A}$ and $\bar{A}'$ define the same bundle if and only if they differ by a globally defined gauge transformation

$$\bar{A} = h^{-1} \bar{A}' h - h^{-1} \bar{\partial} h.$$  \hspace{1cm} (1.2)

In this paper we shall work locally in $\mathcal{M}_\Sigma$ and disregard its global structure.

The cotangent space to $\mathcal{M}_\Sigma$ is naturally identified with the space of $(1, 0)$ $G_C$-valued zero modes of $\bar{\partial} \bar{A}$, i.e. $(1,0)$ forms $J$ obeying

$$\bar{\partial} J - [\bar{A}, J] = 0.$$  \hspace{1cm} (1.3)

The pairing of $J$ to a tangent vector $\delta \bar{A}$ is defined by

$$\langle J, \delta \bar{A} \rangle = \int_{\Sigma} J \delta \bar{A}. $$  \hspace{1cm} (1.4)

Due to the condition (1.3) this definition is consistent with the gauge freedom for $\bar{A}$.

Relating this picture to the WZNW model we treat $\bar{A}$ as the external source and associate to a particular choice of $\bar{A}$ the holomorphic partition function on the Riemann surface $\Sigma$:

$$\Psi[\bar{A}] = \langle e^{\int_{\Sigma} j^a \bar{A}^a} \rangle,$$  \hspace{1cm} (1.5)

where $j^a$ is the WZNW holomorphic current [KZ]. From the Polyakov-Wiegmann identity for the WZNW action $S_{\text{WZNW}}[g]$

$$S_{\text{WZNW}}[g h] = S_{\text{WZNW}}[g] + S_{\text{WZNW}}[h] - 2 \int_{\Sigma} (\bar{\partial} h) h^{-1} g^{-1} \bar{\partial} g$$  \hspace{1cm} (1.6)

it follows that for gauge equivalent forms $\bar{A}$ and $\bar{A}'$ the partition functions are related by a geometrical factor only:

$$\Psi[\bar{A}] = \Psi[\bar{A}'] e^{kS[\bar{A} \to \bar{A}']}, $$  \hspace{1cm} (1.7)

where $k$ is the level of the WZNW theory, $S[\bar{A} \to \bar{A}']$ is a “transition” WZNW action (expressed by certain integrals over the surface and the bulk, independent of $k$ and containing no functional integration). This shows that the WZNW conformal block is a (local) section of a line bundle $\mathcal{L}_k$ over $\mathcal{M}_\Sigma$ which is the $k$-th degree of the level-one bundle (Quillen’s determinant bundle [H$^2$,Q]):

$$\mathcal{L}_k = \mathcal{L}^k_1.$$  \hspace{1cm} (1.8)
In Bernard’s formulation of twisted WZNW theory the principal bundle was determined by constant transition functions along A-cycles (Bernard used twist elements from the compact Lie group $G$, but we extend his construction to the complexified group $G_C$). In our present language this corresponds to $\bar{A}$ concentrated on A-cycles only. Outside A-cycles $\bar{A} = 0$, in the vicinity of the A-cycle $A_i$

$$\bar{A} = (\bar{\theta}_i)\theta_i^{-1},\quad (1.9)$$

where $\theta_i(z, \bar{z})$ is the “step function” equal to 1 on the left of $A_i$ and to $g_i$ (Bernard’s twist) on the right of $A_i$. Actually, as $\theta$ sharpens, the partition function $\Psi[\bar{A}]$ diverges and needs a regularization. It can be shown that Bernard’s definition of the (regularized) twisted partition function $\Psi(g_i)$ results in a regularization again of the form (1.7): $\Psi(g_i) = \Psi[\bar{A}] \exp(kS)$. Therefore this regularization may be absorbed in (1.7) as the choice of local trivialization of $L_k$ and does not affect the further discussion.

Differentiation with respect to the constant twists in Bernard’s picture may be viewed as differentiation on the moduli space $M_\Sigma$. Let $\omega_{ib}(z; w_0)$ form the basis of holomorphic twisted 1-forms:

$$\bar{\partial}_i \omega_{ab}(z) = 0 \quad (z \neq w_0), \quad \oint_{A_j} \omega_{ib}(z) dz = \delta_{ij} \delta_b^a \quad (1.10)$$

with a first-order pole at $z = w_0$. (In this paper we shall keep the notation of [I].) Let $L_i^a$ denote the right-invariant derivative along the $i$-th twist. Then the differential $d_{M_\Sigma}$ on the moduli space $M_\Sigma$ looks like

$$d_{M_\Sigma}(z) = \omega_{ib}(z; w_0) L_i^b \quad (1.11)$$

(which is a differential operator mapping functions on $M_\Sigma$ to (1.0) forms obeying (1.3), i.e. to 1-forms on $M_\Sigma$; it is regular at $z = w_0$ and, moreover, is independent of $w_0$).

Finally, we add to this picture the partition function of the twisted b-c system, described in [B,L,I]. There it has been argued that the equations take the most natural form when written not for the “bare” partition function $\Psi(g)$, but for the product $F(g) = \Psi(g)\Pi(g)$, where $\Pi(g) = \sqrt{Z_{b-c}}$ is the square root of the twisted b-c partition function. The b-c theory is constructed with fermionic fields $b^a(z)$ and $c^a(z)$ with spin 1 and 0 respectively and obeying the OPE

$$b^a(z)c^b(w) = \frac{\delta^{ab}}{z-w} + \text{reg.terms.} \quad (1.12)$$

The b-c current has the form

$$j_{b-c}^a(z) = f^{abc}b^b(z)c^c(z) \quad (1.13)$$

($f^{abc}$ are the structure constants of $G$) and obeys the Kac-Moody algebra at level $2h^*$. Similarly to the WZNW part, we define the b-c partition function as

$$(Z_{b-c})^{a_1 \ldots a_N \dim G}(z_1, \ldots z_N \dim G)[\bar{A}] =$$

$$\langle e^{\int_{\Sigma} f^{a} b_{b-c}^a A^{\bar{a}} (z_1) \cdots b_{a_N \dim G}^{a_N \dim G} (z_N \dim G) \prod_{b} c^b(w_0)} \rangle. \quad (1.14)$$
We need the insertions of $b$ and $c$ fields to eliminate zero modes. Integration of the inserted $b$ fields along the A-cycles would recover the twisted partition function used in [B,L,I]. For our purpose we leave the insertions uncontracted. Since the partition function (1.14) is antisymmetric in $(z_i, a_i)$, is a zero mode of $\bar{\partial}_A$ on $\Sigma \setminus \{w_0\}$ as a function of each variable $z_i$, and has only a simple pole at $z_i = w_0$, it should be viewed as a holomorphic volume form on the moduli space $\mathcal{M}_\Sigma \setminus \{w_0\}$ of $G_C$-bundles over $\Sigma \setminus \{w_0\}$. Sometimes (in genus one) it is more appropriate to work with $\mathcal{M}_\Sigma \setminus \{w_0\}$ instead of $\mathcal{M}_\Sigma$. However, at genus $N > 1$ we may freely convert $Z_{b-c}[\bar{A}]$ into a holomorphic volume form on $\mathcal{M}_\Sigma$ by integrating $\dim G$ out of $\mathcal{N} \dim G$ variables $z_i$ around $w_0$. This will remove the singularity at $z_i = w_0$ and leave $\dim \mathcal{M}_\Sigma = (N - 1) \dim G$ variables $z_i$ as “subscripts” of the holomorphic volume form $Z_{b-c}[\bar{A}]$ on $\mathcal{M}_\Sigma$. If we denote the bundle of holomorphic volume forms (the canonical bundle) on $\mathcal{M}_\Sigma$ as $K = \Omega^{(\dim \mathcal{M}_\Sigma, 0)}(\mathcal{M}_\Sigma)$, then $Z_{b-c}[\bar{A}]$ is a (local) section of the bundle

$$L_{2h^*} \otimes K.$$  \hspace{1cm} (1.15)

One may observe that as a line bundle $K$ is equivalent to $L_{-2h^*}$ [H2] and therefore (1.15) is a trivial bundle. Still we shall keep the two factors of (1.15) separately, since they are realized via different mechanisms: the first factor is due to the anomalous gauge dependence of $Z_{b-c}[\bar{A}]$, while the second one comes from $Z_{b-c}[\bar{A}]$ being explicitly a holomorphic volume form. Now $\Pi = \sqrt{Z_{b-c}}$ is a holomorphic section of $L_{h^*} \otimes K^{1/2}$, and the “dressed” partition function $F = \Psi \Pi$ is a section of

$$L_\kappa \otimes K^{1/2},$$  \hspace{1cm} (1.16)

where $\kappa = k + h^*$.

To summarize our construction, the “dressed” partition function is a (local) section of the bundle (1.16) over $\mathcal{M}_\Sigma$ which is in turn a fiber of a bundle over $\mathcal{C}$. Therefore, the vector spaces $\Gamma(L_\kappa \otimes K^{1/2})$ of (local) holomorphic sections of (1.16) are fibers of a vector bundle over $\mathcal{C}$. The KZB equations is a flat connection on this bundle.

2. Nonstationary Hitchin system.

This section is aimed to review the classical Hitchin system [H1] and recall how the classical limit of the KZB operators coincides with the quadratic Hitchin Hamiltonian [N].

The Hitchin system is a Hamiltonian system on the moduli space $\mathcal{M}_\Sigma$ introduced in the previous section. The Hamiltonians $\mathcal{H}_\mu$ are functions on the cotangent bundle $T^* \mathcal{M}_\Sigma$ equipped with the usual symplectic structure $\omega = dp \wedge dq$. The Hamiltonians $\mathcal{H}_\mu$ are parametrized by the tangent vector $\mu$ to the moduli space of complex structures $\mathcal{C}$; alternatively we may speak of a Hamiltonian $H(z)$ being a quadratic holomorphic differential in $z$ (the usual coupling to the Beltrami differential $\mu(z, \bar{z})$ is assumed: $H_\mu = \int_\Sigma H(z) \mu(z, \bar{z})$). By nonstationary system we understand the collection of Hitchin systems for all possible surfaces $\Sigma \in \mathcal{C}$ which are the different moments of the multidimensional time. At each “time” $\Sigma$ we have its own set of Hamiltonians $\mathcal{H}_\mu$ which govern the motion of the system as “time” changes. If we had an identification of different time slices $\mathcal{M}_\Sigma$, we would have been able to launch the system move in the multi-time $\mathcal{C}$ with the time dependent Hamiltonian $\mathcal{H}_\mu$. The arrangement of such an identification of the fibers $\mathcal{M}_\Sigma$ will be an important issue in our future discussion.
The Hamiltonians $H_\mu$ can be defined as a symplectic quotient of a free Hamiltonian in a certain affine space (see [AWDP] for an analogous procedure over the moduli space of flat connections). We start with the space $A_\Sigma$ of all $G_\mathbb{C}$-valued $(0,1)$ forms $\bar{A}^a$ and the free quadratic Hamiltonian $H_A(z) = J^2(z)/2$ on $T^*A$ ($T^*A$ consists of $(1,0)$ forms $J^a(z)$).* The Poisson bracket of the Hamiltonians at different points $z_1$ and $z_2$ is zero, and these Hamiltonians are invariant under the gauge transformations (1.2).

When we take the symplectic quotient with respect to the action of the gauge group, the space $A_\Sigma$ reduces to the moduli space $M_\Sigma$, its cotangent space $T^*_\bar{A}A$ — to the space of zero modes $J(z)$ (i.e. $J(z)$ obeying (1.3)). The Hitchin Hamiltonian being given by the same expression

$$H(z) = \frac{1}{2} J^2(z)$$

(2.1)

is now a holomorphic quadratic differential. If we parametrize the moduli space $M_\Sigma$ by Bernard’s twists $\{g_i\}$, the tangent space $T^*_{\bar{A}}M_\Sigma$ has a corresponding basis $L^{ia}$ (at this point $L^{ia}$ are not operators yet, they will become operators after quantization) modulo the relation $\sum_i L^{ia} - \sum_i R^{ia} \equiv \sum_i (1 - g_i) a^b L^{ib} = 0$. $J(z)$ is a linear function on the cotangent space $T^*_{\bar{A}}M_\Sigma$:

$$J^a(z) = \omega^a_{ib}(z; w_0) L^{ib}.$$ 

(2.2)

Then the Hitchin Hamiltonian (2.1) exactly reproduces the symbol of the KZB operator from [I]. If we understand the classical limit of the KZB operator as when acting on a rapidly oscillating function, then the “potential” term is negligible (of higher order in “Planck constant”, see the subsequent section for more detail), while the kinetic term reduces to its symbol.

Thus, we explicitly verified that the quadratic Hitchin Hamiltonian is the classical limit of the KZB operator.

3. Multi-time Schrödinger equation.

The KZB equation (on a surface without punctures) [I]

$$\left(\partial_m(z) + \frac{1}{\kappa} \left[ \frac{1}{2} L^{ia} \omega^b_{ia}(z; w_0) \omega^b_{jc}(z; w_0) L^{jc} + U(z) \right] \right) F = 0$$

(3.1)

plays the role of the Schrödinger equation for the Hitchin Hamiltonian (2.1).\(^1\) Here $\partial_m(z)$ is the differential on the moduli space of complex structures $C$, $\kappa^{-1} = (k + h^*)^{-1} = \hbar$ is the “Planck constant”, $U(z)$ is the “potential” term of the KZB operator derived explicitly in [I].

\(^*\) In general, the Hitchin system includes also the higher-order Hamiltonians. The KZB equations correspond to the quadratic part of the Hitchin system. A generalization of the KZB equations to include the higher-order Hamiltonians is an interesting mathematical problem.

\(^1\) Unlike the approach of [O] (specific for the genus $N = 1$ case), we treat the KZB equations as a nonstationary Schrödinger equation.
Returning to the classical system, the Hitchin Hamiltonian (2.1) describes a free particle in a curved space, i.e. it is purely kinetic:

\[ H_\mu = \frac{1}{2} p_A C_{\mu}^{AB} p_B, \]  

where \( p_A \) are the momenta of the particle, \( C_{\mu}^{AB} \) is the inverse mass matrix. Notice that \( C_{\mu}^{AB} \) is space-time dependent; therefore if we naively quantize this Hamiltonian by replacing \( p_A \rightarrow \hbar \partial / \partial x^A \), there is an ambiguity in possible ordering the space derivatives and the space-dependent \( C_{\mu}^{AB} \). Of course, we wish the Hamiltonian operator to be self-conjugate. This leads us to the result

\[ \hat{H}_\mu = \hbar \frac{2}{2} \frac{\partial}{\partial x^A} C_{\mu}^{AB} \frac{\partial}{\partial x^B} + \hbar^2 U(x, t), \]  

where \( U(x, t) \) comprises the ambiguity of quantization. This \( U(x, t) \) is the “potential” term in the KZB operator, but not the real potential in the Hamiltonian, since it is of order of \( \hbar^2 \) and vanishes in the classical limit. In principle, this term should depend on the coordinate frame where the quantization is performed, and in the next section we see that this is indeed the case.

4. Covariance with respect to coordinate change.

To write the KZB connection on the bundle of conformal blocks in the form of the KZB equation (3.1) we need to fix a “coordinate frame”. This amounts to specifying the following three structures:

(i) First, since the “dressed” partition function \( F \) is the square root of a holomorphic volume form (\( K^{1/2} \) factor in (1.16)), one needs a reference holomorphic volume form \( \omega \) on \( M_\Sigma \) to treat \( F \) as a function.

(ii) Similarly, the \( L_\kappa \) factor in (1.16) requires a choice of local trivialization (or, equivalently, gauge fixing).

Once (i) and (ii) are chosen, it fixes a local trivialization of the bundle (1.16) and allows us to regard \( F \) as a function on \( M_\Sigma \).

(iii) Finally, to write a partial differential equation (3.1) we need to specify the meaning of partial derivatives \( \partial_m(z) \). This is equivalent to picking a flat connection on the fiber bundle \( \mathcal{E} \), so that \( \partial_m(z) \) denote a differentiation along this connection. At the same time, such a connection identifies different fibers \( M_\Sigma \), which attributes a meaning to the nonstationary Hitchin system as a Hamiltonian system with a time-dependent Hamiltonian. In Bernard’s formulation [B] this connection was based on fixing the twisting group elements. Two points in different fibers \( M_{\Sigma_1} \) and \( M_{\Sigma_2} \) were identified if they had the same twists \( \{g_i\} \). Then \( \partial_m(z) \) corresponded to changing the complex structure keeping the twisting elements fixed.

As soon as all (i) — (iii) items are chosen, the equations for conformal blocks can be written as partial differential equations (3.1). The freedom of choosing the coordinate frame (i) — (iii) implies two important questions. First, are there any natural or preferable
choices and restrictions for choosing the coordinate frame? Second, are the equations covariant with respect to coordinate frame change?

Of course, we wish that the equations keep the form of a connection with a spectral parameter \( \hbar = 1/\kappa \), i.e.

\[
[\partial_\mu + \hbar (A_{KZB})_\mu] F = 0,
\]

(4.1)

where \( A_{KZB} \) is a second-order differential operator independent of the level \( k \) (or, equivalently, of \( \hbar \)). Bernard’s choice of coordinate frame is based on parametrizing the holomorphic principal bundles by constant \( G_\mathbb{C} \) twists on A-cycles. Since this parametrization is not unique and, moreover, different twists (not related by a global conjugation) may correspond to the same bundle in \( M_\Sigma \) thus defining two different coordinate systems in the vicinity of the same point in \( M_\Sigma \), we expect that the equations (3.1) possess a certain covariance with respect to a coordinate change.

Change of the bundle trivialization (i) — (ii) transforms \( F \) as

\[
F \rightarrow F' = F R e^{\kappa S},
\]

(4.2)

where \( R = (\omega'/\omega)^{1/2} \) describes the change of the reference volume form (i), \( \exp(\kappa S) \) gives the change of the gauge (ii). We require that all the solutions to the KZB equation

\[
[\partial_\mu + \frac{1}{\kappa} (\Delta_\mu + U_\mu)] F = 0
\]

(4.3)

are mapped under (4.2) to the solutions to the KZB equation in the new coordinate frame:

\[
[\partial'_{\mu} + \frac{1}{\kappa} (\Delta'_{\mu} + U'_{\mu})] F' = 0.
\]

(4.4)

Here \( \Delta_\mu \) and \( \Delta'_{\mu} \) are second-order differential operators (their symbols are equal and coincide with the quadratic Hitchin Hamiltonian (2.1), but first-order terms generally differ, since \( \Delta_\mu \) and \( \Delta'_{\mu} \) are self-conjugate with respect to different volume forms \( \omega \) and \( \omega' \), see more discussion below). \( U_\mu \) and \( U'_\mu \) are the “potential” terms of the KZB operators which also depend on the coordinate frame. The connections \( \partial_\mu \) and \( \partial'_{\mu} \) differ by a Lie derivative along vector fields \( v_\mu \) on \( M_\Sigma \):

\[
\partial'_{\mu} = \partial_\mu + L_{v_\mu}.
\]

(4.5)

In addition, we require that in the initial coordinate frame the following two conditions are satisfied. First, the reference form \( \omega \) is invariant with respect to the connection \( \partial_\mu \):

\[
\partial_\mu \omega = 0.
\]

(4.6)

Second, the quadratic differential operator \( \Delta_\mu \) is self-conjugate with respect to the reference form \( \omega \). This is a reflection of the coordinate-independent fact that the KZB operator is self-conjugate on the line bundle \( K^{1/2} \) (where self-conjugacy is naturally defined). Both these conditions are satisfied in Bernard’s coordinate frame. Furthermore, the covariance requirement will enforce these conditions in the new coordinate frame.
By a straightforward substitution (4.2) and comparing terms at equal powers of $\kappa$, we arrive at the following restrictions for the new coordinate frame:

$$\partial'_\mu \log R + \Delta_\mu S = 0, \quad (4.7a)$$

$$U_\mu - U'_\mu = \frac{1}{R} \Delta'_\mu R, \quad (4.7b)$$

$$\partial' S + \frac{1}{2}[[\Delta'_\mu, S], S] = 0, \quad (4.7c)$$

$$\mathcal{L}_{v_\mu} F + [[\Delta'_\mu, S], F] = 0 \quad \text{for any function } F. \quad (4.7d)$$

Using the self conjugacy of $\Delta_\mu$ with respect to $\omega$ and (4.7d), the equation (4.7a) may be rewritten as

$$2\partial'_\mu \log R + \text{div } v_\mu = 0, \quad (4.8)$$

where $\text{div } v_\mu = \frac{1}{\omega} \mathcal{L}_{v_\mu} \omega$ is the divergence of $v_\mu$ with respect to the volume form $\omega$. Using (4.6) this implies that

$$\partial'_\mu \omega' = 0, \quad (4.9)$$

i.e. the reference form must be conserved by the connection $\partial_\mu$ in the new coordinate frame.

The equation (4.7b) indicates that the “potential” term indeed depends on the coordinate frame, which was predicted in Section 3.

Finally, (4.7c) and (4.7d) state that $S$ satisfies the Hamilton-Jacobi equation for the classical Hitchin system and that $v_\mu$ is the velocity corresponding to the classical action $S$. The Hamilton-Jacobi equation (4.7c) appears in the highest order in $\kappa = \hbar^{-1}$, which implies an analogy with the quasiclassical limit of the ordinary quantum mechanics.

The described procedure of changing to a classically moving frame presents an interesting generalization of a Galilean invariance of the Schrödinger equation for a free particle in a flat space. This generalization however has two features special for our problem. First, the space is not flat (the inverse mass matrix is space-time dependent), therefore a “potential” term of the order of $\hbar^2$ appears because of the ambiguity of quantization. This term depends on the coordinate frame via (4.7b). Second, the time is multi-dimensional. It makes the “velocities” $v_\mu$ linearly dependent (since by (4.7d) they are linear combinations of the first space derivatives of $S$). This linear dependence defines a certain integrable contact structure on $\mathcal{E}_N$ which restricts the possible directions of classical motion.

To summarize the results of this section, we observed that to write the KZB equation we need the coordinate frame consisting of a flat connection on $\mathcal{E}_N$ and a trivialization of the bundle (1.16). The KZB equations are covariant with respect to certain changes of the coordinate frame, when the connection and the trivialization are changed consistently. The changes of the connection must obey the classical equations of motion for the Hitchin system (with respect to the initial connection), while the change of trivialization is given by the classical action along the new trajectories. This condition separates a special class of distinguished coordinate frames (Bernard’s twist parametrization being one of them). In these coordinate frames the KZB equation takes the usual form of a flat connection.
with the spectral parameter $1/\kappa$, which may be viewed as a covariance of quantization of the Hitchin system in different coordinate frames.

5. Natural connections on $\mathcal{E}_N$.

In the previous section we observed how starting from a given coordinate frame (understood as the three ingredients (i) — (iii)) to construct a whole family of coordinate frames in accordance to the classical motion of the (nonstationary) Hitchin system. If the KZB equations have the usual form (4.1) in the initial frame, then they preserve this form under coordinate changes within this family. It appears that not all coordinate frames admit a quantization of the Hitchin system of the form (4.1). First of all, we must require that the connection $\partial_\mu$ on $\mathcal{E}_N$ satisfies the two (classical) conditions:

\begin{align}
&[\partial_\mu, \partial_\nu] = 0; \quad (5.1) \\
&\partial_\mu H_\nu = \partial_\nu H_\mu. \quad (5.2)
\end{align}

The first condition is the flatness of the connection. The second one requires that the connection is compatible with the Hitchin Hamiltonians $H_\mu$ (as quadratic forms on the cotangent bundle, these Hamiltonians can be differentiated along the connection $\partial_\mu$). Only if the conditions (5.1) — (5.2) are satisfied may we hope that the classical Hamiltonians $H_\mu$ can be quantized to the operators $(A_{KZB})_\mu$ in (4.1) satisfying the integrability conditions

\begin{align}
\partial_\mu (A_{KZB})_\nu &= \partial_\nu (A_{KZB})_\mu \quad (5.3) \\
[(A_{KZB})_\mu, (A_{KZB})_\nu] &= 0. \quad (5.4)
\end{align}

Naively counting the number of variables and equations in the conditions (5.1) — (5.2) suggests that at sufficiently high genus these conditions are redundant. Fortunately, we know that Bernard’s connection $\partial_\mu$ (keeping the twists unchanged) satisfies (5.1) — (5.2) and, furthermore, admits a quantization (5.3) — (5.4). We conjecture that at sufficiently high genus the conditions (5.1) — (5.2) fix the possible choice of connections $\partial_\mu$ down to the family constructed from Bernard’s connection by classical evolutions as described in the previous section. This selects a family of distinguished (“natural”) connections on the bundle $\mathcal{E}_N$.

Finally, we make a remark how this notion of natural connections can be extended to the case of lower genera (where conditions (5.1) — (5.2) are not restrictive enough), using the idea of compactifying the moduli space of Riemann surfaces so that the moduli of surfaces of lower genera form the boundary of the moduli of surfaces of higher genera. A detailed construction of this compactification may be found elsewhere [PN]. Here we employ only its basic idea to construct the natural connections over the moduli of the surfaces without marked points (punctures) from the natural connections over the moduli of the same surfaces with marked points. Inclusion of marked points increases the dimension of the moduli space, and with sufficiently many marked points the conditions (5.1) — (5.2) will fix the family of natural connections uniquely.
So far in this paper we considered only surfaces without marked points. It has been noted in [1] that the KZB equations on the surface with marked points can be obtained from the equations without marked points by contracting handles of the surface to singular “nodes”. Below we show how to include marked points in the above constructions in a way which appears as the limit of contracting handles of the surface. Namely, if we have a surface $\Sigma$ with marked points $z_1, \ldots, z_M$, then instead of $M\Sigma$ we consider $M\Sigma, \{z_i\}$ which is the moduli space of the $(0,1)$ form $\bar{A}$ together with group elements $g_1, \ldots, g_M$ (located at the points $z_1, \ldots, z_M$) with respect to the gauge transformations

$$\bar{A} \mapsto h^{-1}\bar{A}h - h^{-1}\partial h, \quad g_i \mapsto g_i h(z_i). \quad (5.5)$$

Defined in such a way, this moduli space naturally forms a fiber bundle over $\mathcal{M}_\Sigma$:

$$\mathcal{M}_{\Sigma,\{z_i\}} \rightarrow \mathcal{M}_\Sigma. \quad (5.6)$$

The cotangent vectors are the zero modes of $\bar{A}$ (in the sense of (1.3)) with first-order poles at punctures $z_i$, the pairing being

$$\langle J, \delta(\bar{A}, g_i) \rangle = \int_{\Sigma} J\delta \bar{A} + \sum_i \oint_{z_i} Jg_i^{-1}\delta g_i. \quad (5.7)$$

The holomorphic conformal block is a function of the gauge field $\bar{A}$ and the group elements $g_i$ transforming as (1.7) under gauge transformations. The usual conformal blocks may be recovered by projecting onto different finite-dimensional representations of $G_{CM}$. For example, the usual Knizhik-Zamolodchikov equation will look in our formalism as

$$\left(\frac{\partial}{\partial z_i} + \frac{1}{\kappa} \sum_{j \neq i} \frac{L^i a L^j a}{z_i - z_j} \right) F(g_1, \ldots, g_M) = 0, \quad (5.8)$$

where $F$ is a function of group elements $g_i$ invariant under simultaneous left multiplication of all $g_i$ by the same group element. The whole discussion of the Hitchin system and the coordinate covariance can be extended to the case of surfaces with punctures in a straightforward way.

Suppose now that we have a unique family of natural connections $\partial_\mu$ on the fiber bundle $\mathcal{E}_{N,M}$ of the moduli $\mathcal{M}_{\Sigma,\{z_i\}}$ for surfaces with punctures (up to classical motion, see the previous section). Then select from this family only those connections which preserve the fibers in the bundles (5.6). This subfamily of connections can be pushed down to connections on $\mathcal{E}_N$ which will again satisfy (5.1) — (5.2). Of course, this construction recovers the family induced by Bernard’s connection $\partial_\mu$ and distinguishes these connections as natural on $\mathcal{E}_N$ (where the conditions (5.1) — (5.2) may be insufficient).

In the above treatment many points lack rigor; thus we put it as a conjecture that there exists a unique family of connections on the fiber bundles $\mathcal{E}_{N,M}$ which satisfy (5.1) — (5.2) and which produce connections from the same family at lower genera under degenerating the surface; parametrization by Bernard’s twists defines connections from this family.
6. Conclusion.

The approach of this paper is to learn about the quantization of the Hitchin system from the KZB equations previously derived in the WZNW theory. A more direct way would be to start with the Hitchin system and to set up a procedure to consistently quantize it. Such a theory has been developed in the works on geometric quantization of the moduli spaces of flat connections [AWDP,H2]. According to a theorem of Narasimhan and Seshadri [NS], the moduli space of flat connections on the surface $\mathcal{M}_N$ is isomorphic to the moduli space $\mathcal{M}_\Sigma$ of stable $G_C$-bundles. $\mathcal{M}_N$ has a natural symplectic structure, and fixing a complex structure $\Sigma \in \mathcal{C}$ provides a complex structure on $\mathcal{M}_N$ and, therefore, a Kähler polarization. The result of [AWDP,H2] is that the quantization of $\mathcal{M}_N$ is independent of this polarization. This is achieved by providing a projectively flat connection on holomorphic sections of a certain line bundle over $\mathcal{M}_N$. It was shown by Hitchin [H2] that this geometric quantization provides the quantum operators for the Hamiltonians \((2.1)\). We strongly believe in the uniqueness of the quantization of the Hitchin system, then the connection of [AWDP,H2] must coincide with the KZB connection. This would provide a correspondence between the two apparently different quantization procedures: the geometric quantization and the path integral approach. A dictionary should be developed to translate between the structures outlined in this paper and those appearing in the geometric quantization of flat connections. It would also be interesting to understand the geometric meaning of the "natural" connections on $\mathcal{E}_N$ conjectured in Section 5.

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