On symplectic automorphisms of Hyperkähler fourfolds

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1 Introduction

An automorphism of a Hyperkähler manifold $X$ is symplectic if

$$\varphi^*(\sigma_X) = \sigma_X$$

where $\sigma_X$ is a holomorphic symplectic 2-form on $X$.

Finite abelian groups of symplectic automorphisms of complex K3 surfaces have been classified by Nikulin in [8], in particular one knows that a symplectic automorphism of finite order on a K3 surface over $\mathbb{C}$ has order at most 8.

The present paper will give an example of a Hyperkähler fourfold, in fact a deformation of $K3[2]$, equipped with a symplectic automorphism of order 11.

In the rest of the paper we will call manifolds of $K3[2]$–type all deformations of the Hilbert square of a K3 surface.

Let us recall that for such a manifold we have $b_2 = 23$ and $H^2 \cong U^3 \oplus E_8(-1)^2 \oplus (-2)$ where $U$ is the hyperbolic plane, $E_8(-1)$ is the unique negative definite even unimodular lattice of rank 8 and $(-2)$ is the rank 1 lattice of discriminant $-2$.

Our example is given by the Fano scheme of lines on a cubic fourfold (unique up to projectivities). Fano schemes of lines on cubic fourfolds were first studied in [3], where the authors proved that they are deformations of $K3[2]$.

Let $\varphi \in SL_6(\mathbb{C})$ be the following matrix:

$$\varphi = \text{Diag}(1, \omega, \omega^3, \omega^4, \omega^5, \omega^9), \quad \omega = e^{\frac{2\pi i}{11}}.$$  \hspace{1cm} (1)

Let $\{e_0, \ldots, e_5\}$ be a basis of eigenvectors for $\varphi$ and let $[x_0, x_1, x_2, x_3, x_4, x_5]$ be coordinates on $\mathbb{P}^5_{\mathbb{C}}$. Let

$$X_{KL} = V(x_0^3 + x_1^2x_5 + x_2^2x_4 + x_3^2x_2 + x_4^2x_1 + x_5^2x_3)$$ \hspace{1cm} (2)
And let $F_{Kl} = F(X_{Kl})$ be its fano scheme of lines, we prove

**Theorem 1.1.** Let $\varphi$, $X_{Kl}$ and $F_{Kl}$ be as above, then the following hold:

- $F_{Kl}$ is a Hyperkähler fourfold deformation of $K3[2]$
- $\varphi$ induces on $F_{Kl}$ a symplectic automorphism $\psi$ of order 11

We call this manifold $X_{Kl}$ since it is a 3 to 1 cover of $\mathbb{P}^3$ ramified along a cubic threefold first studied by Klein [6]. We will describe other automorphisms of $X_{Kl}$, they are taken from the same work of Klein and were studied also by Adler [1].

We will also prove some general results concerning symplectic automorphisms on deformations of $K3[2]$. Let $Co_1$ be Conway’s sporadic simple group, the main result is the following:

**Theorem 1.2.** Let $X$ be a Hyperkähler manifold of $K3[2]$–type and let $G$ be a finite group of symplectic automorphisms of $X$, then $G \subset Co_1$.

We recall that Mukai [10] proved an analogous result for $K3$ surfaces: a finite group of symplectic automorphisms is a subgroup of Mathieu’s group $M_{23}$, see also the proof of Kondo [7].

A partial converse to **Theorem 1.2** is provided by **Proposition 2.11** which gives also a computational method to determine possible finite automorphism groups. Then we use this theorem to prove the following result on symplectic automorphisms of order 11:

**Proposition 1.3.** Let $X$ be a deformation of $K3[2]$ and let $\psi : X \to X$ be a symplectic automorphism of order 11. Then $21 \geq h^{1,1}_Z(X) \geq 20$. Moreover $PSL_2(\mathbb{Z}/11) \subset Bir(X)$. Furthermore if $X$ is projective then $h^{1,1}_Z(X) = 21$

The paper is organized as follows: in **Section 2** we recall some results in lattice theory and we prove **Theorem 1.2** and the other general results on symplectic automorphisms.

**Section 3** briefly analyzes polarizations and deformations of Hyperkähler manifolds having a symplectic automorphism of order 11, moreover in it we also compute $NS(X)$ for two minimal polarizations.

In **Section 4** we give the above stated example of an order 11 symplectic automorphism and in **Section 5** we describe more of its symplectic automorphisms.
2 Lattice theory

In this section we give a proof of Proposition 1.3 and a proof of Theorem 1.2 using several results on general lattice theory and on lattices defined by symplectic automorphisms. The interested reader consults [9] for the main results concerning discriminant forms, [5] for a broader treatment of Subsection 2.1 and Subsection 2.2, [11] for proofs of the stated results on lattices defined by symplectic automorphisms.

Let \( L \) be a lattice, i.e. a free \( \mathbb{Z} \) module equipped with an integer valued symmetric non-degenerate bilinear form \((\ ,\ )_L\), we call \( L \) even if \((a,a)_L \in 2\mathbb{Z}\) for all \( a \in L \).

Given an even lattice \( L \) the group \( A_L = L^\vee /L \) is called discriminant group. Let \( l(A_L) \) be the smallest number of generators of \( A_L \). On \( A_L \) there is a well defined quadratic form \( q_{A_L} \) taking values inside \( \mathbb{Q}/2\mathbb{Z} \) which is called discriminant form; moreover we call \( (l_+, l_-) \) the signature of the quadratic form induced by \((\ ,\ )_L\) on \( L \otimes \mathbb{R} \).

It is possible to define the signature \( \text{sign}(q) \) of a discriminant form \( q \) (modulo 8) as the signature modulo 8 of a lattice having that discriminant form. This notion is well defined since 2 lattices \( M, M' \) such that \( q_{A_M} = q_{A_{M'}} \) are stably equivalent, i.e. there exist 2 unimodular lattices \( T, T' \) such that \( M \oplus T \cong M' \oplus T' \).

One more definition we will need is that of the genus of a lattice: two lattices \( M \) and \( M' \) are said to have the same genus if \( M \otimes \mathbb{Z}_p \cong M' \otimes \mathbb{Z}_p \) for all primes \( p \). Notice that there might be several isometry classes in the same genus.

Let \( S \) be a non degenerate sublattice of an unimodular lattice \( L \) (i.e. \( L^\vee = L \)) and let \( M = S^\perp \). Then \( A_M = A_S \) and \( q_{A_M} = -q_{A_S} \).

Let \( G \) be a finite group of isometries of a lattice \( L \), let \( T_G(L) = L^G \) be the invariant lattice and let

\[ S_G(L) = T_G(L)^\perp \]

be the co-invariant lattice.

The following are a simplified version of fundamental results on the existence of lattices and on the existence of primitive embeddings. Both general results were proven by Nikulin [9, Theorem 1.10.1 and 1.12.2]:

**Lemma 2.1.** Suppose the following are satisfied:

\[ \]


- \(\text{sign}(q_T) \equiv t_+ - t_- \mod 8\)

- \(t_+ \geq 0, t_- \geq 0\) and \(t_+ + t_- \geq l(A_T)\)

- There exists a lattice \(T'\) of rank \(t_+ + t_-\) and discriminant form \(q_T\) over the group \(A_T\)

Then there exists an even lattice \(T\) of signature \((t_+, t_-)\), discriminant group \(A_T\) and form \(q_{A_T}\).

**Lemma 2.2.** Let \(S\) be an even lattice of signature \((s_+, s_-)\). The existence of a primitive embedding of \(S\) into some unimodular lattice \(L\) of signature \((l_+, l_-)\) is equivalent to the existence of a lattice \(M\) of signature \((m_+, m_-)\) and discriminant form \(q_{A_M}\) such that the following are satisfied:

- \(s_+ + m_+ = l_+\) and \(s_- + m_- = l_-\)

- \(A_M \cong A_S\) and \(q_{A_M} = -q_{A_S}\)

**Remark 2.3.** Let \(L = U^3 \oplus E_8(-1)^2 \oplus (-2)\) and let \(G \subset O(L)\). Then there exists a primitive embedding \(L \to L' \cong U^4 \oplus E_8(-1)^2\) such that \(G\) extends to a group of isometries of \(L'\) and \(S_G(L) = S_G(L')\).

**Proof.** Let \(x\) be a vector of square 2 and \(v \in L\) a vector of square \(-2\) such that \((v, L) = 2\mathbb{Z}\). Let \(L'\) be the overlattice of \(L \oplus \mathbb{Z}x\) generated by \(L\) and \(x + \frac{v}{2}\) and let us extend the action of \(G\) to \(L'\) by letting \(G\) act as the identity on \(x\). A direct computation shows \(S_G(L) = S_G(L')\).

### 2.1 Niemeier lattices and Leech-type lattices

In this subsection we recall Niemeier list of negative definite even unimodular lattices in dimension 24 and we introduce a class of lattices which will be of fundamental interest in the rest of the section. Detailed information about these lattices can be found in [5, Chapter 16] and in [9, Section 1.14].

**Definition 2.4.** Let \(M\) be a lattice and let \(G \subset O(M)\). Then \(M\) is a Leech type lattice with respect to \(G\) if the following are satisfied:

- \(M\) is negative definite.

- \(M\) contains no vectors of square \(-2\).

- \(G\) acts trivially on \(A_M\).
Moreover we call \((M, G)\) a Leech couple and \(G\) a Leech type group.

Notice that \((\Lambda, Co_0)\) as in Table 2.1 is a Leech couple.

Now we recall Niemeier list of definite even unimodular lattices of dimension 24, usually they are defined as positive definite lattices but we will use them as negative definite ones. All of these lattices can be obtained by specifying a 0 or 24 dimensional negative definite Dynkin lattice such that every semisimple component has a fixed Coxeter number, in Table 2.1 we recall the possible choices. Having the Dynkin lattice \(A\) of the lattice \(N\) we obtain it by adding a certain set of glue vectors, which are a subset \(G(N)\) of \(A^\vee/A\). The precise definition of the glue vectors can be found in [5, Section 4] and we keep the same notation contained therein. Notice that the set of glue vectors forms an additive subgroup of \(A^\vee/A\).

Another fundamental data is what we call maximal Leech-type group, i.e. the maximal subgroup \(G\) of \(\text{Aut}(N)\) such that \((S_G(N), G)\) is a Leech-type couple.

This data is summarized in Table 2.1 let us explain briefly the notation used therein: for the Leech-type group we used standard notation from [2], where \(n\) denotes a cyclic group of order \(n\), \(p^n\) denotes an elementary \(p\)-group of order \(p^n\), \(G.H\) denotes any group \(F\) with a normal subgroup \(G\) such that \(F/G = H\) and \(L_m(n)\) denotes the group \(\text{PSL}_m\) over the finite field with \(n\) elements. \(M_n\) denotes the Mathieu group on \(n\) elements and \(Co_n\) denotes Conway groups.

Regarding the glue codes we kept the notation of [5], hence a glue code \([abc]\) means a vector \((g, h, f)\) where \(g\) is the glue vector of type \(a\), \(h\) is the one of type \(b\) and \(f\) of type \(c\). Moreover \(([abc])\) indicates all glue vectors obtained from cyclic permutations of \(\{a, b, c\}\), hence \([abc], [bca], [cab]\).

It is a well known fact (see [5] Chapter 26)) that all of the Niemeier lattices can be defined as sublattices of \(\Pi_{1,25} \cong U \oplus E_8(-1)^3\) by specifying a primitive isotropic vector \(v\) and setting \(N = (v^\perp \cap \Pi_{1,25})/v\).

**Example 2.5.** Let \(\Pi_{1,25} \subset \mathbb{R}^{26}\) (the first coordinate of \(\mathbb{R}^{26}\) is the positive definite one) be as before and let

\[
\begin{align*}
v &= (17, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, 5) \\
w &= (70, 0, 1, 2, 3, 4, 5, \ldots, 24)
\end{align*}
\]

be two isotropic vectors in the standard basis of \(\mathbb{R}^{26}\). Then

\[
\Lambda \cong (w^\perp \cap \Pi_{1,25})/w
\]

and

\[
N_{15} \cong (v^\perp \cap \Pi_{1,25})/v
\]
| Name  | Dynkin diagram | Maximal Leech-type Group | Coxeter Number | Generating glue code |
|-------|----------------|--------------------------|----------------|----------------------|
| $N_1$ | $D_{24}$       | 1                        | 46             | [1]                  |
| $N_2$ | $D_{16}E_8$    | 1                        | 30             | [10]                 |
| $N_3$ | $E_8^3$        | $S_3$                    | 30             | [000]                |
| $N_4$ | $A_{24}$       | 1                        | 25             | [5]                  |
| $N_5$ | $D_{12}^2$     | 2                        | 22             | [12, 21]             |
| $N_6$ | $A_{17}E_7$    | 1                        | 18             | [31]                 |
| $N_7$ | $D_{10}E_7^2$  | 2                        | 18             | [110, 301]           |
| $N_8$ | $A_{15}D_9$    | 1                        | 16             | [21]                 |
| $N_9$ | $D_{8}^4$      | $S_3$                    | 14             | (122)                |
| $N_{10}$ | $A_{12}^2$   | 2                        | 13             | [15]                 |
| $N_{11}$ | $A_{11}D_7E_6$ | 1                      | 12             | [111]                |
| $N_{12}$ | $E_6^4$       | $S_4$                    | 12             | [1(012)]            |
| $N_{13}$ | $A_{6}D_6$    | 2                        | 10             | [240, 501, 053]      |
| $N_{14}$ | $D_6^4$       | 2.$A_4$                  | 10             | [even perm. of {0, 1, 2, 3}] |
| $N_{15}$ | $A_5^3$       | $S_3$                    | 9              | [1(144)]            |
| $N_{16}$ | $A_5^2D_5^2$ | 2$^2$                    | 8              | [1112, 1721]        |
| $N_{17}$ | $A_5^7$       | of order 12              | 7              | [1(216)]            |
| $N_{18}$ | $A_5D_4$      | $S_4$                    | 6              | [2(024)0, 33001, 30302, 30033] |
| $N_{19}$ | $D_4^6$       | $S_6$                    | 6              | [111111, 0(02332)]  |
| $N_{20}$ | $A_3^7$       | $L_3(5).2$               | 5              | [1(01441)]          |
| $N_{21}$ | $A_3^3$       | $2^3.L_3(7)$             | 4              | [3(2001011)]        |
| $N_{22}$ | $A_3^2$       | $M_{12}$                 | 3              | [2(11211122212)]    |
| $N_{23}$ | $A_3^2$       | $M_{24}$                 | 2              | [1(00000101001100110101111)] |
| $\Lambda$ | $\emptyset$  | $C_{00}$                 | 0              | $\emptyset$         |

### 2.2 The ”holy” construction and automorphisms of the Leech lattice

In this subsection we give a few different constructions of the Leech lattice $\Lambda$ arising from the other Niemeier lattices. These constructions will be instrumental in the proof of Proposition 1.3. The detailed construction is contained in [5, Section 24], in the present paper we just sketch it.
Let $A_n(-1)$ be a negative definite Dynkin lattice defined by

$$A_n = \{(a_1, \ldots, a_{n+1}) \in \mathbb{Z}^{n+1}, \sum a_i = 0\}$$

And let $q_{A_n(-1)} = -q_{A_n}$.

Let $f_j$ be the vector with $-1$ in the $j$-th coordinate and $1$ in the $(j+1)$-th, zero otherwise. Let moreover $f_0 = (1, 0, \ldots, 0, -1)$. Let $g_0 = h^{-1}(-\frac{1}{2}n, -\frac{1}{2}n + 1, \ldots, \frac{1}{2}n)$ where $h$ is the Coxeter number of $A_n$ and let the $g_i$’s be a cyclic permutation of coordinates of $g_0$. Now let $A_n(-1)^m$ be a 24 dimensional lattice and let $h_k = (g_{i_1}, \ldots, g_{i_m})$ where $[i_1i_2\ldots i_m]$ is a glue code obtained from Table 2.1. Let $f^j_i = (0, \ldots, 0, f_i, 0, \ldots, 0)$ where $f_i$ belongs to the $j$-th copy of $A_n(-1)$. Let $m^j_i$ and $n_w$ be integers.

Then the following holds:

$$\sum_{j=1}^m \sum_i m^j_if^j_i + \sum_w n_wh_w, \sum_w n_w = 0$$ (3)

Is isometric to the negative definite Niemeier lattice with Dynkin diagram $A_n^m$.

$$\sum_{j=1}^m \sum_i m^j_if^j_i + \sum_w n_wh_w, \sum_w n_w + \sum_{i,j}^m m^j_i = 0$$ (4)

Is isometric to the Leech lattice $\Lambda$.

Moreover the glue code provides several automorphisms of the Leech lattice, where the action of $t \in G(N)$ is given by sending $h_w$ to $h_{w+t}$.

This construction is really useful to explicit the action of some elements of $Co_1$ on $\Lambda$, namely in the following examples:

**Example 2.6.** Let us apply this construction to the lattice $A_{12}^2$, we then have $G(N) = \mathbb{Z}_{13}$. Let $\phi$ be an automorphism of $\Lambda$ of order 13 generated by a non trivial element $g$ of $G(N)$ on this holy construction. $\phi$ cyclically permutes the roots of both copies of $A_{12}$ and therefore has no fixed points in $\Lambda$.

**Example 2.7.** Let us apply this construction to the lattice $A_{12}^2$ and let us analize an automorphism of order 11: it can be defined by leaving the first copy of $A_2$ fixed and by cyclically permuting the remaining 11. The action is extended accordingly to the glue vectors. This automorphism is defined on both $N_{22}$ and $\Lambda$. Let $\varphi$ be this isometry on $A_{12}^2 \otimes \mathbb{Q}$.

A direct computation shows $T_{\varphi}N_{22}$ is spanned by $f^1_1, f^1_2, \sum_1^{12} f^1_i, \sum_2^{12} f^2_i, \sum_1^{11} g_j$, where $g_j$ are generators for the glue code as in Table 2.1. Keeping the same notation as before one sees that $S_{\varphi}N_{22}$ has rank 20 and is spanned by

$$(f^k_1 - \varphi f^k_1), (f^k_2 - \varphi f^k_2), (g_j - \varphi g_j)$$ (5)
Where $k$ runs from 2 to 12. This vectors satisfy (4), therefore this lattice is contained in $\Lambda$ and, since they are both primitive, $S_\varphi(N_{22}) = S_\varphi(\Lambda)$.

**Example 2.8.** A similar computation can be done for $A_1^{24}$. We use a standard notation where the copies of $A_1$ are indexed by the set

$$\{\infty, 0, 1, \ldots, 22\} = \mathbb{P}^1(\mathbb{Z}/(23)).$$

Here the isometry $\varphi$ of order 11 is defined by the following permutation on the coordinates:

$$(0)(15 7 14 5 10 20 17 11 22 21 19)(\infty)(3 6 12 1 2 4 8 16 9 18 13) \quad (6)$$

As before this isometry preserves both $N_{23}$ and $\Lambda$ and the lattice $S_\varphi(N_{23})$ is generated by the following vectors:

$$(f_1^i - \varphi f_1^i), (f_1^i - \varphi f_1^j), (g_j - \varphi g_j) \quad (7)$$

Where $k$ runs along the indexes contained in the first 11-cycle of (5), $l$ runs along the second one and $j$ along the generators of the glue code contained in Table 2.1.

Once again all of these generators lie also in $\Lambda$ hence $S_\varphi(N_{23}) = S_\varphi(\Lambda)$. A direct computation shows that the lattice $S_{11} = S_\varphi(N_{23})$ is given by the following quadratic form:

$$(\begin{array}{cccccccccccccccc}
-4 & 1 & -2 & -2 & -1 & 1 & -1 & 1 & -1 & -1 & 2 & 1 & -1 & 2 & -1 & -2 & -2 & 2 & 1 & -1 \\
1 & -4 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 2 & -2 & -2 & 0 & 1 & -1 & 0 & 0 & -1 & -2 & 1 \\
-2 & -1 & -4 & -2 & -1 & 1 & 0 & 1 & -1 & 1 & 0 & -1 & 2 & -2 & -1 & -1 & 0 & 0 & 1 & 0 & 1 \\
-2 & -1 & -2 & -1 & 0 & -2 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 2 & -1 & 0 & 1 & 0 & 0 & -1 & -1 \\
-1 & -1 & -1 & 0 & -4 & 1 & -1 & 2 & -2 & -1 & 1 & 0 & -1 & 0 & -2 & -2 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 0 & 1 & -4 & 0 & -1 & 0 & 1 & -2 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & 0 \\
-1 & -1 & -1 & 0 & -2 & -1 & 0 & -4 & 1 & -2 & 1 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -2 \\
1 & 1 & 1 & 0 & 2 & -1 & -1 & -4 & 0 & 0 & -1 & 1 & 1 & 0 & 2 & 2 & 1 & 0 & -1 & 1 & 0 \\
-1 & -1 & 0 & -1 & -2 & 0 & -2 & 0 & -4 & 0 & 0 & 1 & 1 & 0 & -1 & -2 & 0 & 2 & 0 & -2 & -2 \\
-2 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & -4 & 1 & 1 & -2 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\
2 & -1 & 1 & 2 & 1 & -2 & 1 & -1 & 0 & 1 & -4 & -2 & 2 & -1 & 0 & 0 & 0 & -1 & -2 & 1 & -1 \\
1 & -2 & 0 & 1 & 0 & -1 & 1 & 1 & 1 & 1 & -2 & -4 & 1 & 0 & -1 & 0 & -1 & -1 & -2 & 2 & -1 \\
-1 & 2 & -1 & 0 & -1 & 0 & 1 & 1 & -2 & 2 & 1 & -4 & 0 & -1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\
2 & 0 & 2 & 1 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & -4 & 1 & 1 & 1 & 0 & 0 & -1 \\
-1 & -1 & -2 & 0 & -2 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -2 & -4 & -2 & 2 & 0 & -1 \\
-2 & 0 & -1 & 0 & -2 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 2 & 1 & -4 & -2 \\
2 & -2 & 0 & 1 & 1 & 0 & 2 & -1 & 2 & 1 & -1 & 1 & 0 & 1 & 2 & 1 & -4 & 0 & 2 & 0 & -2 & -2 \\
1 & -2 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & -2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 1 & -4 & -1 \\
-1 & 1 & 1 & -1 & -1 & 1 & -2 & 0 & -2 & 0 & 1 & 2 & 0 & -1 & 0 & -1 & 0 & 2 & 1 & 1 & -4 & -1 \\
\end{array})$$

It is worth mentioning that $S_{11}$ is in the same genus of $E_8(-1)^2 \oplus \left(\begin{array}{cc}
-2 & 1 \\
1 & -6 \\
\end{array}\right)^2$ and of $D_{16}^+(-1) \oplus \left(\begin{array}{cc}
-2 & 1 \\
1 & -6 \\
\end{array}\right)^2$, where $D_{16}^+$ is a unimodular overlattice of the Dynkin lattice $D_{16}$. 

8
2.3 General results on finite symplectic automorphisms groups

Let us start with some notation: let $X$ be a Hyperkähler manifold and let $G \subset \text{Aut}(X)$. Then we denote $S_G(X) = S_G(H^2(X, \mathbb{Z}))$ and $T_G(X) = T_G(H^2(X, \mathbb{Z}))$. The following 2 lemmas are contained in [11]:

**Lemma 2.9.** Let $G$ be a finite symplectic automorphism group on a fourfold $X$ of $K3^{[2]}$-type, then the following assertions are true:

1. $S_G(X)$ is non degenerate and negative definite
2. $S_G(X)$ contains no elements with square $-2$
3. $S_G(X) \subseteq \text{Pic}(X)$
4. $G$ acts trivially on $A_{S_G(X)}$

**Lemma 2.10.** Let $L = U^3 \oplus E_8(-1)^2 \oplus (-2)$ and let $G$ be a finite subgroup of $O(L)$. Suppose the following hold:

1. $S_G(L)$ is non degenerate and negative definite.
2. $S_G(L)$ contains no element with square $-2$.

Then there exists a Hyperkähler manifold $X$ of $K3^{[2]}$-type and a subgroup $G' \subset \text{Bir}(X)$ such that $G' \cong G$, $S_G(L) \cong S_{G'}(X)$ and $G_{|H^{2,0}(X)} = \text{Id}$.

**Proposition 2.11.** Let $(S, G)$ be a couple consisting in a Leech-type lattice and its Leech automorphism group. Let moreover $S \subset N$, one of the 24 Niemeier lattices. Suppose that there exists a primitive embedding $S \rightarrow L$. Then $G$ extends to a group of bimeromorphisms on some Hyperkähler manifold $X$ of $K3^{[2]}$-type.

**Proof.** This is an immediate consequence of Lemma 2.10. $G$ acts trivially on $A_S$, therefore we can extend $G$ to a group of isometries of $L$ acting trivially on $S^\perp$. Thus we have $S_G(L) \cong S$. Moreover since $S$ is a Leech-type lattice contained in a negative definite lattice $N$ the other conditions of Lemma 2.10 are satisfied.

We are now ready to prove the main result of this section:
Proof of Theorem 1.2. Let \( b = \text{Rank}(S_G(X)) \), by Lemma 2.9 \( S_G(X) \) has signature \((0, b)\). By Remark 2.3 we have a lattice \( T' \) of signature \((4, 20 - b)\) such that \( A_{T'} = A_{S_G(X)} \) and \( q_{T'} = -q_{A_{S_G(X)}} \). Therefore we can apply Lemma 2.1 obtaining a lattice \( T \) of signature \((0, 24 - b)\) and discriminant \(-q_{A_{S_G(X)}} \).

Corollary 2.12. Let \( \phi \) be a symplectic automorphism of prime order \( p \) on a Hyperkähler fourfold \( X \) of \( K3[2] \)-type. Then \( p \leq 11 \)

Proof. By Theorem 1.2 the order of a symplectic automorphism must divide the order of the group \( Co_1 \). That sorts out all primes apart from 2, 3, 5, 7, 11, 13, 23. An automorphism of order 23 has a co-invariant lattice which is negative definite and of rank 22, therefore it cannot embed into \( H^2(X, \mathbb{Z}) \). This can be explicitly computed using an order 23 element of \( M_{24} \) and letting it act on \( \Lambda \) or on \( N_{23} \). The only Niemeier lattice with an automorphism of order 13 is \( \Lambda \), where all elements of order 13 are conjugate. It is a well known fact that these automorphisms have no fixed points on \( \Lambda \), as in Example 2.6.

Proof of Proposition 1.3. By Theorem 1.2 \( S_\psi(X) \) embeds in a Niemeier lattice \( N \) and \( \psi \) extends to a Leech automorphism of \( N \). By Table 2.1 \( N \) can either be \( N_{22}, N_{23} \) or \( \Lambda \) and, up to conjugacy, there is only one possible choice for \( \psi \subset O(N_{23}) \) or \( \psi \subset O(\Lambda) \) and two possible choices for \( \psi \subset O(N_{22}) \).

However we computed \( S_\psi(N) \) in Example 2.7 and Example 2.8 therefore we immediately have \( 21 \leq h_1^1(X) \leq 20 \) and obviously if \( X \) is projective \( h_2^1(X) = 21 \). Now we wish to give an action of \( L_2(11) = PSL_2(\mathbb{Z}/(11)) \) on \( S_\psi(N) \).

Now by Proposition 2.11 this action of \( L_2(11) \) on cohomology is induced by a group of birationalities isomorphic to \( L_2(11) \).
3 Deformational behaviour

In this section we analyze deformation classes of Hyperkähler manifolds with a symplectic automorphism of order 11 and we look at their possible invariant polarizations. First of all we need an important and well known deformation of a Hyperkähler manifold:

Definition 3.1. Let $X$ be a Hyperkähler manifold with Kähler class $\omega$ and symplectic form $\sigma_X$. Then there exists a family

$$TW_\omega(X) := X \times \mathbb{P}^1$$

$$\downarrow$$

$$S^2 \cong \mathbb{P}^1$$

called Twistor space such that $TW_\omega(X)_{(a,b,c)} = X$ with complex structure given by the Kähler class $a\omega + b(\sigma_X + \overline{\sigma}_X) + c(\sigma_X - \overline{\sigma}_X)$.

Remark 3.2. Let $X$ be a Hyperkähler manifold and let $G \subset Aut(X)$ be a finite symplectic group of automorphisms. Let moreover $\omega$ be a $G$–invariant Kähler class. Then the action of $G$ on $X$ extends to a symplectic action of $G$ on all the fibers of the twistor space associated to $\omega$.

Proof. Every fiber has a Kähler class which is a linear combination of $\sigma_X, \overline{\sigma}_X$ and $\omega$. Since $G$ is symplectic these classes are all $G$–invariant, hence $G \subset Aut(TW_\omega(X))$ for all $t$.

Let $X$ be a Hyperkähler manifold with a symplectic automorphism $\psi$ of order 11 and let $\omega$ be a $\psi$–invariant Kähler class. First of all let us remind that Proposition 1.3 implies that non trivial deformations of $(X, \psi)$ are of maximal dimension 1, moreover the twistor family $TW_\omega(X)$ is naturally endowed with a symplectic automorphism of order 11 as in Remark 3.2. Therefore $TW_\omega(X)$ is already a family of the maximal dimension for such manifolds $(X, \psi)$. Moreover we have that the twistor family $TW_\omega(X)$ is actually a family over the base $\mathbb{P}(T_\psi(X) \otimes \mathbb{R})$ since $T_\psi(X) = \langle \omega, \sigma_X, \overline{\sigma}_X \rangle \cap H^2_\mathbb{Z}(X)$.

Thus what we really need to analyze are the possible lattices $T_\psi(X)$ up to isometry. We have already proved that there exists only one isometry class of lattices $S_\psi(X)$, which is that of (8). However there might be several isomorphism classes of lattices $T_\psi(X)$. In fact Theorem 1.2 and Proposition 2.11 can be used only to compute its genus.

A direct computation shows that there are two such lattices, namely the
Therefore there are 2 distinct families of Hyperkähler manifolds endowed with a symplectic automorphism of order 11, let us call $TW(X_1)$ the first and $TW(X_2)$ the second.

### 3.1 Invariant Polarizations

In this subsection we look at possible invariant polarizations of $TW(X_1)$ and $TW(X_2)$, i.e. at possible primitive vectors inside (11) and (12).

We computed several polarizations in degree up to 24, the following summarizes our results:

**Proposition 3.3.** The minimal degree of an invariant polarization inside $TW(X_1)$ is 2, there are countable polarizations and there are no polarizations in degrees 4, 12, 14, 16, 20. Moreover the least degree of a polarization $f$ such that $(f, L) = 2\mathbb{Z}$ (i.e. $f$ is 2-divisible) is 22.

**Proof.** Let $T_{11}^1$ be as in (11) and let it be generated by vectors $a, b, c$ such that $a^2 = 2, b^2 = 6$ and $c^2 = 22$. A minimal polarization is given by the vector $a$, which is also 2-divisible. A sequence of polarizations is given by the vectors $a + nc$. The rest is just a direct computation. \qed

**Proposition 3.4.** The minimal degree of an invariant polarization inside $TW(X_2)$ is 6, there are countable polarizations and there are no polarizations in degrees 12, 14, 16, 20. Moreover the least degree of a polarization $f$ such that $(f, L) = 2\mathbb{Z}$ is 6.

**Proof.** Let $T_{11}^2$ be as in (12) and let it be generated by vectors $a, b, c$ such that $a^2 = 6, b^2 = 8, c^2 = 8$ and $(b, c) = -3$. A minimal polarization is given by the vector $a$ which is also 2-divisible. A sequence of polarizations is given by the vectors $a + nc$. The rest is just a direct computation. \qed
Corollary 3.5. Let $X$ be a Hyperkähler manifold with a symplectic automorphism $\psi$ of order 11 and an invariant polarization of degree 6 and divisibility 2. Then

$$NS(X) \cong (6) \oplus E_8(-1)^2 \oplus \left( \begin{array}{cc} -2 & 1 \\ 1 & -6 \end{array} \right)^2$$

(13)

And

$$T(X) = \langle \sigma_X, \sigma_X \rangle \cap H^2(X, \mathbb{Z}) \cong \left( \begin{array}{cc} 22 & 33 \\ 33 & 66 \end{array} \right)$$

(14)

Proof. $NS(X)$ is an overlattice of $S_{11} \oplus (6)$, where $S_{11}$ is as in (8). Since there are no non trivial overlattices of it we have $NS(X) = S_{11} \oplus (6)$. A direct computation shows that this lattice is isomorphic to $(6) \oplus E_8(-1)^2 \oplus \left( \begin{array}{cc} -2 & 1 \\ 1 & -6 \end{array} \right)^2$.

Finally $T(X)$ is the orthogonal in $T_\psi(X)$ of its polarization. By Proposition 3.3 and Proposition 3.4 we have $T_\psi(X) = T_{11}^2$ and a direct computation shows $T(X) = \left( \begin{array}{cc} 22 & 33 \\ 33 & 66 \end{array} \right)$.

Corollary 3.6. Let $X$ be a Hyperkähler manifold with a symplectic automorphism $\psi$ of order 11 and an invariant polarization of degree 2. Then

$$NS(X) \cong (2) \oplus E_8(-1)^2 \oplus \left( \begin{array}{cc} -2 & 1 \\ 1 & -6 \end{array} \right)^2$$

(15)

And

$$T(X) = \langle \sigma_X, \sigma_X \rangle \cap H^2(X, \mathbb{Z}) = \left( \begin{array}{cc} 22 & 0 \\ 0 & 22 \end{array} \right)$$

(16)

Proof. $NS(X)$ is an overlattice of $S_{11} \oplus (2)$, where $S_{11}$ is as in (8). Since there are no non trivial overlattices of it we have $NS(X) = S_{11} \oplus (2)$. A direct computation shows that this lattice is isomorphic to $(2) \oplus E_8(-1)^2 \oplus \left( \begin{array}{cc} -2 & 1 \\ 1 & -6 \end{array} \right)^2$.

Finally $T(X)$ is the orthogonal in $T_\psi(X)$ of its polarization. By Proposition 3.3 and Proposition 3.4 we have $T_\psi(X) = T_{11}^1$ and a direct computation shows $T(X) = \left( \begin{array}{cc} 22 & 0 \\ 0 & 22 \end{array} \right)$.
4 The Fano scheme of lines $F_{Kl}$

The main goal of this section is to prove Theorem 1.1 let us start with a few known results about Fano schemes of lines on cubic fourfolds due to Beauville and Donagi.

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold and let $F(X)$ be the scheme parametrizing lines contained in $X$.

Theorem 4.1. [3, Beauville and Donagi] Keep notation as above, then the following hold:

- $F(X)$ is a Hyperkähler manifold
- $F(X)$ is deformation equivalent to $K3^{[2]}
- the Abel-Jacobi map
  \[ \alpha : H^4(X, \mathbb{C}) \rightarrow H^2(F(X), \mathbb{C}) \]  
  is an isomorphism of Hodge structures.

Let $\varphi$ be as in (1), we wish to give a $\varphi$-invariant cubic polynomial on $\mathbb{C}^6$. Inside $S^3((\mathbb{C}^6)^\vee)$ invariant polynomials are spanned by the following monomials:

\[ B = \{x_0^3, x_1^2x_5, x_2^2x_4, x_3^2x_2, x_4^2x_1, x_5^2x_3\}, \]

where $\{x_0, x_1, x_2, x_3, x_4, x_5\}$ is the dual basis of $\{e_0, e_1, e_2, e_3, e_4, e_5\}$.

Let $f$ be a linear combination of these monomials and let $df$ be its differential. A direct computation shows that $df$ has some non trivial zeroes if $f$ lies in the span of some proper subset of $B$.

Furthermore there are polynomials $f$ such that $df$ has only non trivial zeroes, let us choose one of them:

\[ m = x_0^3 + x_1^2x_5 + x_2^2x_4 + x_3^2x_2 + x_4^2x_1 + x_5^2x_3 \]

and let $X_{Kl} = V(m)$ and $F_{Kl} = F(X_{Kl})$.

Proof of Theorem 1.1 $X_{Kl}$ is smooth, therefore we can apply Theorem 4.1 and we obtain that $F_{Kl}$ is a Hyperkähler manifold deformation equivalent to $K3^{[2]}$. Moreover we have the Hodge isomorphism given in (17).

Let $\psi$ be the map induced on $F_{Kl}$ by $\varphi$, using $\alpha$ we have that $\psi$ is symplectic if and only if $\varphi|_{H^{3,1}(X_{Kl})} = Id$.

By the formula in [12, Théorème 18.1] we have

\[ H^{3,1}(X_{Kl}) = <\text{Res}(\frac{\Omega}{m^2})> \]
where $\Omega = \sum_i (-1)^i x_i \, dx_0 \wedge \ldots \wedge dx_i \ldots \wedge dx_5$. Since $\varphi$ acts trivially on both $\Omega$ and $m$ we obtain that $\psi(\sigma_{F_{KL}}) = \sigma_{F_{KL}}$ for any symplectic 2-form $\sigma_{F_{KL}}$ on $F_{KL}$. Finally fixed points of $\varphi$ on $X_{KL}$ are the eigenvectors of $\varphi$ lying on $X_{KL}$, which are the points $[e_1], [e_2], [e_3], [e_4], [e_5]$. A fixed line on $X_{KL}$ must contain 2 fixed points, therefore the fixed points on $F_{KL}$ are those parametrizing lines through those points, namely the five lines $[e_1][e_2], [e_1][e_3], [e_2][e_3], [e_3][e_4], [e_4][e_5]$.

This proves that $\psi$ is not the identity hence it has indeed order 11.

**Remark 4.2.** Proposition 3.3 and Proposition 3.4 imply that $F_{KL} \subset TW(X_2)$, moreover by Corollary 3.5 we obtain its Neron-Severi and transcendental lattices. Moreover, since $h^{1,1}_Z(F_{KL}) = h^{1,1}_C(F_{KL})$, $\psi$ does not lift to any small projective non-trivial deformation of $F_{KL}$.

## 5 $L_2(11)$ acting on $F_{KL}$

In this subsection we give several other symplectic automorphisms of $F_{KL}$, therefore this gives an example to Proposition 1.3.

Let us remark that $X_{KL}$ is a 3 to 1 cover of $\mathbb{P}^4$ ramified along the threefold

$$KA = V(x_1^2x_5 + x_2^2x_4 + x_3^2x_2 + x_4^2x_1 + x_5^2x_3).$$

(19)

Where the covering map is simply the projection

$$[x_0, x_1, x_2, x_3, x_4, x_5] \rightarrow [x_1, x_2, x_3, x_4, x_5]$$

and the covering automorphism group is generated by

$$[x_0, x_1, x_2, x_3, x_4, x_5] \overset{\alpha}{\rightarrow} [\eta x_0, x_1, x_2, x_3, x_4, x_5], \quad \eta = e^\frac{2\pi i}{11}.$$ 

Notice that $\alpha$ acts as multiplication by $\eta$ on $H^{3,1}(X_{KL})$.

Obviously any automorphism of $\mathbb{P}^4$ that preserves $KA$ extends to an automorphism of $X_{KL}$. By the results in [1] and [3] these automorphism span precisely the group $L_2(11) \cong PSL_2(\mathbb{Z}/(11))$, which is a finite simple group of order 660. Hence the automorphism group of $X_{KL}$ contains $\mathbb{Z}/(3) \times L_2(11)$. Now we only need to find generators of this group and to determine whether they act symplectically on $F_{KL}$ or not.

Let $(1 \, 4 \, 2 \, 3 \, 5)$ be a permutation and let $\beta$ be the automorphism it induces on $\mathbb{P}^4$ by permuting the coordinates $[x_1, \ldots, x_5]$. $\beta$ leaves $KA$ invariant, hence it induces an automorphism $\beta$ of order 5 on $F_{KL}$.

Using [15] one obtains that $\beta$ is symplectic on $F_{KL}$. Furthermore a direct
computation on the Jacobian ring of $X_{KL}$ shows that $\text{rk} S_{\beta}(F_{KL}) = 16$.
Let us consider the following exact sequence:

$$1 \to H \to \mathbb{Z}/(3) \times L_2(11) \to \mathbb{C}^*$$  \hspace{1cm} (20)

Where the last map is given by the action of $\mathbb{Z}/(3) \times L_2(11)$ on $H^{2,0}(F_{KL})$ and $H$ is the quotient of $\mathbb{Z}/(3) \times L_2(11)$ by its image in $\mathbb{C}^*$. Therefore $H$ is a normal subgroup of $L_2(11)$, which is simple. Since $\beta \in H$ we have $H = L_2(11)$, therefore $L_2(11)$ acts symplectically on $F_{KL}$.

$L_2(11)$ contains only elements of order 2,3,5,6 and 11 and a direct computation shows the following:

- $\text{rk}(S_f(F_{KL})) = 8$ if $\text{ord}(f) = 2$
- $\text{rk}(S_f(F_{KL})) = 12$ if $\text{ord}(f) = 3$
- $\text{rk}(S_f(F_{KL})) = 16$ if $\text{ord}(f) = 5$
- $\text{rk}(S_f(F_{KL})) = 16$ if $\text{ord}(f) = 6$
- $\text{rk}(S_f(F_{KL})) = 20$ if $\text{ord}(f) = 11$

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