PFA AND PRECIPITOUSNESS OF THE NONSTATIONARY IDEAL

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Abstract. We apply Neeman’s method of forcing with side conditions to show that PFA does not imply the precipitousness of the nonstationary ideal on $\omega_1$.

Introduction

One of the main consequences of Martin’s Maximum (MM) is that the nonstationary ideal on $\omega_1$ ($\text{NS}_{\omega_1}$) is saturated and hence also precipitous. This was already shown by Foreman, Magidor and Shelah in [1], where the principle MM was introduced. It is natural to ask if the weaker Proper Forcing Axiom (PFA) is sufficient to imply the same conclusion. In the 1990s the author adapted the argument of Shelah in [8, Chapter XVII], where PFA is shown to be consistent with the existence of a function $f : \omega_1 \rightarrow \omega_1$ dominating all the canonical functions below $\omega_2$, to show that PFA does not imply the precipitousness of $\text{NS}_{\omega_1}$. This result was also obtained independently by Shelah and perhaps several other people, but since it was never published it was considered a folklore result in the subject.

Some twenty years later Neeman [7] introduced a method for iterating proper forcing by using conditions which consist of two components: the \textit{working part}, which is a function of finite support, and the \textit{side condition}, which is a finite $\in$-chain of models of one of two types. The interplay between the working parts and the side conditions allows us to show that the iteration of proper forcing notions is proper. Neeman used this new iteration technique to give another proof of the consistency of PFA as well as several other interesting applications.

In this paper we adapt Neeman’s iteration technique to give another proof of the consistency of PFA together with $\text{NS}_{\omega_1}$ being non precipitous. Our modification consists of two parts. First, we consider a \textit{decorated} version of the side condition poset. This version is already present in [7]. Its principal virtue is that it guarantees that the generic sequence of models added by the side condition part of the forcing is continuous. The second modification is more subtle. To each condition $p$ we attach the \textit{height function} $\text{ht}_p$, which is defined on certain pairs of ordinals. In order for a condition $q$ to extend $p$ we require that $\text{ht}_q$ extends $\text{ht}_p$. Now, if $G$ is a generic filter, we can define the derived

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height function $h_\alpha$ from which we can read off the functions $h_\alpha$, for $\alpha < \theta$, where $\theta$ is the length of the iteration. Each of these functions is defined on a club in $\omega_1$ and takes values in $\omega_1$. If $U$ is a generic ultrafilter over $\mathcal{P}(\omega_1)/\text{NS}_{\omega_1}$ we can consider $[h_\alpha]_U$, the equivalence class of $h_\alpha$ modulo $U$, as an element of $\text{Ult}(V,U)$. The point is that the family $\{[h_\alpha]_U : \alpha < \theta\}$ cannot be well ordered by $\leq_U$ and hence $\text{Ult}(V,U)$ will not be well-founded. Now, our requirement on the height functions introduces some complications in the proof of Neeman’s lemmas required to show the properness of the iteration. The main change concerns the pure side condition part of the forcing. Our side conditions consist of pairs $(\mathcal{M}_p, d_p)$ where $\mathcal{M}_p$ is the $\in$-chain of models and $d_p$ is the decoration. If a model $M$ occurs in $\mathcal{M}_p$ we can form the restriction $p|M$, which is simply $(\mathcal{M}_p \cap M, d_p \upharpoonright M)$. Then $p|M$ is itself a side condition and belongs to $M$. The problem is that we do not know that $p$ is stronger than $p|M$, simply because $ht_p$ may not extend $ht_{p|M}$. However, for many models $M$ we will be able to find a reflection $q$ of $p$ inside $M$ such that $ht_q$ does extend $ht_p \upharpoonright M$ and we will then use $q$ instead of $p|M$. This requires reworking some of the lemmas of [7]. Since our main changes involve the side condition part of the forcing, we present a detailed proof that after forcing with the pure side condition poset the nonstationary ideal on $\omega_1$ is not precipitous. When we add the working parts we need to rework some of Neeman’s iteration lemmas, but the modifications are mostly straightforward, so we only sketch the arguments and the refer the reader to [7]. Finally, let us mention that precipitousness of ideals in forcing extensions was studied by Laver [6], and while we do not use directly results from that paper, some of our ideas were inspired by [6].

The paper is organized as follows. In §1 we recall some preliminaries about canonical functions and precipitous ideals. In §2 we introduce a modification of the pure side condition forcing with models of two types from [7]. In §3 we prove a factoring lemma for our modified pure side condition poset and use it to show that after forcing with this poset the nonstationary ideal is non precipitous. In §4 we introduce the working parts and show how to complete the proof of the main theorem.

In order to read this paper, a fairly good understanding of [7] is necessary and the reader will be referred to it quite often. Our notation is fairly standard and can be found in [8] and [4] to which we refer the reader for background information on precipitous ideals, proper forcing, and all other undefined concepts. Let us just mention that a family $\mathcal{F}$ of subsets of a set $K$ is called stationary in $K$ if for every function $f : K^{<\omega} \to K$ there is $M \in \mathcal{F}$ which is closed under $f$, i.e. such that $f[M^{<\omega}] \subseteq M$.

1. Preliminaries

We start by recalling the relevant notions concerning precipitous ideals from [5]. Suppose $\mathcal{I}$ is a $\kappa$-complete ideal on a cardinal $\kappa$ which contains all singletons. Let $\mathcal{I}^+$ be the collection of all $\mathcal{I}$-positive subsets of $\kappa$, i.e. $\mathcal{I}^+ = \mathcal{P}(\kappa) \setminus \mathcal{I}$. We consider $\mathcal{I}^+$ as
a forcing notion under inclusion. If $G_I$ is a $V$-generic over $I^+$ then $G_I$ is an ultrafilter on $\mathcal{P}^V(\kappa)$ which extends the dual filter of $I$. We can then form the generic ultrapower $\text{Ult}(V, G_I)$ of $V$ by $G_I$ in the usual way, i.e. it is simply $(V^\kappa \cap V)/G_I$. Recall that $I$ is called precipitous if the maximal condition forces that this ultrapower is well founded. There is a convenient reformulation of this property in terms of games.

**Definition 1.1.** Let $I$ be a $\kappa$-complete ideal on a cardinal $\kappa$ which contains all singletons. The game $G_I$ is played between two players I and II as follows.

\[
\begin{array}{c|cccccc}
I : & E_0 & E_2 & \cdots & E_{2n} & \cdots \\
II : & E_1 & E_3 & \cdots & E_{2n+1} & \cdots
\end{array}
\]

We require that $E_n \in I^+$ and $E_{n+1} \subseteq E_n$, for all $n$. The first player who violates these rules loses. If both players respect the rules, we say that I wins if $\bigcap_n E_n = \emptyset$. Otherwise, II wins.

**Fact 1.2** (Galvin, Jech, Magidor [3]). The ideal $I$ is precipitous if and only if player I does not have a winning strategy in $G_I$. □

We will also need the notion of canonical functions relative to the nonstationary ideal, $\text{NS}_{\omega_1}$. We recall the relevant definitions from [2]. Given $f, g : \omega_1 \to \text{ORD}$ we let $f <_{\text{NS}_{\omega_1}} g$ if $\{\alpha : f(\alpha) < g(\alpha)\}$ contains a club. Since $\text{NS}_{\omega_1}$ is countably complete, the quasi order $<_{\text{NS}_{\omega_1}}$ is well founded. For a function $f \in \text{ORD}^{\omega_1}$, let $||f||$ denote the rank of $f$ in this ordering. It is also known as the Galvin-Hajnal norm of $f$. By induction on $\alpha$, the $\alpha$-th canonical function $f_\alpha$ is defined (if it exists) as the $<_{\text{NS}_{\omega_1}}$-least ordinal valued function greater than the $f_\xi$, for all $\xi < \alpha$. Clearly, if the $\alpha$-th canonical function exists then it is unique up to the equivalence $=_{\text{NS}_{\omega_1}}$. One can show in ZFC that the $\alpha$-th canonical function $f_\alpha$ exists, for all $\alpha < \omega_2$. One way to define $f_\alpha$ is to fix an increasing continuous sequence $(x_\xi)_{\xi < \omega_1}$ of countable sets with $\bigcup_{\xi < \omega_1} x_\xi = \alpha$ and let $f_\alpha(\xi) = o.t.(x_\xi)$, for all $\xi$. The point is that if we wish to witness the non well foundedness of the generic ultrapower we have to work with functions that are above the $\omega_2$ first canonical functions. Our forcing is designed to introduce $\theta$ many such functions, where $\theta$ is the length of the iteration. From these functions we define a winning strategy for I in $G_{\text{NS}_{\omega_1}}$ and implies that $\text{NS}_{\omega_1}$ is not precipitous in the final model.

2. The Side Condition Poset

We start by reviewing Neeman’s side condition poset from [7]. We fix a transitive model $K = (K, \in, \ldots)$ of a sufficient fragment of ZFC, possibly with some additional functions or predicates. Let $S$ denote a collection of countable elementary submodels of $K$ and let $\mathcal{T}$ be a collection of transitive $W \prec K$ such that $W \in K$. We say that the pair $(S, T)$ is appropriate if $M \cap W \in S \cap W$, for every $M \in S$ and $W \in \mathcal{T}$. We are
primarily interested in the case when $K$ is equal to $V_{\theta}$, for some inaccessible cardinal $\theta$, let $S$ consists of all countable submodels of $V_{\theta}$ and $T$ consists of all the $V_{\alpha}$ such that $V_{\alpha} \prec V_{\theta}$ and $\alpha$ has uncountable cofinality. We present the more general version since it will be needed in the analysis of the factor posets of the side condition forcing.

Let us fix a transitive model $K$ of a sufficient fragment of set theory and an appropriate pair $(S, T)$. The side condition poset $M_{S, T}$ consists of finite $\in$-chains $M = \{M_0, \ldots, M_{n-1}\}$ of elements of $S \cup T$, closed under intersection. So for each $k < n$, $M_k \in M_{k+1}$, and if $M, N \in M$, then also $M \cap N \in M$. We will refer to elements of $M \cap S$ as small or countable nodes of $M$, and to the elements of $M \cap T$ as transitive nodes of $M$. We will write $\pi_S(M)$ for $M \cap S$ and $\pi_T(M)$ for $M \cap T$. Notice that $M$ is totally ordered by the ranks of its nodes, so it makes sense to say, for example, that $M$ is above or below $N$, when $M$ and $N$ are nodes of $M$. The order on $M_{S, T}$ is reverse inclusion, i.e. $M \leq N$ iff $N \subseteq M$.

The decorated side condition poset $M_{S, T}^{\text{dec}}$ consists of pairs $p$ of the form $(M_p, d_p)$, where $M_p \in M_{S, T}$ and $d_p : M_p \to K$ is such that $d_p(M)$ is a finite set which belongs to the successor of $M$ in $M_p$, if this successor exists, and if $M$ is the largest node of $M_p$ then $d_p(M) \in K$. Sometimes our $d_p$ will be only a partial function on $M_p$. In this case, we identify it with the total function which assigns the empty set to all nodes on which $d_p$ is not defined. The order on $M_{S, T}^{\text{dec}}$ is given by letting $q \leq p$ iff $M_p \subseteq M_q$ and $d_p(M) \subseteq d_q(M)$, for every $M \in M_p$. Suppose $p \in M_{S, T}^{\text{dec}}$ and $Q \in M_p$. Let $p|Q$ be the condition $(M_p \cap Q, d_p \upharpoonright Q)$. One can check that $p|Q$ is indeed a condition in $M_{S, T}^{\text{dec}}$.

We first observe the following simple fact.

**Lemma 2.1.** Suppose $p$ is a condition in $M_{S, T}^{\text{dec}}$ and $M$ is a model in $S \cup T$ such that $p \in M$. Then there is a condition $p^M$ extending $p$ such that $M$ is the top model of $M_{p, M}$.

**Proof.** We let $M_{p, M}$ be the closure of $M_p \cup \{M\}$ under intersection. Note that if $M \in T$ then all the nodes of $M_p$ are subsets of $M$ and hence $M_{p, M}$ is simply $M_p \cup \{M\}$. On the other hand if $M$ is countable we need to add nodes of the form $M \cap W$, where $W$ is a transitive side model in $M_p$. We define $d_{p, M}$ by letting $d_{p, M}(N) = d_{p}(N)$, if $N \in M_p$, and $d_{p, M}(N) = \emptyset$, if $N$ is one of the new nodes. It is straightforward to check that $p^M$ is as desired.

The main technical results about the (decorated) side condition poset are Corollaries 2.31 and 2.32 together with Claim 2.38 in [7]. We combine them here as one lemma.

**Lemma 2.2 (7).** Let $p$ be a condition in $M_{S, T}^{\text{dec}}$, and let $Q$ be a node in $M_p$. Suppose that $q$ is a condition in $M_{S, T}^{\text{dec}}$ which belongs to $Q$ and strengthens the condition $p|Q$. Then there is $r \in M_{S, T}^{\text{dec}}$ with $r \leq p, q$ such that:

1. $M_r$ is the closure under intersection of $M_p \cup M_q$;
2. $M_r \cap Q = M_q$,
Lemma 2.5. Suppose \( p \in \mathcal{M}_p \) and \( M \in \mathcal{M}_p \). Then \( \{ p \} \cap W \) is a transitive node of \( \mathcal{M}_q \).

\[ \square \]

We now discuss our modification of Neeman’s posets. Let \( \theta = K \cap \text{ORD} \). We will choose \( S \) and \( T \) to be stationary families of subsets of \( K \). The stationary of \( S \) guarantees that \( \omega_1 \) is preserved in the generic extension and the stationarity of \( T \) guarantees that \( \theta \) is preserved. All the cardinals in between will be collapsed to \( \omega_1 \), thus \( \theta \) becomes \( \omega_2 \) in the final model. We plan to simultaneously add \( \theta \)-many partial functions from \( \omega_1 \) to \( \omega_1 \). Each of the partial functions will be defined on a club in \( \omega_1 \) and will be forced to dominate the \( \theta \) first canonical functions in the generic extension. The decorated version of the pure side condition forcing gives us a natural way to represent the canonical function \( f_\alpha \), for cofinally many \( \alpha < \theta \).

Before we introduce our version of the side condition poset let us make a definition.

**Definition 2.3.** Let \( \mathcal{M} \) be a member of \( \mathbb{M}_{S,T} \). We define the partial function \( h_{\mathcal{M}} \) from \( \theta \times \omega_1 \) as follows. The domain of \( h_{\mathcal{M}} \) is the set of pairs \( (\alpha, \xi) \) such that there is a countable node \( M \in \mathcal{M} \) with \( M \cap \omega_1 = \xi \) and \( \alpha \in M \). If \( (\alpha, \xi) \in \text{dom}(h_{\mathcal{M}}) \) we let

\[ h_{\mathcal{M}}(\alpha, \xi) = \max\{ \text{o.t.}(M \cap \theta) : M \in \pi_S(\mathcal{M}), \alpha \in M \text{ and } M \cap \omega_1 = \xi \} . \]

If \( p \in \mathbb{M}_{S,T}^* \) we let \( h_p \) denote \( h_{\mathcal{M}_p} \). We are now ready to define our modified side condition poset \( \mathbb{M}_{S,T}^* \).

**Definition 2.4.** The poset \( \mathbb{M}_{S,T}^* \) consists of all conditions \( p \in \mathbb{M}_{S,T}^{\text{dec}} \) such that

\[ (\ast) \text{ for every } M, N \in \pi_S(\mathcal{M}_p), \text{ if } N \cap \omega_1 \in M, \text{ then o.t.}(N \cap \theta) \in M. \]

The ordering is defined by letting \( q \leq p \) iff \( (\mathcal{M}_q, d_q) \leq_{\mathbb{M}_{S,T}^{\text{dec}}} (\mathcal{M}_p, d_p) \) and \( h_p \subseteq h_q \).

Let us first observe that we have an analog of Lemma 2.1.

**Lemma 2.5.** Suppose \( p \) belongs to \( \mathbb{M}_{S,T}^* \) and \( M \) is a model in \( S \cup T \) such that \( p \in M \). Then there is a condition \( p^M \) extending \( p \) such that \( M \) is the top node of \( \mathcal{M}_{p^M} \).

\[ \square \]

We now establish some elementary properties of conditions in \( \mathbb{M}_{S,T}^* \).

**Lemma 2.6.** Suppose \( p \) is a condition in \( \mathbb{M}_{S,T}^* \) and \( M \in \mathcal{M}_p \). Then \( h_p \parallel M \in M \).

**Proof.** If \( M \) is a transitive node then this is immediate. Suppose \( M \) is countable. We will use the following.

**Claim 2.7.** Suppose \( N \) is a countable node in \( \mathcal{M}_p \) and \( N \cap \omega_1 \in M \). Then \( N \cap M \in M \).

**Proof.** Since \( \mathcal{M}_p \) is closed under intersection we have that \( N \cap M \in \mathcal{M}_p \). Moreover, since \( N \cap \omega_1 \in M \) we have that \( N \cap M \) is below \( M \). If there is no transitive node between \( N \cap M \) and \( M \) then \( N \cap M \in M \). Otherwise, let \( W \) be the least transitive
node above \( N \cap M \). By closure under intersection again, \( M \cap W \in \mathcal{M}_p \). Moreover, \( N \cap M \subseteq W \cap M \) and the inclusion is proper. Therefore, \( M \cap W \) is a countable node above \( N \cap M \) and there is no transitive node between them. Therefore, \( N \cap M \in M \cap W \) and so \( N \cap M \in M \).

Let us say that a node \( N \) of \( \mathcal{M}_p \) is an end node of \( \mathcal{M}_p \) if there is no node in \( \mathcal{M}_p \) which is an end extension of \( N \). The domain of the function \( h_p \) is the union of all the sets of the form \( (N \cap \theta) \times \{ \xi \} \), where \( N \) is a countable end node of \( \mathcal{M}_p \) and \( \xi = N \cap \omega_1 \). Moreover, on \( (N \cap \theta) \times \{ \xi \} \) the function \( h_p \) is constant and equal to o.t.(\( N \cap \theta \)). Now, if \( \xi \in M \) then by Claim \( 2.7 \) \( N \cap M \in M \). Moreover, since \( p \in \mathcal{M}^*_S \) we have that o.t.(\( N \cap \theta \)) \( \in M \). It follows that \( h_p \upharpoonright M \in M \).  

We wish to have an analog of Lemma \( 2.2 \). If \( p \in \mathcal{M}^*_S \) and \( Q \in \mathcal{M}_p \), we can let \( p \upharpoonright Q \) be \( (\mathcal{M}_p \cap Q, d_p \upharpoonright Q) \). It is easy to check that \( p \upharpoonright Q \) is a condition. However, we do not know that \( p \) extends \( p \upharpoonright Q \) since \( h_p \) may not be an extension of \( h_p \upharpoonright Q \). We must refine the notion of restriction in order to arrange this. In order to do this, let us enrich our initial structure \( \mathcal{K} \) by adding predicates for \( S \) and \( T \). Let \( \mathcal{K}^* \) denote the structure \( (K, E, S, T, \ldots) \). Note that our poset \( \mathcal{M}^*_S \) is definable in \( \mathcal{K}^* \). Let \( S^* \) be the collection of all \( M \in S \) that are elementary in \( \mathcal{K}^* \) and let \( T^* \) be the set of all \( W \in T \) that are elementary in \( \mathcal{K}^* \). Note that \( S^* \) (respectively \( T^* \)) is a relative club in \( S \) (respectively \( T \)), hence if \( S \) (respectively \( T \)) is stationary then so is \( S^* \) (respectively \( T^* \)).

Assume \( p \) is a condition and \( Q \) a node in \( p \) which belongs to \( S^* \cup T^* \). Now, we know that \( p \upharpoonright Q \) and \( h_p \upharpoonright Q \) belong to \( Q \). Moreover, \( Q \) is elementary in \( \mathcal{K}^* \) and \( \mathcal{M}^*_S \) is definable in this structure. Therefore, there is a condition \( q \in Q \) such that \( \mathcal{M}_p \cap Q \subseteq \mathcal{M}_q \), \( d_p(R) \subseteq d_q(R) \), for all \( R \in \mathcal{M}_q \cap Q \), and \( h_q \) extends \( h_p \upharpoonright Q \). We will call such \( q \) a reflection of \( p \) inside \( Q \). Note that if \( q \) is a reflection of \( p \) inside \( Q \) then any condition \( r \in Q \) which is stronger than \( q \) is also a reflection of \( p \) inside \( Q \). Let us say that \( p \) reflect to \( Q \) if \( p \upharpoonright Q \) is already a reflection of \( p \) to \( Q \). Finally, let us say that \( p \) is reflecting if \( p \) reflects to \( W \), for all transitive nodes \( W \) in \( \mathcal{M}_p \).

We now have a version of Lemma \( 2.2 \) for our poset.

**Lemma 2.8.** Let \( p \) be a condition in \( \mathcal{M}^*_S \), and let \( Q \) be a node in \( \mathcal{M}_p \) which belongs to \( S^* \cup T^* \). Suppose that \( q \in \mathcal{M}^*_S \) is a reflection of \( p \) inside \( Q \). Then there is \( r \in \mathcal{M}^*_S \) with \( r \leq p, q \) such that:

1. \( \mathcal{M}_r \) is the closure under intersection of \( \mathcal{M}_p \cup \mathcal{M}_q \),
2. \( \mathcal{M}_r \cap Q = \mathcal{M}_q \),
3. The small nodes of \( \mathcal{M}_r \) outside \( Q \) are of the form \( M \) or \( M \cap W \), where \( M \) is a small node of \( \mathcal{M}_p \) and \( W \) is a transitive node of \( \mathcal{M}_q \).

**Proof.** Let \( r \) be the condition given by Lemma \( 2.2 \). We need to check that \( \mathcal{M}_r \) satisfies (*) and \( h_r \) extends \( h_p \) and \( h_q \). Suppose \( N, M \) are countable nodes in \( \mathcal{M}_r \) and \( N \cap \omega_1 \in M \). We need to check that o.t.(\( N \)) \( \in M \). If \( N \) and \( M \) are both in \( \mathcal{M}_p \) or \( \mathcal{M}_q \) this follows.
Lemma 2.11. side condition poset. The following lemma is the main reason we are working with the decorated version of the 

Proof. Note that $(W, W')$ is continuous. We need to show that $M$ is the union of the $\in$-chain $(W, M)$. Let $p \in G$ be a condition such that $M \in M_p$ and $p$ forces that $M$ is a limit member of $(W, W')_G$. Given any $x \in M$ and a condition $q \leq p$ we show that there is $r \leq q$ and a countable node that $R \cap M \cap r \leq M$ such that $x \in R$. We may assume that there is a countable model in $M_q$ between $W$ and $M$. Let $Q$ be the $\in^*$-largest such model. By increasing $d_q(Q)$ if
necessary, we may assume that \( x \in d_q(Q) \). Since \( p \) forces that \( M \) is a limit member of \( \mathcal{M}_G \) so does \( q \). Therefore, there exists \( r \leq q \) such that \( \mathcal{M}_r \) contains a countable node between \( Q \) and \( M \). Let \( R \) be the \( \varepsilon^* \)-least such node. By the definition of the order relation on \( M_{\mathcal{S}, T}^* \) we must have that \( d_q(P) \in R \) and hence \( x \in R \), as desired. 

Note that Lemma 2.11 implies in particular that if \( W' \) is a successor element of \( \mathcal{M}_G \cap T \) then \( W' \) has cardinality \( \omega_1 \) in \( V[G] \). Therefore, if \( \beta = W' \cap \text{ORD} \), one way to represent the \( \beta \)-th canonical function \( f_\beta \) in \( V[G] \) is the following. Let \( W \) be the predecessor of \( W' \) in \( \mathcal{M}_G \cap T \). Since \( \mathcal{S} \) is stationary in \( K \), so is \( \mathcal{S} \cap W' \) in \( W' \). Therefore, \((W, W') \) is an \( \varepsilon \)-chain of length \( \omega_1 \). Let \( \{ M_\xi : \xi < \omega_1 \} \) be the increasing enumeration of this chain. Then we can let \( f_\beta(\xi) = o.t.(M_\xi \cap \beta) \), for all \( \xi \). Note that \( M_\xi \cap \omega_1 = \xi \), for club many \( \xi \).

Now, let \( h_G \) denote \( \bigcup \{ h_p : p \in G \} \). Then \( h_G \) is a partial function from \( \theta \times \omega_1 \) to \( \omega_1 \). Let \( h_{G, \alpha} \) be a partial function from \( \omega_1 \) to \( \omega_1 \) defined by letting \( h_{G, \alpha}(\xi) = h_G(\alpha, \xi) \), for every \( \xi \) such that \((\alpha, \xi) \in \text{dom}(h_G) \). By Lemma 2.11 and the above remarks we have the following.

**Corollary 2.12.** For every \( \alpha < \theta \) the function \( h_{G, \alpha} \) is defined on a club in \( \omega_1 \). Moreover, \( h_{G, \alpha} \) dominates under \( <_{NS, \omega_1} \) all the canonical function \( f_\beta \), for \( \beta < \theta \).

### 3. Factoring the Side Condition Poset

We now let \( K = (V_\theta, \in, \ldots) \), for some inaccessible cardinal \( \theta \). Let \( T \) be the set of all \( \alpha < \theta \) of uncountable cofinality such that \( V_\alpha < K \) and let \( T = \{ V_\delta : \delta \in T \} \). Finally, let \( \mathcal{S} \) be the set of all countable elementary submodels of \( K \). Clearly, the pair \((\mathcal{S}, T)\) is appropriate. Let \( \mathcal{S}^* \) and \( T^* \) be defined as before and let \( T^* = \{ \alpha : V_\alpha \in T^* \} \). We start by analyzing the factor posets of \( M_{\mathcal{S}, T}^* \). Suppose \( \delta \in T^* \) and let \( p_\delta = (\{ V_\delta \}, \emptyset) \). Then, by Lemma 2.3 the map \( i_\delta : M_{\mathcal{S}, T}^* \cap V_\delta \to M_{\mathcal{S}, T}^* \upharpoonright p_\delta \) given by \( i_\delta(p) = (\mathcal{M}_p \cup \{ V_\delta \}, d_p) \) is a complete embedding. Fix a \( V \)-generic filter \( G_\delta \) over \( M_{\mathcal{S}, T}^* \cap V_\delta \). Let \( M_{G_\delta} \) denote \( \bigcup \{ M_p : p \in G_\delta \} \) and let \( h_{G_\delta} \) be the derived height function, i.e. \( h_{G_\delta} = \bigcup \{ h_p : p \in G_\delta \} \). Let \( Q_\delta \) denote the factor forcing \( M_{\mathcal{S}, T}^* \upharpoonright p_\delta \downharpoonleft_{i_\delta}|G_\delta \). We can identify \( Q_\delta \) with the set of all conditions \( p \in M_{\mathcal{S}, T}^* \) such that \( V_\delta \in M_p \), \( p \) reflects to \( V_\delta \) and \( p \upharpoonright V_\delta \in G_\delta \).

We make the following definition in \( V[G_\delta] \).

**Definition 3.1.** Let \( \mathcal{S}_\delta \) be the collection of all \( M \in \mathcal{S} \) such that \( M \not\in V_\delta \), \( M \cap V_\delta \in \mathcal{M}_{G_\delta} \) and \( o.t.(M \cap \theta) \leq h_{G_\delta}(\alpha, M \cap \omega_1) \), for all \( \alpha \in M \cap \delta \).

We also let \( T_\delta = T \setminus (\delta + 1) \) and \( T_\delta = \{ V_\gamma : \gamma \in T_\delta \} \). We define \( \mathcal{S}_\delta^* \) and \( T_\delta^* \) as before. Clearly, the pair \((\mathcal{S}_\delta, T_\delta)\) is appropriate. We show that \( Q_\delta \) is very close to \( M_{\mathcal{S}_\delta, T_\delta}^* \). More precisely, let \( M_{\mathcal{S}_\delta}^* \) consist of all pairs \( p \) of the form \((\mathcal{M}_p, d_p)\) such that \( \mathcal{M}_p \in M_{\mathcal{S}_\delta, T_\delta}^* \), \( d_p : \mathcal{M}_p \cup \{ V_\delta \} \to V_\theta \), \((\mathcal{M}_p, d_p) \upharpoonright M_{\mathcal{S}_\delta, T_\delta} \in M_{\mathcal{S}_\delta, T_\delta}^* \), and \( V_\theta \) and \( d_p(V_\delta) \) belong to the least model of \( \mathcal{M}_p \). So, formally we do not put \( V_\delta \) as the least node of conditions \( p \) in \( M_\delta \),
Lemma 3.2. $Q_\delta$ and $M_\delta^*$ are equivalent forcing notions.

Proof. Given a condition $p \in Q_\delta$, let $\varphi(p) = (M_p \setminus V_{\delta+1}, d_p \upharpoonright (M_p \setminus V_\delta))$. Clearly, the function $\varphi$ is order preserving. To see that $\varphi$ is onto, let $s \in M_\delta^*$. Then $M \cap V_\delta \in M_{G_\delta}$, for all small nodes $M \in M_\delta$. Fix a condition $p \in G_\delta$ such that $M \cap V_\delta \in M_p$, for every such $M$. Define a condition $q$ by letting $M_q = M_p \cup \{V_\delta\} \cup M_\delta$ and $d_q = d_p \cup d_s$. Since every small node of $M_\delta$ is in $S_\delta$ it follows that $ht_q \upharpoonright \delta \times \omega_1 = ht_p$. Therefore, $q \in Q_\delta$ and $\varphi(q) = s$. Finally, note that if $p, q \in Q_\delta$ then $p$ and $q$ are compatible in $Q_\delta$ iff $\varphi(p)$ and $\varphi(q)$ are compatible in $M_\delta^*$. This implies that $Q_\delta$ and $M_\delta^*$ are equivalent forcing notions. □

Corollary 3.3. $Q_\delta$ is $S_\delta^* \cup T_\delta^*$-strongly proper. □

Lemma 3.4. $S_\delta$ is stationary family of countable subsets of $V_\theta$.

Proof. We argue in $V$ via a density argument. Let $\dot{f}$ be a $M_{\delta^*}^* \cap V_\delta$-name for a function from $V_\theta^{\leq \omega}$ to $V_\theta$ and let $p \in M_{\delta^*}^* \cap V_\delta$. We find a condition $q \leq p$ and $M \in S$ such that $q$ forces that $M$ belongs to $S_\delta$ and is closed under $\dot{f}$. For this purpose, fix a cardinal $\theta^* > \theta$ such that $V_{\theta^*}$ satisfies a sufficient fragment of ZFC. Let $M^*$ be an countable elementary submodel of $V_{\theta^*}$ containing all the relevant parameters. It follows that $M \in S^*$, where $M = M^* \cap V_\theta$. Let $p^M$ be the condition given by Lemma 2.5. Since $\delta \in T^*$ we can find a reflection $q$ of $p^M$ inside $V_\delta$. We claim that $q$ and $M$ are as required. To see this, note that, since $M \cap V_\delta \in M_q \cap S^*$, then, by Lemma 2.8 $q$ is $(M \cap V_\delta, M_{\delta^*} \cap V_\delta)$-strongly generic and hence also $(M^*, M_{\delta^*} \cap V_\delta)$-generic. It follows that $q$ forces that $M^*[\dot{G}_\delta] \cap V_\theta = M$, and hence that $M$ is closed under $\dot{f}$. On the other hand, $h_{p^M}(\alpha, M \cap \omega_1) = o.t.(M \cap \theta)$, for all $\alpha \in M \cap \theta$. Since $q$ is a reflection of $p^M$, we have $h_q(\alpha, M \cap \omega_1) = h_{p^M}(\alpha, M \cap \omega_1)$, for all $\alpha \in M \cap \delta$. Therefore, $q$ forces $M$ to belong to $S_\delta$. This completes the argument. □

We need to understand which stationary subsets of $\omega_1$ in $V[G_\delta]$ will remain stationary in the final model. So, suppose $E$ is a subset of $\omega_1$ in $V[G_\delta]$. Let

$S_\delta(E) = \{M \in S_\delta : M \cap \omega_1 \in E\}.$

For $\rho \in T_\delta$ let $S_\delta^\rho(E) = S_\delta(E) \cap V_\rho$. Note that if $M \in S_\delta(E)$ and $\rho \in T_\delta$ then $M \cap V_\rho \in S_\delta^\rho(E)$. Therefore, if $\rho < \sigma$ and $S_\delta^\rho(E)$ is stationary in $V_\rho$ then $S_\delta^\sigma(E)$ is stationary in $V_\rho$. Since $\theta$ is inaccessible, it follows that $S_\delta(E)$ is stationary in $V_\theta$ iff $S_\delta^\theta(E)$ is stationary in $V_\rho$, for all $\rho \in T_\delta$.

Lemma 3.5. The maximal condition in $Q_\delta$ decides if $E$ remains stationary in $\omega_1$. Namely, if $S_\delta(E)$ is stationary in $V_\theta$ then $\Vdash_{Q_\delta} \dot{E}$ is stationary, and if $S_\delta(E)$ is nonstationary then $\Vdash_{Q_\delta} \dot{E}$ is nonstationary.
Lemma 3.7. Recall that if $M$ is stationary in $S$, then $E$ is nonstationary and fix a successor element of $T$, say $\sigma$, such that $S^*_\delta(E)$ is nonstationary in $V_\sigma$. Let $\rho$ be the predecessor of $\sigma$ in $T$ and fix a condition $p \in Q_\delta$ such that $V_\rho, V_\sigma \in M_p$. Pick an arbitrary $V[G_{\delta^*}]$-generic filter $G$ over $Q_\delta$ containing $p$. Then we can identify $G$ with a $V$-generic filter $\bar{G}$ over $M_{S,T}$ which extends $G_{\delta^*}$ and such that $V_\delta \in M_G$. Since $p \in \bar{G}$, we have that $V_\rho$ and $V_\sigma$ are consecutive elements of $M_{\delta^*} \cap T$. By Lemma 2.1, we know that, in $V[\bar{G}]$, $S_{\delta^*} \cap V_\sigma$ contains a club of countable subsets of $V_\sigma$. On the other hand, by our assumption, $S^*_\delta(E)$ is nonstationary. It follows that $E$ is a nonstationary subset of $\omega_1$ in $V[\bar{G}]$. Since $G$ was an arbitrary generic filter containing $p$, it follows that $p \Vdash E$ is nonstationary in $\omega_1$.

Remark 3.6. One can show that if $\delta$ is inaccessible in $V$ then $Q_\delta$ actually preserves stationary subsets of $\omega_1$. To see this note that, under this assumption, for every subset $E$ of $\omega_1$ in $V[G_{\delta^*}]$ there is $\delta^* < \delta$ with $V_{\delta^*} \in M_{G_{\delta^*}}$ such that $E \in V[G_{\delta^*}]$, where $G_{\delta^*} = G_{\delta^*} \cap V_{\delta^*}$. If, in the model $V[G_{\delta^*}]$, $S_{\delta^*}(E)$ is nonstationary there is $\rho < \theta$ such that $S^*_\rho(E)$ is nonstationary. By elementarity of $V_{\delta^*}$ in $V_\delta$ there is such $\rho < \delta$. But then, as in the proof of Lemma 3.5, we would have that $E$ is nonstationary already in the model $V[G_{\delta^*}]$.

Suppose $E \in V[G_{\delta^*}]$ is a subset $\omega_1$ and $\gamma < \delta$. Let

$$S_{\delta}(E, \gamma) = \{M \in S_{\delta}(E) : \gamma, \delta \in M \text{ and } \text{o.t.}(M \cap \theta) < h_{G_{\delta}}(\gamma, M \cap \omega_1)\}.$$ 

Recall that if $M \in S_{\delta}$ then $M \cap V_\delta \in G_{\delta}$. Hence, if $\gamma \in M$ then $(\gamma, M \cap \omega_1) \in \text{dom}(h_{G_{\delta}})$.

Lemma 3.7. Suppose that, in $V[G_{\delta^*}]$, $E$ is a subset of $\omega_1$ such that $S_{\delta}(E)$ is stationary. Then $S_{\delta}(E, \gamma)$ is stationary, for all $\gamma < \delta$.

Proof. Work in $V[G_{\delta^*}]$ and let $\gamma < \delta$ and $f : V_\theta^{<\omega} \rightarrow V_\theta$ be given. We need to find a member of $S_{\delta}(E, \gamma)$ which is closed under $f$. Since $\theta$ is inaccessible, we can first find $\sigma \in T_{\delta}$ such that $V_\sigma$ is closed under $f$. We know that $S_{\delta}(E)$ is stationary, hence we can find $M \in S_{\delta}(E)$ which is closed under $f$ and such that $\gamma, \delta, \sigma \in M$. It follows that $M \cap V_\sigma$ is also closed under $f$. Since $\sigma \in M$ we have that $\text{o.t.}(M \cap \sigma) < \text{o.t.}(M \cap \theta)$. Since $M \in S_{\delta}(E)$ and $\gamma \in M$ we have that $\text{o.t.}(M \cap \theta) \leq h_{G_{\delta}}(\gamma, M \cap \omega_1)$. Finally, $(M \cap V_\sigma) \cap \omega_1 = M \cap \omega_1$. It follows that $M \cap V_\sigma \in S_{\delta}(E, \gamma)$, as desired.

We now consider what happens in the final model $V[G]$, where $G$ is $V$-generic over $M_{S,T}$. For an ordinal $\gamma < \theta$ let $D_\gamma$ denote the domain of $h_{G,\gamma}$. Recall that, by Corollary 2.1, $D_\gamma$ contains a club, for all $\gamma$. Given a subset $E$ of $\omega_1$ and $\gamma, \delta < \theta$ let

$$\varphi(E, \gamma, \delta) = \{\xi \in E \cap D_\gamma \cap D_\delta : h_{G,\delta}(\xi) < h_{G,\gamma}(\xi)\}.$$
Lemma 3.8. Let $G$ be $V$-generic over $\mathbb{M}^*_{\mathcal{S}, \mathcal{T}}$. Suppose, in $V[G]$, that $E$ is a stationary subset of $\omega_1$ and $\gamma < \theta$. Then there is $\delta < \theta$ such that $\varphi(E, \gamma, \delta)$ is stationary.

Proof. Since $\mathbb{M}^*_{\mathcal{S}, \mathcal{T}}$ is $T^*$-proper, we can find $\delta \in T^* \setminus (\gamma + 1)$ such that $V_\delta \in \mathcal{M}_G$ and $E \in V[G_\delta]$, where $G_\delta = G \cap V_\delta$. Since $E$ remains stationary in $V[G]$, it follows that, in $V[G_\delta]$, $S_\delta(E)$ is stationary. Work for a while in $V[G_\delta]$. We claim that the maximal condition in $\mathbb{Q}_\delta$ forces that $\dot{\varphi}(E, \gamma, \delta)$ is stationary, where $\dot{\varphi}(E, \gamma, \delta)$ is the canonical name for $\varphi(E, \gamma, \delta)$. To see this fix a $\mathbb{Q}_\delta$-name $\dot{C}$ for a club in $\omega_1$ and a condition $p \in \mathbb{Q}_\delta$. Let $\theta^* > \theta$ be such that $(V_{\theta^*}, \in)$ satisfies a sufficient fragment of ZFC. We know, by Lemma 3.7, that $S_\delta(E, \gamma)$ is stationary, so we can find a countable elementary submodel $M^*$ of $V_{\theta^*}$ containing all the relevant objects such that $M \in S_\delta(E, \gamma)$, where $M = M^* \cap V_\delta$. Let $q$ be the condition $p^M$ as in Lemma 2.5 (or rather its version for $\mathbb{Q}_\delta$). Since $\dot{C} \in M^*$ and $q$ is $(M^*, \mathbb{Q}_\delta)$-generic, it follows that $q$ forces that $M \cap \omega_1$ belongs to $\dot{C}$. Also, note that the top model of $\mathcal{M}_\eta$ is $M$. Hence $h_\eta(\delta, M \cap \omega_1) = o.t.(M \cap \theta)$. Since $M \in S_\delta(E, \gamma)$ we have that o.t.$(M \cap \theta) < h_{G_\delta}(\gamma, M \cap \omega_1)$. It follows that $q$ forces that $M \cap \omega_1$ belongs to the intersection of $\dot{\varphi}(E, \gamma, \delta)$ and $\dot{C}$, as required.

We now have the following conclusion.

Theorem 3.9. Let $G$ be $V$-generic over $\mathbb{M}^*_{\mathcal{S}, \mathcal{T}}$. Then, in $V[G]$, $\theta = \omega_2$ and $\text{NS}_{\omega_1}$ is not precipitous.

Proof. We already know that $\mathbb{M}^*_{\mathcal{S}, \mathcal{T}}$ is $S^* \cup T^*$-strongly proper. This implies that $\omega_1$ and $\theta$ are preserved. Moreover, by Lemma 2.11, we know that all cardinals between $\omega_1$ and $\theta$ are collapsed to $\kappa_1$. Therefore, $\theta$ becomes $\omega_2$ in $V[G]$. In order to show that $\text{NS}_{\omega_1}$ is not precipitous we describe a winning strategy $\tau$ for Player I in $\mathcal{G}_{\text{NS}_{\omega_1}}$. On the side, Player I will pick a sequence $(\gamma_n)_{n}$ of ordinals $< \theta$. So, Player I starts by playing $E_0 = \omega_1^*$ and letting $\gamma_0 = 0$. Suppose, in the $n$-th inning, Player II has played a stationary set $E_{2n+1}$. Player I applies Lemma 3.8 to find $\delta < \theta$ such that $\varphi(E_{2n+1}, \gamma_n, \delta)$ is stationary. He then lets $\gamma_{n+1} = \delta$ and plays $E_{2n+2} = \varphi(E_{2n+1}, \gamma_n, \gamma_{n+1})$. Suppose the game continues $\omega$ moves and II respects the rules. We need to show that $\bigcap_n E_n$ is empty. Indeed, if $\xi \in \bigcap_n E_n$ then $\xi \in D_{\gamma_n}$, for all $n$, and $h_{G, \gamma_0}(\xi) > h_{G, \gamma_1}(\xi) > \ldots$ is an infinite decreasing sequence of ordinals, a contradiction.

4. The Working Parts

In this section we show how to add the working part to the side condition poset described in §2, which allows us to define a Neeman style iteration. As in [7], if at each stage we choose a proper forcing, the resulting forcing notion will be proper as well. By a standard argument, if we use the Laver function to guide our choices, we obtain PFA in the final model. The point is that the relevant lemmas from §2 and §3 go through almost verbatim and hence we obtain, as before, that $\text{NS}_{\omega_1}$ will be non precipitous in the final model.
Let us now recall the iteration technique from [7]. We fix an inaccessible cardinal \( \theta \) and a function \( F : \theta \rightarrow V_\theta \). Let \( \mathcal{K} \) be the structure \((V_\theta, \in, F)\). Let \( \mathcal{S} \) be the set of all countable elementary submodels of \( \mathcal{K} \) and \( T \) the set of all \( \alpha < \theta \) of uncountable cofinality such that \( V_\alpha \) is an elementary submodel of \( \mathcal{K} \). Let \( \mathcal{T} = \{V_\alpha : \alpha \in T\} \). Define \( S^*, T^* \) and \( T^* \) as before. Note that if \( \alpha \in T^* \) then \( T^* \cap \alpha \) is definable in \( \mathcal{K} \) from parameter \( \alpha \). Hence, if \( M \in \mathcal{S} \) and \( \alpha \in T^* \) then \( M \cap V_\alpha \in S^* \). We will define, by induction on \( \alpha \in T^* \cup \{\theta\} \), a forcing notion \( \mathbb{P}_\alpha \). In general, \( \mathbb{P}_\alpha \) consists of triples \( p \) of the form \((\mathcal{M}_p, d_p, w_p)\) such that \((\mathcal{M}_p, d_p)\) is a reflecting condition in \( M^*_{\mathcal{S}, \mathcal{T}} \) and \( w_p \) is a finite partial function from \( T^* \cap \alpha \) to \( V_\alpha \) with some properties. If \( \alpha < \beta \) are in \( T^* \cup \{\theta\} \) and \( p \in \mathbb{P}_\beta \) we let \( p \upharpoonright \alpha \) denote \((\mathcal{M}_{p \upharpoonright \alpha}, d_{p \upharpoonright \alpha}, (\mathcal{M}_{p \upharpoonright \alpha}, w_p \upharpoonright \alpha)) \). It will be immediate from the definition that \( p \upharpoonright \alpha \in \mathbb{P}_\alpha \). Moreover, since \((\mathcal{M}_p, d_p)\) is reflecting, it will be an extension of \((\mathcal{M}_p \cap V_\alpha, d_p \upharpoonright (\mathcal{M}_p \cap V_\alpha), w_p \upharpoonright \alpha) \). For \( \alpha \in T^* \) we will also be interested in the partial order \( \mathbb{P}_\alpha \cap V_\alpha \). We let \( \dot{G}_\alpha \) denote the canonical \( \mathbb{P}_\alpha \cap V_\alpha \)-name for the generic filter. If \( M \in \mathcal{S} \cup \mathcal{T} \) and \( \alpha \in M \) we let \( \dot{M}[\dot{G}_\alpha] \) be the canonical \( \mathbb{P}_\alpha \cap V_\alpha \)-name for the model \( M[\dot{G}_\alpha] \), where \( \dot{G}_\alpha \) is the generic filter. If \( F(\alpha) \) is a \( \mathbb{P}_\alpha \cap V_\alpha \)-name which is forced by the maximal condition to be a proper forcing notion we let \( \dot{\mathbb{P}}_\alpha \) denote \( F(\alpha) \); otherwise let \( \dot{\mathbb{P}}_\alpha \) denote the \( \mathbb{P}_\alpha \cap V_\alpha \)-name for the trivial forcing. Let \( \leq_{\dot{\mathbb{P}}_\alpha} \) be the name for the ordering on \( \dot{\mathbb{P}}_\alpha \).

We are now ready for the main definition.

**Definition 4.1.** Suppose \( \alpha \in T^* \cup \{\theta\} \). Conditions in \( \mathbb{P}_\alpha \) are triples \( p \) of the form \((\mathcal{M}_p, d_p, w_p)\) such that:

1. \((\mathcal{M}_p, d_p)\) is a reflecting condition in \( M^*_{\mathcal{S}, \mathcal{T}} \).
2. \( w_p \) is a finite function with domain contained in the set \( \{\gamma \in T^* \cap \alpha : V_\gamma \in \mathcal{M}_p\} \).
3. If \( \gamma \in \text{dom}(w_p) \) then:
   a. \( w_p(\gamma) \) is a canonical \( \mathbb{P}_\gamma \cap V_\gamma \)-name for an element of \( \dot{F}_\gamma \).
   b. If \( M \in \mathcal{S} \cap \mathcal{M}_p \) and \( \gamma \in M \) then
   \[
   p \upharpoonright \gamma \forces_{\mathbb{P}_\gamma \cap V_\gamma} w_p(\gamma) \text{ is } (M[\dot{G}_\alpha], \mathbb{P}_\alpha)\text{-generic}.
   \]

We let \( q \leq p \) if \((\mathcal{M}_q, d_q)\) extends \((\mathcal{M}_p, d_p)\) in \( M^*_{\mathcal{S}, \mathcal{T}} \), \( \text{dom}(w_p) \subseteq \text{dom}(w_q) \) and, for all \( \gamma \in \text{dom}(p) \),
\[
q \upharpoonright \gamma \forces_{\mathbb{P}_\gamma \cap V_\gamma} w_q(\gamma) \leq_{\dot{\mathbb{P}}_\gamma} w_p(\gamma).
\]

Our posets \( \mathbb{P}_\alpha \) is almost identical as the posets \( \mathbb{A}_\alpha \) from [7]. The difference is that we have a requirement that the height function \( \text{ht}_p \) of a condition \( p \) is preserved when going to a stronger condition and we also added the decoration \( d_p \). We are restricting ourselves to reflecting conditions \( p \) since we then know that \( p \) is an extension of \( p \upharpoonright \alpha \), for any \( \alpha \in T^* \) such that \( V_\alpha \) is a node in \( \mathcal{M}_p \). Of course, the working part \( w_p \) is defined only for such \( \alpha \). These modifications do not affect the relevant arguments from [7]. We state the main properties of our posets and refer to [7] for the proofs.
Lemma 4.2. Suppose $\beta$ belongs to $T^* \cup \{\theta\}$.

1. Let $p \in P_\beta$ and let $V_\alpha \in \mathcal{M}_p \cap T^*$. Then $p$ is $(V_\alpha, P_\beta)$-strongly generic.
2. Let $p \in P_\beta$, let $V_\alpha \in \mathcal{T}$ and suppose $p \in V_\alpha$. Then $(\mathcal{M}_p \cup \{V_\alpha\}, d_p, w_p)$ is a condition in $P_\beta$.
3. $P_\beta$ is $T^*$-strongly proper.

Proof. This is essentially the same as Lemma 6.7 from [7].

Lemma 4.3. Suppose $\beta \in T^* \cup \{\theta\}$ and $p \in P_\beta$. Let $M \in S$ be such that $p \in M$. Then there is a condition $q \in P_\beta$ extending $p$ such that $M$ is the top model of $q$.

Proof. First, let $M$ be closure of $\mathcal{M}_p \cup \{M\}$ under intersection and let $d$ be the extension of $d_p$ to $\mathcal{M}$ defined by letting $d(N) = \emptyset$, for all $N \in \mathcal{M} \setminus \mathcal{M}_p$. Then $(\mathcal{M}, d) \in \mathcal{M}_S \mathcal{\cap M}$. By Lemma 2.9 we can find a reflecting condition $(\mathcal{M}_q, d_q) \leq (\mathcal{M}, d)$ such that the top model of $\mathcal{M}_q$ is $M$. Now, we need to define $w_q$. If $\alpha \in \text{dom}(w_p)$ then $P_\alpha \cap V_\alpha \in M$. Since $\mathcal{F}_\alpha$ is forced by the maximal condition in $P_\alpha \cap V_\alpha$ to be proper and $w_p(\alpha) \in M$ is a canonical name for a member of $\mathcal{F}_\alpha$, we can fix a canonical $P_\alpha \cap V_\alpha$-name $w_q(\alpha)$ for a member of $\mathcal{F}_\alpha$ such that $p \upharpoonright \alpha$ forces in $P_\alpha \cap V_\alpha$ that $w_q(\alpha)$ extends $w_p(\alpha)$ and is $(M[\dot{G}_\alpha], \dot{\mathcal{F}}_\alpha)$-generic. Then the condition $q = (M_q, d_q, w_q)$ is as required.

Lemma 4.4. Suppose $\beta \in T^* \cup \{\theta\}$ and $p \in P_\beta$. Let $\theta^* > \theta$ be such that $(V_{\theta^*}, \in)$ satisfies a sufficient fragment of ZFC. Let $M^*$ be a countable elementary submodel of $V_{\theta^*}$ containing all the relevant parameters. Let $M = M^* \cap V_\theta$ and suppose $M \in \mathcal{M}_p$. Then $p$ is $(M^*, P_\beta)$-generic.

Proof. This is essentially the same as Lemma 6.11 from [7].

Then, as in [7], we have the following.

Proposition 4.5. Suppose that $\theta$ is supercompact and $F$ is a Laver function on $\theta$. Let $G_\theta$ be a $V$-generic filter over $P_\theta$. Then $V[G_\theta]$ satisfies PFA.

Now, if $\delta \in T^*$ and $G_\delta$ is a $V$-generic filter over $P_\delta \cap V_\delta$, we can define the function $h_{G_\delta}$ and the factor forcing $Q_\delta$ as in §3. Further, we define the set $S_\delta$ in an analogous way to Definition 3.1 and show that it is stationary as in Lemma 3.3. We show, as in Lemma 4.2 that $Q_\delta$ is $T^*_\delta$-strongly proper. By Lemma 4.3 and Lemma 1.4 we also get that, in $V[G_\delta]$, $Q_\delta$ is $S^*_\delta$-proper. For every subset $E$ of $\omega_1$ which belongs to $V[G_\delta]$, we define the set $S_\delta(E)$ as in §3 and prove a version of Lemma 3.5. Then, proceeding in the same way, for every $\gamma < \delta$ we define $S_\delta(E, \gamma)$ and prove an analog of Lemma 3.7. Then, turning to the final model $V[G_\theta]$, we prove an analog of Lemma 3.8. Finally, combining the arguments of Theorem 3.9 and Proposition 4.5 we get the conclusion.

Theorem 4.6. Suppose $\theta$ is supercompact and $F$ is a Laver function on $\theta$. Let $G_\theta$ be $V$-generic over $P_\theta$. Then, in $V[G_\theta]$, PFA holds and $\text{NS}_{\omega_1}$ is not precipitous.
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