Quantum Mechanics of Plancherel Growth

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Abstract

Growth of Young diagrams, equipped with Plancherel measure, follows the automodel equation of Kerov. Using the technology of unitary matrix model we show that such growth process is exactly same as the growth of gap-less phase of Gross-Witten and Wadia (GWW) model. Our analysis also offers an alternate proof of \textit{limit shape} theorem of Vershik-Kerov and Logan-Shepp. We also study fluctuations of random Young diagrams in this paper. We map Young diagrams in automodel class to different shapes of two dimensional phase space droplets of underlying non-interacting fermions. Fluctuation of these Young diagrams correspond to small ripples on the boundaries of such droplets. We quantise this classical system using Hamiltonian dynamics and show that the different modes of these fluctuations satisfy $U(1)$ Kac-Moody algebra. We further construct the Hilbert space of this algebra and find a correspondence between the states in Hilbert space and automodel diagrams. In particular the Kac-Moody primary corresponds to null Young diagram (no box) whereas automodel diagrams are mapped to descendants of Kac-Moody primary.

\textbf{Keywords:} Plancherel growth of Young diagrams, unitary matrix model, Kac-Moody algebra.

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### 1. Introduction

Young diagrams (or sometimes simply diagrams in this paper) play an important role in mathematics as well as in physics. It provides a convenient diagrammatic way to describe the representations of symmetric group and general linear groups. Various properties of these representations can be understood very easily using this diagrammatic technique. There are different notations in literature to depict a Young diagram. In this paper we follow the “English notation”. In this notation, boxes are arranged in horizontal rows with the condition that number of boxes in a row is always less than or equal to that in the row above. In general as one goes to higher and higher dimensional representations the number of boxes in Young diagram increases. It has been observed that irrespective of underlying group or representation the growth of Young diagrams behaves very interestingly.

Young diagrams can be classified in terms of total number of boxes. Let us consider $\mathcal{Y}_k$ to be the set of all Young diagrams with $k$ boxes. Then all the diagrams in $\mathcal{Y}_{k+1}$ can be obtained by adding one box to each diagram in $\mathcal{Y}_k$ in all possible allowed ways. See figure 1. Such process is called growth process of Young diagrams. One can construct all possible Young diagrams at any arbitrary level starting from null diagram ($\emptyset$), means no box.

For a restricted growth process, one can assign a probability to every transition. Denoting a particular Young diagram at level $k$ by $\lambda_k$ we associate a transition probability $P_{\text{transition}}(\lambda_k, \lambda_{k+1})$ for a transition from $\lambda_k$ to $\lambda_{k+1}$

$$P_{\text{transition}}(\lambda_k, \lambda_{k+1}) = \frac{1}{k+1} \frac{\dim \lambda_{k+1}}{\dim \lambda_k}$$ (1.1)

if $\lambda_{k+1}$ is obtained from $\lambda_k$ by adding one box, otherwise $P_{\text{transition}}(\lambda_k, \lambda_{k+1}) = 0$. A growth process, following the above probability measure, is called Plancherel growth process (see [1][2] for a
comprehensive review). Note that the probability to get a diagram at level $k + 1$ from a diagram at level $k$ does not depend on the history of transition from level $k - 1$ to $k$. Thus, the growth process is Markovian.

It was shown by Vershik and Kerov [3] and independently by Logan and Shepp [4] that Young diagrams following Plancherel growth process converge to a universal diagram in the large $k$ limit when normalised (scaled) appropriately such that the area of the diagram is unity. The boundary of such normalised diagram becomes smooth under scaling. A universal Young diagram means the boundary curve takes a particular form, which is called limit shape. The limit shape follows the famous arcsin law [3, 5].

In the continuum (large $k$) limit Kerov introduced a differential model to capture the growth of Young diagrams [1, 5]. He associated a ‘time’ parameter with continuous diagrams to study the evolution of those diagrams with respect to that. It turns out that Young diagrams equipped with Plancherel measure follow a first order partial differential equation. The model was named as automodel [1, 5]. The class of diagrams satisfying such growth or evolution equation is called automodel class. The limit shape is a unique solution of the automodel equation in far future with

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**Figure 1:** Growth of Young diagrams.
∅ as initial condition in far past.

The first goal of this paper is to write down a matrix model which captures the growth of Young diagrams in automodel class. In order to achieve that we define a Young lattice

$$\mathcal{Y} = \bigcup_{k=0}^{\infty} \mathcal{Y}_k.$$  \hspace{1cm} (1.2)

All the members of $\mathcal{Y}_k$ have same number of boxes but different shapes. $\mathcal{Y}_k$ can be thought of as an ensemble of Young diagrams with the same macroscopic variable $k$. Therefore the Young lattice can be thought of as a grand canonical ensemble and one can write down a partition function for the entire lattice

$$Q_y = \sum_{k=0}^{\infty} z^k \mathcal{Z}_y.$$  \hspace{1cm} (1.3)

where $z$ is called fugacity ($z > 0$) and $\mathcal{Z}_y$ is the canonical partition function for $\mathcal{Y}_k$, given by

$$\mathcal{Z}_y = \sum_{\lambda_k} \mathcal{P}(\lambda_k) \delta(k - |\lambda_k|).$$  \hspace{1cm} (1.4)

$\mathcal{P}(\lambda_k)$ is any measure associated with the Young diagram $\lambda_k$. To capture the Plancherel growth process it is natural to take $\mathcal{P}(\lambda_k)$ to be Plancherel measure (2.4). Further discussions on Plancherel measure is deferred till section 2.

It turns out that the above partition function is solvable in large $k$ limit. The large $k$ limit is similar to classical limit in physics. In this limit the partition function is dominated by a single Young diagram. Surprisingly, we find that this dominant diagram has the boundary, exactly same as that of automodel diagram obtained in [3–5]. We also observe that the dominant Young diagrams obtained from (1.4) satisfy similar automodel equation, obtained by Kerov, with time parameter related to an inbuilt parameter of the matrix model. Therefore, the matrix model (1.3) captures the growth of Young diagrams in automodel class for $\mathcal{P}(\lambda_k)$ equal to Plancherel measure.

The most fascinating observation in this paper is the dynamics of large $k$ fluctuations of Young diagrams in automodel class. Large $k$ fluctuations of limit shape diagrams is an interesting subject to study in mathematics [6–17]. However, in this paper we study such fluctuations from a phase space point of view. First, we map different automodel diagrams to different 2 dimensional droplets of an underlying free fermi system. The mapping between automodel diagrams and free fermi droplets follows from the equivalence between the partition functions of automodel growth (1.3) and Gross-Witten and Wadia (GWW) model, which is a unitary matrix model. GWW model has two phases in the large $N$ limit (where $N$ being the rank of unitary matrices) and the automodel class corresponds to gap-less phase of GWW. Thus automodel diagrams can also have an equivalent description in terms of eigenvalue distribution on a unit circle. Since the eigenvalues of unitary matrices in GWW model (or in any generic unitary matrix model) behave like position of free fermions [18], it was shown in [19] and other follow up papers [20, 21] that automodel class can be described in terms of phase space droplets of underlying free fermi theory. These
droplets are similar to Thomas-Fermi droplets at zero temperature. Different automodel diagrams are mapped to different shapes of free fermi droplets. Large $k$ fluctuations of automodel diagrams correspond to small ripples at the boundary of the droplets. To study the dynamics of such fluctuations we construct a single particle Hamiltonian from the shape of the droplets. We employ Hamiltonian dynamics to study the evolution of classical boundary of a droplet and then quantise the system. It turns out that different modes of fluctuations satisfy an abelian Kac-Moody algebra.

We also construct the Hilbert space of this algebra and find a one to one correspondence between the states in Hilbert space and automodel diagrams. In particular the Kac-Moody primary corresponds to null Young diagram (no box) whereas automodel diagrams are mapped to descendants of Kac-Moody primary. We also map the Gaussian fluctuations of Young diagrams to descendant states in the Hilbert space.

2. Plancherel Growth of Young diagrams

To make the growth process meaningful it is customary to assign a probability for each diagram at level $k$ in the Young lattice $\mathcal{Y}$. There is a natural way to assign probability to different diagrams. We count the total number of inequivalent paths one can follow to come to a particular diagram at level $k$ starting from $\emptyset$. See figure. It turns out that the Plancherel measure is proportional to the square of that number. The proportionality constant is fixed by the normalization condition. To calculate the number of paths heading to a diagram $\lambda_k$ we look at growth of Young tableau rather than Young diagrams. Starting from $\emptyset$ we keep on adding one box at each level with increasing number. Therefore the readers can easily convince themselves that at each level $k$ we have different Young tableaux and a particular tableau can be reached from $\emptyset$ by a unique path only. Thus the number of paths available to reach a particular Young diagram $\lambda_k$ is equal to the number of standard Young tableaux $f_{\lambda_k}$ of that given shape. It is well known that,

$$
\sum_{\lambda_k \in \mathcal{Y}_k} (f_{\lambda_k})^2 = k!.
$$

Hence we get the Plancherel measure $P(\lambda_k)$ for a diagram $\lambda_k$

$$
P(\lambda_k) = \frac{f^2_{\lambda_k}}{k!}.
$$

We use this probability to write the partition function for the growth process. The number $f_{\lambda_k}$ is equal to the dimension of the representation $\lambda_k$, i.e.

$$
f_{\lambda_k} = \dim \lambda_k.
$$

and thus we have,

$$
P(\lambda_k) = \frac{(\dim \lambda_k)^2}{k!}.
$$

When a Young lattice is equipped with Plancherel measure, the transition probability between two diagrams $\lambda_k$ and $\lambda_{k+1}$ in that lattice is also fixed and is given by. Therefore
either of the probabilities (1.1 or 2.4) can be used to study the growth process. It was observed in [3, 4] that a Young lattice equipped with Plancherel measure terminates to a universal diagram in the limit \( k \to \infty \) when the diagrams are scaled properly.

### 2.1. The Universal Diagram and Automodel

Although we are using the “English” notation for Young diagrams, but the limit shape of Young diagrams takes a simple form in rotated French notation. A typical shape of Young diagram in French notation is given in fig. 2. The centres of boxes are marked with \((X, Y)\) coordinates. The function \(X(Y)\) specifies a particular shape of Young diagram in this notation. However, it is more convenient to rotate this diagram anti-clock wise by \(\pi/4\) and work in the redefined coordinates

\[
\begin{align*}
  u &= \frac{1}{2}(Y - X) \\
  v &= \frac{1}{2}(Y + X).
\end{align*}
\]

A Young diagram in this notation is depicted in figure 3. For finite number of boxes the function \(v(u)\) is rough and zig-zag i.e. \(v'(u) = \pm 1\). As the number of boxes goes very large we define a rescaled function

\[
\hat{v}_k(u) = \frac{v(u \sqrt{k})}{\sqrt{k}}
\]

Figure 2: Typical structure of a Young diagram in French notation.

\footnote{There is another advantage to draw the Young diagrams in rotated French notation. A transition from \(\lambda_k\) to \(\lambda_{k+1}\) occurs when one keeps a box at any of the minima of rotated diagram. Putting a box at different minima corresponds to different diagrams at \(k + 1\) level. Therefore the transition probabilities are denoted by \(\mu_a\) where \(a\) is the position of a minimum. To find \(\mu_a\) we define two polynomials \(P(x) = \prod_{a=1}^n (x - x_a)\) and \(Q(x) = \prod_{a=1}^{n-1} (x - y_a)\), where \(x_1, \cdots, x_n\) are positions of consecutive minima and \(y_1, \cdots, y_{n-1}\) are consecutive maxima. The transition probability from \(\lambda_k\) to \(\lambda_{k+1}\) by adding a box at \(a^{th}\) minima is given by decomposing the quotient into partial fraction

\[
\sum_{a=1}^n \frac{\mu_a}{x - x_a} = \frac{Q(x)}{P(x)}.
\]
such that the area under the curve is finite and the boundary curve becomes smooth. It was observed by [3, 4] that when the growth process follows Plancherel transition probability (1.1) the asymptotic shape of rescaled Young diagrams converges uniformly to a unique curve given by

$$
\lim_{k \to \infty} \hat{v}_k(u) \equiv \Omega(u) = \begin{cases} 
\frac{2}{\pi} (u \sin^{-1} \frac{u}{2} + \sqrt{4 - u^2}) & \text{if } |u| \leq 2 \\
|u| & \text{if } |u| > 2.
\end{cases} (2.8)
$$

In the continuum limit Kerov introduced [5] charge of a diagram, denoted by $\sigma(u)$ and is given by (we are using the notation that $\hat{v}(u) = \hat{v}_k(u)$ in the large $k$ limit)

$$
\sigma(u) = \frac{1}{2} (\hat{v}(u) - |u|). \quad (2.9)
$$

Therefore,

$$
\sigma'(u) = \begin{cases} 
\frac{1}{2} + \frac{\hat{v}'(u)}{2} & \text{for } u < 0 \\
-\frac{1}{2} + \frac{\hat{v}'(u)}{2} & \text{for } u > 0
\end{cases}. \quad (2.10)
$$

One can define moments of a diagram

$$
p_n = -n \int u^{n-1} d\sigma(u) \quad (2.11)
$$

such that area of a diagram (area covered under the curve $\hat{v}(u)$) is given by $A = (p_2 - p_1^2)/2$. It is convenient to consider a moment generating function

$$
S(x) = \sum_{n=1}^{\infty} \frac{p_n}{n} x^{-n} = \int \frac{d\sigma(u)}{u - x}. \quad (2.12)
$$

The moment generating function as well as the sequence of moments determine the charge and hence the diagram ($\hat{v}(u)$) completely. The moment generating function plays an important role in our large $k$ analysis of partition function.
In [5] Kerov introduced a dynamical model for the growth of Young diagrams. For every continuous Young diagram characterized by the function $\hat{v}(u)$, one can define the function $v(u, t)$, called the automodel tableaux which depends on two variables $u$ and $t$ as
\begin{equation}
\hat{v}(u, t) = \sqrt{t} \frac{\hat{v}(u)}{\sqrt{t}} \quad \text{for} \quad t > 0. \tag{2.13}
\end{equation}

Kerov showed that the Young diagrams, following Plancherel growth, belong to automodel class and satisfy the equation
\begin{equation}
\partial_t \hat{v}(u, t) = \frac{1}{2t} (\hat{v}(u, t) - u \partial_u \hat{v}(u, t)). \tag{2.14}
\end{equation}

In terms of charges the automodel equation is given by,
\begin{equation}
\partial_t \sigma'(u, t) + \frac{u}{2t} \sigma''(u, t) = 0. \tag{2.15}
\end{equation}

With this preliminary discussion on Plancherel growth process of Young diagrams and automodel class we are in a position to write down a partition function for the Young lattice.

3. The Partition Function

The grand canonical partition function for $Y$ is given by
\begin{equation}
Q_Y = \sum_{k=0}^{\infty} z^k \sum_{\lambda} \frac{(\text{dim} \lambda)^2}{k!} \frac{z}{2} \delta(k - |\lambda|), \quad z > 0. \tag{3.1}
\end{equation}

This partition function is related to Poissonised Plancherel measure [26]. The above ensemble sometimes is known as Meixner ensemble in literature [27, 28].

Our goal is to solve this matrix model in the large $k$ limit. In that limit the partition function is dominated by a particular Young diagram and it turns out that the shape of this dominant large $k$ Young diagram falls into the automodel class of Kerov [5] and for a particular value of parameter it becomes limit shape [3–5]. Before we present the calculation to obtain the universal diagram, we show that the partition function (3.1) is remarkably equivalent to the partition function of Gross-Witten-Wadia model and its cousins. This was also observed in [19].

3.1. A connection between Young lattice and Gross-Witten-Wadia model

The Gross-Witten-Wadia model is a well studied unitary matrix model in physics. The partition function for this model is defined over an ensemble of $N \times N$ unitary matrices with a real potential $\text{Tr} U + \text{Tr} U^\dagger$, where the trace has taken over fundamental representation. The partition function of GWW model is given by
\begin{equation}
Z_{GWW} = \int [dU] e^{\frac{z}{2} (\text{Tr} U + \text{Tr} U^\dagger)}, \quad z \geq 0. \tag{3.2}
\end{equation}
Gross and Witten \[29\] (and independently by Wadia \[30\]) studied this matrix model in the context of lattice QCD and observed that the system undergoes a third order phase transition at $\lambda = 2$. Different phases of this model are characterised by the topology of distribution of eigenvalues of unitary matrix $U$ on a unit circle. The strong coupling phase ($\lambda > 2$) corresponds to a gap-less distribution of eigenvalues whereas weak coupling phase ($\lambda < 2$) shows a finite gap in eigenvalue distribution.

A close cousin of GWW model \[31, 32\] is

$$Z_c = \int [dU] e^{i \text{Tr}(U) \text{Tr}U^\dagger}. \tag{3.3}$$

The phase structure and eigenvalue distributions of this model are similar to those of GWW up to a redefinition of parameters: $a \langle \text{Tr}U \rangle = N/\lambda$ \[20\]. Expanding the exponential in (3.3) we get

$$Z_c = \int [dU] \sum_{k=0}^\infty \frac{a^k}{k!} (\text{Tr}U)^k (\text{Tr}U^\dagger)^k. \tag{3.4}$$

Using Frobenius formula for the characters of symmetric group we can write

$$\langle \text{Tr}U \rangle^k = \sum_R \chi_R(1^k) \text{Tr}_R U, \quad \text{and} \quad (\text{Tr}U^\dagger)^k = \sum_R \chi_R(1^k) \text{Tr}_R U^\dagger \tag{3.5}$$

where $\sum_R$ is sum over representations of $U(N)$ (or $SU(N)$) and $\chi_R(1^k)$ is the character of conjugacy class $1^k$ of symmetric group $S_k$ in representation $R$. Finally using the normalization condition for the characters of unitary group

$$\int [dU] \text{Tr}_R U \text{Tr}_R U^\dagger = \delta_{RR'}, \tag{3.6}$$

we arrive at the final expression for $Z_c$ written in terms of sum over representations of $U(N)$ \[19\]

$$Z_c = \sum_{k=0}^\infty \frac{a^k}{k!} \sum_R (\chi_R(1^k))^2 \delta(k - |\lambda_k|). \tag{3.7}$$

It is well know that character of conjugacy class $1^k$ of symmetric group $S_k$ in representation $R$ is equal to the dimension of the representations \[22\]

$$\chi_R(1^k) = \dim R. \tag{3.8}$$

Representations of $U(N)$ can be expressed in terms of Young diagrams. Since $\chi_R(1^k)$ is non-zero only when total number of boxes in the Young diagram is $k$ we have

$$Z_c = \sum_{k=0}^\infty \frac{a^k}{k!} \sum_{\lambda_k} (\dim \lambda_k)^2 \delta(k - |\lambda_k|). \tag{3.9}$$
Thus we see that the partition function for (cousin of) GWW model is same as that of Young lattice with $a$ playing the role of fugacity (see [19] for details). However, there is a small difference between these two partition functions. The sum in (3.9) runs over representations of unitary group where as in (3.1) the sum runs over representations of symmetric group. This difference makes these two systems behave differently for certain range of parameter (fugacity). We shall get back to this issue at appropriate place.

4. Large $k$ analysis of partition function

The large $k$ analysis of partition function (3.1) was explicitly done in [19]. We briefly review the procedure for the readers, not familiar with matrix model techniques (for a more comprehensive treatment of matrix models, see [33–35]). To analyse the partition function (3.1) we denote a valid Young diagram of symmetric group $S_k$ by a set of numbers $\{n_i\}_{i=1}^L$ where $n_i$ denotes the number of boxes in $i^{th}$ row. $L$ is an arbitrary positive integer greater than or equal to the height of the first column. See figure 4. The number of boxes in the first column is less than or equal to $L$. In general, $\exists$ a number $0 < M \leq L$ such that $n_i = 0$ for $i = M + 1, \cdots, L$.

\[
\text{dim} \lambda_k = \frac{k!}{h_1!h_2! \cdots h_L!} \prod_{i<j} (h_i - h_j) \tag{4.1}
\]

where,

\[
h_i = n_i + L - i \tag{4.2}
\]

is the hook lengths of the first box in $i^{th}$ row.

Figure 4: A generic Young diagram in English notation. Here $L$ is an arbitrary positive integer. The number of boxes in the first column is less than or equal to $L$. In general, $\exists$ a number $0 < M \leq L$ such that $n_i = 0$ for $i = M + 1, \cdots, L$. The dimension of a representation $\lambda_k$ of $S_k$ is given by (4.1)
We consider the large $L$ limit of the partition function (3.1). In this limit the hook numbers $h_i \sim L$ (4.2). Therefore we define the following continuous functions to describe Young diagrams at large $L$

\[ n(x) = \frac{n_i}{L}, \quad h(x) = \frac{h_i}{L}, \quad \text{where} \quad x = \frac{i}{L}, \quad x \in [0, 1]. \]  

(4.3)

Functions $n(x)$ or $h(x)$ captures the distribution of boxes in a large $k$ Young diagram. The relation between $n(x)$ and $h(x)$ follows from equation (4.2) and is given by

\[ h(x) = n(x) + 1 - x. \]  

(4.4)

The number of boxes in a Young diagram in the large $L$ limit is given by

\[ k = \sum_{i=1}^{L} n_i \rightarrow L^2 \left[ \int_0^1 dx (h(x) + 1 - x) \right] = L^2 \left[ \int_0^1 d h(x) - \frac{1}{2} \right] = L^2 k' \]  

(4.5)

where

\[ k' = \int_0^1 d h(x) - \frac{1}{2} \]  

(4.6)

is the renormalised box number and is a $O(1)$ quantity. Thus we see that the number of boxes in a Young diagram in the large $L$ limit goes as $\sim O(L^2)$ and hence $L \sim O(\sqrt{k})$. The partition function (3.1) in $L \rightarrow \infty$ limit is given by,

\[ Q_y = \int [Dh(x)] e^{-L^2 S_{\text{eff}}[h(x)]} \]  

(4.7)

where

\[ -S_{\text{eff}}[h(x)] = \int_0^1 dx \int_0^1 dy \ln |h(x) - h(y)| - 2 \int_0^1 d h(x) \ln h(x) + k' \ln(zk') + k' + 1. \]  

(4.8)

In the large $L$ limit the dominant contribution to the partition function comes from the extrema of $S_{\text{eff}}[h(x)]$. Varying $S_{\text{eff}}[h(x)]$ with respect to $h(x)$ we get the saddle point equation

\[ \int \frac{u(h') dh'}{h - h'} = \ln \left( \frac{h}{\xi} \right), \quad \text{where} \quad \xi^2 = zk' \]  

(4.9)

where, Young diagram density $u(h)$ is given by

\[ u(h) = -\frac{\partial x}{\partial h}. \]  

(4.10)

Monotonicity of $h(x)$ implies $0 \leq u(h) \leq 1$. $u(h)$ also satisfies two conditions

\[ \int d h u(h) = 1, \quad \text{and} \quad \int h u(h) d h = k' + \frac{1}{2}. \]  

(4.11)

Our goal is to solve this saddle point equation (4.9) to find Young diagram density such that it satisfies the constraints (4.11).
4.1. Different branches of solutions

All possible large $L$ solutions of (4.9) were thoroughly discussed in [19] and it was observed that (4.9) admits two possible solutions. However, here we look at the problem more carefully keeping the symmetry of the growth process in mind. From the Plancherel measure (2.4) we see that at any level $k$, two Young diagrams related to each other by transposition, have same probability $P(\lambda_k)$. Therefore the large $L$ solution of (4.9) must be invariant under transposition. Young diagrams, symmetric under transposition, are called rectangular diagrams [2, 5].

4.1.1. Rectangular Young diagram - symmetric solution

Following [19], we can take the following ansatz for $u(h)$ to get a rectangular Young diagram

$$ u(h) = \begin{cases} 
1 & h \in [0, p) \\
\tilde{u}(h) & h \in (p, q].
\end{cases} \quad (4.12) $$

To solve the saddle point equation we define a resolvent

$$ H(h) = \int_{h_c}^{h_U} \frac{u(h')dh'}{h - h'} \quad (4.13) $$

After a little algebra, we find that the resolvent $H(h)$ is given by [19]

$$ H(z) = \ln \left[ h \left( h - 1 - \sqrt{(h - 1)^2 - 4\xi^2} \right) \frac{2\xi^2}{2\xi^2} \right] \quad (4.14) $$

The resolvent is same as the moment generating function for the rectangular diagrams defined in (2.12) [5]. The resolvent has a branch cut in the complex $h$ plane. Young diagram density is given by the discontinuity of $H(h)$ about the branch cut

$$ \tilde{u}(h) = \frac{1}{\pi} \cos^{-1} \left[ \frac{h - 1}{2\xi} \right], \quad \text{for} \quad p \leq h \leq q. \quad (4.15) $$

The supports $p$ and $q$ are given by,

$$ p = 1 - 2\xi, \quad q = 1 + 2\xi. \quad (4.16) $$

This particular class of solution exists subject to the following condition

$$ k' = \xi^2. \quad (4.17) $$

Since $p \geq 0$, this solution is valid for $0 \leq \xi \leq 1/2$. From the definition of $\xi (\xi^2 = zk')$ we also see that this solution exists for

either $\xi = k' = 0$ or $z = 1$. \quad (4.18)
Figure 5: A Young diagram in English notation for automodel class \((0 < \xi < 1/2)\).

The case \(\xi = k' = 0\) is trivial. This means there is no box in the Young diagram. The non-trivial solution corresponds to \(z = 1\) (i.e. fugacity is one and hence zero chemical potential). It is easy to check that the Young diagram is invariant under transposition. The height of the first column can be calculated from equation (4.4) and is given by \(2\xi\) which is similar to the length of the first row. Also the function \(\tilde{u}(h)\) is symmetric about body diagonal. A typical Young diagram for this class has been depicted in figure [5]. In this phase the renormalised number of boxes (i.e. \(k'\)) in a Young diagram grows from \(k' = 0\) to \(k' = 1/4\) as \(\xi\) changes from 0 to 1/2. For any value of \(\xi\) between 0 and 1/2 the dominant Young diagram is always symmetric under transposition and hence a rectangular diagram. The limiting value \(\xi = 1/2\) (GWW transition point) corresponds to the distribution
\[
\tilde{u}(h) = \frac{1}{\pi} \cos^{-1}(h - 1). \tag{4.19}
\]
This terminal distribution is same as the universal curve or the limit shape obtained by [3, 4]. Hence we see that the limit shape Young diagram corresponds to GWW transition point in matrix model side.

We calculate Plancherel measure (2.4) for this dominant diagram. Following [19], the Plancherel measure in large \(k\) limit is given by
\[
\frac{1}{L^2} \ln \mathcal{P}_{d_k} = \int_0^\eta dh u(h) \int_0^\eta dh' u(h') \ln|h - h'| - 2 \int_0^\eta u(h)h \ln h dh + k' + 1 + k' \ln k'. \tag{4.20}
\]
Evaluating the right hand side for symmetric solutions (4.12) and (4.15) we get
\[
\mathcal{P}_{d_k} = 1 + O\left(\frac{1}{L}\right). \tag{4.21}
\]
Thus we see that in the large \(k\) (or large \(L\)) limit the symmetric solution (4.12, 4.15) is the maximum probable solution. Probability of having other diagrams is suppressed by powers of \(L\). This gives an alternate proof of limit shape theorem of Vershik-Kerov and Logan-Shepp result [3, 4].
4.1.2. Asymmetric solutions

In large $k$ limit the matrix model (3.3) renders another class of solution \cite{19}. This solution is given by

$$u(h) = \frac{2}{\pi} \cos^{-1}\left(\frac{h + \xi - 1/2}{2\sqrt{\xi h}}\right), \quad \text{for} \quad p \leq h \leq p$$

$$= 0 \quad \text{otherwise}$$

(4.22)

where,

$$\sqrt{p} = \sqrt{\xi} - \frac{1}{\sqrt{2}}, \quad \text{and} \quad \sqrt{q} = \sqrt{\xi} + \frac{1}{\sqrt{2}}$$

(4.23)

and fugacity $z$ (or $a$) is given by

$$z = \frac{4\xi^2}{4\xi - 1}.$$  \hspace{1cm} (4.24)

The solution is valid for $\xi > 1/2$. The Young diagrams for this distribution is not symmetric under transposition.

This is a valid solution in the context of GWW model. In case of GWW model the sum in equation (3.9) was over the representations of unitary group $U(N)$ for which the maximum number of boxes in the first column of a Young diagram is $N$. Therefore, the symmetric representation fails to be a valid solution of GWW when the first column saturates this bound. As a result, GWW model undergoes a third order phase transition at $\xi = 1/2$, known as Gross-Witten-Wadia phase transition.

However we are dealing with partition function (1.3) where the sum is running over the representations of symmetric group. In this case there is no such restriction on $L$ (number of boxes in the first column). Thus we do not see any such phase transition here.

4.2. A connection with automodel

To make a precise connection between automodel class and our solution, we need to set up a dictionary between the variables defined in $(u, v)$ plane and $(h, x)$ plane. The relation between French notation and English notation is $Y = n$ and $X = x$. We use the following transformation between $(n, x)$ and $(u, v)$ so that $u = v = 2$ point is mapped to $n = 1, x = 0$

$$\frac{u}{2} = n - x$$

$$\frac{v}{2} = n + x.$$  \hspace{1cm} (4.25)

Using this mapping one can show that the Young diagram distribution function (4.10) is related to $v'(u)$ in the following way

$$u(h) = \frac{1}{2} - \frac{1}{2} v'(u) \quad \text{with} \quad u = 2(h - 1).$$  \hspace{1cm} (4.26)
One can also check that the terminal diagram (4.19) is exactly same as the limit shape defined in (2.8). Thus we see that the Young diagram density $u(h)$ is related to charge $\sigma(u)$ defined in (2.9) by $u(h) = -\sigma'(u)$. The resolvent (4.14) for this symmetric solution same as the moment generating function for charges (2.12).

We also observe that the symmetric distributions (4.12) for $0 < \xi < 1/2$ satisfies,
\[
\partial_\xi u(h, \xi) + \frac{h - 1}{\xi} \partial_h u(h, \xi) = 0. \tag{4.27}
\]
Since for this branch we have $\xi = k'$, the above equation can be written as,
\[
\partial_{k'} u(h, k') + \frac{h - 1}{2k'} \partial_h u(h, k') = 0. \tag{4.28}
\]
Thus we see that the Young diagram density satisfies the automodel equation (2.15) with $k'$ playing the role of automodel time $t$. This is natural to expect that the renormalised box number $k'$ playing the role of growth parameter $t$ in Kerov’s paper [5]. Hence we conclude that the partition function (3.1), in the limit of large box number, is dominated by Young diagram belonging to the automodel class of Kerov.

5. Fluctuations of automodel diagrams, Kac-Moody algebra and state-diagram correspondence

Gapless phase of GWW model is a classical solution of the model. Study of large $N$ fluctuations or quantum fluctuations of the classical solution is always interesting on its own. Since automodel diagrams are mapped to gapless phase of GWW model, large $N$ fluctuations of classical solution, therefore, correspond to large $k$ fluctuations of automodel diagrams. Such fluctuation of Young diagrams have been under investigation primarily in the mathematics literature [6–17].

Kerov studied the Gaussian fluctuations around the limit shape of Young diagrams (denoted by $\Omega$, as defined in (2.8)) endowed with Plancherel measure in [6]. In [8], Ivanov and Olshanki reconstructed a proof of Kerov’s result on fluctuations around the limit shape from his unpublished work notes, 1999. A rescaled Young diagram defined in (2.7) in the $k \to \infty$ limit takes the form of limit shape $\Omega$. However there can be large $k$ corrections to this result and we call such corrections as fluctuations of limit shape diagram. The central result pertains to large $k$ corrections to the limit shape which can be stated as
\[
\lim_{k \to \infty} \hat{\nu}_k(u) \sim \Omega(u) + \frac{2}{\sqrt{k}} \Delta(u) \tag{5.1}
\]
The sub-leading piece $\Delta(u)$ is a Gaussian process defined for $|u| \leq 2$. More precisely, $\Delta(u)$ is a random trigonometric series given by
\[
\Delta(u) = \Delta(2 \cos \theta) = \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\alpha_n}{\sqrt{n}} \sin(n\theta) ; \quad u = 2 \cos \theta \tag{5.2}
\]
where $\alpha_n$ are independent Gaussian random variables with mean 0 and variance 1. Further investigations has been done towards understanding the central limit theorem for Gaussian fluctuations around the limit shape [10][11]. Fluctuations of random Gaussian and Wishart matrices have been related to the notion of free probability and free cumulants in earlier works [15][17].

Here we take a different route to study the dynamics of such fluctuations. Our approach is quite generic and has importance and implications more on matrix model and gauge theory side. Unitary matrix models in large $N$ ($N$ being the rank of matrices) limit renders different solutions or phases [29][30][36][39]. Such phases are also corroborated by numerical studies of lattice gauge theories in the limit of large number of colours [40][42]. These classical solutions (large $N$ phases of unitary matrix model) can be described in terms of phase space droplets in two dimensions [19]. These droplets are similar to Thomas-Fermi distributions at zero temperature. Therefore, quantum fluctuations ($O(1/N)$) of these classical solutions can be thought of as small ripples on the boundary of these droplets. In this section we study the dynamics of such ripples and show that different modes of fluctuations satisfy an abelian Kac-Moody algebra. Automodel diagrams are captured by unitary matrix model (GWW, in particular) and hence can be represented as free fermi droplets. As a result, large $N$ (large $\sqrt{N}$) fluctuations of automodel diagrams satisfy the same algebra. We further construct the Hilbert space of this algebra and find a one to one correspondence between the states in Hilbert space and automodel diagrams. In particular the Kac-Moody primary corresponds to null Young diagram (no box) whereas automodel diagrams are mapped to descendants of Kac-Moody primary.

Droplet description for classical phases is based on the fact that the partition function of unitary matrix model can also be equivalently written in terms of representations of unitary group. For example GWW model partition function has two descriptions : one in eigenvalue basis (3.2) or (3.3) and the second one in Young diagram basis (3.9). Hence different large $N$ phases can be described either in terms of eigenvalue distributions or Young diagram distributions. There is a one-to-one correspondence between these two descriptions. It is well known that eigenvalues of unitary matrices in a unitary matrix model behave like position of free fermions [18]. On the other hand hook lengths in Young diagram representation are like momenta of these fermions [19][43]. A relation between these two pictures offers droplet or phase space description for different classical phases [20][21]. The phase space is two dimensional and spanned by hook numbers and eigenvalues - $(h, \theta)$.

The gapless phase of GWW model is characterised by the eigenvalue density [19][31][32],

$$\rho_{\text{gapless}}(\theta) = \frac{1}{2\pi} (1 + 2\xi \cos \theta) \quad \text{for} \quad 0 \leq \xi < 1/2.$$  

(5.4)

The gapped phase occurs for $\xi > 1/2$. The eigenvalue density for gapped phase is given by

$$\rho_{\text{gapped}}(\theta) = 2\xi \sqrt{1 - \sin^2 \frac{\theta}{2} \cos \theta} \quad \text{for} \quad \sin^2 \frac{\theta}{2} \leq \frac{1}{2\xi}.$$  

(5.3)

The asymmetric solution (4.22) is mapped to one-gap phase (5.3).
Since the gap-less phase is equivalent to automodel diagrams, we mainly consider the droplet description of this phase and study its quantum fluctuations.

The one-to-one correspondence between Young diagram distributions and eigenvalue distributions allows us a topological way to classify different large $N$ solutions. We define a phase space distribution function $\omega(h, \theta)$ in $(h, \theta)$ plane

$$\omega(h, \theta) = \Theta \left( \frac{(h - h_-(\theta))(h_+(\theta) - h)}{2} \right) \tag{5.5}$$

such that $\omega(h, \theta) = 1$ for $h_-(\theta) < h < h_+(\theta)$ and zero otherwise. The eigenvalue and Young diagram distributions are obtained from $\omega(h, \theta)$ by integrating over $h$ and $\theta$ respectively

$$\rho(\theta) = \frac{1}{2\pi} \int \omega(h, \theta) dh \quad \text{and} \quad u(h) = \frac{1}{2\pi} \int \omega(h, \theta) d\theta. \tag{5.6}$$

Thus, the two dimensional distribution function $\omega(h, \theta)$ captures information about both the distributions. Since eigenvalue and Young diagram densities are normalised, the distribution function $\omega(h, \theta)$ also satisfies the normalisation condition

$$\frac{1}{2\pi} \int \omega(h, \theta) d\theta dh = 1. \tag{5.7}$$

We also define a quantity $S(\theta)$, called momentum density

$$S(\theta) = \frac{1}{2\pi \rho(\theta)} \int h\omega(h, \theta) dh = \frac{h_+(\theta) + h_-(\theta)}{2}. \tag{5.8}$$

Using this definition and eigenvalue distribution defined in (5.6) we have

$$h_{\pm}(\theta) = S(\theta) \pm \pi \rho(\theta). \tag{5.9}$$

Since $\omega^2(h, \theta) = \omega(h, \theta)$, it is actually the shape (i.e. boundary) of this distribution function which captures information about different large $N$ phases of the theory. To find the shape of the distribution we need to find $h_{\pm}(\theta)$ for different phase of the theory. For a generic class of matrix model, it was observed in [20, 21] that $h_{\pm}(\theta)$ is given by

$$h_{\pm}(\theta) = W(\theta) \pm \pi \rho(\theta) \tag{5.10}$$

where the function $W(\theta)$ depends on the matrix model under consideration. For GWW matrix model, which is our current interest, $W(\theta)$ is given by [21]

$$W(\theta) = \frac{1}{2} + \xi \cos \theta. \tag{5.11}$$

Hence for gap-less phase we have,

$$h_+(\theta) = 1 + 2\xi \cos \theta, \quad h_-(\theta) = 0. \tag{5.12}$$
(a) $\xi = 0$ : This droplet depicts the Young diagram with no box $\emptyset$.

(b) $0 < \xi < 1/2$ : This droplet corresponds to generic Young diagram in automodel class.

(c) $\xi = 1/2$ : Droplet corresponds to limit shape.

Figure 6: Droplets for automodel diagrams with $h$ as the radial coordinate and $\theta$ being the angular one.

The distributions for different values of $\xi$ are plotted in figure 6. $\xi = 0$ corresponds to a circular distribution. As we increase $\xi$ the shape of the distribution starts deforming. $\xi = 1/2$ (the last one in the figure) corresponds to the limit shape. The area covered by these distributions is $2\pi$ and independent of $\xi$. Therefore, evolution (with respect to $\xi$) of automodel diagrams (4.27) maps to deformation of these droplets keeping the area constant. One can think of these distributions as incompressible fluid droplets. One important thing to notice here is that the origin ($h = 0$) remains inside the droplet for $0 \leq \xi < 1/2$, i.e. the distribution is single valued. Similar droplet picture exists for one-gap phase also.

5.1. Fluctuations: semiclassical treatment

The quantum fluctuations ($\frac{1}{N}$ corrections) of classical solution ($\frac{1}{\sqrt{\hbar}}$ fluctuations of automodel Young diagrams) corresponds to small ripples on the boundary of classical droplets like in figure 7. To study the dynamics of these fluctuations we need to know how the boundary of these droplets

\begin{equation}
 h(\theta) = \frac{1}{2} + \xi \cos \theta \pm \pi \rho_{gapped}(\theta).
\end{equation}

For one-gap phase ($\xi > 1/2$) the distribution is determined by (19)

It was shown that for this phase the origin remains outside the droplet. Thus if we take the origin out from the 2d plane, $\xi < 1/2$ droplets can not be continuously deformed to $\xi > 1/2$ droplets. In that sense these two phases are topologically different. However, this is not the focus of this current work.
evolve with time. To incorporate dynamics into the picture, we first obtain the single particle Hamiltonian for the underlying fermi system. The distribution functions $\omega(h,\theta)$ for different classical phases are similar to Thomas-Fermi (TF) distribution at zero temperature. Thomas-Fermi distribution at zero temperature is given by

$$\Delta(p,q) = \Theta(\mu - h(p,q)) \quad (5.14)$$

where $\mu$ is chemical potential and $h(p,q)$ is single particle Hamiltonian density. Comparing our phase space distribution (5.5) with TF distribution we find the Hamiltonian density is given by

$$h(h,\theta) = \frac{h^2}{2} - S(\theta)h + \frac{g(\theta)}{2} + \mu, \quad \text{where} \quad g(\theta) = h_+/(\theta)h_-(\theta). \quad (5.15)$$

Total Hamiltonian$^4$ can be obtained by integrating $h(h,\theta)$ over the phase space

$$H_h = \frac{1}{2\pi\hbar} \int d\theta \int dh \omega(h,\theta)h(h,\theta). \quad (5.17)$$

We have taken into account the fact that one state occupies a phase space area of $2\pi\hbar$ in semiclassical approximation. We also need to modify the normalisation of phase space density$^5$

$$\frac{1}{2\pi\hbar} \int d\theta d\phi \omega(h,\theta) = N, \quad \text{with} \quad hN = 1 \quad (5.18)$$

where, $N$ is total number of states available inside a droplet ($N \sim O(\sqrt{k})$). The classical limit corresponds to $\hbar \to 0$, $L \to \infty$ with $hN = 1$.

---

$^4$One can show that integrating over $h$, the total Hamiltonian (without the $\hbar$ factor) is same as the collective field theory Hamiltonian of Jevicki and Sakita $[44]$:

$$H_h = \int d\theta \left( \frac{S^2\rho}{2} + \frac{\pi^2\rho^2}{6} + V_{eff}(\theta)\rho \right) + \mu \quad (5.16)$$

with an effective potential.

$^5$For automodel solution number of boxes in the first column is always less than or equal to $N$, hence we take $L = N$. 

---

Figure 7: Large $k$ fluctuations about classical geometry for $0 < \xi < 1/2$. 
The Hamilton’s equations obtained from the Hamiltonian (5.15) are given by

$$\dot{h} = S'(\theta)h - \frac{g'(\theta)}{2}, \quad \dot{\theta} = h - S(\theta).$$ \hspace{1cm} (5.19)

These are the set of equations for a particle moving on circle under the influence of an effective potential. The above set of equations offers the following solutions for gapless phase \(g(\theta) = 0\)

$$\theta(t) = 2 \tan^{-1}\left(\frac{(2\xi + 1) \tan\left(\frac{1}{4} \sqrt{1 - 4\xi^2} t\right)}{\sqrt{1 - 4\xi^2}}\right), \quad h(t) = \frac{1 - 4\xi^2}{1 - 2\xi \cos\left(\frac{1}{2} \sqrt{1 - 4\xi^2} t\right)}.$$ \hspace{1cm} (5.20)

The solutions are plotted in figure 8. For \(\xi = 0\) the particle is uniformly moving on circle : \(\dot{\theta}(t)\) and \(h(t)\) are constant. As we increase \(\xi\) the particle starts spending more time at \(\theta(t) = \pm \pi\) : momentum is minimum when the particle reaches at \(\pm \pi\). At \(\xi = 1/2\), we get an instanton like solution. Eliminating \(t\) from the above solutions one can find that the phase space trajectory for the particle. The trajectory is given by

$$h(t) = 1 + 2\xi \cos \theta(t)$$ \hspace{1cm} (5.21)
which is the boundary of the classical droplet. Thus we see that shape of large $N$ droplets can be mapped to phase space trajectories of a classical particle moving on a circle under the influence of an effective potential. We use the set of Hamilton’s equations (5.19) to study the evolution of the boundary of classical droplets [45, 46].

Since gapless distributions are single valued ($h_-(\theta) = 0$ always), the boundary is given by $h = h_+(\theta)$ and the boundary evolution is governed by

$$
\dot{h}(\theta) = \frac{h(\theta) h'(\theta)}{2}.
$$

(5.22)

We would like to introduce a Poisson bracket between $h(\theta)$ and $h'(\theta)$ such that the boundary evolution equation can be written as,

$$
\dot{h}(\theta) = \{h(\theta), H_h\}
$$

(5.23)

where $H_h$ is the total Hamiltonian (5.17). Integrating over $h$, $H_h$ can be written as,

$$
H_h = -\frac{1}{2\pi\hbar} \int d\theta' \frac{h^3(\theta')}{12}.
$$

(5.24)

Defining the following Poisson bracket

$$
\{h(\theta), h(\theta')\} = 2\pi\hbar \delta'(\theta - \theta'),
$$

(5.25)

one can check that the equation (5.23) boils down to (5.22).

To quantise the above classical system we promote the Poisson bracket (5.25) to commutation relation

$$
[h(\theta), h(\theta')] = 2\pi\hbar^2 \delta'(\theta - \theta').
$$

(5.26)

We decompose the fluctuations of $h(\theta)$ in Fourier modes

$$
h(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}.
$$

(5.27)

The reality of $h(\theta)$ leads to the condition $f_n^\dagger = f_{-n}$. The commutation relation (5.26) in $h(\theta)$ implies that the different modes of fluctuations satisfy the following commutation relations

$$
[f_m, f_n] = -m\hbar^2 \delta_{m+n,0}.
$$

(5.28)

This is a $U(1)$ Kac-Moody algebra. Demanding the fluctuations to be area preserving leads to the constraint

$$
f_0 = 1.
$$

(5.29)

Non-zero modes do not cost any change in area of the droplets. Redefining $f_n$

$$
f_n = \hbar \sqrt{n} a_n^\dagger \quad \text{for} \quad n \geq 1,
$$

(5.30)

Since $h_- = 0$, we denote boundary of a droplet by $h(\theta)$, instead of $h_+(\theta)$.
it follows from (5.28) that
\[ [a_m, a_n^\dagger] = \delta_{m,n} \quad \text{for } m, n \geq 1 \] (5.31)
which is conventionally known as the Heisenberg algebra.

The representation of (5.31) defines the Hilbert space of the system. We define a ground state \( |0\rangle \) as
\[ a_n|0\rangle = 0, \quad \forall \, n \geq 1. \] (5.32)

A generic excited state can be written as,
\[ |\vec{q}\rangle = \prod_n (a_n^\dagger q_n)|0\rangle \quad \forall \, q_n \in \mathbb{Z}_{\geq 0}. \] (5.33)

5.2. A correspondence between states in Hilbert space and Young diagram

There exists a one to one correspondence between states in the Hilbert space and Young diagrams for the representations of permutation groups via phase space distribution\(^7\). In order to find the same we first define a bilinear of \( h(\theta) \), called “Sugawara stress tensor”
\[ T(\theta) = \frac{h^2(\theta)}{2\hbar^2}. \] (5.34)

The commutation relation (5.26) implies that the stress tensor satisfies the following commutation relation
\[ [T(\theta), T(\theta')] = 2\pi i (T(\theta) + T(\theta')) \delta'(\theta - \theta') \] (5.35)

Decomposing \( T(\theta) \) in Fourier modes
\[ T(\theta) = \sum_{n = -\infty}^{\infty} L_n e^{in\theta} \] (5.36)
we find that the Virasoro generators \( L_m \)s satisfy the Witt algebra
\[ [L_m, L_n] = (n - m)L_{m+n}. \] (5.37)

The Virasoro generators can be written in terms of modes of \( h(\theta) \)
\[ L_m = \frac{1}{2\hbar^2} \sum_{n=0, \mathbb{Z}} f_n f_{m-n}. \] (5.38)

The zero mode, in particular, is given by\(^8\)
\[ L_0 = \frac{1}{2\hbar^2} + \sum_{n=0} na_n^\dagger a_n. \] (5.39)

\(^7\)A mapping between Young diagrams and operators/states was considered in\([47, 48]\). We map states to droplets and hence to Young diagrams.

\(^8\)There is a constant part in \( L_0 \) which is equal to \( \zeta(-1) \). However, one can get rid of this constant part if we consider normal ordered definition of stress tensor.
A generic state $|\vec{q}\rangle$ is an eigenstate of $L_0$. We denote $L_0$ eigenvalue of $|\vec{q}\rangle$ by $h_{\vec{q}}$. From (4.11), it is easy to check that the $L_0$ operator is related to total box number operator and hence $L_0$ eigenvalue $h_{\vec{q}}$ measures the total number of boxes in the corresponding Young diagram associated with a state $|\vec{q}\rangle$.

$$h_{\vec{q}} = k + \frac{N^2}{2}.$$  

(5.40)

For example, ground state $|0\rangle$ has $L_0$ eigenvalue $N^2/2$

$$L_0|0\rangle = \frac{1}{2h^2}|0\rangle$$

(5.41)

and hence, the corresponding diagram has zero box ($\emptyset$). Ground state $|0\rangle$ a primary state of Kac-Moody as well as Virasoro.

To complete the mapping between the states and the Young diagrams we also need to specify the shape of the Young diagram with $k$ boxes corresponding to a state in Hilbert space. Since, a particular Young diagram corresponds to a droplet in phase space, we define an operator $\hat{S}(\theta)$ which captures the shape of the droplet in phase space associated with a state in Hilbert space. From (5.27) we see that an excitation of the ground state by $n^{th}$ mode corresponds to a $\cos n\theta$ deformation of the boundary, therefore we define the “shape” operator

$$\hat{S}(\theta) = 1 + \frac{2h}{\sqrt{k}} \sum_{n>0} \sqrt{n} \cos n\theta a_n^* a_n$$

(5.42)

where $k$ is the total number of boxes. The eigenvalue of $\hat{S}(\theta)$ define the shape function $h(\theta)$ in phase space and hence the distribution of boxes in the corresponding Young diagram. For example $|0\rangle$ state has $\hat{S}(\theta)$ eigenvalue one and therefore gives a circular droplet, which corresponds to $\emptyset$ diagram.

An automodel diagram corresponds to a particular descendant state of $U(1)$ Kac-Moody algebra

$$|\bullet, \xi\rangle = (a_1^+)^{g_1} |0\rangle, \quad \text{with} \quad g_1 = N^2 \xi^2.$$  

(5.43)

This state corresponds to a Young diagram with total number of boxes $N^2 \xi^2$ and the shape of the droplet is given by $h = 1 + 2\xi \cos \theta$.

We consider generic fluctuations of automodel diagrams. The corresponding states in Hilbert space are given by

$$|F\rangle = \prod_n (a_n^*)^{\alpha_n} |\bullet, \xi\rangle, \quad \alpha_n \in \mathbb{Z}_{\geq 0}.$$  

(5.44)

The number of boxes in Young diagram for state $|F\rangle$ is given by

$$N^2 \xi^2 \left(1 + \frac{1}{N^2 \xi^2} \sum_n n \alpha_n\right)$$

(5.45)
and the corresponding shape is given by

\[ 1 + 2\xi \cos \theta + \frac{2\hbar}{\sqrt{k}} \sum_n \sqrt{n} \alpha_n \cos n\theta. \]  

(5.46)

This shape corresponds to a random fluctuations of automodel Young diagrams as \( \alpha_n \)'s are random. One can consider these \( \alpha_n \)'s to be random Gaussian integers with mean zero. Thus these fluctuations are similar to fluctuations given by equation (5.2) except the fact that here \( \alpha_n \)'s are integers.\(^9\)

6. Conclusion

In this paper we show that the growth of Young diagrams equipped with Plancherel measure can be studied through a simple matrix model. We write down a partition function for such growth process and solve the model in the continuum limit. In [5], Kerov introduced a differential model for the growth of Young diagrams, known as the automodel. We find that in continuum limit our one parameter solution falls in the automodel class of Kerov with renormalised box number playing the role of \textit{time}. At the limiting value of the parameter the dominant solution matches with the limit shape of [3, 4]. Our analysis also offers an alternate proof of limit shape theorem of Vershik-Kerov and Logan-Shepp.

We observe that the evolution of Young diagrams in automodel class can be mapped to different shapes of incompressible fluid droplets in two dimensions. Automodel evolution corresponds to area preserving deformation of these fluid droplets. Such identification was possible due to the equivalence between GWW model and automodel partition function [19]. In particular we see that GWW transition point maps to limit shape of [3, 4]. Since eigenvalues of unitary matrices behave like position of free fermions, two dimensional fluid droplets are identified with classical phase space of these free fermions [19, 21]. From this distribution we construct the one particle Hamiltonian. The Hamiltonian describes dynamics of a particle moving on a circle of unit radius under the influence of an effective potential. The classical phase space trajectories of such particle captures the information about the automodel Young diagrams.

In view of the above connection between automodel diagrams and two dimensional phase space droplets, any fluctuations of automodel diagrams correspond to small ripples on the boundary of these droplets. Using the Hamiltonian equations we quantise the dynamics of such ripples and find that different modes of these fluctuations satisfy abelian Kac-Moody algebra. The fluctuations of classical droplets studied in this paper seems quite universal in nature. Edge excitations of fractional quantum Hall fluid also satisfy similar algebra [49]. Fluctuations of large \( N \) fermions in harmonic oscillator potential also belong to the similar class [45, 46, 50]. Since automodel class is equivalent to gapless phase of GWW model, quantum fluctuations of the classical weak

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\(^9\) Equation (5.2) considers fluctuations of \( \hat{v}_k(u) \), where as our Young diagram density is related to derivative of \( \hat{v}_k(u) \). Hence the generic fluctuation (5.46) will match with (5.2) if we take a derivative of the same.
coupling phase of GWW model also falls in this class. We construct Sugawara energy-momentum tensor from $U(1)$ currents and express the Virasoro modes in terms of Kac-Moody modes. It turns out that the zero mode of the Virasoro is proportional to box number operator. We also construct the Hilbert space for the Kac-Moody algebra. It turns out that there is a correspondence between different states in the Hilbert space and Young diagrams. The Kac-Moody primary (which is a Virasoro primary as well) corresponds to $\emptyset$ (null) diagram. Automodel diagrams correspond to descendants of Kac-Moody algebra. Fluctuations about automodel diagrams corresponds to fluctuations about automodel states. The shape of those fluctuations matches with the fluctuations of limit shape studied by [8]. It would be interesting to study these Gaussian fluctuations of limit shape diagrams in the context of states in Hilbert space.

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