Deriving spin-1 quartic interaction vertices from closure of the Poincaré algebra

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Abstract

We derive the quartic interaction vertex of pure Yang-Mills theory by demanding closure of the light-cone Poincaré algebra in four-dimensional Minkowski spacetime. This calculation explicitly shows why structure constants must satisfy the Jacobi identity. We prove that there is no correction to the spin generator, for spin one, at this order. We comment briefly on higher spin fields in this context.
1 Introduction

In light-cone gauge, neither locality nor Poincaré invariance is manifest. Poincaré invariance thus needs to be explicitly verified in light-cone field theories. As we will stress in this paper, this requirement of closure of the Poincaré algebra, apart from being an important check, may be viewed as a first-principles approach to deriving Lagrangians for interacting field theories in light-cone gauge.

The procedure involves starting with an ansatz for the interaction vertices and allowing the Poincaré algebra to completely fix the ansatz. This algebraic approach has many advantages: at first order in the coupling constant, for theories involving fields of odd-spin the algebra requires the introduction of an antisymmetric constant \[1\]. In this paper, we explicitly show that the requirement of Poincaré invariance, at second order in the coupling constant, forces these constants to satisfy the Jacobi identity, signaling the emergence of a gauge group \([1]\).

If we were to adapt this approach, for spins greater than 2, to an AdS\(_4\) background, would the closure of the \(SO(3,2)\) algebra force the introduction of fields with spin larger than the one being considered? In principle, this could yield the elusive action principle corresponding to the Vasiliev equations \([3]\) (which would have to be established order by order in comparison with the Vasiliev model). It remains unclear whether the light-cone formalism presented here circumvents the many covariant no-go results \([4]\), pertaining to higher spin fields in flat spacetime, which assume both locality and Poincaré invariance to be manifest.

In this paper, we extend the symmetry-based approach to deriving interaction vertices, introduced in \([1]\), to second order in the coupling constant for the specific case of spin one. We comment on similar expressions in the spin two and higher spin cases.

2 At order \(\alpha\) : old results

Our metric signature is \((-,+,+,+)+\) and we use light-cone coordinates

\[
x^\pm = x^0 \pm i x^3, \quad x = x^1 + i x^2, \quad \bar{x} = x^1 - i x^2, \quad (1)
\]

the accompanying derivatives being \(\partial^\pm\), \(\partial\) and \(\bar{\partial}\). Light-cone time is \(x^+\) implying that \(\partial^-\), the conjugate momentum, is the light-cone Hamiltonian. Massless fields, in light-cone gauge, have two physical degrees of freedom \(\phi\) and \(\bar{\phi}\) corresponding to the + and − helicity states respectively. The generators of the Poincaré algebra are the momenta

\[
P^- = -i \frac{\partial^-}{\partial^+} = -P_+ \quad P^+ = -i \partial^+ = -P_- \quad P = -i \partial \quad \bar{P} = -i \bar{\partial}. \quad (2)
\]

The \(\frac{1}{\partial^+}\), an artifact of this gauge choice, is an integral operator acting on a field \([5]\)

\[
\frac{1}{\partial^+} \phi(x^-) = \int dy^- \epsilon(y^- - x^-) \phi(y^-), \quad (3)
\]

\(^1\)In \([2]\), similar results for the structure constants were derived using symmetry arguments in a covariant approach, starting with a general local relativistically-invariant Lagrangian.
where $\epsilon$ is the step function. The rotation generators are
\[ J = (x \tilde{\partial} - \bar{x} \partial - \lambda), \quad J^+ = i(x \partial^+ - x^+ \partial), \]
\[ J^{+-} = i(x^- \partial^+ - x^+ \partial^- \bar{\partial}^+) - \lambda \partial^- \partial^+, \quad J^- = i(x \tilde{\partial} - x^- \partial - \lambda \bar{\partial}^+ \partial^-), \]
(4)
and their complex conjugates. $\lambda$ is the spin of the field the generators act on. We work on the surface $x^+ = 0$ to simplify our calculations. The kinematical Poincaré generators do not involve time derivatives and include
\[ P^+, \quad P, \quad \bar{P}, \quad J, \quad J^+, \quad \bar{J}^+ \quad \text{and} \quad J^{+-}. \]
(5)
The dynamical generators are
\[ P^-, \quad J^-, \quad \bar{J}^- \]
and pick up corrections when interactions are switched on. We introduce the Hamiltonian variation
\[ \delta_H \phi \equiv \partial^- \phi = \{ \phi, H \} = \frac{\partial \tilde{\partial}^-}{\partial^+} \phi, \]
(7)
where the last equality only holds for the free theory. In the interacting theory, the $\delta_H$ operator picks up corrections, order by order, in the coupling constant.

In the appendix, we provide a list of all the non-vanishing commutators satisfied by the light-cone Poincaré generators. The basic idea in [1], that we follow in this paper, is to start with an ansatz for the operator $\delta_H \phi$, work through the entire list of Poincaré commutators to refine the ansatz and thus determine the Hamiltonian. For related discussions, see [6, 7].

2.1 Deriving cubic interaction vertices

We briefly review the derivation of cubic interaction vertices for integer $\lambda$. At this order the structure of the Hamiltonian is
\[ \mathcal{H} \sim \alpha \bar{\phi} \phi \phi + \alpha \bar{\phi} \tilde{\phi} \tilde{\phi}. \]
(8)
From [7], we see there are two contributions to $\delta_H \phi$ at this order. The first involves two $\phi$ fields while the second involves a $\phi$ and a $\tilde{\phi}$. We start with the following ansatz for the first variety
\[ \delta_H^2 \phi = \alpha K \partial^{+\mu} \left[ \tilde{\partial}^B \partial^C \tilde{\partial}^{+\rho} \phi \tilde{\partial}^D \partial^E \partial^{+\sigma} \phi \right], \]
where $K$ is a constant and $\mu, \rho, \sigma, B, C, D, E$ are integers to be fixed by the algebra [1]. The commutator of this ansatz with $\delta_{J^{+-}}$ yields [1]
\[ \mu + \rho + \sigma = -1. \]
(10)
\[ \text{The ansatz involving one $\phi$ and one $\tilde{\phi}$ works very similarly.} \]
We commute the ansatz with $\delta J$ to obtain

$$B + D - C - E = \lambda.$$  \hfill (11)

Using (10) and dimensional analysis we then find that

$$B + D = \lambda; \quad C = E = 0.$$  \hfill (12)

The other commutation relations determine the values of $\mu$, $\rho$ and $\sigma$. We thus obtain \[1\]

$$\delta H = \alpha \sum_{n=0}^{\lambda} (-1)^n \left( \frac{\lambda}{n} \right) (\partial^+)^{(\lambda-1)} \left[ \frac{\partial (\lambda-n)}{\partial (\lambda-n)} \phi \frac{\partial^n}{\partial^{n+\lambda}} \phi \right],$$  \hfill (13)

for even $\lambda$.

**Appearance of the “structure constant”**

Interestingly, a non-trivial solution for odd-helicity fields is only possible through the introduction of an antisymmetric three-index object $f^{abc}$,

$$\delta H a = \alpha f^{abc} \sum_{n=0}^{\lambda} (-1)^n \left( \frac{\lambda}{n} \right) (\partial^+)^{(\lambda-1)} \left[ \frac{\partial (\lambda-n)}{\partial (\lambda-n)} \phi \frac{\partial^n}{\partial^{n+\lambda}} \phi \right].$$  \hfill (14)

The same procedure determines $\delta H$ corresponding to the $\alpha \bar{\phi} \phi$ structure. The Hamiltonian, to this order, follows from (13) and (14). The corresponding actions read \[1\]

$$S = \int d^4 x \left( \frac{1}{2} \bar{\phi} \Box \phi + \alpha \sum_{n=0}^{\lambda} (-1)^n \left( \frac{\lambda}{n} \right) \bar{\phi} (\partial^+)^\lambda \left[ \frac{\partial (\lambda-n)}{\partial (\lambda-n)} \phi \frac{\partial^n}{\partial^{n+\lambda}} \phi \right] + c.c. \right),$$  \hfill (15)

for even $\lambda$ and

$$S = \int d^4 x \left( \frac{1}{2} \bar{\phi} \Box \phi^a + \alpha f^{abc} \sum_{n=0}^{\lambda} (-1)^n \left( \frac{\lambda}{n} \right) \bar{\phi}^a (\partial^+)^\lambda \left[ \frac{\partial (\lambda-n)}{\partial (\lambda-n)} \phi^b \frac{\partial^n}{\partial^{n+\lambda}} \phi^c \right] + c.c. \right),$$  \hfill (16)

for odd $\lambda$.

It is interesting to note \[8\] that, at the cubic level, the non-linear dynamical part of the algebra does not restrict the cubic interactions beyond what is required by the kinematical portion of the algebra (see also \[9\]).

### 3 At order $\alpha^2$ : new results

In this section, we extend this formalism to second order in the coupling constant for the specific case of $\lambda = 1$. The two fields in Yang-Mills theory, $A$ and $\bar{A}$, have helicity $+1$ and $-1$ respectively. We also identify $2 \alpha$ with the dimensionless Yang-Mills coupling constant $g$. We will use the following result from the previous section,

$$\delta_H \phi^a = +g f^{abc} \left\{ -A^c \frac{\partial}{\partial^+} A^b + \frac{1}{\partial^{+2}} (\partial^{+2} A^b \partial \bar{A}^c) - \frac{1}{\partial^{+2}} (\partial \partial^+ A^b \bar{A}^c) \right\}.$$  \hfill (17)
We will also need the corrections to the spin generator at this order [1]

\[ \delta_{S^-}^q A^a = -g f^{abc} \frac{1}{\partial^2} \left( \frac{1}{\partial^+} A^c \partial^+ A^b + 3 A^c \partial^+ A^b \right), \]  

(18)

\[ \delta_{S^-}^q A^a = +g f^{abc} \frac{1}{\partial^+} A^b A^c. \]  

(19)

The key commutator involving dynamical generators is

\[ [\delta_{J^-}, \delta_H] A^a = 0. \]  

(20)

In the following computation, we present only terms of the form AAA (terms of the form AAA vanish independently). We note that the other dynamical commutator between \( \delta_{J^-} \) and \( \delta_{J^-} \) does not yield additional information because \( \delta_{J^-} A^a \) at order \( g^2 \) is proportional to \( \delta_H A^a \) (see appendix B in [8] for related discussions). We begin by computing contributions to (20) at order \( g^2 \), from \([\delta_{J^-}, \delta_H g] A^a\). This calculation involves two pieces, orbital and spin,

\[ [\delta_{L^-}, \delta_H g] A^a = [x \delta_{H} g, \delta_{H} g] A^a = -g f^{abc} \frac{1}{\partial^+} (\delta_H g A^b), \]  

(21)

and

\[ [\delta_{S^-}, \delta_H^q] A^a = +g^2 f^{abc} \left\{ f^{bde} \frac{1}{\partial^2} \left( \partial^+ \left( \frac{1}{\partial^+} A^d A^e \right) \partial^+ A^c \right) - f^{bde} \frac{1}{\partial^2} \left( \partial^+ \left( \frac{1}{\partial^+} A^d A^e \right) \partial^+ A^c \right) \right\}, \]  

(22)

The other contribution to (20) is from commutators that involve one generator at order \( g^0 \) and one at order \( g^2 \). Before we evaluate these, we need an ansatz for \( \delta_H^q \). We begin with a very general structure that is the sum of terms of the form [3]

\[ \delta_H^q A^a = +g^2 K f^{abc} f^{cde} \partial^{+\mu} \left[ \partial^B \partial^C \partial^{+\rho} A^b \partial^{+\sigma} \left( \partial^D \partial^E \partial^{+\eta} A^d \partial^F \partial^G \partial^{+\delta} A^e \right) \right], \]  

(23)

where \( K \) is a constant and \( \mu, \rho, \sigma, \eta, \delta, B, C, D, E, F, G \) are integers to be determined by the algebra. We commute this with \( \delta_J \) to find the following conditions.

\[ B + D + F = C + E + G = \lambda - 1. \]  

(24)

\(^{3}One may write down other combinations by moving the derivatives around but these structures can be generated starting from the form in (23).\)

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Thus no transverse derivatives are permitted when $\lambda = 1$, simplifying our ansatz to

$$
\delta_H^2 A^a = +g^2 K f^{abc} f^{cde} \partial^+ \partial^\mu \left[ \partial^+ \partial^\nu A^b \partial^+ \partial^\sigma \left( \partial^+ \partial^\eta A^d \partial^+ \partial^\delta A^e \right) \right].
$$

The commutator with $\delta_{J^+-}$ yields

$$
\mu + \rho + \sigma + \eta + \delta = -1.
$$

The final piece of the computation involves

$$
[\delta_L^2, \delta_H^0] A^a + [\delta_{J^+}, \delta_H^2] A^a,
$$

where we have ignored the spin generator in the first commutator (since it is zero as explained in the next subsection). We find that the following solution satisfies \(20\) and present below the explicit computation of \(27\) for these values (the more general case is far more lengthy but not necessary for the points we wish to make). Any other set of consistent values for these constants is completely equivalent to those above (essentially corresponding to trivial re-writings of the result in \(31\)).

$$
f^{abc} f^{cde} \left[ \frac{-1}{\partial + 2} (\partial^+ \partial A^b \frac{1}{\partial + 2} (\partial^+ \partial A^d)) + \frac{1}{\partial + 2} (\partial^+ \partial A^b \frac{1}{\partial + 2} (\partial^+ \partial A^d)) + \frac{1}{\partial + 2} (\partial^+ A^b \bar{A}^e \partial^+ A^d) \right] + 2 \frac{1}{\partial + 2} (\partial^+ A^b \frac{1}{\partial + 2} (\partial^+ \bar{A}^e \partial^+ A^d)) - \frac{1}{\partial + 2} (\partial^+ A^b \frac{1}{\partial + 2} (\partial^+ \bar{A}^e \partial^+ A^d)) - 2 \frac{1}{\partial + 2} (\partial^+ A^b \frac{1}{\partial + 2} (\partial^+ \bar{A}^e \partial^+ A^d))
$$

$$
+ \frac{1}{\partial + 2} (\partial^+ A^b \frac{1}{\partial + 2} (\partial^+ \bar{A}^e \partial^+ A^d)) - \frac{1}{\partial + 2} (\partial^+ A^b \frac{1}{\partial + 2} (\partial^+ \bar{A}^e \partial^+ A^d)) - 2 \frac{1}{\partial + 2} (\partial^+ A^b \frac{1}{\partial + 2} (\partial^+ \bar{A}^e \partial^+ A^d)) + 4 \frac{1}{\partial + 2} (\partial^+ A^b \frac{1}{\partial + 2} (\partial^+ \bar{A}^e \partial^+ A^d))
$$

$$
+ 2 \frac{1}{\partial + 2} (\partial^+ A^b \frac{1}{\partial + 2} (\partial^+ \bar{A}^e \partial^+ A^d)) - \frac{1}{\partial + 2} (\partial^+ A^b \frac{1}{\partial + 2} (\partial^+ \bar{A}^e \partial^+ A^d)) - 4 \frac{1}{\partial + 2} (\partial^+ A^b \frac{1}{\partial + 2} (\partial^+ \bar{A}^e \partial^+ A^d)).
$$

(29)

**Emergence of a gauge group**

The crucial point here is that the two expressions in \(21\) and \(22\) cancel perfectly against \(29\) if and only if we assume that the $f^{abc}$ introduced in \(14\) satisfy the Jacobi identity\(^4\)

$$
f^{abc} f^{bde} f^{ecd} + f^{abc} f^{bed} f^{ecd} + f^{abc} f^{bce} f^{dcd} = 0.
$$

(30)

Thus, we find

$$
\delta_H^2 A^a = g^2 f^{abc} f^{cde} \left[ \frac{1}{\partial^+} (\partial^+ A^b \frac{1}{\partial + 2} (\partial^+ \bar{A}^e A^d)) - A^b \frac{1}{\partial + 2} (\partial^+ A^e \partial^+ A^d) \right].
$$

(31)

This expression leads to the same quartic interaction vertex obtained by light-cone gauge-fixing the covariant Yang-Mills Lagrangian \(10\).

\(^4\)The Jacobi identity is also necessary to prove that terms of the form $A A A$ vanish.
3.1 The spin generator at order $g^2$

To show that $\delta S^- A^a = 0$, we start by examining the helicities and dimensions involved.

| Quantity | Helicity | Dim $[L]$ |
|----------|----------|------------|
| $x$      | +1       | +1         |
| $\bar{x}$ | −1       | +1         |
| $\partial$ | +1       | −1         |
| $\bar{\partial}$ | −1       | −1         |
| $A$      | +1       | −1         |
| $\bar{A}$ | −1       | −1         |
| $\partial^+$ | 0        | −1         |

At lowest order,

$$\delta S^- A^a = -\frac{\partial}{\partial^+} A^a ,$$

has helicity +2 and a length-dimension of −1. An ansatz at order $g^2$ will take the form

$$\delta S^- A \sim g^2 A A \frac{1}{\partial^{+3}} ,$$

where the derivatives at the end of the expression may be sprinkled on various fields. However, the commutator with $\delta J^+ -$ works only if the number of $\partial^+$'s in the denominator is one greater than that in the numerator (see for example (26)) ruling out this ansatz. No combination of ingredients from the table above, with three fields, has the correct values of helicity, dimension and kinematical commutators. The same argument rules out the possibility of a non-zero $\delta S^- A^a$. This is not surprising since the Poincaré commutator $[\bar{P}, J^-] = -iP^-$ tells us that the spin generator has one less transverse derivative than the Hamiltonian (and the spin one Hamiltonian at order $g^2$ has zero transverse derivatives).

*With this, the construction of the entire light-cone Poincaré algebra for Yang-Mills theory is complete.*

4 Comments

We briefly examine $\delta_H$ for the case of $\lambda = 2$. At lowest order

$$\delta_H^0 h = \frac{\partial \bar{\partial}}{\partial^+} h ,$$

this has a length-dimension of −2 and a helicity of +2. At order $\alpha^2$ we expect the form

$$\delta_H^{\alpha^2} h \sim \alpha^2 hh\bar{h} ,$$

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where \( \alpha \) has the dimensions of length, for a spin two field. The structure in (35) has the correct helicity but the wrong dimension. Using constraints like (26) and dimensional analysis, we conclude that

\[
\delta^2_H h \sim \alpha^2 hh\bar{h} \left( \partial \bar{\partial} \right) \frac{1}{\partial^+} .
\] (36)

Note that we can introduce equal numbers of \( \partial^+ \) and \( \frac{1}{\partial^+} \) through-out the expression resulting in a sum of terms. This matches the structures that appear from gauge-fixing the gravity Lagrangian [11].

At next order, we expect

\[
\delta^3_H h \sim \alpha^3 hh\bar{h}h \partial^2 \frac{1}{\partial^+} + \alpha^3 hh\bar{h}\bar{h} \partial^2 \frac{1}{\partial^+} ,
\] (37)

which again precisely matches the structure written down in [12]. The sheer volume of terms involved in these expressions at orders \( \alpha^2 \) and \( \alpha^3 \) make algebraic computations tedious to work out for spins \( \geq 2 \).

For higher spin fields

\[
\delta^\gamma_H \phi \sim \alpha^2 \phi\phi\bar{\phi} \left( \partial \bar{\partial} \right) \frac{1}{\partial^+} ,
\] (38)

where \( \alpha \) now has length-dimension of \( \lambda - 1 \). The dynamical commutators, in this case, would play a key role in determining whether consistent quartic vertices are allowed in flat spacetime. The corresponding corrections to the spin generator at this order read

\[
\delta^2_{S^-} \phi \sim \alpha^2 \phi\phi\bar{\phi} \partial \left( \partial \bar{\partial} \right) \frac{1}{\partial^+} .
\] (39)



The preceeding section raises the question of whether higher spin fields can have consistent interacting vertices in flat spacetime, within this formalism. For related discussions, see [13]. An obvious next step is to modify this formalism and apply it on an AdS background [14]. Just as the algebra, in flat spacetime, led us to structure constants and the Jacobi identity, it is plausible that the same approach on an AdS background [15] applied to a spin \( > 2 \) field will lead us to re-discover the higher spin tower.

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A Light-cone Poincaré algebra

We define

\[ J^+ = \frac{J^+ + iJ^z}{\sqrt{2}} , \quad \bar{J}^+ = \frac{J^+ - iJ^z}{\sqrt{2}} , \quad J = J^{12} . \]  \hspace{1cm} (40)

The non-vanishing commutators of the Poincaré algebra are

\[
\begin{align*}
[P^-, J^{+-}] &= -iP^- , \\
[P^+, J^{+-}] &= iP^+ , \\
[P, J^-] &= -iP^- , \\
[\bar{P}, J^-] &= -i\bar{P} , \\
[J^-, J^{+-}] &= -iJ^- , \\
[J^-, J^{+ -}] &= -iJ^- , \\
[J^{+-}, J^+] &= -iJ^+ , \\
[J^{+ -}, J^+] &= -iJ^+ , \\
[J^+, J] &= J^+ , \\
[\bar{J}^+, J] &= -\bar{J}^+ .
\end{align*}
\]  \hspace{1cm} (41)
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