ABSTRACT. Kempf [1976] studied proper, G-equivariant maps from equivariant vector bundles over flag manifolds to G-representations V, which he called collapsings. We give a simple formula for the G-equivariant cohomology class on V, or multidegree, associated to the image of a collapsing: apply a certain sequence of divided difference operators to a certain product of linear polynomials, then divide by the number of components in a general fiber. When that number of components is 1, we construct a desingularization of the image of the collapsing. If in addition the image has rational singularities, we can use the desingularization to give also a formula for the G-equivariant K-class of the image, whose leading term is the multidegree.

Our application is to quiver loci and quiver polynomials. Let Q be a quiver of finite type (A, D, or E, in arbitrary orientation), and assign a vector space to each vertex. Let Hom denote the (linear) space of representations of Q with these vector spaces. This carries an action of GL, the product of the general linear groups of the individual vector spaces. A quiver locus Ω is the closure in Hom of a GL-orbit, and its multidegree is the corresponding quiver polynomial. Reineke [2004] proved that every ADE quiver locus is the image of a birational Kempf collapsing (giving a desingularization directly).

Using Reineke’s collapsings, we give formulae for ADE quiver polynomials, previously only computed in type A (though in this case, our formulae are new). In the A and D cases quiver loci are known to have rational singularities [Bobiński-Zwara 2002], so we also get formulae for their K-classes, which had previously only been computed in equioriented type A (and again our formulae are new).

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1. INTRODUCTION

1.1. Kempf collapsings. Let G be a reductive algebraic group, and P a parabolic subgroup. Let Y be a linear representation of G, and Z ≤ Y a P-invariant subspace (or more

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generally, a closed subvariety with at worst rational singularities). In [Ke76], Kempf considers the map

\[ G \times^P Z \to Y \]

here \( G \times^P Z := (G \times Z)/\{[g, v] \sim [gp^{-1}, pv]\} \)

which he calls a **collapsing** of \( G \times^P Z \). This space is the associated fiber bundle with fiber \( Z \) over the homogeneous projective variety \( G/P \). The map \( \kappa \) is proper (it factors as \( G \times^P Z \to G/P \times Y \to Y \)), hence its image \( G \cdot Z \) is closed. When \( \kappa \) is birational, \( \kappa \) serves as a resolution of singularities of the variety \( G \cdot Z \).

Since \( G \cdot Z \) is a \( G \)-invariant subvariety of \( \text{Hom} \), it has a **multidegree** \([G \cdot Z]\), which can be defined as the associated \( G \)-equivariant cohomology class (or Chow class) in the ring \( H^*_G(Y) \). Our first result, Theorem 1 below, is a formula for \([G \cdot Z]\). In Section 3 we recall the properties we need of multidegrees.

If \( T \leq P \) is a maximal torus of \( G \), then \( H^*_G(Y) \) naturally includes into \( H^*_T(Y) = \text{Sym}^*(T^*) \), so it is enough to compute \([G \cdot Z]\) as a \( T \)-equivariant cohomology class. Since \( Z \) is \( P \)-invariant and hence \( T \)-invariant, it too has a \( T \)-multidegree \([Z] \in \text{Sym}^*(T^*) \). This \([Z]\) turns out to be particularly simple for \( Z \) a linear subspace: \([Z]\) is the formal product of the \( T \)-weights in \( Y/Z \), each of which lives in the weight lattice \( T^* \) of \( T \). If none of the \( T \)-weights on \( Y \) are 0, then this product cannot be zero.

For nonlinear \( Z \subseteq Y \), e.g., a union of linear subspaces, there might still be some cancellation giving \([Z] = 0\). This can’t happen (as follows from Theorem D in [KM01]) if all the weights of \( T \) acting on \( Y \) live in an open half-space in \( T^* \); in this very common case \(^1\) any closed \( T \)-invariant scheme \( Z \subseteq Y \) has a nonzero multidegree \([Z]\).

**Theorem 1.** Let \( \kappa : G \times^P Z \to Y \) be a Kempf collapsing, where \( \text{Lie}(P) \) contains all the negative root spaces. Let \( d \) be the number of components in a general fiber of \( \kappa \). Assume that all the weights of \( T \) acting on \( Y \) live in an open half-space in \( T^* \).

Let \( m_0 = [Z] \), and construct a sequence of polynomials \( m_1, m_2, \ldots \) by applying divided difference operators \( \delta_\alpha := \frac{1}{\alpha}(1 - r_\alpha) \) to \( m_0 \), where \( \alpha \) varies over the set of simple roots of \( G \), and \( r_\alpha \) acts on \( \text{Sym}^*(T^*) \) from the reflection action on \( T^* \). Don’t apply a divided difference operator if the result is 0, and only stop when all \( \delta_\alpha \) give the result 0.

This process always terminates after the same number of steps (namely, \( \dim G \times^P Z - \dim G \cdot Z \)) and the last polynomial in this sequence is \( d \) times \([G \cdot Z]\).

In the case that \( \kappa \) is generically finite, the sequence of simple roots can be taken to give a reduced expression for \( w_0w^P_0 \), the product of the long elements of the Weyl groups \( W \) of \( G \) and \( W_P \) of \( P \) respectively. In this case there is an alternate formula

\[
\text{dim } [G \cdot Z] = \sum_{w \in W^P} w \cdot \frac{[Z]}{\prod_{\beta \in \Delta \setminus \Delta_P} \beta}
\]

where \( W^P \) is the set of minimal coset representatives in \( W/W_P \) and \( \Delta \) and \( \Delta_P \) are the sets of roots of \( G \) and \( P \) respectively.

\(^1\)This condition on the weights is not as restrictive as it looks. If \( Z \subseteq Y \) is invariant under rescaling (i.e., is the affine cone over a projective variety), then we can extend the action of \( T \) to \( T \times G_m \) where \( G_m \) acts by dilation, and now all the weights live in \( T^* \times \{1\} \). If \( Z \) is not already rescaling-invariant, we can replace it by the limit subscheme \( Z' := \lim_{t \to 0} t \cdot Z \), and compute the more refined multidegree \([Z'] \in \text{Sym}^*((T \times G_m)^*) \). Afterwards \([Z]\) can be computed as the image of \([Z']\) in \( \text{Sym}^*(T)^* \) (and this image may indeed be zero).
In Section 3 we give an example in which κ does not have connected general fiber.

When κ does have connected fibers and Z and G · Z have rational singularities, we can use the collapsing to compute a more precise invariant than the multidegree, which is the K-polynomial \([G \cdot Z]_Y^κ\). Essentially, this is the numerator of the multigraded Hilbert series of the sheaf \(O_{G,Z}\) on Y; we recall the precise definition in Section 3.

**Theorem 2.** Let \(\kappa : G \times^P Z \to Y\) be a Kempf collapsing whose general fiber is connected (so \(d = 1\) in the notation of Theorem 1), and assume Z and G · Z have rational singularities.

Let \(m_0 = [Z]_Y^κ\), and construct a sequence of Laurent polynomials \(m_1, m_2, \ldots\) by applying Demazure operators \(d_α := (1 - \exp(-α))^{-1}(1 - \exp(-α)r_α)\) to \(m_0\), where α varies over the set of simple roots of G. Don’t bother applying any \(d_α\) that acts as the identity, and only stop when all \(d_α\) act as the identity. The sequence of simple roots can be taken to give a reduced expression for \(w_0w_0^P\). This process terminates after finitely many steps (namely, \(\dim G \times^P Z - \dim G \cdot Z\)).

The last Laurent polynomial in this sequence is the K-polynomial \([G \cdot Z]_Y^κ\). Moreover

\[
[G \cdot Z]_Y^κ = \sum_{w \in \Omega^P} w \cdot \frac{[Z]_Y^κ}{\prod_{\beta \in \Delta \setminus \Delta_\rho}(1 - \exp(-\beta))}.
\]

In the cases where Theorem 2 applies, it implies Theorem 1 by viewing the multidegree as the lowest-order homogeneous component of the K-polynomial.

Kempf worked only with the case that \(Z \subseteq Y\) is a linear subspace (in which case its K-polynomial \([Z]_Y^κ\) is again a very simple product over the weights in \(Y/Z\)), which will also suffice for our main application. It is frequently the case that the weights of T on \(Y\) are distinct, which implies that there are only finitely many \(P\)-invariant linear subspaces \(Z\) on which to apply Kempf’s construction.

1.2. Quiver loci. A quiver \(Q = (Q_0, Q_1)\) is a finite directed graph, which consists of a set \(Q_0\) of vertices and a set \(Q_1\) of directed edges or arrows, such that each arrow \(a \in Q_1\) has a tail \(ta \in Q_0\) and a head \(ha \in Q_0\). A representation \(V\) of \(Q\) is a choice of a vector space \(V(i)\) for each vertex \(i \in Q_0\) and a linear map \(V(a) \in \text{Hom}(V(ta), V(ha))\) for each arrow \(a \in Q_1\). There are obvious notions of isomorphism, direct sum, and indecomposable, for representations of \(Q\). The dimension vector of \(V\) is the map \(Q_0 \to \mathbb{N}\) defined by \(i \mapsto \dim V(i)\). Fix a dimension vector \(d : Q_0 \to \mathbb{N}\) and define

\[
\text{GL} := \text{GL}(Q, d) = \prod_{i \in Q_0} \text{GL}(\mathbb{C}^{d(i)}), \quad \text{Hom} := \text{Hom}(Q, d) = \prod_{a \in Q_1} \text{Hom}(\mathbb{C}^{d(ta)}, \mathbb{C}^{d(ha)}).
\]

A typical element of \(\text{Hom}\) is denoted \(V\), and for \(a \in Q_1\) the \(a\) component is denoted \(V(a)\). The notation comes from thinking of \(V\) as a functor from the free category on \(Q\) to the category \(\text{Vec}\). The group \(\text{GL}\) acts linearly on \(\text{Hom}\) by change of basis: \((g \cdot V)(a) = g(ta)V(a)g(ha)^{-1}\) for all \(g \in \text{GL}, V \in \text{Hom},\) and \(a \in Q_1\). Two points in \(\text{Hom}\) are in the same \(\text{GL}\)-orbit if and only if they define isomorphic representations of \(Q\). The closures of the \(\text{GL}\)-orbits are called quiver loci.\(^2\)

**Theorem.** \(^3\) The action of \(\text{GL}(Q, d)\) on \(\text{Hom}(Q, d)\) has finitely many orbits for all dimension vectors \(d : Q_0 \to \mathbb{N}\), if and only if \(Q\) is a Dynkin quiver, i.e. if the undirected graph underlying \(Q\) is a Dynkin diagram of type \(A_{n \geq 1}, D_{n \geq 4}, E_6, E_7,\) or \(E_8\).

\(^2\)The term “quiver varieties” is already taken, to refer to the hyperkähler quotients \((\text{Hom} \otimes \mathbb{H})//\text{GL}\).
For Q of type A, D, the quiver loci have rational singularities. (To our knowledge the E cases are still open.)

If Q is a Dynkin quiver, each quiver locus Ω ⊆ Hom is the image of a birational linear Kempf collapsing, i.e. there exists a parabolic subgroup P ⊆ GL and a P-invariant linear subspace Z ⊆ Hom such that GL × P Z → GL · Z = Ω is birational.

In Re04, Reineke constructs each Z explicitly using the Auslander-Reiten quiver of Q. We recapitulate this construction precisely in Section 5.

Modulo the construction of a certain ordering, we can state the resulting quiver formulae here. It is well-known [Ga72] that the indecomposable representations of a Dynkin quiver Q are in bijection with the set of positive roots R+ of the root system corresponding to the underlying Dynkin diagram. Fix an ordering R+ = {β1, β2, . . . , βN} of the set of positive roots and write Ij for the (isomorphism class of an) indecomposable representation of Q corresponding to βj. The correspondence Ij ↔ βj is determined as follows: the dimension vector of the indecomposable Ij is given by the expansion βj = ∑i∈Q0 dim Ij(i) αi of the corresponding positive root βj in the basis {αi | i ∈ Q0} of simple roots.

Thus there is a bijection between the GL-orbits in Hom = Hom(Q, d) and the direct sums ⊕N j=1 Ij(mj) where (mj | j = 1, . . . , N) satisfies the obvious dimension condition

\[ \text{for all } i ∈ Q_0, \quad d(i) = ∑_{j=1}^{N} d_j(i), \quad \text{where } d_j(i) := m_j \dim(I_j(i)). \]

Fix such a tuple of multiplicities \( m = (m_j) \) and let Ωm ⊂ Hom be the closure of the corresponding GL-orbit.

Based on m we define a parabolic \( P_m \subset GL \) and a linear subspace Zm ⊂ Hom as follows. For each vertex \( i ∈ Q_0 \) we divide the sets of row and column indices of \( GL(C^{d(i)}) \) into contiguous subsets of sizes \( d_j(i) \) as \( j \) runs from 1 to \( N \). This defines a standard parabolic subgroup \( P_m \subset GL \) whose \( i \)th component (for \( i ∈ Q_0 \)) is the block lower triangular subgroup of \( GL(C^{d(i)}) \) with the given diagonal block sizes.

The decompositions \( d(i) = ∑_{j=1}^{N} d_j(i) \) also induce a block structure on each component Hom(Cd(ta), Cd(ha)) of Hom, whose \((j, j')\) block is a \( d_j(ta) × d_{j'}(ha) \) rectangle. Define the linear subspace Zm ⊂ Hom to be those elements with zeroes in all blocks strictly above the “block diagonal”. This \( Z_m \) is easily seen to be \( P_m \)-invariant. Reineke proves that for certain choices of ordering (built using reduced words for \( w_0 \) adapted to the quiver, as spelled out in Section 5) on \( R_+ \), the Kempf collapsing \( GL × P_m Z_m → Hom \) is birational to \( Ω_m \).

Let \( \{x_k^{(i)} | i ∈ Q_0, k ∈ \{1, . . . , d(i)\}\} \) be a basis for the weight lattice \( T^* \) of the standard maximal torus \( T \) given by the tuples of diagonal matrices in GL. Then the \((k, k')\)th matrix entry in the \( i \)th component of Hom has weight \( x_k^{(ta)} - x_k^{(ha)} \).

**Theorem 3.** Let Q be a Dynkin quiver, and \( \{β1, β2, . . . , βN\} \) a certain order on the set \( R^+ \) of positive roots (constructed explicitly in Section 5). Let Ωm be a quiver locus, with associated multiplicities \( m \), parabolic \( P_m ≤ GL \), and subspace \( Z_m ≤ Hom \).

Then \( [Ωm] \) may be computed by Theorem 1 where \( [Z_m] \) is the product of the weights of all blocks in Hom that are strictly above the “block diagonal”. For types A and D the K-polynomial \( [Ωm]_H^K \) may be computed by Theorem 2, in which \( [Z_m]^K \) is the product of terms of the form \( 1 - e^{-γ} \) where \( γ \) runs over those same weights as in \( [Z_m] \).
Example 1. Let Q be the equioriented $A_n$ quiver:

\[ 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n \]

The simple roots of $A_n$ are given by $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $1 \leq i \leq j \leq n$. Write $I_{ij}$ for the indecomposable representation of $A_n$ corresponding to $\alpha_{ij}$. A suitable ordering on the indecomposables is given by $I_{11}, I_{12}, I_{22}, I_{13}, I_{23}, I_{33}, \ldots, I_{nn}$. Let us consider the specific example $n = 3, d = (2, 3, 2)$, and $\Omega$ given by the GL-orbit closure of $I_{12} \oplus I_{23} \oplus I_{33}$. Geometrically, $\Omega$ is defined by requiring the map $V(2) \rightarrow V(3)$ to have rank $\leq 1$, and the composite map $V(1) \rightarrow V(3)$ to vanish.

The decompositions $d(i) = \sum_{j=1}^{n} d_j(i)$ are

\[ d(1) = 2 + 0 + 0, \quad d(2) = 2 + 1 + 0, \quad d(3) = 0 + 1 + 1, \]

so the parabolic $P_m \subset \text{GL}$ and the linear subspace $Z_m \subset \text{Hom}$ take the following form:

\[ P_m = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ * & * \\ * & * \end{pmatrix} \right\}, \quad Z_m = \left\{ \begin{pmatrix} * & * & 0 \\ 0 & 0 & 0 \\ * & * & * \end{pmatrix} \right\}. \]

By Theorem 1

\[ [Z_m] = (x_1^{(1)} - x_3^{(2)})(x_2^{(1)} - x_3^{(2)}) (x_1^{(1)} - x_1^{(3)})(x_1^{(2)} - x_2^{(3)})(x_2^{(2)} - x_1^{(3)})(x_3^{(2)} - x_2^{(3)}), \]

\[ [\Omega] = \partial_{x_1^{(2)} - x_2^{(3)}} \partial_{x_2^{(2)} - x_3^{(3)}} [Z_m] \]

By Theorem 2

\[ [Z_m]_{\text{Hom}}^K = (1 - e^{-x_1^{(1)} + x_3^{(2)}})(1 - e^{-x_2^{(1)} + x_2^{(3)}})(1 - e^{-x_1^{(2)} + x_3^{(3)}})(1 - e^{-x_2^{(2)} + x_1^{(3)}}), \]

\[ [\Omega]_{\text{Hom}}^K = d_{x_1^{(2)} - x_2^{(3)}} d_{x_2^{(2)} - x_3^{(3)}} d_{x_1^{(3)} - x_2^{(3)}} [Z_m]_{\text{Hom}}^K. \]

We shall work out the multidegree calculation explicitly. Let $a_i = x_1^{(i)}$, $b_i = x_i^{(i)}$, and $c_i = x_i^{(3)}$. We shall use the following properties of $\partial_{\alpha}$: $\partial_{\alpha}(f) = 0$ if $r_{\alpha}(f) = f$, and $\partial_{\alpha}(fg) = \partial_{\alpha}(f)g + r_{\alpha}(f)\partial_{\alpha}(g)$. In particular if $r_{\alpha}(f) = f$ then $\partial_{\alpha}(fg) = f\partial_{\alpha}(g)$. 

\[ 5 \]
Using the notation $\partial_i^a = \partial_{a_i-a_{i+1}}$ (and similarly for $b, c$) we have

$$[\Omega] = \partial_1^b \partial_2^b \partial_3^c (a_1 - b_2) (a_2 - b_3) (b_1 - c_1) (b_2 - c_2) (b_2 - c_2) (b_3 - c_2)$$

$$= \partial_1^b ((a_2 - b_3) (b_1 - c_1) (b_2 - c_2)(b_2 - c_2) + (a_1 - b_2)(b_1 - c_1)(b_2 - c_2)(b_2 - c_2) + (a_1 - b_2)(a_2 - b_2)(b_1 - c_1)(b_2 - c_2) + (a_1 - b_2)(a_2 - b_2)(b_1 - c_1)(b_2 - c_2) + (a_1 - b_2)(a_2 - b_2)(b_1 - c_1)(b_1 - c_2)(b_3 - c_1) = [0] + [(b_1 - c_1)(b_2 - c_2)(b_2 - c_2)] + [(a_1 - b_2)(b_1 - c_1)(b_2 - c_2)(b_2 - c_2)] + [(a_1 - b_1)(b_1 - c_1)(b_1 - c_2)(b_2 - c_2)] + [(a_1 - b_1)(a_2 - b_1)(b_1 - c_1)(b_2 - c_2)(b_3 - c_1) + [(a_1 - b_1)(a_2 - b_1)(b_1 - c_1)(b_2 - c_2)(b_3 - c_1)] + [(a_1 - b_1)(a_2 - b_1)(b_1 - c_2)(b_3 - c_1)] + [(a_1 - b_1)(a_2 - b_1)(b_2 - c_2)(b_3 - c_1)].$$

We check this against the component formula [KMS03. Cor. 6.17], which is a sum over three minimal length lacing diagrams

which give the three tuples of partial permutation matrices

$$\left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \quad \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \quad \left( \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

The formula is then

$$[\Omega] = S_{123}(a; b) S_{3412}(b; c) + S_{132}(a; b) S_{3412}(b; c) + S_{231}(a; b) S_{1342}(b; c) = [(b_1 - c_1)(b_1 - c_2)(b_2 - c_2) + [(a_1 + a_2 - b_1 - b_2)((b_1 - c_1)(b_1 - c_2)(b_2 + b_3 - c_1 - c_2))] + [(a_1 - b_1)(a_2 - b_1)((b_2 - c_1)(b_3 - c_1) + (b_1 - c_2)(b_3 - c_1) + (b_1 - c_2)(b_2 - c_2)])$$

using, say, the pipe dream formula [FK96, KMS03. Thm. 5.3] to evaluate the double Schubert polynomials $S_w(x; y)$.

The multidegrees of quiver loci are particularly important for studying the singularities of composites of differential mappings (see [BF99, FR02, BFR05] and the references therein).

Until now, the only formulae for these multidegrees were in type A. The first such formula was in [BF99], and applied only to the case that the directed arrows are all oriented the same direction. This type A formula has been improved in three ways: it has been made manifestly positive in an appropriate sense, the K-polynomial has been computed...
[KMS03], and the orientation has been generalized [BR04]. Some of these have been combined: the K-polynomial has been computed positively [Bu05, Mi05], and the multidegree has been computed positively for arbitrary orientations [BR04].

Using Theorems 1 and 2 and the rationality of the singularities (from [BZ02]), we give the first formulae for

- the multidegrees of type D and E quiver loci,
- the K-polynomials for type A quiver loci in non-equioriented cases, and
- the K-polynomials for type D quiver loci.

Unfortunately, our formulae are not positive in the senses of [KMS03, Bu05, Mi05]. Some positivity of the answers is expected on very general grounds (e.g. Theorem D in [KM01]).

2. THE BOTT-SAMELSON CRANK

The inductive processes in Theorems 1 and 2 have their geometric origin in the Bott-Samelson crank [BS58]. Fix a Borel subgroup B with $P \geq B \geq T$. For each simple root $\alpha$ of $G$, let $P_\alpha$ be the corresponding minimal parabolic. Then if $f : C \to Y$ is a $B$-equivariant map, the space $P_\alpha \times^B C$ has also a natural $B$-equivariant map to $Y$, which we will call $P_\alpha \times^B f$. We call this functor (on the category of $B$-equivariant maps $f : C \to Y$ to a fixed $G$-space) one turn of the Bott-Samelson crank. By projecting onto the first factor, we see that the resulting space is a $C$-bundle over $P_\alpha/B \cong \mathbb{P}^1$, and in particular $\dim P_\alpha \times^B C = \dim C + 1$. This $C$-bundle is trivial if $f$ is not just $B$- but $P_\alpha$-equivariant, with the projection onto the $C$ factor given by the $B$-quotient of the action map $P_\alpha \times C \to C$; we study this further in Lemma 1 below.

Since we generally turn the crank many times in succession, using a sequence $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$, we will denote products of these functors $P_\alpha \times^B$ by $BS_{\vec{\alpha}} := P_{\alpha_k} \times^B \ldots \times^B P_{\alpha_1} \times^B$. A space $BS_{\vec{\alpha}} \cdot pt$ is a Bott-Samelson manifold. The natural $G$-space for a point to map to $B$-equivariantly is $G/B$, so each Bott-Samelson manifold comes with a Bott-Samelson map to $G/B$.

Seeing a Bott-Samelson manifold as a free quotient by $B$ on the right of $P_{\alpha_k} \times^B \ldots \times^B P_{\alpha_1}$, any Bott-Samelson manifold tautologically carries a principal $B$-bundle. It is sometimes useful to see the space $BS_{\vec{\alpha}}Z$ as the associated $Z$-bundle over the Bott-Samelson manifold $BS_{\vec{\alpha}} \cdot pt$.

**Lemma 1.** Let $G$ act on two varieties $C, Y$ (which need not be linear), and let $f : C \to Y$ be a $G$-equivariant map. Let $\alpha_1, \ldots, \alpha_j$ be a sequence of simple roots.

Then the general fibers of $BS_{\vec{\alpha}}f : BS_{\vec{\alpha}}C \to Y$ have the same number of components as the general fibers of $f$.

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
(BS_{\vec{\alpha}} \cdot pt) \times C & \xrightarrow{\sim} & BS_{\vec{\alpha}}C \\
\downarrow & & \downarrow \\
C & \xrightarrow{f} & Y
\end{array}
\]

The left vertical map is projection onto the second factor, and the right vertical map is $BS_{\vec{\alpha}}f$. If the top map is $([p_k, \ldots, p_1], c) \mapsto [p_k, \ldots, p_1, p_1^{-1} \cdots p_k^{-1}c]$, which is easily seen to be well-defined and an isomorphism, then the diagram commutes.
We can now study the right-hand map $\text{BS}_\alpha f$ by reversing the isomorphism on the top of the diagram. The fibers of the map from the northwest corner to the southeast are just products of the fibers of $f$ with Bott-Samelson manifolds, which are connected.

\textbf{Proposition 1.} Let $G$ act on a scheme $Y$, and $\iota: Z \hookrightarrow Y$ be the inclusion of a $B$-invariant subvariety. (In fact we may as well replace $Y$ by the subvariety $G \cdot Z$.) Let $\mu: G \times^B Z \to G \cdot Z$ be the projective map $[g, z] \mapsto g \cdot z$.

Then there exists a sequence of simple roots $(\alpha_1, \ldots, \alpha_k)$, such that $\text{BS}_\alpha$ is surjective and generically finite, and its degree is the number of components in a general fiber of the map $\mu$.

\textbf{Proof.} We will show there exist two sequences of simple roots $(\alpha_1, \ldots, \alpha_k), (\beta_1, \ldots, \beta_j)$ and a natural commutative diagram

$$
\begin{array}{cccc}
\text{BS}_\alpha Z & \xrightarrow{\text{BS}_\alpha} & G \cdot Z & \xleftarrow{\mu} & G \times^B Z \\
\text{BS}_\beta (G \cdot Z) & \leftarrow & \text{BS}_\beta \text{BS}_\alpha Z \\
\end{array}
$$

in which all maps are onto, the map $\text{BS}_\beta \text{BS}_\alpha Z \to G \times^B Z$ is generically $1:1$, and the map $\text{BS}_\alpha$ is generically finite to one. From this diagram we will derive the conclusions of the proposition.

Let $Z_0 = Z$. Since the subgroups $\{P_\alpha\}$ generate $G$, for each $i$ either $Z_i$ is $G$-invariant or we may pick a simple root $\alpha_i$ such that $Z_i$ is not $P_\alpha$-invariant. Define $Z_i := P_\alpha \cdot Z_{i-1}$.

Since $Z_i$ is the image of the proper map $P_\alpha \times^B Z_{i-1} \to Z_i$, and by inductive assumption $Z_{i-1}$ is closed, reduced, and irreducible, we find $Z_i$ is too. Since $Z_{i-1}$ was not $P_\alpha$-invariant, $Z_i \supset Z_{i-1}$ and $\dim Z_i = \dim Z_{i-1} + 1 = \dim Z + i$. Obviously $G \cdot Z_i = G \cdot Z$, so $Z_i \subseteq G \cdot Z$ is only $G$-invariant if $Z_i = G \cdot Z$. Hence this process stops when $\dim G \cdot Z = \dim Z_k = \dim Z + k$, i.e. $k = \dim G \cdot Z - \dim Z$.

The map $\text{BS}_\alpha: \text{BS}_\alpha Z \to Z_k = G \cdot Z$ is surjective. By dimension count it is generically finite-to-one.

To construct the sequence $(\beta_j)$, consider the $B$-equivariant map $(\text{pt}) \to G/B$ taking a point to the identity coset, and apply $\text{BS}_\beta$ to that. The result $\text{BS}_\beta \cdot \text{pt} \to G/B$ is a Demazure-Hansen resolution $[\text{De74}, \text{Ha73}]$ of a Schubert variety in $G/B$, where the source is a Bott-Samelson manifold. Now select $(\beta_i)$ following the same procedure as was used above, to construct a finite-to-one map $\text{BS}_\beta \text{BS}_\alpha \cdot \text{pt} \to G/B$. In fact the resulting map is generically $1:1$ $[\text{BS82}]$. This obviously extends to a map of $B$-bundles, and our map $\text{BS}_\beta \text{BS}_\alpha Z \to G \times^B Z$ is the corresponding map of associated $Z$-bundles. Consequently it too is generically $1:1$.

To finish setting up the diagram, define

$$
\begin{array}{ll}
\mu: G \times^B Z & \to G \cdot Z \\
[\text{BS}_\beta \text{BS}_\alpha Z] & \to G \times^B Z \\
[\text{BS}_\beta \text{BS}_\alpha Z] & \to \text{BS}_\beta (G \cdot Z) \\
\end{array}
$$

$$
\begin{array}{ll}
[g, z] & \mapsto g \cdot z \\
[g_1, \ldots, g_{j+k}, z] & \mapsto [g_1 g_2 \cdots g_{k+j}, z] \\
[g_1, \ldots, g_{j+k}, z] & \mapsto [g_1 g_2 \cdots g_j, g_{j+1} \cdots g_{j+k} \cdot z] \\
\end{array}
$$

which are all visibly onto and define the commuting square above. It remains to prove our claims about these maps.

Applying Lemma $[\text{I}]$ to the maps $\text{BS}_\alpha: \text{BS}_\alpha Z \to G \cdot Z, \text{BS}_\beta \text{BS}_\alpha: \text{BS}_\beta \text{BS}_\alpha Z \to G \cdot Z$, we see that the general fiber of $\text{BS}_\beta \text{BS}_\alpha Z \to G \cdot Z$ has the same number of
connected components as the general fiber of \( BS_{\bar{\alpha}} : BS_{\bar{\alpha}} Z \to G \cdot Z \), which (since it is generically finite-to-one) is just its degree.

In the case \( Z = \text{pt} \), the following is a standard result about Bott-Samelson manifolds for partial flag manifolds.

**Lemma 2.** Let \( Z \) be a \( B \)-space, and \( \bar{\alpha} \) a list of simple roots whose corresponding reflections \((r_{\alpha_i})\) give a reduced word for the Weyl group element \( w_0w_0^p \) where \( w_0 \) is the long element of \( G \)'s Weyl group and \( w_0^p \) the long element of \( P \)'s. Then the map \( BS_{\bar{\alpha}} : Z \to G \times^P Z \) (constructed by applying \( BS_{\bar{\alpha}} \) to the inclusion \( Z \cong P \times^P Z \mapsto G \times^P Z \) of the fiber over the basepoint) is a birational isomorphism.

**Proof.** These two spaces are \( Z \)-bundles, and the map takes fibers to fibers; as such it is equivalent to check that \( BS_{\bar{\alpha}} : \text{pt} \to G/P \) is a birational isomorphism. Writing this as a composite

\[
BS_{\bar{\alpha}} : \text{pt} \to G/B \to G/P,
\]
the first map is birational, by the assumption of reducedness, to the (opposite) Schubert variety \( \overline{B w_0 B/B} \). The fiber over \( gP \in G/P \) of the second map is \( gP/B = g B w_0^P B/B \). Hence the fiber over \( gP \) of the composite is the intersection

\[
\overline{g B w_0 B/B} \cap \overline{B w_0^P B/B}
\]
which for generic \( g \) is a point, since the \( w_0 \) makes these opposed Schubert varieties.

3. Multidegrees, K-polynomials, and the proofs of Theorems 1 and 2

3.1. Multidegrees and the proof of Theorem 1 Let a torus \( T \) act on a vector space \( Y \). To each \( T \)-invariant subscheme \( Z \subset Y \), we can associate a multidegree \([Z]_Y \) living in the symmetric algebra on the weight lattice \( T^* \) of \( T \), satisfying the following properties:

1. If \( Z = Y = \{0\} \), then \([Z]_Y = 1 \).
2. If as a cycle \( Z = \sum_i m_i Z_i \), where the \( Z_i \) are varieties occurring with multiplicities \( \{m_i\} \), then \([Z]_Y = \sum_i m_i [Z_i]_Y \).
3. If \( H \leq Y \) is a \( T \)-invariant hyperplane, and \( Z \) is a variety, then
   a. if \( Z \not\subset H \), then \([Z]_Y = [Z \cap H]_H \), but
   b. if \( Z \subset H \), then \([Z]_Y = [Z]_H \cdot \text{wt}(Y/H) \), where \( \text{wt}(Y/H) \in T^* \) is the \( T \)-weight on the line \( Y/H \).

The multidegree generalizes the notion of degree of a projective variety \( \mathbb{P}Z \subset \mathbb{P}Y \). If \( T \) is just a circle acting on \( Y \) by rescaling, and \( Z \) is the affine cone (hence \( T \)-invariant) over a projective variety \( \mathbb{P}Z \), then \([Z]_Y = (\deg \mathbb{P}Z) \alpha^\text{codim}_Y Z \) where \( \alpha \) is the generator of \( T^* \). Multidegrees (in \( \text{Sym}(T^*) \)) are a special case of equivariant Chow classes (in \( A_T(Y) \)); since \( Y \) is equivariantly contractible we have \( A_T(Y) = A_T(\text{pt}) \cong \text{Sym}(T^*) \).

It is easy to see that properties (1)-(3) characterize multidegrees. One can show existence in several ways, one being through multigraded Hilbert series, as in the next section. Multidegrees were introduced by [Jo84]. Our reference for them is [MS04].

We only use three results about them. One that follows immediately from the properties above is that for \( Z \leq Y \) a linear subspace, \([Z]_Y \) is the product of the weights in \( Y/Z \). The second is that if all the \( T \)-weights in \( Y \) lie in an open half-space, then \([Z]_Y \neq 0 \) for \( Z \neq 0 \).
Corollary 1. [Jo84, look in BBM] denote the divided difference operator
the degree of the map
\( P \)
\( \text{the normal bundle to the basepoint} \)
\( \prod \)
each
\( wZ \)
By the Weyl-invariance of both sides, the same confirmation holds for the pullback to nonempty. (This follows from Theorem D in [KM01], and is also easily derived from the above properties.) The third is a technical result in equivariant Chow theory:

Lemma 3. Let \( Z \) be a \( P \)-variety and let \( A_T(pt)_{\text{frac}} \) denote the field of fractions of the polynomial ring \( A_T(pt) \). Then we have a formula in the localization \( A_T(G \times^P Z) \otimes_{A_T(pt)} A_T(pt)_{\text{frac}} \) of the equivariant Chow ring \( A_T(G \times^P Z) \):

\[
1 = \sum_{w \in W^P} w \cdot \frac{[Z]_{G \times^P Z}}{\prod_{\beta \in \Delta \setminus \Delta_P} \beta}
\]

where \( [Z]_{G \times^P Z} \in A_T(G \times^P Z) \) is the class induced by the regularly embedded subvariety \( Z \).

Proof. As the map \( G \times^P Z \to G/P \) is \( T \)-equivariant (indeed, \( G \)-equivariant), all the \( T \)-fixed points in \( G \times^P Z \) lie over the \( T \)-fixed points \( \{wP : w \in W^P \} \) in \( G/P \). So we get inclusions \( (G \times^P Z)^T \hookrightarrow \bigcup_{w \in W^P} wZ \hookrightarrow G \times^P Z \).

Then we use the fact, proven in [Br97, section 3.2], that the inclusion of fixed points (the composite of the two above) induces an injective pullback \( A_T(G \times^P Z) \hookrightarrow A_T((G \times^P Z)^T) \). Hence the map \( A_T(G \times^P Z) \hookrightarrow A_T(\bigcup_{w \in W^P} wZ) \equiv \bigoplus_{w \in W^P} A_T(wZ) \) is injective, and to prove the two sides of the formula agree it will suffice to check their images.

Let \( i : Z \hookrightarrow G \times^P Z \) take \( z \mapsto [1, z] \). Then \( i^*[Z]_{G \times^P Z} = i^*1 = \) the equivariant Euler class of the normal bundle of \( Z \) inside \( G \times^P Z \). This normal bundle is the pullback of the normal bundle to the basepoint \( P/P \in G/P \), hence its equivariant Euler class is the product \( \prod_{\beta \in \Delta \setminus \Delta_P} \beta \) of the weights in the tangent space.

Applying \( i^* \) to both sides of the formula, we therefore get \( 1 = \prod_{\beta \in \Delta \setminus \Delta_P} \beta / \prod_{\beta \in \Delta \setminus \Delta_P} \beta \). By the Weyl-invariance of both sides, the same confirmation holds for the pullback to each \( wZ \). Now apply the injectivity above to conclude the formula on \( G \times^P Z \) itself. \( \square \)

This has a well-known corollary due to Joseph:

Corollary 1. [Jo84] look in BBM] Let \( P_\alpha \) act on \( Y \), and \( Z \) be a \( B \)-invariant subscheme. Let \( d \) be the degree of the map \( P_\alpha \times^B Z \to P_\alpha \cdot Z \) unless \( Z \) is \( P_\alpha \)-invariant, in which case let \( d = 0 \). Let \( \partial_\alpha \) denote the divided difference operator \( \frac{1}{\alpha}(1 - r_\alpha) \), acting on \( \text{Sym}^* \bigcirc T^* \). Then

\[
\partial_\alpha [Z]_Y = d [P_\alpha \cdot Z]_Y.
\]

Proof. Let \( L \) denote the Levi factor of \( P_\alpha \) containing \( T \), with semisimple part \( L' \equiv \text{SL}_2 \). Then \( P_\alpha = LB \), so \( P_\alpha \cdot Z = L \cdot Z \). Applying Lemma 3 we learn

\[
\frac{[Z]_{L \times^B Z}}{\alpha} + \frac{r_\alpha \cdot [Z]_{L \times^B Z}}{-\alpha} = 1
\]
as elements of \( A_T(L \times^B L' \times Z) \). Let \( \kappa \) denote the action map \( L \times^B L' \times Z \to Y \), and apply \( \kappa_* \) to both sides:

\[
\frac{[Z]_Y - r_\alpha \cdot [Z]_Y}{\alpha} = \kappa_*(1).
\]

If \( \kappa \) is generically finite of degree \( d \), the right-hand side is \( d [L \cdot Z] \), and otherwise \( 0 \). \( \square \)

(In this corollary we see the reason for \( \text{Lie}(P) \) to contain all the negative root spaces rather than the positive ones; divided difference operators are usually defined for application to Schubert polynomials, which come from Schubert varieties that are \( B_- \)-invariant not \( B \)-invariant.)
**Proof of Theorem** Use Proposition to create a sequence \( (\alpha_i) \). The condition in Proposition on \( (\alpha_i) \) is that \( Z_i \) should grow in dimension at each step, which is the condition that the \( d \) from Corollary is nonzero. By the assumption that all the weights of \( Y \) lie in a half-space, \([P_\alpha \cdot Z]_Y \neq 0\). Hence the dimension grows if and only if \( \partial_\alpha \) does not act as zero. So the conditions on \( (\alpha_i) \) in the theorem’s statement match those used in Proposition.

By Proposition the map \( BS_\alpha Z \to Y \) has image \( G \cdot Z \). The number of components in a general fiber of \( G \times^B Z \to G \cdot Z \) is the degree of the map \( BS_\alpha Z \to G \cdot Z \). That degree is in turn the product of the degrees \( d_i \) of the maps \( P_{\alpha_i} \times^B Z_{i-1} \to Z_i \), since \( BS_\alpha Z \to G \cdot Z \) factors as

\[
\left( \prod_{i=1}^k P_{\alpha_i} \right) Z \mapsto \left( \prod_{i=1}^{k-1} P_{\alpha_i} \right) Z_1 \mapsto \left( \prod_{i=1}^{k-2} P_{\alpha_i} \right) Z_2 \mapsto \cdots \mapsto Z_k = G \cdot Z
\]

where the \( \{Z_i\} \) are as in the proof of Proposition and the \( j \)th map is the associated map of bundles over \( \left( \prod_{i=1}^j P_{\alpha_i} \right) \cdot pt \) to the B-equivariant map \( P_{\alpha_i} \times^B Z_{i-1} \to Z_i \).

Hence by \( k \) applications of Corollary

\[
d \cdot [G \cdot Z]_Y = \left( \prod_{i=1}^k d_i \right) [G \cdot Z]_Y = \left( \prod_{i=1}^k \partial_{\alpha_i} \right) [Z]_Y.
\]

In the case \( \kappa : G \times^P Z \to G \cdot Z \) is generically a finite map, we can use Lemma to know that for \( \bar{\alpha} \) giving a reduced word for \( w_0 \omega_0^p \), the map \( BS_{\bar{\alpha}} : Z \to G \cdot Z \) is also generically finite (with the same degree).

To see the alternate formula, we apply (as in the proof of Joseph’s Lemma) the pushforward \( \kappa_* \) to the equation from Lemma

\[
\kappa_*(1) = \sum_{w \in \mathcal{W}^p} w \cdot \frac{[Z]_Y}{\prod_{\beta \in \Delta \setminus \Delta^p} \beta}.
\]

Since \( \kappa \) is generically finite of degree \( d \), the left-hand side is \( \kappa_*(1) = d \cdot [\text{Im } \kappa]_Y \). □

The first part of this theorem only used Joseph’s Lemma (our Corollary), rather than Lemma directly. This will not be possible in the proof of Theorem where we will use a slightly different approach.

Kempf assumed a condition on \( Z \) that, among other things, forced the general fiber of a collapsing to be connected. While his extremely restrictive condition does not hold in our main application, we will at least have this connectedness, which is not shared by the following example.

**Example 1.** Let \( G = \text{SL}_2(\mathbb{C}) \) act on \( Y = \text{sl}_2(\mathbb{C}) \) via the adjoint action, and let \( Z = b \) be the lower triangular matrices in \( Y \). Let \( T \) be the Cartan subgroup of \( G \) consisting of diagonal matrices, and let \( P = B \) be the lower triangular matrices in \( G \). Then we run into the problem that the weights of \( T \) acting on \( Y \) are \( \alpha, 0, -\alpha \) where \( \alpha \) is the simple root, and do not all lie in a half-space as required to apply the theorem.

To rescue the example, we enlarge \( G \) to \( \text{SL}_2(\mathbb{C}) \times \mathbb{C}^* \), where the latter circle acts by rescaling on \( Y \) and preserves \( Z \). Likewise enlarge \( T \) and \( B \) by this rescaling circle. Now the weights are \( \alpha + a, \alpha, -\alpha + a \) where \( a \) is the generator of the weight lattice of \( \mathbb{C}^* \). Recall
that the multidegree \([Z]\) is the product of the weights not occurring in \(Z\), in this case the one weight \(\alpha + a\).

Then the formula gives \(d [G \cdot Z] = \partial_\alpha (\alpha + a) = 2\). And indeed, \(G \cdot Z = Y\), so \([G \cdot Z] = 1\), while the preimage in \(G \times B Z\) of a typical diagonal matrix \(\text{diag}(t, t^{-1})\) is

\[
\{(g, z) : \text{Ad}(g) \cdot z = \text{diag}(t, t^{-1})\}
\]

which has \(d = 2!\) points, indexed by the permutations of the diagonal entries \(t\) and \(t^{-1}\).

In the very similar example \(G = \text{SL}_3(\mathbb{C})\), with \(Y, Z, P, B, T\) replaced by their \(3 \times 3\) counterparts, the general fiber has \(3!\) points. We have

\[
3! = \partial_\alpha \partial_{\alpha_2} \partial_{\alpha_1}(a + \alpha_1)(a + \alpha_2)(a + \alpha_1 + \alpha_2)
\]

\[
= \sum_{w \in S_3} w \cdot \frac{(a + \alpha_1)(a + \alpha_2)(a + \alpha_1 + \alpha_2)}{\alpha_1 \alpha_2 (\alpha_1 + \alpha_2)}
\]

where \(\alpha_1, \alpha_2\) are the simple roots of \(\text{SL}_3(\mathbb{C})\).

### 3.2. K-polynomials and the proof of Theorem 2

A \(T\)-equivariant coherent sheaf \(F\) on \(Y\) is equivalent to a \(T^*\)-graded module \(\Gamma\) over \(\text{Fun}(Y)\). If we assume that the weights \(\{\lambda_i\}\) of \(T\) on \(Y\) all live in a open half-space of \(T^*\), then each graded piece \(\Gamma_\lambda\) is finite-dimensional, and we can talk about the multigraded Hilbert series \(H(\Gamma; t)\). It is a rational function,

\[
H(\Gamma; t) := \sum_{\lambda \in T^*} \dim(\Gamma_\lambda) \ t^\lambda = \frac{[F]^K_Y}{\prod_{\lambda}(1 - t^{\lambda_1})}
\]

whose numerator one calls the K-polynomial of the sheaf \(F\). If \(Z\) is a subscheme of \(Y\), we will write \([Z]^K_Y\) for the K-polynomial of the structure sheaf of \(Z\). It is a function on \(T\), i.e. an element of the Laurent polynomial ring \(K_T(Y) \cong K_T(pt)\).

We need some results about K-polynomials, corresponding to those we used about multidegrees. The first, easily calculated from the Hilbert series definition, is that the K-polynomial of a linear subspace \(Z \subseteq Y\) is the product \(\prod (1 - t^w)\) where \(w\) varies over the weights of \(Y/Z\). The analogue of Lemma 3 is almost word-for-word the same:

**Lemma 4.** Let \(Z\) be a \(P\)-variety, and let \(K_T(pt)_\text{frac}\) denote the field of fractions of the Laurent polynomial ring \(K_T(pt)\). Then we have a formula in the localization \(K_T(G \times ^P Z) \otimes_{K_T(pt)} K_T(pt)_\text{frac}\) of the equivariant K-ring \(K_T(G \times ^P Z)\):

\[
1 = \sum_{w \in W^p} w \cdot \frac{[Z]_G^K}{\prod_{\beta \in \Delta \setminus \Delta_P}(1 - \exp(-\beta))}
\]

where \([Z]_G^K \in A_T(G \times ^P Z)\) is the class induced by the regularly embedded subvariety \(Z\).

**Proof.** Exactly the same proof holds, except that we need localization in torus-equivariant algebraic K-theory rather than Chow [Th92, Théorème 2.1].

To apply this formula we need to understand the class \(\kappa_i(1) \in K_T(Y)\). The pushforward \(\kappa_i\) in K-theory is defined as the alternating sum of the higher direct images of \(\kappa\), which are difficult to compute in general. An especially easy case is when \(\kappa\) is a birational isomorphism, and both spaces have rational singularities; then

\[
\kappa_* (\mathcal{O}_{G \times ^P Z}) = \mathcal{O}_{G \times Z}, \quad R^i \kappa_* (\mathcal{O}_{G \times ^P Z}) = 0 \quad \forall i > 0
\]
Proof of Theorem 2. Since $\kappa$ has connected fibers, by Proposition 11 the map $BS_{\mathbb{R}} \cdot t$ is a birational isomorphism. Since $Z$ and $G \cdot Z$ have rational singularities, $(BS_{\mathbb{R}} \cdot t)_*|Y = [G \cdot Z]|_Y$ as just explained.

Now we use Lemma 4 to give a formula for $1 \in K_T(BS_{\mathbb{R}} \cdot Z)$, and push it forward using $(BS_{\mathbb{R}} \cdot t)_*$, where $\iota: Z \to Y$ is the inclusion. Unwinding this formula, we get the first formula claimed.

(The reason we didn’t follow the same induction used in the proof of Theorem 1 is that while $Z$ and $G \cdot Z$ have rational singularities, we don’t know that the intermediate spaces constructed in Proposition 1 do (though this seems very likely).)

The proof of the third formula is exactly the same as in Theorem 1 except that we need to invoke rationality of singularities.

Finally, we prove the second formula from the third, using the map $G \times^B Z \to G \times^P Z$. This is a fibration with fibers $P/B$, and the map $\pi: P/B \to pt$ takes $\pi_1(1) = 1$ (the trivial line bundle case of Borel-Weil-Bott). Then we use Lemma 4 to give a formula for $1 \in K_T(G \times^B Z)$, which pushes forward to the desired formula for $[G \cdot Z]|_Y$.

(In $A_T$ rather than $K_T$, the pushforward of 1 along $P/B \to pt$ is zero, which is why there was no analogous formula in Theorem 1.$\square$

4. Quiver Representations

A representation $V$ of a quiver $Q$ is a collection $\{V(i) \mid i \in Q_0\}$ of vector spaces and $\{V(a) \in \text{Hom}_C(V(ta), V(ha)) \mid a \in Q_1\}$ of linear maps. We give the reference [GR92].

4.1. The path algebra $CQ$. A path of length $m > 0$ is a sequence of arrows $p = a_1a_2 \cdots a_m$ such that $ha_i = ta_{i+1}$ for $1 \leq i \leq m - 1$. The tail and head of the path are given by $tp = ta_1$ and $hp = ha_m$ respectively. One should imagine that one starts at the vertex $tp = ta_1$ and walks along the arrow $a_1$ to $ha_1 = ta_2$, then along $a_2$ to $ha_2$, eventually stopping at $ha_m = hp$. For each $i \in Q_0$ there is a path of length zero also denoted $i$, with $hi = ti = i$. If $p$ and $p'$ are paths with $hp = tp'$ then their concatenation $pp'$ is a path. The path algebra $CQ$ of the quiver $Q$ is the associative $C$-algebra with $C$-basis given by the set of paths, and multiplication given by concatenation:

$$p \cdot p' = \begin{cases} pp' & \text{if } hp = tp' \\ 0 & \text{otherwise.} \end{cases}$$

$Q_0$ forms a set of orthogonal idempotents for $CQ$.

4.2. Modules over $CQ$. Let $\text{Mod-}CQ$ be the category of finite-dimensional right $CQ$-modules. The structure of a module $V \in \text{Mod-}CQ$ is determined as follows. From the action of $Q_0$ there is a direct sum decomposition $V \cong \bigoplus_{i \in Q_0} V(i)$ where $V(i) := V \cdot i$. The map $\dim V: Q_0 \to \mathbb{N}$ given by $i \mapsto \dim(V(i))$ is called the dimension vector of $V$. For $i, j \in Q_0$ and $a \in Q_1$ we have $V(i) \cdot a \cdot j = 0$ unless $i = ta$ and $j = ha$. Thus $a$ acts by zero on $V(i)$ for $i \neq ta$ and defines a linear map $V(a) \in \text{Hom}_C(V(ta), V(ha))$. So it is equivalent to work with $CQ$-modules or with representations of $Q$.

Remark 2. We adopt the convention that matrices act on row vectors.
4.3. **Quiver loci and quiver polynomials.** We now change viewpoints, fixing a vector space and the action of the subalgebra \( \mathbb{C}Q_0 \subset \mathbb{C}Q \) on it, but letting the rest of the \( \mathbb{C}Q \)-module structure vary.

Fix a dimension vector \( d : Q_0 \to \mathbb{N} \). Let

\[
\text{Hom} = \text{Hom}(Q, d) = \bigoplus_{a \in Q_1} \text{Hom}_\mathbb{C}(\mathbb{C}^{d(ta)}, \mathbb{C}^{d(ha)})
\]

be the space of all \( \mathbb{C}Q \)-module structures on the vector space \( \bigoplus_{i \in Q_0} \mathbb{C}^{d(i)} \) where \( \mathbb{C}^{d(i)} \) is the image of \( i \in \mathbb{C}Q_0 \). Let \( GL = GL(Q, d) = \prod_{i \in Q_0} GL(d(i), \mathbb{C}) \). The algebraic group \( GL \) acts on \( \text{Hom} \) by change of basis: \( (g \cdot V)(a) = g(ta)V(a)g(ha)^{-1} \) for all \( g \in G, V \in \text{Hom}, \) and \( a \in Q_1 \). It is easy to check that \( V, W \in \text{Hom} \) are isomorphic as elements of \( \text{Mod-} \mathbb{C}Q \) if and only if they are in the same \( GL \)-orbit.

4.4. **Indecomposables and multiplicities.** We want a nice way to index the quiver loci, which are in bijection with the isomorphism classes in \( \text{Mod-} \mathbb{C}Q \). Let \( \text{Indec}_Q \) denote the set of isomorphism classes of indecomposables in \( \text{Mod-} \mathbb{C}Q \). For simplicity of notation, we will sometimes write \( U \) instead of \( [U] \). For \( V \in \text{Mod-} \mathbb{C}Q \) and \( U \in \text{Indec}_Q \), define the multiplicities \( m_U(V) \) of \( V \) by

\[
V \cong \bigoplus_{U \in \text{Indec}_Q} U^{\oplus m_U(V)}.
\]

The multiplicities \( m(V) = (m_U(V) \mid U \in \text{Indec}_Q) \) determine \( V \) up to isomorphism. Let \( \Omega_m := GL \cdot V \) for any \( V \) with multiplicities \( m \). For the equioriented type \( A \) quiver the multiplicities were in \( \text{[KMS03]} \) called the “lace array”.

4.5. **The Auslander-Reiten quiver.** We recall the definition of the Auslander-Reiten quiver \( \Gamma_Q \) associated to the category \( \text{Mod-} \mathbb{C}Q \) \( \text{[ARS95]} \).

A map \( f \) is **irreducible** if for all compositions of maps \( f = gh \) with neither \( g \) nor \( h \) the identity, \( g \) is not a split monomorphism and \( h \) is not a split epimorphism.

The **Auslander-Reiten quiver** \( \Gamma_Q \) of \( Q \) is the directed graph whose vertex set is \( \text{Indec}_Q \) with a directed edge from \( [V] \) to \( [W] \) if and only if there is an irreducible map \( V \to W \).

4.6. **Extensions.** For \( V, W \in \text{Mod-} \mathbb{C}Q \), call \( E \in \text{Mod-} \mathbb{C}Q \) an **extension of** \( V \) **by** \( W \) if there is a short exact sequence \( 0 \to W \to E \to V \to 0 \) of \( \mathbb{C}Q \)-modules. For each \( i \in Q_0 \) choose a basis of \( E(i) \cong W(i) \oplus V(i) \) that consists of a basis of \( W(i) \) followed by a basis of \( V(i) \) and write the linear maps with respect to this basis. With our row-vector conventions of Remark 2, \( E(a) \) has the form

\[
E(a) = \begin{pmatrix} W(a) & 0 \\ * & V(a) \end{pmatrix}.
\]

Let \( E(V, W) \) be the set of extensions of \( V \) by \( W \) with fixed underlying vector space \( V \oplus W \). There is a linear isomorphism

\[
\bigoplus_{a \in Q_1} \text{Hom}_\mathbb{C}(V(ta), W(ha)) \to E(V, W)
\]

whose \( a \)-th component is given by replacing the submatrix * in (3) with the element of \( \text{Hom}_\mathbb{C}(V(ta), W(ha)) \) for \( a \in Q_1 \).
Say that $E, E' \in E(V, W)$ are equivalent if there is a $\mathbb{C}Q$-module isomorphism $E \to E'$ whose restriction to $W$ is the identity and whose induced map $E/W \to E'/W$ is the identity. $\text{Ext}_Q^1(V, W)$ is isomorphic to $E(V, W)$ modulo the above equivalence (see for example [Ro79, Thm. 7.21]).

4.7. The canonical resolution. For $V, W \in \text{Mod-}\mathbb{C}Q$ let $\text{Hom}_Q(V, W)$ be the space of right $\mathbb{C}Q$-module homomorphisms from $V$ to $W$. There is an exact sequence [Ri76]

$$0 \to \text{Hom}_Q(V, W) \xrightarrow{j} \bigoplus_{i \in Q_0} \text{Hom}(V(i), W(i)) \xrightarrow{\oplus_{a \in Q_1} \text{Hom}(V(ta), W(ha))} \bigoplus_{a \in Q_1} \text{Hom}(V(ta), W(ha)) \xrightarrow{p} \text{Ext}_Q^1(V, W) \to 0$$

where $j$ is inclusion, $p$ is induced by the map in (4) and $d_W$ is given by

$$d_W^W(f)_a = V_a f_{ha} - f_{ta} W_a \quad \text{for} \ a \in Q_1.$$

The exactness of (5) gives

$$\dim \text{Hom}_\mathbb{C}(V, W) = \text{rank } d_W^W + \dim \text{Hom}_Q(V, W).$$

4.8. The homological form. Let $V, W \in \text{Mod-}\mathbb{C}Q$. The homological form is defined by

$$\langle V, W \rangle = \sum_{i \geq 0} (-1)^i \dim \text{Ext}_Q^i(V, W).$$

The exact sequence (5) implies that $\text{Mod-}\mathbb{C}Q$ is hereditary (that is, $\text{Ext}_Q^i(V, W) = 0$ for $i \geq 2$) and its exactness gives

$$\langle V, W \rangle = \dim \text{Hom}_Q(V, W) - \dim \text{Ext}_Q^1(V, W)$$

$$= \sum_{i \in Q_0} \dim V(i) \dim W(i) - \sum_{a \in Q_1} \dim V(ta) \dim W(ha)$$

where, for dimension vectors $d, d': Q_0 \to \mathbb{N}$ we write

$$\langle d, d' \rangle = \sum_{i \in Q_0} d(i)d'(i) - \sum_{a \in Q_1} d(ta)d'(ha).$$

4.9. Codimension and Ext. By (7) for $V = W$ and the fact that $\text{Hom}_Q(V, V)$ is the closure of the stabilizer of $V$ in $\text{GL}$, we have

$$\dim \text{GL} - \dim \text{Hom} = \langle V, V \rangle = \dim \text{Hom}_Q(V, V) - \dim \text{Ext}_Q^1(V, V)$$

$$= (\dim \text{GL} - \dim \text{GL} \cdot V) - \dim \text{Ext}_Q^1(V, V).$$

This implies that for $V \in \text{Hom}$, we have

$$\text{codim } \text{GL} \cdot V = \dim \text{Ext}_Q^1(V, V).$$

Let $m$ be a set of multiplicities with $\Omega_m \subset \text{Hom}$. Then

$$\text{codim } \Omega_m = \sum_{U, W \in \text{Indec}_Q} m_U m_W \dim \text{Ext}_Q^1(U, W).$$
5. Quivers of finite type

Let $X_n$ be a simply-laced root system of rank $n$; it is either $A_n$ for $n \geq 1$, $D_n$ for $n \geq 4$, or $E_n$ for $n = 6, 7, 8$, where $n$ is always the number of nodes in the Dynkin diagram. We shall also write $X_n$ for the undirected graph given by its Dynkin diagram.

An orientation of an undirected multigraph is a quiver obtained by choosing directions for the edges of the undirected graph. Orientations of the Dynkin diagrams of simply-laced root systems are called Dynkin quivers.

A quiver $Q$ is of finite type if, for every dimension vector $d : Q_0 \to \mathbb{N}$, there are finitely many isomorphism classes of representations of $Q$ with dimension vector $d$. By Gabriel’s Theorem [Ga72] a quiver is of finite type if and only if it is Dynkin. In this section we shall assume that $Q$ is Dynkin.

5.1. Dimension vectors and roots. We recall some well-known results of Gabriel. There is a bijection from $Q_0$ to the set of simple roots of $X_n$ given by $i \mapsto \alpha_i$. Any dimension vector $d : Q_0 \to \mathbb{N}$ may be viewed as an element of the positive cone of roots $\bigoplus_{i \in Q_0} N\alpha_i$, namely, $\sum_{i \in Q_0} d(i)\alpha_i$. Let $R^+$ be the set of positive roots of $X_n$.

There is a bijection $\text{Indec}_Q \to R^+$ given by $U \mapsto \dim U$. $U$ is indecomposable if and only if $\langle \dim U, \dim U \rangle = 1$.

5.2. Dynkin quivers and orders on $R^+$. Let $s_i$ denote a simple reflection for the Weyl group $W(X_n)$ of $X_n$ and let $w_0 \in W(X_n)$ be the longest element. For $w \in W(X_n)$ let $R(w) \subset Q_0^{(w)}$ denote the set of reduced words for $w$.

Given an orientation $Q$ of $X_n$ and a vertex $i \in Q_0$, let $s_i Q$ be the orientation of $X_n$ given by reversing all arrows with head $i$. Say that a reduced word $a = a_1a_2 \cdots \in R(w_0)$ is adapted to the orientation $Q$ of $X_n$ if $a_j$ is a sink (the tail of no arrow) in $s_{a_{j-1}} \cdots s_{a_2}s_{a_1}Q$ for all $j$. By [BGP73], for every orientation $Q$ of $X_n$, there is a reduced word $a \in R(w_0)$ that is adapted to $Q$.

Each reduced word $a = a_1a_2 \cdots \in R(w_0)$ defines a linear ordering on $R^+$ given by

\[ \gamma_1 < \gamma_2 < \cdots \]

where

\[ \gamma_j = s_{a_1} \cdots s_{a_{j-1}}(\alpha_{a_j}) \]

We use this notation to distinguish the root system of $X_n$ with that of the group $GL$.

\[ \text{Again this notation is to distinguish } s_i \text{ from the reflection } r_i \text{ in the Weyl group of } GL. \]
5.3. **Auslander-Reiten quiver reprise.** There is a combinatorial recipe for the Auslander-Reiten quiver $\Gamma_Q$ of a quiver $Q$ that is an orientation of a Dynkin diagram $X_n$ of type ADE. This is well-known to the experts; see [Be99, Ze02].

The vertices of $\Gamma_Q$ shall be drawn in the plane in rows indexed by the set $Q_0$ and columns indexed by $\mathbb{Z}_{>0}$.

Let $a \in \mathbb{R}(w_0)$ be adapted to $Q$. Let $\gamma_j \in \mathbb{R}^+$ be defined as in (10). Let $c_1 = 1$, and $c_j = c_{j-1}$ unless for some $k < j$ with $c_k = c_{j-1} - 1$ is adjacent in $X_n$ to $\gamma_j$; in this case let $c_j = c_{j-1} + 1$. The vertex $\gamma_j$ is drawn in row $a_j$ and column $c_j$. Draw a directed edge from $\gamma_j$ to $\gamma_k$ if $j < k$, $a_j$ and $a_k$ are adjacent in $X_n$, and $k$ is minimal with this property.

**Example 3.** Let $X_n = D_4$ with orientation $Q$ given below.

```
1 → 2 → 3
  ^   |
  |   |
  4
```

One reduced word adapted to $Q$ is $213423142341$. The corresponding roots have expansions in the simple roots by the following matrix.

$$
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\gamma_5 \\
\gamma_6 \\
\gamma_7 \\
\gamma_8 \\
\gamma_9 \\
\gamma_{10} \\
\gamma_{11} \\
\gamma_{12}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 2 & 1 & 1 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}\cdot
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{pmatrix}
$$

The Auslander-Reiten quiver $\Gamma_Q$ is given by

```
4 → γ4 → γ8 → γ11
  |     |
  |     |
3 → γ3 → γ6 → γ10
  |     |
  |     |
2 → γ1 → γ5 → γ9
  |     |
  |     |
1 → γ2 → γ7 → γ12
```

Since nodes 1, 3, 4 have no connections in $D_4$, the orders they appear in the reduced word $213423142341$ don’t affect the shape of the Auslander-Reiten quiver.

**Remark 4.** For $Q$ an orientation of the Dynkin diagram of a simply-laced root system $X_n$ and $a \in \mathbb{R}(w_0)$ a reduced word adapted to $Q$, let the positive roots (hence the indecomposables) be totally ordered as in (9). Then for $V, W \in \text{Indec}_Q$ we have [Ri84]

$$(11) \quad \text{Ext}^1_{Q}(V, W) = 0 \quad \text{if } V \leq W.$$
5.4. The poset of quiver loci in \( \text{Hom} \).

**Theorem 5.** [Bo96] Let \( Q \) be of finite type and \( V, W \in \text{Mod}-CQ \) with \( \dim V = \dim W \). The following are equivalent:

1. \( GL \cdot V \subseteq GL \cdot W \).
2. \( \dim \text{Hom}_Q(U, V) \leq \dim \text{Hom}_Q(U, W) \) for all \( U \in \text{Indec}_Q \).
3. \( \dim \text{Ext}^1_Q(U, V) \geq \dim \text{Ext}^1_Q(U, W) \) for all \( U \in \text{Indec}_Q \).
4. \( \text{rank } d^V_U \geq \text{rank } d^W_U \) for all \( U \in \text{Indec}_Q \).

Note that the latter three are equivalent for any quiver \( Q \), by (2) and (6).

5.5. The Reineke filtration. We recall a special case of Reineke’s filtration [Re04]. Let \( Q \) be an orientation of a Dynkin diagram \( X_n \) of type ADE, \( a \in R(w_0) \) adapted to \( Q \), with the associated total order \( \leq \) on \( \text{Indec}_Q \). We list the elements of \( \text{Indec}_Q \) in descending order: \( \text{Indec}_Q = \{ \beta_1 > \beta_2 > \cdots > \beta_N \} \) where \( N = |R^+| \); the decreasing indexing is for technical convenience related to our row-vector convention of Remark 2. For short we write \( I_j \) for the indecomposable instead of \( I_{\beta_j} \). Let \( V \in \text{Mod}-CQ \), \( d = \dim V \), \( GL = GL(Q, d) \), \( \text{Hom} = \text{Hom}(Q, d) \). Let \( V \) have multiplicities \( m_1(V) = m_1(V) \) as in (2). For \( 1 \leq j \leq N \) write \( W_j = I_j \oplus V \) and \( V_j = W_1 \oplus \cdots \oplus W_j \). Let \( P \subset GL \) be the parabolic subgroup such that for all \( i \in Q_0 \), the \( i \)-th component \( P(i) \subset GL(C^{d(i)}) \) is the stabilizer of \( V_i \) for all \( 1 \leq j \leq N \). Note that \( P \) has Levi factor \( L = \prod_{i \in Q_0} I_i \otimes GL(W_i(i)) \cong \prod_{i=1}^N GL(W_i) \). Let \( Z, Z' \subset Y := \text{Hom} \) be the coordinate subspaces defined by \( Z(a) = \bigoplus_{j=1}^N \text{Hom}_C(W_j(ta), W_i(ha)) \subset \text{Hom}_C(V(ta), V(ha)) \) and \( Z(a) = \bigoplus_{1 \leq j \leq N} \text{Hom}_C(W_j(ta), W_j(ha)) \). For each \( a \in Q_1 \), \( Z'(a) \) is “block diagonal” and \( Z(a) \) is “block lower triangular” inside the matrices \( \text{Hom}_C(V(ta), V(ha)) \). We claim that

\[
Z = P \cdot V
\]

inside \( \text{Hom} \). By (8) and (11) we have \( \text{codim } \text{Hom}(Q, W_j) \cdot GL(W_j) \cdot W_j = \text{Ext}^1_Q(W_j, W_j) = 0 \), or equivalently, \( \overline{1} \cdot V = Z' \). So it suffices to show

\[
Z = U \cdot Z'
\]

where \( U \) is the unipotent radical of \( P \). But this follows by induction from the definition of \( \text{Ext} \) in Subsection 4.6 combined with the fact that by (11) we have

\[
\text{Ext}^1_Q(W_p, W_q) = 0 \quad \text{for } p < q.
\]

The linear space \( Z \) is the base of our Bott-Samelson induction. Given a quiver locus \( \Omega = \overline{GL \cdot V} \subset \text{Hom} \), we start with \( Z = P \cdot V \subset \text{Hom} \). Then \( \overline{GL \cdot Z} = \Omega \). Since \( Z \) is a coordinate subspace, \( [Z] \in H^*_l(\text{Hom}) \) and \( [O_Z] \in K^*_l(\text{Hom}) \) have simple product formulae. Applying Theorem 1 we obtain divided difference formulae for the multidegree of the quiver locus \( \Omega \). By Theorem 2 for quivers of type AD we obtain divided difference formulae for the \( K \)-polynomial of \( \Omega \).

**Remark.** We use an unnecessarily fine filtration. One may use a directed partition of \( R^+ \) as defined in [Re04] to obtain a coarser filtration of \( V \), which leads to a more efficient divided difference formula.
Example 2. Let \( Q \) be the type \( A_2 \) quiver with dimension vector \((m, n)\). Then \( \text{Hom}(Q, d) = M_{m \times n}(\mathbb{C}) \). For each \( 0 \leq r \leq \min(m, n) \) there is a quiver locus \( \Omega_r \subseteq M_{m \times n}(\mathbb{C}) \) given by the determinantal variety of matrices of rank at most \( r \). Using the reduced word \( s_2s_1s_2 \) we have \( W_1 = I_{\alpha_1}^{\oplus(m-r)}, \quad W_2 = I_{\alpha_1 + \alpha_2}^{\oplus r}, \quad \text{and} \quad W_3 = I_{\alpha_2}^{\oplus(n-r)} \). The indecomposables can be realized by matrices as follows: \( I_{\alpha_1} \) is a \( 1 \times 0 \) matrix, \( I_{\alpha_1 + \alpha_2} \) can be taken to be the \( 1 \times 1 \) identity matrix, and \( I_{\alpha_2} \) is the \( 0 \times 1 \) matrix. With respect to bases adapted to the ordered direct sum \( V = W_1 \oplus W_2 \oplus W_3, \quad V \subseteq M_{m \times n}(\mathbb{C}) \) has the \( r \times r \) identity matrix in its lower left corner and zeroes elsewhere. Then \( P(1) \subseteq \text{GL}(m) \) and \( P(2) \subseteq \text{GL}(n) \) are block lower triangular with diagonal blocks of sizes \((m-r, r)\) and \((r, n-r)\) respectively. We have \( Z = Z' \); both are equal to the linear subspace of \( M_{m \times n} \) where the bottom left \( r \times r \) submatrix is arbitrary and the other entries are zero. Let \( T(m) \subseteq \text{GL}(m) \) and \( T(n) \subseteq \text{GL}(n) \) have weights \( X = (x_1, \ldots, x_m) \) and \( Y = (y_1, \ldots, y_n) \) respectively. Since the parabolics \( P(1) \) and \( P(2) \) are lower triangular, the positive roots of \( P(1) \) have weights \( x_i - x_j \) for \( 1 \leq i < j \leq m \) and those of \( P(2) \) have weight \( y_i - y_j \) for \( 1 \leq i < j \leq n \).

So for \((m, n) = (2, 3)\) and \( r = 1 \) we have

\[
\begin{align*}
[Z] &= (x_1 - y_1)(x_1 - y_2)(x_1 - y_3)(x_2 - y_2)(x_2 - y_3) \\
[\Omega] &= \partial_{x_1 - x_2} \partial_{y_1 - y_2} \partial_{y_2 - y_3} [Z] \\
    &= s_2 [X - Y],
\end{align*}
\]

the double Schur polynomial. In general the multidegree is given by the Giambelli-Thom-Porteous formula \( [\Omega_r] = s_{(m-r) \times (n-r)} [X - Y] \), where the answer is the double Schur polynomial indexed by the \((m-r) \times (n-r)\) rectangle.

6. Beyond ADE Quivers

Let \( Q \) be a quiver, \( d : Q_0 \to \mathbb{N} \) a dimension vector, and \( \text{Hom} \) the associated space of representations. Then as long as \( Q \) has no self-loops (\( ta = ha \) for some edge \( a \)), and no repeated edges (\( ta = tb, \quad ha = hb \) for two edges \( a \neq b \)), the weights of \( T \) on \( \text{Hom} \) are all distinct.

Consequently, there are only finitely many \( T \)-invariant subspaces in \( \text{Hom} \) (precisely \( 2^{\dim \text{Hom}} \)), and hence only finitely many \( B \)-invariant subspaces \( Z \) to which to apply Kempf’s construction. Whereas there may be infinitely many quiver loci. This (and the fact that quiver loci can have bad singularities [Zw02, section 6]) suggests that instead of quiver loci, perhaps the better-behaved objects of study are the GL-sweeps of the \( B \)-invariant subspaces. From this point of view it is merely an accident (and Reineke’s theorem) that in the ADE case, the two notions coincide.

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