WANDERING FATOU COMPONENTS ON $p$-ADIC POLYNOMIAL DYNAMICS

Gabriela Fernández Lamilla

Abstract

We will study perturbations of the polynomials $P_\lambda$, of the form

$$Q_\lambda = P_\lambda + Q$$

in the space of centered monic polynomials, where $P_\lambda$ is the polynomial family defined by

$$P_\lambda(z) = \frac{\lambda}{p} z^p + \left(1 - \frac{\lambda}{p}\right) z^{p+1}$$

with $\lambda \in \Lambda = \{ z : |z - 1| < 1 \}$, studied by Benedetto, who showed that for a dense set of parameters, the polynomials $P_\lambda$ have a wandering disc contained in the filled Julia set.

We will show an analogous result for the family $Q_\lambda$, obtaining the following consequence:

The polynomials $P_\lambda$ belong to $E_{p+1}$ where $E_{p+1}$ denotes the set of polynomials that have a wandering disc in the filled Julia set, in the space of centered monic polynomials of degree $p + 1$. 
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1 Introduction

In complex dynamics there exists an extensive study of polynomials as dynamical systems acting on \( \mathbb{C} \). The orbit of a point \( z_0 \in \mathbb{C} \) under a polynomial \( f \in \mathbb{C}[z] \) is the sequence \( z_0, z_1, z_2, \ldots \) defined by \( z_n = f^n(z_0) \).

Subsets of \( \mathbb{C} \) which are of particular interest are the filled Julia set, which is the set of points with bounded orbit; the Julia set, which corresponds to the boundary of the filled Julia set; and the Fatou set, the complement of the Julia set.

A important result of Sullivan [10] says that for all polynomials \( f \in \mathbb{C}[z] \) there is no wandering component of the Fatou set, i.e. every connected component of the Fatou set is pre-periodic under the action of \( f \) (This result holds also for rational functions but our emphasis will be on polynomials).

Recently the study of iterations of rational functions over \( \mathbb{C} \) has been extended to the study of rational functions with coefficient in the field \( \mathbb{C}_p \) ([1], [3], [8], [9]). This field is the smallest complete algebraically closed extension of \( \mathbb{Q} \) with respect to the \( p \)-adic valuation. The construction of \( \mathbb{C}_p \) is analogous to that of the complex numbers starting with rational numbers and the usual absolute value, some interesting differences arise between \( \mathbb{C} \) and \( \mathbb{C}_p \).

The field \( \mathbb{C}_p \), endowed with the \( p \)-adic valuation, is an ultrametric space, i.e. for all \( x, y \in \mathbb{C}_p \)

\[
|x + y| \leq \max\{|x|, |y|\}.
\]

From the above inequality, known as the strong triangle inequality, it follows that \( \mathbb{C}_p \) is totally disconnected, so the connected component notion used in complex dynamics must be replaced by the concept of infraconnected component (see [5]).

The motivation of this work arises from a result of Benedetto [1], who studied the family of polynomials in \( \mathbb{C}_p[z] \) defined by

\[
P_\lambda(z) = \frac{\lambda}{p} z^p + \left(1 - \frac{\lambda}{p}\right) z^{p+1},
\]

where \( \lambda \in \Lambda = \{ \lambda \in \mathbb{C}_p : |\lambda - 1|_p < 1 \} \), obtaining the following result:

**Theorem (Benedetto).** There is a dense set of parameters \( \lambda \in \Lambda \) such that the polynomial \( P_\lambda \) has a wandering disc contained in the filled Julia set which is not attracted to an attracting cycle.

From the above theorem we conclude that there exist polynomials in \( \mathbb{C}_p[z] \) with wandering infraconnected components of the Fatou set, in contrast with the result of Sullivan for complex rational functions.

In this work, we will study perturbations of the polynomials \( P_\lambda \), of the form

\[
Q_\lambda = P_\lambda + Q,
\]

where \( Q \) is a polynomial with \( ||Q|| \leq \left(\frac{1}{p}\right)^{\frac{1}{p-1}} \) and we will obtain the following result (for the definition of \( ||Q|| \) see Section 2.1).
**Theorem.** There is a dense set of parameters $\lambda \in \Lambda$ such that $Q_\lambda$ has a wandering disc contained in the filled Julia set which is not attracted to an attracting cycle.

Let $Pol_d$ be the space of monic centered polynomials of degree $d \geq 2$ and coefficients in $\mathbb{C}_p$. The parameter space $Pol_d$ is naturally identified with $\mathbb{C}_p^{d-1}$. If we call $E_d$ the set of polynomials in $Pol_d$ that have a wandering disc, from the above theorem, we obtain directly the following consequence.

**Corollary.** For all $\lambda \in \Lambda$, the polynomial $P_\lambda$ belong to the interior of $E_{p+1}$.

In addition, we will prove that the above theorem is also true for a wider class of perturbations of the polynomials $P_\lambda$. In fact, if we consider $R_B$ the set of rational functions without poles in the fixed ball $B = \{z : |z| \leq r\}$ $(r > 1)$ and the subset $R_E^B$ of functions in $R_B$ with a wandering disc, we obtain the following consequence.

**Corollary.** For all $\lambda \in \Lambda$, the polynomial $P_\lambda$ belong to the interior of $R_E^B$.

In sections 2.1 and 2.2 we recall some basic concepts and facts from ultrametric analysis and dynamics. In Section 2.3 we will present in detail some results and techniques used in [7] since they are essential for our study of the perturbation $Q_\lambda$. Our study is not done directly on $Q_\lambda$, but it is more convenient to work with an affinely conjugated map $Q_\lambda^*$. In Section 3.1 we study the behavior of $Q_\lambda^*$ over the filled Julia set. Finally, in Section 3.2 we mimic [1] to study $Q_\lambda^*$ as a function of the parameter $\lambda$ and prove our main result.
2 Preliminaries

In this section we recall definitions and results that are used throughout this work.

The field $\mathbb{C}_p$ endowed with the $p$-adic valuation denoted by $| \cdot |$ is an ultrametric space, i.e. for $z_0, z_1 \in \mathbb{C}_p$ we have that $|z_0 - z_1| \leq \max\{|z_0|, |z_1|\}$. From this inequality and the completeness of $\mathbb{C}_p$ arise interesting topological and geometrical results, some of them are:

(i) The value group of the valuation is the set $|\mathbb{C}_p^*| := \{|z| : z \in \mathbb{C}_p^*\} = \{p^r : r \in \mathbb{Q}\}$.

(ii) The isosceles triangle principle: If $|z_1| \neq |z_2|$, then $|z_1 + z_2| = \max\{|z_1|, |z_2|\}$.

(iii) For $z_0 \in \mathbb{C}_p$, $r \in |\mathbb{C}_p^*|$, the open ball with radius $r$ and center $z_0$ is the set

$$B_r(z_0) = \{z \in \mathbb{C}_p : |z - z_0| < r\}$$

and the closed ball with center $z_0$ and radius $r$ is the set

$$\overline{B}_r(z_0) = \{z \in \mathbb{C}_p : |z - z_0| \leq r\}.$$ 

These are open and closed sets in the topology of $\mathbb{C}_p$. By definition, we have that the diameter of $B$ denoted $\text{diam}(B)$ belong to $|\mathbb{C}_p^*|$, where $B$ is an open or closed ball.

We denote by $O_{\mathbb{C}_p}$ the closed ball $\{z : |z| \leq 1\}$.

(iv) Every point of a ball is a center, that is, if $z_1 \in B_r(z_0)$ (resp. $z_1 \in \overline{B}_r(z_0)$), then $B_r(z_1) = B_r(z_0)$ (resp. $\overline{B}_r(z_1) = \overline{B}_r(z_0)$).

(v) If two balls have not empty intersection then one is contained in the other one.

The properties below are about convergence in ultrametric spaces, some of them are different than in archimedean analysis. Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence in $\mathbb{C}_p$. Then

(vi) If $\lim_{n \to \infty} a_n = a$ and $a \neq 0$, then there exists a $n_0$ such that $|a_n| = |a|$ for all $n > n_0$.

(vii) The series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n \to \infty} a_n = 0$.

(viii) The power series $\sum_{n=1}^{\infty} a_n x^n$ has convergence radius $r := \left(\limsup_{n \to \infty} \sqrt[n]{|a_n|}\right)^{-1}$.

(ix) If $r$ is the convergence radius of $\sum_{n=1}^{\infty} a_n x^n$, then the map

$$x \mapsto \sum_{n=1}^{\infty} a_n x^n$$

is differentiable in $B_r(0)$ and its derivative is

$$x \mapsto \sum_{n=1}^{\infty} n a_n x^{n-1}.$$
2.1 Ultrametric analysis.

Let $B$ be a ball with radius $r$ (open or closed), we denote by $\mathcal{H}(B)$ the ring of power series which converge in $B$. The space $\mathcal{H}(B)$ endowed with the norm

$$\|f\|_B = \sup_{i \geq 0} |a_i| r^i$$

is a complete ultrametric valued ring.

As in the complex case, a rational function can be written as a power series around every point $z_0$ which is not a pole. But observe that if we consider the function $f : C_p \to C_p$ given by

$$f(z) = \begin{cases} 1, & |z| < 1 \\ 0, & |z| \geq 1 \end{cases}$$

we have that also can be written as power series around every point of $C_p$. Hence, it is clear that the idea of holomorphic functions in $C_p$ is different from the one in $C$, which it is defined by a local property. Indeed, a function defined in a subset $X$ of $C_p$ is holomorphic if it is the uniform limit of rational functions without poles in $X$ (see [3]). In this work, we only consider holomorphic functions defined on a ball $B$ and, in this case, the definition coincides with the complex one: a function $f$ is holomorphic in $B$ if and only if $f$ can be written as a convergent power series in $B$. Thus, $\mathcal{H}(B)$ is the space of holomorphic functions in the disc $B$.

Now, we will show an analogous to the Newton’s method, in order to guarantee the existence of roots in a holomorphic function.

**Lemma 2.1 (Hensel).** Let $f \in O_{C_p}[[z]]$. If there exists $z_0 \in O_{C_p}$ with $|f(z_0)| < |f'(z_0)|^2$, then there is an unique root $w$ of $f$ such that $|w - z_0| \leq \frac{|f(z_0)|}{|f'(z_0)|}$.

**Proof:**

We define recursively the sequence,

$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)}, \quad z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

We will show inductively that:

(i) $|f(z_n)| \leq C^n |f'(z_0)|^2$, where $C = \frac{|f(z_0)|}{|f'(z_0)|^2} < 1$.

(ii) $|f'(z_n)| = |f'(z_0)|$.

(iii) $|z_n - z_0| = |z_1 - z_0|$.

Now

$$f(z_1) = f(z_0 + z_1 - z_0) = f(z_0) + f'(z_0)(z_1 - z_0) + d(z_1 - z_0)^2 = d(z_1 - z_0)^2$$

for some $d = d(z_0, z_1) \in O_{C_p}$. Therefore
\[ |f(z_1)| = |d(z_1 - z_0)^2| \leq |z_1 - z_0|^2 = C^2|f'(z_0)|^2. \]

and

\[ f'(z_1) = f'(z_0) + e(z_1 - z_0) \]

for some \( e = e(z_0, z_1) \in \mathcal{O}_\mathbb{C}_p \). Hence

\[ |f'(z_1) - f'(z_0)| \leq |z_1 - z_0| \leq \frac{|f(z_0)|}{f'(z_0)} < |f'(z_0)|. \]

From the previous inequality, and the isosceles triangle principle applied to \(|f'(z_1) - f'(z_0) + f'(z_0)|\) we have that

\[ |f'(z_1)| = |f'(z_0)|. \]

The inductive steps for (i) and (ii) are analogous to the previous one, (iii) is direct consequence of (i) and (ii).

Since \(|f'(z_n)| = |f'(z_0)|\), we have that \(|z_{n+1} - z_n| = \frac{|f(z_n)|}{|f'(z_n)|} \leq C^{2n}|f'(z_0)|\), so \( \{z_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence and if we denote its limit by \( w \) we get, from (i) and (iii), that \( w \) is a root of \( f \) and \(|w - z_0| \leq \frac{|f(z_0)|}{|f'(z_0)|} \).

We now enumerate some interesting properties of holomorphic functions (see [2]).

**Theorem 2.2.** Let \( f(z) = \sum_{i=0}^{\infty} a_i z^i \) with \( a_i \in \mathbb{C}_p \) and \( r \in |\mathbb{C}_p| \) such that \( \lim_{i \to \infty} |a_i|^r = 0 \). Then \( f \) has a root \( \alpha \in \mathbb{C}_p \) with \( |\alpha| = r \) if and only if there exist \( n, m \in \mathbb{Z} \) with \( n < m \) and such that

\[ |a_n|r^n = |a_m|r^m = \sup_{i \geq 0} |a_i|r^i. \] (1)

Moreover if \( n, m \) are the smallest and the greatest integers, respectively, that make \( (1) \) true, then \( f \) has exactly \( m - n \) roots with absolute value \( r \), counting multiplicity.

**Corollary 2.3.** Let \( B \) be a closed ball, \( f \in H(B) \), and \( D \) an open ball (resp. closed) contained in \( B \). Then \( f(D) \) is an open ball (resp. closed).

**Corollary 2.4.** Let \( f \in H(B) \), \( w_0 \in \mathbb{C}_p \), \( \delta \in |\mathbb{C}_p| \) such that \( \overline{B}_\delta(w_0) \subset B \). If \( |f(w) - f(w_0)| = \alpha \) for all \( w \) in \( \{ w : |w - w_0| = \delta \} \), then

\[ f(\{ w : |w - w_0| = \delta \}) = \{ w : |w - f(w_0)| = \alpha \}. \]

**Corollary 2.5.** Let \( B \) be a closed ball and \( f \in H(B) \). Then there exists \( d \in \mathbb{N} \) such that the series \( f - w \) has exactly \( d \) roots in \( B \), counting multiplicity, for all \( w \in f(B) \).

For \( f \in H(B) \), we define the **degree of the map** \( f \) as the number \( d \) from the last corollary.
2.2 Polynomial dynamics over $\mathbb{C}_p$.

This section contains some important definitions and dynamical properties that will be needed later.

Let $P \in \mathbb{C}_p[z]$. For $z \in \mathbb{C}_p$, we define the orbit of $z$, denoted by $O(z)$, as the sequence $\{P^n(z)\}_{n \in \mathbb{N}}$. If $P(z) = z$, we say that $z$ is a fixed point of $P$; if for $z$ there exists $n \in \mathbb{N}$ such that $P^n(z) = z$, we will say $z$ is a periodic point.

Let $P'$ be the formal derivative of $P$, $z_0$ a fixed point of $P$ and $\theta = |P'(z_0)|$. Then:

(i) If $\theta < 1$, we say that $z_0$ is an attracting fixed point.
(ii) If $\theta > 1$, we say that $z_0$ is a repelling fixed point.
(iii) If $\theta = 1$, we say that $z_0$ is an indifferent fixed point.

Other object of study is the filled Julia set denoted by $K(P)$, that correspond to the set of points of $\mathbb{C}_p$ with bounded orbit. Some properties of the filled Julia set are:

(i') $K(P) \neq \emptyset$.
(ii') $K(P)$ is closed and bounded.
(iii') $P^{-1}(K(P)) = K(P)$, i.e. $K(P)$ is completely invariant.

Another important set is the Julia set, $J(P)$, which is the boundary of the filled Julia set. The Julia set can be also defined as follows

\[ \{ z \in \mathbb{C}_p : \text{ for every neighbourhood } U \text{ of } z, \bigcup_{n \in \mathbb{N}} P^n(U) = \mathbb{C}_p \}. \]

Finally, we define the Fatou set as the complement of the Julia set. We denote it by $F(P)$.

We are not going to study just polynomials, so we have to introduce the concept of polynomial like maps. If $U$ and $V$ are open balls in $\mathbb{C}_p$ such that $U \subset V$ and $f : U \to V$ is a holomorphic function of degree $d$ with $d \geq 1$, we say that $(f, U)$ is a polynomial like map of degree $d$.

All the preceding concepts can be also defined, in a similar way, for polynomial like maps. That is, the filled Julia set of $(f, U)$ is the set

\[ K(f, U) = \{ z \in U : f^n(z) \in U \text{ for all } n \in \mathbb{N} \}. \]

The Julia set is

\[ J(f, U) = \partial K(f, U), \]

and the Fatou set is

\[ F(f, U) = U \setminus J(f, U). \]

These new definitions will allow us to study the dynamical behavior of some holomorphic functions restricted to balls.
Let $D$ be a subset of $\mathbb{C}_p$, $a \in D$ and $I_a : \mathbb{C}_p \to \mathbb{R}$ the map defined by $I_a(x) = |x - a|$. We say that $D$ is **infraconnected** if and only if for all $a \in \mathbb{C}_p$ the set $\overline{I_a(D)}$ is an interval (see [5]). In particular, for $(f,U)$, we are interested in understanding the behavior of the filled Julia set. If we consider $B$, the smallest ball that contains $K(f,U)$, then $f^{-n}(B)$ is a collection of disjoint closed balls, named balls of level $n$. Then for $w \in K(f,U)$ there is an unique sequence $\{B_n\}_{n \in \mathbb{N}}$ of nested closed balls, where $B_n$ is a ball of level $n$, such that $w \in \bigcap_{n \in \mathbb{N}} B_n$. The set $C(w) := \bigcap_{n \in \mathbb{N}} B_n$ is the infraconnected component of $K(f,U)$ that contains $w$ (see [4]).

Now let $(f,U)$ be a polynomial like map. We say that $E \subseteq U$ is a **wandering set** if $f^n(E) \cap f^m(E) \neq \emptyset$ only when $n = m$.

Furthermore, if $(f,U)$ and $(g,U)$ are polynomial like maps, if there is a homomorphism $h : U \to U$ such that $g = h^{-1}fh$, we say that $f$ and $g$ are **topologically conjugated**. In these case, if $E$ is a wandering set of $f$, then $h(E)$ is a wandering set of $g$. Therefore, the existence of wandering set is invariant under conjugacy. This fact will turn out to be very important to obtain our results.
2.3 The family of polynomials $P_\lambda$.

For $\lambda \in \Lambda = \{ \lambda \in \mathbb{C}_p : |\lambda - 1| < 1 \}$ let

\[ P_\lambda(z) = \frac{\lambda}{p} z^p + \left(1 - \frac{\lambda}{p}\right) z^{p+1}. \]

**Theorem 2.6 (Benedetto).** There is a dense set of parameters $\lambda \in \Lambda$, such that the polynomial $P_\lambda$ has a wandering disc contained in $K(P_\lambda)$, which is not attracted to an attracting cycle.

Now we will sketch the proof of this theorem (see [7]), paying attention to the techniques which will be important later.

First we notice that $B_{\rho}(0)$, with $\rho = \frac{p - 1}{p - 1}$, is invariant under the action of $P_\lambda$ and that $z = 1$ is a repelling fixed point. Now, we considere $B_1(0)$ and $B_1(1)$, which are neighbourhoods of the fixed ball $B_{\rho}(0)$ and the repelling fixed point $z = 1$ respectively. From the strong triangle inequality and Corollary 2.3, we see that the set $K(P_\lambda)$ is contained in $B_1(0) \cup B_1(1)$. This allows us to define the itinerary of a point $x \in K(P_\lambda)$ as the sequence

$$ \theta_1 \theta_2 \ldots \theta_n \ldots $$

with $\theta_i \in \{0, 1\}$ and $P^i_\lambda(x) \in B_1(\theta_i)$ for $i \in \mathbb{N}$. Furthermore, we obtain that all the points of a ball contained in the filled Julia set have the same itinerary. If this itinerary is not pre-periodic, then $D$ is a wandering disc. In order to find such disc is necessary to study the behavior of the $P_\lambda$ in the filled Julia set. The lemmas below describe such behavior.

We define $S > 0$ by $p^{Sp-1} = \rho$ and the sequence $\{\rho_n\}_{n \in \mathbb{N}}$ by

$$ \rho_0 = 1, \quad pp^n = \rho_{n+1}. $$

**Lemma 2.7.**

1) \text{ Let } m \geq 1, z_0 \text{ and } z_1 \text{ such that } |z_0| = |z_1| = \rho_m. \text{ If } |z_0 - z_1| \leq S, \text{ then: } |P_\lambda(z_0) - P_\lambda(z_1)| \leq \rho_{m-1}|z_0 - z_1|.

2) \text{ If } z_0, z_1 \in B_1(1), \text{ then: } |P_\lambda(z_0) - P_\lambda(z_1)| = p |z_0 - z_1|.

**Proof:**

1) We observe that

\[ P_\lambda(z_0) - P_\lambda(z_1) = \frac{\lambda}{p} \left(p \varepsilon z_0^{p-1} + \cdots + p \varepsilon^{p-1} z_0 + \varepsilon^p\right) + \left(1 - \frac{\lambda}{p}\right) \left((p+1) \varepsilon z_0^{p+1} + \cdots + \varepsilon^{p+1}\right), \]

with $\varepsilon = z_1 - z_0$, since $|\varepsilon| = |z_0 - z_1| \leq S < \rho_m$ and

\[ |P_\lambda(z_0) - P_\lambda(z_1)| \leq |\varepsilon| \max\{|z_0^{p-1}, p |\varepsilon^{p-1}, \rho_{m-1}|, \rho_{m-1}| \} = |\varepsilon| \rho_{m-1}, \]

we have that

\[ |P_\lambda(z_0) - P_\lambda(z_1)| \leq \rho_{m-1} |z_0 - z_1|. \]
2) The proof is straightforward from the previous one and will be omitted. \(\square\)

With the previous lemma it is possible to find a necessary and sufficient condition for the existence of wandering discs in \(K(P_{\lambda})\), this condition is:

**Lemma 2.8.** Let \(\{m_i\}_{i \geq 0}, \{M_i\}_{i \geq 0}\) be two sequences of positive integers such that, for all \(i \geq 0\) we have that \(\rho_{m_i-1} \cdot \ldots \cdot \rho_1 \cdot p_{M_i} \leq 1\). Suppose that for \(\lambda_0 \in \Lambda\) there exists \(x \in K(P_{\lambda_0})\) with itinerary
\[
\{0 \ldots 0, 1 \ldots 1, 0 \ldots 0, 1 \ldots 1, \ldots\},
\]
then the ball \(U = \{z : |z - x| \leq S\}\) is contained in \(K(P_{\lambda_0})\).

Therefore, to prove Theorem 2.6 it suffices to find \(x \in K(P_{\lambda})\) and sequences \(\{M_i\}_{i \in \mathbb{N}}, \{m_i\}_{i \in \mathbb{N}}\) with \(\lim M_i = \infty\) such that the hypothesis of the previous lemma are satisfied. In order to do this we study the function \(P_{\lambda}(z)\) as a function of \(\lambda\).

Now, we will see two lemmas that will allow us to find such sequences \(\{M_i\}_{i \in \mathbb{N}}, \{m_i\}_{i \in \mathbb{N}}\) with \(\lim M_i = \infty\) implying the existence of wandering discs in \(K(P_{\lambda})\).

**Lemma 2.9.** Let \(m \geq 1\), and \(z_0, z_1 \in B_{p-m}(1)\). If \(\lambda_0, \lambda_1 \in \Lambda\) satisfy \(|z_0 - z_1| = |\lambda_0 - \lambda_1|\), then
\[
|P_{\lambda_0}^m(z_0) - P_{\lambda_1}^m(z_1)| = p^m|\lambda_0 - \lambda_1|.
\]

**Proof:** We proceed by induction. From
\[
|P_{\lambda_0}(z_0) - P_{\lambda_0}(z_1)| = p|z_0 - z_1| = p|\lambda_0 - \lambda_1|, \tag{2}
\]
\[
|P_{\lambda_0}(z_1) - P_{\lambda_1}(z_1)| = p|\lambda_0 - \lambda_1||z_1 - 1| < p|\lambda_0 - \lambda_1|, \tag{3}
\]
we have that \(|P_{\lambda_0}(z_0) - P_{\lambda_1}(z_1)| = p|\lambda_0 - \lambda_1|\), therefore the lemma is true for \(m = 1\).

Now, for the inductive step, we suppose that \(z_0, z_1 \in \{z : |z - 1| \leq p^{-m}\}\). By hypothesis we have that
\[
|P_{\lambda_0}^{m-1}(z_0) - P_{\lambda_1}^{m-1}(z_1)| = p^{m-1}|\lambda_0 - \lambda_1|.
\]
\[
|P_{\lambda_1}^{m-1}(z_1) - 1| < p^{-m}.
\]

Therefore,
\[
|P_{\lambda_0}(P_{\lambda_0}^{m-1}(z_0)) - P_{\lambda_0}(P_{\lambda_1}^{m-1}(z_1))| = p^m|\lambda_0 - \lambda_1|, \tag{4}
\]
\[
|P_{\lambda_0}(P_{\lambda_1}^{m-1}(z_1)) - P_{\lambda_1}(P_{\lambda_1}^{m-1}(z_1))| = p|\lambda_0 - \lambda_1||P_{\lambda_1}^{m-1}(z_1) - 1| < p^m|\lambda_0 - \lambda_1|. \tag{5}
\]

From (4) and (5) we obtain
\[
|P_{\lambda_0}^m(z_0) - P_{\lambda_1}^m(z_1)| = p^m|\lambda_0 - \lambda_1|.
\]
\(\square\)
Lemma 2.10. Let \( m \geq 1 \) and \( z_0, z_1 \) with \( |z_0| = |z_1| = \rho_m \) and such that \( |z_0 - z_1| \leq S \). If \( \lambda_0, \lambda_1 \in \Lambda \) are such that

\[
\rho_{m-1} \cdot \ldots \cdot \rho_1 \cdot |z_0 - z_1| < |\lambda_0 - \lambda_1| \leq S,
\]

then

\[
|P_{\lambda_0}^m(z_0) - P_{\lambda_1}^m(z_1)| = |\lambda_0 - \lambda_1|.
\]

Proof: First we will show inductively that, for \( 1 \leq i \leq m \),

\[
|P_{\lambda_0}^i(z_0) - P_{\lambda_1}^i(z_1)| \leq \max\{\rho_{m-1} \cdot \ldots \cdot \rho_{m-i} |z_0 - z_1|, \rho_{m-i} |\lambda_0 - \lambda_1|\}. \tag{6}
\]

Observe that

\[
|P_{\lambda_0}(z_0) - P_{\lambda_0}(z_1)| \leq \rho_{m-1} |z_0 - z_1|, \tag{7}
\]

\[
|P_{\lambda_0}(z_1) - P_{\lambda_1}(z_1)| = \rho_{m-1} |\lambda_0 - \lambda_1|. \tag{8}
\]

Using the ultrametric inequality, (7) and (8), we get (6) for \( i = 1 \).

If we assume (6) as the inductive hypothesis, we have

\[
|P_{\lambda_0}(P_{\lambda_0}^i(z_0)) - P_{\lambda_0}(P_{\lambda_1}^i(z_1))| \leq \rho_{m-i-1} |P_{\lambda_0}^i(z_0) - P_{\lambda_1}^i(z_1)|, \tag{9}
\]

\[
|P_{\lambda_0}(P_{\lambda_1}^i(z_1)) - P_{\lambda_1}(P_{\lambda_1}^i(z_1))| = \rho_{m-i-1} |\lambda_0 - \lambda_1|. \tag{10}
\]

From (9) and (10) we obtain (6) for \( i + 1 \). Notice that for the inductive step from \( m - 1 \) to \( m \), the hypothesis of the lemma gives us that

\[
|P_{\lambda_0}^m(z_0) - P_{\lambda_1}^m(z_1)| = |\lambda_0 - \lambda_1|. \tag*{□}
\]

Let \( \lambda \in \Lambda \) and \( M_0 \in \mathbb{N} \) with \( p^{-M_0} \leq S \), we choose \( m_0 \in \mathbb{N} \) such that

\[
\rho_{m_0-1} \cdot \ldots \cdot \rho_1 p^{M_0} \leq 1.
\]

Now, if we choose \( x \in K(P_{\lambda}) \) with itinerary

\[
\frac{0 \ldots 0 1 1 1 1 1 1 \ldots}{m_0},
\]

we obtain Lemma 2.10 hypothesis with \( z_0 = P_{\lambda_0}^{m_0}(x), z_1 = P_{\lambda_1}^{m_0}(x) \) and \( M = M_1 \), for all \( \lambda_0, \lambda_1 \in \{ z : |\lambda - z| \leq p^{-M_0} \} \), and we have

\[
|P_{\lambda_0}^{m_0+M_0}(x) - P_{\lambda_1}^{m_0+M_0}(x)| = p^{M_0} |\lambda_0 - \lambda_1|.
\]

Hence, there exists \( u_0 \in \Lambda \) with \( P_{u_0}^{m_0+M_0}(x) = 0 \) such that the itinerary of \( x \) for \( P_u \) is

\[
\frac{0 \ldots 0 1 1 1 0 \ldots}{m_0 M_0}.
\]
for all \( w \in \{ z : |z - w_0| < p^{-M_0} \} \).

As before, we choose \( M_1 \) such that \( P^{M_1 - M_0} \leq S \), and \( m_1 \) such that
\[
\rho_{m_1 - 1} \cdots \rho_1 p^{M_1} \leq 1
\]

obtaining that there exists \( \lambda' \in \Lambda \) with \( |\lambda' - w_0| = \rho_{m_1} p^{-M_0} \) such that \( P_{\lambda'}(x) = 1 \).

Therefore, the itinerary of \( x \) for \( P_{\lambda'} \) is
\[
0 \cdots 0 1 \cdots 0 1 1 \cdots
\]

Lemma 2.10 allows us to make this process inductively, obtaining the Lemma 2.8 hypothesis.

3 Results.

In this section, we establish some properties of the perturbations of the polynomials \( P_{\lambda} \). Throughout,
\[
\rho_0 = 1, \quad p \rho_n^p = \rho_{n-1}.
\]

Recall that \( p \rho^p = \rho \) and that \( pS^{p-1} = \rho \). For the rest of this work we fix \( \hat{r} \in |C| \), with \( \hat{r} > 1 \) and \( B = \{ z \in C : |z| \leq \hat{r} \} \). The perturbations are:
\[
Q^*_\lambda(z) = P_{\lambda}(z) + Q(z),
\]

where \( Q \in \mathcal{H}(B) \) with \( \|Q\| < \rho \). For this family we will obtain the following result:

**Theorem 3.1.** There is a dense set of parameter \( \lambda \in \Lambda \) such that the function \( Q^*_\lambda \) has a wandering disc contained in the filled Julia set, which is not attracted to an attracting cycle.

To prove this theorem, we will study a topological conjugation of \( Q^*_\lambda \).

Notice that
\[
p(Q^*_\lambda(z) - z) \in \mathcal{O}_{C_p}[[z]],
\]
in addition
\[
|p Q^*_\lambda(1) - p| = \frac{1}{p} |Q(1)| < \frac{\rho}{p}
\]

and
\[
|p (Q^*_\lambda)'(1) - 1| = \frac{1}{p} |P_{\lambda}'(1) + Q'(1) - 1| = 1.
\]

From Hensel’s Lemma, there is an unique root of \( p (Q^*_\lambda(z) - z) \) in \( B_{r_0}(1) \), where \( r_0 = \frac{|Q(1)|}{p} \). We denote this root by \( z_{\lambda} \) and observe that \( z_{\lambda} \) is a fixed point of \( Q^*_\lambda \).

Now, we define the function
\[
h : \Lambda \rightarrow \left\{ z : |z - 1| \leq \frac{|Q(1)|}{p} \right\}
\]

obtaining that
Proposition 3.2. The function \( h \) is holomorphic in \( \Lambda \).

Proof:
Let \( \{h_n\}_{n \geq 0} \) be the sequence of functions defined recursively as follows:

\[
h_0(\lambda) = 1 \\
h_n(\lambda) = h_{n-1}(\lambda) - \frac{Q^*_\lambda(h_{n-1}(\lambda))}{(Q^*_\lambda)'(h_{n-1}(\lambda))}
\]

Then for all \( n \in \mathbb{N}, h_n \) is a rational function without poles in \( \Lambda \).

As in the proof of Lemma 2.1, we have

\[
|h_n(\lambda) - h(\lambda)| = \left| \sum_{i \geq n} \frac{Q^*_\lambda(h_i(\lambda))}{(Q^*_\lambda)'(h_i(\lambda))} \right| \leq \max_{i \geq n} \left\{ \left| \frac{Q^*_\lambda(h_i(\lambda))}{(Q^*_\lambda)'(h_i(\lambda))} \right| \right\} 
\]

\[
= \max_{i \geq n} |Q^*_\lambda(h_i(\lambda))| < \rho^n.
\]

Hence \( h_n \) converges to \( h \) uniformly in \( \Lambda \). Therefore, \( h \in \mathcal{H}(\Lambda) \).

We may now introduce the affine map

\[
A_\lambda(z) = z + h(\lambda) - 1
\]

we will work with the map

\[
Q_\lambda(z) = A_\lambda^{-1}(Q^*_\lambda(A_\lambda(z))) = P_\lambda(A_\lambda(z)) + Q(A_\lambda(z)) + 1 - h(\lambda).
\]

which is affinely conjugated to \( Q^*_\lambda \). Notice that

\[
Q_\lambda(1) = P_\lambda(A_\lambda(1)) + Q(A_\lambda(1)) + 1 - h(\lambda) = P_\lambda(h(\lambda)) + Q(h(\lambda)) + 1 - h(\lambda) = 1.
\]

Moreover,

\[
|Q_\lambda'(1)| = |P^*_\lambda(A_\lambda(1)) + Q'(A_\lambda(1))| = |P^*_\lambda(h(\lambda)) + Q'(h(\lambda))| = p.
\]

Thus, just as to the polynomials \( P_\lambda, z = 1 \) is a repelling fixed point of \( Q_\lambda \) for all \( \lambda \in \Lambda \).

For the family \( Q_\lambda \) we will obtain the following theorem.
Theorem 3.3. There is a dense set of parameter $\lambda \in \Lambda$ such that the function $Q_\lambda$ has a wandering disc contained in the filled Julia set, which is not attracted to an attracting cycle.

Proof of Theorem 3.3
Recall that $Q_\lambda(z) = A_\lambda^{-1}(Q_\lambda(A_\lambda(z)))$, i.e. $Q_\lambda^*(z) = A_\lambda(Q_\lambda(A_\lambda^{-1}(z)))$.

If $D$ is a ball, then $A_\lambda^{-1}(D)$ and $A_\lambda(D)$ are balls, and, obtaining directly that $K(Q_\lambda, B) = A_\lambda^{-1}(K(Q_\lambda^*, A_\lambda(B)))$, it is sufficient to show that if $D$ is a wandering disk for $Q_\lambda$, then $A_\lambda^{-1}(D)$ is a wandering disk for $Q_\lambda^*$.

Suppose that $D$ is a wandering ball for $Q_\lambda$, i.e. $Q_\lambda^n(D) \cap Q_\lambda^m(D) = \emptyset$ when $n \neq m$. It follows that for $n \neq m$ we have that $A_\lambda^{-1}((Q_\lambda^*)^n(A_\lambda(D))) \cap A_\lambda^{-1}((Q_\lambda^*)^m(A_\lambda(D))) = \emptyset$, then $(Q_\lambda^*)^n(A_\lambda(D)) \cap (Q_\lambda^*)^m(A_\lambda(D)) = \emptyset$. Therefore $A_\lambda(D)$ is a wandering disk for $Q_\lambda^*$.

3.1 Properties of $Q_\lambda(z)$.
The next proposition states a property of the function $h$ that will be used several times.

Proposition 3.4. If $\lambda_0, \lambda_1 \in \Lambda$, then $|h(\lambda_0) - h(\lambda_1)| \leq \rho|\lambda_0 - \lambda_1|$.
Proof:
Let $\lambda_0, \lambda_1 \in \Lambda$. Since

$$Q_{\lambda_0}^*(h(\lambda_0)) - Q_{\lambda_1}^*(h(\lambda_1)) = h(\lambda_0) - h(\lambda_1)$$

we have that

$$|Q_{\lambda_0}^*(h(\lambda_0)) - Q_{\lambda_1}^*(h(\lambda_1))| \leq \frac{|Q(1)|}{p}$$

and from

$$|Q(h(\lambda_0)) - Q(h(\lambda_1))| < \frac{\rho}{p} |h(\lambda_0) - h(\lambda_1)|,$$

it follows that

$$|P_{\lambda_0}(h(\lambda_0)) - P_{\lambda_1}(h(\lambda_1))| < \frac{\rho}{p} |h(\lambda_0) - h(\lambda_1)|.$$

In addition, from

$$|P_{\lambda_0}(h(\lambda_0)) - P_{\lambda_0}(h(\lambda_1))| = p |h(\lambda_0) - h(\lambda_1)| > \frac{\rho}{p} |h(\lambda_0) - h(\lambda_1)|$$

and by isosceles triangle principle, necessarily we have that

$$p |\lambda_0 - \lambda_1| |h(\lambda_1) - 1| = p |h(\lambda_0) - h(\lambda_1)|.$$

Finally from $|h(\lambda_1) - 1| \leq \frac{\rho}{p}$, we have that $|h(\lambda_0) - h(\lambda_1)| \leq \rho|\lambda_0 - \lambda_1|$. 

\square
Lemma 3.5. Let \( z \in \mathbb{C}_p \).

\( i \) If \( \rho < |z| < 1 \), then \( |Q_\Lambda(z)| = p \ |z|^p > |z| \).

\( ii \) If \( |z| \leq \rho \), then \( |Q_\Lambda(z)| \leq \rho \).

\( iii \) If \( |z - 1| < 1 \), then \( |Q_\Lambda(z) - 1| = p \ |z - 1| \).

\( iv \) If \( 1 < |z| < \hat{r} \), then \( |Q_\Lambda(z)| = p \ |z|^{p+1} \).

Proof:

From Proposition 3.4, for every \( \lambda \in \Lambda \) we have that \( h(\lambda) - 1 \leq \frac{2}{p} < \rho \).

\( i \) Since \( \rho < |z| < 1 \), we have that \( |A_\Lambda(z)| = |z| \). In addition, \( |A_\Lambda(z)|^{p+1} < |A_\Lambda(z)|^p \).

Hence \( |P_\Lambda(A_\Lambda(z))| = p \ |A_\Lambda(z)|^p > p\rho^p = \rho \). Furthermore, \( |Q(A_\Lambda(z))| < \rho \) and \( |1 - h(\lambda)| < \rho \), therefore

\[ |Q_\Lambda(z)| = p \ |z|^p. \]

\( ii \) Observe that \( |A_\Lambda(z)| \leq \rho \) since \( |z| \leq \rho \). It follows

\[ |P_\Lambda(A_\Lambda(z))| = p \ |A_\Lambda(z)|^p \leq p\rho^p = \rho. \]

In addition, from \( |Q(A_\Lambda(z))| < \rho \), \( |1 - h(\lambda)| < \rho \) and the strong triangle inequality, we have that

\[ |Q_\Lambda(z)| \leq \rho. \]

\( iii \)

\[ |Q_\Lambda(z) - 1| = |Q_\Lambda(z) - Q_\Lambda(1)| = |P_\Lambda(A_\Lambda(z)) + Q(A_\Lambda(z)) - P_\Lambda(A_\Lambda(1)) - Q(A_\Lambda(1))|. \]

Since \( |A_\Lambda(z) - A_\Lambda(1)| = |z - 1| \), we have that \( |P_\Lambda(A_\Lambda(z)) - P_\Lambda(A_\Lambda(1))| = p \ |z - 1| \).

Moreover, \( |Q_\Lambda(A_\Lambda(z)) - Q_\Lambda(A_\Lambda(1))| \leq \rho \ |z - 1| \). Again, from the strong triangle inequality, we have that

\[ |Q_\Lambda(z) - 1| = p \ |z - 1|. \]

\( iv \) Since \( |z| > 1 \), it follows that \( |P_\Lambda(A_\Lambda(z))| = p \ |z|^{p+1} \). Furthermore, \( |Q(A_\Lambda(z))| < \rho \) and \( |h(\lambda) - 1| < \rho \), therefore

\[ |Q_\Lambda(z)| = p \ |z|^{p+1}. \]

\( \square \)

Recall that \( B \) is the closed ball defined by \( \{ z \in \mathbb{C}_p : |z| \leq \hat{r} \} \), where \( \hat{r} \) is an element of \( |\mathbb{C}_p^\ast| \) chosen in the beginning of this section.

Proposition 3.6. For each \( \lambda \in \Lambda \), \( (Q_\Lambda, B) \) is an polynomial like map of degree \( p + 1 \).  

Proof:

Let \( \lambda \in \Lambda \). From the previous lemma we deduce that \( Q_\Lambda(B) = \{ z : |z| < p \ \hat{r}^{p+1} \} \), we will prove that \( (Q_\Lambda, B) \) is of degree \( p + 1 \).

Since \( |P_\Lambda(A_\Lambda(z)) - P_\Lambda(z)| = p \ |h(\lambda) - 1| < \rho \) we conclude that

\[ Q_\Lambda(z) - P_\Lambda(z) = P_\Lambda(A_\Lambda(z)) - P_\Lambda(z) + Q(A_\Lambda(z)) - h(\lambda) + 1, \]

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using that \(\|Q_\lambda - P_\lambda\| < \rho\), the power series of \(Q_\lambda\) is

\[
Q_\lambda(z) = a_0 + a_1z + \ldots + \left(a_p + \frac{\lambda}{p}\right)z^p + \left(a_{p+1} + 1 - \frac{\lambda}{p}\right)z^{p+1} + a_{p+2}z^{p+2} \ldots
\]

where \(\sup_{i \geq 0} |a_i z^i| < \rho\). From Theorem 2.2 it is possible to count the solutions of \(Q_\lambda(z) - w_0 = 0\).

If \(p \leq |w_0| < p^{p+1}\) and \(f(z_0) = w_0\), then \(|z_0| = \left(\frac{|w_0|}{p}\right)^{p+1}\), by Lemma 3.5 (iv). Therefore \(w_0\) has \(p + 1\) pre-images in \(B\). Hence \((Q_\lambda, B)\) is a polynomial like map of degree \(p + 1\). \(\square\)

**Proposition 3.7.** \(K(Q_\lambda, B) \subset B_1(0) \sqcup B_1(1)\).

**Proof:**

Suppose that \(z \notin B_1(0) \sqcup B_1(1)\).
If \(|z| > 1\) then \(|Q_\lambda(z)| = |\lambda|z^{p+1}\). It follows that there exists \(n \in \mathbb{N}\), such that \(Q^n_\lambda(z) \notin B\).
If \(|z| = 1\) and \(|z - 1| = 1\), then \(|Q_\lambda(z)| = |P_\lambda(A_\lambda(z)) + Q(A_\lambda(z))| = p\). Hence \(z \notin K(Q_\lambda, B)\).
Therefore \(K(Q_\lambda, B) \subset B_1(0) \sqcup B_1(1)\). \(\square\)

This result allow us to define the itinerary of a point in \(K(Q_\lambda, B)\). To simplify notation let \(B_0 = B_1(0)\) and \(B_1 = B_1(1)\).
For any \(z \in K(Q_\lambda, B)\), the itinerary of \(z\) for \(Q_\lambda\) is defined by

\[
\theta_0 \theta_1 \ldots \theta_n \ldots \in \{0, 1\}^{\mathbb{N}} \setminus \{0\}\] where \(Q^n_\lambda(z) \in B_{\theta_n}\) for all \(n \geq 0\).

**Lemma 3.8.** Let \(\lambda \in \Lambda\) and \(D\) a ball contained in \(K(Q_\lambda, B)\). Then:

1) All points in \(D\) have the same itinerary for \(Q_\lambda\).
2) If the common itinerary of points in \(D\) is not pre-periodic, under the one side shift, then \(D\) is a wandering disc which is not attracted to an attracting periodic point.
Proof:

1) We proceed by contradiction. Assume that there exist $z_0$ and $z_1 \in D$ with different itineraries. Then there exists $n_0 \in \mathbb{N}$ such that $Q^{n_0}_\lambda(z_0) \in B_0$ and $Q^{n_0}_\lambda(z_1) \in B_1$. Since $Q^{n_0}_\lambda(D)$ is a ball which has non-trivial intersection with $B_0$ and $B_1$ we have that 
\[ \{ z : |z| \leq 1 \} \subset Q^{n_0}_\lambda(D) \subset K(Q_\lambda, B), \]
obtaining a contradiction with Proposition 3.7.

2) Now, we suppose that $D$ is not a wandering disc, that is, there exist $n > m \geq 0$ such that $Q^n_\lambda(D) \cap Q^m_\lambda(D) \neq \emptyset$. Hence $Q^n_\lambda(z_0) \in Q^m_\lambda(D)$ for some $z_0 \in D$. Therefore $Q^n_\lambda(z_0) \in Q^{n+m}_\lambda(D)$ for all $k \in \mathbb{N}$.

Since every point in $D$ has the same itinerary we conclude that the itinerary of the points in $D$ is pre-periodic with eventual period $n - m$.

We must show that $D$ is not attracted to a periodic orbit. We suppose that there is an attracting periodic point $z_0$ and $s > 0$ such that $B_s(z_0)$ is contained in the attracting basin of $z_0$, and $z_1 \in D$ such that $z_1 \in B_s(z_0)$, from the first part of the proposition we have that every points of $B_s(z_0)$ have a common itinerary, and it is periodic. \( \square \)

From the previous lemma we conclude that in order to prove Theorem 3.8 it is sufficient to find a wandering disc in the filled Julia set of $(Q_\lambda, B)$ whose itinerary is not pre-periodic, for a dense subset in $\Lambda$. Therefore, we need to study the behavior of the points in $K(Q_\lambda, B)$ such that its orbit visits both $B_0$ and $B_1$.

From Lemma 3.6 we know that the open ball $B_{\rho}(0)$ is fixed under the action of $Q_\lambda$ and we have that a point $x \in B$ has itinerary
\[
\underbrace{0 \ldots 0}_{n} 1 \ldots
\]
if and only if $|x| = \rho_n$. Recall $\rho_0 = 1$, $pp_n = \rho_{n-1}$. This crucial fact holds already for the family $P_\lambda \ [3]$.

The following lemma describe the local behavior of $Q_\lambda$ in the set \{ $z : |z| = \rho_n$ \} and in $B_1$.

**Lemma 3.9.** 1. Let $m \geq 1$, $|z_0| = |z_1| = \rho_m$. If $|z_0 - z_1| \leq S$, then
\[
|Q_\lambda(z_0) - Q_\lambda(z_1)| \leq \rho_{m-1}|z_0 - z_1|.
\]

2. If $z_0, z_1 \in B_1(1)$, then:
\[
|Q_\lambda(z_0) - Q_\lambda(z_1)| = p|z_0 - z_1|.
\]

**Proof:**

1) We observe that $|A_\lambda(z_i)| = |z_i| = \rho_m$, hence
\[
|Q(A_\lambda(z_0)) - Q(A_\lambda(z_1))| < p|A_\lambda(z_0) - A_\lambda(z_1)| = p|z_0 - z_1|.
\]
Letting $\epsilon = A_\lambda(z_1) - A_\lambda(z_0)$ we have
\[
P_\lambda(A_\lambda(z_0)) - P_\lambda(A_\lambda(z_1)) = \frac{\lambda}{p}(pA_\lambda(z_0)^{p-1} + \ldots + pA_\lambda(z_0)^{p-1} + \epsilon^p)
\]
\[+ (1 - \frac{\lambda}{p})((p+1)pA_\lambda(z_0)^p + \ldots + \epsilon^{p+1}).\]
Moreover, $|P_\lambda(A_\lambda(z_0)) - P_\lambda(A_\lambda(z_1))| \leq |\epsilon| \max\{|A_\lambda(z_0)|^{p-1}, p, |\epsilon|^{p-1}, \rho_{m-1}\} = |\epsilon| \rho_{m-1}$, since $|\epsilon| = |z_0 - z_1| \leq S < \rho_m = |A_\lambda(z_0)|$.

Therefore

$$|Q_\lambda(z_0) - Q_\lambda(z_1)| \leq \rho_{m-1} |z_0 - z_1|.$$

2) We note that if $y \in B_1$, then $A_\lambda(y) \in B_1$. Now

$$P_\lambda(A_\lambda(z_0)) - P_\lambda(A_\lambda(z_1)) = \frac{\lambda}{p} (p \epsilon A_\lambda(z_0)^{p-1} + \ldots + p \epsilon^{p-1} A_\lambda(z_0) + \epsilon^p)$$

$$+ \left(1 - \frac{\lambda}{p}\right) ((p + 1) \epsilon A_\lambda(z_0)^p + \ldots + \epsilon^{p+1}),$$

where $\epsilon = z_0 - z_1$, thus $|P_\lambda(A_\lambda(z_0)) - P_\lambda(A_\lambda(z_1))| = p |z_0 - z_1|$ since $|\epsilon| < 1$. Furthermore, we have that

$$|Q(A_\lambda(z_0)) - Q(A_\lambda(z_1))| \leq \rho |z_0 - z_1| < p |z_0 - z_1|.$$

Therefore

$$|Q_\lambda(z_0) - Q_\lambda(z_1)| = p |z_0 - z_1|.$$

$\square$

The following lemma gives a sufficient condition for the existence of a wandering disc in $K(Q_\lambda, B)$.

**Lemma 3.10.** Let $\{m_i\}_{i \geq 0}, \{M_i\}_{i \geq 0}$ be sequences of positive integers such that if $i \geq 0$

$$\rho_{m_i - 1} \cdots \rho_1 \cdot p^{M_i} \leq 1.$$

If for $\lambda_0 \in \Lambda$ there exists $z_0 \in K(Q_{\lambda_0})$ with itinerary

$$\begin{align*}
0 &\cdots 0 &1 &\cdots 1 &0 &\cdots 0 &1 &\cdots 1, \\
\begin{array}{cccccccc}
m_0 & M_0 & m_1 & M_1 & m_i & M_i & \ldots
\end{array}
\end{align*}$$

then the closed ball $D = \{|z - z_0| \leq S\}$ is contained in $K(Q_{\lambda_0})$.

If we add the hypothesis $\lim M_i = \infty$, then by Lemma 3.8 we have that $D$ is a wandering disc contained in $K(Q_\lambda, B)$ which is not attracted to an attracting cycle.

**Proof:**

We now define the sequence $\{N_i\}_{i \geq 0}$ recursively:

$$N_0 = 0$$

$$N_i = N_{i-1} + m_{i-1} + M_{i-1}.$$

We will prove inductively that $\text{diam}(Q_{N_0}^{N_i}(D)) \leq S$. For $i = 0$ the claim is true because the definition of $D$. Suppose that $\text{diam}(Q_{N_0}^{N_j}(D)) \leq S$, since $Q_{N_0}^{N_{j+1}}(z_0) \in B_0$ for all $j \in \mathbb{N}$ with $0 \leq j \leq m_i$ and $Q_{N_0}^{N_i + m_i}(z_0) \in B_1$, it follows that $|Q_{N_0}^{N_i}(z_0)| = \rho_{m_i}$. Therefore $|Q_{N_0}^{N_i}(y)| = \rho_{m_i}$ and $Q_{N_0}^{N_i + m_i}(y) \in B_1$ for all $y \in D$.  

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From the first statement of the previous lemma we have that
\[ \text{diam}(Q^{N_i+m_i}(D)) \leq \rho_{m_{i-1}} \cdot \ldots \cdot \rho_1 \text{diam}(Q^{N_i}(D)) \leq p^{-M_i} S. \]

Now, using the second statement of the same lemma, we obtain
\[ \text{diam}(Q^{N_i+1}(D)) \leq p^{M_i} \text{diam}(Q^{N_i+m_i}(D)) \leq S. \]

Therefore \( D \subset K(Q_{\lambda_0}, B) \). \qed
3.2 Parameter selection.

In this section we will prove results that describe the behavior of the iterates $Q^p_\lambda(z)$, not just as a function of $z$ but also as a function of $\lambda$.

**Lemma 3.11.** Let $\lambda_0, \lambda_1 \in \Lambda$.

1) If $z \in B_0$ then $|P_{\lambda_0}(z) - P_{\lambda_1}(z)| = p |z|^p |\lambda_0 - \lambda_1|$.

2) If $z \in B_1$ then $|P_{\lambda_0}(z) - P_{\lambda_1}(z)| = p |\lambda_0 - \lambda_1| |z - 1|$.

**Proof:**

1) 

$$|P_{\lambda_0}(z) - P_{\lambda_1}(z)| = |z^p \left( \frac{\lambda_0 - \lambda_1}{p} \right) - z^{p+1} \left( \frac{\lambda_0 - \lambda_1}{p} \right)| = p |\lambda_0 - \lambda_1| |z - 1|.$$  

2) 

$$|P_{\lambda_0}(z) - P_{\lambda_1}(z)| = |z^p \left( \frac{\lambda_0 - \lambda_1}{p} \right) - z^{p+1} \left( \frac{\lambda_0 - \lambda_1}{p} \right)| = p |z|^p |\lambda_0 - \lambda_1|.$$  

□

**Lemma 3.12.** Let $M \in \mathbb{N}$ and $x_0, x_1 \in \{x : |x - 1| \leq p^{-M}\}$. If the parameters $\lambda_0, \lambda_1 \in \Lambda$ are such that $|\lambda_0 - \lambda_1| = |x_0 - x_1|$, then

$$|Q^M_{\lambda_0}(x_0) - Q^M_{\lambda_1}(x_1)| = p^M |\lambda_0 - \lambda_1|.$$  

**Proof:** First we prove the lemma for $M = 1$.

By the second part of Lemma 3.11 we have

$$|Q_{\lambda_0}(x_0) - Q_{\lambda_0}(x_1)| = p |x_0 - x_1| = p |\lambda_0 - \lambda_1|. \quad (1)$$

Furthermore by Lemma 3.11 and Lemma 3.2 we obtain

$$|P_{\lambda_0}(A_{\lambda_0}(x_1)) - P_{\lambda_0}(A_{\lambda_1}(x_1))| = p |h(\lambda_0) - h(\lambda_1)| < p |\lambda_0 - \lambda_1|. \quad (2)$$

By Lemma 3.11 we conclude that

$$|P_{\lambda_0}(A_{\lambda_1}(x_1)) - P_{\lambda_1}(A_{\lambda_1}(x_1))| = p |\lambda_0 - \lambda_1| |A_{\lambda_1}(x_1) - 1| < p |\lambda_0 - \lambda_1|, \quad (3)$$

and from equations (2) and (3)

$$|P_{\lambda_0}(A_{\lambda_0}(x_1)) - P_{\lambda_1}(A_{\lambda_1}(x_1))| < p |\lambda_0 - \lambda_1|. \quad (4)$$

Moreover,

$$|Q(A_{\lambda_0}(x_1)) - Q(A_{\lambda_1}(x_1))| < p |h(\lambda_0) - h(\lambda_1)| < p |\lambda_0 - \lambda_1|. \quad (5)$$

From (4) and (5) we have that

$$|Q_{\lambda_0}(x_1) - Q_{\lambda_1}(x_1)| < p |\lambda_0 - \lambda_1|. \quad (6)$$

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Finally, from (11) and (13) we obtain that $|Q_{\lambda_0}(x_0) - Q_{\lambda_1}(x_1)| = p|\lambda_0 - \lambda_1|$.  

Now let us prove that the proposition is true for $M+1$. By the inductive hypothesis and the third statement of Lemma 3.9 we have that $Q_{\lambda_0}^M(x_0), Q_{\lambda_1}^M(x_1)$ belong to $B_1$ and using Lemma 3.9 with $z_0 = Q_{\lambda_0}^M(x_0)$ and $z_1 = Q_{\lambda_1}^M(x_1)$ we obtain that

$$|Q_{\lambda_1}(Q_{\lambda_0}^M(x_0)) - Q_{\lambda_0}(Q_{\lambda_1}^M(x_1))| = p^{M+1}|\lambda_0 - \lambda_1|. \tag{7}$$

Furthermore

$$|P_{\lambda_0}(A_{\lambda_0}(Q_{\lambda_0}^M(x_0))) - P_{\lambda_0}(A_{\lambda_1}(Q_{\lambda_0}^M(x_0)))| = p|h(\lambda_0) - h(\lambda_1)| < p|\lambda_0 - \lambda_1|, \tag{8}$$

just as before, from the previous lemma

$$|P_{\lambda_0}(A_{\lambda_0}(Q_{\lambda_0}^M(x_0))) - P_{\lambda_1}(A_{\lambda_0}(Q_{\lambda_0}^M(x_0)))| < p|\lambda_0 - \lambda_1| \tag{9}$$

and

$$|Q(A_{\lambda_0}(Q_{\lambda_0}^M(x_0))) - Q(A_{\lambda_1}(Q_{\lambda_0}^M(x_0)))| < p|h(\lambda_0) - h(\lambda_1)| < p|\lambda_0 - \lambda_1|. \tag{10}$$

The strong triangle principle applied to (8), (9) and (10) gives us

$$|Q_{\lambda_0}(Q_{\lambda_0}^M(x_0)) - Q_{\lambda_1}(Q_{\lambda_0}^M(x_0))| < p|\lambda_0 - \lambda_1|, \tag{11}$$

and from (7) and (11) we conclude that

$$|Q_{\lambda_0}^{M+1}(x_0) - Q_{\lambda_1}^{M+1}(x_1)| = p^{M+1}|\lambda_0 - \lambda_1|. \tag{12}$$

\[\square\]

**Lemma 3.13.** Let $m \in \mathbb{N}$ and let $x_0, x_1$ be such that $|x_0| = |x_1| = \rho_m$ and $|x_0 - x_1| \leq S$. If $\lambda_0, \lambda_1 \in \Lambda$ are such that

$$\rho_{m-1} \cdots \rho_1|x_0 - x_1| < |\lambda_0 - \lambda_1| \leq S,$$

then

$$|Q_{\lambda_0}^m(x_0) - Q_{\lambda_1}^m(x_1)| = |\lambda_0 - \lambda_1|. \tag{13}$$

**Proof:**

We start by inductively proving that if $1 \leq i \leq m$, then

$$|Q_{\lambda_0}^i(x_0) - Q_{\lambda_1}^i(x_1)| \leq \max\{\rho_{m-i} |\lambda_0 - \lambda_1|, \rho_{m-1} \cdots \rho_{m-i} |x_0 - x_1|\}. \tag{14}$$

From the first part of Lemma 3.9 we have

$$|Q_{\lambda_0}(x_0) - Q_{\lambda_0}(x_1)| \leq \rho_{m-1}|x_0 - x_1|. \tag{12}$$

Since $|A_{\lambda_0}(x_1)| = |A_{\lambda_1}(x_1)| = \rho_m$, and $\rho_{m-1} \cdots \rho_1|h(\lambda_0) - h(\lambda_1)| < |\lambda_0 - \lambda_1| \leq S$, we obtain

$$|P_{\lambda_0}(A_{\lambda_0}(x_1)) - P_{\lambda_1}(A_{\lambda_1}(x_1))| \leq \max\{\rho_{m-1}h(\lambda_0) - h(\lambda_1), \rho_{m-1} |\lambda_0 - \lambda_1|\}, \tag{13}$$

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by the equation (14) of Lemma 2.10.
Moreover
\[ |Q(A_{\lambda_0}(x_1)) - Q(A_{\lambda_1}(x_1))| < \rho |h(\lambda_0) - h(\lambda_1)| < \rho |\lambda_0 - \lambda_1|. \] (14)

From inequalities (13) and (14), together with Proposition 3.4 we have
\[ |Q_{\lambda_0}(x_1) - Q_{\lambda_1}(x_1)| \leq \rho_{m-1} |\lambda_0 - \lambda_1|. \] (15)

Now inequalities (12) and (15) give us
\[ |Q_{\lambda_0}(x_0) - Q_{\lambda_1}(x_1)| \leq \max\{\rho_{m-1} |x_0 - x_1|, \rho_{m-1} |\lambda_0 - \lambda_1|\}. \]

Therefore which one is true for \( i = 1 \).

Now suppose \( |Q_{\lambda_0}^i(x_0) - Q_{\lambda_1}^i(x_1)| \leq \max\{\rho_{m-i} |\lambda_0 - \lambda_1|, \rho_{m-1} \cdot \cdots \cdot \rho_{m-i} |x_0 - x_1|\}. \)

Notice that \( |Q_{\lambda_0}(x_0)| = |Q_{\lambda_1}^i(x_1)| = \rho_{m-i} \), therefore, using Lemma 3.11 with \( z_0 = Q_{\lambda_0}(x_0) \) and \( z_1 = Q_{\lambda_1}^i(x_1) \), we obtain that
\[ |Q_{\lambda_0}(Q_{\lambda_0}^i(x_0)) - Q_{\lambda_1}(Q_{\lambda_1}^i(x_1))| \leq \rho_{m-(i+1)} |Q_{\lambda_0}(x_0) - Q_{\lambda_1}(x_1)|. \] (16)

From Lemma 3.11 with \( x = A_{\lambda_0}(Q_{\lambda_0}^i(x_0)) \) we have that
\[ |P_{\lambda_0}(A_{\lambda_0}(Q_{\lambda_0}^i(x_0))) - P_{\lambda_1}(A_{\lambda_1}(Q_{\lambda_1}^i(x_1)))| = \rho |\lambda_0 - \lambda_1| \rho_{m-i} = \rho_{m-(i+1)} |\lambda_0 - \lambda_1|. \] (17)

Using the first part of Lemma 2.10 we obtain that
\[ |P_{\lambda_1}(A_{\lambda_0}(Q_{\lambda_0}^i(x_0))) - P_{\lambda_1}(A_{\lambda_1}(Q_{\lambda_1}^i(x_1)))| \leq \rho_{m-(i+1)} |Q_{\lambda_0}^i(x_0) - Q_{\lambda_1}^i(x_1) + h(\lambda_0) - h(\lambda_1)|,
\]
\[ |Q(A_{\lambda_0}(Q_{\lambda_0}^i(x_0))) - Q(A_{\lambda_1}(Q_{\lambda_1}^i(x_1)))| < \rho |z_{\lambda_0} - z_{\lambda_1}| < \rho |\lambda_0 - \lambda_1|. \]

From the inequalities above and Proposition 3.4 we have that
\[ |Q_{\lambda_0}^{i+1}(x_0) - Q_{\lambda_1}^{i+1}(x_1)| \leq \max\{\rho_{m-(i+1)} |\lambda_0 - \lambda_1|, \rho_{m-1} \cdot \cdots \cdot \rho_{m-(i+1)} |x_0 - x_1|\}. \]

Notice that in the inductive step for \( i = m - 1 \), we have that
\[ |Q_{\lambda_0}^{m-1}(x_0) - Q_{\lambda_1}^{m-1}(x_1)| \leq \max\{\rho_1 |\lambda_0 - \lambda_1|, \rho_{m-1} \cdot \cdots \cdot \rho_1 |x_0 - x_1|\}. \]

From the above inequality we obtain
\[ |Q_{\lambda_1}(Q_{\lambda_0}^{m-1}(x_0)) - Q_{\lambda_1}(Q_{\lambda_1}^{m-1}(x_1))| \leq \rho_1 |Q_{\lambda_0}^{m-1}(x_0) - Q_{\lambda_1}^{m-1}(x_1)| < |\lambda_0 - \lambda_1| \]

and from lemmas 3.11 and 2.7 we have the following inequalities
\[ |P_{\lambda_1}(A_{\lambda_0}(Q_{\lambda_0}^{m-1}(x_0))) - P_{\lambda_1}(A_{\lambda_1}(Q_{\lambda_1}^{m-1}(x_1)))| \leq |Q_{\lambda_0}^{m-1}(x_0) - Q_{\lambda_1}^{m-1}(x_1) + h(\lambda_0) - h(\lambda_1)| < |\lambda_0 - \lambda_1|. \]
\[ |P_{\lambda_0}(A_{\lambda_0}(Q_{\lambda_0}^{m-1}(x_0))) - P_{\lambda_1}(A_{\lambda_0}(Q_{\lambda_0}^{m-1}(x_0)))| = \rho |\lambda_0 - \lambda_1| \rho_p^* = |\lambda_0 - \lambda_1| \]
Moreover
\[ |Q(A_{\lambda_0}(Q_{\lambda_0}^{n-1}(x_0))) - Q(A_{\lambda_1}(Q_{\lambda_0}^{n-1}(x_0)))| < \rho |h(\lambda_0) - h(\lambda_1)| < |\lambda_0 - \lambda_1| \]
and using the four previous inequalities and Proposition 3.11 we have
\[ |Q_{\lambda_0}^n(x_0) - Q_{\lambda_1}^n(x_1)| = |\lambda_0 - \lambda_1|. \]

\[ \square \]

Proposition 3.14. Let \( \lambda \in \Lambda \) and consider \( x \in K(Q_\lambda, B) \) with itinerary
\[ \theta_0 \theta_1 \ldots \theta_{n-1} 1 1 1 \ldots, \]
for \( Q_\lambda \), i.e. \( Q_\lambda^n(x) = 1 \), for some \( n \geq 1 \).
Suppose that there exists \( \epsilon \in (0, 1) \) such that for all \( \lambda_0, \lambda_1 \) in \( \{ \omega : |\omega - \lambda| \leq \epsilon \} \), is true that \( |Q_{\lambda_0}^n(x) - Q_{\lambda_1}^n(x)| = |\lambda_0 - \lambda_1| \).
Let \( M, m \in \mathbb{N} \) be such that \( p^{-M} \leq \epsilon \) and
\[ p^M \cdot \rho_{m-1} \cdot \ldots \cdot \rho_1 < 1. \]
Then there exists \( \lambda' \in \Lambda \) with \( |\lambda - \lambda'| \leq p^{-M} \) such that \( x \) has itinerary
\[ \theta_0 \theta_1 \ldots \theta_{n-1} 1 \ldots 1 0 \ldots 1 1 \ldots \]
for \( Q_{\lambda'} \)
and such that for all pairs of elements \( \lambda_0, \lambda_1 \) in \( \{ \omega : |\omega - \lambda'| \leq S p^{-M} \} \), we have that
\[ |Q_{\lambda_0}^{n+m}(x) - Q_{\lambda_1}^{n+m}(x)| = |\lambda_0 - \lambda_1|. \]

Proof: Let \( \phi : \Lambda \rightarrow \mathbb{C}_p \) be the function defined by \( \phi(w) = Q_w^{n+m}(x) \). By Proposition 3.12 we have that \( \phi \) is holomorphic in \( \Lambda \). Furthermore, by hypothesis, if \( \lambda_0, \lambda_1 \in B_{p^{-M}}(\lambda) \), then \( |Q_{\lambda_0}(x) - Q_{\lambda_1}(x)| = |\lambda_0 - \lambda_1| \leq p^{-M} \).
Applying Lemma 3.12 to \( z_0 = Q_{\lambda_0}^0(x) \) and \( z_1 = Q_{\lambda_1}^0(x) \), we have that
\[ |\phi(\lambda_0) - \phi(\lambda_1)| = p^M |\lambda_0 - \lambda_1|. \quad (18) \]
Then, by Corollary 2.4 we have that \( \phi(\{ w : |w - \lambda| = p^{-M} \}) = \{ w : |w - 1| = 1 \} \). Therefore, there exists \( w_0 \in \Lambda \) such that \( \phi(w_0) = 0 \). If \( w \in \{ z : |z - w_0| < p^{-M} \} \), the itinerary of \( x \) for \( Q_w \) is
\[ \theta_0 \theta_1 \ldots \theta_{n-1} 1 \ldots 1 0 \ldots \]
this is direct consequence of \( 18 \).
Now let us consider the function \( \psi : \Lambda \rightarrow \mathbb{C}_p \) defined by \( \psi(w) = Q_w^{n+m+M}(x) \).
Since $Q_\lambda$ leaves $B_\rho(0)$ fixed, we obtain that $|\psi(w_0)| < \rho$. Now, by (18) we have that $|\phi(w)| = \rho_m$ if $w$ is such that $|w - w_0| = \rho_mp^{-M}$, hence $|\psi(w)| = 1$. Using again Corollary 2.4 we observe that

$$\psi(\{w : |w - w_0| = p^{-M}\rho_m\}) = \{w : |w| = 1\}.$$ 

Therefore, there exists $\lambda'$ such that $\psi(\lambda') = 1$. Thus, the itinerary of $x$ for $Q_{\lambda'}$ is

$$\theta_0 \theta_1 \ldots \theta_{n-1} \underbrace{1 \ldots 10 \ldots 0}_M 1 \ldots$$

Notice that $|\phi(\lambda')| = \rho_m$. If $\lambda_0, \lambda_1$ belong to $\{w : |w - \lambda'| \leq S p^{-M}\}$, then the points $z_0 = \phi(\lambda_0)$ and $z_1 = \phi(\lambda_1)$ are such that $|z_0| = |z_1| = \rho_m$ and $|z_0 - z_1| = \rho_m p^M |\lambda_0 - \lambda_1| \leq S$, by (18). Moreover, by hypothesis, we have

$$\rho_{m-1} \cdots \rho_1 |z_0 - z_1| = \rho_{m-1} \cdots \rho_1 p^M |\lambda_0 - \lambda_1| < |\lambda_0 - \lambda_1|.$$ 

Now, applying Lemma 3.13 we obtain that

$$|Q_{\lambda_0}^{n+M+m}(x) - Q_{\lambda_1}^{n+M+m}(x)| = |\lambda_0 - \lambda_1|.$$ 

\[\square\]

**Proof of Theorem 3.3**

We define the sequence $\{M_i\}_{i \in \mathbb{N}}$ recursively. Choose $M_0 \in \mathbb{N}$ such that $p^{-M_0} \leq S$, and suppose that $M_i$ is already defined. Now choose $M_{i+1} \in \mathbb{N}$ satisfying $p^{M_{i+1} - M_i} \leq S$.

Furthermore, we define $\{m_i\}_{i \in \mathbb{N}}$ such that for each $i \in \mathbb{N}$,

$$\rho_{m_i-1} \cdots \rho_1 \cdot p^{M_i} \leq 1.$$ 

For an arbitrary $\lambda_0 \in \Lambda$ there exists $x \in B_0$ such that $Q_{\lambda_0}^{m_0}(x) = 1$, i.e. its itinerary for $Q_{\lambda_0}$ is

$$0 \ldots 0111 \ldots$$

By Lemma 3.13 for $\lambda \in \Lambda$ with $|\lambda - \lambda_0| \leq S$, we have

$$|Q_{\lambda}^{m_0}(x) - Q_{\lambda_0}^{m_0}(x)| = |\lambda - \lambda_0|.$$ 

Since $\rho_{m_{i+1} - 1} \cdots \rho_1 \cdot p^{M_{i+1}} \leq 1$, we have that

$$p^{M_i} \cdot \rho_{m_{i+1} - 1} \cdots \rho_1 \leq S < 1.$$ 

Therefore for $\lambda = \lambda_0, n = m_0, m = m_1$ and $\epsilon = p^{-M_0}$ the hypothesis of Proposition 3.14 hold. Hence we may consider $\lambda_1$ with $|\lambda_0 - \lambda_1| \leq p^{-M_0}$ and such that the itinerary of $x$ for $Q_{\lambda_1}$ is

$$0 \ldots 01 \ldots 10 \ldots 011 \ldots$$

In view of the second part of Proposition 3.14 for all pairs of elements $\omega_0, \omega_1$ in $\{\omega : |\omega - \lambda_1| \leq S p^{-M_0}\}$ we have that $|Q_{\omega_0}^{m_0+M_0+m_1}(x) - Q_{\omega_2}^{m_0+M_0+m_1}(x)| = |\omega_0 - \omega_1|$, then we can
use this proposition recursively. For the $i$-th step we consider $n = n_0 + M_0 + \ldots + m_{i-1} + M_{i-1} + m_i$, $\lambda = \lambda_{i-1} + \epsilon_i = S^{p^{-M_{i-1}}}$, obtaining $\lambda_i \in \Lambda$ with $|\lambda_i - \lambda_{i-1}| \leq S^{p^{-M_i}}$ and such that the itinerary of $x$ for $Q_{\lambda_i}$ is

$$0 \ldots 0 1 \ldots 1 \ldots 10 \ldots 0 \ldots 11 \ldots$$

By definition $\lim_{i \to \infty} M_i = \infty$, and since $|\lambda_{i+1} - \lambda_i| \leq S^{p^{-M_i}}$, $\{\lambda_i\}_{i \in \mathbb{N}}$ is a Cauchy sequence. If we call its limit $\lambda$ we have that $|\lambda - \lambda_0| \leq p^{-M_0}$ and the itinerary of $x$ for $Q_\lambda$ is

$$0 \ldots 0 1 \ldots 1 \ldots 10 \ldots 0 \ldots 1 \ldots$$

Moreover the sequences $\{M_i\}_{i \in \mathbb{N}}, \{m_i\}_{i \in \mathbb{N}}$ satisfy the hypothesis of Lemma 3.10, therefore $Q_\lambda$ has a wandering disc contained in $K(Q_\lambda, B)$, which is not attracted to an attracting cycle.

Finally, recall that $\lambda_0 \in \Lambda$ and $M_0 \in \mathbb{N}$ were chosen arbitrarily and since $|\lambda - \lambda_0| \leq p^{-M_0}$ we have that for a dense set of parameters $\lambda \in \Lambda$ the function $Q_\lambda$ has a wandering disc in $K(Q_\lambda, B)$ which is not attracted to an attracting cycle.  

$\square$
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