Solution of nonlinear system of equations through homotopy path

A. Dutta\textsuperscript{a,1} and A. K. Das\textsuperscript{b,2}

\textsuperscript{a}Department of Mathematics, Jadavpur University, Kolkata, 700 032, India

\textsuperscript{b}SQC & OR Unit, Indian Statistical Institute, Kolkata, 700 108, India

\textsuperscript{1}Email: aritradutta001@gmail.com

\textsuperscript{2}Email: akdas@isical.ac.in

Abstract

The paper aims to show the equivalency between nonlinear complementarity problem and the system of nonlinear equations. We propose a homotopy method with vector parameter $\lambda$ in finding the solution of nonlinear complementarity problem through a system of nonlinear equations. We propose a smooth and bounded homotopy path to obtain solution of the system of nonlinear equations under some conditions. An oligopolistic market equilibrium problem is considered to show the effectiveness of the proposed homotopy continuation method.

Keywords: Nonlinear complementarity problem, system of nonlinear equations, homotopy function with vector parameter, bounded smooth curve, oligopolistic market equilibrium.

AMS subject classifications: 90C33, 90C30, 15A69.
1 Introduction

A class of problems arising in various fields of sciences, can be studied via the non-linear system of equations using various techniques. In recent years, researchers are interested to solve nonlinear system of equations both analytically and numerically. Several iterative methods have been developed using different techniques such as Taylor’s series expansion, quadrature formulas, homotopy method, interpolation, decomposition and its various modification. For details, see [30], [28], [8], [29], [7] and [34].

Eaves and Saigal [6] formed an important class of globally convergent methods for solving systems of non-linear equations, which is known as homotopy method. Such methods have been used to constructively prove the existence of solutions to many economic and engineering problems. Let $Y, X$ be two topological spaces and $p, q : Y \to X$ be continuous maps. A homotopy from $p$ to $q$ is a continuous function $H : Y \times [0, 1] \to X$ satisfying $H(y, 0) = p(y), H(y, 1) = q(y) \forall \ y \in Y$. If such a homotopy exists, then $p$ is homotopic to $q$ and it is denoted by $p \simeq q$. Let $p, q : R \to R$ any two continuous, real functions, then $p \simeq q$. Now we define a function $H : R \times [0, 1] \to R$ by $H(y, u) = (1 - u)f(y) + u g(y)$. Clearly $H$ is continuous and $H(y, 0) = p(y), H(y, 1) = q(y)$. Thus $H$ is a homotopy between $p$ and $q$. Let $Y, X$ be two topological spaces and $\text{Map}(Y, X)$ be the set of all continuous maps from $Y$ to $X$. Homotopy is an equivalence relation on $\text{Map}(Y, X)$.

The fundamental idea of the homotopy continuation method is to solve a problem by tracing a certain continuous path that leads to a solution to the problem. Thus, defining a homotopy mapping that yields a finite continuation path plays an essential role in a homotopy continuation method. The homotopy method [32] is itself an important class of globally convergent methods. Many homotopy methods are proposed for constructive proof of the existence of solutions to systems of nonlinear equations, nonlinear optimization problems, Brouwer fixed point problems, nonlinear programming, game problem and complementarity problems [33].

We are interested in solving the nonlinear system of equations. The nonlinear
complementarity problem, is identified as an important mathematical programming problem can be converted into nonlinear system of equations. The idea of nonlinear complementarity problem is based on the concept of linear complementarity problem. For recent study on this problem and applications see [4], [25], [18], [20] and references therein. For details of several matrix classes in complementarity theory, see [9], [10], [24], [22], [13], [12], [19], [23], [5] and references cited therein. The problem of computing the value vector and optimal stationary strategies for structured stochastic games for discounted and undiscounted zero-sum games and quadratic Multi-objective programming problem are formulated as linear complementary problems. For details see [14], [15], [21] and [17]. The complementarity problems are considered with respect to principal pivot transforms and pivotal method to its solution point of view. For details see [2], [26], [3] and [27].

In this paper, we consider the modified homotopy continuation method to find the solution of the nonlinear system of equations with vector parameter.

The paper is organized as follows. Section 2 presents some basic notations and results. In section 3, we prove the necessary and sufficient condition, which is a system of nonlinear equations to find the solution of nonlinear complementarity problem. A new homotopy approach with vector parameter is proposed to find the solution of nonlinear system of equations. In the last section, we consider an oligopoly market equilibrium problem [16] is used to illustrate the effectiveness of the proposed homotopy approach with vector parameter.
2 Preliminaries

Consider a function $f : \mathbb{R}^n \to \mathbb{R}^n$, and a vector $z \in \mathbb{R}^n$ such that $f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$ and $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$. The complementarity problem is to find a vector $z \in \mathbb{R}^n$ such that

$$z^T f(z) = 0, \quad f(z) \geq 0, \quad z \geq 0. \quad (2.1)$$

When the function $f$ is a nonlinear function, then it is called nonlinear complementarity problem. For details see [11].

The basic idea of homotopy method is to construct a homotopy continuation path from the auxiliary mapping $g$ to the object mapping $f$. Suppose the given problem is to find a root of the non-linear equation $f(x) = 0$ and suppose $g(x) = 0$ is an auxiliary equation with $g(x_0) = 0$. Then the homotopy function $H : \mathbb{R}^{n+1} \to \mathbb{R}^n$ can be defined as $H(x, \mu) = (1 - \mu)f(x) + \mu g(x)$, $0 \leq \mu \leq 1$ is a parameter. Then we consider the homotopy equation $H(x, \mu) = 0$, where $(x_0, 1)$ is a known solution of the homotopy equation. Our aim is to find the solution of the equation $f(x) = 0$ from the known solution of $g(x) = 0$ by solving the homotopy equation $H(x, \mu) = 0$ varying the values of $\mu$ from 1 to 0.

Now we state some results.

**Lemma 2.1:** (Generalizations of Sard’s Theorem[1]) Let $W \subset \mathbb{R}^n$ be an open set and $f : \mathbb{R}^n \to \mathbb{R}^q$ be smooth. We say $z \in \mathbb{R}^q$ is a regular value for $f$ if $\text{Range} Df(w) = \mathbb{R}^q \forall w \in f^{-1}(z)$, where $Df(w)$ denotes the $n \times q$ matrix of partial derivatives of $f(w)$.

**Lemma 2.2:** (Parameterized Sard Theorem [31]) Let $W \subset \mathbb{R}^p, Z \subset \mathbb{R}^q$ be open sets, and let $\phi : W \times Z \to \mathbb{R}^m$ be a $C^\omega$ mapping, where $\omega > \max\{0, q - m\}$. If $0 \in \mathbb{R}^m$ is a regular value of $\phi$, then for almost all $a \in W, 0$ is a regular value of $\phi_w = \phi(w,.)$. 

4
Lemma 2.3: (The inverse image theorem [31]) Let $\phi : W \subset R^n \to R^q$ be $C^\omega$ mapping, where $\omega > \max\{0, p - q\}$. Then $\phi^{-1}(0)$ consists of some $(p - q)$-dimensional $C^\omega$ manifolds.

Lemma 2.4: (Classification theorem of one-dimensional smooth manifold [35]) One-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval.

Now in the next section we will solve the nonlinear complementarity problem 2.1 using homotopy method with vector parameter.

3 Main results

Theorem 3.1: Let $\phi : R \to R$ be any increasing function such that $\phi(0) = 0$. Then $z$ solves the complementarity problem 2.1 if and only if

$$\phi((f_i(z) - z_i)^2 - \phi(f_i(z)|f_i(z)|) - \phi(z_i|z_i|) = 0 \tag{3.2}$$

Proof. Necessery. For each $i = 1, 2, \ldots, n$, either $z_i = 0, f_i(z) \geq 0$ or $f_i(z) = 0, z_i \geq 0$.

If $z_i = 0, f_i(z) \geq 0$ then $\phi((f_i(z) - z_i)^2 - \phi(f_i(z)|f_i(z)|) - \phi(z_i|z_i|) = \phi((f_i(z))^2) - \phi((f_i(z))^2) = 0$.

If $f_i(z) = 0, z_i \geq 0$ then $\phi((f_i(z) - z_i)^2 - \phi(f_i(z)|f_i(z)|) - \phi(z_i|z_i|) = \phi((z_i)^2) - \phi((z_i)^2) = 0$.

So the solution of (1.1) satisfies (1.2).

Sufficient. (a) To show that $f(z) \geq 0$ assume the contrary, i.e. $f_i(z) < 0$ for some $i = 1, 2, \ldots, n$. Then

$$0 \leq \phi((f_i(z) - z_i)^2) = \phi(f_i(z)|f_i(z)|) + \phi(z_i|z_i|) = \phi(-f_i(z)^2) + \phi(z_i|z_i|) < \phi(z_i|z_i|)$$

This implies that $\phi(z_i|z_i|) > 0 \implies z_i|z_i| > 0 \implies z_i > 0$ and $\phi((f_i(z) - z_i)^2) < \phi(z_i|z_i|) \implies ((f_i(z) - z_i)^2 < z_i|z_i|$. But for $z_i > 0, (z_i)^2 < (f_i(z) - z_i)^2$ [as $f_i(z) - z_i < 0, z_i > 0$ then $f_i(z) - z_i < 0, |f_i(z) - z_i| > z_i$ so $(f_i(z) - z_i)^2 > z_i^2$ ].

So, it contradicts that $f_i(z) < 0$. So $f_i(z) \geq 0$.

(b) To show that $z \geq 0$ interchange the roles of $z_i$ and $f_i(z)$.
(c) From (a) and (b) we have that $z \geq 0$ and $f(z) \geq 0$. To show $z^T f(z) = 0$ assume the contrary $z_i > 0$ and $f_i(z) > 0$ for some $i = 1, 2, \ldots, n$.

If $f_i(z) \geq z_i$, then $\phi((f_i(z) - z_i)^2) < \phi((f_i(z)^2) + \phi((z_i)^2) = \phi(f_i(z)|f_i(z)|) + \phi(z_i|z_i|)$. This contradicts that $\phi((f_i(z) - z_i)^2) = \phi(f_i(z)|f_i(z)|) + \phi(z_i|z_i|)$. Similarly, for $z_i(z) \geq f_i(z)$, interchanging the roles of $z_i$ and $f_i(z)$, we get a contradiction.

Hence it is shown that the complementarity problem of finding a $z \in R^n$ satisfying $z^T f(z) = 0$, $f(z) \geq 0$, $z \geq 0$ where $f : R^n \rightarrow R^n$ is equivalent to the problem of solving system of $n$ nonlinear equations in $n$ variables.

\begin{equation}
\psi_i(z) = \phi((f_i(z) - z_i)^2) - \phi(f_i(z)|f_i(z)|) - \phi(z_i|z_i|) = 0 \quad \forall i \in \{1, 2, \ldots, n\}, \quad (3.3)
\end{equation}

where $\phi$ is an increasing function defined by $\phi : R \rightarrow R$ such that $\phi(x) = x^3$. Now \( \frac{\partial \psi_i}{\partial z_j} = \phi’((f_i(z) - z_i)^2)2(f_i(z) - z_i)(\frac{\partial f_i}{\partial z_j} - \delta_{ij}) - \phi’(f_i(z)|f_i(z)|)2f_i(z)sgn(f_i(z))\frac{\partial f_i}{\partial z_j} - \phi’(z_i|z_i|)2z_i sgn(z_i)\delta_{ij} \)

where

\[ sgn(x) = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0 
\end{cases} \]

**Theorem 3.2:** Let $z$ be the nondegenerate solution of nonlinear complementarity problem [2,1] i.e. $z + f(z) > 0$. Let $J(f(z))$, the Jacobian of $f$ at $z$, have nonsingular principal minors and let $\phi : R \rightarrow R$ be a differentiable strictly increasing function such that $\phi'(y) > 0$ for all $y > 0$ and $\phi(0) = 0$. Then $z$ solves [3,3] and the Jacobian of $\psi$ at $z$, $J(\psi(z))$ is nonsingular.

**Proof.** The Jacobian matrix of $\psi(z)$ is defined by $J(\psi(z)) = \begin{bmatrix}
\frac{\partial \psi_1}{\partial z_1} & \frac{\partial \psi_1}{\partial z_2} & \cdots & \frac{\partial \psi_1}{\partial z_n} \\
\frac{\partial \psi_2}{\partial z_1} & \frac{\partial \psi_2}{\partial z_2} & \cdots & \frac{\partial \psi_2}{\partial z_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \psi_n}{\partial z_1} & \frac{\partial \psi_n}{\partial z_2} & \cdots & \frac{\partial \psi_n}{\partial z_n}
\end{bmatrix}$, where \( \frac{\partial \psi_i}{\partial z_j} = \phi’((f_i(z) - z_i)^2)2(f_i(z) - z_i)(\frac{\partial f_i}{\partial z_i} - 1) - \phi’(f_i(z)|f_i(z)|)2f_i(z)sgn(f_i(z))\frac{\partial f_i}{\partial z_i} - \phi’(z_i|z_i|)2z_i sgn(z_i) \) and \( \frac{\partial \psi_i}{\partial z_j} = \phi’((f_i(z) - z_i)^2)2(f_i(z) - z_i)(\frac{\partial f_i}{\partial z_j}) \).
\( \phi'(f_i(z)|f_i(z)|2f_i(z)sgn(f_i(z))\frac{\partial f_i}{\partial z_j} \) for \( i \neq j \).

Assume that \( f_i(z) = 0 \) for \( i = 1, 2, \ldots, n \), \( n \leq n_1 \) and \( f_i(z) > 0 \) for \( i = n_1 + 1, n_1 + 2, \ldots, n \). Hence by nondegeneracy of \( z, z_i > 0 \) for \( i = 1, 2, \ldots, n_1, n_1 \leq n \), \( z_i = 0 \) for \( i = n_1 + 1, n_1 + 2, \ldots, n \).

\[
\begin{bmatrix}
-\phi'((z_1)^2)2z_1 & 0 & \cdots & 0 \\
0 & -\phi'((z_2)^2)2z_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -\phi'((z_{n_1})^2)2z_{n_1}
\end{bmatrix}
\]

Now \( J(\psi(z)) = J(f(z)) + \begin{bmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix} \begin{bmatrix}
-\phi'((f_{n_1+1}(z))^2)2f_{n_1+1}(z) \\
0 & -\phi'((f_{n_1+2}(z))^2)2f_{n_1+2}(z) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -\phi'((f_n(z))^2)2f_n(z) \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

Since \( J(f(z)) \) has nonsingular principal minors and \( \phi'(y) > 0 \) for \( y > 0 \), the Jacobian of \( \psi(z) \), \( J(\psi(z)) \) is nonsingular.

### 3.1 Homotopy based solution with vector parameter \( \tilde{\lambda} \in R^n \)

The basic idea of homotopy method with \( \tilde{\lambda} \in R^n \) is to construct a multidimensional homotopy path to find the solution of the object function \( \tilde{f}(\tilde{x}) = 0 \) varying each component of \( \tilde{\lambda} \) from 1 to 0. We consider the homotopy function

\( H(\tilde{x}, \tilde{x}^{(0)}, \tilde{\lambda}) = \tilde{f}(\tilde{x}) - \tilde{\lambda}\tilde{f}(\tilde{x}^{(0)}) \), where each component of \( \tilde{f}(\tilde{x}^{(0)}) \), \( f_i(\tilde{x}^{(0)}) \neq 0 \) \( \forall i \in \{1, 2, \ldots, n\} \) and the product term \( \tilde{\lambda}\tilde{f}(\tilde{x}^{(0)}) \) is a componentwise product that is
\[
\tilde{f}(\tilde{x}(0)) = \begin{bmatrix} 
\lambda_1 f_1(\tilde{x}(0)) \\
\lambda_2 f_2(\tilde{x}(0)) \\
\vdots \\
\lambda_n f_n(\tilde{x}(0))
\end{bmatrix}, \text{ where } \tilde{\lambda} = \begin{bmatrix} 
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{bmatrix} \text{ and } \tilde{f}(\tilde{x}(0)) = \begin{bmatrix} 
f_1(\tilde{x}(0)) \\
f_2(\tilde{x}(0)) \\
\vdots \\
f_n(\tilde{x}(0))
\end{bmatrix}.
\]

In this method our main aim is to vary each component \( \lambda_i = \frac{f_i(\tilde{x})}{f(\tilde{x}(0))} \) of \( \tilde{\lambda} \) from 1 to 0.

Now we solve the nonlinear equation [3,2] using the homotopy function with vector parameter \( \tilde{\lambda} = [\lambda_1, \lambda_2, \cdots, \lambda_n]^t \in \mathbb{R}^n \). Let \( \psi_i(z) = \phi(\{f_i(z) - z_i\}) - \phi(\{f_i(z)|f_i(z)| - \phi(z_i|z_i|) = 0 \). Consider the set \( F_\mathcal{H} = \{ \tilde{x} : \tilde{x} \in \mathbb{R}^n \} \) and \( \tilde{F}_\mathcal{H} = \{ (\tilde{x}, \tilde{\lambda}) : (\tilde{x}, \tilde{\lambda}) \in \mathbb{R}^n \times (0, 1]^n \} \). The homotopy function can be given as,

\[
\mathcal{H}(\tilde{x}, \tilde{x}(0), \tilde{\lambda}) = \psi(\tilde{x}) - \tilde{\lambda} \psi(\tilde{x}(0)) = \begin{bmatrix} 
\psi_1(\tilde{x}) - \lambda_1 \psi_1(\tilde{x}(0)) \\
\psi_2(\tilde{x}) - \lambda_2 \psi_2(\tilde{x}(0)) \\
\vdots \\
\psi_n(\tilde{x}) - \lambda_n \psi_n(\tilde{x}(0))
\end{bmatrix}.
\]

Where, \( \tilde{x}(0) \) is the initial known value of the vector \( \tilde{x} \in \mathbb{R}^n \) such that \( \psi_i(\tilde{x}(0)) \neq 0 \forall i \in \{1, 2, \cdots, n\} \) and \( \tilde{\lambda} \in \mathbb{R}^n \). In this modification, \( \tilde{\lambda} \) is taken as a vector.

**Theorem 3.3:** If the Jacobian matrix \( \frac{\partial \psi(\tilde{x}(0))}{\partial \tilde{x}(0)} \) at initial point \( \tilde{x}(0) \) is nonsingular, then for almost all initial points \( \tilde{x}(0) \in \tilde{F}_\mathcal{H} \), 0 is a regular value of the homotopy function \( \mathcal{H} : \mathbb{R}^n \times (0, 1]^n \rightarrow \mathbb{R}^n \) and the zero point set \( \mathcal{H}_{\tilde{x}(0)}^{-1}(0) = \{(\tilde{x}, \tilde{\lambda}) \in \tilde{F}_\mathcal{H} : \mathcal{H}_{\tilde{x}(0)}(\tilde{x}, \tilde{\lambda}) = 0 \} \) contains a smooth curve \( \Gamma_{\tilde{x}(0)} \) starting from \( (\tilde{x}(0), e) \).

**Proof.** The Jacobian matrix of the above homotopy function \( \mathcal{H}(\tilde{x}, \tilde{x}(0), \tilde{\lambda}) \) is denoted by \( D\mathcal{H}(\tilde{x}, \tilde{x}(0), \tilde{\lambda}) \) and we have

\[
D\mathcal{H}(\tilde{x}, \tilde{x}(0), \tilde{\lambda}) = \begin{bmatrix} 
\frac{\partial \mathcal{H}(\tilde{x}, \tilde{x}(0), \tilde{\lambda})}{\partial \tilde{x}} & \frac{\partial \mathcal{H}(\tilde{x}, \tilde{x}(0), \tilde{\lambda})}{\partial \tilde{x}(0)} & \frac{\partial \mathcal{H}(\tilde{x}, \tilde{x}(0), \tilde{\lambda})}{\partial \lambda}
\end{bmatrix}.
\]

For all \( \tilde{x}(0) \in \tilde{F}_\mathcal{H} \) such that \( \psi(\tilde{x}(0)) \neq 0 \) and \( \tilde{\lambda} \in (0, 1]^n \), we have

\[
\frac{\partial \mathcal{H}(\tilde{x}, \tilde{x}(0), \tilde{\lambda})}{\partial \tilde{x}(0)} = \begin{bmatrix} 
-\lambda_1 \frac{\partial \psi_1(\tilde{x}(0))}{\partial x_1} & -\lambda_1 \frac{\partial \psi_1(\tilde{x}(0))}{\partial x_2} & \cdots & -\lambda_1 \frac{\partial \psi_1(\tilde{x}(0))}{\partial x_n} \\
-\lambda_2 \frac{\partial \psi_2(\tilde{x}(0))}{\partial x_1} & -\lambda_2 \frac{\partial \psi_2(\tilde{x}(0))}{\partial x_2} & \cdots & -\lambda_2 \frac{\partial \psi_2(\tilde{x}(0))}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
-\lambda_n \frac{\partial \psi_n(\tilde{x}(0))}{\partial x_1} & -\lambda_n \frac{\partial \psi_n(\tilde{x}(0))}{\partial x_2} & \cdots & -\lambda_n \frac{\partial \psi_n(\tilde{x}(0))}{\partial x_n}
\end{bmatrix}.
\]

Now \( \det \frac{\partial \mathcal{H}(\tilde{x}, \tilde{x}(0), \tilde{\lambda})}{\partial \tilde{x}(0)} = (-1)^n \det J_\mathcal{H}(\tilde{x}(0)) \prod_{i=1}^{i=n} \lambda_i \neq 0 \) for \( \lambda \in (0, 1]^n \), where \( J_\mathcal{H}(\tilde{x}(0)) = \frac{\partial \psi(\tilde{x}(0))}{\partial \tilde{x}(0)} \). Thus \( D\mathcal{H}(\tilde{x}, \tilde{x}(0), \tilde{\lambda}) \) is of full row rank. Therefore, 0 is a regular value of
\( \mathcal{H}(\bar{x}, \bar{x}(0), \bar{\lambda}) \) and \( \mathcal{H}^{-1}_0(0) \) consists of some smooth curves such that \( \mathcal{H}_{\bar{x}(0)}(\bar{x}(0), 1) = 0 \). Hence there must be a smooth curve \( \Gamma_{\bar{x}(0)} \) starting from \( (\bar{x}(0), 1) \).

**Theorem 3.4:** Assume that the function \( f(\bar{x}) \) is either increasing or bounded function. The smooth curve \( \Gamma_{\bar{x}(0)} \) is a bounded curve.

**Proof.** From the homotopy function \( \psi_1 \) we get \( \psi_i(\bar{x}) = \lambda_i \psi_i(\bar{x}(0)) \forall i \in \{1, 2, \ldots, n\}, \) where \( \psi_i(\bar{x}) = \phi((f_i(\bar{x}) - \bar{x}_i)^2) - \phi(f_i(\bar{x})|f_i(\bar{x}))| - \phi(\bar{x}_i|\bar{x}_i|) \). It is clear that \( \lambda_i = [0, 1], \psi_i(\bar{x}) \)

This implies that \( \psi_i(\bar{x}) = \phi((f_i(\bar{x}) - \bar{x}_i)^2) - \phi(f_i(\bar{x})|f_i(\bar{x}))| - \phi(\bar{x}_i|\bar{x}_i|) = \lambda_i \psi_i(\bar{x}(0)) \forall i \in \{1, 2, \ldots, n\} \). Let \( \lambda_i \psi_i(\bar{x}(0)) = k_i \). For \( \lambda_i \in (0, 1), k_i > 0 \).

Hence \( \phi((f_i(\bar{x}) - \bar{x}_i)^2) = \phi(f_i(\bar{x})|f_i(\bar{x}))| + \phi(\bar{x}_i|\bar{x}_i|) + k_i \).

Again \( \forall i \in \{1, 2, \ldots, n\} \) we have \( \psi_i(\bar{x}) = ((f_i(\bar{x}) - \bar{x}_i)^6) - (f_i(\bar{x})|f_i(\bar{x}))|^3 - (\bar{x}_i|\bar{x}_i|)^3 = (f_i(\bar{x}))^6 - 6(f_i(\bar{x}))^5\bar{x}_i + 15(f_i(\bar{x}))^4(\bar{x}_i)^2 - 20(f_i(\bar{x}))^3(\bar{x}_i)^3 + 15(f_i(\bar{x}))^2(\bar{x}_i)^4 - 6(f_i(\bar{x}))(\bar{x}_i)^5 + (\bar{x}_i)^6 - (f_i(\bar{x}))^2(f_i(\bar{x}))| - (\bar{x}_i)^5|(|\bar{x}_i|) | \).

Consider the homotopy path \( \Gamma_{\bar{x}(0)} \) is unbounded. Then there exists a sequence of points \( (\bar{x}(k), \hat{\lambda}(k)) \subset \Gamma_{\bar{x}(0)} \) such that \( \|\bar{x}(k)\| \to \infty \). Then there are two possibilities.

**Case 1:** Let \( \|\bar{x}(k)\| \to \infty \). Then \( \exists \ i \in \{1, 2, \ldots, n\} \) such that \( \bar{x}_i(k) \to -\infty \) as \( k \to \infty \).

Let set \( I_{1\bar{x}} = \{i \in \{1, 2, \ldots, n\} : \lim_{k \to \infty} \bar{x}_i(k) \to -\infty \} \).

Now consider \( \|f(\bar{x}(k))\| < \infty \). Then from above we have \( \forall i \in I_{1\bar{x}}, \) as \( k \to \infty, \)

\( \frac{\psi_i(\bar{x}(k))}{(\bar{x}_i(k))^6} \to 2 \), which contradicts that \( \psi(x) \) is bounded.

Again consider that the nonlinear function \( f(\bar{x}) \) is unbounded. It is noted that \( f(\bar{x}) \)

is an increasing function.

Let \( L_{1f(\bar{x})} = \{l \in \{1, 2, \ldots, n\} : \lim_{k \to \infty} f_l(\bar{x}(k)) \to -\infty \} \) and consider \( \lim_{k \to \infty} \frac{f_l(\bar{x}(k))}{(\bar{x}_i(k))^6} = p \forall i \in I_{1\bar{x}} \cap L_{1f(\bar{x})} \). Therefore \( \forall i \in I_{1\bar{x}} \cap L_{1f(\bar{x})}, \) as \( k \to \infty, \)

\( \frac{\psi_i(\bar{x}(k))}{(\bar{x}_i(k))^6} \to 2p^6 - 6p^5 + 15p^4 - 20p^3 + 15p^2 - 6p + 2 \). From the boundedness of \( \psi(x(k)) \), it is clear that \( 2p^6 - 6p^5 + 15p^4 - 20p^3 + 15p^2 - 6p + 2 = 0 \), has no real solution, which is a contradiction.

Now for all \( l \in L_{1f(\bar{x})}, l \notin I_{1\bar{x}}, \lim_{k \to \infty} \frac{\psi_l(\bar{x}(k))}{f_l(\bar{x}(k))^6} \to 2 \), contradicts the boundedness of the
function $\psi(\tilde{x})$.

**Case 2:** Let $\|\tilde{x}^{(k)}\| \to \infty$. Then $\exists j \in \{1, 2, \cdots, n\}$ such that $\tilde{x}_j^{(k)} \to \infty$ as $k \to \infty$. Let set $I_{2\tilde{x}} = \{j \in \{1, 2, \cdots, n\} : \lim_{k \to \infty} \tilde{x}_j^{(k)} \to \infty\}$.

Let $\tilde{x}_j > 0$, $f_j(\tilde{x}) \geq 0$. Consider that $\lambda_j \in (0, 1]$. Then $\phi((f_j(\tilde{x}) - \tilde{x}_j)^2) = \phi(f_j(\tilde{x})|f_j(\tilde{x})) + \phi(\tilde{x}_j|\tilde{x}_j) + k_j = \phi((f_j(\tilde{x}))^2 + \phi((\tilde{x}_j)^2) + k_j$, where $k_j > 0$. Now $0 \leq \phi((f_j(\tilde{x}) - \tilde{x}_j)^2) \leq \phi((f_j(\tilde{x}))^2 + \phi((\tilde{x}_j)^2) + k_j$. This contradicts that $\phi((f_j(\tilde{x}) - \tilde{x}_j)^2) = \phi((f_j(\tilde{x}))^2 + \phi((\tilde{x}_j)^2) + k_j$.

Therefore if $\psi_j(\tilde{x}) = \lambda_j \psi_j(\tilde{x}(0))$, $\lambda_j \in (0, 1]$ has the solution $\tilde{x}_j > 0$, then $f_j(\tilde{x}) < 0$.

Now consider that $\|f(\tilde{x}(k))\| < \infty$.

Then $\forall j \in I_{2\tilde{x}}$, as $k \to \infty$, $\psi_j(\tilde{x}(k)) \to -6(f_j(\tilde{x}(k))) \to 0$, which is a contradiction.

Therefore considering both the cases we conclude that the homotopy path $\Gamma_{\tilde{x}(0)}$ is bounded for $\lambda_i \in (0, 1] \forall i$.

Note that here we use the vector $\tilde{\lambda}$, such that $\tilde{\lambda} : [0, \infty]^n \to (0, 1]^n$ with $\lambda_i = \exp(-t_i)$.

**Remark 3.1:** Now we trace the homotopy path $\Gamma_{\tilde{x}(0)} \subset H_{\tilde{x}(0)}^{-1}(0) \subset \tilde{F}_H$ from the initial point $(\tilde{x}(0), e)$ until $\tilde{\lambda} \to 0$ and find the solution of the system of nonlinear equation 3.2 under some assumptions. If the homotopy path is bounded and $\tilde{\lambda}$ goes to 0 starting from $e$, the component $\tilde{x}$ give the solution of 3.2. Let $s$ denote the arc length of $\Gamma_{\tilde{x}(0)}$, we can parameterize the homotopy path $\Gamma_{\tilde{x}(0)}$ with respect to $s$ in the following form

$$H_{\tilde{x}(0)}(\tilde{x}(s), \tilde{\lambda}(s)) = 0, \quad \tilde{x}(0) = \tilde{x}(0), \quad \tilde{\lambda}(0) = e.$$ (3.5)

Now differentiating 3.5 with respect to $s$, we obtain the following system of ordinary differential equations with given initial values

$$H'_{\tilde{x}(0)}(\tilde{x}(s), \tilde{\lambda}(s)) \left[ \frac{d\tilde{x}}{ds} \right] = 0, \quad \|\left(\frac{d\tilde{x}}{ds}, \frac{d\tilde{\lambda}}{ds}\right)\|_2 = 1, \quad \tilde{x}(0) = \tilde{x}(0), \quad \tilde{\lambda}(0) = e.$$ (3.6)

and the $\tilde{x}$-component of $(\tilde{x}(s), \tilde{\lambda}(s))$ gives the solution of the complementarity problem for $\tilde{\lambda}(s) = 0$.

**Theorem 3.5:** The homotopy path $\Gamma_{\tilde{x}(0)}$ is determined by the initial value problem 3.6.
Proof. Let $s$ denote the arc length of the homotopy path $\Gamma_{\tilde{x}(0)}$. Now differentiating the homotopy equation (3.5), we get
\[
\frac{\partial H}{\partial \tilde{x}} \frac{d\tilde{x}}{ds} + \frac{\partial H}{\partial \tilde{\lambda}} \frac{d\tilde{\lambda}}{ds} = 0.
\]
Let $\nu = \begin{bmatrix} \tilde{x} \\ \tilde{\lambda} \end{bmatrix}$. As $s$ is the arc length of $\Gamma_{\tilde{x}(0)}$, then $\|\nu'\| = \|\langle \frac{d\tilde{x}}{ds}, \frac{d\tilde{\lambda}}{ds} \rangle\|_2 = 1$. Then we get the following system of ordinary differential equation.

\[
\left[ \begin{array}{c}
\frac{\partial H}{\partial \tilde{x}} \\
\frac{\partial H}{\partial \tilde{\lambda}}
\end{array} \right] \mu = 0, \mu^t \mu = 1, \frac{d\nu}{ds} = \mu, \nu(0) = \begin{bmatrix} \tilde{x}^{(0)} \\ e \end{bmatrix}
\]

(3.7)

Hence from the system (3.7) the first two equations are solvable on $\mu$. Solving the following Cauchy problem, the homotopy curve $\Gamma_{\tilde{x}(0)}$ can be derived.

\[
\frac{d\nu}{ds} = \mu, \nu(0) = \begin{bmatrix} \tilde{x}^{(0)} \\ e \end{bmatrix}.
\]

(3.8)
### 3.2 Algorithm

#### Algorithm 1  
**Homotopy Continuation Method**

**Step 0:** Set $i = i_c = 0$. [$i$ is the Number of Iteration(s) and $i_c$ is the Number of shifting of the Initial Point(s).] Give an initial point $(\tilde{x}^{(0)}, \tilde{\lambda}_0) \in \tilde{F}_H \times \{1\}^n$. Set $\eta_1 = 10^{-12}, c_0 = 50$.

$k_1 = \sqrt{2}, k_2 = 9000$, where the step-length is determined by $k_1^k, k \in Z$ and the limit of the maximum step-length is maintained by $k_1^k \leq k_2$.

Set $\tilde{u}_1 = e, \tilde{p}_1 = \tilde{\lambda}^{-1}(e)$ and $\tilde{p}_0 = \tilde{\lambda}^{-1}(\tilde{0})$, where $e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, vector of all 1’s and

$\tilde{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, vector of all 0’s.

Set $\epsilon_1 = 10^{-16} e, \epsilon_2 = 10^{-10} e$. These are real numbers, used as thresholds for $\tilde{\lambda}$. If $\tilde{\lambda}$ achieves a value $0 \leq \tilde{\lambda} \leq \epsilon_1$, then the algorithm stops with an acceptable solution. If $\tilde{\lambda}$ achieves a value, such that, $\epsilon_1 < \tilde{\lambda} \leq \epsilon_2$, the algorithm stops, declaring that point as probable solution.

**Step 1:** Set $\begin{bmatrix} \tilde{x} \\ \tilde{t} \end{bmatrix} = \begin{bmatrix} \tilde{x}^{(0)} \\ \tilde{p}_1 \end{bmatrix}$. Now calculate the constant $d^{(0)} = \det(\frac{\partial H}{\partial \tilde{x}}(\tilde{x}^{(0)}, \tilde{\lambda}_0))$ and $s^{(0)} = \frac{d\tilde{\lambda}(t)}{dt}(t = p_1)$. If $|d^{(0)}| \leq \epsilon$ or $|s^{(0)}| \leq \epsilon$, then stop else go to step 2 [$\epsilon \to 0$, a threshold].

**Step 2:** Set $c_1 = c_2 = 0$. Calculate the constant $d = \det(\frac{\partial H}{\partial \tilde{x}}(\tilde{x}, \tilde{\lambda})), N = \frac{d\tilde{\lambda}(t)}{dt}$, and $s = Ne$. If $|d| \leq \epsilon$, then stop else go to step 3.
**Step 3:** Determine the unit predictor direction \( \tau^{(n)} \) by the following method: If \( \text{sign}(d) = -\text{sign}(d_0) \), then set \( \tilde{t}_d = N^{-1}(e - \hat{\lambda}(t)) \), else set \( \tilde{t}_d = -N^{-1}\hat{\lambda}(t) \). Calculate
\[
\tilde{w}_d = - \left( \frac{\partial \mathcal{H} (\bar{x}, \hat{\lambda})}{\partial x} \right)^{-1} \left( \frac{\partial \mathcal{H} (\bar{x}, \hat{\lambda})}{\partial \lambda} \right) \tilde{t}_d, \quad \tau^{(n)} = \begin{bmatrix} \bar{x}_n \\ \tilde{t}_d \end{bmatrix} = \frac{1}{\| \tilde{x}_d, \tilde{t}_d \|} \begin{bmatrix} \bar{x}_d \\ \tilde{t}_d \end{bmatrix}, \quad \tilde{r} = \frac{\| \tilde{t}_d \|}{\| \bar{x}_d, \tilde{t}_d \|},
\]
where \( \| \tilde{x}_d, \tilde{t}_d \| = \sqrt{\tilde{x}_d^2 + \tilde{t}_d^2} \). If \( \tilde{r} \leq \eta_1 \), then set \( c_1 = c_1 + 1 \) else reset \( c_1 = 0 \). If \( c_1 < c_0 \), then go to step 4 else, if \( \bar{u}_1 \leq \epsilon_2 \) then, stop with a probable solution else, stop due to non-convergence.

**Step 4:** Choosing step length: Set \( k = 0; \gamma = [\nabla \mu(\bar{x})]^{t} \bar{x}_n \), where \( \mu : R^n \rightarrow R \), is used to increase step length in the descent direction(s). Set this Function, such that, \( \mu(\bar{x}) \leq \mu(\bar{x}) \), and \( \bar{x}, \tilde{\bar{x}} \in \mathcal{F}_H \), where \( \bar{x} \) is the solution of the problem. \( \mu(\bar{x}) \) is taken as
\[
[\mathcal{H}_0(\bar{x})]^t [\mathcal{H}_0(\bar{x})], \quad \mathcal{H}_0(\bar{x}) = \begin{bmatrix} \psi_1(\bar{x}) \\ \psi_2(\bar{x}) \\ \vdots \\ \psi_n(\bar{x}) \end{bmatrix}.
\]
If \( \gamma \geq 0, \bar{x} + \kappa_1^{k+1} \bar{x}_n \in \mathcal{F}_H \), \( \min(\tilde{p}_0, \tilde{p}_1) < \tilde{t} + \kappa_1^{k+1} \tilde{t}_n < \max(\tilde{p}_0, \tilde{p}_1) \), then set \( k = k + 1 \) and go to step 5.
else if \( \gamma < 0, \mu(\bar{x} + \kappa_1^{k+1} \bar{x}_n) < \mu(\bar{x} + \kappa_1^{k} \bar{x}_n), \bar{x} + \kappa_1^{k+1} \bar{x}_n \in \mathcal{F}_H \), \( \min(\tilde{p}_0, \tilde{p}_1) < \tilde{t} + \kappa_1^{k+1} \tilde{t}_n < \max(\tilde{p}_0, \tilde{p}_1) \), then set \( k = k + 1 \), and go to step 5.
else reset \( c_2 = 0 \), and jump to step 6.

**Step 5:** If \( \kappa_1^{k} > \kappa_2 \), then set \( k = k - 1, c_2 = c_2 + 1 \) and go to step 6, else go to step 4.

**Step 6:** If \( c_2 < c_0 \), then go to step 7, else if \( \bar{u}_1 \leq \epsilon_2 \), then stop with probable solution else stop.

**Step 7:** Compute the predictor and corrector point:
\[
\begin{bmatrix} \bar{x}_p \\ \tilde{t}_p \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \tilde{t} \end{bmatrix} + \kappa_1^{k} \begin{bmatrix} \bar{x}_n \\ \tilde{t}_n \end{bmatrix},
\]
\[
\begin{bmatrix} \bar{x}_c \\ \tilde{t}_c \end{bmatrix} = \begin{bmatrix} \bar{x}_p \\ \tilde{t}_p \end{bmatrix} - [J_H(\bar{x}_p, \tilde{t}_p)^+ H(\bar{x}_p, \tilde{t}_p)], \quad \text{where } [J_H(\bar{x}_p, \tilde{t}_p)^+] \text{ is the Moore-Penrose Inverse. If } \text{det(diag}(H(\bar{x}_c, \tilde{p}_0) - H(\bar{x}_c, \tilde{p}_1))) \leq \epsilon, \text{ where } \epsilon \rightarrow 0, \text{ a threshold, then stop else } \bar{u}_s = (\text{diag}(H(\bar{x}_c, \tilde{p}_0) - H(\bar{x}_c, \tilde{p}_1))^{-1} H(\bar{x}_c, \tilde{p}_0). \text{ Now if } (\bar{x}_c, \bar{u}_s) \in \tilde{\mathcal{F}_H}, \text{ jump to step } 10, \text{ else set } k = k - 1 \text{ and go to step 8.}
**Step 8:** Calculate $a = \min(\kappa_2^k, \|\tilde{x} - \bar{x}_c\|)$. If $a \leq \eta_2$, then go to step 9 else jump back to step 5.

**Step 9:** If $\tilde{u}_1 \leq \epsilon_2$, then stop with a Probable Solution else, set $i_c = i_c + 1$ and jump back to Step 1, after changing the Initial Point as, $\bar{x}^{(0)} = \tilde{x}_c$.

**Step 10:** Set $\tilde{t}_s = \tilde{\lambda}^{-1}(\tilde{u}_s)$, $\begin{bmatrix} \tilde{x} \\ \tilde{t} \end{bmatrix} = \begin{bmatrix} \bar{x}_c \\ \tilde{t}_s \end{bmatrix}$, $\tilde{u}_1 = \|\tilde{u}_s\| \sqrt{n}$. If $\tilde{u}_1 \leq \epsilon_1$, then stop with acceptable homotopy solution else set $i = i + 1$ and go to step 2.

**Theorem 3.6:** If the homotopy curve $\Gamma^{(0)}_{\tilde{x}}$ is smooth, then the positive predictor direction $\tau^{(0)}$ at the initial point $\tilde{x}^{(0)}$ satisfies $\text{sign}(\det \begin{bmatrix} \frac{\partial H}{\partial \tilde{\lambda}}(\tilde{x}^{(0)}, 1) \\ e\tau^{(0)} \end{bmatrix}) = (-1)^n \text{sign} \det J_H\tilde{x}^{(0)}$, where $J_H\tilde{x}^{(0)} = \frac{\partial \Psi(\tilde{x}^{(0)})}{\partial \tilde{x}^{(0)}}$.

**Proof.** From equation 3.4 we have

$$H(\tilde{x}, \tilde{x}^{(0)}, \tilde{\lambda}) = \begin{bmatrix} \psi_1(\tilde{x}) - \lambda_1\psi_1(\tilde{x}^{(0)}) \\ \psi_2(\tilde{x}) - \lambda_2\psi_2(\tilde{x}^{(0)}) \\ \vdots \\ \psi_n(\tilde{x}) - \lambda_n\psi_n(\tilde{x}^{(0)}) \end{bmatrix} = 0.$$  

Now $\frac{\partial H}{\partial \tilde{x} \lambda}(\tilde{x} = \tilde{x}^{(0)}, \tilde{\lambda} = e) = \left[ J_H\tilde{x}^{(0)} \quad \Psi(\tilde{x}^{(0)}) \right]$, where $\Psi(\tilde{x}^{(0)}) = \begin{bmatrix} -\psi_1(\tilde{x}^{(0)}) & 0 & 0 & \cdots & 0 \\ 0 & -\psi_2(\tilde{x}^{(0)}) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\psi_n(\tilde{x}^{(0)}) \end{bmatrix}$. Let positive predictor direction be $\tau^{(0)} = \begin{bmatrix} \tilde{t} \\ -e \end{bmatrix} = \begin{bmatrix} (\tilde{R}_1^{(0)})(-1)\tilde{R}_2^{(0)} e \\ -e \end{bmatrix}$, where $\tilde{R}_1^{(0)} = J_H\tilde{x}^{(0)}$ and $\tilde{R}_2^{(0)} = \Psi(\tilde{x}^{(0)})$. Here $\det(\tilde{R}_1^{(0)}) \neq 0$. Therefore, $\det \left[ \frac{\partial H}{\partial \tilde{x} \lambda} \right] = \det \begin{bmatrix} R_1^{(0)} & R_2^{(0)} \\ 0 & -ee^t - ee^t(R_2^{(0)})^t(R_1^{(0)})(-t)(R_1^{(0)})(-1)R_2^{(0)} \end{bmatrix} = \det(R_1^{(0)}) \det(-ee^t - ee^t(R_2^{(0)})^t(R_1^{(0)})(-t)(R_1^{(0)})(-1)R_2^{(0)}) = (-1)^n \det(R_1^{(0)}) \det(ee^t(I + A^tA))$, where $A = (R_1^{(0)})(-1)R_2^{(0)}$. So
sign(det \left[ \frac{\partial H_c}{\partial x}(\tilde{x}^{(0)}, 1) \right] \right) = (-1)^n \text{sign} \det J_H \tilde{x}^{(0)}. \]

4 Numerical Example

To illustrate the use of the above three algorithms presented in the previous section, a numerical example[16] is presented here. Following Murphy et al. [16] the oligopolistic market equilibrium problem follows:

Let there be $n$ firms, which supply a homogeneous product in a noncooperative fashion. Let $P(\tilde{Q})$, $\tilde{Q} \geq 0$ denote the inverse demand, where $\tilde{Q} = \sum_{i=1}^{n} Q_i$, $Q_i \geq 0$ denote the $i$th firm’s supply. Let $c_i(Q_i)$ be the total cost of supplying $Q_i$ units. Now the Nash equilibrium solution is a set of nonnegative output levels $Q_1^*, Q_2^*, \cdots, Q_n^*$, such that $Q_i^*$ is an optimal solution to the following problem $\forall i \in \{1, 2, \cdots, n\}$:

$$\text{maximize } Q_i P(Q_i + \tilde{Q}_i^*) - c_i(Q_i)$$

(4.9)

where $\tilde{Q}_i^* = \sum_{j \neq i} Q_j^*$. Murphy et al. show that if $c_i(Q_i)$ is convex and continuously differentiable $\forall i \in \{1, 2, \cdots, n\}$ and the inverse demand function $P(\tilde{Q})$ is strictly decreasing and continuously differentiable and the industry revenue curve $\tilde{Q}P(\tilde{Q})$ is concave, then

$(Q_1^*, Q_2^*, \cdots, Q_n^*)$ is a Nash equilibrium solution if and only if

$$[P(\tilde{Q}^*) + Q_i^*P'(\tilde{Q}^*) - c_i'(Q_i^*)]Q_i^* = 0$$

(4.10)

$$c_i'(Q_i^*) - P(\tilde{Q}^*) - Q_i^*P'(\tilde{Q}^*) \geq 0$$

(4.11)

$$Q_i^* \geq 0 \ \forall i \in \{1, 2, \cdots, n\}$$

(4.12)

where $\tilde{Q}^* = \sum_{i=1}^{n} Q_i^*$, which is a nonlinear complementarity problem with $f_i(z) = c_i'(Q_i^*) - P(\tilde{Q}^*) - Q_i^*P'(\tilde{Q}^*)$, and $z_i = Q_i^*$.

Note that here the functions $c_i(Q_i)$ and $-\tilde{Q}P(\tilde{Q})$ are convex. So the 1st order derivative of these two functions are increasing function. Hence the function $f_i(z) = c_i'(Q_i^*) - P(\tilde{Q}^*) - Q_i^*P'(\tilde{Q}^*)$ is an increasing function.

Now consider an oligopoly with five firms, each with a total cost function of the
form:

\[ c_i(Q_i) = n_i Q_i + \frac{\beta_i}{\beta_i + 1} L_i^{1/\beta_i} Q_i^{2 \beta_i + 1 / \beta_i} \]  \hfill (4.13)

The demand curve is given by:

\[ \tilde{Q} = 5000 P^{-1.1}, \quad P(\tilde{Q}) = 5000^{1/1.1} \tilde{Q}^{-1/1.1}. \]  \hfill (4.14)
The parameters of the equation \[4.13\] for the five firms are given below:

### Table 1: Value of parameters for five firms

| firm | \(n_i\) | \(L_i\) | \(\beta_i\) |
|------|--------|--------|-----------|
| 1    | 10     | 5      | 1.2       |
| 2    | 8      | 5      | 1.1       |
| 3    | 6      | 5      | 1         |
| 4    | 4      | 5      | 0.8       |
| 5    | 2      | 5      | 0.6       |

To solve this problem using the homotopy method \[3.3\] with vector parameter \(\lambda\), we first take the initial point \(\tilde{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\), \(\tilde{\lambda}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\). After 20 iterations we obtain

\[
\tilde{x} = \begin{bmatrix} 15.429308 \\ 12.498582 \\ 9.663473 \\ 7.165093 \\ 5.132566 \end{bmatrix}, \quad \tilde{\lambda} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]

### 5 Conclusion

In this study we show the equivalency between the nonlinear complementarity problem and the system of nonlinear equations. We introduce a homotopy with vector parameter to find the solution of the system of nonlinear equations as well as the solution of the corresponding nonlinear complementarity problem. In this connection we obtain the sign of the positive tangent direction of the homotopy path. We construct a homotopy continuation method and show that the method reaches to solution
through a bounded and smooth curve under the condition of the nonlinear function corresponding to the nonlinear complementarity problem being either increasing function or bounded function. Finally a real life oligopolistic market equilibrium problem is considered to illustrate our results.

6 Acknowledgment

The author A. Dutta is thankful to the Department of Science and Technology, Govt. of India, INSPIRE Fellowship Scheme for financial support.

References

[1] Shui-Nee Chow, John Mallet-Paret, and J. A. Yorke. Finding zeroes of maps: Homotopy methods that are constructive with probability one *. 2010.

[2] AK Das. Properties of some matrix classes based on principal pivot transform. Annals of Operations Research, 243, 05 2014.

[3] AK Das, R Jana, and Deepmala. On generalized positive subdefinite matrices and interior point algorithm. In Frontiers in Optimization: Theory and Applications, pages 3–16. Springer, 2016.

[4] AK Das, R Jana, and Deepmala. Finiteness of criss-cross method in complementarity problem. In International Conference on Mathematics and Computing, pages 170–180. Springer, 2017.

[5] A Dutta, R Jana, and AK Das. On column competent matrices and linear complementarity problem. In Proceedings of the Seventh International Conference on Mathematics and Computing, pages 615–625. Springer, 2022.

[6] B Curtis Eaves and Romesh Saigal. Homotopies for computation of fixed points on unbounded regions. Mathematical Programming, 3(1):225–237, 1972.
[7] Jin-yan Fan. A modified levenberg-marquardt algorithm for singular system of nonlinear equations. *Journal of Computational Mathematics*, pages 625–636, 2003.

[8] Hossein Jafari and Varsha Daftardar-Gejji. Revised adomian decomposition method for solving a system of nonlinear equations. *Applied Mathematics and Computation*, 175(1):1–7, 2006.

[9] R Jana, AK Das, and A. Dutta. On hidden z-matrix and interior point algorithm. *OPSEARCH*, 56, 09 2019.

[10] R Jana, A. Dutta, and AK Das. More on hidden z-matrices and linear complementarity problem. *Linear and Multilinear Algebra*, 69:1–10, 06 2019.

[11] Stepan Karamardian. The nonlinear complementarity problem with applications, part 1. *Journal of Optimization Theory and Applications*, 4(2):87–98, 1969.

[12] SR Mohan, S.K. Neogy, and AK Das. More on positive subdefinite matrices and the linear complementarity problem. *Linear Algebra and Its Applications*, 338(1-3):275–285, 2001.

[13] SR Mohan, S.K. Neogy, and AK Das. On the classes of fully copositive and fully semimonotone matrices. *Linear Algebra and its Applications*, 323:87–97, 01 2001.

[14] S.R. Mohan, S.K. Neogy, and AK Das. A note on linear complementarity problems and multiple objective programming. *Mathematical Programming. Series A. Series B*, 100, 06 2004.

[15] Prasenjit Mondal, S Sinha, Samir K Neogy, and AK Das. On discounted ar-at semi-markov games and its complementarity formulations. *International Journal of Game Theory*, 45(3):567–583, 2016.

[16] Frederic H Murphy, Hanif D Sherali, and Allen L Soyster. A mathematical programming approach for determining oligopolistic market equilibrium. *Mathematical Programming*, 24(1):92–106, 1982.
Samir K Neogy, AK Das, S Sinha, and A Gupta. On a mixture class of stochastic game with ordered field property. In *Mathematical programming and game theory for decision making*, pages 451–477. World Scientific, 2008.

S.K. Neogy, R Bapat, AK Das, and T Parthasarathy. *Mathematical Programming and Game Theory for Decision Making*. 04 2008.

S.K. Neogy, R. Bapat, AK Das, and T. Parthasarathy. Mathematical programming and game theory for decision making. 11 2021.

S.K. Neogy, Ravindra Bapat, AK Das, and Biswabrata Pradhan. Optimization models with economic and game theoretic applications. *Annals of Operations Research*, 243, 07 2016.

S.K. Neogy and AK Das. Linear complementarity and two classes of structured stochastic games. *Operations Research with Economic and Industrial Applications: Emerging Trends*, eds: SR Mohan and SK Neogy, Anamaya Publishers, New Delhi, India, pages 156–180, 2005.

S.K. Neogy and AK Das. On almost type classes of matrices with q-property. *Linear & Multilinear Algebra - LINEAR MULTILINEAR ALGEBRA*, 53:243–257, 07 2005.

S.K. Neogy and AK Das. Some properties of generalized positive subdefinite matrices. *SIAM J. Matrix Analysis Applications*, 27:988–995, 01 2006.

S.K. Neogy and AK Das. On weak generalized positive subdefinite matrices and the linear complementarity problem. *Linear and Multilinear Algebra*, 61, 07 2013.

S.K. Neogy, AK Das, and R. Bapat. Modeling, computation and optimization. 06 2022.

S.K. Neogy, AK Das, and Abhijit Gupta. Generalized principal pivot transforms, complementarity theory and their applications in stochastic games. *Optimization Letters*, 6:339–356, 02 2012.
[27] S.K. Neogy, AK Das, and Abhijit Gupta. Generalized principal pivot transforms, complementarity theory and their applications in stochastic games. *Optimization Letters*, 6(2):339–356, 2012.

[28] Muhammad Aslam Noor and Muhammad Waseem. Some iterative methods for solving a system of nonlinear equations. *Computers & Mathematics with Applications*, 57(1):101–106, 2009.

[29] Takeo Ojika, Satoshi Watanabe, and Taketomo Mitsui. Deflation algorithm for the multiple roots of a system of nonlinear equations. *Journal of mathematical analysis and applications*, 96(2):463–479, 1983.

[30] K Sayevand and H Jafari. On systems of nonlinear equations: some modified iteration formulas by the homotopy perturbation method with accelerated fourth- and fifth-order convergence. *Applied Mathematical Modelling*, 40(2):1467–1476, 2016.

[31] Xiuyu Wang and Xingwu Jiang. A homotopy method for solving the horizontal linear complementarity problem. *Computational and Applied Mathematics*, 33, 04 2013.

[32] Layne T Watson. Globally convergent homotopy methods: a tutorial. *Applied Mathematics and Computation*, 31:369–396, 1989.

[33] Layne T Watson and Raphael T Haftka. Modern homotopy methods in optimization. *Computer Methods in Applied Mechanics and Engineering*, 74(3):289–305, 1989.

[34] Kiyotaka Yamamura, Hitomi Kawata, and Ai Tokue. Interval solution of nonlinear equations using linear programming. *BIT Numerical Mathematics*, 38(1):186–199, 1998.

[35] X Zhao, S Zhang, and Q Liu. A combined homotopy interior point method for the linear complementarity problem. *Journal of Information and Computational Science*, 7:1589–1594, 07 2010.