GRADED BETTI NUMBERS AND \( h \)-VECTORS OF LEVEL MODULES

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Abstract. We study \( h \)-vectors and graded Betti numbers of level modules up to multiplication by a rational number. Assuming a conjecture on the possible graded Betti numbers of Cohen-Macaulay modules we get a description of the possible \( h \)-vectors of level modules up to multiplication by a rational number. We also determine, again up to multiplication by a rational number, the cancellable \( h \)-vectors and the \( h \)-vectors of level modules with the weak Lefschetz property. Furthermore, we prove that level modules of codimension three satisfy the upper bound of the Multiplicity conjecture of Herzog, Huneke and Srinivasan, and that the lower bound holds if the module, in addition, has the weak Lefschetz property.

1. Introduction

Level algebras, introduced by Stanley [21], often shows up in algebraic geometry and combinatorics, and have in recent years received a lot of attention themselves, especially the problem of describing their \( h \)-vectors. Iarrobino [13] (see also Chipalkatti and Geramita [3]) determined all \( h \)-vectors of level algebras of codimension two, and this result was generalized to level modules by the author [20]. In codimension three, Stanley determined the \( h \)-vectors of Gorenstein algebras [22], that is, level algebras of type one. There are many other results in these directions (see [5] for an overview) but so far the problem of describing the \( h \)-vectors of all level algebras seems out of reach.

In [2], Boij and the author gives a conjecture (Conjecture 3.1 below) on the possible graded Betti numbers of Cohen-Macaulay modules up to multiplication by a positive rational number, and show that this conjecture implies the Multiplicity conjecture of Herzog, Huneke and Srinivasan [10], [12]. The heuristic is that the set of possible graded Betti numbers, or \( h \)-vectors for that matter, is easier to describe, when we consider not only algebras but also modules, and care only for a description up to multiplication by a positive rational number. For example, the \( h \)-vectors of Cohen-Macaulay modules, generated in degree zero, of codimension \( p \), are, up to multiplication by a positive rational number, the finite sequences \( \{ h_i \}_{i \in \mathbb{Z}} \) satisfying

\[
\frac{h_i}{r_i} - \frac{h_{i+1}}{r_{i+1}} \geq 0
\]
for all $i$, where $r_i$ is the dimension, as a vector space, of the graded component of degree $i$ of the polynomial ring in $p$ variables. This should be compared with Macaulay’s description [14] (and Hulett’s generalization to modules of this description and [11]) which precisely characterizes these sequences but which is more combinatorial in nature and more complicated.

In this paper we take as our starting point the conjecture on the possible graded Betti numbers of Boij and the author, and ask what can be said about the $h$-vectors and graded Betti numbers of level modules if this conjecture is assumed to be true. This does not mean that all our results depend on this conjecture, but rather that we are guided by it. In particular, we look for descriptions of $h$-vectors and graded Betti numbers only up to multiplication by a rational number.

In Section 4 we determine the $h$-vectors of level modules which satisfy the condition of the conjecture of Boij and the author. It turns out that, up to multiplication with a positive rational number, these $h$-vectors are precisely those which are non-negative linear combinations of $h$-vectors of extremely compressed level modules of the same socle degree, and moreover, this condition can be expressed as a set of linear inequalities, each in three consecutive entries of the $h$-vector (Theorem 4.7). We also show that the Betti numbers of the linear combinations of extremely compressed level modules, bounds from above the Betti numbers of the level module in this case and even that the Betti numbers of the level module are obtained from these by a sequence of consecutive cancellations (Proposition 4.3). Unfortunately we can not prove that level modules in general satisfy this condition, but the connection with extremely compressed level modules is interesting, and it might be easier prove this result for level modules in general than to prove the conjecture of Boij and the author.

When considering graded Betti numbers up to multiplication by a rational number, the maximal ones, given the $h$-vector, have a simple description. This observation turns out to be very useful and is used in almost all parts of this article.

With this description of maximal Betti numbers in mind, the notion of cancellable $h$-vectors, introduced by Geramita and Lorenzini [6] and also studied by the author in [19], is revisited and we give a description, this time up to multiplication by a rational number, of all cancellable $h$-vectors (Theorem 5.2). In section 6 we study the $h$-vectors, and graded Betti numbers, of artinian level modules with the weak Lefschetz property. In [8] the possible $h$-vectors of algebras with the weak and strong Lefschetz property are determined and a sharp upper bound on the graded Betti numbers of these algebras is given. The possible $h$-vectors are also known for Gorenstein algebras with the weak Lefschetz property [7], but for level algebras in general there are very few results. We determine the possible $h$-vectors of artinian level modules with the weak Lefschetz property up to multiplication by a rational number, and furthermore, we give an upper bound on the graded Betti numbers of these modules (Theorem 6.8). This upper bound is sharp in the sense that it is a rational multiple of
the Betti diagram of some level module with the weak Lefschetz property, and furthermore, the upper bound satisfy the condition of the conjecture of Boij and the author. It turns out that the existence of this upper bound is enough to prove the conjecture of Boij and the author for level modules of codimension three with the weak Lefschetz property which in turn proves the Multiplicity conjecture for these modules (Proposition 7.5).

Finally, in Section 7 we restrict our attention to level modules of codimension three. We observe that the conjecture of Boij and the author implies a strengthening of the Multiplicity conjecture proposed by Zanello in the case of level algebras of codimension three \[17\]. We finish by proving that the upper bound of the Multiplicity conjecture holds for any level module of codimension three (Theorem 7.7).

2. Preliminaries

We begin with the basic definitions. Let \( R = k[x_1, x_2, \ldots, x_n] \) be the polynomial ring in \( n \) variables over a field \( k \). Consider \( R \) as a graded ring by giving each \( x_i \) degree one and let \( m = (x_1, x_2, \ldots, x_n) \) be the unique graded maximal ideal. All \( R \)-modules in this article are assumed to be finitely generated and graded. The \( d \)-th twist of an \( R \)-module \( M \), denoted by \( M^{(d)} \), is defined by \( M^{(d)}_i = M_{i+d} \). If the \( R \)-module \( M \) has a minimal free resolution given by

\[
0 \to \bigoplus_j R(-j)^{\beta_{p,j}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}} \to M \to 0
\]

then \( \beta_{i,j}^R(M) = \beta_{i,j} \) are the graded Betti numbers of \( M \). When the ring in consideration is clear from the context we simply omit the superscript and write \( \beta_{i,j} \) for the Betti numbers of \( M \). From the minimal free resolution of \( M \) we also see that the projective dimension of \( M \) is \( p \) and when \( M \) is Cohen-Macaulay it is equal to the codimension, that is, \( \dim R - \dim M = p \). When considering artinian \( R \)-modules of codimension \( p \) we consequently have \( n = p \). Furthermore, if \( M \) is Cohen-Macaulay then there is an element \( h_M(t) \in \mathbb{Z}[t, t^{-1}] \) such that

\[
h_M(t) = \frac{S_M(t)}{(1-t)^p},
\]

where \( S_M(t) = \sum_{i,j} (-1)^i \beta_{i,j}(M)t^j \). The element \( h_M(t) \) is called the \( h \)-vector of \( M \) and

\[
e(M) = h_M(1)
\]

is called the multiplicity of \( M \). The matrix \( \beta(M) \) with entries \( \beta_{i,j} = \beta_{i,j}(M) \) for each integer \( i \) and \( j \), is called the Betti diagram of \( M \).

The Hilbert function of \( M \) is the function \( H(M, \_): \mathbb{Z} \to \mathbb{Z} \) defined by \( H(M,d) = \dim_k M_d \). If \( M \) is an artinian \( R \)-module with \( h \)-vector \( h = \sum_{i \in \mathbb{Z}} h_i t^i \), then \( H(M,d) = h_d \).

Let \( M \) be a Cohen-Macaulay \( R \)-module of codimension \( p \). We define the dual of \( M \), denoted by \( M^! \), to be the \( R \)-module \( \text{Ext}_R^p(M, R) \). The dual of of \( M \) is
in fact Cohen-Macaulay and its Betti diagram is obtained from that of \( M \) by \( \beta_{i,j}(M') = \beta_{p-i,p-j}(M) \).

The maximal and minimal shifts of degree \( i \) of \( M \) are defined by \( \overline{d}_i(M) = \max\{ j \mid \beta_{i,j}(M) \neq 0 \} \) and \( \underline{d}_i(M) = \min\{ j \mid \beta_{i,j}(M) \neq 0 \} \), respectively. It is well known that when \( M \) is Cohen-Macaulay then both of the sequences \( d_i(M) = \max\{ j \mid \beta_{i,j}(M) \neq 0 \} \) and \( d_i(M) = \min\{ j \mid \beta_{i,j}(M) \neq 0 \} \), respectively. It is well

3. Pure diagrams and consecutive cancellations

We will now explain a conjecture of Boij and the author on the set of possible Betti diagrams of Cohen-Macaulay modules up to multiplication by a rational number. We also explain how Betti diagrams are obtained from, entry by entry, larger ones with the same \( h \)-vector by consecutive cancellations. We refer to [2] for a more detailed exposition of these matters.

We begin by explaining the notion of a diagram. Let \( D \) be a matrix with rational entries satisfying \( D_{i,j} = 0 \) when \( i < 0 \) or \( i > p \) and assume furthermore that \( D \) has only a finite number of non-zero entries. Denote by \( S_D(t) = \sum_{i,j} (-1)^i D_{i,j} t^j \) in \( \mathbb{Q}[t, t^{-1}] \). If \( S_D(t) \) is divisible by \( (1 - t)^p \) we call \( D \) a diagram of codimension \( p \). This definition is motivated by the fact that the Betti diagram of a Cohen-Macaulay module of codimension \( p \) is a diagram of codimension \( p \). Let \( d = (d_0, d_1, \ldots, d_p) \) and \( \underline{d} = (\underline{d}_0, \underline{d}_1, \ldots, \underline{d}_p) \) be two strictly increasing sequences of integers and denote by \( V_{d,\underline{d}} \) the vector space over the rational numbers of all diagrams \( D \), of codimension \( p \), satisfying \( D_{i,j} = 0 \) whenever \( j < \underline{d}_i \) or \( j > d_i \).

An \( R \)-module \( M \) has a pure resolution of type \( d = (d_0, d_1, \ldots, d_p) \) if its minimal free resolution has the form

\[
0 \to R(-d_p)_{\beta_p} \to \cdots \to R(-d_0)_{\beta_0} \to M \to 0,
\]

that is, if the maximal and minimal shifts of \( M \) are equal.

If \( D \) is a diagram whose maximal and minimal shifts both are equal to the strictly increasing sequence of integers \( d = (d_0, d_1, \ldots, d_p) \), then we call \( D \) a pure diagram of type \( d \). The dimension of the vector space \( V_{d,\underline{d}} \), for any strictly increasing sequences \( d \) and \( \underline{d} \), is \( 1 + \sum_{i=0}^p (\overline{d}_i - \underline{d}_i) \) (see [2] Prop. 2.7)), so the dimension of the vector space of all pure diagrams of type \( d \), \( V_{d,d} \), is one. Hence any two pure diagrams of the same type are rational multiples of each other, and we denote by \( \pi(d) \) the pure diagram of type \( d \) satisfying \( \pi(d)_{0,0} = 1 \).

Boij and the author have the following conjecture in [2].

**Conjecture 3.1** (Boij, Söderberg). The Betti diagram of any Cohen-Macaulay \( R \)-module is a non-negative linear combination of pure diagrams and furthermore, any pure diagram is a rational multiple of the Betti diagram of some Cohen-Macaulay \( R \)-module.
Let $M = F/N$ be a Cohen-Macaulay $R$-module, where $F$ is a free $R$-module and $N$ a submodule of $F$. Then there is a lexicographic submodule $L$ of $F$ such that $F/L$ and $M$ have the same $h$-vector (see [14] and [11]). The Betti diagram of $F/L$ is completely determined by the $h$-vector, $h(t)$, of $M$ and of the free module $F$ and we denote this diagram by $\beta_F^{\text{lex}}(h(t))$. The Betti numbers of $M$ are smaller than the Betti numbers of $F/L$, and we even have that $\beta(M)$ is obtained from $\beta_F^{\text{lex}}(h(t))$ by a sequence of consecutive cancellations (see [18]).

A consecutive cancellation in position $(k,l)$ of a diagram $D$ is a diagram $D' = D - bC_{k,l}$, where $b$ is a non-negative rational number, and we assume that $D'$ have no negative entries. Note that a consecutive cancellation does not change the polynomial $S_D(t) = \sum_i (-1)^i D_{i,j} t^j$, that is, $S_D(t) = S_{D'}(t)$, and hence not the $h$-vector or multiplicity of $D$. In short, for any Cohen-Macaulay $R$-module $M$ with $h$-vector $h(t)$ we have that

\begin{equation}
\beta(M) = \beta_F^{\text{lex}}(h(t)) - \sum_{i,j} b_{i,j} C_{i,j}
\end{equation}

for some non-negative integers $b_{i,j}$.

Let $F$ be a free $R$-module with basis $e_1, e_2, \ldots, e_t$ and assume that the basis of $F$ is ordered by $e_1 < e_2 < \cdots < e_t$ where $\deg e_1 \leq \deg e_2 \leq \cdots \leq \deg e_t$. Let $H : \mathbb{Z} \to \mathbb{Z}$ be the Hilbert function of some submodule of $F$. Then, as mentioned above, there is a lexicographic submodule $L$ with $H$ as Hilbert function and we will now describe $L$. The monomials of $F$ are the elements on the form $ue_i$ for some monomial $u$ of $R$ and basis element $e_i$ of $F$. The lexicographic order on monomials of $F$ is given by $ue_i > ve_j$ when $i < j$ or when $i = j$ and $u > v$ in the lexicographic order on monomials of $R$. The lexicographic submodule $L$ is generated in each degree $d$ by the $H(d)$ largest monomials in $F_d$.

For any degree $d$, there are unique integers $k$ and $q$ such that $0 \leq q < H(R, d - \deg e_k)$ and

\begin{equation}
H(d) = \sum_{i=1}^{k-1} H(R, d - \deg e_i) + q.
\end{equation}

For any $F$, $H$ and $d$ denote these numbers by $k(F, H, d) = k$ and $q(F, H, d) = q$ and furthermore let $g(F, H, d) = \deg e_k$. This means that $L$ is generated in degree
where $k = k(F, H, d)$ and $I$ is the ideal generated by the $q(F, H, d)$ largest monomials of degree $d - g(F, H, d)$ in $R$.

In what follows, let $e_{i,j}$, where $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, s$ and $\deg e_{i,j} = \deg e_i$, be a basis of $F^s$ ordered by $e_{i,j} > e_{i',j'}$ when $i < i'$ or when $i = i'$ and $j > j'$, for any integer $s$.

**Lemma 3.2.** Let $N$ be a submodule of $F$ such that the module $M = F/N$ is artinian. Then there is an integer $s$ and a lexicographic submodule $L$ of $F^s$ such that $L$ and $N$ have the same $h$-vector, and $L$ is on the form

$$L = \sum_{j=1}^{s} \sum_{i=1}^{t} \mathfrak{m}^{d_{i,j}} e_{i,j}$$

where $c_{i,j}$ are non-negative integers.

**Proof.** Let $H$ be the Hilbert function of $M$. To prove the lemma it is enough to show that there is a positive integer $s$ such that $q(F^s, sH, d) = 0$ for all non-negative integers $d$. Note that in (3.2), $H(d) = H(F, d)$ implies $q = 0$, and hence that $H(d) = H(F, d)$ implies $q(F, H, d) = 0$. Since $M$ is artinian, $H(d) \neq H(F, d)$ only for a finite number of integers and hence the same holds for $q(F, H, d)$. This means that there exists a positive integer $s$ such that

$$q(F, H, d) = H(S, d - g(F, H, d))$$

is an integer for all $d$. Let $s$ be such an integer.

Fix a degree $d$ and let $k = k(F, H, d)$ and $q = q(F, H, d)$ then we get by multiplying both sides of (3.2) by $s$,

$$sH(d) = s \sum_{i=1}^{k-1} H(R, d - \deg e_i) + sq = \sum_{j=1}^{s} \sum_{i=1}^{k-1} H(R, d - \deg e_{i,j}) + sq$$

and since by (3.3), $sq = s'H(R, d - \deg e_k)$ for some $0 \leq s' < s$ we get

$$sH(d) = \sum_{j=1}^{s} \sum_{i=1}^{k-1} H(R, d - \deg e_{i,j}) + s'H(R, d - \deg e_{k,j}).$$

From the above equation it follows that $q(F^s, sH, d) = 0$ for all non-negative integers $d$. This means that $L$ is generated in degree $d$ by

$$\sum_{j=1}^{s} \sum_{i=1}^{k-1} \mathfrak{m}^{d - \deg e_{i,j}} e_{i,j} + \sum_{i=1}^{s'} \mathfrak{m}^{d - \deg e_{k,j}} e_{i,j},$$

and since this holds for any degree $d$ the lemma follows. □
Proposition 3.3. Let $M = F/N$ be an artinian $R$-module of codimension $p$ with $h$-vector $h(t)$. Then there exists non-negative rational numbers $a_{i,j}$ and an integer $s$ such that the diagram

$$D = \sum_{i,j} a_{i,j} \beta(R/m^{i+1}(-\deg e_i))$$

have $h$-vector $h(t)$ and

$$\frac{1}{s} \beta_{F^s}(sh(t)) = D.$$ 

As a consequence we get that $\beta(M)$ is obtained from $D$ by a sequence of consecutive cancellations, that is,

$$\beta(M) = D - \sum_{i,j} b_{i,j} C^{i,j}$$

for some non-negative rational numbers $b_{i,j}$.

Proof. By Lemma 3.2 there is an integer $s$ and a lexicographic submodule $L$ of $F^s$ such that $F^s/L$ have $h$-vector $sh(t)$ and $L$ is on the form

$$L = \sum_{j=1}^s \sum_{i=1}^t m^{c_{i,j}} e_{i,j}$$

where $c_{i,j}$ are non-negative integers. We get

$$\beta_{F^s}(sh(t)) = \beta(F^s/L)$$

and since then

$$F^s/L = \bigoplus_{j=1}^s \bigoplus_{i=1}^t R/m^{c_{i,j}(-\deg e_{i,j})}$$

we see that

$$\beta_{F^s}(sh(t)) = \sum_{j=1}^s \sum_{i=1}^t R/m^{c_{i,j}(-\deg e_{i,j})}$$

Collecting all the terms in the sum above where the $c_{i,j}$ and $\deg e_{i,j}$ are the same, yields

$$\beta_{F^s}(sh(t)) = \sum_{i,j} a'_{i,j} R/m^{i+1}(-\deg e_i)$$

for some non-negative integers $a'_{i,j}$. Now, let $a_{i,j} = a'_{i,j}/s$ and $D = \beta_{F^s}(sh(t))/s$.

Since the Hilbert function of $M^s$ is $sh(t)$ we get that $\beta(M^s)$ is obtained from $\beta_{F^s}(sh(t)) = sD$ by a sequence of consecutive cancellations, that is,

$$\beta(M^s) = sD - \sum_{i,j} b'_{i,j} C^{i,j}$$
for some non-negative integers \(b_{i,j}\). Now \(\beta(M^s) = s\beta(M)\) so dividing both sides of the equation above with \(s\) shows that \(\beta(M)\) is obtained from \(D\) by a sequence of consecutive cancellations. \(\square\)

When \(F\) is generated in degree zero we omit the subscript \(F\) and simply write
\[
\beta_{\text{lex}}(h(t)) = \beta_{\text{lex}}^F(h(t))
\]
and moreover, in this case there is a simple expression for the numbers \(a_{i,j}\) of Proposition 3.3.

**Proposition 3.4.** Assume that \(F\) is generated in degree zero and let \(M = F/M\) be an artinian \(R\)-module, of codimension \(p\), with \(h\)-vector the polynomial \(h(t) = \sum_{i \in \mathbb{Z}} h_i t^i\) of degree \(c\). Then
\[
\frac{1}{s} \beta_{\text{lex}}(sh(t)) = \sum_{j=0}^{c} a_j \beta(R/m^{j+1})
\]
for some integer \(s\), if and only if
\[
a_j = \frac{h_j}{r_j} - \frac{h_{j+1}}{r_{j+1}}
\]
where \(r_j = \binom{p-1+j}{p-1}\). As a consequence we get that the polynomial \(h(t)\) is a rational multiple of an \(h\)-vector of an artinian \(R\)-module, generated in degree zero, if and only if the \(a_j\)'s above are non-negative.

**Proof.** By Proposition 3.3 there is an integer \(s\) and a diagram
\[
D = \sum_j a_j \beta(R/m^{j+1}),
\]
for some non-negative rational numbers \(a_j\), such that
\[
\frac{1}{s} \beta_{\text{lex}}(sh(t)) = D
\]
Denote by \(h_j\) the \(h\)-vector of \(R/m^{j+1}\) and observe that it is given by
\[
h_j(t) = r_0 + r_1 t + \cdots + r_j t^j.
\]
The \(h\)-vector of \(D\) is then
\[
h_D(t) = \sum_j a_j h_j(t).
\]
Solving the linear equation \(h(t) = \sum_j a_j h_{\pi_j}(t)\) for the rational numbers \(a_j\) gives the unique solution
\[
a_j = h_j/r_j - h_{j+1}/r_{j+1}.
\]
We also see that the \(a_j\)'s are non-negative if \(h(t)\) is the \(h\)-vector of an artinian \(R\)-module generated in degree zero, and furthermore, if the \(a_j\)'s are non-negative
then the module
\[ N = \bigoplus_{j=0}^{c} \bigoplus_{i=1}^{ma_j} R/m^{i+1} \]
where \( m \) is an integer such that \( ma_j \) is an integer for \( j = 0, 1, \ldots, c \), have \( h \)-vector \( h_N(t) = mh(t) \) which shows that \( h \) is a rational multiple of the \( h \)-vector of an artinian \( R \)-module.

4. \( h \)-VECTORS OF LEVEL MODULES

An \( R \)-module \( M \) is level if it is Cohen-Macaulay and its artinian reduction has socle and generators concentrated in single degrees. This can be expressed in terms of the Betti diagram of \( M \). If \( \underline{d} = (d_0, d_1, \ldots, d_p) \) and \( \underline{d} = (d_0, \tilde{d}_1, \ldots, \tilde{d}_p) \) are the minimal and maximal shifts of \( M \), then \( M \) is level if and only if \( d_0 = \tilde{d}_0 \) and \( d_p = \tilde{d}_p \).

We will now show that if \( M \) is level and its Betti diagram is a non-negative linear combination of pure diagrams, which, according to Conjecture 3.1, every Betti diagram of a Cohen-Macaulay module is, then its \( h \)-vector satisfies a certain condition and any polynomial satisfying this condition is, after multiplication by some integer, the \( h \)-vector of a level \( R \)-module. This condition on the \( h \)-vector has to do with extremely compressed level modules, so we begin by explaining this notion.

A compressed level module is an artinian level module of maximal Hilbert function among all artinian level modules with a given codimension, number of generators, socle type and socle degree. (The socle type is the dimension of the socle as a \( k \)-vector space and the socle degree, the degree in which the socle is concentrated). It turns out that this Hilbert function is given by \( i \mapsto \min\{sr_i, tr_{c-i}\} \) where \( s \) is the number of generators, \( t \) is the socle type and \( c \) is the socle degree. For any \( s, t \) and \( c \) there is a level module with this Hilbert function \([1]\), and, as already mentioned, this module is said to be compressed. If in addition \( sr_i = tr_{c-i} \) for some \( i \) then the level module having this Hilbert function is said to be extremely compressed.

**Proposition 4.1.** For any integers \( d_0, j \) and \( c \), such that \( d_0 \leq j \leq c \), there is a Cohen-Macaulay \( R \)-module with pure resolution of type \( d = (d_0, j+1, j+2, \ldots, j+p-1, c+p) \) and the artinian reduction of any Cohen-Macaulay \( R \)-module with a pure resolution of this type is an extremely compressed level module.

This follows from the work on compressed level modules in \([1]\). We include a proof here for the reader’s convenience.

**Proof.** By artinian reduction we may assume that \( p = n \), and hence only consider artinian \( R \)-modules and by shifting the degrees we may assume that \( d_0 = 0 \). Let \( M \) be the compressed level \( R \)-module with socle degree \( c \), \( s \) number of generators and socle type \( t \), where \( s = r_{c-j} \) and \( t = r_j \). Then \( sr_j = tr_{c-j} \) so \( M \) is extremely compressed.
compressed. To see that the minimal free resolution of $M$ is pure of type $d = (0, j + 1, j + 2, \ldots, j + p - 1, c + p)$, consider $d_1(N)$ and $d_{p-1}(N)$. Note that $d_1(N)$ is the lowest degree of a relation among the generators of $M$ and that the Hilbert function of $M$ is that of a free module in degrees lower than, or equal to, $j$, where $j$ is the integer such that $sr_j = tr_{c-j}$. This implies that $d_1(N) = j + 1$. Since the Betti numbers in column $p - 1$, of the Betti diagram of $M$, describes the degrees of the relations among the generators of the dual of $M$, we can use the same argument again and we get $d_{p-1}(N) = p + j - 1$. Now there is only one possibility for the minimal and maximal shifts of $M$, since they are both strictly increasing sequences of integer, they must be the same and equal $d = (0, j + 1, j + 2, \ldots, j + p - 1, c + p)$.

**Definition 4.2.** For any integers $d_0 \leq j \leq c$ we call a pure diagram of type $d = (d_0, j + 1, j + 2, \ldots, j + p - 1, c + p)$ an extremely compressed diagram.

We will now prove a proposition from which the promised description of the $h$-vectors of level modules, up to multiplication by a positive rational number, follows easily, as we will soon see.

**Proposition 4.3.** Let $D$ be a non-negative linear combination of pure diagrams of codimension $p$. Then there is a diagram $E$ which is a non-negative linear combination of extremely compressed diagrams of codimension $p$ such that $D$ is obtained from $E$ by a sequence of consecutive cancellations and furthermore, $E_{0,j} = D_{0,j}$ and $E_{p,j} = D_{p,j}$ for all integers $j$.

We will need the following observation.

**Lemma 4.4.** Let $\pi(d)$ be an extremely compressed diagram of type $d = (d_0, j + 1, j + 2, \ldots, j + p - 1, c + p)$. Then we can write $\pi(d)$ on the form

$$\pi(d) = \sum_j a_j \pi(d_0, j + 1, j + 2, \ldots, j + p - 1, j + p) - \sum_{i,j} b_{i,j} C^{i,j}$$

where $a_i$ and $b_{i,j}$ are non-negative rational numbers.

**Proof.** By Proposition 4.1 there exists an extremely compressed level module, $M$, with a pure resolution of type $d$, and hence $\beta(M) = q\pi(d)$ for some positive integer $q$. By Proposition 3.3 we have

$$\beta(M) = \sum_j a'_j \beta(R/m^{j+1}(-d_0)) - \sum_{i,j} b_{i,j} C^{i,j}$$

for some non-negative rational numbers $a'_j$ and $b_{i,j}$. The lemma follows since $\pi(d) = \beta(M)/q$ and the Betti diagram $\beta(R/m^{j+1}(-d_0))$ is the pure diagram

$$\pi(d_0, d_0 + j + 1, d_0 + j + 2, \ldots, d_0 + j + p - 1, d_0 + j + p)$$

for each integer $j$. □

For the induction step in the proof of Proposition 4.3 and for several other things later, we need a result from [2].
Definition 4.5. For any sequence of integers $d = (d_0, d_1, \ldots, d_{k-1}, d_k, d_{k+1}, \ldots, d_p)$ denote by $\tau_k(d)$ the sequence $(d_0, d_1, \ldots, d_{k-1}, d_{k+1}, \ldots, d_p)$. If $\overrightarrow{d}$ and $\overrightarrow{a}$ are such that $\overrightarrow{d}_k = \overrightarrow{a}_k$ for some integer $k$, then there is an isomorphism of vector spaces (see [2, Lemma 3.3])

$$\phi_k : V_{\overrightarrow{d}} \rightarrow V_{\tau_k(\overrightarrow{d})}$$

given by

$$\phi_k(D)_{i,j} = |\overrightarrow{d}_k - j| \cdot \begin{cases} D_{i,j} & \text{when } i < k \\ D_{i+1,j} & \text{otherwise} \end{cases}$$

for any $D \in V_{\overrightarrow{d}}$. Note that the image of a pure diagram under this map is $\phi_k(\pi(d)) = d_k \pi(\tau_k(d))$, which is a pure diagram of codimension one lower than the pure diagram we started with.

Proof of Proposition 4.3. It is enough to prove the case where $D = \pi(d)$ for some pure diagram of type $d = (d_0, d_1, \ldots, d_p)$. We use induction on the codimension $p$. If $p = 0, 1$ or 2 then any pure diagram is extremely compressed and the assertion follows. Assume now that $p > 2$. Let $\phi_p$ be the linear map from Definition 4.5. By induction we may assume that the assertion of the proposition holds for the codimension $p - 1$ pure diagram $\pi(d_0, d_1, \ldots, d_{p-1})$ and hence that $\phi_p(\pi(d)) = d_p \pi(d_0, d_1, \ldots, d_{p-1})$ is obtained from a non-negative linear combination of extremely compressed diagrams by a sequence of consecutive cancellations, that is,

$$\phi_p(\pi(d)) = \sum_j a_j \pi(d_0, j+1, j+2, \ldots, j+p-2, d_{p-1}) - \sum_{i,j} b_{i,j} C_{i,j}$$

for some non-negative integers $a_i$ and $b_{i,j}$. Note that all these extremely pure diagrams have $d_0$ as their first shift and $d_{p-1}$ as their last, since otherwise we could never have that their linear combination would equal $d_p \pi(d_0, d_1, \ldots, d_{p-1})$ in position $(0, j)$ and $(p-1, j)$ for all $j$ and this holds by the induction assumption. By applying Lemma 4.4 to each of the extremely compressed diagrams in (4.1), and collecting terms with pure diagrams of the same type, we get

$$\phi_p(\pi(d)) = \sum_j a'_j \pi(d_0, j+1, j+2, \ldots, j+p-1) - \sum_{i,j} b'_{i,j} C_{i,j}$$

where and $a'_i$ and $b'_{i,j}$ are non-negative rational numbers. Consider the inverse of $\phi_p$ and note that

$$\phi_p^{-1}(\pi(d_0, j+1, j+2, \ldots, j+p-1)) = \frac{1}{d_p} \pi(d_0, j+1, j+2, \ldots, j+p-1, d_p)$$

Note also that

$$\phi_p^{-1}(C_{i,j}) = \frac{1}{d_p - j} C_{i,j}.$$
Hence, applying $\phi_p^{-1}$ to both sides of (4.2) gives

$$\pi(d) = \phi_p^{-1}\phi_p(\pi(d)) = E - \sum_{i,j} b''_{i,j}C^{i,j}$$

where

$$E = \sum_j a''_j \pi(d_0, j + 1, j + 2, \ldots, j + p - 1, d_p)$$

and $a''_j = a'_j/d_p$ and $b''_{i,j} = b'_{i,j}/(d_p - j)$. We have that $E$ is a non-negative linear combination of extremely compressed diagrams and that $\pi(d)$ is obtained from $E$ by a sequence of consecutive cancellations. It remains to prove that $E_{0,j} = \pi(d_{0,j})$ and $E_{p,j} = \pi(d_{p,j})$ for all integers $j$. Note that $E_{0,j} = 0$ for all $j \neq d_0$ and $E_{p,j} = 0$ for all $j \neq d_p$. So the only entries of $E$ to consider are the one in position $(0,d_0)$ and $(p,d_p)$. To cancel the entry in position $(0,d_0)$ with a consecutive cancellation we need a positive entry in position $(0,d_0-1)$, and by considering the shifts of the pure diagrams in the sum giving $E$ we see that $E_{0,d_0-1} = 0$. Hence $E_{0,d_0} = \pi(d_{0,d_0})$ which shows that $E_{0,j} = \pi(d_{0,j})$ for all integers $j$. That $E_{p,j} = \pi(d_{p,j})$ for all integers $j$ follows in the same way. \[\square\]

Since a cancellation of a Betti diagram does not affect its $h$-vector we see that Proposition 4.3 implies that the $h$-vector of a module, $M$, whose Betti diagram is a non-negative linear combination of pure diagrams is a non-negative linear combination of $h$-vectors of extremely compressed level modules. If in addition, $M$ is level, all of these extremely compressed level modules must be generated in the same degree and have the same socle degree. We will now describe the $h$-vector of such a level module, and we assume for simplicity that it is generated in degree zero.

The $h$-vector of the extremely compressed diagram $\pi_j = \pi(0, j+1, j+2, \ldots, j + p - 1, c + p)$ is

$$h_{\pi_j}(t) = r_0 + r_1 t + \cdots + r_j t^j + \frac{r_j}{r_{c-j}} (r_{c-j-1} t^{j+1} + r_{c-j-2} t^{j+2} + \cdots + r_0 t^c).$$

In fact, let $M$ be the extremely compressed level $R$-module with a pure resolution of type $d = (0, j + 1, j + 2, \ldots, j + p - 1, c + p)$ from the proof of Proposition 4.3. The $h$-vector of this module is $h_N(t) = \sum_{i=0}^c h_i t^i$ where $h_i = \min\{r_{c-j-i}, r_j r_{c-i}\}$ and since $h_{\pi_j}(t) = h_N(t)/r_{c-j}$ we get the desired expression for $h_{\pi_j}(t)$.

**Lemma 4.6.** For any polynomial $h(t) = h_0 + h_1 t + \cdots + h_c t^c$ we have that $h(t) = \sum_{i=0}^c f_i h_{\pi_i}(t)$ for some rational numbers $f_0, f_1, \ldots, f_c$, where $h_{\pi_i}(t)$ is the $h$-vector of the extremely compressed diagram $\pi_i = (0,i+1,i+2,\ldots,i+p-1,c+p),$
if and only if

\[
f_i = r_{c-i} \begin{vmatrix} h_{i-1} & h_i & h_{i+1} \\ r_{i-1} & r_i & r_{i+1} \\ r_{c-i+1} & r_{c-i} & r_{c-i-1} \end{vmatrix}.
\]

**Proof.** This is just a question of solving a system of linear equations. Let \( h_{\pi_j,i} \) be the coefficient of \( t^i \) in the polynomial \( h_{\pi_j}(t) \), for each \( j \), and consider the determinant

\[
\begin{vmatrix} h_{i-1} & h_i & h_{i+1} \\ r_{i-1} & r_i & r_{i+1} \\ r_{c-i+1} & r_{c-i} & r_{c-i-1} \end{vmatrix} = \sum_{j=0}^{c} f_j \begin{vmatrix} h_{\pi_j,i-1} & h_{\pi_j,i} & h_{\pi_j,i+1} \\ r_{i-1} & r_i & r_{i+1} \\ r_{c-i+1} & r_{c-i} & r_{c-i-1} \end{vmatrix}.
\]

Whenever \( i \neq j \) the vector \( (h_{\pi_j,i-1}, h_{\pi_j,i}, h_{\pi_j,i+1}) \) is a multiple of either \((r_{i-1}, r_i, r_{i+1})\) or \((r_{c-i+1}, r_{c-i}, r_{c-i-1})\). (It will be a multiple of the former when \( i < j \) and the latter when \( i > j \)). This means that all determinants in the sum on the right-hand side are zero except for the one in which \( i = j \). We get

\[
\begin{vmatrix} h_{i-1} & h_i & h_{i+1} \\ r_{i-1} & r_i & r_{i+1} \\ r_{c-i+1} & r_{c-i} & r_{c-i-1} \end{vmatrix} = f_i \begin{vmatrix} h_{\pi_{i-1},i-1} & h_{\pi_{i-1},i} & h_{\pi_{i-1},i+1} \\ r_{i-1} & r_i & r_{i+1} \\ r_{c-i+1} & r_{c-i} & r_{c-i-1} \end{vmatrix},
\]

and since \((h_{\pi_{i-1},i-1}, h_{\pi_{i-1},i}, h_{\pi_{i-1},i+1}) = (r_{i-1}, r_i, r_{c-i-1}/r_{c-i})\) we get, by Laplace expansion along the third column, that the determinant on the right-hand side equals

\[
\frac{r_i r_{c-i-1} - r_{i+1}}{r_{c-i}} \begin{vmatrix} r_{i-1} & r_i \\ r_{c-i+1} & r_{c-i} \end{vmatrix} = \frac{1}{r_{c-i}} \begin{vmatrix} r_{i-1} & r_{i+1} \\ r_{c-i+1} & r_{c-i} \end{vmatrix} \begin{vmatrix} r_{c-i+1} & r_{c-i} \\ r_{c-i+1} & r_{c-i} \end{vmatrix}.
\]

\( \square \)

**Theorem 4.7.** Let \( h(t) = \sum_{i \in \mathbb{Z}} h_i t^i \) be a polynomial. Then \( h(t) \) is a rational multiple of the \( h \)-vector of a level \( R \)-module of codimension \( p \), generated in degree zero, whose Betti diagram is a non-negative linear combination of pure diagrams, if and only if

\[
\begin{vmatrix} h_{i-1} & h_i & h_{i+1} \\ r_{i-1} & r_i & r_{i+1} \\ r_{c-i+1} & r_{c-i} & r_{c-i-1} \end{vmatrix} \geq 0
\]

for all integers \( i \), where \( r_i = \left( \frac{p-1+i}{p-1} \right) \).

**Proof.** Assume first that the determinants of the assertion are non-negative for all integers \( i \). Then, using Lemma 4.6, we get that \( h(t) = \sum_{i=0}^{c} f_i h_{\pi_i}(t) \). It follows from a straightforward calculation and the fact that \( r_i \) is an increasing function of \( i \), that

\[
\begin{vmatrix} r_{i-1} & r_i \\ r_{c-i+1} & r_{c-i} \end{vmatrix} \begin{vmatrix} r_i & r_{i+1} \\ r_{c-i-1} & r_{c-i} \end{vmatrix} \geq 0
\]
and hence that the numbers \( f_i \) and the determinants of the assertion have the same sign. Hence, \( f_i \) is positive for \( i = 0, 1, \ldots, c \). Let \( M_i \) be an extremely compressed level module with Betti diagram \( m_i\pi_i \) for some integer \( m_i \), whose existence is guaranteed by Proposition 4.1 and choose an integer \( m \) such that the numbers \( q_i = \frac{km}{m_i} \) are integers for \( i = 0, 1, \ldots, c \). Then the module

\[
M = \bigoplus_{i=0}^{c} M_i^{q_i},
\]

where \( M_i^{q_i} \) is the direct sum of \( M_i \) with itself \( q_i \) number times, is level, its \( h \)-vector is equal to \( mh(t) \) and its Betti diagram, \( \beta(M) = m \sum_{i=0}^{c} f_i \pi_i \), is a non-negative linear combination of pure diagrams.

Assume now that \( m \) is an integer such that \( mh(t) \) is the \( h \)-vector of a level \( R \)-module, \( M \), whose Betti diagram is a non-negative linear combination of pure diagrams. Then by Proposition 4.3 we have

\[
\beta(M) = \sum_{i} f_i \pi(0, j + 1, j + 2, \ldots, j + p - 1, c + p) - \sum_{i,j} b_{ij} C^{ij}
\]

for some non-negative rational numbers \( f_i \). Since then \( h_M(t) = \sum_{i} f_i h_{\pi_i}(t) \), we get by Lemma 4.6 that each number \( f_i \) have the same sign as the determinant in the proposition which shows that these are non-negative. \( \square \)

**Remark 4.8.** When the codimension is two we have that \( r_i = i + 1 \) and hence

\[
\begin{vmatrix}
  h_{i-1} & h_i & h_{i+1} \\
  r_{i-1} & r_i & r_{i+1} \\
  r_{c-i+1} & r_{c-i} & r_{c-i-1}
\end{vmatrix} = \begin{vmatrix}
  h_{i-1} & h_i & h_{i+1} \\
  i & i + 1 & i + 2 \\
  c - i + 2 & c - i + 1 & c - i
\end{vmatrix} = (c + 2) \begin{vmatrix}
  h_{i-1} & h_i & h_{i+1} \\
  0 & 1 & 2 \\
  1 & 1 & 1
\end{vmatrix},
\]

where the second equality follows from the determinant on its left-hand side by row operations on the second and third row. Laplace expansion along the first row of the rightmost determinant above gives

\[
h_{i-1} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} - h_i \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + h_{i+1} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = h_{i-1} - 2h_i + h_{i+1}.
\]

We see that if the codimension is two then the condition given in Theorem 4.7 is equal to \( h_{i-1} - 2h_i + h_{i+1} \geq 0 \) for all \( i \). This is in fact precisely what characterizes the \( h \)-vectors of level modules of codimension two as was shown in [20].

5. **Cancellable \( h \)-vectors**

Geramita and Lorenzini [6] introduced the notion of a cancellable \( h \)-vector of an algebra and this notion was generalized to \( h \)-vectors of modules by the author in [19] in order to use dualization to study level algebras. We will now describe the cancellable \( h \)-vectors up to multiplication by a rational number.
We begin by explaining the notion of a cancellable $h$-vector. Consider the polynomial $h(t) = h_0 + h_1 t + \cdots + h_c t^c$ and assume that it is the $h$-vector of a Cohen-Macaulay $R$-module $M$. Then we have

$$\beta(M) = \beta_{\text{lex}}(h(t)) - \sum_{i,j} b_{i,j} C^{i,j} \quad (5.1)$$

for some non-negative integers $b_{i,j}$. If $M$ is level, of codimension $p$ and generated in degree zero, then its Betti diagram has only one non-zero entry in column $p$, and $\beta$ in degree zero, this means that $\beta$ is cancellable.

Proposition 5.1. Let $h(t) = \sum_{i \in \mathbb{Z}} h_i t^i$ be a polynomial of degree $c$ and let $D = \sum_{j=0}^{c} a_j \pi(0, j+1, j+2, \ldots, j+p)$, where $a_j = h_j/r_j - h_{j+1}/r_{j+1}$. Then

$$D_{p-1,p+j} - D_{p,p+j} = \frac{1}{r_{j+2}} \begin{vmatrix} h_j & h_{j+1} & h_{j+2} \\ r_j & r_{j+1} & r_{j+2} \\ p & 1 & 0 \end{vmatrix}$$

for all $j$ and $h(t)$ is a rational multiple of a cancellable $h$-vector, of codimion $p$, if and only if $D_{p-1,p+j} - D_{p,p+j} \geq 0$, for $j = 0, 1, \ldots, c-1$, and $a_j \geq 0$, for $j = 0, 1, \ldots, c$.

Proof. To compute the entries of the diagram $D$ we need to compute the entries of the pure diagrams $\pi_j = \pi(0, j+1, j+2, \ldots, j+p)$ for each integer $j$. Herzog and Kühl calculated the Betti numbers of a pure resolution of any type [9], and their calculation can be applied directly to pure diagrams (see [2]). Their result states that

$$\pi(d)_{k,d_k} = (-1)^k \prod_{i \neq k} \frac{d_i}{d_i - d_k}$$
for any type $d = (0, d_1, d_2, \ldots, d_p)$. When $d$ is the type of $\pi_j$ we have that $d_i = j + i$ for each $i = 1, 2, \ldots, p$. The formula of Herzog and Kühl now yields

$$(\pi_j)_{p,p+j} = (-1)^p \prod_{i \neq p} \frac{d_i}{d_i - d_p} = \prod_{i=1}^{p-1} \frac{j + i}{p - i} = \frac{(j + p - 1)!}{j!(p-1)!} = \left(\frac{p - 1 + j}{p - 1}\right) = r_j$$

and

$$(\pi_j)_{p-1,p+j-1} = (-1)^p \prod_{i \neq p-1} \frac{j + i}{i - p + 1} = \frac{(j + p - 2)!}{j!(p-2)!} = \frac{(j + p)(p-1)r_j}{p + j - 1} = pr_j - r_{j-1}.$$  

The entries of $D$ that we need are $D_{p,p+j}$ and $D_{p-1,p+j}$ and these entries are given by $D_{p,p+j} = a_j(\pi_j)_{p,p+j}$ and $D_{p-1,p+j} = a_{j+1}(\pi_j)_{p-1,p+j}$. By the above calculation we get

$$D_{p-1,p+j} - D_{p,p+j} = a_{j+1}(pr_j - r_j) - a_j r_j$$

which, using the assumption that $a_j = h_j/r_j - h_{j+1}/r_{j+1}$, in turn gives

$$D_{p-1,p+j} - D_{p,p+j} = \left(\frac{h_{j+1}}{r_{j+1}} - \frac{h_{j+2}}{r_{j+2}}\right) (-r_j + pr_j - 1) = \left(\frac{h_j}{r_j} - \frac{h_{j+1}}{r_{j+1}}\right) r_j$$

$$= - h_j + ph_{j+1} - \frac{pr_{j+1} - r_j}{r_{j+1}} h_{j+2}.$$  

To finish the computation of $D_{p-1,p+j} - D_{p,p+j}$, note that

$$\begin{vmatrix}
  h_j & h_{j+1} & h_{j+2} \\
  r_j & r_{j+1} & r_{j+2} \\
  p & 1 & 0
\end{vmatrix} = - h_j + ph_{j+1} - \frac{pr_{j+1} - r_j}{r_{j+1}} h_{j+2}.$$  

By Proposition 5.3, $h(t)$ is an $h$-vector if and only if $a_j \geq 0$, for $j = 0, 1, \ldots, c$, and furthermore, then it exists an integer $s$ such that $\beta^{\text{lex}}(sh(t)) = sD$. Assume that $D_{p-1,p+j} - D_{p,p+j} \geq 0$, for $j = 0, 1, \ldots, c - 1$ and note that since $D_{p,p+j} = 0$ when $j < 0$ of $j > c$ this means that $D_{p-1,p+j} - D_{p,p+j} \geq 0$, for all $j \neq c$. Then, by definition, $sh(t)$ is cancellable and we see that $h(t)$ is a rational multiple of the cancellable $h$-vector $sh(t)$.

If $h(t)$ is cancellable then, as we will see, any integer multiple of $h(t)$ is cancellable and in particular $sh(t)$ which shows that $D_{p-1,p+j} - D_{p,p+j} \geq 0$ for all $j \neq c$. Let $m$ be an integer and consider the the polynomial $mh(t)$. Then $m\beta^{\text{lex}}((h(t)))$ is the Betti diagram of $(F/L)^m$ where $L$ is the lexicographic submodule such that $F/L$ have $h$-vector $h(t)$, and since $(F/L)^m$ have $h$-vector $mh(t)$ we get

$$m\beta^{\text{lex}}(h(t)) = \beta^{\text{lex}}(mh(t)) - \sum_{i,j} b_{i,j} C^{i,j}.$$
for some non-negative integers $b_{i,j}$. By considering the entries in position $(p-1,j)$ and $(p,j)$ of the equation above, we get

$$m_{i,j}^{\beta_{p,j}^{\text{lex}}}(h(t)) = \beta_{p,j}^{\text{lex}}(mh(t)) - b_{p-1,j}$$

and

$$m_{i,j}^{\beta_{p,j}^{\text{lex}}}(h(t)) = \beta_{p,j}^{\text{lex}}(mh(t)) - b_{p-1,j} - b_{p-2,j}$$

from which it follows that

$$m_{i,j}^{\beta_{p,j}^{\text{lex}}}(h(t)) - m_{i,j}^{\beta_{p,j}^{\text{lex}}}(h(t)) = \beta_{p,j}^{\text{lex}}(mh(t)) - \beta_{p,j}^{\text{lex}}(mh(t)) - b_{p-2,j}.$$ 

Since $b_{p-2,j}$ is non-negative this means that

$$\beta_{p,j}^{\text{lex}}(mh(t)) - \beta_{p,j}^{\text{lex}}(mh(t)) \geq m (\beta_{p,j}^{\text{lex}}(h(t)) - \beta_{p,j}^{\text{lex}}(h(t)))$$

for all $j$, which shows that if $h(t)$ is cancellable then $mh(t)$ is cancellable as well. □

The following proposition uses that the dual of a level module is level and hence that they both have cancellable $h$-vectors which then satisfy the condition of \[5.1\].

For any polynomial $h_0 + h_1 t + \cdots + h_c t^c$ we call the polynomial $h_c + h_{c-1} t + \cdots + h_0 t^c$ its reverse, and we note that if $M$ is a Cohen-Macaulay module with $h$-vector $h(t)$ of degree $c$, then the reverse of $h(t)$ is the $h$-vector of $M'(-c)$.

**Theorem 5.2.** Let $h(t) = \sum_{i \in \mathbb{Z}} h_i t^i$ be a polynomial of degree $c$. Then $h$ is a rational multiple of a cancellable $h$-vector whose reverse also is cancellable if and only if

$$\begin{vmatrix}
  h_{i-1} & h_i & h_{i+1} \\
  r_{i-1} & r_i & r_{i+1} \\
  p & 1 & 0
\end{vmatrix} \geq 0, \quad \begin{vmatrix}
  h_{i-1} & h_i & h_{i+1} \\
  0 & 1 & p \\
  r_{c-i+1} & r_{c-i} & r_{c-i-1}
\end{vmatrix} \geq 0$$

for all $i = 1, 2, \ldots, c$, and

$$\frac{h_i}{r_i} - \frac{h_{i+1}}{r_{i+1}} \geq 0, \quad \frac{h_{i+1}}{r_{c-i+1}} - \frac{h_i}{r_{c-i}} \geq 0$$

for all $i = 0, 1, \ldots, c$, where $r_i = \binom{p+1}{p-1+i}$, and as a consequence all $h$-vectors of level modules, generated in degree zero, satisfy these two conditions.

**Proof.** If $h$ and its reverse are cancellable then conditions of the proposition are satisfied by applying Proposition \[5.1\] to $h$ and its reverse.

Assume now that these conditions are satisfied. Then, by Proposition \[5.1\] there are integers $q$ and $q'$ such that $qh(t)$ and $q'h(t)$, where $h'(t)$ is the reverse of $h(t)$, both are cancellable. Since, as shown in the last part of the proof of Proposition \[5.1\] this means that $qq'h(t)$ and $qq'h'(t)$ both are cancellable, we see that $h(t)$ is a rational multiple of a cancellable $h$-vector whose reverse is also cancellable.

If $h(t)$ is the $h$-vector of a level module $M$, then $h$ is cancellable. Since $M'(-c)$ is level as well, and furthermore, generated in degree zero and have the reverse of $h(t)$ as its $h$-vector, we see that the the reverse of $h(t)$ is cancellable. □
Remark 5.3. By using the Eliahou-Kervaire resolution [1] the following expression is obtained in [19, Proposition 17]

\[
(5.2) \quad \beta_{p-1,p+j}^{\text{lex}}(h(t)) - \beta_{p,p+j}^{\text{lex}}(h(t)) = p \left( h_{j+1} - qr_{j+1} - [s_{(j+2)}]_{-1}^{-1} \right) - h_j + qr_j + [s_{(j+2)}]_{-2}^{-2}
\]

where \( q \) is the quotient and \( s \) the remainder when \( h_{j+2} \) is divided by \( r_{j+2} \) and the expressions \( [s_{(j+2)}]_{-1}^{-1} \) and \( [s_{(j+2)}]_{-2}^{-2} \) are manipulations with the \((j+2)\)-th Macaulay representation of \( s \). We will now rearrange the right-hand side of this equality and see that if we choose the right integer multiple of \( h(t) \) we get the result of Proposition 5.1. The right-hand side of (5.2) equals, after some rearranging,

\[
-h_j + ph_{j+1} - q(pr_{j+1} - r_j) + [s_{(j+2)}]_{-2}^{-2} - p[s_{(j+2)}]_{-1}^{-1}
\]

and using \( q = (h_{j+2} - s)/r_{j+2} \), we see that this equals

\[
(5.3) \quad -h_j + ph_{j+1} - \frac{pr_{j+1} - r_j}{r_{j+2}} h_{j+2} - \left( -[s_{(j+2)}]_{-2}^{-2} + p[s_{(j+2)}]_{-1}^{-1} - \frac{pr_{j+1} - r_j}{r_{j+2}} s \right).
\]

Choose an integer \( m \) such that \( mh_{j+2} \) is divisible by \( r_{j+2} \) for each \( j \). Then the number \( s \) of (5.3), which is the remainder when \( mh_{j+2} \) is divided by \( r_{j+2} \), is \( s = 0 \), and hence

\[
\beta_{p-1,p+j}^{\text{lex}}(mh(t)) - \beta_{p,p+j}^{\text{lex}}(mh(t)) = m \left( -h_j + ph_{j+1} - \frac{pr_{j+1} - r_j}{r_{j+2}} h_{j+2} \right).
\]

The expression above is \( m \) times the difference \( D_{p-1,p+j} - D_{p,p+j} \) of Proposition 5.1, so we get in this way the same description of cancellable \( h \)-vectors up to multiplication by a rational numbers as the one in given in that proposition.

We will now see how the conditions of Theorem 4.7 and Theorem 5.2 behave in some cases. In what follows we restrict the notion of a cancellable \( h \)-vector to those which also have a cancellable reverse. If \( h(t) \) is the \( h \)-vector of a level module we say that \( h(t) \) is level.

Geramita et al. [5] found all \( h \)-vectors of artinian level algebras of codimension three which have either socle degree less than six or socle degree six and type two (the ones of type one are Gorenstein so they are known for any socle degree by Stanley’s result [22]). If \( M \) is level then both \( h(t) \) and its reverse are \( h \)-vectors of modules generated in a single degree, so they both satisfy Macaulay’s condition for modules generated in a single degree. Considering the set of \( h \)-vectors which satisfy, and also have a reverse that satisfy, the condition of Macaulay, we will now see, in one of the cases covered by Geramita et al. how many are level, cancellable and are a rational multiple of a cancellable \( h \)-vector. We will also see how many of them satisfy the condition of Theorem 4.7. Remember that all polynomials that satisfy the condition in Theorem 4.7 are rational multiples of level \( h \)-vectors, and we conjecture the converse to hold, that is, that all \( h \)-vectors that are rational multiples of level \( h \)-vectors satisfy the condition of Theorem 4.7.
**Example 5.4.** There are 148 polynomials on the form \( h(t) = 1 + h_1 t + h_2 t^2 + \cdots + h_5 t^5 + 2t^6 \), with non-negative integer coefficients, which satisfy, and also have a reverse that satisfy, Macaulay’s condition for modules generated in a single degree. Among these are, as mentioned, the level ones and there are 58 of them. Of the 148, 67 satisfy the condition given in Theorem 4.7 that is, they are non-negative linear combinations of \( h \)-vectors of extremely compressed level modules of some fixed socle degree, and all the level ones are among these, which if Conjecture 3.1 is true always will be the case. There are nine \( h \)-vectors, among these 67, which are not level and these are

\[
\begin{align*}
1 + 3t + 4t^2 + 5t^3 + 6t^4 + 4t^5 + 2t^6, & \quad 1 + 3t + 4t^2 + 5t^3 + 6t^4 + 6t^5 + 2t^6, \\
1 + 3t + 5t^2 + 5t^3 + 4t^4 + 3t^5 + 2t^6, & \quad 1 + 3t + 5t^2 + 6t^3 + 7t^4 + 4t^5 + 2t^6, \\
1 + 3t + 5t^2 + 7t^3 + 7t^4 + 4t^5 + 2t^6, & \quad 1 + 3t + 5t^2 + 7t^3 + 9t^4 + 5t^5 + 2t^6, \\
1 + 3t + 6t^2 + 5t^3 + 4t^4 + 3t^5 + 2t^6, & \quad 1 + 3t + 6t^2 + 6t^3 + 5t^4 + 4t^5 + 2t^6, \\
1 + 3t + 6t^2 + 10t^3 + 7t^4 + 5t^5 + 2t^6. & 
\end{align*}
\]

However, since these nine \( h \)-vectors satisfy the condition in Theorem 4.7 they are rational multiples of level \( h \)-vectors. Of the 148 that satisfy Macaulay’s condition for modules generated in a single degree, 71 are cancellable and 116 become cancellable after multiplication by some rational number.

It turns out that the 67 \( h \)-vectors satisfying the condition of Theorem 4.7 they are then rational multiples of level \( h \)-vectors and hence of cancellable \( h \)-vectors, are all cancellable and the cancellable ones that does not satisfy the condition of Theorem 4.7 are

\[
\begin{align*}
1 + 3t + 5t^2 + 7t^3 + 6t^4 + 6t^5 + 2t^6, & \quad 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 6t^5 + 2t^6, \\
1 + 3t + 6t^2 + 9t^3 + 7t^4 + 6t^5 + 2t^6, & \quad 1 + 3t + 6t^2 + 10t^3 + 7t^4 + 6t^5 + 2t^6.
\end{align*}
\]

Considering all the the socle degrees and types covered by Geramita et al. there are only three more examples of cancellable \( h \)-vectors that does not satisfy the condition of Theorem 4.7 and these are

\[
\begin{align*}
1 + 3t + 5t^2 + 6t^3 + 4t^4 + 3t^5, & \quad 1 + 3t + 6t^2 + 6t^3 + 4t^4 + 3t^5, \\
1 + 3t + 6t^2 + 10t^3 + 7t^4 + 6t^5. & 
\end{align*}
\]

Again considering all the cases covered by Geramita et al., there are eight \( h \)-vectors that satisfy the condition of Theorem 4.7 but which are not cancellable. They are

\[
\begin{align*}
1 + 3t + 6t^2 + 8t^3 + 8t^4 + 9t^5, & \quad 1 + 3t + 6t^2 + 9t^3 + 9t^4 + 10t^5, \\
1 + 3t + 6t^2 + 8t^3 + 9t^4 + 11t^5, & \quad 1 + 3t + 6t^2 + 9t^3 + 10t^4 + 12t^5, \\
1 + 3t + 6t^2 + 10t^3 + 11t^4 + 13t^5, & \quad 1 + 3t + 6t^2 + 10t^3 + 12t^4 + 15t^5, \\
1 + 3t + 6t^2 + 6t^3 + 7t^4, & \quad 1 + 3t + 6t^2 + 7t^3 + 9t^4.
\end{align*}
\]
We now take a closer look at one of the $h$-vector above which satisfies the condition of Theorem 4.7, but which is not cancellable.

**Example 5.5.** The polynomial $h(t) = 1 + 3t + 6t^2 + 7t^3 + 9t^4$ satisfies the condition of Theorem 4.7 and hence is a rational multiple of of some level $h$-vector, but it is not cancellable. In fact, the lexicographic ideal $L$ of $R = k[x, y, z]$ such that $R/L$ have the polynomial $h(t)$ as $h$-vector is

$$L = (z^3, yz^2, xz^2, y^4z, xy^3z, x^2y^2z, x^3yz, x^4z, y^5, xy^4, x^2y^3, x^3y^2, x^4y, x^5)$$

and hence $\beta^{\text{lex}}(h(t)) = \beta(R/L)$ and this diagram is in fact given by

$$\beta^{\text{lex}}(h(t)) = \begin{pmatrix} 1 & - & - & - \\ - & - & - & - \\ - & 3 & 3 & 1 \\ - & 11 & 20 & 9 \end{pmatrix},$$

where we have used the convention of writing the entry $\beta^{\text{lex}}_{i,j}(h(t))$ in column $i$ and row $j - i$. We see that $h(t)$ is not cancellable, and hence not level, since there is no way to cancel the Betti number $\beta_{3,5} = 1$ in the last column of $\beta^{\text{lex}}(h(t))$.

The diagram from Proposition 3.4 that bounds from above the Betti numbers of modules with $h$-vector $h(t)$ is

$$D = \frac{3}{10} \beta(R/m^3) + \frac{1}{10} \beta(R/m^4) + \frac{3}{5} \beta(R/m^5) = \begin{pmatrix} 1 & - & - & - \\ - & - & - & - \\ - & 3 & 9/2 & 9/5 \\ - & 3/2 & 12/5 & 1 \\ - & 63/5 & 21 & 9 \end{pmatrix},$$

and since $\beta_{3,5} = 9/5 < \beta_{2,5} = 12/5$ and $\beta_{3,6} = 1 < \beta_{2,6} = 21$ we see that $h(t)$ is a rational multiple of a cancellable $h$-vector. Indeed, $\beta^{\text{lex}}(10h(t)) = 10D$, so $10h(t)$ is cancellable.

In general it is hard to say if a cancellation of a Betti diagram can be realized as the Betti diagram of some module, but in this case, since $h(t)$ is a rational multiple of a level $h$-vector, we know that there is a cancellation of a rational multiple of $D$ such that $\beta_{3,5} = \beta_{3,6} = 0$. We also know that $h(t)$ is a non-negative linear combination of $h$-vectors of extremely compressed level modules of some fixed socle degree, since it satisfies the condition of Theorem 4.7. In fact,

$$h(t) = 1 + 3t + 6t^2 + 7t^3 + 9t^4 =$$

$$\frac{3}{7}(1 + 3t + 6t^2 + 3t^3 + t^4) + \frac{4}{7}(1 + 3t + 6t^2 + 10t^3 + 15t^4)$$

where $1 + 3t + 6t^2 + 3t^3 + t^4$ and $1 + 3t + 6t^2 + 10t^3 + 15t^4$ are the $h$-vectors of some extremely compressed level modules, $M_1$ and $M_2$, with Betti diagrams
\[ \pi(0, 3, 4, 7) \text{ and } \pi(0, 5, 6, 7), \] respectively. The existence of these extremely compressed modules are guaranteed by Proposition 4.1. The corresponding diagram of this linear combination is

\[ E = \frac{3}{7} \pi(0, 3, 4, 7) + \frac{4}{7} \pi(0, 5, 6, 7) = \]

\[ \begin{pmatrix}
1 & - & - & - \\
- & - & - & - \\
- & 7 & 7 & - \\
- & - & - & 1
\end{pmatrix} + \begin{pmatrix}
1 & - & - & - \\
- & - & - & - \\
- & - & - & - \\
- & - & - & -
\end{pmatrix} = \begin{pmatrix}
1 & - & - & - \\
- & - & - & - \\
- & 3 & 3 & - \\
- & 12 & 20 & 9
\end{pmatrix}. \]

We see that \( 7E \), which by Proposition 3.4 is obtained from the diagram \( 7D \) by a sequence of consecutive cancellations, is the Betti diagram of a level module with \( h \)-vector \( 7h(t) \).

6. The weak Lefschetz property

In [8], all the Hilbert functions of artinian algebras with the weak or strong Lefschetz property are determined and a sharp upper bound on the graded Betti numbers of these algebras is given. We give analogous results in the case of level modules with the weak Lefschetz property, but only up to multiplication by a rational number. Furthermore, we show that the upper bound on the graded Betti numbers, in our case, is a non-negative linear combination of pure diagrams and that the graded Betti numbers are obtained from these by a sequence of consecutive cancellations. In Section 7 we will see that this upper bound is enough to prove Conjecture 3.1 for level modules of codimension three with the weak Lefschetz property, which in turn gives the Multiplicity conjecture for these modules.

An artinian \( R \)-module \( M \) has the weak Lefschetz property if there is a linear form \( \ell \) such that \( M_{i-1} \rightarrow M_i \) is either injective or surjective for each \( i \), and such a form is called a Lefschetz element. Let \( M \) be a graded \( R \)-module and \( \ell \) a linear form of \( R \), not necessarily a Lefschetz element. Then we have an exact sequence

\[ 0 \rightarrow P \rightarrow M(-1) \xrightarrow{\times \ell} M \rightarrow Q \rightarrow 0, \]

where the map from \( M(-1) \) to \( M \) is multiplication by \( \ell \), and \( P \) and \( Q \) are the kernel and cokernel, respectively. Then \( P \) and \( Q \) are modules over \( S = R/\ell \) and hence the codimensions of \( P \) and \( Q \) are the codimension of \( M \) minus one. In [15], Migliore and Nagel shows that the exact sequence above gives the following exact sequences of graded \( R \)-modules

\[ \cdots \rightarrow \text{Tor}^R_{i-1}(P, k) \rightarrow \text{Tor}^R_i(M, k) \rightarrow \text{Tor}^S_i(Q, k) \rightarrow \]

\[ \cdots \rightarrow \text{Tor}^S_0(P, k) \rightarrow \text{Tor}^S_1(M, k) \rightarrow \text{Tor}^S_1(Q, k) \rightarrow 0 \]
and
\begin{equation}
0 \to \text{Tor}^S_0(M, k) \to \text{Tor}^R_0(Q, k) \to 0.
\end{equation}

Note that the long exact sequence above yields
\begin{equation}
0 \to \text{Tor}^S_{p-1}(P, k) \to \text{Tor}^R_p(M, k) \to 0.
\end{equation}

These exact sequences imply, as observed by Migliore and Nagel, that
\[ \beta^{R}_{i,j}(M) \leq \beta^{S}_{i,j}(Q) + \beta^{S}_{i-1,j}(P) \]
for all \( i \) and \( j \), with equality when \( \beta^{S}_{i-1,j}(P) = \beta^{S}_{i-2,j}(P) = 0 \) or \( \beta^{S}_{i+1,j}(Q) = \beta^{S}_{i,j}(Q) = 0 \). We now make a further observation. Denote by \( D \) the diagram defined by
\[ D_{i,j} = \beta^{S}_{i,j}(Q) + \beta^{S}_{i-1,j}(P). \]

for all \( i \) and \( j \). The exact sequence (6.1) then yields
\[ \sum_{k=1}^{i} (-1)^{k+1}(D_{k,j} - \beta_{k,j}(M)) \geq 0, \]
for all \( i \), since \( \beta_{i,j}(M) = \dim_k \text{Tor}_i(M, k)_j \) and \( \dim_k \) is additive along exact sequences. This implies not only that \( D \leq \beta(M) \), entry by entry, but also that \( \beta(M) \) is obtained from \( D \) by a sequence of consecutive cancellations. In fact, let
\[ b_{i,j} = \sum_{k=1}^{i-1} (-1)^{k+1}(D_{k,j} - \beta_{k,j}(M)). \]

Then
\[ \beta(M) = \beta(D) - \sum_{i,j} b_{i,j} C^{i,j}. \]

**Lemma 6.1.** Let \( M \) be a level artinian \( R \)-module of codimension \( p \), generated in degree zero, with a Lefschetz element \( \ell \) and let
\[ 0 \to P \to M(-1) \xrightarrow{\times \ell} M \to Q \to 0, \]
be the exact sequence corresponding to the multiplication map given by \( \ell \). Let \( h(t) = \sum_{i \in \mathbb{Z}} h_i t^i \) be the \( h \)-vector of \( M \) and assume that it is a polynomial of degree \( c \). Furthermore, let \( u \) be the smallest integer such that \( h_u \geq h_{u+1} \). Then the dual of \( P \), \( P^\vee \), is generated in a single degree and that degree is \(-c-1\), and hence \( P^\vee(-c-1) \) is generated in degree zero. Furthermore, the \( h \)-vector of \( Q \) is
\[ h_Q(t) = \sum_{i=0}^{u} (h_i - h_{i-1}) t^i, \]
the \( h \)-vector of \( P \) is,
\[ h_P(t) = \sum_{i=u+1}^{c+1} (h_{i-1} - h_i) t^i. \]
The \( h \)-vector of \( P' = P^\vee(-c - 1) \) is,

\[
h_{P'}(t) = \sum_{i=0}^{c-u} (h_{c-i} - h_{c-i+1}) t^i
\]

and for the multiplicities it holds that

\[
e(Q) = e(P) = e(P^\vee(-c - 1)).
\]

**Proof.** The socle degree of \( M \) is \( c \) since the degree of \( h(t) \) is \( c \). This means that \( \text{Tor}^R_0(M, k) \) is concentrated in degree \( c + p \). It follows from (6.3) that \( \text{Tor}^S_{p-1}(P, k) \cong \text{Tor}^R_0(M, k) \) and hence \( \text{Tor}^S_{p-1}(P, k) \) is also concentrated in degree \( c + p \), which shows that \( \beta_{p-1,j}(P) = 0 \) for all \( j \neq c+p \). The graded Betti numbers of the dual of \( P \) are obtained from the ones of \( P \) by \( \beta_{i,j}(P) = \beta_{i-1,j,p-1-j}(P^\vee) \), since \( P \) is of codimension \( p-1 \), and this shows that \( \beta(P^\vee)_{0,j} = 0 \) for all \( j \neq -c-1 \). Hence, \( P^\vee \) is generated in degree \(-c-1\).

Since \( u \) is the smallest integer such that \( h_u \geq h_{u+1} \), we see that \( u \) is also the smallest integer such that the map \( M_i \to M_{i+1} \), given by multiplication by \( \ell \), is surjective. Since \( M \) is level, and hence generated in a single degree, we see that as soon as the map \( M_{i+1} \to M_i \) is surjective it will continue to be in all higher degrees. Hence \( M_{i+1} \to M_i \) is surjective for all \( i \geq u \) and injective for all \( i < u \).

The exact sequence

\[
0 \to P_1 \to M_{i-1} \xrightarrow{\times \ell} M_i \to Q_i \to 0,
\]

then implies that \( Q_i = 0 \) for all \( i > u \) and \( P_i = 0 \) for all \( i \leq u \). It also follows from the exact sequence that

\[
H(Q, i) - H(P, i) = H(M, i) - H(M, i - 1) = h_i - h_{i-1},
\]

and using that \( Q_i = 0 \) for all \( i > u \) and that \( P_i = 0 \) for all \( i \leq u \), we get

\[
h_Q(t) = \sum_{i \in \mathbb{Z}} H(Q, i) t^i = \sum_{i=0}^{u} (h_i - h_{i-1}) t^i
\]

and

\[
h_P(t) = \sum_{i \in \mathbb{Z}} H(P, i) t^i = \sum_{i=u+1}^{c+1} (h_{i-1} - h_i) t^i.
\]

Now, since \( H(P^\vee(-c - 1), i) = H(P^\vee, i - c - 1) = H(P, c + 1 - i) \) we get that the \( h \)-vector of \( P' = P^\vee(-c - 1) \) is

\[
h_{P'}(t) = \sum_{i \in \mathbb{Z}} H(P^\vee(-c - 1), i) t^i = \sum_{i \in \mathbb{Z}} H(P, c + 1 - i) t^i = \sum_{i=0}^{c-u} (h_{c-i} - h_{c+1-i}) t^i.
\]

It remains to prove that \( e(Q) = e(P) = e(P^\vee(i - c - 1)) \). From (6.4) it follows that

\[
h_Q(t) - h_P(t) = (1-t)h_M(t)
\]
and if we put \( t = 1 \) in this equation we get \( h_Q(1) = h_P(1) \). By definition, \( e(Q) = h_Q(1) \) and \( e(P) = h_P(1) \) so we get that \( e(Q) = e(P) \). Since the set of coefficients of \( h_P(t) \) and \( h'_P(t) \) are the same we see that \( h_P(1) = h_P(1) \) which shows that \( e(P) = e(P') = e(P', i - c - 1) \). □

We need the following lemma which is almost a restatement of [2, Lemma 4.1].

**Lemma 6.2.** Let \( F \) and \( G \) be non-negative linear combinations of pure diagrams of codimension \( p - 1 \) such that \( e(F) = e(Q) \) and the maximal shifts of \( F \) are are smaller than the minimal shifts of \( G \), that is, \( d(F) < d(G) \). Then the diagram \( E \) defined by

\[
E_{i,j} = F_{i,j} + G_{i-1,j}
\]

for all integers \( i \) and \( j \), is a non-negative linear combination of pure diagrams of codimension \( p \).

**Proof.** This follows from [2, Lemma 4.1] which states that the diagram \( D \) defined by

\[
D_{i,j} = \pi(d)_{i,j} + \frac{e(\pi(d))}{e(\pi(d'))} \pi(d')_{i-1,j}
\]

for all integers \( i \) and \( j \), is a non-negative linear combination of pure diagrams of codimension \( p \), if the types of the codimension \( p - 1 \) diagrams \( \pi(d) \) and \( \pi(d') \) satisfies \( d < d' \). Assume that \( F = \sum_{i=1}^s F_i \) and \( G = \sum_{i=1}^t G_i \) where \( F_i \) and \( G_i \) are pure diagrams. Then the type of \( F_1 \) is smaller than the type of \( G_1 \), by the assumption on the minimal and maximal shifts of \( F \) and \( G \). Assume furthermore that \( e(F_1) \leq e(G_1) \), the case where \( e(F_1) > e(G_1) \) follows in the same way, then the diagram \( E_1 \) defined by

\[
(E_1)_{i,j} = (F_1)_{i,j} + \frac{e(F_1)}{e(E_1)} (G_1)_{i,j}
\]

for all integers \( i \) and \( j \), is non-negative linear combination of pure diagrams by [2, Lemma 4.1]. The diagram \( E' = E - E_1 \) is now given by the diagrams \( F' = \sum_{i=2}^s F_i \) and

\[
G' = \left( 1 - \frac{e(F_1)}{e(E_1)} \right) G_1 + \sum_{i=2}^t G_i
\]

in the same way as \( E \) is given by \( F \) and \( G \) in the statement of the lemma. The total number of pure diagrams in \( F' \) and \( G' \) are \( s + t - 1 \), that is, one fewer than the total number in \( F \) and \( G \), and furthermore \( e(F') = e(G') \). By induction we may assume that \( E' \) is a non-negative linear combination of pure diagrams and then this is true for \( E \) as well since \( E = E' + E_1 \). □

**Proposition 6.3.** Let \( M \) be a level artinian \( R \)-module of codimension \( p \), generated in degree zero, with the weak Lefschetz property and let the \( h \)-vector of \( M \) be
the degree $c$ polynomial $h(t) = \sum_{i \in \mathbb{Z}} h_i t^i$. Furthermore, let $S = R/\ell$, $s_i = \binom{p-2+i}{p-2}$, $u$ be the smallest integer such that $h_u \geq h_{u+1}$ and

$$F = \sum_{i=0}^{u} f_i \beta^S(S/m^{i+1})$$

where

$$f_i = \frac{h_i - h_{i-1}}{s_i} - \frac{h_{i+1} - h_i}{s_{i+1}}$$

for $i = 0, 1, \ldots, u - 1$ and

$$f_u = \frac{h_u - h_{u-1}}{s_u}$$

and

$$G = \sum_{i=0}^{c-u} g_i \beta^S(S/m^{i+1})$$

where

$$g_i = \frac{h_{c-i} - h_{c-i+1}}{s_i} - \frac{h_{c-i+1} - h_{c-i}}{s_{i+1}}$$

for $i = 0, 1, \ldots, c - u - 1$ and

$$g_{c-u} = \frac{h_u - h_{u+1}}{s_{c-u}}.$$  

Then $f_i \geq 0$, for $i = 0, 1, \ldots, u$, and $g_i \geq 0$, for $i = 0, 1, \ldots, c - u$, and $\beta(M)$ is obtained from the diagram $E$, defined by

$$E_{i,j} = F_{i,j} + G_{p-i,p+c-j},$$

for all integers $i$ and $j$, by a sequence of consecutive cancellations and furthermore, $E$ is non-negative linear combination of pure diagrams of codimension $p$.

**Proof.** Let $\ell$ be a Lefschetz element on $M$, and consider the exact sequence

$$0 \to P \to M(-1) \xrightarrow{\times \ell} M \to Q \to 0,$$

where $P$ and $Q$ are the kernel and cokernel, respectively, of the map $M(-1) \to M$ given by multiplication by $\ell$. Then $P$ and $Q$ are artinian modules over $S = R/\ell$ and hence of codimension $p - 1$. The $h$-vector of $Q$ is, by Lemma 6.1, $\sum_{i=0}^{u} (h_i - h_{i-1}) t^i$ so it follows by Proposition 3.4 that

$$(6.5) \beta^S(Q) = F - \sum_{i,j} h_{i,j}^F C^{i,j}. \tag{6.5}$$

for some non-negative rational numbers $h_{i,j}^F$. In the same way we get, since the $h$-vector of $P^\vee(-c - 1)$ is $\sum_{i=0}^{c-u} (h_{c-i} - h_{c-i+1}) t^i$ and $P^\vee(-c - 1)$, by Lemma 6.1, is generated in degree zero, that

$$(6.6) \beta^S(P^\vee(-c - 1)) = G - \sum_{i,j} h_{i,j}^G C^{i,j}. \tag{6.6}$$
for some non-negative rational numbers $b_{i,j}^G$.

Since $P$ is of codimension $p - 1$ we have by a standard property of dualization that

$$
\beta_{i,j}^S(P) = \beta_{p-1-i,p-1-j}^S(P^\vee)
$$

and since shifting the degrees of $P$ shifts the degrees of its Betti numbers by the same amount we get

$$
\beta_{i,j}^S(P) = \beta_{p-1-i,p+c-j}^S(P^\vee(-c - 1)).
$$

So, if we denote by $G'$ the diagram defined by

$$(6.7)$$

$$
G_{i,j}' = G_{p-1-i,p+c-j}
$$

for all integers $i$ and $j$, we get, by (6.6), that $\beta^S(P)$ is obtained from $G'$ by sequence of consecutive cancellations. Let $D$ and $E$ be the diagrams defined by

$$
D_{i,j} = \beta_{i,j}^S(Q) + \beta_{i-1,j}^S(P)
$$

and

$$
E_{i,j} = F_{i,j} + G_{i-1,j}' = F_{i,j} + G_{p-i,c+p-j},
$$

for all integers $i$ and $j$. Then, since $\beta^S(Q)$ and $\beta^S(P)$ are obtained from $F$ and $G'$, respectively, by a sequence of consecutive cancellations we see that $D$ is obtained from $E$ in this way as well. As noted in the beginning of this section, the long exact sequence (6.1) implies that $\beta(M)$ is obtained from the diagram $D$ by a sequence of consecutive cancellations, and hence $\beta(M)$ is be obtained from $E$ by a sequence of consecutive cancellations.

It remains to prove that $E$ is a non-negative linear combination of pure diagrams of codimension $p$. We will prove this by applying Lemma 6.2 to the diagrams $F$ and $G'$, and thus we have to show that $F$ and $G$ are non-negative linear combinations of pure diagrams and that their minimal and maximal shifts satisfies $\overline{d}(F) < \underline{d}(G)$.

The Betti diagram of $S/m^{i+1}$, for any integer $i$, is the pure diagram $\pi(0, i + 1, i + 2, \ldots, i + p - 1)$ of codimension $p - 1$, and since the rational numbers $f_i$ and $g_i$ are all non-negative we see that the diagrams $F$ and $G$ are non-negative linear combinations of pure diagrams of codimension $p - 1$. We now claim that since $G$ is a non-negative linear combination of pure diagrams, $G'$ is as well. To see this, let $A'$ be the matrix obtained from any diagram $A$, of codimension $p - 1$, by

$$
A'_{i,j} = A_{p-1-i,p+c-j}
$$

for all integers $i$ and $j$, and note that $G'$ defined in (6.7) is obtained from $G$ in this way. Remember that, by definition, $A'$ is a diagram of codimension $p - 1$, if $S_{A'}(t) = \sum_{i,j}(-1)^iA'_{i,j}t^j$ is divisible by $(1 - t)^{p-1}$. Since $S_A(t) = \sum_{i,j}(-1)^iA_{i-j,p+c-j}t^j = (-1)^{p-1}b^{p-c}S_A(t^{-1})$ and this element of $\mathbb{Q}[t, t^{-1}]$ is divisible by $(1 - t)^{p-1}$ if $S_A(t)$ is, we see that $A$ is a diagram of codimension $p - 1$. Furthermore, if $\pi$ is a pure diagram of type $d = (d_0, d_1, \ldots, d_{p-1})$ then, by solving
(p - 1 - i, p + c - j) = (k, d_k) for k = 0, 1, ..., p - 1, we see that the non-zero positions of \( \pi' \) are
\[
(0, p + c - d_{p-1}), (1, p + c - d_{p-1}), \ldots, (p - 1, p + c - d_0)
\]
and hence that \( \pi' \) is a pure diagram of type
\[
d' = (p + c - d_{p-1}, p + c - d_{p-2}, \ldots, p + c - d_0).
\]

The point of all this is that it shows that \( G' \) is a non-negative linear combination of pure diagrams and furthermore, since the maximal shifts of \( G \) are given by the type of the pure diagram \( S/m^{c-u+1} \) and this type is
\[
d = (0, c - u + 1, c - u + 2, \ldots, c - u + p - 1)
\]
we see that the minimal shifts of \( G' \) are
\[
d' = (u + 1, u + 2, \ldots, u + p - 2, c).
\]
The maximal shifts of the diagram \( F \) are given by the type of the diagram \( S/m^u \) and these are
\[
(0, u + 1, u + 2, \ldots, u + p - 1).
\]
Hence, the maximal shifts of \( F \) are smaller than the minimal shifts of \( G' \), that is,
\[
\overline{d}(F) = (0, u + 1, u + 2, \ldots, u + p - 1) < (u + 1, u + 2, \ldots, u + p - 2, c) = \overline{d}(G).
\]
Since \( F \) and \( Q \) have the same \( h \)-vector, by (6.5), \( e(F) = e(Q) \) and for the same reason \( e(G') = e(P'(-c - 1)) \), by (6.6). By Lemma 6.1 \( e(Q) = e(P'(-c - 1)) \) and hence \( e(F) = e(G') \). We can now apply Lemma 6.2 to the diagrams \( F \) and \( G' \) which show that \( E \) is a non-negative linear combination of pure diagrams. □

Example 6.4. Let \( R = k[x, y, z] \) and consider the \( h \)-vector \( h(t) = 1 + 3t + 5t^2 + 6t^3 + 2t^4 \). Then, the \( f_i \)'s of Proposition 6.3 equals \( f_0 = f_1 = 0 \), \( f_2 = 1/3 \), \( f_3 = 5/12 \) and \( f_4 = 1/4 \), and hence the diagram \( F \) of the same proposition equals
\[
F = \frac{1}{3} \beta(S/m^2) + \frac{5}{12} \beta(S/m^3) + \frac{1}{4} \beta(S/m^4) = \begin{pmatrix}
1 & - & - \\
- & 1/3 & 2/3 \\
- & 5/3 & 5/4 \\
- & 5/4 & 1
\end{pmatrix}.
\]
For the \( g_i \)'s, of Proposition 6.3 we get \( g_0 = 0 \) and \( g_1 = 2 \), and hence
\[
G = 2 \cdot \beta(S/m^2) = \begin{pmatrix}
2 & - & - \\
- & 6 & 4
\end{pmatrix}.
\]
The diagram \( G' \), from the proof of Proposition 6.3 is the diagram obtained from \( G \) by rotating it 180 degrees and shifting its rows so that its upper most non-zero
element lies on row \( c \), where \( c \) is the degree of \( h(t) \). In this case we get

\[
G' = \begin{pmatrix}
- & - & - \\
- & - & - \\
4 & 6 & - \\
- & - & 2
\end{pmatrix}.
\]

Now, the diagram \( E \), of Proposition \ref{prop:6.3}, defined by

\[
E_{i,j} = F_{i,j} + G'_{i-1,j} = F_{i,j} + G_{3-i,7-j},
\]

for all integers \( i \) and \( j \), is simply the sum of \( F \) with the diagram obtained from \( G' \) by shifting its columns one position to the right. We get

\[
E = \begin{pmatrix}
1 & - & - & - \\
- & 1 & 2/3 & - \\
- & 5/3 & 5/4 & - \\
- & 21/4 & 7 & - \\
- & - & - & 2
\end{pmatrix}.
\]

Proposition \ref{prop:6.3} states that when \( h(t) = 1 + 3t + 5t^2 + 6t^3 + 2t^4 \) is the \( h \)-vector of an artinian level \( R \)-module with the weak Lefschetz property, then \( E \) is a non-negative linear combination of pure diagrams. For the diagram \( E \) of this example this turns out to be true. We have

\[
E = \frac{4}{21} \pi(0, 2, 3, 7) + \frac{1}{14} \pi(0, 2, 4, 7) +
\frac{13}{84} \pi(0, 3, 4, 7) + \frac{2}{15} \pi(0, 3, 5, 7) + \frac{9}{20} \pi(0, 4, 5, 7).
\]

In fact, Theorem \ref{thm:6.8} says that as soon as the \( f_i \)'s and \( g_i \)'s are non-negative, \( h(t) \) is a rational multiple of the \( h \)-vector of an artinian level \( R \)-module with the weak Lefschetz property, so this was no coincident. This means that \( h(t) = 1 + 3t + 5t^2 + 6t^3 + 2t^4 \) is a rational multiple of a level \( h \)-vector, and a non-negative linear combination of pure diagrams. Hence, by Proposition \ref{prop:4.3} it is a non-negative linear combination of \( h \)-vectors of extremely compressed diagrams. Computing the coefficients of these extremely compressed diagrams with Lemma \ref{lem:4.6} gives the diagram

\[
W = \frac{5}{21} \pi(0, 2, 3, 7) + \frac{11}{42} \pi(0, 3, 4, 7) + \frac{1}{2} \pi(0, 4, 5, 7) = \begin{pmatrix}
1 & - & - & - \\
- & 1 & 5/6 & - \\
- & 11/6 & 11/6 & - \\
- & 35/6 & 7 & - \\
- & - & - & 2
\end{pmatrix}.
\]

where \( h_W(t) = 1 + 3t + 5t^2 + 6t^3 + 2t^4 \). Note that \( W \geq E \), entry by entry, and hence that \( W \) could not be a rational multiple of the Betti diagram of a module with the weak Lefschetz property.
Remark 6.5. We can replace the diagrams $F$ and $G$ of Proposition 6.3 with the diagrams $\beta^{lex}(h_Q)$ and $\beta^{lex}(h_{P'})$, respectively, where $h_Q(t)$ and $h_{P'}(t)$ are the $h$-vectors obtained from the $h$-vector of $M$ in Lemma 6.1. This gives a smaller upper bound than the one given in Proposition 6.3 but we no longer know that this upper bound is a non-negative linear combination of pure diagrams, since we do not know that $\beta^{lex}(h_Q)$ and $\beta^{lex}(h_{P'})$ have this property. In addition, the upper bound of Proposition 6.3 is a linear function of the $h$-vector, so when considering $h$-vectors up to multiplication with a rational number, this upper bound behaves well, something that is not true for one given by $\beta^{lex}(h_Q)$ and $\beta^{lex}(h_{P'})$. In fact, in Theorem 6.8, we will see that there always exists an artinian $R$-module with the weak Lefschetz property having an integer multiple of the upper bound $E$, of Proposition 6.3 as its Betti diagram. So when considering $h$-vectors and Betti diagrams up to multiplication with a rational number this upper bound is in some sense sharp.

We will now show that given an $h$-vector such that the $f_i$'s and $g_i$'s of Proposition 6.3 are all non-negative, we can construct an artinian level $R$-module with the weak Lefschetz property whose $h$-vector is an integer multiple of this $h$-vector, and furthermore, whose Betti diagram is an integer multiple of the maximal one given by the diagram $E$ of Proposition 6.3.

To prove that our construction has the desired Betti diagram we will use the following standard observation, and we include its simple proof.

**Lemma 6.6.** Let $M$ be a graded $R$-module. Then, for any integer $u$,

$$
\beta_{i,j} \left( \bigoplus_{d \leq u} M_d \right) = \beta_{i,j}(M)
$$

for all $j - i < u$.

**Proof.** Let $M' = \bigoplus_{d \leq u} M_d$ and $M'' = \bigoplus_{d > u} M_d$ and consider the exact sequence

$$
0 \to M'' \to M \to M' \to 0
$$

From the associated long exact sequence of Tor

$$
\to \text{Tor}^R_i(M'', k) \to \text{Tor}^R_i(M, k) \to \text{Tor}^R_i(M', k) \to \text{Tor}^R_{i+1}(M'', k)
$$

it follows that $\text{Tor}^R_i(M, k)_j \cong \text{Tor}^R_i(M', k)_j$ whenever

$$
\text{Tor}^R_i(M'', k)_j = \text{Tor}^R_{i+1}(M'', k)_j = 0.
$$

Since $M_i' = 0$ when $i \leq u$ we get that $\text{Tor}^R_i(M', k)_j = 0$ when $j - i \leq u$ and hence that

$$
\text{Tor}^R_i(M', k)_j \cong \text{Tor}^R_i(M, k)_j
$$

when $j - i < u$, which gives the desired equality of the Betti numbers. \qed
By a point in \( \mathbb{P}_k^{p-1} \) we mean a \( k \)-rational point, and such a point \( P \) can be identified with a \( p \)-tuple \( P = (u_0 : u_1 : \cdots : u_{p-1}) \) where \( u_i \in k \). For any set of points \( X \) denote by \( I_X \) the ideal of forms vanishing on \( X \), that is, the ideal generated by
\[
\{ f \in R : f \text{ is homogeneous and } f(P) = 0 \text{ for all } P \in X \}.
\]
The homogeneous coordinate ring of \( X \) is \( R/I_X \) and we denote it by \( A_X \). When \( X \) is a finite set of points, we denote the smallest integer \( d \) such that \( H(A_X, d) = |X| \) by \( \tau(A_X) \).

The construction in the following proposition is more general than we need and we will only apply it to points in general position here.

**Proposition 6.7.** Let \( X \) be a set of distinct points in \( \mathbb{P}_k^{p-1} \) and let \( X = \bigcup_{i=1}^s Y_i \) and \( X = \bigcup_{i=1}^t Z_i \) be partitions of \( X \) and furthermore let \( c \) be an integer such that \( c \geq \max \{ \tau(A_{Y_i}) \}_{i=1}^s + \max \{ \tau(A_{Z_i}) \}_{i=1}^t \). Then there is an artinian \( R \)-module \( M \) of codimension \( p \), with the weak Lefschetz property and Hilbert function given by
\[
H(M, d) = \min \left\{ \sum_{i=1}^s H(A_{Y_i}, d), \sum_{i=1}^t H(A_{Z_i}, c - d) \right\}
\]
and furthermore,
\[
\beta_{i,j}(M) = \sum_{k=1}^s \beta_{i,j}(A_{Y_k}, d) + \sum_{i=1}^t \beta_{p-1,p+c-j}(A_{Z_k}, d)
\]
for all integers \( i \) and \( j \).

**Proof.** Let \( M_X \) be the free \( k[t, t^{-1}] \)-module with basis \( \{ e_P \}_{P \in X} \), that is,
\[
M_X = \bigoplus_{P \in X} k[t, t^{-1}]e_P.
\]
We give \( M_X \) the structure of a graded \( R \)-module by defining the multiplication of any homogeneous element \( f \in R \) and the element \( \sum_{P \in X} a_P e_P \in M_X \), where \( a_P \in k[t, t^{-1}] \), by
\[
f \cdot \sum_{P \in X} a_P e_P = \sum_{P \in X} f(P)t^da_P e_P
\]
where \( d \) is the degree of \( f \). (To make the value of \( f(P) \) well defined we have to choose a fixed coordinate representation for each point \( P \in X \), so we assume that this have been done.) Observe that \( M_X \cong \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(d)) \).

We will now construct a quotient of a submodule of \( M_X \) with the desired Hilbert function. Let \( y_i = \sum_{P \in Y_i} e_P \) for each \( i \), and let \( M_Y \) be the submodule of \( M_X \) generated by \( y_1, y_2, \ldots, y_s \). Then \( M_Y \) is isomorphic to the direct sum
\[
M_Y \cong \bigoplus_{i=1}^s R y_i.
\]
We claim that \( R_{y_i} \cong A_{Y_i} \), where \( A_{Y_i} \) is the homogeneous coordinate ring of \( Y_i \). Consider the \( R \)-module homomorphism \( R \to R_{y_i} \) given by \( f \mapsto f y_i \), for any \( f \in R \), and note that \( f y_i = 0 \) if and only if \( f_0(P) = f_1(P) = \cdots = f_d(P) = 0 \) for all \( P \in Y_i \), where \( f = f_0 + f_1 + \cdots + f_d \) is the decomposition of \( f \) into homogeneous polynomials. Hence, the kernel of \( R \to R_{y_i} \) is the ideal of forms vanishing on all points of \( Y_i \) which implies that \( R_{y_i} \) is isomorphic to the homogeneous coordinate ring of \( Y_i \), that is, \( R_{y_i} \cong A_{Y_i} \). Since \( M_Y \) is the direct sum of the \( R_{y_i} \)'s we get

\[
(6.8) \quad M_Y \cong \bigoplus_{i=1}^s A_{Y_i}.
\]

Let \( M_Z \) be the submodule generated by the elements \( z_i = \sum_{P \in Z_i} e_P \) for \( i = 1, 2, \ldots, t \). Then we see, in the same way as we did for \( M_Y \), that

\[
(6.9) \quad M_Z \cong \bigoplus_{i=1}^t A_{Z_i}.
\]

Furthermore, let \( \tau_Y = \max \{ \tau(A_{Y_i}) \}_{i=1}^s \) and \( \tau_Z = \max \{ \tau(A_{Z_i}) \}_{i=1}^t \). Then

\[
(6.10) \quad H(M_Y, d) = |X|
\]

for all \( d \geq \tau_Y \) and

\[
(6.11) \quad H(M_Z, d) = |X|
\]

for all \( d \geq \tau_Z \) by the definition of \( \tau \).

For the next step we need the \( R \)-module \( \ast \text{Hom}_k(N, k) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_k(N, k) \) defined for any \( R \)-module \( N \). Its grading is given by \( \ast \text{Hom}_k(N, k) \) for each degree \( i \), and multiplication given by letting \( f \cdot \psi, \) for any \( f \in R \) and \( \psi \in \ast \text{Hom}_k(N, k) \), be the function \( m \mapsto \psi(fm) \) for any \( m \in N \). We note that when \( N \) is artinian, which \( M_X, M_Y \) and \( M_Z \) are not, then \( \ast \text{Hom}_k(N, k) \cong N^\vee \), see [20, Proposition 2.5].

We will now show that \( \ast \text{Hom}_k(M_X, k)(d) \cong M_X \) for any integer \( d \). For any \( P \in X \) and integer \( u \), let \( t^{-u}e_P^* \in \ast \text{Hom}_k(M_X, k) \) be defined by

\[
t^{-u}e_P^*(t^ue_Q) = \begin{cases} 1 & \text{if } P = Q \text{ and } u = v, \\ 0 & \text{otherwise.} \end{cases}
\]

Then for any element \( f \in R \) of degree \( d \) we get \( f \cdot t^{-u}e_P^* = f(P)t^{-u+d}e_P^* \) and that \( (t^{-u}e_P^*)_{X \in X} \) is a \( k \)-basis of \( \ast \text{Hom}_k(M_X, k)_{-u} \), in fact, it is the \( k \)-dual of the \( k \)-basis \( (t^ue_P)_{P \in X} \) of \( (M_X)_u \). Hence, the function \( \ast \text{Hom}_k(M_X, k)(d) \to M_X \) defined by \( t^{-u}e_P^* \mapsto t^{u+d}e_P \) is an isomorphism of \( R \)-modules for any integer \( d \).

Note that

\[
(6.12) \quad H(\ast \text{Hom}_k(M_Z, k)(-c), d) = H(M_Z, c - d)
\]
for all $d$, and that we seek a module, $M$, with Hilbert function given by $H(M, d) = \max\{H(M_Y, d), H(M_Z, c - d)\}$, for all $d$. Note also that this Hilbert function is equal to $|X|$ in degree $\tau_Y \leq d \leq c - \tau_Z$ (see Figure 1).

We will now show that $^*\text{Hom}_k(M_Z, k)(-c) \cong M_X/N_Z$ for some submodule $N_Z \subseteq M_X$ and that in fact $N_Z \subseteq M_Y$. The exact sequence

$$0 \to M_Z \to M_X \to M_X/M_Z \to 0$$

gives the exact sequence

$$0 \to ^*\text{Hom}_k(M_X/M_Z, k) \to ^*\text{Hom}_k(M_X, k) \to ^*\text{Hom}_k(M_Z, k) \to 0.$$  

From this exact sequence and the isomorphism $^*\text{Hom}_k(M_X, k)(-c) \cong M_X$ we get

$$0 \to ^*\text{Hom}_k(M_X/M_Z, k)(-c) \to M_X \to ^*\text{Hom}_k(M_Z, k)(-c) \to 0,$$

and hence that $^*\text{Hom}_k(M_Z, k)(-c) \cong M_X/N_Z$ for some submodule $N_Z \subseteq M_X$. To see that $N_Z \subseteq M_Y$ observe first that (6.12) together with the definition of $N_Z$ implies

$$(6.13) \quad H(N_Z, d) = |X| - H(M_Z, c - d).$$

Then it follows from the definition of $\tau_Z$ that the initial degree of $N_Z$ is $c - \tau_Z + 1$. Finally, since $(M_Y)_d = (M_X)_d$ for all $d \geq \tau_Y$ and $c \geq \tau_Y + \tau_Z$, by assumption, we see that $(M_Y)_d = (M_X)_d$ for all $d \geq c - \tau_Z$, and hence, taking into account the initial degree of $N_Z$, that $N_Z \subseteq M_Y$.

We now claim that the artinian $R$-module $M = M_Y/N_Z$ has the weak Lefschetz property and the desired Hilbert function and Betti numbers. We have
the following commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 \\
N_Z & M_Y & M & 0 \\
\varphi & \varphi' & \varphi'' & \\
0 & *\text{Hom}(M_X/M_Z, k)(-c) & M_X & *\text{Hom}_k(M_Z, k)(-c) & 0 \\
\end{array}
\]

(6.14)

where \(\varphi''\) is the unique homomorphism induced by \(\varphi\) and \(\varphi'\). Note that \(\varphi'_d\) is an isomorphism in degrees \(d \geq \tau_Y\) and, as a consequence, \(\varphi''_d\) is also an isomorphism in degrees \(d \geq \tau_Y\).

It follows from (6.13) and the exact sequence in the top row of (6.14) that

\[
H(M, d) = \max\{H(M_Y, d), H(M_Z, c-d)\}
\]

which shows that the Hilbert function of \(M\) is the desired one. Since

\[
\bigoplus_{d \leq \tau_Y} M_d \cong \bigoplus_{d \leq \tau_Y} (M_Y)_d
\]

it follows from Lemma 6.6 that

(6.15)

\[
\beta_{i,j}(M) = \beta_{i,j}(M_Y)
\]

when \(j - i < \tau_Y\). Consider the dual of \(M, M'\), which since \(M\) is artinian equals \(*\text{Hom}_k(M, k)\), and the homomorphism

\[
\varphi'' : M \to *\text{Hom}_k(M_Z, k)(-c)
\]

from (6.14) which is an isomorphism in degrees \(d \geq \tau_Y\). Applying \(*\text{Hom}_k(\_, k)\) to this homomorphism and shifting the degrees by \(-c\) yields a homomorphism

\[
M_Z \to M'(-c)
\]

which is an isomorphism in degrees \(d \leq c - \tau_Y\). Hence we have

\[
\bigoplus_{d \leq c - \tau_Y} M'(-c)_d \cong \bigoplus_{d \leq c - \tau_Y} (M_Z)_d
\]

and it follows from Lemma 6.6 that

(6.16)

\[
\beta_{i,j}(M'(-c)) = \beta_{i,j}(M_Z)
\]

when \(j - i < c - \tau_Y\). Since \(\beta_{i,j}(M) = \beta_{p-i,p+c-j}(M'(-c))\) this means that

(6.17)

\[
\beta_{i,j}(M) = \beta_{i,j}(M_Y) + \beta_{p-i,p+c-j}(M_Z)
\]

when \(j - i > \tau_Y\). From (6.15) and (6.16) and the fact that \(\beta_{i,j}(M_Y) = 0\) when \(j - i > \tau_Y\) and \(\beta_{p-i,p+c-j}(M_Z) = 0\) when \(j - i < \tau_Y\) we get that
for all integers \( i \) and \( j \) satisfying \( j - i \neq \tau_Y \). To show the equality when \( j - i = \tau_Y \), consider the diagram \( D \) defined by

\[
D_{i,j} = \beta_{i,j}(M_Y) + \beta_{p-i,p+c-j}(M_Z)
\]

for all integers \( i \) and \( j \). Then we have seen above that \( h_D(t) = h_M(t) \) and this implies that \( S_D(t) = S_M(t) \) which gives the equality

\[
\sum_{i,j} (-1)^i D_{i,j} t^i = \sum_{i,j} (-1)^i \beta_{i,j}(M) t^i,
\]

and if we remove all terms that are equal by (6.17) we are left with

\[
\sum_{i=0}^p (-1)^i D_{i,u+i} t^{u+i} = \sum_{i=0}^p (-1)^i \beta_{i,u+i}(M) t^{u+i}
\]

which shows that we have equality in (6.17) also when \( j - i = u \), and hence that \( \beta(M) = D \).

It remains to show that \( M \) have the weak Lefschetz property. Let \( \ell \) be a non-zero divisor in \( A_X \), that is, an element in \( R \) such that \( \ell(P) \neq 0 \) for all \( P \in X \). Then \( \ell \) is a non-zero divisor on \( M_X \) and hence also on \( M_Y \) since it is a submodule of \( M_X \). This means that \( (M_Y/N_Z)_{d-1} \xrightarrow{\times \ell} (M_Y/N_Z)_d \) is surjective for all \( d \leq c - \tau_Z \) since \( (N_Z)_d = 0 \) for these values of \( d \). Assume now that \( d > c - \tau_Z \). We have that \( (M_Z)_{c-d-1} \xrightarrow{\times \ell} (M_Z)_{c-d} \) is surjective, since \( M_Z \) is a submodule of \( M_X \), and by applying \( \Hom_k(\_ , k) \) we get that \( \Hom_k((M_Z)_{c-d}, k) \xrightarrow{\ell} \Hom_k((M_Z)_{c-d-1}, k) \) is injective. Since \( (M_Y/N_Z)_d \cong (M_X/N_Z)_d \cong \Hom_k((M_Z)_{c-d}, k) \), when \( d > c - \tau_Z \), we get that \( (M_Y/N_Z)_d \xrightarrow{\ell} (M_Y/N_Z)_{d+1} \) is injective which finishes the proof. \( \square \)

**Theorem 6.8.** Assume that \( k \) is an infinite field and that \( h(t) = \sum_{i \in \mathbb{Z}} h_i t^i \) is a polynomial of degree \( c \) and furthermore, let \( u \) be the smallest integer such that \( h_u \geq h_{u+1} \). Then \( h(t) \) is a rational multiple of the \( h \)-vector of an artinian level \( R \)-module of codimension \( p \), generated in degree zero, having the weak Lefschetz property, if and only if

\[
f_i = \frac{h_i - h_{i-1}}{s_i} - \frac{h_{i+1} - h_i}{s_{i+1}} \geq 0,
\]

for \( i = 0, 1, \ldots, u - 1 \), and

\[
g_i = \frac{h_{c-i} - h_{c-i+1}}{s_i} - \frac{h_{c-i-1} - h_{c-i}}{s_{i+1}} \geq 0,
\]

for \( i = 0, 1, \ldots, c - u - 1 \), where \( s_i = \binom{p-2+i}{p-2} \). Furthermore, if the \( f_i \)'s and \( g_i \)'s are all non-negative, then there exist an artinian level \( R \)-module with the weak Lefschetz property whose Betti diagram is an integer multiple of the diagram \( E \) of Proposition 6.3.
Proof. The necessity of the conditions follows immediately from Proposition 6.3. Assume that the $f_i$'s and $g_i$'s of the proposition are all non-negative and furthermore, let

$$f_u = \frac{h_u - h_{u-1}}{s_u}$$

and

$$g_{c-u} = \frac{h_u - h_{u+1}}{s_{c-u}}.$$ 

Then $f_u \geq 0$ and $g_{c-u} \geq 0$ since, by assumption, $u$ is the smallest integer such that $h_u \geq h_{u+1}$. Let $m$ be an integer such that $mf_i$ is an integer for $i = 0, 1, \ldots, u$ and $mg_i$ is an integer for $i = 0, 1, \ldots, c + u$. We will use Proposition 6.7 to show the existence of an artinian level $R$-module with the weak Lefschetz property and Betti diagram the diagram $mE$, for some integer $m$, where $E$ is the diagram of Proposition 6.3. Then

$$h(t) = h_E(t) = \frac{1}{m}h_M(t)$$

which shows that $h(t)$ is a rational multiple of the $h$-vector of $M$.

Consider a set, $X$, of points in $\mathbb{P}^{p-1}$ in general position. It is a fact that, if $k$ is an infinite field, the condition on $X$ that makes the Hilbert function of $A_X$ given by $H(A_X, d) = \min\{r_d, |X|\}$ is open and non-empty. As a consequence, if we also require that $H(A_Y, d) = \min\{r_d, |Y|\}$ for any subset $Y$ of $X$ the condition is still non-empty and open condition. Assume that $X$ satisfies this condition and that there are two partitions

$$X = \bigcup_{j=0}^{u} \bigcup_{i=1}^{m_{f_i}} Y_{i,j} \quad \text{and} \quad X = \bigcup_{j=0}^{c-u} \bigcup_{i=1}^{m_{g_{i}}} Z_{i,j}$$

such that $|Y_{i,j}| = |Z_{i,j}| = r_j$ for all $i$ and $j$, where $r_j = (p-1+j)$. Then $H(A_{Y_{i,j}}, d) = \min\{r_d, |Y_{i,j}|\} = \min\{r_d, r_j\}$ and $H(A_{Z_{i,j}}, d) = \min\{r_d, |Z_{i,j}|\} = \min\{r_d, r_j\}$ for each integer $d$.

Since $\tau(A_{Y_{i,j}}) = \tau(A_{Z_{i,j}}) = j$, for each integer $j$, we get

$$\max\{\tau(A_{Y_{i,j}}) \mid 0 \leq j \leq u, 1 \leq i \leq m_{f_i}\}$$

and

$$\max\{\tau(A_{Z_{i,j}}) \mid 0 \leq j \leq c - u, 1 \leq i \leq m_{g_j}\} \leq u + c - u = c.$$

Note that the above is an inequality and not an equality since $g_j$ might be zero for some integers $j$. Proposition 6.7 now shows the existence of an artinian $R$-module, $M$, with the weak Lefschetz property and Betti diagram given by

$$\beta_{k,l}(M) = \sum_{j=0}^{u} \sum_{i=1}^{m_{f_i}} \beta_{k,l}(A_{Y_{i,j}}) + \sum_{j=0}^{c-u} \sum_{i=1}^{m_{g_j}} \beta_{p-k,p+c-l}(A_{Z_{i,j}})$$

By the way the $A_{Y_{i,j}}$'s and $A_{Z_{i,j}}$'s are defined we see that

$$\beta^R(A_{Y_{i,j}}) = \beta^S(S/m^{i+1})$$

where $S$ is a graded $R$-module and $m$ is a maximal ideal of $R$. The fact that $\beta^R(A_{Y_{i,j}})$ is a $R$-module with the weak Lefschetz property and Betti diagram given by

$$\beta_{k,l}(M) = \sum_{j=0}^{u} \sum_{i=1}^{m_{f_i}} \beta_{k,l}(A_{Y_{i,j}}) + \sum_{j=0}^{c-u} \sum_{i=1}^{m_{g_j}} \beta_{p-k,p+c-l}(A_{Z_{i,j}})$$

then follows from Proposition 6.7.

Thus, we have shown that $\beta^R(A_{Y_{i,j}}) = \beta^S(S/m^{i+1})$ for all $i, j$. This completes the proof.
and
\[ \beta^R(A_{Z,i,j}) = \beta^S(S/\mathfrak{m}^{i+1}) \]
Hence
\[ \sum_{j=0}^{u} \sum_{i=1}^{m} \beta(A_{Y_i,j}) = m \sum_{i=0}^{u} f_i \beta^S(S/\mathfrak{m}^{i+1}) \]
and we see that this equals \( mF \), where \( F \) is as defined in Proposition 6.3. In the same way we get
\[ \sum_{j=0}^{c-u} \sum_{i=1}^{m} \beta(A_{Z,i,j}) = m \sum_{i=0}^{c-u} g_i \beta^S(S/\mathfrak{m}^{i+1}) \]
which then equals \( mG \), where \( G \) is as defined in Proposition 6.3. And since, by (6.18), \( \beta(M) \) then satisfies
\[ \beta_{i,j}(M) = mF_{i,j} + mG_{p-i,p+c-j} = mE_{i,j}. \]
where \( E \) is as defined in Proposition 6.3. We have shown that \( h(t) \) is a rational multiple of the \( h \)-vector of a level artinian \( R \)-module with the weak Lefschetz property whose Betti diagram is the diagram \( E \) of Proposition 6.3. \( \square \)

Remark 6.9. In codimension two the conditions on the \( f_i \)'s and \( g_i \)'s of Theorem 6.8 are equivalent to
\[ h_{i+1} - 2h_i + h_{i-1} \leq 0. \]
for \( i = 0, 1, \ldots, c \). This follows by straightforward calculation using \( s_i = 1 \). This in turn is equivalent to the condition of Theorem 4.7 in the codimension two case, and furthermore, this precisely describes the level \( h \)-vectors of modules of codimension two (see Remark 4.8).

7. Level modules of codimension three

We prove a result on linear combinations of pure diagrams of codimension three. Combining this result with the result on Lefschetz modules we can prove that the Betti diagram of any level \( R \)-module of codimension three whose artinian reduction has the weak Lefschetz property is a non-negative linear combination of pure diagrams. This in turn proves the Multiplicity conjecture for these module and even a stronger conjecture of Zanello. We finish this section with a proof of the upper bound of the Multiplicity conjecture of Herzog, Huneke and Srinivasan for level modules of codimension three, with or without the weak Lefschetz property.

Zanello has proposed a strengthening of the Multiplicity conjecture in the case of level algebras of codimension three. The conjecture of Zanello [17, Conj. 2.3] is that the Multiplicity conjecture is true not only for the Betti diagram of any level algebra of codimension three but also for any diagram obtained from that diagram by doing all possible cancellations (for level algebras of codimension three this
maximally cancelled, and hence in a sense minimal, diagram is unique). This can
be formulated in terms of the $h$-vector of the level algebra $R/I$, since the minimal
and maximal shifts for the maximally cancelled diagram can be obtained from the
polynomial $S_{R/I}(t) = \sum_{i,j} (-1)^i \beta_{i,j} (R/I)t^j$ and, since $R/I$ is Cohen-Macaulay of
codimension three, $S_{R/I}(t) = (1 - t)^3 h(t)$. Assume that $h(t)$ is of degree $c$ and
$S_{R/I}(t) = \sum_{i=0}^{c+3} \Delta^3 h_i t^i$ and let

i) $n_1 = \min \{i \mid \Delta^3 h_i < 0\}$,

ii) $n_2 = \min \{i > 0 \mid \Delta^3 h_i > 0\}$,

iii) $N_1 = \max \{i \leq c+1 \mid \Delta^3 h_i < 0\}$ and

iv) $N_2 = \max \{i \mid \Delta^3 h_i > 0\}$,

that is, $n_1$ and $n_2$ are the degrees of the first and second sign changes in $S_{R/I}(t)$,
respectively, and $N_1$ and $N_2$ the degrees of the second last and last sign changes.

**Conjecture 7.1** (Zanello). If $R/I$ is level of codimension three then

$$\frac{1}{3!} n_1 n_2 (c + 3) \leq e(R/I) \leq \frac{1}{3!} N_1 N_2 (c + 3).$$

Conjecture \[\ref{conj:7.1}\] has been proved for Gorenstein algebras of codimension three
by Migliore, Nagel and Zanello \[\cite{MiglioreNagelZanello}\].

We will now show that Conjecture \[\ref{conj:7.1}\] holds for any level algebra of codimension
three whose Betti diagram is a non-negative linear combination of pure diagrams.
The reason for this is that this set of diagrams is, as we will see, closed under the
operation of making cancellations. Hence the strengthening proposed by Zanello
follows from Conjecture \[\ref{conj:7.1}\].

We also note that since the Betti diagram of a Gorenstein codimension three
algebra by \[\cite[Theorem 4.3]{Zanello}\] is a non-negative linear combination of pure diagrams
we get an alternative proof, using the technique of pure diagrams, of Zanello’s
stronger conjecture in this case.

**Proposition 7.2.** Let $D$ be a non-negative linear combination of pure diagrams
of codimension three. If $D$ has only one non-zero entry in column zero and
column three, then any consecutive cancellation of $D$ is still a non-negative linear
combination of pure diagrams.

**Proof.** Note that, in codimension one, any consecutive cancellation of a non-
negative linear combination of pure diagrams is still a non-negative linear combination
of pure diagrams. In fact, let $D$ be a diagram of codimension one. Since for any shifts $(d_0, d_1)$ the pure diagram $\pi(d_0, d_1)$ has as its non-zero entries
$\pi(d_0, d_1)_{0,d_0} = 1$ and $\pi(d_0, d_1)_{1,d_1} = 1$ we see that $D$ is a convex combination
of pure diagrams if and only if all entries of $D$ are non-negative and

$$\sum_{i \leq l} D_{0,i} \geq \sum_{i \leq l} D_{1,i+1}$$

for each integer $l$, and furthermore, we see that any cancellation of $D$ still satisfies
these conditions.
Now, denote by \( \overrightarrow{d} = (d_0, d_1, d_2, d_3) \) and \( \overrightarrow{d} = (d_0, d_1, d_2, d_3) \) the maximal and minimal shifts of \( D \), respectively. Then, by the hypothesis on the entries of \( D \), we have \( d_0 = d_0 \) and \( d_3 = d_3 \). This permits us to use the isomorphisms \( \phi_0 \) and \( \phi_3 \) from Definition 4.5. We get an isomorphism

\[
\phi_0 \circ \phi_3 : V_{\overrightarrow{d}} \rightarrow V_{(d_1, d_2)}.
\]

Since this isomorphism preserves non-negative linear combination of pure diagrams the proposition follows from the codimension one case above. \( \square \)

**Corollary 7.3.** Let \( M \) be a level module of codimension three whose Betti diagram, \( \beta(M) \), is a non-negative linear combination of pure diagrams, then any diagram obtained from \( \beta(M) \) by a sequence of consecutive cancellations is a non-negative linear combination of pure diagrams.

**Proof.** Since \( M \) is level, \( \beta(M) \) has only one non-zero entry in column zero and column three, and hence we can apply Proposition 7.2. \( \square \)

**Corollary 7.4.** Conjecture 7.1, and hence the Multiplicity conjecture, is true for any codimension three level algebra whose Betti diagram is a non-negative linear combination of pure diagrams.

**Proof.** Let \( M \) be a level algebra with a Betti diagram, \( \beta(M) \), which is a non-negative linear combination of pure diagrams. The unique diagram, \( D \), obtained from \( \beta(M) \) by doing all possible consecutive cancellations, is then still a non-negative linear combination of pure diagrams, by Corollary 7.3. Then the condition of the Multiplicity conjecture hold for \( D \), since by [2, Proposition 2.8] it holds for any non-negative linear combination of pure diagrams, and this condition for the diagram \( D \) is precisely the condition of Conjecture 7.1 for \( \beta(M) \). \( \square \)

**Proposition 7.5.** Let \( M \) be a level \( R \)-module whose artinian reduction has the weak Lefschetz property. If \( M \) is of codimension three, then its Betti diagram is a non-negative linear combination of pure diagrams which implies

\[
\beta_0(M) \frac{d_1d_2d_3}{3!} \leq e(M) \leq \beta_0(M) \frac{d_1d_2d_3}{3!},
\]

where \( (0, d_1, d_2, d_3) \) and \( (0, d_1, d_2, d_3) \) are the minimal and maximal shifts of \( M \), respectively. Moreover, \( M \) even satisfies the stronger condition of Zanello’s Conjecture 7.7.

**Proof.** By Proposition 6.3, \( \beta(M) \) is obtained from a non-negative linear combination of pure diagrams by a sequence of consecutive cancellations. Hence \( \beta(M) \) is non-negative linear combination of pure diagrams by Corollary 7.3. The two other assertions follows from Corollary 7.4. \( \square \)
7.1. The upper bound of the Multiplicity conjecture. In this section we show that level modules of codimension three satisfies the upper bound of the Multiplicity conjecture of Herzog, Huneke and Srinivasan. The technique we use is based on the description, given in Proposition 3.3 of Betti diagrams as cancellations of non-negative linear combinations of pure diagrams.

We need the following result from [2, Corollary 2.12].

Proposition 7.6. If $E$ is a diagram of codimension $p$ with maximal and minimal shifts $\overrightarrow{d} = (d_0, d_1, \ldots, d_p)$ and $\overrightarrow{d} = (d_0, d_1, \ldots, d_p)$ satisfying $\overrightarrow{d}_{i-1} < \overrightarrow{d}_i$ for all $0 \leq i \leq p$ then $E$ is a non-negative linear combination of pure diagrams.

Note that proposition 7.6 is similar to a result by Herzog and Srinivasan, generalized to modules by Migliore, Nagel and Römer [16], stating that any module with a quasipure minimal free resolution, that is, with maximal and minimal shifts satisfying $d_{i-1} < d_i$ for all $0 \leq i \leq p$, satisfies the Multiplicity conjecture.

The proof of Theorem 7.7 uses the following idea from [2]. The multiplicity of the pure diagram $\pi(0, d_1, d_2, \ldots, d_p)$ is $e(\pi(d)) = d_1d_2\cdots d_p/p!$. Hence, for any diagram $D = \sum d'_\leq d a_{d'} \pi(d')$, where $a_{d'}$ are non-negative integers, we get

$$e(D) = \sum_{d'_\leq d} a_{d'} e(\pi(d')) \leq \sum_{d'_\leq d} a_{d'} e(\pi(d)) = D_{0,0} \frac{d_1d_2\cdots d_p}{p!}.$$

Theorem 7.7. Any level $R$-module, $M$, of codimension three, with maximal shifts given by $\overrightarrow{d} = (0, d_1, d_2, d_3)$, satisfies

$$e(M) \leq \beta_0(M) \frac{d_1d_2d_3}{3!}.$$

Proof. Let $M$ be a level $R$-module of codimension three, with maximal shifts given by $\overrightarrow{d} = (0, d_1, d_2, d_3)$. By artinian reduction we may assume that $M$ is artinian and hence that $R = k[x_1, x_2, x_3]$. Note that $\beta(M)$ in degrees between $d_1 + 1$ and $d_2$ have non-zero entries only in column two, see Figure 2. We will now show that there is a diagram $F = E + D'$ such that $h_F(t) = h_M(t)$ and where $E$ have non-zero entries only in the positions indicated by the grey area in Figure 2 and $D'$ have maximal shifts $(0, d_1 - 2, d_1 - 1, d_1)$.

We start with the description of $\beta(M)$ given in Proposition 3.3. Since $\beta(R/m^j+1)$ is the pure diagram $\pi(0, j + 1, j + 2, j + 3)$ we have, by Proposition 3.3

$$\beta(M) = D - \sum_{i,j} b_{i,j} C^{i,j}.$$

where

$$D = \sum_{j=0}^c a_{j} \pi(0, j + 1, j + 2, j + 3).$$
Figure 2. The Betti diagram of a level module $M$ with maximal shifts $\mathbf{d} = (0, d_1, d_2, d_3)$. Every Betti number outside the outlined area are zero. The grey area indicates the non-zero entries of the diagram $E$ in the proof of Theorem 7.7.

for some non-negative rational numbers $a_j$ and $b_{i,j}$ and $c = d_3 - 3$. Consider the diagrams

$$D' = \sum_{j=0}^{d_1-2} a_j \pi(0, j+1, j+2, j+3)$$

and

$$D'' = \sum_{j=d_1-1}^{c} a_j \pi(0, j+1, j+2, j+3)$$

and note that $D = D' + D''$. We will now see that the diagram, $E$, mentioned in the beginning of this, can be obtained from $D''$ by a sequence of consecutive cancellations. Note that $D''$ is zero in all positions $(i, j)$ such that $j - i < d_1 - 1$ and that the only non-zero entries in the matrix $C^{i,j}$ are $C^{i,j}_{i,j} = 1$ and $C^{i,j}_{i+1,j} = 1$. Now we want to take as many terms $\beta_{i,j}C^{i,j}$ from $\sum_{i,j} b_{i,j} C^{i,j}$ and remove them from $D''$ without getting any negative entries in the result. If we do this we end up with

$$E = D'' - \sum_{j=d_1}^{c} (b_{1,j+1} C^{1,j+1} + b_{2,j+2} C^{2,j+2}).$$

Note that $\beta_{i,j}(M) = E_{i,j}$ whenever $j - i \geq d_1 - 1$ or whenever $i = 3$. The maximal shifts of $E$ are thus the same as those of $\beta(M)$ and $E$ has non-zero entries only in the positions indicated by the grey area in Figure 2. By Proposition 7.6 we get
that \( E \) is a non-negative linear combination of pure diagrams \( \pi(d) \) whose type satisfies \( d \leq (0, \overrightarrow{d}_1, \overrightarrow{d}_2, \overrightarrow{d}_3) \) and hence the same is true for the diagram

\[
F = E + D'.
\]

The diagram \( F \) have the same \( h \)-vector as \( M \), since the cancellations does not affect the \( h \)-vector, and since \( F \) is a non-negative linear combination of pure diagrams \( \pi(d) \) whose type satisfies \( d \leq (0, \overrightarrow{d}_1, \overrightarrow{d}_2, \overrightarrow{d}_3) \) we get, by (7.1),

\[
e(M) = h_M(1) = h_F(1) = e(F) \leq \beta_0(M) \frac{d_1 d_2 d_3}{3!}.
\]

□

What makes the proof of Theorem 7.7 possible is that the upper bound of the Multiplicity conjecture is true for any cancellation of a diagram on the form

\[
\sum_{j=0}^c a_j \pi(0, j + 1, j + 2, j + 3)
\]

for some non-negative integers \( a_0, a_1, \ldots, a_c \), as long as it has only one entry in column three. This means that we can prove Theorem 7.7 without knowing which cancellations of this diagram that actually are Betti diagrams of some module. To see that this is not always the case, we now give two examples. The first, Example 7.8, shows that the lower bound of Multiplicity conjecture cannot be shown to hold with this technique. The second, Example 7.9, shows that we need the module to be level (actually it is enough that all but one of the entries in the last column of its Betti diagram can be numerically cancelled).

Example 7.8. Consider the \( h \)-vector \( h(t) = 16 + 48 t + 21 t^2 + 10 t^3 \). The diagram

\[
D = \sum_{j=0}^3 a_j \pi(0, j + 1, j + 2, j + 3), \quad \text{where} \quad a_j = h_j/r_j - h_{j+1}/r_{j+1},
\]

is then

\[
D = \begin{pmatrix}
16 & - & - \\
- & 75 & 100 & 75/2 \\
- & 25 & 75/2 & 15 \\
- & 15 & 24 & 10
\end{pmatrix}.
\]

This diagram can be cancelled, by consecutive cancellations, to the diagram

\[
\begin{pmatrix}
16 & - & - \\
- & 75 & 75 \\
- & - & - \\
- & 15 & 9 & 10
\end{pmatrix}
\]

with minimal shifts \( d_1 = 2, d_2 = 3 \) and \( d_3 = 6 \). The lower bound of the Multiplicity conjecture for these shifts are

\[
16 \frac{d_1 d_2 d_3}{6} = 16 \frac{2 \cdot 3 \cdot 6}{6} = 96,
\]

while its multiplicity is \( h(1) = 16 + 48 + 21 + 10 = 95. \)
Example 7.9. Now consider the $h$-vector $h(t) = 5 + 15t + 18t^2 + 15t^3$. The diagram $D = \sum_{j=0}^{3} a_j \pi(0, j+1, j+2, j+3)$, where $a_j = h_j/r_j - h_{j+1}/r_{j+1}$, is then

$$D = \begin{pmatrix}
5 & - & - & - \\
- & 12 & 16 & 6 \\
- & 15 & 45/2 & 9 \\
- & 45/2 & 36 & 15
\end{pmatrix}.$$ 

This diagram can be cancelled, by consecutive cancellations, to the diagram

$$\begin{pmatrix}
5 & - & - & - \\
- & 12 & 1 & 6 \\
- & - & - & - \\
- & - & 27 & 15
\end{pmatrix},$$

with maximal shifts $\overline{d}_1 = 2$, $\overline{d}_2 = 5$ and $\overline{d}_3 = 6$. The upper bound of the Multiplicity conjecture for these shifts are

$$\frac{5 \cdot 2 \cdot 5 \cdot 6}{6} = 50,$$

while its multiplicity is $5 + 15 + 18 + 15 = 53$.

Remark 7.10. To prove the lower bound of the Multiplicity conjecture it is enough to consider level modules. In fact, assume that $M$ is an artinian $R$-module of codimension $p$, generated in degree zero, with minimal shifts given by $\underline{d} = (0, d_1, d_2, \ldots, d_p)$. Let $M'$ be the $R$-module

$$M' = \bigoplus_{i=0}^{d_p-p} M_i.$$

The Betti numbers of this module satisfies

$$\beta_{i,j}(M') = \beta_{i,j}(M),$$

for all $j - i < d_p - p$, by Lemma 6.6. This means that the minimal shifts of $M'$ and $M$ are the same and since furthermore

$$e(M) = \sum_i \dim_k M_i \geq e(M') = \sum_{i \leq d_p-p} M_i,$$

we see that if the lower bound of the Multiplicity conjecture holds for $M'$ it holds for $M$ as well, that is,

$$\beta_0(M) \frac{d_1 \cdot d_2 \cdots d_p}{p!} \leq e(M') \leq e(M).$$
7.2. Convexity of the order complex. In [2] we introduced the order complex $\Delta(\Pi_d, \bar{d})$ associated to a pair of strictly increasing sequences $d = (d_0, d_1, \ldots, d_p)$ and $\bar{d} = (\bar{d}_0, \bar{d}_1, \ldots, \bar{d}_p)$. Consider the partial order on strictly increasing sequences of length $p + 1$ defined by $d \leq d'$ if $d_i \leq d'_i$ for each $i = 0, 1, \ldots, p$. The order complex $\Delta(\Pi_d, \bar{d})$ is a geometric realization of the the order complex associated to the partial ordered set of all strictly increasing sequences $d$ such that $\bar{d}_0 \leq d_0 \leq \bar{d}_3$.

Boij and the author conjectured $\Delta(\Pi_d, \bar{d})$ to be a convex set, and noted that this is equivalent to the fact that any non-negative linear combination of pure diagrams can be written as a non-negative linear combination of pure diagrams from the same chain (where the pure diagrams inherits the order from their types).

We will now show that this is true for level algebras of codimension three.

Proposition 7.11. Let $D$ be a non-negative linear combination of pure diagrams of codimension three. If $D$ has only one non-zero entry in column zero and column three, then $D$ is a non-negative linear combination of pure diagrams all from the same chain. In other words, if $\bar{d} = (\bar{d}_0, \bar{d}_1, \bar{d}_2, \bar{d}_3)$ and $\bar{d}' = (\bar{d}_0, \bar{d}_1, \bar{d}_2)$ are strictly increasing sequences of integers such that $\bar{d}_0 = \bar{d}_0'$ and $\bar{d}_3 = \bar{d}_3'$, then $\Delta(\Pi_d, \bar{d})$ is a convex set.

Proof. By [2, Theorem 3.4], $\Delta(\Pi_{\bar{d}', \bar{d}'})$ where $\bar{d}' = (\bar{d}_0, \bar{d}_1, \bar{d}_2)$ and $\bar{d} = (\bar{d}_0, \bar{d}_1, \bar{d}_2, \bar{d}_3)$ is a convex set. The isomorphism

$$\frac{1}{d_3} \phi_3 : V_{d, \bar{d}} \to V_{d', \bar{d}'}$$

of Definition 4.5 preserves convex combinations, and furthermore the image of $\Delta(\Pi_{\bar{d}, \bar{d}})$ under $\frac{1}{d_3} \phi_3$ is $\Delta(\Pi_{d', \bar{d}'} \bar{d}')$. This shows that $\Delta(\Pi_{d, \bar{d}})$ is convex.

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