THE VALUE-DISTRIBUTION OF ARTIN $L$-FUNCTIONS ASSOCIATED WITH CUBIC FIELDS IN CONDUCTOR ASPECT

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Abstract. Arising from the factorizations of Dedekind zeta-functions of cubic fields, we obtain Artin $L$-functions of certain two-dimensional representations. In this paper, we study the value-distribution of such Artin $L$-functions for families of non-Galois cubic fields in conductor aspect. We prove that various mean values of the Artin $L$-functions are represented by integrals involving a density function which can be explicitly constructed. By the class number formula, the result is applied to the study on the distribution of class numbers of cubic fields.

1. Introduction

Let $d$ be a non-square integer such that $d \equiv 0, 1 \mod 4$, namely, a discriminant of a binary quadratic form. The quadratic Dirichlet $L$-function $L(s, \chi_d)$ is the Dirichlet $L$-function of the character $\chi_d(n) = (\frac{d}{n})$, where $(\cdot)$ indicates the Kronecker symbol. The distribution of values $L(s, \chi_d)$ has been studied by various authors. One of the earliest results was obtained by Chowla and Erdős [12]. They proved the existence of a continuous function $F_\sigma$ for $\sigma > 3/4$ such that the limit formula

\[
\lim_{X \to \infty} \frac{\# \{0 < d < X \mid L(\sigma, \chi_d) \leq e^a\}}{X/2} = F_\sigma(a)
\]

holds for any $a \in \mathbb{R}$. Furthermore, $F_\sigma$ is the distribution function of a probability measure on $\mathbb{R}$, that is, $F_\sigma$ is non-decreasing over $\mathbb{R}$ and satisfies $\lim_{t \to -\infty} f(t) = 0$, $\lim_{t \to \infty} f(t) = 1$. Elliott [13,18] studied quadratic Dirichlet $L$-functions for negative discriminants, and obtained a similar result at $\sigma = 1$:

\[
\lim_{X \to \infty} \frac{\# \{0 < -d < X \mid L(1, \chi_d) \leq e^a\}}{X/2} = F_1(a).
\]

It was also shown that $F_1$ possesses a probability density function $Q$ whose Fourier transform is represented as the infinite product

\[
\tilde{Q}(\xi) = \prod_p \left[ \frac{1}{2} \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{p} \right)^{-i\xi} + \frac{1}{2} \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{p} \right)^{-i\xi} + \frac{1}{p} \right],
\]

where $p$ runs through all prime numbers. Some related results were also obtained in [15,17]. These studies led to the idea of comparing the value-distributions of $L(s, \chi_d)$ with a suitable random model, which brought the recent progress in the theory of value-distributions of zeta and $L$-functions; see [19,20,28,29,31] for example.

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The values of quadratic Dirichlet $L$-functions at $s = 1$ are connected with the class numbers. Let $h_d$ denote the class number of a discriminant $d$ in the narrow sense. Put $\epsilon_d = (u_d + v_d\sqrt{d})/2$ for $d > 0$, where $(u_d, v_d)$ is the fundamental solution of the Pell equation $u^2 - dv^2 = 4$. Then we obtain

$$L(1, \chi_d)\sqrt{d} = h_d \log \epsilon_d$$

for $d > 0$ and

$$L(1, \chi_d)\sqrt{|d|} = \pi h_d$$

for $d < -4$ by Dirichlet’s class number formula. Hence, limit formulas (1.1) and (1.2) yield that

$$\sum_{0 < d < X \atop d \equiv 0 \mod 4} h_d \log \epsilon_d \sim \frac{\pi^2}{42\zeta(3)} X^{3/2}$$

and

$$\sum_{0 < -d < X \atop d \equiv 0 \mod 4} h_d \sim \frac{\pi}{42\zeta(3)} X^{3/2}$$

as $X \to \infty$, where $\zeta(s)$ is the usual Riemann zeta-function. Moreover, we have more precise formulas.

$$\sum_{0 < d < X} h_d \log \epsilon_d \sim \frac{\pi^2}{18\zeta(3)} X^{3/2} + O(X \log X),$$

(1.3)

$$\sum_{0 < -d < X} h_d \sim \frac{\pi}{18\zeta(3)} X^{3/2} + O(X \log X)$$

(1.4)

due to Siegel [40]. Recall further that Ayoub [1, 2] studied the case where $d$ varies over the set of discriminants of quadratic fields, and obtained asymptotic formulas similar to (1.3) and (1.4) up to the coefficients of main terms.

1.1. Artin $L$-functions associated with cubic fields. Let $K$ be a cubic field with discriminant $d_K$ which is non-Galois over $\mathbb{Q}$. We denote by $[K]$ the isomorphism class of cubic fields containing $K$. Then we define

$$L^±_3(X) = \{[K] \mid K \text{ is a non-Galois cubic field with } 0 < \pm d_K < X\}$$

and put $N^±_3(X) = \#L^±_3(X)$. Throughout this paper, we write $K \in L^±_3(X)$ instead of $[K] \in L^±_3(X)$ for simplicity. For every $K \in K^±_3(X)$, the Dedekind zeta function $\zeta_K(s)$ is factorized as

$$\zeta_K(s) = L(s, \rho_K)\zeta(s),$$

where $\rho_K$ is the standard representation of the Galois group $\text{Gal}(\widehat{K}/\mathbb{Q}) \simeq S_3$, and $L(s, \rho_K)$ is the attached Artin $L$-function. Here, $\widehat{K}$ indicates the Galois closure of $K$ over $\mathbb{Q}$, and $S_3$ is the symmetric group of degree 3. Let $F = \mathbb{Q}(\sqrt{d_K})$. Then the Galois group $\text{Gal}(\widehat{K}/F)$ is isomorphic to the cyclic group of order 3. We see that $\rho_K$ is the representation induced from the non-trivial character of $\text{Gal}(\widehat{K}/F)$. The Artin $L$-function $L(s, \rho_K)$ is therefore holomorphic over the whole complex plane. Moreover, the strong Artin conjecture is true for $L(s, \rho_K)$, i.e. there exists a cuspidal representation $\pi$ of $GL_2(\mathbb{A}_Q)$ such that $L(s, \rho_K) = L(s, \pi)$ holds. By definition, the value $L(\sigma, \rho_K)$ is real whenever $\sigma$ is a real number.
The purpose of this paper is to study the value-distribution of \( L(s, \rho_K) \) as the cubic field \( K \) varies in the family \( L_3^\pm(X) \). The detailed statements of the results are presented in Section 2. In this section, we pick up two of them for comparison with the above results on quadratic Dirichlet \( L \)-functions.

**Theorem 1.1.** Let \( \sigma > 7/8 \) be a real number. Then there exists a non-negative \( C^\infty \)-function \( C_\sigma \) on \( \mathbb{R} \) such that

\[
\lim_{X \to \infty} \frac{\# \{ K \in L_3^\pm(X) \mid L(\sigma, \rho_K) \leq e^a \}}{N_3^\pm(X)} = \int_{-\infty}^a C_\sigma(x) \frac{dx}{\sqrt{2\pi}}
\]

holds for any \( a \in \mathbb{R} \). Furthermore, the Fourier transform of \( C_\sigma \) is represented as

\[
\tilde{C}_\sigma(\xi) = \prod_p \frac{1}{1 + p^{-1} + p^{-2}} \left[ \frac{1}{6} \left( 1 - \frac{1}{p^\sigma} \right)^{-2i\xi} + \frac{1}{2} \left( 1 - \frac{1}{p^{2\sigma}} \right)^{-i\xi} \right. \\
\left. + \frac{1}{3} \left( 1 + \frac{1}{p^\sigma} + \frac{1}{p^{2\sigma}} \right)^{-i\xi} + \frac{1}{p} \left( 1 - \frac{1}{p^{3\sigma}} \right)^{-i\xi} + \frac{1}{p^2} \right],
\]

where \( p \) runs through all prime numbers.

Limit formula (1.5) gives analogues of (1.1) and (1.2) for the Artin \( L \)-functions.

In this paper, we further evaluate the rate of convergence in (1.5); see Theorem 2.6. Notice that \( C_\sigma \) is similar to Elliott’s density function \( Q \) in view of the infinite product representations of their Fourier transforms. See Theorem 2.1 for more information about the density function \( C_\sigma \).

Put \( D^+ = 4 \) and \( D^- = 2\pi \). By the class number formula, we have

\[
L(1, \rho_K) = \frac{D^+ h_K R_K}{\sqrt{|d_K|}}
\]

where \( h_K \) and \( R_K \) denote the class number and the regulator of a cubic field \( K \), respectively. As analogues of (1.3) and (1.4), we prove the following asymptotic formulas.

**Theorem 1.2.** There exists an absolute constant \( \delta > 0 \) such that

\[
\sum_{K \in L_3^+(X)} h_K R_K = cX^{3/2} + O \left( X^{3/2} \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right),
\]

\[
\sum_{K \in L_3^-(X)} h_K R_K = \frac{6}{\pi} cX^{3/2} + O \left( X^{3/2} \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right),
\]

where \( c \) is a positive constant represented as

\[
c = \frac{\pi^2 \zeta(3)}{432} \prod_p (1 + p^{-2} - 2p^{-3} - 2p^{-4} + 2p^{-6} + p^{-7} - p^{-8}).
\]

By a standard argument using the partial summation, Theorem 1.2 is deduced from (1.6) and some estimate on the first moment of \( L(1, \rho_K) \). More generally, we show an asymptotic formula for the \( z \)-th moment

\[
M_{z, \sigma}^\pm(X) = \sum_{K \in L_3^\pm(X) \setminus E_\sigma(X)} L(\sigma, \rho_K)^z
\]

for \( \sigma > 7/8 \).
for \( \sigma > 7/8 \) with \( z \in \mathbb{C} \), where \( E_\sigma(X) \) is a suitable subset of \( L^2_3(X) \) satisfying at least \( \#E_\sigma(X) = o(X) \) as \( X \to \infty \). See Theorem 2.2 for the strict statement.

1.2. Related topics. The study of this paper is paper is partially motivated by the recent work on “\( M \)-functions” by Ihara–Matsumoto. We recall one of the results from [23] in the number field case. Let \( F \) be \( \mathbb{Q} \) or an imaginary quadratic field. For a prime ideal \( f \) of \( F \), we define \( X(f) \) as the set of all primitive Dirichlet characters \( \chi \) on \( F \) with conductor \( f \). Suppose that the Generalized Riemann Hypothesis (GRH) is true, that is, every Dirichlet \( L \)-function \( L(s, \chi) \) has no zeros in the half-plane \( \text{Re}(s) > 1/2 \). Then \( \log L(s, \chi) \) extends to a holomorphic function on \( \text{Re}(s) > 1/2 \).

Theorem 1.3 (Ihara–Matsumoto [23]). Assume GRH, and denote by \( |dz| \) the measure \( (2\pi)^{-1}dxdy \) with \( z = x + iy \). Then there exists a non-negative \( C^\infty \)-function \( M_\sigma \) on \( \mathbb{C} \) such that

\[
\sum_{\chi \in X(f)} \Phi(\log L(s, \chi)) = \int_{\mathbb{C}} \Phi(z) M_\sigma(z) |dz|
\]

holds for any complex number \( s = \sigma + i\tau \) with \( \sigma > 1/2 \), where \( \Phi \) is any continuous function on \( \mathbb{C} \) satisfying \( \Phi(z) \ll e^{a|z|} \) for some \( a > 0 \).

They also showed that a similar result is valid when \( \log L(s, \chi) \) is replaced with the logarithmic derivative \( (L'/L)(s, \chi) \). Furthermore, several analogous results were proved in [21,22,24], and so on. The construction of the function \( M_\sigma \) was explained in [22] Section 3, and it matches a density function in the classical result of Bohr–Jessen [7,8] on the value-distribution of the Riemann zeta-function. The density functions such as \( M_\sigma \) were named \( M \)-functions by Ihara [21]. Today there are a lot of variants of Theorem 1.3 and the corresponding \( M \)-functions; see the survey of Matsumoto [34]. In particular, the \( M \)-function for the value-distribution of quadratic Dirichlet \( L \)-functions was studied by Mourtada–Murty [38]. The density function \( C_\sigma \) of Theorem 1.1 is regarded as a cubic analogue of Mourtada–Murty’s \( M \)-function. We prove a limit formula similar to (1.8): see Theorem 2.8.

Another topic related to this paper is the work on the Artin \( L \)-function \( L(s, \rho_K) \) due to Cho–Kim. They studied \( L(s, \rho_K) \) not only for cubic fields but also for general \( S_n \)-fields of degree \( n \geq 2 \). Here, a number field \( K \) of degree \( n \) is called an \( S_n \)-field if the Galois group \( \text{Gal}(K/\mathbb{Q}) \) is isomorphic to the symmetric group \( S_n \). Their results were often obtained under the following two conjectures:

- the strong Artin conjecture for \( L(s, \rho_K) \),
- the “counting conjecture” for \( S_n \)-fields; see [11] Conjecture 3.1.

The truth of the former conjecture is known for \( n \leq 4 \), and the latter conjecture is for \( n \leq 5 \). Hence the results for \( n = 2, 3, 4 \) are unconditional. In [11], they proved an asymptotic formula for integral moments of \( \log L(1, \rho_K) \). Here we refer to the result in the cubic case.

Theorem 1.4 (Cho–Kim [11]). Let \( k \) be a positive integer. Then we have

\[
\frac{1}{N}\sum_{\chi \in X} \left( \log L(1, \rho_K) \right)^k = \tilde{r}(k) + O\left( \frac{1}{\log X} \right),
\]

where \( \tilde{r}(k) \) is a positive constant which can be explicitly described.
If we assume that $\tilde{r}(k) \ll c^k \log \log k$ is satisfied for some $c > 1$, the method of moments enables us to show the existence of a continuous function $F$ satisfying

$$\lim_{X \to \infty} \frac{\# \{ K \in L_3^+(X) \mid L(1, \rho_K) \leq e^a \}}{N_3^+(X)} = F(a),$$

where $a$ is a point of continuity of $F$. Then (1.5) refines and generalizes this limit formula. Furthermore, it is remarkable that Theorem 1.1 can be proved without any assumptions on the constants $\tilde{r}(k)$. The original representation of $\tilde{r}(k)$ by Cho–Kim is described in [11, Proposition 5.3], while we obtain another representation $\tilde{r}(k) = \int_{-\infty}^{\infty} x^k C_1(x) \frac{dx}{\sqrt{2\pi}}$ by using the density function $C_\sigma$ of Theorem 1.1. See also Corollary 6.1.

In addition, we recall another result due to Cho–Kim on the distribution of values $L(1, \rho_K)$. Let $\alpha > 0$ be a real number. Then it was proved in [10] that

$$\lim \inf_{X \to \infty} \frac{\# \{ K \in L_3^+(X) \mid |L(1, \rho_K) - \alpha| < \epsilon \}}{N_3^+(X)} > 0$$

for every $\epsilon > 0$. This is a cubic analogue of the denseness result obtained by Mishou–Nagoshi [36]. Let $\sigma > 7/8$. Note that limit formula (1.5) yields

$$\lim_{X \to \infty} \frac{\# \{ K \in L_3^+(X) \mid |L(\sigma, \rho_K) - \alpha| < \epsilon \}}{N_3^+(X)} = \int_{\log(\alpha-\epsilon)}^{\log(\alpha+\epsilon)} C_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

if $\epsilon > 0$ is sufficiently small. Furthermore, the support of the density function $C_\sigma$ equals to $\mathbb{R}$ for $7/8 < \sigma \leq 1$; see Theorem 2.1. Thus the right-hand side of (1.10) is positive. As a result, we recover (1.9) and extend it for $7/8 < \sigma \leq 1$.

The organization of this paper is as follows.

- Section 2 is devoted to presenting the statement of the main results of this paper.
- In Section 3 we collect preliminary lemmas used later.
- The proofs of the main results begin with the study of the density function $C_\sigma$ described in Theorem 1.1. In Section 4.1 we study the random Euler products attached to the Artin $L$-function $L(s, \rho_K)$. Then we explain the construction of the density function $C_\sigma$ in Section 4.2. Several analytic properties of $C_\sigma$ are also proved in Section 4.3.
- Next, we associate the mean value of $\Phi(\log L(\sigma, \rho_K))$ with the integral involving the density function $C_\sigma$, where $\Phi$ is a test function as in Theorem 1.3. We show in Section 5 an asymptotic formula for the $z$-th moment described in (1.7), namely, the mean value of $\Phi(\log L(\sigma, \rho_K))$ in the case $\Phi(x) = e^{izx}$.
- The proofs of the results are completed in Section 6. We prove Theorem 1.1 by the asymptotic formula of the $z$-th moment with $z \in i\mathbb{R}$. As described before, we also deduce Theorem 1.2 from the case $z = 1$ by using the partial summation. Finally, we prove an analogue of Theorem 1.3 for a general test function $\Phi$.
- As well as the work of Ihara–Matsumoto, one can obtain similar results for logarithmic derivatives $(L'/L)(s, \rho_K)$, which are presented in the appendix.
2. Statement of results

To begin with, we set up the notation for counting cubic fields with discriminants not exceeding a given quantity. Based on [39, 41], we introduce the notion of the local specifications of cubic fields as follows. Let

$$\mathcal{A} = \{(111), (21), (3), (1^21), (1^3)\}$$

be the set of symbols. For a prime number \(p\), we write the prime ideal decomposition of \(p\) in \(K\) as \((p) = p_1^{e_1}p_2^{e_2} \cdots p_r^{e_r}\). Then, a cubic field \(K\) is said to satisfy a local specification \(a \in \mathcal{A}\) at \(p\) if

(a) for \(a = (111)\), \(p\) is totally splitting in \(K\), i.e. \((p) = p_1p_2p_3\);
(b) for \(a = (21)\), \(p\) is partially splitting in \(K\), i.e. \((p) = p_1p_2\);
(c) for \(a = (3)\), \(p\) remains inert in \(K\), i.e. \((p) = p_1\);
(d) for \(a = (1^21)\), \(p\) is partially ramified in \(K\), i.e. \((p) = p_1^2p_2\);
(e) for \(a = (1^3)\), \(p\) is totally ramified in \(K\), i.e. \((p) = p_1^3\).

The symbol \(S\) is used to denote a collection of local specifications with the following data: (i) a finite set \(\text{supp}S\) consisting of prime numbers; (ii) an element \(S_p \in \mathcal{A}\) for each \(p \in \text{supp}S\). We say that a cubic field \(K\) satisfies the local specifications \(S = (S_p)_p\) if \(K\) satisfies \(S_p\) at \(p\) for every \(p \in \text{supp}S\). Then we define

$$L_3^+(X, S) = \{K \in L_3^+(X) \mid K \text{ satisfies the local specifications } S\}$$

and put \(N_3^+(X, S) = \#L_3^+(X, S)\). Remark that the set \(\text{supp}S\) may be empty. We define \(L_3^+(X, S) = L_3^+(X)\) in such a case. Let \(p\) be a prime number and \(a \in \mathcal{A}\). Then we define constants \(C_p(a)\) and \(K_p(a)\) as

\[
C_p(a) = \frac{1}{1 + p^{-1} + p^{-2}} \begin{cases} 
1/6 & \text{if } a = (111), \\
1/2 & \text{if } a = (21), \\
1/3 & \text{if } a = (3), \\
1/p & \text{if } a = (1^21), \\
1/p^2 & \text{if } a = (1^3), 
\end{cases}
\]

\[
K_p(a) = \frac{1 - p^{-1/3}}{(1 - p^{-5/3})(1 + p^{-1})} \begin{cases} 
1/6 \cdot (1 + p^{-1/3})^3 & \text{if } a = (111), \\
1/2 \cdot (1 + p^{-1/3})(1 + p^{-2/3}) & \text{if } a = (21), \\
1/3 \cdot (1 + p^{-1}) & \text{if } a = (3), \\
1/p \cdot (1 + p^{-1/3})^2 & \text{if } a = (1^21), \\
1/p^2 \cdot (1 + p^{-1/3}) & \text{if } a = (1^3).
\end{cases}
\]

One can check that both \(\sum_{a \in \mathcal{A}} C_p(a)\) and \(\sum_{a \in \mathcal{A}} K_p(a)\) are equal to 1. We further define

\[
C(S) = \prod_{p \in \text{supp}S} C_p(S_p) \quad \text{and} \quad K(S) = \prod_{p \in \text{supp}S} K_p(S_p),
\]

where the empty product is interpreted as the value 1. Finally, we put

\[
C^\pm(S) = C^\pm C(S) \quad \text{and} \quad K^\pm(S) = K^\pm K(S)
\]
with $C^+ = 1$, $C^- = 3$, $K^+ = 1$, $K^- = \sqrt{3}$. Then, Roberts [39] conjectured that for any local specifications $S = (S_p)_p$ the formula

$$N_3^+(X, S) = C^+(S) \frac{1}{12\zeta(3)} X + K^+(S) \frac{4\zeta(1/3)}{5\Gamma(2/3)\zeta(5/3)} X^{5/6} + o(X^{5/6})$$

holds as $X \to \infty$. This conjecture was proved to be true by Bhargava–Shankar–Tsimerman [24] and Taniguchi–Thorne [41] independently. More precisely, it was shown that

$$N_3^+(X, S) - C^+(S) \frac{1}{12\zeta(3)} X - K^+(S) \frac{4\zeta(1/3)}{5\Gamma(2/3)\zeta(5/3)} X^{5/6} \ll_{\epsilon} X^{7/9+\epsilon} \prod_{p \in \text{supp} S} p^{e_p},$$

where $e_p = 8/9$ for $S_p = (111), (21), (3)$ and $e_p = 16/9$ for $S_p = (121), (1^3)$.

We associate a symbol $a \in A$ with a diagonal matrix $A_a \in M_2(\mathbb{C})$ by putting

$$A_a = \begin{cases} 
\text{diag}(1,1) & \text{if } a = (111), \\
\text{diag}(1,-1) & \text{if } a = (21), \\
\text{diag}(\omega, \omega^2) & \text{if } a = (3), \\
\text{diag}(1,0) & \text{if } a = (1^21), \\
\text{diag}(0,0) & \text{if } a = (1^3),
\end{cases}$$

where $\omega$ is a primitive cube root of unity. Then the Artin $L$-function $L(s, \rho_K)$ has the Euler product representation

$$L(s, \rho_K) = \prod_p \det(I - p^{-s}A_p(K))^{-1}$$

for $\text{Re}(s) > 1$ by definition. Here, $I \in M_2(\mathbb{C})$ is the identity matrix, and $A_p(K)$ is determined by $A_p(K) = A_a$ if $K$ satisfies a local specification $a$ at $p$. According to formula (2.3), we define $\mathcal{X} = (X_p)_p$ and $\mathcal{Y} = (Y_p)_p$ as two sequences of independent random elements on the set $\{A_a \mid a \in A\}$ such that

$$P(X_p = A_a) = C_p(a) \quad \text{and} \quad P(Y_p = A_a) = K_p(a),$$

where $C_p(a)$ and $K_p(a)$ are as in (2.1) and (2.2), respectively. Here, we denote by $P(E)$ the probability of an event $E$. The random Euler products $L(s, \mathcal{X})$ and $L(s, \mathcal{Y})$ are defined as

$$L(s, \mathcal{X}) = \prod_p \det(I - p^{-s}X_p)^{-1} \quad \text{and} \quad L(s, \mathcal{Y}) = \prod_p \det(I - p^{-s}Y_p)^{-1}.$$
(ii) If $\sigma > 1$, the function $C_\sigma$ is compactly supported.

(iii) Let $\sigma > 1/2$. Then the integral
\[ \int_{-\infty}^{\infty} e^{ax} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} \]
is finite for any $a > 0$.

Similarly, for $\sigma > 2/3$, there exists a non-negative $C^\infty$-function $K_\sigma$ such that
\[ \mathbb{P} (\log L(\sigma, \mathcal{Y}) \in A) = \int_A K_\sigma(x) \frac{dx}{\sqrt{2\pi}} \]
holds for all $A \in \mathcal{B}(\mathbb{R})$. Furthermore, it satisfies the following properties.

(i') If $2/3 < \sigma \leq 1$, we have $\text{supp} K_\sigma = \mathbb{R}$, that is, $K_\sigma(x)$ is not identically zero in any interval on $\mathbb{R}$.

(ii') If $\sigma > 1$, the function $K_\sigma$ is compactly supported.

(iii') Let $\sigma > 2/3$. Then the integral
\[ \int_{-\infty}^{\infty} e^{ax} K_\sigma(x) \frac{dx}{\sqrt{2\pi}} \]
is finite for any $a > 0$.

Let $f$ be a function in $L^1(\mathbb{R})$ such that the integral
\[ \int_{-\infty}^{\infty} e^{ax} f(x) \frac{dx}{\sqrt{2\pi}} \]
is finite for any $a > 0$. Throughout this paper, the Fourier–Laplace transform of $f$ is defined as
\[ \tilde{f}(z) = \int_{-\infty}^{\infty} f(x) e^{izx} \frac{dx}{\sqrt{2\pi}} , \]
which presents a holomorphic function on the whole complex plane. By the independence of $\mathcal{X} = (X_p)_p$, the Fourier–Laplace transform of $C_\sigma$ is represented as
\[
\tilde{C}_\sigma(z) = \prod_p \left( \sum_{a \in A} C_p(a) \det (I - p^{-s} A_a)^{-iz} \right) \\
= \prod_p \frac{1}{1 + p^{-1} + p^{-2}} \left[ \frac{1}{6} \left( 1 - \frac{1}{p^\sigma} \right)^{-2iz} + \frac{1}{2} \left( 1 - \frac{1}{p^{2\sigma}} \right)^{-iz} \\
+ \frac{1}{3} \left( 1 + \frac{1}{p^\sigma} + \frac{1}{p^{2\sigma}} \right)^{-iz} + \frac{1}{p} \left( 1 - \frac{1}{p^\sigma} \right)^{-iz} + \frac{1}{p^2} \right]
\]
for any $z \in \mathbb{C}$. In addition, $\tilde{K}_\sigma(z)$ has a similar infinite product representation.

The Grand Riemann Hypothesis (GRH) is used in the sense that every $L(s, \rho_K)$ has no zeros in the half-plane $\text{Re}(s) > 1/2$. We may use the following zero density estimate for $L(s, \rho_K)$ as well. Let $N(\alpha, T; \rho_K)$ count the number of zeros of $L(s, \rho_K)$ satisfying $\text{Re}(\rho) \geq \alpha$ and $|\text{Im}(\rho)| \leq T$ with multiplicity. Then there exists an absolute constant $1/2 < \sigma_1 < 1$ such that the estimate
\[ (2.8) \sum_{K \in L_2^+(X)} N(\sigma_1, (\log X)^3; \rho_K) \ll X^{1-\delta} \]
holds with an absolute constant \( \delta > 0 \). Remark that GRH ensures the truth of (2.8) for any \( 1/2 < \sigma_1 < 1 \). In Section 3.1 we also see that one can choose any \( \sigma_1 > 7/8 \) unconditionally by using the zero-density estimate of Kowalski–Michel [27]. Let \( E(X) \) be a subset of \( L^+_3(X) \) defined as

\[
(2.9) \quad E(X) = \left\{ K \in L^+_3(X) \mid \text{There exists a zero } \rho \text{ of } L(s, \rho_K) \text{ such that } \Re(\rho) \geq \sigma_1 \text{ and } |\Im(\rho)| \leq (\log X)^{3/5} \right\}.
\]

Then we define \( E_\sigma(X) = E(X) \) for \( \sigma_1 < \sigma < 1 \) and \( E_\sigma(X) = \emptyset \) for \( \sigma \geq 1 \). If we assume (2.8), then we obtain the estimate \( \#E_\sigma(X) = O(X^{1-\delta}) \) for \( \sigma > \sigma_1 \). The second and third results are relations between the complex moment \( M^{+3}_{\xi, \sigma}(X) \) of (1.1) and the integrals involving the density functions of Theorem 2.1.

**Theorem 2.2.** Let \( \sigma_1 \) be a real number for which (2.8) holds. Then there exists an absolute constant \( \delta > 0 \) such that

\[
M^{\pm}_{\xi, \sigma}(X) = C^\pm \frac{1}{12\zeta(3)} X \int_{-\infty}^{\infty} e^{\pi x} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} + O \left( X \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right)
\]

holds for \( \sigma > \sigma_1 \) with \( z \in \mathbb{C} \) satisfying \( |z| \leq b_\sigma R_\sigma(X) \), where \( b_\sigma \) is a positive constant, and \( R_\sigma(X) \) is defined as

\[
(2.10) \quad R_\sigma(X) = \left\{ \begin{array}{ll}
(\log X)(\log \log X)^{-2} & \text{if } \sigma \geq 1, \\
(\log X)^{(\sigma-\sigma_1)/(1-\sigma)}(\log \log X)^{-2} & \text{if } \sigma_1 < \sigma < 1.
\end{array} \right.
\]

The implied constant in (2.10) depends only on \( \sigma \).

**Theorem 2.3.** Assume GRH and the upper bound

\[
(2.11) \quad N^\pm_\delta(X, S) - C^\pm(S) \frac{1}{12\zeta(3)} X - K^\pm(S) \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} \\
\ll \epsilon X^{\alpha+\epsilon} \prod_{p \in \text{supp } S} p^\beta
\]

for each \( \epsilon > 0 \) with some constants \( \alpha \) and \( \beta \) such that \( 0 < \alpha < 5/6 \). Put

\[
(2.12) \quad \sigma_2 = \frac{(5 - 6\alpha) + (2 + 2\beta)}{12(1 - \alpha)}.
\]

Then there exists a constant \( \delta = \delta(\sigma) > 0 \) such that

\[
M^{\pm}_{\xi, \sigma}(X) = C^\pm \frac{1}{12\zeta(3)} X \int_{-\infty}^{\infty} e^{\pi x} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} \\
+ K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} X^{5/6} \int_{-\infty}^{\infty} e^{\pi x} K_\sigma(x) \frac{dx}{\sqrt{2\pi}} + O \left( X^{5/6-\delta} \right)
\]

holds for \( \sigma > \max(\sigma_2, 2/3) \) with \( z \in \mathbb{C} \) satisfying \( |z| \leq \tilde{b}_\sigma R_\sigma(X) \), where \( \tilde{b}_\sigma \) is a positive constant, and \( R_\sigma(X) \) is defined as

\[
(2.13) \quad R_\sigma(X) = (\log X)^{a(\sigma)} (\log \log X)^{-1}
\]

with a small constant \( a(\sigma) > 0 \). The implied constant in (2.14) depends only on \( \sigma \).

Recall that (2.12) is true for \( \alpha = 7/9 \) and \( \beta = 16/9 \) by Taniguchi–Thorne [41]. Thus one can take \( \sigma_2 = 53/24 \) at least in Theorem 2.3. Recently, Bhargava–Taniguchi–Thorne [5] improved this estimate by \( \alpha = \beta = 2/3 \), and we have \( \sigma_2 = 13/12 \) further. On the other hand, Cho–Fiorilli–Lee–Södergren [9] proved the lower
bound $\alpha + \beta \geq 1/2$ under GRH, when $\text{supp} \mathcal{S} = \{p\}$. Note that $\alpha + \beta = 1/2$ is equivalent to $\sigma_2 = 2/3$, which means that (2.14) holds for $\sigma > 2/3$.

**Corollary 2.4.** There exists an absolute constant $\delta > 0$ such that

$$\sum_{K \in L_{\pm}^+(X)} (h_K R_K)^r = C^\pm \frac{1}{12 \zeta(3)} \frac{2X^{r/2+1}}{(r+2)(D^\pm)^r} \int_{-\infty}^{\infty} e^{rx} C_1(x) \frac{dx}{\sqrt{2\pi}}$$

$$+ O \left( X^{r/2+1} \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right)$$

holds for any real number $r > -2$, where the implied constant depends only on $r$.

**Corollary 2.5.** Assume GRH and upper bound (2.12) with some constants $\alpha$ and $\beta$ such that $3\alpha + \beta < 5/2$. Then there exists an absolute constant $\delta > 0$ such that

$$\sum_{K \in L_{\pm}^+(X)} (h_K R_K)^r = C^\pm \frac{1}{12 \zeta(3)} \frac{2X^{r/2+1}}{(r+2)(D^\pm)^r} \int_{-\infty}^{\infty} e^{rx} C_1(x) \frac{dx}{\sqrt{2\pi}}$$

$$+ K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)^2\zeta(5/3)} \frac{5X^{r/2+5/6}}{(3r+5)(D^\pm)^r} \int_{-\infty}^{\infty} e^{rx} K_1(x) \frac{dx}{\sqrt{2\pi}}$$

$$+ O \left( X^{r/2+5/6} \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right)$$

holds for any real number $r > -5/3$, where the implied constant depends only on $r$.

We have more consequences of Theorem 2.2 by the methods of probability theory. Define the quantity $D^\pm_{\sigma}(X; a)$ as

$$D^\pm_{\sigma}(X; a) = \# \left\{ K \in L_{\pm}^+(X) \mid L(\sigma, \rho_K) \leq e^a \right\} - \int_{-\infty}^{a} C_{\sigma}(x) \frac{dx}{\sqrt{2\pi}}$$

for $\sigma > 1/2$ and $a \in \mathbb{R}$. Theorem 1.1 asserts that $D^\pm_{\sigma}(X; a) \rightarrow 0$ holds for $\sigma > 7/8$ as $X \rightarrow \infty$. We further evaluate the decay of $D^\pm_{\sigma}(X; a)$ as follows.

**Theorem 2.6.** Let $\sigma_1$ be a real number for which (2.8) holds. Then we obtain

$$\sup_{a \in \mathbb{R}} |D^\pm_{\sigma}(X; a)| \ll \frac{1}{R_{\sigma}(X)},$$

where $R_{\sigma}(X)$ is as in Theorem 2.2.

**Remark 2.7.** Compared with the results in [31, 35], it might be possible that the function $R_{\sigma}(X)$ of Theorems 2.2 and 2.6 is improved by

$$R_{\sigma}(X) = \begin{cases} (\log X)(\log \log X)^{-1} & \text{if } \sigma > 1, \\ (\log X)(\log \log X \log \log X)^{-1} & \text{if } \sigma = 1, \\ (\log X)^{\sigma} & \text{if } \sigma_1 < \sigma < 1. \end{cases}$$

However, we recall that the fact

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{T}^{2T} \frac{1}{p^{-it}} \ dt = \mathbb{E}[\mathcal{T}_p] = 0$$
is used in the method of [31], where $T_p$ is a random variable uniformly distributed on the unit circle in $\mathbb{C}$. In the case of Artin $L$-function $L(s, \rho_K)$, we have a similar fact

$$\lim_{X \to \infty} \frac{1}{N_3^+(X)} \text{tr}(A_p(K)) = \mathbb{E}[\text{tr}(X_p)] = \frac{p^{-1}}{1 + p^{-1} + p^{-2}},$$

where $A_p(K)$ is the matrix as in (2.5), and $X_p$ is a random element satisfying (2.6). However, the non-vanishing of $\mathbb{E}[\text{tr}(X_p)]$ prevents us from applying the totally same method as in [31]. See also [35, condition (1.10) and Lemma 2.10]. Therefore, we adopt in this paper not the method of [31, 35] but another method partially motivated by [13, 23, 29].

Let $\sigma > 1/2$ be a fixed real number. We see that $L(\sigma, \rho_K)$ might be zero or negative if we do not assume GRH. Thus, we define

$$A_\sigma(X) = \{K \in L_3^+(X) \mid L(\alpha, \rho_K) = 0 \text{ for some } \alpha \geq \sigma\}$$

for $1/2 < \sigma < 1$ and $A_\sigma(X) = \emptyset$ for $\sigma \geq 1$. Throughout this paper, we write

$$\sum' = \sum_{K \in L_3^+(X)}_{K \in L_3^+(X) \setminus A_\sigma(X)}.$$

Note that one can define $\log L(\sigma, \rho_K)$ for $K \notin A_\sigma(X)$. Next, we define classes of test functions to describe the statement of an analogue of Theorem 1.3. Let $C(\mathbb{R})$ be the class of all continuous functions on $\mathbb{R}$. Then we define the subclasses of $C(\mathbb{R})$ as

$$C^{\exp}(\mathbb{R}) = \{\Phi \in C(\mathbb{R}) \mid \Phi(x) \ll e^{a|x|} \text{ with some } a > 0\},$$

$$C_b(\mathbb{R}) = \{\Phi \in C(\mathbb{R}) \mid \Phi \text{ is bounded}\}.$$

We further define

$$I(\mathbb{R}) = \{1_A \mid A \text{ is a continuity set of } \mathbb{R}\},$$

where $1_A$ is the indicator function of a set $A \subset \mathbb{R}$, and a Borel set $A$ is called a continuity set of $\mathbb{R}$ if its boundary $\partial A$ has Lebesgue measure zero in $\mathbb{R}$.

**Theorem 2.8.** Let $\sigma_1$ be a real number for which (2.8) holds. Then the limit formula

$$\lim_{X \to \infty} \frac{1}{N_3^+(X)} \sum'_{K \in L_3^+(X)} \Phi(\log L(\sigma, \rho_K)) = \int_{-\infty}^{\infty} \Phi(u)C_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

holds in the following cases:

- $\sigma > 1$ and $\Phi \in C(\mathbb{R}) \cup I(\mathbb{R})$;
- $\sigma = 1$ and $\Phi \in C^{\exp}(\mathbb{R}) \cup I(\mathbb{R})$;
- $\sigma_1 < \sigma < 1$ and $\Phi \in C_b(\mathbb{R}) \cup I(\mathbb{R})$ without assuming GRH;
- $\sigma_1 < \sigma < 1$ and $\Phi \in C^{\exp}(\mathbb{R}) \cup I(\mathbb{R})$ if we assume GRH.

We obtain several results on the logarithmic derivative $(L'/L)(s, \rho_K)$ similar to the above. They are presented in the appendix.
3. Preliminaries

3.1. Properties of the Artin $L$-function. As we mentioned in Section 1.1 there exists a cuspidal representation $\pi$ such that $L(s, \rho_K) = L(s, \pi)$ holds. Hence the Artin $L$-function $L(s, \rho_K)$ is continued to an entire function. The complete $L$-function

$$\Lambda(s, \rho_K) = |d_K|^{s/2} \gamma(s, \rho_K) L(s, \rho_K)$$

satisfies the functional equation $\Lambda(s, \rho_K) = \Lambda(1 - s, \rho_K)$, where the gamma factor is given by

$$\gamma(s, \rho_K) = \begin{cases} \pi^{-s} \Gamma \left( \frac{s}{2} \right)^2 & \text{if } d_K > 0, \\ \pi^{-s} \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{s+1}{2} \right) & \text{if } d_K < 0. \end{cases}$$

Note that $\gamma(s, \rho_K)$ are common over the family $L^+_3(X)$ or $L^*_3(X)$. Then, we obtain the following zero density estimate for $L(s, \rho_K)$.

**Lemma 3.1.** Let $X \geq 1$ and $T \geq 2$. For any $C_0 > 6$, we have

$$\sum_{K \in L^+_3(X)} N(\alpha, T; \rho_K) \ll T^A X^{C_0(1 - \alpha)/(2\alpha - 1)}$$

for any $\alpha \geq 3/4$, where $A > 0$ is an absolute constant. The implied constant depends only on the choice of $C_0$.

**Proof.** Let $S^\pm(X)$ be the set of all cuspidal representations $\pi$ of $GL_2(\mathbb{A}_Q)$ such that $L(s, \pi) = L(s, \rho_K)$ is satisfied for some $K \in L^\pm_3(X)$. Note the Ramanujan–Petersson conjecture is valid for any $\pi \in S^\pm(X)$ since all local roots satisfy $|\alpha_j(p)| \leq 1$ by (2.5). Next, the conductor of $\pi \in S^\pm(X)$ satisfies $\text{Cond}(\pi) \leq X$ due to $\text{Cond}(\pi) = |d_K|$ if $L(s, \pi) = L(s, \rho_K)$. Furthermore, we obtain $\#S^\pm(X) \ll X$, which is deduced from the fact that there exists a one-to-one correspondence between $S^\pm(X)$ and $L^\pm_3(X)$. Finally, the gamma factors in the functional equations of $L(s, \pi)$ are common in $\pi \in S^\pm(X)$. From the above, the desired estimate follows directly if we apply the zero density estimate of Kowalski and Michel [27, Theorem 2] to the family $S^\pm(X)$. \hfill \Box

We check that estimate (2.8) holds for any $7/8 < \sigma_1 < 1$. Taking $C_0 = 6 + \delta$ with $\delta = (8\sigma_1 - 7)/2 > 0$, we obtain

$$\frac{C_0(1 - \sigma_1)}{2\sigma_1 - 1} = 1 - \frac{(8 + \delta)\sigma_1 - (7 + \delta)}{2\sigma_1 - 1} \leq 1 - \delta.$$

Hence Lemma 3.1 ensures the validity of (2.8) with $7/8 < \sigma_1 < 1$.

We often consider the logarithms of $L(s, \rho_K)$ for non-real variables $s$. The branch of $\log L(s, \rho_K)$ is determined as follows. First, we define

$$\log L(s, \rho_K) = \sum_p \sum_{m=1}^{\infty} \frac{\text{tr}(A_p(K)^m)}{m} p^{-ms} = \sum_{n=1}^{\infty} \frac{\Lambda_K(n)}{n} n^{-s}$$

for $\Re(s) > 1$ from Euler product (2.5). Here, the coefficient $\Lambda_K(n)$ is calculated as $\Lambda_K(p^m) = \text{tr}(A_p(K)^m) \log p$ and $\Lambda_K(n) = 0$ unless $n$ is a prime power. Note that
where the implied constant depends only on the choice of $\sigma$ for any $\sigma$.

We define
\[ G_K = D \setminus \bigcup_{\text{Re}(\rho) > 1/2} \{ \sigma + i\text{Im}(\rho) \mid 1/2 < \sigma \leq \text{Re}(\rho) \}, \]
where $\rho$ runs through all possible zeros of $L(s, \rho_K)$ with $\text{Re}(\rho) > 1/2$. Then we extend $\log L(s, \rho_K)$ for $s \in G_K$ by the analytic continuation along the horizontal path from right. The region $G_K$ is adequate to define $\log L(s, \rho_K)$ as a holomorphic function, and we note that the subset $A_\sigma(X)$ defined as (2.16) is represented as
\[ A_\sigma(X) = \{ K \in L_3^+(X) \mid \sigma \notin G_K \}. \]

**Lemma 3.2.** Assume GRH, and let $\sigma_0 > 1/2$ be a real number. Take a complex number $s = \sigma + it$ such that $\sigma \geq \sigma_0$ and $|t| \leq (\log X)^2$. For any cubic field $K \in L_3^+(X)$, we have
\[ \log L(s, \rho_K) \ll \frac{(\log X)^{2-2\sigma}}{\log \log X} + \log \log X, \]
where the implied constant depends only on the choice of $\sigma_0$.

**Proof.** Let $q(s, \rho_K) = |d_K|(|s| + 3)(|s| + 1) + 3$ be the analytic conductor of $L(s, \rho_K)$ in the sense of Iwaniec–Kowalski [25]. Assuming GRH, we have
\[ \log L(s, \rho_K) \ll \frac{(\log q(s, \rho_K))^{2-2\sigma}}{(2\sigma - 1)\log \log q(s, \rho_K)} + \log \log q(s, \rho_K) \]
for any $s = \sigma + it$ with $1/2 < \sigma \leq 5/4$ by [25] Theorems 5.19. Thus the result for $\sigma_0 \leq \sigma \leq 5/4$ follows. The case $\sigma \geq 5/4$ is trivial. Indeed, we obtain the inequality $|\log L(s, \rho_K)| \leq 2\log \zeta(5/4)$ by (3.1) and $|\Lambda_K(n)| \leq 2\Lambda(n)$.

**Lemma 3.3.** Let $\sigma_1$ be a real number for which (2.5) holds. Take a complex number $s = \sigma + it$ such that $\sigma \geq \sigma_1 + 2(\log \log X)^{-1}$ and $|t| \leq (\log X)^2$. For any cubic field $K \in L_3^+(X) \setminus E(X)$, we have
\[ \log L(s, \rho_K) \ll (\log \log X)(\log X)^{(1-\sigma)/(1-\sigma_1)} + \log \log X, \]
where $E(X)$ is the subset defined by (2.9), and the implied constant is absolute.

**Proof.** The proof is based on the method of Barban [3, Lemma 3]. For simplicity, we write $\kappa = (\log \log X)^{-1}$. If $s = \sigma + it \in \mathbb{C}$ satisfies $\sigma \geq 1 + \kappa/2$, then we obtain
\[ \log L(s, \rho_K) \ll \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} n^{-\sigma} \ll \log \log X \]
by formula (3.1). Therefore the result follows in this case, and we let $s = \sigma + it$ with $\sigma_1 + 2\kappa \leq \sigma \leq 1 + \kappa/2$ and $|t| \leq (\log X)^2$ below. Let $z_0 = \kappa^{-1} + \kappa + it \in \mathbb{C}$ and $R = \kappa^{-1} + \kappa - \sigma_1 > 0$. If $K \in L_3^+(X) \setminus E(X)$, then the function $(L'/L)(z, \rho_K)$ is holomorphic on $|z - z_0| < R$ since $L(z, \rho_K)$ has no zeros in this disk. We define
\[ M(r) = \max_{|z - z_0| = r} \left| \frac{L'}{L}(z, \rho_K) \right| \]
for $0 < r < R$. Let $r_1 = \kappa^{-1} - 1$, $r_2 = \kappa^{-1} + \kappa - \sigma$ and $r_3 = \kappa^{-1} - \sigma_1$. Then we have $0 < r_1 < r_2 < r_3 < R$. As a consequence of the Hadamard three circles theorem, we obtain the inequality
\[
M(r_2) \leq M(r_1)^{1-a} M(r_3)^a,
\]
where $a = \log(r_2/r_1)/\log(r_3/r_1)$. We evaluate $M(r_1)$. Since we have $\Re(z) \geq 1 + \kappa$ on the circle $|z - z_0| = r_1$, the estimate $(L'/L)(z, \rho_K) \ll \log \log X$ follows. Therefore we obtain
\[
M(r_1) \leq A \log \log X,
\]
where $A \geq 1$ is an absolute constant. Next, we let $z = x + iy$ be a complex number on the circle $|z - z_0| = r_3$. By [25, Proposition 5.7 (2)], we have
\[
L'(z, \rho_K) = \sum_{|z - \rho| < 1} \frac{1}{z - \rho} + O(|d_K|(|y| + 3))
\]
with an absolute implied constant, where $\rho$ runs through zeros of $L(s, \rho_K)$. Note that the distance between $z$ and $\rho$ is at least $\kappa$ if $K \in L_3^\pm(X) \setminus E(X)$. Furthermore, the number of zeros with $|z - \rho| < 1$ is evaluate as $\ll \log X$ by [25, Proposition 5.7 (1)]. As a result, we obtain $(L'/L)(z, \rho_K) \ll (\log \log X)(\log X)$ by formula (3.4). Therefore we derive
\[
M(r_3) \leq B(\log \log X)(\log X),
\]
where $B \geq 1$ is an absolute constant. Inserting (3.3) and (3.5) to (3.2), we obtain
\[
M(r_2) \leq A^{1-a} B^a (\log \log X)(\log X)^a.
\]
Note that $0 < a < 1$ holds since $r_1 < r_2 < r_3$. Thus we have $A^{1-a} B^a \leq AB$. By the definition of $a$, we further obtain
\[
a = \frac{1 - \sigma}{1 - \sigma_1} + O\left(\frac{1}{\log \log X}\right).
\]
Hence we arrive at the upper bound
\[
\frac{L'}{L}(z, \rho_K) \ll (\log \log X)(\log X)^{(1-\sigma)/(1-\sigma_1)}
\]
in the disk $|z - z_0| \leq r_2$, where the implied constant is absolute. By the choice of the branch, the relation
\[
\log L(s, \rho_K) = \log L(s_0, \rho_K) - \int_{\sigma}^{1+\kappa/2} \frac{L'}{L}(x + it, \rho_K) \, dx
\]
holds with $s_0 = 1 + \kappa/2 + it$. We have $\log L(s_0, \rho_K) \ll \log \log X$ as before. Furthermore, since the horizontal path from $s = \sigma + it$ to $s_0$ is included in $|z - z_0| \leq r_2$, we obtain
\[
\int_{\sigma}^{1+\kappa/2} \frac{L'}{L}(x + it, \rho_K) \, dx \ll (\log \log X)(\log X)^{(1-\sigma)/(1-\sigma_1)}
\]
by (3.6). Hence the desired result follows. \qed

**Lemma 3.4.** For any cubic field $K \in L_3^\pm(X)$, we have
\[
\log L(1, \rho_K) \ll \log \log X,
\]
where the implied constant is absolute.
Proof. Explicit upper and lower bounds for the value
$L(1, \rho_K) = \text{Res}_{s=1} \zeta_K(s)$ were obtained by Louboutin [32, 33]. These results yields the inequalities
\[ \frac{c_K \alpha_K \exp(-c_K \alpha_K/2)}{2 \log |d_K|} \leq L(1, \rho_K) \leq \left( \frac{e}{4 \log |d_K|} \right)^2, \]
where $c_K = 2(\sqrt{2} - 1)^2$ and $\alpha_K = \log |d_K|/\log |d_K|$. We know that $1/4 \leq \alpha_K \leq 1/2$ holds. Hence we conclude that
$-\log \log X - 4 \leq \log L(1, \rho_K) \leq 2 \log \log X + 2$. □

3.2. The $z$-th divisor function. Let $F(w) = \log(1 - w)^{-1}$ with $|w| < 1$. For $z \in \mathbb{C}$, we define $H_r(z)$ as the coefficients of the power series
\[ \exp(zF(w)) = (1 - w)^{-z} = \sum_{r=0}^{\infty} H_r(z)w^r. \]
Then they can be explicitly calculated as $H_0(z) = 1$ and
\[ H_r(z) = \frac{1}{r!}z(z + 1) \cdots (z + r - 1) \]
for $r \geq 1$. The $z$-th divisor function $d_z(n)$ is the multiplicative function determined by $d_z(p^r) = H_r(z)$ for a prime number $p$. By definition, $d_z(n)$ satisfies
\[ \zeta(s)^z = \sum_{n=1}^{\infty} d_z(n)n^{-s} \]
for any $z \in \mathbb{C}$ and $\text{Re}(s) > 1$. If $k$ is a positive integer, then $d_k(n)$ is, as usual, equal to the number of representations of $n$ as the product of $k$ positive integers.

Lemma 3.5. Let $z \in \mathbb{C}$ and $k = |\lfloor z \rfloor| + 1$, where $|x|$ indicates the largest integer less than or equal to $x$. Then we have $|d_z(n)| \leq d_k(n)$ for any $n \geq 1$.

Proof. Since $d_z(n)$ and $d_k(n)$ are multiplicative, it is sufficient to show the inequality in the case $n = p^r$. By (3.8), we have
\[ |d_z(p^r)| \leq \frac{1}{r!}|z|(\lfloor |z| \rfloor + 1) \cdots (\lfloor |z| \rfloor + r - 1) \]
\[ \leq \frac{1}{r!}k(k + 1) \cdots (k + r - 1) = d_k(p^r) \]
due to $|z| \leq |\lfloor z \rfloor| + 1 = k$. Hence we obtain the result. □

Lemma 3.6. Let $\sigma_0$ be a real number. Let $k \in \mathbb{Z}_{\geq 1}$ and $Y \geq 3$. Then there exists an absolute constant $C \geq 1$ such that
\[ \sum_{n=1}^{\infty} d_k(n)n^{-\sigma}e^{-n/Y} \ll (Y^{1-\sigma} + 1)(C \log Y)^{k+1} \]
for $\sigma \geq \sigma_0$, where the implied constant depends only on the choice of $\sigma_0$.

Proof. If $\sigma \geq 1 + (2 \log Y)^{-1}$, then we obtain
\[ \sum_{n=1}^{\infty} d_k(n)n^{-\sigma}e^{-n/Y} \leq \zeta(\sigma)^k \leq (C \log Y)^k \]
by formula (3.9). Thus the result follows in this case. Let \( \sigma_0 \leq \sigma \leq 1 + (2 \log Y)^{-1} \). As a simple application of Fubini’s theorem, we obtain

\[
\sum_{n=1}^{\infty} d_k(n)n^{-\sigma}e^{-n/Y} = \frac{1}{2\pi i} \int_{\text{Re}(w)=c} \zeta(\sigma + w)^k \Gamma(w)Y^w \, dw
\]

for any \( c > \max(1 - \sigma, 0) \). Taking \( c = 1 - \sigma + (\log Y)^{-1} \), we have

\[
\zeta(\sigma + w)^k Y^w \ll \zeta(\sigma + c)^k Y^{1-\sigma} \leq (C \log Y)^k Y^{1-\sigma}
\]

on the line \( \text{Re}(w) = c \). Furthermore, the Stirling formula yields that

\[
\Gamma(w) \ll |v|^{u-1/2} \exp\left(-\frac{\pi}{2}|v|\right)
\]

holds for any \( w = u + iv \in \mathbb{C} \) with \( |v| \geq 1 \). Then the integral of \( \Gamma(w) \) on \( \text{Re}(w) = c \) is estimated as

\[
\int_{\text{Re}(w)=c} \Gamma(w) \, dw \ll \int_{1}^{\infty} v^{1-\sigma_0} e^{-v} \, dv + \max_{|v| \leq 1} |\Gamma(c + iv)|
\]

\[
\ll \sigma_0 1 + \max_{w \in R} |\Gamma(w)|,
\]

where \( R = \{u + iv \in \mathbb{C} \mid (2 \log Y)^{-1} \leq u \leq 2 - \sigma_0, \ |v| \leq 1\} \). Applying the relation \( \Gamma(w) = \Gamma(w + 1)/w \), we obtain

\[
\max_{w \in R} |\Gamma(w)| \leq \frac{\max_{w \in R} |\Gamma(w + 1)|}{\min_{w \in R} |w|}.
\]

Let \( R' = \{u + iv \in \mathbb{C} \mid 1 \leq u \leq 3 - \sigma_0, \ |v| \leq 1\} \). Then we have

\[
\max_{w \in R} |\Gamma(w + 1)| \leq \max_{w \in R'} |\Gamma(w)| \ll \sigma_0 1.
\]

In addition, we note that \( \min_{w \in R} |w| = (2 \log Y)^{-1} \) holds by definition. Hence the upper bound \( \max_{w \in R} |\Gamma(w)| \ll \sigma_0 \log Y \) follows, which deduces

\[
\int_{\text{Re}(w)=c} \Gamma(w) \, dw \ll \sigma_0 \log Y.
\]

By this and (3.10), we obtain

\[
\sum_{n=1}^{\infty} d_k(n)n^{-\sigma}e^{-n/Y} \ll \sigma_0 (C \log Y)^k Y^{1-\sigma} \log Y \leq Y^{1-\sigma} (C \log Y)^k+1
\]

as desired. \( \square \)

Next, we define another arithmetic function \( d_{\varepsilon}(n, \rho_K) \) for which the formula

\[
L(s, \rho_K)^{\varepsilon} = \sum_{n=1}^{\infty} d_{\varepsilon}(n, \rho_K)n^{-s}
\]

is satisfied. Let \( A_n \) be the matrix defined as (2.4). We denote by \( F(w; a) \) the function \( F(w; a) = \log \det(I - wA_n)^{-1} \) for \( |w| < 1 \). Then we define \( H_r(z; a) \) as the coefficients of the power series

\[
\exp(z F(w; a)) = \det((I - wA_n)^{-1})^{-z} = \sum_{r=0}^{\infty} H_r(z; a)w^r
\]
similarly to \( H_r(z) \) of (3.7). Since \( F(w; a) = F(\alpha_aw) + F(\beta_aw) \) holds with the eigenvalues \((\alpha_a, \beta_a)\) of the matrix \( A_a \), we have

\[
H_r(z; a) = \sum_{j=0}^{r} H_j(z)H_{r-j}(z)\alpha_a^j\beta_a^{r-j}.
\]

From the above, we define \( d_z(n, \rho_K) \) as the multiplicative function in \( n \) determined by \( d_z(p^r, \rho_K) = H_r(z; a) \) if \( K \) satisfies a local specification \( a \) at \( p \). Then we have formula (3.12) for any \( z \in \mathbb{C} \) and \( \text{Re}(s) > 1 \).

**Lemma 3.7.** Let \( z \in \mathbb{C} \) and \( k = \lfloor |z| \rfloor + 1 \). Then we have \( \left| d_z(n, \rho_K) \right| \leq d_{2k}(n) \) for any cubic field \( K \in L^\pm_3(X) \) and \( n \geq 1 \).

**Proof.** We consider the case \( n = p^r \) as in the proof of Lemma 3.5. Let \( K \in L^\pm_3(X) \) be a cubic field satisfying a local specification \( a \) at \( p \). Recall that the eigenvalues \((\alpha_a, \beta_a)\) of the matrix \( A_a \) satisfy \( |\alpha_a|, |\beta_a| \leq 1 \). Therefore we deduce from Lemma 3.5 and (3.14) the inequality

\[
|d_z(p^r, \rho_K)| \leq \sum_{j=0}^{r} d_k(p^j)d_k(p^{r-j}).
\]

By (3.7) and \( d_z(p^r) = H_r(z) \), we find that

\[
\sum_{r=0}^{\infty} d_{2k}(p^r)w^r = \left( \sum_{r=0}^{\infty} d_k(p^r)w^r \right)^2 = \sum_{r=0}^{\infty} \sum_{j=0}^{r} d_k(p^j)d_k(p^{r-j})w^r
\]

holds. Comparing the \( r \)-th coefficients, we obtain

\[
d_{2k}(p^r) = \sum_{j=0}^{r} d_k(p^j)d_k(p^{r-j}).
\]

This yields the desired inequality by (3.15). \( \square \)

### 3.3. Results from probability theory

In this subsection, we list several lemmas on random variables and probability measures.

**Lemma 3.8.** Let \((X_n)_n\) be a sequence of independent random variables. If two series

\[
\sum_{n=1}^{\infty} |E[X_n]| \quad \text{and} \quad \sum_{n=1}^{\infty} E[|X_n|^2]
\]

are finite, then \( X_1 + \cdots + X_n \) converges almost surely as \( n \to \infty \).

**Proof.** Put \( \mu_n = E[X_n] \) for \( n \geq 1 \). Then the variance \( \text{Var}(X_n) = \sigma_n^2 \) is given by

\[
\sigma_n^2 = E[|X_n - \mu_n|^2] = E[|X_n|^2] - |\mu_n|^2.
\]

Therefore both \( \sum_{n=1}^{\infty} \mu_n \) and \( \sum_{n=1}^{\infty} \sigma_n^2 \) converge by the assumption that (3.10) are finite. As is well known, this yields the convergence of the random variable \( X_1 + \cdots + X_n \) almost surely; see [6] Theorem 22.6] for example. \( \square \)
Lemma 3.9. Let \((\mu_n)\) be a sequence of probability measures on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\), and let \(\mu\) be a probability measure on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). Denote by \(\phi_n\) and \(\phi\) the characteristic functions

\[
\phi_n(\xi) = \int_{\mathbb{R}} e^{ix\xi} \, d\mu_n(x) \quad \text{and} \quad \phi(\xi) = \int_{\mathbb{R}} e^{ix\xi} \, d\mu(x).
\]

If \(\phi_n(\xi) \to \phi(\xi)\) holds as \(n \to \infty\) uniformly in \(\xi \in [-R, R]\) for any \(R > 0\), then the measure \(\mu_n\) converges weakly to \(\mu\) as \(n \to \infty\), that is, \(\mu_n(A) \to \mu(A)\) as \(n \to \infty\) for any continuity set \(A\) of \(\mathbb{R}\).

Proof. This is just Lévy’s continuity theorem; see [25, Section 3] for example.

Lemma 3.10. Let \((\mu_n)\) be a sequence of probability measures on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\), and let \(\mu\) be a probability measure on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). Suppose that the convolution measure \(\mu_1 \ast \cdots \ast \mu_n\) converges weakly to \(\mu\) as \(n \to \infty\). Then the support of \(\mu\) is represented as

\[
\text{supp } \mu = \lim_{n \to \infty} (\text{supp } \mu_1 + \cdots + \text{supp } \mu_n),
\]

where \(\lim_{n \to \infty}(A_1 + \cdots + A_n)\) is the set of all points \(\alpha \in \mathbb{R}\) such that \(\alpha\) has at least one representation \(\alpha = \lim_{n \to \infty}(\alpha_1 + \cdots + \alpha_n)\) for some sequence \((\alpha_n)\) with \(\alpha_n \in A_n\) for each \(n \geq 1\).

Proof. We have \(\text{supp}(\mu_1 \ast \cdots \ast \mu_n) = \text{supp } \mu_1 + \cdots + \text{supp } \mu_n\). Hence the inclusion

\[
\text{supp } \mu \subseteq \lim_{n \to \infty} (\text{supp } \mu_1 + \cdots + \text{supp } \mu_n)
\]

follows from the assumption \(\mu_1 \ast \cdots \ast \mu_n \to \mu\) as \(n \to \infty\). The opposite inclusion is also proved in [26, Theorem 3].

Lemma 3.11. Let \(\mu\) be a probability measure on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). If the characteristic function \(\phi\) satisfies

\[
\int_{-\infty}^{\infty} |\xi|^k |\phi(\xi)| \, d\xi < \infty
\]

for an integer \(k \geq 0\). Then there exists a non-negative \(C^k\)-function \(D\) on \(\mathbb{R}\) such that

\[
\mu(A) = \int_A D(x) \, \frac{dx}{\sqrt{2\pi}}
\]

holds for all \(A \in \mathcal{B}(\mathbb{R})\).

Proof. If \(k = 0\), then the result follows immediately from Lévy’s inversion formula; see [6, Theorem 26.2]. Moreover, the density function \(D\) is given by

\[
D(x) = \int_{-\infty}^{\infty} \phi(\xi) e^{-ix\xi} \frac{d\xi}{\sqrt{2\pi}} \quad \text{(3.17)}
\]

We obtain the result for \(k > 0\) by differentiating under the integral in (3.17).

Lemma 3.12. Let \(\mu\) and \(\nu\) be probability measures on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). Denote by \(\phi\) and \(\psi\) the characteristic function of \(\mu\) and \(\nu\), respectively. Put \(F(t) = \mu((\infty, t])\) and \(G(t) = \nu((\infty, t])\). Suppose that there exists a non-negative continuous function \(D\) on \(\mathbb{R}\) such that

\[
\nu(A) = \int_A D(x) \, \frac{dx}{\sqrt{2\pi}} \quad \text{(3.18)}
\]
holds for all $A \in B(\mathbb{R})$. Then we have

\begin{equation}
\sup_{t \in \mathbb{R}} |F(t) - G(t)| \ll \frac{1}{R} \sup_{x \in \mathbb{R}} D(x) + \int_{-R}^{R} \left| \frac{\phi(\xi) - \psi(\xi)}{\xi} \right| d\xi
\end{equation}

for any $R > 0$, where the implied constant is absolute.

**Proof.** By equality (3.18), the distribution function $G$ is represented as

\[
G(t) = \int_{-\infty}^{t} D(x) \frac{dx}{\sqrt{2\pi}}.
\]

Therefore $G$ is differentiable, and we see that $G'(t) \ll D(t)$ holds. Finally, we apply Esseen’s inequality [42, Theorem 7.16] for $F$ and $G$ to obtain the conclusion. □

### 3.4. Other preliminary lemmas.

**Lemma 3.13** (Mishou–Nagoshi [36]). Let $H$ be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let $(u_n)_n$ be a sequence in $H$ satisfying

(a) $\sum_{n=1}^{\infty} \|u_n\|^2 < \infty$

(b) $\sum_{n=1}^{\infty} |\langle u_n, u \rangle| = \infty$ for any $u \in H$ such that $\|u\| = 1$.

Then, for any $v \in H$, $k \geq 1$ and $\epsilon > 0$, there exist an integer $N = N(v,k,\epsilon) \geq k$ such that the inequality

\[
\left\| v - \sum_{n=k}^{N} c_n u_n \right\| < \epsilon.
\]

holds with some $c_k, \ldots, c_N \in \{1, -1\}$.

Let $M$ be a non-negative function on $\mathbb{R}$ with the Fourier transform $\tilde{M}$. According to Ihara–Matsumoto [23], we say that $M$ is a good density function on $\mathbb{R}$ if both $M$ and $\tilde{M}$ belong to the class $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and the conditions

\[
\int_{-\infty}^{\infty} M(x) \frac{dx}{\sqrt{2\pi}} = 1 \quad \text{and} \quad M(x) = \int_{-\infty}^{\infty} \tilde{M}(\xi)e^{-ix\xi} \frac{d\xi}{\sqrt{2\pi}}
\]

are satisfied. Then we have the following result.

**Lemma 3.14** (Ihara–Matsumoto [23]). Let $(X_n)_n$ be a sequence of finite sets with probability measures $\omega_n$. We take a function $\ell_n : X_n \to \mathbb{R}$ for each $n \geq 1$. Let $M$ be a good density function on $\mathbb{R}$. Suppose that the condition

\begin{equation}
\lim_{n \to \infty} \sum_{\chi \in X_n} \omega_n \Phi(\ell_n(\chi)) = \int_{-\infty}^{\infty} \Phi(x) M(x) \frac{dx}{\sqrt{2\pi}}.
\end{equation}

is satisfied with the function $\Phi(x) = e^{ix\xi}$ for any $\xi \in \mathbb{R}$, and that the convergence is uniform in $\xi \in [-R,R]$ for any $R > 0$. Then we have the following results:

(i) (3.20) holds for any $\Phi \in C_b(\mathbb{R})$;

(ii) (3.20) holds for any $\Phi \in C(\mathbb{R})$ with $|\Phi(x)| \leq \phi_0(|x|)$, where $\phi_0(r)$ is a continuous non-decreasing function on $[0, \infty)$ which satisfies $\phi_0(r) > 0$ for any
Recall that \( \sum \) convergences of properties on the random Euler products. Random Euler products. 4.1. \((3.22)\)

Proposition 4.1. \((2.7)\)

we obtain the second result by applying Lemma 3.8 again. □

4. The density functions \(C_\sigma\) and \(K_\sigma\)

4.1. Random Euler products. Let \(X = (X_p)_p\) and \(Y = (Y_p)_p\) be the sequences of independent random elements on \(\{A_a \mid a \in A\}\) as in Section 2. We study several properties on the random Euler products \(L(s,X)\) and \(L(s,Y)\) defined by infinite products \((2.7)\).

**Proposition 4.1.** Let \(s = \sigma + it\) be a fixed complex number.

(i) If \(\sigma > 1/2\), then the infinite product on \(L(s,X)\) converges almost surely.

(ii) If \(\sigma > 2/3\), then the infinite product on \(L(s,Y)\) converges almost surely.

**Proof.** The local components of infinite products \((2.7)\) are calculated as

\[
\det (I - p^{-s}X_p)^{-1} = 1 + \text{tr}(X_p)p^{-s} + O(p^{-2\sigma}),
\]

\[
\det (I - p^{-s}Y_p)^{-1} = 1 + \text{tr}(Y_p)p^{-s} + O(p^{-2\sigma}).
\]

Recall that \(\sum_p p^{-2\sigma}\) is convergent for \(\sigma > 1/2\). Then it is sufficient to prove the convergences of \(\sum_p \text{tr}(X_p)p^{-\sigma}\) for \(\sigma > 1/2\) and \(\sum_p \text{tr}(Y_p)p^{-\sigma}\) for \(\sigma > 2/3\) almost surely. Let \(\sigma > 1/2\). We see that

\[
E [\text{tr}(X_p)] = \sum_{a \in A} C_p(a)(\alpha_a + \beta_a) = \frac{p^{-1}}{1 + p^{-1} + p^{-2}},
\]

\[
E [\|\text{tr}(X_p)\|^2] = \sum_{a \in A} C_p(a)(\alpha_a + \beta_a)^2 = \frac{1 + p^{-1}}{1 + p^{-1} + p^{-2}},
\]

where \((\alpha_a, \beta_a)\) are the eigenvalues of the matrix \(A_a\). Hence \(E [\text{tr}(X_p)] \ll p^{-1}\) and \(E [\|\text{tr}(X_p)\|^2] \ll 1\), which yield that

\[
\sum_p |E [\text{tr}(X_p)p^{-\sigma}]| \quad \text{and} \quad \sum_p E [\|\text{tr}(X_p)p^{-\sigma}\|^2]
\]

are finite. Therefore \(\sum_p \text{tr}(X_p)p^{-\sigma}\) converges almost surely by Lemma 3.3 and the first result follows. Let \(\sigma > 2/3\). We also calculate the expected values for \(Y_p\) as

\[
E [\text{tr}(Y_p)] = \frac{1 - p^{-1/3}}{(1 - p^{-5/3})(1 + p^{-1})} \left( p^{-1/3} + p^{-2/3} + O(p^{-1}) \right),
\]

\[
E [\|\text{tr}(Y_p)\|^2] = \frac{1 - p^{-1/3}}{(1 - p^{-5/3})(1 + p^{-1})} \left( 1 + p^{-1/3} + p^{-2/3} + O(p^{-1}) \right)
\]

Thus, we obtain \(E [\text{tr}(Y_p)p^{-\sigma}] \ll p^{-\sigma - 1/3}\) and \(E [\|\text{tr}(Y_p)\|^2] \ll 1\) in this case. Hence we obtain the second result by applying Lemma 3.3 again. □
By Lemma 4.1, \( L(s, \mathcal{X}) \) is a random variable for \( \text{Re}(s) > 1/2 \), and \( L(s, \mathcal{Y}) \) is a random variable for \( \text{Re}(s) > 2/3 \). We also find that

\[
\log L(s, \mathcal{X}) = \sum_p \sum_{m=1}^{\infty} \frac{\text{tr}(\mathcal{X}_p^m)}{m} p^{-ms},
\]

\[
\log L(s, \mathcal{Y}) = \sum_p \sum_{m=1}^{\infty} \frac{\text{tr}(\mathcal{Y}_p^m)}{m} p^{-ms}
\]

are defined for \( \text{Re}(s) > 1/2 \), and for \( \text{Re}(s) > 2/3 \), respectively.

**Proposition 4.2.** Let \( s = \sigma + it \) and \( z \) be complex numbers.

(i) If \( \sigma > 1/2 \), then the expected value

\[
F_s(z) = \mathbb{E} [\exp(z \log L(s, \mathcal{X}))]
\]

is finite and represented as the infinite product

\[
F_s(z) = \prod_p F_{s,p}(z),
\]

where \( F_{s,p}(z) = \mathbb{E} \left[ \det (I - p^{-s} \mathcal{X}_p)^{-z} \right] \). Furthermore, \( 4.1 \) converges uniformly for \( \sigma \geq \sigma_0 \) and \( |z| \leq R \) with any \( \sigma_0 > 1/2 \) and \( R > 0 \).

(ii) If \( \sigma > 2/3 \), then the expected value

\[
G_s(z) = \mathbb{E} [\exp(z \log L(s, \mathcal{Y}))]
\]

is finite and represented as the infinite product

\[
G_s(z) = \prod_p G_{s,p}(z),
\]

where \( G_{s,p}(z) = \mathbb{E} \left[ \det (I - p^{-s} \mathcal{Y}_p)^{-z} \right] \). Furthermore, \( 4.2 \) converges uniformly for \( \sigma \geq \sigma_0 \) and \( |z| \leq R \) with any \( \sigma_0 > 2/3 \) and \( R > 0 \).

**Proof.** For the finiteness of \( F_s(z) \), it is sufficient to show that the expected values

\[
\mathbb{E} [\exp(r \text{Re} \log L(s, \mathcal{X}))] \quad \text{and} \quad \mathbb{E} [\exp(r \text{Im} \log L(s, \mathcal{X}))]
\]

are finite for any \( r \in \mathbb{R} \) since we have

\[
|F_s(z)|^2 = \left| \mathbb{E} [\exp(z \text{Re} \log L(s, \mathcal{X})) \exp(iz \text{Im} \log L(s, \mathcal{X}))] \right|^2
\]

\[
\leq \mathbb{E} [\exp(2 \text{Re}(z) \text{Re} \log L(s, \mathcal{X}))] \cdot \mathbb{E} [\exp(-2 \text{Im}(z) \text{Re} \log L(s, \mathcal{X}))]
\]

by the Cauchy–Schwarz inequality. Recall that the random variable

\[
\log L_y(s, \mathcal{X}) = \sum_{p \leq y} \log \det (I - p^{-s} \mathcal{X}_p)^{-1}
\]

converges to \( \log L(s, \mathcal{X}) \) almost surely as \( y \to \infty \) for \( \text{Re}(s) > 1/2 \) and \( z \in \mathbb{C} \). Thus we have

\[
\mathbb{E} [\exp(r \text{Re} \log L(s, \mathcal{X}))] \leq \liminf_{y \to \infty} \mathbb{E} [\exp(r \text{Re} \log L_y(s, \mathcal{X}))]
\]

\[
= \liminf_{y \to \infty} \prod_{p \leq y} \mathbb{E} \left[ \exp(r \text{Re} \log \det (I - p^{-s} \mathcal{X}_p)^{-1}) \right]
\]
by applying Fatou’s lemma. Here, we also use the independence of \( \mathcal{X} = (\mathcal{X}_p)_p \) in the second equality. Furthermore, we derive

\[
\exp(r \text{ Re } \log \det (I - p^{-s} \mathcal{X}_p)^{-1}) = 1 + r \text{ tr}(\mathcal{X}_p) \text{ Re}(p^{-s}) + O_r(p^{-2\sigma})
\]

by the Taylor series expansion. Since \( \mathbb{E} [\text{tr}(\mathcal{X}_p)] \ll p^{-1} \), we find that the formula

(4.5) \[ \mathbb{E} \left[ \exp(r \text{ Re } \log \det (I - p^{-s} \mathcal{X}_p)^{-1}) \right] = 1 + O_r(p^{-\sigma - 1} + p^{-2\sigma}) \]

holds. Note that \( \sum_p p^{-\sigma - 1} \) and \( \sum_p p^{-2\sigma} \) are convergent for \( \sigma > 1/2 \). Thus the infinite product

\[
\prod_p \mathbb{E} \left[ \exp(r \text{ Re } \log \det (I - p^{-s} \mathcal{X}_p)^{-1}) \right]
\]

is convergent as well. Hence we conclude that \( \mathbb{E} \left[ \exp(r \text{ Re } \log L(s, \mathcal{X})) \right] \) is finite by (4.4). One can show that \( \mathbb{E} \left[ \exp(r \text{ Im } \log L(s, \mathcal{X})) \right] \) is also finite in the same line. These results yield the finiteness of \( F_s(z) \) as described above.

Then we proceed to the proof of (4.1). For \( \text{Re}(s) > 1/2 \) and \( z \in \mathbb{C} \), we have the inequality \( |\log L(s, \mathcal{X}) - \log L_y(s, \mathcal{X})| \leq 1 \) almost surely if \( y \) is large enough. Therefore we derive

\[
|\exp(z \log L_y(s, \mathcal{X}))| \leq \exp(|z|) |\exp(z \log L(s, \mathcal{X}))|
\]

almost surely. Furthermore, one can show that \( \mathbb{E} \left[ |\exp(z \log L(s, \mathcal{X}))| \right] \) is finite in a way similar to the finiteness of \( F_s(z) \). By the dominated convergence theorem, we have

\[
\mathbb{E} \left[ \exp(z \log L(s, \mathcal{X})) \right] = \lim_{y \to \infty} \mathbb{E} \left[ \exp(z \log L_y(s, \mathcal{X})) \right] = \prod_p \mathbb{E} \left[ \det (I - p^{-s} \mathcal{X}_p)^{-z} \right],
\]

where the second inequality is valid due to the independence of \( \mathcal{X} = (\mathcal{X}_p)_p \). Hence we obtain (4.1). Let \( \sigma \geq \sigma_0 \) and \( |z| \leq R \) with \( \sigma_0 > 1/2 \) and \( R > 0 \). Then we obtain

\[
F_{s,p}(z) = 1 + O(p^{-\sigma_0 - 1} + p^{-2\sigma_0})
\]

similarly to (4.3), where the implied constant depends only on \( \sigma_0 \) and \( R \). Thus the uniform convergence of (4.1) follows by noting that \( \sum_p p^{-\sigma_0 - 1} \) and \( \sum_p p^{-2\sigma_0} \) are convergent.

The results for \( G_s(z) \) can be proved similarly to \( F_s(z) \). The difference is just that the expected value \( \mathbb{E} [\text{tr}(\mathcal{Y}_p)] \) is bounded as \( \mathbb{E} [\text{tr}(\mathcal{Y}_p)] \ll p^{-1/3} \). Therefore we omit the proof.

### Proposition 4.3.

Let \( s = \sigma + it \) and \( z \) be complex numbers.

(i) If \( 1/2 < \sigma < 1 \), then there exists an absolute constant \( c_1 > 0 \) such that

\[
|F_s(z)| \leq \exp \left( \frac{c_1}{(2\sigma - 1)(1 - \sigma) \log(|z| + 3)} \right).
\]

(ii) If \( 2/3 < \sigma < 1 \), then there exists an absolute constant \( c_2 > 0 \) such that

\[
|G_s(z)| \leq \exp \left( \frac{c_2}{(3\sigma - 2)(1 - \sigma) \log(|z| + 3)} \right).
\]
Proof. Let $H_r(z)$ and $H_r(z; a)$ be as in (3.9) and (3.13). We have $|H_r(z)| \leq (|z| + 1)^r$ by (3.8). Furthermore, relation (3.14) yields $|H_r(z; a)| \leq (r + 1)(|z| + 1)^r$. Thus we deduce from (3.13) the inequality

$$ (4.6) \quad |\det(I - wA_a)^{1/2} - 1 - z\text{tr}(A_a)w| \leq \sum_{r=2}^{\infty} |H_r(z; a)||w|^r \leq 8(|z| + 1)^2|w|^2 $$

for $|w| \leq (2|z| + 2)^{-1}$ since $H_0(z; a) = 1$ and $H_1(z; a) = z\text{tr}(A_a)$. Let $s = \sigma + it$ with $1/2 < \sigma < 1$ and $z \in \mathbb{C}$. If we put $Q = (4|z| + 4)^{1/\sigma}$, then $|p^{-s}| \leq (2|z| + 2)^{-1}$ is satisfied for prime numbers $p \geq Q$. Therefore we derive

$$ |\det(I - p^{-s}A_a)^{1/2} - 1 - z\text{tr}(A_a)p^{-s}| \leq 8(|z| + 1)^2p^{-2\sigma} $$

for any $p \geq Q$ by (4.6). This deduces the formula

$$ F_{s,p}(z) = \sum_{a \in A} C_p(a) \det(I - p^{-s}A_a)^{1/2} = 1 + \mu + E, $$

where $\mu$ and $E$ satisfy

$$ \mu = \frac{z}{1 + p^{-1} + \sigma p^{-s - 1}} \quad \text{and} \quad |E| \leq 8(|z| + 1)^2p^{-2\sigma}. $$

For $p \geq Q$, we have $|\mu| \leq 1/8$ and $|E| \leq 1/2$. Recall that $\log(1 + w) = w + O(|w|^2)$ holds uniformly in $|w| \leq 5/8$. We obtain

$$ (4.7) \quad \log F_{s,p}(z) = \mu + E + O(|\mu|^2 + |E|^2) \ll (|z| + 1)p^{-2\sigma}, $$

where the implied constant is absolute. By the prime number theorem, we obtain that

$$ (4.8) \quad \sum_{p \geq Q} \log |F_{s,p}(z)| \ll \frac{1}{2\sigma - 1} \frac{(|z| + 3)^{1/\sigma}}{\log(|z| + 3)} $$

with an absolute implied constant. Let $p < Q$. We have

$$ \log \det(I - p^{-s}A_a)^{-1} = \log(1 - \alpha_a p^{-s})^{-1} + \log(1 - \beta_a p^{-s})^{-1} \ll p^{-\sigma}, $$

where $(\alpha_a, \beta_a)$ are the eigenvalues of $A_a$. Hence there exists an absolute constant $c > 0$ such that

$$ |F_{s,p}(z)| \leq \sum_{a \in A} C_p(a) |\det(I - p^{-s}A_a)^{1/2}| \leq \exp(c|z|p^{-\sigma}), $$

is satisfied. Then, we again apply the prime number theorem to obtain

$$ (4.9) \quad \sum_{p > Q} \log |F_{s,p}(z)| \ll \frac{1}{1 - \sigma} \frac{(|z| + 3)^{1/\sigma}}{\log(|z| + 3)} $$

Combining (4.8) and (4.9), we finally arrive at

$$ \prod_{p} |F_{s,p}(z)| \leq \exp \left( \frac{c_1}{(2\sigma - 1)(1 - \sigma) \log(|z| + 3)} \right) $$

with an absolute constant $c_1 > 0$. Hence we obtain the result for $F_s(z)$ by (4.1).
To show the result for \( G_s(z) \), we note that \( \log G_{s,p}(z) \ll (|z| + 1)p^{-\sigma - 1/3} \) holds for \( p \geq Q \) instead of (4.7). This deduces

\[
\sum_{p \geq Q} \log |G_{s,p}(z)| \ll \frac{1}{3\sigma - 2 \log(|z| + 3)} \]

by applying the prime number theorem, where the implied constant is absolute. The remaining work is similar to the case of \( F_s(z) \).

4.2. **Construction of the density functions.** Let \( \sigma > 1/2 \) be a real number. We define probability measures \( \mu_{\sigma} \) and \( \mu_{\sigma,p} \) as

\[
\mu_{\sigma}(A) = \mathbb{P} \left( \log L(\sigma, X) \in A \right),
\mu_{\sigma,p}(A) = \mathbb{P} \left( \log \det \left( I - p^{-\sigma}A_p \right)^{-1} \in A \right)
\]

for \( A \in B(\mathbb{R}) \), where \( p \) is a prime number. In a similar way, we define for \( \sigma > 2/3 \) probability measures \( \nu_{\sigma} \) and \( \nu_{\sigma,p} \) as

\[
\nu_{\sigma}(A) = \mathbb{P} \left( \log L(\sigma, Y) \in A \right),
\nu_{\sigma,p}(A) = \mathbb{P} \left( \log \det \left( I - p^{-\sigma}Y_p \right)^{-1} \in A \right).
\]

Denote by \( \phi_{\sigma} \) and \( \psi_{\sigma} \) be the characteristic functions of \( \mu_{\sigma} \) and \( \nu_{\sigma} \), respectively.

**Proposition 4.4.**

1. Let \( \sigma > 1/2 \) be a real number. There exists a positive constant \( c_1(\sigma) \) depending only on \( \sigma \) such that

\[
|\phi_{\sigma}(\xi)| \leq \exp \left( -c_1(\sigma) \frac{|\xi|^{1/\sigma}}{\log |\xi|} \right)
\]

holds for any \( \xi \in \mathbb{R} \) with \( |\xi| \geq 3 \).

2. Let \( \sigma > 2/3 \) be a real number. There exists a positive constant \( c_2(\sigma) \) depending only on \( \sigma \) such that

\[
|\psi_{\sigma}(\xi)| \leq \exp \left( -c_2(\sigma) \frac{|\xi|^{1/\sigma}}{\log |\xi|} \right)
\]

holds for any \( \xi \in \mathbb{R} \) with \( |\xi| \geq 3 \).

**Proof.** Recall that \( H_0(z; a) = 1 \), \( H_1(z; a) = z \text{tr}(A_a) \) and

\[
H_2(z; a) = z \text{tr}(A_a) + \frac{z^2}{2} \text{tr}(A_a)^2
\]

for the coefficients \( H_r(z; a) \) as in (3.13). Then, similarly to (4.9), we obtain

\[
\left| \det(I - wA_a)^{-z} - 1 - z \text{tr}(A_a)w - z \text{tr}(A_a)w^2 - \frac{z^2}{2} \text{tr}(A_a)^2 w^2 \right| \leq \sum_{r=3}^\infty |H_r(z; a)||w|^r \leq 10(|z| + 1)^3|w|^3
\]

for \( |w| \leq (2|z| + 2)^{-1} \). Let \( \sigma > 1/2 \). The characteristic function \( \phi_{\sigma} \) is represented as

\[
\phi_{\sigma}(\xi) = \mathbb{E} \left[ \exp(i\xi \log L(\sigma, X)) \right] = F_{\sigma}(i\xi)
\]
by using the function $F_4(z)$ of Proposition 3.2. Therefore, formula (4.1) yields

\begin{equation}
\phi_\sigma(\xi) = \prod_p F_{\sigma,p}(i\xi),
\end{equation}

where $F_{\sigma,p}(i\xi)$ is represented as

\begin{equation}
F_{\sigma,p}(i\xi) = \sum_{a \in A} C_p(a) \det(I - p^{-\sigma} A_a)^{-i\xi}.
\end{equation}

Put $Q = (4|\xi| + 4)^{1/\sigma}$ as in the proof of Proposition 3.2. Since $p^{-\sigma} \leq (2|z| + 2)^{-1}$ holds for $p \geq Q$, we deduce from (4.11) that

\begin{equation}
F_{\sigma,p}(i\xi) = 1 + i\mu_1 + i\mu_{2,1} - \frac{1}{2}i\mu_{2,2} + E
\end{equation}

for $p \geq Q$, where $\mu_1, \mu_{2,1}$ and $\mu_{2,2}$ are real numbers given by

\begin{align*}
\mu_1 &= \frac{\xi}{1 + p^{-1} + p^{-2}p^{-\sigma-1}}, \\
\mu_{2,1} &= \frac{\xi}{1 + p^{-1} + p^{-2}(1 + p^{-1})p^{-2\sigma}}, \\
\mu_{2,2} &= \frac{\xi^2}{1 + p^{-1} + p^{-2}(1 + p^{-1})p^{-2\sigma}},
\end{align*}

and $E$ is evaluated as $|E| \leq 10(|\xi| + 1)^3p^{-3\sigma}$. For $p \geq Q$, we have $|\mu_1| \leq 1/8$, $|\mu_{2,1}| \leq 1/16$, $|\mu_{2,2}| \leq 1/16$ and $|E| \leq 5/32$. Hence we deduce from (4.13) the asymptotic formula

\begin{equation}
\log F_{\sigma,p}(i\xi) = i\mu_1 + i\mu_{2,1} - \frac{1}{2}i\mu_{2,2} + O \left(|\mu_1|^2 + |\mu_{2,1}|^2 + |\mu_{2,2}|^2 + |E|^2\right)
\end{equation}

for $p \geq Q$, where the implied constant is absolute. Since $\mu_1, \mu_{2,1}$ and $\mu_{2,2}$ are real, we further obtain

\begin{equation}
\log |F_{\sigma,p}(i\xi)| = -\frac{1}{2} \mu_{2,2} + O \left(|\mu_1|^2 + |\mu_{2,1}|^2 + |\mu_{2,2}|^2 + |E|^2\right) = -\frac{\xi^2}{2} \frac{1 + p^{-1}}{1 + p^{-1} + p^{-2}p^{-2\sigma}} + O \left(\xi^2 p^{-2\sigma - 2} + |\xi|^3 p^{-3\sigma}\right).
\end{equation}

Thus, there exists an absolute constant $A > 0$ such that

\begin{equation}
\log |F_{\sigma,p}(i\xi)| \leq -\frac{\xi^2}{4} p^{-2\sigma} + A \xi^2 p^{-2\sigma - 2} + A |\xi|^3 p^{-3\sigma}
\end{equation}

holds for $p \geq Q$. Put $Q(M) = (M|\xi|)^{1/\sigma}$ with $M \geq 1$. Then we have $Q(M) \geq (3M)^{1/\sigma}$ for $|\xi| \geq 3$. We take a constant $M_\sigma \geq 6$ depending only on $\sigma$ so that

\begin{equation}
Q(M_\sigma)^{-2} \leq \frac{1}{16A} \quad \text{and} \quad A|\xi|Q(M_\sigma)^{-\sigma} \leq \frac{1}{16A}
\end{equation}

are satisfied. We have also $Q(M_\sigma) \geq Q$ due to $M_\sigma \geq 6$. Therefore, inequality (4.14) yields

\begin{equation}
\log |F_{\sigma,p}(i\xi)| \leq -\frac{\xi^2}{8} p^{-2\sigma}
\end{equation}
for \( p \geq Q(M_\sigma) \). By the prime number theorem, we obtain

\[
\sum_{p \geq Q(M_\sigma)} \log |F_{\sigma,p}(i\xi)| \leq -\frac{\xi^2}{8} \sum_{p \geq Q(M_\sigma)} p^{-2\sigma} \leq -c_1(\sigma) \frac{|\xi|^{1/\sigma}}{\log |\xi|},
\]

where \( c_1(\sigma) \) is a positive constant depending only on \( \sigma \). Thus, the inequality

\[
\prod_{p \geq Q(M_\sigma)} |F_{\sigma,p}(i\xi)| \leq \exp \left( -c_1(\sigma) \frac{|\xi|^{1/\sigma}}{\log |\xi|} \right)
\]

follows. Note that \( |F_{\sigma,p}(i\xi)| \leq 1 \) holds for any prime number \( p \). Hence we obtain the desired inequality for \( |\phi_\sigma(\xi)| \) by formula (4.12). The result for \( \psi_\sigma \) can be shown in the same line. \( \square \)

Let \( \sigma > 1/2 \) be a real number. For all \( k \geq 0 \), we obtain

\[
\int_{-\infty}^{\infty} |\xi|^k |\phi(\xi)| \, d\xi < \infty
\]

by Proposition 4.4 (i). Hence we deduce from Lemma 3.11 the existence of a non-negative \( C^\infty \)-function \( C_\sigma \) on \( \mathbb{R} \) such that

\[
\mu_\sigma(A) = \int_A C_\sigma(x) \frac{dx}{\sqrt{2\pi}}
\]

holds for all \( A \in \mathcal{B}(\mathbb{R}) \). The function \( C_\sigma \) is given by

\[
C_\sigma(x) = \int_{-\infty}^{\infty} \phi_\sigma(\xi) e^{-ix\xi} \frac{d\xi}{\sqrt{2\pi}}
\]

by formula (3.17). Furthermore, it can be constructed as an infinite convolution of Schwartz distributions as follows. Let \( C_{\sigma,p} \) denote the Schwartz distribution on \( \mathbb{R} \) such that

\[
\int_{\mathbb{R}} \Phi(x) C_{\sigma,p}(x) \frac{dx}{\sqrt{2\pi}} = \mathbb{E} \left\{ \Phi \left( \log \det \left( I - p^{-\sigma} A_\phi \right)^{-1} \right) \right\}
\]

for each prime number \( p \). In other words, we put

\[
C_{\sigma,p}(x) = \sqrt{2\pi} \sum_{a \in A} C_p(a) \delta \left( x - \log \det \left( I - p^{-\sigma} A_a \right)^{-1} \right),
\]

where \( \delta \) is the Dirac distribution on \( \mathbb{R} \). Denote \( n \)-th prime number by \( p_n \). Then we obtain the identity

\[
\int_{\mathbb{R}} \Phi(x) (C_{\sigma,p_1} \ast \cdots \ast C_{\sigma,p_n})(x) \frac{dx}{\sqrt{2\pi}} = \mathbb{E} \left\{ \Phi \left( \log L_y(\sigma, \mathcal{X}) \right) \right\}
\]

with \( y = p_n \), where \( \log L_y(\sigma, \mathcal{X}) \) is the random variable defined by (4.3). Letting \( n \to \infty \), we have

\[
\int_{\mathbb{R}} \Phi(x) \lim_{n \to \infty} (C_{\sigma,p_1} \ast \cdots \ast C_{\sigma,p_n})(x) \frac{dx}{\sqrt{2\pi}} = \mathbb{E} \left\{ \Phi \left( \log L(\sigma, \mathcal{X}) \right) \right\}
\]

\[
= \int_{\mathbb{R}} \Phi(x) C_\sigma(x) \frac{dx}{\sqrt{2\pi}}
\]

Hence we conclude that the density function \( C_\sigma \) is given by

\[
C_\sigma(x) = \ast_p C_{\sigma,p}(x),
\]
where \(*_p\) stands for the infinite convolution over all prime numbers. We also obtain the density function \(K_{\sigma}\) for \(\sigma > 2/3\) such that

\[
\nu_\sigma(A) = \int_A K_\sigma(x) \frac{dx}{\sqrt{2\pi}}
\]

holds by Lemma 3.11 and Proposition 4.4 (ii). It is a non-negative \(C^\infty\)-function, and we have the infinite convolution representation

\[
K_\sigma(x) = *_p K_{\sigma,p}(x),
\]

where \(K_{\sigma,p}\) is the Schwartz distribution on \(\mathbb{R}\) defined as

\[
K_{\sigma,p}(x) = \sqrt{2\pi} \sum_{a \in A} K_p(a) \delta\left(x - \log \det \left(I - p^{-\sigma} A_a\right)^{-1}\right).
\]

We also remark that the \(M\)-function \(M_{\sigma}\) of Theorem 1.3 is constructed as a similar infinite convolution of Schwartz distributions; see [22, Section 3].

4.3. Analytic properties of the density functions. In the remaining part of Section 4 is devoted to the proof of Theorem 2.1. We prove only the properties of the density function \(C_{\sigma}\) since one can obtain the corresponding properties of \(K_{\sigma}\) by following similar arguments.

4.3.1. Proof of Theorem 2.1 (i). Let \(\sigma > 1/2\) be a real number. Denote by \(\mu_{\sigma}\) and \(\mu_{\sigma,p}\) the probability measures defined as (4.10). Since \(X = (X_p)_p\) is independent, we have

\[
\mu_{\sigma,p_1} \ast \cdots \ast \mu_{\sigma,p_n}(A) = \mathbb{P} \left( \log L_y(\sigma, X) \in A \right)
\]

with \(y = p_n\), where \(p_n\) is the \(n\)-th prime number, and \(\log L_y(\sigma, X)\) is defined as (4.3). Therefore we see that \(\mu_{\sigma,p_1} \ast \cdots \ast \mu_{\sigma,p_n} \to \mu_{\sigma}\) as \(n \to \infty\) by Proposition 4.1. We deduce from Lemma 3.10 the identity

\[
\text{supp } \mu_{\sigma} = \lim_{n \to \infty} (\text{supp } \mu_{\sigma,p_1} + \cdots + \text{supp } \mu_{\sigma,p_n}).
\]

Note that \(\text{supp } C_{\sigma}\) is equal to \(\text{supp } \mu_{\sigma}\). Thus Theorem 2.1 (i) follows if the right-hand side of (4.15) is \(\mathbb{R}\). To prove this, we apply Lemma 3.13 for \(H = \mathbb{R}\) equipped with the usual inner product \(\langle u, v \rangle = uv\). Let \(u_n = p_n^{-\sigma}\) with \(1/2 < \sigma \leq 1\). Then conditions (a) and (b) in Lemma 3.13 are easily checked. Take a real number \(\epsilon > 0\) arbitrarily. Note that we have

\[
\sum_{n=k}^{N} \sum_{m=2}^{\infty} \frac{2}{m} p_n^{-m\sigma} \leq B_\sigma k^{1-2\sigma}
\]

for any \(N \geq k \geq 1\), where \(B_\sigma > 0\) is a constant depending only on \(\sigma\). Then we take an integer \(k = k(\sigma, \epsilon) \geq 2\) so that \(B_\sigma k^{1-2\sigma} < \epsilon\) is satisfied. Let \(x\) be an arbitrary real number. By Lemma 3.13 we obtain an integer \(N = N(\sigma, \epsilon, x) \geq k\) and numbers \(c_k, \ldots, c_N \in \{1, -1\}\) such that

\[
\left| x - \sum_{n=k}^{k-1} \sum_{m=1}^{\infty} \frac{2}{m} p_n^{-m\sigma} - \sum_{n=k}^{N} c_n p_n^{-\sigma} \right| < \epsilon
\]

(4.16)
is satisfied. For $1 \leq n \leq N$, we choose a symbol $a_n \in A$ as follows. For $1 \leq n < k$, take $a_n = (111)$. For $k \leq n \leq N$, take $a_n = (121)$ if $c_n = 1$ and $a_n = (3)$ if $c_n = -1$.

Then (4.16) deduces

$$
| x - \sum_{n=1}^{k-1} \sum_{m=1}^{\infty} \frac{\text{tr}(A_{a_n}^m p_n^{-m})}{m} - \sum_{n=k}^{N} \text{tr}(A_{a_n}) p_n^{-n} | < \epsilon.
$$

By the choice of $k = k(\sigma, \epsilon)$, we finally obtain

$$
| x - \sum_{n=1}^{N} \log \det \left( I - p_n^{-\sigma} A_{a_n} \right)^{-1} | < 2 \epsilon,
$$

which means that $x$ belongs to $\lim_{N \to \infty} (\text{supp} \mu_{\sigma, p_1} + \cdots + \text{supp} \mu_{\sigma, p_N})$. Hence the desired result follows.

4.3.2. Proof of Theorem 2.1 (ii). Let $\sigma > 1$ be a real number. By (4.15), it is sufficient to show that

$$
\lim_{n \to \infty} \left( \text{supp} \mu_{\sigma, p_1} + \cdots + \text{supp} \mu_{\sigma, p_n} \right) \subseteq [-R, R]
$$

with some $R > 0$. We have

$$
\text{supp} \mu_{\sigma, p} = \{ \log \det \left( I - p^{-\sigma} A_a \right) \mid a \in A \}
$$

for every prime number $p$. Since $\sigma > 1$, we have

$$
\sum_{p} \log \det \left( I - p^{-\sigma} A_{a_p} \right) \leq \sum_{p} \sum_{m=1}^{\infty} \frac{\text{tr}(A_{a_p}^m)}{m} p^{-m \sigma} \leq 2 \log \zeta(\sigma)
$$

for any choices of $a_p \in A$. Hence we obtain (4.17) with $R = 2 \log \zeta(\sigma)$.

4.3.3. Proof of Theorem 2.1 (iii). Let $\sigma > 1/2$ and $a > 0$. We see that the equality

$$
\int_{-\infty}^{\infty} e^{ax} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} = E[\exp(a \log L(\sigma, \mathcal{X}))]
$$

is valid. Hence Theorem 2.1 (iii) is just a consequence of Proposition 4.2.

5. Complex moments of $L(\sigma, \rho_K)$

The goal of this section is to complete the proofs of Theorems 2.2 and 2.3 and their corollaries. We fix the branch of $\log L(s, \rho_K)$ as in Section 3.1. Then we define

$$
g_z(s, \rho_K) = L(s, \rho_K)^z
$$

for $z \in \mathbb{C}$, which is holomorphic for $s \in G_K$. If $\text{Re}(s) > 1$, then we obtain

$$
g_z(s, \rho_K) = \sum_{n=1}^{\infty} d_z(n, \rho_K) n^{-s}
$$

by (3.12), where $d_z(n, \rho_K)$ is the multiplicative function of Section 3.2. The function $g_z(s, \rho_K)$ is an analogue of the $g$-function introduced by Ihara–Matsumoto [23].
5.1. **Approximation of the $g$-function.** Throughout this subsection, we write

$$c = \max\{1 - \sigma, 0\} + \frac{1}{\log X} \quad \text{and} \quad \kappa = \frac{1}{\log \log X}$$

for $\sigma > 1/2$ and $X \geq 3$. We define the functions $g^+_{\pm}(\sigma, \rho_K; Y)$ and $g^-_{\pm}(\sigma, \rho_K; Y)$ as

$$g^+_{\pm}(\sigma, \rho_K; Y) = \frac{1}{2\pi i} \int_{L_{\pm}} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w \, dw$$

for $Y \geq 1$, where the integral contours $L_{\pm}$ are described as follows. Let $L^+$ be the vertical line $\text{Re}(w) = c$. Let $L^- = L_1 + \cdots + L_5$ be the contour given by connecting the points $c-i\infty, c-i(\log X)^2, -\kappa-i(\log X)^2, -\kappa+i(\log X)^2, c+i(\log X)^2, c+i\infty$, in order. The connection between $g_z(\sigma, \rho_K)$ and $g^+_{\pm}(\sigma, \rho_K; Y)$ is as follows.

**Proposition 5.1.** Let $\sigma_1$ be a real number for which (2.8) holds, and suppose that $\sigma > \sigma_1 + \kappa$ is satisfied. Take a cubic field $K \in L^+_{\beta}(X) \setminus E(X)$, where $E(X)$ is defined as (2.4). Then the equality

$$g_z(\sigma, \rho_K) = g^+_{\pm}(\sigma, \rho_K; Y) - g^-_{\pm}(\sigma, \rho_K; Y)$$

holds for any $z \in \mathbb{C}$ and $Y \geq 1$.

**Proof.** We have $\text{Re}(\sigma + w) > 1$ for $w \in L^+$ by the choice of $c$. Hence formula (5.1) yields that $|g_z(\sigma + w, \rho_K)|$ is bounded for $w \in L^+$. Furthermore, $|\Gamma(w)|$ is rapidly decreasing on $L^+$ as $|\text{Im}(w)| \to \infty$ by (3.11). As a result, $g_z(\sigma + w, \rho_K)\Gamma(w)Y^w$ is absolutely integrable on the contour $L^+$. Then we shift the contour to $L^-$. Remark that the function $g_z(s, \rho_K)$ is holomorphic in the region $\text{Re}(s) > \sigma_1$, $|\text{Im}(s)| < (\log X)^3$ if $K \in L^+_{\beta}(X) \setminus E(X)$. Hence we do not encounter any pole of the integrand except for the simple pole at $w = 0$ while shifting the contour. The residue at $w = 0$ is equal to $g_z(\sigma, \rho_K)$. Therefore, we obtain

$$\frac{1}{2\pi i} \int_{L^+} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w \, dw$$

$$= \frac{1}{2\pi i} \int_{L^-} g_z(\sigma + w, \rho_K)\Gamma(w)Y^w \, dw + g_z(\sigma, \rho_K)$$

as desired. \hfill \Box

Then, we show several properties on the functions $g^+_{\pm}(\sigma, \rho_K; Y)$ and $g^-_{\pm}(\sigma, \rho_K; Y)$.

**Proposition 5.2.** Let $\sigma > 1/2$ be a real number. Take a cubic field $K \in L^+_{\beta}(X)$ arbitrarily. Then the function $g^+_{\pm}(\sigma, \rho_K; Y)$ is represented as

$$g^+_{\pm}(\sigma, \rho_K; Y) = \sum_{n=1}^{\infty} d_z(n, \rho_K)n^{-\sigma}e^{-n/Y}$$

for any $z \in \mathbb{C}$ and $Y \geq 3$. If $Y = X^\eta$ with $\eta > 0$, then we have also

$$g^+_{\pm}(\sigma, \rho_K; Y) \ll X^\eta \quad \text{for any } z \in \mathbb{C} \quad \text{such that } |z| \leq R_\sigma(X),$$

where $R_\sigma(X)$ is given by (2.11). The implied constant in (5.2) depends only on $\sigma$ and $\eta$. 

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Proof. Recall that we have \( \Re(\sigma + w) > 1 \) for \( w \in L^+ \). Hence formula (5.1) yields

\[
g_z^+(\sigma, \rho_K; Y) = \frac{1}{2\pi i} \int_{\Re(w)=c} g_z(\sigma + w, \rho_K) \Gamma(w) Y^w \, dw
\]

\[
= \sum_{n=1}^{\infty} d_z(n, \rho_K) n^{-\sigma} \frac{1}{2\pi i} \int_{\Re(w)=c} \Gamma(w) (n/Y)^{-w} \, dw
\]

\[
= \sum_{n=1}^{\infty} d_z(n, \rho_K) n^{-\sigma} e^{-n/Y}.
\]

Put \( k = |z| + 1 \). By Lemmas 3.6 and 3.7, we further obtain

\[
|g_z^+(\sigma, \rho_K; Y)| \leq \sum_{n=1}^{\infty} d_{2\kappa}(n)n^{-\sigma} e^{-n/Y} \ll (Y^{1-\sigma} + 1)(C \log Y)^{2k+1},
\]

where the implied constant is absolute. Hence (5.2) follows if we let \( Y = X^\eta \) with \( \eta > 0 \) and \( |z| \leq R_\sigma(X) \).

Proposition 5.3. Let \( \sigma_1 \) be a real number for which (2.3) holds, and suppose that \( \sigma \geq \sigma_1 + 3\kappa \) is satisfied. Take a cubic field \( K \in L_3^+(X) \setminus E(X) \), where \( E(X) \) is defined as (2.9). If \( Y = X^\eta \) with \( \eta > 0 \), then there exists a constant \( b(\eta) > 0 \) depending only on \( \eta \) such that

\[
g_z^-(\sigma, \rho_K; Y) \ll \exp \left( -\frac{\eta}{2} \frac{\log X}{\log \log X} \right)
\]

holds for any \( z \in \mathbb{C} \) satisfying \( |z| \leq b(\eta)R_\sigma(X) \), where \( R_\sigma(X) \) is given by (2.11), and the implied constant is absolute.

Proof. We divide the integral contour \( L^- \) into \( L_1, L_2, \ldots, L_5 \) as above. Then we have \( \Re(\sigma + w) \geq 1 + (\log X)^{-1} \) on \( L_1 \) and \( L_5 \). Hence formula (5.1) is available to deduce

\[
\log L(\sigma + w, \rho_K) \ll \log \log X
\]

for \( w \in L_1 \cup L_5 \). If \( w \) lies on \( L_2 \cup L_3 \cup L_4 \), then we have \( \Re(\sigma + w) \geq \sigma_1 + 2\kappa \) and \( |\Im(\sigma + w)| \leq (\log X)^2 \). Since \( K \in L_3^+(X) \setminus E(X) \), we have

\[
\log L(\sigma + w, \rho_K) \ll (\log \log X)(\log X)^{(1-\sigma)/(1-\sigma_1)} + \log \log X
\]

by Lemma 3.6 in this case. As a result, we obtain for any \( w \in L^- \) the upper bound

\[
(5.3) \quad g_z(\sigma + w, \rho_K) \ll \exp \left( Ab(\eta) \frac{\log X}{\log \log X} \right)
\]

due to \( |z| \leq b(\eta)R_\sigma(X) \), where \( A > 0 \) is an absolute constant. Note that \( Y^w \) satisfies

\[
(5.4) \quad Y^w \ll \begin{cases} 
X^\eta & \text{if } w \text{ lies on } L_1, L_2, L_4, L_5, \\
\exp \left( -\eta \frac{\log X}{\log \log X} \right) & \text{if } w \text{ lies on } L_3.
\end{cases}
\]

Applying (3.11), we also obtain

\[
(5.5) \quad \frac{1}{2\pi i} \int_{L_j} \Gamma(w) \, dw \ll \begin{cases} 
\exp \left( -(\log X)^2 \right) & \text{if } j = 1, 2, 4, 5, \\
\log \log X & \text{if } j = 3.
\end{cases}
\]
Combining (5.3), (5.4) and (5.5), we obtain the desired estimate on \( g_z(\sigma, \rho_K; Y) \) if we choose \( b(\eta) > 0 \) as a suitably small constant.

As an application of Propositions 5.1, 5.2 and 5.3, we obtain the following asymptotic formula for the \( g \)-function \( g_z(\sigma, \rho_K) \).

**Corollary 5.4.** Let \( \sigma_1 \) be a real number for which (2.14) holds. Take a cubic field \( K \in L^+_3(X) \setminus E(X) \), where \( E(X) \) is defined as (2.14). If \( Y = X^n \) with \( \eta > 0 \), then there exists a constant \( 0 < b(\eta) \leq 1 \) depending only on \( \eta \) such that

\[
g_z(\sigma, \rho_K) = \sum_{n=1}^{\infty} d_z(n, \rho_K)n^{-\sigma} e^{-\eta Y} + O\left( \exp\left( -\frac{\eta \log X}{2 \log \log X} \right) \right)
\]

holds for \( \sigma > \sigma_1 \) with \( z \in \mathbb{C} \) satisfying \( |z| \leq b(\eta) R_\sigma(X) \), where \( R_\sigma(X) \) is given by (2.11), and the implied constant depends only on \( \sigma \) and \( \eta \).

One can improve the upper bound on \( g_z(\sigma, \rho_K; Y) \) of Proposition 5.3 if GRH is assumed. Indeed, we obtain the following result.

**Proposition 5.5.** Assume GRH. Let \( \sigma > 1/2 \) be a real number. Choose a positive real number \( \kappa_1(\sigma) \) depending on \( \sigma \) such that \( \sigma - \kappa_1(\sigma) > 1/2 \) is satisfied. Take a cubic field \( K \in L^+_3(X) \) arbitrarily. If \( Y = X^n \) with \( \eta > 0 \), then there exists an absolute constant \( A \geq 1 \) such that

\[
g_z(\sigma, \rho_K; Y) \ll X^{-\eta \kappa_1(\sigma)} \exp\left( \frac{A \log X}{\log \log X} \right)
\]

holds for any \( z \in \mathbb{C} \) satisfying \( |z| \leq \tilde{R}_\sigma(X) \), where \( \tilde{R}_\sigma(X) \) is given by (2.15) with \( a(\sigma) = \min(2\sigma - 2\kappa_1(\sigma) - 1, 2/3) > 0 \). Here, the implied constant depends only on the choice of \( \kappa_1(\sigma) \).

**Proof.** Denote by \( \tilde{L} \) the contour \( \tilde{L}_1 + \cdots + \tilde{L}_5 \) which is obtained by connecting the points \( c - i\infty, c - i(\log X)^2, -\kappa_1(\sigma) - i(\log X)^2, -\kappa_1(\sigma) + i(\log X)^2, c + i\infty \), in order. Then we have

\[
g_z(\sigma, \rho_K; Y) = \frac{1}{2\pi i} \int_{L^-} g_z(\sigma + w, \rho_K) \Gamma(w) Y^w dw = \frac{1}{2\pi i} \int_{L^+} g_z(\sigma + w, \rho_K) \Gamma(w) Y^w dw
\]

since GRH yields the absence of any pole of \( g_z(\sigma + w, \rho_K) \) while shifting the contour. We have

\[
\log L(\sigma + w, \rho_K) \ll \log \log X
\]

for \( w \in \tilde{L}_1 \cup \tilde{L}_5 \). Furthermore, we deduce from Lemma 3.2 that the upper bound

\[
\log L(\sigma + w, \rho_K) \ll \frac{(\log X)^{2 - 2\sigma + 2\kappa_1(\sigma)}}{\log \log X} + \log \log X
\]

is valid for \( w \in \tilde{L}_2 \cup \tilde{L}_3 \cup \tilde{L}_4 \). Then, by the choice of \( a(\sigma) \), we obtain

\[
g_z(\sigma + w, \rho_K) \ll \exp\left( \frac{A \log X}{\log \log X} \right)
\]
for any \( z, w \in \mathbb{C} \) with \( |z| \leq \tilde{R}_\sigma(X) \) and \( w \in \tilde{L} \). Similarly to (5.4) and (5.5), \( Y^w \) and the integral on \( \Gamma(w) \) are estimated as

\[
Y^w \ll \begin{cases} 
X^\eta & \text{if } w \text{ lies on } \tilde{L}_1, \tilde{L}_2, \tilde{L}_4, \tilde{L}_5, \\
X^{-\eta \kappa_1(\sigma)} & \text{if } w \text{ lies on } \tilde{L}_3,
\end{cases}
\]

\[
\frac{1}{2\pi i} \int_{L_j} \Gamma(w) \, dw \ll \begin{cases} 
\exp\left(-\left(\log X\right)^2\right) & \text{if } j = 1, 2, 4, 5, \\
1 & \text{if } j = 3.
\end{cases}
\]

Hence we obtain the conclusion by these estimates. \( \square \)

By using Propositions 5.1, 5.2 and 5.5 we obtain another asymptotic formula for \( g_z(\sigma, \rho_K) \) under GRH.

**Corollary 5.6.** Assume GRH. Let \( \sigma > 1/2 \) be a real number. Choose a positive real number \( \kappa_1(\sigma) \) depending on \( \sigma \) such that \( \sigma - \kappa_1(\sigma) > 1/2 \) holds. Take a cubic field \( K \in L_3^\pm(X) \) arbitrarily. If \( Y = X^\eta \) with \( \eta > 0 \), then there exists a constant \( 0 < \tilde{b}(\eta) \leq 1 \) depending only on \( \eta \) such that

\[
g_z(\sigma, \rho_K) = \sum_{n=1}^{\infty} d_z(n, \rho_K)n^{-\sigma}e^{-n/Y} + O\left(X^{-\eta \kappa_1(\sigma)}\exp\left(A\frac{\log X}{\log \log X}\right)\right)
\]

holds for \( \sigma > 1/2 \) with \( z \in \mathbb{C} \) satisfying \( |z| \leq \tilde{b}(\eta)\tilde{R}_\sigma(X) \), where \( \tilde{R}_\sigma(X) \) is given by (2.15) with \( a(\sigma) = \min(2\sigma - 2\kappa_1(\sigma) - 1, 2/3) > 0 \). Here, the constant \( A \geq 1 \) is absolute, and the implied constant depends only on \( \sigma \) and \( \eta \).

**5.2. Calculation of the complex moments.** Let \( H_r(z; a) \) be the coefficients in power series (3.13). Recall that \( d_z(n, \rho_K) \) is defined as the multiplicative function satisfying \( d_z(p^r, \rho_K) = H_r(z; a) \) if \( K \) satisfies a local specification \( a \) at \( p \). In this subsection, we calculate the \( z \)-th moment

\[
M_{z, \sigma}^\pm(X) = \sum_{K \in L_3^\pm(X), E_\sigma(X)} L(\sigma, \rho_K)^z = \sum_{K \in L_3^\pm(X) \setminus E_\sigma(X)} g_z(\sigma, \rho_K),
\]

where \( E_\sigma(X) \) is the subset defined by \( E_\sigma(X) = E(X) \) for \( \sigma_1 < \sigma < 1 \) and \( E_\sigma(X) = \emptyset \) for \( \sigma \geq 1 \) as in Section 2. The following proposition is a key tool for the calculation of \( M_{z, \sigma}^\pm(X) \).

**Proposition 5.7.** Let \( \alpha \) and \( \beta \) be constants for which (2.12) holds. For \( z \in \mathbb{C} \), we denote by \( \lambda_z(n) \), \( \mu_z(n) \) and \( \nu_z(n) \) the multiplicative functions satisfying

\[
\lambda_z(p^r) = \sum_{a \in A} C_p(a)H_r(z; a), \\
\mu_z(p^r) = \sum_{a \in A} K_p(a)H_r(z; a), \\
\nu_z(p^r) = \sum_{a \in A} |H_r(z; a)|,
\]
respectively. Then we have the asymptotic formula
\[
\sum_{K \in L^\pm_3(X)} d_z(n, \rho_K) = C^\pm \frac{1}{12\zeta(3)} X \lambda_z(n) + K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)^3 \zeta(5/3)} X^{5/6} \mu_z(n) + O \left( X^{\alpha+\epsilon} \nu_z(n) n^\beta \right)
\]
for each \(\epsilon > 0\), where the implied constant depends only on \(\epsilon\).

**Proof.** The result for \(n = 1\) follows from (2.12) with \(\text{supp} S = \emptyset\). Thus we consider the case \(n > 1\) below. Let \(n = p_1^{a_1} \cdots p_k^{a_k}\) be the prime factorization of \(n\). For \(\mathfrak{a} = (a_1, \ldots, a_k) \in A^k\), we denote by \(S(\mathfrak{a})\) the local specifications of cubic fields such that \(\text{supp} S(\mathfrak{a}) = \{p_1, \ldots, p_k\}\) and \(S_{p_j} = a_j\) for \(j = 1, \ldots, k\). Then
\[
d_z(n, \rho_K) = H_{r_1}(z; a_1) \cdots H_{r_k}(z; a_k)
\]
holds if \(K\) satisfies the local specifications \(S(\mathfrak{a})\). Dividing the condition \(K \in L^\pm_3(X)\) into \(K \in L^\pm_3(X, S(\mathfrak{a}))\) for \(\mathfrak{a} = (a_1, \ldots, a_k) \in A^k\), we have
\[
\sum_{K \in L^\pm_3(X)} d_z(n, \rho_K) = \sum_{\mathfrak{a} \in A^k} \sum_{K \in L^\pm_3(X, S(\mathfrak{a}))} d_z(n, \rho_K) = \sum_{\mathfrak{a} \in A^k} N^\pm_3(X, S(\mathfrak{a})) \prod_{j=1}^k H_{r_j}(z; a_j).
\]
Then we apply (2.12) to obtain
\[
N^\pm_3(X, S(\mathfrak{a})) = C^\pm \frac{1}{12\zeta(3)} X \prod_{j=1}^k C_p(a_j) + K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)^3 \zeta(5/3)} X^{5/6} \prod_{j=1}^k K_p(a_j)
\]
\[\quad + O_\epsilon \left( X^{\alpha+\epsilon} \prod_{j=1}^k p_j^\beta \right).
\]
From the above, we deduce
\[
\sum_{K \in L^\pm_3(X)} d_z(n, \rho_K) = C^\pm \frac{1}{12\zeta(3)} X \sum_{\mathfrak{a} \in A^k} \prod_{j=1}^k C_p(a_j) H_{r_j}(z; a_j)
\]
\[\quad + K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)^3 \zeta(5/3)} X^{5/6} \sum_{\mathfrak{a} \in A^k} \prod_{j=1}^k K_p(a_j) H_{r_j}(z; a_j)
\]
\[\quad + O_\epsilon \left( X^{\alpha+\epsilon} \sum_{\mathfrak{a} \in A^k} \prod_{j=1}^k |H_{r_j}(z; a_j)| p_j^\beta \right).
\]
Hence we obtain the conclusion by noting that \(\lambda_z(n)\) satisfies
\[
\lambda_z(n) = \prod_{j=1}^k \left( \sum_{a_j \in A} C_{p_j}(a_j) H_{r_j}(z; a_j) \right) = \sum_{\mathfrak{a} \in A^k} \prod_{j=1}^k C_p(a_j) H_{r_j}(z; a_j),
\]
and that similar equalities are valid for \(\mu_z(n)\) and \(\nu_z(n)\). \(\square\)
Let $X = (X_p)_p$ and $Y = (Y_p)_p$ denote the sequences of independent random elements on $\{A_a \mid a \in A\}$ as in Section 2. Furthermore, we define $F_s(z)$ and $G_s(z)$ as

$$F_s(z) = \mathbb{E}[\exp(z \log L(s, X))] \quad \text{and} \quad F_s(z) = \mathbb{E}[\exp(z \log L(s, Y))]$$

as in Proposition 4.2. If we define $F_{s,p}(z) = \mathbb{E}[\exp(z \log L_s^p)]$ and $G_{s,p}(z) = \mathbb{E}[\exp(z \log L_s^p)]$ for any prime number $p$, then we have by (3.13) the identities

$$F_{s,p}(z) = \sum_{a \in A} C_p(a) \det(I - p^{-s}A_a)^{-z} = \sum_{r=0}^\infty \lambda_z(p^r)p^{-rs}; \quad G_{s,p}(z) = \sum_{a \in A} K_p(a) \det(I - p^{-s}A_a)^{-z} = \sum_{r=0}^\infty \mu_z(p^r)p^{-rs}.$$

Hence we derive

(5.6) $$F_s(z) = \prod_p \left( \sum_{r=0}^\infty \lambda_z(p^r)p^{-rs} \right) = \sum_{n=1}^\infty \lambda_z(n)n^{-s}$$

for $\text{Re}(s) > 1/2$ by formula (4.1), and similarly,

(5.7) $$G_s(z) = \prod_p \left( \sum_{r=0}^\infty \mu_z(p^r)p^{-rs} \right) = \sum_{n=1}^\infty \mu_z(n)n^{-s}$$

for $\text{Re}(s) > 2/3$ by formula (4.2).

5.2.1. **Proof of Theorem 2.2** Let $\sigma_1$ be a real number for which (2.8) holds. Let $Y = X^\eta$ with $\eta > 0$ chosen later. By Corollary 5.4 there exists a constant $0 < b(\eta) \leq 1$ depending only on $\eta$ such that

(5.8) $$\sum_{K \in L_3^+(X) \cap E(X)} g_z(\sigma, \rho_K) = S_1 - S_2 + O\left(X \exp\left(-\frac{\eta \log X}{2 \log \log X}\right)\right)$$

holds for $z \in \mathbb{C}$ satisfying $|z| \leq b(\eta)R_\sigma(X)$, where

(5.9) $$S_1 = \sum_{K \in L_3^+(X)} \sum_{n=1}^\infty d_z(n, \rho_K)n^{-\sigma}e^{-n/Y}, \quad S_2 = \sum_{K \in E(X)} \sum_{n=1}^\infty d_z(n, \rho_K)n^{-\sigma}e^{-n/Y}.$$ 

Here, $E(X)$ is the subset of $L_3^+(X)$ defined by (2.9), and $R_\sigma(X)$ is as in (2.11).
The main term comes from $S_1$. Recall that (2.12) holds for $\alpha = 7/9$ and $\beta = 16/9$ due to Taniguchi–Thorne [41]. By Proposition 5.7, we obtain

$$S_1 = C^{\pm} \left( \frac{1}{12 \zeta(3)} X \left( \sum_{n=1}^{\infty} \lambda_n(n) n^{-\sigma} e^{-n/Y} \right) + K^{\pm} \frac{4(1/3)}{3 \Gamma(2/3) \zeta(5/3)} X^{5/6} \left( \sum_{n=1}^{\infty} \mu_n(n) n^{-\sigma} e^{-n/Y} \right) + O \left( X^{7/9 + \epsilon} \sum_{n=1}^{\infty} \nu_n(n) n^{-\sigma + 16/9} e^{-n/Y} \right) \right)$$

for any $\epsilon > 0$. Note that formula (5.6) yields

$$\sum_{n=1}^{\infty} \lambda_n(n) n^{-\sigma} e^{-n/Y} = \frac{1}{2\pi i} \int_{\text{Re}(w) = \epsilon} F_{\sigma+w}(z) \Gamma(w) Y^w \, dw$$

for $\sigma > \sigma_1$ and $c > 0$. The function $F_{\sigma+w}(z)$ is a holomorphic function in $w$ on the half-plane $\text{Re}(w) > 1/2 - \sigma$ by Proposition 4.2 (i). Hence, by shifting the integral contour to left, we obtain

$$(5.10) \quad \sum_{n=1}^{\infty} \lambda_n(n) n^{-\sigma} e^{-n/Y} = F_\sigma(z) + \frac{1}{2\pi i} \int_{\text{Re}(w) = -\kappa_1} F_{\sigma+w}(z) \Gamma(w) Y^w \, dw$$

with $0 < \kappa_1 < \sigma - 1/2$. We choose the parameter $\kappa_1$ as

$$\kappa_1 = \begin{cases} \frac{1}{2} \min(\sigma - \sigma_1, 1 - \sigma) & \text{if } \sigma_1 < \sigma < 1, \\ \sigma - 1 + (\log \log X)^{-1} & \text{if } \sigma \geq 1 \end{cases}$$

so as to keep $1/2 < \text{Re}(\sigma+w) < 1$ on the vertical line $\text{Re}(w) = -\kappa_1$. Then we apply Proposition 4.3 (i) to obtain

$$F_{\sigma+w}(z) \ll \exp \left( c_1(\sigma) \left( |z| + 3 \right)^{1/(\sigma - \kappa_1)} \log(|z| + 3) \right)$$

for $\text{Re}(w) = -\kappa_1$, where $c_1(\sigma)$ is a positive constant depending only on $\sigma$. Therefore, there exists a small constant $b_\sigma(\eta)$ with $0 < b_\sigma(\eta) \leq b(\eta)$ such that

$$F_{\sigma+w}(z) \ll \exp \left( \frac{\eta \log X}{2 \log \log X} \right)$$

holds for any $z, w \in \mathbb{C}$ satisfying $|z| \leq b_\sigma(\eta) R_\sigma(X)$ and $\text{Re}(w) = -\kappa_1$. For $Y = X^\eta$, we have

$$Y^w \ll \exp \left( -\eta \frac{\log X}{\log \log X} \right)$$

on the line $\text{Re}(w) = -\kappa_1$. Furthermore, the integral on $\Gamma(w)$ is estimated as

$$\frac{1}{2\pi i} \int_{\text{Re}(w) = -\kappa_1} \Gamma(w) \, dw \ll \log \log X$$
by (3.11). The above implied constants depend at most on $\sigma$ and $\eta$. Combining these estimates, we deduce from (5.10) the asymptotic formula

\begin{equation}
\sum_{n=1}^{\infty} \lambda_n(n) e^{-n/Y} = F_\sigma(z) + O\left(\exp\left(-\frac{\eta}{2} \log X\right)\right)
\end{equation}

for $\sigma > \sigma_1$. Next, we obtain $|\mu_n| \leq d_2(n)$ and $|\nu_n| \leq d_2(n)$ with $k = |z|$ + 1 by Lemma 3.7. Thus Lemma 3.6 yields the upper bound by using the density function $C_{\lambda}$.

By (5.11), (5.12) and (5.13), we calculate the first term $S_1$ as

\begin{equation}
S_1 = C^+ \frac{1}{12\zeta(3)} X \int_{-\infty}^{\infty} e^{zx} C_{\sigma}(x) \frac{dx}{\sqrt{2\pi}} + O\left(X \exp\left(-\frac{\eta}{2} \log X\right)\right)
\end{equation}

for $\sigma > \sigma_1$ and $z \in \mathbb{C}$ such that $|z| \leq b_\sigma(X)$ since $F_\sigma(z)$ is represented as

\begin{equation}
F_\sigma(z) = \mathbb{E}[\exp(z \log L(\sigma, \lambda))] = \int_{-\infty}^{\infty} e^{zx} C_{\sigma}(x) \frac{dx}{\sqrt{2\pi}}
\end{equation}

by using the density function $C_{\sigma}$ of Theorem 2.1.

The second term $S_2$ is estimated as follows. First, we obtain by Proposition 5.2 the upper bound

\begin{equation}
\max_{K \in L_3^\infty(X)} \left|\sum_{n=1}^{\infty} d_z(n, \rho_K) n^{-\sigma} e^{-n/Y}\right| \ll X^\eta
\end{equation}

for $\sigma > \sigma_1$ and $|z| \leq R_\sigma(X)$. Furthermore, by the definition of $E(X)$, zero-density estimate (2.8) deduces

\begin{equation}
\# E(X) \ll \sum_{K \in L_3^\infty(X)} N(\sigma_1, (\log X)^3; \rho_K) \ll X^{1-\delta}
\end{equation}

with an absolute constant $\delta > 0$. Hence we have

\begin{equation}
S_2 \ll \# E(X) \max_{K \in L_3^\infty(X)} \left|\sum_{n=1}^{\infty} d_z(n, \rho_K) n^{-\sigma} e^{-n/Y}\right| \ll X^{1-\delta/2}
\end{equation}
if we let \( \eta < \delta/2 \). As a result, we deduce from (5.8) the formula

\[
(5.15) \quad \sum_{\kappa \in L^+_3(X) \setminus E(X)} g_z(\sigma, \rho_K) = C^\pm \frac{1}{12\zeta(3)} X \int_{-\infty}^{\infty} e^{zx} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} + O \left( X \exp \left( -\delta' \frac{\log X}{\log \log X} \right) \right)
\]

for \( \sigma > \sigma_1 \) and \( |z| \leq b_\sigma(\eta) R_\sigma(X) \), where \( \delta' \) is an absolute positive constant.

Note that the left-hand side of (5.15) is equal to \( M^\pm_{x,\sigma}(X) \) for \( \sigma_1 < \sigma < 1 \) since we have \( E_\sigma(X) = E(X) \) in this case. Hence Theorem 2.2 follows if \( \sigma_1 < \sigma < 1 \). To complete the proof, we consider the case \( \sigma \geq 1 \). We see that

\[
M^\pm_{x,\sigma}(X) = \sum_{\kappa \in L^+_3(X) \setminus E(X)} g_z(\sigma, \rho_K) + \sum_{\kappa \in E(X)} g_z(\sigma, \rho_K)
\]

since we have \( E_\sigma(X) = \emptyset \) in this case. Thus Theorem 2.2 is deduced from if we evaluate the second sum as

\[
(5.16) \quad \sum_{\kappa \in E(X)} g_z(\sigma, \rho_K) \ll X \exp \left( -\delta' \frac{\log X}{\log \log X} \right)
\]

with some absolute constant \( \delta' > 0 \). Recall that \( |\log L(\sigma, \rho_K)| \leq 2 \log \zeta(\sigma) \) holds for \( \sigma > 1 \). Thus, there exists a positive constant \( b_\sigma(\delta) \) for \( \delta > 0 \) such that

\[
|g_z(\sigma, \rho_K)| \leq \exp \left( \frac{\delta \log X}{2 \log \log X} \right)
\]

holds for \( |z| \leq b_\sigma(\delta) R_\sigma(X) \) in the case \( \sigma > 1 \). Furthermore, we see that this is true even for \( \sigma = 1 \) by using Lemma 5.3. Together with (5.14), we finally obtain upper bound (5.16) for \( \sigma \geq 1 \). Hence the proof of Theorem 2.2 is completed.

5.2.2. Proof of Theorem 2.2. Let \( Y = X^\eta \) with a parameter \( \eta > 0 \) determined later. First, we remark that the subset \( E(X) \) of (2.9) is empty for any \( \sigma > 1/2 \) under GRH. Hence \( E_\sigma(X) \) remains empty for \( \sigma > 1/2 \), and thus, there exists a constant \( 0 < \tilde{b}(\eta) \leq 1 \) such that

\[
(5.17) \quad M^\pm_{x,\sigma}(X) = S_1 + O \left( X^{1-\eta \kappa_1(\sigma)} \exp \left( A \frac{\log X}{\log \log X} \right) \right)
\]

holds for \( |z| \leq \tilde{b}(\eta) \tilde{R}_\sigma(X) \) by Corollary 5.6 where \( S_1 \) is as in (5.9), and \( \kappa_1(\sigma) \) is a real number chosen later such that \( 1/2 < \sigma - \kappa_1(\sigma) < 1 \). By Proposition 5.7 we calculate \( S_1 \) as

\[
(5.18) \quad S_1 = C^\pm \frac{1}{12\zeta(3)} X \left( \sum_{n=1}^{\infty} \lambda_z(n)n^{-\sigma} e^{-n/Y} \right) + K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)\zeta(5/3)} X^{5/6} \left( \sum_{n=1}^{\infty} \mu_z(n)n^{-\sigma} e^{-n/Y} \right) + O \left( X^{\alpha + \varepsilon} \sum_{n=1}^{\infty} \nu_z(n)n^{-\sigma + \beta} e^{-n/Y} \right),
\]
where $\alpha$ and $\beta$ are constants such that $0 < \alpha < 5/6$. We deduce from (5.6) and (5.7) the identities
\[
\sum_{n=1}^{\infty} \lambda_z(n)n^{-\sigma}e^{-n/Y} = F_\sigma(z) + \frac{1}{2\pi i} \int_{\text{Re}(w)=-\kappa_1(\sigma)} F_{\sigma+w}(z)\Gamma(w)Y^w \, dw,
\]
\[
\sum_{n=1}^{\infty} \mu_z(n)n^{-\sigma}e^{-n/Y} = G_\sigma(z) + \frac{1}{2\pi i} \int_{\text{Re}(w)=-\kappa_2(\sigma)} G_{\sigma+w}(z)\Gamma(w)Y^w \, dw
\]
for any $\sigma > 2/3$ similarly to (5.10), where $\kappa_2(\sigma)$ is a real number chosen later such that $2/3 < \sigma - \kappa_2(\sigma) < 1$. Then, we apply Proposition 4.3 (i), (ii) to evaluate $F_{\sigma+w}(z)$ on $\text{Re}(w) = -\kappa_1(\sigma)$ and $G_{\sigma+w}(z)$ on $\text{Re}(w) = -\kappa_2(\sigma)$. Let $z$ be a complex number satisfying $|z| \leq \tilde{b}_\sigma(\eta)\tilde{R}_\sigma(X)$ with some constant $\tilde{b}_\sigma(\eta)$ such that $0 < \tilde{b}_\sigma(\eta) \leq \tilde{b}(\eta)$. We have
\[
F_{\sigma+w}(z) \ll \exp \left( c_1(\sigma)\tilde{b}_\sigma(\eta)\frac{\log X}{\log \log X} \right)
\]
on $\text{Re}(w) = -\kappa_1(\sigma)$ by noting that $a(\sigma) \leq 2\sigma - 2\kappa_1(\sigma) - 1$, and
\[
G_{\sigma+w}(z) \ll \exp \left( c_2(\sigma)\tilde{b}_\sigma(\eta)\frac{\log X}{\log \log X} \right)
\]
on $\text{Re}(w) = -\kappa_2(\sigma)$ by noting that $a(\sigma) \leq 2/3$. Here, $c_1(\sigma)$ and $c_2(\sigma)$ are positive constants depending only on $\sigma$. These upper bounds yield
\[
\sum_{n=1}^{\infty} \lambda_z(n)n^{-\sigma}e^{-n/Y} = F_\sigma(z) + O \left( X^{-\eta\kappa_1(\sigma)} \exp \left( \frac{\log X}{\log \log X} \right) \right),
\]
\[
\sum_{n=1}^{\infty} \mu_z(n)n^{-\sigma}e^{-n/Y} = G_\sigma(z) + O \left( X^{-\eta\kappa_2(\sigma)} \exp \left( \frac{\log X}{\log \log X} \right) \right)
\]
if we let the parameter $\tilde{b}_\sigma(\eta)$ be small enough. Furthermore, we have
\[
\sum_{n=1}^{\infty} \nu_z(n)n^{-\sigma+\beta}e^{-n/Y} \ll (X^{\eta(1-\sigma+\beta)} + 1) \exp \left( \frac{\log X}{\log \log X} \right)
\]
by Lemma 3.6 since $|\nu_z(n)| \leq d_{2k}(n)$ with $k = \|z\| + 1$. Hence we deduce from (5.18) that $S_1$ is calculated as
\[
(5.19) \quad S_1 = C^\pm \frac{1}{12\zeta(3)} X F_{\sigma}(z) + K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)\zeta(5/3)} X^{5/6} G_{\sigma}(z) + E,
\]
where the error term $E$ satisfies
\[
E \ll \left( X^{1-\eta\kappa_1(\sigma)} + X^{5/6-\eta\kappa_2(\sigma)} + X^\alpha + \eta(1-\sigma+\beta)+\epsilon + X^{\alpha+\epsilon} \right) \exp \left( A\frac{\log X}{\log \log X} \right)
\]
\[
\ll X^{5/6+\epsilon} \left( X^{-\eta\kappa_1(\sigma)-1/6} + X^{-\eta\kappa_2(\sigma)} + X^{-(5/6-\alpha)+\eta(1-\sigma+\beta)} + X^{-(5/6-\alpha)} \right).
\]
with the implied constants depend at most on $\sigma$, $\eta$ and $\epsilon$. Then we show that it is evaluated as $E \ll X^{5/6-\delta}$ with some $\delta = \delta(\sigma) > 0$ as follows.

(a) Let $\sigma \geq 1 + \beta$. In this case, we choose the constants $\kappa_1(\sigma)$ and $\kappa_2(\sigma)$ satisfying $1/2 < \sigma - \kappa_1(\sigma) < 1$ and $2/3 < \sigma - \kappa_2(\sigma) < 1$ arbitrarily. Then,
we take a real number \( \eta \) so that \( \eta \kappa_1(\sigma) \geq 1/3 \) is satisfied. In this setting, we obtain

\[
E \ll X^{5/6+\epsilon} \left( X^{-1/6} + X^{-\eta \kappa_2(\sigma)} + X^{-(5/6-\alpha)} \right) \ll X^{5/6+\epsilon-\delta}
\]

with some constant \( \delta = \delta(\sigma) > 0 \) depending only on \( \sigma \).

(b) Let \( \max(\sigma_2, 2/3) < \sigma < 1 + \beta \), where \( \sigma_2 \) is the real number of (5.13). In this case, we need to choose \( \kappa_1(\sigma) \) and \( \eta \) more carefully. Note that the inequality

\[
\sigma - 1 < \frac{1-\sigma+\beta}{5-6\alpha} < \sigma - \frac{1}{2}
\]

holds by the definition \( \sigma_2 \). Thus we choose a constant \( \kappa_1(\sigma) > 0 \) so that

\[
\frac{1-\sigma+\beta}{5-6\alpha} < \kappa_1(\sigma) < \sigma - \frac{1}{2}
\]

is satisfied. Therefore we obtain

\[
\frac{1}{2} < \sigma - \kappa_1(\sigma) < 1 \quad \text{and} \quad -\frac{(1-\alpha)\kappa_1(\sigma)}{1-\sigma+\beta+\kappa_1(\sigma)} + \frac{1}{6} < 0
\]

by the choice of \( \kappa_1(\sigma) \). Then, we choose a real number \( \eta \) as

\[
\eta = \frac{1-\alpha}{1-\sigma+\beta+\kappa_1(\sigma)} > 0
\]

to satisfy \( X^{-\eta \kappa_1(\sigma)+1/6} = X^{-\eta \kappa_1(\sigma)+1/6} \). Finally, we take \( \kappa_2(\sigma) > 0 \) such that \( 2/3 < \sigma - \kappa_2(\sigma) < 1 \) arbitrarily. From the above, we conclude

\[
E \ll X^{5/6+\epsilon} \left( X^{-\eta \kappa_1(\sigma)+1/6} + X^{-\eta \kappa_2(\sigma)} + X^{-(5/6-\alpha)} \right) \ll X^{5/6+\epsilon-\delta},
\]

where \( \delta = \delta(\sigma) > 0 \) is a constant depending only on \( \sigma \).

Therefore, we obtain \( E \ll X^{5/6-\delta/2} \) in both cases if we let \( \epsilon = \delta/2 \). Recall that \( F_\sigma(z) \) and \( G_\sigma(z) \) are represented as

\[
F_\sigma(z) = \int_{-\infty}^{\infty} e^{zx} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} \quad \text{and} \quad G_\sigma(z) = \int_{-\infty}^{\infty} e^{zx} K_\sigma(x) \frac{dx}{\sqrt{2\pi}}.
\]

Hence we derive Theorem 2.3 by (5.17) and (5.19).

5.3. Applications of the class number formula. Let \( r \) be a real number. For an integer \( d > 0 \), we define

\[
f_{r,\pm}(d) = \sum_{K \in \mathcal{L}^\pm_d(X), |d_K| = d} \left( \frac{h_K R_K}{\sqrt{|d_K|}} \right)^r.
\]

Then we obtain

\[
\sum_{K \in \mathcal{L}^\pm_d(X)} (h_K R_K)^r = \sum_{0 < d \leq X} f_{r,\pm}(d) d^{r/2}
\]

\[
= F_{r,\pm}(X) X^{r/2} - \frac{r}{2} \int_1^X F_{r,\pm}(y) y^{r/2-1} dy
\]
by the partial summation, where the function \( F_{r,\pm}(y) \) is given by

\[
F_{r,\pm}(y) = \sum_{0 \leq d \leq y} f_{r,\pm}(d) = \sum_{K \in L_{\pm}(y)} \left( \frac{h_K R_K}{\sqrt{|d_K|}} \right)^r.
\]

**5.3.1. Proof of Corollary 2.4.** By formula (1.8), we have

\[
F_{r,\pm}(y) = \frac{1}{(D^{\pm})^r} \sum_{K \in L_{\pm}(y)} L(1, \rho_K)^r.
\]

Hence we deduce from Theorem 2.2 the formula

\[
F_{r,\pm}(y) = C^{\pm}(r) y + O \left( y \exp \left( -\delta \frac{\log y}{\log \log y} \right) \right)
\]

with an absolute constant \( \delta > 0 \), where we write

\[
C^{\pm}(r) = \frac{1}{12 \zeta(3)} \frac{1}{(D^{\pm})^r} \int_{-\infty}^{\infty} e^{rx} C_1(x) \frac{dx}{\sqrt{2\pi}},
\]

and the implied constant depends only on \( r \). Let \( r > -2 \) be a fixed real number.

Inserting (5.21) to formula (5.20), we derive

\[
\sum_{K \in L_{\pm}(X)} (h_K R_K)^r = C^{\pm}(r) X^{r/2+1} - \frac{r}{2} \int_{1}^{X} C^{\pm}(r) y^{r/2} dy + O \left( X^{r/2+1} \exp \left( -\frac{\log X}{\log \log X} \right) \right)
\]

as desired.

**Remark 5.8.** If we let \( r = 1 \) in Corollary 2.4, then we have

\[
\sum_{K \in L_{\pm}(X)} h_K R_K = C^{\pm}/D^{\pm} + 18 \zeta(3) X^{3/2} + O \left( X^{3/2} \exp \left( -\frac{\log X}{\log \log X} \right) \right),
\]

where \( = \mathbb{E} [\exp (z \log L(s, X))] \) as in Proposition 4.2 (i). Here, we have \( C^+/D^+ = 1/4 \) and \( C^-/D^- = 3/(2\pi) \). Furthermore, we obtain \( F_1(1) = \prod_p F_{1,p}(1) \) by formula (4.1), where \( F_{1,p}(1) \) is calculated as

\[
F_{1,p}(1) = \frac{1}{1+p^{-1}+p^{-2}} \left[ \frac{1}{6} (1-p^{-1})^{-2} + \frac{1}{2} (1-p^{-2})^{-1} + \frac{1}{3} (1+p^{-1}+p^{-2})^{-1} + \frac{1}{p} (1-p^{-1})^{-2} + \frac{1}{p^2} \right]
\]

\[
= (1-p^{-3})^{-2} (1-p^{-2})^{-1} \left( 1 + p^{-2} - 2p^{-3} - 2p^{-4} + 2p^{-6} + p^{-7} - p^{-8} \right).
\]

Thus we derive the asymptotic formula

\[
\sum_{K \in L_{\pm}(X)} h_K R_K = C^{\pm}/D^{\pm} 4c X^{3/2} + O \left( X^{3/2} \exp \left( -\frac{\log X}{\log \log X} \right) \right)
\]

with \( c > 0 \) described in Theorem 1.2.
5.3.2. Proof of Corollary 2.5. If we assume that (2.12) holds with some $\alpha$ and $\beta$ such that $3\alpha + \beta < 5/2$, then Theorem 2.3 is available at $\sigma = 1$. Thus we obtain

\[ F_{r,\pm}(y) = C^\pm(r)y + K^\pm(r)y^{5/6} + O\left(y^{5/6}\exp\left(-\delta\frac{\log y}{\log \log y}\right)\right) \]

similarly to (5.21), where $C^\pm(r)$ is as in the proof of Corollary 2.4, and

\[ K^\pm(r) = K^\pm \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)} \left(\frac{D^\pm}{3}\right)^{5/6} \sqrt{2\pi} \]

By (5.20), we have

\[ \sum_{K \in L_3(X)} (h_K R_K)^r = C^\pm(r) \frac{2r+2}{r+2} X^{r/2+1} + K^\pm(r) \frac{5}{3r+5} X^{r/2+5/6} \]

\[ + O \left(X^{r/2+5/6}\exp\left(-\delta\frac{\log X}{\log \log X}\right)\right) \]

if $r/2 + 5/6 > 0$, i.e. $r > -5/3$. Hence we obtain the desired result.

6. Completion of the proofs

The remaining work is to complete the proofs of Theorems 2.6 and 2.8. We begin with showing the following corollary of Theorem 2.2 which is used in the proof of Theorem 2.6.

**Corollary 6.1.** Let $k$ be a positive integer. Let $\sigma_1$ be a real number for which (2.8) holds. Then there exists an absolute constant $\delta > 0$ such that

\[ \frac{1}{N_3(X)} \sum_{K \in L_3(X) \setminus E_\sigma(X)} (\log L(\sigma, \rho_K))^k \]

\[ = \int_{-\infty}^\infty x^k C_1(x) \frac{dx}{\sqrt{2\pi}} + O\left(\exp\left(-\delta\frac{\log X}{\log \log X}\right)\right) \]

holds for $\sigma > \sigma_1$, where the implied constant depends only on $k$ and $\sigma$. Here, $E_\sigma(X)$ is the subset defined by $E_\sigma(X) = E(X)$ for $\sigma_1 < \sigma < 1$ and $E_\sigma(X) = \emptyset$ for $\sigma \geq 1$ as in Section 2.

**Proof.** We define two holomorphic functions $f$ and $g$ as

\[ f(z) = \frac{1}{N_3(X)} \sum_{K \in L_3(X) \setminus E_\sigma(X)} \exp(z \log L(\sigma, \rho_K)), \]

\[ g(z) = \int_{-\infty}^\infty e^{zx} C_1(x) \frac{dx}{\sqrt{2\pi}} \]
for $z \in \mathbb{C}$. Let $\epsilon > 0$. Then it is deduced from Theorem 2.2 that

$$|f(z) - g(z)| \leq A(\sigma) \exp\left(-\delta \frac{\log X}{\log \log X}\right)$$

holds in the range $|z| < \epsilon$, where $A(\sigma)$ is a positive constant depending on $\sigma$. Hence, as a consequence of Cauchy’s integral formula, we obtain

$$\frac{1}{N_3^+(X)} \sum_{K \in L_3^+(X) \setminus E_\sigma(X)} (\log L(\sigma, \rho_K))^k - \int_{-\infty}^{\infty} x^k C_1(x) \frac{dx}{\sqrt{2\pi}}$$

$$= \frac{d^k}{dz^k}(f(z) - g(z)) \bigg|_{z=0} \ll_{k,\sigma} \exp\left(-\delta \frac{\log X}{\log \log X}\right),$$

which completes the proof. 

6.1. **Proof of Theorem 2.6.** We apply Lemma 3.12 for the probability measures $\mu$ and $\nu$ defined as

$$\mu(A) = \frac{\#\{K \in L_3^+(X) \setminus E_\sigma(X) \mid \log L(\sigma, \rho_K) \in A\}}{\#(L_3^+(X) \setminus E_\sigma(X))},$$

$$\nu(A) = \int_A C_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

for $A \in \mathcal{B}(\mathbb{R})$. Note that the characteristic function of $\mu$ is calculated as

$$\phi(\xi) = \frac{1}{\#(L_3^+(X) \setminus E_\sigma(X))} \sum_{K \in L_3^+(X) \setminus E_\sigma(X)} L(\sigma, \rho_K)^{i\xi},$$

$$= \frac{1}{N_3^+(X)} M_{\xi,\sigma}^+(X) + O(X^{-\delta})$$

since $\#E_\sigma(X) \ll X^{1-\delta}$ by zero-density estimate (2.8). Furthermore, the characteristic function of $\nu$ is given by

$$\psi(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} C_\sigma(x) \frac{dx}{\sqrt{2\pi}}.$$

Thus Theorem 2.2 yields

$$\phi(\xi) - \psi(\xi) \ll \exp\left(-\delta \frac{\log X}{\log \log X}\right)$$

for $|\xi| \leq b_\sigma R_\sigma(X)$, where the implied constant depends only on $\sigma$. Here, $b_\sigma$ is a positive constant, and $R_\sigma(X)$ is given by (2.11). Put $R = b_\sigma R_\sigma(X)$ and $r = (\log X)^{-2}$. Then we obtain

$$\int_{r}^{R} \frac{|\phi(\xi) - \psi(\xi)|}{\xi} d\xi \ll \exp\left(-\delta \frac{\log X}{\log \log X}\right) \int_{r}^{R} \frac{1}{\xi} d\xi$$

$$\ll \exp\left(-\frac{\delta}{2} \frac{\log X}{\log \log X}\right)$$

by (6.1). We have also

$$\int_{-R}^{-r} \frac{|\phi(\xi) - \psi(\xi)|}{\xi} d\xi \ll \exp\left(-\frac{\delta}{2} \frac{\log X}{\log \log X}\right).$$
Let $-r < \xi < r$. Then, we recall that $e^{i\theta} = 1 + O(|\theta|)$ holds uniformly for $\theta \in \mathbb{R}$. Therefore $\phi(\xi)$ is approximated as

\[
\phi(\xi) = 1 + O\left(\frac{1}{N_3^+(X)} \sum_{K \in \mathcal{L}_3^+(X)} |\xi||\log L(\sigma, \rho_K)|\right)
\]

\[
= 1 + O\left(\left|\xi\right| \left(\frac{1}{N_3^+(X)} \sum_{K \in \mathcal{L}_3^+(X)} (\log L(\sigma, \rho_K))^2\right)^{1/2}\right)
\]

\[
= 1 + O\left(\left|\xi\right| \left(\int_{-\infty}^{\infty} x^2 C_1(x) \frac{dx}{\sqrt{2\pi}}\right)^{1/2}\right)
\]

by the Cauchy–Schwarz inequality and Corollary 6.1 with $k = 2$. In a similar way, we derive

\[
\psi(\xi) = 1 + O\left(\left|\xi\right| \left(\int_{-\infty}^{\infty} x^2 C_1(x) \frac{dx}{\sqrt{2\pi}}\right)^{1/2}\right).
\]

Then we see that $\phi(\xi) - \psi(\xi)$ is estimated as $\phi(\xi) - \psi(\xi) \ll |\xi|$, which yields

\[
\int_{-r}^{r} \left|\frac{\phi(\xi) - \psi(\xi)}{\xi}\right| d\xi \ll \int_{-r}^{r} d\xi \ll (\log X)^{-2}.
\]

Combining (6.2), (6.3) and (6.4), we obtain that the integral in the right-hand side of (3.19) is estimated as

\[
\int_{-R}^{R} \left|\frac{\phi(\xi) - \psi(\xi)}{\xi}\right| d\xi \ll (\log X)^{-2}.
\]

Recall that $\sup_{a \in \mathbb{R}} C_\sigma(x) \ll_\sigma 1$. As a result, we drive by Lemma 3.12 the upper bound

\[
\sup_{t \in \mathbb{R}} |F(t) - G(t)| \ll \frac{1}{R_{\sigma}(X)} + (\log X)^{-2} \ll \frac{1}{R_{\sigma}(X)},
\]

where $F(t) = \mu((\infty, t])$ and $G(t) = \nu((\infty, t])$, and the implied constant depends only on $\sigma$. To end the proof, we see that the quantity $D_{\sigma}^+(X; a)$ is represented as

\[
D_{\sigma}^+(X; a) = \frac{\# \left\{ K \in \mathcal{L}_3^+(X) \mid L(\sigma, \rho_K) \leq e^a \right\}}{N_3^+(X)} - G(a)
\]

\[
= F(a) - G(a) + E,
\]

where

\[
E = \frac{\# \left\{ K \in \mathcal{L}_3^+(X) \mid L(\sigma, \rho_K) \leq e^a \right\}}{N_3^+(X)} - \frac{\# \{ K \in \mathcal{L}_3^+(X) \mid E_{\sigma}(X) \mid L(\sigma, \rho_K) \leq e^a \}}{\#(\mathcal{L}_3^+(X) \setminus E_{\sigma}(X))}.
\]

We have $E \ll X^{-\delta}$ by using $\#E_{\sigma}(X) \ll X^{1-\delta}$. Hence we obtain the conclusion

\[
\sup_{a \in \mathbb{R}} \left|D_{\sigma}^+(X; a)\right| \ll \sup_{a \in \mathbb{R}} |F(a) - G(a)| + X^{-\delta} \ll \frac{1}{R_{\sigma}(X)}.
\]
6.2. **Proof of Theorem 2.8**

Let \( A_\sigma(X) \) be the set of (2.16) for \( \sigma_1 < \sigma < 1 \) and \( A_\sigma(X) = \emptyset \) for \( \sigma \geq 1 \). Then we have \( \#A_\sigma(X) \ll X^{1-\delta} \) with some \( \delta > 0 \) by zero-density estimate (2.8). Therefore, limit formula (2.17) follows if we obtain

\[
\lim_{X \to \infty} \frac{1}{\#(L_{\sigma/3}(X) \setminus A_\sigma(X))} \sum_{K \in L_{\sigma/3}(X)} \Phi(\log L(\sigma, \rho_K)) = \int_{-\infty}^{\infty} \Phi(u) C_\sigma(x) \frac{dx}{\sqrt{2\pi}}.
\]

Let \( \mu_{\sigma, X} \) and \( \mu_{\sigma} \) be the probability measures on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) defined as

\[
\mu_{\sigma, X}(A) = \frac{1}{\#(L_{\sigma/3}(X) \setminus A_\sigma(X))} \sum_{K \in L_{\sigma/3}(X) \setminus A_\sigma(X)} L(\sigma, \rho_K)^i \xi,
\]

where

\[
\Phi(\xi) = \int_{-\infty}^{\infty} e^{ix\ell_n(K)} \frac{dx}{\sqrt{2\pi}}.
\]

Then the characteristic functions are given by

\[
\phi_{\sigma, X}(\xi) = \phi_{\sigma}(\xi) + O(X^{-\delta}),
\]

where \( C_\sigma(x) = \int_{-\infty}^{\infty} \tilde{C}_\sigma(\xi) e^{-ix\xi} \frac{d\xi}{\sqrt{2\pi}} \) satisfies

\[
\int_{-\infty}^{\infty} |\tilde{C}_\sigma(\xi)| \frac{d\xi}{\sqrt{2\pi}} < \infty
\]

by Proposition 4.4 (i). Therefore both \( C_\sigma \) and \( \tilde{C}_\sigma \) belong to the class \( L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). Furthermore, we have

\[
C_\sigma(x) = \int_{-\infty}^{\infty} \tilde{C}_\sigma(\xi) e^{-ix\xi} \frac{d\xi}{\sqrt{2\pi}}
\]

by inversion formula (3.17). Hence \( C_\sigma \) is a good density function in the sense of Ihara–Matsumoto [23]. Since we have

\[
\sum_{K \in X_n} \omega_n \exp(i\xi \ell_n(K)) = \frac{1}{N_{3/4}(X)} M_{i\xi, \sigma}(X) + O(X^{-\delta}),
\]

we obtain Theorem 2.8 in the case \( \Phi \in I(\mathbb{R}) \).
Theorem [2.2] yields that (3.20) is satisfied with \( \Phi(x) = e^{i\xi x} \) for any \( \xi \in \mathbb{R} \), and that the convergence is uniform in \( \xi \in [-R, R] \) for any \( R > 0 \). Then we prove that (3.20) holds with general test functions as follows.

- **Let** \( \sigma > 1 \) and \( \Phi \in C(\mathbb{R}) \). We define a continuous function \( \phi_0 \) on \([0, \infty)\) as

\[
\phi_0(r) = \max_{|x| \leq r} |\Phi(x)|,
\]

which is non-decreasing and satisfies \( \phi_0(r) > 0 \) for any \( r \in [0, \infty) \) and \( \phi_0(r) \to \infty \) as \( r \to \infty \). To obtain (3.20) in this case, it is sufficient to check that conditions (3.21) and (3.22) are satisfied. We have \( |\log L(\sigma, \rho_K)| \leq 2\log \zeta(\sigma) \) for \( \sigma > 1 \). Thus, there exists a constant \( A_\sigma > 0 \) depending only on \( \sigma \) such that

\[
\phi_0 \left( |\ell_n(K)| \right) \leq \phi_0(A_\sigma)
\]

holds for any \( K \in X_n \), which implies that condition (3.21) is satisfied. Then, we recall that \( C_\sigma \) is compactly supported on \( \mathbb{R} \) by Theorem 2.1 (ii). Hence condition (3.22) is also satisfied.

- **Let** \( \sigma > 1 \) and \( \Phi \in C^{\exp}(\mathbb{R}) \). In this case, we define the function \( \phi_0 \) as

\[
\phi_0(r) = Ce^{ar}
\]

for \( r \in [0, \infty) \), where \( C \) is a positive constant. Then we have

\[
\sum_{K \in X_n} \omega_n \phi_0 \left( |\ell_n(K)| \right)^2 = \frac{C}{N_3^{\pm}(X)} M_{2a,1}^\pm (X)
\]

since \( E_1(X) = A_1(X) = \emptyset \). Therefore, Theorem 2.2 implies that condition (3.21) is satisfied. Note that condition (3.22) is also valid due to Theorem 2.1 (iii). Hence Lemma 3.14 (ii) yields that (3.20) holds for any continuous function \( \Phi \) satisfying \( \Phi(x) \ll e^{a|x|} \).

- **Let** \( \sigma_1 < \sigma < 1 \) and \( \Phi \in C_b(\mathbb{R}) \). By Lemma 3.14 (i), we obtain (3.20) in this case.

- **Let** \( \sigma_1 < \sigma < 1 \) and \( \Phi \in C^{\exp}(\mathbb{R}) \). We take \( \phi_0(r) = Ce^{ar} \) with a constant \( C > 0 \) as before. Then we have

\[
\sum_{\chi \in X_n} \omega_n \phi_0 \left( |\ell_n(\chi)| \right)^2 = \frac{C}{N_3^{\pm}(X)} M_{2a,\sigma}^\pm (X)
\]

since \( E_\sigma(X) = A_\sigma(X) = \emptyset \) under GRH. Hence we deduce from Theorem 2.2 that condition (3.21) holds. As described before, condition (3.22) is valid. As a result, Lemma 3.14 (ii) yields formula (3.20) in this case.

We finally note that formula (3.20) implies (6.5) by the setting of \( X_n, \omega_n \) and \( \ell_n \). Hence we complete the proof of Theorem 2.8.

**Appendix: Results on the logarithmic derivative**

In the above sections, we proved several results on values \( \log L(\sigma, \rho_K) \). We obtain similar results for \( (L'/L)(\sigma, \rho_K) \), which are listed in this section. Note that the value \( (L'/L)(1, \rho_K) \) is connected with the Euler–Kronecker constant \( \gamma_K \) defined by

\[
\gamma_K = \lim_{s \to 1} \left( \frac{\zeta_K'}{\zeta_K}(s) + \frac{1}{s-1} \right).
\]
By definition, $\gamma_Q$ is equal to Euler’s constant $\gamma = 0.577\ldots$. Thus we obtain

$$\frac{L'}{L}(1, \rho_K) = \gamma_K - \gamma$$

for any non-Galois cubic field $K$ since $L(s, \rho_K) = \zeta_K(s)/\zeta(s)$. The proofs for the results in this section are omitted unless we need special remarks arising from the difference between $\log L(\sigma, \rho_K)$ and $(L'/L)(\sigma, \rho_K)$. Let $X = (X_p)_p$ and $Y = (Y_p)_p$ be the sequences of independent random elements on $\{A_a | a \in A\}$ as in Section 2. Then we define the random Euler products $L(s, X)$ and $L(s, Y)$ as in (2.7).

**Theorem A.2.** For $\sigma > 1/2$, there exists a non-negative $C^\infty$-function $C_\sigma$ such that

$$\mathbb{P}\left(\frac{L'}{L}(\sigma, X) \in A\right) = \int_A C_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

holds for all $A \in \mathcal{B}(\mathbb{R})$. Furthermore, it satisfies the following properties.

(i) If $1/2 < \sigma \leq 1$, we have $\text{supp} C_\sigma = \mathbb{R}$, that is, $C_\sigma(x)$ is not identically zero in any interval on $\mathbb{R}$.

(ii) If $\sigma > 1$, the function $C_\sigma$ is compactly supported.

(iii) Let $\sigma > 1/2$. Then the integral

$$\int_{-\infty}^{\infty} e^{ax} C_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

is finite for any $a > 0$.

Similarly, for $\sigma > 2/3$, there exists a non-negative $K^\infty$-function $K_\sigma$ such that

$$\mathbb{P}\left(\frac{L'}{L}(\sigma, Y) \in A\right) = \int_A K_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

holds for all $A \in \mathcal{B}(\mathbb{R})$. Furthermore, it satisfies the following properties.

(i') If $2/3 < \sigma \leq 1$, we have $\text{supp} K_\sigma = \mathbb{R}$, that is, $K_\sigma(x)$ is not identically zero in any interval on $\mathbb{R}$.

(ii') If $\sigma > 1$, the function $K_\sigma$ is compactly supported.

(iii') Let $\sigma > 2/3$. Then the integral

$$\int_{-\infty}^{\infty} e^{ax} K_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

is finite for any $a > 0$.

Denote by $\mathcal{E}_\sigma(X)$ the subset of $L^+_\sigma(X)$ such that $\mathcal{E}_\sigma(X) = E(X)$ for $\sigma_1 < \sigma \leq 1$ and $\mathcal{E}_\sigma(X) = \emptyset$ for $\sigma > 1$, where $\sigma_1$ is a real number for which (2.8) holds, and $E(X)$ is defined by (2.9). Then we define $\mathcal{M}^{\pm}_{z, \sigma}(X)$ as

$$\mathcal{M}^{\pm}_{z, \sigma}(X) = \sum_{K \in L^+_\sigma(X) \setminus \mathcal{E}_\sigma(X)} \exp \left( z \frac{L'}{L}(\sigma, \rho_K) \right)$$

as an analogue of the $z$-th moment $M^{\pm}_{z, \sigma}(X)$ of (1.7).

**Theorem A.3.** Let $\sigma_1$ be a real number for which (2.8) holds. Then there exists an absolute constant $\delta > 0$ such that

$$\mathcal{M}^{\pm}_{z, \sigma}(X) = C_1^{\pm} \frac{1}{12\zeta(3)} X \int_{-\infty}^{\infty} e^{az} C_\sigma(x) \frac{dx}{\sqrt{2\pi}} + O \left( X \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right)$$

(A.1)
holds for $\sigma > \sigma_1$ with $z \in \mathbb{C}$ satisfying $|z| \leq b_\sigma R_\sigma(X)$, where $b_\sigma$ is a positive constant, and $R_\sigma(X)$ is defined as in (2.11). The implied constant in (A.1) depends only on $\sigma$.

Note that the subset $\mathcal{E}_\sigma(X)$ differs from $E_\sigma(X)$ at $\sigma = 1$. We hereby explain the reason why this difference arises. In the same line as (5.15), one can show the asymptotic formula
\[
\sum_{K \in E_1(X)} \exp \left( \frac{L'}{L} \sigma K \right) = C^\pm \frac{1}{12 \zeta(3)} X \int_{-\infty}^\infty e^{x\sigma} \mathcal{C}_\sigma(x) \frac{dx}{\sqrt{2\pi}} + O \left( X \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right)
\]
for $\sigma > \sigma_1$ and $|z| \leq b_\sigma R_\sigma(X)$, where $\delta$ is an absolute positive constant. To prove formula (A.1) at $\sigma = 1$ with $E_1(X) = \emptyset$, one needs to obtain the upper bound
\[
\sum_{K \in E(X)} \exp \left( \frac{L'}{L} \sigma K \right) \ll X \exp \left( -\delta \frac{\log X}{\log \log X} \right)
\]
for $|z| \leq b_\sigma R_\sigma(X)$ with $\sigma = 1$. This is true for $\sigma > 1$ due to $(L'/L)(\sigma, \rho_K) \ll 1$ and $\#E(X) \ll X^{1-\delta}$. However, we know just
\[
\frac{L'}{L}(1, \rho_K) \ll \log X
\]
for $K \in L_1^+(X)$ without any assumptions, which prevents us from obtaining (A.2) at $\sigma = 1$. A related difficulty on studying the distribution of values $(L'/L(1, \chi_d)$ was discussed in the Ph.D. thesis of Mourtada [37]. See also Lamzouri [30] for more information about the Euler–Kronecker constants of quadratic fields.

**Theorem A.4.** Assume GRH and upper bound (2.12) for each $\epsilon > 0$ with some constants $\alpha$ and $\beta$ such that $0 < \alpha < 5/6$. Let $\sigma_2$ be as in (2.13). Then there exists a constant $\delta = \delta(\sigma) > 0$ such that
\[
M_{z, \sigma}^\pm(X) = C^\pm \frac{1}{12 \zeta(3)} X \int_{-\infty}^\infty e^{x\sigma} \mathcal{C}_\sigma(x) \frac{dx}{\sqrt{2\pi}}
\]
\[
+ \frac{4 \zeta(1/3)}{5 \Gamma(2/3) \zeta(5/3)} X^{5/6} \int_{-\infty}^\infty e^{x\sigma} \mathcal{K}_\sigma(x) \frac{dx}{\sqrt{2\pi}} + O \left( X^{5/6 - \delta} \right)
\]
holds for any $\sigma > \max(\sigma_2, 2/3)$ with $z \in \mathbb{C}$ satisfying $|z| \leq \tilde{b}_\sigma \tilde{R}_\sigma(X)$, where $\tilde{b}_\sigma$ is a positive constant, and $\tilde{R}_\sigma(X)$ is defined as in (2.15). The implied constant in (A.3) depends only on $\sigma$.

**Corollary A.5.** There exists an absolute constant $\delta > 0$ such that
\[
\sum_{K \in L_2^+(X) \setminus E(X)} \exp(z \cdot \gamma_K) = C^\pm \frac{1}{12 \zeta(3)} X e^{\sigma z} \int_{-\infty}^\infty e^{x\gamma_1} \mathcal{C}_{\gamma_1}(x) \frac{dx}{\sqrt{2\pi}}
\]
\[
+ O \left( X \exp \left( -\delta \frac{\log X}{\log \log X} \right) \right)
\]
holds for any $z \in \mathbb{C}$ with $|z| \leq b_1 R_1(X)$, where the implied constant is absolute.
Corollary A.6. Assume GRH and upper bound (2.12) with some constants $\alpha$ and $\beta$ such that $3\alpha + \beta < 5/2$. Then there exists an absolute constant $\delta > 0$ such that

$$\sum_{K \in L_3^+(X)} \exp(z \cdot \gamma_K) = C_0 \int_{-\infty}^{\infty} e^{xz} C_1(x) \frac{dx}{\sqrt{2\pi}}$$

$$+ K \exp\left(-\delta \log X \frac{\log \log X}{\log X}\right)$$

holds for any $z \in \mathbb{C}$ with $|z| \leq b_1 R_1(X)$, where the implied constant is absolute. Define the quantity $D_0^+(X; a)$ as

$$D_0^+(X; a) = \sum_{L_3^+(X)} \frac{\sum_{K \in L_3^+(X)} \frac{L'_{\sigma}(\sigma, \rho_K)}{\Lambda_{\sigma}(X)}}{\Lambda_0^+(X)} - \int_{-\infty}^{a} C_0(x) \frac{dx}{\sqrt{2\pi}}$$

for $\sigma > 1/2$ and $a \in \mathbb{R}$.

Theorem A.7. Let $\sigma_1$ be a real number for which (2.8) holds. Then we obtain

$$\sup_{a \in \mathbb{R}} |D_0^+(X; a)| \ll \frac{1}{R_\sigma(X)},$$

where $R_\sigma(X)$ is as in (2.11).

Finally, we present a result similar to Theorem A.4. There exists a difference from the result for $L(\sigma, \rho_K)$ when we consider the case $\sigma = 1$ by the same reason as in Theorem A.4. For this, we define a subclass of $C(S)$ as

$$C_{\text{poly}}(\mathbb{R}) = \{ \Phi \in C(\mathbb{R}) \mid \Phi(x) \ll |x|^a \text{ with some } a > 0 \}.$$ 

Theorem A.8. Let $\sigma_1$ be a real number for which (2.8) holds. Then the limit formula

$$\lim_{X \to \infty} \frac{1}{N_3^+(X)} \sum_{K \in L_3^+(X)} \Phi\left(\frac{L'_{\sigma}(\sigma, \rho_K)}{\Lambda_{\sigma}(X)}\right) = \int_{-\infty}^{\infty} \Phi(u) C_\sigma(x) \frac{dx}{\sqrt{2\pi}}$$

holds in the following cases:

- $\sigma > 1$ and $\Phi \in C(\mathbb{R}) \cup I(\mathbb{R})$;
- $\sigma = 1$ and $\Phi \in C_{\text{poly}}(\mathbb{R}) \cup I(\mathbb{R})$;
- $\sigma_1 < \sigma < 1$ and $\Phi \in C_\sigma(\mathbb{R}) \cup I(\mathbb{R})$ without assuming GRH;
- $\sigma_1 < \sigma \leq 1$ and $\Phi \in C_{\exp}(\mathbb{R}) \cup I(\mathbb{R})$ if we assume GRH.

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