Dynamics of a magnetized Bianchi I universe with vacuum energy

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Abstract
We make use of a flat, axisymmetric Bianchi I metric to investigate the effects of a magnetic field upon the dynamics of the universe for the case in which the accompanying fluid is a cosmological constant and derive two exact solutions to the dynamical equations for this situation. We examine the behaviour of the scale factor perpendicular and parallel to the field lines, $A(t)$ and $W(t)$ respectively, and find the expected behaviour. The field has the strongest effect when $A(t)$ is small, decelerating collapse perpendicular to the field lines, due to magnetic pressure, and accelerating collapse along the field lines, due to magnetic tension, while the vacuum energy dominates at late time, driving accelerated expansion.

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1. Introduction
The Bianchi models, which describe homogeneous but anisotropic spacetimes, have recently undergone a resurgence in popularity, motivated in part by attempts to explain small but significant anisotropies in the cosmic microwave background (CMB), as seen in data from satellites such as WMAP (e.g. [1]). Such models can admit large-scale magnetic fields which, if of sufficient strength, may be able to affect the overall expansion rate of the universe. The axisymmetric Bianchi I model is appropriate for examining the case of a universe which is permeated by a large-scale, homogeneous magnetic field. This model has been studied in the literature for the case when the field is accompanied by a perfect fluid with $0 \leq \gamma \leq 1$, where $\gamma$ is the barotropic index of the fluid. The case where the accompanying fluid is a cosmological constant, $\Lambda$, which has $\gamma = -1$, has, however, historically been ignored. This is primarily because these models were under investigation in the late 1960s, when a cosmological constant was not generally considered to be a relevant or realistic constituent of the universe. Since then, we have learned that the universe is currently dominated by dark energy, which may
be in the form of a cosmological constant [2–5], and that the universe underwent exponential expansion for a short period, known as inflation, during which time the inflaton acted as an effective cosmological constant [6] (for a recent review, see [7]). Hence models containing a cosmological constant are now of significant interest.

The observed level of isotropy in the CMB places tight constraints upon the strength of any Hubble scale magnetic field, and the upper limit on such a field is found to be of the order of $10^{-9}$ G [8]. It is important to remember, however, that these and other similar limits quote the adiabatically expanded, present-day equivalent value of the field strength. The strength of a magnetic field is proportional to the density of its field lines and so, assuming the field is frozen into the plasma that expands with the Hubble flow,

$$ B \propto \frac{1}{a^2(t)}, \quad (1) $$

where $B$ is the magnetic field strength and $a(t)$ is the scale factor of the universe. Thus although equivalent to a very weak field at the present time, these limits equate to a much stronger field at very early times, when the scale factor is small. In particular, the exponential rate of increase of the scale factor during inflation means that a significant field could exist during the early stages of inflation before being diluted away and becoming consistent with later constraints. An understanding of the effect of a magnetic field upon the dynamics of the universe in the presence of a cosmological constant may, therefore, have implications for models of the universe which contain a magnetic field generated before or during inflation (e.g. [9–14]).

In this paper we use the axisymmetric Bianchi I model to look at the effects of a large-scale magnetic field upon the dynamics of the universe in the presence of a cosmological constant. In section 2 we outline the axisymmetric Bianchi I model and give the fundamental dynamical equations that describe its evolution in the presence of both a perfect, barotropic fluid and a large-scale magnetic field. In section 3 we briefly review the main findings which arise from existing solutions to these equations from the literature, with particular emphasis upon the general behaviour and effects of the magnetic field in these solutions. In section 4 we derive two exact solutions for a magnetized universe with vacuum energy and in section 5 we examine the results of those solutions for various specific cases and in certain limits. Finally, in section 6 we discuss our findings.

2. Axisymmetric Bianchi I model

The assumption that the universe is both homogeneous and isotropic restricts the spacetime metric to the Friedmann–Robertson–Walker (FRW) model. Relaxation of the condition of isotropy allows extra degrees of freedom and the Bianchi models are the resulting set of homogeneous but anisotropic spacetimes, first classified by Bianchi in 1918. A good review of anisotropic spacetimes, which gives details of the Bianchi classifications, can be found in [15]; see also [16].

The Bianchi I model describes a universe in which the scale factor is not restricted to be the same in each direction, as it is in the FRW model. We wish to consider a flat universe in which the anisotropy is introduced by a large-scale homogeneous magnetic field. Such a magnetic field will impose a single preferred direction in space, along the field lines, and so the anisotropy of the spacetime will be axisymmetric. Hence we use the flat, axisymmetric Bianchi I metric,

$$ ds^2 = -dt^2 + A^2(t)(dx^2 + dy^2) + W^2(t) \, dz^2, \quad (2) $$
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as the basis for describing such a universe. This metric is homogeneous, geometrically flat
and has two equivalent transverse directions, x and y, and one different longitudinal direction,
z, along which we assume the magnetic field is orientated ($B = B_z$ only). $A(t)$ is then
the scale factor in the transverse direction, perpendicular to the magnetic field, while $W(t)$
is the scale factor in the longitudinal direction, along the field lines. In the limit where
$A(t) = W(t) = a(t)$, (2) reduces to the flat FRW metric.

We assume that there is no electric field, that the magnetic field is orientated along the z
axis and that the universe is filled with a perfect, barotropic fluid with equation of state

$$p_f = \gamma \rho_f,$$

where $\rho$ is the energy density, $p$ is the pressure and a subscript ‘f’ indicates the fluid. The
combined stress–energy tensor for both the magnetic field and the fluid is then given by

$$T^{\mu\nu} = \text{diag}(−\rho_f − ρ_B, p_f + ρ_B, p_f + ρ_B, p_f − ρ_B),$$

where a subscript ‘B’ indicates the magnetic field. The Einstein field equation,

$$G_{\mu\nu} = 8\pi T_{\mu\nu},$$

then leads to three equations which describe the dynamics of the spacetime [17, 18]:

$$16\pi \rho_B = \frac{\dddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 + \frac{\dot{A}}{A} \frac{\dot{W}}{W} − \frac{\ddot{W}}{W},$$

$$16\pi \rho_f = -\frac{\dddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 + 5\frac{\dot{A}}{A} \frac{\dot{W}}{W} + \frac{\ddot{W}}{W}$$

and

$$16\pi \gamma \rho_f = -3\frac{\dddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 - \frac{\dot{A}}{A} \frac{\dot{W}}{W} − \frac{\ddot{W}}{W}. $$

Assuming conservation of energy,

$$T^{\mu\nu;\mu} = 0,$$

where a semicolon indicates a covariant derivative, for the fluid and the magnetic field leads
to two equations [17],

$$\rho_f = \frac{\mu}{8\pi} \frac{1}{A^2 W^{1+\gamma}},$$

and

$$\rho_B = \frac{\beta}{8\pi} \frac{1}{A^4},$$

respectively, where $\mu$ and $\beta$ are positive constants.

Equations (6)–(8), (10) and (11) describe the dynamics of a Bianchi I universe which
contains a magnetic field of energy density $\rho_B$ and a perfect, barotropic fluid of energy density
$\rho_f$ and equation of state given by (3). It is possible to solve these equations to find $A(t)$ and
$W(t)$ which then describe the expansion of the universe in the transverse and longitudinal
directions, respectively.
3. General properties

The equations from section 2 have been used in the literature to investigate the effects of a large-scale uniform magnetic field on the dynamics of the universe when the accompanying fluid has $0 \leq \gamma \leq 1$ (e.g. [17–19]). A wide range of scenarios has been investigated; here we restrict ourselves to considering those in a flat Bianchi I cosmology. For example, [19] contains analytic solutions for the case of dust ($\gamma = 0$) and radiation ($\gamma = 1/3$), while [17] also finds analytic solutions for $\gamma = 1$ and for $1/3 \leq \gamma \leq 1$. In addition, they find a special case of the $\gamma = 1$ solution in which $\mu = 0$ and hence there is no fluid—the pure magnetic case. For general properties of these solutions, see [20].

Both [17, 19] find that the asymptotic nature of these solutions varies depending on the values of the various constants. They can collapse isotropically or anisotropically to a point singularity, collapse in the longitudinal direction thus forming a pancake singularity, or collapse in the transverse direction thus forming a cigar singularity. In general the existing solutions follow several ‘rules of thumb’ that describe the effect of the magnetic field upon the dynamical evolution of the universe, which are given in [19]. These are:

(i) If $\rho_B \ll \rho_f$ and the model is not approaching a singularity then the magnetic field has negligible effect upon the dynamics.

(ii) In general, the magnetic field accelerates expansion (or decelerates collapse) in the transverse direction owing to magnetic pressure, which resists compression of the field lines, and decelerates expansion (or accelerates collapse) in the longitudinal direction owing to tension in the field lines.

(iii) Near a singularity of infinite density ($\rho_f \to \infty$), if $\rho_B/\rho_f$ (which is always positive) $\to 0$ as the singularity is approached then the magnetic field has negligible effect upon the dynamics.

(iv) Near a singularity of infinite density, if $\rho_B/\rho_f$ does not approach 0 then one of two cases occur. Either (a) $\rho_B/\rho_f \to \text{const}$, in which case the fluid and magnetic field jointly determine the dynamics, or (b) the magnetic field causes rapid expansion in the transverse direction (see point (ii)), and this change in the dynamics causes $\rho_B/\rho_f$ to approach zero.

It is worth commenting on the magnetic Bianchi models with dust or radiation. In these cases there is a significant effect on the evolution of the shear in that it falls off much more slowly than in a model without the magnetic field. This is why the cosmic microwave background gives a strong constraint [8] on homogeneous magnetic fields [21].

4. Solutions with vacuum energy

The existing solutions, as discussed in section 3, concentrate on cases where the fluid has $0 \leq \gamma \leq 1$. In this section we present two solutions to the dynamical equations which describe a magnetized Bianchi I universe containing a cosmological constant, $\gamma = -1$, one of which is physically interesting. These belong to a general class of solutions that has been discussed in the literature before, but from a different perspective [22] to that discussed here; see also [20] for these solutions in a different form.

When $\gamma = -1$, $\rho_f = \rho_\Lambda$, the energy density of the cosmological constant, or vacuum energy. Equations (6)–(8) then become

$$16\pi\rho_B = \frac{\dot{A}}{A} + \left(\frac{A}{\dot{A}}\right)^2 - \frac{\dot{A}}{A} \frac{\dot{W}}{W} - \frac{\ddot{W}}{W},$$

(12)
\[ 16\pi \rho_\Lambda = -\frac{\dddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 + 5 \frac{\ddot{A}}{A} W + \frac{\dot{W}}{W} \]  

(13)

and

\[ -16\pi \rho_\Lambda = -3 \frac{\dddot{A}}{A} - \left(\frac{\dot{A}}{A}\right)^2 - \frac{\ddot{A}}{A} W - \frac{\dot{W}}{W} \]  

(14)

respectively, and from (10), the vacuum energy is given by

\[ \rho_\Lambda = \frac{\mu}{8\pi}. \]  

(15)

The left-hand sides of (13) and (14) are now equal and opposite. There are two ways in which the right-hand sides of these equations can be equal and opposite, and these give rise to our two solutions. The easiest route is to assume that all derivatives of \(A\) are equal to zero, which leads us to conclude that \(A = \text{constant}\). Substituting this into (14) gives

\[ \frac{\dot{W}}{W} = 2\mu. \]  

(16)

Substituting this result into (12) and (13) and adding these equations immediately shows that either \(\rho_B, \rho_\Lambda\) or \(\mu\) must all be identically zero (the empty universe case), or at least one must be negative, in which case this solution is unphysical.

An alternative solution can be found by equating (13) and (14) which, after some cancellation, reduces to

\[ \frac{\dot{W}}{W} = \frac{\dddot{A}}{A} \]  

(17)

from which we can immediately see that

\[ W = k A, \]  

(18)

where \(k\) is a constant of integration. Substituting (10) and (11) back into (12) and (14), adding these equations and substituting (20) into the result, leads to

\[ \frac{\beta}{A^4} + \mu = k^2 \left( \frac{W}{A} \right)^2 + 2 \frac{WW'}{A}. \]  

(19)

where a prime denotes differentiation with respect to \(A\). The constant \(k\) can be removed from this equation by scaling both \(\beta\) and \(\mu\) by a factor of \(k^2\). Hence the value of \(k\) does not affect the form of the solution and so we are free to set it to unity, giving

\[ W = \dot{A} \]  

(20)

and

\[ \frac{\beta}{A^4} + \mu = \left( \frac{W}{A} \right)^2 + 2 \frac{WW'}{A}. \]  

(21)

Equation (21) can now be solved to find \(W(A)\), giving

\[ W = \frac{1}{3A} (3\mu A^4 - 9\beta + 9c A)^{1/2}, \]  

(22)

where \(c\) is a constant of integration, which, from (20), leads to

\[ t = \int 3A(3\mu A^4 - 9\beta + 9c A)^{-1/2} dA. \]  

(23)

Equations (20) and (23) form our second solution, which is of more physical interest than the first. Unfortunately, it is not easy to integrate (23) analytically, making it difficult to find an explicit expression for \(t(A)\) and hence \(A(t)\) and \(W(t)\). Instead, in the next section we consider the form of \(A\) in different limits and use this to gain an insight into the form of \(A(t)\) and \(W(t)\) and hence into the expansion history of the universe.
5. Specific cases

From (22) we can make deductions about the form of $W(A)$, and hence of $A(t)$, depending upon the value of the constants $\mu$, $\beta$ and $c$ and in different limits. In this section we consider first the empty universe case, $\mu$, $\beta = 0$, before examining the effect of the magnetic field and vacuum energy independently. Finally, we look at the combined case, $\mu$, $\beta > 0$.

5.1. Empty universe ($\mu$, $\beta = 0$)

When $\mu$, $\beta = 0$, (22) reduces to

$$W = \left( \frac{c}{A} \right)^{1/2},$$

(24)

giving

$$A = c^{1/3} \left( \frac{3}{2} t \right)^{2/3}.$$  

(25)

Substituting this back into (24) gives

$$W = \left( \frac{3}{2c} t \right)^{-1/3}.$$  

(26)

Coordinate freedom in the metric given in equation (2) allows us to scale away the factors of $c$ in both $A(t)$ and $W(t)$, leaving

$$A = \left( \frac{3}{2} t \right)^{2/3}$$

(27)

and

$$W = \left( \frac{3}{2} t \right)^{-1/3},$$

(28)

respectively.

Equations (27) and (28) describe the evolution of the empty universe in the transverse and longitudinal directions respectively, as illustrated in figure 1. In this case the universe emerges from an infinitely long cigar singularity ($A = 0$, $W = \infty$) at $t = 0$. As $t$ increases, $W$ decreases asymptotically to zero while $A$ continues to increase, leading to another singularity at $t = \infty$. This final singularity is an infinitely wide pancake ($A = \infty$, $W = 0$).

Reassuringly, this solution is one of two well-known possible solutions to the empty axisymmetric Bianchi I model, also known as the Kasner universe (see, e.g., [15, 20, 23]). The other possible solution was the special, empty universe case of our first solution. We can see directly from (6)–(8) that $A =$ constant, $W =$ constant is also a solution when $\mu$, $\beta = 0$. This is the case of a static empty universe and it is this which reduces to the static solution to the empty FRW model when $A(t) = W(t)$.

5.2. Pure magnetic ($\mu = 0$)

When $\mu = 0$, we have the pure magnetic case, with a magnetic field but no vacuum energy. In this case (22) reduces to

$$W = \frac{(cA - \beta)^{1/2}}{A},$$

(29)
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Figure 1. Evolution of (a) $A(t)$ and (b) $W(t)$ for the empty universe case.

from which we can see that $c$ must be a positive constant for this solution to be physical. Hence

$$t = \frac{2}{3} (2\beta + cA) \sqrt{cA - \beta}.$$  \hfill (30)

The only real solution to this equation is given by

$$A = \frac{1}{c} \left( \beta + \left(\frac{1}{2} x - \frac{2\beta}{x}\right)^2 \right),$$  \hfill (31)

where

$$x = (6tc^2 + 2\sqrt{16\beta^3 + 9c^4t^2})^{1/3}.$$  \hfill (32)

Substituting this back into (29) gives

$$W = \frac{c\left(\frac{1}{2} x - \frac{2\beta}{x}\right)^{1/2}}{\beta + \left(\frac{1}{2} x - \frac{2\beta}{x}\right)^2}.$$  \hfill (33)

Both $A$ and $W$ again have constant factors which can be scaled away due to coordinate freedom in (2); however, the factors of $c$ included in $x$ cannot be removed in the same fashion.

Although we are able to solve for $A(t)$ and $W(t)$ exactly, it is instructive to consider independently the limits of large and small $A$ in (29) in order to understand the behaviour of $A$ and $W$ at early and late times.

For small $A$ ($A \sim \beta/c$), the magnetic field has a profound effect upon the dynamics of the universe. As we approach the initial singularity it prevents collapse in the transverse direction, due to magnetic pressure, and accelerates collapse in the longitudinal direction, due to tension in the field lines. This causes a bounce at a pancake singularity ($A > 0$, $W = 0$) at $t = 0$. The value of $A$ at the initial singularity is determined by $\beta$ as expected, since a stronger magnetic field will provide a greater pressure resisting the collapse. The strength of the field also affects...
Figure 2. Evolution of (a) $A(t)$ and (b) $W(t)$ for the pure magnetic case plotted for $c = 1$ and varying values of $\beta$, as follows: solid, $\beta = 1$; dot-dashed, $\beta = 2$; dashed, $\beta = 5$; dotted, $\beta = 10$.

the tension in the field lines; hence the maximum value of $W$ and the time at which this is reached also depends upon $\beta$.

For large $A$ ($A \gg \beta/c$), the magnetic field becomes negligible and (29) again reduces to the empty universe case. Hence, far from the initial singularity the magnetic field has no effect, as would be expected for large values of $A$, since $\rho_B$ is very small here. We therefore have a second pancake singularity at $t = \infty$ (mirrored at $t = -\infty$), exactly as for the empty universe.

We can also see from these limits that the value of $c$ influences the point at which the magnetic regime moves into the empty universe regime, with this happening at smaller values of $A$ (and hence earlier times) for larger values of $c$.

It is interesting to note that the behaviour of $A$ and $W$ in this case is qualitatively the same as for the axisymmetric pure magnetic case in [17], which was found from the solutions for $A(t)$ and $W(t)$ when $\gamma = 1$. This solution is therefore probably the same as theirs for this case, as would be expected, but arrived at via a different route.

5.3. Pure $\Lambda$ ($\beta = 0$)

When $\beta = 0$ we have the pure $\Lambda$ case, with vacuum energy but no magnetic field. In this case, (22) reduces to

$$W = \frac{(3\mu A^4 + 9cA)^{1/2}}{3A},$$

which gives

$$t = \int \frac{3A}{(3\mu A^4 + 9cA)^{1/2}} \, dA.$$  \hspace{1cm} (35)

This expression is not easy to integrate using analytical techniques. It is clear, however, that, in the denominator of (35), the $A$ term will dominate in the limit of small $A$, while the $A^4$ term
will dominate in the limit of large $A$. These two regimes meet when
\[ A = (3c/\mu)^{1/3}. \] (36)

Hence we can consider the behaviour of $A(t)$ and $W(t)$ in these two limits.

In the limit of small $A$ ($A \ll (3c/\mu)^{1/3}$), (34) reduces to the empty universe case, as described in section 5.1. In the limit of large $A$ ($A \gg (3c/\mu)^{1/3}$) it reduces to
\[ W = \sqrt{\frac{\mu}{3}} A, \] (37)
which gives
\[ A = e^{\sqrt{\mu/3}t}. \] (38)

Substituting this back into (37) then gives
\[ W = \sqrt{\frac{\mu}{3}} e^{\sqrt{\mu/3}t}, \] (39)
and so both $A(t)$ and $W(t)$ increase exponentially in this limit (see figure 3).

This analysis suggests that the universe emerges from an infinitely long cigar singularity at $t = 0$, as in the empty universe, with $A$ increasing steadily and $W$ decreasing rapidly. $W$ then reaches a minimum value, after which it begins to increase. At large $t$, both $A$ and $W$ increase exponentially, leading to an infinitely large, ever expanding universe without a final singularity. This exponential increase in $W$ and $A$ at late times can be understood as the effect of the vacuum energy driving an accelerated expansion. The accelerated expansion begins when $A \sim (3c/\mu)^{1/3}$; hence the larger the value of $\mu$, and the higher the energy density of the cosmological constant, the earlier its effect is seen, as expected. The higher the value of $c$, on the other hand, the longer the empty universe regime persists.
5.4. Combined case

For the combined case, $\mu, \beta > 0$, which contains both a magnetic field and vacuum energy, we cannot integrate $t(A)$, so must again look at the limits where different terms dominate in (22). Although mathematically it is possible to consider two regimes, domination by the $-9\beta + 9cA$ term at small $A$ and by the $3\mu A^4$ term at large $A$, it is again instructive to split the early regime into two. Hence we consider three possible regimes as follows:

(i) $A \sim \beta/c$ and $A \ll (3c/\mu)^{1/3}$. The square root in (22) is dominated by $9cA - 9\beta$. The magnetic field dominates as in the early pure magnetic case.

(ii) $A \gg \beta/c$ and $A \ll (3c/\mu)^{1/3}$. The square root in (22) is dominated by the $9cA$ term.

Neither magnetic field nor cosmological constant affect the evolution, which proceeds as for an empty universe.

(iii) $A \gg \beta/c$ and $A \gg (3c/\mu)^{1/3}$. The square root in (22) is dominated by the $3\mu A^4$ term.

The vacuum energy dominates the expansion as in the late pure $\Lambda$ case.

Hence the universe has a bounce at a finite pancake singularity ($A > 0, W = 0$) at $t = 0$, from which $A$ and $W$ both increase. Depending upon the values of $\mu, \beta$ and $c$ it may then enter a regime where $W$ decreases while $A$ continues to increase, as in the empty universe case. Finally, the vacuum energy comes to dominate causing both $A$ and $W$ to increase exponentially. The behaviour of $A(t)$ and $W(t)$ for this case is illustrated in figure 4.

This behaviour is unsurprising; we expect the influence of the magnetic field to be greater at smaller values of $A$, since the magnetic field strength is proportional to $1/A^2$, and the cosmological constant to dominate at later times, since its energy density remains constant as the universe expands. Hence as $A$ increases, the effect of the magnetic field is diminished and the cosmological constant comes to dominate once the magnetic field has been diluted away.

The time at which these regimes cross depends upon the values of $\mu, \beta$ and $c$. The higher the value of $\beta$, the stronger the magnetic field and the longer it will continue to influence the dynamics of the universe. Similarly, the higher the value of $\mu$, the higher the energy density of the cosmological constant and the earlier it will come to dominate. If both $\mu$ and $\beta$ are sufficiently large compared to $c$, the magnetic regime will move directly into the $\Lambda$ regime.

Figure 4. Evolution of (a) $A(t)$ and (b) $W(t)$ for the combined magnetic and cosmological constant case.
and there will be no empty universe regime. We leave the magnetic regime at $A \gg \beta/c$ and enter the $\Lambda$ regime at $A \gg (3c/\mu)^{1/3}$; thus there is no empty universe regime if these two limits cross, i.e. if

$$\mu \beta^3 \gg 3c^4.$$  \hfill (40)

6. Discussion

We have obtained two exact solutions to the dynamical equations for an axisymmetric Bianchi I universe containing a magnetic field and a perfect, barotropic fluid for the previously unconsidered case of a cosmological constant with $\gamma = -1$. The first of these solutions is unphysical, leading to imaginary values of the transverse scale factor, $A(t)$, if $\beta > 0$. The second solution is more physically interesting, and we have used it to investigate the effect of a magnetic field on the dynamics of the spacetime in the presence of vacuum energy.

The magnetic field has the strongest effect upon the dynamics at early times, when $A$ is small. It tends to decelerate collapse in the transverse direction and accelerate collapse in the longitudinal direction as the initial singularity is approached. The vacuum energy, on the other hand, comes to dominate the expansion at late time and causes accelerated expansion in both the transverse and longitudinal directions.

Additionally, we find that the shape of the initial singularity depends upon the presence or absence of the magnetic field, with the field transforming it from a cigar to a pancake singularity, and that the existence of a final singularity depends upon the presence or absence of the cosmological constant, which prevents the final singularity from occurring. It should be noted that all the singularities we find are physical, as can be seen from the Ricci curvature scalar, which becomes infinite in each case.

These results are unsurprising; we expect the effect of the magnetic field to be greatest when $\rho_B$ is highest, which occurs at low values of $A$, and that of the cosmological constant to be greatest at late times, since its energy density is not diluted by cosmological expansion. We also expect magnetic pressure to decelerate collapse (or accelerate expansion) in the transverse direction, while magnetic tension accelerates collapse (or decelerates expansion) in the longitudinal direction, and the vacuum energy to drive accelerated expansion. Our results clearly follow the ‘rules of thumb’ for the effect of the magnetic field given by [19], which were outlined in section 3.

Finally, we find that the constant $c$ affects the initial rate of expansion, which is greater for higher values of $c$. In addition, the empty universe regime will become dominant at earlier times, and persist for longer, for larger values of $c$.

Since our model is homogenous and spatially flat, magnetic tension effects arise from the negative magnetic pressure along the field lines. In more general magnetic cosmologies there is also the possibility that tension arises from inhomogeneities. The implications of these other effects have been discussed elsewhere in the literature [24–27], but they are of a physically different origin and should not be confused with those we discuss here.

Our model cannot be compared directly to the observable universe, which contains significant quantities of dust and radiation as well as dark energy and which is not currently observed to have any significant anisotropy which would indicate the presence of a large-scale magnetic field. It is possible, however, that the universe may have contained a significant strength magnetic field at early times and so these results may have implications for the evolution of the universe during inflation, when the inflaton acts as an effective cosmological constant. It is clear from our results that the effect of the exponential expansion at late times is to drastically dilute the magnetic field, $\rho_B \to 0$, and to isotropize the initially highly
anisotropic universe. Similar processes may have operated in the early universe and hence any magnetic field created before or during inflation could have had a significant impact upon the dynamics of the universe before being diluted away in the same fashion.

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