Greatest solutions of equations in $\text{CLL}_R$ and its application

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Abstract

This paper explores the process calculus $\text{CLL}_R$ furtherly. First, we prove that for any equation $X =_{RS} t_X$ such that $X$ is strongly guarded in $t_X$, $\langle X | X = t_X \rangle$ is the largest solution w.r.t $\sqsubseteq_{RS}$. Second, we encode a fragment of action-based CTL in $\text{CLL}_R$.

Keywords: $\text{CLL}_R$, Solution of equations, Action-based CTL

1. Introduction

It is well-known that process algebra and temporal logic take different standpoint for looking at specifications and verifications of reactive and concurrent systems, and offer complementary advantages [16]. To take advantage of these two paradigms when designing systems, a few of theories for heterogeneous specifications have been proposed, e.g., [4, 5, 6, 8, 10, 11, 12, 15]. Among them, Lüttgen and Vogler propose the notion of logic labelled transition system (Logic LTS or LLTS for short), which combines operational and logical styles of specification in one unified framework [10, 11, 12]. In addition to usual process operators (e.g., CSP-style parallel composition, hiding, etc) and logic operators (disjunction and conjunction), some standard temporal logic operators, such as “always” and “unless”, are also integrated into this framework [12], which allows ones to freely mix operational and logic operators when designing systems.

Lüttgen and Vogler's approach is entirely semantic, and doesn't provide any kind of syntactic calculus. Recently, we propose a LLTS-oriented process calculus $\text{CLL}_R$, and establish the uniqueness of solutions of equations in $\text{CLL}_R$ under a certain circumstance [17].

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This paper works on CLL$_R$ furtherly. Our main contributions include:

(1) We will show that, without the assumption that $X$ does not occur in the scope of any conjunction in $t$, the given equation $X =_{RS} t$ may have more than one consistent solution. This answers conjecture in [17] negatively. Under the hypothesis that $X$ is strongly guarded in a given (open) term $t$, it is shown that the recursive process $\langle X | X = t \rangle$ is indeed the greatest (w.r.t $\sqsubseteq_{RS}$) consistent solution of the equation $X =_{RS} t$ whenever consistent solutions exist.

(2) We encode a temporal logic language action-based CTL [12] in CLL$_R$ so that safety properties could be described directly without resorting to complicated settings [12], which are used to embed temporal logic operators into LLTS.

The rest of this paper is organized as follows. The calculus CLL$_R$ and its semantics are recalled in the next section. Section 3 show that for any given equation $X =_{RS} t$ such that $X$ is strongly guarded in $t$, $\langle X | X = t \rangle$ is the greatest solution w.r.t $\sqsubseteq_{RS}$. We encode action-based CTL in Section 4. The paper is concluded with Section 5, where a brief discussion is given.

2. Preliminaries

The purpose of this section is to fix our notation and terminology, and to introduce some concepts that underlie our work in all other parts of the paper.

2.1. Logic LTS and ready simulation

Let $Act$ be the set of visible action names ranged over by $a$, $b$, etc., and let $Act_\tau$ denote $Act \cup \{\tau\}$ ranged over by $\alpha$ and $\beta$, where $\tau$ represents invisible actions. A labelled transition system with predicate is a quadruple $(P, Act_\tau, \rightarrow, F)$, where $P$ is a set of states, $\rightarrow \subseteq P \times Act_\tau \times P$ is the transition relation and $F \subseteq P$.

As usual, we write $p \xrightarrow{\alpha} q$ (or, $p \xrightarrow{\tau} q$) if $\exists q \in P. p \xrightarrow{\alpha} q$ ($\exists q \in P. p \xrightarrow{\tau} q$, resp.). The ready set $\{\alpha \in Act_\tau | p \xrightarrow{\alpha}\}$ of a given state $p$ is denoted by $I(p)$. A state $p$ is stable if $p \xrightarrow{\tau} q$. We also list some useful decorated transition relations:

- $p \xrightarrow{\alpha}_F q$ if $p \xrightarrow{\alpha} q$ and $p, q \notin F$;
- $p \xrightarrow{\gamma}_F q$ if $p(\xrightarrow{\gamma})^* q$, where $(\xrightarrow{\gamma})^*$ is the transitive and reflexive closure of $\xrightarrow{\gamma}$;
- $p \xrightarrow{\gamma}$ if $\exists r, s \in P. p \xrightarrow{\gamma} r \xrightarrow{\gamma} s \xrightarrow{\gamma} q$;
- $p \Rightarrow | q$ if $p \xrightarrow{\gamma}_F q$ with $\gamma \in Act_\tau \cup \{\epsilon\}$;
- $p \Rightarrow_F q$ if there exists a sequence of $\tau$-transitions from $p$ to $q$ such that all states along this sequence, including $p$ and $q$, are not in $F$; the decorated transition $p \Rightarrow_F q$ may be defined similarly;
- $p \Rightarrow_F | q$ if $p \Rightarrow_F q$ with $\gamma \in Act_\tau \cup \{\epsilon\}$.

Notice that the notation $p \Rightarrow q$ in [11,12] has the same meaning as $p \Rightarrow_F | q$ in this paper, while $p \Rightarrow | q$ in this paper does not involve any requirement on $F$-predicate.

**Definition 2.1 (Logic LTS [11]).** An LTS $(P, Act_\tau, \rightarrow, F)$ is an LLTS if, for each $p \in P$, 

\[\text{...}\]
\[(LTS1)\] \(p \in F\) if \(\exists \alpha \in I(p) \forall q \in P(p \xrightarrow{\alpha} q \text{ implies } q \in F)\);

\[(LTS2)\] \(p \in F\) if \(\nexists q \in P.p \xrightarrow{\tau} q\).

Moreover, an LTS \((P, Act, \rightarrow, F)\) is \(\tau\)-pure if, for each \(p \in P\), \(p \xrightarrow{\tau} \) implies \(\nexists a \in Act.p \xrightarrow{a}\).

Compared with usual LTSs, it is one distinguishing feature of LLTS that it involves consideration of inconsistencies. The main motivation behind such consideration lies in dealing with inconsistencies caused by conjunctive composition. Formally, the predicate \(F\) is used to denote the set of all inconsistent states that represent empty behaviour that cannot be implemented [12]. The condition (LTS1) formalizes the backward propagation of inconsistencies, and (LTS2) captures the intuition that divergence (i.e., infinite sequences of \(\tau\)-transitions) should be viewed as catastrophic. For more intuitive ideas and motivation about inconsistency, the reader may refer [10, 11].

A variant of the usual notion of weak ready simulation [1, 9] is adopted to capture the refinement relation in [11, 12]. It has been proven that such kind of ready simulation is the largest precongruence w.r.t parallel composition and conjunction which satisfies the desired property that an inconsistent specification can only be refined by inconsistent ones (see Theorem 21 in [11]).

**Definition 2.2 (Ready simulation on LLTS [11])**. Let \((P, Act, \rightarrow, F)\) be a LLTS. A relation \(R \subseteq P \times P\) is a stable ready simulation relation, if for any \((p, q) \in R\) and \(a \in Act\):

\[(RS1)\] both \(p\) and \(q\) are stable;
\[(RS2)\] \(p \notin F\) implies \(q \notin F\);
\[(RS3)\] \(p \xrightarrow{a} F\) \(|p'\) implies \(\exists q'q \xrightarrow{a} F\) \(|q'\) and \((p', q') \in R\);
\[(RS4)\] \(p \notin F\) implies \(I(p) = I(q)\).

We say that \(p\) is stable ready simulated by \(q\), in symbols \(p \sqsupseteq_{RS} q\), if there exists a stable ready simulation relation \(R\) with \((p, q) \in R\). Further, \(p\) is ready simulated by \(q\), written \(p \sqsupseteq_{RS} q\), if \(\forall p' (p \xrightarrow{\tau} F\) \(|p'\) implies \(\exists q'q \xrightarrow{\tau} F\) \(|q'\) and \(p' \sqsupseteq_{RS} q'\)). The kernels of \(\sqsubseteq_{RS}\) and \(\sqcup_{RS}\) are denoted by \(\approx_{RS}\) and \(=_{RS}\) resp.. It is easy to see that \(\approx_{RS}\) itself is a stable ready simulation relation and both \(\sqsubseteq_{RS}\) and \(\sqcup_{RS}\) are pre-order.

2.2. The calculus \(\text{CLL}_R\) and its operational semantics

This subsection introduces the LLTS-oriented process calculus \(\text{CLL}_R\) presented in [17]. Let \(V_{AR}\) be an infinite set of variables. The terms of \(\text{CLL}_R\) can be given by the following BNF grammar

\[t ::= 0 | \bot | (\alpha.t) | (t \boxtimes t) | (t \vee t) | (t \wedge t) | (t \parallel A t) | X | (Z|E)\]

where \(X \in V_{AR}, \alpha \in Act, A \subseteq Act\) and recursive specification \(E = E(V)\) with \(V \subseteq V_{AR}\) is a set of equations \(\{X = t|X \in V\}\) and \(Z\) is a variable in \(V\) that acts as the initial variable.
Most of these operators are from CCS \cite{13} and CSP \cite{7}: $0$ is the process capable of doing no action; \texttt{a}.$t$ is action prefixing; $\boxempty$ is non-deterministic external choice; $\parallel$ is a CSP-style parallel composition. $\perp$ represents an inconsistent process with empty behavior. $\lor$ and $\land$ are logical operators, which are intended for describing logical combinations of processes.

For any term $\langle Z|E \rangle$ with $E = E(V)$, each variable in $V$ is bound with scope $E$. This induces the notion of free occurrence of variable, bound (and free) variables and $\alpha$-equivalence as usual. A term $t$ is a process if it is closed, that is, it contains no free variable. The set of all processes is denoted by $T(\Sigma_{\text{CLL}_R})$.

Unless noted otherwise we use $p,q,r$ to represent processes. Throughout this paper, as usual, we assume that recursive variables are distinct from each other and no recursive variable has free occurrence; moreover we don’t distinguish between $\alpha$-equivalent terms and use $\equiv$ for both syntactical identical and $\alpha$-equivalence. In the sequel, we often denote $\langle X|\{X = t_X\} \rangle$ briefly by $\langle X|X = t_X \rangle$.

For any recursive specification $E(V)$ and term $t$, the term $\langle t|E \rangle$ is obtained from $t$ by simultaneously replacing all free occurrences of each $X \in V$ by $\langle X|E \rangle$, that is, $\langle t|E \rangle = t\{X|E\}/X : X \in V \}$. For example, consider $t \equiv X \cdot a.\langle Y|Y = X \boxempty Y \rangle$ and $E(\{X\}) = \{X = t_X\}$ then $\langle t|E \rangle \equiv \langle X|X = t_X \rangle \cdot a.\langle Y|Y = \langle X|X = t_X \rangle \boxempty Y \rangle$. In particular, for any $E(V)$ and $t \equiv X$, $\langle t|E \rangle \equiv \langle X|E \rangle$ whenever $X \in V$ and $\langle t|E \rangle \equiv X$ if $X \notin V$.

A context $C_{\tilde{X}}$ is a term whose free variables are in some $n$-tuple distinct variables $\tilde{X} = (X_1, \ldots, X_n)$ with $n \geq 0$. Given $\tilde{p} = (p_1, \ldots, p_n)$, the term $C_{\tilde{X}}\{p_1/X_1, \ldots, p_n/X_n\}$ ($C_{\tilde{X}}\{\tilde{p}/\tilde{X}\}$ for short) is obtained from $C_{\tilde{X}}$ by replacing $X_1$ by $p_1$ for each $i \leq n$ simultaneously.

An occurrence of $X$ in $t$ is strongly (or, weakly) guarded if such occurrence is within some subexpression $a.t_1$ with $a \in \text{Act}$ ($\tau.t_1$ or $t_1 \lor t_2$ resp.). A variable $X$ is strongly (or, weakly) guarded in $t$ if each occurrence of $X$ is strongly (weakly resp.) guarded. A recursive specification $E(V)$ is guarded if for each $X \in V$ and $Z = t_2 \in E(V)$, each occurrence of $X$ in $t_2$ is (weakly or strongly) guarded. As usual, we assume that all recursive specifications considered in the remainder of this paper are guarded. SOS rules of $\text{CLL}_R$ are listed in Table \ref{tab:cllr} where $a \in \text{Act}$, $\alpha \in \text{Act}_\tau$ and $A \subseteq \text{Act}$. All rules are divided into two parts:

Operational rules specify behaviours of processes. Negative premises in Rules $R_{a2}$, $R_{a3}$, $R_{a13}$ and $R_{a14}$ give $\tau$-transition precedence over visible transitions, which guarantees that the transition model of $\text{CLL}_R$ is $\tau$-pure. Rules $R_{a9}$ and $R_{a10}$ illustrate that the operational aspect of $t_1 \lor t_2$ is same as internal choice in usual process calculus. Rule $R_{a6}$ reflects that conjunction operator is a synchronous product for visible transitions. The operational rules of the other operators are as usual.

Predicate rules specify the inconsistency predicate $F$. Rule $Rp_1$ says that $\perp$ is inconsistent. Hence $\perp$ cannot be implemented. While $0$ is consistent and implementable. Thus $0$ and $\perp$ represent different processes. Rule $Rp_3$ reflects that if both two disjunctive parts are inconsistent then so is the disjunction. Rules $Rp_4 - Rp_9$ describe the system design strategy that if one part is incon-
sistent, then so is the whole composition. Rules $R_{p10}$ and $R_{p11}$ reveal that a stable conjunction is inconsistent whenever its conjuncts have distinct ready sets. Rules $R_{p13}$ and $R_{p15}$ are used to capture (LTS2) in Def. 2.1. Intuitively, these two rules say that if all stable $\tau$-descendants of $z$ are inconsistent, then $z$ itself is inconsistent.

**Operational rules**

\[
\begin{align*}
R_{a1} & : \alpha, x_1 \to x_1 \\
R_{a4} & : x_1 \sqcap x_2 \to y_1 \sqcap x_2 \\
R_{a7} & : x_1 \sqcap x_2 \to y_1 \sqcap x_2 \\
R_{a9} & : x_1 \lor x_2 \to x_1 \\
R_{a11} & : x_1 \parallel A x_2 \to y_1 \parallel A x_2 \\
R_{a13} & : x_1 \parallel A x_2 \to y_1 \parallel A x_2 \\
R_{a15} & : x_1 \parallel A x_2 \to y_1 \parallel A x_2 \\
R_{a2} & : x_1 \to y_1, x_2 \to y_2 \\
R_{a5} & : x_1 \to y_1, x_2 \to y_2 \\
R_{a8} & : x_1 \lor x_2 \to x_1 \lor x_2 \\
R_{a10} & : x_1 \lor x_2 \to x_1 \lor x_2 \\
R_{a12} & : x_1 \parallel A x_2 \to x_1 \parallel A x_2 \\
R_{a14} & : x_1 \parallel A x_2 \to x_1 \parallel A x_2 \\
R_{a16} & : (t_X | E) \to y (X = t_X \in E)
\end{align*}
\]

**Predicative rules**

\[
\begin{align*}
R_{p1} & : F \to (t_F) \\
R_{p4} & : x_1 F \to x_2 F \\
R_{p7} & : x_1 \parallel A x_2 F \\
R_{p10} & : x_1 \to y_1, x_2 \to y_2, x_1 \lor x_2 \to y_2 \\
R_{p12} & : x_1 \lor x_2 \to y_1 \lor x_2 \to y_2 \\
R_{p14} & : (t_X | E) F (X = t_X \in E)
\end{align*}
\]

\[
\begin{align*}
R_{p2} & : x_1 F, x_2 F \to x_1 \parallel x_2 F \\
R_{p5} & : x_1 \parallel x_2 F \to x_1 \parallel x_2 F \\
R_{p8} & : x_1 \parallel x_2 F \\
R_{p11} & : x_1 \parallel x_2 F \\
R_{p13} & : \{y : x_1 \land x_2 \to y\}
\end{align*}
\]

Table 1: SOS rules of $\text{CLL}_R$

It has been shown that $\text{CLL}_R$ has the unique stable transition model $M_{\text{CLL}_R}$ \[17\], which exactly consists of all positive literals of the form $t \Rightarrow t'$ or $tF$ that are provable in $\text{Strip}(\text{CLL}_R, M_{\text{CLL}_R})$. Here $\text{Strip}(\text{CLL}_R, M_{\text{CLL}_R})$ is the stripped version \[2\] of $\text{CLL}_R$ w.r.t $M_{\text{CLL}_R}$. Each rule in $\text{Strip}(\text{CLL}_R, M_{\text{CLL}_R})$ is of the form $\text{pprem}(r)$ for some ground instance $r$ of rules in $\text{CLL}_R$ such that $M_{\text{CLL}_R} \vDash \text{pprem}(r)$, where $\text{pprem}(r)$ (or, $\text{pprem}(r)$) is the set of negative (positive resp.)
a consistent solution of this equation. First we show that a proof tree of Strip\( a.X \)
\( (6) \)

Theorem 2.4. (Unique solution). For any \( p, q / \) and \( R \)

\( 0 \)

More on solutions of equations in CLL

Observation 3.1. Consider the equation \( X = t_X \) where \( t_X \equiv ((Y | Y = a.Y) \land a.X) \lor ((Z | Z = b.Z) \land b.X) \). In the following, we show that \( X | X = a.X \) is a consistent solution of this equation. First we show that \( X | X = a.X \in F \). Contrarily, assume that \( X | X = a.X \in F \). Then the last rule applied in the proof tree of \( \text{Strip}(CLL_R, M_{CLL_R}) \vdash X | X = a.X F \) is

\[
\frac{a. (X | X = a.X) F \quad \{r F : (X | X = a.X) \not\models r\}}{(X | X = a.X) F}
\]

Lemma 2.3. Let \( p \) and \( q \) be any two processes. Then

\( (1) \) \( p \lor q \in F \) if \( p, q \in F \);

\( (2) \) \( p \land q \in F \) if \( p \in F \) for each \( \alpha \in \text{Act}_\tau \);

\( (3) \) \( p \land q \in F \) if \( p \not\in F \) or \( q \not\in F \) with \( \land \in \{ \ulcorner, \urcorner \} \);

\( (4) \) \( p \not\in F \) or \( q \not\in F \) implies \( p \land q \in F \);

\( (5) \) \( 0 \not\in F \) and \( \bot \not\in F \);

\( (6) \) \( \langle X | E \rangle \in F \) iff \( \langle X | E \rangle \in F \) for each \( X \) with \( X = t_X \in E \).

Theorem 2.4. \( \text{LTS}(CLL_R) \) is a \( \tau \)-pure LLTS. Moreover if \( p \in F \) and \( \tau \in \mathcal{I}(p) \) then \( \forall q (p \not\rightarrow q \implies q \in F) \).

Theorem 2.5 (precongruence). If \( p \not\in RS q \) then \( C_X \{p/X\} \not\in RS C_X \{q/X\} \).

3. More on solutions of equations in CLL

In [17], the following theorem has been obtained.

Theorem (Unique solution). For any \( p, q \not\in F \) and \( t_X \) where \( X \) is strongly guarded and does not occur in the scope of any conjunction, if \( p = RS t_X \{p/X\} \) and \( q = RS t_X \{q/X\} \) then \( p = RS q \). Moreover \( X | X = t_X \) is the unique consistent solution (modulo \( = RS \)) of the equation \( X = RS t_X \) whenever consistent solutions exist.

As we know, temporal operators could be described in equational style, represented by fixpoint of some equations [3]. Such style requires us to remove the special requirement (i.e. \( X \) does not occur in the scope of any conjunction) occurring in Theorem Unique Solution. In the following, we give a negative answer for this removement by providing a counterexample:

Observation 3.1. Consider the equation \( X = t_X \) where \( t_X \equiv ((Y | Y = a.Y) \land a.X) \lor ((Z | Z = b.Z) \land b.X) \). In the following, we show that \( X | X = a.X \) is a consistent solution of this equation. First we show that \( X | X = a.X \not\in F \). Contrarily, assume that \( X | X = a.X \in F \). Then the last rule applied in the proof tree of \( \text{Strip}(CLL_R, M_{CLL_R}) \vdash X | X = a.X F \) is

\[
\frac{a. (X | X = a.X) F \quad \{r F : (X | X = a.X) \not\models r\}}{(X | X = a.X) F}
\]
It is not difficult to see that every proof tree of \( \langle X \mid X = a.X \rangle F \) has proper sub-tree with root \( \langle X \mid X = a.X \rangle F \), this contradicts the well-foundedness of proof tree, as desired. Second we show that \( \langle X \mid X = a.X \rangle \) indeed is a solution of \( X = RS t_X \). Clearly, due to Rules \( Rp_{10} \) and \( Rp_{11} \), \( \langle Z \mid Z = b.Z \rangle \wedge \langle X \mid X = a.X \rangle \in F \), which is the unique \( b \)-derivative of \( \langle Z \mid Z = b.Z \rangle \wedge b.\langle X \mid X = a.X \rangle \). Hence \( \langle Z \mid Z = b.Z \rangle \wedge b.\langle X \mid X = a.X \rangle \) \( \in F \) by Condition (LTS1) in Def. \( \ref{def:2.1} \) and Theorem \( \ref{thm:2.3} \). Moreover we also have \( \langle X \mid X = a.X \rangle =_{RS} \langle Y \mid Y = a.Y \rangle \wedge a.\langle X \mid X = a.X \rangle \). Therefore \( \langle X \mid X = a.X \rangle =_{RS} t_X \langle X \mid X = a.X \rangle /X \rangle \). Similarly, \( \langle X \mid X = b.X \rangle \) is another consistent solution. However, \( \langle X \mid X = a.X \rangle \neq_{RS} \langle X \mid X = b.X \rangle \).

In the remainder of this section, we intend to show that the recursive process \( \langle X \mid X = t \rangle \) captures the extreme solution of the equation \( X = t \). To this end, a number of results in \( \ref{ref:17} \) are listed below.

**Lemma 3.2.** If \( C_X \langle p/X \rangle \rightarrow r \) then

1. either there exists \( C'_X \) such that \( r \equiv C'_X \langle p/X \rangle \) and \( C_X \langle q/X \rangle \rightarrow C'_X \langle q/X \rangle \) for any \( q \),
2. or there exist \( C'_{X,Z} \) and \( p' \) such that \( p \rightarrow p' \), \( r \equiv C'_{X,Z} \langle p/X, p'/X \rangle \) and \( C_X \langle q/X \rangle \rightarrow C'_{X,Z} \langle q/X, q'/Z \rangle \) for any \( q \rightarrow q' \).

**Lemma 3.3.** Let \( a \in Act \). If \( C_X \langle p/X \rangle \rightarrow r \) then there exists \( C'_{X,Y} \) such that

1. \( r \equiv C'_{X,Y} \langle p/X, p'/Y \rangle \) for some \( p' \) with \( p \rightarrow p' \) for each \( Y \in \tilde{Y} \), and
2. if \( C_X \langle q/X \rangle \) is stable and for each \( Y \in \tilde{Y} \), \( q \rightarrow q'_Y \), then \( C_X \langle q/X \rangle \rightarrow C'_{X,Y} \langle q/X, q'_Y /Y \rangle \).

**Lemma 3.4.** Let \( X \) be guarded in \( C_X \). If \( C_X \langle p/X \rangle \rightarrow r \) then there exists \( B_X \) such that \( r \equiv B_X \langle p/X \rangle \) and \( C_X \langle q/X \rangle \rightarrow B_X \langle q/X \rangle \) for any \( q \).

**Lemma 3.5.** If \( C_X \langle p/X \rangle \rightarrow r \) then there exist stable \( C'_{X,Y} \) and stable \( p'_Y \) for each \( Y \in \tilde{Y} \) such that

1. \( p \rightarrow \leftarrow p' \) for each \( Y \in \tilde{Y} \) and \( r \equiv C'_{X,Y} \langle p/X, p'_Y /Y \rangle \);
2. for any \( q \) such that \( q \rightarrow \leftarrow q'_Y \) and \( q \rightarrow \leftarrow q'_Y \) for each \( Y \in \tilde{Y} \) then \( C_X \langle q/X \rangle \rightarrow \leftarrow C'_{X,Y} \langle q/X, q'_Y /Y \rangle \); (3) if \( X \) is strongly guarded in \( C_X \) then so is it in \( C'_{X,Y} \), and \( \tilde{Y} = \emptyset \).

Before giving the main result of this section, we prove a lemma concerning \( F \)-predicate.

**Lemma 3.6.** If \( X \) is strongly guarded in \( t_X \) and \( p \subseteq_{RS} t_X \{p/X\} \) then for any \( C_Y \), \( C_Y \{t_X \{p/X\}/Y\} \notin F \) implies \( C_Y \{\langle X \mid X = t_X \rangle /Y\} \notin F \).

**Proof.** Clearly, by Lemmas \( \ref{lem:3.2} \), \( \ref{lem:3.3} \), and \( \ref{lem:3.4} \), we get

\[
C_Y \{t_X \{p/X\}/Y\} \rightarrow \text{iff } C_Y \{\langle X \mid X = t_X \rangle /Y\} \rightarrow \text{ for any } C_Y. \tag{3.6.1}
\]

Set \( \Omega \triangleq \{B_Y \{\langle X \mid X = t_X \rangle /Y\} : B_Y \{t_X \{p/X\}/Y\} \notin F\} \). Clearly, it suffices to prove that \( F \cap \Omega = \emptyset \). Conversely, suppose that \( F \cap \Omega \neq \emptyset \). Due to the well-foundedness of proof trees, to complete the proof, it is sufficient to show that,
for each $C_Y \{ \langle X | X = t_X \rangle / Y \} \in \Omega$, any proof tree for $\text{Strip}(\text{CLL}, M_{\text{CLL},r}) \vdash C_Y \{ \langle X | X = t_X \rangle / Y \} F$ has a proper subtree with root $sF$ for some $s \in \Omega$. We shall prove this as follows. Let $\mathcal{T}$ be any proof tree of $C_Y \{ \langle X | X = t_X \rangle / Y \} F$. It is a routine case analysis based on the last rule applied in $\mathcal{T}$. We treat only non-trivial three cases and leave the others to the reader.

Case 1. $C_Y \equiv Y$.

Then $C_Y \{ \langle X | X = t_X \rangle / Y \} \equiv \langle X | X = t_X \rangle$. So the last rule applied in $\mathcal{T}$ is $\langle (X | X = t_X) \rangle F$ or $\langle (X | X = t_X) \rangle \vdash t_X F$.

For the former, since $C_Y \{ t_X \{ p / X \} / Y \} \not\equiv \langle X | X = t_X \rangle$, we have $t_X \{ t_X \{ p / X \} / X \} \not\equiv F$ due to Theorem 2.5. Hence $\langle t_X | X = t_X \rangle \equiv t_X \{ \langle X | X = t_X \rangle / X \} \not\equiv F$. For the latter, we treat the non-trivial subcase that $\langle X | X = t_X \rangle \not\equiv t_X$. Since $t_X \{ p / X \} \not\equiv F$, $t_X \{ p / X \} \not\equiv t_X \{ \langle X | X = t_X \rangle / X \}$ for some $s$. For this transition, since $X$ is strongly guarded in $t_X$, by Lemma 3.5 there exist a stable $t_X'$ with strongly guarded $X$ such that $s \equiv t_X \{ p / X \}$ and $t_X \{ \langle X | X = t_X \rangle / X \} \not\equiv t_X \{ \langle X | X = t_X \rangle / \Omega \}$. Further, by Lemma 3.3 $\langle X | X = t_X \rangle \not\equiv t_X \{ \langle X | X = t_X \rangle / X \}$ due to $\langle X | X = t_X \rangle \not\equiv t_X$. Moreover $t_X \{ t_X \{ p / X \} / X \} \not\equiv F$ because of $s \equiv t_X \{ p / X \} \not\equiv F$ and $p \equiv t_X \{ t_X \{ p / X \} / X \}$. Hence $t_X \{ \langle X | X = t_X \rangle / X \} \in \Omega$, as desired.

Case 2. $C_Y \equiv \langle Z | E \rangle$.

The last rule applied in $\mathcal{T}$ is one of following two cases: $\langle (Z | E) \{ \langle X | X = t_X \rangle / Y \} F \rangle$ or $\langle (Z | E) \{ \langle X | X = t_X \rangle / Y \} \not\equiv \langle (Z | E) \{ (X | X = t_X) / Y \} F \rangle$.

By Lemma 2.3(6), the former is easy to handle and omitted. Next we treat the latter. Since $C_Y \{ t_X \{ p / X \} / Y \} \not\equiv F$, $C_Y \{ t_X \{ p / X \} / Y \} \not\equiv t_X \{ \langle Z | E \rangle / X \} \not\equiv F$ for some $s$. For this transition, by Lemma 3.5 there exist stable $C_{Y,W}^{t_X}$ and $s_{W}^{t_X}$ such that $s \equiv C_{Y,W}^{t_X} \{ t_X \{ p / X \} / Y, s_{W}^{t_X} / W \}$ and $t_X \{ p / X \} \not\equiv s_{W}^{t_X}$ for each $W \in \tilde{W}$. Further, for each $t_X \{ p / X \} \not\equiv s_{W}^{t_X}$, there exists stable $t_X^{W}$ with strongly guarded $X$ such that $s_{W}^{t_X} \equiv t_X^{W} \{ p / X \}$ and $t_X \{ \langle X | X = t_X \rangle / X \} \not\equiv t_X^{W} \{ \langle X | X = t_X \rangle / \Omega \}$. So, by Lemma 3.4 $\langle X | X = t_X \rangle \not\equiv t_X^{W} \{ \langle X | X = t_X \rangle / X \}$ for each $W \in \tilde{W}$ and hence $C_Y \{ \langle X | X = t_X \rangle / Y \} \not\equiv C_{Y,W}^{t_X} \{ \langle X | X = t_X \rangle / Y, t_X^{W} \{ \langle X | X = t_X \rangle / \tilde{W} \} \} \equiv u$.

Since $s \equiv C_{Y,W}^{t_X} \{ t_X \{ p / X \} / Y, t_X^{W} \{ p / X \} / \tilde{W} \} \not\equiv F$ and $p \equiv t_X \{ t_X \{ p / X \} / Y \} \not\equiv F$, we get $C_{Y,W}^{t_X} \{ t_X \{ p / X \} / Y, t_X^{W} \{ t_X \{ p / X \} / \tilde{W} \} \not\equiv F$, which implies $u \in \Omega$, as desired.

Case 3. $C_Y \equiv B_Y \wedge D_Y$.

We split the argument into the following four subcases.

Case 3.1. $B_Y \{ \langle X | X = t_X \rangle / Y \} F$.

Since $C_Y \{ t_X \{ p / X \} / Y \} \not\equiv F$, $B_Y \{ t_X \{ p / X \} / Y \} \not\equiv F$ by Lemma 2.3. So, $B_Y \{ \langle X | X = t_X \rangle / Y \} \not\equiv F$, as desired.
Case 3.2. \( \frac{D_Y(Y)}{C_Y(Y)} \) with \( D_Y(Y) \neq \emptyset \) and \( C_Y(Y) \neq \emptyset \).

By (3.6.1), a contradiction arises due to \( C_Y(p/X) \not\in F \).

Case 3.3. \( \frac{r \in F \{ C_Y(Y) \} \rightarrow r'}{C_Y(Y)} \).

Similar to the second case of Case 2, omitted.

Case 3.4. \( \frac{C_Y(Y)}{C_Y(Y) \rightarrow F s} \) for some \( s \).

In the following, we treat two cases based on \( \alpha \).

Case 3.4.1. \( \alpha = \tau \).

For (3.6.2), by Lemma 3.52, either \( s \equiv C'_\tau \{ p/X \} \) for some \( C'_\tau \) such that \( C'_\tau \{ q/Y \} \rightarrow C'_\tau \{ q/Y \} \) for any \( q \), or there exist \( s' \) and \( C'_{\tau, Z} \) such that \( s \equiv C'_{\tau, Z} \{ p/X \} \) and \( C'_{\tau, Z} \{ p/X \} \rightarrow s' \). For the former, it is trivial.

Next we treat the later. For \( t_X(p/X) \rightarrow s' \), since \( X \) is strongly guarded in \( t_X \), by Lemma 3.3 there exists \( t'_X \) such that \( s' \equiv t'_X \{ p/X \} \) and \( t_X \{ X = t_X \} \rightarrow t'_X \{ X = t_X \} \). Then \( (X = t_X) \rightarrow t'_X \{ X = t_X \} \) and hence \( C_Y \{ X = t_X \} \rightarrow C'_{\tau, Z} \{ X = t_X \} \). Since \( p \in RS t_X \{ p/X \} \) and \( s \equiv C'_{\tau, Z} \{ p/X \} \not\in F \), we get \( C'_{\tau, Z} \{ p/X \} \not\in F \). Clearly, \( u \in \Omega \), as desired.

Case 3.4.2. \( \alpha \in Act \).

For (3.6.2), by Lemma 3.3, \( s \equiv C'_{\tau, \bar{Z}} \{ p/X \}, \bar{s}_Z \) for some \( C'_{\tau, \bar{Z}} \) and \( \bar{s}_Z \) such that \( t_X \{ p/X \} \rightarrow \bar{s}_Z \) for each \( Z \in \bar{Z} \). Since \( X \) is strongly guarded in \( t_X \), for each \( t_X \{ p/X \} \rightarrow \bar{s}_Z \), by Lemma 3.3 there exists \( t'_X \) such that \( s'_Z \equiv t'_X \{ p/X \} \) and \( t_X \{ X = t_X \} \rightarrow t'_X \{ X = t_X \} \). Then \( (X = t_X) \rightarrow t'_X \{ X = t_X \} \) for each \( Z \in \bar{Z} \) and hence \( C_Y \{ X = t_X \} \rightarrow C'_{\tau, \bar{Z}} \{ X = t_X \} \). Since \( p \in RS t_X \{ p/X \} \) and \( s \equiv C'_{\tau, \bar{Z}} \{ p/X \} \not\in F \), by Theorem 2.9 we get \( C'_{\tau, \bar{Z}} \{ p/X \} \not\in F \). Clearly, \( u \in \Omega \), as desired.

Next we recall an equivalent formulation of \( \sqsubseteq_{RS} \) and an up-to technique.

**Definition 3.7.** A relation \( R \subseteq T(S_{CLL_\bar{R}}) \times T(S_{CLL_\bar{R}}) \) is an alternative ready simulation relation, if for any \( (p, q) \in R \) and \( a \in Act \)

(RSi) \( p \triangleleft_F p' \) implies \( q \triangleleft_F q' \) and \( (p', q') \in R \);

(RSi) \( p \triangleleft_F p' \) and \( p, q \) stable implies \( q \triangleleft_F q' \) and \( (p', q') \in R \);

(RSi) \( p \not\in F \) and \( p, q \) stable implies \( I(p) = I(q) \).

We write \( p \sqsubseteq_{ALT} q \) if there exists an alternative ready simulation relation \( R \) with \( (p, q) \in R \).
Definition 3.8 (ALT up to ≰RS). A relation \( R \subseteq T(\Sigma_{CLL}) \times T(\Sigma_{CLL}) \) is an alternative ready simulation relation up to \( \sqsubseteq \), if for any \((p,q) \in R\) and \( a \in \text{Act}\):

(ALT-upo-1) \( p \xrightarrow{a}_F | p' \) implies \( \exists q'. q \xrightarrow{a}_F | q' \) and \( p' \sqsubseteq _{RS} R \sqsubseteq _{RS} q' \);

(ALT-upo-2) \( p \xrightarrow{a}_F | p' \) and \( p,q \) stable implies \( \exists q'. q \xrightarrow{a}_F | q' \) and \( p' \sqsubseteq _{RS} R \sqsubseteq _{RS} q' \);

(ALT-upo-3) \( p \notin F \) and \( p,q \) stable implies \( I(p) = I(q) \).

It has been proved that \( \sqsubseteq _{RS} = \sqsubseteq _{ALT} \) and if \( R \) is an alternative ready simulation relation up to \( \sqsubseteq \), then \( R \sqsubseteq _{RS} \). With these results, we could prove the next lemma.

Lemma 3.9. Let \( X \) be strongly guarded in \( t_X \). If \( p \sqsubseteq _{RS} t_X\{p/X\} \) then \( t_X\{p/X\} \sqsubseteq _{RS} \langle X | X = t_X \rangle \).

Proof. Set \( R \triangleq \{(By\{t_X\{p/X\}/Y\},By\{\langle X | X = t_X \rangle /Y\}\}\}. It is sufficient to prove that \( R \) is an alternative ready simulation relation up to \( \sqsubseteq _{RS} \). Let \( \langle C_Y\{t_X\{p/X\}/Y\},C_Y\{\langle X | X = t_X \rangle /Y\}\rangle \in R \). By Lemma 3.5 and 3.6 (ALT-upo-3) holds clearly. Next we handle the other two clauses.

(ALT-upo-1) Assume \( C_Y\{t_X\{p/X\}/Y\} \xrightarrow{a}_F | s \). For this transition, by Lemma 3.5 \( s = C'_{Y,Z}\{t_X\{p/X\}/Y,s'_Z/Z\} \) for some stable \( C'_{Y,Z} \) and \( s'_Z \) such that \( t_X\{p/X\} \xrightarrow{a}_F | s'_Z \) for each \( Z \in \tilde{Z} \). Further, for each \( t_X\{p/X\} \xrightarrow{a}_F | s'_Z \), since \( X \) is strongly guarded in \( t_X \), there exists stable \( t'_X \) with strongly guarded \( X \) such that \( s'_Z \equiv t'_X\{p/X\} \) and \( t_X\{\langle X | X = t_X \rangle /X\} \xrightarrow{a}_F | t'_X\{\langle X | X = t_X \rangle /X\} \). So \( \langle X | X = t_X \rangle \xrightarrow{a}_F | t'_X\{\langle X | X = t_X \rangle /X\} \) for each \( Z \in \tilde{Z} \) and hence \( C_Y\{\langle X | X = t_X \rangle /Y\} \xrightarrow{a}_F | C'_{Y,Z}\{\langle X | X = t_X \rangle /Y,t'_X\{\langle X | X = t_X \rangle /X\}/Z\} \equiv u \). Since \( s \equiv C'_{Y,Z}\{t_X\{p/X\}/Y,t'_X\{\langle X | X = t_X \rangle /X\}/\tilde{Z}\} \notin F \) and \( p \sqsubseteq _{RS} t_X\{p/X\} \), by Lemma 3.4 and Theorem 2.3 we obtain \( s \sqsubseteq _{RS} C'_{Y,Z}\{t_X\{p/X\}/Y,t'_X\{\langle X | X = t_X \rangle /X\}/\tilde{Z}\} \notin F \), which implies \( u \notin F \) by Lemma 3.6. Clearly \( C_Y\{\langle X | X = t_X \rangle /Y\} \xrightarrow{a}_F | u \) by Lemma 2.4 and \( s \sqsubseteq _{RS} \tilde{R}u \), as desired.

(ALT-upo-2) Assume that \( C_Y\{t_X\{p/X\}/Y\} \) and \( C_Y\{\langle X | X = t_X \rangle /Y\} \) are stable and \( C_Y\{t_X\{p/X\}/Y\} \xrightarrow{a}_F | s \). Then \( C_Y\{t_X\{p/X\}/Y\} \xrightarrow{a}_F r \xrightarrow{a}_F | s \) for some \( r \). For the \( a \)-transition, by Lemma 3.5 \( r \equiv C'_{Y,Z}\{t_X\{p/X\}/Y,r'_Z/Z\} \) for some \( C'_{Y,Z} \) and \( r'_Z \) such that \( t_X\{p/X\} \xrightarrow{a} r'_Z \) for each \( Z \in \tilde{Z} \). Since \( X \) is strongly guarded in \( t_X \), for each \( t_X\{p/X\} \xrightarrow{a}_F r'_Z \), by Lemma 3.4 there exists \( t'_X \) such that \( r'_Z \equiv t'_X\{p/X\} \) and \( t_X\{\langle X | X = t_X \rangle /X\} \xrightarrow{a}_F | t'_X\{\langle X | X = t_X \rangle /X\} \). Then \( \langle X | X = t_X \rangle \xrightarrow{a}_F | t'_X\{\langle X | X = t_X \rangle /X\} \) for each \( Z \in \tilde{Z} \) and hence \( C_Y\{\langle X | X = t_X \rangle /Y\} \xrightarrow{a}_F | C'_{Y,Z}\{\langle X | X = t_X \rangle /Y,t'_X\{\langle X | X = t_X \rangle /X\}/\tilde{Z}\} \equiv v \). Let \( u \equiv C'_{Y,Z}\{t_X\{p/X\}/Y,t'_X\{\langle X | X = t_X \rangle /X\}/\tilde{Z}\} \). Since \( p \sqsubseteq _{RS} t_X\{p/X\} \), by
Theorem 2.35: We have \( r \equiv C'_{Y, Z} \{ p/X \} / Y, t'_{Y, Z} \{ p/X \} / Z \} \subset_{RS} u \). Hence since \( r \Rightarrow_F s \), we have \( u \Rightarrow_F t \) and \( s \subset_{RS} t \) for some \( t \). Since \( u \Rightarrow_F \), by (ALT-upto-1), \( v \Rightarrow_F \) for some \( t' \) such that \( t \subset_{RS} R \subset_{RS} t' \). Therefore, by Lemma 3.6, \( C_Y \{ (X|X = t_X)/Y \} \Rightarrow_F \) \( t' \) and \( s \subset_{RS} R \subset_{RS} t' \).

Now with the previous lemma, it is not difficult to get

**Theorem 3.10.** For any equation \( X =_{RS} t_X \) such that \( X \) is strongly guarded in \( t_X \), if consistent solution exists then \( (X|X = t_X) \) is the greatest consistent solution.

### 4. Encoding ACTL in CLL

In [12], Lüttgen and Vogler introduce a fragment of action-based CTL (ACTL for short), embed it into LLTS and present the desired compatibility result between logical satisfaction and \( \sqsubseteq_{RS} \). In this section, we recall their ACTL and encode it in CLL under the hypothesis that \( Act \) is finite.

**Definition 4.1.** The action-based CTL is defined by BNF:

\[
\phi ::= tt \mid ff \mid en(a) \mid dis(a) \mid \phi \lor \phi \mid \phi \land \phi \mid [a] \phi \mid \mathcal{A} \phi \mid \phi W \phi
\]

where \( a \in Act \). \( T(\Sigma_{ACTL}) \) denotes the set of all terms in ACTL.

\( en(a) \) and \( dis(a) \) denote enabledness and disabledness of action \( a \) resp. \( [a] \), \( \mathcal{A} \) and \( W \) are usual next, always and weak until operators. For more motivations and intuitions about these operators, the reader may refer to [12].

Before encoding formulas of ACTL in CLL, we introduce some useful notations. Given \( n \) terms \( t_i (0 \leq i \leq n-1) \) in \( T(\Sigma_{CLL}) \), the general external choice \( \bigvee_{i<n} t_i \) and disjunction \( \bigvee_{i<n} t_i \) are defined recursively as:

\[
\square t_i \triangleq 0, \square t_i \triangleq t_0, \text{ and } \square t_i \triangleq (\square t_i) \square t_k \text{ for } k \geq 1;
\]

\[
\bigvee_{i<n} t_i \triangleq t_0, \text{ and } \bigvee_{i<n} t_i \triangleq (\bigvee_{i<n} t_i) \lor t_k \text{ for } k \geq 1.
\]

The general conjunction \( \bigwedge_{i<n} t_i \) is defined similarly as disjunction.

Given a term \( \phi \) in \( T(\Sigma_{ACTL}) \), the encoding of \( \phi \), denoted by \( \mathcal{E}(\phi) \), is defined as:

\[
\mathcal{E}(tt) \triangleq (X|X = \bigvee_{A \subseteq Act, a \in A} \Box a.X) \quad \mathcal{E}(ff) \triangleq \bot
\]

\[
\mathcal{E}(en(a)) \triangleq \bigvee_{a \in A \subseteq Act, b \in A} \Box b.\mathcal{E}(tt) \quad \mathcal{E}(dis(a)) \triangleq \bigvee_{a \notin A \subseteq Act, b \in A} \Box b.\mathcal{E}(tt)
\]

\[
\mathcal{E}([a] \phi) \triangleq [a](\mathcal{E}(\phi)) \quad \mathcal{E}(\phi_1 \lor \phi_2) \triangleq \mathcal{E}(\phi_1) \lor \mathcal{E}(\phi_2)
\]

\[
\mathcal{E}(\phi_1 \land \phi_2) \triangleq \mathcal{E}(\phi_1) \land \mathcal{E}(\phi_2) \quad \mathcal{E}(\mathcal{A} \phi) \triangleq (X|X = \mathcal{E}(\phi) \land (\bigwedge_{a \in Act} [a](X)))
\]

\[
\mathcal{E}(\phi_1 W \phi_2) \triangleq (X|X = \mathcal{E}(\phi_2) \lor (\mathcal{E}(\phi_1) \land (\bigwedge_{a \in Act} [a](X))))
\]
where \([a] \triangleq \lambda x. (\bigvee_{a \in A \subseteq Act} (\Box b. E(tt)) \Box a.x)) \lor (\bigvee_{a \notin A \subseteq Act} (\Box b. E(tt)))\), intuitively, \([a]\) says “along \(a\)-transition, it is necessary that . . . ”.

Therefore, if we want to check a specification \(p \in T(\Sigma_{CLL_R})\) satisfies some desired property \(\phi \in T(\Sigma_{ACTL})\), we only check whether \(p \sqsubseteq_{RS} E(\phi)\) or \(p \land E(\phi) =_{RS} \bot\) holds.

**Theorem 4.2.** \(p \models \phi\) if and only if \(p \sqsubseteq_{RS} E(\phi)\).

5. Conclusions and discussion

This paper works on LLTS-oriented process calculus CLL\(_R\) furtherly. We show that for any given equation \(X =_{RS} t\) such that \(X\) is strongly guarded in \(t\), \(\langle X|X = t\rangle\) is the largest consistent solution w.r.t \(\sqsubseteq_{RS}\) if consistent solutions exist. Moreover we also encode a temporal logic language ACTL in CLL\(_R\).

For further work, it is very interesting to study the structure of the solution space \(\{p : p \sqsubseteq_{RS} t|X\{p/X\}\}\) if \(X\) is strongly guarded in \(t|X\).

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