TOPOLOGICAL FREENESS FOR HILBERT BIMODULES

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Abstract. It is shown that topological freeness of Rieffel’s induced representation functor implies that any $C^*$-algebra generated by a faithful covariant representation of a Hilbert bimodule $X$ over a $C^*$-algebra $A$ is canonically isomorphic to the crossed product $A \rtimes_X Z$. An ideal lattice description and a simplicity criterion for $A \rtimes_X Z$ are established.

Introduction

The topological freeness is a condition expressed in terms of the dual system allowing to relate the ideal structure of the crossed product to that of the original algebra. In particular, it implies that every faithful representation of the $C^*$-dynamical system integrates to a faithful representation of the reduced crossed product. The idea behind this notion probably goes back to works of W. Arveson in late 60’s of XX century, and for the first time was explicitly formulated by D. P. O’Donovan, see [14, Thm. 1.2.1]. It is closely related with the properties of the Connes spectrum [15] and with proper outerness [7, 3]. The role of topological freeness for $C^*$-dynamical systems with arbitrary discrete group actions on commutative $C^*$-algebras was clarified in [10, Thm. 4.4] and for arbitrary $C^*$-algebras in [3, Thm. 1]. Independently, and even earlier, in connection with investigation of spectral properties of functional operators, similar results were obtained by A. B. Antonevich, A. V. Lebedev and others, see [2, Cor. 12.17] and [2, pp. 225, 226] for the corresponding survey. These results were improved to cover the case of partial actions in [8, Thm. 2.6] and [12, Thm. 3.7].

In the present paper we prove a statement that generalizes all the aforementioned theorems in the case $G = Z$ and which is formulated in terms of the crossed product $A \rtimes_X Z$, introduced in [11], of a Hilbert bimodule $X$. Thus potentially, by passing to the core $C^*$-algebra, see [11, Thm. 3.1], it may be applied to all the $C^*$-algebras equipped with a semi-saturated circle action and thereby to all relative Cuntz-Pimsner algebras [13]. As a corollary of our main result we provide an ideal lattice description (in the case the dual system is free) and a simplicity criterion for the algebras considered.

Conventions. In essence we follow the notation and conventions adopted in [11]. For maps $\gamma: A \times B \to C$ such as inner products, multiplications or representations we denote by $\gamma(A,B)$ the closed linear span of the set $\{\gamma(a,b) \in C : a \in A, b \in B\}$. An ideal in a $C^*$-algebra is a closed two-sided one, and $[\pi]$ stands for the unitary equivalence class of a representation $\pi$.

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1. Hilbert bimodules, their crossed products and dual partial
dynamical systems

Let $A$ be a $C^*$-algebra and $X$ be an $A$-$A$-Hilbert bimodule as introduced in [5] 1.8. More specifically, $X$ is both left and right Hilbert module over $A$ with left and right $A$-valued inner products $\langle x, y \rangle_L$ and $\langle x, y \rangle_R$ satisfying the so-called imprimitivity condition:

$$x \cdot \langle y, z \rangle_R = \langle x, y \rangle_L \cdot z,$$

for all $x, y, z \in X$.

A simple but crucial observation is that $X$ may treated as an imprimitivity $I_L - I_R$-bimodule where

$$I_L = \langle X, X \rangle_L, \quad I_R = \langle X, X \rangle_R$$

are ideals in $A$. Hence the induced representation functor $X$-$\text{Ind} = X$-$\text{Ind}_{I_R}^{I_L}$ factors through to a homeomorphism $\hat{h} : \hat{I}_R \to \hat{I}_L$ between the spectra of $I_L$ and $I_R$:

$$\hat{h}(\pi) := [X$-$\text{Ind}_{I_L}^{I_R} \pi],$$

cf. e.g. [10] Thm. 3.29, Cor. 3.32, 3.33].

Thus identifying the spectra $\hat{I}_L$ and $\hat{I}_R$ with open subsets of the spectrum of $A$, we may and we will treat $\hat{h}$ as a mapping between open subsets of $\hat{A}$.

**Definition 1.1.** We call the partial homeomorphism $\hat{h}$ of $\hat{A}$ described above a partial homeomorphism dual to the Hilbert bimodule $X$.

A covariant representation of $(A, X)$ [1] Defn. 2.1] is a pair $(\pi_A, \pi_X)$ of representations into algebra of all bounded linear operators $B(H)$ on a Hilbert space $H$ such that all module operations become the ones inherited form $B(H)$, i.e.

$$\pi_X(ax) = \pi_A(a)\pi_X(x), \quad \pi_X(xa) = \pi_X(x)\pi_A(a),$$

$$\pi_A((x, y)_R) = \pi_X(x)^*\pi_X(y), \quad \pi_A((x, y)_L) = \pi_X(x)\pi_X(y)^*,$$

for all $a \in A, x, y \in X$. We say that $(\pi_A, \pi_X)$ is faithful if $\pi_A$ is faithful (then $\pi_X$ is automatically isometric). A crossed product of $A$ by $X$, see [1] Defn. 2.4], is the $C^*$-algebra $A \rtimes_X Z$ universal with respect to covariant representations of $(A, X)$. We denote by $\pi_A \rtimes_X \pi_X$ the representation of $A \rtimes_X Z$ corresponding to $(\pi_A, \pi_X)$ and call it an integrated form of $(\pi_A, \pi_X)$.

The interior tensor power $X^\otimes_n, n \geq 1$, of $X$ is naturally an $A$-$A$ Hilbert bimodule which, as follows from the lemma below, embeds into $A \rtimes_X Z$.

**Lemma 1.2.** Suppose $(\pi_A, \pi_X)$ is a covariant representation of $X$. Then for every $n \in \mathbb{N}$, the mapping

$$X^\otimes_n \ni x_1 \otimes x_2 \otimes ... \otimes x_n \xrightarrow{\pi_X^\otimes_n} \pi_X(x_1)\pi_X(x_2)\cdots\pi_X(x_n).$$

yields a well defined linear homomorphism such that the pair $(\pi_A, \pi_X^\otimes_n)$ is a covariant representation of the tensor product Hilbert bimodule $X^\otimes_n$. In particular, the linear span of the spaces

$$\pi_X^\otimes_n(X^\otimes_n), \quad \pi_A(A), \quad \pi_X^\otimes_n(X^\otimes_n)^*, \quad n \in \mathbb{N},$$

forms a dense $^*-$subalgebra of $(\pi_A \rtimes_X \pi_X)(A \rtimes_X Z) = C^*(\pi_A(A) \cup \pi_X(X)).$

**Proof.** Apply for instance [1] Lem. 2.7, see also [1] Lem. 2.5. □
The next lemma shows how to iterate representation \([\pi] \in \hat{A}\) under \(\hat{\nu}\) using representations of \(A \times X \mathbb{Z}\). Roughly, in order to determine \(\hat{\nu}^n([\pi])\) it suffices to extend \(\pi\) (say, acting in a Hilbert space \(K\)) to any representation \(\nu : A \times X \mathbb{Z} \to \mathcal{B}(H), K \subset H\), and then determine representation \(\nu|_A\) of \(A\) acting in the subspace \(\nu(X^\otimes K)\).

**Lemma 1.3.** Suppose \((\pi_A, \pi_X)\) is covariant representation of \(X\) in a Hilbert space \(H\) and let \(\pi\) be an irreducible summand of \(\pi_A\) acting on a Hilbert subspace \(K\). The representation \(\pi_n : A \to \mathcal{B}(\pi_X \otimes^n (X^\otimes K))\) where
\[
\pi_n(a) := \pi_A(a)|_{\pi_X \otimes^n (X^\otimes K)}, \quad a \in A,
\]
is non-zero if and only if \([\pi]\) belongs to the domain of \(\hat{\nu}^n\), and then
\[
[\pi_n] = \hat{\nu}^n([\pi]).
\]

**Proof.** We recall that \(X - \text{Ind}(\pi_A)(a)(x \otimes h) = (ax) \otimes h\) where \(X \otimes_k H\) is the tensor product Hilbert space with the inner product satisfying \(\langle x_1 \otimes_h h_1, x_2 \otimes_h h_2 \rangle = \langle h_1, \pi((x_1, x_2)_R) h_2 \rangle\). In particular, one sees that \([X^\otimes \text{Ind}(\pi)] = \hat{\nu}^n([\pi]).\) Since \((\pi_A, \pi_X \otimes^n)\) is a covariant representation of \(X^\otimes n\) we have
\[
\|\pi_X \otimes^n(x)h\|^2 = \langle \pi_X \otimes^n(x)h, \pi_X \otimes^n(x)h \rangle = \langle h, \pi_A((x, x)_R) h \rangle = \|x \otimes h\|^2.
\]
Accordingly, the mapping \(\pi_X \otimes^n(x)h \mapsto x \otimes h, x \in X^\otimes n, h \in K\), extends by linearity and continuity to a unitary operator \(V : \pi_X \otimes^n(X^\otimes K) \to X^\otimes n \otimes \pi K\), which intertwines \(\pi_n\) and \(X^\otimes \text{Ind}(\pi)\) because
\[
V \pi_n(a) \pi_X \otimes^n(x)h = V \pi_X \otimes^n(ax)h = (ax \otimes h) = \hat{\nu}^n(\pi)(a) V \pi_X \otimes^n(x)h.
\]

\(\square\)

2. The main result and its corollaries

The goal of the paper could be stated as follows.

**Definition 2.1.** We say that a partial homeomorphism \(\varphi\) of a topological space, i.e. a homeomorphism between open subsets, is topologically free if for any \(n > 0\) the set \(F_n = \{x : \varphi^n(x) = x\}\) (contained in the domain of \(\varphi^n\)) has empty interior.

**Theorem 2.2.** If the partial homeomorphism \(\hat{h}\) is topologically free, then every faithful covariant representation \((\pi_A, \pi_X)\) of \(X\) integrates to faithful representation \((\pi_A \times \pi_X)\) of \(A \times X \mathbb{Z}\).

**Remark 2.3.** The map \(\hat{h}\) is a lift of the partial homeomorphism \(h : \text{Prim}(I_R) \to \text{Prim}(I_L)\) of \(\text{Prim}(A)\) where \(h(\ker \pi) := \ker X - \text{Ind}(\pi), [\pi] \in \hat{A}\). Actually \(h\) is the restriction of the so-called Rieffel isomorphism between the ideal lattices of \(I_R\) and \(I_L\) where
\[
h(J) = \langle XJ, X \rangle_L, \quad h^{-1}(K) = \langle X, KX \rangle_R,
\]
cf. [10, 3.3]. Plainly, topological freeness of the Rieffel homeomorphism \(h\), treated as a partial homeomorphism of \(\text{Prim}(A)\) implies the topological freeness of \(\hat{h}\). However, the converse implication is not true.
An equivalent form of Theorem 2.2 states that if the partial homeomorphism \( \hat{h} \) is topologically free, then every non-trivial ideal in \( A \times_X \mathbb{Z} \) leaves an "imprint" in \( A \) – has a non-trivial intersection with \( A \). By specifying these imprints one may determine the ideal structure of \( A \times_X \mathbb{Z} \). To this end we adopt the following definition, cf. [8] Defn. 2.7, 2.8).

**Definition 2.4.** We say that a set \( V \) is *invariant* under a partial homeomorphism \( \varphi \) with a domain \( \Delta \) if

\[
\varphi(V \cap \Delta) = V \cap \varphi(\Delta).
\]

If there are no non-trivial closed invariant sets, then \( \varphi \) is called *minimal*, and \( \varphi \) is said to be *free*, if it is topologically free on every closed invariant set (in the Hausdorff space case this amounts to requiring that \( \varphi \) has no periodic points).

Similarly to topological freeness, cf. Remark 2.3, the freeness of \( \hat{h} \) is a stronger condition than freeness of \( h \). However, the minimality of \( \hat{h} \) and \( h \) are equivalent, and moreover using (2.1) one sees that, if \( I \) is an ideal in \( A \), then the open set \( \hat{I} \) in \( \hat{A} \) is \( \hat{h} \)-invariant if and only if

\[
IX = XI.
\]

Ideals satisfying (2.2) are called \( X \)-invariant in [9], and \( X \)-invariant and saturated in [11]. It is known, see [9, 10.6] or [11, Thm. 7.11], that the map

\[
A \times_X \mathbb{Z} \ni J \mapsto J \cap A \in A
\]

defines a homomorphism from the lattice of ideals in \( A \times_X \mathbb{Z} \) onto the lattice of ideals satisfying (2.3). When restricted to *gauge invariant ideals*, i.e. ideals preserved under the *gauge circle action* \( \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \ni \lambda \mapsto \gamma_\lambda \in \text{Aut}(A \times_X \mathbb{Z}) \) given by

\[
\gamma_\lambda(a) = a, \quad a \in A, \quad \gamma_\lambda(x) = \lambda x, \quad x \in X,
\]

homomorphism (2.3) is actually an isomorphism. Thus if one is able to show that all ideals in \( A \times_X \mathbb{Z} \) are gauge invariant, one obtains a complete description of the ideal structure of \( A \times_X \mathbb{Z} \).

**Theorem 2.5** (ideal lattice description). *If the partial homeomorphism \( \hat{h} \) is free, then all ideals in \( A \times_X \mathbb{Z} \) are gauge invariant and the map*

\[
J \mapsto J \cap A
\]

*is a lattice isomorphism between ideals in \( A \times_X \mathbb{Z} \) and open invariant sets in \( \hat{A} \).*

**Proof.** It suffices to show that the map (2.5) is injective. To this end suppose that \( J \) is an ideal in \( A \times_X \mathbb{Z} \), let \( J_0 = J \cap A \) and denote by \( \langle J_0 \rangle \) the ideal generated by \( J_0 \) in \( A \times_X \mathbb{Z} \). Clearly, \( \langle J_0 \rangle \subset J \) and to prove that \( \langle J_0 \rangle = J \) we consider the homomorphism

\[
\Psi : A \times_X \mathbb{Z} \to A/J_0 \times_X A/J_0
\]

arising from the composition of the quotient maps and the universal covariant representation of \( (A/J_0, X/XJ_0) \). Then, cf. for instance [11] Thm. 6.20), ker \( \Psi = \langle J_0 \rangle \) and we claim that \( \psi(J) \cap A/J_0 = \{0\} \). Indeed, if \( b \in \psi(J) \cap A/J_0 \), then \( b = \Psi(a) \) for some \( a \in J \) and \( b = \Psi(a_1) \) for some \( a_1 \in A \). Thus \( a - a_1 \in \ker \psi \subset J \) and it follows that \( a_1 \) itself is in \( J \). But then \( a_1 \in J \cap A = J_0 \), so \( b = \Psi(a_1) = 0 \), which proves our claim. The system dual to \( (A/J_0, X/XJ_0) \) naturally identifies with \( (\hat{A} \setminus \hat{J}_0, \hat{h}) \) and thus it is topologically free by freeness of
(\widehat{A}, \widehat{h})$. Hence Theorem 2.2 implies that $\Psi(J)$ is trivial in $A/J_0 \times_\chi X J_0, \mathbb{Z}$. Hence $J = \langle J_0 \rangle = \ker \Psi$. 

**Corollary 2.6** (simplicity criterion). *If the partial homeomorphism $\widehat{h}$ is topologically free and minimal, then $A \rtimes_\chi \mathbb{Z}$ is simple.*

### 3. Proof of Theorem 2.2

We denote by $X_n, n \in \mathbb{Z}$, the fibers of the $\mathbb{Z}$-bundle constructed in [1, p. 3046-3047]. In particular, $X_0 = A$ and for $n > 0$, $X_n = X^{\otimes n}$ and $X_{-n} = X^{\otimes |n|}$, where $\tilde{X}$ is the Hilbert $I_R - I_L$-bimodule dual to the $I_L - I_R$-bimodule $X$. The $C^*$-algebra bundle operations equip each pair $(A, X_n)$, $n \in \mathbb{Z}$, with a Hilbert bimodule structure. For all $n \in \mathbb{Z}$ we put

$$D_n := \langle X_n, X_n \rangle_R = X_n^* \cdot X_n$$

where “$\cdot$” is the $C^*$-algebra bundle multiplication (then, in particular, $D_0 = A$ and $D_{-n} = \langle X_n, X_n \rangle_L$). We notice that $X_n$ is a $D_{-n} - D_n$-imprimitivity bimodule and the partial homeomorphism of $\widehat{A}$ given by the induced representation functor $X_n - \text{Ind}^D_n$ coincides with the $n$-th power $\widehat{h}^n$ of $\widehat{h}$, and in particular, $\widehat{D}_n$ is a natural domain of $\widehat{h}^n$.

A covariant representation $(\pi_A, \pi_X)$ of $(A, X)$ yields covariant representations $(\pi_A, \pi_{X_n})$ of $(A, X_n)$ for all $n \in \mathbb{Z}$, cf. Lemma 1.2, [1, Lem. 2.7] or [11, Thm. 3.12]. The copy of $A$ embedded into $A \rtimes \mathbb{Z}$ is a fixed point algebra for the gauge circle action (2.4). Thus, in the language of circle actions, it seems to be a part of a $C^*$-algebra folklore and follows for instance from [6, Lem. 2.11] or [4, Lem. 2.2] that the representation $(\pi_A \times \pi_X)$ of $A \rtimes_\chi \mathbb{Z}$ is faithful if and only if $\pi_A$ is faithful and the formula

$$\mathcal{E} \left( \sum_{k=-n}^{n} \pi_{X_k}(a_k) \right) = \pi_A(a_0),$$

where $a_k \in X_k, k = 0, \pm 1, \ldots, \pm n$, defines a mapping (conditional expectation) $\mathcal{E}$ from the $C^*$-algebra $C^*(\pi_A(A), \pi_X(X))$ generated by $\pi_A(A)$ and $\pi_X(X)$ onto the $C^*$-algebra $\pi_A(A)$. Therefore Theorem 2.2 follows immediately from Lemma 3.2 below, and among the technical instruments of the proof of this latter statement we use the following simple fact, see e.g. [2] Lem. 12.15).

**Lemma 3.1.** Let $B$ be a $C^*$-subalgebra of an algebra $B(H)$ and $P_1, P_2 \in B'$ be two orthogonal projections such that the restrictions of $B$ to $H_1 = P_1 H$ and $H_2 = P_2 H$ are both irreducible representations. Then $P_1 \neq P_2$ if and only if $P_1 \perp P_2$.

**Lemma 3.2.** Let the Rieffel homeomorphism $\widehat{h}$ be topologically free. Assume that $A$ and $X$ are embedded in $B(H)$ so that the module actions and inner products become inherited from $B(H)$ (then the whole $\mathbb{Z}$-bundle $\{X_n \}_{n \in \mathbb{Z}}$ embeds into $B(H)$) and let $b$ be an operator of the form

$$(3.1) \quad b = \sum_{k=-n}^{n} a_k \quad \text{where} \quad a_k \in X_k, \; k = 0, \pm 1, \ldots, \pm n.$$ 

Then for every $\varepsilon > 0$ there exists an irreducible representation $\pi : A \to B(H_{\varepsilon})$ such that for any representation $\nu : C^*(A, X) \to B(H_{\varepsilon})$ that extends $\pi$ ($H_{\pi} \subset H_{\nu}$) we have
Proof. Let \( \varepsilon > 0 \). Since for every \( a \in A \) the function \( [\pi] \rightarrow \|\pi(a)\| \) is lower semicontinuous on \( \hat{A} \) and attains its upper bound equal to \( \|a\| \), cf. for instance\cite{16} App. A, there exists an open set \( U \subset \hat{A} \) such that
\[
\|\pi(a_0)\| > \|a_0\| - \varepsilon \quad \text{for every} \quad [\pi] \in U.
\]

By topological freeness of \( \hat{h} \) the set \( F_n! = \{ [\pi] \in \hat{D}_n! : \hat{h}^n([\pi]) = [\pi] \} \) has empty interior and thus we may find \([\pi] \in U\) such that all the points \( \hat{h}^k([\pi]), k = 0, 1, ..., n \) are distinct (if they are defined, i.e. if \( \pi(D_k) \neq 0 \)). Let \( \nu \) be any extension of \( \pi \) up to a representation of \( C^*(A, X) \) and denote by \( H_\pi \) and \( H_\nu \) the corresponding representation spaces for \( \pi \) and \( \nu \): \( H_\pi \subset H_\nu \). Item (i) follows from the choice of \( \pi \).

To prove (ii) we need to show that for the orthogonal projection \( P_\pi : H_\nu \rightarrow H_\pi \) and any element \( a_k \in X_k, k \neq 0 \), of the sum (3.1) we have
\[
P_\pi \nu(a_k) P_\pi = 0.
\]

We fix \( k \neq 0 \) and consider two different possible positions of \( \pi \).

If \( \pi \notin \hat{D}_k \cap \hat{D}_{-k} \), then either \( \pi(D_k) = 0 \) or \( \pi(D_{-k}) = 0 \). By Hewitt-Cohen Theorem (see, for example,\cite{16} Prop. 2.31) operator \( a_k \) may be presented in a form \( a_k = d_- a d_+ \) where \( d_\pm \in D_\pm k, a \in X_k \), and thus
\[
P_\pi \nu(a_k) P_\pi = P_\pi \nu(d_- a d_+) P_\pi = P_\pi \pi(d_-) P_\pi \nu(a) P_\pi \pi(d_+) P_\pi = 0.
\]

Suppose then that \( \pi \in \hat{D}_k \cap \hat{D}_{-k} \). Accordingly, \( \pi \) may be treated as an irreducible representation for both \( D_k \) and \( D_{-k} \). We will use Lemma 3.1 where the role of \( \pi_1 \) is played by \( P_\pi \) and \( \pi_2 \) is the orthogonal projection onto the space
\[
H_2 := \nu(X_k) H_\pi.
\]

Clearly, \( P_\pi \in \nu(A)' \) and to see that \( P_2 \in \nu(A)' \) it suffices to note that for \( a \in A \), \( x \in X_k \) and \( h \in H_\pi \) we have \( \nu(a) \nu(x) h = \nu(ax) h \in H_2 \), that is \( \nu(a) P_2 = P_2 \nu(a) P_2 \), since then using the same relation for \( \nu(a^*) \) one gets \( \nu(a) P_\pi = P_\pi \nu(a) \). Moreover, by Lemma 1.3 for the representation \( \pi_2 : A \rightarrow L(H_2) \) given by \( \pi_2(a) = \nu(a)|_{H_2} \), we have \( \pi_2 \cong X_k \cdot \text{Ind}(\pi) \), or equivalently
\[
[\pi_2] = \hat{h}^k([\pi]).
\]

Consequently, \( \pi \) and \( \pi_2 \) may be treated as irreducible representations of \( D_{-k} \), and by the choice of \( \pi \) these representations are different (actually even not equivalent). Hence, by Lemma 3.1
\[
P_\pi \cdot P_\pi = 0
\]

from which we have
\[
P_\pi \nu(a_k) P_\pi = P_\pi \cdot P_2 \nu(a_k) P_\pi = 0.
\]

\[\square\]
References

[1] B. Abadie, S. Eilers, and R. Exel. Morita equivalence for crossed products by Hilbert $C^*$-bimodules. Trans. Amer. Math. Soc., 350(8):3043–3054, 1998.
[2] A. B. Antonevich and A. V. Lebedev. Functional differential equations: I. $C^*$-theory. Longman Scientific & Technical, Harlow, Essex, England, 1994.
[3] R. J. Archbold and Spielberg J. S. Topologically free actions and ideals in discrete $C^*$-dynamical systems. Proc. Edinburgh Math. Soc., 37(2):119–124, 1993.
[4] S. Boyd, N. Keswani, and I. Raeburn. Faithful representations of crossed products by endomorphisms. Proc. Amer. Math. Soc., 118:427–436, 1993.
[5] L. G. Brown, J. Mingo, and N. Shen. Quasi-multipliers and embeddings of hilbert $C^*$-modules. Canad. J. Math., 71:1150–1174, 1994.
[6] S. Doplicher and J. E. Roberts. Duals of compact lie groups realized in the Cuntz algebras and their actions on $C^*$-algebras. J. Funct. Anal., 74:96–120, 1987.
[7] G. A. Elliot. Some simple $C^*$-algebras constructed as crossed products with discrete outer automorphism groups. Publ. Res. Inst. Math. Sci., 13:299–311, 1980.
[8] R. Exel, M. Laca, and J. Quigg. Partial dynamical systems and $C^*$-algebras generated by partial isometries. J. Operator Theory, 47:169–186, 2002.
[9] T. Katsura. Ideal structure of $C^*$-algebras associated with $C^*$-correspondences. Pacific J. Math., 230:107–146, 2007.
[10] S. Kawamura and J. Tomiyama. Properties of topological dynamical systems and corresponding $C^*$-algebras. Tokyo J. Math., 13:251–257, 1990.
[11] B. K. Kwaśniewski. $C^*$-algebras generalizing both relative Cuntz-Pimsner and Doplicher-Roberts algebras. accepted in Trans. Amer. Math. Soc., arXiv:0906.4382.
[12] A. V. Lebedev. Topologically free partial actions and faithful representations of partial crossed products. Funct. Anal. Appl., 39:207–214, 2005.
[13] P. S. Muhly and B. Solel. Tensor algebras over $C^*$-correspondences (representations, dilations, and $C^*$-envelopes). J. Funct. Anal., 158:389–457, 1998.
[14] D. P. O’Donovan. Weighted shifts and covariance algebras. Trans. Amer. Math. Soc., 208:1–25, 1975.
[15] D. Olesen and G. K. Pedersen. Applications of the Connes spectrum to $C^*$-dynamical systems. J. Funct. Anal., 30:179–197, 1978.
[16] I. Raeburn and D. P. Williams. Morita equivalence and continuous-trace $C^*$-algebras. Amer. Math. Soc., Providence, 1998.