Structure of Chern-Simons Scattering Amplitudes from Topological Equivalence Theorem and Double-Copy

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Abstract

We study the mechanism of topological mass-generation for 3d Chern-Simons (CS) gauge theories, where the CS term can retain the gauge symmetry and make gauge boson topologically massive. Without CS term the 3d massless gauge boson has a single physical transverse polarization state, while adding the CS term converts it into a massive physical polarization state and conserves the total physical degrees of freedom. We newly formulate the mechanism of topological mass-generation at $S$-matrix level. For this, we propose and prove a new Topological Equivalence Theorem (TET) which connects the $N$-point scattering amplitude of the gauge boson’s physical polarization states ($A_{P}^{a}$) to that of the transverse polarization states ($A_{T}^{a}$) under high energy expansion. We present a general 3d power counting method on the leading energy dependence of $N$-point scattering amplitudes in both topologically massive Yang-Mills (TMYM) and topologically massive gravity (TMG) theories. With these, we uncover a general energy cancellation mechanism for $N$-gauge boson scattering amplitudes which predicts the cancellation $E^{4} \rightarrow E^{4-N}$ at tree level. Then, we compute the four-point amplitudes of $A_{P}^{a}$’s and of $A_{T}^{a}$’s, with which we explicitly demonstrate the TET and establish such energy cancellations. We further extend the double-copy approach and construct the four-point massive graviton amplitude of the TMG theory from the massive gauge boson amplitude of the TMYM theory. With these, we newly uncover striking large energy cancellations $E^{12} \rightarrow E^{1}$ in the four-graviton amplitude of the TMG, and establish its new correspondence to the leading energy cancellations $E^{4} \rightarrow E^{0}$ in the four-gauge boson amplitude of the TMYM.

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1 Introduction

Chern-Simons gauge theories in (2+1)-dimensional (3d) spacetime play an important role in studying modern quantum field theories for particle physics and condensed matter physics [1][2][3]. Such 3d theories can always contain gauge-invariant mass terms of gauge bosons through the topological mass-generation à la Chern-Simons (CS) [4], without invoking the conventional Higgs mechanism [5] in the 4d standard model (SM).

In this work, we study the dynamics of 3d topological mass-generation for the (Abelian and non-Abelian) gauge bosons $A_{\mu}^a$. A spin-1 massless gauge boson in 3d contains only one physical degree of freedom (DoF) which is the transversely polarized state $A_T^a$. Including the gauge-invariant topological CS term converts this massless transverse polarization state $A_T^a$ into a massive physical state $A_P^a$. We newly formulate this 3d topological mass-generation mechanism at $S$-matrix level. For this, we propose and prove a new Topological Equivalence Theorem (TET) which connects the $N$-point scattering amplitudes of the physical polarization states of massive gauge bosons ($A_P^a$) to the scattering amplitudes of the corresponding transversely polarized gauge boson states ($A_T^a$) under high energy expansion. This differs essentially from the conventional equivalence theorem (ET) [7] in the 4d SM because the 3d gauge bosons acquire gauge-invariant topological mass-term without invoking the conventional Higgs mechanism [5]. We note that the Kaluza-Klein ET (KK-ET) [8][9][10] was formulated for the compactified 5d Yang-Mills theories which realize a geometric Higgs mechanism with the 5th component of 5d gauge field converted to the longitudinal component of the corresponding 4d massive KK gauge boson. But our TET also has essential difference from the KK-ET, because the 5d gauge symmetry is spontaneously broken by compactification down to the 4d residual gauge symmetry of the massless zero-modes and the induced KK gauge boson mass-term is not gauge-invariant. In contrast, the 3d CS term for the topological mass-generation of gauge bosons can be manifestly gauge-invariant and the 3d gauge symmetry is unchanged before and after including the CS term.

We present a general 3d power counting method to count the leading energy-power dependence of the $N$-point scattering amplitudes in both topologically massive Yang-Mills (TMYM) theory and topologically massive gravity (TMG) theory. Using the TET and power counting method for the 3d TMYM theory, we uncover that despite the individual diagrams in a given $N$-particle scattering amplitude of on-shell physical gauge bosons ($N \geq 4$) having leading energy dependence of $E^4$ at tree level, they have to cancel down to $E^{4-N}$ in the full tree-level amplitude. We will prove that the TET provides a general theoretical mechanism to guarantee such nontrivial energy cancellations: $E^4 \rightarrow E^{4-N}$, without invoking any conventional Higgs boson. For the massive 4-gauge boson scattering amplitudes at tree level, we will demonstrate explicitly

\footnote{The conventional Higgs mechanism [5] is sometimes also called Brout-Englert-Higgs (BEH) mechanism or Anderson-Higgs mechanism [6] in the literature.}
the large energy cancellations of $E^4 \rightarrow E^0$ under high energy expansion.

Furthermore, using the scattering amplitude of topologically massive gauge bosons in the 3d TMYM theory, we will reconstruct the topologically massive graviton scattering amplitude of the 3d TMG theory, by extending the conventional double-copy method of Bern-Carrasco-Johansson (BCJ) \cite{11, 12} for massless gauge/gravity theories to the current 3d topologically massive gauge/gravity theories. The BCJ method was inspired by the Kawai-Lewellen-Tye (KLT) \cite{13} relation which connects the product of open string amplitudes to that of the closed string at tree level. Analyzing the properties of the heterotic string and open string amplitudes can prove and refine parts of the BCJ conjecture \cite{14}. Many studies appeared in the literature to test the double-copy conjecture in massless gauge/gravity field theories \cite{12}, and some recent works attempted to extend the double-copy method to the 4d massive YM theory versus Fierz-Pauli-like massive gravity \cite{15}, to the KK-inspired effective gauge theory with extra global U(1) \cite{18}, and to the compactified 5d KK gauge/gravity theories and KK string theories \cite{19}. Double-copies of three- and two-algebra gauge theories were considered previously for the 3d supersymmetric theories \cite{21}, \cite{22}, \cite{23}, and some double-copy analyses for the amplitudes with matter fields in 3d CS gauge theory as well as the study of 3d covariant color-kinematics duality appeared very recently \cite{24}, \cite{25}, \cite{26}.

We stress that the topological mass-generation for gauge bosons and gravitons in the 3d TMYM and TMG theories can be realized in a fully gauge-invariant way under the path integral formulation, which is important for the successful double-copy construction in the massive gauge/gravity theories. In this work, we will use an extended double-copy approach to construct the massive four-graviton amplitude of the TMG theory from the corresponding massive four-gauge boson amplitude of the TMYM theory with properly improved kinematic numerators. Our findings newly demonstrate a series of strikingly large energy cancellations, $E^{12} \rightarrow E^1$, in the massive four-graviton amplitude under high energy expansion. With these we establish a new correspondence between the two types of leading energy cancellations in the massive scattering amplitudes: $E^4 \rightarrow E^0$ in the TMYM theory and $E^{12} \rightarrow E^1$ in the TMG theory.

This paper is organized as follows. In section 2, we study the mechanism of the topological mass-generation in the 3d CS gauge theories at the Lagrangian level via path integral formulation. We identify the conversion of transverse polarization state $A^a_T$ in the massless theory into the massive physical polarization state $A^a_P$ under the topological mass-generation of the CS gauge theories. In section 3.1, we propose and prove the new TET which connects the $N$-point $A^a_P$-amplitudes to the corresponding $A^a_T$-amplitudes under high energy expansion. Using the TET, we newly formulate the mechanism of topological mass-generation at $S$-matrix level. Then, in section 3.2, we present the general 3d power counting rules on the leading energy dependence of the $N$-point scattering amplitudes in both the CS gauge theories and the TMG theory. Using
the TET and the power counting rule, we prove in section 3.3 a general energy cancellation mechanism for the $N$-gauge boson scattering amplitudes which predicts the cancellation $E^4 \rightarrow E^{4-N}$ at tree level. For sections 4.1-4.2, we first compute the four-point matter-induced gauge boson amplitudes, and then compute the pure four-gauge boson amplitudes for the $A^a_\mu$-states and $A^a_\nu$-states in the Abelian and non-Abelian CS gauge theories. These analyses explicitly demonstrate the TET for the first time, and newly establish the energy cancellations $E^2 \rightarrow E^0$ of the four-point amplitudes with just two gauge bosons (in either Abelian or non-Abelian CS theories) and the energy cancellations $E^4 \rightarrow E^0$ of the four-gauge boson amplitudes (in TMYM theories). In section 4.3, we analyze the perturbative unitarity bounds for both the TMYM theory and the TMG theory. We demonstrate that their partial wave amplitudes can exhibit good high energy behaviors. In section 5, we further extend the double-copy approach and construct the massive four-graviton amplitude of the TMG from the massive four-gauge boson amplitude of the TMYM. With these, we newly uncover strikingly large energy cancellations in the four-graviton amplitude: $E^{12} \rightarrow E^1$, and establish its new correspondence to the leading energy cancellation $E^4 \rightarrow E^0$ in the massive four-gauge boson amplitude of the TMYM. We conclude in section 6. Finally, we provide more derivations and formulas in Appendices A-E which are used for the analyses in the main text.

2 Topological Mass Generation in Chern-Simons Gauge Theories

We consider the 3d topological massive gauge theories including the Chern-Simons (CS) Lagrangian with Abelian or non-Abelian gauge symmetry, where the former may be denoted as Topologically Massive QED (TMQED) and the latter as Topologically Massive Yang-Mills (TMYM) theory. In either case, the CS Lagrangian provides a gauge-invariant topological mass-term for the 3d gauge bosons. The 3d TMQED and TMYM Lagrangians have their gauge sectors take the following forms:

$$\mathcal{L}_{\text{TMQED}} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} \tilde{m} \varepsilon^{\mu\rho\sigma} A_\mu \partial_\nu A_\rho,$$  

$$\mathcal{L}_{\text{TMYM}} = -\frac{1}{2} \text{tr} F_{\mu\nu}^2 + \tilde{m} \varepsilon^{\mu\rho\sigma} \text{tr} \left( A_\mu \partial_\nu A_\rho - \frac{i2g}{3} A_\mu A_\nu A_\rho \right),$$

where the non-Abelian gauge field $A_\mu = A^a_\mu T^a$, and its field strength $F_{\mu\nu} = F_{\mu\nu}^a T^a$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ and $T^a$ denotes the generator of the non-Abelian group $\text{SU}(N)$. The gauge coupling $g$ has mass-dimension $\frac{1}{2}$. The gauge boson acquires a topological mass $m = |\tilde{m}|$ from the CS term, and the ratio $s = \tilde{m}/m = \pm 1$ corresponds to its spin projection $\pm 1$. The mass parameter $\tilde{m}$ is related to the CS level $n = 4\pi \tilde{m}/g^2 \in \mathbb{Z}$ [2][3]. The CS terms in Eq.(2.1) violate the discrete symmetries $P, T$ and $CP$.

For the TMQED (2.1a), the action $\int d^3x \mathcal{L}_{\text{TMQED}}$ is gauge-invariant up to a total derivative
which vanishes at the boundary for trivial topology. For the TMYM theory (2.1b), under the
gauge transformation \( A_\mu \rightarrow A'_\mu = U^{-1}A_\mu U + \frac{i}{g}U^{-1}\partial_\mu U \), the action changes by

\[
\Delta S_{\text{TMYM}} = 2\pi nw + \int d^3x \left[ i\tilde{m}g^{-1}\varepsilon^{\mu\nu\rho}\partial_\nu \text{tr}(\partial_\mu UU^{-1}A_\rho) \right],
\]

(2.2a)

\[
w = \frac{1}{24\pi^2} \int d^3x \left[ \varepsilon^{\mu\nu\rho} \text{tr}(U^{-1}\partial_\mu UUU^{-1}\partial_\nu UU^{-1}\partial_\rho U) \right],
\]

(2.2b)

where \( w \) is the winding number which follows from the homotopy group \( \Pi_3[SU(N)] \cong \mathbb{Z} \) [3]. Hence, Eq.(2.2a) will not contribute to the path integral since \( e^{i2\pi nw} = 1 \), and the second term is a total derivative (similar to the Abelian case).

With the path integral formulation, we can add the covariant gauge-fixing term and the Faddeev-Popov ghost term:

\[
\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi} (\mathcal{F}^a)^2, \quad \mathcal{F}^a = \partial^\mu A^a_\mu,
\]

(2.3a)

\[
\mathcal{L}_{\text{FP}} = \bar{c}^a \partial^\mu (\delta^{ab} \partial_\mu - gC^{abc} A^c_\mu) c^b,
\]

(2.3b)

where \( C^{abc} \) is the gauge group structure constant and \( (c^a, \bar{c}^a) \) denote the Faddeev-Popov ghost and anti-ghost fields. Eq.(2.3) can be reduced to the Abelian case by simply setting \( C^{abc} = 0 \) and \( A^a_\mu = A_\mu \). So, hereafter we need not to specify the Abelian case unless needed. The quantized CS action \( \int d^3x (\mathcal{L} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}) \) is BRST-invariant (Becchi-Rouet-Stora-Tyutin), with which we can derive the relevant BRST identities.

The equation of motion (EOM) for the massive gauge boson \( A^a_\mu \) can be derived from the quadratic part of the CS action,

\[
[\eta^{\mu\nu} \partial^2 + (\xi^{-1} - 1)\partial^\mu \partial^\nu + \tilde{m} \varepsilon^{\mu\nu\rho} \partial_\rho ] A^a_\nu = 0,
\]

(2.4)

which describes the propagation of the free field \( A^a_\mu \). For the on-shell wave solution \( A^a_\mu \sim \epsilon_\mu(p)e^{-ipx} \) with \( p^\mu A^a_\mu = 0 \), the polarization vector should satisfy the equation

\[
(m\eta^{\mu\nu} - is\varepsilon^{\mu\nu\rho} p_\rho) \epsilon_\nu(p) = 0,
\]

(2.5)

under the on-shell condition \( p^2 = -m^2 \) and with \( s = \tilde{m}/m = \pm 1 \). The 3d Poincaré group ISO(2,1) contains the proper Lorentz group SO(2,1) and translations. The little group is \( \mathbb{Z}_2 \otimes \mathbb{R} \) for massless particles and SO(2) for massive particles [27]. The Poincaré algebra is characterized by two Casimir operators \( (P^2, W) = (P_\mu P^\mu, P_\mu J^\mu) \), where \( W \) is the Pauli-Lubanski pseudoscalar and the angular momentum \( J^\mu \) can be generally expressed as [28]:

\[
J^\mu = -i\varepsilon^{\mu\nu\alpha} p_\nu \frac{\partial}{\partial p_\alpha} - s \frac{p^\mu + \eta^\mu m}{p \cdot \eta - m},
\]

(2.6)
with \( \eta^\mu = (1, 0, 0) \). Thus, in the rest frame it gives \( W = P \cdot J = -s m \). We see that the spin is a pseudoscalar and takes the values \( s = \pm 1 \) for gauge fields \( A^a_\mu \). The polarization state with either \( s = +1 \) or \( s = -1 \) is physically equivalent. (More discussions on the gauge boson polarization vector are given in Appendix A.)

Note that the 3d massless gauge field can be viewed as a scalar field of spin-0 with one physical degree of freedom (DoF) [27][28]. Including the CS term does not add any new field, and the total physical DoF remains as one because the physical DoF of \( A^a_\mu \) should be conserved [28][29].

For the on-shell one-particle state, the 3d massless gauge boson has a single (transverse) physical polarization state \( A^a_T = \epsilon^a_\mu A^a_\mu \). As the physical DoF is conserved, the CS term could only convert the massless \( A^a_T \) state into a massive physical polarization state \( A^a_P = \epsilon^a_\mu A^a_\mu \).

For the on-shell gauge boson in the rest frame with momentum \( p^\mu = (m, 0, 0) \equiv \bar{p}^\mu \), the physical polarization vector \( \epsilon^\mu_P(\bar{p}) \) can be solved from Eq.(2.5):

\[
\epsilon^\mu_P(\bar{p}) = \frac{1}{\sqrt{2}} (0, 1, -is), \quad (2.7)
\]

in agreement with [29][30]. Then, by making a Lorentz boost we can express \( \epsilon^\mu_P(p) \) in the moving frame:

\[
\epsilon^\mu_P(p) = \frac{1}{\sqrt{2}} \left( \frac{ip_1 + sp_2}{m} + \frac{p_1(ip_1 + sp_2)}{m(m - p_0)}, \frac{ip_1 + sp_2}{m(m - p_0)} \right), \quad (2.8)
\]

which agrees with [30] up to an overall factor \( i \). The on-shell physical polarization vector \( \epsilon^\mu_P(p) \) obeys the conditions \( \epsilon^\mu_P \epsilon^\mu_P = 1 \) and \( p^\mu \epsilon^\mu_P = 0 \). We can express the general momentum \( p^\mu \) in a familiar form \( p^\mu = E(1, \beta s_\theta, \beta c_\theta) \), where the notations \( p^0 = -p_0 = E, (s_\theta, c_\theta) = (\sin \theta, \cos \theta), \beta = \sqrt{1 - m^2/E^2} \), and \( \theta \) denotes the angle between the moving direction and \( y \)-axis. With these, we can rewrite the polarization vector (2.8) as follows:

\[
\epsilon^\mu_P(p) = \frac{1}{\sqrt{2}} \left( \bar{E} \beta, \bar{E} s_\theta + is c_\theta, \bar{E} c_\theta - is s_\theta \right), \quad (2.9)
\]

where \( \bar{E} = E/m \) and we have removed an irrelevant overall phase factor. Inspecting the structure of the physical polarization vector (2.9), we derive the following general decomposition:

\[
\epsilon^\mu_P = \frac{1}{\sqrt{2}} (\epsilon^\mu_T + \epsilon^\mu_L), \quad (2.10)
\]

which contains the transverse and longitudinal polarization vectors \((\epsilon^\mu_T, \epsilon^\mu_L)\) of the massive gauge boson \( A^a_\mu \),

\[
\epsilon^\mu_T = (0, is c_\theta, -is s_\theta), \quad \epsilon^\mu_L = \bar{E}(\beta, s_\theta, c_\theta). \quad (2.11)
\]

Thus, we have the relation for the on-shell polarization states of \( A^a_\mu \):

\[
A^a_P = \frac{1}{\sqrt{2}} (A^a_T + A^a_L), \quad (2.12)
\]
where \((A^a_\mu, A^a_T, A^a_L) = (\epsilon^a_\mu, \epsilon^a_T, \epsilon^a_L)A^a_\mu\). The gauge boson \(A^a_\mu\) also has an unphysical scalar polarization state \(A^a_S = \epsilon_S^a A^a_\mu\) with \(\epsilon_S^a = p^a/m\). It is important to note that the polarization vectors \((\epsilon^a_\mu, \epsilon^a_T, \epsilon^a_L)\) are all enhanced by energy and scale as \(\mathcal{O}(E/m)\) under the high energy expansion. The 3d gauge boson \(A^a_\mu\) has 3 possible polarization states in total, including 1 physical polarization and 2 unphysical polarizations. In the massless case \((m = 0)\), \(A^a_\mu\) contains 1 physical transverse polarization state \(A^a_T = \epsilon^a_T A^a_\mu\) and 2 unphysical (longitudinal, scalar) polarization states \((A^a_\mu, A^a_S) = (\epsilon^a_L A^a_\mu, \epsilon_S^a A^a_\mu)\) with \(\epsilon^a_L + \epsilon_S^a \propto p^a\). On the other hand, for the massive case with CS term \((m \neq 0)\), \(A^a_\mu\) includes 1 physical polarization state \(A^a_P\) as in Eq.(2.12) and 2 orthogonal unphysical polarization states:

\[
A^a_X = \epsilon^a_X A^a_\mu = \frac{1}{\sqrt{2}} (A^a_T - A^a_L), \\
A^a_S = \epsilon_S^a A^a_\mu, 
\]

where \(\epsilon^a_X = (\epsilon^a_T - \epsilon^a_L)/\sqrt{2}\) and \(\epsilon^a_S = p^a/m\). The three polarization vectors obey the orthogonal conditions \(\epsilon_X \cdot \epsilon_X = \epsilon_L \cdot \epsilon_L = \epsilon_S \cdot \epsilon_S = 0\). We see that adding the CS term for gauge boson \(A^a_\mu\) dynamically generates a new physical polarization state \(A^a_P\) of spin-1 (which has mass \(m\) and is composed of \(A^a_T + A^a_L\)), and converts its orthogonal combination \(A^a_X \propto (A^a_T - A^a_L)\) into the unphysical state, while the scalar-polarization state \(A^a_S = \epsilon_S^a A^a_\mu\) (with \(\epsilon_S^a \propto p^a\)) remains unphysical as constrained by the gauge-fixing function \(\mathcal{F}^a = -ip^a A^a_\mu\) in Eq.(2.13a).

The above mechanism of 3d topological mass-generation might be called a “topological Higgs mechanism” to resemble the dynamical conversion of \((A^a_T + A^a_L)\) into the massive physical state \(A^a_P\) of the gauge field \(A^a_\mu\), while making the orthogonal combination \(A^a_X \propto (A^a_T - A^a_L)\) be an unphysical “Goldstone boson” state. However, for the reasons given below, the “topological Higgs mechanism” is not the most appropriate name for the 3d topological mass-generation. We stress that the mechanism of topological mass-generation of gauge bosons differs from the conventional Higgs mechanism \([5]\) in essential ways: (i) topological CS mass-term automatically holds the exact gauge symmetry in the path integral formulation, without invoking any spontaneous gauge symmetry breaking by the vacuum of Higgs potential; (ii) before including the CS term, the transverse \(A^a_T\) is the physical polarization state and is exactly massless as ensured by the gauge symmetry; while after including the CS term, \(A^a_T\) combines with \(A^a_L\) to form the massive physical state \(A^a_P\) and makes its orthogonal combination \(A^a_X\) become unphysical; hence there is no spontaneous symmetry breaking invoked to generate massless Goldstone boson, nor is there any extra physical Higgs boson component; (iii) the massive physical gauge boson state \(A^a_P\) is converted from the massless transverse polarization state \(A^a_T\) combined with the longitudinal polarization state \(A^a_L\) via Eqs.(2.12) and (2.13a). The single physical degree of freedom of \(A^a_\mu\) is conserved before and after adding the CS term, through the topological conversion \(A^a_\mu \rightarrow A^a_\mu\). Taking the massless limit \(m \rightarrow 0\), we see that the massive state \(A^a_P\) disappears and the massless state \(A^a_T\) is released to be the physical transverse polarization, while the longitudinal state \(A^a_L\)
becomes fully unphysical again. As we will demonstrate shortly, in the high energy limit the scattering amplitudes of the physical polarization states \( A^a_P \) equal the corresponding amplitudes of the transverse polarization states \( A^a_T \), which means that the \( A^a_P \) state remembers its origin of \( A^a_T \) state under the limit \( m/E \to 0 \).

### 3 Topological Equivalence Theorem for Chern-Simons Gauge Theories

As shown above, for the 3d topological gauge theories (2.1), the Chern-Simons (CS) Lagrangian generates a topological mass for gauge boson \( A^a_\mu \) by converting the massless transverse polarization state \( A^a_T \) (combined with the longitudinal polarization state \( A^a_L \)) into the massive physical polarization state \( A^a_P \). In this section, we formulate the mechanism of topological mass-generation at the \( S \)-matrix level by newly proposing and proving a general Topological Equivalence Theorem (TET), which quantitatively connects the \( N \)-point scattering amplitudes of \( A^a_P \)'s to the corresponding amplitudes of the \( A^a_T \)'s in the high energy limit \( m/E \to 0 \).

#### 3.1 Topological Equivalence Theorem for Topological Mass Generation

Inspecting the quantized CS Lagrangians (2.1) and (2.3) and following the method in Refs. [31] [7], we can derive the following Slavnov-Taylor-type identity in momentum space:

\[
\langle 0| F^{a_1}(p_1) F^{a_2}(p_2) \cdots F^{a_N}(p_N) \Phi |0 \rangle = 0 ,
\]

(3.1)

which is based on the 3d gauge symmetry, where \( F^a \) is the gauge-fixing function defined in Eq.(2.3a), and the symbol \( \Phi \) denotes any other on-shell physical fields after the Lehmann-Symanzik-Zimmermann (LSZ) amputation. Since the function \( F^a \) contains only a single gauge field \( A^a_\mu \) having no mixing with any other field, it is straightforward to amputate each external \( F^a \) line by the LSZ reduction. We impose the on-shell condition \( p_j^2 = -m^2 \) for each external momentum. In the momentum space, we can express the gauge-fixing function \( F^a = -i p^\mu A^a_\mu = -i m A^a_S \). We also deduce \( v^\mu \equiv \epsilon^\mu_L - \epsilon^\mu_S = \mathcal{O}(m/E) \). With Eq.(2.10), we can express the scalar polarization vector \( \epsilon^\mu_S \) as

\[
\epsilon^\mu_S = \sqrt{2} \epsilon^\mu_P - (\epsilon^\mu_T + v^\mu) .
\]

(3.2)

Thus, we derive the following formula for the gauge-fixing function:

\[
F^a = -i \sqrt{2} m (A^a_P - \Omega^a) ,
\]

(3.3a)

\[
\Omega^a = \frac{1}{\sqrt{2}} (A^a_T + v^a) ,
\]

(3.3b)

where \( (A^a_P, A^a_T) = (\epsilon^\mu_P, \epsilon^\mu_T) A^a_\mu \) and \( v^a = v^\mu A^a_\mu \) with \( v^\mu = \epsilon^\mu_L - \epsilon^\mu_S = \mathcal{O}(m/E) \). With these and Eq.(3.1) after the LSZ reduction, we can derive the following TET identity which connects two
scattering amplitudes:

\[ \mathcal{T}[A^a_1, \ldots, A^a_N, \Phi] = \mathcal{T}[\Omega^a_1, \ldots, \Omega^a_N, \Phi], \]  

(3.4)

where we have made use of the fact that an amplitude including one or more external \( F \) lines plus any other external on-shell physical fields (such as \( A^a_\mu \) and/or \( \Phi \)) must vanish due to the identity (3.1). Thus, we can expand the TET identity as follows:

\[ \mathcal{T}[A^a_1, \ldots, A^a_N, \Phi] = \mathcal{T}[\tilde{A}^a_1 T, \ldots, \tilde{A}^a_N T, \Phi] + \mathcal{T}_v, \]  

(3.5a)

\[ \mathcal{T}_v = \sum_{j=1}^{N} \mathcal{T}[\tilde{v}^a_1, \ldots, \tilde{v}^a_j, \tilde{A}^a_{j+1} T, \ldots, \tilde{A}^a_N T, \Phi], \]  

(3.5b)

where for convenience we have adopted the notations \( \tilde{A}^a_\mu = \frac{1}{\sqrt{2}} A^a_\mu \) and \( \tilde{v}^a = \frac{1}{\sqrt{2}} v^a \). Under the high energy expansion, the residual term behaves as \( \mathcal{T}_v = \mathcal{O}(m/E) \ll 1 \) due to the suppression factor \( v^\mu \). Thus, we can derive the Topological Equivalence Theorem (TET):

\[ \mathcal{T}[A^a_1, \ldots, A^a_N, \Phi] = \mathcal{T}[\tilde{A}^a_1 T, \ldots, \tilde{A}^a_N T, \Phi] + \mathcal{O}\left(\frac{m}{E}\right). \]  

(3.6)

The TET (3.6) states that any \( A^a_\mu \)-scattering amplitude equals the corresponding \( A^a_T \)-scattering amplitude in the high energy limit. We note that different from the conventional equivalence theorem (ET) \cite{31,32} for the case of the SM Higgs mechanism, the right-hand-side (RHS) of Eq.(3.5) or Eq.(3.6) receives no multiplicative modification factor at loop level. This is because in the present case both \( A^a_\mu \) and \( A^a_T \) belong to the same gauge field \( A^a_\mu \) and the LSZ reduction on the left-hand-side (LHS) of Eq.(3.1) becomes much simpler.

Finally, we note that our present formulation of the TET (3.6) in the 3d CS gauge theories differs essentially from the conventional ET \cite{7} in the 4d SM because the 3d gauge bosons acquire gauge-invariant topological mass-term without invoking the conventional Higgs mechanism \cite{5}. We also note that the KK-ET \cite{8,9,10} for the compactified 5d Yang-Mills theories formulates the geometric Higgs mechanism at \( S \)-matrix level where the 5th component of 5d gauge field is converted to the longitudinal component of the corresponding 4d massive KK gauge boson. But our TET has essential difference from the KK-ET because the 5d gauge symmetry is spontaneously broken down to the 4d residual gauge symmetry of zero-modes by the boundary conditions of compactification and the induced KK gauge boson mass-term is not gauge-invariant. On the contrary, the 3d CS term for the topological mass-generation of gauge bosons can be manifestly gauge-invariant, and the inclusion of CS term does not change the gauge symmetry of the 3d theory.

\footnote{The 4d ET in the presence of the Higgs-gravity interactions was established in Ref. \cite{33} which can be applied to studying cosmological models (such as the Higgs inflation \cite{34,33,35}) or to testing self-interactions of weak gauge bosons and Higgs bosons \cite{33,36}.}
3.2 Power Counting Method for 3d Chern-Simons Theories

In this subsection, we develop a general energy power counting method for the scattering amplitudes in the 3d topologically massive gauge and gravity theories. We also present a general energy power counting rule on the $d$-dimensional scattering amplitudes in Appendix B.

We note that Weinberg proposed a power counting method for the 4d ungauged nonlinear $\sigma$-model as an effective theory of low energy QCD\cite{37}. The extensions of Weinberg’s power counting method to the compactified 5d Kaluza-Klein (KK) gauge theories and 5d KK gravity theory were recently given in Ref.\cite{19}.\footnote{Weinberg’s power counting rule was also extended previously\cite{7}\cite{38} to the 4d gauge theories including the SM, the SM effective theory (SMEFT), and the electroweak chiral Lagrangian.}

Weinberg’s power counting method includes the following key points: (i). For an $S$-matrix element $S$, its total mass-dimension $D_S$ is determined by the number of external states ($E$) and the number of spacetime dimensions, $D_S = 4 - E$, in the 4d field theories. (ii). Consider the scattering amplitude $S$ having scattering energy $E$ much larger than all the masses of the internal propagators as well as the masses of the external states. Thus, for the $E$-independent coupling constants contained in the amplitude $S$, their total mass-dimension $D_C$ can be counted directly according to the type of vertices therein. Based on these, the total energy-power dependence $D_E$ of the amplitude $S$ is given by $D_E = D_S - D_C$. We note that for our following derivation in 3d spacetime (or the general derivation in $d$-dimensional spacetime in Appendix B), we should modify the formula of $D_S$ in point (i) accordingly. As for the point (ii), it should hold for any high energy scattering with energy $E$ much larger than the involved particle masses. The nontrivial energy-dependence from the polarization vectors (tensors) of the gauge bosons (gravitons) can be taken into account accordingly. Keeping these in mind, we will construct the new power counting rules for the 3d topologically massive gauge and gravity theories.

Consider a general scattering $S$-matrix element $S$ having $E$ external states and $L$ loops ($L \geq 0$) in the $(2+1)d$ spacetime. Thus, we can deduce that the amplitude $S$ has a mass-dimension:

\[ D_S = 3 - \frac{1}{2} E, \tag{3.7} \]

where the number of external states $E = E_B + E_F$, with $E_B$ ($E_F$) being the number of external bosonic (fermionic) states. We note that the above Eq.(3.7) agrees to the $d = 3$ case of our general formula (B.1) in Appendix B. For the fermions, we only consider the SM fermions whose masses are much smaller than the scattering energy $E$. We denote the number of vertices of type-$j$ as $V_j$. Each vertex of type-$j$ contains $d_j$ derivatives, $b_j$ bosonic lines, and $f_j$ fermionic lines. Then, the energy-independent effective coupling constant in the amplitude $S$ has its total
mass-dimension given by
\[ D_C = \sum_j V_j \left( 3 - d_j - \frac{1}{2} b_j - f_j \right). \] (3.8)

For each Feynman diagram in the scattering amplitude \( S \), we denote the number of the internal lines as \( I = I_B + I_F \) with \( I_B \) (\( I_F \)) being the number of the internal bosonic (fermionic) lines. Thus, we have the following general relations:
\[ L = 1 + I - V, \quad \sum_j V_j b_j = 2I_B + \mathcal{E}_B, \quad \sum_j V_j f_j = 2I_F + \mathcal{E}_F, \] (3.9)
where \( V = \sum_j V_j \) is the total number of vertices in a given Feynman diagram. With these, we can derive the following leading energy dependence \( D_E = D_S - D_C \) from Eqs.(3.7)-(3.9):
\[ D_E = 2(1 - V) + L + \sum_j V_j \left( d_j + \frac{1}{2} f_j \right). \] (3.10)

Furthermore, we have the following relations:
\[ \sum_j V_j d_j = V_d, \quad \sum_j V_j f_j = 2V_F, \quad V = \sum_j V_j = V_3 + V_4, \quad V_3 = V_d + V_F + \bar{V}_3, \] (3.11)
where \( V_d \) denotes the number of all cubic vertices including one partial derivative and \( \bar{V}_3 \) denotes the number of bosonic cubic vertices having no partial derivative.

Then, we consider the topologically massive CS gauge theories. In such gauge theories, we have the relation \( 2I + \mathcal{E} = 3V_3 + 4V_4 \). With these, we can derive the following power counting rule on the leading energy-power dependence of a general scattering amplitude:
\[ D_E = (\mathcal{E}_A - \mathcal{E}_v) + (4 - \mathcal{E} - \bar{V}_3) - L, \] (3.12)
where \( \mathcal{E}_A \) is the number of external gauge boson states with physical polarizations \( A^a_{\mu} = \epsilon^a_{\mu} A^a_{\mu} \), and \( \mathcal{E}_v \) denotes the number of external gauge bosons \( v^a = v_{\mu} A^a_{\mu} \). In Eq.(3.12), the terms \( (\mathcal{E}_A - \mathcal{E}_v) \) arise from the high energy behaviors \( \epsilon^a_{\mu} = \mathcal{O}(E/m) \) and \( v^a = \epsilon^a_{\mu} - \epsilon^a_{S} = \mathcal{O}(m/E) \).

For the sake of later applications, we further consider the 3d topologically massive gravity (TMG) and derive the energy power counting rule for general scattering amplitudes of massive gravitons. The graviton self-interaction vertices from the gravitational CS term (5.1) (cf. Sec. 5) always contain 3 partial derivatives and contribute to the leading energy dependence of the graviton scattering amplitudes, which correspond to \( d_j = 3 \) and \( f_j = 0 \) in Eq.(3.10). Thus, we have \( \sum_j V_j d_j = 3V_{d3} \) and \( V = V_{d3} \) in such leading diagrams, where \( V_{d3} \) denotes the number of vertices containing 3 partial derivatives. Hence, the leading energy dependence of the pure graviton scattering amplitudes in (2+1)d arise from the Feynman diagrams containing the CS graviton vertices with 3 derivatives, and can be derived as follows:
\[ D_E = 2\mathcal{E}_{hp} + (2 + V_{d3} + L), \] (3.13)
where \( \mathcal{E}_{hp} \) denotes the number of external graviton states with physical polarizations \( (h_P = \epsilon^{\mu\nu}_P h_{\mu\nu}) \) and the physical graviton polarization tensor scales as \( \epsilon^{\mu\nu}_P = \mathcal{O}(E^2/m^2) \). For the leading tree-level diagrams composed of the cubic CS vertices with \( d_j = 3 \), we derive a relation \( \mathcal{E}_{hp} = 2 + V_3 \). Hence, using Eq.(3.13), we can deduce the leading energy dependence of such tree-level diagrams:

\[
D_E^0 = 3\mathcal{E}_{hp}.
\]

(3.14)

For instance, the leading four-graviton scattering amplitudes of the TMG theory contain individual leading energy terms of \( E^{12} \) at the tree level. We will analyze these further in section 5.

### 3.3 Energy Cancellations for Topological Scattering Amplitudes

In this subsection, we will apply our power counting rule (3.12) to analyze the leading energy dependence of the pure gauge boson scattering amplitudes in the 3d topological massive CS gauge theory. We also note that because the 3d CS theory is super-renormalizable, the leading energy dependence of a given amplitude is always given by the diagrams having \( L = 0 \) (tree level) and \( \nabla_3 = 0 \). Thus, given the external states of an amplitude, its maximal energy dependence is realized at tree level:

\[
D_{E}^{\max} = (\mathcal{E}_A - \mathcal{E}_v) + (4 - \mathcal{E}),
\]

(3.15)

with \( L = 0 \) and \( \nabla_3 = 0 \).

According to Eq.(3.15), the scattering amplitudes of pure gauge bosons \( (A^a_P) \) with the number of external states \( \mathcal{E} = \mathcal{E}_A = N \) and \( \mathcal{E}_v = 0 \) can receive leading individual contributions of \( \mathcal{O}(E^4) \) at the tree level. For the pure \( A^a_T \)-amplitudes with \( \mathcal{E} = \mathcal{E}_A = N \) and \( \mathcal{E}_A = \mathcal{E}_v = 0 \), its individual leading contributions scale like \( \mathcal{O}(E^{4-N}) \) at the tree level. With these, we find that our TET identity (3.5a) guarantees the energy cancellation in the \( N \)-gauge boson \( (A^a_P) \) scattering amplitude on its LHS:

\[
E^4 \rightarrow E^{4-N}.
\]

(3.16)

This is because on the RHS of Eq.(3.5a) the corresponding pure \( N \)-gauge boson \( (A^a_T) \) amplitude scales as \( \mathcal{O}(E^{4-N}) \) and the residual term \( T_v \) (with \( \mathcal{E}_v \geq 1 \)) scales no more than \( \mathcal{O}(E^{3-N}) \).

We can readily generalize this result to up to \( L \)-loop level and deduce the following energy cancellations based on Eq.(3.5a) and Eq.(3.12):

\[
\Delta D_E = D_E[NA^a_P] - D_E[NA^a_T] = N.
\]

(3.17)

Hence, the TET identity (3.5) [or the TET (3.6)] provides a general mechanism which guarantees the nontrivial energy cancellations in Eq.(3.16) or Eq.(3.17).

Before concluding the current section 3, we discuss further the conversion of physical degrees of freedom during the 3d topological mass-generation, in comparison with that realized during
the 5d geometric mass-generation under the Kaluza-Klein (KK) compactification. For the 3d gauge theories, before including the CS term, the massless gauge boson $A_{\mu}^a$ has only 1 physical transverse polarization state $A_{T}^a$; while after including the CS term, the gauge boson acquires a topological mass and generates a single physical polarization state $A_{P}^a$ (by absorbing the massless state $A_{T}^a$ combined with the longitudinal state $A_{L}^a$), without invoking the conventional spontaneous gauge symmetry breaking. Hence, this topological mass-generation mechanism leads to the conversion of the physical states: $A_{T}^a \rightarrow A_{P}^a$, which conserves the physical degree of freedom: $1 = 1$, as we explained earlier. In consequence, we observe that both the Lagrangians (2.1a)-(2.1b) and the gauge boson propagator (A.11a) indeed have a smooth massless limit $m \rightarrow 0$, which is similar to the massive KK gauge theories [8]-[10]. Based upon this mechanism of the topological mass-generation, we have newly established the TET (3.6) which connects a given $A_{P}^a$-amplitude to the corresponding $A_{T}^a$-amplitude under the high energy expansion.

In comparison, we note that the 5d geometric mass-generation for the KK gauge bosons $A_{n}^{\mu a}$ is realized by absorbing ("eating") the corresponding 5th components $A_{n}^{5a}$ of the 5d gauge fields $\hat{A}_{3-M}^a$ at each KK level-$n$ [8][9]. The 5th components $A_{n}^{5a}$ may be regarded as a kind of "geometric Goldstone bosons" due to the KK compactification, although they do not arise from a separate scalar Higgs potential and differs essentially from the conventional Higgs mechanism [5]. The 5d massless gauge boson $\hat{A}_{3-M}^a$ has 3 physical transverse polarizations and after KK compactification each 4d massive KK gauge boson $A_{0}^{\mu a}$ has 2 transverse polarizations plus 1 longitudinal polarization (from absorbing $A_{n}^{5a}$). So, the physical degrees of freedom are conserved before and after the KK mass generation: $3 = 2 + 1$; and this corresponds to the conversion of one physical degree of freedom at each KK level-$n$: $A_{5}^{a n} \rightarrow A_{L}^{a n}$. This geometric mass generation of KK gauge bosons leads to the KK Equivalence Theorem (KK-ET) which connects the high-energy scattering amplitudes of the longitudinal KK gauge bosons $A_{L}^{a n}$ to that of the corresponding KK Goldstone bosons $A_{n}^{5a}$ [8][10].

4 Topological Scattering Amplitudes and Energy Cancellations

In this section, we present explicit calculations of the four-particle scattering amplitudes in the topologically massive gauge theories including the Abelian QED (2.1a) and the non-Abelian TMYM theory (2.1b). With these, we newly demonstrate the energy cancellation of $E^2 \rightarrow E^0$.

---

4 Besides, the study of the geometric mass-generation of 5d KK gravitons and its gravitational equivalence theorem (GRET) were presented recently in Ref. [19], where the KK graviton field $h_{\mu \nu}^{n}$ becomes massive by absorbing the scalar-component $h_{55}^{n}$ and vector-component $h_{\mu 5}^{n}$ from compactification of the 5d graviton field $\hat{h}_{M N}$. Note that before compactification the massless 5d graviton $\hat{h}_{M N}$ has 5 physical degrees of freedom and after compactification the massive KK graviton $h_{\mu \nu}^{n}$ contains the physical states with helicities $\lambda = \pm 2, \pm 1, 0$. We see that the physical degrees of freedom are conserved before and after the KK graviton mass-generation: $5 = 2 + 2 + 1$.  

---

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Figure 1: Feynman diagrams for the scattering processes $\phi^-\phi^+ \to A_P A_P$ ($\phi^-\phi^+ \to A_T A_T$) and $\phi^- A_P \to \phi^- A_P$ ($\phi^- A_T \to \phi^- A_T$) in 3d topological massive scalar QED.

for the $A_P$-amplitudes in the TMQED and the energy cancellation of $E^4 \to E^0$ for the pure $A_T$-amplitudes in the TMYM theory, under high energy expansion. Then, we verify for the first time that the TET (3.6) holds for both the Abelian and non-Abelian CS gauge theories.

4.1 Topologically Massive QED and Scattering Amplitudes

In this subsection, we consider two realizations of the topologically massive QED, namely, the topologically massive scalar QED (TMSQED) and the topologically massive spinor QED (TMQED). We will compute the scattering amplitudes in these two models and uncover the nontrivial energy cancellations in these amplitudes. Then, we will demonstrate explicitly that the TET (3.6) holds in each model.

4.1.1 Scattering Amplitudes of Topologically Massive Scalar QED

We first consider the TMSQED, which is composed by the scalar QED plus the Chern-Simons term (2.1a). The Lagrangian contains a scalar sector:

$$L_S = -(D_\mu \phi)^* (D^\mu \phi) - m^2_\phi |\phi|^2 - \lambda |\phi|^4,$$

where we choose the metric tensor $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, 1, 1)$ and denotes the complex scalar field by $\phi$. The covariant derivative is defined as $D_\mu = \partial_\mu + i e A_\mu$. In the charge eigenstates, we have $(\phi^-, \phi^+) = (\phi, \phi^*)$, with $\phi^-$ ($\phi^+$) denoting the scalar electron (scalar positron).

In the following, we compute and analyze two types of scattering processes $\phi^- \phi^+ \to A_P A_P$ ($\phi^- \phi^+ \to A_T A_T$) and $\phi^- A_P \to \phi^- A_P$ ($\phi^- A_T \to \phi^- A_T$) at tree level, where the relevant Feynman diagrams are shown in Fig. 1.

For the annihilation processes $\phi^- \phi^+ \to A_P A_P$ and $\phi^- \phi^+ \to A_T A_T$, we find that under the high energy expansion and by using the power counting rule (3.12), the scattering amplitude
Table 1: Energy cancellations in the scattering amplitudes of 3d topologically massive scalar QED, $\mathcal{T}[\phi^-\phi^+\rightarrow A_P A_P] = \mathcal{T}_{PP}[(a)] + \mathcal{T}_{PP}[(b)] + \mathcal{T}_{PP}[(c)]$ and $\mathcal{T}[\phi^- A_P^0 \rightarrow \phi^- A_P^0] = \mathcal{T}_{PP}[(d)] + \mathcal{T}_{PP}[(e)] + \mathcal{T}_{PP}[(f)]$, where $E = E/m$ and $(s_\theta, c_\theta) = (\sin \theta, \cos \theta)$ with $\theta$ being the scattering angle. Each full amplitude equals the sum of individual diagrams $(a) + (b) + (c)$ and $(c) + (d) + (e)$, respectively, as shown in Fig. 1.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Amplitude} & \times \bar{E}^2 & \times \bar{E}^1 & \text{Amplitude} & \times \bar{E}^2 & \times \bar{E}^1 \\
\hline
\mathcal{T}_{PP}[(a)] & -e^2(1+c_\theta) & -2e^2 & \mathcal{T}_{PP}[(d)] & -2e^2 & 0 \\
\mathcal{T}_{PP}[(b)] & -e^2(1-c_\theta) & i2e^2 s_\theta & \mathcal{T}_{PP}[(e)] & e^2(1+c_\theta) & i2e^2 s_\theta \\
\mathcal{T}_{PP}[(c)] & 2e^2 & 0 & \mathcal{T}_{PP}[(f)] & e^2(1-c_\theta) & -i2e^2 s_\theta \\
\hline
\text{Sum} & 0 & 0 & \text{Sum} & 0 & 0 \\
\hline
\end{array}
\]

Thus, we can make high energy expansions for both amplitudes as follows:

\[
\begin{align*}
\mathcal{T}[\phi^- \phi^+ \rightarrow A_P A_P] &= \mathcal{T}_{PP}^{(2)} \bar{E}^2 + \mathcal{T}_{PP}^{(1)} \bar{E}^1 + \mathcal{T}_{PP}^{(0)} \bar{E}^0 + \mathcal{O}(\bar{E}^{-1}), \\
\mathcal{T}[\phi^- A_P^0 \rightarrow A_T A_T] &= \mathcal{T}_{TT}^{(0)} \bar{E}^0 + \mathcal{O}(\bar{E}^{-1}),
\end{align*}
\]

where $\bar{E} = E/m$ and $E$ denotes the energy of the scalar electron (positron). For simplicity, we set the scalar mass $m_\phi \simeq 0$. The amplitude $\mathcal{T}[\phi^- \phi^+ \rightarrow A_P A_P]$ contains the contributions of the Feynman diagrams $(a)-(c)$ of Fig. 1. According to Eq.(4.2a), we compute the amplitude at each order of the high energy expansion, which is given by the sum of the three diagrams $(a)-(c)$. As shown in Table 1, we demonstrate explicitly that the sum of diagrams $(a)-(c)$ vanishes at the $\mathcal{O}(\bar{E}^2)$ and $\mathcal{O}(\bar{E}^1)$:

\[
\begin{align*}
\mathcal{T}_{PP}^{(2)}[(a) + (b) + (c)] &= 0, \\
\mathcal{T}_{PP}^{(1)}[(a) + (b) + (c)] &= 0.
\end{align*}
\]

Furthermore, we compute both amplitudes $\mathcal{T}[\phi^- \phi^+ \rightarrow A_P A_P]$ and $\mathcal{T}[\phi^- A_P^0 \rightarrow A_T A_T]$ at the $\mathcal{O}(\bar{E}^0)$ and obtain:

\[
\mathcal{T}_{PP}^{(0)}[\phi^- \phi^+ \rightarrow A_P A_P] = \left(1 + \frac{1}{2}\mathcal{T}_{TT}^{(0)}[\phi^- \phi^+ \rightarrow A_T A_T]\right) = e^2.
\]

Without losing generality, we set the spin $s = \bar{m}/m = +1$ in the above calculations and afterwards.

Similarly, we compute the Compton scattering amplitudes $\mathcal{T}[\phi^- A_P \rightarrow \phi^- A_P]$ and $\mathcal{T}[\phi^- A_T \rightarrow \phi^- A_T]$. Then, we make the following high energy expansions for both amplitudes:

\[
\begin{align*}
\mathcal{T}[\phi^- A_P \rightarrow \phi^- A_P] &= \mathcal{T}_{PP}^{(2)} \bar{E}^2 + \mathcal{T}_{PP}^{(1)} \bar{E}^1 + \mathcal{T}_{PP}^{(0)} \bar{E}^0 + \mathcal{O}(\bar{E}^{-1}), \\
\mathcal{T}[\phi^- A_T \rightarrow \phi^- A_T] &= \mathcal{T}_{PP}^{(0)} \bar{E}^0 + \mathcal{O}(\bar{E}^{-1}).
\end{align*}
\]
As shown in Table 1, we demonstrate explicitly that the sum of the three diagrams (d)-(f) of Fig. 1 vanishes at the $O(E^2)$ and $O(E^3)$:

$$
\mathcal{T}^{(2)}_{\phi P}[(d) + (e) + (f)] = 0,
$$

(4.6a)

$$
\mathcal{T}^{(1)}_{\phi P}[(d) + (e) + (f)] = 0.
$$

(4.6b)

Moreover, we find that both amplitudes $\mathcal{T}[\phi^- A_P \to \phi^- A_P]$ and $\mathcal{T}[\phi^- A_T \to \phi^- A_T]$ are nonzero and equal at the $O(E^0)$:

$$
\mathcal{T}^{(0)}_{\phi P}[(\phi^- A_P \to \phi^- A_P)] = \frac{1}{2} \mathcal{T}^{(0)}_{\phi T}[(\phi^- A_T \to \phi^- A_T)] = -e^2.
$$

(4.7)

Finally, from Eqs. (4.3) and (4.6) together with Eqs. (4.4) and (4.7), we derive

$$
\mathcal{T}[\phi^- \phi^+ \to A_PA_P] = \frac{1}{2} \mathcal{T}[\phi^- \phi^+ \to A_TA_T] + O\left(\frac{m}{E}\right),
$$

(4.8a)

$$
\mathcal{T}[\phi^- A_P \to \phi^- A_P] = \frac{1}{2} \mathcal{T}[\phi^- A_T \to \phi^- A_T] + O\left(\frac{m}{E}\right),
$$

(4.8b)

which explicitly verify the TET (3.6) for the topologically massive scalar QED.

### 4.1.2 Scattering Amplitudes of Topologically Massive Spinor QED

In this subsection, we consider the topologically massive QED (TMQED) which includes the gauge sector Lagrangian (2.1a) (with Chern-Simons term) and the following matter Lagrangian,

$$
\mathcal{L}_f = \bar{\psi}(\gamma^\mu D_\mu - m_f)\psi,
$$

(4.9)

where the covariant derivative is defined as $D_\mu = \partial_\mu + ieA_\mu$ and the gamma matrices are given by $(\gamma^0, \gamma^1, \gamma^2) = (i\sigma^2, \sigma^1, \sigma^3)$. We define the 3d Dirac spinors and solve the 3d Dirac equation in Appendix C.

Then, we analyze the amplitudes of the annihilation process $e^+e^- \to A_PA_P$ ($e^+e^- \to A_TA_T$) and the Compton scattering $e^-A_P \to e^-A_P$ ($e^-A_T \to e^-A_T$). The relevant Feynman diagrams at tree level are shown in Fig. 2. Using the power counting rule (3.12), we find that the scattering amplitudes $\mathcal{T}[e^+e^- \to A_PA_P]$ and $\mathcal{T}[e^-A_P \to e^-A_P]$ have leading contributions scale as $E^2$, while the scattering amplitudes $\mathcal{T}[e^+e^- \to A_TA_T]$ and $\mathcal{T}[e^-A_T \to e^-A_T]$ have leading contributions scale as $E^0$. Thus, we can make the following high energy expansions:

$$
\mathcal{T}[e^-e^+ \to A_PA_P] = \mathcal{T}^{(2)}_{eP} E^2 + \mathcal{T}^{(1)}_{eP} E^1 + \mathcal{T}^{(0)}_{eP} E^0 + O(E^{-1}),
$$

(4.10a)

$$
\mathcal{T}[e^-e^+ \to A_TA_T] = \mathcal{T}^{(0)}_{eT} E^0 + O(E^{-1}),
$$

(4.10b)

$$
\mathcal{T}[e^-A_P \to e^-A_P] = \mathcal{T}^{(2)}_{eP} E^2 + \mathcal{T}^{(1)}_{eP} E^1 + \mathcal{T}^{(0)}_{eP} E^0 + O(E^{-1}),
$$

(4.10c)

$$
\mathcal{T}[e^-A_T \to e^-A_T] = \mathcal{T}^{(0)}_{eT} E^0 + O(E^{-1}),
$$

(4.10d)
where $\bar{E} = E/m$ and $E$ denotes the energy of the incoming electron (positron). For simplicity, we set the electron mass $m_e \simeq 0$.

Then, we explicitly compute the above scattering amplitudes. We find that all the $O(E^2)$ and $O(E^1)$ terms cancel exactly in each amplitude and the final results actually behave as $O(E^0)$. We present these cancellations explicitly in Table 2. Hence, we have

$$T^{(2)}_{PP}[(a) + (b)] = 0, \quad T^{(1)}_{PP}[(a) + (b)] = 0; \quad (4.11a)$$

$$T^{(2)}_{eP}[(c) + (d)] = 0, \quad T^{(1)}_{eP}[(c) + (d)] = 0. \quad (4.11b)$$

Finally, we derive the remaining amplitudes of $O(E^0)$ as follows:

$$T^{(0)}_{PP}[e^- e^+ \rightarrow A_p A_p] = \frac{1}{2} T^{(0)}_{TT}[e^- e^+ \rightarrow A_T A_T] = ie^2 \cot \theta, \quad (4.12a)$$

$$T^{(0)}_{eP}[e^- A_p \rightarrow e^- A_p] = \frac{1}{2} T^{(0)}_{eT}[e^- A_T \rightarrow e^- A_T] = ie^2 \frac{(3 + c_g)(1 + c_g + s_g)}{4(1 + c_g)(1 + s_g)^2}. \quad (4.12b)$$

For completeness, we also summarize in Appendix D the full tree-level amplitudes (without high energy expansion) for the scattering processes discussed above and in sections 4.1 and 4.2.1. These exact formulas can provide self-consistency checks for the corresponding expanded scattering amplitudes given in the main text and will also be useful for future studies.

From the above Eqs. (4.12a)-(4.12b), we deduce the following relations under the high energy expansion:

$$T[e^- e^+ \rightarrow A_p A_p] = \frac{1}{2} T[e^- e^+ \rightarrow A_T A_T] + O\left(\frac{m}{E}\right), \quad (4.13a)$$

$$T[e^- A_p \rightarrow e^- A_p] = \frac{1}{2} T[e^- A_T \rightarrow e^- A_T] + O\left(\frac{m}{E}\right). \quad (4.13b)$$

These verify explicitly that the TET (3.6) does hold, as expected from our general formulation of the TET in section 3.1. We observe that the TET identity (3.5) [or the TET (3.6)] provides a general mechanism which guarantees the exact energy cancellations of the $O(E^2)$ and $O(E^1)$ contributions in the $A_p$-amplitude and matches the corresponding $A_T$-amplitude of $O(E^0)$.

Before concluding this subsection, we further present an exact verification of the TET identity (3.4) or (3.5a) without taking the high energy limit and by considering the simplest case of $N = 1$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Scattering processes $e^+ e^- \rightarrow A_p A_p$ ($e^+ e^- \rightarrow A_T A_T$) via Feynman diagrams (a)-(b) and $e^- A_p \rightarrow e^- A_p$ ($e^- A_T \rightarrow e^- A_T$) via Feynman diagrams (c)-(d) in 3d topologically massive spinor QED.}
\end{figure}
Table 2: Energy cancellations in the scattering amplitudes of 3d topologically massive spinor QED, $\mathcal{T}[e^+e^-\rightarrow A_\mathbf{F}A_\mathbf{P}] = \mathcal{T}_\mathbf{PP}[(a)] + \mathcal{T}_\mathbf{PP}[(b)]$ and $\mathcal{T}[e^-A_\mathbf{P} \rightarrow e^-A_\mathbf{P}] = \mathcal{T}_\mathbf{eP}[(c)] + \mathcal{T}_\mathbf{eP}[(d)]$, where the notations are defined as $E = E/m$ and $(s_\theta, c_\theta) = (\sin \theta, \cos \theta)$ with $\theta$ denoting the scattering angle. Each full amplitude equals the sum of individual diagrams $(a)+(b)$ and $(c)+(d)$, respectively, as shown in Fig. 2.

For the scattering process $e^-e^+ \rightarrow A_\mathbf{P}A_\mathbf{P}$, we apply the TET identity (3.5) to just one external state of $A_\mathbf{P}$:

$$\mathcal{T}[e^-e^+ \rightarrow A_\mathbf{P}A_\mathbf{P}] = \mathcal{T}[e^-e^+ \rightarrow \tilde{A}_\mathbf{T}A_\mathbf{P}] + \mathcal{T}[e^-e^+ \rightarrow \tilde{v}A_\mathbf{P}],$$

(4.14)

where $\tilde{A}_\mathbf{T} = \frac{1}{\sqrt{2}} A_\mathbf{T}$ and $\tilde{v} = \frac{1}{\sqrt{2}} v = \frac{1}{\sqrt{2}} \epsilon^\mu A_\mu$. Using the basic relation of polarization vectors in Eq.(3.2), we can rewrite the above TET identity (4.14) as follows:

$$\mathcal{T}[e^-e^+ \rightarrow A_\mathbf{S}A_\mathbf{P}] = 0,$$

(4.15)

where $A_\mathbf{S} = \epsilon_\mu^\nu A_\mu$ is the unphysical scalar polarization state of the photon. As we explained above Eq.(3.2), the gauge-fixing function in momentum space can be expressed as $\mathcal{F} = -imA_\mathbf{S}$. Thus, the above TET identity (4.15) is equivalent to

$$\mathcal{T}[e^-e^+ \rightarrow \mathcal{F}A_\mathbf{P}] = 0,$$

(4.16)

which is just the simplest $N=1$ case of the Slavnov-Taylor-type identity (3.1). Hence, to verify the TET identity (3.5) in the case of $N=1$, we only need to prove explicitly that the identity (4.15) holds at the tree level.

The tree-level scattering process $e^-e^+ \rightarrow A_\mathbf{S}A_\mathbf{P}$ contains the same type of diagrams $(a)-(b)$ via $(t, u)$-channels, as shown in Fig. 2. Then, we compute directly the contributions of the $(t, u)$-channels as follows:

$$\mathcal{T}_t[e^-e^+ \rightarrow A_\mathbf{S}A_\mathbf{P}] = -\mathcal{T}_u[e^-e^+ \rightarrow A_\mathbf{S}A_\mathbf{P}] = \sqrt{2}e^2(iE^2s_\theta + Ec_\theta),$$

(4.17)

which ensures that the full amplitude vanishes:

$$\mathcal{T}[e^-e^+ \rightarrow A_\mathbf{S}A_\mathbf{P}] = \mathcal{T}_t + \mathcal{T}_u = 0.$$
4.2 Topologically Massive QCD and Scattering Amplitudes

In this subsection, we study four-point scattering amplitudes in the 3d topologically massive QCD (TMQCD) with non-Abelian gauge group SU($N$). This is also called the topologically massive YM (TMYM) theory in Sec. 2 for the pure gauge sector without matter fields. We will not discriminate these two terminologies hereafter. The Lagrangian of the TMQCD can be written as follows:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \frac{m}{2}\varepsilon^{\mu\nu\rho}A_\mu^a\partial_\nu A_\rho^a + \frac{gm}{6}C^{abc}\varepsilon^{\mu\nu\rho}A_\mu^aA_\nu^bA_\rho^c + \sum_{i,j=1}^{N}\bar{\psi}_i(\gamma_\mu D_{ij}^\mu - m_\varphi\delta_{ij})\psi_j,$$

where $D_{ij}^\mu = \delta_{ij}\partial^\mu - igA^{a\mu}T^a_{ij}$ and $(i, j)$ denote the color indices of the quarks. We will compute the scattering amplitudes of the quark-antiquark annihilation and the pure gauge boson scattering, from which we uncover the nontrivial energy cancellations. Then, we will demonstrate explicitly that the TET (3.6) holds for the non-Abelian TMQCD.

4.2.1 Scattering Amplitudes of Quark-Antiquark Annihilation

In this subsection, we analyze the scattering amplitudes of quark-antiquark annihilation processes $q\bar{q} \rightarrow A_\mu^aA_\mu^b$ and $q\bar{q} \rightarrow A_\mu^aA_T^b$, which include three Feynman diagrams as shown in Fig. 3. The non-Abelian cubic gluon vertex generates the $s$-channel diagram of Fig. 3(a) which is absent in the $e^-e^+$ annihilation process of the TMQED as shown Fig. 2(a)-(b).

Applying the power counting rule (3.12), we find that the high-energy scattering amplitude $\mathcal{T}[q\bar{q} \rightarrow A_\mu^aA_\mu^b]$ has leading contributions scale as $E^2$, while the amplitude $\mathcal{T}[q\bar{q} \rightarrow A_\mu^aA_T^b]$ scales as $E^0$. Thus, we can make the following high energy expansions:

$$\mathcal{T}[q\bar{q} \rightarrow A_\mu^aA_\mu^b] = \mathcal{T}_{pp}^{(2)}\bar{E}^2 + \mathcal{T}_{pp}^{(1)}\bar{E}^1 + \mathcal{T}_{pp}^{(0)}\bar{E}^0 + \mathcal{O}(\bar{E}^{-1}),$$

$$\mathcal{T}[q\bar{q} \rightarrow A_\mu^aA_T^b] = \mathcal{T}_{TT}^{(0)}\bar{E}^0 + \mathcal{O}(\bar{E}^{-1}),$$

where $\bar{E} = E/m$ and $E$ denotes the energy of the incoming quark (anti-quark). For simplicity, we set the quark mass $m_q \simeq 0$. Then, we explicitly compute these scattering amplitudes, and find that the summed contributions in each amplitude cancel exactly at $\mathcal{O}(E^2)$ and $\mathcal{O}(E^1)$, respectively.
respectively. The final net results could only behave as $O(E^0)$. We present these cancellations explicitly in Table 3. From these, we deduce

$$T_{pp}^{(2)}[(a)+(b)+(c)] = 0,$$

$$T_{pp}^{(1)}[(a)+(b)+(c)] = 0,$$

where we have applied the commutation relation $[T^a, T^b] = iC^{abc}T^c$ to the sum of the diagrams $(b)+(c)$, which further cancels the contribution of the diagram $(a)$ at $O(E^2)$ and $O(E^1)$ respectively.

Next, we compute the remaining $q\bar{q}$ annihilation amplitudes at $O(E^0)$ and derive the following results:

$$T_{pp}^{(0)}[q_i \bar{q}_j \rightarrow A_P^a A_P^b] = \frac{1}{2} T_{TT}^{(0)}[q_i \bar{q}_j \rightarrow A_T^a A_T^b] = \frac{g^2}{4} \left[ \frac{s_{2\theta}}{1+c_\theta} (T^a_{jk} T^b_{ki}) + \frac{s_{2\theta}}{1-c_\theta} (T^b_{jk} T^a_{ki}) \right].$$

We may further define the color-singlet states of the SU($N$) gauge group:

$$|0\rangle_q = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |q_j \bar{q}_j\rangle, \quad |0\rangle_{A_p} = \frac{1}{\sqrt{2(N^2-1)}} \sum_{a=1}^{N^2-1} |A_P^a A_P^a\rangle, \quad |0\rangle_{A_T} = \frac{1}{\sqrt{2(N^2-1)}} \sum_{a=1}^{N^2-1} |A_T^a A_T^a\rangle.$$

Then, we compute the $q\bar{q}$ annihilation amplitudes of the color-singlet states at $O(E^0)$:

$$T_{pp}^{(0)}[|0\rangle_q \rightarrow |0\rangle_{A_p}] = \frac{1}{2} T_{TT}^{(0)}[|0\rangle_q \rightarrow |0\rangle_{A_T}] = ig^2 f(N) \cot \theta,$$

where we have defined the function $f(N) = \frac{1}{2\sqrt{2}} \sqrt{(N^2-1)/N}$. We note that for the color-singlet initial and final states, the $s$-channel contribution vanishes due to $C^{aac} = 0$, and the sum of the $(t,u)$-channel contributions just differs from the Abelian case of TMQED by an overall factor $(g^2/e^2)f(N)$. This relation holds even without making the high energy expansion, namely,

$$T_{pp}[|0\rangle_q \rightarrow |0\rangle_{A_p}] = \frac{g^2}{e^2} f(N) T[e^{-} e^{+} \rightarrow A_P A_P],$$

$$T_{TT}[|0\rangle_q \rightarrow |0\rangle_{A_T}] = \frac{g^2}{e^2} f(N) T[e^{-} e^{+} \rightarrow A_T A_T].$$

Table 3: Energy cancellations in the scattering amplitude $T[q_i \bar{q}_j \rightarrow A_P^a A_P^b] = T_{pp}[(a)] + T_{pp}[(b)] + T_{pp}[(c)]$ of 3d topologically massive QCD, where the relevant Feynman diagrams (a)-(c) are shown in Fig. 3.
After the high energy expansion, only the $\mathcal{O}(E^0)$ amplitudes survive for the TMQCD and TMQED, as shown in Eq.(4.24) and Eq.(4.12a) which obey the above relations (4.25).

Finally, from Eqs.(4.20)(4.21) and Eqs.(4.22)(4.24), we derive the following equivalence relation under the high energy expansion:

$$
\mathcal{T}[q_i \bar{q}_j \rightarrow A^a_\mu A^b_\mu] = \frac{1}{2} \mathcal{T}[q_i \bar{q}_j \rightarrow A^a_\mu A^b_\mu] + \mathcal{O}\left(\frac{m}{E}\right),
$$

(4.26)

which explicitly realizes the TET (3.6).

### 4.2.2 Pure Gauge Boson Scattering Amplitudes

In this subsection, we study the four-particle amplitudes of the pure gauge boson scattering processes $A^a_\mu A^b_\mu \rightarrow A^c_\mu A^d_\mu$ and $A^a_\tau A^b_\tau \rightarrow A^c_\tau A^d_\tau$ in the 3d non-Abelian topologically massive YM (TMYM) theory, where the gauge field $A^a_\mu$ belongs to the adjoint representation of the SU($N$) gauge group. The relevant Feynman diagrams are shown in Fig.4.

We see that the four-gauge boson scattering amplitudes $\mathcal{T}[A^a_\mu A^b_\mu \rightarrow A^c_\mu A^d_\mu]$ and $\mathcal{T}[A^a_\tau A^b_\tau \rightarrow A^c_\tau A^d_\tau]$ receive contributions from the contact diagram and the pole diagrams via (s, t, u)-channels. The kinematics of such four-particle elastic scattering processes is defined in Appendix A. Using the power counting rule (3.12), we find that for the scattering amplitude $\mathcal{T}[4A^a_\mu]$ the leading contributions of each diagram in Fig.4 scale as $E^3$, while for the scattering amplitude $\mathcal{T}[4A^a_\tau]$ the individual leading contributions scale as $E^0$. Hence, using the TET identity (3.5) [or the TET (3.6)], we predict that the $A^a_\mu$-amplitude should contain exact energy cancellations at the $\mathcal{O}(E^1)$, $\mathcal{O}(E^2)$, and $\mathcal{O}(E^3)$, respectively. This is because the leading energy-dependence of the $A^a_\mu$-amplitude must match that of the corresponding $A^a_\tau$-amplitude of $\mathcal{O}(E^0)$ on the RHS of the TET identity (3.5) [or the TET (3.6)].

Then, we compute explicitly the full scattering amplitude of $A^a_\mu$’s at tree level and present it in a compact form:

$$
\mathcal{T}[4A^a_\mu] = g^2 \left( \frac{C_s N_s}{s-m^2} + \frac{C_t N_t}{t-m^2} + \frac{C_u N_u}{u-m^2} \right),
$$

(4.27)
where the color factors are defined as usual \((C_s, C_t, C_u) = (C^{abc}C^{cde}, C^{ade}C^{bce}, C^{ace}C^{dbe})\) with \(C^{abc}\) denoting the structure constants of the gauge group. The numerators \((\mathcal{N}_s, \mathcal{N}_t, \mathcal{N}_u)\) take the following form:

\[
\mathcal{N}_s = \frac{4m^2 - s}{16m^3 s^2} \left[ 4ms^\frac{3}{2}(5m^2 + 4s) c_\theta + i(4m^4 + 29m^2 s + 3s^2) s_\theta \right],
\]

\[
\mathcal{N}_t = -\frac{c_{\theta/2}}{16m^3} \left( s^{\frac{1}{2}} + i2m \tan \frac{\theta}{2} \right)^2 \times \\
\left\{ 4m[13m^2 - 3s + (8m^2 - s) c_\theta] c_{\theta/2} + i s^{\frac{1}{2}}[22m^2 - 3s + (20m^2 - 3s) c_\theta] s_{\theta/2} \right\},
\]

\[
\mathcal{N}_u = \frac{s_{\theta/2}}{16m^3} \left( s^{\frac{1}{2}} - i2m \cot \frac{\theta}{2} \right)^2 \times \\
\left\{ 4m[13m^2 - 3s - (8m^2 - s) c_\theta] s_{\theta/2} - i s^{\frac{1}{2}}[22m^2 - 3s - (20m^2 - 3s) c_\theta] c_{\theta/2} \right\}.
\]

We note that in the \((2+1)d\) spacetime there is a kinematic exchange symmetry between the scattering amplitudes of \(t\)-channel and \(u\)-channel, namely, their numerators obey the relation \(\mathcal{N}_u(\pi + \theta) = -\mathcal{N}_t(\theta)\). We have verified that our numerators (4.28b)-(4.28c) indeed satisfy this kinematic exchange symmetry.

We note that each term on the RHS of Eq. (4.27) scales as \(E^3\) at most because summing up each contribution of the contact diagram with the corresponding pole diagram already cancels \(O(E^4)\) terms, as we show in Table 4. For the high energy scattering with \(E \gg m\), we expand the full amplitudes in terms of \(1/s_0\), where \(s_0 = 4E^2\beta^2\) and \(\bar{s}_0 = 4\bar{E}^2\beta^2\) with \(\bar{E} = E/m\). Thus, we can explicitly demonstrate the exact energy cancellations at each order of \(E^n\) \((n = 4, 3, 2, 1)\), respectively. We summarise our findings in Table 4, from which we prove the following exact energy cancellations:

\[
\mathcal{T}^{(4)}_{c_1} + \mathcal{T}^{(4)}_j = 0,
\]

\[
\sum_j \left( \mathcal{T}^{(3)}_{c_1} + \mathcal{T}^{(3)}_j \right) = -i24s_\theta \bar{s}_0^\frac{3}{2} c_0 (C_s + C_t + C_u) = 0,
\]

\[
\sum_j \left( \mathcal{T}^{(2)}_{c_1} + \mathcal{T}^{(2)}_j \right) = -128c_\theta \bar{s}_0 c_0 (C_s + C_t + C_u) = 0,
\]

\[
\sum_j \left( \mathcal{T}^{(1)}_{c_1} + \mathcal{T}^{(1)}_j \right) = -i304s_\theta \bar{s}_0^\frac{5}{2} c_0 (C_s + C_t + C_u) = 0,
\]

where \(j \in (s, t, u)\), \(c_0 = g^2/128\), and the superscript \((n)\) in the amplitudes \((\mathcal{T}^{(n)}_{c_1}, \mathcal{T}^{(n)}_j)\) denotes the contributions at the \(O(E^n)\) with \(n = 1, 2, 3, 4\). The amplitudes \(\mathcal{T}_c\) (contributed by the contact diagram) and \(\mathcal{T}_j\) (contributed by the gauge-boson-exchange in each channel-\(j\)) are given by the sums:

\[
\mathcal{T}_c = \sum_{j,n} \mathcal{T}^{(n)}_{c_1}, \quad \mathcal{T}_j = \sum_n \mathcal{T}^{(n)}_j,
\]
The factor is defined as decomposed into three sub-amplitudes according to their color factors, where

\[ j \in (s, t, u) \quad \text{and} \quad n = 4, 3, 2, \cdots. \]  

From Table 4 and Eq. (4.29a), the \( O(E^4) \) contributions cancel exactly between the contact diagram and the pole diagrams in each channel of \( j \in (s, t, u) \). Furthermore, it is striking to see that the sum of each \( O(E^n) \) contributions \( (n = 3, 2, 1) \) also cancel exactly because of the Jacobi identity \( C_s + C_t + C_u = 0 \), as shown in Table 4 and Eqs. (4.29b)-(4.29d).

After all these energy cancellations, we systematically derive the remaining scattering amplitude at \( O(E^0) \). We also compute the amplitude \( T[4A_T^a] \) which contains terms no more than \( O(E^0) \) by the direct power counting. Thus, we present both scattering amplitudes expanded to \( O(E^0) \) as follows:

\[
T_0[4A_T^a] = g^2 \left[ C_s \left( \frac{-9c_\theta}{4} \right) + C_t \left( \frac{-1-9c_\theta-4c_{2\theta}}{4(1+c_\theta)} \right) + C_u \left( \frac{1-9c_\theta+4c_{2\theta}}{4(1-c_\theta)} \right) \right],
\]

\[
T_0[4\bar{A}_T^a] = g^2 \left[ C_s \left( \frac{-c_\theta}{4} \right) + C_t \left( \frac{3-c_\theta}{4(1+c_\theta)} \right) + C_u \left( \frac{-3-c_\theta}{4(1-c_\theta)} \right) \right],
\]

where we have denoted \( \bar{A}_T^a = \frac{1}{\sqrt{2}} A_T^a \) as before. Comparing the two amplitudes above, we find that they differ by an amount:

\[
T_0[4A_T^a] - T_0[4\bar{A}_T^a] = -2g^2c_\theta(C_s + C_t + C_u) = 0,
\]

which vanishes identically because of the Jacobi identity \( C_s + C_t + C_u = 0 \). This demonstrates

| Amplitude | \( \times s_0^2 \) | \( \times s_0^{3/2} \) | \( \times s_0 \) | \( \times s_0^{1/2} \) |
|-----------|----------------|----------------|----------------|----------------|
| \( T_{cs} \) | \( 8s_\theta C_s \) | \( i32s_\theta C_s \) | \( 64c_\theta C_s \) | \( i64s_\theta C_s \) |
| \( T_{ct} \) | \( -(5+4c_\theta-c_{2\theta}) C_t \) | \( -i8(2s_\theta-s_{2\theta}) C_t \) | \( -32(c_\theta-c_{2\theta}) C_t \) | \( -i16(2s_\theta-5s_{2\theta}) C_t \) |
| \( T_{cu} \) | \( (5-4c_\theta-c_{2\theta}) C_u \) | \( -i8(2s_\theta+s_{2\theta}) C_u \) | \( -32(c_\theta+c_{2\theta}) C_u \) | \( -i16(2s_\theta+5s_{2\theta}) C_u \) |
| \( T_s \) | \( -8s_\theta C_s \) | \( -i56s_\theta C_s \) | \( -192c_\theta C_s \) | \( -i368s_\theta C_s \) |
| \( T_t \) | \( (5+4c_\theta-c_{2\theta}) C_t \) | \( -i8(s_\theta+s_{2\theta}) C_t \) | \( -32(3c_\theta+c_{2\theta}) C_t \) | \( -i16(17s_\theta+5s_{2\theta}) C_t \) |
| \( T_u \) | \( -(5-4c_\theta-c_{2\theta}) C_u \) | \( -i8(s_\theta-s_{2\theta}) C_u \) | \( -32(3c_\theta-c_{2\theta}) C_u \) | \( -i16(17s_\theta-5s_{2\theta}) C_u \) |
| Sum | 0 | 0 | 0 | 0 |

Table 4: Energy cancellations in the four-gauge boson scattering amplitude of 3d non-Abelian TMYM theory, \( T[A_p^a A_P^b \rightarrow A_p^c A_P^d] = T_c + T_s + T_t + T_u \), where the amplitude from contact diagram is further decomposed into three sub-amplitudes according to their color factors, \( T_c = T_{cs} + T_{ct} + T_{cu} \). The energy factor is defined as \( \bar{s}_0 = 4E^2/32 \) and \( E = E/m \). A common overall factor \( (g^2/128) \) in each amplitude is not displayed for simplicity.
explicitly the TET for the four-gauge boson scattering in the non-Abelian TMYM theory:

$$\mathcal{T}[A_{p}^{a}A_{p}^{b}\rightarrow A_{p}^{c}A_{p}^{d}] = \mathcal{T}[A_{T}^{a}A_{T}^{b}\rightarrow A_{T}^{c}A_{T}^{d}] + \mathcal{O}\left(\frac{m}{E}\right),$$

which confirms the general TET (3.6).

Before concluding this subsection, we stress that the present study has well understood and justified the structure of our gauge boson scattering amplitude (4.27) order by order under the high energy expansion, including the exact energy cancellations at each $\mathcal{O}(E^n)$ with $n = 4, 3, 2, 1$ and the proof of the TET relation (4.32) at $\mathcal{O}(E^0)$.

4.3 Unitarity Bounds on TMYM and TMG Theories

In this subsection, we analyze the partial wave amplitudes of the 3d topologically massive gauge boson scattering and the 3d topologically massive graviton scattering (Sec. 5). We will demonstrate that the partial wave amplitudes for either the topologically massive gauge boson scattering or the topologically massive graviton scattering have high energy behaviors no larger than $\mathcal{O}(E^0)$, so they can respect the perturbative unitarity bound.

For an SU($N$) gauge theory, we define the gauge-singlet one-particle state:

$$|0\rangle_{A_{p}} = \frac{1}{\sqrt{2(N^2-1)}} \sum_{a=1}^{N^2-1} |A_{p}^{a}A_{p}^{a}\rangle.$$ (4.34)

Thus, we compute the scattering amplitude for the gauge-singlet state as follows:

$$\mathcal{T}[|0\rangle_{A_{p}}\rightarrow |0\rangle_{A_{p}}] = \frac{g^2 N}{2} \left(\frac{N_{t}'}{t-m^2} - \frac{N_{u}'}{u-m^2}\right).$$ (4.35)

In $d$-dimensions, the partial wave amplitude takes the following general form [39]:

$$a_{\ell}(s) = \frac{s^{d/2-2}}{C_{\ell}^{\nu}(1)\lambda_{d}^{2}} \int_{0}^{\pi} d\theta \left[(\sin\theta)^{d-3} C_{\ell}^{\nu}(\cos\theta) \mathcal{T}_{el}\right],$$ (4.36)

where $C_{\ell}^{\nu}(x)$ is the Gegenbauer polynomial and

$$\nu = \frac{1}{2} (d-3), \quad \lambda_{d} = 2\Gamma\left(\frac{1}{2} d-1\right) (16\pi)^{d/2-1}.$$ (4.37)

The partial wave $a_{\ell}$ should satisfy the unitarity condition $\text{Im}(a_{\ell}) \geq |a_{\ell}|^2$, leading to [39][40]:

$$|a_{\ell}| \leq 1, \quad |\text{Re}(a_{\ell})| \leq \frac{1}{2}, \quad |\text{Im}(a_{\ell})| \leq 1.$$ (4.38)

For the present study, we have $d=3$. Thus, we can compute the real part of the $s$-wave amplitude (4.36) as follows:

$$\text{Re}(a_{0}) = \frac{1}{8\pi \sqrt{s}} \int_{0}^{\pi} d\theta \text{Re}(\mathcal{T}[|0\rangle_{A_{p}}\rightarrow |0\rangle_{A_{p}}]).$$
\[-\frac{g^2 N (16m^4 + 24m^2 s + s^2)}{32\sqrt{s} (s - 4m^2)^2} \simeq -\frac{N g^2}{32\sqrt{s}}. \quad (4.39)\]

The imaginary part \( \Im (a_0) \) has collinear divergences around \( \theta = 0, \pi \) of the integral. After adding an angular cut on the scattering angle (\( \delta \leq \theta \leq \pi - \delta \)) to remove the collinear divergences of the integral, we find that \( \Im (a_0) \) vanishes. Eq.(4.39) shows that for high energy scattering the leading partial wave amplitude \( \Re (a_0) \) scales as \( E^{-1} \), which has good high energy behaviors. This is expected, because the 3d TMYM theory is gauge-invariant and has a super-renormalizable gauge coupling \( g \) with mass-dimension \( +\frac{1}{2} \). Applying the unitarity condition (4.38) to s-wave amplitude (4.39), we derive the following perturbative unitarity bound:

\[ \sqrt{s} \geq \frac{g^2 N}{16}, \quad (4.40) \]

which puts a lower limit on the scattering energy, in addition to the requirement of high energy expansion \( \sqrt{s} \gg m \).

Next, we study the perturbative unitarity bound for the TMG theory in Sec.5. Using the high-energy graviton scattering amplitude in Eqs.(5.13)-(5.16), we compute the partial wave amplitudes of its real and imaginary parts as follows:

\[ \Re (a_0) \simeq -\frac{15\kappa^2 m^2}{2048 \pi \sqrt{s}} \frac{(3 \cos \delta - \cos 3\delta)}{\sin^3 \delta} \simeq -\frac{15\kappa^2 m^2}{1024 \pi \delta^3 \sqrt{s}}, \quad (4.41a) \]

\[ \Im (a_0) \simeq -\frac{247\kappa^2 m}{49152 \pi} \frac{(3 \cos \delta - \cos 3\delta)}{\sin^3 \delta} \simeq -\frac{247\kappa^2 m}{24576 \pi \delta^3}, \quad (4.41b) \]

where we put an angular cut on the scattering angle (\( \delta \leq \theta \leq \pi - \delta \)) to remove the collinear divergences of the integral. We see that both \( \Re (a_0) \) and \( \Im (a_0) \) exhibit good high-energy behaviors since they scale as \( E^{-1} \) and \( E^0 \), respectively. Imposing the perturbative unitarity condition on the s-wave amplitude (4.41), we derive the unitarity bounds on the real and imaginary parts:

\[ \sqrt{s} \geq \frac{15\kappa^2 m^2}{1024 \pi \delta^3}, \quad m \leq \frac{49152 \pi \delta^3}{247\kappa^2}, \quad (4.42) \]

where the first condition places a lower bound on the scattering energy proportional to \( \kappa^2 m^2 \). The second condition puts an upper bound on the graviton mass \( m \), proportional to \( 1/\kappa^2 \) which characterizes the ultraviolet cutoff scale of the TMG gravity since the 3d gravitational coupling \( \kappa^2 = 16\pi G = 2/M_{\text{Pl}} \) has a negative mass-dimension \(-1\), where \( G \) and \( M_{\text{Pl}} \) denote the 3d Newton constant and Planck mass respectively.

### 5 Structure of Topological Graviton Amplitude from Double-Copy

In this section, we study the extended double-copy construction of the massive graviton amplitude in the 3d topologically massive gravity (TMG) from the massive gauge boson amplitude in
the 3d TMYM theory. Our focus is to analyze the structure of massive graviton scattering amplitudes under high energy expansion and newly uncover a series of striking energy-cancellations of the graviton amplitudes in connection to the corresponding gauge boson amplitudes through the extended massive double-copy construction. In section 5.1, starting from the 3d action of the TMG we will analyze the equation of motion (EOM) of the 3d graviton field and identify the physical polarization state of the graviton. Then, in section 5.2 we will extend the conventional double-copy method for massless gauge/gravity theories\cite{11}\cite{12} to the 3d topologically massive gauge/gravity theories. For this we will improve the original massive four-gauge-boson scattering amplitude (4.27)-(4.28) by proper choice of the gauge transformation on its kinematic numerators. With these we can reconstruct the correct four-graviton scattering amplitude in the TMG theory. We stress that a key feature of the 3d TMYM and TMG theories is that these theories can realize the topological mass-generation of gauge bosons and gravitons in a fully gauge-invariant way under the path integral formulation, which is important for the successful double-copy construction in the 3d massive gauge/gravity theories.

5.1 3d Topologically Massive Gravity

In this subsection, we first introduce the 3d action of the TMG. Then, we analyze the equation of motion of the 3d graviton field and identify the physical polarization state of the graviton. We note that even though the 3d massless Einstein gravity has no physical content\cite{1}\cite{41}\cite{42}, including the gravitational Chern-Simons term can make the TMG fully nontrivial. The TMG action contains the conventional Einstein-Hilbert term and the gravitational Chern-Simons term\cite{1}:

\[ S_{\text{TMG}} = -\frac{2}{\kappa^2} \int d^3 x \left[ \sqrt{-g} R - \frac{1}{2\tilde{m}} \varepsilon^{\mu\nu\rho} \Gamma^\alpha_{\rho\beta} \left( \partial_\mu \Gamma^\beta_{\alpha\nu} + \frac{2}{3} \Gamma^\beta_{\mu\gamma} \Gamma^\gamma_{\nu\alpha} \right) \right], \tag{5.1} \]

where the 3d gravitational coupling constant \( \kappa \) is connected to the Planck mass \( M_{\text{Pl}} \) via \( \kappa = 2/\sqrt{M_{\text{Pl}}} \) with \( M_{\text{Pl}} = 1/(8\pi G) \) and \( G \) as the Newton constant. The parameter \( \tilde{m} \) in Eq.(5.1) will provide the graviton mass \( m = |\tilde{m}| \), as shown in Eq.(E.6). Under the weak field expansion \( g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \) and the linearized diffeomorphism transformation \( h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \), the change of the gravitational Chern-Simons term in Eq.(5.1) gives a total derivative, so the action is diffeomorphism invariant under the path integral formulation.

The nonlinear EOM is derived as follows\cite{42}:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{m} C_{\mu\nu} = 0, \tag{5.2} \]

where the Cotton tensor \( C_{\mu\nu} = \varepsilon_{\rho}^{\mu\alpha} \nabla_\rho (R_{\sigma\nu} - \frac{1}{2} g_{\sigma\nu} R) \) is symmetric and traceless. In Eq.(5.2) and the discussions hereafter, we use the positive branch of the mass parameter \( \tilde{m} > 0 \), which corresponds to the graviton with spin \( s = +2 \)\cite{1}. Then, we can expand the metric tensor around
the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ and impose the transverse-traceless condition for the graviton field. With these, we obtain the linearized EOM from Eq.(5.2):

$$\eta_{\mu\alpha} h_{\nu\beta} + \frac{1}{2m} \left( \varepsilon_{\mu\rho\alpha} h_{\nu\beta} + \varepsilon_{\nu\rho\beta} h_{\mu\alpha} \right) \partial^\rho = 0 . \quad (5.3)$$

We may denote the operator in the square brackets of Eq.(5.3) as

$$\hat{O}_{\mu\nu\alpha\beta} = \eta_{\mu\alpha} h_{\nu\beta} + \frac{1}{2m} \left( \varepsilon_{\mu\rho\alpha} h_{\nu\beta} + \varepsilon_{\nu\rho\beta} h_{\mu\alpha} \right) \partial^\rho . \quad (5.4)$$

Then, we act the operator $\hat{O}_{\mu\nu\alpha\beta}$ twice on the graviton field and impose the transverse-traceless condition on the physical graviton state, which leads to

$$\left( \partial^2 - m^2 \right) \partial^2 h_{\mu\nu} = 0 . \quad (5.5)$$

This shows that the graviton field obeys a Klein-Gordon-like equation and carries the physical mass $m$.

Alternatively, we can “square” the EOM of the TMYM theory (2.5) and arrive at

$$\left( \eta_{\mu\alpha} h_{\nu\beta} - \frac{i}{m} \varepsilon_{\mu\rho\alpha} p^\rho + \frac{i}{m} \varepsilon_{\nu\sigma\beta} p^\sigma - \frac{\varepsilon_{\mu\rho\alpha} \varepsilon_{\nu\sigma\beta} p^\rho p^\sigma}{m^2} \right) \epsilon^\alpha \epsilon^\beta = 0 , \quad (5.6)$$

where in the second row we have used the relations:

$$\epsilon_{\mu\nu} = \epsilon_{\mu} \epsilon_{\nu} , \quad p^\mu \epsilon_{\mu} = 0 , \quad \epsilon_{\mu} = 0 , \quad \varepsilon_{\mu\rho\alpha} \epsilon_{\nu\sigma\beta} p^\rho p^\sigma = \left( \eta_{\mu\beta} h_{\nu\alpha} - \eta_{\mu\nu} h_{\alpha\beta} \right) p^2 + \eta_{\alpha\beta} p_\mu p_\nu + \eta_{\mu\nu} p_\alpha p_\beta - \eta_{\nu\alpha} p_\mu p_\beta - \eta_{\mu\beta} p_\nu p_\alpha , \quad (5.7)$$

with the momentum $p^\mu$ obeying the on-shell condition $p^2 = -m^2$. Thus, we see that Eq.(5.6) coincides with Eq.(5.3) where the graviton field is expressed in the plane wave form $h_{\mu\nu} = \epsilon_{\mu\nu} e^{-ip \cdot x}$, with $\epsilon_{\mu\nu} = \epsilon_{\mu} \epsilon_{\nu}$. The graviton polarization tensor $\epsilon_{\mu\nu}^{(\mu)}$ is the solution of the EOM (5.6), where the subscript “$p$” indicates that it corresponds to the physical polarization state of the graviton $h_{\mu\nu}$. Then, we impose the following gauge-fixing term:

$$\mathcal{L}_{GF} = \frac{1}{\xi} (\mathcal{F}_\mu)^2 , \quad \mathcal{F}_\mu = \partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h . \quad (5.8)$$

With the above and making the weak field expansion, we can derive the graviton propagator, which we present in Eq.(E.6) of Appendix E.
5.2 Double-Copy Construction of Graviton Amplitude and Energy Cancellations

In this subsection, we extend the conventional double-copy method for massless gauge/gravity theories [11][12] and construct the massive four-graviton scattering amplitude in the 3d TMG theory from the four-gauge boson amplitude in the 3d TMYM theory. Our focus is to analyze the structure of massive graviton scattering amplitudes under high energy expansion and newly uncover a series of striking energy-cancellations of the graviton amplitudes in connection to the corresponding gauge boson amplitudes.

We examine the kinematic numerators \( \{ \mathcal{N}_j \} \) of the original massive four-gauge-boson scattering amplitude (4.27)-(4.28), and find that they violate the kinematic Jacobi identity due to \( \sum_j \mathcal{N}_j \neq 0 \). Then, we analyze the gauge boson amplitude (4.27) and find that it is invariant under the following generalized gauge transformation:

\[
\mathcal{N}_j \to \mathcal{N}_j' = \mathcal{N}_j + \Delta (s_j - m^2),
\]

where \( j \in (s, t, u) \) and \( \Delta \) is a coefficient. Thus, by requiring the gauge-transformed numerators \( \{ \mathcal{N}_j' \} \) to satisfy the kinematic Jacobi identity \( \sum_j \mathcal{N}_j' = 0 \), we can determine the coefficient \( \Delta \) as follows:

\[
\Delta = \frac{-i \csc \theta}{32 m^2} \left[ (16 m^4 s^2 - i + 8 m^2 s^2 - 3 s \frac{3}{2} ) - (16 m^4 s^2 - i - 24 m^2 s^2 - 3 s \frac{3}{2} ) c_{2\theta} + i 16 m s s_{2\theta} \right].
\]

Then, we apply the gauge transformation (5.9) to the numerators (4.28) and further derive the following new kinematic numerators \( \{ \mathcal{N}_s', \mathcal{N}_t', \mathcal{N}_u' \} \):

\[
\mathcal{N}_s' = \frac{i \csc \theta}{8 m s^2} \left[ 8 m^4 + 26 m^2 s - 7 s^2 - (8 m^4 + 18 m^2 s + s^2 ) c_{2\theta} - i m s \frac{1}{2} (20 m^2 + 7 s) s_{2\theta} \right],
\]

\[
\mathcal{N}_t' = \frac{-i \csc \theta}{32 m s^2} \left[ 16 m^4 + 52 m^2 s - 14 s^2 + (16 m^4 + 104 m^2 s - 15 s^2 ) c_{2\theta} - 2 (8 m^4 + 18 m^2 s + s^2 ) c_{2\theta}
- (16 m^4 + 24 m^2 s + s^2 ) c_{2\theta} + i m s \frac{1}{2} (176 m^2 - 20 s) s_{2\theta} - i m s \frac{1}{2} (40 m^2 + 14 s) s_{2\theta}
- i m s \frac{1}{2} (32 m^2 + 8 s) s_{2\theta} \right],
\]

\[
\mathcal{N}_u' = \frac{-i \csc \theta}{32 m s^2} \left[ 16 m^4 + 52 m^2 s - 14 s^2 - (16 m^4 + 104 m^2 s - 15 s^2 ) c_{2\theta} - 2 (8 m^4 + 18 m^2 s + s^2 ) c_{2\theta}
+ (16 m^4 + 24 m^2 s + s^2 ) c_{2\theta} - i m s \frac{1}{2} (176 m^2 - 20 s) s_{2\theta} - i m s \frac{1}{2} (40 m^2 + 14 s) s_{2\theta}
+ i m s \frac{1}{2} (32 m^2 + 8 s) s_{2\theta} \right],
\]

which nicely obey the kinematic Jacobi identity \( \sum_j \mathcal{N}_j' = 0 \) by construction. With the above, we can reexpress the gauge boson amplitude (4.27) as follows:

\[
\mathcal{T}[4 A^0_p] = g^2 \left( \frac{C_s \mathcal{N}_s'}{s - m^2} + \frac{C_t \mathcal{N}_t'}{t - m^2} + \frac{C_u \mathcal{N}_u'}{u - m^2} \right).
\]
As a consistency check, we have also verified that the gauge-transformed numerators \((5.11b)-(5.11c)\) satisfy the kinematic exchange symmetry between the \(t\)-channel and \(u\)-channel, namely, \(\mathcal{N}'_u(\pi + \theta) = -\mathcal{N}'_t(\theta)\).

For the four-particle scattering amplitudes of massive physical gravitons \(h_p = \epsilon^\mu_\nu h_{\mu
u}\) in the 3d TMG theory, we can use the power counting rule (3.13) or (3.14) of section 3.2 to count the energy-dependence of the graviton scattering amplitudes and find that the leading individual contributions to the tree-level amplitudes scale as \(E^{12}\). But, using the extended double-copy approach for massive scattering amplitudes, we will prove that such leading contributions of \(\mathcal{O}(E^{12})\) in the four-graviton scattering amplitudes must cancel down to \(\mathcal{O}(E^1)\), which gives the striking large cancellations of \(\mathcal{O}(E^{12}) \to \mathcal{O}(E^1)\) by an energy power of \(E^{11}\).

For this purpose, we extend the conventional double-copy method with the color-kinematics duality for massless gauge/gravity theories [11][12] to the 3d topologically massive gauge/gravity theories. Using our improved massive-gauge-boson amplitude (5.12) of 3d TMYM theory and the color-kinematics duality \(C_j \to \mathcal{N}'_j\), we construct the full scattering amplitude of physical gravitons, \(\mathcal{M}[h_p h_p \to h_p h_p] \equiv \mathcal{M}[4h_p]\), in the 3d TMG theory:

\[
\mathcal{M}[4h_p] = \frac{\kappa^2}{16} \left[ \frac{(\mathcal{N}')^2_s}{s-m^2} + \frac{(\mathcal{N}')^2_t}{t-m^2} + \frac{(\mathcal{N}')^2_u}{u-m^2} \right],
\]

where we have made the gauge-gravity coupling conversion \(g \to \kappa/4\). (The same conversion factor will work for the 4d double-copy of the massless gauge/gravity theories [11] if the same normalization of color factors is adopted.) Then, substituting the improved kinematic numerators (5.11) into Eq.(5.13), we derive the following compact formula for the four-graviton scattering amplitude after significant simplifications:

\[
\mathcal{M}[4h_p] = -\frac{\kappa^2 m^2 (P_0 + P_2 c_{20} + P_4 c_{49} + P_6 c_{69} + \bar{P}_2 s_{20} + \bar{P}_4 s_{49} + \bar{P}_6 s_{69}) \csc^2 \theta}{4096 (3 + \bar{s}_0)(4 + \bar{s}_0)^{3/2}(2 + \bar{s}_0 - \bar{s}_0 c_\theta)(2 + \bar{s}_0 + \bar{s}_0 c_\theta)},
\]

where \((P_j, \bar{P}_j)\) are expressed as polynomial functions of the variable \(\bar{s}_0 = s_0/m^2\),

\[
P_0 = -4(7992\bar{s}_0^2 + 4767\bar{s}_0^3 + 692\bar{s}_0^4)(4 + \bar{s}_0)^{1/2},
\]
\[
P_2 = (-221184 - 304278\bar{s}_0 - 114048\bar{s}_0^2 - 10928\bar{s}_0^3 + 505\bar{s}_0^4)(4 + \bar{s}_0)^{1/2},
\]
\[
P_4 = 4(55296 + 45312\bar{s}_0 + 13208\bar{s}_0^2 + 1563\bar{s}_0^3 + 58\bar{s}_0^4)(4 + \bar{s}_0)^{1/2},
\]
\[
P_6 = -(98304 + 57344\bar{s}_0 + 11264\bar{s}_0^2 + 832\bar{s}_0^3 + 17\bar{s}_0^4)(4 + \bar{s}_0)^{1/2},
\]
\[
\bar{P}_2 = i(-442368 - 663552\bar{s}_0 - 300672\bar{s}_0^2 - 46048\bar{s}_0^3 + 540\bar{s}_0^4 + 475\bar{s}_0^5),
\]
\[
\bar{P}_4 = i4(110592 + 104448\bar{s}_0 + 36880\bar{s}_0^2 + 5828\bar{s}_0^3 + 372\bar{s}_0^4 + 5\bar{s}_0^5),
\]
\[
\bar{P}_6 = -i(196608 + 139264\bar{s}_0 + 35328\bar{s}_0^2 + 3776\bar{s}_0^3 + 148\bar{s}_0^4 + \bar{s}_0^5).
\]
It is striking to see that the above LO graviton amplitude actually scales as 

\[ \mathcal{O}(m E) \]

under high energy expansion. Because the direct application of our power counting rule (3.14) to the double-copy graviton amplitude (5.13) gives the scaling behavior 

\[ \mathcal{M}_0[4h_F] = \mathcal{O}(m^{-2}E^4) \]

we can expect from Eq.(5.16) the further nontrivial energy cancellations of \( E^4 \rightarrow E^1 \), which we will analyze in the following paragraph.

Next, we make high energy expansion for the reconstructed graviton amplitude (5.14)-(5.15) and derive the following leading order (LO) result:

\[
\mathcal{M}_0[4h_F] = -\frac{i\kappa^2}{2048} m s_0^{\frac{1}{2}} (494c_d + 19c_{39} - c_{59}) \csc^3 \theta.
\]  

(5.16)

It is striking to see that the above LO graviton amplitude actually scales as \( \mathcal{O}(m E) \) under high energy expansion. Because the direct application of our power counting rule (3.14) to the double-copy graviton amplitude (5.13) gives the scaling behavior \( \mathcal{M}_0[4h_F] = \mathcal{O}(m^{-2}E^4) \), we can expect from Eq.(5.16) the further nontrivial energy cancellations of \( E^4 \rightarrow E^1 \), which we will analyze in the following paragraph.

Inspecting the expressions of \( \mathcal{N}_j, \mathcal{N}_j' \) in Eqs.(4.28) and (5.11), we find that they contain individual leading terms scaling as \( (E^5, E^3) \), respectively. This shows that the gauge transformation (5.9) leads to an energy cancellation of \( E^5 \rightarrow E^3 \) in each of the transformed numerators \( \mathcal{N}_j' \). This has an important impact on the energy dependence of the double-copy amplitude (5.13), namely, in each channel of \( \mathcal{N}_j'^2/(s_j - m^2) \), it contains a leading energy term behaving as \( E^4 \), rather than \( E^8 \) from \( \mathcal{N}_j'/(s_j - m^2) \). In comparison with the leading energy-dependence of the individual diagrams contributing to the tree-level four-graviton amplitude which scales as \( E^{12} \) by the direct power counting rule (3.14), our double-copy construction (5.13) demonstrates that the four-graviton amplitude could have a leading energy dependence of \( E^4 \) at most in each channel. Hence, this double-copy construction guarantees a nontrivial large energy cancellation in the original four-graviton scattering amplitude: \( E^{12} \rightarrow E^4 \), which cancels the leading energy dependence by a large power factor of \( E^8 \).

Furthermore, it is striking to see that our summed full amplitude (5.13) actually scales as \( E^1 \) under high energy expansion as shown in the above Eq.(5.16). We can demonstrate explicitly this large energy cancellation of \( E^4 \rightarrow E^1 \), which includes exact cancellations of the energy power terms at each order of \( E^4, E^3, \) and \( E^2 \). We summarize our findings on these exact energy

| Amplitude | \( \propto s_0^2 \) | \( \propto s_0^{3/2} \) | \( \propto \bar{s}_0 \) |
|-----------|----------------|----------------|----------------|
| \( \mathcal{M}_d \) | \( \frac{99+28c_{39}+c_{49}}{1-c_{29}} \) | \( -i14(15c_{39} + c_{39}) \csc \theta \) | \( -\frac{2(75+326c_{39}+47c_{49})}{1-c_{29}} \) |
| \( \mathcal{M}_t \) | \( \frac{99+28c_{39}+c_{49}}{4(1-c_{29})} \) | \( i(102+105c_{39}+70c_{29}+7c_{39}+4c_{49}) \csc \theta \) | \( \frac{75-107c_{39}+326c_{29}+206c_{39}+47c_{49}+31c_{59}}{1-c_{29}} \) |
| \( \mathcal{M}_u \) | \( \frac{99+28c_{39}+c_{49}}{4(1+c_{29})} \) | \( i(-102+105c_{39}-70c_{29}+7c_{39}-4c_{49}) \csc \theta \) | \( \frac{75+107c_{39}+326c_{29}-268c_{39}+47c_{49}-31c_{59}}{1-c_{29}} \) |
| Sum | 0 | 0 | 0 |

Table 5: Exact energy cancellations at each order of \( E^4, E^3, \) and \( E^2 \) in our four-graviton scattering amplitude (5.13) by double-copy construction. Here the notations \( (s_0, \bar{s}_0) \) are defined by \( \bar{s}_0 = s_0/m^2 \) and \( s_0 = 4E^2\beta^2 \). A common overall factor \((\kappa^2m^2/1024)\) in each amplitude is not displayed for simplicity.
cancellations of the full amplitude \((5.13)\) into Table 5. We may understand the reason for such energy cancellations of \(E^4 \rightarrow E^1\) as follows. We first note that an \(S\)-matrix element \(S\) with \(E\) external states and \(L\) loops \((L > 0)\) in the \((2+1)\)d spacetime has mass-dimension \(D_S = 3 - \frac{1}{2} E\) as given by Eq.\((3.7)\). Thus, the four-graviton amplitude \(\mathcal{M}[4h_p]\) has mass-dimension \(D_M = 1\).

At tree level it contains the gravitational coupling \(\kappa^2\) of mass-dimension \(-1\). Hence, we can express the four-graviton amplitude \(\mathcal{M}[4h_p] = \kappa^2 \mathcal{M}[4h_p]\), where \(\mathcal{M}[4h_p]\) has mass-dimension equal \(+2\). The tree-level amplitude \(\mathcal{M}[4h_p]\) contains only two parameters \((E, m)\), each of which has mass-dimension \(+1\). With these we can deduce the scaling behavior \(\mathcal{M}[4h_p] \propto m^{n_1} E^{n_2}\) with \(n_1 + n_2 = 2\), under high energy expansion. Hence, for the energy terms of \(E^{n_2}\) with \(n_2 = 4, 3, 2\), we deduce the mass-power factor \(n_1 = -2, -1, 0\), respectively. This means that in the massless limit \(m \rightarrow 0\), the physical graviton amplitude \(\mathcal{M}[4h_p]\) would go to infinity (for \(n_2 \geq 3\)) or remain constant (for \(n_2 = 2\)). But, in the \(m \rightarrow 0\) limit, the 3d graviton field becomes unphysical and has no physical degree of freedom \([1][41][42]\); so the amplitude \(\mathcal{M}[4h_p]\) should vanish because the physical graviton \(h_p\) no longer exists in the massless limit. This means that the \(m^{n_1} E^{n_2}\) terms with \(n_1 = -2, -1, 0\) should vanish, and the physical amplitude \(\mathcal{M}[4h_p]\) has to start with the leading behavior of \(m^{1} E^{1}\) under high energy expansion. Thus, it is expected that the energy cancellations should hold at each order of \(E^1, E^3,\) and \(E^2\), in agreement with what we firstly uncovered by explicit calculations in Table 5.

In summary, using our double-copied graviton amplitudes in Eqs.\((5.13)-(5.16)\) and Table 5 we have uncovered a new type of large energy cancellations in the original four-graviton scattering amplitude at tree level for the 3d TMG theory:

\[
\mathcal{O}(E^{12}) \rightarrow \mathcal{O}(E^1), \quad \text{(for } E_{hp} = 4 \text{ in 3d TMG).} \tag{5.17}
\]

Furthermore, with this extended double-copy construction, we have established a new correspondence between the two types of leading energy cancellations in the massive scattering amplitudes: \(E^4 \rightarrow E^0\) in the TMYM theory and \(E^{12} \rightarrow E^1\) in the TMG theory. We also note that in Eq.\((5.17)\) the exact energy cancellations in the four-graviton amplitude down by a large power of \(E^{11}\) are even much more severe than the energy cancellations \((E^{10} \rightarrow E^2)\) in the massive four-longitudinal KK graviton scattering amplitudes of the compactified 5d KK Einstein gravity found before by explicit calculations \([43][44]\) and by the double-copy construction \([19]\).

In passing, during the finalization of the present paper in this summer, we became aware of a recent new paper \([45]\) which directly calculated the graviton amplitude of the 3d TMG with very lengthy expressions in its Eq.\((C.1)\) where all the polarization tensors (vectors) take symbolic forms. We have quantitatively compared our full graviton amplitude \((5.13)\) (by double-copy construction) with their Eq.\((C.1)\) and find agreement after substituting explicitly all the polarization formulas into Eq.\((C.1)\) and making substantial simplifications of Eq.\((C.1)\). This comparison gives an independent verification of our double-copy result. Our work has fully different focus from
and our analyses differ from [45] in several essential ways. (i). The main part of our work (Secs. 2-4) is to analyze the mechanism of topological mass-generation of gauge bosons and uncover nontrivial energy cancellations in the gauge boson scattering amplitudes. These were not studied by [45]. (ii). For this purpose, we newly formulated the 3d topological mass-generation mechanism at S-matrix level, and newly proposed and proved the TET for N-point gauge boson amplitudes in Sec. 3. We further verified the TET explicitly by computing the four-point scattering amplitudes of various high-energy processes for both Abelian and non-Abelian gauge theories (with and without matter fields) in Secs. 4.1-4.2. These were not considered by [45]. (iii). Our whole study on the gauge boson amplitudes and the double-copied graviton amplitudes has focused on the structure of the scattering amplitudes under high energy expansion and on the mechanism of nontrivial large energy cancellations as summarized in Tables 1-5 and Eq. (5.17). For this, we newly constructed the general 3d power counting method in Sec. 3.2, and used it together with the TET to prove the nontrivial energy cancellations for N-point gauge boson amplitudes in Sec. 3.3. These were not covered by [45]. (iv). We also note that Eq. (C.2) of [45] further gave more compact formula of the 4-graviton amplitude in a very different Briet coordinate system and cannot be directly compared to our double-copied graviton amplitude (5.14)-(5.15). Their 4-gauge boson amplitude in Eq. (C.3) was also given in the Briet coordinate system and cannot be directly compared to our Eq. (5.11)-(5.12). The Eqs. (4.12)-(4.13) of [45] gave 4-gauge boson amplitude with all polarization vectors in symbolic forms. We have further confirmed with the authors of [45] that our gauge boson amplitude (4.27) and their Eq. (4.13) are in good agreement after substituting all the polarization formulas into their Eq. (4.13) and after taking into account the notational difference in defining the Mandelstam variables.\(^5\) We stress that the parts of our study for the four-gauge boson amplitudes and double-copied four-graviton amplitudes have focused on analyzing their structures of energy-dependence and on uncovering the striking large energy cancellations in these amplitudes as well as the mechanism of such energy cancellations. These new findings were not covered by [45] whose independent study had fully different focus. We also note that the structures of our non-Abelian gauge boson amplitudes (4.27) and (5.12) are well understood and justified by nontrivial self-consistency checks as we explained at the end of Sec. 4.2.2 and showed in Tables 4-5.

6 Conclusions

Understanding the mechanism of topological mass-generation and the structure of the scattering amplitudes in the 3d Chern-Simons (CS) gauge and gravity theories is important for applying

\(^5\) After posting this paper to arXiv:2110.05399, we had helpful discussions with the authors of Ref. [45]. We are glad to thank them for the comparison between their Eq. (4.13) and our Eq. (4.28) which confirms the good agreement between the independent analyses on both sides.
the modern quantum field theories to particle physics and condensed matter physics [1] [2] [3]. In 3d spacetime the existence of the CS actions for gauge bosons and gravitons is theoretically unavoidable and compelling. This generates gauge-invariant topological mass-terms for gauge bosons and gravitons without invoking the conventional Higgs mechanism [5] and leads to good high energy behaviors for the scattering amplitudes of topologically massive gauge bosons and gravitons.

In this work, we systematically studied the mechanism of the topological mass-generations in 3d CS gauge theories and newly formulated it at the $S$-matrix level. For this, we proposed and proved a new Topological Equivalence Theorem (TET) for understanding the structure of the scattering amplitudes of physical gauge bosons ($A_\mu^a$) in the topologically massive gauge theories. We newly uncovered the nontrivial large energy cancellations in the $N$-point gauge boson scattering amplitudes for both the Abelian and non-Abelian CS gauge theories. We further used an extended double-copy approach to analyze the structure of the graviton scattering amplitudes in the 3d topologically massive gravity (TMG) theory, with which we constructed the massive physical graviton scattering amplitudes from that of the corresponding massive physical gauge bosons in the topologically massive Yang-Mills (TMYM) theory. From these, we newly uncovered a series of striking large energy cancellations in the four-point physical graviton scattering amplitudes, which ensure such massive scattering amplitudes to have good high energy behaviors and obey the perturbative unitarity bounds. We summarize these new findings more explicitly as follows.

In section 2, we analyzed the dynamics of topological mass-generation in the 3d CS gauge theories. In such dynamics, including the CS term (2.1) automatically converts the gauge boson’s transverse polarization state $A_T^a$ (combined with its longitudinal polarization state $A_L^a$) into the massive physical polarization state $A_P^a \propto (A_T^a + A_L^a)$ as given in Eq.(2.12), while making its orthogonal combination $A_X^a \propto (A_T^a - A_L^a)$ in Eq.(2.13a) become an unphysical state. This topological mass-generation mechanism has essential difference from the conventional Higgs mechanism [5], because the CS term generates gauge-invariant mass term of $A_\mu^a$ and no spontaneous gauge symmetry breaking and Higgs boson are invoked.

In section 3, we newly proposed and proved a TET to formulate the topological mass-generation of gauge bosons at the $S$-matrix level, which quantitatively connects the $N$-point scattering amplitudes of physical gauge bosons $A_P^a$ to the amplitudes of the corresponding transverse gauge bosons $A_T^a$ in the high energy limit. For this, we established the general TET identity (3.5), with which we derived the TET (3.6) under high energy expansion. We presented a new energy power counting rule (3.12) to count the leading energy dependence of general $N$-point scattering amplitudes for the 3d topologically massive gauge theories and another new power counting rule (3.13) [and (3.14)] to count the leading energy dependence of $N$-point scattering
amplitudes for the 3d TMG theory. (A generalized power counting method in \(d\)-dimensions is given in Appendix B.) With these, we demonstrated that our TET identity (3.5) provides a general mechanism of nontrivial energy cancellations in the \(N\)-point \(A_P^a\)-amplitudes because the net energy dependence of a given \(A_P^a\)-amplitude must match that of the leading \(A_T^a\)-amplitude on the RHS of Eq.(3.5a). For the high-energy scattering of \(N\)-gauge bosons \(A_P^a\) (with \(N \geq 4\)), the TET identity (3.5) [or TET (3.6)] guarantees the nontrivial large energy cancellations in the \(A_P^a\)-amplitude: \(E^4 \rightarrow E^{4-N}\).

In section 4, we explicitly demonstrated the TET (3.6) for the first time by using various high-energy four-particle scattering amplitudes in both the Abelian and non-Abelian topologically massive CS gauge theories. In sections 4.1.1-4.1.2, we computed the scattering amplitudes of the annihilation processes \(\phi^- \phi^+ \rightarrow A_P A_P\) (\(\phi^- \phi^+ \rightarrow A_T A_T\)) and Compton scattering \(\phi^- A_P \rightarrow \phi^- A_P\) (\(\phi^- A_T \rightarrow \phi^- A_T\)) in the topologically massive scalar QED (TMSQED), as shown in Fig.1. In parallel, we computed the scattering amplitudes of annihilation processes \(e^- e^+ \rightarrow A_P A_P\) (\(e^- e^+ \rightarrow A_T A_T\)) and Compton scattering \(e^- A_P \rightarrow e^- A_P\) (\(e^- A_T \rightarrow e^- A_T\)) in the topologically massive spinor QED (TMQED), as shown in Fig.2. From these analyses, we newly uncovered the nontrivial energy cancellations of \(E^2 \rightarrow E^0\) in each \(A_P^a\)-amplitude, which are summarized in Tables 1-2 and in Eqs.(4.3)(4.6) and Eqs.(4.11a)-(4.11b). We further computed the remaining nonzero scattering amplitudes of \(\mathcal{O}(E^0)\) and proved explicitly the validity of the TET as in Eqs.(4.8) and (4.13).

Next, in section 4.2 we studied the structure of scattering amplitudes in the non-Abelian topologically massive QCD (TMQCD). In section 4.2.1, we computed the quark-antiquark annihilation processes \(q_i \bar{q}_j \rightarrow A_P^a A_P^b\) (\(q_i \bar{q}_j \rightarrow A_T^a A_T^b\)) for the TMQCD, which are shown in Fig.3 and contain additional \(s\)-channel diagram induced by the non-Abelian cubic vertex. We uncovered nontrivial energy cancellations of \(E^2 \rightarrow E^0\) in the \(A_P^a\)-amplitude as summarized by Table 3 and Eq.(4.21). We further computed the remaining nonzero \(A_P^a\)-amplitude and \(A_T^a\)-amplitude of \(\mathcal{O}(E^0)\), and proved explicitly that the TET holds for the TMQCD as in Eq.(4.26). Then, in section 4.2.2, we systematically analyzed the four-gauge boson scattering amplitudes of \(A_P^a A_P^b \rightarrow A_P^c A_P^d\) and \(A_T^a A_T^b \rightarrow A_T^c A_T^d\) in the TMQCD, which are shown in Fig.4. We newly uncovered the nontrivial large energy cancellations of \(E^4 \rightarrow E^0\) in the \(A_P^a\)-amplitude, at each order of \((E^4, E^3, E^2, E^1)\) respectively, which are summarized in Table 4 and Eqs.(4.29a)-(4.29d). We further computed the remaining nonzero \(A_P^a\)-amplitude and \(A_T^a\)-amplitude at \(\mathcal{O}(E^0)\) as given in Eqs.(4.31a)-(4.31b). With these and the Jacobi identity, we proved explicitly that the TET indeed holds for the four-gauge boson scattering amplitudes of the TMQCD, as shown in Eq.(4.33). Finally, in section 4.3, we analyzed the perturbative unitarity of the TMYM and TMG theories. We found that the partial wave amplitudes (4.39) and (4.41) exhibit good high energy behaviors as they scale as \(E^{-1}\) or \(E^0\) in the high energy limit. This is expected for the 3d TMYM theory because its gauge
coupling has mass-dimension $+\frac{1}{2}$ and thus is super-renormalizable. This issue becomes much more nontrivial for the 3d TMG theory since its Newton constant $G \propto \kappa^2$ has mass-dimension $-1$. But, we should expect this theory to exhibit good high energy behavior because the massive physical graviton scattering amplitudes in the TMG theory can be reconstructed from the corresponding massive gauge boson amplitudes in the TMYM theory via extended double-copy approach, as shown in section 5.

In section 5, we extended the conventional double-copy approach and constructed the massive four-graviton scattering amplitude of the 3d TMG theory by using the massive four-gauge boson amplitude of the 3d TMYM theory. We found that the reconstructed tree-level four-graviton scattering amplitude could only scale as $E^1$ under high energy expansion. We made the gauge transformation (5.9) on the kinematic numerators $N_j$ of Eq.(4.28) in the four-gauge boson scattering amplitude (4.27) such that the new numerators $N_j'$ in Eq.(5.11) obey the Jacobi identity. The gauge transformation of $N_j \rightarrow N_j'$ leads to the energy cancellations of $E^5 \rightarrow E^3$ in each kinematic numerator. This determines the individual leading energy dependence of the reconstructed graviton amplitude (5.13) to be no more than $\mathcal{O}(E^4)$. By explicit computations, we further uncovered new energy cancellations of $E^4 \rightarrow E^1$ in the graviton scattering amplitude (5.13) which are summarized in Table 5. Then, we computed the remaining nonzero graviton amplitude as in Eq.(5.16) which scales as $\mathcal{O}(E^1)$ only. In contrast, applying the general power counting rule (3.14) we found that the individual contributions to the four-graviton amplitude have leading energy dependence behave as $E^{12}$. From these together, we demonstrated a new type of striking large energy cancellations in the four-graviton scattering amplitude as in Eq.(5.17), $\mathcal{O}(E^{12}) \rightarrow \mathcal{O}(E^1)$, for the 3d TMG theory. Furthermore, with the extended double-copy construction, we established a new correspondence between the two types of leading energy cancellations in the massive scattering amplitudes: $E^4 \rightarrow E^0$ in the TMYM theory and $E^{12} \rightarrow E^1$ in the TMG theory.

Our present findings are highly nontrivial and encouraging. We already generally proved our TET for the $N$-point gauge boson scattering amplitudes (section 3.1) and uncovered the new mechanism of large energy cancellations for these $N$-point amplitudes by using the TET and the general energy power counting method in 3d (sections 3.2-3.3). It would be also interesting to extend our current explicit calculations and analyses to the $N$-point scattering amplitudes of gauge bosons (gravitons) in the TMYM (TMG) theories with more external states (such as $N=5$ and $N=6$). Since the 3d topologically massive CS gauge theories are super-renormalizable and have good high energy behaviors, it would be desirable to extend the present tree-level analyses up to loop levels. We will pursue such extended studies in future works.
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Appendix:

A Kinematics and Feynman Rules of 3d CS Gauge Theories

In this Appendix, we present definitions of relevant kinematic variables for the four-particle scattering process and the Feynman rules for the 3d topologically massive Chern-Simons gauge theories.

For the present analysis, we choose the following metric signature and the rank-3 anti-symmetric tensor:

\[ \eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, 1, 1), \quad \varepsilon^{012} = -\varepsilon_{012} = 1. \] (A.1)

Thus, we have the momentum on-shell condition \( p^2 = -m^2 \).

For the \( 2 \rightarrow 2 \) elastic scattering process, the momenta in the center-of-mass frame can be defined as follows:

\[
\begin{align*}
p_1^\mu &= E(1, 0, \beta), & p_2^\mu &= E(1, 0, -\beta), \\
p_3^\mu &= E(1, \beta s_\theta, \beta c_\theta), & p_4^\mu &= E(1, -\beta s_\theta, -\beta c_\theta),
\end{align*}
\] (A.2)

where we have defined \( \beta = \sqrt{1 - m^2/E^2} \). Thus, we further define the Mandelstam variables:

\[
\begin{align*}
s &= -(p_1 + p_2)^2 = 4E^2, \\
t &= -(p_1 - p_4)^2 = -\frac{1}{2} s \beta^2 (1 + c_\theta), \\
u &= -(p_1 - p_3)^2 = -\frac{1}{2} s \beta^2 (1 - c_\theta).
\end{align*}
\] (A.3)

For convenience of our analysis, we can use the relation \( E^2 = E^2 \beta^2 + m^2 \) to define another set of Mandelstam variables \( (s_0, t_0, u_0) \):

\[
\begin{align*}
s_0 &= 4E^2 \beta^2, & t_0 &= -\frac{1}{2} s_0 (1 + c_\theta), & u_0 &= -\frac{1}{2} s_0 (1 - c_\theta).
\end{align*}
\] (A.4)

The summations of \( s, t, u \) and \( (s_0, t_0, u_0) \) obey the conditions:

\[
\begin{align*}
s + t + u &= 4m^2, & s_0 + t_0 + u_0 &= 0.
\end{align*}
\] (A.5)
In the rest frame with 3-momentum \( p^\mu = (m, 0, 0) \equiv \tilde{p}^\mu \), we can solve Eq. (2.5) and derive the polarization vector:

\[
\epsilon^\mu(\tilde{p}) = \frac{1}{\sqrt{2}} (0, 1, -i s),
\]

(A.6)

where \( s = \tilde{m}/m = \pm 1 \) and \( m = |\tilde{m}| \). We note that the in the rest frame the gauge boson polarization vector has zero time-component and its two possible forms are not independent due to the relation \( \epsilon^\mu_2 = is\epsilon^\mu_1 \). Furthermore, by choosing the orthonormal basis \( \epsilon^\mu_1 = (1, 0) \) and \( \epsilon^\mu_2 = (0, 1) \) in a plane, we can define a polarization basis:

\[
\epsilon^\mu_\pm = \frac{1}{\sqrt{2}} (\epsilon^\mu_1 \pm i \epsilon^\mu_2) = \frac{1}{\sqrt{2}} (1, \pm i).
\]

(A.7)

Thus, in the rest frame the spatial part of the polarization vector \( \epsilon^\mu(\tilde{p}) \) can be decomposed in terms of the basis \( \{ \epsilon^\mu_\pm \} \):

\[
\epsilon^\mu(\tilde{p}) = \epsilon_+ \epsilon^\mu_+ + \epsilon_- \epsilon^\mu_-,
\]

(A.8)

where the coefficients \( (\epsilon_+, \epsilon_-) \) satisfy \( (\epsilon_+, \epsilon_-) = (0, 1) \) for \( s = +1 \) and \( (\epsilon_+, \epsilon_-) = (1, 0) \) for \( s = -1 \) [30]. So, it is clear that in the 3d Chern-Simons gauge theory, the case of \( s = +1 \) (or, \( s = -1 \) only allows one physical polarization state \( \epsilon_- \) (or, \( \epsilon_+ \)) of the gauge boson, as expected.

After taking the Lorentz boost along an arbitrary direction, the gauge boson momentum can be generally written as \( p^\mu = E(1, \beta s_\theta, \beta c_\theta) \). Thus, we can Lorentz-boost the rest-frame polarization vector (A.6) to the following general polarization vector:

\[
\epsilon^\mu_\mu(p) = \frac{1}{\sqrt{2}} \left( \frac{ip_1+s_2}{m}, i + \frac{p_1(ip_1+s_2)}{m(m-p_0)}, \frac{s}{m(m-p_0)} \right),
\]

(A.9)

where \( \epsilon^\mu_\mu = -(\epsilon^\mu_\mu)^* \). Thus, by substituting Eq. (A.2) into Eq. (A.9), we derive the following explicit formulas of the physical polarization vectors:

\[
\begin{align*}
\epsilon^\mu_1 &= \frac{s}{\sqrt{2}} (\bar{E} \beta, is, \bar{E}) , \quad &\epsilon^\mu_2 &= -\frac{s}{\sqrt{2}} (\bar{E} \beta, -is, -\bar{E}) , \\
\epsilon^\mu_3 &= \frac{se^{is\theta}}{\sqrt{2}} (\bar{E} \beta, \bar{E}s_\theta + isc_\theta, \bar{E}c_\theta - is s_\theta) , \quad &\epsilon^\mu_4 &= -\frac{se^{is\theta}}{\sqrt{2}} (\bar{E} \beta, -\bar{E}s_\theta - isc_\theta, -\bar{E}c_\theta + is s_\theta) ,
\end{align*}
\]

(A.10)

where we have defined \( \bar{E} = E/m \).

For \( \tilde{m} > 0 \), the propagators for the Abelian and non-Abelian topological gauge theories can be derived as follows:

\[
D_{\mu\nu}(p) = -i \left[ \frac{1}{p^2 + m^2} \left( \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{im \varepsilon_{\mu\nu\rho} p^\rho}{p^2} \right) + \xi \frac{p_\mu p_\nu}{p^4} \right],
\]

(A.11a)

\[
D_{\mu\nu}^{ab}(p) = \delta^{ab} D_{\mu\nu}(p).
\]

(A.11b)

In Eq. (A.11a), the pole \( p^2 = 0 \) is unphysical, for which the EOM (2.5) becomes \( \tilde{m} \varepsilon^{\mu\nu} p_\rho \epsilon_{\nu} = 0 \). It can be solved as \( \epsilon^\mu = f(p) p^\mu \), but it can be eliminated by the freedom of gauge transformations [29]. Hence, the massless mode is a pure gauge artifact [29][46]. Furthermore, we can
derive the following Feynman rule of the cubic gauge boson vertex:

\[ V^{abc}_{\mu\nu\rho}(p_1, p_2, p_3) = g C^{abc} \left[ \eta_{\mu\nu} (p_1 - p_2)_{\rho} + \eta_{\nu\rho} (p_2 - p_3)_{\mu} + \eta_{\rho\mu} (p_3 - p_1)_{\nu} + \text{i} m \epsilon_{\mu\nu\rho} \right] . \] (A.12)

The quartic gauge boson vertex is similar to that of the 4d QCD.

**B General Power Counting Method in \( d \)-Dimensions**

In this Appendix, extending the 3d power counting method of section 3.2, we present a general power counting formula for the \( d \)-dimensional spacetime.

In \( d \)-dimensions, we derive the mass-dimension of a given \( S \)-matrix element \( S \) as follows:

\[ D_S = d - \frac{d - 2}{2} E , \] (B.1)

where \( E \) denotes the total number of external states as before. We see that the general formula (B.1) reduces to \( D_S = 4 - E \) for \( d = 4 \) and \( D_S = 3 - \frac{1}{2} E \) for \( d = 3 \), respectively. Then, we can deduce the mass-dimension of all the coupling constants in the \( S \)-matrix element \( S \):

\[ D_C = \sum_j V_j \left( d - d_j - \frac{d - 2}{2} b_j - \frac{d - 1}{2} f_j \right) , \] (B.2)

where \( V_j \) denotes the number of vertices of type-\( j \), and the quantities \( (d_j, b_j, f_j) \) denote the numbers of (partial derivatives, bosonic fields, fermions) in each vertex of type-\( j \), respectively. We have the following general relations for each Feynman diagram which contributes to the amplitude \( S \),

\[ L = 1 + I - V , \quad V = \sum_j V_j , \quad \sum_j V_j b_j = 2 I_B + E_B , \quad \sum_j V_j f_j = 2 I_F + E_F , \] (B.3)

where \( L \) denotes the number of loops of a given diagram, \( (I_B, E_B) \) denote the numbers of (internal, external) bosonic lines in this diagram, and \( (I_F, E_F) \) denote the numbers of (internal, external) fermionic lines in the same diagram. With these, we derive the leading energy dependence of the amplitude \( S \):

\[ D_E = D_S - D_C = 2(1 - V) + (d - 2)L + \sum_j V_j (d_j + \frac{1}{2} f_j) . \] (B.4)

Then, we consider the \( d \)-dimensional gauge theories (including Chern-Simons term when allowed). By imposing the relations (3.11), we derive the leading energy dependence of the amplitude \( S \):

\[ D_E = (E_A - E_v) + (4 - E - \overline{V}_3) + (d - 4)L , \] (B.5)
where $E_A$ is the total number of external states of the physical gauge bosons and $E_v$ denotes number of the external states of gauge bosons $v^a = v_\mu A^{a\mu}$ with the factor $v^\mu = \epsilon_L^\mu - \epsilon_S^\mu$.

Next, we can apply the power counting formula (B.4) to the topologically massive gravity (TMG) theory. For this, we can derive the leading energy-dependence of a pure graviton scattering amplitude $S$ in the TMG theory, which corresponds to setting $d_j = 3$ and $f_j = 0$ in Eq.(B.4):

$$D_E = 2E_A + V_3 + 2 + (d - 2)L.$$  \hspace{1cm} (B.6)

where $V_3$ denotes the number of cubic vertices from the CS term in the TMG action. For instance, we can check that for the TMG theory of $d = 3$, the above Eqs.(B.5) and (B.6) just reduce to the power counting formulas (3.12) and (3.13), which we derived for the 3d TMYM and TMG theories in section 3.2.

\section{C Dirac Spinors in (2+1)d Spacetime}

The anti-symmetric and symmetric commutation relations for the gamma matrices in (2+1)d spacetime are given by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad [\gamma^\mu, \gamma^\nu] = 2\epsilon^{\mu\nu\alpha}\gamma_\alpha, \hspace{1cm} (C.1)$$

where we can choose the gamma matrices as the Pauli matrices\cite{21}:

$$\gamma^0 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \hspace{1cm} (C.2)$$

The Dirac equation in the 3d spacetime is derived as follows\cite{21}\cite{47}:

$$(\not{\partial} - m_f)\psi = 0, \hspace{1cm} (C.3)$$

with $\not{\partial} = \gamma^\mu \partial_\mu$. Its solution takes the plane wave form $\psi \sim u(p)e^{-ip\cdot x} + v(p)e^{ip\cdot x}$. Thus, the spinors $(u, v)$ satisfy the momentum-space equations:

$$(\slashed{p} - im_f)u = 0, \quad (\slashed{p} + im_f)v = 0. \hspace{1cm} (C.4)$$

Then, solving Eq.(C.4) gives the spinor solutions for particle and anti-particle:

$$u = \frac{1}{\sqrt{-p_0 + p_1}} \begin{pmatrix} p_2 + im_f \\ -p_0 + p_1 \end{pmatrix}, \quad v = \frac{1}{\sqrt{-p_0 + p_1}} \begin{pmatrix} p_2 - im_f \\ -p_0 + p_1 \end{pmatrix}. \hspace{1cm} (C.5)$$

They obey the following spinor identities:

$$u\bar{u} = -\slashed{p} - im_f, \quad \bar{u}u = -i2m_f, \quad v\bar{v} = -\slashed{p} + im_f, \quad \bar{v}v = i2m_f, \hspace{1cm} (C.6)$$

where $\bar{u} = u^\dagger \gamma^0$ and $\bar{v} = v^\dagger \gamma^0$.  

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D Topological Scattering Amplitudes with Matter Fields

In this Appendix, we present the full amplitudes of the scattering processes discussed in section 4. For the notational convenience of the following scattering amplitudes, we have defined the parameters \( \beta_\pm = 1 \pm \beta \) with \( \beta = \sqrt{1-m^2/E^2} \).

The scattering amplitudes of pair annihilation in the topologically massive scalar QED take the following forms:

\[
\mathcal{T}[\phi^- \phi^+ \to A_P A_P] = -\frac{2e^2 [1 - \bar{E}^4 + \bar{E}^2 (1 + \bar{E}^2) c_{2\theta} + i 2 \bar{E}^3 s_{2\theta}]}{(1 - 2\bar{E}^2)^2 + 4 \bar{E}^2 (1 - \bar{E}^2) c_\theta^2}, \tag{D.1a}
\]
\[
\mathcal{T}[\phi^- \phi^+ \to A_T A_T] = -\frac{2e^2 (1 - 2\bar{E}^4 + 2\bar{E}^4 c_{2\theta})}{(1 - 2\bar{E}^2)^2 + 4 \bar{E}^2 (1 - \bar{E}^2) c_\theta^2}. \tag{D.1b}
\]

The Compton scattering amplitudes in the topologically massive scalar QED are given by

\[
\mathcal{T}[\phi^- A_P \to \phi^- A_P] = \frac{e^2}{1 - \beta(1 + \beta_\pm) - 2\beta^2 c_\theta} \times \]
\[
\{(2 - \bar{E}^2 \beta_\pm^2) \beta_\pm^2 - [1 - \beta(1 + \beta_\pm) + \bar{E}^2 \beta_\pm^2 (1 - 2\beta^2)] c_\theta - i 2 \bar{E} \beta_\pm (1 - 2\beta) s_\theta \}, \tag{D.2a}
\]
\[
\mathcal{T}[\phi^- A_T \to \phi^- A_T] = \frac{2e^2 [2\beta_\pm^2 - (1 - 2\beta - 2\beta^2) c_\theta]}{1 - \beta(1 + \beta_\pm) - 2\beta^2 c_\theta}. \tag{D.2b}
\]

The scattering amplitudes of pair annihilation in the topologically massive spinor QED are derived as follows:

\[
\mathcal{T}[e^- e^+ \to A_P A_P] = \frac{2e^2 \bar{E}^2 [2(1 - \bar{E}^2) + 2 \bar{E}^2 c_{2\theta} + i \bar{E} (1 + \bar{E}^2) s_{2\theta}]}{(1 - 2\bar{E}^2)^2 + 4 \bar{E}^2 (1 - \bar{E}^2) c_\theta^2}, \tag{D.3a}
\]
\[
\mathcal{T}[e^- e^+ \to A_T A_T] = \frac{i 4e^2 \bar{E}^4 s_{2\theta}}{(1 - 2\bar{E}^2)^2 + 4 \bar{E}^2 (1 - \bar{E}^2) c_\theta^2}. \tag{D.3b}
\]

The Compton scattering amplitudes in the topologically massive spinor QED:

\[
\mathcal{T}[e^- A_P \to e^- A_P] = \frac{ie^2 \beta (1 + \tan \frac{\theta}{2})}{2\beta_\pm [1 - \beta(1 + \beta_\pm) - 2\beta^2 c_\theta] (1 + s_\theta) \frac{1}{2}} \left[ 1 + \bar{E}^2 - 4 \beta \beta_+ - \bar{E}^2 \beta_+^2 (2 - \beta^2) - 4 \beta_+ c_\theta - (1 + \bar{E}^2 \beta_+^2 c_{2\theta} - i 4 \bar{E} \beta_+ s_\theta - i 2 \bar{E} \beta_+ s_{2\theta} \right], \tag{D.4a}
\]
\[
\mathcal{T}[e^- A_T \to e^- A_T] = \frac{i 2e^2 \beta (1 - 2\beta - c_\theta) (1 + c_\theta + s_\theta)}{\beta_+ [1 - \beta(1 + \beta_\pm) - 2\beta^2 c_\theta] (1 + s_\theta) \frac{1}{2}}. \tag{D.4b}
\]

The scattering amplitudes of pair annihilation via the color-singlet channel in the topologically massive QCD are connected to that of the TMQED according to Eq. (4.25) in section 4.2.1:

\[
\mathcal{T}_{PP}[|0\rangle_q \to |0\rangle_{A_P}] = \frac{g^2}{e^2} f(N) \mathcal{T}[e^- e^+ \to A_P A_P], \tag{D.5a}
\]
\[ T_{\text{TT}}[\langle 0 \rangle_q \rightarrow \langle 0 \rangle_{A_T}] = \frac{g^2}{e^2} f(N) \, T[\epsilon^- e^+ \rightarrow A_T A_T]. \]  

(D.5b)

where the function \( f(N) = \frac{1}{2\sqrt{2}} \sqrt{(N^2 - 1)/N} \).

E Graviton Propagator and Scattering Amplitude in TMG

From the action (5.1) together with the gauge-fixing term (5.8), we derive the quadratic term of the graviton fields:

\[ S_{\text{TMG}} = \int d^3 x \frac{1}{2} h^{\mu\nu} D^{-1}_{\mu\nu\alpha\beta} h_{\alpha\beta}, \]  

(E.1)

where the inverse of the graviton propagator \( D^{-1}_{\mu\nu\alpha\beta} \) takes the following form:

\[ D^{-1}_{\mu\nu\alpha\beta} = (1 - \frac{1}{2\xi}) \eta_{\mu\nu} \eta_{\alpha\beta} - \frac{1}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}) \partial^2 + \left( \frac{1}{\xi} - 1 \right) (\eta_{\mu\nu} \partial_\alpha \partial_\beta + \eta_{\alpha\beta} \partial_\mu \partial_\nu) \]

\[ + \frac{1}{2} \left( \frac{1}{\xi} - 1 \right) (\eta_{\mu\alpha} \partial_\rho \partial_\beta + \eta_{\mu\beta} \partial_\rho \partial_\alpha + \eta_{\nu\alpha} \partial_\rho \partial_\beta + \eta_{\nu\beta} \partial_\rho \partial_\alpha) \]

\[ + \frac{1}{2m} \left[ \epsilon_{\mu\rho\alpha} (\partial_\nu \partial_\beta \partial^\rho - \eta_{\nu\beta} \partial^2 \partial^\rho) (\mu \leftrightarrow \nu) \right]. \]  

(E.2)

Then, transforming Eq.(E.2) into momentum space and imposing the normalization condition

\[ D^{-1}_{\mu\nu\alpha\beta} D^{\alpha\beta\rho\sigma} = \frac{i}{2} (\delta^\rho_\mu \delta^\sigma_\nu + \delta^\rho_\nu \delta^\sigma_\mu), \]  

(E.3)

we can derive the massive graviton propagator as follows:

\[ D_{\mu\nu\alpha\beta} = \frac{i \Delta_{\mu\nu\alpha\beta}}{2(p^2 + m^2)}, \]  

(E.4)

where the numerator is given by

\[ \Delta_{\mu\nu\alpha\beta} = -\eta_{\mu\nu} \eta_{\alpha\beta} - \frac{m^2}{p^2} (2\eta_{\mu\nu} \eta_{\alpha\beta} - \eta_{\mu\alpha} \eta_{\nu\beta} - \eta_{\mu\beta} \eta_{\nu\alpha}) - \frac{1}{p^2} (\eta_{\mu\nu} P_{\alpha} P_{\beta} + \eta_{\alpha\beta} P_{\mu} P_{\nu}) \]

\[ - \frac{1}{p^4} P_{\mu} P_{\nu} P_{\alpha} P_{\beta} + \frac{\xi (p^2 + m^2) - m^2}{p^4} (\eta_{\mu\alpha} P_{\nu} P_{\beta} + \eta_{\mu\beta} P_{\nu} P_{\alpha} + \eta_{\nu\alpha} P_{\mu} P_{\beta} + \eta_{\nu\beta} P_{\mu} P_{\alpha}) \]

\[ + \frac{im p^\rho}{2p^2} (\epsilon_{\rho\mu\alpha} \eta_{\nu\beta} + \epsilon_{\rho\mu\beta} \eta_{\nu\alpha} + \epsilon_{\rho\nu\alpha} \eta_{\mu\beta} + \epsilon_{\rho\nu\beta} \eta_{\mu\alpha}) \]

\[ - \frac{im p^\rho}{2p^4} (\epsilon_{\rho\mu\alpha} P_{\nu} P_{\beta} + \epsilon_{\rho\mu\beta} P_{\nu} P_{\alpha} + \epsilon_{\rho\nu\alpha} P_{\mu} P_{\beta} + \epsilon_{\rho\nu\beta} P_{\mu} P_{\alpha}). \]  

(E.5)

Using the notation \( P_{\mu\nu} = \eta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \), we can further express the propagator (E.4)-(E.5) into the following form:

\[ D_{\mu\nu\alpha\beta}(p) = -\frac{i}{2(p^2 + m^2)} (P_{\mu\alpha} P_{\nu\beta} + P_{\mu\beta} P_{\nu\alpha} - P_{\mu\nu} P_{\alpha\beta}) + \frac{i}{2p^2} (P_{\mu\alpha} P_{\nu\beta} + P_{\mu\beta} P_{\nu\alpha} - 2P_{\mu\nu} P_{\alpha\beta}) \]

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\[- \frac{i}{p^4} (\eta_{\mu\nu} p_\mu p_\nu + \eta_{\alpha\beta} p_\mu p_\nu) + \frac{i\xi}{2p^4} (\eta_{\mu\alpha} p_\nu p_\beta + \eta_{\mu\beta} p_\nu p_\alpha + \eta_{\nu\alpha} p_\mu p_\beta + \eta_{\nu\beta} p_\mu p_\alpha) \]
\[- \frac{m p^\rho}{4p^2(p^2 + m^2)} (\varepsilon_{\rho\mu\alpha} P_{\nu\beta} + \varepsilon_{\rho\mu\beta} P_{\nu\alpha} + \varepsilon_{\rho\nu\alpha} P_{\mu\beta} + \varepsilon_{\rho\nu\beta} P_{\mu\alpha}). \quad (E.6)\]

Under the Landau gauge \( \xi = 0 \), the above propagator reduces to
\[ D_{\mu\nu;\alpha\beta}^{\xi=0}(p) = - \frac{i}{2(p^2 + m^2)} (P_{\mu\alpha} P_{\nu\beta} + P_{\mu\beta} P_{\nu\alpha} - P_{\mu\nu} P_{\alpha\beta}) + \frac{i}{2p^2} (P_{\mu\alpha} P_{\nu\beta} + P_{\mu\beta} P_{\nu\alpha} - 2P_{\mu\nu} P_{\alpha\beta}) \]
\[- \frac{m p^\rho}{4p^2(p^2 + m^2)} (\varepsilon_{\rho\mu\alpha} P_{\nu\beta} + \varepsilon_{\rho\mu\beta} P_{\nu\alpha} + \varepsilon_{\rho\nu\alpha} P_{\mu\beta} + \varepsilon_{\rho\nu\beta} P_{\mu\alpha}) - \frac{i}{p^4} (\eta_{\mu\nu} p_\alpha p_\beta + \eta_{\alpha\beta} p_\mu p_\nu). \quad (E.7)\]

If the last term above is removed by contracting with a conserved current or on-shell physical graviton polarization, the propagator \((E.7)\) agrees with the result of Ref. [1].

Next, we note that in section 5 we reconstructed the four-graviton scattering amplitude in Eqs.(5.14)-(5.15) which is expressed in terms of the energy variable \( \bar{s}_0 = s_0/m^2 \). For the sake of comparison, we further reexpress the four-graviton amplitude in terms of the Mandelstam variable \( \bar{s} = s/m^2 \), which is connected to \( \bar{s}_0 = s_0/m^2 \) via \( \bar{s} = \bar{s}_0 + 4 \). Thus, from Eqs.(5.14)-(5.15), we derive the following equivalent expressions:
\[ \mathcal{M}[4h_p] = \frac{\kappa^2 m^2 (Q_0 + Q_2 c_{2\theta} + Q_4 c_{4\theta} + Q_6 c_{6\theta} + \bar{Q}_2 s_{2\theta} + \bar{Q}_4 s_{4\theta} + \bar{Q}_6 s_{6\theta}) \csc^2 \theta}{4096 (1 - \bar{s}) \bar{s}^{3/2} [2 - \bar{s} - (4 - \bar{s})c_\theta][2 - \bar{s} + (4 - \bar{s})c_\theta]}, \quad (E.8)\]
where \((Q_j, \bar{Q}_j)\) are expressed as polynomial functions of the variable \( \bar{s} = s/m^2 \),
\[ Q_0 = (256 + 49088\bar{s} - 68880\bar{s}^2 + 25220\bar{s}^3 - 2768\bar{s}^4)\bar{s}^{1/2}, \]
\[ Q_2 = (-768 - 45568\bar{s} + 65568\bar{s}^2 - 19008\bar{s}^3 + 505\bar{s}^4)\bar{s}^{3/2}, \]
\[ Q_4 = 4(192 - 176\bar{s} + 20\bar{s}^2 + 635\bar{s}^3 + 55\bar{s}^4)\bar{s}^{5/2}, \]
\[ Q_6 = -(256 + 2816\bar{s} + 2912\bar{s}^2 + 560\bar{s}^3 + 17\bar{s}^4)\bar{s}^{3/2}, \]
\[ \bar{Q}_2 = i(1280 - 256\bar{s} + 21312\bar{s}^2 - 8960\bar{s}^3 + 475\bar{s}^4)\bar{s}, \]
\[ \bar{Q}_4 = i(320 - 544\bar{s} + 676\bar{s}^2 + 272\bar{s}^3 + 5\bar{s}^4)\bar{s}, \]
\[ \bar{Q}_6 = -i(1280 + 3584\bar{s} + 1568\bar{s}^2 + 128\bar{s}^3 + \bar{s}^4)\bar{s}. \]

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