Radiative Correction to the Casimir Force on a Sphere

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March 3, 2018

Abstract

The first radiative correction to the Casimir energy of a perfectly conducting spherical shell is calculated. The calculation is performed in the framework of covariant perturbation theory with the boundary conditions implemented as constraints. The formalism is briefly reviewed and its use is explained by deriving the known results for two parallel planes.

The ultraviolet divergencies are shown to have the same structure as those for a massive field in zeroth order of $\alpha$. In the zeta–functional regularization employed by us no divergencies appear.

If the radius of the sphere is large compared to the Compton wavelength of the electron the radiative correction is of order $\alpha/(R^2 m_e)$ and contains a logarithmic dependence on $m_e R$. It has the opposite sign but the same order of magnitude as in the case of two parallel planes.

1 Introduction

The Casimir effect is one of the basic effects in Quantum Electrodynamics (QED) and provided the first example of Nullpunktsenergie in a field theory. Proposed in 1948 and qualitatively verified in 1958, it has only recently been tested quantitatively [1]. Based on the attractive force that is exerted on the conducting plates, Casimir even proposed a model for the electron. This model looked quite promising until Boyer showed a repulsive force in the first calculation for a sphere in 1968. In a large number of calculations for different geometries the Casimir force was found to be repulsive for configurations whose sizes in different directions are close to one another (a cube, for instance) and attractive when at least one of the sizes is much larger than the others (a long cylinder, for instance). Nevertheless, a general understanding of this property is still missing.
Radiative corrections have rarely been considered because the expected effect, being proportional to the fine structure constant $\alpha$ and at least to the ratio of the Compton wavelength $\lambda_c$ of the electron to a typical geometric size $L$ of the boundaries, is too small to be directly observable. Nevertheless, there is a general interest in their consideration, mainly having in mind the applicability of the general methods of quantum field theory. For example, its covariant formulation in the presence of explicitly non-covariant boundaries and the generalization of the perturbation expansion of a gauge theory are among the important issues to be addressed in this context. Boundary conditions necessitate a modification of the renormalization procedure as well. The interaction of the quantum fields with the boundary leads to ultraviolet divergencies in vacuum graphs that cannot simply be discarded by normal ordering. Last but not least the question for the order of the geometrical effects is raised, i.e. the question for the leading power of $\lambda_c/L$ contributing to the effective action. Additional attention to radiative corrections arises in the bag model of QCD where they are not a priori negligible.

The calculation of the ground state energy can be performed starting from the basic relation

$$E_0 = \frac{1}{2} \sum_{(n)} E_{(n)},$$

(1)

where $E_{(n)}$ are the one particle energies of the considered system and the sum runs over the corresponding spectrum. This form, modified by some regularization (e.g., the zeta functional one with $E_0^{\text{reg}} = (1/2) \sum_{(n)} E_{(n)}^{1-2s}, \ (s > 3/2)$), is best suited for most of the calculations for different boundary conditions and geometries. From the point of view of general quantum field theory it is connected with the vacuum expectation value of the energy-momentum tensor $<0|\hat{T}_{\mu\nu}|0>$ by $E_0 = \int d^4x <0|\hat{T}_{00}|0>$ (leaving aside modifications due to translational invariant directions). By means of the well known relations $\hat{T}_{00} = \frac{\partial}{\partial x^\mu} \hat{\phi} (x) \frac{\partial}{\partial x^\nu} \hat{\phi} (x)$ (real scalar field), $<0|\hat{\phi} (x) \hat{\phi} (y)|0> = iD(x,y)$ and $<0|\hat{T}_{00}|0> = i\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} D(x,y)|_{y=x}$ it is qualified as a one loop contribution and can be represented as the Feynmann graph $\otimes$, where $\phi$ is the vertex factor corresponding to $T_{\mu\nu}$. Using this language it is straightforward to incorporate radiative corrections as higher loops.

For instance, in $\varphi^4$-theory we would have $\otimes$. In QED there are two contributions to the energy-momentum tensor, one resulting from the electromagnetic field $\otimes$ and one from the spinor field $\otimes$. The corresponding two
Currently there is no established formalism for the calculation of radiative corrections in the presence of boundaries, although a number of attempts have been undertaken $[2, 3]$. In the present paper we use the formalism of general covariant perturbation theory, including the standard theory of renormalization with the 'minimal' modifications needed and calculate the first radiative correction to the Casimir energy for a perfectly conducting spherical shell. The basic ideas are taken from an earlier work $[5]$, however the time passed shows that the required modifications are not simple.

The conductor boundary conditions the electromagnetic field has to fulfil on some boundary $S$ read

$$n^\mu F_{\mu \nu}^*(x)_{|_{x \in S}} = 0,$$

where $F_{\mu \nu}^* = \epsilon_{\mu \nu \alpha \beta} F^{\alpha \beta}$ is the dual field strength tensor and $n^\mu$ is the (outer) normal of $S$. Being the idealization of a physical interaction (with the conductor), they are formulated in terms of the field strengths and thus gauge–invariant. Now, for well known reasons, it is desirable to perform the quantization of QED in terms of the gauge potentials $A_\mu(x)$. Obviously, the boundary conditions (2) do not unambiguously imply boundary conditions for all components of $A_\mu(x)$ as it is required in order to obtain a selfadjoint wave operator.

There are two possibilities to proceed. The first one is to impose boundary conditions on the potentials in such a way that the conditions (2) are satisfied and a selfadjoint wave operator is provided. Electric and magnetic boundary conditions are commonly used for this purpose. These conditions are stronger than (2) but they are not gauge invariant. When now requiring BRST invariance the ghosts become boundary dependent too and contribute to physical quantities like the Nullpunktsenergie. This was first observed in the papers $[6]$ and has later also been noticed in connection with some models in quantum cosmology $[7]$. The common understanding is that the ghost contributions cancel those from the unphysical photon polarizations, for recent discussions see $[8, 9]$.

In a second approach one considers the boundary conditions (2) as constraints when quantizing the potentials $A_\mu(x)$ as it was first done in $[5]$. In that case, explicit gauge invariance is kept. There is no need to impose any additional conditions. The conditions (2) appear to be incorporated in a 'minimal' manner. In that respect, this second approach resembles the so called dyadic formalism which has succesfully been used in the calculation of the Casimir energy in spherical geometry $[10]$.

There are two ways to put this approach into practice. The first way is to solve the constraints explicitly. For this purpose one has to introduce a basis of polarization vectors $E_\mu^s$ (instead of the commonly used $e_\mu^s$) such that only two amplitudes (in our notation those with $s = 1, 2$) of the corresponding decompo-
of the electromagnetic potential have to satisfy boundary conditions. The other two amplitudes (in our notation those with \( s = 0, 3 \)) remain free. In the case where the surface \( S \) consists of two parallel planes such a polarization basis can be constructed explicitly \[ 1 ]

\[
E_1^\mu = \left( \begin{array}{c} 0 \\ i\partial_x^2 \\ -i\partial_x^1 \\ 0 \end{array} \right) \frac{1}{\sqrt{-\partial_x^0}}, \quad E_2^\mu = \left( \begin{array}{c} -\partial^2_x \\ -\partial_{x_0}\partial_{x_1} \\ -\partial_{x_0}\partial_{x_2} \\ 0 \end{array} \right) \frac{1}{\sqrt{-\partial^2_{x_0} + \partial^2_{x_1}}}, \\
E_3^\mu = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right), \quad E_0^\mu = \left( \begin{array}{c} 0 \\ -i\partial_{x_0} \\ -i\partial_{x_1} \\ -i\partial_{x_2} \end{array} \right) \frac{1}{\sqrt{-\partial^2_{x_0} + \partial^2_{x_1}}}.
\]

\((E_1 s g^{\mu\nu} E_2^\nu = g^{st})\). Inserting the decomposition (3) with these polarization vectors into the boundary conditions (2), we find

\[
a_1(x) = a_2(x) = 0 \quad \text{for} \quad x \in S ,
\]

whereas the amplitudes \( a_s(x) \) with \( s = 0, 3 \) are unaffected by the boundary conditions. In this way, roughly speaking half of the photon polarizations feel the boundary (in \[ 12 ] \) they have been shown to be the physical ones in the sense of the Gupta–Bleuler quantization procedure) and half of them do not. However, such an explicit decomposition, which simultaneously diagonalizes both, the action and the boundary conditions can only be found in the simplest case. The problem is that for a non–planar surface \( S \) the polarizations \( E_1^\mu \) become position dependent. To the authors’ knowledge, their existence in the general case is still an open issue.

Here we follow the second way to realize the boundary conditions as constraints, namely we use Lagrange multipliers or, equivalently, restrict the integration space in the functional integral approach. The generating functional of the Green functions in QED reads

\[
Z(J, \bar{\eta}, \eta) = C \int DA \, D\bar{\psi} \, D\psi \prod_\nu \prod_{x \in S} \delta \left( n^\nu F_{\mu\nu}^s(x) \right) \exp \left\{ iS \left[ A_\mu, \psi, \bar{\psi} \right] \right\} + i \int dx \left( A_\mu(x) J_\mu(x) + \bar{\psi}(x) \eta(x) + \bar{\eta}(x) \psi(x) \right),
\]

where the integration runs over all fields with the usual asymptotic behavior. The functional delta function restricts the integration space to such potentials \( A_\mu \) that the corresponding field strengths satisfy the boundary conditions (2).
This approach had been used in [5] in the case of plane parallel boundaries. It was shown to result in a new photon propagator and an otherwise unaltered covariant perturbation theory of QED. As it will be seen in the next section, the boundary conditions (2) appear to be incorporated with a 'minimal' disturbance of the standard formalism, completely preserving gauge invariance (as well as the gauge fixing procedure) and Lorentz covariance as far as possible. An additional advantage of this approach is the fact that the resulting formulas for practical calculations turn out to be rather simple.

The spinor field deserves a special discussion with respect to the boundary conditions. We do not impose boundary conditions on the electron and consider the electromagnetic field and the spinor field on the entire Minkowski space with the conducting surface placed in it. In the case of $S$ being a sphere we thus consider the interior and the exterior region together. In general, the surface $S$ need not be closed. Only the electromagnetic field obeys boundary conditions on the surface $S$. The electron penetrates it freely, it does not feel the surface. As for a physical model one can think of a very thin metallic surface which does not scatter the electrons but reflects the electromagnetic waves. If the thickness of the metallic surface (e.g. 1 $\mu$m) is small compared to the radiation length of the metal (e.g. 1.43 cm for copper [18]), this approximation is well justified. However, the radiative corrections are of order $(\alpha \lambda_c / L)$ with respect to the Casimir force itself and thus too small to be directly observable. A discussion of the validity of this approximation seems therefore a bit academic.

We remark, that the situation is different in the case of the bag model. The boundary conditions of the gluon field and of the spinor field are connected by means of the equation of motion (this is because the field strength tensor enters the boundary conditions rather than the dual field strength tensor in (2)). Also, we cannot a priori expect the radiative corrections to be very small. However, a detailed discussion of this subject goes beyond the scope of the present paper. Nevertheless, we would like to regard the investigation of the radiative corrections in QED as a necessary step towards the corresponding calculations in QCD.

For the needs of renormalization we consider the quantum fields embedded in a classical system, given by the surface $S$ with a corresponding classical energy. For example, the classical energy of a sphere takes the form

$$E_0^{\text{class}} = pV + \sigma S + FR + k + \frac{h}{R},$$

where $V = \frac{4}{3} \pi R^3$ and $S = 4\pi R^2$ are the volume and the surface area, respectively. The basic idea is to renormalize the divergencies of the ground state energy by a redefinition of the corresponding parameters $p, \sigma, F, k, h$ in the classical part of the system. For recent comprehensive discussions we refer to the papers [13, 14]. The general structure of the potentially divergent contributions for an arbitrary surface $S$ is completely known from the heat kernel expansion. Explicit formulas for the sphere are given in, e.g., [13]. When one considers the quantum fields
in the interior as well as in the exterior of the sphere (as we do), the divergent contributions with odd powers of the radius $R$ cancel. Therefore, it is sufficient to keep only the two contributions

$$E_0^{\text{class}} = \sigma S + k$$

in the classical energy. Hence, as a hypothesis we expect divergent contributions proportional to $R^2$ and $R^0$ (i.e., proportional to a constant) when including the radiative correction. Indeed, this turns out to be the case (see section 5). The corresponding redefinition of $\sigma$ and $k$ now depends on the fine structure constant $\alpha$ and the electron mass $m$ (see eqs. (74) and (75) below). Besides these boundary induced divergencies, the ordinary free Minkowski space vacuum contributions which are usually discarded by normal ordering, are expected to appear as well. In line with the common interpretation, their removal is understood as a renormalization of the cosmological constant.

It is interesting to observe the following fact. For dimensional reasons, the divergent contributions (8) to the ground state energy are proportional to positive powers of the mass, $\sigma \sim m^3$ and $k \sim m$. Consequently, they vanish for massless fields, e.g. the pure electromagnetic field. However, this well known picture is not preserved when including the first radiative correction. Due to the interaction a dimensional parameter, namely the electron mass, is introduced and the divergent contributions proportional to $R^2$ and $R^0$ do no longer vanish. The necessity to consider the general structure (8) of the classical energy in QED becomes evident.

In the case of plane parallel plates of separation $L$, the ground state energy $E_0$ including the radiative correction $E_0^{(1)}$ to leading order in $\lambda_c/L$ is given by

$$E_0^{\text{plates}} \equiv E_0^{(0)} + E_0^{(1)} = -\frac{\pi^2}{720} \frac{1}{L^3} + \frac{\pi^2}{2560} \frac{\alpha \lambda_c}{L^4}.$$  

Let us remark that the first loop correction is of order $\alpha$ and that the first possible contribution in the small ratio $\lambda_c/L$ is present. In fact, once this is known, the remaining task consists in the calculation of the number $\pi^2/2560$. The calculations that have appeared in the literature, show different results. Xua [3] implements periodic boundary conditions for the potentials. In his result, all powers of $\lambda_c/L$ are absent leaving only an exponentially small contribution. In the paper by Kong and Ravndal [2] the same boundary conditions as here are studied with the help of a different method, namely some effective field theory and the Euler–Heisenberg Lagrangian. Their result, $E_0^{(1)} \sim (\alpha^2/L^3)(\lambda_c/L)^4$, is of one order $\alpha$ smaller than (9). The renormalization has also been discussed in the paper [4].

In view of the result (9) the radiative correction in leading order of $\lambda_c/R$ for a conducting sphere of radius $R$ could be expected as

$$E_0^{\text{sphere}} = \frac{0.092353}{2R} + \text{const} \cdot \frac{\alpha \lambda_c}{R^2},$$
where the first term is Boyer’s result (in the formulation of [10]). The correction is calculated in section 5. We find

\[ E^{(1)}_0 = -7.5788 \cdot 10^{-4} \frac{\alpha \lambda_c}{R^2} \ln(mR) - 6.4833 \cdot 10^{-3} \frac{\alpha \lambda_c}{R^2}. \]  

(11)

Besides the term proportional to \(1/R^2\) it exhibits an additional logarithmic dependence \(\ln(mR)/R^2\) (\(m\) is the mass of the electron).

The result (9) allows for an interesting intuitive interpretation. Since no boundary conditions have been imposed on the spinors, a photon can pass the surface via the creation and subsequent annihilation of a virtual electron–positron pair. So the separation of the plates is effectively enlarged due to the radiative corrections. In this sense, the result (9) can be viewed as a ‘renormalization’ of the distance \(L\) separating the plates

\[ L \to L \left(1 + \frac{3}{32} \frac{\alpha \lambda_c}{L}\right) \]  

(12)

to leading order in \(\lambda_c/L\). This was first noticed in [5]. As it was shown in [11], the same ‘distance renormalization’ results when considering the radiative corrections to the photon states between the plates. This picture can also be adopted in the case of a conducting sphere. The radiative correction increases the effective radius \(R\) according to

\[ R \to R \left[1 + \frac{\alpha \lambda_c}{R} \left(1.4040 \cdot 10^{-1} + 1.6413 \cdot 10^{-2} \ln(mR)\right)\right] \]  

(13)

to leading order in \(\lambda_c/R\).

The paper is organized as follows. In the next section we present the procedure of quantization with constraints and in the third section we derive the general structure of the radiative corrections. In the fourth section we rederive the results for parallel planes in order to explain the use of the formalism in details. The fifth section contains the main part of the paper, namely the calculation of the radiative correction for spherical geometry. We conclude the paper with a summary and two technical appendices.

### 2 Quantization

We start the quantization procedure with the generating functional \(Z(J, \bar{\eta}, \eta)\) represented by the path integral (8). The perturbation expansion is obtained by means of

\[ Z(J, \bar{\eta}, \eta) = \exp \left[iS_{\text{int}} \left(\frac{\delta}{\delta J}, -\frac{\delta}{\delta \bar{\eta}}, \frac{\delta}{\delta \eta}\right)\right] Z^{(0)}(J, \bar{\eta}, \eta) \]  

(14)

with \(S_{\text{int}}(A, \bar{\psi}, \psi) = \int dx \bar{\psi}(x) \tilde{A}(x)\psi(x)\) and the problem is reduced to the calculation of the generating functional for the non–interacting theory.
Before proceeding further we rewrite the boundary conditions (2) in the following way. Let \(E^s_\mu(x)\) \((s = 1, 2)\) be the two polarization vectors in (3) with the properties

\[
\begin{align*}
\frac{\partial}{\partial x_\mu} E^s_\mu(x) &= 0, \\
n^\mu E^s_\mu(x) &= 0
\end{align*}
\]  

for \(x \in S\) (assuming \(\partial_\mu n_\nu(x) = \partial_\nu n_\mu(x)\)). They span a space of transversal vectors tangential to the surface \(S\). Note that due to (16), there is no derivative acting outside the tangential space, i.e. no normal derivative in (15). Without loss of generality, we assume the normalization

\[
E^s_\mu \partial^\mu g_{\mu \nu} E^t_\nu = -\delta^{st}.
\]

We remark that the transformations \(A_\mu \rightarrow A_\mu + \partial_\mu \varphi(x)\) and \(A_\mu \rightarrow A_\mu + n_\mu \varphi(x)\) respect the boundary conditions (2), i.e. if \(A_\mu(x)\) satisfies the boundary conditions, the transformed potential also does. The invariance under the first transformation simply says that the boundary conditions are gauge independent. The second means that the projection of \(A_\mu\) onto the normal \(n_\mu\) of \(S\) is unaffected by the boundary conditions. We therefore conclude that the boundary conditions (2) can equivalently be expressed as

\[
E^s_\mu A_\mu(x) |_{x \in S} = 0 \quad (s = 1, 2).
\]

Explicit examples for such polarization vectors \(E^s_\mu\) (besides \(E^s_\mu\) with \(s = 1, 2\) in (4) for parallel plates) are

\[
\begin{align*}
E^1_\mu &= \begin{pmatrix} i\partial_{x_3} \\
0 \\
0 \\
i\partial_{x_0}
\end{pmatrix} \frac{1}{\sqrt{-\partial^2_{x_0} + \partial^2_{x_3}}}, \\
E^2_\mu &= \begin{pmatrix} -\frac{x^2 L}{R} & \frac{2}{R} \partial^2_{x_0} L & \frac{2}{R} \partial^2_{x_3} L \\
0 & \frac{2}{R} \partial^2_{x_0} - \partial^2_{x_3} \\
0 & \frac{2}{R} \partial^2_{x_0} - \partial^2_{x_3} \\
i\partial_{x_0} \partial_{x_3} L
\end{pmatrix} \frac{1}{\sqrt{-\partial^2_{x_0} + \partial^2_{x_3}} \sqrt{-\partial^2_{x_0} + \partial^2_{x_3} + L^2}}
\end{align*}
\]

for a cylinder with axis along the \(x_3\)-direction, radius \(R\) and the orbital momentum operator \(L = -x_2 \partial_{x_1} + x_1 \partial_{x_2}\), and

\[
\begin{align*}
E^1_\mu &= \begin{pmatrix} 0 \\
0 \\
L \\
iR \partial^2_{x_0} (\vec{n} \times \vec{L})
\end{pmatrix} \frac{1}{\sqrt{L^2}} \\
E^2_\mu &= \begin{pmatrix} 0 \\
L \\
iR \partial^2_{x_0} (\vec{n} \times \vec{L})
\end{pmatrix} \frac{1}{\sqrt{L^2} \sqrt{-R^2 \partial^2_{x_0} - L^2}}
\end{align*}
\]

for a sphere of radius \(R\) with normal vector \(\vec{n} = \vec{x}/|\vec{x}|\) and angular momentum operator \(\vec{L} = i\vec{x} \times \vec{\partial}_x\). The properties (15) and (16) as well as the normalization
are easily checked. Evidently, these explicit polarization vectors do not contain normal derivatives. It should be noticed that the polarization $E_\mu$ in the examples (15) and (19) coincides with that of the standard approach (and the dyadic approach) whereas $E_\mu^2$ contains the combination $\vec{n} \times \vec{L}$ instead of $\vec{\nabla} \times \vec{L}$.

Now, with the boundary conditions in the form (17) we rewrite the delta functions in (3) and represent them as functional Fourier integrals

$$\prod_{x \in S} \delta \left( E_\mu^s(x) A^\mu(x) \right) = \int \mathcal{D}b_s \exp \left\{ i \int_{z \in S} dz \ b_s(z) E_\mu^s(z) A^\mu(z) \right\} \quad (s = 1, 2),$$

(20)

where $b_s(z)$ is a field ‘living’ on the surface $\mathcal{S}$. It corresponds to the Lagrange multiplier in the canonical quantization approach. According to (14) we turn to functions in (6) and represent them as functional Fourier integrals

$$Z^{(0)}(J, \eta, \eta) = C \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}b \exp \left\{ i[S^{(0)}(A) + S^{(0)}(\bar{\psi}, \psi)] + \int dx \ (A_\mu(x) J^\mu(x) + \bar{\psi}(x) \eta(x) + \bar{\eta}(x) \psi(x)) \right\}$$

(21)

with $S^{(0)}(A) = \frac{1}{2} \int dx \ A_\mu(x) K^{\mu\nu} A_\nu(x)$, $S^{(0)}(\bar{\psi}, \psi) = \int dx \ \bar{\psi}(x) \left( i\delta - m \right) \psi(x)$ and $K^{\mu\nu} = g^{\mu\nu} \partial^2 - (1 - 1/\alpha) \partial^\mu \partial^\nu$ (hereafter, the summation over double latin indices from 1 to 2 is understood). Since the boundary conditions are gauge–invariant, they do not interfere with the integration over the gauge group. Therefore, the quantization of the gauge field in covariant gauge with gauge parameter $\alpha$ proceeds in a standard manner, e.g., with the help of the Faddeev–Popov procedure. In order to diagonalize the quadratic form in the exponential of (21) we substitute

$$A_\mu(x) \rightarrow A_\mu(x) - \int dy \ D_{\mu\nu}(x - y) J^\nu(y) - \int_{z \in S} dz \ D_{\mu}^\nu(x - z) E_\nu^s(z) b_s(z)$$

where

$$D_{\mu\nu}(x - y) = \left( g_{\mu\nu} - (1 - \alpha) \partial_{x_\mu} \partial_{x_\nu} / \partial_x^2 \right) D(x - y),$$

(22)

$$D(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{\epsilon^{ik(x-y)}}{-k^2 - i\epsilon} \quad (\epsilon > 0)$$

(23)

are the photon propagator in covariant gauge and the causal propagator, respectively. The result is

$$S^{(0)}(A) + \int_{z \in S} dz \ b_s(z) E_\mu^s(z) A^\mu(z) + \int dx \ A_\mu(x) J^\mu(x)$$

$$\rightarrow \frac{1}{2} \int dx \ A_\mu(x) K^{\mu\nu} A_\nu(x) - \frac{1}{2} \int dx \int dy \ J^\mu(x) D_{\mu\nu}(x - y) J^\nu(y)$$

$$- \int dx \int_{z \in S} dz \ J^\mu(x) D_{\mu}^\nu(x - z) E_\nu^s(z) b_s(z)$$

$$- \frac{1}{2} \int_{z \in S} dz \int_{z' \in S} dz' \ b_s(z) \tilde{K}^{st}(z, z') b_t(z')$$

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with
\[ \tilde{K}^{st}(z, z') \equiv E^s_\mu(z) D^{\mu\nu}(z - z') E^t_\nu(z'), \quad z, z' \in S. \tag{24} \]

The newly defined object \( \tilde{K}^{st}(z, z') \) is in fact the projection of the propagator \( D_{\mu\nu}(z - z') \) on the surface \( S \) and, with respect to the Lorentz indices, into the tangential subspace spanned by the polarization vectors \( E^s_\mu(x), s = 1, 2 \). Further we need to define the inversion \( \tilde{K}^{-1st}(z, z') \) of this operation
\[ \int_S dz'' \tilde{K}^{-1st}(z, z'') \tilde{K}^{st}(z'', z') = \delta_S(z - z') \delta_{st}, \tag{25} \]

where \( \delta_S(z - z') \) is the delta function with respect to the integration over the surface \( S \), \( \delta_{st} \) is the usual Kronecker symbol. The quadratic form in (21) is finally diagonalized by the substitution
\[ b_s(z) \rightarrow b_s(z) - \int_S dz' \int dx \tilde{K}^{-1st}(z, z') E^t_\nu(z') D^{\mu\nu}(z' - x), J^\nu(x) \]

and we obtain
\[
S^{(0)}(A) + \int_{z \in S} dz b_s(z) E^s_\mu(z) A^\mu(z) + \int dx A_\mu(x) J^\mu(x) \\
\rightarrow \frac{1}{2} \int dx A_\mu(x) K^{\mu\nu} A_\nu(x) - \frac{1}{2} \int_S dz \int_S dz' b_s(z) \tilde{K}^{st}(z, z') b_s(z') \\
- \frac{1}{2} \int dx \int dy J^\mu(x) S^{\mu\nu}(x, y) J^\nu(y)
\]

with the new photon propagator
\[
S^{\mu\nu}(x, y) \equiv D_{\mu\nu}(x - y) - \tilde{D}_{\mu\nu}(x, y) \\
= D_{\mu\nu}(x - y) - \int_S dz \int_S dz' D_{\mu\nu}(x - z) E^s_\mu(z) \tilde{K}^{-1st}(z, z') E^t_\nu(z') D^{\mu\nu}(z' - y). \tag{26}
\]

Now, the integration over all fields can be carried out and we arrive at
\[
Z^{(0)}(J, \bar{\eta}, \eta) = C (\det K)^{-\frac{1}{2}} (\det \tilde{K})^{-\frac{1}{2}} \\
\times \exp \left[ \frac{1}{2} \int dx \int dy J^\mu(x) S^{\mu\nu}(x, y) J^\nu(y) \\
+ \frac{1}{i} \int dx \int dy \overline{\psi}(x) S(x - y) \psi(y) \right], \tag{27}
\]

where \( S(x - y) \) is the usual electron propagator.

The generating functional (27) together with relation (14) defines the perturbation theory of QED in the conventional way. Apart from the factor \( (\det K)^{-\frac{1}{2}} \) due to the integration over the field \( b_s(z) \) \( (z \in S) \), the boundary conditions
manifest themselves in the modified photon propagator \( \Phi(26) \). Our calculation of the radiative corrections relies on this representation of the boundary dependent propagator. As a consequence of \( \Phi(27) \) the usual language of Feynmann graphs can be adopted. It may be observed with the help of \( \Phi(15) \) that the boundary dependent part of the propagator \( \Phi(26) \) and its part depending on the gauge parameter \( \alpha \) are orthogonal.

Equation \( \Phi(27) \) was first derived in \[5\] in a more complicated way for the case of two parallel planes. Here it is derived for a general surface \( S \). However, the main problem in a practical calculation consists in finding the inverse of \( \Phi(27) \).

The representation \( \Phi(26) \) allows for a simple interpretation of the boundary dependent part \( \Phi(26) \) of the photon propagator. It can be viewed as describing the free propagation of a photon from the point \( x \) to a point \( z \in S \) on the surface \( S \), a subsequent propagation on \( S \) from \( z \) to \( z' \) mediated by \( \Phi(27) \) which is non–local in \( z \) and \( z' \), and another free propagation from \( z' \) to \( y \). In this way, the interaction of a free photon with the boundary \( S \) is accounted for, as symbolically indicated in fig. 1.

![Figure 1: Illustration of the boundary dependent part \( \Phi(26) \) of the photon propagator.](image)

### 3 Radiative Corrections

Using the above derived representation \( \Phi(27) \) we find the effective action by standard methods in the form

\[
\Gamma(A, \psi, \bar{\psi}) = \frac{i}{2} \text{Tr} \log K + \frac{i}{2} \text{Tr} \log \Phi + S^{(0)}(A) + S^{(0)}(\bar{\psi}, \psi) - i \sum \{1PI graphs\} .
\]  

(28)

The Feynman rules are obtained from the usual ones (i.e. without boundary conditions) by replacing the photon propagator by its modified analogue \( \Phi(26) \). Since the boundary conditions are static, the effective action is proportional to the total time \( T \) and the ground state energy is given by

\[
E_0 = -\frac{1}{T} \Gamma(A, \psi, \bar{\psi}) \bigg|_{A=0, \psi=0} .
\]  

(29)

If there are translationally invariant directions (e.g., parallel to the plates), the relevant physical quantity is the energy density and we have to divide by the
corresponding volume too. Graphically, the ground state energy is given by 1PI graphs without external legs. The only two–loop diagram contributing to the effective action is \( \bullet \). So up to two loop order we find the following expression for the ground state energy

\[
E_0 \equiv E_0^{(0)} + E_0^{(1)} = -\frac{i}{2T} \text{Tr} \log K - \frac{i}{2T} \text{Tr} \log \bar{K}
\]

\[
+ \frac{i}{2T} \int dx \int dy \left[ D_{\nu\mu}(x - y) - D_{\nu\mu}(x, y) \right] \Pi^{\mu\nu}(y - x),
\]

where

\[
\Pi^{\mu\nu}(x - y) = -ie^2 \text{Tr} \gamma^\mu S^c(x - y) \gamma^\nu S^c(y - x)
\]

is the known polarization tensor. The term proportional to \( \text{Tr} \log K \) and the contribution of the free propagator \( D_{\mu\nu}(x - y) \) just constitute the two loop ground state energy of QED without boundary conditions. These terms do not depend on the geometry. According to the renormalization scheme outlined above, their removal can be absorbed in a redefinition of the cosmological constant. Therefore, they will be omitted in the following. It may be observed that the remaining contributions are independent of the gauge parameter \( \alpha \), i.e. the gauge dependence of the ground state energy is completely contained in its free part. Let us note that the representation (30) which was first used in [19] is much simpler than the one previously applied in [5] where the ground state energy was expressed in terms of the energy-momentum tensor.

Next we insert the definition (26) of \( \bar{D}_{\nu\mu}(x, y) \) into the expression (30) for the radiative correction and obtain (still for a general surface \( S \))

\[
E_0^{(1)} = -\frac{i}{2T} \int dz \int_{S} dz' \bar{K}^{-1st}(z, z') \bar{\Pi}^{ts}(z', z),
\]

where the abbreviation

\[
\bar{\Pi}^{ts}(z', z) = \int dx \int dy \left( E^t_{\nu\mu}(y) D_{\nu\mu}(z' - y) \Pi^{\nu\nu}(y - x) D_{\mu\nu}(x - z) E^s_{\mu\nu}(z) \right)_{z,z'\in S}
\]

has been introduced. Using the transversality of \( \Pi_{\mu\nu}(x) \),

\[
\Pi_{\mu\nu}(x) = (g_{\mu\nu} \partial_x^2 - \partial_{x_\nu} \partial_{x_\nu}) \Pi(x^2),
\]

this expression simplifies to

\[
\bar{\Pi}^{ts}(z', z) = \int dx \int dy \left( E^t_{\mu\nu}(z') D(z' - y) g^{\mu\nu} \partial_y^2 \Pi((y - x)^2) D(x - z) E^s_{\mu\nu}(z) \right)_{z,z'\in S}.
\]

Introducing the Fourier transform of \( \Pi(x) \)

\[
\Pi(x^2) = \int \frac{d^4q}{(2\pi)^4} e^{iqx} \bar{\Pi}(q^2)
\]
we rewrite it in the final form
\[
\tilde{\Pi}^{ts}(z', z) = \mathcal{E}^{\mu \nu}_\mu (z') \int \frac{d^4q}{(2\pi)^4} e^{iq(z-z')} g^{\mu\nu} \left( \frac{\tilde{\Pi}(q^2)}{-q^2} \right) E^{s}_\nu(z)_{z,z' \in S},
\] (35)
which is best suited for the calculation of the radiative correction in particular geometries. There is yet another important conclusion to be drawn from the representation (35). The scalar part \(\tilde{\Pi}(q^2)\) of the polarization tensor is known to possess a logarithmic divergence which is independent of \(q\) (for example, in Pauli–Villars regularization it is of the form \(-(2\alpha/3\pi) \log(M/m)\) where \(M\) is the regularizing mass). However, if only a \(q\)-independent constant is added to \(\tilde{\Pi}\), i.e. \(\tilde{\Pi}(q^2) \rightarrow \tilde{\Pi}(q^2) + c\), the quantity \(\tilde{\Pi}^{ts}(z', z)\) changes according to \(\tilde{\Pi}^{ts}(z', z) \rightarrow \tilde{\Pi}^{ts}(z', z) + c \tilde{K}^{st}(z', z)\). Then it may be verified with the help of (25) that the corresponding change in the ground state energy is a simple constant which is independent of the geometry. Consequently, the removal of the divergence of the polarization tensor can be interpreted as a renormalization of the cosmological constant in complete analogy to the free Minkowski space contributions discussed above. As a result only the finite, renormalized part of the polarization tensor needs to be taken into account when calculating the boundary dependent part of the radiative correction \(E_0^{(1)}\).

4 The ground state energy in plane geometry

In this section we wish to demonstrate the calculational techniques needed in the spherical case on a simple example first. For this purpose we rederive the results for the ground state energy \(E_0^{(0)}\) and the radiative correction \(E_0^{(1)}\) in the geometry of two parallel conducting planes. In this case, the surface \(S\) consists of two pieces. A coordinization of the planes is given by \(z = \{x_\alpha, x_3 = a_i\}\), where the subscript \(i = 1, 2\) distinguishes the two planes and \(\alpha = 0, 1, 2\) labels the directions parallel to them (they are taken perpendicular to the \(x_3\)-axis intersecting them at \(x_3 = a_i\), \(|a_1 - a_2| \equiv L\) is the distance between them). The polarizations \(E^{s}_\mu (s = 1, 2)\) are given by (11) and do not depend on \(x_\alpha\) or \(i\). Therefore they commute with the free photon propagator (22). Inserting them into (24) yields the operator \(\tilde{K}^{st}\) in the form
\[
\tilde{K}^{st}(z, z') = -\delta^{st} \Delta(x - x')_{z,z' \in S}.
\] (36)
We proceed with deriving a special representation of the scalar propagator. It is obtained by performing the integration over \(k_3\) in eq. (23)
\[
D(x - x') = \int \frac{d^3k_\alpha}{(2\pi)^3} \frac{e^{ik_\alpha(x^{\alpha} - x'^{\alpha}) + i\Gamma|x_3 - x'_3|}}{-2i\Gamma},
\] (37)
with \(\Gamma = \sqrt{k_0^2 - k_1^2 - k_2^2 + i\epsilon}\). Substituting \(x_3 = a_i\) and \(x'_3 = a_j\) we get
\[
\tilde{K}^{st}(z, z') = -\delta^{st} \int \frac{d^3k_\alpha}{(2\pi)^3} \frac{i}{2\Gamma} h_{ij} e^{ik_\alpha(x^{\alpha} - x'^{\alpha})}
\] (38)
where the abbreviation
\[ h_{ij} = e^{i\Gamma|a_i - a_j|} \quad (i, j = 1, 2) \] (39)
has been introduced. With (38) we have achieved a mode decomposition of the operator \( \bar{K}^{st} \) on the surface \( S \). As an advantage of this representation the inversion of \( \bar{K}^{st} \), defined by (25), is now reduced to the algebraic problem of inverting the (2x2)–matrix \( h_{ij} \). The inverse of \( h_{ij} \) is given by
\[ h_{ij}^{-1} = \frac{i}{2\sin\Gamma L} \begin{pmatrix} e^{-i\Gamma L} & -1 \\ -1 & e^{-i\Gamma L} \end{pmatrix}_{ij} \] (40)
and we get
\[ K^{-1st}(z, z') = -\delta_{st} \int \frac{d^3k_\alpha}{(2\pi)^3} \frac{2\Gamma}{i} h_{ij}^{-1} e^{ik_\alpha(x^a - x'^a)}. \] (41)

After inserting this expression into (26) we find the photon propagator for the electromagnetic field in covariant gauge with conductor boundary conditions on two parallel planes as proposed in [5].

Next we substitute (38) into (30) and obtain for the ground state energy density per unit area
\[ E^{(0)}_0 = \frac{-i}{2T V_{\parallel}} \int d^3x_\alpha \int \frac{d^3k_\alpha}{(2\pi)^3} \text{Tr} \log \left(-\delta_{st} \frac{i}{2\Gamma} h_{ij} \right), \] (42)
where we have divided by the volume \( V_{\parallel} \) of the translational invariant directions parallel to the planes. We perform the Wick rotation \( k_0 \to ik_0 \) (thereby \( \Gamma \to i\gamma \equiv i\sqrt{k_0^2 + k_1^2 + k_2^2} \)) and calculate the \( '\text{Tr} \log' \)
\[ \text{Tr} \log \left(-\delta_{st} \frac{1}{2\gamma} h_{ij} \right) = \log \det \left(-\delta_{st} \frac{1}{2\gamma} h_{ij} \right) = 2 \log \left(1 - e^{-2\gamma L} \right) - 4 \log(2\gamma). \]

The term \(-4 \log(2\gamma)\) yields a distance independent contribution and will be dropped. In this way the known result for the Casimir energy is reproduced
\[ E^{(0)}_0 = \int \frac{d^3k_\alpha}{(2\pi)^3} \log \left(1 - e^{-2\gamma L} \right) = -\frac{\pi^2}{720L^3}. \]

Now we turn to the radiative correction. The polarization vectors \( E^{s}_\mu \) (\( s = 1, 2 \)) commute with \( \Pi \) and we find
\[ \Pi^{st}(z - z') = \delta_{st} \int \frac{d^4q}{(2\pi)^4} e^{iq(z - z')} \frac{\tilde{\Pi}(q^2)}{q^2} \bigg|_{z, z' \in S}. \] (43)

We substitute this expression into the radiative correction (32) and get
\[ E^{(1)}_0 = \sum_{i,j=1}^2 \int \frac{d^4k}{(2\pi)^4} \Gamma h_{ij}^{-1} e^{-ik_3(a_j - a_i)} \frac{\tilde{\Pi}(k^2)}{k^2} \] (44)
with $\Gamma = \sqrt{k_\alpha k^\alpha + i\epsilon}$ and $k^2 = k_\mu k^\mu$. Using (40) the last equation takes the form

$$E_0^{(1)} = 2i \int \frac{d^4k}{(2\pi)^4} \frac{\Gamma}{\sin \Gamma L} \left( e^{-i\Gamma L} - \cos k_3 L \right) \frac{\tilde{\Pi}(k^2)}{k^2}.$$  \hspace{1cm} (45)

By means of the trivial relation $\exp(-i\Gamma L) = \exp(i\Gamma L) - 2i \sin \Gamma L$ we separate again a distance independent contribution that will be omitted. Now we perform the Wick rotation and obtain the final result for the radiative correction to the ground state energy to order $\alpha$ in the geometry of two parallel conducting planes

$$E_0^{(1)} = 2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma}{\sinh \gamma L} \left( e^{-\gamma L} - \cos k_3 L \right) \frac{\tilde{\Pi}(k^2)}{k^2}.$$  \hspace{1cm} (46)

It is interesting to study the radiative correction (46) in the limit $\lambda_c/L << 1$. For this purpose we transform the integration path of the $k_3$-integration. Due to $\tilde{\Pi}(k_3^2 + \gamma^2)$ there is a cut with branch point $k_3 = i\sqrt{4m^2 + \gamma^2}$ in the upper half of the complex $k_3$-plane. The discontinuity of the one loop vacuum polarization $\tilde{\Pi}(k^2)$ across the cut, $\text{disc} \tilde{\Pi}(k^2) = \tilde{\Pi}(k^2 + i\epsilon) - \tilde{\Pi}(k^2 - i\epsilon)$, is well known

$$\text{disc} \tilde{\Pi}(k^2) = -\frac{2i}{3} \alpha \sqrt{1 - \frac{4m^2}{k^2}} \left( 1 + \frac{2m^2}{k^2} \right).$$  \hspace{1cm} (47)

As shown in fig. 2 we move the integration contour towards the imaginary axis so that the cut is enclosed. The result can be written in the form

$$E_0^{(1)} = \frac{i}{\pi L^2} \int \frac{d^3k}{(2\pi)^3} \frac{\gamma}{\sinh \gamma L} \int_1^\infty dk_3 \frac{\text{disc} \tilde{\Pi} \left( \frac{k_3^2}{4m^2} \right)}{k_3} \left( e^{-\gamma} - e^{-\sqrt{4m^2 L^2 k_3^2 + \gamma^2}} \right).$$  \hspace{1cm} (48)

In the limit $mL >> 1$, the second term in the round brackets is exponentially suppressed and can be neglected. The desired series in inverse powers of $mL$ is now simply achieved by expanding the square root in the denominator

$$E_0^{(1)} = \frac{i}{2\pi mL^4} \int \frac{d^3k}{(2\pi)^3} \frac{\gamma e^{-\gamma L}}{\sinh \gamma L} \int_1^\infty dk_3 \frac{\text{disc} \tilde{\Pi} \left( \frac{k_3^2}{4m^2} \right)}{k_3^2} \left( 1 + O \left( \frac{\gamma^2}{mLk_3^2} \right) \right).$$  \hspace{1cm} (49)

---

**Figure 2: Integration path in the $k_3$-plane**

- $\Im k$ along $\gamma L$
- $\Re k$ along $i\sqrt{4m^2 + q^2}$
After some elementary integrations and with the help of
\[ \int_{1}^{\infty} dk_{3} \frac{\text{disc} \tilde{\Pi} \left( \frac{k_{3}^{2}}{4m^{2}} \right)}{k_{3}^{3}} = -\frac{3i\pi}{16} \alpha \] (50)
we find the leading order contribution to the radiative correction
\[ E_{0}^{(1)} = \frac{\pi^{2} \alpha}{2560mL^{4}} + O \left( \frac{1}{m^{2}L^{5}} \right) \] (51)
in agreement with [5] and [19].

5 The radiative correction in spherical geometry

This section is devoted to the calculation of the radiative correction to the ground state energy for a perfectly conducting spherical shell of radius $R$. Appropriate coordinates $z \in S$ are given by $z = \{ z_{0}, r_{z}, \theta, \phi \}$ where $r_{z}, \theta, \phi$ are the usual spherical coordinates. In order to calculate the radiative correction according to the representation (32) we need to know $\tilde{K}^{st}(z,z')$ and $\tilde{\Pi}^{st}(z,z')$, i.e. the projections of the scalar propagator $D(x - y)$ and its product with the vacuum polarization, $(D\Pi)(x - y) \equiv \int dx' D(x - x')\Pi(x' - y)$, with respect to the polarization vectors $E_{\mu}^{s}$ of (19).

Due to the rotational symmetry it proves convenient to express these quantities in terms of spherical waves. Using the expansion of a plane wave in spherical waves
\[ e^{iqx} = 4\pi \sum_{lm} i^{l} j_{l}(qr_{x})Y_{lm}^{*}(\theta_{x}, \phi_{x})Y_{lm}(\theta_{z}, \phi_{z}) , \]

the scalar propagator (23) can be written as
\[ D(x - y) = \sum_{lm} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(x_{0} - y_{0})} Y_{lm}(\theta, \phi) d_{l}(\omega; r_{x}, r_{y}) Y_{lm}^{*}(\theta_{y}, \phi_{y}) \] (52)
with
\[ d_{l}(\omega; r_{x}, r_{y}) = i|\omega|j_{l}(|\omega|r_{<})h_{l}^{(1)}(\omega|r_{>}) , \] (53)
where $j_{l}$ and $h_{l}^{(1)}$ are the spherical Bessel functions and $Y_{lm}(\theta, \phi)$ are the surface harmonics. $r_{<}(r_{>})$ denotes the smaller (larger) one of the radii $r_{x}$ and $r_{y}$. The corresponding spherical expansion of $(D\Pi)(x - y)$ is obtained from (22) by replacing the quantity $d_{l}(\omega; r_{x}, r_{y})$ by $\Pi_{l}(\omega; r_{x}, r_{y})$ whose explicit form is
\[ \Pi_{l}(\omega; r_{x}, r_{y}) = -\frac{2}{\pi} \int_{0}^{\infty} dq \bar{q}^{2} j_{l}(qr_{x}) j_{l}(qr_{y}) \tilde{\Pi}(\omega^{2} - q^{2})/\omega^{2} - q^{2} \]
\[ = \frac{i}{\sqrt{r_{x}r_{y}}} \int_{4m^{2}}^{\infty} \frac{dq^{2}}{2\pi q^{2}} \text{disc} \tilde{\Pi} (q^{2}) I_{l+\frac{i}{2}}(pr_{<}) K_{l+\frac{i}{2}}(pr_{>}) \] (54)
with \( p = \sqrt{q^2 - \omega^2} \). \( I_{l+\frac{1}{2}}, K_{l+\frac{1}{2}} \) are the modified Bessel functions.

The polarization vector \( E_{\mu}^{11} \) is proportional to the angular momentum operator \( \vec{L} \) (it corresponds to the TM-mode). It therefore commutes with \( D(z - z') \) and \( (D\Pi)(z - z') \). Consequently, \( \tilde{K}^{st}(z,z') \) and \( \tilde{\Pi}^{st}(z,z') \) are diagonal in \( s,t \) and their 11–components are given by

\[
\tilde{K}^{11}(z,z') = - \sum_{l \geq 1, m = -\infty}^{\infty} \int \frac{d\omega}{2\pi} e^{i\omega(z_0 - z_0')} Y_{lm}(\theta, \phi) dl(\omega; R, R) Y_{lm}^*(\theta', \phi') \tag{55}
\]

and

\[
\tilde{\Pi}^{11}(z,z') = - \sum_{l \geq 1, m = -\infty}^{\infty} \int \frac{d\omega}{2\pi} e^{i\omega(z_0 - z_0')} Y_{lm}(\theta, \phi) \Pi_l(\omega; R, R) Y_{lm}^*(\theta', \phi') , \tag{56}
\]

respectively. Due to the projection with the angular momentum operator \( \vec{L} \) the contribution with \( l = 0 \) is absent.

The second polarization vector \( E_{\mu}^{22} \) contains the operator \( \vec{n} \times \vec{L} \) (it is analogous but not identical to the TE–mode). This operator does not commute with either \( D(z - z') \) or \( (D\Pi)(z - z') \). Substituting the polarization vector \( E_{\mu}^{22} \) of (19) into (24) we find

\[
\tilde{K}^{22}(z,z') = E_{\mu}^{12}(z) \eta^{\mu\nu} D(z - z') E_{\mu}^{22}(z') \]

\[
= \sum_{l \geq 1, m = -\infty}^{\infty} \int \frac{d\omega}{2\pi} \frac{e^{i\omega(z_0 - z_0')}}{l(l + 1)[\omega^2 R^2 - l(l + 1)]} dl(\omega; R, R) \]

\[
\times \left[ l^2 (l + 1)^2 Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') - \omega^2 R^2 (\vec{n} \times \vec{L}) Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi')(\vec{n} \times \vec{L}) \right]. \tag{57}
\]

The method of reduced matrix elements which is known from elementary quantum mechanics [17] provides a plain way to simplify the last expression. After a straightforward computation we arrive at

\[
\tilde{K}^{22}(z,z') = \sum_{l \geq 1, m = -\infty}^{\infty} \int \frac{d\omega}{2\pi} e^{i\omega(z_0 - z_0')} Y_{lm}(\theta, \phi) f_l(\omega; R, R) Y_{lm}^*(\theta', \phi') \tag{58}
\]

with

\[
f_l(\omega; R, R) = \frac{l(l + 1) dl(\omega; R, R) - \omega^2 R^2}{\omega^2 R^2 - l(l + 1)} \tag{59}
\]

Again, the corresponding result for \( \tilde{\Pi}^{22}(z,z') \) is obtained from \( \tilde{K}^{22}(z,z') \) by replacing \( dl(\omega; R, R) \rightarrow \Pi_l(\omega; R, R) \). Since \( \tilde{K}^{st}(z,z') \) is diagonal in \( s, t \) as well as in
the quantum numbers \( \{\omega, l, m\} \) its inverse is readily found. We get

\[
K^{-11}(z, z') = \sum_{l \geq 1, m} -\frac{1}{2\pi R^2} e^{i\omega(z_0 - z'_0)} Y_{lm}(\theta_z, \phi_z) \frac{1}{d_l(\omega; R, R)} Y_{lm}^*(\theta_{z'}, \phi_{z'}),
\]

\[
K^{-12}(z, z') = \sum_{l \geq 1, m - \infty} \frac{d\omega}{2\pi R^2} e^{i\omega(z_0 - z'_0)} Y_{lm}(\theta_z, \phi_z) \frac{1}{f_l(\omega; R, R)} Y_{lm}^*(\theta_{z'}, \phi_{z'}). \tag{60}
\]

Inserting these expressions into (28) we find the photon propagator in covariant gauge with conductor boundary conditions on a sphere. We note that it has a somewhat unusual form because it seems not to contain radial derivatives (corresponding to Neumann boundary conditions). However, such derivatives are implicitly present in the terms proportional to \( d_{l+1} \) in \( f_l \) (59). Utilizing the recursion relations for the spherical Bessel functions, \( f_l \) can be cast into the form

\[
f_l(\omega; R, R) = -ik \frac{(kR j_l(kR)')'(kR j_l^{(1)}(kR))'}{\omega^2 R^2 - l(l + 1)}, \quad (k = |\omega|).
\]

In fact, substituting \( K^{'st}(z, z') \) with the last expression into (31) and performing the Wick rotation \( \omega \rightarrow iy \) we obtain

\[
E_0 = \frac{1}{2R} \sum_{l=1}^{\infty} (2l + 1) \int_{-\infty}^{\infty} \frac{dy}{2\pi} \log \left(1 - \lambda_l^2(x)\right) \tag{61}
\]

with \( x = |y| \) and \( \lambda_l = (s_l c_{l'})' = (x I_{l+\frac{3}{2}}(x)K_{l+\frac{3}{2}}(x))' \). This mode sum representation was first derived in [10] with the help of the dyadic formalism. We therefore reproduce the known result for the Casimir energy of a conducting sphere.

Now we return to the calculation of the radiative correction. Substituting \( K^{-1st}(z, z') \) and \( \Pi^{st}(z, z') \) into \( E_0^{(1)} \) (32) and making use of the orthogonality of the surface harmonics one finds

\[
E_0^{(1)} = -\frac{i}{2} \sum_{l \geq 1, m} \int \frac{d\omega}{2\pi} \left[ \left(l + \frac{1}{2}\right)^2 - \omega^2 R^2 \right]^{-\frac{3}{2}} \times \Pi_l(\omega; R, R) \frac{l(2l + 1)\Pi_l - \omega R^2 [l\Pi_{l+1} + (l + 1)\Pi_{l-1}]}{l(l + 1)d_l - \omega^2 R^2 [l(l+1) + (l + 1)d_{l-1}]}, \tag{62}
\]

where we have introduced a regularizing factor which is removed by taking the limit \( \epsilon \rightarrow 0 \) in the end. Next we perform the Wick rotation \( \omega \rightarrow ik \). Thereby we have from (53)

\[
d_i(ik; R, R) = \frac{1}{R} I_{l+\frac{3}{2}}(|k| R)K_{l+\frac{3}{2}}(|k| R) \tag{63}
\]

and from (54)

\[
\Pi_l(ik; R, R) = \frac{i}{R} \int_{4m^2}^{\infty} \frac{dq^2}{2\pi q^2} \text{disc}\tilde{\Pi}(q^2) I_{l+\frac{3}{2}}(p R)K_{l+\frac{3}{2}}(p R) \tag{64}
\]
with \( p = \sqrt{q^2 + k^2} \). Finally, we substitute \( d_l \) \((63)\) and \( \Pi_l \) \((64)\) into the radiative correction \((62)\) and obtain

\[
E^{(1)}_0 = \frac{i}{2\pi R} \sum_{l=1}^{\infty} \int_{0}^{\infty} \frac{d\nu}{\pi} \frac{\nu}{(k^2 + \nu^2)^{\frac{3}{2}}} \int_{4m^2R^2}^{\infty} \frac{dq^2}{q^2} \text{disc} \left( \frac{q^2}{R^2} \right) \left( B_1 + B_2 \right),
\]

(65)

where the abbreviations \( \nu \equiv l + \frac{1}{2} \),

\[
B_1 = \frac{I_\nu(p)K_\nu(p)}{I_\nu(k)K_\nu(k)}
\]

(66)

and

\[
B_2 = \frac{I_\nu(p)K_\nu(p) + \frac{k^2}{2\nu} \left[ \frac{1}{\nu+\frac{1}{2}} I_{\nu+1}(p)K_{\nu+1}(p) + \frac{1}{\nu-\frac{1}{2}} I_{\nu-1}(p)K_{\nu-1}(p) \right]}{I_\nu(k)K_\nu(k) + \frac{k^2}{2\nu} \left[ \frac{1}{\nu+\frac{1}{2}} I_{\nu+1}(k)K_{\nu+1}(k) + \frac{1}{\nu-\frac{1}{2}} I_{\nu-1}(k)K_{\nu-1}(k) \right]},
\]

(67)

have been introduced.

Evidently, the expression \((65)\) for the radiative correction to the ground state energy of a conducting sphere contains ultraviolet divergencies. Hence, it needs to be renormalized. In order to isolate the ultraviolet divergent terms we employ the uniform asymptotic expansion of the Bessel functions for \( \nu \to \infty \) and \( k \to \infty \) with fixed

\[
s \equiv \left( 1 + \frac{k^2 + q^2}{\nu^2} \right)^{-\frac{1}{2}}, \quad t \equiv \left( 1 + \frac{k^2}{\nu^2} \right)^{-\frac{1}{2}}.
\]

(68)

The corresponding asymptotic expansion of the quantities \( B_1 \) and \( B_2 \) reads

\[
B_{1,2} = \frac{s}{t} \sum_{n=0}^{\infty} \sum_{n \leq i, j \leq 3n} X_{nij}^{(1,2)} \frac{s^{2l}t^{2j}}{\nu^{2n}},
\]

(69)

where only even powers contribute. The coefficients \( X_{nij}^{(1,2)} \) are listed in Appendix A.

Now we insert the expansions \((69)\) into \((65)\) and perform the integration over \( k \):

\[
\int_{-\infty}^{\infty} \frac{dk}{\nu} \epsilon^{-1+2j} s^{1+2i} = \frac{\nu \Gamma \left( \frac{\epsilon-1}{2} + i + j \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} + i + j \right)}
\]

\[
\times \, _2F_1 \left( \frac{1}{2} + i, \frac{\epsilon - 1}{2} + i + j; \frac{\epsilon}{2} + i + j; -\frac{q^2}{\nu^2} \right).
\]

In the next step we utilize a Mellin–Barnes representation of the hypergeometric function and rewrite the last expression as

\[
\int_{-\infty}^{\infty} \frac{dk}{\nu} \epsilon^{-1+2j} s^{1+2i} = \frac{\nu \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} + i \right)}
\]

\[
\times \int_{c} \frac{d\sigma}{2\pi i} \frac{\Gamma \left( \frac{1}{2} + i + \sigma \right) \Gamma \left( \frac{\epsilon-1}{2} + i + j + \sigma \right) \Gamma(-\sigma)}{\Gamma \left( \frac{1}{2} + i + j + \sigma \right)} \left( \frac{q^2}{\nu^2} \right)^{\sigma},
\]

(70)
where the integration contour \( c \) goes from \(-i\infty\) to \(i\infty\) parallel to the imaginary axis with \(-\frac{1}{2} < \Re \sigma < 0\). The sum over the orbital momentum \( l \) can now be carried out. The result is a Hurwitz zeta function:

\[
\sum_{l=1}^{\infty} \nu^{2-\epsilon-2n-2\sigma} = \zeta_H(2n + 2\sigma + \epsilon - 2; \frac{3}{2}).
\] (71)

Here and in the previous steps we have assumed \( \Re \epsilon > 3 \) so that \( \Re \epsilon + 2\Re \sigma + 2n - 2 > 1 \) is satisfied. Finally, we put (47) into the radiative correction (65) where we are able to perform the \( q \)-integration as well

\[
\int_{-1}^{1} \frac{dq^2}{q^2} q^{2\sigma} \sqrt{1 - \frac{1}{q^2}} \left( 1 + \frac{1}{2q^2} \right) = \frac{3\sqrt{\pi}}{4} \frac{(1 - \sigma)\Gamma(-\sigma)}{\Gamma(\frac{\sigma}{2} - \sigma)},
\] (72)

(we have \( \Re \sigma < 0 \) along the integration contour \( c \)). As a result of all these manipulations we obtain the representation

\[
E^{(1)}_0 = \frac{\alpha}{8\pi R} \sum_{nij} \left( X^{(1)}_{nij} + X^{(2)}_{nij} \right) \frac{1}{\Gamma(i + \frac{1}{2})} \times \int_{c} d\sigma \frac{\Gamma\left( \frac{1}{2} + i + \sigma \right) \Gamma\left( \frac{\sigma}{2} + i + j + \sigma \right) \Gamma^2(-\sigma)(1 - \sigma)}{\Gamma\left( \frac{\sigma}{2} + i + j + \sigma \right) \Gamma\left( \frac{\sigma}{2} - \sigma \right)} \times \left( 4m^2R^2 \right)^{\sigma} \zeta_H\left( 2n + 2\sigma + \epsilon - 2; \frac{3}{2} \right).
\] (73)

With this representation of \( E^{(1)}_0 \) at our disposal, we are in a position to separate the ultraviolet divergent contributions. They are identified with those poles of the integrand of (73) that cross the integration contour when removing the regularization, i.e. in the limit \( \epsilon \to 0 \). All such poles are contained in the addends \( n = 0, 1 \). Namely, the zeta function possesses single poles at \( \sigma = (3 - \epsilon)/2 \) (\( n = 0 \)) and \( \sigma = (1 - \epsilon)/2 \) (\( n = 1 \)), respectively while the second Gamma function in the numerator of (73) contributes a simple pole at \( \sigma = (1 - \epsilon)/2 \) when \( n = 0 \). The addends with \( n \geq 2 \) only contain poles that remain in the half–plane \( \Re \sigma < 0 \) when taking the limit \( \epsilon \to 0 \). The residues of the poles passing the integration contour for \( \epsilon \to 0 \) give extra contributions to \( E^{(1)}_0 \). These additional contributions are

\[
E^{\text{div,1}}_0 = -\frac{16}{9\pi} \alpha m^3 R^2
\] (74)

from \( \sigma = (3 - \epsilon)/2 \) and

\[
E^{\text{div,2}}_0 = -\frac{4}{15\pi} \alpha m
\] (75)

from \( \sigma = (1 - \epsilon)/2 \). As we have anticipated, the ultraviolet divergent contributions are proportional to \( R^2 \) and \( R^0 \). Hence their removal can be interpreted as a redefinition of the parameters \( \sigma \) and \( k \) in the energy of the classical system (7).
The fact that the ultraviolet divergent contributions only appear as finite residues is a peculiarity of the zeta–functional regularization.

The renormalized radiative correction to the ground state energy is now given by

\[
E_{0}^{(1)\text{ren}} \equiv E_{0}^{(1)}(\epsilon \to 0) - E_{0}^{\text{div,1}} - E_{0}^{\text{div,2}}
\]

\[
= \frac{\alpha}{8\pi R} \sum_{n=0}^{\infty} \sum_{ij} \left( X_{nij}^{(1)} + X_{nij}^{(2)} \right) \frac{1}{\Gamma(i + \frac{1}{2})} \int_{c} \frac{d\sigma}{2\pi i} \frac{\Gamma\left(\frac{1}{2} + i + \sigma\right) \Gamma\left(\frac{1}{2} - i + j + \sigma\right) \Gamma^{2}(-\sigma)(1 - \sigma)}{\Gamma(i + j + \sigma) \Gamma\left(\frac{3}{2} - \sigma\right)} 
\]

\[
\times \left( 4m^{2}R^{2} \right)^{\sigma} \zeta_{H} \left( 2n + 2\sigma - 2; \frac{3}{2} \right).
\]

(76)

At the first glance \(E_{0}^{(1)\text{ren}}\) looks just like \(E_{0}^{(1)}\) with \(\epsilon = 0\). The difference is that the poles at \(\sigma = (3 - \epsilon)/2\) and \(\sigma = (1 - \epsilon)/2\) are now located on the right of the integration contour. The structure of the asymptotic series \(E_{0}^{(1)\text{ren}}\) is well known. Shifting the integration contour to the left yields an expansion of \(E_{0}^{(1)\text{ren}}\) in inverse powers of \(mR\) (likewise, an expansion for \(mR << 1\) is obtained by moving it to the right).

Having separated the ultraviolet divergencies with the help of representation \((73)\) we return to eq. \((65)\). When inserting \(E_{0}^{(1)}\) \((73)\) into \((76)\) the limit \(\epsilon \to 0\) cannot directly be taken. Therefore we subtract the first two terms \((n = 0, 1)\) of the asymptotic expansion \((73)\),

\[
B_{\text{as}} \equiv s \sum_{l=1}^{1} \sum_{n=0}^{\infty} \sum_{i,j} \left( X_{nij}^{(1)} + X_{nij}^{(2)} \right) \frac{s^{2l} t^{2j}}{2n}.
\]

(77)

from the integrand and add them again. In this way we split \(E_{0}^{(1)\text{ren}}\) according to

\[
E_{0}^{(1)\text{ren}} = E_{f} + E_{\text{as}}
\]

(79)

into

\[
E_{f} = \frac{i}{2\pi R} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left( \frac{\nu}{k^{2} + \nu^{2}} \right) \int_{4m^{2}R^{2}}^{\infty} \frac{dq^{2}}{q^{2}} \text{disc}\Pi \left( \frac{q^{2}}{R^{2}} \right) \left( B_{1} + B_{2} - B_{\text{as}} \right)
\]

(80)

and

\[
E_{\text{as}} = \frac{i}{2\pi R} \sum_{l=1}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left( \frac{\nu}{k^{2} + \nu^{2}} \right) \int_{4m^{2}R^{2}}^{\infty} \frac{dq^{2}}{q^{2}} \text{disc}\Pi \left( \frac{q^{2}}{R^{2}} \right) B_{\text{as}} \right] - E_{0}^{\text{div,1}} - E_{0}^{\text{div,2}}.
\]

(81)
In the first contribution, $E_f$, the limit $\epsilon \to 0$ can be carried out under the signs of the integral and the sum because they converge now. This form is suited for a numerical evaluation, if necessary. In the second contribution, $E_{as}$ we pass again to the Mellin-Barnes representation and observe that $E_{as}$ is identical with the first two addends ($n = 0, 1$) of the asymptotic series in (77).

Representation (79) is the final result for the renormalized radiative correction of order $\alpha$ to the ground state energy of a conducting sphere.

Next we turn to the interesting physical situation where the radius $R$ of the sphere is large compared to the Compton wavelength of the electron, i.e. $mR >> 1$. In other words, we are interested in the leading order term when expanding $E^{(1)\text{ren}}_0$ in inverse powers of $mR$. The representation (77) serves us as a starting point. As already mentioned above, the expansion in inverse powers of $mR$ is achieved by shifting the integration contour to the left. The leading term of the desired expansion is given by the residue at $\sigma = -1/2$. A table of all addends contributing a pole at $\sigma = -1/2$ is displayed in Appendix B. We observe that the addends with $n = 0, 1, 2$ contain also double poles which lead to the appearance of logarithmic contributions whereas the terms with $n > 2$ show only simple poles. It is therefore technically convenient to calculate the contributions with $n \leq 2$ and $n > 2$ separately. Accordingly, we split the leading order term into two pieces, $E^{(1)}_0 = E_{n \leq 2} + E_{n > 2} + O(m^{-2}R^{-3})$. The residues contributing to $E_{n \leq 2}$ are calculated by standard methods and we find

$$E_{n \leq 2} = -\frac{\alpha}{2\nu \pi m^2} \log(mR) - \frac{\alpha}{8\nu \pi m R^2} \left\{ \frac{1447573}{16934400} + \frac{31}{5376} \right\}$$

$$+ \frac{3\nu}{4480} \log{2} - \frac{3}{4} \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \left( X_{n0j}^{(1)} + X_{n0j}^{(2)} \right) \frac{t^{2j-1}}{\nu^{2n}}.$$  \hspace{1cm} (82)

The contribution $E_{n > 2}$ can only be calculated numerically. For this purpose, we cast it into the more convenient form

$$E_{n > 2} = \frac{3\alpha}{32\pi m R^2} \sum_{\nu \geq 1} \nu^2 \int_0^\infty \nu^2 \left\{ B_1^0 + B_2^0 - \sum_{n=0}^{2} \sum_{j=0}^{\infty} \left( X_{n0j}^{(1)} + X_{n0j}^{(2)} \right) \frac{t^{2j-1}}{\nu^{2n}} \right\},$$  \hspace{1cm} (83)

where we have introduced the notation

$$B_1^0 = \frac{1}{2 \nu} \frac{1}{I_\nu(k)K_\nu(k)}$$  \hspace{1cm} (84)

and

$$B_2^0 = \frac{t^{-2} \left( 1 - \frac{t^2}{4\nu^2} \right) / \left( 1 - \frac{1}{4\nu^2} \right)}{2\nu I_\nu(k)K_\nu(k) + k^2 \left( \frac{1}{\nu+\frac{1}{2}} I_{\nu+1}(k) K_{\nu+1}(k) + \frac{1}{\nu-\frac{1}{2}} I_{\nu-1}(k) K_{\nu-1}(k) \right)}.$$  \hspace{1cm} (85)
Expression (83) can also directly be obtained from (80). For this purpose, the term with \( n = 2 \) of the asymptotic expansion (69) has to be subtracted from the integrand in addition to \( B_{as} \). Then, the leading term of the expansion of the integrand in inverse powers of \( q^2 \) yields \( E_{n>2} \).

The remaining task is the numerical evaluation of \( E_{n>2} \). The result is

\[
E_{n>2} = -\frac{\alpha}{mR^2} \cdot 9.8230 \cdot 10^{-6},
\]

(86)

where contributions up to \( l = 50 \) have been taken into account (the sum over \( l \) converges as \( \sum_l \nu^{-3}, (\nu = l + 1/2) \)).

So we finally obtain the radiative correction in leading order \( 1/(mR) \)

\[
E^{(1)}_0 = -\frac{\alpha}{mR^2} \left( 7.5788 \cdot 10^{-4} \log mR + 6.4833 \cdot 10^{-3} \right) + O \left( \frac{1}{m^2 R^3} \right).
\]

(87)

6 Summary

In the present paper we have calculated the radiative correction to the ground state energy of a perfectly conducting spherical shell. Being proportional to the fine structure constant \( \alpha \), it is small compared to the Casimir energy itself. In a realistic physical situation where the radius \( R \) of the sphere is much larger than the Compton wavelenght \( \lambda_c \) of the electron, the radiative correction is proportional to the ratio \( \lambda_c/R \). In contrast to the case of two parallel conducting planes, the radiative correction for a sphere exhibits a logarithmic dependence on \( \lambda_c/R \).

The calculations have been performed in the framework of general covariant perturbation theory with the boundary conditions incorporated as constraints. A brief review of quantization with boundary conditions in this formalism has been supplied. As a valuable advantage of this approach, the ghost degrees of freedom are not affected by the boundary conditions so that they need not be taken into account. The use of the formalism has been demonstrated by recalculating the Casimir energy including radiative correction for two perfectly conducting planes. The Casimir energy of a conducting sphere could be reproduced as well. As expected from the known heat kernel expansion the removal of the geometry–dependent ultraviolet divergencies can be absorbed in a redefinition of the parameters (7) of the external system represented by the surface \( S \). In the calculation of the vacuum graph to order \( \alpha \) the electron mass \( m \) and the charge \( e \) need not be renormalized.

A thorough investigation of the renormalization of gauge theories in the presence of boundary conditions beyond the vacuum structure as well as the analogous calculations within the MIT bag model seems to be an interesting future perspective.

The authors thank K. Kirsten and D. Vassilevich for helpful discussions.
A  The coefficients $X_{nij}^{(1,2)}$

The coefficients $X_{nij}^{(1,2)}$ in formula (69) are:

$$X_{000}^{(1,2)} = 1 ,$$

$$X_{1ij}^{(1)} = \begin{pmatrix}
0 & -\frac{1}{8} & \frac{3}{4} & -\frac{5}{8} \\
-\frac{1}{8} & 0 & 0 & 0 \\
-\frac{3}{4} & 0 & 0 & 0 \\
-\frac{5}{8} & 0 & 0 & 0
\end{pmatrix} ,
X_{1ij}^{(2)} = \begin{pmatrix}
0 & -\frac{1}{8} & -\frac{3}{4} & \frac{7}{8} \\
\frac{1}{8} & 0 & 0 & 0 \\
-\frac{3}{4} & -\frac{3}{2} & 0 & 0 \\
\frac{7}{8} & 0 & 0 & 0
\end{pmatrix} ,
$$

and

$$X_{2ij}^{(1)} = \begin{pmatrix}
0 & 0 & -\frac{25}{128} & \frac{139}{32} & -\frac{1039}{64} & \frac{663}{32} & -\frac{1105}{128} \\
0 & -\frac{1}{64} & \frac{3}{32} & -\frac{5}{64} & 0 & 0 & 0 \\
\frac{27}{128} & \frac{3}{32} & -\frac{9}{16} & \frac{15}{32} & 0 & 0 & 0 \\
-\frac{145}{32} & -\frac{5}{64} & \frac{15}{32} & -\frac{25}{64} & 0 & 0 & 0 \\
\frac{1085}{64} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{693}{32} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1155}{128} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} ,
$$

$$X_{2ij}^{(2)} = \begin{pmatrix}
0 & 0 & \frac{23}{128} & -\frac{143}{32} & \frac{1181}{64} & -\frac{819}{32} & \frac{1463}{128} \\
0 & -\frac{1}{64} & -\frac{3}{32} & \frac{7}{64} & 0 & 0 & 0 \\
-\frac{21}{128} & \frac{21}{32} & -\frac{3}{4} & \frac{57}{32} & -\frac{21}{16} & 0 & 0 \\
\frac{125}{32} & -\frac{545}{64} & -\frac{15}{32} & -\frac{25}{64} & 0 & 0 & 0 \\
-\frac{595}{64} & \frac{105}{4} & 0 & 0 & 0 & 0 & 0 \\
-\frac{63}{32} & -\frac{315}{16} & 0 & 0 & 0 & 0 & 0 \\
\frac{1155}{128} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} .$$

B  The pole structure of $E_0^{(1)}_{\text{ren}}$ in (77)

Here we list all addends of (77) which contribute a pole at $\sigma = -\frac{1}{2}$ when shifting the integration path to the left. They result from the zeta function and the first
two Gamma functions in the numerator of (77):

\[
\begin{array}{cccccc}
 n & i & j & \Gamma(i) & \Gamma(i + j - 1) & \zeta_H(2n - 3; 3/2) \\
0 & 0 & 0 & \times & \times & \text{(double pole)} \\
1 & 0 & 1 & \times & \times & \text{(double pole)} \\
1 & 0 & 2 & \times & & \\
1 & 0 & 3 & \times & & \\
1 & 1 & 0 & & \times & \\
2 & 0 & 2\ldots 6 & \times & \times & \text{(double poles)} \\
2 & 2 & 0\ldots 4 & & \times & \\
2 & 4 & 0\ldots 2 & & \times & \\
2 & 6 & 0 & & \times & \\
3\ldots \infty & 0 & n\ldots 3n & \times & & \\
\end{array}
\]

The residues can be calculated by standard methods. The double poles yield the logarithm \(\log mR\).
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