SPECIAL FUNCTIONS FOR HYPEROCTAHEDRAL GROUPS USING 
BOSONIC, TRIGONOMETRIC SIX-VERTEX MODELS

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Abstract. Recent works have sought to realize certain families of orthogonal, symmetric polynomials as partition functions of well-chosen classes of solvable lattice models. Many of these use Boltzmann weights arising from the trigonometric six-vertex model $R$-matrix (or generalizations or specializations of these weights). In this paper, we seek new variants of bosonic models on lattices designed for type B/C root systems, whose partition functions match the zonal spherical function in type C. Under general assumptions, we find that this is possible for all highest weights in rank 2 and 3, but not for higher rank.

1. Introduction

A number of recent papers have studied symmetric function theory and related special functions using solvable lattice models, including [4, 5, 8, 10, 11, 13, 24]. The word “solvable” here indicates that the models satisfy certain local relations, principally the “Yang-Baxter equations” or “RTT relations” (to be described in more detail in the subsequent sections). Such relations are known to arise naturally from modules for quantum groups and it is an interesting problem to associate families of symmetric functions to quantum group modules via the partition functions of solvable lattice models in this way. In the present paper, we explore this connection when the resulting $R$-matrix is, up to scaling, the trigonometric $R$-matrix for the six-vertex model coming from the standard module for the quantum group $U_q(\hat{sl}_2)$ and the lattices are constructed to reflect symmetries of the Weyl group in types B/C – the so-called hyperoctahedral groups.

The two-dimensional lattice models we consider will consist primarily of finite rectangular arrays of vertices on a square lattice, with four edges (two vertical and two horizontal) adjacent to each vertex. Models for classical Cartan types outside of type $A$ may have additional boundary vertices which roughly reflect the embeddings of classical groups into the general linear group and are inspired by Kuperberg’s models for symmetry classes of alternating sign matrices [16]. These will be defined precisely in Section 3.

Admissible states of these models for given boundary data will consist of paths of particles beginning from the bottom boundary, moving upward and leftward, and exiting out the left boundary. There’s an interesting dichotomy here in whether we allow superposition of particles along vertical edges - such models are called “bosonic” models when superposition is allowed, and “fermionic” when not allowed. Superposition of particles on horizontal edges will always be forbidden in this context, in keeping with our requirement that the $R$-matrix in the RTT-relation involving a pair of adjacent rows is a six-vertex model.

We may keep track of initial particle locations along the bottom boundary using an integer partition $\lambda$ whose parts $\lambda_i$ are associated to column indices where particles appear. In particular the parts of $\lambda$ are weakly decreasing in the bosonic case and strictly decreasing in the fermionic case. All other boundaries will be regular, so $\lambda$ suffices to determine the boundary conditions. An example of an admissible state in a type $A$ bosonic model is pictured in Figure 1 below. Note in particular the superposition of two particles on a lone edge in the column labeled ‘2’ at the bottom, as depicted by the particle number in the box. Let $S_\lambda$ denote the set of all admissible states corresponding to the bottom boundary in the model. Full details on these models will be provided in Sections 2 and 3.

Broadly speaking, the goal is to represent families of symmetric functions indexed by partitions $\lambda$ as generating functions on the admissible states $S_\lambda$. These generating functions come from choices of local weights, the “Boltzmann weights” at each vertex depending on occupancy in adjacent edges. Taking the product of weights at all vertices in the admissible state, and summing these products over all admissible states in $S_\lambda$, results in a generating function over states called the “partition function,” denoted here as $Z(S_\lambda)$. As indicated in the example above, each row of the model is associated to an indeterminate $x_i$ (the so-called “spectral parameters”) and the Boltzmann weights of vertices in the row will be allowed to depend
on these $x_i$. In this way, we will see that $Z(S_\lambda)$ will be a polynomial in the parameters $x_i$ according to the choice of Boltzmann weights.

The symmetric functions we seek to represent as partition functions of lattice models are a particular family of orthogonal polynomials in several variables $x := (x_1, \ldots, x_r)$ indexed by $\lambda \in \Lambda_r$ associated to affine root systems. More precisely, they are one parameter specializations of the two parameter Macdonald polynomials $P_\lambda^{(\epsilon)}(x)$ defined in Section 5.7 of [18]. If the affine Weyl group associated to the root system is $W = W_0 \rtimes \Lambda_r$ with $W_0$ the finite Weyl group and $\Lambda_r$ the weight lattice of rank $r$, then the family of polynomials in $r$ variables $x$ and a linear character $\epsilon$ of $W_0$. In our specialization, the $P_\lambda^{(\epsilon)}$ are constructed from certain Hecke algebra symmetrizers (cf. (5.5.6) in [18]) built from $\epsilon$ as sums over $W_0$, and acting on monomials $x^\lambda = x_1^{\lambda_1} \cdots x_r^{\lambda_r}$. Our primary example will be the case where $\epsilon$ is the trivial character, where $P_\lambda^{(\epsilon)}$ is the usual two parameter Macdonald polynomial and in our specialization to one parameter $q$ is Macdonald’s zonal spherical function.\footnote{Macdonald polynomials are commonly written in terms of parameters $q$ and $t$, though conventions differ about their roles. Our conventions match those of $p$-adic representation theory. In particular, the usual role of $t$ in the literature is played by $q$ for us, owing to its connection with the cardinality of the residue field of the $p$-adic field which is typically notated with the same letter. No confusion will arise since our polynomials have a single deformation parameter and its role will always appear through explicit formulas like (1.1).} In type $A$, this is also known as the Hall-Littlewood polynomial. For any root system and associated finite Weyl group $W_0$, it takes the form:

\begin{equation}
P_\lambda(x_1, \ldots, x_r; q) = \frac{1}{Q_\lambda(q)} \sum_{\sigma \in W_0} \sigma \left( x^\lambda \prod_{\alpha \in \Phi^+} \frac{1 - q x^{-\alpha}}{1 - x^{-\alpha}} \right),
\end{equation}

for some polynomial $Q_\lambda$. For other choices of linear character $\epsilon$, we achieve similar averaging formulas over $W_0$; see for example Section 8 of [19] for a discussion of this. Attempting to use lattice models derived from quantum group modules to represent Macdonald polynomials made from Hecke algebras is perhaps natural in light of Jimbo’s generalization of Schur-Weyl duality [18]. Making this connection precise in the language of partition functions, viewed as matrix coefficients of quantum group modules, is an interesting open question. At present, we don’t even know whether solvable lattice models exist for all such polynomials in one deformation parameter (nevermind two). This paper is an attempt to understand this question better for classical groups, and we begin by reviewing what is known.

In type $A$, the connections between such orthogonal polynomials and solvable lattice models are well developed. The Weyl groups $W_0$ of simply laced types have just two linear characters - the sign and trivial characters - which produce spherical Whittaker functions and Hall-Littlewood polynomials, respectively. Families of solvable lattice models whose partition functions are spherical Whittaker functions and the aforementioned Hall-Littlewood polynomials appeared previously in the literature in [8] and [24], respectively. The Whittaker functions use a fermionic lattice model while those for Hall-Littlewood polynomials are bosonic. In fact, this connection between symmetric functions and lattice models extends to the nonsymmetric analogues. Indeed if we distinguish each path in an admissible state of the model by recording its color, then the resulting lattice models with a fixed set of colors along the boundary realize the non-symmetric analogues above. These are the non-symmetric Hall-Littlewood polynomials and Iwahori-Whittaker functions, which were studied in [4] and [6], respectively. These latter Whittaker functions arise in representation theory of $p$-adic algebraic groups; further specializing the one parameter in the Boltzmann weights, they degenerate.
to so-called “Demazure atoms” as in [19], originally called “standard bases” by Lascoux and Schützenberger in [17], and their connection to lattice models is explored in [7].

In types $B$ and $C$, less is known. There are results for spherical Whittaker functions — that is, $P_{\lambda}^{(\epsilon)}$ with $\epsilon$ the sign character — in [4] using fermionic lattice models with bends along one side and bivalent vertices at each bend (Figure 5 gives one such example). In [24], bosonic models with the same underlying lattice shape are shown to give certain two-parameter generalizations of polynomials $P_{\lambda}^{(\epsilon)}$ where $\epsilon$ is the character of $W_0$ sending long roots to 1 and short roots to $-1$. The two parameters, called $\gamma$ and $\delta$, arise from a pair of “bonus” columns introduced into the lattice models next to the bends. The Boltzmann weights for vertices in each of these two extra columns depend on one of these parameters, but setting $\gamma = \delta = 0$ is equivalent to omitting these two bonus columns entirely from the model. In this case, according to (64) in [24], one obtains $P_{\lambda}^{(\epsilon)}$ for this character $\epsilon$ (see Theorem 4.2). If instead one chooses $\gamma$ and $\delta$ with $\gamma + \delta = 0$ and $\gamma \delta = q$, this results in the zonal spherical function in type $C$. So it is natural to ask which special functions are achievable from a solvable lattice model with bends in which there are no “bonus” columns, and for which the rectangular (tetravalent) vertices satisfy a Yang-Baxter equation with the trigonometric six-vertex model. This is the main question we try to answer in the present paper. In particular, we address whether one can obtain the zonal spherical function in these non-simply-laced types via such solvable lattice models.

As we alluded to above, the lattice models in types $B$ and $C$ consist almost entirely of tetravalent vertices, together with a set of bivalent vertices appearing as bends connecting pairs of adjacent rows along one side of the model. The example in Figure 5 shows one such admissible state in rank two. Thus our question becomes one of exploring choices of these Boltzmann weights at the bend vertices (or “bend weights,” for short) which simultaneously preserve the solvability of the model and achieve our desired special functions as partition functions of the model. In doing so, we are forced to consider bend weights which are not uniformly defined in terms of spectral parameters. For example in rank two models, the function defining the Boltzmann weight for the top bend may differ from the one for the bottom bend. Details are given in Section 3.

In later sections of the paper, we provide conditions on the bend weights to guarantee the solvability of the model. This leads to the following result.

**Main Theorem.** Let $B_\lambda$ be a solvable lattice model of type $B/C$ with boundary conditions corresponding to a dominant weight $\lambda$, and having Boltzmann weights from the bosonic, trigonometric six-vertex model at all tetravalent vertices. Then the existence of bend weights such that $Z(B_\lambda) = P_{\lambda}(x;q)$, the zonal spherical function in type $C$, for all dominant weights $\lambda$ depends on the rank as follows:

- For rank $r \leq 3$, there exists a choice of Boltzmann weights for the bend vertices that realizes the zonal spherical function at dominant weights $\lambda$. These bend weights depend on the row, and are not uniformly dependent on the (spectral) parameters $x_i$.
- In rank $r \geq 4$, no such choice of Boltzmann weights for bend vertices exists.

We briefly outline the contents of each section and their role in leading up to this main result. In Section 2 we fix notation, recall the definition of triangular weights and review the concept of solvability (the existence of a Yang-Baxter equation for tetravalent vertices) for bosonic models in type $A$. In Section 3 we extend this notion of solvability to type $B/C$ models. See Definition 3.2 for a precise definition of solvability for lattice models of types $B/C$. The definition ensures that the row-to-row transfer matrices (taken in pairs of rows connected by a bend vertex) commute, as shown in Lemmas 3.5 and 3.6. Solvability in these types requires certain additional relations, as has been long understood in lattice models of this geometry [10, 14, 16, 22, 24]. Following Kuperberg, these relations tend to be named after animals — fish, caduceus, etc. We prove necessary conditions on bend weights to satisfy the required relations in a series of lemmas in Section 3. In particular, we find new solutions to the caduceus relation (e.g., Lemma 3.2) using non-uniform choices of bend weights. Section 4 provides methods for explicit evaluations of solvable lattice models of various ranks, as the analysis is rather different in each case, and results in the main theorem above. In the process, we evaluate all solvable lattice models in rank at most 3 with trigonometric six-vertex model weights at tetravalent vertices.

Our main theorem handles the most general set of bend weights that is consistent with the paradigm of partition functions as matrix coefficients for quantum group modules — in particular, we require the caduceus
equation for bend weights to be consistent with Cherednik and Sklyanin’s reflection equation and we require monomial dependence in the spectral parameters \( x_i \), consistent with formal group laws for parametrized Yang-Baxter equations from families of affine quantum group representations. Thus any solution for bend weights outside of our assumptions would represent a break from this point of view. It is possible that our analysis could be used as a starting point for developing combinatorial solutions outside of this paradigm, and one might hope the resulting algebraic structures would be of fundamental interest. This work was partially supported by NSF grant DMS-2101392 (Brubaker).

2. Bosonic Models for Cartan Type A

We begin by describing the general features of six-vertex lattice models in type \( A \), all of which will be needed in subsequent sections on other classical Cartan types. Each lattice model is indexed by an integer partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r) \) (with \( \lambda_r \geq 0 \)) which determines its size, shape, and boundary conditions. Given such a \( \lambda \), each lattice model will be built from two-dimensional grids of tetravalent vertices with \( n \) rows, numbered 1 to \( n \) from bottom to top, and \( \lambda_1 + 1 \) columns, numbered 0 to \( \lambda_1 \) from left to right. An admissible state of such a model is a configuration of \( r \) paths along the lattice edges, with the \( i \)-th path starting from the \( \lambda_i \)-th column along the bottom boundary edges and ending on an edge at the far left side of the diagram. At each vertex, paths are allowed to either move up or left. Paths are allowed to overlap without restriction along vertical edges, but any given horizontal edge can allow at most one path. Alternatively, we could view columns between each row as recording the position of a family of particles (or a finite window of interest among a semi-infinite collection of particles) and the admissible states record a discrete time evolution of these particles with each new row. The fact that columns may contain multiple paths then translates as the superposition of these particles, so we refer to the associated models as “bosonic” (as opposed to “fermionic” where no superposition is permitted). We will denote the set of all such (admissible) states as \( \mathcal{A}_\lambda \).

Ultimately we will attach a polynomial to each particular model. To do this, each vertex \( v \) is assigned a Boltzmann weight \( B(v) \); that weight will be a function depending on the edge configurations adjacent to that vertex \( v \) and on a transcendental parameter \( x_i \) corresponding to the row \( i \) in which the vertex occurs. For an admissible state \( S \) of the system, its weight \( \text{wt}(S) \) is then defined as the product of weights over all its vertices: \( \text{wt}(S) = \prod_{v \in S} B(v) \). The partition function \( Z(\mathcal{A}_\lambda) \) for the model is then recovered by summing these weights over all states of the model:

\[
Z(\mathcal{A}_\lambda) = \sum_{S \in \mathcal{A}_\lambda} \text{wt}(S).
\]

This is a slight abuse of notation, as we’ve continued to use \( \mathcal{A}_\lambda \) rather than introduce a new notation for the set of admissible states together with an associated collection of Boltzmann weights. We repeat this often in what follows when the underlying Boltzmann weights are clear from context.

Thus to specify the partition function \( Z(\mathcal{A}_\lambda) \) it remains to define the Boltzmann weights. The Boltzmann weights at each vertex are given in the table in Figure 2. They agree with the weights in Equation (35) of [24] and arise naturally from evolution operators on bosonic Fock space made by creation and annihilation of particles (see Section 2 of [24] for details). For us, the key point is that these weights satisfy certain relations known as “Yang-Baxter equations" or “RTT relations," which we address shortly below.

![Figure 2. Rectangular lattice Boltzmann weights - the boxed integer \( m \) denotes the multiplicity of paths along the associated vertical edge.](image)

This choice of Boltzmann weights gives the following explicit evaluations of the partition function:
Theorem 2.1 (Wheeler-Zinn-Justin [24], Thm. 2). For any partition $\lambda$ and weights as in Figure 2, then
\[
Z(A_\lambda) = (x_1 \cdots x_r) P_\lambda(x; q)
\]
where $P_\lambda$ denotes the Hall-Littlewood polynomial as in (1.1).

To prove this, one must demonstrate that the partition function $Z(A_\lambda)$ satisfies certain symmetries. In this case, we want to ensure that $Z(A_\lambda)$ is left unchanged after an exchange of variables $x_i \leftrightarrow x_{i+1}$. Recalling that the weights depend on $x_i$ in row $i$, this is equivalent to saying the partition function is symmetric upon switching the roles of the rows $i$ and $i + 1$, a property known as “commuting transfer matrices” for the row-to-row transfer matrix.

Following Baxter [1], one may do this by taking advantage of certain “local symmetries” enjoyed by these Boltzmann weights. Consider an auxiliary family of Boltzmann weights for the six so-called “R-vertices,” or $R$-matrix weights depicted in Figure 3 below. Their graphical depiction reflects the fact that each such vertex is the intersection of two horizontal edges, and hence each adjacent edge may admit at most one path. Their Boltzmann weights are a scaled version of the entries in the trigonometric $R$-matrix of the six-vertex model, now depending on the pair of parameters $x_j$ and $x_k$ on the respective rows. In addition to providing the explicit description of each weight in the bottom row of the table, the middle row gives an abstract name for each vertex, to be used in later computations.

| $a_2(k,j)$ | $a_1(k,j)$ | $b_3(k,j)$ | $b_1(k,j)$ | $c_2(k,j)$ | $c_1(k,j)$ |
|------------|------------|------------|------------|------------|------------|
| $1$        | $1$        | $q(x_k-x_j)/x_k-q x_j$ | $(1-q)x_j/x_k-q x_j$ | $(1-q)x_k/x_k-q x_j$ |

Figure 3. $R$-matrix weights from the trigonometric six-vertex model

Proposition 2.2 (Yang-Baxter equation, RTT relation). For any choice of edge labels $\alpha, \beta, \gamma, \delta, \epsilon, \phi$ and Boltzmann weights as in Figures 2 and 3, one has the equality of partition functions
\[
Z \begin{pmatrix} x_j \\ x_k \end{pmatrix} = Z \begin{pmatrix} x_j \\ x_k \end{pmatrix}.
\]

This may be checked explicitly, as the weights in Figure 2 are uniformly expressed in terms of the number of particles $m$. Such an identity was asserted in Equation 3.3 of [3] and in Equation (14) of [24].

In addition to the previous result, it will be useful to know the following result; compare Equation (11) in [24]. Its proof is a direct computation and left to the reader.

Proposition 2.3 (Unitarity relation). For any choice of edge labels $\alpha$ and $\beta$ and using Boltzmann weights as in Figure 3, one has
\[
Z \begin{pmatrix} x_j \\ x_k \end{pmatrix} = 1.
\]

We say that a type $A$ model is “solvable” if its weights possess a solution $R$ to the Yang-Baxter equation above. It implies that the row-to-row transfer matrices commute, via the now familiar “train argument,” and hence the partition function $Z(A_\lambda)$ is symmetric under the exchange of variables $x_i \leftrightarrow x_{i+1}$. A graphical depiction of this argument is shown in Figure 4, which highlights both the repeated use of the Yang-Baxter equation and that the symmetry of $Z$ hinges on the fact that the $R$ matrix weights $a_1(k, k+1) = a_2(k, k+1)$
for all $k$. Note that the figure doesn’t attempt to depict “generic” boundary conditions (in-coming paths) according to $\lambda$ along the bottom boundary.

In the above figure, and throughout the remainder of the paper, we write “$\text{wt}(S)$” when evaluating the Boltzmann weight of a single admissible state or lattice configuration and we write $\mathcal{Z}(S)$ when evaluating the partition function of the set of admissible states $S$.

3. Models for Cartan Type $B$ and $C$

Our results pertain to solvable lattice models for root systems of Cartan type $B$ and $C$, so named because solvability implies an action of the Weyl group of type $B/C$ on the partition function. These models are similar to the Type $A$ models presented in Section 2 in many ways. Our Type $B/C$ model has vertices in a rectangular lattice dictated by a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ – with $\lambda_1 + 1$ columns numbered in ascending order from left to right, and with $2r$ rows whose associated spectral parameters are labeled in pairs $x_i^{-1} =: x_i$ and $\pi_i$ in ascending order from bottom to top. The lattice is then augmented with a set of $r$ bivalent vertices connecting the rows corresponding to the spectral parameters $x_i$ and $\pi_i$ for each $i$, forming u-turn bends along the left-hand boundary of the model.

An admissible state in the rank $r$ type $B/C$ model is a configuration of $r$ paths, with the $i$th path starting from the $\lambda_i$-th edge along the bottom and moving upward and leftward through the lattice, ending at a bivalent vertex along the u-turn bend. An example of an admissible state in rank 2 is shown in Figure 5. The set of all admissible configurations for the lattice with boundary conditions determined by $\lambda$ will be denoted $\mathcal{B}_\lambda$. It is natural to use these admissible states to represent special functions for type $B/C$ root systems. Indeed, there is a natural bijection between admissible states in $\mathcal{B}_\lambda$ and certain arrays of interleaving integers generalizing Gelfand-Tsetlin patterns in that they arise naturally from branching rules for symplectic and odd orthogonal groups (see [20] for details and references).

There is further motivation for these geometries from quantum groups, where Boltzmann weights of tetravalent vertices represent matrix coefficients in $R$-matrices in $\text{End}(V(x) \otimes W)$ for a pair of quantum group
modules $V(x), W$ representing the horizontal and vertical edges, respectively, while the bivalent vertices at the u-turn bends represent so-called $K$-matrices in $\text{Hom}(V(x_i) \otimes V(\bar{x}_i), \mathbb{C})$. These have been featured in solvable lattice models for symmetric functions previously, for example in [16, 22, 2, 24]. We require the bare minimum set of conditions on the bend weights to guarantee that the resulting partition function has hyperoctahedral symmetry - these are the so-called “fish” and “caduceus” relations to be presented in this section after fully introducing the lattice model. Our “caduceus” relation will turn out to be equivalent to the “reflection equation” of Sklyanin [21]; see also Section 2.7 of [24] where the $K$-matrix is referred to as the boundary covector.

At each of the tetravalent vertices in the rectangular part of the lattice model, we use the same Boltzmann weights from Figure 2, just as in the Type $A$ model. The remaining Boltzmann weights for the bivalent vertices at each bend remain to be determined. We have recorded the possible admissible configurations in Figure 6, but have only represented those weights with very generic labels. In particular, for any row index $j \in [1, r]$ we allow four possible configurations with respective bend weights $A_j, B_j, C_j, D_j$ as in Figure 6, which are each functions (potentially distinct for each $j$) taking values in the polynomial ring $\mathbb{Z}[q](x_k, \bar{x}_k)$ where $x_k$ is the associated spectral parameter for the pair of rows. We will momentarily determine conditions on these bend weights in order that the type $B/C$ models are solvable.

Just as in type $A$, when the choice of Boltzmann weights is understood for a set of admissible configurations $B_\lambda$, we refer to the set of states with associated weights as the “model” associated to $\lambda$ and continue to refer to the model as $B_\lambda$.

![Figure 6. Generic Bend Weights](image)

With these definitions in place, we can again create a partition function by summing the weights of all admissible configurations:

$$Z(B_\lambda) = \sum_{S \in B_\lambda} \text{wt}(S).$$

As in the case of $Z(A_\lambda)$, we may evaluate $Z(B_\lambda)$ by understanding its behavior under action of the Weyl group. In order to exhibit this action, we need both the Yang-Baxter equation (Proposition 2.2) as well as two additional identities: the Fish relation (Lemma 3.1) and the Caduceus relation (Lemmas 3.2, 3.3, and 3.4). These lemmas establish all non-trivial bend weight choices among monomials in the $x^{\pm 1}$ in the Laurent polynomial ring $\mathbb{Z}[q](x_1^{\pm 1}, \ldots, x_r^{\pm 1})$ which achieve these relations. The degree $\text{deg}(M)$ of a monomial $M$ in $\mathbb{Z}[q](x^{\pm 1})$ is the integer exponent of $x$. In each of the subsequent lemmas, it is sometimes convenient to identify the set of edge labels (thus far described by paths) with the non-negative integers. In particular, horizontal edges can only have $\{0, 1\}$ (no path, one path, respectively) as potential labels.

**Lemma 3.1** (Fish relation). Let the $R$-matrix Boltzmann weights be given as in Figure 3 with spectral parameters $x$ and $\bar{x}$. We further assume that the bend weights are monomial in $x, \bar{x}$ with coefficients in $\mathbb{C}[q]$. Then the quantities

$$Z \left( \begin{array}{c} j & \bigotimes & \bar{x} \\ x \end{array} \right) \quad \text{and} \quad \text{wt} \left( \begin{array}{c} j & \bigotimes & \bar{x} \\ x \end{array} \right)$$

are proportional with proportionality constant $F$ in $\mathbb{Z}[q](x^{\pm 1})$ independent of the choice of boundary labels $\alpha$ and $\beta$ in $\{0, 1\}$ if and only if one of the following conditions on the Boltzmann weights at bivalent vertices in Figure 6 hold:

1. Either $\text{deg}(B_j(x, \bar{x})) = \text{deg}(C_j(x, \bar{x}))$ and either:
the functions \( B \) and \( C \), in which case the constant of proportionality \( F = 1 \). In this case, 
\[ \text{deg}(A_j(x, \bar{x})) = \text{deg}(D_j(x, \bar{x})) = 0; \]
\[ \text{deg}(B_j(x, \bar{x})) = -q C_j(x, \bar{x}) \text{ with } F = (x^2 - q)/(1 - qx^2). \]

In this case, \( A_j(x, \bar{x}) = D_j(x, \bar{x}) = 0 \).

(2) Or \( \text{deg}(B_j(x, \bar{x})) = 2 \) and either:

(a) \( B_j(x, \bar{x}) = -q C_j(x, \bar{x}) \) with \( F = (x^2 - q)/(1 - qx^2) \), and then \( A_j(x, \bar{x}) = D_j(x, \bar{x}) = 0 \); or

(b) \( B_j(x, \bar{x}) = q^2 C_j(x, \bar{x}) \) with \( F = 1 \) and then \( \text{deg}(A_j(x, \bar{x})) = \text{deg}(D_j(x, \bar{x})) = 0 \).

Proof. In what follows, we abbreviate \( A_j(x, \bar{x}) \) by \( A_j(x) \) and \( A_j(\bar{x}, x) \) by \( A_j(\bar{x}) \), with similar abbreviations for the functions \( B_j, C_j \) and \( D_j \). Let’s begin by considering the case where \( B_j, C_j \neq 0 \). In the case \( (\alpha, \beta) = (1, 0) \), we have

\[
Z \left( \begin{array}{c}
\bar{1} \\
\bar{0} \end{array} \middle| \begin{array}{c}
\bar{1} \\
\bar{0} \\
x \end{array} \right) / \text{wt} \left( \begin{array}{c}
\bar{1} \\
\bar{0} \end{array} \middle| \begin{array}{c}
\bar{1} \\
\bar{0} \\
x \end{array} \right) = \left( C_j(x) \cdot \frac{(1 - q)x^2}{1 - qx^2} + B_j(x) \cdot \frac{1 - x^2}{1 - qx^2} \right) / C_j(\bar{x}),
\]

where the Boltzmann weights \( \frac{(1 - q)x^2}{1 - qx^2} \) and \( \frac{1 - x^2}{1 - qx^2} \) come from the twisted weights in Figure 3. Similarly, in the case \( (\alpha, \beta) = (0, 1) \), the ratio is

\[
\left( C_j(x) \cdot \frac{q(1 - x^2)}{1 - qx^2} + B_j(x) \cdot \frac{1 - q}{1 - qx^2} \right) / B_j(\bar{x}).
\]

Equating these two expressions and clearing denominators, we require

\[
[C_j(x)(1 - q)x^2 + B_j(x)(1 - x^2)] B_j(\bar{x}) = C_j(\bar{x}) \left[ q(1 - x^2)C_j(x) + B_j(x)(1 - q) \right].
\]

Since we have assumed that \( B_j(x) \) and \( C_j(x) \) are monomial in \( x \), then a simple degree comparison of the two sides shows that the possibilities above (equal degree, or degree differing by 2 as stated) are the only possibilities.

Suppose first that \( \text{deg}(C_j(x, \bar{x})) = \text{deg}(B_j(x, \bar{x})) \). Since our required identity is a homogeneous quadratic in \( B_j \) and \( C_j \) we may scale and assume, without loss of generality, that the common degree is 0. Equating the coefficients of \( x^2 \) and equating the constant terms on either side of (3.4), both turn out to produce the same condition:

\[
(C_j(x)q + B_j(x))(C_j(x) - B_j(x)) = 0.
\]

Plugging in any solution to the above to (3.2) or (3.3) gives the respective ratios \( F \) in Cases (1a) and (1b) of the Lemma.

If instead \( \text{deg}(C_j(x, \bar{x})) = \text{deg}(B_j(x, \bar{x})) - 2 \) then we may again scale the monomial weights so that \( C_j(x) = \bar{c} e \) for some \( c \in \mathbb{C}[q] \) and \( B_j(x) = bx \) for some \( b \in \mathbb{C}[q] \). Substituting these expressions into (3.4) we again find that both the \( x^2 \) coefficient and the constant term produce the same required identity on \( b, c \):

\[
(b + c)(b - qc) = 0.
\]

If \( b = -c \), then substituting into (3.3) and simplifying, the resulting constant of proportionality \( F = (x^2 - q)/(1 - qx^2) \). The case of \( b = qc \) gives \( F = 1 \).

Observe that if \( B_j = 0 \), then (3.3) implies that \( C_j \) must also be 0 for the quantities in (3.1) to be proportional. Similarly, if \( C_j = 0 \), then \( B_j \) must be 0 in order for the ratio of terms \( F \) in (3.1) to be a constant independent of \( \alpha \) and \( \beta \).

Lastly, we consider the cases where \( (\alpha, \beta) = (1, 1) \) or \( (0, 0) \). Only bends of type \( A_j \) and \( D_j \) arise here. If \( (\alpha, \beta) = (1, 1) \), then \( F \cdot D_j(\bar{x}) = (D_j(x) \cdot 1) \) and one may easily see the conditions on \( D(x) \) in the statement of the Lemma follow for the two cases for \( F \). In particular, if \( F = (x^2 - q)/(1 - qx^2) \), then no such monomial \( D(x) \) can satisfy \( F = D_j(x)/D_j(\bar{x}) \) which forces \( D(x) = 0 \). Similar conditions hold for \( A_j(x) \) following from the \( (0, 0) \) case.

The Fish relation is needed to prove that \( Z(\mathcal{B}_3) \) is symmetric under \( x_i \leftrightarrow x_i \). In order to prove that \( Z(\mathcal{B}_3) \) is symmetric under the remaining relations \( x_i \leftrightarrow x_{i+1} \), however, we will need to use an additional relation that we call the Caduceus relation (to be described below). That relation involves interactions between bends, and for this reason one is forced to consider how the fish relations for various bends are related to each other (or not). For this reason, we introduce the following

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Definition 3.1. A weighting scheme for which each bend satisfies the same subcase of the Fish relation (Lemma 3.1) is said to be of uniform regime. If different bends satisfy different subcases of the Fish relation, we say the model is mixed regime.

It is sometimes convenient to be explicit about which condition of the Fish relation is satisfied for a model of uniform regime, in which case we will say that the model is uniform in regime \( R \) (where \( R \) is some condition from \( \{1(a), 1(b), 2(a), 2(b)\} \) as enumerated in Lemma 3.1).

The lure of mixed regimes is that they offer more flexibility; unfortunately, this also makes them more complex to study. For the duration of this paper, we will focus on analyzing the behavior of uniform regime models.

The Caduceus relation gives algebraic conditions that are described via a case analysis of the number of particles allowed on the boundary (always between 0 and 4, as the caduceus has 4 boundary edges). We will treat the cases where there are one, two, or three particles allowed on the boundary in separate lemmas below. Note that our diagrams show only configurations in rank two, but the methods apply generally to any pair of adjacent bends in a lattice of arbitrary rank.

Lemma 3.2 (Caduceus relation, 2 particles). Let the \( R \)-matrix Boltzmann weights be given as in Figure 3, and assume that the bend weights are monomial with coefficients in \( \mathbb{C}[q] \) of uniform regime. Then the quantities

\[
Z \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \quad \text{and} \quad \text{wt} \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \cdot \text{wt} \left( \begin{array}{c} 1 \\ \end{array} \right)
\]

are proportional with proportionality constant \( F \in \mathbb{Z}[q, q^{-1}] \) independent of the choice of boundary labels \( \alpha, \beta, \gamma, \delta \) in \( \{0, 0, 1, 1\} \) if and only if one of the following regime-dependent conditions hold:

1. when the weights are regime 1(a), we have \( C_j \in \mathbb{C}[q] \) for \( j \in \{1, 2\} \), and

\[
q^2 (A_1 D_2 - C_1 C_2) - (A_2 D_1 - C_1 C_2) = 0,
\]

in which case \( F = 1 \);

2. when the weights are regime 2(b), we have \( B_j(x) = q m_j x, C_j(x) = m_j \tilde{x}, \) where \( m_j \in \mathbb{C}[q] \) for \( j \in \{1, 2\} \), and

\[
q^3 m_2 m_1 + q^2 D_2 A_1 - q m_2 m_1 - A_2 D_1 = 0,
\]

in which case \( F = 1 \);

3. when the weights are regime 1(b) the Caduceus relation automatically holds, in which case \( F = \frac{(C_1(x_1) C_2(x_2))/C_1(x_1) C_2(x_1))}{(C_1(x_2) C_2(x_2))} \); and

4. when the weights are regime 2(a), the Caduceus relation automatically holds, in which case \( F = \frac{(C_1(x_1) C_2(x_2))/C_1(x_1) C_2(x_1))}{(C_1(x_2) C_2(x_1))} \).

Proof. As an example, consider the case \( (\alpha, \beta, \gamma, \delta) = (1, 0, 1, 0) \). Note that there are eight different states with these boundary conditions, depicted below with Boltzmann weights from Figures 3 and 6.
and hence, in order to have $\bar{q}$ is a constant independent of boundary conditions in this case. If (3.7) is satisfied, then equating them yields

Next, consider the case where the weights are regime 2(b), so that $B_1(x_1)C_2(x_2)c_2(1, 2)c_1(T, 2)b_1(T, \bar{2})$

(Cases have the same denominator, so we can calculate the ratio of terms from (3.5),

$$F_j(x_1) = 0$$

Using this fact in the analysis of the remaining cases, it can be shown that satisfying (3.7) is sufficient to ensure that $F$ is a constant independent of boundary conditions in this case. If (3.7) is satisfied, then $F = 1$. 

\begin{align*}
A_1(x_1)D_2(x_2)c_2(1, 2)b_2(T, \bar{2})a_1(T, \bar{2}) + C_1(x_1)C_2(x_2)c_2(1, 2)c_1(T, 2)b_1(T, \bar{2}) +
B_1(x_1)C_2(x_2)c_2(1, 2)a_1(T, \bar{2})b_1(T, \bar{2}) + C_1(x_1)B_2(x_2)b_1(1, 2)a_2(1, \bar{2})a_1(T, \bar{2})b_2(T, \bar{2}) +
B_1(x_1)B_2(x_2)b_1(1, 2)c_2(1, \bar{2})a_1(T, \bar{2})b_1(1, \bar{2})b_1(T, \bar{2}) + D_1(x_1)A_2(x_2)b_1(1, 2)b_1(1, \bar{2})c_1(T, 2)b_1(T, \bar{2})
\end{align*}
Now suppose we are in regime 1(b), so that \( B_j(x) = -qC_j(x) \) and \( A_j = D_j = 0 \). Then for \((1,0,1,0)\),
\[
F = C_1(x_1)C_2(x_2)(a_2(1,2)b_1(1,\overline{2})b_2(T,\overline{T})a_1(T,\overline{T}) + c_2(1,2)c_1(1,\overline{2})c_2(T,\overline{T}) - qc_2(1,2)a_1(1,\overline{2})c_1(T,\overline{T})b_1(T,\overline{T}) \\
- qb_1(1,2)a_2(1,\overline{2})c_1(T,\overline{T}) + q^2b_1(1,2)c_2(1,\overline{2})c_1(T,\overline{T})b_1(T,\overline{T}))/(C_1(x_2)C_2(x_1)) = (C_1(x_1)C_2(x_2))/(C_1(x_2)C_2(x_1)),
\]
and for \((1,0,0,1)\),
\[
F = C_1(x_1)C_2(x_2)(c_2(1,2)a_1(1,\overline{2})b_2(T,\overline{T}) - qc_2(1,2)a_1(1,\overline{2})a_1(T,\overline{T})b_1(T,\overline{T}) \\
+ q^2b_1(1,2)c_2(1,\overline{2})a_1(T,\overline{T}))/(C_1(x_2)C_2(x_1)) = (C_1(x_1)C_2(x_2))/(C_1(x_2)C_2(x_1)).
\]

Upon checking the four remaining cases, it can be verified that \( F = (C_1(x_1)C_2(x_2))/(C_1(x_2)C_2(x_1)) \) in this case, regardless of the values of \( \alpha, \beta, \gamma, \delta \).

Finally, consider regime 2(a), where \( B_j(x) = -x^2C_j(x) \) and \( A_j = D_j = 0 \). In this case, if the boundary conditions are \((1,0,1,0)\), then
\[
F = C_1(x_1)C_2(x_2)(a_2(1,2)b_1(1,\overline{2})b_2(T,\overline{T})a_1(T,\overline{T}) + c_2(1,2)c_1(1,\overline{2})c_2(T,\overline{T}) - x^2c_2(1,2)a_1(1,\overline{2})c_1(T,\overline{T})b_1(T,\overline{T}) \\
- x^2b_1(1,2)a_2(1,\overline{2})c_1(T,\overline{T}) + x^4b_1(1,2)c_2(1,\overline{2})c_1(T,\overline{T})b_1(T,\overline{T}))/(C_1(x_2)C_2(x_1)) = (C_1(x_1)C_2(x_2))/(C_1(x_2)C_2(x_1)).
\]

If the boundary conditions are \((1,0,0,1)\), then
\[
F = C_1(x_1)C_2(x_2)(c_2(1,2)a_1(1,\overline{2})b_2(T,\overline{T}) - x^2c_2(1,2)a_1(1,\overline{2})a_1(T,\overline{T})b_1(T,\overline{T}) \\
- x^2b_1(1,2)a_2(1,\overline{2})a_1(T,\overline{T}) + x^4b_1(1,2)c_2(1,\overline{2})a_1(T,\overline{T})b_1(T,\overline{T}))/(C_1(x_2)C_2(x_1)) = (C_1(x_1)C_2(x_2))/(C_1(x_2)C_2(x_1)).
\]

Upon checking the four remaining cases, it can be verified that \( F = (C_1(x_1)C_2(x_2))/(C_1(x_2)C_2(x_1)) \) in this case, regardless of the values of \( \alpha, \beta, \gamma, \delta \).

\( \square \)

**Lemma 3.3** (Caduceus relation, 3 particles). Let the \( R \)-matrix Boltzmann weights be given as in Figure 3, and assume that the bend weights are monomial with coefficients in \( \mathbb{C}[q] \) of uniform regime. Then the quantities

\[
(3.8) \quad \mathcal{Z} \begin{pmatrix} 2 & 1 \\ x_1 & x_2 \end{pmatrix} \quad \text{and} \quad \text{wt} \begin{pmatrix} 2 & 1 \\ \pi_1 & \pi_2 \end{pmatrix} \quad \text{wt} \begin{pmatrix} 2 & 1 \\ x_1 & x_2 \end{pmatrix} \]

are proportional with proportionality constant \( F \in \mathbb{Z}[q,q^{-1}][x_1^{\pm 1},x_2^{\pm 2}] \) independent of the choice of boundary labels \( \alpha, \beta, \gamma, \delta \) in \( \{0,1,1,1\} \) if and only if one of the following the following regime-dependent conditions hold:

1. when the weights are regime 1(a) and \( C_1, C_2 \) are both nonzero, we have \( C_1, C_2 \in \mathbb{C}[q] \), and \( C_2D_1 = C_1D_2 \), in which case \( F = 1 \);
2. when the weights are regime 2(b), we have \( B_j(x) = qm_jx, C_j(x) = m_j\bar{x} \), where \( m_j \in \mathbb{C}[q] \), and \( m_2D_1 = q^2m_1D_2 \), in which case \( F = 1 \); and
3. when the weights are either regime 1(b), regime 2(a), or regime 1(a) with \( C_1 = C_2 = 0 \), then the quantities in question are all zero, in which case they are vacuously proportional.

**Proof.** As an example, consider the case \((\alpha, \beta, \gamma, \delta) = (1,1,1,0)\). Note that there are three different states with these boundary conditions, depicted below with Boltzmann weights from Figures 3 and 6.
In this case, the ratio of terms from (3.8), \( F \), is equal to
\[
(C_1(x_1)D_2(x_2)aw(1, 2)aw(1, \overline{a})c_2(\overline{a}, 2)c_2(\overline{a}, 2) + D_1(x_1)C_2(x_2)aw(1, 2)aw(1, \overline{a})c_2(\overline{a}, 2)b_1(\overline{a}, \overline{2})) + B_1(x_1)D_2(x_2)aw(1, 2)b_1(\overline{a}, 2)b_1(\overline{a}, \overline{2})/C_1(x_1)D_2(x_2).
\]

Similarly, in the case \((1, 1, 0, 1)\), one can show that
\[
F = (C_1(x_1)D_2(x_2)aw(1, 2)aw(1, \overline{a})c_2(\overline{a}, 2)b_2(\overline{a}, \overline{2}) + B_1(x_1)D_2(x_2)aw(1, 2)aw(1, \overline{a})c_2(\overline{a}, 2)b_1(\overline{a}, \overline{2}) + D_1(x_1)C_2(x_2)aw(1, 2)b_1(\overline{a}, 2)b_1(\overline{a}, \overline{2})c_2(\overline{a}, 2) + D_1(x_1)C_2(x_2)aw(1, 2)c_1(\overline{a}, 2)b_2(\overline{a}, \overline{2}) + D_1(x_1)B_2(x_2)b_1(1, 2)aw(1, \overline{a})b_1(\overline{a}, 2)b_2(\overline{a}, \overline{2})/C_1(x_1)D_2(x_2).
\]

In order for a set of Boltzmann weights to be of uniform regime, the bivalent weights must fall into one of the four cases described in Lemma 3.1. If the bivalent weights satisfy cases 1(b), 2(a), or 1(a) with \( C_1 = C_2 = 0 \) of Lemma 3.1 then the result follows trivially, so suppose first that the weights are uniform of regime 1(a) with both \( C_1 \) and \( C_2 \) nonzero: \( B_j(x) = C_j(x) \) and \( A_1, A_2, D_1, D_2 \) are polynomials in \( q \). By equating the ratio of terms from (3.8) and simplifying, we have
\[
\frac{C_1(x_1)D_2(x_2) - C_2(x_2)D_1(x_1)}{C_1(x_1)D_2(x_2)} = \frac{(1 - q^2)(q^2 - 1)(x_1 - x_2)^{-1}(x_1 - x_2)^{-1}}{(x_1 - q x_2)^{-1}(x_1 - q x_2)^{-1} - (x_1 - x_2)^{-1}} = 0.
\]

In order to have the ratio of terms from (3.8) be a constant independent of boundary conditions, it is necessary that \( C_2(x_2)D_1 = C_1(x_1)D_2 \); in particular, since \( \deg(D_j) = 0 \), it must be the case that \( \deg(C_j) = 0 \). Examining the remaining two cases, the condition \( C_2D_1 = C_1D_2 \) is easily shown to be sufficient. One can quickly verify that \( F = 1 \) in this case.

Suppose then that the model is uniform of regime 2(b): \( B_j(x) = q x^2 C_j(x) \) and \( A_1, A_2, D_1, D_2 \in \mathbb{C}[q] \). Calculating the ratio of terms from (3.8) in the cases \((1, 1, 1, 0)\) and \((1, 1, 0, 1)\) and equating them yields
\[
\frac{q^2 x_1 C_1(x_1)D_2(x_2) - x_2 C_2(x_2)D_1(x_1)}{q x_2 C_2(x_2)D_2(x_2)} = \frac{(1 - q^2)(1 - q^2) x_1(x_1 - x_2) x_1(x_1 - x_2)}{x_1(x_1 - x_2)(x_1 - q x_2)(x_1 - x_2)(x_1 - q x_2)}.
\]

Since \( \deg(D_j) = 0 \), it must be the case that \( \deg(C_j) = -1 \). Using this fact, it is easy to check that \( F = 1 \) in all four cases.

\[\square\]

Lemma 3.4 (Caduceus relation, 1 particle). Let the R-matrix Boltzmann weights be given as in Figure 3, and assume that the bend weights are monomial with coefficients in \( \mathbb{C}[q] \) of uniform regime. Then the quantities

\[
(3.9)
\]

are proportional with proportionality constant \( F \in \mathbb{Z}[q, q^{-1}](x_1^{1}, x_2^{1}) \) independent of the choice of boundary labels \( \alpha, \beta, \gamma, \delta \) in \( \{0, 0, 0, 1\} \) if and only if one of the following regime-dependent conditions hold:

1. when the weights are regime 1(a) and \( C_1, C_2 \) are both nonzero, we have \( C_1, C_2 \in \mathbb{C}[q] \), and \( C_2A_1 = C_1A_2 \), in which case \( F = 1 \);
2. when the weights are regime 2(b), we have \( B_j(x) = q m_j x \), \( C_j(x) = m_j x \) where \( m_j \in \mathbb{C}[q] \), and \( m_1 A_2 = q^2 m_2 A_1 \), in which case \( F = 1 \); and
3. when the weights are either regime 1(b), regime 2(a), or regime 1(a) with \( C_1 = C_2 = 0 \), then the quantities in question are all zero, in which case they are vacuously proportional.

The proof of Lemma 3.4 is analogous to the proof of Lemma 3.3 and is left to the reader.

Now we are ready to define an analogous notion of solvability for these lattice models with bends. This isn’t a standard definition in the literature, as this bend geometry is somewhat exceptional, but the following conditions ensure that we can perform an analogous “train argument” to the type A case in Figure 4 for models with bends.
Definition 3.2. We say that a type $B/C$ model is solvable (with respect to the trigonometric six-vertex model) if it satisfies the following three local identities:

- the Yang-Baxter equation (2.2) for all pairs $j, k$
- uniform regime in the Fish relation (3.1)
- the Caduceus relations (3.5), (3.8), (3.9) for all pairs $j, k$

Remark. Whereas our definition relies on these three families of relations, work in [24] instead focuses on the Yang-Baxter and Fish relations together with the reflection equation of Cherednik and Sklyanin (see for example [21]); for a diagrammatic presentation of this relation close to our point of view, see Equation (23) in Section 2.7 of [24]. Given our choice of $R$-matrix weights from Figure 3, and in particular the unitarity relation in Proposition 2.3, the reflection equation is equivalent to our Caduceus relations.

General solutions to the reflection equation (and hence the Caduceus relations) for the trigonometric $R$-matrix are explored in [12] where the weight of the bend vertices is encoded in a so-called $K$-matrix, a notation that we’ll use in subsequent sections. We’ve chosen to reprove the caduceus relations here, assuming the fish relation, because our subsequent analysis depending on the rank will require us to make distinctions between the “one-,” “two-,” and “three-particle” cases as presented above, while a solution to the reflection equation would assume all three are satisfied. Note also that we’re assuming the Fish relation in our proof, simplifying some of the analysis.

The solvability of the type $B/C$ model ensures that the partition function inherits functional equations under the action of the Weyl group of type $B/C$, otherwise known as the hyperoctahedral group with generating simple reflections

\begin{equation}
    s_i : (x_1, \ldots, x_i, x_{i+1}, \ldots, x_r) \mapsto (x_1, \ldots, x_{i+1}, x_i, \ldots, x_r) \quad \text{for } 1 \leq i \leq r - 1
\end{equation}

\begin{equation}
    s_r : (x_1, \ldots, x_{r-1}, x_r) \mapsto (x_1, \ldots, x_{r-1}, x_r^{-1})
\end{equation}

We demonstrate this hyperoctahedral symmetry of the partition function using the respective train arguments for the associated lattice models with bends in the next two lemmas.

Lemma 3.5. Let $B_\lambda$ be a type $B/C$ solvable model with tetravalent Boltzmann weights from the trigonometric six-vertex model. Then if the bend weights in row $i$ satisfy condition 1(a) or 2(b) of Lemma 3.1, $Z(B_\lambda)$ is invariant under $x_i \leftrightarrow \overline{x}_i$. If instead the bend weights in row $i$ satisfy condition 1(b) or 2(a) of Lemma 3.1, then $(x_i^{-1} - qx_i)Z(B_\lambda)$ is invariant under the inversion $x_i \leftrightarrow \overline{x}_i$.

Proof. We begin by adding a single $R$ vertex at the right edge of the ice between the rows with parameters $x_i$ and $\overline{x}_i$, as depicted below in diagram (3.11). Since no paths exit through the right edge of the ice, the only admissible $R$ vertex is of type $a_1(k,j)$ in Figure 3 with $(k,j) = (i,i)$. Hence, we have the equality of partition functions given below (depicted in the case where $i = 1$, for simplicity):

\begin{equation}
    Z(B_\lambda) = Z\begin{pmatrix}
        2 & \bullet \\
        1 & \bullet \\
    \end{pmatrix}
\end{equation}

From here, we can repeatedly apply the Yang-Baxter equation, so that the partition function for diagram (3.11) is equal to

\begin{equation}
    Z\begin{pmatrix}
        2 & \bullet \\
        1 & \bullet \\
    \end{pmatrix} = \cdots = Z\begin{pmatrix}
        2 & \bullet \\
        1 & \bullet \\
    \end{pmatrix} = F \cdot Z\begin{pmatrix}
        2 & \bullet \\
        1 & \bullet \\
    \end{pmatrix}
\end{equation}

where we have used an application of the Fish relation (Lemma 3.1) in the final equality, and the respective cases result from the two possibilities for the constant of proportionality $F$ in the fish relation. \qed
Lemma 3.6. Let $\mathcal{B}_\lambda$ be a type $B/C$ solvable model with tetravalent Boltzmann weights from the trigonometric six-vertex model. Then if the associated caduceus constants $F = 1$ for all simple transpositions $s_i = (i \ i + 1)$ with $1 \leq i \leq r - 1$, then $Z(\mathcal{B}_\lambda)$ is a symmetric function under permutation of the $\{x_i\}$.

Proof. Similarly to the proof of Lemma 3.5, we start by adding four $R$ vertices at the right edge of the ice, as depicted below in rank two for simplicity:

$$Z(x; \mathcal{B}_\lambda) = Z \left( \begin{array}{c} \begin{array}{c} x_2 \\ \overline{\tau}_2 \\ x_1 \\ \overline{\tau}_1 \\ 1 \\ 2 \end{array} \\ \begin{array}{c} 2 \\ \overline{\tau}_2 \\ x_1 \\ \overline{\tau}_1 \\ 1 \\ x_2 \end{array} \end{array} \right) = Z \left( \begin{array}{c} \begin{array}{c} x_1 \\ \overline{\tau}_1 \\ x_2 \\ \overline{\tau}_2 \\ 1 \\ 2 \end{array} \\ \begin{array}{c} 2 \\ \overline{\tau}_1 \\ x_2 \\ \overline{\tau}_2 \\ 1 \\ x_1 \end{array} \end{array} \right).$$

Note that, as in the proof of Lemma 3.5, the boundary conditions along the right edge of the diagram mean that all four of these $R$ vertices are of type $a_1(k,j)$ in Figure 3 with $(k,j)$ in $\{1, \overline{1}, 2, \overline{2}\}$. From here, we can repeatedly apply the Yang-Baxter equation (2.2), so that the partition function in (3.13) is equal to

$$Z \left( \begin{array}{c} \begin{array}{c} x_1 \\ \overline{\tau}_1 \\ x_2 \\ \overline{\tau}_2 \\ 1 \\ 2 \end{array} \\ \begin{array}{c} 1 \\ \overline{\tau}_2 \\ x_1 \\ \overline{\tau}_1 \\ 2 \\ x_2 \end{array} \end{array} \right) = Z \left( \begin{array}{c} \begin{array}{c} x_1 \\ \overline{\tau}_1 \\ x_2 \\ \overline{\tau}_2 \\ 1 \\ 2 \end{array} \\ \begin{array}{c} 2 \\ \overline{\tau}_1 \\ x_2 \\ \overline{\tau}_2 \\ 1 \\ x_1 \end{array} \end{array} \right),$$

where we apply the Caduceus relation (Lemma 3.2, 3.3, or 3.4) in the final equality, noting that the constant of proportionality $F$ is equal to 1 in all cases, to obtain the desired result.

Further generalizations of the prior Lemma are possible using other constants of proportionality from the caduceus relations in Lemmas 3.2, 3.3, and 3.4, but we refrain from stating them to focus on the case of symmetric functions. Indeed, combining the previous two lemmas, we may immediately conclude the following result, noting that the subcases involved imply that the caduceus constants $F$ are all equal to one.

Proposition 3.7. Let $\mathcal{B}_\lambda$ be a type $B/C$ solvable model with tetravalent Boltzmann weights from the trigonometric six-vertex model. If the bend weights are uniform of case 1(a) or 2(b) in Lemma 3.1, then $Z(x; \mathcal{B}_\lambda)$ is invariant under the usual action of the hyperoctahedral group on $x$ given above.

4. General Methods for Closed Form Solutions of Solvable $B/C$ Models

We can now take advantage of solvability, providing a general strategy for evaluating the rank $r$ partition functions for lattice models with bends, mimicking the approach taken in Section 3.4 of [24]. In particular, we adopt the Dirac bra-ket notation to describe the partition function:

$$Z(\mathcal{B}_\lambda) = \langle \mathbf{K}|\mathbf{C}_0\mathbf{C}_1\cdots\mathbf{C}_\lambda|\varnothing \rangle,$$

where the ket $|\varnothing \rangle$ denotes the right-hand boundary of $2r$ empty rows, viewed as a column vector with $2r$ components. The operators $\mathbf{C}_i$ denote the column transfer matrix in column $i$, and the $\mathbf{K}$ operator in $\langle \mathbf{K}|$ denotes the partition function of the resulting $r$ caps along the left boundary. If we let $W_1$ and $W_{\overline{1}}$ denote the two-dimensional vector space in rows $i$ and $\overline{i}$, respectively, with basis indexed by particle and hole, then $|\varnothing \rangle$ represents the (column) vector of $2r$ holes in $W_1 \otimes W_{\overline{1}} \otimes \cdots \otimes W_1 \otimes W_{\overline{1}}$ and each column transfer matrix $\mathbf{C}_i$ is an element of $\text{End}(W_{r} \otimes \cdots \otimes W_1)$. In [24], the analysis of this braket is greatly simplified by the use of a matrix $\mathbf{F}_r := F_{r,\overline{r},\ldots,\overline{1},1}$ in $\text{End}(W_{r} \otimes \cdots \otimes W_1)$ such that

$$\mathbf{F}_r \mathbf{R}_\sigma = \mathbf{F}$$

where $\mathbf{R}_\sigma$ is the $R$-matrix that acts on the left of a column of particles by a tangle of crossings (i.e., $R$-matrices of simple transpositions), taking row $i$ to row $\sigma(i)$, and $\mathbf{F}_r := F_{\sigma(r),\sigma(\overline{r}),\ldots,\sigma(\overline{1}),\sigma(1)}$. For example,
in the case \( \sigma = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \) in two-line permutation notation, then \( R_\sigma \) has the form:

\[
\begin{align*}
\sigma(2) & \quad 2 \\
\sigma(2) & \quad 2 \\
\sigma(1) & \quad 1 \\
\sigma(1) & \quad 1 \\
\sigma(\top) & \quad \top
\end{align*}
\]

It is not immediately clear that \( R_\sigma \) is well-defined. In order for the value of \( R_\sigma \) to be consistent regardless of how one writes \( \sigma \) as a product of simple reflections, one needs to ensure that products of associated \( R \)-matrices satisfy the defining relations of the symmetric group. The Yang-Baxter equations (Proposition 2.2) give the braid relations, while the unitarity relation (Proposition 2.3) shows that \( R \)-matrices for simple reflections square to the identity, ensuring that crossing strands and crossing back result in no change.

An explicit expression for \( F_{r, \bar{r}, \ldots, 1, \bar{1}} \) is given by making some small changes to an analogous computation in [24, Appendix A]. To state this result, we will assume the ordering \( r \succ \bar{r} \succ \cdots \succ 1 \succ \bar{1} \), for any given \( i \in I \setminus \{r\} \) we will write \( i + 1 \) for the immediate successor of \( i \) in \( I \) (e.g., \( 1 + 1 = 2 \) and \( 2 + 1 = 2 \)).

\[
F_{r, \bar{r}, \ldots, 1, \bar{1}} = \sum_{i \in I} \left( \prod_{i < j \leq r} E_{22}^j \prod_{1 < j \leq i} E_{11}^k \right) + \sum_{i \in I} \sum_{\rho \in S_{i+1}} \left( \prod_{i < j \leq r} E_{\rho(j)} \prod_{1 < j \leq i} E_{\rho(k)} \right) R_{r, \bar{r}, \ldots, 1, \bar{1}}
\]

where \( E_{kk} \) denotes the elementary \( 2 \times 2 \) matrix with entry 1 at position \((k, k)\) and 0 elsewhere, as an endomorphism of the vector space \( W_i \), and the final sum is over permutations \( \rho \) with a single inversion \( \rho(i) \succ \rho(i+1) \). As noted in [24], one can determine \( F \) by identifying a natural operator

\[
F^* = \sum_{\rho \in S_{i+1}} R_{r, \bar{r}, \ldots, 1, \bar{1}} \left( \prod_{i < j \leq r} E_{\rho(j)} \prod_{1 < j \leq i} E_{\rho(k)} \right)
\]

such that \( FF^* \) is an explicit diagonal matrix. Specifically, if we define

\[
[\Delta_{kl}(x_k, x_l)]_{i, j|k, l} = \delta_{k, i}\delta_{j, l}b_{i, j}(x_k, x_l), \quad b_{i, j}(x_k, x_l) = \begin{cases} 1, & \text{if } i = k, i = l, \\
(x_l - x_k)/(x_l - x_k), & \text{if } i < k, i < l, \\
(x_k - x_l)/(x_l - x_k), & \text{if } i > k, i < l,
\end{cases}
\]

then we have \( F_{r, \bar{r}, \ldots, 1, \bar{1}} = \prod_{i < j \leq r} \Delta_{kl}(x_k, x_l) \), and so we have \( F_{r, \bar{r}, \ldots, 1, \bar{1}}^{-1} = F^*_{r, \bar{r}, \ldots, 1, \bar{1}} \prod_{i < j \leq r} \Delta_{kl}^{-1}(x_k, x_l) \). Using this explicit formula, one can check that

\[
F_{r, \bar{r}, \ldots, 1, \bar{1}}^{-1} |\varnothing\rangle = |\varnothing\rangle.
\]

With these facts in hand, we can return to analyzing the bracket in (4.1). We find

\[
\langle K|C_0 \cdots C_{\lambda_r}|\varnothing\rangle = \langle K|F^{-1}FC_0F^{-1}F \cdots C_{\lambda_r}F^{-1}|\varnothing\rangle = \langle K|F^{-1}\tilde{C}_0 \cdots \tilde{C}_{\lambda_r}|\varnothing\rangle
\]

where we have used (4.4) and introduced the notation \( \tilde{C}_i := FC_iF^{-1} \) for a new family of column operators, the so-called “twisted” column operators. The Boltzmann weights for the column operators \( C_i \) are no longer local, but rather depend on the configurations above and below it in the same column. Nevertheless, the weights may be given explicitly in Figure 7.

Here the Boltzmann weight in the second column of the table depends on a set \( A_{j, i} \), defined as the set of indices \( k \neq j \) in \( I \) such that the vertex \( v_{k, i} \) in row \( k \) and column \( i \) is of the type appearing in the rightmost column of Figure 7 (i.e., those vertices where a path enters from the left and moves downward). This set is the only way in which the above Boltzmann weights fail to be local. The interested reader can find an example of the weight associated to a particular configuration in rank 3 in Figure 8.

Thus we’ve reduced the calculation of the partition function for general rank to a pair of computations according to (4.6), a computation of \( \langle K|F^{-1} \) (graphically represented as a set of braids attached to bend
vertices) and \( \tilde{C}_0 \cdots \tilde{C}_{\lambda_r} | \varnothing \) (graphically represented as a partition function consisting entirely of tetravalent vertices in the rectangular lattice). As we will see in subsequent sections, explicit formulas for the former are very dependent on the choice of weights for the bend vertices in a given rank, while the latter may be evaluated in general. In particular, we will see that \( \langle K | F^{-1} \rangle \) consists of partition functions whose right-hand boundary has paths exiting in each row of the top half of the model and no paths in rows in the bottom half of the model; for ease of notation, we will use \( \hat{u} \) to denote the vector with this occupancy, so that for example in rank three we have

\[
\hat{u} = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & & \bullet \end{pmatrix}.
\]

Since the states in \( \langle K | F^{-1} \rangle \) must pair with those in \( \tilde{C}_0 \cdots \tilde{C}_{\lambda_r} | \varnothing \), this greatly reduces the number of cases we must consider in the latter ket (i.e., the possible left-hand boundaries resulting from this ket). The right-hand side of Figure 8 illustrates one such non-zero configuration in this process.

As it will be useful in subsequent sections, we will now compute the term \( \tilde{C}_0 \cdots \tilde{C}_{\lambda_r} | \varnothing \).

**Theorem 4.1.** Let \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_r) \) be any partition. Then

\[
\langle \hat{u} | \tilde{C}_0 \cdots \tilde{C}_{\lambda_r} | \varnothing \rangle = \sum_{\sigma \in S_r / S_r^\lambda} y_{\sigma(1)}^{\lambda_1+1} \cdots y_{\sigma(r)}^{\lambda_r+1} \prod_{i,j: \lambda_j > \lambda_i} \frac{y_{\sigma(j)} - qy_{\sigma(i)}}{y_{\sigma(j)} - y_{\sigma(i)}},
\]

where \( S_r^\lambda \) is the subgroup of \( S_r \) that stabilizes \( \lambda \) and \( y_1, \ldots, y_r \) denote the spectral parameters for the top \( r \) rows of the lattice (in descending order).

**Proof.** To begin, consider the path of a given particle in a given state of \( \langle \hat{u} | \tilde{C}_0 \cdots \tilde{C}_{\lambda_r} | \varnothing \rangle \). This particle emanates from the bottom boundary along column \( \lambda_1 \), and, because the Boltzmann weight in the third column of Figure 7 is 0, makes a single turn to exit via the left boundary along row \( j \), where \( j \) is one of
the top \( r \) rows of the lattice. Since any of the \( r \) particles can exit via the left boundary of any of the top \( r \) rows, we can see that there is a bijection between states of \( \langle \bar{u} \rangle \mathbf{C}_0 \cdots \mathbf{C}_{\lambda_c} | \emptyset \rangle \) and elements of \( S_r/S_r^\lambda \). As an example, consider the state below:

\[
\begin{pmatrix}
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\bar{u} & \bar{v} & \bar{w} & \bar{x} \\
\bar{v} & \bar{w} & \bar{x} & \bar{y} \\
\bar{w} & \bar{x} & \bar{y} & \bar{z} \\
\bar{x} & \bar{y} & \bar{z} & \bar{t}
\end{array}
\end{pmatrix}
\]

If we instead consider the state where the particle emanating from column \( \lambda_i \) exits along row \( \sigma(i) \) (rather than row \( i \)), for a given \( \sigma \in S_r/S_r^\lambda \), then we see that the Boltzmann weight for this state is \( y_{\sigma(1)} \cdots y_{\sigma(r)} \prod_{i,j:|\lambda_j|>\lambda_i} \frac{y_j - qy_i}{y_j - y_i} \). Summing over all possible states, we see that

\[
\langle \bar{u} \rangle \mathbf{C}_0 \cdots \mathbf{C}_{\lambda_c} | \emptyset \rangle = \sum_{\sigma \in S_r/S_r^\lambda} y_{\sigma(1)} \cdots y_{\sigma(r)} \prod_{i,j:|\lambda_j|>\lambda_i} \frac{y_j - qy_i}{y_j - y_i}.
\]

Following [24] (see also [23]), we note that \( \langle \bar{u} \rangle \mathbf{C}_0 \cdots \mathbf{C}_{\lambda_c} | \emptyset \rangle \) can be expressed as a sum over \( S_r \):

\[
\langle \bar{u} \rangle \mathbf{C}_0 \cdots \mathbf{C}_{\lambda_c} | \emptyset \rangle = \frac{1}{c_\lambda(q)} \sum_{\sigma \in S_r} y_{\sigma(1)} \cdots y_{\sigma(r)} \prod_{i,j:|\lambda_j|>\lambda_i} \frac{y_j - qy_i}{y_j - y_i},
\]

where \( c_\lambda(q) = \prod_{i=1}^{r} \prod_{\lambda_j \geq \lambda_i} \frac{1}{1-q^{\lambda_j}} \) and \( m_i(\lambda) \) is the number of \( \lambda_j \) equal to \( i \) for each \( i \geq 0 \).

Before proceeding to evaluate the partition functions for various ranks under the scheme described above, we first include a result from [24] concerning the partition function that arises from the bend weight values \( C_j = 1, B_j = -q, \) and \( A_j = D_j = 0 \).

**Theorem 4.2** (Wheeler-Zinn-Justin, Thm. 3 and Rmk. 5 in [24]). Let \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r) \) be a partition. With rectangular Boltzmann weights as in Figure 2 and bend weights \( C_j = 1, B_j = -q, \) and \( A_j = D_j = 0 \) we have,

\[
Z(B_\lambda) = c_\lambda(q)^{-1} \prod_{i=1}^{r} x_i \cdot \prod_{\alpha \in \Phi^+ \setminus \alpha_{\text{short}}} (1 - qx^{-\alpha}) \cdot \sum_{w \in W} \left( \prod_{i=1}^{r} x_i^{\lambda_i} \cdot \prod_{\alpha \in \Phi^+ \setminus \alpha_{\text{long}}} \frac{1}{1-x^{-\alpha}} \cdot \prod_{\alpha \in \Phi^+} (1 - qx^{-\alpha}) \right),
\]

with \( c_\lambda(q) \) defined as above.

This result is the special case of (64) in [24] where two parameters that determine weights associated to certain “bonus columns” are both set to 0; for further context, see Section 1.

5. Rank One Solvable Models

A rank one lattice consists of a single pair of rows, connected by a bend. In these lattices, our family of boundary conditions permits only a single particle to enter along the bottom and exit at the bend. Since the caduceus relation is not relevant to this case, we will say that a model is “rank one solvable” if it satisfies the Yang-Baxter equation and Fish relation. Lemma 3.1 gives the necessary conditions, reducing to four possible cases of solvable rank one models. We continue to use the abbreviated notation \( B_j(x) \) for \( B_j(x, \bar{x}) \), etc., in the following results.

**Theorem 5.1.** Let the rank \( r = 1 \) and let \( \lambda = (\lambda_1) \) be a one-part partition. If \( B_\lambda \) is a solvable rank one model, then

1. if \( B_1(x) = C_1(x) \), we have \( Z(B_\lambda) = C_1(x_1) \sum_{w \in W_0} w \left( x_1^{\lambda_1+1} \cdot \frac{1 - qx^{-\alpha_1}}{1 - x^{-\alpha_1}} \right) \);

2. if \( B_1(x) = -qC_1(x) \), we have \( Z(B_\lambda) = C_1 \left( \frac{1 - qx^{-\alpha_1}}{1 - x^{-\alpha_1}} \right) \sum_{w \in W_0} (-1)^{\ell(w)} w \left( x_1^{\lambda_1+1} \right) \);
Lemma 3.1), then
\[ W_2 \frac{j}{w} \]

We’ll prove this result by verifying that both components of \( \mathcal{Z}(\mathcal{B}_\lambda) \) and \( \mathcal{Z}(\mathcal{B}_\lambda) = x_1 C_1(x_1) \sum_{w \in W_0} w \left( x_1^\lambda, \frac{1 - q x^{-\alpha_i}}{1 - x^{-\alpha_i}} \right) \).

Here \( B_1(x) \) and \( C_1(x) \) refer to bend weights as in Figure 6. Moreover, \( W_0 \) denotes the Weyl group in rank one, so two elements with \( \ell(w) \) the length.

This may seem like a grandiose way of expressing the sum of two terms, but we mean to suggest expressions that will reappear in higher rank in subsequent sections. In particular, the reader can see that rank one solvable models with u-turn bend split into two families according to the constant of proportionality \( F \) appearing in the Fish relation (Lemma 3.1). Members of the family all give, up to a simple explicit factor, one of the two orthogonal polynomials of Macdonald associated to characters of this two-element \( W_0 \) described in the introduction. Cases (1a) and (2b) have \( F = 1 \) and give rise to the symmetric Hall-Littlewood polynomials and cases (1b) and (2a) have \( F = (x^2 - q)/(1 - qx^2) \) and give the \( q \)-antisymmetric deformations of Weyl’s character formula. The proof requires the following simple lemma according to the values of \( F \), and written in the language of our general approach detailed in the previous section.

**Lemma 5.2.** In rank one, if the Fish constant of proportionality is \( F = 1 \) (i.e., in cases (1a) and (2b) of Lemma 3.1), then
\[
\langle K | F^{-1} = \left[ C_1(x_1) \frac{x_1 - q T_1}{x_1 - T_1} \right] \cdot (1 \otimes (-1)_T) + \left[ C_1(x_1) \frac{\bar{x}_1 - q x_1}{x_1 - x_1} \cdot \begin{cases} 1 & \text{if } B_1(x) = C_1(x) \\ x_1^2 & \text{if } B_1(x) = x_1 C_1(x) \end{cases} \right] \cdot (1 \otimes 1)_T.
\]

If instead \( F = (x^2 - q)/(1 - qx^2) \) (i.e., in cases (1b) and (2a) of Lemma 3.1), then
\[
\langle K | F^{-1} = \left[ C_1(x_1) \frac{x_1 - q T_1}{x_1 - T_1} \right] \cdot (1 \otimes (-1)_T) + \left[ C_1(x_1) \frac{\bar{x}_1 - q x_1}{x_1 - x_1} \cdot \begin{cases} -1 & \text{if } B_1(x) = -q C_1(x) \\ -x_1^2 & \text{if } B_1(x) = -x_1 C_1(x) \end{cases} \right] \cdot (1 \otimes 1)_T.
\]

Here
\[
\langle 1 \otimes (-1)_T = 1 \quad \text{and} \quad (-1)_T \otimes 1 = 1 \quad \text{both particles at } x_1.
\]

**Proof.** We’ll prove this result by verifying that both components of \( \langle K | F^{-1} \) have the form given on the right of (5.1). Suppose that \( j_1, j_2 \in \{0, 1\} \), where 1 denotes the presence of a particle at a vertex, and 0 denotes the lack of a particle. In the case \( j_1 = 1 \) and \( j_2 = 0 \), by (4.2) and (4.3), we have
\[
\langle (K | F^{-1})^{j_1 j_2} = \left( \langle K | R_{11}^{11} \cdot \Delta_{11}^{-1}(x_1, x_1) \right)^{j_1 j_2} = \left( \langle K | R_{11}^{11} \right)^{j_1 j_2} \cdot b_{j_1, j_2}(x_1, x_1) = \left( \langle K | R_{11}^{11} \right)^{j_1 j_2} \cdot x_1 - q T_1 \over x_1 - T_1.
\]

Since
\[
\langle (K | R_{11}^{11})^{j_1 j_2} = \text{wt} \left( 1 \quad \text{both particles at } x_1 \right) = C_1(x_1),
\]
we have \( \langle (K | F^{-1})^{j_1 j_2} = C_1(x_1) \frac{x_1 - q T_1}{x_1 - T_1} \).

\(^2\)We say “\( q \)-antisymmetric” as setting \( q = 1 \) gives the usual antisymmetric alternator. This is precisely the family of orthogonal polynomials associated to the sign character of \( W_0 \) by Macdonald. As written, case (1b) exactly matches the Casselman-Shalika formula for a \( p \)-adic Whittaker function and recovers the Weyl character formula at \( q = 0 \).
On the other hand, when \( j_1 = 0 \) and \( j_1 = 1 \), we have

\[
((K|R_{11}^{11}))^{j_1j_1} = Z \begin{pmatrix} 1 & \bullet & \circ & \bar{x}_1 \\ \circ & \circ & \circ & 1 \end{pmatrix} = \text{wt} \begin{pmatrix} 1 & \bullet & \circ & \bar{x}_1 \\ \circ & \circ & \circ & 1 \end{pmatrix} + \text{wt} \begin{pmatrix} 1 & \bullet & \circ & \bar{x}_1 \\ \circ & \circ & \circ & 1 \end{pmatrix} = C_1(x_1) \begin{cases} 1 & \text{if } B_1(x) = C_1(x), \\ x_1^2 & \text{if } B_1(x) = qx^2C_1(x), \end{cases}
\]

and hence evaluating \((K|F^{-1})^{j_1j_1} = ((K|R_{11}^{11}))^{j_1j_1} \cdot B_{j_1j_1}(\bar{x}_1, x_1)\) gives the desired result. We leave the cases with constant of proportionality \( F = (x^2 - q)/(1 - qx^2) \) by similar computation to the reader. □

\textbf{Proof of Theorem 5.1.} We prove cases (1a) and (2b), noting that Case (1b) follows from results of Wheeler and Zinn-Justin (c.f. [24] Section 3.3 Remark 5). Case (2a) is similar and left to the reader.

By (4.1) and (4.6), we have \( Z(\mathcal{B}_\lambda) = (K|F^{-1}\bar{C}_0 \cdots \bar{C}_{\lambda_r} |\emptyset) \). Note that the rank one case of Theorem 4.1 gives \( \langle x | C_0 \cdots C_{\lambda_r} |\emptyset \rangle = y_1^{\lambda_1+1} \), where \( y_1 \) represents the spectral parameter for the particle entering on the top row of the lattice. Combining this with Lemma 5.2 we see for example that when \( B_1(x) = C_1(x) \),

\[
Z(\mathcal{B}_\lambda) = (K|F^{-1}\bar{C}_0 \cdots \bar{C}_{\lambda_r} |\emptyset) = C_1(x_1) \frac{x_1 - qx_1}{x_1 - \bar{x}_1} \cdot \lambda_1^{\lambda_1+1} + C_1(x_1) \frac{\bar{x}_1 - qx_1}{\bar{x}_1 - x_1} \cdot \lambda_1^{\lambda_1+1}.
\]

The case of \( B_1(x) = qx^2C_1(x) \) follows by a similarly straightforward substitution. □

\section{6. Rank Two Solvable Models}

If we restrict ourselves to the rank two case, then the general caduceus relation has fewer cases. Indeed, the only case of the caduceus we need to treat is the case with two particles along the boundary, namely Lemma [3.2] owing to the fact that our family of boundary conditions in rank two permits only two particles to enter along the bottom and exit at the bends. Parroting our definition from the prior section for this special case, we record the following:

\textbf{Definition 6.1.} A type \( B/C \) lattice model with two pairs of rows is said to be \textbf{rank two solvable} if it satisfies the Yang-Baxter equation, is uniform regime in the Fish relation (Lemma 3.2), and satisfies the two particle Caduceus relation of Lemma 3.2.

One can systematically treat each subcase from Lemma 3.1 and the two particle Caduceus relation, and the resulting partition functions have uniform expressions within each subcase, much as we saw for rank one models in the prior section. For example, subcase (1b) of Lemma 3.1 rules out double bend configurations (i.e., \( A_j = D_j = 0 \)) and all such weights lead to variants of the Wheeler-Zinn-Justin model [24]. For this reason, we will not consider the partition functions in this case here. Instead we focus on Subcases (1a) and (2b) of Lemma 3.1 where the Fish and Caduceus constants \( F \) are equal to one, leading to partition functions symmetric under the action of the hyperoctahedral group. (In this result, recall the definition of \( s_2 \) from Equation 3.10.)

\textbf{Theorem 6.1.} Let rank \( r = 2 \), and let \( \lambda = (\lambda_1 \geq \lambda_2) \) be a partition. Suppose our Boltzmann weights are from a solvable, type \( B/C \) lattice model for the trigonometric six-vertex model with \( R \)-weights as in [3]. Suppose further that the uniform Fish relation constant of proportionality is one; that is, assume either uniform in regime 1(a) (so that \( C_i = B_i \) for \( i \in \{1,2\} \)) or uniform in regime 2(b) (so that \( C_i = m_i \bar{x} \) for
Following the approach taken in [24], we'll verify that each component of \( \mathbb{F} \) in (6.1) does not need to keep track of the associated spectral parameters in the above expression. We have chosen to write (6.1) on the right of (6.1). To begin, then, suppose that \( i \in \{1, 2\} \). Then one has

\[
\mathbb{F}(x_i, \pi_i) = \frac{\kappa}{c \lambda(q)} \sum_{w \in W} w \left( x_1^{\lambda_1+1}, x_2^{\lambda_2+1}, \prod_{a \in \Phi^+} \frac{1-q x_{-a}}{1-x_{-a}} \right) + A_1 D_2 \left( x_1^{\lambda_1+1}, x_2^{\lambda_2+1}, \prod_{a \in \Phi^+} \frac{1-q x_{-a}}{1-x_{-a}} \right)
\]

where we have written \( c \lambda(q) = \left\{ \begin{array}{ll} 1 & \text{if } \lambda_1 > \lambda_2 \\ 1 + q & \text{if } \lambda_1 = \lambda_2 \end{array} \right\} \), \( \delta_i = \left\{ \begin{array}{ll} 1 & \text{in regime 1(a)} \\ 0 & \text{in regime 2(b)} \end{array} \right\} \) for \( i \in \{1, 2\} \), and \( \kappa = \left\{ \begin{array}{ll} C_1 C_2 & \text{in regime 1(a)} \\ m_1 m_2 & \text{in regime 2(b)} \end{array} \right\} \).

In order to prove Theorem 6.1, we will follow the approach outlined in the previous section: first we will compute \( (K| \mathbb{F}^{-1}) \) (in Lemma 6.2), and then we will combine this with Theorem 4.1 to complete the evaluation.

**Lemma 6.2.** Let rank \( r = 2 \), and suppose our Boltzmann weights are from a solvable, type B/C lattice model for the trigonometric six-vertex model with R-weights as in [3]. Suppose further that the uniform Fish relation constant of proportionality is one. Then, with notation as in Section 4 (\( K \) is uniform), \( (K| \mathbb{F}^{-1}) = \sum_{\{\epsilon_1, \epsilon_2\} \in \{\pm 1\}^2} \left( C_1(x_1^{\epsilon_1}) C_2(x_2^{\epsilon_2}) + A_1(x_1^{\epsilon_1}) D_2(x_2^{\epsilon_2}) \cdot \prod_{\{\epsilon_1, \epsilon_2\} \in \{\pm 1\}^2} \right) \cdot \left( 1-q x_{-1} \right) \cdot \left( \epsilon_1 \right) \cdot \left( \epsilon_2 \right) \cdot \left( \epsilon_1 \right) \cdot \left( \epsilon_2 \right)
\]

where

\[
\langle 1 |_{\pi_i} \rangle = \left( \begin{array}{c} x_i \\ \pi_i \end{array} \right), \quad \langle -1 |_{\pi_i} \rangle = \left( \begin{array}{c} x_i \\ \pi_i \end{array} \right), \quad \langle 1 |_{\pi_i} \rangle \cdot \langle 1 |_{\pi_i} \rangle = \left( \begin{array}{c} x_i \\ \pi_i \end{array} \right), \quad \langle 1 |_{\pi_i} \rangle \cdot \langle 1 |_{\pi_i} \rangle = \left( \begin{array}{c} x_i \\ \pi_i \end{array} \right)
\]

Note that the assumption on solvability with Fish constant one ensures that \( A_1 \) and \( D_2 \) have degree 0, so there’s no need to keep track of the associated spectral parameters in the above expression. We have chosen to retain them to track the contributions from each permutation in the proof below. However, we do make use of the independence of spectral parameter in the subsequent proof of Theorem 6.1.

**Proof.** Following the approach taken in [24], we’ll verify that each component of \( (K| \mathbb{F}^{-1}) \) (in Lemma 6.2) is uniform, and that each component of \( (K| \mathbb{F}^{-1}) \) on the right of (6.1). To begin, then, suppose that \( j_1, j_2, j_3, j_4 \in \{0, 1\} \), where 1 denotes the presence of a particle at a vertex, and 0 denotes the lack of a particle. By (4.2) and (4.3), we have

\[
\langle (K| \mathbb{F}^{-1}) \rangle_{j_2 j_3 j_4} = \langle (K| \mathbb{F}^{-1}) \rangle_{j_2 j_3 j_4} \cdot \prod_{k<l} b^{-1}_{j_{kl}}(x_k, x_l)
\]

20
where \( \rho \) is a permutation of \( \{1, 1, \overline{2}, 2\} \) such that \( j_{\rho(1)} = j_{\rho(\overline{1})} = 0 \) and \( j_{\rho(2)} = j_{\rho(\overline{2})} = 1 \).

Since the \( R \)-matrix weights \( a_1(k, j) \) and \( a_2(k, j) \) are both equal to 1, it suffices to consider permutations \( \rho \) such that \( \rho(2) \geq \rho(\overline{2}) \) and \( \rho(1) \geq \rho(\overline{1}) \). This leaves us with six cases: \( \rho = (2, \overline{2}, 1, \overline{1}), (2, 1, \overline{2}, \overline{1}), (\overline{2}, 1, 2, \overline{1}), (\overline{2}, 1, 2, 1), (1, \overline{1}, 2, \overline{1}). \) Recall that we are writing the permutation \( \rho \) in “one-line” notation where the components indicate the respective images of \( (2, \overline{2}, 1, \overline{1}). \)

First, suppose that \( \rho = (2, \overline{2}, 1, \overline{1}) \) (and hence \( j_2 = j_1 = 1 \), and \( j_1 = j_\overline{1} = 0 \)). Then \( (\overline{K} | R_{2211}^\rho)^{j_2 j_\overline{2} j_1 j_\overline{1}} = A_1(x_1) D_2(x_2); \) diagrammatically, we have

\[
Z \begin{pmatrix} 2 & \vdots & x_2 \\ 1 & \vdots & x_1 \\ \overline{x}_2 & \vdots & \overline{x}_1 \end{pmatrix} = \text{wt} \begin{pmatrix} 2 & \vdots & x_2 \\ 1 & \vdots & x_1 \\ \overline{x}_2 & \vdots & \overline{x}_1 \end{pmatrix} = A_1(x_1) D_2(x_2).
\]

Hence,

\[
(\overline{K} | F^{-1})^{j_2 j_\overline{2} j_1 j_\overline{1}} = A_1(x_1) D_2(x_2) \cdot \prod_{k,l \in \{1, \overline{1}, \overline{2}, 2\}} b_{j_k,j_l}^{-1}(x_k, x_l) = A_1(x_1) D_2(x_2) \cdot \prod_{\{x_1, x_2\} \in \{1, \overline{1}\}} x_2^q - x_1^q
\]

Next, suppose that \( \rho = (2, 1, \overline{2}, \overline{1}) \) (so \( j_2 = j_1 = 1 \), and \( j_\overline{2} = j_\overline{1} = 0 \)). Note that

\[
(6.2) \prod_{k,l \in \{1, \overline{1}, \overline{2}, 2\}} b_{j_k,j_l}^{-1}(x_k, x_l) = \prod_{k,l \in \{\rho(2), \rho(\overline{2})\}} \frac{x_k - q \overline{x}_l}{x_k - \overline{x}_l} \quad \text{when} \quad \rho(2) \neq \rho(\overline{2}),
\]

so that, in these cases,

\[
(6.3) \quad (\overline{K} | F^{-1})^{j_2 j_\overline{2} j_1 j_\overline{1}} = (\overline{K} | R_{2211}^\rho)^{j_2 j_\overline{2} j_1 j_\overline{1}} \cdot \prod_{k,l \in \{\rho(2), \rho(\overline{2})\}} \frac{x_k - q \overline{x}_l}{x_k - \overline{x}_l}.
\]

We have

\[
(\overline{K} | R_{2211}^\rho)^{j_2 j_\overline{2} j_1 j_\overline{1}} = Z \begin{pmatrix} 2 & \vdots & x_2 \\ 1 & \vdots & x_1 \\ \overline{x}_2 & \vdots & \overline{x}_1 \end{pmatrix} = \text{wt} \begin{pmatrix} 2 & \vdots & x_2 \\ 1 & \vdots & x_1 \\ \overline{x}_2 & \vdots & \overline{x}_1 \end{pmatrix} + \text{wt} \begin{pmatrix} 2 & \vdots & x_2 \\ 1 & \vdots & x_1 \\ \overline{x}_2 & \vdots & \overline{x}_1 \end{pmatrix}.
\]

Hence,

\[
(\overline{K} | F^{-1})^{j_2 j_\overline{2} j_1 j_\overline{1}} = (\overline{K} | R_{2211}^\rho)^{j_2 j_\overline{2} j_1 j_\overline{1}} \cdot \prod_{k,l \in \{2, 1\}} x_k - q \overline{x}_l \overline{x}_l - \overline{x}_k

= \left(C_1(x_1) C_2(x_2) \cdot \frac{x_1 - \overline{x}_2}{x_1 - q \overline{x}_2} + A_1(x_1) D_2(x_2) \cdot \frac{(1 - q) \overline{x}_2}{x_1 - q \overline{x}_2} \right).
\]

For \( \rho = (2, 1, \overline{2}, 1) \), consider

\[
(\overline{K} | R_{2211}^\rho)^{j_2 j_\overline{2} j_1 j_\overline{1}} = Z \begin{pmatrix} 2 & \vdots & x_2 \\ 1 & \vdots & x_1 \\ \overline{x}_2 & \vdots & \overline{x}_1 \end{pmatrix} = Z \begin{pmatrix} 2 & \vdots & x_2 \\ 1 & \vdots & x_1 \\ \overline{x}_2 & \vdots & \overline{x}_1 \end{pmatrix},
\]

where the second equality comes from the fish equation, noting that the associated constant of proportionality is assumed to be one. Applying the same analysis as in the \( \rho = (2, 1, \overline{2}, 1) \) case, we see that
\[
((\mathbf{K}|F^{-1})_{ij}^{j_1j_2j_3j_4} = ((\mathbf{K}|R_{2211}^{\rho})_{ij}^{j_1j_2j_3j_4} \cdot \prod_{k,l \in \{2,1,1\}} \frac{x_k - q_{l}}{x_k - \overline{x}_l} \\
= \left( C_1(\overline{x}_1)C_2(x_2) \cdot \frac{x_1 - \overline{x}_2}{x_1 - q_{x_2}} + A_1(\overline{x}_1)D_2(x_2) \cdot \frac{(1 - q_{x_2})\overline{x}_2}{x_1 - q_{x_2}} \right) \cdot \prod_{k,l \in \{2,1,1\}} \frac{x_k - q_{l}}{x_k - \overline{x}_l} \\
= \left( C_1(\overline{x}_1)C_2(x_2) + A_1(\overline{x}_1)D_2(x_2) \cdot \frac{(1 - q_{x_2})\overline{x}_2}{x_1 - \overline{x}_2} \right) \cdot \frac{x_1 - \overline{x}_2}{x_1 - q_{x_2}} + \frac{x_2 - q_{x_2}}{x_2 - \overline{x}_2} \cdot \frac{x_2 - q_{x_2}}{x_2 - x_1}.
\]

The cases where \(\rho = (2,1,2,1)\) and \(\rho = (2,1,1,2)\) can be checked in a very similar manner to the method used in the previous case, using one and two instances of the fish relation, respectively. In the case of \(\rho = (2,1,2,1)\), we find
\[
((\mathbf{K}|R_{2211}^{\rho})_{ij}^{j_1j_2j_3j_4} = \left( C_1(x_1)C_2(\overline{x}_2) \cdot \frac{x_1 - x_2}{x_1 - q_{x_2}} + A_1(x_1)D_2(\overline{x}_2) \cdot \frac{(1 - q_{x_2})x_2}{x_1 - q_{x_2}} \right),
\]
while for \(\rho = (2,1,1,2)\), we obtain
\[
((\mathbf{K}|R_{2211}^{\rho})_{ij}^{j_1j_2j_3j_4} = \left( C_1(\overline{x}_1)C_2(x_2) \cdot \frac{x_1 - x_2}{x_1 - q_{x_2}} + A_1(\overline{x}_1)D_2(x_2) \cdot \frac{(1 - q_{x_2})x_2}{x_1 - q_{x_2}} \right),
\]
and their overall contribution can again be computed as in (6.3). Finally, suppose that \(\rho = (1,1,2,2)\), then, by the Caduceus relation for two particles (Lemma 3.2), we find
\[
\mathcal{Z} \left( \begin{array}{cc} 2 \\ 1 \end{array} \right) \left( \begin{array}{cc} x_1 \\ \overline{x}_1 \\ x_2 \\ \overline{x}_2 \end{array} \right) = \mathcal{Z} \left( \begin{array}{cc} 2 \\ 1 \end{array} \right) \left( \begin{array}{cc} x_1 \\ \overline{x}_1 \\ x_2 \\ \overline{x}_2 \end{array} \right) = \text{wt} \left( \begin{array}{cc} 2 \\ 1 \end{array} \right) \left( \begin{array}{cc} x_1 \\ \overline{x}_1 \\ x_2 \\ \overline{x}_2 \end{array} \right) = A_1(x_2)D_2(x_1).
\]

Note in particular that the caduceus constant of proportionality is also forced to be one, by our assumption on the Fish constants of proportionality. With the spectral parameters swapped as a consequence of the Caduceus relation, we have
\[
((\mathbf{K}|F^{-1})_{ij}^{j_1j_2j_3j_4} = A_1(x_2)D_2(x_1) \cdot \prod_{k,l \in \{1,1,1,2\}} b_{jk}^{-1}(x_k, x_l) = A_1(x_2)D_2(x_1) \cdot \prod_{\{i_1, i_2\} \in \{\pm 1\}^2} \frac{x_1^{i_1} - q_{x_2^{i_2}}}{x_1^{i_1} - x_2^{i_2}}. \quad \square
\]

**Proof of Theorem 6.1.** First, by (4.1) and (4.6), we have \(Z(\mathcal{E}_\lambda) = (\mathbf{K}|F^{-1}\mathcal{C}_0 \cdots \mathcal{C}_\lambda \ | \emptyset)\). In rank 2, Theorem 4.1 tells us that
\[
(\hat{u} | \mathcal{C}_0 \cdots \mathcal{C}_\lambda \ | \emptyset) = \sum_{\sigma \in S_2} y_{\sigma(1)}^{\lambda_1} y_{\sigma(2)}^{\lambda_2} \cdot \sum_{\sigma_{(1)} = \emptyset, \sigma_{(2)} = \emptyset} \frac{y_{\sigma(1)} - y_{\sigma(2)}}{y_{\sigma(1)} - y_{\sigma(2)}} \quad \text{if } \lambda_1 > \lambda_2
\]
\[
\quad \sum_{\sigma \in S_2} y_{\sigma(1)}^{\lambda_1} y_{\sigma(2)}^{\lambda_2} \cdot \sum_{\sigma_{(1)} = \emptyset, \sigma_{(2)} = \emptyset} \frac{y_{\sigma(1)} - y_{\sigma(2)}}{y_{\sigma(1)} - y_{\sigma(2)}} \quad \text{if } \lambda_1 = \lambda_2,
\]
where \(y_1\) represents the spectral parameter for the particle entering on the top row of the lattice, and \(y_2\) represents the spectral parameter particle entering on the second row from the top. Combining this with Lemma 6.2 in the case where \(\lambda_1 \neq \lambda_2\), we see that \(Z(\mathcal{E}_\lambda) = (\mathbf{K}|F^{-1}\mathcal{C}_0 \cdots \mathcal{C}_\lambda \ | \emptyset)\) equals
\[
\sum_{\{i_1, i_2\} \in \{\pm 1\}^2} \left[ \left( \frac{x_1^{i_1} - q_{x_1^{i_2}}}{x_1^{i_1} - \overline{x}_1^{i_2}} \cdot \frac{x_2^{i_2} - q_{x_2^{i_1}}}{x_2^{i_2} - \overline{x}_2^{i_1}} \cdot \frac{x_2^{i_2} - q_{x_2^{i_1}}}{x_2^{i_2} - x_1^{i_1}} \cdot \left( C_1(x_1^{i_1})C_2(x_2^{i_2}) + A_1D_2 \cdot \frac{(1 - q_{x_2^{i_1}})\overline{x}_2^{i_2}}{x_1^{i_1} - \overline{x}_2^{i_2}} \right) \right) \cdot \left( x_1^{i_1, \lambda_1} x_2^{i_2, \lambda_2} + x_1^{i_1} x_2^{i_2} \right) \cdot \left( x_1^{i_1} x_2^{i_2} - q_{x_2^{i_1}} - q_{x_2^{i_2}} \overline{x}_1^{i_2} \overline{x}_2^{i_1} \right) \right] \\
+ \sum_{\{i_1, i_2\} \in \{\pm 1\}^2} \left[ \left( \frac{x_1^{i_1} - q_{x_1^{i_2}}}{x_1^{i_1} - \overline{x}_1^{i_2}} \cdot \frac{x_2^{i_2} - q_{x_2^{i_1}}}{x_2^{i_2} - \overline{x}_2^{i_1}} \cdot \frac{x_2^{i_2} - q_{x_2^{i_1}}}{x_2^{i_2} - x_1^{i_1}} \cdot \left( C_1(x_1^{i_1})C_2(x_2^{i_2}) + A_1D_2 \cdot \frac{(1 - q_{x_2^{i_1}})\overline{x}_2^{i_2}}{x_1^{i_1} - \overline{x}_2^{i_2}} \right) \right) \cdot \left( x_1^{i_1, \lambda_1} x_2^{i_2, \lambda_2} + x_1^{i_1} x_2^{i_2} \right) \cdot \left( x_1^{i_1} x_2^{i_2} - q_{x_2^{i_1}} - q_{x_2^{i_2}} \overline{x}_1^{i_2} \overline{x}_2^{i_1} \right) \right] \\
+ \sum_{\{i_1, i_2\} \in \{\pm 1\}^2} \left[ \left( \frac{x_1^{i_1} - q_{x_1^{i_2}}}{x_1^{i_1} - \overline{x}_1^{i_2}} \cdot \frac{x_2^{i_2} - q_{x_2^{i_1}}}{x_2^{i_2} - \overline{x}_2^{i_1}} \cdot \frac{x_2^{i_2} - q_{x_2^{i_1}}}{x_2^{i_2} - x_1^{i_1}} \cdot \left( C_1(x_1^{i_1})C_2(x_2^{i_2}) + A_1D_2 \cdot \frac{(1 - q_{x_2^{i_1}})\overline{x}_2^{i_2}}{x_1^{i_1} - \overline{x}_2^{i_2}} \right) \right) \cdot \left( x_1^{i_1, \lambda_1} x_2^{i_2, \lambda_2} + x_1^{i_1} x_2^{i_2} \right) \cdot \left( x_1^{i_1} x_2^{i_2} - q_{x_2^{i_1}} - q_{x_2^{i_2}} \overline{x}_1^{i_2} \overline{x}_2^{i_1} \right) \right],
\]

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where we have used the fact that when the Fish constant of proportionality is one, so we are uniform of case 1(a) or 2(b) in Lemma 3.1 then \( \deg(A_1) = \deg(D_2) = 0 \) according to Lemma 3.2 and so we may drop the spectral dependence (i.e., the dependence on \( x_i \)) in the statement of Lemma 0.2.

To analyze the case where \( \lambda_1 \neq \lambda_2 \) further, we now separate the calculation into cases – that where the fish relation is uniform of case 1(a) and of 2(b). In case 1(a), Lemma 3.2 ensures \( C_i \) is independent of spectral parameter, so we may naturally write the partition function above as

\[
Z(B_\lambda) = C_1 C_2 \sum_{w \in W} w \left( x_1^{\lambda_1+1} x_2^{\lambda_2+1}, \prod_{\alpha \in \Phi^+} \frac{1 - q x^{-\alpha}}{1 - x^{-\alpha}} \right) + A_1 D_2 \left[ \sum_{w \in W} w \left( x_1^{\lambda_1+1} x_2^{\lambda_2+1}, \frac{x_1 - q x_2}{x_1 - x_2}, \frac{x_2 - q x_1}{x_2 - x_1}, (1 - q) x_2, \frac{x_1 - q x_1}{x_1 - x_1}, \frac{x_2 - q x_2}{x_2 - x_2} \right) \right] + \sum_{w \in W \setminus \langle s_2 \rangle} w \left( x_1^{\lambda_1-\lambda_2}, \frac{x_1 - q x_2}{x_1 - x_2}, \frac{x_1 - q x_1}{x_1 - x_1}, x_2 - q x_2, \frac{x_1 - q x_2}{x_1 - x_2}, \frac{x_1 - q x_1}{x_1 - x_1} \right) \right].
\]

If instead we are uniform of case 2(b), recall that Lemma 3.2 ensures that \( C_i(x) = m_i \bar{x} \) for some \( m_i \in \mathbb{C}[q] \). One may then check that:

\[
Z(B_\lambda) = m_1 m_2 \sum_{w \in W} w \left( x_1^{\lambda_1} x_2^{\lambda_2}, \prod_{\alpha \in \Phi^+} \frac{1 - q x^{-\alpha}}{1 - x^{-\alpha}} \right) + A_1 D_2 \left[ \sum_{w \in W} w \left( x_1^{\lambda_1+1} x_2^{\lambda_2+1}, \frac{x_1 - q x_2}{x_1 - x_2}, \frac{x_2 - q x_1}{x_2 - x_1}, (1 - q) x_2, \frac{x_1 - q x_1}{x_1 - x_1}, \frac{x_2 - q x_2}{x_2 - x_2} \right) \right] + \sum_{w \in W \setminus \langle s_2 \rangle} w \left( x_1^{\lambda_1-\lambda_2}, \frac{x_1 - q x_2}{x_1 - x_2}, \frac{x_1 - q x_1}{x_1 - x_1}, x_2 - q x_2, \frac{x_1 - q x_2}{x_1 - x_2}, \frac{x_1 - q x_1}{x_1 - x_1} \right) \right].
\]

Case 2(b) is handled similarly where again \( C_i(x) = m_j \bar{x} \) for some \( m_j \in \mathbb{C}[q] \), giving the result in this case.

On the other hand, if \( \lambda_1 = \lambda_2 \), then \( Z(B_\lambda) \) is equal to

\[
\sum_{\{c_1, c_2\} \in \{\pm 1\}^2} \frac{x_1^{c_1} - q x_2^{c_1}}{x_1^{c_1} - x_2^{c_1}} \cdot \frac{x_1^{c_2} - q x_2^{c_2}}{x_1^{c_2} - x_2^{c_2}} \cdot \frac{x_1^{c_1} - q x_2^{c_1}}{x_1^{c_1} - x_2^{c_1}} \cdot \left( C_1(x_1^{c_1}) C_2(x_2^{c_2}) + A_1 D_2 (\frac{1 - q x_2^{c_2}}{x_1^{c_1} - x_2^{c_2}}) \right) \cdot \left( x_1^{c_1} \right)^{\lambda_1+1} \left( x_2^{c_2} \right)^{\lambda_2+1} + A_1 D_2 \prod_{\{c_1, c_2\} \in \{\pm 1\}^2} \frac{x_1^{c_1} - q x_2^{c_1}}{x_1^{c_1} - x_2^{c_1}} \cdot A_1 D_2 \prod_{\{c_1, c_2\} \in \{\pm 1\}^2} \frac{x_1^{c_2} - q x_2^{c_2}}{x_1^{c_2} - x_2^{c_2}}.
\]

Again, assuming we are in case 1(a) of the Fish relation, then the weights \( C_i \) are independent of the \( x_i \) and we may write:

\[
Z(B_\lambda) = \frac{C_1 C_2}{1 + q} \sum_{w \in W} w \left( x_1^{\lambda_1+1} x_2^{\lambda_2+1}, \prod_{\alpha \in \Phi^+} \frac{1 - q x^{-\alpha}}{1 - x^{-\alpha}} \right) + A_1 D_2 \left[ \sum_{w \in W} w \left( x_1^{\lambda_1+1} x_2^{\lambda_2+1}, \frac{x_1 - q x_2}{x_1 - x_2}, \frac{x_2 - q x_1}{x_2 - x_1}, (1 - q) x_2, \frac{x_1 - q x_1}{x_1 - x_1}, \frac{x_2 - q x_2}{x_2 - x_2} \right) \right] + \sum_{w \in W \setminus \langle s_2 \rangle} w \left( x_1^{\lambda_1-\lambda_2}, \frac{x_1 - q x_2}{x_1 - x_2}, \frac{x_1 - q x_1}{x_1 - x_1}, x_2 - q x_2, \frac{x_1 - q x_2}{x_1 - x_2}, \frac{x_1 - q x_1}{x_1 - x_1} \right) \right].
\]

In particular, when the bend weights are uniform in Case 2(b), there’s a particularly nice choice of weights realizing the B/C Hall-Littlewood polynomials for all partitions in rank two.

**Corollary 6.3.** With Boltzmann weights from the trigonometric six-vertex model and bend weights (labeled as in Figure 6) satisfying

\[
C_i(x) = \bar{x}, \quad A_1 D_2 = 0, \quad A_2 D_1 = q^3 - q,
\]


then for every dominant weight \((\lambda_1, \lambda_2)\) (i.e., pairs with \(\lambda_1 \geq \lambda_2\)), the partition function \(Z(B)\) is the type \(B/C\) Hall-Littlewood polynomial:

\[
Z(B) = \frac{1}{c_\lambda(q)} \sum_{w \in W} w \left( x_{\lambda_1}^1 x_{\lambda_2}^2 \prod_{\alpha \in \mathbb{Z}} \frac{1 - q x^{-\alpha}}{1 - x^{-\alpha}} \right) \quad \text{where} \quad c_\lambda(q) = \begin{cases} \ 1 & \text{if } \lambda_1 > \lambda_2 \\ 1 + q & \text{if } \lambda_1 = \lambda_2 \end{cases}.
\]

7. Rank Three Solvable Models

In contrast to the rank two case, solvability in rank three requires the full generality of all the Lemmas on the caduceus relation featured in Section 3. Indeed, since Definition 3.2 is more restrictive than the definition of a “rank two solvable” model, there will be fewer solvable models in rank three. As the next result shows, these additional restrictions lead to many more refined conditions on the possible bend weights. Remarkably, we show as a special case of Theorem 7.2 that one such choice still results in rank three \(B/C\) Hall-Littlewood polynomials for all partitions.

**Proposition 7.1.** The only possible choices of bend weights to complete the trigonometric six-vertex model to a solvable \(B/C\) model in rank three are as follows:

1. \(B_j(x) = -q C_j(x)\) and \(A_j(x) = D_j(x) = 0\) for all \(j\).
2. \(B_j(x) = -x^2 C_j(x)\) and \(A_j(x) = D_j(x) = 0\) for all \(j\).
3. \(A_j, B_j, C_j, D_j \in \mathbb{C}[q]\), with \(B_j = C_j\) for \(j = 1, 2, 3\), and \(A_i C_j D_k = C_1 C_2 C_3\) for all arrangements \((i, j, k)\) of \([1, 2, 3]\), or
4. \(A_j, B_j, C_j, D_j \in \mathbb{C}[q]\), with \(B_j = C_j\) for \(j = 1, 2, 3\), and
   \[
   \begin{cases}
   A_1, D_3 = 0 \\
   A_3 C_2 D_1 = A_3 C_1 D_2 = A_2 C_3 D_1 = (1 - q^2) C_1 C_2 C_3,
   \end{cases}
   \]
5. \(A_j, B_j, C_j, D_j \in \mathbb{C}[q]\), with \(B_j = C_j\) for \(j = 1, 2, 3\), and
   \[
   \begin{cases}
   A_3, D_1 = 0 \\
   A_1 C_2 D_3 = A_2 C_1 D_3 = A_1 C_3 D_2 = (1 - q^2) C_1 C_2 C_3,
   \end{cases}
   \]
6. \(B_j(x) = q m_j x, C_j(x) = m_j \bar{x},\) where \(m_j \in \mathbb{C}[q]\) and \(A_j, D_j \in \mathbb{C}[q]\) for \(j = 1, 2, 3\) and \(q^{2(k-1)} m_i A_j D_k = q m_1 m_2 m_3\) for all arrangements \((i, j, k)\) of \([1, 2, 3]\), or
7. \(B_j(x) = q m_j x, C_j(x) = m_j \bar{x},\) where \(m_j \in \mathbb{C}[q]\) and \(A_j, D_j \in \mathbb{C}[q]\) for \(j = 1, 2, 3\) and
   \[
   \begin{cases}
   A_1, D_3 = 0 \\
   m_1 A_3 D_2 = m_3 A_2 D_1 = (q^3 - q) m_1 m_2 m_3 \\
   m_2 A_3 D_1 = (q^3 - q^2) m_1 m_2 m_3,
   \end{cases}
   \]
8. \(B_j(x) = q m_j x, C_j(x) = m_j \bar{x},\) where \(m_j \in \mathbb{C}[q]\) and \(A_j, D_j \in \mathbb{C}[q]\) for \(j = 1, 2, 3\) and
   \[
   \begin{cases}
   A_3, D_1 = 0 \\
   m_1 A_2 D_3 = m_3 A_1 D_2 = (q^{-1} - q) m_1 m_2 m_3 \\
   m_2 A_1 D_3 = (q^{-3} - q^{-1}) m_1 m_2 m_3,
   \end{cases}
   \]

**Proof.** To begin, recall that a type \(B/C\) model is of uniform regime and satisfies the two particle caduceus relation if and only if the model satisfies one of the four conditions listed in Lemma 3.2. Note that cases (1) and (2) above correspond to cases (3) and (4) of Lemma 3.2, respectively. By Lemma 3.1, \(A_j(x) = D_j(x) = 0\) in these cases, so that the caduceus condition with 3 particles and 1 particle is trivially satisfied. Hence, the choice of bend weights described cases (1) and (2) complete the trigonometric six-vertex model to a solvable \(B/C\) model in rank three.

Suppose instead that a given model has bivalent weights as described in case (1) of Lemma 3.2. In this case, we have the following set of constraints on the bivalent weights from Lemmas 3.2, 5.3, and 5.4:

\[
\begin{align*}
(i) \quad & C_3 (q^2 (A_1 D_2 - C_1 C_2) - (A_2 D_1 - C_1 C_2)) = 0 \\
(ii) \quad & C_1 (q^2 (A_2 D_3 - C_2 C_3) - (A_3 D_2 - C_2 C_3)) = 0 \\
(iii) \quad & A_3 C_1 D_2 = A_3 C_2 D_1 \\
(iv) \quad & A_1 C_2 D_3 = A_1 C_3 D_2 \\
(v) \quad & D_3 C_1 A_2 = D_3 C_2 A_1 \\
(vi) \quad & D_1 C_2 A_3 = D_1 C_3 A_2.
\end{align*}
\]

(7.1)
Recall that in this case, $A_j, B_j, C_j, D_j \in \mathbb{C}[q]$ and $B_j = C_j$ for $j = 1, 2, 3$. Note that (iii) and (iv) (resp. (v) and (vi)) include an extra factor of $A_j$ (resp. $D_j$) when compared with the result from Lemma 3.3 (resp. Lemma 3.4) due to the presence of the third bivalent vertex in rank three. For similar reasons, there is a factor of $C_j$ in (i) and a factor of $C_1$ in (ii).

Before we begin our case analysis, we note that if $C_j = 0$ for any $j$, then every state in the associated model has weight 0, since the only states in the model that don’t have a factor of $C_j$ would have a weight of 0 by (iii)-(vi), and hence we may assume that $C_j \neq 0$ for all $j$.

First, suppose that $A_1 C_2 D_3 \neq 0$ and $A_3 C_2 D_1 \neq 0$. This case will lead us to scenario (3) as a result of (iii)-(vi), and hence we may assume that $A_1 \neq 0$ for all $j$. Notice that in this case, the Fish constant of proportionality is scenario (5) above.

Now, suppose that $A_1 C_2 D_3 = 0$. Then, by (iv) and (v), respectively, we have $A_1 C_3 D_2 = 0$ and $A_2 C_1 D_3 = 0$. Hence, by (i) and (ii) we have $A_2 C_3 D_1 = (1 - q^2) C_1 C_2 C_3$ and $A_3 C_1 D_2 = (1 - q^2) C_1 C_2 C_3$. From this, we see that $A_2, D_3 \neq 0$. Since $A_1 C_2 D_3 = 0$, we must have $A_1 = 0$, and, since $A_2 C_1 D_3 = 0$, we must have $D_3 = 0$. Summarizing, if $A_1 C_2 D_3 = 0$, then

$$\begin{cases} A_1, D_3 = 0 \\ A_2 C_2 D_1 = A_3 C_1 D_2 = A_2 C_3 D_1 = (1 - q^2) C_1 C_2 C_3, \end{cases}$$

which is scenario (4) above.

If, instead, $A_3 C_2 D_1 = 0$, then due to the symmetry within conditions (i)-(vi), we have

$$\begin{cases} A_3, D_1 = 0 \\ A_3 C_2 D_3 = A_2 C_1 D_3 = A_3 C_2 D_2 = (1 - q^2) C_1 C_2 C_3, \end{cases}$$

which is scenario (5) above. Finally, note that if $A_1 C_2 D_3 = 0$ and $A_3 C_2 D_1 = 0$, then $A_1, D_1 = 0$, so by (i) $(1 - q^2) C_1 C_2 C_3 = 0$, which leads to a trivial model, as discussed at the beginning of the proof.

Cases (6), (7), and (8) emerge from an analysis of rank three models that satisfy case (2) of Lemmas 3.2, 3.3 and 3.4 in much the same way that cases (3), (4), and (5) were established above.

Note that if $C_j(x) = 1$ in scenario (1), then we recover the model from [24]. Meanwhile, scenario (3) is equivalent to having $A_j = B_j = C_j = D_j = 1$, since in this case every state receives the same weight contribution from its bivalent vertices. In Theorem 7.2 we show that the zonal spherical function in type C can be realized as the partition function of a model from scenario (4) or as the partition function of a model from scenario (7).

**Theorem 7.2.** Let rank $r = 3$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3)$. If $B_\lambda$ is solvable then

(1) $Z(B_\lambda) = \frac{C_1 C_2 C_3}{c_\lambda(q)} \sum_{w \in W} w \left( x_1^{\lambda_1+1} x_2^{\lambda_2+1} x_3^{\lambda_3+1} \prod_{a \in \Phi^+} \frac{1 - qx^{-\alpha}}{1 - x^{-\alpha}} \right)$ if $B_\lambda$ has bivalent weights as in case (4) of Proposition 7.1

or

(2) $Z(B_\lambda) = x_1 x_2 x_3 \frac{C_1(x_1) C_2(x_2) C_3(x_3)}{c_\lambda(q)} \sum_{w \in W} w \left( x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \prod_{a \in \Phi^+} \frac{1 - qx^{-\alpha}}{1 - x^{-\alpha}} \right)$ if $B_\lambda$ has bivalent weights as in case (7) of Proposition 7.1

where $c_\lambda(q) = \begin{cases} 1 & \text{if } \lambda_1 > \lambda_2 \geq \lambda_3 \\ 1 + q & \text{if } \lambda_1 \geq \lambda_2 \geq \lambda_3 \text{ or } \lambda_1 > \lambda_2 = \lambda_3 \\ (1 + q)(1 + q + q^2) & \text{if } \lambda_1 = \lambda_2 = \lambda_3. \end{cases}$

**Lemma 7.3.** Let rank $r = 3$. If the bivalent weights satisfy condition (4) or (7) of Proposition 7.1, then the Fish constant of proportionality $F = 1$, then

$$\langle K \rangle F^{-1} = \sum_{\{\epsilon_1, \epsilon_2, \epsilon_3\} \in \{\pm 1\}^3} C_1(x_1^{\epsilon_1}) C_2(x_2^{\epsilon_2}) C_3(x_3^{\epsilon_3}) \prod_{3 \geq k \geq l \geq 1} \left( \frac{x_k^{\epsilon_k} - qx_l^{\epsilon_l}}{x_k^{\epsilon_k} - x_l^{\epsilon_l}} \right)^3 \otimes (\epsilon_i | i \rangle \otimes (-\epsilon_i | i \rangle),$$
Proof. We will prove \((\ref{7.2})\) by verifying that each component of \(\langle K | \mathbf{F}^{-1} \rangle \) has the form given on the right of \((\ref{7.2})\) in a manner similar to our approach in the proof of Lemma \((\ref{3.2})\). Suppose that \(j_1, j_2, j_3, j_2, j_2, j_3 \in \{0, 1\} \), where 1 denotes the presence of a particle at a vertex, and 0 denotes the lack of a particle. By Equations \((\ref{4.2})\) and \((\ref{4.3})\), we have

\[
\langle K | \mathbf{F}^{-1} \rangle^{j_3 j_3 j_2 j_2 j_1 j_1} = \langle K | \mathbf{R}^p_{\mathcal{R}_{33211}} \prod_{k,l \in \{1, 2, 2, 3, 3\}} \Delta^{-1}_{kl}(x_k, x_l) \rangle^{j_3 j_3 j_2 j_2 j_1 j_1} = \langle K | \mathbf{R}^p_{\mathcal{R}_{33211}} \rangle^{j_3 j_3 j_2 j_2 j_1 j_1} \prod_{k,l \in \{1, 2, 2, 3, 3\}} b^{-1}_{j_k j_l}(x_k, x_l)
\]

where \(\rho\) is a permutation of \(\{1, 2, 2, 3, 3\}\) such that \(j_{\rho(2)} = j_{\rho(1)} = 0 \) and \(j_{\rho(3)} = j_{\rho(3)} = j_{\rho(2)} = 1\).

Since the \(R\)-matrix weights \(a_1(k, j)\) and \(a_2(k, j)\) are both equal to 1, it suffices to consider permutations \(\rho\) such that \(\rho(j) \geq \rho(l)\) for \(j = 1, 2, 3\). Thus, we have twenty cases to consider: one for each selection of three elements from the set above. For twelve of these cases, \(\langle K | \mathbf{F}^{-1} \rangle^{j_3 j_3 j_2 j_2 j_1 j_1} = 0\), which is why the right hand side of \((\ref{7.2})\) only has eight terms in it.

First, let’s consider the non-zero cases. Let \(\rho_0 = (3, 2, 1, \bar{3}, \bar{2}, 1)\). Since we are assuming that the bivalent weights satisfy either condition \((\ref{4})\) or condition \((\ref{7})\), we have that \(D_3 = A_1 = 0\), and hence there is only one way to fill the corresponding diagram to yield a non-zero weight. Explicitly, \(\langle K | \mathbf{R}^p_{\mathcal{R}_{33211}} \rangle^{j_3 j_3 j_2 j_2 j_1 j_1} = \) is equal to

\[
Z \left( \begin{array}{cccc}
 x_3 \\
 \bar{x}_3 \\
 x_2 \\
 \bar{x}_2 \\
 x_1 \\
 \bar{x}_1 \\
 \end{array} \right) = \text{wt} \left( \begin{array}{cccc}
 x_3 \\
 \bar{x}_3 \\
 x_2 \\
 \bar{x}_2 \\
 x_1 \\
 \bar{x}_1 \\
 \end{array} \right) = B_1 B_2 B_3 \left( \begin{array}{cccc}
 \frac{x_1 - \bar{x}_2}{x_1 - q \bar{x}_2}, \frac{x_1 - \bar{x}_3}{x_1 - q \bar{x}_3}, \frac{x_2 - \bar{x}_3}{x_2 - q \bar{x}_3}. \\
 \end{array} \right)
\]

Additionally, we have

\[
(\ref{7.3}) \prod_{k \leq l} b^{-1}_{j_k j_l}(x_k, x_l) = \prod_{k \leq l} \frac{x_k - q^{-1} x_l}{x_k - x_l} \quad \text{when } \rho(3) = \pm 3, \rho(3) = \pm 2, \rho(2) = \pm 1.
\]

Thus, for \(\rho = \rho_0\),

\[
\langle K | \mathbf{F}^{-1} \rangle^{j_3 j_3 j_2 j_2 j_1 j_1} = C_1(x_1)C_2(x_2)C_3(x_3) \frac{x_1 - \bar{x}_2}{x_1 - q \bar{x}_2}, \frac{x_1 - \bar{x}_3}{x_1 - q \bar{x}_3}, \frac{x_2 - \bar{x}_3}{x_2 - q \bar{x}_3}, \prod_{k \leq l} \frac{x_k - q^{-1} x_l}{x_k - x_l},
\]

\[
= C_1(x_1)C_2(x_2)C_3(x_3) \prod_{3 \leq k \leq l} \frac{x_k - q^{-1} x_l}{x_k - x_l}.
\]

Note that if \(\rho = (3^\epsilon_3, 2^\epsilon_2, 1^\epsilon_1, \bar{3}^\epsilon_3, \bar{2}^\epsilon_2, \bar{1}^\epsilon_1)\), for some \(\epsilon_1, \epsilon_2, \epsilon_3 \in \{|\pm 1\}|\), then, after applying the Fish equation (Lemma \((\ref{3.1})\) and carrying out the same analysis as in the \(\rho_0\) case above, we have that

\[
\langle K | \mathbf{F}^{-1} \rangle^{j_3 j_3 j_2 j_2 j_1 j_1} = C_1(x_1^\epsilon_3)C_2(x_2^\epsilon_2)C_3(x_3^\epsilon_3) \prod_{3 \leq k \leq l} \frac{x_k^\epsilon_3 - q^{-1} x_l^\epsilon_3}{x_k^\epsilon_3 - x_l^\epsilon_3}.
\]

In the other twelve cases, \(\langle K | \mathbf{R}^p_{\mathcal{R}_{33211}} \rangle^{j_3 j_3 j_2 j_2 j_1 j_1} = 0\) so that \(\langle K | \mathbf{F}^{-1} \rangle^{j_3 j_3 j_2 j_2 j_1 j_1} = 0\). Let’s explain why this is the case. First, recall that \(D_3 = A_1 = 0\). Hence, \(\langle K | \mathbf{R}^p_{\mathcal{R}_{33211}} \rangle^{j_3 j_3 j_2 j_2 j_1 j_1} = 0\) in the following six cases:

\[
\rho = (3, \bar{3}, 2, \bar{3}, 1, \bar{1}), (3, \bar{3}, 1, 2, \bar{3}, \bar{1}), (3, 2, \bar{3}, 1, \bar{1}), (3, \bar{3}, \bar{2}, 1, \bar{1}), (3, \bar{3}, 1, 2, \bar{3}, 1), (\bar{3}, \bar{2}, \bar{2}, 3, 1, \bar{1}).
\]

One application of the Caduceus relation (Lemmas \((\ref{3.2}) ((\ref{3.3}) \& \(\ref{3.4}))\) shows us that \(\langle K | \mathbf{R}^p_{\mathcal{R}_{33211}} \rangle^{j_3 j_3 j_2 j_2 j_1 j_1} = 0\) for \(\rho = (3, 1, \bar{3}, 3, \bar{2})\) and \((2, \bar{3}, 1, 3, 3, \bar{1})\), and a subsequent application of the Fish relation takes care of \(\rho = (\bar{3}, 1, 1, 3, 3, \bar{1})\). Finally, if \(\rho = (2, 1, \bar{1}, 3, \bar{3}, \bar{2})\), it can be verified directly that
Proposition 8.1. The only possible choices of bend weights to complete the trigonometric six-vertex model to a solvable B/C model in rank \( r > 3 \) are as follows:

1. \( B_j(x) = -qC_j(x) \) and \( A_j(x) = D_j(x) = 0 \) for all \( j \).
2. \( B_j(x) = -x^2C_j(x) \) and \( A_j(x) = D_j(x) = 0 \) for all \( j \).
3. \( A_j, B_j, C_j, D_j \in \mathbb{C}[q] \), with \( B_j = C_j \) and

\[
\frac{A_i}{C_i} \cdot \frac{D_j}{C_j} \cdot \prod_{k=1}^{r} C_k = \prod_{k=1}^{r} C_k \text{ for any } i \neq j.
\]
(4) $B_j(x) = q m_j x$, $C_j(x) = m_j x$, where $m_j \in \mathbb{C}[q]$ and $A_j, D_j \in \mathbb{C}[q]$ for all $j$ and
\[
\frac{A_i}{C_i} \cdot \frac{D_j}{C_j} \cdot \prod_{k=1}^{r} C_k = q^{1-2(j-i)} \prod_{k=1}^{r} C_k \text{ for any } i \neq j.
\]

(5) $B_j = C_j = 0$, $A_j, D_j \in \mathbb{C}[q]$ with $q^2 A_j D_{j+1} = A_{j+1} D_j$ for all $j$, and $r$ is even.

Proof. As in the proof of Proposition 7.1, we begin by noting that a type $B/C$ model of uniform regime satisfies the two particle caduceus relation if and only if it satisfies one of the four conditions listed in Lemma 3.2. Note that scenarios (1) and (2) above correspond to cases (3) and (4) of Lemma 3.2, respectively. Since the bivalent weights in scenarios (1) and (2) above trivially satisfy the three particle caduceus relation and the one particle caduceus relation, we see that these choices of bend weights complete the trigonometric six-vertex model to a solvable $B/C$ model in rank $r > 3$.

Suppose instead that a given model has bivalent weights as described in case (1) of Lemma 3.2, such that $C_j \neq 0$ for each $j$. (Later on, we will show that if $C_j = 0$ for some $j$ then the model in question must fall into scenario (5) above.) Then, in order to have a solvable $B/C$ model in rank $r > 3$, the following constraints on the bivalent weights must be satisfied:

\[
\begin{align*}
\text{(i) } & q^2 (A_j D_{j+1} - C_j C_{j+1}) = (A_{j+1} D_j - C_j C_{j+1}) \quad \text{for } j \in [1, r-1], \\
\text{(ii) } & A_j C_{j+k} = A_j C_{k+1} D_k \quad \text{for } j \in [1, r], k \in [1, r-1], j \neq k, k+1 \\
\text{(iii) } & C_k A_{k+1} D_j = A_k C_{k+1} D_j \quad \text{for } j \in [1, r], k \in [1, r-1], j \neq k, k+1
\end{align*}
\]

Constraint (i) is a restatement of the result of Lemma 3.2. Constraint (ii) is a consequence of Lemma 3.3. Note that this constraint contains an extra factor of $A_j$ when compared with the result from Lemma 3.3 due to particle conservation. For example, if $r = 4$, we have $D_1 C_2 A_3 C_4 = C_1 D_2 A_3 C_4$ and $D_1 C_2 A_4 C_3 = C_1 D_2 C_3 A_4$ both by Lemma 3.3. Similarly, constraint (iii) is a consequence of Lemma 3.4.

Since $C_j \in \mathbb{C}[q]$ and $C_j \neq 0$ for each $j$, (ii) and (iii) imply that,
\[
A_i D_j = v_{ij}(q) A_i D_k \quad \text{if } 1 \leq i < j \leq n \text{ and } 1 \leq k < \ell \leq n
\]
where $v_{ij}(q)$ is some rational function in $q$. As an example when $r = 4$, we can repeatedly apply (ii) and (iii) to establish the following equalities:
\[
A_1 D_2 C_3 C_4 = A_1 C_2 D_3 C_4 = A_1 C_2 C_3 D_4 = C_1 A_2 C_3 D_4 = C_1 C_2 A_3 D_4.
\]

Similarly,
\[
A_1 D_j = w_{ij}(q) A_k D_j \quad \text{if } i > j \text{ and } k > \ell
\]
for some rational function $w_{ij}(q)$.

If $A_1 C_2 D_3 \neq 0$ and $A_3 C_2 D_1 \neq 0$, then, using (8.2) and (8.3) and an argument analogous to the one given to address scenario (3) in Proposition 7.1, one can be used to prove that
\[
\frac{A_i}{C_i} \cdot \frac{D_j}{C_j} \cdot \prod_{k=1}^{r} C_k = \prod_{k=1}^{r} C_k
\]
for any $i \neq j$, which is scenario (3) above. On the other hand, it is clear that a type $B/C$ model satisfying the condition above along with the assumption $B_j = C_j$ for all $j$ is solvable.

Now suppose towards a contradiction that $A_1 D_2 = 0$. Then, by (8.2), $A_3 D_4 = 0$ as well. From (i), we have that $A_2 D_1 = (1 - q^2) C_1 C_2$ and $A_4 D_3 = (1 - q^2) C_3 C_4$, and hence $A_2, A_4, D_1, D_3 \neq 0$. On the other hand, since $A_1 D_2 = 0$, $A_2 D_3$ must be $0$ as well, which is a contradiction. Similarly, if we assume that $A_2 D_1 = 0$ and conditions (i), (ii), and (iii), then we reach a contradiction. Thus, in order to have a solvable rank $r > 3$ model, we must have $A_j \neq 0$ and $D_j \neq 0$ for all $j$.

If instead we consider a model with bivalent weights as described in case (2) of Lemma 3.2, such that $C_j \neq 0$ for any $j$, then an analysis similar to the one given above where the bivalent weights satisfy case (1) of Lemma 3.2 leads us to scenario (4) in the statement of this proposition.

Finally, observe that, if $C_j = 0$ for some $j$, then one can use (ii) and (iii) to show that every state that contains a factor of $B_k$ or $C_k$ for any $k$ must evaluate to zero. For example, if $r = 4$ and $C_1 = 0$, then
\[
A_1 D_2 C_3 C_4 = A_1 C_2 D_3 C_4 = C_1 A_2 D_3 C_4 = 0.
\]
Hence, the only non-zero states must consist only of vertices with weight $A_j$ or $D_j$; furthermore, such states occur only if the rank $r$ is even. Thus, if $C_j = 0$ for some $j$, then the model in question falls into scenario (5). □

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