Almost Sharp Global Well-Posedness for a class of Dissipative and Dispersive Equations on $\mathbb{R}$ in Low Regularity Sobolev Spaces

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Abstract

In this paper we obtain global well-posedness in low order Sobolev spaces of higher order KdV type equations with dissipation. The result is optimal in the sense that the flow-map is not $C^2$ in rougher spaces. The solution is shown to be smooth for positive times.

Keywords: KdV-Burger's equation equation, Benney-Lin equation, Ostrovsky-Stepanyams-Tsimring equation, KdV-Kuramoto-Sivashinsky equation, Low regularity, Bourgain space. MSC 35Q53.

1 Introduction

In this article we consider the global well-posedness of the following initial value problem

$$\begin{cases}
\partial_t u + 2u\partial_x u + H(D)u - iP(D)u = 0, & t > 0 \\
u(0, x) = u_0(x)
\end{cases}$$

(1.1)

where $f(D)u(x) = \mathcal{F}^{-1}_\xi[f(\xi)\hat{u}](x)$ and $H$ and $P$ are real valued functions and the initial value $u_0$ is real-valued. The number 2 in front of the non-linearity is inessential and is there for convenience only. It appears for specific choices of parameters e.g. in $[6]$ as a model of nonlinear acoustic waves.

For $P(\xi) = \xi^3$ and $H(\xi) = \xi^2$ it is called the Korteweg-de Vries-Burgers equation. In $[1]$ it was proven that this equation is globally well-posed in $L^2$. In $[7]$ this result was extended to Sobolev spaces with $s > -1$ using Bourgain spaces. In $[8]$ well-posedness was proven at the endpoint $s = -1$ by "besovification" of the Bourgain space.

For $P(\xi) = -\xi^5 + \xi^3$ and $H(\xi) = \xi^4 - \xi^2$ it is named the Benney-Lin equation. Local well-posedness was obtained for $s > -7/3$ in $[10]$. In $[2]$ the result was strengthened to be globally well-posed for $s > -2$.

For $P(\xi) = \xi^3$ and $H(\xi) = |\xi| + |\xi|^3$ we get the Ostrovsky-Stepanyams-Tsimring equation first derived in $[9]$. Global well-posedness was proved in $[4]$ for $s \geq 0$. Local well-posedness was proved in $[12]$ for $s > -3/4$ and for $s > -1$ in $[11]$. In $[5]$ the local result was pushed down to $s > -3/2$ by Dix’s method. We will prove global well-posedness for the same range of Sobolev spaces.
For $P(\xi) = \xi^3$ and a more general form of $H$, local well-posedness was obtained for $s > -3/4$ in [3]. We extend this result to more general dissipative terms and to $s > -\deg H/2$.

For $P(\xi) = \xi^3$ and $H(\xi) = \xi^2 + \xi^4$ it is called the Korteweg-de Vries-Kuramoto-Sivashinsky equation. The best result so far is by [3] where local well-posedness for $s > -3/4$ was obtained. We will prove global well-posedness for $s > -2$.

In this work we will treat all these equations in a unified way using Bourgain’s method, as extended by [7], by proving global well-posedness for any odd and positively homogeneous function, $P$, with $\deg P \geq 3$, and any $H$ bounded for small $|\xi|$ and $H(\xi) \sim (\xi)^{\deg H}$ for large $|\xi|$ and $\deg H = \deg P - 1$. Or, simpler, if $\deg H > \deg P - 1$ and $\deg H \geq 3$.

2 Definitions and Notations

The spatial or temporal Fourier transforms of a function, $u$, will be denoted by $\hat{u}(\xi)$ or $\hat{u}(\tau)$ and the space-time transformed function by $\tilde{u}$. Defining the linear solution map $S(t)$ by $\tilde{S}(t) = e^{(-H(\xi) + iP(\xi))t}$, the Duhamel principle gives

$$u(t) = S(t)u_0 - \int_0^t S(t-r)\partial_x(u^2(r))\,dr$$ (2.1)

We will now localize (2.1) in the time variable. Define a symmetric cut off function $\theta \in C_0^\infty(\mathbb{R})$ in time by requiring

$$\theta = 1 \quad \text{on} \quad [-1/2, 1/2], \quad \text{supp} \theta \subset [-1, 1]$$

and let $\theta_T(t) = \theta(t/T)$ for all $t \in \mathbb{R}$. Denote the characteristic function for the non-negative reals as $\chi_+(t)$. We will first prove local existence and uniqueness by a fixed point argument applied to the localized version of (2.1), namely

$$u(t) = \theta_T(t)S(t)u_0 - \chi_+(t)\theta_T(t)\int_0^t S(t-s)\partial_x(\theta_T^2(s)u^2(s))\,ds$$ (2.2)

for some short enough time span $T$. Here $S(t)$ is the modified solution operator defined by $\mathcal{F}_x S(t) = e^{-H|x|+iP(t)}$. It is clear that for $t \in [0, 1/2 \min(1, T)]$ any solution to (2.2) will solve (2.1). Then an argument showing that the $L^2$-norm in $x$ grows at most exponentially in time gives global well-posedness.

Let $\tau = \tau_1 + \tau_2$ and $\xi = \xi_1 + \xi_2$. We will sometimes write $P_i = P(\xi_i)$ and similarly for $H$. Define

$$\sigma = i(\tau - P(\xi)) + H(\xi), \quad \sigma_i = i(\tau_i - P(\xi_i)) + H(\xi_i) \quad \text{for} \quad i \in \{1, 2\}$$

For any function, $Q$, with $Q(0) = 0$, we define its resonance function, $R(Q)$, by

$$R(Q) = R(Q)(\xi_1, \xi_2) = Q(\xi_1) + Q(\xi_2) - Q(\xi)$$

Then we have

$$\sigma - \sigma_1 - \sigma_2 = i[P(\xi_1) + P(\xi_2) - P(\xi)] - [H(\xi_1) + H(\xi_2) - H(\xi)] = iR(P) - R(H)$$

It will be convenient to use the coordinates

$$\nu = (\xi, \tau) \quad \nu_1 = (\xi_1, \tau_1) \quad \nu_2 = \nu - \nu_1$$

Define $s_c = -1/2 \max(\deg P - 1, \deg H)$. We now state the main Theorem. Assume that the following conditions hold:

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• $P$ is a linear combination of odd and positively homogeneous functions with the highest order equal to $\deg P$.

• $H = H(\xi)$ is bounded for small $|\xi|$ and $H(\xi) \sim \langle \xi \rangle^{\deg H}$ for large $|\xi|$.

• Either $\deg P \geq 3$ and $\deg H = \deg P - 1$ or $\deg H \geq 3$ and $\deg H > \deg P - 1$.

**Theorem 2.1** The initial value problem (1.1) is globally well-posed in $H^s$ if $s > s_c$.

### 3 $X^{s,b}$-spaces

The linear symbol of the equation is $H(\xi) + i(\tau - P(\xi))$ and we therefore define the corresponding Bourgain spaces $X^{s,b}$ as the closure of the Schwarz functions under the norm

$$
\|u\|_{X^{s,b}} = \| \langle \xi \rangle^s (i(\tau - P(\xi)) + H(\xi))\hat{u}\|_{L_{\tau,\xi}^2}.
$$

Define the unitary operator $U(t)$ by $U(t)g = F^{-1}_x [e^{itP(\xi)}\hat{g}]$. A simple calculation then shows that

$$
\|u\|_{X^{s,b}} = \| \langle \xi \rangle^s (i(\tau - H)\hat{u}\|_{L_{\tau,\xi}^2} \sim \|u\|_{X^{s,b}} + \|u\|_{H^{\alpha,\beta}_{t,x}}^{\max(b, \deg H)}
$$

where $X^{s,b}$ is a classical Bourgain space based on the dispersive part of the symbol and $H^{\alpha,\beta}_{t,x}$ is a Sobolev space.

The (time-) localized Bourgain space $X^{s,b}_T$, for some $0 \leq T \leq 1$, consists of all $u \in S'(\mathbb{R}^2)$ coinciding with some $w \in X^{s,b}(\mathbb{R}^2)$ on $[0, T]$ and having finite norm

$$
\|u\|_{X^{s,b}_T} = \inf_{w \in X^{s,b}} \{ \|w\|_{X^{s,b}} : w(t) = u(t), t \in [0, T] \}.
$$

### 4 Various estimates for the free and forcing term

The following results follows from fairly standard methods of proof in the literature or are easy to prove and proofs are omitted.

**Lemma 4.1** If $b \geq 2$ then

$$
\int \frac{dt}{\langle t \rangle (a - bt)} \lesssim \frac{1}{\max(a, b)^{1-\delta}} \tag{4.1}
$$

**Lemma 4.2** (Stability w.r.t. localization.) Let $\eta \in S(\mathbb{R})$ and $b \in \mathbb{R}$. Then

$$
\|\eta(t)u\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b}}
$$

where the implied constant is $\|\eta\|_{H_b^{1,1}}$.

**Lemma 4.3** For any $s \in \mathbb{R}$

$$
\|\theta(t)S(t)u_0\|_{X^{s,1/2}} \lesssim \|u_0\|_{H^s}
$$

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Proposition 4.4 Let $s \in \mathbb{R}$. Then for $0 < \delta < 1/2$ we have

$$\left\| \chi_+(t)\theta(t) \int_0^t S(t-s)F(s)\, ds \right\|_{X^{s,1/2}} \lesssim \|F\|_{X^{s,-1/2+\delta}}$$

Proposition 4.5 If $h = h(t, x)$ has compact support in $t \in [-T, T]$ then for any $0 \leq \delta < 1/2$

$$\left\| \frac{\tilde{h}(\tau, \xi)}{(\iota(\tau - P(\xi)) + H(\xi))^\delta} \right\|_{L^2_{\tau, \xi}} \lesssim T^\delta \|h\|_{L^2} \tag{4.2}$$

5 The bilinear estimate

It is easy to show that any odd and positively homogeneous function must be a proportional to $F(x) = \text{sgn}(x) |x|^\alpha$. If $\alpha = 1$ then its resonance function vanishes. For $\alpha > 1$ the following estimates on the resonance function hold.

Proposition 5.1 For $F$ as above with $\alpha > 1$ and resonance function $R$ we have

$$|R| \sim \min(|\xi_1|, |\xi_2|, |\xi|) \cdot \max(|\xi_1|, |\xi_2|)^{\alpha-1}$$

Proof: Since $F$ is odd the resonance function satisfies

$$R(-\xi_1, -\xi_2) = -R(\xi_1, \xi_2)$$

and the right hand side of the conclusion has the same symmetry it is enough to check $\xi_1 > 0$. The second symmetry is

$$R(\xi_1, \xi_2) = R(\xi_2, \xi_1)$$

which also holds for the right hand side and we may therefore restrict attention to the cone $C \subset \mathbb{R}^2$ defined by $0 < |\xi_2| < \xi_1$. In $C$ we have $|\xi_2| \lesssim |\xi_1| \sim |\xi_1 + \xi_2|$. Divide $C$ into $C^+ = C \cap \{\xi_2 \geq 0\}$ and $C^- = C \cap \{\xi_2 \leq 0\}$. Split these further dyadically as

$$C^+_m = \left\{ (x_1, x_2) \in C^+ : 2^{-m-1} \leq \frac{|x_2|}{x_1} \leq 2^{-m-1} \right\} \quad \text{so that} \quad C^\pm = \bigcup_{m=0}^\infty C^\pm_m$$

Let $(\xi_1, \xi_2) \in C$ be fixed. Then there exists an $m \geq 0$ such that $(\xi_1, \xi_2) \in C^+_m \cup C^-_m$, i.e.,

$$2^{-m-1} \leq \frac{|\xi_2|}{\xi_1} \leq 2^{-m}$$

For this $m$ we have two line segments $L^+_m = C^+_m \cap \{x_1 = \xi_1\}$ and $L^-_m = C^-_m \cap \{x_1 = \xi_1\}$. On a line segment $\xi_2 = c\xi_1$ and if $\xi_1 > 0$ and $|c| \leq 1$ then by homogeneity

$$R(\xi_1, c\xi_1) = F(\xi_1) + \text{sgn}(c)|c|^\alpha F(\xi_1) - (1 + c)^\alpha F(\xi_1)$$

$$= \xi_1^\alpha \left[ 1 + \text{sgn}(c)|c|^\alpha - (1 + c)^\alpha \right] \tag{5.1}$$
5.0.1 \((\xi_1, \xi_2) \in C^+\)

Parameterize \(L_m^+\) be letting \(x_2 = \theta \xi_1, \theta \in [2^{-m-1}, 2^{-m}]\). In this case, \(c > 0\), and

\[
R(\xi_1, x_2) = \xi_1^\alpha [1 + \theta^\alpha - (1 + \theta)^\alpha] = \xi_1^\alpha h_\alpha(\theta)
\]

Differentiating \(h_\alpha\) gives

\[
\begin{align*}
h'_\alpha(\theta) &= \alpha[(\theta^\alpha - (1 + \theta)^\alpha] < 0 \\
h''_\alpha(\theta) &= \alpha \cdot (\alpha - 1) \cdot [(\theta^\alpha - (1 + \theta)^\alpha] - 1 \\
\end{align*}
\]

Hence, on the line segment \(L_m^+\), \(h_\alpha\) is non-positive, decreasing and if \(\alpha \geq 2\) also concave. We get

\[-(2^\alpha - 2) \theta = h(1) \theta \leq h_\alpha(\theta) \leq h_\alpha(0) \theta = -\alpha \theta\]

i.e., \(|h_\alpha(\theta)| \sim \theta \sim 2^{-m} \sim \xi_2/\xi_1\) and it holds that

\[|R| \sim \xi_1^\alpha \xi_2 = \xi_1^{\alpha-1} \xi_2 \tag{5.2}\]

If, on the other hand, \(\alpha < 2\) then \(h''_\alpha > 0\) and \(h_\alpha\) is convex. Thus,

\[-\alpha \theta = h'_\alpha(0) \theta \leq h_\alpha(\theta) \leq h_\alpha(1) \theta = -(2^\alpha - 2) \theta\]

and \(|h_\alpha(\theta)| \sim \theta\) as before so (5.2) still holds. Since \(|\xi_2| \lesssim |\xi_1| \sim |\xi_1 + \xi_2|\) the result follows in this case.

5.0.2 \((\xi_1, \xi_2) \in C^-\)

The symmetry

\[
R(\xi_1, \xi_2) = R(\xi_1, -\xi_1 - \xi_2)
\]

is also satisfied by the right hand side of the conclusion in \(C^-\), which shows that it is enough to consider the cone defined by vectors \((\xi_1, \xi_2)\) in the cone \(-\xi_1/2 < \xi_2 < 0\). This means that we may assume \(m \geq 1\) in the dyadic decomposition of \(C^-\). Parameterize the line segment \(L_m^-\) by setting \(x_2 = -\theta \xi_1, \theta \in [2^{-m-1}, 2^{-m}]\). Then \(c < 0\) in (5.1) and

\[
R(\xi_1, x_2) = \xi_1^\alpha [1 - \theta^\alpha - (1 - \theta)^\alpha] = \xi_1^\alpha g_\alpha(\theta)
\]

Differentiating gives

\[
\begin{align*}
g'_\alpha(\theta) &= \alpha[(1 - \theta)^\alpha - (1 - \theta)^{-\alpha}] \geq 0 \\
g''_\alpha(\theta) &= -\alpha(\alpha - 1)[(1 - \theta)^{\alpha-2} + \theta^{\alpha-2}] < 0
\end{align*}
\]

for all cones \(C_m^-, m \geq 1\). Hence, \(g_\alpha\) is non-negative, non-decreasing, and concave. Thus, when \(\theta \leq 1/2\),

\[
c\theta = 2(1 - 2(1/2)^\alpha) \theta = \frac{g_\alpha(1/2)^\alpha}{1/2} \theta \leq g_\alpha(\theta) \leq g_\alpha'(0) \theta = \alpha \theta
\]

for some \(c > 0\) since \(\alpha > 1\). Hence, \(g_\alpha(\theta) \sim \theta \sim \xi_2/\xi_1\) and the result follows as before.
Proposition 5.2 Let $F_i = \text{sgn}(x)|x|^{d_i}$, $i = 1, \ldots, n$ with degrees $1 < d_1 \leq d_2 \leq \cdots \leq d_n$ and resonance functions $R_1, \ldots, R_n$. Let $P$ be a linear combination, $P = \sum a_i F_i$, with $a_n \neq 0$ and resonance function $R$. There is a constant $c$ such that $\max(|\xi_1|, |\xi_2|) \geq c$ implies

$$|R(\xi_1, \xi_2)| \sim |R_n(\xi_1, \xi_2)|$$

We hereby define $\deg P = d_n$.

Proof: We will use the estimates in Proposition 5.1 and since possibly different $F_i$’s but with the same degree get the estimate we may assume $d_1 < d_2 < \cdots < d_n$. Note that if at least one of $|\xi_1|$ and $|\xi_2|$ is large then at least two of $|\xi_1|$, $|\xi_2|$, and $|\xi_1 + \xi_2|$ are large. Inspection gives

$$|R_{i+1}| \sim |R_i| \cdot \max(|\xi_1|, |\xi_2|)^{d_{i+1}-d_i}$$

so that $|R_i| \lesssim |R_n|^{1/c^{d_n-d_i}}$. Therefore, by linearity,

$$R = \sum_{i=1}^{n} a_i R_i = a_n \left( R_n + \sum_{i=1}^{n-1} \frac{a_i}{a_n} R_i \right)$$

and by the triangle inequality and for some large $c$

$$|R| \gtrsim |R_n| - \sum_{i=1}^{n-1} \left| \frac{a_i}{a_n} \right| \cdot |R_i| \gtrsim |R_n| \left( 1 - \sum_{i=1}^{n-1} \left| \frac{a_i}{a_n} \right| \frac{1}{c^{d_n-d_i}} \right) \gtrsim |R_n|$$

from which the conclusion follows since the upper estimate is trivial. □

By the triangle inequality

$$|\sigma_{\max}| = \max(|\sigma|, |\sigma_1|, |\sigma_2|) \geq \frac{1}{3}(|\sigma| + |\sigma_1| + |\sigma_2|)$$

$$\geq \frac{1}{3}(|\sigma| + |\sigma_1 + \sigma_2|) \geq \frac{1}{3}(|\sigma - \sigma_1 - \sigma_2|)$$

we get the lower bound

$$|\sigma_{\max}| \geq |R(P)(\xi_1, \xi_2)| \sim \min(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|) \cdot \max(|\xi_1|, |\xi_2|)^{\deg P - 1}$$

when $\max(|\xi_1|, |\xi_2|)$ is large enough.

We will let $s_c = -1/2 \max(\deg P - 1, \deg H)$.

Theorem 5.3 Let $H = H(\xi)$ be bounded for small $|\xi|$ and $H(\xi) \sim <\xi>^{\deg H}$, $\deg H = \deg P - 1$ for large $|\xi|$, $s \in (s_c, -1/2)$. Then for all $u$ and $v$ with compact support in $[-T, T]$

$$\|\partial_x(uv)\|_{X^{s,-1/2+\delta}} \lesssim T^\delta \|u\|_{X^{s+1/2}} \|v\|_{X^{s+1/2}}$$
We must bound

\[ I_1 = \int_{\nu} \langle \xi_1 \rangle^{-2s} \frac{\langle \xi \rangle^2 \langle \xi_2 \rangle^{-2s}}{\langle \sigma \rangle^{1-\delta}} \, d\nu_1 \]

The second estimate is similar and results in the need to bound

\[ I = \int_{\nu} \frac{\langle \xi \rangle^2 \langle \xi_2 \rangle^{-2s}}{\langle \sigma \rangle^{1-\delta}} \, d\nu \]

By symmetry we can always assume that \( |\sigma_2| \leq |\sigma_1| \). Below we will denote the denominator by \( N = \langle \sigma_1 \rangle \langle \sigma \rangle^{1-\delta} \langle \sigma_2 \rangle \).
We split $\mathbb{R}^4$ in 7 parts as

\[
A = \{ (\nu, \nu_1) \in \mathbb{R}^4 : |\xi_1| \lesssim 1 \text{ or } |\xi_2| \lesssim 1 \} \\
B_1 = \{ (\nu, \nu_1) \in \mathbb{R}^4 : |\xi_1|, |\xi_2| \geq 1, |\xi| \leq 2|\xi_1|, |\xi_2| \leq 1/2|\xi_1|, |\sigma_2| \leq |\sigma| \leq |\sigma_1| \} \\
B_2 = \{ (\nu, \nu_1) \in \mathbb{R}^4 : |\xi_1|, |\xi_2| \geq 1, |\xi| \leq 2|\xi_1|, |\xi_2| > 1/2|\xi_1|, |\sigma_2| \leq |\sigma| \leq |\sigma_1| \} \\
C_1 = \{ (\nu, \nu_1) \in \mathbb{R}^4 : |\xi_1|, |\xi_2| \geq 1, |\xi| > 2|\xi_1|, |\sigma_1|, |\sigma_2| \leq |\sigma| \} \\
C_2 = \{ (\nu, \nu_1) \in \mathbb{R}^4 : |\xi_1|, |\xi_2| \geq 1, |\xi| > 2|\xi_1|, |\sigma_1|, |\sigma_2| \leq |\sigma| \} \\
D_1 = \{ (\nu, \nu_1) \in \mathbb{R}^4 : |\xi_1|, |\xi_2| \geq 1, |\xi| > 2|\xi_1|, |\sigma_1|, |\sigma_2| \leq |\sigma| \} \\
D_2 = \{ (\nu, \nu_1) \in \mathbb{R}^4 : |\xi_1|, |\xi_2| \geq 1, |\xi| > 2|\xi_1|, |\sigma|, |\sigma_2| \leq |\sigma_1| \}
\]

We begin by estimating (5.5) on $A$. Since the integrand in the $\nu_1$-integral is symmetric in $\xi_1$ and $\xi_2$ we may assume that $|\xi_1| \lesssim 1$. Then $\langle \xi \rangle \sim \langle \xi_2 \rangle$ and $I_1$ simplifies to

\[
I_1 \sim |\xi|^2 \sup_{|\xi_1| \lesssim 1} \int \frac{d\tau_1}{\langle \sigma_1 \rangle \langle \sigma \rangle^{1-4\delta} \langle \sigma_2 \rangle} = \frac{|\xi|^2}{\langle \sigma \rangle^{1-4\delta}} \sup_{|\xi_1| \lesssim 1} \int \frac{d\tau_1}{\langle \sigma_1 \rangle \langle \sigma_2 \rangle} \tag{5.6}
\]

If also $\xi$ is bounded it is trivial that $I_1$ is bounded. We therefore assume $|\xi| \geq 1$. The integrand in $\tau_1$ is

\[
\int \frac{d\tau_1}{\langle i(\tau_1 - P) + H_1 \rangle \langle i(\tau - \tau_1 - P_2) + H_2 \rangle} \sim \int \frac{d\tau_1}{\langle \tau \rangle \langle \tau - \tau_1 - P + H \rangle} \\
\quad \sim \int \frac{d\tau_1}{\langle \tau \rangle \langle \tau - \tau_1 - P \rangle \langle H \rangle} \\
\quad = \int \frac{ds}{\langle \tau - P - s \rangle \langle |s| + \langle H \rangle \rangle} \\
\quad = \int \frac{dt}{\langle \tau - P - \langle H \rangle t \rangle \langle t \rangle}
\]

so by Lemma [L.1] we get the bound

\[
\frac{1}{\max(|\tau - P|, \langle H \rangle)^{1-\delta}} \sim \frac{1}{\sigma^{1-\delta}}
\]

Hence, since $\deg H > 1$,

\[
I_1 \lesssim \frac{|\xi|^2}{\langle \sigma \rangle^{2-5\delta}} \lesssim 1
\]

In all of $B_1, B_2, C_1, C_2, D_1, D_2$ we will have $|\xi_{\min}| \lesssim |\xi_{\max 1}| \sim |\xi_{\max 2}|$ using the notation that $\xi_{\min}$, $\xi_{\max 1}$, and $\xi_{\max 2}$ being the smallest and the two larger of $|\xi|, |\xi_1|, |\xi_2|$. Then

\[
|\sigma_{\max}| \lesssim |\xi_{\min}| \cdot |\xi_{\max}|^{\deg H - 1} \tag{5.7}
\]

Now consider the set $B_1$. We have $|\xi_1| \sim |\xi|$ which simplifies $I_1$ to

\[
I_1 \sim \int_{B_1} \frac{|\xi|^2 \langle \xi_2 \rangle^{2s}}{\langle \sigma_1 \rangle \langle \sigma \rangle^{1-4\delta} \langle \sigma_2 \rangle} d\nu_1
\]
By (5.7), $\langle \sigma \rangle \gtrsim \langle \xi_2 \rangle \langle \xi_1 \rangle^{\deg P - 1}$ and we have the estimate
\[
N \geq \langle \sigma \rangle^{1-4\delta} \langle \xi_2 \rangle^{1-4\delta} \langle \sigma_2 \rangle^{1+4\delta} \gtrsim \langle \xi_1 \rangle^{(1-4\delta) \deg H} \langle \xi_2 \rangle^{1-4\delta} \langle \xi_1 \rangle^{(1-4\delta)(\deg P - 1)} \langle \tau_2 - P_2 \rangle^{1+4\delta}
= \langle \xi_1 \rangle^{(1-4\delta)(2(\deg P - 1))} \langle \xi_2 \rangle^{1-4\delta} \langle \tau_2 - P_2 \rangle^{1+4\delta}
\]
(5.8)

Since $\langle \xi_2 \rangle^{-2s-1+4\delta} \lesssim \langle \xi \rangle^{-2s-1+4\delta}$ we get
\[
I_1 \lesssim \int_{|\xi_1| \sim |\xi_2|} \frac{\langle \xi \rangle^2 \langle \xi_2 \rangle^{2s} \langle \xi_1 \rangle^{-4s}}{\langle \sigma_1 \rangle^{1-4\delta} \langle \sigma_2 \rangle} d\nu \lesssim \langle \xi \rangle^{-2s-2(\deg P - 2) + 4\delta(2 \deg P - 1)} \lesssim 1
\]
whenever $-(2 \deg P - 4) - 2s < 0$ and $\delta$ is small, so that we require
\[
s > -(\deg P - 2) \geq \nu
\]
(5.9)

which is fulfilled since $\deg P \geq 3$.

Now consider the set $B_2$. Here $|\xi_1| \sim |\xi_2|$ which simplifies $I$ to
\[
I \sim \int |\xi|^2 \langle \xi_2 \rangle^{2s} \langle \xi_1 \rangle^{-4s} \langle \sigma_1 \rangle^{1-4\delta} \langle \sigma_2 \rangle \ d\nu
\]
(5.10)

By (5.7), $\langle \sigma \rangle \gtrsim |\xi| \langle \xi_1 \rangle^{\deg P - 1}$ and similar to (5.8) we get
\[
N \gtrsim \langle \xi_1 \rangle^{(1-4\delta)(2(\deg P - 1))} |\xi|^{1-4\delta} \langle \tau_2 - P_2 \rangle^{1+4\delta}
\]
This gives
\[
I \lesssim \int_{|\xi| \sim |\xi_1|} \frac{|\xi|^{1+4\delta} \langle \xi_2 \rangle^{2s} \langle \xi_1 \rangle^{-4s-2(\deg P - 1)} + 8\delta(\deg P - 1)}{\langle \tau_2 - P_2 \rangle^{1+4\delta}} d\nu
\]
\[
\lesssim \langle \xi_1 \rangle^{-4s-2(\deg P - 2) + 8\delta(\deg P - 1)} \int_{|\xi| \sim |\xi_1|} \langle \xi \rangle^{2s + 1 + 4\delta} d\xi
\]
If $s < -1$ and $\delta$ is small enough then
\[
I \lesssim \langle \xi_1 \rangle^{-4s-2(\deg P - 1) + 4\delta(2 \deg P - 2)} \lesssim 1
\]
for $-4s-2(\deg P - 1) < 0$, i.e., $s > \nu$. If $s \geq -1$ then
\[
I \lesssim \langle \xi_1 \rangle^{-2s-2(\deg P - 4) + 4\delta(2 \deg P - 1)} \lesssim 1
\]
for $-2s-2(\deg P - 2) < 0$, i.e.,
\[
s > -(\deg P - 2) \geq \nu
\]
(5.11)

For the set $C_1$ we have, as in $B_1$, $|\xi| \sim |\xi_1|$. $I$ can be simplified to
\[
I \sim \int_{\nu} \frac{|\xi|^2 \langle \xi_2 \rangle^{-2s}}{\langle \sigma_1 \rangle \langle \sigma \rangle^{1-4\delta} \langle \sigma_2 \rangle} d\nu
\]
and \((\sigma_1) \geq \langle \xi_1 \rangle \langle \xi \rangle^{\deg P - 1}\) gives
\[
N \geq \langle \sigma_1 \rangle^{1 - 4\delta} \langle \sigma \rangle^{1 - 4\delta} \langle \sigma_2 \rangle^{1 + 4\delta} \geq \langle \xi_2 \rangle^{1 - 4\delta} \langle \xi \rangle^{(1 - 4\delta)(\deg P - 1)} \langle \xi \rangle^{1 - 4\delta} \deg H (\tau_2 - P_2)^{1 + 4\delta} = \langle \xi_2 \rangle^{1 - 4\delta} \langle \xi \rangle^{(1 - 4\delta)2(\deg P - 1)} (\tau_2 - P_2)^{1 + 4\delta}
\]
Since \((\xi_2)^{-2s - 1 + 4\delta} \lesssim \langle \xi \rangle^{-2s - 1 + 4\delta}\) we get
\[
I \lesssim \int_{|\xi| \sim |\xi_1|} \frac{|\xi|^2 \langle \xi \rangle^{-2s - 1 + 4\delta}}{\langle \xi \rangle^{(1 - 4\delta)2(\deg P - 1)} (\tau_2 - P_2)^{1 + 4\delta}} d\nu \lesssim \langle \xi \rangle^{-2s - 2(\deg P - 2) + 4\delta(2 \deg P - 1)}
\]
which is bounded when (5.11) holds and \(\delta\) is small.

Now consider the set \(C_2\). As in \(B_2\), \(|\xi_1| \sim |\xi_2|\) and \(I\) simplifies as in (5.10). Since \((\sigma_1) \geq |\xi| \langle \xi \rangle^{\deg P - 1}\) the denominator is bounded by
\[
N \geq \langle \sigma_1 \rangle^{1 - 5\delta} \langle \sigma \rangle^{1 + \delta} \langle \sigma_2 \rangle \geq |\xi|^{1 - 5\delta} \langle \xi_1 \rangle^{(1 - 5\delta)(\deg P - 1)} \langle \tau - P \rangle^{1 + \delta} \langle \xi \rangle^{\deg H}
\]
This gives
\[
I \lesssim \int_{|\xi| \leq |\xi_1|} \frac{|\xi|^{1 + 5\delta} \langle \xi \rangle^{2s - 4s - 2(\deg P - 1) + 5\delta(2 \deg P - 2)}}{(\tau - P)^{1 + 4\delta}} d\nu 
\lesssim \langle \xi \rangle^{-4s - 2(\deg P - 2) + 5\delta(2 \deg P - 1)} \int_{|\xi| \leq |\xi_1|} \langle \xi \rangle^{1 + 2s + 5\delta} d\xi
\]
which is similar to the case \(B_2\).

In \(D_1\) we have \(|\xi| \sim |\xi_2|\) so that \(I_1\) simplifies to
\[
I_1 \sim \int \frac{|\xi|^2 \langle \xi_1 \rangle^{-2s}}{\langle \sigma_1 \rangle^{1 - 4\delta} \langle \sigma_2 \rangle} d\nu_1 \tag{5.12}
\]
Since \((\sigma) \geq \langle \xi_1 \rangle \langle \xi \rangle^{\deg P - 1}\) we get
\[
N \geq \langle \sigma_1 \rangle^{1 + \delta} \langle \sigma \rangle^{1 - 5\delta} \langle \sigma_2 \rangle \geq \langle \tau_1 - P_1 \rangle^{1 + \delta} \langle \xi_1 \rangle^{1 - 5\delta} \langle \xi \rangle^{(1 - 5\delta)(\deg P - 1)} \langle \xi \rangle^{\deg H} = \langle \tau_1 - P_1 \rangle^{1 + \delta} \langle \xi_1 \rangle^{1 - 5\delta} \langle \xi \rangle^{(1 - 5\delta)(2 \deg P - 2)} \tag{5.13}
\]
Since \(s < 0\) we get the bound
\[
I_1 \lesssim \int_{|\xi| \leq |\xi_1|} \frac{\langle \xi \rangle^{-2(\deg P - 2) + 5\delta(2 \deg P - 2)} \langle \xi_1 \rangle^{-2s - 1 + 5\delta}}{(\tau_1 - P_1)^{1 + \delta}} d\nu_1
\lesssim \langle \xi \rangle^{-2(\deg P - 2) + 5\delta(2 \deg P - 2)} \int_{|\xi| \leq |\xi_1|} \langle \xi_1 \rangle^{-2s - 1 + 5\delta} d\xi_1 \lesssim \langle \xi \rangle^{-2s - 2(\deg P - 2) + 5\delta(2 \deg P - 1)}
\]
which is bounded if \(s > s_c\).

In the set \(D_2\) \(|\xi| \sim |\xi_2|\) holds as in \(D_1\) and we estimate (5.12). Since \((\sigma_1) \geq \langle \xi_1 \rangle \langle \xi \rangle^{\deg P - 1}\) we have
\[
N \geq \langle \sigma_1 \rangle^{1 - 4\delta} \langle \sigma \rangle^{1 - 4\delta} \langle \sigma_2 \rangle^{1 + 4\delta} \geq \langle \xi_1 \rangle^{1 - 4\delta} \langle \xi \rangle^{(1 - 4\delta)(\deg P - 1)} \langle \xi \rangle^{(1 - 4\delta) \deg H} (\tau_2 - P_2)^{1 + 4\delta} = \langle \xi_1 \rangle^{1 - 4\delta} \langle \xi \rangle^{(1 - 4\delta)(2 \deg P - 2)} (\tau_2 - P_2)^{1 + 4\delta}
\]
Since $s < 0$

$$I_1 \lesssim \int_{|\xi_1| \leq |\xi|} \frac{\langle \xi \rangle^{-2(\deg P - 2) + 4\delta(2 \deg P - 2)} \langle \xi_1 \rangle^{-2s - 1 + 4\delta}}{\langle \tau_2 - P_2 \rangle^{1 + \delta}} \, d\nu_1$$

$$\lesssim \langle \xi \rangle^{-2(\deg P - 2) + 4\delta(2 \deg P - 2)} \int_{|\xi_1| \leq |\xi|} \langle \xi_1 \rangle^{-2s - 1 + 4\delta} \, d\xi_1 \sim \langle \xi \rangle^{-2s - (2 \deg P - 4) + 4\delta(2 \deg P - 1)}$$

which is similar to the case $D_1$.

The case where $H$ dominates is much simpler and doesn’t require any resonance function estimates.

**Theorem 5.4** Let $H = H(\xi)$ be bounded for small $|\xi|$ and $H(\xi) \sim \langle \xi \rangle^{\deg H}$, $\deg H \geq \max(3, \deg P - 1)$ for large $|\xi|$, $s \in (s_c - 1/2)$. Then for all $u$ and $v$ with compact support in $[-T, T]$

$$\|\partial_x (uv)\|_{X^{s, -1/2 + \delta}} \lesssim T^\delta \|u\|_{X^{s, 1/2}} \|v\|_{X^{s, 1/2}}$$

**Proof:** We proceed as in Theorem 5.3. Since $\deg H > 1$ the estimate on $A$ is clear. In $B_1$, $|\xi_2| \lesssim |\xi_1|$ and $s \leq 0$ yielding

$$I_1 \sim \int_{|\xi_1| \leq |\xi|} \frac{|\xi|^2 |\xi_2|^{-2s}}{(\xi_1)^{1 - 4\delta} \deg H (\xi_1)^{1 - 4\delta} \deg H (\tau_2 - P_2)^{1 + \delta}} \, d\nu_1$$

$$\lesssim \int_{|\xi_1| \leq |\xi|} \langle \xi_1 \rangle^{2 - 2s - 2 \deg H + 8\delta \deg H} \, d\xi_1 \sim \langle \xi_1 \rangle^{3 - 2s - 2 \deg H + 8\delta \deg H} \lesssim 1$$

if $s > 3/2 - \deg H$ which is fulfilled since $s > s_c$ and $\deg H \geq 3$.

In $B_2$ $I$ is bounded by

$$\int_{B_2} \langle \xi_1 \rangle^{1 - 4\delta} \deg H \langle \xi_2 \rangle^{\deg H (\tau - P)^{1 + \delta}} \, d\nu \lesssim \langle \xi_1 \rangle^{-4s - 2 \deg H + 5\delta \deg H} \int_{|\xi| \leq |\xi_1|} \langle \xi \rangle^{2 + 2s} \, d\xi$$

If $s \geq -3/2$ then

$$I \lesssim \langle \xi_1 \rangle^{3 - 2s - 2 \deg H + 5\delta \deg H}$$

which is similar to the case $B_1$. If $s < -3/2$ then the integral converges and $I \lesssim 1$ since $s > s_c$.

In $C_1$, $|\xi_2| \lesssim |\xi|$ and $s \leq 0$ which gives

$$I_1 \sim \int_{|\xi| \leq |\xi_1|} \frac{|\xi|^2 |\xi_2|^{-2s}}{(\xi_1)^{1 - 4\delta} \deg H (\xi_1)^{1 - 4\delta} \deg H (\tau_2 - P_2)^{1 + \delta}} \, d\nu_1 \lesssim \langle \xi_1 \rangle^{3 - 2s - 2 \deg H + 5\delta \deg H}$$

which again is similar to $B_1$.

In $C_2$ we bound $I$ as

$$\int_{C_2} \langle \xi_1 \rangle^{1 - 4\delta} \deg H \langle \xi_2 \rangle^{\deg H (\tau - P)^{1 + \delta}} \, d\nu \lesssim \langle \xi_1 \rangle^{-4s - 2 \deg H + 5\delta \deg H} \int_{|\xi| \leq |\xi_1|} \langle \xi \rangle^{2 + 2s} \, d\xi$$

We now conclude by arguing exactly as in $B_2$. 

□
5.1 Local well-posedness

**Theorem 5.5** Let \( s > s_c \) and assume \( u_0 \in H^s \). Then there exists a unique solution \( u \in X^{s,1/2} \) to \((1.1)\) on \([0,T]\), where \( T \sim \|u_0\|_{H^s}^{-2/\delta} \) for some small \( \delta > 0 \). The solution depends continuously on initial data.

**Proof:** For any \( u_0 \in H^s \) define a ball, \( X^{s,1/2}(M) \), in \( X^{s,1/2} \) by
\[
X^{s,1/2}(M) = \left\{ u \in X^{s,1/2} \mid \|u\|_{X^{s,1/2}} \leq M \right\}
\]
where \( M \) is a number to be determined later. Define an operator \( \Pi \) on \( X^{s,1/2}(M) \) by
\[
\Pi(u) = \theta(t)S(t)u_0 - \chi_+(t)\theta(t) \int_0^t S(t-s)\partial_x(\theta_T^2(s)u^2(s)) \, ds
\]
By Lemma \[4.3\], Proposition \[4.1\], Theorem \[5.3\] and Lemma \[1.2\],
\[
\|\Pi(u)\|_{X^{s,1/2}} \leq \|\theta(t)S(t)u_0\|_{X^{s,1/2}} + \|\chi_+\theta \int_0^t S(t-s)\partial_x(\theta_T^2(s)u^2(s)) \, ds\|_{X^{s,1/2}} \leq C \|u_0\|_{H^s} + C \|u_0\|_{H^s} + T^\delta \|\theta_T u\|^2_{X^{s,1/2}} \leq C \|u_0\|_{H^s} + T^\delta \|u\|^2_{X^{s,1/2}} \quad (5.14)
\]
For \( M > 2C\|u_0\|_{H^s} \), we have
\[
\|\Pi(u)\|_{X^{s,1/2}} \leq M/2 + T^\delta M^2 = M(1/2 + T^\delta M)
\]
and hence \( \Pi \) maps into \( X^{s,1/2}(M) \) if \( T \lesssim M^{-1/\delta} \). Since \( u^2 - v^2 = (u-v)(u+v) \) the same calculation apart from the free term gives
\[
\|\Pi(u) - \Pi(v)\|_{X^{s,1/2}} = \left\| \chi_+ \theta \int_0^t S(t-s)\partial_x(\theta_T^2(s)(u-v)(u+v)(s)) \, ds \right\|_{X^{s,1/2}} \leq C \|\partial_x(\theta_T(u-v)(u+v))\|_{X^{s,1/2} + 2/\delta} \leq C T^\delta \|\theta_T(u-v)\|_{X^{s,1/2}} \|\theta_T(u+v)\|_{X^{s,1/2}} \leq C T^\delta \|u-v\|_{X^{s,1/2}} \|u+v\|_{X^{s,1/2}} \leq C 2MT^\delta \|u-v\|_{X^{s,1/2}} \quad (5.15)
\]
which shows that \( \Pi \) is a contraction for small enough \( T \)'s. We have \( T \lesssim \|u_0\|_{H^s}^{-1/\delta} \). By the contraction mapping principle there exists a unique solution to \((1.1)\) in \( X^{s,1/2}(M) \).

As for the dependence on initial data, let \( u \) and \( v \) be solutions with initial data \( u_0 \) and \( v_0 \), respectively. By \[5.14\] and \[5.15\],
\[
\|u-v\|_{X^{s,1/2}} = \|\Pi(u) - \Pi(v)\|_{X^{s,1/2}} \leq \|\theta(t)S(t)(u_0 - v_0)\|_{X^{s,1/2}} + \frac{1}{2} \|u-v\|_{X^{s,1/2}} \leq C \|u_0 - v_0\|_{H^s} + \frac{1}{2} \|u-v\|_{X^{s,1/2}}
\]
for small enough \( T \)'s so that
\[
\|u-v\|_{X^{s,1/2}} \lesssim \|u_0 - v_0\|_{H^s}
\]
which proves continuity with respect to initial data.

\[\square\]
5.2 Uniqueness in the larger localized space \(X_{T^*}^{s,1/2}\)

Let \(u \in X^{s,1/2}(M)\) be the solution to the localized equation (2.2) and \(\hat{v} \in X_{T^*}^{s,1/2}\) a solution to (2.1) with identical initial data. Fix a \(T^* \in (0, T)\) and a \(v \in X^{s,1/2}\) which agrees with \(\hat{v}\) on \([0, T^*]\). Then \(v\) solves (2.2) for \(t \in [0, T^*]\). Now, \(u - v \in X^{s,1/2}\) and hence \(u - v \in X_{T^*}^{s,1/2}\) so for every \(\varepsilon > 0\) there exists a \(w \in X^{s,1/2}\) that agrees with \(u - v\) on \([0, T^*]\) and

\[
\|w\|_{X_{T^*}^{s,1/2}} \leq \|u - v\|_{X_{T^*}^{s,1/2}} + \varepsilon
\]  

(5.16)

For \(u, v,\) and \(w\) define

\[
\ddot{w} = \theta_{2T^*} \chi_+(t) \int_0^t S(t-r) \partial_x (\theta_{2T^*}^2 (r) w(r)(u(r) + v(r))) \, dr
\]

\[
= \theta_{2T^*} \chi_+(t) \int_0^t S(t-r) \partial_x (\theta_{2T^*}^2 (r) (u^2(r) - v^2(r))) \, dr
\]

\[
= \theta_{2T^*} \chi_+(t) \int_0^t S(t-r) \partial_x (\theta_{2T^*}^2 (r) v^2(r)) \, dr = v(t) - u(t) = w(t)
\]

on \([0, T^*]\). By the very definition of the localized norm, Proposition 4.4, Theorem 5.3, Lemma 4.2, and (5.16)

\[
\|u - v\|_{X_{T^*}^{s,1/2}} \leq \|\dot{w}\|_{X_{T^*}^{s,1/2}} \lesssim \|\partial_x (\theta_{2T^*}^2 w(u + v))\|_{X_{T^*}^{s-1/2,\delta}}
\]

\[
\lesssim T^* \|\theta_{2T^*} w\|_{X_{T^*}^{s,1/2}} \|\theta_{2T^*} (u + v)\|_{X_{T^*}^{s,1/2}}
\]

\[
\lesssim T^* \|w\|_{X_{T^*}^{s,1/2}} \|(u + v)\|_{X_{T^*}^{s,1/2}}
\]

\[
\lesssim T^* \|w\|_{X_{T^*}^{s,1/2}} 2 \max(\|u\|_{X_{T^*}^{s,1/2}}, \|v\|_{X_{T^*}^{s,1/2}})
\]

\[
\leq c T^* \max(\|u\|_{X_{T^*}^{s,1/2}}, \|v\|_{X_{T^*}^{s,1/2}}) \left(\|u - v\|_{X_{T^*}^{s,1/2}} + \varepsilon\right)
\]

For \(T^*\) so small that

\[
c T^* \max(\|u\|_{X_{T^*}^{s,1/2}}, \|v\|_{X_{T^*}^{s,1/2}}) \leq 1/2
\]

we get

\[
\|u - v\|_{X_{T^*}^{s,1/2}} \leq \varepsilon
\]

and since the choice of \(T^*\) is independent on \(\varepsilon, u = v\) on \([0, T^*]\). This independence implies that the argument can be restarted at \(T^*\) and up to \(\min(2T^*, T)\) and similarly beyond \(2T^*\). After a finite number of iterations we conclude that \(u = v\) on \([0, T]\).

\[\square\]
5.3 Continuity in time

Proposition 5.6 Let $s \in \mathbb{R}$ and $\delta > 0$. For all $F \in X^{s,-1/2+\delta}$,

$$t \mapsto \int_0^t S(t-r)F(r) \, dr \in C([0, \infty), H^{s+\delta, \text{deg} H})$$

Proof: Let $J^{-s} = \mathcal{F}_x^{-1}[(\xi)^s]$ be the Sobolev potential and set $G = U(-t)J^{-s}[F]$ so that $\|G\|_{X^{s,b}_t} = \|F\|_{X^{s,b}}$. Since $\hat{S}(t) = \hat{U}(t)\hat{V}(t)$ we can write

$$\langle \xi \rangle^s \mathcal{F} \left[ \int_0^t S(t-r)F(r) \, dr \right] = \hat{\bar{U}}(t) \int_0^t \hat{\bar{V}}(t-r) \hat{\bar{U}}(-r) \hat{F}(r) \, dr = \hat{\bar{U}}(t) \int_0^t \hat{\bar{V}}(t-r) \hat{G}(r) \, dr = \hat{\bar{U}}(t) I(t, \xi)$$

Hence we may assume $s = 0$. It is easy to show that

$$I(t, \xi) = \int_0^t \hat{\bar{V}}(t-r) \hat{G}(r) \, dr = \int_\tau \hat{\bar{G}}(\tau, \xi) \frac{e^{i\tau t} - e^{-H(\xi)t}}{H(\xi) + i\tau} \, d\tau = \int_\tau g_{\xi,t}(\tau) \, d\tau \quad (5.17)$$

We have $g_{\xi,t}(\tau) \in L^1_\tau$ for a.e. $(\xi, t)$. In fact, if $|H| \geq 1$ then by Cauchy-Schwartz’s inequality

$$\int_\tau |g_{\xi,t}(\tau)| \, d\tau \lesssim \int_\tau \frac{|\hat{\bar{G}}(\tau, \xi)|}{|H(\xi) + i\tau|} \langle \tau \rangle^{1/2-\delta} \, d\tau \leq \|\langle \tau \rangle^{1/2+\delta} \hat{\bar{G}}(\tau, \xi)\|_{L^2_\tau} \left( \int_\tau \frac{\langle \tau \rangle^{1-2\delta}}{|H(\xi) + i\tau|^2} \, d\tau \right)^{1/2} \leq \|\langle \tau \rangle^{1-2\delta} \hat{\bar{G}}(\tau, \xi)\|_{L^2_\tau} \langle H \rangle^{-\delta} < \infty \quad (5.18)$$

uniformly in $t$ since

$$\int_\tau \frac{\langle \tau \rangle^{1-2\delta}}{|H(\xi) + i\tau|^2} \, d\tau \lesssim \langle H \rangle^{-2\delta}$$

If $|H| < 1$ then we split $I$ into $I_1$ and $I_2$ as

$$I(t, \xi) = \left( \int_{|\tau| \leq 1} + \int_{|\tau| > 1} \right) g_{\xi,t}(\tau) \, d\tau = I_1(t, \xi) + I_2(t, \xi)$$

For $I_2$ we reuse the calculation in (5.18) to get

$$|I_2(t, \xi)| \left( \int_{|\tau| > 1} \frac{\langle \tau \rangle^{1-2\delta}}{|\tau|^2} \, d\tau \right)^{1/2} \lesssim \|\langle \tau \rangle^{1-2\delta} \hat{\bar{G}}(\tau, \xi)\|_{L^2_\tau} \quad (5.19)$$

For $I_1$ we note that the function $f(t) = e^{i\tau t} - e^{-Ht}$, $t \in [0, 1]$, goes through the origin and has a derivative uniformly bounded as $|f'(t)| \leq |\tau| + |H|e^{iH} \lesssim 1$ and thus $|f(t)| \lesssim |t|$. On the other hand, a
Taylor expansion shows that \(|f(t)| \lesssim |t||\tau + H| + C t^{2} (H^{2} + \tau^{2}) \lesssim |t||\tau + H|(1 + |t||\tau + H|) \lesssim |t||\tau + H|.

By Cauchy-Schwartz’s inequality this gives

\[|I_{1}(t, \xi)| \lesssim |t| \int_{|\tau| \leq 1} |\tilde{G}(\tau, \xi)| \, d\tau \lesssim |t| \cdot \|\langle \tau + H \rangle^{-1/2+\delta} \tilde{G}(\tau, \xi)\|_{L_{x}^{2}}\]

and hence \(g_{\xi,t}\) is in \(L_{x}^{1}\) for a.e. \((\xi, t)\) even in this case. By continuity in \(t\) of \(g_{\xi,t}\), \(I\) is continuous in \(t\) for a.e. \(\xi\). In addition, \((5.18)\) holds generally. Since \(U\) is unitary and by \((5.19)\) we have the uniform bound

\[
\left\| \int_{0}^{t} S(t - r) F(r) \, dr \right\|_{H^{4}\cdot \text{deg} H} = \left\| I(t, \cdot) \right\|_{H^{4}\cdot \text{deg} H} \lesssim \left\| \langle \tau + H \rangle^{-1/2+\delta} \tilde{G}\|_{L_{x}^{2}}(H)^{-\delta} \|_{L_{x}^{2}} \right\|_{L_{x}^{2}} \sim \left\| \langle \tau + H \rangle^{-1/2+\delta} \tilde{G}(\tau, \xi)\|_{L_{x}^{2}} \right\|_{L_{x}^{2}} = \|F\|_{X^{0,-1/2+\delta}}
\]

The result now follows from the Lebesgue domination theorem.

\[\square\]

**Proposition 5.7** If \(u_{0} \in H^{s}\) then

\[
t \mapsto S(t)u_{0} \in C([0, \infty), H^{s}) \cap C^{\infty}((0, \infty), H^{\infty})
\]

**Proof:** The map \(t \mapsto \langle \xi \rangle^{s}\hat{S}(t)\hat{u}_{0}, t \in [0, \infty), \) is continuous for a.e. \(\xi\) and also approximately bounded by \(\langle \xi \rangle^{s}\hat{u}_{0}\) in \(L^{2}\). By Lebesgue domination theorem the map is in \(C([0, \infty), H^{s})\). If \(t > 0\) then

\[
\langle \xi \rangle^{s}\hat{S}(t)\hat{u}_{0} = \chi_{\{H \leq 0\}} \langle \xi \rangle^{s-s}\hat{S}(t)\langle \xi \rangle^{s}\hat{u}_{0} + \chi_{\{H > 0\}} \langle \xi \rangle^{s-s}\hat{S}(t)\langle \xi \rangle^{s}\hat{u}_{0}
\]

Note that \(d^{n}/dt^n\hat{S}(t) = (iP - H)^{n}\hat{S}(t)\). Since the set \(\{H \leq 0\}\) is bounded \(\langle \xi \rangle^{s-s}(iP - H)^{n}\hat{S}(t)\| \lesssim 1\). Also, for some positive constant \(c_{s}\)

\[
\chi_{\{H > 0\}} \langle \xi \rangle^{s-s}(iP - H)^{n}\hat{S}(t) \lesssim \chi_{\{H > 0\}} \langle \xi \rangle^{s-n} e^{-ct} \langle \xi \rangle^{\text{deg} H} \in L_{x}^{2}
\]

It follows that the map \(t \mapsto \langle \xi \rangle^{s}\hat{S}(t)\hat{u}_{0}\) is in \(C^{n}((0, \infty), H^{s})\) for every \(n \in \mathbb{N}\) and \(s \in \mathbb{R}\).

\[\square\]

**Proposition 5.8** Let \(u_{0} \in H^{s}\). Then \(u \in C^{\infty}((0, T), H^{\infty})\).

**Proof:** For every \(t \in (0, T)\) there is a subsequence \(0 = t_{0} < t_{1} < t_{2} < \ldots < t_{n} \nearrow t\). Let \(u_{n}\) be the sequence of solutions starting at \(t = t_{n}\) with initial value \(u_{n-1}(t_{n})\) for \(n \geq 1\) and \(u_{0}(t, x)\) the solution starting at \(t = t_{0} = 0\) with initial value \(u_{0}(x)\). Uniqueness implies that \(u_{n}(t, x) = u_{0}(t, x)\) for \(t \geq t_{n}\). By Lemmas \((5.6)\) and \((5.7)\) \(u_{1} \in C((t_{1}, T), H^{s+2\delta\cdot \text{deg} H})\), \(u_{2} \in C((t_{2}, T), H^{s+3\delta\cdot \text{deg} H})\) and so on. It follows that \(u \in C((t, T), H^{\infty})\) for every \(t > 0\) which proves the Lemma. By \((2.2)\) and Lemma \((5.7)\) it is seen that \(\partial_{t} u \in C((0, T), H^{\infty})\), i.e., \(u \in C^{1}((0, T), H^{\infty})\). By differentiating \((2.2)\) it follows immediately that \(u \in C^{2}((0, T), H^{\infty})\). Iteration gives the result.

\[\square\]
5.4 Global well-posedness

The following result shows that the $L^2$-norm in $x$ grows not faster than exponentially in time. Hence, lower order Sobolev norms never explode and, since the maximal time in the local existence result only depends on the size of the Sobolev-norm of the initial value, the proof can be iterated an arbitrary number of times. This gives global well-posedness.

**Theorem 5.9** Let $H(\xi) = \beta(\xi) + a_0|\xi|^{\deg H}$ for some $\beta \in L^\infty$ and $a_0 > 0$. Then the $L^2$-norm in $x$ of the solution grows at most exponentially.

**Proof:** We will show that the $L^2$-norm in space of the solution grows at most exponentially in time thereby preventing lower order Sobolev norms from exploding. By the regularity of the solution, the original equation (1.1) is satisfied. Since the initial value is real the solution must also be real. Multiply the equation by $u$ and integrate in $x$,

$$\frac{1}{2} \partial_t \|u\|_{L^2}^2 - i \int u P(D) u \, dx + \int u H(D) u \, dx + 2 \int u^2 \partial_x u \, dx = 0 \quad (5.20)$$

Fix $t$ and let $\varphi_n \in C_0(\mathbb{R})$ converge to $u = u(t)$ in $L^\infty$. By sublinearity the maximal function, $M(u)$, of $u$ satisfies

$$M(u) = M(u - \varphi_n + \varphi_n) \leq M(u - \varphi_n) + M(\varphi_n)$$

so that

$$M(u) - M(\varphi_n) \leq M(u - \varphi_n)$$

Changing places of the functions and using that $M(-f) = M(f)$ gives

$$|M(u) - M(\varphi_n)| \leq M(u - \varphi_n)$$

Since $M : L^\infty \mapsto L^\infty$ trivially

$$\|M(u) - M(\varphi_n)\|_{L^\infty} \leq \|M(u - \varphi_n)\|_{L^\infty} \leq \|u - \varphi_n\|_{L^\infty} < \varepsilon$$

for $n$ large enough. Hence

$$M(u) \leq \varepsilon + M(\varphi_n)$$

For every $n$ we have $\varphi_n \lesssim 1_{\text{supp} \varphi_n}$ so that

$$M(\varphi_n)(x) \lesssim M(1_{\text{supp} \varphi_n})(x) \lesssim \langle x \rangle^{-1}$$

where the implied constant depends on $n$. Since $M(u)$ controls $u$ pointwise a.e.

$$|u(x)| \leq M(u)(x) \leq \varepsilon + M(\varphi_n)(x) \lesssim \varepsilon + \langle x \rangle^{-1}$$

and by continuity of $u$ this holds everywhere. Since $\varepsilon$ is arbitrary, $\lim_{|x| \to \infty} |u(t,x)| = 0$ for every $t > 0$. This yields that last term in (5.20) is

$$2 \int u^2 \partial_x u \, dx = \frac{2}{3} \int \partial_x (u^3) \, dx = 0$$
By Parseval’s relation
\[ \int uH(D)u \, dx = \int H(\xi)|\hat{u}|^2 \, d\xi \]
is real since \( H \) is. Similarly,
\[ i \int uP(D)u \, dx = i \int P(\xi)|\hat{u}|^2 \, d\xi \]
is imaginary since \( P \) is real and thus this term disappears since all other terms in \((5.20)\) are real. Hence,
\[ \frac{1}{2} \partial_t \|u\|^2_{L^2} = - \int H(\xi)|\hat{u}|^2 \, d\xi = - \int \beta(\xi)|\hat{u}|^2 \, d\xi - a_0 \int |\xi|^\deg H|\hat{u}|^2 \, d\xi \leq \|\beta\|_{L^\infty} \|u\|_{L^2} \]
Integrating this differential inequality yields that for \( t \geq T/2 \)
\[ \|u(t)\|^2_{L^2} \leq \|u(T/2)\|^2_{L^2} e^{c(t-T/2)} \]
which ends the proof.

\[\Box\]

6 Ill-posedness

For a contraction argument to hold we need a space \( Y_T \hookrightarrow C([0,T], H^s) \) together with the inequalities (1) and (2) below. The following Theorem shows that for \( s < s_c \) such a space does not exist.

**Theorem 6.1** Let \( s < s_c \) and \( T > 0 \). Then there does not exist a space \( Y_T \hookrightarrow C([0,T], H^s) \) such that

\[ (1) \quad \|S(t)\phi\|_{Y_T} \lesssim \|\phi\|_{H^s} \quad \forall \phi \in H^s \]
\[ (2) \quad \left\| \int_0^t S(t-s)\partial_x u^2(s) \, ds \right\|_{Y_T} \lesssim \|u\|^2_{Y_T} \quad \forall u \in Y_T \]

**Proof:** Suppose, on the contrary, that a space such as \( Y_T \) above exists. For \( \phi \in H^s \) set \( u(t) = S(t)\phi \). With this choice of \( u \) we use the continuous embedding, (2), and (1) to get
\[ \left\| \int_0^t S(t-s)\partial_x u^2(s) \, ds \right\|_{H^s} \lesssim \left\| \int_0^t S(t-s)\partial_x u^2(s) \, ds \right\|_{Y_T} \lesssim \|u\|^2_{Y_T} \lesssim \|\phi\|^2_{H^s} \]
for every \( \phi \in H^s \). Define a sequence \( \phi_n \) by its Fourier transforms as
\[ \hat{\phi}_n(\xi) = n^{-s} \gamma^{-1/2}(1_{I_n}(\xi) + 1_{I_n}(-\xi)) \]
where \( 1_A \) is the characteristic function of the set \( A \), \( 1 \sim \gamma \), and \( I_n = [n, n + 2\gamma] \). Then \( \|\phi_n\|_{H^s} \sim 1 \). Define two sequences by
\[ u_{1,n}(t) = S(t)\phi_n \]
\[ u_{2,n}(t) = \int_0^t S(t-s)\partial_x u_{1,n}^2(s) \, ds \]

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Taking the Fourier transform in $x$ of $u_{2,n}(t, x)$ yields
\[
    \mathcal{F}_x[u_{2,n}(t)] = \int_0^t e^{(t-s)(iP-H)} i \xi \{ \mathcal{F}_x[u_{1,n}(s)] * \mathcal{F}_x[u_{1,n}(s)] \} \, d\xi \, ds
\]
(6.1)

The convolution in $\xi$ is
\[
(\mathcal{F}_x[u_{1,n}(s)] * \mathcal{F}_x[u_{1,n}(s)])(\xi) = (e^{i\xi P - sH} \hat{\phi}_n) * (e^{i\xi P - sH} \hat{\phi}_n)(\xi)
= \int \hat{\phi}_n(\xi_1) \hat{\phi}_n(\xi_2) e^{i\xi P - sH} \, d\xi_1 \, d\xi_2
\]

By the Fubini theorem and the notation that $R$ is the resonance function for $P$
\[
(\mathcal{F}_x[u_{2,n}(t)])(\xi) = \int_0^t e^{(t-s)(iP-H)} i \xi \int \hat{\phi}_n(\xi_1) \hat{\phi}_n(\xi_2) e^{i\xi P - sH} \, d\xi_1 \, d\xi_2
= i \xi \tilde{S}(t) \int \hat{\phi}_n(\xi_1) \hat{\phi}_n(\xi_2) \int_0^t e^{i\xi R(P) - sR(H)} \, ds \, d\xi_1 \, d\xi_2
= i \xi \tilde{S}(t) \int \hat{\phi}_n(\xi_1) \hat{\phi}_n(\xi_2) e^{i\xi R(P) - tR(H)} \, dt \, d\xi_1 \, d\xi_2
\]

Note that $|\xi| > 2\gamma$ implies that $\hat{\phi}_n(\xi_1) \hat{\phi}_n(\xi_2) = 0$, i.e., $\text{supp} \mathcal{F}[u_{2,n}(t)] \subset [-2\gamma, 2\gamma]$. Thus, since $\gamma \sim 1$
implies $(\xi)^* \sim 1$,
\[
    ||u_{2,n}(t)||_{H^s} \sim \int_{-2\gamma}^{2\gamma} |\xi|^2 e^{-2tH(\xi)} \left| \int \hat{\phi}_n(\xi_1) \hat{\phi}_n(\xi_2) e^{i\xi R(P) - tR(H)} - 1 \, d\xi_1 \right|^2 \, d\xi
\]
\[
\leq \frac{1}{n^{4s}\gamma} \int_{-2\gamma}^{2\gamma} |\xi|^2 \left| \int_{K_n} e^{i\xi R(P) - t(H_1 + H_2) - e^{-tH}} - e^{-tH} \, d\xi_1 \right|^2 \, d\xi
\]

where
\[
K_n = K_n(\xi) = \{ \xi_1 \in I_n : -\xi - \xi_1 \in I_n \} \cup \{ -\xi_1 \in I_n : \xi - \xi_1 \in I_n \}
\]

If $|\xi| \leq \gamma$ then $|K_n| \geq 2\gamma$. By Proposition 5.1
\[
|\mathcal{R}[R(P)]| \sim \gamma n^{\deg P - 1}
\]
and it is trivial that $|R(H)| \sim n^{\deg H}$. Note that, for large $n$ and since $|H(\xi)| \geq c_1 n^{\deg H}$ and $|H(\xi)| \leq c_2$ we have
\[
\Re \left( e^{i\xi R(P) - t(H_1 + H_2)} - e^{-tH} \right) \leq e^{-tH} c_1 n^{\deg H} - e^{-c_2 t} \leq -\frac{1}{2} e^{-c_2 t}
\]

An estimate of the imaginary part gives the exact same bound. This gives
\[
||u_{2,n}(t)||_{H^s}^2 \gtrsim \frac{1}{n^{4s}\gamma (\gamma n^{\deg P - 1} + n^{\deg H})} \int_{-\gamma}^{\gamma} |\xi|^2 e^{-c_2 t} \, d\xi
\]
\[
\gtrsim \frac{1}{n^{4s+\max(\deg P - 1, \deg H)}} \rightarrow \infty
\]
as $n \rightarrow \infty$ when $s < -1/2 \max(\deg P - 1, \deg H)$.

Remark As in \[\square\] this can be used to prove that the mapping
\[
H^s \ni u_0 \mapsto u \in C([0, T], H^s)
\]
is not $C^2$ at the origin.
References

[1] Biagioni, H. A., Linares, F. (1997). On the Benney-Lin and Kawahara Equations. *Journal of Mathematical Analysis and Applications*. **211**: 131-152.

[2] Chen, W., Li, J. (2008). On the low regularity of the Benney-Lin equation. *Mathematical Analysis and Applications*. No. 339: 1134-1147.

[3] Carvajal, X., Panthee, M. (2012). Well-posedness of KdV type equations. *Electron. J. Differential Equations*. No. 40: 1-15.

[4] Carvajal, X., Scialom, M. (2005). On the well-posedness for the generalized Ostrovsky, Stepanyams and Tsimring equation. *Nonlinear Analysis*. **62**:1277-1287.

[5] Esfahani, A. (2013). Sharp well-posedness of the Ostrovsky, Stepanyams and Tsimring equation. *Mathematical Communications* **18**: 323-335.

[6] Korsunsky, S. (1997). Nonlinear waves in dispersive and dissipative systems with coupled fields. *Pitman Monographs and Surveys in Pure and Applied Mathematics 83*.

[7] Molinet, L., Ribaud, F. (2002). On the low regularity of the Korteveg-de Vries-Burgers equation. *International Mathematics Research Notices* No. 37: 19792005.

[8] Molinet, L., vento, S. (2011). Sharp ill-posedness and well-posedness results for the KdV-Burgers equation: the real line case. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 10, no. 3: 531560.

[9] Ostrowsky, L. A., Stepanyants, Y. A., Tsimring, L. S. (1983). Radiation instability in a stratified shear flow. *J. Non-Linear Mechanics* **19**, No. 2: 151-161.

[10] Zhao, X. Q., Cui, S. B. (2009). On Cauchy problem of the Benney-Lin Equation with Low Regularity Initial Data. *Acta Mathematica Sinica, English Series* **25**, No. 12: 2157-2168.

[11] Zhao, X. Q., Cui, S. B. (2009). Local well-posedness of the Ostrovsky Stepanyams Tsimring equation in Sobolev spaces negative indices. *Nonlinear Analysis*. **70**: 3483-3501.

[12] Zhao, X. Q., Cui, S. B. (2008). Well-posedness of the Cauchy problem for Ostrovsky Stepanyams Tsimring equation with low regularity data. *J. Math. Anal. Appl.* No. 334: 778-787.