THE STEADY STATE SOLUTIONS TO THERMOHALINE CIRCULATION EQUATIONS

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ABSTRACT. In the article, we study the existence and the regularity of the steady state solutions to thermohaline circulation equations. Firstly, we obtain a sufficient condition of the existence of weak solutions to the equations by acute angle theory of weakly continuous operator. Secondly, we prove the existence of strong solutions to the equations by ADN theory and iteration procedure. Furthermore, we study the generic property of the solutions by Sard-Smale theorem and the existence of classical solutions by ADN theorem.

1. Introduction. In this paper, we investigate the existence and the regularity of steady state solutions to the following initial-boundary problem of thermohaline circulation equations involving unknown functions \((u, T, S, p)\) at \((x, t) = (x_1, x_2, x_3, t) \in \Omega \times (0, \infty)\),

\[
\frac{\partial u}{\partial t} = Pr(\Delta u - \nabla p) + Pr(\bar{R}S - \text{sign}(S_0 - S_1)\bar{R}S)\vec{k} - (u \cdot \nabla)u, \tag{1}
\]

\[
\frac{\partial T}{\partial t} = \Delta T + u_3 - (u \cdot \nabla)T + Q, \tag{2}
\]

\[
\frac{\partial S}{\partial t} = Lc\Delta S + \text{sign}(S_0 - S_1)u_3 - (u \cdot \nabla)S, \tag{3}
\]

\[\text{div}u = 0, \tag{4}\]

where the unknown functions \(u = (u_1, u_2, u_3)\), \(T, S, p\) denote respectively velocity field, temperature, salinity and pressure, and \(Q(x)\) is heat source. \(\vec{k} = (0, 0, 1)\).

All of the parameters \(Pr, R, \bar{R}, Lc, S_0, S_1\) are positive constants. The region \(\Omega = D \times (0, 1)\) is bounded open set of \(R^3\), and \(D = \{(x_1, x_2)|x_1^2 + x_2^2 < r^2\}\) is a circular domain.

We consider the following boundary condition for the problem (1)-(4):

\[
u_n = 0, \frac{\partial u_3}{\partial n} = \frac{\partial T}{\partial n} = \frac{\partial S}{\partial n} = 0, \text{ on } \partial D, \tag{5}
\]

\[
u_3 = T = S = 0, \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0, \text{ at } x_3 = 0, 1. \tag{6}
\]

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Here $u_n$ and $u_3$ are normal and tangential component of the velocity field on the vertical side of cylinder $\Omega$ respectively.

The thermohaline circulation equations (1)-(4) describe the motion and the states of the thermohaline circulation. The thermohaline circulation is one important source of internal climate variability and the greatest oceanic current on the earth. The thermohaline circulation is also called the ocean conveyor belt, the great ocean conveyor, or the global conveyor belt. It works in a fashion similar to a conveyor belt transporting enormous volume of cold, salty water from the North Atlantic to the North Pacific, and bringing warmer fresher water in return. Physically speaking, the buoyancy fluxes at the ocean surface give rise to gradients in temperature and salinity, which produce, in turn, density gradients. These gradients are, overall, sharper in the vertical than in the horizontal and are associated therefore with an overturning or thermohaline circulation.

The thermohaline circulation varies on time-scales of decades or longer, so there have been extensive observational, physical and numerical studies\cite{1, 2, 3, 7, 10, 18, 19}. Henk A. Dijkstra and Michael Ghil \cite{1} reviewed recent theoretical and numerical results that helped explain the physical processes governing the large-scale ocean circulation and its intrinsic variability by applying systematically the methods of dynamical systems theory. Ma and Wang studied dynamic stability and transitions by using the above equations (1)-(4) \cite{10}. The equations governing the thermohaline circulation is the Navier-Stokes equations with the Coriolis force generated by the earth’s rotation, coupled with the first law of thermodynamics. Ma and Wang have received the equations (1)-(4) by nondimensional process and have obtained the condition for the transitions of the thermohaline circulation in \cite{10}, where they considered the equations on the square region $\Omega = (0, L_1) \times (0, L_2) \times (0, 1)$ because of eigenvalue problem. They gave the trivial steady state solution because of the neglect of the heat source. However, the heat source is a important factor for the thermohaline circulation, so we consider the heat source. Furthermore, we discuss the equations in the region $\Omega = D \times (0, 1)$ because the thermohaline circulation is periodic, where $D$ is a section and $[0, 1]$ is a length of a period. So the tube region $\Omega$ is $C^\infty$.

In this paper, we study the steady state solutions to the thermohaline circulation equations. The steady state solution is a special state of evolution equations and time-independent solution. The steady state solutions to the thermohaline circulation equations are significant to understand the dynamical behavior of the thermohaline circulation and are the main directions and important content in studying thermohaline circulation equations. The steady state solutions of other system have been studied in \cite{5, 6, 9, 15}. In paper \cite{6}, the author discussed the existence of the equilibrium solutions to the semilinear reaction diffusion system by fixed point theorem. In papers \cite{5} and \cite{15}, the authors discussed the existence of stationary solutions to the Navier-Stokes equations by Galerkin method and the generic properties by Sard-Smale theorem.

We discuss the existence, regularity and generic property of the steady state solutions to thermohaline circulation equations (1)-(4) with the boundary condition (5)-(6). That is to say that we discuss the following equations:

\begin{align}
\Delta u - \nabla p + (RT - \text{sign}(S_0 - S_1)\tilde{R}S)\tilde{k} - \frac{1}{Fr}(u \cdot \nabla)u &= 0 \quad x \in \Omega, \quad \text{(7)}
\end{align}

\begin{align}
\Delta T + u_3 - (u \cdot \nabla)T + Q &= 0, \quad x \in \Omega, \quad \text{(8)}
\end{align}
\begin{equation}
L_c \Delta S + \text{sign}(S_0 - S_1)u_3 - (u \cdot \nabla)S = 0, \quad x \in \Omega,
\end{equation}
\begin{equation}
div u = 0, \quad x \in \Omega,
\end{equation}
\begin{equation}
u_n = 0, \quad \frac{\partial u_3}{\partial n} = \frac{\partial T}{\partial n} = \frac{\partial S}{\partial n} = 0, \quad \text{on} \quad \partial \Omega,
\end{equation}
\begin{equation}
u_3 = T = S = 0, \quad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0, \quad x_3 = 0, 1.
\end{equation}

The paper is organized as follows. In section 2, we present preliminary results. In section 3, using an iteration procedure, we prove that these equations (7)-(12) possess steady state solutions in $W^{2,q}(\Omega, \mathbb{R}^3) \times W^{1,q}(\Omega), q \geq 2$ by acute angle theory of weakly continuous operator and ADN theorem. In section 4, we study the regularity and generic property of the solutions by Sard-Smale theorem and ADN theorem.

2. Preliminaries. Firstly, we present the eigenvalue results of the elliptic operator.

**Lemma 2.1.** For the eigenvalue equations
\[ \begin{aligned}
-\Delta u(x_1, x_2, x_3) &= \lambda u(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in D \times (0, 1), \\
u_n &= 0, \quad \frac{\partial u_3}{\partial n} = 0, \quad \text{on} \quad \partial D, \\
u_3 &= 0, \quad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0, \quad x_3 = 0, 1,
\end{aligned} \]
have eigenvalue $\{\lambda_k\}_{k=1}^\infty$, and $0 < \lambda_1 \leq \lambda_2 \leq \cdots, \lambda_k \to \infty, \quad \text{as} \quad k \to \infty$.

Secondly, we consider with Stokes equations
\[ \begin{aligned}
-\mu \Delta u + \nabla p &= f(x), \\
div u &= 0, \\
u_n &= 0, \quad \frac{\partial u_3}{\partial n} = 0, \quad \text{on} \quad \partial D, \\
u_3 &= 0, \quad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0, \quad x_3 = 0, 1.
\end{aligned} \]

**Lemma 2.2.** [15, 16] (ADN theory of Stokes equation) (1) Let $f \in C^{k,\alpha}(\Omega, \mathbb{R}^n)$, $k \geq 0$. If $(u, p) \in C^{2,\alpha}(\Omega, \mathbb{R}^n) \times C^{1,\alpha}(\Omega)$ is a solution of the equations (13), then the solution $(u, p) \in C^{k+2,\alpha}(\Omega, \mathbb{R}^n) \times C^{k+1,\alpha}(\Omega)$, and
\[ \|u\|_{C^{k+2,\alpha}} + \|p\|_{C^{k+1,\alpha}} \leq C(\|f\|_{C^{k,\alpha}} + \|(u, p)\|_{C^n}), \]
where $C > 0$ depends on $\mu, n, k, \alpha, \Omega$.

(2) Let $f \in W^{k,p}(\Omega, \mathbb{R}^n), k \geq 0$. If $(u, p) \in W^{2,p}(\Omega, \mathbb{R}^n) \times W^{1,p}(\Omega, \mathbb{R}^n)(1 < p < \infty)$ is a solution of the equations (13), then the solution $(u, p) \in W^{k+2,p}(\Omega, \mathbb{R}^n) \times W^{k+1,p}(\Omega, \mathbb{R}^n)$, and
\[ \|u\|_{W^{k+2,p}} + \|p\|_{W^{k+1,p}} \leq C(\|f\|_{W^{k,p}} + \|(u, p)\|_{L^p}), \]
where $C > 0$ depends on $\mu, n, k, \alpha, \Omega$.

Furthermore, we introduce the theory of weakly continuous operator.

Let $X$ be a linear space, $X_1$, $X_2$ two Banach spaces, $X_1$ separable, and $X_2$ reflexive. Let $X \subset X_2$. There exists a linear mapping
\[ L : X \to X_1, \]
which is one to one and dense.

**Definition 2.3.** A mapping $F : X_2 \to X_1^*$ is called weakly continuous provided
\[ \lim_{n \to \infty} \langle F(u_n), v \rangle = \langle F(u_0), v \rangle, \quad \forall v \in X_1, \]
for all $\{u_n\} \subset X_2, u_n \to u_0$ in $X_2$. 
Lemma 2.4. [13] (Acute angle theory) Let $F : X_2 \to X_1^*$ be weakly continuous, and $\Omega \subset X_2$ be bounded open set, $0 \in \Omega$. If
\[ < F(u), Lu > \geq 0, \quad \forall u \in \partial \Omega \cap X, \]
then the equation $F(u) = 0$ has a solution in $X_2$.

Definition 2.5. Let $X, Y$ be two separable Banach spaces, and $F : X \to Y$ be $C^1$. $F$ is called a Fredholm operator provided the derivative operator $DF : X \to Y$ is a Fredholm operator for all $x \in X$.

Lemma 2.6. [11, 12] (Sard-Smale Theorem). Let $F : X \to Y$ be a $C^1$ Fredholm operator with zero index. Then regular value of $F$ is dense in $Y$. If $p \in Y$ is critical value of $F$, then $F^{-1}(p)$ is discrete set.

3. Existence of steady state solution. In this paper, we will introduce the spaces as follows
\[ X = \{ \Phi = (u, T, S) \in C^\infty(\Omega, R^3) | \Phi \text{ satisfy } (10) - (12) \}, \]
\[ H_1 = \{ \Phi = (u, T, S) \in H^1(\Omega, R^3) | \Phi \text{ satisfy } (10) - (12) \}. \]

We write $L = I : X \to H_1$, which is a containing mapping.

Theorem 3.1. If $\lambda_1 \geq \max\{|\tilde{R} - 1| + |R + 1|, \frac{|\tilde{R} - 1|}{e_{c_5}}\}$, where $\lambda_1$ is the first eigenvalue of elliptic equations (2.1), then for $Q \in L^2(\Omega)$ the equations (7)-(12) have a weak solution $(u, T, S) \in H_1$.

Proof. Define $F : H_1 \to H_1^*$, for all $\Psi = (v, W, Z) \in H_1$,
\[ < F\Phi, \Psi > = \int_{\Omega} |\nabla u \cdot \nabla v - (RT - \text{sign}(S_0 - S_1)\tilde{R}S)v_3 + \frac{1}{Pr} (u \cdot \nabla)u \cdot v + \nabla T \cdot \nabla W - u_3W + (u \cdot \nabla)TW - QW + L_c \nabla S \cdot \nabla Z - \text{sign}(S_0 - S_1)u_3Z + (u \cdot \nabla)SZ |dx. \]

First, we prove that the definition $F$ is reasonable. In other words, we will show that $\| F\Phi \|_{H_1^*} < \infty, \forall \Psi = (v, W, Z) \in H_1$. By using the Hölder inequality, we can get
\[ \int_{\Omega} |\nabla u \cdot \nabla v|dx \leq \left( \int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla v|^2 dx \right)^{\frac{1}{2}} \leq \| u \|_{H^1(\Omega, R^3)} \| v \|_{H^1(\Omega, R^3)}, \] (14)
and
\[ \int_{\Omega} |\nabla T \cdot \nabla W|dx \leq \left( \int_{\Omega} |\nabla T|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla W|^2 dx \right)^{\frac{1}{2}} \leq \| T \|_{H^1(\Omega)} \| W \|_{H^1(\Omega)}, \] (15)
and
\[ \int_{\Omega} |L_c \nabla S \cdot \nabla Z|dx \leq L_c \left( \int_{\Omega} |\nabla S|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla Z|^2 dx \right)^{\frac{1}{2}} \leq L_c \| S \|_{H^1(\Omega)} \| Z \|_{H^1(\Omega)}. \] (16)

According to the Hölder inequality and the Sobolev imbedding theorem [17], we have
\[ \int_{\Omega} |-(RT - \text{sign}(S_0 - S_1)\tilde{R}S)v_3 - u_3W - QW - \text{sign}(S_0 - S_1)u_3Z|dx \]
\[
\| F\Phi \| + \| F \Psi \| \\
= \int_\Omega (u \cdot \nabla) v - (RT - \text{sign}(S_0 - S_1) \tilde{R} S)v_3 + \frac{1}{\tau} (u \cdot \nabla) u \cdot v + \nabla T \cdot \nabla W \\
- u_3 W + (u \cdot \nabla) TW - QW + Lc \nabla S \cdot \nabla Z - \text{sign}(S_0 - S_1) u_3 Z + (u \cdot \nabla) S Z \text{d}x \\
\leq \int_\Omega |\nabla u \cdot \nabla v| \text{d}x + \int_\Omega (u \cdot \nabla) |v| \text{d}x + (RT - \text{sign}(S_0 - S_1) \tilde{R} S)v_3 - u_3 W - QW \\
- \text{sign}(S_0 - S_1) u_3 Z \text{d}x + \frac{1}{\tau} \int_\Omega (u \cdot \nabla) u \cdot v \text{d}x + \int_\Omega |\nabla T \cdot \nabla W| \text{d}x \\
+ \int_\Omega (u \cdot \nabla) TW \text{d}x + Lc \int_\Omega |\nabla S \cdot \nabla Z| \text{d}x + \int_\Omega (u \cdot \nabla) S Z \text{d}x \\
\leq \| u \|_{L^2(\Omega, \mathbb{R}^2)} \| v \|_{H^1(\Omega)} + C_1 \| T \|_{H^1(\Omega)} + \tilde{R} \| S \|_{H^1(\Omega)} \| v \|_{H^1(\Omega, \mathbb{R}^2)} \\
+ C_1 \| u \|_{H^1(\Omega)} \| W \|_{H^1(\Omega)} + C_1 \| Q \|_{L^2(\Omega)} \| W \|_{H^1(\Omega)} + C_1 \| u \|_{H^1(\Omega, \mathbb{R}^2)} \| Z \|_{H^1(\Omega)} \\
+ \frac{C_2}{\tau} \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega, \mathbb{R}^2)} + \| T \|_{H^1(\Omega)} \| W \|_{H^1(\Omega)} + C_3 \| u \|_{H^1(\Omega, \mathbb{R}^2)} \| T \|_{H^1(\Omega)} \| W \|_{H^1(\Omega)} \\
+ \| u \|_{H^1(\Omega, \mathbb{R}^2)} \| T \|_{H^1(\Omega)} + \| u \|_{H^1(\Omega, \mathbb{R}^2)} \| S \|_{H^1(\Omega)} \| \Psi \|_{H^1(\Omega)} \\
\leq C_1 \| T \|_{H^1(\Omega)} + \tilde{R} \| S \|_{H^1(\Omega)} \| v \|_{H^1(\Omega, \mathbb{R}^2)} \\
+ C_1 \| u \|_{H^1(\Omega)} \| W \|_{H^1(\Omega)} + C_1 \| Q \|_{L^2(\Omega)} \| W \|_{H^1(\Omega)} + C_1 \| u \|_{H^1(\Omega, \mathbb{R}^2)} \| Z \|_{H^1(\Omega)} \\
+ \frac{C_2}{\tau} \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega, \mathbb{R}^2)} + \| T \|_{H^1(\Omega)} \| W \|_{H^1(\Omega)} + C_3 \| u \|_{H^1(\Omega, \mathbb{R}^2)} \| T \|_{H^1(\Omega)} \| W \|_{H^1(\Omega)} \\
+ \| u \|_{H^1(\Omega, \mathbb{R}^2)} \| T \|_{H^1(\Omega)} + \| u \|_{H^1(\Omega, \mathbb{R}^2)} \| S \|_{H^1(\Omega)} \| \Psi \|_{H^1(\Omega)} .
As a result, we get
\[
\|F\Phi\|_{H^1} = \sup_{|\Psi|_{H^1} \leq 1} \langle F\Phi, \Psi \rangle \\
\leq C (\|\Phi\|_{H^1(\Omega, R^3)} + \|T\|_{H^1(\Omega)} + \|S\|_{H^1(\Omega)} + \|Q\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega, R^3)}^2) \\
+ \|u\|_{H^1(\Omega, R^3)} \|T\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega, R^3)} \|S\|_{H^1(\Omega)}) < \infty,
\]
which implies that the definition \( F \) is reasonable.

Second, we show \( \langle F\Phi, \Phi \rangle \geq 0 \). According to the Young inequality, we have
\[
\langle F\Phi, \Phi \rangle = \int_{\Omega} [\nabla u \cdot \nabla u - (RT - \text{sign}(S_0 - S_1) R S) u_3 + \frac{1}{\rho_r}(u \cdot \nabla) u \cdot u + \nabla T \cdot \nabla T - u_3 T - (u \cdot \nabla) T \cdot T - QT + L_c \nabla S \cdot \nabla S - \text{sign}(S_0 - S_1) u_3 S + (u \cdot \nabla) S \cdot S] \, dx \\
= \int_{\Omega} [\|\nabla u\|^2 + \|\nabla T\|^2 + L_c |\nabla S|^2] \\
+ \text{sign}(S_0 - S_1)(R - 1) u_3 S - (R + 1) Tu_3 - QT \, dx \\
\geq \int_{\Omega} [\|\nabla u\|^2 + \|\nabla T\|^2 + L_c |\nabla S|^2] - \|R - 1\| u_3 |S| - |R + 1| u_3 |T| - |Q| |T| \, dx \\
\geq \int_{\Omega} [\|\nabla u\|^2 + \|\nabla T\|^2 + L_c |\nabla S|^2] dx - \frac{1}{2} \|R - 1\| \int_{\Omega} |u|^2 + |S|^2 dx - \frac{1}{2} |R + 1| \int_{\Omega} |u|^2 \\
+ |T|^2 dx - \epsilon \int_{\Omega} |T|^2 dx - \frac{\epsilon^2}{4} \int_{\Omega} |Q|^2 dx \\
= \frac{1}{2} \int_{\Omega} [\|\nabla u\|^2 + \|\nabla T\|^2 + L_c |\nabla S|^2] dx + \frac{1}{2} \int_{\Omega} [\|\nabla u\|^2 - (|R - 1| + |R + 1|)|u|^2] \, dx \\
+ \frac{1}{2} \int_{\Omega} [\|\nabla T\|^2 - |R + 1| |T|^2] dx + \frac{1}{4} \int_{\Omega} [L_c |\nabla S|^2 - |R - 1| |S|^2] dx - \epsilon \int_{\Omega} |T|^2 dx \\
- \frac{\epsilon^2}{4} \int_{\Omega} |Q|^2 dx.
\]

If \( \lambda_1 \geq \max\{|R - 1| + |R + 1|, \frac{|R - 1|}{L_c}\} \), we can deduce that
\[
\langle F\Phi, \Phi \rangle \geq C_1 \int_{\Omega} [\|\nabla u\|^2 + \|\nabla T\|^2 + |\nabla S|^2] dx - \epsilon \int_{\Omega} |T|^2 dx - \frac{\epsilon^2}{4} \int_{\Omega} |Q|^2 dx.
\]
Let \( \epsilon > 0 \) and be appropriate small. Then we have
\[
\langle F\Phi, \Phi \rangle \geq C_1 \int_{\Omega} [\|\nabla u\|^2 + \|\nabla T\|^2 + |\nabla S|^2] dx - C_2 \int_{\Omega} |Q|^2 dx.
\]

Consequently,
\[
\langle F\Phi, \Phi \rangle \geq C_1 \|\Phi\|_{H^1}^2 - C_3.
\]

Then there exists an appropriate large constant \( M \) such that
\[
\langle F\Phi, \Phi \rangle \geq 0, \quad \forall \Phi \in \partial B_M \cap X.
\]

At last, we prove that \( F \) is weakly continuous. Assume \( \Phi_k \to \Phi \) in \( H_1 \), we have from the Sobolev compactness embedding theorem \([8, 14, 17]\)
\[
\Phi_k \to \Phi \text{ in } L^p(\Omega, R^5), 1 \leq p < 6.
\]
As $u_k \to u$ in $H^1(\Omega, R^3)$, and $u \cdot v \in L^2(\Omega)$, it follows that

$$
\lim_{k \to \infty} \int_{\Omega} \left[ (u \cdot \nabla)u_k - (u \cdot \nabla)u \right] \cdot v \, dx = 0. \quad (22)
$$

Consequently, combining the general Hölder inequality and (21), we can get

$$
\lim_{k \to \infty} \left| \int_{\Omega} \left[ (u_k \cdot \nabla)u_k - (u \cdot \nabla)u_k \right] \cdot v \, dx \right| \leq \lim_{k \to \infty} \int_{\Omega} |u_k - u| \cdot |\nabla u_k| \cdot |v| \, dx
$$

$$
\leq \lim_{k \to \infty} \left( \int_{\Omega} |u_k - u|^4 \, dx \right)^{\frac{1}{4}} \left( \int_{\Omega} |\nabla u_k|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |v|^4 \, dx \right)^{\frac{1}{4}} = 0. \quad (23)
$$

Then,

$$
\lim_{k \to \infty} \int_{\Omega} [(u_k \cdot \nabla)u_k - (u \cdot \nabla)u] \cdot v \, dx = 0. \quad (24)
$$

Therefore,

$$
\lim_{k \to \infty} \int_{\Omega} [(u_k \cdot \nabla)u_k] \cdot v \, dx = \int_{\Omega} [(u \cdot \nabla)u] \cdot v \, dx. \quad (25)
$$

Similarly, we can prove

$$
\lim_{k \to \infty} \int_{\Omega} [(u_k \cdot \nabla)T_k]W \, dx = \int_{\Omega} [(u \cdot \nabla)T]W \, dx, \quad (26)
$$

and

$$
\lim_{k \to \infty} \int_{\Omega} [(u_k \cdot \nabla)S_k]Z \, dx = \int_{\Omega} [(u \cdot \nabla)S]Z \, dx. \quad (27)
$$

Combining (21)-(27), we have

$$
\lim_{k \to \infty} \langle F \Phi_k, \Psi \rangle = \lim_{k \to \infty} \int_{\Omega} \left[ \nabla u_k \cdot \nabla v - (R T - \operatorname{sign}(S_0 - S_1) \tilde{R} S_k) v_3 \right.
+ \frac{1}{P} (u_k \cdot \nabla)u_k \cdot v + \nabla T_k \cdot \nabla W - u_{k3} W + (u_k \cdot \nabla)T_k W - Q W + L_e \nabla S_e \cdot \nabla Z
$$
$$
- \operatorname{sign}(S_0 - S_1) u_{k3} Z + (u_k \cdot \nabla)S_k Z \big| \, dx
$$
$$
\quad = \int_{\Omega} \left[ \nabla u \cdot \nabla v - (R T - \operatorname{sign}(S_0 - S_1) \tilde{R} S) v_3 \right. \left. + \frac{1}{P} (u \cdot \nabla)u \cdot v + \nabla T \cdot \nabla W - u_{3} W \right.
+ (u \cdot \nabla)T W - Q W + L_e \nabla S \cdot \nabla Z - \operatorname{sign}(S_0 - S_1) u_{3} Z + (u \cdot \nabla)S Z \big| \, dx
$$
$$
= \langle F \Phi, \Psi \rangle.
$$

Hence, $F : H_1 \to H_1^*$ is weakly continuous. So, the equations (7)-(12) have a weak solution $\Phi = (u, T, S) \in H_1(\Omega, R^8)$ from the Lemma 2.4.

**Theorem 3.2.** If $\lambda_1 \geq \max\{|\tilde{R} - 1| + |R + 1|, \frac{|\tilde{R} - 1|}{L_e} \}$, where $\lambda_1$ is the first eigenvalue of elliptic equations (2.1), and $\Phi = (u, T, S)$ is the weak solution to equations (7)-(12), then, $(\Phi, p) \in W^{2,q}(\Omega, R^8) \times W^{1,q}(q \geq 2)$ for all $Q \in L^q(\Omega)(q \geq 2)$.

**Proof.** First, according to the Hölder inequality and the Sobolev imbedding theorem [17], we have

$$
\int_{\Omega} |(u \cdot \nabla)u|^{\frac{3}{2}} \, dx \leq \int_{\Omega} |u|^{\frac{3}{2}} |\nabla u|^{\frac{3}{2}} \, dx \leq (\int_{\Omega} |\nabla u|^2 \, dx)^{\frac{1}{2}} (\int_{\Omega} |u|^q \, dx)^{\frac{1}{2}}.
$$

Then, $(u \cdot \nabla)u \in L^{3}(\Omega, R^3)$.
For the Stokes equations
\[
\begin{aligned}
&-\Delta u + \nabla p = f_1, \\
div u = 0, \\
u_n = 0, \frac{\partial u}{\partial n} = 0, \text{ on } \partial D, \\
u_3 = 0, \frac{\partial u_3}{\partial x_3} = 0, \quad x_3 = 0, 1,
\end{aligned}
\]
(28)
since \(f_1 = (RT - \text{sign}(S_0 - S_1)i\bar{S})\tilde{k} - \frac{1}{\rho_c}(u \cdot \nabla)u \in L^\frac{7}{2}(\Omega, R^3)\), by the Lemma 2.2, the Stokes equations (28) have a solution \((u, p) \in W^{2, \frac{7}{2}}(\Omega, R^3) \times W^{1, \frac{7}{2}}(\Omega)\).

According to the Hölder inequality and the Sobolev imbedding theorem, we have
\[
\int_{\Omega} |(u \cdot \nabla)T|^\frac{7}{2} dx \leq \int_{\Omega} |u|^\frac{7}{2} |\nabla T|^\frac{7}{2} dx \leq (\int_{\Omega} |\nabla T|^2 dx)^\frac{7}{4}(\int_{\Omega} |u|^6 dx)^\frac{1}{4}.
\]
Thus, \((u \cdot \nabla)T \in L^\frac{7}{2}(\Omega)\).

For the elliptic equations
\[
\begin{aligned}
&-\Delta T = f_2, \\
\frac{\partial T}{\partial n} = 0, \text{ on } \partial D, \\
T = 0, \quad x_3 = 0, 1,
\end{aligned}
\]
(29)
since \(f_2 = u_3 - (u \cdot \nabla)T + Q \in L^\frac{7}{2}(\Omega)\), by the theory of linear elliptic equation [4], the equations (29) have a solution \(T \in W^{2, \frac{7}{2}}(\Omega)\).

By the Hölder inequality and the Sobolev imbedding theorem, we have
\[
\int_{\Omega} |(u \cdot \nabla)S|^\frac{7}{2} dx \leq \int_{\Omega} |u|^\frac{7}{2} |\nabla S|^\frac{7}{2} dx \leq (\int_{\Omega} |\nabla S|^2 dx)^\frac{7}{4}(\int_{\Omega} |u|^6 dx)^\frac{1}{4}.
\]
Thus, \((u \cdot \nabla)S \in L^\frac{7}{2}(\Omega)\).

For the elliptic equations
\[
\begin{aligned}
&-\Delta S = f_3, \\
\frac{\partial S}{\partial n} = 0, \text{ on } \partial D, \\
S = 0, \quad x_3 = 0, 1,
\end{aligned}
\]
(30)
since \(f_3 = \frac{1}{\rho_c}(\text{sign}(S_0 - S_1)u_3 - (u \cdot \nabla)S) \in L^\frac{7}{2}(\Omega)\), by the theory of linear elliptic equation, the equations (30) have a solution \(S \in W^{2, \frac{7}{2}}(\Omega)\).

Second, from the Sobolev imbedding theorem, we can deduce that \(W^{2, \frac{7}{2}}(\Omega) \hookrightarrow W^{1, 3}(\Omega)\), and \(W^{2, \frac{7}{2}}(\Omega) \hookrightarrow L^q(\Omega), \quad 1 \leq q < \infty\).

Then,
\[
\int_{\Omega} |(u \cdot \nabla)u|^\frac{7}{2} dx \leq \int_{\Omega} |u|^\frac{7}{2} |\nabla u|^\frac{7}{2} dx \leq (\int_{\Omega} |\nabla u|^3 dx)^\frac{7}{4}(\int_{\Omega} |u|^{15} dx)^\frac{1}{4}.
\]
Thus, \((u \cdot \nabla)u \in L^\frac{7}{2}(\Omega)\), therefore, \(f_1 \in L^\frac{7}{2}(\Omega, R^3)\). According to the Lemma 2.2, the Stokes equations (28) have a solution \((u, p) \in W^{2, \frac{7}{2}}(\Omega, R^3) \times W^{1, \frac{7}{2}}(\Omega)\).
According to the Hölder inequality and the Sobolev imbedding theorem, we have
\[
\int_{\Omega} |(u \cdot \nabla)T|^2\,dx \leq \int_{\Omega} |u|^2 |\nabla T|^2\,dx \leq \left( \int_{\Omega} |\nabla T|^3\,dx \right)^{\frac{2}{3}} \left( \int_{\Omega} |u|^{15}\,dx \right)^{\frac{2}{15}}.
\]

Then, \((u \cdot \nabla)T \in L^{\frac{5}{2}}(\Omega)\), therefore, \(f_2 \in L^{\frac{5}{2}}(\Omega)\). From the theory of linear elliptic equation, the equations (29) have a solution
\[
T \in W^{2,\frac{5}{2}}(\Omega).
\]

According to the Hölder inequality and the Sobolev imbedding theorem, we have
\[
\int_{\Omega} |(u \cdot \nabla)S|^2\,dx \leq \int_{\Omega} |u|^2 |\nabla S|^2\,dx \leq \left( \int_{\Omega} |\nabla S|^3\,dx \right)^{\frac{2}{3}} \left( \int_{\Omega} |u|^{15}\,dx \right)^{\frac{2}{15}}.
\]

Then, \((u \cdot \nabla)S \in L^{\frac{5}{2}}(\Omega)\), therefore, \(f_3 \in L^{\frac{5}{2}}(\Omega)\). From the theory of linear elliptic equation, the equations (30) have a solution
\[
S \in W^{2,\frac{5}{2}}(\Omega).
\]

Third, from the Sobolev imbedding theorem, we have
\[
W^{2,\frac{5}{2}}(\Omega) \hookrightarrow W^{1,\frac{15}{2}}(\Omega) \hookrightarrow L^{15}(\Omega),
\]
then, \(T, S \in L^{15}(\Omega)\), and
\[
\int_{\Omega} |(u \cdot \nabla)u|^3\,dx \leq \int_{\Omega} |u|^3 |\nabla u|^3\,dx \leq \left( \int_{\Omega} |u|^6\,dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla u|^6\,dx \right)^{\frac{1}{2}}.
\]

Consequently, \((u \cdot \nabla)u \in L^3(\Omega, R^3)\). Therefore, \(f_1 \in L^3(\Omega, R^3)\). Hence, according to the Lemma 2.2, the Stokes equations (28) have a solution
\[
(u, p) \in W^{2,3}(\Omega, R^3) \times W^{1,3}(\Omega).
\]

Similarly, we have
\[
T \in W^{2,3}(\Omega) \quad \text{and} \quad S \in W^{2,3}(\Omega).
\]

By doing the same procedures as above, the equations have a solution \((u, T, S, p) \in W^{2,q}(\Omega, R^3) \times W^{1,q}(\Omega),\) \(q \geq 2\). □

4. Regularity of steady state solution.

**Theorem 4.1.** If \(\lambda_1 \geq \max\{|\tilde{R} - 1| + |R + 1|, |R - 1|\}\), where \(\lambda_1\) is the first eigenvalue of elliptic equations (2.1), then there exists a dense open set \(\mathcal{R} \subset L^3(\Omega), \forall q, 3 \leq q < \infty,\) the solution to equations (7)-(12) is finite for all \(Q \in \mathcal{R}.

**Proof.** First, we know that the solution \((u, T, S, p) \in W^{2,3}(\Omega, R^3) \times W^{1,3}(\Omega)\) as the \(Q \in L^3(\Omega)\) from Theorem 3.2. We will prove the following estimate for (7)-(12)
\[
||u||_{W^{2,3}} + ||T||_{W^{2,3}} + ||S||_{W^{2,3}} + ||p||_{W^{1,3}} \leq C(||Q||_{L^3} + 1)^4.
\]
Since \(\Phi = (u, T, S)\) is the solution of equations (7)-(12), then \(\langle F\Phi, \Phi \rangle = 0\), that is
\[
\int_{\Omega} [||\nabla u||^2 + ||\nabla T||^2 + L_u ||\nabla S||^2 + \text{sign}(S_0 - S_1)(\tilde{R} - 1)u_3 S - (R + 1)Tu_3 - QT] \,dx = 0.
\]
Thus, according to the Young inequality, we have

\[ \int_\Omega \|\nabla u\|^2 + |\nabla T|^2 + L_e |\nabla S|^2 \, dx \]

\[ = \int_\Omega [-\text{sign}(S_0 - S_1)(\tilde{R} - 1)u_3 S + (R + 1)Tu_3 + QT] \, dx. \]

\[ \leq \int_\Omega [\tilde{R} - 1]|u_3||S| \, dx + \int_\Omega |R + 1||u_3||T| \, dx + \int_\Omega |Q||T| \, dx \]

\[ \leq \frac{1}{2}[\tilde{R} - 1]\int_\Omega |u_3|^2 \, dx + \frac{1}{2}[\tilde{R} - 1] \int_\Omega |S|^2 \, dx + \frac{1}{2}|R + 1| \int_\Omega |u_3|^2 \, dx \]

\[ + \frac{1}{2}|R + 1| \int_\Omega |T|^2 \, dx + \epsilon \int_\Omega |T|^2 \, dx + \frac{1}{4}\epsilon^{-1} \int_\Omega |Q|^2 \, dx \]

\[ \leq \int_\Omega \frac{1}{2}(2R + 1)u_3^2 + \frac{3}{2}(2R + 1)S^2 + \frac{3}{2}(R + 1)|T|^2 \]

\[ + \epsilon|T|^2 + \frac{1}{4}\epsilon^{-1}|Q|^2 \, dx. \]

Thus, we can deduce that

\[ \frac{1}{2} \int_\Omega \|\nabla u\|^2 + |\nabla T|^2 + L_e |\nabla S|^2 \, dx \]

\[ + \frac{1}{2} \int_\Omega \|\nabla u\|^2 - (|\tilde{R} - 1| + |R + 1|)|u|^2 \, dx + \frac{1}{2} \int_\Omega |\nabla T|^2 - |R + 1||T|^2 \, dx \]

\[ + \frac{1}{2} \int_\Omega L_e |\nabla S|^2 - |\tilde{R} - 1||S|^2 \, dx \]

\[ \leq \epsilon \int_\Omega |T|^2 \, dx + \frac{1}{4}\epsilon^{-1} \int_\Omega |Q|^2 \, dx. \]

Taking an appropriate constant \( \epsilon \), we have

\[ \int_\Omega \|\nabla u\|^2 + |\nabla T|^2 + |\nabla S|^2 \, dx \leq C\|Q\|_{L_2}^2. \]

By using the H"{o}lder inequality, we obtain

\[ \|Q\|_{L_2} \leq C\|Q\|_{L_3} \leq C(\|Q\|_{L_3} + 1). \]

Consequently, we have

\[ \int_\Omega \|\nabla u\|^2 + |\nabla T|^2 + |\nabla S|^2 \, dx \leq C(\|Q\|_{L_3} + 1)^2. \quad (31) \]

Using the Sobolev imbedding theorem and (31), we obtain

\[ \|u\|_{L_6} \leq C\|u\|_{L_\infty} \leq C\|\nabla u\|_{L_2} \leq C(\|Q\|_{L_3} + 1). \quad (32) \]

According to the Gagliardo-Nirenberg inequality and the Young inequality, we have

\[ \|\nabla u\|_{L_\infty} \leq C(\|D^2 u\|_{L_3}^{\frac{3}{2}} \|\nabla u\|_{L_2}^{\frac{1}{2}}) \leq \epsilon \|D^2 u\|_{L_3} + C\epsilon^{-2} \|u\|_{H^1}. \quad (33) \]

Similarly, we have

\[ \|\nabla T\|_{L_\infty} \leq C(\|D^2 T\|_{L_3}^{\frac{3}{2}} \|\nabla T\|_{L_2}^{\frac{1}{2}}) \leq \epsilon \|D^2 T\|_{L_3} + C\epsilon^{-2} \|T\|_{H^1}, \quad (34) \]

and

\[ \|\nabla S\|_{L_\infty} \leq C(\|D^2 S\|_{L_3}^{\frac{3}{2}} \|\nabla S\|_{L_2}^{\frac{1}{2}}) \leq \epsilon \|D^2 S\|_{L_3} + C\epsilon^{-2} \|S\|_{H^1}. \quad (35) \]

Combining the H"{o}lder inequality and (32)-(35), we deduce

\[ \|(u \cdot \nabla)u\|_{L_3} + \|(u \cdot \nabla)T\|_{L_3} + \|(u \cdot \nabla)S\|_{L_3} \]

\[ \leq \|u\|_{L_6}(\|\nabla u\|_{L_6} + \|\nabla T\|_{L_6} + \|\nabla S\|_{L_6}) \]
and we can obtain
\[ \leq C(||Q||_{L^3} + 1)[\epsilon(||D^2 u||_{L^3} + ||D^2 T||_{L^3} + ||D^2 S||_{L^3})] \]
\[ + \epsilon^{-2} ||u||_{H^1} + ||T||_{H^1} + ||S||_{H^1}] \]

As \( u, T, S \) are solutions of the equations (28), (29) and (30), from the Lemma 2.2, \( L^p \) theorem and theory of linear elliptic equation, we have
\[ ||u||_{W^{2,3}} + ||T||_{W^{2,3}} + ||S||_{W^{2,3}} + ||p||_{W^{1,3}} \]
\[ \leq C(||f_1||_{L^3} + ||f_2||_{L^3} + ||f_3||_{L^3}) \]
\[ \leq C(||T||_{L^3} + ||S||_{L^3} + ||(u \cdot \nabla) u||_{L^3} + ||u_3||_{L^3} + ||(u \cdot \nabla) T||_{L^3} + ||Q||_{L^3} \]
\[ + ||(u \cdot \nabla) S||_{L^3} \]
\[ \leq C(||u||_{L^3} + ||T||_{L^3} + ||S||_{L^3}) + C(||Q||_{L^3} + 1)[\epsilon(||D^2 u||_{L^3}) \]
\[ + ||D^2 T||_{L^3} + ||D^2 S||_{L^3}] + C \epsilon^{-2} (||Q||_{L^3} + 1)(||u||_{H^1} + ||T||_{H^1} + ||S||_{H^1}) \]
\[ + C||Q||_{L^3} \]

Taking \( C(||Q||_{L^3} + 1) = \frac{1}{2} \), then \( \epsilon^{-2} = 4C^2(||Q||_{L^3} + 1)^2 \). Thus, we have
\[ \frac{1}{2}(||u||_{W^{2,3}} + ||T||_{W^{2,3}} + ||S||_{W^{2,3}}) + ||p||_{W^{1,3}} \leq C(||Q||_{L^3} + 1) + C(||Q||_{L^3} + 1)^4 \]

Consequently, we have
\[ ||u||_{W^{2,3}} + ||T||_{W^{2,3}} + ||S||_{W^{2,3}} + ||p||_{W^{1,3}} \leq C(||Q||_{L^3} + 1)^4 \]  \( (36) \)

Second, we prove the following estimate
\[ ||u||_{W^{2,q}} + ||T||_{W^{2,q}} + ||S||_{W^{2,q}} + ||p||_{W^{1,q}} \leq C(||Q||_{L^3} + 1)^{12}, \quad \forall q, 3 < q < \infty \]

From the Sobolev imbedding theorem, we have
\[ W^{2,3}(\Omega) \hookrightarrow W^{1,3}(\Omega) \hookrightarrow L^{2q}(\Omega), \quad \forall q, 3 < q < \infty \]

Then, combing the Hölder inequality and (36), we deduce that
\[ ||u||_{L^{2q}} \leq C||u||_{W^{2,3}} \leq C(||Q||_{L^3} + 1)^4 \leq C(||Q||_{L^3} + 1)^4 \]  \( (37) \)

Moreover, according to the Gagliardo-Nirenberg inequality and the Young inequality, we can obtain
\[ ||\nabla u||_{L^{2q}} \leq C(||D^2 u||_{L^3}^{\frac{1}{2}} ||\nabla u||_{L^3}^{\frac{1}{2}}) \leq \epsilon(||D^2 u||_{L^3} + C \epsilon^{-1} ||\nabla u||_{L^3}) \]
\[ \leq \epsilon ||D^2 u||_{L^3} + C \epsilon^{-1} ||u||_{W^{1,3}} \leq \epsilon ||D^2 u||_{L^3} + C \epsilon^{-1} ||u||_{W^{2,3}} \]  \( (38) \)

Similarly, we have
\[ ||\nabla T||_{L^{2q}} \leq C(||D^2 T||_{L^3}^{\frac{1}{2}} ||\nabla T||_{L^3}^{\frac{1}{2}}) \leq \epsilon ||D^2 T||_{L^3} + C \epsilon^{-1} ||\nabla T||_{L^3} \]
\[ \leq \epsilon ||D^2 T||_{L^3} + C \epsilon^{-1} ||T||_{W^{1,3}} \leq \epsilon ||D^2 T||_{L^3} + C \epsilon^{-1} ||T||_{W^{2,3}} \]  \( (39) \)

and
\[ ||\nabla S||_{L^{2q}} \leq C(||D^2 S||_{L^3}^{\frac{1}{2}} ||\nabla S||_{L^3}^{\frac{1}{2}}) \leq \epsilon ||D^2 S||_{L^3} + C \epsilon^{-1} ||\nabla S||_{L^3} \]
\[ \leq \epsilon \|D^2 S\|_{L^\Phi} + C \epsilon^{-1} \|S\|_{W^{1,3}} \leq \epsilon \|D^2 S\|_{L^\Phi} + C \epsilon^{-1} \|S\|_{W^{2,3}}. \] (40)

Then, combining the Hölder inequality and (37)-(40), we have

\[ \|(u \cdot \nabla)u\|_{L^\Phi} + \|(u \cdot \nabla)T\|_{L^\Phi} + \|(u \cdot \nabla)S\|_{L^\Phi} \]
\[ \leq \|u\|_{L^{2\Phi}} \left( \|\nabla u\|_{L^{2\Phi}} + \|\nabla T\|_{L^{2\Phi}} + \|\nabla S\|_{L^{2\Phi}} \right) \]
\[ \leq C(\|Q\|_{L^\Phi} + 1)^4 \epsilon \left( \|D^2 u\|_{L^\Phi} + \|D^2 T\|_{L^\Phi} + \|D^2 S\|_{L^\Phi} \right) + C \epsilon^{-1} (\|Q\|_{L^\Phi} + 1)^8. \]

As \( u, T, S \) are solutions of the equations (28), (29) and (30), from the Lenma 2.2, \( L^p \) theorem and theory of linear elliptic equation, we have

\[ \|u\|_{W^{2,\Phi}} + \|T\|_{W^{2,\Phi}} + \|S\|_{W^{2,\Phi}} + \|p\|_{W^{1,\Phi}} \]
\[ \leq C(\|f_1\|_{L^\Phi} + \|f_2\|_{L^\Phi} + \|f_3\|_{L^\Phi}) \]
\[ \leq C(\|T\|_{L^\Phi} + \|S\|_{L^\Phi} + \|(u \cdot \nabla)u\|_{L^\Phi} + \|u_3\|_{L^\Phi} + \|(u \cdot \nabla)T\|_{L^\Phi} + \|Q\|_{L^\Phi}) \]
\[ + \|(u \cdot \nabla)S\|_{L^\Phi} \]
\[ \leq C(\|u\|_{L^{2\Phi}} + \|T\|_{L^{2\Phi}} + \|S\|_{L^{2\Phi}}) + C(\|Q\|_{L^\Phi} + 1)^4 \epsilon \left( \|D^2 u\|_{L^\Phi} + \|D^2 T\|_{L^\Phi} + \|D^2 S\|_{L^\Phi} \right) + C \epsilon^{-1} (\|Q\|_{L^\Phi} + 1)^8 + C(\|Q\|_{L^\Phi} + 1). \]

Taking \( C \epsilon(\|Q\|_{L^\Phi} + 1)^4 = \frac{1}{2} \), we have

\[ \epsilon^{-1} = 2C(\|Q\|_{L^\Phi} + 1)^4. \]

Therefore, we have

\[ \frac{1}{2}(\|u\|_{W^{2,\Phi}} + \|T\|_{W^{2,\Phi}} + \|S\|_{W^{2,\Phi}}) + \|p\|_{W^{1,\Phi}} \leq C(\|Q\|_{L^\Phi} + 1)^4 \]
\[ + C(\|Q\|_{L^\Phi} + 1)^{12} + C(\|Q\|_{L^\Phi} + 1), \]

which implies that

\[ \|u\|_{W^{2,\Phi}} + \|T\|_{W^{2,\Phi}} + \|S\|_{W^{2,\Phi}} + \|p\|_{W^{1,\Phi}} \leq C(\|Q\|_{L^\Phi} + 1)^{12}. \]

Third, we introduce the mappings

\[ F = L + H : W^{2,\alpha}(\Omega, \mathbb{R}^5) \times W^{1,\Phi}(\Omega) \to L^\Phi(\Omega, \mathbb{R}^5), \]

\[ L(u, T, S, p) = \begin{pmatrix} -\Delta u + \nabla p \\ -\Delta T \\ -\Delta S \end{pmatrix}, \]

\[ H(u, T, S, p) = \begin{pmatrix} -(RT - \text{sign}(S_0 - S_1) \hat{R}S) \hat{k} + \frac{1}{r} (u \cdot \nabla)u \\ -u_3 + (u \cdot \nabla)T \\ -\frac{1}{r} (\text{sign}(S_0 - S_1)) u_3 + (u \cdot \nabla)S \end{pmatrix}. \]
Let

\[ f(x) = \begin{pmatrix} 0 \\ Q(x) \\ 0 \end{pmatrix} \in L^q(\Omega, R^5). \]

Then, the equations (7)-(12) can be rewritten as

\[ F(u, T, S, p) = f(x). \]

It is obvious that the mappings \( F : W^{2,q}(\Omega, R^5) \times W^{1,q}(\Omega) \rightarrow L^q(\Omega, R^5) \) is completely continuous field.

So, the \( F \) is a Fredholm operator with zero index. According to the Lemma 2.6, the regular value of \( F \) is dense in \( \mathbb{N} \subset L^q(\Omega, R^5) \), and \( F^{-1}(f) \) is discrete in \( W^{2,q}(\Omega, R^5) \times W^{1,q}(\Omega) \) for all \( f \in \mathbb{N} \). Therefore, according to the theory of linear elliptic equations (2.1) and \( \Phi = (u, T, S) \) is the solution of equations (7)-(12), then

1. the equations have a classical solution \((u, T, S, p) \in C^{2,\alpha}(\Omega, R^5) \times C^{1,\alpha}(\Omega)\)
   for all \( Q \in C^\alpha, \ 0 \leq \alpha < 1 \).
2. If \( Q \in C^\infty(\Omega) \), then the solution \((u, T, S, p) \in C^\infty(\Omega, R^5)\).

Proof. (1) Since \( C^\alpha(\Omega) \hookrightarrow L^q(\Omega) \), for all \( 1 \leq q \leq \infty \), then, \( Q \in L^q(\Omega) \) for all \( Q \in C^\alpha(\Omega) \). According to Theorem 3.2, the equations (7)-(12) have a strong solution \((u, T, S, p) \in W^{2,q}(\Omega, R^5) \times W^{1,q}(\Omega)\).

Using the Sobolev imbedding theorem, we have \( W^{2,q}(\Omega) \hookrightarrow C^{1,\alpha}(\Omega), \ \alpha = 1 - \frac{3}{q} \).

Then,

\[ (u, T, S) \in C^{1,\alpha}(\Omega, R^5). \]

Thus,

\[ (u \cdot \nabla)u \in C^\alpha(\Omega). \]

Therefore, \( f_1 \in C^\alpha \). As a result, the Stokes equations (28) have a solution

\[ (u, p) \in C^{2,\alpha}(\Omega) \times C^{1,\alpha}(\Omega). \]

Furthermore, we have

\[ (u \cdot \nabla)T \in C^\alpha(\Omega). \]

Therefore, \( f_2 \in C^\alpha \). As a result, according to the theory of linear elliptic equation, the equations (29) have a solution

\[ T \in C^{2,\alpha}(\Omega). \]

Similarly, we can prove \( S \in C^{2,\alpha}(\Omega) \). Consequently, the equations (7)-(12) have a classical solution \((u, T, S, p) \in C^{2,\alpha}(\Omega, R^5) \times C^{1,\alpha}(\Omega)\).

(2) Since \( Q \in C^\infty(\Omega) \), it follows that \( Q \in W^{k,q}(\Omega) \)(k is arbitrary integer). According to theorem 3.2, we have \((u, T, S, p) \in W^{2,q}(\Omega, R^5) \times W^{1,q}(\Omega)\). Consequently, we have \((u, T, S, p) \in W^{k+2,q}(\Omega, R^5) \times W^{k+1,q}(\Omega)\)(k is arbitrary integer) from the Lemma 2.2. Using the Sobolev imbedding theorem, it follows that \((u, T, S, p) \in C^{k+1,\alpha}(\Omega, R^5) \times C^{k,\alpha}(\Omega) \)(k is arbitrary integer). Then \((u, T, S, p) \in C^\infty(\Omega, R^5). \]

\[ \square \]
Remark. (i) The condition $\lambda_1 \geq \max\{|\hat{R} - 1| + |\hat{R} + 1|, \frac{|\hat{R} - 1|}{\lambda_2}\}$ is a sufficient, not necessary condition. In fact, if the condition does not hold, the equations (7)-(12) may have a solution for some $Q$.

(ii) If $Q = 0$, it is obvious that the equations (7)-(12) have a trivial classical solution $(u, T, S, p) = 0$ as Ma and Wang pointed out in [10].

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