Noiseless Quantum Circuits for the Peres Separability Criterion

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In this Letter we give a method for constructing sets of simple circuits that can determine the spectrum of a partially transposed density matrix, without requiring either a tomographically complete POVM, or the addition of noise to make the spectrum non-negative. These circuits depend only on the dimension of the Hilbert space and are otherwise independent of the state.

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Introduction There has recently been much interest in measuring functions of the density matrix directly, without performing full state tomography. The non-local properties of density matrices have attracted particular attention and there has been a series of papers exploring this possibility [1, 2, 3, 4, 5, 6]. These typically rely on a combination of the Structural Physical Approximation (SPA) [2] followed by measuring the spectrum of the resulting density operator in the method in [2] (but see also [1, 2, 8]). The SPA is a method for modifying a map which is not a completely positive map (CP map) so that it becomes one. This is done by forming a convex mixture of the original map with another map (SPA) [7] followed by measuring the spectrum of the resulting CP-map [2].

For a more detailed discussion of the properties of this map and its implementation as a positive operator-valued measure, see [2].

This approach was partly inspired by the interferometer circuits in [10] and the analysis in [11]. The proof in [10] showed that circuits of the form in figure 1 where the density matrices undergo the evolution \( \rho \to U \rho U^\dagger \) conditioned on the state of the control qubit, can measure in the modulation of the interference pattern of that qubit,

\[
\text{Tr}(U \rho) = v e^{i \alpha}
\]

where \( v \) is the visibility, and \( \alpha \) is a phase shift. This is significant, because it enables us to characterise the effect of maps defined by a unitary map on the kets only, without being constrained to perform \( U^\dagger \) on the bra vectors.

If we make \( \rho = \rho \otimes \cdots \otimes \rho \) from multiple copies of \( \rho \), an \( m \times m \) density matrix of interest, we can use this circuit to measure the moments of the density matrix, \( \text{Tr}(\rho^2) \cdots \text{Tr}(\rho^m) \) by choosing \( U \) to be the cyclic shift (2)

\[
V^{(k)}|\varphi_1\rangle|\varphi_2\rangle \cdots |\varphi_k\rangle = |\varphi_k\rangle|\varphi_1\rangle \cdots |\varphi_{k-1}\rangle,
\]

acting on the kets of the basis only, thus measuring

\[
\text{Tr}_k \rho \otimes |k\rangle V^{(k)} = \text{Tr}_k \rho^k = \sum_{i=1}^{m} \lambda_i^k,
\]

from which we can determine the eigenspectrum of the density matrix [1, 2, 11].

Let us begin by considering polynomial local-unitary-invariants of pure states. These can be constructed as follows. Begin by writing

\[
|\psi_{ij\ldots k}\rangle|\psi_{pq\ldots r}\rangle = |\psi_{ij\ldots k}\rangle|\psi_{pq\ldots r}\rangle
\]

where the tensor indices denote the various subsystems in the usual way. Now contract every “downstairs” (or “input”) index with one of the “upstairs” (or “output”) indices labelling the same subsystem, (but not necessarily on a neighbouring term) until no free indices remain. These functions are therefore invariant under local basis changes by construction.

In [12] Rains proved that there was a one-to-one correspondence between the set of all local polynomial invariants and sets of permutations that permute the downstairs indices corresponding to each subsystem with other downstairs indices corresponding to the same subsystem. (Or equivalently, perform the inverse permutation on the upstairs indices.) Rains’ work was motivated by the need to find new shadow enumerators of quantum codes and it was not until [13] that these invariants were given a physical interpretation in terms of the expectation values of operators. In [13] the methods of [2] were extended to
include arbitrary permutations, thus recovering Rains’ construction for the full set of polynomial invariants.

The preceding papers seem to share the assumption that the only way to measure the effect of a map on \( \rho \) is to cause the density matrix to actually undergo that map as a physical evolution: the map must be physically implementable. For maps that are not in this class one must therefore use the SPA [7]. (However, we will argue later that a more subtle measurement model is implied by certain results in some of these papers.)

This assumption makes intuitive sense and one might think that it is therefore “obviously” true; except, like some other “obviously true” assertions about quantum mechanics, it is not. To illustrate this, we will give a method for constructing a set of unitary circuits that can measure the spectrum of \( \rho^{T_2} \) for any \( \rho \), without adding noise of any kind. The structure of these circuits depends only on the dimension of \( \rho \) and does not require a tomographically complete POVM at any stage.

Measuring the Kempe Invariant In [14] a method for constructing circuits to measure polynomial invariants was presented. Although the authors do not mention this fact, this class of circuits includes a set that can measure the effects of some non-positive maps obtained by extending maps that are positive, but not CP maps to act on larger spaces.

Consider polynomial invariants of pure states of three qubits. A 6th order invariant (in \( \psi \)) that is algebraically independent of all the lower-order invariants was discovered by Kempe [15] and can be written (with a little rearranging) as

\[
I_6 = \psi_{ktp} \psi_{inp} \psi_{imq} \psi_{jfq} \psi_{jnr} \psi_{kmr}.
\]

This index-based notation is rather opaque. Figure 2 is the same invariant drawn as a summation diagram: the linked nodes represent summations; each upstairs-downstairs pair of nodes after each copy of \( \psi \) denotes the indices for one local subsystem (qubit), just as with tensor index notation. The triangles represent indices labelling qubit 1; the squares, those for qubit 2 and the open circles correspond to qubit 3.

![FIG. 2: The summation diagram for the Kempe invariant.](image1)

We can always choose to order the copies of \( \rho \) in the invariant so that one subsystem is traced out; this is just another manifestation of a theorem due to Schrödinger [16, 17, 18], which is also known as the GHJW theorem [19, 20, 21]. The traced out party is represented by the open circles. This way of writing invariants highlights the Rains correspondence between invariants and permutations of the indices for each subsystem [12]. Consider the following three qubit invariant:

This is a three qubit invariant but we are interested in bipartite entanglement here. If we ignore the third qubit altogether, we can see that this invariant is \( \text{Tr}((\rho_{12}^{T_2})^3) \), where the subscripts 1, 2 are no longer indices, but now refer to qubits 1 and 2 and the two longest lines represent the final matrix trace. We are free to drop the third qubit because the indices for particle 3 are always traced out in equation (6) and their links in the summation diagram (Fig. 2) are never exchanged between copies of \( \rho \).

We can now connect figure 2 to the partial transpose map. One way to obtain \( \rho \) from \( \rho^{T_2} \) is to take each upstairs-downstairs pair of indices for particle 2 and swap the upstairs index with the downstairs one. This performs the partial transpose operation on each copy of \( \rho \).

\[
I_6 = \text{Tr}((\rho_{12}^{T_2})^3),
\]

where the symbol \( T_2 \) denotes the partial transpose with respect to qubit 2. In other words, equation (6) is the third moment of the partial transpose of \( \rho \).

We can measure the Kempe invariant because the summation in Fig. 2 can be obtained from that in Fig. 3 in two equivalent ways. The other way is via the Rains construction, using a cyclic permutation acting on the downstairs indices for qubit 2 to reproduce the “contraflow” summation pattern in Fig. 2. All the indices are summed out in upstairs-downstairs pairs, their labels have no physical significance and so these two operations produce the same invariant. Any invariant of the form in \( \psi \) with no free indices and all summations on upstairs-downstairs pairs can be generated using Rains’ construction [12].

A circuit can be obtained for this invariant by using the construction in [14]. This is done by rotating the summation diagram in figure 2 by 90° clockwise and omitting the \( \rho \) symbols (see Fig. 4). The downstairs indices now correspond to input rails and the upstairs indices to output rails.

![FIG. 4: The Kempe invariant, in the notation from [14].](image2)

To construct the circuit, insert the wiring in Fig. 4 as the controlled-\( U \) in Fig. 10.
Measuring the spectrum of $\rho^{T_2}$  It can be seen that we can also write the invariants $\text{Tr}((\rho_{12}^{T_2})^2)$ and $\text{Tr}((\rho_{12}^{T_2})^4)$ in a similar form, so we can construct circuits to evaluate these moments. We can therefore determine the eigenspectrum of $\rho_{12}^{T_2}$ using the recipe given in [2,11] (or alternatively, reconstructing the characteristic polynomial for $\rho_{12}^{T_2}$ from the values of $\text{Tr}((\rho_{12}^{T_2})^k)$). Thus we do not need to use the SPA to determine the spectrum of $\rho^{T_2}$. The circuits for the second, third and fourth moments are shown in Figs. 5-7 respectively, with the permutations decomposed into controlled-SWAP (CSWAP) gates.

These are the only circuits required for two qubit states; we do not need a circuit for $\text{Tr}(\rho_{12}^{T_2})$ because the partial transpose is a trace preserving map, so the first moment is already known to be 1. In fact, the only unitary circuit we can write down that is first order in $\rho$, or indeed $\rho_{12}^{T_2}$ is the circuit for the trace norm. This is also the only polynomial invariant that is quadratic in $\psi$.

Discussion  We have exhibited a class of circuits that can measure the spectrum of $\rho^{T_2}$, without needing to perform a tomographically complete POVM or add noise. While we have only given the circuits for the two qubit case, the construction generalizes to bipartite systems of any finite dimension in the obvious way; all that is required is two cyclic permutations of opposite handedness acting on the rails for each subsystem. Fiurášek [11] has shown that the SPA for the partial transpose for two qubits is a tomographically complete CP map. It is an open problem whether this is generally true for the partial transpose on larger systems, but the method given in this paper never needs a tomographically complete POVM to characterize the effects of the partial transpose on $\rho$, regardless of its dimension. These circuits are surprisingly simple: if we can implement an eigenspectrum measurement by the method in [1,2] then we can measure the spectrum of $\rho^{T_2}$ with a set of circuits that uses the same set of primitives. The complexity of the networks also scales the same way as those for eigenspectrum circuits; for a general $d_1 \times d_2$ dimensional mixed bipartite state, the circuits consume the same number of copies per run, $\frac{1}{2}d_1^2d_2^2 + \frac{1}{2}d_1d_2 - 1$.

What may be more interesting about these circuits is that they disprove our intuition that the only characterizable map is a physically implementable map. This statement is still true if we only have access to one copy of the state at a time, as this method for measuring the effects of maps needs multiple copies in order to work; since we cannot implement the map directly, we must characterise it via its moments. It is also worth noting that having access to only two copies at a time is not much use either; the circuit for $\text{Tr}((\rho_{12}^{T_2})^2)$ is identical to that for $\text{Tr}(\rho_{12}^{T_2})$. Thus the partial transpose may also preserve $\text{Tr}(\rho^2)$.

If we can manipulate at least $d_1d_2$ copies at once, we can perform the Peres test [22]. This is a necessary and sufficient condition for separability for two qubit states, although it is not sufficient to test for separability in general [23]. The entire minimal generating set for the ring of polynomial invariants described by Rains [12,14] is necessary to completely separate the orbits of the state under local unitary operations in general. It is also sufficient [24,27,26]. However, there is no known upper bound on the order of the invariants needed to construct a complete minimal generating set. One way to see this is to notice that this paper has only examined invariants whose summation pattern has period 1 in the number of copies of the density matrix it passes through before it repeats. There is no a priori reason why there might not be invariants corresponding to longer-period summation patterns that cannot be written as products of terms with period 1 and thus could have a much larger order.

This is not the only reason why this problem is hard, but it is enough to show that a general density operator may need circuits in more than $d_1d_2$ copies in order to characterize its non-local properties completely. We
cannot say how many copies we would need to be able to process at once without first finding the minimal generating set of invariants for the system. This is a hard problem and all we can say in general is that this number is finite.

If we assume that we can use unitary circuits that can manipulate any finite number of copies at once, the class of measurable maps is considerably larger than the set of all unitary maps. Consider a density matrix that we have the option of pre-processing according to

$$\rho^{\otimes n} \rightarrow V \rho^{\otimes n} V^\dagger$$

(8)

for any unitary V. If our interferometer circuit contains a controlled-U of our choice, we can now measure any

$$\rho^{\otimes n} \rightarrow W \rho^{\otimes n} V^\dagger$$

(9)

where $W = UV$, which is a much larger class of operations than is accessible by equation (8) alone. What is not immediately obvious is that this class of operations is qualitatively different from the usual unitary operations. This is because we now have the ability to operate on the ket basis of the density matrix without having to perform the adjoint operation on the bra basis.

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