Baryon-Pion Couplings from Large-$N_c$ QCD

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Abstract

We derive a set of consistency conditions for the pion-baryon coupling constants in the large-$N_c$ limit of QCD. The consistency conditions have a unique solution which are precisely the values for the pion-baryon coupling constants in the Skyrme model. We also prove that non-relativistic $SU(2N_f)$ spin-flavor symmetry (where $N_f$ is the number of light flavors) is a symmetry of the baryon-pion couplings in the large-$N_c$ limit of QCD. The symmetry breaking corrections to the pion-baryon couplings vanish to first order in $1/N_c$. Consistency conditions for other couplings, such as the magnetic moments are also derived.
In this paper, we will study the baryon-pion coupling constants in the large-$N_c$ limit. We derive a set of consistency conditions for the baryon-pion couplings that must be satisfied by large-$N_c$ QCD. These consistency conditions completely determine all the pion-baryon coupling constants in terms of one overall normalization constant. The consistency conditions require that there be an infinite tower of degenerate baryon states with $I = J$. The ratios of the pion-baryon couplings of this baryon tower are precisely those given by the Skyrme model, or by the non-relativistic quark model, which is known to be equivalent to the Skyrme model in the large-$N_c$ limit. This implies that the large-$N_c$ limit of QCD has a $SU(2N_f)$ spin-flavor symmetry, where $N_f$ is the number of light flavors. This result is unusual in that the $SU(2N_f)$ symmetry is only a symmetry of the baryon sector of large-$N_c$ QCD, but not of the meson sector. After this work was completed, we discovered that the first consistency condition proved in this paper using pion-nucleon scattering was derived earlier using the same method by Gervais and Sakita. They also realized that there is a contracted $SU(4)$ algebra in the large-$N_c$ limit of QCD. We believe that the other consistency conditions derived in this paper, the connection with chiral perturbation theory, and the study of $1/N_c$ corrections (which will be presented elsewhere) are new.

The large-$N_c$ counting rules for meson-baryon scattering were analysed by Witten, who showed that meson-baryon scattering amplitudes at fixed energy must be of order one. The incident pion can couple to any of the quarks in the baryon with an amplitude $1/\sqrt{N_c}$. If the emitted pion couples to the same quark as the incident pion, there are $N_c$ choices for the quark, so the net amplitude is order one. If the emitted pion couples to a different quark, then there are $N_c^2$ ways to choose the two quarks, but there must also be at least a single gluon exchange between the two quarks to transfer energy from the incident pion to the final pion. The gluon produces a suppression of $1/N_c$, so again, the amplitude is of order one. The large-$N_c$ counting rules show that the axial vector coupling constant $g_A$ of a baryon is of order $N_c$, since the axial current can be inserted on any of the $N_c$ quark lines. It is also easy to see that the pion decay constant $f_\pi$ is of order $\sqrt{N_c}$. The pion-baryon vertex is proportional to $g_A q/f_\pi$, where $q$ is the pion momentum, and grows as $\sqrt{N_c}$ at fixed pion energy. The consistency conditions follow from this simple result.

Let us assume that the large-$N_c$ QCD baryon spectrum contains a state with $I = J = 1/2$, which we will call the nucleon. The nucleon is infinitely heavy, and can be treated as a static fermion interacting with the pion. The pion-nucleon scattering amplitude is
dominated by the graphs of Fig. 1. The axial current matrix element in the nucleon can be written as

$$\langle N | \bar{\psi} \gamma^i \gamma_5 \tau^a \psi | N \rangle = g N_c \langle N | X^{ia} | N \rangle,$$

(1)

where $\langle N | X^{ia} | N \rangle$ and $g$ are of order one. The coupling constant $g$ has been factored out so that the normalization of $X^{ia}$ can be chosen conveniently. $X^{ia}$ is an operator (or $4 \times 4$ matrix) defined on nucleon states which has a finite large-$N_c$ limit. The pion-nucleon scattering amplitude for $\pi^a(q) + N(k) \rightarrow \pi^b(q') + N(k')$ is (neglecting the third graph which is suppressed by $1/N_c^2$)

$$-i q^i q'^j \frac{N_c^2 g^2}{f_\pi} \left[ \frac{1}{q^0} X^{jb} X^{ia} - \frac{1}{q'^0} X^{ia} X^{jb} \right],$$

(2)

where the amplitude is written in the form of an operator acting on nucleon states. Both initial and final nucleons are on-shell, so $q^0 = q'^0$. The product of the $X$’s in eq. (2) then sums over the possible spins and isospins of the intermediate nucleon. Since $f_\pi \sim \sqrt{N_c}$, the overall amplitude is of order $N_c$, which violates unitarity, and also contradicts the large-$N_c$ counting rules of Witten. Thus large-$N_c$ QCD with a $I = J = 1/2$ nucleon multiplet interacting with a pion is an inconsistent field theory. There must be other states that cancel the order $N_c$ amplitude in eq. (2) so that the total amplitude is order one, and consistent with unitarity. One can then generalize $X^{ai}$ to be an operator on this degenerate set of baryon states, with matrix elements equal to the corresponding axial current matrix elements. With this generalization, the form of eq. (2) is unchanged. Thus we obtain the first consistency condition for QCD,

$$[X^{ia}, X^{jb}] = 0,$$

(3)

so that the axial currents are represented by a set of operators $X^{ia}$ that commute in the large-$N_c$ limit. The solution of the constraint is non-trivial, because we also have the commutation relations

$$[J^i, X^{jb}] = i \epsilon_{ijk} X^{kb}, \quad [I^{a}, X^{jb}] = i \epsilon_{abc} X^{jc},$$

$$[J^i, J^j] = i \epsilon_{ijk} J^k, \quad [I^{a}, I^{b}] = i \epsilon_{abc} I^c, \quad [I^{a}, J^i] = 0,$$

(4)

since $X^{ia}$ has spin one and isospin one. It is simple to prove that eqs. (3) and (4) form a non-semisimple Lie Algebra, and have no non-trivial finite dimensional representations. 

3
The solutions of the consistency condition eq. (3) requires that there be an infinite tower of degenerate baryon states, and also determines the ratios of the pion-baryon couplings between the states. We will make one simplifying assumption in this letter; the degenerate baryon spectrum consists only of states \( I = J = 1/2, 3/2, \ldots \), where the sequence can be finite or infinite. (This assumption is not necessary \([7]\).) The states will be denoted by \( |J, J_3, I_3⟩ \). The reduced matrix element of \( X^{ai} \) between baryon states can be written as

\[
\langle J', m', \alpha' | X^{ia} | J, m, \alpha \rangle = X(J, J') \sqrt{\frac{2J + 1}{2J' + 1}} \left( \begin{array}{c} J \\ m \\ i \end{array} \right) \left( \begin{array}{c} J' \\ 1 \\ a \end{array} \right), \tag{5}
\]

where \( X(J, J') \) is a reduced matrix element. The normalization constant has been chosen so that \( X(J, J') = X(J', J) \). Since \( X^{ia} \) is a tensor with spin one and isospin one, it can only couple states with \( \Delta J = 0, \pm 1 \), and the independent reduced matrix elements are \( X(J, J) \) and \( X(J, J + 1) \). Taking the matrix element of eq. (3) between \( |J, m, \alpha⟩ \) and \( |J', m', \alpha'⟩ \), and inserting a complete set of intermediate states gives

\[
0 = \sum_{J_1, m_1, \alpha_1} \langle J', m', \alpha' | X^{ia} | J_1, m_1, \alpha_1 \rangle \langle J_1, m_1, \alpha_1 | X^{jb} | J, m, \alpha \rangle - (ia \leftrightarrow jb). \tag{6}
\]

We will first show that eq. (6) has a unique solution Consider the case \( J' = J + 1 \), where only intermediate states \( J_1 = J, J + 1 \) contribute. The condition in eq. (6) reduces to

\[
0 \sim X(J, J) X(J, J + 1) + X(J, J + 1) X(J + 1, J + 1), \tag{7}
\]

where we have only shown the structure of the reduced matrix elements that contribute, and neglected all numerical factors, signs, etc. Similarly, the diagonal matrix element \( J' = J \) of eq. (6) has the form

\[
0 \sim X(J, J - 1)^2 + X(J, J)^2 + X(J, J + 1)^2. \tag{8}
\]

If we are given \( X(J, J) \) and \( X(J - 1, J) \), eq. (8) determines \( X(J, J) \) and eq. (7) determines \( X(J + 1, J + 1) \), so that all the reduced matrix elements are uniquely determined by recursion starting from the lowest ones. If we assume that the lowest state is \( J = 1/2 \), all the reduced matrix elements are determined in terms of \( X(1/2, 1/2) \), since \( X(1/2, -1/2) = 0 \). Since eq. (6) is homogeneous in \( X(J, J') \), the starting value \( X(1/2, 1/2) \) can be set equal to one, which determines the solution uniquely up to an overall normalization constant. To construct eq. (7) and (8) explicitly, substitute eq. (5) into eq. (6), and project onto the
pion-nucleon scattering channel with spin \( H \) and isospin \( K \) by multiplying the equation by
\[
\begin{pmatrix} J & 1 \\ m & i \end{pmatrix} H \begin{pmatrix} J & 1 \\ \alpha & a \end{pmatrix} K \begin{pmatrix} J' & 1 \\ m' & j \end{pmatrix} H' \begin{pmatrix} J' & 1 \\ \alpha' & b \end{pmatrix} K'.
\]
The resulting equation for the reduced matrix elements can be written as
\[
X(J, H) X(J', H) \delta_{HK} = (2H + 1) \sum_{J_1} (2J_1 + 1) \begin{pmatrix} J_1 & J \end{pmatrix} H \begin{pmatrix} J_1 & J' \end{pmatrix} X(J, J_1) X(J', J_1). \tag{9}
\]
The solution to this equation has been shown to be unique, so it is sufficient to verify that \( X(J, J) = X(J, J + 1) = 1 \) for all \( J \) is a solution to this set of equations, using the symmetry properties of the \( 6j \)-symbols and the identity \([10]\)
\[
\sum_{K} (2K + 1)(2H + 1) \begin{pmatrix} J_1 & J_2 & H \\ J_3 & J_4 & K \end{pmatrix} \begin{pmatrix} J_1 & J_2 & L \\ J_3 & J_4 & K \end{pmatrix} = \delta_{HL}. \tag{10}
\]
A similar result can also be proved for the infinite tower \( I = J = 0, 1, \ldots \) if one starts from the lowest state \( I = J = 0 \). The two possible towers correspond to the large-\( N_c \) limit of QCD with \( N_c \) odd and even, respectively. The solution eq. (7) for the pion-baryon couplings in large-\( N_c \) QCD is identical to the results obtained in the Skyrme model and the non-relativistic quark model in the large-\( N_c \) limit.

The algebra in eq. (3) and (4) is a contracted SU(4) algebra. Consider the embedding \( SU(4) \rightarrow SU(2) \otimes SU(2) \) where \( 4 \rightarrow (2, 2) \). If the generators of \( SU(2) \otimes SU(2) \) in the defining representation are \( I^a \), and \( J^i \), the \( SU(4) \) generators in the defining representation are \( J^i \otimes 1, 1 \otimes I^a \) and \( J^i \otimes I^a \), which we will call \( I^a, J^i \) and \( G^{ia} \) respectively. (The properly normalized \( SU(4) \) generators are \( I^a/\sqrt{2}, J^i/\sqrt{2} \) and \( \sqrt{2} G^{ia} \).) The algebra of large-\( N_c \) QCD is given by taking the limit
\[
X^{ia} = \lim_{N_c \to \infty} \frac{G^{ia}}{N_c}, \tag{11}
\]
(up to an overall normalization of the \( X^{ia} \)). The commutation relations of \( SU(4) \),
\[
\begin{align*}
[J^i, J^j] &= i \epsilon_{ijk} J^k, & [I^a, I^b] &= i \epsilon_{abc} I^c, \\
[I^a, G^{ia}] &= i \epsilon_{abc} G^{jc}, & [J^i, G^{jb}] &= i \epsilon_{ijk} G^{kb}, \\
[I^a, J^i] &= 0, & [G^{ia}, G^{jb}] &= \frac{i}{4} \epsilon_{ijk} \delta_{ab} J^k + \frac{i}{4} \epsilon_{abc} \delta_{ij} I^c,
\end{align*}
\tag{12}
\]
turn into the commutation relations eqs. (3)–(4) in the large-$N_c$ limit. The non-trivial irreducible representations of the contracted algebra are obtained by taking the limit of $SU(4)$ representations for which $G^{ia}$ is of order $N_c$, so that $X^{ia}$ is finite. We have constructed the simplest such representation explicitly above, and it corresponds to taking the $N_c \to \infty$ limit of the completely symmetric $SU(4)$ tensor with $N_c$ boxes. Thus we see that the large-$N_c$ limit of QCD has a contracted $SU(4)$ symmetry in the baryon sector. This is sufficient to prove that the predictions for the pion-baryon couplings are identical to those of the Skyrme model or non-relativistic quark model, and exhibit $SU(4)$ symmetry. This explains the origin of the non-relativistic $SU(4)$ symmetry of baryons in QCD.

There is an alternative approach to deriving consistency conditions for large-$N_c$ QCD that leads to a condition equivalent to eq. (3) as well as to a second consistency condition. Consider the one-loop radiative corrections in the baryon sector, given by the diagrams of fig. 2. The one loop renormalization of the pion-baryon coupling constants is proportional to

$$
\delta \left( gN_c X^{ia} \right) \sim \frac{g^2 N_c^2}{16 \pi^2 f_\pi^2} \left[ X^{jb}, [X^{jb}, X^{ia}] \right] m_\pi^2 \log \frac{m_\pi^2}{\mu^2},
$$

so that a term of order $N_c$ gets a correction of order $N_c^2$. The pion mass is a free parameter of the theory, and is of order $\sqrt{\Lambda_{QCD}} m_q$ and finite in the large-$N_c$ limit. Thus the stability of the pion-baryon couplings under an infinitesimal shift in the light quark mass, or equivalently, stability under radiative corrections gives the consistency condition

$$
[X^{jb}, [X^{jb}, X^{ia}] ] = 0,
$$

which is a cubic relation between the $X$’s. One can show that this consistency condition also leads to a set of recursion relations for the reduced matrix elements of $X$, with a unique solution that is the same as that obtained earlier. The details will be given elsewhere. Thus one-loop chiral perturbation theory in the large-$N_c$ limit implies that there must be an infinite tower of degenerate baryon states of the form $I = J = 1/2, 3/2, \ldots$, with pion-baryon couplings which are precisely those given by the Skyrme model. In addition, the one-loop corrections must cancel exactly at leading order in $N_c$.

The quark counting rules of Witten do not lead to any violations of unitarity or consistency conditions in the large-$N_c$ limit. In a quark picture, one sums over all possible intermediate quark states, which is equivalent to summing over all intermediate baryon states. The singular large-$N_c$ behavior of the pion-baryon scattering amplitude or of one-loop chiral perturbation theory arises because the intermediate state was projected onto
a single baryon state, the nucleon. It is this projection that leads to an inconsistency in the large-$N_c$ limit. Large-$N_c$ QCD requires the existence of an infinite tower of degenerate resonances which must be included as intermediate states for chiral perturbation theory to be consistent. The loop graphs cancel exactly to leading order as $N_c \to \infty$. It was recently suggested [11] that $\Delta$ resonances must be included as intermediate states in baryon chiral perturbation theory for a consistent expansion, and that there is a large cancellation between intermediate $\Delta$ and nucleon states. The computation was done for $N_c = 3$ and the physical values of the $\pi NN$, $\pi N\Delta$ and $\pi\Delta\Delta$ couplings. It was also found that the best fit values for these couplings were remarkably close to the values obtained using non-relativistic $SU(6)$ symmetry, and that the large cancellation occurred only for values of the pion-couplings near the $SU(6)$ values, but not for generic couplings. It was suggested that the baryon-pion couplings should have a non-relativistic $SU(6)$ symmetry [12]. These results are exact in the large-$N_c$ limit of QCD. We will also show that the leading corrections to the $SU(6)$ relations are $1/N_c^2$ [13], which explains why the values of the pion-baryon couplings observed experimentally are so close to their $SU(6)$ values.

The wave-function renormalization graph also gives a mass shift of the baryons proportional to
\[
\frac{g^2 N_c^2}{16\pi f^2} X^{ia} X^{ia} m_\pi^3.
\] (15)
This mass shift cannot break the degeneracy of the baryon spectrum, or else the pion-baryon coupling consistency equations eq. (3) or (14) cannot be satisfied. Thus we require that
\[
X^{ia} X^{ia} = C_X + O \left( \frac{1}{N_c^2} \right),
\] (16)
where $C_X$ is a constant. This is the third consistency condition that must be satisfied by large-$N_c$ QCD. It is a trivial check to see that the solution eq. (5) with $X(J,J') = 1$ satisfies this consistency condition, with $C_X = 3$.

The consistency conditions eq. (3) and (14) are equivalent if one also assumes eq. (16). Eq. (14) can be written in the form
\[
0 = [X^{jb}, [X^{jb}, X^{ia}]] = X^{ja} X^{ja} X^{ia} + X^{ja} X^{ja} X^{jb} - 2 X^{ia} X^{ja} X^{ja},
\]
which leads to an equivalent form for eq. (14)
\[
X^{ja} X^{ia} X^{ja} = C_X X^{ia}.
\] (17)
Now assume eq. (16) and eq. (17), and define $M_{ia,jb} = i [X^{ia}, X^{jb}]$. Then
\[
\sum_{ia,jb} [M_{ia,jb}]^2 = \sum_{ia} X^{ia} X^{jb} X^{ia} + X^{jb} X^{ia} X^{ia} X^{jb} - X^{jb} X^{ia} X^{ja} - X^{ia} X^{jb} X^{ia} X^{ja} = 0,
\]
using eq. (16) and eq. (17). But $M_{ia,jb}$ is a Hermitian operator, and so $\sum_{ia,jb} [M_{ia,jb}]^2 = 0$ implies that $M_{ia,jb} = 0$ for all $ia$ and $jb$. This proves eq. (3). To prove the converse, assume eq. (3) and eq. (16). Then eq. (17) follows trivially by multiplying eq. (3) by $X^{jb}$ and summing over $jb$.

We can also look at other operators in the large-$N_c$ limit of QCD such as magnetic moments, or mass splittings. These operators can be treated as a vertex operator $V\cdots$, where the indices on $V\cdots$ depend on its spin and isospin transformation properties. The stability of insertions of the vertex operator to one-loop chiral corrections implies the consistency condition
\[
[X^{ia}, [X^{ia}, V\cdots]] = 0 \tag{18}
\]
which can be written using eq. (16) as
\[
X^{ia} V X^{ia} = C_X V. \tag{19}
\]

The operator $= (X^{ia})_{\alpha\beta} (X^{ia})_{\lambda\tau}$ is a linear operator, and eq. (19) implies that allowed vertex operators in the large-$N_c$ limit must be eigenstates of the Casimir operator $X^{ia} X^{ia}$ with eigenvalue $C_X$. An example is the isovector magnetic moment vertex $\mu V$. It must satisfy eq. (19) and has the same spin and isospin as the pion vertex $X$. Therefore, $\mu^{iq}$ must be a constant times $X^{ia}$, as in the Skyrme model. The same result can be derived by looking at the $N_c$ dependence of pion photoproduction. The scattering formalism also provides a somewhat stronger version of eq. (18) when $V$ is an isospin-splitting mass insertion $\Delta M$. Expanding the energy denominators $q^0$ and $q'^0$ in eq. (3) and generalizing to scattering from an arbitrary bayon yields
\[
[X^{ia}, [X^{jb}, \Delta M]] = 0, \tag{20}
\]
with an error of order $1/N_c$. Unlike the axial couplings, the single commutator $[X^{ia}, \Delta M]$ need not vanish.

The results presented here can be generalized to an arbitrary number of flavors $N_f$. The consistency conditions eqs. (3) (16) (20) are still valid, and prove that large-$N_c$ QCD
has a contracted $SU(2N_f)$ symmetry. The recursion relations such as eq. (9) are more complicated, because the isospin $6j$-symbol is replaced by the flavor $6j$-symbol for $SU(N_f)$.

In conclusion, we have derived a set of consistency conditions that must be satisfied by large-$N_c$ QCD, and have proved that large-$N_c$ QCD has a contracted $SU(4)$ symmetry (for $N_f = 2$). The solutions of the consistency equations lead to a unique (minimal) solution for the pion-baryon coupling constants, which are identical to those of the Skyrme model or non-relativistic quark model, and also to the existence of an infinite tower of baryon resonances. The $1/N_c$ corrections to the pion-baryon coupling constant ratios can be shown to vanish [6]. We have also briefly discussed consistency conditions for other operators, and the generalization to an arbitrary number of flavors. The details will be presented elsewhere. The methods used here have been used to confirm the soliton picture of baryons containing a heavy quark [13] in which the baryon is treated as a bound state of a soliton and a heavy meson [14] [15] [16], and to prove that the leading correction to the baryon mass splittings is proportional to $J^2$ [17].

We would like to thank E. Jenkins for helpful discussions. This work was supported in part by DOE grant DOE-FG03-90ER40546 and by a PYI award PHY-8958081.
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Figure Captions

Fig. 1. Diagrams contributing to $\pi N$ scattering. Fig 1(c) is suppressed by $1/N_c$ relative to figs. 1(a) and 1(b).

Fig. 2. Diagrams contributing to the one-loop corrections of the pion-baryon couplings.
