ON THE COMPLEXITY OF BACKWARD SMOOTHING
ALGORITHMS

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Abstract. In the context of state-space models, backward smoothing algorithms rely on a backward sampling step, which by default has a $O(N^2)$ complexity (where $N$ is the number of particles). An alternative implementation relying on rejection sampling has been proposed in the literature, with stated $O(N)$ complexity. We show that the running time of such algorithms may have an infinite expectation. We develop a general framework to establish the convergence and stability of a large class of backward smoothing algorithms that may be used as more reliable alternatives. We propose three novel algorithms within this class. The first one mixes rejection with multinomial sampling; its running time has finite expectation, and close-to-linear complexity (in a certain class of models). The second one relies on MCMC, and has deterministic $O(N)$ complexity. The third one may be used even when the transition of the model is intractable. We perform numerical experiments to confirm the good properties of these novel algorithms.

1. Introduction

1.1. Background. A state-space model is composed of an unobserved Markov process $X_0, \ldots, X_T$ and observed data $Y_0, \ldots, Y_T$. Given $X_0, \ldots, X_T$, the data $Y_0, \ldots, Y_T$ are independent and generated through some specified emission distribution $Y_t|X_t \sim f_t(dy_t|x_t)$. These models have wide-ranging applications (e.g. in biology, economics and engineering). Two important inference tasks related to state-space models are filtering (computing the distribution of $X_t$ given $Y_0, \ldots, Y_t$) and smoothing (computing the distribution of the whole trajectory $(X_0, \ldots, X_t)$, again given all data until time $t$). Filtering is usually carried out through a particle filter, that is, a sequential Monte Carlo algorithm that propagates $N$ weighted particles (realisations) through Markov and importance sampling steps; see Chopin and Papaspiliopoulos (2020) for a general introduction to state-space models (Chapter 2) and particle filters (Chapter 10).

This paper is concerned with smoothing algorithms that approximate the smoothing distributions with empirical distributions based on the output of a particle filter (i.e. the locations and weights of the $N$ particles at each time step). A simple example is genealogy tracking (initially introduced by Kitagawa, 1996), which keeps track of the ancestry (past states) of each particle. This smoother suffers from degeneracy: for $t$ large enough, all the particles have the same ancestor at time 0.

The forward filtering backward smoothing (FFBS) algorithm (Godsill et al., 2004) has been proposed as a solution to this problem. Starting from the filtering approximation at time $t$, the algorithm samples successively particles at times $t - 1$, $t - 2$, etc. using backward kernels. The naive implementation has an $O(N^2)$ cost. However, if the Markov transition density is bounded, a rejection sampling-based
scheme can be used (Douc et al., 2011). Its complexity is shown to be $O(N)$ under restrictive assumptions on the model.

In many applications, one is mainly interested in approximating smoothing expectations of additive functions of the form

$$\mathbb{E} \left[ \psi_0(X_0) + \psi_1(X_0, X_1) + \cdots + \psi_t(X_{t-1}, X_t) \mid Y_0, \ldots, Y_t \right]$$

for some functions $\psi_0, \ldots, \psi_t$. Such expectations can be approximated on-line in $O(N^2)$ time by a procedure described in Del Moral et al. (2010). Inspired by this, the particle-based, rapid incremental smoother (PaRIS) algorithm of Olsson and Westerborn (2017) replaces some of the calculations with an additional layer of Monte Carlo approximation. Again, it is possible to employ rejection sampling at this level and get an $O(N)$ cost under strong assumptions.

There are three problems however with the rejection versions of FFBS and PaRIS. First, the $O(N)$ claim does not hold for realistic models (as we shall demonstrate theoretically and numerically). Second, their running time is random (we elaborate below why this is a drawback). Third (and this problem also apply to the $O(N^2)$ versions of FFBS and PaRIS), they require the Markov transition of the model to be tractable, which is not the case for certain models of practical interest. These three problems are deeply linked to the way the backward sampling step operates.

1.2. Motivation and structure. We will therefore propose new methods to address these issues. Since backward sampling is central to a wide variety of algorithms (e.g. FFBS, forward-additive smoothing, PaRIS), Section 2 presents them in a unified framework. We show how they can all be expressed in terms of discrete backward kernels, which are essentially random $N \times N$ matrices. We specify how these matrices are used differently for off-line and on-line smoothing scenarios. Importantly, we state generic sufficient conditions which ensure that the resulting algorithms are consistent and stable as $T \to \infty$.

Having at hand the necessary theoretical framework, the rest of the article is spent on methodological innovations. We first closely look at the use of rejection sampling and realise that in many models, the resulting execution time may have an infinite expectation; see Section 3. In highly parallel computing architectures, each processor only handles one or a small number of particles. As such, the heavy-tailed nature of the execution time means that a few machine might prevent the whole system from moving forward. In all computing architectures, an infinite mean running time makes it difficult to know when a program will stop, even after having performed few pilot runs. We introduce a hybrid rejection sampling procedure which fixes this problem and leads to a nearly $O(N)$ algorithm (up to some log factor) in an important class of practical models; again see Section 3.

To make the execution time fully linear and deterministic, we propose in Section 4.1 backward kernels based on MCMC (Markov chain Monte Carlo). Convergence and stability of the resulting algorithm are inherited from the general framework developed in Section 2. MCMC methods require evaluation of the likelihood and thus cannot be applied to models with intractable transition densities. In Section 4.2, we show how the use of forward coupling can replace the role of backward MCMC steps in these scenarios. This makes it possible to obtain stable performance in both on-line and off-line scenarios.
Section 5 illustrates the aforementioned algorithms in both on-line and off-line uses. We highlight how hybrid and MCMC samplers lead to a more user-friendly (i.e. smaller, less random and less model-dependent) execution time than the pure rejection sampler. We also apply our smoother for intractable densities to a continuous-time diffusion process with discretisation. Further numerical examples, in particular those concerning Markov jump processes, are planned to be included in a future version of this paper. We observe that our procedure can indeed prevent degeneracy as $T \to \infty$, provided that some care is taken to build couplings with good performance.

Finally, Section 6 concludes the paper with final practical recommendations and further research directions.

1.3. Related work. Interestingly, the potential issue with the complexity of FFBS-reject has been suggested by their authors of this method. Indeed, Proposition 1 of Douc et al. (2011) implies that the asymptotic complexity of FFBS-reject might tend to infinity in some situations. However, we show that the infiniteness might already happen at finite values of $N$. Moreover, we propose concrete solutions to fix this problem and analyse them both theoretically and practically.

Figure 1 of Olsson and Westerborn (2017) and the accompanying discussion provide an excellent intuition on the stability of smoothing algorithms based on the support size of the backward kernels. We formalise these insights and use them to construct new efficient and stable algorithms.

The heavy-tailed distribution of the running time of FFBS-reject have been remarked in Taghavi et al. (2013), who proposed a hybrid algorithm combining multinomial and rejection sampling. However, theoretical analysis of the complexity was not performed and the extension to online smoothing was not considered. (The PaRIS algorithm had not be invented then.) Using MCMC steps (started at the previous ancestor) instead of rejection sampling has been explored in Bunch and Godsill (2013). Again, consistency and stability were not formally proved and the article was limited to the offline scenario. Gloaguen et al. (2019) briefly mention the use of MCMC in PaRIS algorithm, without doing further theoretical or numerical analyses. Moreover, since their MCMC chains do not start at the previous ancestors, a large number of steps are necessary to get a correct algorithm. As we shall see, our procedure requires only one MCMC step.

Another way to reduce the computation time is to perform the expensive backward sampling steps at certain times $t$ only. For other values of $t$, the naive genealogy tracking smoother is used instead. This idea has been recently proposed by Mastrototaro et al. (2021), who also provided a practical recipe for deciding at which values of $t$ the backward sampling should take place and derived corresponding theoretical results.

Smoothing in models with intractable transition densities is very challenging. If these densities can be estimated accurately, the algorithms proposed by Gloaguen et al. (2019) permit to attack this problem. A case of particular interest is diffusion models, where unbiased transition density estimators are provided in Beskos et al. (2006); Fearnhead et al. (2008). More recently, Yonekura and Beskos (2022) use a special bridge path-space construction to overcome the unavailability of transition densities when the diffusion (possibly with jumps) must be discretised.

Our smoother for intractable models are based on a general coupling principle that is not specific to diffusions. We only require users to be able to simulate their
dynamics (e.g. using discretisation in the case of diffusions) and to manipulate random numbers in their simulations so that dynamics starting from two different points can meet with some probability. Our method does not directly provide an estimator for the gradient of the transition density with respect to model parameters and thus cannot be used in its current form to perform maximum likelihood estimation (MLE) in intractable models; whereas the aforementioned work have been able to do so in the case of diffusions. However, the main advantage of our approach lies in its generality beyond the diffusion case. Furthermore, modifications allowing to perform MLE are possible and might be explored in further work specifically dedicated to the parameter estimation problem.

The idea of coupling has been incorporated in the smoothing problem in a different manner by Jacob et al. (2019). There, the goal is to provide offline unbiased estimates of the expectation under the smoothing distribution. Coupling and more generally ideas based on correlated random numbers are also useful in the context of partially observed diffusions via the multilevel approach (Jasra et al., 2017).

In this work, we consider smoothing algorithms that are based on a unique pass of the particle filter. Offline smoothing can be done using repeated iterations of the conditional particle filter (Andrieu et al., 2010). Another approach to smoothing consists of using an additional information filter (Fearnhead et al., 2010), but it is limited to functions depending on one state only. Each of these algorithmic families has their own advantages and disadvantages, of which a detailed discussion is out of the scope of this article (see however Nordh and Antonsson, 2015).

2. General structure of smoothing algorithms

2.1. Notations. Measure-kernel-function notations. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two measurable spaces with respective \( \sigma \)-algebras \( \mathcal{B}(\mathcal{X}) \) and \( \mathcal{B}(\mathcal{Y}) \). The following definitions involve integrals and only make sense when they are well-defined. For a measure \( \mu \) on \( \mathcal{X} \) and a function \( f : \mathcal{X} \to \mathbb{R} \), the notations \( \mu f \) and \( \mu(f) \) refer to \( \int f(x)\mu(dx) \). A kernel (resp. Markov kernel) \( K \) is a mapping from \( \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \) to \( \mathbb{R} \) (resp. \([0,1]\)) such that, for \( B \in \mathcal{B}(\mathcal{Y}) \) fixed, \( x \mapsto K(x,B) \) is a measurable function on \( \mathcal{X} \); and for \( x \) fixed, \( B \mapsto K(x,B) \) is a measure (resp. probability measure) on \( \mathcal{Y} \). For a real-valued function \( g \) defined on \( \mathcal{Y} \), let \( Kg : \mathcal{X} \rightarrow \mathbb{R} \) be the function \( Kg(x) := \int g(y)K(x,dy) \). We sometimes write \( K(x,g) \) for the same expression. The product of the measure \( \mu \) on \( \mathcal{X} \) and the kernel \( K \) is a measure on \( \mathcal{Y} \), defined by \( \mu K(B) := \int K(x,B)\mu(dx) \).

Other notations. \bullet The notation \( X_{0:t} \) is a shorthand for \( (X_0,\ldots,X_t) \) \bullet We denote by \( \mathcal{M}(W^{1:N}) \) the multinomial distribution supported on \( \{1,2,\ldots,N\} \). The respective probabilities are \( W_1,\ldots,W_N \). If they do not sum to 1, we implicitly refer to the normalised version obtained by multiplication of the weights with the appropriate constant \bullet The symbol \( \Rightarrow \) means convergence in probability \bullet The geometric distribution with parameter \( \lambda \) is supported on \( \mathbb{Z}_{\geq 1} \), has probability mass function \( f(n) = \lambda(1-\lambda)^{n-1} \) and is noted by \( \text{Geo}(\lambda) \) \bullet Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two measurable spaces. Let \( \mu \) and \( \nu \) be two probability measures on \( \mathcal{X} \) and \( \mathcal{Y} \) respectively. The \( n \) times product measure \( \mu \otimes \nu \) is defined via \( (\mu \otimes \nu)(h) := \int\int h(x,y)\mu(dx)\nu(dy) \) for bounded functions \( h : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \). If \( X \sim \mu \) and \( Y \sim \nu \), we sometimes note \( \mu \otimes \nu \) by \( X \otimes Y \).
2.2. Feynman-Kac formalism and the bootstrap particle filter. Let $X_{0:T}$ be a sequence of measurable spaces and $M_{1:T}$ be a sequence of Markov kernels such that $M_t$ is a kernel from $X_{t-1}$ to $X_t$. Let $X_{0:T}$ be an unobserved inhomogeneous Markov chain with starting distribution $X_0 \sim \mathcal{M}_0(dx_0)$ and Markov kernels $M_{1:T}$; i.e. $X_t|X_{t-1} \sim M_t(X_{t-1}, dx_t)$ for $t \geq 1$. We aim to study the distribution of $X_{0:T}$ given observed data $Y_{0:T}$. Conditioned on $X_{0:T}$, the data $Y_0, \ldots, Y_T$ are independent and

$$Y_t|X_{0:T} \equiv Y_t|X_t \sim f_t(\cdot|X_t)$$

for a certain emission distribution $f_t(dy_t|x_t)$. Assume that there exists dominating measures $\tilde{\lambda}_t$ not depending on $x_t$ such that

$$f_t(dy_t|x_t) = f_t(y_t|x_t)\tilde{\lambda}_t(dy_t).$$

The distribution of $X_{0:t}|Y_{0:t}$ is then given by

$$(1) \quad Q_t(dx_{0:t}) = \frac{1}{L_t} \mathcal{M}_0(dx_0) \prod_{s=1}^t M_s(x_{s-1}, dx_s) G_s(x_s)$$

where $G_s(x_s) := f(y_s|x_s)$ and $L_t > 0$ is the normalising constant. Moreover, $Q_{-1} := \mathcal{M}_0$ and $L_{-1} := 1$ by convention. Equation (1) defines a Feynman-Kac model (Del Moral, 2004). It does not require $M_t$ to admit a transition density, although herein we only consider models where this assumption holds. Let $\lambda_t$ be a dominating measure on $X_t$ in the sense that there exists a function $m_t$ (not necessarily tractable) such that

$$(2) \quad M_t(x_{t-1}, dx_t) = m_t(x_{t-1}, x_t)\lambda_t(dx_t).$$

A special case of the current framework are linear Gaussian state space models. They will serve as a running example for the article, and some of the results will be specifically demonstrated for models of this class. The rationale is that many real-world dynamics are partly, or close to, Gaussian. The notations for linear Gaussian models are given in Appendix A.1 and we will refer to them whenever this model class is discussed.

Particle filters are algorithms that sample from $Q_t(dx_t)$ in an on-line manner. In this article, we only consider the bootstrap particle filter (Gordon et al., 1993) and we detail its notations in Algorithm 1. Many results in the following do apply to the auxiliary filter (Pitt and Shephard, 1999) as well, and we shall as a rule indicate explicitly when it is not the case.

We end this subsection with the definition of two sigma-algebras that will be referred to throughout the paper. Using the notations of Algorithm 1, let

$$(3) \quad \mathcal{F}_t := \sigma(X_{0:t}^{1:N}, A_{1:t}^{1:N}), \quad \mathcal{F}_t^- := \sigma(X_{0:t}^{1:N}).$$

2.3. Backward kernels and off-line smoothing. In this subsection, we first describe three examples of backward kernels, in which we emphasise both the random measure and the random matrix viewpoints. We then formalise their use by stating a generic off-line smoothing algorithm.
Then, recursively for $W^n_m$ manner. More precisely, it starts by simulating index $\frac{\omega_0^n}{N^{\ell_0^n}}$ for $n = 1, \ldots, N$

Set $\ell_0^n \leftarrow \sum_{n=1}^N \omega^n / N$

Set $W_0^n \leftarrow \frac{\omega_0^n}{N \ell_0^n}$ for $n = 1, \ldots, N$

For $t \leftarrow 1$ to $T$

Resample. Simulate $A_t^{1:N}$ i.i.d. $M(W_t^{1:N})$

Move. Simulate $X^n_t \sim M_t(X^n_{t-1}, dx_t)$ for $n = 1, \ldots, N$

Reweight. Set $\omega^n_t \leftarrow G_t(X^n_t)$ for $n = 1, 2, \ldots, N$

Set $\ell_t^n \leftarrow \sum_{n=1}^N \omega^n_t / N$

Set $W_t^n \leftarrow \frac{\omega^n_t}{N \ell_t^n}$ for $n = 1, 2, \ldots, N$

Output: For all $t \geq 0$ and function $\varphi : X_t \rightarrow \mathbb{R}$, the quantity

$$\sum_{n=1}^N W^n_t \varphi(X^n_t)$$

approximates $\int \varphi_t(dx_0,1)\varphi(x_t)$ and the quantity

$\ell_t^n$ approximates $L_t/L_{t-1}$



**Algorithm 1:** Bootstrap particle filter

**Input:** Feynman-Kac model (1)

Simulate $X_{t-1}^{1:N}$ i.i.d. $M(W_{t-1}^{1:N})$

Set $\omega^n_0 \leftarrow G_0(X^n_0)$ for $n = 1, \ldots, N$

Set $\ell_0^n \leftarrow \sum_{n=1}^N \omega^n_0 / N$

Set $W_0^n \leftarrow \omega^n_0 / N \ell_0^n$ for $n = 1, \ldots, N$

For $t \leftarrow 1$ to $T$

Resample. Simulate $A_t^{1:N}$ i.i.d. $M(W_t^{1:N})$

Move. Simulate $X^n_t \sim M_t(X^n_{t-1}, dx_t)$ for $n = 1, \ldots, N$

Reweight. Set $\omega^n_t \leftarrow G_t(X^n_t)$ for $n = 1, 2, \ldots, N$

Set $\ell_t^n \leftarrow \sum_{n=1}^N \omega^n_t / N$

Set $W_t^n \leftarrow \omega^n_t / N \ell_t^n$ for $n = 1, 2, \ldots, N$

Output: For all $t \geq 0$ and function $\varphi : X_t \rightarrow \mathbb{R}$, the quantity

$$\sum_{n=1}^N W^n_t \varphi(X^n_t)$$

approximates $\int \varphi_t(dx_0,1)\varphi(x_t)$ and the quantity

$\ell_t^n$ approximates $L_t/L_{t-1}$



**Example 1** (FFBS algorithm). Once Algorithm 1 has been run, the FFBS procedure generates a trajectory approximating the smoothing distribution in a backward manner. More precisely, it starts by simulating index $\mathcal{I}_T \sim M(W_T^{1:N})$ at time $T$. Then, recursively for $t = T, \ldots, 1$, given indices $\mathcal{I}_t:T$, it generates $\mathcal{I}_{t-1} \in \{1, \ldots, N\}$ with probability proportional to $W_{t-1}^m(X_{t-1}^m, X_t^m)$. The smoothing trajectory is returned as $(X_T^m, \ldots, X_T^m)$. Formally, given $\mathcal{F}_T$, the indices $\mathcal{I}_0:T$ are generated according to the distribution

$$\mathcal{M}(W_t^{1:N})(\mathcal{I}_t)\left[B_t^{N,FFBS}(i_t, \mathcal{I}_{t-1})B_{t-1}^{N,FFBS}(i_{t-1}, \mathcal{I}_{t-2})\ldots B_1^{N,FFBS}(i_1, \mathcal{I}_0)\right]$$

where the (random) backward kernels $B_t^{N,FFBS}$ are defined by

$$B_t^{N,FFBS}(i_t, \mathcal{I}_{t-1}) := \sum_{n=1}^N \frac{W^n_{t-1}m_t(X^n_{t-1}, X^n_t)\delta_n}{\sum_{k=1}^N W^n_{t-1}m_t(X^n_{t-1}, X^n_t)}.$$

More simply, we can also look at these random kernels as random $N \times N$ matrices of which entries are given by

$$B_t^{N,FFBS}[i_t, i_{t-1}] := \frac{W_{i_t-1}^{i_t}m_t(X_{i_t-1}^{i_t}, X_{i_t}^{i_t})}{\sum_{k=1}^N W_{i_t-1}^{i_t}m_t(X_{i_t-1}^{i_t}, X_{i_t}^{i_t})}.$$

We will need both the kernel viewpoint (4) and the matrix viewpoint (5) in this paper as the better choice depends on the context.

**Example 2** (Genealogy tracking). It is well known that Algorithm 1 already gives a by-product an approximation of the smoothing distribution. This information can be extracted from the genealogy, by first simulating index $\mathcal{I}_T \sim M(W_T^{1:N})$ at time $T$, then successively appending ancestors until time 0 (i.e. setting sequentially $\mathcal{I}_{t-1} \leftarrow \mathcal{A}_t^{i_t}$). The smoothed trajectory is returned as $(X_T^0, \ldots, X_T^T)$. More formally, conditioned on $\mathcal{F}_T$, we simulate the indices $\mathcal{I}_0:T$ according to

$$\mathcal{M}(W_t^{1:N})(\mathcal{I}_t)\left[B_t^{N,GT}(i_t, \mathcal{I}_{t-1})B_{t-1}^{N,GT}(i_{t-1}, \mathcal{I}_{t-2})\ldots B_1^{N,GT}(i_1, \mathcal{I}_0)\right]$$
where GT stands for “genealogy tracking” and the kernels $B_{t}^{N,GT}$ are simply

$$B_{t}^{N,GT}(i_t, d_{t-1}) := \delta_{A_{t}^n}(d_{t-1}).$$

Again, it may be more intuitive to view this random kernel as a random $N \times N$ matrix, the elements of which are given by

$$\hat{B}_{t}^{N,GT}[i_t, i_{t-1}] := 1 \{i_{t-1} = A_{t}^n\}.$$

**Example 3 (MCMC backward samplers).** In Example 2, the backward variable $\mathcal{I}_{t-1}$ is simply set to $A_{t}^{T}$. On the contrary, in Example 1, we need to launch a simulator for the discrete measure $W_{t-1}^{n}m(X_{t-1}, X_{t}^{T})$. Interestingly, the current value of $A_{t}^{T}$ is not taken into account in that simulator. Therefore, a natural idea to combine the two previous examples is to apply one (or several) MCMC steps to $A_{t}^{T}$ and assign the result to $\mathcal{I}_{t-1}$. The MCMC algorithm operates on the space $\{1, 2, \ldots, N\}$ and targets the invariant measure $W_{t-1}^{n}m(X_{t-1}, X_{t}^{T})$. If only one independent Metropolis-Hastings (MH) step is used and the proposal is $\mathcal{M}(W_{t-1}^{1:N})$, the corresponding random matrix $\hat{B}_{t}^{N,IMH}$ has values

$$\hat{B}_{t}^{N,IMH}[i_t, i_{t-1}] = W_{t-1}^{i_{t-1}} \min \left(1, m_t(X_{t-1}^{i_{t-1}}, X_{t}^{i_t})/m_t(X_{t-1}^{A_{t}^n}, X_{t}^{i_t}) \right)$$

if $i_{t-1} \neq A_{t}^n$, and

$$\hat{B}_{t}^{N,IMH}[i_t, A_{t}^n] = 1 - \sum_{n \neq A_{t}^n} \hat{B}_{t}^{N,IMH}[i_t, n].$$

This third example shows that some elements of the matrix $\hat{B}_{t}^{N,IMH}$ might be expensive to calculate. If several MCMC steps are performed, all elements of $\hat{B}_{t}^{N,IMH}$ will have non-trivial expressions. Still, simulating from $\hat{B}_{t}^{N,IMH}(i_t, d_{t-1})$ is easy as it amounts to running a standard MCMC algorithm. MCMC backward samples are studied in more details in Section 4.1.

We formalise how off-line smoothing can be done given random matrices $\hat{B}_{1:T}^{N}$; see Algorithm 2. Note that in the above examples, our matrices $\hat{B}_{t}^{N}$ are $\mathcal{F}_T$-measurable (i.e. they depend on particles and indices up to time $t$), but this is not necessarily the case in general (i.e. they may also depend on additional random variables, see Section 2.5). Furthermore, Algorithm 2 describes how to perform smoothing using the matrices $\hat{B}_{1:T}^{N}$, but does not say where they come from. At this point, it is useful to keep in mind the above three examples. In Section 2.4, we will give a general recipe for constructing valid matrices $\hat{B}_{t}^{N}$ (i.e. those that give a consistent algorithm).

Algorithm 2 simulates, given $\mathcal{F}_T$ and $\hat{B}_{1:T}^{N}$, $N$ i.i.d. index sequences $\mathcal{I}_{0:T}^{n}$, each distributed according to

$$\mathcal{M}(W_{T}^{1:N})(d_{T}) \prod_{i=T}^{1} B_{i}^{N}(i_t, d_{t-1}).$$

Once the indices $\mathcal{I}_{0:T}^{1:N}$ are simulated, the $N$ smoothed trajectories are returned as $(X_{0}^{\mathcal{I}_{0}^{n}}, \ldots, X_{T}^{\mathcal{I}_{T}^{n}})$. Given $\mathcal{F}_T$ and $\hat{B}_{1:T}^{N}$, they are thus conditionally i.i.d. and their
The conditional distribution is described by the $x_{0:T}$ component of the joint distribution

$$
\hat{Q}_T^N(dx_{0:T}, d\gamma_{0:T}) := \mathcal{M}(W_{1:T}^1)(d\gamma_T) \left[ \prod_{t=T}^{1} B_t^N(i_t, d\gamma_{t-1}) \right] \left[ \prod_{t=T}^{0} \delta_{X_t^{\gamma_t}}(dx_t) \right].
$$

Throughout the paper, the symbol $\hat{Q}_T^N$ will refer to this joint distribution, while the symbol $Q_T^N$ will refer to the $x_{0:T}$-marginal of $\hat{Q}_T^N$ only. This allows the notation $Q_T^N \varphi$ to make sense, where $\varphi = \varphi(x_0, \ldots, x_T)$ is a real-valued function defined on the hidden states.

### 2.4. Validity and convergence

The kernels $B_t^{N,FFBS}$ and $B_t^{N,GT}$ are both valid backward kernels to generate convergent approximation of the smoothing distribution (Del Moral, 2004; Douc et al., 2011). This subsection shows that they are not the only ones and gives a sufficient condition for a backward kernel to be valid. It will prove a necessary tool to build more efficient $B_t^N$ later in the paper.

Recall that Algorithm 1 outputs particles $X_{0:T}^1$, weights $W_{1:T}^1$, and ancestor variables $A_{1:T}^1$. Imagine that the $A_{1:T}^1$ were discarded after filtering has been done and we wish to simulate them back. We note that, since the $X_{0:T}^1$ are given, the $T \times N$ variables $A_{1:T}^1$ are conditionally i.i.d. We can thus simulate them back from

$$
p(a_t^n|x_{0:t}^1) = p(a_t^n|x_{t-1}^1, a_t^n) \propto w_{t-1}^{a_t^n} m_t(x_{t-1}^{a_t^n}, x_t^n).
$$

This is precisely the distribution of $B_t^{N,FFBS}(n, \cdot)$. It turns out that any other invariant kernel that can be used for simulating back the discarded $A_{1:T}^1$ will lead to a convergent algorithm as well. For instance, $B_t^{N,GT}(n, \cdot)$ (Example 2) simply returns back the old $A_t^n$, unlike $B_t^{N,FFBS}(n, \cdot)$ which creates a new version. The kernel $B_t^{N,IMH}(n, \cdot)$ (Example 3) is somewhat an intermediate between the two. We formalise these intuitions in the following theorem. It is stated for the bootstrap particle filter, but as a matter of fact, the proof can be extended straightforwardly to auxiliary particle filters as well.

**Assumption 1.** For all $0 \leq t \leq T$, $G_t(x_t) > 0$ and $\|G_t\|_\infty < \infty$.

**Theorem 1.** We use the same notations as in Algorithms 1 and 2 (in particular, $B_t^N$ denotes the transition matrix that corresponds to the considered kernel $B_t^N$).

Assume that for any $1 \leq t \leq T$, the random matrix $\hat{B}_t^N$ satisfies the following conditions:
There are no complications in the backward kernel for any $\varepsilon > (8)$, and the measure is illustrated by a graphical model in Figure 1. (See Bishop 2006, Chapter 8 for the formal definition of graphical models and how to use them.) By “typical”, we mean that Theorem 1 technically allows for more complicated relations, but the aforementioned figure captures the most essential cases.

Then under Assumption 1, there exists constants $C_T > 0$ and $S_T < \infty$ such that, for any $\varepsilon > 0$ and function $\varphi = \varphi(x_0, \ldots, x_T)$:

$$
\mathbb{P}\left(\frac{\sqrt{N}}{S_T \|\varphi\|_\infty} \left| Q_T^N \varphi - Q_T \varphi \right| \geq \varepsilon\right) \leq 2C_T e^{-\varepsilon^2/2}
$$

where $Q_T^N$ is defined by (7).

A typical relation between variables defined in the statement of the theorem is illustrated by a graphical model in Figure 1. (See Bishop 2006, Chapter 8 for the formal definition of graphical models and how to use them.) By “typical”, we mean that Theorem 1 technically allows for more complicated relations, but the aforementioned figure captures the most essential cases.

Then under Assumption 1, there exists constants $C_T > 0$ and $S_T < \infty$ such that, for any $\varepsilon > 0$ and function $\varphi = \varphi(x_0, \ldots, x_T)$:

$$
\mathbb{P}\left(\frac{\sqrt{N}}{S_T \|\varphi\|_\infty} \left| Q_T^N \varphi - Q_T \varphi \right| \geq \varepsilon\right) \leq 2C_T e^{-\varepsilon^2/2}
$$

where $Q_T^N$ is defined by (7).

A typical relation between variables defined in the statement of the theorem is illustrated by a graphical model in Figure 1. (See Bishop 2006, Chapter 8 for the formal definition of graphical models and how to use them.) By “typical”, we mean that Theorem 1 technically allows for more complicated relations, but the aforementioned figure captures the most essential cases.

Theorem 1 is a generalisation of Douc et al. (2011, Theorem 5). Its proof thus follows the same lines (Appendix E.1). However, in our case the measure $Q_T^N(dx_{0:T})$ is no longer Markovian. This is because the backward kernel $B^N_{t_1}(i_t, d_{i_{t-1}})$ does not depend on $X^N_{t_1}$ alone, but also possibly on its ancestor and extra random variables.

As we have seen in (7), $Q_T^N$ is fundamentally a discrete measure of which the support contains $N^{T+1}$ elements. As such, $Q_T^N \varphi$ cannot be computed exactly in general and must be approximated using $N$ trajectories $(X^N_{t_0}, \ldots, X^N_{t_T})$ simulated via Algorithm 2. Theorem 1 is thus completed by the following corollary, which is an immediate consequence of Hoeffding inequality (Appendix E.13).

**Corollary 1.** Under the same setting as Theorem 1, we have

$$
\mathbb{P}\left(\frac{\sqrt{N}}{(S_T + 1) \|\varphi\|_\infty} \left| \frac{1}{N} \sum_n \varphi(X^N_{t_0}, \ldots, X^N_{t_T}) - Q_T^N \varphi \right| \geq \varepsilon\right) \leq 2(C_T + 1)e^{-\varepsilon^2/2}.
$$

### 2.5. Generic on-line smoother.

As we have seen in Section 2.3 and Section 2.4, in general, the expectation $Q_T^N \varphi$, for a real-valued function $\varphi = \varphi(x_0, \ldots, x_T)$ of the hidden states, cannot be computed exactly due to the large support ($N^{T+1}$ elements) of $Q_T^N$. Moreover, in certain settings we are interested in the quantities $Q_t^N \varphi_t$ for different functions $\varphi_t$. They cannot be approximated in an on-line manner without more assumptions on the connection between $\varphi_{t-1}$ and $\varphi_t$. If the family

\[ X_{t-1}^{N-1} \rightarrow A^n_t \]

\[ \downarrow \]

\[ X^n_t \]

\[ \downarrow \]

\[ \hat{B}^N_t(n, \cdot) \rightarrow J^n_t \]

**Figure 1.** Relation between variables described in Theorem 1.

- given $F_{t-1}$ and $B^N_{1:t-1}$, the variables $(X^n_t, A^n_t, \hat{B}^N_t(n, \cdot))$ for $n = 1, \ldots, N$ are i.i.d. and their distribution only depends on $X_{t-1}^{N-1}$, where $\hat{B}^N_t(n, \cdot)$ is the $n$-th row of matrix $B^N_t$;
- if $J^n_t$ is a random variable such that $J^n_t \mid X_{t-1}^{N-1}, X^n_t, \hat{B}^N_t(n, \cdot) \sim B^N_t(n, \cdot)$, then $(J^n_t, X^n_t)$ has the same distribution as $(A^n_t, X^n_t)$ given $X_{t-1}^{N-1}$.
\((\varphi_t)\) is additive, i.e. there exists functions \(\psi_t\) such that
\[\varphi_t(x_{0:t}) := \psi_0(x_0) + \psi_1(x_0, x_1) + \cdots + \psi_t(x_{t-1}, x_t)\]
then we can calculate \(Q^N t \varphi_t\) both exactly and on-line. The procedure was first described in Del Moral et al. (2010) for the kernel \(Q^N,FFBS\) (i.e. the measure defined by (7) and the random kernels \(B^N,FFBS\), but we will use the idea for other kernels as well. In this subsection, we first explain the principle of the method, then discuss its computational complexity and the link to the PaRIS algorithm (Olsson and Westerborn, 2017).

**Principle.** For simplicity, we start with the special case \(\varphi_t(x_{0:t}) = \psi_0(x_0)\). Equation (7) and the matrix viewpoint of Markov kernels then give
\[
Q^N_t \varphi_t = [W^1_t \ldots W^N_t] \hat{B}^N_t \hat{B}^N_{t-1} \ldots \hat{B}^N_1 \begin{bmatrix} \psi_0(X^1_t) \\ \vdots \\ \psi_0(X^N_t) \end{bmatrix}.
\]
This naturally suggests the following recursion formula to compute \(Q^N_t \varphi_t\):
\[
Q^N_t \varphi_t = [W^1_t \ldots W^N_t] \hat{S}^N_t
\]
with \(\hat{S}^N_0 = [\psi_0(X^1_0) \ldots \psi_0(X^N_0)]^T\) and
\[
\hat{S}^N_t := \hat{B}^N_t \hat{S}^N_{t-1}.
\]
In the general case where functions \(\varphi_t\) are given by (9), simple calculations (Appendix E.2) show that (10) is replaced by
\[
\hat{S}^N_t := \hat{B}^N_t \hat{S}^N_{t-1} + \mathrm{diag}(\hat{\psi}^N_t)
\]
where the \(N \times N\) matrix \(\hat{\psi}^N_t\) is defined by
\[
\hat{\psi}^N_t[i_{t-1}, i_t] := \psi_t(X^i_{t-1}, X^i_t)
\]
and the operator \(\mathrm{diag} : \mathbb{R}^{N \times N} \to \mathbb{R}^N\) extracts the diagonal of a matrix. This is exactly what is done in Algorithm 3.

**Algorithm 3:** Generic on-line smoother for additive functions (one step)

**Input:** Particles \(X^i_{t-1}\) and weights \(W^i_{t-1}\) at time \(t-1\); the \(N \times 1\) vector \(\hat{S}^N_{t-1}\) (see text); additive function (9)

Generate \(X^i_t\) and \(W^i_t\) according to the particle filter (Algorithm 1)

Calculate the random matrix \(\hat{B}^N_t\) (see Section 2.3 and Section 2.4)

Create the \(N \times 1\) vector \(\hat{S}^N_t\) according to (11). More precisely:

for \(i_t \leftarrow 1\) to \(N\) do

\[
\hat{S}^N_t[i_t] \leftarrow \sum_{i_{t-1}} \hat{B}^N_t[i_t, i_{t-1}] \left( \hat{S}^N_{t-1}[i_{t-1}] + \psi_t(X^i_{t-1}, X^i_t) \right)
\]

**Output:** Estimate \(\sum_n W^n_t \hat{S}^N_t[n]\) of \(Q_t(\varphi_t)\); particles \(X^i_t\), weights \(W^i_t\) and vector \(\hat{S}^N_t\) for the next step.
Computational complexity and the PaRIS algorithm. Equations (10) and (11) involve a matrix-vector multiplication and thus require, in general, $O(N^2)$ operations to be evaluated. When $\hat{B}_t^N \equiv \hat{B}_t^{N,\text{FFBS}}$, Algorithm 3 becomes the $O(N^2)$ on-line smoothing algorithm of Del Moral et al. (2010). The $O(N^2)$ complexity can however be lowered to $O(N)$ if the matrices $\hat{B}_t^N$ are sparse. This is the idea behind the PaRIS algorithm (Olsson and Westerborn, 2017), where the full matrix $\hat{B}_t^{N,\text{FFBS}}$ is unbiasedly estimated by a sparse matrix $\hat{B}_t^{N,\text{PaRIS}}$. More specifically, for any integer $N > 1$, for any $n \in \{1, \ldots, N\}$, let $J_t^{n,1}, \ldots, J_t^{n,N}$ be conditionally i.i.d. random variables simulated from $B_t^{N,\text{FFBS}}(n, \cdot)$. The random matrix $\hat{B}_t^{N,\text{PaRIS}}$ is then defined as

$$
\hat{B}_t^{N,\text{PaRIS}}[n, m] := \frac{1}{N} \sum_{\tilde{n}=1}^{\tilde{N}} I \left\{ J_t^{\tilde{n}, \tilde{n}} = m \right\}
$$

and the corresponding random kernel is

$$
B_t^{N,\text{PaRIS}}(n, dm) = \frac{1}{N} \sum_{\tilde{n}=1}^{\tilde{N}} \delta_{J_t^{\tilde{n}, \tilde{n}}}(dm).
$$

The following straightforward proposition establishes the validity of the $B_t^{N,\text{PaRIS}}$ kernel as well as the corresponding $O(N)$ complexity of (10) and (11), provided that $\tilde{N}$ is fixed as $N \to \infty$.

**Proposition 1.** The matrix $\hat{B}_t^{N,\text{PaRIS}}$ has only $O(N\tilde{N})$ non-zero elements out of $N^2$. It is an unbiased estimate of $B_t^{N,\text{FFBS}}$ in the sense that

$$
E\left[ \hat{B}_t^{N,\text{PaRIS}} | \mathcal{F}_t \right] = B_t^{N,\text{FFBS}}.
$$

Moreover, the sequence of matrices $\hat{B}_t^{N,\text{PaRIS}}$ satisfies the two conditions of Theorem 1.

It is important to remark that the $O(N)$ complexity only refers to the cost of computing the recursions (10) and (11). It does not include the cost of generating the matrices $\hat{B}_t^{N,\text{PaRIS}}$ themselves, i.e., the operations required to simulate the indices $J_t^{n, \tilde{n}}$. In Olsson and Westerborn (2017) it is argued that such simulations have an $O(N)$ cost using the rejection sampling method whenever the transition density is both upper and lower bounded. Section 3 investigates the claim when this hypothesis is violated.

2.6. Stability. When $\hat{B}_t^N \equiv \hat{B}_t^{N,\text{GT}}$, Algorithms 2 and 3 reduce to the genealogy tracking smoother (Kitagawa, 1996). The matrix $\hat{B}_t^{N,\text{GT}}$ is indeed sparse, leading to the well-known $O(N)$ complexity of this on-line procedure. As per Theorem 1, smoothing via genealogy tracking is convergent at rate $O(N^{-1/2})$ if $T$ is fixed. When $T \to \infty$ however, all particles will eventually share the same ancestor at time 0 (or any fixed time $t$). Mathematically, this phenomenon is manifested in two ways: (a) for fixed $t$ and function $\phi_t : \mathcal{X}_t \to \mathbb{R}$, the error of estimating $E[\phi_t(X_t)|Y_{0:T}]$ grows linearly with $T$; and (b) the error of estimating $E\left[ \sum_{t=0}^{T} \psi_t(x_{t-1}, x_t) | Y_{0:T} \right]$ grows quadratically with $T$. These correspond respectively to the degeneracy for the fixed marginal smoothing and the additive smoothing problems; see also the introductory section of Olsson and Westerborn (2017) for a discussion. The random matrices $\hat{B}_t^{N,\text{GT}}$ are therefore said to be unstable as $T \to \infty$, which is not the case...
for $\hat{B}_t^{N,\text{FFBS}}$ or $\hat{B}_t^{N,\text{PaRIS}}$. This subsection gives sufficient conditions to ensure the stability of a general $\hat{B}_t^N$.

The essential point behind smoothing stability is simple: the support of $B_t^{N,\text{FFBS}}(n, \cdot)$ or $B_t^{N,\text{PaRIS}}(n, \cdot)$ for $N \geq 2$ contains more than one element, contrary to that of $B_t^{N,\text{GGT}}(n, \cdot)$. This property is formalised by (13). To explain the intuitions, we use the notations of Algorithm 2 and consider the estimate

$$N^{-1} \left( \psi_0(X_0^{t_1}) + \cdots + \psi_0(X_0^{t_N}) \right)$$

of $\mathbb{E} \{ \psi_0(X_0) | Y_0, T \}$ when $T \to \infty$. The variance of the quantity above is a sum of $\text{Cov}(\psi_0(X_0^{t_1}), \psi_0(X_0^{t_2}))$ terms. It can therefore be understood by looking at a pair of trajectories simulated using Algorithm 2.

At final time $t = T$, $T_{k_1}^1$ and $T_{k_2}^2$ both follow the $\mathcal{M}(W_1^N)$ distribution. Under regularity conditions (e.g. no extreme weights), they are likely to be different, i.e., $\mathbb{P}(T_{k_1}^1 = T_{k_2}^2) = \mathcal{O}(1/N)$. This property can be propagated backward: as long as $T_{k_1}^1 \neq T_{k_2}^2$, the two variables $T_{k_1-1}^1$ and $T_{k_2-1}^2$ are also likely to be different, with however a small $\mathcal{O}(1/N)$ chance of being equal. Moreover, as long as the two trajectories have not met, they can be simulated independently given $\mathcal{F}_T$ (the sigma algebra defined in (3)). In mathematical terms, under the two hypotheses of Theorem 1, given $\mathcal{F}_T$ and $T_{k_1}^{1,2}$, it can be proved that the two variables $T_{k-1}^1$ and $T_{k-1}^2$ are independent if $T_{k_1}^1 \neq T_{k_2}^2$ (Lemma 2, Appendix E.3).

Since there is an $\mathcal{O}(1/N)$ chance of meeting at each time step, if $T \gg N$, it is likely that the two paths will meet at some point $t \gg 0$. When $T_{k_1}^1 = T_{k_2}^2$, the two indices $T_{k-1}^1$ and $T_{k-2}^2$ are both simulated according to $B_t^N(T_{k_1}^1, \cdot)$. In the genealogy tracking algorithm, $B_t^{N,\text{GGT}}(i, \cdot)$ is a Dirac measure, leading to $T_{k_1}^1 = T_{k_2}^2$ almost surely. This spreads until time 0, so $\text{Corr}(\psi_0(X_0^{t_1}), \psi_0(X_0^{t_2}))$ is almost 1 if $T \gg N$.

Other kernels like $B_t^{N,\text{FFBS}}$ or $B_t^{N,\text{PaRIS}}$ do not suffer from the same problem. For these, the support size of $B_t^N(T_{k_1}^1, \cdot)$ is greater than one and thus there is some real chance that $T_{k-1}^1 \neq T_{k-1}^2$. If that does happen, we are again back to the regime where the next states of the two paths can be simulated independently. Note also that the support of $B_t^N(T_{k_1}^1, \cdot)$ does not need to be large and can contain as few as 2 elements. Even if $T_{k-1}^1$ might still be equal to $T_{k-1}^2$ with some probability, the two paths will have new chances to diverge at times $t-2$, $t-3$ and so on. Overall, this makes $\text{Corr}(\psi_0(X_0^{t_1}), \psi_0(X_0^{t_2}))$ quite small (Lemma 4, Appendix E.3).

We formalise these arguments in the following theorem, whose proof (Appendix E.3) follows them very closely. The price for proof intuitiveness is that the theorem is specific to the bootstrap filter, although numerical evidence (Section 5) suggests that other filters are stable as well.

**Assumption 2.** The transition densities $m_t$ are upper and lower bounded:

$$\underline{M}_t \leq m_t(x_{t-1}, x_t) \leq \overline{M}_t$$

for constants $0 < \underline{M}_t < \overline{M}_t < \infty$.

**Assumption 3.** The potential functions $G_t$ are upper and lower bounded:

$$\underline{G}_t \leq G_t(x_t) \leq \overline{G}_t$$

for constants $0 < \underline{G}_t < \overline{G}_t < \infty$. 
Remark. Since Assumption 2 implies that the $X_i$'s are compact, Assumption 1 automatically implies Assumption 3 as soon as the $G_i$'s are continuous functions.

Theorem 2. We use the notations of Algorithms 1 and 2. Suppose that Assumptions 2 and 3 hold and the random kernels $B_i^n$ satisfy the conditions of Theorem 1. If, in addition, for the pair of random variables $(J_t^{n,1}, J_t^{n,2})$ whose distribution given $X_{t-1}^N, X_t^n$ and $B_i^n(n, \cdot)$ is defined by $B_i^n(n, \cdot) \otimes B_i^n(n, \cdot)$, we have

$$P \left( J_t^{n,1} \neq J_t^{n,2} \mid X_{t-1}^N, X_t^n \right) \geq \varepsilon_S$$

for some $\varepsilon_S > 0$ and all $t$, $n$, $m$; then there exists a constant $C$ not depending on $T$ such that:

- fixed marginal smoothing is stable, i.e. for $s \in \{0, \ldots, T\}$ and a real-valued function $\phi_s : X_s \to \mathbb{R}$ of the hidden state $X_s$, we have

$$E \left[ \left( \int Q_T^N (dx_s) \phi_s(x_s) - E[\phi_s(X_s)\mid Y_{0:T}] \right)^2 \right] \leq \frac{C \Vert \phi_s \Vert_\infty^2}{N};$$

- additive smoothing is stable, i.e. for $T \geq 2$ and the function $\varphi_T$ defined in (9), we have

$$E \left[ (Q_T^N (\varphi_T) - Q_T (\varphi_T))^2 \right] \leq \frac{C \sum_{i=0}^{T-1} \Vert \psi_i \Vert_\infty^2}{N} \left( 1 + \sqrt{\frac{T}{N}} \right)^2.$$

The $(1 + \sqrt{T/N})^2$ term in (15) first appeared in Dubarry and Le Corff (2013, Theorem 3.1) (which we used in our proof) and we do not know whether it can be dropped. However, it does not affect the scaling of the algorithm. Indeed, with or without it, the inequality implies that in order to have a constant error in the additive smoothing problem, one only has to take $N = O(T)$ (instead of $N = O(T^2)$ without backward sampling). Moreover, from an asymptotic point of view, we always have $\sigma^2(T) = O(T)$ regardless of the presence of the $(1 + \sqrt{T/N})^2$ term, where

$$\sigma^2(T) := \lim_{N \to \infty} N E \left[ (Q_T^N (\varphi_T) - Q_T (\varphi_T))^2 \right].$$

Theorem 2 is stated under strong assumptions (similar to those used in Chopin and Papaspiliopoulos 2020, Chapter 11.4, and slightly stronger than Douc et al. 2011, Assumption 4). On the other hand, it applies to a large class of backward kernels (rather than only FFBS), including the new ones introduced in the forthcoming sections.

3. Sampling from the FFBS Backward Kernels

Sampling from the FFBS backward kernel lies at the heart of both the FFBS algorithm (Example 1) and the PaRIS one (Section 2.5). Indeed, at time $t$, they require generating random variables distributed according to $B_i^n, FFBS (i_t, d_{i-1})$ for $i_t$ running from 1 to $N$. Since sampling from a discrete measure on $N$ elements requires $O(N)$ operations (e.g. via CDF inversion), the total computational cost becomes $O(N^2)$. To reduce this, we start by considering the subclass of models satisfying the following assumption, which is much weaker than Assumption 2.

Assumption 4. The transition density $m_t(x_{t-1}, x_t)$ is strictly positive and upper bounded, i.e. there exists $\tilde{M}_h > 0$ such that $0 < m_t(x_{t-1}, x_t) \leq \tilde{M}_h, \forall (x_{t-1}, x_t)$.
The hidden state assumption 5 is an additional weak point of rejection-based algorithms. Filters and see that the performance could change from one type to another, which other filtering algorithms. In Section 5, we shall employ different types of particle not the bootstrap particle filter. At each time \( t \), let \( \tau_t^{\text{PaRIS}} \) be the number of rejection

\begin{algorithm}
\textbf{Algorithm 4:} Pure rejection sampler for simulating from \\
\( B_t^{N,\text{FFBS}}(i_t, d_{i_t-1}) \)

\textbf{Input:} Particles \( X_{i_t-1}^{1:N} \) and weights \( W_{i_t-1}^{1:N} \) at time \( t-1 \); particle \( X_{i_t}^t \) at time \( t \); constant \( \bar{M}_h \); pre-initialised \( \mathcal{O}(1) \) sampler for \( \mathcal{M}(W_{i_t-1}^{1:N}) \)

\textbf{repeat} \\
\( \mathcal{I}_{t-1} \sim \mathcal{M}(W_{i_t-1}^{1:N}) \) using the pre-initialised \( \mathcal{O}(1) \) sampler \\
\( U \sim \text{Unif}[0,1] \)
\textbf{until} \( U \leq m_t(X_{i_t-1}^{1}, X_{i_t}^t) / \bar{M}_h \)

\textbf{Output:} \( \mathcal{I}_{t-1} \), which is distributed according to \( B_t^{N,\text{FFBS}}(i_t, d_{i_t-1}) \).
\end{algorithm}

The present subsection intends to fill this gap. Our main focus is the PaRIS algorithm of which the presentation is simpler. Results for the FFBS algorithm can be found in Appendix B. We restrict ourselves to the case where \( \mathcal{X}_t = \mathbb{R}^{d_t} \), although extensions to other non compact state spaces are possible. Only the bootstrap particle filter is considered, and results from this section do not extend trivially to other filtering algorithms. In Section 5, we shall employ different types of particle filters and see that the performance could change from one type to another, which is an additional weak point of rejection-based algorithms.

**Assumption 5.** The hidden state \( X_t \) is defined on the space \( \mathcal{X}_t = \mathbb{R}^{d_t} \). The measure \( \lambda_t(dx_t) \) with respect to which the transition density \( m_t(x_{t-1}, x_t) \) is defined (cf. (2)) is the Lebesgue measure on \( \mathbb{R}^{d_t} \).

This assumption together with the condition \( m_t(x_{t-1}, x_t) > 0 \) of Assumption 4 ensures that the state space model is “truly non-compact”. Indeed, if \( m_t(x_{t-1}, x_t) \) is zero whenever \( x_{t-1} \notin C_{t-1} \) or \( x_t \notin C_t \), where \( C_{t-1} \) and \( C_t \) are respectively two compact subsets of \( \mathbb{R}^{d_{t-1}} \) and \( \mathbb{R}^{d_t} \), then we are basically reduced to a state space model where \( \mathcal{X}_{t-1} = C_{t-1} \) and \( \mathcal{X}_t = C_t \).

**3.1. Complexity of PaRIS algorithm with pure rejection sampling.** We consider the PaRIS algorithm (i.e. Algorithm 3 using the \( B_t^{N,\text{PaRIS}} \) kernels). Algorithm 5 provides a concrete description of the resulting procedure, using the bootstrap particle filter. At each time \( t \), let \( \tau_t^{\text{PaRIS}} \) be the number of rejection
trials required to sample from $B_t^{N,\text{FFBS}}(n, dm)$. We then have

\begin{equation}
\tau_t^{n, \text{PaRIS}} | \mathcal{F}_{t-1}, X_t^n \sim \text{Geo} \left( \frac{\sum_t W_{t-1}^t m_t(X_{t-1}^t, X_t^n)}{M_h} \right)
\end{equation}

with $M_h$ defined in Assumption 4.

\textbf{Algorithm 5:} Concrete implementation of PaRIS algorithm (i.e. Algorithm 3 with the $B_t^{N, \text{PaRIS}}$ backward kernel) using the bootstrap particle filter

\textbf{Input:} Particles $X_{t-1}^{1:N}$; weights $W_{t-1}^{1:N}$; vector $S_{t-1}^N$ in $\mathbb{R}^N$; pre-initialised sampler for $M(W_{t-1}^{1:N})$; function $\psi_t$ (cf. (9)); user-specified parameter $\tilde{N}$

\textbf{for $n \leftarrow 1$ to $N$ do}
\begin{itemize}
  \item $A^n_t \sim M(W_{t-1}^{1:N})$ (⋆)
  \item $X^n_t \sim M_t(X_{t-1}^n, dx_t)$
  \item Simulate $J_{t}^{\tilde{N},n}$ i.i.d. $B_t^{N, \text{FFBS}}(n, dm')$ using either the pure rejection sampler (Algorithm 4) or the hybrid rejection sampler (Algorithm 6)
  \item $S_t^n[n] \leftarrow \tilde{N}^{-1} \sum_{\tilde{N}=1}^{\tilde{N}} \{ S_{t-1}^N[J_{t}^{\tilde{N},n}] + \psi_t(X_{t-1}^n, X_t^n) \}$
\end{itemize}

\textbf{for $n \leftarrow 1$ to $N$ do}
\begin{itemize}
  \item $W^n_t \leftarrow G_t(X^n_t) / \sum_i G_t(X^n_i)$
  \item $\mu^n_t \leftarrow \sum_{n=1}^{N} W^n_t S^n_t(n)$
  \item Initialise a sampler for $M(W_{t}^{1:N})$
\end{itemize}

\textbf{Output:} Estimate $\mu^n_t$ of $E[\varphi(X_{0:t})| Y_{0:t}]$; particles $X_{t}^{1:N}$; weights $W_{t}^{1:N}$; vector $S_{t}^N$ in $\mathbb{R}^N$ and pre-initialised sampler $M(W_{t}^{1:N})$ for the next iteration

By exchangeability of particles, the expected cost of the PaRIS algorithm at step $t$ is proportional to $N \tilde{N} \text{E}[^{1, \text{PaRIS}}_t]$, where $\tilde{N}$ is a fixed user-chosen parameter. Occasionally, $X_t^n$ falls into an unlikely region of $\mathbb{R}^d$ and the acceptance rate becomes low. In other words, $\tau_t^{1, \text{PaRIS}}$ is a mixture of geometric distribution, some components of which might have a large expectation. Unfortunately, these inefficiencies add up and produce an unbounded execution time in expectation, as shown in the following proposition.

\textbf{Proposition 2.} Under Assumptions 4 and 5, the version of Algorithm 5 using the pure rejection sampler satisfies $\text{E}[^{1, \text{PaRIS}}_t] = \infty$, where $\tau_t^{1, \text{PaRIS}}$ is defined in (16).
3.2. Hybrid rejection sampling.

The basic observation is that, for a single rejection sampling procedure, we propose a hybrid rejection sampling scheme. The measurable space on trails drawn so far. The proof, which is not established in the following proposition, where we actually allow the difference between the non-parallel and parallel computing.

For the proof of Proposition 2, it is clear that the quantity \( \sum_n W_{t-1}^n m_t(X_{t-1}^n, x_t) \) will play a key role in the upcoming developments. We thus define it formally.

**Definition 1.** The true predictive density function \( r_t \) and its approximation \( r_t^N \) are defined as

\[
\begin{align*}
    r_t(x_t) & := \frac{(Q_{t-1} M_t)(dx_t)}{\lambda_t(dx_t)} \\
    r_t^N(x_t) & := \sum W_{t-1}^n m_t(X_{t-1}^n, x_t)
\end{align*}
\]

where the first equation is understood in the sense of the Radon-Nikodym derivative and the density \( m_{t-1}(x_{t-1}, x_t) \) is defined with respect to the dominating measure \( \lambda_t(dx_t) \) on \( \mathcal{X}_t \) (cf. (2)).

### 3.2. Hybrid rejection sampling.

To solve the aforementioned issues of the pure rejection sampling procedure, we propose a hybrid rejection sampling scheme. The basic observation is that, for a single \( m_t \), direct simulation (e.g. via CDF inversion) of \( B_t^{N,FFBS}(i_t, di_{t-1}) \) costs \( O(N) \). Thus, once \( K = O(N) \) rejection sampling trials have been attempted, one should instead switch to a direct simulation method. In other words, it does not make sense (at least asymptotically) to switch to direct sampling after \( K \) trials if \( K \ll O(N) \) or \( K \gg O(N) \). The validity of this method is established in the following proposition, where we actually allow \( K \) to depend on trials drawn so far. The proof, which is not an immediate consequence of the validity of ordinary rejection sampling, is given in Appendix E.4.

**Proposition 3.** Let \( \mu_t(x) \) and \( \mu_1(x) \) be two probability densities defined on some measurable space \( \mathcal{X} \) with respect to a dominating measure \( \lambda(dx) \). Suppose that there exists \( C > 0 \) such that \( \mu_1(x) \leq C \mu_0(x) \). Let \( (X_1, U_1), (X_2, U_2), \ldots \) be a sequence of
i.i.d. random variables distributed according to \( \mu_0 \otimes \text{Unif}[0,1] \) and let \( X^* \sim \mu_1 \) be independent of that sequence. Put

\[
K^* := \inf \left\{ \, n \in \mathbb{Z}_{\geq 1} \text{ such that } U_n \leq \frac{\mu_1(X_n)}{\mathcal{C} \mu_0(X_n)} \, \right\}
\]

and let \( K \) be any stopping time with respect to the natural filtration associated with the sequence \( \{(X_n,U_n)\}_{n=1}^{\infty} \). Let \( Z \) be defined as

\[
Z := \begin{cases} 
X_{K^*} & \text{if } K^* \leq K \\
X^* & \text{otherwise.}
\end{cases}
\]

Then \( Z \) is \( \mu_1 \)-distributed.

Proposition 3 thus allows users to pick \( K = \alpha N \), where \( \alpha > 0 \) might be chosen somehow adaptively from earlier trials. In the following, we only consider the simple rule \( K = N \), which does not induce any loss of generality in terms of the asymptotic behaviour and is easy to implement. The resulting iteration is described in Algorithm 6.

\[\text{Algorithm 6: Hybrid rejection sampler for simulating from } B_{i_t}^{N,\text{FFBS}}(i_t, d_{i_t-1})\]

\textbf{Input:} Particles \( X_{i_t-1}^{1:N} \) and weights \( W_{i_t-1}^{1:N} \) at time \( t-1 \); particle \( X_{i_t}^{i_t} \) at time \( t \); constant \( \bar{M}_h \); pre-initialised \( O(1) \) sampler for \( \mathcal{M}(W_{i_t-1}^{1:N}) \)

\( \text{accepted} \leftarrow \text{False} \)

\textbf{for} \( i \leftarrow 1 \) \textbf{to} \( N \) \textbf{do}

\[I_{i_t-1} \sim \mathcal{M}(W_{i_t-1}^{1:N}) \text{ using the pre-initialised } O(1) \text{ sampler}
\]

\[U \sim \text{Unif}[0,1] \]

\[\text{if } U \leq m_t(X_{i_t-1}^{i_t}, X_{i_t}^{i_t})/\bar{M}_h \text{ then}
\]

\( \text{accepted} \leftarrow \text{True} \)

\( \text{break} \)

\textbf{if} \( \text{not accepted} \) \textbf{then}

\[I_{i_t-1} \sim \mathcal{M}(W_{i_t}^{1:N} m(X_{i_t-1}^{i_t}, X_{i_t}^{i_t})) \]

\[\text{Output: } I_{i_t-1}, \text{ which is distributed according to } B_{i_t}^{N,\text{FFBS}}(i_t, d_{i_t-1}).\]

When applied in the context of Algorithm 5, Algorithm 6 gives a smoother of expected complexity proportional to

\[N \bar{N} \mathbb{E} \text{[min} (\tau_t^{1,\text{PaRIS}})]\]

at time \( t \), where \( \tau_t^{1,\text{PaRIS}} \) is defined in (16)). This quantity is no longer infinite, but its growth when \( N \to \infty \) might depend on the model. Still, in all cases, it remains strictly larger than \( \mathcal{O}(N) \) and strictly smaller than \( \mathcal{O}(N^2) \). Perhaps more surprisingly, in linear Gaussian models (see Appendix A.1 for detailed notations), the smoother is of near-linear complexity (up to log factors). The following two theorems formalise these claims.

||
Assumption 6. The predictive density \( r_t \) of \( X_t \) given \( Y_{0:t-1} \) and the potential function \( G_t \) are continuous functions on \( \mathbb{R}^{d_t} \). The transition density \( m_t(x_{t-1}, x_t) \) is a continuous function on \( \mathbb{R}^{d_{t-1}} \times \mathbb{R}^{d_t} \).

Theorem 3. Under Assumptions 1, 4, 5 and 6, the version of Algorithm 5 using the hybrid rejection sampler (Algorithm 6) satisfies \( \lim_{N \to \infty} \mathbb{E}[\min(\tau_t^1, \text{PaRIS}, N)] = \infty \) and \( \lim_{N \to \infty} \mathbb{E}[\min(\tau_t^1, \text{PaRIS}, N)]/N = 0 \), where \( \tau_t^1, \text{PaRIS} \) is defined in (16).

Theorem 4. We assume the same setting as Theorem 3. In linear Gaussian state space models (Appendix A.1), we have \( \mathbb{E}[\min(\tau_t^1, \text{PaRIS}, N)] = \mathcal{O}(\log N)^{d_t/2} \).

4. Alternative backward kernels

4.1. MCMC Backward Kernels. This subsection analyses and extends the MCMC backward kernel defined in Example 3. As we remarked there, the matrix \( \tilde{B}_t^{N, \text{IMH}} \) is not sparse and even has some expensive-to-evaluate entries. We thus reserve it for use in the off-line smoother (Algorithm 2) whereas in the on-line scenario (Algorithm 3), we use its PaRIS-like counterpart

\[
\tilde{B}_t^{N, \text{IMHP}}[i_t, \tilde{i}_{t-1}] := \frac{1}{N} \sum_{\tilde{n}=1}^{N} 1\{\tilde{i}_{t-1} = \tilde{j}_t^{i_t, \tilde{n}}\}
\]

where \( \tilde{j}_t^{i_t, 1:N} \) is an independent Metropolis-Hastings chain started at \( J_t^{i_t, 1} := A_t^{i_t} \), targeting the measure \( B_t^{N, \text{FPBIS}}(i_t, d_{i_{t-1}}) \) and using the proposal distribution \( \mathcal{M}(W_{t-1}^{1:N}) \). The validity and the stability of \( \tilde{B}_t^{N, \text{IMHP}} \) and \( \tilde{B}_t^{N, \text{IMHP}} \) are established in the following simple proposition (proved in Appendix E.9). For simplicity, only the case \( \bar{N} = 2 \) is examined, but as a matter of fact, the proposition remains true for \( \bar{N} \geq 2 \).

Proposition 4. The kernels \( B_t^{N, \text{IMH}} \) and \( B_t^{N, \text{IMHP}} \) with \( \bar{N} = 2 \) satisfy the hypotheses of Theorem 1 and, under Assumptions 2 and 3, those of Theorem 2. Hence, their respective uses in Algorithms 2 and 3 guarantee a convergent and stable smoother.

The advantages of independent Metropolis-Hastings MCMC kernels compared to the rejection samplers of Section 3 are the dispensability of specifying an explicit \( M_t \) and the deterministic nature of the execution time. It is not hard to imagine situations where some proposal smarter than \( \mathcal{M}(W_{t-1}^{1:N}) \) would be beneficial. However, we only consider that one here, mainly because it already performs satisfactorily in our numerical examples.

4.2. Dealing with intractable transition densities.

4.2.1. Intuition and formulation. The purpose of backward sampling is to re-generate, for each particle, a new ancestor that is different from that of the filtering step. However, backward sampling is infeasible if the transition density \( m_t(x_{t-1}, x_t) \) cannot be calculated. To get around this, we modify the particle filter so that each particle might, in some sense, have two ancestors right from the forward pass.

Consider the standard PF (Algorithm 1). Among the \( N \) resampled particles \( X_{t-1}^{1:N} \), let us track two of them, say \( x_{t-1} \) and \( x'_{t-1} \) for simplicity. The move step of Algorithm 1 will push them through \( M_t \) using independent noises, resulting in \( x_t \) and \( x'_{t} \). (that is, given \( x_{t-1} \) and \( x'_{t-1} \), we have \( x_t \sim M_t(x_{t-1}, \cdot) \) and \( x'_{t} \sim M_t(x'_{t-1}, \cdot) \) such that \( x_t \) and \( x'_{t} \) are independent). Thus, for e.g. linear Gaussian
models, we have $\mathbb{P}(x_t = x_t') = 0$. However, if the two simulations $x_t \sim M_t(x_{t-1}, \cdot)$ and $x_t' \sim M_t(x_{t-1}', \cdot)$ are done with specifically correlated noises, it can happen that $\mathbb{P}(x_t = x_t') > 0$. The joint distribution $(x_t, x_t')$ given $(x_{t-1}, x_{t-1}')$ is called a coupling of $M_t(x_{t-1}, \cdot)$ and $M_t(x_{t-1}', \cdot)$; the event $x_t = x_t'$ is called the meeting event and we say that the coupling is successful when it occurs. In that case, the particle $x_t$ automatically has two ancestors $x_{t-1}$ and $x_{t-1}'$ at time $t - 1$ without needing any to perform a backward sampling step.

The precise formulation of the modified forward pass is detailed in Algorithm 7. It consists of building in an on-line manner the backward kernels $B_{t,i}$ (where ITR stands for “intractable”). The main interest of this algorithm lies in the fact that while the function $m_t$ may prove impossible to evaluate, it is usually possible to make $x_t$ and $x_t'$ meet by correlating somehow the random numbers used in their simulations. One typical example which this article focuses on is the coupling of continuous-time processes, but it is useful to keep in mind that Algorithm 7 is conceptually more general than that.

**Algorithm 7**: Modified forward pass for smoothing of intractable models (one time step)

**Input**: Feynman-Kac model (1), particles $X_{t-1}^{1:N}$ and weights $W_{t-1}^{1:N}$ that approximate the filtering distribution at time $t - 1$

for $n \leftarrow 1$ to $N$ do
  Resample. Simulate $(A_t^{n,1}, A_t^{n,2})$ such that marginally each component is distributed according to $\mathcal{M}(W_{t-1}^{1:N})$
  Move. Simulate $(X_t^{n,1}, X_t^{n,2})$ such that marginally the two components are distributed respectively according to $M_t(X_{t-1}^{A_t^{n,1}}, dx_t)$ and $M_t(X_{t-1}^{A_t^{n,2}}, dx_t)$
  Choose $L \sim \text{Uniform}\{1, 2\}$
  Set $X_t^n \leftarrow X_t^{n,L}$
  Calculate backward kernel.
  if $X_t^{n,1} = X_t^{n,2}$ then
    $$B_{t,i}^{N,\text{ITR}}(n, di_{i-1}) \leftarrow \left(\delta\left\{A_t^{n,1}\right\} + \delta\left\{A_t^{n,2}\right\}\right)/2$$
  else
    $$B_{t,i}^{N,\text{ITR}}(n, di_{i-1}) \leftarrow \delta\left\{A_t^{n,L}\right\}$$
  Reweight. Set $\omega_t^n \leftarrow G_t(X_t^n)$ for $n = 1, 2, \ldots, N$
  Set $\ell_t^n \leftarrow \sum_{n=1}^{N} \omega_t^n / N$
  Set $W_t^n \leftarrow \omega_t^n / N\ell_t^n$ for $n = 1, 2, \ldots, N$

**Output**: Particles $X_{t}^{1:N}$ and weights $W_{t}^{1:N}$ that approximate the filtering distribution at time $t$; backward kernel $B_{t,i}^{N,\text{ITR}}$ that can be used in either Algorithm 2 or 3

4.2.2. **Validity and stability.** The consistency of Algorithm 7 follows straightforwardly from Theorem 1. To produce a stable routine however, some conditions
must be imposed on the couplings \((A_{t}^{n,1}, A_{t}^{n,2})\) and \((X_{t}^{n,1}, X_{t}^{n,2})\). We want \(A_{t}^{n,1}\) to be different from \(A_{t}^{n,2}\) as frequently as possible. On the contrary, we aim for a coupling of the two distributions \(M_{t}(X_{t-1}^{A_{t}^{n,1}, \cdot})\) and \(M_{t}(X_{t-1}^{A_{t}^{n,2}, \cdot})\) with high success rate so as to maximise the probability that \(X_{t}^{n,1} = X_{t}^{n,2}\).

**Assumption 7.** There exists an \(\varepsilon_{A} > 0\) such that

\[
P(A_{t}^{n,1} \neq A_{t}^{n,2} | X_{t-1}^{1:N}) \geq \varepsilon_{A}.
\]

**Assumption 8.** There exists an \(\varepsilon_{D} > 0\) such that

\[
P(X_{t}^{n,2} = X_{t}^{n,1} | X_{t-1}^{1:N}, A_{t}^{n,1}, A_{t}^{n,2}, X_{t}^{n,1}) \geq \varepsilon_{D} \left( 1 \wedge \frac{m_{t}(X_{t-1}^{A_{t}^{n,2}}, X_{t}^{n,1})}{m_{t}(X_{t-1}^{A_{t}^{n,1}}, X_{t}^{n,1})} \right).
\]

The letters A and D in \(\varepsilon_{A}\) and \(\varepsilon_{D}\) stand for “ancestors” and “dynamics”. Assumption 8 means that the user-chosen coupling of \(M_{t}(X_{t-1}^{A_{t}^{n,1}, \cdot})\) and \(M_{t}(X_{t-1}^{A_{t}^{n,2}, \cdot})\) must be at least as \(\varepsilon_{D}\) times as efficient as their maximal couplings. For details on this interpretation, see Proposition 10 in the Appendix. In Lemma 13, we also show that in spite of its appearance, Assumption 8 is actually symmetric with regards to \(X_{t}^{n,1}\) and \(X_{t}^{n,2}\).

We are now ready to state the main theorem of this subsection (see Appendix E.11 for a proof).

**Theorem 5.** The kernels \(B_{t}^{N,\text{ITR}}\) generated by Algorithm 7 satisfy the hypotheses of Theorem 1. Thus, under Assumption 1, Algorithm 7 provides a consistent smoothing estimate. If, in addition, the Feynman-Kac model (1) satisfies Assumptions 2 and 3 and the user-chosen couplings satisfy Assumptions 7 and 8, the kernels \(B_{t}^{N,\text{ITR}}\) also fulfil (13) and the smoothing estimates generated by Algorithm 7 are stable.

4.2.3. Good ancestor couplings. It is notable that Assumption 7 only considers the event \(A_{t}^{n,1} \neq A_{t}^{n,2}\), which is a pure index condition that does not take into account the underlying particles \(X_{t-1}^{A_{t}^{n,1}}\) and \(X_{t-1}^{A_{t}^{n,2}}\). Indeed, if smoothing algorithms prevent degeneracy by creating multiple ancestors for a particle, we would expect that their separation (i.e. that they are far away in the state space \(\mathcal{X}_{t-1}\), e.g. \(\mathbb{R}^{d}\)) is critical to the performance. Surprisingly, it is unnecessary: two very close particles (in \(\mathbb{R}^{d}\)) at time \(t - 1\) may have ancestors far away at time \(t - 2\) thanks to the mixing of the model.

We advise choosing an ancestor coupling \((A_{t}^{n,1}, A_{t}^{n,2})\) such that the distance between \(X_{t-1}^{A_{t}^{n,1}}\) and \(X_{t-1}^{A_{t}^{n,2}}\) is small. It will then be easier to design a dynamic coupling of \(M_{t}(X_{t-1}^{A_{t}^{n,1}, \cdot})\) and \(M_{t}(X_{t-1}^{A_{t}^{n,2}, \cdot})\) with a high success rate. Furthermore, simulating the dynamic coupling with two close rather than far away starting points can also take less time when, for instance, the dynamic involves multiple intermediate steps, but the two processes couple early. One way to achieve an ancestor coupling with the aforementioned property is to first simulate \(A_{t}^{n,1} \sim \mathcal{M}(W_{t-1}^{1:N})\), then move \(A_{t}^{n,1}\) through an MCMC algorithm which keeps invariant \(\mathcal{M}(W_{t-1}^{1:N})\) and set the result to \(A_{t}^{n,2}\). It suffices to use a proposal looking at indices whose underlying particles are close (in \(\mathbb{R}^{d}\)) to \(X_{t-1}^{A_{t}^{n,1}}\). Finding nearby particles are efficient if they are first sorted using the Hilbert curve, hashed using locality-sensitive hashing or put in a
KD-tree (see Samet, 2006, for a comprehensive review). In the context of particle filters, such techniques have been studied for different purposes in Gerber and Chopin (2015), Jacob et al. (2019) and Sen et al. (2018).

4.2.4. Conditionally-correlated version. In Algorithm 7, the ancestor pairs \((A^n_{t,1}, A^n_{t,2})\) are conditionally independent given \(F_{t-}\) and the same holds for the particles \((X^n_t)_{n=1}^N\). These conditional independences allow easier theoretical analysis, in particular, the casting of Algorithm 7 in the framework of Theorems 1 and 2. However, they are not optimal for performance in two important ways: (a) they do not allow keeping both \(X^n_{t,1}\) and \(X^n_{t,2}\) when the two are not equal, and (b) the set of ancestor variables \((A^n_{t,1})_{n=1}^N\) is multinomially resampled from \(\{1, 2, \ldots, N\}\) with weights \(W_{1:t-1}\). We know that multinomial resampling is not the ideal scheme, see Appendix C.1 for discussion.

Consequently, in practice, we shall allow ourselves to break free from conditional independence. The resulting procedure is described in Algorithm 9 (Appendix C). Despite a lack of rigorous theoretical support, this is the algorithm that we will use in Section 5 since it enjoys better performance and it constitutes a fair comparison with practical implementations of the standard particle filter, which are mostly based on alternative resampling schemes.

5. Numerical experiments

5.1. Linear Gaussian state-space models. Linear Gaussian models constitute a particular class of state space models. They are characterised by Markov dynamics that are Gaussian and observations that are projection of hidden states plus some Gaussian noises. Appendix A.1 defines, for different components of these models, the notations that we shall use here. In this section, we consider an instance described in Guarniero et al. (2017), where the matrix \(F_X\) satisfies \(F_X[i,j] = \alpha^{1+|i-j|}\) for some \(\alpha\). We consider the problem with \(\dim X = \dim Y = 2\) and the observations are noisy versions of the hidden states with \(C_Y\) being \(\sigma_Y^2\) times the identity matrix of size 2. Unless otherwise specified, we take \(\alpha = 0.4\) and \(\sigma_Y^2 = 0.5\).

In this section, we focus on the performance of different online smoothers based on either genealogy tracking, pure/hybrid rejection sampling or MCMC. Rejection-based online smoothing amounts to the PaRIS algorithm, for which we use \(\tilde{N} = 2\) for the \(B_t^{N,\text{PaRIS}}\) kernel. We take \(T = 3000\) and simulate the data from the model. The benchmark additive function is simply

\[
\varphi_t(x_{0:t}) = \sum_{s=0}^t x_s(0)
\]

where \(x_s(0)\) is the first coordinate of the \(\mathbb{R}^2\) vector \(x_s = [x_s(0), x_s(1)]\). For a study of offline smoothers (including FFBS), see Appendix D.1. In all programs here and there, we choose \(N = 1000\) and use systematic resampling for the forward particle filters (see section C.1). Regarding MCMC smoothers, we employ the kernels \(B_t^{N,\text{IMH}}\) or \(B_t^{N,\text{IMHP}}\) consisting of only one MCMC step. All results are based on 150 independent runs.

Although our theoretical results are only proved for the bootstrap filter, we stress throughout that some of them extend to other filters as well. Therefore, we will also consider guided particle filters in the simulations. An introduction to this topic can be found in Chopin and Papaspiliopoulos (2020, Chapter 10.3.2), where
the expression for the optimal proposal is also provided. In linear Gaussian models, this proposal is fully tractable and is the one we use.

To present efficiently the combination of two different filters (bootstrap and guided) and four different algorithms (naive genealogy tracking, pure/hybrid rejection and MCMC) we use the following abbreviations: “B” for bootstrap, “G” for guided, “N” for naive genealogy tracking, “P” for pure rejection, “H” for hybrid rejection and “M” for MCMC. For instance, the algorithm referred to as “BM” uses the bootstrap filter for the forward pass and the MCMC backward kernels to perform smoothing. Furthermore, the letter “R” will refer to the rejection kernel whenever the distinction between pure rejection and hybrid rejection is not necessary. (Recall that the two rejection methods produce estimators with the same distribution.)

Figure 2 shows the squared interquartile range for the online smoothing estimates \( Q_t(\phi_t) \) with respect to \( t \). It verifies the rates of Theorem 2, although linear Gaussian models are not strongly mixing in the sense of Assumptions 2 and 3: the grid lines hint at a variance growth rate of \( O(T) \) for the MCMC and reject-based smoothers and of \( O(T^2) \) for the genealogy tracking ones. Unsurprisingly guided filters have better performance than bootstrap.

Figure 3 (and its zoomed-in, Figure 4) show box-plots of the execution time (divided by \( NT \)) for different algorithms over 150 runs. By execution time, we mean the number of Markov kernel transition density evaluations. We see that the bootstrap particle filter coupled with pure rejection sampling has a very heavy-tailed execution time. This behaviour is expected as per Proposition 2. Using the guided particle filter seems to fare better, but Figure 5 (for the same model but with \( \sigma_y^2 = 2 \)) makes it clear that this cannot be relied on either. Overall, these results highlight two fundamental problems with pure rejection sampling:
the computational time has heavy tails and depends on the type of forward particle filter being used.

On the other hand, hybrid rejection sampling, despite having random execution time in principle, displays a very consistent number of transition density evaluations over different independent runs. Thus it is safe to say that the algorithm has a virtually deterministic execution time. The catch is that the average computational load (which is around 16 in Figure 4) cannot be easily calculated beforehand. In any case, it is much larger than the value 1 of MCMC smoothers (since only 1 MCMC step is performed in the kernel $B_t^{N,1MHP}$); whereas the performance (Figure 2) is comparable.
The bottom line is that MCMC smoothers should be the default option, and one MCMC step seems to be enough. If for some reason one would like to use rejection-based methods, using hybrid rejection is a must.

5.2. Lotka-Volterra SDE. Lotka-Volterra models (originated in Lotka, 1925 and Volterra, 1928) describe the population fluctuation of species due to natural birth and death as well as the consumption of one species by others. The emblematic case of two species is also known as the predator-prey model. In this subsection, we study the stochastic differential equation (SDE) version that appears in Hening and Nguyen (2018). Let \( X_t = (X_t(0), X_t(1)) \) represent respectively the populations of the prey and the predator at time \( t \) and let us consider the dynamics

\[
\begin{align*}
\text{d}X_t(0) &= \left[ \beta_0 X_t(0) - \frac{1}{2} \tau_0 [X_t(0)]^2 - \tau_1 X_t(0) X_t(1) \right] \text{d}t + X_t(0) \text{d}E_t(0) \\
\text{d}X_t(1) &= \left[ -\beta_1 X_t(1) + \tau_1 X_t(0) X_t(1) \right] \text{d}t + X_t(1) \text{d}E_t(1)
\end{align*}
\]

where \( E_t = \Gamma W_t \) with \( W_t \) being the standard Brownian motion in \( \mathbb{R}^2 \) and \( \Gamma \) being some \( 2 \times 2 \) matrix. The parameters \( \beta_0 \) and \( \beta_1 \) are the natural birth rate of the prey and death rate of the predator. The predator interacts with (eats) the prey at rate \( \tau_1 \). The quantity \( \tau_0 \) encodes intra-species competition in the prey population. The \( \frac{1}{2} \) in its parametrisation is to line up with the Lotka Volterra jump process in \( \mathbb{Z}^2 \) (to be included in a future version of this paper), where the population sizes are integers and the interaction term becomes \( \tau_0 X_t(0)[X_t(0) - 1]/2 \).

The state space model is comprised of the process \( X_t \) and its noisy observations \( Y_t \) recorded at integer times. The Markov dynamics cannot be simulated exactly, but can be approximated through (Euler) discretisation. Nevertheless, the Euler transition density \( m^F_t(x_{t-1}, \cdot) \) remains intractable (unless the step size is exactly 1). Thus, the algorithms presented in Subsection 4.2 are useful. The missing bit is a method to efficiently couple \( m^F_t(x_{t-1}, \cdot) \) and \( m^F_t(x'_{t-1}, \cdot) \), which we carefully describe in Appendix D.2.1.
We consider the model with \( \tau_0 = 1/800 \), \( \tau_1 = 1/400 \), \( \beta_0 = 0.3125 \) and \( \beta_1 = 0.25 \). The matrix \( \Gamma \) is such that the covariance matrix of \( E_1 \) is
\[
\begin{bmatrix}
1/100 & 1/200 \\
1/200 & 1/100
\end{bmatrix}.
\]
The observations are recorded on the log scale with Gaussian error of covariance matrix \[
\begin{bmatrix}
0.04 & 0.02 \\
0.02 & 0.04
\end{bmatrix}.
\]
The distribution of \( X_0 \) is two-dimensional normal with mean \( [100, 100] \) and covariance matrix \[
\begin{bmatrix}
100 & 50 \\
50 & 100
\end{bmatrix}.
\]
This choice is motivated by the fact that the preceding parameters give the stationary population vector \( [100, 100] \). According to Hening and Nguyen (2018), they also guarantee that neither animal goes extinct almost surely as \( t \to \infty \).

By discretising (19) with time step \( \delta = 1 \), one can get some very rough intuition on the dynamics. For instance, per second there are about 31 preys born. Approximately the same number die (to maintain equilibrium), of which 6 die due to internal competition and 25 are eaten by the predator. The duration between two recorded observations corresponds more or less to one-third generation of the prey and one-fourth generation of the predator. The standard deviation of the variation due to environmental noise is about 10 individuals per observation period, for each animal.

Again, these intuitions are highly approximate. For readers wishing to get more familiar with the model, Appendix D.2.2 contains real plots of the states and the observations; as well as data on the performance of different smoothing algorithms for moderate values of \( T \). We now showcase the results obtained in a large scale problem where \( T = 3000 \) and the data is simulated from the model.

We consider the additive function
\[
\varphi_t(x_{0:t}) := \sum_{s=0}^{t} [x_s(0) - 100].
\]

Figure 6 represents using box plots the distributions of the estimators for \( Q_T(\varphi_T) \) using either the genealogy tracking smoother (with systematic resampling; see Appendix C.1) or Algorithm 9. Our proposed smoother greatly reduces the variance, at a computational cost which is empirically 1.5 to 2 times greater than the naive method. Since we used Hilbert curve to design good ancestor couplings (see Section 4.2.3), coupling of the dynamics succeeds 80% of the time. As discussed in the aforementioned section, starting two diffusion dynamics from nearby points make them couple earlier, which reduces the computational load afterwards.

Figure 7 plots with respect to \( t \) the squared interquartile range of the two methods for the estimation of \( Q_t(\varphi_t) \). Grid lines hint at a quadratic growth for the genealogy tracking smoother (as analysed in Olsson and Westerborn, 2017, Sect. 1) and a linear growth for the kernel \( B_t^{W,ITRC} \) (as described in Theorem 2).

Finally, Figure 21 (Appendix D.2.2) shows properties of the effective sample size (ESS) ratio for this model. In a nutshell, while being globally stable (between 40% and 70%), it has a tendency to drift towards near 0 from time to time due to unusual data points. At these moments, resampling kills most of the particles and aggravates the degeneracy problem for the naive smoother. As we have seen in the above figures, systematic resampling is not enough to mitigate this in the long run.
Intractable smoother
Naive smoother

4500
5000
5500
6000
6500
7000
7500

Value of additive func

Figure 6. Box plot of estimators (over 50 independent runs with $N = 1000$ particles) for $Q_T(\phi_T)$ in the Lotka-Volterra SDE model with $T = 3000$. They are calculated using either the naive genealogy tracking smoother or our smoother developed for intractable models (Algorithm 9).

50 51 52 53 54 55
Time

Squared interquantile range

Intractable smoother
Naive smoother

Figure 7. Squared interquartile range for the genealogy tracking smoother and our proposed one. Same context as in Figure 6

6. Conclusion

6.1. Practical recommendations. Our first recommendation does not concern the smoothing algorithm per se. It is of paramount importance that the particle filter used in the preliminary filtering step performs reasonably well, since its output defines the support of the approximations generated by the subsequent smoothing algorithm. (Standard recommendations to obtain good performance from a particle filter are to increase $N$, or to use better proposal distributions, or both.)
Then, if the transition density is tractable, we recommend the MCMC smoother by default. One to two MCMC steps seem to be sufficient in most situations. If one still prefers to use rejection sampling (see below for a possible motivation), it is safe to say that there is no reason not to use the hybrid method.

Although the assumptions under which we prove the stability of the smoothing estimates are strong, the general message still holds. The Markov kernel and the potential functions must make the model forget its past in some ways. Otherwise, we get an unstable model for which no smoothing methods can work. The rejection sampling – based smoothing algorithms can therefore serve as the ultimate test. Since they simulate exactly independent trajectories given the skeleton, there is no hope to perform better, unless one switches to another family of smoothing algorithms.

For intractable models, the key issue is to design couplings with high meeting probability. Fortunately, the inherent chaos of the model makes it possible to choose two very close starting points for the dynamics and thus easy to obtain a reasonable meeting probability.

If further difficulties persist, there is a practical (and very heuristic) recipe to test whether one coupling of $M_t(x, \cdot)$ and $M_t(x', \cdot)$ is close to optimal. It consists in approximating $M_t(x, \cdot)$ and $M_t(x', \cdot)$ by Gaussian distributions and deduce the optimal coupling rate from their total variation distance. There is no closed formula for the total variation distance between two Gaussian distributions in high dimensions. However, it can be reliably estimated using the geometric interpretation of the total variation distance being one minus the area of the intersection created by the corresponding density graphs. In this way, one can get a very rough idea of to what extent a certain coupling realises the meeting potential that the two distributions have. If the coupling seems good and the trajectories still look degenerate, it can very well be that the model is unstable.

6.2. Further directions. The major limitation of our work is the exclusive theoretical analysis under the bootstrap particle filter. Moreover, we require that the $N$ new particles generated at step $t$ are conditionally independent given previous particles at time $t-1$. This excludes practical optimisations like systematic resampling and Algorithm 9.

Finally, the backward sampling step is also used in other algorithms (in particular Particle Markov Chain Monte Carlo, see Andrieu et al., 2010) and it would be interesting to see to what extent our techniques can be applied there.

6.3. Data and code. The code used to run numerical experiments is available at https://github.com/hai-dang-dau/backward-samplers-code. The algorithms will soon be implemented in the particles package at https://github.com/nchopin/particles.

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### Appendix A. Additional notations

This section defines new notations that do not appear in the main text (except notations for linear Gaussian models) but are used in the Appendix.

#### A.1. Linear Gaussian models

Let $\dim X$ and $\dim Y$ be two strictly positive integers and $F_X$ and $F_Y$ be two full-rank matrices of sizes $\dim X \times \dim X$ and $\dim Y \times \dim X$ respectively. Let $C_X$ and $C_Y$ be two symmetric positive definite matrices of respective sizes $\dim X \times \dim X$ and $\dim Y \times \dim Y$. A linear Gaussian state space model has the underlying Markov process defined by

$$X_t|X_{0:t-1} \sim \mathcal{N}(F_X X_{t-1}, C_X),$$

where $X_0$ also follows a Gaussian distribution; and admits the observation process

$$Y_t|X_t \sim \mathcal{N}(F_Y X_t, C_Y).$$

The predictive ($X_t$ given $Y_{0:t-1}$), filtering ($X_t$ given $Y_{0:t}$) and smoothing ($X_t$ given $Y_{0:T}$) distributions are all Gaussian and their parameters can be explicitly calculated via recurrence formulas (Kalman, 1960; Kalman and Bucy, 1961). We shall denote their respective mean vectors and covariance matrices by $(\mu_{\text{pred}}^t, \Sigma_{\text{pred}}^t)$, $(\mu_{\text{filt}}^t, \Sigma_{\text{filt}}^t)$ and $(\mu_{\text{smth}}^t, \Sigma_{\text{smth}}^t)$. In particular, the starting distribution $X_0$ is $\mathcal{N}(\mu_0^\text{pred}, \Sigma_0^\text{pred})$.

#### A.2. Total variation distance

Let $\mu$ and $\nu$ be two probability measures on $\mathcal{X}$. The total variation distance between $\mu$ and $\nu$, sometimes also denoted $\text{TV}(\mu, \nu)$, is defined as $||\mu - \nu||_{\text{TV}} := \sup_{f: X \to [0,1]} |\mu(f) - \nu(f)|$. The definition remains valid if $f$ is restricted to the class of indicator functions on measurable subsets of $\mathcal{X}$. It implies in particular that $|\mu(f) - \nu(f)| \leq \|f\|_{\text{osc}} \text{TV}(\mu, \nu)$.

Next, we state a lemma summarising basic properties of the total variation distance and defining coupling-related notions (see, e.g. Proposition 3 and formula (13) of Roberts and Rosenthal (2004)). While the last property (covariance bound) is not in the aforementioned reference and does not seem popular in the literature, its proof is straightforward and therefore omitted.

**Lemma 1.** *The total variation distance has the following properties:*
(Alternative expressions.) If \( \mu \) and \( \nu \) admit densities \( f(x) \) and \( g(x) \) respectively with reference to a dominating measure \( \lambda \), we have
\[
TV(\mu, \nu) = \frac{1}{2} \int |f(x) - g(x)| \lambda(dx) = 1 - \int \min(f(x), g(x)) \lambda(dx).
\]

(Coupling inequality & maximal coupling.) For any pair of random variables \((M, N)\) such that \( M \sim \mu \) and \( N \sim \nu \), we have
\[
P(M \neq N) \geq TV(\mu, \nu).
\]
There exist pairs \((M^*, N^*)\) for which equality holds. They are called maximal couplings of \( \mu \) and \( \nu \).

(Contraction property.) Let \((X_n)\) be a Markov chain with invariant measure \( \mu^* \). Then
\[
TV(X_n, \mu^*) \geq TV(X_{n+1}, \mu^*).
\]

(Covariance bound.) For any pair of random variables \((M, N)\) such that \( M \sim \mu \) and \( N \sim \nu \) and real-valued functions \( h_1 \) and \( h_2 \), we have
\[
|\text{Cov}(h_1(M), h_2(N))| \leq 2 \|h_1\|_\infty \|h_2\|_\infty TV((M, N), \mu \otimes \nu).
\]

A.3. Cost-to-go function. In the context of the Feynman-Kac model (1), define the associated cost-to-go function \( H_{T,T} \) as (see e.g. Chopin and Papaspiliopoulos (2020, Chapter 5))
\[
(20) \quad H_{T,T}(x_i) := \prod_{s=t+1}^T M_{s-1}(x_{s-1}, dx_s) G_s(x_s).
\]
This function bridges \( Q_T(dx_i) \) and \( Q_T(dx_i) \), since \( Q_T(dx_i) \propto Q_T(dx_i) H_{T,T}(x_i) \).

A.4. The projection kernel. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two measurable spaces. The projection kernel \( \Pi^{(\mathcal{X},\mathcal{Y})}_\mathcal{X} \) is defined by
\[
\Pi^{(\mathcal{X},\mathcal{Y})}_\mathcal{X}((x, y), dx^*) := \delta_x(dx^*).
\]
In particular, for any function \( g : \mathcal{X} \to \mathbb{R} \) and measure \( \mu(dx, dy) \) defined on \( \mathcal{X} \times \mathcal{Y} \), we have
\[
(\Pi^{(\mathcal{X},\mathcal{Y})}_\mathcal{X} g)(x, y) = g(x)
\]
\[
(\mu \Pi^{(\mathcal{X},\mathcal{Y})}_\mathcal{X})(g) = \iint g(x) \mu(dx, dy) = \int g(x) \mu(dx)
\]
where the second identity shows the marginalising action of \( \Pi^{(\mathcal{X},\mathcal{Y})}_\mathcal{X} \) on \( \mu \). In the context of state space models, we define the shorthand
\[
\Pi^{(X_0,\ldots,X_T)} := \Pi^{(\mathcal{X},\mathcal{Y})}_{X_0}.
\]

A.5. Other notations. For a real number \( x \), let \( \lfloor x \rfloor \) be the largest integer not exceeding \( x \). The mapping \( x \mapsto \lfloor x \rfloor \) is called the floor function \( \bullet \) The Gamma function \( \Gamma(a) \) is defined for \( a > 0 \) and is given by \( \Gamma(a) := \int_{\mathbb{R}} e^{-x} x^{a-1} dx \bullet \) Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two measurable spaces. Let \( K(x, dy) \) be a (not necessarily probability) kernel from \( \mathcal{X} \) to \( \mathcal{Y} \). The norm of \( K \) is defined by \( \|K\|_\infty := \sup_{f: \mathcal{X} \to \mathcal{Y}, f \neq 0} \|Kf\|_\infty / \|f\|_\infty \). In particular, for any function \( f : \mathcal{X} \to \mathcal{Y} \), we have \( \|Kf\|_\infty \leq \|K\|_\infty \|f\|_\infty \bullet \) Let \( X_n \) be a sequence of random variables. We say that \( X_n = O_p(1) \) if for any \( \varepsilon > 0 \), there exists \( M > 0 \) and \( N_0 \), both depending on \( \varepsilon \), such that \( P(|X_n| \geq M) \leq \varepsilon \) for all \( n \geq N_0 \). For a strictly positive deterministic sequence \( a_n \), we say that
Appendix B. FFBS complexity for different rejection schemes

B.1. Framework and notations. The FFBS algorithm is a particular instance of Algorithm 2 where $B_{n,FFBS}^t$ kernels are used. If backward simulation is done using pure rejection sampling (Algorithm 4), the computational cost to simulate the $t-1$-th index of the $n$-th trajectory has conditional distribution

$$\tau_{n,FFBS}^t \mid F_T, T_{n,t} \sim \text{Geo} \left( \frac{\sum_i W_{t-i-1} m_t (X_{i,t-1}, X_{t-1}^T)}{\lambda_t} \right).$$

At this point, it would be useful to compare this formula with (16) of the PaRIS algorithm. The difference is subtle but will drive interesting changes to the way rejection-based FFBS behaves.

If hybrid rejection sampling (Algorithm 6) is to be used instead, we are interested in the distribution of $\min(\tau_{n,FFBS}^t, N)$, for reasons discussed in Subsection 3.2. In a highly parallel setting, it is preferable that the distribution of individual execution times, i.e. $\tau_{n,FFBS}^t$ or $\min(\tau_{n,FFBS}^t, N)$, are not heavy-tailed. In contrast, for non-parallel hardware, only cumulative execution times, i.e. $\sum_{n=1}^N \tau_{n,FFBS}^t$ or $\sum_{n=1}^N \min(\tau_{n,FFBS}^t, N)$, matter. Even though the individual times might behave badly, the cumulative times could be much more regular thanks to effect of the central limit theorem, whenever applicable. Nevertheless, studying the finiteness of the $k$-th order moment of $\tau_{1,FFBS}^t$ is still a good way to get information about both types of execution times, since it automatically implies $k$-th order moment (in)finiteness for both of them.

B.2. Execution time for pure rejection sampling. We show that under certain circumstances, the execution time of the pure rejection procedure has infinite expectation. Proposition 1 in Douc et al. (2011) hints that the cost per trajectory for FFBS-reject might tend to infinity when $N \to \infty$. In contrast, we show that infinite expectation might very well happen for finite sample sizes. We first give the statement for general state space models, then focus on their implications for Gaussian ones. In particular, while infinite expectations occur only under certain configurations, infinite higher moments happen in all linear Gaussian models with non-degenerate dynamics.

**Theorem 6.** Using the setting and notations of Appendix B.1, under Assumptions 1 and 4, we have $E[\tau_{1,FFBS}^t] = \infty$ whenever

$$\int_{X_t} G_t(x_t) H_{t:T}(x_t) \lambda_t(dx_t) = \infty$$

where the cost-to-go function $H_{t:T}$ is defined in (20) and the measure $\lambda_t$ is defined in (2).

**Theorem 7.** Using the setting and notations of Appendix B.1, we consider linear Gaussian models and their notations defined in Appendix A.1. Then we have
\[ E[(\tau_t^{1,\text{FFBS}})^k] = \infty \text{ whenever } k \text{ is greater than a certain } k_0 \text{ being the smallest eigenvalue of the matrix } \text{Id} + C_X^{1/2} \left((\Sigma^\text{mth})^{-1} - (\Sigma^\text{pred})^{-1}\right) C_X^{1/2}. \]

The proofs of the two assertions are given in Appendix E.7. We now look at how they are manifested in concrete examples. The first remark is that for technical reasons, Theorem 7 gives no information on the finiteness of \( E[\tau_t^{1,\text{FFBS}}] \) for \( k = 1 \) (since \( k_0 \) is already greater than or equal to 1 by definition). To study the finiteness of \( E[\tau_t^{1,\text{FFBS}}] \), we thus turn to Theorem 6.

**Example 4.** In linear Gaussian models, the integral of Theorem 6 is equal to

\[ \int N(y_t|F_Y x_t, C_Y) \prod_{s=t+1}^T N(x_s|F_X x_{s-1}, C_X)N(y_s|F_Y x_s, C_Y)dx_{t:T} \]

where the notation \( N(\mu, \Sigma) \) refers to the density of the normal distribution. The integrand is proportional to \( \exp[-0.5(Q(x_{t:T})) - R(x_{t:T})] \) for some quadratic form \( Q(x_{t:T}) \) and linear form \( R(x_{t:T}) \). The integral is finite if and only if \( Q \) is positive definite. In our case, this means that there is no non-trivial root for the equation \( Q(x_{t:T}) = 0 \), which is equivalent to

\[
\begin{align*}
F_Y x_s &= 0, \forall s = t, \ldots, T \\
F_X x_{s-1} &= x_s, \forall s = t + 1, \ldots, T.
\end{align*}
\]

Put another way, \( E[\tau_t^{1,\text{FFBS}}] \) is infinite whenever the intersection

\[ \bigcap_{k=0}^{T-t} \text{Ker}(F_Y F_X^k) = \bigcap_{k=0}^{T-t} F_X^{-k}(\text{Ker}(F_Y)) \]

contains other things than the zero vector. A common and particularly troublesome situation is when \( F_X = c \text{Id} \) for some \( c > 0 \) (but \( C_X \) can be arbitrary) and the dimension of the states (\( \dim X \)) is greater than that of the observations (\( \dim Y \)). Then the above intersection remains non-trivial no matter how big \( T - t \) is. Thus, \( E[\tau_t^{1,\text{FFBS}}] \) has no expectation for any \( t \). In general, the problem is less severe as successive intersections will shrink the space quickly to \( \{0\} \). Consequently, Theorem 6 only points out infiniteness of \( E[\tau_t^{1,\text{FFBS}}] \) for \( t \) close to \( T \). The bad news however will come from higher moments, as seen in the below example.

We will now focus on a simple but particularly striking example. Our purpose here is to illustrate the concepts as well as to show that their implications are relevant even in small, familiar settings. More advanced scenarios are presented in Section 5 devoted to numerical experiments.

**Example 5.** We consider two one-dimensional Gaussian state-space models: they both have \( F_X = 0.5, C_X = 1, X_0 \sim N(0, C_X^2/(1-F_X^2)) \) and \( T = 3 \). The only difference between them is that one has \( \sigma_y^2 := C_Y = 0.5^2 \) and another has \( \sigma_y^2 = 3^2 \).

We are interested in the execution times \( \tau_1^{n,\text{FFBS}} \) at time \( t = 1 \) (i.e. the rejection-based simulation of indices \( \mathcal{I}_0^n \) at time \( t = 0 \)). Theorem 7 then gives \( k_0 \approx 1.14 \) for \( \sigma_y = 3 \) and \( k_0 \approx 5 \) for \( \sigma_y = 0.5 \). The first implication is that in both cases, \( \tau_1^{n,\text{FFBS}} \) is a heavy-tailed random variable and therefore FFBS-reject is not a viable option in a highly parallel setting. But an interesting phenomenon happens in the sequential hardware scenario where one is rather interested in the cumulative execution time, i.e. \( \sum_{n=1}^N \tau_1^{n,\text{FFBS}} \), or equivalently, the mean number of trials per particle. In the
**Figure 8.** Box plots for the mean number of trials per particles to simulate indices at time 0, for models described in Example 5 and for FFBS algorithms based on pure and hybrid rejection sampling. The figure is obtained by running bootstrap particle filters with $N = 500$ over 1500 independent executions.

**Figure 9.** Zoom of Figure 8 to $0 \leq y \leq 8$

$\sigma_y = 3$ case, non-existence of second moment prevents the cumulative regularisation effect of the central limit theorem. This is not the case for $\sigma_y = 0.5$, in which the cumulative execution time actually behaves nicely (Figures 8 and 9). However, the most valuable message from this example is perhaps that the performance of FFBS-reject depends in a non-trivial (hard to predict) way on the model parameters.
B.3. Execution time for hybrid rejection sampling. Formula (21) suggests defining the limit distribution \( \tau_t^{\infty, \text{FFBS}} \) as

\[
\tau_t^{\infty, \text{FFBS}} | X_t^{\infty, \text{FFBS}} \sim \text{Geo} \left( \frac{r_t(X_t^{\infty, \text{FFBS}})}{M_h} \right)
\]

where \( X_t^{\infty, \text{FFBS}} \sim \mathcal{Q}_T(dx_t) \) and \( r_t \) given in Definition 1. These quantities provide the following characterisation of the cumulative execution time for the hybrid FFBS algorithm (proved in Section E.8).

**Theorem 8.** Under Assumptions 1 and 4 and the setting of Section B.1, we have

\[
\sum_{n=1}^{N} \min(\tau_t^n, \text{FFBS}_t, N) = \mathcal{O}_p \left( E[\min(\tau_t^{\infty, \text{FFBS}}, N)] \right)
\]

where the notation \( \mathcal{O}_p \) is defined in Appendix A.

This theorem admits the following corollary for linear Gaussian models (also proved in Section E.8).

**Corollary 2.** For linear Gaussian models (Appendix A.1), if smoothing is performed using the hybrid rejection version of the FFBS algorithm, the mean execution time per particle at time step \( t \) is \( \mathcal{O}_p(\log^{d_t/2} N) \) where \( d_t \) is the dimension of \( X_t \).

The bound \( \mathcal{O}_p(\log^{d_t/2} N) \) is actually quite conservative. For instance, with either \( \sigma_y = 0.5 \) or \( \sigma_y = 3 \), the model considered in Example 5 admits \( E[\tau_t^{\infty, \text{FFBS}}] < \infty \). (Gaussian dynamics can be handled using exact analytic calculations and enables to verify the claim straightforwardly.) Theorem 8 then gives an execution time per particle of order \( \mathcal{O}_p(1) \) for hybrid FFBS, which is better than the \( \mathcal{O}_p(\sqrt{\log N}) \) predicted by Corollary 2. Yet another unsatisfactory point of the result is its failure to make sense of the spectacular improvement brought by hybrid rejection sampling over the ordinary procedure in the \( \sigma_y = 3 \) case (see Figure 8). As explained in Example 5, this is connected to the variance of \( E[\tau_t^{\infty, \text{FFBS}}] \) and not merely the expectation; so a study of second order properties of \( \sum_{n} \min(\tau_t^n, \text{FFBS}_t, N) \) would be desirable.

**Appendix C. Conditionally-correlated versions of particle algorithms**

C.1. **Alternative resampling schemes.** In Algorithm 1, the indices \( A_t^{1:N} \) are drawn conditionally i.i.d. from the multinomial distribution \( \mathcal{M}(W_t^{1:N}) \). They satisfy

\[
E \left[ \sum_{j=1}^{N} 1_{A_{t-1}^j = i} F_{t-1} \right] = NW_{t-1}^i
\]

for any \( i = 1, \ldots, N \). There are other ways to generate \( A_t^{1:N} \) from \( W_t^{1:N} \) that still verify this identity. We call them unbiased resampling schemes, and the natural one used in Algorithm 1 multinomial resampling.

The main motivation for alternative resampling schemes is performance. We refer to Chopin (2004); Douc et al. (2005); Gerber et al. (2019) for more details, but would like to mention that the theoretical studies of particle algorithms using other resampling schemes are more complicated since \( X_t^{1:N} \) are no longer i.i.d. given
We use systematic resampling (Carpenter et al., 1999) in our experiments. See Algorithm 8 for a succinct description and Chopin and Papaspiliopoulos (2020, Chapter 9) for efficient implementations in $O(N)$ running time.

Algorithm 8: Systematic resampling

**Input:** Weights $W_{t}^{1:N}$ summing to 1
Generate $U \sim \text{Uniform}[0, 1]$

for $n \leftarrow 1$ to $N$ do

Set $A_{n}^{t}$ to the unique index $k$ satisfying

$$W_1 + \cdots + W_{k-1} \leq nU < W_1 + \cdots + W_k$$

**Output:** Resampled indices $A_{t}^{1:N}$

C.2. Conditionally-correlated version of Algorithm 7. In this part, we present an alternative version of Algorithm 7 that does not create conditionally i.i.d. particles at each time step. The procedure is detailed in Algorithm 9. It creates on the fly backward kernels $B_{t}^{N,1:NC}$ (for “intractable, conditionally correlated”). It involves a resampling step which can be done in principle using any unbiased resampling scheme. Following the intuitions of Subsection 4.2.3 and the notations of Algorithm 9, we want a scheme such that in most cases, $X_{t-1}^{A_{t-1}^{k-1}} \neq X_{t-1}^{A_{t-1}^{k}}$ but the Euclidean distance between $X_{t-1}^{A_{t-1}^{k-1}}$ and $X_{t-1}^{A_{t-1}^{k}}$ is small. Algorithm 10 proposes such a method (which we name the Adjacent Resampler). It can run in $O(N)$ time using a suitably implemented linked list.

Appendix D. Additional information on numerical experiments

D.1. Offline smoothing in linear Gaussian models. In this section, we study offline smoothing for the linear Gaussian model specified in Section 5.1. Since offline processing requires storing particles at all times $t$ in the memory, we use $T = 500$ here instead of $T = 3000$. Apart from that, the algorithmic and benchmark settings remain the same.

Figure 10 plots the squared interquartile range of the estimators $Q_{T}(\phi_{t})$ with respect to $t$, for different algorithms. For small $t$, the function $\phi_{t}$ only looks at states close to time 0, whereas for bigger $t$, recent states less affected by degeneracy are also taken into account. In all cases though, we see that MCMC and rejection-based smoothers have superior performance.

Figure 11 shows box plots of the averaged execution times (per particle $N$ per time $t$) based on 150 runs. The observations are comparable to those in Section 5.1. We see a performance difference between the rejection-based smoothers using the bootstrap and the guided filters. Both have an execution time that is much more variable than hybrid rejection algorithms. The latter still need around 10 times more CPU load than MCMC smoothers, for essentially the same precision.

We now take a closer look at the reason behind the performance difference between the bootstrap filter and the guided one when pure rejection sampling is used.
Algorithm 9: Conditionally-correlated version of Algorithm 7

**Input:** Feynman-Kac model (1), particles $X_{t-1}^{1:N}$ and weights $W_{t-1}^{1:N}$ that approximate $Q_{t-1}(dx_{t-1})$

Resample $A_{t}^{1:N}$ from \{1, 2, ..., $N$\} with weights $W_{t-1}^{1:N}$ using any resampling scheme (such as the Adjacent Resampler in Algorithm 10)

for $k \leftarrow 1$ to $N/2$ do

Move. Simulate $X_{t}^{2k-1}$ and $X_{t}^{2k}$ such that marginally, $X_{t}^{2k-1} \sim M_t(X_{t-1}^{A_{t}^{2k-1}}, \cdot)$ and $X_{t}^{2k} \sim M_t(X_{t-1}^{A_{t}^{2k}}, \cdot)$

Calculate backward kernel.

if $X_{t}^{2k-1} = X_{t}^{2k}$ then

Set $B_{t}^{N, ITRC}(2k-1, \cdot) \leftarrow (\delta \{A_{t}^{2k-1}\} + \delta \{A_{t}^{2k}\})/2$

Set $B_{t}^{N, ITRC}(2k, \cdot) \leftarrow (\delta \{A_{t}^{2k-1}\} + \delta \{A_{t}^{2k}\})/2$

else

Set $B_{t}^{N, ITRC}(2k-1, \cdot) \leftarrow \delta \{A_{t}^{2k-1}\}$

Set $B_{t}^{N, ITRC}(2k, \cdot) \leftarrow \delta \{A_{t}^{2k}\}$

Reweight. Set $\omega_{t}^{n} \leftarrow G_t(X_{t}^{n})$ for $n = 1, 2, ..., N$

Set $\ell_{t}^{N} \leftarrow \sum_{n=1}^{N} \omega_{t}^{n}/N$

Set $W_{t}^{n} \leftarrow \omega_{t}^{n} N \ell_{t}^{N}$ for $n = 1, 2, ..., N$

**Output:** Particles $X_{t}^{1:N}$ and weights $W_{t}^{1:N}$ that approximate $Q_{t}(dx_{t})$; backward kernel $B_{t}^{N, ITRC}$ for use in Algorithms 2 and 3

Algorithm 10: The Adjacent Resampler

**Input:** Particles $X_{t-1}^{1:N}$, weights $W_{t-1}^{1:N}$

Sort the particles $X_{t-1}^{1:N}$ using the Hilbert curve. Let $s \leftarrow [s_1 \ldots s_N]$ be the corresponding indices

Resample from \{1, ..., $N$\} with weights $W_{t-1}^{1:N}$ using systematic resampling (Carpenter et al., 1999; Gerber et al., 2019), then let $f : \{1, ..., N\} \rightarrow Z$ be the function defined by $f(i)$ being the number of times the index $s_i$ was resampled. Obviously $\sum_{i=1}^{N} f(i) = N$

Initialise $i \leftarrow 1$

for $n \leftarrow 1$ to $N$ do

Set $A_{t}^{n} \leftarrow s_i$

Update $f(i) \leftarrow f(i) - 1$

Let $\Omega_1$ be the set $\{\min \{\ell > i \mid f_{\ell} > 0\}\}$ (which has one element if the minimum is well-defined and zero element otherwise)

Let $\Omega_2$ be the set $\{\max \{\ell < i \mid f_{\ell} > 0\}\}$ (which has one element if the maximum is well-defined and zero element otherwise)

If $\Omega_1 \cup \Omega_2$ is not empty, update $i \leftarrow \arg\max f|_{\Omega_1 \cup \Omega_2}$ (see section A.5 for the restriction notation). If there is more than one argmax, pick one randomly

**Output:** Resampled indices $A_{t}^{1:N}$
Figure 10. Squared interquartile range of the estimators of \( Q_T(\varphi_t) \) with respect to \( t \), for different algorithms applied to the model of Section D.1. See Section 5.1 for the meaning of the acronyms in the legend.

Figure 11. Box plots of the number of transition density evaluations divided by \( NT \) for different algorithms in the offline linear Gaussian model of Section D.1.

Figure 12 shows the effective sample size (ESS) of both filters as a function of time. We can see that there is an outlier in the data around time \( t = 40 \). Figure 13 boxplots the execution times divided by \( N \) at \( t = 40 \) for the pure rejection sampling algorithm, whereas Figures 14 and 15 do the same for \( t = 38 \) and \( t = 42 \). The root of the problem is now clear: at most times \( t \) there is very few difference between the execution times of the bootstrap and the guided filters. However, if an outlier is present in the data, the guided filter suddenly requires a very high number of transition density evaluation in the rejection sampler. This gives yet another reason to avoid using pure rejection sampling.
D.2. Lotka-Volterra SDE.

D.2.1. Coupling of Euler discretisations. Consider the SDE

\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t \]

and two starting points \( X^A_0 \) and \( X^B_0 \) in \( \mathbb{R}^d \). We wish to simulate \( X^A_1 \) and \( X^B_1 \) such that the transitions from \( X^A_0 \) to \( X^A_1 \) and \( X^B_0 \) to \( X^B_1 \) both follow the Euler-discretised version of the equation, but \( X^A_1 \) and \( X^B_1 \) are correlated in a way that increases, as much as we can, the probability that they are equal. Algorithm 11 makes it clear that it all boils down to the coupling of two Gaussian distributions.
Lindvall and Rogers (1986) propose the following construction: if two diffusions $X^A_t$ and $X^B_t$ both follow the dynamics of (22), that is,

\begin{align*}
\mathrm{d}X^A_t &= b(X^A_t)\mathrm{d}t + \sigma(X^A_t)\mathrm{d}W^A_t \\
\mathrm{d}X^B_t &= b(X^B_t)\mathrm{d}t + \sigma(X^B_t)\mathrm{d}W^B_t
\end{align*}

and the two Brownian motions are correlated via

\begin{equation}
\mathrm{d}W^B_t = [\mathrm{Id} - 2u(X^A_t, X^B_t)u(X^A_t, X^B_t)\top] \mathrm{d}W^A_t
\end{equation}

where $\mathrm{Id}$ is the identity matrix and the vector $u$ is defined by

\[
u(x, x') = \frac{\sigma(x')^{-1}(x - x')}{\|\sigma(x')^{-1}(x - x')\|_2},
\]
Algorithm 11: Coupling of two Euler discretisations

**Input:** Functions \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d} \), two starting points \( X_0^A \) and \( X_0^B \) at time 0, number of discretisation step \( N_{\text{dist}} \)

Initialise \( X^A \leftarrow X_0^A \)

Initialise \( X^B \leftarrow X_0^B \)

Set \( \delta \leftarrow 1/N_{\text{dist}} \)

for \( i \leftarrow 1 \) to \( N_{\text{dist}} \) do

Simulate \((\tilde{X}^A, \tilde{X}^B)\) from a coupling of

\[ \mathcal{N}(X^A + \delta b(X^A), \delta \sigma(X^A)\sigma(X^A)^\top) \]

and

\[ \mathcal{N}(X^B + \delta b(X^B), \delta \sigma(X^B)\sigma(X^B)^\top), \]

such as Algorithm 14

Update \((X^A, X^B) \leftarrow (\tilde{X}^A, \tilde{X}^B)\)

Set \((X_1^A, X_1^B) \leftarrow (X^A, X^B)\)

**Output:** Two endpoints \( X_1^A \) and \( X_1^B \) at time 1, obtained by passing \( X_0^A \) and \( X_0^B \) in a correlated manner through a discretised version of (22)

then under some regularity conditions, the two diffusions meet almost surely. (Note two special features of (23): it is valid because the term in the square bracket is an orthogonal matrix; and it ceases to be well-defined once the two trajectories have met.) Simulating the meeting time \( \tau \) turns out to be very challenging. The Euler discretisation (Algorithm 11 + Algorithm 12) has a fixed step size \( \delta \), and there is zero probability that \( \tau \) is of the form \( k\delta \) for some integer \( k \). Since the coupling transform is deterministic, the two Euler-simulated trajectories will never meet. Figure 16 depicts this difficulty in the special case of two Brownian motions in dimension 1 (i.e. \( b(x) \equiv 0 \) and \( \sigma \equiv 1 \)). Under this setting, (23) means that the two Brownian increments are symmetric with respect to the midpoint of the segment connecting their initial states. Note that the two dashed lines do cross at two points, but using them as meeting points is invalid: since they are not part of the discretisation but the result of some heuristic “linear interpolation”, it would change the distribution of the trajectories.

Algorithm 12: Lindvall-Rogers coupling of two Gaussian distributions

**Input:** Two vectors \( \mu^A, \mu^B \) in \( \mathbb{R}^d \) and two \( d \times d \) matrices \( \sigma^A \) and \( \sigma^B \)

Calculate \( u \leftarrow (\sigma^B)^{-1}(\mu^A - \mu^B) \)

Normalise \( u \leftarrow u/\|u\|_2 \)

Simulate \( W^A \sim \mathcal{N}(0, \text{Id}) \)

Set \( W^B \leftarrow (\text{Id} - 2uu^\top)W^A \)

Set \( X^A \leftarrow \mu^A + \sigma^A W^A \)

Set \( X^B \leftarrow \mu^B + \sigma^B W^B \)

**Output:** Two correlated points \( X^A \) and \( X^B \) marginally distributed according to \( \mathcal{N}(\mu^A, \sigma^A(\sigma^A)^\top) \) and \( \mathcal{N}(\mu^B, \sigma^B(\sigma^B)^\top) \) respectively
We therefore need some coupling that has a non-zero meeting probability at each δ-step. This can be achieved by the rejection maximal coupling (Algorithm 13, see also, e.g. Roberts and Rosenthal, 2004) as well as the recently proposed coupled rejection sampler (Corenflos and Särkkä, 2022). However, they all make use of rejection sampling in one way or another, which renders the execution time random. We wish to avoid this if possible. The reflection-maximal coupling (Bou-Rabee et al., 2020; Jacob et al., 2020) has deterministic cost and optimal meeting probability, but is only applicable for two Gaussian distributions of the same covariance matrix, which is not our case.

**Algorithm 13:** Rejection maximal coupler for two distributions

**Input:** Two probability distributions \(f^A\) and \(f^B\)

- Simulate \(X^A \sim f^A\)
- Simulate \(U^A \sim \text{Uniform}[0, f^A(X^A)]\)

if \(U^A \leq f^B(X^A)\) then
  - Set \(X^B \leftarrow X^A\)
else
  repeat
  - Simulate \(X^B \sim f^B\)
  - Simulate \(U^B \sim \text{Uniform}[0, f^B(X^B)]\)
  until \(U^B > f^A(X^B)\)

**Output:** Two maximally-coupled realisations \(X^A\) and \(X^B\), marginally \(f^A\)-distributed and \(f^B\)-distributed respectively
As suggested by Figure 16, the discretised Lindvall-Rogers coupling (Algorithm 12) is actually great for bringing together two faraway trajectories. Only when they start getting closer that it misses out. At that moment, the two distributions corresponding to the next $\delta$-step have non-negligible overlap and would preferably be coupled in the style of Algorithm 13. We propose a modified coupling scheme that acts like Algorithm 12 when the two trajectories are at a large distance and behaves as Algorithm 13 otherwise.

The idea is to preliminarily generate a uniform draw in the “overlapping zone” of the two distributions (if they are close enough to make that easy). Next, we perform Algorithm 12 and then, any of the two simulations belonging to the overlapping zone will be replaced by the aforementioned preliminary draw (if it is available). The precise mathematical formulation is given in Algorithm 14 and the proof in Appendix E.12.

**Algorithm 14:** Modified Lindvall-Rogers (MLR) coupler of two Gaussian distributions

**Input:** Two vectors $\mu^A$ and $\mu^B$ in $\mathbb{R}^d$, two $d \times d$ matrices $\sigma^A$ and $\sigma^B$ Let $f^A$ and $f^B$ be respectively the probability densities of $\mathcal{N}(\mu^A, \sigma^A(\sigma^A)^\top)$ and $\mathcal{N}(\mu^B, \sigma^B(\sigma^B)^\top)$

Simulate $X^A$ and $X^B$ from Algorithm 12

Simulate $U \sim \text{Uniform}[0, 1]$

Set $U^A \leftarrow U f^A(X^A)$ and $U^B \leftarrow U f^B(X^B)$

Simulate $Y \sim f^A$ and $V \sim \text{Uniform}[0, f^A(Y)]$

if $V \leq f^B(Y)$ then

if $U^A \leq f^B(X^A)$ then update $(X^A, U^A) \leftarrow (Y, V)$

if $U^B \leq f^A(X^B)$ then update $(X^B, U^B) \leftarrow (Y, V)$

**Output:** Two correlated random vectors $X^A$ and $X^B$, distributed marginally according to $\mathcal{N}(\mu^A, \sigma^A(\sigma^A)^\top)$ and $\mathcal{N}(\mu^B, \sigma^B(\sigma^B)^\top)$

Algorithm 14 has a deterministic execution time, but it does not attain the optimal coupling rate. Yet, as $\delta \to 0$, we see empirically that it still recovers the oracle coupling time defined by (23) (although we did not try to prove this formally). In Figure 17, we couple two standard Brownian motions starting from $a = 0$ and $b = 1.5$ using Algorithm 14 with different values of $\delta$. It is known, by a simple application of the reflection principle (Lévy, 1940; see also Chapter 2.2 of Mörters and Peres, 2010), that the reflection coupling (23) succeeds after a $\text{Levy}(0, (b - a)^2/4)$-distributed time. We therefore have to deal with a heavy-tailed distribution and restrict ourselves to the interval $[0, 5]$. We see that the law of the meeting time is stable and convergent as $\delta \to 0$. Thus, at least empirically, Algorithm 14 does not suffer from the instability problem as $\delta \to 0$, contrary to a naive path space augmentation approach (see Yonekura and Beskos, 2022 for a discussion).

D.2.2. Supplementary figures. Figure 18 plots a realisation of the states and data with parameters given in Subsection 5.2, for a relatively small scale dataset ($T = \ldots$)
While the periodic trait seen in classical deterministic Lotka-Volterra equations is still visible (with a period of around 20), it is clear that here random perturbations have added considerable chaos to the system. Figures 19 and 20 show respectively the performances of the naive genealogy tracking smoother and ours (Algorithm 9) on the dataset of Figure 18. Our smoother has successfully prevented the degeneracy phenomenon, particularly for times close to 0. Figure 21 shows, in two different ways, the properties of effective sample sizes (ESS) in the \( T = 3000 \) scenario (see Section 5.2).

**APPENDIX E. PROOFS**

E.1. **Proof of Theorem 1 (general convergence theorem).** In line with (7), we define the distribution \( Q_t^N(dx_{0:t}) \) for \( t < T \) as the \( x_{0:t} \) marginal of the joint distribution

\[
Q_t^N(dx_{0:t}, di_{0:t}) := \mathcal{M}(W_t^{1:N})(di_t) \left[ \prod_{s=t}^{1} B^N_s(i_s, di_{s-1}) \right] \left[ \prod_{s=t}^{0} \delta_{X^*_s}(dx_s) \right].
\]

The proof builds up on an inductive argument which links \( Q_t^N \) with \( Q_{t-1}^N \) through new innovations at time \( t \). More precisely, we have the following fundamental proposition, where \( F^+_t \) is defined as the smallest \( \sigma \)-algebra containing \( F_t \) and \( \hat{B}_{1:t}^N \).

**Proposition 5.** \( Q_t^N \) is a mixture distribution that admits the representation

\[
Q_t^N(dx_{0:t}) = (t^N_t)^{-1} N^{-1} \sum_n G_t(x_t)K^N_t(n, dx_{0:t})
\]
Figure 18. A realisation of the Lotka-Volterra SDE with parameters described in Section 5.2. The stationary point of the system is $[100, 100]$.

Figure 19. Smoothing trajectories for the dataset of Figure 18 using the naive genealogy tracking smoother ($B_{N,GT}^{i}$ kernels) with systematic resampling (see Section C.1). We took $N = 100$ and randomly plotted 30 smoothing trajectories.

where $t_{i}^{N}$ is defined in Algorithm 1 and $K_{i}^{N}(n, dx_{0:t})$ is a certain probability measure satisfying

\begin{equation}
\mathbb{E} \left[ K_{i}^{N}(n, dx_{0:t}) \mid \mathcal{F}_{t-1}^{+} \right] = Q_{t-1}^{N}(dx_{0:t-1}) M_{t}(x_{t-1}, dx_{t}).
\end{equation}

In other words, for any (possibly random) function $\varphi_{t}^{N} : X_{0} \times \cdots \times X_{t} \to \mathbb{R}$ such that $\varphi_{t}^{N}(x_{0:t})$ is $\mathcal{F}_{t-1}^{+}$-measurable, we have

\begin{equation}
\mathbb{E} \left[ \int K_{i}^{N}(n, dx_{0:t}) \varphi_{t}^{N}(x_{0:t}) \mid \mathcal{F}_{t-1}^{+} \right] = \int Q_{t-1}^{N}(dx_{0:t-1}) M_{t}(x_{t-1}, dx_{t}) \varphi_{t}^{N}(x_{0:t}).
\end{equation}
Moreover, $\int K^N_t(n, dx_0, t) \varphi^N_t(x_0, s), n = 1, \ldots, N$ are i.i.d. given $\mathcal{F}^T_{t-1}$.

The proof is postponed until the end of this subsection. This proposition gives the expression (25) for $Q^N_t$, which is easier to manipulate than (24) and which highlights, through (26), its connection to $Q^N_{t-1}$. To further simplify the notations,
In other words, for real-valued functions \( \phi_t \), we have

\[
L_{t_1:t_2}(x^0_t, d\sigma_t) = \delta_{x^0_{t_1}}(d\sigma_t) \prod_{s=t_1+1}^{t_2} M_s(x_{s-1}, d\sigma_s)G_s(x_s).
\]

In other words, for real-valued functions \( \phi_t = \phi_t(x_0, \ldots, x_t), \) we have

\[
L_{t_1:t_2}(x^0_{t_1}, \phi_t) = \int \phi_t(x^0_t, \ldots, x^t_{t_1}, x_{t_1+1}, \ldots, x_{t_2}) \prod_{s=t_1+1}^{t_2} M_s(x_{s-1}, d\sigma_s)G_s(x_s).
\]

The usefulness of these kernels will come from the simple remark

\[ \text{Corollary 3. Let } Q \text{ be a (possibly random) function such that } \phi_t^N(x_{0:t}) \text{ is uniformly non-negative or uniformly bounded. Then}
\]

\[ Q_t^N \phi_t^N = \frac{N^{-1} \sum_n \tilde{K}_t^N(n, \phi_t^N)}{N^{-1} \sum_n K_t^N(n, 1)}, \]

where \( \tilde{K}_t^N(n, \cdot) \) is a certain random kernel such that

- \( \mathbb{E} \left[ \tilde{K}_t^N(n, \phi_t^N) \bigg| \mathcal{F}_{t-1}^+ \right] = (Q_{t-1}^N L_{t-1:t}) \phi_t^N; \)
- \( N^{-1} \sum_n \tilde{K}_t^N(n, 1) = \ell_t^N; \)
- \( \left( \tilde{K}_t^N(n, \phi_t^N) \right)_{n=1,\ldots,N} \) are i.i.d. given \( \mathcal{F}_{t-1}^+; \)
- almost surely, \( \left| \tilde{K}_t^N(n, \phi_t^N) \right| \leq \| \phi_t^N \| \| G_t \| \) if \( \phi_t^N \) is uniformly bounded and \( \tilde{K}_t^N(n, \phi_t^N) \geq 0 \) if \( \phi_t^N \) is uniformly non-negative.

These statements are valid for \( t = 0 \) under the convention \( Q_{-1}^N L_{-1:0} = Q_{-1}^N \) and \( \mathcal{F}_{-1}^+ \) being the trivial \( \sigma \)-algebra.

**Proof.** Put \( \tilde{K}_t^N(n, \phi_t^N) := \int G_t(x_t)K_t^N(n, d\sigma_{0:t})\phi_t^N(x_{0:t}) \) where \( K_t^N \) is defined in Proposition 5. Then

\[
Q_t^N(\phi_t^N) = \frac{N^{-1} \sum_n \int G_t(x_t)K_t^N(n, d\sigma_{0:t})\phi_t^N(x_{0:t})}{\ell_t^N}
\]

Since \( Q_t^N \) is a probability measure, applying this identity twice yields

\[
Q_t^N(\phi_t^N) = \frac{Q_t^N(\phi_t^N)}{Q_t^N(1)} = \frac{N^{-1} \sum_n \tilde{K}_t^N(n, \phi_t^N)}{N^{-1} \sum_n K_t^N(n, 1)}.
\]

The remaining points are simple consequences of the definition of \( \tilde{K}_t^N \) and \( L_{t-1:t} \).
The corollary hints at a natural induction proof for Theorem 1.

Proof of Theorem 1. The following calculations are valid for all \( T \geq 0 \), under the convention defined at the end of Corollary 3. They will prove (8) for \( T = 0 \) and, at the same time, prove it for any \( T \geq 1 \) under the hypothesis that it already holds true for \( T - 1 \). Let \( \varphi_T = \varphi_T(x_0, \ldots, x_T) \) be a real-valued function on \( X_0 \times \cdots X_T \).

Write
\[
(28) \quad \sqrt{N}(\tilde{Q}_T^N \varphi_T - Q_T \varphi_T) = \sqrt{N} \left( \frac{N^{-1} \sum_n \tilde{K}_T^N(n, \varphi_T) - Q_{T-1} L_{T-1:T} \varphi_T}{Q_{T-1} L_{T-1:T} \varphi_T} \right)
\]
where the rewriting of \( Q_T \varphi_T \) is a consequence of \( Q_T \propto Q_{T-1} L_{T-1:T} \). We will bound this difference by Hoeffding’s inequalities for ratios (see Appendix E.13 for notations, including the definition of sub-Gaussian variables that we shall use below).

We have
\begin{itemize}
  \item that \( \sqrt{N}(N^{-1} \sum \tilde{K}_T^N(n, \varphi_T) - Q_{T-1} L_{T-1:T} \varphi_T) \) is \((1, \| \varphi_T \|_\infty \| G_T \|_\infty)\)-sub-Gaussian conditioned on \( F_{T-1}^+ \) because of Theorem 9 (and thus unconditionally, by the law of total expectation);
  \item and that \( \sqrt{N}(N^{-1} \tilde{Q}_{T-1}^N L_{T-1:T} \varphi_T - Q_{T-1} L_{T-1:T} \varphi_T) \) is sub-Gaussian with parameters
    \[
    (C_{T-1}, S_{T-1} \| L_{T-1:T} \|_\infty \| \varphi_T \|_\infty)
    \]
    if \( T \geq 1 \) by induction hypothesis. The quantity is equal to 0 if \( T = 0 \).
\end{itemize}

This permits to apply Lemma 16, which results in the sub-Gaussian properties of
\begin{itemize}
  \item the quantity \( \sqrt{N}(N^{-1} \sum \tilde{K}_T^N(n, \varphi_T) - Q_{T-1} L_{T-1:T} \varphi_T) \), with parameters \((1 + C_{T-1}, S_{T-1} \| \varphi_T \|_\infty)\), for a certain constant \( S_{T-1} \);
  \item and the quantity \( \sqrt{N}(N^{-1} \sum \tilde{K}_T^N(n, \| G_T \|_\infty) - Q_{T-1} L_{T-1:T} \varphi_T) \), which is a special case of the former one, with parameters \((1 + C_{T-1}, S_{T-1}')\).
\end{itemize}

Finally, we invoke Proposition 11 and deduce the sub-Gaussian property of (28) with parameters
\[
\left(2 + 2C_{T-1}, 2 \frac{S_{T-1} \| \varphi_T \|_\infty}{Q_{T-1} L_{T-1:T} \varphi_T} \right)
\]
which finishes the proof. \( \square \)

Proof of Proposition 5. From (24), we have
\[
\tilde{Q}_t^N(dx_{0:t}) = \sum_{i_t} \tilde{Q}_t^N(di_{i_t}) \tilde{Q}_t^N(dx_{0:t}|i_t)
\]
\[
= (\ell_t^N)^{-1} N^{-1} \sum_{i_t} G_t(X_{i_t}^t) \tilde{Q}_t^N(dx_{0:t}|i_t)
\]
\[
= (\ell_t^N)^{-1} N^{-1} \sum_{i_t} G_t(x_t) \tilde{Q}_t^N(dx_{0:t}|i_t)
\]
since \( \tilde{Q}_t^N(dx_{0:t}|i_t) \) has a \( \delta_{X_{i_t}^t}(dx_t) \) term. In fact, the identity
\[
\tilde{Q}_t^N(dx_{0:t}, di_{t-1}i_t) = \delta_{X_{i_t}^t}(dx_t) B_t^N(i_t, di_{i_t-1}) \tilde{Q}_{t-1}^N(dx_{0:t-1}|i_{t-1})
\]
follows directly from the backward recursive nature of Algorithm 2, and thus
\[
(29) \quad \tilde{Q}_t^N(dx_{0:t}|i_t) = \delta_{X_{i_t}^t}(dx_t) \int_{i_{t-1}} B_t^N(i_t, di_{i_t-1}) \tilde{Q}_{t-1}^N(dx_{0:t-1}|i_{t-1}).
\]
The $\bar{Q}^N_{t-1}(dx_{0:t-1}|i_{t-1})$ term is $\mathcal{F}_{t-1}^+$-measurable. We shall calculate the expectation of $\delta_{X^t_{i_{t-1}}}(dx_{t})B^N_t(i_{t}, di_{t-1})$ given $\mathcal{F}_{t-1}^+$. The following arguments are necessary for formal verification, but the result (30) is natural in light of the ancestor regeneration intuition explained in Section 2.4.

Let $f^N_t : \{1, \ldots, N\} \times \mathcal{X}_t \rightarrow \mathbb{R}$ be a (possibly random) function such that $f^N_t(i_{t-1}, x_t)$ is $\mathcal{F}_{t-1}^+$-measurable. Let $J^N_t$ be a random variable such that given $\mathcal{F}_{t-1}^+$, $X^t_{i_t}$ and $\bar{B}^N_t(i_{t}, \cdot)$, $J^N_t$ is $B^N_t(i_{t}, di_{t-1})$-distributed. This automatically makes $J^N_t$ satisfy the second hypothesis of Theorem 1. Additionally, by virtue of its first hypothesis, the distribution of $(J^N_t, A^N_t)$ is the same given either $\mathcal{F}_{t-1}^+$ or $X^1_{i-1}$ (see also Figure 1). We can now write

$$\mathbb{E}\left[\int f^N_t(i_{t-1}, x_t)\delta_{X^t_{i_{t-1}}}(dx_t)B^N_t(i_{t}, di_{t-1}) \bigg| \mathcal{F}_{t-1}^+\right]$$

$$= \mathbb{E}\left[\int f^N_t(i_{t-1}, X^t_{i})B^N_t(i_{t}, di_{t-1}) \bigg| \mathcal{F}_{t-1}^+\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[f^N_t(J^N_t, X^t_{i}) \big| \mathcal{F}_{t-1}^+, X^t_{i}, \bar{B}^N_t(i_{t}, \cdot)\right] \bigg| \mathcal{F}_{t-1}^+\right]$$

$$= \mathbb{E}\left[f^N_t(J^N_t, X^t_{i}) \big| \mathcal{F}_{t-1}^+\right] \text{ by the law of total expectation}$$

$$= \mathbb{E}\left[f^N_t(A^N_t, X^t_{i}) \big| \mathcal{F}_{t-1}^+\right] \text{ by the second hypothesis of Theorem 1}$$

$$= \int f^N_t(i_{t-1}, x_t)\mathcal{M}(W^{1:N}_{t-1})(di_{t-1})M_t(X^t_{i-1}, dx_t).$$

This equality means that

$$\mathbb{E}\left[\delta_{X^t_{i_{t-1}}}(dx_t)B^N_t(i_{t}, di_{t-1}) \bigg| \mathcal{F}_{t-1}^+\right] = \mathcal{M}(W^{1:N}_{t-1})(di_{t-1})M_t(X^t_{i-1}, dx_t),$$

Now, put

$$K^N_t(i_{t}, dx_{0:t}) := \bar{Q}^N_{t-1}(dx_{0:t}|i_{t}).$$

From (29) and (30), we have

$$\mathbb{E}\left[K^N_t(i_{t}, dx_{0:t}) \big| \mathcal{F}_{t-1}^+\right] = \int_{i_{t-1}} \mathcal{M}(W^{1:N}_{t-1})(di_{t-1})M_t(X^t_{i-1}, dx_t)\bar{Q}^N_{t-1}(dx_{0:t-1}|i_{t-1})$$

$$= M_t(x_{t-1}, dx_t)\int_{i_{t-1}} \mathcal{M}(W^{1:N}_{t-1})(di_{t-1})\bar{Q}^N_{t-1}(dx_{0:t-1}|i_{t-1})$$

$$= M_t(x_{t-1}, dx_t)\mathbb{E}[\delta_{X^t_{i_{t-1}}}(dx_{t-1})]$$

which finishes the proof.

### E.2. Proof of Equation (11) (online smoothing recursion).

**Proof.** Using (7) and the matrix notations, the distribution $\bar{Q}^N_t(d\bar{s}_t)$ can be represented by the $1 \times N$ vector

$$\bar{q}_t^N := [W^1_t \ldots W^N_t] \bar{B}^N_t \ldots \bar{B}^N_{t+1}.$$ 

Defining the $N \times N$ matrix $\bar{\psi}^N_s$ as

$$\bar{\psi}^N_s[i_{s-1}, i_s] := \psi_s(X^t_{i_{s-1}}, X^t_{i_s}),$$

we can express the recursion as

$$\bar{Q}^N_{t-1}(dx_{0:t-1}|i_{t-1}) = \int_{i_{t-1}} \mathcal{M}(W^{1:N}_{t-1})(di_{t-1})M_t(X^t_{i-1}, dx_t)\bar{Q}^N_{t-1}(dx_{0:t-1}|i_{t-1}).$$
we have
\[ \mathbb{E}_{Q^N_t} [\psi_s(X_{s-1}, X_s)] = \sum_{i_s, i_{s-1}} q^N_{si_s}[1, i_s] \dot{B}^N_s[i_s, i_{s-1}] \dot{\psi}^N_s[i_{s-1}, i_s] \]
\[ = \sum_{i_s} q^N_{si_s}[1, i_s] (\dot{B}^N_s \dot{\psi}^N_s)[i_s, i_s] \]
\[ = q^N_{s1} \text{diag}(\dot{B}^N_s \dot{\psi}^N_s). \]

Therefore,
\[ Q^N_t \varphi_t = \sum_{s=0}^t [W^1_t \ldots W^N_t] \dot{B}^N_t \ldots \dot{B}^N_{s+1} \text{diag}(\dot{B}^N_s \dot{\psi}^N_s) \]
from which follows the recursion
\[
\begin{align*}
Q^N_t \varphi_t &= [W^1_t \ldots W^N_t] \dot{S}^N_t, \\
\dot{S}^N_t &= \dot{B}^N_t \dot{S}^N_{t-1} + \text{diag}(\dot{B}^N_t \dot{\psi}^N_s).
\end{align*}
\]
This is exactly (11).

\[ \square \]

E.3. **Proof of Theorem 2 (general stability theorem).** The following lemma describes the simultaneous backward construction of two trajectories \( I^1_{0:T} \) and \( I^2_{0:T} \) given \( F^-_T \).

**Lemma 2.** We use the same notations as in Algorithms 1 and 2. Suppose that the hypotheses of Theorem 1 are satisfied. Then, given \( I^1_{0:T}, I^2_{0:T} \) and \( F^-_T \),

- if \( I^1_t \neq I^2_t \), the two variables \( I^1_{t-1} \) and \( I^2_{t-1} \) are conditionally independent and their marginal distributions are respectively \( B^{N,\text{FFBS}}_t(I^1_t, \cdot) \) and \( B^{N,\text{FFBS}}_t(I^2_t, \cdot) \);
- if \( I^1_t = I^2_t \), under the aforementioned conditioning, the two variables \( I^1_{t-1} \) and \( I^2_{t-1} \) are both marginally distributed according to \( B^{N,\text{FFBS}}_t(I^1_t, \cdot) \). Moreover, if (13) holds, we have

\[ F \left( I^1_{t-1} \neq I^2_{t-1 \mid I^1_{0:T}, I^2_{0:T}, F^-_T} \right) \mathbb{I}_{I^1_t = I^2_t} \geq \varepsilon_S \mathbb{I}_{I^1_t = I^2_t}. \]

In particular, the sequence of variables \( (I^1_{t-1}, I^2_{t-1})_{t=0}^T \) is a Markov chain given \( F^-_T \).

**Proof.** To simplify the notations, let \( \tilde{b}_t^N \) denote the \( \mathbb{R}^n \) vector \( \tilde{B}^N_t(n, \cdot) \). The relation between variables generated by Algorithm 2 is depicted as a graphical model in Figure 22. We consider

\[ p(\tilde{b}^{1:N}_{t-1}, i_{t-1}^{1:2} | \mathcal{F}^-_T, i_{t}^{1:2}) = p(\tilde{b}^{1:N}_{t-1} | \mathcal{F}^-_T, i_{t}^{1:2}) p(i_{t-1}^{1:2} | \tilde{b}^{1:N}_{t-1}, \mathcal{F}^-_T, i_{t}^{1:2}) \]
\[ = p(\tilde{b}^{1:N}_{t-1} | \tilde{x}^{1:N}_{t-1}, x_t^N) p(i_{t-1}^{1:2} | \tilde{b}^{1:N}_{t-1}, i_{t}^{1:2}) \]
\[ = \prod_n p(\tilde{b}_n^N | \tilde{x}^{1:N}_{t-1}, x_t^N) \tilde{b}_t^{i_t^1} (i_{t-1}^1) \tilde{b}_t^{i_t^2} (i_{t-1}^2). \]

(by properties of graphical models, see Figure 22)

\[ = \prod_n p(\tilde{b}_n^N | \tilde{x}^{1:N}_{t-1}, x_t^N) \tilde{b}_t^{i_t^1} (i_{t-1}^1) \tilde{b}_t^{i_t^2} (i_{t-1}^2). \]

The distribution of \( i_{t-1}^1 \) given \( \mathcal{F}^-_T \) and \( i_{t-1}^{1:2} \) is thus the \( i_{t-1}^1 \)-marginal of

\[ p(\tilde{b}_t^{i_t^1} | x_{t-1}^{i_{t-1}^1}, x_t^i) \tilde{b}_t^{i_t^1} (i_{t-1}^1), \]

and
which is exactly the distribution of $p(j_{i_1}^{1:2} | x_{i_1}^{1:N}, x_{i_2}^{1:N})$ where the $J$’s are defined in the statement of Theorem 1. By the second hypothesis of that theorem, the aforementioned distribution is equal to $p(a_{i_1}^{1:2} | x_{i_1}^{1:N}, x_{i_2}^{1:N})$, which is in turn no other than $B_{i_1}^{N, FFBS}(i_1, \cdot)$. Moreover, if $i_1^{1:2} \neq i_2^{1:2}$, (32) straightforwardly implies the conditional independence of $i_{i_1}^{1:2}$ and $i_{i_2}^{1:2}$. When $i_1^{1:2} = i_2^{1:2}$, the distribution of $i_{i_1}^{1:2}$ given $F_{i_1}$ and $i_{i_2}^{1:2}$ is the $i_{i_1}^{1:2}$-marginal of

$$p(\hat{b}_{i_1}^{i_1} | x_{i_1}^{1:N}, x_{i_2}^{1:N}) \hat{b}_{i_1}^{i_1} (i_{i_1}^{1} \hat{b}_{i_1}^{i_1} (i_{i_1}^{1})), $$

Thus, we can apply (13) for $n = i_1$, where $i_{i_1}^{1:2}$ here plays the role of $J_{i_1}^{1:2}$ there. Equation (31) is now proved. \hfill \Box

As Lemma 2 describes the distribution of two trajectories, it immediately gives the distribution of a single trajectory.

**Corollary 4.** Under the same settings as in Lemma 2, given $F_{i_1}$, the distribution of $I_{i_0;i_1}$ is

$$\mathcal{M}(W_{i_1}^{1:N})(di_{i_1})B_{i_1}^{N, FFBS}(i_{i_1}, di_{i_1-1}) \cdots B_{i_1}^{N, FFBS}(i_1, di_0).$$

Note that the corollary applies even if the backward kernel used in Algorithm 2 is *not* the FFBS one. This is due to the conditioning on $F_{i_1}$ and the second hypothesis of Theorem 1.
Proof of Theorem 2. First of all, we remark that as per Algorithm 2, using index variables \( I_{\Theta_T} \) adds a level of Monte Carlo approximation to \( Q_T^N(dx_{0:T}) \). Therefore

\[
\mathbb{E} \left[ (Q_T^N(\varphi_T) - Q_T(\varphi_T))^2 \right] = \mathbb{E} \left[ \left( \frac{1}{N} \sum_{n=1}^{N} \varphi_T(X^{T_0}_0, \ldots, X^{T_T}_T) - Q_T(\varphi_T) \right)^2 \right]
\]

(34)

\[
= \mathbb{E} \left[ (Q_T^{N, \text{FFBS}}(\varphi_T) - Q_T(\varphi_T))^2 \right] + \mathbb{E} \left[ \text{Var} \left( \frac{1}{N} \sum_{n=1}^{N} \varphi_T(X^{T_0}_0, \ldots, X^{T_T}_T | F_T) \right) \right]
\]

where the ultimate inequality is justified by the law of total expectation and Corollary 4. (Note that \((I_{\Theta_T})^N_{n=1}\) are identically distributed but not necessarily independent given \(F_T\)). Using Lemma 4 (stated and proved below) and putting \( \rho := 1 - M_1/M_2 \), we have

\[
\text{Var} \left( \frac{1}{N} \sum_{n=1}^{N} \varphi_T(X^{T_0}_0, \ldots, X^{T_T}_T | F_T) \right)
\]

\[
= \text{Var} \left( \frac{1}{N} \sum_{n=1}^{N} \sum_{l=0}^{T} \psi_l(X^{T_{l-1}}_{l-1}, X^{T_T}_t | F_T) \right)
\]

\[
= \frac{1}{N^2} \sum_{n,m \leq N} \sum_{s,t \leq T} \text{Cov} \left( \psi_n(X^{T_{n-1}}_{n-1}, X^{T_T}_t), \psi_s(X^{T_{s-1}}_{s-1}, X^{T_T}_t) \right) | F_T \right)
\]

\[
\leq \frac{2}{N^2} \sum_{n,m \leq N} \sum_{s,t \leq T} \|\psi_n\|_\infty \|\psi_s\|_\infty \rho^{(t-s)}
\]

(35)

\[
+ \frac{4}{N^2} \sum_{n,m \leq N} \sum_{s,t \leq T} \frac{\bar{C}}{N} \|\psi_n\|_\infty \|\psi_s\|_\infty \rho^{(t-s)}
\]

\[
= \left( \sum_{n,m \leq N} 2 \|\psi_n\|_\infty \|\psi_s\|_\infty \rho^{(t-s)} \right) \frac{(2\bar{C} + 1)N - 2\bar{C}}{N^2}
\]

\[
\leq \left( \sum_{n,m \leq N} \left( \|\psi_n\|_\infty^2 + \|\psi_s\|_\infty^2 \right) \rho^{(t-s)} \right) \frac{2\bar{C} + 1}{N} \leq \frac{4(2\bar{C} + 1)}{N(1 - \rho)} \sum \|\psi_n\|_\infty^2.
\]

We now look at the first term of (34). In the fixed marginal smoothing case, for any \( s \in \mathbb{Z}_+, s \leq T \) and any function \( \phi_s : \mathcal{X}_s \rightarrow \mathbb{R} \), Douc et al. (2011) proved that

\[
\mathbb{P} \left( \left| Q_T^{N, \text{FFBS}}(\varphi_T) - Q_T(\varphi_T) \right| \geq \varepsilon \right) \leq B' e^{-C'N \varepsilon^2 / \|\phi_s\|_\infty^2}
\]

for \( \varphi_T(x_{0:T}) = \phi_s(x_s) \) and constants \( B' \) and \( C' \) not depending on \( T \). Using \( \mathbb{E}[\Delta^2] = \int_0^\infty \mathbb{P}(\Delta^2 \geq t)dt \), the inequality implies

\[
\mathbb{E} \left[ (Q_T^{N, \text{FFBS}}(\varphi_T) - Q_T(\varphi_T))^2 \right] \leq \frac{B' \|\phi_s\|_\infty^2}{C'N}.
\]

(36)
for \( \varphi_T(x_{0:T}) = \phi_s(x_s) \). In the additive smoothing case, Dubarry and Le Corff (2013) proved that, for \( T \geq 2 \),

\[
\mathbb{E} \left[ \left| Q_{T}^{N,\text{FFBS}}(\varphi_T) - Q_{T}(\varphi_T) \right|^2 \right] \leq \frac{C'}{N} \left( \sum_{t=0}^{T} \|\psi_t\|_\infty^2 \right) \left( 1 + \sqrt{\frac{T}{N}} \right)^2.
\]

Equations (36), (37), (35) and (34) conclude the proof. \( \square \)

The following lemma quantifies the backward mixing property induced by Assumption 2.

**Lemma 3.** Under the same setting as Theorem 2, we have

\[
\text{TV} \left( B_{t}^{N,\text{FFBS}}(m, \cdot), B_{t}^{N,\text{FFBS}}(n, \cdot) \right) \leq 1 - \frac{\bar{M}_t}{\bar{M}_h}
\]

for all \( m, n \in \{1, \ldots, N\} \) and \( t \in \{1, \ldots, T\} \).

**Proof.** We have

\[
1 - \text{TV} \left( B_{t}^{N,\text{FFBS}}(m, \cdot), B_{t}^{N,\text{FFBS}}(n, \cdot) \right) = \sum_{i=1}^{N} \min \left( \frac{G_{t-1}(X_{i-1}^{m})m_{t}(X_{i-1}^{m}, X_{i}^{m})}{\sum_{j=1}^{N} G_{t-1}(X_{j-1}^{m})m_{t}(X_{j-1}^{m}, X_{j}^{m})}, \frac{G_{t-1}(X_{i-1}^{n})m_{t}(X_{i-1}^{n}, X_{i}^{n})}{\sum_{j=1}^{N} G_{t-1}(X_{j-1}^{n})m_{t}(X_{j-1}^{n}, X_{j}^{n})} \right) \]

by Lemma 1 (Appendix A.2)

\[
\geq \frac{\sum_{i=1}^{N} G_{t}(X_{i-1}^{m})\bar{M}_t}{\sum_{j=1}^{N} G_{t}(X_{j-1}^{m})\bar{M}_h} \quad \text{by Assumption 2}
\]

\[
= (\frac{\bar{M}_t}{\bar{M}_h}).
\]

\( \square \)

**Lemma 4.** Under the same settings as in Theorem 2, for any \( m, n \in \{1, \ldots, N\} \) and \( s, s' \in \{0, \ldots, T\} \), we have

\[
\text{Cov} \left( \psi_{s}(X_{s-1}^{m}, X_{s}^{m}), \psi_{s'}(X_{s'-1}^{n}, X_{s'}^{n}) \middle| \mathcal{F}_{T} \right) \leq \frac{2\bar{C}/N}{1 - \frac{\bar{M}_t}{\bar{M}_h}} \|\psi_{s}\|_\infty \|\psi_{s'}\|_\infty \times \begin{cases} 2\bar{C}/N & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}
\]

where \( \bar{C} = \bar{C}(\bar{M}_t, \bar{M}_h, \bar{G}_t, \bar{G}_h, \varepsilon_S) \) is a constant that does not depend on \( T \) (and which arises in the formulation of Lemma 5). If \( s \) or \( s' \) is equal to 0, we adopt the natural convention \( \psi_0(x_{-1}, x_0) := \psi_0(x_0) \).

**Proof.** We first handle the case \( m \neq n \). Without loss of generality, assume that \( m = 1, n = 2 \) and \( s \geq s' \). The covariance bound of Lemma 1 yields

\[
\text{Cov} \left( \psi_{s}(X_{s-1}^{1}, X_{s}^{1}), \psi_{s'}(X_{s'-1}^{2}, X_{s'}^{2}) \middle| \mathcal{F}_{T} \right) \leq 2 \|\psi_{s}\|_\infty \|\psi_{s'}\|_\infty \text{TV} \left( (T_{s-1:1}^{1}, T_{s-1:1}^{2}) \middle| \mathcal{F}_{T} \right) \text{TV} \left( (T_{s'-1:2}^{2}, T_{s'-1:2}^{2}) \middle| \mathcal{F}_{T} \right) \right).
\]
We shall bound this total variation distance via the coupling inequality of Lemma 1 (Appendix A.2). The idea is to construct, in addition to \( I_{0:T}^1 \) and \( I_{0:T}^2 \), two trajectories \( I_{0:T}^{0:1} \) and \( I_{0:T}^{0:2} \) i.i.d. given \( \mathcal{F}_T \) such that each of them is conditionally distributed according to \( I_{0:T}^1 \) (cf. Corollary 4). To make the coupling inequality efficient, it is desirable to make \( I_{0:T}^1 \) and \( I_{0:T}^2 \) as similar as possible (same thing for \( I_{0:T}^{0:1} \) and \( I_{0:T}^{0:2} \)).

The detailed construction of the four trajectories \( I_{0:T}^1, I_{0:T}^2, I_{0:T}^{0:1} \) and \( I_{0:T}^{0:2} \) given \( \mathcal{F}_T \) is described in Algorithm 15. In particular, we ensure that \( \forall t \geq s - 1 \), we have \( I_t^1 = I_t^{1:0} \). For \( t \leq s - 1 \), if \( I_t^2 = I_t^{2:0} \), it is guaranteed that \( I_t^2 = I_t^{2:0} \) holds \( \forall \ell \leq t \).

The rationale for different coupling behaviours between the times \( t \geq s - 1 \) and \( t \leq s - 1 \) will become clear in the proof: the former aim to control the correlation between two different trajectories \( m = 1 \) and \( n = 2 \) and result in the \( \tilde{C}/N \) term of (38); the latter are for bounding the correlation between two times \( s \) and \( s' \) and result in the \( (1 - M_t/M_h)|s-s'|^{-1} \) term of the same equation.

---

**Algorithm 15:** Sampler for the variables \( I_{0:T}^1, I_{0:T}^2, I_{0:T}^{0:1} \) and \( I_{0:T}^{0:2} \) (see proof of Lemma 4)

**Input:** Feynman-Kac model (1), variables \( X_{0:T}^{1:N} \) from the output of Algorithm 1, integer \( s \geq 0 \) (see statement of Lemma 4)

Sample \( \mathcal{I}_T^1, \mathcal{I}_T^2 \overset{\text{i.i.d.}}{\sim} \mathcal{M}(W_{0:T}^{1:N}) \)

Set \( \mathcal{I}_T^1 \leftarrow \mathcal{I}_T^1 \) and \( \mathcal{I}_T^2 \leftarrow \mathcal{I}_T^2 \)

for \( t \leftarrow T \) to 1 do

if \( \mathcal{I}_T^1 \neq \mathcal{I}_T^2 \) then

for \( k \in \{1, 2\} \) do

Sample \( (\mathcal{I}_{t-1}^k, \mathcal{I}_{t-1}^k) \) from any maximal coupling of \( B_t^{N, \text{FFBS}}(\mathcal{I}_t^k, \cdot) \) and \( B_t^{N, \text{FFBS}}(\mathcal{I}_t^k, \cdot) \) (cf. Lemma 1)

else

Sample the \( \mathbb{R}^N \) vector \( \mathcal{B}_N^t(\mathcal{I}_t^k, \cdot) \) from \( p(\mathcal{b}_N^t(i_t^k, \cdot)|x_{t-1}^{1:N}, x_t^i) \)

Sample \( \mathcal{I}_{t-1}^1, \mathcal{I}_{t-1}^2 \overset{\text{i.i.d.}}{\sim} \mathcal{B}_N^t(\mathcal{I}_t^k, \cdot) \)

Set \( k \leftarrow 1, t \leftarrow 2 \) if \( t \geq s \) and \( k \leftarrow 2, t \leftarrow 1 \) otherwise

Sample \( \mathcal{I}_{t-1}^k \sim B_t^{N, \text{FFBS}}(\mathcal{I}_t^k, \cdot) \) such that \( (\mathcal{I}_{t-1}^k, \mathcal{I}_{t-1}^k) \) is any maximal coupling of \( B_t^{N, \text{FFBS}}(\mathcal{I}_t^k, \cdot) \) and \( B_t^{N, \text{FFBS}}(\mathcal{I}_t^k, \cdot) \) given \( \mathcal{I}_{t-2}, \mathcal{I}_{t-1}^2 \) and \( \mathcal{F}_T \) (\( \ast \) - see text for validity of this step)

Sample \( \mathcal{I}_{t-1}^k \sim B_t^{N, \text{FFBS}}(\mathcal{I}_t^k, \cdot) \)

end if

end for

Output: Four trajectories \( I_{0:T}^1, I_{0:T}^2, I_{0:T}^{0:1}, I_{0:T}^{0:2} \) to be used in the proof of Lemma 4

---

The correctness of Algorithm 15 is asserted by Lemma 2. Step \( \ast \) is valid because that lemma states that the distribution of \( I_T^{k:0} \) given \( \mathcal{F}_T \), \( I_{t:T}^{1:2} \) and \( I_{t:T}^{2:1} \) is \( B_t^{N, \text{FFBS}}(\mathcal{I}_t^k, \cdot) \). Furthermore, we note that \( (R_{T-1})_{t=0} \) where

\[
R_t := (\mathcal{I}_t^1, \mathcal{I}_t^2, \mathcal{I}_t^{1:0}, \mathcal{I}_t^{2:0})
\]

is a Markov chain given \( \mathcal{F}_T^- \).
From (39), applying the coupling inequality of Lemma 1 gives

\begin{equation}
\text{Cov}\left(\psi_s(X_{s-1}^{I^1_s}, X_{s}^{I^1_s}, \psi_s'(X_{s-1}^{I^2_s}, X_{s}^{I^2_s}) \big| F_T\right) \\
\leq 2 \|\psi_s\|_{\infty} \|\psi_s'\|_{\infty} \mathbb{P}\left( (I^1_{s-1:s}, I^2_{s-1:s'}) \neq (I^3_{s-1:s}, I^4_{s-1:s'}) \big| F_T\right) \\
= 2 \|\psi_s\|_{\infty} \|\psi_s'\|_{\infty} \mathbb{P}\left( I^2_{s-1:s'} \neq I^4_{s-1:s'} \big| F_T\right)
\end{equation}

where the last equality results from the construction of Algorithm 15. The sub-case \( s = s' \) following directly from Lemma 5, we now focus on the sub-case \( s \geq s' + 1 \). For all \( t \leq s - 1 \),

\begin{equation}
\mathbb{P}\left( I^2_{t-1} \neq I^2_{t} \big| F_T\right) \\
= \mathbb{P}\left( I^2_{t-1} \neq I^2_{t}, I^2_{t} \neq I^2_{t} \big| F_T\right) \\
\text{by construction of Algorithm 15} \\
= \mathbb{E}\left[ \mathbb{P}\left( I^2_{t-1} \neq I^2_{t}, I^2_{t} \neq I^2_{t} \big| R_t, F_T\right) \big| F_T\right] \\
\text{by the law of total expectation} \\
= \mathbb{E}\left[ \text{TV}\left( B^{N,FFBS}_{t'} (I^2_{t}, \cdot), B^{N,FFBS}_{t'} (I^2_{t}, \cdot) \right) \right] \mathbb{1}\{ I^2_{t} \neq I^2_{t}\} \big| F_T\right] \\
\leq \left( 1 - \frac{M_t}{M_h} \right) \mathbb{P}\left( I^2_{t} \neq I^2_{t} \big| F_T\right) \text{ by Lemma 3.}
\end{equation}

Thus

\begin{equation}
\mathbb{P}\left( I^2_{s'-1:s'} \neq I^2_{s'-1:s'} \big| F_T\right) \\
= \mathbb{P}\left( I^2_{s'} \neq I^2_{s'} \big| F_T\right) \text{ by construction of Algorithm 15} \\
\leq \left( 1 - \frac{M_t}{M_h} \right)^{s-s'-1} \mathbb{P}\left( I^2_{t-1} \neq I^2_{t-1} \big| F_T\right) \text{ by applying (41) recursively} \\
\leq \left( 1 - \frac{M_t}{M_h} \right)^{s-s'-1} \frac{C}{N} \text{ by Lemma 5.}
\end{equation}

which, combined with (40) finishes the proof for the current sub-case \( s \geq s' + 1 \). It remains to show (38) when \( m = n \). The proof follows the same lines as in the case \( m \neq n \), although we shall briefly outline some arguments to show how the factor \( \frac{C}{N} \) disappeared. The case \( s = s' \) being trivial, suppose that \( s \geq s' + 1 \) and without loss of generality that \( m = n = 3 \). To use the coupling tools of Lemma 1, we construct trajectories \( I^3_{0:T}, I^3_{0:T} \) and \( I^4_{0:T} \) via Algorithm 16 and write, in the spirit of (40):

\begin{equation}
\text{Cov}\left( \psi_s(X_{s-1}^{I^3_{s-1}}, X_{s}^{I^3_{s}}, \psi_s'(X_{s-1}^{I^4_{s-1}}, X_{s}^{I^4_{s}}) \big| F_T\right) \\
\leq 2 \|\psi_s\|_{\infty} \|\psi_s'\|_{\infty} \mathbb{P}\left( (I^3_{s-1:s}, I^4_{s-1:s'}) \neq (I^3_{s-1:s}, I^4_{s-1:s'}) \big| F_T\right) \\
= 2 \|\psi_s\|_{\infty} \|\psi_s'\|_{\infty} \mathbb{P}\left( I^3_{s} \neq I^4_{s} \big| F_T\right)
\end{equation}
for which, when plugged into (42), finishes the proof. □

Lemma 5. For $I^3_s$ and $I^{s*}_s$ defined by the output of Algorithm 15, we have

$$\mathbb{P}(I^3_s \neq I^{s*}_s | F_T) \leq \tilde{C}/N,$$

and

$$\mathbb{P}(I^2_{s-1} \neq I^{2*}_{s-1} | F_T) \leq \tilde{C}/N, \text{ if } s \geq 1,$$

for some constant $\tilde{C} = \tilde{C}(M_t, M_h, G_t, G_h, \varepsilon_\delta)$.

Proof. Define $A_t := \mathbb{1}\{I^2_t \neq I^{2*}_t\}$, $B_t := \mathbb{1}\{I^2_t = I^{2*}_t\}$ and $\Gamma_t := A_tB_t$ and recall that $R_t := (I^1_t, I^2_t, I^{3*}_t, I^{4*}_t)$. The sequence $(R_T)_{t=0}^T$ is a Markov chain given $F_T$, but this is not necessarily the case for the sequence $(\Gamma_T)_{t=0}^T$ of Bernoulli random variables. Nevertheless, Lemma 6 below shows that one can get bounds on two-step “transition probabilities” for $(\Gamma_T)$, i.e. the probabilities under $F_T$ that $\Gamma_{t-2} = 1$ given $\Gamma_t$ and $R_t$. This motivates our following construction of actual Markov chains.

Algorithm 16: Sampler for the variables $I^3_{0:T}$, $I^{4*}_{0:T}$ and $I^{4*}_{0:T}$ (see proof of Lemma 4)

**Input:** Feynman-Kac model (1), variables $X_{0:T}$ from the output of Algorithm 1, integer $s \geq 0$ (see statement of Lemma 4)

Sample $I^3_T, I^{4*}_T \overset{i.i.d.}{\sim} \mathcal{M}(W_{T}^{1:N})$

Set $T_s := I^3_T$

for $t \leftarrow T$ to 1 do

if $t \geq s$ then

Sample $I^3_{t-1} \sim B^N_{FFBS}(I^3_t)$ and $I^{4*}_{t-1} \sim B^N_{FFBS}(I^{4*}_t)$

Set $T_{t-1} := I^3_{t-1}$

else

Sample $(I^3_{t-1}, I^{4*}_{t-1})$ from a maximal coupling of $B^N_{FFBS}(I^3_t)$ and $B^N_{FFBS}(I^{4*}_t)$

Sample $I^{4*}_{t-1} \sim B^N_{FFBS}(I^{3*}_t)$

**Output:** Three trajectories $I^3_{0:T}$, $I^{4*}_{0:T}$ and $I^{4*}_{0:T}$ to be used in the proof of Lemma 4

where the last equality follows from the construction of Algorithm 16 and the hypothesis $s \geq s' + 1$. For all $t \leq s - 1$, the inequality

\[
\mathbb{P}(I^3_s \neq I^{4*}_s | F_T) \leq \left(1 - \frac{M_t}{M_h}\right)^{s-s'-1} \mathbb{P}(I^3_s \neq I^{4*}_s | F_T)
\]

can be proved using the same techniques as those used to prove (41): applying Lemma 3 given $(I^3_{s+1}, I^{4*}_{s+1}, I^{4*}_{s+1})$ then invoking the law of total expectation. Repeatedly instantiating (43) gives

\[
\mathbb{P}(I^3_s \neq I^{4*}_s | F_T) \leq \left(1 - \frac{M_t}{M_h}\right)^{s-s'-1} \mathbb{P}(I^3_{s-1} \neq I^{4*}_{s-1} | F_T)
\]

which, when plugged into (42), finishes the proof. □

\[
\text{Lemma 5. For } I^3_s \text{ and } I^{s*}_s \text{ defined by the output of Algorithm 15, we have}
\]

$$\mathbb{P}(I^3_s \neq I^{s*}_s | F_T) \leq \tilde{C}/N, \text{ and}
\]

$$\mathbb{P}(I^2_{s-1} \neq I^{2*}_{s-1} | F_T) \leq \tilde{C}/N, \text{ if } s \geq 1,$$

for some constant $\tilde{C} = \tilde{C}(M_t, M_h, G_t, G_h, \varepsilon_\delta)$. 

Proof. Define $A_t := \mathbb{1}\{I^1_t \neq I^{1*}_t\}$, $B_t := \mathbb{1}\{I^2_t = I^{2*}_t\}$ and $\Gamma_t := A_tB_t$ and recall that $R_t := (I^1_t, I^2_t, I^{3*}_t, I^{4*}_t)$. The sequence $(R_T)_{t=0}^T$ is a Markov chain given $F_T$, but this is not necessarily the case for the sequence $(\Gamma_T)_{t=0}^T$ of Bernoulli random variables. Nevertheless, Lemma 6 below shows that one can get bounds on two-step “transition probabilities” for $(\Gamma_T)$, i.e. the probabilities under $F_T$ that $\Gamma_{t-2} = 1$ given $\Gamma_t$ and $R_t$. This motivates our following construction of actual Markov chains.
approximating the dynamic of $\Gamma_t$. Let $\Gamma^*_t$ and $\Gamma^*_{t-1}$ be two independent Bernoulli random variables given $\mathcal{F}_T^-$ such that

\begin{equation}
\begin{aligned}
\mathbb{P} (\Gamma^*_t = 1 | \mathcal{F}_T^-) &= \mathbb{P} (\Gamma_T = 1 | \mathcal{F}_T^-) \\
\mathbb{P} (\Gamma^*_{t-1} = 1 | \mathcal{F}_T^-) &= \mathbb{P} (\Gamma_{T-1} = 1 | \mathcal{F}_T^-).
\end{aligned}
\end{equation}

Let $\Gamma^*_T, \Gamma^*_{T-2}, \Gamma^*_{T-4}; \ldots$ and $\Gamma^*_{T-1}, \Gamma^*_{T-3}; \ldots$ be two homogeneous Markov chains given $\mathcal{F}_T^-$ with the same transition kernel $T^\ast$ defined by

\begin{equation}
\begin{aligned}
\mathbb{P}_{\mathcal{F}_T^-} (\Gamma^*_{t-2} = 1 | \Gamma^*_t = 1) &= 1 - \frac{2}{N} \left( \frac{\hat{G}_h \hat{M}_h}{\hat{G}_\ell \hat{M}_\ell} \right)^2 : = \tilde{c}^2_{11} \\
\mathbb{P}_{\mathcal{F}_T^-} (\Gamma^*_{t-2} = 1 | \Gamma^*_t = 0) &= \frac{\hat{M}_\ell \varepsilon S}{2\hat{M}_h} =: \tilde{c}^2_{01}
\end{aligned}
\end{equation}

where for two events $E_1, E_2$, the notation $\mathbb{P}_{\mathcal{F}_T^-} (E_1 | E_2)$ is the ratio between $\mathbb{P} (E_1, E_2 | \mathcal{F}_T^-)$ and $\mathbb{P} (E_2 | \mathcal{F}_T^-)$. We shall now prove by backward induction the following statement:

\begin{equation}
\mathbb{P} (\Gamma_t = 1 | \mathcal{F}_T^-) \geq \mathbb{P} (\Gamma^*_t = 1 | \mathcal{F}_T^-), \forall t \geq s - 1.
\end{equation}

Firstly, (46) holds for $t = T$ and $t = T - 1$. Now suppose that it holds for some $t \geq s + 1$ and we wish to justify it for $t - 2$. By Lemma 6,

\begin{equation}
\begin{aligned}
\mathbb{P} (\Gamma_{t-2} = 1 | R_t, \mathcal{F}_T^-) 1_{\Gamma_t = 1} \geq \tilde{c}^2_{11} \Gamma_t = 1 \\
\mathbb{P} (\Gamma_{t-2} = 1 | R_t, \mathcal{F}_T^-) 1_{\Gamma_t = 0} \geq \tilde{c}^2_{01} \Gamma_t = 0.
\end{aligned}
\end{equation}

Applying the law of total expectation gives

\begin{equation}
\begin{aligned}
\mathbb{P} (\Gamma_{t-2} = 1 | \mathcal{F}_T^-) &\geq \tilde{c}^2_{11} \mathbb{P} (\Gamma_t = 1 | \mathcal{F}_T^-) + \tilde{c}^2_{01} \mathbb{P} (\Gamma_t = 0 | \mathcal{F}_T^-) \\
&= \left( \tilde{c}^2_{11} - \tilde{c}^2_{01} \right) \mathbb{P} (\Gamma_t = 1 | \mathcal{F}_T^-) + \tilde{c}^2_{01} \\
&\geq \left( \tilde{c}^2_{11} - \tilde{c}^2_{01} \right) \mathbb{P} (\Gamma^*_t = 1 | \mathcal{F}_T^-) + \tilde{c}^2_{01}
\end{aligned}
\end{equation}

if $N$ is large enough, by induction hypothesis

\begin{equation}
\mathbb{P} (\Gamma^*_{t-2} = 1 | \mathcal{F}_T^-)
\end{equation}

and (46) is now proved. To finish the proof of the lemma, it is necessary to lower bound its right hand side. We start by controlling the distribution $\Gamma^*_t$ for $t = T$ and $t = T - 1$. We have

\begin{equation}
\begin{aligned}
\mathbb{P} (\Gamma^*_T = 1 | \mathcal{F}_T^-) &= \mathbb{P} (\Gamma_T = 1 | \mathcal{F}_T^-) \text{ by (44)} \\
&= 1 - \mathbb{P} (A_T = 0 | \mathcal{F}_T^-) \text{ as } B_T = 1 \text{ by Algorithm 15} \\
&= 1 - \sum_{i=1}^N \mathbb{P} (i_T = i | \mathcal{F}_T^-) \\
&= 1 - \sum_{i=1}^N \left( \frac{G(X^i_T)}{\sum_{j=1}^N G(X^j_T)} \right)^2 \\
&\geq 1 - \frac{1}{N} \left( \frac{\hat{G}_h}{\hat{G}_\ell} \right)^2 \text{ by Assumption 3}
\end{aligned}
\end{equation}
\( \mathbb{P} (\Gamma_{T-1}^* = 1 | \mathcal{F}_T^c) \geq \mathbb{P} (\Gamma_T = 1, \Gamma_{T-1} = 1 | \mathcal{F}_T^c) \)

\[
= \mathbb{E} \left[ \mathbb{P} (\Gamma_T = 1, \Gamma_{T-1} = 1 | R_T, \mathcal{F}_T^c) | \mathcal{F}_T^c \right]
\]

by the law of total expectation

\[
= \mathbb{E} \left[ \mathbb{P} (\Gamma_{T-1} = 1 | R_T, \mathcal{F}_T^c) 1_{\Gamma_T = 1} | \mathcal{F}_T^c \right]
\]

(48)

\[
\geq \left[ 1 - \frac{1}{N} \left( \frac{\tilde{G}_h \tilde{M}_h}{G_t M_t} \right)^2 \right] \mathbb{P} (\Gamma_T = 1 | \mathcal{F}_T^c) \quad \text{via (53)}
\]

\[
\geq \left[ 1 - \frac{1}{N} \left( \frac{\tilde{G}_h \tilde{M}_h}{G_t M_t} \right)^2 \right] \left[ 1 - \frac{1}{N} \left( \frac{\tilde{G}_h}{G_t} \right)^2 \right].
\]

The contraction property of Lemma 1 makes it possible to relate the intermediate distributions \( \Gamma_t^* | \mathcal{F}_T^c \) to the end point ones \( \Gamma_{T-1}^* | \mathcal{F}_T^c \) and \( \Gamma_T^* | \mathcal{F}_T^c \). More specifically, (45) and Lemma 1 lead to

(49) \( \text{TV}(\Gamma_t^* | \mathcal{F}_T^c, \mu^*) \leq \max \{ \text{TV}(\Gamma_{T-1}^* | \mathcal{F}_T^c, \mu^*), \text{TV}(\Gamma_T^* | \mathcal{F}_T^c, \mu^*) \} \)

where \( \mu^* \) is the invariant distribution of a Markov chain with transition matrix \( \tilde{\mathcal{C}}^2 \), namely

(50)

\[
\begin{cases}
\mu^*(\{0\}) = \frac{\tilde{\mathcal{C}}_{19}}{\tilde{\mathcal{C}}_{19} + \tilde{\mathcal{C}}_{29}}, \\
\mu^*(\{1\}) = 1 - \mu^*(\{0\}).
\end{cases}
\]

Furthermore, an alternative expression of the total variation distance given in Lemma 1 implies that the total variation distance between two Bernoulli distributions of parameters \( p \) and \( q \) is \( |p - q| \). Combining this with (49), the triangle inequality and the rough estimate \( \max(a, b) \leq a + b \forall a, b \geq 0 \), we get

\[
\mathbb{P} (\Gamma_T^* = 0 | \mathcal{F}_T^c) \leq 3\mu^*(\{0\}) + \mathbb{P} (\Gamma_T^* = 0 | \mathcal{F}_T^c) + \mathbb{P} (\Gamma_{T-1}^* = 0 | \mathcal{F}_T^c) \leq \tilde{C} / N
\]

where \( \tilde{C} = \tilde{C}(M_t, M_h, G_t, \tilde{G}_h, \varepsilon_S) \). The last inequality is straightforwardly derived by plugging respectively (50), (47) and (48) into the three terms of the preceding sum. This combined with (46) finishes the proof. \( \square \)

**Lemma 6.** For \( s \) defined in the statement of Lemma 4; \( A_t, B_t \) and \( R_t \) defined in the proof of Lemma 5 and all \( t \geq s + 1 \), we have

\[
\mathbb{P} (A_{t-2} B_{t-2} = 1 | R_t, \mathcal{F}_T^c) 1_{A_t B_t = 1} \geq \frac{1 - \frac{2}{N} \left( \frac{\tilde{G}_h M_h}{G_t M_t} \right)^2}{2 M_h} \quad \text{\(1_{A_t B_t = 1} \)}.
\]

\[
\mathbb{P} (A_{t-2} B_{t-2} = 1 | R_t, \mathcal{F}_T^c) \geq \frac{M_t \varepsilon_S}{2 M_h}
\]

where the inequalities hold for \( N \) large enough, i.e., \( N \geq N_0 = N_0(M_t, M_h, G_t, \tilde{G}_h, \varepsilon_S) \).
Proof: We start by showing the following three inequalities for all $t \geq s$ and $N$ sufficiently large:

\begin{align}
&\mathbb{P} \left( A_{t-1} = 1 \mid R_t, \mathcal{F}_T^- \right) \geq \varepsilon_S; \\
&\mathbb{P} \left( A_{t-1}B_{t-1} = 1 \mid R_t, \mathcal{F}_T^- \right) \mathbb{1}_{A_{t-1} = 1} \geq (\bar{M}_t/2\bar{M}_h)\mathbb{1}_{A_{t-1} = 1}; \\
&\mathbb{P} \left( A_{t-1}B_{t-1} = 1 \mid R_t, \mathcal{F}_T^- \right) \mathbb{1}_{A_{t-1}B_{t-1} = 1} \geq \left[ 1 - \frac{1}{N} \left( \frac{\bar{G}_h \bar{M}_h}{G_t M_t} \right)^2 \right] \mathbb{1}_{A_t B_t = 1}.
\end{align}

For (51), we have

\begin{align}
\mathbb{P} \left( A_{t-1} = 1 \mid R_t, \mathcal{F}_T^- \right) \mathbb{1}_{A_{t-1} \neq 1} = \mathbb{P} \left( T_{t-1}^1 \neq T_{t-1}^2 \mid R_t, \mathcal{F}_T^- \right) \mathbb{1}_{T_{t-1}^1 \neq T_{t-1}^2} \\
= \left[ 1 - \sum_i \mathbb{P} \left( T_{t-1}^1 = T_{t-1}^2 = i \mid R_t, \mathcal{F}_T^- \right) \right] \mathbb{1}_{T_{t-1}^1 \neq T_{t-1}^2} \\
\geq \left[ 1 - \frac{1}{N} \left( \frac{\bar{G}_h \bar{M}_h}{G_t M_t} \right)^2 \right] \mathbb{1}_{A_{t-1} = 1} \quad \text{by Assumptions 2 and 3.}
\end{align}

Combining (54) and (55) yields (51) for $N$ large enough. To prove (52), we write

\begin{align}
\mathbb{P} \left( A_{t-1}B_{t-1} = 1 \mid R_t, \mathcal{F}_T^- \right) \mathbb{1}_{A_{t-1} = 1} \\
= \left[ 1 - \mathbb{P} \left( A_{t-1}B_{t-1} = 0 \mid R_t, \mathcal{F}_T^- \right) \right] \mathbb{1}_{A_{t-1} = 1} \\
\geq \left[ 1 - \mathbb{P} \left( A_{t-1} = 0 \mid R_t, \mathcal{F}_T^- \right) - \mathbb{P} \left( B_{t-1} = 0 \mid R_t, \mathcal{F}_T^- \right) \right] \mathbb{1}_{A_{t-1} = 1} \\
= \left[ \mathbb{P} \left( A_{t-1} = 1 \mid R_t, \mathcal{F}_T^- \right) + \mathbb{P} \left( B_{t-1} = 1 \mid R_t, \mathcal{F}_T^- \right) - 1 \right] \mathbb{1}_{A_{t-1} = 1}.
\end{align}

We analyse the second term in the above expression. We have

\begin{align}
\mathbb{P} \left( B_{t-1} = 1 \mid R_t, \mathcal{F}_T^- \right) \mathbb{1}_{A_{t-1} = 1} \\
= \mathbb{P} \left( T_{t-1}^2 = T_{t-1}^2 \mid R_t, \mathcal{F}_T^- \right) \mathbb{1}_{T_{t-1}^1 \neq T_{t-1}^2} \\
= \left[ 1 - \text{TV} \left( B_{t,i}^{N,\text{FFBS}}(T_{t-1}^2, \cdot), B_{t,i}^{N,\text{FFBS}}(T_{t-1}^2, \cdot) \right) \right] \mathbb{1}_{A_{t-1} = 1} \\
\geq (\bar{M}_t/\bar{M}_h)\mathbb{1}_{A_{t-1} = 1} \quad \text{by construction of Algorithm 15}
\end{align}

Plugging (55) and (57) into (56) yields

\begin{align}
\mathbb{P} \left( A_{t-1}B_{t-1} = 1 \mid R_t, \mathcal{F}_T^- \right) \mathbb{1}_{A_{t-1} = 1} \geq \left( \frac{1}{N} \left( \frac{\bar{G}_h \bar{M}_h}{G_t M_t} \right)^2 + \frac{\bar{M}_t}{\bar{M}_h} \right) \mathbb{1}_{A_{t-1} = 1}
\end{align}

and thus (52) follows if $N$ is large enough. The inequality (53) is justified by combining (55), the simple decomposition $\mathbb{1}_{A_{t-1}B_{t-1} = 1} = \mathbb{1}_{A_{t-1} = 1} \mathbb{1}_{B_{t-1} = 1}$ and the fact that Algorithm 15 guarantees $B_{t-1} = 1$ if $A_t = B_t = 1$. 
We can now deduce the two inequalities in the statement of the Lemma. The first one is a straightforward double application of (53):

\[
\mathbb{P}\left( A_{t-2}B_{t-2} = 1 \mid R_t, \mathcal{F}_T^\perp \right) \mathbb{1}_{A_tB_t=1} \\
\geq \mathbb{P}\left( A_{t-2}B_{t-2} = 1, A_{t-1}B_{t-1} = 1 \mid R_t, \mathcal{F}_T^\perp \right) \mathbb{1}_{A_tB_t=1} \\
= \mathbb{E}\left[ \mathbb{P}\left( A_{t-2}B_{t-2} = 1, A_{t-1}B_{t-1} = 1 \mid R_{t-1}, R_t, \mathcal{F}_T^\perp \right) \mid R_t, \mathcal{F}_T^\perp \right] \mathbb{1}_{A_tB_t=1} \\
\text{by the law of total expectation} \\
= \mathbb{E}\left[ \mathbb{P}\left( A_{t-2}B_{t-2} = 1 \mid R_{t-1}, \mathcal{F}_T^\perp \right) \mathbb{1}_{A_{t-1}B_{t-1}=1} \mid R_t, \mathcal{F}_T^\perp \right] \mathbb{1}_{A_tB_t=1} \\
\text{since } (R_{T-\ell})_{\ell=0}^T \text{ is Markov given } \mathcal{F}_T^\perp \\
\geq \mathbb{E}\left[ \left( 1 - \frac{1}{N} \left( \frac{\bar{G}_h \bar{M}_{\ell}}{\bar{G}_l \bar{M}_T} \right)^2 \right) \mathbb{1}_{A_{t-1}B_{t-1}=1} \mid R_t, \mathcal{F}_T^\perp \right] \mathbb{1}_{A_tB_t=1} \\
\geq \left[ 1 - \frac{1}{N} \left( \frac{\bar{G}_h \bar{M}_{\ell}}{\bar{G}_l \bar{M}_T} \right)^2 \right]^2 \mathbb{1}_{A_tB_t=1} \geq \left( 1 - \frac{2}{N} \left( \frac{\bar{G}_h \bar{M}_{\ell}}{\bar{G}_l \bar{M}_T} \right)^2 \right) \mathbb{1}_{A_tB_t=1}.
\]

Finally, we have

\[
\mathbb{P}\left( A_{t-2}B_{t-2} = 1 \mid R_t, \mathcal{F}_T^\perp \right) \\
\geq \mathbb{P}\left( A_{t-2}B_{t-2} = 1, A_{t-1} = 1 \mid R_t, \mathcal{F}_T^\perp \right) \\
= \mathbb{E}\left[ \mathbb{P}\left( A_{t-2}B_{t-2} = 1 \mid R_{t-1}, \mathcal{F}_T^\perp \right) \mathbb{1}_{A_{t-1}=1} \mid R_t, \mathcal{F}_T^\perp \right] \\
\text{using law of total expectation and the Markov property as above} \\
\geq \frac{M_{\ell}}{2M_h} \mathbb{P}\left( A_{t-1} = 1 \mid R_t, \mathcal{F}_T^\perp \right) \text{ by (52)} \\
\geq \frac{M_{\ell}}{2M_h} \varepsilon_S \text{ by (51)}
\]

and the second inequality is proved. \(\square\)

### E.4. Proof of Proposition 3 (hybrid rejection validity).

**Proof.** Put \(Z_n := (X_n, U_n C \mu_0(X_n))\). Then \(Z_n\) is uniformly distributed on

\[
\mathcal{G}_0 := \{(x, y) \in \mathcal{X} \times \mathbb{R}_+, y \leq C \mu_0(x)\}.
\]

The proof would be done if one could show that, given \(K^* \leq K\), the variable \(Z_{K^*}\) is uniformly distributed on

\[
\mathcal{G}_1 := \{(x, y) \in \mathcal{X} \times \mathbb{R}_+, y \leq \mu_1(x)\}.
\]
Indeed, this would mean that for any continuous bounded function $\psi$, we have

$$
\mathbb{E}[\psi(\tau^t_{1, \text{PaRIS}})] = \mathbb{E}\left[ (\text{Geo}^* \psi) \left( \frac{r^N_t(X_1^t)}{M_h} \right) \right] \to \mathbb{E}\left[ (\text{Geo}^* \psi) \left( \frac{r_t(X^\infty_{1, \text{PaRIS}})}{M_h} \right) \right] = \mathbb{E}[\psi(\tau^\infty_{t, \text{PaRIS}})] .
$$

Note that $K^*$ is, by definition, the first time index where the sequence $(Z_n)$ touches $G_1$. Let $B$ be any subset of $G_1$. We have

$$
P \left( Z_{K^*} \in B \mid K^* \leq K \right) \propto P(Z_{K^*} \in B, K^* \leq K)
$$

$$
= \sum_{k^* = 1}^{\infty} P \left( Z_{k^*} \in B, K^* = k^*, K \geq k^* \right)
$$

$$
= \sum_{k^* = 1}^{\infty} P \left( Z_{k^*} \in B, Z_{1:k^* - 1} \notin G_1, K > k^* - 1 \right)
$$

(58)

$$
= \sum_{k^* = 1}^{\infty} P(Z_{k^*} \in B)P \left( Z_{1:k^* - 1} \notin G_1, K > k^* - 1 \right) \text{ since } K \text{ stopping time}
$$

$$
= P(Z_1 \in B) \sum_{k^* = 1}^{\infty} P \left( Z_{1:k^* - 1} \notin G_1, K > k^* - 1 \right)
$$

$$
\propto P(Z_1 \in B) \propto P \left( Z_1 \in B \mid Z_1 \in G_1 \right).
$$

By considering the special case $B = G_1$, we see that the constant of proportionality between the first and the last terms of (58) must be 1, from which the proof follows.

E.5. Proof of Theorem 3 (hybrid algorithm’s intermediate complexity).

From (16), one may have the correct intuition that as $N \to \infty$, $\tau^1_{t, \text{PaRIS}}$ tends in distribution to that of the variable $\tau^\infty_{t, \text{PaRIS}}$ defined as

(59)

$$
\tau^\infty_{t, \text{PaRIS}} \mid X^\infty_{t, \text{PaRIS}} \sim \text{Geo} \left( \frac{r_t(X^\infty_{t, \text{PaRIS}})}{M_h} \right)
$$

where $X^\infty_{t, \text{PaRIS}} \sim Q_{t-1} M_t(dx_t)$ is distributed according to the predictive distribution of $X_t$ given $y_{0:t-1}$ and $r_t$ is the density of $X^\infty_{t, \text{PaRIS}}$ with respect to the Lebesgue measure (cf. Definition 1). The following proposition formalises the connection between $\tau^1_{t, \text{PaRIS}}$ and $\tau^\infty_{t, \text{PaRIS}}$.

**Proposition 6.** We have $\tau^1_{t, \text{PaRIS}} \Rightarrow \tau^\infty_{t, \text{PaRIS}}$ as $N \to \infty$.

**Proof.** From (16) and Definition 1 one has

(60)

$$
\tau^1_{t, \text{PaRIS}} \mid X^1_{t}, F_{t-1} \sim \text{Geo} \left( \frac{r^N_t(X_1^t)}{M_h} \right).
$$

In light of (59), it suffices to establish that

(61)

$$
\frac{r^N_t(X_1^t)}{M_h} \Rightarrow \tau_t(X^\infty_{t, \text{PaRIS}}).
$$

Indeed, this would mean that for any continuous bounded function $\psi$, we have

$$
\mathbb{E}[\psi(\tau^1_{t, \text{PaRIS}})] = \mathbb{E} \left[ (\text{Geo}^* \psi) \left( \frac{r^N_t(X_1^t)}{M_h} \right) \right] \to \mathbb{E} \left[ (\text{Geo}^* \psi) \left( \frac{r_t(X^\infty_{t, \text{PaRIS}})}{M_h} \right) \right] = \mathbb{E}[\psi(\tau^\infty_{t, \text{PaRIS}})].
$$
where \( \text{Geo}^\ast \) is the geometric Markov kernel that sends each \( \lambda \) to the geometric distribution of parameter \( \lambda \), i.e. \( \text{Geo}^\ast(\lambda, dx) = \text{Geo}(\lambda) \). To this end, write

\[
\begin{align*}
    r_t^N(X_{t_1}^1) - r_t(X_{t_1}^1) &= \frac{\sum_n G_{t-1}(X_{t-1}^n)m_t(X_{t-1}^n, X_{t_1}^1)}{\sum_n G_{t-1}(X_{t-1}^n)} - r_t(X_{t_1}^1) \\
    &= \frac{\sum_n N^{-1}G_{t-1}(X_{t-1}^n) \left[m_t(X_{t-1}^n, X_{t_1}^1) - r_t(X_{t_1}^1)\right]}{N^{-1}\sum_n G_{t-1}(X_{t-1}^n)}.
\end{align*}
\]

(62)

We study the mean squared error of the numerator:

\[
\begin{align*}
    \mathbb{E} \left\{ \frac{1}{N} \sum_n G_{t-1}(X_{t-1}^n) \left[m_t(X_{t-1}^n, X_{t_1}^1) - r_t(X_{t_1}^1)\right] \right\}^2 \\
    &= \frac{1}{N} \mathbb{E} \left\{ G_{t-1}(X_{t-1}^1)^2 \left[m_t(X_{t-1}^1, X_{t_1}^1) - r_t(X_{t_1}^1)\right]^2 \right\} \\
    &\quad + \frac{N(N-1)}{N^2} \mathbb{E} \left\{ G_{t-1}(X_{t-1}^1)G_{t-1}(X_{t_1}^1) \left[m_t(X_{t-1}^1, X_{t_1}^1) - r_t(X_{t_1}^1)\right] \right. \\
    &\quad \left. \times \left[m_t(X_{t-1}^1, X_{t_1}^1) - r_t(X_{t_1}^1)\right] \right\} \\
\end{align*}
\]

where we have again used the exchangeability induced by step (*) of Algorithm 5. The first term obviously tends to 0 as \( N \to \infty \) by Assumptions 4 and 1. The second term also vanishes asymptotically thanks to Lemma 7 below and Assumption 6. Assumption 1 also implies that the denominator of (62) converges in probability to some constant, via the consistency of particle approximations, see e.g. Del Moral (2004) or Chopin and Papaspiliopoulos (2020). Thus, \( r_t^N(X_{t_1}^1) - r_t(X_{t_1}^1) \to 0 \) by Slutsky’s theorem. Moreover, \( r_t(X_{t_1}^1) \Rightarrow r_t(X_{\infty, 	ext{ParIS}}^1) \) by the continuity of \( r_t \) and the consistency of particle approximations. Using again Slutsky’s theorem yields (61).

The following lemma is needed to complete the proof of Proposition 6 and is related to the propagation of chaos property, see Del Moral (2004, Chapter 8).

**Lemma 7.** We have \( (X_{t-1}^1, X_{t-1}^2, X_{t_1}^1) \Rightarrow Q_{t-2}M_{t-1} \otimes Q_{t-2}M_{t-1} \otimes Q_{t-1}M_t \).

**Proof.** For vectors \( u, v, \) and \( w \), we have, by the symmetry of the distribution of particles:

\[
\begin{align*}
    \mathbb{E} \left[ \exp \left\{ iuX_{t-1}^1 + ivX_{t-1}^2 + iwX_{t_1}^1 \right\} \right] \\
    &= \mathbb{E} \left[ \left( \frac{1}{N} \sum e^{iuX_{t-1}^n} \right) \left( \frac{1}{N} \sum e^{ivX_{t-1}^n} \right) \left( \frac{1}{N} \sum e^{iwX_{t-1}^n} \right) \right] \\
    &\quad - \frac{N}{N^2} \mathbb{E} \left[ e^{iuX_{t-1}^1}e^{ivX_{t-1}^1} \left( \frac{1}{N} \sum e^{iwX_{t-1}^n} \right) \right] \\
    &\quad + \frac{N}{N^2} \mathbb{E} \left[ e^{iuX_{t-1}^1}e^{ivX_{t-1}^2} \left( \frac{1}{N} \sum e^{iwX_{t-1}^n} \right) \right].
\end{align*}
\]

Note that

\[
\frac{1}{N} \sum e^{iuX_{t-1}^n} \overset{a.s.}{\Rightarrow} Q_{t-2}M_{t-1}(\exp(\{iu\}))
\]

and

\[
\frac{1}{N} \sum e^{iwX_{t-1}^n} \overset{a.s.}{\Rightarrow} Q_{t-1}M_t(\exp(\{iw\}))
\]


The dominated convergence theorem, applicable since $|e^{iu}| \leq 1$ for $u \in \mathbb{R}$, finishes the proof.

**Proof of Theorem 3.** First of all,

$$
\mathbb{E}[\tau_{t,PaRIS}^{\infty}] = \mathbb{E} \left[ \frac{\tilde{M}_h}{\tilde{r}_t(X_t^{\infty,PaRIS})} \right] = \int_{X_t} \frac{\tilde{M}_h}{\tilde{r}_t(x_t)} r_t(x_t) dx_t = \infty
$$

by Assumption 5. Next, for any $x \in \mathbb{R} \setminus \mathbb{Z}$ and $N > x$,

$$
P \left( \min(\tau_{t,PaRIS}^1, N) \leq x \right) = P \left( \tau_{t,PaRIS}^1 \leq x \right) \rightarrow P \left( \tau_{t,PaRIS}^{\infty} \leq x \right)
$$

by Proposition 6. Thus, by Portmanteau theorem,

$$
\min(\tau_{t,PaRIS}^1, N) \Rightarrow \tau_{t,PaRIS}^{\infty}.
$$

Altogether, we have

$$
limit \inf_{N \to \infty} \mathbb{E} \left[ \min(\tau_{t,PaRIS}^1, N) \right] = limit \inf_{N \to \infty} k \mathbb{P} \left( \min(\tau_{t,PaRIS}^1, N) = k \right) 
\geq \sum k \mathbb{P} \left( \tau_{t,PaRIS}^{\infty} = k \right) \text{ by Fatou's lemma}
= \sum k \mathbb{P} \left( \tau_{t,PaRIS}^{\infty} = \frac{k}{N} \right) \text{ by (64)}
= \infty \text{ by (63)}
$$

and

$$
limit \inf_{N \to \infty} \frac{1}{N} \mathbb{E} \left[ \min(\tau_{t,PaRIS}^1, N) \right] = limit \inf_{N \to \infty} \mathbb{E} \left[ \min \left( \frac{\tau_{t,PaRIS}^1}{N}, 1 \right) \right] \to 0
$$

since $\tau_{t,PaRIS}^1 \Rightarrow \tau_{t,PaRIS}^{\infty}$ implies that the sequence of random variables

$$
\min \left( \frac{\tau_{t,PaRIS}^1}{N}, 1 \right)
$$

converges to 0 in distribution while being bounded between 0 and 1.

**E.6. Proof of Theorem 4 (hybrid PaRIS near-linear complexity).** The following proposition shows that the real execution time for the hybrid algorithm is asymptotically at most of the same order as the “oracle” hybrid execution time.

**Proposition 7.** We have

$$
limit \sup_{N \to \infty} \frac{\mathbb{E} \left[ \min(\tau_{t,PaRIS}^1, N) \right]}{\mathbb{E} \left[ \min(\tau_{t,PaRIS}^{\infty}, N) \right]} < \infty.
$$

**Proof.** Put

$$
z(N) := \frac{1 - (1 - \lambda)^N}{\lambda} = \sum_{n=0}^{N-1} (1 - \lambda)^n.
$$

One can quickly verify (using the memorylessness of the geometric distribution for example) that $z(N) = \mathbb{E} [\min(G, N)] | G \sim Geo(\lambda)$. It will be useful to keep in
mind the elementary estimate $z^N(\lambda) \leq \min(N, \lambda^{-1})$. We can now write

$$
\mathbb{E}\left[ \min(\tau_t^{\text{ParIS}}, N) \right] = \mathbb{E}\left[ z^N \left( \frac{r_t^N(X_t^1)}{M_h} \right) \right] \quad \text{(by (60))}
\leq \mathbb{E}\left[ \int_{X_t} z^N \left( \frac{r_t^N(x_t)}{M_h} \right) r_t^N(x_t) \lambda_t(dx_t) \right]
\leq c_t \left( \int_{X_t} z^N \left( \frac{r_t(x_t)}{M_h} \right) r_t(x_t) \lambda_t(dx_t) + b_t \right) \quad \text{by Lemma 8}
\leq c_t \left( \mathbb{E}\left[ \min(\tau_t^{\infty, \text{ParIS}}, N) \right] + b_t \right)
$$

from which the proposition is immediate. \hfill \square

**Lemma 8.** In addition to notations of Algorithm 1, let the function $z^N$ be defined as in (65) and the functions $r_t$ and $r_t^N$ be defined as in Definition 1. Let $\phi_t : X_t \to \mathbb{R}_{>0}$ be a bounded non-negative deterministic function. Then, under Assumptions 1 and 4, there exist constants $b_t$ and $c_t$ depending only on the model such that

$$
\mathbb{E}\left[ \int_{X_t} z^N \left( \frac{r_t^N(x_t)}{M_h} \right) r_t^N(x_t) \phi_t \right] \leq c_t \left( \int_{X_t} z^N \left( \frac{r_t(x_t)}{M_h} \right) r_t \phi_t + b_t \| \phi_t \|_\infty \right)
$$

where for brevity, we shortened the integration notation (e.g. dropping $\lambda_t(dx_t)$, dropping $x_t$ from $\phi(x_t)$, etc.) whenever there is no ambiguity.

**Proof.** We have

$$
\mathbb{E}\left[ \int_{X_t} z^N \left( \frac{r_t^N(x_t)}{M_h} \right) r_t^N \phi_t \right] \leq \mathbb{E}\left[ \int_{X_t} z^N \left( \frac{r_t^N(x_t)}{M_h} \right) \right] \mathbb{E}\left[ r_t^N(x_t) \right] \phi_t
$$

using Fubini’s theorem and the concavity of $\lambda \mapsto \lambda z^N(\lambda)$ on $[0, 1]$. By a well-known result on the bias of a particle filter (which is in fact the propagation of chaos in the special case of $q = 1$ particle), we have:

$$
|\mathbb{E}\left[ r_t^N(x_t) \right] - r_t(x_t)| = \left| \mathbb{E}\left[ \sum W_t^n m_t(X_{t-1}^n, x_t) \right] - r_t(x_t) \right|
\leq \mathbb{E}\left[ m_t \left( X_{t-1}^1, x_t \right) \right] - Q_{t-1} (m_t (\bullet, x_t))
\leq b_t M_h \frac{1}{N}
$$
for some constant $b_t$. We next show that such a bias does not change the asymptotic behavior of $z^N$. More precisely,

$$z^N \left( \mathbb{E} \left[ \frac{r_t^N(x_t)}{M_h} \right] \right) \leq z^N \left( \frac{r_t(x_t) - b_t}{N} \right)$$

$$= \sum_{n=0}^{N-1} \left( 1 - \frac{r_t(x_t)/\bar{M}_h + b_t/N}{1 - r_t(x_t)/\bar{M}_h} \right)^n \left( 1 - \frac{r_t(x_t)}{\bar{M}_h} \right)^n$$

(67)

$$\leq \sum_{n=0}^{N-1} \left( 1 + \frac{b_t}{N (1 - r_t(x_t)/\bar{M}_h)} \right)^N \left( 1 - \frac{r_t(x_t)}{\bar{M}_h} \right)^n$$

$$\leq \exp \left( \frac{b_t}{1 - r_t(x_t)/\bar{M}_h} \right) z^N \left( \frac{r_t(x_t)}{\bar{M}_h} \right)$$

$$\leq e^{2b_t} z^N \left( \frac{r_t(x_t)}{\bar{M}_h} \right)$$

if $x_t$ is such that $r_t(x_t)/\bar{M}_h \leq 1/2$. In contrast, if $r_t(x_t)/\bar{M}_h \geq 1/2$, then provided that $N \geq 6b_t$, we have

(68) $z^N \left( \mathbb{E} \left[ \frac{r_t^N(x_t)}{M_h} \right] \right) \leq z^N \left( \frac{r_t(x_t) - b_t}{N} \right) \leq z^N \left( \frac{1}{3} \right) \leq 3z^N \left( \frac{r_t(x_t)}{\bar{M}_h} \right)$.

Putting together (67) and (68), we have, for $N \geq 6b_t$,

$$z^N \left( \mathbb{E} \left[ \frac{r_t^N(x_t)}{M_h} \right] \right) \leq (e^{2b_t} + 3) z^N \left( \frac{r_t(x_t)}{\bar{M}_h} \right)$$

and so, by (66),

$$\mathbb{E} \left[ \int_{X_t} z^N \left( \frac{r_t^N(x_t)}{M_h} \right) r_t^N \phi_t \right] \leq (e^{2b_t} + 3) \int_{X_t} z^N \left( \frac{r_t(x_t)}{M_h} \right) \mathbb{E} [r_t^N(x_t)] \phi_t$$

$$= (e^{2b_t} + 3) \mathbb{E} \left[ z^N \left( \frac{r_t^N(X_t)}{M_h} \right) \phi_t (X_t) \right].$$

Again, using the result on the bias of a particle filter,

$$\left| \mathbb{E} \left[ z^N \left( \frac{r_t(X_t)}{M_h} \right) \phi_t (X_t) \right] - \int_{X_t} z^N \left( \frac{r_t(x_t)}{M_h} \right) r_t \phi_t \right| \leq b_t \| z^N \|_\infty \| \phi_t \|_{\infty} = b_t \| \phi_t \|_{\infty}$$

which, together with the previous inequality, implies the desired result. □

**Proposition 8.** In linear Gaussian state space models, we have

$$\mathbb{E} \left[ \min(r_t^{\infty,\text{PaRIS}}, N) \right] = O \left( (\log N)^{d_t/2} \right).$$

**Proof.** Let $\mu_t$ and $\Sigma_t$ be such that $X_t^{\infty,\text{PaRIS}} \sim \mathcal{N}(\mu_t, \Sigma_t)$. Then

$$\log(r_t(X_t^{\infty,\text{PaRIS}})/\bar{M}_h) = b'_t - W_t$$

where $b'_t$ is some constant and

$$W_t := \frac{(X_t^{\infty,\text{PaRIS}} - \mu_t)^\top \Sigma_t^{-1} (X_t^{\infty,\text{PaRIS}} - \mu_t)}{2} \sim \text{Gamma} \left( \frac{d_t}{2}, 1 \right).$$
We have

\[
\mathbb{E}\left[ \min(\tau_{t}^{\infty, \text{PaRIS}}, N) \right] = \mathbb{E}\left[ z^{N} \left( \frac{r_{t}(X_{t}^{\infty, \text{PaRIS}})}{M_{h}} \right) \right] = \mathbb{E}\left[ z^{N} (e^{b_{t} - W_{t}}) \right]
\]

\[
= \int_{0}^{\infty} z^{N} (e^{b_{t} - w}) \frac{w^{d_{t}/2 - 1}e^{-w}}{\Gamma(d_{t}/2)} \, dw
\]

\[
\leq \int_{0}^{\log N} e^{w - b_{t}} w^{d_{t}/2 - 1}e^{-w} \frac{dw}{\Gamma(d_{t}/2)} + \int_{\log N}^{\infty} Nw^{d_{t}/2 - 1}e^{-w} \, dw
\]

using the bound \( z^{N}(\lambda) \leq \min(N, 1/\lambda) \). The first term is of order \( O(\log d_{t}/2 N) \) by elementary calculus, while the second term is of order \( O(\log d_{t}/2 - 1 N) \) using asymptotic properties of the incomplete Gamma function, see Olver et al. (2010, Section 8.11). □

**Proof of Theorem 4.** The theorem is a straightforward consequence of Proposition 7 and Proposition 8. □

**E.7. Proof of Theorems 6 and 7 (pure rejection FFBS complexity).** We start with a useful remark linking the projection kernels \( \Pi \) and the cost-to-go functions defined in Appendix A with the \( L \)-kernels formulated in (27). The proof is simple and therefore omitted.

**Lemma 9.** We have

\[
L_{t,T}(x_{0:t}, \mathbb{1}) = H_{t,T}(x_{t}) \text{ for all } x_{0:t}. \text{ Moreover, for any function } \phi_{t} : \mathcal{X}_{t} \to \mathbb{R}, \text{ we have}
\]

\[
L_{t,T} \Pi_{t}^{0:T} \phi_{t} = \Pi_{t}^{0:T}(\phi_{t} \times H_{t,T}).
\]

Theorems 6 and 7 both rely on an induction argument wrapped up in the following proposition.

**Proposition 9.** We use the notations of Algorithm 2. Let \( Q_{t}^{N} \) be defined as in (24), where the \( B_{t}^{N} \) kernels can be \( B_{t}^{N, \text{FFBS}} \) or any other kernels satisfying the hypotheses of Theorem 1. Suppose that Assumption 1 holds. Let \( f_{t}^{N} : \mathcal{X}_{t} \to \mathbb{R}_{\geq 0} \) be a (possibly random) function such that \( f_{t}^{N}(x_{t}) \) is \( \mathcal{F}_{t-1} \)-measurable. Then the following assertions are true:

(a) Suppose that \( \mathbb{E}\left[ \int_{\mathcal{X}_{t}} \left\{ r_{t}^{N} \times f_{t}^{N} \times G_{t} \times H_{t,T} \right\} (x_{t}) \lambda_{t}(dx_{t}) \right] = \infty \), where \( r_{t}^{N} \) and \( \lambda_{t} \) are defined in Definition 1. Then

\[
\mathbb{E}\left[ \int Q_{T}^{N}(dx_{t}) f_{t}^{N}(x_{t}) \right] = \infty.
\]

(b) Suppose that \( \int_{\mathcal{X}_{t}} \left\{ r_{t}^{N} \times f_{t}^{N} \times G_{t} \times H_{t,T} \right\} (x_{t}) \lambda_{t}(dx_{t}) \overset{P}{\to} 0 \). Then

\[
\int Q_{T}^{N}(dx_{t}) f_{t}^{N}(x_{t}) \overset{P}{\to} 0.
\]

**Proof.** **Part (a).** We shall prove by induction the statement

\[
\mathbb{E}\left[ Q_{s}^{N} L_{s:T} \Pi_{t}^{0:T} f_{t}^{N} \right] = \infty, \forall \ t - 1 \leq s \leq T.
\]
For $s = t - 1$, it follows from part (a)’s hypothesis and Lemma 9. Indeed,

$$Q_{t-1}^N L_{t-1:T} \Pi_{t_1}^{0:T} f_t^N$$

$$= Q_{t-1}^N L_{t-1:T} \Pi_{t_1}^{0:T} f_t^N = Q_{t-1}^N L_{t-1:T} \Pi_{t_1}^{0:T} (f_t^N \times H_{t:T})$$

$$= \int_{X_{t-1} \times X_t} Q_{t-1}^N (dx_{t-1}) m_t(x_{t-1}, x_t) \lambda_t(dx_t) G_t(x_t)(f_t^N \times H_{t:T})(x_t)$$

For $s \geq t$, we have

$$\mathbb{E} \left[ Q_s^N L_{s:T} \Pi_t^{0:T} f_t^N \right] = \mathbb{E} \left[ \frac{N^{-1} \sum \hat{K}_s^N(n, L_{s:T} \Pi_t^{0:T} f_t^N)}{\ell_s^N} \right] \text{ by Corollary 3}$$

$$\geq \frac{1}{\|G_s\|_{\infty}} \mathbb{E} \left[ N^{-1} \sum \hat{K}_s^N(n, L_{s:T} \Pi_t^{0:T} f_t^N) \right]$$

by Assumption 1 and definition of $\ell_s^N$ (see Algorithm 1)

$$\geq \frac{1}{\|G_s\|_{\infty}} \mathbb{E} \left[ Q_{s-1}^N L_{s-1:T} \Pi_t^{0:T} f_t^N \right]$$

by Corollary 3 and law of total expectation

$$= \mathbb{E} \left[ Q_{s-1}^N L_{s-1:T} \Pi_t^{0:T} f_t^N \right] = \infty \text{ (induction hypothesis).}$$

**Part (b).** Similar to part (a), we shall prove by induction the statement

$$Q_s^N L_{s:T} \Pi_t^{0:T} f_t^N \overset{p}{\to} 0, \forall t - 1 \leq s \leq T.$$  

Again, by Corollary 3, this quantity is equal to

$$\frac{N^{-1} \sum \hat{K}_s^N(n, L_{s:T} \Pi_t^{0:T} f_t^N)}{\ell_s^N},$$

and the expectation of the numerator given $F_{s-1}$ is $Q_{s-1}^N L_{s-1:T} \Pi_t^{0:T} f_t^N$, which tends to 0 in probability by induction hypothesis. Lemma 12 (see below at the end of the section), the classical result $\ell_s^N \overset{p}{\to} \ell_s := Q_{s-1} M_s(G_s)$ and Stutsky’s theorem concludes the proof. \hfill \Box

**Proof of Theorem 6.** By (21), we have $\mathbb{E}[\tau_t^{1,FFBS}] = \mathbb{E}[\int Q_T^{N,FFBS} (dx_t) f_t^N (x_t)]$ where

$$f_t^N (x_t) = \frac{\bar{M}_h}{\sum W_{t-1} m_t(X_{t-1}^n, x_t)} = \frac{\bar{M}_h}{\tau_t^N}$$

with $\tau_t^N$ given in Definition 1. Proposition 9(a) gives a sufficient condition for $\mathbb{E}[\tau_t^{1,FFBS}] = \infty$ to hold, namely

$$\int_{X_t} (\tau_t^N \times f_t^N \times G_t \times H_{t:T})(x_t) \lambda_t(dx_t) = \infty,$$

which is equivalent to the hypothesis of the theorem. \hfill \Box

**Proof of Theorem 7.** We use notations from Definition 1 and Appendix A.1. We note $\mathcal{N}(x|\mu, \Sigma)$ the density of the specified normal distribution at point $x$. Using
Lemma 10, Proposition 9 and (21), we have
\[ E[(r_t^{1,FFBS})] = \infty \iff E\left[\frac{1}{r_t^N(X_{t_i}^{x_i})^k}\right] = \infty \]
\[ \iff E\left[\int Q_t^N(dx_t) \frac{1}{r_t^N(x_t)^k}\right] = \infty \]
\[ \iff E\left[\int Q_t^N(dx_t) \frac{r_t G_t H_{t,T}}{(r_t^N)^k}(x_t)\lambda_t(dx_t)\right] = \infty \]
\[ \iff \int X_t r_t G_t H_{t,T}(x_t)\lambda_t(dx_t) = \infty \text{ almost surely} \]
\[ \iff \int X_t r_t G_t H_{t,T}(x_t)^{k-1} \lambda_t(dx_t) = \infty \text{ a.s.} \]

The theorem then follows from elementary arguments, by noting that \( r_t^N \) is a mixture of \( N \) Gaussian distributions with covariance matrix \( C_X \). □

**Lemma 10.** Let \( L \) be a \([0,1]\)-valued random variable. Suppose \( X \) is another random variable such that \( X|L \sim \text{Geo}(L) \). Then for any real number \( k > 0 \),
\[ E[X^k] = \infty \iff E[L^{-k}] = \infty. \]

**Proof.** By the definition of \( X \), we have
\[ E[X^k] = E\left[\sum_{x=1}^{\infty} x^k(1 - L)^{x-1} L \right]. \]

A natural idea is then to approximate the sum by the integral \( \int_0^\infty x^k(1 - L)x^{-1} Ldx \), from which one easily extracts the \( L^{-k} \) factor. This is however technically laborious, since the function \( x \mapsto x^k(1 - L)x^{-1} L \) is not monotone on the whole real line. It is only so starting from a certain \( x_0 \) which itself depends on \( L \). We would therefore rather write
\[ E[X^k] = \int_0^\infty \mathbb{P}(X^k \geq x)dx = \int_0^\infty \mathbb{P}(X \geq x^{1/k})dx \]
\[ = \int_0^\infty \mathbb{E}\left[(1 - L)^{|x^{1/k}}|\right]dx \]
where the two integrands are equal Lebesgue–almost-everywhere
\[ = \mathbb{E}\left[\int_0^\infty \exp\left(-|\log(1 - L)||x^{1/k}|\right)dx\right] \]
with the natural interpretation of expressions when \( L = 1 \). Using \( u \sim v \) as a shorthand for “\( u \) and \( v \) are either both finite or both infinite”, we have
\[ E[X^k] \sim \mathbb{E}\left[\int_0^\infty \exp\left(-|\log(1 - L)||x^{1/k}|\right)dx\right] \text{ by Lemma 11} \]
\[ = k \Gamma(k) \left[\frac{1}{|\log(1 - L)|^k}\right] \sim E\left[\frac{1}{L^k}\right] \text{ by Lemma 11 again}. \]

The following lemma is elementary. Its proof is therefore omitted.
Lemma 11. Let $L$ be a $[0,1]$-valued random variable and let $f_1$ and $f_2$ be two continuous functions from $[0,1]$ to $\mathbb{R}$. Suppose that $\limsup_{t \to 0^+} f_1(t)/f_2(t)$ and $\limsup_{t \to 0^+} f_2(t)/f_1(t)$ are both finite. Then $E[f_1(L)]$ is finite if and only if $E[f_2(L)]$ is so.

Lemma 12. Let $Z_1, Z_2, \ldots$ be non-negative random variables. Suppose that there exist $\sigma$-algebras $\mathcal{F}_1, \mathcal{F}_2, \ldots$ such that $E[Z_n | \mathcal{F}_n] \to 0$. Then $Z_n \mathbb{P} \to 0$.

Proof. Fix $\varepsilon > 0$. By Markov’s inequality, $\mathbb{P}(Z_n \geq \varepsilon | \mathcal{F}_n) \leq \varepsilon^{-1} E[Z_n | \mathcal{F}_n]$. Therefore, the $[0,1]$-bounded random variable $\mathbb{P}(Z_n \geq \varepsilon | \mathcal{F}_n)$ tends to 0 in probability, hence also in expectation. The law of total expectation then gives $\mathbb{P}(Z_n \geq \varepsilon) \to 0$, which, by varying $\varepsilon$, establishes the convergence of $Z_n$ to 0 in probability. □

E.8. Proof of Theorem 8 and Corollary 2 (hybrid FFBS complexity).

Proof of Theorem 8. According to Janson (2011, Lemma 3), it is sufficient to show that

$$\sum_n \min(\tau_{t, \text{FFBS}}^n, N) \mathbb{P} \to 0$$

for any deterministic sequence $\alpha_N$ such that $\alpha_N / E[\min(\tau_{t, \text{FFBS}}^\infty, N)] \to \infty$. By Lemma 12, we can take expectation with respect to $\mathcal{F}_T$ to derive a sufficient condition, namely

$$\int_{\mathcal{X}_t} \alpha_N^{-1} z^N \left( \frac{r_t^N(x_t)}{M_h} \right) Q_T^N(dx_t) \mathbb{P} \to 0 \text{ with } z^N \text{ defined in (65)}$$

$$\iff \int_{\mathcal{X}_t} \alpha_N^{-1} z^N \left( \frac{r_t^N(x_t)}{M_h} \right) r_t^N \times G_t \times H_{t:T} \mathbb{P} \to 0 \text{ by Proposition 9(b)}$$

$$\iff \mathbb{E} \left[ \int_{\mathcal{X}_t} \alpha_N^{-1} z^N \left( \frac{r_t^N(x_t)}{M_h} \right) r_t^N \times G_t \times H_{t:T} \right] \to 0$$

$$\iff \int_{\mathcal{X}_t} \alpha_N^{-1} z^N \left( \frac{r_t(x_t)}{M_h} \right) r_t \times G_t \times H_{t:T} \mathbb{P} \to 0 \text{ by Lemma 8}$$

$$\iff \frac{\mathbb{E}[\min(\tau_{t, \text{FFBS}}^\infty, N)]}{\alpha_N} \to 0.$$ 

The proof is now complete. □

Proof of Corollary 2. We have, using the cost-to-go, the $z^N$ functions and the $\tau_{t, \text{PaRIS}}^\infty$ distribution defined respectively in (20), (65) and (59):

$$E[\min(\tau_{t, \text{FFBS}}^\infty, N)] = \int_{\mathcal{X}_t} z^N \left( \frac{r_t(x_t)}{M_h} \right) Q_T(dx_t)$$

$$= [(Q_{t-1} M_t)(G_t H_{t:T})]^{-1} \int_{\mathcal{X}_t} z^N \left( \frac{r_t(x_t)}{M_h} \right) (G_t H_{t:T})(x_t)(Q_{t-1} M_t)(dx_t)$$

$$\leq \|G_t H_{t:T}\|_{\infty} [(Q_{t-1} M_t)(G_t H_{t:T})]^{-1} \int_{\mathcal{X}_t} z^N \left( \frac{r_t(x_t)}{M_h} \right) (Q_{t-1} M_t)(dx_t)$$

$$= \|G_t H_{t:T}\|_{\infty} [(Q_{t-1} M_t)(G_t H_{t:T})]^{-1} E[\min(\tau_{t, \text{PaRIS}}^\infty, N)].$$

Proposition 8 then finishes the proof. □
E.9. Proof of Proposition 4 (MCMC kernel properties).

Proof. To show that a certain kernel $B^N_t$ satisfies (13), we look at two conditionally i.i.d. simulations $J^{n,1}_t$ and $J^{n,2}_t$ from $B^N_t(n, \cdot)$ and lower bound the probability that they are different. For the kernel $B^{N,IMH}_t$, the variables $J^{n,1}_t$ and $J^{n,2}_t$ both result from one step of MH applied to $A^n_t$. Let $J^{n,1}_t$ and $J^{n,2}_t$ be the corresponding MH proposals. A sufficient condition for $J^{n,1}_t \neq J^{n,2}_t$ is that $J^{n,1}_t \neq J^{n,2}_t$ and the two proposals are both accepted. The acceptance rate is at least $\bar{S} \geq 1$ from one step of MH applied to $A^n_t$. Let $\tilde{J}^{n,1}_t$ and $\tilde{J}^{n,2}_t$ be the corresponding MH proposals. A sufficient condition for $\tilde{J}^{n,1}_t \neq \tilde{J}^{n,2}_t$ is that $\tilde{J}^{n,1}_t \neq \tilde{J}^{n,2}_t$ and the two proposals are both accepted. The acceptance rate is at least $M_t/M_h$ by Assumption 2 and the probability that $\tilde{J}^{n,1}_t \neq \tilde{J}^{n,2}_t$ is

$$1 - \sum_{n=1}^N (W^n_{t-1})^2 \geq 1 - \frac{1}{N} \left( \frac{\bar{G}_h}{G_t} \right)^2$$

by Assumption 3. Thus (13) is satisfied for $\varepsilon_S = \bar{M}_t/2\bar{M}_h$ for $N$ large enough. Similarly, the probability that $\tilde{J}^{n,1}_t \neq \tilde{J}^{n,2}_t$ for the $B^{N,IMHP}_t$ kernel with $N = 2$ can be lower-bounded via the probability that $\tilde{J}^{n,1}_t \neq \tilde{J}^{n,2}_t$ (where $\tilde{J}^{n,1}_t$ and $\tilde{J}^{n,2}_t$ are defined in (17)). Thus using the same arguments, (13) is satisfied here for $\varepsilon_S = \bar{M}_t/4\bar{M}_h$. \qed

E.10. Conditional probability of maximal couplings. In general, there exist multiple maximal couplings of two random distributions (i.e. couplings that maximise the probability of equality of the two variables). However, they all satisfy a certain conditional probability property stated in the following lemma. Its proof, which we were unable to find in the literature, is obvious in the discrete case but requires lengthier arguments in the continuous one.

Proposition 10. Let $X_1$ and $X_2$ be two random variables with densities $f_1$ and $f_2$ with respect to some dominating measure defined on a space $\mathcal{X}$. Then, the following inequality holds almost surely:

$$\mathbb{P}(X_2 = X_1|X_1) \leq 1 + \frac{f_2(X_1)}{f_1(X_1)}.$$  

Moreover, the equality occurs almost surely if and only if $X_1$ and $X_2$ form a maximal coupling.

Proof. Let $h$ be any non-negative test function from $\mathcal{X}$ to $\mathbb{R}$. Putting

$$A_1 := \{ x \in \mathcal{X} | f_1(x) \geq f_2(x) \}$$

$$A_2 := \{ x \in \mathcal{X} | f_2(x) \geq f_1(x) \},$$

we have

$$\mathbb{E}[\mathbb{P}(X_2 = X_1|X_1)h(X_1)] = \mathbb{E}[\mathbb{1}_{X_2 = X_1}h(X_1)]$$

$$= \mathbb{E}[\mathbb{1}_{X_2 = X_1}\mathbb{1}_{X_2 \in A_1}h(X_1)] + \mathbb{E}[\mathbb{1}_{X_2 = X_1}\mathbb{1}_{X_1 \in A_2}h(X_1)]$$

$$= \mathbb{E}[\mathbb{1}_{X_2 = X_1}\mathbb{1}_{X_2 \in A_1}h(X_2)] + \mathbb{E}[\mathbb{1}_{X_2 = X_1}\mathbb{1}_{X_1 \in A_2}h(X_1)]$$

$$\leq \mathbb{E}[\mathbb{1}_{X_2 \in A_1}h(X_2)] + \mathbb{E}[\mathbb{1}_{X_1 \in A_2}h(X_1)]$$

$$= \int h(x)f_2(x)h_1(x)\,dx = \mathbb{E} \left[ \left( 1 + \frac{f_2(X_1)}{f_1(X_1)} \right) h(X_1) \right].$$

The inequality (69) is now proved almost-surely. As a result, almost-sure equality occurs if and only if the expectation of the two sides of (69) are equal, which means, via Lemma 1, that the two variables are maximally coupled. \qed
The following lemma establishes the symmetry of Assumption 8. Again, its statement is obvious in the discrete case, though some work is needed to rigorously justify the continuous one.

**Lemma 13.** Let $X_1$ and $X_2$ be two random variables of densities $f_1$ and $f_2$ w.r.t. some dominating measure defined on some space $\mathcal{X}$. Suppose that almost-surely

$$\mathbb{P}(X_2 = X_1 | X_1) \geq \varepsilon \left( 1 \wedge \frac{f_2(X_1)}{f_1(X_1)} \right)$$

for some $\varepsilon > 0$. Then almost-surely,

$$\mathbb{P}(X_1 = X_2 | X_2) \geq \varepsilon \left( 1 \wedge \frac{f_1(X_2)}{f_2(X_2)} \right).$$

**Proof.** We introduce a non-negative test function $h_2 : \mathcal{X} \to \mathbb{R}$ and write

$$\mathbb{E}[\mathbb{P}(X_1 = X_2 | X_2)h(X_2)] = \mathbb{E}[\mathbb{1}_{X_1 = X_2} h(X_2)] = \mathbb{E}[\mathbb{1}_{X_2 = X_1} h(X_1)]$$

$$= \mathbb{E}[\mathbb{P}(X_2 = X_1 | X_1)h(X_1)]$$

$$\geq \mathbb{E} \left[ \varepsilon \left( 1 \wedge \frac{f_2(X_1)}{f_1(X_1)} \right) h(X_1) \right]$$

$$= \int \varepsilon f_1 \wedge f_2(x) h(x) dx$$

$$= \mathbb{E} \left[ \varepsilon \left( 1 \wedge \frac{f_1(X_2)}{f_2(X_2)} \right) h(X_2) \right]$$

which implies the desired result. \qed

**E.11. Proof of Theorem 5** (intractable kernel properties).

**Proof.** Let $J_t^n$ be a random variable such that

$$J_t^n | X_{t-1}^N, X_t^n, \hat{B}_t^{N,ITR}(n, \cdot) \sim B_t^{N,ITR}(n, \cdot).$$

By construction of Algorithm 7, given $X_{t-1}^N$, the couple $(J_t^n, X_t^n)$ has the same distribution as $(A_t^{n,L}, X_t^{n,L})$. Thus, $B_t^{N,ITR}$ satisfies the hypotheses of Theorem 1.
Lemma 14. We use the notations of Algorithm 7. Under Assumptions 2 and 7, we have
\[ \mathbb{P} \left( A_t^{n,1} \neq A_t^{n,2} \mid X_t^{n,1}, X_t^{1:N} \right) \]
\[ = \frac{1}{2} \mathbb{P} \left( X_t^{n,1} = X_t^{n,2}, A_t^{n,1} \neq A_t^{n,2} \mid X_t^n = x_t, X_t^{1:N} = x_t^{1:N} \right) \]
\[ = \mathbb{P} \left( X_t^{n,1} = X_t^{n,2}, A_t^{n,1} \neq A_t^{n,2} \mid L = 1 \right) \]
\[ = \frac{1}{2} \mathbb{P} \left( X_t^{n,1} = X_t^{n,2}, A_t^{n,1} \neq A_t^{n,2} \mid L = 1, X_t^n = x_t, X_t^{1:N} = x_t^{1:N} \right) \]
\[ = \frac{1}{2} \sum_{a_t^{n,1} \neq a_t^{n,2}} \mathbb{P} \left( X_t^{n,2} = x_t \mid A_t^{n,1} = a_t^{n,1}, A_t^{n,2} = a_t^{n,2}, X_t^n = x_t, X_t^{1:N} = x_t^{1:N} \right) \times \]
\[ \mathbb{P} \left( A_t^{n,1} = a_t^{n,1}, A_t^{n,2} = a_t^{n,2} \mid X_t^{n,1} = x_t, X_t^{1:N} = x_t^{1:N} \right) \]
\[ \geq \frac{1}{2} \frac{\tilde{M}_t}{M_h} \sum_{a_t^{n,1} \neq a_t^{n,2}} \mathbb{P} \left( A_t^{n,1} = a_t^{n,1}, A_t^{n,2} = a_t^{n,2} \mid X_t^{n,1} = x_t, X_t^{1:N} = x_t^{1:N} \right) \]
\[ \geq \frac{1}{2} \frac{\tilde{M}_t}{M_h} \varepsilon_A \text{ by Lemma 14.} \]

The proof is complete. \( \square \)

Lemma 14. We use the notations of Algorithm 7. Under Assumptions 2 and 7, we have
\[ \mathbb{P} \left( A_t^{n,1} \neq A_t^{n,2} \mid X_t^{n,1}, X_t^{1:N} \right) \geq \frac{\tilde{M}_t}{M_h} \varepsilon_A. \]

Proof. We write (and define new notations along the way):
\[ \pi(a_t^{n,1}, a_t^{n,2}) := p(a_t^{n,1}, a_t^{n,2} \mid X_t^{n,1}, X_t^{1:N}) \]
\[ \propto p(a_t^{n,1}, a_t^{n,2} \mid X_t^{1:N}) \mathbb{P}(X_t^{n,1}) \]
\[ = p(a_t^{n,1}, a_t^{n,2} \mid X_t^{1:N}) \phi(a_t^{n,1}) \]
\[ =: \pi_0(a_t^{n,1}, a_t^{n,2}) \phi(a_t^{n,1}). \]

Thus
\[ \mathbb{P} \left( A_t^{n,1} \neq A_t^{n,2} \mid X_t^{n,1}, X_t^{1:N} \right) = \int \mathbb{1} \left\{ a_t^{n,1} \neq a_t^{n,2} \right\} \pi(a_t^{n,1}, a_t^{n,2}) \]
\[ = \int \mathbb{1} \left\{ a_t^{n,1} \neq a_t^{n,2} \right\} \pi_0(a_t^{n,1}, a_t^{n,2}) \phi(a_t^{n,1}) \]
\[ \geq \int \mathbb{1} \left\{ a_t^{n,1} \neq a_t^{n,2} \right\} \pi_0(a_t^{n,1}, a_t^{n,2}) \frac{\tilde{M}_t}{M_h} \]
by the boundedness of the function \( \phi \) between \( \tilde{M}_t \) and \( M_h \). From this, we get the desired result by virtue of Assumption 7. \( \square \)
E.12. **Validity of Algorithm 14 (modified Lindvall-Rogers coupler).** Recall that generating a random variable is equivalent to uniformly simulating under the graph of its density (see e.g. Robert and Casella, 2004, The Fundamental Theorem of Simulation, chapter 2.3.1). Algorithm 14’s correctness is thus a direct corollary of the following intuitive lemma.

**Lemma 15.** Let $S_A$ and $S_B$ be two subsets of $\mathbb{R}^d$ with finite Lebesgue measures. Let $A$ and $B$ be two not necessarily independent random variables distributed according to $\text{Uniform}(S_A)$ and $\text{Uniform}(S_B)$ respectively. Denote by $S_0$ the intersection of $S_A$ and $S_B$; and by $C$ a certain $\text{Uniform}(S_A)$-distributed random variable that is independent from $(A,B)$. Define $A^*$ and $B^*$ as

$$A^* = \begin{cases} C & \text{if } (A,C) \in S_0 \times S_0 \\ A & \text{otherwise} \end{cases}$$

and

$$B^* = \begin{cases} C & \text{if } (B,C) \in S_0 \times S_0 \\ B & \text{otherwise} \end{cases}$$

Then $A^* \sim \text{Uniform}(S_A)$ and $B^* \sim \text{Uniform}(S_B)$.

**Proof.** Given $(A,C) \in S_0 \times S_0$, the two variables $A$ and $C$ have the same distribution (which is $\text{Uniform}(S_0)$). Thus, the definition of $A^*$ implies that $A$ and $A^*$ have the same (unconditional) distribution. The same argument applies to $B$ and $B^*$ notwithstanding the asymmetry in the definition of $C$. □

E.13. **Hoeffding inequalities.** This appendix proves a Hoeffding inequality for ratios, which helps us to bound (28). It is essentially a reformulation of Douc et al. (2011, Lemma 4) in a slightly more general manner.

**Definition 2.** A real-valued random variable $X$ is called $(C,S)$-sub-Gaussian if

$$\mathbb{P}
\left( \frac{|X|}{S} > t \right) \leq 2Ce^{-t^2/2}, \forall \ t \geq 0.$$

This definition is close to other sub-Gaussian definitions in the literature, see e.g. Vershynin (2018, Chapter 2.5). It basically means that the tails of $X$ decreases at least as fast as the tails of the $\mathcal{N}(0,S^2)$ distribution, which is itself $(1,S)$-sub-Gaussian. The following result is classic.

**Theorem 9** (Hoeffding’s inequality). Let $X_1, \ldots, X_N$ be $N$ i.i.d. random variables with mean $\mu$ and almost surely contained between $a$ and $b$. Then $N^{1/2}(\sum X_i/N - \mu)$ is $(1,(b-a)/2)$-sub-Gaussian.

The following lemma is elementary from Definition 2. The proof is omitted.

**Lemma 16.** Let $X$ and $Y$ be two (not necessarily independent) random variables. If $X$ is $(C_1,S_1)$-sub-Gaussian and $Y$ is $(C_2,S_2)$-sub-Gaussian, then $X+Y$ is $(C_1+C_2,S_1+S_2)$-sub-Gaussian.

We are ready to state the main result of this section.

**Proposition 11** (Hoeffding’s inequality for ratios). Let $a_N$, $b_N$, $a^*$, $b^*$ be random variables such that $\sqrt{N}(a_N-a^*)$ is $(C_a,S_a)$-sub-Gaussian and $\sqrt{N}(b_N-b^*)$ is...
\((C_b, S_b)\)-sub-Gaussian. Then \(\sqrt{N} \left( a_N/b_N - a^*/b^* \right)\) is sub-Gaussian with parameters \((C^*, S^*)\) where

\[
\begin{align*}
C^* &= C_a + C_b \\
S^* &= \left\| \frac{1}{b^*} \right\|_\infty (S_a + S_b \left\| \frac{a}{b_N} \right\|_\infty).
\end{align*}
\]

The terms with inf-norm can be infinite if the corresponding random variables are unbounded.

Proof. We have

\[
\left| \sqrt{N} \left( \frac{a_N}{b_N} - \frac{a^*}{b^*} \right) \right| \leq \left| \sqrt{N} \left( \frac{a_N}{b_N} - \frac{a_N}{b^*} \right) \right| + \left| \sqrt{N} \left( \frac{a_N}{b^*} - \frac{a^*}{b^*} \right) \right|
\]

\[
= \left| \frac{a_N}{b_N} \right| \left| \frac{1}{b^*} \right| \left| \sqrt{N} (b_N - b^*) \right| + \left| \frac{1}{b^*} \right| \left| \sqrt{N} (a_N - a^*) \right|
\]

by which the proposition follows from Lemma 16. \(\square\)