Static vacuum spacetimes with prescribed multipole moments

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Abstract. This paper gives sufficient conditions on a sequence of multipole moments for a static spacetime to exist with precisely these moments. We outline the proof, which is constructive in the sense that a metric having prescribed multipole moments up to a given order can be calculated. These sufficient conditions agree with already known necessary conditions, and hence this completes the proof of a long standing conjecture due to Geroch.

1. Introduction
There are various definitions of multipole moments, [6]. For static asymptotically flat spacetimes, the standard definition was given by Geroch [1], and this definition was later extended to the stationary case by Hansen [3]. In [1], Geroch made two conjectures, where the second reads

Conjecture 2: Given any set of multipole moments, subject to the appropriate convergence condition, there exists a static spacetime of Einstein’s equations having precisely those moments.

This conjecture, see theorem 1 below, was proved in [5]; here we only give the most important lemmas used and a short outline of the arguments used in the proof. For earlier and related results, we also refer to [5].

Basically, the result stated in theorem 1 is that the necessary conditions in [4] are also sufficient, i.e., if they are satisfied, there exists a static asymptotically flat vacuum spacetime with these moments. In short, this was possible to prove by casting the metric in a special form, which ensured both correct moments and also uniqueness of relevant variables. This determines the metric formally, and by referring to a result by Friedrich, [2], theorem 1 follows.

2. Multipole moments of stationary spacetimes
Since we refer to a result concerning stationary spacetimes from [4], we will quote the definition of multipole moments given by Hansen in [3], which is an extension to stationary spacetimes of the definition for static spacetimes by Geroch [1]. We thus consider a stationary spacetime \((M, g_{ab})\) with signature \((-+,+,+,+)\) and with a timelike Killing vector field \(\xi^a\). We let \(\lambda = -\xi^a\xi_a\) be the norm, and define the twist \(\omega\) through \(\nabla_a \omega = \epsilon_{abcd}\xi^b\nabla^c\xi^d\). If \(V\) is the 3-manifold of trajectories, the metric \(g_{ab}\) induces the positive definite metric

\[ h_{ab} = \lambda g_{ab} + \xi_a \xi_b \]
on $V$. It is required that $V$ is asymptotically flat, i.e., there exists a 3-manifold $\hat{V}$ and a conformal factor $\Omega$ satisfying

(i) $\hat{V} = V \cup i^0$, where $i^0$ is a single point

(ii) $\hat{h}_{ab} = \Omega^2 h_{ab}$ is a smooth metric on $\hat{V}$

(iii) At $i^0$, $\Omega = 0$, $\hat{D}_a \Omega = 0$, $\hat{D}_a \hat{D}_b \Omega = 2 \hat{h}_{ab}$.

where $\hat{D}_a$ is the derivative operator associated with $\hat{h}_{ab}$. $i^0$ is referred to as spacelike infinity\(^1\).

On $M$, and/or $V$, one defines the scalar potential

$$\phi = \phi_M + i \phi_J, \quad \phi_M = \frac{\lambda^2 + \omega^2 - 1}{4 \lambda}, \quad \phi_J = \frac{\omega}{2 \lambda}. \quad (1)$$

The multipole moments of $M$ are then defined on $\hat{V}$ as certain derivatives of the scalar potential $\hat{\phi} = \phi / \sqrt{\Omega}$ at $i^0$. More explicitly, following [3], let $\hat{R}_{ab}$ denote the Ricci tensor of $\hat{V}$, and let $P = \hat{\phi}$. Define the sequence $P, P_{a_1}, P_{a_1 a_2}, \ldots$ of tensors recursively:

$$P_{a_1 \ldots a_n} = C[\hat{D}_{a_1} P_{a_2 \ldots a_n} - \left(\frac{n - 1}{2}\right)(2n - 3) \hat{R}_{a_1 a_2} P_{a_3 \ldots a_n}], \quad (2)$$

where $C[\cdot]$ stands for taking the totally symmetric and trace-free part. The multipole moments of $M$ are then defined as the tensors $P_{a_1 \ldots a_n}$ at $i^0$.

In [1], a slightly different setup and a different potential is used, but it is known [7] that the potential used there and (1) with $\omega = 0$ produce the same moments.

3. Static spacetimes with prescribed multipole moments

Consider $\mathbb{R}^3$ with Cartesian coordinates $\mathbf{r} = (x, y, z) = (x^1, x^2, x^3)$, and let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index. With the convention that, in terms of components, $P_\alpha^0 = P_\alpha^{01 \ldots 122 \ldots 233 \ldots 3}$, we have the following theorem.

**Theorem 1.** Let $P^0, P_\alpha^0, P_{a_1 a_2}^0, \ldots$ be a sequence of real valued totally symmetric and trace free tensors on $\mathbb{R}^3$, and let $P_{01}, P_{11}^0, P_{111}^0, \ldots$ be the corresponding components with respect to the Cartesian coordinates $\mathbf{r} = (x, y, z)$. If $u(\mathbf{r}) = \sum |\alpha| \geq 0 \frac{\partial}{\partial \alpha} P_\alpha^0$ converges in a neighbourhood of the origin in $\mathbb{R}^3$, there exists a static asymptotically flat vacuum spacetime having the moments $P^0_0, P^0_{a_1}, P^0_{a_1 a_2}, \ldots$.

Note that we do not require the monopole $P^0_0 = \hat{\phi}(0)$ to be non-zero\(^2\). The proof, however, will first be carried out under the assumptions $P^0_0 \neq 0$, and this condition will then be relaxed in Section 3.7.

3.1. Outline of the proof

As mentioned in Section 1, it is possible to reduce the coordinate freedom in the metric components, and simultaneously establish a direct link between the potential $\phi$ and the desired moments. This will be done in Section 3.4, where the link is expressed in theorem 6.

In Section 3.5, we will address the conformal field equations from Section 3.3, and further restrict the form of the metric. This restriction ensures that the resulting conformal field equations, (4), (after splitting into regular and non-regular parts) have smooth extensions to $i^0$.

\(^1\) $i^0$ is also used in a four-dimensional context.

\(^2\) Due to the definition of $\phi$, $P^0_0 = -m$, where $m$ is the mass of the spacetime.
These tensor equations will then be written as a set of scalar equations, and if a certain subset of these equations are satisfied, the full metric will be formally determined (see lemma 7).

In Section 3.6 we will address the full set of equations, as well as the issue of convergence of the power series derived. By referring to a result by Friedrich, [2], convergence of the power series will be concluded. It will also be seen that the full set of equations are satisfied.

Finally, in Section 3.7, the condition that the monopole is nonzero will be relaxed.

3.2. Notation
Small Latin letters $a, b, \ldots$ refer to abstract indices, as in Section 2. Small Latin letters $i, j, k, \ldots$ are numerical indices and refer to components with respect to the normal coordinates $(x, y, z) = (x^1, x^2, x^3)$ introduced below. Since these components refer to this particular coordinate system only, the equations will not be tensor equations. With this said, we will still use $=$ instead of $\equiv$.

Almost all variables\(^3\) are assumed to be formally analytic, i.e., they admit a formal power series expansion (again in terms of the chosen coordinates), so that, for a tensor, $A_{ijk}$ say, we can write

$$A_{ijk} = \sum_{n=0}^{\infty} A_{ijk}^\langle n \rangle$$

where $A_{ijk}^\langle n \rangle$ denotes polynomials in $(x, y, z)$ which are homogeneous of order $n$ (and where the summation may be formal). Both $\eta_{ij}$ and $\eta^{ij}$ denote the identity matrix, and by $[A_{ij}]$ we denote the trace of $A_{ij}$, i.e., $\eta^{ij}A_{ij}$. We also use $\partial_i = \frac{\partial}{\partial x^i}$ and the operator $D = x^j \frac{\partial}{\partial x^j}$.

With $r = \sqrt{x^2 + y^2 + z^2}$, $f_1 \equiv f_2 \pmod{r^2}$ means that $f_1(x, y, z) - f_2(x, y, z) = r^2 g(x, y, z)$ for some (formally analytic) function $g$. When $f \equiv 0 \pmod{r^2}$, so that $f = r^2 g$ for some formally analytic function $g$, we also use the shorter notation $r^2[f]$.

In the proof of lemma 8 we will refer to [2] and hence use some of the notation there.

3.3. The field equations
Apart from having the correct multipole moments, we must ensure that the metric describes a static vacuum spacetime. We will formulate our equations on the 3-manifold $(V, h_{ab})$ defined in Section 2, starting with the field equations from [1]. However, the 3-manifolds in [1] and [3] are defined slightly differently, and the relations imply the following.

Starting with a static spacetime $(M, g_{ab})$ with timelike killing vector $\xi^a$, we put $0 < \lambda = -\xi^a \xi_a$ and $\Psi = 1 - \sqrt{\lambda}$. From [1] we consider\(^4\) a 3-surface $V_G$ orthogonal to $\xi^a$. In terms of the metric on $V_G$: $(h_G)_{ab} = g_{ab} + \xi_a \xi_b / \lambda$, the field equations are, [1],

$$\begin{cases} (D_G)^\alpha (D_G)_a \Psi = 0 \\ (R_G)_{ab} = \frac{1}{\Psi^2} (D_G)_a (D_G)_b \Psi \end{cases} \iff \begin{cases} (R_G)^\alpha_a = 0 \\ (R_G)_{ab} = \frac{1}{\sqrt{\lambda}} (D_G)_a (D_G)_b \sqrt{\lambda} \end{cases}$$

(3)

where $(D_G)_a$ is the derivative operator and $(R_G)_{ab}$ is the Ricci tensor associated with $(V_G, (h_G)_{ab})$.

On the 3-manifold $V$ on the other hand, the metric is $h_{ab} = \lambda g_{ab} + \xi_a \xi_b$, i.e., $h_{ab} = \lambda (h_G)_{ab}$, and with $\hat{h}_{ab} = \Omega^2 h_{ab}$, this implies that $\hat{h}_{ab} = (\sqrt{\lambda} \Omega)^2 (h_G)_{ab}$. We can now express equations (3) on $(\hat{V}, \hat{h}_{ab})$ using as conformal factor $\Omega = \sqrt{\lambda} \Omega$. Using the properties of conformal transformations,

\(^3\) The exception is $r = \sqrt{x^2 + y^2 + z^2}$.

\(^4\) In [1], the potential $\psi = -\Psi$ is used.
We will now return to the original recursion (2). By combining the previous lemmas, and using Taylor’s theorem, we have the following theorem, which allows for the direct connection between the moments $P_{a_1\ldots a_n}$ and the potential $P = \hat{\phi} = \sum_{|\alpha|\geq 0} \frac{r^\alpha}{\alpha!} P_0^\alpha$.

**Theorem 6.** Let $(x^1, x^2, x^3)$ be normal coordinates on $\hat{V}$, $\hat{h}_{ij} - \eta_{ij}$ satisfy $r^2|\eta^{ij}(\hat{h}_{ij} - \eta_{ij}) = r^2|\eta^{ij} \hat{h}_{ij} = r^2|\eta^{ij} \hat{h}_{ij}$, and let $P_{a_1\ldots a_n}$ be defined by the recursion (2). Then

$$x^{i_1} \ldots x^{i_n} P_{i_1\ldots i_n} \equiv x^{j_1} \ldots x^{i_n} \partial_{i_1} \ldots \partial_{i_n} P \pmod{r^2}$$

In short, if the coordinates $(x, y, z)$ are such that $\hat{h}_{ij}$ satisfies the conditions of lemma 4, and if the potential $P = \hat{\phi}$ is expressed in these coordinates, it is possible to specify the multipole moments.

### 3.5. The conformal field equations and uniqueness of the metric

In this section, we will address the conformal field equations (4), using a metric satisfying that conditions of lemma 4. Since (4) contains $\lambda$ which not formally smooth, it can be split into a regular and non-regular part (again, see [5]). To ensure smooth extensions of these to $i^0$, the form of $\hat{h}_{ij}$ will be restricted further, namely into the form

$$\hat{h}_{ij} = \eta_{ij} + f(x, y, z)(x_ix_j - r^2\eta_{ij}) + r^2\gamma_{ij}, \quad \gamma_{ij}x^i = 0. \quad (5)$$

Thus the task is to find a conformal factor $\hat{\Omega}$ and a metric $\hat{h}_{ij}$, defined in a neighbourhood of $i^0$, which both produces the desired multipole moments and also satisfies (4).
Under this metric, a set of scalar equations equivalent to (4) can be derived. The scalar components formed from (4), which are defined in [5], are denoted by $S, t_{00}, t_{11}, t_{22}, t_{33}, t_{01}, t_{02}, t_{03}$, and the vanishing of these is equivalent to (4) being satisfied.

In terms of these scalar quantities, one can prove the following lemma.

**Lemma 7.** Suppose that the metric components $h_{ij}$ takes the form (5). Then the equations

$$
S = 0, \quad t_{11} = 0, \quad t_{22} = 0, \quad t_{33} = 0
$$

determines the metric $h_{ij}$ as formal power series.

It is noteworthy that these equations give the metric components of the appropriate order explicitly as a polynomial expression in the desired/prescribed multipole moments and metric components of lower order.

### 3.6. Convergence of the metric

In this section we will assure that the series expansion for $h_{ij}$, which was found in the previous section, converges and also solves the full conformal field equations. This is the content of the following lemma.

**Lemma 8.** Suppose that $h_{ij}$ is a formal power series for the metric of the form (5), producing the moments of theorem 1. If

$$
S = 0, \quad t_{11} = 0, \quad t_{22} = 0, \quad t_{33} = 0
$$

then

$$
t_{00} = 0, \quad t_{0k} = 0, k = 1, 2, 3
$$

and the power series is convergent in a neighbourhood of $i^0$.

This will conclude the proof of theorem 1 when the monopole is nonzero.

To prove of this lemma, relies on a result by Friedrich, namely theorem 1.1 in [2], where a related property is investigated. In [2] it is proven that under rather similar settings, there exists a metric which instead of prescribed multipole moments, produces a set of prescribed null data, where also growth conditions for the null data in order for the series expansion of the metric to converge are given. These null data are proven to be in a one-to-one correspondence with the family of multipole moments with non-zero monopole, but since this correspondence is rather implicit, the actual conditions on the multipole moments (as required by the conjecture by Geroch) are not clear.

Using the notation in [2], the condition is roughly that there exist $M, r$ such that

$$
\left| C[\nabla_{a_1} \ldots \nabla_{a_3} R_{a_2 a_1}](i^0) \right| \leq \frac{M!}{r^p}, \quad a_1, \ldots a_3, b, c = 1, 2, 3, \quad p = 0, 1, 2, \ldots
$$

(6)

where $R_{a_2 a_1}$ is the Ricci tensor of [2], and the bold indices indicate that we are evaluating tensor expressions with respect to a certain frame. It is possible to prove that the conformal factor relating $h_{ab}$ and the metric used in [2] is smooth, and this also, starting with $\hat{\phi}$, allows us to specify a potential with desired moments in [2]. Finally, using induction, it is possible to show that the estimates which are known to hold for $\hat{\phi}$, namely that for some $M, \tilde{r}$, $|\partial_\alpha \hat{\phi}(0)| \leq \frac{p! M}{\tilde{r}^p}$, $p = |\alpha| = 0, 1, 2, \ldots$, implies the estimates in (6), for some constants $M, r$. From this lemma 8 and hence theorem 1 follow when $P^0 \neq 0$.  

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3.7. The case $P^0 = 0$

To complete the proof, the condition that $P^0 \neq 0$, i.e., $m \neq 0$ must be relaxed. This will be possible due to the fact that the conformal factor $\hat{\Omega}$ is well behaved for $m = 0$. Thus, suppose that we have a sequence of totally symmetric and trace-free tensors $P^0, P^0, P^0, \ldots$ as in theorem 1, and where $P^0 = 0$. In particular, it is assumed that $u(r) = \sum_{\alpha} r^P_\alpha$, $P^0$ converges in some polydisc $U = \{ (x, y, z) : |x| < d, |y| < d, |z| < d \}$. By replacing only the monopole, i.e., putting $P^0 = m_0 > 0$, the corresponding sequence $m_0, P^0, P^0, \ldots$ corresponds to the function $\hat{u}(r) = m_0 + \sum_{\alpha} r^P_\alpha$, which also converges in $U$. Thus, by the arguments given so far there exists convergent, in a polydisc $V$ say, power series

$$\hat{h}_{ij} = \sum_{|\alpha|\geq 0} (c_{ij})_\alpha r^\alpha$$

for the metric components; furthermore we also know that there is a static spacetime having the multipole moments $m_0, P^0, P^0, \ldots$. Now, from the recursion producing the metric, it is seen that each coefficient $(c_{ij})_\alpha$ is a polynomial in $m_0$, and hence the metric components $\hat{h}_{ij}$ can be regarded as a power series in the four variables $(m, x, y, z)$:

$$\hat{h}_{ij} = \sum_{|\beta|\geq 0} (d_{ij})_{\beta} m^{\beta_1} x^{\beta_2} y^{\beta_3} z^{\beta_4},$$

where the multi-index $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$. Since this series converges for $m = m_0$, $(x, y, z) \in V$, it also converges for $|m| < m_0$, $(x, y, z) \in V$. We can now choose $m = 0$ and still have convergence of $\hat{h}_{ij}$, $\kappa$ and $g_{ij}$. In particular, with $m = 0$ the multipole moments will be the initially desired sequence $P^0, P^0, P^0, \ldots$ where $P^0 = 0$.

4. Discussion

In this paper, we have outlined the proof of a long standing conjecture by Geroch, [1]. The full arguments are found in [5]. The conditions given are simple and natural. In essence, each allowed set of multipole moments is connected to a harmonic function on $\mathbb{R}^3$, defined in a neighbourhood of $0$. The proof is constructive in the sense that an explicit metric having prescribed moments up to a given order can be calculated.

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