Multi-form solitary wave solutions of the KdV-Burgers-Kuramoto equation

Clovis Taki Djeumen Tchaho, Hugues Martial Omanda, Gaston N’tchayi Mbourou, Jean Roger Bogning and Timoléon Crépin Kofané

Abstract

This work is dedicated to the construction of solitary wave solutions of the KdV-Burgers-Kuramoto equation. The peculiarity of the solutions obtained for this purpose is that they result from the combination of solitary waves of the bright and dark type thus generating multi-form solutions which are also called hybrid solitary waves. The Bogning-Djeumen Tchaho-Kofané method is used to obtain the results. The reliability and feasibility of these results are tested using numerical simulations.

1. Introduction

The universe in its complex formation is constituted by physical systems whose deployments generate nonlinear and exciting phenomena which arouse in a growing way the curiosity and the desire of comprehension by human beings. Thus, the human being in general and the physicist in particular model the dynamics of nonlinear phenomena by mathematical equations of all outputs, among which are the nonlinear differential equations. They vary most often according to the physical system studied [1–15]. If one thing is to get these equations, in order to analyze and understand the dynamics of these physical systems, another thing is to solve them and get solutions that are closer to reality. But the general observation is that nonlinear partial differential equations (NPDEs) are for the most difficult to integrate. This justifies the proliferation of techniques used by researchers to provide solutions to these different equations and the multiplicity of solutions sometimes resulting from the resolution of a single NPDE. It is in this perspective that physicists and mathematicians in recent years are multiplying the approaches and methods to study analytically and numerically these equations. This last decade and very recently new methods and approaches have been developed [16–28]. This approach will not stop because of many phenomena whose dynamics still escape our understanding. In this work, we subscribe to this approach to build multiple solitary wave solutions of the KdV-Burgers-Kuramoto equation (KBK) given by

\[ u_t + uu_x + \alpha u_{xxx} + \beta u_{xxxx} + \gamma u_{xxxxx} = 0, \]

where \( u_{xxxx} \) is the stability and energy dissipation term, \( u_{xxx} \) represents the dispersion term, \( u_{xx} \) accounts for the instability and energy production term, \( u_x \), the dominant nonlinear term and \( u(x, t) \) a function characterizing some physical processes in unstable systems [4], with \( \beta \) a parameter which measures the relative importance of dispersion, \( \alpha, \gamma \) are real constants [1], \( x \) the coordinate, \( t \) the time. It can be used to describe unstable draft waves in plasma [4], long waves in a viscous fluid flowing down along an inclined plane and turbulent cascade model in a barotropic atmosphere. For \( \beta = 0 \), equation (1) is referred too as the Kuramoto-Sivashinsky equation (KSE), which is a canonical nonlinear evolution equation arising in a variety of physical contexts [15].
The solutions of KBK equation possess their actual physical applications. The peculiarity of the solutions sought in this work comes from the fact that they result from the combination of solitary wave packages [29, 30].

To achieve our goal, the work that we propose in this article is organized as follows: in section 2, we return to the Bogning-Djeumen Tchaho-Kofané method that will be used to determine the solutions of the KBK equation. Section 3 proposes the different analytical solutions coupled to numerical simulations in order to verify the reliability and practical feasibility of these analytically results. Section 4 is devoted to discussions on the results obtained. Finally we end the work with a conclusion.

2. The Bogning-Djeumen Tchaho-Kofané method (BDKm)

It is a method developed on the theory of construction of nonlinear partial differential equation solutions [31–42]. This method is interested particularly the NPDEs of the form [31–42]

\[ H(u, u_t, u_{xx}, u_{tx}, u_{xxx}, u_{xxt}, u_{xxxx}, ..., |u|^2, (u|u|^2)_t, ..., |u|^3, (u|u|^2)_t, ...,) = 0, \]

where \( u(x, t) \) is an unknown function to be determined, \( H \) is some function of \( u \) and its derivatives with respect to \( x \) and \( H \) includes the highest order derivatives and the nonlinear terms. Under the travelling wave transformation \( u(x, t) = \psi(\xi), \xi = x - \nu t \) where \( \nu \) is the wave speed, equation (2) is transformed to the following ordinary differential equation (ODE)

\[ H_{ODE}(\psi, \psi', \psi'', ..., |\psi|^2, ...) = 0, \]

where \( \psi' \), \( \psi'' \) represent respectively the first and second derivatives of the envelope \( \psi \) with respect to \( \xi \). We construct the solutions of equation (3) in the form

\[ \psi(\xi) = \sum_{ij} b_{ij} \left( \frac{\sinh \alpha \xi^j}{\cosh \alpha \xi^j} \right), \]

where \( \alpha \) is a real constant and \( b_{ij} \) are the unknown constants to be determined. Thus, inserting equation (4) into equation (3) permits to have the main equation under the form

\[ \sum_{ijm} F(b_{ij}, \alpha, \nu) + \sum_{ijn} G(b_{ij}, \alpha, \nu) \sinh \alpha \xi + \sum_{ijk} V(b_{ij}, \alpha, \nu) \cosh \alpha \xi + \sum_{ijl} T(b_{ij}, \alpha, \nu) \sinh \alpha \xi \cosh \alpha \xi + \sum_{ijkl} W(b_{ij}, \alpha, \nu) = 0, \]

where \( i, j, k, l \) are positive natural integers and \( n, m \) the real numbers [29, 30]. Through equation (5), we obtain, the coefficients \( b_{ij}, \alpha, \nu \) and some constraints that may result, by identify at zero the functions \( F(b_{ij}, \alpha, \nu), G(b_{ij}, \alpha, \nu), V(b_{ij}, \alpha, \nu), T(b_{ij}, \alpha, \nu), W(b_{ij}, \alpha, \nu) \). Thus, the ansatz given in equation (4) can be supported by equation (2) as a solution. It is worth mentioning that, inside the sets of obtained equations, priority is given to the equations which are to the powers \( \frac{1}{(\cosh \alpha \xi)^n} \) and \( \frac{\sinh \alpha \xi}{(\cosh \alpha \xi)^m} \) from the most elevated. This method was initially used to propose solitary solutions of the modified Kuramoto–Sivashinsky equation [31, 32, 35] while, it was recently used in [40–42] to construct solitary waves which propagate through transmission media such as electrical lines. The BDKm thereafter permitted to successfully obtain pulse solutions of the simplified Korteweg–de Vries (KdV) equation and nonlinear Schrödinger equation [36, 37]. In the following we applied this method in resolution of KBK equation.

3. Results

3.1. Analytical solitary wave solutions

In this subsection, we apply the BDKm to construct the new analytical solitary wave solutions of the KBK equation. Using the transformation \( u(x, t) = \psi(\xi), \xi = x - \nu t \) where \( \nu \) is the wave speed, we obtain from equation (1), the travelling wave equation

\[ \nu \psi' - \psi' + \alpha \psi' - \beta \psi'' - \gamma \psi''' = 0, \]

where \( \psi' \), \( \psi'' \) and \( \psi''' \) denote first, second, third and fourth order partial derivatives of \( \psi \) with respect to \( \xi \), respectively. We focus on the case \( \gamma / \delta = 0 \) in equation (1). Based on the BDKm largely explained in [31–42], equation (6) may have the following ansatz (which can be supported by equation (1) as a solution)

\[ \psi(\xi) = a \text{sech}^\theta \xi \tan \theta \xi + b \text{sech}^\theta \xi \tan h \theta \xi + c \text{sech}^\theta \xi \tan h^2 \theta \xi, \]

where \( a, b, c \) are real constants to be determined later, \( h \), the inverse of the width of each component (hybrid solitary wave) of the ansatz. The concept of solitary multi-form wave is explained by the fact that each term of
equation (7) constitutes a package [29, 30] of analytical forms of solitary waves, or solitary waves belonging to a classical family of solitary waves. To be fixed with respect to this explanation, we can notice that sech$^2\theta\xi$ tanh $\theta\xi$ is simple and can not undergo another transformation while [29, 30],

$$\text{sech}^2\theta\xi \tanh \theta\xi = \text{sech}^2\theta\xi - 2\text{sech}^2\theta\xi + \text{sech}^2\theta\xi$$  \hspace{1cm} (8)

and

$$\text{sech}^2\theta\xi \tanh \theta\xi = \text{sech}^2\theta\xi \sinh \theta\xi - 2\text{sech}^2\theta\xi \sinh \theta\xi + \text{sech}^2\theta\xi \sinh \theta\xi.$$  \hspace{1cm} (9)

The transformations of equations (8) and (9) clearly show that equation (8) is constituted of a bright wave package and equation (9) is constituted of a package solitary waves of dark type. Thus, the insertion of equation (7) into equation (6) leads to the coefficient range equation $a$, $b$ and $c$ whose identification of the coefficients of terms in $\text{sech}^j\theta\xi$ ($j = 2; 4; 6; \ldots; 14$) and $\sinh \theta\xi \text{sech}^i\theta\xi$ ($i = 1$ and $k = 3; 5; 7; \ldots; 15$) at zero leads to series of following equations the term in $\text{sech}^4\theta\xi$,

$$-250bc = 0,$$  \hspace{1cm} (10)

. the term in $\text{sech}^2\theta\xi$,

$$560bc - 110ab = 0,$$  \hspace{1cm} (11)

. the term in $\text{sech}^0\theta\xi$,

$$280ab + (5040 \beta - 3024 \theta^3 \gamma)b - 940bc = 0,$$  \hspace{1cm} (12)

. the term in $\text{sech}^4\theta\xi$,

$$2100^3\beta a + (5880 \theta^3 \gamma + 420^2 \alpha - 4480 \beta^3 \gamma)b + (7 \theta^3 \nu - 6720 \beta \gamma)c + 760ab - 20bc = 0,$$  \hspace{1cm} (13)

. the term in $\text{sech}^6\theta\xi$,

$$(5 \theta^3 \nu - 260 \theta \gamma^3) a - (760^2 \alpha + 3496 \theta \gamma^3) b + (7960 \beta \gamma^3 - 16 \theta \nu)c + 66ab - 290bc = 0,$$  \hspace{1cm} (14)

. the term in $\text{sech}^8\theta\xi$,

$$(640 \beta^3 - 40 \theta \nu) a + (6320 \theta^2 \gamma + 38 \theta^2 \alpha) b + (11 \theta^3 \nu - 188 \beta \gamma)c + 40bc = 0,$$  \hspace{1cm} (15)

. the term in $\sinh \theta\xi \text{sech}^2\theta\xi$,

$$(8 \beta^2 - 2 \theta \nu) c - (4 \beta^2 \alpha + 16 \theta \nu^2) b = 0,$$  \hspace{1cm} (16)

. the term in $\sinh \theta\xi \text{sech}^4\theta\xi$,

$$-7 \theta c^2 = 0,$$  \hspace{1cm} (17)

. the term in $\sinh \theta\xi \text{sech}^6\theta\xi$,

$$30 \theta c^2 - 12 \theta ac = 0,$$  \hspace{1cm} (18)

. the term in $\sinh \theta\xi \text{sech}^8\theta\xi$,

$$30 \theta ac - 5 \theta a^2 - 50 \theta c^2 = 0,$$  \hspace{1cm} (19)

. the term in $\sinh \theta\xi \text{sech}^{10}\theta\xi$,

$$4 \theta a^2 - 1680 \theta^4 \gamma a + 40 \theta c^2 + 5376 \theta^4 \gamma c - 240 \theta ac = 0,$$  \hspace{1cm} (20)

. the term in $\sinh \theta\xi \text{sech}^{12}\theta\xi$,

$$(1560 \theta^4 \gamma + 30 \theta^2 \alpha) a - 150 \theta c^2 + (960 \theta^2 \alpha - 4776 \theta^4 \gamma)c + 60 \theta ca = 0,$$  \hspace{1cm} (21)

. the term in $\sinh \theta\xi \text{sech}^{14}\theta\xi$,

$$(-256 \theta^4 \gamma - 16 \theta^2 \alpha) a + 20 \theta c^2 + (7520 \theta^4 \gamma + 440 \theta^2 \alpha)c = 0,$$  \hspace{1cm} (22)

. the term in $\sinh \theta\xi \text{sech}^{16}\theta\xi$,

$$(-16 \theta^4 \gamma - 4 \theta^2 \alpha)c = 0,$$  \hspace{1cm} (23)

from which expansion coefficients $a$, $b$ and $c$ can be determined under certain conditions satisfied by parameters $\alpha$, $\beta$, $\theta$ and $\gamma$. One notices that, from equation (10), it is easy to determine

$$b = 0, \text{ or } c = 0,$$  \hspace{1cm} (24)

So, in order to obtain nontrivial solutions, the set of algebraic equations about the expansion coefficients $a$, $b$ and $c$, can be solve according to the following important cases which give rise to different families of solutions: $b \neq 0$, and $c \neq 0; b = 0$, and $c = 0; b = 0$, and $c \neq 0; b = 0$, and $c = 0$. However, when one fixes one of the values of these constants, the major terms or coefficients vary. For instance, if $c = 0$, equations (10) and (11) lead to trivial solutions and consequently only the equation obtaining from the term in $\text{sech}^{10}\theta\xi$ and $\text{sech}^{8}\theta\xi$ give nontrivial solutions.
3.1.1. First family; case: \( b = 0, \) and \( c \neq 0 \)
From equations (11), (12) and (13), the expansion coefficients \( a, b \) and \( c \) can be determined as

\[
a = \frac{28224\theta^2\gamma^2 - 4704\theta^2\beta}{89},
\]

\[
c = \frac{5544\theta^3\gamma - 924\theta^2\beta}{89},
\]

and

\[
b = \frac{183456\theta^3\beta^2 + (3234\theta^3\beta - 19404\theta^2\gamma)\nu - 1100736\theta^2\gamma\beta}{1328628\theta^2\gamma^2 - 1977649\beta^2 + 1869\alpha},
\]

with \( \gamma = \frac{1977649\beta^2 - 1869\alpha}{1328628\theta^2\gamma^2}. \) Thus, the first family of solutions can be written as

\[
\psi(\xi) = a \text{sech}^4\theta\xi \tanh \theta\xi + b \text{sech}^2\theta\xi \tanh^2 \theta\xi + \frac{5544\theta^3\gamma - 924\theta^2\beta}{89} \text{sech}^2\theta\xi \tanh^3 \theta\xi,
\]

where \( a \) and \( b \) are given by equations (25) and (27), respectively. Here, it is important to emphasize that equation (28) is the prototype of hybrid analytical solitary wave solutions resulting from the superposition of two sub-families solutions bright and dark (see figures 1 and 2).

3.1.2. Second family; case: \( b = 0, \) and \( c = 0 \)
Under the condition: \( b = 0, c = 0, \) the first set of algebraic equations about the expansion of coefficients \( a, b \) and \( c \) in terms \( \text{sech}^j \theta\xi \) \((j = 2; 4; 6; \ldots; 14)\) leads to \( a = 0 \) and consequently we obtain trivial solutions which are not important. As explained above, the second set of algebraic equations in terms \( \sinh \theta\xi \text{sech}^i \theta\xi \) \((i = 1 \text{ and } k = 3; 5; 7; \ldots; 15)\) is that enables to obtain the values of constant \( a. \) So, only equation in the term \( \sinh \theta\xi \text{sech}^i \theta\xi \) permits to obtain nontrivial solutions and one obtains the following algebraic equation

\[
4\theta a^2 - 1680\theta^4\gamma a = 0.
\]

From equation (29) it is easy to find

\[
a = 0, \text{ or } a = 420\theta^3\gamma.
\]

Thus, the second family solutions (nontrivial solutions) of equation (1) are

\[
\psi(\xi) = 420\theta^3\gamma \text{sech}^4\theta\xi \tanh \theta\xi.
\]

This result shows that the KBK equation has hybrid solitary wave solutions which are a product of the sech function to power four and tanh function (bright and kink product). One may see this by considering the wave profiles obtained in section 3.2. (figure 3).

3.1.3. Third family; case: \( b = 0, \) and \( c = 0 \)
If: \( b = 0, \) and \( c = 0, \) the first set of algebraic equations about the expansion of coefficients \( a, b \) and \( c \) in the terms \( \text{sech}^j \theta\xi \) \((j = 2; 4; 6; \ldots; 14)\) leads to \( a = c = 0 \) and consequently we obtain trivial solutions which are not important. As explained above, the second set of algebraic equations in terms \( \sinh \theta\xi \text{sech}^i \theta\xi \) \((i = 1 \text{ and } k = 3; 5; 7; \ldots; 15)\) is that enables to obtain the values of constants \( a \) and \( c. \) Thus, from equation (17) it is easy to find \( c = 0 \) which is the case of the second family solutions obtained in subsection. 3.2 while, the combination of equations (18) and (19) leads to \( a = c = 0 \) which are trivial solutions and consequently only the equations obtained from the terms in \( \sinh \theta\xi \text{sech}^i \theta\xi \) to the terms in \( \sinh \theta\xi \text{sech}^i \theta\xi \) give the expressions of constants \( a \) and \( c. \) So from equation (23) we obtain parameters satisfying the following formula

\[
\alpha = -4\theta^2\gamma.
\]

So substituting equation (32) into equation (22) results the algebraic equation

\[
a = \frac{c^2 + 288\theta^3\gamma c}{96\theta^2\gamma}.
\]

By taking into account equation (33) into equation (21) with \( \alpha = -4\theta^2\gamma, \) we obtain the quadratic equation in unknown coefficient \( c \)

\[
c^2 + 288\theta^3\gamma c - 864\theta^6 - 288\theta^4\gamma^2 = 0.
\]

Equation (34) is a quadratic equation in \( c. \) So, we derive the discriminant

\[
\Delta = (750^2 + 1)\theta^4\gamma^2.
\]

One notices that \( \Delta \geq 0 \) due to the pair powers of \( \theta \) and \( \gamma. \) Consequently it is easy to obtain the coefficient \( c \) as
\[ c = \frac{-288\theta^2\gamma \pm \theta^2\gamma\sqrt{75\theta^2 + 1}}{2}. \]  

(36)

So the third family solutions of equation (1) can be written under the form
\[ \psi = a\text{sech}^4\theta\xi \tanh \theta\xi + \frac{-288\theta^2\gamma \pm \theta^2\gamma\sqrt{75\theta^2 + 1}}{2} \tanh^3\theta\xi\text{sech}^2\theta\xi, \]

(37)

where \( a \) is given by equation (33) which takes into account equation (36). One can notice that in the case of this third family, it is a question of multiform analytical solutions induced by the combination of two sub-families of hybrid solutions of dark and dark types respectively which, according to the values of parameters \( a, c, \alpha, \theta \) and \( \gamma \) in turn generates the multi-form solutions of the bright-dark type or another hybrid solitary wave solution (see figure 4).

3.1.4. Fourth family; case: \( b \neq 0 \), and \( c = 0 \)

If \( b \neq 0 \), and \( c = 0 \), equations (10) and (11) lead to the trivial solutions which are not important. So, from the remaining part of range equation of coefficients, we obtain the term in \( \text{sech}^{10}\theta\xi \)
\[ 28\theta ab + (5040\beta^3 - 3024\theta^4\gamma)b = 0, \]

(38)

. the term in \( \text{sech}^9\theta\xi \)
\[ 210\theta^3\beta a + (5880\theta^5\gamma + 420\theta^2\gamma - 448\theta^3\beta)b + 76\theta ab = 0, \]

(39)

. the term in \( \text{sech}^8\theta\xi \)
\[ (50\nu - 260\theta^3\beta)a - (76\theta^2\gamma + 3496\theta^4\gamma)b + 60\theta ab = 0, \]

(40)

. the term in \( \text{sech}^7\theta\xi \)
\[ (64\theta^3\beta - 40\nu)a + (632\theta^4\gamma + 38\theta^2\gamma)b = 0, \]

(41)

. the term in \( \text{sech}^6\theta\xi \)
\[ -(4\theta^2\gamma + 16\theta^4\gamma)b = 0, \]

(42)

From equation (42), it is easy that \( \alpha, \beta \) and \( \theta \) satisfy the following condition
\[ \alpha = -4\theta^2\gamma, \]

(43)

One notices that this result is the same as in equation (32). The resolution of equation (38) and equation (39), according to equation (43) permits to find
\[ a = \frac{756\theta^5\gamma - 126\theta^3\beta}{7}, \]

(44)

and
\[ b = \frac{6615\theta^2\beta^2 - 3969\theta^3\beta\gamma}{24360\theta\gamma - 3178\beta}, \]

(45)

with the constraint \( \gamma = \frac{15893}{121008} \). Taking into account equations (44) and (45) in equations (40) and (41), we obtain the condition satisfy by \( \nu, \beta, \theta \) and \( \gamma \) as follows
\[ \nu = \frac{88\theta^3\beta + 16\theta^2\beta^2}{18\theta\gamma + \beta}, \]

(46)

with \( \beta = -18\theta\gamma \). So the fourth family of solutions of equation (1) can be written as
\[ \psi(\xi) = \frac{756\theta^5\gamma - 126\theta^3\beta}{7} \text{sech}^9\theta\xi \tanh \theta\xi + \frac{6615\theta^2\beta^2 - 3969\theta^3\beta\gamma}{24360\theta\gamma - 3178\beta} \text{sech}^7\theta\xi \tanh^3\theta\xi, \]

(47)

with \( \gamma = \frac{15893}{121008} \) and \( \beta = -18\theta\gamma \). Equation (47) is the family of multi-form analytical solutions resulting from the superposition of two sub-families solutions of the dark and bright types respectively.

It should be noted that all families of analytical solutions obtained in this section are different from those found in [11, 17, 18, 24–28, 34]

3.2. Numerical simulations
Now, we test the feasibility, reliability and even robustness of some of the solutions obtained in this work through intense numerical simulations that will allow their observations during laboratory propagation tests. Thus, they may be useful in detecting so many new phenomena that are simultaneously involved in non-linearity, dissipation, dispersion and instability. To achieve this, we used the MATLAB toolbox pdepe [43] which
Figure 1. Stable dynamics of a multi-form solutions given by equation (28): dark-bright solitary waves: Top row: (Left): 
\[ a = -0.00018605; b = -0.02; c = -0.5; \theta = 0.02; \beta = \gamma = 0.01; \alpha = -0.226; \nu = 16.0441. \] 
(Left): \[ b = 0.01; c = 0.5; \alpha = 0.5069. \] Other parameters as on (Left). Bottom row: (Left): 
\[ a = 0.0008; b = 0.6; c = 0.00015714; \alpha = 0.2; \beta = -0.0336; \nu = 128.04657. \] Other parameters as in top row. (Right): \[ a = -0.0007; b = -0.5; c = 0.00013750; \beta = 0.0343; \nu = -56.8304. \] Other parameters as on (Left).

Figure 2. Robust dynamics of dark-bright solitary waves from equation (28): Top row: (Left): 
\[ a = -0.00068237; b = 0.05; \alpha = -0.04; \beta = -0.01; \alpha = 0.4374; \gamma = -0.3523. \] Other parameters as in figure 1 Top row: (Left). (Right): \[ b = -0.051; c = 0.041; \alpha = -0.2724. \] Other parameters as in figure 1 Top row: (Left). Bottom row: (Left): \[ a = -0.0013; b = 0.4; c = 0.3; \gamma = -0.4231; \alpha = 0.0466; \nu = -18.0441. \] Other parameters as in figure 1 top row: (Left). (Right): \[ a = 0.0013; b = -0.5; c = -0.4; \alpha = -0.2174; \gamma = 0.4231; \beta = -0.01. \] Other parameters as in figure 1 top row: (Left).
solves initial-boundary value problems for parabolic-elliptic PDEs in 1-D with zero flux boundary conditions. We also used large enough spatial grids in order to limit reflections at the boundaries which could bring spurious effects. These boundary conditions well suit the profiles of the solutions investigated in this work as there are not identical at the two boundaries, instead of the well known periodic boundary conditions which imply that when a wave passes through one end of the computational spatial grid it reappears on the opposite end with the same properties.

It is worth mentioning that, all the curves obtained in this section were from the same approach. That is to say, one fixes some parameters of the system modeled by equation (1) and those of the wave given by the ansatz (7) and one deduces the others. As an example, in figure 1 Top row (Left), when fixed the parameters $b; c; \theta; \gamma; \beta$ to

![Figure 3](image1.png)

**Figure 3.** Stable dynamics of a bright-dark solitary waves given by equation (31): (Left): $a = -0.000 \, 8; b = c = 0; \gamma = -0.004 \, 8; \alpha = -0.02; \nu = -3.2$. Other parameters as in figure 2 bottom row-(right). (Right): $a = -0.000 \, 873 \, 6; \gamma = 0.005 \, 2; \beta = -0.012; \nu = -4.2$. Other parameters as on the (Left).

![Figure 4](image2.png)

**Figure 4.** Stable spatiotemporal evolution of a bright-dark solitary waves given by equation (37): Top row: (Left): $a = 0.000 \, 8; b = 0; \epsilon = -0.034 \, 6; \gamma = 30; \alpha = -0.048; \nu = -2$. Other parameters as in figure 2 bottom row: (left). (Right): $a = -0.009 \, 2; \gamma = 8; \nu = -5; \alpha = -0.012 \, 8$. Other parameters as on (left). Bottom row: (Left): $c = -0.002; \gamma = 1.75; \alpha = -0.002 \, 8$. Other parameters as on Top row: (left). (Right): $c = 0.019 \, 6; \alpha = -0.027 \, 2; \gamma = -17$. Other parameters as in top row: (Right):
−0.02; −0.5; 0.02; 0.01; 0.01, respectively, then derived \( a = −0.000 186 05; \alpha = −0.226 0; \nu = 16.044 1 \) from equations (25), (26), (27) respectively. These profiles are different from those obtained in [17, 18, 24–28, 34].

4. Discussions

The numerical study carried out allowed to obtain profiles of the multi-form solitary wave solutions or hybrid solitary wave solutions resulting from the combination of solitary wave packages [29, 30] of the dark solitary wave family and the bright solitary wave family. The concept package is used here because on the one hand the second term of equation (7) \( \text{sech}^2 \theta \xi \tan h^4 \theta \xi \) can be decomposed into a sum of analytic sequences specific to bright solitary waves and on the other hand the third term \( \text{sech}^2 \theta \xi \tan h^2 \theta \xi \) can also be broken down into a sum of analytic sequences specific to solitary waves of the dark type. We also use the notion of solitary wave family because in a common way, the sech analytic sequence is known as that which indicates the fundamental solitary wave of bright type and the analytical sequence tanh is known as that which indicates the fundamental solitary wave of dark type. In contrast to the previous work and the different representations of solitary wave profiles obtained in the past [17, 18, 24–28, 34], the analytical sequence \( \text{sech}^2 \theta \xi \tan h^4 \theta \xi \) is that of a family of solitary waves of bright type or of the large family \( \text{sech}^n \theta \xi (n > 0) \) and therefore the representative is \( \text{sech} \theta \xi \) and the analytic representation \( \text{sech}^2 \theta \xi \tan h^2 \theta \xi \) of that of a dark type solitary wave or the great family \( \sinh \theta \xi \text{sech}^2 \theta \xi (n > 0) \) and therefore the representative is \( \text{tanh} \theta \xi \). Figures 1 and 2 show the hybrid solitary wave profiles resulting from the combination of the solitary wave packets of the dark family and the bright family, the analytical sequence of which is given by equation (28). One notices that, the solutions obtained in the bottom row of figure 1 are symmetric with respect to the plane \( \gamma = 0 \). In contrast, figure 3 shows the hybrid character of one of these families represented by the first term of equation (7) \( \text{sech}^2 \theta \xi \tan h \theta \xi \). Further on, figure 4 presents hybrid solitary wave profiles derived from the combination of solitary wave packets of the dark and bright family whose analytic sequence is given by equation (37). One also notices that, by setting \( \gamma = −1 \), then derived from equations (32) and (34) \( \alpha = −0.016; \nu = 0.001 2 \), other parameters as in figure 4 Top row (Right), one obtains the same profile as in figure 3, (Right), which we did not consider necessary to plot again. When focusses on figure 4, it is easy to observe that \( \gamma \) increases \( \gamma \in \{−17; 1.75; 8; 30\} \) with the structure formation of a multi-form solitary wave solution from Bottom row (Right) to Top row (Left). Thus, this observation allows to choose in a simple manner the solution profile (already obtained in the above figures) by acting on the \( \gamma \) value. So, it would be useful and would also save time in the laboratory during the study of propagation test. One hopes that hybrid solitary wave may in general found its application field in telecommunication technologies and will further revolutionize the energy and information transport.

5. Conclusion

In this work, we have investigated approximate ‘New’ multi-form solitary wave solutions of the KdV-Burgers-Kuramoto equation. By using the straightforward Bogning-Djoumen Tchaho-Kofane method based on the construction of solitary wave solutions of certain types of Nonlinear Partial Differential equations (which are an appropriate models to describe many phenomena which are simultaneously involved in non-linearity, dissipation, dispersion and instability (self-excitation), especially at the description of turbulence process), we have proposed multi-form solitary wave solutions that consist of a combination of hybrid solitary waves called bright–dark solitary wave profiles. One reliable feature of the solutions proposed here is the possibility to alter the amplitude of each individual hybrid solitary wave or the possibility to alter the value of \( \gamma \) allowing to generate ‘new’ multi-form structures. The analytical predictions are confirmed by the numerical integrations of equation (1). By further numerical simulations, we have also shown that different robust multi-form structures might be obtained when one inserts as initial conditions the combined hybrid solitary waves proposed in equation (7) with relative values of amplitudes \( a; b \) and \( c \). Some solutions expressed as travelling waves (solitary waves, periodic waves and so on) of the KKB equation have been found in [11, 17, 18, 24–28, 34] which are different from those obtained in the present work. According to the obtained profiles in figure 4, one could numerically track as a function of \( \gamma \) the bifurcations of the solutions. So that our result may be useful in detecting many new phenomena that are simultaneously involved in non-linearity, dissipation, dispersion and instability. However, many issues suggest that there are the dynamic properties of the analytical solutions which remain to be unveiled in details such as the bifurcation theory for dynamical systems may be used to precise exact explicit parametric representations of the dynamical solutions [28]. In the same vein, further development and analysis should be used to unveiled many awaiting issues.
Acknowledgments

The authors gratefully acknowledge support from the both Ministries of Higher Education of Cameroon and Gabon through their programs of support to research, which enable us to carry out this work. The constructive criticisms of the reviewers have been of a considerable contribution to the improvement of the quality of this work. That they find here the sincere thanks of the authors.

ORCID iDs

Clovis Taki Djeumen Tchaho  https://orcid.org/0000-0003-2599-8148

References

[1] Kawahara T 1983 Formation of saturated solitons in a nonlinear dispersive systems with instability and dissipation Phys. Rev. Lett. 51 381–3
[2] Kuramoto Y 1980 Instability and turbulence of wave fronts diffusion in reaction systems Prog. Theor. Phys. 63 1885–903
[3] Conte R and Musette M 1989 Painlevé analysis and Backlund transformation in the Kuramoto-Sivashinsky equation J. Phys. A: Math. Gen. 22 169–77
[4] Sivashinsky G I 1983 Instabilities, pattern formation and turbulence in flames Annu. Rev. Fluid Mech. 15 179–99
[5] Kuramoto Y and Tsuzuki T 1976 Persistent propagation of concentration waves in dissipative media far from thermal equilibrium Prog. Theor. Phys. 55 356–69
[6] Fu Y and Liu Z 2010 Persistence of travelling fronts of KdV-Burgers-Kuramoto equation Appl. Math. Comput. 216 2199–206
[7] Helal M A and Seadawy A R 2009 Variational method for the derivative nonlinear Schrödinger equation with computational applications Phys. Scr. 80 035004
[8] Khater A H, Callebaut D K and Seadawy A R 2006 General soliton solutions for nonlinear dispersive waves in convective type instabilities Phys. Scr. 74 384–93
[9] Seadawy A R, Arsalsh M and Lu D 2017 Stability analysis of new exact travelling-wave solutions of new coupled KdV and new coupled Zakharov-Kuznetsov systems Eur. Phys. J. Plus 132 162
[10] Seadawy A R 2017 Two-dimensional interaction of a shear flow with a free surface in a stratified fluid and its solitary-wave solutions via mathematical methods Eur. Phys. J. Plus 132 518
[11] Kudryashov N A 1990 Exact solutions for the generalized Kuramoto-Sivashinsky equation Phys. Lett. A 147 287
[12] Sayed S M, Elhamahmy O O and Gharib G M 2008 Travelling wave solutions for the KdV-Burgers-Kuramoto equation and the nonlinear Schrödinger equation which describe pseudospherical surfaces J. Appl. Math. 57 678310
[13] Hossen M R, Roshid H-O and Ali M Z 2018 Characteristics of the solitary waves with interaction phenomena in a (2+1)-dimensional Breaking Soliton equation Phys. Lett. A 382 1268–74
[14] Belobo Belobo D, Ben-Bolie G H and Kofané T C 2015 Dynamics of kink, antikink, bright, generalized Jacobi elliptic function solutions of matter-wave condensates with time-dependent two- and three-body interactions Phys. Rev. E 91 042902
[15] Medenov J and Simões A 2001 Heuristic model for the energy spectrum of phase turbulence Phys. Rev. E 64 057301
[16] Kudryashov N A and Soukharev M B 2009 Popular ansatz methods and solitary wave solutions of the Kuramoto-Sivashinsky equation Regul. Chaotic Dyn. 14 407–19
[17] Xie S M, Zhu S and Su K 2009 Solving the KdV-Burgers-Kuramoto equation by a combination method Int. J. Modern Phys. B 23 2101–6
[18] Kim J M and Chun C 2012 New exact solutions to the KdV-Burgers-Kuramoto equation with the exp-function method Abstr. Appl. Analys. 892420 10
[19] Roshid H-O and Rahman M A 2014 The exp(−f(φ(x))) -expansion method with application in the (1+1)-dimensional classical Boussinesq equations Results in Physics 4 150–5
[20] Seadawy A R 2015 Approximation solutions of derivative nonlinear schroedinger equation with computational applications by variational method Eur. Phys. J. Plus 130 182
[21] Hossen M B, Roshid H-O and Ali M Z 2017 Modified double sub-equation method for finding coximation solutions to the (1+1)-dimensional nonlinear evolution equations Int. J. Appl. Comput. Math. 3 1–19
[22] Harun-Or-Roshid 2019 Multi-soliton of the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation and KdV equation Computational Methods for Differential Equations 7 86–95
[23] Roshid H-O 2019 Kink type travelling wave solutions of right-hand non-commutative burgers equations via extended –expansion method Physical Science International Journal 21 1–6
[24] Fu Z T, Liu S K and Liu S D 2005 New exact solutions to the KdV-Burgers-Kuramoto equation Chaos Soliton Fractals 23 609–16
[25] Jiang L, Cheng X, Fu Z T, Liu S K and Liu S D 2006 Periodic solutions to KdV-Burgers-Kuramoto equation Commun. Theor. Phys. (Beijing, China) 45 815–8
[26] Fu Z T, Liu S D and Liu S K 2005 New exact solutions to the KdV-Burgers-Kuramoto equation Chaos Soliton Fractals 23 609
[27] Zhang S 2006 New exact solutions of the KdV-Burgers-Kuramoto equation Phys. Rev. Lett. A 358 414–20
[28] Yang L, Lu X and Tang X 2015 Exact travelling wave solutions for the generalized KdV-Burgers-Kuramoto equation J. Math. Sci. Adv. Appl. 31 1–13
[29] Bogning J R 2019 Mathematics for Nonlinear Physics: Solitary Waves in the Center of Resolutions of Dispersive Nonlinear Partial Differential equations (USA: Dorrance Publishing Co.)
[30] Bogning J R 2019 Mathematics for Nonlinear Physics: The Implicit Bogning Functions and Applications (Germany: Lambert Academic Publishing)
[31] Djeumen Tchaho C T, Bogning J R and Kofané T C 2010 Construction of the analytical solitary wave solutions of modified Kuramoto-Sivashinsky equation by the method of identification of coefficients of the hyperbolic functions Far East J. Dyn. Syst. 14 14–7
[32] Djeumen Tchaho C T, Bogning J R and Kofané T C 2011 Multi-Soliton solutions of the modified Kuramoto-Sivashinsky equation by the BDK method Far East J. Dyn. Syst. 15 83–98
[33] Bogning J R, Djeumen Tchaho C T and Kofané T C 2012 Construction of the soliton solutions of the Ginzburg–Landau equations by the new Bogning-Djeumen Tchaho-Kofané method Phys. Scr. 85 025013–7
[34] Djeumen Tchaho C T, Omanda H M and Belobo Belobo D 2018 Hybrid solitary waves for the generalized Kuramoto-Sivashinsky equation Eur. Phys. J. Plus 133 387
[35] Djeumen Tchaho C T, Bogning J R and Kofané T C 2012 Modulated soliton solution of the modiﬁed Kuramoto–Sivashinsky’ equation Amer. J. Comput. Appl. Math. 2 218–24
[36] Bogning J R, Djeumen Tchaho C T and Kofané T C 2012 Generalization of the Bogning-Djeumen Tchaho-Kofané Method for the construction of the solitary waves and the survey of the instabilities Far East J. Dyn. Sys. 20 101–11
[37] Djeumen Tchaho C T 2015 New method of construction of the solitary wave solutions of some physical nonlinear partial differential equations Doctorat/Ph, D Thesis University of Yaounde I (Cameroon)
[38] Njikue R, Bogning J R and Kofané T C 2018 Exact bright and dark solitary wave solutions of the generalized higher-order nonlinear Schrödinger equation describing the propagation of ultra-short pulse in optical fiber J. Phys. Commun. 2 025030
[39] Bogning J R 2018 Exact solitary wave solutions of the (3+1) modiﬁed B-type Kadomtsev-Petviashvili family equations Amer. J. Comput. Appl. Maths. 8 85–92
[40] Tiague Takongmo G and Bogning J R 2018 Construction of solutions in the Shape (Pulse; Pulse) and (Kink; Kink) of a set of two equations modeled in a nonlinear inductive electrical line with crosslink capacitor Amer. J. Cirt., Syst. Signal Proc. (AIS) 4 28–35
[41] Tiague Takongmo G and Bogning J R 2018 Construction of Breather soliton solutions of a modeled equation in a discrete nonlinear electrical line and the survey of modulational Instability J. Phys. Commun. 2 115007
[42] Tiague Takongmo G and Bogning J R 2018 Coupled soliton solutions of modeled equations in a Noguchi electrical line with crosslink capacitor J. Phys. Commun. 2 105016
[43] Skeel R D and Berzins M 1990 A method for the spatial discretization of parabolic equations in one space variable SIAM J. Sci. Stat. Comput. 11 1–32