Renormalization of the energy-momentum tensor in noncommutative complex scalar field theory

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Abstract

We study the renormalization of dimension four composite operators and the energy-momentum tensor in noncommutative complex scalar field theory. The proper operator basis is defined and it is proved that the bare composite operators are expressed via renormalized ones with the help of an appropriate mixing matrix which is calculated in the one-loop approximation. The number and form of the operators in the basis and the structure of the mixing matrix essentially differ from those in the corresponding commutative theory and in noncommutative real scalar field theory. We show that the energy-momentum tensor in the noncommutative complex scalar field theory is defined up to six arbitrary constants. The canonically defined energy-momentum tensor is not finite and must be replaced by the "improved" one, in order to provide finiteness. Suitable "improving" terms are found. Renormalization of dimension four composite operators at zero momentum transfer is also studied. It is shown that the mixing matrices are different for the cases of arbitrary and zero momentum transfer. The energy-momentum vector, unlike the energy-momentum tensor, is defined unambiguously and does not require "improving", in order to be conserved and finite, at least in the one-loop approximation.

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1 Introduction

The study of noncommutative field theories has attracted much attention lately, due to their profound links with the string theory [1] and remarkable properties in classical and quantum domains (see e.g. the reviews [9–11]).

There exist two general aspects of renormalization procedure in any field theory. Firstly, the renormalization of Green functions or effective action and secondly, the renormalization of composite operators (see e.g. [2] for a discussion of this problem in the commutative theories). The problem of renormalization of Green’s functions (i.e. fields, masses, and coupling constants) was studied for many noncommutative field theories to different approximation orders (see e.g. [3–8] and the reviews [9–11]).

The present paper is devoted to the problem of renormalization of composite operators and the energy-momentum tensor in noncommutative complex scalar field theory. The analogous problem in noncommutative real field theory was considered in [12]. As we will see, the renormalization of composite operators in noncommutative complex scalar field theory essentially differs from that in noncommutative real field theory.\(^1\)

A noncommutative field theory is usually constructed from the corresponding commutative theory, by replacing the pointwise product of the fields with the star one

\[
f \cdot g \to (f \ast g)(x) = \exp\left(\frac{i}{2} \theta^\mu{}_{\nu} \partial'_\mu \partial'_\nu\right) f(x+u)g(x+v)\bigg|_{u=v=0} \neq (g \ast f)(x), \tag{1}\]

where the constants \(\theta^\mu{}_{\nu}\) are noncommutativity parameters with dimension of a length squared. The star product (1) is noncommutative, so, in contrast to the commutative field theories, there is a problem of ordering of the fields in the Lagrangian of noncommutative theory. In the noncommutative real scalar field theory there was only one kind of field and this problem was absent. Therefore in that case both commutative and noncommutative theories had one coupling constant. In the case of noncommutative complex scalar field theory we have two kinds of fields and the problem of fields ordering arises. Therefore we have to take into account all possible ways of field ordering. The action of the theory is the Lagrangian integrated over the whole space-time. However, when we integrate the star product (1) over the whole space-time, we can prove, integrating by parts, the following consequences:

\[
\int d^4xf \ast g = \int d^4x f \cdot g = \int d^4x g \ast f, \tag{2}\]

\[
\int d^4xf_1 \ast f_2 \ast \cdots \ast f_N = \int d^4x f_2 \ast \cdots \ast f_N \ast f_1. \tag{3}\]

Eq. (2) leads us to conclude that the free part of an action in noncommutative theory is the same as in the corresponding commutative model, and from eq. (3) we see that interaction terms which differ in the Lagrangian by a cyclic permutation are the same in the action. For example, in the theory of noncommutative complex scalar field theory which we shall study, there are two different interaction terms [7, 11] and the action may be written as

\[
S = \int d^4x \left(\partial'_\mu \phi^* \ast \partial'_\mu \phi - m^2 \phi^* \ast \phi - \frac{\lambda_a}{4!} \phi^* \ast \phi \ast \phi \ast \phi - \frac{\lambda_b}{4!} \phi^* \ast \phi \ast \phi \ast \phi \ast \phi \right). \tag{4}\]

\(^1\)Problem of constructing the classical energy-momentum tensor in noncommutative field theories is discussed in [13–18].
Here there are two possibilities of ordering the operators in the interaction terms, therefore we can introduce in general two independent coupling constants, in contrast to the commutative case, where there is only one interaction and only coupling constant. This distinguishes the case of noncommutative complex scalar field theory from the real one and makes the consideration of the renormalization of composite operators in this theory also interesting.

In the present paper we study this problem and compare it with analogous problems both in noncommutative real scalar field theory and in commutative complex scalar field theory.

The paper is organized as follows. In the next section we derive the classical energy-momentum tensor of the noncommutative complex scalar field theory which follows from the Noether’s theorem and discuss some points concerning its derivation in the noncommutative case. In section 3 we present the general renormalization structure of dimension four composite operators and then in section 4 we renormalize these operators in the one-loop approximation. In section 5 we find that the energy-momentum tensor is divergent and, in order to make it finite, we need to add “improving” terms to it. These “improving” terms make the energy-momentum tensor traceless (apart from being finite), but the latter is conserved in the massless case only. Also we study the renormalization of composite operators at zero momentum transfer. This problem is considered in section 6. In section 7 we construct the energy-momentum vector of the theory which follows from the Neother’s procedure and find it to be conserved and finite in the one-loop approximation. In the summary we briefly discuss our results.

2 Classical energy-momentum tensor

In this section we define the energy-momentum tensor of the noncommutative complex scalar field theory. The action (4) is invariant under the global translation $x'^\mu = x^\mu + \varepsilon^\mu$, $\varepsilon^\mu = \text{const}$

$$\delta S = \int d^4x' L'(x') - \int d^4x L(x) = 0.$$ (5)

The Lagrangian (as well as the field functions) is changed under this transformation both because of changing the form of the field functions\(^2\) (the first line of (5)) $\delta \phi^A(x) = \phi^A(x) - \phi^A(x) = -\varepsilon^\mu \partial_\mu \phi^A$ and because of changing the argument of the field functions $\delta \phi^A(x) = \phi^A(x') - \phi^A(x) = \varepsilon^\mu \partial_\mu \phi^A$ (the second line of (5)). Therefore we can rewrite (5) as follows:

$$\delta S = \int d^4x' \left[ L'(x') - L(x') \right] + \int d^4x' L(x') - \int d^4x L(x) = 0.$$ (6)

Since $d^4x' = d^4x$, the last line of (5) to the first order in $\varepsilon^\mu$ reads

$$\int d^4x \left[ L(x + \varepsilon) - L(x) \right] = \int d^4x \varepsilon^\mu \partial_\mu L.$$ (7)

\(^2\)We have introduced the index $A$ at the fields for the generality of analysis of the energy-momentum tensor. For the theory under consideration we put $\phi^1 = \phi$, $\phi^2 = \phi^*$. 


As far as the first line of (6) is concerned, we transform it as follows (changing $x' \to x$):

$$\bar{\delta}S = \int d^4x \left[ L'(x) - L(x) \right]$$

$$= \int d^4x \left[ \frac{\partial L}{\partial \phi^A} \bar{\delta} \phi^A + \frac{\partial L}{\partial \phi^A_{\mu}} \delta \phi^A_{\mu} \right]. \quad (8)$$

It should be noted here that in calculating this expression (and the equation of motion in the following) we have used the cyclic property (3) and therefore the integration region in the noncommutative directions is not arbitrary. Since $\bar{\delta} \phi^A_{\mu} = \phi^A_{\mu}' - \phi^A_{\mu} = (\phi^A - \phi^A)_{,\mu} = (\bar{\delta} \phi^A)_{,\mu}$ and using the equations of motion we have

$$\bar{\delta}S = \int d^4x \left[ \frac{\partial L}{\partial \phi^A_{\mu}} \bar{\delta} \phi^A_{\mu} \right] = -\int d^4x \varepsilon^{\nu} \partial_{\nu} \left( \frac{\partial L}{\partial \phi^A_{\mu}} \delta \phi^A \right). \quad (9)$$

Collecting together (7), (9) and using the arbitrariness of $\varepsilon^{\nu}$ we find that

$$\int d^4x \partial_{\mu} \left[ \frac{\partial L}{\partial \phi^A_{,\mu}} \partial_{\nu} \phi^A - \eta_{\mu\nu} L \right] = 0. \quad (10)$$

In the case of spatial noncommutativity $\theta^{ij} = 0$ there are no time derivatives in the star product (1) and therefore the properties (2), (3) still hold when the integration is performed in space coordinates only. Therefore in eq. (10) the time integration region is arbitrary and we can write that

$$\int d^3x \partial_{\mu} \left[ \frac{\partial L}{\partial \phi^A_{,\mu}} \partial_{\nu} \phi^A - \eta_{\mu\nu} L \right] = 0 \quad (11)$$

and so, the quantity

$$\int d^3x \left[ \frac{\partial L}{\partial \phi^A_{,\mu}} \partial_{\nu} \phi^A - \eta_{\mu\nu} L \right]$$

is conserved. As a result, in noncommutative field theories there are only global conservation laws of the energy-momentum and only in the case of spatial noncommutativity $\theta^{ij} = 0$.

Substituting the Lagrangian of the theory under consideration (4) in (11) we find

$$\int \partial^\mu \tilde{T}_{\mu
u} \, d^3x = 0, \quad (13)$$

$$\tilde{T}_{\mu
u} = \partial_{\mu} \phi^* \ast \partial_{\nu} \phi + \partial_{\nu} \phi^* \ast \partial_{\mu} \phi - \eta_{\mu\nu} \left( \partial_{\alpha} \phi^* \ast \partial^\alpha \phi - m^2 \phi^* \ast \phi \right.$$  

$$\left. - \frac{\lambda_a}{4!} \phi^* \ast \phi \ast \phi^* \ast \phi \right) \quad (14)$$

The expression (13) is the basis for defining the energy-momentum tensor of the theory. Since the properties (2), (3) are still valid, we must consider all possibilities of field ordering in the energy-momentum tensor which lead us to (14). As a result, in contrast
to the commutative case, the energy-momentum tensor of the noncommutative complex scalar field theory cannot be defined unambiguously and we may write it only as follows:

\[ T_{\mu\nu} = c_1 \left( \partial_\mu \phi^* \star \partial_\nu \phi + \partial_\nu \phi^* \star \partial_\mu \phi \right) + (1 - c_1) \left( \partial_\mu \phi \star \partial_\nu \phi^* + \partial_\nu \phi \star \partial_\mu \phi^* \right) \]

\[ - \eta_{\mu\nu} \left( \partial_\alpha \phi^* \star \partial^\alpha \phi + (1 - c_2) \partial_\alpha \phi \star \partial^\alpha \phi^* \right) \]

\[ + \eta_{\mu\nu} m^2 \left( c_3 \phi^* \star \phi + (1 - c_3) \phi \star \phi^* \right) \]

\[ + \eta_{\mu\nu} \frac{\lambda_a}{4!} \left( c_4 \phi^* \star \phi \star \phi \star \phi + (1 - c_4) \phi \star \phi^* \star \phi \star \phi \right) \]

\[ + \eta_{\mu\nu} \frac{\lambda_b}{4!} \left( c_5 \phi \star \phi \star \phi \star \phi + (1 - c_5 - c_6) \phi \star \phi \star \phi \star \phi \right), \quad (15) \]

with \( c_1, \ldots, c_6 \) being arbitrary real numbers, which define all possibilities of field ordering. Also we have demanded here that the energy-momentum tensor (15) be symmetric and real. If we substitute (15) in (13), we find that all the arbitrary constants are cancelled.

As a result, we see that the energy-momentum tensor of the noncommutative complex scalar field theory is defined up to six arbitrary real constants. This separates the theory under consideration both from the corresponding commutative theory and from the noncommutative theory of real scalar field, where the energy-momentum tensors are defined unambiguously.

3 General renormalization structure of dimension four composite operators

In order to construct the renormalized energy-momentum tensor, we have to renormalize all the composite operators which enter into it. Before starting an explicit calculation of the renormalization of composite operators, let us describe the general situation. As well known (see e.g. [2]), in order to renormalize some composite operator we need, in general, to take into account all composite operators with the same mass dimension, tensor structure, and symmetries. One can show that it is sufficient to renormalize the operators which constitute a basis of such operators [2].

The operators which enter into the energy-momentum tensor (15) are hermitian operators with mass dimension four. Let us construct a corresponding operator basis. From a general point of view it is convenient also to include into the basis, together with other operators, the operators which are not renormalized and are related to the field equations in terms of bare (with index 0) or renormalized quantities\(^3\)

\[ L_0 = (\partial^2 + m_0^2)\phi_0 + 2 \frac{\lambda_{a0}}{4!} \phi_0 \star \phi_0 \star \phi_0 \star \phi_0 + \frac{\lambda_{b0}}{4!} \phi_0 \star \phi_0 \star \phi_0 \star \phi_0 + \frac{\lambda_{b0}}{4!} \phi_0 \star \phi_0 \star \phi_0 \star \phi_0, \quad (16) \]

\[ L = (\partial^2 + m^2)\phi + 2 \frac{\mu^4-d \lambda_a}{4!} \phi \star \phi \star \phi \star \phi + \frac{\mu^4-d \lambda_b}{4!} \phi \star \phi \star \phi \star \phi + \frac{\mu^4-d \lambda_{ab}}{4!} \phi \star \phi \star \phi \star \phi. \quad (17) \]

\(^3\)We use dimensional regularization and keep the renormalized coupling constants \( \lambda_a \) and \( \lambda_b \) to be dimensionless, therefore we introduce by standard way an arbitrary parameter \( \mu \) of mass dimension.
We construct the basis of the scalar hermitian composite operators with mass dimension four as follows:

\[
Q_0 = \begin{pmatrix}
\phi_0^* \phi_0 \phi_0^* \phi_0 \\
\phi_0 \phi_0^* \phi_0 \phi_0^* \\
\phi_0^* \phi_0^* \phi_0 \phi_0 \\
\phi_0 \phi_0^* \phi_0^* \phi_0 \\
\phi_0^* \phi_0 \phi_0^* \phi_0^* + \\
\phi_0 \phi_0^* \phi_0^* \phi_0^* \phi_0 + \\
m_0^2 \phi_0^* \phi_0 \phi_0^* \\
m_0^2 \phi_0^* \phi_0^* \phi_0 \\
\partial^2(\phi_0^* \phi_0) \\
\partial^2(\phi_0 \phi_0^*) \\
\phi_0^* L_0 + L_0^* \phi_0 \\
\phi_0 \phi_0^* L_0^* + L_0 \phi_0^*
\end{pmatrix}
\]

and

\[
[Q] = \begin{pmatrix}
[\phi^* \phi \phi^* \phi] \\
[\phi \phi^* \phi \phi^*] \\
[\phi^* \phi \phi^* \phi] \\
[\phi \phi^* \phi \phi^*] + \\
[\phi \phi^* \phi \phi^* \phi] \\
[\phi^* \phi \phi^* \phi] \\
m_0^2 [\phi^* \phi] \\
m_0^2 [\phi \phi^*] \\
\partial^2[\phi^* \phi] \\
\partial^2[\phi \phi^*] \\
[\phi^* L + L^* \phi] \\
[\phi L^* + L \phi^*]
\end{pmatrix},
\]

for the bare and the renormalized operators respectively. Due to the noncommutativity of the multiplication rule (11), the operator basis (18) contains more operators in comparison with the corresponding commutative theory because we may order the fields in composite operators in different ways. For example, the hermitian operator \((\phi^* \phi)^2\) in the commutative theory is a prototype for five different hermitian operators in the noncommutative theory (18). This situation is typical for any noncommutative field theory. These operator basis are mixed by renormalization. By dimensional and symmetry analysis it may be shown that no more operators are required. We will write the relation (18) in the form

\[
Q_0 = Z[Q],
\]

with \(Z\) being a mixing matrix. In the next section we calculate this mixing matrix \(Z\) and some more renormalization relations for composite operators, which we will use for the renormalization of the energy-momentum tensor in the one-loop approximation.

4 One-loop renormalization of dimension four composite operators

In this section we carry out the one-loop renormalization of all composite operators entering into the expression for the energy-momentum tensor (15) of the noncommutative complex scalar field theory. The general procedure of constructing the renormalized operators which is valid for both commutative and noncommutative theories was described in [12]. Here we follow this procedure. Before starting to renormalize the composite operators we need to renormalize all parameters of the theory. The one-loop renormalization of the field, the mass, and the coupling constants of the model may be found as a particular case of [8] and reads

\[
\phi_0 = \phi,
\]

\[
m_0^2 = \left(1 - \frac{1}{(d-4)(4\pi)^2} \frac{2\lambda_0 + \lambda_b}{3!}\right) m^2,
\]

\[
\lambda_{a0} = \frac{\mu^{4-d} \lambda_a}{4!} - \frac{2\mu^{1-d} \lambda_a}{(d-4)(4\pi)^2} \frac{4\lambda_{a0}^2 + \lambda_b^2}{(4!)^2},
\]

where the renormalized coupling constant \(\lambda_{a0}\), the mass \(m_0^2\), and the field \(\phi_0\) are the renormalized versions of these quantities.
Let us consider the renormalization of the composite operator $m^2 \phi^* \phi$. Following the procedure described in [12], in order to renormalize this operator we should add to the action (4) the term
\[
\begin{align*}
  m^2 \int d^d x J^* \phi^* \phi, 
\end{align*}
\]
with $J$ being some arbitrary function (source), and then calculate all divergent one particle irreducible diagrams linear in $J$. There are six types of such diagrams which are shown in Figure 1. Performing the Fourier transform of the fields $\phi^*$, $\phi$ and the source $\tilde{J}$
\[
\begin{align*}
  \phi^*(x) &= \int \left( \frac{dp_1}{2\pi} \right)^d e^{i p_1 x} \tilde{\phi}^*(p_1) \equiv \int_{p_1} e^{i p_1 x} \tilde{\phi}^*(p_1), \\
  \phi(x) &= \int_{p_2} e^{i p_2 x} \tilde{\phi}(p_2), \\
  \tilde{J}(x) &= \int_k e^{i k x} \tilde{J}(k),
\end{align*}
\]
we get the following expression in momentum space for the first two diagrams (which correspond to the $\lambda_a$ interaction term) in Figure 1
\[
\begin{align*}
  2i m^2 \int_{kp_1p_2} (2\pi)^d &\delta((k + p_1 + p_2)\tilde{J}(k)\tilde{\phi}^*(p_1)\phi(p_2) \times \\
  &\times \left\{ e^{-i \frac{p_1 \theta p_2}{2}} \int_p \frac{1}{(p^2 - m^2)((p + k)^2 - m^2)} e^{i p \theta k} \\
  &+ e^{-i \frac{p_2 \theta p_1}{2}} \int_p \frac{1}{(p^2 - m^2)((p + k)^2 - m^2)} e^{i p \theta k} \right\}. \quad (28)
\end{align*}
\]

The expression (28) corresponds to diagram $a$ in Figure 1 and has a UV divergence at any external momenta $k$, $p_1$, $p_2$ (the so-called "planar" diagram). The expression (29) corresponds to diagram $b$ (so-called "non-planar" diagram) and displays UV/IR mixing [4]: its divergence depends on the value of $\theta^{\mu\nu}k_{\nu}$. If $\theta^{\mu\nu}k_{\nu} = 0$, then we have a UV divergence as in the commutative theory. If $\theta^{\mu\nu}k_{\nu} \neq 0$ then the integral (29) is UV finite, but if we were to put $k_{\nu} = 0$ after carrying out the integration, then we would find it to be divergent (IR divergence). We suppose that $\theta^{\mu\nu}k_{\nu} \neq 0$, so only the expression (28) contains a UV divergence and for its subtraction we need to add a counterterm in the effective action.
An analogous consideration is valid for the other four diagrams in Figure 1. We have for them the following expressions in momentum space:

\[
i \frac{\lambda_k}{4!} m^2 \int_{kp_1p_2} (2\pi)^4 \delta(k + p_1 + p_2) \bar{J}(k) \bar{\phi}_r(p_1) \bar{\phi}(p_2) \times \\
x \left\{ e^{-\frac{i}{2} p_2 \theta p_1} \int_p \frac{1}{(p^2 - m^2)((p + k)^2 - m^2)} e^{-ip \theta p_2} \right. \\
+ e^{-\frac{i}{2} p_2 \theta p_1} \int_p \frac{e^{-ip \theta p_1}}{(p^2 - m^2)((p + k)^2 - m^2)} e^{ip \theta k} \\
+ e^{-\frac{i}{2} p_2 \theta p_1} \int_p \frac{e^{ip \theta k}}{(p^2 - m^2)((p + k)^2 - m^2)} \right\}.
\]

The expression (30) corresponds to diagram c (planar diagram) and is UV divergent at any external momenta. Expressions (31), (32), (33) correspond to diagrams d, e, f in the figure. These diagrams are non-planar and their divergences depend on values of \( \theta^{\mu\nu} p_{2\nu}, \theta^{\mu\nu} p_{1\nu}, \theta^{\mu\nu} k_{\nu} \) respectively. This situation is analogous to that when we consider diagram b in Figure 1. As in that case, we suppose that \( \theta^{\mu\nu} p_{2\nu} \neq 0, \theta^{\mu\nu} p_{1\nu} \neq 0, \theta^{\mu\nu} k_{\nu} \neq 0 \), so these diagrams have no UV divergences. Using dimensional regularization we find the UV divergent parts of (28) and (30):

\[
\frac{1}{(d - 4)(4\pi)^2} \frac{\lambda_a}{3!} m^2 \int d^4x J \star \phi^* \star \phi
\]

\[
\frac{1}{(d - 4)(4\pi)^2} \frac{2\lambda_b}{4!} m^2 \int d^4x J \star \phi \star \phi^*
\]

respectively. As a result we get the following expression connecting the bare and renormalized operators:

\[
m^2_0 \phi^*_0 \star \phi_0 = \left( 1 - \frac{1}{(d - 4)(4\pi)^2} \frac{\lambda_a + \lambda_b}{3!} \right) m^2 [\phi^* \star \phi] \\
+ \frac{1}{(d - 4)(4\pi)^2} \frac{2\lambda_b}{4!} m^2 [\phi \star \phi^*].
\]

From (36) we see that there is operator mixing here: in order to renormalize the operator \( m^2_0 \phi^*_0 \star \phi_0 \), we have to take into account the operator \( m^2 [\phi \star \phi^*] \) besides \( m^2 [\phi^* \star \phi] \) which are different due to noncommutativity of the multiplication (1). Also the expression (36) differs from the corresponding renormalization relation in the commutative theory where, as it may easily be shown, one has

\[
m^2_0 \phi^*_0 \phi_0 = m^2 [\phi \star \phi].
\]

This situation is similar to that in the noncommutative real scalar field theory: in both cases the corresponding renormalization relations in noncommutative and commutative theories have different form.

In complete analogy we may calculate the renormalization of the operator \( m^2_0 \phi_0 \star \phi^*_0 \) which differs from the previously renormalized operator by the following exchange of the
divergences and their contribution to the effective action is suppose that external momenta are not zero, therefore only planar diagrams have UV distributions to the generating functional of one particle irreducible diagrams. As usual we contain both vertices with \( \lambda \) and renormalize all divergent one particle irreducible diagrams linear in \( J \). In the following we shall ignore the order of lines in vertices, which in noncommutative theories commonly are depicted in the proper cyclic order, reflecting the order of the fields in the action and the property \( [3] \). Therefore, for example, we could have shown all diagrams represented in Figure 2 with the help of only one of them, usually \( a \) or \( c \).

Now let us consider the renormalization of the operator \( \partial_\mu \phi^* \star \partial_\nu \phi \). According the procedure described in \([12]\) we should add to the action \([1] \) the term

\[
\int d^d x \ J^{\mu\nu} \star \partial_\mu \phi^* \star \partial_\nu \phi
\]

and renormalize all divergent one particle irreducible diagrams linear in \( J^{\mu\nu} \). These diagrams are shown in Figure 2. As was explained above these diagrams are concise and contain both vertices with \( \lambda_a \) and \( \lambda_b \) interactions and both planar and nonplanar contributions to the generating functional of one particle irreducible diagrams. As usual we suppose that external momenta are not zero, therefore only planar diagrams have UV divergences and their contribution to the effective action is

\[
i \mu^{d-4} \int_{kp_1p_2} (2\pi)^d \delta(k + p_1 + p_2) \tilde{J}^{\mu\nu}(k) \tilde{\phi}_*(p_1) \tilde{\phi}(p_2) \left( 2 \frac{\lambda_a}{4!} e^{-\frac{i}{2} p_1 \theta p_2} + \frac{\lambda_b}{4!} e^{\frac{i}{2} p_1 \theta p_2} \right) \times \int_p \frac{(p\mu - k_\mu) p_\nu}{(p^2 - m^2)(p - k)^2 - m^2}
\]

for diagram \( a \) and

\[
i \mu^{8-2d} \int_{kp_1p_2p_3p_4} (2\pi)^d \delta(k + p_1 + p_2 + p_3 + p_4) \tilde{J}^{\mu\nu}(k) \tilde{\phi}_*(p_1) \tilde{\phi}(p_2) \tilde{\phi}_*(p_3) \tilde{\phi}(p_4) \times \left\{ 4 \left( \frac{\lambda_a}{4!} \right)^2 e^{-\frac{i}{2} p_1 \theta(p_2 + p_3 + p_4)} e^{-\frac{i}{2} p_2 \theta(p_1 + p_4)} e^{-\frac{i}{2} p_3 \theta p_4} + \right. \]

\[
+ \left. \left( \frac{\lambda_b}{4!} \right)^2 e^{-\frac{i}{2} p_2 \theta(p_1 + p_3 + p_4)} e^{-\frac{i}{2} p_1 \theta(p_3 + p_4)} e^{-\frac{i}{2} p_4 \theta p_3} + \right. \]

\[
+ 2 \frac{\lambda_a \lambda_b}{4!} e^{-\frac{i}{2} p_2 \theta(p_1 + p_3 + p_4)} e^{-\frac{i}{2} p_1 \theta(p_3 + p_4)} e^{-\frac{i}{2} p_4 \theta p_3} + \right. \]

\[
+ \left. 2 \frac{\lambda_a}{4!} \frac{\lambda_b}{4!} e^{-\frac{i}{2} p_2 \theta(p_1 + p_3 + p_4)} e^{-\frac{i}{2} p_1 \theta(p_3 + p_4)} e^{-\frac{i}{2} p_4 \theta p_3} \right) \times \int_p \frac{(p\mu - k_\mu) p_\nu}{(p^2 - m^2)(p - k)^2 - m^2}
\]
\[ + 2 \frac{\lambda_a \lambda_b}{4!} e^{-\frac{1}{2} p_1 \theta(p_2 + p_3 + p_4)} e^{-\frac{1}{2} p_2 \theta(p_3 + p_4)} e^{-\frac{1}{2} p_4 \theta(p_3)} \times \]
\[ \times \int_p \frac{(p_\mu - k_\mu) p_\nu}{(p^2 - m^2)((p + p_3 + p_4)^2 - m^2)((p - k)^2 - m^2)} \]
\[ + i \mu^{8-2d} \left( \frac{\lambda_b}{4!} \right)^2 \int_{k p_1 p_2 p_3 p_4} \delta(k + p_1 + p_2 + p_3 + p_4) \tilde{J}^{\mu \nu}(k) \tilde{\phi}_\nu(p_1) \tilde{\phi}(p_2) \tilde{\phi}_\nu(p_3) \tilde{\phi}(p_4) \times \]
\[ \times e^{\frac{1}{2} p_1 \theta(p_2 + p_3 + p_4)} e^{-\frac{1}{2} p_2 \theta(p_3 + p_4)} e^{-\frac{1}{2} p_3 \theta(p_2 + p_4)} \times \]
\[ \times \int_p \frac{(p_\mu - k_\mu) p_\nu}{(p^2 - m^2)((p + p_2 + p_4)^2 - m^2)((p - k)^2 - m^2)} \] (41)

for diagram b. From (40) and (41) UV divergences may easily be extracted and the result is

\[ \frac{2}{(d - 4)(4\pi)^2} \frac{\lambda_a}{4!} \int d^4 x \phi^* \phi \phi \phi \left[ \frac{1}{6} \eta_{\mu \nu} \partial^2 + m^2 \eta_{\mu \nu} + \frac{1}{3} \partial^2_{\mu \nu} \right] J^{\mu \nu} \]
\[ + \frac{1}{(d - 4)(4\pi)^2} \frac{\lambda_b}{4!} \int d^4 x \phi^* \phi \phi \phi \left[ \frac{1}{6} \eta_{\mu \nu} \partial^2 + m^2 \eta_{\mu \nu} + \frac{1}{3} \partial^2_{\mu \nu} \right] J^{\mu \nu} \] (42)

for expression (40) and

\[ \frac{2\mu^{4-d}}{(d - 4)(4\pi)^2} \left( \frac{\lambda_a}{4!} \right)^2 \eta_{\mu \nu} \int d^4 x J^{\mu \nu} \phi^* \phi \phi \phi \phi \]
\[ + \frac{1/2 \mu^{4-d}}{(d - 4)(4\pi)^2} \left( \frac{\lambda_b}{4!} \right)^2 \eta_{\mu \nu} \int d^4 x J^{\mu \nu} \phi^* \phi \phi \phi \phi \]
\[ + \frac{\mu^{4-d}}{(d - 4)(4\pi)^2} \frac{\lambda_a \lambda_b}{4!} \eta_{\mu \nu} \int d^4 x J^{\mu \nu} \phi^* \phi \phi \phi \phi \]
\[ + \frac{\mu^{4-d}}{(d - 4)(4\pi)^2} \frac{\lambda_a \lambda_b}{4!} \eta_{\mu \nu} \int d^4 x J^{\mu \nu} \phi^* \phi \phi \phi \phi \]
\[ + \frac{1/2 \mu^{4-d}}{(d - 4)(4\pi)^2} \left( \frac{\lambda_b}{4!} \right)^2 \eta_{\mu \nu} \left[ \phi^* \phi \phi \phi \phi + \phi^* \phi \phi \phi \phi \right] \] (43)

for expression (41). Using (42) and (43) we get the one-loop renormalization relation for the operator \( \partial_\mu \phi^* \partial_\nu \phi \)

\[ \partial_\mu \phi^*_0 \partial_\nu \phi_0 = [\partial_\mu \phi^* \partial_\nu \phi] \]
\[ + \frac{2}{(d - 4)(4\pi)^2} \frac{\lambda_a}{4!} \left( \frac{1}{6} \eta_{\mu \nu} \partial^2 + \frac{1}{3} \partial^2_{\mu \nu} + \eta_{\mu \nu} m^2 \right) [\phi^* \phi] \]
\[ + \frac{1}{(d - 4)(4\pi)^2} \frac{\lambda_b}{4!} \left( \frac{1}{6} \eta_{\mu \nu} \partial^2 + \frac{1}{3} \partial^2_{\mu \nu} + \eta_{\mu \nu} m^2 \right) [\phi^* \phi^*] \]
\[ + \frac{2\mu^{4-d}}{(d - 4)(4\pi)^2} \left( \frac{\lambda_a}{4!} \right)^2 \eta_{\mu \nu} [\phi^* \phi \phi \phi] \]
\[ + \frac{1/2 \mu^{4-d}}{(d - 4)(4\pi)^2} \left( \frac{\lambda_b}{4!} \right)^2 \eta_{\mu \nu} [\phi^* \phi \phi^* \phi \phi + \phi^* \phi^* \phi \phi \phi] \]
\[ + \frac{\mu^{4-d}}{(d - 4)(4\pi)^2} \frac{\lambda_a \lambda_b}{4!} \eta_{\mu \nu} [\phi^* \phi \phi^*] \] (44)
The renormalization of the operator \( \partial_\mu \phi \star \partial_\nu \phi^* \) may be found by exchanging \( \phi \leftrightarrow \phi^* \) in (44).

\[
\partial_\mu \phi_0 \star \partial_\nu \phi^*_0 = [\partial_\mu \phi \star \partial_\nu \phi^*] \\
+ \frac{2}{(d-4)(4\pi)^2} \frac{\lambda_a}{4!} \left( \frac{1}{6} \eta_{\mu\nu} \partial^2 + \frac{1}{3} \partial^2_{\mu\nu} + \eta_{\mu\nu} m^2 \right) [\phi \star \phi^*] \\
+ \frac{1}{(d-4)(4\pi)^2} \frac{\lambda_b}{4!} \left( \frac{1}{6} \eta_{\mu\nu} \partial^2 + \frac{1}{3} \partial^2_{\mu\nu} + \eta_{\mu\nu} m^2 \right) [\phi^* \star \phi] \\
+ \frac{2\mu^{4-d}}{(d-4)(4\pi)^2} \left( \frac{\lambda_a}{4!} \right)^2 \eta_{\mu\nu} [\phi \star \phi^* \star \phi \star \phi^*] \\
+ \frac{1/2\mu^{4-d}}{(d-4)(4\pi)^2} \left( \frac{\lambda_b}{4!} \right)^2 \eta_{\mu\nu} \left( \left[ \phi^* \star \phi \star \phi \star \phi \right] + \left[ \phi \star \phi \star \phi \star \phi \right] \right) \\
+ \frac{\mu^{4-d}}{(d-4)(4\pi)^2} \frac{\lambda_a \lambda_b}{4!} \eta_{\mu\nu} \left( \left[ \phi \star \phi \star \phi \star \phi \right] + \left[ \phi \star \phi \star \phi \star \phi \right] \right). \tag{45}
\]

Let us consider the renormalization of the operator \( \phi^* \star \phi \star \phi^* \star \phi^* \). For this purpose we add to the action (4) the following term:

\[
\mu^{4-d} \int d^d x \ J \star \phi^* \star \phi \star \phi^* \star \phi 
\]

and calculate all one-loop UV divergent contributions to the effective action linear in \( J \). The diagrams which we need are shown in Figure 3. Also we notice that in eq. (46) in order to keep the source \( J \) to be dimensionless, an arbitrary parameter \( \mu \) with dimension of mass was introduced. The UV divergences which arise from these diagrams are

\[
\frac{6m^2}{(d-4)(4\pi)^2} \int d^d x \ J \star \phi^* \star \phi 
\]

for diagram a and

\[
\frac{\mu^{4-d}}{(d-4)(4\pi)^2} \frac{\lambda_a}{2} \int d^d x \ J \star \phi^* \star \phi \star \phi^* \star \phi \\
+ \frac{\mu^{4-d}}{(d-4)(4\pi)^2} \frac{2\lambda_b}{4!} \int d^d x \ J \star \left[ \phi^* \star \phi^* \star \phi \star \phi + \phi^* \star \phi \star \phi \star \phi + \phi \star \phi^* \star \phi \star \phi \right] \tag{48}
\]

for diagram b, respectively. The result of the renormalization of the operator under consideration is

\[
\frac{\lambda_{\alpha 0}}{4!} \phi^*_0 \star \phi_0 \star \phi^*_0 \star \phi_0 = \mu^{4-d} \left( \frac{\lambda_a}{4!} + \frac{1}{(d-4)(4\pi)^2} \frac{4\lambda_a^2 - 2\lambda_b^2}{4!} \right) [\phi^* \star \phi \star \phi \star \phi] \\
+ \frac{2\mu^{4-d}}{(d-4)(4\pi)^2} \frac{\lambda_a \lambda_b}{4!} \frac{1}{4!} \left( \left[ \phi^* \star \phi^* \star \phi \star \phi \right] + \left[ \phi^* \star \phi \star \phi \star \phi \right] + \left[ \phi \star \phi^* \star \phi \star \phi \right] \right) \\
+ \frac{1}{(d-4)(4\pi)^2} \frac{\lambda_a}{4!} \frac{6m^2}{4!} [\phi^* \star \phi]. \tag{49}
\]
Exchanging $\phi \leftrightarrow \phi^*$ in (49) we find the renormalization of the operator $\phi \ast \phi^* \ast \phi \ast \phi^*$

\[
\frac{\lambda_{ab}}{4!} \phi_0 \ast \phi_0^* \ast \phi_0 \ast \phi_0^* = \mu^{4-d} \left( \frac{\lambda_a}{4!} + \frac{1}{(d-4)(4\pi)^2} \frac{4\lambda_a^2 - 2\lambda_b^2}{4!} \right) [\phi \ast \phi^* \ast \phi \ast \phi^*]
\]

\[
+ \frac{2\mu^{4-d}}{(d-4)(4\pi)^2} \lambda_a \lambda_b \left( \left[ \phi \ast \phi^* \ast \phi^* \right] + \left[ \phi \ast \phi^* \ast \phi \right] + \left[ \phi^* \ast \phi \ast \phi^* \right] \right)
\]

\[
+ \frac{1}{(d-4)(4\pi)^2} \frac{\lambda_a}{4!} 6m^2 [\phi \ast \phi^*].
\]  

(50)

Carrying out analogous calculations, we may find the renormalization relations for the remaining composite operators which enter into the energy-momentum tensor (15)

\[
\frac{\lambda_{ab}}{4!} \phi_0 \ast \phi_0^* \ast \phi_0 \ast \phi_0 = \mu^{4-d} \frac{\lambda_b}{4!} \left( 1 + \frac{1}{(d-4)(4\pi)^2} \frac{-4\lambda_a + 2\lambda_b}{4!} \right) [\phi \ast \phi^* \ast \phi \ast \phi]
\]

\[
+ \frac{2\mu^{4-d}}{(d-4)(4\pi)^2} \left( \frac{\lambda_b}{4!} \right)^2 [\phi \ast \phi^* \ast \phi \ast \phi]
\]

\[
+ \frac{1}{(d-4)(4\pi)^2} \frac{\lambda_b}{4!} 2m^2 [\phi \ast \phi^*],
\]  

(51)

\[
\frac{\lambda_{ab}}{4!} \phi_0 \ast \phi_0 \ast \phi_0 \ast \phi_0^* = \mu^{4-d} \frac{\lambda_b}{4!} \left( 1 + \frac{1}{(d-4)(4\pi)^2} \frac{-4\lambda_a + 2\lambda_b}{4!} \right) [\phi \ast \phi \ast \phi^* \ast \phi^*]
\]

\[
+ \frac{2\mu^{4-d}}{(d-4)(4\pi)^2} \left( \frac{\lambda_b}{4!} \right)^2 [\phi \ast \phi \ast \phi \ast \phi^*]
\]

\[
+ \frac{1}{(d-4)(4\pi)^2} \frac{\lambda_b}{4!} 2m^2 [\phi \ast \phi],
\]  

(52)

\[
\frac{\lambda_{ab}}{4!} \phi_0 \ast \phi_0^* \ast \phi_0 \ast \phi_0^* = \mu^{4-d} \frac{\lambda_b}{4!} [\phi \ast \phi \ast \phi \ast \phi^*]
\]

\[
+ \frac{2\mu^{4-d}}{(d-4)(4\pi)^2} \left( \frac{\lambda_b}{4!} \right)^2 \left( \left[ \phi \ast \phi \ast \phi \ast \phi \right] + \left[ \phi \ast \phi \ast \phi \ast \phi^* \right] \right)
\]

\[
+ \frac{1}{(d-4)(4\pi)^2} \frac{\lambda_b}{4!} 2m^2 \left( \left[ \phi \ast \phi \right] + \left[ \phi \ast \phi^* \right] \right),
\]  

(53)

\[
\frac{\lambda_{ab}}{4!} \phi_0 \ast \phi_0 \ast \phi_0 \ast \phi_0 = \mu^{4-d} \frac{\lambda_b}{4!} [\phi \ast \phi \ast \phi \ast \phi]
\]

\[
+ \frac{2\mu^{4-d}}{(d-4)(4\pi)^2} \left( \frac{\lambda_b}{4!} \right)^2 \left( \left[ \phi \ast \phi \ast \phi \ast \phi \right] + \left[ \phi \ast \phi \ast \phi \ast \phi^* \right] \right)
\]

\[
+ \frac{1}{(d-4)(4\pi)^2} \frac{\lambda_b}{4!} 2m^2 \left( \left[ \phi \ast \phi \right] + \left[ \phi \ast \phi^* \right] \right).
\]  

(54)

Using the renormalization relations which we have found in this section we can define the mixing matrix $Z$ (19) in the one-loop approximation. In view of its huge size we write it in explicit form in Table 1 at page 13 of the article.

Now we turn to the renormalization of the energy-momentum tensor (15) of the theory.
$$Z = \begin{pmatrix} 
1 - \frac{1/2 \lambda_a}{(4\pi)^2 \varepsilon} & 0 & -\frac{2 \lambda_b}{(4\pi)^2 4! \varepsilon} & 0 & -\frac{2 \lambda_b}{(4\pi)^2 4! \varepsilon} & -\frac{6 \mu - \varepsilon}{(4\pi)^2 \varepsilon} & 0 & 0 & 0 & 0 \\
0 & 1 - \frac{1/2 \lambda_a}{(4\pi)^2 \varepsilon} & 0 & -\frac{2 \lambda_b}{(4\pi)^2 4! \varepsilon} & -\frac{2 \lambda_b}{(4\pi)^2 4! \varepsilon} & 0 & -\frac{6 \mu - \varepsilon}{(4\pi)^2 \varepsilon} & 0 & 0 & 0 \\
-\frac{2 \lambda_b}{(4\pi)^2 4! \varepsilon} & 0 & 1 - \frac{\lambda_a + \lambda_b}{(4\pi)^2 3! \varepsilon} & 0 & 0 & 0 & -\frac{2 \mu - \varepsilon}{(4\pi)^2 \varepsilon} & 0 & 0 & 0 \\
0 & -\frac{2 \lambda_b}{(4\pi)^2 4! \varepsilon} & 0 & 1 - \frac{\lambda_a + \lambda_b}{(4\pi)^2 3! \varepsilon} & 0 & 0 & -\frac{2 \mu - \varepsilon}{(4\pi)^2 \varepsilon} & 0 & 0 & 0 \\
-\frac{\lambda_b}{(4\pi)^2 3! \varepsilon} & 0 & -\frac{\lambda_b}{(4\pi)^2 3! \varepsilon} & 0 & 0 & 0 & 1 - \frac{8 \lambda_a + 2 \lambda_b}{(4\pi)^2 4! \varepsilon} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\lambda_a + \lambda_b}{2 \lambda_b} (4\pi)^2 4! \varepsilon & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\lambda_a + \lambda_b}{(4\pi)^2 3! \varepsilon} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 - \frac{\lambda_a}{2 \lambda_b} (4\pi)^2 4! \varepsilon & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\lambda_a}{(4\pi)^2 3! \varepsilon} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{pmatrix}$$

\[\varepsilon = 4 - d\]

Table 1: The mixing matrix in the one-loop approximation
5 One-loop renormalization of the energy-momentum tensor

The purpose of this section is to construct the finite operator of the energy-momentum tensor of the theory under consideration. Using the result of the previous section, we can see that the bare energy-momentum tensor

\[ T_{0\mu\nu} = c_1 \left( \partial_{\mu} \phi^* \ast \partial_{\nu} \phi + \partial_{\nu} \phi^* \ast \partial_{\mu} \phi \right) + (1 - c_1) \left( \partial_{\mu} \phi_0 \ast \partial_{\nu} \phi^* + \partial_{\nu} \phi_0 \ast \partial_{\mu} \phi^* \right) \]

\[ - \eta_{\mu\nu} \left( c_2 \partial_{\alpha} \phi^* \ast \partial^\alpha \phi \ast + (1 - c_2) \partial_{\alpha} \phi_0 \ast \partial^\alpha \phi^* \right) \]

\[ + \eta_{\mu\nu} m_0^2 \left( c_3 \phi^*_0 \ast \phi_0 + (1 - c_3) \phi_0 \ast \phi^*_0 \right) \]

\[ + \eta_{\mu\nu} \frac{\lambda_{a0}}{4!} \left( c_4 \phi^*_0 \ast \phi_0 \ast \phi^*_0 \ast \phi_0 \ast (1 - c_4) \phi_0 \ast \phi^*_0 \ast \phi_0 \right) \]

\[ + \eta_{\mu\nu} \frac{\lambda_{b0}}{4!} \left( c_5 \phi^*_0 \ast \phi_0 \ast \phi_0 \ast \phi^*_0 \ast \phi_0 \right) \]

\[ + \eta_{\mu\nu} \frac{\lambda_{b0}}{4!} \left( \frac{1 - c_5 - c_6}{2} \right) \left( \phi^*_0 \ast \phi_0 \ast \phi_0 \ast \phi^*_0 \ast \phi_0 + \phi_0 \ast \phi^*_0 \ast \phi^*_0 \ast \phi_0 \phi_0 \right) \]

(55)

is not finite in the one-loop approximation. Substituting the expressions which relate the bare operators entering into the bare energy-momentum tensor with the renormalized ones \([38], [39], [11], [17], [19] b1\) we find its divergences

\[ T_{0\mu\nu} = \left[ T_{\mu\nu} \right] + \]

\[ + \partial^2_{\mu\nu}[\phi^* \ast \phi] \frac{1/3}{(d - 4)(4\pi)^2 4!} \left( 2(1 - c_1)\lambda_b + 4c_1\lambda_a \right) \]

\[ + \partial^2_{\mu\nu}[\phi \ast \phi^*] \frac{1/3}{(d - 4)(4\pi)^2 4!} \left( 2c_1\lambda_b + 4(1 - c_1)\lambda_a \right) \]

\[ + \eta_{\mu\nu} \partial^2[\phi^* \ast \phi] \frac{1/3}{(d - 4)(4\pi)^2 4!} \left( \lambda_a(c_1 - 3c_2) + \lambda_b(-2 - c_1 + 3c_2) \right) \]

\[ + \eta_{\mu\nu} \partial^2[\phi \ast \phi^*] \frac{1/3}{(d - 4)(4\pi)^2 4!} \left( 2\lambda_a(-2 - c_1 + 3c_2) + \lambda_b(c_1 - 3c_2) \right) \]

\[ + \eta_{\mu\nu} m^2[\phi^* \ast \phi] \frac{2}{(d - 4)(4\pi)^2 4!} \left( \lambda_a(2c_1 - 4c_2 - 2c_3 + 3c_4) \right. \]

\[ + \lambda_b(-1 - 2c_1 + 4c_2 + 2c_3 - c_4) \]

\[ \left. + \lambda_b(-1 + c_1 - 2c_2 + 3c_3 - c_5) \right) \]

\[ + \eta_{\mu\nu} m^2[\phi \ast \phi^*] \frac{2}{(d - 4)(4\pi)^2 4!} \left( \lambda_a(-1 - 2c_1 + 4c_2 + 2c_3 - 3c_4) \right. \]

\[ + \lambda_b(-1 + c_1 - 2c_2 + 3c_3 - c_5) \]

\[ + \eta_{\mu\nu}[\phi^* \ast \phi \ast \phi^* \ast \phi] \frac{\mu^{d-4}}{(d - 4)(4\pi)^2(4!)^2} \times \]

\[ \times \left( 4\lambda_a^2(c_1 - 2c_2 + c_4) + \lambda_b^2(1 - c_1 + 2c_2 - 2c_4 - 2c_6) \right) \]

\[ + \eta_{\mu\nu}[\phi \ast \phi^* \ast \phi \ast \phi^*] \frac{\mu^{d-4}}{(d - 4)(4\pi)^2(4!)^2} \times \]

\[ \times \left( 4\lambda_a^2(-c_1 + 2c_2 - c_4) + \lambda_b^2(c_1 - 2c_2 + 2c_4 - 2c_5) \right) \]
Here $[T_{\mu\nu}]$ is a finite quantity

$$
[T_{\mu\nu}] = c_1 \left( [\partial_\mu \phi^* \partial_\nu \phi] + [\partial_\nu \phi^* \partial_\mu \phi] \right) + (1 - c_1) \left( [\partial_\mu \phi \partial_\nu \phi^*] + [\partial_\nu \phi \partial_\mu \phi^*] \right)
- \eta_{\mu\nu} \left( c_2 [\partial_\alpha \phi^* \partial^\alpha \phi] + (1 - c_2) [\partial_\alpha \phi \partial^\alpha \phi^*] \right)
+ \eta_{\mu\nu} m^2 \left( c_3 [\phi^* \phi] + (1 - c_3) [\phi \phi^*] \right)
+ \eta_{\mu\nu} \mu^{4-d} \frac{\lambda_a}{4!} \left( c_4 [\phi^* \phi \phi^* \phi] + (1 - c_4) [\phi \phi^* \phi \phi^* \phi] \right)
+ \eta_{\mu\nu} \mu^{4-d} \frac{\lambda_b}{4!} \left( c_5 [\phi^* \phi \phi^* \phi \phi^* \phi] + c_6 [\phi \phi^* \phi \phi^* \phi \phi^* \phi] \right)
+ \eta_{\mu\nu} \mu^{4-d} \frac{\lambda_b}{4!} \frac{1 - c_5 - c_6}{2} \left( [\phi^* \phi \phi^* \phi^* \phi^* \phi + [\phi \phi^* \phi^* \phi^* \phi^* \phi] \right). (57)
$$

In order to make the energy-momentum tensor finite, we add to it all possible real terms having the same mass dimensions and symmetry with arbitrary coefficients, to be determined

$$
T_{0\mu\nu} = d_1 \left( \partial_\mu \phi_0^* \partial_\nu \phi_0 + \partial_\nu \phi_0^* \partial_\mu \phi_0 \right) + d_2 \left( \partial_\mu \phi_0 \partial_\nu \phi_0^* + \partial_\nu \phi_0 \partial_\mu \phi_0^* \right)
+ \eta_{\mu\nu} m^2 \left( d_3 \phi_0^* \phi_0 + d_4 \phi_0 \phi_0^* \right)
+ d_5 \eta_{\mu\nu} \partial^2 (\phi_0^* \phi_0) + d_6 \eta_{\mu\nu} \phi_0 \partial^2 (\phi_0^* \phi_0)
+ d_7 \left( (\partial^2_{\mu\nu} \phi_0^*) \phi_0 + \phi_0^* (\partial^2_{\mu\nu} \phi_0) \right) + d_8 \left( (\partial^2_{\mu\nu} \phi_0) \phi_0^* + \phi_0 \phi_0^* (\partial^2_{\mu\nu} \phi_0^*) \right)
+ d_9 \eta_{\mu\nu} \left( L_0^* \phi_0 + \phi_0^* \phi_0^* \phi_0 \right) + d_10 \eta_{\mu\nu} \left( L_0^* \phi_0^* + \phi_0 \phi_0^* \phi_0^* \right)
+ \eta_{\mu\nu} \frac{\lambda_{0a}}{4!} \left( d_{11} \phi_0^* \phi_0 \phi_0^* \phi_0 + d_{12} \phi_0 \phi_0^* \phi_0^* \phi_0 + d_{13} \phi_0 \phi_0^* \phi_0 \phi_0^* \phi_0 + d_{14} \phi_0^* \phi_0 \phi_0^* \phi_0 \phi_0^* \phi_0 \right)
+ \eta_{\mu\nu} \frac{\lambda_{0b}}{4!} \left( d_{15} \phi_0^* \phi_0 \phi_0^* \phi_0^* \phi_0^* \phi_0 + \phi_0 \phi_0^* \phi_0 \phi_0^* \phi_0 \phi_0^* \phi_0 \phi_0 \right). (58)
$$

Demanding that the expression be finite in the one-loop approximation we find some restrictions on the arbitrary coefficients. From these restrictions we can determine only some of them, while the others are still arbitrary. Substituting the found coefficients back into (58) we get the general expression for the finite operator in the energy-momentum tensor of noncommutative complex scalar field theory

$$
T_{0\mu\nu}^{fin} = c_1 \left( \partial_\mu \phi_0^* \partial_\nu \phi_0 + \partial_\nu \phi_0^* \partial_\mu \phi_0 \right) + (1 - c_1) \left( \partial_\mu \phi_0 \partial_\nu \phi_0^* + \partial_\nu \phi_0 \partial_\mu \phi_0^* \right)
$$
resulting "improved" energy-momentum tensor is finite in the one-loop approximation. For simplicity we make the coefficients standing before these finite operators to be zero by choosing

\[ \eta_{\mu \nu} \ \lambda_{0} \ \frac{1}{4!} \left( (c_{2} - c_{1}/2) \phi_{0}^{*} \phi_{0} + (1/2 + c_{1}/2 - c_{2}) \phi_{0} \phi_{0}^{*} \right) \]

Let us notice that the ordering coefficients do not enter into the general expression for the traceless operators which are finite in the one-loop approximation:

\[ S_{10\mu \nu} = \frac{1}{2} \left( \partial_{\mu} \delta_{0}^{*} \phi_{0} + \partial_{\nu} \delta_{0}^{*} \phi_{0} \right) \]

\[ S_{20\mu \nu} = \frac{1}{2} \left( \partial_{\mu} \phi_{0} + \partial_{\nu} \phi_{0} \phi_{0}^{*} + \partial_{\mu} \phi_{0}^{*} \phi_{0} \right) \]

where we have introduced the following notation for the traceless operators which are finite in the one-loop approximation:

\[ S_{10\mu \nu} = \frac{1}{2} \left( \partial_{\mu} \delta_{0}^{*} \phi_{0} + \partial_{\nu} \delta_{0}^{*} \phi_{0} \right) \]

\[ S_{20\mu \nu} = \frac{1}{2} \left( \partial_{\mu} \phi_{0} + \partial_{\nu} \phi_{0} \phi_{0}^{*} + \partial_{\mu} \phi_{0}^{*} \phi_{0} \right) \]

Let us notice that the ordering coefficients do not enter into the general expression for the finite energy-momentum tensor of the theory. As far as the undefined arbitrary coefficients are concerned, the operators standing after them are finite in the one-loop approximation. For simplicity we make the coefficients standing before these finite operators to be zero by choosing \( d_{1}, d_{2}, d_{9}, d_{10} \) in a proper way. The resulting "improved" energy-momentum tensor is

\[ T_{0\mu \nu}^{I} = c_{1} \left( \partial_{\mu} \phi_{0}^{*} \phi_{0} + \partial_{\nu} \phi_{0} \phi_{0}^{*} \right) + \left( 1 - c_{1} \right) \left( \partial_{\mu} \phi_{0} \phi_{0}^{*} + \partial_{\mu} \phi_{0} \phi_{0}^{*} \right) \]

\[ + \eta_{\mu \nu} \left[ \left( c_{2} - c_{1}/2 \right) \phi_{0}^{*} \phi_{0} + (1/2 + c_{1}/2 - c_{2}) \phi_{0} \phi_{0}^{*} \right] \]

\[ + \eta_{\mu \nu} \left[ (c_{2} - c_{1}/2) \phi_{0}^{*} \phi_{0} \phi_{0}^{*} \phi_{0} + (1 + c_{1} - 2c_{2}) \phi_{0} \phi_{0}^{*} \phi_{0} \phi_{0}^{*} \right] \]

\[ + \eta_{\mu \nu} \left[ (c_{2} - c_{1}/2) \phi_{0}^{*} \phi_{0}^{*} \phi_{0} \phi_{0} + (1/2 + c_{1}/2 - c_{2}) \phi_{0} \phi_{0}^{*} \phi_{0} \phi_{0}^{*} \right] \]

\[ + \eta_{\mu \nu} \left[ \left( (c_{2} - c_{1}/2) \phi_{0}^{*} \phi_{0}^{*} \phi_{0} \phi_{0} + (1 + c_{1} - 2c_{2}) \phi_{0} \phi_{0}^{*} \phi_{0} \phi_{0}^{*} \right) \right] \]
If we consider the commutative limit $\theta^{\mu \nu} \rightarrow 0$ of this expression, we get the improved energy-momentum tensor of the commutative complex scalar field theory up to the mass term. Let us also notice that the expression for the "improved" energy-momentum tensor (62) is traceless unlike the commutative case. This situation is completely analogous to the case of real field theory [12]: in the noncommutative case the "improved" energy-momentum is traceless, and in the commutative limit it coincides with the "improved" energy-momentum tensor of the corresponding commutative theory up to the mass term.

Let us check if the "improved" energy-momentum tensor (62) leads to global conserved quantities. For this end we calculate its divergence

$$\partial^\nu T_{0 \mu \nu}^I = c_1 \partial^\nu \left\{ \partial_\mu \phi_0^* \partial_\nu \phi_0 \right\} + c_1 \partial^\nu \left\{ \partial_\nu \phi_0^* \partial_\mu \phi_0 \right\} + c_2 \partial_\mu \left\{ \partial_\alpha \phi_0^* \partial^\alpha \phi_0 \right\}$$

$$+ m_0^2 \left( c_2 - c_1/2 \right) \partial_\mu \left\{ \phi_0^* \phi_0 \right\} + \frac{1}{4} \left( c_2 - c_1 \right) \partial_\mu \partial_\nu \left\{ \phi_0^* \phi_0 \right\}$$

$$+ \frac{\lambda_{00}}{4!} \left( 2c_2 - c_1 \right) \partial_\mu \left\{ \phi_0^* \phi_0 \right\} + \frac{\lambda_{00}}{4!} \left( c_2 - c_1/2 \right) \partial_\mu \left\{ \phi_0^* \phi_0 \right\}$$

$$+ \frac{\lambda_{00}}{4!} \left\{ \left\{ \phi_0^* \phi_0^* \right\} \partial_\mu \phi_0 + \left\{ \phi_0 \phi_0^* \right\} \partial_\mu \phi_0 \right\}$$

$$+ \frac{1}{4} \frac{\lambda_{00}}{4} \partial_\mu \left\{ \left\{ \phi_0 \phi_0^* \right\} \partial_\mu \phi_0 + \left\{ \phi_0 \phi_0^* \right\} \partial_\mu \phi_0 \right\}$$

$$- \frac{m_0^2}{2} \partial_\mu \left( \phi_0 \phi_0^* \right).$$

(63)

with $\{A, B\} = A \star B - B \star A$ being the Moyal bracket. In the case of spatial noncommutativity $\theta^0 = 0$ the Moyal bracket is a spatial divergence $\{A, B\} = \partial^C C_i$, with $C_i$ being some functions of the fields of the theory. So, the expression (63) is a spatial divergence only in the case when $\mu$ is a spatial index because of the last line. In this case after integration over space coordinates we have $\int \partial^\nu J_{\mu} d^3x = 0$, and $J_{i0}$ are conserved. The quantity $J_{00}$ is conserved if the mass of the field is zero. The same situation occurs in the noncommutative theory of a real scalar field [12]: the energy of the field is conserved in the massless case only.

It is interesting to note that the "improved" energy-momentum tensor (62) depends on the two arbitrary ordering coefficients $c_1$ and $c_2$ which do not influence its renormalizability, tracelessness and conservation conditions.
6 Renormalization of the composite operators at zero momentum transfer

The purpose of this section is to renormalize scalar hermitian composite operators of the theory under consideration at zero momentum transfer. As we have seen in the previous section (see e.g. (28), (29)), in noncommutative field theories the UV divergence of a diagram depends on the value of external momenta, so we need different counterterms if the momentum transfer ($k$ in our formula) is equal to zero. This situation is typical for any noncommutative field theory (see the discussion of this problem in noncommutative real scalar field theory in [12]). Since we may expand any operator on some basis it is sufficient to study renormalization of those operators which constitute a basis. We may take as a basis the operators (18) which are integrated over the whole space-time. However, because of the cyclic property (3) the number of independent operators is reduced. Also the operators which are a total divergence disappear when we integrate them over the whole space-time. We choose the following operators as a basis:

\[
\begin{pmatrix}
\int d^d x \phi_0^* \phi_0 \\
\int d^d x \phi_0^* \phi_0 \phi_0^* \phi_0 \\
\int d^d x \phi_0^* \phi_0 \phi_0^* \phi_0 \\
\int d^d x (\phi_0^* L_0 + L_0^* \phi_0)
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\int d^d x \phi^* \phi^* \phi^* \phi \\
\int d^d x \phi^* \phi^* \phi^* \phi \\
\int d^d x \phi^* \phi^* \phi^* \phi \\
\int d^d x (\phi^* L + L^* \phi)
\end{pmatrix}
\]

for the bare and renormalized operators respectively. We see that the number of independent operators is reduced in comparison with the case of arbitrary momentum transfer (18). Composite operators at zero momentum transfer are also mixed by renormalization and we may write

\[
Q_0^{(0)} = Z^{(0)} [Q^{(0)}]
\]

with $Z^{(0)}$ being a mixing matrix for the basis of composite operators at zero momentum transfer (64) which usually differs from $Z$ in (19) in noncommutative field theories.

In this section we calculate this mixing matrix $Z^{(0)}$ in the one-loop approximation.

Let us consider the renormalization of the operator $m^2 \int d^d x \phi_0^* \phi_0$. This operator may be obtained from (24) taking $J(x) = 1$ or $\tilde{J}(k) = (2\pi)^d \delta(k)$ in momentum space. By dimensional analysis we may show that the same diagrams as in the case of arbitrary momentum transfer may contain UV divergences. They are shown in Figure 1. But unlike that case, now we have a vanishing external momentum $k$. Therefore (see the discussion after formulae (29)), in addition to integrals (28), (30), also integrals (29), (33) become UV divergent. As a result the UV divergences of the diagrams shown in Figure 1 are

\[
\frac{1}{(d-4)(4\pi)^2} \frac{\lambda_a}{3} m^2 \int d^d x \phi^* \phi,
\]

\[
\frac{1}{(d-4)(4\pi)^2} \frac{\lambda_b}{3!} m^2 \int d^d x \phi \phi^*,
\]

and we have the following renormalization relation for the operator under consideration

\[
m_0^2 \int d^d x \phi_0^* \phi_0 = m^2 [\int d^d x \phi^* \phi].
\]

This renormalization relation is similar to that of the commutative field theory (34). The same situation occurs in the theory of real scalar field theory: the renormalization
relation for composite operators at zero momentum transfer in the noncommutative theory is similar to the corresponding renormalization relation in the commutative theory [12].

As far as the remaining composite operators in (64) are concerned, the same situation occurs. Since one of the external momenta is zero the number of UV divergent diagrams becomes bigger. The renormalization relation for the composite operators at zero momentum transfer changes, in comparison with the case of arbitrary momentum transfer

\[ \int d^d x \phi^* \phi \phi^* \phi = \left( 1 + \frac{2/3 \lambda_a}{(d-4)(4\pi)^2} \right) \left[ \int d^d x \phi^* \phi \phi^* \phi \right] + \frac{1/3 \lambda_b}{(d-4)(4\pi)^2} \left[ \int d^d x \phi^* \phi \phi^* \phi \right] + \frac{8 \mu^{d-4}}{(d-4)(4\pi)^2} m^2 \left[ \int d^d x \phi^* \phi^* \phi \right], \]  

(69)

\[ \int d^d x \phi^* \phi^* \phi^* \phi = \left( 1 + \frac{2 \lambda_a + \lambda_b}{(d-4)(4\pi)^2} \right) \left[ \int d^d x \phi^* \phi \phi^* \phi \right] + \frac{1/6 \lambda_b}{(d-4)(4\pi)^2} \left[ \int d^d x \phi^* \phi \phi^* \phi \right] + \frac{4 \mu^{d-4}}{(d-4)(4\pi)^2} m^2 \left[ \int d^d x \phi^* \phi^* \phi \right]. \]  

(70)

As a result we have that the mixing matrix \( Z^{(0)} \) in the one-loop approximation is

\[ Z^{(0)} = \begin{pmatrix} 1 + \frac{2/3 \lambda_a}{(d-4)(4\pi)^2} & \frac{1/3 \lambda_b}{(d-4)(4\pi)^2} & \frac{8 \mu^{d-4}}{(d-4)(4\pi)^2} & 0 \\ \frac{1/6 \lambda_b}{(d-4)(4\pi)^2} & 1 + \frac{2 \lambda_a + \lambda_b}{(d-4)(4\pi)^2} & \frac{8 \mu^{d-4}}{(d-4)(4\pi)^2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]  

(71)

In the next section we study the energy-momentum vector which follows from the Noether’s procedure.

7 Renormalization of the energy-momentum vector

The purpose of this section is to construct the finite and conserved energy-momentum vector of noncommutative complex scalar field theory. From expression (13) we may define it as

\[ P_\mu = \int \left( \partial_\mu \phi^*_0 \phi_0 + \partial_0 \phi^*_0 \phi_0 - \eta_{0\mu} \partial_0 \phi^*_0 \phi_0 + \eta_{0\mu} m_0^2 \phi^*_0 \phi_0 + \eta_{0\mu} \frac{\lambda_{a0}}{4!} \phi^*_0 \phi_0 \phi^*_0 \phi_0 + \eta_{0\mu} \frac{\lambda_{b0}}{4!} \phi^*_0 \phi_0 \phi^*_0 \phi_0 \right) d^{d-1}x. \]  

(72)

As we noted above, since \( \theta^{0i} = 0 \), we have no time derivatives in the star product (1), and consequently, the properties (2) and (3) are still valid in the case of spatial integration only. Therefore we have no problem of field ordering, and the energy-momentum vector is defined unambiguously. It is evident that \( P_\mu \) is conserved in time \( \partial \bar{\partial} P_\mu = 0 \). Now we show that this operator is finite, at least in the one-loop approximation.
In order to prove the finiteness of the energy-momentum vector (72), we need to renormalize each of the six composite operators appearing in its expression. As an example, let us consider the renormalization of the operator $m_0^2 \int \phi_0^* \phi_0 \, d^{d-1}x$. In order to renormalize such an operator, we put $J(x) = \delta(x^0 - t)$ in the integrand (72). In momentum space we have that $J \sim \delta(k)$. In the case of spatial noncommutativity this leads to $\theta_{\mu \nu} k_{\nu} = 0$ and, apart from (28) and (30), also expressions (29) and (33) become UV divergent. This is the appearance of the UV/IR mixing: the divergences of a diagram depend on whether we put some of external momenta to zero before or after regularization is removed. As a result we have the following renormalization relation

$$m_0^2 \int \phi_0^* \phi_0 \, d^{d-1}x = m^2 \left[ \int \phi^* \phi \, d^3x \right]. \quad (73)$$

For the remaining composite operators similar calculations give

$$\int \partial_\mu \phi^*_0 \partial_\nu \phi_0 \, d^{d-1}x = \left[ \int \partial_\mu \phi^*_0 \partial_\nu \phi \, d^3x \right]$$

$$+ \frac{2}{(d-4)(4\pi)^2} \left[ \frac{2\lambda_a + \lambda_b}{4!} \left( \frac{1}{6} \eta_{\mu \nu} \delta_{ab}^2 + \frac{1}{3} \eta_{\nu \rho} \delta_{ab} \right) \right] \left[ \int \phi^* \phi \, d^3x \right]$$

$$+ \frac{\mu^{4-d}}{(d-4)(4\pi)^2} \left[ 4 \frac{\lambda_a^2 + \lambda_b^2}{(4!)^2} \eta_{\mu \nu} \right] \left[ \int \phi^* \phi \, d^3x \right]$$

$$+ \frac{\mu^{4-d}}{(d-4)(4\pi)^2} \frac{\lambda_a \lambda_b}{4!} \eta_{\mu \nu} \left[ \int \phi^* \phi \, d^3x \right], \quad (74)$$

$$\frac{\lambda_{ab}}{4!} \int \phi_0^* \phi_0^* \phi_0^* \phi_0 \, d^{d-1}x =$$

$$\mu^{4-d} \left( \frac{\lambda_a}{4!} + \frac{1}{(d-4)(4\pi)^2} \frac{8\lambda_a^2 - 2\lambda_b^2}{(4!)^2} \right) \left[ \int \phi^* \phi \phi^* \phi \, d^3x \right]$$

$$+ \frac{8 \mu^{4-d}}{(d-4)(4\pi)^2} \frac{\lambda_a \lambda_b}{4!} \left[ \int \phi^* \phi \phi^* \phi \, d^3x \right]$$

$$+ \frac{8}{(d-4)(4\pi)^2} \frac{\lambda_a}{4!} m^2 \left[ \int \phi^* \phi \, d^3x \right], \quad (75)$$

$$\frac{\lambda_{ab}}{4!} \int \phi_0^* \phi_0^* \phi_0^* \phi_0 \, d^{d-1}x = \frac{4 \mu^{4-d}}{(d-4)(4\pi)^2} \left( \frac{\lambda_b}{4!} \right)^2 \left[ \int \phi^* \phi \phi^* \phi \, d^3x \right]$$

$$+ \frac{4}{(d-4)(4\pi)^2} \frac{\lambda_b}{4!} \left[ \int \phi^* \phi \phi^* \phi \, d^3x \right]$$

$$+ \frac{4}{(d-4)(4\pi)^2} \frac{\lambda_a}{4!} m^2 \left[ \int \phi^* \phi \, d^3x \right]. \quad (76)$$

Substituting (73, 76) in (72) we find that the energy-momentum vector (72) is finite in the one-loop approximation

$$P_\mu = \left[ \int \partial_\mu \phi^*_0 \partial_0 \phi \, d^3x \right] + \left[ \int \partial_0 \phi^*_0 \partial_\mu \phi \, d^3x \right] - \eta_{0\mu} \left[ \int \phi^* \phi \phi^* \phi \, d^3x \right]$$

$$+ \eta_{0\mu} m^2 \left[ \int \phi^* \phi \, d^3x \right] + \eta_{0\mu} \frac{\mu^{4-d} \lambda_a}{4!} \left[ \int \phi^* \phi \phi^* \phi \, d^3x \right]$$

$$+ \eta_{0\mu} m^2 \left[ \int \phi^* \phi \, d^3x \right] + \eta_{0\mu} \frac{\mu^{4-d} \lambda_b}{4!} \left[ \int \phi^* \phi \phi^* \phi \, d^3x \right]$$

$$+ \eta_{0\mu} m^2 \left[ \int \phi^* \phi \, d^3x \right]. \quad (77)$$
\[ + \eta_{0\mu} \frac{\mu^{4-d} \lambda_b}{4!} \left[ \int \phi^* \star \phi^* \star \phi \star \phi \, d^3 x \right]. \tag{77} \]

This situation is similar to that in the noncommutative scalar field theory [12]: the energy-momentum vectors which follow from the Noether’s theorem are finite in both theories and do not require improving, but the energy-momentum tensors must be improved in order to be finite, and this improving makes them conserved only in the massless case.

8 Summary

In this paper we have derived with the help of the Noether’s procedure the classical energy-momentum tensor of the noncommutative complex scalar field theory. It was shown that it cannot be defined unambiguously and its expression is defined up to six arbitrary ordering constants.

Next we have considered the renormalization of dimension four composite operators of the theory and have found that the renormalization of any composite operators of the theory demands to take into account all composite operators with the same mass dimension. This phenomenon is called operator mixing and is typical for the renormalization of composite operators of any theory. The proper bases of hermitian scalar operators have been constructed both for the bare and renormalized operators. Due to the noncommutativity the number of the operators in the bases is larger, in comparison with the commutative theory. The mixing matrix which expresses the bare operators of the basis in term of the renormalized ones is calculated in the one-loop approximation.

We considered the renormalization of the energy-momentum tensor which follows from the Noether’s theorem and found it to be divergent in the one-loop approximation. In order to make it finite we have to add ”improving” terms to it. The expression for the ”improved” energy-momentum tensor has been calculated and shown to be, apart from traceless, conserved in the massless case only. Besides, some ordering constants do not enter into the expression for the ”improved” energy-momentum tensor.

The renormalization of the composite operators at zero momentum transfer was also considered. The number of operators in the bases of such operators is reduced in comparison with the case of arbitrary momentum transfer, although it is bigger than in the corresponding commutative case due to the noncommutativity. The mixing matrix for the case of zero momentum transfer was calculated in the one-loop approximation. Finally we find that, as in the case of noncommutative real scalar field theory [12], the energy-momentum vector which follows from the Noether’s theorem is conserved and finite in the one-loop approximation.

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References

[1] N. Seiberg and E. Witten, "String theory and noncommutative geometry", JHEP 9909 (1999) 032, hep-th/9908142

[2] J.C. Collins, "Renormalization", Cambridge University Press, 1984.

[3] Iouri Chepelev, Radu Roiban, "Renormalization of Quantum Field Theories on Noncommutative $R^d$, I. Scalars", JHEP 0005 (2000) 037, hep-th/9911098, "Convergence Theorem for Non-commutative Feynman Graphs and Renormalization", JHEP 0103 (2001) 001, hep-th/0008090

[4] S. Minwalla, M. Van Raamsdonk, N. Seiberg, "Noncommutative Perturbative Dynamics", JHEP 0002 (2000) 020, hep-th/9912072

[5] I. Ya. Aref’eva, D. M. Belov, A. S. Koshelev, "Two-Loop Diagrams in Noncommutative $\phi^4$ theory", Phys. Lett. B476 (2000) 431-436, hep-th/9912075

[6] A. Micu, M.M. Sheikh-Jabbari, "Noncommutative $\Phi^4$ Theory at Two Loops", JHEP 0101 (2001) 025, hep-th/0008057

[7] I.Ya. Aref’eva, D.M. Belov, A.S. Koshelev, "A Note on UV/IR for Noncommutative Complex Scalar Field", hep-th/0001215

[8] I.L. Buchbinder, V.A. Krykhtin, "One-loop renormalization of general noncommutative Yang-Mills field model coupled to scalar and spinor fields", Int. J. Mod. Phys. A18 (2003) 3057, hep-th/0207086

[9] M.R. Douglas, N.A. Nekrasov, "Noncommutative Field Theory", Rev. Mod. Phys. 73 (2001) 977-1029, hep-th/0106048

[10] R. Szabo, "Quantum Field Theory on Noncommutative Spaces", hep-th/0109162

[11] I.Ya. Aref’eva, D.M. Belov, A.A. Giryavets, A.S. Koshelev, P.B. Medvedev, "Noncommutative Field Theories and (Super)String Field Theories", hep-th/0111208

[12] S. Bellucci, I.L. Buchbinder, V.A. Krykhtin, "Renormalization of the energy-momentum tensor in noncommutative scalar field theory", Nucl. Phys. B665 (2003) 402-424. hep-th/0303186

[13] A. Gerhold, J. Grimstrup, H. Grosse, L. Popp, M. Schweda, R. Wulkenhaar, "The energy-momentum tensor on noncommutative spaces - some pedagogical comments", hep-th/0012112

[14] Yuji Okawa, Hirosi Ooguri, "Energy-momentum Tensors in Matrix Theory and in Noncommutative Gauge Theories", hep-th/0103124
[15] Mohab Abou-Zeid, Harald Dorn, "Comments on the Energy-Momentum Tensor in Non-Commutative Field Theories", Phys. Lett. B514 (2001) 183-188, hep-th/0104244.

[16] J.M. Grimstrup, B. Kloibock, L. Popp, V. Putz, M. Schweda, M. Wickenhauser, "The Energy-Momentum Tensor in Noncommutative Gauge Field Models", hep-th/0210288.

[17] Ashok Das and J. Frenkel, "On the energy-momentum tensor in non-commutative gauge theories", Phys.Rev. D67 (2003) 067701, hep-th/0212122.

[18] Subir Ghosh, "Space-Time Symmetries in Noncommutative Gauge Theory: A Hamiltonian Analysis", hep-th/0310155.