A General Theory of the Stochastic Linear Bandit and Its Applications

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Recent growing adoption of experimentation in practice has led to a surge of attention to multiarmed bandits as a technique to reduce the opportunity cost of online experiments. In this setting, a decision-maker sequentially chooses among a set of given actions, observes their noisy rewards, and aims to maximize her cumulative expected reward (or minimize regret) over a horizon of length $T$. In this paper, we introduce a general analysis framework and a family of algorithms for the stochastic linear bandit problem that includes well-known algorithms such as the optimism-in-the-face-of-uncertainty-linear-bandit (OFUL) and Thompson sampling (TS) as special cases. Our analysis technique bridges several streams of prior literature and yields a number of new results. First, our new notion of optimism in expectation gives rise to a new algorithm, called sieved greedy (SG) that reduces the overexploration problem in OFUL. SG utilizes the data to discard actions with relatively low uncertainty and then choosing one among the remaining actions greedily. In addition to proving that SG is theoretically rate optimal, our empirical simulations show that SG outperforms existing benchmarks such as greedy, OFUL, and TS. The second application of our general framework is (to the best of our knowledge) the first polylogarithmic (in $T$) regret bounds for OFUL and TS, under similar conditions as the ones by Goldenshluger and Zeevi (2013). Finally, we obtain sharper regret bounds for the $k$-armed contextual MABs by a factor of $\sqrt{k}$.

Key words: Stochastic Linear Bandit, Contextual Bandit, Thompson Sampling, Optimism in the Face of Uncertainty, Greedy Algorithms

1. Introduction

Recently, multiarmed bandit (MAB) experiments have received extensive attention due to their potential for reducing the opportunity cost of running online experiments (Scott 2010, 2015, Johari et al. 2017). Specifically, MAB experiments allow adaptive adjustments to the design of the experiments, based on partially available data during the experiment. The MAB approach was first motivated by the cost of experimentation in clinical trials (Thompson 1933, Lai and Robbins 1985).

More formally, in a MAB problem, a decision-maker, also known as the policy or algorithm, sequentially chooses actions from given action sets and receives rewards corresponding to the selected actions. The goal is to maximize the cumulative reward throughout the experimentation...
periods, by utilizing the *history* of previous observations. Alternatively, the aim is to choose a policy that minimizes the cumulative *regret*, which is the difference between the highest achievable reward by a clairvoyant decision-maker who knows the expected reward of each action relative to the reward obtained by the policy. This paper considers a variant of this problem, called *stochastic linear bandit*, in which all actions are elements of \( \mathbb{R}^d \) for a positive integer \( d \) and the expected value of the reward depends on the actions via a linear function. This class of problems includes the well-known subclass of \( k \)-armed contextual MABs as a special case, when the action sets are allowed to be time-dependent.

Since its introduction by Abe and Long (1999), the linear bandit problem has attracted a great deal of attention. Several algorithms based on the idea of optimism or upper confidence bound (UCB), due to Lai and Robbins (1985), have been proposed and analyzed. Notable examples are by Auer (2003), Dani et al. (2008), Rusmevichientong and Tsitsiklis (2010), Abbasi-Yadkori et al. (2011), and Lattimore (2015). The best algorithm in this class is the optimism-in-the-face-of-uncertainty-linear-bandit (OFUL) algorithm of Abbasi-Yadkori et al. (2011) with the regret \( O(d\sqrt{T \log T}) \) that matches the best lower bound due to Dani et al. (2008) up to logarithmic factors.

A second line of research examines the performance of Thompson sampling (TS) or posterior sampling, a Bayesian heuristic due to Thompson (1933) that employs the posterior distribution of the reward function to balance exploration and exploitation and reduce regret. Russo and Van Roy (2014), Dong and Van Roy (2018) proved an \( O(d\sqrt{T \log T}) \) upper bound for the *Bayesian* regret of TS, thereby indicating its near-optimality.

In addition, when there is a deterministic gap \( \delta \) between the expected rewards of the top two actions, OFUL and TS are shown to have a regret with a \( \log(T)/\delta \) dependence in \( T \) instead of a \( \sqrt{T \log T} \) one. But this bound is not applicable when \( \delta \) is exactly zero. In fact, this happens for the well-known subclass of linear \( k \)-armed contextual MABs. Hence, the prior \( \log T/\delta \) bounds for OFUL or TS are not applicable. One needs a more general (probabilistic) notion of the gap to study these types of problems. This is in fact the subject of the third stream of research, pioneered by Goldenshluger and Zeevi (2013), that leverages a so-called *margin condition* to model probabilistic reward \( \delta \). They showed that the best lower bound for the contextual MAB is logarithmic in \( T \), and proposed a variant of the \( \epsilon \)-greedy algorithm that achieves this bound. This idea was extended by Bastani and Bayati (2020) to settings where contexts are high-dimensional (i.e., \( d \) becomes very large). However, both of these papers propose algorithms that require an input parameter \( h \) to adjust for the probabilistic gap. *It is an open problem whether such logarithmic (in \( T \)) bounds for OFUL and TS, that do not take any gap parameter as input can be proved, under the same*
conditions as in (Goldenshluger and Zeevi 2013). In addition, while the first two streams of the aforementioned research were mostly united by the results of Russo and Van Roy (2014) and Abeille et al. (2017) that connected OFUL and TS, there was a disconnect between them and the third stream of research.

Contributions. In this paper we propose an analysis framework for the stochastic linear bandit problem that bridges all three aforementioned streams of literature and yields a number of new results. To be explicit, the main contributions of this paper are as follows:

1. We propose a general family of algorithms, called randomized OFUL (ROFUL), for the stochastic linear bandit problem and prove that they are rate optimal (their regret matches the best lower bound). We also show that OFUL and TS are special cases of this family of algorithms and that our regret bound for ROFUL recovers previously known rate-optimal regret bounds for OFUL and TS in Bayesian and frequentist settings, respectively.

2. Most importantly, we employ the margin assumption of Goldenshluger and Zeevi (2013) to model a probabilistic gap that allows δ to be zero, in order to obtain a polylogarithmic (in $T$) regret bound for OFUL and TS when the action sets are independently drawn from an unknown distribution; this includes the contextual MAB problem as in (Goldenshluger and Zeevi 2013). To the best of our knowledge, this result had not been known for OFUL and TS.

3. Our analysis of ROFUL naturally leads us to introduce a new rate-optimal policy, Sieved Greedy (SG), that leverages data to reduce the overexploration problem in OFUL and TS. A key technique to achieve this is to employ a more general form of the optimism principle that we introduce, called optimism in expectation, that allows data-driven exploration by focusing only on actions with sufficient uncertainty, and then choosing one of them greedily. Our empirical simulations show that SG outperforms greedy, OFUL, and TS.

While SG has the same spirit as recent literature on algorithms that put more emphasis on greedy decision-making (Bastani et al. 2017, Kannan et al. 2018, Raghavan et al. 2018, Hao et al. 2019, Bayati et al. 2020), it relies on the notion of optimism in expectation that is fundamentally a different idea compared to the ones powering the prior literature (e.g., covariate diversity or large number of arms). Investigating potential connections between all these algorithms is a tantalizing direction for future research.

4. Motivated by the fact that the $k$-armed $d$-dimensional contextual MAB problem is a special case of the $kd$-dimensional stochastic linear bandit, see Abbasi-Yadkori (2012) for the reduction, we formulate a slightly more general version of the stochastic linear bandit that we refer to as the grouped linear bandit (GLB). Then, using the structure of GLB, we obtain sharper regret bounds (by a factor $\sqrt{k}$) for our general ROFUL algorithm. Therefore, we obtain sharper regret
bounds for OFUL and TS than those that can be obtained by directly applying the existing approaches studied by Abbasi-Yadkori et al. (2011), Russo and Van Roy (2014), Agrawal and Goyal (2013), and Abeille et al. (2017).

1.1. Other literature

Some of the components in our analysis of the ROFUL algorithm have similarities with prior literature (Srinivas et al. 2010, Russo and Van Roy 2016, Kirschner and Krause 2018). Specifically, our notion of uncertainty complexity is similar to the notion of maximum information by Kirschner and Krause (2018). We discuss the differences between the two in Remark 3.1, but in summary our approach provides regret bounds for the more general probabilistic δ setting as well. In addition, our notion of gain rate is similar to the notion of information ratio by Russo and Van Roy (2016). We discuss the differences between the two in Remark 4.3, but in summary Russo and Van Roy (2016) consider a Bayesian setting while we consider both Bayesian and frequentist settings.

1.2. Organization

We introduce notation and the problem formulation in Section 2. Then we introduce uncertainty complexity and its connection to regret in Sections 3 and 4. Our ROFUL algorithm and its regret analysis are presented in Section 5. In Section 6, we first demonstrate how OFUL and TS are special cases of ROFUL and then introduce our rate-optimal SG algorithm, which is empirically compared with existing benchmarks in Section 7. Finally, in Section 8, we provide extensions of our results to obtain polylogarithmic regret bounds for OFUL and TS and sharper bounds for the k-armed contextual bandit problem.

2. Problem Formulation and Notation

For any positive integer $n$, we denote $\{1, 2, \ldots, n\}$ by $[n]$. The transpose of a vector $A$ is denoted by $A^\top$. For positive semidefinite matrix $\Sigma \in \mathbb{R}^{n \times n}$ and for any vector $A \in \mathbb{R}^n$, notation $\|A\|_\Sigma$ refers to $\sqrt{A^\top \Sigma A}$. We also use notation $I_d$ to denote the $d$-by-$d$ identity matrix.

Let $\mathcal{A}$ be the set of all possible actions and let $(\mathcal{A}_t)_{t=1}^T$ be a sequence of $T$ random subsets of $\mathcal{A}$, where $T \in \mathbb{N}$ will be referred to as the time horizon. A policy $\pi$ sequentially interacts with this environment in $T$ rounds. At time $t \in [T]$, the action set $\mathcal{A}_t$ is revealed to the policy and it chooses an action $\tilde{A}_t \in \mathcal{A}_t$ and receives a stochastic reward $\mathcal{R}(\tilde{A}_t)$. We also assume that $\mathcal{A} \subset \mathbb{R}^d$ and is bounded; that is, there exists a positive constant $a$ such that $\|A\|_2 \leq a$ for all $A \in \mathcal{A}$. Moreover, we assume there exists a random vector $\Theta^* \in \mathbb{R}^d$ for which

$$
E[\mathcal{R}(A) | \Theta^*] = \langle \Theta^*, A \rangle,
$$

(2.1)
for all $A \in \mathcal{A}_t$ where $\langle \cdot, \cdot \rangle$ is the standard dot product on $\mathbb{R}^d$. We also assume that there exists a positive parameter $\theta$ such that distribution of $\Theta^*$ satisfies

$$\forall \xi > 0 , \quad P(\|\Theta^*\|_2 \geq \theta + \xi) \leq e^{-\xi^2/2}.$$  \hspace{1cm} (2.2)

For example, if $\Theta^*$ is bounded then Eq. (2.2) easily holds. Another important setting where Eq. (2.2) holds is when $\Theta^*$ has a normal distribution.

Next, we introduce the notation

$$M_t(A) := \langle \Theta^*, A \rangle,$$

and assume that there exists (random) optimal action $A^*_t \in \mathcal{A}_t$ such that the following holds almost surely for all $A \in \mathcal{A}_t$,

$$M_t(A^*_t) \geq M_t(A).$$  \hspace{1cm} (2.3)

Now, consider the sequence of $\sigma$-algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \cdots \subseteq \mathcal{F}_{t-1}$ that encode history of observations up to time $t$ and are defined by

$$\mathcal{F}_0 := \sigma(\emptyset) \quad \text{and} \quad \mathcal{F}_{t-1} := \sigma(A_1, \tilde{A}_1, R(\tilde{A}_1), \ldots, A_{t-1}, \tilde{A}_{t-1}, R(\tilde{A}_{t-1}), A_t).$$

In this model, a policy $\pi$ is formally defined as a deterministic function that maps $\mathcal{F}_{t-1}$ to an element of $\mathcal{A}_t$.

Moreover, for each chosen action $\tilde{A}_t \in \mathcal{A}_t$, its stochastic reward $R(\tilde{A}_t)$ is equal to $\langle \Theta^*, \tilde{A}_t \rangle + \varepsilon_t$ where, conditional on $\mathcal{F}_{t-1}$, the noise random variable $\varepsilon_t$ has mean zero and is $\sigma^2$-sub-Gaussian. Specifically, $\mathbb{E}[|\varepsilon_t| | \mathcal{F}_{t-1}] < \infty$ and

$$\mathbb{E}[\exp(\kappa \varepsilon_t) | \mathcal{F}_{t-1}] \leq \exp \left( \frac{\sigma^2 \kappa^2}{2} \right), \quad \forall \kappa \in \mathbb{R}.$$  

The performance measure for evaluating the policies is the standard cumulative Bayesian regret defined as

$$\text{Regret}(T,\pi) := \sum_{t=1}^{T} \mathbb{E} \left[ \sup_{A \in \mathcal{A}_t} M_t(A) - M_t(A^*_t) \right].$$

The expectation is taken with respect to the entire randomness in our model, including the prior distribution of $\Theta^*$. Although we assumed that $\Theta^*$ is random, and we described a Bayesian formulation of regret, our model and results include the deterministic $\Theta^*$ setting as well. This can be achieved by considering the prior distribution to be the distribution with a point mass at $\Theta^*$. 

Action sets. Action sets and their structure play key roles in this paper which require introducing a number of important notions associated with them. We start by defining the extremal points of an action set.

**Definition 2.1 (Extremal points)** For an action set $\mathcal{A}$, define its extremal points $E(\mathcal{A})$ to be all $A' \in \mathcal{A}$ that are not a convex combination of other actions in $\mathcal{A}$, i.e., actions in $\mathcal{A}$ for which one cannot find actions $\{A_1, \cdots, A_n\} \subseteq \mathcal{A} \setminus \{A'\}$ and coefficients $\{c_1, \cdots, c_n\} \subset [0, 1]$ satisfying

$$A' = \sum_{i=1}^{n} c_i A_i \quad \text{and} \quad \sum_{i=1}^{n} c_i = 1.$$ 

The importance of this definition is that all the algorithms studied in this paper choose only extremal points in action sets, because of the linearity assumption on the mean reward as stated in Eq. (2.1). This observation implies that the rewards attained by any of these algorithms, when provided with the action set $\mathcal{A}$, belong to the reward profile of $\mathcal{A}$ defined by

$$\Pi_A := \{M_t(A) : A \in E_A\}.$$ 

Recall from Eq. (2.3) that $M_t(A^*_i)$ is the maximum attainable reward of an action set $\mathcal{A}$. Building on this, we define gap of an action set $\mathcal{A}$ as

$$\Delta_A := M_t(A^*_i) - \sup(\Pi_A \setminus \{M_t(A^*_i)\}).$$ 

Moreover, for any $z \geq 0$, we define

$$A^z_t := \{A \in \mathcal{A}_t : M_t(A) \geq M_t(A^*_i) - z\}.$$ 

In the sequel, we may simplify the above notation and use subscript $t$ instead of $\mathcal{A}_t$. For instance, $\Delta_t$ refers to $\Delta_{\mathcal{A}_t}$. We now define a gapped problem as follows:

**Definition 2.2 (Gapped problem)** We call a linear bandit problem gapped if for some positive numbers $\delta$ and $q_\delta$, the following inequality holds:

$$\mathbb{P}(\Delta_t \leq \delta) \leq q_\delta \quad \text{for all } t \in [T],$$

where the probability is calculated with respect to the randomness of the action sets. Moreover, for a fixed gap level $\delta$, we define $G_t$ to be the indicator of the event $\{\Delta_t \geq \delta\}$.

**Remark 2.1** The above notion of gap is more general than the well-known notion of gap in the literature, as in (Abbasi-Yadkori et al. 2011), which is a deterministic concept. Specifically, we do not assume that the probability $q_\delta$ is equal to 0, for a fixed $\delta > 0$.

**Remark 2.2** All problems are gapped for all $\delta > 0$ and $q_\delta = 1$ since Eq. (2.4) will be trivially satisfied. This observation will help us obtain gap-independent bounds.
3. Uncertainty Complexity

In this section we introduce the notion of uncertainty structure, which will be a key parameter in obtaining regret bounds in subsequent sections. We also calculate this parameter in three examples to help build intuition.

By uncertainty structure, we simply refer to a sequence of functions $V_t : (\mathcal{F}_{t-1}, A) \mapsto \mathbb{R}$, where $A \in \mathcal{A}_t$. By a slight abuse of notation, for any policy $\pi$, we define expected uncertainty to be

$$V(\pi) := \mathbb{E} \left[ \sum_{t=1}^{T} V_t(A_t) \right].$$

Finally, for a set of policies $P$, the uncertainty complexity is defined as

$$K := \sup_{\pi \in P} V(\pi).$$

Note that uncertainty complexity is not a unique quantity for a given problem as the choice of functions $V_t$ can vary. We will see in the following sections that any uncertainty structure, together with an associated gain rate that is defined in Section 4, can be used to provide an upper bound for the regret of any policy. However, the quality of the regret bound does depend on the choice of uncertainty structure.

In order to get a better of sense of uncertainty complexity, in the remainder of this section we provide upper bounds for the uncertainty complexity of several well-known problems. We then use these bounds in Section 6 to derive rate-optimal regret bounds for OFUL and variants of TS. Overall, the optimal selection of an uncertainty structure is an interesting and challenging research question, but one that is well beyond the scope of this paper.

Remark 3.1 The above notion of uncertainty complexity is similar to the notion of maximum information by Kirschner and Krause (2018), see their Eq. (2). The main difference is that we do not require the essential supremum of $\sum_t V_t$ to exist. This makes our analysis simpler; see, e.g., the second paragraph on page 6 of (Kirschner and Krause 2018). Also, in contrast to Kirschner and Krause (2018), our proof technique provides regret bounds for the (generalized) gapped version of the problem as well.

Example 3.1 (Unstructured linear bandit) Let $\lambda$ be a positive and fixed real number and, for any $t \in [T]$, define

$$\Sigma_t := \left( \frac{1}{\lambda} I_d + \frac{1}{\sigma^2} \sum_{s=1}^{t} A_s A_s^\top \right)^{-1}. \quad (3.1)$$
Then, we choose the following uncertainty structure:

$$V_t(A) := \min \left\{ \sigma^2, \|A\|_\Sigma^{-1} \right\}.$$ 

Lemmas 10 and 11 of Abbasi-Yadkori et al. (2011) essentially prove that

$$K \leq 2\sigma^2 d \log \left( 1 + \frac{T\alpha^2\lambda}{d\sigma^2} \right). \quad (3.2)$$

Example 3.2 (Bayesian linear bandit with fixed finite action sets) Consider a finite action set \( \mathcal{A} = \{A_1, A_2, \ldots, A_k\} \) that does not change over time. In other words, \( A_t = \mathcal{A} \) for all \( t \in [T] \) almost surely. Following a similar notation as Russo and Van Roy (2016), for all \( j \in [k] \), we let

$$\alpha_{t,j} := \Pr(A^* = A_j | F_{t-1})$$

and

$$\mu_{t,j} := \mathbb{E}[\Theta^* | F_t, A^* = A_j] = \frac{1}{\alpha_{t,j}} \mathbb{E}[\Theta^* : I(A^* = A_j) | F_{t-1}].$$

Now, defining \( \mu_t := \mathbb{E}[\Theta^* | F_{t-1}] \), we consider the following uncertainty functions:

$$V_t(A) := \|A\|_{\Gamma_t}^2,$$

where

$$\Gamma_t := \sum_{j=1}^k \alpha_j (\mu_{t,j} - \mu_t)(\mu_{t,j} - \mu_t)^\top. \quad (3.3)$$

The analysis of Russo and Van Roy (2016) implies that

$$K \leq 2\sigma^2 H(A^*), \quad (3.4)$$

where \( H(A^*) \) is the entropy of \( A^* \). For completeness, we provide a slightly modified version of their proof in Section A.

Example 3.3 (Bayesian linear bandit with normal prior and noise) In this example, we focus on the Bayesian setting in which \( \Theta^* \sim \mathcal{N}(0, \lambda I_d) \), and at round \( t \), the reward of selecting action \( \tilde{A}_t \) is given by \( R(\tilde{A}_t) = \langle \Theta^*, \tilde{A}_t \rangle + \epsilon_t \) where \( \epsilon_t \sim \mathcal{N}(0, \sigma^2) \) is independent of \( F_{t-1} \). However, we allow the action sets to change over time and also allow the action sets to have infinite size. Inspired by the previous example, we define

$$V_t(A) := \|A\|_{\Gamma_t}^2,$$
where

\[ \Gamma_t := \text{Cov}(E[\Theta^* | F_{t-1}, A_t^*] | F_{t-1}) \]  

(3.5)

It is easy to see that in the setting of Example 3.2, the above definition is equivalent to Eq. (3.3).

We now use a different technique to bound the uncertainty complexity. Notice that the normality assumption yields

\[ \Gamma_t \approx \text{Cov}(\Theta^* | F_{t-1}) = \Sigma_t. \]

Therefore, Example 3.1 implies that

\[ K \leq 2\sigma^2 d \log \left( 1 + \frac{T \alpha^2 \lambda}{d \sigma^2} \right). \]

(3.6)

4. Regret Bound and Gain Rate

In this section, building on the notion of uncertainty structure, we introduce the notion of gain rate of any policy and then use that to obtain an upper bound for the regret.

Definition 4.1 (Gain rate) Let \( \delta > 0 \) be fixed. We say that a policy \( \pi \) has gain rate \( G_\delta > 0 \) with respect to an uncertainty structure \( \{\mathcal{V}_t\}_{t \geq 1} \) if

\[ \mathbb{E} \left[ \left( M_t(A_t^*) - M_t(\tilde{A}_t) \right)^2 : \mathbb{I} \left( M_t(A_t^*) - M_t(\tilde{A}_t) \geq \delta \right) \right] \leq \frac{1}{G_\delta} \mathbb{E} \left[ \mathcal{V}_t(\tilde{A}_t) \right] + \frac{D_\delta}{2t^2}, \]

(4.1)

for all \( t \in [T] \).

Remark 4.1 The constant \( D_\delta \) is meant to account for very unlikely cases where the observations deviate from generic cases (this will be formalized by tail bounds). In most cases, \( D_\delta \) can be set to 0 or 1.

We are ready now to state a general result on the regret of any policy for any gap level \( \delta \) that relies on uncertainty complexity \( K \), gain rate \( G_\delta \), and \( D_\delta \).

Theorem 4.1 Given an uncertainty structure \( \{\mathcal{V}_t\}_{t \geq 1} \), gap level \( \delta \), and associated parameter \( q_\delta \), the regret of any policy \( \pi \) satisfies

\[ \text{Regret}(T, \pi) \leq \frac{K}{\delta G_\delta} + \frac{D_\delta}{\delta} + T \delta q_\delta. \]

(4.2)
Remark 4.2 (Problem-independent bound) In most examples of this paper we will prove the following stronger variant of Eq. (4.1):
\[
\mathbb{E} \left[ \left( M_t(A_t^*) - M_t(\tilde{A}_t) \right)^2 \right] \leq \frac{1}{G} \mathbb{E} \left[ V_t(\tilde{A}_t) \right] + \frac{D}{2\delta^2}.
\] (4.3)
This inequality implies that the gain rate is not a function of $\delta$, which means that the regret bound in Eq. (4.2) holds for any $\delta$. Therefore, one can take the infimum of the right-hand side of Eq. (4.2) over $\delta$ to get a $\delta$-independent regret bound
\[
\text{Regret}(T, \pi) \leq \inf_{\delta > 0} \left\{ \frac{K}{G} + \frac{D}{\delta} + T\delta q_3 \right\}
\leq \inf_{\delta > 0} \left\{ \frac{K}{G} + \frac{D}{\delta} + T\delta \right\}
= 2 \sqrt{\left( \frac{K}{G} + D \right) T}.
\] (4.4)
Remark 4.3 The above notion of gain rate is similar to the notion of information ratio of Russo and Van Roy (2016). Specifically, if $\{V_t\}_{t \geq 1}$ is defined as in Example 3.2 and $D = 0$, then $1/G$ becomes the information ratio. We also note that Russo and Van Roy (2016) consider a Bayesian setting while our gain rate is defined for both Bayesian and frequentist settings.

Proof of Theorem 4.1. Let $B_t$ be the shorthand for the indicator function $\mathbb{I} \left( M_t(A_t^*) - M_t(\tilde{A}_t) \geq \delta \right)$. We then have
\[
\mathbb{E} \left[ \mathcal{R}(A_t^*) - \mathcal{R}(\tilde{A}_t) \right] = \mathbb{E} \left[ M_t(A_t^*) - M_t(\tilde{A}_t) \right]
= \mathbb{E} \left[ \left( M_t(A_t^*) - M_t(\tilde{A}_t) \right) \cdot B_t \right] + \mathbb{E} \left[ \left( M_t(A_t^*) - M_t(\tilde{A}_t) \right) \cdot (1 - B_t) \right]
\leq \frac{1}{\delta} \mathbb{E} \left[ \left( M_t(A_t^*) - M_t(\tilde{A}_t) \right)^2 \cdot B_t \right] + \mathbb{E} \left[ \left( M_t(A_t^*) - M_t(\tilde{A}_t) \right) \cdot (1 - B_t) \right]
\leq \frac{1}{\delta G_\delta} \mathbb{E} [V_t(\tilde{A}_t)] + \frac{D_\delta}{2\delta^2} + \mathbb{E} \left[ \left( M_t(A_t^*) - M_t(\tilde{A}_t) \right) \cdot (1 - B_t) \right].
\]
By summing both sides of the above inequality over $t$, we get
\[
\text{Regret}(T, \pi) = \sum_{t=1}^{T} \mathbb{E} \left[ \mathcal{R}(A_t^*) - \mathcal{R}(\tilde{A}_t) \right]
\leq \sum_{t=1}^{T} \left\{ \frac{1}{\delta G_\delta} \mathbb{E} [V_t(\tilde{A}_t)] + \frac{D_\delta}{2\delta^2} + \mathbb{E} \left[ \left( M_t(A_t^*) - M_t(\tilde{A}_t) \right) \cdot (1 - B_t) \right] \right\}
\leq \frac{V(\pi)}{\delta G_\delta} + \frac{D_\delta}{\delta} + \sum_{t=1}^{T} \mathbb{E} \left[ \left( M_t(A_t^*) - M_t(\tilde{A}_t) \right) \cdot (1 - B_t) \right]
\leq \frac{K}{\delta G_\delta} + \frac{D_\delta}{\delta} + \sum_{t=1}^{T} \mathbb{E} \left[ \left( M_t(A_t^*) - M_t(\tilde{A}_t) \right) \cdot (1 - B_t) \right].
\] (4.5)
Finally, note that
\[
\sum_{t=1}^{T} \mathbb{E}\left[\left(M_t(A_t^*) - M_t(\tilde{A}_t)\right) \cdot (1 - B_t)\right] \leq \sum_{t=1}^{T} \mathbb{E}\left[\delta \cdot (1 - B_t)\right]
\]
\[
= \sum_{t=1}^{T} \mathbb{E}\left[\delta \cdot (1 - B_t)I(\Delta_t < \delta)\right]
\]
\[
\leq \sum_{t=1}^{T} \mathbb{E}\left[\delta \cdot I(\Delta_t < \delta)\right]
\]
\[
\leq \sum_{t=1}^{T} \delta q_\delta
\]
\[
= T\delta q_\delta.
\]

This completes the proof of Theorem 4.1. \(\Box\)

5. ROFUL Algorithm

In this section we generalize the well-known optimism principle that is at the core of the OFUL algorithm of Abbasi-Yadkori et al. (2011). Specifically, we introduce the new notion of optimism in expectation, which allows us to propose a more general and more flexible version of OFUL, which we call the randomized OFUL (ROFUL) algorithm. We then show how optimism in expectation for a policy leads to a high gain rate and, hence, a small regret bound. This allows us to prove a regret bound for ROFUL. In the next section we will show that, in addition to OFUL, Thompson sampling (TS) is also a special case of ROFUL. We will also see that our regret bound for ROFUL leads to a unified proof of rate optimality for both OFUL and TS.

Before executing the above plan, let us start with a few definitions.

Definition 5.1 (Confidence bounds) Confidence bounds are real-valued functions \(L_t(\cdot)\) and \(U_t(\cdot)\) such that, with probability at least \(1 - t^{-3}\),
\[
M_t(A) \in [L_t(A), U_t(A)] \quad \text{for all} \quad A \in \mathcal{A}_t.
\]

Also, \(\mathbb{T}_t^*\) refers to the indicator function for the event that \(M_t(A) \in [L_t(A), U_t(A)]\) holds for all \(A \in \mathcal{A}_t\).

Definition 5.2 (Baseline) For confidence bounds \(L_t(\cdot)\) and \(U_t(\cdot)\), the baseline \(B_t\) at time \(t\) is defined by,
\[
B_t := \sup_{A \in \mathcal{A}_t} L_t(A).
\]
Next, we state an assumption that allows us to provide results in situations where $\Theta^*$ is unbounded. In most of the prior literature $\|\Theta^*\|_2$ is bounded almost surely, which results in the exclusion of normal priors. The assumption allows us to overcome this constraint.

**Assumption 5.1** For any constant $\rho \in [T^{-2}, 1]$, let $B_\rho$ refer to the family of all Bernoulli random variables $Z$ such that $\mathbb{E}[Z] = \rho$. Assume that,

$$
\sup_{Z \in B_\rho} \mathbb{E} \left[ \left( \sup_{A \in A_t} M_t(A) - \inf_{A \in A_t} M_t(A) \right)^2 \cdot Z \right] \leq \frac{D \cdot \rho}{4}.
$$

Note that random variables $Z$ in $B_\rho$ can be correlated with $(\sup_{A \in A_t} M_t(A) - \inf_{A \in A_t} M_t(A))$.

The expression $(\sup_{A \in A_t} M_t(A) - \inf_{A \in A_t} M_t(A))$ in Assumption 5.1 is the maximum attainable regret of any policy at time $t$. Applying the Cauchy–Schwarz inequality, we can see that a sufficient condition for Assumption 5.1 to hold is that

$$
\sup_{Z \in B_\rho} \mathbb{E}[\|\Theta^*\|_2^2 \cdot Z] \leq \frac{D \cdot \rho}{4a^2}.
$$

For example, in the special case where $\|\Theta^*\|_2 \leq 1$ almost surely, the parameter $D$ can be set to $4a^2$.

We are ready now to introduce the ROFUL algorithm.

**ROFUL Algorithm.** ROFUL receives a *worth function* $\tilde{M}_t(\cdot)$ that maps each arm $A \in A_t$ and each history instance $F_{t-1}$ into a real number. The policy then chooses the action with the highest worth. Algorithm 1 presents the pseudocode of ROFUL.

**Algorithm 1** Randomized OFUL

**Require:** Worth functions $\{\tilde{M}_t(\cdot)\}_{t \geq 1}$.

1: **for** $t = 1, 2, \cdots$ **do**
2: \hspace{1em} Observe $A_t$,
3: \hspace{1em} $\tilde{A}_t \leftarrow \arg \max_{A \in A_t} \tilde{M}_t(A)$
4: **end for**

**Regret of ROFUL.** The ROFUL algorithm as formulated in Algorithm 1 may not perform well, unless the worth functions $\tilde{M}_t(\cdot)$ are well behaved. We formally define what “well behaved” means by introducing two conditions of *reasonableness* and *optimism*. Intuitively, an algorithm that explores too much or too little incurs a high regret. Reasonableness and optimism are mechanisms for controlling these potential flaws, respectively. To define these notions rigorously, we assume that for each action $A$ we are given upper and lower confidence bounds $U_t(A) \geq L_t(A)$, where as in
Definition 5.1, the interval $[L_t(A), U_t(A)]$ contains $M_t(A)$ with high probability. In Section 6, we provide examples of these confidence bounds for several examples of problems.

We are now ready to define the reasonableness for worth functions.

**Definition 5.3 (Reasonableness)** Given confidence bounds $L_t(\cdot)$ and $U_t(\cdot)$, a worth function $\tilde{M}_t(\cdot)$ is called reasonable if, with probability at least $1 - t^{-3}$,

$$\tilde{M}_t(A) \in [L_t(A), U_t(A)] \quad \text{for all } A \in \mathcal{A}_t.$$  

Moreover, the notation $\tilde{T}_t$ refers to the indicator function for the event that $\tilde{M}_t(A) \in [L_t(A), U_t(A)]$ holds for all $A \in \mathcal{A}_t$.

As we saw in Definition 5.1, the confidence bounds are such that for each arm $A$, the true mean reward $M_t(A)$ lies in the confidence interval $[L_t(A), U_t(A)]$ with high probability. Therefore, reasonableness ensures that the action chosen by ROFUL is close to the best action that ensures that ROFUL does not explore actions unnecessarily.

Next, we define optimism in expectation, which guarantees that ROFUL explores sufficiently.

**Definition 5.4 (Optimism in expectation)** Given confidence bounds $L_t(\cdot)$ and $U_t(\cdot)$, a worth function $\tilde{M}_t(\cdot)$ is called optimistic with parameter $p \in (0, 1]$, if

$$E\left[\left(M_t(A^*_t) - B_t\right)^2 \cdot \tilde{T}_t\right] \leq \frac{1}{p} E\left[\left(\tilde{M}_t(A_t) - B_t\right)^2 \cdot \tilde{T}_t\right].$$  

Note that $T_t^*$ and $\tilde{T}_t$ are defined as in Definition 5.1 and Definition 5.3, respectively.

Figure 1 shows an illustration of the confidence bounds, the baseline, the interval used in optimism, and the worth functions.

The above notion requires the ROFUL algorithm to avoid paying the price of pure optimism (as OFUL does). Specifically, OFUL ensures that the inequality

$$\left(M_t(A^*_t) - B_t\right)^2 \cdot T_t^* \leq \left(M_t(A_t) - B_t\right)^2 \cdot \tilde{T}_t$$

holds almost surely since for OFUL (as we will see in Section 6) the worth function is $\tilde{M}_t(A_t) = U_t(A_t)$. However, the analysis of ROFUL shows that all we need is that the inequality holds in expectation and up to a constant $p$. In Section 6.5, we will leverage the above intuition and introduce our sieved greedy (SG) algorithm that selects actions more greedily than OFUL while maintaining OFUL’s regret guarantees up to a constant. The core idea behind SG is to use data to stay as close as possible
Figure 1  Illustration of the building blocks of the ROFUL algorithm. Specifically, confidence bounds, baseline, worth functions, and optimism are shown when \( A_t = \{A_1, A_2, \ldots, A_5\} \).

to the greedy policy while ensuring that the selected action \( \tilde{A}_t \) satisfies the optimism-in-expectation condition.

Next, we show that the gain rate of ROFUL can be controlled by \( p \), when the worth functions are reasonable and optimistic in expectation with parameter \( p \).

**Theorem 5.1 (Gain rate of ROFUL)** Assume that \( \tilde{M}_t(\cdot) \) is reasonable and optimistic in expectation (with parameter \( p \)). Also assume that Assumption 5.1 holds with constant \( D \); then we have

\[
\mathbb{E} \left[ \left( M_t(A_t^*) - M_t(\tilde{A}_t) \right)^2 \right] \leq \frac{2}{p} \mathbb{E} \left[ \left( U_t(\tilde{A}_t) - L_t(\tilde{A}_t) \right)^2 \right] + \frac{D}{2T^2}.
\]

Before proving Theorem 5.1, we state our main regret bound for ROFUL which is a corollary of Theorem 4.1, Remark 4.2, and Theorem 5.1.

**Corollary 5.1 (Regret of ROFUL)** If one defines an uncertainty structure by

\[
V_t(A) = \left( U_t(A) - L_t(A) \right)^2,
\]

Theorem 4.1 implies the following gap-dependent regret bound, for any \( \delta \) and \( q_\delta \) as in Definition 2.2:

\[
\text{Regret}(T, \pi^{\text{ROFUL}}) \leq \frac{2K}{\delta p} + \frac{D}{\delta} + T\delta q_\delta,
\]

and (by Remark 4.2) the following gap-independent regret bound:

\[
\text{Regret}(T, \pi^{\text{ROFUL}}) \leq 2 \sqrt{\left( \frac{2K}{p} + D \right) T}.
\]
Proof of Theorem 5.1. Define indicator variable $T_t$ as $T_t := \tilde{T}_t \cdot \tilde{T}_t^*$. Since these are indicator variables, we have $1 - T_t \leq (1 - \tilde{T}_t) + (1 - \tilde{T}_t^*)$. Therefore, by the definition of $\tilde{T}_t$ and of $T_t^*$, we obtain that,

$$E[1 - T_t] \leq E[1 - \tilde{T}_t] + E[1 - \tilde{T}_t^*] \leq 2t^{-3} \leq 2t^{-2}. $$

Using Assumption 5.1 we obtain

$$E\left[\left(M_t(A_t^*) - M_t(\tilde{A}_t)\right)^2 \cdot (1 - T_t)\right] \leq \frac{D}{2t^2}. \quad (5.2)$$

On the other hand,

$$E\left[\left(M_t(A_t^*) - M_t(\tilde{A}_t)\right)^2 \cdot T_t\right] \leq E\left[\left(M_t(A_t^*) - L_t(\tilde{A}_t)\right)^2 \cdot T_t\right]
\leq 2E\left[\left(M_t(A_t^*) - B_t\right)^2 \cdot T_t^* + \left(B_t - L_t(\tilde{A}_t)\right)^2 \cdot T_t\right]
\leq \frac{2}{p}E\left[\left(M_t(A_t^*) - B_t\right)^2 \cdot \tilde{T}_t + \left(B_t - L_t(\tilde{A}_t)\right)^2 \cdot \tilde{T}_t\right]
\leq \frac{2}{p}E\left[\left(U_t(\tilde{A}_t) - B_t\right)^2 \cdot \tilde{T}_t + \left(B_t - L_t(\tilde{A}_t)\right)^2 \cdot \tilde{T}_t\right]
\leq \frac{2}{p}E\left[\left(U_t(\tilde{A}_t) - L_t(\tilde{A}_t)\right)^2 \cdot \tilde{T}_t\right]. \quad (5.3)$$

The result now follows by summing both sides of Eq. (5.2) and Eq. (5.3). \qed

6. Examples of ROFUL and Sieved Greedy

The goal of this section is to demonstrate tangible examples of the ROFUL algorithm that may have seemed rather abstract up to this point. First, in Sections 6.1 to 6.4 we show that OFUL and variations of TS are special cases of ROFUL, which helps us to recover known regret bounds for them via our machinery from Sections 3 to 5. Then, in Section 6.5, motivated by our notion of optimism in expectation and its role in the regret of ROFUL, we introduce a new algorithm (sieved greedy) that enjoys similar theoretical guarantees to those of OFUL, but tends to make more greedy decisions, and hence achieves better empirical performance.

6.1. Worst-case analysis of OFUL

As our first example, we study the OFUL algorithm of Abbasi-Yadkori et al. (2011). First, building on the notation from Eq. (3.1), we define

$$\tilde{\Theta}_t := \Sigma_t \left(\frac{1}{\sigma^2} \sum_{s=1}^{t} \tilde{A}_s \mathcal{R}(\tilde{A}_s)\right).$$
Using Theorem 1 of Abbasi-Yadkori et al. (2011), we realize that
\[ P\left( \| \Theta^* - \hat{\Theta}_t \|_{\Sigma_t^{-1}} \geq \rho \right) \leq \frac{1}{t^3}, \]
where
\[ \rho := \sqrt{d \log \left( 1 + \frac{T a^2}{d \sigma^2} \right) + 7 \log T + \frac{1}{\sqrt{\lambda}} \left( \theta + \sqrt{7 \log T} \right)} \]
(6.1)
Therefore, we can apply the Cauchy–Schwartz inequality and conclude that, for all \( A \in \mathcal{A}_t \), the following functions satisfy the confidence bounds definition:
\[ L_t(A) := \langle \hat{\Theta}_t, A \rangle - \rho \| A \|_{\Sigma_t} \quad \text{and} \quad U_t(A) := \langle \hat{\Theta}_t, A \rangle + \rho \| A \|_{\Sigma_t}. \]
(6.2)
Moreover, OFUL can be written as an instance of ROFUL as follows:
\[ \tilde{M}_t(A) := U_t(A). \]
Reasonableness follows from the definition of this worth function and \( \tilde{T}_t \) will be always equal to 1.
For optimism, note that, whenever \( T_t^* = 1 \), we have
\[ \tilde{M}_t(\tilde{A}_t) = U_t(\tilde{A}_t) \]
\[ \geq U_t(A_t^*) \]
\[ \geq M_t(A_t^*) \]
\[ \geq B_t. \]
We thus get
\[ \mathbb{E}\left[ \left( M_t(A_t^*) - B_t \right)^2 \cdot T_t^* \right] \leq \mathbb{E}\left[ \left( \tilde{M}_t(\tilde{A}_t) - B_t \right)^2 \cdot \tilde{T}_t \right]. \]
This in turn implies that the optimism in expectation holds with \( p = 1 \). Using Corollary 5.1 together with Eq. (3.2) leads to the following gap-dependent bound:
\[ \text{Regret}(T, \pi_{\text{OFUL}}) \leq \frac{16 \sigma^2 \rho^2 d}{\delta} \log \left( 1 + \frac{T a^2}{d \sigma^2} \right) + \frac{D}{\delta} + T \delta q_\delta, \]
and the following gap-independent bound:
\[ \text{Regret}(T, \pi_{\text{OFUL}}) \leq 2 \sqrt{16 \sigma^2 \rho^2 d \log \left( 1 + \frac{T a^2}{d \sigma^2} \right) + D} T. \]
Note that, if we ignore logarithmic factors, this bound is \( O(d \sqrt{T}) \) since \( \rho \) is \( O(\sqrt{d}) \) and \( D \) is constant.
6.2. Bayesian analysis of TS

We obtain a Bayesian regret upper bound for TS similar to the one proved by Russo and Van Roy (2014). Let \( \tilde{\Theta}_t, \rho, L_t, \) and \( U_t \) be given as in Section 6.1. Unlike in the previous section where \( \Theta^* \) was fixed, here we assume that \( \Theta^* \) is also drawn from a prior distribution. Define the worth function by

\[
\tilde{M}_t(A) := \langle \tilde{\Theta}_t, A \rangle,
\]

where \( \tilde{\Theta}_t \) is a sample drawn from the posterior distribution of \( \Theta^* \) at time \( t \) that is used in TS. Therefore, \( \tilde{\Theta}_t \) and \( \Theta^* \) are exchangeable, given \( F_{t-1} \); this, together with the definition of TS, gives

\[
E \left[ \left( M_t(A^*_t) - B_t \right)^2 \cdot T^*_t \mid F_{t-1} \right] = E \left[ \left( \tilde{M}_t(\tilde{A}_t) - B_t \right)^2 \cdot \tilde{T}_t \mid F_{t-1} \right],
\]

almost surely. This implies optimism in expectation with \( p = 1 \). For reasonableness, we can leverage the same argument and obtain

\[
P \left( \forall A \in \mathcal{A}_t : \tilde{M}_t(A) \in [L_t(A), U_t(A)] \right) = E \left[ P \left( \forall A \in \mathcal{A}_t : \tilde{M}_t(A) \in [L_t(A), U_t(A)] \mid F_{t-1} \right) \right] = E \left[ P \left( \forall A \in \mathcal{A}_t : M_t(A) \in [L_t(A), U_t(A)] \mid F_{t-1} \right) \right] \leq \frac{1}{t^3}.
\]

Hence, we obtain the same gap-dependent bound of

\[
\text{Regret}(T, \pi^{TS}) \leq \frac{16 \sigma^2 \rho^2 d}{\delta} \log \left( 1 + \frac{T a^2}{d \sigma^2} \right) + \frac{D}{\delta} + T \delta q_s,
\]

and the same gap-independent bound of

\[
\text{Regret}(T, \pi^{TS}) \leq 2 \sqrt{16 \sigma^2 \rho^2 d \log \left( 1 + \frac{T a^2}{d \sigma^2} \right) + D} T.
\]

6.3. Worst-case analysis of TS

In this section, we study the worst-case (frequentist) regret of TS with inflated posterior variance. We recover the same bounds as the ones by Agrawal and Goyal (2013) and Abeille et al. (2017). Algorithm 2 shows the pseudocode for this instance of TS. We also make the additional assumption that \( |\mathcal{A}_t| \leq n \) for all \( t \).

Due to the inflated variance, we need to redefine \( L_t(\cdot) \) and \( U_t(\cdot) \). Specifically, let

\[
\rho' := \max \left\{ \rho, \, \sqrt{\min \{2d + 12 \log(T), 6 \log(2nT)\}} \right\},
\]
Algorithm 2 Linear Thompson sampling with inflated posterior

Require: Inflation rate $\iota$.

1: Initialize $\Sigma_1 \leftarrow \frac{1}{\lambda} I_d$ and $\hat{\Theta}_1 \leftarrow 0$

2: for $t = 1, 2, \cdots$ do

3: Observe $A_t$

4: Sample $\tilde{\Theta}_t \sim \mathcal{N}(\hat{\Theta}_t, \iota^2 \Sigma_t)$

5: $\tilde{A}_t \leftarrow \arg \max_{A \in A_t} \langle A, \hat{\Theta}_t \rangle$

6: Observe reward $R(\tilde{A}_t)$

7: $\Sigma_t^{-1} \leftarrow \Sigma_t^{-1} + \frac{1}{\sigma^2} \tilde{A}_t \tilde{A}_t^\top$

8: $\hat{\Theta}_{t+1} \leftarrow \Sigma_t^{-1} \left( \Sigma_t^{-1} \hat{\Theta}_t + \frac{1}{\sigma^2} \tilde{A}_t R(\tilde{A}_t) \right)$

9: end for

and define

$$L_t(A) := \langle \hat{\Theta}_t, A \rangle - \rho' ||A||_{\Sigma_t} \quad \text{and} \quad U_t(A) := \langle \hat{\Theta}_t, A \rangle + \rho' ||A||_{\Sigma_t}.$$

As $\rho' \geq \rho$, we infer that $L_t(\cdot)$ and $U_t(\cdot)$ satisfy the confidence bounds condition (Definition 5.1). We note that this definition replaces the $\rho^2$ term in $K$ with $\rho'^2$. Next, we prove that the worth function given by

$$\bar{M}_t(A) = \langle \tilde{\Theta}_t, A \rangle$$

is reasonable. This is achieved by the following lemma, which is proved in Section A.

**Lemma 6.1** For all $t \in [T]$, we have

$$\mathbb{P}\left( \forall A \in A_t : \bar{M}_t(A) \in [L_t(A), U_t(A)] \right) \geq 1 - \frac{1}{2t^3}.$$

In order to derive our regret bound, we also need to verify the optimism in expectation assumption. Whenever $\langle \tilde{\Theta}_t - \Theta^*, A_t^* \rangle \geq -\rho ||A_t^*||_{\Sigma_t}$, we have

$$\bar{M}_t(\tilde{A}_t) - M_t(A_t^*) \geq M_t(A_t^*) - M_t(A_t^*)$$

$$= \langle \tilde{\Theta}_t - \Theta^*, A_t^* \rangle$$

$$= \langle \tilde{\Theta}_t - \hat{\Theta}_t, A_t^* \rangle + \langle \hat{\Theta}_t - \Theta^*, A_t^* \rangle$$

$$\geq \langle \tilde{\Theta}_t - \hat{\Theta}_t, A_t^* \rangle - \rho ||A_t^*||_{\Sigma_t}.$$

Since $\langle \tilde{\Theta}_t - \hat{\Theta}_t, A_t^* \rangle$ is distributed as $\mathcal{N}\left(0, \iota^2 ||A_t^*||_{\Sigma_t}^2\right)$, we can deduce that

$$\mathbb{P}\left( \bar{M}_t(\tilde{A}_t) \geq M_t(A_t^*) \right) \geq \Phi\left( -\frac{\rho}{\iota} \right) \mathbb{P}\left( \langle \tilde{\Theta}_t - \Theta^*, A_t^* \rangle \geq -\rho ||A_t^*||_{\Sigma_t} \right)$$
Finally, since $T'_i = 1$ and $	ilde{T}_i = 1$ imply $M_t(A'_i) \geq B_t$ and $\tilde{M}_t(A'_i) \geq B_t$, respectively, it follows that

$$\mathbb{P}\left((\tilde{M}_t(A'_i) - B_t)^2 \cdot \tilde{T}_i \geq (M_t(A'_i) - B_t)^2 \cdot T'_i \mid F_{i-1}\right) \geq \frac{1}{2} \Phi \left(-\frac{\rho}{\ell}\right) - \frac{1}{T^3}.$$  

Therefore, for sufficiently large $T$, we have

$$\mathbb{P}\left((\tilde{M}_t(A'_i) - B_t)^2 \cdot \tilde{T}_i \geq (M_t(A'_i) - B_t)^2 \cdot T'_i \mid F_{i-1}\right) \geq \frac{1}{4} \Phi \left(-\frac{\rho}{\ell}\right).$$

Noting that $(M_t(A'_i) - B_t)^2 \cdot T'_i$ is deterministic conditional on $F_{i-1}$, we have

$$\mathbb{E}\left[(\tilde{M}_t(A'_i) - B_t)^2 \cdot \tilde{T}_i \right] \geq \frac{1}{4} \Phi \left(-\frac{\rho}{\ell}\right) \cdot \mathbb{E}\left[(M_t(A'_i) - B_t)^2 \cdot T'_i \right].$$

Therefore, optimism in expectation holds with $p = \Phi(-\rho/\ell)/2$.

Thus, similar to Section 6.2, Corollary 5.1 gives a gap-dependent bound of

$$\text{Regret}(T, \pi_{TS}) \leq \frac{32\sigma^2 \rho'd}{\delta \Phi \left(-\frac{\rho}{\ell}\right)} \log \left(1 + \frac{T a^2}{d\sigma^2}\right) + \frac{D}{\delta} + T\delta q_\delta,$$

and a similar gap-independent bound of

$$\text{Regret}(T, \pi_{TS}) \leq 2\sqrt{\frac{32\sigma^2 \rho'd}{\delta \Phi \left(-\frac{\rho}{\ell}\right)} \log \left(1 + \frac{T a^2}{d\sigma^2}\right) + D} \cdot T.$$  

### 6.4. Bayesian analysis of TS with finitely many arms

Following the same technique as in the previous example, we can prove a sharper regret bound for TS in the normal prior and normal noise setting. The main idea is to use smaller confidence bounds. More precisely, write

$$\rho'' := \sqrt{6 \log(2nT)}$$

and then define

$$L_t(A) := \langle \tilde{\Theta}_t, A \rangle - \rho'' \|A\|_{\Sigma_t} \quad \text{and} \quad U_t(A) := \langle \tilde{\Theta}_t, A \rangle + \rho'' \|A\|_{\Sigma_t}.$$  

Using the same techniques as in the proof of reasonableness in the previous example, we can show that these functions satisfy the confidence bounds condition and that the worth function defined by Eq. (6.3) is reasonable with respect to these confidence bounds. This yields an $O(\sqrt{dT \log(T) \log(nT)})$ regret bound that is sharper than the well-known $O(d \log(T) \sqrt{T})$ regret bound. The only comparable result that we are aware of is the $O(\sqrt{dTH(A^*)})$ bound provided by Russo and Van Roy (2016). Although their bound does not require normality and is sharper than ours, it does not allow changing action sets as does our bound.
Algorithm 3 Sieved-Greedy (SG)

Require: $\rho$, $\lambda$, $\sigma$, $\alpha$.

1: Initialize $\Sigma_1 \leftarrow \frac{1}{\lambda} I_d$ and $\hat{\Theta}_1 \leftarrow 0$
2: for $t = 1, 2, \cdots$ do
3: Observe $A_t$
4: Define $L_t(A) := \langle \hat{\Theta}_t, A \rangle - \rho \| A \|_{\Sigma_t}$ and $U_t(A) := \langle \hat{\Theta}_t, A \rangle + \rho \| A \|_{\Sigma_t}$.
5: Construct $A'_{t} := \left\{ A \in A_t : U_t(A) \geq \alpha \left( \max_{A' \in A_t} U_t(A') - B_t \right) + B_t \right\}$
6: $\tilde{A}_t \leftarrow \arg \max_{A \in A'_t} \langle \hat{\Theta}_t, A \rangle$
7: Observe reward $R(\tilde{A}_t)$
8: $\Sigma_{t+1}^{-1} \leftarrow \Sigma_t^{-1} + \frac{1}{\sigma^2} \tilde{A}_t \tilde{A}_t^T$
9: $\hat{\Theta}_{t+1} \leftarrow \Sigma_{t+1}^{-1} \left( \Sigma_t^{-1} \hat{\Theta}_t + \frac{1}{\sigma^2} \tilde{A}_t R(\tilde{A}_t) \right)$
10: end for

6.5. Toward a better use of data: sieved greedy (SG)

In this section, we present a novel algorithm that enjoys the same regret bound as the one we proved for OFUL. This new policy, nonetheless, tends to make more greedy decisions. As we will see in Section 7, this algorithm achieves a similar cumulative regret to that of the best policy in each scenario.

This algorithm receives a sieving-rate parameter $\alpha$ as input. Then, at time $t$, this algorithm first discards all the actions that lack sufficient uncertainty, i.e., that satisfy

$$U_t(A) < \alpha \left( \max_{A' \in A_t} U_t(A') - B_t \right) + B_t.$$
Denoting the set of remaining (sieved) actions by $A_t'$, we note that the algorithm makes a greedy decision over $A_t'$, i.e.,

$$\tilde{A}_t \in \arg \max_{A \in A_t'} \langle \hat{\Theta}_t, A \rangle.$$ 

Therefore, we call the algorithm sieved greedy (SG). When $\alpha = 1$, this algorithm is identical to OFUL and $\alpha = 0$ leads to the greedy algorithm. Algorithm 3 shows the pseudocode for SG and Figure 2 is an illustration of how SG works.

We show that this algorithm is also an instance of ROFUL. To do so, we introduce the following worth function:

$$\tilde{M}_t(A) := \begin{cases} U_t(A) & \text{if } A = \tilde{A}_t, \\ L_t(A) & \text{otherwise}. \end{cases} \quad (6.4)$$

We need to show that the ROFUL algorithm with this worth function chooses the same action as SG. Notice that

$$\sup_{A \in A_t \setminus \{\tilde{A}_t\}} L_t(A) \leq B_t \leq U_t(\tilde{A}_t).$$

Next, the reasonableness of the worth function is evident from the definition of $\tilde{M}_t(\cdot)$. For the optimism, note that

$$E \left[ (M_t(A_t^*) - B_t)^2 \cdot T_t^* \right] \leq E \left[ \sup_{A \in A_t} U_t(A) - B_t \right]^2 \leq \frac{1}{\alpha^2} E \left[ (M_t(\tilde{A}_t) - B_t)^2 \right].$$

Therefore, optimism in expectation holds with $p = \alpha^2$, which means that Corollary 5.1 gives the gap-dependent bound of

$$\text{Regret}(T, \pi_{SG}) \leq \frac{16\sigma^2 \rho^2 d}{\delta \alpha^2} \log \left( 1 + \frac{T \alpha^2}{d \sigma^2} \right) + \frac{D}{\delta} + T \delta q_s,$$

and the gap-independent bound of

$$\text{Regret}(T, \pi_{SG}) \leq 2 \sqrt{\frac{16\sigma^2 \rho^2 d}{\alpha^2} \log \left( 1 + \frac{T \alpha^2}{d \sigma^2} \right) + D} \cdot T.$$ 

**Remark 6.1 (Sieved version of general ROFUL)** Here we showed that SG is an instance of ROFUL. However, as shown in Eq. (6.4), this reduction is general. Specifically, $\tilde{A}_t$ need not to be selected greedily. In fact, any action that is selected from the set of sieved actions can be replaced with $\tilde{A}_t$, and the above regret analysis of SG stays valid. This means one can apply the sieving idea
to any instance of ROFUL, including TS and OFUL. We expect SG to outperform such “sieved TS” or “sieved OFUL”, at least empirically, because it makes more greedy decisions. But, there could be other circumstances, under which, sieved TS or sieved OFUL may be more preferred. For example, a decision-maker may prefer TS as it is a randomized policy, and in such a scenario, she can use sieved TS, with the same theoretical guarantees as SG, while maintaining a TS-based policy.

7. Numerical Simulations

In this section, we compare the performance of OFUL, TS, greedy, and SG (with sieving rates 0, 2, 0.5, and 0.8) in two scenarios. In each scenario, the unknown parameter vector $\Theta^*$ is first sampled from $\mathcal{N}(0, I_d)$, where $d = 120$. Then, in round $t$, a set of $n = 10$ actions is generated. More precisely:

**Scenario I.** A random vector $V_t$ is picked uniformly at random on the sphere of radius 5 in $\mathbb{R}^{12}$. Then, the action $A_{t,i} \in \mathbb{R}^{120}$ for $i = 1, 2, \ldots, 10$ is constructed by copying $V_t$ into the $i$-th block of size 12. Although $d = 120$, this scenario is equivalent to a 10-armed 12-dimensional contextual bandit problem with a shared feature vector $V_t$, embedded in the linear bandit framework, as explained by Abbasi-Yadkori (2012).

**Scenario II.** Motivated by the more general linear bandit problem, each action is chosen uniformly at random on the sphere of radius 5 in $\mathbb{R}^{120}$. Each policy $\pi$ chooses an action $\tilde{A}_\pi^t \in A_t$ and receives the reward $R(\tilde{A}_\pi^t) = \langle \Theta^*, \tilde{A}_\pi^t \rangle + \varepsilon_t$, where $\varepsilon_t$ is a sequence of i.i.d. standard normal random variables. We run each experiment for $T = 10,000$ rounds and repeat this procedure 50 times. The average regret of each policy (and error bars of width $2 \times$ SD in each direction) is shown in Figure 3. As is clear from the plots, in Scenario I, TS is the best policy and SG has a very similar performance, while greedy performs very poorly. But in Scenario II, greedy and SG achieve a substantially better performance compared to OFUL and TS. We also see that the performance of SG is generally less dependent on the sieving rate $\alpha$. Specifically, in Scenario II, all versions have the same performance as greedy. In Scenario I, while all versions outperform OFUL and greedy, $\alpha = 0.5$ slightly outperforms the other two variants and nearly ties with TS.

These results underscore that SG inherits beneficial properties of both greedy and OFUL. It performs similar to greedy when greedy works well, but does not prematurely drop potentially optimal arms, which causes greedy to perform very poorly sometimes.

8. Improved Bounds for Two Important Subproblems

We can strengthen our regret bounds for ROFUL for two important special cases of the stochastic linear bandit problem. Specifically, in Section 8.1, motivated by the $k$-armed contextual bandit
problem, we show how our proof technique allows improving all regret bounds of Section 6 by a factor $\sqrt{k}$. Then, in Section 8.2, making similar (generalized gap and margin) assumptions to those by Goldenshluger and Zeevi (2013) and Bastani and Bayati (2020), we obtain polylogarithmic regret bounds for ROFUL and the obtain the first such results for OFUL and TS.

8.1. Grouped linear bandit

Here we focus on improving our previous regret bounds for a family of subproblems. Although these improvements are mainly motivated by the special case of the $k$-armed contextual bandit, we formulate a slightly more general case of the stochastic linear bandit which we will refer to as the grouped linear bandit.

**Definition 8.1 (Grouped linear bandit)** Let $k$ and $d$ be two integers and let $(Z_j)_{j=1}^k$ be a sequence of $d$-dimensional subspaces of $\mathbb{R}^{kd}$ such that each vector $v \in \mathbb{R}^{kd}$ can be uniquely decomposed to $v = \sum_{j=1}^k z_j$, where $z_j \in Z_j$, i.e., $\mathbb{R}^{kd} = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k$. Then, a grouped linear bandit (GLB) problem simply refers to a linear bandit problem in which $\mathcal{A} \subseteq \bigcup_{j=1}^k Z_j$.

As mentioned above, the GLB formulation is meant to capture the specific structure in the contextual setting. In fact, a $k$-armed $d$-dimensional contextual bandit problem can be modeled as a $kd$-dimensional linear bandit one, as discussed by Abbasi-Yadkori (2012). However, the GLB problem also includes the original linear bandit problem if we assume that $k = 1$. Notice that in the GLB problem we also have $\Theta^* \in \mathbb{R}^{kd}$, which in turn implies that the number of parameters is $kd$ (rather than $d$). Therefore, our previous problem-independent regret bounds from Section 6 would
be $O(kd\sqrt{T})$ for this $kd$-dimensional problem. As we will see shortly, this bound can be tightened by a factor of $\sqrt{k}$ to $O(d\log(T)\sqrt{kT})$.

The key observation is that the radius of the confidence set can be shrunk to

$$
\eta := \sqrt{d\log\left(1 + \frac{T a^2}{kd\sigma^2}\right) + 7\log(kT) + \frac{1}{\sqrt{\lambda}} \left(\theta + \sqrt{7\log(kT)}\right)}. \quad (8.1)
$$

Note that $\rho$ as defined in Eq. (6.1) for this problem is given by

$$
\rho := \sqrt{kd\log\left(1 + \frac{T a^2}{kd\sigma^2}\right) + 7\log T + \frac{1}{\sqrt{\lambda}} \left(\theta + \sqrt{7\log T}\right)},
$$

which is worse than $\rho$ as defined in Eq. (8.1) by an asymptotic factor of $\sqrt{k}$ as $T$ grows large. Specifically, we can show that the following functions satisfy the confidence bounds definition:

$$
L_t(A) := \langle \hat{\Theta}_t, A \rangle - \eta \| A \|_{\Sigma_t} \quad \text{and} \quad U_t(A) := \langle \hat{\Theta}_t, A \rangle + \eta \| A \|_{\Sigma_t}.
$$

This can be shown by noting that when $A \in Z_j$, then $\langle \hat{\Theta}_t - \Theta^*, A \rangle$ can be bounded by applying Theorem 1 of Abbasi-Yadkori et al. (2011) in a $d$-dimensional rather than $kd$-dimensional setting. Combining this with the union bound, we obtain

$$
P\left( \exists A \in A_t : M_t(A) \notin [L_t(A), U_t(A)] \right) \leq \sum_{j=1}^k P\left( \text{For some } A \in Z_j, \ M_t(A) \notin [L_t(A), U_t(A)] \right) \leq \sum_{j=1}^k P\left( \text{For some } A \in Z_j, \ |\langle \hat{\Theta}_t - \Theta^*, A \rangle| \geq \eta \| A \|_{\Sigma_t} \right) \leq \frac{1}{T^3}.
$$

By the same argument as in Section 6.1, we get

$$
\text{Regret}(T, \pi^{\text{OFUL}}) \leq \frac{16\sigma^2 \eta^2 kd}{\delta} \log\left(1 + \frac{T a^2}{kd\sigma^2}\right) + \frac{D}{\delta} + T\delta q_\delta,
$$

and by tuning $\delta$ as before we get

$$
\text{Regret}(T, \pi^{\text{OFUL}}) \leq 2\sqrt{\left[16\sigma^2 \eta^2 kd \log\left(1 + \frac{T a^2}{kd\sigma^2}\right) + D\right] T}.
$$

Moreover, in the Bayesian setting, one can use these confidence bounds to prove a similar Bayesian regret bound for TS with the proper update rule. In the frequentist setting, on the other hand, this idea can be used to show that

$$
\text{Regret}(T, \pi^{\text{TS}}) \leq \frac{32\sigma^2 \eta^2 kd}{\delta \Phi(-\sqrt{\frac{\eta}{\delta}})} \log\left(1 + \frac{T a^2}{kd\sigma^2}\right) + \frac{D}{\delta} + T\delta q_\delta,
$$
and by tuning $\delta$ as before,
\[
\text{Regret}(T, \pi^{TS}) \leq 2 \sqrt{\left[ \frac{32\sigma^2\eta'^2kd}{\Phi \left( -\frac{2}{\eta'} \right)} \log \left( 1 + \frac{T\eta'^2}{kd\sigma^2} \right) + D \right] T}
\]

where
\[
\eta' := \max \left\{ \eta, \sqrt{\min \left\{ 2kd + 12\log(T), 6\log(2nT) \right\}} \right\}.
\]

This shows that the posterior variance inflation can be reduced by a factor of $O(\sqrt{k})$ in TS as $T$ grows large.

### 8.2. Polylogarithmic regret bounds

In this subsection we provide regret bounds for ROFUL when confidence sets are defined such that they grow with $T$ polylogarithmically, under additional assumptions. Our assumptions are similar to those made by Goldenshluger and Zeevi (2013) and Bastani and Bayati (2020). Throughout this section we consider special classes of ROFUL where confidence intervals (Definition 5.1) are defined as in Eq. (6.2) with varying definitions of $\rho$. This special class includes all examples of Section 6 such as OFUL, TS, and SG algorithms.

To state the first assumption, recall the gap parameters $\Delta_t$ and $\delta$ from §2.

**Assumption 8.1 (Margin condition)** There exists constants $c_0, t_0 > 0$ such that
\[
\mathbb{P}(\Delta_t \leq z) \leq c_0 z \tag{8.2}
\]
for all $0 \leq z \leq \delta$ and $t \in [T]$ with $t \geq t_0$.

Before stating the next condition, we need to define a notion of near-optimal space for the family of GLB problems.

**Definition 8.2 (Near-optimal space)** Consider a GLB problem as defined in Definition 8.1. Let $l \leq k$ be the smallest number such that there exists $\mathcal{I} \subseteq [k]$ with $|\mathcal{I}| = l$ and
\[
\mathbb{P}(A^t_{\mathcal{I}} \subseteq \bigoplus_{j \in \mathcal{I}} Z_j) = 1 \quad \text{for all } t \in [T].
\]
We define near-optimal space $\mathcal{W}$ as $\mathcal{W} := \bigoplus_{j \in \mathcal{I}} Z_j$, and, with a slight abuse of notation, we also treat $\mathcal{W}$ as the projection of $\mathbb{R}^{kd}$ onto the subspace $\mathcal{W}$. 
**Remark 8.1** The main purpose of this notion is to handle suboptimal arms in the special case of a $k$-armed contextual bandit. One might harmlessly assume that $W = \mathbb{R}^{kd}$, or equivalently, assuming it is the identity function if viewed as an operator, and follow the rest of this section.

The next assumption demands the selected actions to be diverse in the near-optimal space. Specifically, recall the inverse covariance matrix $\Sigma_t$ of the actions chosen by a policy from Eq. (3.1).

**Assumption 8.2 (Linear expansion)** We say that linear expansion holds for a policy if
\[
P\left(\|W^\top \Sigma_t W\|_{\text{op}} \geq \frac{c_2}{t}\right) \leq \frac{c_1}{2t^2}
\]
for some constants $c_1, c_2 > 0$ and all $t \in [T]$ with $t \geq t_0$. We denote the indicator variable for the event $\|W^\top \Sigma_t W\|_{\text{op}} < c_2/t$ by $L_t$.

We will show in Lemma 8.2 that ROFUL satisfies the linear expansion assumption, under a variant of the optimism-in-expectation assumption from Section 5 as well as a certain diversity assumption. This fact, combined with the following lemma, leads to our main result of this section which is presented as Corollary 8.1. The next lemma operates on the same setting as Theorem 4.1 with additional assumptions on the reasonableness of the worth functions, the margin condition, and the linear expansions.

**Lemma 8.1** Consider an uncertainty structure $\{V_t\}_{t \geq 1}$ with uncertainty complexity $K$, gap level $\delta$, associated parameter $q_\delta$, as well as gain-rate parameters $G_\delta$ and $D_\delta$. Also, assume that worth functions of Algorithm 1 (policy $\pi_{\text{ROFUL}}$) are reasonable (Definition 5.3), and that the margin condition (Assumption 8.1) and the linear expansion condition (Assumption 8.2) hold. Then, the cumulative regret of $\pi_{\text{ROFUL}}$ satisfies the following inequality:
\[
\text{Regret}(T, \pi_{\text{ROFUL}}) \leq \frac{K}{\delta G_\delta} + \frac{D_\delta}{\delta} + (t_0q_\delta + c_1 + 1) + 16a^2\rho^2 c_2c_0 \log(T).
\]
The proof of Lemma 8.1 is given in Section A.3.

**Linear expansion and ROFUL.** In what follows, we will show that under a certain diversity condition (Assumption 8.3) and a generalization of the optimism assumption in Section 5, the ROFUL algorithm satisfies linear expansion.

**Assumption 8.3 (Diversity condition)** We say that a GLB problem satisfies the diversity condition with parameter $\upsilon$ if $A_t$ is independent of $\sigma(A_1, \tilde{A}_1, R(\tilde{A}_1), \ldots, A_{t-1}, \tilde{A}_{t-1}, R(\tilde{A}_{t-1}))$ and
\[
\lambda_{\text{min}}(E[Y_t]) \geq \upsilon \quad \text{and} \quad \left\|E[(Y_t - E[Y_t])^2] \right\|_{\text{op}} \leq \psi^2 \quad \text{for all } t \in [T],
\]
where $Y_t := W^\top A_t^* A_t^\top W \cdot G_t = A_t^* A_t^\top \cdot G_t$. 

Assumption 8.3 is similar to Assumption A3 of Goldenshluger and Zeevi (2013) and Assumption 4 of Bastani and Bayati (2020).

**Definition 8.3 (Optimism in probability)** We say that the worth function $\tilde{M}_t(\cdot)$ is optimistic in probability if for some $\omega$ and $p$ in $(0, 1]$ we have

$$
P\left(\left(\tilde{M}_t(A^*_t) - B_t\right) \cdot T^*_t \leq \frac{1}{\omega} \left(\tilde{M}_t(\tilde{A}_t) - B_t\right) \cdot \tilde{T}_t \bigg| \mathcal{F}_{t-1}\right) \geq \frac{p}{\omega^2},
$$

almost surely.

**Remark 8.2** A slightly stronger version of Eq. (8.4) is that

$$
P\left(\left(\tilde{M}_t(A^*_t) - B_t\right) \cdot T^*_t \leq \frac{1}{\omega} \left(\tilde{M}_t(\tilde{A}_t) - B_t\right) \cdot \tilde{T}_t \bigg| \Theta^*, \mathcal{F}_{t-1}\right) \geq \frac{p}{\omega^2},
$$

almost surely. It is worth noting that this stronger condition also implies optimism in expectation (Definition 5.4). First, note that $\omega(\tilde{M}_t(A^*_t) - B_t) \cdot T^*_t$ is a deterministic function of $(\Theta^*, \mathcal{F}_{t-1})$.

Therefore, we have

$$
\frac{p}{\omega^2} \cdot \omega^2 \left(\tilde{M}_t(A^*_t) - B_t\right)^2 \cdot T^*_t \leq \mathbb{E}\left[\left(\tilde{M}_t(\tilde{A}_t) - B_t\right)^2 \cdot \tilde{T}_t \bigg| \Theta^*, \mathcal{F}_{t-1}\right].
$$

This proves optimism in expectation since,

$$
\mathbb{E}\left[\left(\tilde{M}_t(A^*_t) - B_t\right)^2 \cdot T^*_t\right] = \frac{1}{p} \mathbb{E}\left[p(\tilde{M}_t(A^*_t) - B_t)^2 \cdot T^*_t\right] \\
\leq \frac{1}{p} \mathbb{E}\left[\mathbb{E}\left[\left(\tilde{M}_t(\tilde{A}_t) - B_t\right)^2 \cdot \tilde{T}_t \bigg| \Theta^*, \mathcal{F}_{t-1}\right]\right] \\
= \frac{1}{p} \mathbb{E}\left[\tilde{M}_t(\tilde{A}_t) - B_t\right]^2 \cdot T^*_t].
$$

It is worthwhile to mention that in the worst-case analysis of an algorithm, $\Theta^*$ is a deterministic constant and, therefore, Eq. (8.4) and Eq. (8.5) are equivalent. Nevertheless, the stronger condition Eq. (8.5) need not hold when $\Theta^*$ is drawn from a prior distribution. An example of this situation is the Bayesian analysis of TS in which Eq. (5.1) and Eq. (8.4) hold simultaneously, although Eq. (8.5) fails to hold.

Now, we are ready to state our result that ROFUL satisfies the linear expansion assumption if its worth functions are optimistic in probability.

**Lemma 8.2 (ROFUL satisfies linear expansion)** If the diversity condition (Assumption 8.3) holds and the worth functions of ROFUL are optimistic in probability (Definition 8.3), then ROFUL satisfies the linear expansion condition (Assumption 8.2) with

$$
c_1 := 6, \quad c_2 := \frac{1}{4pu}, \quad t_0 := \max\left\{k, d, t'_0, 3t''_0 \log(t''_0)\right\},
$$
where
\[ t'_0 := \frac{16a^2\rho^2kd}{p\nu\omega^2\delta^2} \cdot \log \left( \lambda + \frac{T^2a^2}{d} \right), \quad t''_0 = \frac{32\psi^2 + 16va^2/3}{p\nu^2}. \]

The proof of Lemma 8.2 is given in Section A.4.

Next, we state the main result of this section, which directly follows from Lemma 8.1 and Lemma 8.2.

**Corollary 8.1** Consider a GLB problem that satisfies the margin condition (Assumption 8.1) and the diversity condition (Assumption 8.3). Also, assume that the worth functions of Algorithm 1 (denoted by policy \( \pi^{\text{ROFUL}} \)) are reasonable (Definition 5.3) and optimistic in probability (Definition 8.3). Then the cumulative regret of \( \pi^{\text{ROFUL}} \) satisfies the following inequality:

\[
\text{Regret}(T, \pi^{\text{ROFUL}}) \leq Kd + Dd + \delta(t_0q_3 + c_2 + 1) + 16a^2\rho^2c_2\log(T),
\]

where constants \( c_1, c_2, \) and \( t_0 \) are defined as in Lemma 8.2.

**Remark 8.3** Note that in terms of dependence in \( T \), by Corollary 8.1 we prove a regret bound that is \( O(\log^2(T)) \) under similar conditions as the ones by Goldenshluger and Zeevi (2013) Bastani and Bayati (2020). Since OFUL and TS are special cases of ROFUL, this immediately provides an \( O(\log^2(T)) \) regret bound for OFUL and TS as well. To the best of our knowledge, these results are new.

**Remark 8.4** Corollary 8.1 also holds when a more general \( \tau \)-margin condition for \( \tau > 0 \), as by Goldenshluger and Zeevi (2009) Bastani et al. (2017) that replaces Eq. (8.2) with

\[
P(\Delta_t \leq z) \leq c_0 z^\tau \quad \text{(8.6)}
\]

is satisfied. In this case, the term \( 16a^2\rho^2c_2\log(T) \) in the regret bound would be replaced by a term of order

\[
O\left( (a\rho)^{\tau + 1}\left( 1 + \int_1^T t^{-\frac{\tau + 1}{2}} dt \right) \right),
\]

through the same proof technique.

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Appendix A: Additional Proofs

A.1. Proof of Eq. (3.4)

Proof. Noting that
\[ E[R(A_i) \mid F_{t-1}, A^* = A_j] = \langle \mu_{t,j}, A_i \rangle \quad \text{and} \quad E[R(A_i) \mid F_{t-1}] = \langle \mu_i, A_i \rangle, \]
we get, by Lemma 3 of Russo and Van Roy (2016), that for all \( i \in [k] \),
\[ A_i^\top (\mu_{t,j} - \mu_i)(\mu_{t,j} - \mu_i)^\top A_i = (\mu_{t,j} - \mu_i)^2 \leq 2\sigma^2 D_{\text{kl}}(\mathbb{P}(R(A_i) \mid F_{t-1}, A^* = A_j) \mid \mathbb{P}(R(A_i) \mid F_{t-1})). \]
This in turn implies that
\[ V_t(A_i) \leq 2\sigma^2 \sum_{j=1}^{k} \alpha_j D_{\text{kl}}(\mathbb{P}(R(A_i) \mid F_{t-1}, A^* = A_j) \mid \mathbb{P}(R(A_i) \mid F_{t-1})) \]
\[ = 2\sigma^2 I(A^*; R(A_i) \mid F_{t-1}). \]
For any policy \( \pi \in \mathcal{P} \), we have
\[ E[V_t(\tilde{A}_i)] = E\left[E\left[V_t(\tilde{A}_i) \mid F_{t-1}, \tilde{A}_i\right]\right] \]
\[ = E\left[\sum_{i=1}^{k} \mathbb{P}(\tilde{A}_i = A_i \mid F_{t-1}) \cdot V_t(A_i)\right] \]
\[ \leq 2\sigma^2 E\left[\sum_{i=1}^{k} \mathbb{P}(\tilde{A}_i = A_i \mid F_{t-1}) \cdot I(A^*; R(A_i) \mid F_{t-1})\right]. \]
Using the assumption that \( \tilde{A}_i \) is independent of \( A^* \) conditional on \( F_{t-1} \), we can write
\[ E[V_t(\tilde{A}_i)] \leq 2\sigma^2 E\left[\sum_{i=1}^{k} \mathbb{P}(\tilde{A}_i = A_i \mid F_{t-1}) \cdot I(A^*; R(\tilde{A}_i) \mid F_{t-1}, \tilde{A}_i = A_i)\right] \]
\[ = 2\sigma^2 E\left[I(A^*; R(\tilde{A}_i) \mid F_{t-1}, \tilde{A}_i)\right] \]
\[ \leq 2\sigma^2 E\left[I(A^*; R(\tilde{A}_i) \mid F_{t-1}, \tilde{A}_i) + I(A^*; \tilde{A}_i \mid F_{t-1})\right] \]
\[ \leq 2\sigma^2 E\left[I(A^*; (R(\tilde{A}_i), \tilde{A}_i) \mid F_{t-1})\right]. \]
Therefore, by summing up both sides of the above inequalities, we get
\[ V(\pi) = E\left[\sum_{t=1}^{T} V_t(\tilde{A}_i)\right] \]
\[ \leq 2\sigma^2 E\left[\sum_{t=1}^{T} I(A^*; (R(\tilde{A}_i), \tilde{A}_i) \mid F_{t-1})\right] \]
\[ \leq 2\sigma^2 \sum_{t=1}^{T} \mathbb{E} \left[ I(A^* ; (\mathcal{R} (\tilde{A}_t), \tilde{A}_t) | \mathcal{F}_{t-1}) \right] \]
\[ = 2\sigma^2 \mathcal{H}(A^*). \]

\begin{flushright}
\[ \square \]
\end{flushright}

\section*{A.2. Proof of Lemma 6.1}

First, we state the following lemma.

\begin{lemma}
If \( X \sim \chi^2_d \), then for all positive constants \( \gamma \), we have \( \mathbb{P}(X \geq 2d + 3\gamma) \leq \exp(-\gamma) \).
\end{lemma}

\begin{proof}
The proof follows directly from applying Lemma 1 of Laurent and Massart (2000) which gives
\[ \mathbb{P}(X \geq 2d + 3\gamma) \leq \mathbb{P}(X \geq d + 2\sqrt{d\gamma} + 2\gamma) \leq \exp(-\gamma). \]
\end{proof}

\begin{flushright}
\[ \square \]
\end{flushright}

\begin{proof}[Proof of Lemma 6.1] First, assume that \( 6 \log 2nT \geq 2d + 12 \log T \). Since \( \iota^{-1} \Sigma^{-\frac{1}{2}}_t (\tilde{\Theta}_t - \hat{\Theta}_t) \sim \mathcal{N}(0, \mathbf{I}_d) \), it follows from Lemma A.1 (with \( \gamma = 4 \log T \)) that
\[ \mathbb{P}\left( \|\tilde{\Theta}_t - \hat{\Theta}_t\|_{\Sigma_t^{-1}} \geq \iota^2 [2d + 12 \log T] \right) \leq 1/T^4 \leq 1/(2T^3). \]
Therefore, combining this with \( \log 2nT \geq 2d + 12 \log T \), we have
\[ \mathbb{P}\left( \forall A \in \mathcal{A}_t : \tilde{M}_t(A) \in [L_t(A), U_t(A)] \right) = \mathbb{P}\left( \forall A \in \mathcal{A}_t : |(\tilde{\Theta}_t - \hat{\Theta}_t, A)| \leq \rho' \|A\|_{\Sigma_t} \right) \geq \mathbb{P}\left( \|\tilde{\Theta}_t - \hat{\Theta}_t\|_{\Sigma_t^{-1}} \leq \iota^2 (2d + 12 \log T) \right) \geq 1 - 1/(2T^3). \]

In the finite action set case, we provide a different bound using the union bound. For each \( A \in \mathcal{A}_t \), note that \( (\tilde{\Theta}_t - \hat{\Theta}_t, A) \sim \mathcal{N}(0, \iota^2 \|A\|^2_{\Sigma_t}) \). Hence, we have,
\[ \mathbb{P}\left( \forall A \in \mathcal{A}_t : \tilde{M}_t(A) \in [L_t(A), U_t(A)] \right) = \mathbb{P}\left( \forall A \in \mathcal{A}_t : |(\tilde{\Theta}_t - \hat{\Theta}_t, A)| \leq \rho' \|A\|_{\Sigma_t} \right) \geq 1 - n \cdot \Phi\left(-\frac{\rho'}{\iota}\right) \geq 1 - 1/(2T^3), \]
where in the last step we used the fact that \( \Phi(-x) \leq \exp(-x^2/2)/(x\sqrt{2\pi}) \) for all positive \( x \). \( \square \)
A.3. Proof of Lemma 8.1

**Proof.** The main idea is to refine the proof of Theorem 4.1. We first recall Eq. (4.5):

\[
\text{Regret}(T, \pi^{\text{ROFUL}}) \leq \frac{K}{\delta G_\delta} + \frac{D_\delta}{\delta} + \sum_{t=1}^{T} \mathbb{E}\left[\left(\mathbf{M}_t(A^*_t) - \mathbf{M}_t(\tilde{A}_t)\right) \cdot (1 - \mathbb{B}_t)\right].
\] (A.1)

where \(\mathbb{B}_t\) denotes \(\mathbb{I}(\mathbf{M}_t(A^*_t) - \mathbf{M}_t(\tilde{A}_t) \geq \delta)\). We next improve the upper bound for each individual term in the above sum. For \(t \leq t_0\), where \(t_0\) is defined as in Assumption 8.2, we use our previous bound

\[
\mathbb{E}\left[\left(\mathbf{M}_t(A^*_t) - \mathbf{M}_t(\tilde{A}_t)\right) \cdot (1 - \mathbb{B}_t)\right] \leq \delta q_\delta.
\]

Next, we consider \(t > t_0\). Whenever \(\mathbb{B}_t = 0\), we have \(\mathbf{M}_t(\tilde{A}_t) > \mathbf{M}_t(A^*_t) - \delta\), which in turn implies that \(A^*_t, \tilde{A}_t \in \mathbf{W}\). By recalling the indicator variable \(T_t\) defined in the proof of Theorem 5.1, and provided that \((1 - \mathbb{B}_t)\mathbb{I}_t T_t = 1\), we have

\[
\mathbf{M}_t(A^*_t) - \mathbf{M}_t(\tilde{A}_t) \leq U_t(A^*_t) - L_t(\tilde{A}_t)
\]

\[
\leq L_t(A^*_t) - U_t(\tilde{A}_t) + 2\rho(\|A^*_t\|_{\Sigma_t} + \|\tilde{A}_t\|_{\Sigma_t})
\]

\[
\leq \tilde{M}_t(A^*_t) - \tilde{M}_t(\tilde{A}_t) + 2\rho(\|A^*_t\|_{\Sigma_t} + \|\tilde{A}_t\|_{\Sigma_t})
\]

\[
\leq 2\rho(\|A^*_t\|_{\Sigma_t} + \|\tilde{A}_t\|_{\Sigma_t})
\]

\[
\leq \frac{4a\rho \sqrt{c_2}}{t}.
\]

In the above, (a) holds since \(T_t^* = 1\), (b) follows from \(T_t = 1\), (c) uses the fact that ROFUL chooses the action with maximum worth \(\tilde{M}_t(\cdot)\), and (d) is a consequence of \(\mathbb{I}_t = 1\). Now, using this inequality, we can write

\[
\mathbb{E}\left[\left(\mathbf{M}_t(A^*_t) - \mathbf{M}_t(\tilde{A}_t)\right) \cdot (1 - \mathbb{B}_t)\right] = \mathbb{E}\left[\left(\mathbf{M}_t(A^*_t) - \mathbf{M}_t(\tilde{A}_t)\right) \cdot (1 - \mathbb{B}_t)(1 - \mathbb{L}_t T_t + \mathbb{L}_t T_t)\right]
\]

\[
\leq \mathbb{E}\left[\left(\mathbf{M}_t(A^*_t) - \mathbf{M}_t(\tilde{A}_t)\right) \cdot (1 - \mathbb{B}_t)\mathbb{L}_t T_t\right] + \delta \mathbb{P}(\mathbb{L}_t T_t = 0)
\]

\[
\leq \frac{4a\rho \sqrt{c_2}}{\sqrt{t}} \mathbb{P}\left(\mathbb{L}_t T_t = 1, \Delta_t < \frac{4ac_2 \rho}{\sqrt{t}}\right) + \frac{\delta(c_1 + 1)}{2t^2}
\]

\[
\leq \frac{4a\rho \sqrt{c_2}}{\sqrt{t}} \mathbb{P}\left(\Delta_t < \frac{4ac_2 \rho}{\sqrt{t}}\right) + \frac{\delta(c_1 + 1)}{2t^2}
\]

\[
\leq \frac{16a^2 \rho^2 c_2 c_0}{t^2} + \frac{\delta(c_1 + 1)}{2t^2},
\]

where the last step uses Assumption 8.1. This inequality in combination with Eq. (A.1) yields

\[
\text{Regret}(T, \pi^{\text{ROFUL}}) \leq \frac{K}{\delta G_\delta} + \frac{D_\delta}{\delta} + \delta t_0 q_\delta + \sum_{t=t_0+1}^{T} \left\{ \frac{16a^2 \rho^2 c_2 c_0}{t} + \frac{\delta(c_1 + 1)}{2t^2} \right\}
\]

\[
\leq \frac{K}{\delta G_\delta} + \frac{D_\delta}{\delta} + \delta(t_0 q_\delta + c_1 + 1) + 16a^2 \rho^2 c_2 c_0 \log(T),
\]

which is the desired result. \(\Box\)
A.4. Proof of Lemma 8.2

Proof. For any \( t \in [T] \), let \( O_t \) be a Bernoulli random variable with
\[
P(O_t = 1 | F_{t-1}) = \frac{p}{\omega^2}
\]
almost surely such that
\[
O_t = 1 \quad \text{and} \quad T_t = 1 \implies M_t(A^*_t) - B_t \leq \frac{1}{\omega} (\tilde{M}_t(A_t) - B_t).
\]
(A.2)
The existence of this random variable is guaranteed by the optimism-in-probability assumption.

Next, for \( t \in [T] \), we have
\[
\sum_{i=t}^{t} \tilde{A}_i \tilde{A}_i^\top \cdot (1 - B_i) G_i O_i \leq \Sigma_i^{-1}.
\]
Moreover, it follows from the definition of \( B_i \) and \( G_i \) that
\[
\tilde{A}_i \tilde{A}_i^\top \cdot (1 - B_i) G_i = A^*_i A_i^* \cdot (1 - B_i) G_i
\]
for all \( i \in [T] \). Therefore, we get
\[
\sum_{i=t}^{t} A^*_i A_i^* \cdot (1 - B_i) G_i O_i \leq \Sigma_i^{-1}.
\]
By recalling \( \Upsilon_i = A^*_i A_i^* \cdot G_i \), we have
\[
\|W^\top \Sigma_i W\|_{op} = \lambda_{\min} \left( W^\top \Sigma_i^{-1} W \right)
\geq \lambda_{\min} \left( \sum_{i=t}^{t} \Upsilon_i \cdot (1 - B_i) O_i \right)
\geq \lambda_{\min} \left( \sum_{i=t}^{t} \Upsilon_i \cdot O_i \right) - \lambda_{\max} \left( \sum_{i=t}^{t} \Upsilon_i \cdot B_i O_i \right).
\]
We now bound each term separately. Next, we prove that the smallest singular value of \( \sum_{i=t}^{t} \Upsilon_i \cdot O_i \) grows linearly with high probability. Using the noncommutative Bernstein’s inequality for \( s \geq 0 \), (e.g., Theorem 1.4 of Tropp (2012)), we get
\[
P\left( \left\| \sum_{i=t}^{t} (\Upsilon_i \cdot O_i - \mathbb{E} [\Upsilon_i \cdot O_i]) \right\|_{op} \geq s \right) \leq kd \cdot \exp \left( -\frac{s^2}{2 t p \psi^2 + sa^2/3} \right).
\]
Setting \( s := tp \psi / 2 \) and applying the triangle inequality yields
\[
P\left( \left\| \sum_{i=t}^{t} \Upsilon_i \cdot O_i \right\|_{op} \leq \frac{tp \psi}{2} \right) \leq kd \cdot \exp \left( -\frac{tp \psi^2}{8 \psi^2 + 4va^2/3} \right).
\]
(A.3)
Our next goal is to prove an upper bound for the largest singular value of \( \sum_{i=\frac{t}{2}}^{t} \mathbf{\Gamma}_i \cdot \mathbf{B}_i \). We apply the following bound:

\[
\lambda_{\max} \left( \sum_{i=\frac{t}{2}}^{t} \mathbf{\Gamma}_i \cdot \mathbf{B}_i \right) \leq \mathbf{a}^2 \sum_{i=\frac{t}{2}}^{t} \mathbf{B}_i \mathbf{O}_i.
\]

Using Eq. (A.2), we can deduce that, whenever \( T_t = 1 \) and \( \mathbf{B}_i \mathbf{O}_i = 1 \), we have

\[
\delta \leq \mathbf{M}_i (\mathbf{A}_i) - \langle \mathbf{\Theta}^\star, \mathbf{\tilde{A}_i} \rangle
\]

\[
\leq \mathbf{M}_i (\mathbf{A}_i) - \mathbf{L}_i (\mathbf{\tilde{A}_i})
\]

\[
= \mathbf{M}_i (\mathbf{A}_i) - \mathbf{B}_i + \mathbf{B}_i - \mathbf{L}_i (\mathbf{\tilde{A}_i})
\]

\[
\leq \frac{1}{\omega \delta} (\mathbf{M}_i (\mathbf{\tilde{A}_i}) - \mathbf{B}_i) + \mathbf{B}_i - \mathbf{L}_i (\mathbf{\tilde{A}_i})
\]

\[
\leq \frac{1}{\omega \delta} (\mathbf{M}_i (\mathbf{\tilde{A}_i}) - \mathbf{B}_i) + \mathbf{B}_i - \mathbf{L}_i (\mathbf{\tilde{A}_i})
\]

\[
= \frac{1}{\omega \delta} (\mathbf{U}_i (\mathbf{\tilde{A}_i}) - \mathbf{L}_i (\mathbf{\tilde{A}_i}))
\]

\[
\leq \frac{2}{\omega \delta} \| \mathbf{\tilde{A}_i} \|_{\Sigma_{i-1}}.
\]

Therefore, we can write

\[
\mathbf{T}_i \mathbf{B}_i \mathbf{O}_i \leq \left( \frac{2 \rho}{\omega \delta} \cdot \| \mathbf{\tilde{A}_i} \|_{\Sigma_{i-1}} \right)^2 < \frac{4 \rho^2}{\omega^2 \delta^2} \cdot \| \mathbf{\tilde{A}_i} \|_{\Sigma_{i-1}}^2.
\]

Next, Lemma 10 and Lemma 11 in Abbasi-Yadkori et al. (2011) yield

\[
\left( \prod_{i=\frac{t}{2}}^{t} \mathbf{T}_i \right) \sum_{i=\frac{t}{2}}^{t} \mathbf{B}_i \mathbf{O}_i \leq \sum_{i=\frac{t}{2}}^{t} \mathbf{T}_i \mathbf{B}_i \mathbf{O}_i
\]

\[
\leq \sum_{i=1}^{t} \mathbf{T}_i \mathbf{B}_i \mathbf{O}_i
\]

\[
\leq \frac{4 \rho^2}{\omega^2 \delta^2} \cdot \sum_{i=1}^{t} \| \mathbf{\tilde{A}_i} \|_{\Sigma_{i-1}}^2
\]

\[
\leq \frac{4 \rho^2 k d}{\omega^2 \delta^2} \cdot \log \left( \frac{\lambda + \frac{t \mathbf{a}^2}{d}}{2} \right).
\]

Hence, it is a direct consequence of Eq. (A.3) that, for any \( t \geq t_0 \geq \frac{16 \mathbf{a}^2 \rho^2 k d}{\rho \psi^2 + 4 \mathbf{va}^2/3} \cdot \log \left( \frac{\lambda + \frac{t \mathbf{a}^2}{d}}{2} \right) \), we get

\[
\mathbb{P} \left( \| \mathbf{W}^\top \mathbf{\Sigma}_t \mathbf{W} \|_{\text{op}} \geq \frac{4}{t \rho \psi^2} \right) \leq k d \cdot \exp \left( -\frac{t \rho \psi^2}{8 \psi^2 + 4 \mathbf{va}^2/3} \right) + \mathbb{P} \left( \prod_{i=\frac{t}{2}}^{t} = 0 \right) \quad (A.4)
\]

\[
\leq k d \cdot \exp \left( -\frac{t \rho \psi^2}{8 \psi^2 + 4 \mathbf{va}^2/3} \right) + \frac{2}{t^2} \quad (A.5)
\]
We prove that for sufficiently large \( t \), the right-hand side of the above inequality is bounded above by \( 3/t^2 \). This is equivalent to
\[
\frac{8\psi^2 + 4va^2/3}{pu^2} \leq \frac{t}{\log(t^2kd)} = \frac{t}{2\log(t) + \log(kd)}.
\]
Using Lemma A.2 below, we infer that this is satisfied for all \( t \geq t_0 \). \( \square \)

**Lemma A.2** Let \( a \geq 3 \) be given. Then, for all \( t \geq 3a \log(a) \), we have \( \frac{t}{\log(t)} \geq a \).

**Proof.** First, note that \( f : t \mapsto \frac{t}{\log(t)} \) is an increasing function of \( t \) for all \( t \geq e \). To see this, we compute the derivative of \( f \) as follows:
\[
f'(t) = \frac{\log(t) - 1}{\log^2(t)} \geq 0.
\]
Next, setting \( t_0 = 3a \log(a) \), we have
\[
f(t_0) = \frac{3a \log(a)}{\log(3a \log(a))} \geq \frac{3a \log(a)}{\log(a^3)} = a.
\] \( \square \)