The topological AC effect on noncommutative phase space

Kang Li\textsuperscript{a,c,1}, Jianhua Wang\textsuperscript{b,c,2}

\textsuperscript{a} Department of Physics, Hangzhou Teachers College, Hangzhou, 310036, P.R.China
\textsuperscript{b} Department of Physics, Shaanxi University of Technology, Hanzhong, 723001, P.R. China
\textsuperscript{c} The Abdus Salam International Center for Theoretical Physics, Trieste, Italy

Abstract

The Aharonov-Casher (AC) effect in non-commutative (NC) quantum mechanics is studied. Instead of using the star product method, we use a generalization of Bopp’s shift method. After solving the Dirac equations both on noncommutative space and noncommutative phase space by the new method, we obtain the corrections to AC phase on NC space and NC phase space respectively.

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1 Introduction

In recent years, there has been an increasing interest in the study of physical effects on noncommutative space, apart from field theory, non-commutative quantum mechanics has recently attract much attentions, because the effects of the space non-commutativity may become significant in the extreme situation such as in the string scale or at the Tev and higher energy level. There are many papers devoted to the study of various aspects of quantum mechanics on NC space with usual (commutative) time coordinate [1]-[6]. For example, the topological AB in NC space and even NC phase space have studied in [2]-[4]. In this work, other than the method employed in [5], we develop a new method to obtain the corrections to the topological phase of the AC effect both on NC space and NC phase space, where we know that in a commutative space the line spectrum does not depend on the relativistic nature of the dipoles. The article is organized as follows: in section 2, by using the Lagrangian formulation, we discuss the AC effect on a commutative space in 2+1 dimensions. In section 3, the AC effect on a noncommutative space is studied, and the correction to AC phase on NC space is obtained, the AC effect on a non-commutative phase space will be discussed in section 4, and a most generalized formula for holonomy in NC phase space is given in this section, at last, some remarks are given in the last section.

\textsuperscript{1}kangli@hztc.edu.cn
\textsuperscript{2}jianhuawang59@yahoo.com.cn
2 Description of AC effect in 2+1 commutative space time

To begin with, let’s give a brief review of AC effect in 2 + 1 commutative space time. The Lagrangian for a neutral particle of spin-1/2 with an anomalous magnetic dipole moment $\mu_m$ interacting with the electromagnetic field has the form

$$L = \bar{\psi}i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - \frac{1}{2}\mu_m \bar{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu}. \quad (1)$$

The last term in the Lagrangian is responsible for the AC effect.

We restrict the particle moves on a plane (say $x-y$ plane), then the problem can be treated in 2 + 1 space time. We use the following conventions for the 2+1 dimensional metric $g_{\mu\nu}$ and the anti-symmetric tensor $\epsilon_{\mu\nu\alpha}$:

$$g_{\mu\nu} = \text{diag}(1, -1, -1) \quad \text{and} \quad \epsilon_{012} = +1. \quad (2)$$

Other than to use $2 \times 2$ matrices satisfying the 2+1 dimensional Dirac algebra, we will use 3 four component Dirac matrices which can describe spin up and down in the notional $z$ direction for a particle and for its anti-particle. In 2+1 dimensions these Dirac matrices satisfy the following relation [7]:

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} - i\gamma^0 \sigma^{12} \epsilon^{\mu\nu\lambda} \gamma_\lambda. \quad (3)$$

A particular representation is

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}. \quad (4)$$

Then the interaction term in the Lagrangian can be written as

$$\bar{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu} = F^{\mu\nu} \gamma^0 \sigma^{12} \epsilon_{\mu\nu\lambda} \gamma^\lambda \psi, \quad (5)$$

with

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 \\ E^1 & 0 & -B^3 \\ E^2 & B^3 & 0 \end{pmatrix}, \quad (6)$$

where $E^i$ and $B^i$ are the electric and magnetic fields, respectively. The indices “1” and “2” indicate the coordinates on the $x-y$ plane along the $x$ and $y$ directions. The index “3” indicates the $z$ direction. The Lagrangian now can be written as

$$L = \bar{\psi}i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - \frac{1}{2}\mu_m \bar{\psi} \sigma^{12} \epsilon_{\mu_\alpha\beta} F^{\alpha\beta} \gamma^\mu \psi. \quad (7)$$

By using E-L equation, the Dirac equation of motion for a spin half neutral particle with a magnetic dipole moment $\mu_m$ is

$$(i\gamma_\mu \partial^\mu - (1/2)\gamma^0 \sigma^{12} \mu_m \epsilon_{\mu_\alpha\beta} F^{\alpha\beta} \gamma^\mu - m)\psi = 0, \quad (8)$$
and the solution will have the form

$$\psi = e^{-\frac{i}{2} \gamma^0 \sigma^{12} \mu_m \int^x \epsilon_{\mu\alpha\beta} F^{\alpha\beta} dx^\mu} \psi_0,$$

where $\psi_0$ is the solution for electromagnetic field free case. The phase in Eq.(9) is called AC phase, we write it as

$$\phi_{AC} = -\frac{1}{2} \gamma^0 \sigma^{12} \mu_m \int^x \epsilon_{\mu\alpha\beta} F^{\alpha\beta} dx^\mu.$$ (10)

The AC phase above is the general AB phase for a spin-1/2 neutral particle passing through an electromagnetic field. If we consider a situation of the standard AC configuration [8]-[9], i.e. the particle moves on a plane under the influence of an pure electric field produced by a uniformly charged infinitely long filament perpendicular to the plane, then we have

$$\phi_{AC} = -\gamma^0 \sigma^{12} \mu_m \int^x \epsilon_{0ij} F^{0i} dx^j = \gamma^0 \sigma^{12} \mu_m \int^x (\hat{k} \times \vec{E}) \cdot d\vec{x} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \int^x (\vec{\mu}_m \times \vec{E}) \cdot d\vec{x},$$ (11)

where $\hat{k}$ is the unit vector in $z$ direction and we assume that the magnetic dipole moment is always along this direction, i.e. $\vec{\mu}_m = \mu_m \hat{k}$.

### 3 The AC Effect on noncommutative space

On the noncommutative space the coordinate and momentum operators satisfy the following commutation relations (we take $\hbar = c = 1$ unit)

$$[\hat{x}_i, \hat{x}_j] = i\Theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\delta_{ij},$$ (12)

where $\Theta_{ij}$ is an element of an antisymmetric matrix, it is very small related to the energy scale and it represents the non-commutativity of the NC space, $\hat{x}_i$ and $\hat{p}_i$ are the coordinate and momentum operators on a NC space.

Just like the static Schrödinger equation on NC space[4], the Dirac equation (8) for a spin half neutral particle with a magnetic dipole moment $\mu_m$, on NC space, can be written as

$$(i\gamma_\mu \partial^\mu - (1/2) \gamma^0 \sigma^{12} \mu_m \epsilon_{\mu\alpha\beta} F^{\alpha\beta} \gamma^\mu - m) \ast \psi = 0,$$ (13)

i.e. simply replace usual product with a star product (Moyal-Weyl product), the Dirac equation in usual commuting space will change into the Dirac equation on NC space. The star product between two functions is defined by,

$$(f \ast g)(x) = e^{\frac{i}{2} \Theta_{ij} \partial_i \partial_j} f(x_i)g(x_j) = f(x)g(x) + i \frac{1}{2} \Theta_{ij} \partial_i f \partial_j g|_{x_i = x_j} + \mathcal{O}(\Theta^2).$$ (14)

here $f(x)$ and $g(x)$ are two arbitrary functions.

Some features of AC effect on noncommutative space has been studied in [5] by using the star calculation, but it is still meaningful to study it again by using the method gave in reference
[4], i.e. through a generalized Bopp’s shift, and the method can be easily generalized to NC phase space which will be discussed in the next section.

On NC space the star product can be replaced by a Bopp’s shift, i.e the star product can be changed into ordinary product by shifting coordinates $x_\mu$ with

$$\hat{x}_\mu = x_\mu - \frac{1}{2} \Theta_{\mu \nu} p^\nu.$$  \hspace{1cm} (15)

Now, let us consider the noncommutative Dirac equation (13), to replace the star product with ordinary product, equivalent to the Bopp’s shift, the $F_{\mu \nu}$ must, up to the first order of the NC parameter $\Theta$, be shifted by

$$F_{\mu \nu} \rightarrow \hat{F}_{\mu \nu} = F_{\mu \nu} + \frac{1}{2} \Theta^{\alpha \beta} p_\alpha \partial_\beta F_{\mu \nu}.$$  \hspace{1cm} (16)

Then the Dirac equation for AC problem on NC space has the form

$$(i \gamma_\mu \partial_\mu - (1/2) \gamma^0 \sigma_{12} \mu_m \epsilon_{\mu \alpha \beta} \hat{F}^{\alpha \beta} \gamma^\mu - m) \psi = 0.$$  \hspace{1cm} (17)

So, on NC space, the AC phase has the form:

$$\hat{\phi}_{AC} = -\frac{1}{2} \gamma^0 \sigma_{12} \mu_m \int x^\nu \epsilon_{\mu \alpha \beta} \hat{F}^{\alpha \beta} d\nu$$

$$= -\frac{1}{2} \gamma^0 \sigma_{12} \mu_m \int x^\nu \epsilon_{\mu \alpha \beta} F^{\alpha \beta} d\nu - \frac{1}{4} \gamma^0 \sigma_{12} \mu_m \int x^\nu \epsilon_{\mu \alpha \beta} \Theta^{\sigma \tau} p_\sigma \partial_\tau F^{\alpha \beta} d\nu.$$  \hspace{1cm} (18)

This is the general AC phase for a spin-1/2 neutral particle moving in a general electromagnetic field.

In the standard AC configuration, the momentum on NC space can be written as

$$p_l = m v_l + (\vec{E} \times \vec{\mu})_l + \mathcal{O}(\theta),$$  \hspace{1cm} (19)

where $\vec{\mu} = \mu_m \vec{\sigma}$, insert equation (19) to (18) and notice that

$$F^{\alpha \beta} \longrightarrow F^{0i} \text{ and } \Theta^{ij} = \Theta^{ji}, \Theta^{0\mu} = \Theta^{\mu 0} = 0.$$  \hspace{1cm} (20)

3In the standard AC configuration the Hamiltonian in commuting space has the form [8]-[9]:

$$H = \frac{1}{2m} \vec{\sigma} \cdot (\vec{p} - i \mu_m \vec{E}) \vec{\sigma} \cdot (\vec{p} + i \mu_m \vec{E}).$$

In the region $\vec{\nabla} \cdot \vec{E} = 0$ (if the particle do not reach at the charged filament, this is always true), then the equation above can be recast as

$$H = \frac{1}{2m} (\vec{p} - \vec{E} \times \vec{\mu})^2 - \frac{\mu^2 E^2}{2m},$$

then the velocity operator can be gotten

$$v_l = \frac{\partial H}{\partial p_l} = \frac{1}{m} [p_l - (\vec{E} \times \vec{\mu})_l].$$

4
we have

\[ \hat{\phi}_{AC} = \phi_{AC} + \delta\phi_{NCS}, \]  

(21)

where \( \phi_{AC} \) is the AC phase on commuting space given by (11), the additional phase \( \delta\phi_{NCS} \), related to the non-commutativity of space, is given by

\[ \delta\phi_{NCS} = -\frac{1}{2}\gamma^0\sigma^{12}\mu_m \int^x \epsilon_{\alpha\beta}\epsilon^{\alpha\beta} [mv_\alpha + (\vec{E} \times \vec{\mu})_\alpha] \partial_\beta F^{0i} dx^\mu \]

\[ = \frac{1}{2}\gamma^0\sigma^{12}\mu_m \epsilon^{ij} \int^x [k_j + (\vec{E} \times \vec{\mu})_j] (\partial_i E^2 dx^1 - \partial_i E^1 dx^2), \]  

(22)

where \( k_j = mv_j \), and the result here coincides with the result given in reference [5], where the tedious star product calculation has been used.

The first term is a velocity dependent correction and does not have the topological properties of the commutative AC effect and could modify the phase shift. The second term is a correction to the vortex and does not contribute to the line spectrum.

### 4 The AC effect on noncommutative phase space

We have discussed the AC effect on NC space, where space-space do not commute but with momentum-momentum commuting. The Bose-Einstein statistics in noncommutative quantum mechanics requires both space-space and momentum-momentum non-commutativity, we call the NC space with momenta non-commuting is NC phase space. So study the physics on NC phase space is very important. On NC phase space, the momentum \( \hat{p}_i \)'s commutation relation in (12) should be replaced with

\[ [\hat{p}_i, \hat{p}_j] = i\Theta_{ij}, \]  

(23)

where \( \Theta \) is the antisymmetric matrix, its elements represent the non-commutativity of the momenta. The Dirac equation (8) on NC phase space can also be written as:

\[ (-\gamma_\mu p^\mu - (1/2)\gamma^0\sigma^{12}\mu_m \epsilon_{\mu\alpha\beta} F^{\alpha\beta} - m) \ast \psi = 0, \]  

(24)

but here, the star product in Eqs. (24) defines,

\[ (f \ast g)(x, p) = e^{2i\alpha \Theta_{ij}\partial_i \partial_j} f(x, p)g(x, p) \]

\[ = f(x, p)g(x, p) + \frac{i}{2\alpha^2} \Theta_{ij} \partial_i \partial_j f \partial_j g|_{x_i = x_j} + \frac{i}{2\alpha^2} \Theta_{ij} \partial_i \partial_j f \partial_i g|_{p_i = p_j} + \mathcal{O}(\Theta^2), \]  

(25)

where \( \mathcal{O}(\Theta^2) \) stands the second and higher order terms of \( \Theta \) and \( \tilde{\Theta} \). The star product in Dirac equation on NC phase space can be placed by the usual product from the two steps, first we need to replace \( x_i \) and \( p_i \) with a generalized Bopp’s shift as

\[ x_\mu \rightarrow \alpha x_\mu - \frac{1}{2\alpha} \Theta_{\mu\nu} p_\nu, \]

\[ p_\mu \rightarrow \alpha p_\mu + \frac{1}{2\alpha} \Theta_{\mu\nu} x_\nu, \]  

(26)
and also need the partner of shift in Eq.(16) in NC phase space as,

\[ F_{\mu \nu} \rightarrow \tilde{F}_{\mu \nu} = \alpha F_{\mu \nu} + \frac{1}{2\alpha} \Theta^{\alpha \beta} p_\alpha \partial_\beta F_{\mu \nu}. \]  

(27)

The Dirac equation (24) then read

\[ \{-\alpha \gamma^\mu p_\mu - \frac{1}{2\alpha} \gamma^\mu \Theta_{\mu \nu} x_\nu - (1/2) \gamma^0 \sigma^{12} \mu_m \epsilon_{\mu \alpha \beta} \gamma^{\alpha \beta} - \frac{1}{2\alpha} \Theta^{\tau \sigma} p_\tau \partial_\sigma F_{\mu \nu} \} \psi = 0. \]

(28)

Because \( \alpha \neq 0 \), so the above Dirac equation can be recast to

\[ \{-\gamma^\mu p_\mu - \frac{1}{2\alpha^2} \gamma^\mu \Theta_{\mu \nu} x_\nu - (1/2) \gamma^0 \sigma^{12} \mu_m \epsilon_{\mu \alpha \beta} [F_{\tau \sigma}^{\alpha \beta} + \frac{1}{2\alpha} \Theta^{\tau \sigma} p_\tau \partial_\sigma F_{\mu \nu}^{\alpha \beta}] \gamma^{\mu} - m' \} \psi = 0, \]

(29)

where \( m' = m/\alpha \). The solution to (29) is

\[ \psi = e^{i\tilde{\varphi}_{AC} \psi_0}, \]

(30)

where \( \psi_0 \) is the solution of Dirac equation for free particle with mass \( m' \), and the \( \tilde{\varphi}_{AC} \) stands AC phase in NC phase space, and it has the form below,

\[ \dot{\tilde{\varphi}}_{AC} = -\frac{1}{2} \gamma^0 \sigma^{12} \mu_m \int x^\mu \epsilon_{\mu \alpha \beta} F_{\alpha \beta}^{0 \mu} + \frac{1}{2\alpha^2} \Theta_{\mu \nu} \Theta_{\mu \nu} - \frac{1}{4\alpha^2} \gamma^0 \sigma^{12} \mu_m \int \epsilon_{\mu \alpha \beta} \Theta^{\tau \sigma} p_\tau \partial_\sigma F_{\alpha \beta}^{0 \mu} \]

(31)

Equation (31) is the general AC phase in noncommutative phase space. For the standard AC case i.e. particle moves in a pure static electric field, then the AC phase reduces to

\[ \dot{\tilde{\varphi}}_{AC} = \dot{\phi}_{AC} - \frac{1}{2\alpha^2} \int x^\mu \Theta_{\mu \nu} x_\nu = \frac{1}{2\alpha^2} \gamma^0 \sigma^{12} \mu_m \int \epsilon_{\mu \alpha \beta} [m' v_\alpha + (\vec{E} \times \vec{\mu})_\alpha] \partial_\beta F^{0 i} \]

\[ = \dot{\phi}_{AC} - \frac{1}{2\alpha^2} \int x^\mu \Theta_{\mu \nu} x_\nu + \frac{1}{2\alpha^2} \gamma^0 \sigma^{12} \mu_m \theta \epsilon^{ij} \int x^k [k'_j + (\vec{E} \times \vec{\mu})_j] (\partial_i E^2 dx^1 - \partial_i E^1 dx^2), \]

(32)

where \( k'_j = m' v_j \), \( p_i = m' v_i + (\vec{E} \times \vec{\mu})_i + O(\theta) \) has been applied and we omit the second order terms in \( \theta \). Equation (32) can also be written as

\[ \dot{\tilde{\varphi}}_{AC} = \dot{\phi}_{AC} + \delta \dot{\phi}_{NCS} + \delta \dot{\phi}_{NCPS}, \]

(33)

where \( \dot{\phi}_{AC} \) is the AC phase on commuting space ( see Eq. 11), \( \delta \dot{\phi}_{NCS} \) is the space-space non-commuting contribution to the AC phase on NC space (see Eq. 22), and the last term \( \delta \dot{\phi}_{NCPS} \) is given by

\[ \delta \dot{\phi}_{NCPS} = -\frac{1}{2\alpha^2} \int x^\mu \tilde{\Theta}_{\mu \nu} x_\nu + \frac{1}{2\alpha^3} \gamma^0 \sigma^{12} \mu_m \theta \epsilon^{ij} \int x^k [k'_j (\partial_i E^2 dx^1 - \partial_i E^1 dx^2) - \frac{1}{2\alpha^2} \gamma^0 \sigma^{12} \mu_m \theta \epsilon^{ij} \int x^k (\vec{E} \times \vec{\mu})_j (\partial_i E^2 dx^1 - \partial_i E^1 dx^2), \]

(34)

which represents the non-commutativity of the momenta. When \( \alpha = 1 \), which will lead to \( \tilde{\Theta}_{\mu \nu} = 0 \) [6], then the AC phase on NC phase space will return to the AC phase on NC space, i.e. \( \delta \dot{\phi}_{NCPS} = 0 \) and equations (31) and (32) will change into equations (18) and (21) respectively.
5 Conclusion remarks

In this paper, we study the AC effect on both noncommutative space and noncommutative phase space. Instead of doing tedious star product calculation, we use the "shift" method, i.e. the star product in Dirac equation can be replaced by Bopp's shift and together with the shift we defined in (16) for NC space and (27) for NC phase space. These shifts is exact equivalent to the star product. The additional AC phase (22) in NC space is exact the same as in reference [5] shows the correctness of our method. Our results of AC phase in NC phase space, especially the new term (34) which comes from the momenta non-commutativity is totally new result of this paper.

The method we use in this paper may also be employed to other physics problem on NC space and NC phase space. The further study on the issue will be reported in our forthcoming papers.

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