Background field approach to electromagnetic properties of baryons

Tim Ledwig
Institut für Kernphysik, Universität Mainz, D-55099 Mainz, Germany

We investigate the self-energies of particles in an external magnetic field $B$. The dependence is generally of the type $\sqrt{P(B)}$ with $P$ a polynomial in $B$ and the participating masses. The non-analytic point depends on the mass and charge constellation, is unproblematic for stable particles but constrains the linear energy shift approximation for resonances. We recover an earlier reported condition on when the energy can be expanded in $B$ and derive two more conditions. Further, we obtain the $B$ dependent self-energies of the nucleon and $\Delta(1232)$-isobar in the $SU(2)$ covariant chiral perturbation theory.

I. INTRODUCTION

Electromagnetic (EM) properties of particles are fundamental observables for their internal structure. For hadrons it is still not possible to reveal this structure analytically from QCD first principles. Two prominent ways to obtain information on baryon EM properties are chiral perturbation theories ($\chi$PT) and lattice QCD (lQCD) which both allow to investigate the quark/pion mass dependence of observables. At present, finite volume lQCD results are mainly compared to infinite volume $\chi$PT ones \cite{1} where discrepancies in the small pion mass region are seen \cite{2}.

EM finite volume effects on the $\chi$PT side are therefore of interest, however, reveal certain subtleties. Special care has to be taken for the decomposition of the vector-current matrix element in form factors as well as for their Lorentz invariance \cite{6}. An alternative approach, e.g. to investigate the magnetic moment, is to use the particle’s self-energy in an external EM field. This was done in \cite{7} for pions and nucleons in the non-relativistic heavy baryon $\chi$PT.

One motivation for this work is to derive the corresponding covariant infinite volume $\chi$PT self-energies of the nucleon and $\Delta(1232)$ that would be needed for a future finite volume study. Another one is to investigate further the non-analytic EM field dependence of self-energies \cite{8}. We see that beside the found condition for expanding a resonance self-energy in the field strength there exist two more. For this, we modernize the EM background field technique (BFT) of \cite{9} and apply it to stable and unstable particles.

In 1958 C. M. Sommerfield used the BFT to obtain the electron’s AMM up to the fourth order, $\kappa_e = \frac{e^2}{8\pi^2} - 0.328 \frac{e^2}{4\pi^2}$, correctly for the first time. However, it is the well known three point function method, i.e. the one-photon approximation, that became the preferred method to calculate EM moments in field theories. We will use both techniques to check our formulas. It turns out that the self-energies of particles, stable as well as unstable, generally depend non-analytically on the magnetic field $B$. This is e.g. seen for the nucleon and $\Delta(1232)$-isobar in the covariant $SU(2)$ B$\chi$PT with $N - \pi$ loops where the $MS$ renormalized results with $\mu = m_\pi/M_N$ are:

$$\Sigma_p (B) = \frac{M_N C_N}{24 (1 - B)^2} \left[ n_1 + n_2 \ln \mu + \frac{n_3}{\sqrt{4\mu^2 - (B + \mu^2)^2}} \arccos \frac{B + \mu^2}{2\mu} \right], \quad (1)$$

$$\Sigma_{\Delta^+} (B) = \frac{M_\Delta C_\Delta}{(1 - B)^2} \left[ d_1 + d_2 \ln \mu + d_3 \ln r + d_4 \ln (1 - B) + \frac{(B - 1) d_5 \arctanh \omega_1}{\sqrt{\lambda^2 + 4\mu^2 B}} + \frac{(B - 1) d_6 \arctanh \omega_2}{\sqrt{\lambda^2 + B (B - 2 + 2\tau^2 + 2\mu^2)}} + d_7 \ln (r^2 - B) \right], \quad (2)$$

with certain polynomials $n_i = n_i (B, \mu)$, $d_i = d_i (B, \mu, r)$, $\lambda^2 = \lambda^2 (\mu, r)$ and further non-analytic functions $\omega_1 = \omega_1 (B, \mu, r)$ and notations given later. One difference of the self-energies is that the square root in $\Sigma_p (B)$ can be expanded in a weak magnetic field $B$ for all pion masses $m_\pi > 0$ whereas for the $\Delta(1232)$-isobar only if the condition

$$\frac{eB}{2M_\Delta} \ll |M_\Delta - (M_N + m_\pi)| \quad (3)$$

*Electronic address: ledwig@kph.uni-mainz.de*
between the nucleon, Δ(1232) and pion masses is met. This is a general situation for unstable particles and was discussed in [8] within a simpler field theory. In the work [8] only the one Feynman graph leading to the above condition was investigated whereas we derive here the remaining two conditions coming from the two more Feynman graphs.

In the next section we will give necessary notations for the BFT and will use them in the third section to investigate the self-energies of stable and unstable particles. In the forth and fifth section we apply the BFT to the nucleon and Δ(1232)-isobar and derive the above expressions.

II. FIELD EQUATIONS IN PRESENCE OF AN EXTERNAL EM FIELD

We consider spin-1/2 and spin-3/2 fields moving in a constant electromagnetic field given by the potential $A_\mu(x) = -\frac{1}{2} F_{\mu\nu} x^\nu$ with $F^{\mu\nu} = \partial^{[\mu} A^{\nu]}$ as the electromagnetic field strength tensor. The Dirac equation for a particle $\Psi(x)$ of mass $M$ with the EM minimal substitution is:

$$\left[\Pi - M\right] \Psi_A(x) = 0,$$

where we take $e > 0$ and define the momentum $\Pi_\mu = i \partial_\mu - e A_\mu(x)$. This operator is non-commutative with

$$[\Pi_\mu, \Pi_\nu] = \frac{1}{i} F_{\mu\nu}.$$

The spin 3/2 field $\psi(x)$ satisfies the equation

$$\left[\Pi - M\right] \Psi^\mu_A(x) = 0,$$

with the subsidiary conditions

$$\gamma_\mu \Psi^\mu_A(x) = 0,$$
$$\Pi_\mu \Psi^\mu_A(x) = 0.$$

Because of the non-commutativity we have to symmetrize occurring expressions and use for the propagators:

$$\frac{1}{\Pi - M} = \frac{1}{2} \left[ (\Pi + M) \frac{1}{\Pi^2 - M^2 + F} + \frac{1}{\Pi^2 - M^2 + F} (\Pi + M) \right],$$

where we introduce the notation $F = \frac{1}{2} \gamma^\alpha F_{\alpha\beta} \gamma^\beta$. With this we calculate the particle’s self-energy $\Sigma(F)$ and obtain its anomalous magnetic moment (AMM) $\kappa$ through the linear energy shift:

$$\langle \Psi_A | \Sigma(F) | \Psi_A \rangle = \langle \Sigma(0) \rangle - \langle F \rangle \frac{\kappa}{2M} + O(F^2).$$

In Fig. 1 we list all Feynman graphs used in this work for the BFT and three point function method. The upper row shows all types of graphs that appear to the one-loop level in the BFT, tadpole graphs not considered.

For better reading we use the notation $\vec{B} = M^2 B \vec{e}_z$, i.e. $B = |\vec{B}|/M^2$, with which the linear approximation to the self-energy reads:

$$\Sigma(B) = \Sigma(0) - M \frac{1}{2} \kappa B + O(B^2),$$
$$\Sigma(-B) - \Sigma(B) = M \kappa B + O(B^3),$$

where in the last equation the $B^2$ terms cancel out. The self-energy formulas in this work omit some, but not all, $O(B^2)$ terms. In particular non-analytic $B$ structures are preserved in the BFT which are not in the one-photon approximation Eq. (11).
with the covariant derivative $D\gamma$.

Figure 1: Upper row: Feynman graphs contributing to the self-energy in the presence of an external electromagnetic field. The blue lines indicate which loop-internal particle is affected by the field. Middle and lower row: Feynman graphs contributing to the three point function method to obtain the AMM. Single solid lines correspond to stable particles (e.g. the nucleon), double solid lines to unstable particles (e.g. the $\Delta (1232)$), dashed lines to (pseudo-) scalar particles (e.g. the pion) and the blue cross to the coupling of the photon.

III. SCALAR COUPLINGS AND ESTIMATION OF $O(B^2)$ EFFECTS

We use the notations of the appendix and consider two spin $1/2$ fields $\Psi_1$, $\Psi_2$ interacting with a scalar field $\phi$:

\[ L = \sum_{a=1}^{2} \bar{\Psi}_a (iD_a - M_a) \Psi_a + \frac{1}{2} (D_\mu \phi) (D^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \sum_{a,b=1}^{2} g \bar{\Psi}_a \Gamma_{ab} \Psi_b \phi , \]

with the covariant derivative $D_\mu = \partial_\mu + i q e A_\mu$, $q$ as the charge of the field with $e > 0$ and $\Gamma_{ab}$ either 1 or $\gamma_5$. In the upper row of Fig. 1 we show all the types of self-energy graphs that can occur for a charged or uncharged external particle $\Psi_{ex}$. The corresponding expressions $\Sigma_1 (B)$ with a loop-internal particle $\Psi_{in}$ are:

\[ \langle \Psi_{ex} | \Sigma_1 (F) | \Psi_{ex} \rangle = \frac{g^2}{i} \langle \Psi_{ex} | \int d\Gamma_1 \frac{1}{[I - M_{in} + i\varepsilon]} \Gamma_1 \frac{1}{(\Pi - l^2 - m^2 + i\varepsilon)} | \Psi_{ex} \rangle , \]

\[ \langle \Psi_{ex} | \Sigma_2 (F) | \Psi_{ex} \rangle = \frac{g^2}{i} \langle \Psi_{ex} | \int d\Gamma_2 \frac{1}{[\Pi - I - M_{in} + i\varepsilon]} \Gamma_2 \frac{1}{l^2 - m^2 + i\varepsilon} | \Psi_{ex} \rangle , \]

\[ \langle \Psi_{ex} | \Sigma_3 (F) | \Psi_{ex} \rangle = \frac{g^2}{i} \langle \Psi_{ex} | \int d\Gamma_3 \frac{1}{[\Pi - I - M_{in} + i\varepsilon]} \Gamma_3 \frac{1}{(\Pi + l^2 - m^2 + i\varepsilon)} | \Psi_{ex} \rangle , \]

with $\Pi_\mu^a = i \partial_\mu + e A_\mu (x)$. For a third-spin projection of $+1/2$ we can write these expressions in dimensional regularization as:

\[ \Sigma_i = \frac{-g^2}{(4\pi)^2} M_{ex} \int_\alpha_{i-1}^{1-\alpha_i} dz [s_5 (z + \alpha_i) + r] \left[ L + \ln (z^2 - \lambda_i^2 - i\varepsilon) + (\delta_{i1} + \delta_{i2}) \ln (1 - B) \right] + O (B^2) , \]

with $\alpha_i$, $\lambda_i$ given below and $s_5 = -1$ for $\Gamma_i = \gamma_5$ and $+1$ for $\Gamma_i = 1$. We see that for $B = 0$ these expressions are the same and for $B \neq 0$ several different logarithms occur. The formula Eq. 17 omissions some, but not all, $B^2$ terms and the integrated solution for the real and imaginary parts are:
\[
\text{Re} \Sigma_i = \frac{-g^2}{(4\pi)^2} M_{ex} \left[ + (s_5 \alpha_i + r) (\beta_i \ln(\beta_i^2 - \lambda_i^2) + \alpha_i \ln(\alpha_i^2 - \lambda_i^2) - 2) \\
+ s_5 \frac{1}{2} (\alpha_i^2 - \beta_i^2 + (\beta_i^2 - \lambda_i^2) \ln(\beta_i^2 - \lambda_i^2) - (\alpha_i^2 - \lambda_i^2) \ln(\alpha_i^2 - \lambda_i^2)) \\
+ (\delta_{i1} + \delta_{i2}) \ln (1 - B) \left( s_5 \alpha_i + r + \frac{s_5}{2} \left( \beta_i^2 - \alpha_i^2 \right) \right) + (s_5 \alpha_i + r) \Omega_i \right], \quad (18)
\]
\[
\text{Im} \Sigma_i = \frac{g^2 \pi}{(4\pi)^2} M_{ex} (s_5 \alpha_i + r) 2 \lambda_i
\]

with \( \beta_i = 1 - \alpha_i \) and

\[
\Omega_i = \begin{cases} 
2 \sqrt{-\lambda_i^2} \left( \arctan \frac{\beta_i}{\sqrt{-\lambda_i^2}} + \arctan \frac{\alpha_i}{\sqrt{-\lambda_i^2}} \right) & \lambda_i^2 < 0 \\
2 \sqrt{\lambda_i^2} \left( \arctanh \frac{\beta_i}{\sqrt{\lambda_i^2}} + \arctanh \frac{\alpha_i}{\sqrt{\lambda_i^2}} \right) & \lambda_i^2 > 0
\end{cases}, \quad (20)
\]

\[
\alpha_0 = \frac{1}{2} (1 + r^2 - \mu^2) \quad \lambda_0^2 = \alpha_0^2 - r^2, \quad (21)
\]
\[
\alpha_1 = \frac{1}{2 (1 - B)} (1 + r^2 - \mu^2 - B) \quad \lambda_1^2 = \alpha_1^2 - \frac{r^2}{1 - B}, \quad (22)
\]
\[
\alpha_2 = \frac{1}{2 (1 - B)} (1 + r^2 - \mu^2 - 2B) \quad \lambda_2^2 = \alpha_2^2 - \frac{r^2 - B}{1 - B}, \quad (23)
\]
\[
\alpha_3 = \frac{1}{2} (1 + r^2 - \mu^2 - B) \quad \lambda_3^2 = \alpha_3^2 - r^2 + B. \quad (24)
\]

The self-energies obtain imaginary parts if \( \lambda_i^2 \) becomes positive together with \( -\lambda_i < \beta_i \) and/or \( \lambda_i > -\alpha_i \). In addition, the non-analytic contributions \( \Omega_i \) give constrains on when the self-energies can be expanded for small \( B \). These constrains can generically be written as:

\[
\sqrt{f_i(\mu, r)B + \lambda_0^2} \to |B| < \left| \frac{\lambda_0^2}{f_i(\mu, r)} \right|, \quad (25)
\]

for a certain function \( f_i \) depending on the type of graph. One of these constrains, coming from the first graph in Fig. [1] was investigated in [3] for the situation of \( M_{ex} > M_{in} \). In the following we investigate the remaining two as well as further applications of Eq. (17).

1. Nucleon-pion system

The first example is the nucleon-pion \((N, \pi)\) system with pseudo-scalar couplings, \( \Gamma_{aN\pi} = g_a \gamma_5 \). For this we have \( s_5 = -1, r = 1, M_{ex} = M_N, m = m_\pi \) and write for the proton and neutron energies:

\[
\Sigma_p(B) = 2 \Sigma_1(B) + \Sigma_2(B), \quad (26)
\]
\[
\Sigma_n(B) = \Sigma_3(0) + 2 \Sigma_3(B), \quad (27)
\]

with

\[
\Sigma_i = -\frac{g^2}{(4\pi)^2} M_N \int_0^1 dz \left( 1 - z \right) \left[ L + \ln \left( \frac{z\mu^2 + (1 - z)^2 + B_i - i\varepsilon}{z\mu^2 + (1 - z)^2 + B_i - i\varepsilon} \right) \right], \quad (28)
\]

and \( B_1 = z (1 - z) B, B_2 = -(1 - z)^2 B \) and \( B_3 = -(1 - z) B \). It is easy to check that we get from this form the same AMM as obtained from the usual three point function method:

\[
\kappa_p = 2 \kappa_1 + \kappa_2, \quad \kappa_n = -2 \kappa_1 + 2 \kappa_2, \quad (29)
\]
\[
\kappa_1 = \frac{g^2}{(4\pi)^2} \int_0^1 dz \frac{2z (1 - z)^2}{z\mu^2 + (1 - z)^2}, \quad \kappa_2 = \frac{g^2}{(4\pi)^2} \int_0^1 dz \frac{-2 (1 - z)^3}{z\mu^2 + (1 - z)^2}. \quad (30)
\]
Figure 2: Energy shift of the proton (left) and neutron (right) with pseudo-scalar coupling for $m_\pi = 139$ MeV. The upper row corresponds to the shift with $\Sigma(B) - \Sigma(0)$ and the lower row to $\Sigma(-B) - \Sigma(B)$. The linear solid green lines correspond to the linear approximation Eq. (11) while the curved solid red lines to Eqs. (26,27). The solid blue line shows the result of Eq. (31). The dotted lines correspond to the imaginary parts.

According to this, we can also write for the neutron and iso-vector self-energies:

$$
\Sigma_n(B) = 3\Sigma_1(0) - 2\Sigma_1(B) + 2\Sigma_2(B),
$$

$$
\Sigma_v(B) = 4\Sigma_1(B) - \Sigma_2(B),
$$

where the difference of Eq. (31) to Eq. (27) is of order $B^2$.

In Fig. 2 we plot the nucleon energy shifts together with the linear approximation with $M_N = 939$ MeV and $m_\pi = 139$ MeV. We see that the signs of the AMM are in agreement with phenomenology, i.e. $\kappa_p > 0$ and $\kappa_n < 0$. To estimate $O(B^2)$ effects, we use the difference of the red and blue lines in the lower right neutron graph. The combination $\Sigma(-B) - \Sigma(B)$ does not have $O(B^2)$ contributions whereas the difference of the expressions Eq. (27) and Eq. (31) is of $O(B^2)$. Plotting the same results for various pion masses shows that the $B^2$ effects are small for magnetic field strengths of $|B| < \frac{1}{5}M_N^2$ for pion masses larger than $m_\pi = 100$ MeV and for $|B| < \frac{1}{6}M_N^2$ with pion masses around $m_\pi = 600$ MeV. Within these region the Eqs. (11,27,31) give approximately the same results. We estimate therefore that the self-energy formulas are applicable for magnetic field strengths of $|B| < \frac{1}{5}M_N^2$ with pion masses between the physical point $m_\pi = 140$ MeV up to $m_\pi = 600$ MeV. This is the applied pion mass range in lattice QCD calculations.

In addition we have also for the nucleon the non-analytic $B$ expressions, Eq. (20), which would constrain a definition of the magnetic moment by the linear energy shift. These constrains read:
\[ \Sigma_{1,3} : \quad |B| < |\mu (\mu - 2)| , \quad (33) \\
\Sigma_{1,3} : \quad |B| < |\mu (\mu + 2)| , \quad (34) \\
\Sigma_2 : \quad |B| < \frac{1}{4} \mu^2 - 1| , \quad (35) \]

which can always be fulfilled for \(0 < \mu < 2\).

Further, we obtain for the nucleon self-energy an imaginary part when a stronger magnetic field is applied. The imaginary parts come only from \(\Sigma_1\) and \(\Sigma_3\) and read:

\[ \text{Im}\Sigma_i = M_N \frac{g^2 \pi}{(4\pi)^2} (1 - \alpha_i) 2\lambda_i , \quad (36) \]

for \(B \leq -\mu (2 + \mu)\) in \(\Sigma_1\) and \(B \geq \mu (2 - \mu)\) in \(\Sigma_3\).

The nucleon results of this section are obtained from the general expression Eq. (17) for particles with Yukawa couplings as in the Lagrangian Eq. (13) and are transcriptable for other stable particles.

2. Nucleon-pion-resonance system

For the second example we include a resonance \((R)\) of mass \(M_R\) with the coupling \(\Gamma^{NR\pi}_a = ig_a\) and get with \(s_5 = 1\) and \(r = M_N/M_R\) the self-energies:

\[ \Sigma_i = \frac{g^2}{(4\pi)^2} M_R \int_0^1 dz \left( z + r \right) \left[ L + \ln \left( z\mu^2 - z(1 - z) + (1 - z)r^2 + B_i - i\varepsilon \right) \right] + \mathcal{O}(B^2) . \quad (37) \]

From this, we also recover the results of the three point method:

\[ \kappa_1 = \frac{g^2}{(4\pi)^2} \int_0^1 dz \frac{2z(1 - z)(-z - r)}{z\mu^2 - z(1 - z) + (1 - z)r^2 - i\varepsilon} , \quad \kappa_2 = \frac{g^2}{(4\pi)^2} \int_0^1 dz \frac{2(1 - z)^2(z + r)}{z\mu^2 - z(1 - z) + (1 - z)r^2 - i\varepsilon} . \quad (38) \]

Since the resonance is unstable we have the following imaginary parts for the self-energies:

\[ \text{Im}\Sigma_i = -M_R \frac{g^2 \pi}{(4\pi)^2} (r + \alpha_i) 2\lambda_i \]

(39) for \(\alpha_i > \lambda_i\). Explicitly, these parts are present for magnetic fields of

\[ \begin{align*}
\Sigma_1 : & \quad B < -\sqrt{(1 - r^2 - \mu^2)^2 - 4\lambda_0^2 + 1 - r^2 - \mu^2} , \\
\Sigma_2 : & \quad B > -\frac{\lambda_0^3}{\mu^2} , \\
\Sigma_3 : & \quad B > +\sqrt{(1 - r^2 + \mu^2)^2 - 4\lambda_0^2 - 1 + r^2 + \mu^2} .
\end{align*} \quad (40) \]

(41) (42)

In the case of \(\Sigma_2\) and \(\Sigma_3\) with a magnetic field of \(B \geq r^2\) we have \(\lambda_1 > \alpha_1\), Eq. (39) has do be altered accordingly and an additional cusp is present at \(B = r^2\).

In Fig. 3 we show all three possible self-energies for the parameters \(M_N = 939\) MeV, \(M_R = 1232\) MeV and \(m_\pi = 139\) MeV together with the linear approximations. For the graph \(\Sigma_1\) we see the linear behavior near \(B = 0\) and a cusp appearing according to Eq. (40). This graph was investigated in \([8]\). In the case of \(\Sigma_2\) we see only one cusp at \(B = r^2 \approx 0.6\) since for the present mass constellation Eq. (41) is fulfilled for all \(|B| < 1\) and an imaginary part is steadily present. However, choosing \(m_\pi\) closer to the mass-gap \(M_R - M_N\) the second cusp appears on the \(B < 0\) side. The occurrence of such two cusps can be seen for \(\Sigma_3\). The cusp for \(B < 0\) is due to the imaginary part from Eq. (42) and the one for \(B > 0\) due to \(B \geq r^2\). As we approach with \(m_\pi\) the mass-gap \(M_R - M_N\), i.e. \(\mu = 1 - r\), all cusps corresponding to Eqs. (40), (41) converge on \(B = 0\) where we have \(1 - r - \mu = 0\). For a \(m_\pi\) mass larger than the mass gap the resonance will not be unstable anymore and we get similar results as in the previous example.
Figure 3: Self-energies of a resonance in an external magnetic field $B$. The linear green lines correspond to the linear approximation Eq. (11) while the curved red lines to Eqs. (37). The solid lines show the real parts and the dotted lines the imaginary parts.

The explicit conditions to expand the self-energies for small $B$ due to the non-analytic contributions $\Omega_i$, Eq. (20), are:

\[ \Sigma_1 : \quad |B| < 2|1 - r - \mu|, \quad (43) \]
\[ \Sigma_2 : \quad |B| < \frac{2r}{1 - r}|1 - r - \mu|, \quad (44) \]
\[ \Sigma_3 : \quad |B| < 2r|1 - r - \mu|, \quad (45) \]

for the parameters $r < 1$, $\mu > 0$ and $\mu < 1 + r$. The first condition was found in [8]. Which condition is the most strict one on $B$ depends on the charge of the external particle and the actual values of the participating masses. For $\mu < r$ and $r > 1/2$ it is the first one.

IV. NUCLEON SELF-ENERGY

We apply now the BFT to a more involved situation, namely, the nucleon self-energy in the SU(2) chiral perturbation theory of [10]:

\[ \mathcal{L}_{N\pi} = \overline{N} (iD - M_N) N - \frac{g_A}{2f_\pi} \overline{N} \tau^a \left( D^{ab} \pi^b \right) \gamma_5 N + \frac{1}{2} \left( D^{ab}_\mu \pi^b \right) \left( D^{\mu}_a \pi^a \right) - \frac{1}{2} m_\pi^2 \pi^a \pi^a. \quad (46) \]

The covariant derivatives are given in the appendix and the nucleon axial-vector and the pion decay constants are $g_A = 1.27$ and $f_\pi = 92.4$ MeV. We consider the two graphs $\Sigma_1$ and $\Sigma_2$ in Fig. 1 and obtain for the nucleon the following unrenormalized self-energies in $d = 4 - 2\varepsilon$ dimensions:

\[ \Sigma^{N}_1 (B) = i \left( \frac{g_A}{2f_\pi} \right)^2 M_N \int_0^1 dz \left[ - M_N^2 (1 - z)^3 J_2 + (-6 + 3z) J_1 + (3 - z) \varepsilon J_1 + Bz^2 (3 - z) J_2 \right], \quad (47) \]
\[ \Sigma^{N}_2 (B) = i \left( \frac{g_A}{2f_\pi} \right)^2 M_N \int_0^1 dz \left[ - M_N^2 (1 - z)^3 J_2 + (-6 + 3z) J_1 + (3 - z) \varepsilon J_1 \
+ B \left( 2 (1 - z)^4 M_N^2 J_3 + (3 - 8z + 6z^2 - z^3) J_2 - (3 - 4z + z^2) \varepsilon J_2 \right) \right]. \quad (48) \]

The loop integrals $J_i = J_i (\mathcal{M}_N)$ with $\mathcal{M}_N = zm_\pi^2 + (1 - z)^2 M_N^2 + z (1 - z) B$ are listed in the appendix. The expressions $\Sigma_1$ and $\Sigma_2$ only differ by $B$ dependent terms and we write for the nucleon self-energies:

\[ \Sigma_p (B) = 2\Sigma_1 (B) + \Sigma_2 (B), \quad (49) \]
\[ \Sigma_n (B) = 3\Sigma_1 (0) - 2\Sigma_1 (B) + 2\Sigma_2 (B). \quad (50) \]
Figure 4: Proton self-energy as function of a magnetic field $B$ for $m_\pi = 139$ MeV and $M_N = 939$ MeV. The solid red line is the result of Eq. (51) and the solid linear green line the linear approximation $\Sigma_p(B) = M_N - \frac{\kappa}{2M_N}B$ with $\kappa = 1.73$. The dashed line is the imaginary part of Eq. (51).

By integrating the Feynman parameter we get the $\hat{M}S$ renormalized proton self-energy

$$
\Sigma_p(B) = \frac{M_NC_N}{24(1-B)} \left[ n_1 + n_2 \ln \mu + \frac{n_3}{\sqrt{4\mu^2 - (B+\mu)^2}} \arccos \frac{B+\mu}{2\mu} \right],
$$

(51)

$$
n_1(B,\mu) = \begin{cases} 
36 - 156B + 236B^2 - 135B^3 + 12B^4 + 7B^5 & (52)
\end{cases}
$$

$$
n_2(B,\mu) = -24B^3 + 42B^4 - 12B^5 + 96\mu^2 + (-144 + 348B)\mu^4 + (36 - 72B)\mu^6 + O(B^2),
$$

(53)

$$
n_3(B,\mu) = -24B^2 + 42B^3 - 12B^4 + 120\mu^2 + (-36 + 72B)\mu^4 + O(B^2),
$$

(54)

with $C_N = \left( \frac{2M_NC_N}{4\pi f^2} \right)^2$. We give here only terms up to $O(B^2)$ or constant in $\mu$ and list in the appendix the full coefficients $n_i(B,\mu)$. By setting $B = 0$ we recover the normal nucleon $\hat{M}S$ renormalized B\(\chi\)PT self-energy [11, 12]:

$$
\Sigma_p(0) = \frac{3}{2}M_NC_N \left[ 1 + 2\mu^2 - 2\mu^3 \sqrt{1 - \frac{\mu^2}{4}} \arccos \frac{\mu}{2} - \mu^4 \ln \mu \right].
$$

(55)

The chiral expansion of the nucleon mass to order $p^3$ is

$$
M_N(B) = \tilde{M}_N - 4\tilde{c}_1 m_\pi^2 + \Sigma_N^{(3)}(B) - \frac{\kappa_N}{2M_N}B,
$$

(56)

where $\tilde{M}_N$, $\tilde{c}_1$, and $\kappa_N$ are the low-energy constants for the nucleon mass and its AMM. The $\mu^0$ and $\mu^2$ terms in Eq. (55) break the usual power counting scheme [10] where e.g. the EOMS renormalization scheme [13] is one way to deal with this problem. The general behavior of the self-energy in this section is similar to the nucleon self-energy of the last section for pseudo-scalar $N - \pi$ couplings. Especially the non-analytic $B$ term is the same and we get again an imaginary part for $B < -2\mu - \mu^2$.

In Fig. 4 we show the proton self-energy as function of the magnetic field $B$ for the phenomenological values. In case of the linear approximation we use the AMM obtained from the three-point method, i.e. defined by $\kappa = F_2(0)$ via the matrix element

$$
\langle N(p')|\bar{\Psi}(0)\gamma^\mu\Psi(0)|N(p)\rangle = \bar{u}(p') \left[ \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M_N} F_2(q^2) \right] u(p),
$$

(57)

with $q = p' - p$ as the momentum transfer. The explicit results for the nucleon graphs in the second row of Fig. 4 are:
\[
\kappa_1^N = \frac{1}{i} \left( \frac{gAM}{2f_\pi} \right)^2 \int_0^1 dz 2z \left[ (-6 + 12z - 4z^2) J_2 + (3 - 4z + z^2) \varepsilon J_2 - 2M_N^2 (1 - z)^4 J_3 \right] \tag{58}
\]
\[
\kappa_2^N = \frac{1}{i} \left( \frac{gAM}{2f_\pi} \right)^2 \int_0^1 dz 2 \left[ (3 - 14z + 15z^2 - 4z^3) J_2 - (3 - 7z + 5z^2 - z^3) \varepsilon J_2 + 2M_N^2 (1 - z)^5 J_3 \right] \tag{59}
\]

with \( J_i = J_i (M) \) and \( M = zm^2 + (1 - z)^2 M_N^2 \). Extracting the AMM from the above self-energy yields identical results where in the case of \( \kappa_1^N \) this can be seen literally even before the Feynman parameter integration:

\[
\kappa_1^{N (BFT)} = \frac{1}{i} \left( \frac{gAM}{2f_\pi} \right)^2 \int_0^1 dz 2z \left[ (-6 + 9z - 3z^2) J_2 + z (3 - z) J_2 + (3 - 4z + z^2) \varepsilon J_2 - 2M_N^2 (1 - z)^4 J_3 \right] \tag{60}
\]

The two \( J_2 \) integrals add up to the same expression in Eq. (58). However, in the case of the three-point function method, e.g., the \( J_2 \) contributions come purely from tensor integrals whereas in the case of the BFT method it is a combination of tensor and scalar loop-integrals. As we see for the infinite volume case this difference is not important.

V. **DELTA(1232) SELF-ENERGY**

We consider now the \( \Delta^+ \) (1232)-isobar and concentrate on the graphs that give its decay width. We take the following Lagrangian \[1] for its decay width:

\[
\mathcal{L}_{\Delta^=} = \overline{\Delta}_\mu (i\gamma^{\mu\nu}D_\lambda - M_\Delta \gamma^{\mu\nu}) \Delta_\nu + i\frac{h_A}{2f_\pi M_\Delta} \overline{\Delta}_\mu \gamma^{\mu\nu\lambda} (D_\mu \Delta_\nu) (D_\nu \pi^b) + h.c. , \tag{61}
\]

and obtain the relevant self-energies as:

\[
\Sigma_1^\Delta (B) = \frac{1}{i} M_\Delta \left( \frac{h_A}{2f_\pi} \right)^2 \frac{1}{2} \int_0^1 dz \left[ (z + r) J_1 - B (z + r) z^2 J_2 \right] , \tag{62}
\]
\[
\Sigma_2^\Delta (B) = \frac{1}{i} M_\Delta \left( \frac{h_A}{2f_\pi} \right)^2 \frac{1}{2} \int_0^1 dz \left[ (z + r) J_1 - B (z + r) (1 - z)^2 J_2 \right] , \tag{63}
\]
\[
M_\Delta = zm^2 + (1 - z)^2 B_1 , \tag{64}
\]

with \( B_1 = +z (1 - z) B \) and \( B_2 = -(1 - z)^2 B \) and all other definitions given in the appendix. Integrating these expressions yield

\[
\Sigma_1^\Delta (B) \cdot \frac{144}{M_\Delta C_\Delta} = A_1 (B) + 48 \left[ 2 (r + \alpha_1) \lambda_1^2 + B (3 r + \alpha_1) \alpha_1^2 + (\alpha_1 - r) \lambda_1^3 \right] \Omega_1 (B) , \tag{65}
\]
\[
\Sigma_2^\Delta (B) \cdot \frac{144}{M_\Delta C_\Delta} = A_2 (B) + 48 \left[ 2 (r + \alpha_2) \lambda_2^2 + B (3 r + 3 \alpha_2 - 6 r \alpha_2 - (3 \alpha_2^2 + 1) (2 r - \alpha_2) \lambda_2^3 \right] \Omega_2 (B) , \tag{66}
\]
\[
\Omega_i (B) = \sqrt{\lambda_i^2 \left( \arctanh \frac{\beta_i}{\sqrt{\lambda_i^2}} + \arctanh \frac{\alpha_i}{\sqrt{\lambda_i^2}} \right)} , \tag{67}
\]

with \( C_\Delta = \left( \frac{h_A M_\Delta}{8f_\pi} \right)^2 \) and the analytic parts \( A_i \) also listed in the appendix. With these two expressions we can also write the self-energy of the different iso-spin states as:

\[
\Sigma_{\Delta^{++}} (B) = -\Sigma_1^\Delta (0) + \Sigma_1^\Delta (B) + \Sigma_2^\Delta (B) , \tag{68}
\]
\[
\Sigma_{\Delta^+} (B) = \frac{1}{3} \Sigma_1^\Delta (B) + \frac{2}{3} \Sigma_2^\Delta (B) , \tag{69}
\]
\[
\Sigma_{\Delta^0} (B) = \Sigma_1^\Delta (0) - \frac{1}{3} \Sigma_1^\Delta (B) + \frac{1}{3} \Sigma_2^\Delta (B) , \tag{70}
\]
\[
\Sigma_{\Delta^-} (B) = 2 \Sigma_1^\Delta (0) - \Sigma_1^\Delta (B) . \tag{71}
\]
For the phenomenological parameters together with the linear approximation as obtained from the three-point function method. The cusp comes from the \( \Sigma_1 \) while the second cusp of \( \Sigma_2 \) is not present for \( m_\pi = 139 \text{ MeV} \), however it emerges on the \( B < 0 \) side for larger pion masses and both cusps fall on \( B = 0 \) for \( m_\pi = M_\Delta - M_N \). The imaginary parts read

\[
\begin{align*}
\text{Im} \Sigma_1 (B) \cdot \frac{3}{\pi M_\Delta C_\Delta} &= -2 (1 - B) (r + \alpha_1) \lambda_1^2 - B \lambda_1 \left[ 3 \alpha_1 \left( \alpha_1^2 + \lambda_1^2 \right) + r \left( 3 \alpha_1^2 + \lambda_1^2 \right) \right], \\
\text{Im} \Sigma_2 (B) \cdot \frac{3}{\pi M_\Delta C_\Delta} &= -2 (1 - B) (r + \alpha_2) \lambda_2^2 - B \lambda_2 \left[ -6 \alpha_2^2 - 3 \alpha_2^4 - 2 \lambda_2^2 + 3 \alpha_2 (1 + \lambda_2^2) + r \left( 3 - 6 \alpha_2 + 3 \alpha_2^2 + \lambda_2^2 \right) \right],
\end{align*}
\]

where the \( \Delta^+ (1232) \) decay width, experimentally given by \( \Gamma_\Delta \approx 120 \text{ MeV} \), is obtained from \( \Gamma = -2 \text{Im} \Sigma (B = 0) \). The slope of the imaginary part at \( B = 0 \) in Fig. 5 is consistent with \cite{14} and we obtain a vanishing decay width at:

\[
B = - \frac{(1 - 2r + r^2 - \mu^2) (1 + 2r + r^2 - \mu^2)}{2 (1 + r - 2r^3 + 3r^4 + 2r \mu^2 + 3 \mu^4 - 2r^2 (1 + 3 \mu^2))} + O (B^2). \quad (74)
\]

The magnetic moment from the three-point function method is again defined through the vector matrix element by \( \kappa = F_2 (0) \) and

\[
\langle \Delta (p) | \bar{\Psi} (0) \gamma^\mu \Psi (0) | \Delta (p') \rangle = - \bar{u}_\alpha (p') \left[ F_1 \gamma^\mu + \frac{i \sigma^{\mu \nu} q_\nu}{2 M_\Delta} F_2 \right] g^{\alpha \beta} + \left[ F_3 \gamma^\rho + \frac{i \sigma^{\rho \nu} q_\nu}{2 M_\Delta} F_4 \right] q^{\alpha} q^{\beta} \right] u_{\beta} (p), \quad (75)
\]

with the results \cite{5}:

\[
\begin{align*}
\kappa_1^\Delta &= \frac{1}{i} \left( \frac{h_\lambda M_\Delta}{2 f_\pi} \right)^2 \int_0^1 dz \int_0^1 2z \left[ \frac{1}{2} \right] \frac{1}{2} + \frac{1}{2} z - \frac{1}{2} r + z r \right] J_2, \\
\kappa_2^\Delta &= \frac{1}{i} \left( \frac{h_\lambda M_\Delta}{2 f_\pi} \right)^2 \int_0^1 dz \int_0^1 2 (1 - z) \left[ (1 + r) (1 - z) - (1 - z)^2 \right] J_2,
\end{align*}
\]

with \( J_i = J_i (M_\Delta) \) and \( M_\Delta = z \mu^2 + (1 - z) r^2 - z (1 - z) \). The results of the BFT method agree in this form literally. From \( \sqrt{\lambda_1^2} \) in Eq. (67) we obtain the condition

\[
\frac{eB}{2 M_\Delta} \ll \left| M_\Delta - (M_N + m_\pi) \right| \quad (78)
\]

for expanding the \( \Delta (1232) \)-isobar self-energies in small magnetic fields \cite{8}.
VI. SUMMARY

We investigated the self-energies of particles placed in a constant electromagnetic (EM) field. Explicitly, we applied the EM background field technique (BFT) once to the situation of stable and unstable spin-1/2 particles coupling to (pseudo-) scalar fields and once to the nucleon and $\Delta$ (1232)-isobar baryons in the $SU(2)$ chiral perturbation theory ($B\chi$PT). We obtained the self-energies of these particles as function of the external constant magnetic field $B$ and calculated from these the anomalous magnetic moments (AMM) by the linear energy shift. We summarize our findings as:

- Self-energies of Dirac particles coupling to (pseudo-) scalar fields: We investigated all three types of Feynman graphs that can appear in the one-loop BFT. The self-energies generally depend non-analytically on $B$ where for stable particles the actual non-analytic points are unproblematic for defining the AMM by the linear energy shift. These points depend on the mass and charge constellations in the loop and give three different conditions for resonances on when their self-energies can be expanded for weak $B$. One of these conditions was reported earlier. The self-energy formulas contain those $O(B^2)$ terms that keep the non-analytical dependence intact, in contrast to the one-photon approximation. However, they omit some $O(B^2)$ contributions where we estimated for the nucleon-pion system with pseudo-scalar couplings that these contributions are small for magnetic fields of $|B| < \frac{1}{5} M_N^2$ with $M_N$ as the nucleon mass.

- Self-energies of the nucleon and $\Delta$ (1232)-isobar in $B\chi$PT: We derived the formulas for the nucleon and $\Delta$ (1232)-isobar self-energies depending on the pion mass and the magnetic field. We recover the expressions as obtained from the three point function method as well as the condition on when the $\Delta$ (1232)-isobar magnetic moment is well defined by the linear energy shift. Further, we saw that the AMM expressions have different tensor- and scalar-loop integral combinations depending on whether the AMM is derived by the three point function method or from the self-energy. In the infinite volume these combinations add up to the same AMM expressions.

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Appendix A: Notation

We use the following notations: $d\ell = d^n l / (4\pi)^n$ with $n = 4 - 2\epsilon$ dimensions,

$$\mu = m/M_{ex}, \quad r = M_{in}/M_{ex}, \quad B = B/M_{ex}^2.$$  \hspace{1cm} (A1)

The covariant derivatives are:

$$D_{\mu}^a a^b = \delta^{ab} \partial_{\mu} \pi^b + i e Q_{a\pi} A_{\mu} \pi^b,$$  \hspace{1cm} (A2)

$$D_{\mu} N = \partial_{\mu} N + i e Q_{N} A_{\mu} N + \frac{i}{4f_\pi^2} \epsilon^{abc} \tau^a \pi^b (\partial_{\mu} \pi^c),$$  \hspace{1cm} (A3)

$$D_{\mu} \Delta_{\nu} = \partial_{\mu} \Delta_{\nu} + i e Q_{\Delta} A_{\mu} \Delta_{\nu} + \frac{i}{2f_\pi^2} \epsilon^{abc} \tau^a \pi^b (\partial_{\mu} \pi^c),$$  \hspace{1cm} (A4)

with the operators $Q_N = (1 + \tau^3)/2$ and $Q_{a\pi} = -i \epsilon^{ab3}$. The results for the loop integration in infinite volume and dimensional regularization with $L = -\frac{1}{\epsilon} + \gamma_E + \ln \frac{M^2}{4\pi}\Lambda^2$ is:

$$J_1 (M) = -i \left( \frac{m_{sc}}{4\pi^2} \right)^2 M \left[ L - 1 + \ln M \right], \quad J_2 (M) = \frac{i}{(4\pi)^2} \left[ L + \ln M \right], \quad J_3 (M) = \frac{i}{(4\pi)^2} \frac{1}{2m_{sc}^2} \frac{1}{M}.$$  \hspace{1cm} (A5)

The corresponding parts of the nucleon and $\Delta$ (1232) self-energies are:
\begin{align}
n_1(B, \mu) &= 36 - 156B + 236B^2 - 135B^3 + 12B^4 + 7B^5 \\
&\quad + (72 - 252B + 321B^2 - 186B^3 + 45B^4) \mu^2 + (6B^2 - 6B^3) \mu^4 \tag{A6}
\end{align}
\begin{align}
n_2(B, \mu) &= -24B^3 + 42B^4 - 12B^5 + (96\beta - 180\beta^2 + 60\beta^3 + 12\beta^4 - 12\beta^5) \mu^2 \\
&\quad + (-144 + 348B - 270B^2 + 120B^3 - 18B^4) \mu^4 + (36 - 72B + 12B^2) \mu^6 + 6B^2 \mu^8 \tag{A7}
\end{align}
\begin{align}
n_3(B, \mu) &= -24B^2 + 42B^3 - 12B^4 + (120B - 294B^2 + 216B^3 - 60B^4) \mu^2 \\
&\quad + (-36 + 72B - 24B^2 + 6B^3) \mu^4 - 6B^2 \mu^6 \tag{A8}
\end{align}

\begin{align}
A(B) &= 27 + 40r - 68\alpha - 120r\alpha + 42\alpha^2 + 120r\alpha^2 + 12\alpha^3 - 54\lambda^2 - 168r\lambda^2 - 60\alpha\lambda^2 \\
&\quad + (-18 + 48\alpha - 36\alpha^2 + 36\lambda^2 - 24r (1 - 3\alpha + 3\alpha^2 - 3\lambda^2)) \ln (1 - B) \\
&\quad + (-24\alpha^3 - 6\alpha^4 + 72\alpha\lambda^2 + 36\alpha^2\lambda^2 + 184) \ln (\alpha^2 - \lambda^2) \\
&\quad + (-18 + 48\alpha - 36\alpha^2 + 6\alpha^4 + 36\lambda^2 - 36\alpha^2\lambda^2 - 18\lambda^4 + 24r(\alpha - 1)((\alpha - 1)^2 - 3\lambda^2)) \ln (\beta^2 - \lambda^2) \tag{A9}
\end{align}
\begin{align}
A_1(B) &= \left[ \frac{4}{3} + 3(3 - 4\alpha + 3\alpha^2 - 3\lambda^2 + r (4 - 6\alpha + 6\alpha^2 - 6\lambda^2)) \ln (1 - B) \\
&\quad + (48\alpha^3 + 244\alpha^4 + 72\alpha^2\lambda^2) \ln (\alpha^2 - \lambda^2) - 12(4\alpha + 2\alpha^2 + 3(\lambda^2 - 1) + \alpha^2 (6\lambda^2 - 3) \\
&\quad + r (6(\alpha - \alpha^2 + \lambda^2) - 4 + 4\alpha^3)) \ln (\beta^2 - \lambda^2) \right]_{\alpha, \beta, \lambda} \tag{A10}
\end{align}
\begin{align}
A_2(B) &= \left[ \frac{4}{3} + 3(3 - 4\alpha + 3\alpha^2 - 3\lambda^2 + r (4 - 6\alpha + 6\alpha^2 - 6\lambda^2)) \ln (1 - B) \\
&\quad + (72\alpha - 36\alpha^2 - 72\alpha^2 - 48\alpha^3 + 48\alpha^4 + 36\alpha^2 - 72\alpha^2 - 144\alpha^2 + 72\alpha^2\lambda^2) \ln (\alpha^2 - \lambda^2) \\
&\quad - 24(-1 + \alpha)^2 (-1 + 2r(-1 + \alpha) + \alpha^2 + 3\lambda^2) \ln (\beta^2 - \lambda^2) \right]_{\alpha, \beta, \lambda} \tag{A11}
\end{align}

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