Towards an effective description of holographic vortex dynamics

Yu-Kun Yan,1, Shanquan Lan,2 Yu Tian,1,3 Peng Yang,1 Shunhui Yao,1 and Hongbao Zhang4

1 School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
2 Department of Physics, Lingshan Normal University, Zhanjiang 524048, China
3 Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China
4 Department of Physics, Beijing Normal University, Beijing 100875, China

Although holographic duality has been regarded as a complementary tool in helping understand the non-equilibrium dynamics of strongly coupled many-body systems, it still remains a remarkable challenge how to confront its predictions quantitatively with the real experimental scenarios. By taking a right evolution scheme for the holographic superfluid model and matching the holographic data with the phenomenological dissipative Gross-Pitaevskii models, we find that the holographic dissipation mechanism can be well captured by the Landau form, which is expected to open up novel avenues for facilitating the quantitative test of the holographic predictions against the upcoming experimental data. Our result also provides a prime example how holographic duality can help select proper phenomenological models by invalidating the claim made in the previous literature that the Keldysh self-energy can serve as an effective description of the holographic dissipation in superfluids.

Introduction.—For the non-equilibrium dynamics of strongly interacting quantum systems where the quasiparticle picture does not apply and the perturbation method fails, developing its theoretical description remains an important task1, 2. Gratefully, holographic duality3, 5, also known as Anti-de-Sitter space/conformal field theory correspondence, has provided a powerful insight into the universal behaviors of strongly coupled dynamics through the classical theory of gravity with one additional dimension. In particular, a variety of bottom-up gravitational models have been proposed to address the strongly correlated condensed matter systems6, 12. But nevertheless, associated with these bottom-up holographic models, there exists a significant deficiency, namely, the effective Lagrangians of the dual boundary systems are generically unknown, which makes it a notoriously difficult challenge to compare the holographic prediction with the experimental data.

Faced up with this challenge, the authors in 13 have recently made a first stab at constructing the corresponding phenomenological model for the non-linear dynamics of holographic superfluids by matching the condensate profile of a single static vortex as well as the vortex dipole motion between the holographic superfluid and the dissipative Gross-Pitaevskii equation, where the dissipation comes from the Keldysh self-energy. They claimed that the vortex dynamics of the holographic superfluid could be well quantified by this equation. Thus one could relatively easily test the holographic vortex dynamics in the experimentally accessible superfluids such as strongly coupled cold atomic gases and thin Helium films.

Wonderful though such a strategy is, their claim is doubtful, because unlike the well behaved holographic superfluids14, the above dissipative Gross-Pitaevskii equation, albeit extensively used in numerical simulation for finite temperature superfluids, has an entirely unphysical dispersion relation, which makes the long-wavelength sound mode non-propagating. As a matter of fact, there is a fatal problem with their evolution scheme for the holographic superfluids, where one of the bulk equations of motion is severely violated, making their claim invalid. In this Letter, we adopt a correct holographic evolution scheme and match our holographic data not only with the aforementioned dissipative Gross-Pitaevskii equation but also with the other one motivated by Landau’s two fluid model for superfluidity, whose dispersion relation behaves as well as the holographic superfluids. As a result, we find that the dissipation mechanism in our holographic superfluid can be well fitted by the Landau form rather than the Keldysh one claimed in 13. Among others, this presents us a prime example how holographic duality can help select proper phenomenological models. On the other hand, as alluded to above, with our finding, the holographic superfluid model with four bulk dynamical variables and one adjustable boundary value can be described effectively by only one dynamical variable with three adjusted parameters in one less dimension, which will make the quantitative comparison of the holographic predictions with the real upcoming experimental data much easier and much more efficient.

Holographic setups and evolution schemes.—In the probe limit, where the holographic superfluid is implemented by the Abelian Higgs model with the Lagrangian density given by 6, 8

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |D_\mu \Phi|^2 - m^2 |\Phi|^2, \]

(1)
on top of the (3+1) dimensional planar Schwarzschild AdS black hole in the Eddington-Finkelstein coordinates

\[ ds^2 = \frac{L^2}{z^2} (-f(z) dt^2 + dx^2 + dy^2 - 2dtdz), \]

(2)
where \( D_\mu = \nabla_\mu - i A_\mu \) and \( f(z) = 1 - (z/z_h)^5 \) with \( z_h \) the location of the black hole horizon. The correspond-
ing dynamics is governed by the following equations of motion
\[ \nabla_{\mu} F^\mu_{\nu} = i(\Phi^* D^\nu \Phi - \Phi (D^\nu \Phi)^*), \]
\[ D_\mu D^\mu \Phi - m^2 \Phi = 0, \tag{3} \]
where the asterisk denotes the complex conjugation. In what follows, we set \( L_{\text{AdS}} = 1 \). In addition, for simplicity but without loss of generality, we take \( m^2 = -2 \). With the choice of the axial gauge \( A_t = 0 \), the above equations of motion can be decomposed into the constraint equation
\[ 0 = -\partial_z^2 A_t + \partial_z \partial \cdot A - 2 \text{Im}(\phi^* \partial_z \phi), \tag{4} \]
and the evolution equations
\[
\partial_t \partial_z \phi = \partial_z \left( \frac{f(z)}{2} \partial_z \phi \right) + \frac{1}{2} \partial^2 \phi - i A \cdot \partial \phi + i A_t \partial_z \phi \\
- \frac{i}{2} (\partial \cdot A - \partial_z A_t) \phi - \frac{1}{2} (z + A^2) \phi, \tag{5}
\]
\[
\partial_t \partial_z A = \partial_z \left( \frac{f(z)}{2} \partial_z A \right) - |\phi|^2 A + \text{Im}(\phi^* \partial_z \phi) \\
+ \frac{1}{2} [\partial \partial_z A_t + \partial^2 A - \partial \partial \cdot A], \tag{6}
\]
\[
\partial_t \partial_z A_t = \partial^2 A_t - \partial_t \partial \cdot A + f(z) \partial_z \partial \cdot A - 2 A_t |\phi|^2 \\
+ 2 \text{Im}(\phi^* \partial_t \phi) - 2 f(z) \text{Im}(\phi^* \partial_z \phi), \tag{7}
\]
where \( \phi = \Phi/z \) and \( A = (A_x, A_y) \).

By holography, the temperature of the dual boundary system is given by the Hawking temperature \( T = 3/(4 \pi z_h) \), and the chemical potential is related to the boundary data of the bulk field \( A_t \) as \( \tilde{\mu} = A_t|_{z=0} \). Due to the scaling symmetry, one can set \( z_h = 1 \) once and for all. Accordingly, it turns out when the chemical potential is higher than the critical value \( \tilde{\mu}_c = 4.064 \), the bulk complex scalar field will spontaneously condensate, which signals the transition to the superfluid phase on the boundary. The corresponding order parameter \( \psi \) can be read off from the boundary data of \( \phi \) according to the holographic dictionary. It is noteworthy that holography provides a natural built-in mechanism to account for the irreversible finite temperature dissipation by geometrizing the excitations absorbed by the black hole. To investigate the non-equilibrium dynamics of the holographic superfluid such as the vortex dipole dynamics, one is required to perform a full non-linear numerical simulation of the bulk equations of motion.

To this end, we would first like to impose the boundary conditions onto the other bulk fields as follows
\[ \phi|_{z=0} = 0, \quad A_t|_{z=0} = \tilde{\mu}, \quad A|_{z=0} = 0. \tag{8} \]
We evolve \( \phi \) and \( A \) by Eq.\((5)\) and Eq.\((6)\) but solve \( A_t \) by the constraint equation with an extra boundary condition \( \partial_z A_t|_{z=0} = -\rho \), where \( \rho \) corresponds to the boundary particle number density. The dynamics of \( \rho \) is controlled by the current conservation law, which is simply Eq.\((7)\) evaluated at the boundary. Thus by implementing the current conservation during our evolution, Eq.\((7)\) is guaranteed to hold automatically elsewhere in the whole bulk. Different from ours, the authors in \cite{13} take \( A_t|_{z=z_h} = 0 \) as the extra boundary condition and simply disregard Eq.\((7)\). In order to compare our numerical simulation with that in \cite{13}, we prepare the initial data by essentially the same strategy albeit a little bit rough. Namely, the initial \( \phi \) is constructed by multiplying the static and homogeneous background \( \phi(z) \) with a vortex dipole profile in the \( x \) plane. In addition, the initial value for \( A \) is set to zero. As a result, the initial \( A_t \) with \( A_t|_{z=z_h} = 0 \) and \( A_t|_{z=0} = \tilde{\mu} \) can be obtained by solving the constraint equation followed by the initial \( \rho \) extracted through \( \rho = -\partial_z A_t|_{z=0} \). We demonstrate the numerical results for the vortex dipole motion by our scheme (S1) and theirs (S2) with the exactly same initial data in Fig.\(1\). As one can see from (a), the relative distance \( D \) of the vortex dipole starts to exhibit different behaviors at the vortex dipole annihilation stage such that it takes relatively more time for their vortex dipole
to annihilate than ours. Thus in comparison with ours, their scheme produces a kind of repulsive force in addition to the familiar Magnus force between the vortex dipole when closed to each other. Moreover, as illustrated in (b), it is from the very beginning that the center position $R$ of their vortex dipole moves forward in a totally different pace from ours. As a result, the trajectory for their vortex dipole is quite distinct from ours, which is displayed in (c).

As detailed in Supplemental Material, their evolution scheme leads inevitably to not only the violation of the salient current conservation law but also the breakdown of Eq.(7) into the whole bulk, which makes their results unreliable. Therefore, below we shall match the holographic data obtained by our numerical scheme with the dissipative Gross-Pitaevskii equations.

The dissipative Gross-Pitaevskii models and dispersion relations.—As to the Bose-Einstein condensates (BEC) in the dilute cold atom gases at nearly zero temperature, the behaviour of order parameter $\psi$ can be successfully described by Gross-Pitaevskii equation (GPE) [15]. However, GPE cannot describe BEC at finite temperature. In order to account for the finite temperature effect, one is required to introduce the dissipative terms. For our purpose, we consider two such dissipative Gross-Pitaevskii equations (DGPEs), which can be written in the dimensionless form as follows

$$\partial_t \psi = -\frac{(i + \gamma)}{\tau} (-\frac{\nabla^2}{2} \psi + \mu(|\psi|^2 - 1)\psi),$$  (9)

$$\partial_t \psi + i\gamma \partial_t |\psi|^2 = -\frac{i}{\tau} \left[-\frac{\nabla^2}{2} \psi + \mu(|\psi|^2 - 1)\psi\right].$$  (10)

Here the parameter $\tau$ controls the characteristic time scale of dynamics, and $\mu$ is the chemical potential, from which the dimensionless healing length is given by $\xi = (2\mu)^{-1/2}$. The dissipative parameter $\gamma$ in Eq.(9) is suspected to be determined by the Keldysh self-energy through the fluctuation-dissipation theorem [16][18]. So we call this equation as KGPE. On the other hand, we denote Eq.(10) with $\eta$ the dissipative parameter as LGPE, as it was phenomenologically motivated by Landau’s requirement that the second law of thermodynamics hold in his two fluid model for superfluidity [19][20]. Although both equations are widely used to attempt modeling the finite temperature BEC, they are supposed to predict different behaviors as it should be the case. Among others, here we point out that the linear response theory of KGPE has already displayed a qualitatively distinct behavior from LGPE, which, to our best knowledge, has not been noticed before. As such, let us consider the linear perturbations

$$\delta \psi = p e^{-i\omega t + ik \cdot x} + \bar{p} e^{i\omega t - ik \cdot x},$$  (11)
on top of the static and homogeneous equilibrium configuration $\psi = 1$, where $p$ and $\bar{p}$ are independent of each other. Then as usual, the dispersion relation can be obtained by substituting Eq.(11) into the linearized equation of motion as an eigenvalue problem. For KGPE, the resulting dispersion relation is

$$\omega(k) = \pm \frac{\sqrt{k^2 + 4k^2 \mu - 4\gamma^2 \mu^2}}{2\tau} - \frac{i\gamma(k^2 + 2\mu)}{2\tau},$$  (12)

when $\gamma = 0$, the long wave limit gives rise to the familiar dispersion relation for the sound mode. While in the presence of dissipation, we have

$$\omega_+ (k) = -i\frac{1}{2\tau\gamma}(1 + \gamma^2)k^2,$n$$

$$\omega_- (k) = \mp i\frac{2\gamma \mu}{\tau} + \frac{1}{2\tau} (1 - \gamma^2) k^2$$  (13)
as $k \to 0$. It is obvious that the sound mode behaves abnormally as the sound speed becomes zero.

Different from KGPE, the dispersion relation for LGPE produces the normal behavior of the sound mode as

$$\omega(k) = \frac{\sqrt{\eta}}{\tau} k - \frac{i\eta}{2\tau} k^2$$  (14)
at small $k$.

Note that in our holographic superfluid, the long wave behavior of the sound mode is similar to that in LGPE rather than KGPE. So it is reasonable to expect that our holographic superfluid is presumably well matched by LGPE rather than KGPE claimed in [13].

Matching procedure and relevant results.—In order to quantify how well the above two models serve as a phenomenological description of the holographic vortex dynamics, we employ the same matching procedure as that in [13]. Namely we first determine the healing length in both models by fitting the order parameter profile for the holographic vortex of winding number 1 with the form $|\psi|^2 \sim \frac{s^2}{2\tau^2 + s^2}$. Then we intend to fit the holographic vortex dipole trajectory by adjusting the corresponding dissipation parameter. Finally, the parameter $\tau$ is fixed by tracking the real time evolution of the vortex dipole.

We demonstrate our relevant results by focusing on a typical example, namely the holographic superfluid at $\bar{\mu} = 4.5$. As illustrated in Fig[2], the resulting holographic vortex can be well fitted by both models with the same healing length $\xi = 1.0$. On the other hand, as shown in Fig[3], the corresponding holographic trajectory can be better modeled by LGPE with $\eta = 1.85$ till the vortex dipole annihilation than KGPE, which starts to display an apparent deviation from the holographic behavior when the vortex dipole get contacted with each other. Similarly, as one can see in Fig[3] the real time evolution of the holographic vortex dipole can also be better captured by LGPE with $\tau = 2.35$ all the way to the annihilation stage than KGPE, which fails to describe the real time dynamics of the vortex dipole when get closed to each other.
FIG. 2. The matching results between the holographic superfluid at $\tilde{\mu} = 4.5$ and DGPEs. In (a), the normalized condensate profile of a single static holographic vortex is well fitted by both DGPEs, where the black dotted line is used to identify the vortex size. In (b), the holographic vortex dipole trajectory is fitted by both KGPE and LGPE, where the black dotted line indicates the location where the vortex dipole get contacted with each other.

TABLE I. The best fitting parameters in DGPEs for the holographic superfluid at $\tilde{\mu} = 4.5$ and $\tilde{\mu} = 6$.

|          | $\tilde{\mu}$ | $\xi$ | $\eta$ | $\gamma$ | $\tau$ |
|----------|---------------|-------|--------|----------|--------|
| DGPEs    |               |       |        |          |        |
| LGPE     | 4.5           | 1.00  | 1.85   | /        | 2.35   |
| LGPE     | 6             | 0.44  | 1.51   | /        | 4.70   |
| KGPE     | 4.5           | 1.00  | /      | 0.129    | 2.22   |
| KGPE     | 6             | 0.44  | /      | 0.085    | 4.50   |

Similar matching results apply to the holographic superfluid at other chemical potentials. Here we only list the resulting best fitting parameters in Table I for $\tilde{\mu} = 4.5$ and $\tilde{\mu} = 6$.

Discussions.—By adopting the correct numerical simulation scheme for our holographic superfluid, we find that the holographic vortex dipole dynamics can be well matched by LGPE all the way down to the vortex dipole annihilation. Compared with LGPE, KGPE matches up with our holographic data only when the vortex dipole are far apart from each other, while an apparent deviation occurs when the vortex dipole get close to each other. Together with the observation that KGPE displays a dramatically abnormal dispersion relation, qualitatively distinct from the well behaved one shared by both LGPE and the holographic superfluid, we are convinced that the reasonable phenomenological model for our holographic superfluid should be LGPE rather than KGPE. This thus invalidates the claim made recently by the authors in [13], where a defective evolution scheme for the holographic numerical simulation is invoked. In this regard, our result presents a prime example how a proper phenomenological dissipative model can be selected through the lens of holography.

On the other hand, although the holographic superfluid model is superior to those phenomenological models such as DGPEs in the sense that it offers a first principles description of non-equilibrium dissipative dynamics at finite temperature, not only do DGPEs live in one less dimension, but also involve only one dynamical variable. Thus it is much easier and much more efficient for
one to perform a large scale of numerical simulations using DGPEs once the undetermined parameters are fixed. Now according to our matching result, LGPE is selected by holography to serve as the appropriate phenomenological model for the vortex dynamics, so we can use it to greatly facilitate the quantitative confrontation of our holographic predictions with the real experimental data. In particular, with the recent experimental progress in the vortex dynamics\cite{21}, we expect our results can be verified by the upcoming experiments in the future.

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\[\text{\[1\]} S. Sachdev and B. Keimer, Phys. Today \textbf{64}, 29 (2011)\]
\[\text{\[2\]} H. Liu and J. Sonner, Nat Rev Phys \textbf{2}, 615 (2022)\]
\[\text{\[3\]} J. M. Maldacena, Int. J. Theor. Phys. \textbf{38}, 1113 (1999)\]
\[\text{\[4\]} S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B \textbf{428}, 105 (1998)\]
\[\text{\[5\]} E. Witten, Adv. Theor. Math. Phys. \textbf{2}, 253 (1998)\]
\[\text{\[6\]} S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Phys. Rev. Lett. \textbf{101}, 031601 (2008)\]
\[\text{\[7\]} S. A. Hartnoll, A. Lucas, and S. Sachdev, \textit{Holographic quantum matter} (The MIT Press, Cambridge, MA 2018)\]
\[\text{\[8\]} C. P. Herzog, P. K. Kovtun, and D. T. Son, Phys. Rev. D \textbf{76}, 066002 (2009)\]
\[\text{\[9\]} J. McGreevy, Adv. High Energy Phys. \textbf{2010}, 723105 (2010)\]
\[\text{\[10\]} A. Adams, P. M. Chesler, and H. Liu, Science \textbf{341}, 368 (2013)\]
\[\text{\[11\]} J. Casalderrey-Solana, H. Liu et al, \textit{Gauge/String Duality, Hot QCD and Heavy Ion Collisions} (Cambridge University Press, Cambridge, England, 2014)\]
\[\text{\[12\]} J. Zaanen, Y. W. Sun, Y. Liu et al, \textit{Holographic Duality in Condensed Matter Physics} (Cambridge University Press, Cambridge, England, 2015)\]
\[\text{\[13\]} P. Wittmer, C. M. Schmied, T. Gasenzer, and C. Ewerz, Phys. Rev. Lett. \textbf{127}, 101601 (2020)\]
\[\text{\[14\]} I. Amado, D. Arean, A. Jimenez-Alba, K. Landsteiner, L. Melgar, and I. S. Landea, JHEP \textbf{02}, 63 (2014)\]
\[\text{\[15\]} E. P. Gross, J. Math. Phys. \textbf{4} 195 (1963); L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. \textbf{13}, 451 (1961)\]
\[\text{\[16\]} H. T. C. Stoof, Phys. Rev. Lett. \textbf{78}, 768 (1997)\]
\[\text{\[17\]} H. T. C. Stoof, J. Low Temp Phys. \textbf{114}, 11 (1999)\]
\[\text{\[18\]} R. A. Duine and H. T. C. Stoof, Phys. Rev. A \textbf{65}, 013603 (2001)\]
\[\text{\[19\]} L. D. Landau and E. M. Lifshitz, \textit{Fluid Mechanics} (Pergamon Press, London, 1987)\]
\[\text{\[20\]} N. N. Carlson, Physics D \textbf{98}, 183 (1996)\]
\[\text{\[21\]} Y. P. Sachkou, C. G. Baker, G. I. Harris et al, Science \textbf{366}, 1480 (2019)\]
SUPPLEMENTAL MATERIAL

The viability of the evolution schemes can be examined by checking the degree of violation of Eq. (7) and its convergence during the evolution. As such, we define the error function by

$$E = \partial_t (S - \partial_z A_t),$$

where $S$ is solved numerically according to $\partial_t S = FA_t$. For the convenience of comparison, below we shall denote our evolution scheme as $S_1$ and that in [13] as $S_2$, respectively.

![Graph](image1)

**FIG. 4.** The temporal evolution of the maximal numerical error at the black hole horizon in $S_1$ and $S_2$.

![Graph](image2)

**FIG. 5.** The profile of $E$ at the black hole horizon in $S_1$ on the left and $S_2$ on the right, where the top panels is for the initial stage, the middle for the intermediate stage and the bottom for the annihilation stage. It turns out that the maximal error occurs in the neighbourhood of vortices.

As a demonstration, we first plot the temporal evolution of the maximal error $E_{max}$ at the horizon in Fig. 4 for $\tilde{\mu} = 4.5$, where the number of grid points in the $(x, y, z)$-direction is taken as $128 \times 128 \times 32$. As one can see, although
$E_{\text{max}}$ is appreciable in the beginning due to the aforementioned rough initial data, the black hole horizon offers a natural damping mechanism to make it die away quickly to order of $10^{-4} \sim 10^{-5}$ in $S_1$. In particular, when the vortex dipole annihilate, $E_{\text{max}}$ is further decreased into order of $10^{-7}$, which indicates that the maximal error lies in the location of vortices. Such an observation is further confirmed in Fig.5. While in $S_2$, not only does $E_{\text{max}}$ die down simply to order of $10^{-2} \sim 10^{-3}$, but also stop to increase gradually during the later evolution. Moreover, it experiences a sharper increase right before the vortex dipole annihilation. This suggests that $S_2$ is not as applicable as $S_1$ for one to investigate superfluid dynamics, especially the vortex dynamics under consideration. In support of such a suspicion, we further examine the convergence of the error function by increasing the grid points. By comparing Fig.5 and Fig.6, one can see that as the number of the grid points is increased, the numerical error in $S_1$ decreases dramatically while that in $S_2$ keeps almost unchanged.