ON THE SHAPE CONLEY INDEX THEORY OF SEMIFLOWS
ON COMPLETE METRIC SPACES

JINTAO WANG AND DESHENG LI*
Department of Math., School of Science, Tianjin University
Tianjin 300072, China

JINQIAO DUAN
Department of Applied Math., Illinois Institute of Technology
Chicago IL 60616, USA
and
School of Math. and Statistics, Huazhong University of Science and Technology
Wuhan 430074, China

(Communicated by Jiangong You)

Abstract. In this work we develop the shape Conley index theory for local semiflows on complete metric spaces by using a weaker notion of shape index pairs. This allows us to calculate the shape index of a compact isolated invariant set \( K \) by restricting the system on any closed subset that contains a local unstable manifold of \( K \), and hence significantly increases the flexibility of the calculation of shape indices and Morse equations. In particular, it allows to calculate shape indices and Morse equations for an infinite dimensional system by using only the unstable manifolds of the invariant sets, without requiring the system to be two-sided on the unstable manifolds.

1. Introduction. The Conley index theory is of crucial importance in performing the qualitative analysis of differential equations. It was first developed by C. Conley and his group for flows on locally compact spaces in 1970s [2]. Later on Rybakowski et al. extended the theory to local semiflows on complete metric spaces [14], so that it can be successfully applied to infinite dimensional dynamical systems generated by PDEs.

The basic idea of the Conley index theory is as follows. Let \( K \) be a compact isolated invariant set of a semiflow \( \Phi \) on a complete metric space \( X \). Using appropriate homotopies induced by the flow it can be shown that all the pointed quotient spaces \( (N/E, [E]) \) of index pairs \( (N, E) \) have the same homotopy type. (An index pair is a pair of suitable closed sets \( (N, E) \), where \( N \) is an isolating neighborhood of \( K \), and \( E \) is an exit set of \( N \).) Thus one can define the homotopy Conley index \( h(K) \) of \( K \) to be the homotopy type of the pointed space \( (N/E, [E]) \).

In general invariant sets may have very complicated topological structures. For such dynamical objects shape is another suitable concept to describe their topological quantities. It can be seen as a generalization of homotopy, and was first invented

2010 Mathematics Subject Classification. 37B30, 37D45, 37C75.

Key words and phrases. Complete metric space, local semiflow, shape index pair, shape index, Morse equation.

* Corresponding author. Supported by NSF grants of China 11071185, 11471240.
by Borsuk [1]. As spaces that have the same homotopy type have the same shape, for a compact isolated invariant set \( K \) one can immediately define the shape index \( s(K) \) as

\[
s(K) = \text{Sh}(N/E, [E]),
\]

where \((N, E)\) is a Conley index pair of \( K \), and \( \text{Sh}(\cdot) \) denotes the Borsuk’s shape functor. Theoretically the shape Conley index works equally well as the homotopy one. In the case where \( X \) is a compact smooth manifold and \( \Phi \) is a flow, Robbin and Salamon [13] introduced a certain *intrinsic topology* for the unstable manifold \( W^u(K) \) of \( K \). They proved that the shape index of \( K \) can be directly defined via the unstable manifold \( W^u(K) \) equipped with this intrinsic topology without using index pairs. This brings us several advantages. First, it enables us to define Conley index for discrete dynamical systems. Second, it allows to calculate shape indices and Morse equations of invariant sets by using only the unstable manifolds of the invariant sets and their Morse sets. Robbin and Salamon’s approach to the shape index theory was further developed in the works of Mrozek [12] and Sánchez-Gabites [15] for dynamical systems on locally compact spaces.

Situations in the case of non-locally compact spaces seem to be more complicated. In [8] Kapitanski and Rodnianski studied the shape of attractors of semiflows on complete metric spaces. They proved that the global attractor of a semiflow shares the shape of the phase space. Based on this elegant result a shape Morse theory was developed for attractors, in which the evaluation of the coefficients of Morse equations is made in terms of the unstable manifolds of Morse sets. This work was extended to isolated invariant sets of flows on locally compact metric spaces in a recent paper [16] by Sanjurjo. The author also addressed semiflows on non-locally compact spaces (see [16], Section 6). Let \( \Phi \) be a given semiflow on a complete metric space \( X \), and \( K \) be compact isolated invariant set of \( \Phi \). It was shown that if \( \Phi \) is two-sided on the unstable manifold \( W^u(K) \), then the shape index \( s(K) \) can be successfully calculated via its unstable manifold. Note that we do have many important examples of infinite dimensional systems arising from applications in which the corresponding semiflows are two-sided on the unstable manifolds of invariant sets (see e.g. [7, 18]).

In this present work we are mainly interested in a quite general situation, in which the semiflow may fail to be two-sided on the unstable manifolds. We show that the shape index \( s(K) \) and Morse equations of an isolated invariant set \( K \) can be calculated by using either the Conley index pairs or local unstable manifolds of \( K \) and its Morse sets. In fact, we can even calculate \( s(K) \) by using a suitable closed set containing a local unstable manifold of \( K \). Our strategy is very much like the one as we are in the situation of the classical Conley index theory.

Let \( \Phi \) be an *asymptotically compact* local semiflow on a complete metric space \( X \). First, instead of the usual Conley index pairs we define a new type of shape index pairs for isolated invariant sets. Roughly speaking, a *shape index pair* \((N, E)\) of a compact isolated invariant set \( K \) is a pair of closed sets that enjoys the following properties:

1. \( N \setminus E \) is strongly admissible with \( E \) being an exit set of \( N \);
2. \( K \) is the maximal compact invariant set in \( N \setminus E \); and
3. \( N \setminus E \) contains a local unstable manifold of \( K \).
One can easily verify that each bounded Conley index pair defined as in [14] is a shape index pair defined as above. On the other hand, it is clear that shape index pairs can be constructed by using local unstable manifolds and their appropriate sections. Then using the fundamental results in [8] and [16] etc. we prove that the pointed spaces \((N/E, [E])\) have the same shape for all the shape index pairs \((N, E)\) of a compact isolated invariant set \(K\). This allows us to define the shape Conley index \(s(K)\) to be the shape \(\text{Sh}(N/E, [E])\) of the pointed space \((N/E, [E])\) by using any shape index pair \((N, E)\). Such an approach has an obvious advantage. That is, the calculation of the shape index of \(K\) can be reduced on any closed set of \(X\) that contains a local unstable manifold of \(K\). This significantly increases the flexibility of the calculation of indices and Morse equations.

This paper is organized as follows. In Section 2 we collect some basic notions and results in the theory of topology and dynamical systems, and in Section 3 we verify the continuity property of quotient flows defined on quotient spaces of Ważewski pairs. Section 4 is the central part of this article, in which we introduce the concept of shape index pairs and define shape Conley indices for isolated invariant sets. Section 5 consists of some discussions on Morse equations. Section 6 consists of a simple illustration example.

2. Preliminaries. In this section we collect some basic notions and results in the theory of topology and dynamical systems.

2.1. HEP and homotopy equivalence. Let \(X\) be a topological space.

Given a closed subset \(A\) of \(X\), the pair \((X, A)\) is said to have the homotopy extension property (HEP for short), if for any space \(Y\) and continuous map \(F : X \times \{0\} \cup A \times I \rightarrow Y\), there exists a continuous map \(\tilde{F} : X \times I \rightarrow Y\) such that \(\tilde{F}\) is an extension of \(F\).

Proposition 2.1. [14] \((X, A)\) has the HEP if and only if \(A\) is a strong deformation retract of one of its open neighborhoods.

Let \(A\) and \(B\) be two closed subsets of \(X\). Following Rybakowski [14] (see Chap. 1, Sec. 1.6), we define the quotient space \(B/A\) as follows.

If \(A \neq \emptyset\), then the space \(B/A\) is obtained by collapsing \(A\) to a single point \([A]\) in \(W := A \cup B\). If \(A = \emptyset\), we choose any point \(p \notin B\) and define \(B/A\) to be the space \(B \cup \{p\}\) equipped with the sum topology. In the latter case we still use the notation \([A]\) to denote the base point \(p\).

The result below is a pointed version of the corresponding one in [6], Pro. 0.17.

Proposition 2.2. Let \(e \in A \subset X\). Suppose the pair \((X, A)\) has the HEP, and that \(e\) is a strong deformation retract of \(A\). Then \((X/A, [A]) \simeq (X, e)\).

As a simple consequence, we have

Corollary 2.1. Let \(A\) and \(B\) be two closed subsets of \(X\). Suppose \((X, A)\) has the HEP and that \(B\) is a strong deformation retract of \(A\). Then

\[
(X/A, [A]) \simeq (X/B, [B]).
\]

2.2. Local semiflow. Let \(X\) be a complete metric space with metric \(d(\cdot, \cdot)\). For any \(M, N \subset X\), define

\[
d(M, N) = \inf \{d(x, y) : x \in M, y \in N\}.
\]
Given $\varepsilon > 0$, denote by $B(M, \varepsilon)$ the $\varepsilon$-neighborhood of $M$, 
$$B(M, \varepsilon) = \{ x \in X : \ d(x, M) < \varepsilon \}.$$ 

**Definition 2.1.** A *local semiflow* $\Phi$ on $X$ is a continuous map $\Phi : D(\Phi) \to X$, where $D(\Phi)$ is an open subset of $\mathbb{R}^+ \times X$, and $\Phi$ enjoys the following properties:

1. for each $x \in X$, there exists $0 < T_x \leq \infty$ such that
   $$(t, x) \in D(\Phi) \iff 0 \leq t < T_x;$$
2. $\Phi(0, x) = x$ for all $x \in X$;
3. if $(t + s, x) \in D(\Phi)$, where $t, s \in \mathbb{R}^+$, then $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$.

The number $T_x$ in (1) is called the maximal existence time of $\Phi(t, x)$, and $D(\Phi)$ is called the domain of $\Phi$.

In the case when $D(\Phi) = \mathbb{R}^+ \times X$, we simply call $\Phi$ a *global semiflow*.

Let $\Phi$ be a given local semiflow on $X$. For notational convenience, we will rewrite $\Phi(t, x)$ as $\Phi(t)x$.

A subset $N$ of $X$ is said to be admissible (with respect to $\Phi$), if for any sequences $x_n \in N$ and $t_n \to +\infty$ with $\Phi([0, t_n])x_n \subset N$ for all $n$, the sequence of the end points $\Phi(t_n)x_n$ has a convergent subsequence.

Since $X$ may be an infinite dimensional space, to overcome the difficulty due to the lack of compactness of $X$, we always assume that $\Phi$ is asymptotically compact, that is, each bounded subset $B$ of $X$ is admissible. It is well known that this condition is naturally satisfied by many important examples from applications.

**Proposition 2.3.** [10] Let $x \in X$. Then for any $0 < T < T_x$, there exists a $\delta > 0$ such that $T < T_y$ for $y \in B(x, \delta)$. Furthermore, for any $\varepsilon > 0$, we have

$$d(\Phi(t)y, \Phi(t)x) < \varepsilon, \quad \forall t \in [0, T], \ y \in B(x, \delta),$$

provided $\delta$ is sufficiently small.

A solution (trajectory) on an interval $J \subset \mathbb{R}^1$ is a map $\gamma : J \to X$ satisfying

$$\gamma(t) = \Phi(t-s)\gamma(s), \quad \forall s, t \in J, \ s \leq t.$$ 

A full solution $\gamma$ is a solution defined on the whole line $\mathbb{R}^1$.

2.3. *Attractors.* Let $M$ and $B$ be two subsets of $X$. We say that $M$ attracts $B$, if $T_x = \infty$ for all $x \in B$ and moreover, for any $\varepsilon > 0$ there exists $T > 0$ such that

$$\Phi(t)B \subset B(M, \varepsilon), \quad t > T.$$ 

A compact invariant set $A \subset X$ is said to be an attractor of $\Phi$, if it attracts a neighborhood $U$ of itself.

Let $A$ be an attractor. Set

$$\Omega(A) = \{ x \in X : A \text{ attracts } x \}.$$ 

$\Omega(A)$ is called the attraction basin of $A$. One can easily verify that $\Omega(A)$ is open. Moreover, $A$ attracts each compact subset of $\Omega(A)$. If $\Omega(A) = X$, then we simply call $A$ the global attractor of $\Phi$.

A continuous function $\phi : \Omega := \Omega(A) \to \mathbb{R}^+$ is called a Lyapunov function of $A$, if $\phi(x) \equiv 0$ on $A$, and

$$\phi(\Phi(t)x) < \phi(x), \quad \forall x \in \Omega \setminus A, \ t > 0.$$
Proposition 2.4. \( \mathcal{A} \) has a radially unbounded Lyapunov function \( \phi \) on \( \Omega \) with
\[
\phi(x) \geq d(x, A), \quad \text{for all } x \in \Omega.
\]

Proof. We infer from Li [9] that \( \mathcal{A} \) has a radially unbounded Lyapunov function \( \psi \) on \( \Omega \). Following the procedure in [8] one can also construct a Lyapunov function \( \xi \) of \( \mathcal{A} \) on \( \Omega \) such that \( \phi(x) \geq d(x, A) \) for all \( x \in \Omega \). Now setting \( \phi = \psi + \xi \), we immediately obtain a Lyapunov function of \( \mathcal{A} \) as desired.

Proposition 2.5. Let \( e \in X \). Suppose that the singleton \( \mathcal{A} = \{e\} \) is an attractor. Let \( U \supset A \) be a positively invariant subset of \( \Omega(A) \). Then \( \mathcal{A} \) is a strong deformation retract of \( U \).

Proof. Define \( F : U \times [0, 1] \to U \) as follows: \( F(1, x) = e \) for all \( x \in U \), and
\[
F(t, x) = \Phi(t/(1-t))x, \quad \forall t \in [0, 1), \ x \in U.
\]

Then \( F \) is a strong deformation retraction from \( U \) to \( \mathcal{A} \).

3. Ważewski pairs and quotient flows. From now on we also assume that \( X \) is separable. Therefore all the quotient spaces involved in this work are completely metrizable [10].

Let \( A \) be a subset of \( X \). For each \( x \in A \), denote by \( t_A(x) \) the maximal time \( \tau \) such that \( \Phi(t)x \) stays in \( A \) before \( \tau \),
\[
t_A(x) = \sup\{t \geq 0 : \Phi([0, t])x \subset A\}.
\]

(3.1)

\( A \) is called strongly admissible, if it is admissible, and moreover, \( \Phi \) does not explode in \( A \) (that is, we have \( T_x = \infty \) whenever \( \Phi(0, T_x)x \subset A \)).

Let \( N, E \) be two closed subsets of \( X \). \( E \) is said to be \( N \)-positively invariant, if for all \( x \in E \cap N \) and \( t \geq 0 \), we have \( \Phi([0, t])x \subset E \).

\( E \) is called an exit set of \( N \), if
(1) \( E \) is \( N \)-positively invariant; and
(2) for any \( x \in N \) with \( t_N(x) < T_x \), there exists \( t \leq t_N(x) \) such that \( \Phi(t)x \in E \).

Definition 3.1. The pair \( (N, E) \) is called a Ważewski pair, if
(1) \( E \) is an exit set of \( N \); and
(2) \( N \setminus E \) is strongly admissible.

Let \( (N, E) \) be a Ważewski pair. Consider the quotient space \( N/E \). For notational simplicity, we denote \( [A] = \pi(A) \) for any \( A \subset W := N \cup E \), where \( \pi : W \to N/E \) is the quotient map. (Recall that \( N/E = N \cup \{p\} \) when \( E = \emptyset \), where \( p \notin N \). In this case \( \pi \) is actually the identity map \( \text{id}_N \) on \( N \).) Define a quotient flow \( \Phi \) of \( \Phi \) on \( N/E \) as follows:
If \( \bar{x} = [E] \), then
\[
\bar{\Phi}(t)\bar{x} \equiv \bar{x}
\]
for \( t \in \mathbb{R}^+ \); and if \( \bar{x} = [x] \) for some \( x \in H := N \setminus E \), then

\[
\tilde{\Phi}(t)\bar{x} = \begin{cases} 
\Phi(t)x, & \text{for } t < t_H(x); \\
E, & \text{for } t \geq t_H(x).
\end{cases}
\]

Since \( E \) is \( N \)-positively invariant, it can be easily seen that \( \tilde{\Phi} \) is well defined.

**Lemma 3.2.** \( \tilde{\Phi} \) is continuous on \( \mathbb{R}^+ \times N/E \). Furthermore, \( N/E \) is admissible.

**Proof.** We may assume \( N \cap E \neq \emptyset \); otherwise \( N \) is a positively invariant set, and the proof of the lemma then becomes trivial.

Let us first verify the continuity of \( \tilde{\Phi} \) at any point \((t_0, \bar{x}) \in \mathbb{R}^+ \times N/E \). We split the argument into two cases.

1. “\( \bar{x} = [E] \)”\). In this case we show that for any \( b > 0 \) and open neighborhood \( \tilde{V} \) of \( [E] \) in \( N/E \), there is an open neighborhood \( \tilde{U} \) of \( [E] \) such that

\[
\tilde{\Phi}([0, b])\tilde{U} \subset \tilde{V},
\]

which implies the continuity of \( \tilde{\Phi} \) at \((t_0, \bar{x}) \).

Since \( \pi^{-1}(\tilde{V}) \) is relatively open in \( W = N \cup E \), one can pick an open neighborhood \( V \) of \( E \) in \( X \) such that \( \pi^{-1}(\tilde{V}) = V \cap W \). Consequently \( \tilde{V} = [V \cap W] \). For each \( x \in H \), denote \( b_x = \min\{b, t_H(x)\} \). For convenience, we also assign \( b_x = 0 \) for \( x \notin H \). Because \( \tilde{\Phi} \) does not explode in \( H \), it is easy to see that \( b_x < t_x \leq \infty \) for all \( x \). To prove (3.2), by the definition of \( \tilde{\Phi} \) we only need to show that there exists a neighborhood \( U \) of \( E \) in \( X \) such that

\[
\Phi([0, b_y])y \subset V, \quad \forall y \in U \cap W. \tag{3.3}
\]

Let \( x \in E \). If \( x \notin H \), then \( d(x, H) > 0 \). Take an \( r_x > 0 \) sufficiently small so that \( B(x, r_x) \cap H = \emptyset \). Then \( b_y = 0 \) for \( y \in B(x, r_x) \). Hence

\[
\Phi([0, b_y])y = y \in E \subset V, \quad \forall y \in B(x, r_x) \cap W. \tag{3.4}
\]

Now assume \( x \in E \cap H \). As \( x \in E \) and \( \Phi([0, b_x])x \subset H \subset N \), by the \( N \)-positive invariance of \( E \) we see that \( \Phi([0, b_x])x \subset E \subset V \). Further by continuity of \( \Phi \) one can pick a \( \delta > 0 \) with \( t_\delta := b_x + \delta < T_x \) such that \( \Phi([0, t_\delta])x \subset V \). It then follows by Pro. 2.3 that there exists \( r_x > 0 \) such that \( T_y > t_\delta \) for all \( y \in B(x, r_x) \). Moreover,

\[
\Phi([0, t_\delta])B(x, r_x) \subset V. \tag{3.5}
\]

If \( b_x = b \), then since \( b_y \leq b \) for all \( y \), by (3.5) we have

\[
\Phi([0, b_y])y \subset \Phi([0, b])y \subset \Phi([0, t_\delta])y \subset V, \quad \forall y \in B(x, r_x). \tag{3.6}
\]

On the other hand, if \( b_x < b \), we infer from the definition of \( b_x \) that \( b_x = t_H(x) \). Hence \( t_H(x) < b < t_\delta \). Therefore there exists \( t_H(x) < \tau \leq t_\delta \) such that \( \Phi(\tau)y \notin H \). We may assume that \( r_x \) is sufficiently small so that

\[
\Phi(\tau)B(x, r_x) \cap H = \emptyset.
\]

Then \( t_H(y) < \tau \leq t_\delta \) for each \( y \in B(x, r_x) \). Thus \( b_y = \min\{b, t_H(y)\} \leq t_\delta \). Again by (3.5), we have

\[
\Phi([0, b_y])y \subset \Phi([0, t_\delta])y \subset V, \quad y \in B(x, r_x). \tag{3.7}
\]

Set \( U = \bigcup_{x \in E} B(x, r_x) \). Then by (3.4), (3.6) and (3.7) we see that \( U \) is a neighborhood of \( E \) satisfying (3.3).
(2) “$\tilde{x} \neq [E]$”. In this case there exists a unique $x \in N \setminus E$ such that $\tilde{x} = [x]$. If $\Phi(t_0)\tilde{x} \neq [E]$, we infer from the definition of $\Phi$ that $\Phi(t)x \in N \setminus E$ for all $t \in [0, t_0]$. Since $E$ is closed, one can find a $\delta > 0$ sufficiently small such that $\Phi([0, t_0 + \delta])x \cap E = \emptyset$. Further by Pro. 2.3 there exists $r > 0$ such that $\Phi([0, t_0 + \delta])B(x, r) \cap E = \emptyset$.

Thus by the definition of $\Phi$ we have $\tilde{\Phi}(t)y = [\Phi(t)y]$ for all $y \in B(x, r)$ and $t \in [0, t_0 + \delta]$, and the continuity of $\Phi$ at $(t_0, \tilde{x})$ follows immediately from that of $\Phi$.

Now suppose that $\tilde{\Phi}(t_0)\tilde{x} = [E]$. Let $\tilde{V}$ be an open neighborhood of $[E]$ in $N/E$. We show that there exists a neighborhood $\tilde{U}$ of $\tilde{x}$ and a neighborhood $I$ of $t_0$ such that

$$\tilde{\Phi}(I)\tilde{U} \subset \tilde{V},$$

which verifies the continuity of $\Phi$ at $(t_0, \tilde{x})$.

Fix a number $b > t_0$. Then by (3.2) there exists an open neighborhood $\tilde{V}_0$ of $[E]$ such that

$$\tilde{\Phi}([0, b])\tilde{V}_0 \subset \tilde{V}.$$ (3.9)

Take an open neighborhood $V_0$ of $E$ in $X$ such that $\tilde{V}_0 = [V_0 \cap W]$. As $\Phi(t_{N\setminus E}(x))x \in E$, one can find a number $s < t_{N\setminus E}(x)$ such that $\Phi(s)x \in V_0 \setminus E$. Note that $t_{N\setminus E}(x) \leq t_0$ (recall $\tilde{\Phi}(t_0)\tilde{x} = [E]$). Thus $I = [s, b]$ is a neighborhood of $t_0$.

Pick an $\varepsilon > 0$ sufficiently small so that $\Phi(s)B(x, \varepsilon) \subset V_0 \setminus E$. Then for each $y \in B(x, \varepsilon) \cap W := U$, by the $N$-positive invariance of $E$ we necessarily have $\Phi([0, s])y \subset N \setminus E$. Consequently one concludes that

$$t_H(y) \geq t_{N\setminus E}(y) \geq s, \quad y \in U.$$ Therefore

$$\tilde{\Phi}(s)[y] = [\Phi(s)y] \in \tilde{V}_0, \quad \forall y \in U.$$ (3.10)

Combining this with (3.9) we deduce that

$$\tilde{\Phi}([s, b])\tilde{U} \subset \tilde{V},$$

where $\tilde{U} = [U]$. This proves what we desired in (3.8).

Now let us examine the admissibility of $N/E$ for $\tilde{\Phi}$. It suffice to show that, for any sequences $\tilde{x}_n \in N/E$ and $t_n \to \infty$, the sequence $\tilde{\Phi}(t_n)\tilde{x}_n$ has a convergent subsequence. We may assume $\Phi(t_n)\tilde{x}_n \neq [E]$ for all $n$; otherwise we are done. Then for each $n$ there exists $x_n \in N \setminus E$ such that $\tilde{x}_n = [x_n]$. Note that $\Phi([0, t_n])x_n \subset N \setminus E \subset H$. Because $H$ is admissible (with respect to $\Phi$), $\Phi(t_n)x_n$ has a convergent subsequence $\Phi(t_{n_k})\tilde{x}_{n_k}$. Consequently $\tilde{\Phi}(t_{n_k})\tilde{x}_{n_k}$ converges in $N/E$. □

In the following argument we denote by $I(A)$ the maximal invariant set in $A$ for any $A \subset X$. In general $I(A)$ may not be compact. However, if we assume $A$ is closed and admissible, then one can easily verify that $I(A)$ is compact.

Lemma 3.3. Suppose that $I(H) \cap E = \emptyset$. Then the equilibrium $[E]$ is an attractor of $\tilde{\Phi}$ in $N/E$.

Proof. The proof is a slight modification of that for the same conclusion in [10], Lemma 3.7. We omit the details. □

4. Shape index. In this section we introduce the notion of shape index pairs and define shape Conley indices for isolated invariant sets.
4.1. **Shape.** The exposition of the basic notions and results on shape theory given here is adapted from [8, 16]. For details we refer the interested reader to [1, 3] and [11], etc.

We call a topological space $P$ an absolute neighborhood retract (ANR for short) provided, for any embedding $i : P \to P_0$ of $P$ into a (metrizable) space $P_0$, there exists a neighborhood $U$ of $i(P)$ in $P_0$ such that $i(P)$ is a retract of the neighborhood $U$. It is known that every metric space can be embedded into an ANR as a closed subspace.

Let $X$ and $Y$ be two metric spaces. Suppose that $X$ and $Y$ are subsets of ANRs $P$ and $Q$, respectively. Denote by $U(X, P)$ (resp. $U(Y, Q)$) the set of all open neighborhoods of $X$ (resp. $Y$) in $P$ (resp. $Q$). Let $f = \{ f : U \to V \}$ be a collection of continuous maps from $U \in U(X, P)$ to $V \in U(Y, Q)$. We call $f$ a mutation from $U(X, P)$ to $U(Y, Q)$, if the following conditions are fulfilled:

(i) For every $V \in U(Y, Q)$ there exists a map $f \in f$ such that $f(U) \subset V$ for some $U \in U(X, P)$.

(ii) If $f : U \to V$ is in the collection $f$, then the restriction $f|_{U_1} : U_1 \to V_1$ is also in $f$ for any two neighborhoods $U_1 \subset U$ and $V_1 \supset V$.

(iii) If there are two maps $f_1, f_2 : U \to V$ in $f$, then there exists a neighborhood $U_1 \subset U$ such that the restrictions of $f_1$ and $f_2$ to $U_1$ are homotopy equivalent.

An example of a mutation is the trivial mutation $\text{id}_{U(X, P)}$ which is comprised of the identity maps $\text{id} : U \to U$.

**Definition 4.1.** A mutation $f : U(X, P) \to U(Y, Q)$ is said to be a shape equivalence between $X$ and $Y$, if there exists a mutation $g : U(Y, Q) \to U(X, P)$ such that the compositions $g \circ f : U(X, P) \to U(X, P)$ and $f \circ g : U(Y, Q) \to U(Y, Q)$ are homotopic to the trivial mutations $\text{id}_{U(X, P)}$ and $\text{id}_{U(Y, Q)}$, respectively.

We say that $X$ and $Y$ are shape equivalent, if there exists a shape equivalence $f$ between them.

**Remark 4.1.** That the composition $g \circ f$ is homotopic to the trivial mutation $\text{id}_{U(X, P)}$ (notated as $g \circ f \simeq \text{id}_{U(X, P)}$) means that, whenever $f : U \to V \in f$ and $g : V \to U' \in g$, there exists $U'' \subset U \cap U'$ so that $g \circ f|_{U''} : U'' \to U'$ is homotopic to the identity map.

**Remark 4.2.** One may think shape as a generalization of homotopy, which can be seen from that spaces belonging to the same homotopy type have the same shape.

For pointed spaces $(X, x)$ and $(Y, y)$, we define a (pointed) mutation

$$f : U(X, x; P) := U(X, P) \to U(Y, y; Q) := U(Y, Q)$$

as a mutation from $U(X, P)$ to $U(Y, Q)$ such that $f(x) = y$ for each $f \in f$.

In a similar manner as in Def. 4.1, one can define shape equivalence for pointed spaces. We omit the details.

The following result is a pointed version of one of the main results in [8], and can be found in [17]. See also [4, 5] for relevant results.

**Theorem 4.2.** Let $Y$ be a complete metric space, and $G$ be a global semiflow on $Y$. Suppose that $G$ has a global attractor $A$, and that the system has an equilibrium $e \in A$. Then the inclusion $i : (A, e) \to (X, e)$ induces a shape equivalence.
4.2. **Shape index.** For any \( M \subset X \), we denote by \( \omega(M) \) the \( \omega \)-limit set of \( M \),
\[
\omega(M) = \{ y : \text{there exist } x_n \in M \text{ and } t_n \to \infty \text{ such that } \Phi(t_n)x_n \to y \}.
\]
For a solution \( \gamma \) on \((a, \infty)\) (resp. \((-\infty, a)\)), one can also define its \( \omega \)-limit set \( \omega(\gamma) \) (resp. \( \alpha \)-limit set \( \alpha(\gamma) \)) in a similar manner. We omit the details.

Let \( N \) be a subset of \( X \), and \( K \subset N \) be a compact invariant set. Define the *local unstable manifold* of \( K \) in \( N \)
\[
W^u_N(K) = \{ x : \text{there is a solution } \gamma \text{ on } (-\infty, 0] \text{ with } \gamma(0) = x \text{ and } \gamma((-\infty, 0]) \subset N \text{ such that } \alpha(\gamma) \subset K \}.
\]
If \( N = X \) then we simply write \( W^u_N(K) \) as \( W^u(K) \). \( W^u(K) \) is called the *unstable manifold* of \( K \).

**Definition 4.3.** Let \( K \subset X \) be a compact isolated invariant set of \( \Phi \). A Ważewski pair \((N, E)\) is said to be a *shape index pair* of \( K \), if
1. there is a closed neighborhood \( U \) of \( K \) such that \( W^u_U(K) \subset H \); and
2. \( K = I(H) \), where \( H = N \setminus E \).

**Remark 4.3.** We easily check that a Conley index pair in the terminology of Rybakowski [14] is a shape index pair defined as above.

We are now in position to define the shape Conley index of \( K \) via shape index pairs introduced here.

**Definition 4.4.** Let \((N, E)\) be a shape index pair of \( K \). Then the *shape index* \( s(\Phi, K) \) of \( K \) is defined as
\[
s(\Phi, K) = \text{Sh}(N/E, [E]).
\]
As the flow \( \Phi \) is clear, in what follows we simply rewrite \( s(\Phi, K) \) as \( s(K) \). The result below indicates that shape indices of isolated invariant sets are well defined.

**Theorem 4.5.** The shape index \( s(K) \) is independent of the choice of index pairs.

Before proving this result, let us first give some auxiliary results. Let \((N, E)\) be a shape index pair of \( K \), and \( \bar{\Phi} \) be the quotient flow on \( N/E \) defined as in Section 3.

**Lemma 4.6.** \( \bar{\Phi} \) has a global attractor \( A \) in \( N/E \). Moreover,
\[
A = W^u([K]) \cup \{[E]\}. \tag{4.11}
\]
**Proof.** We infer from Lemma 3.2 that \( N/E \) is admissible for the quotient flow \( \bar{\Phi} \). Hence one can easily verify that \( A = \omega(N/E) \) is compact and is precisely the global attractor of \( \bar{\Phi} \). We show that (4.11) holds, thus completing the proof of the lemma.

We infer from the admissibility of \( N/E \) that the maximal invariant set \( I(N/E) \) in \( N/E \) is necessarily compact. Since the global attractor of a system, if exists, is necessarily the maximal compact invariant set of the system, we deduce that \( A = I(N/E) \). Hence to prove (4.11) it suffices to check that
\[
I(N/E) = W^u([K]) \cup \{[E]\} := \mathcal{M}. \tag{4.12}
\]
It is clear that \( \mathcal{M} \subset I(N/E) \). Therefore to prove (4.12) there remains to verify the converse inclusion \( I(N/E) \subset \mathcal{M} \).

Let \( \bar{x} \in I(N/E) \). Then there is a full solution \( \bar{\gamma} \) of \( \bar{\Phi} \) with \( \bar{\gamma}(0) = \bar{x} \). If \( \bar{x} = [E] \), then clearly \( \bar{x} \in \mathcal{M} \), and hence we are done. So we assume \( \bar{x} \neq [E] \). In such a
case one necessarily has $\tilde{\gamma}(t) \neq [E]$ for all $t \in (-\infty, 0]$. By the definition of the quotient flow there exists a solution $\gamma$ of $\Phi$ on $(-\infty, 0]$ contained in $N \setminus E$ such that $\tilde{\gamma}(t) = [\gamma(t)]$ for all $t \in (-\infty, 0]$. As $H := N \setminus E$ is admissible under the flow $\Phi$, it is easy to deduce that $\alpha(\gamma)$ is a nonempty compact invariant set of $\Phi$. Since $K$ is the maximal compact invariant set of $\Phi$ in $H$, we find that $\alpha(\gamma) \subset K$. Consequently $\gamma(t) \in W^u(K)$ for all $t \leq 0$. This implies that $\tilde{\gamma}(t) \in W^u([K])$ for $t \leq 0$. In particular, 
\[ \tilde{x} = \tilde{\gamma}(0) \in W^u([K]) \subset M. \]
Hence $I(N/E) \subset M$. 
\[ \square \]
Lemma 4.7. Let $\mathcal{A}$ be the global attractor of $\tilde{\Phi}$ given in Lemma 4.6. Then 
\[ (\mathcal{A}, [E]) \cong (W^u_N(K)/E, [E]). \] (4.13)
Proof. We first show that 
\[ \mathcal{A} = [W^u_N(K)] \cup \{ [E] \}. \] (4.14)
Let $\tilde{x} \in \mathcal{A}$. We may assume $\tilde{x} \neq [E]$. Then there is a full solution $\tilde{\gamma}$ for $\tilde{\Phi}$ such that $\tilde{\gamma}(0) = \tilde{x}$. Note that $\tilde{\gamma}(t) \neq [E]$ for all $t \in (-\infty, 0]$. Define $\gamma(t) = \gamma^{-1}(\tilde{\gamma}(t))$ for $t \in (-\infty, 0])$. We infer from the definition of $\tilde{\Phi}$ that $\gamma$ is a solution of $\Phi$ on $(-\infty, 0]$. Clearly $\gamma(t) \in N \setminus E$ for all $t \leq 0$. Hence $\gamma$ is contained in $W^u_N(K)$. Consequently 
\[ \tilde{x} = [\gamma(0)] \in [W^u_N(K)]. \]
Thus we see that $\mathcal{A} \subset [W^u_N(K)] \cup \{ [E] \}$.
Now suppose $\tilde{x} \in [W^u_N(K)] \cup \{ [E] \}$. If $\tilde{x} = [E]$, it is clear that $\tilde{x} \in \mathcal{A}$. So we assume that $\tilde{x} \neq [E]$. Then there exists $x \in W^u_N(K) \setminus E$ such that $\tilde{x} = [x]$. Let $\gamma$ be a solution of $\Phi$ on $(-\infty, 0]$ contained in $W^u_N(K)$ such that $\gamma(0) = x$. Then $\tilde{\gamma} = \gamma \circ \gamma$ is a solution of $\Phi$ on $(-\infty, 0]$ with $\alpha(\tilde{\gamma}) = \alpha(\gamma) \subset [K]$. Hence $\tilde{x} = \tilde{\gamma}(0) \in W^u([K])$. By Lemma 4.6 one deduces that $\tilde{x} \in \mathcal{A}$. Therefore we have $[W^u_N(K)] \cup \{ [E] \} \subset \mathcal{A}$, which completes the proof of (4.14).
It is easy to see that 
\[ ([W^u_N(K)] \cup \{ [E] \}, [E]) \cong (W^u_N(K)/E, [E]). \]
Thus by (4.14) one immediately concludes the validity of (4.13). 
\[ \square \]
Lemma 4.8. If $W^u_N(K) \cap E = \emptyset$, then $W^u(K) = K$.
Proof. It can be assumed that $K \neq \emptyset$; otherwise we have $W^u(K) = K = \emptyset$, which completes the proof of the lemma.
We only need to verify that $W^u(K) \subset K$. Let $x \in W^u(K)$. Then there exists a solution $\gamma$ on $(-\infty, 0]$ with $\gamma(0) = x$ and $\gamma((-\infty, 0]) \subset W^u(K)$. We extend $\gamma$ to a solution on $(-\infty, T_x)$ (still denoted by $\gamma$), where $T_x$ is the maximal existence time of $\Phi(t)x$. We claim that 
\[ \gamma((-\infty, T_x)) \subset H, \] (4.15)
where $H = \overline{N \setminus E}$. Indeed, if this was not the case, then one should have $t_H(x) < T_x$. Now it is easy to see that 
\[ \gamma(t_H(x)) = \Phi(t_H(x))x \in E \cap N. \]
However, this contradicts the assumption that $W^u_N(K) \cap E = \emptyset$ (as $\gamma((-\infty, t_H(x))] \subset W^u_N(K)$) and proves our claim.
Because $\Phi$ does not explode in $H$, by (4.15) we deduce that $T_x = \infty$. It also follows by the admissibility of $H$ that $\gamma$ is bounded on $\mathbb{R}^3$. Thus by the maximality
of $K$ in $H$ one concludes that $\gamma(\mathbb{R}^1) \subset K$. In particular, $x = \gamma(0) \in K$. Hence we see that $W^u(K) \subset K$. \hfill \square

**Lemma 4.9.** For any open neighborhood $U$ of $K$, there is a closed neighborhood $F$ of $E$ in $W = N \cup E$ with $K \cap F = \emptyset$ such that

1. $(N, F)$ is a shape index pair and has the HEP;
2. $W^u_N(K) \setminus F \subset U$; and
3. $(N/E, [E]) \simeq (N/F, [F])$.

**Proof.** If $N \cap E = \emptyset$, then $N$ is positively invariant, and $K$ is an attractor of $\Phi$ restricted on $N$. Hence $W^u_N(K) = K$. In such a case $F = E$ fulfills all the requirements of the lemma.

Assume that $N \cap E \neq \emptyset$. Let $\rho$ be a metric on $N/E$ such that $N/E$ is a complete metric space. By Lemma 3.3, $[E]$ is an attractor of $\tilde{\Phi}$. We also infer from (4.14) that $[W^u_N(K)] = A$, where $A$ is the global attractor of $\tilde{\Phi}$ in $N/E$. Noticing that $\tilde{U} = [U \cap W]$ is an open neighborhood of $[K]$ in $N/E$, one deduces that $M := [W^u_N(K)] \setminus \tilde{U}$ is a compact subset of $N/E$. It is easy to see that $[E]$ attracts each point in $M$. Hence $M \subset \Omega = \Omega([E])$.

Let $\phi$ be a radially unbounded Lyapunov function of $[E]$ on $\Omega$ given by Pro. 2.4. For any $c \in \mathbb{R}^1$, denote by $\phi^c$ the level set of $\phi$ defined by

$$\phi^c = \{ x \in \Omega : \phi(x) \leq c \}.$$ 

Pick a positive number $a > \max_{\bar{x} \in M} \phi(\bar{x})$ and define $F = \pi^{-1}(\phi^a)$, where $\pi$ is the quotient map from $W$ to $N/E$. Then $F$ is a closed neighborhood of $E$ in $W$; see Fig. 1. We show that $F$ satisfies (2). Indeed, we infer from $M \subset \phi^a$ that

$$[W^u_N(K)] \setminus \phi^a \subset [W^u_N(K)] \setminus M.$$ 

Thereby

$$W^u_N(K) \setminus F = (W^u_N(K) \cup E) \setminus F$$

$$= \pi^{-1}([W^u_N(K)] \setminus \pi^{-1}(\phi^a)) = \pi^{-1}([W^u_N(K)] \setminus \phi^a)$$

$$\subset \pi^{-1}([W^u_N(K)] \setminus M) = \pi^{-1}([W^u_N(K)] \cap \tilde{U}) \subset U.$$
We need to prove that (4.17) holds for $W$. We observe that $W(K) \subset \mathbb{N} \setminus F$, where $V = U \setminus F$. Therefore $(N, E)$ is a shape index pair of $K$.

To complete the proof of the lemma, there remains to check the HEP of $(N, F)$ and the third conclusion (3).

For this purpose, we fix a number $b > a$ and set $F' = \pi^{-1}(\text{int } \phi^b)$. Then $F'$ is an open neighborhood of $F$ in $W$. We claim that $F$ is a strong deformation retract of $F'$. Indeed, define $T : \phi^b \to \mathbb{R}^+$ as follows:

$$T(\bar{x}) := \left\{ \begin{array}{ll}
sup \{ t \geq 0 : \bar{\Phi}(0, t)\bar{x} \subset \phi^b \setminus \phi^a \} & , \bar{x} \in \phi^b \setminus \phi^a; \\
0, & , \bar{x} \in \phi^a.
\end{array} \right.$$ 

By very standard argument (see e.g. Rybakowski [14]) it can be shown that $T$ is continuous. Set

$$h(x, s) = \Phi(s T(\bar{x})) x, \quad \forall x \in F', \ s \in [0, 1],$$

where $\bar{x} = [x]$. Then $h$ is a strong deformation retraction from $F'$ to $F$, hence the claim holds true. Now it follows by Theorem 2.1 that $(N, F)$ has the HEP.

Since $\pi(F) \subset \Omega$ is positively invariant under the quotient flow $\bar{\Phi}$, by Pro. 2.5 $\pi(F)$ is a strong deformation retract of $\pi(F)$. Further by Corollary 2.1 we deduce that

$$(N/F, [F]) \cong ((N/E)/\pi(F), [\pi(F)]) \simeq (N/E, [E]),$$

which completes the proof of (3).

\[\square\]

**Proof of Theorem 4.5.** Let $(N_1, E_1)$ and $(N_2, E_2)$ be two shape index pairs of $K$. We need to prove that

$$\text{Sh}(N_1/E_1, [E_1]) = \text{Sh}(N_2/E_2, [E_2]).$$

Let $N = N_1 \cap N_2$, and $E = E_1 \cup E_2$. One can easily verify that $(N, E)$ is a shape index pair of $K$. We show that

$$\text{Sh}(N_k/E_k, [E_k]) = \text{Sh}(N/E, [E]), \quad k = 1, 2,$$

hence (4.16) holds true.

If $W^s(K) = K$, then $W_{\mathcal{A}}^s(K) = K$. Denote by $\bar{\Phi}_k$ the quotient flow of $\Phi$ on $N_k/E_k$. Let $\mathcal{A}_k$ be the global attractor of $\bar{\Phi}_k$ on $N_k/E_k$. Lemma 4.6 asserts that $\mathcal{A}_k = \pi_k(K) \cup \{ [E_k] \}$, where $\pi_k(\cdot)$ denotes the quotient map from $W_k = N_k \cup E_k$ to $N_k/E_k$. By Theorem 4.2 we have

$$\text{Sh}(N_k/E_k, [E_k]) = \text{Sh}(\mathcal{A}_k, [E_k])$$

$$= \text{Sh}(\pi_k(K) \cup \{ [E_k] \}, [E_k]) = \text{Sh}(K \cup \{ p \}, p),$$

where $p \notin K$, from which (4.17) immediately follows.

Henceforth we assume that $W^s(K) \neq K$. Then by Lemma 4.8 we deduce that $W_{\mathcal{A}}^s(K) \cap E' \neq \emptyset$ for all index pairs $(N', E')$ of $K$. In what follows we show that (4.17) holds for $k = 1$.

Set $E' = W_{\mathcal{A}}^s(K) \cap E$, and define

$$\Sigma_1 = \{ y \in N_1 : \text{ there exist } x \in E' \text{ and } t \geq 0 \text{ such that } \Phi([0, t])x \subset N_1, \Phi(t)x = y \};$$

(4.18)
see Figure 2. It is easy to check that $E^u$ and $\Sigma_1$ are $N$-positively invariant and $N_1$-positively invariant, respectively. Moreover,

$$W^u_N(K) \cup \Sigma_1 = W^u_{N_1}(K).$$  \hspace{1cm} (4.19)

We have

**Lemma 4.10.** There exists a neighborhood $U$ of $K$ such that $\Sigma_1 \cap U = \emptyset$.

**Proof.** We argue by contradiction and suppose the contrary. Then for each $n$ one could find an $x_n \in E^u$ and $t_n \geq 0$ with $\Phi([0,t_n])x_n \subset N_1$ such that

$$y_n = \Phi(t_n)x_n \in B(K,1/n).$$  \hspace{1cm} (4.20)

Since $K$ is compact, one can find a $\delta > 0$ such that $B(K,\delta) \cap E = \emptyset$. Therefore it can be assumed that $y_n \not\in E$ for all $n$. By the definition of $E^u$, for each $n$ there is a solution $\gamma_n : (-\infty,0] \to X$ contained in $W^u_N(K)$ such that $\gamma_n(0) = x_n$. Note that $\gamma_n$ can be extended to a solution on $(-\infty,t_n]$ (still denoted by $\gamma_n$) by simply setting $\gamma_n(t) = \Phi(t)x_n$. We claim that $\gamma_n((-\infty,t_n]) \cap E_1 = \emptyset$ for all $n$. Indeed, if $\gamma_n(t) \in E_1$ for some some $t \leq t_n$, then by the $N_1$-invariance of $E_1$ one should have $\gamma_n([t,t_n]) \subset E_1 \subset E$. In particular, $y_n = \gamma_n(t_n) \in E$, which leads to a contradiction and proves our claim.

Because $N_1 \setminus E_1 := H_1$ is strongly admissible and $\gamma_n((-\infty,t_n]) \subset H_1$, we can assume that $x_n \to x_0$ and $y_n \to y_0$. Then $x_0 \in E^u = W^u_N(K) \cap E$, and $y_0 \in K$. Let $\gamma^- : (-\infty,0] \to X$ be a solution contained in $W^u_N(K)$ such that $\gamma^-(0) = x_0$. Now two cases may occur.

(1) $\{t_n\}$ is bounded. In this case we may assume $t_n \to \tau$. Passing to the limit in $\gamma_n$ one obtains a solution $\gamma^+$ on $[0,\tau]$ with $\gamma^+([0,\tau]) \subset H_1$ such that

$$\gamma^+(0) = x_0, \quad \gamma^+(\tau) = y_0.$$

Since $y_0 \in K$, we can extend $\gamma^+$ to a solution on $[0,\infty)$ in $H_1$ by setting $\gamma^+(t) = \Phi(t-\tau)y_0$ for $t > \tau$. Now define

$$\gamma(t) = \begin{cases} 
\gamma^-(t), & t \leq 0; \\
\gamma^+(t), & t \geq 0.
\end{cases} \hspace{1cm} (4.21)$$

Then $\gamma$ is a full solution in $H_1$. As $K$ is the maximal compact invariant set in $H_1$, we necessarily have $x_0 = \gamma(0) \in K$. However, this contradicts the fact that $x_0 \in E$.

(2) $\{t_n\}$ is unbounded. In this case by passing to the limit in $\gamma_n$ one can directly obtain a solution $\gamma^+$ on $[0,\infty)$ in $H_1$ with $\gamma^+(0) = x_0$. Define a solution $\gamma$ as in (4.21). Then $\gamma$ is a full solution in $H_1$, and hence $x_0 = \gamma(0) \in K$. This again leads to a contradiction.

Now let us proceed to prove Theorem 4.5. Let $U$ be the neighborhood of $K$ given by Lemma 4.10. We may assume $U$ is open. Then by Lemma 4.9 there is a closed neighborhood $F$ of $E$ in $W = N \cup E$ with $K \cap F = \emptyset$ such that $(N,F)$ is a shape index pair and has the HEP; see Figure 2. Moreover,

$$W^u_N(K) \setminus F \subset U, \quad (N/E,[E]) \simeq (N/F,[F]).$$  \hspace{1cm} (4.22)

It follows by the second relation in (4.22) that

$$\text{Sh}(N/E,[E]) = \text{Sh}(N/F,[F]).$$
On the other hand, by Theorem 4.2 one deduces that $\text{Sh}(N/F, [F]) = \text{Sh}(A', [F])$, where $A'$ is the global attractor of the quotient flow on $N/F$. Therefore by Lemma 4.7 we have

$$\text{Sh}(N/E, [E]) = \text{Sh}(A', [F]) = \text{Sh}(W^u_N(K)/F, [F]) = \text{Sh}(W^u_N(K)/F, [F^u]),$$

(4.23)

where $F^u = W^u_N(K) \cap F$.

Let $\Sigma = F^u \cup \Sigma_1$ (the grey-colored part in Fig. 3), where $\Sigma_1$ is defined as in (4.18). We claim that

$$W^u_N(K) \cap \Sigma = F^u.$$  (4.24)

Indeed, if $y \in W^u_N(K) \cap \Sigma_1$ then since $\Sigma_1 \cap U = \emptyset$ and $W^u_N(K) \setminus F \subset U$, we necessarily have $y \in F$. Hence $y \in F^u$. Thus $W^u_N(K) \cap \Sigma_1 \subset F^u$. Consequently

$$W^u_N(K) \cap \Sigma = F^u \cup (W^u_N(K) \cap \Sigma_1) = F^u,$$

which completes the proof of (4.24). Now we have

$$(W^u_N(K)/F^u, [F^u]) = (W^u_N(K)/(W^u_N(K) \cap \Sigma), [W^u_N(K) \cap \Sigma]) \cong (W^u_N(K)/\Sigma, [\Sigma]) \cong ((W^u_N(K) \cup \Sigma)/\Sigma, [\Sigma]) \cong (W^u_{N_1}(K)/\Sigma, [\Sigma]).$$

Therefore by (4.23) it holds that

$$\text{Sh}(N/E, [E]) = \text{Sh}(W^u_{N_1}(K)/\Sigma, [\Sigma]).$$  (4.25)

Consider the quotient space $X_1 := W^u_{N_1}(K)/E^u_1$ along with the quotient flow $\tilde{\Phi}_1$ on $X_1$, where $E^u_1 = W^u_{N_1}(K) \cap E_1$. The quotient flow lemma in Section 3 asserts that $[E^u_1]$ is an attractor of $\tilde{\Phi}_1$. Denote $q$ the quotient map from $W^u_{N_1}(K)$ to $X_1$. Then

$$(W^u_{N_1}(K)/\Sigma, [\Sigma]) \cong (X_1/q(\Sigma), [q(\Sigma)]).$$  (4.26)

see Figure 4.

Since $q(\Sigma)$ is positively invariant and contained in the attraction basin of $[E^u_1]$ and $[E^u_1] \in q(\Sigma)$, by Pro. 2.5 we know that $[E^u_1]$ is a strong deformation retract.
of \( q(\Sigma) \). Because \((N, F)\) has the HEP, it is easy to see that \((W^u_N(K), F^u)\) has the HEP as well. Consequently we know that \((W^u_{N_1}(K), \Sigma)\) has the HEP. This implies that \((A_1, q(\Sigma))\) has the HEP. Therefore by Corollary 2.1 we have
\[
\left( X_1 / q(\Sigma), [q(\Sigma)] \right) \simeq \left( X_1, [E_1^u] \right) \cong \left( W^u_{N_1}(K) / E_1, [E_1] \right).
\]
(The last relation “\( \cong \)” in the above equation is due to that \( E_1^u = W^u_{N_1}(K) \cap E_1 \).) Combining this with (4.25) and (4.26), one finds that
\[
\text{Sh}(N/E, [E]) = \text{Sh} (W^u_{N_1}(K) / E_1, [E_1])
= (\text{by Lemma } 4.7) = \text{Sh} (A_1, [E_1]),
\]
where \( A_1 \) is the global attractor of the quotient flow on \( N_1 / E_1 \). Further by Theorem 4.2 we conclude that
\[
\text{Sh}(N/E, [E]) = \text{Sh} (A_1, [E_1]) = \text{Sh} (N_1 / E_1, [E_1]).
\]
Likewise, it can be shown that (4.17) holds for \( k = 2 \).

\[\square\]

Remark 4.4. Since Conley index pairs are naturally shape index pairs, one immediately concludes that shape index in the terminology here possesses continuation (homotopy) property. We omit the details.

5. Morse equations. In this section we pay some attention to Morse equations. Our results extend the corresponding ones in [8] from attractors to isolated invariant sets.

5.1. Morse decompositions of invariant sets. For the reader’s convenience, we recall briefly the definition of Morse decompositions of invariant sets.

Let \( K \) be a compact invariant set. Then the restriction \( \Phi|_K \) of \( \Phi \) on \( K \) is a semiflow on \( K \). A set \( A \subseteq K \) is called an attractor of \( \Phi \) in \( K \), if it is an attractor of \( \Phi|_K \).

Let \( A \) be an attractor of \( \Phi \) in \( K \). Set
\[
A^* = \{ x \in K : \omega(x) \cap A = \emptyset \}.
\]
\( A^* \) is called the repeller dual to \( A \) relative to \( K \). Accordingly, \( (A, A^*) \) is called an attractor-repeller pair in \( K \).

Definition 5.1. An ordered collection \( \mathcal{M} = \{ M_1, \cdots, M_n \} \) of subsets \( M_k \subseteq K \) is called a Morse decomposition of \( K \), if there exists an increasing sequence \( 0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = K \) of attractors in \( K \) such that
\[
M_k = A_k \cap A_{k-1}^*, \quad 1 \leq k \leq n.
\]
The attractor sequence of \( A_k \) \((k = 0, 1, \cdots, n)\) is often called the Morse filtration of \( K \), and each \( M_k \) is called a Morse set of \( K \).

Remark 5.5. It is well known that if \( K \) is isolated, then so are the Morse sets \( M_k \).

Let \( \mathcal{A} \) be an attractor with a Morse decomposition \( \mathcal{M} = \{ M_1, \cdots, M_n \} \). A continuous function \( \phi \) on the attraction basin \( \Omega = \Omega(\mathcal{A}) \) is called a Morse-Lyapunov function (M-L function for short) of \( \mathcal{M} \) on \( \Omega \), if
\begin{enumerate}
\item \( \phi \) is constant on each Morse set \( M_k \); and
\item \( \phi \) is strictly decreasing along each solution in \( \Omega \setminus (\cup_{1 \leq k \leq n} M_k) \).
\end{enumerate}

An M-L function \( \phi \) of \( \mathcal{M} \) is said to be a strict M-L function, if in addition one has \( \phi(M_k) < \phi(M_l) \) whenever \( k < l \).
Proposition 5.6. [9] Each Morse decomposition \( \mathcal{M} \) of an attractor \( \mathcal{A} \) has a non-negative radially unbounded strict M-L function \( \phi \) on \( \Omega(\mathcal{A}) \).

5.2. Morse equations. Let \( K \) be a compact isolated invariant set. Then the pointed spaces \((N/E, [E])\) are shape equivalent for all the shape index pairs \((N, E)\) of \( K \). By virtue of the shape invariance of \( \check{C}ech \) homologies, both \( \check{H}_q(N/E, [E]) \) and \( \check{H}^q(N/E, [E]) \) (with a fixed coefficient group \( G \)) are independent of shape index pairs. This allows us to define the \( \check{C}ech \) homology index \( \check{H}_*(s(K)) \) and \( \check{C}ech \) cohomology index \( \check{H}^*(s(K)) \) of \( K \), respectively, to be the \( \check{C}ech \) homology theory \( \check{H}_*(N/E, [E]) \) and cohomology theory \( \check{H}^*(N/E, [E]) \) of any shape index pair \((N, E)\) of \( K \).

Now suppose \( K \) has a Morse decomposition \( \mathcal{M} = \{M_1, \cdots, M_n\} \) with the corresponding Morse filtration \( \emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = K \). Let \((N, E)\) be a shape index pair of \( K \), and \( \Phi \) be the quotient flow on \( N/E \) defined as in Section 3. Then as in Lemma 4.6 it can be shown that \( \mathcal{M} = \{M_0, M_1, \cdots, M_n\} \) forms a Morse decomposition of the global attractor \( \mathcal{A} \) of \( \Phi \), where

\[
\check{M}_0 = \{[E]\}, \quad \check{M}_k = \pi(M_k) (1 \leq k \leq n).
\]

Here \( \pi : W := N \cup E \rightarrow N/E \) is the quotient map.

Let \( \phi \) be a radially unbounded strict Morse-Lyapunov function of \( \mathcal{A} \) corresponding to the Morse decomposition \( \mathcal{M} \), and let \( a_k = \phi(M_k) \) \((k = 0, 1, \cdots, n)\). Then \( a_0 < a_1 < \cdots < a_n \). For each \( k \geq 0 \) we fix a number \( b_k \) with \( a_k < b_k < n_{k+1} \) (we assign \( a_{n+1} = \infty \)). Set \( N_k = \phi^{b_k} \). Clearly \( N_0 \subset N_1 \subset \cdots \subset N_n \). It is easy to verify that \((N_k, N_{k-1})\) is a shape index pair of \( M_k \) for each \( k \geq 1 \). Consequently \((N_k, N_{k-1})\) is a shape index pair of \( M_k \), where

\[
N_k = \pi^{-1}(\tilde{N}_k), \quad k = 0, 1, \cdots, n.
\]

Similarly we also know that \((N_k, N_0)\) is a shape index pair of \( A_k \) for \( k \geq 1 \). Since \( N_0 \subset N_1 \subset \cdots \subset N_n \), by very standard argument (see e.g. [16]) one can obtain the following Morse equation

\[
\sum_{k=1}^n \sum_{q=0}^\infty t^q \text{rank} \hat{H}_q(N_k, N_{k-1}) = \sum_{q=0}^\infty t^q \text{rank} \hat{H}_q(N_n, N_0) + (1 + t)Q(t), \tag{5.27}
\]

where

\[
Q(t) = \sum_{k=1}^n \sum_{q=1}^\infty t^{q-1} \text{rank} \partial_{q,k}, \tag{5.28}
\]

and \( \partial_{q,k} \) is the boundary operator from \( \hat{H}_q(N_k, N_{k-1}) \) to \( \hat{H}_{q-1}(N_{k-1}, N_0) \). Here we have assumed that all the relative homology groups have finite ranks (results in this line can be found in [8, 13] etc.).

As \((N_k, N_{k-1})\) and \((N_k, N_0)\) are shape index pairs of \( M_k \) and \( A_k \), respectively, we have

\[
\hat{H}_*(N_k, N_{k-1}) = \hat{H}_*(s(M_k)), \quad \hat{H}_*(N_k, N_0) = \hat{H}_*(s(A_k)).
\]

Hence (5.27) can be rewritten as follows:

\[
\sum_{k=1}^n \sum_{q=0}^\infty t^q \text{rank} \hat{H}_q(s(M_k)) = \sum_{q=0}^\infty t^q \text{rank} \hat{H}_q(s(K)) + (1 + t)Q(t). \tag{5.29}
\]
For any compact isolated invariant set $M$, set
\[ p(t, s(M)) = \sum_{q=0}^{\infty} t^q \text{rank} \hat{H}_q(s(M)). \]
$p(t, s(M))$ is called the formal Poincaré polynomial of $s(M)$. Now the Morse equation (5.29) can be restated in terms of formal Poincaré polynomials:
\[ \sum_{k=1}^{n} p(t, s(M_k)) = p(t, s(K)) + (1 + t)Q(t). \]  
\[ (5.30) \]

**Remark 5.6.** Similar results remain valid for Čech cohomologies. We omit the details.

6. An example. In this section we give an easy example for the computation of shape indices, which may help the reader to have a better understanding to the concept of shape index pairs introduced here.

Consider the parabolic problem:
\[ \begin{align*}
    u_t - \Delta u + f(u) & = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \\
    u(x, 0) & = u_0(x), \quad \text{in } \Omega, \\
    u(x, t) & = 0, \quad \text{on } \partial \Omega \times \mathbb{R}^+, \\
    u(x, t) & = u_0(x), \quad \text{in } \Omega,
\end{align*} \]  
\[ (6.31) \]
where $\Omega$ is bounded domain in $\mathbb{R}^m$, and
\[ f(s) = s^{2p-1} + b_1 s + b_0, \quad \text{where } p > 1. \]

It is well known (see e.g. [18]) that for each $u_0 \in H = L^2(\Omega)$, the problem (6.31) has a unique solution $u \in C(\mathbb{R}^+; L^2(\Omega))$ with
\[ u \in L^2(0, T; H_0^1(\Omega)) \cap L^{2p}(0, T; L^{2p}(\Omega)), \quad \forall T > 0. \]
Moreover, the solution operator $u_0 \to u(t)$ generates a global semiflow $\Phi$ on $L^2(\Omega)$. $\Phi$ is asymptotically compact and has a global attractor $A$ in $L^2(\Omega)$. If the system has only a finite number of equilibria: $e_1, e_2, \cdots, e_n$, then $\mathcal{M} = \{e_1, e_2, \cdots, e_n\}$ forms a Morse decomposition of $A$. Suppose that we want to write out explicitly the Morse equation of $A$. Then one has to make out all the homology groups $\hat{H}_q(s(e_k))$ and $\hat{H}_q(s(A))$. As $A$ is the global attractor, we infer from [8] that
\[ s(A) = \text{Sh}(\Phi, p) = \text{Sh}(\{p\}, p). \]
(The second equality in the above equation is due to the fact that $H$ is contractible.) Hence we find that $\hat{H}_q(s(A)) = 0$.

Now let us try to calculate $\hat{H}_q(s(e_k))$. If $e_k$ is hyperbolic, then $\hat{H}_q(s(e_k))$ is completely determined by the local unstable manifold of $e_k$, and the calculation of $\hat{H}_q(s(e_k))$ is somewhat trivial. The situation in the case when $e_k$ is not hyperbolic seems to be complicated. In such a case the local unstable manifolds of $e_k$ usually remain unknown. Noticing that the system has a natural Morse-Lyapunov function $J$ defined as
\[ J(u) = \frac{1}{2} ||u||^2 + \int_{\Omega} g(u) dx, \]  
\[ (6.32) \]
where
\[ ||u|| = \left( \int_{\Omega} |\nabla u|^2 dx \right)^{1/2}, \quad g(s) = \frac{1}{2p} s^{2p} + \frac{b_1}{2} s^2 + b_0 s, \]
one may wish to find appropriate Conley index pairs via the level sets $J^c$ of the function $J$ to calculate the index $s(e_k)$. Unfortunately $J$ is only defined on a subspace
\[ V = H_0^1(\Omega) \cap L^{2p}(\Omega). \]

To overcome this difficulty, the usual way was to restrict the system on the attractor \( \mathcal{A} \) and think of \( \mathcal{A} \) as the phase space. (It can be shown that \( \mathcal{A} \) is a compacta in \( V \).) But to do so, one first need to calculate the attractor \( \mathcal{A} \). Another drawback is that the continuity of the semigroup \( \Phi \) in the topology of \( V \) can be hardly examined. Here we advocate to use shape index pairs defined as in Section 4.

For clarity, we may assume that
\[ J(e_1) < J(e_2) < \cdots < J(e_n). \]

Pick two numbers \( \alpha \) and \( \beta \) with
\[ J(e_{k-1}) < \alpha < J(e_k) < \beta < J(e_{k+1}). \]

We claim that \((J^\beta, J^\alpha)\) is a shape index pair of \( e_k \). To see this, we first show that for any \( c \in \mathbb{R}^1 \), the level set \( J^c \) is closed in \( H \). Indeed, let \( u_n \in J^c \), and \( u_n \rightarrow u_0 \) in \( H \). Since \( J(u_n) \leq c \) for all \( n \), the sequence \( u_n \) is bounded in both the spaces \( H_0^1(\Omega) \) and \( L^{2p}(\Omega) \). Hence we can extract a subsequence of \( u_n \), still denoted by \( u_n \), such that
\[ u_n \rightarrow v \text{ (in } H_0^1(\Omega)\text{), and } u_n \rightarrow w \text{ (in } L^{2p}(\Omega)\text{).} \]

Of course one has \( u_0 = v = w \). Thus \( u_0 \in H_0^1(\Omega) \cap L^{2p}(\Omega) \). Pick a subsequence \( u_{n_k} \) such that both the sequences \( ||u_{n_k}|| \) and \( ||u_{n_k}||_{L^{2p}(\Omega)} \) converge. Then
\[
c \geq \lim \inf_{n \rightarrow \infty} J(u_n) \geq \lim \inf_{n \rightarrow \infty} J(u_{n_k})
\]
\[
= \frac{1}{2} \lim \inf_{n \rightarrow \infty} ||u_{n_k}||^2 + \frac{1}{2 p} \lim \inf_{n \rightarrow \infty} ||u_{n_k}||_{L^{2p}(\Omega)}^2
\]
\[
+ \lim \inf_{n \rightarrow \infty} \int_{\Omega} \left( \frac{b}{2} u_{n_k}^2 + b_0 u_{n_k} \right) \, dx
\]
\[
\geq \frac{1}{2} ||u_0||^2 + \frac{1}{2 p} ||u_0||_{L^{2p}(\Omega)}^2 + \lim \inf_{n \rightarrow \infty} \int_{\Omega} \left( \frac{b}{2} u_{n_k}^2 + b_0 u_{n_k} \right) \, dx
\]
\[
= \frac{1}{2} ||u_0||^2 + \frac{1}{2 p} ||u_0||_{L^{2p}(\Omega)}^2 + \int_{\Omega} \left( \frac{b}{2} u_0^2 + b_0 u_0 \right) \, dx = J(u_0).
\]

Therefore \( u_0 \in J^c \), which completes the proof of the closedness of \( J^c \).

As \( J^J = J^\beta \setminus J^\alpha \) is a bounded subset of \( H \), it follows by the asymptotic compactness of \( \Phi \) that \( J^J \) is strongly admissible. (Recall that \( \Phi \) is a global semiflow, hence no solutions explode.) Since both \( J^\alpha \) and \( J^\beta \) are positively invariant for \( \Phi \), clearly \( J^\alpha \) is \( J^\beta \)-positively invariant. The verification of that \( J^J \) contains a local unstable manifold of \( e_k \) and that \( J^\alpha \) is an exit set of \( J^\beta \) is also trivial. We omit the details. Hence we see that \((J^\beta, J^\alpha)\) is indeed a shape index pair.

**Remark 6.7.** Note that the pair \((J^\beta, J^\alpha)\) in the above argument may fail to be a Conley index pair of \( e_k \), because \( J^J \) is in general not a neighborhood of \( e_k \).

**REFERENCES**

[1] K. Borsuk, *Theory of Shape*, Monografie Matematyczne, Tom 59 [Mathematical Monographs, Vol. 59], PWN–Polish Scientific Publishers, Warsaw, 1975.

[2] C. Conley, *Isolated Invariant Sets and the Morse Index*, Regional Conference Series in Mathematics, 38, American Mathematical Society, Providence, RI, 1978.

[3] J. Dydak and J. Segal, *Shape Theory: An Introduction*, Lecture Notes in Math., 688, Springer-Verlag, Berlin, 1978.

[4] A. Giraldo, M. A. Morón, F. R. Ruiz del Portal and J. M. R. Sanjurjo, *Shape of global attractors in topological spaces*, *Nonlinear Anal.*, 60 (2005), 837–847.

[5] A. Giraldo, R. Jiménez, M. A. Morón, F. R. Ruiz del Portal and J. M. R. Sanjurjo, *Pointed shape and global attractors for metrizable spaces*, *Topology Appl.*, 158 (2011), 167–176.

[6] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
SHAPE CONLEY INDEX THEORY OF SEMIFLOWS

[7] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, 840, Springer-Verlag, Berlin-New York, 1981.

[8] L. Kapitanski and I. Rodnianski, *Shape and Morse theory of attractors*, Comm. Pure Appl. Math., 53 (2000), 218–242.

[9] D. Li, Morse theory of attractors via Lyapunov functions, preprint, arXiv:1003.1576.

[10] D. Li, G. Shi and X. Song, A linking theory for dynamical systems with applications to PDEs, arXiv:1312.1868v3.

[11] S. Mardešić and J. Segal, *Shape Theory — The Inverse System Approach*, North-Holland Mathematical Library, 26, North–Holland, Amsterdam–New York, 1982.

[12] M. Mrozek, Shape index and other indices of Conley type for local maps on locally compact Hausdorff spaces, Fund. Math., 145 (1994), 15–37.

[13] J. W. Robbin and D. Salamon, *Dynamical systems, shape theory and the Conley index*, Ergodic Theory Dynam. Systems, 8 (1988), 375–393.

[14] K. P. Rybakowski, *The Homotopy Index and Partial Differential Equations*, Springer-Verlag, Berlin, 1987.

[15] J. J. Sánchez-Gabites, An approach to the Conley shape index without index pairs, Rev. Mat. Complut., 24 (2011), 95–114.

[16] J. M. R. Sanjurjo, Morse equation and unstable manifolds of isolated invariant sets, Nonlinearity, 16 (2003), 1435–1448.

[17] J. M. R. Sanjurjo, Shape and Conley index of attractors and isolated invariant sets, Progress Nonlinear Diff. Eqns., 75 (2008), 393–406.

[18] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd edition, Springer-Verlag, New York, 1997.

Received December 2014; revised June 2015.

E-mail address: wangjt425@tju.edu.cn
E-mail address: lidsmath@tju.edu.cn
E-mail address: duanjq@gmail.com