Introduction.

The purpose of this note is to give an effective version of Matsusaka’s theorem on a smooth projective surface. Matsusaka’s general theorem states that given an ample line bundle $A$ on a smooth projective variety (of arbitrary dimension) $X$ there exists a constant $k_0$, depending only on the coefficients of the Hilbert polynomial of $A$, such that $kA$ is very ample for $k \geq k_0$. In case $X$ is a curve of genus $g$, then $A$ is ample if and only if $\deg(A) > 0$, and in this case $kA$ is very ample if $k \geq \frac{2g+1}{\deg(A)}$; for an arbitrary surface and ample divisor $A$ (in any characteristic) the theorem was proven by Matsusaka and Mumford in [MM] and the general case by Matsusaka in [Ma].

The original proof of Matsusaka’s theorem depends on the boundedness of certain invariants of certain varieties and divisors in a bounded family; the constant $k_0$ cannot be effectively computed. Matsusaka’s original approach is non-cohomological, but as pointed out by Lieberman and Mumford [LM], it suffices to find an integer $k_0$ such that for any polarized variety $(X, A)$ with given Hilbert polynomial, and any pair of points $x, y \in X$

$$H^1(X, \mathcal{O}_X(kA) \otimes \mathfrak{m}_x \otimes \mathfrak{m}_y) = 0 \text{ for } k \geq k_0.$$ 

Recently, Siu [S] gave an effective version of Matsusaka’s theorem in all dimensions. His method uses the strong Morse inequality, the numerical criterion for very ampleness of Demailly and Nadel’s vanishing theorem. Unfortunately, his lower bound depends doubly exponentially on the dimension of the variety; in this sense Siu’s bound is more of theoretical interest than of practical utility. In this note, we focus only on the case of surfaces, and obtain a result which is essentially optimal, as is shown by an example of Xiao. Our version of Matsusaka’s theorem is the following:

**Theorem.** Let $A$ be an ample divisor on a nonsingular projective algebraic surface $X$. If

$$k > \frac{1}{2} \left[ \frac{(A \cdot (K_X + 4A) + 1)^2}{A^2} + 3 \right],$$

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then $kA$ is very ample.

Following Siu, the argument starts by recalling a classical lemma guaranteeing that some multiple of the difference of suitable divisors has a lot of sections. Next, we use a technique of Ein and Lazarsfeld —very much motivated by some of Demailly’s analytic ideas (c.f. [D]) — to produce some auxiliary divisors with almost isolated singularities. Then, using the cohomological techniques pioneered by Kawamata, Reid and Shokurov in connection with the minimal model program (c.f. [CKM] or [KMM]), we are able to conclude with the Kawamata-Viehweg vanishing theorem for fractional divisors. Along the way we give a greatly simplified proof of the main lemma of Lazarsfeld, Ein and Nakamaye [L] required for this technique, which potentially opens the door to higher dimensional extensions.

At the end of this note, we reproduce an example of Gang Xiao, kindly communicated to me by Lawrence Ein, which shows that this version of Matsusaka’s theorem is essentially optimal.

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§1. Ein-Lazarsfeld-Nakamaye’s lemma.

Consider a nonsingular projective surface $X$ and let $B$ be a big divisor on $X$, i.e. suppose there exists a constant $\beta > 0$ such that $h^0(X, \mathcal{O}_X(nB)) \geq \frac{n^2}{2}\beta$ for $n \gg 0$ (recall that if in addition $B$ is nef, this condition is equivalent to $B^2 > 0$).

For $n > 0$, let

$$nB = M_n + F_n$$

be the decomposition of $nB$ in it’s moving part $M_n$ and it’s fixed part $F_n$, so that $M_n$ is nef (and big) and
Note that as a consequence of Bertini’s theorem, the general element of the complete linear system $|M_n|$ is reduced.

Ein-Lazarsfeld-Nakamaye’s technique to construct a divisor linearly equivalent to $nB$ with (almost) isolated singularities, consists in finding first a lower bound on the self-intersection of $M_n$ in terms of the number of sections of $\mathcal{O}_X(nB)$, and then use the Hodge index theorem to find an upper bound on the coefficients of the fractional divisor $\frac{1}{n}F_n$. More precisely, we first need the following (c.f. [L]):

**Lemma (Ein-Lazarsfeld-Nakamaye).** Suppose that for $n \gg 0$, there is a constant $\beta > 0$ with

$$h^0(X, \mathcal{O}_X(nB)) \geq \frac{n^2}{2} \beta + o(n).$$

Then $M_n^2 \geq n^2 \beta + o(n)$ for $n \gg 0$.

**Proof.** Let $H$ be a very ample divisor on $X$, with the property that $K_X + H$ is very ample. A general element of the linear system $|K_X + H|$ defines, for every $n$, an inclusion $\mathcal{O}_X(M_n) \subseteq \mathcal{O}_X(K_X + H + M_n)$, so in particular

$$h^0(X, \mathcal{O}_X(K_X + H + M_n)) \geq h^0(X, \mathcal{O}_X(M_n)).$$

Since $H + M_n$ is nef and big, the Kawamata-Viehweg vanishing theorem implies that $\chi(X, \mathcal{O}_X(K_X + H + M_n)) = h^0(X, \mathcal{O}_X(K_X + H + M_n))$, and hence it follows from Riemann-Roch that

$$\frac{1}{2}((K_X + H + M_n) \cdot (H + M_n)) + o(n) \geq h^0(X, \mathcal{O}_X(M_n)).$$

But $M_n \cdot (K_X + H) \sim o(n)$, in fact, since $K_X + H$ is very ample

$$nB \cdot (K_X + H) \geq M_n \cdot (K_X + H);$$

the result now follows from (1). \(\Box\)
Now suppose that $N$ is an arbitrary nef and big divisor on $X$. Then $F_n \cdot N \geq 0$ and the Hodge index theorem implies that
\[
\frac{1}{n} F_n \cdot N = B \cdot N - \frac{1}{n} M_n \cdot N \\
\leq B \cdot N - M^2_n \sqrt{n^2},
\]
so in the situation of the previous lemma
\[
(2) \quad 0 \leq \frac{1}{n} F_n \cdot N \leq B \cdot N - \sqrt{\beta N^2}
\]
and in particular an upper bound on the right hand side of (2) imposes numerical conditions on the coefficients of $F_n$ and in consequence on the divisor $[\frac{1}{n} F_n]$.

§2. Proof of theorem.

Following Beltrametti and Sommese [BS], recall that a divisor $L$ is $\kappa$-jet ample if, given any $r$ positive integers $\kappa_1, \ldots, \kappa_r$ with $\kappa + 1 = \sum_{i=1}^r \kappa_i$, and any $r$ distinct points $\{x_1, \ldots, x_r\} \subseteq X$, the evaluation map
\[
H^0(X, \mathcal{O}_X(L)) \longrightarrow H^0(X, \mathcal{O}_X(L) \otimes \mathcal{O}_Z)
\]
is surjective, where $Z = \kappa_1 x_1 + \cdots + \kappa_r x_r$. In particular $L$ is globally generated (resp. very ample) if and only if $L$ is 0-jet ample (resp. 1-jet ample).

The purpose now is to prove the following generalized effective version of Matsusaka’s Big theorem.

**Theorem.** Let $A$ be an ample divisor on a nonsingular projective surface $X$. If
\[
k > \frac{1}{2} \left[ \frac{(A \cdot (K_X + 4A) + 1)^2}{A^2} + (\kappa^2 + 4\kappa - 2) \right]
\]
then $kA$ is $\kappa$-jet ample.
For the proof, let $\kappa_1, \ldots, \kappa_r$ be $r$ positive integers with $\kappa + 1 = \sum_{i=1}^{r} \kappa_i$, and let \( \{x_1, \ldots, x_r\} \subseteq X \) be any $r$ distinct points. Let $Z$ denote the 0-cycle $Z = \sum_{i=1}^{r} \kappa_i x_i$ and let
\[
m_Z = m_{x_1}^{\kappa_1} \otimes \cdots \otimes m_{x_r}^{\kappa_r}
\]
where $m_{x_i}$ denotes the maximal ideal at $x_i$. Then for $kA$ to be $\kappa$-jet ample it suffices to show that
\[
H^1(X, \mathcal{O}_X(kA) \otimes m_Z) = 0.
\]

To this end, let $f : Y \to X$ be the blow-up of $X$ at $x_1, \ldots, x_r$, with corresponding exceptional divisors $E_1, \ldots, E_r \subseteq Y$. Using the Leray spectral sequence, (3) is then equivalent to
\[
H^1(Y, \mathcal{O}_Y(f^*(kA) - \sum_{i=1}^{r} \kappa_i E_i)) = 0.
\]

Let $k$ be a positive integer. In order to apply Ein-Lazarsfeld-Nakayama’s lemma, we need to find a lower bound on the number of sections of $n(kA - K_X)$ for $n \gg 0$. Writing
\[
kA - K_X = (k + 4)A - (K_X + 4A),
\]
the question reduces to the case of the difference of two ample divisors (a result of Fujita [F] asserts that $K_X + 4A$ is ample whenever $A$ is ample). In the case of surfaces, this is (an easy) well known consequence of Riemann-Roch (c.f. [Mu]).

**Lemma 1.** Let $D$ and $E$ be ample divisors on a nonsingular projective surface $X$. If $D^2 - 2D \cdot E > 0$ then $h^0(X, \mathcal{O}_X(n(D - E))) \neq 0$ for $n \gg 0$. In fact, for $n \gg 0$
\[
h^0(X, \mathcal{O}_X(n(D - E))) \geq \frac{n^2}{2}(D^2 - 2D \cdot E) + o(n).
\]
In particular, if $k$ is a positive integer such that
\[(k+4)A^2 - 2A \cdot (K_X + 4A) > 0 ,\]
then for $n \gg 0$ we have that
\[h^0(X, \mathcal{O}_X(n(kA - K_X))) \geq \frac{n^2}{2}((k+4)^2A^2 - 2(k+4)A \cdot (K_X + 4A)) + o(n) .\]
Write $\beta(k) = (k+4)A^2 - 2A \cdot (K_X + 4A)$ and $\overline{\beta}(k) = (k+4)\beta(k)$.

Next we apply Ein-Lazarsfeld-Nakamaye's lemma on the blow-up $Y$ of $X$ at $x_1, \ldots, x_r$. For this, consider the divisor
\[B = f^*(kA - K_X) - \sum_{i=1}^{r}(\kappa_i + 1)E_i .\]

**Lemma 2.** With notation as above, if $k$ is such that $\beta(k) > 0$ then
\[(7) \quad h^0(Y, \mathcal{O}_Y(nB)) \geq \frac{n^2}{2}(\beta(k) - (\kappa + 2)^2) + o(n) .\]

**Proof.** Recall that the integers $\kappa_1, \ldots, \kappa_r$ are such that $\sum_{i=1}^{r}\kappa_i = \kappa + 1$; under this condition, a direct computation shows (c.f. [BS]) that
\[(8) \quad (\kappa + 2)^2 \geq \sum_{i=1}^{r}(\kappa_i + 1)^2 + (r - 1)^2 .\]
On the other hand, we have that
\[h^0(Y, \mathcal{O}_Y(nB)) \geq h^0(X, \mathcal{O}_X(n(kA - K_X))) - \sum_{i=1}^{r}\ell(\mathcal{O}_X/\mathfrak{m}_x^{n(\kappa_i+1)})\]
and that $\ell(\mathcal{O}_X/\mathfrak{m}_x^{\kappa_i}) = \frac{n(\kappa_i+1)+1}{2}$, so it follows from (5) and (8) that
\[h^0(Y, \mathcal{O}_Y(nB)) \geq \frac{n^2}{2}(\beta(k) - \sum_{i=1}^{r}(\kappa_i + 1)^2) + o(n) \]
\[\geq \frac{n^2}{2}(\beta(k) - (\kappa + 2)^2) + o(n) .\quad \square\]
In particular for $B$ to be big it suffices by (7) that

\begin{equation}
\beta(k) - (\kappa + 2)^2 > 0 .
\end{equation}

Suppose that $B$ is big, that is, suppose that $k$ is such that (9) holds. Then Ein-Lazarsfeld-Nakamaye’s lemma implies that

\[
\frac{1}{n} F_n \cdot f^* A \leq A \cdot (kA - K_X) - \sqrt{\beta(k) - (\kappa + 2)^2 \sqrt{A^2}} .
\]

**Lemma 3.** Suppose that $k$ is such that

\[
k > \frac{1}{2} \left[ \frac{(A \cdot (K_X + 4A) + 1)^2}{A^2} + (\kappa^2 + 4\kappa - 4) \right] .
\]

Then $\beta(k) - (\kappa + 2)^2 > 0$ and

\[
A \cdot (kA - K_X) - \sqrt{\beta(k) - (\kappa + 2)^2 \sqrt{A^2}} < 1 .
\]

**Proof.** This is a direct computation. \square

Now suppose that

\[
k > \frac{1}{2} \left[ \frac{(A \cdot (K_X + 4A) + 1)^2}{A^2} + (\kappa^2 + 4\kappa - 4) \right] .
\]

Then $\frac{1}{n} F_n \cdot f^* A < 1$ and hence the irreducible components of the divisor $\lceil \frac{1}{n} F_n \rceil$ are exceptional, that is,

\begin{equation}
\lceil \frac{1}{n} F_n \rceil = \sum_{i=1}^{r} \eta_i E_i \quad (\eta_i \geq 0) .
\end{equation}

Fix $n \gg 0$, and let $D$ be a general divisor on the linear system $|nB|$ so that if $D = M + F_n$ with $F_n$ the fixed part of $nB$ then $M$ is reduced. The divisor

\[
f^*((k + 1)A - K_X) - \sum_{i=1}^{r} (\kappa_i + 1) E_i - \frac{1}{n} D
\]

7.
being numerically equivalent to \( f^* A \) is nef and big. Using (10) and the fact that \( M \) is reduced, the Kawamata-Viehweg vanishing theorem then implies the vanishing of the higher cohomology groups of the divisor

\[
K_Y + [f^*((k + 1)A - K_X)] - \sum_{i=1}^{r} (\kappa_i + 1)E_i - \frac{1}{n}D = (k + 1)f^*A - \sum_{i=1}^{r} (\kappa_i + \eta_i)E_i,
\]

and in particular we get that

\[
(11) \quad H^1(Y, f^*\mathcal{O}_X((k + 1)A) \otimes \mathcal{O}_Y(-\sum_{i=1}^{r} (\kappa_i + \eta_i)E_i)) = 0.
\]

But (11) implies (4) and it follows that \((k + 1)A\) is \(\kappa\)-jet ample. This completes the proof of the theorem.

As an immediate consequence, we get the following effective version of Matsusaka’s theorem:

**Corollary.** Let \( A \) be an ample divisor on a nonsingular projective surface. If

\[
k > \frac{1}{2} \left[ \frac{(A \cdot (K_X + 4A) + 1)^2}{A^2} + 3 \right]
\]

then \( kA \) is very ample.

The following example of Xiao shows that the lower bound on (12) is essentially optimal.

**Example (Xiao).** Let \( X_n \) be the ruled surface over \( \mathbb{P}^1 \) defined by \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \) with section \( C_0 \) corresponding to \( \mathcal{O}_{X_n}(C_0) = \mathcal{O}_{X_n}(1) \) and fiber \( f \). Let \( C_1 \sim C_0 + nf \) be a section with \( C_0 \cap C_1 = \emptyset \) and let \( D \) be an irreducible nonsingular curve on the linear system \( |2C_1 + nf| \) (c.f. [Ha, III.2]). For the surface \( X \) consider the double cover \( \phi: X \to X_n \) of \( X_n \) branched along \( D \), and for the ample divisor \( A \) consider \( \phi^*(C_1 + f) \). Note that, for \( n \geq 6 \), \( \phi^*(C_0) \subseteq X \) is a hyperelliptic curve of genus \( \frac{n}{2} - 1 \) and that \( A|_{\phi^*(C_0)} \) is an ample divisor of degree 2 on \( \phi^*(C_0) \); in particular, if \( kA \) is very ample then

\[
(13) \quad kA \cdot \phi^*(C_0) > \frac{n}{2} + 1.
\]
On the other hand $A^2 = 2(n+2)$ and $A \cdot K_X = n - 6$, which implies by (13) that if $kA$ is very ample then

$$k > \frac{1}{2} \left[ \frac{(A \cdot K_X)^2 + 8(A \cdot K_X)}{A^2} + 1 \right].$$

**Remark.** Using the same method as above, with a different lower bound on the number of sections of $n(kA - K_X)$ (namely $\beta(k) = (kA - K_X)^2$), one can show the following effective variant of Matsusaka’s theorem:

**Theorem*. Let $A$ be an ample divisor on a nonsingular projective surface. If

$$k > \frac{1}{2} \left[ \frac{(A \cdot K_X + 1)^2}{A^2} - K_X^2 + (\kappa^2 + 4\kappa + 6) \right]$$

then $kA$ is $\kappa$-jet ample.

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School of Mathematics
Institute for Advanced Study
Princeton, NJ 08540
E-mail: gfb@math.ias.edu