Hydrodynamic Approach to Vortex Lifetime in Trapped Bose Condensates

Emil Lundh and P. Ao

Department of Theoretical Physics, Umeå University, S-90187 Umeå, Sweden

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We study a vortex in a two-dimensional, harmonically trapped Bose-Einstein condensate at zero temperature. Through a variational calculation using a trial condensate wave function and a nonlinear Schrödinger Lagrangian, we obtain the effective potential experienced by a vortex at an arbitrary position in the condensate, and find that an off-center vortex will move in a circular trajectory around the trap center. We find the frequency of this precession to be smaller than the elementary excitation frequencies in the cloud.

We also study the radiation of sound from a moving vortex in an infinite, uniform system, and discuss the validity of this as an approximation for the trapped case. Furthermore, we estimate the lifetime of a vortex due to imperfections in the trapping potential.

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I. INTRODUCTION

The prospect of creating quantized vortices in trapped Bose-Einstein condensed gases (BEC’s) has been an intensely discussed and studied subject in the last few years [1–7]. Despite the considerable interest in this area of BEC studies, many of the most fundamental questions have yet to be answered, such as those concerning the stability and lifetime of such a state [2,4–7]. In this article, we study the properties of a vortex in a trapped BEC from a hydrodynamic point of view. We confine the discussion to two-dimensional systems at zero temperature, and furthermore employ the limit of a condensate which is large in comparison with the size of the vortex.

We will thus only be concerned with a system whose properties can be described by a nonlinear Schrödinger equation. The system that we have in mind is the dilute Bose gas, which is governed by the Gross-Pitaevskii equation [8,9] at temperatures sufficiently low that the system may be described as a superfluid. It should, however, be noted that this type of equation is applicable to a wider class of systems than just zero-temperature dilute gases [10,11].

The stability of a vortex in a BEC is limited by several factors. At finite temperatures, the vortex may be destroyed due to collisions with thermal excitations [12]. This will not be the subject of this study. Second, the vortex may decay spontaneously even at zero temperature through the excitation of modes (or, equivalently, the emission of phonons) in the cloud [2,4–5]. Third, deviations from spherical symmetry in the trapping potential will also limit the lifetime of the vortex. The two latter processes will be the subject of this paper.

The paper is organized as follows. Sections II–IV are concerned with the motion of an off-center vortex and its decay through phonon emission. In section II, we introduce the model and the trial assumptions for the density and velocity distributions, and analyze the motion of an off-center vortex. In Sec. III, the parallel case of a precessing vortex in an infinite, homogeneous system is analyzed, and the power loss due to the radiation of sound is calculated. This can be thought of as a “semiclassical” approximation to the trapped case considered here. In Sec. IV, the validity of this approximation is discussed and a lower bound for the lifetime of a vortex is arrived at. In Sec. V, we find the characteristic time for destruction of a vortex due to deviations from cylindrical symmetry in the trapping potential, and finally, in Section VI, the results are summarized and discussed.

II. CIRCULAR MOTION

In the case of a harmonic trapping potential, the equation for the condensate wave function $\psi(\vec{r})$, whose squared modulus gives the superfluid density distribution $\rho(\vec{r})$, reads

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} m \omega_z^2 r^2 + U_0 |\psi(\vec{r})|^2 \right] \psi(\vec{r}) = \mu \psi(\vec{r}),$$

(1)

where $\mu$ is the chemical potential and $U_0 = 4\pi \hbar^2 a/m$ is the effective interaction potential, with $a$ being the $s$-wave scattering length. Assuming the kinetic energy to be negligible, we obtain the so-called Thomas-Fermi approximation [13] for the wave function for a non-rotating cloud,
\[ \psi_{\text{TF}}(\vec{r}) = \sqrt{\rho_0} \left( 1 - \frac{r^2}{R^2} \right)^{1/2}, \]  

with an associated density distribution \( \rho_{\text{TF}} = |\psi_{\text{TF}}|^2 \). Here, the central density \( \rho_0 = \mu/U_0 \) and the Thomas-Fermi radius \( R = (2\mu/m\omega^2)^{1/2} \). For a two-dimensional system, the wave function \( \psi(\vec{r}) \) is normalized according to

\[ \int d^2\vec{r} |\psi(\vec{r})|^2 = \nu, \]  

where \( \nu \) is the number of particles per unit length. As a measure of the influence of the inter-particle interactions on the system’s properties, we define the dimensionless parameter

\[ \gamma \equiv \nu a. \]  

The Thomas-Fermi approximation is a valid one when \( \gamma \) is large, and in that limit we have for a two-dimensional system

\[ \mu = 2\hbar \omega \sqrt{\gamma}, \]
\[ R = a_{\text{osc}} \gamma^{1/4}, \]

where \( a_{\text{osc}} = (\hbar/m\omega)^{1/2} \) is the oscillator length.

We now turn to the problem of a cloud containing a singly quantized vortex at the position \( \vec{r}_0 \). In the limit of large \( \gamma \), the density distribution of the cloud will not be appreciably affected by the presence of a vortex, except in a region whose size is comparable to the healing length \( \xi \), defined as

\[ \xi = \sqrt{\frac{\hbar^2}{2mpU_0}} = \frac{1}{(8\pi \rho_0 a)^{1/2}}. \]

The healing length gives the length scale over which the wave function for a vortex in a homogeneous Bose gas increases from zero to its bulk value \[8,9,14]. For an untrapped system, the density \( \rho \) is the value of the density far from the vortex core. In the case of a trapped system, \( \rho \) must be taken to be the local Thomas-Fermi density at the point \( \vec{r}_0 \) in the absence of a vortex, \( \rho(\vec{r}_0) \), thus defining a local healing length

\[ \xi(\vec{r}_0) = \frac{1}{(8\pi \rho(\vec{r}_0) a)^{1/2}} = \frac{\xi_0}{\sqrt{1 - \frac{r_0^2}{R^2}}}, \]

where \( \xi_0 = \xi(0) \) is the value of the healing length in the center. The velocity distribution in a Bose-condensed system is given by \( \hbar/m \) times the gradient of the phase of the wave function. For a positively oriented vortex in an infinite, uniform system, it is known to be

\[ \vec{v}_{\text{uni}}(\vec{r}) = \frac{\hbar}{m} \nabla \phi, \]

where \( \phi \) is the polar angle relative to the position of the vortex, which gives

\[ \vec{v}_{\text{uni}}(\vec{r}) = \frac{\hbar}{m} \frac{\hat{z} \times (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^2}, \]

\( \hat{z} \) being the unit vector in the \( z \) direction.

The velocity field is altered due to the boundary of the system and due to the spatially varying density. The existence of a boundary requires that the normal velocity vanishes there, and for homogeneous systems the recipe is to introduce a negatively oriented image vortex at the point \( \vec{r}_1 \equiv \vec{r}_0 R^2/r_0^2 \), giving the velocity field

\[ \vec{v}_{\text{uni}}(\vec{r}) \]

\[1\text{The existence of sharp boundary is an artefact of the Thomas-Fermi approximation. Although this approximation never holds at the boundary, we do have that for sufficiently large clouds, } \rho(R) \text{ is small, while } \nabla \rho(R) \text{ is nonnegligible, which justifies the neglect of the first term in the equation for stationary flow, } \rho \nabla \cdot \vec{v} + \vec{v} \cdot \nabla \rho = 0. \text{ Hence the radial velocity has to (approximately) vanish at } r = R. \]
\begin{equation}
\vec{v}_0(\vec{r}) = \frac{\hbar}{m} \hat{z} \times (\vec{r} - \vec{r}_0) - \frac{\hbar}{m} \hat{z} \times (\vec{r} - \vec{r}_1).
\tag{7}
\end{equation}

If the system has a density gradient, as in the present case, the condition for stationary flow, $\nabla \cdot (\rho \vec{v}) = 0$, is not automatically fulfilled. Writing the velocity field as

$$\vec{v} = \vec{v}_0 + \vec{v}_1,$$

with $\vec{v}_0$ taken from Eq. (7), we get an equation for the correction $\vec{v}_1$:

$$\rho \nabla \cdot \vec{v}_1 + \vec{v}_0 \cdot \nabla \rho + \vec{v}_1 \cdot \nabla \rho = 0, \tag{8}$$

since the divergence of $\vec{v}_0$ vanishes. An approximate solution to this equation, valid close to the center of the system, may be found by treating $\nabla \rho$ as small, whereupon $\vec{v}_1$ can also be expected to be a small correction, and the third term on the left-hand side of Eq. (8) can be discarded. We then have

$$\nabla \cdot \vec{v}_1(\vec{r}) = f(\vec{r}),$$

where

$$f(\vec{r}) = -\frac{\vec{v}_0(\vec{r}) \cdot \nabla \rho(\vec{r})}{\rho(\vec{r})},$$

with the boundary condition that the normal velocity vanish for $r = R$. Since we consider a Bose condensate, the velocity has to be a potential flow, $\vec{v}_1(\vec{r}) = \nabla \phi_1(\vec{r})$, and we have the equation for $\phi_1$,

$$\nabla^2 \phi_1(\vec{r}) = f(\vec{r}) \text{ for } r \leq R,$$

$$\frac{\partial \phi_1(\vec{r})}{\partial r} = 0 \text{ for } r = R$$

whose solution is written in terms of the Green’s function for the Neumann problem on a disk of radius $R$

$$G_N(\vec{r}', \vec{r}) = \frac{1}{2\pi} \ln \frac{R}{|\vec{r}' - \vec{r}|} + \frac{1}{2\pi} \ln \frac{R}{|\vec{r}' - \vec{r}^2|} + \frac{1}{2\pi} \ln \frac{R}{r}$$

as

$$\phi_1(\vec{r}) = \int d^2 r' f(\vec{r}') G_N(\vec{r}', \vec{r}). \tag{9}$$

One can easily obtain higher-order velocity terms in $\nabla \rho$, if one writes

$$\vec{v} = \vec{v}_0 + \vec{v}_1 + \vec{v}_2 + ...$$

One immediately finds that

$$\phi_{n+1}(\vec{r}) = -\int d^2 r' \frac{\vec{v}_n(\vec{r}') \cdot \nabla \rho(\vec{r}')}{\rho(\vec{r})} G_N(\vec{r}', \vec{r}),$$

where $\vec{v}_n = \nabla \phi_n$, $n = 1, 2, ...$. When the density varies over a scale larger than the healing length, higher order corrections are small. We will be content in this paper to retain only the zeroth-order term.

The energy per unit length of a two-dimensional system described by a nonlinear Schrödinger equation is

$$E[\psi, \vec{r}_0] = \int d^2 r' \left( \frac{\hbar^2}{2m} |\nabla \psi|^2 + \frac{1}{2} m \omega_r^2 r^2 |\psi|^2 + \frac{U_0}{2} |\psi|^4 \right), \tag{10}$$

where we have made explicit the dependence of $E$ on the vortex coordinate $\vec{r}_0$. The change in energy of the system due to the presence of a vortex only shows up in the kinetic-energy term, as long as its effect on the density profile is neglected. We shall denote this additional energy by $U_{\text{eff}}$, since one may regard it as an effective potential for the vortex, depending on the vortex position $\vec{r}_0$. In terms of the velocity field $\vec{v}(\vec{r})$ it is written

$$U_{\text{eff}}(\vec{r}_0) = \frac{m}{2} \int d^2 r' \rho(\vec{r}) v(\vec{r})^2. \tag{11}$$

3
Using the lowest-order approximation \( \delta_0 \) for the velocity, and employing the Thomas-Fermi wave function \( \psi_0 \), the result is (cf. [3])

\[
U_{\text{eff}}(\vec{r}_0) = \frac{\pi \hbar^2 \rho_0}{2m} \left[ 1 - \frac{r_0^2}{R^2} \right] \ln \left( \frac{R^2}{\xi_0^2} \right) + \left( \frac{R^2}{r_0^2} + 1 - 2 \frac{r_0^2}{R^2} \right) \ln \left( 1 - \frac{r_0^2}{R^2} \right).
\]  

(12)

To obtain this result, one needs to exclude the vortex core, of size \( \xi(r_0) \) around the vortex position, from the radial integral for the integrals to converge. The first term dominates for small \( r_0 \) and is independent of the details of the model.

The above analysis shows that as long as the system is not subjected to an external rotational constraint, the vortex will experience an effective potential which decreases with the distance from the trap center, except in a small region close to the boundary, where \( U_{\text{eff}} \) has a local minimum due to the unphysical behaviour of the Thomas-Fermi wave function at \( r = R \). This minimum is located a distance \( \delta \) from the edge, given by

\[
\delta = \frac{1}{2e} \left( \frac{\xi_0}{R} \right)^{2/3}.
\]  

(13)

It is interesting to note that the healing length does not set the length scale here. Apart from constants of order unity, this “boundary thickness” is the same as the cut-off length \( \delta \) of Ref. [13], which comes about when computing the kinetic energy of a Thomas-Fermi cloud.

We now turn to the motion of the vortex in this simple model. We assume that the vortex coordinate \( \vec{r}_0 \) may have a time dependence, but that all other parameters of the system remain stationary. Solving the time-dependent counterpart to the equation (1) is equivalent to minimizing the action obtained from the Lagrangian

\[
L[\psi, \vec{r}_0, \dot{\vec{r}}_0] = T[\psi, \vec{r}_0, \dot{\vec{r}}_0] - E[\psi, \vec{r}_0],
\]  

(14)

where the kinetic term

\[
T[\psi, \vec{r}_0, \dot{\vec{r}}_0] = \int d^2 \vec{r} \frac{i \hbar}{2} \left[ \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right].
\]  

(15)

A straightforward calculation, remembering that the gradient of the phase of \( \psi \) is the velocity field, yields

\[
T = \hbar \cdot \frac{\dot{\vec{r}}_0 \times \vec{r}_0}{r_0^2} (\vec{\nu}(\vec{r}_0) - \nu),
\]  

(16)

where \( \vec{\nu}(\vec{r}_0) = 2\pi \int_0^r \rho(r) \, dr \) is the number of particles per unit length inside the circle of radius \( r_0 \). The Euler-Lagrange equation for \( \vec{r}_0 \) and \( \dot{\vec{r}}_0 \) will finally yield, for the radial \( (r_0) \) and azimuthal \( (\phi_0) \) components respectively,

\[
\ddot{r}_0 = 0;
\]

\[
\dot{\phi}_0 = \frac{F(r_0)}{\hbar \partial \vec{\nu}/\partial r_0},
\]

where we have defined \( F = -\partial U_{\text{eff}}/\partial r_0 \). We see that in this model, an off-center vortex executes an orbiting motion around the center with an angular frequency \( \omega = \dot{\phi}_0 \). Since \( \partial \vec{\nu}/\partial r_0 = 2\pi r_0 \rho(r_0) \), we finally obtain

\[
\omega = \frac{F(r_0)}{2\pi \hbar r_0 \rho(r_0)}.
\]

In the case of a Thomas-Fermi profile, \( F \) is obtained by differentiating Eq. (12):

\[
F(r_0) = \frac{\pi \hbar^2 \rho_0 r_0}{mR^2} g(r_0/R),
\]  

(17)

where

\[
g(x) = 2 \ln \left( \frac{R}{\xi} \right) + \left( \frac{1}{x^4} + 2 \right) \ln \left( 1 - x^2 \right) + \frac{1}{x^2} + 2,
\]  

(18)

which yields the final result for the frequency of precession of a vortex in a Thomas-Fermi cloud,

\[
\omega = \frac{\hbar}{2mR^2 \left( 1 - \frac{r_0^2}{R^2} \right)} g(r_0/R).
\]  

(19)

The same expression has been obtained previously by different approaches [17][18]. Nevertheless, the above treatment allows a smooth generalization to incorporate the phonon effect.
We now turn to the problem of a vortex in an infinite system, exercising (e. g. under the influence of an external force) circular motion.

It has previously been shown \[19,20\] how any homogeneous superfluid described by a nonlinear Schrödinger-type energy functional is equivalent to \((2+1)\)-dimensional electrodynamics, with vortices playing the role of charges and sound corresponding to electromagnetic radiation. For a fluid, which in the absence of vortices has the density \(\rho_0\) (note that the use of this symbol is not the same as in the preceding section), with the local fluid velocity \(\vec{v}(\vec{r},t)\) and density \(\rho(\vec{r},t)\), and (possibly) containing a vortex at the position \(\vec{r}_0(t)\), moving at a velocity \(\dot{\vec{r}}_0(t) = \vec{v}_v(t)\), we define the “vortex charge” \(q_v = -\hbar\sqrt{2\pi\rho_0/m}\), “vortex density” \(\rho_v(\vec{r},t) = \delta(2)(\vec{r} - \vec{r}_0(t))\), and the corresponding “vortex current” \(\vec{j}_v(\vec{r},t) = q_v\rho_v(\vec{r},t)\vec{v}_v(\vec{r},t)\). The speed of sound is \(c = \sqrt{U_0\rho_0/m}\). We then have the analogous Maxwell equations:

\[
\begin{align*}
\nabla \cdot \vec{b} &= 0, \\
\nabla \cdot \vec{e} &= 2\pi q_v\rho_v, \\
\nabla \times \vec{e} + \frac{1}{c} \frac{\partial \vec{b}}{\partial t} &= 0, \\
\nabla \times \vec{b} - \frac{1}{c} \frac{\partial \vec{e}}{\partial t} &= \frac{2\pi}{c} q_v\vec{j}_v,
\end{align*}
\]

where we have defined

\[
\begin{align*}
\vec{e}(\vec{r},t) &= \sqrt{\frac{2\pi m}{\rho_0}} \rho(\vec{r},t) \hat{z} \times \vec{v}(\vec{r},t), \\
\vec{b}(\vec{r},t) &= \sqrt{\frac{2\pi m}{\rho_0}} c \hat{z} \rho(\vec{r},t).
\end{align*}
\]

The “no magnetic monopole” law is clear from the definition of \(\vec{b}\); the Coulomb law states how the presence of vortices create a rotational current, the Faraday law is equivalent to the continuity equation for the fluid, and the counterpart to Ampère’s law derives from the Josephson-Anderson relation implied by the Euler equation. The energy of the system is

\[
E = \frac{1}{4\pi} \int d^2\vec{r} \left( \vec{e}^2(\vec{r},t) + \vec{b}^2(\vec{r},t) \right),
\]

and correspondingly the Poynting vector

\[
\vec{\sigma}(\vec{r},t) = \frac{c}{2\pi} \vec{e}(\vec{r},t) \times \vec{b}(\vec{r},t). \tag{20}
\]

Electromagnetic potentials, \(\vec{a}(\vec{r},t)\) and \(\varphi(\vec{r},t)\), are defined in the usual way, and within a Lorentz gauge we recover the usual wave equations

\[
\begin{align*}
\left( \nabla^2 - \frac{1}{c} \frac{\partial^2}{\partial t^2} \right) \varphi(\vec{r},t) &= -2\pi \rho_v(\vec{r},t), \\
\left( \nabla^2 - \frac{1}{c} \frac{\partial^2}{\partial t^2} \right) \vec{a}(\vec{r},t) &= -\frac{2\pi}{c} \vec{j}_v(\vec{r},t),
\end{align*}
\]

which have the exact solution in \((2+1)\) dimensions \[21\].

\[
\begin{align*}
\varphi(\vec{r},t) &= -\int dt' d^2\vec{r}' \frac{\theta \left( t - t' - \frac{|\vec{r} - \vec{r}'|}{c} \right)}{\sqrt{(t - t')^2 - \frac{|\vec{r} - \vec{r}'|^2}{c^2}}} \rho_v(\vec{r}',t'), \\
\vec{a}(\vec{r},t) &= -\frac{1}{c} \int dt' d^2\vec{r}' \frac{\theta \left( t - t' - \frac{|\vec{r} - \vec{r}'|}{c} \right)}{\sqrt{(t - t')^2 - \frac{|\vec{r} - \vec{r}'|^2}{c^2}}} \vec{j}_v(\vec{r}',t'). \tag{21}
\end{align*}
\]
The integrals for the vector potential’s x and y components can now be done exactly, yielding Bessel functions:

\[ a_x(\vec{r}', t) = \frac{q v c}{c} \int_{-\infty}^{t-r/c} \frac{dt' \sin \omega t'}{\sqrt{(t-t')^2 - \frac{c^2}{c^2}}} = -\frac{qv}{2c} \left[ N_0 \left( \frac{\omega r}{c} \right) \sin \omega t + J_0 \left( \frac{\omega r}{c} \right) \cos \omega t \right], \]

\[ a_y(\vec{r}', t) = -\frac{q v c}{c} \int_{-\infty}^{t-r/c} \frac{dt' \cos \omega t'}{\sqrt{(t-t')^2 - \frac{c^2}{c^2}}} = -\frac{qv}{2c} \left[ -N_0 \left( \frac{\omega r}{c} \right) \cos \omega t + J_0 \left( \frac{\omega r}{c} \right) \sin \omega t \right], \]

and finally

\[ \vec{a}(\vec{r}', t) = \frac{\pi q v}{2c} \left[ \vec{v}_v(t) N_0 \left( \frac{\omega r}{c} \right) + \hat{z} \times \vec{v}_v(t) J_0 \left( \frac{\omega r}{c} \right) \right]. \] (23)

Finally, we utilize the asymptotic formulas for the Bessel functions at large arguments, which yields

\[ \vec{b}(\vec{r}) = q v \sqrt{\frac{\pi \omega}{2c^3 r}} \left[ \vec{v}_v(t) \times \hat{r} \sin \left( \frac{\omega r}{c} - \frac{3\pi}{4} \right) + (\hat{z} \times \vec{v}_v(t)) \times \hat{r} \cos \left( \frac{\omega r}{c} - \frac{3\pi}{4} \right) \right]. \] (24)

It is not necessary to calculate the electric field \( \vec{e}(\vec{r}) \) in order to obtain the Poynting vector, since at large \( r \), the field is locally that of a plane wave, in which case we have

\[ \vec{\sigma} = \frac{c}{2\pi} \vec{b}^2 \hat{r}. \] (25)

On integrating around the circle with radius \( r \) we get the power radiated by a vortex exercising circular motion in an infinite system:

\[ P = \int_0^{2\pi} r d\theta \vec{r} \cdot \vec{\sigma} = \frac{\pi q_v^2 \omega v^2}{4c^2} = \frac{\pi q_v^2 \omega^3 r_0^2}{4c^2}. \] (26)
IV. LIFETIME OF A VORTEX

In Sec. II we found that an off-center vortex performs a circular motion. In the preceding section we saw how, in an infinite and homogeneous system, such motion excites sound waves, which carry away energy from the vortex. In a trapped cloud, the effect of such radiation would be that the vortex move outward towards regions of lower potential energy $U_{\text{eff}}(\vec{r}_0)$, until it finally escapes from the cloud [2,7].

The application of the results of Sec. III on the trapped case may be thought of as a semiclassical approximation. One condition for this approximation to hold is that the precession frequency of the vortex match the attainable excitation frequencies of the cloud, as seen in Eq. (24), where the moving vortex excites sound waves which have the same frequency as the precession.

This requirement, however, is not met in the present case. In a harmonically trapped cloud containing a vortex, all but one of the mode frequencies are greater than or equal to the trap frequency $\omega_t$ [1,22]; the single low-lying mode is identical to an off-center displacement of the vortex [4,6,23].

Comparing the precession frequency with the trap frequency, we find, using Eq. (4),

$$\frac{\omega}{\omega_t} = \frac{g(r_0/R)}{8\gamma^{1/2} \left(1 - \frac{r_0^2}{R^2}\right)^{3/2}}$$

which is always less than one, except very close to the boundary.

We conclude that the semiclassical approximation is never valid for a vortex in a harmonically trapped cloud. This does not, however, necessarily imply that the vortex is stable; only that its lifetime is longer than that implied by the semiclassical approximation.

The results of the two preceding sections can therefore be utilized to calculate a lower bound for the vortex lifetime. The power dissipated from the vortex by phonon emission, Eq. (26), is to be set equal to the rate of motion downhill the potential gradient:

$$P = \frac{dE}{dt} = \frac{Fdr_0}{dt}.$$

We note that $P$ and $F$ are functions of $r_0$. Rearranging terms, we obtain the time $\tau_p$ for the vortex to move from $r_0 = \xi$ to $r_0 = R - \delta$:

$$\tau_p = \int_{\xi}^{R-\delta} \frac{dr_0}{\frac{F(r_0)}{P(r_0)}}$$

This quantity is a lower bound for the lifetime of a vortex originally residing at a distance $\xi$ from the trap center. The upper cutoff, $R - \delta$, is needed in order to avoid unphysical boundary effects, as discussed in connection with Eq. (13). The subscript $p$ is introduced to indicate that this time scale is associated with the radiation of phonons.

Inserting Eqs. (19), (17) and (26) into (27) we get

$$\tau_p = \frac{16m^2\rho_0U_0R^4}{\pi\hbar^3} I,$$

where the dimensionless integral $I$ equals

$$I = \int_{\xi/R}^{1-\delta/R} dx \frac{(1 - x^2)^{3/2}}{x(g(x))^2},$$

with $g(x)$ given by Eq. (18). An exact result for the integral $I$ is easily obtained numerically; the result is shown in Fig. 1, and will be discussed shortly. An estimate can be obtained by noting that the function $g(x)$ for strong coupling is approximately equal to $2\ln(R/\xi) + 1$ over a large range of values of $x$, and that the lowest-order term in $x$ dominates the numerator, whereupon one gets $I \approx \ln(R/\xi)/(2\ln(R/\xi) + 1)^2$ and

$$\tau_p \approx \frac{16m^2\rho_0U_0R^4}{\pi\hbar^3 (2\ln(R/\xi) + \frac{1}{2})^2}.$$  

(28)

This is, in fact, the case for a larger class of trapping potentials, including all power-law potentials and the square well.
Finally, we insert the Thomas-Fermi results (4) to cast the above result in terms of the parameter $\gamma$:

$$
\tau_p \approx \frac{1}{\omega_t} \frac{128 \gamma^{3/2} \ln(4\sqrt{\gamma})}{\pi (\ln(4\sqrt{\gamma}) + \frac{1}{4})^2}.
$$

(29)

**FIG. 1.** Lower-bound estimates for the lifetime of the vortex, multiplied by the trap frequency $\omega_t$. The solid line shows the time scale $\tau_p$ connected with phonon radiation, integrated numerically, while the dashed line is the quantity $\tau_{\delta M}$ connected with broken rotational symmetry, with the parameter $\epsilon$ set to 0.001.

**V. BROKEN ROTATIONAL SYMMETRY: CHANGE OF ANGULAR MOMENTUM**

The other factor which at zero temperature may limit the lifetime of a vortex is deviations from rotational symmetry of the trapping potential. Even very small irregularities in the magnetic or electric fields used to trap the condensed gases in experiments may affect the possibility to keep a vortex in the system.

We therefore consider a BEC in a somewhat deformed trap. The total torque due to inhomogeneities in the density and the trap is

$$
\vec{M} = \int d^2 \vec{r} \rho(\vec{r}) \vec{r} \times \nabla V(\vec{r}),
$$

(30)

where $\rho(\vec{r})$ and $V(\vec{r})$ are the actual (not exactly cylindrically symmetric) density and external potential, respectively. A natural unit for measuring the torque would be the potential energy associated with the trapping potential, which has the same units:

$$
U_{osc} = \int d^2 \vec{r} \rho(\vec{r}) V(\vec{r}),
$$

whose value for a harmonic-oscillator trap and a Thomas-Fermi density profile equals

$$
U_{osc} = \frac{\pi}{12} \rho_0 m \omega_t^2 R^4.
$$

Writing

$$
M = \vec{M} \cdot \hat{z} = \epsilon U_{osc},
$$

we can use $\epsilon$ as a (approximately independent of coupling strength) measure of the relative distortion of the trap and the density profile. Assuming $\epsilon$ to be approximately constant over time, the time scale for the destruction of an initially centrally placed vortex is
\[ \tau_M = \frac{L_0}{M}, \]

where \( L_0 \) is the value of the angular momentum for a system with a central vortex; this is equal to \( \hbar \) times the number of particles \( \nu \) per unit length; and so the time for the vortex to move out is

\[ \tau_M = \frac{6\hbar}{\epsilon m \omega^2 R^2}. \]

VI. CONCLUSIONS

We have arrived at two estimates for vortex lifetime: one connected with phonon radiation, and one due to broken rotational symmetry. The former, \( \tau_p \), is an increasing function of coupling strength \( \gamma \), whereas the latter, \( \tau_M \), decreases with increasing \( \gamma \). This leaves us with a “window” at moderate values of coupling strength, where both time scales are reasonably large.

The lower-bound time scales \( \tau_p \) and \( \tau_M \) are plotted as functions of coupling strength in Fig. 1. Both are to be considered as lower bounds, since the assumptions applied in deriving them are very pessimistic. The parameter \( \epsilon \) is taken to be \( 10^{-3} \), which is an upper bound for the trap deformations attainable experimentally \[24\]. Trap frequencies are often around \( 100 \, s^{-1} \) in experiments; thus \( \tau_p \) will be longer than one second (which is a typical order of magnitude for condensate lifetimes) as long as \( \gamma \) is greater than unity, and \( \tau_M \) is longer than one second for all \( \gamma \leq 5000 \). This leaves us with a large parameter range, easily attainable experimentally, for which a vortex at zero temperature can be considered as long-lived; considering the conservative assumptions made here, the actual region of vortex stability is probably much larger than this analysis indicates.

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