SYMmetric F-CONJECTURE FOR \( g \leq 35 \)

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Abstract. We prove the symmetric F-conjecture describing the ample cone of \( \overline{M}_{0,g}/S_g \) and \( \overline{M}_g \) for \( g \leq 35 \).

1. Introduction

A divisor on the moduli space \( \overline{M}_{g,n} \) of stable \( n \)-pointed genus \( g \) curves is \( F \)-nef if it pairs with all 1-dimensional boundary strata (\( F \)-curves) non-negatively. The (symmetric) F-conjecture says that every \( (S_n \text{-invariant}) \) \( F \)-nef divisor is nef \([GKM02, \text{Conjecture (0.2)}]\). The strong F-conjecture (or, Fulton’s conjecture) for \( \overline{M}_{0,n} \) says that every \( F \)-nef divisor is equivalent to an effective \( \mathbb{Q} \)-linear combination of the standard boundary divisors; it is known up to \( n \leq 7 \) \([Lar12]\) and false for \( n \geq 12 \) \([Pix13]\).

A breakthrough Bridge Theorem of Gibney-Keel-Morrison \([GKM02, \text{Theorem (0.3)}]\) reduces the F-conjecture for \( \overline{M}_g \) to the symmetric F-conjecture for \( \overline{M}_{0,g} \). In this note, we prove the symmetric F-conjecture for \( \overline{M}_{0,g} \) for \( g \leq 35 \) in all characteristics. Previously, the symmetric F-conjecture was proved for \( g \leq 24 \) by Gibney \([Gib09]\). A technical statement of the main result is in Theorem 1, with the main ingredient of the proof being Lemma 4, which is a special case of our earlier \([Fed14, \text{Prop. 6.0.6}]\). However, no familiarity with loc.cit. is assumed.

2. Main result

For an integer partition \( \lambda \vdash n \) of length \( k \), there is a closed immersion \( b_{\lambda}: \overline{M}_{0,k} \to \overline{M}_{0,n} \) where \( b_{\lambda}(C, \{p_i\}_{i=1}^k) \) is obtained from \( C \) by attaching a fixed \( (\lambda_i + 1) \)-pointed (maximally degenerate) rational curve to \( p_i \), and stabilizing. An \( S_n \text{-invariant} \) line bundle \( L \in \text{Pic}(\overline{M}_{0,n})^{S_n} \) is called stratally effective boundary if the pullback \( b_{\lambda}^*L \) is an effective boundary (that is, an effective \( \mathbb{Q} \)-linear combination of boundary divisors) on \( \overline{M}_{0,|\lambda|} \) for all partitions \( \lambda \). By a standard argument \([Mor07, \text{Effective Dichotomy, p.39}]\), a stratally effective boundary divisor is nef. The main result of this note is:

**Theorem 1.** An \( F \)-nef \( S_n \text{-invariant} \) line bundle \( L \) on \( \overline{M}_{0,n} \) is stratally effective boundary if \( b_{\lambda}^*L \) is an effective boundary for all strict partitions \( \lambda \) of \( n \).

Since \( n \) has a strict partition of size > \( k \) only if \( n \geq (k+1)(k+2)/2 \), we obtain:

**Corollary 2.** Suppose the strong F-conjecture holds for \( \overline{M}_{0,m} \) for all \( m \leq k \). Then the symmetric F-conjecture is true for \( \overline{M}_{0,n} \) for all \( n \leq (k+1)(k+2)/2 - 1 \).

In particular, since the strong F-conjecture holds for \( \overline{M}_{0,m} \) for all \( m \leq 7 \), by \([Lar12]\) for \( m = 7 \) and \([FG03]\) for \( m \leq 6 \), we conclude that:

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Corollary 3. Suppose $g \leq 35$. The symmetric $F$-conjecture holds for $\overline{M}_{0,g}$ in any characteristic, and for $\overline{M}_g$ in any characteristic except $2$.

The characteristic restriction in the second part is from [GKM02, Theorem (0.3)].

3. Proof of Theorem 1

Theorem 1 follows from a special case of [Fed14, Prop. 6.0.6], which we now present:

Lemma 4 (Ascent of effectivity). Let $L$ be an $F$-nef $S_n$-invariant line bundle on $\overline{M}_{0,n}$. Suppose $\lambda \vdash n$ is a non-strict partition of size $k$, say, with $\lambda_{k-1} = \lambda_k$. Set $\mu := (\lambda_1, \ldots, \lambda_{k-2}, 2\lambda_{k-1}) \vdash n$. If $b_\mu L$ is an effective boundary on $\overline{M}_{0,k-1}$, then $b_\lambda L$ is an effective boundary on $\overline{M}_{0,k}$.

Preliminaries: Every divisor $D$ on $\overline{M}_{0,m}$ can be written as

$$D = -\sum_{I \sqcup J = [m]} b_{I,J} \Delta_{I,J},$$

where the sum is taken over all 2-part partitions of $[m] := \{1, \ldots, m\}$. Such a partition is proper if $|I|, |J| \geq 2$. Proper partitions enumerate the boundary divisors, and the non-proper 2-part partitions of $[m]$ correspond to the cotangent line bundles via a standard convention that $\Delta_{\{i\},[m]\setminus\{i\}} := -\psi_i$ for $i = 1, \ldots, m$.

The ambiguity in writing $D$ as in (5) is completely described by Keel’s relations in $\text{Pic}(\overline{M}_{0,m})$ (see [AC98, Theorem 2.2(d)] and [Kee92]). We use the following formulation:

Lemma 6 (Effective Boundary Lemma [Fed14, Lemma 2.3.3]). We have

$$-\sum_{I \sqcup J = [m]} b_{I,J} \Delta_{I,J} = \sum_{I \sqcup J = [m]} c_{I,J} \Delta_{I,J} \in \text{Pic}(\overline{M}_{0,m}) \otimes \mathbb{Q},$$

if and only if there is a function $w : \text{Sym}^2 \{1, \ldots, m\} \to \mathbb{Q}$ such that for every 2-part partition $I \sqcup J = [m]$, we have:

$$\sum_{i \in I, j \in J} w(i,j) = c_{I,J} + b_{I,J}.$$ 

In particular, a divisor $D = -\sum_{I \sqcup J = [m]} b_{I,J} \Delta_{I,J}$ is an effective boundary on $\overline{M}_{0,m}$ if and only if there exists a function $w$ such that

$$\sum_{i \in I, j \in J} w(i,j) \geq b_{I,J},$$

for all partitions $I \sqcup J = [m]$, with equality holding for all non-proper partitions.

Proof of Lemma 4. Let $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$ be a function such that

$$L = -\sum_{I \sqcup J = [n]} f(|I|) \Delta_{I,J},$$

where the sum is taken over all 2-part partitions of $[n]$. If $L = \sum_{i=2}^{[n/2]} c_i \Delta_i$ in the standard basis of $\text{Pic}(\overline{M}_{0,n})^{S_n}$, then we can take $f(i) = f(n-i) = -c_i$ for all $i = 2, \ldots, [n/2]$, and $f(0) = f(1) = f(n-1) = 0.$
The function $f$ is symmetric, that is $f(a) = f(n - a)$ for all $a \in \mathbb{Z}/n\mathbb{Z}$. By F-nefness of $L$, for every 4-part integer partition $n = a + b + c + d$, corresponding to the F-curve $F(a, b, c, d) \subset \overline{M}_{0,n}/S_n$, we have an F-inequality:

$$(8) \quad L \cdot F(a, b, c, d) = f(a) + f(b) + f(c) + f(d) - f(a + b) - f(b + c) - f(a + c) \geq 0.$$ 

By the assumption,

$$b^*_\mu L = \sum_{I \sqcup J = [k - 1]} f(\sum_{t \in I} \Delta_{I,J})$$

is an effective boundary on $\overline{M}_{0,k-1}$. By Lemma 6, there is a function $\bar{w}: \text{Sym}^2[k - 1] \to \mathbb{Q}$ such that for every 2-part partition $I \sqcup J = [k - 1]$, we have:

$$(9) \quad \sum_{i \in I, j \in J} \bar{w}(i,j) \geq f(\sum_{t \in I} \mu_t),$$

with equality holding for all non-proper partitions.

Define $w: \text{Sym}^2[k] \to \mathbb{Q}$ by

$$w(i,j) = \begin{cases} \bar{w}(i,j) & \text{if } i, j \in \{1, \ldots, k - 2\} \\ \bar{w}(i,j)/2 & \text{if } i \in \{k - 1, k\} \text{ and } j \notin \{k - 1, k\} \end{cases}$$

$$w(k - 1, k) = f(\lambda_k) - \frac{1}{2} f(2\lambda_k) = f(\lambda_k) - \frac{1}{2} f(\mu_{k-1}).$$

Since

$$b^*_\lambda L = \sum_{I \sqcup J = [k]} f(\sum_{t \in I} \lambda_t) \Delta_{I,J},$$

Lemma 6 implies that $b^*_\lambda L$ is an effective boundary on $\overline{M}_{0,k}$ once we establish the following:

Claim 10. For all 2-part partitions $I \sqcup J = [k]$, we have

$$(11) \quad \sum_{i \in I, j \in J} w(i,j) \geq f(\sum_{t \in I} \lambda_t),$$

with equality holding for all non-proper partitions.

We consider three cases:

Case 1: If $k - 1$ and $k$ belong to the same part, say $J$, then

$$\sum_{i \in I, j \in J} w(i,j) = \sum_{i \in I, j \in \{1, \ldots, k - 1\} \setminus I} \bar{w}(i,j) \geq f(\sum_{t \in I} \mu_t) = f(\sum_{t \in I} \lambda_t),$$

where we used Inequality (9). In particular, the equality holds when $I$ is a singleton.

Case 2: If $J = \{k - 1\}$, or $J = \{k\}$,

$$\sum_{i \in I, j \in J} w(i,j) = w(k - 1, k) + \frac{1}{2} \sum_{j \in \{1, \ldots, k - 2\}} \bar{w}(j,k-1) = f(\lambda_k) - \frac{1}{2} f(2\lambda_k) + \frac{1}{2} f(\mu_{k-1}) = f(\lambda_k).$$
Case 3: Suppose $I = I' \cup \{k-1\}$ and $J = J' \cup \{k\}$, where $I' \sqcup J' = \{1, \ldots, k-2\}$. Then

$$
\sum_{i \in I, j \in J} w(i, j) = f(\lambda_k) - \frac{1}{2} f(2\lambda_k) + \frac{1}{2} \sum_{i \in I', j \in J \cup \{k-1\}} \bar{w}(i, j) + \frac{1}{2} \sum_{i \in I' \cup \{k-1\}, j \in J'} \bar{w}(i, j)
$$

$$\geq f(\lambda_k) - \frac{1}{2} f(2\lambda_k) + \frac{1}{2} f(\sum_{t \in I'} \lambda_t) + \frac{1}{2} f(\sum_{t \in J'} \lambda_t),$$

where we used Inequality (9).

Denoting $A := \sum_{t \in I'} \lambda_t$, $B := \sum_{t \in J'} \lambda_t$, and $a := \lambda_{k-1} = \lambda_k$, we have $A + B + 2a = n$, and (12) translates into

$$
\sum_{i \in I, j \in J} w(i, j) \geq f(a) - \frac{1}{2} f(2a) + \frac{1}{2} f(A) + \frac{1}{2} f(B).
$$

Applying the F-inequality (8):

$$
L \cdot F(A, B, a, a) = f(A) + f(B) + 2f(a) - 2f(A + a) - f(2a) \geq 0,
$$

we conclude that

$$
\sum_{i \in I, j \in J} w(i, j) \geq f(A + a) = f\left(\sum_{t \in I} \lambda_t\right),
$$

as desired. \(\square\)

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