Quantum mechanics on thin cylinders

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ABSTRACT

We discuss the quantum mechanics of particles of arbitrary statistics on an infinite cylinder with and without a magnetic field perpendicular to the surface. In the presence of a magnetic field, the translational symmetry along the cylinder is broken down to a discrete one by the Aharonov-Bohm effect. For interacting fermions in a strong field we get an effective one-dimensional lattice model that in a limit can be mapped on an Ising chain. We also show that a system of anyons on a cylinder are, in a certain limit closely related to the 1-dimensional (integrable) Sutherland model. By order of magnitude estimates we demonstrate that none of these effects are likely to be experimentally observed with present techniques.

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1 Introduction and summary

It is now well known that the quantum theory of identical particles in two
dimensions allows for fractional statistics \[1, 2, 3\]. The statistics is charac-
terized by an angle \(\theta\) that interpolates continuously between bosons \((\theta = 0)\) and
fermions \((\theta = \pi)\). For particles moving on topologically non-trivial surfaces,
there are additional parameters that can be thought of as magnetic fluxes
through the non-contractible loops on the surface. In this case the quantum
mechanics is more complicated, and in general there is an interplay between
the phases due to (fractional) statistics, and the Aharonov-Bohm phases due
to the magnetic fluxes. In particular, for surfaces with handles there are re-
strictions on the possible values of \(\theta\) \[4\]. The simplest example of a non-simply
connected two-dimensional base space is the cylinder, and the aim of this paper
is to analyze the quantum mechanics in this case in some detail.

Since there has been some confusion concerning the possible values of \(\theta\) on
the cylinder \[3\], section 2 contains a brief analysis of the corresponding braid
group. We show that there are no restrictions on \(\theta\), and that there is a single
additional arbitrary angle, \(\phi\), corresponding to the magnetic flux through the
cylinder. We then construct the wave functions corresponding to any values
of \(\theta\) and \(\phi\), thus explicitly showing that both \(\theta\) or \(\phi\) are arbitrary.

Next we consider particles with charge \(q\) on a cylinder (of radius \(R\)) with a
homogenous magnetic field (of strength \(B\)) perpendicular to the surface. This
system looks translationally invariant, but we show that due to the Aharonov-
Bohm effect only the discrete symmetry \(x \rightarrow x + n \ell^2/R\) \((n\ \text{an integer})\) is left,
where \(\ell = (qB)^{-\frac{1}{2}}\) is the magnetic length. This is easily understood; a net
flux of \(2\pi RB\) leaves the cylinder through the surface per unit length, so the
magnetic flux inside the cylinder, which the particles moving on the surface feel
via the Aharonov-Bohm effect, decreases linearly along the axis. If this flux is
zero at some \(x_0\), then adding an integer number \(n\) of flux quanta \(\Phi_0 = 2\pi/q\)
does not change the phases on the surface (and hence not the physics), but
shifts \(x_0 \rightarrow x_0 + n \ell^2/R\) corresponding to the discrete symmetry just mentioned.
linearly related to \(x_0\) and the reflects the periodicity \(\phi \rightarrow \phi + 2\pi n\). The
symmetry is also manifest in the wave functions.

Could effects of breaking translational invariance be detected in experi-
ments? Could we e.g. imagine making a small cylinder in the lab, or perhaps
even using naturally occurring cylinders like buckytubes \[6\]? These questions
are addressed in section 5. Leaving aside the (very hard) technical problem of
how to bring the magnetic field out through the surface of a cylinder, (there is
clearly no effect if this is not the case) we make order of magnitude estimates
of the parameters required to detect any effect. The conclusions are negative;
with present magnetic fields we cannot hope even to insert sufficient amount of flux in a cylinder thin enough to allow for the detection of broken translational symmetry.

Since our system is characterized by two energy scales, $1/mR^2$ and $qB/m$ (or length scales $R$ and $\ell$), it is in general quite complicated and it is of obvious interest to study different limits where one or both of these scales are large compared to other relevant scales like the interparticle interaction energy and thermal excitation energies. In these situations it is natural to think in terms of dimensional reduction, and describe the system by an effective one-dimensional model.

The limit $R$ small and $R \ll \ell$ is rather trivial for bosons or fermions; the particles are simply restricted to move along the cylinder and the resulting theory is again bosons or fermions but on a line. For anyons the situation is a bit more complicated since the corresponding one-dimensional (1d) model also exhibits fractional statistics in a sense that was originally discussed in [1]. In section 4 we use the methods in [7] to derive the statistics parameter $\eta$ of the dimensionally reduced system.

The limit $\ell$ small (i.e. $qB$ large) and $\ell \ll R$ is also rather trivial and simply corresponds to confining the particles to the lowest Landau level. As for the infinite plane we can write down the full anyonic lowest Landau level spectrum and this is done in section 4.3.

The most interesting limit, which we shall study in some detail, is the combination of the two above where $R$ and $\ell$ both are small. In fact there will be a new relevant length scale $a = \ell^2/R$. We show that in this limit a system of interacting fermions is described by a one dimensional lattice model, explicitly exhibiting the discrete symmetry $x \rightarrow x + a$ discussed above. We discuss some properties of this lattice model and show that for any interaction, in the ”sharp” lattice limit $qB \rightarrow \infty$, $R \rightarrow 0$ and $\ell^2/R \rightarrow a$, we obtain a one-dimensional Ising chain.

For anyons moving in an infinite plane penetrated by a magnetic field, the subset of the spectrum (wave functions and energies) corresponding to anyonic continuation of all fermi states in the lowest Landau level, is exactly known [8, 9]. In the spirit of the above, we expect this subset to correspond to a one dimensional model. This is indeed the case, and it has been shown [7, 10, 11] that anyons in the lowest Landau level, with the conserved angular momentum acting as the ‘Hamiltonian’, is equivalent to the Calogero model i.e. to particles on a line interacting via the two body potential $\frac{1}{2}m\omega^2(x_i - x_j)^2 + g/(x_i - x_j)^2$ [12, 13]. In section 4.3 we ask whether a similar construction is possible on the cylinder. Indeed we find that in the ”sharp” lattice limit there is an equivalent integrable 1d model, similar to the Calogero model, but
defined on a circle and with the interparticle interaction $g/\sin^2(x_i - x_j)$. This model, first studied by Sutherland [14], has the same eigenvalues and the same degeneracies as the anyons on the cylinder. We also make some comments about the wave functions, and conclude with a speculation concerning the corresponding system on a torus.

2 Quantum mechanics on a cylinder

Following Leinaas and Myrheim [1], we quantize a system of free particles by constructing wave functions that form a unitary representation of the first homotopy group of the classical configuration space. In the plane this leads to fractional statistics with an arbitrary phase $e^{i\theta}$ associated with the interchange of the coordinates for two particles. On topologically more complicated two dimensional surfaces there are additional generators of the first homotopy group, corresponding to moving particles around non-contractible loops on the surface. Einarsson has shown [4, 15], that for closed surfaces this imposes restrictions on the statistical phase and introduces new phases associated with transport around the handles of the surfaces. He also showed that for anyons ($\theta \neq 0, \pi$) this requires the wave function to have more than one component. In the case of the cylinder there is, in addition to the particle exchange generators, a set of generators associated to paths which take a particle once around the cylinder. When quantizing there are two independent phases to be determined; $e^{i\theta}$ determining the statistics and $e^{i\phi}$ which can be interpreted as an Aharonov-Bohm phase from a constant flux flowing through the cylinder. There is no restriction on either of these phases i.e. neither on the statistics, nor on the flux. The analysis of the braid group on the cylinder is well known but since there has been some confusion concerning the resulting statistics we will briefly describe the construction. For a general discussion we refer to [16].

The classical configuration space of $N$ hard core identical particles on the cylinder is

$$Q^N = (C^N - \Delta)/P^N$$

where $C = S^1 \times R$ is the cylinder, $\Delta = \{x \in C^N \mid x_i = x_j \text{ some } i \neq j\}$ is the set of points of particle coincidence and $P^N$ the permutation group of $N$ particles. The first homotopy group of $Q^N$ is generated by two classes of generators; $\sigma_i$ which exchanges particle $i$ with particle $i+1$, and $\rho_k$ which takes particle $k$ once around the cylinder leaving particles $1..k-1$ to the left and
particles $k+1..N$ to the right of the loop (Fig 1a). The generators $\{\sigma_i, \rho_k\}$ obey the set of relations

$$\sigma_k \sigma_l = \sigma_l \sigma_k \quad |k-l| \neq 1$$  \hspace{1cm} (2)

$$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}$$  \hspace{1cm} (3)

$$\rho_i \rho_j = \rho_j \rho_i$$  \hspace{1cm} (4)

$$\rho_{i+1} = \sigma_i \rho_i \sigma_i$$  \hspace{1cm} (5)

Relation (5), displayed in Fig 1b, shows that we need only one generator, say $\rho_1$, among the $\{\rho_k\}$.

Since higher dimensional representations of the generators correspond to particles with some internal 'statistical' degree of freedom \[1\] we restrict ourselves to one dimensional representations of the braid group (scalar anyons). Hence we represent the generators $\{\sigma_i, \rho_k\}$ with phases, and according to (3) the $\sigma_i$s must all be represented by the same 'statistical' phase which we customarily denote by $e^{i\theta}$. Writing $\rho_1 = e^{i\phi}$, relation (6) immediately implies

$$\rho_k = \rho_1 e^{i(2k-1)\theta} = e^{i(\phi + 2(k-1)\theta)} \quad .$$  \hspace{1cm} (6)

The remaining relations (2) and (4) give no further restriction on $\theta$ or $\phi$. While the interpretation of $\theta$ as the statistics angle is clear, there has been some confusion as to the meaning of relation (6). To clarify this point we recall that if we use single-valued wave functions, the phases representing the elements in the fundamental group are Wilson loops

$$W[\Gamma] = e^{i \int \Gamma \cdot dx A}$$  \hspace{1cm} (7)

where the vector potential $A$ is the connection that defines parallel transport of position eigenstates on the Hilbert space of wave functions. The phases $e^{i\theta}$ originate from a 'statistical' gauge potential, while the phases associated with the $\rho_i$s, where the integration is around the cylinder, are related to the flux through the cylinder. (In a real experimental setup with charged particles, this is of course the real magnetic flux.) It is now easy to understand the $k$-dependence in (6); assume that a flux $\phi$ leaves the left end of the cylinder and that a statistical flux $2\theta$ flows through each anyon (Fig. 2). The path which takes the $k^{th}$ particle around the cylinder will now enclose a total flux $\phi_{tot} = \phi + (k-1)2\theta$, resulting in the phase factor (6).

\[1\] With a slight, but common, abuse of notation we use the same symbols for the generators and their representations

\[2\] In ref. [5] the authors identify the generators $\rho_1$ and $\rho_N^{-1}$, and erroneously conclude
For anyons in a plane, characterized by an arbitrary statistical angle $\theta$, there exist a singular gauge transformation which maps a system of bosons (with single valued wave functions) interacting via a $\theta$-dependent gauge potential, onto a system of non-interacting (i.e. no gauge potential) anyons described by multi valued wave functions \[17, 3\]. The configuration space is the set of unordered $N$-tuples of complex numbers $x = \{z_1..z_N\}$ (where $z_k = x_k + iy_k$) of which no two are identical. In this notation the transformation reads,

$$\Psi \rightarrow \Psi_\theta = \left(\frac{\gamma}{|\gamma|}\right)\nu \Psi \equiv g\Psi$$

$$A \rightarrow A + g^{-1}dg$$

where $\nu = \theta/\pi$ and

$$\gamma = \prod_{j<k}(z_j - z_k)$$

leading to multi valued wave functions of the form

$$\Psi(z_1..z_N) = \gamma^\nu \Psi_B(z_1..z_N)$$

where $\Psi_B$ is single valued (bosonic).

We now show that a similar construction is possible also on the cylinder. On a cylinder of radius $R$, the points $z_k$ and $z_k + 2\pi i R$ are identified. The configuration space $Q^N$ in \[1\] is again unordered $N$-tuples of complex numbers but with the identification of $z_k$ with $z_k + 2\pi i R$. The gauge transformation corresponding to (8) reads,

$$\Psi \rightarrow \Psi_{\theta,\phi} = \left(\frac{\gamma_c}{|\gamma_c|}\right)^{\nu_1} \exp\left(i\frac{\nu_2}{2R} \sum_i\right)\Psi,$$

where $\nu_1 = \theta/\pi$, $\nu_2 = [\phi - (N - 1)]/\pi$ and

$$\gamma_c = \prod_{j<k} \sinh\left(\frac{z_j - z_k}{2R}\right).$$

that there is a constraint on the possible statistics of particles moving on the cylinder. It should be clear from the analysis of the braid group and the subsequent discussion that this identification is incorrect. In \[11\] we also explicitly construct wave functions on the cylinder for any value of the statistical parameter $\theta$.

\[3\] A mean field Chern-Simons description of anyons on a cylinder has been studied in ref. \[18\].
The factor $\gamma_c$, which generalizes $\gamma$ to the cylinder, has correct periodicity to ensure that the gauge transformation \( (\ref{11}) \) is well defined on the cylinder. For relative distances much smaller than $R$, $\gamma_c$ approaches $\gamma$ and the dependence on the phase $\phi$ disappears. Thus we recover the result \( (\ref{8}) \) appropriate to the infinite plane.

To check that \( (\ref{11}) \) indeed is correct we calculate the phases representing the generators $\sigma_i$ and $\rho_i$ using the multi-valued wave functions \( (\ref{11}) \). First we exchange particle $j$ with particle $j+1$ counter clockwise by writing $z_j - z_{j+1} \equiv z = |z|e^{i\phi}$ and increasing the argument $\phi$ with $\pi$. Under this transformation $\gamma_c \rightarrow e^{i\pi\nu_1} \gamma_c$ while the center of mass piece in the exponential and the bosonic part $|\gamma_c|^{-\nu_1} \Psi_B$ are invariant so that

\[
\Psi_{\nu_1,\nu_2} \overset{\sigma_j}{\rightarrow} e^{i\pi\nu_1} \Psi_{\nu_1,\nu_2}
\] (13)

as required. To get $\rho_1$ we consider $z_j \rightarrow z_j + 2\pi i R$ with $Re(z_j) < Re(z_k)$ all $k \neq j$ (take particle $j$ once around the cylinder with all other particles to the right of the path). Then

\[
\begin{align*}
\gamma_{\nu_1} & \rightarrow \rho_1 \\
\frac{e^{i\pi\nu_2} \sum_i (z_i - \bar{z}_i)}{i} & \rightarrow e^{i\pi\nu_1} e^{i\pi\nu_2} \frac{e^{i\pi\nu_1} \sum_i (z_i - \bar{z}_i)}{i} \\
\Psi_{\nu_1,\nu_2} & \rightarrow e^{i\pi\nu_2 + (N-1)\nu_1} \Psi_{\nu_1,\nu_2}
\end{align*}
\] (14)

so that

\[
\Psi_{\nu_1,\nu_2} \overset{\rho_1}{\rightarrow} e^{i\pi\nu_2 + (N-1)\nu_1} \Psi_{\nu_1,\nu_2}
\] (15)

which completes the demonstration.

3 Cylinder with constant $\vec{B}$ field

3.1 General considerations

Let us now put a homogeneous magnetic field $B$ perpendicular to the surface of the cylinder. The vector potential

\[
\vec{A}(x, y) = (x - c)\hat{y}
\] (16)

where $c$ is an arbitrary constant, generates the required magnetic field and obeys the periodicity condition $\vec{A}(x, y) = \vec{A}(x + 2\pi R, y)$. One might think that

\[\text{If we use a description where the gauge potential is zero the phases are encoded in the multi-valued wave functions; when particle coordinates are transformed along a closed path $\Psi$ is to pick up the corresponding phase.} \]
c has no physical significance, since it can be shifted by a gauge transformation \( \vec{A} \to \vec{A} + \vec{\nabla} \Lambda \) with \( \Lambda = y \cdot \text{const.} \). This conclusion is incorrect however, since \( \Lambda \) is not single valued on the cylinder. Calculating the Wilson loop \( W[\Gamma] \) for (16) around the cylinder for fixed \( x \) (Fig. 3a)

\[
\exp(iq \oint_{\Gamma_x} d\vec{r} \cdot \vec{A}) = e^{2\pi i q R B (x - c)}
\]

we see that a change in \( c \) is equivalent to a change in the value of the Wilson loop. Indeed, defining \( e^{i\phi} \) as the Wilson loop around \( \Gamma_{x=0} \) we have

\[
\phi = -2\pi R q B c.
\]

To understand (18) it is again instructive to consider a real magnetic field configuration in three dimensions as in Fig. 3b. Then the integral in (17) measures the flux \( \Phi(x) \) flowing through the cylinder at \( x \). From the figure it is clear that \( \Phi(x) \) is linear in \( x \) since a flux of \( 2\pi R B \) leaves the cylinder per unit length by crossing the cylinder surface and thus \( c \) is the position where the net flux through the cylinder is zero. The relation (17) implies that translation invariance is broken by the Aharonov-Bohm effect, since the phase-factor is a physical observable. It is also clear that the translational symmetry is broken down to the discrete symmetry \( x \to x + na \), which is also manifest in the wave functions. Note, that adding an integer number \( n \) of flux quanta through the cylinder (i.e. \( \Phi(x) \to \Phi(x) + \frac{2\pi}{q} n \)) leaves the Wilson loop (17) invariant. Thus this does not change the physics on the surface but shifts \( c \to c + na \).

These considerations emphasizes the importance of boundary conditions. The cylinder can be viewed as an infinite strip where \( 0 \leq y < 2\pi R \), with periodic boundary conditions in \( y \). If we instead had imposed, say, hard wall boundary conditions (or confined \( y \) close to a constant by some potential), translation invariance would have remained unbroken. It is also necessary that the cylinder surface is penetrated by a net flux. To see this, consider a magnetic field which is independent of \( x \) but with an arbitrary \( y \)-dependence. Then if (and only if)

\[
\int_0^{2\pi R} dy B(y) = 0
\]

we can pick the gauge

\[
A_x = \int_0^y dy' B(y')
\]

which gives a fully translationally invariant Hamiltonian and wave functions;

\[
H = \frac{1}{2m} \left[ (i \partial_x + q A_x(y))^2 - \partial_y^2 \right]
\]

\[
\psi(x, y) = e^{ikx} f(y)
\]
with $k$ continuous.

### 3.2 One-particle solutions

Choosing $c=0$ in (16) the Hamiltonian is

$$
\mathcal{H}_0 = -\frac{1}{2m} \sum_j \left[ \vec{\nabla}_j - iq\vec{A}_j \right]^2 ,
$$

$$
\vec{A}_j = B x_j \hat{y}
$$

where $j$ labels the particles. Measuring energy in units of the cyclotron frequency $\omega_c = \frac{qB}{m}$, the one particle eigenstates of (22) are

$$
\psi_{m,n}(x,y) = N_m e^{-\frac{x-k_n\ell^2}{R^2}^2} e^{ik_ny} H_m(x-k_n\ell^2) ,
$$

$$
\begin{align*}
& m = 0, 1, 2.. ; \quad n = 0, \pm 1, \pm 2.. , \\
& k_n = \frac{n}{R} ,
\end{align*}
$$

where $H_m$ are Hermite polynomials, the integers $m$ labels the Landau levels with energies $m + \frac{1}{2}$ and the $N_m$’s are normalization constants. The quantization of the wave number $k$ is from demanding single valued wave functions, $\Psi(x,y) = \Psi(x,y+2\pi R)$. The properly normalized lowest Landau level ($m=0$) states

$$
\psi_n(x,y) = \left( \frac{1}{4R^2 \pi^{3} l^2} \right)^{\frac{1}{4}} e^{i\frac{2\pi n y}{R}} e^{-\frac{1}{R^2} (x-na)^2} .
$$

are gaussians centered at positions $x_n = na$. Notice that one cannot form any translationally invariant linear combination of these states. Indeed, any translated state $\psi(x-\alpha,y)$ will be a superposition of states from different Landau levels and hence no longer an eigenstate to (22). Instead (24) reflects the discrete symmetry $x \rightarrow x+a$ discussed above.

### 4 Dimensional reduction

#### 4.1 General discussion

As discussed in the introduction, the presence of the two length scales $R$ and $\ell$ allow several possibilities for dimensional reduction, and we will start by recalling some facts about dimensional reduction in anyon systems. For a more comprehensive treatment, see [4]. Naturally we would expect that the one
dimensional model would "remember" the statistics of the original particles. The situation is, however, a bit complicated since there is no unique definition of statistics in 1+1 dimensions. One possibility is to observe that for hard core particles, we can choose a particular ordering of the particles thus reducing the configuration space of \( N \) particles to \( \mathbb{R} \times \mathbb{R}^{N-1} \) where \( \mathbb{R}_+ \) is the the real half line. Specifying the statistics now corresponds to choosing a boundary condition that guarantees unitarity at the coincidence points. As usual it is sufficient to study the two body problem, and in the (relative) coordinate \( x \) unitarity is ensured by the class of boundary conditions

\[
\Psi(0) = \eta \Psi'(0)
\]  

which makes the probability current vanish at \( x = 0 \). With this definition \( \eta = 0 \) corresponds to bosons and \( \eta^{-1} \) to fermions. This approach has been pursued in [7] where it is shown that when 2\( d \) anyons are confined by some strong potential to move only along a line, there is a non-trivial connection between the statistical angle \( \theta \) of the anyons and the statistical parameter \( \eta \) of the resulting effective 1\( d \) system. This also holds true for imposing periodic boundary conditions at \( y = 0 \) and \( y = 2\pi R \), which is of course nothing but the \( R \ll \ell \) limit of the cylinder problem studied in this paper. A calculation very similar to that in [7] gives, for small \( R \),

\[
\eta^{-1} = R \cot^2 \left( \frac{\theta}{2} \right) \frac{\zeta(3)}{\pi^2} \sim 0.823 R \cot^2 \left( \frac{\theta}{2} \right)
\]  

(26)

to first order in \( \cot \left( \frac{\theta}{2} \right) \) (\( \zeta(z) \) being the Riemann zeta-function). The flux \( \Phi \) through the cylinder does not enter (26) since for particles with identical charge, only the center of mass movement is affected by \( \Phi \); no flux flows in the 'relative' cylinder. Would we instead consider particles with different charges (i.e. an anyon anti-anyon pair) the relation (26) will have a non-trivial \( \Phi \) dependence.

In the presence of a strong magnetic field there is another possible dimensional reduction corresponding to the restriction to the lowest Landau level. In the limit \( \ell \ll R \) the effects of the finite radius of the cylinder can be neglected and the general analysis in [7] is applicable also to the cylinder case. However, as mentioned in the introduction, there is non-trivial combination of the two dimensional reduction discussed above, namely

\[
qB \rightarrow \infty \quad R \rightarrow 0
\]  

(27)
\[
\frac{\ell^2}{R} \to a
\]

where \(a\) will be identified as the lattice constant in the resulting discrete model. First we study the limit (27) for interacting fermions and then for non-interacting anyons.

4.2 Fermions on the cylinder

If the Landau energy \(\bar{\hbar}\omega_c\) is much larger than the typical interaction energy between the fermions the mixing of higher Landau levels in the lowest energy states will be suppressed. We thus restrict the Hilbert space to the lowest Landau level states. The fermionic operators \(c_n^\dagger\) and \(c_n\) obey the canonical anticommutation relations

\[
[c_i, c_j^\dagger]_+ = \delta_{ij}
\]

and create and annihilate a fermion in the \(\psi_n\) state respectively. Adding a generic two body interaction \(\sum_{i \neq j} V(\vec{r}_i - \vec{r}_j)\) to (22) the full second quantized Hamiltonian becomes

\[
\hat{H} = \frac{1}{2} \sum_n c_n^\dagger c_n + \frac{1}{2} \sum_{i,j,k,l} V_{ij,kl} c_i^\dagger c_j^\dagger c_l c_k; \quad (29)
\]

\[
V_{ij,kl} = \int d\vec{r}_1 d\vec{r}_2 \psi_i^*(\vec{r}_1) \psi_j^*(\vec{r}_2) V(\vec{r}_1 - \vec{r}_2) \psi_k(\vec{r}_1) \psi_l(\vec{r}_2). 
\]

Fourier expanding in the \(y\)-direction

\[
V(\vec{r}) = \sum_{m=-\infty}^{\infty} V^m(x) e^{im\bar{y}/R} \quad (30)
\]

the Hamiltonian (29) becomes

\[
\hat{H} = \frac{1}{2} \sum_n c_n^\dagger c_n + \frac{1}{2} \sum_{i,j,m} e^{-m^2\frac{a^2}{2R}} V^m_{i-j} c_i^\dagger c_j^\dagger c_{i+m} c_{i-m} \quad (31)
\]

where

\[
V^m_s = \frac{1}{\sqrt{8\pi}} \int_{-\infty}^{\infty} dx e^{-\frac{(x-s-m)^2}{8a^2}} V^m(x). \quad (32)
\]

\(^5\)We assume that the number of particles is not sufficient to fill one Landau level, i.e. the filling factor is less than one.

\(^6\)The fermions are taken to be spinless. If spin is included the \(c_n\) operators come with an additional spin index and a term corresponding to a Zeeman term in the Hamiltonian.
The first term in (31) counts the total number of fermions giving the magnetic zero point energy $N/2$ which acts like a chemical potential. The second term induces pair-wise hopping; the fermion at site $i$ moves $m$ positions to the right while the fermion at site $j$ moves $m$ positions to the left. The hopping amplitude $V_{i,j}^m$ depends on the distance $i - j$ between the fermions and the hopping distance $m$, see Fig.4. The exact form of $V_s^m$ depends on the potential $V(\vec{r})$ but the general form of the interaction part of the Hamiltonian is easy to appreciate; since in (23) the lattice position $n$ of a particle is related to the canonical momentum $p_y$ through $p_y = k_n = n/R$ any single particle hopping is forbidden by conservation of total $P_y = \sum p_y$ and hence the linear momentum along the cylinder is quenched by the magnetic field. For the same reason any two particle interaction must move the two particles the same number of lattice sites but in opposite directions. From the translational invariance of $V(\vec{r}_1 - \vec{r}_2)$ it finally follows that the amplitude depends only on the hopping distance $m$ and the particle separation $i - j$. The conservation of $P_y$ hence implies that there can be no net charge transport along the cylinder.

In the limit (27), all $m \neq 0$ contributions get exponentially damped and we are left with,

$$\hat{H} = \frac{1}{2} \sum_i n_i + \frac{1}{2} \sum_{i \neq j} V_{i-j}^0 n_i n_j, \quad (33)$$

so any distinct distribution of fermions on the lattice sites is an eigenstate. The system is classical in the sense that the kinetic energy only contributes the constant $N/2$ (which vanishes when $\hbar \to 0$) so that the Hamiltonian is purely potential and lacks dynamics. From general arguments follows that at finite temperature all lattice sites are occupied with equal probability and the correlation function $C_{ij} = \langle n_in_j \rangle - \langle n_i \rangle \langle n_j \rangle$ decays exponentially, i.e. there is no spontaneous symmetry breaking. This will however not destroy the lattice structure itself since the lattice is not formed dynamically by inter-particle forces but instead enforced by the external gauge potential. Indeed, shifting any combination of the eigenstates (24) by a fraction of the lattice constant $a$ will involve the mixing of higher Landau level states and thus increase the energy. The periodicity of the lattice hence persists at finite temperature and may be detected as a periodic charge density.

The Hamiltonian (33) is of course nothing but the familiar lattice gas model and if only nearest neighbour interaction is included we recover the 1d Ising chain through the identification $S_i^z = n_i - \frac{1}{2}$. 

\[12\]
4.3 Anyons on the cylinder

It is easily shown that any function

\[ \Psi(x_j, y_j) = f(z_1, \ldots z_N) \exp \left[ -\frac{1}{2\ell^2} \sum_j x_j^2 \right] \]  

(34)

where \( f(z_j) \) is holomorphic in \( z_j = x_j + iy_j \) is an eigenfunction to the Hamiltonian (22) with energy \( Nh\omega_c/2 \). From this and (11) we can immediately find the anyonic wave functions corresponding to the lowest Landau level on a cylinder:

\[ \Psi\{n_j\}(x_j, y_j) = \gamma_c S \left\{ \exp \sum_{j=1}^N \left[ \frac{n_j}{R} y_j + \frac{1}{2Ra}(x_j - an_j)^2 \right] \right\} \]  

(35)

Here \( \gamma_c \) is given by (12) and \( S \) means symmetrization of the particle indices. It is interesting to ask if there is some effective one dimensional model corresponding to these solutions. If the anyons move in an infinite plane this is indeed the case and the equivalent model is the Calogero model, i.e. particles interacting via pairwise harmonic and \( 1/x^2 \) forces\[12, 13\]. The Calogero model is integrable, and the spectrum (energies\[10\] and wave functions\[11\]) of the Calogero Hamiltonian is the same as that for the conserved total angular momentum operator for the \( N \) anyon system\[14\].

In the case of a cylinder there is in general no such correspondence, but we shall show that in the limit (27) the anyon problem is equivalent to another integrable one-dimensional quantum system, namely the Sutherland model on a circle. The Sutherland model is defined by the Hamiltonian

\[ H_{Suth} = -\sum_{j=1}^N \partial_{\vartheta_j}^2 + \sum_{j \neq k}^N \frac{g}{\sin^2(\vartheta_j - \vartheta_k)} \]  

(36)

where the angular variables \( \vartheta_j \) range from 0 to \( 2\pi \)\[14, 20\]. The spectrum of (36) is given by

\[ E(n_j) = 4 \left\{ \sum_j n_j^2 + \lambda \sum_{j>k}^N (n_j - n_k) + \frac{\lambda^2}{12} N(N^2 - 1) \right\} \]  

(37)

7 With anyons in the lowest Landau level we mean anyonic states continuing Bose states where all particles are in the lowest Landau level\[1\].

8 Note that it is the conserved total angular momentum operator in the lowest Landau level anyon system which corresponds to the Hamiltonian of the Calogero model. The spectrum of the Hamiltonian in the former system is of course trivial since all lowest Landau level states have the same energy of \( Nh\omega_c/2 \).
where $\lambda = \frac{1}{2}[1 + \sqrt{1 + 2g}]$ and the quantum numbers $n_j$ are ordered integers with $n_j > n_k$ for $j > k$ \[14\]. To understand the connection to anyons on a cylinder it is useful to consider the guiding center coordinates for the cylinder problem, defined as

$$
X_j = x_j + \frac{m}{qB} v_{y_j} = \ell^2 p_{y_j} \quad (38)
$$

$$
Y_j = y_j - \frac{m}{qB} v_{x_j} = y_j - \ell^2 p_{x_j}
$$

with the commutator

$$
[X_j, Y_k] = -i\ell^2 \delta_{jk} \quad (39)
$$

In the plane, using radial gauge, the conserved (canonical) angular momentum $L$, can be expressed as $L = \frac{1}{2\ell^2} \sum_i (X_i^2 + Y_i^2) - H/\omega_c$, and in the lowest Landau level, the last term is just $1/2$. This amounts to observing that, in the radial gauge, the guiding center radius $R_i^2 = X_i^2 + Y_i^2$ is essentially the angular momentum, and thus a constant of motion. It is this conserved quantity that, can be reinterpreted as the Hamiltonian for the equivalent one dimensional model - in this case the Calogero model. On the cylinder, the (canonical) angular momentum is not conserved, but we have another conserved quantity, namely the total momentum in the $y$-direction, which according to (38) is nothing but the $x$-component of the guiding center coordinate. Note that since $Y_j \in [0, 2\pi]$, $X_j$ is quantized,

$$
X_i^{n_i} = n_i \frac{\ell^2}{R} \quad . \quad (40)
$$

Now we have a very natural guess for the 'Hamiltonian' for the equivalent one dimensional model, namely

$$
H_e = \frac{1}{2\ell^2} \sum_{j=0}^{N} X_j^2 = -\frac{\ell^2}{2} \sum_{j=0}^{N} \frac{\partial^2}{\partial Y_j^2} = -\frac{\ell^2}{2} \sum_{j=0}^{N} \frac{\partial^2}{\partial y_j^2} \quad . \quad (41)
$$

$H_e$ commutes with the Hamiltonian and it is the $Y_i \to 0$ limit of the quantity $L = \frac{1}{2\ell^2} \sum_i (X_i^2 + Y_i^2)$ discussed above. The eigenfunctions of $H_e$ in the $Y$ representation (where $X_j = (i\ell^2) \frac{\partial}{\partial y_j}$) are given by

$$
\Psi_{(n_j)}(Y_j) = N \exp \sum_{j=1}^{N} \left[ i \frac{n_j}{R} Y_j \right] = N \exp \sum_{j=1}^{N} \left[ i \frac{n_j}{R} y_j - in_j \frac{\ell^2}{R} p_{x_j} \right] \quad (42)
$$
In the \((x, y)\) representation this becomes

\[
\Psi_{\{n_j\}}(x_j, y_j) = N \exp \left( \sum_{j=1}^{N} \left[ \frac{i n_j}{R} y_j \right] + \frac{\nu}{2R} y_j - y_k \right) \prod_{j=1}^{N} \delta(x_j - n_j a) \quad (43)
\]

Note that these wave functions can also be obtained from (23) by taking the limit (27), and using the representation of the delta function as a limit of a gaussian.

We now turn to the anyon problem. If we order the \(x\) coordinates so \(x_j < x_k\) for \(j < k\), it is easy to show that up to an unimportant overall phase factor

\[
\gamma_c \to \prod_{j>k} e^{\frac{x_j - x_k}{2R}} e^{i\nu \frac{y_j - y_k}{2R}} \quad \text{for } R \to 0 \quad (44)
\]

so the anyonic wave function (35) becomes

\[
\Psi_{\{n_j\}}(x_j, y_j) = N \exp \left( \sum_{j=1}^{N} \left[ \frac{i n_j}{R} y_j \right] + \frac{\nu}{2R} y_j - y_k \right) \prod_{j=1}^{N} \delta(x_j - n_j a) \quad (45)
\]

After some combinatorics, it follows that (45) are indeed eigenfunctions for the effective Hamiltonian \(H_e\) in (41) and that the spectrum is

\[
E_{\{n_j\}} = \frac{\ell^2}{2R^2} \left[ \sum_{j=1}^{N} n_j^2 + \nu \sum_{j>k} (n_j - n_k) + \nu^2 N(N^2 - 1) \right] \quad (46)
\]

which is exactly the spectrum (37) obtained by Sutherland, if we identify \(\lambda = \nu\) and measure "energy" in units of \(\frac{\ell^2}{2R^2}\). Note that it follows from the delta function in (45) and the assumption of the ordering of the \(x_j\) that the ordering of the quantum numbers \(n_j\) is the same as in the Sutherland spectrum (37).

When it came to energy eigenvalues and degeneracies, it was fairly easy to establish the equivalence between the Calogero model and anyons in the lowest Landau level, both for the infinite plane and the cylinder. It is much harder to obtain the corresponding connection between the wave functions. In fact, this has been done only recently [11] in the case of the plane, using an operator approach which relies heavily on that the spectrum can be generated by step operators. Since the spectrum of the Sutherland model is not equally spaced we cannot expect any simple generalization of this method to be applicable, and we have not been able to understand the connection between the wave functions. Nevertheless, the equivalence of the spectra strongly indicates that
in the limit (27), anyons on a cylinder is just another representation of the Sutherland model, and with the advantage of having very simple wave functions. Finally we note that the proof of the equivalence between the anyon system and the Calogero model given in [11] is based on the observation that they are just two equivalent representations of the same algebra. Thus the proof of equivalence did not require the knowledge of the wave functions. In [21] Polychronakos has given the algebra underlying the Sutherland model, and it would be interesting to check whether the same algebra is realized by the anyon problem considered above. It is also interesting to speculate what happens if we compactify both directions, i.e. consider anyons on a torus. In this case there are only a finite number of states in the lowest Landau level, so we expect the corresponding equivalent one-dimensional model to be a lattice model with a finite number of points determined by the total flux out of the cylinder. In ref. [7] we guessed that such a model would be of the type considered by Haldane [22, 23] and Shastry [24], who constructed a lattice version of the Sutherland model. The results of this paper supports that guess, and we can furthermore assert that the correspondence between anyons on a torus and a one dimensional lattice model should be looked for in the limit (27).

5 Order of magnitude estimates and experimental implications

In this section we ask whether any of the effects above could be detected experimentally. Having in mind the Buckytubes [6] we may also ask whether a microscopic cylindrical topology could affect any magnetic properties of a macroscopic sample put in an external magnetic field. Recall from the end of section 3.1 that in order to get flux induced breakdown of the translational symmetry there must be a net magnetic flux passing through the cylinder surface. It should be obvious that a homogeneous magnetic field will not do and that what is needed is rather something like the field at the end of a thin solenoid. This almost immediately rules out the possibility to see any effect in a system of Buckytubes since this would require a magnetic field which is strongly inhomogeneous on the length scale of 1 nm. \[9\] Ignoring for a moment the technical difficulties of actually taking the magnetic field out through the surface of the cylinder we now estimate on which scales a lattice can form.

\[9\] In fact, even to get a quantum of magnetic flux into the buckytube would require a magnetic field \( \sim 100T \).
For the lattice to be significant we must require that $\ell < a$, i.e. $R < \ell$ or $qBR^2 < 1$. For given radius $R$ this puts an upper bound on the field strength $B_\perp$. For the lattice structure to be of dynamical significance the magnetic energy must in addition be larger than (or at least of the same magnitude as) the typical (Coulomb) interaction energy between the particles and for the Landau level structure to be robust against thermal fluctuations $kT \leq \hbar \omega_c$ which puts an absolute lower bound on $B_\perp$. Finally one magnetic flux quantum per lattice site must be inserted at the end of the cylinder in order to be taken out homogeneously along the cylinder. Ignoring the difficulties of extracting the magnetic flux smoothly through the cylinder surface these conditions summarize to

\[
\begin{align*}
\pi R^2 B_\parallel &\geq N\Phi_0 \quad \text{(47)} \\
R &\leq a \quad \text{(48)} \\
kT &\leq \hbar \omega_c \quad \text{(49)} \\
n &\leq \rho_{LL} \
\end{align*}
\]

Here $B_\parallel$ is the field strength injecting a flux $\Phi = \pi R^2 B_\parallel$ into the cylinder, $N$ is the number of lattice sites and $n$ the charge carrier density. Of course, we also assume that the gap to transverse excitations is big enough for the system to be effectively two-dimensional. Starting with the lower bound on the magnetic field and comparing with the QHE the Landau level structure is certainly of importance at magnetic field strengths around $0.01 - 1$T and temperatures below 1K. Taking $R \sim a_0$, the largest possible value of $R$, creating a lattice of the order of a couple of hundred lattice sites would require

\[
B_\parallel \sim 100B_\perp . \quad \text{(51)}
\]

Thus $B_\parallel \sim 10$T when radius $R \sim a \sim 10^{-7}$m. An even smaller radius, permitted by (48) would increase $B_\parallel$ through (17). This effectively rules out the possibility for a lattice structure to form on buckytubes (where the typical radii are of the order of one nm). While the radius $R \sim a_0 \sim 10^{-7}$m, implying a layer thickness $\Delta z \sim 10\AA$, is not unrealistic the difficulty in taking out the injected flux through the surface of the cylinder makes an experiment presently impossible, since already the required $B_\parallel$ is on the limit of accessible magnetic fields.

\footnote{If the electron system is to be described by a 1d lattice model this condition sharpens to $kT \ll \hbar \omega_c$ so that the lowest Landau level approximation is valid. Nevertheless the weaker condition $kT \leq \hbar \omega_c$ is sufficient for the Landau level structure to survive and a charge density to form.}
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Figure captions

**Fig 1a:** The generators to the braid group on the cylinder; $\sigma_i$ exchanges particles $i$ and $i + 1$ counter clockwise and $\rho_k$ takes one particle once around the cylinder, leaving $k - 1$ particles to the left of the path.

**Fig 1b:** The trajectory corresponding to the group element $\sigma_i \rho_i \sigma_i$ can continuously be transformed into the trajectory representing $\rho_{i+1}$, thus implying the relation (5).

**Fig 2:** If the phases associated with the braid group generators are represented by Wilson loops, the phase $e^{i\theta}$ is induced by a statistical magnetic flux of $2\theta$ tied to each particle. From the picture it is clear that the flux enclosed by the path $\rho_k$ depends on $k$ as in (5).

**Fig 3a:** The path $\Gamma_x$.

**Fig 3b:** A real magnetic field configuration generating a constant magnetic flux through the surface of the cylinder. The flux, as measured by the Wilson loop $\Gamma_x$, increases linearly in $x$ and $x = c$ is the point where this is zero.

**Fig 4:** The two particle hopping induced by $V_{i-j}^m$. 

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