On SUSY-QM, fractal strings and steps towards a proof of the Riemann hypothesis

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Abstract

We present, using spectral analysis, a possible way to prove the Riemann’s hypothesis (RH) that the only zeroes of the Riemann zeta-function are of the form \(s = 1/2 + i\lambda_n\). A supersymmetric quantum mechanical model is proposed as an alternative way to prove the Riemann’s conjecture, inspired in the Hilbert-Polya proposal; it uses an inverse eigenvalue approach associated with a system of \(p\)-adic harmonic oscillators. An interpretation of the Riemann’s fundamental relation \(Z(s) = Z(1 - s)\) as a duality relation, from one fractal string \(L\) to another dual fractal string \(L'\) is proposed.

1 Introduction

Riemann’s outstanding hypothesis (RH) that the non-trivial complex zeroes of the zeta-function \(\zeta(s)\) must be of the form \(s = 1/2 \pm i\lambda_n\), remains one of the open problems in pure mathematics. The zeta-function has a relation with the number of prime numbers less than a given quantity and the zeroes of zeta are deeply connected with the distribution of primes [12]. The spectral properties of the zeroes are associated with the random statistical fluctuations of the energy levels (quantum chaos) of a classical chaotic system [1]. Montgomery [3] has shown that the two-level correlation function of the distribution of the zeroes is the same expression obtained by Dyson using random matrices techniques corresponding to a Gaussian unitary ensemble. See also [1], [8], [24] and [9]. An extensive compilation on zeta related papers can be found at [26].

One can consider a \(p\)-adic stochastic process having an underlying hidden Parisi-Sourlas supersymmetry, as the effective motion of a particle in a potential which can be expanded in terms of an infinite collection of \(p\)-adic harmonic oscillators with fundamental (Wick-rotated imaginary) frequencies \(\omega_p = i \ln p\) (\(p\) is a prime) and whose harmonics are \(\omega_{p,n} = i \ln p^n\) (See [2]). This \(p\)-adic harmonic oscillator potential allowed to determine a one-to-one correspondence between the amplitudes of oscillations \(a_n\) (and phases) with the imaginary parts of the zeroes of zeta \(\lambda_n\), after solving a inverse eigenvalue problem.
Pitkänen [16] proposed an strategy for proving the Riemann hypothesis inspired by orthogonality relations between eigenfunctions of a non-Hermitian operator that describes superconformal transformations. In his approach the states orthogonal to a vacuum state correspond to the zeroes of Riemann zeta. However, a proof was not given.

The contents of this work are the following. In section 2 we consider the SUSY-QM model approach to the Hilbert-Polya proposal. In section 3 we define a different operator than Pitkänen’s, expressed in terms of the Jacobi theta series, and we find the orthogonality relations among its eigenfunctions, and finally we prove some theorems and present a novel approach to prove the RH. In section 4 a discussion of the fractal string construction, in relation to the Riemann zeta-function, given by Lapidus and Frankenhuyzen and a possible generalization is presented. Finally, some concluding remarks concerning the multifractal distribution of prime numbers found by Wolf and the distribution of lengths of fractal strings are provided.

2 A supersymmetric potential

The Hilbert-Polya idea to prove the RH is the following [3]. If the zeroes of the Riemann zeta function are $s_n = 1/2 + i\lambda_n$, then it must exist a Hermitian operator $\hat{T}$ such that the $s_n$ are complex eigenvalues of the operator $1/2 + i\hat{T}$, in other words, the real values $\lambda_n$ are eigenvalues of $\hat{T}$. Here we propose a way to construct such operator by using SUSY-QM arguments.

One of us [2], was able to consider a $p$-adic stochastic process having an underlying hidden Parisi-Sourlas supersymmetry, as the effective motion of a particle in a potential which can be expanded in terms of an infinite collection of $p$-adic harmonic oscillators with fundamental (Wick-rotated imaginary) frequencies $\omega_p = i\ln p$ ($p$ is a prime) and whose harmonics are $\omega_{p,n} = i\ln p^n$. This $p$-adic harmonic oscillator potential allowed to determine a one-to-one correspondence between the amplitudes of oscillations $a_n$ (and phases) with the imaginary parts of the zeroes of zeta, $\lambda_n$, after solving a inverse eigenvalue problem.

In SUSY-QM two iso-spectral operators $\hat{H}^{(+)}$ and $\hat{H}^{(-)}$ are defined in terms of the so called SUSY-QM potential. Here we use the SUSY-QM model proposed in [2] based on the pioneering work of B. Julia [26], where the zeta-function and its fermionic version were related to the partition function of a system of $p$-adic oscillators in thermal equilibrium at a temperature $T$. The fermionic zeta-function has zeroes at the same positions of the ordinary Riemann function plus a zero at $1/2 + i0$, this zero is associated to the SUSY ground state. See also the reference [26].

We propose an ansatz for the following antisymmetric SUSY QM potential (a “$p$-adic Fourier expansion”):

$$\Phi(x; a, b) = \prod_p \sum_j (a_j^{(p)} p^{jx} + b_j^{(p)} p^{-jx}) - \prod_p \sum_j (a_j^{(p)} p^{-jx} + b_j^{(p)} p^{jx}),$$  \hspace{1cm} (1)
$p = \text{primes, } j = \text{naturals, } a \equiv \{a_j^{(p)}\} \text{ and } b \equiv \{b_j^{(p)}\}$. This comes from the following SUSY Schrödinger equation associated with the $\hat{H}^{(+)}$ Hamiltonian \[10\],
\[
\left( \frac{\partial}{\partial x} + \Phi \right) \left( - \frac{\partial}{\partial x} + \Phi \right) \psi^{(+)}_n(x) = \lambda^{(+)}_n \psi^{(+)}_n(x),
\]
where we set $\hbar = 2m = 1$. SUSY imposes that $\Phi(x; a, b)$ is antisymmetric in $x$ so it must vanish at the origin. Hence, $\Phi^2(x; a, b)$ is an even function of $x$ so the left/right turning points obey: $x^{(n)}_L = -x^{(n)}_R$ for all orbits, for each $n = 1, 2, \ldots$ We define $x_n = x^{(n)}_R$.

The quantization conditions using the fermionic phase path integral calculation (which is not the same as the WKB and sometimes is called the Comtet, Bandrauk and Campbell formula \[10\]) are, set $\hbar = 2m = 1$, so all quantities are written in dimensionless variables for simplicity,
\[
I_n(x_n; \lambda_n; a, b) \equiv 4 \int_0^{x_n} dx \left[ \lambda_n - \Phi^2(x; a, b) \right]^{1/2} = \pi n,
\]
for $n = 1, 2, \ldots$ and $\lambda_n$ are imaginary parts of the zeroes of zeta.

The second set of equations are given by the definition of the turning points of the bound state orbits:
\[
\Phi^2(x_n; a, b) = \lambda_n; \quad n = 1, 2, \ldots
\]
So, from the two sets of equations (3) and (4) we get what we are looking for: Amplitudes of $p$-adic harmonic oscillators, $(a, b)$, and (right) turning points $x_n$, depending on all the $\lambda_n$.

It is very plausible that due to some hidden symmetry of the inverse scattering problem there may be many solutions for the amplitudes of the $p$-adic harmonic oscillators and turning points; i.e. many different SUSY potentials $\Phi(x; a, b)$ do the job for many different sets $(a, b)$ and $x_n$. Numbers related by a symmetry which leaves fixed the eigenvalues of the SUSY QM model = imaginary parts of the zeroes of zeta (fixed points). We are not concerned with this case now only with proving that a solution exists.

We emphasize that the integral quantization condition for the energy of the orbitals of the SUSY QM model based on the fermionic phase space approximation to the path integral Eq. (3) are only valid for shape-invariant potentials $V_\pm(x)$ of the partner Hamiltonians $H_\pm$. Only for those cases one has that the energy levels obtained from this approximation happen to be exact. Since imposing shape-invariance amount to an additional constraint one can just forget about this restriction and write down the integral conditions based on a fermionic path integral calculation that yields nice approximate results. It is far easier to solve the inverse eigenvalue problem in this way than to try to invert the Shrödinger equations! Therefore, since we are working in the opposite direction, and we do not wish to impose an impossible constraint on the SUSY potential, and we impose that the energy levels are the imaginary parts of the
zeroes, then one concludes that the amplitudes of the \( p \)-adic harmonic oscillator SUSY QM potential, Eq. (1), \((a,b)\) are only approximate by our method. The numerical results will be given in a forthcoming publication. In the next section we propose an approach for a direct proof of the RH. It is based on the idea of relating the non-trivial zeroes of the \( \zeta \) with orthogonalities between eigenfunctions of a conveniently chosen operator. See [14], [21] and [22].

3 A way to prove the Riemann conjecture

The essence of our proposal is based in finding the appropriate \( D_1 \) operator

\[
D_1 = -\frac{d}{d \ln t} + \frac{dV}{d \ln t} + k,
\]

whose eigenvalues \( s \) are complex-valued, and its eigenfunctions are given by

\[
\psi_s(t) = t^{-s+k}e^{V(t)}.
\]

Notice that \( D_1 \) is not self adjoint with eigenvalues given by complex valued numbers \( s \).

Also we define a partner operator of \( D_1 \) as follows,

\[
D_2 = \frac{d}{d \ln t} + \frac{dV}{d \ln t} + k.
\]

For this operator we have

\[
D_2\psi_s(t) = (-s + 2k + 2t\frac{dV}{dt})\psi_s(t).
\]

The key of our approach relies in choosing the \( V \) to be related to the Bernoulli string spectral counting function, given by a Jacobi theta series,

\[
e^{2V(t)} = \sum_{n=1}^{\infty} e^{-\pi n^2 t}.
\]

The Jacobi’s theta series is deeply connected to the statistics of Brownian motion and integral representations of the Riemann zeta-function [23].

Then we are considering a family of \( D_2 \) operators, each characterized by two real numbers \( k \) and \( l \) to be chosen conveniently. Notice also that \( D_1 \) is invariant under scale transformations of \( t \) and \( F = e^V \), due to \( dV/(d \ln t) = d \ln F/(d \ln t) \) [14]. The \( D_1 \) defined in [16] has \( k = 0 \) and a different definition of \( F \).

Let’s recall the functional equation of the Riemann zeta-function [15],

\[
Z(s) \equiv \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \equiv Z(1-s).
\]

We define the inner product as follows:

\[
\langle f|g \rangle = \int_0^\infty f^* g \frac{dt}{t}.
\]
With this definition, the inner product of two $D_1$ eigenfunctions is

$$\langle \psi_{s_1} | \psi_{s_2} \rangle = \alpha \int_0^\infty e^{2V - t} \, dt = \frac{\alpha}{l} Z \left[ \frac{2}{l} (2k - s_{12}) \right],$$

(12)

where we have defined $s_{12} = s_1^* + s_2 = x_1 + x_2 + i(y_2 - y_1)$ and used the expressions (9) and (10). $\alpha$ is a constant to be conveniently chosen so that the inner product in the critical domain is semi positive definite. The measure of integration $d\ln t$ is also scale invariant. The integral is performed after the change of variables $t^l = x$, which gives $dt/t = \left(\frac{1}{l}\right)^{1/l} dx/x$, and using the result of equation (13), given by Voronin and Karatsuba’s book [15].

We recall that $Z$ is the fundamental Riemann function, expressed in terms of the Jacobi theta series, $\omega(x) = \sum_{n=1}^\infty \exp(-\pi n^2 x)$ (see Karatsuba and Voronin [19]),

$$\int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2 x} x^{s/2-1} \, dx =$$

$$= \int_0^\infty x^{s/2-1} \omega(x) \, dx \left(\frac{1}{s} + \int_1^{\infty} (x^{s/2-1} + x^{(1-s)/2-1}) \omega(x) \, dx \right) = Z(s) = Z(1-s).$$

(13)

The right side is defined for all $s$, i.e., this formula gives the analytic continuation of the function $Z(s)$ onto the entire complex $s$-plane [13].

Having the relation $(\alpha/l)Z[(2/l)(2k - s)] = (\alpha/l)Z[1 - 2/l(2k - s)]$, from $Z[(2/l)(2k - s)]$ we can obtain the other expression simply by taking $k \rightarrow k - l/4$, $l \rightarrow -l$ and $\alpha \rightarrow -\alpha$. If, and only if, $8k - 4 = l$, then the above discrete transformations become: $k \rightarrow 1 - k$, $l \rightarrow -l$ and $\alpha \rightarrow -\alpha$. The importance of the relation $8k - 4 = l$ will be seen shortly.

From (12) we obtain the “norm” of any state characterized by $s = x + iy$,

$$\langle \psi_s | \psi_s \rangle = \frac{\alpha}{l} Z \left[ \frac{4}{l} (k - x) \right].$$

(14)

The “norm” of the vectors is the same for all $s$ having the same $x$. We will choose the domain of the values of $s$, $1 - s$ such that they fall inside the critical domain: $0 < Re(s) < 1$ and $0 < Re(1 - s) < 1$. We will see soon why this is crucial. Recall that at the boundaries, $s = 0$ and $s = 1$ we have that $Z(s) = \infty$.

We exclude $Re(s) = 0$ and $Re(s) = 1$ because Vallée de la Poussin-Hadamard theorem says there are no zeroes of $\zeta$ at $x = 0$ and $x = 1$ [13].

In particular, for the critical line $x = 1/2$, the value of the “norm” is $(\alpha/l)Z[2(2k-1)/l]$. Since we will choose that $l = 4(2k-1)$, the “norm” becomes
\((\alpha/l)Z(1/2) = (\alpha/l)(-3.97...),\) for all states in the critical line, independently of the chosen value of \(k.\) This forces the value of \(\alpha\) to be negative, in the critical domain, and for this reason we shall fix it to be equal to \(-l.\)

A recently published report by Elizalde, Moretti and Zerbini \[21\], which contains some comments about the first version of this paper, considers in detail the consequences of equation \((13).\) One of them is that equation \((15)\) lose its original meaning as a scalar product. Despite of this, we will loosely referring this map as a scalar product. The Hilbert space inner product property is not required so that the eigenvalues can be also negative. The states have real norm squared, which need not however be positive definite. Hermiticity requirement implies that the states are orthogonal to the reference state and correspond to the zeros at the critical line. The problem is whether there could be also zeros outside the critical line but inside the critical strip.

Also we must caution the reader that our arguments do not rely on the validity of the zeta-function regularization procedure \[20\] nor the analytic extension of the \(\zeta\), which precludes a rigorous interpretation of the right hand side of \((13)\) like a scalar product. We simply can replace the expression “scalar product of \(\psi_{s_1}\) and \(\psi_{s_2}\)” by the map \(S\) defined as

\[
S: \mathbb{C} \otimes \mathbb{C} \rightarrow \mathbb{C}
\]

\[
(s_1, s_2) \mapsto S(s_1, s_2) = -Z\left(\frac{s_1^* + s_2}{2}\right).
\] (15)

In other words, our arguments do not rely on an evaluation of the integral \(\langle \psi_{s_1} | \psi_{s_2}\rangle\), but only in the mapping \(S(s_1, s_2)\).

Now we describe our proposal to prove the RH. From now, we shall set \(k = 1,\) \(l = 4.\)

**Th. 1.** If \(s = x + iy\), where \(0 < x < 1\) and \(\lambda_n\) is such that \(\zeta(1/2 + i\lambda_n) = 0,\) then the states \(\psi_s\) and \(\psi_{1-s}\) are orthogonal.

**Proof:** From \((13)\) it follows that

\[
\langle \psi_s | \psi_{1-s}\rangle = -Z(1/2 + i\lambda_n) = 0.
\] (16)

Figure 1 represents those orthogonal states.

**Th. 2.** Any pair of states \(s_1\) and \(s_2\) symmetrically localized with respect to the vertical line \(x = 1/2,\) and such that \(y_1 - y_2 = 2\lambda_n,\) are orthogonal.

**Proof:** We can always find an \(s = x + iy\) such that \(\langle \psi_{s_1} | \psi_{s_2}\rangle = \langle \psi_s | \psi_{1-s}\rangle = 0.\) This follows straightforwardly, by using \((13),\) \(s_1^* + s_2 = s^* + 1 - s = 1 - 2iy.\) Then, \(x_1 + x_2 = 1\) and \(y_2 - y_1 = -2y.\) By equating the arguments of the \(Z,\) it follows, from theorem 1, that, \(\langle \psi_s | \psi_{1-s}\rangle = 0\) only if \(s = x + iy\). Therefore, \(y = \lambda_n,\)

\[x_1 + x_2 = 1\]

\[y_1 - y_2 = 2\lambda_n,\] as we intended to prove. Figure 2 represents those states and we see that for any point \(x + iy\) of the complex plane within the critical strip, there is a doubly infinity family of orthogonal states to it, given by \(1 - x + i(y \pm 2\lambda_m)\) where \(\zeta(1/2 + \lambda_m) = 0.\) As a special case we have that the orthogonal states to the reference state \(1/2 + i0\) are located in the critical line.

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Th. 3. The scalar product of two arbitrary states $s_1$ and $s_2$ within the critical strip is the same as the scalar product of a third state $s_3$ of the critical strip with the reference state $1/2 + i0$.

Proof: The statement follows directly from the definition (12) written for the pairs $s_1$ and $s_2$, and $1/2 + i0$ and $s_3$. By using $\langle \psi_{s_1} | \psi_{s_2} \rangle = \langle \psi_{1/2+i0} | \psi_{s_1}^* + s_2 - 1/2 \rangle$, yields directly $s_3 = s_1^* + s_2 - 1/2$.

Let’s recall that $Z(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$ and $\langle \psi_{s_1} | \psi_{s_2} \rangle = -Z(1 - s_1^* / 2 - s_2 / 2)$ for the chosen values of $k$ and $l$. This scalar product can be rewritten as $\langle \psi_{1/2+i0} | \psi_{s_1}^* + s_2 - 1/2 \rangle$. If we define $s = 1 - s_1^* / 2 - s_2 / 2$, so that $s_1^* + s_2 - 1/2 = 3/2 - 2s$, then the scalar product between any states in the critical domain can be rewritten as,

$$\langle \psi_{s_1} | \psi_{s_2} \rangle = \langle \psi_{1/2+i0} | \psi_{3/2-2s} \rangle = -Z(s). \quad (17)$$

Due to the fact that $Z(s) = 0$ if and only if $\zeta(s) = 0$, therefore $\zeta(s) = 0$ if and only if $\psi_{3/2-2s}$ is orthogonal to $\psi_{1/2+i0}$. Then the RH is equivalent to the following statement: The orthogonal states to the reference state are $1/2 \pm 2i \lambda_n$.

Now let’s see an interesting consequence of theorem 1. If we define the superposition

$$| \Psi \rangle = | \psi_s \rangle + | \psi_{1-s} \rangle, \quad (18)$$

and evaluate the “norm” of $\Psi$, we note that the interference terms are real valued

$$\langle \psi_s | \psi_{1-s} \rangle + \langle \psi_{1-s} | \psi_s \rangle = -Z \left( \frac{1}{2} + iy \right) - Z \left( \frac{1}{2} - iy \right). \quad (19)$$

The arguments of the $Z$’s are in the critical line. If $y = \lambda_n$, for all these states, the norm-squared of the sum equals the sum of the norm-squares. This is more precise, like the Pythagoras rule. See [16]. Therefore, the destructive interfering states are distributed on infinite horizontal lines, each of them labelled by one of the zeroes of the $\zeta$. The interference is destructive when $y$ in $s = 1/2 + iy$ is such that $y = \pm \lambda_n$, with $\lambda_n$ one of the zeroes of the $\zeta$.

Since on the critical line the norm-squared given by (14) is the same for all the states labeled by $1/2 \pm iy$, we can visualize all those states as living on a sphere and having the minimal “norm”.

Notice the form of (13). We get as we should a quantity plus its complex conjugate for the interference which gives us a real valued number. This makes perfect sense because the second inner product must be the complex conjugate of the first inner product since we have reversed the order of the inner product.

Summing up, the interference of the $\psi_n$ with $\psi_{1-s}$ is destructive if and only if the values of $y_n = \lambda_n$. In such case the norm-squared of the superposition equals the sum of the norm-squares of its constituents, this is roughly speaking Pythagoras theorem. For Pythagorean rational phases and the Riemann conjecture (see [17]).

Now we explore some consequences of the mapping given by equation (13). Let’s suppose that $s = x + iy$ is a generic state orthogonal to the reference state. Equation (12) yields,

$$\langle \psi_{1/2+i0} | \psi_s \rangle = -Z(3/4 - s/2) = -Z(1/4 + s/2), \quad (20)$$

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together with their complex conjugates. It follows that if \( x + iy \) is orthogonal to \( 1/2 + i0 \), also orthogonal to this state are: \( 1 - x + iy, 1 - x - iy \) and \( x - iy \).

The arguments of the \( Z \) in each single one of those orthogonality relations must be of the form \( 3/4 - x/2 \pm iy/2 \) and \( 1/4 + x/2 \pm iy/2 \). From the definition of \( Z \), there exists a zero if and only if \( \zeta(x' + iy') = 0 \). We are going to determine relations between \( s' = x' + iy' \) and \( s = x + iy \) such that the symmetries of the \( Z \) and the orthogonality condition of states to the reference state are satisfied.

From the properties of the Riemann \( \zeta \)-function it follows that if \( s' \) is a zero, then there are three another zeroes, located at \( s'^* \), \( 1 - s' \) and \( 1 - s'^* \). See figure 3. The 4 arguments of the \( Z \) appearing in the orthogonality conditions must be related to the location of the 4 zeroes of the \( \zeta \). There are 24 orthogonality pairwise combinations, easily identified if we maintain fixed the inner rectangle of figure 3 and perform all permutations of the labels of the orthogonal states.

If one studies all the possible families of mappings due to all the possibly \( 4! = 24 \) permutations of the vertices of the outer rectangle one will inevitably introduce constraints among the four vertices of the same rectangle; i.e. like \( s^* = 1 - s \), and this will trivially lead to the RH. This is unacceptable.

There are 4 relations of \( s \) with \( s' \) of the form \( s = -1/2 + 2x' \pm 2iy' \) or \( s = 3/2 - 2x' \pm 2iy' \), that we call generic cases. They are both reflection-symmetric of each other. These simply state a correspondence between the position of the orthogonal states and hypothetical zeroes located in any point of the critical strip.

4 of the relations lead to an identification of the vertices of the rectangle symmetrically located respect to the critical line, that is \( s' = 1 - s'^* \) or \( s'^* = 1 - s' \), which implies the trivial result that those vertices are located on the critical line and the imaginary parts are related by \( y = \pm 2y' \). These 8 cases are compatible with the existence of zeroes in the critical line.

The remaining 12 relations lead to an identification of all the four vertices of the rectangle at the point \( s' = 1/2 + i0 \) which correspond to the state \( s = 1/2 + i0 \). These cases do not correspond to any orthogonality relation, due to in fact represent the inner product of the reference state with itself.

It can not be discarded the presence of orthogonal states to the reference located outside the critical line, the generic cases, and in consequence the RH can not be deduced from this analysis.

Notice that the “norm” of the state corresponding to \( s = 0 \) is infinite and that the states orthogonal to the \( s = 0 \) are those states whose \( s = 1 + iy_n = 1 + 2i\lambda_n \). The steps to this proof of the RH did not rely on the same reductio ad absurdum argument proposed by Pitkänen (16). In our approach, the RH is a consequence of the symmetries of the orthogonal states as we intend to prove next, then Th. 4 implies a proof of the RH assuming these symmetries.

Th. 4. The symmetries of the orthogonal states shown in figure 3 are preserved for any map \( S \), equation (15), which give rise to \( Z(as + b) \), if \( a \) and \( b \) are such that \( 2b + a = 1 \).

Proof: If \( s = x + iy \) is orthogonal to a reference state, then the Riemann zeta has zeroes at \( s' = x' + iy' \), \( s'^* \), \( 1 - s' \) and \( 1 - s'^* \). If we equate \( as + b = s' \), then also \( as'^* + b = s'^* \). Now, \( 1 - s' \) can be equated to \( a(1 - s) + b \) and \( 1 - s'^* \)
can be equated to $a(1 - s^*) + b$ if and only if if $2b + a = 1$.

Notice that our election $a = -2/l$, $b = (4k - 1)/l$ is compatible with this symmetry if $k$ and $l$ are related by $l = 4(2k - 1)$. Reciprocally, if we assume that the orthogonal states have the symmetries of figure 3, then $a$ and $b$ must be related by $2b + a = 1$, which gives rise to a very specific relation between $k$ and $l$, obtained from $a + 2b = 1$; $a, b$ real. The two generic cases correspond to taking $k = 1, l = 4$ and $k = 0, l = -4$ respectively. It is clear that a map with arbitrary values of $a$ and $b$ does not preserve the mentioned symmetries. Theorem 4 contains a genuine proof of the RH assuming that the invariance under the double symmetry is true.

We have a family of $D_1$ operators, each labelled by $(k, l)$. Since we can set $l = 4(2k - 1)$ due to the constraint $1 = a(k, l) + 2b(k, l)$ imposed by the double-reflection symmetry, we can parametrize the eigenfunctions as $\psi_s^{(k)}$, where $k$ and $s$ are continuous variables. Let’s consider two of those operators, $D_1^{(k_1)}$ and $D_1^{(k_2)}$. It must exist a one to one correspondence between eigenfunctions of this pair of operators at any given point $s$. We can see that to prove the RH is equivalent to the following statement:

St. (i). If $t = 1/2 + i0$ is the reference states and if $s_1$ and $s_2$ are both points having $x_1 = x_2 = x$, then if $\psi_s^{(k_1)}$ is orthogonal to $\psi_t^{(k_1)}$, then $\psi_s^{(k_2)}$ is also orthogonal to $\psi_t^{(k_2)}$. In other words, the orthogonality of states (with the same $x$) to the reference state is independent of $l$ and $k$.

Due to the fact that the inner products of the states of (i) with the corresponding reference state are given by $Z[(2/l_1)(2k_1 - t^* - s_1)]$ and $Z[(2/l_2)(2k_2 - t^* - s_2)]$, then the orthogonality gives rise to two zeroes of the Riemann zeta. Those zeroes are of the form $s_1' = x' + iy_1'$ and $s_2' = x' + iy_2'$, that is, they are to be located on the same vertical line.

This immediately allows us to write $s_1' = (2/l_1)(2k_1 - t^* - s_1)$ and $s_2' = (2/l_2)(2k_2 - t^* - s_2)$, whose real parts are respectively $x_1' = (2/l_1)(2k_1 - 1/2 - x)$ and $x_2' = (2/l_2)(2k_2 - 1/2 - x)$. If, and only if, $x = 1/2$ and $l_1 = 4(2k_1 - 1)$, $l_2 = 4(2k_2 - 1)$, then it follows that $x_1' = x_2' = x = 1/2$, $y_1' = (-2/l_1)y_1$, $y_2' = (-2/l_2)y_2$. Notice that the values of $y_1, y_2$ do depend on $(k, l)$. $y_1' = \lambda_1$ and $y_2' = \lambda_2$ are the imaginary parts of the two zeroes of $\zeta$. Then (i) is equivalent to the RH.

These arguments can be generalized for all $(k, l)$ obeying $l = 4(2k - 1)$ as follows. From $s' = a(k, l)s + b(k, l)$ and $a + 2b = 1$ we can write $s' = a(s - 1/2) + 1/2$. So $x' = 1/2 + a(x - 1/2)$ and, for a fixed $x$ the position of a zero could continuously change by changing $(k, l)$. If the zeroes are supposed to form a discrete set, then one must have that $x = 1/2$, so that $x' = 1/2$ is the only consistent value one can have.

Another way of rephrasing this is saying that the family of the $D_1^{(k,l)}$ operators yields a continuous family of pairs $(x, x')$ having the double-reflection symmetry $x \rightarrow 1 - x, x' \rightarrow 1 - x'$, from which one arrives at the constraint $1/2 = a(k, l)/2 + b(k, l)$. This means that all the curves given by $x' = a(k, l)x + b(k, l)$ must contain the common point in the $x - x'$ plane given by $(1/2, 1/2)$, for
all values of \((k,l)\), obeying \(l = 4(2k - 1)\). Also, if \(\psi_s^{(k)}\) is orthogonal to the reference state \(1/2 + i0\) then \(s'\) is a zero of the \(\zeta\), and the real parts are related by \(x' = a(x - 1/2) + 1/2\). Due to the statement (i) this real part must to be independent of \(k\) namely, independent of \(a\). This can be satisfied only if the orthogonal state satisfies \(x = 1/2\), which gives \(x' = 1/2\), the RH.

Let us summarize what we have found so far, for \(0 < \Re(s) < 1\). When we choose \(k = 1, l = 4\) then we have the following results (i), (ii), (iii). (i) The interference between \(\psi_s\) and \(\psi_{1-s}\) is destructive at the horizontal lines defined by \(0 < x < 1\) and \(y = \pm \lambda_n\). (ii) If two states \(s_1 = 1/2 + iy_1\) and \(s_2 = 1/2 + iy_2\) on the critical line are orthogonal, then \(y_1 - y_2 = \pm 2\lambda_n\) for one given \(n\). (iii) Any pair of orthogonal states must obey \(x_1 + x_2 = 1\) and \(y_2 - y_1 = 2\lambda_n\). And when we assume the double reflection symmetry, then for all \((k,l)\) obeying \(8k - 4 = l\) we have: (iv) The RH is a consequence of the assumption that the orthogonal states to the reference state have the same symmetry like the zeroes of \(\zeta\) have.

4 The Riemann fractal string and its dual

Finally, we recall that the \(e^{2V}\) defined by (9), expressed in terms of \(x = t^4\), is nothing else than the energy partition function of a Bernoulli string (whose standing waves have for wave vectors integral multiples of the inverse of length \(L\)).

Several new relations of Riemann zeta-function with the spectral properties of different physical systems have been found during the last years. The authors discuss the Epstein zeta-functions \(\zeta_{ep}(s)\) when dealing with the Laplace operator in square/rectangular domains. The counting function of the spectral eigenvalues of 2D-Laplacian gives the Epstein zeta. Those functions also appear in the physics of \(p\)-branes moving in hyperbolic spaces (negative constant curvature); i.e. in the calculation of the effective potential of toroidal \(p\)-branes living/embedded in target hyperbolic spaces \(\mathbb{H}^5\).

The spectrum of toroidal \(p\)-branes and branes moving in hyperbolic spaces of constant negative curvature, is linked to the Epstein zeta-functions \(\zeta_{ep}(s)\).

\[
\zeta_{ep}(s) = \sum_{\{n\}} \frac{1}{(n_1^2 + n_2^2 + \ldots + n_{p+1}^2)^{s/2}},
\]

where \(\{n\} \equiv \{n_1, n_2, \ldots n_{p+1}\}\) is a non-zero integer vector.

The Riemann zeta is a special case of the Epstein zeta and has a relation to the spectrum of fractal strings (Alain Connes had urged to use a different word to avoid confusion with string theory, suggests call them “fractal harps” instead). Two functions describe the fractal string, the geometric length counting function \(Z_L(s)\) and the frequency counting function \(Z_F(s)\) that are related by the fundamental relation involving the Riemann zeta,

\[
Z_F(s) = Z_L(s)\zeta(s).
\]

Let’s imagine a chain made out of many links of sizes \(l_1, l_2, \ldots l_j, \ldots\). The natural frequencies associated with the oscillations of each one of those links
are given by: \( \frac{1}{l_1}; \frac{1}{l_2}, \ldots \frac{1}{l_j}, \ldots \) and the excitation states are just integer multiples of \( \frac{1}{l_j}; \frac{k}{l_j} \) for \( k = 1, 2, \ldots j, \ldots \) There are described by the geometric length counting function \( Z_L(s) \) and the frequency counting function \( Z_F(s) \). The boundaries of the fractal strings are Cantor sets. See [1] and [7].

Since we have selected the \( V(t) \) of Eq. (9) to be related to the Bernoulli string spectral counting function we must continue with the fractal strings duality discussion. It is not a coincidence that choosing \( V(t) \) to be related to the Bernoulli string was telling us something!

The length counting function is the sum,

\[
Z_L(s) = \sum_{j=1}^{\infty} l_j^s.
\]

(23)

The frequency counting function is the double sum,

\[
Z_F(s) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} k^{-s} l_j^s.
\]

(24)

And we always have \( Z_F(s) = \zeta_{gen}(s) Z_L(s) \). In the case of a fractal string it reduces to Eq. (22). The book is entirely based in choosing particular examples of fractals which fixed the lengths \( l_j \) a priori which enable to evaluate the sums explicitly. The authors of [13] were able to find the complex dimensions of the fractal strings by finding the location of the poles of the \( Z_L(s) \). It was essential that the poles do not coincide with the zeroes of \( \zeta(s) \).

Let us explain how one obtains the complex dimensions of the Cantor string. The sequence of lengths must always be decreasing: \( l_1 > l_2 > \ldots \) or at most equal. Let’s take the Cantor set as an example; there is 1 segment of length equal to \( 1/3; 2 \) segments of length \( 1/9; \ldots \) \( 2^n \) segments of \( l_n = (1/3)^n \) and so forth. The degeneracy of each link of length \( l_n \) is \( w_n = 2^n \). So the geometric counting function of the Cantor string is:

\[
Z_L(s) = \sum_n (l_n)^s = \sum_n 2^n \left( \frac{1}{3} \right)^{ns} = \frac{1}{1 - 2^{1-s} 3^{-s}}.
\]

(25)

so the poles of this sum yield the complex dimension: \( 1 = e^{i2n\pi} = 2(3)^{-s}; \)
taking the logarithm on both sides yields \( s_n = (ln 2/ln 3) \pm i(2n\pi/ln 3) \) for the complex dimensions for \( n = 0, 1, 2, \ldots \).

The real part is the standard dimension of the Cantor set. The imaginary parts are periodic whose period is \( 2\pi/ln 3 \).

We propose that the physical meaning of these complex valued dimensions may be relevant for a theory of quantum gravity, because of interference of complex dimensions.

It is very important to insist on the condition \( 0 < \Re s < 1 \) since a fractal string of real valued dimension \( D \), embedded in a space of \( R^d \), must satisfy the constraint: \( d - 1 < D < d \).
The Cantor string, the golden string and the Fibonacci string are defined respectively for the finite number of scaling ratios given by

\[ r_1 = r_2 = 1/3; \quad r_1 = 1/2, \quad r_2 = 1/2^{1+\phi}, \quad \phi = 0.618...; \quad r_1 = 1/2, \quad r_2 = 1/3. \]  

(26)

We shall construct the Riemann fractal string (RFS) by different procedures than the authors [13], as the fractal boundaries of the open 2-D domain.

The duality relation of the Riemann zeta \( Z(s) = Z(1-s) \) only makes sense if, and only if, we write the continuum limit for the length counting function

\[ Z_L(s) = \int_0^\infty l(x)^s d\mu(x) \quad \text{and} \quad Z_L(1-s) = \int_0^\infty \tilde{l}(x)^{1-s} d\tilde{\mu}(x), \]  

(27)

and the frequency counting function

\[ Z_F(s) = \int_0^\infty f(x)^{-s} d\mu(x) \quad \text{and} \quad Z_F(1-s) = \int_0^\infty \tilde{f}(x)^{-1+s} d\tilde{\mu}(x), \]  

(28)

where \((l(x), \tilde{l}(x)), (f(x), \tilde{f}(x)), (\mu(x), \tilde{\mu}(x))\) are the respective complex-valued, and their complex conjugates, lengths, frequencies and measures associated with the continuum limit of the RFS and their dual, that in the general case are complex valued maps from the real line to the complex plane. The measure \(\mu(x)\) is such that \(\int_0^\infty d\mu(x) = 1\).

In reference [8] a detailed analysis of the vector calculus and contour integrals on fractal curves (boundaries of a bounded domain in the complex plane) and interfaces is given. This was attained by constructing pseudo-measures of integration based on iterated function systems. Expressions for the contour integrals are obtained by means of a suitable renormalization procedure (the length of a fractal contour is infinite) and the solution of the Dirichlet problem on bounded two-dimensional domains possessing fractal boundaries is given. These analytical tools allow us to find in principle the complex valued functions appearing in equation (29) and to define the integrals in (27,28).

From (27) and (28) one can obtain the generalization of the Lapidus and Frankenhuysen result, our proposal for RFS,

\[ \frac{Z_F}{Z_L} = \frac{\int_0^\infty f(x)^{-s} d\mu(x)}{\int_0^\infty l(x)^s d\mu(x)} = \zeta(s), \]  

(29)

with analogous relation for the dual string. Lapidus and Frankenhuysen results are recovered by using a Dirac delta distribution for the measure.

When \(s\) is one of the zeroes of the \(\zeta\), the equation (29) implies that the geometric length counting function has a simple pole or that the frequency counting function has a simple zero. On basis of this, we define the Riemann fractal string as such string whose geometric length counting function has a simple pole precisely at the zeroes of the Riemann zeta-function.
5 Concluding remarks

In a previous work [4] we found very suggestive relations of the golden mean and the distribution of the imaginary parts of the zeroes of zeta. We pointed out that this could be related to the multifractal character of the prime numbers distribution (See [25]). Here we observed yet another evidence of the multifractal distribution of primes in the similarities in Wolf’s formulae with those of Lapidus and Frankenhuysen:

\[ p(S_i) \sim l_i, \quad p(S_i)^q \sim l_i^q, \quad q \sim s, \quad \chi_q(l) \sim Z_L(s), \quad l \sim L, \quad (30) \]

where \( L \) is the initial length where one begins to construct the fractal string by defining \( N \) finite segments of lengths \( l_j = r_j L, \ j = 1, 2, \ldots, N \) so that what is left is \( L(1 - R) \) where \( R = \sum_{j=1}^N r_j < 1 \). One chooses \( L \) so that \( L(1 - R) = 1 \) which means that the first length of the iteration has unit length. From Wolf’s paper we have that \( p(S_i) \) is the measure of the set \( S_i \), the \( \chi_q \) are the moments or partition functions, which scale like a \( L^{\tau(q)} \).

The Riemann zeta is deeply connected to fractal strings (whose boundaries are Cantor sets) and it is no wonder why it is related to quantum chaos, random matrix models, random walks, Brownian motion, etc.

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Figure 1: Lines between dots and crosses represent pairwise orthogonal states. Dots represent zeroes of the $\zeta$. Crosses on the critical line represent states orthogonal to the $1/2+i0$. Crosses on the $Re(s) = 1$ represent states orthogonal to the $0+i0$. Crosses on the $Re(s) = 0$ represent states orthogonal to the $1+i0$. The states $1/2 + i\lambda_n$ are orthogonal to $1/2 - i\lambda_n$. On the critical line, pairs of dots and/or pairs of crosses are mutually orthogonal. Notice that for simplicity we are representing the orthogonalities of states having only $x = 0$, $x = 1/2$ and $x = 1$. Here we are referring the particular case $k = 1$, $l = 4$. 
Figure 2: The state associated to a point $x + iy$ is orthogonal to a doubly infinite series of states at $1 - x + i(y + 2\lambda_n)$, for $n = 1, 2, \ldots \infty$. Here we are referring the particular case $k = 1, l = 4$. 
Figure 3: The dots represent generic zeroes of the $\zeta$. The crosses represent generic states orthogonal to the reference state $1/2 + i0$. The numbers $3/4 - x/2 - iy/2$, etc, are the arguments of $Z$ appearing in the orthogonality relations between states orthogonal to the reference state. Due to the functional equation of the Riemann zeta-function (10), these arguments are just the average values between $1/2 + i0$ and those orthogonal states. Here we are referring the particular case $k = 1, l = 4$. 