SYMMETRY OF POSITIVE SOLUTIONS FOR SYSTEMS OF FRACTIONAL HARTREE EQUATIONS

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Abstract. In this paper, we deal with a system of fractional Hartree equations. By means of a direct method of moving planes, the radial symmetry and monotonicity of positive solutions are presented.

1. Introduction. Consider the system of the fractional Hartree equations in the form

\[
\begin{aligned}
(-\Delta)^s u + u &= (K \ast v^2)v, \quad \text{in } \mathbb{R}^n, \\
(-\Delta)^s v + v &= (K \ast u^2)u, \quad \text{in } \mathbb{R}^n,
\end{aligned}
\]  

(1)

where \((-\Delta)^s\) is the fractional operator defined by

\[
(-\Delta)^s u = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} dy \tag{2}
\]

Here, \(C_{n,s}\) is a positive constant (see [8, 22, 9]) and the symbol “\(\ast\)” stands for the convolution in \(\mathbb{R}^n\) with \(0 < s < 2\) and \(n \geq 3\). In order to ensure integral (2) to be well-defined, we require positive functions \(u, v\) satisfy the following conditions

\((U_1)\) \(u, v > 0, u, v \in C^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap \mathcal{L}_s(\mathbb{R}^n)\), where

\[
\mathcal{L}_s(\mathbb{R}^n) = \left\{ w \mid w \in L^{1,1}_{\text{loc}}(\mathbb{R}^n), \int_{\mathbb{R}^n} \frac{|w(x)|}{1 + |x|^{n+s}} dx < \infty \right\}.
\]

The goal of this study is to investigate the qualitative properties of positive solutions of system (1). A solution pair \((u, v)\) is called positive, if both \(u\) and \(v\) are positive. In recent years, there has been a great deal of interest in using the fractional Laplacian to model diverse physical phenomena and pure mathematical problems such as the obstacle problem [5, 24], minimal surfaces [4, 7], phase transition [1], anomalous diffusion [3, 16, 21, 27] and mathematical finance [12]. In particular, the fractional Laplacian can be understood as the infinitesimal generator of a stable Levy process [26]. The nonlocal nature of these operators makes them difficult to investigate. To overcome this difficulty, the extension method was
introduced by Caffarelli and Silvestre [6] and this method can reduce the associated nonlocal problems into local cases in the higher dimensions. Another powerful approach is the integral equation methods [10]. After establishing the equivalence between a fractional equation and its corresponding integral equation, one can apply the method of moving planes in integral forms and regularity lifting to obtain the symmetry and regularity of solutions for the fractional equations. These methods have been widely applied to study the fractional Laplacian equations and a series of fruitful results have been obtained [8, 2, 11, 10] and the references therein.

Liu [19] considered the following single fractional Hartree equation

\[ (-\Delta)^s u + u = (K * u^2)u, \quad \text{in } \mathbb{R}^n. \]  

(3)

By using the method of moving planes, it shows that any positive solution of equation (3) must be radially symmetric and strictly decreasing in \( \mathbb{R}^n \). For different functions \( K \), there are lots of results for equation (3). For example, if we denote \( v = K * u^2 \), then equation (3) becomes

\[ (-\Delta)^s u + u = vu, \quad \text{in } \mathbb{R}^n. \]  

(4)

In particular, if \( v \) is a solution of the following equation

\[ -\Delta v = u^2, \quad \text{in } \mathbb{R}^n, \]  

(5)

and \( K \) is the fundamental solution of the Laplace equation, then by using (4) and (5) it implies

\[ (-\Delta)^s u + u = \left( \frac{1}{|x|^{n-2s}} * u^2 \right) u, \quad \text{in } \mathbb{R}^n, \]  

(6)

which is a special case of (3). For \( n = 3, s = 2 \), (6) is extensively studied [6, 15]. It is shown that (6) admits ground state solutions and all ground state solutions of (6) must be radially symmetric and nonincreasing. Dai et al [14] considered equation (3) for \( K(x) = \frac{1}{|x|^{2s}} \), which is the following fractional order static Hartree equation

\[
\begin{cases}
(-\Delta)^s u = \left( \frac{1}{|x|^{n-2s}} * u^2 \right) u, & \text{in } \mathbb{R}^n, \\
u \in C^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap L^s(\mathbb{R}^n), & u(x) > 0, \quad \text{in } \mathbb{R}^n,
\end{cases}
\]  

(7)

where \( 0 < s < \min \{ 2, \frac{n}{2} \} \) and \( n \geq 2 \). By using the direct moving plane method for nonlocal nonlinearity [9], they proved that all positive solutions of (7) are radially symmetric about some point in \( \mathbb{R}^n \) with the certain explicit forms. In addition, the classification and regularity results for weak solutions of the fractional order static Hartree equation (7) were presented in [13].

Equation (7) arises in the Hartree-Fock theory of the nonlinear Schrödinger equations [18, 25]. The solution \( u(x, t) \) of (7) is also a ground state or a stationary solution to the following \( H^s \)-critical focusing fractional order dynamic Schrödinger-Hartree equation

\[ i\partial_t u + (-\Delta)^s u = \left( \frac{1}{|x|^{n-2s}} * u^2 \right) u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n. \]  

(8)

The Schrödinger-Hartree equation (8) has many applications in the quantum theory of large systems of non-relativistic bosonic atoms and molecules, and the associated dynamical equations of type (8) can be seen in [17, 20, 23]. Physically, the ground state solution can be regarded as a crucial criterion or threshold for the global well-posedness and scattering in the focusing case, so the properties of solutions of system (1) play an important and fundamental role in the study of the focusing...
fractional order Schrödinger-Hartree equations (8). In [19], it was focused on the case of $0 < s < 2$ in (3) and for the more general function $K$, it shows that the positive solution of (3) is radially symmetric and strictly monotone decreasing about some point in $\mathbb{R}^n$. In this study we generalize these results of a single fractional equation to the case of a system, and show that the same conclusion holds for positive solutions of system (1) under certain conditions.

We assume that the kernel function $K$ in system (1) is radially symmetric, $K(x) = k(|x|)$, and the function $k$ is differentiable in $(0, \infty)$ and satisfies the following conditions:

(K1) $k(t) > 0$ in $(0, \infty)$ and there exists a constant $c$ such that

$$k(t) \leq \frac{c}{t^{n-2}}, \quad t > 0.$$  

(K2) $k$ is strictly decreasing in $(0, \infty)$ and there exists a positive constant $c$ such that

$$|k'(t)| \leq \frac{c}{t^{n-1}}, \quad t > 0.$$  

Moreover, we assume $u, v$ satisfy the following condition

(U1) $u(x) = o(|x|^{-1}), \ v(x) = o(|x|^{-1})$ as $|x| \to \infty$, and the integrals $\int_{\mathbb{R}^n} \frac{u(y)}{|y|^{n-1}} dy, \ \int_{\mathbb{R}^n} \frac{v(y)}{|y|^{n-1}} dy$ are convergent.

Let us summarize our main result as follows.

**Theorem 1.1.** Assume that $(u, v)$ is a positive solution of system (1). Assume that conditions (U1), (U2), (K1) and (K2) hold. Then $u, v$ are radially symmetric and strictly decreasing about some point in $\mathbb{R}^n$, i.e., there exist $x_0 \in \mathbb{R}^n$ and two strictly decreasing functions $f$ and $g$ in $[0, \infty)$ such that $u(x) = f(|x-x_0|)$ and $v(x) = g(|x-x_0|)$.

Throughout this paper, $c$ denotes a generic constant which may be different from line to line.

2. **Proof of main result.** In this section, we always assume that $(u, v)$ is a positive solution of system (1) and conditions (U1), (U2), (K1) and (K2) hold. As mentioned in the Introduction, we shall use the method of moving planes for the fractional Laplacian equations, developed by Chen et al [9] to prove Theorem 1.1. First of all, for the reader’s convenience, we present a few technical lemmas and notations, some of which have been introduced in [19].

**Lemma 2.1.** $(K * v^2)(x) \to 0$ as $|x| \to \infty$.

**Proof.** We divide $\mathbb{R}^n$ into two parts:

$$A_1 = \left\{ y \ | \ y \in \mathbb{R}^n, \ |x-y| \leq \frac{1}{2}|x| \right\},$$

$$A_2 = \left\{ y \ | \ y \in \mathbb{R}^n, \ |x-y| \geq \frac{1}{2}|x| \right\},$$

and estimate the integral $K * v^2 = \int_{\mathbb{R}^n} K(x-y)v^2(y)dy$ over these two sets, respectively. For the set $A_1$, we have

$$|y| \geq |x| - |x-y| \geq |x| - \frac{1}{2}|x| = \frac{1}{2}|x|.$$
Thus by the conditions \((K_1)\) and \((U_2)\), we get
\[
\int_{A_1} K(x-y)v^2(y)\,dy \leq c \int_{|x-y| \leq \frac{1}{2}|x|} \frac{v^2(y)}{|x-y|^{n-2}}\,dy \\
\leq c \sup_{|y| \geq \frac{1}{2}|x|} v^2(y) \int_{|x-y| \leq \frac{1}{2}|x|} \frac{dy}{|x-y|^{n-2}} \\
= c|x|^2 \sup_{|y| \geq \frac{1}{2}|x|} v^2(y) = o(1), \text{ as } |x| \to \infty.
\]

For the set \(A_2\), we have
\[
|y| \leq |x-y| + |x| \leq 3|x-y|.
\]

Let \(R > 0\), it follows from conditions \((K_1)\) and \((U_2)\) that
\[
\int_{A_2} K(x-y)v^2(y)\,dy \\
\leq c \int_{|x-y| \geq \frac{1}{2}|x|} \frac{v^2(y)}{|x-y|^{n-2}}\,dy \\
= c \int_{|y| \leq R, |x-y| \geq \frac{1}{2}|x|} \frac{v^2(y)}{|x-y|^{n-2}}\,dy + c \int_{|y| \geq R, |x-y| \geq \frac{1}{2}|x|} \frac{v^2(y)}{|x-y|^{n-2}}\,dy \\
\leq c \int_{|y| \leq R} v^2(y)\,dy + c \int_{|y| \geq R} \frac{v^2(y)}{|y|^{n-1}}\,dy.
\]

We choose \(R\) large enough so that the integral \(\int_{|y| \geq R} \frac{v^2(y)}{|y|^{n-1}}\,dy\) is small enough, and fix \(R\). Then \(\int_{|y| \leq R} v^2(y)\,dy \to 0\) as \(|x| \to \infty\) and \(\int_{A_2} K(x-y)v^2(y)\,dy \to 0\) as \(|x| \to \infty\).

As usual, to apply the method of moving planes, for \(\lambda \in \mathbb{R}\), we take
\[
T_\lambda = \{x \mid x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, x_1 = \lambda\}
\]
to be the moving plane, take
\[
\Sigma_\lambda = \{x \mid x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, x_1 < \lambda\}
\]
to be the region to the left of the plane, and take
\[
x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n)
\]
to be the reflection of \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) about the plane \(T_\lambda\). We denote \(u_\lambda(x) = u(x^\lambda), \ v_\lambda(x) = v(x^\lambda), \ U_\lambda(x) = u_\lambda(x) - u(x)\) and \(V_\lambda(x) = v_\lambda(x) - v(x), x \in \mathbb{R}^n\).

**Lemma 2.2.** Assume \(\lambda \leq 0, U_\lambda \geq 0, V_\lambda \geq 0\ in \Sigma_\lambda\). Then either \(U_\lambda \equiv 0, V_\lambda \equiv 0\) or \(U_\lambda > 0, V_\lambda > 0\ in \Sigma_\lambda\).

**Proof.** By \((1)\), we know that \(U_\lambda\) satisfies
\[
(-\Delta)^2 U_\lambda + U_\lambda = (K \ast v^2)\lambda v_\lambda - (K \ast v^2)v. \tag{11}
\]
It follows from the right hand side of \((11)\) that
\[
(K \ast v^2)\lambda v_\lambda - (K \ast v^2)v = (K \ast v^2)(v_\lambda - v) + ((K \ast v^2)\lambda - K \ast v^2) v_\lambda \\
= (K \ast v^2)\lambda + ((K \ast v^2)\lambda - K \ast v^2) v_\lambda. \tag{12}
\]
For $\lambda \leq 0$, $x \in \Sigma_{\lambda}$, we have

$$(K * v^2)_{\lambda} - K * v^2 = \int_{\Sigma_{\lambda}} (K(x - y) - K(x^\lambda - y)) (v^2(y) - v^2(y)) dy$$

$$= \int_{\Sigma_{\lambda}} (K(x - y) - K(x^\lambda - y)) (V_{\lambda}(y) + 2v(y)) V_{\lambda}(y)dy.$$  \hspace{1cm} (13)

Since $|x - y| < |x^\lambda - y|$ for $x, y \in \Sigma_{\lambda}$, according to condition $(K_2)$ we have $K(x - y) > K(x^\lambda - y)$ for $y \in \Sigma_{\lambda}$. By the assumption $V_{\lambda} \geq 0$ in $\Sigma_{\lambda}$, by (13) we have

$$(K * v^2)_{\lambda} - K * v^2 \geq 0 \quad \text{in} \quad \Sigma_{\lambda}. \hspace{1cm} (14)$$

Suppose that there exists $x_0 \in \Sigma_{\lambda}$ such that $U_{\lambda}(x_0) = 0$, then by (11), (12) and (14) we have

$$(-\Delta)^s U_{\lambda}(x_0) = (-\Delta)^s U_{\lambda}(x_0) + U_{\lambda}(x_0) \geq (K * v^2)(x_0) V_{\lambda}(x_0) \geq 0. \hspace{1cm} (15)$$

From the definition of the fractional Laplacian (2) and the antisymmetry of $U_{\lambda}$, we have

$$(-\Delta)^s U_{\lambda}(x_0) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{U_{\lambda}(x_0) - U_{\lambda}(y)}{|x_0 - y|^{n+s}} dy$$

$$= C_{n,s} P.V. \left( \int_{\Sigma_{\lambda}} \frac{U_{\lambda}(x_0) - U_{\lambda}(y)}{|x_0 - y|^{n+s}} dy + \int_{\Sigma_{\lambda}} \frac{U_{\lambda}(x_0) - U_{\lambda}(y^\lambda)}{|x_0 - y^\lambda|^{n+s}} dy \right)$$

$$= -C_{n,s} P.V. \int_{\Sigma_{\lambda}} \left( \frac{1}{|x_0 - y|^{n+s}} - \frac{1}{|x_0 - y^\lambda|^{n+s}} \right) U_{\lambda}(y) dy. \hspace{1cm} (16)$$

Since $\frac{1}{|x_0 - y|^{n+s}} - \frac{1}{|x_0 - y^\lambda|^{n+s}} > 0$ and $U_{\lambda}(y) \geq 0$ for $y \in \Sigma_{\lambda}$, by (15)-(16) we obtain $U_{\lambda} \equiv 0$ in $\Sigma_{\lambda}$, and thus $U_{\lambda}(x) \equiv 0$, $(-\Delta)^s U_{\lambda}(x) \equiv 0$ in $\mathbb{R}^n$. By (11) and (14) we derive

$$0 = (-\Delta)^s U_{\lambda}(x) + U_{\lambda}(x)$$

$$= (K * v^2) V_{\lambda} + ((K * v^2)_{\lambda} - K * v^2) v_{\lambda}$$

$$\geq (K * v^2) V_{\lambda}.$$  

Due to $K * v^2 > 0$ and $V_{\lambda} \geq 0$, we obtain $V_{\lambda} \equiv 0$. \hfill \square

**Lemma 2.3.** Suppose that $\lambda \leq 0$, $x_0 = (x_0^0, x_0^1, \cdots, x_0^n) \in \Sigma_{\lambda}$, and

$$U_{\lambda}(x_0) = \min \{ \min_{x \in \Sigma_{\lambda}} U_{\lambda}(x), \min_{x \in \Sigma_{\lambda}} V_{\lambda}(x) \} < 0. \hspace{1cm} (17)$$

Then we have

$$\frac{a}{(\lambda - x_0^1)^s} + 1 \leq (K * v^2)(x_0) + 2v(x_0) \int_{\Sigma_{\lambda}} (K(x_0 - y) - K(x_0^\lambda - y)) v(y) dy, \hspace{1cm} (18)$$

where

$$a = 2C_{n,s} \int_{\Sigma_{\lambda}} \frac{dy}{|e - y|^{n+s}} \quad \text{and} \quad e = (1,0,\cdots,0).$$

**Proof.** Since $U_{\lambda}$ satisfies equation (11), we claim that

$$(-\Delta)^s U_{\lambda}(x_0) \leq \frac{a}{(\lambda - x_0^1)^s} U_{\lambda}(x_0). \hspace{1cm} (19)$$
In fact, by the definition of the fractional Laplacian (2) and the antisymmetry of $U_\lambda$, we have
\[
(-\Delta)^{\frac{s}{2}} U_\lambda(x_0) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^{n+s}} \, dy
= C_{n,s} P.V. \left( \int_{\Sigma_\lambda} \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0 - y|^{n+s}} \, dy + \int_{\Sigma_\lambda} \frac{U_\lambda(x_0) - U_\lambda(y^\lambda)}{|x_0 - y^\lambda|^{n+s}} \, dy \right)
\leq C_{n,s} \left( \int_{\Sigma_\lambda} \frac{U_\lambda(x_0) - U_\lambda(y)}{|x_0|^{n+s}} \, dy + \int_{\Sigma_\lambda} \frac{U_\lambda(x_0) + U_\lambda(y)}{|x_0|^{n+s}} \, dy \right)
= 2C_{n,s} U_\lambda(x_0) \int_{\Sigma_\lambda} \frac{dy}{|x_0 - y^{n+s}|},
\]
where $x_0 = (x_0^1, x_0^2, \ldots, x_0^n)$, $x_0^\lambda = (2\lambda - x_0^1, x_0^2, \ldots, x_0^n) = (\lambda - x_0^1)e + y_0$, $e = (1, 0, \ldots, 0)$, and $y_0 = (\lambda, x_0^2, \ldots, x_0^n)$. Make a change of variables in (20) as $y = (\lambda - x_0^1)\bar{y} + y_0, \ y \in \Sigma_\lambda, \ \bar{y} \in \Sigma_0$.

Then we get
\[
\int_{\Sigma_\lambda} \frac{dy}{|x_0^\lambda - y|^{n+s}} = \frac{1}{(\lambda - x_0^1)^s} \int_{\Sigma_0} \frac{dy}{|e - y|^{n+s}}.
\]
Clearly, (19) follows from (20)-(21) immediately.

Now let us estimate (20)-(21) immediately.

Case 1. $V_\lambda(x_0) = v_\lambda(x_0) - v(x_0) \leq 0$.

By (13) and (17), at $x = x_0$ we have
\[
(K * v^2)\lambda v_\lambda - (K * v^2)v
= (K * v^2) V_\lambda + ((K * v^2)_\lambda - K * v^2) v_\lambda
= (K * v^2)(x_0) V_\lambda(x_0)
+ v_\lambda(x_0) \int_{\Sigma_\lambda} (K(x_0 - y) - K(x_0^\lambda - y)) (V_\lambda(y) + 2v(y)) V_\lambda(y) \, dy
\geq (K * v^2)(x_0) U_\lambda(x_0) + 2v(x_0) \int_{\Sigma_\lambda} (K(x_0 - y) - K(x_0^\lambda - y)) v(y) \, dy \cdot U_\lambda(x_0).
\]  

By (11), (19) and (22), we have
\[
\frac{a}{(\lambda - x_0^1)^s} U_\lambda(x_0) + U_\lambda(x_0)
\geq (K * v^2)(x_0) U_\lambda(x_0) + 2v(x_0) \int_{\Sigma_\lambda} (K(x_0 - y) - K(x_0^\lambda - y)) v(y) \, dy \cdot U_\lambda(x_0).
\]
Due to $U_\lambda(x_0) < 0$, we obtain (18).

Case 2. $V_\lambda(x_0) = v_\lambda(x_0) - v(x_0) \geq 0$.

In this case, by (13) and (17), at $x = x_0$ we deduce
\[
(K * v^2)\lambda v_\lambda - (K * v^2)v
\geq ((K * v^2)_\lambda - (K * v^2)) v
\]  

(23)
We estimate the integrals over these three sets separately.

\[ \int_{\Sigma} (K(x_0 - y) - K(x_0^\lambda - y)) (V_\lambda(y) + 2v(y))V_\lambda(y)dy \cdot v(x_0) \]

\[ \geq 2v(x_0) \int_{\Sigma} (K(x_0 - y) - K(x_0^\lambda - y)) v(y)dy \cdot U_\lambda(x_0). \]

By (11), (19) and (23), we have

\[ \frac{a}{(\lambda - x_1^0)^s}U_\lambda(x_0) + U_\lambda(x_0) \geq 2v(x_0) \int_{\Sigma} (K(x_0 - y) - K(x_0^\lambda - y)) v(y)dy \cdot U_\lambda(x_0) \]

and

\[ \frac{a}{(\lambda - x_1^0)^s} + 1 \leq 2v(x_0) \int_{\Sigma} (K(x_0 - y) - K(x_0^\lambda - y)) v(y)dy \]

\[ \leq (K \ast v^2)(x_0) + 2v(x_0) \int_{\Sigma} (K(x_0 - y) - K(x_0^\lambda - y)) v(y)dy. \]

Hence, we complete the proof. \( \square \)

To estimate the integral on the right hand side of inequality (18), we have the following lemma.

**Lemma 2.4.** There exists a constant \( c \) independent of \( \lambda \) such that for \( \lambda \leq 0, x \in \Sigma_\lambda \), there holds

\[ \int_{\Sigma} (K(x - y) - K(x^\lambda - y)) v(y)dy \leq c|x|. \]  

Moreover, we have

\[ v(x) \int_{\Sigma} (K(x - y) - K(x^\lambda - y)) v(y)dy \leq c. \]  

\[ v(x) \int_{\Sigma} (K(x - y) - K(x^\lambda - y)) v(y)dy \to 0, \quad \text{as } |x| \to \infty. \]

**Proof.** For \( \lambda \leq 0 \), we have \( 0 \not\in \Sigma_\lambda \). We divide \( \Sigma_\lambda \) into three parts

\[ \Sigma_\lambda^1 = \left\{ y \mid y \in \Sigma_\lambda, \ |x - y| \leq \frac{1}{2}|x| \right\}, \]

\[ \Sigma_\lambda^2 = \left\{ y \mid y \in \Sigma_\lambda, \ |x - y| \geq \frac{1}{2}|x|, \ |y| \leq 100|x| \right\}, \]

and

\[ \Sigma_\lambda^3 = \left\{ y \mid y \in \Sigma_\lambda, \ |x - y| \geq \frac{1}{2}|x|, \ |y| \geq 100|x| \right\}. \]

We estimate the integrals over these three sets separately.

For \( \Sigma_\lambda^1 = \left\{ y \mid y \in \Sigma_\lambda, \ |x - y| \leq \frac{1}{2}|x| \right\} \), we have

\[ |y| \geq |x| - |x - y| \geq |x| - \frac{1}{2}|x| = \frac{1}{2}|x|, \]

and by condition \((U_2)\) there exists a constant \( c \) such that \( v(y) \leq \frac{c}{|y|} \). Thus we can deduce

\[ \int_{\Sigma_\lambda^1} (K(x - y) - K(x^\lambda - y)) v(y)dy \leq \int_{|x-y| \leq \frac{1}{2}|x|} K(x - y)v(y)dy \]

\[ \leq \frac{c}{|x|} \int_{|x-y| \leq \frac{1}{2}|x|} \frac{dy}{|x-y|^{n-2}} = c|x|. \]
Finally, for $\Sigma_3^3 = \{ y \mid y \in \Sigma, |x-y| \geq \frac{1}{2}|x|, |y| \leq 100|x| \}$, by conditions (U_2) and (K_1), we have

$$\int_{\Sigma_3^3} (K(x-y) - K(x^\lambda - y)) v(y)dy \leq \int_{\Sigma_3^3} K(x-y) v(y)dy$$

$$\leq \int_{\{|x-y| \geq \frac{1}{2}|x|, |y| \leq 100|x|\}} \frac{v(y)}{|x-y|^{\alpha-2}} dy$$

$$\leq \frac{c}{|x|^{\alpha-2}} \int_{|y| \leq 100|x|} \frac{dy}{|y|} = c|x|.$$  \hfill (28)

Finally, for $\Sigma_3^3 = \{ y \mid y \in \Sigma, |x-y| \geq \frac{1}{2}|x|, |y| \geq 100|x| \}$, it follows from the Lagrange mean value theorem and condition (K_2) that

$$\int_{\Sigma_3^3} (K(x-y) - K(x^\lambda - y)) v(y)dy = \int_{\Sigma_3^3} (k(|x-y|) - k(|x^\lambda - y|)) v(y)dy$$

$$= \int_{\Sigma_3^3} k'(||x_t - y||) \frac{(x_t - y, x - x^\lambda)}{|x_t - y|} v(y)dy$$

$$\leq c|x - x^\lambda| \int_{\Sigma_3^3} \frac{v(y)}{|x_t - y|^{\alpha-1}} dy,$$  \hfill (29)

where $t = t(y) \in (0,1)$ and $x_t = tx + (1-t)x^\lambda$. For $\lambda \leq 0$, $x \in \Sigma$, we have $|x^\lambda| \leq |x|$, and thus

$$|x_t - y| \geq |y| - t|x| - (1-t)|x^\lambda| \geq |y| - |x| \geq \frac{99}{100}|y|.$$  \hfill (30)

It follows from (29) and condition (U_2) that

$$\int_{\Sigma_3^3} (K(x-y) - K(x^\lambda - y)) v(y)dy \leq c|x| \int_{|y| \geq 100|x|} \frac{v(y)}{|y|^{\alpha-2}} dy \leq c|x|.  \hfill (30)$$

As we see, (24) follows from (27)-(28) and (30), (25)-(26) follow from (24) and condition (U_2), and $v(x) = o(|x|^{-1})$ as $|x| \to \infty$. \hfill \Box

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We use the method of moving planes to prove that there exists $\lambda^* \in \mathbb{R}$ such that $U_{\lambda^*} \equiv 0$ and $V_{\lambda^*} \equiv 0$. We process the proof by three steps.

Step 1. There exists $\Lambda < 0$ such that $U_{\lambda} \geq 0$, $V_{\lambda} \geq 0$ in $\Sigma_{\lambda}$ for $\lambda < \Lambda$.

If it is not true that $U_{\lambda} \geq 0$, $V_{\lambda} \geq 0$ in $\Sigma_{\lambda}$, then $U_{\lambda}$ or $V_{\lambda}$ may be negative in $\Sigma_{\lambda}$.

Without loss of generality, we assume that (17) holds, that is, there exists $x_0 \in \Sigma_{\lambda}$ such that

$$U_{\lambda}(x_0) = \min_{x \in \Sigma} \{ \min_{x \in \Sigma} U_{\lambda}(x), \min_{x \in \Sigma} V_{\lambda}(x) \} < 0.$$  \hfill (18)

By Lemma 2.3, (18) holds. Given $\varepsilon > 0$, by Lemmas 2.1 and 2.4, there exists $\Lambda < 0$ such that for $\lambda \leq \Lambda$, $x_0 \in \Sigma_{\lambda}$ there hold

$$(K \ast v^2)(x_0) \leq \varepsilon,$$

$$v(x_0) \int_{\Sigma_{\lambda}} (K(x_0-y) - K(x_0^\lambda - y)) v(y)dy \leq \varepsilon,$$

since $x_0 \in \Sigma_{\lambda}$ and $|x_0| \geq |\lambda| \geq |\Lambda|$. Thus by (17) we obtain

$$\frac{a}{(\lambda - x_0)^{\alpha}} + 1 \leq \varepsilon + 2\varepsilon,$$
which is a contradiction for sufficiently small \( \varepsilon \). Hence, for \( \lambda \leq \Lambda < 0 \), we must have \( U_\lambda \geq 0 \), \( V_\lambda \geq 0 \) in \( \Sigma_\lambda \).

**Step 2.** Let

\[
\lambda_0 = \sup \{ \lambda | \lambda < 0, \forall \mu < \lambda, U_\mu \geq 0, V_\mu \geq 0 \text{ in } \Sigma_\mu \}.
\]

Then we have \( U_{\lambda_0} \equiv 0 \), \( V_{\lambda_0} \equiv 0 \), provided \( \lambda_0 < 0 \).

For \( \lambda < \lambda_0 \), we have \( U_\lambda \geq 0 \), \( V_\lambda \geq 0 \) in \( \Sigma_\lambda \). Since \( U_\lambda, V_\lambda \) continuously depend on \( \lambda \), we have \( U_{\lambda_0} \geq 0 \), \( V_{\lambda_0} \geq 0 \) in \( \Sigma_{\lambda_0} \). If \( \lambda_0 < 0 \), we consider \( \lambda \in (\lambda_0, 0] \). Assume for some \( \lambda \in (\lambda_0, 0] \) that

\[
\min_{x \in \Sigma_\lambda} \{ \min_{x \in \Sigma_\lambda} U_\lambda(x), \min_{x \in \Sigma_\lambda} V_\lambda(x) \} < 0.
\]

Again we can assume that there exists \( x_0 = (x_1^0, x_2^0, \ldots, x_n^0) \in \Sigma_\lambda \) such that (17)-(18) hold, i.e.

\[
U_\lambda(x_0) = \min_{x \in \Sigma_\lambda} \{ \min_{x \in \Sigma_\lambda} U_\lambda(x), \min_{x \in \Sigma_\lambda} V_\lambda(x) \} < 0,
\]

\[
a \frac{1}{(\lambda - x_i^0)^s} + 1 \leq (K * v^2)(x_0) + 2v(x_0) \int_{\Sigma_\lambda} (K(x_0 - y) - K(x - y)) v(y) \, dy.
\]

In view of (26), Lemmas 2.4 and 2.1, there exists \( R > 0 \) such that for \( x_0 \in \Sigma_\lambda \), \( |x_0| \geq R \), we have

\[
(K * v^2)(x_0) + 2v(x_0) \int_{\Sigma_\lambda} (K(x_0 - y) - K(x_0^\lambda - y)) v(y) \, dy \leq \frac{1}{2},
\]

and then

\[
a \frac{1}{(\lambda - x_i^0)^s} + 1 \leq \frac{1}{2}.
\]

This is a contradiction. Hence, if \( \lambda \in (\lambda_0, 0] \), then the negative minimum point \( x_0 \) must satisfy \( |x_0| \leq R \).

For \( \lambda < 0, x \in \Sigma_\lambda \), by Lemma 2.4 and inequality (25), there exists \( M \) such that

\[
(K * v^2)(x) + 2v(x) \int_{\Sigma_\lambda} (K(x - y) - K(x_0^\lambda - y)) v(y) \, dy \leq M.
\]

Choose \( \delta > 0 \) such that \( \frac{a}{\delta^s} \geq M \) and \( \lambda_0 + \delta < 0 \). Denote \( \delta_1 = \frac{1}{2} \delta \) and consider the bounded closed set

\[
B = \{ x \mid |x| \leq R, x \in \Sigma_{\lambda_0 - \delta_1} \},
\]

where \( R \) is defined as before with the property that the negative minimum point \( x_0 \) satisfies \( |x_0| \leq R \). Since \( U_{\lambda_0} \geq 0 \), \( V_{\lambda_0} \geq 0 \) in \( \Sigma_{\lambda_0} \), by Lemma 2.1 either \( U_{\lambda_0} \equiv 0 \), \( V_{\lambda_0} \equiv 0 \) in \( \Sigma_{\lambda_0} \), we are done; or \( U_{\lambda_0} > 0 \), \( V_{\lambda_0} > 0 \) in \( \Sigma_{\lambda_0} \). In the bounded closed set \( B \subset \Sigma_{\lambda_0} \), there exists \( \beta > 0 \) such that \( U_{\lambda_0}(x) \geq \beta \), \( V_{\lambda_0}(x) \geq \beta \) for \( x \in B \). By the continuity, there exists \( \delta_2 \leq \frac{1}{2} \delta \) such that

\[
U_\lambda(x) \geq \frac{1}{2} \beta > 0, \quad V_\lambda(x) \geq \frac{1}{2} \beta > 0 \text{ for } \lambda_0 \leq \lambda \leq \lambda_0 + \delta_2, \ x \in B.
\]

Therefore, for \( \lambda \in [\lambda_0, \lambda_0 + \delta_2] \), the negative minimum point \( x_0 \) must be in the set \( \Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta_1} \).

For \( x_0 \in \Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta_1} \), \( x_0 = (x_1^0, x_2^0, \ldots, x_n^0) \), \( \lambda \in (\lambda_0, \lambda_0 + \delta_2] \), we have

\[
\lambda_0 - \delta_1 \leq x_1^0 \leq \lambda \leq \lambda_0 + \delta_2.
\]

Therefore, \( \lambda - x_1^0 \leq \delta_1 + \delta_2 \leq \delta \). By (18) we have

\[
a \frac{1}{\delta^s} + 1 \leq \frac{a}{(\lambda - x_1^0)^s} + 1.
\]
Proposition 3.1. Let $x \in \mathbb{R}^N$.

The proof is similar to that of Lemma 2.2. Suppose $\delta > 0$.

Proof. Strictly decreasing in the later case.

Example 3: there exists $\gamma > 0$.

Example 1: $u(\cdot) = 0$ in $\Omega$.

Then $u(\cdot) = 0$ in $\Omega$.

Step 3. Proof of the Theorem.

We move the plane $T_\lambda$ from the right to the left, by applying the method of moving planes. Using an analogous process to Step 1, we can prove that there exists $\Lambda^+ > 0$ such that $U_\lambda \leq 0$, $V_\lambda \leq 0$ in $\Sigma_\lambda$ for $\lambda \geq \Lambda^+$. Similar to the definition of $\lambda_0$ in Step 2, we define

$$\lambda_0^+ = \inf \{ \lambda \mid \lambda > 0, \forall \mu > \lambda, U_\mu \leq 0, V_\mu \leq 0 \text{ in } \Sigma_\mu \}.$$

By the continuity of $U_\lambda$ and $V_\lambda$ on $\lambda$, we know $U_{\lambda_0^+} \leq 0$ and $V_{\lambda_0^+} \leq 0$. Similar to Step 2, we can prove $U_{\lambda_0^+} = 0$ and $V_{\lambda_0^+} \equiv 0$ in $\Sigma_{\lambda_0^+}$. Provided $\lambda_0^+ > 0$.

Finally, if $\lambda_0 = 0$, $\lambda_0^+ = 0$, then we have both $U_0 \equiv 0$, $V_0 \equiv 0$ and $U_0 \leq 0$, $V_0 \leq 0$ in $\Sigma_0$. Hence, we have $U_0 \equiv 0$, $V_0 \equiv 0$ in $\Sigma_0$. Anyway, we find $\lambda^+ \in \mathbb{R}$ such that $U_{\lambda^+} \equiv 0$, $V_{\lambda^+} \equiv 0$ in $\Sigma_{\lambda^+}$ (hence in $\mathbb{R}^n$).

By changing the normal of moving planes we can find $x_0 \in \mathbb{R}^n$ and functions $f, g$ such that $u(x) = f(|x - x_0|)$ and $v(x) = g(|x - x_0|)$. Let $x = x_0 + te$, $t \geq 0$.

Then $f(t) = u(x_0 + te)$ is strictly decreasing in $t \in [0, \infty)$. The same is true for $g(t)$, $t \geq 0$.

3. Remarks and extensions. (1) Remark on the condition $(U_2)$.

We give some examples of functions, which satisfy condition $(U_2)$.

Example 1: $u(x) = o(|x|^2)$ as $|x| \to \infty$.

Example 2: there exists $\gamma > 0$ such that $u(x) = O(|x|^{-(1+\gamma)})$ as $|x| \to \infty$.

Example 3: there exists $\gamma > 0$ such that $u(x) = O(|x|^{-1} \ln |x|^{-(1+\gamma)})$ as $|x| \to \infty$.

(2) Nonnegative solutions of system (1)

For nonnegative solutions of system (1), we have the following proposition.

Proposition 3.1. Let $(u, v)$ be a nonnegative solution pair of system (1). Then either $u \equiv 0$, $v \equiv 0$ or $u > 0$, $v > 0$ in $\mathbb{R}^N$. Consequently, $u, v$ are symmetric and strictly decreasing in the later case.

Proof. The proof is similar to that of Lemma 2.2. Suppose $u \geq 0$ and there exists $x_0 \in \mathbb{R}^N$ with $u(x_0) = 0$. Let $u$ satisfy system (1). It follows that

$$0 \leq (K \ast u^2)(x_0)v(x_0)$$

$$= (-\Delta)u^2u(x_0) + u(x_0)$$

$$= (-\Delta)u^2u(x_0) + C_{N, \alpha} P.V. \int_{\mathbb{R}^N} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+\alpha}} \, dy$$

$$= -C_{N, \alpha} P.V. \int_{\mathbb{R}^N} \frac{u(y)}{|x_0 - y|^{N+\alpha}} \, dy.$$ 

So, $u \equiv 0$. Now by system (1) we get

$$0 \leq (K \ast u^2)v = (-\Delta)u^2u + u = 0.$$

Hence, we have $v \equiv 0$. 

\qed
(3) Other types of systems of fractional Laplacian equations

As an example, we consider the following system of the fractional Laplacian equations
\[
\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}} u + u &= (K * u^2)v, \quad \text{in } \mathbb{R}^N, \\
(-\Delta)^{\frac{\alpha}{2}} v + v &= (K * v^2)u, \quad \text{in } \mathbb{R}^N.
\end{aligned}
\] (32)

In parallel with Theorem 1.1, we have the following theorem.

**Theorem 3.2.** Assume that \((u, v)\) is a positive solution of system (32). Assume that conditions \((U_1), (U_2), (K_1), (K_2)\) hold. Then \(u, v\) are radially symmetric and strictly decreasing about some point in \(\mathbb{R}^N\), i.e. there exist \(x_0 \in \mathbb{R}^N\) and two strictly decreasing functions \(f, g\) in \([0, \infty)\) such that \(u(x) = f(|x - x_0|)\) and \(v(x) = g(|x - x_0|)\).

The proof of Theorem 3.1 is closely similar to that of Theorem 1.1, so we omit it. Instead of Lemma 2.3, we have the following similar result.

**Lemma 3.3.** Suppose that \(\lambda \leq 0, x_0 = (\xi_1, \xi_2, \cdots, \xi_N) \in \Sigma_\lambda\) and
\[
U_\lambda(x_0) = \min \{ \min_{x \in \Sigma_\lambda} U_\lambda(x), \min_{x \in \Sigma_\lambda} V_\lambda(x) \} < 0.
\]
Then we have
\[
\frac{a}{(\lambda - \xi_1)^{\alpha}} + 1 \leq (K * u^2)(x_0) + 2v(x_0) \int_{\Sigma_\lambda} (K(x_0 - y) - K(x_0^\lambda - y)) v(y) dy.
\]

(4) Fractional Laplacian equations

Finally, let us consider the single fractional Laplacian equation
\[
(-\Delta)^{\frac{\alpha}{2}} u + u = (K * u^2)u, \quad \text{in } \mathbb{R}^N.
\] (33)
Assume that \(u\) is a positive solution of equation (33). Then the pair \((u, u)\) is a positive solution of system (1) (and system (32)). As a result, we have the following corollary.

**Corollary 3.4.** Assume that \(u\) is a positive solution of equation (33). Assume that conditions \((U_1), (U_2), (K_1), (K_2)\) hold. Then \(u\) is radially symmetric and strictly decreasing about some point in \(\mathbb{R}^N\), i.e. there exist \(x_0 \in \mathbb{R}^N\) and a strictly decreasing function \(f\) in \([0, \infty)\) such that \(u(x) = f(|x - x_0|)\).

Corollary 3.4 has been proved in [19] under the strong assumption by using the function as Example 1, i.e. \(u(x) = o(|x|^{-2})\) as \(|x| \to \infty\) instead of the weaker assumption \((U_2)\).

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