KBSM of the product of a disk with two holes and $S^1$

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**Abstract**

We introduce diagrams and Reidemeister moves for links in $F \times S^1$, where $F$ is an orientable surface. Using these diagrams we compute (in a new way) the Kauffman Bracket Skein Modules (KBSM) for $D^2 \times S^1$ and $A \times S^1$, where $D^2$ is a disk and $A$ is an annulus. Moreover, we also find the KBSM for the $F_{0,3} \times S^1$, where $F_{0,3}$ denotes a disk with two holes, and thus show that the module is free.

1. Introduction

Skein modules, as invariants of 3-manifolds, were introduced by J. Przytycki [8] and V. Turaev [10] in 1987 and have become important algebraic structures for studying 3-dimensional manifolds $M^3$ and knot theory in $M^3$. The Kauffman Bracket Skein Module (KBSM) is the most extensively studied skein module. We recall its definition here for the purposes of further sections.

**Definition 1.1.** Let $M^3$ be an oriented 3-manifold, $R$ a commutative ring with identity, and $A \in R$ a unit of $R$. A framed link is an embedded annulus in which the central curve of the annulus determines an unframed link. Let $L_{fr}$ be the set of ambient isotopy classes of unoriented framed links in $M^3$, including the empty link which we denote by $\emptyset$. We denote by $RL_{fr}$ the free $R$-module with basis $L_{fr}$ (we choose some ordering of the set $L_{fr}$). Let $SL_{fr}$ denote the submodule of $RL_{fr}$ generated by local relations shown

(K1): $L_+ = AL_0 + A^{-1}L_\infty,$

(K2): $L \cup T_1 = (-A^2 - A^{-2})L,$

where $T_1$ is the trivial framed link of one component (the trivial framed knot) and the triple $L_+, L_0, L_\infty$ is presented in Fig. 1. Then the Kauffman Bracket Skein Module, $S_{2,\infty}(M^3; R, A)$, of $M^3$ is defined to be $S_{2,\infty}(M^3; R, A) = RL_{fr}/SL_{fr}$.

The local relations are the ones that arise in the definition of the Kauffman bracket polynomial of a link. By local relations we mean that the three links $L_+, L_0, L_\infty$ are three framed links that are identical outside some small neighborhood.
It can quickly be seen that \( L^{(1)} = -A^3 L \) (where \( L^{(1)} \) is \( L \) with a positive twist) in \( S_{2,\infty}(M^3; R, A) \) which we call the framing relation.

As it was shown in [9] and [5] \( S_{2,\infty}(S^2; R, A) \) is free cyclic, \( S_{2,\infty}(S^1 \times I; R, A) \) and \( S_{2,\infty}(S^1 \times S^1 \times I; R, A) \) are free modules, generated by infinite sets, whereas \( S_{2,\infty}(S^2 \times S^1; R, A) \) has torsion. However, the KBSM has been found for considerably large classes of 3-dimensional manifolds [9,5,3,4,1], and the following problem was proposed in [7] (see Problem 4.4, p. 446): Find KBSM of the 3-manifold that is obtained as the product of a disk with two holes and \( S^1 \). Let \( F_{g,k} \) denote an orientable surface of genus \( g \) with \( k \) boundary components. In this paper, we compute \( S_{2,\infty}(F \times S^1; R, A) \), where \( F \) is a disk \( D^2 = F_{0,1} \), annulus \( A = F_{0,2} \), and disk with two holes \( F_{0,3} \). The modules \( S_{2,\infty}(F \times S^1; R, A) \) for \( F = D^2 \) and \( F = F_{0,2} \) were shown to be free in [9], however the methods developed for computing them will be used in our latter computations of \( S_{2,\infty}(F_{0,3} \times S^1; R, A) \). Our main result (see Theorem 5.3) stating that \( S_{2,\infty}(F_{0,3} \times S^1; R, A) \) is free solves, in particular, Problem 4.4 of J. Przytycki [7, p. 447] and supports the conjecture (see Conjecture 4.3 [7, p. 446]) that if every closed incompressible surface in \( M^3 \) is parallel to the boundary of the 3-manifold \( \partial M^3 \) then \( S_{2,\infty}(M^3; R, A) \) of \( M^3 \) is torsion free.\(^1\)

2. Diagrams of links in \( F \times S^1 \)

Let \( F \) be an orientable surface (possibly with boundary) and \( I = [0, 1] \). Let \( f : F \times I \to F \times S^1 \) be given by \( f(x, y) = (x, e^{2\pi i y}) \). Using an argument of general position we can assume that all links \( L \) in \( F \times S^1 \) are transversal to \( F \times \{1\} \). Then each \( f^{-1}(L) \) consists of embedded arcs in \( F \times I \) with all endpoints coming in pairs of the form \([x, 0], (x, 1)\] where \( x \in F \).

Let \( p \) denote the projection of \( F \times I \) onto \( F \times \{0\} = F \), given by \( p(x, y) = (x, 0) \). If \( D = p(f^{-1}(L)) \) then \( D \) consists of immersed curves in \( F \). A point in \( D \) that corresponds to a pair of endpoints of \( f^{-1}(L) \) is called a dot in \( D \). Again, applying an argument of general position, we may assume that all dots in \( D \) are distinct and that there are only transversal double points in \( D \) that are disjoint from dots. Near each of the double points of \( D \) we label two branches as upper and lower according to their corresponding values of the arguments \( y \) in the second coordinate. Moreover, while the link \( L \) crosses the surface \( f(F \times \{1\}) = f(F \times \{0\}) \) near the point \( (x_0, 1) \in F \times S^1 \) of the intersection) the corresponding arc component containing \( (x_0, 1) \in F \times I \) has the \( y \) values close to \( 1 \) and increasing in coordinate \( y \). Therefore, the corresponding dot in \( D \) is passed in the unique direction (determined by the increasing values of \( y \)). Hence, each link in \( F \times S^1 \) determines uniquely an assignment of arrows at dots in \( D \). Now, a diagram of \( L \) in \( F \times S^1 \) consists of \( D \) together with additional information regarding over and under branches (for double points of \( D \)) and an assignment of arrows (for dots in \( D \)). An example, showing the construction of the diagram of a link \( L \) in \( F_{0,3} \times S^1 \) is shown in Fig. 2.

Two links in \( F \times S^1 \) are isotopic if their diagrams differ by a finite sequence of "Reidemeister moves". These moves are obtained while we consider the resolution of generic singularities for the diagrams: cusps, tangency points, and triple points give us the classical Reidemeister moves \( \Omega_1, \Omega_2 \) and \( \Omega_3 \); double dots give us the fourth move \( \Omega_4 \); and dots combined with double points give us the fifth move \( \Omega_5 \). All moves are shown in Fig. 3. The geometric interpretation for the \( \Omega_4 \) and \( \Omega_5 \)-moves is shown in Fig. 4. We call the moves \( \Omega_2, \Omega_3, \Omega_4, \) and \( \Omega_5 \) regular Reidemeister moves.\(^2\)

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\(^1\) As it was shown in [5] and [11] incompressible (non-boundary parallel) surface can cause torsion in KBSM. Therefore, our result concerning \( S_{2,\infty}(F_{0,3} \times S^1; \mathbb{Z}(A^{2n}), A) \) also suggests that torsion in the KBSM cannot be caused by the existence of an immersed torus.

\(^2\) Analogy with the classical case of links in \( S^3 \).
3. Kauffman bracket skein module of $D^2 \times S^1$

Let $L$ be a link in $S^1 \times D^2$ and $D$ its diagram. Each double point of $D$ (crossing of $D$) can be equipped with a positive or a negative marker corresponding to the horizontal and vertical smoothings (see Fig. 5).

A state $s$ of the diagram $D$ is the choice of a marker for each crossing of $D$. Let $p(s)$ (resp. $n(s)$) be the number of crossings with positive (resp. negative) markers in $s$. Let $D(s)$ be the diagram obtained from $D$ by smoothing all crossings in $D$ according to the markers determined by $s$ and removing all pairs of opposite arrows by the $\Omega_4$-move. To each component of $D(s)$ is assigned an integer that gives the number of arrows on this component (arrows giving the counterclockwise orientation of the component are counted as positive, and those giving the clockwise orientation are counted as negative) as shown in Fig. 6. We will refer to such a component as a component with $n$ arrows.

A component $c$ of $D(s)$ is called trivial if there are no arrows on it and every connected component lying inside the disk bounded by $c$ also has no arrows. Let us denote by $|s|$ the number of trivial components of $D(s)$ and by $D'(s)$ the diagram obtained from $D(s)$ by removing all trivial components.

**Definition 3.1.** The Kauffman bracket of $D$ is given by the following sum taken over all states $s$ of $D$:

$$\langle D \rangle = \sum_s A^{p(s)}(-1)^{n(s)}(-A^2 - A^{-2})^{\mid s \mid} \cdot D'(s).$$
Lemma 3.2. The Kauffman bracket is preserved by $\Omega_2$, $\Omega_3$, and $\Omega_4$ moves.

Proof. The proof for the invariance of the bracket $\langle \rangle$ under $\Omega_2$ and $\Omega_3$-moves is analogous to the classical case (Kauffman bracket for classical diagrams [6]), and invariance of $\langle \rangle$ under $\Omega_4$-move follows directly from its definition. □

The bracket $\langle \rangle$ is not invariant under the $\Omega_5$-move. For this reason, we introduce a "refined version" of the bracket which is unchanged under the $\Omega_5$-move. Let us denote by $x$ the diagram $\bigcirc$ and by $\mathbb{N} = \{0, 1, 2, \ldots \}$ the set of all natural numbers. We show that the $S_{2,\infty}(D^2 \times S^1; R, A)$ is a free $R$-module, generated by $\{x^n | n \in \mathbb{N}\}$, where $x^n$ stands for $\bigcirc \bigcirc \ldots \bigcirc$ ($n$ copies of $x$, and $x^0 = \emptyset$).

Lemma 3.3. In the skein module $S_{2,\infty}(D^2 \times S^1; R, A)$ the following identity holds:

$$\bigcirc = A^{-6} \bigcirc.$$

Proof. Using the fact that the removal of a positive kink contributes $-A^3$ in the KBSM, and the removal of a negative kink contributes $-A^{-3}$ (just as it is for the classical case of the Kauffman bracket [6]), we have:

$$\bigcirc = -A^{-3} \bigcirc \bigcirc = -A^{-3} \bigcirc \bigcirc = A^{-6} \bigcirc.$$

The above calculation finishes our argument. □

Let $D_n$ be the diagram that consists of a single component with no crossings and $n$ arrows. Using the calculation shown in Fig. 7 and applying Lemma 3.3, we can express $D_n$ in $S_{2,\infty}(D^2 \times S^1; R, A)$ as the combination of $D_{n-1}$ and $D_{n-2}$. Therefore, the following recursion holds true in $S_{2,\infty}(D^2 \times S^1; R, A)$:

$$D_n = -A^{-2}x D_{n-1} - A^{-2} D_{n-2}.$$  \hspace{1cm} (3.1)

Let $D_{n,k}$ be the diagram with no crossings and consisting of $k + 1$ components, where $k$ of these components (corresponding to $x^k$) are encircled by a single component with $n$ arrows. We show in Fig. 8 how to express $D_{n,k}$ as a combination of $D_{n+1,k-1}$ and $D_{n,k-1}$. Hence, the following recursion holds true in $S_{2,\infty}(D^2 \times S^1; R, A)$:

$$D_{n,k} = (-A^4 + 1) D_{n+1,k-1} + A^{-2}x D_{n,k-1}.$$  \hspace{1cm} (3.2)
The recursive relations (3.1) and (3.2) motivate the following definition.

**Definition 3.4.** Let $P_n$ ($n \in \mathbb{Z}$) be polynomials in $x$ and coefficients in the ring $R$ defined inductively by $P_0 = -A^2 - A^{-2}$, $P_1 = x$, and:

$$P_n = -A^{-2}xP_{n-1} - A^2P_{n-2},$$

where the last relation is also used to define $P_n$ for all negative $n$.

Let $P_{n,k}$ ($n \in \mathbb{Z}, k \in \mathbb{N}$) be polynomials in $x$ and coefficients in $R$ defined inductively by: $P_{n,0} = P_n$, and for $k \neq 0$:

$$P_{n,k} = (-A^4 + 1)P_{n+1,k-1} + A^{-2}xP_{n,k-1}.$$  

We have for instance that $P_{-1} = A^{-6}x$. Now we are ready to define the refined Kauffman bracket.

**Definition 3.5.** Let $D$ be a diagram, $s$ be a state of $D$, and let $D'(s)$ denote the corresponding diagram with no crossings and no trivial components. The refined Kauffman bracket polynomial $\langle D \rangle_r \in R[x]$ is defined inductively as follows:

(i) First, we replace by $P_{n}$’s the most nested components (no components in the disks they bound) of $D'(s)$ that have $n$ arrows, which results in replacing such components by a linear combination of some $x^k$’s.

(ii) Second, we replace by $P_{n,k}$ each component with $n$ arrows that encircles only $x^k$.

The first and second steps are then repeated until $D'(s)$ is expressed as a polynomial in $x$ which we denote by $\langle D'(s) \rangle_r$. The refined Kauffman bracket of $D$ is given by the following sum taken over all states $s$ of the diagram $D$:

$$\langle D \rangle_r = \sum_s A^{p(s)-n(s)}(-A^2 - A^{-2})^{|s|}\langle D'(s) \rangle_r.$$  

We notice that the refined Kauffman bracket was defined in such a way that it clearly satisfies relations (3.1) and (3.2). Therefore, we have the following identities:

$$\langle D_n \rangle_r = -A^{-2}x\langle D_{n-1} \rangle_r - A^2\langle D_{n-2} \rangle_r \quad \text{and} \quad \langle D_{n,k} \rangle_r = (-A^4 + 1)\langle D_{n+1,k-1} \rangle_r + A^2x\langle D_{n,k-1} \rangle_r.$$  

It also follows from the definition that for the refined bracket $\langle \cdot \rangle_r$, the above two relations hold true even when they occur in a disk outside of which there are components which are not involved in the three diagrams appearing in the relations. We show that $\langle \cdot \rangle_r$ satisfies a generalized version of the relation (3.1).

Let $D_1, D_2, D_u$, and $D_d$ be four diagrams which are the same outside a small disk but which differ inside the disk as it is shown in Fig. 9.

**Lemma 3.6.** The refined Kauffman bracket satisfies:

1. $\langle D_u \rangle_r = -A^{-2}\langle D_r \rangle_r - A^2\langle D_d \rangle_r$;
2. $\langle D_d \rangle_r = -A^4\langle D_r \rangle_r - A^{-2}\langle D_u \rangle_r$.

**Proof.** We observe that it suffices to prove (1) and (2) for diagrams with no crossings and no trivial components. In $D'(s)$ let us consider only the components of $D'(s)$ that are inside the component $C$ to which the vertical segment shown in Fig. 9 belongs. Suppose that $\langle \cdot \rangle_r$ is applied to these components so that they are replaced by the linear combination of some $x^k$’s (polynomials in $x$). Therefore, it suffices to prove (1) and (2) in the case when there is only one $x^k$ inside $C$.

(1) First, let us assume that in $D_r$ the circle $x$ is outside relative to the component $C$. When computing $\langle \cdot \rangle_r$ any circles inside $C$ are first pushed outside using the relation (3.2). This modifies $D_u$, $D_r$ and $D_d$ in the same way and by the same linear combinations. Thus, we can assume that there are no $x$-components inside $C$. Considering only $C$ and the circles appearing in Fig. 9 (since contribution of the others is the same when computing $\langle \cdot \rangle_r$), assuming that there are $(n-1)$ arrows in the direction of the arrow on $D_u$ on the part of the arc $C$ that is not shown in the diagram, we notice that:

$$D_u = D_n, \quad D_d = D_{n-2} \quad \text{and} \quad D_r = xD_{n-1}, \quad \text{for some } n \in \mathbb{Z}.$$
Since the refined bracket $\langle \ \rangle_r$ satisfies relation (3.1), we have:

$$\langle D_u \rangle_r = -A^{-2}x\langle D_{n-1} \rangle_r - A^2\langle D_{n-2} \rangle_r.$$ 

which gives (1).

Suppose now that in $D_r$ the circle $x$ is inside relative to $C$. As before, when computing $\langle \ \rangle_r$, one pushes the $x$'s out of $C$ except for the circle in $D_r$, assuming that there are $(n+1)$ arrows on $C$ pointing in the direction opposite to the direction of $D_u$. In this case we observe that:

$$D_u = D_n, \quad D_d = D_{n+2}, \quad D_r = D_{n+1,1} \quad \text{and} \quad D_l = xD_{n+1}, \quad \text{for some } n \in \mathbb{Z}.$$ 

Relation (3.2) for $\langle \ \rangle_r$ gives:

$$\langle D_r \rangle_r = (-A^4 + 1)\langle D_d \rangle_r + A^{-2}\langle D_l \rangle_r$$

which is equivalent to:

$$-A^{-4}\langle D_l \rangle_r = (-A^2 + A^{-2})\langle D_d \rangle_r - A^{-2}\langle D_r \rangle_r.$$ 

Since $\langle \ \rangle_r$ satisfies relation (3.1), we also have:

$$\langle D_d \rangle_r = -A^{-2}\langle D_l \rangle_r - A^2\langle D_u \rangle_r,$$

which, in turn, is equivalent to:

$$\langle D_u \rangle_r = -A^{-4}\langle D_l \rangle_r - A^{-2}\langle D_d \rangle_r.$$ 

In the last equation, substituting for $-A^{-4}\langle D_l \rangle_r$ from the equation before, we obtain:

$$\langle D_u \rangle_r = (-A^2 + A^{-2})\langle D_d \rangle_r - A^{-2}\langle D_r \rangle_r - A^{-2}\langle D_d \rangle_r = -A^2\langle D_d \rangle_r - A^{-2}\langle D_r \rangle_r,$$

which proves the identity (1).

(2) From part (1) one has

$$\langle D_u \rangle_r = -A^{-2}\langle D_l \rangle_r - A^2\langle D_d \rangle_r,$$

which is equivalent to:

$$\langle D_d \rangle_r = -A^{-4}\langle D_l \rangle_r - A^{-2}\langle D_u \rangle_r.$$ 

Now, we observe that after rotating the diagrams in Fig. 9 by $\pi$, we have $D_u$ is switched with $D_d$ and $D_r$ is switched with $D_l$. Relabelling them accordingly in the last equation gives (2). \qed

**Proposition 3.7.** The refined Kauffman bracket $\langle \ \rangle_r$ is unchanged under regular Reidemeister moves. Therefore, $S_{2, \infty}(D^2 \times S^1; R, A)$ is a free $R$-module with basis $\{x^n \mid n \in \mathbb{N}\}$.

**Proof.** By the result of Lemma 3.2, the Kauffman bracket $\langle \ \rangle$ is preserved by $\Omega_2$, $\Omega_3$, and $\Omega_4$-moves, thus the same holds true for the refined Kauffman bracket $\langle \ \rangle_r$. It remains to study the case when diagrams $D$ and $D'$ differ by an $\Omega_2$-move. We may assume that there is only one crossing in these diagrams, with all others being smoothed using the relation (K1). Moreover, we may assume that inside the disks bounded by the component with the crossing all other components are already expressed as polynomials in $x$. Moreover, as the contribution of components outside the component (on which the move is performed) is the same for both $D$ and $D'$, we can additionally assume that there are no such components. Thus, we need to check the invariance of $\langle \ \rangle_r$ in the two cases shown in Figs. 10 and 11.

In Fig. 10 one needs to verify that $\langle D \rangle_r = \langle D' \rangle_r$. Applying Lemma 3.6 for $D$ and $D'$, one pushes the $x$'s in the parts with $m$ arrows (at the expense of getting diagrams with $m + 1$ and $m - 1$ arrows). Analogously, the same happens for the part with $n$ arrows. Therefore, it is sufficient to check the case when $k = 0$ and $l = 0$. Now, applying Lemma 3.6 again for both diagrams, one may push out an arrow from the part with $n$ arrows, at the expense of some exterior $x$ and diagrams with
\[ l = n \quad \text{induces a natural order of the} \]

written as a linear combination of diagrams shown in Fig. 12. Moreover, an order of the two boundary components in

\[ \partial A \]

4. Kauffman bracket skein module of \( A \times S^1 \)

Let \( A = F_{0,2} \) denote an annulus. The Kauffman bracket for diagrams of links in \( A \times S^1 \) is defined in a way that is analogous to the previous case (see Definition 3.1). However, we notice that after smoothing all of the crossings in such a diagram, there are now two types of components: ones bounding a disk (called the \( x \)-type) or ones parallel to the boundary \( \partial A = S^1 \cup S^1 \) components (called the \( y \)-type).

The Kauffman bracket is first refined in the way analogous to the one introduced in Definition 3.5. Using the refined bracket for the \( x \)-type components enclosed between two successive components of the \( y \)-type allows us to express the \( x \)-type components as linear combinations of \( x^n \) (\( n \geq 0 \)) over \( R \). Thus, the refined bracket \( \langle D \rangle \) for the diagram \( D \) can be written as a linear combination of diagrams shown in Fig. 12. Moreover, an order of the two boundary components in \( \partial A \) induces a natural order of the \( y \)-type components and all terms \( x^n \) enclosed by any two successive \( y \)-type components. We
use the ordering from the “interior” $S^1$ to the “exterior” $S^1$ of $\partial A$ for all of our diagrams. Therefore, diagrams that constitute terms of the polynomial $\langle D \rangle_r$ can be encoded uniquely by words of the form

$$x^{n_1}y_{m_1}x^{n_2} \cdots y_{m_k}x^{n_l},$$

where $n_i \in \mathbb{N}$, $m_i \in \mathbb{Z}$, and $y_1$ stands for the $y$-type component with $l$ arrows (where the sign of $l$ satisfies the previously introduced convention that counterclockwise orientation is counted as positive). For example the diagram shown in Fig. 12 is encoded by the word $x^4y_{-1}x^2y_2x^6$.

For our convenience, we identify such diagrams with words.

Every word $w = y_{m_1}y_{m_2} \cdots y_{m_k}x^n$, where $m_i = 0$ or $1$; $n \in \mathbb{N}$ will be called semi-reduced. For a given word $w$ we define inductively the semi-reduced refined bracket $\langle w \rangle_{srr}$ corresponding to the diagram of $w$, as a linear combination of semi-reduced words.

1. If $w$ is semi-reduced, we set $\langle w \rangle_{srr} = w$.

If $w$ is not semi-reduced then $w$ contains the letter $x$ before some $y_n$ or the letter $y_n$ where $n \in \mathbb{Z} \setminus \{0, 1\}$. We write $w = w_1w_2$, where $w_1$ is semi-reduced and contains no $x$.

2. If $w_2$ starts with the letter $x$ then $w_2 = x^ky_nw_3$ for some word $w_3$. Using the calculation shown in Fig. 8, we obtain the following relation in the KBSM:

$$w_1x^ky_nw_3 = (A^4 + 1)w_1x^{k-1}y_{n+1}w_3 + A^{-2}w_1x^{k-1}y_nxw_3,$$

and in this case we define

$$\langle w \rangle_{srr} = (A^4 + 1)w_1\langle x^{k-1}y_{n+1}w_3 \rangle_{srr} + A^{-2}w_1\langle x^{k-1}y_nxw_3 \rangle_{srr}.$$  

3. If $w_2$ starts with $y_n$, where $n > 1$ (that is, $w_2 = y_nw_3$ for some word $w_3$), using the calculation shown in Fig. 7 we obtain the following relation in the KBSM:

$$w_1y_nw_3 = -A^{-2}w_1y_{n-1}xw_3 - A^2w_1y_{n-2}w_3,$$

and accordingly, we define

$$\langle w \rangle_{srr} = -A^{-2}w_1\langle y_{n-1}xw_3 \rangle_{srr} - A^2w_1\langle y_{n-2}w_3 \rangle_{srr}.$$  

4. If $n < 0$ then the last relation is used to express the word containing $y_n$ with words containing $y_{n+1}$ and $y_{n+2}$ and we define $\langle w \rangle_{srr}$ accordingly.

It can easily be seen that the inductive definition of $\langle w \rangle_{srr}$ results with a linear combination of semi-reduced words.

**Definition 4.1.** Let $D$ be a diagram and let $\langle D \rangle_r$ be a linear combination of words $w$ with coefficients in $R$, that is, $\langle D \rangle_r = \sum_w p_w w$, $p_w \in R$. We define the semi-reduced refined Kauffman bracket of the diagram $D$ as follows:

$$\langle D \rangle_{srr} = \sum_w p_w \langle w \rangle_{srr}.$$

Now, we prove another generalized version of Lemma 3.6.

**Lemma 4.2.** The semi-reduced refined Kauffman bracket satisfies the following properties:

1. $\langle D_u \rangle_{srr} = -A^{-2} \langle D_l \rangle_{srr} - A^2 \langle D_d \rangle_{srr}$;
2. $\langle D_u \rangle_{srr} = -A^{-4} \langle D_l \rangle_{srr} - A^{-2} \langle D_d \rangle_{srr}$

where the diagrams $D_r$, $D_l$, $D_u$, and $D_d$ are shown in Fig. 9.
Proof. If the vertical segment shown in Fig. 9 is a part of the x-type component, then the proof is the same as for Lemma 3.6 and, moreover, the identities (1) and (2) hold true even for the refined Kauffman bracket $⟨⟩_{r}$. For the other case (when the vertical line segment is a part of the y-type component) we observe, as before in the proof of Lemma 3.6, that the relations (3.1) and (3.2) hold true for $⟨⟩_{srr}$, where instead of $D_{n}$ we take $y_{n}$, and any configuration of the remaining components. Indeed, the components between the first boundary component $S^{1}$ of $\partial A$ and $y_{n}$ are reduced while computing $⟨⟩_{srr}$ to a point when there is some $x^{k}$ to be pushed through the $y_{n}$. The rest of the proof is the same as for Lemma 3.6: one pushes all x’s, or all but one, through $y_{n}$, and then (1) follows from definition of $⟨⟩_{srr}$. The property (2) is a consequence of (1) exactly for the same reasons as in the proof of Lemma 3.6. □

Since the Kauffman bracket $⟨⟩$ is invariant under $\Omega_{2}$, $\Omega_{3}$, and $\Omega_{4}$-moves it follows that the same is true for all other brackets we defined. To check the invariance of $⟨⟩_{srr}$ under the $\Omega_{5}$-move one may, as before, consider diagrams where all crossings are smoothed except for the crossing directly involved in the move. In $S^{1} \times A$ we have five versions (types) of such moves shown in Fig. 13, where only the first boundary component $S^{1}$ of $\partial A$ is shown and, instead of presenting the actual $\Omega_{5}$-move, we show only the type of a component on which this move is performed.

For the moves of type I and II shown in Fig. 13 (where smoothing in both ways gives components bounding disks in A) $⟨⟩_{r}$ is invariant by Proposition 3.7 and therefore the same holds true for $⟨⟩_{srr}$.

**Proposition 4.3.** The semi-reduced refined Kauffman bracket, $⟨⟩_{srr}$, is invariant under $\Omega_{5}$-moves of type III and IV shown in Fig. 13.

Proof. For the move of type III the situation is shown in Fig. 14. We may assume that $⟨⟩_{srr}$ is applied locally inside of the disks and annuli that are introduced later after all crossings are smoothed. Applying Lemma 4.2, we can push all the components $x^{k}$ outside of the component on which the $\Omega_{5}$-move is applied and then, using the same lemma again, we can also reduce $n$ to 0 or 1. Moreover, we can disregard all the components inside and outside the component on which $\Omega_{5}$-move is applied (see Fig. 14) since they equally contribute to $⟨D⟩_{srr}$ and $⟨D′⟩_{srr}$.

For $n = 0$, smoothing both sides gives the following relation

$$-A^{-3}y_{m+1} = Ay_{m}A^{-6}x + A^{-1}y_{m-1}$$

which, after multiplying by $-A^{3}$ gives the relation (3.1) which holds for $⟨⟩_{srr}$.

For $n = 1$, we have:

$$Ay_{m+2} + A^{-1}y_{m+1}x = -A^{3}y_{m}$$

which, after multiplying by $A^{-1}$ becomes again the relation (3.1) which holds for $⟨⟩_{srr}$.

For type IV the situation is presented in Fig. 15.

As in the previous case, we reduce $n$ to 0 or −1 and disregard all components except the one on which the $\Omega_{5}$-move is applied. We consider again both cases below.

For $n = 0$, after smoothing the crossing in $D'$ and an application of relations (3.2) and (3.1), which hold for $⟨⟩_{srr}$, we obtain

$$Ay_{m} + A^{-1}xy_{m} = Ay_{m-1} + (-A^{3} + A^{-1})y_{m+1} + A^{-3}y_{m}x$$

which is the same as the result of the smoothing of the crossing in $D$. 

![Fig. 13. Five types of $\Omega_{5}$ moves.](image)

![Fig. 14. Invariance of $⟨⟩_{srr}$ under the $\Omega_{5}$-move of type III.](image)
For $n = -1$, smoothing the crossing in $D$ and using relations (3.2) and (3.1), which hold for $\langle \rangle_{srr}$, gives

$$AA^{-6}xy_{m+1} + A^{-1}y_{m+2} = (-A^{-1} + A^{-5})y_{m+2} + A^{-7}y_{m+1}x + A^{-1}y_{m+2}$$

$$= -A^{-7}y_{m+1}x - A^{-3}y_{m} + A^{-7}y_{m+1}x = -A^{-3}y_{m},$$

which is the same as the result of smoothing of the crossing in $D'$. \(\Box\)

In order to show the invariance under the $\Omega_5$-move of all five types shown in Fig. 13, we need another refinement where the semi-reduced words are expressed as the reduced words or words of the form $y_{m_1}y_{m_2}\cdots y_{m_k}x^n$, where the $m_1 = m_2 = \cdots = m_k = 0$, and $m_{k+1}$ is 0 or 1, $n \in \mathbb{N}$. To simplify our notations we set $y_0 = y$ and $y_1 = y'$, thus the reduced words have the form $y_kx^n$ or $y_ky'y^n$.

Let $w$ be a semi-reduced word (and at the same time the diagram represented by this word). We define inductively the reduced refinement $\langle w \rangle_{rr}$ as a linear combination of reduced words.

1. If $w$ is reduced, we set $\langle w \rangle_{rr} = w$.
2. Otherwise $w$ must contain a subword of the form $y'y$ or $y'y'$.
3. Suppose that $w = y^k y' y_{2}$. In the KBSM the word $w$ satisfies the following relation:

$$y^k y' y_{2} = (-A^{-4} + 1)y^k x_{2} w_{2} + A^2 y^k y y_{2} w_{2}$$

as it is shown in Fig. 16, and we accordingly define

$$\langle w \rangle_{rr} = (-A^{-4} + 1)y^k |(x_{2} w_{2})_{srr}\rangle_{rr} + A^2 y^k y |y' y_{2} w_{2}\rangle_{rr}.$$
Again, we see that $\langle w \rangle_{rr}$ can be expressed by the word containing more $y$’s at the beginning than $w$ (and the number of components of the $y$-type remains same) and the words containing less $y$-type components.

It can easily be seen that such an inductive definition of $\langle w \rangle_{rr}$ results in a linear combination of the reduced words as needed.

**Definition 4.4.** Let $D$ be a diagram and let $\langle D \rangle_{sr}$ be a linear combination of the semi-reduced words $w$, that is, $\langle D \rangle_{sr} = \sum_w P_w w$ for some $P_w \in R$. We define the reduced refined Kauffman bracket of $D$ by setting

$$\langle D \rangle_{rr} = \sum_w P_w \langle w \rangle_{rr}.$$ 

**Proposition 4.5.** The reduced refined Kauffman bracket $\langle \_ \rangle_{rr}$ is invariant under all regular Reidemeister moves. Therefore, $S_{2, \infty}(A \times S^1; R, A)$ is a free $R$-module with basis that consists of all reduced words.

**Proof.** Being a refinement of the previous bracket, it remains to show that $\langle \_ \rangle_{rr}$ is invariant under Reidemeister $\Omega_2$-move of the type $V$ (see Fig. 13). The situation is shown in Fig. 18.

Applying Lemma 4.2 we push out (of the component on which the $\Omega_2$-move is applied) all $x$-type components, and reduce $n$ to 0 or 1. The components that appear between the component in Fig. 18 and the second boundary component $S^1$ of $\partial A$ can be disregarded since their contribution to $\langle \_ \rangle_{rr}$ is the same for $\langle D \rangle_{rr}$ and $\langle D' \rangle_{rr}$ and comes after the contribution of the component on which the $\Omega_2$-move is applied. Therefore, we can assume (for the simplicity of our proof) that there are no such components. For the components of the $y$-type between the first boundary component $S^1$ of $\partial A$ and the component with the crossing appearing in Fig. 18, we start applying $\langle \_ \rangle_{rr}$ and arrive at the situation where these components form $y^k$ or $y^l y^m$. Thus, we have two cases to address:

**Case 1.** The $y$-type components form $y^k$. Then we have

$$\langle D \rangle_{rr} = A\langle y^k y_{m-1} y_m \rangle_{rr} + A^{-1}\langle y^k p_{m-n+1} \rangle_{rr}$$

and

$$\langle D' \rangle_{rr} = A\langle y^k p_{m-n-1} \rangle_{rr} + A^{-1}\langle y^k y_n y_{m-1} \rangle_{rr}.$$ 

For simplicity, we omit $y^k$ at the beginning of each word. If $n = 1$ then applying Lemma 4.2 we reduce $m$ to 1 or 2.

If $n = m = 1$ then from the definition of $\langle \_ \rangle_{rr}$ it follows that

$$\langle D' \rangle_{rr} = A p_{-1} + A^{-1} (y_1 y_0)_{rr} = A^{-5} x + (A^{-5} + A^{-1}) x + A y y' = A y y' + A^{-1} x = \langle D \rangle_{rr}.$$ 

If $n = 1$ and $m = 2$ then we have:

$$\langle D' \rangle_{rr} = A (-A^2 - A^{-2}) + A^{-1} (y'y)_n = (-A^2 - A^{-1}) - A^{-3} x^2 + (2A^3 + 2A^{-1}) + A (y y_2)_{rr} \nonumber
\quad = A^3 + A^{-1} - A^{-3} x^2 + A (y y_2)_{rr} = A (y y_2)_{rr} + A^{-1} (p_2)_{rr} = \langle D \rangle_{rr}.$$ 

If $n = 0$ one reduces $m$ to 0 or 1.

If $n = m = 0$ then using relations (3.1) and (3.2) while computing $\langle \_ \rangle_{sr}$ we have

$$\langle D \rangle_{rr} = A (y_1 y)_n + A^{-1} x = -A^{-3} (y y y)_{rr} - A^{-1} (y y)'_{rr} + A^{-1} x$$

$$= -A^{-3} (A^4 + 1) y y' - A^{-5} y y x - A^{-1} (A^{-4} + 1) x - A y y' + A^{-1} x$$

and respectively,

$$\langle D' \rangle_{rr} = A (p_{-1})_{rr} + A^{-1} (y y - 1)_{rr} = A^{-5} x - A^{-3} y y' - A^{-5} y y x.$$

![Fig. 18. Invariance of $\langle \_ \rangle_{rr}$ under the Reidemeister moves $\Omega_2$ of type $V$.](image-url)
Therefore, we have
\[ \langle D \rangle_{rr} = \langle D' \rangle_{rr}. \]

If \( n = 0 \) and \( m = 1 \) then, we have
\[
\langle D \rangle_{rr} = A(y_{-1}y_1)_{rr} + A^{-1}(P_2)_{rr} = -A^{-1}(y'y')_{rr} - A^{-3}(xyy')_{rr} - A^{-3}x^2 + A^3 + A^{-1} \\
= A^{-3}x^2 - 2A^3 - 2A^{-1} - A(yy)_y^{-1} - A^{-3}(-A^4 + 1)(yy)_y^{-1} - A^{-3}yy'x - A^{-3}x^2 + A^3 + A^{-1} \\
= -A^3 - A^{-1} - A^{-3}(yy)_y^{-1} - A^{-5}yy'x = -A^3 - A^{-1} + A^{-5}yy'x + A^{-3}yy - A^{-5}yy'x \\
= -A^3 - A^{-1} + A^{-1}yy
\]
and respectively, for \( \langle D' \rangle_{rr} \) we have
\[
\langle D' \rangle_{rr} = A(-A^2 - A^{-2}) + A^{-1}yy = -A^3 - A^{-1} + A^{-1}yy.
\]
Therefore, again it follows that
\[ \langle D \rangle_{rr} = \langle D' \rangle_{rr}. \]

**Case 2.** The \( y \)-type components form \( y^ky' \). The situation is shown in Fig. 19.

We apply the \( \Omega_2 \)-move followed by the \( \Omega_2 \)-move on both \( D \) and \( D' \) and observe that the refined bracket \( \langle \_ \rangle_{rr} \) is unchanged for both diagrams. This is clear for the \( \Omega_2 \)-move. For the \( \Omega_2 \)-move shown in Fig. 19 one verifies the invariance of \( \langle \_ \rangle_{rr} \) by considering all smoothings of the two crossings that are not involved in the move. After doing so, one obtains moves that are not of the type \( V \) and a move of type \( V \) to which Case 1 applies (because inside the component \( y' \) there is only \( y^k \)). Now, we apply the desired \( \Omega_2 \)-move between \( D \) and \( D' \) that we just transformed by the two Reidemeister moves in Fig. 19. The \( \Omega_2 \)-move leaves \( \langle \_ \rangle_{rr} \) unchanged because again, by considering all smoothings of the two crossings that are not involved in this move, we obtain moves that are not of type \( V \) and a move of type \( V \) to which Case 1 applies. \( \square \)

As it was shown by Przytycki [9] the basis for the KBBSM of \( A \times S^1 \) consists of all framed simple closed curves on torus and their parallel copies (including the empty curve). Another basis for the KBBSM of \( A \times S^1 \) was found by Frohman and Gelca [2]. The basis of all reduced words found by us provides yet another alternative for a basis of this skein module.

### 5. Kauffman bracket skein module of \( F_{0,3} \times S^1 \)

Recall that we denoted the disk with two holes by \( F_{0,3} \). The Kauffman bracket, for the case \( F_{0,3} \times S^1 \), is defined again just as in Definition 3.1. After smoothing all crossings in a diagram of the link \( L \) we can now encounter four types of components: ones bounding a disk (of the \( x \)-type) or ones parallel to one of the three boundary components \( S^1 \) of \( \partial F_{0,3} \). We represent \( F_{0,3} \) as the disk \( D^2 \) with two smaller disks removed, one on the left denoted by \( D^2_a \) and one on the right denoted by \( D^2_b \). Components parallel to \( \partial D^2_a \) and \( \partial D^2_b \) and \( \partial D^2 \) are called, respectively, of the \( y \)-type, \( z \)-type and \( t \)-type. As in the case of \( A \times S^1 \), diagrams can be expressed using words. An example of a diagram corresponding to the word \( y_0xyy_z0z_0x^2z_2z_0x \) is shown on the left in Fig. 20. Note that for the components of \( t \)-type the order is from the circle \( \partial D^2 \) into the interior and the arrows corresponding to the clockwise orientation are counted as positive. In such words components of the \( y \)-type are always written before the components of the \( z \)-type and the \( t \)-type, and components of the \( z \)-type are always written before components of the \( t \)-type. If \( x^0 \) appears at the end of the word, it is in between the \( y \), \( z \) and \( t \) components. Otherwise, if it is before—say the \( y \)-type component—then it is placed somewhere in between the \( y \)-type components as it is shown on the left in Fig. 20. As before, to simplify our notations we set \( y_0 = y \), \( y_1 = y' \), \( z_0 = z \), \( z_1 = z' \), \( t_0 = t \) and \( t_1 = t' \).

The bracket \( \langle \_ \rangle \) is constructed exactly like for \( D^2 \times S^1 \) and \( A \times S^1 \), and the constructions of \( \langle \_ \rangle_{31} \) and \( \langle \_ \rangle_{rr} \) are done analogously as for \( A \times S^1 \) by considering separately the components of types \( y \), \( z \), and \( t \). The \( x \)-type components and the arrows are pushed by this procedure towards the area between the components of the three types. In this way \( \langle \_ \rangle \) is constructed and allows us to expresses diagrams as the linear combination of the words \( y^aw^bz^ct^d \) where \( a, b, c, d \in \mathbb{N} \) and \( a', b', c' \in \{0, 1\} \). Such words, as before, are called reduced. An example, represented by \( y'z^2tt'x^2 \), appears on the right in Fig. 20.
The refined reduced bracket \( \langle \_ \rangle_{\Omega} \) is invariant under \( \Omega_2 \)-moves shown in Fig. 13 (where the additional hole \( D^2 \) is not shown and is outside of the moves) using the same arguments as in the proofs for \( D^2 \times S^1 \) and \( A \times S^1 \). Fig. 21 shows some of the types of \( \Omega_2 \)-moves for which \( \langle \_ \rangle_{\Omega} \) is invariant, and three new types of moves, for which we introduce new refinements of the bracket necessary for us to show the invariance under the \( \Omega_2 \)-move. We first construct the refinement which will be invariant under the \( \Omega_2 \)-move for all old-type diagrams and some new types of moves. Then, finally, we construct a refinement which will be invariant under all \( \Omega_2 \)-moves, therefore letting us define the map from \( \Omega_2,F_0,3 \times S^1; R,A \) to a free \( R \)-module with an explicit basis.

We notice that in the KBSM it is possible for arrows to jump between components of type \( y, z \) and \( t \). Using this idea, we define the quasi-final bracket \( \langle w \rangle_{af} \) for the reduced words \( w \). A reduced word is called quasi-final if it has at most one prime (i.e. \( y', z' \) or \( t' \)) and after the occurrence of such a prime only \( x^0 \) can follow it. For instance, the reduced word \( yzz'tx^2 \) is not quasi-final because after \( z' \) we have \( t \) that follows it, whereas the reduced word \( yy'y'x \) is quasi-final. Letting \( w \) be a reduced word, we define the quasi-final bracket \( \langle w \rangle_{af} \) inductively:

1. If \( w \) is a quasi-final word then we set, \( \langle w \rangle_{af} = w \).

Otherwise there are several cases to consider depending on the form of the reduced word \( w \) and values for \( a, b, c, d \in \mathbb{N} \):

2. If \( w = y^a y^b t^c t' c' x^d \), where \( c' = 0 \) or \( 1 \) and \( b > 0 \), then in the KBSM, as shown on the top part of Fig. 22, the word \( w \) satisfies the following relation:

\[
y^a y^b t^c t' c' x^d = A^2 y^{a+1} z^{b-1} z_{-1} t^c t' c' x^d + 2 y^a z^{b-1} t^c t' c' x^d t + A^{-2} y^a z^{b-1} t^c t' c' x^d t x.
\]

3. If \( w = y^a y^b z^c t^c x^d \), where \( c' = 0 \) or \( 1 \), then in the KBSM, as shown on the bottom part of Fig. 22, the word \( w \) satisfies the following relation:

\[
y^a y^b z^c t^c x^d = A^2 y^{a+1} z^{b+1} t^c x^d + 2 y^a z^b t^c x^d t + A^{-2} y^a z^b t^c x^d t x^{-1} x.
\]

4. If \( w = y^a y^c x^d \), where \( c > 0 \), then in the KBSM, as shown on the top of Fig. 23, the word \( w \) satisfies the following relation:

\[
y^a y^c x^d = A^2 y^{a+1} c^{-1} x^{-1} x^d + 2 y^a x^d z^c x^d x + A^{-2} y^a x^d z^c x^d.
\]
Fig. 23. Quasi-final bracket $\langle w \rangle_{qf}$ for $w = y^a y^t x^d$ (top) and $w = y^a y^t x^d$ (bottom).

Fig. 24. Passing the arrow from the component of $y$-type to one of $t$-type.

(5) If $w = y^a y^t x^d$, then in the KBSM, as shown on the bottom of Fig. 23, the word $w$ satisfies the following relation:

$$y^a y^t x^d = A^2 y^a + 1 + 1 y^d + 2 y^a x^d z t^c - 1 + A^{-2} y^a x^d z_{-1} t^c - 1 x.$$  

For cases (6) and (7) similar identities to the ones shown in Fig. 23 hold, with the roles of $y$ and $z$ components switched.

(6) If $w = y^b z^b y^t x^d$, where $c > 0$, then in the KBSM the word $w$ satisfies the following relation:

$$y^b z^b y^t x^d = A^2 y^b z^{b+1} t^{-1} x^d + 2 y^b x^d y^b t^c - 1 + A^{-2} y^a z^b x^d z_{-1} t^c - 1 x.$$  

(7) If $w = y^b z^b y^t x^d$, then in the KBSM the word $w$ satisfies the following relation:

$$y^b z^b y^t x^d = A^2 y^b z^{b+1} t^{-1} x^d + 2 y^b x^d y^b t^c + A^{-2} y^a z^b x^d z_{-1} t^c - 1 x.$$  

Thus, in each of the six cases (2)–(7) we can express the reduced word $w$ in the form $A^2 P + 2Q + A^{-2} R$ for some appropriate diagrams (different in each case) $P$, $Q$ and $R$. We define the quasi-final Kauffman bracket of $D$ by setting

$$\langle D \rangle_{qf} = A^2 \langle P \rangle_{qf} + 2\langle Q \rangle_{qf} + A^{-2}\langle R \rangle_{qf}.$$  

To see that this inductive definition terminates resulting with the linear combination of the quasi-final words, we notice that in $Q$ and $R$ the sum of components of types $y$, $z$ and $t$ is decreased by one. Moreover, for $P$ an arrow is moved from the $y$-type component to the component of $z$- or $t$-types, and from component of the $z$-type to component of $t$-type, yielding finally a quasi-final diagram.

**Definition 5.1.** Let $D$ be a diagram and let $\langle D \rangle_{qf}$ be a linear combination of some reduced words $w$, that is, $\langle D \rangle_{qf} = \sum_{w} P_{w} w$ for $P_{w} \in R$. We define the quasi-final Kauffman bracket of $D$ by setting:

$$\langle D \rangle_{qf} = \sum_{w} P_{w} \langle w \rangle_{qf}.$$  

In the KBSM the relation shown in Fig. 24 holds.
By passing the arrow from the component of y-type to the t-type component directly we obtain another relation presented in Fig. 25.

The elements X, Y and Z have less components of the y, z or t-types. Setting equal the last terms of the two equations above and rearranging the terms gives the relation in the KBSM as in Fig. 26.

Analogously, when there is an arrow on the t-type component, there are two ways of moving the arrow from the y to the t-type component: first via z as shown in Fig. 27 or directly as shown in Fig. 28.

Setting equal the last terms of these two equations and rearranging terms gives a relation in the KBSM that is shown in Fig. 29.

A reduced word is called final if it is quasi-final and does not contain components of all the 4 types (x, y, z, and t). We define inductively the final bracket \( \langle \rangle_f \) as follows:

1. If \( w \) is final then we set \( \langle w \rangle_f = w \).
2. If \( w = y^a z^b t^c x^d + 1 \) then

\[
\langle w \rangle_f = -2A^2\langle y^a z^b t^{-1} x^d \rangle_f + A^2\langle (Z - Y - X) \rangle_f.
\]
where

\[
Z = 2y^{a1}z^{b1}x^{d}z^{c}t^{-1} + A^{-2}y^{a1}z^{b1}x^{d}zt^{-1}x,
\]
\[
X = 2y^{a1}z^{b1}t'c'x't' + A^{-2}y^{a1}z^{b1}t'x'tx.
\]

(3) If \( w = y^{a}z^{b}t'c't'x'^{d+1} \) then

\[
(w) = -2A^{4}(y^{a}z^{b}t'^{1}x'^{1})_f + A^{2}((Y' + X')_qf)_f,
\]

where

\[
Z' = 2y^{a1}z^{b1}x^{d}z't' + A^{-2}y^{a1}z^{b1}x't'x',
\]
\[
X' = 2y^{a1}z^{b1}t't'x't' + A^{-2}y^{a1}z^{b1}t't'x'tx.
\]

In \( X, Y, Z, X', Y' \) and \( Z' \) the total number of \( y\)-, \( z\)- and \( t\)-types decreases, and in the other terms the number of the components of the \( x\)-type decreases, so the induction results with words that are final.

**Definition 5.2.** Let \( D \) be a diagram and \( (D)_qf \) be a linear combination of some quasi-final words \( w \), that is, \((D)_qf = \sum_w P_w w \) for some \( P_w \in R \). Define the final Kauffman bracket of \( D \) by:

\[
(D)_f = \sum_w P_w \langle w \rangle_f.
\]

**Remark 1.** By definition, the quasi-final bracket \( \langle \ \rangle_qf \) preserves all the equalities appearing in Fig. 24. However, \( \langle \ \rangle_qf \) does not preserve the first equality in Fig. 25. But the final bracket \( \langle \ \rangle_f \) preserves this equality since it satisfies, by its definition, the relation shown in Fig. 26. Analogously, \( \langle \ \rangle_qf \) preserves all equalities in Fig. 27 and does not preserve the equality in Fig. 28. However, again \( \langle \ \rangle_f \) preserves this equality since, by the definition, it satisfies the relation in Fig. 29.

To show the invariance of \( \langle \ \rangle_f \) under \( \Omega_3\)-moves of all three types, first note that Lemma 4.2 can clearly be extended from \( A \times S^1 \) to the case of \( F_{0.3} \times S^1 \) (one just considers the components of \( y \), \( z \) or \( t\)-type separately).

**Theorem 5.3.** The final Kauffman bracket \( \langle \ \rangle_f \) is invariant under all regular Reidemeister moves. Therefore \( S_{2, \infty}(F_{0.3} \times S^1; R, A) \) is a free \( R \)-module with basis that consists of all final words.

**Proof.** As before, we assume that, except for the crossing at which \( \Omega_3\)-move is to be applied, all other crossings for both diagrams \( D \) and \( D' \) were smoothed and all trivial components were already removed. Let \( p \) be the number of components of \( y \), \( z \) or \( t\)-types in \( D \) (and also in \( D' \)) without counting the component with the crossing. The proof of invariance of \( \langle \ \rangle_f \) under the \( \Omega_3\)-move is done by induction on the number \( p \). We first show this invariance for moves of types I–III when \( p = 0 \).

Consider the \( \Omega_3\)-move of type I shown in Fig. 30. Applying Lemma 4.2, we push all \( x\)-type components and all arrows out of the component that is involved in the \( \Omega_3\)-move, thus it is sufficient to consider the situation when there are \( d \) of \( x\)-type components outside, where \( d \in \mathbb{N} \) and \( m \) and \( n \) are 0 or 1.

If \( m = n = 0 \) then using relation (3.1), which holds already for \( \langle \ \rangle_{sr} \), we have:

\[
(D)_f = A[x^d \cdot t_{-1}]_f + A^{-1}[y^z x^d]_f - A^{-1}[x^d t]_f + A^{-1}[y^z x^d]_f
\]
\[
= -A^{-1}[x^d t]_f + A^{-1}[x^d t]_f + A^{-1}[y^z x^d]_f + A^{-1}[y^z x^d]_f
\]
\[
= A^{-1}[x^d t]_f + A^{-1}[y^z x^d]_f
\]
\[
= (D')_f.
\]
components. The situation for the components of the possible smoothings of crossings except the one at which the \( p \langle \rangle \) bracket moves under which \( \Omega \) moves, is shown in Fig. 31.

Fig. 31. Invariance of \( \langle \rangle \) under the \( \Omega_2 \)-move of type I.

\[
\langle D \rangle_f = A(x^d t_{-2})_f + A^{-1}(y'z'x^d)_f = -A^{-1}(x^d t)_f - A^{-3}(x^d t_{-1}x)_f + A(yz^d)_f + 2A^{-1}(x^d t)_f + A^{-3}(x^d t_{-1}x)_f \\
= A(yz^d)_f + A^{-1}(x^d t)_f = \langle D \rangle_f.
\]

If \( m = 1 \) and \( n = 0 \), we have:

\[
\langle D \rangle_f = A(x^d t_{-2})_f + A^{-1}(y'z'x^d)_f = -A^{-1}(x^d t)_f - A^{-3}(x^d t_{-1}x)_f - A(yz^d)_f - A^{-3}(y'zx^{d+1})_f \\
= -A(yz^d)_f - 2A^{-1}(x^d t)_f - A^{-3}(x^d t_{-1}x)_f - A^{-3}(y'zx^{d+1})_f + A^{-1}(x^d t)_f \\
= -A^{-1}(y'z'x^d)_f - A^{-3}(y'zx^{d+1})_f + A^{-1}(x^d t)_f = A(y'z_{-1}x^d)_f + A^{-1}(x^d t)_f \\
= \langle D \rangle_f.
\]

If \( m = 1 \) and \( n = 1 \), we have:

\[
\langle D \rangle_f = A(x^d t_{-3})_f + A^{-1}y'z'x^d) = -A^{-1}(x^d t_{-1})_f - A^{-3}(x^d t_{-2}x)_f - A(yz^d)_f - A^{-3}(y'zx^{d+1})_f \\
= -A^{-1}(x^d t_{-1})_f + A^{-3}(x^d t_{-2}x)_f + A^{-7}(x^d t_{-1}x^2)_f - A(yz^d)_f - A^{-1}(y'zx^{d+1})_f \\
- 2A^{-3}(x^{d+1} t)_f - A^{-5}(x^{d+1} t_{-1}x)_f = -A^{-1}(x^d t_{-1})_f + A^{-5}(x^d t_{-1}x^2)_f + A^{-7}(x^d t_{-1}x^2)_f \\
+ A^{3}(y z_{-1}x^d)_f - 2A^{-3}(x^{d+1} t)_f + A^{-1}(x^d t_{-1}x^2)_f + A^{-5}(x^d t_{-1}x^2)_f - A^{-7}(x^d t_{-1}x^2)_f \\
= -A^{-1}(x^d t_{-1})_f + A^{3}(y z_{-1}x^d)_f + 2A(x^d t')_f - 2A^{-3}(x^d t')_f - 2A^{-5}(x^d t'x)_f + A^{-1}(x^d t')_f \\
= A^{3}(y z_{-1}x^d)_f + 2A(x^d t')_f + A^{-1}(x^d t'x)_f - A^{-1}(x^d t_{-1})_f + 2A^{-1}(x^d t_{-1})_f \\
= A(y'x^d)_f + A^{-1}(x^d t_{-1}) = \langle D \rangle_f.
\]

Now, if \( p = 0 \), the situation is similar for the \( \Omega_2 \)-moves of types II and III. In the formulas one just has to permute \( z \) with \( t \), keeping \( y \) (for the type II), or permute \( y \) with \( z \) and \( z \) with \( t \) (for the type III). Note also that in these cases quasi-final bracket \( \langle \rangle \) is unchanged just like the final bracket \( \langle \rangle \).

Now, by induction, let us assume that the final bracket \( \langle \rangle \) is invariant under the \( \Omega_2 \)-moves of types I-III that involve less than \( p \) components of \( y, z \) or \( t \)-type (counting without the component with the crossing). Let \( D \) and \( D' \) have \( p \) such components. The situation for the \( \Omega_2 \)-move of type I is shown in Fig. 31.

We may assume that the bracket \( \langle \rangle \) is applied until all possible arrows appear only on the most external component of the type \( t \) (not shown in Fig. 31) and on the components shown in Fig. 31. Using Lemma 4.2 we also assume that \( k \) and \( l \) are equal to 0 or 1. If both of them are equal to 0 then we can use the same argument as in the case \( p = 0 \) since the interior components of the \( y \)-type and \( z \)-type and all the components of the \( t \)-type do not play any role in the previous calculation.

Suppose now that \( k = 1 \) and \( l = 0 \). In that case we apply the \( \Omega_2 \)-move followed by the \( \Omega_5 \)-move to both diagrams \( D \) and \( D' \) to obtain a situation when we can show easily that \( \langle \rangle \) is the same for both diagrams. These moves are shown in Fig. 32.

The \( \Omega_2 \)-move does not change any of the brackets. The effect of applying the \( \Omega_2 \)-move can be analyzed by looking at all possible smoothings of crossings except the one at which the \( \Omega_2 \)-move is applied. For all such smoothings we obtain three moves under which \( \langle \rangle \) is invariant and one move of type III but with one less component of \( y, z \) or \( t \)-type, so by the
induction hypothesis \( \langle \rangle_f \) is invariant under this move of type \( \text{III} \). Thus, if \( D_1 \) is obtained from \( D \) by the application of the two Reidemeister moves mentioned before and, in a similar way, \( D'_1 \) is obtained from \( D' \), we have

\[
\langle D \rangle_f = \langle D_1 \rangle_f \quad \text{and} \quad \langle D' \rangle_f = \langle D'_1 \rangle_f.
\]

Now the \( \Omega_5 \)-move between \( D_1 \) and \( D'_1 \) is again expressed by smoothing the two crossings not involved in the \( \Omega_5 \)-move yielding three \( \Omega_5 \)-moves under which \( \langle \rangle_f \) is invariant and the \( \Omega_5 \)-move of type \( \text{I} \) for which \( k = 0 \) instead of 1 (and \( l \) remains equal to 0). Thus we have \( \langle D_1 \rangle_f = \langle D'_1 \rangle_f \) using the proof of the preceding case \( k = l = 0 \); therefore, it follows that \( \langle D \rangle_f = \langle D' \rangle_f \). The same proof works for \( k = 0 \) and \( l = 1 \). Having established these cases, one applies analogous arguments for the case \( k = l = 1 \).

Consider now the \( \Omega_5 \)-move of type \( \text{II} \) between diagrams \( D \) and \( D' \) which involve \( p \) components of \( y \), \( z \) or \( t \)-type. The situation is shown in Fig. 33.

Applying \( \langle \rangle_{fr} \) one arrives at the case \( k = 0 \) or \( k = 1 \). If \( k = 0 \), then the proof is as before: if there are no extra arrows except on the \( x \)-type components and the ones shown in Fig. 33, then the situation is as in the case \( p = 0 \); if there are such arrows, then the proof is like for the case of \( \Omega_5 \)-move of type \( \text{I} \), where one used two Reidemeister moves to decompose the \( \Omega_5 \)-move into moves for which \( \langle \rangle_f \) is invariant.

If \( k = 1 \) the similar arguments can be applied again; the situation is shown in Fig. 34.

Namely, we observe that the \( \Omega_5 \)-move leaves the bracket \( \langle \rangle_f \) unchanged for both \( D \) and \( D' \). The \( \Omega_5 \)-move is expressed by smoothing the crossings yielding three moves for which \( \langle \rangle_{fr} \) is preserved and one \( \Omega_5 \)-move of type \( \text{I} \) with the number \( p \) unchanged, for which we already showed that the bracket \( \langle \rangle_f \) is unchanged. Finally, after these two Reidemeister moves, the original \( \Omega_5 \)-move of type \( \text{II} \) is expressed using three moves for which the bracket \( \langle \rangle_{fr} \) is unchanged and the \( \Omega_5 \)-move of type \( \text{II} \) for which \( k = 0 \).

It remains to show the invariance of \( \langle \rangle_f \) under the \( \Omega_5 \)-move of type \( \text{III} \); such a case is illustrated in Fig. 35.

Again, there are few cases to consider. First, if there are no components of the type \( z \) (i.e., neither \( z \) nor \( z' \) present in the words), then the proof is as before: either there are no arrows except on the \( x \)-type components and the ones shown in Fig. 35, in which case the calculation is the same as for the case \( p = 0 \); or there are such arrows on the components of \( y \) or \( t \)-type, in which case the proof is the same as it was in that case for \( \Omega_5 \)-move of types \( \text{I} \) and \( \text{II} \).

It remains to check the invariance of \( \langle \rangle_f \) when the \( z \)-type components appear in both diagrams \( D \) and \( D' \). Applying Lemma 4.2, we can assume that \( k, l, \) and \( m \) are equal to 0 or 1, and all \( x \)-type components are in between the components of the three other types and there are no extra arrows.

Suppose that \( k = 0 \). In the computations of \( \langle D \rangle_f \) and \( \langle D' \rangle_f \) one proceeds as in the case \( p = 0 \) except for the situations when an arrow appearing on an \( y \)-type component is moved to become an arrow on a \( t \)-type component. By definition of \( \langle \rangle_{df} \) this has to be done via a \( z \)-type component. However, by Remark 1, for \( \langle \rangle_f \) the same result is obtained if the arrow is moved directly from the \( y \)-type component to the \( t \)-type component. Thus, for the final bracket \( \langle \rangle_f \) the component of the \( z \)-type plays no role in the calculations which are then like for the case \( p = 0 \).

The final case to check is when \( k = 1 \). Again, we decompose the \( \Omega_5 \)-move of type \( \text{III} \) into other moves for which the invariance of \( \langle \rangle_f \) has already been established. This is shown in Fig. 36.
The \( \Omega_5 \)-move shown in Fig. 36 does not change \( \langle \rangle_f \) since it can be decomposed by smoothing the crossings not involved in the move into three moves for which \( \langle \rangle_f \) is invariant and an \( \Omega_5 \)-move of type I. After this, the desired \( \Omega_5 \)-move is applied and, considering the smoothings of the crossings that are not involved in the move, it can be expressed using three moves (for which \( \langle \rangle_f \) is invariant) and the \( \Omega_5 \)-move of type III with \( k = 0 \) and for which, as it was shown above, the bracket \( \langle \rangle_f \) is invariant. This finishes our proof. \( \square \)

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