Dynamic Scaling in Rotating Turbulence: A Shell Model Study

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We investigate the scaling form of appropriate time-scales extracted from time-dependent correlation functions in rotating, turbulent flows. In particular, we obtain precise estimates of the dynamic exponents $z_p$, associated with the time-scales, and their relation with the more commonly measured equal-time exponents $\zeta_p$. These theoretical predictions, obtained by using the multifractal formalism, are validated through extensive numerical simulations of a shell model for such rotating flows.

Many aspects of turbulence are understood through $p$-th order correlation functions of velocity increments across suitably defined length scales $r$ which lie in the so-called inertial ranges of the flow [1–3]. In simple terms, the inertial range is well-separated from, and lie between, the system-dependent energy injection scale $L$ and dissipation scale $\eta$ of the turbulent flow. We now know that there exist power-laws [3–5] in these correlators—typically called structure functions—and a universality of the associated scaling exponents $\zeta_p$ which are perhaps universal for a given class of turbulent flows but may well vary for different forms of turbulence. Thus, the evidence favouring the universality of such exponents in fully developed, homogeneous and isotropic [6–8], passive-scalar [9–10], magnetohydrodynamic (MHD) [11–13], two-dimensional [14–17] and indeed rotating turbulence [18–21], to name a few, is overwhelming; nevertheless, the values of $\zeta_p$ are known to be different and specific to each of these turbulent flows.

The algebraic nature of these structure functions, and indeed the universality of the exponents, also reminds one of the behaviour of correlation functions near a critical point [22], for example in spin systems. However, for the turbulent flows that we are familiar with this analogy is limited [24]. This is because in fully developed turbulence an infinite set of exponents are required to fully characterise different-order structure functions in the inertial range as opposed to the simple scaling one is familiar with in critical phenomena. Such a complexity, which can be rationalised through a multifractal description of turbulent flows, has also meant that unlike in critical phenomenon where studies of static and dynamic correlators [24] went more or less hand-in-hand, the study of time-dependent structure functions is more recent in turbulence.

Nevertheless, over the past decades there has been a concerted effort to generalise the dynamic-scaling ansatz in the critical phenomenon—namely the dynamic scaling exponent $z$ associated with the relaxation time $\tau$ near a critical point—and obtain estimates of the multiscale nature of time-dependent structure functions in turbulence. These investigations have, however, been limited to homogeneous, isotropic turbulence in two and three dimensions or for the case of passive-scalar turbulence [10] [25–29]. In particular, these studies demonstrate that just like the case of equal-time exponents $\zeta_p$, there exists an infinite set of (universal) exponents $z_p$ whose values depend on the kind of relaxation time fished out from the order-$p$, time-dependent structure functions. Perhaps the most important success of these studies was the generalisation of the Frisch–Parisi multifractal formalism [30] for the velocity field $u$ to derive (linear) bridge relations [26, 31, 32] connecting the dynamic $z_p$ and equal-time $\zeta_p$ exponents and establish the notion of dynamic multifractal scaling.

The complex nature of time correlations in these systems is intrinsic. But what happens when there is an external global time-scale governing the statistical nature of the turbulent flow itself? Indeed for such systems, it is difficult to separate the hierarchy of dynamics intrinsic to the system and the time-correlations set by the imposed time-scale making the study of the nature of dynamic (multi)scaling, when such effects are at play, non trivial.

One of the more natural and ubiquitous examples of turbulence with an imposed global time-scale is that of rotating turbulence [33–36], observed in geophysical phenomena [37] including oceanic and atmospheric flows [38], astrophysical phenomena [39], and many engineering applications. When the Coriolis force dominates over the nonlinear term, strongly rotating, but mildly turbulent, three-dimensional flows tend to become two-dimensional, consistent with the Taylor–Proudman theorem [35, 40]. However, when the flow becomes turbulent, the nonlinear effects can no longer be ignored [20, 41–46]; in fact, the nonlinear interactions among the inertial waves play an important role in developing the quasi-two-dimensional behaviour of rotating turbulence [47, 53].

In such rotating turbulent flows, the addition of a global rotation rate $\Omega$ through the Coriolis force sets a unique time scale $1/\Omega$. At the level of statistics, we know [54] that this time-scale leads to a characteristic length scale in the problem: The Zeman scale $\ell_\Omega = \sqrt{\varepsilon/\Omega^2}$, where $\varepsilon$ is the mean kinetic energy dissipation. The role of this global rotation, via the Zeman scale, in determining the equal-time statistics of three-dimensional rotating turbulence has been exten-
sively studied. In particular, we know that unlike homogeneous and isotropic turbulence, in the limit of large Reynolds numbers, when \( L \gg \ell_\Omega \gg \eta \), the equal-time (longitudinal) structure functions of the (projected) velocity increments \( \delta u(r) = (u(x + r) - u(x)) \cdot \hat{f} \) in the inertial range \( L \gg r \gg \eta \) show a dual scaling; \( \hat{f} \) is the unit vector along the separation vector \( r \).

More precisely, defining the \( p \)-th order, equal-time structure function \( S_p(r) = (|\delta u(r)|^p) \), the equal-time exponents are extracted via the power-law \( S_p(r) = r^{\zeta_p} \) in the inertial range. The rotation-induced Zeman scale results in two different classes of exponents: Theoretical estimates suggest that for the rotation-dominated larger scales \( L \gg r \gg \ell_\Omega \), the exponents \( \zeta_p = p/2 \); however, at smaller scales \( \ell_\Omega \gg r \gg \eta \), which are less sensitive to the Coriolis effects, \( \zeta_p = p/3 \) as is the case for fully developed three-dimensional turbulence \([18, 21]\).

A consequence of this is that kinetic energy spectrum \( E(k) \) also displays dual-scaling \([54, 57]\): \( E(k) = k^{-2} \) and \( E(k) = k^{-5/3} \) for wavenumbers smaller or larger, respectively, than the Zeman wavenumber \( k_0 = 1/\ell_\Omega \). This dual scaling of the energy spectra is seen remarkably well in shell models, such as the one we use here, as shown in Fig. 1 of Ref. \([21]\). Of course, measurements suggest strong intermittency corrections to this simple dimensional form. Thus, in rotating turbulence—just like in homogeneous, isotropic turbulence—there exists multi-scaling at the level of equal-time statistics.

But, is there a similar multiscaling for the dynamic correlators in such systems? This remains a somewhat open question because while different aspects of Lagrangian turbulence of rotating flows have been studied \([41, 58-61]\), studies of dynamic correlators are sparse \([62, 63]\). Furthermore, these studies \([62, 63]\) use an Eulerian approach to measure the second-order dynamic correlation function which, as we know from insights developed in non-rotating turbulence \([20]\), can lead to an oversimplification and mask an underlying multiscaling as we illustrate below.

In the much simpler non-rotating, homogeneous, isotropic turbulent flow, a na"ive calculation of dynamic scaling within the Eulerian framework—in a manner similar to what is done for equal-time structure functions—yields a trivial dynamic exponent of unity because the sweeping effect dominates and thus, linearly couples the temporal and spatial scales. Indeed, this “sweeping” effect leads to the simpler dynamic exponents for the Eulerian time-dependent correlation functions in rotating turbulence as well as reported by Favier, Godeferd and Cambon \([62]\).

Thus, unlike for equal-time structure functions, special care must be taken which eliminates this sweeping effect in order to obtain non-trivial dynamic (multi-)scaling exponents. This can be done through the Lagrangian or the quasi-Lagrangian framework \([26, 28, 32, 64, 66]\). While the former allows us to measure the structure functions of temporal velocity increments \( \delta u(\tau) = u(t + \tau) - u(t) \), the latter is especially useful as it allows us to obtain time-dependent structure functions for velocity increments and hence, in the limiting case, recovers the (Eulerian) equal-time structure functions \([32, 67]\) as we now show. The quasi-Lagrangian velocity field

\[
v(r_0, t_0 | x, t_0 + t) \equiv u(x + R_L(r_0, t_0 | t_0 + t), t_0 + t)
\]

is measured along the Lagrangian trajectory \( R_L(r_0, t_0 | t_0 + t) \) of a fluid particle starting at \((r_0, t_0)\). This allows us to define the (quasi-Lagrangian) velocity increments \( \delta v(r, t) = [v(r_0, t_0 | x + r, t_0 + t) - v(r_0, t_0 | x, t_0 + t)] \cdot \hat{f} \) and thence the time-dependent structure function \( \mathcal{F}_p(r, t_1, \ldots, t_p) = (|\delta v(r, t_1) \delta v(r, t_2) \ldots \delta v(r, t_p)|^p) \). By setting \( t_1 = t_2 = \cdots = t_p = t \), the quasi-Lagrangian time-dependent structure function is written simply as \( \mathcal{F}_p(r, t) \) with the obvious identity \( \mathcal{F}_p(r, t = 0) \equiv S_p(r) \).

The quasi-Lagrangian structure function also lends itself to an adaptation of the Frisch-Parisi multifractal formalism \([3, 30]\) for the equal-time structure function. Assuming a multifractal description of rotating turbulence, the velocity field ought to possess a range of (universal) scaling exponents \( h \in \mathcal{I} \equiv (h_{\min}, h_{\max}) \), each of which corresponds to a fractal set \( \Sigma_h \) of dimension \( D(h) \). This allows one to write down the velocity increments \( \delta u(x, r)/u_L \sim (r/L)^h \), where \( u_L \) is the velocity associated with the large length scale of the flow. Given the multifractal description, for individual increments it is important to keep track of the point \( x \) at which the increments are taken because the increment picks up different scaling exponents \( h \) for every \( x \in \Sigma_h \).

Such a prescription allows us to define the equal-time structure function in terms of the scaling exponents \( h \) and the measure \( d\mu(h) \) which gives the weight of the contributing fractal sets:

\[
S_p(r) \propto u_L^p \int_{\mathcal{I}} d\mu(h) \left( \frac{r}{L} \right)^{ph + 3 - D(h)}.
\]

Formally, the measured scaling exponents \( \zeta_p \) are then extracted through a saddle-point calculation.

We now extend the equal-time formalism for time-dependent structure functions

\[
\mathcal{F}_p(r, t) \propto u_L^p \int_{\mathcal{I}} d\mu(h) \left( \frac{r}{L} \right)^{ph + 3 - D(h)} \mathcal{G}^{p,h} \left( \frac{t}{\tau_{p,h,\Omega(r)}} \right),
\]

where \( \tau_{p,h,\Omega}(r) \) is the characteristic scale-dependent time scale of the flow. The scaling function \( \mathcal{G}^{p,h} \) is unity at \( t = 0 \) and its integral is assumed to exist to allow us to define the integral-time scale:

\[
\tau_p^f(r) = \frac{1}{S_p(r)} \int_0^\infty \mathcal{F}_p(r, t) dt \sim r^{z_p}.
\]

In order to proceed further and calculate the dynamic exponent \( z_p \), we make reasonable assumptions on the time-scale \( \tau_{p,h,\Omega} \). The phenomenology of rotating turbulence suggests that in the rotation-dominated regime \( L \gg r \gg \ell_\Omega \), to leading order, the time-scale is set by the rotation rate \( \Omega \) and hence \( \tau_{p,h,\Omega} \propto 1/\Omega \). On the
other side of the Zeman scale $\ell_\Omega \gg r \gg \eta$ however, we expect $\tau_{p,h,\Omega} \equiv \tau_{p,h} \propto r^{1-h}$ consistent with the ideas of homogeneous and isotropic turbulence.

By using standard tools to evaluate the integral in Eq. (4), we eventually obtain (see, e.g., Refs. 26, 32)

$$z_p \sim \begin{cases} 
1 + (\zeta_p - 1), & \ell_\Omega \gg r \gg \eta; \\
0, & L \gg r \gg \ell_\Omega.
\end{cases} \quad (5)$$

Furthermore, the same analysis suggests that for $L \gg r \gg \ell_\Omega$, the integral-time scale $\tau^p(r) \propto 1/\Omega$. Indeed, this form is perhaps not entirely surprising given the scale-independent form of $\tau_{p,h,\Omega}$.

Our predictions suggest that in rotating turbulence, the time-dependent structure functions also show dual scaling consistent with what we know from equal-time measurements. Indeed, the bridge relation connecting the integral-time scale based dynamic exponent $z_p$ for scales where turbulent fluctuations swamp the effect of rotation is identical to what happens in three-dimensional turbulent flows 29. On the other hand, the dynamic structure functions are scale-independent ($z_p = 0$) as soon as rotation is dominant. (It is perhaps useful to keep in mind that although to leading order our analysis shows the integral-time scale in the rotation-dominated regime is scale-independent, the structure functions themselves are not as we show below.)

Are our results surprising? The surprise and apparent contradiction arises when we examine the dynamics in terms of local turn-over time-scales of the flow $\tau^\text{local}(r) \sim r/\delta u(r)$. For scales smaller than the Zeman scale by using $\delta u(r) \sim r^{1/3}$, we obtain the local $p$-independent dynamic exponent $z^\text{local} = 2/3$. This exponent is exactly the same as what we obtain from Eq. (5) in the absence of intermittency correction, i.e., $\zeta_p = p/3$. However, for scales larger than $\ell_\Omega$, a similar analysis yields $z^\text{local} = 1/2$ since $E(k) \sim k^{-2}$. This result is in sharp contradiction with the exponent $z_p = 0$ as obtained above (5) but also, crucially, suggests that in the rotation-dominant regime, the dynamic structure function is scale-dependent.

This begs the question as to which of these two approaches are correct and how is this contradiction resolved. Indeed, how valid are our theoretical predictions 5 when confronted with data from simulations?

While formally quasi-Lagrangian structures are well-defined, measurements from direct numerical simulations (DNSs) of the three-dimensional Navier–Stokes equation are still a challenge 16, 29. Fortunately, this problem of circumventing sweeping through a quasi-Lagrangian description was solved 26 by adopting a shell model approach. Indeed, by construction, shell models are dynamical systems for (complex) variables which resemble velocity increments and sweeping is eliminated by restricting the coupling between modes which are only nearest or next-nearest neighbours. Remarkably, such a dynamical systems approach to turbulence 3, 6, 68–70 does capture the essential multifractal and cascade processes of fully developed turbulence as was recognised since the pioneering works of Obukhov 71, Desnepansky and Novikov 72, Gledzer 73, and Ohkitani and Yamada 74 and then generalised to several other single and multiphase flows 9, 13, 16, 27, 75–87. Moreover, shell models, although structurally isotropic, reproduce and predict many properties of the rotating turbulence, e.g., two-dimensionalisation, the dual scaling of energy spectrum, and the scaling of equal-time structure functions 21, 50, 88.

Thus, given the question at hand, it is natural for us to approach this problem with a shell model for rotating turbulence. Such models are constructed on a logarithmically-spaced lattice of wavenumbers $k_n = k_0 \lambda^n$; we use the conventional choices of $k_0 = 1/16$ and $\lambda = 2$ in our study. Associated with each shell $n$ is a complex variable $u_n$ which mimics velocity increments over a scale $k_n \sim 1/r$ in the Navier–Stokes equation. By retaining only the nearest and next-nearest neighbour couplings in the nonlinear (convolution) term of the Navier–Stokes equation, the shell model equations are coupled ordinary differential equations

$$\frac{du_n}{dt} = -\nu k_n^2 u_n + f_n - i\Omega u_n + i[ak_{n+1}u_{n+1} + bk_nu_{n+1}u_{n-1} + ck_{n-1}u_{n-1}u_{n-2}]^* \quad (6)$$

with shell numbers running from 1 to $N$. The asterisk in the equation denotes a complex conjugation and $i \equiv \sqrt{-1}$ and, as noted before, the nonlinear couplings are limited ensuring the absence of direct coupling of large and small scales effectively eliminating sweeping effects. The shell model, in the absence of viscosity ($\nu = 0$) and external forcing $f_n = 0$, conserves energy, helicity, and phase-space, through a proper choice of the (real) coefficients $a, b$ and $c$; we use, as is common, $a = 1, b = -1/2$, and $c = -1/2$.

In our simulations of the shell model we choose $N = 27$ shells, $\nu = 10^{-9}$ and an exponential fourth-order Runge-Kutta scheme for time-marching with a sufficiently small time-step $\delta t = 10^{-5}$ given the stiffness of these coupled ordinary differential equations. We use an external forcing on shells $n = 2$ and 3 with $f_n = \epsilon(1 + i)(\delta_{n,2}/u^*_n + \delta_{n,3}/2u^*_n)$, where $\epsilon$ specifies the energy input rate and, in our simulations, $\epsilon = 0.01$; this specific form ensures a zero helicity input rate and the energy flux free from period 3 oscillations 92. We initialise our velocity field $(u_n = \sqrt{k_n} \exp(i\theta))$ for $n \leq 4$ and $u_n = \sqrt{k_n} \exp(-k_n^2) \exp(i\theta)$ for $n \geq 5$, where $\theta \in [0, 2\pi]$ is a random phase) and drive the system to a statistically steady state before turning on the Coriolis term.

We characterise rotating, turbulent flows not only by the (large-scale) Reynolds number $Re \equiv U_{\text{rms}}/k_{0}\nu$, where the root-mean-square velocity $U_{\text{rms}} = (\sum_n |u_n|^2)^{1/2}$ [93], but also by the Rossby number $Ro \equiv U_{\text{rms}}k_0/\Omega$, which is a measure of the relative strength of the nonlinearity to the Coriolis force and the Zeman wavenumber $k_\Omega = \sqrt{\Omega^2/\zeta}$. We use, in our simulations, Reynolds number $Re \sim 10^9$ and $Ro = \infty (\Omega = 0; k_\Omega = 0)$, 0.809 ($\Omega = \sqrt{\Omega^2/\zeta}$).
FIG. 1. Representative plots for the evolution of normalised time-dependent fourth order structure function $F_4(k_n, t)$ versus time which is normalised by the Kolmogorov time-scale $\tau_\eta$ for (a) $Ro = 0.161$ and (b) $Ro = 0.043$. The black, red, blue and magenta coloured lines correspond to $n = 11$, $n = 12$, $n = 14$ and $n = 16$ respectively. (c) Plots of the relative spectral energy density $E_4(k_n, \omega)$ for $Ro = 0.043$ versus $k_n$ for different harmonics $\omega$; the vertical dashed line represents the Zeman wavenumber.

| $p$ | $\zeta_p$ | $z_p$ (Eq. [5]) | $Ro = 0.809$ | $Ro = 0.232$ | $Ro = 0.161$ | $Ro = 0.043$ |
|-----|-----------|-----------------|---------------|---------------|---------------|---------------|
| 1   | 0.379 ± 0.006 | 0.621 ± 0.006 | 0.63 ± 0.01 | 0.64 ± 0.02 | 0.65 ± 0.03 | 0.67 ± 0.07 |
| 2   | 0.707 ± 0.005 | 0.672 ± 0.008 | 0.673 ± 0.009 | 0.68 ± 0.01 | 0.68 ± 0.01 | 0.67 ± 0.06 |
| 3   | 1.0 | 0.707 ± 0.005 | 0.719 ± 0.009 | 0.703 ± 0.009 | 0.72 ± 0.01 | 0.73 ± 0.04 |
| 4   | 1.267 ± 0.007 | 0.733 ± 0.007 | 0.72 ± 0.01 | 0.72 ± 0.01 | 0.746 ± 0.008 | 0.76 ± 0.02 |
| 5   | 1.51 ± 0.02 | 0.75 ± 0.02 | 0.72 ± 0.02 | 0.74 ± 0.01 | 0.761 ± 0.008 | 0.79 ± 0.02 |
| 6   | 1.74 ± 0.03 | 0.77 ± 0.03 | 0.76 ± 0.02 | 0.75 ± 0.02 | 0.78 ± 0.01 | 0.80 ± 0.03 |

TABLE I. We summarise, for $k_n \gg k_\Omega$, our results for the dynamic exponents $z_p$ (column 3) calculated through the bridge relations [5] from the equal-time exponents $\zeta_p$ obtained through ESS [89–91] (column 2) for different orders $p$ (column 1). Columns 4 - 6 lists the dynamic exponents for different Rossby numbers obtained directly from our shell model simulations. (We note the marginal increase in the error bars and mean exponents, while still being consistent with the theoretical prediction, as $Ro \rightarrow 0$ is likely due to the shrinking of the inertial range $k_n \gg k_\Omega$ as $k_\Omega$ becomes larger with decreasing Rossby numbers.)

0.1; $k_\Omega = 0.3$; $0.232 (\Omega = 0.5$; $k_\Omega = 3.5$), $0.161 (\Omega = 1.0$; $k_\Omega = 10.0)$, and $0.043 (\Omega = 5.0$; $k_\Omega = 111.8$).

From the statistically steady velocity field of the rotating turbulent flow, it is simple to define the shell model analogue of the order-$p$, scale-dependent, quasi-Lagrangian normalised time-dependent structure function as

$$F_p(k_n, t) = \mathcal{R}[\frac{\langle |u_n(t_0)u^*_n(t_0 + t)|^p \rangle^{1/p}}{\langle |u_n(t_0)|^p \rangle}]$$

where $\mathcal{R}$ denotes the real part of the function and the angular brackets an average of over different time origins $t_0$. We choose integer values of $p$ between 1 and 6 in this study.

In Fig. 1(a) we show representative plots of the fourth-order time-dependent structure function $F_4(k_n, t)$, for $Ro = 0.161$, and different shell numbers which are all greater than the Zeman scale and hence much less influenced by the effects of rotation. As one would expect, the correlations decay much faster for higher wavenumbers than for lower wavenumbers.

To bring out the effect of rotation clearly, we go to a lower value of the Rossby number (hence a higher value of the Zeman wavenumber). Figure 1(b) shows such a plot for $Ro = 0.043$ for the same wavenumbers as in panel (a). However, for such a low value of $Ro$, shell numbers $n = 11$ and 12, corresponding to wavenumbers close to the Zeman scale, are clearly affected by the Coriolis force. This is clearly seen in the conspicuous oscillatory profile of the structure function.

These oscillations arise, as already shown through the direct numerical simulations of Eulerian time-dependent correlators in Refs. [12], [23], at rotation-dominated scales $k \lesssim k_\Omega$ because of the presence of the Coriolis term. Indeed, the formal solution of Eq. [6] ought to have a dominant harmonic $\sim \exp(-i\Omega t)$, in addition to the contributions of viscosity and the non-linearity; this oscillatory factor of course becomes vanishingly small when $k_n \gg k_\Omega$. Consequently, for $k_n \lesssim k_\Omega$ (Fig. 1(b), $n = 11$ and 12), the time-dependent structure functions $F_p(k_n, t)$ have an oscillatory profile with a dominant harmonic of angular frequency $\sim p\Omega/2$. For wavenumbers much $k_n \gg k_\Omega$, the nonlinearity of the dynamical systems ensures a mixing of the harmonics of different scales results in several sub- and super-harmonics in the system eventually washing away the clear oscillatory profile seen for $k_n \lesssim k_\Omega$.

This picture is easily validated, through a Fourier decomposition, from measurements of the spectral relative energy content $E_p(k_n, \omega) \equiv$
The Fourier transform of \[ \frac{\hat{F}_p(k_n, \omega)}{\sum_n \hat{F}_p(k_n, \omega)} \] is the Fourier transform of \( F_p(k_n, t) \). Figure 1(c) illustrates such an analysis, for \( p = 4 \) and \( R = 0.043 \) corresponding to the structure functions in Fig. 1(b), which clearly shows that while the energy is maximally contained, for \( k_n \lesssim k_{11} \), in the \( \omega \sim 10 \) (corresponding to \( Ro = 0.043 \)) mode, at wavenumbers \( k_n \gg k_\Omega \) the energy is distributed more uniformly amongst the other harmonics that we calculate.

From the time-dependent structure functions of the sort shown in Fig. 1(a) and (b), we define the \( p \)-th order, shell-model analogue of the integral-time scale \( T_p^I(k_n) \equiv \int_0^\infty F_p(k_n, t) dt \); in practice (to avoid contamination from statistical noise at long times [20]), the upper limit of the integral is restricted to times when \( F_p(k_n, t) \) has reached a value of 0.6 and we have checked that our results are insensitive if this limit is varied between 0.4 and 0.8.

In Fig. 2 we show loglog plots of \( T_2^I(k_n) \) and \( T_4^I(k_n) \) vs \( k_n \) for \( Ro = 0.043 \). Clearly, for \( k_n \lesssim k_{11} \), the plateau in \( T_p^I(k_n) \) leads to a dynamic exponent \( z_p = 0 \) as indicated by the dashed best fit lines. In the lower inset, we plot the values of these plateau for different orders vs \( 1/\Omega \); the dashed line fits for each order shows clearly that the theoretical prediction from the multifractal analysis \( T_p^I \propto 1/\Omega \), for \( k_n \lesssim k_{11} \) holds. However for \( k_n \gg k_{11} \), the integral-time scale seems to be clearly a power-law with \( z_p = 0 \) which extends over a decade as shown by the black lines which best fit the data. From plots such as those we extract, through a least-square fit, \( z_p \) (for different values of \( Ro \)) from 500 different measurements; in Table 1 we quote the mean of these exponents and their standard deviations as error bars. To further illustrate the quality of the scaling range for higher wavenumbers, in the upper inset we show a loglog plot of the fourth vs the second-order integral time-scale in a manner reminiscent of the extended self-similarity (ESS) technique used for equal-time measurements. This representation shows a clear scaling with the best fit (black line) slope \( z_4/z_2 \approx 1.09 \), consistent with what one would obtain from the multifractal theory. While this ESS-like approach is convincing, we would advice caution in over-interpreting the role of such an extended self-similarity for dynamic exponents in the absence of a theory analogous to what is known for equal-time structure functions [89-91].

Comparing the different columns in Table 1, it is clear that the bridge relation [5] are indeed satisfied for all Rossby numbers for wavenumbers \( k_n \gg k_{11} \). Furthermore, in the rotation dominated scales \( k_n \lesssim k_{11} \), we find (within error bars) \( z_p = 0 \), again consistent with our theoretical prediction [6].

In this paper we have addressed the issue of dynamic scaling in rotating turbulence by using the tools of the Frisch-Parisi multifractal formalism and then validated our predictions through detailed numerical simulations of a shell model which factors in the Coriolis force. By adopting a quasi-Lagrangian approach, our work complements earlier (Eulerian) studies [62, 63] of time-dependent correlation functions in such flows. We uncover a new set of exponents [3], and associated bridge relations and find, unsurprisingly, for wavenumbers larger than the Zeman scale, even strongly rotating flows show dynamic multiscaling which is completely consistent to what has been known [20, 28]. Surprisingly, for wavenumbers which are dominated by the rotation, the relevant time-scales are scale-independent and hence \( z_p = 0 \) in sharp contrast to estimates from local time-scale arguments. This is because at such scales rotation is the dominant mechanism when compared with those imposed by the nonlinear term. Hence naively one would expect that the dominant time-scale here would be \( \sim 1/\Omega \); the multifractal approach picks out this dominant time-scale over all others. This is perhaps because at these scales a local turnover time approach fails to factor in the time-scale imposed on the flow by rotation which dominates over the intrinsic (and local) time-scales arising in the flow itself. Thus a comparison of the time-scales which emerge from arguments based on the turnover time with those from the multifractal model shows a greater disparity at rotation dominated scales than those which are not underlining the singular nature of the Coriolis force especially when it comes to dynamic correlators. Furthermore, curiously the intermittency corrections seen in the equal-time measurements seem to be absent from the dynamics altogether. To the best of our knowledge this
is the only example of a turbulent flow where such a de-
coupling of a fundamental feature of turbulence happens
when we move from the statics to the dynamics and de-
serves a more rigorous investigation in the future.

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