Suppression of timing errors in short overdamped Josephson junctions

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The influence of fluctuations and periodical driving on temporal characteristics of short overdamped Josephson junction is analyzed. We obtain the standard deviation of the switching time in the presence of a dichotomous driving force for arbitrary noise intensity and in the frequency range of practical interest. For sinusoidal driving the resonant activation effect has been observed.

The mean switching time and its standard deviation have a minimum as a function of driving frequency. As a consequence the optimization of the system for fast operation will simultaneously lead to minimization of timing errors.

The Rapid Single Flux Quantum (RSFQ) electronic devices are promising candidates to built a petaflop computer due to high operating frequencies of RSFQ elements close to 1 THz [1]. Moreover they are of particular interest in solid-state quantum information processing. They may be used both for realization of qubits and for characterization of a macroscopic quantum behavior, e.g., readout electronics for quantum computing [2]. The processes going on in RSFQ devices are based on a re-production of quantum pulses due to spasmodic changing by 2\pi of the phase difference of damped Josephson junctions (JJ). The major restriction in the development of RSFQ logic circuits is given by influence of fluctuations [1, 3, 4, 5]. Moreover the operating temperatures of the high-Tc superconductors lead to higher noise levels by increasing the probability of associated errors. Recently lots of investigations were performed to study pulse jitter and timing errors in RSFQ circuits [4, 5]. The Timing errors is one of reasons limiting a 16-bit RSFQ microprocessor prototype clock frequencies to 20GHz [1], instead of theoretically predicted hundreds GHz. Therefore, investigation of possible ways for suppression of time jitter of transmitting signals is a very important problem from fundamental and technological points of view. An RSFQ circuit in fact has long timing loops, so the time jitter produces a noise-induced digital accumulative effect in Josephson transmission lines [4, 5]. It is also important for different branches of physics, where a nonlinear element is driven either from an external source or from another element, as in neural networks. For RSFQ circuits three different types of digital errors may be identified.

First, storage errors may occur during the passive storage of data: fluctuations can induce a 2\pi phase flip and thus switch the quantizing loop into the neighboring flux state. The analytical description of the mean time of such noise-induced flips, valid for arbitrary noise intensity, has been presented in [2]. Second, decision errors may be produced in the two junction comparator. This situation has been studied in detail in the linearized overdamped JJ model [7]. If an RSFQ circuit operates at high speed, another type of noise-induced error becomes important. Due to noise the time interval between input and output pulses fluctuates, producing ”timing” error. Timing errors have also been studied in the linearized overdamped JJ model [4]. It is known that in noisy nonlinear systems with a metastable state, the resonant activation (RA) and noise enhanced stability (NES) phenomena may be observed. These effects may play positive and negative role in accumulation of fluctuational errors in RSFQ logic devices: the RA phenomenon minimizes timing errors, while the NES phenomenon increases the switching time. These effects however were not still observed in previous investigations due to the use of linearized models [4, 7]. Moreover the limiting frequencies of RFSQ devices and possible optimizations, in order to increase working frequencies and reduce timing errors in RSFQ circuits, is an open question.

This letter is aimed to answer this question by investigating nonlinear noise properties of an overdamped JJ, subjected to periodic driving, to understand possible ways of RSFQ circuits optimization for high-frequency operation with minimal timing errors. To this end we consider the dynamics of a short overdamped JJ, under a current I, given by the Langevin equation [2]

\begin{equation}
\omega_c^{-1} \frac{d \varphi(t)}{dt} = -\frac{du(\varphi)}{d\varphi} + i_F(t),
\end{equation}

where \( \varphi \) is the order parameter phase difference, \( u(\varphi) = 1 - \cos \varphi - i(t)\varphi \), \( i(t) = i_0 + f(t) \), and

\begin{equation}
\omega_c = 2eR_N I_c/h \quad \text{is the characteristic frequency of the JJ}, \quad R_N \quad \text{is the normal state resistance of a JJ}, \quad e \quad \text{is the electron charge and} \quad h \quad \text{is the Planck constant}. \quad \text{Let, initially, the JJ is biased by a current smaller than the critical one} \quad (i_0 < 1), \quad \text{and the junction is in the superconductive state. The current pulse} \quad f(t), \quad \text{such that} \quad i(t) > 1, \quad \text{switches the junction into the resistive state. However, an output}
\end{equation}
pulse will be born not immediately, but later on. Such a
time is the switching time, which is a random quantity
characterized by mean value and standard deviation. As
an example of a driving with sharp fronts we consider the
dichotomous signal \( f(t) = A\sin(\omega t) \), and as an ex-
ample of a driving with smooth fronts – sinusoidal signal
\( f(t) = A\sin(\omega t) \). In spite that we consider the peri-
odic driving, that makes the results more evident, it is
obvious, that below the cut-off frequency the switching
occurs during half of the first period, so other periods
are not important. Besides, our adiabatic analysis may
easily be generalized to an arbitrary form of \( f(t) \). In
computer simulations we set \( \omega_c = 1 \) and, therefore,
in plots \( \omega \) is normalized to \( \omega_c \). The mean switching
time (MST) \( \tau = \langle t \rangle = \int_0^\infty tw(t)dt \), and its standard deviation
(SD) \( \sigma \), may be introduced as characteristic time scales of
the evolution of the probability \( P(t) = \int_{\varphi_1}^{\varphi_2} W(\varphi, t)d\varphi \) to
find the phase within one period of the potential profile
\( Eq.(2) \)

\[
\frac{\partial P(\varphi_0,t)}{\partial t} + \frac{P(\varphi_0,t)}{\omega_c} = 0.
\]

We choose therefore \( \varphi_2 = \pi \), \( \varphi_1 = -\pi \) and the initial
distribution at the bottom of a potential well. Let us

\[
w(t) = \frac{\partial P(\varphi_0,t)}{\partial t} + \frac{P(\varphi_0,t)}{\omega_c}.
\]

![FIG. 1: The MST \( \tau(\omega) \) and SD \( \sigma(\omega) \) as functions of
the frequency for dichotomous driving, for two values of noise
intensity: \( \gamma = 0.2, 0.02 \), and \( i_0 = 0.5 \), \( i = 1.5 \). The results
of computer simulations are: \( \tau(\omega) \) (solid line) and \( \sigma(\omega) \) (di-
amonds and circles). Dashed line are the theoretical results
given by Eqs. 4, 5.](image)

first consider the case of dichotomous driving. From the
Fokker-Planck equation corresponding to the Langevin
Eq. 1, we calculate the MST and its SD. In Fig. 1 we
report the behaviors of MST \( \tau(\omega) \) and SD \( \sigma(\omega) \) as a func-
tion of the driving frequency for two values of noise intensity:
\( \gamma = 0.2, 0.02 \). For dichotomous driving both MST
and SD do not depend on the driving frequency below a
certain cut-off frequency, above which the characteristics
degrad. In the frequency range \( 0 \div 0.2\omega_c \), therefore, we
can describe the effect of dichotomous driving by time

characteristics in a constant potential. The exact ana-
lytical expression of MST and its asymptotic expansion are: for arbitrary \( \gamma \)

\[
\tau_c(\varphi_0) = f_1(\varphi_2) - f_1(\varphi_0) + \gamma [f_2(\varphi_2) + f_2(\varphi_0)] + \ldots,
\]

and for \( \gamma \ll 1 \)

\[
f_c(\varphi_0) = f_1(\varphi_2) - f_1(\varphi_0) + \gamma [f_2(\varphi_2) + f_2(\varphi_0)] + \ldots,
\]

Following ref. 10, the exact expression for \( \tau_c = \langle t^2 \rangle \) in
a time-constant potential may be derived

\[
\tau_{c2}(\varphi_0) = \frac{2}{\gamma\omega_c^2} H(\varphi_0) \int_{\varphi_1}^{\varphi_2} e^{u(x)/\gamma} H(x)dx - \frac{2}{(\omega_c^2)} H(\varphi_0) \int_{\varphi_1}^{\varphi_2} e^{u(x)/\gamma} dx,
\]

where \( H(x) = \int_{\varphi_1}^{\varphi_2} e^{u(y)/\gamma} dy \int_{\varphi_1}^{\varphi_2} e^{u(y)/\gamma} dy \),
and \( \tau_c(\varphi_0) \) is given by 11. The asymptotic expression of \( \sigma = \sqrt{\tau_{c2} - \tau_c^2} \) in the small noise limit \( \gamma \ll 1 \) is

\[
\sigma(\varphi_0) = \frac{1}{\omega_c} \sqrt{2\gamma [F(\varphi_0) + f_3(\varphi_0)] + \ldots},
\]

\[
F(\varphi_0) = f_1(\varphi_2)f_2(\varphi_2) - 2f_1(\varphi_0)f_2(\varphi_0) + f_1(\varphi_2)f_2(\varphi_2) + f_1(\varphi_2) - f_1(\varphi_0) + \frac{3}{2} [\sin(\varphi_2 - \varphi_1)]
\]

\[
f_3(\varphi_0) = \int_{\varphi_0}^{\varphi_2} \frac{\cos(x) f_1(x)}{(\sin(x) - 1)^2} dx.
\]

The comparison between the asymptotic theoretical re-
results (Eqs. 3, 8) and simulations is reported in Fig. 1. The
agreement is very good within the frequency range
\( 0 \div 0.2\omega_c \). It is interesting to see that the SD of the switch-
ting time scales as square root of noise intensity. The de-
pendencies of the MST and its SD on the bias current
are presented in Fig. 2 for \( \gamma = 0.001 \). The agreement be-
tween theoretical results and simulations is very good for
all the bias current values investigated. In the low noise
limit Eq. 11 gives actually the same results of ref. 11.
In the high noise limit Eq. 3, since the largest contribu-
tion to the MST comes from the deterministic term \( \tau_c(\varphi_0) = \frac{1}{\omega_c} [f_1(\varphi_2) - f_1(\varphi_0)] \).
However the Eq. 3, in some cases, significantly devi-
ates from the linearized calculations. For \( \gamma = 0.001 \) and
\( i = 1.2 \), \( \sigma = 0.4\omega_c^{-1} \) in ref. 11, while we get \( \sigma = 0.436\omega_c^{-1} \).
For larger current \( i = 1.5 \), the discrepancy is larger:
\( \sigma = 0.06\omega_c^{-1} \) in 11, and we get \( \sigma = 0.14\omega_c^{-1} \). In the
inset of Fig. 2 we report the behaviour of SD (Eq. 8)
as a function of noise intensity $\gamma$. For noise intensity values up to $\gamma = 0.05$ the agreement with computer simulations is very good. Therefore not only low temperature ($\gamma \leq 0.001$), but also high temperature devices may be described by Eqs. (6), (8). If noise intensity is rather large, the phenomenon of NES may be observed: the MST increases with the noise intensity, as it may be easily seen from Eq. (8). In the design of large arrays of RSFQ elements, operating at high frequencies, it is very important to consider this effect, otherwise it may lead to malfunctions due to the accumulation of digital errors.

Now let us consider the case of sinusoidal driving. The corresponding time characteristics may be derived using the modified adiabatic approximation [4, 11]

$$P(\varphi_0, t) = \exp \left\{ - \int_0^t \frac{1}{\tau_c(\varphi_0, t')} dt' \right\},$$

with $\tau_c(\varphi_0, t')$ given by (1). For dichotomous driving the value of initial current $i_0$ has a weak effect on temporal characteristics, as in ref. [4], while it is important for sinusoidal driving, since it also defines the potential barrier height. We focus now on the current value $i = 1.5$, because $i = 1.2$ is too small for high frequency applications. In Fig. 3 the MST as a function of the driving frequency for different values of bias current is shown. For smaller $i_0$ the switching time is larger, since $\varphi_0 = \arcsin(i_0)$ depends on $i_0$. On the other hand, the bias current $i_0$ must be not too large, since it will lead, in absence of driving, to the reduction of the mean life time of superconductive state, i.e. to increasing storage errors (Eq. (1)). Therefore, there must be an optimal value of bias current $i_0$, giving minimal switching time and acceptably small storage errors. Storage errors, in fact, are acceptably small up to $i_0 = 0.99$ for $\gamma = 0.001$ [4]. We observe the phenomenon of resonant activation: MST has a minimum as a function of driving frequency. The approximation [4] does not describe the resonant activation effect at high frequencies, but it works rather well below 0.1 $\omega_c$, that is enough for practical applications. Moreover, it is interesting to see that near the minimum the MST has a very weak dependence on the noise intensity, i.e. in this signal frequency range the noise is effectively suppressed. We observe also the NES phenomenon. There is a frequency range, around $0.2 - 0.4 \omega_c$ for $i_0 = 0.5$ and around $0.3 - 0.5 \omega_c$ for $i_0 = 0.8$, where the switching time increases with the noise intensity. The NES effect increases for smaller $i_0$ because the potential barrier disappears for a short time interval within the driving period $T = 2\pi/\omega$ [11] and the potential is more flat, so noise has more chances to prevent the phase to move down and delay switching process. This effect may be avoided, if the operating frequency does not exceed $0.2 \omega_c$. Besides (see Fig. 4) the SD also increases above $0.2 \omega_c$. The plots of SD as a function of driving frequency for $\gamma = 0.02, i = 1.5$ and different values of $i_0$ are shown in Fig. 4. The approximation [4] is not so good for SD as for MST, even if the qualitative behaviour of SD is recovered. We see that the minimum of $\sigma(\omega)$, for $\gamma = 0.02$, is located near the corresponding minimum for $\tau(\omega)$ in Fig. 3. For the SD the optimal frequency range, where the noise induced error will be minimal, is from 0.1 to 0.3 for the considered range of parameters. It is interesting to see that, near the minimum, the SD for sinusoidal driving actually coincides with SD for dichotomous driving (Eq. (8)). This means that even in the case of smooth driving, the limiting value of SD may nearly be reached.
but the RSFQ circuit must be properly optimized.

Finally we note that close location of minima of MST and its SD means that optimization of RSFQ circuit for fast operation will simultaneously lead to minimization of timing errors in the circuit, which is the main result of this paper.

In the present paper we reported an analytical and numerical analysis of influence of fluctuations and periodic driving on temporal characteristics of the JJ. For dichotomous driving the analytical expression of standard deviation of switching time works in practically interesting frequency range and for arbitrary noise intensity. For sinusoidal driving the resonant activation effect has been observed in the considered system: mean switching time has a minimum as a function of driving frequency. Near this minimum the standard deviation of switching time takes also a minimum value. The RA phenomenon was observed very recently in the underdamped Josephson tunnel junction [12]. Our theoretical investigation could motivate experimental work in overdamped JJs as well. Utilization of this effect in fact allows to suppress time jitter in practical RSFQ devices and, therefore, allows to significantly increase working frequencies of RSFQ circuits. Our study is not only important to understand the physics of fluctuations in a Josephson junction, to improve the performance of complex digital systems, but also in nonequilibrium statistical mechanics of dissipative systems, where noise assisted switching between metastable states takes place [8, 9, 11, 12].

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\[ \text{FIG. 4: The SD vs frequency for } f(t) = A \sin(\omega t) \text{ and } \gamma = 0.02. \text{ Computer simulations - dash-dotted line: } i_0 = 0.3; A = 1.2, \text{ short-dashed line: } i_0 = 0.5; A = 1, \text{ long-dashed line: } i_0 = 0.8; A = 0.7. \text{ The MST is given by crosses for comparison } (i_0 = 0.8; A = 0.7). \text{ Formula (8) - solid line. Formula (9) - diamonds.} \]