The Little-Hopfield model on a Random Graph

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Abstract. We study the Hopfield model on a random graph in scaling regimes where
the average number of connections per neuron is a finite number and where the spin
dynamics is governed by a synchronous execution of the microscopic update rule (Little-
Hopfield model). We solve this model within replica symmetry and by using bifurcation
analysis we prove that the spin-glass/paramagnetic and the retrieval/paramagnetic
transition lines of our phase diagram are identical to those of sequential dynamics.
The first-order retrieval/spin-glass transition line follows by direct evaluation of our
observables using population dynamics. Within the accuracy of numerical precision
and for sufficiently small values of the connectivity parameter we find that this
line coincides with the corresponding sequential one. Comparison with simulation
experiments shows excellent agreement.

1. Introduction

Since the first analytical work describing pattern recall was presented and the theoretical
foundations describing the operation of neural networks were subsequently set, the
progress in the field of attractor neural network models has been advancing revealing
interesting properties. Early analytical attempts to solve non-trivial neural network
systems were analytically constrained to the study of fully-connected ones. From a
real (i.e. biological) point of view one would ideally prefer to study models of sparse
connectivity. Subsequently, theoretical work turned (among other directions) to sparse
models, and, in particular, to the so-called extremely diluted ones [1]. These could be
still solved exactly owing to the simplification of considering system sizes exponentially
bigger than the average connectivity per neuron (while both of these numbers were
eventually sent to infinity).

One direction in developing the neural network theory further towards realism
implies moving away from the simplifications made in extremely diluted systems and
considering finite degrees of connectivity. This however appears to be a highly non-
trivial step ahead since at an early stage of solving the relevant equations one is
confronted with more than the traditional two observables (the magnetisation and the
overlap), in fact, one is required to consider an order-parameter function.

Although initial attempts to solve such models showed the underlying analytical
complexity [2, 3], the more refined mathematical tools were only relatively recently
The Little-Hopfield model on a Random Graph

developed and in fact for applications different than neural networks (for instance, for optimisation problems [4] or error-correcting codes [5]). The first study which applied the finite-connectivity methodology to neural network problems has been very recently presented in [6], where phase diagrams were presented for a variety of synaptic kernels. There, the authors study neural networks in which the microscopic neuronal dynamics is a sequential execution of an update rule (describing alignment to post-synaptic potentials). In this paper we examine the effect of choosing a \textit{synchronous} microscopic dynamics in Hopfield models of finite connectivity, i.e. one in which neurons are updated in parallel at each time step. These two different types of dynamics are known to share very interesting properties. For instance, in simple ferromagnetic models one can prove that thermodynamic observables become identical [7]. It is yet unclear to what extent the two types of dynamical models share these common equilibrium features. For example, it is known that the phase diagram of the synchronous Hopfield model changes [8, 9] whereas that of the Sherrington-Kirkpatrick model remains unaffected [10].

This paper is organised as follows: in the following section we present definitions of our model. In section 3 we derive the saddle point equations for the free energy and make our replica-symmetric ansatz. In section 4 we give expressions of the free energy in terms of the replica symmetric order function whereas our results and phase diagrams are given in section 5.

2. Definitions

We study neural network models of $N$ binary neurons $\sigma = (\sigma_1, \ldots, \sigma_N)$ with $\sigma_i = -1$ representing ‘at rest’ and $\sigma_i = 1$ the ‘firing’ state. The microscopic neuron dynamics is a stochastic alignment to ‘local fields’ (the post-synaptic potentials) in which updates in neuronal states are made for all $i \in \{1, \ldots, N\}$ in a fully synchronous way at each discrete time step:

\[
\text{Prob}[\sigma_i(t + 1) = \pm 1] = \frac{1}{2}[1 \pm \tanh(\beta h_i(\sigma(t)))]
\]

where $h_i(\sigma(t)) = \sum_j J_{ij} \sigma_j(t)$. The parameter $\beta \in [0, \infty)$ controls the amount of thermal noise in the dynamics with $\beta = \infty$ corresponding to a fully deterministic execution of (1) and $\beta = 0$ to a fully random execution. The quantities $J_{ij}$ describe interaction couplings. If one expresses (1) in terms of the microscopic state probabilities $p_t(\sigma)$:

\[
p_{t+1}(\sigma) = \sum_{\sigma'} W[\sigma; \sigma'] p_t(\sigma') \quad W[\sigma; \sigma'] = \prod_{i=1}^{N} \frac{e^{\beta \sigma_i h_i(\sigma')}}{2 \cosh[\beta \sigma_i h_i(\sigma')]} \tag{2}
\]

then, for any finite $\beta$ and finite $N$ the process (2) is ergodic and evolves to a unique distribution $p_\infty(\sigma)$. It can be shown that this is a unique equilibrium state (obeying

\footnote{This difference seems to be an artifact of the replica symmetric approximation. The two phase diagrams become identical within full Replica Symmetry Breaking [8], although it is unclear at which stage of the breaking scheme this occurs.}
detailed balance) if and only if $J_{ij} = J_{ji}$ $\S$. At equilibrium, the microscopic state probabilities acquire the form $p_\infty \sim \exp[-\beta H(\sigma)]$, where

$$H(\sigma) = -\frac{1}{\beta} \sum_i \log[2 \cosh(\beta h_i(\sigma))]$$

(3)

is the Hamiltonian. For the interactions $J_{ij}$ we now take:

$$J_{ij} = \frac{c_{ij}}{c} \sum_{\mu=1}^p \xi^\mu_i \xi^\mu_j$$

(4)

This corresponds to storing $p$ memories $\xi_i = (\xi^1_i, \ldots, \xi^p_i) \in \{-1, 1\}^p$ among the synapses in a Hebbian-type way. The variables $c_{ij} \in \{0, 1\}$ with $c_{ij} = c_{ji}$ represent dilution, while $c = \sum_j c_{ij}$ (for all $i$) corresponds to the average number of connections per neuron. Models of the type (4) with $c \to \infty$ (while $c \sim \log N$) are known as extremely diluted and due to their simplicity have been studied extensively (see for instance \footnote{[1]} \footnote{[2]} \footnote{[3]} and references therein). What is less known are properties of these systems in the non-trivial scaling regime of finite connectivity where $c \sim \mathcal{O}(1)$ (with the probability $c/N \to 0$).

Due to the complicated methodology required such systems have only recently been studied in \footnote{[6]} where a thorough bifurcation analysis was performed and phase diagrams were derived. For the distribution of the random variable $c_{ij}$ we will consider

$$P(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + \left(1 - \frac{c}{N}\right) \delta_{c_{ij},0}$$

(5)

for all pairs $(i, j)$. All thermodynamic quantities will have to be averaged over (5). Note that due to the system’s sparse connectivity the number of patterns can only be finite.

To derive observable quantities we will calculate the free energy per neuron $f = -\lim_{N \to \infty} \frac{1}{\beta N} \left\langle \log \sum_\sigma e^{-\beta H(\sigma)} \right\rangle_{\{c_{ij}\}}$, with $\left\langle \cdots \right\rangle$ denoting average over the distribution of dilution variables. As in all synchronous dynamics models, the evaluation of the free energy is greatly simplified by introducing an extra set of spins so that $f$ can be rewritten as

$$f = -\lim_{N \to \infty} \frac{1}{\beta N} \left\langle \log \sum_{\sigma \tau} e^{-\beta \mathcal{H}(\sigma, \tau)} \right\rangle_{\{c_{ij}\}} \quad \mathcal{H}(\sigma, \tau) = -\sum_{ij} \sigma_i J_{ij} \tau_j$$

(6)

Equation (6) will be the starting point for our analysis.

3. Saddle-point equations

In order to calculate the free energy (6) we begin by invoking the replica identity

$$\langle \log Z \rangle = \lim_{n \to 0} \frac{1}{n} \log \langle Z^n \rangle.$$ 

One can then take the average over the dilution variables resulting in

$$f = -\lim_{N \to \infty} \lim_{n \to 0} \frac{1}{\beta N n} \log \sum_{\sigma^1 \ldots \sigma^n} \sum_{\tau^1 \ldots \tau^n} \exp \left[ \frac{c}{2N} \sum_{ij} \left( e^\frac{\beta}{n} (\epsilon_i \epsilon_j) \sum_{\alpha} (\sigma^\alpha_i \tau^\alpha_j + \sigma^\alpha_j \tau^\alpha_i) - 1 \right) \right]$$

(7)

$\S$ Note that unlike sequential models where detailed balance requires the exclusion of self-interactions, this is not here a prerequisite. However, due to the scaling regime of finite connectivity terms originating from the self-interacting part will for $N \to \infty$ give a vanishing contribution to thermodynamic quantities.
where $\xi_i \cdot \xi_j = \sum_{\mu} \xi^\mu_i \xi^\mu_j$. Note that at this stage an exponential has appeared as an argument of another exponential. Here, effectively, the inner-most exponential introduces an infinite number of observables. However, unlike systems of full-connectivity or of extreme dilution where only the linear and quadratic observables (order parameters) survive in the limit $N \to \infty$, all cumulants play here a role. They can be re-cast in an order-parameter function. This complication is a direct consequence of the scaling regime of finite connectivity.

It is convenient now to use the concept of sublattices [15] which divide the space of sites into $2^p$ groups $I_\xi = \{i|\xi_i = \xi\}$. We define $\sigma_i = (\sigma^1_i, \ldots, \sigma^n_i)$ and abbreviate the averages over the sublattices by $\langle F(\xi) \rangle_\xi = \sum_\xi p_\xi F(\xi)$ with the probabilities $p_\xi \equiv |I_\xi|/N$. Then the free energy becomes:

$$f = -\lim_{N \to \infty} \frac{1}{\beta N} \log \sum_{\sigma^1 \ldots \sigma^n} \sum_{\tau^1 \ldots \tau^n} \exp \left[ \frac{C}{2N} \sum_{\sigma \sigma'} \sum_{\tau \tau'} \sum_{\xi \xi'} \left( e^{\frac{\alpha}{c}(\xi \xi') \sum_{\alpha} (\sigma_{\alpha} \tau'_{\alpha} + \sigma'_{\alpha} \tau_{\alpha})} - 1 \right) \right] \times \sum_{i \in I_\xi} \delta_{\sigma_i, \delta_{\tau_i}} \sum_{j \in I_{\xi'}} \delta_{\sigma'_{j}, \delta_{\tau'_{j}}}$$

(8)

We now see that an order-function has emerged which we introduce into our expression via

$$1 = \int \prod_{\xi \sigma \tau} \left\{ dP_\xi(\sigma, \tau) \delta \left[ P_\xi(\sigma, \tau) - \frac{1}{|I_\xi|} \sum_{i \in I_\xi} \delta_{\sigma_i, \delta_{\tau_i}} \right] \right\}$$

(9)

In the above expression we replace the $\delta$-function by its integral representation (which introduces the conjugate order function parameter $\hat{P}_\xi(\sigma, \tau)$). We then perform the trace over the spin variables and take the limit $N \to \infty$ in our equations resulting in the extremisation problem:

$$f = -\lim_{n \to 0} \frac{1}{\beta n} \text{Extr}_{(P, \hat{P})} \left\{ \sum_{\sigma \tau} P_\xi(\sigma, \tau) \hat{P}_\xi(\sigma, \tau) \right\}_\xi + \langle \log[\sum_{\sigma \tau} e^{-\hat{P}_\xi(\sigma, \tau)}] \rangle_\xi$$

(10)

$$+ \frac{C}{2} \sum_{\sigma \sigma'} \sum_{\sigma'' \tau''} P_\xi(\sigma, \tau) P_\xi(\sigma', \tau') \left[ e^{\frac{\alpha}{c}(\xi \xi') \sum_{\alpha} (\sigma_{\alpha} \tau'_{\alpha} + \sigma'_{\alpha} \tau_{\alpha})} - 1 \right]_{\xi \xi''}$$

(11)

Variation with respect to the densities $P_\xi(\sigma, \tau)$ and $\hat{P}_\xi(\sigma, \tau)$ gives the self-consistent equation:

$$P_\xi(\sigma, \tau) = \frac{\exp \left[ c \langle \sum_{\sigma'} P_\xi(\sigma', \tau') \left( e^{\frac{\alpha}{c}(\xi \xi') \sum_{\alpha} (\sigma_{\alpha} \tau'_{\alpha} + \sigma'_{\alpha} \tau_{\alpha})} - 1 \right) \right]_\xi}{\sum_{\sigma'' \tau''} \exp \left[ c \langle \sum_{\sigma'''} P_\xi(\sigma''', \tau''') \left( e^{\frac{\alpha}{c}(\xi \xi'') \sum_{\alpha} (\sigma'_{\alpha} \tau''_{\alpha} + \sigma''_{\alpha} \tau'_{\alpha})} - 1 \right) \right]_{\xi''}}$$

(12)

and also allows us to eliminate $\hat{P}_\xi(\sigma, \tau)$ from the expression of the free energy which now becomes:

$$f = \lim_{n \to 0} \frac{1}{\beta n} \text{Extr}_{(P, \xi)} \left\{ \frac{C}{2} \sum_{\sigma \tau \sigma'} \sum_{\tau'} P_\xi(\sigma, \tau) P_\xi(\sigma', \tau') \left( e^{\frac{\alpha}{c}(\xi \xi') \sum_{\alpha} (\sigma_{\alpha} \tau'_{\alpha} + \sigma'_{\alpha} \tau_{\alpha})} - 1 \right) \right\}_{\xi \xi'}$$

(13)
This expression requires knowledge of the densities \( P_{\xi}(\sigma, \tau) \). However, the evaluation of the self-consistent equation (12) is an impossible task unless a further simplification is made. To proceed further we will make an assumption about the form of the densities \( P_{\xi}(\sigma, \tau) \). In the spirit of replica symmetry (RS) we will consider that permutation of spins within different replicas leave the order parameters invariant. Here however, due to the presence of two species of spins we will also require that within the same replica group permutation of the states of \( \sigma_\alpha \) and \( \tau_\alpha \) also leave for all \( \alpha \) \( P_{\xi}(\sigma, \tau) \) invariant:

\[
P_{\xi}(\sigma, \tau) = \int \frac{dhdrdt}{\mathcal{N}(h, r, t)^n} W_{\xi}(h, r, t) e^{\beta h \sum_\alpha \sigma_\alpha + \beta r \sum_\alpha \tau_\alpha + \beta t \sum_\alpha \sigma_\alpha \tau_\alpha} \tag{14}
\]

where \( \mathcal{N}(h, r, t) \) is the corresponding normalisation constant

\[
\mathcal{N}(h, r, t) = 4 \cosh(\beta h) \cosh(\beta r) \cosh(\beta t) + 4 \sinh(\beta h) \sinh(\beta r) \sinh(\beta t) \tag{15}
\]

Let us finally turn to our system’s macroscopic observables. Our replicated sublattice overlaps will be given by

\[
m^{\lambda \alpha}_\lambda = \langle \xi^\mu m^{\lambda \alpha}_\xi \rangle_{\xi} \tag{16}
\]

with the sublattice magnetisations defined as

\[
m^{\lambda \alpha}_\xi = \sum_{\sigma \tau} P_{\xi}(\sigma, \tau) \lambda^\alpha \tag{17}
\]

with \( \lambda = \sigma, \tau \). In RS the above quantities are the same for all values of the replica index. Thereafter we will drop for notational simplicity the index \( \alpha \). The observables (16) will be generated from the densities \( P_{\xi}(\sigma, \tau) \). In general, from \( P_{\xi}(\sigma, \tau) \) one can evaluate all higher order observables

\[
L^{\alpha_1 \cdots \alpha_m \gamma_1 \cdots \gamma_\ell}_{\xi} = \sum_{\sigma \tau} P_{\xi}(\sigma, \tau) \sigma^{\alpha_1} \cdots \sigma^{\alpha_m} \tau^{\gamma_1} \cdots \tau^{\gamma_\ell} \tag{18}
\]

Working out the simplest of these using the RS ansatz (14) gives:

\[
m^{\mu}_\sigma \equiv \langle \xi^\mu \sum_{\sigma \tau} P_{\xi}(\sigma, \tau) \sigma^{\alpha_1} \rangle_{\xi} \\
= \langle \xi^\mu \int dhdrdt W_{\xi}(h, r, t) \frac{\tanh(\beta h) + \tanh(\beta r) \tanh(\beta t)}{1 + \tanh(\beta t) \tanh(\beta r) \tanh(\beta h)} \rangle_{\xi} \tag{19}
\]

\[
q_{\sigma \sigma} \equiv \langle \sum_{\sigma \tau} P_{\xi}(\sigma, \tau) \sigma^{\alpha_1} \sigma^{\alpha_2} \rangle_{\xi} \\
= \langle \int dhdrdt W_{\xi}(h, r, t) \left[ \frac{\tanh(\beta h) + \tanh(\beta r) \tanh(\beta t)}{1 + \tanh(\beta t) \tanh(\beta r) \tanh(\beta h)} \right]^2 \rangle_{\xi} \tag{20}
\]

and similar expressions follow for the observables \( m^{\mu}_\tau, q_{\tau \tau} \) and \( q_{\sigma \tau} \).
4. The self-consistent equation and the free energy

4.1. Derivation of the self-consistent equation

In order to arrive at a self-consistent equation for the densities $W_\xi(h, r, t)$ we first substitute the RS ansatz (14) into (12). Then, following [6], we isolate the occurrences of quantities of the form $\sum_\alpha \sigma_\alpha$, $\sum_\alpha \tau_\alpha$ and $\sum_\alpha \sigma_\alpha \tau_\alpha$ by inserting

$$1 = \sum_{m_\alpha = -\infty}^{\infty} \int_0^{2\pi} \frac{d\hat{m}_\sigma}{2\pi} e^{i\hat{m}_\sigma (m_\sigma - \sum_\alpha \sigma_\alpha)}$$

$$1 = \sum_{m_\tau = -\infty}^{\infty} \int_0^{2\pi} \frac{d\hat{m}_\tau}{2\pi} e^{i\hat{m}_\tau (m_\tau - \sum_\alpha \tau_\alpha)}$$

$$1 = \sum_{q = -\infty}^{\infty} \int_0^{2\pi} \frac{dq}{2\pi} e^{iq (q - \sum_\alpha \sigma_\alpha \tau_\alpha)}$$

After some algebra we can take the limit $n \to 0$ in our equations and arrive at:

$$\int dW_\xi(h, r, t) e^{\beta h m_\sigma + \beta r m_\tau + \beta q} = \exp \left[ c \left( \int dW_\xi'(h', r', t') \left( e^{m_\alpha K_1(h', r', t'; \xi - \xi') + m_\tau K_2(h', r', t'; \xi - \xi') + q K_3(h', r', t'; \xi - \xi') - 1} \right) \right] \xi$$

where we have used the notation $dW_\xi(h, r, t) = dhdrdt W_\xi(h, r, t)$ and we introduced

$$K_1(h', r', t'; \xi - \xi') = \frac{1}{4} \log \frac{\Omega_{+++\Omega_{+-}+\Omega_{-+}}}{\Omega_{+-+\Omega_{-++\Omega_{-}}}}$$

$$K_2(h', r', t'; \xi - \xi') = \frac{1}{4} \log \frac{\Omega_{++\Omega_{-+}+\Omega_{+-}}}{\Omega_{+-\Omega_{+-+}}+\Omega_{-++\Omega_{-}}$$

$$K_3(h', r', t'; \xi - \xi') = \frac{1}{4} \log \frac{\Omega_{++\Omega_{-+}+\Omega_{+-}}}{\Omega_{+-\Omega_{+-+}}+\Omega_{-++\Omega_{-}}$$

with

$$\Omega_{\sigma r} \equiv \Omega_{\sigma r}(h', r', t'; \xi - \xi') = 4 \cosh[\beta h' + \sigma \frac{\beta}{c} (\xi - \xi')] \cosh[\beta r' + \tau \frac{\beta}{c} (\xi - \xi')] \cosh[\beta t'] +$$

$$+ 4 \sinh[\beta h' + \sigma \frac{\beta}{c} (\xi - \xi')] \sinh[\beta r' + \tau \frac{\beta}{c} (\xi - \xi')] \sinh[\beta t']$$

Performing an inverse Fourier transform to (24), expanding the right-hand side and integrating over the magnetisations $m_\sigma, m_\tau$ and the overlap $q$ we obtain

$$W_\xi(h, r, t) = \sum_{k=0}^\infty \frac{e^{-c_k}}{k!} \left( \prod_{l=1}^k \int dW_\xi(h_l, r_l, t_l) \right) \delta \left[ h - \frac{1}{\beta} \sum_{l=1}^k K_1(h_l', r_l', t_l'; \xi - \xi') \right]$$

$$\times \delta \left[ t - \frac{1}{\beta} \sum_{l=1}^k K_2(h_l', r_l', t_l'; \xi - \xi') \right] \delta \left[ r - \frac{1}{\beta} \sum_{l=1}^k K_3(h_l', r_l', t_l'; \xi - \xi') \right]$$

Expression (29) is the final result from which to evaluate the densities $W_\xi(h, r, t)$. Although appearing a daunting numerical task it can be evaluated using the simple
algorithm of ‘population dynamics’ as described e.g. in [16]. This relies on the local ‘tree-like’ structure of finitely-connected networks of spins and it consists of having a large population of triplets \((h_l, r_l, t_l)\) which are to be updated for a large number of iteration steps in the following way: One first selects a number \(k\) from a Poisson distribution of mean \(c\). Then, one chooses \(l = 1, \ldots, k\) triplets \((h_l, r_l, t_l)\) and \(l\) sublattices \(\xi^l\) at random and calculates the expressions appearing in the delta functions. Finally, one selects a new triplet \((h, r, t)\) and a new sublattice \(\xi\) at random and updates them using the calculated expressions.

To get an idea about the profile of the densities \(W_\xi(h, r, t)\) we have applied the previous algorithm to study the marginal densities \(W^h_\xi(h) = \int d r d t W_\xi(h, r, t)\), \(W^r_\xi(r) = \int d h d t W_\xi(h, r, t)\) and \(W^t_\xi(t) = \int d h d r W_\xi(h, r, t)\). Firstly, we observe that close to the zero-temperature limit the marginals \(W^h_\xi(h)\) and \(W^r_\xi(r)\) become a collection of delta peaks; in this limit due to the absence of thermal noise the effective fields \(\{h, r\}\) become identical to the true local fields. Since the average number of connections per neuron is here a finite number and the couplings are discrete, the local fields are a multiple integer \([3]\) of \(1/c\) (see figure 1 for typical profiles of the marginal densities).

As one moves away from the \(T = 0\) regime the profile of the marginal densities \(W^h_\xi(h)\) and \(W^r_\xi(r)\) takes a non-trivial form until the paramagnetic phase is reached where one observes that \(W^h_\xi(x) = W^r_\xi(x) = W^t_\xi(x) = \delta(x)\), as it should. We have numerically evaluated the above marginals for several regions of the parameter space and it is interesting to note that \(W^h_\xi(h)\) and \(W^r_\xi(r)\) are identical (within the accuracy of numerical

\[\|\text{In fact, for even number of patterns } p \text{ we have that the local fields present peaks at } 2l/c \text{ with } l \text{ integer, while for odd } p \text{ we have } l/c.\]

Figure 1. Typical profiles of the marginal densities \(W^h_\xi(h)\) for \(p = 6, c = 4, T = 0.1\) (left picture, corresponding to the spin-glass region of figure 2) and for \(p = 2, c = 5, T = 0.1\) (right picture, corresponding to the retrieval region of figure 2). Notice that since \(p\) is even the peaks are located at \(h = 2l/c\) with \(l\) integer.
precision) \( \mathbb{Q} \) whereas \( W_{\xi}^{t}(t) = \delta(t) \). The latter implies that our RS ansatz can be simplified further. However this result relies on numerical observations and a rigorous analytical proof appears somewhat hard (one cannot rely on an induction proof since at the first iteration step of \( \mathbb{Q} \) the fields \( t \) must take a non-zero value). Nevertheless, it is interesting to note that setting \( t = 0 \) to \( \mathbb{Q} \) completely decouples the density \( W_{\xi}(h, r, t) \) to two identical densities

\[
Q_{\xi}(x) = \sum_{k=0}^{\infty} \frac{e^{-c}c^{k}}{k!} \left\langle \cdots \int \prod_{l=1}^{k} dx_{l} Q_{\xi}(x_{l}) \right\rangle \times \delta \left[ x - \frac{1}{\beta} \sum_{l=1}^{k} \mathrm{ath} \left( \tanh[\beta x_{l}] \tanh[\frac{\beta}{c}(\xi_{l})] \right) \right] \xi^{i} \cdots \xi^{i} \tag{30}
\]

with \( x = \{h, r\} \). Note that this equation is now the same as the one that follows from the analysis of \( \mathbb{Q} \), as it should.

4.2. The free energy

Let us now express the free energy \( \mathbb{Q} \) as function of the densities \( W_{\xi}(h, r, t) \). The first term comprising \( \mathbb{Q} \) (energetic part) can be calculated without difficulty. One uses the RS ansatz \( \mathbb{Q} \) to replace the distributions \( P_{\xi}(\sigma, \tau) \) by the densities \( W_{\xi}(h, r, t) \) and subsequently one performs the spins summations and takes the limit \( n \rightarrow 0 \). The second term is slightly more complicated. First we expand the exponential function. This replicates the traces over the spins and sublattice variables. Inserting then for each of the replicated densities the RS ansatz \( \mathbb{Q} \) leads to

\[
\left\langle \log \sum_{\sigma_{\tau}} \left[ c \left( \sum_{\sigma_{\tau}} P_{\xi}(\sigma_{\tau}) \left( e^{\frac{\beta}{c}(\xi_{\tau}) \sum_{\alpha}(\sigma_{\alpha}, \tau_{\alpha}) - 1 \right) \right) \right] \right\rangle_{\xi} = \left\langle \log \sum_{k=0}^{\infty} \frac{e^{-c}c^{k}}{k!} \left\langle \cdots \int \prod_{l=1}^{k} dW_{\xi}(h_{l}, r_{l}, t_{l}) \right\rangle \right\rangle_{\xi} \times \left\langle \prod_{\alpha=1}^{n} \sum_{\sigma_{\alpha}, \tau_{\alpha}}^{k} \left[ c^{\frac{1}{2}}(\xi_{\alpha})^{(\sigma_{\alpha}, \tau_{\alpha})} \right] \right\rangle_{\xi^{i}} \cdots \xi^{i} \tag{31}
\]

Performing the spin summations in the above equation and using the simple expression \( F(\sigma_{\alpha}, \tau_{\alpha}) = \sum_{\sigma_{\tau}} \delta_{\sigma_{\alpha}, \sigma_{\tau}} \delta_{\tau_{\alpha}, \tau_{\tau}} F(\sigma, \tau) \) to relocate the occurrences of \( \sigma_{\alpha} \) and \( \tau_{\alpha} \), we can take the limit \( n \rightarrow 0 \). The final result for the free energy \( \mathbb{Q} \) is then

\[
\begin{align*}
f = & -\frac{1}{\beta} \sum_{k=0}^{\infty} \frac{e^{-c}c^{k}}{k!} \left\langle \cdots \int \prod_{l=1}^{k} dW_{\xi}(h_{l}, r_{l}, t_{l}) \right\rangle \log \left[ \sum_{\sigma_{\tau}=\pm}^{k} \prod_{l=1}^{k} \frac{\Omega_{\sigma_{\tau}}(h_{l}, r_{l}, t_{l}; \xi_{l})^{(\xi_{l})}}{N(h_{l}, r_{l}, t_{l})} \right] \xi^{i} \cdots \xi^{i} \\
+ & \frac{c}{2\beta} \left\langle \int dW_{\xi}(h, r, t) dW_{\xi}(h', r', t') \right\rangle \end{align*}
\]

\( \mathbb{Q} \) This is of course expected since the hamiltonian \( \mathbb{Q} \) is invariant under the interchange of the two spin species.
The Little-Hopfield model on a Random Graph

5. Phase Diagrams

5.1. Bifurcation Analysis

The numerical observations concerning the densities \( W_\xi(h, r, t) \) in section 4 support the equivalence between the parallel and sequential versions of the finite-\( c \) Hopfield model. In this section we will show further that one can analytically prove that the location of the second-order transitions between paramagnetic/spin-glass and paramagnetic/retrieval phases is identical to those of sequential dynamics.

Our bifurcation analysis is similar in spirit to [6]. First we note that \( W_\xi(h, r, t) = \delta(h)\delta(r)\delta(t) \) solves (24) for all \( \xi \). This solution, as can be easily confirmed, corresponds to the high-temperature paramagnetic state where no recall is possible and \( m_\xi^h = m_\xi^h = 0 \). As the temperature is lowered from \( T = \infty \) we expect non-trivial solutions to appear. To determine the transition temperature at which these occur we first assume that close to the transition \( \int dW_\xi(h, r, t)h^\ell = O(\epsilon^\ell) \), \( \int dW_\xi(h, r, t)r^\ell = O(\epsilon^\ell) \) and \( \int dW_\xi(h, r, t)t^\ell = O(\epsilon^\ell) \). We can then expand equation (24) and identify term by term each expression in the resulting power series in the left- and right-hand sides of (24).

To second order, this one-to-one correspondence firstly indicates that integrals carrying the field \( t \) must vanish. We then obtain:

\[
\int dW_\xi(h, r, t) h = c\langle \int dW_\xi(h, r, t) h \, \tanh[\beta \langle \xi \cdot \xi' \rangle] \rangle \xi' \tag{33}
\]

\[
\int dW_\xi(h, r, t) r = c\langle \int dW_\xi(h, r, t) r \, \tanh[\beta \langle \xi \cdot \xi' \rangle] \rangle \xi' \tag{34}
\]

\[
\int dW_\xi(h, r, t) h^2 - \left[ \int dW_\xi(h, r, t) h \right]^2 = c\langle \int dW_\xi(h, r, t) h^2 \, \tanh[\beta \langle \xi \cdot \xi' \rangle] \rangle \xi' \tag{35}
\]

\[
\int dW_\xi(h, r, t) r^2 - \left[ \int dW_\xi(h, r, t) r \right]^2 = c\langle \int dW_\xi(h, r, t) r^2 \, \tanh[\beta \langle \xi \cdot \xi' \rangle] \rangle \xi' \tag{36}
\]

These four equations mark the different types of transitions away from the paramagnetic state. The first pair of equations will give us the transition from \( m_\xi^h = 0 \) and \( m_\xi^r = 0 \) respectively to a non-trivial recall (ferromagnetic) state whereas the second pair will give us the transition to the spin-glass state. The resulting similarity in the first and second pair of the above equations is a consequence of the equivalence of the two species of spins in our system.

To determine now the critical parameter values for which these transitions occur we need to evaluate the highest temperature for which the following \( 2^p \times 2^p \) matrices have an eigenvalue equal to 1:

\[
M_{\xi\xi'} = cp_\xi \tanh[\beta \langle \xi \cdot \xi' \rangle] \quad Q_{\xi\xi'} = cp_\xi \tanh^2[\beta \langle \xi \cdot \xi' \rangle] \tag{37}
\]
These equations are identical to those found in [6]. An elegant construction of the eigenvalues of matrices of the above form (which relies on exploiting the properties $M\xi\xi' = M(\xi \cdot \xi')$ and $Q\xi\xi' = Q(\xi \cdot \xi')$) has already been given in [15]. Here we state the final result determining the transition lines, identical to that found in [6]:

\[ P \rightarrow R : \quad \frac{c}{2p} \sum_{m=0}^{p} \binom{p}{m} (p - 2m) \tanh[\beta \frac{c}{c}(p - 2m)] = 1 \quad (38) \]

\[ P \rightarrow SG : \quad \frac{c}{2p} \sum_{m=0}^{p} \binom{p}{m} \tanh^2[\beta \frac{c}{c}(p - 2m)] = 1 \quad (39) \]

In general, the bifurcation analysis can not be applied to determine the spin-glass/retrieval transition line which must be computed directly from equations (29) and (19). This numerical task becomes increasingly difficult for large values of the connectivity parameter $c$ (since one is required to evaluate $2^p$ sublattice densities where $p \approx c$ close to the spin-glass/retrieval transition). For sufficiently small values of the connectivity parameter $c$ (up to $c = 7$) we find that this transition line is identical to that of sequential dynamics. It is interesting to note that within replica-symmetric considerations, the $c \rightarrow \infty$ (extremely-diluted) phase diagrams of sequential and parallel Hopfield models are not identical, indicating that a critical $c^*$ exists above which $W_t^\xi(t)$ is no longer given by $\delta(t)$. Determining this transition is a challenging numerical task.

5.2. Results and comparison with simulations

By numerically solving equations [19] and (29) we find that only one magnetisation component (also called pattern overlap) $m^\lambda_\alpha$ (16) provides non-trivial solutions (i.e. only one pattern is retrieved while the others are zero.). For notational simplicity, we will from now on denote these non-zero solutions simply as $m_\sigma$ and $m_\tau$.

Plotting solutions of (38) and (39) in the $(\alpha, T)$ plane (with $\alpha = p/c$) results in the phase diagram of figure 2. It has been plotted for $c \in \{2, 3, 5, 7, 10\}$ and it consists of three phases: a high-temperature paramagnetic phase with all observables identically zero, a retrieval phase with $m_\sigma, m_\tau \neq 0$ and $q_{\lambda\lambda'} = 0$ (with $\lambda = \sigma, \tau$) and a spin-glass phase with $m_\sigma = m_\tau = 0$ and $q_{\lambda\lambda'} \neq 0^+$. For $T \leq 1$ the spin-glass/retrieval transition is obtained numerically by direct evaluation of our observables.

We see that already for $c = 10$ (upper line) the phase diagram resembles closely the one derived in the extremely-diluted model [1] where the transition lines are given by $T = 1$ for $\alpha \leq 1$ ($P \rightarrow R$) and $T = \sqrt{\alpha}$ for $\alpha \geq 1$ ($P \rightarrow SG$). From the perspective of neural network operation, this shows that sparsely connected models retain the same qualitative features for a wide variety of synaptic connections per neuron and can retrieve information even for surprisingly small values of connections.

In figure 2 we present solutions of the sublattice overlap $m_\sigma$ [19] and $m_\tau$ showing the second-order transition from retrieval to paramagnetic states for the simplest non-trivial + note that due to the inequality $q_{\alpha_1\alpha_2} \geq q_{\alpha_1\alpha_2\alpha_3\alpha_4} \geq \cdots \geq 0$ we only need to examine $q_{\alpha_1\alpha_2}$ to identify the spin-glass phase.
Figure 2. Phase diagram of the finite-connectivity Hopfield model with synchronous dynamics in the $\alpha, T$ plane. Left: The paramagnetic/retrieval and paramagnetic/spin-glass transition lines as obtained from the bifurcation analysis. (from lower to upper lines $c = \{2, 3, 5, 7, 10\}$). Right: $c = 7$ phase diagram with the spin-glass/retrieval transition line (obtained by direct evaluation of the pattern overlaps $19$ using population dynamics). Markers correspond to integer $p$ values (lines are simply put as guides to the eye). On the retrieval/spin-glass transition line $m_\sigma = m_\tau = 0$.

The Little-Hopfield model on a Random Graph

11

0 1 2 3 4 5
0 0.5 1 1.5

$P$

$SG$

$R$

$\alpha = p/c$

On the retrieval/spin-glass transition line $m_\sigma = m_\tau = 0$. These have been drawn for $c = \{3, 5, 7\}$ showing the effect of different degrees of connectivity on the retrieval success. To verify our results we have performed simulation experiments on the dynamical process $11$ for a system size of $N = 10,000$ neurons (markers of figure $3$). These appear to be in very good agreement with our theory. The simulation experiments also show that while Hebbian-type couplings $4$ lead to fixed-point stationary solutions, anti-Hebbian ones where $J_{ij} = -\frac{c_{ij}}{\xi_i \cdot \xi_j}$ lead to 2-cycles with $m_\sigma(t) = -m_\sigma(t + 1)$ (and similarly for $m_\tau$). Also, evaluation of the free energy $32$ as a function of temperature $T$ shows that it is a monotonically decreasing function indicating that the entropy remains positive even for low temperatures.

We have also compared our pattern overlaps with those derived from the analysis of $6$ (which has as a starting the Ising ‘sequential’ Hamiltonian) and within the accuracy of numerical precision we have found that observables of the two systems are identical. This result, although somewhat expected given both knowledge from earlier neural network studies and, of course, the identity of our bifurcation results with those of $6$, is also a bit surprising: in the process of solving our equations we introduced a 3D RS ‘effective-field’ distribution (compared to the 1D sequential one of $6$). As it turns out however the system in the process of updating fields by iteration of $29$ effectively discards those which describe species-correlations and factorises $W_\xi(h, r, t)$ to $W_\xi(h, r) \delta(t)$. Henceforth, our 3D RS field-distribution reduces to a 2D one. The equivalent treatment of $\sigma$ and $\tau$ then ensures (as also in analytically simpler models) that equilibrium observables between sequential and synchronous systems are identical.
Figure 3. Pattern overlaps $m_\sigma = m_\tau = m$ as functions of $T = 1/\beta$ calculated from equations (19) and (29). Parameter values are: $p = 2$ and $c = \{3, 5, 7\}$ (from lower to upper lines). Markers correspond to simulation experiments of $N = 10,000$ neurons.

(this relation however ceases to exist for a certain large value $c_*$ since for $c \to \infty$ the sequential and parallel phase diagrams are different [8]).

Figure 4. Pattern overlap $m_\sigma^\mu = m_\tau^\mu = m^\mu$ as a function of $\alpha$ (with $c = 7$) for $T = 0.1$ (circles) and $T = 0.7$ (triangles). Here we can see the first-order transition from retrieval to the spin-glass phase at $\alpha = 6/7$ and $\alpha = 7/7$ for $T = 0.1$ and $T = 0.7$ respectively.

In figure 4 we have plotted the pattern overlaps $m_\sigma^\mu$ and $m_\tau^\mu$ as a function of $\alpha$ for different values of the temperature (note that $m_\sigma^\mu = m_\tau^\mu$). We have set the average connectivity to $c = 7$ while varying the number of patterns $p$ (therefore the number of pattern overlaps $m^\mu$ also changes with $\alpha$). From this graph we can complete the phase diagram (figure 2), determining the location of the first-order transition from retrieval to spin-glass states.
6. Discussion

In this paper we have studied equilibrium properties of attractor neural network models with finite connectivity in which neurons operate in a parallel way. This work is motivated by the interesting properties that the two types of dynamical models share in simpler (e.g. fully-connected) scenarios: There, and for a surprisingly large number of universality classes, one can prove analytically that the equilibrium states following from the two different dynamical rules are identical. Our starting point here is the Peretto Hamiltonian \[3\]. We have followed on the footsteps of \[6\] to derive the transition lines in our phase diagram and expressions for the (sublattice) overlaps. The resulting phase diagram plotted in the \((\alpha, T)\) plane then consists of three phases: a high-temperature paramagnetic phase, a retrieval and a spin-glass phase. The transitions from the paramagnetic phase have been determined analytically while the first-order retrieval/spin-glass transition has been computed using population dynamics \[16\]. Under our replica-symmetric considerations we have shown that the retrieval properties of the parallel finite-connectivity Hopfield model are identical to those of the sequential one. Comparison with numerical simulations for large system sizes shows excellent agreement.

Many questions remain to be answered for neural networks of finite connectivity. Using the same framework (as initiated by \[6\]) one can proceed further with the study of multi-state e.g. \(Q\)-Ising or Blume-Emery-Griffiths neural networks. A different approach would be to study systems which consist of two species of operating units (as, effectively, here) but which explicitly interact with one another. This would lead us to an Ashkin-Teller-type neural network in which the phase diagram can be different. Non-trivial extensions would be to study the validity of the RS solution following the schemes in \[19, 20\]. These will be the subject of a future work.

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