Privacy Amplification in Quantum Key Distribution: Pointwise Bound \textit{versus} Average Bound

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In order to be practically useful, quantum cryptography must not only provide a guarantee of secrecy, but it must provide this guarantee with a useful, sufficiently large throughput value. The standard result of generalized privacy amplification yields an upper bound only on the average value of the mutual information available to an eavesdropper. Unfortunately this result by itself is inadequate for cryptographic applications. A naive application of the standard result leads one to incorrectly conclude that an acceptable upper bound on the mutual information has been achieved. It is the pointwise value of the bound on the mutual information, associated with the use of some specific hash function, that corresponds to actual implementations. We provide a fully rigorous mathematical derivation that shows how to obtain a cryptographically acceptable upper bound on the actual, pointwise value of the mutual information. Unlike the bound on the average mutual information, the value of the upper bound on the pointwise mutual information and the number of bits by which the secret key is compressed are specified by two different parameters, and the actual realization of the bound in the pointwise case is necessarily associated with a specific failure probability. The constraints amongst these parameters, and the effect of their values on the system throughput, have not been previously analyzed. We show that the necessary shortening of the key dictated by the cryptographically correct, pointwise bound, can still produce viable throughput rates that will be useful in practice.

I INTRODUCTION

Quantum cryptography has been heralded as providing an important advance in secret communications because it provides a guarantee that the amount of mutual information available to an eavesdropper can unconditionally be made arbitrarily small. Any practical realization of quantum key distribution that consists only of sifting, error correction and authentication will allow some information leakage, thus necessitating privacy amplification. Of course, one might contemplate carrying out privacy amplification after executing a classical key distribution protocol. In the absence of any assumed conditions on the capability of an eavesdropper, it is not possible to deduce a provable upper bound on the leaked information in the classical case, so that the subsequent implementation of privacy amplification would produce nothing, i.e., the “input” to the privacy amplification algorithm cannot be bounded, and as a result neither can the “output.” In the case of quantum key distribution, however, the leaked information associated with that string which is the input to the privacy amplification algorithm can be bounded, and this can be done in the absence of any assumptions about the capability of an eavesdropper. This bound is not good enough for cryptography, however. Nevertheless, this bound on the input allows one to prove a bound on the output of privacy amplification, so that one deduces a final, unconditional upper bound on the mutual information available to an eavesdropper. Moreover this bound can be made arbitrarily small, and hence good enough for cryptography, at the cost of suitably shortening the final string.

Except that as usually presented this is not exactly true. The above understanding is usually presented in connection with the standard result of generalized privacy amplification given in [1], which applies only to the average value of the mutual information. The average is taken with respect to a set of elements, namely, the universal class of hash functions introduced by Carter and Wegman [2]. The actual implementation of privacy amplification, however, will be executed by software and hardware that selects a particular hash function. The bound on the average value of the mutual information does not apply to this situation: it does not directly measure the amount of mutual information available to an eavesdropper in practical quantum cryptography.

In this paper we calculate cryptographically acceptable pointwise bounds on the mutual information which can be achieved while still maintaining sufficiently high throughput rates. In contrast to a direct application of the privacy amplification result of [1], we must also consider and bound a probability of choosing an unsuitable hash function and relate this to cryptographic properties of the protocol and the throughput rate. The relation between average bounds and pointwise bounds of random variables is not new and follows from elementary probability theory, as was also noticed in [4].
II PRIVACY AMPLIFICATION

In ideal circumstances, the outcome of a $k$-bit key-exchange protocol is a $k$-bit key shared between Alice and Bob which is kept secret from Eve. Perfect secrecy means that from Eve’s perspective the shared key is chosen uniformly from the space of $k$-bit keys. In practice, one can only expect Eve’s probability distribution for the shared key be close to uniform in the sense that its Shannon entropy is close to its largest possible value $k$. Moreover, because quantum key-exchange protocols implemented in practice inevitably leak information to Eve, Eve’s distribution of the key is too far from uniform to be usable as a real random variable because quantum key-exchange protocols implemented in a keyspace of smaller bitsize.

Privacy amplification is the process of obtaining a nearly uniformly distributed key in an image of smaller sizes. We review the standard assumptions of the underlying probability model of $\Omega$: $\Omega$ is the underlying sample space with probability measure $P$. Expectation of a real random variable $X$ with respect to $P$ is denoted $\mathbb{E}X$. $W$ is a random variable with key material known jointly to Alice and Bob and $V$ is a random variable with Eve’s information about $W$. $W$ takes values in some finite keyspace $\mathcal{W}$. The function $P_{w}(w) = P(W = w)$ for $w \in \mathcal{W}$. Eve’s distribution having observed a value $v$ of $V$ is the conditional probability $P_{w|v}(w) = P(W = w|V = v)$ on $\mathcal{W}$. In the discussion that follows, $v$ is fixed and accordingly we denote Eve’s distribution of Alice and Bob’s shared key given $v$ by $P_{Eve}$. $H$ and $R$ denote Shannon and Renyi entropies of random variables defined on $\mathcal{W}$ relative to $P_{Eve}$.

Definition II.1 Suppose $\mathcal{Y}$ is a keyspace. If $\alpha$ is a positive real number, a mapping $\gamma : \mathcal{W} \rightarrow \mathcal{Y}$ is an $\alpha$ strong uniformizer for Eve’s distribution iff $H(\gamma) = \sum_{y \in \mathcal{Y}} P_{Eve}(\gamma^{-1}(y)) \log_{2} P_{Eve}(\gamma^{-1}(y)) \geq \log_{2} |\mathcal{Y}| - \alpha$.

If $\gamma$ is an $\alpha$ strong uniformizer, then we obtain a bound on the mutual information between Eve’s data $V$ and the image of the hash transformation $Y$ as follows:

$$I(Y, V) = I(Y) - H(Y|V) = \log_{2} |\mathcal{Y}| - H(\gamma) \leq \alpha \ .$$

Definition II.2 Let $\Gamma$ be a random variable with values in $\mathcal{W}^{\mathcal{W}}$ (space of functions $\mathcal{W} \rightarrow \mathcal{Y}$) which is conditionally independent of $W$ given $V = v$ i.e. $P(\Gamma = \gamma \ and \ W = w|V = v) = P(\Gamma = \gamma|V = v) P(W = w|V = v)$, $\Gamma$ is an $\alpha > 0$ average uniformizer for Eve’s distribution iff

$$E(H(\Gamma)) \geq \log_{2} |\mathcal{Y}| - \alpha \ .$$

where $H(\Gamma) = H(\Gamma(z))$. $H(\Gamma) = H(\Gamma(z))$. $H(\Gamma(z))$. $H(\Gamma) = H(\Gamma(z))$.

If $\Gamma$ is an $\alpha$ average uniformizer, the bound is on the mutual information averaged over the set $\Gamma$:

$$I(Y, \Gamma V) = I(Y) - H(Y|\Gamma V) = \log_{2} |\mathcal{Y}| - E(H(\Gamma)) \leq \alpha \ .$$

Uniformizers are produced stochastically. Notice that by the conditional stochastic independence assumption, $z$ can be assumed to vary independently of $w \in \mathcal{W}$ with the law $P_{Eve}$.

Proposition II.3 Suppose $\Gamma$ is an $\alpha$ average uniformizer. Then for every $\beta > 0$, $\Gamma(\omega)$ is a $\beta$ strong uniformizer for $\omega$ outside a set of probability $\frac{1}{\beta}$.

Proof. Note that for any $\gamma : \mathcal{W} \rightarrow \mathcal{Y}$, $H(\gamma)$ is at most $\log_{2} |\mathcal{Y}|$. Thus $\log_{2} |\mathcal{Y}| - H(\Gamma)$ is a nonnegative random variable. Applying Chebychev’s inequality to $\log_{2} |\mathcal{Y}| - H(\Gamma)$, it follows that for every $\beta > 0$,

$$P(\log_{2} |\mathcal{Y}| - \beta \geq H(\Gamma)) \leq \frac{1}{\beta} E(\log_{2} |\mathcal{Y}| - H(\Gamma))$$

$$= \frac{1}{\beta} (\log_{2} |\mathcal{Y}| - E(H(\Gamma))) \leq \frac{1}{\beta} \alpha .$$

The random variable $\Gamma$ is strongly universal $\alpha$ if for all $x \neq x' \in X$,

$$P\{y : \Gamma(x) = \Gamma(x')\} \leq \frac{1}{|\mathcal{Y}|} .$$

The following is the main result of [1]:

Proposition II.4 (BBCM Privacy Amplification). Suppose $\Gamma$ is a universal $\alpha$ family of mappings $\mathcal{W} \rightarrow \mathcal{Y}$ conditionally independent of $W$. Then $\Gamma$ is an $\alpha > 0$ average uniformizer for $X$.

III PRACTICAL RESULTS

We will refer to the inequality that provides the upper bound on the average value of the mutual information as the average privacy amplification bound, or APA, and we will refer to the inequality that provides the upper bound on the actual, or pointwise mutual information as the pointwise privacy amplification bound, or PPA.

In carrying out privacy amplification we must shorten the key by the number of bits of information that have potentially been leaked to the eavesdropper. Having taken that into account, we denote by $g$ the additional number of bits by which the key length will be further shortened to assure sufficient secrecy, i.e., the additional bit subtraction amount, and we refer to $g$ as the privacy amplification subtraction parameter. With this definition of $g$, Bennett et al. [1] show as a corollary of II.4 that the set of Carter-Wegman hash functions is an $2^{-g} \ln 2$ average uniformizer. We thus have for the APA bound
we may define the failure probability $P$ as the case of PPA we need the parameter $g$ except on a set of probability

Wegman hash functions are 2 pointwise bound parameter we have for the PPA bound on 

The upper bound on the average of the mutual information, the inequality

In the case of APA the quantity $g$ plays a dual role: in addition to representing the number of additional subtraction bits, as above for APA, but the upper bound on the pointwise mutual information is now given in terms of a different quantity $g'$, which we refer to as the pointwise bound parameter. Also in the case of PPA we need the parameter $g''$, which we refer to as the pointwise probability parameter, in terms of which we may define the failure probability $P_f$. This definition is motivated by [12], from which we find that the Carter-Wegman hash functions are $2^{-g'/\ln 2}$ strong uniformizers except on a set of probability

We therefore define the pointwise probability parameter as

Thus the quantities $g$, $g'$ and $g''$ are not all independent, and are constrained by equation [4]. In terms of these parameters we have for the PPA bound on $I$, the actual value of the mutual information, the inequality

where the associated failure probability $P_f$ is given by

The failure probability is not even a defined quantity in the APA case, but it plays a crucial role in the PPA case. Thus, the bound on the pointwise mutual information is directly determined by the value of the parameter $g'$, with respect to which one finds a tradeoff between $g$, the number of additional compression bits by which the key is shortened, and $g''$, the negative logarithm of the corresponding failure probability.

IV APPLICATION OF POINTWISE BOUND

Operationally, it will usually be the case in practice that end-users of quantum key distribution systems will be first and foremost constrained to ensure that a given upper bound on the pointwise mutual information available to the enemy is realized.

To appreciate the significance of the distinction between the PPA and APA results, we will consider an illustrative example that shows how reliance on the APA bound can lead to complete compromise of cryptographic security. We begin with the APA case. As noted above, in the case of APA the privacy amplification subtraction parameter, which we will now denote by $g_{APA}$ to emphasize the nature of he bound, directly specifies both the upper bound on $\langle I \rangle$ and also the number of bits by which the key needs to be shortened to achieve this bound. Without loss of generality we take the value of the privacy amplification subtraction parameter to be given by $g_{APA} = 30$, which means that, in addition to the compression by the number of bits of information that were estimated to have been leaked, the final length of the key will be further shortened by an additional 30 bits. This results in an upper bound on the average mutual information given by $\langle I \rangle \leq 2^{-30}/\ln 2 \simeq 1.34 \times 10^{-9}$, which we take as the performance requirement for this example. While this might appear to be an acceptable bound, the fact that it applies only to the average of the mutual information of course means that it is not the quantity we require.

We turn to the PPA case, with respect to which we will now refer to the privacy amplification subtraction parameter as $g_{PPA}$. In order to discuss the PPA bound we must select appropriate values amongst $g_{PPA}$, $g'$ and $g''$. In the APA case discussed above, the bound on the (average) mutual information and the number of subtraction bits are both specified by the same parameter $g_{APA}$. In the PPA case, the number of subtraction bits and the parameter that specifies the bound on the (pointwise) mutual information are not the same. To achieve the same value for the upper bound on $I$ as we discussed for the upper bound on $\langle I \rangle$ above, we must select $g' = 30$ as the value of the pointwise bound parameter. From eq.(8) this indeed yields the required inequality $I \leq 2^{-30}/\ln 2 \simeq 1.34 \times 10^{-9}$. However, with respect to this requirement on the value of the mutual information, i.e., the required final amount of cryptographic secrecy, there are a denumerable set (since bits are discrete) of different amounts of compression of the key that are possible to select, each associated with a corresponding failure probability, $P_f$, in the form of ordered pairs $(g_{PPA}, g'')$ that satisfy the constraint given by $g_{PPA} = g' + g''$ (cf eq.(4)).

Our starting point was the secrecy performance requirement that must be satisfied. On the basis of the
APA analysis above, one might conclude that in order to achieve the required secrecy performance constraint it is sufficient to shorten the key by 30 bits. However in the PPA case, satisfying the same performance requirement and shortening the key by 30 bits means choosing identical values for the privacy amplification subtraction parameter \((g_{PPA} = 30)\) and the pointwise bound parameter \((g' = 30)\). However, we note from eq. \((7)\) that in the case of the PPA bound, \(g_{PPA}\) and \(g'\) become the same only when \(g'' = 0\), which corresponds to 100% failure probability on the upper bound. This is clearly cryptographically useless!

This example emphasizes the importance of assuring a sufficiently small failure probability in addition to a sufficiently small upper bound on the mutual information. As we see from the above example, the APA result provides no information about the correct number of subtraction bits that are required in order to achieve a specified upper bound on the pointwise mutual information with a suitable failure probability, for which it is essential to use the PPA result instead. In Figure 1 we have plotted the failure probability as a function of the upper bound on the mutual information, for a family of choices of \(g_{PPA}\) values. Returning to the example discussed above for the APA bound, we see that if we need to achieve an upper bound on \(I\) of about \(10^{-9}\), we may do so with a failure probability of about (coincidentally) \(10^{-9}\), at the cost of shortening the final key by 60 bits: the secrecy is dictated by the pointwise bound parameter value of \(g' = 30\), which is effected by choosing \(g_{PPA} = 60\), corresponding to \(P_f \simeq 10^{-9}\). Smaller upper bounds can obviously be obtained, with suitable values of the failure probability, at the cost of further shortening of the key.

We assume the use of an attenuated, pulsed laser, with Alice located on a low earth orbit satellite at an altitude of 300 kilometers and Bob located at mean sea level, with the various system parameters corresponding to those for Scenario \((i)\) in Section 5.3.2 in \(\ddagger\), except that here the source of the quantum bits operates at a pulse repetition frequency (PRF) of 1 MHz, and we specifically assume that the enemy does not have the capability to make use of prior shared entanglement in conducting eavesdropping attacks. We see that the additional cost incurred in subtracting the amount required to achieve the required mutual information bound and failure probability reduces the throughput rate by an amount that is likely to be acceptable for most purposes. For instance, for a source PRF of 1 MHz we find that the throughput rate with a value of \(g_{PPA} = 30\) is 5614 bits per second. With a subtraction amount of \(g_{PPA} = 60\) the throughput rate drops to 5563 bits per second \(\ddagger\).

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\text{Effective Throughput of Secret Vernam Cipher for Different } g \text{-values}
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\[
\text{Failure Probability versus Secrecy Bound for various Privacy Amplification Compressions}
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\[
\text{Figure 2}
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\[
\text{Figure 1}
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\[\text{V CONCLUSIONS}\]

The significance and proper implementation of privacy amplification in quantum cryptography are clarified by our analysis. By itself the bound on the average value of the mutual information presented in \(\ddagger\) does not allow one to determine the values of parameters required to bound the actual, pointwise value of the mutual information. Those parameters must satisfy a constraint, which in turn implies a constraint on the final throughput of secret key material. We have rigorously derived the cryptographically meaningful upper bound on the pointwise mutual information associated with the use of some specific privacy amplification hash function, and shown that the corresponding requirements on the shortening of the key still allow viable throughput values.

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[5] The difference between the two throughput values is about 50 bits per second, because an additional 30 bits are subtracted per processing block, and in the example presented there are about 1.6 blocks per second. See reference [3] for a discussion of processing block size.