Functional Description of $S^1 \times S^2$ and $S^3$ Gowdy Cosmologies

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Abstract. We will briefly review the classical formulation of the 3-handle $S^1 \times S^2$ and 3-sphere $S^3$ Gowdy cosmological models coupled with massless scalar fields and their exact (non-perturbative) quantization by defining suitable Schrödinger functional representations in terms of appropriate probability spaces. We will pay special attention to the construction of closed expressions for the corresponding quantum time evolution propagators.

1. Introduction

Two-Killing symmetry reductions of General Relativity are appealing testing grounds to probe the features of gravity in its quantum regime, and also to deepen our knowledge in the axiomatic structure of quantum field theory in curved spacetimes. This is due to their exact solvability –both classical and quantum mechanical–, the existence of local degrees of freedom, and the invariance under a certain (restricted) class of diffeomorphisms. These models differ from each other in the action of the biparametric and abelian isometry group $G^{(2)}$ and the topologies of the spatial sections $\Sigma$ compatible with the group action (see Table 1 and [1]).

The so-called Gowdy cosmological models correspond to the group $G^{(2)} = U(1) \times U(1)$ acting on symmetric compact slices [2]. The possible spatial topologies are those of the 3-torus $T^3$, the 3-handle $S^1 \times S^2$, the 3-sphere $S^3$, or the lens spaces $L(p, q)$. The 3-torus topology has been profusely analyzed in the past and is, by far, the preferred choice to discuss quantization issues [3, 4, 5, 6]. Here, we will restrict our attention on the remaining spatial topologies, the 3-handle and the 3-sphere, which are less known than the 3-torus one but equally relevant in cosmology and quantum gravity due to the fact that they have both initial and final singularities. The lens spaces $L(p, q)$ can be studied by imposing discrete symmetries on the 3-sphere case and, in fact, the general arguments presented for $S^3$ remain valid for them. Among other interesting issues, these models are privileged frameworks to discuss the canonical quantization of cyclic universes, some fundamental aspects of quantum field theory in curved spacetimes –particularly, the unitary implementability of the dynamics–, and also the application of modern differential geometry techniques to the Hamiltonian formulation of field theories.

We will closely rely on references [7, 8, 9, 10], where the reader will find the rigorous classical and quantum treatments for these models coupled to massless scalar fields.
Table 1. Spatial topologies compatible with the abelian biparametric Lie group $G^{(2)}$, with a smooth, effective, and proper action on the spatial sections of a globally hyperbolic spacetime $(M^4 \simeq \mathbb{R} \times \Sigma, g^{(4)})$. The action of the group, unique up to automorphisms of $G^{(2)}$ and diffeomorphisms of $\Sigma$, can be free or have degenerate orbits.

| Group $G^{(2)}$ | Manifold $\Sigma$ | Action | Name of the model |
|-----------------|-------------------|--------|------------------|
| $U(1) \times U(1)$ | $\mathbb{R}^2 \times S^1$ | Not free | Schmidt model |
| | $\mathbb{R} \times T^2$ | Free | $T^3$ Gowdy model |
| | $T^3$ | Free | $S^1 \times S^2$ Gowdy model |
| | $S^2 \times S^1$ | Not free | $S^3$ Gowdy model |
| | $S^3$ | Not free | |
| $\mathbb{R} \times U(1)$ | $\mathbb{R}^3$ | Not free | Cylindrical gravitational waves |
| | $\mathbb{R} \times S^2$ | Free | Cylindrical wormhole |
| | $\mathbb{R} \times T^2$ | Free | |
| $\mathbb{R}^2$ | $\mathbb{R}^3$ | Free | |
| | $\mathbb{R}^2 \times S^1$ | Free | |

2. Group action

Consider a smooth, effective, and proper action of the biparametric Lie Group $U(1) \times U(1)$ on a compact, connected, and oriented 3-manifold $\Sigma$. As we explained in the Introduction, the spatial manifold $\Sigma$ is then restricted to have the topology of a 3-torus $T^3$, a 3-handle $S^1 \times S^2$, a 3-sphere $S^3$, or the lens spaces $L(p,q)$. Moreover, the action of the group is unique up to automorphisms of $G^{(2)}$ and diffeomorphisms of $\Sigma$ [11, 12]. We will focus our attention on the 3-handle and 3-sphere Gowdy models. Let start by defining the smooth, effective, and proper action $\varrho$ of $U(1) \times U(1) = \{(e^{\theta_1}, e^{\theta_2}) \mid x_1, x_2 \in \mathbb{R} \mod 2\pi, \alpha = 1, 2\}$ on $\Sigma = S^1 \times S^2 = \{(e^{i\kappa}, e^{i\sigma \sin \theta, \cos \theta}) \mid \theta \in [0, \pi], \kappa, \sigma \in \mathbb{R} \mod 2\pi\}$ by

$$
\varrho((e^{\theta_1}, e^{\theta_2}), (e^{i\kappa}, e^{i\sigma \sin \theta, \cos \theta})) := (e^{i(x_1 + \kappa)}, e^{i(x_2 + \sigma)} \sin \theta, \cos \theta).
$$

The tangent vectors corresponding to the orbits of the two $U(1)$ commuting subgroup factors of $U(1) \times U(1)$ are

$$(ie^{i\kappa}, 0, 0), \quad (0, ie^{i\sigma \sin \theta}, 0).$$

Both fields commute. The first one is never zero but the latter vanishes at $\theta = 0$ and $\theta = \pi$. For the 3-sphere case, we take Hopf coordinates $\Sigma = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, $z_1 = e^{i\sigma \sin (\theta/2)}$, $z_2 = e^{i\kappa \cos (\theta/2)}$, with $\theta \in [0, \pi]$ and $\kappa, \sigma \in \mathbb{R} \mod 2\pi$, and define the action

$$
\varrho((g_1, g_2), (z_1, z_2)) := (e^{i(x_1 + \kappa)} \sin (\theta/2), e^{i(x_2 + \kappa)} \cos (\theta/2)).
$$

Here, the commuting tangent vectors corresponding to the action of the two $U(1)$ subgroup factors are $(iz_1, 0)$ and $(0, iz_2)$. They alternatively vanish at the circles $(0, e^{i\kappa})$ and $(e^{i\sigma}, 0)$.

3. Einstein field equations

Here, we briefly review the construction of the 3-handle and 3-sphere Gowdy models coupled to massless scalar fields. We will assume that all fields are smooth. We consider 4-manifolds $M^4$ diffeomorphic to $\mathbb{R} \times \Sigma$, endowed with Lorentzian metrics $g^{(4)}$. The spacetimes $(M^4, g^{(4)})$ are considered to be globally hyperbolic. We further require $U(1) \times U(1)$ to act by isometries on the
spatial slices Σ of $M^4$, obtaining in this way the so-called $S^1 \times S^2$ and $S^3$ Gowdy cosmological models. We will focus on the linearly polarized cases, where the isometry group is generated by the pair of mutually orthogonal, commuting, spacelike, and globally defined, hypersurface-orthogonal Killing vector fields $κ := (∂/∂κ)$ and $σ := (∂/∂σ)$.

The 3-handle and 3-sphere Gowdy models coupled to massless scalar fields are determined by solving the Einstein-Klein-Gordon equations

$$\text{Ric}(g^{(4)}) = dφ \otimes dφ, \quad □^{(4)} φ = 0,$$

corresponding to (1+3)-dimensional gravity minimally coupled to a massless scalar field, $φ : M^4 → ℝ$, symmetric under the diffeomorphisms generated by $U(1) \times U(1)$. Here, $\text{Ric}(g^{(4)})$ and $□^{(4)}$ denote the Ricci tensor and the d’Alembertian associated with $g^{(4)}$, respectively. As usual, the exterior derivative and the tensor product are denoted by $d$ and $\otimes$, and we use units such that $8πG_N = 1$, where $G_N$ is the Newton constant. In order to get a simplified, lower dimensional description we perform a Geroch reduction [13] with respect to one of the Killing fields, say $κ$. Let $\tilde{M}^4$ denote the set of points of $M^4$ in which $κ$ does not vanish. Consider the space of orbits $M^3 = \tilde{M}^4/U(1)$. In the present situation, hypersurface orthogonality allows us to view $M^3$ as an embedded submanifold of $\tilde{M}^4$, everywhere orthogonal to the closed orbits of $κ$, endowed with the induced metric

$$g^{(3)} := g^{(4)} − \frac{1}{\lambda_κ} K_κ \otimes K_κ,$$

where $λ_κ := g^{(4)}(κ, κ) > 0$ and $K_κ$ is the one-form field defined by $K_κ(·) = g^{(4)}(κ, ·)$. We can reintroduce the previously removed points $M^4 \setminus M^4$, that correspond to the symmetry axis, as a boundary where the fields must satisfy certain regularity conditions. In the so called linearly polarized case, the twist of the Killing fields vanishes, i.e.

$$K_κ ∧ dK_κ = 0,$$

and similarly for $σ$. As a consequence, the field equations can be written as those corresponding to a set of massless scalar fields coupled to (1+2)-gravity by performing the conformal transformation $g := λ_κ g^{(3)}$. The system (1) is then equivalent to

$$\text{Ric}(g) = \frac{1}{2} \sum \phi_i \otimes dφ_i, \quad □φ_i = 0, \quad L_σ g = 0, \quad L_σ φ_i = 0, \quad i = 1, 2,$$

where $\text{Ric}(g)$ and $□$ are, respectively, the Ricci tensor and the d’Alembertian associated with $g$, all of them are 3-dimensional objects defined on $M^3$, and $L_σ$ denotes the Lie derivative with respect $σ$. Here, we have defined the scalar fields $ϕ_i := log λ_κ$ and $ϕ_2 := √2ϕ$, and we must remember that we have the additional symmetry generated by the remaining Killing vector field $σ$. Note that the equations (2) are formally symmetric under the exchange of the gravitational and matter sectors. However, it is important to realize that these fields are subject to different regularity conditions in the gravitational and matter sectors that effectively break the symmetry among them. These are the so-called polar constraints [7], which are necessary ingredients to ensure the consistency of the models as they guarantee the differentiability of the other constraints.

4. Hamiltonian formalism and reduced phase space

Now, we can proceed to obtain the Hamiltonian formalisms of these models within the (1+2)-dimensional scheme provided by (2). These equations can be derived from the standard (1+2)
Einstein-Hilbert Lagrangian coupled to massless scalar fields [7]. The singular nature of the Lagrangian function forces us to carefully apply the Dirac-Bergmann theory of constrained systems. As expected for closed universes, one finds that the physical dynamical variables are restricted to belong to first class (i.e., coisotropic) constrained submanifolds of the phase spaces, where the Hamiltonian functions identically vanish. In order to isolate the true physical degrees of freedom and recover a dynamics for these models we perform a deparameterization (partial gauge fixation process) by imposing appropriate extra conditions on the phase space variables. This allows us to describe its reduced phase space by quadratic nonautonomous Hamiltonian systems.

An interesting feature of both the 3-handle and 3-sphere cases is the fact that after the deparameterization process there are no Dirac constraints left. This is in contrast with the situation for the 3-torus topology, where in addition to the dynamics generated by the time-dependent Hamiltonian there is an additional constraint in the system that must be appropriately taken into account.

Apart from some global modes present in these models, whose quantization follows readily in terms of standard position and momentum operators with dense domains in $L^2(\mathbb{R})$, the dynamics of both the gravitational and matter local degrees of freedom in the linearly polarized 3-handle and 3-sphere Gowdy models obeys the same field equations. These are the Klein-Gordon equations corresponding to massless scalar fields evolving in the same fixed background spacetime $((0, \pi) \times S^2, \hat{g})$ conformally equivalent to the (1+2)-dimensional Einstein universe. Explicitly,

$$
\hat{g} = \sin^2 t \left( -dt \otimes dt + \gamma \right),
$$

where $\gamma$ is the unit round metric in the 2-sphere that, using spherical coordinates $(\theta, \sigma) \in (0, \pi) \times (0, 2\pi)$ on $S^2$, has the form $\gamma = d\theta \otimes d\theta + \sin^2 \theta d\sigma \otimes d\sigma$. The field equations are then given by

$$
\ddot{\varphi}_i + \cot t \dot{\varphi}_i - \Delta \varphi_i = 0, \quad L_\sigma \varphi_i = 0
$$

where, in the Hamiltonian formalism, the scalar fields $\varphi_i$ are related to the original fields $\phi_i$ through nontrivial symplectic transformations [7] and $\Delta$ denotes the Laplace-Beltrami operator on the round 2-sphere $(S^2, \gamma)$. We have to impose an extra symmetry condition: invariance under the diffeomorphisms generated by the Killing vector field $\sigma = (\partial/\partial \sigma)$. Concerning the exact (non-perturbative) quantization of these models, there are subtleties associated with the unitary implementability of the quantum dynamics. First, it is shown in [8] that it is not possible to encode the quantum time evolution in a unitary operator for any $SO(3)$-invariant Fock/Schrödinger representation when the systems are written in terms of the dynamical variables encoded in the fields $\varphi_i$ that naturally appear in their Hamiltonian formalisms. This is in conflict with the axiomatic structure of the quantum theory, and makes it difficult to properly perform a probabilistic interpretation for these models. However, it is possible to overcome this problem by performing a redefinition of the fields at the classical level involving precisely the conformal factor $\sin t$ mentioned above [8]. This type of field redefinitions was used for the first time in [5] to deal with the problem of the unitary implementability of the quantum dynamics for the vacuum 3-torus model, and it was generalized and reinterpreted later on geometric grounds for the remaining topologies [8]. Specifically, we define the rescaled fields

$$
\xi_i := \sqrt{\sin t} \varphi_i
$$

satisfying

$$
-\ddot{\xi}_i + \Delta \xi_i = \frac{1}{4} (1 + \csc^2 t) \xi_i, \quad L_\sigma \xi_i = 0.
$$

(3)
A way to understand what is going on is to realize that the singular behavior introduced by the conformal factors is translated, in terms of the redefined fields, into the behavior of a singular, time-dependent, potential term for the rescaled fields. The time evolution can now be implemented unitarily as a direct consequence of the fact that, in spite of being singular at some instants of time, these potentials are well behaved enough as functions of the time variable in a definite sense that will be clarified below. There is another difficulty in the quantization process: The quantum counterpart of the classical Hamiltonian does not contain the vacuum state in its domain. As a consequence, the action of the operator is not defined on the dense subspace of states with a finite number of particles. This difficulty can be overcome right from the start by describing the classical dynamics through a positive definite Hamiltonian that is related to the original one by a symplectic transformation. Taking the above prescriptions into account, both gravitational and matter local degrees of freedom of the systems under consideration, whose dynamics is governed by (3), can be described by the same nonautonomous Hamiltonian system \((0, \pi) \times \Gamma, dt, \omega, H(t)\), mathematically described by an infinite-dimensional cosymplectic space. Here, \(\Gamma := \mathcal{C} \times \mathcal{C}\) is the space of Cauchy data, where the classical configuration space \(\mathcal{C}\) is identified with the Fréchet space of rapidly decreasing real sequences \(q := (q_\ell : \ell \in \mathbb{N}_0)\). The space \(\Gamma\) is endowed with the natural symplectic structure

\[
\omega((q_1, p_1), (q_2, p_2)) := \sum_{\ell \in \mathbb{N}_0} (p_\ell q_{2\ell} - p_{2\ell} q_\ell), \quad (q_1, p_1), (q_2, p_2) \in \Gamma.
\]

The nonautonomous Hamiltonian \(H(t) : \Gamma \to \mathbb{R}_+\) derived from (3) is given by a positive-definite quadratic form on \(\Gamma\) that can be diagonalized as a sum of time-dependent harmonic oscillator Hamiltonians. Explicitly,

\[
H(t, q, p) := \frac{1}{2} \sum_{\ell \in \mathbb{N}_0} (p_\ell^2 + \omega_\ell^2(t) q_{\ell}^2), \quad t \in (0, \pi),
\]

where the time-dependent squared frequencies \(\omega_\ell^2(t)\) are given by

\[
\omega_\ell^2(t) = \ell(\ell + 1) + (1 + \csc^2 t)/4, \quad \ell \in \mathbb{N}_0.
\]

The time evolution is implemented by the symplectic transformations

\[
\begin{pmatrix}
q_{Ht}(t, t_0) \\
p_{Ht}(t, t_0)
\end{pmatrix} = T_c(t, t_0) \cdot \begin{pmatrix}
q_{c}(t, t_0) \\
p_{c}(t, t_0)
\end{pmatrix},
\quad T_c(t, t_0) := \begin{pmatrix}
c_\ell(t, t_0) & s_\ell(t, t_0) \\
\dot{c}_\ell(t, t_0) & \dot{s}_\ell(t, t_0)
\end{pmatrix}, \quad \ell \in \mathbb{N}_0,
\]

where, for each \(\ell \in \mathbb{N}_0\), \(c_\ell(\cdot, t_0)\) and \(s_\ell(\cdot, t_0)\) are the solutions to the time-dependent harmonic oscillator equation

\[
\ddot{u} + \omega_\ell^2(t) u = 0
\]

such that \(c_\ell(t_0, t_0) = 1 = s_\ell(t_0, t_0), s_\ell(t_0, t_0) = 0 = \dot{c}_\ell(t_0, t_0)\). Here, the dot denotes partial derivative with respect to the first argument. Explicitly,

\[
c_\ell(t, t_0) = \frac{1}{2} \sqrt{\frac{\sin t}{\sin t_0}} \mathcal{D}(\omega_\ell^{-1}/2)(\cos t)((1 + \omega')^2/2(\cos t_0) - \omega_\ell^2(\omega_\ell^{-1}/2)(\cos t_0)) - \frac{1}{2} \sqrt{\frac{\sin t}{\sin t_0}} \mathcal{D}(\omega_\ell^{-1}/2)(\cos t)((1 + \omega')^2/2(\cos t_0) - \omega_\ell^2(\omega_\ell^{-1}/2)(\cos t_0)),
\]

\[
s_\ell(t, t_0) = \sqrt{\sin t \sin t_0}(\mathcal{D}(\omega_\ell^{-1}/2)(\cos t)\mathcal{D}(\omega_\ell^{-1}/2)(\cos t) - \mathcal{D}(\omega_\ell^{-1}/2)(\cos t_0)\mathcal{D}(\omega_\ell^{-1}/2)(\cos t_0)),
\]

where \(\omega'_\ell := \sqrt{1 + 4\ell(\ell + 1)}\).
5. Schrödinger functional representation

In order to exactly quantize the systems under consideration, we introduce Schrödinger representations \([14, 15, 16, 9]\), where quantum state vectors act as functionals \(\Psi : \mathcal{F} \to \mathbb{C}\) belonging to suitable \(L^2\)-spaces \(\mathcal{H}_\alpha = L^2(\mathcal{F}, \sigma(\text{Cyl}(\mathcal{F})), d\mu_\alpha)\) endowed with Gaussian measures.\(^1\) The quantum configuration space is given by a proper distributional extension of the classical configuration space \(\mathcal{F}\). In this case, it suffices to consider the dual \(\mathcal{F}\), i.e., the linear space of slowly increasing real sequences. The Gaussian measure \(\mu_\alpha\) is defined on the cylinder sets \(\sigma(\text{Cyl}(\mathcal{F}))\) on \(\mathcal{F}\). Finally, the configuration observables act as multiplication operators, whereas the canonically conjugate momenta differ from the usual ones in terms of derivatives by the appearance of multiplicative terms which are necessary to ensure their self-adjointness; specifically, for cylindrical functions \(\Psi\),

\[
(Q_\ell \Psi)(\mathbf{q}) = q_\ell \Psi(\mathbf{q}), \quad (P_\ell \Psi)(\mathbf{q}) = -i \frac{\partial \Psi}{\partial q_\ell} (\mathbf{q}) + \frac{\beta_\ell}{\alpha_\ell} q_\ell \Psi(\mathbf{q}).
\]

The complex sequences \(\alpha\) and \(\beta\) must satisfy

\[
\alpha_\ell \beta_\ell - \beta_\ell \bar{\alpha}_\ell = i, \quad \forall \ell \in \mathbb{N}_0,
\]

by virtue of the CCR, \([Q_\ell, P_\ell] = i \delta(\ell, \ell') \mathbf{1}\) and \([Q_\ell, Q_\ell'] = 0 = [P_\ell, P_\ell']\). The functional form of \(\alpha\) and \(\beta\) is crucial to ensure the unitary implementation of the dynamics. Specifically, this is guaranteed for \(\alpha\) and \(\beta\) sequences with the asymptotic expansions \([8]\)

\[
\alpha_\ell = \frac{1}{\sqrt{2(|\ell| + 1)}} \exp(i\eta_\ell) + O(|\ell|^{-3/2}), \quad \beta_\ell = -i \sqrt{\frac{(|\ell| + 1)}{2}} \exp(i\eta_\ell) + O(|\ell|^{-1/2}),
\]

where \(\eta\) is an arbitrary real-valued sequence. Moreover, all \(SO(3)\)-invariant representations for which the dynamics is well-defined and unitarily equivalent \([8]\). In what follows, we will simply take

\[
\alpha_\ell = \frac{1}{\sqrt{2(|\ell| + 1)}}, \quad \beta_\ell = -i \sqrt{\frac{(|\ell| + 1)}{2}}. \tag{7}
\]

We can rewrite the configuration and momentum operators in terms of annihilation and creation operators \(a_\ell\) and \(a_\ell^*\), with \([a_\ell, a_{\ell'}^*] = \delta(\ell, \ell') \mathbf{1}\) and \([a_\ell, a_{\ell'}] = 0 = [a_{\ell}, a_{\ell'}^*]\), such that

\[
Q_\ell = \alpha_\ell a_\ell + \bar{\alpha}_\ell a_\ell^*, \quad P_\ell = \beta_\ell a_\ell + \bar{\beta}_\ell a_\ell^* \leftrightarrow a_\ell = -i \beta_\ell Q_\ell + i \bar{\alpha}_\ell P_\ell, \quad a_\ell^* = i \beta_\ell Q_\ell - i \alpha_\ell P_\ell.
\]

Note that the annihilation operators are given by the derivatives

\[
a_\ell = \frac{\partial}{\partial q_\ell},
\]

and, hence, the vacuum state \(\Psi_0 \in \mathcal{H}_\alpha\) satisfying \(a_\ell \Psi_0 = 0\) for all \(\ell \in \mathbb{N}_0\) is given by the unit constant functional \(\Psi_0(\mathbf{q}) = 1\) up to multiplicative phase. The quantum time evolution, characterized by a unitary operator \(U(t, t_0)\), is univocally determined \textit{up to phase} by the evolution of these creation and annihilation operators

\[
U^{-1}(t, t_0) a_\ell U(t, t_0) = A_\ell(t, t_0) a_\ell + B_\ell(t, t_0) a_\ell^*, \quad U^{-1}(t, t_0) a_\ell^* U(t, t_0) = \bar{B}_\ell(t, t_0) a_\ell + \bar{A}_\ell(t, t_0) a_\ell^*.
\]

\(^1\) Given the infinite-dimensionality of the classical configuration space, it is not possible to define nontrivial Lebesgue-type translation invariant measures \(\mu_\alpha\), but rather probability ones.
and the evolution of the vacuum state

\[ U(t, t_0)\Psi_0 = \langle \Psi_0 | U(t, t_0)\Psi_0 \rangle \exp \left( \frac{1}{2} \sum_{\ell \in \mathbb{N}_0} \frac{B_\ell(t_0, t)}{A_\ell(t_0, t)} \vartheta_\ell^2 \right) \Psi_0 , \]

with

\[ |\langle \Psi_0 | U(t, t_0)\Psi_0 \rangle| = \prod_{\ell \in \mathbb{N}_0} \frac{1}{\sqrt{|A_\ell(t_0, t)|}} . \]

Here, \( A_\ell(t_0, t) \) and \( B_\ell(t_0, t) \) are the Bogoliubov coefficients given by

\[
A_\ell(t, t_0) := i \left( \dot{s}_\ell(t, t_0) \alpha_\ell \beta_\ell - c_\ell(t, t_0) \beta_\ell \alpha_\ell + \dot{c}_\ell(t, t_0) |\alpha_\ell|^2 - s_\ell(t, t_0) |\beta_\ell|^2 \right) ,
\]

\[
B_\ell(t, t_0) := i \left( (\dot{s}_\ell(t, t_0) - c_\ell(t, t_0)) \alpha_\ell \beta_\ell + \dot{c}_\ell(t, t_0) \alpha_\ell^2 - s_\ell(t, t_0) \beta_\ell^2 \right) ,
\]

satisfying \( A_\ell(t_0, t) = \overline{A_\ell(t_0, t)} \), \( B_\ell(t_0, t) = -\overline{B_\ell(t_0, t)} \), and \( |A_\ell(t_0, t)|^2 - |B_\ell(t_0, t)|^2 = 1 \), for all \( \ell \in \mathbb{N}_0 \). In particular, \( A_\ell(t_0, t) \neq 0 \) for all \( t, t_0 \in (0, \pi) \). Due to the unitary implementability of the dynamics, the square summability of the sequence \( (B_\ell(t_0, t) : \ell \in \mathbb{N}_0) \) is guaranteed [17] and, hence, the action of \( U(t, t_0) \) is well defined over states with finite number of particles. The phase of the expectation value (8), though being irrelevant to answer most of the physical questions, can be explicitly calculated once a quantum Hamiltonian has been fixed. Given the quadratic nature of the classical Hamiltonian (4), its quantum counterpart coincides with the operator directly promoted from the classical expression modulo an arbitrary \( t \)-dependent real term proportional to the identity which encodes the choice of \( U(t, t_0) \), i.e.,

\[
H(t) = \frac{1}{2} \sum_{\ell \in \mathbb{N}_0} (P_\ell^2 + \omega_\ell^2(t) Q_\ell^2 + 2 \vartheta_\ell(t) 1) ,
\]

where the sequence \( \vartheta_\ell(t) \in C^0(I) \), \( \ell \in \mathbb{N}_0 \), may be employed to avoid the appearance of infinite phases. Analogously to the one-dimensional case, when the dynamics is unitarily implementable, we define the time evolution propagator through the relation

\[
|\langle \Psi_0 | U(t, t_0)\Psi(t) \rangle| = \int_\mathcal{P} K_{\alpha \beta}(\Psi(t); \Psi(t_0)) d\mu_{\alpha}(\Psi(t_0)) ,
\]

where a straightforward calculation formally provides

\[
K_{\alpha \beta}(\Psi(t); \Psi(t_0)) = \prod_{\ell \in \mathbb{N}_0} \sqrt{2\pi} |\alpha_\ell| \exp \left( \frac{i}{2} \left( \frac{\beta_\ell q_{0\ell}^2 - \bar{\beta}_\ell q_{0\ell}^2}{\alpha_\ell} \right) \right) K_\ell(q_t, t; q_{0\ell}, t_0) \exp \left( -i \int_{t_0}^t d\tau \vartheta_\ell(\tau) \right) ,
\]

with \( K_\ell \) denoting the propagator in the standard Schrödinger representation corresponding to the one-dimensional oscillator of squared frequency \( \omega_\ell^2(t) \) defined in (5), whose construction is analyzed in [10]. For the sequences (7), this equation provides the expectation value

\[
|\langle \Psi_0 | U(t, t_0)\Psi_0 \rangle| = \prod_{\ell \in \mathbb{X}} \frac{1}{\sqrt{|A_\ell(t_0, t)|}} \exp \left( i \left( \sigma_\ell(t, t_0) - \int_{t_0}^t d\tau \vartheta_\ell(\tau) \right) \right) ,
\]

with

\[
\sigma_\ell(t, t_0) = -\frac{1}{2} \arctan \left( \frac{|\ell| + 1}s_\ell(t, t_0) - (|\ell| + 1)^{-1}\dot{c}_\ell(t, t_0) \right) c_\ell(t, t_0) + \dot{s}_\ell(t, t_0)
\]

for times \( t \) close to \( t_0 \). Taking the asymptotic behavior of the \( c_\ell \) and \( s_\ell \) functions defined in (6) into account, it is easy to check that normal ordering allows only the cancelation of the phases at high frequencies, for which \( \vartheta_\ell(t) \sim -|\ell|/2 \) as \( \ell \to +\infty \). This is in contrast with the situation for the Minkowskian quantum field theory generalized to a spacetime \( \mathbb{R} \times \mathbb{T}^3 \) with closed spatial sections, where normal ordering allows the cancelation of all phases.
6. Conclusions

We have performed the classical Hamiltonian formulation and the exact canonical quantization of the 3-handle and 3-sphere Gowdy models in mathematically rigorous terms. At the classical level, we have carefully applied the Dirac-Bergmann theory of constrained systems and modern symplectic geometry techniques in order to attain appealing reduced phase space descriptions for these systems. Here, the dynamics is governed by nonautonomous Hamiltonians, where the time dependence reflects the appearance of both initial and final singularities. Concerning the quantization process, we have analyzed the construction of Schrödinger functional representations in terms of appropriate probability spaces, the unitarity of the time evolution, and the uniqueness of the representation, obtaining closed expressions for the evolution operators of these models for the first time. These systems provide a privileged framework to discuss quantization issues related to cyclic universes, and they are expected to have interesting applications in quantum cosmology, particularly, within the loop quantum cosmology program.

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