ON THE GENERALIZATION OF THE DARBOUX THEOREM

KAVEH EFTEKHARINASAB

ABSTRACT. We provide sufficient conditions for the existence of Darboux charts on weakly symplectic bounded Fréchet manifolds by using the Moser’s trick.

1. INTRODUCTION

The Darboux theorem has been extended to weakly symplectic Banach manifolds by using Moser’s method, see [1]. The essence of this method is to obtain an appropriate isotopy generated by a time dependent vector field that provides the chart transforming of symplectic forms to constant ones. In order to apply this method to more general context of Fréchet manifolds we need to establish the existence of the flow of a vector field which in general does not exist. One successful approach to the differential geometry in Fréchet context is in terms of projective limits of Banach manifolds (see [3]). In this framework, a version of the Darboux theorem is proved in [4].

Another approach to Fréchet geometry is to use the stronger notion of differentiability (see [2]). This differentiability leads to a new category of generalized manifolds, the so call bounded (or $MC^k$) Fréchet manifolds. In this paper we prove that in that context the flow of a vector field exists (Theorem 2.2) and we will apply the Moser’s method to obtain the Darboux theorem (Theorem 3.1).

The obtained theorem might be useful to study the topology of the space of Riemannian metrics $\mathcal{M}$ as it has the structure of a bounded Fréchet manifold. Theorem [§48.9, 7] asserts that if $(M, \sigma)$ is a smooth weakly symplectic convenient manifold which admits smooth partitions of unity in $C^\infty_\sigma(M, \mathbb{R})$, and which admits ‘Darboux chart’, then the symplectic cohomology equals to the De Rham cohomology: $H^k_\sigma(M) = H^k_{DR}(M)$.

The manifold $\mathcal{M}$ admits smooth partition of unity in $C^\infty_\sigma(M, \mathbb{R})$ (it follows from Theorem [7, Theorem 16.10] and Definition [7, Definition 16.1]) so it is interesting to ask if it has a Darboux chart. This, in turn, rises other question: how to construct on Fréchet manifolds weak symplectic forms. It is known that (see [5]) expect Hilbert manifolds an infinite dimensional manifold may not admit a Lagrangian splitting so in general the Weinstein’s construction ([10]) is not applicable. Moreover, the Marsden’s idea to construct a symplectic
form on a manifold by using the canonical form on its cotangent bundle also is not applicable as there is no natural smooth vector bundle structure on the cotangent bundle [8, Remark I.3.9]. It seems that a symplectic form on $\mathcal{M}$ might arise from a weak Riemannian metric and complex structure, however, that would require different version of the Darboux theorem. This is an interesting topic for further studies.

2. Bounded differentiability

In this section we prove the existence of the local flow of a $MC^k$-vector field, we refer to [2] for more details on bounded Fréchet geometry. We denote by $(F, \rho)$ a Fréchet space whose topology is defined by a complete translational-invariant metric $\rho$. We consider only metrics with absolutely convex balls. Note that every Fréchet space admits such a metric, cf [2]. One reason to choose this particular metric is that a metric with this property can give us a collection of seminorms that defines the same topology. More precisely:

**Theorem 2.1** ([6], Theorem 3.4). Assume that $(F, \rho)$ is a Fréchet space and $\rho$ is a metric with absolutely convex balls. Let $B^\rho_\epsilon(0) := \{ y \in F \mid \rho(y, 0) < \frac{1}{\epsilon} \}$, and suppose $U_i$’s, $i \in \mathbb{N}$, are convex subsets of $B^\rho_\epsilon(0)$. Define the Minkowski functionals

$$\| v \|_i := \inf \{ \epsilon > 0 \mid \epsilon \in \mathbb{R}, \frac{1}{\epsilon} \cdot v \in U_i \}.$$  

These Minkowski functionals are continuous seminorms on $F$. A collection $\{ \| v \|_i \}_{i \in \mathbb{N}}$ of these seminorms gives the topology of $F$.

In the sequel we assume that a Fréchet space $F$ is graded with the collection of seminorms $\| v \|_i := \sum_{k=1}^{n} \| v \|_k$ that defines its topology.

Let $(E, g)$ be another Fréchet space. Let $\mathcal{L}_{g,\rho}(E, F)$ be the set of all linear maps $L : E \to F$ such that

$$\text{Lip}(L)_{g,\rho} := \sup_{x \in E \setminus \{0\}} \frac{\rho(L(x), 0)}{g(x, 0)} < \infty.$$  

The transversal-invariant metric

$$D_{g,\rho} : \mathcal{L}_{g,\rho}(E, F) \times \mathcal{L}_{g,\rho}(E, F) \longrightarrow [0, \infty), \ (L, H) \mapsto \text{Lip}(L - H)_{g,\rho},$$  

on $\mathcal{L}_{\rho,\rho}(E, F)$ turns it into an Abelian topological group. Let $U$ an open subset of $E$, and $P : U \to F$ a continuous map. If $P$ is Keller-differentiable, $dP(p) \in \mathcal{L}_{\rho,\rho}(E, F)$ for all $p \in U$, and the induced map $dP(p) : U \to \mathcal{L}_{\rho,\rho}(E, F)$ is continuous, then $P$ is called bounded differentiable. We say $P$ is $MC^0$ and write $P^0 = P$ if it is continuous. We say $P$ is an $MC^1$ and write $P^{(1)} = P'$ if it is bounded differentiable. We define for $(k > 1)$ maps of class $MC^k$
For all $f$ property that on each open subset $U \subset X$ of a vector field, integral curves. However, a $k$-vector field, \cite{2, Proposition 5.1} \cite{2, Corollary 5.1} any two such curves are equal on the intersection of their domains.

Within this framework we define $MC^k$ (bounded) Fréchet manifolds, $MC^k$-maps of manifolds and tangent bundles and their $MC^k$-vector fields. A $MC^k$-vector field $X$ on a $MC^k$-Fréchet manifold $M$ is a $MC^k$-section of the tangent bundle $\pi_{TM} : TM \to M$, i.e. a $MC^k$ map $X : M \to TM$ with $\pi_{TM} \circ X = \text{id}_M$. We write $\mathcal{V}(M)$ for the space of all vector fields on $M$. If $f \in MC^\infty(M, E)$ is a smooth function on $M$ with values in a Fréchet space $E$ and $X \in \mathcal{V}(M)$, then we obtain a smooth function on $M$ via

$$X.f := d f \circ X : M \to E.$$ 

For $X, Y \in \mathcal{V}(M)$, there exists a unique a vector field $[X, Y] \in \mathcal{V}(M)$ determined by the property that on each open subset $U \subset M$ we have

$$[X, Y].f = X.(Y.f) - Y.(X.f)$$

for all $f \in MC^\infty(U, \mathbb{R})$, see \cite{9, Lemma II.3.1}.

A vector field on an infinite dimensional Fréchet manifold may have no, one or multiple integral curves. However, a $MC^k$-vector field always has a unique integral curve.

**Proposition 2.1.** \cite{2, Proposition 5.1} Let $U \subseteq F$ be open and let $X : U \to F$ be a $MC^k$-vector field, $k \geq 1$. Then for $p_0 \in U$, there is an integral curve $\ell : I \to F$ at $p_0$. Furthermore, any two such curves are equal on the intersection of their domains.

**Corollary 2.1.** \cite{2, Corollary 5.1} Let $U \subseteq F$ be open and let $X : U \to F$ be a $MC^k$-vector field, $k \geq 1$. Let $F_1(p_0)$ be the solution of $\ell'(t) = X(\ell(t))$, $\ell(t_0) = p_0$. Then there is an open neighborhood $U_0$ of $p_0$ and a positive real number $\alpha$ such that for every $q \in U_0$ there exists a unique integral curve $\ell(t) = F_1(q)$ satisfying $\ell(0) = q$ and $\ell(t) = X(\ell(t))$ for all $t \in (-\alpha, \alpha)$.

**Theorem 2.2.** Let $X$ be a $MC^k$-vector field on $U \subset F$, $k \geq 1$. There exists a real number $\alpha > 0$ such that for each $x \in U$ there exists a unique integral curve $\ell_x(t)$ satisfying $\ell_x(0) = x$ for all $t \in I = (-\alpha, \alpha)$. Furthermore, the mapping $F : I \times U \to F$ given by $F_t(x) = F(t, x) = \ell_x(t)$ is of class $MC^k$.

**Proof.** The first part of the proof follows from Corollary \cite{3,4}. We now prove the second part.

Let $x, y \in U$ be arbitrary and define the maps $\varphi_n(t) := \| F(t, x) - F(t, y) \|_F^n$, $\forall n \in \mathbb{N}$. Since $X$ is $MC^k$, so it is globally Lipschitz. Let $\beta > 0$ be its Lipschitz constant we then have

$$\varphi(t) = \| \int_0^t \left( X(F(s, x)) - X(F(s, y)) \right) ds + x - y \|_F \leq \| x - y \|_F^n + \beta \int_0^t \varphi(s) ds, \quad \forall n \in \mathbb{N}$$
Thus, by Gronwall’s inequality we obtain
\[ \| F(t, x) - F(t, y) \|_F^n \leq e^{\beta |t|} \| x - y \|_F^n, \forall n \in \mathbb{N}. \] (2.2)

Thereby \( F \) is Lipschitz continuous in the second variable and is jointly continuous.

Now, define \( F(t, x) \in \mathcal{L}_p(F) \) to be the solution of the equations
\[ \frac{d F(t, x)}{dt} = d X(F(t, x)) \circ F(t, x), \quad F(0, x) = \text{id}, \] (2.3)
where \( d X(F(t, x)) : F \to F \) is derivative of \( X \) with respect to \( x \) at \( F(t, x) \). By Proposition 2.1, \( F(t, x) \) exits and is well defined. Since the vector field \( F \mapsto d X(F(t, x)) \circ F \) on \( \mathcal{L}_p(F) \) is Lipschitz in \( F \), uniformly in \((t, x)\) in a neighborhood of every \((t_0, x_0)\), by the above argument it follows that \( F(t, x) \) is continuous in \((t, x)\). We show that \( d F(t, x) = F(t, x) \). Fix \( t \in I \), for \( h \in U \) define \( \psi(s, h) = F(s, x + h) - F(s, x) \), then
\[
\psi(t, h) - F(t, x)(h) = \int_0^t \left( X(F(s, x + h)) - X(F(s, x)) \right) ds - \int_0^t \left[ d X(F(s, x)) \circ F(s, x) \right](h) ds
\]
\[
= \int_0^t d X(F(s, x))((\psi(s, h) - F(s, x)(h))) ds
\]
\[
+ \int_0^t \left( X(F(s, x + h)) - X(F(s, x)) \right) ds
\]
\[
- d X(F(s, x))(F(s, x + h) - F(s, x)) ds
\]
Since \( X \) is \( MC^k \), for given \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( \| h \|_F^n < \delta (\forall n \in \mathbb{N}) \) yields that the second term is less than
\[ \int_0^t \varepsilon \| F(s, x + h) - F(s, x) \|_F^n, \quad \forall n \in \mathbb{N}, \] (2.4)
but by (2.2) this integral is less than \( B \varepsilon \sup_{n \in \mathbb{N}} \| h \|_F^n \) for some positive constant \( B \). Thus, by Gronwall’s inequality we obtain
\[ \| \psi(t, h) - F(t, h)(h) \|_F^n \leq \varepsilon C \| h \|_F^n, \quad \forall n \in \mathbb{N}, \]
where \( C \) is a positive constant. Whence \( d F(t, x)(h) = F(t, x)(h) \). Thus, both partial derivatives of \( F(t, x) \) exist and are continuous so \( F(t, x) \) is \( C^1 \). Moreover, \( F \) is globally Lipschitz and \( x \mapsto F(\cdot, x) \) is continuous therefore \( F(t, x) \) is \( MC^1 \). By induction on \( k \) we prove that \( F(t, x) \) is of class \( MC^k \). By definition of \( F(t, x) \)
\[ \frac{d}{dt} F(t, x) = X(F(t, x)) \] (2.5)
so
\[ \frac{d}{dt} \frac{d}{dt} F(t, x) = d X(F(t, x)) \left( X(F(t, x)) \right) \] (2.6)
and
\[ \frac{d}{dt} dF(t, x) = dX(F(t, x))(dF(t, x)). \] (2.7)

The right-hand sides are $MC^{k-1}$, so are the solutions by induction. Thus $F(t, x)$ is $MC^k$. \qed

3. Darboux charts

In general for a Fréchet manifold differential forms cannot be defined as the sections of its cotangent bundle since in general we can not define a manifold structure on the cotangent bundle, see [8, Remark I.3.9]. To define differential forms we follow the approach of Neeb [8].

**Definition 3.1.** Let $M$ be a bounded Fréchet manifold. A $p$-form $\omega$ on $M$ is a function $\omega$ which associates to each $x \in M$ a $p$-linear alternating map $\omega_x : T_x(M)^p \to \mathbb{R}$ such that in local coordinates the map $(x, v_1, \ldots, v_p) \mapsto \omega_x(v_1, \ldots, v_p)$ is smooth. We write $\Omega^p(M, \mathbb{R})$ for the space of $p$-forms on $M$ and identify $\Omega^p(M, \mathbb{R})$ with the space $C^\infty(M, \mathbb{R})$ of smooth functions on $M$.

The exterior differential $dR : \Omega^p(M, \mathbb{R}) \to \Omega^{p+1}(M, \mathbb{R})$ is determined uniquely by the property that for each open subset $U \subset M$ we have for $X_0, \ldots, X_p \in \mathcal{V}(U)$ in the space $C^\infty(U, \mathbb{R})$ the identity
\[ (dR \omega)(X_0, \ldots, X_p) := \sum_{i=0}^p (-1)^i X_i \omega(X_0, \ldots, \dot{X}_i, \ldots, X_p) \]
\[ + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \dot{X}_i, \ldots, \dot{X}_j, \ldots, X_p). \] (3.1)

Let $\sigma \in \Omega^p(M, \mathbb{R})$, $Y \in \mathcal{V}(M)$ and $F_t$ the local flow of $Y$. We define the usual Lie derivative for by
\[ \mathcal{L}_Y \omega = \frac{d}{dt} (F_t^* \omega) \big|_{t=0}, \]
which of course coincides by
\[ (\mathcal{L}_Y \omega)(X_1, \ldots, X_p) = Y.\omega(X_1, \ldots, X_p) - \sum_{j=1}^p \omega(X_1, \ldots, [Y, X_j], \ldots, X_p) \]
for $X_i \in \mathcal{V}(U)$, $U \subset M$ open. For each $X \in \mathcal{V}(M)$ and $p \geq 1$ we define a linear map
\[ i_X : \Omega^p(M, \mathbb{R}) \to \Omega^{p-1}(M, \mathbb{R}) \quad \text{with} \quad (i_X \omega)_x = i_{X(x)} \omega_x, \]
where $(i_x \omega_x)(v_1, \ldots, v_{p-1}) := \omega_x(v, v_1, \ldots, v_{p-1})$. For $\omega \in \Omega^0(M, \mathbb{R}) = C^\infty(M, \mathbb{R})$, we put $i_X \omega := 0$. For $X, Y \in \mathcal{V}(M)$, we have on $\Omega(M, \mathbb{R})$ the Cartan formulas ([8, Proposition I.4.3]):
\[ [\mathcal{L}_X, i_Y] = i_{[X,Y]}, \quad \mathcal{L}_X = dR \circ i_X + i_X \circ dR \quad \text{and} \quad \mathcal{L}_X \circ dR = dR \circ \mathcal{L}_X. \] (3.3)
Definition 3.2. Let $M$ be a bounded Fréchet manifold. We say that $M$ is weakly symplectic if there exists a closed smooth 2-form $\omega$ ($d_dR\omega = 0$) such that it is weakly non-degenerate i.e. for all $x \in M$ and $v_x \in T_x M$

$$\omega_x(v_x, w_x) = 0$$

(3.4)

for all $w_x \in T_x M$ implies $v_x = 0$.

The Darboux theorem is a local result so we consider the case where $M$ is an open set $U$ of $F$. For the simplicity we suppose that $0 \in U$.

Let $F'_b$ be the strong dual of $F$ and define the map $\omega^\#: F \to F'_b$ by

$$\langle w, \omega^\#(v) \rangle = \omega_x(w, v),$$

where $\langle \cdot, \cdot \rangle$ is a duality pairing. Condition (3.4) implies that $\omega^\#$ is injective.

Let $x \in U$ be fixed and define $H_x := \{\omega_x(y, .) \mid y \in F\}$, this is a subset of $F'_b$ and its topology is induced from it. Henceforth we assume that all Fréchet spaces are reflexive.

Lemma 3.1. $\omega^\#: F \to H_x$ is an isomorphism.

Proof. Obviously $\omega^\#$ is injective. The space $F$ is reflexive so it is distinguished and therefore the strong dual is barrelled therefore by open mapping theorem and the restriction of the domain to $H_x$ the inverse is continuous. Since $F$ is reflexive the natural embedding $\iota: E \to E'', x \to \hat{x}$ where $\hat{x}(\ell) = \ell(x)$ for $\ell \in F'$ is onto. Assume that $\omega^\#$ is not surjective and $R$ is its range, i.e. $R \neq H_x$. Form the continuity of the inverse it follows that $R$ is closed. By the Hahn-Banach theorem there is a $0 \neq \phi \in F''$ such that $\phi(R) = \{0\}$. Let $\phi = \iota(v)$. Then for any $w \in F$,

$$\omega(v, w) = \langle v, \omega^\#(w) \rangle = \phi(\omega^\#(w)) = 0.$$ 

Therefore $v = 0$ and so $\phi = 0$ which is the contradiction. \hfill \Box

We will need the following result.

Lemma 3.2. [9 (Poincaré Lemma) II.3.5] Let $E$ be locally convex, $V$ a sequentially complete space and $U \subset E$ an open subset which is star-shaped with respect to $0$. Let $\omega \in \Omega^{K+1}(U, V)$ be a $V$-valued closed $(k+1)$-form. Then $\omega$ is exact. Moreover, $\omega = d_{dR}\alpha$ for some $\alpha \in \Omega^K(U, V)$ with $\alpha(0) = 0$ given by

$$\alpha(x)(v_1, \cdots v_k) = \int_0^1 t^k \omega(tk)(x, v_1, \cdots v_k) dt.$$ 

Theorem 3.1. Let $(M, \omega)$ be a weakly symplectic smooth bounded Fréchet manifold modeled on $F$. Let $\omega_t = \omega_0 + t(\omega - \omega_0)$ for $t \in [0, 1]$. Suppose that following hold
There exits an open star-shaped neighborhood $U$ of zero such that for all $x \in U$ the map $\omega^t_x : F \to H_x$ is isomorphism for each $t$.

(2) for $x \in U$ the map $(\omega^t_x)^{-1} : H_x \to E$ is smooth for each $t$.

Then there exists a coordinate chart $(V, \varphi)$ around zero such that $\varphi^* \omega = \omega_0$.

**Proof.** By Lemma 2.2 there exist 1-form $\alpha$ such that $d_{dR} \alpha = \omega - \omega_0$ and $\alpha(0) = 0$. Consider a time-dependent vector field $X_t : U \to F$ such that

\[ i_{X_t} \omega^t = -\alpha. \]

By Condition 1 for $x \in U$ and all $t$ we have $\omega^t_x$ is isomorphism hence

\[ X_t(x) = (\omega^t_x)^{-1}(\alpha_x) \]

is well defined and it is smooth by Condition 2. Thus, by Theorem 2.2 there exists the smooth isotopy $F_t$ on $F$ generated by $X_t$ and for $t \in [0, 1]$ it satisfies

\[ F_t^* \omega^t = \omega_0 \] (3.5)

To solve (3.5) we need to solve

\[ \frac{d}{dt} F_t^* \omega^t = 0. \] (3.6)

We have by product rule of derivative and the Cartan formula

\[ \frac{d}{dt} F_t^* \omega^t = F_t^* (\mathcal{L}_{X_t} \omega^t) + F_t^* \frac{d}{dt} \omega^t \] (3.7)

\[ = F_t^* (\frac{d}{dt} \omega^t - d_{dR} (i_{X_t} \omega^t)) \] (3.8)

\[ = F_t^* (-d\alpha + \omega_0 - \omega) = 0. \] (3.9)

Thus, $F_1^* \omega_1 = F_1^* \omega_0$ so $F_1^* \omega = \omega_0$. \[ \square \]

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Topology lab., Institute of Mathematics of NAS of Ukraine, Tereshchenkivska st. 3, Kyiv, 01601 Ukraine

E-mail address: kaveh@imath.kiev.ua