Generalized area law under multiparameter rotating black hole spacetime

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Abstract
We study the statistical mechanics of quantum scalar fields under multiparameter rotating black hole spacetime in arbitrary $D$ dimensions. The method of analysis is general because the metric does not depend on explicit black hole solutions. A generalized Stefan–Boltzmann law for scalar fields is derived by properly considering the allowed energy region. Then a generalized area law for scalar field entropy is derived by introducing an invariant regularization parameter in Rindler spacetime. The derived area law is applied to Kerr–AdS black holes in four and five dimensions. The thermodynamic implications are also discussed.

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1. Introduction

Recently, many candidates of rotating black holes have been observed. Such black holes are well described by axisymmetric solutions of Einstein’s field equations [1–3]. Furthermore, rotating black holes in higher dimensions [4] and in the (anti)-de Sitter spacetime of (negative) cosmological terms have had great theoretical interest in view of string theory, M-theory and AdS/CFT correspondence [5–7].

Matter fields may be absorbed into black holes by strong gravity, and the information of the matter fields may be transformed to the black holes. To satisfy the first and second laws of thermodynamics, the black holes themselves have their own entropy proportional to the area of the horizon, known as black hole thermodynamics [8–11]. Black holes attract matter fields around themselves and into their horizon. On the other hand, black holes may radiate matter fields by strong gravitational acceleration [12–14]. Therefore, it is worthwhile studying the matter field contribution to black hole thermodynamics as a quasi-equilibrium state around black holes.
The scalar field contribution to entropy under Schwarzschild spacetime has been extensively studied, and the area law was shown to hold [15–19]. Due to superradiant stability and/or instability [20], deriving the area law for the scalar field under rotating black hole spacetime in $(2+1)$ and $(3+1)$ dimensions was difficult. For Bañados–Teitelboim–Zanelli (BTZ) black hole spacetime in $(2+1)$ dimensions [21–24] and Kerr black hole spacetime in $(3+1)$ dimensions [25–27], additional divergences appeared in the statistical sum of the quantum states of the scalar fields, so the non-rotating limit could not be taken. Another problem in BTZ spacetime was that the results depended on the method of calculation in addition to the above difficulties. We studied scalar field contribution to rotating black hole entropy in single-parameter cases in our previous paper [28]. We considered energy restriction for scalar fields in statistical integration, which defines the Boltzmann’ factor well. We also introduced a zenithal angle dependent regularization parameter in original spacetime, from which we derived area law for a scalar field. The obtained area law of entropy is applied to BTZ and Kerr black hole spacetimes, where two methods are adopted to confirm the results: a semiclassical method and an Euclidean path integral method.

In this paper, we extend our previous method for scalar field entropy [28] to multiparameter rotating black hole spacetime in arbitrary $D$ dimensions. To explain our strategy, we first express the definition of the coordinate and the properties of the metric. We set the $D$-dimensional polar coordinate as

$$x^\mu = (x^0, x^1, x^2, \ldots, x^{D-1}) = (t, \varphi_1, \ldots, \varphi_p, \theta_1, \ldots, \theta_q, r).$$

where the number of azimuthal angles ($\varphi_a$) is $p$ and zenithal angles ($\theta_m$) is $q$, where $p + q + 2 = D$. We use suffix notation, $\mu, \nu = 0, \ldots, D-1$ for full $D$-dimensional spacetime, $i, j = 1, \ldots, D-1$ for special components $a, b = 1, \ldots, p$ for azimuthal angle components ($\varphi$), and $m, n = 1, \ldots, q$ for zenithal angle components ($\theta$) in the following. A background metric form is minimally required and is in a possible general form:

$$ds^2 = \sum_{\mu, \nu = 0}^{D-1} g_{\mu\nu} x^\mu x^\nu = g_{tt} dt^2 + 2 \sum_{a = 1}^{p} g_{\varphi_a} dt d\varphi_a + \sum_{i,j=1}^{D-1} h_{ij} dx^i dx^j$$

$$= g_{tt} dt^2 + 2 \sum_{a = 1}^{p} g_{\varphi_a} dt d\varphi_a + \sum_{a,b = 1}^{p} g_{\varphi_a \varphi_b} d\varphi_a d\varphi_b + \sum_{m,n = 1}^{q} g_{\theta_m \theta_n} d\theta_m d\theta_n + g_{rr} dr^2,$$  \hspace{1cm} (1.2)

where special metric components are defined as $h_{ij} := (g_{\varphi_a \varphi_b}, g_{\theta_m \theta_n}, g_{rr})$ with the block diagonal form with respect to $\varphi_a, \theta_m$ and $r$. The basic properties of the metric are the following:

**Metric property (1):** the off-diagonal metrics are those between the time and azimuthal components $g_{\varphi a}$ in equation (1.2). The angular velocities with respect to azimuthal angles $\varphi_a$ at an arbitrary position $x^a$ and at the black hole horizon $r = r_H$ are expressed by these off-diagonal metric components as

$$\Omega_a := \frac{g_{\varphi a}}{g^{tt}}, \quad \Omega_{t a} := \frac{g_{\varphi a}}{g^{tt}} \bigg|_{r = r_H} (a = 1, \ldots, p).$$ \hspace{1cm} (1.3)

Note that in general the usual expression in single-parameter rotation black hole cases for angular velocity $-g_{\varphi a}/g_{\varphi a \varphi a}$ is not applicable in multiparameter cases.

**Metric property (2):** all metrics are assumed not to depend on time $t$ and azimuthal angles $\varphi_a$. This property induces the existence of $(p + 1)$ Killing vectors:

$$\xi_a = \partial_t \xi_{\varphi_a} = \partial_{\varphi_a} \quad (a = 1, \ldots, p).$$ \hspace{1cm} (1.4)
The existence of Killing vectors implies the conservation of total energy $H$ and azimuthal angular momentum $P_{\phi a}$ of the scalar field:

$$H := - \int_\Sigma (\xi_\mu) T^{\mu t} d\Sigma_t, \quad P_{\phi a} := \int_\Sigma (\xi_{\phi a}) T^{\mu t} d\Sigma_t$$

where the energy–momentum tensor is $T^{\mu\nu} = -(2/\sqrt{-g})(\delta I_{\text{scalar}}/\delta g^{\mu\nu})$ and scalar field action $I_{\text{scalar}}$ will be defined in section 2.

**Metric property (3):** horizon $r_H$ is defined as the position of simple zeros for the inverse of contravariant time component $1/g^{tt}$ and covariant radial one $1/g_{rr}$:

$$\frac{1}{g^{tt}} \simeq \left. \frac{\partial_r (1/g^{tt})}{g^{tt}(\theta)} \right|_{r_H} \times (r - r_H), \quad \frac{1}{g_{rr}} \simeq \left. \frac{\partial_r (1/g_{rr})}{g_{rr}(\theta)} \right|_{r_H} \times (r - r_H).$$

The coefficients $\left. \partial_r (1/g^{tt}) \right|_{r_H}$ and $\left. \partial_r (1/g_{rr}) \right|_{r_H}$ should be positive definite in order to define the temperature well.

Under these metric properties, we studied the statistical mechanics for scalar fields under multiparameter rotating black hole spacetime by adopting the Euclidean path integral method. From metric properties (1) and (2), we can rigorously derive energy restriction for scalar fields, which is important for performing energy integration in statistical mechanics. The obtained thermodynamic quantities of scalar fields are in the form of a generalized Stefan–Boltzmann law. To evaluate the entropy of scalar fields, Rindler spacetime is naturally introduced using the definition of the horizon (metric property (3)), which is the invariant under radial coordinate transformation. The temperature on the horizon and the ultraviolet cutoff parameter are introduced in an angular independent way in Rindler spacetime. The ultraviolet cutoff for non-rotating black holes was originally introduced in the brick wall model by ’t Hooft [16].

The generalized area law of the scalar field contribution to black hole entropy is derived naturally using the metric properties and the ultraviolet cutoff.

The generalized area law is applied to Kerr–(anti)-de Sitter (Kerr–AdS) black holes in four and five dimensions. Kerr–AdS black holes in higher dimensional spacetime are interesting because they lead to deeper understanding of black hole thermodynamics. The obtained thermodynamic quantities for scalar fields are shown to satisfy the thermodynamic (Gibbs–Duhem) relation and the first law of thermodynamics. It is interesting to note that they also satisfy thermodynamics in the language of black hole variables. This implies that thermodynamics for scalar fields can be consistently considered to be the thermodynamics for the black hole itself.

The organization of this paper is as follows. In section 2, the statistical mechanics for scalar fields is studied, and a generalized area law is derived. In section 3, the generalized area law is applied to Kerr–AdS black hole spacetime in $(3+1)$ and $(4+1)$ dimensions. The relation between thermodynamics for scalar fields and black holes is studied in section 4. Summary and discussions are given in the last section.

We adopt units so that $c = \hbar = k_B = G = 1$ unless otherwise specified.

### 2. Generalized area law for scalar fields

In this section, we study the statistical mechanics for a quantum scalar field under multiparameter rotating black hole spacetime in arbitrary $D$ dimensions.
2.1. Generalized Stefan–Boltzmann law

The matter action for scalar field $\Phi$ of mass $m$, with metric \((1.2)\), is

$$I_{\text{scalar}} = \int d^dx \sqrt{-g} L_{\text{scalar}}(x)$$  \hspace{1cm} \text{(2.1)}

And the canonical momentum of the scalar field is defined by

$$\Pi_{\Phi_1}(x) := \frac{\partial L_{\text{scalar}}(x)}{\partial \dot{\Phi}_1(x)} = -g^{tt} \dot{t} \Phi_1(x) - \sum_{a=1}^{p} g^{\psi_a} \dot{\psi}_a \Phi_1(x)$$  \hspace{1cm} \text{(2.2)}

And the quantization condition is given by

$$[\Phi_1(x), \Pi_\Phi(y)]|_{t=t'} = i \delta^{D-1}(x - y) \sqrt{-g}$$  \hspace{1cm} \text{(2.3)}

A new Killing vector, $\eta$, is introduced by linearly combining \((1.4)\) to account for the rotating geometry effect, as

$$\eta := \xi_t + \sum_{a=1}^{p} \Omega_{\psi_a} \xi_{\psi_a}.$$  \hspace{1cm} \text{(2.4)}

Its quadratic form is shown to satisfy the identical relation (see derivation in appendix A) as

$$\eta^2 = g_{tt} + 2 \sum_{a=1}^{p} g_{\psi_a} \Omega_{\psi_a} + \sum_{a,b=1}^{p} g_{\psi_a \psi_b} \Omega_{\psi_a} \Omega_{\psi_b}$$

$$= \frac{1}{g_{tt}} \sum_{a,b=1}^{p} g_{\psi_a \psi_b} \left( (\Omega_{\psi_a} - \Omega_a) (\Omega_{\psi_b} - \Omega_b) \right)$$  \hspace{1cm} \text{(2.5)}

where $\Omega_a$ and $\Omega_{\psi_a}$ are angular velocities defined in equation (1.3). This relation shows that the new Killing vector $\eta$ is null and is future directed on the horizon. Corresponding to new Killing vector $\eta$, a new conserved quantity is introduced that combines the energy and the angular momenta defined in equation (1.5) as

$$H = \sum_{a=1}^{p} \Omega_{\psi_a} P_{\psi_a} = -\int_E \eta^\mu T_{\mu \nu} d\Sigma = \int d^Dx \sqrt{-g} \mathcal{H}'.$$  \hspace{1cm} \text{(2.6)}

Newly defined Hamiltonian density $\mathcal{H}'$ satisfies the identity relation (see derivation in appendix B) as

$$\mathcal{H}' := \Pi_\Phi \dot{\Phi} - L_{\text{scalar}} + \sum_{a=1}^{p} \Omega_{\psi_a} \Pi_{\psi_a} \Phi$$

$$= \frac{1}{2} \left( -\Pi_\Phi^2 + \sum_{i,j=1}^{D-1} h^{ij} \dot{\phi}_i \dot{\phi}_j + m^2 \Phi^2 \right) + \sum_{a=1}^{p} (\Omega_{\psi_a} - \Omega_a) \Pi_{\psi_a} \Phi.$$  \hspace{1cm} \text{(2.7)}

In the identity relation, $h^{ij}$ denote the contravariant components of the special metric defined as

$$\sum_{j=1}^{D-1} h_{jk} h^{jk} = \delta^k_j \quad \text{with} \quad h^{ij} := \left( g_{\psi_a \psi_b} - \frac{g_{\psi_a} g_{\psi_b}}{g_{tt}}, g_{\psi_a \theta_j}, g_{\psi_a r} \right)$$  \hspace{1cm} \text{(2.8)}

where covariant components $h_{ij}$ are in equation (1.2). Newly introduced energy $H = \sum_{a=1}^{p} \Omega_{\psi_a} P_{\psi_a}$ in (2.6) is positive definite near horizon $r \simeq r_H$:
\[ H - \sum_{a=1}^{p} \Omega_{\text{Hu}} P_{\phi a} = \int d^{D-1}x \sqrt{-g} \mathcal{H}' \geq 0. \] (2.9)

In the following, we adopt near horizon approximation \( \Omega_{\text{Hu}} \simeq \Omega_{a} \) near \( r \simeq r_{H}. \)

Next, we consider the partition function of temperature \( T = 1/\beta \) as

\[ Z = \text{Tr} \left[ \exp \left( -\beta \left( H - \sum_{a=1}^{p} \Omega_{\text{Hu}} P_{\phi a} \right) \right) \right]. \] (2.10)

The exponent of the Boltzmann factor \( \sum_{a=1}^{p} \Omega_{\text{Hu}} P_{\phi a} \) is understood, considering the multiparameter rotation effect according to the Hartle–Hawking vacuum [15] and is positive definite near the horizon, as shown in equation (2.9), which ensures that the partition function is well-defined. To calculate the partition function, we express it in the Euclidean path integral form as

\[ Z = \int [\mathcal{D} \Phi \mathcal{D} \Pi \mathcal{G}^{1/2}] \exp \left( \int_{0}^{\beta} d\tau \int d^{D-1}x \sqrt{\mathcal{G}} \mathcal{E} \left( i \Pi \partial \tau - H' \right) \right) \] (2.11)

where Euclidean time \( \tau = it \), Euclidean metric \( g_{\tau \tau} = -g_{tt}, g_{\tau \phi a} = -ig_{t \phi a} \), and determinant \( \mathcal{G} \) are used. The periodic boundary condition for scalar fields is required in the integration: \( \Phi(x, \tau = \beta) = \Phi(x, \tau = 0) \).

After performing the momentum field \( \Pi \) integration, the partition function becomes

\[ Z = \int [\mathcal{D} \Phi \mathcal{G}^{1/4} (g^{\tau \tau})^{1/2}] \exp \left( -\int_{0}^{\beta} d\tau \int d^{D-1}x \sqrt{\mathcal{G}} \mathcal{E} g_{\tau \tau} \frac{\beta}{2} \Phi \Phi \right) \] (2.12)

where \( \mathcal{K} \) denotes the kernel

\[ \mathcal{K} := -\partial_{\tau}^{2} - \frac{1}{g^{\tau \tau}} \sum_{i,j=1}^{D-1} \frac{1}{\hbar} \partial_{i} (\sqrt{h} h^{ij} \partial_{j}) - m^{2} \] (2.13)

in the optical space of the metric: \( ds^{2} = d\tau^{2} + g^{\tau \tau} \sum_{i,j=1}^{D-1} h_{ij} dx^{i} dx^{j} \). The determinant of \( h_{ij} \) is denoted by \( h \). Subsequently, scalar field \( \Phi \) integration is performed, and free energy is obtained using heat kernel representation [29]:

\[ \beta F = -\ln Z = -\frac{1}{2} \text{Tr} \int_{0}^{\beta} \frac{ds}{s} \exp \left( -s \mathcal{K} \right). \] (2.14)

The trace of the heat kernel consists of two parts: the Euclidean time part and the space part. The Euclidean time part is evaluated by using the eigenfunction of \( -i\partial_{\tau} \) as

\[ \text{Tr} \exp \left( s \partial_{\tau}^{2} \right) = \int_{0}^{\beta} d\tau \sum_{\ell = -\infty}^{\infty} \frac{1}{\beta} \exp \left( -s \left( \frac{2\pi \ell}{\beta} \right)^{2} \right) = \sum_{\ell = -\infty}^{\infty} \frac{\beta}{(4\pi s)^{1/2}} \exp \left( -\frac{\beta^{2} n^{2}}{4s} \right) \] (2.15)

where the Poisson’s summation formula is used. The space part in the trace is calculated by the asymptotic expansion method

\[ \text{Tr} \exp \left( \frac{s}{g^{\tau \tau}} \sum_{i,j=1}^{D-1} \frac{1}{\hbar} \partial_{i} (\sqrt{h} h^{ij} \partial_{j}) - m^{2} \right) = \frac{1}{(4\pi s)^{(D-1)/2}} \sum_{k=0}^{\infty} \bar{B}_{k} (-s)^{k} \exp \left( -\frac{sm^{2}}{g^{\tau \tau}} \right) \] (2.16)
where $\bar{B}_k$ are the coefficient functions of the asymptotic expansion. The explicit lower order contributions are

$$
\bar{B}_0 = \int d^{D-1}x (g^{\tau\tau})^{(D-1)/2} h^{1/2},
$$

$$
\bar{B}_1 = \left( \frac{1}{4} \frac{D-2}{D-1} - \frac{1}{6} \right) \int d^{D-1}x (g^{\tau\tau})^{(D-1)/2} h^{1/2} \bar{R} \tag{2.17}
$$

where bar notation denotes quantities in optical space.

The free energy in the lowest order is expressed by multiplying the two trace parts:

$$
F = -\frac{1}{\beta} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\bar{B}_0}{\bar{B}^{D/2}} \exp \left( -\frac{m^2 n^2}{4s} \right) \bar{v},
$$

$$
U = F + \beta^{-1} S = \frac{\xi(D)(D-1)\Gamma(D/2)}{\pi^{D/2} \beta^D} \bar{V} \tag{2.22}
$$

where $\bar{V}$ denotes the volume of optical space:

$$
\bar{V} = \bar{B}_0 \times 1 = \int d^{D-1}x (g^{\tau\tau})^{(D-1)/2} h^{1/2}. \tag{2.20}
$$

Entropy and internal energy are obtained by

$$
S = -\beta^2 \frac{\partial F}{\partial \beta} = \frac{\xi(D)D\Gamma(D/2)}{\pi^{D/2} \beta^{D-1}} \bar{v}, \tag{2.21}
$$

where the integration variable has been changed to $t = \beta^2 n^2/(4s)$. Vacuum energy term ($n = 0$) is subtracted in the sum. The contribution of the scalar field mass term is very small near the horizon. After neglecting the mass term, a compact expression for the free energy is obtained as

$$
F = -\frac{\xi(D)}{\pi^{D/2} \beta^D} \bar{V} \tag{2.19}
$$

2.2. Generalized area law

We first introduce Rindler spacetime, which is invariant under radial coordinate transformation, where the temperature on the horizon will be defined. In order to do this, we rearrange the Euclidean line element in a diagonal form with respect to modified azimuthal angles $d\phi_a - \Omega_a dr$ (see derivation in appendix C) as

$$
dr^2 = g^{\tau\tau} dr^2 + 2 \sum_{a=1}^{p} g_{\tau\phi_a} dr d\phi_a + \sum_{m,n=1}^{q} g_{\theta_m\theta_n} d\theta_m d\theta_n + g_{rr} dr^2,
$$

$$
= \frac{1}{g^{\tau\tau}} dr^2 + \sum_{a,b=1}^{p} g_{\psi_a\psi_b} (d\phi_a - \Omega_a dr) (d\phi_b - \Omega_b dr) + \sum_{m,n=1}^{q} g_{\theta_m\theta_n} d\theta_m d\theta_n + g_{rr} dr^2. \tag{2.23}
$$
Using new radial coordinate
\[ R(r) := \int_{r_H}^{r} \frac{dr}{\sqrt{g_{rr}}} \simeq \frac{2}{\sqrt{\beta_H}} \sqrt{r - r_H} \]
and the definition of horizon (1.6), the line element (2.23) is expressed near the horizon in the form of the Rindler spacetime as
\[ ds^2 \simeq -\frac{1}{4} \frac{\partial_r}{g^{tt}(\theta)} \frac{\partial_r}{g_{rr}(\theta)} \, R^2 \, dr^2 + dR^2 + (\varphi, \theta \text{ terms}). \]  

The temperature on horizon \((T_H = 1/\beta_H)\) is defined by requiring no conical singularity and no angle dependence in the Rindler spacetime:
\[ 2\pi \frac{\beta_H}{2} = \frac{\partial_r(1/g^{tt})}{2\sqrt{g_{rr}/g^{tt}}} \bigg|_{r_H} = \left( \frac{1}{4} \frac{\partial_r}{g^{tt}(\theta)} \frac{\partial_r}{g_{rr}(\theta)} \right)^{1/2} \bigg|_{r_H} = \text{indep. on } \theta. \]  

Next we consider the volume of optical space \(\bar{V}\) in (2.20). As \(\bar{V}\) diverges on the horizon, we introduce an invariant regularization parameter for small distance \(\epsilon_{\text{inv}}\) and large distance \(L_{\text{inv}}\) in Rindler spacetime as
\[ \epsilon_{\text{inv}} := \int_{r_H}^{r_H + \epsilon_{\text{inv}}(\theta)} \, \text{d}r \, (g_{rr}(\theta))^{1/2} \quad \text{and} \quad L_{\text{inv}} := \int_{r_H}^{r_H + L_{\text{inv}}(\theta)} \, \text{d}r \, (g_{rr}(\theta))^{1/2}. \]  

Regularization parameters \(\epsilon_{\text{inv}}\) and \(L_{\text{inv}}\) should be angle independent and invariant under radial coordinate transformation (instead, those in the original spacetime \(\epsilon(\theta), L(\theta)\) become angle dependent).

The optical volume \(\bar{V}\) is composed of radial interaction part and angle interaction part. Using these regularization parameters, the radial integration part in \(\bar{V}\) is evaluated in Rindler spacetime near the horizon as
\[ \int_{r_H + \epsilon(\theta)}^{r_H + L(\theta)} \, \text{d}r \, (g^{tt})^{D/2-1/2}(g_{rr})^{1/2} \bigg|_{r_H} = \int_{\epsilon_{\text{inv}}}^{L_{\text{inv}}} \, \text{d}R \, (g^{tt})^{D/2-1/2} \bigg|_{r_H} \]
\[ \approx \int_{\epsilon_{\text{inv}}}^{L_{\text{inv}}} \left( \frac{\partial_r}{g^{tt}(\theta)} \right)^{1/2} (r - r_H) \, \text{d}r \]
\[ \approx \left( \frac{\beta_H}{2\pi} \right)^{D/2-1/2} \frac{1}{(D - 2)\epsilon_{\text{inv}}^{D-2}}, \]  

where the temperature on the horizon (2.25) is used. The contribution from large regularization parameter \((L_{\text{inv}})\) is negligible under the magnitude ordering: \(\epsilon_{\text{inv}} \ll L_{\text{inv}}\). Angular integration in \(\bar{V}\) gives the black hole area on the horizon:
\[ A_H := \int \prod_{a=1}^{p} \text{d}\varphi_a \prod_{m=1}^{q} \text{d}\theta_m \sqrt{\det(g_{\varphi_a \varphi_a})} \det(g_{\theta_m \theta_m}) \bigg|_{r_H}. \]  

Combining equations (2.27) and (2.28), the optical volume \(\bar{V}\) is evaluated:
\[ \bar{V} = \left( \frac{\beta_H}{2\pi} \right)^{D-1} A_H \frac{1}{(D - 2)\epsilon_{\text{inv}}^{D-2}}. \]  

Then the generalized entropy formula (2.21) is obtained as
\[ S = \frac{\zeta(D)D\Gamma(D/2 - 1)}{2^D A^{3D/2-1}} \frac{A_H}{\epsilon_{\text{inv}}^{D-2}}. \]
We obtained the generalized area law for quantum scalar fields under multiparameter rotating black hole spacetime without using the explicit expression of black hole solutions. The expression of area law under multiparameter cases has the same form as that under non-rotating and single-parameter rotating cases [28]. This means that the non-rotating limit of the generalized law can be taken smoothly.

We also note that superradiant modes are considered in our entropy expression (2.30) because the scalar field energy $E$ can be negative if angular contribution $\sum_{a=1}^{\rho} \Omega_a P_{\psi_a}$ is negative and satisfies our energy restriction (2.9):

$$H - \sum_{a=1}^{\rho} \Omega_a P_{\psi_a} \geq 0.$$

3. Applications

In this section, we apply the generalized area law under multiparameter rotating black hole spacetime to cases of Kerr–AdS black holes. We first study four-dimensional Kerr–AdS spacetime as a single-parameter rotating case and next five-dimensional Kerr–AdS spacetime as a double-parameter rotating case.

3.1. Four-dimensional Kerr–AdS black hole spacetime

The four-dimensional Kerr–AdS metric is given by Carter [30] as

$$ds^2_{D=4} = -\frac{\Delta}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\Xi} \, d\psi \right)^2 + \frac{\rho^2}{\Delta} \left( dr^2 + \rho^2 \sin^2 \theta \, d\theta^2 + \frac{\Delta_0 \sin^2 \theta}{\rho^2} \left( a \, dt - \frac{r^2 + a^2}{\Xi} \, d\psi \right)^2 \right),$$

(3.1)

with

$$\Delta = (r^2 + a^2)(1 + r^2 \ell^2) - 2Mr, \quad \Delta_0 = 1 - a^2 \ell^2 \cos^2 \theta, \quad \Xi = 1 - a^2 \ell^2,$$

(3.2)

where $M$, $a$, and $\Lambda = -3\ell^{-2}$ denote black hole mass, its angular momentum per unit mass and the cosmological term, respectively. Metric (3.1) is expressed in the general form to apply the general area law (2.30):

$$ds^2_{D=4} = g_{tt} \, dt^2 + 2 g_{t\psi} \, dt \, d\psi + g_{\psi\psi} \, d\psi^2 + g_{\theta\theta} \, d\theta^2 + g_{rr} \, dr^2,$$

(3.3)

where metric components are given by

$$g_{tt} = \frac{1}{\rho^2}(-\Delta + \Delta_0 a^2 \sin^2 \theta), \quad g_{\psi\psi} = \frac{a \sin^2 \theta}{\rho^2 \Xi} (\Delta - (r^2 + a^2) \Delta_0),$$

$$g_{\theta\theta} = \frac{\sin^2 \theta}{\rho^2 \Xi^2} (-\Delta a^2 \sin^2 \theta + (r^2 + a^2)^2 \Delta_0), \quad g_{rr} = \frac{\rho^2}{\Delta}, \quad g_{\theta\theta} = \frac{\rho^2}{\Delta_0}.$$

(3.4)

To evaluate the temperature and the angular velocity, we need to know some contravariant components of the metric:

$$g^{tt} = \frac{g_{\psi\psi}}{\Gamma}, \quad g^{t\psi} = \frac{g_{tt}}{\Gamma}, \quad g^{\psi\psi} = -\frac{g_{\theta\theta}}{\Gamma}, \quad g^{\theta\theta} = -\frac{g_{rr}}{\Gamma},$$

(3.5)

with $\Gamma := g_{tt} g_{\psi\psi} - g_{t\psi}^2 = -\Delta_0 \sin^2 \theta / \Xi^2$. Horizon $r_H$ is defined as the larger zero of $1/g^{rr}$ as well as $1/g^{t\psi}$, which gives $\Delta(r = r_H) = 0$. The angular velocity and the temperature on
the horizon are calculated as

$$\Omega_H = \frac{g^\ell_{\ell}}{g^{\theta\theta}} \bigg|_{r_H} = - \frac{g_{\theta\theta}}{g_{\ell\ell}} \bigg|_{r_H} = \frac{a \Xi}{r_H^2 + a^2}, \quad (3.6)$$

$$\frac{2\pi}{\beta_H} = \frac{\partial r}{2(r^2 + a^2)} \bigg|_{r_H} = \frac{r_H^2}{2(r_H^2 + a^2)} \left( 1 + 3r_H^2 \ell - 2 \ell - 2 + \frac{a^2}{r_H^2} \right). \quad (3.7)$$

Angular velocity $\Omega_H$ is in the same form as that given by Hawking, Hunter and Taylor-Robinson [5]. Other definitions of angular velocity given by Gibbons, Perry and Pope [6] will be discussed in the next section. The entropy of a four-dimensional Kerr–AdS black hole is obtained from the general expression (2.30):

$$S_{D=4} = \frac{1}{360\pi} \frac{A_H}{\epsilon_{inv}}, \quad (3.8)$$

with the area on horizon $A_H = 4\pi \left( r_H^2 + a^2 \right)/\Xi$. In the limit of no cosmological term ($\ell^2 \to 0$), the resultant area law (3.8) recovers that of the previous work in Kerr black hole cases [28].

### 3.2. Five-dimensional Kerr–AdS black hole spacetime

In five-dimensional Einstein’s field equation, a double-parameter rotating black hole solution exists. The metric of the Kerr–AdS black hole in five dimensions is given by Hawking, Hunter and Taylor-Robinson [5] as

$$ds^2_{D=5} = -\frac{\Delta}{\rho^2} \left( dr - \frac{a \sin^2 \theta}{\Xi_a} d\varphi_a = \frac{b \cos^2 \theta}{\Xi_b} d\varphi_b \right)^2 + \frac{\Delta_b \sin^2 \theta}{\rho^2} \left( a dr - \frac{(r^2 + a^2)}{\Xi_a} d\varphi_a \right)^2 + \frac{\Delta_a \cos^2 \theta}{\rho^2} \left( b dr - \frac{(r^2 + b^2)}{\Xi_b} d\varphi_b \right)^2 + \rho^2 \Delta d\theta^2 + \frac{\rho^2}{\Delta_b} d\theta^2 + \frac{(1 + r^2 \ell^2)}{r^2 \rho^2} \left( ab dr - \frac{b(r^2 + a^2)}{\Xi_a} \sin^2 \theta \ d\varphi_a = \frac{a(r^2 + b^2)}{\Xi_b} \cos^2 \theta \ d\varphi_b \right). \quad (3.9)$$

where

$$\Delta = \frac{1}{r^2} (r^2 + a^2)(r^2 + b^2)(1 + r^2 \ell^2) - 2M,$$

$$\Delta_a = (1 - a^2 \ell^2 \cos^2 \theta - b^2 \ell^2 \sin^2 \theta),$$

$$\Delta_b = (1 - a^2 \ell^2 \cos^2 \theta + b^2 \ell^2 \sin^2 \theta),$$

$$\rho^2 = (r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta),$$

$$\Xi_a = (1 - a^2 \ell^2), \quad \Xi_b = (1 - b^2 \ell^2).$$

Quantities $M, a, b$ and $\Lambda = -6\ell^2$ denote the mass of the black hole, two angular velocities per unit mass, and the cosmological term. We rewrite the metric in a general form to apply the generalized area law (2.30):

$$ds^2_{D=5} = g_{\ell\ell} \ dr^2 + 2\left( g_{\theta\varphi_a} \ dr \ d\varphi_a + g_{\theta\varphi_b} \ dr \ d\varphi_b \right) + g_{\varphi_a \varphi_a} \ d\varphi_a^2 + g_{\varphi_b \varphi_b} \ d\varphi_b^2 + 2g_{\varphi_a \varphi_b} \ d\varphi_a \ d\varphi_b + g_{\theta\theta} \ d\theta^2 + g_{rr} \ dr^2. \quad (3.11)$$
where metric components are
\[ g_{tt} = \frac{1}{\rho^2} (-\Delta + \Delta_\phi (a^2 \sin^2 \theta + b^2 \cos^2 \theta) + \Delta_\ell a^2 b^2), \]
\[ g_{t\phi_a} = \frac{a \sin^2 \theta}{\rho^2 \Xi_a} (\Delta - A (\Delta_\phi + \Delta_\ell b^2)), \]
\[ g_{t\phi_b} = \frac{b \cos^2 \theta}{\rho^2 \Xi_b} (\Delta - B (\Delta_\phi + \Delta_\ell a^2)), \]
\[ g_{\phi_a \phi_a} = \frac{\sin^2 \theta}{\rho^2 \Xi_a} (\Delta a^2 \sin^2 \theta + A^2 (\Delta_\phi + \Delta_\ell b^2 \sin^2 \theta)), \]
\[ g_{\phi_b \phi_b} = \frac{\cos^2 \theta}{\rho^2 \Xi_b} (\Delta b^2 \cos^2 \theta + B^2 (\Delta_\phi + \Delta_\ell a^2 \cos^2 \theta)), \]
\[ g_{\phi_a \phi_b} = \frac{ab \sin^2 \theta \cos^2 \theta}{\rho^2 \Xi_a \Xi_b} (-\Delta + AB \Delta_\ell), \]
with \( A = r^2 + a^2, B = r^2 + b^2, \Delta_\ell = r^{-2} + \ell^{-2}. \) We need the contravariant components of the metric with respect to time and the azimuthal angle to obtain the angular velocity and temperature (see derivation in appendix D):
\[ g^{tt} = \frac{1}{r^2 \rho^2 \Delta_\phi} (Ab^2 \cos^2 \theta + Ba^2 \sin^2 \theta - \ell^{-2} a^2 b^2 (A \cos^2 \theta + B \sin^2 \theta)) - \frac{1}{r^2 \rho^2 \Delta} A^2 B^2, \tag{3.13} \]
\[ g^{t\phi_a} = \frac{a \Xi_a}{r^2 \rho^2} \left( \frac{1}{\Delta_\phi} (B - \ell^{-2} b^2 (A \cos^2 \theta + B \sin^2 \theta) - \frac{1}{r^2 \Delta} AB^2) \right), \tag{3.14} \]
\[ g^{t\phi_b} = \frac{b \Xi_b}{r^2 \rho^2} \left( \frac{1}{\Delta_\phi} (A - \ell^{-2} a^2 (A \cos^2 \theta + B \sin^2 \theta) - \frac{1}{r^2 \Delta} A^2 B) \right). \tag{3.15} \]

Horizon \( r_\text{H} \) is defined as the larger zero of inverse metric components \( 1/g^{tt} \) and \( 1/g_{rr}, \) which is given by the root of \( \Delta: \)
\[ \Delta (r = r_\text{H}) = 0 \quad \iff \quad (r_\text{H}^2 + a^2)(r_\text{H}^2 + b^2)(1 + r_\text{H}^2 \ell^{-2}) = 2Mr_\text{H}^2. \tag{3.16} \]
The two angular velocities on the horizon are obtained:
\[ \Omega_{\phi_a} \equiv \frac{g^{\phi_a}}{g^{tt}} \bigg|_{r=r_\text{H}} = \frac{a \Xi_a}{r_\text{H}^2 + a^2}, \quad \Omega_{\phi_b} \equiv \frac{g^{\phi_b}}{g^{tt}} \bigg|_{r=r_\text{H}} = \frac{b \Xi_b}{r_\text{H}^2 + b^2}, \tag{3.17} \]
which coincide with those by Hawking, Hunter and Taylor-Robinson [5]. Note that generally in multiparameter cases, angular velocities \( \Omega_{\phi_a} \equiv g^{\phi_a}/g^{tt} \) and/or \( \Omega_{\phi_b} \equiv g^{\phi_b}/g^{tt} \) do not equal \( -g_{t\phi_a}/g_{\phi_a \phi_a} \) and/or \( -g_{t\phi_b}/g_{\phi_b \phi_b} \) generally in multiparameter cases.

The inverse Hawking temperature on horizon (2.25) is given as
\[ \Theta_\text{H} = \frac{2\pi}{\Delta_\phi} = \frac{r_\text{H}^2 \Delta}{2AB} \bigg|_{r=r_\text{H}} = r_\text{H} (1 + r_\text{H}^2 \ell^{-2}) \left( \frac{1}{r_\text{H}^2 + a^2} + \frac{1}{r_\text{H}^2 + b^2} \right) - \frac{1}{r_\text{H}}. \tag{3.18} \]
The entropy of the five-dimensional Kerr–AdS black hole is obtained from the general expression (2.30):
\[ S_{D=5} = \frac{\xi (5) A_{\text{H}}}{2^6 \pi^6 \ell^{-6}}, \tag{3.19} \]
with the area of the five-dimensional Kerr–AdS black hole:
\[ A_{\text{H}} = \frac{2\pi^2 (r_\text{H}^2 + a^2)(r_\text{H}^2 + b^2)}{r_\text{H} \Xi_a \Xi_b}. \tag{3.20} \]
We obtained the quantum scalar field contribution to the entropy in four- and five-dimensional spacetime, which will help to understand cases in general higher-dimensional rotating black hole spacetime.

4. Relation between thermodynamics for scalar fields and black holes

In this section we study the thermodynamics of scalar fields obtained in section 2 based on the standard statistical method under general form of metrics and compare them with corresponding black hole thermodynamics.

First we can show that the statistical (Gibbs–Duhem) relation for a quantum scalar field holds as

\[ TS = U + pV, \]

(4.1)

where the volume of optical space \( V \) is given in equation (2.20), entropy \( S \) in equation (2.21), internal energy \( U \) in equation (2.22), and pressure \( p \) is given by

\[ p := -\frac{\partial F}{\partial V} = \frac{\xi(D)\Gamma(D/2)}{\pi^{D/2}\beta_H^D}. \]

(4.2)

The first law of thermodynamics also holds as

\[ TdS = dU + pdV. \]

(4.3)

The statistical relation (4.1) and the first law of thermodynamics (4.2) are related through linear scaling properties. These thermodynamic relations hold because we derived the thermal quantities based on the standard statistical mechanics method.

To compare thermodynamics for the scalar field to those for black holes, we define the effective gravitational constant of the scalar field:

\[ G' = \frac{\pi^{3D/2-1}2^{D-2}}{4\xi(D)\Gamma(D/2-1)}, \]

(4.4)

which normalizes the entropy as \( S = A_H/4G' \).

Alternatively, we can express thermal quantities in the language of black hole variables instead of scalar field variables. In the case of four-dimensional Kerr–AdS black holes (see subsection 3.1), the statistical relation in the language of black holes is obtained:

\[ TS = \frac{E_H}{2} - \Omega_H J + X_H \ell^{-2}, \]

(4.5)

where black hole energy \( E_H \), angular momentum \( J \) and the conjugate variable to cosmological term \( X_H \) are defined:

\[ E_H = \frac{M}{G'}, \quad J = \frac{Ma}{G'}, \quad X_H = \frac{r_H(r_H^2 + a^2)}{2G'}. \]

(4.6)

Temperature \( T \) and entropy \( S \) in equation (4.5) are identical to those in equation (4.1). The first law of thermodynamics can also be expressed in the language of black holes as

\[ TdS = dE_G - \Omega_G dJ - X_G d\ell^{-2}, \]

(4.7)

where black hole energy \( E_G \), angular velocity \( \Omega_G \) and conjugate variable to cosmological term \( X_G \) are defined:

\[ E_G = \frac{M}{G'}, \quad \Omega_G = \frac{a(1 + r_H^2\ell^{-2})}{r_H + a^2}, \]

\[ X_G = \frac{r_H(r_H^2 + a^2)}{2G'} \left( 1 + \frac{a^2}{2r_H^2(1 - r_H^2\ell^{-2})} \right). \]

(4.8)
The numerical factors in front of each thermal quantity in equation (4.5) are from the scaling property: \(1/2\) for energy, 1 for thermal and angular terms, and \(-1\) for conjugate variable to the cosmological term. The differences between \(E_H, \Omega_H\) and \(X_H\) in equation (4.5) and \(E_G, \Omega_G\), and \(X_G\) in equation (4.7) are due to the nonlinearity in the scaling transformation in all quantities.

We should note the difference between the angular velocity defined by Hawking, Hunter and Taylor-Robbinson \(\Omega_H\) in equation (3.6) [5] and Gibbons, Perry and Pope \(\Omega_G\) in equation (4.8) [6]. Angular velocity \(\Omega_H\) is considered to be measured relative to a rotating frame at infinity, while angular velocity \(\Omega_G\) is measured relative to a non-rotating frame at infinity:

\[
\Omega_G := \Omega_H - \Omega_\infty = \frac{a \left(1 + \frac{r_H^2}{r_H^2 + a^2}\right)}{\beta_H} \quad (4.9)
\]

with \(\Omega_\infty = -a \ell^{-2}\). Gibbons, Perry and Pope [6] stressed the importance of holding the first law of thermodynamics in which \(\Omega_G\) and \(E_G\) are used in equation (3.8).

A naive interpretation of the conjugate variable to cosmological term \(X_H\) is the cosmological term in the action:

\[
-\frac{1}{16\pi G} \int d^4x \sqrt{-g} 2\Lambda = \beta_H \frac{r_H^2}{2\Xi G} (r_H^2 + a^2) \ell^{-2} \quad (4.10)
\]

where the coefficient factor is gravitational constant \(G\) instead of \(G'\) in \(X_H\) in equation (4.7). For cases of the constant cosmological term, \(d\ell^{-2} = 0\), the first law (4.7) reduces to that given by Gibbons, Perry and Pope with the replacement: \(G \rightarrow G'\). We include the cosmological contribution in the first law of the thermodynamics to maintain the scaling transformation.

By this correspondence, thermodynamics for the quantum scalar fields around black holes is consistent with that for the black holes themselves.

5. Summary and discussions

We studied the statistical mechanics for quantum scalar fields under multiparameter rotating black hole spacetime in arbitrary \(D\) dimensions. We obtained the generalized area law for scalar field entropy around black holes under the general form of the metric (see metric properties (1)–(3) in Introduction). Here the background metric form in equation (1.2) is understood as the possible general form taking multi-rotating effects into the static spherical symmetry coordinate frame.

The contribution to scalar field entropy is extremely dominant near the horizon in the optical volume \(\bar{V}\) (2.20). We introduce regularization parameter \(\epsilon_{\text{inv}}\) in Rindler spacetime to control divergence on the horizon. The large contribution near the horizon naturally leads the area law of entropy for the scalar fields.

Thermodynamics for the scalar fields can alternatively be interpreted in the language of black hole variables, where the gravitational constant is replaced by an effective one: \(G \rightarrow G'\).

In deriving the area law for the quantum scalar field under multiparameter rotating black hole spacetime, we postulated the quasi-equilibrium state for quantum scalar fields. In connection with this, we should comment on the superradiant instability for Kerr black hole spacetime in \((3 + 1)\) dimensions [31–33]. Superradiant effects occur even in our analysis, because the scalar field energy \(E\) can be negative if scalar field angular momentum is in the inverse direction with respect to the direction of black hole angular momentum. The relation between our quasi-equilibrium treatment and the superradiant instability phenomena will be studied in more detail in our future work.
The quantum effects of gravity near the horizon, which become crucial for deriving black hole entropy, will be an additional future problem.

Related to black hole thermodynamics, conserved quantities in general relativity increase in importance [34] and interest.

Appendix A. Derivation of null potency of $\eta$ in (2.5)

In this appendix, we calculate the square of newly introduced Killing vector $\eta (= \xi_t + \sum_{a=1}^p \Omega_{H_a} \xi_{\psi_a})$ and show its null potency. First we note the orthogonality relation among metrics:

$$g_{tt} g_{tt} + \sum_{a=1}^p g_{\psi_a \psi_a} = 1, \quad g_{t\psi_a} t^{\psi_a} + \sum_{b=1}^p g_{\psi_a \psi_b} = 0. \quad (A.1)$$

Using orthogonality relations, we obtain metric–angular velocity relations:

$$g_{tt} + 2 \sum_{a=1}^p g_{\psi_a} \Omega_{H_a} + \sum_{a,b=1}^p g_{\psi_a \psi_b} \Omega_{H_a} \Omega_{H_b} = \frac{1}{g^{tt}},$$

$$-2 \sum_{a=1}^p g_{\psi_a} \Omega_{H_a} - \sum_{a,b=1}^p g_{\psi_a \psi_b} \Omega_{H_a} \Omega_{H_b} = \sum_{a,b=1}^p g_{\psi_a \psi_b} \Omega_{H_a} \Omega_{H_b}, \quad (A.3)$$

$$\sum_{a=1}^p g_{\psi_a} \Omega_{H_a} = - \sum_{a,b=1}^p g_{\psi_a \psi_b} \Omega_{H_a} \Omega_{H_b}. \quad (A.4)$$

Further using these relations, we evaluate the square of $\eta$ in the following as

$$\eta^2 = g_{tt} + 2 \sum_{a=1}^p g_{\psi_a} \Omega_{H_a} + \sum_{a,b=1}^p g_{\psi_a \psi_b} \Omega_{H_a} \Omega_{H_b}$$

$$= g_{tt} + 2 \sum_{a=1}^p g_{\psi_a} \Omega_{H_a} + \sum_{a,b=1}^p g_{\psi_a \psi_b} \Omega_{H_a} \Omega_{H_b}$$

$$+ 2 \sum_{a=1}^p g_{\psi_a} \Omega_{a} + \sum_{a,b=1}^p g_{\psi_a \psi_b} \Omega_{a} \Omega_{b} - 2 \sum_{a=1}^p g_{\psi_a} \Omega_{a} - \sum_{a,b=1}^p g_{\psi_a \psi_b} \Omega_{a} \Omega_{b}$$

$$= \frac{1}{g^{tt}} + \sum_{a,b=1}^p g_{\psi_a \psi_b} (\Omega_{H_a} - \Omega_a)(\Omega_{H_b} - \Omega_b). \quad (A.5)$$

The last equality in equation (A.5) is the expression of $\eta^2$ in equation (2.5), which shows the null potency at horizon position $r = r_H$, where $\Omega_{H_a} = \Omega_a$ and $1/g^{tt} = 0$.

Appendix B. Derivation of non-negativity of $\mathcal{H}'$ in (2.7)

Here we show the non-negativity of newly introduced energy density $\mathcal{H}'$ defined in equation (2.7). For this purpose, we insert the explicit expression of scalar fields in Lagrangian
density (2.1) into $\mathcal{H}'$ and rewrite it as

$$\mathcal{H}' = \Pi \partial_t \Phi - L_{\text{scalar}} + \sum_{a=1}^{p} \Omega_{\Phi a} \Pi \partial_{\Phi a} \Phi$$

$$= \Pi \partial_t \Phi + \frac{1}{2} \left( g^{tt} (\partial_t \Phi)^2 + 2 \sum_{a=1}^{p} g^{a\psi} \partial_{\psi a} \Phi \Phi_{\Phi a} + \sum_{a,b=1}^{p} g^{a\psi b} \partial_{\psi a} \Phi \partial_{\psi b} \Phi \right) + (r, \theta \text{ and mass terms}) + \sum_{a=1}^{p} \Omega_{\Phi a} \Pi \partial_{\psi a} \Phi$$

$$= \frac{1}{2} \left( -g^{tt} (\partial_t \Phi)^2 + \sum_{a,b=1}^{p} g^{a\psi b} \partial_{\psi a} \Phi \partial_{\psi b} \Phi + (r, \theta \text{ and mass terms}) \right) + \sum_{a=1}^{p} \Omega_{\Phi a} \Pi \partial_{\psi a} \Phi.$$

(B.1)

After eliminating the time derivative terms by means of momentum $\Pi$ in (2.2), we obtain

$$\mathcal{H}' = \frac{1}{2} \left( -\frac{1}{g^{tt}} \left( \Pi + \sum_{a=1}^{p} g^{a\psi} \partial_{\psi a} \Phi \right)^2 + \sum_{a,b=1}^{p} g^{a\psi b} \partial_{\psi a} \Phi \partial_{\psi b} \Phi \right) + (r, \theta \text{ and mass terms}) + \sum_{a=1}^{p} \Omega_{\Phi a} \Pi \partial_{\psi a} \Phi$$

$$= \frac{1}{2} \left( -\frac{1}{g^{tt}} \Pi^2 + \sum_{a,b=1}^{p} \tilde{g}^{a\psi b} \partial_{\psi a} \Phi \partial_{\psi b} \Phi + (r, \theta \text{ and mass terms}) \right) + \sum_{a=1}^{p} (\Omega_{\Phi a} - \Omega_a) \Pi \partial_{\psi a} \Phi,$$

(B.2)

where $\tilde{g}^{a\psi b}$ is the inverse metric of azimuthal angle components $g_{a\psi b}$ defined by

$$g_{a\psi b} \tilde{g}^{a\psi b} = \delta^e_e, \quad \tilde{g}^{a\psi b} := g^{a\psi b} - \frac{g^{a\psi a} g^{\psi b}}{g^{tt}}.$$

(B.3)

More conveniently, we introduce the contravariant components of special metric $h^{ij}$ defined by

$$h^{ij} := \left( g^{\psi a} g^{a\psi} \frac{1}{g_{\psi a\psi a}} \right) = \left( \tilde{g}^{a\psi} \tilde{g}_{a\psi} \right).$$

(B.4)

which is the expression of the contravariant metric component in (2.8). The desired form for $\mathcal{H}'$ of equation (2.7) is obtained using these quantities as

$$\mathcal{H}' = \frac{1}{2} \left( -\frac{\Pi^2}{g^{tt}} + \sum_{i,j=1}^{D-1} h^{ij} \partial_i \Phi \partial_j \Phi + \mu^2 \Phi^2 \right) + \sum_{a=1}^{p} (\Omega_{\Phi a} - \Omega_a) \Pi \partial_{\psi a} \Phi,$$

(B.5)

where this new Hamiltonian density is expressed as positive definite under the near horizon approximation $\Omega_{\Phi a} \simeq \Omega_a$. 
Appendix C. Derivation of diagonal line element form (2.23)

Using the definition of angular velocity (1.3) and orthogonality relation (A.1), we obtain another metric–angular momentum identity relation:

\[ g_{tt} = \frac{1}{g_{tt}} + \sum_{a,b=1}^{p} g_{\phi_a\phi_b} \Omega_{\phi_a} \Omega_{\phi_b} \]

\[ g_{t\phi_a} = - \sum_{b=1}^{p} g_{\phi_a\phi_b} \Omega_{\phi_b} \]

(C.1)

Then line element (1.2) is rewritten using identity relations as

\[ ds^2 = g_{tt} d\tau^2 + 2 \sum_{a=1}^{p} g_{t\phi_a} d\tau d\phi_a + (r, \theta \text{ terms}) \]

\[ = \left( \frac{1}{g_{tt}} + \sum_{a,b=1}^{p} g_{\phi_a\phi_b} \Omega_{\phi_a} \Omega_{\phi_b} \right) d\tau^2 - 2 \sum_{a,b=1}^{p} g_{\phi_a\phi_b} \Omega_{\phi_a} d\phi_a + (r, \theta \text{ terms}) \]

\[ + \left( r, \theta \text{ terms} \right), \]

which completes the derivation of equation (2.23) and shows the diagonal form of the line element with respect to modified azimuthal angles \( \phi_a - \Omega_a \).

Appendix D. Contravariant metric for Kerr–AdS black holes in five dimensions in (3.13)–(3.15)

We derive the contravariant metric components with respect to time and azimuthal angles in five dimensions, which are defined in general form as

\[ g'''' = \frac{\gamma_t}{\Gamma}, \quad g''^{\phi_a} = \frac{\gamma_{\phi_a}}{\Gamma}, \quad g''^{\phi_b} = \frac{\gamma_{\phi_b}}{\Gamma}, \]

(D.1)

where

\[ \gamma_t = g_{\phi_a\phi_a} g_{\phi_b\phi_b} - g_{\phi_a\phi_b}^2 \]

\[ \gamma_{\phi_a} = g_{\phi_a} g_{\phi_b\phi_b} - g_{\phi_a\phi_b} g_{\phi_b} \]

\[ \gamma_{\phi_b} = g_{\phi_b} g_{\phi_a\phi_a} - g_{\phi_a\phi_b} g_{\phi_a} \]

(D.2)

and the determinant of the time and azimuthal angle components is defined as

\[ \Gamma = g_{tt} g_{\phi_a\phi_a} g_{\phi_b\phi_b} + 2 g_{t\phi_a} g_{\phi_a\phi_b} g_{\phi_b} - g_{t\phi_a}^2 g_{\phi_b\phi_b} - g_{t\phi_b}^2 g_{\phi_a\phi_a} - g_{\phi_a\phi_b}^2 g_{tt} \]

\[ = g_{tt} \gamma_t + g_{t\phi_a} \gamma_{\phi_a} + g_{t\phi_b} \gamma_{\phi_b}. \]

(D.3)

Inserting the explicit five-dimensional Kerr–AdS black hole solution (3.12), factors \( \gamma \) in (D.2) are expressed as

\[ \gamma_t \times \frac{r^2 \rho^2 \Xi_a^2 \Xi_b^2}{\sin^2 \theta \cos^2 \theta} = r^2 \Delta (-Ab^2 \cos^2 \theta - B\alpha^2 \sin^2 \theta) \]

\[ + \ell^{-2}a^2b^2(A \cos^2 \theta + B \sin^2 \theta)) + \Delta_0 A^2 B^2, \]

(D.4)
\[ \gamma_\alpha \times \frac{r^2 \rho^2 \Xi_a \Xi_b}{a \sin^2 \theta \cos^2 \theta} = r^2 \Delta (-B + \epsilon^{-2} b^2 (A \cos^2 \theta + B \sin^2 \theta)) + \Delta_a AB^2, \]  
(D.5)

\[ \gamma_\alpha \times \frac{r^2 \rho^2 \Xi_a \Xi_b}{b \sin^2 \theta \cos^2 \theta} = r^2 \Delta (-A + \epsilon^{-2} a^2 (A \cos^2 \theta + B \sin^2 \theta)) + \Delta_a A^2 B, \]  
(D.6)

and \( \Gamma \) in (D.3) is expressed as

\[ \Gamma = -\frac{\sin^2 \theta \cos^2 \theta r^2 \Delta \Delta_a}{\Xi_a \Xi_b}. \]  
(D.7)

The contravariant metrics of time and azimuthal components are obtained using equations (D.1)–(D.7):

\[ g^{tt} = \frac{1}{r^4 \rho^2 \Delta} (A b^2 \cos^2 \theta + B a^2 \sin^2 \theta - \epsilon^{-2} a^2 b^2 (A \cos^2 \theta + B \sin^2 \theta)) - \frac{1}{r^4 \rho^2 \Delta} A^2 B^2, \]  
(D.8)

\[ g^{\phi_a} = \frac{a \Xi_a}{r^2 \rho^2} \left( \frac{1}{\Delta} (B - \epsilon^{-2} b^2 (A \cos^2 \theta + B \sin^2 \theta)) - \frac{1}{r^2 \Delta} A B^2 \right), \]  
(D.9)

\[ g^{\phi_b} = \frac{b \Xi_b}{r^2 \rho^2} \left( \frac{1}{\Delta} (A - \epsilon^{-2} a^2 (A \cos^2 \theta + B \sin^2 \theta)) - \frac{1}{r^2 \Delta} A^2 B \right), \]  
(D.10)

which are expressions (3.13)–(3.15).

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