Discrete Differential Calculus on (Co-)Simplicial Complexes and Generalized (Co-)Homology

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Abstract

Let $V$ be a finite set. Let $K$ be a simplicial complex with its vertices in $V$. In this paper, we discuss some differential calculus on $V$. We construct some generalized homology groups of $K$ by using the differential calculus on $V$. Moreover, we define a co-simplicial complex to be the complement of a simplicial complex in the complete hypergraph on $V$. Let $L$ be a co-simplicial complex with its vertices in $V$. We construct some generalized cohomology groups of $L$ by using the differential calculus on $V$.

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1 Introduction

Simplicial complexes play an important and fundamental role in algebraic topology. So far, topologists have developed the homology and cohomology theory for simplicial complexes. We refer to [23, Chapter 1] and [19, Section 2.1] for a systematic introduction to the simplicial homology theory. We also refer to [23, Section 42, Chapter 5] and [19, Section 3.1 and Section 3.2] for an introduction to the simplicial cohomology theory. On the other hand, since 1950’s, topologists have developed the simplicial homotopy theory (for example, we may refer to [10, 11, 12, 22, 30]), which has been found to have significant applications in various topics in algebraic and geometric topology (for example, we refer to [5, 20, 25] for some of such applications). In simplicial homotopy theory, simplicial complexes are the fundamental models for simplicial sets.

The notion of hypergraphs is a higher dimensional generalization of the notion of graphs (cf. [11, 24]). In a graph, an edge consists of two vertices while in a hypergraph, a hyperedge is allowed to be consisted of $n$-vertices for any $n \geq 1$. From a topological point of view, a hypergraph can be obtained by deleting some non-maximal faces in a simplicial complex (cf. [3, 24]) while a simplicial complex is a special hypergraph with no non-maximal faces missing. The complete hypergraph $\Delta[V]$ on a finite set $V$ has its set of the hyperedges as all the non-empty subsets of $V$ (cf. Definition 5). A simplicial complex with all of its vertices in $V$ has its set of the simplices as a subset of $\Delta[V]$. The complement of the set of the simplices in $\Delta[V]$ is called a co-simplicial complex (cf. Definition 8 and Proposition 2.1).
Differential calculus is an important tool in (co)homology theory. In some textbooks in algebraic topology (for example, [2, 21]), the methods of differential calculus have been applied to the (co)homology theory of differentiable manifolds and fibre bundles. During the 1990s, A. Dimakis and F. Müller-Hoissen [7, 8, 9] initiated the study of discrete differential calculus on discrete sets with a motivation from theoretical physics. During the 2010s, A. Dimakis and F. Müller-Hoissen [7, 8, 9], Alexander Grigor’yan, Yong Lin and Shing-Tung Yau [13], Alexander Grigor’yan, Yong Lin, Yuri Muranov and Shing-Tung Yau [14, 15, 16] and Alexander Grigor’yan, Yuri Muranov and Shing-Tung Yau [17, 18] developed the discrete differential calculus methods on discrete sets and applied the methods to the study of digraphs.

In this paper, we apply the method of the (discrete) differential calculus and give some generalized homology for simplicial complexes as well as generalized cohomology for co-simplicial complexes.

Let $V$ be a finite set. Let $K$ be a simplicial complex whose the set of the vertices is a subset of $V$. Let $n \geq 0$. Let $v_0 v_1 \ldots v_n$ be an $n$-simplex of $K$. The usual boundary operator (cf. [19, p. 105], [23, p. 28]) is given by

$$\partial_n(v_0 v_1 \ldots v_n) = \sum_{i=0}^{n} (-1)^i v_0 \ldots \hat{v}_i \ldots v_n.$$  \hspace{1cm} (1.1)

We generalize the usual boundary operator and define a weighted boundary operator

$$\frac{\partial}{\partial v}(v_0 v_1 \ldots v_n) = \sum_{i=0}^{n} (-1)^i \delta(v, v_i)v_0 \ldots \hat{v}_i \ldots v_n$$

with respect to any fixed vertex $v \in V$. Note that

$$\partial_n = \sum_{v \in V} \frac{\partial}{\partial v}.$$  

We take the exterior algebra $T^*(V)$ generated by the $\frac{\partial}{\partial v}$’s for all $v \in V$. We prove in Subsection 4.2 that for any $t \geq 0$ and any $\alpha \in T^{2t+1}_*(V)$, there is a generalized homology group of $K$ with respect to $\alpha$. Moreover, we prove in Theorem 4.6 that for any $s \geq 0$ and any $\beta \in T^s_*(V)$, the element $\beta$ induces a homomorphism between the corresponding generalized homology groups.

We point out that the generalized homology groups which will be investigated in Subsection 4.2 are generalizations of the weighted homology groups investigated by Robert. J. MacG. Dawson [6], Shiquan Ren, Chengyuan Wu and Jie Wu in [27, 28] and Shiquan Ren, Chengyuan Wu, Jie Wu and Kelin Xia [29] for weighted simplicial complexes. Let $f$ be a real function on $V$. We take $t = 1$ and

$$\alpha = \sum_{v \in V} f(v) \frac{\partial}{\partial v}$$

in Definition 21 Subsection 4.2 Then the generalized homology groups of the simplicial complex $K$ with respect to $\alpha$, which will be investigated in Subsection 4.2 give the weighted homology groups of the weighted simplicial complex $(K, f)$ which have been investigated in [27, 28, 29].
On the other hand, let $L$ be a co-simplicial complex whose the set of the vertices is a subset of $V$. For any $v \in V$, we consider the adjoint linear map $dv$ of the element $\partial v$ in $T_*(V)$. We define $T^*(V)$ as the exterior algebra generated by the $dv$’s for all $v \in V$. We prove in Subsection 4.3 that for any $t \geq 0$ and any $\omega \in T^{2t+1}(V)$, there is a generalized cohomology group of $L$ with respect to $\omega$. Moreover, we prove in Theorem 4.8 that for any $s \geq 0$ and any $\mu \in T^{2s}(V)$, the element $\mu$ induces a homomorphism between the generalized cohomology groups.

The remaining part of this paper is organized as follows. In Section 2, we introduce the definitions of hypergraphs, simplicial complexes and co-simplicial complexes. In Section 3 as a preparation for Section 4, we discuss some differential calculus for paths on discrete sets. In Section 4, we define the generalized homology groups for simplicial complexes in Definition 21 and define the generalized cohomology groups for co-simplicial complexes in Definition 22. We prove Theorem 4.6 and Theorem 4.8. Finally, in Section 5, we give some examples for Section 4.

2 Hypergraphs, Simplicial Complexes and Co-Simplicial Complexes

In this section, we review the definitions of hypergraphs and simplicial complexes. We give the definition of co-simplicial complexes. We explain the topological meanings of these definitions by introducing the geometric realizations of hypergraphs, simplicial complexes and co-simplicial complexes.

2.1 Combinatorial Models of Hypergraphs, Simplicial Complexes, and Co-Simplicial Complexes

Let $V$ be a discrete set whose elements are called vertices. Suppose $V$ has a total order $\prec$. Let $n \geq 0$ be a non-negative integer.

Definition 1. An $n$-hyperedge on $V$ is a sequence

$$\sigma^{(n)} = v_0v_1 \ldots v_n \quad (2.1)$$

where $v_0 \prec v_1 \prec \cdots \prec v_n$ are vertices in $V$. For simplicity, an $n$-hyperedge is also called a hyperedge and $\sigma^{(n)}$ in (2.1) is also denoted as $\sigma$.

Remark 1: By Definition 1, a 0-hyperedge on $V$ is just a single vertex $v_0$ in $V$ and a 1-hyperedge on $V$ is just an edge $v_0v_1$ in the complete graph on $V$.

Definition 2. The complete $n$-uniform hypergraph $\Delta_n(V)$ on $V$ is the collection of all the possible $n$-hyperedges on $V$. In other words, $\Delta(V)$ consists of all the subsets of $V$ with $n$-vertices:

$$\Delta_n(V) = \{v_0v_1 \ldots v_n \mid v_0, v_1, \ldots, v_n \in V \text{ and } v_0 \prec v_1 \prec \cdots \prec v_n\}.$$
Remark 2: In particular, let $n = 2$ in Definition 2. Then the complete 2-uniform hypergraph $\Delta_2(V)$ is just the complete graph on $V$.

Definition 3. An $n$-uniform hypergraph $\mathcal{H}^{(n)}$ on $V$ is a collection of some of the $n$-hyperedges on $V$. In other words, $\mathcal{H}^{(n)}$ consists of some of the subsets of $V$ with $n$-vertices:

$$\mathcal{H}^{(n)} \subseteq \{v_0v_1 \cdots v_n \mid v_0, v_1, \cdots, v_n \in V \text{ and } v_0 \prec v_1 \prec \cdots \prec v_n \}.$$ 

Definition 4. A hypergraph on $V$ is a disjoint union

$$\mathcal{H} = \bigcup_{n \geq 0} \mathcal{H}^{(n)} \quad (2.2)$$

where $\mathcal{H}^{(n)}$ is an $n$-uniform hypergraph on $V$ for each $n \geq 0$.

Definition 5. The complete hypergraph $\Delta[V]$ on $V$ is the collection of all the possible hyperedges on $V$. In other words, $\Delta[V]$ consists of all the non-empty finite subsets of $V$.

Remark 3: It is direct that we have a disjoint union

$$\Delta[V] = \bigcup_{n \geq 0} \Delta_n(V).$$

Definition 6. Let $H_1$ and $H_2$ be two hypergraphs on $V$. The complement of $H_1$ in $H_2$ is defined to be a hypergraph $H_2 \setminus H_1$ on $V$ by

$$H_2 \setminus H_1 = \{\sigma \text{ is a hyperedge on } V \mid \sigma \in H_2 \text{ and } \sigma \notin H_1\}.$$ 

Definition 7. A simplicial complex (pl. simplicial complexes) $K$ on $V$ is a hypergraph on $V$ such that for any hyperedge $\sigma \in K$ and any non-empty subset $\tau \subseteq \sigma$, we always have $\tau \in K$. A hyperedge in a simplicial complex is also called a simplex (pl. simplices).

Definition 8. A co-simplicial complex (pl. co-simplicial complexes) $L$ on $V$ is a hypergraph on $V$ such that for any hyperedge $\sigma \in L$ and any hyperedge $\tau$ on $V$ satisfying $\sigma \subseteq \tau$, we always have $\tau \in L$. A hyperedge in a co-simplicial complex is also called a co-simplex (pl. co-simplices).

Remark 4: From Definition 3, Definition 7 and Definition 8, it is direct that

- for any $n \geq 1$, an $n$-uniform hypergraph is not a simplicial complex;
- for any $n \leq \#V - 1$ where $\#V$ is the cardinality of $V$ (here $\#V$ can be either finite or infinite), an $n$-uniform hypergraph is not a co-simplicial complex.

Remark 5: From Definition 5, Definition 7 and Definition 8, it is direct that

- the complete hypergraph $\Delta[V]$ is a simplicial complex on $V$;
- the complete hypergraph $\Delta[V]$ is a co-simplicial complex on $V$.

Proposition 2.1. Let $\Delta[V]$ be the complete hypergraph on $V$. Let $K$ be a simplicial complex on $V$. Let $L$ be a co-simplicial complex on $V$. Then both the followings are satisfied:
(i). $\Delta[V] \setminus K$ is a co-simplicial complex on $V$;
(ii). $\Delta[V] \setminus L$ is a simplicial complex on $V$.

Proof. (i). Let $\sigma \in \Delta[V] \setminus K$. Let $\tau$ be a hyperedge on $V$ such that $\sigma \subseteq \tau$. In order to prove that $\Delta[V] \setminus K$ is a co-simplicial complex, it suffices to prove $\tau \in \Delta[V] \setminus K$. Suppose to the contrary, $\tau \notin \Delta[V] \setminus K$. Then $\tau \in K$. Since $K$ is a simplicial complex and $\sigma \subseteq \tau$, we have $\sigma \in K$. This contradicts $\sigma \in \Delta[V] \setminus K$. Therefore, $\tau \in \Delta[V] \setminus K$, which implies that $\Delta[V] \setminus K$ is a co-simplicial complex.

(ii). Let $\sigma \in \Delta[V] \setminus L$. Let $\tau$ be a hyperedge on $V$ such that $\tau \subseteq \sigma$. In order to prove that $\Delta[V] \setminus L$ is a simplicial complex, it suffices to prove $\tau \in \Delta[V] \setminus L$. Suppose to the contrary, $\tau \notin \Delta[V] \setminus L$. Then $\tau \in L$. Since $L$ is a co-simplicial complex and $\tau \subseteq \sigma$, we have $\sigma \in L$. This contradicts $\sigma \in \Delta[V] \setminus L$. Therefore, $\tau \in \Delta[V] \setminus L$, which implies that $\Delta[V] \setminus L$ is a co-simplicial complex.

Example 2.2. Consider the set $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$. Then

(i). $\sigma^{(3)} = v_0v_2v_4v_5$ is 3-hyperedge on $V$;
(ii). $H^{(2)} = \{v_0v_2v_3, v_1v_2v_3, v_1v_3v_5, v_2v_4v_5\}$ is a 2-uniform hypergraph on $V$;
(iii). $H = \{v_0, v_0v_1, v_0v_2, v_1v_2, v_2v_3v_4v_5\}$ is a hypergraph on $V$;
(iv). $\Delta[V] = \{v_i \mid 0 \leq i \leq 5\} \cup \{v_iv_j \mid 0 \leq i < j < k \leq 5\} \cup \{v_iw_0v_kv_l \mid 0 \leq i < j < k < l \leq 5\} \cup \{v_iv_jw_0v_kv_l \mid 0 \leq j < k < l < s \leq 5\} \cup \{v_0v_1v_2v_3v_4v_5\}$;
(v). $K = \{v_0, v_0v_1, v_0v_2, v_1v_2, v_0v_1v_2\}$ is a simplicial complex on $V$;
(vi). $L = \{v_0v_1v_2v_4, v_0v_1v_2v_3v_5, v_0v_1v_2v_3v_4, v_0v_1v_2v_4v_5, v_0v_1v_2v_3v_4v_5\}$ is a co-simplicial complex on $V$.

Example 2.3. Consider the set $V = \mathbb{Z}$ of all the integers. Then

(i). for any $p \in \mathbb{Z}$ and any $q \geq 0$, the sequence $p(p + 1)\ldots(p + q)$ of subsequent integers is a $q$-hyperedge on $V$;
(ii). for any $q \geq 0$, the collection $H^{(q)} = \{p(p + 1)\ldots(p + q) \mid p \equiv 1 \pmod{3}\}$ of sequences of subsequent integers is a $q$-uniform hypergraph on $V$;
(iii). the collection $H = \{p(p + 1)\ldots(p + q) \mid p \equiv 1 \pmod{3} \text{ and } 2 \leq q \leq 5\}$ of sequences of subsequent integers is a hypergraph on $V$;
(iv). $\Delta[V] = \{i_0 \in \mathbb{Z}\} \cup \{i_0i_1 \in \mathbb{Z} \times \mathbb{Z} \mid i_0 < i_1\} \cup \{i_0i_1i_2 \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \mid i_0 < i_1 < i_2\} \cup \cdots$;
(v). the collection $K = \{p(p + 1)\ldots(p + q) \mid p \in \mathbb{Z} \text{ and } 0 \leq q \leq 5\}$ of sequences of subsequent integers is a simplicial complex on $V$;
(vi). the collection $L = \{p(p + 1)\ldots(p + q) \mid p \in \mathbb{Z} \text{ and } q > 5\}$ of sequences is a co-simplicial complex on $V$.
2.2 Geometric Realizations of Hypergraphs, Simplicial Complexes, and Co-Simplicial Complexes

Throughout this subsection, we suppose $V$ is a finite set. Let $N = \#V$ be the number of the vertices in $V$. We enumerate the vertices in $V$ as $v_0, v_1, \ldots, v_N$ such that $v_0 \prec v_1 \prec \cdots \prec v_N$.

Consider an Euclidean space $\mathbb{R}^N$ with an orthonormal basis $e_1, e_2, \ldots, e_N$. For each $1 \leq i \leq N$, take the point

$$p_i = (0, \cdots, 0, 1, 0, \cdots, 0)$$

in $\mathbb{R}^N$ where the $i$-th coordinate is 1 and all the other coordinates are 0. Then the vector from 0 to $p_i$ in $\mathbb{R}^N$ is the unit vector $e_i$. Define a map

$$\rho : V \rightarrow \mathbb{R}^N$$

by $\rho(v_i) = p_i$ for $i = 1, 2, \ldots, N$.

**Definition 9.** Let $0 \leq k_0 \leq N$. Let $\sigma^{(0)} = v_{k_0}$ be a 0-hyperedge (which is just a vertex) on $V$. The geometric realization of $\sigma$, denoted by $|\sigma|$, is the point $\rho(v_{k_0})$ in $\mathbb{R}^N$.

**Definition 10.** Let $1 \leq n \leq N - 1$. Let $0 \leq k_0 < k_1 < \cdots < k_n \leq N$. Let $\sigma^{(n)} = v_{k_0} v_{k_1} \cdots v_{k_n}$ be an $n$-hyperedge on $V$. The geometric realization of $\sigma$, denoted by $|\sigma|$, is a subset of $\mathbb{R}^N$ given by

$$|\sigma| = \left\{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid \text{there exist } 0 < t_0, t_1, \ldots, t_n < 1 \right. \right.$$  

$$\left. \text{such that } \sum_{i=0}^{n} t_i = 1 \text{ and } (x_1, x_2, \ldots, x_N) = \sum_{i=0}^{n} t_i \rho(v_{k_i}) \right\}.$$  

**Lemma 2.4.** Let $V$ be a finite set consisting of $N$ vertices. Let $1 \leq n \leq N - 1$. Let $\sigma$ be an $n$-hyperedge on $V$. Then the geometric realization $|\sigma|$ is an open subset of an $n$-hyperplane in $\mathbb{R}^N$.

**Proof.** Suppose $\sigma^{(n)} = v_{k_0} v_{k_1} \ldots v_{k_n}$. Let $P(\rho(\sigma))$ be the $n$-dimensional hyperplane in $\mathbb{R}^N$ affinely spanned by the $(n+1)$-points $\rho(v_{k_0}), \rho(v_{k_1}), \ldots, \rho(v_{k_n})$. We observe that the geometric realization $|\sigma|$ is the interior of the convex hull of the subset

$$\rho(\sigma) = \{\rho(v_{k_0}), \rho(v_{k_1}), \ldots, \rho(v_{k_n})\}$$  

of $P(\rho(\sigma))$. Thus $|\sigma|$ is an open subset of the $n$-hyperplane $P(\rho(\sigma))$ in $\mathbb{R}^N$. \hfill $\square$

**Lemma 2.5.** Let $V$ be a finite set consisting of $N$ vertices. Let $\sigma$ and $\tau$ be two distinct hyperedges on $V$. Then $|\sigma| \cap |\tau| = \emptyset$.

**Proof.** Suppose $\sigma = v_{k_0} v_{k_1} \ldots v_{k_n}$ and $\tau = u_{l_0} u_{l_1} \ldots u_{l_m}$. Let $p \in |\sigma|$ and $q \in |\tau|$. Then the coordinate of $p$ in $\mathbb{R}^N$ is

$$(0, \cdots, 0, t_0, 0, \cdots, 0, t_1, 0, \cdots)$$  

where

$$t_0 < t_1$$  

and

$$t_0 < t_1$$  

for $0 \leq i < j \leq n$. Thus $p \neq q$, so $|\sigma| \cap |\tau| = \emptyset$. \hfill $\square$
• for each $0 \leq i \leq n$, the $k_i$-th coordinate is $t_i > 0$ such that $t_0 + t_1 + \cdots + t_n = 1$;

• all the other coordinates are 0.

Similarly, the coordinate of $q$ in $\mathbb{R}^N$ is

$$(0, \cdots, 0, s_0, 0, \cdots, 0, s_1, 0, \cdots)$$

where

• for each $0 \leq j \leq m$, the $l_j$-th coordinate is $s_j > 0$ such that $s_0 + s_1 + \cdots + s_m = 1$;

• all the other coordinates are 0.

Since $\sigma \neq \tau$, we have $(k_0, k_1, \ldots, k_n) \neq (l_0, l_1, \ldots, l_m)$. Consequently, the coordinates of $p$ given by (2.4) do not equal to the coordinates of $q$ given by (2.5), which implies $p \neq q$. Therefore, $|\sigma| \cap |\tau| = \emptyset$.

**Definition 11.** Let $\mathcal{H}$ be a hypergraph on $V$. The geometric realization of $\mathcal{H}$, denoted by $|\mathcal{H}|$, is a subset of $\mathbb{R}^N$ given by

$$|\mathcal{H}| = \bigcup_{\sigma \in \mathcal{H}} \, |\sigma|.$$  

That is, the geometric realization $|\mathcal{H}|$ is the union of the geometric realizations $|\sigma|$ for all the hyperedges $\sigma \in \mathcal{H}$.

**Remark 6:** By Lemma 2.5, the union in (2.6) is a disjoint union.

**Lemma 2.6.** Let $V$ be a finite set consisting of $N$ vertices. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypergraphs on $V$. Let $\mathcal{H}_2 \setminus \mathcal{H}_1$ be the complement hypergraph of $\mathcal{H}_1$ in $\mathcal{H}_2$. Then the geometric realizations of $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_2 \setminus \mathcal{H}_1$ satisfy

$$|\mathcal{H}_2 \setminus \mathcal{H}_1| = |\mathcal{H}_2| \setminus |\mathcal{H}_1|.$$  

**Proof.** It is direct to verify

$$|\mathcal{H}_2 \setminus \mathcal{H}_1| = \bigcup_{\sigma \in \mathcal{H}_2 \setminus \mathcal{H}_1} |\sigma| = \left( \bigcup_{\sigma \in \mathcal{H}_2} |\sigma| \right) \setminus \left( \bigcup_{\sigma \in \mathcal{H}_1} |\sigma| \right) = |\mathcal{H}_2| \setminus |\mathcal{H}_1|.$$  

Here the second equality follows from Lemma 2.5 (or Remark 6) while the first and the third equalities follow from Definition [11].

**Proposition 2.7.** Let $V$ be a finite set consisting of $N$ vertices. Let $\mathcal{K}$ be a simplicial complex on $V$. Then the geometric realization $|\mathcal{K}|$ is a closed subset of $\mathbb{R}^N$.

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1We write $(k_0, k_1, \ldots, k_n) = (l_0, l_1, \ldots, l_m)$ if $n = m$ and $k_i = l_i$ for each $0 \leq i \leq m$. We write $(k_0, k_1, \ldots, k_n) \neq (l_0, l_1, \ldots, l_m)$ otherwise.
Proof. Let $K$ be a simplicial complex. To prove that $|K|$ is a closed subset of $\mathbb{R}^N$, we take $x$ to be a point in the boundary of $|K|$. It suffices to prove $x \in |K|$. Since $x$ is in the boundary of $|K|$, it follows that there exists $\sigma \in K$ such that $x$ is in the boundary of $|\sigma|$.

CASE 1. $\sigma$ is a 0-simplex.

Then $x$ is the single point in $|\sigma|$, which implies $x \in |K|$.

CASE 2. $\sigma$ is an $n$-simplex for $n \geq 1$.

Then $x$ is in the boundary of the convex hull of $\rho(\sigma)$ (recall $\rho(\sigma)$ is given by (2.3) in the proof of Lemma 2.4). Thus there exists a non-empty proper subset $\tau$ of $\sigma$ such that $x$ is in the convex hull of $\rho(\tau)$. Hence without loss of generality, we assume $\tau$ is the smallest subset of $\sigma$ such that $x$ is in the convex hull of $\rho(\tau)$. Since $K$ is a simplicial complex, we have that $\tau$ is a simplex in $K$.

SUBCASE 2.1. $\tau$ is a 0-simplex.

Then $x$ is the single point in $|\tau|$, which implies $x \in |K|$.

SUBCASE 2.2. $\tau$ is an $m$-simplex for some $1 \leq m \leq n - 1$.

Then $x$ is in the convex hull of $\rho(\tau)$. Moreover, for any non-empty proper subset $\delta$ of $\tau$, since $\tau$ is assumed to be smallest, we have that $x$ is not in the convex hull of $\rho(\delta)$. It follows that $x$ is in the interior of the the convex hull of $\rho(\tau)$. Therefore, $x \in |\tau|$, which implies $x \in |K|$.

Summarizing the above cases, we have $x \in |K|$. Thus $|K|$ is a closed subset of $\mathbb{R}^N$. □

Corollary 2.8. Let $V$ be a finite set consisting of $N$ vertices. Let $L$ be a co-simplicial complex on $V$. Then the geometric realization $|L|$ is an open subset of the geometric realization $|\Delta[V]|$.

Proof. Since $\Delta[V]$ is a simplicial complex (cf. Remark 5), it follows from Proposition 2.7 that $|\Delta[V]|$ is a closed subset of $\mathbb{R}^N$. By Proposition 2.1, the complement hypergraph $\Delta[V] \setminus L$ is a simplicial complex on $V$. Thus also by Proposition 2.7, the geometric realization $|\Delta[V] \setminus L|$ is a closed subset of $\mathbb{R}^N$ as well. In addition, since each simplex of $\Delta[V] \setminus L$ is a simplex of $\Delta[V]$, it follows that $|\Delta[V] \setminus L|$ is a closed subset of $|\Delta[V]|$. Note that $\Delta[V]$ is the disjoint union of $L$ and $\Delta[V] \setminus L$. Consequently, it follows from Lemma 2.8 that $|L|$ is the complement of $|\Delta[V] \setminus L|$ in $|\Delta[V]|$. Therefore, $|L|$ is an open subset of $|\Delta[V]|$. □

Example 2.9. Let $V = \{v_0, v_1, v_2\}$.

(i). The complete hypergraph on $V$ is

$$\Delta[V] = \{v_0, v_1, v_2, v_0v_1, v_1v_2, v_0v_2, v_0v_1v_2\}.$$ 

The geometric realization $|\Delta[V]|$ is in the following figure:
(ii). Let the hypergraph $H = \{v_1, v_2, v_0v_1, v_0v_1v_2\}$. The geometric realization $|H|$ is in the following figure:

![Hypergraph](image1)

(iii). Let the simplicial complex $K = \{v_0, v_1, v_2, v_0v_1, v_0v_2, v_1v_2\}$. Then $\Delta[V \setminus K] = \{v_0v_1v_2\}$ is a co-simplicial complex. The geometric realizations $|K|$ and $|\Delta[V \setminus K]|$ are in the following figure:

![Simplicial Complex](image2)

From the figure we can see that $|K|$ is closed in $|\Delta[V]|$, $|\Delta[V \setminus K]|$ is open in $|\Delta[V]|$, and $|\Delta[V]|$ is the disjoint union of $|K|$ and $|\Delta[V \setminus K]|$.

(iv). Let the co-simplicial complex $L = \{v_0v_1, v_0v_2, v_0v_1v_2\}$. Then $\Delta[V \setminus L] = \{v_0, v_1, v_2, v_1v_2\}$ is a simplicial complex. The geometric realizations $|L|$ and $|\Delta[V \setminus L]|$ are in the following figure:

![Cosimplicial Complex](image3)

From the figure we can see that $|L|$ is open in $|\Delta[V]|$, $|\Delta[V \setminus L]|$ is closed in $|\Delta[V]|$, and $|\Delta[V]|$ is the disjoint union of $|L|$ and $|\Delta[V \setminus L]|$.

## 3 Differential Calculus for Paths on Discrete Sets

In this section, we review the definitions of the paths and the elementary paths on a discrete set (cf. [14]). By applying some discrete differential calculus, we construct certain chain complexes and co-chain complexes for the space of paths on a discrete set.

### 3.1 Paths on Discrete Sets

Throughout this section, we let $V$ be a discrete set. Let $n \geq 0$ be a non-negative integer.

**Definition 12.** (cf. [14] Definition 2.1). An elementary $n$-path on $V$ is an ordered sequence $v_0v_1 \ldots v_n$ of $n + 1$ vertices in $V$. Here for any $0 \leq i < j \leq n$, we do not require $v_i \prec v_j$, $v_j \prec v_i$, or $v_i \neq v_j$. 


Definition 13. (cf. [13] Definition 2.2). A formal linear combination of elementary $n$-paths on $V$ with coefficients in the real numbers $\mathbb{R}$ is called an $n$-path on $V$.

Notation 1. (cf. [14] Subsection 2.1). Denote by $\Lambda_n(V)$ the vector space of all $n$-paths. Then any element in $\Lambda_n(V)$ is of the form
\[ \sum_{v_0, v_1, \ldots, v_n \in V} r_{v_0 v_1 \ldots v_n} v_0 v_1 \ldots v_n, \quad r_{v_0 v_1 \ldots v_n} \in \mathbb{R}. \]

Notation 2. Letting $n$ run over all non-negative integers, we have a graded vector space $\Lambda^*(V) = \bigoplus_{n=0}^{\infty} \Lambda_n(V)$.

Notation 3. For each $n \geq 0$, we have a canonical inner product
\[ \langle \cdot , \cdot \rangle : \Lambda_n(V) \times \Lambda_n(V) \rightarrow \mathbb{R} \]
on $\Lambda_n(V)$ by
\[ \langle u_0 u_1 \ldots u_n , v_0 v_1 \ldots v_n \rangle = \prod_{i=0}^{n} \delta(u_i, v_i). \quad (3.1) \]

Remark 7: It follows from (3.1) that
\begin{itemize}
  \item if $u_0 u_1 \ldots u_n$ and $v_0 v_1 \ldots v_n$ are identically the same elementary $n$-path, then
    \[ \langle u_0 u_1 \ldots u_n , v_0 v_1 \ldots v_n \rangle = 1; \]
  \item if $u_0 u_1 \ldots u_n$ and $v_0 v_1 \ldots v_n$ are not the same elementary $n$-path, then
    \[ \langle u_0 u_1 \ldots u_n , v_0 v_1 \ldots v_n \rangle = 0. \]
\end{itemize}

3.2 Partial Derivatives on Path Spaces

Definition 14. For any $v \in V$, we define the partial derivative of $\Lambda_*(V)$ with respect to $v$ to be a sequence of linear maps
\[ \frac{\partial}{\partial v} : \Lambda_n(V) \rightarrow \Lambda_{n-1}(V), \quad n \geq 0 \]
by letting
\[ \frac{\partial}{\partial v}(v_0 v_1 \ldots v_n) = \sum_{i=0}^{n} (-1)^i \delta(v, v_i) v_0 \ldots \hat{v}_i \ldots v_n. \quad (3.2) \]

Here in (3.2), for any vertices $u, v \in V$, we use the notation $\delta(u, v) = 1$ if $u = v$ and $\delta(u, v) = 0$ if $u \neq v$. We extend (3.2) linearly over $\mathbb{R}$.

Remark 8: By Definition 14 for any distinct vertices $v_0, v_1, \ldots, v_n$ in $V$ we have the followings:
• if \( v_i = v \) for some \( 0 \leq i \leq n \), then
\[
\frac{\partial}{\partial v}(v_0v_1 \ldots v_n) = (-1)^i v_0 \ldots \hat{v}_i \ldots v_n;
\]
• if \( v_i \neq v \) for any \( 0 \leq i \leq n \), then
\[
\frac{\partial}{\partial v}(v_0v_1 \ldots v_n) = 0.
\]

**Lemma 3.1.** ([26 Lemma 2.7]). For any \( u, v \in V \), we have
\[
\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}(v_0v_1 \ldots v_n) = \frac{\partial}{\partial u} \left( \sum_{j=0}^{n} (-1)^j \delta(v, v_j) v_0 \ldots \hat{v}_j \ldots v_n \right)
\]
Proof. Since both \( \frac{\partial}{\partial u} \) and \( \frac{\partial}{\partial v} \) are linear, it follows that both \( \frac{\partial}{\partial u} \circ \frac{\partial}{\partial v} \) and \( \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u} \) are linear as well. Hence in order to prove the identity (3.3) as linear maps from \( \Lambda_n(V; R) \) to \( \Lambda_{n-1}(V; R) \), we only need to verify the identity (3.3) on an elementary \( n \)-path \( v_0v_1 \ldots v_n \). By the definition (3.2), we have
\[
\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}(v_0v_1 \ldots v_n) = \frac{\partial}{\partial u} \left( \sum_{j=0}^{n} (-1)^j \delta(v, v_j) v_0 \ldots \hat{v}_j \ldots v_n \right)
\]
\[
= \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \frac{\partial}{\partial u}(v_0 \ldots \hat{v}_j \ldots v_n)
\]
\[
= \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \sum_{i=0}^{j-1} (-1)^i \delta(u, v_i)(v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n)
\]
\[
+ \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \sum_{i=j+1}^{n} (-1)^{i-1} \delta(u, v_i)(v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n)
\]
\[
= \sum_{0 \leq i < j \leq n} (-1)^{i+j} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n)
\]
\[
+ \sum_{0 \leq j < i \leq n} (-1)^{i+j-1} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \hat{v}_j \ldots \hat{v}_i \ldots v_n).
\]

Similarly,
\[
\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}(v_0v_1 \ldots v_n) = \sum_{0 \leq i < j \leq n} (-1)^{i+j} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n)
\]
\[
+ \sum_{0 \leq j < i \leq n} (-1)^{i+j-1} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \hat{v}_j \ldots \hat{v}_i \ldots v_n).
\]

Therefore, for any elementary \( n \)-path \( v_0v_1 \ldots v_n \) on \( V \), we have
\[
\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}(v_0v_1 \ldots v_n) + \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}(v_0v_1 \ldots v_n) = 0.
\]

Consequently, by the linear property of \( \frac{\partial}{\partial u} \circ \frac{\partial}{\partial v} \) and \( \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u} \), we obtain (3.3). \( \square \)

**Notation 4.** We denote \( \frac{\partial}{\partial u} \circ \frac{\partial}{\partial v} \) as \( \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial u} \) for any \( u, v \in V \).

**Definition 15.** We consider the exterior algebra
\[
T_e(V) = \bigwedge \left( \frac{\partial}{\partial v} \mid v \in V \right)
\]
and call it the differential algebra on \( V \).
We have the following observations:

- The differential algebra $T_*(V)$ is a direct sum
  \[ T_*(V) = \bigoplus_{k=0}^{\infty} T_k(V); \]

- $T_0(V) = \mathbb{R}$ while for each $k \geq 1$, the space $T_k(V)$ is the vector space spanned by all the following elements
  \[ \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} \wedge \cdots \wedge \frac{\partial}{\partial v_k}, \quad v_1, v_2, \ldots, v_k \in V \]
  modulo the relation
  \[ \frac{\partial}{\partial v_1} \wedge \cdots \wedge \frac{\partial}{\partial v_i} \wedge \cdots \wedge \frac{\partial}{\partial v_k} = - \frac{\partial}{\partial v_1} \wedge \cdots \wedge \frac{\partial}{\partial v_{i+1}} \wedge \cdots \wedge \frac{\partial}{\partial v_k} \]
  for any $1 \leq i \leq k-1$;

- The exterior product
  \[ \wedge : \ T_k(V) \times T_l(V) \rightarrow T_{k+l}(V), \quad k, l \geq 1, \]
  is the composition of linear maps. It is given by
  \[ (\frac{\partial}{\partial v_1} \wedge \cdots \wedge \frac{\partial}{\partial v_k}) \wedge (\frac{\partial}{\partial u_1} \wedge \cdots \wedge \frac{\partial}{\partial u_l}) = \frac{\partial}{\partial v_1} \wedge \cdots \wedge \frac{\partial}{\partial v_k} \wedge \frac{\partial}{\partial u_1} \wedge \cdots \wedge \frac{\partial}{\partial u_l}, \]
  which extends bi-linearly over $\mathbb{R}$.

- For any $k \geq 1$ and any $\alpha \in T_k(V)$, we have that $\alpha$ gives a sequence of linear maps
  \[ \alpha_n : \Lambda^n(V) \rightarrow \Lambda_{n-k}(V), \quad n \geq 0. \quad (3.4) \]
  Here we adopt the notation that $\Lambda^{-n}(V) = 0$ for any $n \geq 0$. Precisely, if we write
  \[ \alpha = \sum_{v_1, v_2, \ldots, v_k \in V} r_{v_1, v_2, \ldots, v_k} \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} \wedge \cdots \wedge \frac{\partial}{\partial v_k}, \quad r_{v_1, v_2, \ldots, v_k} \in \mathbb{R}, \]
  then for any elementary $n$-path $u_0 u_1 \ldots u_n$ on $V$ with $n \geq k$, we have\(^2\)
  \[ \alpha(u_0 u_1 \ldots u_n) = \sum_{v_1, v_2, \ldots, v_k \in V} r_{v_1, v_2, \ldots, v_k} \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} \wedge \cdots \wedge \frac{\partial}{\partial v_k} (u_0 u_1 \ldots u_n) \]
  \[ = \sum_{0 \leq i_1 < i_2 < \ldots < i_k \leq n} \sum_{\sigma \in S_k} \sum_{u_{i_1}, u_{i_2}, \ldots, u_{i_k} \in V} (-1)^{i_1 + i_2 + \cdots + i_k} u_{i_1} \ldots \widetilde{u}_{l_1} \ldots \widetilde{u}_{l_2} \ldots \widetilde{u}_{l_k} \ldots u_n. \]

Here $S_k$ is the permutation group on $k$-letters and for any permutation $\sigma \in S_k$, we use $\text{sgn}(\sigma)$ to denote the signature of $\sigma$.

---

\(^2\)The expression of $\alpha(u_0 u_1 \ldots u_n)$ follows from the following two observations:

(i). for any $0 \leq i_1 < i_2 < \ldots < i_k \leq n$, by applying (3.3) for $k$-times, we have
  \[ \frac{\partial}{\partial u_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial u_{i_k}} (u_0 u_1 \ldots u_n) = (-1)^{i_1 + i_2 + \cdots + i_k} u_0 \ldots \widetilde{u}_{i_1} \ldots \widetilde{u}_{i_2} \ldots \widetilde{u}_{i_k} \ldots u_n; \]

(ii). for any $\sigma \in S_k$, by applying (3.3) iteratively, we have
  \[ \frac{\partial}{\partial u_{\sigma(1)}} \wedge \frac{\partial}{\partial u_{\sigma(2)}} \wedge \cdots \wedge \frac{\partial}{\partial u_{\sigma(k)}} = \text{sgn}(\sigma) \frac{\partial}{\partial u_1} \wedge \frac{\partial}{\partial u_2} \wedge \cdots \wedge \frac{\partial}{\partial u_k}. \]
3.3 Partial Differentiations on Path Spaces

**Definition 16.** For any \( v \in V \), we define the *partial differentiation* \( dv \) with respect to \( v \) to be a sequence of linear maps

\[
dv : \Lambda_n(V) \rightarrow \Lambda_{n+1}(V), \quad n \geq 0,
\]

such that \( dv \) is the adjoint linear map of \( \frac{\partial}{\partial v} \) for each \( n \geq 0 \). Precisely, for any \( n \geq 0 \), any \( \xi \in \Lambda_n(V) \), and any \( \eta \in \Lambda_{n+1}(V) \), we have

\[
\langle \frac{\partial}{\partial v}(\eta), \xi \rangle = \langle \eta, dv(\xi) \rangle.
\]  

(3.5)

The next lemma gives an explicit formula for \( dv \).

**Lemma 3.2.** (\cite[Lemma 2.10]{ref}). For any \( n \geq 1 \), any \( v \in V \), and any elementary \((n-1)\)-path \( u_0u_1 \ldots u_{n-1} \) on \( V \), we have

\[
dv(u_0u_1 \ldots u_{n-1}) = \sum_{i=0}^{n} (-1)^i u_0u_1 \ldots \hat{v}_i \ldots u_{i-1}v_{i+1} \ldots u_{n-1}.
\]  

(3.6)

**Proof.** In (3.5), we take \( \eta \) to be an elementary \( n \)-path \( v_0v_1 \ldots v_n \in \Lambda_n(V) \) and take \( \xi \) to be an elementary \((n-1)\)-path \( u_0u_1 \ldots u_{n-1} \in \Lambda_{n-1}(V) \). Then

\[
\langle v_0v_1 \ldots v_n, dv(u_0u_1 \ldots u_{n-1}) \rangle = \langle \frac{\partial}{\partial v}(v_0v_1 \ldots v_n), u_0u_1 \ldots u_{n-1} \rangle
\]  

\[
= \sum_{i=0}^{n} (-1)^i \delta(v, v_i) v_0 \ldots \hat{v}_i \ldots v_n u_0u_1 \ldots u_{n-1}
\]  

\[
= \sum_{i=0}^{n} (-1)^i \delta(v, v_i) \prod_{j=0}^{i-1} \delta(v_j, u_j) \prod_{j=i}^{n-1} \delta(v_{j+1}, u_j).
\]  

Consequently, we have

\[
dv(u_0u_1 \ldots u_{n-1}) = \sum_{v_0, v_1, \ldots, v_n \in V} \langle v_0v_1 \ldots v_n, dv(u_0u_1 \ldots u_{n-1}) \rangle v_0v_1 \ldots v_n
\]  

\[
= \sum_{v_0, v_1, \ldots, v_n \in V} \left( \sum_{i=0}^{n} (-1)^i \delta(v, v_i) \prod_{j=0}^{i-1} \delta(v_j, u_j) \prod_{j=i}^{n-1} \delta(v_{j+1}, u_j) \right) v_0v_1 \ldots v_n
\]  

\[
= \sum_{i=0}^{n} (-1)^i u_0u_1 \ldots \hat{v}_i \ldots u_{i-1}v_{i+1} \ldots u_{n-1}.
\]  

(3.7)

We obtain (3.6). \( \square \)

The next corollary gives the case \( n = 1 \) in Lemma 3.2.

**Corollary 3.3.** For any \( u, v \in V \) we have \( dv(u) = vu - uv \). \( \square \)

Similar with the proof of Lemma 3.1 it is direct to verify the next lemma.
Lemma 3.4. ([26 Lemma 2.7]). For any $u, v \in V$ we have

\[ du \circ dv = -dv \circ du. \]  

(3.8)

Proof. For any $n \geq 0$ and any elementary $n$-path $v_0v_1 \ldots v_n \in \Lambda_n(V)$, by (3.4), we have

\[
du \circ dv(v_0v_1 \ldots v_n) = du \left( \sum_{i=0}^{n+1} (-1)^i v_0 \ldots v_{i-1}v_i \ldots v_n \right) 
\]

\[ = \sum_{i=0}^{n+1} (-1)^i du(v_0 \ldots v_{i-1}v_i \ldots v_n) \]

\[ = \sum_{i=0}^{n+1} (-1)^i \left( \sum_{j=0}^{i-1} (-1)^j v_0 \ldots v_{j-1}uv_j \ldots v_{i-1}v_i \ldots v_n \right) 
+ (-1)^i v_0 \ldots v_{i-1}uv_i \ldots v_n + (-1)^{i+1} v_0 \ldots v_{i-1}uv_i \ldots v_n 
+ \sum_{j=i+1}^{n+1} (-1)^{i+1} v_0 \ldots v_{i-1}uv_i \ldots v_jv_{j-1} \ldots v_n \right) 
\]

while

\[
dv \circ du(v_0v_1 \ldots v_n) = \sum_{i=0}^{n+1} (-1)^i \left( \sum_{j=0}^{i-1} (-1)^j v_0 \ldots v_{j-1}uv_j \ldots v_{i-1}v_i \ldots v_n 
+ (-1)^j v_0 \ldots v_{i-1}uv_i \ldots v_n + (-1)^{i+1} v_0 \ldots v_{i-1}uv_i \ldots v_n 
+ \sum_{j=i+1}^{n+1} (-1)^{i+1} v_0 \ldots v_{i-1}uv_i \ldots v_jv_{j-1} \ldots v_n \right) 
\]

Thus

\[ du \circ dv(v_0v_1 \ldots v_n) = -dv \circ du(v_0v_1 \ldots v_n) \]

for any $n \geq 0$ and any elementary $n$-path $v_0v_1 \ldots v_n \in \Lambda_n(V)$. Consequently, we obtain (3.8).

\[ \square \]

Notation 5. We denote $du \circ dv$ as $du \wedge dv$ for any $u, v \in V$.

Definition 17. We consider the exterior algebra

\[ T^*(V) = \bigwedge \left( dv \mid v \in V \right) \]

and call it the co-differential algebra on $V$.

We have the following observations:

\[ \text{An alternative proof for Lemma 3.4 follows from Lemma 3.1 directly: Let } u, v \in V. \text{ For any any } n \geq 0, \text{ any } \xi \in \Lambda_n(V) \text{ and any } \eta \in \Lambda_{n+2}(V), \text{ we have} \]

\[ \langle \eta, du \wedge dv(\xi) \rangle = \langle \frac{\partial}{\partial u}(\eta), dv(\xi) \rangle = \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial u}(\eta), \xi \rangle \]

\[ = -\langle \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}(\eta), \xi \rangle = -\langle \frac{\partial}{\partial v}(\eta), du(\xi) \rangle = -\langle \eta, dv \wedge du(\xi) \rangle. \]

This implies (3.8). Nevertheless, the proof for Lemma 3.4 in the main-body consolidates (3.6) in Lemma 3.2.

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• $T^*(V)$ is a direct sum

$$T^*(V) = \bigoplus_{k=0}^{\infty} T^k(V).$$

• $T^0(V) = \mathbb{R}$ while for each $k \geq 1$, the space $T^k(V)$ is spanned by

$$dv_1 \wedge dv_2 \wedge \cdots \wedge dv_k, \quad v_1, v_2, \ldots, v_k \in V$$

modulo the relation

$$dv_1 \wedge \cdots \wedge dv_i \wedge dv_{i+1} \wedge \cdots \wedge dv_k = -dv_1 \wedge \cdots \wedge dv_i \wedge dv_i \wedge \cdots \wedge dv_k$$

for any $1 \leq i \leq k - 1$.

• For any $k \geq 1$ and any $\omega \in T^k(V)$, we have that $\omega$ gives a sequence of linear maps

$$\omega_n : \Lambda_n(V) \to \Lambda_{n+k}(V), \quad n \geq 0.$$  \hfill (3.9)

**Definition 18.** Let $k \geq 1$, $\alpha \in T_k(V)$ and $\omega \in T^k(V)$. We say that $\alpha$ and $\omega$ are adjoint to each other if for any $n \geq 0$, any $\xi \in \Lambda_n(V)$ and any $\eta \in \Lambda_{n+k}(V)$, the identity

$$\langle \alpha(\eta), \xi \rangle = \langle \eta, \omega(\xi) \rangle$$

is satisfied.

**Proposition 3.5.** Let $k \geq 1$ be any positive integer. Let $\alpha \in T_k(V)$ be given by

$$\alpha = \sum_{v_1, v_2, \ldots, v_k \in V} r_{v_1, v_2, \ldots, v_k} \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} \wedge \cdots \wedge \frac{\partial}{\partial v_k}, \quad r_{v_1, v_2, \ldots, v_k} \in \mathbb{R}.$$  \hfill (3.10)

Suppose $\omega \in T^k(V)$ is adjoint to $\alpha$. Then $\omega$ is given by

$$\omega = \text{sgn}(k) \sum_{v_1, v_2, \ldots, v_k \in V} r_{v_1, v_2, \ldots, v_k} dv_1 \wedge dv_2 \wedge \cdots \wedge dv_k, \quad r_{v_1, v_2, \ldots, v_k} \in \mathbb{R}$$

where $\text{sgn}(k) = 1$ if $k \equiv 0, 1$ modulo 4 and $\text{sgn}(k) = -1$ if $k \equiv 2, 3$ modulo 4.

**Proof.** Let $n \geq 0$, $\xi \in \Lambda_n(V)$, and $\eta \in \Lambda_{n+k}(V)$. Then we have

$$\langle \alpha(\eta), \xi \rangle = \sum_{v_1, v_2, \ldots, v_k \in V} r_{v_1, v_2, \ldots, v_k} \left( \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} \wedge \cdots \wedge \frac{\partial}{\partial v_k} \right) \langle \eta, \xi \rangle$$

$$= \sum_{v_1, v_2, \ldots, v_k \in V} r_{v_1, v_2, \ldots, v_k} \left( \frac{\partial}{\partial v_2} \wedge \cdots \wedge \frac{\partial}{\partial v_k} \right) \langle \eta, dv_1(\xi) \rangle$$

$$= \sum_{v_1, v_2, \ldots, v_k \in V} r_{v_1, v_2, \ldots, v_k} \left( \frac{\partial}{\partial v_3} \wedge \cdots \wedge \frac{\partial}{\partial v_k} \right) \langle \eta, dv_2 \wedge dv_1(\xi) \rangle$$

$$= \cdots$$

$$= \sum_{v_1, v_2, \ldots, v_k \in V} r_{v_1, v_2, \ldots, v_k} \langle \eta, dv_k \wedge \cdots \wedge dv_1(\xi) \rangle$$

$$= \sum_{v_1, v_2, \ldots, v_k \in V} r_{v_1, v_2, \ldots, v_k} \text{sgn}(k) \langle \eta, dv_1 \wedge dv_2 \wedge \cdots \wedge dv_k(\xi) \rangle.$$
The last equality follows from the fact that the permutation \((k, k-1, \ldots, 1)\) of \((1, 2, \ldots, k)\) has the signature
\[
\text{sgn}\left(1, 2, \ldots, k \atop k, k-1, \ldots, 1\right) = (-1)^{(k-1)(k-2)+\cdots+1} = (-1)^{\frac{k(k-1)}{2}}
\]
for any \(k \geq 2\) and the permutation \((k, k-1, \ldots, 1)\) of \((1, 2, \ldots, k)\) has the signature 1 for \(k = 1\). In other words, the permutation \((k, k-1, \ldots, 1)\) of \((1, 2, \ldots, k)\) has the signature \((-1)^{k}k^{(k-1)/2}\) for any \(k \equiv 0, 1 \pmod{4}\) and the signature \(-1\) for \(k \equiv 2, 3 \pmod{4}\). Therefore, we have that the \(\omega\) given by (3.10) is adjoint to \(\alpha\). The proposition follows.

### 3.4 Some Chain Complexes and Co-Chain Complexes on Path Spaces

**Proposition 3.6.** Let \(t \geq 0\) be a non-negative integer. Let \(\alpha \in T_{2t+1}(V)\) and \(\omega \in T^{2t+1}(V)\). Then for any \(0 \leq q \leq 2t\), we have a chain complex

\[
\cdots \xrightarrow{\alpha} \Lambda_{n(2t+1)+q}(V) \xrightarrow{\alpha} \Lambda_{(n-1)(2t+1)+q}(V) \xrightarrow{\alpha} 0
\]

and a co-chain complex

\[
\cdots \xleftarrow{\omega} \Lambda_{n(2t+1)+q}(V) \xleftarrow{\omega} \Lambda_{(n-1)(2t+1)+q}(V) \xleftarrow{\omega} 0.
\]

**Proof.** Let \(t \geq 0\). Let \(\alpha \in T_{2t+1}(V)\) and \(\omega \in T^{2t+1}(V)\). Let \(0 \leq q \leq 2t\). Note that for each \(n \geq 0\), the maps

\[
\alpha : \Lambda_{n(2t+1)+q}(V) \longrightarrow \Lambda_{(n-1)(2t+1)+q}(V)
\]

and

\[
\omega : \Lambda_{n(2t+1)+q}(V) \longrightarrow \Lambda_{(n+1)(2t+1)+q}(V)
\]

are well-defined. By the anti-symmetric property of exterior algebras, we have

\[
\alpha \land \alpha = (-1)^{(2t+1)^2} \land \alpha, \quad \omega \land \omega = (-1)^{(2t+1)^2} \omega \land \omega.
\]

Since \((2t+1)^2\) is odd, we have

\[
\alpha \circ \alpha = \alpha \land \alpha = 0, \quad \omega \circ \omega = \omega \land \omega = 0.
\]

Thus for any \(0 \leq q \leq 2t\), we have the chain complex as well as the co-chain complex as given in the proposition.

**Notation 6.** Let \(0 \leq q \leq 2t\). Let \(\alpha \in T_{2t+1}(V)\) and \(\omega \in T^{2t+1}(V)\). We adopt the following notations:
(i). denote the chain complex in Proposition 3.6 as 

$$\Lambda^*(V, \alpha, q) = \{ \Lambda_{n(2t+1)+q}(V), \alpha \}_{n \geq 0};$$

(ii). denote the co-chain complex in Proposition 3.6 as 

$$\Lambda^*(V, \omega, q) = \{ \Lambda_{n(2t+1)+q}(V), \omega \}_{n \geq 0}.$$

**Notation 7.** For any integer \( m \), there is a unique integer \( \lambda \) (not necessarily non-negative) and a unique integer \( 0 \leq q \leq 2t \) such that \( m = \lambda(2t+1) + q \). We adopt the following notations:

(i). denote the chain complex

$$\cdots \xrightarrow{\alpha} \Lambda_{(n+\lambda)(2t+1)+q}(V) \xrightarrow{\alpha} \Lambda_{(n-1+\lambda)(2t+1)+q}(V) \xrightarrow{\alpha} \cdots$$

as

$$\Lambda_*(V, \alpha, m) = \{ \Lambda_{(n+\lambda)(2t+1)+q}(V), \alpha \}_{n \geq 0};$$

(ii). denote the co-chain complex

$$\cdots \xleftarrow{\omega} \Lambda_{(n+\lambda)(2t+1)+q}(V) \xleftarrow{\omega} \Lambda_{(n-1+\lambda)(2t+1)+q}(V) \xleftarrow{\omega} \cdots$$

as

$$\Lambda^*(V, \omega, m) = \{ \Lambda_{(n+\lambda)(2t+1)+q}(V), \omega \}_{n \geq 0}.$$ 

Here in both (i) and (ii), we use the notation \( \Lambda_k(V) = 0 \) for \( k < 0 \).

**Proposition 3.7.** Let \( t, s \geq 0 \) be non-negative integers. Let \( m \in \mathbb{Z} \). Let \( \alpha \in T_{2t+1}(V) \) and \( \omega \in T^{2t+1}(V) \). Let \( \beta \in T_{2s}(V) \) and \( \mu \in T^{2s}(V) \). Then \( \beta \) gives a chain map

$$\beta : \Lambda_*(V, \alpha, m) \longrightarrow \Lambda_*(V, \alpha, m - 2s)$$

and \( \mu \) gives a co-chain map

$$\mu : \Lambda^*(V, \omega, m) \longrightarrow \Lambda^*(V, \omega, m + 2s).$$

**Proof.** Note that as linear maps,

$$\beta : \Lambda_{(n+\lambda)(2t+1)+q}(V) \longrightarrow \Lambda_{(n+\lambda)(2t+1)+q-2s}(V)$$

and

$$\mu : \Lambda_{(n+\lambda)(2t+1)+q}(V) \longrightarrow \Lambda_{(n+\lambda)(2t+1)+q+2s}(V)$$

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are well-defined. By the anti-symmetric property of exterior algebras, we have (cf. [4] p, 53, Anticommutative Law]

\[ \alpha \wedge \beta = (-1)^{2s(2t+1)} \beta \wedge \alpha = \beta \wedge \alpha. \]

That is,

\[ \alpha \circ \beta = \beta \circ \alpha. \]

Thus \( \beta \) is a chain map from \( \Lambda_s(V, \alpha, m) \) to \( \Lambda_s(V, \alpha, m - 2s) \). Moreover, we also have (cf. [4] p, 53, Anticommutative Law]

\[ \omega \wedge \mu = (-1)^{2s(2t+1)} \mu \wedge \omega = \mu \wedge \omega. \]

That is,

\[ \omega \circ \mu = \mu \circ \omega. \]

Thus \( \mu \) is a co-chain map from \( \Lambda^*(V, \omega, m) \) to \( \Lambda^*(V, \omega, m + 2s) \). The proposition follows.

\[ \square \]

4 Generalized (Co-)Homology for (Co-)Simplicial Complexes

In this section, we define the generalized homology groups for simplicial complexes and the generalized cohomology groups for co-simplicial complexes. We prove that any element \( \beta \in T_{2s}(V) \), where \( s \geq 0 \), induces homomorphisms between the generalized homology groups for the simplicial complexes on \( V \). We also prove that any element \( \mu \in T_{2s}(V) \), where \( s \geq 0 \), induces homomorphisms between the generalized cohomology groups for the co-simplicial complexes on \( V \).

4.1 Some Auxiliaries

Throughout this section, we \( V \) be a finite set. Let \( \Delta[V] \) be the complete hypergraph on \( V \). For each \( n \geq 0 \), let

\[ C_n(\Delta[V]; \mathbb{R}) = \operatorname{Span}_\mathbb{R}\{\sigma(n) \in \Delta[V]\} \]

be the vector space consisting of all the linear combinations of the \( n \)-hyperedges on \( V \). Consider the direct sum

\[ C_*\left( \Delta[V]; \mathbb{R} \right) = \bigoplus_{n \geq 0} C_n(\Delta[V]; \mathbb{R}). \]  \( (4.1) \)

Note that since \( V \) is assumed to be a finite set, the direct sum in the right-hand side of \( (4.1) \) is a finite sum.
Lemma 4.1. Let \( t \geq 0 \). Let \( m \in \mathbb{Z} \). Suppose \( m = \lambda(2t + 1) + q \) where \( \lambda \in \mathbb{Z} \) and \( 0 \leq q \leq 2t \). Then for any \( \alpha \in T_{2t+1}(V) \), the graded vector space

\[
C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R}), \quad n \geq 0
\]

equipped with the boundary map \( \alpha \) gives a sub-chain complex of \( \Lambda_*(V, \alpha, m) \), which will be denoted as \( C_*(\Delta[V], \alpha, m) \).

Proof. For each \( n \geq 0 \), the vector space \( C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R}) \) is a subspace of the vector space \( \Lambda_{(n+\lambda)(2t+1)+q}(V) \). Hence in order to prove that \( \text{(4.2)} \) equipped with \( \alpha \) is a sub-chain complex of \( \Lambda_*(V, \alpha, m) \), it suffices to prove that the map

\[
\alpha : C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R}) \longrightarrow C_{(n-1+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R})
\]

(4.3)
is well-defined for each \( n \geq 0 \). This follows from the observation that for any \( [(n+\lambda)(2t+1)+q] \)-simplex

\[
v_0 v_1 \ldots v_{(n+\lambda)(2t+1)+q} \in C_{(n+\lambda)(2t+1)+q}(\Delta[V]; R)
\]

and any

\[
\alpha = \frac{\partial}{\partial u_1} \land \frac{\partial}{\partial u_2} \land \cdots \land \frac{\partial}{\partial u_{2t+1}}
\]

where \( u_1, u_2, \ldots, u_{2t+1} \in V \) and \( u_1 < u_2 < \cdots < u_{2t+1} \), we have

\[
\alpha(v_0 v_1 \ldots v_{(n+\lambda)(2t+1)+q}) \in C_{(n-1+\lambda)(2t+1)+q}(\Delta[V]; R).
\]

By a calculation of linear combinations, it follows that the map \( \text{(4.3)} \) is well-defined. Therefore, the graded vector space \( \text{(4.2)} \) equipped with \( \alpha \) is a sub-chain complex of \( \Lambda_*(V, \alpha, m) \).

Definition 19. For any \( n \geq 0 \) and any elementary \( n \)-path \( v_0 v_1 \ldots v_n \) on \( V \), we call \( v_0 v_1 \ldots v_n \) a cyclic elementary \( n \)-path if there exists \( 0 \leq i < j \leq n \) such that \( v_i = v_j \).

Definition 20. Let \( \mathcal{O}_n(V) \) be the vector space spanned by all the cyclic elementary \( n \)-paths on \( V \). Then \( \mathcal{O}_n(V) \) consists of all the linear combinations of the cyclic elementary \( n \)-paths on \( V \). We call an element in \( \mathcal{O}_n(V) \) a cyclic \( n \)-path on \( V \).

Lemma 4.2. Let \( t \geq 0 \). Let \( m \in \mathbb{Z} \). Suppose \( m = \lambda(2t + 1) + q \) where \( \lambda \in \mathbb{Z} \) and \( 0 \leq q \leq 2t \). Then for any \( \omega \in T^{2t+1}(V) \), the graded vector space

\[
\mathcal{O}_{(n+\lambda)(2t+1)+q}(V), \quad n \geq 0
\]

equipped with the co-boundary map \( \omega \) gives a sub-co-chain complex of \( \Lambda^*(V, \omega, m) \), which will be denoted as \( \mathcal{O}^*(V, \omega, m) \).

Proof. It suffices to verify that the map

\[
\omega : \mathcal{O}_{(n+\lambda)(2t+1)+q}(V) \longrightarrow \mathcal{O}_{(n+1+\lambda)(2t+1)+q}(V)
\]

(4.5)
is well-defined for each $n \geq 0$. This follows from the observation that for any cyclic elementary 
$(n + \lambda)(2t + 1) + q$-path 

$v_0v_1 \cdots v_{(n+\lambda)(2t+1)+q} \in \mathcal{O}_{(n+\lambda)(2t+1)+q}(V)$

and any 

$\omega = du_1 \wedge du_2 \wedge \cdots \wedge du_{2t+1}$

where $u_1, u_2, \ldots, u_{2t+1} \in V$ and $u_1 < u_2 < \cdots < u_{2t+1}$, we have 

$\omega(v_0v_1 \cdots v_{(n+\lambda)(2t+1)+q}) \in \mathcal{O}_{(n+\lambda)(2t+1)+q}(V)$. By a calculation of linear combinations, it follows that the map (4.5) is well-defined. Therefore, the graded vector space (4.4) equipped with $\omega$ is a sub-co-chain complex of $\Lambda^*(V, \omega, m)$. □

**Lemma 4.3.** Let $t \geq 0$. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t + 1) + q$ where $\lambda \in \mathbb{Z}$ and $0 \leq q \leq 2t$. Then for any $\omega \in T^{2t+1}(V)$, the graded vector space 

$$C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R}), \quad n \geq 0$$

equipped with the co-boundary map $\omega$ gives a quotient co-chain complex $\Lambda^*(V, \omega, m)$ which will be denoted as $C^*(\Delta[V], \omega, m)$. □

**Proof.** Note that the canonical inclusion of the sub-co-chain complex $\mathcal{O}^*(V, \omega, m)$ into the co-chain complex $\Lambda^*(V, \omega, m)$ gives a quotient co-chain complex $\Lambda^*(V, \omega, m)/\mathcal{O}^*(V, \omega, m)$ on the other hand, for each $n \geq 0$, the quotient vector space

$$\Lambda_{(n+\lambda)(2t+1)+q}(V)/\mathcal{O}_{(n+\lambda)(2t+1)+q}(V)$$

is canonically isomorphic to the vector space $C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R})$. Therefore, the quotient co-chain complex $\Lambda^*(V, \omega, m)/\mathcal{O}^*(V, \omega, m)$ is given by the graded vector space (4.6) equipped with the co-boundary map $\omega$. The lemma follows. □

With the help of Proposition 3.7, the next proposition follows.

**Proposition 4.4.** Let $t, s \geq 0$ be non-negative integers. Let $m \in \mathbb{Z}$. Let $\alpha \in T_{2t+1}(V)$ and $\omega \in T^{2t+1}(V)$. Let $\beta \in T_{2s}(V)$ and $\mu \in T^{2s}(V)$. Then $\beta$ is a chain map 

$$\beta : C_*(\Delta[V], \alpha, m) \longrightarrow C_*(\Delta[V], \alpha, m - 2s)$$

and $\mu$ is a co-chain map 

$$\mu : C^*(\Delta[V], \omega, m) \longrightarrow C^*(\Delta[V], \omega, m + 2s).$$

**Proof.** By a similar argument in the proof of Lemma 4.1, it can be verified that as a linear map, $\beta$ in (4.7) is well-defined. Thus by Proposition 3.7 and Lemma 4.1 it follows that $\beta$ in (4.7) is a chain map. On the other hand, by a similar argument in the proof of Lemma 4.3, it can be verified that as a linear map, $\mu$ in (4.8) is well-defined. Thus by Proposition 3.7 and Lemma 4.3 it follows that $\omega$ in (4.8) is a co-chain map. □
4.2 Generalized Homology for Simplicial Complexes

Let $\mathcal{K}$ be a simplicial complex with its vertices in $V$.

**Notation 8.** For each $n \geq 0$, let $C_n(\mathcal{K}; \mathbb{R})$ be the (real) vector space consisting of all the linear combinations of the $n$-simplices in $\mathcal{K}$.

**Theorem 4.5.** Let $t, s \geq 0$. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1)+q$ where $\lambda \in \mathbb{Z}$ and $0 \leq q \leq 2t$.

Then

(i). for any $\alpha \in T_{2t+1}(V)$, the graded vector space

$$C_{(n+\lambda)(2t+1)+q}(\mathcal{K}; \mathbb{R}), \quad n \geq 0$$

(4.9)

equipped with the chain map $\alpha$ gives a sub-chain complex of $C_*(\Delta[V], \alpha, m)$, which will be denoted as $C_*(\mathcal{K}, \alpha, m)$;

(ii). for any $\beta \in T_{2s}(V)$, there is an induced chain map

$$\beta : C_*(\mathcal{K}, \alpha, m) \rightarrow C_*(\mathcal{K}, \alpha, m-2s).$$

(4.10)

**Proof.** We prove (i) and (ii) subsequently.

(i). For each $n \geq 0$, the vector space $C_{(n+\lambda)(2t+1)+q}(\mathcal{K}; \mathbb{R})$ is a subspace of the vector space $C_{(n+\lambda)(2t+1)+q}(\Delta[V]; \mathbb{R})$. Hence in order to prove that the graded vector space (4.9) equipped with the chain map $\alpha$ is a sub-chain complex of $C_*(\Delta[V], \alpha, m)$, it suffices to prove that the map

$$\alpha : C_{(n+\lambda)(2t+1)+q}(\mathcal{K}; \mathbb{R}) \rightarrow C_{(n-1+\lambda)(2t+1)+q}(\mathcal{K}; \mathbb{R})$$

(4.11)

is well-defined for each $n \geq 0$. This follows from the observation that for any $[(n+\lambda)(2t+1)+q]$-simplex

$$v_0v_1 \ldots v_{(n+\lambda)(2t+1)+q} \in C_{(n+\lambda)(2t+1)+q}(\mathcal{K}; R)$$

and any

$$\alpha = \frac{\partial}{\partial u_1} \wedge \frac{\partial}{\partial u_2} \wedge \cdots \wedge \frac{\partial}{\partial u_{2t+1}}$$

where $u_1, u_2, \ldots, u_{2t+1} \in V$ and $u_1 \prec u_2 \prec \cdots \prec u_{2t+1}$, we have

$$\alpha(v_0v_1 \ldots v_{(n+\lambda)(2t+1)+q}) \in C_{(n-1+\lambda)(2t+1)+q}(\mathcal{K}; R).$$

By a calculation of linear combinations, it follows that the map (4.11) is well-defined. Therefore, the graded vector space (4.9) equipped with the chain map $\alpha$ is a sub-chain complex of $C_*(\Delta[V], \alpha, m)$.

(ii). Similar with the verification that the map $\alpha$ in (4.11) is well-defined for each $n \geq 0$, we can prove that the map

$$\beta : C_{(n+\lambda)(2t+1)+q}(\mathcal{K}; \mathbb{R}) \rightarrow C_{(n+\lambda)(2t+1)+q-2s}(\mathcal{K}; \mathbb{R})$$

is well-defined for each $n \geq 0$. Therefore, with the help of (4.7) in Proposition 4.4, we have that $\beta$ gives a chain map in (4.10). \qed
Definition 21. Let $t \geq 0$. Let $\alpha \in T_{2t+1}(V)$. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1) + q$ where $\lambda \in \mathbb{Z}$ and $0 \leq q \leq 2t$. Let $K$ be a simplicial complex with its vertices in $V$. For each $n \geq 0$, we define the $n$-th generalized homology group $H_n(K, \alpha, m)$ of $K$ with respect to $\alpha$ and $m$ to be the $n$-th homology group

$$H_n(K, \alpha, m) = H_n(C_*(K, \alpha, m))$$

$$= \frac{\text{Ker}(\alpha : C_{n+1}(K; \mathbb{R}) \rightarrow C_{n+1}(K; \mathbb{R}))}{\text{Im}(\alpha : C_{n+1}(K; \mathbb{R}) \rightarrow C_{n+1}(K; \mathbb{R}))}$$

of the chain complex $C_*(K, \alpha, m)$.

The next theorem follows from Theorem 4.5 and Definition 21 immediately.

Theorem 4.6 (Main Result I). Let $t, s \geq 0$. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1) + q$ where $\lambda \in \mathbb{Z}$ and $0 \leq q \leq 2t$. Then for any $\alpha \in T_{2t+1}(V)$ and $\beta \in T_{2s}(V)$, there is an induced homomorphism

$$\beta_* : H_n(K, \alpha, m) \rightarrow H_n(K, \alpha, m-2s), \quad n \geq 0$$

(4.12)

of the generalized homology groups.

Proof. Apply the homology functor to the chain complex in Theorem 4.5 (i) and the chain map in Theorem 4.5 (ii). We obtain the homomorphism $\beta_*$ of the generalized homology groups in (4.12). \qed

4.3 Generalized Cohomology for Co-Simplicial Complexes

Let $L$ be a co-simplicial complex with its vertices in $V$.

Notation 9. For each $n \geq 0$, let $C_\omega(L; \mathbb{R})$ be the (real) vector space consisting of all the linear combinations of the $n$-co-simplices in $L$.

Theorem 4.7. Let $t, s \geq 0$. Let $m \in \mathbb{Z}$. Suppose $m = \lambda(2t+1) + q$ where $\lambda \in \mathbb{Z}$ and $0 \leq q \leq 2t$. Then

(i). for any $\omega \in T^{2t+1}(V)$, the graded vector space

$$C_{(n+\lambda)(2t+1)+q}(L; \mathbb{R}), \quad n \geq 0$$

(4.13)

equipped with the co-boundary map $\omega$ gives a sub-co-chain complex of $C^*(\Delta[V], \omega, m)$, which will be denoted as $C^*(L, \omega, m)$;

(ii). for any $\mu \in T^{2s}(V)$, there is an induced co-chain map

$$\mu : C^*(L, \omega, m) \rightarrow C^*(L, \omega, m+2s).$$

(4.14)
Proof. We prove (i) and (ii) subsequently.

(i). For each \( n \geq 0 \), the vector space \( C_{(n+\lambda)(2t+1)+q}(\mathcal{L};\mathbb{R}) \) is a subspace of the vector space \( C_{(n+\lambda)(2t+1)+q}(\Delta[V];\mathbb{R}) \). Hence in order to prove that the graded vector space (4.13) equipped with the co-boundary map \( \omega \) is a sub-co-chain complex of \( C^*(\Delta[V],\omega,m) \), it suffices to prove that the map
\[
\omega: \quad C_{(n+\lambda)(2t+1)+q}(\mathcal{L};\mathbb{R}) \longrightarrow C_{(n+1+\lambda)(2t+1)+q}(\mathcal{L};\mathbb{R}) \tag{4.15}
\]
is well-defined for each \( n \geq 0 \). This follows from the observation that for any \( [(n+\lambda)(2t+1)+q]\)-co-simplex
\[
v_0 v_1 \cdots v_{(n+\lambda)(2t+1)+q} \in C_{(n+\lambda)(2t+1)+q}(\mathcal{L};\mathbb{R})
\]
and any
\[
\omega = du_1 \wedge du_2 \wedge \cdots \wedge du_{2t+1}
\]
where \( u_1, u_2, \ldots, u_{2t+1} \in V \) and \( u_1 < u_2 < \cdots < u_{2t+1} \), we have
\[
\omega(v_0 v_1 \cdots v_{(n+\lambda)(2t+1)+q}) \in C_{(n+1+\lambda)(2t+1)+q}(\mathcal{L};\mathbb{R}).
\]
By a calculation of linear combinations, it follows that the map (4.15) is well-defined. Therefore, the graded vector space (4.13) equipped with the co-boundary map \( \omega \) is a sub-co-chain complex of \( C^*(\Delta[V],\omega,m) \).

(ii). Similar with the verification that the map \( \omega \) in (4.15) is well-defined for each \( n \geq 0 \), we can prove that the map
\[
\mu: \quad C_{(n+\lambda)(2t+1)+q}(\mathcal{L};\mathbb{R}) \longrightarrow C_{(n+1+\lambda)(2t+1)+q+2s}(\mathcal{L};\mathbb{R})
\]
is well-defined for each \( n \geq 0 \). Therefore, with the help of (4.18) in Proposition 4.4, we have that \( \mu \) gives a co-chain map in (4.14).

Definition 22. Let \( t \geq 0 \). Let \( \omega \in T^{2t+1}(V) \). Let \( m \in \mathbb{Z} \). Suppose \( m = \lambda(2t+1)+q \) where \( \lambda \in \mathbb{Z} \) and \( 0 \leq q \leq 2t \). Let \( \mathcal{L} \) be a co-simplicial complex with its vertices in \( V \). For each \( n \geq 0 \), we define the \( n \)-th generalized cohomology group \( H^n(\mathcal{L},\omega,m) \) of \( \mathcal{L} \) with respect to \( \omega \) and \( m \) to be the cohomology group
\[
H^n(\mathcal{L},\omega,m) := H^n(C^*(\mathcal{L},\omega,m))
\]
\[
= \frac{\text{Ker} \left( \omega: C_{(n+\lambda)(2t+1)+q}(\mathcal{L};\mathbb{R}) \longrightarrow C_{(n+1+\lambda)(2t+1)+q}(\mathcal{L};\mathbb{R}) \right)}{\text{Im} \left( \omega: C_{(n-\lambda)(2t+1)+q}(\mathcal{L};\mathbb{R}) \longrightarrow C_{(n+\lambda)(2t+1)+q}(\mathcal{L};\mathbb{R}) \right)}
\]
of the co-chain complex \( C^*(\mathcal{L},\omega,m) \).

The next theorem follows from Theorem 4.7 and Definition 22 immediately.

Theorem 4.8 (Main Result II). Let \( t, s \geq 0 \). Let \( m \in \mathbb{Z} \). Suppose \( m = \lambda(2t+1)+q \) where \( \lambda \in \mathbb{Z} \) and \( 0 \leq q \leq 2t \). Then for any \( \omega \in T^{2t+1}(V) \) and \( \mu \in T^{2s}(V) \), there is an induced homomorphism
\[
\mu_*: \quad H^n(\mathcal{L},\omega,m) \longrightarrow H^n(\mathcal{L},\omega,m+2s), \quad n \geq 0 \tag{4.16}
\]
of the generalized cohomology groups.

Proof. Apply the cohomology functor to the co-chain complex in Theorem 4.7 (i) and the co-chain map in Theorem 4.7 (ii). We obtain the homomorphism $\mu^*$ of the generalized cohomology groups in (4.16). □

5 Examples

We give some examples for Theorem 4.5, Theorem 4.6, Theorem 4.7 and Theorem 4.8.

Example 5.1. Let $V$ be any finite set. Then we have the followings.

(i) Any element $\alpha \in T_1(V)$ can be expressed as

$$\alpha = \sum_{v \in V} f(v) \frac{\partial}{\partial v} \tag{5.1}$$

for some function $f : V \to \mathbb{R}$. Let $\mathcal{K}$ be a simplicial complex with its vertices in $V$. Then for any $n \geq 0$ and any $n$-simplex $v_0v_1 \ldots v_n$ in $\mathcal{K}$, we have

$$\alpha(v_0v_1 \ldots v_n) = \sum_{v \in V} f(v) \frac{\partial}{\partial v}(v_0v_1 \ldots v_n)$$

$$= \sum_{v \in V} f(v) \sum_{i=0}^{n} (-1)^i \delta(v, v_i)v_0 \ldots \hat{v}_i \ldots v_n$$

$$= \sum_{i=0}^{n} (-1)^i \left( \sum_{v \in V} \delta(v, v_i)f(v) \right)v_0 \ldots \hat{v}_i \ldots v_n$$

$$= \sum_{i=0}^{n} (-1)^i f(v_i)v_0 \ldots \hat{v}_i \ldots v_n.$$  

In [27, 28, 29], the $\alpha$ given in (5.1) is called the $f$-weighted boundary operator on $\mathcal{K}$ and the $(\alpha, 0)$-homology of $\mathcal{K}$ is denoted as the weighted homology $H_*(\mathcal{K}, f)$ of the weighted simplicial complex $(\mathcal{K}, f)$. Particularly, if $f$ takes the constant value 1 for all $v \in V$, then $\alpha$ is the usual boundary operator $\partial_*$ given in (1.1) and $H_*(\mathcal{K}, f)$ is the usual homology $H_*(\mathcal{K})$ (cf. [23, Chapter 1] and [19, Section 2.1]) of $\mathcal{K}$.

(ii) Any element $\omega \in T_1(V)$ can be expressed as

$$\omega = \sum_{v \in V} f(v)dv \tag{5.2}$$

for some function $f : V \to \mathbb{R}$. Let $\mathcal{L}$ be a co-simplicial complex with its vertices in $V$. Then for any $n \geq 0$ and any $n$-co-simplex $v_0v_1 \ldots v_n$ in $\mathcal{L}$, we have

$$\omega(v_0v_1 \ldots v_n) = \sum_{v \in V} f(v)dv(v_0v_1 \ldots v_n)$$

$$= \sum_{v \in V} f(v) \sum_{i=0}^{n+1} (-1)^i v_0v_1 \ldots \hat{v}_{i-1}v_{i+1} \ldots v_n$$

$$= \sum_{i=0}^{n+1} (-1)^i \left( \sum_{v \in V} f(v)v_0 \ldots v_{i-1}v_{i+1} \ldots v_n \right).$$  

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Similar with (i), we call the $\omega$ given in (5.3) the $f$-weighted co-boundary operator on $L$ and denote the $(\omega, 0)$-cohomology of $L$ as $H^*(L, f)$. Particularly, if $f$ takes the constant value 1 for all $v \in V$, then we denote the $\omega$ as $d_\ast$ denote the $H^*(L, f)$ as $H^*(L)$.

Example 5.2. Let $V = \{v_0, v_1, v_2\}$. Let $f : V \to \mathbb{R}$ be a function on $V$.

(i). Let

$$K = \{v_0, v_1, v_2, v_0v_1, v_0v_2, v_1v_2\}$$

be a simplicial complex with its vertices in $\mathbb{R}$ value and denote the $\omega_\ast$ given in (5.2) the $L$-weighted co-boundary operator on $L$. Particularly, if $f$ takes the constant value 1 for all $v \in V$, then we denote the $\omega$ as $d_\ast$ denote the $H^*(L, f)$ as $H^*(L)$.

Then we have

$$C_0(K; \mathbb{R}) = \text{Span}_\mathbb{R}\{v_0, v_1, v_2\},$$

$$C_1(K; \mathbb{R}) = \text{Span}_\mathbb{R}\{v_0v_1, v_0v_2, v_1v_2\},$$

$$C_n(K; \mathbb{R}) = 0 \text{ for all } n \geq 2.$$

Let $t = 1$. Let

$$\alpha = \sum_{v \in V} f(v) \frac{\partial}{\partial v}$$

$$= f(v_0) \frac{\partial}{\partial v_0} + f(v_1) \frac{\partial}{\partial v_1} + f(v_2) \frac{\partial}{\partial v_2}.$$

With the help of Example 5.1 (i), we have

$$\alpha(v_0) = \alpha(v_1) = \alpha(v_2) = 0,$$

$$\alpha(v_0v_1) = f(v_0)v_1 - f(v_1)v_0,$$

$$\alpha(v_0v_2) = f(v_0)v_2 - f(v_2)v_0,$$

$$\alpha(v_1v_2) = f(v_1)v_2 - f(v_2)v_1,$$

$$\alpha(v_0v_1v_2) = f(v_0)v_1v_2 - f(v_1)v_0v_2 + f(v_2)v_0v_1.$$

Note that

$$\dim \text{Ker}\left(\alpha : C_0(K; \mathbb{R}) \to 0\right) = 3$$

and

$$\dim \text{Im}\left(\alpha : C_1(K; \mathbb{R}) \to C_0(K; \mathbb{R})\right) = \begin{cases} 2, & \text{if } f(v_i) \neq 0 \text{ for some } i = 0, 1, 2; \\ 0, & \text{if } f(v_0) = f(v_1) = f(v_2) = 0. \end{cases}$$

Thus

$$H_0(K, f) = H_0(K, \alpha, 0) = \begin{cases} \mathbb{R}, & \text{if } f(v_i) \neq 0 \text{ for some } i = 0, 1, 2; \\ \mathbb{R}^3, & \text{if } f(v_0) = f(v_1) = f(v_2) = 0. \end{cases}$$

Note that

$$\dim \text{Ker}\left(\alpha : C_1(K; \mathbb{R}) \to C_0(K; \mathbb{R})\right) = \begin{cases} 1, & \text{if } f(v_i) \neq 0 \text{ for some } i = 0, 1, 2; \\ 3, & \text{if } f(v_0) = f(v_1) = f(v_2) = 0. \end{cases}$$
and

\[ \dim \text{Im}(\alpha : C_2(K; \mathbb{R}) \to C_1(K; \mathbb{R})) = 0. \]

Thus

\[ H_1(K, f) = H_1(K, \alpha, 0) = \begin{cases} \mathbb{R}, & \text{if } f(v_i) \neq 0 \text{ for some } i = 0, 1, 2; \\ \mathbb{R}^3, & \text{if } f(v_0) = f(v_1) = f(v_2) = 0. \end{cases} \]

- Let \( s = 1 \). Let

\[ \beta = b_{01} \frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_1} + b_{02} \frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_2} + b_{12} \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2}. \]

Then

\[ \beta(v_i) = 0 \quad \text{for } 0 \leq i \leq 2 \]

and

\[ \beta(v_i, v_j) = 0 \quad \text{for } 0 \leq i < j \leq 2. \]

Thus the induced homomorphism \( \beta \) between the homology groups is identically zero.

(ii). Let

\[ \mathcal{L} = \{v_0v_1, v_0v_2, v_0v_1v_2\} \]

be a co-simplicial complex with its vertices in \( V \). Then we have

\[ C_0(\mathcal{L}; \mathbb{R}) = 0, \]
\[ C_1(\mathcal{L}; \mathbb{R}) = \text{Span}_\mathbb{R}\{v_0v_1, v_0v_2\}, \]
\[ C_2(\mathcal{L}; \mathbb{R}) = \text{Span}_\mathbb{R}\{v_0v_1v_2\}, \]
\[ C_n(\mathcal{L}; \mathbb{R}) = 0 \quad \text{for all } n \geq 3. \]

- Let \( t = 1 \). Let

\[ \omega = \sum_{v \in V} f(v)dv = f(v_0)dv_0 + f(v_1)dv_1 + f(v_2)dv_2 + f(v_3)dv_3. \]

With the help of Example 5.1 (ii), we have

\[ \omega(v_0v_1) = f(v_2)v_0v_1v_2, \]
\[ \omega(v_0v_2) = -f(v_1)v_0v_1v_2, \]
\[ \omega(v_0v_1v_2) = 0. \]

Note that

\[ \dim \ker(\omega : C_1(\mathcal{L}; \mathbb{R}) \to C_2(\mathcal{L}; \mathbb{R})) = \begin{cases} 1, & \text{if } f(v_i) \neq 0 \text{ for some } i = 1, 2; \\ 2, & \text{if } f(v_1) = f(v_2) = 0. \end{cases} \]
where

\[
\text{dim} \text{Im}\left( \omega : C_1(\mathcal{L}; \mathbb{R}) \rightarrow C_2(\mathcal{L}; \mathbb{R}) \right) = \begin{cases} 
1, & \text{if } f(v_i) \neq 0 \text{ for some } i = 1, 2; \\
0, & \text{if } f(v_1) = f(v_2) = 0.
\end{cases}
\]

Thus

\[
H^1(\mathcal{L}, f) = H^1(\mathcal{L}, \omega, 0) = \begin{cases} 
\mathbb{R}, & \text{if } f(v_i) \neq 0 \text{ for some } i = 1, 2; \\
\mathbb{R}^2, & \text{if } f(v_1) = f(v_2) = 0
\end{cases}
\]

and

\[
H^2(\mathcal{L}, f) = H^2(\mathcal{L}, \omega, 0) = \begin{cases} 
0, & \text{if } f(v_i) \neq 0 \text{ for some } i = 1, 2; \\
\mathbb{R}, & \text{if } f(v_1) = f(v_2) = 0.
\end{cases}
\]

Moreover,

\[
H^n(\mathcal{L}, f) = H^n(\mathcal{L}, \omega, 0) = 0
\]

for any \( n \neq 1, 2. \)

- Let \( s = 1. \) Let

\[
\mu = u_{01}dv_0 \wedge dv_1 + u_{02}dv_0 \wedge dv_2 + u_{12}dv_1 \wedge dv_2.
\]

Then

\[
\mu(v_0v_1v_2) = mu(v_0v_1) = \mu(v_0v_2) = 0.
\]

Thus the induced homomorphism \( \mu_\ast \) between the cohomology groups is identically zero.

**Example 5.3.** Let \( V = \{v_0, v_1, v_2, v_3\}. \) Let \( t = 1. \) Then any \( \alpha \in T_3(V) \) can be expressed as

\[
\alpha = f(v_0, v_1, v_2) \frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} + f(v_0, v_1, v_3) \frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_3} + f(v_0, v_2, v_3) \frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial v_3} + f(v_1, v_2, v_3) \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial v_3}
\]

where

\[
f : V \times V \times V \rightarrow \mathbb{R}
\]

is a real function on the 3-fold Cartesian product of \( V. \) By Proposition 3.2, the adjoint \( \omega \in T^3(V) \) of \( \alpha \) is given by

\[
\omega = -f(v_0, v_1, v_2)dv_0 \wedge dv_1 \wedge dv_2 - f(v_0, v_1, v_3)dv_0 \wedge dv_1 \wedge dv_3 - f(v_0, v_2, v_3)dv_0 \wedge dv_2 \wedge dv_3 - f(v_1, v_2, v_3)dv_1 \wedge dv_2 \wedge dv_3.
\]

Let \( s = 1. \) Then any \( \beta \in T_2(V) \) can be expressed as

\[
\beta = g(v_0, v_1) \frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_1} + g(v_0, v_2) \frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_2} + g(v_0, v_3) \frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_3} + g(v_1, v_2) \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} + g(v_1, v_3) \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_3} + g(v_2, v_3) \frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial v_3}.
\]
where
\[ g \colon V \times V \rightarrow \mathbb{R} \]
is a real function on the 2-fold Cartesian product of \( V \). By Proposition 3.3, the adjoint \( \mu \in T^2(V) \) of \( \beta \) is given by
\[
\mu = -g(v_0, v_1)dv_0 \wedge dv_1 - g(v_0, v_2)dv_0 \wedge dv_2 - g(v_0, v_3)dv_0 \wedge dv_3 \\
- g(v_1, v_2)dv_1 \wedge dv_2 - g(v_1, v_3)dv_1 \wedge dv_3 - g(v_2, v_3)dv_2 \wedge dv_3.
\]
Consider the complete hypergraph
\[
\Delta [V] = \{ v_0, v_1, v_2, v_3, v_0v_1, v_0v_2, v_0v_3, v_1v_2, v_1v_3, v_2v_3, \\
v_0v_1v_2, v_0v_1v_3, v_0v_2v_3, v_1v_2v_3, v_0v_1v_2v_3 \}.
\]
Then \( \Delta [V] \) is a simplicial complex and is also a co-simplicial complex.

- By a direct calculation,
\[
\alpha(v_i) = 0, \quad i = 0, 1, 2, 3, \\
\alpha(v_iv_j) = 0, \quad 0 \leq i < j \leq 3, \\
\alpha(v_iv_jv_k) = 0, \quad 0 \leq i < j < k \leq 3, \\
\alpha(v_0v_1v_2v_3) = (-1)^{0+1+2}f(v_0, v_1, v_2)v_3 + (-1)^{0+1+3}f(v_0, v_1, v_3)v_2 \\
+ (-1)^{0+2+3}f(v_0, v_2, v_3)v_1 + (-1)^{1+2+3}f(v_1, v_2, v_3)v_0 \\
= -f(v_0, v_1, v_2)v_3 + f(v_0, v_1, v_3)v_2 - f(v_0, v_2, v_3)v_1 \\
+ f(v_1, v_2, v_3)v_0.
\]
It follows that
\[
\dim \text{Im} \left( \alpha : C_3(\Delta [V]; \mathbb{R}) \rightarrow C_0(\Delta [V]; \mathbb{R}) \right) \\
= \begin{cases} 
1, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;} \\
0, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3
\end{cases}
\]
or equivalently,
\[
\dim \text{Ker} \left( \alpha : C_3(\Delta [V]; \mathbb{R}) \rightarrow C_0(\Delta [V]; \mathbb{R}) \right) \\
= \begin{cases} 
0, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;} \\
1, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3
\end{cases}
\]
Consequently,
\[
H_0(\Delta [V], \alpha, 0) = \begin{cases} 
3, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;} \\
4, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3
\end{cases}
\]
and

\[ H_1(\Delta[V], \alpha, 0) = \begin{cases} 0, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;} \\ 1, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3. \end{cases} \]

By a similar calculation, we have

\[ H_1(\Delta[V], \alpha, 0) = \mathbb{R}^6, \quad H_2(\Delta[V], \alpha, 0) = \mathbb{R}^4. \]

Moreover,

\[ H_n(\Delta[V], \alpha, 0) = 0 \quad \text{for any} \quad n \neq 0, 1, 2, 3. \]

- It is direct that

\[ \beta \circ \alpha(v_i) = 0 \quad \text{for any} \quad 0 \leq i \leq 3, \]

\[ \beta \circ \alpha(v_i v_j) = 0 \quad \text{for any} \quad 0 \leq i < j \leq 3, \]

\[ \beta \circ \alpha(v_i v_j v_k) = 0 \quad \text{for any} \quad 0 \leq i < j < k \leq 3, \text{ and } \]

\[ \beta \circ \alpha(v_0 v_1 v_2 v_3) = 0. \]

Therefore, the induced homomorphism \( \beta_* \) between the homology groups is the zero map.

- By a direct calculation,

\[
\begin{align*}
\omega(v_0) &= -f(v_1, v_2, v_3)dv_1 \wedge dv_2 \wedge dv_3(v_0) \\
&= f(v_1, v_2, v_3)v_0v_1v_2v_3, \\
\omega(v_1) &= -f(v_0, v_2, v_3)dv_0 \wedge dv_1 \wedge dv_3(v_1) \\
&= -f(v_0, v_2, v_3)v_0v_1v_2v_3, \\
\omega(v_2) &= -f(v_0, v_1, v_3)dv_0 \wedge dv_1 \wedge dv_2(v_2) \\
&= f(v_0, v_1, v_3)v_0v_1v_2v_3, \\
\omega(v_3) &= -f(v_0, v_1, v_2)dv_0 \wedge dv_1 \wedge dv_2(v_3) \\
&= -f(v_0, v_1, v_2)v_0v_1v_2v_3, \\
\omega(v_i v_j) &= 0, \quad 0 \leq i < j \leq 3, \\
\omega(v_i v_j v_k) &= 0, \quad 0 \leq i < j < k \leq 3, \\
\omega(v_0 v_1 v_2 v_3) &= 0.
\end{align*}
\]
It follows that

\[
\dim \text{Im} \left( \omega : C_0(\Delta[V]; \mathbb{R}) \rightarrow C_3(\Delta[V]; \mathbb{R}) \right) = \begin{cases} 
1, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;}
0, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3
\end{cases}
\]

or equivalently,

\[
\dim \text{Ker} \left( \omega : C_0(\Delta[V]; \mathbb{R}) \rightarrow C_3(\Delta[V]; \mathbb{R}) \right) = \begin{cases} 
3, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;}
4, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3
\end{cases}
\]

Consequently,

\[
H^0(\Delta[V], \omega, 0) = \begin{cases} 
3, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;}
4, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3
\end{cases}
\]

and

\[
H^3(\Delta[V], \omega, 0) = \begin{cases} 
0, & \text{if } f(v_i, v_j, v_k), 0 \leq i < j < k \leq 3, \text{ are not all zero;}
1, & \text{if } f(v_i, v_j, v_k) = 0 \text{ for any } 0 \leq i < j < k \leq 3.
\end{cases}
\]

By a similar calculation, we have

\[
H^1(\Delta[V], \omega, 0) = \mathbb{R}^6, \quad H^2(\Delta[V], \omega, 0) = \mathbb{R}^4.
\]

Moreover,

\[
H^n(\Delta[V], \omega, 0) = 0
\]

for any \(n \neq 0, 1, 2, 3\).

• It is direct to see that the induced homomorphism \(\mu_*\) between the cohomology groups is the zero map.

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References

[1] C. Berge, Graphs and hypergraphs. North-Holland Mathematical Library, Amsterdam, 1973.
[2] Raoul Bott, Loring W. Tu, Differential forms in algebraic topology. Springer-Verlag Berlin and Heidelberg, 1982.

[3] Stephane Bressan, Jingyan Li, Shiquan Ren, Jie Wu, The embedded homology of hypergraphs and applications, *Asian Journal of Mathematics* 23 (3), 479-500, 2019.

[4] S. S. Chern, W. H. Chen, K. S. Lam, Lectures on differential geometry. World Scientific, 2000.

[5] F. R. Cohen, J. Wu, On braid groups and homotopy groups, *Geometry and Topology Monographs* 13, 169-193, 2008.

[6] Robert. J. MacG. Dawson, Homology of weighted simplicial complexes, *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 31 (3), 229-243, 1990.

[7] A. Dimakis, F. Müller-Hoissen, Differential calculus and gauge theory on finite sets, *Journal of Physics A: Mathematical and General* 27 (9), 3159-3178, 1994.

[8] A. Dimakis, F. Müller-Hoissen, Discrete differential calculus: Graphs, topologies, and gauge theory, *Journal of Mathematical Physics* 35 (12), 6703-6735, 1994.

[9] A. Dimakis, F. Müller-Hoissen, Discrete Riemannian geometry, *Journal of Mathematical Physics* 40 (3), 1518-1548, 1999.

[10] Edward B. Curtis, Simplicial homotopy theory, *Advance in mathematics* 6, 107-209, 1971.

[11] S. Eilenberg, J. A. Zilber, Semi-simplicial complexes and singular homology, *Annals of Mathematics* 51, 499-513, 1950.

[12] Goerss, Paul G., Jardine, John, Simplicial homotopy theory. Birkhäuser Basel, 2009.

[13] Alexander Grigor’yan, Yong Lin, Shing-Tung Yau, Torsion of digraphs and path complexes, arXiv: 2012.07302v1, 2020.

[14] Alexander Grigor’yan, Yong Lin, Yuri Muranov, Shing-Tung Yau, Homologies of path complexes and digraphs, arXiv: 1207.2834, 2013.

[15] Alexander Grigor’yan, Yong Lin, Yuri Muranov, Shing-Tung Yau, Homotopy theory for digraphs, *Pure and Applied Mathematics Quarterly*, 10 (4), 619-674, 2014.

[16] Alexander Grigor’yan, Yong Lin, Yuri Muranov, Shing-Tung Yau, Cohomology of digraphs and (undirected) graphs, *Asian Journal of Mathematics*, 15 (5), 887-932, 2015.

[17] Alexander Grigor’yan, Yuri Muranov, Shing-Tung Yau, Homologies of digraphs and Künnefeh formulas, *Communications in Analysis and Geometry*, 25, 969-1018, 2017.

[18] Alexander Grigor’yan, Yuri Muranov, Shing-Tung Yau, Path complexes and their homologies, *Journal of Mathematical Sciences*, 248 (5), 564-599, 2020.

[19] Allen Hatcher, Algebraic topology. Cambridge University Press, 2002.
[20] Fengchun Lei, Fengling Li, Jie Wu, On simplicial resolutions of framed links, *Transactions of The American Mathematical Society* **366** (6), 3075-3093, 2014.

[21] Ib H. Madsen, Jørgen Tornehave, From calculus to cohomology: De Rham cohomology and characteristic classes. Cambridge University Press, 1997.

[22] John Milnor, The geometric realization of a semi-simplicial complex, *Annals of Mathematics, 2nd Ser.*, **65** (2), 357-362, 1957.

[23] J.R. Munkres, Elements of algebraic topology. Addison-Wesley Publishing Company, California, 1984.

[24] A.D. Parks and S.L. Lipscomb, Homology and hypergraph acyclicity: a combinatorial invariant for hypergraphs. Naval Surface Warfare Center, 1991.

[25] Fedor Pavutnitskiy, Jie Wu, A simplicial James-Hopf map and decompositions of the unstable Adams spectral sequence for suspensions, *Algebraic and Geometric Topology* **19** (1), 77-108, 2019.

[26] Shiquan Ren, Simplicial-like identities for the Paths and the regular paths on discrete sets (unpublished manuscript), arXiv 2107.09868, 2021.

[27] Shiquan Ren, Chengyuan Wu, Jie Wu, Weighted persistent homology, *Rocky Mountain Journal of Mathematics* **48** (8), 2661-2687, 2018.

[28] Shiquan Ren, Chengyuan Wu, Jie Wu, Computational tools in weighted persistent homology, *Chinese Annals of Mathematics, Ser. B* **42** (2), 237-258, 2021.

[29] Chengyuan Wu, Shiquan Ren, Jie Wu, Kelin Xia, Discrete Morse theory for weighted simplicial complexes, *Topology and its Applications* **270**, Article 107038, 2020.

[30] Jie Wu, Simplicial objects and homotopy groups. Braids, 31-181, Lecture Notes Series, Institute of Mathematical Sciences, National University of Singapore, **19**, World Scientific Publishing, Hackensack, NJ, 2010.

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