Transient chaos under coordinate transformations in relativistic systems

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Abstract

We use the Hénon-Heiles system as a paradigmatic model for chaotic scattering to study the Lorentz factor effects on its transient chaotic dynamics. In particular, we focus on how time dilation occurs within the scattering region by measuring the time in a non-inertial clock comoving with the particle. We observe that the several events of time dilation that the particle undergoes exhibit sensitivity to initial conditions. However, the structure of the singularities appearing in the escape time function remains invariant under coordinate transformations. This occurs because the singularities are closely related to the chaotic saddle, which is a time measure independent set. Using a Cantor set approach, we show that the fractal dimension of the escape time function is relativistic invariant. Finally, we relate the fractality in phase space to the unpredictability of the particle final destination. In order to quantify this fractality, we compute the fractal dimensions of the escape time functions as measured with inertial and non-inertial clocks, by means of the uncertainty dimension algorithm. We conclude that, from a mathematical point of view, chaotic transient phenomena are equally predictable in the proper and the inertial reference frames and that transient chaos is coordinate invariant.

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I. INTRODUCTION

Chaotic scattering in open Hamiltonian systems is a fundamental part of the theoretical study of dynamical systems. There are many applications in numerous fields of physics, such as the interaction between the solar wind and the magnetosphere tail [1], the simulation in several dimensions of the molecular dynamics [2], the modeling of chaotic advection of particles in fluid mechanics [3], or the analysis of the escaping mechanism from a star cluster or a galaxy [4, 5], to name a few.

A scattering phenomenon can be described as a process in which a particle travels freely from a remote region and encounters an obstacle, often described in terms of a potential, which affects its evolution. Finally, the particle leaves the interaction region and continues its journey freely. Depending on the nature of the potential, this interaction can be nonlinear, possibly leading the particle to perform transient chaotic dynamics, i.e., chaotic dynamics with a finite lifetime [6, 7]. Scattering processes are typically studied by means of the scattering functions, which relate the particle states at the beginning of its evolution and when the interaction with the potential has already taken place. Thus, nonlinear interactions can make these functions exhibit self-similar arrangements of singularities, which hinder the system predictability [8]. Transient chaos is a manifestation of the presence in phase space of a chaotic set called non-attracting chaotic set, also called chaotic saddle [9]. This phenomenon can be found in a wide variety of situations [10], as for example the dynamics of decision making, the doubly transient chaos of undriven autonomous mechanical systems or even in the sedimentation of volcanic ash.

There have been numerous efforts to characterize chaos in relativistic systems in an observer-independent manner [11]. It has been rigorously demonstrated that the sign of the Lyapunov exponents is invariant under coordinate transformations that satisfy four minimal conditions [12]. More specifically, such conditions consider that a valid coordinate transformation has to leave the system autonomous, its phase space bounded, the invariant measure normalizable and the domain of the new time parameter infinite [12]. As a consequence, chaos is a property of relativistic systems independent of the choice of the coordinate system in which they are described. In other words, homoclinic tangles cannot be untangled by means of coordinate transformations. We shall utilize the Lorentz transformations along this paper (we delve into the details in Sec. III), which satisfy this set of conditions [13].
Moreover, the Lorentz transformations also preserve the spectrum of generalized dimensions, except for the information dimension \[^{14}\].

Despite the fact that the sign of the Lyapunov exponents is invariant, the precise values of these exponents, which indicate “how chaotic” a dynamical system is, are noninvariant. Therefore, this lack of invariance leaves some room to explore how coordinate transformations affect the unpredictability in dynamical systems with transient chaos. In the present work we analyze the structure of singularities of the scattering functions and its fractal dimension under a valid coordinate transformation. In particular, we compute the escape time function as measured in an inertial clock and another non-inertial clock, respectively. Computing this fractal dimension helps us to characterize the unpredictability in open Hamiltonian systems, since it enables to infer the dimension of the chaotic saddle \[^{15}\]. Indeed, this purely geometrical method has been proposed as an independent-observer procedure to determine whether the system behaves chaotically \[^{16}\].

The Lorentz factor effects on the dynamical properties of the system have already been addressed in relativistic chaotic scattering \[^{17, 18}\]. These works focus on special relativity and study the dynamical regime, the decay law, the basin topology and the fractal dimension of scattering functions. In this paper, we focus on how changes of reference frame affect these typical phenomena of chaotic scattering. We describe the model in Sec. II, which consists on a relativistic version of the Hénon-Heiles system. Two well-known scattering functions are explored in Sec. III, such as the exit through which the particle escapes and its escape time. Subsequently, the invariance under a coordinate transformation of the fractal dimension is demonstrated in Sec. IV. We quantify the unpredictability of the escape times and analyze the effect of such a reference frame modification in Sec. V. We conclude with a discussion of the main results and findings of the present work in Sec. VI.

II. MODEL DESCRIPTION

The Hénon-Heiles system was proposed in 1964 to study the existence of a third integral of motion in galactic models with axial symmetry \[^{19}\]. We can consider a single particle whose total mechanical energy in the Newtonian approximation is

\[
E_N = \frac{p^2}{2m} + V(x, y),
\] (1)
where \( p = (p, q) \) is the momentum, \( m \) the mass of the particle (both of them determine the system kinetic energy) and \( V \) is the potential energy that only depends on the particle position. The total energy \( E_N \) is conserved along the trajectory described by the particle, which is launched from the interior of the potential well, within a finite region of the phase space called the scattering region. The potential of the Hénon-Heiles system is written as

\[
V(x, y) = \frac{1}{2}(x^2 + y^2) + \left(x^2 y - \frac{1}{3} y^3\right).
\] (2)

When the energy is above a threshold value called the escape energy \( E_e \), i.e., when \( E_N > E_e = 1/6 \), the potential well exhibits three exits due to its triangular symmetry, as visualized in Fig. 1. We call Exit 1 the exit located at the top \( (y \to +\infty) \), Exit 2 the one located downwards to the left \( (x \to -\infty, y \to -\infty) \) and Exit 3 the one at the right \( (x \to +\infty, y \to -\infty) \). One of the characteristics of open Hamiltonian systems with escapes is the existence of highly unstable periodic orbits known as Lyapunov orbits [20], which are placed near the exits. In fact, when a trajectory crosses through a Lyapunov orbit with its momentum vector pointing towards the outside of the scattering region, the particle escapes to infinity and never returns back to the scattering region. Finally, to conclude this description we recall that the system presents four fixed points: a stable fixed point at the potential minimum and three saddle points near the exits.

The value of \( E_N \) determines both the possibility of escaping and also the dynamical

![FIG. 1: (a) The three-dimensional representation of the Hénon-Heiles potential \( V(x, y) = \frac{1}{2}(x^2 + y^2) + (x^2 y - \frac{1}{3} y^3) \). (b) The isopotential curves show that the Hénon-Heiles system is open and has triangular symmetry. If the energy of the particle is higher than a threshold value \( E_e \), related to the potential saddle points, there exist bounded and unbounded orbits. Following these latter trajectories the particle leaves the scattering region through any of the three exits.](image-url)
regime to which the particle belongs. Indeed, regarding the energy $E_N$, we can distinguish two dynamical open regimes in which escapes are allowed: the nonhyperbolic regime for $E_N \in (E_e = 1/6, E_d \approx 0.2309)$ and the hyperbolic regime when $E_N \in [E_d \approx 0.2309, +\infty)$, where $E_d$ is the energy value for which the destruction of the KAM islands [21] occurs. The transition from the nonhyperbolic to the hyperbolic regime has been recently measured in [22] by using the basin entropy [23]. The only difference between these regimes is the coexistence of the KAM tori together with the chaotic saddle. The KAM tori consist of invariant sets of uncountable stable quasiperiodic trajectories usually located inside the scattering region. Its presence has crucial consequences on the dynamics of the particle, as for example the delay of escaping trajectories due to their stickiness [24]. This latter property results in an algebraic decay of the survival probability of a particle in the scattering region. Moreover, the particle can be trapped in the KAM tori forever describing a quasiperiodic orbit inside the scattering region. When the KAM islands destruction takes place, the stable and unstable manifolds of the chaotic saddle are never tangent and every saddle point is hyperbolic [9]. Consequently, the decay law becomes exponential.

When the speed of the particle is comparable to the speed of light, the relativistic effects have to be taken into consideration [25]. In the present work we consider a relativistic version of the Hénon-Heiles system and, therefore, a particle of rest mass $m$ whose dynamics is governed by the conservative Hamiltonian

$$H = \gamma mc^2 + V(x, y),$$  
(3)

where $c$ is the value of the speed of light and $\gamma$ is the Lorentz factor. This is defined as

$$\gamma = \sqrt{1 + \frac{\mathbf{v}^2}{m^2 c^2}} = \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - \beta^2}},$$  
(4)

where $\mathbf{v}$ is the velocity vector of the particle and $\beta = |\mathbf{v}|/c$ the ratio between the speed of the particle and the speed of light. The Lorentz factor $\gamma$ and $\beta$ are two equivalent ways to express how large is the speed of the particle compared to the speed of light. These two factors vary in the range $\gamma \in [1, +\infty)$ and the range $\beta \in [0, 1)$, respectively. For convenience, along this work we shall use $\beta$ as a parameter and, without loss of generality, set the value of the rest mass of the particle equal to one ($m = 1$) hereafter. Hamilton’s canonical equations
can be derived from Eq. (3), yielding the equations of motion

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p} = \frac{p}{\gamma}, \\
\dot{p} &= -\frac{\partial H}{\partial x} = -x - 2xy, \\
\dot{y} &= \frac{\partial H}{\partial q} = \frac{q}{\gamma}, \\
\dot{q} &= -\frac{\partial H}{\partial y} = y^2 - x^2 - y.
\end{align*}
\]  

(5)

Although the complete phase space is four-dimensional, the conservative Hamiltonian constrains the dynamics to a three-dimensional manifold of the phase space, known as the energy shell.

Some recent works aim at isolating the effects of the variation of the Lorentz factor \( \gamma \) (or, equivalently, \( \beta \)) from the remaining system variables in the context of relativistic chaotic scattering [17, 18]. In order to accomplish this, they modify the initial value of \( \beta \) and use it as the only parameter of the dynamical system. Since \( \beta \) is a quantity that depends on \( |v| \) and \( c \), they choose to vary the numerical value of \( c \) for each computation. Needless to say, the real value of the speed of light \( c \) remains constant during the trajectory of the particle. It is just the system of units that is permanently changing and, consequently, its numerical value. On the contrary, with this change, all the other model parameters, which keep the same numerical values are being modified. In this manner, we consider a constant initial value of the speed of the particle \( |v_0| \) with a different value of \( \beta \) in every simulation.

The fundamental reason for deciding to increase the kinetic energy of the system by reducing the numerical value of the speed of light is simply as follows. If we keep the Hénon-Heiles potential constant and increase the speed of the particle to values close to the speed of light, the potential will be in a much lower energy regime compared to the kinetic energy of the particle. Therefore, the potential becomes negligible and the interaction between them becomes irrelevant. Consequently, each time we select a value of the speed of light we are scaling the system, and with it the potential is rescaled as well. This sequence of systems with increasing value of \( \beta \) represents potentials with the morphology of the Hénon-Heiles system, but at different scales. In this way, the effects of the Lorentz factor are isolated from the other system variables, because \( \gamma \) is the only parameter that differentiates all these systems at comparison.

Henceforth, we have arbitrarily chosen \( |v_0| = \sqrt{2 \cdot E_N} \approx 0.583 \), which corresponds to the open nonhyperbolic regime with energy \( E_N = 0.17 \), close to the escape energy \( E_e \) in the nonrelativistic approximation. Thus, we analyze how the relativistic parameter \( \beta \), as its value increases, affects the dynamical properties starting from the nonhyperbolic regime.
Subsequently, the units of the problem change inevitably and, for instance, if the simulation is carried out for a small $\beta$, where $|v_0| \ll c$, the initial speed of the particle only represents a very low percentage of the speed of light. In this case, we recover the Newtonian scheme and the classical version of the Hénon-Heiles system, with $|v_0| \approx 0.583$ m/s and $m = 1$ kg, when $c = 3 \cdot 10^8$ m/s. On the contrary, if the simulation takes place with a value of $\beta$ near one, the speed of the particle represents a high percentage of the speed of light and the system is reduced to the Planck units range.

The total KAM islands destruction occurs abruptly near the critical value $\beta_c \approx 0.4$ and the topology of the phase space becomes less fractal [18], since we also establish $|v_0| \approx 0.583$. We focus on the hyperbolic regime in this work, because the relativistic phenomena are more relevant for high values of $\beta$. For this reason, the simulations are run for values of $\beta \in [0.5, 1)$ and by means of a fixed step fourth-order Runge-Kutta method [26], evolving the particles from the minimum potential $(x_0, y_0) = (0, 0)$. We recall that the initial values of the momentum $(p_0, q_0)$ depend on the chosen initial value of $\beta$ and, therefore, this computational technique (to vary the value of $\beta$ fixing $|v_0|$) is an ideal method to control which dynamical regime the relativistic particle evolves in. For example, a particle trapped in the KAM tori can escape if the initial value of $\beta$ is high enough, as depicted in Fig. 2.

![FIG. 2: The evolution of a particle launched within the scattering region from the same initial condition $(x_0, y_0, \theta_0) = (0, 0.6, \pi)$ for different values of $\beta$. The shooting angle of the particle is formed by the initial velocity vector and the positive $x$-axis. (a) For a very low $\beta$ (Newtonian scheme), the particle is trapped in the KAM tori and describes a bounded quasiperiodic trajectory. (b) The value of $\beta$ is large enough to destroy the KAM tori and the particle leaves the scattering region following a trajectory typical of transient chaos. (c) Finally, a larger value of $\beta$ than in (b) makes the particle escape faster.](image-url)
III. ESCAPE TIMES IN INERTIAL AND NON-INERTIAL FRAMES

The scattering functions enable us to represent the relation between input and output dynamical states of the particle, i.e., how the interaction of the particle with the potential takes place. As our study focuses on the hyperbolic regime, the particle escape from the potential well always befalls. If this interaction is linear, the underlying dynamics will be regular and, therefore, the scattering functions are smooth. However, if the interactions are nonlinear and the dynamics is chaotic, the scattering function presents singularities produced by the chaotic saddle.

The potential of the Hénon-Heiles makes the particle describe chaotic trajectories before converging to a specific dynamical state. These specific final states are identified with the allowed channels through which the particle escapes to infinity. In order to verify the sensitivity of the system to the three exits, we launch particles from the potential minimum \((x_0, y_0) = (0, 0)\), varying slightly the shooting angle \(\theta\) that is formed by the initial velocity vector \(\mathbf{v}_0\) and the positive \(x\)-axis, as shown in Fig. 3(a). The value of its kinetic energy remains bounded, because it bounces back and forth against the potential well before escaping. Hereafter, we shall refer to this fact as bounded dynamics. Then, we propose a new escape criterion based on the fact that the dynamics followed by the particle is bounded inside the potential well. The escape happens at the time \(t_e\) when the Lorentz factor of the particle surpasses a critical maximum value \(\gamma_c\), so that \(\gamma(t_e) > \gamma_c\). As in the Hamiltonian system the mechanical energy is conserved, this critical value of escape is defined by the kinetic energy that the particle has at the potential minimum, therefore \(\gamma_c = 1/\sqrt{1 - \beta^2}\), where we recall that \(\beta\) is the initial speed. In summary, given a value of \(\beta\), this critical value of the Lorentz factor is reached when the kinetic energy is maximum and the particle dynamics is bounded. We show a simple example of escape in Fig. 3(b). This new escape criterion is computationally very affordable compared to the Lyapunov orbit criterion that is usually implemented in previous works. In addition, our proposed escape criterion also includes all the escapes that take place when the Lyapunov orbit criterion is considered.

A particle launched with \(\theta = \pi/2\) escapes directly towards the Exit 1 (red) as shown in Fig. 4(a), whereas if it is launched with \(\theta = 5\pi/6\), the particle bounces against the potential well placed between Exit 1 and Exit 2 (green) and escapes through the Exit 3 (blue). The whole structure of exits in between is apparently fractal. Nonetheless, the exit function
becomes smoother when the value of $\beta$ increases, but it is never completely smooth. On the other hand, the stable manifold as computed from the boundary between the exit basins is shown in Fig. 4(b). We recall that the chaotic saddle is an invariant set formed by the intersection of the stable and unstable manifolds, and we emphasize that the concept invariant here implies that a trajectory that has a point belonging to an invariant set can never leave the mentioned set. However, to distinguish this invariance from the invariance of a set after a coordinate transformation, we refer to it as invariant under coordinate transformations or relativistic invariant set.

The escape time of a particle is defined by the time it spends evolving inside the scattering region before escaping to the infinity. In nonrelativistic systems, the particular clock in which the time is measured is irrelevant since time is absolute. However, here we consider two time quantities: the time $t$ that is measured by an inertial reference frame at rest, and the proper time $\tau$ as measured by a non-inertial reference frame co-moving with the particle. This proper time is simply the time measured by a clock attached to the particle.

We remark that the proper time is a spacetime invariant under Lorentz transformations. As is well known, any uniformly moving clock runs slow by a factor $\sqrt{1 - \beta^2}$ when compared to the other identically constructed and synchronized clocks positioned at different points

![FIG. 3: (Color online) (a) Each of the exits is identified with a different color, such that Exit 1 (red), Exit 2 (green) and, finally, Exit 3 (blue). In order to avoid redundant results due to the triangular symmetry of the well, we only let the particle evolve from the angular region $\theta_0 \in [\pi/2, 5\pi/6]$ (black dashed lines). (b) The set of points $(x, y)$ where $\gamma(x, y) = \gamma_c$ is depicted in brown, and corresponds to the proposed escape criterion. The Lyapunov orbit (LO) associated with the Exit 1 is also drawn (black). The trajectory of the example starts from the initial condition $(x_0, y_0, \theta_0) = (0, 0, 2.23)$ with $\beta = 0.5$ and escapes through the Exit 1.](image)
FIG. 4: (Color online) (a) The scattering function of the exits given the parameter map \((\beta \in [0.5, 0.99], \theta_0 \in [\pi/2, 5\pi/6])\) in the hyperbolic regime. The initial conditions \((2000 \times 2000)\) that lead the particle to escape through the different exits are identified with three colors, such that Exit 1 (red), Exit 2 (green) and Exit 3 (blue). (b) The stable manifold of the open system, i.e., the boundary of the exit regions. For the sake of clarity, we emphasize that the points that belong to this manifold are the initial conditions \(\theta_0\) located in the boundary of the exit regions given a value of \(\beta\).

which are at rest in some inertial frame. Therefore, we assume that, at any instant of time, the clock of the accelerating particle advances at the same rate as an inertial clock that momentarily had the same velocity [27]. In this manner, given an infinitesimal time interval \(dt\), the particle’s clock will measure a time interval

\[
d\tau = \frac{dt}{\gamma(t)},
\]

where \(\gamma(t)\) is the particle’s Lorentz factor at the instant of time \(t\). We recall that \(\gamma(\tau) = 0\) for the proper observer. Since the Lorentz factor is a quantity defined in \(\gamma(t) \in [1, +\infty)\), the proper time interval always obeys that \(d\tau \leq dt\), which is just the mathematical statement of the twin paradox. When the particle’s velocities is very close to the speed of light, the time dilation phenomenon takes place, so that the time of the particle’s clock runs more slowly in comparison to clocks at rest in the potential. It is important to bear in mind that, since we are studying the system in the context of special relativity, it is assumed that the potential does not affect the clocks’ rate. In other words, we assume that all the clocks placed at rest in the potential are ticking at the same rate.
Without loss of generality, Eq. (6) can be expressed as an integral in the form
\[
\tau_e = \int_0^{t_e} \frac{dt}{\gamma(t)}.
\] (7)

As it has been shown above, if the dynamics is bounded along the particle’s trajectory before escaping, the Lorentz factor value also has a finite value \(\gamma(t) \in [1, \gamma_c]\). For this reason, we assume that there exists an average value of the Lorentz factor along the particle’s trajectory given a value of \(\beta\). In general, it can be estimated as a phase space average \(\bar{\gamma} = \langle \gamma(t) \rangle\), and depends on the particle’s distribution of configurations \(\rho(p, q, t)\) in the \(N\)-dimensional phase space, where \(q = (x, y)\) is the vector of the spatial coordinates. In order to perform comparisons with our numerical results, we can estimate \(\bar{\gamma}\) as the arithmetic mean between the maximum and minimum values of the bounded Lorentz factor given a value of \(\beta\)
\[
\bar{\gamma}(\beta) = \frac{1 + \gamma_c}{2} = \frac{1 + \sqrt{1 - \beta^2}}{2\sqrt{1 - \beta^2}}.
\] (8)

Applying this definition to Eq. (7), we can estimate
\[
\tau_e \approx \frac{1}{\bar{\gamma}} \int_0^{t_e} dt = \frac{t_e}{\bar{\gamma}}.
\] (9)

In this manner, the escape time \(t_e\) and the escape proper time \(\tau_e\) should obey an approximate linear relation as the dynamics remains bounded. Consequently, the time difference measured by the two clocks also increases linearly as the time \(t_e\) goes by
\[
\delta t_e = t_e - \tau_e = \frac{1 - \sqrt{1 - \beta^2}}{1 + \sqrt{1 - \beta^2}} t_e,
\] (10)

where the slope is always positive and less than one because the factor \(\sqrt{1 - \beta^2}\) is in \((0, 1]\).

The scattering function of escape times \(t_e\) appears in Fig. 5 employing the same shooting method and parameter map as for the exit function. Likewise as the fractality, the values of the escape time decrease as \(\beta\) increases due to the fact that the particle’s velocities become closer to the speed of light and then its energy increases as well. The structure of singularities is again associated to the stable manifold. Therefore, the fractality of the escape time function is also closely related to the exit function. This is a first evidence that the fractality must be an observer-independent feature of the escape time function, because the exit through which the particle escapes does not depend on changes of the clock in which we measure the escape time. Moreover, the escape proper time function exhibits a
similar structure of singularities because of the linear relation described in Eq. 9. In order to show this similarity, we plot both escape time functions versus the initial shooting angle in Figs. 5(b) and 5(c) for \( \beta = 0.5 \) and \( \beta = 0.8 \), respectively. We solve the integral shown in Eq. (7) using the Simpson’s rule \[28\] to compute the proper times. Despite being almost identical structures, the dilation time phenomenon reduces the escape times \( \tau_e(\theta_0) \), obtaining always \( \tau_e(\theta_0) \leq t_e(\theta_0) \). This effect of time reduction increases when the value of \( \beta \) becomes higher. There exist peaks of escape time in the initial conditions \( \theta_0 \) that belong to the stable

![Graphs showing escape times and time differences for different values of \( \beta \)](image)

**FIG. 5**: (Color online) (a) The scattering function of escape times \( t_e \) in logarithmic scale given the parameter map \( (\beta \in [0.5, 0.99], \theta_0 \in [\pi/2, 5\pi/6]) \) in the hyperbolic regime. The two black dashed lines corresponds to the subfigures (b) and (c), which show the scattering function of escape time \( t_e(\theta_0) \) (blue) and \( \tau_e(\theta_0) \) (red) for \( \beta = 0.5 \) and \( \beta = 0.8 \), respectively. (d, e) The time difference function \( \delta t_e(\theta_0) \) (black) for the same values of \( \beta \) indicated above.
manifold because a delay in converging to a definitive exit takes place.

Importantly, we observe in Figs. 5(d) and 5(e) that the time difference function $\delta t_e(\theta_0)$ also preserves the fractal structure. The reason for that is that the longer the time the particle spends in the well, the more travels from the center to the wells and back. If we think of each of these travels as an example of a twin paradox journey, we get an increasing time dilation for particles that spend more time. Moreover, since these times are sensitive to modifications in the initial conditions, so it is the time dilation effect. In this manner, sensitivity to initial conditions is here translated into sensitivity to time dilation. In this respect, we could introduce what might be called the triplet’s paradox. In this case an additional third sibling leaves the planet and comes back to the starting point having a different age than their two other siblings, because of the sensitivity to initial conditions. In particular, this phenomenon illustrates how chaotic dynamics affects typical relativistic phenomena and explains the fact that a fractal structure is also obtained in Figs. 5(d) and 5(e).

IV. INVARIANCE OF THE FRACTAL DIMENSION

The chaotic saddle and the stable manifold are self-similar fractal sets when the underlying dynamics is hyperbolic [9]. This fact is reflected in the peaks’ structure of the escape time functions, which is present at any scale of initial conditions. In this sense, the escape time functions share with the Cantor set some properties with regard to their singularities and, therefore, to their fractal dimensions. It is possible to study the fractal dimensions of the escape time functions in terms of a Cantor-like set [29, 30].

In this manner, we can build a Cantor-like set to schematically represent the escape of particles launched from different initial conditions $\theta_0$. We consider that a certain fraction $\eta_t$ of particles escapes from the scattering when a minimal characteristic time $t_0$ has elapsed. If these particles were launched from initial conditions located in a segment centered in the original interval, two new identical segments are created and the trajectories that began in those segments do not escape at least by a time $t_0$. Analogously, a same fraction of particles $\eta_t$ of the two surviving segments has escaped by a time $2t_0$. If we continue this iterative procedure for $3t_0$, $4t_0$ and so on, we obtain a Cantor-like set of Lebesgue measure zero, as
shown in Fig. 6(a), with associated fractal dimension $d_t$ that can be computed as

$$d_t = \frac{\ln 2}{\ln 2 - \ln (1 - \eta_t)}. \quad (11)$$

On the other hand, if the escape times are measured by the non-inertial clock, a fraction of particles $\eta_r$ escapes every time $\tau_0$ and, therefore, the associated fractal dimension $d_r$, can be correspondingly expressed as

$$d_r = \frac{\ln 2}{\ln 2 - \ln (1 - \eta_r)}. \quad (12)$$

We use the normalized probability densities of particle survival time within the scattering region, $P(t)$ and $P(\tau)$, to study qualitatively the values of $\eta_t$ and $\eta_r$. Then, measuring the escape times in an inertial clock we obtain the probability of particle’s survival

$$P(t) = \sigma_t e^{-\sigma_t t}. \quad (13)$$

We then consider the relation for the change between random variables $P(t)dt = P(\tau)d\tau$, to obtain the decay law of particles according to a non-inertial clock

$$P(\tau) = \gamma \sigma_t e^{-\sigma_t(\tau)}.$$

We can further assume that the hyperbolic dynamics implies that the decay law follows an approximately exponential behavior in any reference frame. Mathematically, this means that the probability of decay can be written in a non-inertial frame as $P(\tau) = \sigma_\tau e^{-\sigma_\tau \tau}$. This assumption is justified because of the underlying Cantor-like set structure and, as we show below, holds very well. The exponents $\sigma_t$ and $\sigma_\tau$ quantify how fast the normalized probabilities of survival within the well decrease. The higher the value of $\beta$, the sooner that the particle escapes in the non-inertial frame. Thus, these exponents exhibit a directly proportional growth with $\beta$, as we observe in Fig. 6(b). Moreover, $\sigma_t < \sigma_\tau$ is satisfied because the escapes always happen faster when measured the time in the proper clock, as already shown in Sec. 3. As previously done, we can also make the assumption that a particle follows a bounded evolution inside the well and then the average value of its Lorentz factor is $\bar{\gamma}$ (see Eq. (8)). Following this reasoning in Eq. (11) we can estimate

$$P(\tau) = \sigma_\tau e^{-\sigma_\tau \tau} \approx \bar{\gamma} \sigma_t e^{-\bar{\gamma} \sigma_t \tau}, \quad (15)$$

with $\sigma_\tau \approx \bar{\gamma} \sigma_t$. This latter approximation is nicely satisfied, as shown in Fig. 6(b) what indicates that our approach of bounded dynamics in open-Hamiltonian systems is adequate.
The fractions of particles that escape each time \( t_0 \) and \( \tau_0 \) can be calculated as

\[
\eta_t = \frac{P(t_0) - P(2t_0)}{P(t_0)} = 1 - e^{-\sigma t_0},
\]
\[
\eta_\tau = \frac{P(\tau_0) - P(2\tau_0)}{P(\tau_0)} \approx 1 - e^{-\bar{\sigma} \tau_0}.
\]  

(16)

We can now replace \( \bar{\sigma}_\tau = \bar{\gamma} \sigma_t \) into Eq. (16), approximate \( t_0 \approx \bar{\gamma} \tau_0 \) and, using Eqs. (11) and (12), obtain that \( d_t \approx d_\tau \). These results show that the fractal dimension of escape time functions is connected to the decay law. Finally, since our finding \( d_t \approx d_\tau \) holds for the clock of any particular trajectory, the present results are a strong indication that transient chaos in open Hamiltonian systems is relativistic invariant under non-inertial coordinate transformations and agrees with a similar conclusion shown in [12], where it is demonstrated that relativistic chaos is coordinate invariant. Using the results of such work, we provide a demonstration of this invariance right ahead.

FIG. 6: (Color online) (a) A self-similar Cantor set associated with an escape time function in the hyperbolic regime. A constant fraction of particles \( \eta_t = 1/3 \) escapes when the elapsed time is a multiple of the characteristic time \( t_0 \). (b) The evolution of the exponent \( \sigma \) versus \( \beta \) in the hyperbolic regime: \( \sigma_t(\beta) \) (blue), \( \sigma_\tau(\beta) \) (red) and, finally, \( \bar{\sigma}_\tau(\beta) \) (black triangles). Given a value of \( \beta, \sigma_t \) and \( \sigma_\tau \) have been calculated by a linear fitting between the quantities \( \ln P(t) \) and \( t \).
V. PERSISTENCE OF TRANSIENT CHAOS UNDER COORDINATE TRANSFORMATIONS

The scattering functions exhibit a fractal structure, which is responsible for the unpredictability in foreseeing the particle’s final dynamical state. We quantify this unpredictability by computing the fractal dimension $d$ associated with these functions. The self-similar structure of escape times in the hyperbolic regime involves fractal dimensions less than the unity ($d < 1$) and implies that the system exhibits sensitivity to initial conditions [30]. In order to calculate this fractal dimension, we make use of the uncertainty dimension algorithm described in Refs. [30, 31]. We recall that $d$ is also defined as the uncertainty dimension because of the method to compute it.

As the fractality is similar in all the regions of the phase space, we compute the fractal dimension using the shooting method previously described in Sec. III. We launch a particle from the potential minimum with a random shooting angle $\theta_0$ in the interval $[\pi/2, 5\pi/6]$ and measure the escape times $t_e(\theta_0)$ and $\tau_e(\theta_0)$, and the exit $\varepsilon(\theta_0)$ through escapes. Then, we carry out again the same procedure from a slightly different shooting angle $\theta_0 + \epsilon$, where $\epsilon$ can be considered a small perturbation, and calculate the quantities $t_e(\theta_0 + \epsilon)$, $\tau_e(\theta_0 + \epsilon)$ and $\varepsilon(\theta_0 + \epsilon)$. We say that an initial condition $\theta_0$ is uncertain in measuring, e.g, the escape time $t_e$, if the difference between the escape times $|t_e(\theta_0) - t_e(\theta_0 + \epsilon)|$ is higher than given time. This time is usually associated with the integration step $h$ of the numerical method employed to perform the computations.

In order to clarify the criterion adopted along this work, $h$ is the minimum time that empirically both clocks can measure and, therefore, that both observers can notice. We establish a fixed systematic error in the measure of time in any reference frame along the work, by setting $h = 0.005$. Thus, empirically, the differences between the escape times displayed on the screen of the clocks can be only multiples of $h$, so that the time differences $\Delta t_e(\theta_0), \Delta \tau_e(\theta_0) \geq 2h$ imply an uncertain $\theta_0$. However, the time in the non-inertial clock typically has more decimal digits than the step $h$, since it is the result of a computation by means of Eq. [7]. In this manner, we consider $3h/2$ as the criterion of an uncertain initial condition because the time differences $3h/2 \leq \Delta \tau_e(\theta_0) < 2h$ can be rounded up to $2h$ and can be noticed empirically by the non-inertial observer as uncertain. Moreover, the time $3h/2$ is higher than the systematic error of the measured differences of escape times, which
is $\sqrt{2}h$. Therefore, an initial condition $\theta_0$ is uncertain in measuring the escape time $t_e$ if

$$\Delta t_e(\theta_0) = |t_e(\theta_0) - t_e(\theta_0 + \epsilon)| > 3h/2,$$

where $h$ is the integration step of the numerical method, as mentioned above. Analogously, an initial condition $\theta_0$ is uncertain in measuring the escape time $\tau_e$ if

$$\Delta \tau_e(\theta_0) = |\tau_e(\theta_0) - \tau_e(\theta_0 + \epsilon)| > 3h/2.$$

Finally, an initial condition is uncertain with respect to the exit through which the particle escapes if $e(\theta_0) \neq e(\theta_0 + \epsilon)$.

We can also approximate the escape times differences using Eq. (9) and approximate the expression

$$\Delta \tau_e(\theta_0) \approx \frac{\Delta t_e(\theta_0)}{\bar{\gamma}}.$$

Subsequently, we can estimate $\Delta \tau_e(\theta_0) < \Delta t_e(\theta_0)$ on average. Thus, given the same criterion $3h/2$ in both reference frames, on the one hand, there will be some uncertain initial conditions $\theta_0$ in the inertial clock ($\Delta t_e(\theta_0) > 3h/2$) that become certain in the non-inertial clock ($\Delta \tau_e(\theta_0) < 3h/2$). We show a scheme in Fig. 7(a) to understand better this physical effect on the unpredictability of the escape times. It is easy to see that this effect is caused by the limited resolution of the hypothetical clocks. If this resolution was infinite ($h \rightarrow 0$), the initial condition $\theta_0$ shown in Fig. 7(a) would be uncertain with respect to escape times in both clocks. In addition, this effect takes place for high values of $\beta$ because it is proportional to the Lorentz factor, as we observe in Figs. 7(b) and 7(c).

As shown in previous works [30], the fraction of uncertain initial conditions behaves as

$$f(\epsilon) \sim \epsilon^{1-d}.$$

Taking decimal logarithms in Eq. (20), we obtain

$$\log_{10} \frac{f(\epsilon)}{\epsilon} \sim -d \log_{10} \epsilon.$$
times. Given the shooting method previously explained, these differences depend generally on the angular perturbations \( \epsilon \) and the parameter \( \beta \). On the one hand, when angular perturbations become small, the trajectories described by the particle are more similar on average, and then the differences of escape times become also smaller. On the other hand, as the value of \( \beta \) increases, the fractality and the escape times on average decrease, and with these, their differences as well. In summary, when a finite resolution of clocks is considered, it is adequate to make use of parameter values of \( \epsilon \) and \( \beta \) where the time differences can be distinguished. For this reason, the linear fittings have been calculated using the following

\[
\Delta t_e(\theta_0) > \frac{3h}{2} \quad \Delta \tau_e(\theta_0) < \frac{3h}{2}
\]

FIG. 7: (Color online) (a) A scheme to visualize the physical effect of a reference frame modification on the unpredictability of the escape times. (b, c) Uncertain initial conditions \( \theta_0 \) located over the stable manifold in the hyperbolic regime. We show the results for fifty equally spaced values of \( \beta \in [0.5, 0.99] \) and only two values of perturbation (b) \( \log_{10} \epsilon = -4 \) and (c) \( \log_{10} \epsilon = -6 \), respectively.
specific range of angular perturbations $\log_{10} \epsilon \in [-6, -1]$.

If the clocks’ resolution is small, it is generally more predictable to determine the exit through which the particle escapes than it is to measure exactly its escape time. Therefore, there is a greater probability of obtaining an uncertain initial condition when measuring the escape times. In this regard, there is a greater number of uncertain conditions concerning escape times than in relation to exits, as we observe in Fig. 7(b). This fact can be explained because the former ones are located outside and over the stable manifold, and the uncertain conditions regarding exits can only be located on the stable manifold by definition for any small perturbation $\epsilon$. As a consequence, the uncertainty dimensions obey the inequation $d_e < d_t, d_\tau$. Virtually, for almost every $\beta$ in Fig. 8(b), we obtain again the computational result $d_t \approx d_\tau$, which entails that transient chaos in open Hamiltonian systems is relativistic invariant under coordinate transformations. Nonetheless, in a very energetic regime, where the time dilation phenomenon is manifestly intense in our system, we observe that the physical effect explained above causes a small difference between the fractal dimensions associated with the escape time functions.

![Graphs](image_url)

**FIG. 8:** (Color online) (a) The linear fittings between $\log_{10} f(\epsilon)/\epsilon$ and $\log_{10} \epsilon \in [-6, -1]$ for $\beta = 0.55$ and $\beta = 0.95$, respectively. We show the linear fittings associated with the exit function (green), the escape time function (blue) and the escape proper time function (red). (b) Fractal dimensions $d$ with errors (standard deviations) computed by the uncertainty dimension algorithm versus twenty five equally spaced values of $\beta \in [0.5, 0.98]$ are depicted in the same color code as in (a). All these results have been calculated by setting a fixed time step $h = 0.005$. 

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Finally, from a purely mathematical point of view, if we consider extremely small resolutions for our clocks ($h \to 0$), we can set a range of extremely small angular perturbations ($\epsilon \to 0$) and, consequently, the uncertain conditions regarding escape times will be located over the stable manifold in any clock. An example of how the uncertain conditions are located closer to the stable manifold when the angular perturbations become smaller can be found in Figs. 7(b) and 7(c). Thus, the geometric and observer-independent nature of the fractality caused by the chaotic saddle is reflected into the values of the fractal dimensions. It is expected that in this limit the equality $d_e = d_t = d_r$ holds.

This equality extends the very important statement that relativistic chaos is coordinate invariant to transient chaos as well. The result provided in [12] showing that the signs of the Lyapunov exponents of a chaotic dynamical system are invariant under coordinate transformations can be perfectly extended to transient chaotic dynamics. For this purpose, it is only required to consider a chaotic trajectory on the chaotic saddle, which meets the necessary four conditions described in [12]. Since the sign of the Lyapunov exponents of a trajectory on the chaotic saddle are also invariant, it is therefore evident that the existence of transient chaotic dynamics can not be avoided by considering suitable changes of the reference frame. We believe that this analytical result is at the basis of the results arising from all the numerical explorations performed in the previous sections.

VI. CONCLUSIONS

The Hénon-Heiles Hamiltonian has been extensively studied in the context of chaotic scattering in open Hamiltonian systems. However, only a few works have focused on relativistic chaotic scattering. In particular, we have concentrated our efforts on the impact of the Lorentz factor on the system’s dynamics and the phenomena of transient chaos. Firstly, we have proposed that the particle follows a bounded dynamics within the scattering region until it escapes when its Lorentz factor reaches a critical value associated with the maximal kinetic energy. Based on this later assumption, we have provided a new computationally and very affordable escape criterion of particles from the potential well that includes all the escapes that take place when the Lyapunov orbit criterion is considered.

Scattering functions, such as the exit function or the escape time function, exhibit a fractal structure of singularities as a consequence of the presence of the chaotic saddle. Then, as
the origin of the singularities of the escape times is geometric, the fractality of the escape times must be observer-independent. In order to verify this, we have measured the escape times from two different reference frames: an inertial reference frame fixed in the potential well and another non-inertial frame attached to the particle. In this way, we have observed that the phenomenon of time dilation does not affect the structure of singularities typical of the escape times and, therefore, a change of reference frame does not affect the fractality of the escape time function. In fact, we have provided a linear relation between escape times measured by both reference frames. As a consequence, the time differences between them, which is produced by the events of time dilation that take place inside the well, also preserve the fractal structure. Thus, typical relativistic phenomena behave chaotically.

The fractal structure of the escape times as measured in both reference frames is self-similar in the hyperbolic regime, which implies that there exists sensitivity to initial conditions at any scale. This feature allows us to relate the fractal dimensions of the escape time function and a Cantor-like set of Lebesgue measure zero. Then, we have proved that this fractal dimension is, on average, relativistic invariant under coordinate transformations and, together with it, we have suggested that transient chaos in open Hamiltonian systems is invariant as well.

Finally, in order to quantify this fractal dimension, we have used the uncertainty dimension algorithm. We have pointed out a physical effect on the unpredictability of escape times by which the particle’s final dynamical state is more likely to be predictable in the non-inertial reference frame, even though from a mathematical point of view this unpredictability is reference frame independent. This effect is caused by the limited resolution of the clocks and it takes place only when the value of the particle’s velocity is close to the speed of light. However, from a theoretical point of view, the main conclusion of the present work is that transient chaos is coordinate invariant. This statement extends the universality of occurrence of chaos and fractals under coordinate transformations to the realm of transient chaotic phenomena as well.
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