On the transfer of some $t$–locally properties

Omar Ouzzaouit$^1$, Ali Tamoussit$^2$

$^1$Department of Mathematics, The Higher School of Education and Training, Ibn Zohr University, Agadir, Morocco
$^2$Department of Mathematics, The Regional Center for Education and Training Professions Souss Massa, Inezgane, Morocco

Abstract

In this paper, we study the transfer of some $t$–locally properties which are stable under localization to $t$–flat overrings of an integral domain $D$. We show that $D$, $D[X]$, $D\langle X \rangle$, $D\left[ X \right]_{\mathcal{N}v}$ and $D\left( X \right)$ are simultaneously $t$–locally P$v$MDs (resp., $t$–locally Krull, $t$–locally G-GCD, $t$–locally Noetherian, $t$–locally Strong Mori). A complete characterization of when a pullback is a $t$–locally P$v$MD (resp., $t$–locally GCD, $t$–locally G-GCD, $t$–locally Noetherian, $t$–locally Strong Mori, $t$–locally Mori) is given. As corollaries, we investigate the transfer of some $t$–locally properties among domains of the form $D + XK[X]$, $D + XK[[X]]$ and amalgamated algebras. A particular attention is devoted to the transfer of almost Krull and locally P$v$MD properties to integral closure of a domain having the same property.

Mathematics Subject Classification (2020). 13A15, 18A30, 13C11, 13F05, 13F20

Keywords. $t$–Flat overring, ($t$–)Nagata ring, Serre conjecture ring, pullback construction

1. Introduction

It is convenient to begin by recalling some definitions and notation. Let $D$ be an integral domain with quotient field $K$. For a nonzero fractional ideal $I$ of $D$, we let $I^{-1} := \{ x \in K \mid xI \subseteq D \}$. On $D$ the $v$–operation is defined by $I_v = (I^{-1})^{-1}$; the $t$–operation is defined by $I_t := \bigcup J_v$, where $J$ ranges over the set of all nonzero finitely generated ideals contained in $I$; and the $w$–operation is defined by $I_w := \{ x \in K \mid xJ \subseteq I \}$ for some nonzero finitely generated ideal $J$ of $D$ with $J^{-1} = D$ for all nonzero fractional ideals $I$ of $D$. A nonzero ideal $I$ of $D$ is divisorial (or $v$–ideal) (resp., $t$–ideal, $w$–ideal) if $I_v = I$ (resp., $I_t = I$, $I_w = I$). In general, for each nonzero fractional ideal $I$ of $D$, $I \subseteq I_w \subseteq I_t \subseteq I_v$, and the inclusions may be strict (cf. [14, Proposition 1.2]). So, $v$–ideals are $t$–ideals and $t$–ideals are $w$–ideals. For $*$ = $t$ or $w$, a $*$–ideal which is also prime is called a $*$–prime ideal, $*$–maximal ideal is an ideal that is maximal among the

*Corresponding Author.
Email addresses: ouzzaouitomar@gmail.com; o.ouzzaouit@uiz.ac.ma (O. Ouzzaouit),
tamoussit2009@gmail.com (A. Tamoussit)
Received: 08.07.2020; Accepted: 14.01.2021
proper $+$-ideals and let $\text{Max}(D)$ denote the set of all $+$-maximal ideals of $D$. Notice that $w\text{-Max}(D) = t\text{-Max}(D)$ and each height-one prime is $t$–prime.

An integral domain $D$ is said to be a Prüfer $v$–multiplication domain (for short, PvMD) (resp., $t$–almost Dedekind domain) if $D_m$ is a valuation domain (resp., a DVR) for each $t$–maximal ideal $m$ of $D$. Trivially, Krull domains and almost Dedekind domains are $t$–almost Dedekind domains and $t$–almost Dedekind domains are PvMDs. An integral domain $D$ is a Strong Mori domain (for short, SM domain) (resp., Mori domain) if it satisfies the ascending chain condition (acc) on integral $w$–ideals (resp., $v$–ideals) of $D$. Clearly, Noetherian domains and Krull domains are SM and SM domains are Mori.

An integral domain $D$ is a GCD domain (resp., generalized GCD domain (for short, G-GCD domain)) if the intersection of any two (integral) principal ideals (resp., invertible ideals) of $D$ is still principal (resp., invertible). Notice that valuation domains are GCD domains, GCD domains are G-GCD domains and G-GCD domains form a subclass of PvMDs.

In this paper, we begin by the study of the transfer of some $t$–locally properties which are stable under localization to a $t$–flat overring of a domain $D$. Then we give several applications, namely for $t$–almost Dedekind domains. Among other results, we show that every $t$–linked overring of a $t$–almost Dedekind domain which is not a field is also a $t$–almost Dedekind domain. In our second major result we prove that for any integral domain $D$, the domains $D, D[X], D\langle X \rangle, D(X)$ and $D[X], D\langle X \rangle$ are simultaneously $t$–locally PvMDs (resp., $t$–locally Krull, $t$–locally G-GCD, $t$–locally Noetherian, $t$–locally SM). By the way, we treat a relevant case when $D$ is a $t$–locally G-GCD domains. Next we establish necessary and sufficient conditions for a pullback construction to be $t$–locally PvMD (resp., $t$–locally GCD, $t$–locally G-GCD). As additional applications we recover the cases of domains of the form $D + XK[X], D + XK[[X]]$ and amalgamated algebras. Then we extend [14, Theorem 3.11] to $t$–locally Noetherian (resp., $t$–locally SM) domains arising from pullback constructions. Finally, while dealing with the integral closure of an integral domain, we show that the converse of [15, Theorem 2.13] holds with less hypotheses. Moreover, we investigate the transfer of the locally PvMD property to the integral closure $\overline{D}$ of an integrally closed domain $D$ in an algebraic field extension of its quotient field, and we prove that $D$ is a locally PvMD if and only if so is $\overline{D}$.

2. Main results

Let $(P)$ denote a property of integral domains. An integral domain $D$ is said to be locally $(P)$ (resp., $t$–locally $(P)$) if $D_m$ is $(P)$ for each maximal ideal (resp., $t$–maximal ideal) $m$ of $D$. Notice that in domains that are Prüfer or of dimension one, $t$–locally $(P)$ coincides with locally $(P)$.

By an overring of $D$ we mean any domain $R$ between $D$ and the quotient field of $D$. Recall from [12] that an overring $R$ of $D$ is said to be $t$–flat over $D$ if $R_m = D_{m \cap D}$, for each $t$–maximal ideal $m$ of $R$, or equivalently $R_p = D_{p \cap D}$, for each $t$–prime ideal $p$ of $R$ (cf. [5, Theorem 2.6]).

**Proposition 2.1.** Let $(P)$ be a property of integral domains which is stable under localization. Then, for any integral domain $D$, the following statements are equivalent:

1. $D$ is $t$–locally $(P)$;
2. $D_p$ is $(P)$ for each $t$–prime ideal $p$ of $D$;
3. Each $t$–flat overring of $D$ is also $t$–locally $(P)$.

**Proof.** (1) $\Rightarrow$ (2) Assume that $D$ is $t$–locally $(P)$ and let $p$ be a $t$–prime ideal of $D$. Then there exists a $t$–maximal ideal $m$ of $D$ such that $p \subseteq m$. It follows from [2, Lemma 1] that $D_p = (D_m)_p$. Hence, $D_p$ is a $(P)$ domain as a localization of the $(P)$ domain $D_m$. 


(2) ⇒ (3) Let $R$ be a (proper) $t$–flat overring of $D$ and $q$ be a $t$–maximal ideal of $R$. Then, by [5, Lemma 1.2], $p := q \cap D$ is a $t$–prime ideal of $D$, and hence $D_p = R_q$ is a $(P)$ domain. Thus, $R$ is $t$–locally $(P)$.

(3) ⇒ (1) Straightforward.

Similarly, using [16, Theorem 2], it easy to prove an analogue of the previous result when dealing with flat overrings of a locally $(P)$ domain.

**Corollary 2.2.** Let $(P)$ denote one of the following properties: GCD, Krull, PrMD, G-GCD, Noetherian, SM or Mori. Then, $D$ is a $t$–locally $(P)$ domain if and only if every $t$–flat overring of $D$ is also $t$–locally $(P)$.

For $t$–almost Dedekind domains, we get a more interesting result. Recall that an overring $R$ of $D$ is $t$–linked over $D$ if, for each nonzero finitely generated ideal $I$ of $D$ such that $I^{-1} = D$, we have $(IR)^{-1} = R$. Notice that every $t$–flat overring is $t$–linked.

**Corollary 2.3.** Let $D$ be a $t$–almost Dedekind domain which is not a field. Then, each $t$–locally overring of $D$ is also $t$–almost Dedekind.

**Proof.** Let $R$ be a (proper) $t$–linked overring of $D$. Since any $t$–almost Dedekind domain is a PrMD, it follows from [12, Proposition 2.10] that $R$ is $t$–flat over $D$ and then, by Proposition 2.1, $R$ is a $t$–almost Dedekind.

**Corollary 2.4.** Let $D$ be a $t$–almost Dedekind domain. We have:

1. If $R = \cap_\alpha D_\alpha$, with each $D_\alpha$ is a $t$–linked overring of $D$, then $R$ is a $t$–almost Dedekind domain.

2. If $T$ is an overring of $D$ and $p$ is a $t$–prime ideal of $D$, then $T_{D,p}$ is a $t$–almost Dedekind domain.

3. The complete integral closure of $D$ is a $t$–almost Dedekind domain.

**Proof.** Follows from Corollary 2.3 and [4, Propositions 2.2(b), 2.9, and Corollary 2.3].

Now, let $X$ be an indeterminate over an integral domain $D$. For each polynomial $f \in D[X]$, we denote by $c(f)$ the content of $f$, that is, the ideal of $D$ generated by the coefficients of $f$. The sets $U = \{f \in D[X] | f \text{ is monic}\}$, $S = \{f \in D[X] | c(f) = D\}$ and $N_v = \{f \in D[X] | c(f)_v = D\}$ are multiplicatively closed subsets of $D[X]$. The localization $D\langle X \rangle := D[X]/U$ (resp., $D(X) := D[X]/S$, $D[X]_{N_v}$) is called the Serre conjecture (resp., the Nagata, the $t$–Nagata) ring of $D$. Note that $D[X] \subseteq D(X) \subseteq D(X)_{N_v}$.

**Theorem 2.5.** Let $(P)$ denote one of the following properties: PrMD, Krull, G-GCD, Noetherian or SM. Then, for any integral domain $D$, the following statements are equivalent:

1. $D$ is a $t$–locally $(P)$ domain;
2. $D[X]$ is a $t$–locally $(P)$ domain;
3. $D(X)$ is a $t$–locally $(P)$ domain;
4. $D(X)$ is a $t$–locally $(P)$ domain;
5. $D[X]_{N_v}$ is a $t$–locally $(P)$ domain;
6. $D[X]_{N_v}$ is a locally $(P)$ domain.

The proof of this theorem requires the following preparatory lemmas.

**Lemma 2.6.** Let $D$ be an integral domain. Then:

1. $\text{Max}(D[X]_{N_v}) = t\text{-Max}(D[X]_{N_v}) = \{m[X]_{N_v} | m \in t\text{-Max}(D)\}$.
2. For each $t$–maximal ideal $m$ of $D$, we have: $D[X]_m = (D[X]_{N_v})_m[X]_{N_v} = D(X)_m D(X) = D_m(X)$.
3. For each $t$–maximal ideal $Q$ of $D[X]$, we have either: $Q \cap D = (0)$, or $Q \cap D$ is a $t$–maximal ideal of $D$ and $Q = (Q \cap D)[X]$. 


Proof. (1) [11, Propositions 2.1 and 2.2].
(2) [2, Lemma 2].
(3) [7, Proposition 2.2]. \hfill \square

Lemma 2.7. Let \( D \) be an integral domain with quotient field \( K \). Then, \( D(\bar{x}) \) is a \( \text{PvMD} \) (resp., Krull, \( G\text{-GCD} \), Noetherian, SM) if and only if \( D \) has the same property.

Proof. It is well known that \( D \) is a Krull (resp., \( G\text{-GCD} \)) domain if and only if \( D(\bar{x}) \) has the same property [1, Theorem 5.2(1)] (resp., [1, Theorem 5.1(1)]).

Now, if \( D \) is a \( \text{PvMD} \) (resp., a Noetherian domain, an SM domain), then so is \( D(\bar{x}) \) and hence its localization \( D(\bar{x}) \) has the same property. Conversely, assume that \( D(\bar{x}) \) is a \( \text{PvMD} \) and let \( \mathfrak{m} \) be a \( t \)-maximal ideal of \( D \). By [11, Corollary 2.3(2)], \( \mathfrak{m}D(\bar{x}) \) is a \( t \)-prime ideal of \( D(\bar{x}) \), and then \( D(\bar{x})_{\mathfrak{m}D(\bar{x})} = D(\bar{x})|_{\mathfrak{m}X} = D_{\mathfrak{m}}(\bar{x}) \) is a valuation domain. Thus, \( D_{\mathfrak{m}} \) is a valuation domain since \( D_{\mathfrak{m}} = D_{\mathfrak{m}}(\bar{x}) \cap K \). Therefore, \( D \) is a \( \text{PvMD} \). Next, assume that \( D(\bar{x}) \) is a Noetherian domain and let \( I \) be an ideal of \( D \). Then, \( ID(\bar{x}) \) is finitely generated and so is \( I \) [1, Theorem 2.2(2)]. Lastly, assume that \( D(\bar{x}) \) is an SM domain and let \( \mathfrak{m} \) be a \( t \)-maximal ideal of \( D \). By [11, Corollary 2.3(2)], \( \mathfrak{m}D(\bar{x}) \) is a \( t \)-prime ideal of \( D(\bar{x}) \), and then \( D(\bar{x})_{\mathfrak{m}D(\bar{x})} = D(\bar{x})|_{\mathfrak{m}X} = D_{\mathfrak{m}}(\bar{x}) \) is a Noetherian domain. Hence, \( D_{\mathfrak{m}} \) is a Noetherian domain and thus \( D \) is \( t \)-locally Noetherian. On the other hand, since \( D = D(\bar{x}) \cap K \), \( D \) is a Mori domain as an intersection of two Mori domains. Therefore, \( D \) is an SM domain. \hfill \square

Proof of Theorem 2.5. (1) \( \Rightarrow \) (2) Let \( Q \) be a \( t \)-maximal ideal of \( D[\bar{x}] \) and set \( P = Q \cap D \). If \( P = (0) \) then \( D[\bar{x}]_Q = K[\bar{x}]_{K[\bar{x}]} \) is a DVR, where \( K \) is the quotient field of \( D \). If \( P \neq (0) \) then, by Lemma 2.6(3), \( Q = P[\bar{x}] \) and \( P \) is a \( t \)-maximal ideal of \( D \). Hence, \( D_P \) is a \( (\mathcal{P}) \) domain, so by Lemmas 2.6(2) and 2.7, \( D[\bar{x}]_Q = D[\bar{x}]_{P[\bar{x}]} = D_P(\bar{x}) \) is a \( (\mathcal{P}) \) domain. Therefore, \( D(\bar{x}) \) is a \( t \)-locally \( (\mathcal{P}) \) domain.

(2) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (4) follows from Corollary 2.1, since \( D(\bar{x}) = D[\bar{x}]_{\mathcal{U}} \) is a localization of \( D[\bar{x}] \) and \( D(\bar{x}) \) is a localization of \( D \).

(4) \( \Rightarrow \) (1) Let \( \mathfrak{m} \) be a \( t \)-maximal ideal of \( D \). Then, by [11, Corollary 2.3(2)], \( \mathfrak{m}D(\bar{x}) \) is a \( t \)-prime ideal of \( D(\bar{x}) \). Hence, \( D(\bar{x})_{\mathfrak{m}D(\bar{x})} = D(\bar{x})|_{\mathfrak{m}X} = D_{\mathfrak{m}}(\bar{x}) \) is a \( (\mathcal{P}) \) domain. Thus, by Lemma 2.7, \( D_{\mathfrak{m}} \) is a \( (\mathcal{P}) \) domain. Therefore, \( D \) is a \( t \)-locally \( (\mathcal{P}) \) domain.

(1) \( \Rightarrow \) (6) Let \( Q \) be a maximal ideal of \( D[\bar{x}]_{N_v} \). By Lemma 2.6(1), \( Q = \mathfrak{m}[\bar{x}]_{N_v} \) for some \( t \)-maximal ideal \( \mathfrak{m} \) of \( D \). As \( D[\bar{x}]_{N_v}|_{\mathfrak{m}[\bar{x}]_{N_v}} = D[\bar{x}]|_{\mathfrak{m}X} = D_{\mathfrak{m}}(\bar{x}) \) and \( D_{\mathfrak{m}} \) is a \( (\mathcal{P}) \) domain, it follows from Lemma 2.7 that \( D[\bar{x}]_{N_v}|_{\mathfrak{m}[\bar{x}]_{N_v}} \) is a \( (\mathcal{P}) \) domain and hence, \( D[\bar{x}]_{N_v} \) is locally \( (\mathcal{P}) \).

(6) \( \Leftrightarrow \) (5) This equivalence follows from Lemma 2.6(1).

(5) \( \Rightarrow \) (1) Let \( \mathfrak{m} \) be a \( t \)-maximal ideal of \( D \). Then, \( \mathfrak{m}[\bar{x}]_{N_v} \) is a \( t \)-maximal ideal of \( D[\bar{x}]_{N_v} \), and hence \( D[\bar{x}]_{N_v}|_{\mathfrak{m}[\bar{x}]_{N_v}} = D[\bar{x}]|_{\mathfrak{m}X} = D_{\mathfrak{m}}(\bar{x}) \) is a \( (\mathcal{P}) \) domain. Thus, by Lemma 2.7, \( D_{\mathfrak{m}} \) is a \( (\mathcal{P}) \) domain and hence, \( D \) is \( t \)-locally \( (\mathcal{P}) \).

For the case of \( G\text{-GCD} \) domains we have a more precise result.

Proposition 2.8. For any integral domain \( D \), the following statements are equivalent:

1. \( D \) is a \( t \)-locally \( G\text{-GCD} \) domain;
2. \( D[\bar{x}] \) is a \( t \)-locally \( G\text{-GCD} \) domain;
3. \( D(\bar{x}) \) is a \( G\text{-GCD} \) domain;
4. \( D[\bar{x}]_{N_v} \) is a \( G\text{-GCD} \) domain.

Proof. The proof is similar to the proof of the above theorem by using the fact that \( D \) is a \( G\text{-GCD} \) domain if and only if \( D(\bar{x}) \) is a \( G\text{-GCD} \) domain (cf. [1, Theorem 5.1(1)]) \hfill \square

To avoid unnecessary repetition, let us fix some notation for the remainder of this paper.

Let \( T \) be an integral domain, \( \mathfrak{M} \) a maximal ideal of \( T \), \( K \) the residue field \( T/\mathfrak{M} \), \( \varphi : T \to K \) is the natural projection, \( D \) a proper subring of \( K \). Let \( R := \varphi^{-1}(D) \) be the
pullback arising from the following diagram of canonical homomorphisms:

\[
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
T & \xrightarrow{\varphi} & K.
\end{array}
\]

We shall refer to this as a pullback diagram of type (\(\Box\)).

**Proposition 2.9.** Let \((\mathcal{P})\) denote one of the following properties: \(PrMD\), GCD or G-GCD. Then, for a pullback diagram of type (\(\Box\)), \(R\) is a \(t\)-locally \((\mathcal{P})\) domain if and only if \(qf(D) = K\), \(D\) and \(T\) are \(t\)-locally \((\mathcal{P})\) domains and \(T_{\mathfrak{m}}\) is valuation.

**Proof.** Recall that \(R\) is a \((\mathcal{P})\) domain if and only if \(qf(D) = K\), \(D\) and \(T\) are \((\mathcal{P})\) domains and \(T_{\mathfrak{m}}\) is valuation (cf. \([6, \text{Theorems 4.1 and 4.2(a-b)}]\)). Assume that \(R\) is a \(t\)-locally \((\mathcal{P})\) domain. Since \(\mathfrak{m}\) is a \(t\)-prime ideal of \(R\), \(R_{\mathfrak{m}}\) is a \((\mathcal{P})\) domain.

Let \(Q\) be a \(t\)-maximal ideal of \(T\). If \(Q = \mathfrak{m}\), then we localize the previous diagram at \(\mathfrak{m}\) to obtain the following pullback:

\[
\begin{array}{ccc}
R_{\mathfrak{m}} & \longrightarrow & D_{\varphi(\mathfrak{m})} \\
\downarrow & & \downarrow \\
T_{\mathfrak{m}} & \xrightarrow{\varphi} & K.
\end{array}
\]

It follows from \([6, \text{Theorems 4.1 and 4.2(a-b)}]\) that \(qf(D) = K\) and \(T_{\mathfrak{m}}\) is a \((\mathcal{P})\) domain. If \(Q \neq \mathfrak{m}\), then \(P := Q \cap R\) is a \(t\)-maximal ideal of \(R\) and hence \(T_Q = R_P\) is a \((\mathcal{P})\) domain. Thus, \(T\) is a \(t\)-locally \((\mathcal{P})\) domain.

Let \(P\) be a \(t\)-maximal ideal of \(D\) and set \(Q := \varphi^{-1}(P)\). Considering the following pullback:

\[
\begin{array}{ccc}
R_Q & \longrightarrow & D_P \\
\downarrow & & \downarrow \\
T_{\mathfrak{m}} & \xrightarrow{\varphi} & K.
\end{array}
\]

By \([6, \text{Theorems 4.1 and 4.2(a-b)}]\), \(D_P\) is a \((\mathcal{P})\) domain and \(T_{\mathfrak{m}}\) is a valuation domain.

Conversely, let \(Q\) be a \(t\)-maximal ideal of \(R\). If \(Q = \mathfrak{m}\), then, by \([6, \text{Theorems 4.1 and 4.2(a-b)}]\), \(R_{\mathfrak{m}}\) is a \((\mathcal{P})\) domain. If \(Q \neq \mathfrak{m}\), then there is only one \(t\)-maximal ideal \(P\) of \(T\) such that \(P \cap R = Q\) (cf. \([9, \text{Theorem 2.6(1)}]\)), and hence \(R_Q = T_P\) is a \((\mathcal{P})\) domain since \(T\) is a \(t\)-locally \((\mathcal{P})\) domain. Thus, \(R\) is a \(t\)-locally \((\mathcal{P})\) domain. \(\square\)

From \([3]\) we introduce the definition of amalgamated algebras along an ideal as follows:

Let \(A\) and \(B\) be two rings, \(f : A \rightarrow B\) a ring homomorphism and \(J\) an ideal of \(B\). The following subring of \(A \times B:\)

\[A \triangleright J = \{(a, f(a) + j) | a ∈ A \text{ and } j ∈ J\},\]

is called the *amalgamation* of \(A\) with \(B\) along \(J\) with respect to \(f\).

**Corollary 2.10.** Let \(A\) and \(B\) be two integral domains, \(J\) a maximal ideal of \(B\) and \(f : A \rightarrow B\) a ring homomorphism such that \(f^{-1}(J) = \{0\}\). Let \((\mathcal{P})\) denote one of the following properties: \(PrMD\), GCD or G-GCD. Then, \(A \triangleright J\) is a \(t\)-locally \((\mathcal{P})\) domain if and only if \(qf(A) = B/J\), \(A\) and \(B\) are \(t\)-locally \((\mathcal{P})\) domains and \(B_J\) is valuation.

**Proof.** By \([3, \text{Proposition 4.2}\]\), we have the following pullback:

\[
\begin{array}{ccc}
A \triangleright J & \longrightarrow & A \\
\downarrow & & \downarrow \tilde{f} \\
B & \xrightarrow{\varphi} & B/J,
\end{array}
\]

where \(\tilde{f} = \varphi \circ f\). The result follows immediately by applying Proposition 2.9. \(\square\)

Now, we recover the case of simple amalgamation.
Corollary 2.11. Let $A$ be an integral domain, $I$ a maximal ideal of $A$ and let $(\mathcal{P})$ denote one of the following properties: PeMD, GCD or G-GCD. Then, $A \triangleright I$ is a $t$–locally $(\mathcal{P})$ domain if and only if $A$ is a field.

Proof. Take $B = A$ and $f = \text{Id}_A$ in Corollary 2.10.

Corollary 2.12. Let $K$ be a field, $X$ an indeterminate over $K$, $D$ a subring of $K$ and let $(\mathcal{P})$ denote one of the following properties: PeMD, GCD or G-GCD. If $R$ is an integral domain of the form $D + XK[X]$ or $D + XK[[X]]$, then $R$ is a $t$–locally $(\mathcal{P})$ domain if and only if so is $D$ and $\text{qf}(D) = K$.

Proof. Let $T = K[X]$ (resp., $T = K[[X]]$) and $\mathfrak{M} = XK[X]$ (resp., $\mathfrak{M} = XK[[X]]$). Then, $T$ is a PID with $T/\mathfrak{M} \cong K$ and $T_{\mathfrak{M}}$ is a DVR. Thus the conclusion follows from Proposition 2.9.

We now study the transfer of the $t$–locally Noetherian (resp., the $t$–locally SM) notion to pullbacks. In fact, we extend [14, Theorem 3.11] to $t$–locally Noetherian (resp., $t$–locally SM) domains.

Proposition 2.13. For a pullback diagram of type (□), $R$ is a $t$–locally Noetherian domain (resp., a $t$–locally SM domain) if and only if $T$ is a $t$–locally Noetherian domain (resp., a $t$–locally SM domain), $T_{\mathfrak{M}}$ is Noetherian, $D = k$ is a field, and $[K : k]$ is finite. In particular, if $T$ is local, then $R$ is a $t$–locally Noetherian domain (resp., a $t$–locally SM domain) if and only if $R$ is Noetherian.

Proof. Assume that $R$ is $t$–locally Noetherian. If $D$ is not a field, then $D$ has a nonzero $t$–maximal ideal $P$. Set $Q = \varphi^{-1}(P)$. Then $Q$ is a $t$–maximal ideal of $R$ and so $R_Q$ is Noetherian. Now consider the following pullback:

$$
\begin{array}{ccc}
R_Q & \longrightarrow & D_P \\
\downarrow & & \downarrow \\
T_S & \longrightarrow & K,
\end{array}
$$

where $S = R \setminus Q$. Necessarily $D_P = k$ is a field, which is absurd. Thus $D = k$ is a field. Now that $D = k$ is a field implies that $\mathfrak{M}$ is a maximal ideal of $R$ which is divisorial and so it is a $t$–maximal ideal. Localizing at $\mathfrak{M}$, we obtain the following pullback:

$$
\begin{array}{ccc}
R_{\mathfrak{M}} & \longrightarrow & k \\
\downarrow & & \downarrow \\
T_{\mathfrak{M}} & \longrightarrow & K.
\end{array}
$$

So that $R_{\mathfrak{M}}$ is Noetherian implies that $T_{\mathfrak{M}}$ is Noetherian and $[K : k]$ is finite.

Conversely, let $Q$ be a $t$–maximal ideal of $R$. Then, we distinguish the following two possible cases:

Case 1: $Q = \mathfrak{M}$. Since $T_{\mathfrak{M}}$ is Noetherian, $D = k$ is a field, and $[K : k]$ is finite, it follows from [8, Theorem 4.12] that $R_{\mathfrak{M}}$ is Noetherian.

Case 2: $Q \neq \mathfrak{M}$. Then there is a unique $t$–maximal ideal $P$ of $T$ such that $P \cap R = Q$ and hence $R_Q = T_P$ is a Noetherian domain because $T$ is a $t$–locally Noetherian domain.

Therefore, $R$ is a $t$–locally Noetherian domain.

The case of $t$–locally SM domains is similar to the previous case by using [14, Theorem 3.11].

For the particular case, we have $T = T_{\mathfrak{M}}$ and so the conclusion follows from [8, Theorem 4.12].

Corollary 2.14. Let $A$ and $B$ be two integral domains, $J$ a maximal ideal of $B$ and $f : A \to B$ a ring homomorphism such that $f^{-1}(J) = \{0\}$. Then, $A \triangleright f^! J$ is a $t$–locally Noetherian domain (resp., a $t$–locally SM domain) if and only if $B$ is a $t$–locally Noetherian domain (resp., a $t$–locally SM domain), $B_J$ is Noetherian, $A$ is a field, and $[B/J : A]$ is finite.
Proof. It follows from [3, Proposition 4.2] and Proposition 2.13. □

Corollary 2.15. Let $K$ be a field, $X$ an indeterminate over $K$, $D$ a subring of $K$. If $R$ is an integral domain of the form $D + XK[X]$ or $D + XK[[X]]$, then the following statements are equivalent.

1. $R$ is a t–locally Noetherian domain;
2. $R$ is a t–locally SM domain;
3. $D = k$ is a field and $[K : k]$ is finite;
4. $R$ is Noetherian.

By adapting the proof of Proposition 2.13 and using [8, Theorem 4.18], we get the following:

Proposition 2.16. For a pullback diagram of type $(\square)$, $R$ is a t–locally Mori domain if and only if $T$ is a t–locally Mori domain, $T_{\mathfrak{m}}$ is Mori, and $D = k$ is a field.

In [15, Theorem 2.13], Pirtle established that if $D$ is an almost Krull domain, i.e., a locally Krull domain, with quotient field $K$, then the integral closure of $D$ in a finite field extension of $K$ is also almost Krull. Next, we show that the converse holds without the finiteness condition.

Proposition 2.17. Let $D$ be an integrally closed domain with quotient field $K$, let $L$ be an algebraic field extension of $K$ and let $\overline{D}$ be the integral closure of $D$ in $L$. If $\overline{D}$ is an almost Krull domain then so is $D$.

Proof. Let $p$ be a prime ideal of $D$ and $q$ be a prime ideal of $\overline{D}$ lying over $p$. Since $\overline{D}$ is almost Krull, $\overline{D}_q$ is a Krull domain and then, by [10, Theorem 1], $\overline{D}_q \cap K = D_p$ is also a Krull domain. That is, $D$ is an almost Krull domain. □

In the case of locally PvMDs we get a stronger result.

Proposition 2.18. Let $D$ be an integrally closed domain with quotient field $K$, let $L$ be an algebraic field extension of $K$ and let $\overline{D}$ be the integral closure of $D$ in $L$. Then, $D$ is a locally PvMD if and only if so is $\overline{D}$.

Proof. Assume that $D$ is a locally PvMD and let $M$ be a maximal ideal of $\overline{D}$. Set $P = M \cap D$ and $S = D \setminus P$. Then, $D_P$ is a PvMD. Since $\overline{D}$ is the integral closure of $D$ in $L$, $\overline{D}_S$ is the integral closure of $D_P$ in $L$ and hence it follows from [13, Theorems 4.4 and 4.6] that $\overline{D}_S$ is a PvMD. Thus we deduce from the equality $\overline{D}_M = (\overline{D}_S)_M \overline{D}_S$ that $\overline{D}_M$ is a PvMD. Therefore, $\overline{D}$ is a locally PvMD. The converse is similar to that of the proof of the previous proposition. □

Acknowledgment. The authors would like to thank the anonymous referee for careful reading of the manuscript and for detecting some typographical errors.

References

[1] D.D. Anderson, D.F. Anderson and R. Markanda, The rings $R(X)$ and $R(X)$, J. Algebra 95, 96-115, 1985
[2] J.T. Arnold, On the ideal theory of the Kronecker function ring and the domain $D(X)$, Canad. J. Math. 21, 558-563, 1969.
[3] M. D'Anna, C.A. Finocchiaro and M. Fontana, Amalgamated algebras along an ideal, in: Commutative Algebra and Applications, Walter de Gruyter, Berlin, 155-172, 2009.
[4] D.E. Dobbs, E. Houston, T. Lucas and M. Zafrullah, t–Linked overrings and Prüfer $v$–multiplication domains, Comm. Algebra 17 (11), 2835-2852, 1989.
[5] S. El Baghdadi, On a class of Prüfer $v$–multiplication domains, Comm. Algebra 30, 3723-3742, 2002.
[6] M. Fontana and S. Gabelli, *On the class group and the local class group of a pullback*, J. Algebra **181**, 803-835, 1996.

[7] M. Fontana, S. Gabelli and E. Houston, *UMT-domains and domains with Prüfer integral closure*, Comm. Algebra **26**, 1017-1039, 1998.

[8] S. Gabelli and E. Houston, *Coherent-like conditions in pullbacks*, Michigan Math. J. **44**, 99-122, 1997.

[9] S. Gabelli and E.G. Houston, *Ideal theory in pullbacks*, in: Non-Noetherian Commutative Ring Theory, in: Math. Appl. **520**, 199-227, Kluwer Academic, Dordrecht, 2000.

[10] W. Heinzer, *Some properties of integral closure*, Proc. Am. Math. Soc. **18**, 749-753, 1967.

[11] B.G. Kang, *Prüfer v–multiplication domains and the ring $R[[X]]_{N_v}$*, J. Algebra **123**, 151-170, 1989.

[12] D.J. Kwak and Y.S. Park, *On t–flat overrings*, Chinese J. Math. **23**, 17-24, 1995.

[13] F. Lucius, *Rings with a theory of greatest common divisors*, Manuscripta Math. **95**, 117-136, 1998.

[14] A. Mimouni, *TW–domains and Strong Mori domains*, J. Pure Appl. Algebra **177**, 79-93, 2003.

[15] E.M. Pirtle, *Integral domains which are almost Krull*, J. Sci. Hiroshima Univ. Ser. A-I Math. **32** (2), 441-447, 1968.

[16] F. Richman, *Generalized quotient rings*, Proc. Amer. Math. Soc. **16**, 794-799, 1965.