LIMIT THEOREMS FOR SELF-INTERSECTING TRAJECTORIES IN 
\( \mathbb{Z} \)-EXTENSIONS

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Abstract. We investigate the asymptotic properties of the self-intersection numbers for \( \mathbb{Z} \)-extensions of chaotic dynamical systems, including the \( \mathbb{Z} \)-periodic Lorentz gas and the geodesic flow on a \( \mathbb{Z} \)-cover of a negatively curved compact surface. We establish a functional limit theorem.

1. Introduction

The self-intersection number of a flow \( (\varphi_h^0)_{s \geq 0} \) up to time \( t \) is the number \( N_t \) of couple of times \( (s, u) \in [0, t] \) with \( s \neq u \) such that \( \varphi_0^s \) and \( \varphi_0^u \) have the same position (but maybe not the same velocity).

Asymptotic properties of the self-intersection number of trajectories of the unit geodesic flow on a negatively curved compact surface have been studied by Lalley in [13] (see also the appendix B for a new approach of this result). In this finite measure case, the self-intersection number, normalised by \( t^2 \) converges almost surely to a constant \( e'_I \), corresponding to the expectation of the intersection number of two independent trajectories of unit length. In this article, we investigate this question in infinite measure, and more specifically the case of \( \mathbb{Z}^d \)-extension of chaotic probability preserving dynamical system. In [16], Pène studied the case of the \( \mathbb{Z}^2 \)-periodic Lorentz gas (which is a \( \mathbb{Z}^2 \)-extension of the Sinai billiard). For this model, the self-intersection number (normalized by \( \log t \)) converges almost everywhere to a constant. For completeness, let us indicate that in the easy case of \( \mathbb{Z}^2 \)-extensions of chaotic systems, again an almost everywhere convergence holds (with normalization in \( t \), see appendix A).

The present paper is mainly devoted to the case of systems modeled by \( \mathbb{Z} \)-extension of a chaotic dynamical system, which will exhibit a very different behavior from the two previous studied cases (finite measure and \( \mathbb{Z}^2 \)-extension). Instead of an almost everywhere convergence to a constant, we establish a result of convergence in distribution (in the strong sense) to a random variable. Motivated by the study of models enjoying the same properties as the \( \mathbb{Z} \)-periodic Lorentz gas with finite horizon (with domain \( \mathcal{R}_0 \) contained in a cylinder \( \mathbb{T} \times \mathbb{R} \)) and as the unit geodesic flow on a \( \mathbb{Z} \)-cover of an hyperbolic surface, we establish a result in a natural general context including these two models. As a consequence, we establish the following result. Let us recall that these two models describe the displacement of point particles moving at unit speed in some domain \( \mathcal{R}_0 \), and thus are given by flows defined on the unit tangent bundle \( T^1 \mathcal{R}_0 \) of the domain \( \mathcal{R}_0 \) (up to some identification in the case of the Lorentz gas). Furthermore, the Lorentz gas flow preserves the Lebesgue measure on \( T^1 \mathcal{R}_0 \) and the geodesic flow preserves the Liouville measure on \( T^1 \mathcal{R}_0 \).
Theorem 1.1. For the Z-periodic Lorentz gas with finite horizon (resp. for the unit geodesic flow on a Z-cover \( R_0 \) of a compact negatively curved surface), the family of normalized self-intersection number \((\frac{1}{m}N_t)_{t \geq 0}\) converges strongly in distribution\(^1\) with respect to the Lebesgue measure (resp. with respect to the Liouville measure) to \( e_1 \int_{\tilde{B}} L_1^2(x)dx \), where \( e_1 \) corresponds to the Lalley constant (expectation of the intersections number modulo \( Z \) between two independent trajectories of unit length) and where \((L_t)_{t}\) is the continuous version of the local time of the one dimensional Brownian motion \((\tilde{B}_t)_{t}\) limit, as \( m \to +\infty \) of the "discretized" (in \( Z \)) position of \((\nu^0_n)_{n}\), normalized by \( \sqrt{m} \) (see Proposition 5.15 for rigorous definitions of the quantity \( e_1 \) as well as the Brownian motion \( B \)).

For the Lorentz gas, it is also natural to investigate the question of the self-intersection number \( \nu_n \) until the \( n \)-th collision with an obstacle. The following result will appear as an intermediate result in our proof.\(^2\)

Theorem 1.2. For the Z-periodic Lorentz gas with finite horizon (recall \( R_0 \subset \mathbb{T} \times \mathbb{R} \)), \((\frac{1}{m}L_n)_{n \geq 0}\) converges strongly in distribution to \( e_1 \int_{\tilde{B}} L_1^2(x)dx \), where \( e_1 \) denotes the mean number of intersections modulo \( Z \) of two randomly chosen independent trajectories generated before hitting the Poincaré section. \((L_t)_{t}\) is a continuous version of the local time of the brownian motion \((B_t)_{t}\) limit, as \( m \to +\infty \), of the vertical position (in \( \mathbb{R} \)) of \((T^{[mt]}(\cdot))\), normalized by \( \sqrt{m} \), where \( T^{[mt]} \) is the configuration at the \([mt]\)-th collision time (see Theorem 3.7 for a precised definition of \( e_1 \) and \( B \)).

The behavior of \( \nu_n \) is itself related through the Z-extension structure to the behavior of a dynamical random walk on \( Z \) (Birkhoff sum) for which the asymptotic properties of self intersection and local times appear as a consequence of the combination of a general result by [9] (generalizing [12]) and probabilistic theorems resulting from the Nagaev-Guivarc’h perturbation method. This combined with fine decorrelation properties (i.e. mixing local limit theorem with nice error term) on such systems provide the product in the limit given in the theorems.

The crucial point in our study is indeed that the systems we are interested can be modeled by Z-extension of chaotic probability preserving dynamical systems enabling the establishment of mixing local limit theorem. Indeed Ergodic properties of Z-extension systems (see [7],[20],[14] and [15]) are closely related to those of their finitely measured fundamental domain, such as the Sinai billiard (for the Lorentz gas) or the unit geodesic flow on a compact negatively curved surface. Ergodicity of the Sinai billiard has been proved by Sinai in [21] whereas Bunimovich, Sinai and Chernov studied the central limit theorem (see [5], [6]) and Young stated exponential mixing properties through Young towers in [26]. Similar properties adapted to continuous flows have been proven for geodesic flows on hyperbolic compact surfaces with Liouville measure by Hopf, Bowen and Ratner ([3],[18]).

Let us indicate that, due to the fact that we study Z-extension instead of \( Z^2 \)-extensions, specific difficulties appear compared to the previous work [16]. First, for Z-extensions, the error term in the local limit theorem is more delicate to deal with since it is not summable. This results in different estimates and a very different kind of result (convergence to a random variable and not almost sure convergence to a constant). For the considered Z-extensions

\(^1\)The strong convergence in distribution with respect to some (finite or \( \sigma \)-finite) measure \( m \) means the convergence in distribution to the same random variable (with same distribution) with respect to any probability measure \( P \) absolutely continuous with respect to \( m \).

\(^2\)We also prove an analogous result for the self-intersections number of the geodesic flow until the \( n \)-th passage to the Poincaré section.
(unlike the $\mathbb{Z}^2$-periodic Lorentz gas), the number of intersections of two geodesic segments is not limited to 0 or 1, we have to take in account the possibility of multiple intersections. The fact that the convergence holds in distribution and not almost everywhere complicates the passage from discrete time (Theorem 1.2) to continuous time (Theorem 1.1). Indeed, it is not enough to have a convergence result for $\nu_n$, we need a functional version of this result. The fact that we have to control the gap with a random variable leads to more delicate estimates. Luckily, at some points, we were able to shorten some arguments since a control in $L^1$ is enough, but for most terms the control in $L^2$ is the most reasonable way to get the control in $L^1$. The assumptions highlighted in the present article are simpler than the properties of billiards used in [16] (e.g. our proof does not require a quantitative control of fast returns close to the initial position). An additional effort is made here to express the quantities appearing in Theorem 1.2 in terms of the flow, with e.g. the appearance in the limit of the Lalley constant. Furthermore, we establish a result under general and natural assumptions verified both in the case of the $\mathbb{Z}$-periodic Lorentz gas and in the case of the geodesic flow over a $\mathbb{Z}$-cover. This result, because of its generality, can be applied to other flows represented by $\mathbb{Z}$-extensions over a hyperbolic flow.

The article is organized as follows. Section 2 presents and describes the aforementioned dynamical systems and some key decorrelation result they satisfy. The main results are stated under abstract conditions for a general class of dynamical systems in Section 3 and applied in section 4 to the $\mathbb{Z}$-periodic Lorentz gas and the $\mathbb{Z}$-periodic geodesic flow. Section 5 is dedicated to the proof of the main results.

2. Examples

2.1. The one-dimensional Lorentz gas with finite horizon. The one dimensional Lorentz gas is a $\mathbb{Z}$-periodic billiard flow $(\mathcal{M}_0, \varphi^0_t, \mathcal{L}_0)$ describing the behavior of a point particle moving at unit speed in a domain $\mathcal{R}_0$ corresponding to a cylindrical surface dotted of open convex obstacles $(O_m + l)_{1 \leq m \leq I, l \in \mathbb{Z}}$ periodically placed according to $l \in \mathbb{Z}$ ($I$ being a finite set) with $C^3$ boundary, and non zero curvature. The point particle goes straight inside $\mathcal{R}_0$ and bounces against the obstacles according to the Snell-Descartes reflection law. Formally, we define the set of allowed positions

$$\mathcal{R}_0 := (\mathbb{T} \times \mathbb{R}) \setminus \bigcup_{m,l} (O_m + l)$$

where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. $\mathcal{M}_0$ is then the phase space i.e the set of couple positions/unit-speed on $\mathcal{R}_0$ :

$$\mathcal{M}_0 := \mathcal{R}_0 \times \mathbb{S}^1 / \sim$$

where $\sim$ identifies incident and reflected vectors, i.e. it identifies elements $(q, v) \in \partial \mathcal{R}_0 \times \mathbb{S}^1$ satisfying $\langle v, n(q) \rangle \geq 0$ (outgoing vector) with $(q, v')$ where $v' = v - 2 \langle v, n(q) \rangle n(q)$ with $n(q)$ denoting the unit normal vector to $\partial \mathcal{R}_0$ in $q$ directing into $\mathcal{R}_0$. The Lorentz gas flow

$$\varphi^0 : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$$

$$(t, x) \to \varphi^0_t(x)$$

is defined as the flow associating the couple position/unit-speed $(q_0, v_0) \in \mathcal{M}_0$ the new couple $(q_t, v_t)$ after time $t$. If no obstacle is met before time $t$ then

$$\varphi^0_t(q_0, v_0) = (q_0 + tv_0, v_0).$$

And if $q_0 + tv_0 \in \partial \mathcal{R}_0$ then

$$q_t = q_0 + tv_0$$
and 

\[ v_0 = v_0 - 2(v_0, n(q))n(q), \]

where again \( n(q) \) stands for the unit normal vector to \( \partial \mathcal{R}_0 \) in \( q \) directed into \( \mathcal{R}_0 \). This flow preserves the Lebesgue measure \( \mathcal{L}_0 \) on \( \mathcal{M}_0 \). The \( \mathbb{Z} \)-periodic Lorentz gas \( (\mathcal{M}_0, \varphi^0_t, \mathcal{L}_0) \) is said to be in finite horizon if the following roof function \( \tau^0 : \mathcal{M}_0 \to \mathbb{R} \) (free flight) is finite on any \( x \in \mathcal{M}_0 \) where \( \tau^0 \) is defined by

\[ \tau^0(q,v) = \inf\{t > 0, q + tv \in \partial \mathcal{R}_0\} \]

with \( \pi_{\mathcal{R}} \) the projection on position coordinates. This model can easily be modeled \( (\mathcal{M}_0, \varphi^0_t, \mathcal{L}_0) \) as a suspension flow (formal definition is recalled in section 3) with roof function \( \tau^0 \) over the billiard map \( (M,T,\mu,\varphi^0_t) \) defined as follow :

\[ M := \bigcup_{i \leq l \leq i+1} \{(q,v), q \in \partial \mathcal{O}_i + l, v \in \mathbb{S}, \langle v, n(q) \rangle \geq 0\} \]

the set of unitary vectors leaving the obstacles, \( T : \mathcal{M}_0 \to M \) is defined for \( x \in M \) by

\[ Tx := \varphi^0_{\tau^0(x)}(x), \]

and the measure \( \mu \) given by

\[ \mu(dr,d\theta) := \cos(\theta)drd\theta. \]

where \( r \) holds for the curvilinear coordinates with direction cosine on the boundary of the obstacles \( \partial \mathcal{O}_i + l \) and \( \theta \) the angle \( \langle v, n(q) \rangle \).

Notice that \( T \) is actually invertible, we denote by \( \bar{T} := (T|_M)^{-1} \) its inverse defined on \( M \).

By \( \mathbb{Z} \)-periodicity of the configuration of obstacles, \( \mathbb{Z} \) acts on \( M \) by the translation \( \mathcal{D} : M \to M \) of the position coordinates defined by \( \mathcal{D}(q,v) = (q + (0,1), v) \). Since \( \mu \) and \( T \) are themselves invariant by \( \mathcal{D} \), \( (M,T,\mu) \) passes itself by quotient into a probability preserving dynamical system \( (\overline{M},\overline{T},\overline{\mu}) \) which is the discrete-time Sinai billiard and which may be described as :

- \( \overline{M} := \bigcup_{i \leq l \leq i+1}\{(q,v), q \in \partial \mathcal{O}_i, v \in \mathbb{S}, \langle v, n(q) \rangle \geq 0\} \)
- \( \overline{\pi}(dr,d\theta) := \frac{1}{2\sum_{i=1}^{\infty}|\partial \mathcal{O}_i|} \cos(\theta)drd\theta \) is the probability measure on \( \overline{M} \) obtained through rescaling.
- For \( x = (q,v) \in \overline{M} \), denoting \( (q',v') := T(x), \overline{T}(x) = (q' \mod \mathbb{Z}, v') \) (where \( \mod \mathbb{Z} \) means \( \mod \{0\} \times \mathbb{Z} \)).

Define on \( (\overline{M},\overline{T},\overline{\mu}) \) the following step function \( \phi : \overline{M} \to \mathbb{Z} \) which for any \( l \in \mathbb{Z} \) associate to any \( x \in \overline{M} \) the element \( \phi(x) \) such that

\[ T \circ \mathcal{D}^l = \mathcal{D}^{l+\phi(x)} \circ \overline{T}. \]

We remind here some useful facts on the Sinai Billiard \( (\overline{M},\overline{T},\overline{\mu}) \) and the step function \( \phi \) defined above :

- Sinai shew in [4] the ergodicity of \( (\overline{M},\overline{T},\overline{\mu}) \).
- the set \( R_0 := \{(q,v) \in \overline{M}, \langle v, n(q) \rangle = 0\} \) of tangent vectors to an obstacle generates a congruent sequence \( (\xi^k_{l,k})_{l,k \in \mathbb{Z}} \) where \( \xi^k_l \) is the partition of \( \overline{M} \setminus R_{i,k} \) into connected components with \( R_i := T^i(R_0) \) for \( i \in \mathbb{Z} \) and \( R_{i,k} = \bigcup_{l=i}^{i+k} R_l \). The sets \( R_{i,k} \) are finite union of curves \( C^4 \) and according to [17], Lemma A.1], there are constants \( C > 0 \) and \( 0 < a < 1 \) such that for any \( k \in \mathbb{N} \) and any element \( A \in \xi^k_{l,k} \),

\[ \text{diam}(A) \leq Ca^k. \]
• $\phi$ is measurable and constant on elements of $\ell^0_{\ell^{-1}}$, and thanks to the finite horizon hypothesis, $\phi$ is bounded on $\overline{M}$. In addition, $\phi$ has zero mean on $\overline{M}^0$.

2.2. The Geodesic flow on a $\mathbb{Z}$-cover of a negatively curved surface. The Geodesic flow on a surface describes the evolution of a point particle moving at unit speed on the surface the geodesic defined by its initial position and speed. The case where the surface is hyperbolic and compact is widely studied and provide typical example of Anosov flow. We remind here some definitions and notations kept through this section on the notion of geodesic flow and $\mathbb{Z}$-cover.

**Hypotheses 2.1** (Geodesic flow on a compact negatively curved surface $\mathcal{R}$). Let $\mathcal{R}$ be a $C^3$-negatively curved oriented connected surface. Let $(\mathcal{M}, \varphi_t, L)$ be the dynamical system, where $\mathcal{M} := T^1 \mathcal{R}$ is the unit tangent bundle over $\mathcal{R}$, where $(\varphi_t : T^1 \mathcal{R} \to T^1 \mathcal{R})_t$ is the geodesic flow and $L$ the Liouville measure on $T^1 \mathcal{R}$ (which is $\varphi_t$-invariant).

**Hypotheses 2.2** ($\mathbb{Z}$-cover $\mathcal{R}_0$ of $\mathcal{R}$). Let $\mathcal{R}$ be as in Hypotheses 2.1. Let $\mathcal{R}_0$ be a connected surface such that there is some onto map $p : \mathcal{R}_0 \to \mathcal{R}$ such that

(1) for any $x \in \mathcal{R}$, there is a neighborhood $V_x$ such that $p$ is isometric from any connected component $C \subset p^{-1}(V_x)$ onto its image.

(2) The group

$$Cell(\mathcal{R}_0, p) := \{D' : \mathcal{R}_0 \to \mathcal{R}_0, D' \text{ est une isométrie vérifiant } p \circ D' = p\}$$

is isomorphic to $\mathbb{Z}$

(3) for any $x \in \mathcal{R}$, there is some $x_0 \in \mathcal{R}_0$ such that

$$p^{-1}(x) = \{D(x_0) : D' \in Cell(\mathcal{R}_0, p)\}.$$ 

In this section, we consider the geodesic flow system $(\mathcal{M}_0, \varphi_t^0, L_0)$ on a $\mathbb{Z}$-cover $\mathcal{R}_0$ of $\mathcal{R}$. Assuming hypotheses 2.1 and 2.2, $(\mathcal{M}_0, \varphi_t^0, L_0)$ can be defined the following way : $\mathcal{M}_0 := T^1 \mathcal{R}_0$ is actually a $\mathbb{Z}$-cover of a negatively curved oriented compact connected $C^2$ surface $T^1 \mathcal{R}$. According to point (2) of hypotheses 2.2, let $D \in Cell(\mathcal{R}_0, p)$ be an element satisfying $Cell(\mathcal{R}_0, p) := \{D^n, n \in \mathbb{Z}\}$ which then passes onto an isometry on $T^1 \mathcal{R}_0$.

The geodesic flow $\varphi_t^0$ on $T^1 \mathcal{R}_0$ is characterized by the one on $T^1 \mathcal{R}$ via $p :$

$$dp \circ \varphi_t^0 = \varphi_t \circ dp,$$

and $L_0$ is still defined as the Liouville measure on the $\mathbb{Z}$-cover.

In what follow we recall how $(\mathcal{M}_0, \varphi_t^0, L_0)$ can be seen as a $\mathbb{Z}$-extension of the $C^2$ geodesic flow $(\mathcal{M}, (\varphi_t)_t, L)$ over $\mathcal{R}$. Furthermore the geodesic flow $(\mathcal{M}, (\varphi_t)_t, L)$ over $\mathcal{R}$ can be represented as a special flow over a dynamical system isomorphic to a mixing subshift of finite type. Some geometric details in this construction will be useful to ensures that next section result applies.

Bowen and Ratner’s work in [3] and [18] led to the construction of arbitrarily small Markov partitions made from rectangles and Hasselblatt proved in [11] that their boundary are $C^1$ thus allowing the identification of $(T^1 \mathcal{R}, \varphi, L)$ with a suspension flow over a probability preserving dynamical system $(\overline{M}, \overline{T}, \overline{\mu})$ isomorphic to a mixing subshift of finite type. The set $\overline{M}$ is a subset of $T^1 \mathcal{R}$ and the map $\overline{T}$ corresponds to the first return map to $\overline{M}$ and can be defined on the whole unit tangent bundle $T^1 \mathcal{R}$ by setting

$$\overline{T}x = \varphi_{\tau(x)}(x),$$

$^3$To see that, notice that $\overline{T}^{-1} = \kappa \circ \overline{T} \circ \kappa$ and $\phi = -\phi \circ \kappa \circ \overline{T}$, where $\kappa$ is the involution given by $\kappa(q, v_{\text{reflected}}) = (q, -v_{\text{incident}})$. 
where \( \tau \) is the return time map defined by \( \tau(x) := \inf\{t > 0, \varphi_t(x) \in \overline{M}\} \). Furthermore the Poincaré section \( \overline{M} := \bigcup_{i=1}^{d} \Pi_i \subset T^1\mathcal{R} \) can be chosen so that it satisfies the following properties (given a fixed \( \delta \) smaller than the injectivity radius of \( \mathcal{R} \)).

**Properties 2.3.**
- The sets \( \Pi_i \) are pairwise disjoints and each \( \Pi_i \) is a connected subset of a two-dimensional disk contained in \( \pi^{-1}_\mathcal{R}(D_i) \) whose diameter is less than \( \delta/2 \) and where \( D_i \) is a geodesic segment, and where \( \pi_\mathcal{R} : T^1\mathcal{R} \to \mathcal{R} \) is the canonical projection.
- The sets \( \Pi_i \) are transverse to the flow, and there is some \( \eta > 0 \) such that for all \( i \), in local coordinates, for every \( q \in \gamma_i \), \( \{\theta, \exp_{x_0}^{-1}(q, \theta) \in \Pi_i\} \) is contained within \( (\pi/2 - \eta, \pi/2 + \eta) \).
- For all \( x \in \overline{M} \), \( \varphi_{[0, \frac{\pi}{2}]}(x) \cap \overline{M} \neq \emptyset \)

Note that these properties ensure in particular that two distinct geodesic trajectories \( \varphi_{[0,\tau(x)]}(x) \) and \( \varphi_{[0,\tau(y)]}(y) \) (with \( x,y \in \overline{M} \), intersect each other at most once. The importance of having such bound will be fully stated in Section 3.

Denoting \( R_i := \{x \in T^1\mathcal{R}, T^1x \in \partial\overline{M}\} \). As shown in next proposition, \( \mathcal{T} \) is \( C^1 \) on \( T^1\mathcal{R}\backslash R_{-1} \) and bijective when restricted to the Poincaré section \( \overline{M} \) over itself with reciprocal function given by

\[
\mathcal{T}x = \varphi_{-\hat{\tau}(x)}(x)
\]

where \( \hat{\tau}(x) := \inf\{t > 0, \varphi_t^{-1}(x) \in \overline{M}\} \).

**Proposition 2.4.** The application \( \mathcal{T} \) as defined above is \( C^1 \) on \( T^1\mathcal{R}\backslash R_{-1} \) with \( \overline{M} \) as above.

**Proof.** Fix \( x_0 \in T^1\mathcal{R}\backslash R_{-1} \), by construction of the Poincaré section, the trajectory \( (\varphi_t(x_0))_{t \in \mathbb{R}} \) necessarily crosses the section. The latter being transverse to the flow, \( \tau(x) \) is well defined : There is some \( y_0 \in \overline{M} \) and \( \tau(x_0) \) such that

\[
\varphi_\tau(x_0)(x_0) = y_0
\]

Passing to local coordinates by \( \exp_{x_0} \), the above constraint gives

\[
\varphi_\tau(x_0)(x_0) - y_0 = 0.
\]

Then apply the implicit functions theorem : Define \( \psi(x,t,y) = \varphi_t(x) - y \) (which is \( C^1 \)) where locally \( t \) and \( y \) are described one and two dimensional spaces whereas \( x \) is in dimension 3. Since \( \partial_3 \psi(x,t_0,v_0) = (f(y_0), 1) \) where \( f(y_0) \) and \( v_0 \) are free vectors (the section being transverse to the flow), denoting \( x_0 = (q_0,v_0) \), \( \partial_2 \psi = \partial_1 (g_0 + tv_0, v_0) = (v_0, 0) \) is still invertible.

Deduce that \( \partial_{2,3} \psi(x_0, t_0, y_0) \) is isomorphic to \( \begin{pmatrix} v_0 & f(y_0) \\ 0 & 1 \end{pmatrix} \) which is invertible. The implicit function theorem then applies, there is a \( C^1 \) function \( f \) defined on a neighborhood of \( x_0 \) such that \( f(x_0) = (\tau(x_0), y_0) \) and

\[
\psi(x,f(x)) = 0.
\]

Thus

\[
\varphi(p_1(f(x)), x) \in \overline{M}.
\]

where \( p_1 \) is the projection onto the first coordinate. Thus \( \mathcal{T}(x) = \varphi(p_1(f(x)), x) \) is \( C^1 \) on \( T^1\mathcal{R} \).

The invariant measure \( \overline{\mu} \) is defined as the renormalized probability measure from the measure \( \mu_0 \) characterized through Borel sets \( A \times [0,s) \) for \( s > 0 \) and \( A \subset \overline{M} \) by the following relation for any measurable \( f \) on \( \mathcal{M} \)

\[
\int_A \int_0^s f \circ \varphi_t(q) d\lambda(s) d\mu_0(q) = \int_{\varphi_{[0,s]}(A)} f(x) dL(x).
\]
Here is a following well known property (see chapter 19 from book [19])

**Proposition 2.5.** Let $\mathcal{R}$ be a compact Riemannian surface and $L$ the Liouville measure on the unitary bundle $T^1\mathcal{R}$. Let $\overline{M}$ be as described above, then $T^1\mathcal{R}$ is locally diffeomorphic to an open set in $\overline{M} \times \mathbb{R}$ (by $(x,t) \mapsto \varphi_t(x)$ for $(x,t) \in \overline{M} \times \mathbb{R}$) and $L$ coincides in local coordinates with the measure $\mu_0 \otimes \lambda$ where $\mu_0$ is supported on $\overline{M}$ and given in its local curvilinear coordinates $(r, \phi)$ by

$$d\mu_0(r, \phi) = \cos \theta drd\theta.$$ 

($r$ being the curvilinear position and $\theta$ the angle with the normal line to the disk $D_1$ at the point with curvilinear position $r$).

According to the Bowen and Ratner constructions (see [18],[3]), there is some $\overline{M}$ satisfying Properties 2.3 such that $(\overline{M}, T, \mu)$ is isomorphic to a mixing subshift of finite type $(\Sigma, \sigma, \nu)$ where $\nu$ is a Gibbs measure with potential $h \in H(\Sigma)$ ($H(\Sigma)$ being the set of Hölder functions).

For $i, j \in \mathbb{N}$, $i \leq j$, define $R_{i,j} := \bigcup_{i \leq k \leq j} R_i$ and $\xi_i^j := \overline{M} \setminus R_{i,j}$. Elements of $\xi_i^j$ make cylinders in the Bowen and Ratner constructions and thus satisfy

$$\exists C > 0, 0 < a < 1, \text{ s.t } \forall k \in \mathbb{N}, \forall A \in \xi_{i-k}^i, \text{ diam}(A) \leq Ca^k.$$ 

By definition of $\mathcal{M}_0$ as $\mathbb{Z}$-cover, $\overline{M}$ may be lifted in a set $M$ within $\mathcal{M}_0$. Define the dynamical system $(M, T, \mu)$ with $T$ given by

$$Tx := \varphi^0_{\tau^0(x)}(x)$$

and $\tau^0$ standing for $\tau^0(x) := \inf\{t > 0, \varphi_t(x) \in \mathcal{M}\}$. The measure $\mu_0$ on $\overline{M}$ is lifted into a measure $\mu$ on $M$ by the following characterization on small enough open sets $A$ in $M$ by $\mu(dp(A)) = \mu_0(A)$. Fix some section $s: \mathcal{R} \rightarrow \mathcal{R}_0$ of the $\mathbb{Z}$-cover $p$ (i.e $p \circ s = id$) and define the step function $\phi: \overline{M} \rightarrow \mathbb{Z}$ through the following characterization for $x \in \overline{M}$,

$$D^{\phi(x)} \circ s(Tx) = T \circ s(x).$$

As defined, this function $\phi$ is constant on elements of $\xi_{0,1}$. $(\mathcal{M}_0, \varphi^0, \mathcal{L}_0)$ is then a suspension flow with roof function $\tau^0$ over $(M, T, \mu)$ which is itself a $\mathbb{Z}$-periodic extension of $(\overline{M}, T, \Gamma \overline{p})$ through step function $\phi$ and rescaling $\Gamma := \mu_0(\overline{M})$ (see next section for formal definition). Such structure along with the invariance of the Liouville measure $\mathcal{L}_0$ under the flow $\varphi^0$ ensures that $\phi: \overline{M} \rightarrow \mathbb{Z}$ has zero mean.

3. **General results.**

Let $(\overline{M}, T, \overline{p})$ be a probability preserving dynamical system, $\phi: \overline{M} \rightarrow \mathbb{Z}$ a step function, $\tau: \overline{M} \rightarrow \mathbb{R}_+$ a roof function and $\Gamma > 0$ a normalizing constant. Remind the following definitions.

**Definition 3.1.** The $\mathbb{Z}$-extension of a probability preserving dynamical system $(\overline{M}, T, \overline{p})$ by $\phi$ is a probability preserving dynamical system $(\overline{M} \times \mathbb{Z}, T', \sum_{n \in \mathbb{Z}} \overline{p} \otimes \delta_n)$ whose transformation $T'$ is defined by

$$T'(x, n) := (Tx, n + \phi(x)). \hspace{1cm} (1)$$

We will assume that $(M, T, \mu/\Gamma)$ is the $\mathbb{Z}$-extension of $(\overline{M}, T, \overline{p})$ by $\phi$. 
Remark 3.2. Given an extension \((M, T, \mu)\), there exists a automorphism \(D\) on \(M\) satisfying
\[ T \circ D = D \circ T, \]
where \(D\) corresponds to the application \((x, k) \mapsto (x, k + 1)\) seen on \(\overline{M} \times \mathbb{Z}\). Identifying \(\overline{M}\) with \(\overline{M} \times \{0\}\), it may be said that \(\overline{M} \subset M\) and relation (1) can be rewritten as
\[ T(x) = D^{\nu(x)}(T(x)) \quad \forall x \in \overline{M}. \]

Definition 3.3. Given a dynamical system \((M, T, \mu)\) and a roof function \(\tau : M \to \mathbb{R}_+\), a suspension flow over \((M, T, \mu)\) is a system \((M_\tau, \varphi_t, \lambda')\) where
- \(M_\tau := \{(x, s), x \in M, 0 \leq s \leq \tau(x)\}\)
- for all \((x, s) \in M_\tau\), \(\varphi_t(x, s) := (T^{\tau(x)} + t + s - \sum_{k=0}^{n(x)-1} \tau \circ T^k(x))\) where \(n_t(x) := \sup \left\{ n, \sum_{k=0}^{n-1} \tau \circ T^k(x) \leq t \right\} \)
- \(d\lambda'(x, s) = d\mu(x)ds\) for all \((x, s) \in M_\tau\) is an invariant measure.

Remark 3.4. Here again, \(M\) may be identified with \(M \times \{0\} \subset M_0\).

Throughout this section, \((M_0, \varphi^0, \mathcal{L}_0)\) denotes a measure preserving dynamical system isomorphic to a suspension flow with roof function \(\tau : M \to \mathbb{R}_+\) over a system \((M, T, \mu)\). We assume furthermore that \((M, T, \Gamma^{-1}\mu)\) is a \(Z\)-extension of the probability preserving dynamical system \((\overline{M}, \overline{T}, \overline{\mu})\) by \(\phi : \overline{M} \to \mathbb{Z}\) and that \(\tau \circ D = \tau\), where \(D\) stands for an associated automorphism.

Let \(\mathcal{R}\) be a set (corresponding to the set of positions in our examples), and fix a projection \(\pi_R : M_0 \to \mathcal{R}\). Denote by \(N_i\) the number of self intersections of the trace of the flow \(\pi_R \circ \varphi^0\) :
\[ N_i(x) := |\{(s, u), 0 \leq s < u \leq t : \pi_R(\varphi_s(x)) = \pi_R(\varphi_u(x))\}| \]

In the discrete case \((M, T, \mu)\), we also introduce and study the following natural quantity \(\nu_n\) denoting the number of self-intersection up to the \(n^{th}\) crossing of the Poincaré Section :
\[ \nu_n(x) := \sum_{k \geq 1} \sum_{0 \leq s < t \leq n-1} \left( V_{k}^{(\tau^i(x))} \circ T^i(x) + \sum_{i=0}^{n-1} \nu_1(T^i(x)) \right) \]
where \(\nu_1 := |\{(s, t) \in [0, \tau(x]) : s < t : \pi_R(\varphi_s(x)) = \pi_R(\varphi_t(x))\}|\) and \(V_k^{(x)}\) is the set
\[ V_k^{(x)} := \{y \in M, |[y] \cap [x]| = k\}. \]
with \([x] := \pi_R(\varphi_{0, \tau(x)}(x))\) standing for the set of points in \(\mathcal{R}\) within the trajectory from \(x \in M\) to \(Tx\).

Let \(D > 0\), for \(i, j \in \mathbb{Z}\) such that \(i < j\), we suppose we have a congruent family \((\xi^j_i)_{i, j}\) of partitions of \(\overline{M}\) and thus of \(M\) such that \((\xi^j_i)_{i, j}, \phi, (\overline{M}, \overline{T}, \overline{\mu})\) and \(D\) satisfy the following assumptions :
(a) \((\overline{M}, \overline{T}, \overline{\mu})\) is an ergodic dynamical system and \(\phi\) is constant along the elements of \(\xi^0\).
(b) Starting from the Poincaré section \(M\), distinct trajectories cross just a bounded number of times before reaching the Poincaré section again :
for all \(x \in M\),
\[ \mu(\{y \in M, |[y] \cap [x]| > D\}) = 0 \]
and for all \(k \geq 1\),
\[ \mu(\{y \in M, |[y] \cap [T^k y]| > D\}) = 0. \]
In addition, \( V_1 \) satisfies \(|V_1| \leq D\) and \( \phi \) satisfies a “finite horizon” hypothesis:
\[
\forall x \in M, \forall k \in \mathbb{Z}, |\phi(x)| \leq D \quad \text{and} \quad V_k(x) \subset \bigcup_{N=-D}^{D} D^N M.
\]
(c) \( D^{-1}V_k^{(D)} = V_k^{(x)} \)
(d) \( S_n := \sum_{k=0}^{n-1} \phi \circ T^k \) satisfies
\[
\|S_n\|_{L^2(\mathcal{M})} = O(n^{1/2})
\]
(e) \( \phi \) satisfies a local limit theorem with decorrelation: For any \( \sqrt{\frac{n}{\pi}} > p > 1 \), there is some \( C > 0 \) such that for all \( N \in \mathbb{Z} \), and all sets \( A, B \subset M \) where \( A \) is a union of elements from \( \xi^{-k} \) and \( B \) is an union of sets from \( \xi^{-\infty} \) and for all \( n \geq 2k + 1 \):
\[
\left| \mathcal{P}(A \cap (S_n = N) \cap T^{-n}(B)) - \frac{e^{-\frac{n^2}{2(n-2k)}}}{(2\pi\Sigma)^{1/2}(n-2k)^{1/2}} \mathcal{P}(A)\mathcal{P}(B) \right| \leq \frac{Cn^{1/p}}{n-2k}
\]
(f) \( \left( \frac{S_n}{n^{1/2}} \right)_n \) converges in distribution towards a Brownian motion \( (B_t) \), with variance \( \Sigma \) and the local time \( N_n(l) := \sum_{k=0}^{n-1} 1_{S_k=l} \) (for \( l \in \mathbb{Z} \)) associated to \( S_n \) satisfies
\[
E_\mathcal{P} \left( |N_n(x) - N_n(y)|^2 \right) = O\left(n^{1/2}|x-y|\right).
\]
Where the upper bound is uniform in \( x, y \in \mathbb{Z} \) and \( n \in \mathbb{N} \).
(g) Denote for \( A \subset M \)
\[
A^{[n]} := \bigcup_{Z \in \xi^{-n}, Z \cap A \neq \emptyset} Z,
\]
and
\[
A^{[n]}^- := \bigcup_{Z \in \xi^{-n}, Z \subset A} Z.
\]
There are some constants \( 1 \leq \alpha > 0 \), \( C > 0 \) and \( \alpha > 0 \) such that for \( \epsilon > 0 \) and \( k \in \mathbb{N} \), there is a \( \mu \)-essential partition \( \mathcal{P}_\epsilon \) of \( M \) made of at most \( C \left[ \frac{1}{\epsilon} \right]^{2\alpha} \) sets \( A \subset M \) satisfying
\[
\mathcal{P}(A) \leq C\epsilon^{2\alpha}
\]
and
\[
\mu((\partial A)^{[n]}) \leq C\epsilon^\alpha n^\alpha.
\]
In addition, for \( A \in \mathcal{P}_\epsilon \), there is some set \( B_A^{[k]} \) described as union of elements of \( \xi^{-k} \), such that for all \( x, y \in A \)
\[
\bigcup_{m \geq 1} (V_m^x)^{[k]} \Delta V_m^y \subset B_A^{[k]}
\]
and satisfying
\[
\mu(B_A^{[k]}) \leq C(\epsilon + a^k)^\alpha.
\]
\textbf{Remark 3.5.} Hypothesis (e) is a local limit theorem for \( S_n \) with decorrelation. Hypothesis (f) frequently derives from hypothesis (e).

\textbf{Definition 3.6} (Strong convergence in distribution). Given a measurable space \( (M, \mathcal{B}) \) endowed with a finite or \( \sigma \)-finite measure \( \mu \), a sequence \( (V_n)_{n \in \mathbb{N}} \) of measurable functions on \( M \) is said to converge strongly in distribution, with respect to \( \mu \), to a random variable \( R \) defined on
a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) if for all probability measure \(P\) absolutely continuous with respect to \(\mu\) \((P \ll \mu)\),

\[
V_n \overset{\mathcal{L}}{\underset{n \to \infty}{\rightarrow}} R,
\]

where \(\mathcal{L}_P\) stands for the convergence in distribution with respect to \(P\).

We then write \(V_n \overset{\mathcal{L}}{\underset{n \to \infty}{\rightarrow}} R\).

**Theorem 3.7.** Let \(T > 0\). Under the above hypotheses,

\[
\left( \frac{1}{n^{3/2}} \nu(nt) \right)_{t \in [0,T]} \overset{\mathcal{L}}{\underset{n \to \infty}{\rightarrow}} \left( e_I \Gamma^{-2} \int_R (L_t)^2 dx \right)_{t \in [0,T]},
\]

where \((L_s)_{s \geq 0}\) is a continuous version of the local time associated of the Brownian motion \(B\) from hypothesis \((f)\) and where

\[
e_I := \int_{\mathbb{M} \times \mathbb{M}} |[y] \cap [x]| d\mu(y) d\mu(x).
\]

Recall that \(L_t(a)\) is defined almost surely for \(a \in \mathbb{R}\) and \(t \geq 0\) by

\[
L_t(a) := \lim_{t \to 0} \frac{1}{2t} \int_0^t 1_{a-t \leq \tau \leq a+\epsilon} ds,
\]

**Remark 3.8.** Introduce an additional hypothesis,

\((b')\) for all \(i \in \mathbb{Z}\), for every \(x\), \(\pi_R([x]) \cap \pi_R([D^i(x)]) = \emptyset\).

The function \(D\) defined on \(\mathbb{M}\) may be extended into a function \(D_0\) on \(\mathbb{M}_0\) satisfying

\[
D_0(\varphi^0_0(x)) = \varphi^0_0(D_0(x)).
\]

Hypothesis \((b')\) leads to the notion of “projection” modulo \(\mathbb{Z}\) \(\pi_R\) on \(\mathbb{M}_0\) given by

\[
\pi_R(x) = \pi_R(x)
\]

for \(x \in \mathbb{M}\) and satisfying \(\pi_R(D_0x) = \pi_R(x)\) for all \(x \in \mathbb{M}_0\). \(\Gamma^{-2}e_I\) can then be rewritten

\[
\Gamma^{-2}e_I := \int_{\mathbb{M} \times \mathbb{M}} |\pi_R(\varphi^0_0(\tau(x))) \cap \pi_R(\varphi^0_0(\tau(x)))(y)| d\pi(y) d\pi(x).
\]

Denote by \(\pi\) the projection of \(\mathcal{M}\) onto \(\overline{\mathcal{M}}\).

**Theorem 3.9.** Let \(T > 0\). On \((\mathcal{M}_0, \varphi^0_0, \mathcal{L}_0)\), adding assertion \((b')\) to the hypotheses of previous theorem then

\[
\left( \frac{1}{t^{1/2}} N_{ts} \right)_{s \in [0,T]} \overset{\mathcal{L}}{\underset{t \to \infty}{\rightarrow}} \left( e'_I \int_R \tilde{L}_s^2(x) dx \right)_{s \in [0,T]},
\]

with

\[
e'_I := \int_{\mathbb{M} \times \mathbb{M}} |\pi_R(\varphi^0_0(x)) \cap \pi_R(\varphi^0_0(y))| dL(y) dL(x)
\]

the constant appearing in \([13]\) corresponding to the expectation of the number of intersections between \(\pi_R(\varphi^0_{[0,1]}(x))\) and \(\pi_R(\varphi^0_{[0,1]}(y))\) for \(x\) and \(y\) randomly chosen on \((\mathcal{M}, \varphi_1, \mathcal{L})\) and \((\tilde{L}_s)_{s \geq 0}\) is defined as the local associated to a Brownian motion \(\tilde{B}\) which can be seen as

\[
\left( \frac{1}{t^{1/2}} S_{nt} \circ \pi(.) \right)_{t \to \infty} \overset{\mathcal{L}}{\underset{t \to \infty}{\rightarrow}} (\tilde{B}_u)_{u \in [0,T]}.
\]
4. Application: Lorentz gas and geodesic flows

Both systems presented in introduction, the $\mathbb{Z}$-periodic Lorentz gas and the $\mathbb{Z}$-periodic geodesic flows on a negatively curved surface, satisfy Theorems 3.9 and 3.7 on the self intersections number of the trajectories on $\mathcal{R}_0$. This section checks the different hypotheses required.

In both cases $\mathcal{R}$ denotes the set of position coordinates and $\pi_{\mathcal{R}}$ the projection along speed vectors $\pi_{\mathcal{R}}(q,v) := q$. Only the constant $\alpha$ in the hypotheses differ from one system to another and corresponds to the Hölder regularity of $T$ on connected components of $M\setminus R_{-1,0}$ in each system (i.e $\alpha = 1/2$ for the Lorentz gas and 1 for the geodesic flow).

Hypothesis (a): Both systems are suspension flows over an hyperbolic system $(\mathcal{M}, T, \mu)$ which is then ergodic (see [3] and [18] for the geodesic flow or [4] for the Lorentz gas). $\phi$ is constant on the respective partition $\xi_{0}^{-1}$ as defined for each system in section 2.

Hypothesis (b): For both systems, since the trajectories are in both cases geodesics the distance between two intersections are bounded below and their length being uniformly bounded above, the number of self intersections $| x \cap y |$ is uniformly bounded by some $D > 0$. The case of non finite intersections between $y$ and $[Ty]$ only occurs for periodic orbits which lay in a set of null measure, $\mu(\{y, |y\cap[T^k y]| > D\}) = 0$ for all $k$. Besides, orbits on the continuous dynamical system $(\mathcal{M}_0, \varphi^0_t, \mathcal{L}_0)$ always hit the Poincaré section $\mathcal{M}$ in bounded time, thus for all $x \in \mathcal{M}$,

$$|\phi(x)| \leq D$$

and $V_k(x)$, the set of elements whose trajectory crosses the one from $x$, is also bounded.

Hypotheses (c) and (b') are satisfied in both cases thanks to the explicit $\mathbb{Z}$-periodicity of our systems and the projection $\pi_{\mathcal{R}}$ on position coordinates.

Hypothesis (d) holds on both systems and is a straightforward consequence of the perturbation Theorem from [10, 23].

Hypothesis (e) is a consequence of the perturbation theorems from [10, 23] which have been adapted into local limit theorem with decorrelation by Yassine in [25] in the case of geodesic flows on negatively curved surface, and by Pène and Saussol in [17] for planar Lorentz gas (which can be straightforwardly projected to the decorrelation result as stated in hypothesis (e)).

Hypothesis (f) is a known result for both systems but may be seen as a direct consequence of the perturbation Theorem and [8].

Let's show that hypothesis (g) is satisfied:

First construct a suitable partition of $\overline{\mathcal{M}}$:

In both systems, since $\mathcal{M}$ can be embedded in some compact set, let $\mathcal{U}$ be a finite atlas of open sets $O$ with diameter lesser than the injectivity radius denoted by $R$. Denote $P_\mathcal{U}$ the $\overline{\mathcal{P}}$ essential partition made of the interior of the elements of the finite partition $\bigwedge_{O \in \mathcal{U}} O$. To each element $A \in P_\mathcal{U}$ one may associate $x_A \in A$ and a chart $(O_A, \exp_A)$ such that $\exp_A(x_A) = 0 \in O_A$. 


Now, fixing $m > 0$, let $\tilde{\mathcal{P}}_m$ be the $\mathcal{P}$ essential partition made of non empty elements

$$\exp_A^{-1} \left( A \cap \left[ \frac{i - 1 + \epsilon}{m^{\alpha}}, \frac{i + 1 + \epsilon}{m^{\alpha}} \right] \times \left[ \frac{j - 1 + \epsilon}{m^{\alpha}}, \frac{j + 1 + \epsilon}{m^{\alpha}} \right] \right)$$

for $|i| \leq \frac{3m^\alpha}{4}$.

Such partition is finite and denoting by $N$ the number of elements in $P_\mathcal{U}$,

$$|\tilde{\mathcal{P}}_m| \leq N |2R|m^{2\alpha}.$$  

From $\tilde{\mathcal{P}}_m$ we construct $\mathcal{P}_m$ whose elements are given by the connected components of

$$A \setminus R_{-1,0}$$

for $A \in \mathcal{P}_m$ where $R_{-1,0} := \bigcup_{B \in \mathcal{P}_0} \partial B$. By construction we get

$$\overline{p}(A) \leq C \frac{1}{m^{2\alpha}}.$$  

The boundaries of the partition $\xi_0^1$ consists in a finite number of $C^1$ curves of finite length which cut $\mathcal{X}$ into finitely many connected components. This means, on the one hand, that the number of elements in $\mathcal{P}_m$ grows still uniformly in $O(m^{2\alpha})$ and, on the other hand, that the boundary of $A \setminus R_{-1,0}$ is made of uniformly regular curves and so its length can be bounded above by some value of order $\frac{1}{m^{\alpha}}$.

Let us remind now that for both systems, there is a constant $0 < a < 1$ such that the elements of $B \in \xi_0^k$ are of diameter

$$\text{diam}(B) = O(a^k),$$

uniformly in $k$. And so for $A \in \mathcal{P}_m$,

$$\overline{p}(\partial A)^{[k]} = O \left( a^k \frac{1}{m^{2\alpha}} \right).$$

Let $A \in \mathcal{P}_m$. The set $A$ is a connected component of $B(x_A, \epsilon) \setminus R_{-1,0}$. Let $x_A$ be a fixed element of $A$ and $\epsilon > 0$. For all $y \in A$,

$$V_k^{(x_A)} \Delta V_k^{(y)} \subset B_A,$$

where $B_A := \tilde{T}^{-1}(\pi_R^{-1} \pi_R A) \cup \tilde{T}^{-1} \left( \pi_R^{-1} \pi_R (T(A)) \right) \cup \pi_R^{-1} \pi_R A \cup \pi_R^{-1} \pi_R (T(A))$. Thanks to the regularity of $T$ and of $\tilde{T}$ ($1/2$-Holder for billiard map and Lipschitz for geodesic flow on connected components of $\xi_0^1$), such a set can be controled as follows

$$B_A \subset \tilde{T}^{-1} \left( \pi_R^{-1} \pi_R \left( B(T(x_A), C\epsilon^\alpha) \cup B(x_A, \epsilon) \right) \right) \cup \pi_R^{-1} \pi_R \left( B(x_A, \epsilon) \cup B(T(x_A), \epsilon^\alpha) \right).$$

In other words, a trajectory $[z]$ hits $[y]$ as many times as $[x_A]$ except if one of its extremities is either between $y$ and $x_A$ or between $Ty$ and $Tx_A$.

Observe that, for $y \in A$,

$$\bigcup_{m \in \mathbb{N}} (V_m^{(x_A)})^{[k]} \Delta V_m^{(y)} \subset (B_A)^{[k]}$$

for any $k$ arbitrarily large.

Noticing that $V_m^{(y)} \setminus V_m^{(x_A)} \subset B_A$,

$$V_m^{(y)} \setminus (V_m^{(x_A)})^{[k]} \subset (V_m^{(y)} \setminus V_m^{(x_A)})^{[k]} \subset B_A^{[k]}$$
and,

\[
\begin{align*}
(V_m^{(x_A)})^{[k]} \backslash V_m^{(y)} & \subset (V_m^{(x_A)})^{[k]} \backslash V_m^{(x_A)} \cup V_m^{(x_A)} \backslash V_m^{(y)} \\
& \subset (\partial V_m^{(x_A)})^{[k]} \cup V_m^{(x_A)} \backslash V_m^{(y)}.
\end{align*}
\]

Where \(\partial V_m^{(x_A)}\) stands for the boundary seen within \(\bigcup_{B \in E_0} B\) which is

\[
\partial V_m^{(x_A)} = \tilde{T}^{-1}(\pi_R^{-1} \pi_R(x_A)) \cup \tilde{T}^{-1}(\pi_R^{-1} \pi_R(T(x_A))) \cup \pi_R^{-1} \pi_R(x_A) \cup \pi_R^{-1} \pi_R(T(x_A)) \subset B_A.
\]

And so

\[
(V_m^{(x_A)})^{[k]} \backslash V_m^{(y)} \subset B_A^{[k]}.
\]

Thanks to (3), \(B_A^{[k]}\) is bounded from above by

\[
B_A^{[k]} \subset \tilde{T}^{-1}(\pi_R^{-1} \pi_R(B(T(x), C(\epsilon^a + C a^k))) \cup B(x, \epsilon + C a^k)) \cup \pi_R^{-1} \pi_R B(x, \epsilon) \cup B(T(x), \epsilon^a + C a^k)).
\]

And thus, since in both systems the measure is absolutely continuous with respect to the Lebesgue (or Liouville) measure with a bounded density, \(\mu(B_A^{[k]}) = O (\epsilon + C_2 a_k)^n\).

Hence we have proved that Both systems satisfy the hypotheses of Theorems 3.7 and 3.9.

Now is given some lemma providing explicit value for the constant \(e_I\) in both systems:

**Lemma 4.1.** For \(x \in M\), recall

\[
e_I := \int_{M \times M} [y] \cap [x] d\mu(y) d\mu(x) = \int_M E_{\mu} \left( \sum_{k=1}^D k 1_{V_k^{(x)}} \right) d\mu(x).
\]

\(e_I\) satisfies in both systems

\[
e_I = 4\Gamma E_{\pi(\tau)},
\]

where \(\Gamma := \mu(\overline{M})\).

**Proof.** Let’s build a dynamical system \((M', T', \mu')\) from \((M, T, \mu)\) by adding the abstract obstacle \([x] := \{ \pi_R(x), \pi_R(T(x)) \} \times S\) to the Poincaré section. The measure \(\mu'\) coincides with \(\mu\) on \(M = M' \backslash [x] \times S\) and is still given in local coordinates by \(\mu(dr, d\theta) = \cos(\theta)drd\theta\).

Denoting for \(y \in M\),

\[
\phi_M(y) := \inf\{n \in \mathbb{N}^*, T^nx \in M\}
\]

then for all \(y \in V_k^{(x)}\),

\[
\phi_M(y) = k + 1.
\]

Kac’s formula then gives,

\[
\mu'(M') = \int_M \phi_M d\mu' = \int_M \phi_M d\mu = \mu(M) + \int_M \sum_{k=1}^D k 1_{V_k^{(x)}} d\mu.
\]

And so

\[
E_{\mu} \left( \sum_{k=1}^D k 1_{V_k^{(x)}} \right) = \mu'(M') - \mu'(M) = \mu'([x] \times S) = 4\tau(x).
\]

\(\square\)
5. Proof of theorem 3.7.

From now on, for $n \in \mathbb{N}^*$, let $\mathcal{A}_n := \mathcal{P}_{\lceil n/20n \rceil}$ and fix for each $A \in \mathcal{A}_n$ some $x_A \in \overline{A}$.

Introduce $k_n := \lfloor \log(n)^2 \rfloor$.

Let us show that it follows from hypothesis (c) that $\nu_n$ is $\mathcal{D}$ invariant (periodic). So it will enough to study directly $\nu_n$ over $(\overline{M}, \overline{T}, \overline{\mu})$:

Indeed for $x \in \overline{M}$, the orbit may be rewritten $T^i(x) = \mathcal{D}^{S_i(x)}(\overline{T}^i(x))$. So $\nu_n$ may be rewritten as a function in $\overline{M}$ in a way underlying the role of $S_n$ and $\overline{T}^n$ in it behavior:

Thanks to hypothesis (b),

$$T^i(x) \in V_{k}^{(\overline{T}^i(x))}$$

if and only if

$$T^i(x) \in \bigcup_{N = -D}^{D} \left( \mathcal{D}^{(N + S_i(x))}(\overline{M}) \cap V_{k}^{(\overline{T}^i(x))} \right)$$

if and only if

$$\left\{ \begin{array}{ll}
S_i(x) = S_j(x) + N \\
T^i(x) \in \overline{M} \cap \mathcal{D}^{-(N + S_i(x))}(V_{k}^{(\overline{T}^i(x))})
\end{array} \right.$$

in other words, with hypothesis (d),

$$\left\{ \begin{array}{ll}
S_i(x) = S_j(x) + N \\
\overline{T}^i(x) \in \overline{M} \cap \mathcal{D}^{-N}(V_{k}^{(\overline{T}^i(x))})
\end{array} \right. \quad (4)$$

So $\nu_n$ can be fully expressed on $(\overline{M}, \overline{T}, \overline{\mu})$ with comparisons between $\overline{T}$, $\overline{T}^j$ and $S_i, S_j$:

$$\nu_n(x) := \sum_{N = -D}^{D} \sum_{k = 1}^{D} \sum_{0 \leq i < j \leq n-1} 1_{\{S_i = S_j + N\}} 1_{\mathcal{D}^{-N}(V_{k}^{(\overline{T}^j(x))}) \cap \overline{M}} \circ \overline{T}^i(x) + \sum_{i = 0}^{n-1} \nu_1(\overline{T}^i x)$$

in order to make our hypothesis of decorrelation (e) working, we adapt $\nu_n$ into some approximation according to the partition $\xi_{-k_n}$:

$$\nu_n(x) := \sum_{N = -D}^{D} \sum_{k = 1}^{D} 2k \sum_{0 \leq i < j \leq n-1} 1_{\{S_i = S_j + N\}} \sum_{A \in \mathcal{A}_n} 1_{A} \circ \overline{T}^i(x) 1_{D^{-N}(V_{k}^{(\overline{T}^i(x))}) \cap \overline{M}} \circ \overline{T}^i(x).$$

Let’s give here some sketch of the proof: In (4), hypothesis (f) states that the Birkhoff sum $\mathcal{S}_n$ acts like a random walk whereas the decorrelation in hypothesis (e) states that the asymptotic expectation of $e_{(i,j)} := \sum_{A \in \mathcal{A}_n} 1_{A} \circ \overline{T}^i(x) 1_{D^{-N}(V_{k}^{(\overline{T}^i(x))}) \cap \overline{M}} \circ \overline{T}^i(x)$ is $e_I$ (along with some control on its fluctuations). So separating $\mathcal{V}_n$ into an increasing offset part and its fluctuations we get

$$\mathcal{V}_n := \sum_{N = -D}^{D} \sum_{0 \leq i < j \leq n-1} 1_{\{S_i = S_j + N\}} e_I + \sum_{N = -D}^{D} \sum_{0 \leq i < j \leq n-1} 1_{\{S_i = S_j + N\}} (e_I - e_{(i,j)})$$

$$\mathcal{V}_n \simeq \sum_{N = -D}^{D} \sum_{l \in \mathbb{Z}} N_n(N + l)N_n(l)e_I$$
where \(N_n(N) := \sum_{i=0}^{n-1} 1\{S_i = N\}\) is the local time of the ”random walk” \((S_n)_n\). Combining known results in probability theory [9] with estimates established via the Nagaev-Guivarc’h perturbation method, the local time of \((S_n)_n\) converges in distribution to the one of a Brownian motion
\[
\frac{1}{n^{1/2}} N_{\lfloor n^{1/2} \rfloor} \xrightarrow{n \to \infty} (L_t(\cdot))_t.
\]
And thus giving that \(\sum_{t \in \mathbb{Z}}\) acts like an integral on a process with discrete values,
\[
\frac{1}{n^{1/2}} V_{\lfloor n^{1/2} \rfloor} \xrightarrow{n \to \infty} \int_{\mathbb{R}} L_t^2(x) \, dx.
\]
which should prove the theorem.

Here is the rigorous proof, we make sure here that \(V_n\) is a good approximation of \(\nu_n\):

**Proposition 5.1.** Under the hypotheses of theorem 3.7,
\[
\text{(1) } \|\nu_n - \nu_n\|_p = o(n^{3/2})
\]
\[
\text{(2) } \lim_n \sum_{A \in \mathcal{A}_n} \sum_{k=1}^{D} k \mu(A^{(k_n)}) \mu(V_k^{(x_A)}) = e_I
\]
where \(\Gamma := \mu(\mathcal{M})\) and \(e_I\) is a constant given in theorem 3.7 .

**Proof.** The proof of this statement actually follows the ideas of the work by Pène [16] adapted to the broader scope of our hypotheses of section 3. The main difference with the work in [16] resides in the fact that two trajectories starting from the Poincaré section \(M\) may intersect several times before reaching the Poincaré section again. The main difference with the work in [16] resides in the fact that two trajectories starting from the Poincaré section \(M\) may intersect several times before reaching the Poincaré section again. For \(x \in \mathcal{M}\), we adapt the quantity \(E_{i,n}^{k,x} := \{y \in \mathcal{M}, |[\pi_R(y), \pi_R(Ty)] \cap [\pi_R(x), \pi_R(Tx)]| = k\}\) into an integrable one and then into a measurable one with respect to the families of partitions \((\xi^{-m}_{-m})_{m \in \mathbb{N}}\). Denote
\[
E_{i,n}^{k,x} := \left\{ x \in \mathcal{M} : x \in T^{-n}(V_k^{T^k x}) \right\}
\]
\[
= \bigcup_{N=-D}^{D} \left\{ x \in \mathcal{M} : S_n(x) = N + S_i(x), x \in T^{-n}(D^{-N}(V_k^{T^k x}) \cap \mathcal{M}) \right\}.
\]
For \(x \in \mathcal{M}\) and \(n \in \mathcal{M}\), \(\nu_n(x)\) can be rewritten
\[
\nu_n(x) = \sum_{k=1}^{D} k \sum_{0<i<j<n-1} 1_{E_{i,j}^{k,x}}(x) + \sum_{i=0}^{n-1} \nu_1(T^i x).
\]

From now on, we denote \(V_m^{(x_A)} := V_m^{(x_A)} \) for \(A \in \mathcal{A}_n\). \(A \cap E_{0,n}^k\) may be approximated by \(A^{(k_n)} \cap T^{-n}(V_k^{(A)} \cap \mathcal{M})\):
\[
\left( A^{(k_n)} \cap T^{-n}(V_k^{(A)}) \right) \Delta \left( A \cap E_{0,n}^{k,x} \right) \subset ((A)^{(k_n)} \cap (A) \cap T^{-n}(B_A^{(k_n)} \cup (V_k^{(A)})^{(k_n)})) \cup (A^{(k_n)} \cap T^{-n}(B_A^{(k_n)}))
\]
In order to apply hypothesis (e) on sets with shape $A' \cap T^{-n}B'$, we divide them according to copies $D^N\overline{M}$ of $\overline{M}$

$$A' \cap T^{-n}B' = \bigcup_{N=-D}^{D} A' \cap \overline{T}^{-n}(D^{-N}B' \cap \overline{M}) \cap (S_n = N).$$

When $E \subset \overline{M}$, $\mu(E) = \Gamma\overline{\mu}(E)$. Thus hypothesis (e) gives

$$\mu(A^{[k_n]} \cap T^{-n}B_A^{[k_n]}) = \sum_{N=-D}^{D} \Gamma\overline{\mu}(A^{[k_n]} \cap \{S_n = N\} \cap \overline{T}^{-n}(D^{-N}(B_A^{[k_n]} \cap \overline{M}))) \leq \sum_{N=-D}^{D} \left( \frac{e^{-\frac{3k_n^2}{2(2\pi\sum(n-2k_n))^{1/2}n}}} {\pi(A^{[k_n]})\mu(B_A^{[k_n]}) + c\beta n^{-1/2}(n-2k_n)^{-1}} \right)$$

(5)

And according to upper bounds in (g), since $\overline{\mu}((\partial A)^{[k_n]}) = o(n^{-\beta})$ for all $\beta > 0$,

$$\overline{\mu}(A^{[k_n]}) \leq \overline{\mu}(A) + \overline{\mu}(\partial A)^{[k_n]} = O(n^{-1/10})$$

$$\mu(B_A^{[k_n]}) = O(n^{-1/20}).$$

And so the second term of (5) is in $O(n^{-1+\frac{1}{p}})$ with $p > 1$ (hypothesis (e)), thus giving an upper bound uniform in the choice of $A$

$$\mu(A^{[k_n]} \cap T^{-n}B_A^{[k_n]}) = O(n^{-13/20}).$$

The same reasoning (limit local theorem with decorrelation in hypothesis (e) and then upper bounds from hypothesis (g)) on $(\partial A)^{[k_n]} \cap T^{-n}(B_A^{[k_n]} \cup (V^{(A)})^{[k_n]})$ gives

$$\mu((\partial A)^{[k_n]} \cap T^{-n}(B_A^{[k_n]} \cup (V^{(A)})^{[k_n]})) = O(n^{-13/20}).$$

with uniform upper bound in the choice of $A$. And so

$$\mu \left( \left( A^{[k_n]} \cap T^{-n}(V^{(A)})^{[k_n]} \right) \Delta(A \cap E_{0,n}^{k}) \right) = O(n^{-13/20})$$

(6)

Then the first point of proposition 5.1 holds: Indeed, according to hypothesis (b), $\sum_{i=0}^{n-1} \nu_1(T^ix) = O(n)$ and since for all $x \in \overline{M}$, $1_{E_{k,i,j}}(x) = 1_{E_{0,j-i}} \circ \overline{T}^i(x)$,

$$\nu_n(x) = \sum_{k=1}^{D} \sum_{i<j=1}^{n} 1_{E_{k,i,j}}(x) + \sum_{i=0}^{n-1} \nu_1(T^ix)$$

$$= \sum_{k=1}^{D} \sum_{i<j=1}^{n} 1_{E_{0,j-i}} \circ \overline{T}^i(x) + O(n)$$

$$= \sum_{k=1}^{D} \sum_{i<j=1}^{n} \sum_{A \in \mathcal{A}_n} 1_A \circ \overline{T}^i(x)1_{E_{0,j-i}} \circ \overline{T}^i(x) + O(n).$$
Relation (6), comparisons series-integrals and \(|A_n| = O(n^{1/10})\) then ensure that
\[
\|\nu_n - \nu_n\|_{L^1} = \sum_{k=1}^{D} k \sum_{i<1}^{n} \sum_{A \in A_n} \mu \left( \left( A^{[k_n]} \cap T^{-(j-1)}(V_k^{(xA)}) \right) \Delta(A \cap E_{i,j}^k) \right) + O(n) \\
= O(n) + O(n^{29/20}) = o(n^{3/2}).
\]

Let's show the second point of the proposition:

Remind that
\[
e_I := \int_{M \times M} \left| \pi_R(y), \pi_R(T(y)) \right| \left\{ \pi_R(x), \pi_R(T(x)) \right\} d\mu(y)d\mu(x) = \int_{M} D \sum_{k=1}^{D} k\mu(V_k^{(x)})d\mu(x).
\]

Since
\[
\left| \sum_{k=1}^{D} k\mu(A^{[k_n]})\mu(V_k^{(xA)}) - \int_{A} D \sum_{k=1}^{D} k\mu(V_k^{(x)})d\mu(x) \right| \leq \sum_{k=1}^{D} k \left( \mu((\partial A)^{[k_n]}) + \mu(A^{[k_n]})\mu(B_{A}^{[k_n]}) \right)
\]
\[
\leq C\mu(A)n^{-1/20}.
\]

Summing over all \(A \in A_n\) the conclusion is reached:
\[
e_I := \int_{M} D \sum_{k=1}^{D} k\mu(V_k^{(x)})d\mu(x) = \lim_{n} \sum_{A \in A_n} D \sum_{k=1}^{D} k\mu(A^{[k_n]})\mu(V_k^{(xA)}).
\]



5.1. control in \(L^1\) norm. Using the local limit result with decorrelation (hypothesis (e)), we will prove in the following lemma that the self intersections number behaves closely, in the \(L^1\)-sense, to a constant times the self intersections \(\sum_{t \in \mathbb{Z}} N_n^2(l)\) of \((S_n)_n\), where
\[
N_n(l) := \sum_{i=0}^{n-1} 1_{S_i = l}.
\]

**Proposition 5.2.** Keeping notations and hypotheses from theorem 3.7,
\[
\frac{1}{n^{3/2}} \left\| \nu_n - \Gamma^{-2} e_I \sum_{x \in \mathbb{Z}} N_n^2(x) \right\|_{L^1_M} \to 0.
\]

Thanks to Proposition 5.1 it is enough to study the \(L_M^{2/n}\) convergence of
\[
\nu_n := \sum_{|N| \leq D} \sum_{A \in A_n} \sum_{k=1}^{D} \sum_{0 \leq i < j \leq n-1} k1_{A^{[k_n]}} L^{1/2} C_{A,n}^{k} \circ L^{1/2} \sum_{x \in \mathbb{Z}} N_n(x)N_n(x + N).
\]

Where \(C_{A,N}^{k,n} := D^{-N}((V_k^{(A)})^{[k_n]}) \cap M\).

The proof of Proposition 5.2 will be given in Section 5.3 and will rely on two technical steps.

Step 1 (proved in Lemma 5.5) will show the \(L_M^{2/n}\) convergence of
\[
\left( \frac{1}{n^{3/2}} \nu_n - \frac{1}{n^{3/2}} \sum_{|N| \leq D} \sum_{k=1}^{D} \sum_{A \in A_n} P_n(A^{[k_n]})P_k(C_{A,N}^{k,n}) \sum_{x \in \mathbb{Z}} N_n(x)N_n(x + N) \right) \to 0.
\]

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And Step 2 (proved in Lemma 5.6) consists in proving the asymptotic identification in $L^1_{\mu}$

$$E_{\mu}(\frac{1}{n^{3/2}} \left| \sum_{|N| \leq D} \sum_{k} \sum_{A \in A_n} \sum_{0 \leq i, j \leq n-1} \mu(A^{[kn]} \mu(C_{A,N}^k) 1_{\{S_{j-i} = N\}} \circ T^i - \sum_{k=1}^{D} \sum_{A \in A_n} \sum_{|N| \leq D} \mu(A^{[kn]} \mu(C_{A,N}) \sum_{x \in \mathbb{Z}} N_n(x)^2) \right|) \rightarrow 0.$$ 

The probabilistic convergence of the local time $N_n(.)$ would enable us to conclude.

5.2. Technical Lemma. In order to prove proposition 5.2, we establish the following decorrelation lemma using hypothesis (e). Notice that the error made in the decorrelation involved in (e) are not summable but, applying it iteratively, we still obtain decorrelation result with an adequate error term in $o(n^3)$.

Lemma 5.3. Given integers $N, N' \leq D$, and functions $f, g, f', g'$ so that one of them could be written as $1_A - \mu(A)$ where $A$ is a union of elements from $\xi_{k_n}^{\alpha}$, the other being linear combination of $1_B$ with $B \in \xi_{k_n}$, then

$$\sum_{0 \leq i, j, k, l \leq n-1} E_{\mu}(\left( f \circ T^{i-j} 1_{S_{j-i} = N} g \right) \circ T^i \left( g' f' \circ T^{k-l} 1_{S_{k-l} = N'} \right) \circ T^k) = o(n^3),$$

with $o(n^3)$ uniform in $\|f\|_\infty$, $\|f'\|_\infty$, $\|g\|_\infty$ and $\|g'\|_\infty$.

In order to prove this, we use the intermediate lemma,

Lemma 5.4. There is a constant $C_0 > 0$ such that for all $a \geq 1$,

$$\sum_{|r| \leq a} e^{-r^2/a} \leq C_0 a^{1/2}.$$

Proof. Comparison series/integrals through Riemann series, give for all $r \geq 1$,

$$e^{-r^2/a} \leq \int_{r-1}^r e^{-t^2/a} dt,$$

summing over $r$

$$\frac{1}{(\pi a)^{1/2}} \sum_{1 \leq |r| \leq a} e^{-r^2/a} \leq \frac{1}{(\pi a)^{1/2}} \int_{-\infty}^{\infty} e^{-t^2/a} dt.$$ 

The change of variable $u = \frac{t}{a^{1/2}}$ gives the conclusion. \hfill \square

Proof of Lemma 5.3. In order to prove the lemma it is enough to consider characteristic functions $1_A$ where $A$ is a union of elements of $\xi_{k_n}$ and since one of the function is supposed to have zero mean, to decompose

$$\sum_{0 \leq i, j, k, l \leq n-1} E_{\mu}(\left( f \circ T^{i-j} 1_{S_{j-i} = N} g \right) \circ T^i \left( g' f' \circ T^{k-l} 1_{S_{k-l} = N'} \right) \circ T^k) = \sum_{0 \leq i, j, k, l \leq n-1} a_{i,j,k,l} \mu(g) \mu(f) \mu(g') \mu(f') + o(n^3) \quad (8)$$

The dominating term would then vanished after recombination thanks to the zero mean of one of the function $f, f', g$ or $g'$.

In order to obtain this using hypothesis (e) it is necessary to fix an order between the indexes.
Since the proof goes roughly the same way for each ordering of the indexes, we only treat the case when \(0 \leq i \leq k \leq j \leq l\), the two other cases are done in appendix D.

Here we consider only the indexes such that at most one of the terms \(k - i, j - k, l - k\) goes below \(2k_n\), other configurations being too scarce to matter in the total sum. First suppose \(k - i \geq 2k_n + 1\) and using the \(T\)-invariance of \(\overline{\rho}\) and hypothesis (e),

\[
E(\overline{\rho}((f \circ T^{j-i}1_{S_{j-i=N}}) \circ T^{j-k}1_{S_{l-k=N'}}) \circ T^k)
\]

= \[\sum_{|r| \leq \min(k-i+1,l-k+1,j-l+1)D} E(\overline{\rho}(g^j \circ T^{j-i}1_{S_{j-i=N}} \circ T^{j-k}1_{S_{l-k=N'}}) \circ T^k)
\]

= \[\sum_{|r| \leq \min(k-i+1,l-k+1,j-l+1)D} \left(\overline{\rho}(g^j)\overline{\rho}(A_r)\frac{e^{-(N-r)^2/(2\Sigma(k-i-2k_n))}}{(2\pi\Sigma)^{1/2}(k-i-2k_n)^{1/2}} + \frac{ck_n\overline{\rho}(A_r)^{1/p}}{k-i-2k_n}\right)\tag{9}
\]

where \(A_r := g^j \circ T^{j-k}1_{S_{j-i=N \cap N'}} \circ T^{j-k}1_{S_{l-k}=r}\). When \(k-i \leq 2k_n\), the expectancy is bounded above by \(\sum_{|r| \leq \min(k-i+1,j-k+1,l-j+1)D} \overline{\rho}(A_r)\).

Passing from first to second line in (9) is due to the following decomposition,

\[
T^{-i}(S_{j-i} = N) \cap T^{-k}(S_{l-k} = N')
\]

= \[\bigcup_{r \in \mathbb{Z}} T^{-i}(S_{j-i} = N-r) \cap T^{-k}(S_{l-k} = r) \cap T^{-j}(S_{l-j} = N'-r)\].

Since \(\phi\) is bounded above by \(D\),

\[
\|S_n\|_\infty \leq nD
\]

And thus, \(\{S_{j-i} = N-r, S_{j-k} = r, S_{l-k} = N'-r\}\) is non empty if and only if \(N-r \leq (k-i)D, r \leq (j-k)D\) and \(N'-r \leq (l-k)D\) which means

\[
T^{-i}(S_{j-i} = N) \cap T^{-k}(S_{l-k} = N')
\]

= \[\bigcup_{r \leq \min(k-i+1,l-k+1,j-l+1)D} T^{-i}(S_{j-i} = N-r) \cap T^{-k}(S_{j-k} = r) \cap T^{-j}(S_{l-j} = N'-r)\].

Giving the sum in \(r\) in (9).

Hypothesis (e) also applies on the expression of \(A_r\) whenever \(j-k \geq 2k_n\):

\[
\overline{\rho}(A_r) = \overline{\rho}(g^j 1_{S_{j-k}=r} (f \circ T^{j-i}1_{S_{j-i=N}}) \circ T^j)
\]

= \[\overline{\rho}(g^j)\overline{\rho}(B_r)\frac{e^{-r^2/(2\Sigma(j-k-2k_n))}}{(2\pi\Sigma)^{1/2}(j-k-2k_n)^{1/2}} + \frac{ck_n\overline{\rho}(B_r)^{1/p}}{j-k-2k_n}\]  

\[\leq (1 + ck_n\overline{\rho}(B)^{1/p}/(j-k-2k_n)^{1/2})\tag{10}
\]

where \(B_r := f \circ T^{j-k}1_{S_{j-k}=N'-r}\). When \(j-k \leq 2k_n\), \(\overline{\rho}(A_r) \leq \overline{\rho}(B_r)\). Then applying again hypothesis (e) on \(B_r\) whenever \(l-j \geq 2k_n+1\),
\[
\mathfrak{P}(B_r) = \frac{\mathfrak{P}(f^i)\mathfrak{P}(f)e^{-(N'-r)^2/(2\Sigma(l-k)2k_n))}}{(2\pi\Sigma)^{1/2}(l-j-2k_n)^{1/2}} \pm \frac{ck_n\mathfrak{P}(f)^{1/\mu}}{1 + ck_n} + \frac{ck_n\mathfrak{P}(f)^{1/\mu}}{1 + ck_n} \leq \frac{(l-j-2k_n)^{1/2}}{\mu}.
\]

(12)

Thus, when \(k - i \leq 2k_n\), the next sum is bounded above thanks to inequalities (11) and (13):

\[
\sum_{i=0}^{n-1} \sum_{k=i}^{n-1} \sum_{j=k}^{n-1} \sum_{l=j}^{n-1} E_{\mathfrak{P}}\left((f \circ T^{-j-i}_{1_{S_{j-i}=N}}) \circ T^i \left(g' f' \circ T^{-k}_{1_{S_{l-k}=N'}} \circ T^k\right) \right) \leq \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \sum_{j=k}^{n-1} \sum_{l=j}^{n-1} \mathfrak{P}(A_r) \]

(11) gives \(\mathfrak{P}(A_r) \leq \frac{1+c_k}{(j-k-2k_n)^{1/2}} \leq \frac{1+c_k}{(j-k-2k_n)^{1/2}}\) whenever \(j - k \geq 2k_n + 1\), and thus

\[
\sum_{i=0}^{n-1} \sum_{k=i}^{n-1} \sum_{j=k}^{n-1} \sum_{l=j}^{n-1} \mathfrak{P}(A_r) \leq \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \sum_{j=k}^{n-1} \sum_{l=j}^{n-1} \frac{1+c_k}{(j-k-2k_n)^{1/2}} + 2k_n^2 n^2 = o(n^3). \]

The same reasoning after using (9) when \(k \geq i + 2k_n\) gives

\[
E_{\mathfrak{P}}((f \circ T^{-j-i}_{1_{S_{j-i}=N}}) \circ T^i \left(g' f' \circ T^{-k}_{1_{S_{l-k}=N'}} \circ T^k\right) \leq 2D \min(k-i+1, l-k+1, j-l+1) \frac{1+c_k}{(k-i-2k_n)^{1/2}}. \]

And so we get the same upper bound for the partial sums

\[
\sum_{i=0}^{n-1} \sum_{k=i+2k_n}^{n-1} \sum_{j=k}^{n-1} \sum_{l=j}^{n-1} E_{\mathfrak{P}}\left((f \circ T^{-j-i}_{1_{S_{j-i}=N}}) \circ T^i \left(g' f' \circ T^{-k}_{1_{S_{l-k}=N'}} \circ T^k\right) \right) = o(n^3) \]

and

\[
\sum_{i=0}^{n-1} \sum_{k=i+2k_n}^{n-1} \sum_{j=k}^{n-1} \sum_{l=j}^{n-1} E_{\mathfrak{P}}\left((f \circ T^{-j-i}_{1_{S_{j-i}=N}}) \circ T^i \left(g' f' \circ T^{-k}_{1_{S_{l-k}=N'}} \circ T^k\right) \right) = o(n^3). \]

The only terms left are those for which \(k - i > 2k_n\), \(j - k > 2k_n\) and \(l - k > 2k_n\). (11) in \(A_r\), the decorrelation in (9) and then the upper bound (13) in the first error term from \(A_r\) gives the following expansion:
\[ \mu(g)\mu(A_r) e^{-(N-r)^2/(2\Sigma)(k-i-2k_n))} \pm \frac{ck_n\mu(A_r)^{1/p}}{k-i-2k_n} \]

\[ = \mu(g) \left( \mu(g')\mu(B_r) e^{-r^2/(2\Sigma)(j-k-2k_n))} \pm \frac{ck_n\mu(B_r)^{1/p}}{j-k-2k_n} \right) \]

\[ = \frac{ck_n}{k-i-2k_n} \left( 1 + \frac{ck_n}{j-k-2k_n} \right)^{1/p} \]

\[ = e^{-(N-r)^2/(2\Sigma(k-i-2k_n))} \pm \frac{ck_n(1 + ck_n)^{1/p}}{(j-k-2k_n)^{(l-j-2k_n)1/(2p)}} \]

\[ + 2 \frac{ck_n}{k-i-2k_n} \left( 1 + \frac{ck_n}{l-j-2k_n} \right)^{1/p} \left( \frac{\mu(f)\mu(f')e^{-(N-r)^2/(2\Sigma(l-j))}}{(l-j-2k_n)^{1/2}} \right)^{1/p^2} + \frac{ck_n\mu(f)^{1/p}}{(l-j-2k_n)^{1/p^2}} \]

(14)

(15)

Except for the main term (14) which has the shape desired in (8), all the error terms give an upper bound in \(o(n^3)\) when summed over \(0 \leq i \leq k \leq j \leq l \leq n-1\) and \(r \leq D \min(k-i, j-k, l-j)\). Here the proof is given for the first term (the one with exponential factor) in (15) other terms are treated the same way:

denoting \(u = k-i, v = j-k\) and \(w = l-j\), and recalling that \(p < \frac{3}{7}\),

\[ \sum_{i=0}^{n-1} \sum_{u=k_n}^{n-1} \sum_{v=k_n}^{n-1} \sum_{w=k_n}^{n-1} \sum_{r=0}^{w} k^2_n e^{-(N-r)^2/(2p^2\Sigma w)} \]

\[ = \sum_{i=0}^{n-1} \sum_{u=k_n}^{n-1} \sum_{v=k_n}^{n-1} \sum_{w=k_n}^{n-1} O \left( \frac{k^2_n w^{1/2}}{uw^{1/(2p^2)}} \right) \]

\[ = \sum_{i=0}^{n-1} \sum_{u=k_n}^{n-1} \sum_{v=k_n}^{n-1} \sum_{w=k_n}^{n-1} O \left( \frac{w^{1/2}}{uw^{1/(2p^2)}} \right) \]

\[ = O(k^2_n (\ln n)^{3/2} n^{1-\frac{1}{2p}} n^{1-\frac{1}{2p^2}}) \]

This error term is in \(o(n^3)\) which concludes the lemma.

\[ \square \]

5.3. Proof of the proposition 5.2. Step 1:

Lemma 5.5. The sequence

\[ \left( \frac{1}{n^{3/2}} \left( V_n - \sum_{k=1}^{D} \sum_{|N|\leq D} \sum_{x \in \mathbb{Z}} \sum_{A \in \mathcal{A}_n} \mu(A^{[k_n]}\mu(C_{A,N})N_n(x + N)N_n(x) \right) \right)_{n \in \mathbb{N}^+} \]

converges in \(L^2\) toward 0.
Proof. Noticing that
\[
\sum_{|N| \leq D} N_n(x + N) N_n(x) = n + 2 \sum_{1 \leq i < j \leq n} \sum_{|N| \leq D} 1_{S_{j-i}=N} \circ T^{j-i},
\]
and using the inequalities of convexity, enable to approximate the quantity as follows.

\[
E_\mathcal{P} \left( \left( \mathcal{V}_n - \sum_{k=1}^{D} \sum_{|N| \leq D} \sum_{A \subseteq A_n} \mathcal{P}(A^{[k_n]}) \mathcal{P}(C^k_{A,N}) N_n(x + N) N_n(x) \right)^2 \right) \leq
\]
\[
2E_\mathcal{P} \left( \left( \sum_{k=1}^{D} \sum_{|N| \leq D} \sum_{A \subseteq A_n} \sum_{0 \leq i < j \leq n-1} \left( (1_{A^{[k_n]} - \mathcal{P}(C^k_{A,N})} \circ T^{j-i}) 1_{S_{j-i} = N} \circ T^{j-i} \right)^2 + o(n^2) \right) \right) \leq 4(E_1 + E_2) + o(n^2),
\]

where

\[
E_1 := E_\mathcal{P} \left( \left( \sum_{k=1}^{D} \sum_{|N| \leq D} \sum_{A \subseteq A_n} \sum_{0 \leq i < j \leq n-1} \left( (1_{A^{[k_n]} - \mathcal{P}(C^k_{A,N})} \circ T^{j-i}) 1_{S_{j-i} = N} \circ T^{j-i} \right)^2 \right) \right)
\]

and

\[
E_2 := E_\mathcal{P} \left( \left( \sum_{k=1}^{D} \sum_{|N| \leq D} \sum_{A \subseteq A_n} \sum_{0 \leq i < j \leq n-1} \left( \mathcal{P}(A^{[k_n]}) (1_{C^k_{A,N}} - \mathcal{P}(C^k_{A,N})) \circ T^{j-i} 1_{S_{j-i} = N} \circ T^{j-i} \right)^2 \right) \right).
\]

In both expressions, the terms within the sum may be written as

\[
f g \circ T^{j-i} 1_{S_{j-i} = N} \circ T^{j-i}.
\]

with \((f, g) := (1_{C^k_{A,N}} - \mathcal{P}(C^k_{A,N})) 1_{C^k_{A,N}}\) for \(E_1\) and \((f, g) := (\mathcal{P}(A^{[k_n]})) 1_{C^k_{A,N}} - \mathcal{P}(C^k_{A,N}) 1\) for \(E_2\).

So the expression to estimate for \(N\) and \(N'\) fixed are of the kind of lemma 5.3 :

\[
\sum_{A \subseteq A_n} \sum_{A' \subseteq A_n} \sum_{0 \leq i \leq j, i \leq k \leq l \leq n-1} E_\mathcal{P} \left( f g \circ T^{j-i} 1_{S_{j-i} = N} \left( f' g' \circ T^{j-i} 1_{S_{j-i} = N'} \right) \right).
\]

Since one of the function \(f, g, f'\) or \(g'\) has zero mean, the conclusion of the lemma holds and the expression is in \(o(n^3)\):

\[
E_1 = o(n^3) \text{ and } E_2 = o(n^3).
\]

This shows lemma 5.5.

\[\square\]

Step 2:
Lemma 5.6. For \( l \in \mathbb{Z} \),
\[
E_{\pi} \left( \sum_{x \in \mathbb{Z}} N_n^2(x) - N_n(x)N_n(x + l) \right) = o(n^{3/2}).
\]

Proof. Using Cauchy-Schwarz inequality and the upper bound \( E_{\pi}(|N_n(x) - N_n(y)|^2) = O(n^{1/2}|x - y|) \) from hypothesis (f):
\[
E_{\pi} \left( \sum_{x \in \mathbb{Z}} N_n^2(x) - N_n(x)N_n(x + l) \right) \leq E_{\pi} \left( \sum_{x \in \mathbb{Z}} |N_n(x)|^2 \right)^{1/2} E_{\pi} \left( \sum_{x \in \mathbb{Z}} |N_n(x) - N_n(x + l)|^2 \right)^{1/2}.
\]

Show that \( E_{\pi} \left( \sum_{x \in \mathbb{Z}} |N_n(x) - N_n(x + l)|^2 \right) = o(n^{3/2}) \) by fixing \( a \in \left[ \frac{1}{3}, \frac{1}{2} \right] \) and applying hypothesis (f),
\[
E_{\pi} \left( \sum_{x \in \mathbb{Z}} |N_n(x) - N_n(x + l)|^2 \right) \leq n^2 \left( \sup_{0 \leq k \leq n} |S_k| \geq n^{a+1/2} \right) + E_{\pi} \left( \sum_{-n^{a+1/2} \leq x \leq n^{a+1/2}} |N_n(x) - N_n(x + l)|^2 \right)
\]
\[
\leq n^2 \frac{1}{n^{2a+1}} E_{\pi} \left( \sup_{0 \leq k \leq n} (S_k^2) \right) + O \left( n^{a+1/2}|l|n^{1/2} \right).
\]

We apply the following theorem by Billingsley [1, p. 102] to \( (X_i = \phi \circ T^i)_i \) which follows the relation from hypothesis (e), \( E_{\pi}(|S_n|^2) = O(n) \).

Theorem 5.7 ([1]). Given \( (X_i)_{i \in \mathbb{N}} \) a sequence of centered random variables such that there are constants \( \alpha \geq 1 \) and \( v \geq 1 \) and there is a sequence \( (u_i)_{i \in \mathbb{N}} \) of non negative numbers satisfying for all \( a, n \geq 1 \)
\[
E \left( \left( \sum_{i=1}^{a+n} X_i \right)^a \right) \leq \left( \sum_{i=1}^{a+n} u_i \right)^v,
\]
then
\[
E_{\pi} \left( \sup_{0 \leq k \leq n} \left| \sum_{i=1}^{a+n} X_i \right|^{a} \right) \leq (\log_2(4n))^a \left( \sum_{i=1}^{a+n} u_i \right)^v.
\]

This theorem gives a constant \( K > 0 \) such that the following inequality holds for all \( n \geq 0 \)
\[
E_{\pi} \left( \sup_{0 \leq k \leq n} (S_k^2) \right) \leq Kn(\log(n)).
\]

It follows from the previous inequality that
\[
E_{\pi} \left( \sum_{x \in \mathbb{Z}} |N_n(x) - N_n(x + l)|^2 \right) = O(n^2 \frac{1}{n^{2a+1}} n(\log(n)) + O(n^{a+1/2}|l|n^{1/2}) = o(n^{3/2}).
\]
Moreover, it follows from the local limit theorem in hypothesis (e) that

\[
E_\mathcal{P} \left( \sum_{x \in \mathbb{Z}} N_n(x)^2 \right) = E_\mathcal{P} \left( \sum_{x \in \mathbb{Z}} \left( \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 1_{S_i=x} 1_{S_j=x} \right) \right) \\
= \sum_{i<j} \sum_{x \in \mathbb{Z}} 2E_\mathcal{P}(1_{S_{i-1}=0} \circ T^j 1_{S_i=x}) + O(n) \\
= 2 \sum_{i<j} E_\mathcal{P}(1_{S_{i-1}=0}) + O(n) \\
\leq 2 \sum_{i<j} Cn^{-1/2} + O(n) \\
= O(n^{3/2}).
\]

We reach the conclusion

\[
E_\mathcal{P} \left( \left| \sum_{x \in \mathbb{Z}} N_n^2(x) - N_n(x)N_n(x + 1) \right| \right) = o(n^{3/2}).
\]

\[\Box\]

Proof of proposition 5.2. Proposition 5.1 alongside the above lemmas 5.5 and 5.6 give the following result

\[
E_\mathcal{P} \left( \frac{1}{n^{3/2}} \left| V_n - \sum_{|N| \leq D} \sum_{A \in \mathcal{A}_n} \mu(A) \mu(C^A_{N,N}) \sum_{x \in \mathbb{Z}} N_n^2(x) \right| \right) \to 0.
\]

\[\Box\]

The convergence of \( \left( \frac{1}{n^{3/2}} V_{[tn]} \right)_{n \in \mathbb{N}} \) in the f.d.d. (finite dimensional distributions) sense is then strongly linked to the convergence of the f.d.d of \( \left( \frac{1}{n^{3/2}} \sum_{x \in \mathbb{Z}} N_{[tn]}^2(x) \right)_{n \in \mathbb{N}} \). In order to prove the convergence of the latter, we just need to check that hypotheses (e) and (f) ensure the assumptions of the following proposition (proposition 2.1 of [9])

**Proposition 5.8.** Let \((S_n)_{n \in \mathbb{N}}\) be a random walk on \(\mathbb{Z}\) and denote \(N_n(a) := \sum_{i=0}^{n-1} 1_{S_i=a}\) its local time satisfying

1. The sequence of processes \( \frac{1}{n^{3/2}} S_{[nt]} \) converges in distribution according to metric \(J_1\) toward some Brownian motion \((B_t)_{t \geq 0}\) with local time \((L_t)_{t \geq 0}\),
2. \(\sup_{a \in \mathbb{Z}} \|n^{-1/2} N_n(n^{1/2}a)\|_{L^2} < \infty\),
3. \(\limsup_{n \to \infty} \lim_{n \to \infty} \|n^{-1/2} N_n(n^{1/2}a) - n^{-1/2} N_n(n^{1/2}(a + b))\|_{L^2} = 0\).

Then the finite dimensional distributions (f.d.d) of the sequence of processes \(\left( N_{[nt]}(. \right)_{t \geq 0}\) converge to those of \((L_t(.))_{t \geq 0}\) in the space \(L^p(\mathbb{R}), \| \cdot \|_{L^p}\), where \((L_t(.))_{t \geq 0}\) stands for a local time of Brownian motion \((B_t)_{t \geq 0}\).

**Corollary 5.9.** The family of processes \( \left( \frac{1}{n^{3/2}} V_{[tn]} \right)_{n \in \mathbb{N}} \) converges in the f.d.d sense toward \(\Gamma^{-2} e_1 \int_\mathbb{R} L_t^2(x) dx \) for \(t \geq 0\).
Proof. Assumption (1) of Proposition 5.8 derives from hypothesis (f), which also ensures that assumption (3) holds:

\[
\limsup_{b \to 0} \lim_{n \to \infty} \|n^{-1/2}N_n([n^{-1/2}a]) - n^{-1/2}N_n([n^{-1/2}(a + b)])\|_2^2 = \\
\limsup_{b \to 0} \lim_{n \to \infty} O([n^{-1/2}|n^{-1/2}a| - n^{-1/2}|n^{-1/2}(a + b)|]) = o(1).
\]

Assumption (2) of Proposition 5.8 is then satisfied using hypothesis (e) and comparisons series/integrals,

\[
E_{\overline{\mathcal{P}}}(n^{-1}|N_n(a)|^2) = \frac{2}{n} \sum_{0 \leq i \leq j \leq n-1} E_{\overline{\mathcal{P}}}(1_{S_{j-i}=0} \circ \overline{\mathcal{P}} 1_{S_{i}=[n^{-1/2}a]}) = O \left( \frac{2}{n} \sum_{i=k_n+1}^{n-1} \sum_{a=0}^{n-1} \overline{\mathcal{P}}(S_a = 0) (i - k_n)^{1/2} \right) = O(1).
\]

Where \(O\) is taken uniformly in \(a\) and \(n\).

Thus the sequence of processes \(\left( (N_{\lfloor nt \rfloor}(x))_{t \in \mathbb{R}_+} \right)_{n \in \mathbb{N}}\) converges in the f.d.d sense towards the process \((L_t(.))_{t \in \mathbb{R}_+}\) where \(L_t(.)\) is seen as an element of \((L^2)\). Since the function \(L(.) \to \int_{\mathbb{R}} L^2(x)dx\) is continuous on \(L^2(\mathbb{R})\), we get the following convergence in the f.d.d sense

\[
\left( \sum_{x \in \mathbb{Z}} N_{\lfloor nt \rfloor}(x) \right)_{t \in \mathbb{R}_+} \overset{L^2}{\to} \left( \int_{\mathbb{R}} L^2(x)dx \right)_{t \in \mathbb{R}_+}.
\]

To conclude, notice that equation (7) page 17 gives the almost sure convergence of

\[
\sum_{k=1}^{D} k \sum_{|N| \leq D} \sum_{A \in \mathcal{A}_n} \overline{\mathcal{P}}(A[k_n]) \overline{\mathcal{P}}(C_{A,N}^k) = \Gamma^{-2} \sum_{k=1}^{D} k \sum_{|N| \leq D} \sum_{A \in \mathcal{A}_n} \mu(A[k_n]) \mu(C_{A,N}^k)
\]

towards \(\Gamma^{-2} e_I\), and thus Slutsky lemma gives the conclusion of the convergence in the f.d.d sense of \(\left( \frac{1}{n^{3/2}} \nu_{\lfloor nt \rfloor} \right)_{t \geq 0}\) toward \(\Gamma^{-2} e_I \int_{\mathbb{R}} L^2_I(x)dx\)\(\)_{t \geq 0}.

\[\square\]

5.4. **Proof of Theorems 3.7 and 3.9.** We now focus on proving the convergence in law of the self intersections processes in both discrete and continuous time. In what follows, \((M_T, \varphi^T, \mu_T)\) is a suspension flow over \((M,T,\mu)\) with root function \(\tau\) which is isomorphic to \((M_0, \varphi^0, \mathcal{L}_0)\).

Denote in this subsection, \(\mu_0(.) := \overline{\mathcal{P}}(. \cap M)\) a probabilistic measure on \(M\).

**Remark 5.10.** Let \(f\) be a bounded continuous function on \((D[0,\infty), J_1)\) the Skorohod space with metric \(J_1\) (see appendix C),

\[
E_{\mu_0}(f(\frac{1}{n^{3/2}} \nu_n)) = \int_{M} f(\frac{1}{n^{3/2}} \nu_n(x))d\overline{\mathcal{P}}(x) = E_{\overline{\mathcal{P}}}(f(\frac{1}{n^{3/2}} \nu_n))\).
\]

Denote in what follows \(\nu_n^0(t) := (\lfloor nt \rfloor + 1 - nt) \nu_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) \nu_{\lfloor nt \rfloor+1}\) the continuous process from \(\nu_n\). Both processes are close to each other. Indeed

\[
\frac{1}{n^{3/2}} \|\nu_n^0(.) - \nu_{\lfloor . \rfloor}\|_{\infty, [0,T]} \leq \frac{1}{n^{3/2}} (nt - \lfloor nt \rfloor)(2\lfloor nt \rfloor + 1) D,
\]

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\[\square\]
and thus $\frac{1}{n^{1/2}} \|\nu^0_n(t) - \nu_{[nt]}\|_{\infty, [0,T]} \rightarrow 0$, $\mu_0$ almost surely. Which implies that convergence for the $J_1$-metric (resp. for the f.d.d.) of $\left( \frac{1}{n^{1/2}} \nu^0_n(t) \right)_{t \in [0,T]}$ is equivalent to the convergence $\left( \frac{1}{n^{1/2}} \nu_{[nt]} \right)_{t \in [0,T]}$ to the same limit according to measure $\mu_0$.

Studying the continuous process enables the use of the following lemma.

**Lemma 5.11.** Let $((\nu^0_n(t))_{t \in [0,T]}))_{n \in \mathbb{N}}$ be a sequence of continuous non-decreasing processes on $[0,T]$ converging in the f.d.d sense toward some process $X$ continuous on $[0,T]$. Then $((\nu^0_n(t))_{t \in [0,T]}))_{n \in \mathbb{N}}$ converges in distribution on $C^0([0,T], \mathbb{R})$.

**Proof.** The process $(\nu^0_n(t))_{t \in [0,T]}$ is non-decreasing, thus it satisfies Skorohod criterion (see theorems 3.1 and 3.2 of [22]) for $M_1$ metric (see C),

$$w_{\delta, H}(\nu^0_n) = 0$$

with

$$w_{\delta, H}(Z) := \sup_{0 < t < t_1 < t_2 \leq \delta} M(Z(t_1), Z(t), Z(t_2)).$$

and $M$ being the oscillation function

$$M(x_1, x_2, x_3) = \begin{cases} 0 & \text{if } x_2 \in [x_1, x_2] \\ \min(|x_1 - x_2|, |x_3 - x_2|) & \text{otherwise} \end{cases}$$

Since $(\nu^0_n(\cdot))_{n \in \mathbb{N}}$ satisfies the Skorohod criteria and converges in f.d.d sense to $X$, according to Theorems 3.1 and 3.2 from [22], it converges in distribution to $X$ for the $M_1$ metric.

Let $f$ be a bounded uniformly continuous function on $C^0([0,T])$ (space of continuous functions from $[0,T]$ to $\mathbb{R}$) with supremum norm $\| \cdot \|_{\infty}$. The topology induced by $M_1$ on $C^0([0,T])$ is equivalent to the induced topology for supremum norm. Thus $f$ is still continuous on this space with $M_1$ metric. The space $C^0([0,T])$ being dense in the completion of $D([0,T])$ for $M_1$ metric, $f$ may be uniquely extended into bounded continuous function $\tilde{f}$ on $D([0,T])$ for $M_1$. Thus convergence in distributions of $(\nu^0_n(\cdot))_{n \in \mathbb{N}}$ to $X$ implies

$$E_\mu(f(\nu^0_n)) = E_\mu(\tilde{f}(\nu^0_n)) \rightarrow E(\tilde{f}(X)) = E(f(X)).$$

We conclude that $(\nu^0_n(\cdot))_{n \in \mathbb{N}}$ converges in distribution to $X$. \hfill \Box

**Proof of Theorem 3.7.** It follows from Corollary 5.9 along with the previous lemma that $\left( \frac{1}{n^{1/2}} \nu^0_n(t) \right)_{t \in [0,T]}$ and $\left( \frac{1}{n^{1/2}} \nu_{[nt]} \right)_{t \in [0,T]}$ converges in distribution, relatively to the $J_1$-metric, to $(\Gamma^{-2} \int L^2_{\nu}(x)dx)_{t \in [0,T]}$ for the measure $\mu$ on $\mathcal{M}$ proving Theorem 3.7, due to [27]. \hfill \Box

**Proof of Theorem 3.9.** This result follows directly from the following results. In particular it is a direct consequence of Propositions 5.13 (limit theorem with limit expressed in terms of the Poincaré section) and 5.15 (intrinsic expression of the limit). \hfill \Box

We remind here that $n_t$ is the number of passages by the Poincaré section up to time $t$:

$$n_t(x) = \sup \left\{ n, \sum_{k=0}^{n-1} \tau \circ T^k(x) \leq t \right\}.$$ 

for $n \in \mathbb{N}$, and $x \in M$. 

Since \((n_t)_{t \geq 0}\) is non-decreasing in \(t\) and not bounded, Birkhoff theorem on \((M, T, \mu)\) gives, for \(\mu\)-almost every \(x \in M \subset M\)

\[
\lim_{t \to \infty} \frac{t}{n_t} = \lim_{t \to \infty} \frac{1}{n_t} \sum_{k=0}^{n_t} \tau \circ T^k_{p.s} = E_{\mu}(\tau). \tag{16}
\]

The convergence then holds \(\mu_0\)-almost surely on \(M\).

**Proposition 5.12.** For any \(S \in \mathbb{R}_+\), the process \((\frac{1}{\mu_{\tau}} \nu_{s \uparrow \tau} \circ \pi)_{s \in [0, \bar{s}]}\) (defined on the suspension system \((M, \varphi^\tau, \mu_\tau)\)) strongly converges in distribution (for the \(J_1\) metric), with respect to \(\mu_\tau\), to \((E_{\mu}(\cdot)^{-3/2} \Gamma^{-2} e_1 \int L_s^2(x)dx)_{s \in [0, \bar{s}]}\) when \(t\) goes to infinity.

**Proof.** Due to Theorem 3.7 and comparing \((\frac{1}{\mu_{\tau}} \nu_{[s, \tau]}(x))_{s \in [0, \bar{s}]}\) and \((\frac{1}{\mu_{\tau}} \nu_{[s, 1]}(x))_{s \in [0, \bar{s}]}\), the process \((\frac{1}{\mu_{\tau}} \nu_{[s, \tau]}(x))_{s \in [0, \bar{s}]}\) still converges as \(t\) goes to infinity towards \((\Gamma^{-2} e_1 \int R L_s^2(x)dx)_{s \in [0, \bar{s}]}\).

According to [2, Theorem 3.9], the convergence in distribution (for the \(J_1\)-metric) with respect to measure \(\mu_0\) of \((\frac{1}{\mu_{\tau}} \nu_{[s, t]}(x))_{s \in [0, \bar{s}]\}}\) along with the \(\mu_0\)-almost sure convergence of \((\frac{\mu_0}{\nu_{\tau}})_{s \in [0, \bar{s}]\}}\) to \((\frac{1}{\mu_{\tau}} \nu_{s \uparrow \tau})_{s \in [0, \bar{s}]\}}\) implies the following convergence in distribution

\[
\lim_{t \to \infty} \frac{1}{3/2} \nu_{n_t} L_{\mu_0} = \Gamma^{-2} e_1 \int R L_s^2(x)dx.
\]

Thanks to the occupation time formula, and the self similarity of brownian motion \(\int R L_s^2(x)dx\) satisfies for any measurable function \(f\) on \(R\):

\[
\int R f(x) L_{s/E_{\mu}(\tau)}(x)dx = \int_0^{s/E_{\mu}(\tau)} f(B_t)dt
\]

\[
= E_{\mu}(\tau)^{-1} \int_0^{s} f(B_t)dt
\]

\[
= E_{\mu}(\tau)^{-1} \int_0^{s} f \left( \frac{1}{E_{\mu}(\tau)} \right)^{1/2} B_t' dt
\]

\[
= E_{\mu}(\tau)^{-1} \int_0^{s} f \left( \frac{x}{E_{\mu}(\tau)} \right)^{1/2} L_s^2(x)dx
\]

\[
= \int R f(u) E_{\mu}(\tau)^{-1/2} L_s^2(u E_{\mu}(\tau)^{1/2})du.
\]

where \(B'\) is a Brownian motion with the same distribution as \(B\) and \((L_s')_{s \geq 0}\) an associated local time. Thus \(L_s/E_{\mu}(\tau)\) is \(E_{\mu}(\tau)^{-1/2} L_s (u E_{\mu}(\tau)^{1/2})\) almost surely. Therefore

\[
\lim_{t \to \infty} \frac{1}{3/2} \nu_{n_t} L_{\mu_0} = \Gamma^{-2} e_1 \int R L_s^2(x)dx = E_{\mu}(\tau)^{-1} \Gamma^{-2} e_1 \int R L_s^2(x)dx
\]

\[
= E_{\mu}(\tau)^{-3/2} \Gamma^{-2} e_1 \int R L_s^2(x)dx.
\]

Since

\[
\left\| \left( \frac{1}{3/2} \nu_{n_t}) \circ T - \frac{1}{3/2} \nu_{n_t} \right|_{\infty, [0, \bar{s}]} \right\| \to 0,
\]

the family of processes \((\frac{1}{3/2} \nu_{n_t})_{s \geq 0})_{t \geq 0}\) satisfies the assumption of Theorem 1 from [27] for probability measure \(\mu_0\), thus it strongly converges in distribution \((J_1)\) on the system \((M, T, \mu)\)
to the process $E_P(\tau)^{-3/2}\Gamma^{-2}e_I \int_\mathbb{R} L^2_s(x)dx$ when $t$ tends to infinity.

Let now show the convergence in law for the family of processes $\left(\left(\frac{1}{t^{3/2}}\nu_{n_I} \circ \pi\right)_{s \geq 0}\right)_{t \geq 0}$ on the suspension flow $(M_\tau, \varphi^\tau, \mu_\tau)$:

Let $P$ be a probability measure absolutely continuous according to $\mu_\tau$ and $F$ a continuous function on $\mathbb{R}$, denoting $dP(x, u) = g(x, u)d\mu_\tau(x, u)$:

$$E_{\mu_\tau}\left(f\left(\left(\frac{1}{t^{3/2}}\nu_{n_I} \circ \pi\right)_{s \geq 0}\right) \right) = \int_M \int_0^{\tau(x)} f\left(\left(\frac{1}{t^{3/2}}\nu_{n_I} \circ \pi(x, u)\right)_{s \geq 0}\right) g(x, u)d\lambda d\mu$$

$$= \int_M f\left(\left(\frac{1}{t^{3/2}}\nu_{n_I}(x, 0)\right)_{s \geq 0}\right) \int_0^{\tau(x)} g(x, u)d\lambda d\mu.$$

Since $\int_0^{\tau(x)} g(x, u)d\lambda d\mu$ is a probability measure on $M$ absolutely continuous with respect to the measure $\mu$. The strong convergence in law of $\left(\left(\frac{1}{t^{3/2}}\nu_{n_I}\right)_{s \geq 0}\right)_{t \geq 0}$ over $(M, T, \mu)$ gives the conclusion for $S > 0$,

$$E_{\mu_\tau}\left(f\left(\left(\frac{1}{t^{3/2}}\nu_{n_I}\right)_{s \in [0, S]}\right) \right) \xrightarrow{t \to \infty} E_{\mu_\tau}\left(f\left(\left(\Gamma^{-2}e_I \int_\mathbb{R} L^2_s(x)dx\right)_{s \in [0, S]}\right) \right).$$

In other words, the sequence of processes $\left(\left(\frac{1}{t^{3/2}}\nu_{n_I}\right)_{s \in [0, S]}\right)_{t \geq 0}$ strongly converges in law on $(M_\tau, \varphi, \mu_\tau)$ to $(E_P(\tau)^{-3/2}\Gamma^{-2}e_I \int_\mathbb{R} L^2_s(x)dx)_{s \in [0, S]}$. \hfill $\square$

We are now ready to prove the following result which is very close to Theorem 3.9.

**Proposition 5.13.** Under the assumptions of Theorem 3.9. For any $S > 0$,

$$\left(\frac{1}{t^{3/2}} N_{t_I}\right)_{s \in [0, S]} \xrightarrow{L^2_{\mathbb{E}}} \left(E_P(\tau)^{-3/2}\Gamma^{-2}e_I \int_\mathbb{R} L^2_s(x)dx\right)_{s \in [0, S]}.$$

**Proof of Proposition 5.13.** Let just remind that the number of self intersections after the last collision $n_I$, $\nu_{n_I}$ gives a bound over the number $N_I$ of self intersections up to $t$:

$$\nu_{n_I}(x) \leq N_I(x, s) \leq \nu_{n_I+1}(T) \leq \nu_{n_I}(T) + 2(n_I + 1)D. \quad (17)$$

Where the upper bound is given by hypothesis (b). Equation (17) ensures that for any probability measure $P$ absolutely continuous with respect to $\mu_\tau$, and any $S > 0$,

$$E_P\left(\|\nu_{n_I}\|_{s \geq 0} - \|N_{t_I}\|_{s \geq 0}(x, u)\|_{\infty, [0, S]}\right) \to 0.$$

Thus, it follows from Proposition 5.12, combined with the Slutsky theorem and (16) (ensuring the convergence of $(\frac{1}{n_I})$ that

$$\lim_{t \to \infty} \left(\frac{1}{t^{3/2}} N_{t_I}\right)_{s \in [0, S]} = \lim_{t \to \infty} \left(\frac{1}{t^{3/2}} \nu_{n_I}\right)_{s \in [0, S]} \xrightarrow{L^2_{\mathbb{E}}} \left(E_P(\tau)^{-3/2}\Gamma^{-2}e_I \int_\mathbb{R} L^2_s(x)dx\right)_{s \in [0, S]}.$$

\hfill $\square$
5.5. **Intrinsic expression of the limit.** Here we express the limit appearing in Proposition 5.13 in terms of the flow, and will make appear Lalley’s constant. This will conclude the proof of Theorem 3.9.

Since \((\mathcal{M}, \varphi, L)\) is isomorphic to the suspension flow \((\mathcal{M}_\tau, \varphi^\tau_0, \varphi^\tau_0/E_\varphi(\tau))\) with roof function \(\tau\), we use the latest notations in what follows.

**Lemma 5.14.** Thanks to hypothesis (b’) and remark 3.8, \(e_I\) satisfies

\[
e_I = \Gamma^2 \int_{\mathcal{M}} \int_{\mathcal{M}} |\bar{\pi}(\varphi^{0}_{[0,\tau(x)]}(x)) \cap \bar{\pi}(\varphi^{0}_{[0,\tau(y)]}(y))|d\bar{\mu}(x)d\bar{\mu}(y).
\]

**Proof.**

\[
e_I := \int_{\mathcal{M}} \int_{\mathcal{M}} |[x] \cap [y]|d\mu(x)d\mu(y)
= \Gamma^2 \int_{\mathcal{M}} \sum_{k \in \mathbb{Z}} |[x] \cap [D^k y]|d\bar{\mu}(x)d\bar{\mu}(y)
= \int_{\mathcal{M}} \sum_{k \in \mathbb{Z}} |\pi(\varphi^{0}_{[0,\tau(x)]}(x)) \cap \pi(\varphi^{0}_{[0,\tau(y)]}(y))|d\bar{\mu}(x)d\bar{\mu}(y).
\]

Then for any \(x, y \in \mathcal{M}\), along with hypothesis (b’),

\[
\sum_{k \in \mathbb{Z}} |[x] \cap [D^k y]| = \sum_{k \in \mathbb{Z}} |\{s \in [0, \tau(y)], \pi(\varphi^0_s(D^k y)) \in [x]\}|
= |\bigcup_{k \in \mathbb{Z}} \{s \in [0, \tau(y)], \pi(\varphi^0_s(D^k y)) \in [x]\}|
= |\{s \in [0, \tau(y)], \exists k \in \mathbb{Z}, \pi(\varphi^0_s(D^k y)) \in \pi(\varphi^0_{[0,\tau(x)]}(x))\}|
= |\{s \in [0, \tau(y)], \exists u \in \varphi^0_{[0,\tau(x)]}(x), \pi(\varphi^0_s(y)) = \pi(u)\}|
= |\bar{\pi}(\varphi^0_{[0,\tau(x)]}(x)) \cap \bar{\pi}(\varphi^0_{[0,\tau(y)]}(y))|.
\]

and thus,

\[
e_I := \Gamma^2 \int_{\mathcal{M}} \int_{\mathcal{M}} |\bar{\pi}(\varphi^{0}_{[0,\tau(x)]}(x)) \cap \bar{\pi}(\varphi^{0}_{[0,\tau(y)]}(y))|d\bar{\mu}(x)d\bar{\mu}(y).
\]

\(\square\)

**Proposition 5.15.** Assuming hypotheses from Theorem 3.9,

\[
E_{\varphi}(\tau)^{-3/2}T^{-2}e_I \int_{\mathbb{R}^2} L_x^2(x)dx = e'_I \int_{\mathbb{R}} \tilde{L}_x^2(x)dx
\]

where

\[
e'_I = \int_{\mathcal{M}_0 \times \mathcal{M}} |\pi(\varphi^0_{[0,1]}(x)) \cap \pi(\varphi^0_{[0,1]}(y))|d\mu(y)dL(x)
= \int_{\mathcal{M} \times \mathcal{M}} |\bar{\pi}(\varphi^0_{[0,1]}(x)) \cap \bar{\pi}(\varphi^0_{[0,1]}(y))|d\bar{\mu}(x)d\bar{\mu}(y)
\]

is the mean intersections number between \(\bar{\pi}(\varphi^0_{[0,1]}(x))\) and \(\bar{\pi}(\varphi^0_{[0,1]}(y))\) for \(x, y\) taken independently according to distribution \(L\), and where \((\tilde{L}_x)_{x \geq 0}\) is a continuous version of the local time of the Brownian motion \(\tilde{B}\) seen as the limit in distribution of \(\left\{\frac{1}{\sqrt{T}}S_{\sqrt{T}u} \circ \pi(.\right)_{u \in [0, T]}\).
Proof. The Birkhoff relation (16) page 27 and hypothesis (f) ensure the convergence in distribution (for the $J_1$-metric) of $(n^{-1/2}S_{[nt]})_{t \geq 0}$ to $(B_t)_{t \geq 0}$ with respect to the measure $\mu$. Noticing that for $S > 0$, in one hand we have
\[ \left\| t^{-1/2}S_{[st]} - t^{-1/2}S_{[s]} \right\|_{\infty,[0,S]} \to 0 \]
and on the other hand
\[ \left\| 1 - \left[ \frac{t}{\Gamma} \right]^{-1/2} S_{[s]} \right\| \xrightarrow{\text{proba}} 0, \]
The process $(t^{-1/2}S_{nts})_{s \in [0,S]}$ converges in law to the Brownian motion $(B_s)_{s \in [0,S]}$ as $t \to \infty$. Then for any $S > 0$, theorem (3.9) from [2] provides the convergence in distribution (for the $J_1$-metric) with respect to the probability measure $\bar{\mu}$:
\[ (t^{-1/2}S_{nts})_{s \in [0,S]} \xrightarrow{s \to \infty} (B_s/E_{\bar{\mu}}(\tau))_{s \in [0,S]} \]
\[ (t^{-1/2}S_{nts})_{s \in [0,S]} \text{ converge to } (\tilde{B}_s)_{s \in [0,S]} := (E_{\bar{\mu}}(\tau)^{-1/2}B_s)_{s \in [0,S]} . \]

Here we study the local time $\tilde{L}$ and its related Brownian motion $\tilde{B}$: for any measurable function $f$ on $\mathbb{R}$, the occupation time formula and the self similarity of the Brownian motion give
\[ \int_0^t f(E_{\bar{\mu}}(\tau)^{-1/2}B_t)dt = \int_{\mathbb{R}} f\left(\frac{x}{E_{\bar{\mu}}(\tau)^{1/2}}\right) L_s(x)dx = \int_{\mathbb{R}} f(u)E_{\bar{\mu}}(\tau)^{1/2}L_s(E_{\bar{\mu}}(\tau)^{1/2}u)du. \]
Thus $\tilde{L}_s = E_{\bar{\mu}}(\tau)^{1/2}L_s(E_{\bar{\mu}}(\tau)^{1/2}u)$ almost surely
\[ \int_{\mathbb{R}} \tilde{L}_s^2(x)dx = \int_{\mathbb{R}} E_{\bar{\mu}}(\tau) L_s^2\left(E_{\bar{\mu}}(\tau)^{1/2}u\right)du = E_{\bar{\mu}}(\tau)^{1/2} \int_{\mathbb{R}} L_s^2(x)dx, \]
and
\[ E_{\bar{\mu}}(\tau)^{-3/2} \Gamma^{-2} e_I \int_{\mathbb{R}} \tilde{L}_s^2(x)dx = E_{\bar{\mu}}(\tau)^{-2} \Gamma^{-2} e_I \int_{\mathbb{R}} \tilde{L}_s^2(x)dx. \]
We conclude using the next lemma.

**Lemma 5.16.**

\[ E_{\bar{\mu}}(\tau)^{-2} \Gamma^{-2} e_I = e'_I \]

*Proof.* Here we show that $\Gamma^{-2} e_I$ coincides with the measure
\[ \int_{\mathcal{M}} \int_0^{\tau(x)} \int_0^{\tau(y)} |\pi_{\mathcal{R}}(\tilde{\varphi}^x_{[0,1]}(x,t)) \cap \pi_{\mathcal{R}}(\tilde{\varphi}^y_{[0,1]}(y,s))|dtd\pi(x)dsd\pi(y) \]
\[ = E_{\bar{\mu}}(\tau)^2 \int_{\mathcal{M}} \int_0^{\tau(x)} |\pi_{\mathcal{R}}(\varphi_{[0,1]}(x,t)) \cap \pi_{\mathcal{R}}(\varphi_{[0,1]}(y,s))|dLdL. \]
The last equation above gives an expression of $\frac{e_I}{E_{\bar{\mu}}(\tau)^2}$ as the mean number of intersections between the ”trajectories” of two flows running up to time 1 with starting point taken randomly.

Here let $B$ be a measurable set such that for $\mathcal{P}_x$-almost all $x$
\[ |\{s \in [0,1], \tilde{\varphi}^x_s(x,t) \in B\}| \leq D. \]
Denote for any \( x \in M \), 
\[
F(x) := \{s \in [0, \tau(x)], \varphi_s^x(x) \in B\}.
\]
and for \((x,t) \in M, r\), 
\[
f(x,t) := \{s \in [0, 1], \varphi_s^x(x,t) \in B\}.
\]
Suppose both functions are bounded by some constant \( D \), let’s show that 
\[
E_{\pi}(F) = E_{\pi,r}(f).
\]
Birkhoff relation (16) page 27 on \( n_t \) along with the Birkhoff theorem applied to \( F \) ensures that for \( \pi \)-almost all \( x \in M \),
\[
\frac{1}{t} \sum_{k=0}^{n_t-1} F \circ T^k(x) \rightarrow_{t \to \infty} \frac{E_{\pi}(F)}{E_{\pi}(\tau)}.
\]
The left hand side sum corresponds to
\[
\left\{ s \in [0, \sum_{k=0}^{n_t} \tau \circ T^k(x)], \varphi_s^x(x) \in B \right\}.
\]
whereas
\[
\frac{1}{T} \int_0^T f \circ \varphi_s^x(x,t) ds \rightarrow_{N \to \infty, \pi ps} \frac{E_{\pi}(f)}{E_{\pi}(\tau)}
\]
where left hand side integral means, by definition of \( f \),
\[
\int_0^T f \circ \varphi_s^x(x,t) ds = \sum_{s \in [0,\infty[} \lambda([0,T] \cap [s-1,s]) = |\{s \in [1,T], \varphi_s^x(x,t) \in B\}| \pm 2D.
\]
Thus, for \((x,u) \in M, r\)
\[
\left| \int_0^t f \circ \varphi_s^x(x,u) ds - \sum_{k=0}^{n_t-1} F \circ T^k \circ \tau \right|
\leq 4D + \left| \left\{ s \in [0,u] \cup \sum_{k=0}^{n_t} \tau \circ T^k(x), t + u, \varphi_s^x(x,u) \in B \right\} \right|
\leq 4D + 2[u]D
\leq 6D.
\]
Taking the limit as \( t \to \infty \) in the above Birkhoff sum we get
\[
\frac{E_{\pi}(F)}{E_{\pi}(\tau)} = \frac{E_{\pi,r}(f)}{E_{\pi,r}(\tau)}.
\]
Thus taking \( B := \pi^{-1}_R(\pi_R(\varphi_{[0,\tau(y)][y]}(y))) \) and applying Fubini’s theorem :
\[
\int_M \int_M |\pi_R(\varphi_{[0,\tau(x)]}(x)) \cap \pi_R(\varphi_{[0,\tau(y)][y]}(y))| d\mu(x) d\mu(y)
= \int_M \int_0^{\tau(x)} |\pi_R(\varphi_{[0,1]}(x,s)) \cap \pi_R(\varphi_{[0,\tau(y)][y]}(y))| dt d\mu(x) d\mu(y)
= \int_M \int_0^{\tau(x)} |\pi_R(\varphi_{[0,1]}(x,s)) \cap \pi_R(\varphi_{[0,\tau(y)][y]}(y))| d\mu(y) dt d\mu(x)
\]
Then, choosing $B := \pi^{-1}_R(\pi_R(\varphi_{[0,1]}(x)))$, the previous reasoning gives:

$$\int_M \int_0^{r(x)} \int_M |\pi_R(\varphi_{[0,1]}(x, s)) \cap \pi_R(\varphi_{[0,1]}(y))(y))| d\mu(y) dt d\mu(x) = \int_M \int_0^{r(x)} \int_M |\pi_R(\varphi_{[0,1]}(x, s)) \cap \pi_R(\varphi_{[0,1]}(y))(y))| d\mu(y) dt d\mu(x).$$

Lemma 5.16 thus gives the identification

$$E_{\pi}(\tau)^{-1/2 \Gamma - 2}e_k \int_R L_2^2(x) dx = e'_k \int_R \tilde{L}^2_2(x) dx.$$

**Appendix A. Self-intersections for $\mathbb{Z}^3$ extensions**

This section is devoted to a quick investigation of the stochastic behavior of self intersections on $\mathbb{Z}^d$-extensions of chaotic systems when $d \geq 3$ with theorem A.2 stated on the extended settings of group extensions defined as follows.

**Definition A.1.** Let $(\mathcal{M}, \varphi, L)$ be a probability preserving ergodic flow and $(G, +)$ an infinite countable commutative group equipped with counting measure $g$. Let $h_t : R \times \mathcal{M} \to G$ be a cocycle, (i.e $h_{t+s}(x) = h_t(x) + h_s(\varphi_t(x))$). on définit la The $G$-extension over $(\mathcal{M}, \varphi, L)$ is the dynamical system $(\mathcal{M}_0, \varphi^0, L_0)$ with $\mathcal{M}_0 := \mathcal{M} \times G$, a flow $\varphi^0$ defined for any $(x, a) \in \mathcal{M}_0$ by,

$$\varphi^0_t(x, a) := (\varphi_t(x), a + h_t(x)),$$

and a measure $L_0$ defined as the formal sum $L_0 := \sum_{g \in G} L \otimes g$.

$G$-extensions conveniently extends the notion of $\mathbb{Z}^d$-extension over special flows as stated in definitions 3.1 and 3.3 to any commutative countable group. The following theorem then gives some asymptotic behavior of the number $N_t$ of self intersections which is only relevant in the case of a non recurrent system. The corollary A.3 derived from that theorem states the almost sure convergence of $N_t$ in the case of $\mathbb{Z}^d$-extensions over chaotic suspension flows.

**Theorem A.2.** Let $(G, +)$ be a countable commutative group, $\mathcal{R}$ a set, $(\mathcal{M}_0, \varphi^0, L_0)$ a $G$-extension over the probability preserving flow $(\mathcal{M}, \varphi, L)$ and $\pi_R : \mathcal{M}_0 \to \mathcal{R}$ an application such that for any $(x, a)$ and $(x', a') \in \mathcal{M}_0$,

$$\pi_R(x, a) = \pi_R(x', a') \Leftrightarrow \pi_R(x, 0) = \pi_R(x', a' - a).$$

Then the quantity defined for $t \geq 0$ and $y \in \mathcal{M}_0$ by

$$N_t(y) := \{|(s, u) \in [0, t]^2 : s \neq u, \pi_R(\varphi^0_u(y)) = \pi_R(\varphi^0_u(y))\}|$$

satisfies:

$$N_t \frac{L_0^{-p \cdot p}}{t} E_L(I),$$

where

$$I(x) := \{|(s, u) \in [0, 1]^2 : s \neq u, \pi_R(\varphi^0_u(x, 0)) = \pi_R(\varphi^0_u(x, 0))\}|$$

and

$$+ 2\{|(s, u) \in [0, 1] \times [1, +\infty] : s \neq u, \pi_R(\varphi^0_u(x, 0)) = \pi_R(\varphi^0_u(x, 0))\}.$$
(1) $G := \mathbb{Z}^d$ with $d \geq 1$

(2) There is $M > 0$ such that for almost all $x \in \mathcal{M}$ and all $k \in \mathbb{N}$, 

$$\{((s,u) \in [0,1]\times[k,k+1]: s \neq u, \pi_R(\varphi^0_s(x,0)) = \pi_R(\varphi^0_u(x,0))\} \leq M\{h_k(x) \leq M\}.$$ 

(3) the time spent in any neighborhood is bounded:

$$\sum_{k \geq 0} L(|h_k| \leq M) < \infty.$$ 

Then $E_L(\iota)$ is finite.

Corollary A.3 applies for some chaotic suspension flow over a $\mathbb{Z}^d$-extensions $(\mathcal{M}, T, \varphi)$ with step function $\phi$ when the latter is non recurrent. The recurrence of the system is then equivalent to the recurrence in $\mathbb{Z}^d$ of the Birkhoff sum $(\sum_{i=0}^n \phi \circ T^i)_n$. An example of such systems on which corollary A.3 applies is the geodesic flow over $\mathbb{Z}^d$-cover of negatively curved compact surfaces (see [23]).

**Proof of theorem A.2.** We prove the theorem by approximating the quantity $\mathcal{N}_t$ by the sum of blocs describing the number of intersections for trajectories of length 1, defined in the following way : for any $(x,a) \in \mathcal{M}_0$,

$$J_{k,m}(x,a) := \{(s,u) \in [m,m+1]\times[k,k+1]: s \neq u, \pi_R(\varphi^0_s(x,0)) = \pi_R(\varphi^0_u(x,0))\},$$

and for $t > 0$,

$$\mathcal{N}_t(x,a) \leq \sum_{k,m=0}^{[t]} J_{k,m}(x,a) \quad (19)$$

$$\leq \sum_{k=0}^{[t]} J_{k,k}(x,a) + 2 \sum_{0 \leq k < m \leq [t]} J_{k,m}(x,a). \quad (20)$$

On the other hand, for $N \in \mathbb{N}^*$ and $t \geq N$,

$$\mathcal{N}_t(x,a) \geq \sum_{k,m=0}^{[t]-1} J_{k,m}(x,a)$$

$$\geq \sum_{k=0}^{[t]-1} J_{k,k} + 2 \sum_{k=0}^{[t]-N} \sum_{m=k+1}^{k+N} J_{k,m}(x,a). \quad (21)$$

Then relation (18), ensures that for $s,u \in \mathbb{R}$, and $k \in \mathbb{N}$,

$$\pi_R(\varphi^0_s \circ \varphi^0_k(x,0)) = \pi_R(\varphi^0_u \circ \varphi^0_k(x,0))$$

$$\Rightarrow \pi_R(\varphi_s \circ \varphi_k(x), h_k(x) + h_s \circ \varphi_k(x)) = \pi_R(\varphi_u \circ \varphi_k(x), h_k(x) + h_u \circ \varphi_k(x))$$

$$\Rightarrow \pi_R(\varphi_s(\varphi_k(x), 0)) = \pi_R(\varphi_u(\varphi_k(x), 0)).$$

Thus for any $(x,a) \in \mathcal{M}_0$, and $m \geq k$,

$$J_{k,m}(x,a) = J_{0,m-k}(\varphi^0_k(x,a)) = J_{0,m-k}(\varphi_k(x,0)).$$

---

4When this Birkhoff sum behave as a random walk on $\mathbb{Z}^d$, then it is transient whenever $d \geq 3$. 

---
Applying the above relation to sums (21) and (20), then for $N \in \mathbb{N}^*$, $t \geq N$, and $(x, a) \in \mathcal{M}_0$,

$$
\frac{1}{t} \left( \sum_{k=0}^{t-1} J_{0,0}(\varphi_k(x), 0) + 2 \sum_{k=0}^{t-N} \sum_{m=1}^{N} J_{0,m}(\varphi_k(x), 0) \right)
\leq \frac{N_t(x, a)}{t}
\leq \frac{1}{t} \sum_{k=0}^{t} \left( J_{0,0}(\varphi_k(x), 0) + 2 \sum_{m \geq 1} J_{0,m}(\varphi_k(x), 0) \right).
$$

Notice that

$$
\int_0^{[t]+1} J_{0,m}(\varphi_t \circ \varphi_1^{-1}(x), 0) \geq \sum_{k=0}^{[t]} J_{0,m}(\varphi_k(x), 0) \geq \int_0^{[t]-1} J_{0,m}(\varphi_t(x), 0).
$$

Since $(\mathcal{M}, \varphi, L)$ is ergodic, Birkhoff theorem applies: for any $N \in \mathbb{N}^*$, and $L_0$-a.e $(x, a)$,

$$
E_L(I_N) \leq \liminf_{t \to \infty} \frac{N_t(x, a)}{t} \leq \limsup_{t \to \infty} \frac{N_t(x, a)}{t} \leq E_L(I),
$$

where $I_N := J_{0,0}(., 0) + 2 \sum_{m=1}^{N} J_{0,m}(., 0)$. Since $I = J_{0,0}(., 0) + 2 \sum_{m=1}^{N} J_{0,m}(., 0)$, monotonous convergence applies

$$
\lim_{N \to \infty} E_L(I_N) = E_L(I).
$$

Which concludes the theorem. 

**Appendix B. Self-intersections for finitely measured systems.**

The following theorem is a slight adaptation of Lalley’s study of the asymptotic behaviour of self-intersections from geodesic flows on compact negatively curved Riemannian surfaces (see theorem 1.1 from [13]). In particular it allows us to state that the same behavior occurs for the self intersection of the flow on a Sinai Billiard.

Let introduce some lightened settings from the section 3 which translate the notion of self-intersections in the case of probabilistic dynamical systems.

**Hypotheses B.1 (Settings).** Let $(\mathcal{M}, \varphi_t, \nu)$ be a suspension flow over the ergodic probabilistic dynamical system $(\mathcal{M}, T, \mu)$ with roof function $\tau : \mathcal{M} \mapsto \mathbb{R}$. Let $\mathcal{R}$ be some set and $\pi_\mathcal{R} : \mathcal{M} \to \mathcal{R}$ an associated function. As in section 3 we define the number $N_t(x)$ of self intersections for a trajectory starting from $x \in \mathcal{M}$ up to time $t \in \mathbb{R}^+$ as

$$
N_t(x) := |\{(s, u) \in [0, t]^2 : s \neq u, \pi_\mathcal{R}(\varphi_s(y)) = \pi_\mathcal{R}(\varphi_u(y))\}|. \quad (22)
$$

Identifying $\overline{\mathcal{M}}$ with $\overline{\mathcal{M}} \times \{0\} \subset \mathcal{M}$, we identify the number of self intersections of a trajectory starting from $x$ up to the $n^{th}$ reflection as

$$
\nu_n(x) := \sum_{k \geq 1} \sum_{0 \leq i < j \leq n-1} 1_{\pi_i(\varphi_j(x))} \circ T^i(x) + \sum_{i=0}^{n-1} \nu_1(T^i(x)), \quad (23)
$$

identifying $\overline{\mathcal{M}}$ with $\overline{\mathcal{M}} \times \{0\} \subset \mathcal{M}$, we identify the number of self intersections of a trajectory starting from $x$ up to the $n^{th}$ reflection as

$$
\nu_n(x) := \sum_{k \geq 1} \sum_{0 \leq i < j \leq n-1} 1_{\pi_i(\varphi_j(x))} \circ T^i(x) + \sum_{i=0}^{n-1} \nu_1(T^i(x)), \quad (23)
$$

identifying $\overline{\mathcal{M}}$ with $\overline{\mathcal{M}} \times \{0\} \subset \mathcal{M}$, we identify the number of self intersections of a trajectory starting from $x$ up to the $n^{th}$ reflection as

$$
\nu_n(x) := \sum_{k \geq 1} \sum_{0 \leq i < j \leq n-1} 1_{\pi_i(\varphi_j(x))} \circ T^i(x) + \sum_{i=0}^{n-1} \nu_1(T^i(x)), \quad (23)
$$
LIMIT THEOREMS FOR SELF-INTERSECTING TRAJECTORIES IN $\mathbb{Z}$-EXTENSIONS

where $\nu_1 := \{((s, t) \in [0, \tau(x)]^2 : 0 \leq s < t \leq \tau(x), s \neq t : \pi_R(\varphi^0_s(x)) = \pi_R(\varphi^0_t(x))\}$ and $V_k^{(x)}$ is the following subset of $\overline{M}$,

$$V_k^{(x)} := \{y \in \overline{M}, |y| \cap [x] = k\},$$

with $[x] := \pi_R(\varphi_{[0,\tau(x)]}(x))$. Suppose that $(\overline{M}, \overline{T}, \overline{\nu})$ and $\tau$ satisfy the following hypotheses :

(a) Trajectories cross just finitely many times between two reflection :

for all $x \in M$,

$$\mu(\{y \in M, [x] \cap [y] > D\}) = 0$$

and for all $k \geq 1$,

$$\mu(\{y \in M, [y] \cap [\overline{T}^k y] > D\}) = 0.$$ 

In addition, $V_1$ satisfies $|V_1| \leq D$

(b) There is some congruent family of partitions $(\xi_k)_{k \in \mathbb{N}}$ of $\overline{M}$ and a real valued sequence $(\epsilon_k)_{k \in \mathbb{N}}$ converging to 0 such that for any fixed $k \in \mathbb{N}^*$ and any $A \in \xi_k^{−1}$, there is some subset $B_A \subset \overline{M}$ which can be written as a union of elements of the partition $\xi_k^{−1}$ and satisfies for any $x, y \in \overline{M}$ and any $m \in \mathbb{N}^*$,

$$(V_m^{(x)})^{[k]} \Delta V_m^{(y)} \subset B_A$$

and satisfying

$$\overline{\nu}(B_A) \leq \epsilon_k.$$

The suspension flow $(\mathcal{M}, \varphi_t, \nu)$ derived from the Sinai billiard or the geodesic flows over a compact negatively curved surface presented in section 2 for example satisfy these settings and thus the following theorem.

**Theorem B.2.** Let $(\mathcal{M}, \varphi_t, \nu)$ be a suspension flow over an ergodic dynamical system $(\overline{M}, \overline{T}, \overline{\nu})$ with roof function $\tau$ satisfying the settings B.1. Then the quantity $\nu_n$ defined in (23) satisfies

$$\frac{1}{n^2} \nu_n \xrightarrow{n \to \infty} \epsilon_I,$$

(24)

Where $\epsilon_I$ follows a similar definition as in lemma 4.1:

$$\epsilon_I := \int_{\overline{M} \times \overline{M}} |y| \cap [x]|d\overline{\nu}(y)d\overline{\nu}(x).$$

Fix the following probability measure $L := \frac{\nu - \nu^0}{\nu^0}$, then the self-intersections $N_I$ defined in (22) satisfies the following almost sur convergence on $(\mathcal{M}, \varphi_t, L)$ :

$$\frac{1}{t^2} N_I \xrightarrow{t \to \infty} \epsilon'_I, $$

(25)

Where $\epsilon'_I$ is defined similarly as in theorem 3.9 by:

$$\epsilon'_I := \int_{\mathcal{M} \times \mathcal{M}} |\pi_R(\varphi^0_{[0,1]}(x)) \cap \pi_R(\varphi^0_{[0,1]}(y))|dL(y)dL(x).$$

Lalley proved similar result to (25) in the case of the geodesic over a negatively curved compact manifold in [13] using proposition 2.10. However this proposition does not apply on measurably bounded function.
$f : \overline{M} \times \overline{M} \to \mathbb{R}$ as the one counting the number of intersections. So we use the following proposition instead which would adapt Lalley’s when $f$ counts the number of intersections.

**Proposition B.3.** Let $(\overline{M}, \overline{T}, \overline{\nu})$ be a probabilistic ergodic dynamical system and let $(\mathcal{A}_n)_{n \in \mathbb{N}}$ be a sequence of essential partitions of $\overline{M}$. Suppose the function $F : \overline{M} \times \overline{M} \to \mathbb{R}$ satisfies the following relation

$$
\sum_{A, B \in \mathcal{A}_n} \overline{\mu}(A)\overline{\mu}(B) \left( \sup_{A \times B} F - \inf_{A \times B} F \right) \xrightarrow{n \to \infty} 0,
$$

then

$$
\frac{1}{n^2} \sum_{0 \leq i, j \leq n-1} F(\overline{T}^i x, \overline{T}^j x) \xrightarrow{a.e. n \to \infty} \int_{\overline{M} \times \overline{M}} F(x, y) d\overline{\mu}(x) d\overline{\mu}(y).
$$

**Proof.** For any $x, y \in \overline{M}$,

$$
\sum_{A, B \in \mathcal{A}_n} 1_A(x)1_B(y) \inf_{A \times B} F \leq F(x, y) \leq \sum_{A, B \in \mathcal{A}_n} 1_A(x)1_B(y) \sup_{A \times B} F.
$$

Thus summing along every couple $(T^i x, T^j x)$ for $0 \leq i, j \leq n - 1$,

$$
\sum_{A, B \in \mathcal{A}_n} \inf_{A \times B} F \left( \frac{1}{n} \sum_{k=0}^{n-1} 1_A(T^i x) \right) \left( \frac{1}{n} \sum_{k=0}^{n-1} 1_B(T^j x) \right) \leq \frac{1}{n^2} \sum_{0 \leq i, j \leq n-1} F(\overline{T}^i x, \overline{T}^j x) \leq \sum_{A, B \in \mathcal{A}_n} \sup_{A \times B} F \overline{\mu}(A)\overline{\mu}(B).
$$

Since $(\overline{M}, \overline{T}, \overline{\nu})$ is ergodic, Birkhoff theorem then gives the following limit for almost any $x \in \overline{M}$:

$$
\sum_{A, B \in \mathcal{A}_n} \inf_{A \times B} F \overline{\mu}(A)\overline{\mu}(B) \leq \lim_{n \to \infty} \frac{1}{n^2} \sum_{0 \leq i, j \leq n-1} F(\overline{T}^i x, \overline{T}^j x) \leq \sum_{A, B \in \mathcal{A}_n} \sup_{A \times B} F \overline{\mu}(A)\overline{\mu}(B).
$$

The assumption (26) then leads to the conclusion,

$$
\sum_{0 \leq i, j \leq n-1} F(\overline{T}^i x, \overline{T}^j x) \xrightarrow{p.s.} \int F d\overline{\mu} d\overline{\mu}.
$$

□

**Proof of theorem B.2.** First let’s prove (24), the quantity $\nu_n$ introduced in (23) can be written, for any $x \in \overline{M}$ and any $n \in \mathbb{N}$, as

$$
\nu_n(x) = \sum_{0 \leq i, j \leq n-1} f(\overline{T}^i x, \overline{T}^j x) + \sum_{i=0}^{n-1} \nu_1(T^i x),
$$

where $f(x, y) := \sum_{k \geq 1} k 1_{V_k^{(i)}}(y)$.

Let $\mathcal{A}_n := \mathcal{E}_{\leq n}$, then for any $A, B \in \mathcal{A}_n$, $f$ is constant on $A \times B$ whenever $B \not\subset B_A$. When $B \subset B_A$ then the setting (a) tell us that $f$ is bounded by some constant $D$. Thus,

$$
\sum_{A, B \in \mathcal{A}_n} \overline{\mu}(A)\overline{\mu}(B) \left( \sup_{A \times B} f - \inf_{A \times B} f \right) \leq D \sum_{A \in \mathcal{T}_m} \overline{\mu}(A)\overline{\mu}(B_A).
$$

\footnote{For example, the function $f(x, y) := 1_{\{x \in T^k y\}}(x)$ satisfies
\[\frac{1}{n} \sum_{i, j=0}^n f(T^i x, T^j x) = 1\].}
Hypothesis (b) then ensures that the right hand side converge to 0 when \( n \) goes to infinity. Thus proposition B.3 applies on \( f \) for the partition sequence \( (A_n)_{n \in \mathbb{N}} \):
\[
\frac{1}{n^2} \nu_n \xrightarrow{n \to \infty} \int_{M \times M} f(x, y) \, d\mu d\mu.
\]
This last integral corresponds to the term \( e_I \).

To prove (25), we reintroduce the number \( n_t(x) \) of reflection on the Poincaré section \( M \) up to some time \( t \in \mathbb{R} \) for a flow starting at \( x \in M \) defined as
\[
n_t(x) := \sup\{n, \sum_{k=0}^{n-1} \tau \circ T^k(x) \leq t\}.
\]
For the same reason as in (16) page 27, the following convergence stands:
\[
\lim_{t \to \infty} \frac{t}{n_t(x)} = \lim_{t \to \infty} \frac{1}{n_t(x)} \sum_{k=0}^{n_t(x)-1} \tau \circ T^k \xrightarrow{\text{P. a.s.}} E_{\mu}(\tau).
\] (27)

In addition, \( \nu_{n_t} \) gives the following upper and lower bounds of \( \mathcal{N}_t \) for any \( x \in M \) and \( s \in [0, \tau(x)) \):
\[
\frac{1}{t^2} (\nu_{n_t}(x) - D) \leq \frac{\mathcal{N}_t(x, s)}{t^2} \leq \frac{1}{t^2} \nu_{n_t+2}(x).
\]
Thus, (27) and the above inequality lead to the following convergence,
\[
\frac{\mathcal{N}_t}{t^2} \xrightarrow{\text{L- a.s.}} \frac{e_I}{E_{\mu}(\tau)}.
\]

**Appendix C. Metric \( M_1 \) et \( J_1 \) on Skorohod space**

This section gives short definitions on metrics \( J_1 \) and \( M_1 \) on the Skorohod space \( D([0, T]) \) (i.e. the set of right continuous and left limited functions on \( [0, T] \) for \( T \in \mathbb{R}^+ \)). Much more details on these metrics may be found in the books [2] or [24].

**Definition C.1.** Let \( T > 0 \), the \( J_1 \) metric on the Skorohod space \( D([0, T]) \) is for all \( x, y \in D([0, T]) \),
\[
d_J(x, y) := \inf_{\lambda \in \Lambda} \{\|\lambda \circ x - y\|_{\infty} \vee \|\lambda - \text{Id}_{[0,T]}\|_{\infty}\},
\]
where \( \Lambda \) is the set of homeomorphisms on \([0, T]\).

Such metric is thinner than metric \( M_1 \) defined in what follow :

**Definition C.2.** For any \( x \in D([0, T]) \) define the complete graph \( \Gamma_x \) of \( x \) by
\[
\Gamma_x := \{(\alpha x(t^-) + (1 - \alpha)x(t), t), t \in [0, T], \alpha \in [0, 1]\}.
\]
On \( \Gamma_x \) fix an ordering relation by stating that \((g, r) \leq (h, s)\) if and only if \( r < s \) or \( r = s \) and \( |x(r^-) - g| \leq |x(r^-) - h| \).

We call parametric representation \( \Gamma_x \) any non decreasing continuous function \((g, r)\) from \([0, 1]\) into \( \Gamma_x \). Denote \( \Pi(x) \) the set of parametric representation on \( \Gamma_x \).

Metric \( M_1 \) is defined as follow.
Definition C.3. For any $x_1, x_2 \in D([0,T])$, the $M_1$ metric is defined by the distance $d_M$ built through parametric representations:

$$d_M(x_1, x_2) := \inf_{(g_r, r_2) \in \Pi(x_2), j \in \{1, 2\}} \{\|g_1 - g_2\|_\infty \lor \|r_1 - r_2\|_\infty\}.$$

### Appendix D. Technical lemma

Let’s prove that

$$\sum_{0 \leq i,j,k,l \leq n-1} E_{\mathbf{\mu}} \left( (f \circ T_i^{-1} 1_{S_{j-i=N}} g) \circ T^i \left( g' f' \circ T_j^{-k} 1_{S_{l-k=N'}} \right) \circ T^j \right) = \sum_{0 \leq i,j,k,l \leq n-1} a_{i,j,k,l} \mathbf{\mu}(g) \mathbf{\mu}(f) \mathbf{\mu}(g') \mathbf{\mu}(f') + o(n^3)$$

when $0 \leq i \leq j \leq k \leq l \leq n - 1$ and then when $0 \leq i \leq k \leq l \leq j \leq n - 1$.

**First case**: $0 \leq i \leq j \leq k \leq l \leq n - 1$. Then

$$\text{Suppose first } \min(j - i, k - j, l - k) \geq 2k_n + 1,$

and apply the invariance of $\mathbf{\mu}$ by $T$.}

$$E_{\mathbf{\mu}} \left( (f \circ T_i^{-1} 1_{S_{j-i=N}} g) \circ T^i \left( g' f' \circ T_j^{-k} 1_{S_{l-k=N'}} \right) \circ T^j \right) = \sum_{|r| \leq (k-j+1)D} E_{\mathbf{\mu}} \left( (f \circ T_i^{-1} 1_{S_{j-i=N}} g) 1_{S_{h-j=r}} \circ T^i \left( g' f' \circ T_j^{-k} 1_{S_{l-k=N'}} \right) \circ T^j \right)$$

$$= \sum_{|r| \leq (k-j+1)D} E_{\mathbf{\mu}} \left( 1_{S_{j-i=N}} g \left( f1_{S_{j-i=N}} g' \circ T_j^{-k} 1_{S_{l-k=N'}} \right) \circ T^j \right) + o(n^3)$$

(28) Since $\|S_n\|_\infty \leq nD$, the sum is done $r \leq (k-j+1)D$. Applying several times hypothesis (e) as in the proof of lemma 5.3, we obtain

$$E_{\mathbf{\mu}} \left( 1_{S_{j-i=N}} g \left( f1_{S_{j-i=N}} g' \circ T_j^{-k} 1_{S_{l-k=N'}} \right) \circ T^j \right) = e^{-Nj/(2(2k+1))} (2\pi \Sigma)^{1/2} \mu(f_{A_r}) \mu(f_{A_i}) \mu(f_{A_l}) \mu(f_{A_k}) + \frac{k_n \mu(f_{A_r})^{1/p}}{(j-i-2k_n)^{1/2}}$$

with $f_{A_r} := f1_{S_{j-i=N}} (g' f' \circ T_j^{-k} 1_{S_{l-k=N'}}) \circ T^j$. Then

$$\mu(f_{A_r}) = \mu(f) \mu(f_{B_r}) = e^{-r^2/(2(2k+1))} (2\pi \Sigma)^{1/2} \frac{k_n \mu(f_{B_r})^{1/p}}{k-j-2k_n}$$

$$\leq \frac{(1 + c_k \mu(f_{B_r})^{1/p})}{(k-j-2k_n)^{1/2}}$$

With $f_{B_r} := g' f' \circ T_j^{-k} 1_{S_{l-k=N'}}$. And finally

$$\mu(f_{B_r}) = \frac{\mu(f') \mu(f_{B_r}) e^{-N^{2}/(2(2k+1))}}{(2\pi \Sigma)^{1/2} (l-k-2k_n)^{1/2}} \pm \frac{ck_n \mu(f')^{1/p}}{(l-k-2k_n)^{1/2}}$$

injecting these upper bounds in (28)
\[ E_{\mathcal{P}}(1_{S_{j-i,N}} g(f' f^l \circ T^{j-i} 1_{S_{j-k=N'}}) \circ T^k) \]
\[ = \frac{e^{-N^2/(2 \Sigma (l-k-2k_n))} \mathbb{P}(g)}{(2 \pi \Sigma)^{1/2} (j-k-2k_n)^{1/2}} \frac{e^{-r^2/(2 \Sigma (k-j-2k_n))}}{(2 \pi \Sigma)^{1/2} (k-j-2k_n)^{1/2}} \]
\[ \pm \frac{e^{-N^2/(2 \Sigma (l-k-2k_n))} \mathbb{P}(f) \ c k_n \mathbb{E}(f')^{1/p}}{(2 \pi \Sigma)^{1/2} (l-k-2k_n) (2 \pi \Sigma)^{1/2} (k-j-2k_n)^{1/2}} \]
\[ \pm \frac{e^{-N^2/(2 \Sigma (l-k-2k_n))} \mathbb{P}(f) \ c k_n \mathbb{E}(f')^{1/p}}{(2 \pi \Sigma)^{1/2} (l-k-2k_n)^{1/2} (2 \pi \Sigma)^{1/2} (j-i-2k_n)^{1/2}} \]
\[ \pm \frac{e^{-N^2/(2 \Sigma (l-k-2k_n))} \mathbb{P}(f) \ c k_n \mathbb{E}(f')^{1/p}}{(l-k-2k_n)^{1/2} (2 \pi \Sigma)^{1/2} (k-j-2k_n)^{1/2}} \]
\[ \pm \frac{e^{-N^2/(2 \Sigma (l-k-2k_n))} \mathbb{P}(f) \ c k_n \mathbb{E}(f')^{1/p}}{(l-k-2k_n)^{1/2} (2 \pi \Sigma)^{1/2} (j-i-2k_n)^{1/2}} \]

Treating each term through comparison series/integral we obtain an upper bound in \( O(n^3) \).

Denote \( u := j-i \), \( v := k-j \) and \( w := l-k \) and suppose one of them exactly is lower than \( 2k_n + 1 \). Suppose without loss of generality \( u \geq 2k_n + 1 \) and \( w \leq 2k_n + 1 \). Then applying hypothesis (e) on \( j-i \),

\[ |E_{\mathcal{P}}((f \circ T^{j-i} 1_{S_{j-i,N}} g) \circ T^k (g' f' \circ T^{j-i} 1_{S_{j-k=N'}}) \circ T^k)| \]
\[ \leq E_{\mathcal{P}}(f \circ T^{j-i} 1_{S_{j-i,N}} g) \]
\[ \leq \frac{e^{-N^2/(2 \Sigma (j-i-2k_n))} \mathbb{P}(g) \mathbb{E}(f)}{(2 \pi \Sigma)^{1/2} (j-i-2k_n)^{1/2}} \]
\[ + \frac{e^{-r^2/(2 \Sigma (k-j-2k_n))}}{(2 \pi \Sigma)^{1/2} (k-j-2k_n)^{1/2}} \]
\[ + \frac{e^{-N^2/(2 \Sigma (l-k-2k_n))} \mathbb{P}(f) \ c k_n \mathbb{E}(f')^{1/p}}{(2 \pi \Sigma)^{1/2} (j-i-2k_n)^{1/2}} \]
\[ + \frac{e^{-N^2/(2 \Sigma (l-k-2k_n))} \mathbb{P}(f) \ c k_n \mathbb{E}(f')^{1/p}}{(l-k-2k_n)^{1/2} (2 \pi \Sigma)^{1/2} (k-j-2k_n)^{1/2}} \]
\[ + \frac{e^{-N^2/(2 \Sigma (l-k-2k_n))} \mathbb{P}(f) \ c k_n \mathbb{E}(f')^{1/p}}{(l-k-2k_n)^{1/2} (2 \pi \Sigma)^{1/2} (j-i-2k_n)^{1/2}} \]
\[ \leq \frac{1}{(2 \pi \Sigma)^{1/2} (u-2k_n)^{1/2}} + c \frac{k_n}{(u-2k_n)} \]

Then summing over all the permitted indices give an upper bound in \( o(n^3) \).

third case \( 0 \leq i \leq k \leq l \leq j \)

This case is done exactly the same way as the case done in the proof of lemma 5.3 once the following decomposition made :

\[ E_{\mathcal{P}}((f \circ T^{j-i} 1_{S_{j-i,N}} g) \circ T^d (g' f' \circ T^{d-k} 1_{S_{j-k=N'}}) \circ T^k) \]
\[ = \sum_{r \leq \min(k-i,l-k,j-l)} E_{\mathcal{P}}(1_{S_{j-i,N}} g(f' \circ T^{j-i} 1_{S_{j-k=N'}}) \circ T^k) \]
\[ = \sum_{r \leq \min(k-i,l-k,j-l)} \left( \frac{\mathbb{P}(g) \mathbb{P}(A_r)}{(2 \pi \Sigma)^{1/2} (k-i-2k_n)^{1/2}} + c \frac{k_n \mathbb{P}(A_r)^{1/p}}{k-i-2k_n} \right) \]
\[ \text{with } A_r := g'(f' \circ T^{j-i} 1_{S_{j-i=N'-N}}) \circ T^{j-i} \]

Then
\[ \bar{\mu}(A_r) = \bar{\mu} \left( g'1_{S_{j-1}=N'} \left( f'f \circ \bar{T}^{j-1}1_{S_{j-1}=N-N'-r} \right) \circ \bar{T}^{-k} \right) \]
\[ = \bar{\mu}(g') \bar{\mu}(B_r) \frac{e^{-N'^2/(2\Sigma(l-k-2k_n))}}{(2\pi \Sigma)^{1/2}(l-k-2k_n)^{1/2}} \pm \frac{ck_n \bar{\mu}(B_r)^{1/p}}{l-k-2k_n} \]
\[ \leq (1 + c k_n) \bar{\mu}(B_r)^{1/p} \frac{1}{(l-k-2k_n)^{1/2}} \]
with \( B_r = f'f \circ \bar{T}^{j-1}1_{S_{j-1}=N-N'-r} \) satisfying
\[ \bar{\mu}(B_r) = \frac{\bar{\mu}(f') \bar{\mu}(f) \exp(-N'-N'^2/(2\Sigma(j-l-2k_n)))}{(2\pi \Sigma)^{1/2}(j-l-2k_n)^{1/2}} \pm \frac{ck_n \bar{\mu}(f)^{1/p}}{j-l-2k_n} \]
\[ \leq \frac{1 + c k_n}{(j-l-2k_n)^{1/2}}. \]

Introducing these comparisons in (30) and making comparisons series/integral, we obtain an upper bound in \( o(n^3) \).

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