Inequalities for solutions to some nonlinear equations

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Abstract

Let $F$ be a nonlinear Frechet differentiable map in a real Hilbert space. Condition sufficient for existence of a solution to the equation $F(u) = 0$ is given, and a method (dynamical systems method, DSM) to calculate the solution as the limit of the solution to a Cauchy problem is justified under suitable assumptions.

1 Introduction

In this paper Theorem 1 gives a method for proving existence of the solution to nonlinear equation $F(u) = 0$ and for computing it. Our method (dynamical systems method: DSM) consists of solving a suitable Cauchy problem which has a global solution $u(t)$ such that $y := u(\infty)$ does exist and $F(y) = 0$. Theorem 1 generalizes Theorem 1 in [1] and its proof is based on the idea in [1]. The result of Example 1 in Section 3 below was obtained in [5]. The result of Example 5 was essentially obtained in [8], where the existence of the solution was assumed, while in Theorem 1 below the existence of the solution is proved. In Example 6 some of the results from [4] are obtained. In Remark 2.4 in [8] some of the examples we discuss in Section 3 were mentioned. In [1] DSM was developed for solving ill-posed operator equations with not necessarily monotone operators and for constructing convergent iterative methods for their solution.

Let $F$ be a nonlinear Frechet differentiable map in a real Hilbert space. Consider the equation

$$F(u) = 0. \quad (1.1)$$

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Let \( \Phi(t, u) \) be a map, continuous in the norm of \( H \) and Lipschitz with respect to \( u \) in the ball \( B := \{u : ||u - u_0|| \leq R, u \in H\} \). Weaker conditions, which guarantee local existence and uniqueness of the solution to (1.6) below, would suffice. Assume that:

\[
(F'(u)\Phi(t, u), F(u)) \leq -g_1(t)||F(u)||^2 \quad \forall u \in B, \tag{1.2}
\]

and

\[
||\Phi(t, u)|| \leq g_2(t)||F(u)|| \quad \forall u \in B, \tag{1.3}
\]

where \( g_j, j = 1, 2, \) are positive functions on \( R_+ := [0, \infty) \), \( g_2 \) is continuous, \( g_1 \in L^1_{loc}(R_+) \),

\[
\int_0^\infty g_1 dt = +\infty, \tag{1.4}
\]

and

\[
G(t) := g_2(t) \exp(-\int_0^t g_1 ds) \in L^1(R_+). \tag{1.5}
\]

**Remark:** Sometimes the assumption (1.3) can be used in the following modified form:

\[
||\Phi(t, u)|| \leq g_2(t)||F(u)||^b \quad \forall u \in B, \tag{1.3'}
\]

where \( b > 0 \) is a constant. The statements and proofs of Theorems 1-3 in Sections 1 and 2 can be easily adjusted to this assumption.

Consider the following Cauchy problem:

\[
\dot{u} = \Phi(t, u), \quad u(0) = u_0, \quad \dot{u} := \frac{du}{dt}. \tag{1.6}
\]

Finally, assume that

\[
||F(u_0)|| \int_0^\infty G(t) dt \leq R. \tag{1.7}
\]

The above assumptions on \( F, \Phi, g_1 \) and \( g_2 \) hold throughout and are not repeated in the statement of Theorem 1 below. By global solution to (1.6) we mean a solution defined for all \( t \geq 0 \).

**Theorem 1.** Under the above assumptions problem (1.6) has a global solution \( u(t) \), there exists the strong limit \( y := \lim_{t \to \infty} u(t) = u(\infty), \ F(y) = 0, u(t) \in B \) for all \( t \geq 0 \), and the following inequalities hold:

\[
||u(t) - y|| \leq ||F(u_0)|| \int_t^\infty G(x) dx, \tag{1.8}
\]

and

\[
||F(u(t))|| \leq ||F(u_0)|| \exp(-\int_0^t g_1(x) dx). \tag{1.9}
\]

Theorem 1 generalizes Theorem 1 in [1].

In Section 2 proof of Theorem 1 is given, and two other theorems are proved, and in Section 3 examples of applications are presented. In Section 4 a linear equation and in Section 5 a nonlinear equation are discussed.
2 Proof of Theorem 1 and additional results.

The assumptions about \( \Phi \) imply local existence and uniqueness of the solution \( u(t) \) to (1.6). To prove global existence of \( u \), it is sufficient to prove a uniform with respect to \( t \) bound on \( ||u(t)|| \). Indeed, if the maximal interval of the existence of \( u(t) \) is finite, say \([0, T)\), and \( \Phi(t, u) \) is locally Lipschitz with respect to \( u \), then \( ||u(t)|| \to \infty \) as \( t \to T \). Let \( g(t) := ||F(u(t))|| \). Since \( H \) is real, one uses (1.6) and (1.2) to get \( gg = (F'(u)\dot{u}, F) \leq -g_1(t)g^2 \), so \( \dot{g} \leq -g_1(t)g \), and integrating one gets (1.9), because \( g(0) = ||F(u_0)|| \). Using (1.3), (1.6) and (1.9), one gets:

\[
||u(t) - u(s)|| \leq g(0) \int_s^t G(x)dx, \quad G(x) := g_2(x) \exp(-\int_0^x g_1(z)dz). \tag{2.1}
\]

Because \( G \in L^1(R_+) \), it follows from (2.1) that the limit \( y := \lim_{t \to \infty} u(t) = u(\infty) \) exists, and \( y \in B \) by (1.7). From (1.9) and the continuity of \( F \) one gets \( F(y) = 0 \), so \( y \) solves (1.1). Taking \( t \to \infty \) and setting \( s = t \) in (2.1) yields estimate (1.8). The inclusion \( u(t) \in B \) for all \( t \geq 0 \) follows from (2.1) and (1.7). Theorem 1 is proved. \( \square \)

If condition (1.2) is replaced by

\[
(F'\Phi, F) \leq -g_1(t)||F||^a, \quad 0 < a < 2, \tag{2.2}
\]

then the proof of Theorem 1 yields \( g^{1-a} \dot{g} \leq -g_1(t) \), so

\[
0 \leq g(t) \leq [g^{2-a}(0) - (2-a) \int_0^t g_1(s)ds]^{1/a}. \tag{2.3}
\]

If (1.4) holds, then (2.3) implies \( g(t) = 0 \) for all \( t \geq T \), where \( T \) is defined by the equation:

\[
g^{2-a}(0) - (2-a) \int_0^T g_1(s)ds = 0. \tag{2.4}
\]

Thus \( ||F(u(t))|| = 0 \) for \( t \geq T \). So, by (1.3), \( \Phi = 0 \) for \( t \geq T \). Thus, by (1.6), \( u(t) = u(T) \) for \( t \geq T \). Therefore \( y := u(T) \) solves equation (1.1), \( F(y) = 0 \), and \( ||u(T) - u(0)|| \leq ||F(u_0)|| \int_0^T g_2ds \). If \( ||F(u_0)|| \int_0^T g_2ds \leq R \), then \( u(t) \in B \) for all \( t \geq 0 \).

We have proved:

**Theorem 2.** If (1.2) is replaced by (2.2), (1.4) holds, and \( ||F(u_0)|| \int_0^T g_2ds \leq R \), where \( T \) is defined by (2.4), then equation (1.1) has a solution in \( B = \{u : ||u - u_0|| \leq R\} \), the solution \( u(t) \) to (1.6) exists for all \( t > 0 \), \( u(t) \in B \), \( u(t) := y \) for \( t \geq T \), and \( F(y) = 0 \), \( y \in B \).

If (2.2) holds with \( a > 2 \), and (1.4) holds, then a similar calculation yields:

\[
0 \leq g(t) \leq [g^{-(a-2)}(0) + (a-2) \int_0^t g_1(s)ds]^{1/a} := h(t) \to 0 \quad t \to \infty, \quad \tag{2.5}
\]

because of (1.4). Assume that

\[
\int_0^\infty g_2(s)h(s)ds \leq R. \tag{2.6}
\]
Then (1.3) and (1.6) yield \(|u(t) - u(0)| \leq R, |u(t) - u(\infty)| \leq \int_0^\infty g_2(s)h(s)ds \to 0\) as \(t \to \infty\). Therefore an analog of Theorem 1 is obtained:

**Theorem 3.** If (2.2) holds with \(a > 2\), (1.4) and (2.6) hold, then the solution \(u(t)\) to (1.6) exists for all \(t > 0\), \(u(t) \in B\), there exists \(u(\infty) := y\), and \(F(y) = 0\), \(y \in B\).

3 Applications.

If \(g_j = c_j, j = 1, 2\), and \(c_j > 0\) are constants, then (1.4) and (1.5) hold, \(\int_0^\infty Gdx = c_2/c_1\), so condition (1.7) is:

\[
\frac{c_2}{c_1}||F(u_0)|| \leq R, \quad c_j = c_j(R), \quad j = 1, 2.
\] (3.1)

Let us give examples of applications of Theorem 1 using its simplified version with \(g_j = c_j > 0, j = 1, 2\).

**Example 1. Continuous Newton-type method.**

Let \(\Phi = -[F'(u)]^{-1}F(u)\), and assume

\[
||[F'(u)]^{-1}|| \leq m_1 = m_1(R), \quad \forall u \in B.
\] (3.2)

Assumption (3.2) holds in all the examples below. It implies ”well-posedness” of equation (1.1). The case of ill-posed equation (1.1), when (3.2) is not valid, was studied in [2] for linear selfadjoint nonnegative-definite operator, in [3] for nonlinear monotone hemicon-
tinuous operator, and in [1] for some other cases. Then \(c_1 = 1, c_2 = m_1, \Phi\) is locally Lipschitz if, for example, one assumes

\[
||F''(u)|| \leq M_2, \quad \forall u \in B,
\] (3.3)

where \(M_2 = M_2(R)\) is a positive constant, and (3.1) is:

\[
m_1(R)||F(u_0)|| \leq R.
\] (3.4)

In the examples below condition (3.3) is assumed and not repeated. As we have mentioned in Section 1, this condition can be weakened.

**Conclusion 1:** By Theorem 1, inequality (3.4) implies existence of a solution \(y\) to (1.1) in \(B\), global existence and uniqueness of the solution \(u(t)\) to (1.6), convergence of \(u(t)\) to \(y\) as \(t \to \infty\), and the error estimate (1.9). Condition (3.4) is always satisfied if equation (1.1) has a solution \(y\) and if \(u_0\) is chosen sufficiently close to \(y\).

**Example 2. Continuous simple iterations method.**

Let \(\Phi = -F'(u)\), and assume \(F'(u) \geq c_1(R) > 0\) for all \(u \in B\). Then \(c_2 = 1, c_1 = c_1(R)\), and (3.1) is:

\[
[c_1(R)]^{-1}||F(u_0)|| \leq R.
\] (3.5)

If this inequality holds, then Conclusion 1 holds with (3.5) replacing (3.4).

**Example 3. Continuous gradient method.**
Let $\Phi = -[F']^*F$ and assume (3.2). Then $c_2 = M_1(R)$, because $||[F'(u)]^*|| = ||F'(u)|| \leq M_1(R)$, and $(F'\Phi, F) = -||[F'(u)]^*F||^2 \leq -m_1^2||F||^2$, so $c_1 = m_1^2$, where $m_1$ is the constant from (3.2). Here we have used the estimates $||f|| = ||A^{-1}Af|| \leq ||A^{-1}||||Af||, ||Af|| \geq ||A^{-1}||^{-1}||f||$, with $A := F'(u)$ and $f = F(u)$.

If this inequality holds, then Conclusion 1 holds with (3.6) replacing (3.4).

**Example 4. Continuous Gauss-Newton method.**

Let $\Phi = -([F']^*F)^{-1}[F']^*F$, and assume (3.2). Then $c_1 = 1, c_2 = m_1^2M_1$, and (3.1) is:

$$M_1m_1^2||F(u_0)|| \leq R.$$  \hspace{1cm} (3.7)

If this inequality holds, then Conclusion 1 holds with (3.7) replacing (3.4).

**Example 5. Continuous modified Newton method.**

$\Phi = -[F'(u_0)]^{-1}F(u)$, and assume $|[F(u_0)]^{-1}| \leq m_0$. Then $c_2 = m_0$. To find $c_1$, let us note that:

$$(F'\Phi, F) = -||F(u)||^2 - ((F'(u) - F'(u_0))[F(u_0)]^{-1}F, F) \leq 0.5||F(u)||^2,$$

provided that

$$|((F'(u) - F'(u_0))[F(u_0)]^{-1}F, F)| \leq M_2Rm_0||F(u)||^2 \leq 0.5||F(u)||^2,$$

that is, $M_2Rm_0 \leq 0.5$. If $R = (2M_2m_0)^{-1}$, then the last inequality becomes an equality. Choosing such $R$, one has $c_2 = m_0, c_1 = 0.5$, and (3.1) is: $2m_0||F(u_0)|| \leq (2M_2m_0)^{-1}$, that is,

$$4m_0^2M_2||F(u_0)|| \leq 1.$$  \hspace{1cm} (3.8)

If this inequality holds, then Conclusion 1 holds with (3.8) replacing (3.4).

**Example 6. Descent methods.**

Let $\Phi = -f/||f||$ $h$, where $f = f(u(t))$ is a differentiable functional $f : H \to [0, \infty)$, and $h$ is an element of $H$. One has $\dot{f} = (f', \dot{u}) = -f$. Thus $f = f_0e^{-t}$, where $f_0 := f(u_0)$. Assume $||\Phi|| \leq c_2|f|^b, b > 0$. Then $||\dot{u}|| \leq c_2|f_0|^b e^{-bt}$. Therefore $u(\infty)$ does exist, $f(u(\infty)) = 0$, and $||u(\infty) - u(t)|| \leq ce^{-bt}, c = const > 0$.

If $h = f'$, and $f = ||F(u)||^2$, then $f'(u) = 2[F']^*(u)F(u), \Phi = -\frac{f'}{||f||} f'$, and (1.6) is a descent method. For this $\Phi$ one has $c_1 = \frac{1}{2}, c_2 = \frac{m_1}{2}$, where $m_1$ is defined in (3.2). Condition (3.1) in this example is condition (3.4).

If (3.4) holds, then Conclusion 1 holds.

We have obtained some results from [4]. Our approach is more general than the one in [4], since the choices of $f$ and $h$ do not allow one, e.g., to obtain $\Phi$ used in Example 5.
4 Remark about linear equations.

The following result was proved in [2]: if equation $Ay = f$ in a Hilbert space has a solution $y$, and $A \geq 0$ is a linear selfadjoint operator, then the global solution $u$ to the regularized Cauchy problem

$$\dot{u} = -Au - \alpha(t)u + f, \quad u(0) = u_0,$$

has a limit $\lim_{t \to \infty} u(t) = u(\infty)$, and $A(u(\infty)) = f$. In [2] $u_0 \in H$ is arbitrary, $\alpha > 0$ is a continuously differentiable, monotonically decaying to zero as $t \to \infty$, function on $R_+$, $\int_0^\infty \alpha dt = +\infty$, and $\alpha^{-2}\dot{\alpha} \to 0$ as $t \to \infty$.

Unfortunately the author of [7] was not aware of the paper [2]. Some of the proofs in [7] are close to these in [2]. The author thanks Dr. Ya. Alber for pointing out references [2] and [3].

If $\alpha > 0$, $\dot{\alpha} \leq 0$, and $\alpha^{-2}|\dot{\alpha}| \leq c$, where $c = \text{const}$, then $\alpha^{-1}(t) - \alpha^{-1}(0) \leq ct$, so $\alpha(t) \geq [ct + \alpha^{-1}(0)]^{-1}$ and consequently $\int_0^\infty \alpha dt = +\infty$ (cf [7]). Therefore the condition $\int_0^\infty \alpha dt = +\infty$ in [2] can be dropped.

In this Section we give a new derivation of the result in [2] under weaker assumptions about $\alpha$, and show that the regularization in (4.1) is not necessary.

First, let us prove that the regularization in (4.1) is not necessary: the result holds with $\alpha = 0$. Below $\to$ denotes strong convergence in $H$.

The solution to (4.1) with $\alpha = 0$ is $u(t) = U(t)u_0 + \int_0^t U(t-s)f ds$, where $U(t) := \exp(-tA)$. If $E_t \lambda$ is the resolution of the identity of the selfadjoint operator $A$, then $U(t)u_0 = \int_\lambda^\infty e^{-\lambda}dE_t \lambda u_0 \to Pu_0$ as $t \to \infty$, where $P$ is the operator of the orthogonal projection on $N$, and $N$ is the null-space of $A$. Also $\int_0^t U(t-s)f ds = \int_0^\infty (1-e^{-\lambda})dE_t \lambda y \to y - Py$ as $t \to \infty$, by the dominated convergence theorem. Thus, $u(\infty) = y - Py + Pu_0$ and $A(u(\infty)) = f$. □

Consider now the case $0 < \alpha \to 0$ as $t \to \infty$. If $h(t) := \exp(\int_0^t \alpha(s)ds)$, and $u$ solves (4.1), then

$$u(t) = h^{-1}(t)U(t)u_0 + h^{-1}(t) \int_0^\infty \exp(-t\lambda) \int_0^t e^{s\lambda}h(s)ds \lambda dE_t \lambda y. \quad (4.2)$$

Using L'Hospital's rule one checks that

$$\lim_{t \to \infty} \frac{\lambda \int_0^t e^{s\lambda}h(s)ds}{e^{t\lambda}h(t)} = \lim_{t \to \infty} \frac{\lambda e^{t\lambda}h(t)}{\lambda e^{t\lambda}h(t) + e^{t\lambda}h(t)\alpha(t)} = 1 \quad \forall \lambda > 0. \quad (4.3)$$

From (4.2), (4.3), and the dominated convergence theorem, one gets $u(\infty) = y - Py$. The first term on the right-hand side of (4.2) tends to zero as $t \to \infty$ (even if $Pu_0 \neq 0$), if $h(\infty) = \infty$. To apply the dominated convergence theorem, one checks that $\frac{\lambda \int_0^t e^{-(t-s)\lambda}h(s)ds}{h(t)} = \frac{\lambda \int_0^t e^{-\lambda}h(t-s)ds}{h(t)} \leq 1$ for all $t > 0$ and all $\lambda > 0$, where the inequality $0 < h(t-s) \leq h(t)$, valid for $s \geq 0$, was used. □
Our derivation uses less restrictive assumptions on $\alpha$ than in [2] and [7]: we do not assume differentiability of $\alpha$, and the property $\lim_{t \to \infty} \alpha^{-2} \alpha' = 0$. The property $\int_0^\infty \alpha dt = +\infty$, which is equivalent to $h(\infty) = \infty$, was used above only to prove that $\lim_{t \to \infty} h^{-1}(t) U(t) u_0 = 0$. If $\int_0^\infty \alpha dt := q < \infty$, then $u(\infty) = y - Py + e^{-q} Pu_0$, and $Au(\infty) = f$, so that the basic conclusion holds without the assumption $h(\infty) = \infty$.

Finally, let us prove a typical for ill-posed problems result: the rate of convergence $u(t) \to y$ can be as slow as one wishes, it is not uniform with respect to $f$. Assume $\alpha = 0$, but the proof is essentially the same for $0 < \alpha \to 0$ as $t \to \infty$. Assume that $A > 0$ is compact, and $A \varphi_j = \lambda_j \varphi_j$, $(\varphi_j, \varphi_m) = \delta_{jm}$. Then (4.2) with $y = y_m := \varphi_m$ and $u_0 = 0$ yields $u(t) = \varphi_m(1 - e^{-t\lambda_m})$. Thus $u(\infty) = y$, but for any fixed $T > 0$, however large, one can find $m$ such that $||u(T) - y_m|| > 0.5$, that is, convergence is not uniform with respect to $f$.

5 Remark about nonlinear equations.

In this Section we give a short and simple proof of the basic result in [3], and close a gap in the proof in [3], where it is not explained why one can apply the L’Hospital rule the second time.

The assumptions in [3] are: the operator $A$ is monotone (possibly nonlinear), defined on all of $H$, hemicontinuous, problem (4.1) has a unique global solution, equation $A(y) = f$ has a solution, $\alpha(t) > 0$ decays monotonically to zero, $\lim_{t \to \infty} \alpha \alpha^{-2} = 0$, and $\alpha$ is convex.

We refer below to these assumptions as A3). If A3) hold, the basic result, proved in [3], is the existence of $u(\infty) := y$, and the relation $A(y) = f$. In [3], p. 184, under the additional assumption (1.24) from [3], the global existence of the solution to (4.1) is proved. Actually, the assumption about global existence of the solution to (4.1) can be dropped altogether: in [6], p.99, it is proved that A3) (and even weaker assumptions) imply that problem (4.1) has a unique global solution.

Let us give a short proof of the basic result from [3]. It is well known that A3) imply that the problem $A(v_{\alpha}) + \alpha v_{\alpha} - f = 0$, for any fixed number $\alpha > 0$, has a unique solution, there exists $\lim_{\alpha \to 0} v_{\alpha} := y$, $A(y) = f$, and $||y|| < \delta/2$ for all $\alpha > \alpha_0$, $\lim_{\alpha \to 0} \alpha \alpha^{-2} = 0$. Let $w := u - v_{\alpha}$, where $u$ solves (4.1) and $v_{\alpha}$ does not depend on $t$. Then $\dot{w} = -[A(u) - A(v_{\alpha}) + \alpha(t)(u - v_{\alpha}) + (\alpha(t) - \alpha)v_{\alpha}]$. Multiply this by $w$, use the monotonicity of $A$, and let $||w|| := g$. Then $\dot{g} \leq -\alpha(t) g^2 + c|\alpha(t) - \alpha| g$, $c = ||y||$. Indeed, multiply $A(v_{\alpha}) + \alpha v_{\alpha} - A(y) = 0$ by $v_{\alpha} - y$ and use monotonicity of $A$ to get $\alpha(v_{\alpha} v_{\alpha} - y) \leq 0$. Thus $||v_{\alpha}|| \leq ||y||$, so $c = ||y||$.

Since $\alpha(t)$ is convex, one has $|\alpha(t) - \alpha| \leq |\dot{\alpha}(t)|(t_\alpha - t)$, where $t_\alpha \geq t$ is defined by the equation $\alpha = \alpha(t_\alpha)$, $\lim_{\alpha \to 0} t_\alpha = \infty$. Thus, $\dot{g} \leq -\alpha(t) g^2 + c|\dot{\alpha}(t)|(t_\alpha - t)g$, and, taking
\[ u(0) = v_\alpha, \text{ one gets} \]

\[ g(t_\alpha) \leq ce^{- \int_0^{t_\alpha} \alpha(x) dx} \int_0^{t_\alpha} e^{\int_0^t \alpha(x) dx} |\dot{\alpha}(s)| (t_\alpha - s) ds. \]  \hspace{1cm} (5.1)

We prove below that

\[ \lim_{t \to \infty} \alpha(t) e^{\int_0^t \alpha(s) ds} = \infty. \]  \hspace{1cm} (5.2)

This allows one to apply twice L'Hospital's rule to the right-hand side of (5.1), and to get: \( \lim_{t_\alpha \to 0} g(t_\alpha) = \lim_{t_\alpha \to \infty} \frac{\dot{\alpha}(t_\alpha)}{\alpha(t_\alpha) + \alpha^2(t_\alpha)} = 0. \) Now, \( ||u(t_\alpha) - y|| \leq ||u(t_\alpha) - v_\alpha|| + ||v_\alpha - y||, \) \( ||v_\alpha - y|| \leq \delta/2, \) and \( ||u(t_\alpha) - v_\alpha|| \leq \delta/2, \) for sufficiently large \( t_\alpha. \) Since \( \delta > 0 \) is arbitrarily small, it follows that \( \lim_{t \to \infty} ||u(t) - y|| = 0. \)

Let us prove (5.2). From our assumptions about \( \alpha, \) it follows that for all sufficiently large \( t, \) one has \(-\dot{\alpha} \alpha^{-2} \leq c, \) where \( 0 < c < 1, \) so \( \alpha(t) \geq (c_1 + t)^{-1}b, \) where \( b := c^{-1} > 1, c_1 > 0 \) is a constant, and \( e^{\int_0^t \alpha(s) ds} \geq (c_1 + t)^b. \) Thus, (5.2) holds. \( \square \)

If one assumes additionally that \( A \) is Frechet differentiable, then the proof is shorter. Namely, let \( h(t) := ||A(u(t)) + \alpha(t)u(t) - f|| := ||\psi||. \) Then \( h = -((A'(u(t)) + \alpha(t))\psi, \psi) \leq -\alpha(t) h^2, \) because \( A' \geq 0 \) due to the monotonicity of \( A. \) Thus \( h(t) \leq \phi(t), \) where \( \phi(t) := h(0)e^{-\int_0^{t_\alpha} \alpha(s) ds}. \) As we proved in Section 4, the assumptions on \( \alpha(t) \) imply \( \alpha(t) \geq (c_1 t + c_2)^{-1}, \) where \( c_1 \) and \( c_2 \) are positive constants, and \( c_1 \) can be chosen so that \( 0 < c_1 < 1, \) due to the assumption \( \lim_{t \to \infty} \dot{\alpha} \alpha^{-2} = 0. \) Therefore \( \int_0^t \phi(t) dt < \infty. \) From (4.1) one gets: \( ||\dot{u}|| \leq \phi(t). \) Because \( \int_0^\infty \phi(t) dt < \infty, \) it follows that \( u(\infty) := y \) exists, and \( ||u(\infty) - u(t)|| \leq \int_t^\infty \phi(s) ds. \) Finally, \( A(y) = f \) because \( h(\infty) = 0 = ||A(y) - f||. \) Any choice of \( \alpha, \) for which \( \int_0^\infty \phi(t) dt < \infty, \) is sufficient for the above argument. \( \square \)

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