Magnetic resonance, especially spin echo, in spinor Bose–Einstein condensates

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Abstract.
Magnetic resonance, especially NMR and ESR, has been studied in magnetic materials for a long time, having been used in various fields. Spin echo is a typical phenomenon in magnetic resonance. The magnetic resonance should be applied to spinor Bose–Einstein condensates (BECs). We numerically study spin echo of a spinor BEC in a gradient magnetic field by calculating the spin-1 two-dimensional Gross–Pitaevskii equations, obtaining the recovery of the signal of the spins, which is called spin echo. We will discuss the relation between the spin echo and the Stern–Gelrach separation in the system.

1. Introduction
Magnetic resonance has been one of important branches in condensed matter system. Particularly the discovery of spin echo in 1950 by Hahn was epoch-making in the field of magnetic resonance [1]. Hahn observed the recovery of a free induction decay signal of spins by using the combination of two \( \pi/2 \) pulses, naming the phenomenon “spin echo”. Then Carr and Purcell proposed well-known spin echo for the \( \pi/2–\pi \) pulses sequence [2]. Spin echo has contributed to development of magnetic resonance as a method of measuring the relaxation time on magnetic resonance imaging (MRI).

In low temperature physics, particularly in superfluid \(^3\)He, NMR has made significant contribution for revealing the superfluid phases and discovering many vortices and textures [3]. It would be natural that we apply magnetic resonance to a spinor Bose–Einstein condensate (BEC) [4]. Thus we numerically study spin echo in spin-1 BECs by calculating two-dimensional Gross–Pitaevskii equations.

Before the discussion, we review spin echo briefly. Initially, spins with gyromagnetic ratio \( \gamma \) are polarized to the \( z \) axis under a magnetic field \( \mathbf{H}_0 \). The spins resonate with a \( \pi/2 \) pulse, which is the rotational magnetic field \( \mathbf{H}_{\text{rot}} = H_1 \cos(\gamma H t) \hat{x} + H_1 \sin(\gamma H t) \hat{y} \) for \( t_{\pi/2} = \pi/(2 \gamma H_1) \), and tilt to the \( -y \) axis with the precessing with Larmor frequency \( \omega_L = \gamma H \). When the static magnetic field has inhomogeneity, the Larmor frequency for each spin depends on space like \( \omega_L(r) = \gamma H(r) \). Then the precessions lose the coherence after the \( \pi/2 \) pulse. Immediately applying a \( \pi \) pulse, which is the rotational magnetic field for \( t_{\pi} = 2t_{\pi/2} \), at \( t = \tau + t_{\pi/2} \), where \( \tau \) is the interval between the \( \pi/2 \) and \( \pi \) pulses, spins reverse the direction. Then the precession gradually recovers the coherence. At \( \tau \) after the pulse, the spins complete the coherence on the \( y \) axis again. This recovery of the coherence is called spin echo.
2. Formulation

In order to obtain the spin echo in a $^{87}$Rb BEC trapped by the harmonic potential of the frequency $\omega_\perp$, we study the two-dimensional spin-1 Gross–Pitaevskii (GP) equations for magnetic sublevels $\alpha$ of atoms

$$i\hbar \frac{\partial \psi_\alpha}{\partial t} = \left( -\frac{\hbar^2}{2M} \nabla^2 + V_{\text{trap}} - \mu \right) \psi_\alpha - g\mu_B H_i S_{i\alpha\beta} \psi_\beta + c_0 \psi_\beta^* \psi_\alpha + c_2 S_i S_{i\alpha\beta} \psi_\beta. \quad (1)$$

Here $V_{\text{trap}} = M\omega^2_\perp (x^2 + y^2)/2$ is the trapping potential, $\mu$ the chemical potential, $g$ the Landé’s $g$-factor, $\mu_B$ the Bohr magneton, and

$$S_i = \psi_\alpha^* S_{i\alpha\beta} \psi_\beta \quad (2)$$

is $i = \{x, y, z\}$ component of the spin density vector $\mathbf{S}$ represented by the spin matrices $S_{i\alpha\beta}$. The short range interaction constants are $c_0 = 4\pi\hbar^2(a_0 + 2a_2)/3M$ and $c_2 = 4\pi\hbar^2(a_2 - a_0)/3M$ given by the s-wave scattering lengths $a_0$ and $a_2$. For $^{87}$Rb, the parameters satisfy the relation $c_0 \gg -c_2$. We will investigate the spin echo under the gradient magnetic field $\mathbf{H} = (Gx + H_0)\hat{z}$ with a constant $G$.

The dynamics starts from the stationary states in the gradient magnetic field of the magnitude $0.98H_0 < |\mathbf{H}| < 1.02H_0$. Applying $\mathbf{H}_{\text{rot}}$ of $H_1 = 0.1H_0$ and the frequency $\gamma H_0$, we calculate the GP equations by using the Crank–Nicolson method, obtaining the spin echo numerically.

3. Spin echo in spinor BECs

Spin echo is clearly found in Fig. 1, showing the dynamics of $\mathbf{S}$ projected on a $x$-$y$ plane for $\omega_\perp \tau = 5$. Just after the $\pi/2$ pulse, the spins are oriented to -$y$ axis (Fig. 1(a)), and start precessing on the plane. Then, the precessions gradually diffuse because the Larmor frequency is spatially dependent through the gradient magnetic field (Fig. 1(b), (c)). At $t = \tau + t_{\pi/2}$ (Fig. 1(d)), the $\pi$ pulse is applied, which reverses the direction of the spin (Fig. 1(e)). Then, the spins gradually become coherent (Fig. 1(f), (g)), eventually refocusing at $t_{\text{peak}} = 2\tau + t_{\pi/2} + t_{\pi}$ (Fig. 1(h)). The dynamics from the dephasing to the rephasing must be spin echo. After the echo,

![Figure 1](image-url)
the spins start to defocus again. These features are represented in the time development of the expectation values $\langle \hat{S}_{y,z} \rangle = \int d\mathbf{r} \psi^* \alpha S^*_{\alpha\beta} \psi_{\beta}$ in Fig. 2. The signal of $\langle \hat{S}_y \rangle$ gradually decays from $t_{\pi/2}$ (a) to $t_{\pi/2} + t_{\pi/2}$ (d), increasing from $t_{\pi/2} + t_{\pi/2} + t_{\pi}$ (e) to $2t_{\pi/2}$ (h). The signal of $\langle \hat{S}_z \rangle$ shows that spins after the $\pi/2$ pulse and the $\pi$ pulse tilt to the $x$–$y$ plane.

The recovery of spin echo is not perfect, which can be understood by comparing (a) with (h) in both Figs. 1 and 2. This is understood the equation of motion of spin

$$\frac{\partial S_i}{\partial t} = \frac{1}{M} \nabla \cdot \left( S_i^J_{\alpha\beta} J_{\alpha\beta} \right) + \gamma |\mathbf{S} \times \mathbf{H}|_i,$$

which is derived from Eqs. (1) and (2). Here $J_{\alpha\beta} = \hbar/2i(\psi^*_{\beta} \nabla \psi_{\alpha} - \psi_{\alpha} \nabla \psi^*_{\beta})$ is the current of the momentum density. If the condensates are uniform i.e. $\nabla \psi_{\alpha} = 0$, the spin echo would show the perfect recovery as shown in general texts of magnetic resonance [6]. In the system, the effect of the gradient term cannot be ignored because the condensates are trapped and the components $\psi_{\pm1}$ separate like the Stern–Gelrach experiment [5] in the gradient magnetic field. Especially the Stern–Gelrach separation affects the spin echo.

Figure 3 shows the characteristic dynamics of components $|\psi_{\alpha}|^2$ for $\omega_{\perp} \tau = 12.5$ during the spin echo. The condensates $\psi_{\pm1}$ after the $\pi/2$ pulse (Fig. 3(b)) become to separate from the trap center because the force due to the gradient magnetic field acts on them oppositely along the $x$ axis (Fig. 3(c)). Then applying the $\pi$ pulse changes the $\psi_1$ and $\psi_{-1}$ components to $\psi_{-1}$ and $\psi_1$ respectively (Fig. 3(d)), which reverses $\mathbf{S}$. The $\pi$ pulse breaks the symmetry of the distribution of the condensates because of the gradient magnetic field. After the pulse, the gradient magnetic field move $\psi_{\pm1}$ to the center (Fig. 3(e)), making them overlap each other. The overlap, however, is not perfect because of the repulsive interaction between the condensates, which is given by the $c_0$ term of Eq. (1) (Fig. 3(f)). The imperfect overlap would prevent the spins from completing the coherence. Thus the spin echo is strongly relevant to unique dynamics of the phase separation of the system.

**Figure 2.** The black line and the white circles represent the time development of the $\langle S_y \rangle/\hbar$ and $\langle S_z \rangle/\hbar$. The symbols (a) - (h) are same timing in Fig. 1. The gray zones represent the interval of the pulses.
The dynamics of density profiles $|\tilde{\psi}_\alpha(y=0)|^2 = a_h^2|\psi_\alpha(y=0)|^2/N$ with $a_h = \sqrt{\hbar/M\omega_\perp}$ and $\tilde{x} = x/a_h$ for $\omega_\perp \tau = 12.5$. The dash, solid and dot lines show $|\tilde{\psi}_1|^2$, $|\tilde{\psi}_0|^2$ and $|\tilde{\psi}_{-1}|^2$ respectively.

4. Conclusion
We have numerically realized spin echo in a trapped $^{87}\text{Rb}$ BEC by calculating the spin-1 two-dimensional GP equations. We have investigated how the spin echo is affected by the Stern–Gelrach separation. The spin echo would be utilized for measuring spin-spin relaxation time, which is given by a magnetic dipole-dipole interaction, in spinor dipolar BECs. The physics is discussed in a recent paper of ours [7]. The study of spin echo of spinor BECs will help the field of magnetic resonance in the cold atomic BECs system develop.

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