NUMERICAL ALGORITHM FOR FINDING BALANCED METRICS ON VECTOR BUNDLES

REZA SEYYEDALI

Abstract. In [D3], Donaldson defines a dynamical system on the space of Fubini-Study metrics on a polarized compact Kähler manifold. Sano proved that if there exists a balanced metric for the polarization, then this dynamical system always converges to the balanced metric ([S]). In [DKLR], Douglas, et. al., conjecture that the same holds in the case of vector bundles. In this paper, we give an affirmative answer to their conjecture.

1. Introduction

In [D3], Donaldson defines a dynamical system on the space of Fubini-Study metrics on a polarized compact Kähler manifold. Sano proved that if there exists a balanced metric for the polarization, then this dynamical system always converges to the balanced metric ([S]). In [DKLR], Douglas, et. al., conjecture that the same holds in the case of vector bundles. In this paper, we give an affirmative answer to their conjecture.

Let \((X, \omega)\) be a Kähler manifold of dimension \(n\) and \(E\) be a very ample holomorphic vector bundle on \(X\). Let \(h\) be a Hermitian metric on \(E\). We can define a \(L^2\)-inner product on \(H^0(X, E)\) by

\[
\langle s, t \rangle = \int_X h(s, t) \frac{\omega^n}{n!}.
\]

Let \(s_1, ..., s_N\) be an orthonormal basis for \(H^0(X, E)\) with respect to this \(L^2\)-inner product. The Bergman kernel of \(h\) is defined by

\[
B(h) = \sum s_i \otimes s_i^h.
\]

Note that \(B(h)\) does not depend on the choice of the orthonormal basis \(s_1, ..., s_N\).

A metric \(h\) is called balanced if \(B(h)\) is a constant multiple of the identity. By the theorem of Wang ([W], Theorem 1.1]), we know that the existence of balanced metrics is closely related to the stability of the vector bundle \(E\). Indeed \(E\) admits a unique (up to a positive constant) balanced metric if and only if the Gieseker point of \(E\) is stable. On the
other hand, a balanced metric is unique (up to a constant) provided
the bundle is simple (cf. Lemma 2.7 below). Let $K$ and $M$ be the space
of Hermitian metrics on $E$ and Hermitian inner product on $H^0(X, E)$
respectively. Following Donaldson ([D2]), one can define the following
maps

- Define
  \[ \text{Hilb} : K \to M \]
  by
  \[ \langle s, t \rangle_{\text{Hilb}(h)} = \frac{N}{V} \int_X \langle s(x), t(x) \rangle_h \omega^n, \]
  where $N = \dim(H^0(X, E))$ and $V = \text{Vol}(X, \omega)$. Note that
  Hilb only depends on the volume form $\omega^n$.

- For the metric $H$ in $M_E$, $FS(H)$ is the unique metric on $E$
such that $\sum s_i \otimes s_i^{FS(H)} = I$, where $s_1, ..., s_N$ is an orthonormal basis
  for $H^0(X, E)$ with respect to $H$. This gives the map $FS : M \to K$.

- Define a map
  \[ T : M \to M \]
  by $T(H) = \text{Hilb} \circ FS(H)$. Notice that this map $T$ is called
generalized $T$-operator in [DKLR].

It is easy to see that a metric $h$ is balanced if and only if $\text{Hilb}(h)$ is
a fixed point of the map $T$.

The main theorem of this paper is the following

**Theorem 1.1.** Suppose that $E$ is simple and admits a balanced metric.
Then for any $H_0 \in M_E$, the sequence $T^n(H_0)$ converges to $H_\infty$, where
$H_\infty$ is a balanced metric on $E$.

Our proof follows Sano’s argument in [S] with the necessary modifi-
cations for the bundle case.

In order to prove the theorem, we consider the functional $Z$ that is
used by Wang ([W]) and Phong, Sturm ([PS]) in order to study the
existence and uniqueness of balanced metrics on holomorphic vector
bundles. The key property of this functional is that its critical points
are balanced metrics. In the first section we recall some properties of
the functionals $Z$ and $\tilde{Z}$. In the second section, we give an appropriate
notation of boundedness for subsets of $M$ which is defined in [S]. It is
easy to see that any bounded sequence has a convergent subsequence
after a suitable rescaling of the sequence. Therefore in order to prove
that the sequence $H_n = T^n(H)$ converges, we need to show that $H_n$
is bounded. On the other hand, existence of a balanced metric implies
that $\tilde{Z}$ is bounded from below and proper in a suitable sense. Hence it shows that $\tilde{Z}(H_n)$ is bounded. Now properness of $\tilde{Z}$ implies that $H_n$ is bounded.

Acknowledgements: I am sincerely grateful to Richard Wentworth for introducing me the subject and many helpful discussions and suggestions on the subject. I would also like to thank him for all his help, support and encouragement.

2. Balanced metrics on vector bundles

As before, let $(X, \omega)$ be a Kähler manifold and $E$ be a very ample holomorphic vector bundle on $X$. Using global sections of $E$, we can map $X$ into $G(r, H^0(X, E)^*)$. Indeed, for any $x \in X$, we have the evaluation map $H^0(X, E) \rightarrow E_x$, which sends $s$ to $s(x)$. Since $E$ is globally generated, this map is a surjection. So its dual is an inclusion of $E^*_x \hookrightarrow H^0(X, E)^*$, which determines a $r$-dimensional subspace of $H^0(X, E)^*$. Therefore we get an embedding $i : X \hookrightarrow G(r, H^0(X, E)^*)$. Clearly we have $i^* U_r = E^*$, where $U_r$ is the tautological vector bundle on $G(r, H^0(X, E)^*)$, i.e. at any $r$-plane in $G(r, H^0(X, E)^*)$, the fibre of $U_r$ is exactly that $r$-plane. A choice of basis for $H^0(X, E)$ gives an isomorphism between $G(r, H^0(X, E)^*)$ and the standard $G(r, N)$, where $N = \dim H^0(X, E)$. We have the standard Fubini-Study hermitian metric on $U_r$, so we can pull it back to $E$ and get a hermitian metric on $E$. Using $i^* h_{FS}$ and $\omega$, we get an $L^2$ inner product on $H^0(X, E)$. The embedding is called balanced if $\int_X \langle s_i, s_j \rangle_{\omega} = C \delta_{ij}$. We can formulate this definition in terms of maps $\Hilb$ and $FS$.

Definition 2.1. A $\omega$ balanced metric on $E$ is a pair $(h^*, H^*)$ so that

$$\Hilb(h^*) = H^* , \quad FS(H^*) = h^*$$

Fixing a nonzero element $\Theta \in \bigwedge^N H^0(X, E)$, We can define the determinant of any element in $M$. Thus we can define a map

$$\log \det : M \rightarrow \mathbb{R}$$

A different choice of $\Theta$ only changes this map by an additive constant. Also, we define a functional $I : K \rightarrow \mathbb{R}$ again unique up to an additive constant. Fix a background metric $h_0$ and consider a path $h_t = e^{\phi t} h_0$ in $K$ then

$$\frac{dI}{dt} = \int_X tr(\dot{\phi}) \ d\text{Vol}_\omega$$

This functional is a part of Donaldson’s functional. We define:

$$Z = -I \circ FS : M \rightarrow \mathbb{R}$$
We have the following scaling identities:

\[
\text{Hilb}(e^{\alpha}h) = e^{\alpha}\text{Hilb}(h), \\
\text{FS}(e^{\alpha}h) = e^{\alpha}\text{FS}(h), \\
I(e^{\alpha}h) = I(h) + \alpha rV,
\]

where \(\alpha\) is a real number.

Following Donaldson, we define:

\(\tilde{Z} = Z + \frac{rV}{N} \log \det\).

So \(\tilde{Z}\) is invariant under constant scaling of the metric.

This functional \(Z\) is studied by Wang in [W] and Phong and Sturm in [PS]. They consider this as a functional on \(SL(N)/SU(N)\). In order to see this, we observe that there is a correspondence between \(M\) and \(GL(N)/U(N)\). Let fix an element \(H_0 \in M\) and an orthonormal basis \(s_1, \ldots, s_N\) for \(H^0(X, E)\) with respect to \(H_0\). Now for any \(H \in M\) we assign \([H(s_i, s_j)] \in GL(N)\). Notice that change of the orthonormal basis only changes this matrix by multiplication by elements of \(U(N)\). So we get a well define element of \(GL(N)/U(N)\). The subset

\[M_0 = \{H \in M | \det[H(s_i, s_j)] = 1\}\]

corresponds to \(SL(N)/SU(N)\).

We recall the definition of the Gieseker point of the bundle \(E\). We have a natural map

\[T(E) : \bigwedge^r H^0(X, E) \to H^0(X, \text{det}(E))\]

which for any \(s_1, \ldots, s_r\) in \(H^0(X, E)\) is defined by

\[T(E)(s_1 \wedge \ldots \wedge s_r)(x) = s_1(x) \wedge \ldots \wedge s_r(x)\].

Since \(E\) is globally generated, \(T(E)\) is surjective. We can view \(T(E)\) as an element of \(\text{Hom}(\bigwedge^r H^0(X, E), H^0(X, \text{det}(E)))\). This is called the Gieseker point of \(E\). Notice that fixing a basis for \(H^0(X, E)\) gives an isomorphism between \(\bigwedge^r H^0(X, E)\) and \(\bigwedge^r \mathbb{C}^N\). Hence, there is a natural action of \(GL(N)\) on \(\text{Hom}(\bigwedge^r H^0(X, E), H^0(X, \text{det}(E)))\). Phong-Sturm ([PS]) and Wang ([W]) prove that \(Z\) is convex along geodesics of \(SL(N)/SU(N)\) and its critical points are corresponding to balanced metrics on \(E\). Phong and Sturm prove the following

**Theorem 2.1.** ([PS] Theorem 2) There exists a \(SU(N)\)-invariant norm \(||.||\) on \(\text{Hom}(\bigwedge^r H^0(X, E), H^0(X, \text{det}(E)))\) such that for any \(\sigma \in SL(N)\)
\[ Z(\sigma) = \log \frac{||\sigma.T(E)||^2}{||T(E)||^2} \]

**Remark 2.2.** In [W], Wang proves a slightly weaker version of Theorem 2.1. He proves that for any norm \(||\cdot||\) on \(\text{Hom}(\wedge^r H^0(X, E), H^0(X, \text{det}(E)))\), there exists positive constants \(c\) and \(c'\) such that

\[ Z(\sigma) \geq c \log ||\sigma.T(E)||^2 + c'. \]

**Theorem 2.3.** ([W, Lemma 3.5], [PS, Lemma 2.2]) The functional \(Z\) is convex along geodesics of \(M\).

The Kempf-Ness theorem ([KN]) shows that \(Z\) is proper and bounded from below if \(T(E)\) is stable under the action of \(SL(N)\).

The following is an immediate consequence of the above theorem and the fact that balanced metrics are critical points of \(Z\). Also notice that \(\tilde{Z}\) is invariant under the scaling of a metric by a positive real number.

**Theorem 2.4.** Assume that \(H_0\) is a balanced metric on \(E\). Then \(\tilde{Z}|_{M_0}\) is proper and bounded from below. Moreover \(\tilde{Z}(H) \geq \tilde{Z}(H_0)\) for any \(H \in M\).

**Lemma 2.5.** For any \(H \in M\), we have

\[ \text{Tr}(T(H)H^{-1}) = N \]

**Proof.** We define \(h = FS(H)\). Let \(s_1, ..., s_N\) be an \(H\)-orthonormal basis. We have,

\[ \sum s_i \otimes s_i^* = I \]

Therefore,

\[ r = \text{Tr}(\sum s_i \otimes s_i^*) = \sum |s_i|^2_h. \]

Integrating the above equation concludes the lemma.

\[ \square \]

**Lemma 2.6.** For any \(H \in M\),

- \(Z(H) \geq Z(T(H))\).
- \(\log \det(H) \geq \log \det(T(H))\).
- \(\tilde{Z}(H) \geq \tilde{Z}(T(H))\).

**Proof.** Put \(h = FS(H)\), \(H' = \text{Hilb} \circ FS(H)\) and \(h' = FS(H') = e^{\varphi} h\). Let \(s_1, ..., s_N\) be an \(H'\)-orthonormal basis. We have,

\[ \sum s_i \otimes s_i^* = e^{-\varphi}. \]

\[ 5 \]
\[
\int_X tr(-\varphi) = \int_X \log \det(e^{-\varphi}) \leq \int_X \log \left(\frac{tr(e^{-\varphi})}{r}\right)^r
\]

\[
= r \int_X \log(tr(e^{-\varphi})) - rV \log r \leq rV \log \left(\frac{1}{V} \int_X tr(e^{-\varphi})\right) - rV \log r
\]

\[
= rV \log \left(\frac{1}{V} \sum_i |s_i|^2_{h_i}\right) - rV \log r = 0
\]

This shows the first inequality. For the second one, Lemma 2.5 implies that \(tr(H'H^{-1}) = N\). Using the arithmetic-geometric mean inequality, we get

\[
\det(H'H^{-1})^{\frac{1}{N}} \leq \frac{tr(H'H^{-1})}{N} = 1.
\]

This implies that \(\log \det(H'H^{-1}) \leq 0\). The third inequality is obtained by summing up the first two.

\[\square\]

A bundle \(E\) is called simple if \(\text{Aut}(E) \simeq \mathbb{C}^*\). We will also need the following

**Lemma 2.7.** Suppose that \(E\) is simple and admits a balanced metric. Then the balanced metric is unique up to a positive constant.

**Proof.** Let \(H_\infty\) be a balanced metric on \(E\) and \(s_1, ..., s_N\) be an orthonormal basis of \(H^0(X, E)\) with respect to \(H_\infty\). This basis gives an embedding \(\iota : X \to Gr(r, N)\) such that \(\iota^*U_r = E\), where \(U_r \to Gr(r, N)\) is the universal bundle over the Grassmannian. Let assume that \(H\) is another element of \(M_0\). Therefore, there exists an element \(a \in su(N)\) such that \(e^{ia}.H_\infty = H\). The one parameter family \(\{e^{ita}\}\) gives a one parameter family of automorphism of \((Gr(r, N), U_r)\) and therefore gives a one parameter family in \(\text{Aut}(X, E)\). From lemma 3.5 in [W], we have

\[(2.4) \quad \frac{d^2}{dt^2} Z(e^{ita}) = \int_{\iota(X)} ||\tilde{a}||^2 dvol_X,\]

where \(\tilde{a}\) is the vector field on \(Gr(r, N)\) generated by the infinitesimal action of \(a\) and \(||\tilde{a}||\) is the Fubini-Study metric on \(Gr(r, N)\). Suppose that \(H\) is a balanced metric. Therefore it is a minimum for the functional \(Z\). This implies that

\[\frac{d^2}{dt^2} Z(e^{ita}) = 0.\]

Hence \(2.4\) implies that \(\tilde{a} \equiv 0\). This implies that the one parameter family \(\{e^{ita}\}\) fixes \(X\) and therefore it is a one parameter family of
endomorphisms of $E$. Thus, simplicity of $E$ implies that $e^{ita} = I_E$ which basically means that $a = 0$ and therefore $H = H_\infty$.

3. Proof of Theorem

In this section, we follow Sano’s argument in ([S, Section 3]) very closely. Let $s_1, ..., s_N$ be a basis for $H^0(E)$. Using this basis, we can view elements of $M$ as $N \times N$ matrices. Now using this identification, we state the following definition stated in Sano ([S]).

**Definition 3.1.** A subset $U \subseteq M$ is called bounded if there exists a number $R > 1$, satisfying the following:

For any $H \in U$, there exists a positive number $\gamma_H$ so that

\begin{equation}
\gamma_H \leq \min \frac{|H(\xi)|}{|\xi|} \leq \max \frac{|H(\xi)|}{|\xi|} \leq \gamma_H R.
\end{equation}

Note that boundedness does not depend on the choice of the basis. Also notice that $\min \frac{|H(\xi)|}{|\xi|}$ is the smallest eigenvalue of the matrix $[H(s_i, s_j)]$ and $\max \frac{|H(\xi)|}{|\xi|}$ is the largest eigenvalue of the matrix $[H(s_i, s_j)]$.

From the definition, one can see that $U$ is bounded if and only if there exists $R > 1$ satisfying the following:

For any $H \in U$, there exists a positive number $\gamma_H$ so that

\[ ||[H(s_i, s_j)]||_{op} \leq \gamma_H R \text{ and } ||[H(s_i, s_j)]^{-1}||_{op} \leq \gamma_H^{-1} R. \]

**Proposition 3.1.** Any bounded sequence $H_i$ has a subsequence $H_{n_i}$ such that $\gamma_{n_i}^{-1}H_{n_i}$ converges to some point in $M$. Here $\gamma_i = \gamma_{H_i}$ in definition 3.1.

**Proof.** The sequence $\gamma_{n_i}^{-1}H_{n_i}$ is a bounded sequence in the space of $N \times N$ matrices with respect to the standard topology. Hence the proposition follows from the fact that the closure of bounded sets are compact.

Notice that the standard topology in the space of $N \times N$ matrices is induced by the standard Euclidean norm. Since all norms on a finite dimensional vector space are equivalent, we can use the operator norm on the space of $N \times N$ matrices. Therefore a sequence $\{H_\alpha\}$ in $M$ converges to $H \in M$ if and only if

\[ ||H_\alpha(s_i, s_j) - H(s_i, s_j)||_{op} \to 0 \text{ as } \alpha \to 0. \]
Lemma 3.2. The set $U \subseteq M$ is bounded if and only if there exists a number $R > 1$ so that for any $H \in U$, we have
\[
\frac{1}{R} \leq \min \frac{|\tilde{H}(\xi)|}{|\xi|} \leq \max \frac{|\tilde{H}(\xi)|}{|\xi|} \leq R,
\]
where $\tilde{H} = (\det(H))^{-\frac{1}{N}} H$.

Proof. Assume that $U$ is bounded. So by definition there exists a number $R > 1$, satisfying the following:

For any $H \in U$, there exists a positive number $\gamma_H$ so that
\[
\frac{\gamma_H}{R} \leq \min \frac{|H(\xi)|}{|\xi|} \leq \max \frac{|H(\xi)|}{|\xi|} \leq \gamma_H R
\]
Let $H$ be an element of $U$. Without loss of generality we can assume that $H(s_i, s_j) = e^{\lambda_i} \delta_{ij}$ and $\lambda_1 \leq \ldots \leq \lambda_N$. For any $i$, we have
\[
\frac{\gamma_H}{R} \leq e^{\lambda_i} \leq \gamma_H R.
\]
This implies that $\gamma_H \leq Re^{\lambda_i}$ and $\gamma_H \geq R^{-1} e^{\lambda_i}$. Therefore
\[
e^{\lambda N} \leq \gamma_H R \leq R^2 e^{\lambda_i},
\]
and
\[
e^{\lambda_1} \geq \gamma_H R^{-1} \geq R^{-2} e^{\lambda_i},
\]
for any $1 \leq i \leq N$. Hence
\[
(det(H))^{-\frac{1}{N}} e^{\lambda_N} = e^{\lambda_N - \frac{\sum_{i} \lambda_i}{N}} = \left( \prod e^{\lambda_N - \lambda_i} \right) \frac{1}{N} \leq R^2.
\]
and
\[
(det(H))^{-\frac{1}{N}} e^{\lambda_1} = e^{\lambda_1 - \frac{\sum_{i} \lambda_i}{N}} = \left( \prod e^{\lambda_1 - \lambda_i} \right) \frac{1}{N} \geq R^{-2}.
\]

Let $H_0$ be an element in $M$. We define the sequence $\{H_n\}$ by $H_n = \text{Hilb} \circ FS(H_{n-1})$.

Lemma 3.3. If $\{H_n\}$ is a bounded sequence in $M$, then $det(H_n)$ is bounded and
\[
det(H_{n+1}H_n^{-1}) \to 1 \text{ as } n \to \infty.
\]

Proof. $\tilde{Z}(H_n)$ is bounded since the sequence $\{H_n\}$ is bounded. On the other hand, lemma 2.6 implies that the sequences $Z(H_n)$ and $\log det(H_n)$ are decreasing. So, $\log det(H_n)$ is bounded and decreasing. Hence, it converges to some real number. This implies that $det(H_{n+1}H_n^{-1}) \to 1$ as $n \to \infty$. 

\[ \square \]
Lemma 3.4. Assume \( \{H_n\} \) is a bounded sequence in \( M \). Let \( H \) be a fixed element of \( M \) and \( s_1^{(l)}, ..., s_N^{(l)} \) be an orthonormal basis with respect to \( H \), so that the matrix \( [H(s_i^{(l)}, s_j^{(l)})] \) is diagonal. Then

\[
\frac{N}{\sqrt{r}} \int_X |s_i^{(l)}|_h^2 \, dvol_X \to 1 \quad \text{as} \quad l \to \infty,
\]

where \( h_n = FS(H_n) \).

Proof. Let \( \hat{s}_1^{(l)}, ..., \hat{s}_N^{(l)} \) be an orthonormal basis with respect to \( H \), so that \( H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_j^{(l)}) \) is diagonal. Hence

\[
\det[H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_j^{(l)})] = \prod_{i=1}^{N} H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_1^{(l)}).
\]

Lemma 3.3 implies that

\[
\det[H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_j^{(l)})] \to 1.
\]

On the other hand, Lemma 2.6 implies that

\[
\text{tr}[H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_j^{(l)})] = N.
\]

We define \( A_l(i) = H_{l+1}(\hat{s}_i^{(l)}, \hat{s}_i^{(l)}) \). Therefore, we have

\[
(3.2) \quad \prod_{i=1}^{N} A_l(i) \to 1 \quad \text{as} \quad l \to \infty,
\]

\[
(3.3) \quad \sum_{i=1}^{N} A_l(i) = N, \quad \text{for any} \ 1 \leq l \leq N.
\]

We claim that for any \( i \),

\[
(3.4) \quad A_l(i) \to 1 \quad \text{as} \quad l \to \infty.
\]

Suppose that for some \( 1 \leq \alpha \leq N \), \( \{A_l(\alpha)\} \) does not converge to 1 as \( l \to \infty \). This means that there exists a positive number \( \epsilon > 0 \) and a subsequence \( \{A_{l_q}(\alpha)\} \) such that

\[
(3.5) \quad |A_{l_q}(\alpha) - 1| \geq \epsilon.
\]

On the other hand, (3.3) implies that \( A_l(i) \leq N \) since \( A_l(i) \geq 0 \) and therefore the sequences \( \{A_l(i)\} \) are bounded for any \( 1 \leq i \leq N \). Hence there exist nonnegative numbers \( A(1), ..., A(N) \) and a subsequence \( \{l_{q_j}\} \) so that

\[
(3.6) \quad A_{l_{q_j}}(i) \to A(i) \quad \text{as} \quad j \to \infty.
\]
Therefore, (3.2), (3.3) and (3.6) imply that
\[ \prod_{i=1}^{N} A(i) = 1 \quad \text{and} \quad \sum_{i=1}^{N} A(i) = N. \]
By arithmetic-geometric mean inequality, we always have
\[ \left( \prod_{i=1}^{N} A(i) \right)^{\frac{1}{N}} \leq \frac{1}{N} \sum_{i=1}^{N} A(i) \]
and equality holds if and only if all \( A_i \)'s are equal. Since equality holds in this case, we conclude that \( A(1) = \ldots = A(N) = 1 \). In particular
\[ A_{l,j}(\alpha) \to 1 \quad \text{as} \quad j \to \infty, \]
which contradicts (3.5). This implies that \( H_{l+1}(\bar{s}_l^{(i)}, \bar{s}_l^{(j)}) \to 1 \) for all \( i \).

On the other hand, there exists \( [a_{l,j}] \in U(N) \) such that \( s_i^{(l)} = \sum_{j=1}^{N} a_{l,j} s_j \). Since \( U(N) \) is compact, we can find a subsequence of \( [a_{l,j}] \) which converges to an element of \( U(N) \). Without loss of generality, we can assume that there exists \( [a_{ij}] \in U(N) \) such that \( a_{l,j} \to a_{ij} \) as \( l \to \infty \). We have,
\[ H_{l+1}(s_i^{(l)}, s_i^{(l)}) = \sum_{j=1}^{N} a_{ij} a_{ik} H_{l+1}(\bar{s}_j^{(l)}, \bar{s}_k^{(l)}) \to \sum_{j=1}^{N} |a_{ij}|^2 = 1 \]

Proposition 3.5. (cf. [S, Proposition ] ) If \( \{H_{n}\} \) is a bounded sequence in \( M \), then for any \( H \in M \) and any \( \epsilon > 0 \),
\[ \tilde{Z}(H) > \tilde{Z}(H_n) - \epsilon, \]
for sufficiently large \( n \).

Proof. Let \( s_1^{(l)}, \ldots, s_N^{(l)} \) be an orthonormal basis with respect to \( H_l \) such that \( H(s_i^{(l)}, s_j^{(l)}) = \delta_{ij} e^{\lambda_i^{(l)}} \). We fix a positive integer \( l \). Define \( H_l(s_i^{(l)}, s_j^{(l)}) = \delta_{ij} e^{\lambda_i^{(l)}} \). We have \( H_0 = H_l \) and \( H_1 = H \). Let \( f_l(t) = f(t) = \tilde{Z}(H_l) \). We have
\[ f(1) - f(0) = \int_0^1 f'(t) \, dt = \int_0^1 \left( f'(0) + \int_0^t f''(s) \, ds \right) \, dt \]
\[ = f'(0) + \int_0^1 \int_0^1 f''(s) \, ds \, dt \geq f'(0), \]
since \( \tilde{Z} \) is convex along geodesics. On the other hand, we have
\[ f'(t) = \frac{d}{dt} \left( -I(FS(H_t)) + \frac{V_{rN}}{N} \log \det(H_t) \right) \]
\[-\int_X \frac{d}{dt}(FS(H_t))\,dvol_X + \frac{Vr}{N} \sum \lambda_i^{(l)} \]

Therefore,

\[(3.8) \quad f'_l(0) = -\int_X \left( \sum \lambda_i^{(l)} |s_i^{(l)}|_h^2 \right) dvol_X + \frac{Vr}{N} \sum \lambda_i^{(l)}, \]

where \( h_l = FS(H_l) \).

We have that \( e^{-\lambda_i^{(l)}} s_1^{(l)}, ..., e^{-\lambda_i^{(l)}} s_N^{(l)} \) is an orthonormal basis with respect to \( H \) for any \( l \). Hence lemma 3.2 implies that there exists \( R > 1 \) so that

\[
\frac{(\det(H_i))^{\frac{1}{N}}}{R} < H_l(e^{-\lambda_i^{(l)}} s_1^{(l)}, e^{-\lambda_i^{(l)}} s_1^{(l)}) < (\det(H_i))^{\frac{1}{N}} R, 
\]

for any \( i \) and \( l \). Therefore

\[
\frac{1}{N} \log(\det(H_i)) - \log R < -\lambda_i^{(l)} < \frac{1}{N} \log(\det(H_i)) + \log R.
\]

This implies that \( \{\lambda_i^{(l)}\} \) is bounded since \( \{\det(H_i)\} \) is bounded by Lemma 3.3. Hence (3.8) implies that \( f'_l(0) \to 0 \), as \( l \to \infty \).

\[ \square \]

**Corollary 3.6.** If \( \{H_n\} \) is a bounded sequence in \( M \), then

\[ \bar{Z}(H_n) \to \inf \{ \bar{Z}(H) \mid H \in M \}. \]

**Proof of Theorem 1.1.** As before, fix \( H_0 \in M \) and an orthonormal basis \( s_1, ..., s_N \) for \( H^0(X, E) \) with respect to the metric \( H_0 \). As in Section 2, let

\[ M_0 = \{ H \in M \mid \det[H(s_i, s_j)] = 1 \}. \]

Assume that there exists a balanced metric on \( E \). Since the balanced metric is unique up to a positive constant, there exists a unique balanced metric \( H_\infty \in M_0 \). As before, for any \( H \in M \), we define

\[ \tilde{H} = (\det H)^{-\frac{1}{N}} H. \]

Clearly \( \tilde{H} \in M_0 \) and

\[ \tilde{Z}(\tilde{H}) = \tilde{Z}(H) = Z(\tilde{H}). \]

Since there exists a balanced metric on \( E \), theorem 2.4 implies that the functional \( Z|_{M_0} \) is proper and bounded from below. Hence the sequence \( Z(\tilde{H}_n) \) is a bounded sequence in \( \mathbb{R} \) since the sequence \( \tilde{Z}(H_n) = Z(\tilde{H}_n) \)
is decreasing. Therefore the sequence \( \{ \tilde{H}_n \} \) is bounded in \( M_0 \) since \( Z_{|M_0} \) is proper. We claim that

\[
\tilde{H}_n \rightarrow H_\infty \quad \text{as} \quad n \rightarrow \infty.
\]

Suppose that the sequence \( \{ \tilde{H}_n \} \) does not converge to \( H_\infty \). Then there exists \( \epsilon > 0 \) and a subsequence \( \{ H_{n_j} \} \) such that

\[
||\tilde{H}_{n_j} - H_\infty||_{op} \geq \epsilon.
\]  (3.9)

On the other hand, we know that the sequence \( \{ \tilde{H}_{n_j} \} \) is bounded. Therefore there exist a subsequence \( \{ \tilde{H}_{n_{jq}} \} \) and an element \( \hat{H} \in M \) such that

\[
\tilde{H}_{n_{jq}} \rightarrow \hat{H} \quad \text{as} \quad q \rightarrow \infty.
\]

Therefore,

\[
1 = \det[\tilde{H}_{n_{jq}}(s_\alpha, s_\beta)] \rightarrow \det[\hat{H}(s_\alpha, s_\beta)] \quad \text{as} \quad q \rightarrow \infty,
\]

which implies that \( \hat{H} \in M_0 \). Now, corollary 3.6 implies that

\[
\tilde{Z}(\tilde{H}_{n_{jq}}) = \tilde{Z}(H_{n_{jq}}) \rightarrow \inf\{ \tilde{Z}(H) \mid H \in M \}.
\]

Hence,

\[
\tilde{Z}(\hat{H}) = \inf\{ \tilde{Z}(H) \mid H \in M \}.
\]

This implies that \( \hat{H} \) is a balanced metric and therefore \( H_\infty = \hat{H} \) by lemma 2.7. This contradicts (3.9). Thus \( \tilde{H}_n \rightarrow H_\infty \) as \( q \rightarrow \infty \).

Now lemma 3.3 implies that \( \log \det(H_n) \) is bounded. The sequence \( \{ \log \det(H_n) \} \) is bounded and decreasing. Therefore there exists \( b \in \mathbb{R} \) such that

\[
\log \det(H_n) \rightarrow b \quad \text{as} \quad n \rightarrow \infty.
\]

Hence \( \det(H_n) \) converges to the positive real number \( e^b \). Thus

\[
H_n \rightarrow e^{\frac{b}{n}} H_\infty \quad \text{as} \quad n \rightarrow \infty.
\]

\( \square \)

References

[D1] S. K. Donaldson, Scalar curvature and projective embeddings. I, J. Differential Geom. 59 (2001), no. 3, 479–522.

[D2] S. K. Donaldson, Scalar curvature and projective embeddings. II, Q. J. Math. 56 (2005), no. 3, 345–356.

[D3] S. K. Donaldson, Some numerical results in complex differential geometry arXiv:math/0512625v1 [math.DG]

[G] D. Gieseker, On the moduli of vector bundles on an algebraic surface, Ann. of Math. (2) 106 (1977), no. 1, 45–60.
[DKLR] M. R. Douglas; R. L. Karp; S. Lukic and R. Reinbacher, Numerical solution to the Hermitian Yang-Mills equation on the Fermat quintic. J. High Energy Phys. 2007, no. 12, 083, 24 pp.

[KN] G. Kempf and L. Ness, The length of vectors in representation spaces, in Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), 233–243, Lecture Notes in Math., 732, Springer, Berlin.

[L] H. Luo, Geometric criterion for Gieseker-Mumford stability of polarized manifolds, J. Differential Geom. 49 (1998), no. 3, 577–599.

[PS] D. H. Phong and J. Sturm, Stability, energy functionals, and Kähler-Einstein metrics, Comm. Anal. Geom. 11 (2003), no. 3, 565–597.

[S] Y. Sano, Numerical algorithm for finding balanced metrics, Osaka J. Math. 43 (2006), no. 3, 679–688.

[W] X. Wang, Balance point and stability of vector bundles over a projective manifold, Math. Res. Lett. 9 (2002), no. 2-3, 393–411.

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218

E-mail address: seyyedali@math.jhu.edu