Distribution of Energy Levels
of a Quantum Free Particle on a
Surface of Revolution

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Abstract. We prove that the error term $\Lambda(R)$ in the Weyl asymptotic formula

$$\#\{E_n \leq R^2\} = \frac{\text{Vol } M}{4\pi} R^2 + \Lambda(R),$$

for the Laplace operator on a surface of revolution $M$ satisfying a twist hypothesis, has the form $\Lambda(R) = R^{1/2} F(R)$ where $F(R)$ is an almost periodic function of the Besicovitch class $B^2$, and the Fourier series of $F(R)$ in $B^2$ is $\sum_\gamma A(\gamma) \cos(\gamma R - \phi)$ where the sum goes over all closed geodesics on $M$, and $A(\gamma)$ is computed through simple geometric characteristics of $\gamma$. We extend this result to surfaces of revolution, which violate the twist hypothesis and satisfy a more general Diophantine hypothesis. In this case we prove that $\Lambda(R) = R^{2/3} \Phi(R) + R^{1/2} F(R)$, where $\Phi(R)$ is a finite sum of periodic functions and $F(R)$ is an almost periodic function of the Besicovitch class $B^2$. The Fourier series of $\Phi(R)$ and $F(R)$ are computed.
1. Introduction

Let $M$ be a two-dimensional smooth compact manifold which is homeomorphic to a sphere, and which is a surface of revolution in $\mathbb{R}^3$, with an axis $A$ and poles $N$ and $S$ (see Fig. 1). The geodesic flow on $M$ is a classical integrable system due to the Clairaut integral,

$$r \sin \alpha = \text{const}.$$  \hfill (1.1)

In the present work we are interested in high energy levels of the corresponding quantum system,

$$-\Delta u_n = E_n u_n.$$  \hfill (1.2)

Let $s$ be the normal coordinate (the length of geodesic) along meridian and

$$r = f(s), \quad 0 \leq s \leq L,$$  \hfill (1.3)

be the equation of $M$, where $r$ is the radial coordinate. Then

$$\Delta = f(s)^{-1} \frac{\partial}{\partial s} \left( f(s) \frac{\partial}{\partial s} \right) + f(s)^{-2} \frac{\partial^2}{\partial \varphi^2},$$  \hfill (1.4)

where $\varphi$ is the angular coordinate.

We will assume that $f(s)$ has a simple structure, so that

$$f'(s) \neq 0, \quad s \neq s_{\text{max}}; \quad f''(s_{\text{max}}) \neq 0,$$  \hfill (1.5)

where

$$f(s_{\text{max}}) = \max_{0 \leq s \leq L} f(s) \equiv f_{\text{max}}.$$  \hfill (1.5)

For normalization we put $f_{\text{max}} = 1$. Another assumption on $M$ is the following twist hypothesis.

Consider the equator on $M$,

$$\gamma_E = \{s = s_{\text{max}}, \quad 0 \leq \varphi \leq 2\pi\},$$

(we do not assume that $M$ is symmetric with respect to $\gamma_E$, but we still call $\gamma_E$ the equator keeping a visual interpretation of objects on $M$), and a geodesic $\gamma$ which starts at $x_0 = (s = s_{\text{max}}, \varphi = 0) \in \gamma_E$ at some angle $-\pi/2 < \alpha_0 < \pi/2$ to the direction to the north. The Clairaut integral on $\gamma$ is $I = \sin \alpha_0$, and we can parametrize $\gamma$ by $-1 < I < 1$: $\gamma = \gamma(I)$. It follows from the Clairaut
integral that $\gamma(I)$ oscillates between two parallels, $s = s_+$ and $s = s_-$, where $f(s_-) = f(s_+) = I$, so $\gamma(I)$ intersects $\gamma_E$ infinitely many times. Let $x_n$ be the $n$–th intersection of $\gamma$ with $\gamma_E$, $n \in \mathbb{Z}$. Define

$$\tau(I) = |\gamma[x_0, x_2]|,$$

the length of $\gamma$ between $x_0$ and $x_2$, and

$$\omega(I) = (2\pi)^{-1}(\varphi(x_2) - \varphi(x_0)),$$

the phase of $\gamma$ between $x_0$ and $x_2$ (see Fig. 1). Observe that $\omega(I)$ is defined mod 1. To define $\omega(I)$ uniquely, we choose a continuous branch of $\omega(I)$ starting at $\omega(0) = 0$. Then $\omega(-I) = -\omega(I)$ and for $I \geq 0$,

$$\omega(I) = \pi^{-1} \int_{s_-}^{s_+} \frac{d\varphi}{ds} ds - 1. \quad (1.6)$$

Define $\tau(1) = \lim_{I \to 1^-} \tau(I)$ and $\omega(1) = \lim_{I \to 1^-} \omega(I)$.

It is easy to see that a finite geodesic $\gamma$ with the Clairaut integral $0 < I < 1$ is closed iff $\omega(I)$ is rational. More precisely, let $n(\gamma)$ denote the number of revolutions of a closed geodesic $\gamma$ around the axis $A$ and $m(\gamma)$ denote the number of oscillations of $\gamma$ along meridian. Then

$$\omega(I) = (n(\gamma)/m(\gamma)) - 1.$$ 

To facilitate formulation of subsequent results we take the convention that a finite geodesic $\gamma$ with $I = 1$, which goes along the equator, is closed iff both $n(\gamma)$ and

$$m(\gamma) \equiv \frac{n(\gamma)}{\omega(1) + 1}$$

are integers.

**Twist Hypothesis (TH).** $\omega'(I) \neq 0$, $\forall I \in [0, 1]$.

To illustrate TH consider an ellipsoid of revolution,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1.$$ 

If $a < b$ (oblong ellipsoid), then $\omega'(I) > 0$, if $a > b$ (oblate ellipsoid), then $\omega'(I) < 0$; so TH holds in both cases. Curves (a) and (b) on Fig.2 are the graphs of $\omega(I)$ found with the help of computer for the ellipsoids of revolution with $a = 1$, $b = 2$ and $a = 2$, $b = 1$, respectively. The cross–sections of the ellipsoids are shown in the lower part of Fig.2. TH is violated for a sphere ($a = b$), when
\( \omega'(I) \equiv 0. \) TH can be also violated for a bell-like shape of \( M \) shown on Fig.2 (see the cross-section (c) and the graph (c) of \( \omega(I) \) on this figure) and in some other cases.

Let

\[ N(R) = \# \{ E_n \leq R^2 \} \quad (1.7) \]

be the counting function of \( E_n \). Then the Weyl law says that

\[ N(R) = \frac{\text{Vol} M}{4\pi} R^2 + \Lambda(R) \]

where \( \Lambda(R) = o(R^2), \ R \to \infty \). A general estimate of Hörmander [Hör1] gives

\[ \Lambda(R) = O(R). \]

This estimate is sharp for \( S^2 \) and some other degenerate surfaces for which closed geodesics cover a set of positive Liouville measure in the phase space. If the Liouville measure of the union of all closed geodesics in the phase space is 0, then as was shown by Duistermaat and Guillemin [DG],

\[ \Lambda(R) = o(R). \]

For surfaces of negative curvature Selberg and Bérard [Bér] proved a better estimate:

\[ \Lambda(R) = O(R/\log R), \]

and it is a very difficult open problem to show that \( \Lambda(R) = O(R^{1-\varepsilon}) \) for some \( \varepsilon > 0 \), even in the case of constant negative curvature (see recent works [Sar], [LS] and [HR] where statistics of eigenvalues and eigenfunctions of the Laplace operator on surfaces of constant negative curvature is discussed).

For a flat torus \( \Lambda(R) \) reduces to the error term of the classical circle problem, and the best estimate here is due to Huxley [Hux]:

\[ \Lambda(R) = O(R^{46/73}(\log R)^{315/146}). \]

A well-known conjecture of Hardy [Har1]

\[ \Lambda(R) = O(R^{(1/2)+\varepsilon}), \ \forall \varepsilon > 0, \]

is probably also a very difficult open problem. On the other hand Hardy proved [Har2] that

\[ \limsup_{R \to \infty} R^{-1/2}|\Lambda(R)| = \infty, \]
so \((1/2) + \varepsilon\) is the best possible exponent.

Colin de Verdière [CdV1,CdV2] proved that for a generic surface of revolution of simple structure

\[ \Lambda(R) = O(R^{2/3}). \quad (1.8) \]

We prove in the present paper the following result:

**Theorem 1.1.** Assume that \(M\) is a surface of revolution of simple structure, and \(M\) satisfies \(TH\). Then

\[ N(R) = \frac{\text{Vol } M}{4\pi} R^2 + R^{1/2} F(R), \quad (1.9) \]

where \(F(R)\) is an almost periodic function of the Besicovitch class \(B^2\), and the Fourier series of \(F(R)\) in \(B^2\) is

\[ F(R) = \sum_{\text{closed geodesics } \gamma} A(\gamma) \cos(|\gamma| R - \phi), \quad (1.10) \]

where summation goes over all closed (in general, multiple) oriented geodesics \(\gamma \neq 0\) on \(M\), \(\phi = (\pi/2) + (\pi/4) \text{sgn } \omega'(I)\), and

\[ A(\gamma) = \pi^{-1}(-1)^{m(\gamma)} |\omega'(I)|^{-1/2} m(\gamma)^{-3/2} \]

\[ = \pi^{-1}(-1)^{m(\gamma)} |\omega'(I)|^{-1/2} \tau(I)^{3/2} |\gamma|^{-3/2}, \quad I = I(\gamma). \quad (1.11) \]

In Theorem 1.2 we extend Theorem 1.1 to the case when \(TH\) is violated. In this case we introduce

**Diophantine Hypothesis** (DH). Assume that \(\omega(I)\) has at most finitely many critical points

\(0 < I_1 < \cdots < I_K < 1\) (so that \(I = 0, 1\) are not critical) with \(\omega'(I_k) = 0\), and \(\omega''(I_k) \neq 0\), \(k = 1, \ldots, K\). Assume, in addition, that for every \(k = 1, \ldots, K\), \(\omega(I_k)\) is either rational or Diophantine in the sense that \(\exists 1 > \zeta > 0 \text{ and } C > 0\) such that

\[ \left| \frac{\omega(I_k) - p}{q} \right| \geq \frac{C}{q^{2+\zeta}}, \quad \forall \frac{p}{q} \in \mathbb{Q}. \quad (1.12) \]

**Theorem 1.2.** Assume that \(M\) is a surface of revolution of simple structure and DH holds. Then

\[ N(R) = \frac{\text{Vol } M}{4\pi} R^2 + R^{2/3} \sum_{k: \omega(I_k) \in \mathbb{Q}} \Phi_k(R) + R^{1/2} F(R), \quad (1.13) \]
where $\Phi_k(R)$ are bounded periodic functions, and $F(R)$ is an almost periodic function of the Besicovitch class $B^2$. The Fourier series of $\Phi_k(R)$ is

$$\Phi_k(R) = (1/2)3^{-2/3}\Gamma(2/3)\pi^{-4/3}\tau(I_k)^{4/3}\sum_{\gamma: I(\gamma)\neq I_k} (-1)^{m(\gamma)}|\gamma|^{-4/3}\sin(|\gamma|R),$$

and the Fourier series of $F(R)$ is

$$F(R) = \sum_{\gamma: I(\gamma)\neq I_1,\ldots,I_k} A(\gamma)\cos(|\gamma|R - \phi(\gamma)),$$

where $\phi(\gamma) = (\pi/2) + (\pi/4)\text{sgn}\omega'(I), I = I(\gamma)$, and $A(\gamma)$ is given in (1.11).

In the works [H-B], [BCDL], [Ble1], [Ble2] and [BL] some general results were proved on the existence and properties of a limit distribution of any almost periodic function of the Besicovitch class $B^2$ (see especially Theorems 4.1–4.3 in [Ble1] and Theorems 3.1, 3.3 in [Ble2]). Theorem 1.2 combined with these results lead us to the following

**Corollary.** Assume that $M$ is a surface of revolution of simple structure and $M$ satisfies TH or, more generally, DH. Then the normalized error function $F(R)$ in the formulas (1.9) and (1.13) has a limit distribution $\nu(dt)$, i.e., for every bounded continuous function $g(t)$ on the line,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(F(R)) dR = \int_{-\infty}^\infty g(t) \nu(dt).$$

If, in addition, the lengths of all primitive closed geodesics on $M$ with $I \geq 0$ are linearly independent over $\mathbb{Z}$, then $\nu(dt)$ is absolutely continuous and the density function $p(t) = \nu(dt)/dt$ is an entire function of $t$ which satisfies on the real axis the estimates

$$0 \leq p(t) \leq C\exp(-\lambda t^4), \quad C, \lambda > 0,$$

$$P(-t), 1 - P(t) \geq C'\exp(-\lambda' t^4), \quad t \geq 0; \quad C', \lambda' > 0; \quad P(t) = \int_{-\infty}^t p(t') dt'.$$

Observe that Theorem 1.1 is a particular case of Theorem 1.2, so we need to prove only Theorem 1.2. The plan of the remainder of the paper is as follows. In Section 2 we present a theorem of Colin de Verdière and show how with the help of this theorem to reduce Theorem 1.2 to a lattice–point problem for the Bohr–Sommerfeld quasi–classical approximation. Section 3 is auxiliary: here we investigate the asymptotics at infinity of the Fourier transform of the characteristic function of a plane domain with non–degenerate inflection points and angular points. In Section 4 we prove the lattice–point version of Theorem 1.2 for the number of lattice points inside a dilated plane oval with finitely many points of inflection. Section 5 is technical, and here we prove some lemmas used in Section 4. In Section 6 we prove Theorem 1.2. Finally in an Appendix we prove a 2/3–estimate for ovals with semicubic singularity. This estimate is used in the main part of the paper.
2. Quasi–Classical Approximation

Colin de Verdière proved in [CdV2] the following result:

**Theorem CdV.** If $M$ is a surface of revolution of simple structure then

\[
\text{Spectrum } (-\Delta) = \{ E_{kl} = Z(k + (1/2), l); \ k, l \in \mathbb{Z}, |l| \leq k \} \tag{2.1}
\]

with $Z(p) = Z(p_1, p_2) \in C^\infty(\mathbb{R}^2)$ such that

\[
Z(p) = Z_2(p) + Z_0(p) + O(|p|^{-1}), \quad |p| \to \infty, \tag{2.2}
\]

where

\[
Z_2(p), Z_0(p) \in C^\infty(\mathbb{R}^2 \setminus \{0\}); \quad Z_2(p) > 0, \quad p \neq 0,
\]

and

\[
Z_j(\lambda p) = \lambda^j Z_j(p), \quad \forall \lambda > 0, p \in \mathbb{R}^2, \quad j = 0, 2.
\]

In addition, in the sector $\{ p_1 \geq |p_2| \}$, $Z_2(p)$ satisfies the equation

\[
\pi^{-1} \int_a^b \sqrt{Z_2(p) - p_2^2 f^{-2}(s)} ds = p_1 - |p_2|, \tag{2.4}
\]

where $a, b$ are the turning points, i.e.,

\[
Z_2(p) - p_2^2 f^{-2}(s) = 0 \quad \text{for} \quad s = a, b. \tag{2.5}
\]

It is to be noted that the Bohr–Sommerfeld quantization rule is

\[
\pi^{-1} \int_a^b \sqrt{E_{kl} - l^2 f^{-2}(s)} ds = k + (1/2) - |l|, \quad k \geq |l|, \tag{2.6}
\]

which is equivalent to the approximation

\[
E_{kl} = Z_2(k + (1/2), l). \tag{2.7}
\]

(2.1) implies that

\[
N(R) = \#\{ E_{kl} \leq R^2 \} = \#\{(k, l): Z(k + (1/2), l) \leq R^2, |l| \leq k \}. \tag{2.8}
\]
Define

$$N_{BS}(R) = \#\{(k, l) : Z_2(k + (1/2), l) \leq R^2, |l| \leq k\} \quad (2.9)$$

(BS stands for Bohr–Sommerfeld).

**Theorem 2.1.**

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |N(R) - N_{BS}(R)|^2 R^{-1} dR = 0. \quad (2.10)$$

**Proof.** Define

$$M = \{n = (k + (1/2), l) : k, l \in \mathbb{Z}; |l| \leq k\}. \quad (2.11)$$

Then

$$N(R) - N_{BS}(R) = \sum_{n \in M} (\chi(n; R) - \chi_{BS}(n; R)), \quad (2.12)$$

where $\chi(p; R)$ and $\chi_{BS}(p; R)$ are the characteristic functions of the domains $\{Z(p) \leq R^2\}$ and $\{Z_2(p) \leq R^2\}$, respectively. Hence

$$\frac{1}{T} \int_0^T |N(R) - N_{BS}(R)|^2 R^{-1} dR = \sum_{n, n' \in M} I(n, n') \quad (2.13)$$

with

$$I(n, n') = \frac{1}{T} \int_0^T (\chi(n; R) - \chi_{BS}(n; R)) (\chi(n'; R) - \chi_{BS}(n'; R)) R^{-1} dR. \quad (2.14)$$

Now, $\exists C > 0$ such that

$$\chi(n; R) - \chi_{BS}(n; R) = 0 \quad \text{if} \quad \text{dist } \{n, \Gamma_R\} > C|n|^{-1}, \quad (2.15)$$

where

$$\Gamma_R = \{p \in \mathbb{R}^2 : Z_2(p) = R^2\} = R\Gamma, \quad \Gamma = \{p \in \mathbb{R}^2 : Z_2(p) = 1\}. \quad (2.16)$$

Indeed,

$$\sup_{p \in \mathbb{R}^2} |Z(p) - Z_2(p)| = C_0 < \infty, \quad (2.17)$$

so if $Z(p) = R^2$ then for $|\varepsilon| < 1/2$,

$$|Z_2(p(1 + \varepsilon)) - R^2| \geq |Z_2(p(1 + \varepsilon)) - Z_2(p)| - |Z_2(p) - Z(p)| \geq C_1|p|^2\varepsilon - C_0.$$

Hence if $\varepsilon = C|p|^{-2}$ with $C = 2C_0C_1^{-1}$, then

$$\pm (Z_2(p(1 \pm \varepsilon)) - R^2) \geq CC_1 - C_0 = C_0 > 0.$$
and \( \exists \varepsilon_0 < \varepsilon \) such that \( Z_2(p(1 + \varepsilon_0)) = R^2, \) \( |p\varepsilon_0| \leq C|p|^{-1} \), which implies (2.15).

Define \( X(p) = (Z_2(p))^{1/2} \). Then \( X(p) > 0 \) and

\[
X(\lambda p) = \lambda X(p) \quad \forall \lambda > 0, p \in \mathbb{R}^2.
\]

(2.15) implies that

\[
I(n, n') = 0 \quad \text{if} \quad |X(n) - X(n')| > C_0|n|^{-1}.
\]

(2.18)

In addition, (2.14), (2.15) imply that

\[
I(n, n') = 0 \quad \text{if} \quad X(n) \geq 2T,
\]

and \( \forall n, n' \),

\[
|I(n, n')| \leq CT^{-1}|n|^{-2}.
\]

(2.20)

From (2.18)–(2.20)

\[
\sum_{n, n' \in M} I(n, n') \leq C \sum_{n \in M: X(n) \leq 2T} T^{-1}|n|^{-2} \sum_{n' \in M: |X(n) - X(n')| \leq C_0|n|^{-1}} 1.
\]

Due to the 2/3–estimate of Sierpinski–Landau–Randol–Colin de Verdière [Sie], [Lan], [Ran], [CdV1],

\[
\sum_{n' \in M: |X(n) - X(n')| \leq C_0|n|^{-1}} 1 \leq C|n|^{2/3},
\]

hence

\[
\sum_{n, n' \in M} I(n, n') \leq CT^{-1} \sum_{n \in M: X(n) \leq 2T} |n|^{-4/3} \leq C_0T^{-1/3}.
\]

(2.21)

Since (2.13) and (2.21) imply (2.10), Theorem 2.1 is proved.

By (2.9) \( N_{BS}(R) \) is the number of lattice points \( (k + (1/2), l) \) of a shifted square lattice in the sectorial domain

\[
\Omega(R) = \{ p \in \mathbb{R}^2: Z_2(p) \leq R^2, |p_2| \leq p_1 \}.
\]

Observe that \( \Omega(R) \) is \( \Omega = \Omega(1) \) dilated with the coefficient \( R \), so the problem of finding the asymptotics of \( N_{BS}(R) \) as \( R \to \infty \) reduces to a lattice–point problem about the asymptotics of the number of lattice points inside \( R\Omega \), where \( \Omega \) is a sectorial domain between diagonals \( p_2 = \pm p_1 \) which is bounded by the curve \( \Gamma = \{ Z_2(p) = 1 \} \). By (2.4) \( \Gamma \) is the graph of the function

\[
p_1 = |p_2| + \pi^{-1} \int_a^b \sqrt{1 - p_2^2f^{-2}(s)} ds.
\]

Fig.3 shows the curve \( \Gamma \) found with the help of a computer for the surfaces of revolution presented on Fig.2 above.

Sections 3–5 below are devoted to the lattice–point problem for arbitrary sectorial domain between diagonals which is bounded by a “generic” curve \( \Gamma \).
3. Asymptotics of Fourier Transform of Nonconvex Domains

In this section we consider the following auxiliary problem. Let $\Omega$ be a sectorial domain on a plane, which is bounded by two segments $[0, z_0]$, $[0, z_1]$ and by a smooth curve $\Gamma$ which goes from $z_0$ to $z_1$ (see Fig.4). We will assume that $\Gamma$ has at most finitely many points of inflection where the curvature $\sigma(p)$ vanishes, and all the points of inflection are non–degenerate, i.e., $\frac{d\sigma}{ds} \neq 0$ at these points. We are interested in the asymptotics of the Fourier transform

$$\tilde{\chi}(\xi) = \int_{\Omega} \exp(ip\xi) dp,$$

as $|\xi| \to \infty$. With the help of a partition of unity this problem can be reduced to a series of local problems of the following type.

Let $\Gamma$ be a smooth curve near some point $p_0 \in \Gamma$ and $\chi(p) = 1$ from one side of $\Gamma$ and $\chi(p) = 0$ from the other side of $\Gamma$ (locally). Let $\varphi(p) \in C_0^\infty$ be a $C^\infty$–function with compact support near $p^0$. Assume in addition that $\varphi(p) = 1$ in the vicinity of $p^0$. Then the local problem is: what is the asymptotics of

$$\tilde{\chi}(\xi) = \int_{\mathbb{R}^2} \varphi(p)\chi(p) \exp(ip\xi) dp,$$

as $|\xi| \to \infty$?

We will be also interested in the case when $\Gamma$ has an angular point at $p^0$, and we will consider in a sequence the following cases: (a) $\Gamma$ is either convex or concave near $p^0$; (b) $\Gamma$ has an inflection point at $p^0$; (c) $\Gamma$ has an angular point at $p^0$ with one side which is either convex or concave and with the other side which is a straight ray; (d) $\Gamma$ is an angle between two straight rays; and (e) $\Gamma$ is a straight line (see Fig.5).

In the case (a) the answer is the following well–known lemma. Let $\Gamma_0 \subset \Gamma$ be an open arc on $\Gamma$ such that

$$p^0 \in \Gamma_0 \subset \{ p \in \Gamma : \varphi(p) = 1 \},$$

and

$$V = \{ \xi \in \mathbb{R}^2 \setminus \{0\} : \exists p = p(\xi) \in \Gamma_0 \text{ such that } n_\Gamma(p) = |\xi|^{-1}\xi \},$$

where $n_\Gamma(p)$ is the vector of normal to $\Gamma$ at $p \in \Gamma$ which looks in the direction where $\chi(p) = 0$. Define

$$Y(\xi) = \xi \cdot p(\xi), \quad \xi \in \mathbb{R}^2 \setminus \{0\}. \quad (3.1)$$
For the sake of brevity we will denote $\sigma(p(\xi))$ by $\sigma(\xi)$. We will assign a sign to the curvature $\sigma(p)$, so that $\sigma(p) > 0$ if the region $\{\chi(p) = 1\}$ is convex near $p \in \Gamma$, and $\sigma(p) < 0$ if this region is concave near $p$. If $p_2 = f(p_1)$ is the equation of $\Gamma$ near some $\hat{p} \in \Gamma$ in the coordinate system with an orthonormal basis $e_1, e_2$ such that $e_2 = n_{\Gamma}(\hat{p})$, then

$$\sigma(\hat{p}) = -f''(\hat{p}_1).$$

**Lemma 3.1** (see [Hla]). If $\sigma(p) \neq 0$ on $\Gamma_0$ then

$$\chi(\xi) = (2\pi)^{-1/2}|\xi|^{-3/2}|\sigma(\xi)|^{-1/2} \cos(iY(\xi) - \phi)) + O(|\xi|^{-5/2}), \quad \text{if } \xi \in V,$$

where

$$\phi = (\pi/2) + (\pi/4) \text{sgn } \sigma(\xi).$$

Assume now that $p^0$ is a non-degenerate point of inflection, i.e., $\sigma(p^0) = 0$ and $\frac{d\sigma}{ds}(p^0) \neq 0$. In this case $p^0$ is the turning point for $n_{\Gamma}(p)$, so that for small deviations of the direction of $\xi$ from $n_{\Gamma}(p^0)$ we have either two points $p = p_\pm(\xi)$ with $n_{\Gamma}(p) = |\xi|^{-1}\xi$ or no such point at all. Define for $\xi \in V$,

$$\theta(\xi) = \begin{cases} 1 & \text{if } \exists p \in \Gamma_0 \text{ with } n_{\Gamma}(p) = |\xi|^{-1}\xi, \\ 0 & \text{otherwise.} \end{cases}$$

To be definite in the choice of $p_{\pm}(\xi)$ we will assume that $\sigma(p_+(\xi)) > 0$ and $\sigma(p_-(\xi)) < 0$. Let $\sigma_{\pm}(\xi) = \sigma(p_{\pm}(\xi))$ and $Y_{\pm}(\xi) = \xi \cdot p_{\pm}(\xi)$. Let $\alpha(\xi)$ be the angular coordinate of $\xi$. Let finally

$$\text{Ai}(t) = \int_{-\infty}^{\infty} \exp(it\tau + i\tau^3/3)d\tau$$

be the Airy function. Recall that $\text{Ai}(t) \in C^\infty(\mathbb{R}^1)$ and when $t \to \infty$,

$$\text{Ai}(-t) = \pi^{-1/2}t^{-1/4}\cos(\zeta - (\pi/4)) + O(t^{-7/4}),$$

$$\text{Ai}'(-t) = \pi^{-1/2}t^{1/4}\sin(\zeta - (\pi/4)) + O(t^{-5/4}), \quad \zeta = (2/3)t^{3/2},$$

$$|\text{Ai}(t)|, |\text{Ai}'(t)| \leq C \exp(-t),$$

which implies that for $t \neq 0$,

$$|\text{Ai}(t) - \theta(-t)\pi^{-1/2}|t|^{-1/4}\cos(\zeta - (\pi/4))| \leq C|t|^{-7/4},$$

$$|\text{Ai}'(t) - \theta(-t)\pi^{-1/2}|t|^{1/4}\sin(\zeta - (\pi/4))| \leq C|t|^{-5/4},$$

(3.6)
where $\theta(t) = 1$ for $t \geq 0$ and $\theta(t) = 0$ for $t < 0$.

**Lemma 3.2.** Assume that $\sigma(p^0) = 0$ and $\frac{d\sigma}{ds}(p^0) \neq 0$. Then there exist real valued functions $a(\alpha), b(\alpha)$ near $\alpha_0 = \alpha(n_\Gamma(p^0))$ such that $a(\alpha_0) = 0$, $a'(\alpha_0) \neq 0$, $b(\alpha_0) = Y(n_\Gamma(p^0))$ and

\[
\tilde{\chi}(\xi) = -i \exp\left(i|\xi|b(\alpha(\xi))\right)\left\{|\xi|^{-4/3}\text{Ai}\left(|\xi|^{2/3}a(\alpha(\xi))\right)u(\alpha(\xi)) + |\xi|^{-5/3}\text{Ai}'\left(|\xi|^{2/3}a(\alpha(\xi))\right)v(\alpha(\xi))\right\} + O(|\xi|^{-7/3}), \quad \text{if } \xi \in V,
\]

where $u(\alpha), v(\alpha)$ are $C^\infty$ functions and

\[
u(\alpha_0) = \left|\frac{1}{2} \frac{d\sigma}{ds}(p^0)\right|^{1/3}.
\]

**Corollary.** If $\alpha(\xi) \neq \alpha_0$ then

\[
|\chi(\xi) - \theta(\xi)(\chi^+(\xi) + \chi^-(\xi))| \leq C|\xi|^{-5/2}|\alpha(\xi) - \alpha_0|^{-7/4}
\]

where

\[
\chi^\pm(\xi) = |\xi|^{-3/2}2\pi\sigma_\pm(\xi)^{-1/2} \exp(iY_\pm(\xi) - \phi_\pm)), \quad \phi_\pm = (\pi/2) \pm (\pi/4).
\]

**Proof of Corollary.** From (3.6) and (3.7) we obtain that for $\alpha(\xi) = \alpha \neq \alpha_0$,

\[
\chi_m(\xi) = -i \exp\left(i|\xi|b(\alpha)|\xi|^{-3/2}\theta(\xi)\left[u_0(\alpha)\cos(|\xi|a_0(\alpha) - (\pi/4)) + v_0(\alpha)\sin(|\xi|a_0(\alpha) - (\pi/4))\right]
+ O(|\xi|^{-5/2}|\alpha - \alpha_0|^{-7/4})
\]

with some $a_0(\alpha), u_0(\alpha)$ and $v_0(\alpha)$. On the other hand, Lemma 3.1 gives us that for a fixed $\alpha(\xi) = \alpha \neq \alpha_0$,

\[
\chi_m(\xi) = \theta(\xi)(\chi_m^+(\xi) + \chi_m^-(\xi)) + O(|\xi|^{-5/2}), \quad |\xi| \to \infty
\]

Comparing (3.11) with (3.12) we obtain that the main terms in these two asymptotics coincide, hence

\[
\chi_m(\xi) = \theta(\xi)(\chi_m^+(\xi) + \chi_m^-(\xi)) + O(|\xi|^{-5/2}|\alpha - \alpha_0|^{-7/4}),
\]

which proves Corollary.

**Proof of Lemma 3.2.** Consider an orthonormal basis $e_1, e_2$ on the plane with $e_2 = n_\Gamma(p^0)$. Let $p = p^0 + p_1e_1 + p_2e_2$, $\xi = \xi_1e_1 + \xi_2e_2$ and $p_2 = f(p_1)$ be the equation of $\Gamma$ near $p^0$. Observe that
\[ f(0) = f'(0) = f''(0) = 0 \] and we can choose the direction of \( e_1 \) in such a way that \( f''(0) = \frac{ds}{ds}(p^0) \).

Integrating by parts in \( p_2 \) we obtain that

\[
\chi(\xi) = \int_{\mathbb{R}^2} \varphi(p)\chi(p)\exp(ip\xi)dp = -i\xi^{-1}_2 \int_{-\infty}^{\infty} \varphi(p_1, f(p_1)) \exp(ip_1\xi_1 + if(p_1)\xi_2) dp_1
\]

\[ + O(|\xi|^{-N}) = -i\xi_2^{-1} \int_{-\infty}^{\infty} \varphi(p_1, f(p_1)) \exp[i|\xi|\Phi(p_1, \alpha(\xi))] dp_1 + O(|\xi|^{-N}), \]

where \( \Phi(p_1, \alpha) = p_1 \sin(\alpha - \alpha_0) + f(p_1)\cos(\alpha - \alpha_0) \) is a \( C^\infty \) real valued function with

\[
\Phi = \frac{\partial \Phi}{\partial p_1} = \frac{\partial^2 \Phi}{\partial p_1^2} = 0, \quad \frac{\partial^3 \Phi}{\partial p_1^3} = \frac{d\sigma}{ds}(p^0), \quad \text{and} \quad \frac{\partial^2 \Phi}{\partial p_1 \partial \alpha} \neq 0,
\]

at \( p_1 = 0, \alpha = \alpha_0 \). (3.7) follows now from Theorem 7.7.18 in [Hör2]. Lemma 3.2 is proved.

Let us turn now to angular points. So, assume that \( \Gamma \) is a smooth curve near \( p^0 \in \Gamma \) and \( L \) is a straight line which intersects \( \Gamma \) at \( p^0 \) transversally. Then near \( p^0 \), \( \Gamma \) and \( L \) divide the plane into four parts. Let \( \chi(p) = 1 \) in one of these parts and \( \chi(p) = 0 \) in the remainder. Again we are interested in asymptotics of \( \tilde{\chi}(\xi) \) as \( |\xi| \to \infty \).

Define the auxiliary functions

\[
P_{\pm}(y) = (2\pi)^{-1/2} \exp(\pm i\pi/4) \int_{-\infty}^{y} \exp(\mp it^2/2) dt,
\]

which are \( C^\infty \) bounded functions on \( \mathbb{R}^1 \) such that

\[
P_{\pm}(-\infty) = 0, \quad P_{\pm}(0) = 1/2, \quad P_{\pm}(\infty) = 1;
\]

\[
P_+(y) = P_-(y), \quad P_+(y) + P_+(-y) = 1; \quad |P_{\pm}(y) - \theta(y)| \leq C(1 + |y|)^{-1}
\]

**Lemma 3.3.** Assume \( \sigma(p) \neq 0 \) in vicinity of \( p^0 \in \Gamma \) and \( L \) intersects \( \Gamma \) at \( p^0 \) transversally. Then there exists \( C^\infty \) real valued functions \( a(\alpha) \) and \( b(\alpha) \) near \( \alpha_0 = \alpha(n_\Gamma(p^0)) \) such that \( a(\alpha_0) = 0, a'(\alpha_0) \neq 0, b(\alpha_0) = Y(n_\Gamma(p^0)) \) and

\[
\tilde{\chi}(\xi) = |\xi|^{-3/2} \exp[i|\xi|b(\alpha(\xi)) - i\phi]P_\zeta(|\xi|^{1/2}a(\alpha(\xi)))u(\alpha(\xi)) + O(|\xi|^{-2}), \quad \text{if} \ \xi \in V,
\]

where \( \phi = (\pi/2) + (\pi/4) \text{sgn} \sigma(p^0), \ \zeta = \text{sgn} \sigma(p^0) \) and \( u(\alpha) \) is a \( C^\infty \) function near \( \alpha_0 \) with \( u(\alpha_0) = |2\pi\sigma(p^0)|^{-1/2} \).

**Corollary.** If \( \xi \in V \) and \( \alpha(\xi) \neq \alpha_0 = \alpha(n_\Gamma(p^0)) \), then

\[
|\tilde{\chi}(\xi) - |\xi|^{-3/2}2\pi\sigma(\xi)|^{-1/2} \theta(\xi) \exp[i(Y(\xi) - \phi)]| \leq C|\xi|^{-2}|\alpha(\xi) - \alpha_0|^{-1},
\]

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where \( \theta(\xi) = 1 \) if \( \exists p = p(\xi) \in \Gamma \) with \( n_\Gamma(p) = |\xi|^{-1}|\xi| \), and \( \theta(\xi) = 0 \) otherwise.

**Proof of Corollary.** From (3.14), (3.15), we obtain that for \( \alpha(\xi) = \alpha \neq \alpha_0 \)

\[
\tilde{\chi}(\xi) = |\xi|^{-3/2} \exp[i|\xi|b(\alpha) - i\phi]\theta(\xi)u(\alpha) + O(|\xi|^{-2}|\alpha - \alpha_0|^{-1}), \quad \xi \in V
\]

On the other hand, when \( \alpha(\xi) = \alpha \neq \alpha_0 \) is fixed, Lemma 3.1 proves that

\[
\tilde{\chi}(\xi) = |\xi|^{-3/2}[2\pi\sigma(\xi)]^{-1} \theta(\xi) \exp[i(Y(\xi) - \phi)] + O(|\xi|^{-2}), \quad |\xi| \to \infty, \quad \xi \in V.
\]

Comparing these two asymptotics we conclude that the main terms in them coincide, and therefore (3.16) holds.

**Proof of Lemma 3.3.** Consider a basis \( e_1, e_2 \) on the plane, where \( e_1 \) is a unit tangent vector to \( \Gamma \) at \( p^0 \) and \( e_2 \) is a unit tangent vector to \( L \) at \( p^0 \). Let \( e_1^\perp, e_2^\perp \) be a dual basis, \( e_1^\perp \cdot e_j = \delta_{ij} \), and \( p = p^0 + p_1e_1 + p_2e_2 \), \( \xi = \xi_1e_1^\perp + \xi_2e_2^\perp \). Then

\[
\tilde{\chi}(\xi) = \int_{\mathbb{R}^2} \varphi(p)\chi(p) \exp(ip\xi)dp = J \exp(ip_1^0\xi)i\xi^{-1}_2 \int_0^\infty \varphi(p_1, f(p_1)) \exp(ip_1\xi_1 + if(p_1)\xi_2)dp_1 + O(|\xi|^{-N})
\]

where \( p_2 = f(p_1) \) is the equation of \( \Gamma \) and \( J \) is the Jacobian. Observe that \( p_1\xi_1 + f(p_1)\xi_2 = |\xi|\Phi(p_1; \alpha(\xi)) \) where \( \Phi(p_1, \alpha) \) is a \( C^\infty \) real valued function with

\[
\Phi = \frac{\partial \Phi}{\partial p_1} = 0, \quad \frac{\partial^2 \Phi}{\partial p_1^2} \neq 0, \quad \frac{\partial^2 \Phi}{\partial p_1 \partial \alpha} \neq 0,
\]

at \( p_1 = 0, \alpha = \alpha_0 \).

By theorem 7.5.13 in [Hör2] there exists a \( C^\infty \) change of variable \( t = t(p_1, \alpha) \) with \( t(0, \alpha_0) = 0, \frac{\partial t}{\partial p_1}(0, \alpha_0) \neq 0 \) such that \( \Phi(p_1, \alpha) = -\zeta(t^2/2) + b(\alpha), b(\alpha_0) = 0, \) where \( \zeta = \text{sgn} \sigma(p^0) \). Hence

\[
I(\xi) = \int_0^\infty \varphi(p_1, f(p_1)) \exp(i|\xi|\Phi(p_1, \alpha))dp_1
\]

\[
= \exp(i|\xi|b(\alpha)) \int_{t(0, \alpha)}^\infty \psi(t; \alpha) \exp(-i|\xi|t^2/2) \frac{dp_1}{dt} dt,
\]

where \( \psi(t, \alpha) = \varphi(p_1, f(p_1)), p_1 = p_1(t, \alpha) \). Observe that \( \psi(t, \alpha) = 1 \) near \( (0, \alpha_0) \). Let

\[
\frac{dx_1}{dt} = u_0(\alpha) + tu_1(\alpha, t)
\]

Then

\[
I(\xi) = \exp(i|\xi|b(\alpha)) u_0(\alpha) \xi^{-1/2}P_\zeta(-t(0, \alpha)\xi^{1/2}) + O(|\xi|^{-1})
\]
From (3.17), (3.18) we obtain (3.15). Lemma 3.3 is proved.

**Lemma 3.4.** Let \( \chi(p) = 1 \) between two rays \( L_1 = \{ p^0 + \lambda e_1, \lambda \geq 0 \} \) and \( L_2 = \{ p^0 + \lambda e_2, \lambda \geq 0 \} \), where \( e_1, e_2 \) are linearly independent vectors, and \( \chi(p) = 0 \) otherwise. Then

\[
|\tilde{\chi}(\xi)| \leq C|\xi|^{-1}[(1 + |\xi \cdot e_1|)^{-1} + (1 + |\xi \cdot e_2|)^{-1}], \quad (3.20)
\]

and for \( \lambda \to \pm \infty \),

\[
\tilde{\chi}(\lambda e_j^\perp) = C_j \lambda^{-1} + O(|\lambda|^{-2}), \quad j = 1, 2; \quad e_i^\perp \cdot e_j = \delta_{ij}. \quad (3.21)
\]

**Proof.** Let \( p = p^0 + p_1 e_1 + p_2 e_2 \), \( \xi = \xi_1 e_1^\perp + \xi_2 e_2^\perp \). Without loss of generality we may assume that \( |\xi_2| \geq |\xi_1| \). Then integrating by parts in \( p_2 \) we obtain that

\[
\tilde{\chi}(\xi) = i\xi_2^{-1} \exp(ip^0 \xi) \int_0^\infty \varphi(p_1, 0) \exp(ip_1 \xi_1) dp_1 + O(|\xi|^{-2}).
\]

Integrating now by parts in \( p_1 \) we obtain (3.20), while setting \( \xi_1 = 0 \) we obtain (3.21). Lemma 3.4 is proved.

**Lemma 3.5.** Let \( \chi(p) = 1 \) from one side of the line \( L = \{ p^0 + \lambda e, \lambda \in \mathbb{R} \} \) and \( \chi(p) = 0 \) from the other side of \( L \). Then

\[
|\tilde{\chi}(\xi)| \leq C|\xi|^{-1}(1 + |\xi \cdot e|)^{-1}, \quad \tilde{\chi}(\lambda e_j^\perp) = C \lambda^{-1} \exp(i\lambda p^0 \cdot e_j^\perp) + O(|\lambda|^{-2}), \quad |\lambda| \to \infty.
\]

The proof of Lemma 3.5 is similar to the proof of Lemma 3.4.
4. Lattice–Point Problem

Let $Z(p) \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ be a $C^\infty$ positive function homogeneous of degree 2. Define

$$N(R) = \# \{ n = (n_1, n_2) \in \mathbb{Z}^2 : Z(n_1 + (1/2), n_2) \leq R^2, |n_2| \leq n_1 \},$$

$$\Omega(R) = \{ p \in \mathbb{R}^2 : Z(p) \leq R^2, |p_2| \leq p_1 \},$$

$$A(R) = AR^2 = \text{Area} \Omega(R).$$

(4.1)

The lattice–point problem we are interested in is to evaluate $N(R) - A(R)$ as $R \to \infty$.

Let

$$\Gamma = \{ p \in \mathbb{R}^2 : Z(p) = 1, |p_2| \leq p_1 \},$$

and $z_0, z_1 \in \Gamma$ be the endpoints of $\Gamma$ with $z_{01} = -z_{02} > 0$ and $z_{11} = z_{12} > 0$. For $p \in \Gamma$ denote by $n_\Gamma(p)$ the vector of outer normal to $\Gamma$ at $p$. Observe that

$$n_\Gamma(p) = |\text{grad } Z(p)|^{-1} \text{grad } Z(p),$$

and

$$p \cdot \text{grad } Z(p) = 2Z(p) > 0,$$

(4.2)

hence

$$p \cdot n_\Gamma(p) = 2|\text{grad } Z(p)|^{-1} \text{grad } Z(p) > 0.$$  

(4.3)

Denote by $\sigma(p)$ the curvature of $\Gamma$ at $p \in \Gamma$ with a sign, so that $\sigma(p) > 0$ if $\Omega$ is convex near $p$, $\sigma(p) < 0$ if $\Omega$ is concave near $p$. In what follows we assume the following

**Hypothesis D.** (i) $\sigma(p) \neq 0$ everywhere on $\Gamma$ except, maybe, a finite set $W = \{w_1, \ldots, w_K\}$, $z_0, z_1 \notin W$, and

$$\frac{d\sigma}{ds}(w_k) \neq 0, \quad k = 1, \ldots, K,$$

(4.5)

where $s$ is the natural coordinate on $\Gamma$. (ii) $\forall w_k \in W$ the vector $\nu_k = n_\Gamma(w_k)$ is either rational, i.e., $n \cdot \nu_k = 0$ for some $n \in \mathbb{Z}^2$, $n \neq 0$, or Diophantine in the sense that $\exists 1 > \zeta > 0$ and $C > 0$ such that

$$|n \cdot \nu_k| > \frac{C}{|n|^{1+\zeta}}, \quad \forall n \in \mathbb{Z}^2, n \neq 0.$$  

(4.6)

Without loss of generality we may assume that

$$0 = s(z_0) < s(w_1) < \cdots < s(w_K) < s(z_1),$$

(4.7)
where $s(p)$ is the natural coordinate of $p \in \Gamma$.

We call $\xi \in \mathbb{R}^2 \setminus \{0\}$ rational if $n \cdot \xi = 0$ for some $n \in \mathbb{Z}^2 \setminus \{0\}$. It is to be noted that $\xi$ is rational iff the set

$$L(\xi) = (\mathbb{Z}^2 \setminus \{0\}) \cap \{\lambda \xi, \lambda \in \mathbb{R}\}$$

is non-empty. Let

$$\Gamma_{\text{rat}} = \{p \in \Gamma: n_{\Gamma}(p) \text{ is rational}\}.$$

For $p \in \Gamma$ define

$$Y(p) = p \cdot n_{\Gamma}(p).$$

By (4.3) $Y(p) > 0$.

**Theorem 4.1.** Assume Hypothesis D holds. Then

$$N(R) = AR^2 + R^{2/3} \sum_{k: \nu_k \text{ is rational}} \Phi_k(R) + R^{1/2}F(R),$$

where $\Phi_k(R)$ are continuous periodic functions,

$$\Phi_k(R) = (1/2)3^{-2/3}\Gamma(2/3)\pi^{-4/3} \left|\frac{d\sigma}{ds}(w_k)\right|^{-1/3} \sum_{n \in L(\nu_k)} (-1)^{n_1} |n|^{-4/3} \sin(2\pi Y(w_k)|n|R)$$

and $F(R) \in B^2$. The Fourier series of $F(R)$ is

$$F(R) = \pi^{-1} \sum_{p \in \Gamma_{\text{rat}} \setminus W} \theta(p)|\sigma(p)|^{-1/2} \sum_{n \in L(n_{\Gamma}(p))} (-1)^{n_1} |n|^{-3/2} \cos(2\pi Y(p)|n|R - \phi(p)),$$

where

$$\theta(p) = \begin{cases} 1 & \text{if } p \neq z_0, z_1, \\ (1/2) & \text{if } p = z_0, z_1, \end{cases}$$

and $\phi(p) = (\pi/2) + (\pi/4) \text{sgn } \sigma(p)$.

We need Theorem 4.1 to prove our main Theorem 1.2. For the sake of completeness we want to formulate another theorem which is not needed for the proof of Theorem 1.2, but which is of interest by itself.

Let $\alpha \in \mathbb{R}^2$ be a fixed point on the plane. Define

$$N(R; \alpha) = \#\{n \in \mathbb{Z}^2: Z(n + \alpha) \leq R^2\}.$$

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Theorem 4.2. Assume Hypothesis D holds. Then

\[ N(R; \alpha) = AR^2 + R^{2/3} \sum_{k: \nu_k \text{ is rational}} \Phi_k(R; \alpha) + R^{1/2} F(R; \alpha), \]

where \( \Phi_k(R; \alpha) \) are continuous periodic functions of \( R \),

\[ \Phi_k(R; \alpha) = \left( \frac{1}{2} \right) 3^{-2/3} \Gamma(2/3) \pi^{-4/3} \int_{ds(w_k)}^{1/3} |n|^{-4/3} \sin(2\pi(Y(w_k)|n|R - n \cdot \alpha)) \]

and \( F(R; \alpha) \in B^2 \) in \( R \). The Fourier series of \( F(R; \alpha) \) is

\[ F(R; \alpha) = \pi^{-1} \sum_{p \in \Gamma_{\text{rat}} \setminus W} |\sigma(p)|^{-1/2} \sum_{n \in L(n_{\Gamma}(p))} |n|^{-3/2} \cos(2\pi Y(p)|n|R - \phi(p, n, \alpha)), \quad \text{(4.12)} \]

where \( \phi(p, n, \alpha) = (\pi/2) + (\pi/4) \text{sgn}(p) + 2\pi n \cdot \alpha \).

Theorem 4.2 is a generalization of Theorem 1.1 in [Ble1] to non-convex domains.

Proof of Theorem 4.1. \( N(R) \) can be written as

\[ N(R) = \sum_{n \in \mathbb{Z}^2} \chi(n + \alpha; R), \quad \alpha = (1/2, 0), \quad \text{(4.13)} \]

where \( \chi(p; R) \) is the characteristic function of \( \Omega(R) \). Define for \( \delta > 0 \),

\[ N_\delta(R) = \sum_{n \in \mathbb{Z}^2} \chi_\delta(n + \alpha; R), \quad \text{(4.14)} \]

where

\[ \chi_\delta(p; R) = \int_{\Omega(R)} \delta^{-2} \varphi(\delta^{-1}(p - p')) dp' = \chi(\cdot; R) * (\delta^{-2} \varphi(\delta^{-1}\cdot))(p), \quad \text{(4.15)} \]

and

\[ \varphi(p) \in C^\infty_0(\mathbb{R}^2); \quad \varphi(p) = \varphi_0(|p|); \quad \varphi(p) \geq 0; \quad \int_{\mathbb{R}^2} \varphi(p) dp = 1; \quad \varphi(p) = 0 \quad \text{when} \quad |p| \geq 1. \]

Lemma 4.3. For \( T > 1 \),

\[ \frac{1}{T} \int_1^{T} |N_\delta(R) - N(R)|^2 R^{-1} dR \leq C\delta T(T^{-1/3} + \delta). \quad \text{(4.16)} \]

Proof of this and all subsequent lemmas is given in the next section. For what follows we put

\[ \delta = T^{-1}. \quad \text{(4.17)} \]
In this case (4.16) reduces to
\[ \frac{1}{T} \int_{1}^{T} |N_{\delta}(R) - N(R)|^{2} R^{-1} dR \leq CT^{-1/3}. \] (4.18)

By the Poisson summation formula
\[ N_{\delta}(R) - AR^{2} = R^{2} \sum_{n \in \mathbb{Z}^{2} \setminus \{0\}} \hat{\varphi}(2\pi n\delta) \hat{\chi}(2\pi nR)(-1)^{n_{1}}, \] (4.19)
where
\[ \hat{\chi}(\xi) = \int_{\Omega} \exp(ip\xi) dp, \quad \Omega = \{ p \in \mathbb{R}^{2} : Z(p) \leq 1, |p_{2}| \leq p_{1} \}. \]

Let us consider a partition of unity on the projective line \( \mathbb{P}^{1} \),
\[ \sum_{l=1}^{L} \psi_{l}(\xi) = 1, \quad 0 \leq \psi_{l}(\xi) \leq 1, \quad \psi_{l}(\xi) \in C^{\infty}(\mathbb{P}^{1}), \]
and lift it to \( \mathbb{R}^{2} \setminus \{0\} \) putting
\[ \psi(\lambda \xi) = \psi(\xi) \quad \forall \lambda \in \mathbb{R}^{1} \setminus \{0\}. \]

Let
\[ \Gamma^{(l)} = \{ p \in \Gamma : n_{\Gamma}(p) \in \text{supp} \psi_{l}(\xi) \}, \]
and \( \Gamma_{j}^{(l)}, j = 1, \ldots, J_{l} \), be connected components of \( \Gamma^{(l)} \). Without loss of generality we may assume that each \( \Gamma_{j}^{(l)} \) contains at most one singular point \( z_{0}, z_{1}, w_{1}, \ldots, w_{K} \).

Then, for a given \( l \) consider a partition of unity in the \( p \)-plane,
\[ \sum_{m=1}^{M} \chi_{m}(p) = 1, \quad 0 \leq \chi_{m}(p) \leq 1, \quad \chi_{m}(p) \in C^{\infty}(\Omega), \]
such that for each \( \Gamma_{j}^{(l)} \), \( 1 \leq j \leq J_{l} \), \( \exists \) a unique \( \chi_{m}(p) \) which is \( \not\equiv 0 \) on \( \Gamma_{j}^{(l)} \). Without loss of generality we may also assume that for each \( m \), \( \chi_{m}(p) \) contains at most one singular point.

Define
\[ E_{lm}(R) = R^{2} \sum_{n \in \mathbb{Z}^{2} \setminus \{0\}} \psi_{l}(n) \hat{\varphi}(2\pi n\delta) \hat{\chi}_{m}(2\pi nR)(-1)^{n_{1}}, \] (4.20)
where
\[ \hat{\chi}_{m}(\xi) = \int_{\Omega} \chi_{m}(p) \exp(ip\xi) dp. \]
where

\[ \phi \]

Now we evaluate \( E_{lm}(R) \). Let \( \Gamma_m = \Gamma \cap \text{supp} \chi_m \).

**Lemma 4.4** \((\chi_m\text{ out of }\Gamma^{(l)})\). If \( \Gamma_m \cap \Gamma^{(l)} = \emptyset \) then

\[
\sup_{1 \leq R \leq T} |E_{lm}(R)| \leq C \log^2 T.
\]

Next we consider the case when \( \Gamma_j^{(l)} \subset \Gamma_m \) and \( \Gamma_m \) contains no singular point. In this case \( \forall \xi \in \text{supp} \psi \exists \) a unique \( p(\xi) \in \Gamma_j^{(l)} \) such that \( n_{\Gamma}(p(\xi)) = |\xi|^{-1} \xi \). Denote by

\[
Y_0(\xi) = \xi \cdot n_{\Gamma}(p(\xi)), \quad \sigma_0(\xi) = \sigma(p(\xi)).
\]

**Lemma 4.5** \((\text{regular } \chi_m \text{ on } \Gamma)\). Assume that \( z_0, z_1, w_1, \ldots, w_K \notin \Gamma_m \) and \( \Gamma_j^{(l)} \subset \Gamma_m \). Then

\[
E_{lm}(R) = R^{1/2} F_{lm}(R), \quad \text{where } F_{lm}(R) \in B^2 \text{ and}
\]

\[
F_{lm}(R) = \pi^{-1} \sum_{n \in \mathbb{Z}^{2k}} \psi_l(n)|n|^{-3/2} |\sigma_0(n)|^{-1/2} \cos(2\pi Y_0(n) R - \phi),
\]

where \( \phi = (\pi/2) + (\pi/4) \text{ sgn } \sigma(p), p \in \Gamma_m \).

Assume now that some inflection point \( w_k \) lies inside \( \Gamma_j^{(l)} \) and \( \Gamma_j^{(l)} \subset \Gamma_m \). Then for every \( \xi \in \text{supp} \psi_l \setminus \{ \lambda \nu_k, \lambda \in \mathbb{R} \} \) two possibilities exist: either \( \exists \) two points \( p_{\pm}(\xi) \in \Gamma_j^{(l)} \) such that \( n_{\Gamma}(p_{\pm}(\xi)) = \pm |\xi|^{-1} \xi \), or there is no such point at all. Define the function \( \theta(\xi) \) which is equal to 1 in the first case, and which is equal to 0 in the second case. For the sake of definiteness we will assume that \( \pm \sigma(p_{\pm}(\xi)) > 0 \). Denote by

\[
Y_{\pm}(\xi) = \xi \cdot n_{\Gamma}(p_{\pm}(\xi)), \quad \sigma_{\pm}(\xi) = \sigma(p_{\pm}(\xi)).
\]

**Lemma 4.6** \((\chi_m \text{ on inflection point})\). If \( \nu_k = n_{\Gamma}(w_k) \in \Gamma_j^{(l)} \subset \Gamma_m \) and \( \nu_k \) is rational then

\[
E_{lm}(R) = R^{2/3} \Phi_{lm}(R) + R^{1/2} F_{lm}(R),
\]

where \( \Phi_{lm}(R) \) is a periodic continuous function, which is given in (4.10), and \( F_{lm}(R) \in B^2 \). The Fourier series of \( F_{lm}(R) \) is

\[
F_{lm}(R) = \pi^{-1} \sum_{\pm} \sum_{n \in \mathbb{Z}^{2k} \setminus \{ \lambda \nu_k, \lambda \in \mathbb{R} \}} \psi_l(n) \theta(n)|n|^{-3/2} |\sigma_{\pm}(n)|^{-1/2} \cos(2\pi Y_{\pm}(n) R - \phi_{\pm}),
\]

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where $\phi_{\pm} = (\pi/2) \pm (\pi/4)$.

If $\nu_k$ is Diophantine then $E_{lm}(R) = R^{1/2}F_{lm}(R)$ where $F_{lm}(R) \in B^2$, and the Fourier series of $F_{lm}(R)$ coincides with (4.25).

**Lemma 4.7** (angular $\chi_m$). If $n_{\Gamma}(z_0) \in \Gamma_{j}^{(l)} \subset \Gamma m$ then $E_{lm}(R) = R^{1/2}F_{lm}(R)$, where $F_{lm}(R) \in B^2$, and the Fourier series of $F_{lm}(R)$ is

$$F_{lm}(R) = \pi^{-1} \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \psi(n) \theta(n)|n|^{-3/2} |\sigma_0(n)|^{-1/2} \cos(2\pi Y_0(n) - \phi), \quad (4.26)$$

where

$$\theta(n) = \begin{cases} 
1 & \text{if } n = n_{\Gamma}(p) \text{ for some } p \in \Gamma_{j}^{(l)} \setminus \{z_0\}, \\
(1/2) & \text{if } n = n_{\Gamma}(z_0), \\
0 & \text{otherwise,}
\end{cases} \quad (4.27)$$

and $\phi = (\pi/2) + (\pi/4) \text{sgn}(z_0)$.

Proof of Lemmas 4.3–4.7 is given in the next section.

**End of the proof of Theorem 4.1.** Let us fix $l$. By Lemma 4.4,

$$\lim_{T \to \infty} \frac{1}{T} \int_1^T |R^{-1/2}E_{lm}(R)|^2 dR = 0,$$

and by Lemmas 4.5–4.7,

$$\sum_{m : \Gamma_m \cap \Gamma_{l}^{(l)} \neq \emptyset} E_{lm}(R) = R^{2/3} \sum_{m \in I_1(l)} \Phi_{lm}(R) + R^{1/2} \sum_{m \in I_2(l)} F_{lm}(R),$$

where

$I_1(l) = \{m : \exists k, j \text{ such that } w_k \in \Gamma_{j}^{(l)} \subset \Gamma_m \text{ and } \nu_k \text{ is rational}\}, \quad I_2(l) = \{m : \Gamma_m \cap \Gamma_{l}^{(l)} \neq \emptyset\}$.

Making a summation over $l$ we obtain that if we define

$$\Phi(R) = \sum_{l=1}^{L} \sum_{m \in I_1(l)} \Phi_{lm}(R), \quad F(R) = \sum_{l=1}^{L} \sum_{m \in I_2(l)} F_{lm}(R),$$

then

$$\lim_{T \to \infty} \frac{1}{T} \int_1^T |N_{\delta}(R) - AR^2 - R^{2/3} \Phi(R) - R^{1/2} F(R)|^2 R^{-1} dR = 0.$$ 

Lemma 4.3 implies that the same is true for $N(R)$:

$$\lim_{T \to \infty} \frac{1}{T} \int_1^T |N(R) - AR^2 - R^{2/3} \Phi(R) - R^{1/2} F(R)|^2 R^{-1} dR = 0.$$
Since \( F_{lm}(R) \in B^2 \), \( F(R) \in B^2 \) as well (see [Bes]). The Fourier series for \( \Phi(R) \) and \( F(R) \) are the sum of the Fourier series for \( \Phi_{lm}(R) \) and \( F_{lm}(R) \), respectively. This proves the formulas (4.11), (4.12). Theorem 4.1 is proved.

Theorem 4.2 is proved in the same way.

5. Proof of Lemmas

Proof of Lemma 4.3. We have:

\[
N_\delta(R) - N(R) = \sum_{n \in \mathbb{Z}^2} (\chi_\delta(n + \alpha; R) - \chi(n + \alpha; R)).
\]

The support of \( \chi_\delta(p; R) - \chi(p; R) \) is concentrated in the \( \delta \)-neighborhood of \( \partial \Omega(R) \). Observe that \( \partial \Omega(R) \) consists of the curve \( R\Gamma \) and two segments \([0, Rz_0] \) and \([0, Rz_1] \) oriented along the diagonals \( p_1 \pm p_2 = 0 \). If \( \delta \) is small, then the \( \delta \)-neighborhood of \([0, Rz_0] \cup [0, Rz_1] \) does not contain points \( n + \alpha, n \in \mathbb{Z}^2, \alpha = (1/2, 0) \), and it does not contribute to \( N_\delta(R) - N(R) \).

Now,

\[
I = \frac{1}{T} \int_0^T |N_\delta(R) - N(R)|^2 R^{-1} dR = \sum_{n,n'} I_\delta(n,n'),
\]

where

\[
I_\delta(n,n') = \frac{1}{T} \int_0^T (\chi_\delta(n + \alpha; R) - \chi(n + \alpha; R))(\chi_\delta(n' + \alpha; R) - \chi(n' + \alpha; R)) R^{-1} dR.
\]

Observe that \( I_\delta(n,n') = 0 \) unless both \( n + \alpha \) and \( n' + \alpha \) lie in \( \delta \)-neighborhood of \( R\Gamma \) for some \( R \), hence \( \exists C > 0 \) such that

\[
I_\delta(n,n') = 0, \quad \text{if either } |X(n + \alpha) - X(n' + \alpha)| \geq C\delta \quad \text{or } X(n + \alpha) \geq CT,
\]

where \( X(p) = (Z(p))^{1/2} \). In addition, for all \( n,n' \),

\[
I_\delta(n,n') \leq CT^{-1} \delta X(n)^{-1} \leq C_0 T^{-1} \delta |n|^{-1}.
\]

Therefore,

\[
I \leq \sum_{n: X(n+\alpha) \leq CT} C_0 T^{-1} \delta |n|^{-1} \sum_{n': |X(n+\alpha) - X(n'+\alpha)| \leq C\delta} 1
\]

By 2/3-estimate (see, e.g., [CdV1])

\[
\sum_{n': |X(n+\alpha) - X(n'+\alpha)| \leq C\delta} 1 \leq C(n^{2/3} + \delta |n|),
\]

Therefore,
hence

\[ I \leq C_1 T^{-1} \delta \sum_{n: X(n+\alpha) \leq CT} |n|^{-1} \left( |n|^{2/3} + \delta |n| \right) \leq C_2 \delta T (T^{-1/3} + \delta). \]

Lemma 4.3 is proved.

**Proof of Lemma 4.4.** Let us consider two cases: when \(0 \in \text{supp}\chi_m(p)\) and when \(z_0 \in \Gamma_m\) (or \(z_1 \in \Gamma_m\) and \(\Gamma_m \cap \Gamma(l) = \emptyset\); all the other cases are simpler. Due to (4.20),

\[ E_{lm} = \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} K(n) \]

with

\[ K(n) = \tilde{\varphi}(2\pi n \delta) \psi(n) R^2 \tilde{\chi}_m(2\pi Rn)(-1)^{n_1}. \]  

(5.1)

By Lemma 3.4, if \(0 \in \text{supp}\chi_m(p)\) then

\[ R^2 |\tilde{\chi}_m(R\xi)| \leq C |\xi|^{-1} \max\{R^{-1} + |\xi_1 + \xi_2|^{-1}, R^{-1} + |\xi_1 - \xi_2|^{-1}\}, \]

so if \(n \in \Lambda\), where

\[ \Lambda = \{n \in \mathbb{Z}^2: |n_1 + n_2| \geq |n_1 - n_2| > 0\}, \]

then

\[ R^2 |\tilde{\chi}_m(2\pi Rn)| \leq C |n|^{-1} |n_1 - n_2|^{-1}. \]

Therefore \(I_0 = \sum_{n \in \Lambda} K(n)\) is estimated as

\[ I_0 \leq C \sum_{n \in \Lambda} |\tilde{\varphi}(2\pi n \delta)| \cdot |n|^{-1} |n_1 - n_2|^{-1}. \]

Since \(\varphi(p) \in C_0^\infty(\mathbb{R}^2)\),

\[ |\tilde{\varphi}(\xi)| \leq C (1 + |\xi|)^{-5}, \]  

(5.2)

and

\[ \sum_{|n| > T^2} |\tilde{\varphi}(2\pi n \delta)| \leq C_0, \quad \delta = T^{-1}. \]

On the other hand

\[ \sum_{n \in \Lambda, |n| \leq T^2} |n|^{-1} |n_1 - n_2|^{-1} \leq C \log^2 T, \]  

(5.3)

Hence \(I_0 \leq C \log^2 T\). The sum over \(|n_1 - n_2| \geq |n_1 + n_2| > 0\) is estimated similarly hence we obtain

\[ \left| \sum_{n: n_1 - n_2 \neq 0, n_1 + n_2 \neq 0} K(n) \right| \leq C \log^2 T. \]
By Lemma 3.4
\[ \sum_{n: \, n_1 = n_2 \neq 0} K(n) = C_1 R \sum_{n: \, n_1 = n_2 \neq 0} \tilde{\varphi}(2\pi n \delta)\psi_l(n)n_1^{1-1}(-1)^{n_1} + O(1). \]

Since \( \tilde{\varphi}(\xi) \) and \( \psi_l(\xi) \) are even functions,
\[ \sum_{n: \, n_1 = n_2 \neq 0} \tilde{\varphi}(2\pi n \delta)\psi_l(n)n_1^{1-1}(-1)^{n_1} = 0, \]
and thus
\[ \sum_{n: \, n_1 = n_2 \neq 0} K(n) = O(1). \]

Similarly,
\[ \sum_{n: \, n_1 = -n_2 \neq 0} K(n) = O(1). \]

As a result,
\[ |E_{lm}(R)| = \left| \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} K(n) \right| \leq C \log^2 T, \]
which was stated.

Assume now that \( z_0 \in \Gamma_m \) and \( \Gamma_m \cap \Gamma^{(l)} = \emptyset \). Let \( \Omega_0 \) be the angle with the vertex at \( z_0 \) and the sides which go along \([z_0, 0]\) and the tangent vector to \( \Gamma \) at \( z_0 \), so that \( \Omega_0 \) is a linear approximation to \( \Omega \) near \( z_0 \). Assume for the sake of definiteness that \( \sigma(z_0) > 0 \). Then \( \Omega_0 \supset \Omega \) near \( z_0 \). Let
\[ \tilde{\chi}^{(0)}_m(\xi) = \int_{\Omega_0} \chi_m(p) \exp(ip\xi) dp. \]

Then
\[ \tilde{\chi}^{(0)}_m(\xi) - \tilde{\chi}_m(\xi) = \int_{\Omega_0 \setminus \Omega} \chi_m(p) \exp(ip\xi) dp. \]
is estimated as follows. Integrating in the direction orthogonal to \( \xi \) we obtain that
\[ \tilde{\chi}^{(0)}_m(\xi) - \tilde{\chi}_m(\xi) = \int_{t_0}^{t_1} \overline{\chi}_m(t) \exp(it|\xi|) dt, \]
where \( \overline{\chi}_m(t) \) is equal to zero in vicinity of \( t_1 \) and
\[ \overline{\chi}_m(t) = a_1(t - t_0)^2 + a_2(t - t_0)^3 + \ldots \]
in vicinity of \( t_0 \). This implies that \( |\tilde{\chi}^{(0)}_m(\xi) - \tilde{\chi}_m(\xi)| \leq C(1 + |\xi|)^{-3} \), hence
\[ \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} (K(n) - K^{(0)}(n)) = O(R^{-1}), \]
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where
\[ K^{(0)}(n) = \tilde{\phi}(2\pi n \delta)\psi_1(n)R^2\chi_m^{(0)}(2\pi Rn)(-1)^{n_1}. \]

Now, \( \Omega_0 \) is an angular domain. Using the same arguments as in the case \( 0 \in \text{supp} \chi_m \), we obtain that
\[ \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} K^{(0)}(n) = O(\log^2 T). \]

This proves that
\[ \sup_{1 \leq R \leq T} |E_{lm}(R)| \leq C \log^2 T. \]

Lemma 4.4 is proved.

We omit proof of Lemma 4.5 and pass now to more complicated Lemmas 4.6, 4.7. Lemma 4.5 is proved similarly, with some simplifications (see also the proof of Theorem 3.1 in [Ble1]).

**Proof of Lemma 4.6.** Assume \( \nu_k \in \Gamma_j \subset \Gamma_m \). Let us split \( E_{lm}(R) \) into two parts:
\[ E^{(1)}_{lm}(R) = \sum_{n \in \mathbb{Z}^2 \cap L, n \neq 0} K(n), \quad E^{(2)}_{lm}(R) = \sum_{n \in \mathbb{Z}^2 \setminus L} K(n), \] (5.4)

where \( K(n) \) is defined in (5.1) and
\[ L_\gamma = \{ \xi \in \mathbb{R}^2 : \text{dist}(\xi, L) \leq \gamma \}, \quad L = \{ \xi = \lambda \nu_k, \ \lambda \in \mathbb{R}^1 \}. \]

with some \( \gamma > 0 \) which will be chosen later. Let us evaluate \( E^{(1)}_{lm}(R) \).

Assume first that \( \nu_k \) is rational. Then we can choose \( \gamma > 0 \) such that \( L_\gamma \setminus L \) contains no integer points. In this case
\[ E^{(1)}_{lm}(R) = \sum_{n \in L(\nu_k)} \tilde{\phi}(2\pi n \delta)R^2\tilde{\chi}(2\pi n R)(-1)^{n_1}. \]

By Lemma 3.5, if \( \xi \in L \) then
\[ \tilde{\chi}(\xi) = -i \exp(iY(\xi))|\xi|^{-4/3}\text{Ai}(0)|\sigma'(w_k)|^{-1/3} + O(|\xi|^{-5/3}), \quad \text{Ai}(0) = 3^{-2/3} \Gamma^{-1}(2/3), \]

hence
\[ E^{(1)}_{lm}(R) = R^{2/3} \Phi_{lm}(R; \delta) + O(R^{1/3}), \]

with
\[ \Phi_{lm}(R; \delta) = \sum_{n \in L(\nu_k)} \tilde{\phi}(2\pi n \delta)(-i) \sin(2\pi Y(n)R) \times |2\pi n|^{-4/3}\text{Ai}(0)|\sigma'(w_k)|^{-1/3}(-1)^{n_1}. \]
Since
\[ \sum_{q=1}^{\infty} |1 - \tilde{\varphi}(2\pi qr\delta)|q^{-4/3} \leq C\delta^{1/3} = CT^{-1/3}, \]
we obtain that
\[ E_{lm}^{(1)}(R) = R^{2/3}\Phi_{lm}(R) + O(R^{1/3} + R^{2/3}T^{-1/3}), \quad (5.5) \]
where
\[ \Phi_{lm}(R) = (1/2)3^{-2/3}\pi^{-4/3}\Gamma^{-1}(2/3)|\sigma'(w_k)|^{-1/3} \sum_{n \in L(\nu_k)} (-1)^n |n|^{-4/3} \sin(2\pi Y(n)R) \quad (5.6) \]
is a periodic function of \( R \).

Assume now that \( \nu_k \) is Diophantine,
\[ |n \cdot \nu_k| \geq C|n|^{-1-\zeta}, \quad 0 < \zeta < 1. \quad (5.7) \]
In this case we put \( \gamma = 1 \). Define
\[ E_{lm}^{(1)}(R; N) = \sum_{n \in \mathbb{Z}^{2} \cap L_1, 0 < |n| \leq N} K(n). \]
Let us prove that
\[ \sup_{1 \leq R < \infty} R^{-1/2}|E_{lm}^{(1)}(R) - E_{lm}^{(1)}(R; N)| \leq CN^{(\zeta - 1)/4}. \quad (5.8) \]
Indeed, \( |\text{Ai}(y)| \leq C|y|^{-1/4} \) and \( |a_k(\alpha)| \geq C_0|\alpha - \alpha(\nu_k)| \), hence (3.7) implies that
\[ |\tilde{\chi}_m(\xi)| \leq C|\xi|^{-3/2}|\alpha(\xi) - \alpha(\nu_k)|^{-1/4} \]
and
\[ |K(n)| \leq CR^{1/2}|n|^{-3/2}|\alpha(n) - \alpha(\nu_k)|^{-1/4}. \]
Therefore
\[ R^{-1/2}|E_{lm}^{(1)}(R) - E_{lm}^{(1)}(R; N)| = R^{-1/2}\left| \sum_{n \in \mathbb{Z}^{2} \cap L_1, |n| > N} K(n) \right| \leq C \sum_{n \in \mathbb{Z}^{2} \cap L_1, |n| > N} |n|^{-3/2}|\alpha(n) - \alpha(\nu_k)|^{-1/4}. \quad (5.9) \]
Due to the Diophantine condition (5.7),
\[ |\alpha(n) - \alpha(\nu_k)| \geq C|n|^{-(2+\zeta)}, \]
Let $J \geq N$. Order all $n \in \mathbb{Z}^2 \cap L_1$ with $1 \leq |n| \leq 2J$ and $n \cdot \nu_k > 0$ in the increasing order of $|n \cdot \nu_k| = |n_2 \nu_{k1} - n_1 \nu_{k2}|$:

$$|n^{(1)} \cdot \nu_k| \leq |n^{(2)} \cdot \nu_k| \leq \ldots$$

Then (5.7) implies that

$$|n^{(1)} \cdot \nu_k| \geq C_0 J^{-(1+\zeta)}$$

and

$$|n^{(j+1)} \cdot \nu_k| \geq |n^{(j)} \cdot \nu_k| + C_0 J^{-(1+\zeta)}, \quad j \geq 1,$$

so that

$$|n^{(j)} \cdot \nu_k| \geq C_0 j J^{-(1+\zeta)},$$

hence

$$|\alpha(n^{(j)}) - \alpha(\nu_k)| \geq C_1 j J^{-(2+\zeta)}$$

and

$$\sum_{n \in \mathbb{Z}^2 \cap L_1, J \leq |n| \leq 2J} |n|^{-3/2} |\alpha(n) - \alpha(\nu_k)|^{-1/4} \leq C J^{-3/2} J^{(2+\zeta)/4} \sum_{j=1}^{C_0 J} j^{-1/4} \leq C_1 J^{(\zeta-1)/4}.$$

This implies

$$\sum_{n \in \mathbb{Z}^2 \cap L_1, N > |n|} |n|^{-3/2} |\alpha(n) - \alpha(\nu_k)|^{-1/4} \leq C N^{(\zeta-1)/4},$$

and hence (5.8) follows from (5.9). Let us evaluate now $E^{(2)}_{lm}(R)$.

From (3.9)

$$|E^{(2)}_{lm}(R) - E^+_{lm}(R) - E^-_{lm}(R)|$$

$$\leq C R^2 \sum_{n \in \mathbb{Z}^2 \setminus L_\gamma} \psi_l(n) |\tilde{\varphi}(2\pi n\delta)| |Rn|^{-5/2} |\alpha(n) - \alpha(\nu_k)|^{-7/4} \quad (5.11)$$

where

$$E^\pm_{lm}(R) = \sum_{n \in \mathbb{Z}^2 \setminus L_\gamma} K^\pm(n)$$

and

$$K^\pm(n) = R^{1/2} \pi^{-1} \psi_l(n) \tilde{\varphi}(2\pi n\delta) \theta(n) |n|^{-3/2} |\sigma^\pm(n)|^{-1/2} (-1)^{n_1} \cos(2\pi R Y^\pm(n) - \phi^\pm). \quad (5.12)$$
The RHS in (5.11) is estimated as follows. Let \( d(\xi) = \text{dist}(\xi, L) \). Then
\[
\text{RHS} \leq CR^{-1/2} \sum_{n \in \mathbb{Z}^2 \setminus L_\gamma} \psi_1(n) |\tilde{\varphi}(2\pi n\delta)| \cdot |n|^{-5/2} d(n)^{-7/4} |n|^{7/4}
\]
\[
= CR^{-1/2} \sum_{n \in \mathbb{Z}^2 \setminus L_\gamma} \psi_1(n) |\tilde{\varphi}(2\pi n\delta)| \cdot |n|^{-3/4} d(n)^{-7/4}.
\]
Since for \( p = 1, 2, \ldots, \)
\[
\sum_{n \in \mathbb{Z}^2 \setminus L_\gamma, p \leq |n| \leq p+1} \psi_1(n) d(n)^{-7/4} \leq C,
\]
and
\[
\sup_{p \leq |n| \leq p+1} |\tilde{\varphi}(2\pi n\delta)| \leq C(1 + p\delta)^{-5},
\]
we obtain
\[
\text{RHS} \leq CR^{-1/2} \sum_{p=1}^{\infty} (1 + p\delta)^{-5} p^{-3/4} \leq C_0 R^{-1/2} \delta^{-1/4} = C_0 R^{-1/2} T^{1/4},
\]
so that
\[
|E^{(2)}_{lm}(R) - E^+_{lm}(R) - E^-_{lm}(R)| \leq CR^{-1/2} T^{1/4}. \tag{5.13}
\]
Define
\[
F^\pm_{lm}(R) = R^{-1/2} E^\pm_{lm}(R) = R^{-1/2} \sum_{n \in \mathbb{Z}^2 \setminus L_\gamma} K^\pm(n),
\]
\[
F^\pm_{lm}(R; N, \delta) = R^{-1/2} \sum_{n \in \mathbb{Z}^2 \setminus L_\gamma, |n| \leq N} K^\pm(n) \tag{5.14}
\]
The central point in our proof is

**Lemma 5.1.** For all \( N, T \geq 1, \)
\[
\frac{1}{T} \int_0^T |F^\pm_{lm}(R) - F^\pm_{lm}(R; N, \delta)|^2 dR \leq C(N^{-1/3} + T^{-1/4}), \quad \delta = T^{-1}.
\]

We will give the proof of Lemma 5.1 below, in the end of this section, and now let us derive Lemma 4.6 from Lemma 5.1.

Assume that \( \nu_k \) is rational. Define
\[
F_{lm}(R) = R^{-1/2}(E_{lm}(R) - R^{2/3} \Phi_{lm}(R))
\]
with \( \Phi_{lm}(R) \) given in (5.6). Then by (5.5)
\[
F_{lm}(R) = R^{-1/2} E^{(2)}_{lm}(R) + O(R^{-1/6}), \quad 1 \leq R \leq T,
\]

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and by (5.13)

\[ R^{-1/2} E^{(2)}_{lm}(R) = F^{+}_{lm}(R) + F^{-}_{lm}(R) + O(R^{-1/4} T^{1/4}), \quad 1 \leq R \leq T, \]

so that

\[ F_{lm}(R) = F^{+}_{lm}(R) + F^{-}_{lm}(R) + O(R^{-1/6} + R^{-1/4} T^{1/4}), \quad 1 \leq R \leq T, \]

which implies that

\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \left| F_{lm}(R) - F^{+}_{lm}(R) - F^{-}_{lm}(R) \right|^2 dR = 0. \tag{5.15}
\]

Define

\[ F_{lm}^{\pm}(R; N) = \sum_{n \in \mathbb{Z}^2 \backslash L_r, |n| \leq N} K_{\pm}(n), \]

where

\[ K_{\pm}(n) = \pi^{-1} \psi_l(n) \theta(n)|n|^{-3/2} |\sigma_{\pm}(n)|^{-1/2} (-1)^{n_1} \cos(2\pi R \gamma(n) - \phi_{\pm}). \tag{5.16} \]

Observe that \( F_{lm}^{\pm}(R; N) \) is a finite trigonometric sum, hence

\[ |F_{lm}^{\pm}(R; N, \delta) - F_{lm}^{\pm}(R; N)| \leq C(N) \delta = C(N) T^{-1}. \]

Therefore Lemma 5.1 implies that

\[
\limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \left| F_{lm}^{\pm}(R) - F_{lm}^{\pm}(R; N) \right|^2 dR \leq C N^{-1/3},
\]

hence from (5.15) we deduce that

\[
\limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \left| F_{lm}(R) - F_{lm}^{+}(R; N) - F_{lm}^{-}(R; N) \right|^2 dR \leq C N^{-1/3}. \tag{5.17}
\]

This implies that \( F_{lm}(R) \in B^2 \) and (4.19) is the Fourier expansion of \( F_{lm}(R) \). For rational \( \nu_k \) Lemma 4.6 is proved.

In the case of Diophantine \( \nu_k \) we define \( F_{lm}(R) = R^{-1/2} E_{lm}(R) \). Then

\[ F_{lm}(R) = F_{lm}^{(1)}(R) + F_{lm}^{(2)}(R), \quad F_{lm}^{(j)}(R) = R^{-1/2} E_{lm}^{(j)}(R), \quad j = 1, 2, \]

and similarly to (5.17) we have that

\[
\limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \left| F_{lm}^{(2)}(R) - F_{lm}^{+}(R; N) - F_{lm}^{-}(R; N) \right|^2 dR \leq C N^{-1/3} \tag{5.18}
\]

with \( F_{lm}^{\pm}(R; N) \) defined in (5.16).
Then, (5.8) implies that

$$\limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} |F_{lm}^{(1)}(R) - F_{lm}^{(1)}(R; N)|dR \leq C N^{(\zeta-1)/4},$$  \hspace{1cm} (5.19)

where

$$F_{lm}^{(1)}(R; N) = R^{3/2} \sum_{n \in L_{\gamma}, 0 < |n| \leq N} \psi(n) \hat{\varphi}(2\pi n \delta) \hat{\chi}(2\pi n R)(-1)^{n}.$$  

This is a finite sum and from (3.9) we obtain that

$$|F_{lm}^{(1)}(R; N) - \hat{F}_{lm}^{+}(R; N) - \hat{F}_{lm}^{-}(R; N)| \leq C(N)(R^{-1} + T^{-1})$$  \hspace{1cm} (5.20)

with

$$\hat{F}_{lm}^{\pm}(R) = \sum_{n \in \mathbb{Z}^{2} \cap L_{\gamma}, 0 < |n| \leq N} K_{\pm}(n).$$

It follows from (5.18)–(5.20) that

$$\limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} |F_{lm}(R) - F_{lm}^{+}(R; N) - \hat{F}_{lm}^{+}(R; N)$$
$$- F_{lm}^{-}(R; N) - \hat{F}_{lm}^{-}(R; N)|^{2}dR \leq C N^{(\zeta-1)/4},$$

which proves Lemma 4.6 for Diophantine $\nu_k$.

**Proof of Lemma 5.1.** To simplify notations we will omit $\pm$ in sub– and superscripts. From (5.12) and (5.14)

$$\frac{1}{T} \int_{0}^{T} |F_{lm}(R) - F_{lm}(R; N, \delta)|^{2}dR = \pi^{-2} \sum_{n, n' \in \mathbb{Z}^{2} \setminus L_{\gamma}; |n|, |n'| > N} \psi(n)\psi(n')$$
$$\hat{\varphi}(2\pi n \delta)\hat{\varphi}(2\pi n' \delta)\theta(n)\theta(n'|n|^{-3/2}|n'|^{-3/2} |\sigma(n)|^{1/2}|\sigma(n')|^{1/2}I(n, n'),$$  \hspace{1cm} (5.21)

where

$$I(n, n') = \frac{1}{T} \int_{0}^{T} \cos(2\pi Y(n)R - \phi(n))\cos(2\pi Y(n')R - \phi(n'))dR.$$  

Observe that

$$|I(n, n')| \leq \min\{1, T^{-1}|Y(n) - Y(n')|^{-1}\}.$$  \hspace{1cm} (5.22)

In addition,

$$|\hat{\varphi}(2\pi n \delta)| \leq C(1 + Y(n)\delta)^{-5}, \quad C|n| \leq Y(n) \leq C'|n|,$$

and the RHS of (5.21) is symmetric in $n, n'$. This implies that

$$\frac{1}{T} \int_{0}^{T} |F_{lm}(R) - F_{lm}(R; N, \delta)|^{2}dR \leq C_{0} \sum_{n, n' \in \mathbb{Z}^{2} \setminus L_{\gamma}; Y(n) \geq Y(n') > \beta N} H(n, n'),$$  \hspace{1cm} (5.23)
with
\[ H(n, n') = \psi_l(n) \psi_l(n')(1 + Y(n)\delta)^{-5} \theta(n) \theta(n') Y(n)^{-3/2} Y(n')^{-3/2} \]
and \( \beta > 0 \). First we estimate \( \sum_{(n, n') \in S(N)} H(n, n') \) with
\[
S(N) = \{ n, n' \in \mathbb{Z}^2 \setminus L_\gamma; Y(n) \geq Y(n') > \beta N; Y(n) - Y(n') \geq 1 \}. \tag{5.24}
\]
Let us fix some \( n \) with \( Y(n) \geq \beta N \) and define the layers
\[
S(N, n, j) = \{ n' : n' \in \mathbb{Z}^2 \setminus L_\gamma; Y(n') \geq \beta N; j + 1 \geq Y(n) - Y(n') \geq j \},
\]
\[ 1 \leq j \leq Y(n) - \beta N. \]
Observe that \( |\sigma(n)| \leq C|\alpha(n) - \alpha(\nu_k)|^{-1/2} \), hence
\[
\sum_{n' \in S(N, n, j)} \psi_l(n') \theta(n') |\sigma(n')|^{1/2} \leq C \text{ Area } S(N, n, j) \leq C_0 (Y(n) - j)
\]
and therefore
\[
\sum_{n' \in S(N, n)} \psi_l(n') \theta(n') |\sigma(n')|^{1/2} Y(n')^{-3/2} T^{-1} (Y(n) - Y(n'))^{-1} \leq C \sum_{j=1}^{Y(n) - 1} (Y(n) - j)^{-1/2} T^{-1} j^{-1} \leq C_0 T^{-1} Y(n)^{-1/2} \log Y(n),
\]
where \( S(N, n) = \bigcup_j S(N, n, j) \). Making a summation in \( n \) we obtain now that
\[
\sum_{(n, n') \in S(N)} H(n, n') \leq C T^{-1} \sum_{n \in \mathbb{Z}^2 \setminus L_\gamma; Y(n) > \beta N} \psi_l(n)(1 + Y(n)\delta)^{-5} \times Y(n)^{-2} \log Y(n) |\sigma(n)|^{-1/2}.
\]
Define the layers
\[
S_j = \{ n \in \mathbb{Z}^2 \setminus L_\gamma, j \leq Y(n) \leq j + 1 \}, \quad j > \beta N.
\]
Then
\[
\sum_{n \in S_j} \psi_l(n) \theta(n) |\sigma(n)|^{-1/2} \leq C \text{ Area } S_j \leq C_0 j,
\]
hence
\[
\sum_{(n, n') \in S(N)} H(n, n') \leq C T^{-1} \sum_{j=1}^{\infty} (1 + j\delta)^{-5} j^{-1} \log j \leq C_0 T^{-1} |\log \delta|^{2} = C_0 T^{-1} \log^2 T. \tag{5.25}
\]
It remains to estimate \( \sum_{(n,n') \in S_0(N)} H(n,n') \) where
\[
S_0(N) = \{ n, n' \in \mathbb{Z}^2 \setminus L_\gamma; \ Y(n) \geq Y(n') > \beta N; \ 1 \geq Y(n) - Y(n') \geq 0 \}.
\]
Let us fix some \( n \) with \( T \geq Y(n) > \beta N \). Define the layers
\[
S_0(N,n,j) = \{ n' \in \mathbb{Z}^2 \setminus L_\gamma; \ Y(n') > \beta N; \ (j+1)T^{-1} \geq Y(n) - Y(n') \geq jT^{-1}, \quad j = 0,1,\ldots,T. \]

To estimate the sum over \( S_0(N,n,j) \) we use the following

**Lemma 5.2 (2/3–estimate).** Let
\[
Y(\xi) = |\xi|f(|\alpha(\xi) - \alpha_0|^{1/2}), \quad (5.26)
\]
where \( f(t) \in C^\infty([0,\varepsilon]), \varepsilon > 0, f(t) > 0, f'(0) = 0, f'''(0) \neq 0 \). Then if \( \psi(\alpha) \in C^\infty([\alpha - 0, \alpha + \varepsilon]) \) with \( \text{supp} \psi(\alpha) \) near \( \alpha_0 \) then
\[
\sum_{n \in \Pi(R), \ \text{dist}(n,L)>1} \psi(\alpha(n))|\alpha(n) - \alpha_0|^{-1/2} \leq CR^{2/3}, \quad (5.27)
\]
where
\[
\Pi(R) = \{ n \in \mathbb{Z}^2; \ \alpha_0 \leq \alpha(n) \leq \alpha_0 + \varepsilon; \ R \leq Y(n) \leq R + R^{-1/3} \}
\]
and \( L = \{ \xi; \ \alpha(\xi) = \alpha_0 \} \).

**Remark.** If we step away from \( \alpha_0 \) putting \( \alpha_0 + \varepsilon_0 \leq \alpha(n) \leq \alpha_0 + \varepsilon, \varepsilon_0 > 0 \), in \( \Pi(R) \) (instead of \( \alpha_0 \leq \alpha(n) \leq \alpha_0 + \varepsilon \)), then (5.27) reduces to a well–known 2/3–estimate (see, e.g., [CdV1]).

Proof of Lemma 5.2 is given in Appendix to the paper.

With the help of Lemma 5.2 we obtain that if \( T^3 \geq Y(n) \) then
\[
\sum_{n' \in S_0(N,n,j)} \psi_l(n')\theta(n')|\sigma(n')|^{-1/2} \leq CY(n)^{2/3}
\]
(observe that \( |\sigma(n')|^{-1/2} \leq C|\alpha(n') - \alpha(\nu_k)|^{-1/4} \leq C|\alpha(n') - \alpha(\nu_k)|^{-1/2} \), and
\[
\sum_{n' \in S_0(N,n,j)} \psi_l(n')\theta(n')|\sigma(n')|^{-1/2}Y(n')^{-3/2} \min\{1, T^{-1}(Y(n) - Y(n'))^{-1} \}
\]
\[
\leq CY(n)^{-3/2}Y(n)^{2/3}T^{-1}j^{-1}Y(n) = CY(n)^{1/6}T^{-1}j^{-1}, \quad \text{if} \quad T \geq j \geq 2,
\]
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and $\leq CY(n)^{-5/6}$ if $j = 0, 1$. Hence

$$\sum_{n' \in S_0(N,n)} \psi(n') \theta(n') |\sigma(n')|^{-1/2} Y(n')^{-3/2} \min \{1, T^{-1}(Y(n) - Y(n'))^{-1}\}$$

$$\leq C(Y(n)^{1/6} T^{-1} \log T + Y(n)^{-5/6}),$$

where

$$S_0(N, n) = \{n' \in \mathbb{Z}^2 : Y(n') > \beta N; \ 1 \geq Y(n) - Y(n') \geq 0\}.$$ 

Making a summation over $n$, we obtain

$$\sum_{(n, n') \in S_0(N); Y(n) \leq T^3} H(n, n') \leq C \sum_{n \in \mathbb{Z}^2 \setminus L_\gamma; T^3 \geq Y(n) \geq \beta N} \psi(n)(1 + Y(n) \delta)^{-5} \theta(n) Y(n)^{-3/2} |\sigma(n)|^{-1/2} (Y(n)^{1/6} T^{-1} \log Y(n) + Y(n)^{-5/6}) \equiv CI_0.$$ 

Consider the layers

$$S_j = \{n \in \mathbb{Z}^2 \setminus L_\gamma; j + 1 \geq Y(n) \geq j, \ T^3 \geq j \geq \beta N\}.$$ 

The width of $S_j$ is of order of 1, and a simple argument shows that

$$\sum_{n \in S_j} \psi(n) \theta(n) |\sigma(n)|^{-1/2} \leq C j.$$ 

This implies

$$I_0 \leq C \sum_{j = \beta N}^{T^3} (1 + j \delta)^{-5} j^{-3/2} j (j^{1/6} T^{-1} \log j + j^{-5/6})$$

$$\leq C_0 (\delta^{-2/3} |\log \delta| T^{-1} + N^{-1/3}) = C_0 (T^{-1/3} \log T + N^{-1/3}).$$ 

Thus

$$\sum_{(n, n') \in S_0(N); Y(n) \leq T^3} H(n, n') \leq C(T^{-1/3} \log T + N^{-1/3}) \quad (5.28)$$

The final step is to estimate

$$\sum_{(n, n') \in S_0(N); Y(n) \geq T^3} H(n, n')$$

and this is quite simple. Since for $(n, n') \in S_0(N)$,

$$H(n, n') \leq C(1 + Y(n) \delta)^{-5} Y(n)^{-3} |\sigma(n)|^{-1/2} |\sigma(n')|^{-1/2}$$

and

$$\sum_{n': Y(n) \geq Y(n') \geq Y(n) - 1; n' \notin L_\gamma} |\sigma(n')|^{-1/2} \leq CY(n),$$

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we obtain
\[
\sum_{\substack{(n,n') \in S_0(N) : Y(n) \geq T^3}} H(n, n') \leq C \sum_{\substack{(n,n') \in S_0(N) : Y(n) \geq T^3}} (1 + Y(n)\delta)^{-5} Y(n)^{-2} |\sigma(n)|^{-1/2} \leq C_0 \sum_{j=1}^{\infty} (1 + j\delta)^{-5} j \leq C_1 T^{-2}.
\]

(5.29)

From (5.23), (5.25), (5.28) and (5.29) Lemma 5.1 follows.

**Proof of Lemma 4.7.** The proof of Lemma 4.7 is similar in main steps to the proof of Lemma 4.6. Assume \( \nu_0 = n_\Gamma(z_0) \in \Gamma_j \subset \Gamma_m \). By (4.15)
\[
F_{lm}(R) = \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} K(n).
\]
with
\[
K(n) = R^{3/2} \tilde{\varphi}(2\pi n\delta) \psi_l(n) \tilde{\chi}_m(2\pi Rn)(-1)^{n_1}
\]
Define
\[
L = \{ \lambda \nu_0, \quad \lambda \in \mathbb{R} \}, \quad L_1 = \{ n \in \mathbb{Z}^2 : \text{dist}(n, L) \leq 1 \},
\]
\[
F_{lm}^{(1)}(R) = \sum_{n \in L_1, n \neq 0} K(n), \quad F_{lm}^{(2)}(R) = \sum_{n \notin L_1} K(n),
\]
so that \( F_{lm}(R) = F_{lm}^{(1)}(R) + F_{lm}^{(2)}(R) \). By (3.15)
\[
|K(n)| \leq C |n|^{-3/2},
\]
hence
\[
\sum_{n \in L_1, |n| \geq N} |K(n)| \leq CN^{-1/2}.
\]
(5.31)

Define
\[
K_0(n) = \pi^{-1} |n|^{-3/2} \tilde{\varphi}(2\pi n\delta) |\sigma_0(n)|^{-1/2} \theta(n) \cos(2\pi RY_0(n) - \phi).
\]
(5.32)

By (3.18)
\[
\sum_{n \notin L_1, |n| \geq N} |K(n) - K_0(n)| \leq CR^{-1/2} \sum_{n \notin L_1, |n| \geq N} |\tilde{\varphi}(2\pi n\delta)| \cdot |n|^{-2} |\alpha(n) - \alpha_0|^{-1} \leq R^{-1/2} \log^2 T.
\]
(5.33)

The following lemma holds:
Lemma 5.3. For all $N, T \geq 1$,

$$\frac{1}{T} \int_1^T \left| \sum_{n \notin \mathbb{L}_1, |n| \geq N} K_0(n) \right|^2 dR \leq C(N^{-1/3} + T^{-1/4}).$$

The omit the proof of this lemma since it basically the same as the proof of Lemma 3.3 in [Ble1] (see also the proof of Lemma 5.1 above, where a similar statement was proved in a more complicated situation).

Lemma 3.3 implies that for a fixed $n \in \mathbb{Z}^2 \setminus \{0\}$, $K(n)$ converges to $K_0(n)$ as $R \to \infty$, hence

$$\lim_{T \to \infty} \frac{1}{T} \int_1^T \left| \sum_{0<|n|<N} (K(n) - K_0(n)) \right|^2 dR = 0,$$

and by Lemma 5.3,

$$\lim_{T \to \infty} \frac{1}{T} \int_1^T \left| \sum_{0<|n|<N} K(n) - K_0(n) \right|^2 dR = 0.$$

This proves Lemma 4.7.

6. Energy Levels and Closed Geodesics

In this section we prove Theorem 1.2. Let

$$N(R) = \# \{ E_n \leq R^2 \}, \quad N_{BS}(R) = \# \{ n \in \mathbb{Z}^2 : Z_2(n_1 + (1/2), n_2) \leq R^2, |n_2| \leq n_1 \} \quad (6.1)$$

and

$$\Gamma = \{ p \in \mathbb{R}^2 : Z_2(p) = 1, |p_2| \leq p_1 \}. \quad (6.2)$$

Recall that DH is defined by (1.12) and Hypothesis D by (4.5), (4.6).

Lemma 6.1. DH implies Hypothesis D for $\Gamma$.

Proof. By (2.4) $\Gamma$ is the graph of the function

$$p_1 = g(p_2) \equiv |p_2| + \pi^{-1} \int_a^b (1 - p_2^2 f^{-2}(s))^{1/2} ds, \quad f(a) = f(b) = |p_2|, \quad |p_2| \leq f_{\text{max}}. \quad (6.3)$$

g($p_2$) is an even $C^\infty$ function, so we will assume $p_2 \geq 0$. The function

$$\frac{dp_1}{dp_2} = 1 - \pi^{-1} \int_a^b p_2 f^{-2}(s)(1 - p_2^2 f^{-2}(s))^{-1/2} ds \quad (6.4)$$
has a nice geometric interpretation.

**Proposition 6.2.**

\[
\frac{dp_1}{dp_2} \bigg|_{p_2 = I} = -\omega(I), \quad I \geq 0. \tag{6.5}
\]

**Proof.** An equation of \( \gamma(I) \) is

\[
\frac{d\varphi}{dl} = I f^{-2}(s), \quad \frac{ds}{dl} = (1 - l^2 f^{-2}(s))^{1/2}, \tag{6.6}
\]

where \( l \) is the normal coordinate on \( \gamma \). Hence

\[
\frac{d\varphi}{ds} = I f^{-2}(s)(1 - l^2 f^{-2}(s))^{-1/2}
\]

and

\[
\omega(I) = \pi^{-1} \int_a^b \frac{d\varphi}{ds} ds - 1 = \pi^{-1} \int_a^b I f^{-2}(s)(1 - l^2 f^{-2}(s))^{-1/2} ds - 1.
\]

Comparing this with (6.4) we obtain (6.5). Proposition 6.2 is proved.

Observe that an inflection point on \( \Gamma \) is characterized by \( \frac{d^2 p_1}{dp_2^2} = 0 \). By (6.5) this is equivalent to \( \omega'(I) = 0 \). Similarly, the nondegeneracy of the inflection point is characterized by \( \omega''(I) \neq 0 \). In addition, the Diophantine condition (4.6) is equivalent to (1.12), hence DH implies Hypothesis D. Lemma 6.1 is proved.

Lemma 6.1 implies that Theorem 4.1 holds for \( N_{BS}(R) \). Our goal now is to find geometric interpretation of frequencies and amplitudes in formulas (4.11) (4.12).

Consider a finite geodesic \( \gamma \) which starts at \( x_0 = (s_{\text{max}}, 0) \) at some angle \(-\pi/2 \leq \alpha \leq \pi/2\) to the direction to the north. \( \gamma \) is uniquely determined by \( I = \sin \alpha \) and \( l = |\gamma|, \gamma = \gamma(I, l) \). Let \( G \) be the set of all \( \gamma(I, l), -1 \leq I \leq 1, l > 0 \).

Assume that \( \Gamma \) is defined as in (6.3). Define two maps:

\[
p: G \to \Gamma, \quad \xi: G \to \mathbb{R}^2 \setminus \{0\}, \tag{6.6}
\]

where

\[
p: \gamma = \gamma(I, l) \to p(\gamma) = (g(I), I), \quad \xi: \gamma = \gamma(I, l) \to \xi(\gamma) = (l^{-1}(I), \omega(I)l^{-1}(I)). \tag{6.7}
\]

**Proposition 6.3.** \( p(\gamma) \) and \( \xi(\gamma) \) satisfy

\[
n_\Gamma(p(\gamma)) = |\xi(\gamma)|^{-1} \xi(\gamma) \tag{6.8}
\]
and
\[ p(\gamma) \cdot \xi(\gamma) = (2\pi)^{-1} |\gamma|. \] (6.9)

**Proof.** (6.5) implies that \( n_\Gamma(p(\gamma)) \) is collinear with the vector \((1, \omega(I))\) as well as \( \xi(\gamma) \), hence (6.8) follows. To prove (6.9) observe that the both sides of (6.9) depend linearly on \(|\gamma|\), so it is sufficient to prove (6.9) in the particular case when \(|\gamma| = \tau(I)\). In this case (6.9) reduces to
\[ g(I) + I \tau(I) = (2\pi)^{-1} \tau(I). \] (6.10)

Since
\[ g(I) = I + \pi^{-1} \int_a^b (1 - I^2 f^{-2}(s))^{1/2} ds, \quad \omega(I) = \pi^{-1} \int_a^b I f^{-2}(s)(1 - I^2 f^{-2}(s))^{-1/2} ds - 1, \]
and by (6.6)
\[ \tau(I) = 2 \int_a^b \frac{dl}{ds} ds = 2 \int_a^b (1 - I^2 f^{-2}(s))^{-1/2} ds, \]
(6.10) follows. Proposition 6.3 is proved.

(6.8) implies that \( \xi(\gamma) \in L_+(n_\Gamma(p(\gamma))) = \{ \lambda n_\Gamma(p(\gamma)), \lambda > 0 \} \).

Hence we can define the map \( \pi: G \to N_+\Gamma \), where
\[ N_+\Gamma = \cup_{p \in \Gamma} L_+(n_\Gamma(p)), \]
as \( \pi: \gamma \to (p(\gamma), \xi(\gamma)) \). Observe that \( \pi \) is one–to–one.

**Proposition 6.4.** \( \gamma \in G \) is a closed geodesic iff \( \xi(\gamma) \in \mathbb{Z}^2 \). In this case
\[ Y(p(\gamma))|\xi(\gamma)| = (2\pi)^{-1} |\gamma|, \] (6.11)
\[ |\sigma(p(\gamma))|^{-1/2} |\xi(\gamma)|^{-3/2} = |\omega'(I)|^{-1/2} \tau(I)^{3/2} |\gamma|^{-3/2}, \] (6.12)
\[ \text{sgn} \sigma(p(\gamma)) = \text{sgn} \omega'(I), \quad \gamma = \gamma(I, l). \] (6.13)

**Proof.** Observe that \( \gamma(I, l) \), \( I \neq 0 \), is a closed geodesic with \( n_1 \) revolutions around the axis and \( n_2 \) oscillations along the meridian iff \( l = |\gamma| = n_2 \tau(I) \) and \( \omega(I) = (n_1/n_2) - 1 \), so that
\(\xi(\gamma) = (n_2, n_1 - n_2) \in \mathbb{Z}^2\). Similarly, \(\gamma = \gamma(0, l)\) is a closed geodesic iff \(l = |\gamma| = n_2 \tau(I)\), so that \(\xi(\gamma) = (n_2, 0)\). This proves the first part of Proposition 6.4.

To prove (6.11) let us notice that \(Y(p(\gamma))|\xi(\gamma)| = p(\gamma) \cdot \xi(\gamma)\), hence (6.11) follows from (6.9).

Let us prove (6.12). We have:
\[
\sigma(I) = -\frac{g''(I)}{(1 + (g'(I))^2)^{3/2}},
\]
and since \(g'(I) = -\omega(I)\),
\[
\sigma(p(\gamma)) = \frac{\omega'(I)}{(1 + \omega(I)^2)^{3/2}}.
\] (6.14)

On the other hand, by (6.7)
\[
|\xi(\gamma)| = |\gamma|\tau(I)^{-1}(1 + \omega(I)^2)^{1/2},
\]
hence
\[
|\sigma(p(\gamma))|^{-1/2}|\xi(\gamma)|^{-3/2} = |\omega'(I)|^{-1/2}(1 + \omega(I)^2)^{3/4}|\gamma|^{-3/2}\tau(I)^{3/2}(1 + \omega(I)^2)^{-3/4}
\]
\[
= |\omega'(I)|^{-1/2}\tau(I)^{3/2}|\gamma|^{-3/2}.
\]
(6.12) is proved. (6.13) follows from (6.14). Proposition 6.4 is proved.

Proof of Theorem 1.2. From Lemma 6.1 and Theorem 4.1 we obtain that
\[
N_{BS}(R) = AR^2 + R^{2/3} \sum_{k: \frac{n_{\Gamma}}{(w_k)} \text{ is rational}} \Phi_k(R) + R^{1/2}F(R),
\]
where \(\Phi_k(R)\) are periodic continuous functions and \(F(R) \in \mathcal{B}^2\). In addition, the Fourier series of \(\Phi_k(R)\) and \(F(R)\) are given in formulas (4.11) and (4.12), respectively. From Theorem 2.1 we obtain now that
\[
N(R) = AR^2 + R^{2/3} \sum_{k: \frac{n_{\Gamma}}{(w_k)} \text{ is rational}} \Phi_k(R) + R^{1/2}\hat{F}(R),
\]
where
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |F(R) - \hat{F}(R)|^2 dR = 0.
\]
This implies that \(\hat{F}(R) \in \mathcal{B}^2\) as well and the Fourier series of \(F(R)\) and \(\hat{F}(R)\) coincide. If we substitute formulas of Lemma 6.4 into (4.11), (4.12) we obtain (1.14), (1.15). Theorem 1.2 is proved.
Appendix. Proof of Lemma 5.2

For the sake of definiteness we will assume that $f'''(0) > 0$. Let $\Gamma^* = \{Y(\xi) = 1\} \cap \{\alpha_0 \leq \alpha(\xi) \leq \alpha_0 + \varepsilon\}$ and $\xi^0 = \Gamma^* \cap \{\alpha(\xi) = \alpha_0\}$. Consider a basis $e_1, e_2$ on the plane such that $e_1 = \xi^0$ and $e_2$ is parallel to the tangent vector to $\Gamma^*$ at $\xi_0$ (see Fig.6). In the basis $e_1, e_2$ the equation of $\Gamma^*$ has the form $\xi_1 = h(\xi^1_1^{1/2})$, where $h(t) \in C^\infty$, $h(0) = 1$, $h'(0) = h''(0) = 0$, $h'''(0) > 0$. Let us choose the length of $e_2$ in such a way that $h'''(0) = 2$, so that $h(t) = 1 + t^3/3 + \ldots$ for small $t$. Let $e^1_+, e^2_+$ be a dual basis to $e_1, e_2$. Let $0 \leq \lambda(t) \leq 1$ be a $C^\infty$ function on a line which is equal to 0 near 0, and which is equal to 1 when $t > 1$.

We will show that

$$I(R) \equiv \sum_{n \in \mathbb{Z}^2 \cap \Pi(R)} \lambda(n^+_2)\psi(n^+_2)(n^+_2)^{-1/2} \leq C R^{1/6},$$

(A.1)

where $n^+_j = n \cdot e^+_j$, $j = 1, 2$. Observe that $|\alpha(\xi) - \alpha_0|^{-1/2} \leq C_0 |\xi^+_2|^{-1/2} R^{1/2}$ when $\xi \in \Pi(R)$, hence from (A.1) Lemma 5.2 follows.

Let $Q(\xi; R) = \lambda(\xi^+_2)\psi(R^{-1}\xi^+_2)|\xi^+_2|^{-1/2}$, and $Q_0(\xi; R) = Q(\xi; R)$ if $\xi \in \Pi(R)$ and $Q_0(\xi; R) = 0$ otherwise, so that

$$I(R) = \sum_{n \in \mathbb{Z}^2} Q_0(n; R).$$

Let

$$Q_\delta(\xi; R) = \int_{\Pi(R)} Q(\eta; R)\delta^{-2}\varphi(\delta^{-1}(\xi - \eta))d\eta = Q_0(\cdot; R) * (\delta^{-2}\varphi(\delta^{-1}\cdot))(\xi), \hspace{1cm} \delta > 0,$$

where $\varphi(\xi) \in C^\infty_0(\mathbb{R}^2)$, $\varphi(\xi) \geq 0$, $\varphi(\xi) = 0$ if $|\xi| > 1$ and $\int_{\mathbb{R}^2} \varphi(\xi)d\xi = 1$, and

$$I_\delta(R) = \sum_{n \in \mathbb{Z}^2} Q_\delta(n; R).$$

Put

$$\delta = R^{-1/3}.$$

(A.2)

Observe that

$$Q_0(\xi; R) \leq C Q_\delta(\xi; R),$$

so $I(R) \leq C I_\delta(R)$ and to prove (A.1) it is sufficient to show that

$$I_\delta(R) \leq C R^{1/6}.$$

(A.3)
By the Poisson summation formula

\[ I_δ(R) = \sum_{n \in \mathbb{Z}^2} \hat{\varphi}(2\pi n δ) \hat{Q}(2πn; R). \]  

(A.4)

The term \( n = 0 \) is

\[ \hat{Q}(0; R) = \int_{\Pi(R)} Q(\xi; R) d\xi \leq CR^{-1/3} \int_0^{εR} |\xi_2^\perp|^{-1/2} d\xi_2^\perp \leq C_0 R^{1/6}, \]

hence we may consider only \( n \neq 0 \). Let \( p_1 = p \cdot e_1, \ p_2 = p \cdot e_2 \). If \( p_2 \geq γ|p_1|, \ γ > 0 \), then

\[ \hat{Q}(p; R) = \int_{\Pi(R)} Q(\xi; R) \exp(ip\xi) d\xi \]

(A.5)

can be estimated in the following way.

First we integrate in (A.5) by the lines \( |p|^{-1} p \cdot \xi = c \) and then in \( c \), so that

\[ \hat{Q}(p; R) = \int_{-∞}^{∞} \exp(i|p|c) S(c; R) dc, \]

where

\[ S(c; R) = \int_{\Pi(R) \cap \{|p|^{-1} p \xi = c\}} \lambda(\xi_2^\perp) \psi(R^{-1} \xi_2^\perp) \xi_2^\perp^{-1/2} d\xi. \]

If \( ε \ll γ \), then the lines \( |p|^{-1} p \cdot \xi = c \) cross \( \Pi(R) \) transversally, which implies that

\[ S(c; R) = R^{-1/3} λ_0(c - c_0) \psi_0(R^{-1}(c - c_0); R)|c - c_0|^{-1/2}, \]

where \( c_0 = R|p|^{-1} p \cdot ξ^0, \ λ_0(t) \in C^∞, \ λ_0(t) = 0 \) in vicinity of 0, and \( λ_0(t) = 1 \) in vicinity of \( ∞ \) and \( ψ_0(t; R) \in C^∞_0([0, ∞)) \) has a limit in \( C^∞_0 \)-topology as \( R \rightarrow ∞ \). Therefore

\[ |\hat{Q}(p; R)| \leq CR^{-1/3}|p|^{-5} \]

and

\[ \left| \sum_{n : |n \cdot e_2| ≥ γ|n \cdot e_1|, n \neq 0} \hat{Q}(2πn; R) \right| \leq CR^{-1/3}. \]  

(A.6)

The main difficulty is to estimate \( \hat{Q}(p; R) \) when \( |p_2| < γp_1, γ \geq 0 \). We have:

\[ \hat{Q}(p; R) = J_0 ∞ dt \exp(ip_2 t) λ(t) \psi(R^{-1} t) t^{-1/2} \int_a^b ds \exp(ip_1 s), \]

\[ t = \xi_2^\perp, \ s = \xi_1^\perp, \ a = h(t^{1/2}; R), \ b = h(t; R + R^{-1/3}), \]  

(A.7)
where $J$ is Jacobian and $h(t; R) = Rh(R^{-1}t)$. Let
\[
\tilde{Q}_0(p; R) = J \int_0^\infty dt \exp(ip_2t)\psi(R^{-1}t)t^{-1/2} \int_0^{b} ds \exp(ip_1s),
\] (A.8)
The difference $\tilde{Q}_1(p; R) = \tilde{Q}_0(p; R) - \tilde{Q}(p; R)$ can be estimated as follows.

Observe that
\[
\left| \int_0^\infty dt \exp(ip_2t)(1 - \lambda(t))\psi(R^{-1}t)t^{-1/2} \right| \leq C(1 + p_2)^{-1/2},
\]
hence
\[
|\tilde{Q}_1(p : R)| \leq C(1 + p_2)^{-1/2} R^{-1/3}
\]
and
\[
\left| \sum_{n : |n \cdot e_2| < \gamma|n \cdot e_1|, \ n \neq 0} \tilde{Q}_1(2\pi n; R) \right| \leq CR^{-1/3} \sum_{n \in \mathbb{Z}^2} |\tilde{\varphi}(2\pi n\delta)|(1 + |n \cdot e_2|)^{-1/2}
\]
\[
\leq C_0 R^{-1/3} \delta^{-3/2} = C_0 R^{1/6}.
\]
Thus it remains to estimate a similar sum with $\tilde{Q}_0(2\pi n; R)$.

Let us integrate in $s$ in (A.8) and make the change of variable $u = (R^{-1}t)^{1/2}$. This gives
\[
\tilde{Q}_0(p; R) = 2J(ip_1)^{-1} R^{1/2} (W(p; R) - U(p; R)),
\]
where
\[
U(p; R) = \int_0^\infty du \exp(ip_2 u^2)\psi(u^2) \exp(iR p_1 h(u)),
\]
\[
W(p; R) = \int_0^\infty du \exp(ip_2 u^2)\psi(u^2) \exp[iR(1 + R^{-4/3}) p_1 h((1 + R^{-4/3})u)].
\] (A.9)

Let us evaluate first
\[
U(p; R) = \int_0^\infty du \exp[ip_1(yu^2 + h(u))]\psi(u^2), \quad y = p_2/p_1.
\] (A.10)
There exists a $C^\infty$ change of variable $T = T(u, y)$ such that $T(0) = 0$, $\frac{\partial T}{\partial u}(0) = \frac{\partial T}{\partial y}(0) = 1$ and
\[
yu^2 + h(u) = b(y) + a(y)T + T^3/3,
\] (A.11)
where $a(y), b(y) \in C^\infty$, $a(0) = b(0) = 0$ (see [Hör2]). In addition,
\[
a(y) + T^2(0, y) = 0,
\]
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which follows if we differentiate both sides of (A.11) at $u = 0$. After this change of variable $U(p; R)$ reduces to

$$U(p; R) = \exp(iRp_1b(y)) \int_{c(y)}^{\infty} \exp(iRp_1(-c^2(y)T + T^3/3))\psi_0(T, y) \frac{\partial u}{\partial T}(T, y)\,dT, \quad (A.12)$$

with $c(y) = T(0, y)$ and $\psi_0(T, y) = \psi(t^2(T, y))$. Following [Hör2] let us divide $(\partial u/\partial T)(T, y)$ by $-c^2(y) + T^2$ with a remainder:

$$\frac{\partial u}{\partial T}(T, y) = r(T, y)(-c^2(y) + T^2) + r_0(y) + r_1(y)T.$$

Then the first term after substitution into (A.12) allows integration by parts, which gives an extra $(Rp_1)^{-1}$, and the other two terms give the main contribution to $U(p; R)$:

$$U(p; R) = \exp(iRp_1b(y))\{(Rp_1)^{-1/3}V(c(y)(Rp_1)^{1/3})r_0(y) + (Rp_1)^{-2/3}V_0(c(y)(Rp_1)^{1/3})r_1(y)\} + O((Rp_1)^{-1}),$$

(A.13)

where

$$V(x) = \int_x^{\infty} \exp[i(-x^2T + T^3/3)]dT, \quad V_0(x) = \int_x^{\infty} T \exp[i(-x^2T + T^3/3)]dT.$$

The method of stationary phase gives the asymptotics of $V(x), V_0(x)$ when $|x| \to \infty$, and from this asymptotics we obtain $|V(x)| \leq C(1 + |x|)^{-1/2}$ and $|V_0(x)| \leq C(1 + |x|)^{1/2}$. Therefore

$$|U(p; R)| \leq C|Rp_1|^{-1/3}\min\{1, (|p_2/p_1||Rp_1|^{1/3})^{-1/2}\} = C\min\{R^{-1/2}|p_2|^{-1/2}, R^{-1/3}|p_1|^{-1/3}\}.$$  

A similar estimate holds for $W(p; R)$ and finally we obtain

$$|\tilde{Q}_0(p; R)| \leq C|p_1|^{-1}\min\{|p_2|^{-1/2}, R^{1/6}|p_1|^{-1/3}\}.$$  

Hence

$$\left| \sum_{|n \cdot e_2| \leq 1} \tilde{\varphi}(2\pi n\delta)\tilde{Q}_0(2\pi n : R) \right| \leq CR^{1/6} \sum_{|n \cdot e_1| \leq 1} |n \cdot e_1|^{-4/3} \leq C_0R^{1/6},$$

and

$$\left| \sum_{1 < |n \cdot e_2| \leq \gamma |n \cdot e_1|} \tilde{\varphi}(2\pi n\delta)\tilde{Q}_0(2\pi n : R) \right| \leq C \sum_{1 < |n \cdot e_2| \leq \gamma |n \cdot e_1|} |\tilde{\varphi}(2\pi n\delta)||n \cdot e_1|^{-1}|n \cdot e_2|^{-1/2} \leq C_0\delta^{-1/2} = C_0R^{1/6}.$$

Lemma 5.2 is proved.

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Figures Captions

Fig. 1. Geodesic on a surface of revolution.

Fig. 2. The phase function $\omega(I)$ for different surfaces of revolution. Cross-sections of the surfaces of revolution are shown in the lower part of the figure.

Fig. 3. Curve $\Gamma$ for the surfaces of revolution shown on Fig. 2.

Fig. 4. Sectorial domain with points of inflection on the boundary.

Fig. 5. Local structures of $\Gamma$.

Fig. 6. Basis $e_1, e_2$. 