Inflationary cosmology from STM theory of gravity

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Abstract

I study the power-law and de Sitter expansions for the universe during inflation from the STM theory of gravity. In a de Sitter expansion the additional dimension is related to the cosmological constant Λ = 3/ψ^2. I find from experimental data that the mass of the inflaton field is m^2 = 2/(3ψ^2). The interesting in this case is that the inflaton field fluctuations are related to the fifth coordinate. In power-law expansion, the fifth coordinate (ψ) appears to be a dimensionless constant. Here, the ψ-value depends on the initial conditions. I find the 5D line element for this inflationary expansion, which is a function of the classical component of the inflaton. But the more important result here obtained is that in both cases there isn’t dynamical compactification during inflation.

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I. INTRODUCTION

The extra dimension is already known to be of potential importance for cosmology. There is a class of five-dimensional (5D) cosmological models which reduce to the usual four-dimensional ones, on hypersurfaces defined by setting the value of the extra coordinate equal to a constant [1]. In these models, physical properties such as the mean energy density and pressure of matter are well defined consequences of how the extra coordinate enters the metric [2]. That is, matter is explained as the consequence of geometry in five dimensions. The physics of this follows from a mathematical result, which is that Einstein’s equations of general relativity with matter are a subset of the Kaluza-Klein (KK) equations in apparent vacuum. Hence, cosmology is a subject of pure geometry in five-dimensional Kaluza-Klein gravity known as space-time-matter (STM) theory. This theory is supported by more local astrophysics [3]. There is a canonical class of five-dimensional metrics, most often discussed in the form due to Gross with Perry [4] and Davidson with Owen [5]. During the last years

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there were many attempts to construct a consistent brane cosmology [6]. However, after a detailed investigation of this possibility a series of no-go theorems have been proven [7]. There is a main difference between STM and Brane-World (BW) [8] formalisms. In the STM theory of gravity we don’t need to insert any matter into the 5D manifold by hand, as is commonly done in the BW formalism. In the STM theory of gravity the 5D metric is an exact solution of the 5D field equations in apparent vacuum [11]. The interesting here, is that matter appears in four dimensions without any dimensional compactification, but induced by the 5D vacuum conditions.

On the other hand, inflationary cosmology is one of the most reliable concepts in modern cosmology. The first model of inflation was proposed by A. Starobinsky in 1979 [12]. A much simpler inflationary model with a clear motivation was developed by Guth in the 80’s [13], in order to solve some of the shortcomings of the big bang theory, and in particular, to explain the extraordinary homogeneity of the observable universe. However, the universe after inflation in this scenario becomes very inhomogeneous. Following a detailed investigation of this problem, A. Guth and E. Weinberg concluded that the old inflationary model could not be improved [14].

These problems were sorted out by A. Linde in 1983 with the introduction of chaotic inflation [15]. In this scenario inflation can occur in theories with potentials such as $V(\varphi) \sim \varphi^n$. It may begin in the absence of thermal equilibrium in the early universe, and it may start even at the Planckian density, in which case the problem of initial conditions for inflation can be easily solved [16]. According to the simplest versions of chaotic inflationary theory, the universe is not a single expanding ball of fire produced in the big bang, but rather a huge eternally growing fractal. It consists of many inflating balls that produce new balls, which produce more new balls, ad infinitum.

BW cosmology has been studied very recently within [9] and without inflation [10]. However, inflationary cosmology from the STM theory, still remains without a detailed study. The aim of this paper is the study of inflation from the STM theory of gravity. More exactly, I am interested in the study of de Sitter and power-law expansions for the universe during inflation from the STM theory developed by P. Wesson [17]. This paper is organized as follows. In Sec. II, I study the STM basic equations. In Sec. III, I develop the classical and quantum field dynamics during the inflationary stage using the semiclassical approach developed in [18]. Furthermore, de Sitter and power-law expansions for the universe are examined from the STM theory of gravity and the results are contrasted with experimental data. Finally, in Sec. IV some final comments are developed.

II. BASIC STM EQUATIONS

Following the idea suggested by Wesson and co-workers [11], in this section I develop the induced 4D equation of state from the 5D vacuum field equations, $G_{AB} = 0$ ($A, B = 0, 1, 2, 3, 4$), which give the 4D Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$). In particular, we consider a 5D spatially isotropic and flat spherically-symmetric line element

$$ds^2 = -e^{\alpha(\psi,t)}dt^2 + e^{\beta(\psi,t)}dr^2 + e^{\gamma(\psi,t)}d\psi^2,$$  

(1)
where \( dr^2 = dx^2 + dy^2 + dz^2 \) and \( \psi \) is the fifth coordinate. We assume that \( e^\alpha, e^\beta \) and \( e^\gamma \) are separable functions of the variables \( \psi \) and \( t \). The equations for the relevant Einstein tensor elements are

\[
G_{00}^0 = -e^{-\alpha} \left[ \frac{3\dot{\beta}^2}{4} + \frac{3\dot{\beta}\dot{\gamma}}{4} \right] + e^{-\gamma} \left[ \frac{3\dot{\beta}^2}{2} + \frac{3\dot{\beta}\dot{\gamma}}{2} - \frac{3\gamma\dot{\beta}}{4} \right],
\]

\[
G_{04}^0 = e^{-\alpha} \left[ \frac{3\dot{\beta}}{2} + \frac{3\dot{\beta}\dot{\alpha}}{4} - \frac{3\dot{\beta}\dot{\gamma}}{4} \right],
\]

\[
G_{i}^i = -e^{-\alpha} \left[ \ddot{\beta} + \frac{3\dot{\beta}^2}{4} + \frac{\dot{\gamma}^2}{4} + \frac{\dot{\beta}\dot{\gamma}}{2} - \frac{\dot{\alpha}\dot{\beta}}{2} - \frac{\dot{\alpha}\dot{\gamma}}{4} \right]
+ e^{-\gamma} \left[ \ddot{\beta} + \frac{3\dot{\beta}^2}{4} + \frac{\dot{\alpha}^2}{4} + \frac{\dot{\beta}\dot{\alpha}}{2} - \frac{\dot{\gamma}\dot{\beta}}{2} - \frac{\dot{\gamma}\dot{\alpha}}{4} \right],
\]

\[
G_{4}^4 = e^{-\alpha} \left[ \frac{3\ddot{\beta}}{2} + 3\dot{\beta}^2 - 3\dot{\alpha}\dot{\beta} \right] + e^{-\gamma} \left[ \frac{3\dot{\beta}^2}{4} + \frac{3\dot{\beta}\dot{\alpha}}{4} \right],
\]

where the overstar and the overdot denote respectively \( \frac{\partial}{\partial \psi} \) and \( \frac{\partial}{\partial t} \), and \( i = 1, 2, 3 \). Following the convention \((-,-,+,-,+\)) for the 4D metric, we define \( T_{00} = -\rho_t \) and \( T_{11} = p \), where \( \rho_t \) is the total energy density and \( p \) is the pressure. The 5D-vacuum conditions \( G_{AB}^A = 0 \) are given by [17]

\[
8\pi G \rho_t = \frac{3}{4} e^{-\alpha} \ddot{\beta}^2,
\]

\[
8\pi G p = e^{-\alpha} \left[ \frac{\dot{\alpha}\dot{\beta}}{2} - \ddot{\beta} - \frac{3\dot{\beta}^2}{4} \right],
\]

\[
e^\alpha \left[ \frac{3\dot{\beta}^2}{4} + \frac{3\dot{\beta}\dot{\alpha}}{4} \right] = e^{-\gamma} \left[ \frac{\ddot{\beta}}{2} + \frac{3\dot{\beta}^2}{2} - \dot{\alpha}\dot{\beta} \right].
\]

Hence, from eqs. (6) and (7) and taking \( \dot{\alpha} = 0 \), we obtain the equation of state for the induced matter

\[
p = -\left( \frac{4}{3} \frac{\ddot{\beta}}{\dot{\beta}^2} + 1 \right) \rho_t.
\]

Notice that for \( \ddot{\beta}/\dot{\beta}^2 \leq 0 \) and \( |\ddot{\beta}/\dot{\beta}^2| \ll 1 \) (or zero), this equation describes an inflationary universe. The equality \( \ddot{\beta}/\dot{\beta}^2 = 0 \) corresponds with a 4D de Sitter expansion for the universe.

In this paper I explore the possibility to obtain inflation for the metrics (1) with the restrictions

\[
\alpha \equiv \alpha(\psi); \quad \beta \equiv \beta(\psi, t); \quad \gamma \equiv \gamma(t),
\]

where \( e^\beta \) is a separable function of \( \psi \) and \( t \). The conditions (10) imply that \( \dot{\alpha} = \dot{\gamma} = 0 \). Furthermore, in that follows we restrict our study to inflationary models on 5D metrics.
where all the coordinates are independent. The choice (10) implies that only the spatial
sphere and the the fifth coordinate have respectively squared sizes \( e^\beta \) and \( e^{\gamma} \) that evolve
with time.

**III. INFLATIONARY UNIVERSE**

The dynamics of a scalar field minimally coupled to gravity is described by the Lagrangian
density

\[
L(\phi, \dot{\phi}) = -\sqrt{-g} \left[ \frac{R}{16\pi} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right],
\]

where \( R \) is the 4D scalar curvature and \( g \) is the determinant of the 4D metric tensor. If
the spacetime has a Friedman-Robertson-Walker (FRW) metric, \( ds^2 = -d\tau^2 + a^2(\tau)dr^2 \), the
resulting equations of motion for the field operator \( \phi \) and the Hubble parameter, are

\[
\ddot{\phi} - \frac{1}{a^2} \nabla^2 \phi + 3H \dot{\phi} + V'(\phi) = 0,
\]

\[
H^2 = \frac{4\pi}{3M_p^2} \left\langle \dot{\phi}^2 + \frac{1}{a^2} (\nabla \phi)^2 + 2V(\phi) \right\rangle,
\]

where, in the following the overdot represents the derivative with respect to \( \tau \) and \( V'(\phi) \equiv \frac{dV}{d\phi} \). The expectation value is assumed to be a constant function of the spatial variables for
consistency with the FRW metric. The discussion of the inflationary stage is usually done
through the classical analysis of the above equations, which in the case of a homogeneous
scalar field and null spatial curvature reduce to a system of two first order equations, even
without any slow-roll assumption [19]. The period in which \( \ddot{a} > 0 \) and thus the universe
has an accelerated expansion is the inflationary stage, and models are discarded or not
depending on if they provide enough inflation or not. When the inflation ends the field
starts oscillating rapidly and its potential energy is converted into thermal energy. This is
the general scheme of the inflationary scenario without considering the quantum effects. In
the usual formulation of this approach the slow-roll regime is assumed. Instead, we avoid
here the use of such an assumption and consider a consistent semiclassical expansion of the
theory. To obtain this we decompose the inflaton operator in a classical field \( \phi_c \) plus a
quantum correction \( \phi, \phi = \phi_c + \phi \), such that \( \langle \phi \rangle = \phi_c \) and \( \langle \phi \rangle = 0 \). The field
\( \phi_c \) is defined as the solution to the classical equation of motion

\[
\ddot{\phi}_c + 3H \dot{\phi}_c + V'(\phi_c) = 0,
\]

where we have assumed that the classical field is spatially homogeneous, in agreement with
the hypothesis of an inflationary regime. The evolution of the quantum operator \( \phi \) is given by

\[
\ddot{\phi} - \frac{1}{a^2} \nabla^2 \phi + 3H \dot{\phi} + \sum_n \frac{1}{n!} V^{(n+1)}(\phi_c) \phi^n = 0.
\]

At the same time the squared Hubble parameter can be expanded as
\[ H^2 = H_c^2 + \frac{4\pi}{3M_p^2} \left( \dot{\phi}^2 + \frac{1}{a^2} (\nabla \phi)^2 + \sum_n \frac{1}{n!} V^{(n+1)}(\phi_c)\phi^n \right), \]  

(16)

where

\[ H_c^2 = \frac{4\pi}{3M_p^2} \left[ \phi_c^2 + 2V(\phi_c) \right], \]  

(17)

is the classical Hubble parameter. If the quantum fluctuations are small, hence the inflation is driven by the classical field. The effect of the classical field \( \phi_c \) is to change the environment in which the \( \phi \) field evolves, and in particular the mass of the \( \phi \) particles. Making a first order expansion in eq. (16) for \( \phi \), we obtain \( H \equiv H_c = \dot{a}/a \) in eq.(15), which reduces to

\[ \ddot{\phi} - \frac{1}{a^2} \nabla^2 \phi + 3H_c \dot{\phi} + V'(\phi_c) \phi = 0. \]  

(18)

From eqs. (14) and (17), we can describe the classical dynamics of the Hubble parameter and the inflaton field by the relations

\[ \dot{\phi}_c = -\frac{M_p^2}{4\pi} H_c', \]  

(19)

\[ \dot{H}_c = H_c' \dot{\phi}_c = -\frac{M_p^2}{4\pi} (H_c')^2. \]  

(20)

In these terms the potential accounting for such a dynamics is given by:

\[ V(\phi_c) = \frac{3M_p^2}{8\pi} \left( H_c^2 - \frac{M_p^2}{12\pi} (H_c')^2 \right). \]  

(21)

Equations (19) and (20) define the classical evolution of spacetime, determining the relation between the classical potential and the inflationary regimes. On the other hand eq. (8) defines the quantum dynamics of the field \( \phi \), whose classical character was studied in [18,20]. In this framework, the total energy density and the pressure (neglecting the terms \( \langle \dot{\phi}^2 \rangle /2 \), because they are very small on cosmological scales), are given by

\[ \langle \rho_t \rangle = \frac{\phi_c^2}{2} + V(\phi_c), \]  

(22)

\[ \langle p \rangle = \frac{\phi_c^2}{2} - V(\phi_c), \]  

(23)

such that the equation of state for supercooled inflation is

\[ \langle p \rangle = -\left( \frac{2H_c}{3H_c^2} + 1 \right) \langle \rho_t \rangle. \]  

(24)

From eqs. (9) and (24) one obtains the Hubble parameter as a function of \( \beta \) for inflationary models with \( \gamma = \psi = 0 \)

\[ H_c = \dot{\beta}/2. \]  

(25)
Furthermore, from eqs. (25) and (21) we can write the scalar potential for inflationary models that follows from the 5D metric (1)

$$V[\phi_c(t)] = \frac{3M_p^2}{2\pi} \left[ \dot{\beta}^2 + \frac{2}{3} \ddot{\beta} \right].$$

(26)

The equation (25) shows that models with $H_c = \text{const.}$ (like a de Sitter one) are described with potentials

$$V[\phi_c(t)] = \frac{3M_p^2}{2\pi} \dot{\beta}^2.$$  

(27)

To map the eq. (8) in a wave equation, we can make the transformation $\chi = e^{\frac{3}{2} \int dt H_c} \phi = a^{\frac{3}{2}} \phi$

$$\ddot{\chi} - \frac{1}{a^2} \nabla^2 \chi - \frac{k_0^2}{a^2} \chi = 0,$$  

(28)

where $k_0^2 = a^2 \left( \frac{2}{3} H_c^2 + \frac{3}{2} \dot{H}_c - V' \right)$. We now have a scalar field with a time-dependent mass. It can be described in terms of a set of modes

$$\chi = \frac{1}{(2\pi)^{d/2}} \int d^3k \left[ a_k e^{i\vec{k} \cdot \vec{r}} \xi_k(t) + h.c. \right],$$

(29)

where we have made explicit use of the spatial homogeneity of the FRW metric. The operators $a_k$ and $a_k^\dagger$ satisfy the canonical commutators:

$$[a_k, a_k^\dagger] = \delta(\vec{k} - \vec{k}'),$$

$$[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0,$$

(30)  

(31)

and the equation of motion for the modes $\xi_k(t)$ are given by

$$\ddot{\xi}_k + \omega_k^2 \xi_k = 0,$$  

(32)

with $\omega_k^2 = \frac{1}{a^2} (k^2 - k_0^2)$. Here, $k_0(\tau)$ separates the large scale (unstable) sector ($k^2 < k_0^2$) and the short scale (stable) one ($k^2 > k_0^2$). When $k_0$ surpasses $k$, the temporal oscillation of the mode ceases. During the expansion of the universe, more and more new fluctuations suffer this transformation. These quantum fluctuations with wave number below $k_0$ are generally interpreted as inhomogeneities superimposed to the classical field $\phi_c$. They are responsible for the density inhomogeneities generated during the inflation.

In order that $\chi$ and $\dot{\chi}$ satisfy a canonical commutation relation, $[\chi(\vec{r}, t), \dot{\chi}(\vec{r}', t)] = i\delta(\vec{r} - \vec{r}')$, the normalization of the modes $\xi_k(t)$ must be chosen

$$\xi_k \xi_k^* - \dot{\xi}_k \xi_k^* = i,$$  

(33)

where the asterisk denotes the complex conjugate. In the following subsections I will study two particular cases of inflation from the STM theory of gravity; de Sitter and power-law expansions.
A. de Sitter expansion

The special case $e^\alpha = \psi^2$ in eq. (26) [see eqs. (25) and (27)], with $\ddot{\beta}/\dot{\beta}^2 = 0$, $\dot{\gamma} = 0$ in eq. (1), gives a de Sitter expansion for which $\langle \rho_t + p \rangle = 0$, so that $\phi_c = \phi_0$, $V(\phi_c) = V_0$ and $H_c(\tau) = H_0$ are constant [18]. It implies that

$$H_0^2 = \frac{8\pi}{3M_p^2} V_0. \quad (34)$$

This case corresponds exactly to a scalar field $\phi$ with a mass $m^2 = \left( \frac{dV}{d\phi^2} \right)_{\phi = \phi_0}$ in a de Sitter background with a constant Hubble parameter $H_0$ and a scale factor $a(t) \sim e^{H_0 \tau}$. For this model the cosmological constant $\Lambda$ gives the vacuum energy density

$$\langle \rho_t \rangle = \frac{\Lambda}{8\pi G}, \quad (35)$$

such that $\Lambda$ is related with the fifth coordinate by means of

$$\Lambda = \frac{3}{\psi^2}. \quad (36)$$

The 5D-metric for a de Sitter expansion of the universe is [1]

$$dS^2 = -d\tau^2 + \psi^2 e^{-2\sqrt{\Lambda/(3\psi^2)} \tau} dr^2 + d\psi^2. \quad (37)$$

Hence, the 4D de Sitter inflationary universe can be seen as a 5D metric with the fifth constant coordinate and unitary size. The equation (35) shows that the vacuum on the metric (37) induces the cosmological constant $\Lambda$, which depends on the value of the fifth coordinate and generates the expansion of the universe.

On the other hand, the wave number $k_0$ is

$$k_0 = H_0 a(\tau) \sqrt{\frac{9}{4} - \frac{m^2}{H_0^2}}. \quad (38)$$

The equation of motion for the modes in a de Sitter expansion can be written explicitly

$$\ddot{\xi}_k(\tau) + \left[ k^2 e^{-2H_0 \tau} - \frac{9}{4} H_0^2 + m^2 \right] \xi_k(\tau) = 0, \quad (39)$$

and its general solution is

$$\xi_k(\tau) = A_1 \mathcal{H}^{(1)}_\nu \left[ \frac{k}{H_0 a} \right] + A_2 \mathcal{H}^{(2)}_\nu \left[ \frac{k}{H_0 a} \right], \quad (40)$$

where $\mathcal{H}^{(1)}_\nu$ and $\mathcal{H}^{(2)}_\nu$ are the Hankel functions and $\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H_0^2}} < \frac{3}{2}$. The long time behavior of the mean square of $\phi$ reproduces the exact value as calculated by standard quantum field methods [22]. For a massive inflaton (i.e., with $m/H_0 \ll 1$), once we taken into account the normalization condition for the modes (33), we have
\[ \langle \phi^2 \rangle |_{SH} \approx \frac{a^{-(3-2\nu)}(\tau)}{2^{3-2\nu} \pi^3} \Gamma^2(\nu) H_0^{2\nu-1} \int_0^{k_0} \frac{dk}{k} k^{3-2\nu}, \]  

where $\Gamma(\nu)$ is the gamma function with $\nu$-argument. Hence, the scale invariant spectrum $n_s = 1$ (i.e., $\nu = 1/2$) implies that $m^2 = 2H_0^2 = 2/(3\psi^2)$, and

\[ \langle \phi^2 \rangle |_{SH} \approx \frac{\Gamma^2(1/2)}{24\pi^3} \psi^{-2}, \]  

where $\Gamma$ is the gamma function. Hence, the fifth coordinate also should be related to the inflaton fluctuations for a scale invariant power spectrum.

### B. Power-law inflation

To study a power-law expansion of the universe, we can make $e^\alpha = \psi^s$, $e^\beta = a^2(t)\psi^r$ and $e^\gamma = A t^n$, in eq. (1). Hence, the 5D-line element can be written as

\[ dS^2 = -\psi^s dt^2 + a^2(t)\psi^r dr^2 + At^n d\psi^2, \]  

where the scale factor is $a \sim t^p$. From the 5D-vacuum conditions (6-8), we find

\[ s = n = 2, \quad r = \frac{2p}{p-1}, \quad A = (p-1)^{-2}, \]  

such that we obtain the Ponce de Leon metric [1]

\[ dS^2 = -\psi^2 dt^2 + a^2(t)\psi^2 \tau^{-1} dr^2 + (p-1)^{-2} t^2 d\psi^2. \]  

This metric is an exact solution of the 5D field equations in apparent vacuum, which in terms of the Ricci tensor are $R_{AB} = 0$. The induced metric $h_{\alpha\beta}$ on these hypersurfaces is given by

\[ ds^2 = h_{\alpha\beta} dx^\alpha dx^\beta = -d\tau^2 + a^2(\tau) d\psi^2, \]  

where $\tau = \psi t$ is the cosmic time which corresponds to the spatially flat FRW metric. With this representation, the scale factor is

\[ \alpha(\tau) = \psi^{\frac{p(2-p)}{p-1}} \tau^p. \]  

I am interested in the study of power-law inflationary dynamics from the metric (45). In this case $p > 1$, and the equation of state if matter is interpreted as a perfect fluid, is

\[ \langle p \rangle = -\left(\frac{3p-2}{3p}\right) \langle \rho_t \rangle. \]  

From eqs. (20) and (26), one obtains

\[ (H_c')^2 = -\frac{2\pi}{M_p^2} \beta. \]  

8
Replacing in eq. (21) the scalar potential can be obtained as a function of $\beta(\psi, t)$ Furthermore the constant $p$ is related with the properties of matter

$$p = \frac{2}{3} \left( \frac{<\rho_t>}{<\rho_t + p>} \right),$$

and the temporal evolution for $\phi_c(\tau)$ can be obtained from eq. (19)

$$\phi_c(\tau) = -\frac{M_p}{2} \sqrt{\frac{p}{\pi}} \ln \left[ \frac{H_0}{p \tau} \right].$$

Hence, due to $\tau = \psi t$, we obtain the value of the constant $\psi$

$$\psi = \frac{p}{H_0 t_0} e^{-\sqrt{\frac{2}{3}p} \sqrt{\phi_0}},$$

where $(H_0, t_0, \phi_0)$ are respectively the values of $(H_c, t, \phi_c)$ when inflation starts. This implies that $\psi$ should be strongly dependent of initial conditions. Note that $p$ is related to the properties of matter such as the energy density $\rho_t$ and pressure $p$. In the induced-matter interpretation of KK theory, these are the result of the geometry in five dimensions, insofar as they are functions of the extra part of the metric and partial derivatives of the metric coefficients with respect to the extra coordinate. The field equations of the theory in terms of the Ricci tensor are $R_{AB} = 0$. These appear to describe empty five-dimensional space. But they in fact contain the Einstein equations $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$.

Finally, taking into account the results above, the Ponce de Leon metric (45) can be written as

$$dS^2 = -d\tau^2 + \psi^{2p-1} \left( \frac{p}{H_0 \psi} \right)^{2p} e^{-4\sqrt{\frac{p}{3}}M_p \phi_c(\tau)} d\tau^2 + \left( \frac{1}{H_0 \psi} \right)^{2} e^{-4\sqrt{\frac{2}{3}p} \phi_c(\tau)} d\psi^2,$$

where $(p, \phi_c, \psi)$ are given respectively by eqs. (50), (51) and (52). Notice the $\phi_c(\tau)$-dependence of the metric (53), which make the difference between power-law inflationary scenarios and other cosmological models with $p < 1$, studied in [1]. From eq. (53) can be view that the size of the fifth coordinate grows with $\tau$, but the 5D vacuum here required implies that $\psi = \text{const}$. Hence, for a power-law inflationary expansion, the 4D universe that expands from a metastable 4D vacuum with a scalar potential given by eq. (26), is really a 5D universe in a true vacuum with a metric given by eq. (53). The differential equation (32), has the solution

$$\xi_k(\tau) = A_1 \sqrt{\frac{\tau}{\tau_0}} H^{(1)}_{\nu} \left[ k \tau^{1-p} H_0 \right] + A_2 \sqrt{\frac{\tau}{\tau_0}} H^{(2)}_{\nu} \left[ k \tau^{1-p} H_0 \right],$$

where $H^{(1)}_{\nu}$ and $H^{(2)}_{\nu}$ are the Hankel functions and $\nu = \sqrt{\frac{2}{4}p^2 - \frac{15}{2}p + \frac{9}{4}}/(p - 1)$. From the normalization condition (33), we obtain the adiabatic vacuum

$$\xi_k(\tau) = \sqrt{\frac{\tau}{\tau_0}} \sqrt{\frac{\pi}{2}} H^{(2)}_{\nu} \left[ k \tau^{1-p} H_0 \right].$$
The Hankel functions take the small-argument limit \( H_{\nu}^{(2,1)}[x] \bigg|_{x \ll 1} \simeq \frac{(x/2)^{\nu}}{\Gamma(1+\nu)} \pm \frac{i}{\pi} \Gamma(\nu) (x/2)^{-\nu} \). Hence, the super Hubble squared fluctuations \( \langle \phi^2 \rangle \big|_{SH} \) are

\[
\langle \phi^2 \rangle \big|_{SH} \simeq \frac{4^{\nu-1} \Gamma^2(\nu)(p-1)^{2\nu-1}}{\pi H_0^{3+2\nu}} \frac{\tau^{-p(3-2\nu)-2\nu+1}}{\tau_0^{-p(3-2\nu)+1}} \int_0^{k_0} \frac{dk}{k} k^{3-2\nu}, \tag{56}
\]

where \( k_0(\tau) = \frac{\psi(p(2-p)}{\nu-p-1} \left[ \frac{9}{2} p^2 - \frac{15}{2} p + 1 \right]^{1/2} \tau^{1-p} \) and \( \Gamma(\nu) \) is the gamma function with \( \nu \)-argument. The integral controlling the presence of infrared divergences is \( \int_0^{k_0} dk k^{2(1-\nu)} \), which has a power spectrum \( P_{\langle \delta \phi \rangle^2} \sim k^{3-2\nu} \). We can evaluate the expression (56), and we obtain

\[
\langle \phi^2 \rangle \big|_{SH} \simeq \frac{4^{\nu-1} \Gamma^2(\nu) \left[ \frac{2}{p^2} - \frac{15}{2} p + 1 \right]^{1/2} \left( \frac{3-2\nu}{2} \right) (p-1)^{2\nu-1} \psi(p(2-p)(3-2\nu)}{\pi H_0^{3+2\nu} \tau_0^{-p(3-2\nu)}} \tau^{1-2\nu}. \tag{57}
\]

The spectral index \( n_s = 3/2 - \nu \simeq 1 \) agrees with the experimental data [21] for \( \nu \simeq 1/2 \) (i.e., for \( p \simeq 3.08 \)). For this value of \( \nu \), we obtain a time independent \( \langle \phi^2 \rangle \big|_{SH} \), so that the SH metric fluctuations in (53) remains constant for a scale invariant power spectrum.

**IV. FINAL COMMENTS**

In this work we studied the evolution of the early universe from the STM theory of gravity. In this framework inflation should be a consequence of the expansion generated by a cosmological “constant” (variable in power-law inflation and constant in a de Sitter expansion) induced by a generalized 5D spatially flat, isotropic homogeneous background metric where the fifth dimension becomes constant as a consequence of the 5D vacuum state.

The condition to obtain an inflationary expansion is

\[
\langle p \rangle = -\left( \frac{4}{3} \frac{\ddot{\beta}}{\dot{\beta}^2} + 1 \right) \langle \rho_i \rangle \simeq -\langle \rho_i \rangle ,
\]

or respectively

\[
\frac{\ddot{\beta}}{\dot{\beta}^2} \ll 1, \quad \frac{\ddot{\beta}}{\dot{\beta}^2} = 0,
\]

for power-law and de Sitter expansions where \( \dot{\beta} = 2H_c \).

In the models here studied all of the components of the Riemann-Christoffel tensor for the 5D metric are zero. Despite this, the model’s 4D part is not flat, since the 4D Ricci scalar is non zero. Hence, we see that while the inflationary universe may be curved in 4D, it is flat in 5D. Thus, the 5D STM theory of gravity describes an inflationary universe consisting of localized singular sources embedded in a globally 5D flat cosmology.

We have restricted our study to 4D spatially flat FRW metrics, which are relevant to inflationary cosmology. We studied the power-law and de Sitter models of inflation. We find that a) for a de Sitter expansion the situation appears to be different. In this case the
additional dimension is related to the cosmological constant $\Lambda = 3/\psi^2$ and the Hubble parameter is $H_0 = 1/\sqrt{3\psi^2}$. Hence, we inquire that $\psi$ has $G^{1/2}$ unities. From the experimental data obtained from BOOMERANG-98, MAXIMA-1 and COBE DMR [21], we obtain that $m^2\psi^2 = 2/3$. This implies that, in a de Sitter expansion with a 5D vacuum state, $\psi$ gives nearly the inverse of the mass of the inflaton field. Hence, for $m^2 \simeq (10^{-8} - 10^{-12}) M_p^2$, the resulting fifth coordinate will be of the order of $\psi^2 \simeq (10^8 - 10^{12}) M_p^{-2}$. Note that the inflaton field fluctuations are also related to the fifth coordinate [see eq. (42)]. Hence, in the induce-matter interpretation of KK theory, matter field fluctuations also appear as a consequence of the five dimensional geometry. b) For the power-law expansion, the fifth coordinate appears to be a dimensionless constant, which, in principle, could be considered as 1 (notice that we are considering $\hbar = 1$). The interesting here is that the $\psi$-value depends on the initial conditions during power-law inflation [see eq. (52)]. Furthermore, the main result of this paper is the metric (53) which describes the 5D line element in power-law inflation. Notice the dependence with the spatially isotropic component of the inflaton field, $\phi_c$. Furthermore, it is very important the fact that the squared size of the fifth coordinate $e^\gamma$ grows with time in a power-law inflationary universe.

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