Extreme fluctuations of the relative velocities between droplets in turbulent airflow

Ewe-Wei Saw,1,2 Gregory P. Bewley,1 Eberhard Bodenschatz,1 Samriddhi Sankar Ray,3 and Jérémie Bec2
1Max Planck Institute for Dynamics and Self-Organization (MPIDS), 37077 Göttingen, Germany
2Laboratoire J.-L. Lagrange, Université de Nice-Sophia Antipolis, CNRS, Observatoire de la Côte d’Azur, 06300 Nice, France
3International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bangalore 560012, India

(Dated: 8 July 2014)

We compare experiments and direct numerical simulations to evaluate the accuracy of the Stokes-drag model, which is used widely in studies of inertial particles in turbulence. We focus on statistics at the dissipation scale and on extreme values of relative particle velocities for moderately inertial particles (St < 1). The probability distributions of relative velocities in the simulations were qualitatively similar to those in the experiments. The agreement improved with increasing Stokes number and decreasing relative velocity. Simulations underestimated the probability of extreme events, which suggests that the Stokes drag model misses important dynamics. Nevertheless, the scaling behavior of the extreme events in both the experiments and the simulations can be captured by the same multi-fractal model.

In warm clouds (where the cloud particles are liquid water), turbulence in the airflow enhances the collision rate of the water droplets. It thus influences the evolution of droplet sizes and the timescale for rain formation.2-4 Two mechanisms are at play: preferential concentration, which is due to a combination of dissipative dynamics and non-trivial correlations between the fluid flow and particle positions, and very large approach velocities, which have been explained in terms of the sling effect and the subsequent formation of caustics.5-8 Many questions remain open regarding the impact of such phenomena on the coalescence rate of droplets. Whilst it is generally accepted that turbulence increases the rate at which particles collide, collisions that are too violent can cause particle fragmentation.9 To produce reliable models for coalescence efficiencies, a key issue is to understand how often this occurs. Such considerations are decisive for unravelling the impact of turbulence on the size distribution of droplets in clouds.

Contemporary theories and simulations of heavy particle dynamics in turbulent flows predominantly assume point particles coupled to the flow through a linear Stokes drag. This simplified framework may be justified when the particles are (a) smaller than the smallest scales of the flow, (b) made of material much denser than the fluid (‘heavy’ particles), and (c) far apart. Clearly the last assumption fails when particles come close enough to collide. There are several other corrections to Stokes drag missing from this framework and it is not clear when they are needed to capture the real dynamics. In addition to hydrodynamic interactions between nearby particles, the corrections include the Basset history force, nonlinear drag and the added mass effect. Recent studies, for instance, suggest that turbulence tends to suppress preferential concentration and caustic formation.10-11 To find out the extent to which a model that includes Stokes drag alone is quantitatively descriptive, we compare experimental data from heavy droplets in a turbulent flow to results from direct numerical simulations (DNS) that match the conditions of the experiment, but with point particles coupled to the flow through Stokes drag. We then investigate the scaling of the particles’ relative velocities with respect to their spatial separation. This scaling is instrumental to predict collisional velocities at small scales from the statistics at larger scales that are more easily measured. Finally, we compare both experiment and DNS with recent theoretical results and investigate the nature of the transition from tracer-like statistics at low relative velocities to the particle-inertia dominated statistics at large relative velocities.

The experiment is described in detail in Ref. 6 and only an overview is given here. Nearly homogeneous and isotropic turbulent flows are generated in a 1 m-diameter acrylic sphere by 32 randomly pulsating jets. Each jet is made up of an audio-speaker capped by a conical nozzle.12 The homogeneous and isotropic region was about 10 cm in diameter and at the center of the apparatus. We ran the experiment under three different conditions, with the Taylor micro-scale Reynolds numbers, $R_\lambda$, being 160, 170 and 190 and kinetic energy dissipation rates ($\varepsilon$) 0.45 ± 0.05, 1.2 ± 0.1 and 3.2 ± 0.2 $m^2/s^3$, respectively (the corresponding Kolmogorov dissipative micro-scales, $\eta$, were 300, 230 and 180 $\mu$m). Droplets are produced with a spinning disc device13 that eject bi-disperse drops with diameters 6.8$\mu$m and 19$\mu$m and standard deviations of 2 $\mu$m and 4 $\mu$m, respectively. The Stokes numbers for the droplets are defined with respect to the Kolmogorov time-scale as $St = \tau_p/\tau_\eta$ where $\tau_\eta = \sqrt{\nu/\varepsilon}$ is the Kolmogorov timescale and $\tau_p = (2/9)(\rho_p/\rho_f)a^2/\nu$ is the particle viscous
response time ($\rho$ and $\rho_f$ are the particle and the fluid densities, respectively, $a$ the particle radius and $\nu$ the fluid kinematic viscosity). In order of increasing $R_\lambda$ for the flows studied, the large (small) droplets have Stokes number of values 0.19 (0.02), 0.31 (0.04) and 0.51 (0.06). The motion of the droplets are measured by an imaging of their shadows projected by white light sources into two cameras fitted with macro lenses, at a frame-rate of 15kHz ($>30/\tau_\eta$) and a spatial resolution of $3 \mu m/pixel$ ($<\eta/50$). The three-dimensional positions of the droplets are determined by stereoscopic Lagrangian Particle Tracking.

The DNS are performed by using a pseudo-spectral parallel solver for the fluid velocity $u$ obtained from the incompressible Navier–Stokes equation. Turbulence was sustained in a statistically stationary regime by holding constant the energy content of the lowest Fourier modes. We use $512^3$ grid points with $\nu = 1.5 \times 10^{-4}$ (corresponding to $R_\lambda = 180$) to approximately match the Reynolds numbers of the experiments. The droplets are approximated by individual point particles whose trajectories $X(t)$ solve the Stokes equation

$$\ddot{X} = -\frac{1}{\tau_p} [X - u(X,t)] + g,$$

where the dots designate time derivatives and $g$ the acceleration of gravity. The fluid velocity at each particle position is obtained by cubic interpolation from the grid points. As described above, the point-particle approach is expected to be valid when the particle size is much smaller than $\eta$ and its Reynolds number is much less than unity. Furthermore, the particles in this model do not modify or perturb the flow, which may be valid when their volume fraction is small.

Of fundamental importance to the problem of turbulence-induced collisions between particles are the statistics of the longitudinal component of their relative velocity $v^\parallel$ when the particles are close to each other. In Fig.1 we show the probability density function (PDF) of $v^\parallel$ between two particles, conditioned on different values, $r$, of their separation. The plots are organized into four Stokes number groups: $St = 0.05, 0.2, 0.3$ and 0.5. We note that in some groups the experimental and simulated Stokes numbers differ by a small amount (i.e., for the $St = 0.05$ group the experimental $St$ was 0.04 and for the $St = 0.2$ group the experimental and DNS values were 0.19 and 0.24 respectively). There is general agreement in the trends and shapes of the distributions. For example, all the distributions can be approximated by stretched-exponentials whose concavity grows more pronounced with increasing $St$ and decreasing $r$. This is qualitatively consistent with what is known about the velocity distributions of fluid particles, which grow more stretched with decreasing scale. It is relevant to note that Ref. predicts compressed exponential distributions for very large Stokes numbers. The distributions we measure are stretched rather than compressed, and the implication is that the large $St$ limit taken in the theory does not accurately describe the intermediate $St$ dynamics studied here.

Both experiments and DNS show an increase in the amplitude of the left tail with $St$, manifest in an increased skewness. This signifies that particles with larger inertia approached one another more violently on average than lower inertia particles. This observation is consistent with the sling effect, where inertial particles fly towards each other with relative velocities much higher than that of the background fluid, as has been observed in Ref. The faster approach should enhance their collision rate. The increased skewness of the distributions for the cases of higher inertia seen in the inset of Fig.1 can be rationalized as the effect of damping, by viscous drag, of the relative velocities of particles in a sling event. As is clear in the inset, the advection dominated cores of the PDF do not change with $St$ while the tails grow wider with increasing $St$, which makes the distributions more concave than that of fluid tracers. This observation is consistence with the existence of a velocity scale that separates the fluid-advection-dominated core of the PDFs from the inertia-dominated tails. Such a velocity scale is likely $r/\tau_p$ in the light of the analysis in a previous work.

Quantitatively, we found the differences between experiments and simulations to be less than about 15% in the core of the distributions. Similarly, we found excellent agreement in the tails of the distributions, but only for the largest Stokes number ($St = 0.5$), the smallest scale ($r < 2\eta$), and for the left side of the distributions corresponding to approaching particle pairs. In other cases, the experimental tails of the PDFs increasingly deviate from the simulated ones as one moves to higher relative velocities. The discrepancy is larger in the right tails, corresponding to separating pairs, where in the worst case the experimental data is about 5 times above the simulated data. In the left tails, the discrepancy is less severe, but worsens with decreasing $St$, so that the largest discrepancy is a factor of two.

In the case of $St = 0.5$ (Fig.1), the discrepancy in the right tails seems at first glance to contradict the good agreement observed for the left tails. Here, effects beyond linear Stokes drag maybe at play (e.g., the Basset history force, the added mass and nonlinear drag forces). For example, there is some indication...
in recent numerical simulations that the history force plays a important role under some conditions\textsuperscript{11} in any case, we could not find a clear explanation for the discrepancies, despite considering several possibilities including measurement uncertainty. To capture its influence, we characterized the measurement noise and added it to the DNS data. This however resulted only in a negligible widening of the tails of the distributions (the r.m.s. of the noise was in the data about 10\% of $v_\parallel$). We also evaluated the accuracy of the method used to estimate $\varepsilon$, in the experiment by applying the same method to the DNS data i.e., by using $\langle |v_\parallel|^2 \rangle = \varepsilon r^2/(15 \nu)$ for $r < 5\eta$ and on particles of $St = 0.05$). This resulted in very good agreement (within 5\%) with the direct measure of $\varepsilon$ in DNS, and so gave strong support to the $\varepsilon$ reported in the experiment. We ruled out the possibility of a Reynolds number effect by comparing DNS data at increasing Reynolds numbers (with resolution up to $2048^3$ grid points, corresponding to $R_\lambda = 460$). This exercise also addressed, partially, the possibility of small-scale turbulence statistics being non-universal and thus different in the DNS and in the experiment. We also explored the possibility of inaccuracy of $\nu$ in the experiment by reprocessing the experimental data with a modified $\nu$ (±30\%) and found no clear improvement. The droplets’ Reynolds numbers ($u_\parallel a/\nu$) were of the order of 0.1 on average, so that the effect of non-linear drag on the droplets were typically negligible. We note that given the conditions of our experiment, and specifically since $a/\eta$...
was of the order of 0.1, the history force term stands next to the Stokes drag in the hierarchy of importance amongst the various forces on the droplets. In summary, the influences of nonlinear forces, hydrodynamic interactions, and non-universal turbulence statistics merit further study.

To address the problem of droplet collision-coalescence in clouds, one needs to characterize droplet relative velocities at contact, which is typically of the order of 100 times smaller than \( \eta \). To that end, it is important to understand how droplet relative velocities scale with vanishing \( r \) interactions, and non-universal turbulence statistics merit further study.

![Figure 2](image)

**FIG. 2.** (Color online) (a) Rescaled probability distributions of the longitudinal velocity difference for \( St = 0.5 \), with \( \beta = 2.1 \), conditioned on different separations \( r \) for both the experimental (symbols) and DNS (solid lines) data, where green corresponds to \( r = 1 - 1.6 \eta \), blue to \( r = 3 - 3.6 \eta \), and red to \( r = 5 - 5.6 \eta \). Inset: \( r \)-scaling of the distribution function; collapse is attained by \( r \times p(v^||r) \) and \( (1/r) \times v^||/\eta \). (b) Rescaled conditional probability distributions for \( St = 0.3 \), with \( \beta = 2.2 \), including experimental (symbols) and numerical (solid lines) data, with the colors as in (a). Inset: plots of \( \ln \Pr(v^||/\eta < 5 r) \) (denoted as \( \ln \Pr_{5} \)) versus \( \ln(r/\eta) \) for different \( St \) from the experiment. The values of \( \beta \) were obtained as the linear slope of each of these curves. From the top curve to the bottom one, \( St = 0.5 \) (blue), \( 0.3 \) (red), \( 0.2 \) (green) and \( 0.04 \) (gold crosses) with the corresponding \( \beta \) denoted as \( \beta_{St=0.5} = 2.1 \), \( \beta_{St=0.3} = 2.2 \) and \( \beta_{St=0.2} = 2.7 \). Note that in the case of \( St = 0.3 \), if we exclude the first point from the left (which has an exceptionally high value) we find a different value of \( \beta_{St=0.3} \), namely 2.5. As a control, the highly curved \( St = 0.04 \) data are shown. Unambiguous values of \( \beta \) could not be obtained at such low \( St \).

The exponent \( \beta \) corresponds exactly to the saturated value \( \xi_{\infty} \) of the scaling exponents of the structure functions of particle relative velocities in the limit of large order \( p \) \( (\xi, (|v^||p | r) \propto r^{\xi_{\infty}} \) for all sufficiently large \( p \)). The collapse to a scale-independent form occurs for large velocity differences, namely \( |v^|| \gg r/\tau_{p} \). This condition corresponds to a traveling time over a distance \( r \) that is much shorter than the particle response time, so that damping is negligible. Under these conditions particle pairs move ballistically, which is related to the sling effect\(^{16} \). Gustavsson and Mehlig\(^{21,22} \) predict that \( \xi_{\infty} = 3 - D_{2} = c_{1} \) asymptotically for large \( St \), where \( D_{2} \) is the fractal (correlation) dimension of inertial particle clusters and \( c_{1} \) the corresponding exponent of the radial distribution function. In the case of \( St = 0.5 \), \( \xi_{\infty} \approx 0.7 \) according to the values for \( D_{2} \) and \( c_{1} \) given in Ref. 24 and elsewhere. This value is significantly different from what we measure: \( \xi_{\infty}(St = 0.5) \approx 2.1 \). In other words, the theory does not accurately describe the behaviors of droplets in the range of intermediate \( St \), which play a crucial role in the warm rain formation in clouds.

We turn our attention to the scaling exponents, \( \xi_{p} \), of the relative velocity statistics, whose values for asymptotically large \( p \) we reported above. By analyzing the relative dynamics of heavy particles in delta-correlated random flows, Gustavsson and Mehlig\(^{22} \) argued that the distribution of scaling exponents is bi-fractal, i.e.,

\[
\xi_{p} = \begin{cases} 
    p & \text{if } p \leq \xi_{\infty} \\
    \xi_{\infty} & \text{if } p > \xi_{\infty}
\end{cases}
\]
The measurement of $h$ and of $S(h)$ requires some attention because their definitions include the undetermined scales $\ell$ and $v_\ell$. Particular definitions of $\ell$ and $v_\ell$ do not alter the values of $h$ and $S(h)$ in the limit $r \to 0$, but we cannot reach this limit in practice. The explicit dependence of $S(h)$ on $\ell$ and $v_\ell$ can be eliminated by using the formula $\ln[p(h | r_2)/p(h | r_1)]/\ln(r_2/r_1)$. There is however no such stratagem...
to make $h$ independent of $\ell$ and $v_\ell$. A given choice, say $\ell'$ and $u'_\ell$, leads to a measurement of the scaling exponent $h' = \ln |v_\ell|/\ln(v/r/\ell')$ that for any finite $r$ differs from reference choices of $\ell$ and $v_\ell$ by $h' = h + [\ln(\ell/\ell') + \ln(v/v'_\ell)]/\ln(v/r/\ell')$. Thus, we must choose definitions for $\ell'$ and $u'_\ell$.

The main panel of Fig. 3 shows $S(h')$ obtained from our experiments and DNS for $\ell' = 10\eta$ and $u'_\ell = u_\eta$, and with $r$ going from $\sim 5\eta$ to $\eta$. The y-axis intercept for the case of $r \sim \eta$, albeit noisy, gives roughly the value deduced from Fig. 2, namely $\xi_\infty \approx 2.1$. The location of the minimum shifts towards $(1, 0)$ as $r$ decreases. The vertical displacement is partly due to the normalization factor present in $p(h \mid r)$, which itself involves some $r$ dependence and can be compensated by subtracting from $S(h')$ the value $S_{\min} = S(h'_{\min})$ of its minimum. The origin of the horizontal displacement can be twofold: it is either due to finite-$r$ deviations from the limiting form of $S(h)$ or to a mismatch in the definition of $h'$ due to our arbitrary choice of $\ell'$ and $v'_\ell$. The DNS data were consistent with $h'_{\min} \approx 1 + C/\ln(r/\ell')$, giving a strong support to the second scenario.

In order to probe the limiting form of $S$ close to its minimum at vanishing $r$, we show in the inset of Fig. 3: the rate functions $S(h)$ from the DNS with their minima translated to $(0, 0)$, for $r \approx 0.15\eta$ to $1.05\eta$. The excellent collapse of these curves around their minima suggests that they have reached their final limiting form at $r \lesssim \eta$. The frozen curvature around the minimum, for about a decade in $r$, indicates that $S(h)$ is convex and thus supports the view that the statistics are multi-fractal, and not bifractal.

To summarize, we evaluated the accuracy of the Stokes drag model for the advection of inertial particles in turbulent flow by comparing the results from DNS with experimental measurements. Focussing on large (longitudinal) relative velocities, we found that DNS reproduced all qualitative trends of the experiments. Furthermore, accurate quantitative agreements were found for inertia-dominated regimes ($St = 0.5$, $v_\parallel \lesssim -r/\tau_p$). Discrepancies up to a factor of 5 were found for regimes less influenced by particle inertia (that is, for separating particles or for small $St$). Further analysis did not support trivial explanations for such discrepancies, which implies that the discrepancies could have been caused either by corrections to the Stokes drag model, such as the Basset history force or hydrodynamic interactions between particles, or by small-scale non-universality of the turbulence (DNS and experiment have different large scale energy injection schemes). Where the data agree, they consistently show that for inertial particles and at dissipative scales of turbulence, the tails of the probability density function of $v_\parallel$ scale as a power law of $r$. This is consistent with the saturation of the scaling exponents of the moments of velocities differences found in previous studies. Furthermore, the frozen convexity of the rate function, $S(h)$, at small $r$ is consistent with multi-fractal statistics of velocities differences.

Several questions remain open. We observed that the functional form of the velocity difference PDF (stretched-exponential like) itself depends on the value of the Stokes number. At the moment, the only theoretical predictions pertain to the limit of large Stokes numbers and rely on Gaussian statistics of turbulent velocity differences at large scales. Such arguments do not straightforwardly extend to moderate values of the Stokes numbers for which the contribution of inertial-range and dissipative-scale statistics cannot be neglected. For that reason, we expect the intermittency of turbulent velocity statistics to play an important role, leading to non-trivial Reynolds number dependencies of particle relative velocity and collision statistics. These questions will be addressed in future work.

We acknowledge Poh Yee Lim for help with the experiments; Holger Homann for help with the simulation; M. Cencini, B. Mehlig and S. Musacchio for crucial discussions. This work received funding from the Max Planck Society (Germany) and the European Research Council under the European Community’s Seventh Framework Program (FP7/2007-2013, Grant Agreement no. 240579). SSR acknowledges the support of the Indocenter for Applied Mathematics (IFCAM) and from the EADS Corporate Foundation Chair awarded to ICTS-TIFR and TIFR-CAM. Computations were performed on the IBM Blue Gene/P computer JUGENE at the FZ Jülich was made available through the PRACE project PRA031 and on the “mésocentre de calcul SIGAMM”.

1G. Falkovich, A. Fouxon, and M. Stepanov, Nature 419, 151 (2002).
2R. Shaw, Annu. Rev. Fluid Mech. 35, 183 (2003).
3E. Balkovsky, G. Falkovich, and A. Fouxon, Phys. Rev. Lett. 86, 2790 (2001).
4J. Bec, L. Biferale, M. Cencini, A. Lanotte, S. Musacchio, and F. Toschi, Phys. Rev. Lett. 98, 84502 (2007).
5E.-W. Saw, R. A. Shaw, S. Ayyalasomayajula, P. Y. Chuang, and A. Gylfason, Phys. Rev. Lett. 100, 214501 (2008).
6G. P. Bewley, E.-W. Saw, and E. Bodenschatz, New J. Phys. 15, 083051 (2013).
7M. Wilkinson, B. Mehlig, and V. Bezuglyy, Phys. Rev. Lett. 97, 48501 (2006).
8G. Falkovich and A. Pumir, J. Atmos. Sci. 64, 4497 (2007).
9M. Orme, Prog. Energy Combust. Sci. 23, 65 (1997).
10R. J. Hill, Phys. Fluids 17, 037103 (2005).
11A. Daitche and T. Tél, Phys. Rev. Lett. 107, 244501 (2011).
12K. Chang, G. P. Bewley, and E. Bodenschatz, J. Fluid Mech. 692, 464 (2012).
13W. H. Walton and W. C. Prewett, Proc. Phys. Soc. B 62, 341 (1949).
14N. Ouellette, H. Xu, M. Bourgoin, and E. Bodenschatz, New J. Phys. 8, 109 (2006).
15D. Gottlieb and S. A. Orszag, *Numerical analysis of spectral methods*, Vol. 2 (SIAM, 1977).
16S. Chen and X. Shan, Comput. in Phys. 6, 643 (1992).
17P. Kailasnath, K. Sreenivasan, and G. Stolovitzky, Phys. Rev. Lett. 68, 2766 (1992).
18K. Gustavsson, B. Mehlig, M. Wilkinson, and V. Uski, Phys. Rev. Lett. 101, 174503 (2008).
19M. R. Maxey and J. J. Riley, Phys. Fluids 26, 883 (1983).
20A. Celani, A. Lanotte, A. Mazzino, and M. Vergassola, Phys. Rev. Lett. 84, 2385 (2000).
21J. Bec, L. Biferale, M. Cencini, A. Lanotte, and F. Toschi, J. Fluid Mech. 646, 527 (2010).
22K. Gustavsson and B. Mehlig, Phys. Rev. E 84, 045304 (2011).
23K. Gustavsson and B. Mehlig, J. Turbul. 15, 34 (2013).
24E.-W. Saw, J. P. Salazar, L. R. Collins, and R. A. Shaw, New J. Phys. 14, 105030 (2012).
25U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, UK, 1996).
26G. Paladin and A. Vulpiani, Phys. Rep. 156, 147 (1987).
27C. Meneveau and K. Sreenivasan, J. Fluid Mech. 224, 429 (1991).
28M. Broniatowski and P. Mignot, Stat. & Prob. Lett. 54, 125 (2001).
29W. van de Water and P. Schram, Phys. Rev. A 37, 3118 (1988).