Continuous-variable entanglement test in driven quantum contacts

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(Dated: December 1, 2017)

The standard entanglement test using the Clauser-Horne-Shimony-Holt inequality is known to fail in mesoscopic junctions at finite temperatures. Since this is due to the bidirectional particle flow, a similar failure is expected to occur in an ac-driven contact. We develop a continuous-variable entanglement test suitable for electrons and holes that are created by the ac drive. The generalized Bell inequality is violated in junctions with low conductance or small number of transport channels and with ac voltages which create few electron-hole pairs per cycle. Our ac-entanglement test depends on the total number of electron-hole pairs and on the distribution of probabilities of pair creations similar to the Fano factor.

Quantum mechanics in fact is a nonlocal theory. In 1935 Einstein, Podolsky, and Rosen put forward the EPR paradox [1] to question the completeness of quantum mechanics which disobeys, in their view apparent, the principle of local realism. They argued that the quantum mechanics should be completed with some hidden variables. In 1964 Bell derived an inequality [2] to actually test the principle of local realism. Remarkably, he found that in some cases the physical results based on the principle of local realism are inconsistent with the predictions of quantum mechanics. In later years, the experiments on quantum entanglement [3] proved the existence of nonlocality, including the recent loophole-free tests [4–6]. Besides their fundamental importance, quantum entanglement and nonlocality have attracted a lot of attention over the past two decades in the context of quantum computation [7], teleportation [8], and cryptography [9]. While most of the tests are performed using quantum optics, it still remains a challenge to detect entanglement within the Fermi sea in mesoscopic systems [10–13]. Some concepts have even been tested experimentally [14–19].

The most common way to detect the entanglement is to test for the violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [20]. In mesoscopic junctions as shown in Fig. 1, the CHSH inequality reads $|C(a, b) + C(a', b') - C(a', b) - C(a, b')| \leq 2$ where $C(n_A, n_B) = \langle (N_A^\dagger - N_B^\dagger)(N_B^\dagger - N_A^\dagger) \rangle / \langle (N_A^\dagger + N_B^\dagger)(N_B^\dagger + N_A^\dagger) \rangle$ are the spin correlators and $N_j^\sigma$ are the numbers of quasiparticles with spin projections $\sigma = \uparrow, \downarrow$ in direction $n_j$ detected in the lead $j = A, B$. The spin correlators can be expressed in terms of the average current $I_0$ and current noise power $S_0$ [20]. $C(n_A, n_B) = -n_A \cdot n_B S_0 / (S_0 + 2 \tau I_0^2)$, where $\tau$ is the measurement time. What is peculiar about this result is that it reduces to $C(n_A, n_B) = -n_A \cdot n_B$ in an ac driven system with no dc bias ($I_0 = 0$). In that case, the CHSH inequality can always be maximally violated with the left hand side equal to $2\sqrt{2}$ if the angles between the spin polarization directions in the leads are chosen as $\theta_{ab} = \theta_{\omega b} = \theta_{ab'} = \pi/4$ and $\theta_{a'b'} = 3\pi/4$. However, this violation does not mean entanglement because it cannot single out the entangled pair if there are many entangled pairs. In addition, it is implausible that the violation is independent of the amplitude and the frequency of the ac voltage and even the shape of the drive. The reason why CHSH inequality fails to reveal the entanglement is because it requires unidirectional particle flow in the leads, the condition which is not satisfied when we apply ac voltage to the system. Indeed, ac drive leads to fluctuations with $|N_j^\dagger - N_J^\dagger| > |N_j^\dagger + N_J^\dagger|$ which are forbidden in the Bell test [21]. The same problem occurs in a dc biased junction at finite temperatures [22]. To detect entanglement in an ac driven junction, a generalized Bell test is needed which is free from this restriction.

In this paper, we study two different setups for a generalized Bell test in ac driven systems at low temperatures. The setup shown in Fig. 1(a) is a normal junction with transmission eigenvalues $\{T_n\}$ biased by the periodic ac voltage $V(t)$ with frequency $\omega$ and zero average. The second setup, shown in Fig. 1(b), is an ac-driven super-
conductor (S) – normal-metal (N) beam splitter which emits entangled pairs of electrons or holes. In both cases, the spin-polarized particles along directions ±a and ±b are detected in the leads A (Alice) and B (Bob). The test observables are the differences between the numbers of spin-up and spin-down particles \( \hat{N}_A^+ - \hat{N}_A^- \) and \( \hat{N}_B^+ - \hat{N}_B^- \). In contrast to the dc bias which at zero temperature leads to unidirectional electron transport \([23, 24]\), the ac drive generates electron-hole pairs in the junction \([25, 26]\). The particle flow in the leads is therefore bidirectional which renders the standard Bell test inapplicable. In this paper, we derive a generalized Bell inequality for the proposed setups,

\[
\sqrt{3}(2 - \sqrt{2})/4 \leq \sqrt{1 - \langle A^2 \rangle /\langle A \rangle},
\]

(1)

whose violation implies the presence of entangled particles in the system. Expressing the moments \( \langle A^2 \rangle \) and \( \langle A \rangle \) in terms of transmission eigenvalues and properties of the drive, we find

\[
\sqrt{2} - 1/4 \leq \sqrt{X_i}/(1 + 6X_i).
\]

(2)

Here, \( X_i = \sum_{n,k} p_{n,k}^i - \sum_{n,k} p_{n,k}^{(i/2)} / \sum_{n,k} p_{n,k}^i \) where \( i = 1 \) (2) stands for the normal (superconducting) junction shown in Fig. 1. The probabilities \( p_{n,k}^i = p_k T_n R_n \) and \( p_{n,k}^{(2)} = p_k R_n^2 / 4 \), where \( R_n = 1 - T_n \) and \( R_n^A = T_n^A/(2 - T_n) \) are normal and Andreev reflection coefficients; \( p_k \) \( (k = 1, 2, \ldots) \) are the probabilities of electron-hole pair creations which depend on the details of the drive \([25, 26]\).

To derive Eqs. (1) and (2), we first obtain the statistics of \( A \) and \( B \) in the junctions at hand. The statistics is computed by using the circuit theory of quantum transport. We assign the counting fields \( \chi_A^+ \) and \( \chi_B^\sigma \) (\( \sigma = \uparrow, \downarrow \)) to the spin-polarized leads of Alice and Bob, see Fig. 1. Since we are only interested in the differences between the numbers of spin-up and spin-down particles, we can set \( \chi_A^+ = -\chi_A^- = \chi_A \) and \( \chi_B^\sigma = -\chi_B^- = \chi_B \), where \( \chi_A \) and \( \chi_B \) are the counting fields which determine the statistics of \( A \) and \( B \). For the cumulant generating functions \( S_1 \) and \( S_2 \) of the normal and the superconducting junction shown in Fig. 1 we find \([28]\)

\[
S_1(\chi_A, \chi_B) = M \sum_{n,k} \ln \left( 1 + p_{n,k}^i \right) \left( 1 + \alpha \beta a \cdot b \left( e^{\imath \alpha \chi_A - \imath \beta \chi_B} - 1 \right) \right)
\]

(3)

with \( M = \tau \omega / \pi \). In the reminder of the paper we set \( M = 1 \) which is the optimal value for the Bell test. Indeed, as the measuring time is increased the number of particles detected also increases which will destroy the entanglement test. Experimentally, \( M = 1 \) can be achieved by measuring the cumulants of \( \hat{A} \) and \( \hat{B} \) and dividing the result by \( \tau \omega / \pi \).

Next we apply the generalized Bell inequality \([23]\)

\[
|\langle AB(A^2 + B^2) \rangle + \langle A'B'(A^2 + B^2) \rangle | - |\langle AB'(A^2 + B^2) \rangle - \langle A'B'(A^2 + B^2) \rangle | \leq |\langle A^4 \rangle + \langle B^4 \rangle | + |\langle B^4 \rangle | - \langle B^4 \rangle | = \sum_{C,D,E} \sqrt{|C|^2} \sqrt{|D|^2} (|D|^2 - E^2)^2.
\]

(4)

Here, the summation is taken over \( C, D, E \in \{A, A', B, B'\} \) and * denotes restriction \( D \neq C \) and \( E \neq C, D, D' \). Importantly, inequality in Eq. (4) does not require charge quantization or unidirectional particle flow and also does not pose restrictions on the possible values of the observables \( A, A', B, B' \). As with any Bell test, the symbol ' in Eq. (4) denotes two different settings of the Alice and Bob measuring apparatus. In this case these are the two choices of spin-polarization directions \( a, a' \) for Alice and \( b, b' \) for Bob. The cumulants of \( A \) and \( B \) are obtained by taking derivatives of the cumulant generating function in Eq. (3) with respect to the counting fields. From the cumulants we can find the moments which appear in Eq. (4). We obtain the following relations which are valid for both setups: \( \langle A^0 \rangle = \langle B^0 \rangle = 0 \), \( \langle A^0 A' \rangle = \langle B^0 B' \rangle \), \( \langle A^0 B^0 A' B' \rangle = \langle A^0 B^0 A' B' \rangle = 1 \), \( \langle A^0 B^0 A' B' \rangle = 1 \). Substituting these relations in Eq. (4), we obtain \( |B(a, b, a', b')| \leq 2 + 2 \sqrt{3}/\sqrt{1 - \langle A^2 \rangle /\langle A \rangle} \sum_{c,d} \sqrt{1 - (c \cdot d)^2} \) where \( c \in \{a, a'\} \), \( d \in \{b, b'\} \), and \( B(a, b, a', b') = a \cdot b + a' \cdot b + a \cdot b' - a' \cdot b' \). Choosing the angles between spin-polarization directions \( \theta_{ab} = \theta_{a'b'} = \pi/4 \) and \( \theta_{ab'} = 3\pi/4 \) such that \( |B| \) is maximal, the generalized Bell inequality reduces to Eq. (1). Finally, after computing the moments \( \langle A^2 \rangle = \partial^2_{\chi A} S_1 |_{\chi = 0} \) and \( \langle A^4 \rangle = 3 \langle A^2 \rangle^2 + \partial^4_{\chi A} S_1 |_{\chi = 0} \) we recover Eq. (2).

Equations (1) and (2) represent a generalized Bell test suitable for driven systems shown in Fig. 1 where the particle flow in the leads is bidirectional due to presence of electron-hole pairs created by the drive. When the number of transport channels is large, Eq. (2) reduces to \( (\sqrt{2} - 1)/4 \leq 1/\sqrt{6} \) which is not violated, as one might expect. The same is true when many electron-hole pairs are created in the system \( (p_k = 1 \text{ for } k = 1, \ldots, N_{ch}) \). To violate Eq. (2), the contribution of an entangled pair has to be singled out from the rest. This is achieved in the junction of low conductance or small number of transport channels and with few particles created per voltage cycle. In the tunnel limit, Eq. (2) reduces to \( (\sqrt{2} - 1)/4 \leq \beta_1 \) where

\[
\beta_1 = \sqrt{G \sum_k p_k - (1 - F_p)(1 - F)}
\]

(5)
The distribution of probabilities of electron-hole pair creations $p_k$ and (b) parameter $F_p = \sum_k p_k (1-p_k)/\sum_k p_k$ as a function of the amplitude $V_0$ for harmonic drive $V(t) = V_0 \cos(\omega t)$. Test of a generalized Bell inequality for different Fano factors: (c) normal junction in the tunnel limit [see Eq. (3) and Fig. (a)] and (d) SN beam splitter geometry [see Eq. (6) and Fig. (b)]. The generalized Bell inequality is violated for junction conductances and applied voltages that are below the solid curves shown in (c) and (d). The effective charge is $e^* = e$ for the normal and $e^* = 2e$ for the SN junction. Results for a biharmonic voltage $V(t) = V_0 [0.8 \cos(\omega t) + 0.1 \cos(3\omega t)]$ which approximates triangle-wave drive are shown in the panels (e)-(h) for comparison.

and

$$\beta_2 = \frac{1}{2} \sqrt{\frac{G_S}{2G_Q} \sum_k p_k - (1-F_p)(1-F_S/2)}$$

for the normal junction and the SN beam splitter, respectively. Here, $G = G_Q \sum_n T_n$ ($G_S = 2G_Q \sum_n R_n^A$) and $F = \sum_n T_n R_n/\sum_n T_n$ ($F_S = 2 \sum_n R_n^A (1-R_n^S)/\sum_n R_n^A$) are the conductance and the Fano factor of the normal (superconducting) junction, and $G_Q = e^2/\pi$. The distribution of probabilities of electron-hole pair creations is characterized by $F_p = \sum_k p_k (1-p_k)/\sum_k p_k$, in analogy with the Fano factor for transmission eigenvalues. The Bell test is analyzed in Fig. 2 as a function of the junction conductance, the amplitude of applied voltage, and different Fano factors that are close to the tunnel limit. We find that small deviations from the Poissonian statistics, that is, the presence of open channels, helps in violating the generalized Bell inequality. In addition, Bell test also depends on the details of the drive through the probabilities $p_k$ and distribution $F_p$ of electron-hole pair creations. We compare the results for a simple harmonic drive [panels (a)-(d)] and a biharmonic drive which approximates a triangle-wave bias [panels (e)-(h) in Fig. 2]. The latter drive creates fewer electron-hole pairs per cycle which makes the entanglement observable in a wider range of junction conductances and drive amplitudes. The extent of a generalized Bell inequality violation is shown in Fig. 4 for harmonic drive.

Inequality (2) can also be violated beyond tunnel limit in a quantum point contact with only a few discrete transport channels $T_n$. In case of a single-channel contact, the parameters $X_i$ in Eq. (2) read

$$X_1 = TR(\sum_k p_k + F_p - 1)$$

for the normal junction and

$$X_2 = (R_A/4)(\sum_k p_k + F_p - 1)$$

for the SN beam splitter. The generalized Bell test is shown in Fig. 4 as a function of the harmonic drive amplitude: (a) normal junction with $G/G_Q = 0.02$ and (b) SN beam splitter with $G_S/2G_Q = 0.04$. Inequality is violated in the shaded regions in which $\beta_1$ and $\beta_2$ (solid lines) are smaller than $(\sqrt{2} - 1)/4$ (dashed line).
of the contact transparency, amplitude of the drive, and for different shapes of the drive voltage. In the normal junction, violation of Eq. (2) occurs mostly in contacts that are either open or closed. This is because the probability of a successful Bell test is proportional to $TR$ which corresponds to one particle from a pair being transmitted to Bob while the other one is reflected to Alice, or vice versa. If this probability is sufficiently low, the signal from one entangled pair is not obscured by other pairs and Eq. (2) is violated. Similarly, the Bell test in SN beam splitter geometry is realized by injection of entangled pairs of electrons or holes with probability $R_A$ towards Alice and Bob. Violation of Eq. (2) occurs mainly in SN contacts with lower $R_A$ in which the signal from one entangled pair is singled out from the rest. As regards the shape of the drive, we note that the square-wave drive creates more electron-hole pairs per cycle than harmonic, sawtooth, or triangle-wave drive of the same amplitude, which decreases the chances of entangled pair detection. In the single-channel contact with low ac voltage, Eq. (2) is violated for any contact transparency. This is expected because in this case there is at most one electron-hole pair created and detected per voltage cycle, see Fig. (2a),(e).

In the following we analyze a normal-metal beam splitter analogous to the superconducting one shown in Fig. (b). It has been found in [21] that the normal beam splitter is not suitable for detection of entangled electron spin singlets injected by a dc voltage [22]. We show that this setup is inefficient in a driven case as well. For the cumulant generating function we find $\Delta_N(\chi_A, \chi_B) = M \sum_{n,k} \ln[1 + (p_k T_n R_n/2)(e^{i\chi_A} + e^{-i\chi_A} + e^{i\chi_B} + e^{-i\chi_B} - 4) + (p_k T_n^2/8) \sum_{\alpha,\beta = \pm 1}(1 + \alpha \beta \mathbf{a} \cdot \mathbf{b})(e^{i\alpha \chi_A - i\beta \chi_B} - 1)]$. The charge transfer statistics consists of two parts. The spin-dependent part is proportional to $p_k T_n$ and is related to the Bell-type events in which both electron and hole from a pair are transmitted to Alice and Bob. The spin-independent part is proportional to $p_k T_n R_n$ and describes events in which only one particle is transmitted to Alice or Bob while the other one is reflected back at the contact. These events carry no information on the particle spin correlations and lead to current fluctuations which obscure entanglement detection. Therefore, violation of a generalized Bell inequality can be achieved only in a normal beam splitter of large transparency. For a single-channel junction with transmission coefficient $T$, Eq. (11) reduces to $1 - 1/\sqrt{2} \leq 2R/(1 + 3X_p/2) + 2\sqrt{(2R + X_p/2)/(1 + 3X_p/2)}$, where $X_p = \sum_k p_k + F_p - 1$. This inequality is violated for $T > 0.99$ when Bell-type events dominate over particle reflections at the contact, see Fig. (6).

In conclusion, we have developed a continuous-variable entanglement test which does not require charge quantization, single-particle detection, or unidirectional particle flow and is therefore suitable for entanglement detection in ac driven systems. We have shown that the entanglement between electrons and holes from electron-hole pairs can be probed using a 4-terminal normal-metal junction while the SN beam splitter can be used to reveal the entanglement of Cooper pairs emitted or absorbed by the superconductor. The success of entanglement detection relies on the ability to differentiate between the overall current fluctuations and the specific current correlations coming from the entangled particle pairs. This can be achieved in quantum contacts with low conductance or small number of transport channels and with ac drive which creates few electron-hole pairs per cycle. We have studied the feasibility of the entanglement test for cosine, square, sawtooth, and triangle drive. The ac drive affects the entanglement test through the total number of electron-hole pairs and the distribution of the pair creation probabilities. Finally, we have found that the normal-metal beam-splitter geometry is not suitable for the electron-hole entanglement test as it requires contacts of a very large transparency.

We gratefully acknowledge the support from DFG through SFB 767 and the Serbian Ministry of Science Project No. 171027.

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SUPPLEMENTAL MATERIAL FOR "CONTINUOUS-V ARIABLE ENTANGLEMENT TEST IN A DRIVEN QUANTUM CONTACT"

Here we present the derivation of Eq. (3) for the cumulant generating functions $S_i$ of the normal junction and the superconductor – normal-metal beam splitter.

NORMAL-METAL JUNCTION

The junction is schematically depicted in Fig. 1(a). There are 4 leads: 2 left leads for Alice and 2 right leads for Bob. We assume the leads are perfectly spin filtered in directions $\pm \mathbf{a}$ and $\pm \mathbf{b}$ where $|\mathbf{a}| = |\mathbf{b}| = 1$. We count charges that enter each of the leads and assign the counting field $\chi_j$ ($j = A, B$; $\sigma = \uparrow, \downarrow$) to them. The leads are characterized by the Keldysh-Green’s functions

$$
\hat{G}_L(\chi_A) = e^{-i\chi_A \tau_3/2} \hat{G}_L(0) e^{i\chi_A \tau_3/2},
$$

$$
\hat{G}_R(\chi_B) = e^{-i\chi_B \tau_3/2} \hat{G}_R(0) e^{i\chi_B \tau_3/2},
$$

where

$$
\hat{G}_I(0) = \begin{pmatrix} 1 & 2\hbar i \\ 0 & -1 \end{pmatrix}.
$$

Here, $\tau_3$ is the Pauli matrix in Keldysh space and $f_l = (1 - h_l)/2$ ($l = L, R$) are the generalized nonequilibrium distribution functions which depend on two time or energy indices. For an ac driven junction, we can use a $2 \times 2$ representation of $h_L$ and $h_R$ in energy space [25]

$$
h_L = \begin{pmatrix} e^{-i\omega_k} & 0 \\ 0 & e^{i\omega_k} \end{pmatrix}, \quad h_R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

where $p_k = \sin^2(\alpha_k/2)$ ($k = 1, 2, \ldots$) are the probabilities of electron-hole pair creations. Note that the Green’s functions $\hat{G}_I$ are scalars in the spin space.

For simplicity, we assume that the conductances of the spin filters are much larger than the one of the central junction $g = (e^2/\pi) \sum_n T_n$. In this case, the electron which arrives at Alice or Bob will enter the spin-polarized leads without backscattering. Since $A$ is strongly coupled to the spin-filtering leads and only weakly coupled to $B$, we can obtain the Green’s function $\hat{G}_A$ of the node $A$ with the node $B$ disconnected. In addition, we assume that the spin filters have the same conductance and that an electron can enter any of the spin-filtering leads with equal probability. The Green’s function $\hat{G}_A$ is obtained from the matrix current conservation $[(1 + \mathbf{a} \cdot \hat{\sigma}) \hat{G}_L(\chi_A^\uparrow) + (1 - \mathbf{a} \cdot \hat{\sigma}) \hat{G}_L(\chi_A^\downarrow)] \hat{G}_A = 0$ and the normalization condition $\hat{G}_A^2 = \frac{1}{1 + \mathbf{b} \cdot \hat{\sigma}}$. Here, $\hat{\sigma}$ is the vector of Pauli matrices in spin space. We obtain

$$
\hat{G}_A(\chi_A^\uparrow, \chi_A^\downarrow) = \frac{1 + \mathbf{a} \cdot \hat{\sigma}}{2} \hat{G}_L(\chi_A^\uparrow) + \frac{1 - \mathbf{a} \cdot \hat{\sigma}}{2} \hat{G}_L(\chi_A^\downarrow),
$$

and similarly for the node $B$,

$$
\hat{G}_B(\chi_B^\uparrow, \chi_B^\downarrow) = \frac{1 + \mathbf{b} \cdot \hat{\sigma}}{2} \hat{G}_R(\chi_B^\uparrow) + \frac{1 - \mathbf{b} \cdot \hat{\sigma}}{2} \hat{G}_R(\chi_B^\downarrow).
$$

Cumulant generating function for the normal-metal junction is given by

$$
S_1(\chi) = \frac{1}{2} \sum_n \text{Tr} \ln \left( \hat{I} + \frac{T_n}{4}(\{\hat{G}_A, \hat{G}_B\} - 2) \right),
$$

where the trace is taken over Keldysh, spin, and energy indices. Since Alice and Bob are measuring the differences between the spin-up and spin-down charges, $\hat{A} = N_A^\uparrow - N_A^\downarrow = \int dt (\hat{I}_A^\uparrow - \hat{I}_A^\downarrow)/e$ and $\hat{B} = N_B^\uparrow - N_B^\downarrow = \int dt (\hat{I}_B^\uparrow - \hat{I}_B^\downarrow)/e$, we can set $\chi_A^\uparrow = \chi_A$, $\chi_A^\downarrow = -\chi_A$, $\chi_B^\uparrow = \chi_B$, and $\chi_B^\downarrow = -\chi_B$, where $\chi_A$ and $\chi_B$ are the counting fields related to the statistics of $\hat{A}$ and $\hat{B}$.

To compute $S_1$, we proceed as follows. We note that $\hat{G}_A$ and $\hat{G}_B$ are $8 \times 8$ matrices in Keldysh × spin × energy space. Cumulant generating function $S_1$ can be written as a sum $S_1 = S_1^+ + S_1^-$, where

$$
S_1^\pm(\chi_A, \chi_B) = \frac{1}{2} \sum_n \text{Tr} \ln \left( \hat{I} \pm \sqrt{T_n}/2 (\hat{G}_A - \hat{G}_B) \right).
$$

It turns out that $\pm \sqrt{T_n}/2 (\hat{G}_A - \hat{G}_B)$ have the same eigenvalues, hence $S_1^+ = S_1^-$. Moreover, the eigenvalues are pairwise equal to each other, $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \lambda_5 = \lambda_6$, and $\lambda_7 = \lambda_8$. We obtain that the product $\lambda_1 \lambda_3 \lambda_5 \lambda_7 = 1 + 2p_k T_n R_n [\cos(\chi_A) \cos(\chi_B) - 1 + \mathbf{a} \cdot \mathbf{b} \sin(\chi_A) \sin(\chi_B)]$. Therefore, the cumulant generating function reads

$$
S_1(\chi_A, \chi_B) = \frac{T\omega}{\pi} \sum_{n,k} \ln \left( 1 + 2p_k T_n R_n \times [\cos(\chi_A) \cos(\chi_B) - 1 + \mathbf{a} \cdot \mathbf{b} \sin(\chi_A) \sin(\chi_B)] \right).
$$

This expression reduces to Eq. (3) of the main text.

SUPERCONDUCTOR – NORMAL-METAL BEAM SPLITTER

The junction consists of a superconductor ($S$) coupled to the normal-metal lead ($N$) through the central junction characterized by transmission eigenvalues $\{T_n\}$. The normal lead is split into 4 outgoing terminals with spin filters along directions $\pm \mathbf{a}$ (Alice) and $\pm \mathbf{b}$ (Bob), see Fig. 1(b). As before, we assume strong coupling of the spin filters to the node $N$ and neglect the backscattering. We also assume that the spin-filtering leads have the same conductance. The cumulant generating function is given by

$$
S_2(\chi) = \frac{1}{4} \sum_n \text{Tr} \ln \left( 1 + \frac{T_n}{4}(\{G_S(0), G_N(\chi)\} - 2) \right),
$$

where $G_S(0)$ and $G_N(\chi)$ are the Green’s functions of the superconductor and normal-metal leads, respectively.
where the trace is taken in electron-hole (Nambu), Keldysh, spin, and energy indices.

At energies and temperatures much smaller than the gap, the Green’s function $G_S$ of the superconductor is given by

$$G_S(0) = \tilde{\tau}_2 \otimes \tilde{1} = \begin{pmatrix} 0 & -i\tilde{1} \\ i\tilde{1} & 0 \end{pmatrix},$$

(17)

where the block-matrix structure is in electron-hole space. The Green’s function $G_N(\chi)$ of the node $N$ in the normal terminal is given by the matrix current conservation

$$\left[ \left( \tilde{G}_{Ac} + \tilde{G}_{Be} \right) 0 \right]_{\chi} \cdot \tilde{G}_N = 0$$

(18)

and the normalization condition $G^2_N = 1$. Here, the Green’s functions for electrons and holes are given by

$$\tilde{G}_{Ac} = \frac{1 + a \cdot \hat{\sigma}}{2} \tilde{G}_c(\chi_A^\dagger) + \frac{1 - a \cdot \hat{\sigma}}{2} \tilde{G}_c(\chi_A),$$

(19)

$$\tilde{G}_{Be} = \frac{1 + b \cdot \hat{\sigma}}{2} \tilde{G}_e(\chi_B^\dagger) + \frac{1 - b \cdot \hat{\sigma}}{2} \tilde{G}_e(\chi_B),$$

(20)

and

$$\tilde{G}_{Ah} = \frac{1 - a \cdot \hat{\sigma}}{2} \tilde{G}_h(-\chi_A^\dagger) + \frac{1 + a \cdot \hat{\sigma}}{2} \tilde{G}_h(-\chi_A),$$

(21)

$$\tilde{G}_{Bh} = \frac{1 - b \cdot \hat{\sigma}}{2} \tilde{G}_h(-\chi_B^\dagger) + \frac{1 + b \cdot \hat{\sigma}}{2} \tilde{G}_h(-\chi_B).$$

(22)

We note that the counting fields and spin-polarization vectors for holes are the opposite from the ones for electrons. The counting fields are incorporated via the gauge transform $\tilde{G}_{e,h}(\chi) = e^{-i\chi \tilde{\tau}_3 \otimes \tilde{\tau}_1/2} \tilde{G}_{e,h}(\chi) e^{i\chi \tilde{\tau}_3 \otimes \tilde{\tau}_1/2}$, with

$$\tilde{G}_e(0) = \begin{pmatrix} 1 & 2U\hbar U^\dagger \\ 0 & -1 \end{pmatrix}, \quad \tilde{G}_h(0) = \begin{pmatrix} 1 & 2U^\dagger \hbar U \\ 0 & -1 \end{pmatrix}.$$  

(23)

Here, $U(t', t'') = \exp[-i \int_0^{t'} eV(t)dt] \delta(t' - t'')$ is unitary operator which takes into account the time-dependent drive 23.

From Eq. (18) we obtain

$$G_N = \begin{pmatrix} \tilde{G}_{Ne} & 0 \\ 0 & -\tilde{G}_{Nh} \end{pmatrix}, \quad \tilde{G}_{Ne,h} = \frac{\tilde{G}_{Ac,h} + \tilde{G}_{Be,h}}{2}.$$  

(24)

Substituting this result in Eq. (16) and after taking the trace over electron-hole indices we find

$$S_2 = \frac{1}{2} \sum_n \text{Tr} \ln \left( \tilde{1} + \frac{\sqrt{R_n^A}}{2} \left( \tilde{G}_{Ne}(2V(t)) - \tilde{G}_{Nh}(0) \right) \right).$$

(25)

Here, the remaining trace is taken over Keldysh, spin, and energy indices and $R_n^A = T_n^2/(2 - T_n)$ are the coefficients of Andreev reflections. In Eq. (25) we have also performed a gauge transform under the trace and ascribed the effect of the voltage drive to the electron part of the Green’s function, which leads to an effective voltage (or charge) doubling 29. After diagonalization of the operator in Eq. (25), we obtain

$$S_2(\chi_A, \chi_B) = \frac{\tau \omega}{\pi} \sum_{n,k} \ln \left( 1 + \frac{p_k R_n^A}{4} \right) \times \sum_{a,\beta = \pm 1} \frac{1 + \alpha \beta a \cdot b}{2} \left( e^{i\alpha \chi_A - i\beta \chi_B} - 1 \right).$$

(26)

This expression coincides with Eq. (3) of the main text.