Hopf Parametric Adjoint Objects through a 2-adjunction of the type \( \text{Adj-Mnd} \)

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Abstract

In this article Hopf parametric adjunctions are defined and analysed within the context of the 2-adjunction of the type \( \text{Adj-Mnd} \). In order to do so, the definition of adjoint objects in the 2-category of adjunctions and in the 2-category of monads for \( \text{Cat} \) are revised and characterized. This article finalises with the application of the obtained results on current categorical characterization of Hopf Monads.

In memory of Lecter and Cosmo

Introduction

In 2002, I. Moerdijk [5] characterized the liftings of a monoidal structure to the category of Eilenberg-Moore algebras, for a related initial monad. This characterization lead to the definition of a opmonoidal monad. In 2011, A. Bruguieres, S. Lack and A. Virelizier [1] characterized the liftings of a closed monoidal structure through the concept of a Hopf monad. These two examples will be analysed in the context of higher category theory.

This article belong to a series where 2-adjunctions of the type \( \text{Adj-Mnd} \) are applied to classical monad theory. In this installment, the author analyse adjoint objects and parametric adjunctions within this context.

On the last subject, that of parametric adjunctions, this article is mainly based upon the ideas laid out in the seminal article of A. Bruguieres et. al. [1]. In order to apply the 2-adjunction \( \text{Adj-Mnd} \), the ideas are developed into a 2-categorical framework, cf. [3]. It is in this framework that the Hopf monad concept, for a monoidal closed structure, is extended to the concept of Hopf 1-cells and adjoint parametric objects on certain 2-categories.

Without any further ado, the structure of the article is given.

In chapter 1, the 2-categorical structure needed for the rest of the article is given, namely the construction of the 2-adjunction \( \Phi^m_E \dashv \Psi^m_E \).

In chapter 2, adjoints objects in the 2-categories \( \text{Adj}_E(\text{Cat}) \) and \( \text{Mnd}(\text{Cat}) \) are revised and characterized. The characterization of such objects is done based on [1] which is suitable for the 2-categorical context of the article.

In chapter 3, the concept of (left) Hopf 1-cells is defined within the 2-category \( \text{Adj}_E(\text{Cat}) \) and it is used in order to construct the Hopf parametric adjoint objects in that 2-category. The concept of antipode is

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In chapter 4, the concept of Hopf 1-cells is provided within the 2-category $\text{Mnd}(\text{Cat})$ and it is used in order to construct the corresponding Hopf parametric adjoint objects for that 2-category.

In chapter 5, the condition for being Hopf 1-cell and the related structure, that of a parametric adjoint object, is analysed through the 2-adjunction. The condition for a 1-cell to be Hopf 1-cell is preserved and then Hopf 1-cells, in each 2-category, are compared using the 2-adjunction. At the end, using the isomorphism of categories, induced by the 2-adjunction, a bijection of Hopf parametric adjoint objects is found.

In chapter 6, remaining concepts and statements are done in order to get the main theorem of the article which gives a bijection between Hopf parametric liftings, mimicking those liftings for the closure of a monoidal structure [1], and certain parametric adjoint objects is given. This chapter finalises with the corresponding application to Hopf monads in a monoidal category which was the main inspiration for this extension to a 2-categorical context.

A list of the notations and conventions taken in this article is given as follows. Consider an adjunction $L \dashv R$, its unit and counit are denoted as $\eta_L$ and $\varepsilon_R$, respectively. This notation might be complicated but refrain one from running out of, and $a posteriori$ very needed, the finite set of greek letters. Nevertheless, the notation will be simplifyed whenever possible. For example, if the adjunction comes from a free-forgetful case, i.e. $F^\leftarrow U$, the unit can be written as $\eta^{F\leftarrow U}$ or when a parametric adjunction is involved, on $P$, $F_P \dashv G_P$ its unit can be written as $\eta^{F\leftarrow G}$. The direction of the adjunction $L \dashv R$ will be taken as the direction of its left adjoint function $L$, therefore the domain category of the adjunction is the domain category of $L$. The triangular identity given by $\varepsilon^{F\leftarrow G} : L \circ \eta^{F\leftarrow G} = 1_L$ will be refered to as the triangular identity associated to $L$.

For the 2-category $\text{Cat}$, of small categories and functors, the notation $\mathcal{C}$ will be used instead. The notation $1^\ast$ will be used for cases like $1_P E$ whenever the context allows it. Also, in the cartesian monoidal structure for $\text{Cat}$, whenever possible $L \times P$ will be understood as $L \times 1_P$, for example.

In the proof of the communativity for a diagram, arguments based on the naturality property of a certain transformation will ommited whenever possible, in order to spare for the numerous details. The pasting composition of 2-cells will be denoted as $\cdot_p$.

1 The 2-category context

The 2-category context needed for this article is the following 2-adjunction

$$\Phi^m_E : \text{Adj}_R(\text{Cat}) \rightarrow \text{Mnd}(\text{Cat})$$

the subindex $E$ refers to the Eilenberg-Moore objects, since $\text{Cat}$ admits the construction of algebras, and the superindex $M$ refers to the monad case [3]. Whenever possible, one or both indexes will be dropped.

1.1 The 2-category $\text{Adj}_R(\mathcal{C})$

The n-cells for the domain 2-category, $\text{Adj}_R(\mathcal{C})$, are the following:

i) The 0-cells are adjunctions $L \dashv R : \mathcal{C} \rightarrow \mathcal{X}$.

ii) The 1-cells are of the form $(J, V, \lambda^{JW}, \rho^{JV}) : L \dashv R \rightarrow L \dashv R$ and depicted as
where

are mates and such that $\rho^{JV}$ is an isomorphism. The inverse of $\rho^{JV}$ will be denoted as $\delta^{JV}$ or $\rho^{J^2}$.

Because of the previous, the notation can be shorten to $(J, V, \lambda^{JV})$ or even to $(J, V) : L \rightarrow R \rightarrow L \rightarrow R$, whenever the left mate is understood or unimportant. Since the right mate is an isomorphism, the 2-category will be denoted as $\text{Adj}_R(C)$.

Note: In general, the mate of a natural transformation $\vartheta : LF \rightarrow GL$ might be denoted as

$$\mu_{\vartheta} = \varepsilon^L_p \circ \vartheta \circ \eta^L = R \varepsilon^R \circ R \vartheta \circ \eta^R \circ F R : FR \rightarrow RG.$$
\[
\psi^B \circ \mu^T B = B \mu^S \circ \psi^B S \circ T \psi^B \\
\psi^B \circ \eta^T B = B \eta^S
\]

these equations might be referred to as the compatibility, with the product and the unit of the monads, conditions.

iii) The 2-cells \( \theta : (A, \psi^A) \rightarrow (B, \psi^B) \) are just natural transformations \( \theta : A \rightarrow B : C \rightarrow D \) such that the following equation takes place

\[
\psi^B \circ T \theta = \theta S \circ \psi^A
\]

1.3 The 2-functor \( \Phi^{mE} \)
The 2-functor \( \Phi^{mE} : \text{Adj}_R(C) \rightarrow \text{Mnd}(C) \) is defined on \( n \)-cells as follows

i) For the 0-cell, \( L \dashv R : C \rightarrow X \), \( \Phi^{mE}(L \dashv R) = (C, RL, R \varepsilon L S L, \eta R L) \). That is to say, the monad induced on the domain of the adjunction.

ii) For the 1-cell, \( (J, V, \lambda) : L \dashv R \rightarrow L \dashv R \),

\[
\Phi^{mE}(J, V, \lambda) = (J, \hat{\lambda} \cdot V L \circ R \lambda) : (C, RL) \rightarrow (D, RL)
\]

it will be useful to provide the following notation, \( \Phi(\lambda) = \hat{\lambda} \cdot V L \circ R \lambda \).

iii) For the 2-cell, \( (\alpha, \beta) \), \( \Phi^{mE}(\alpha, \beta) = \alpha \).

1.4 The 2-functor \( \Psi^{mE} \)
The 2-functor \( \Psi^{mE} : \text{Mnd}(C) \rightarrow \text{Adj}_R(C) \) can be constructed if the initial 2-category admits the construction of algebras [6], which is certainly the case for \( \text{Cat} \). It is defined on \( n \)-cells as follows.

i) For \( (C, S) \), \( \Psi^{mE}(C, S) = F^S \dashv U^S : C \rightarrow C^S \). The category \( C^S \) is the Eilenberg-Moore category for the monad \( S \), on \( C \), and the adjunction is the usual free-forgetful adjunction.

ii) For \( (B, \psi^B) : (C, S) \rightarrow (D, T) \),

\[
\Psi^{mE}(B, \psi^B) = (B, \hat{B}, \lambda B) : F^S \dashv U^S \rightarrow F^T \dashv U^T
\]

as in

\[
\begin{array}{ccc}
C & \xrightarrow{B} & D \\
\downarrow{F^S} & & \downarrow{F^T} \\
C^S & \xrightarrow{\hat{B}} & D^T \\
\downarrow{U^S} & & \downarrow{U^T} \\
C & \xrightarrow{\hat{B}} & D
\end{array}
\]

where the functor \( \hat{B} : C^S \rightarrow D^T \) is defined as

\[
\hat{B}(M, k_M) = \left( BU^S(M, k_M), BU^S \varepsilon U^S (M, k_M) \cdot \psi^B U^S (M, k_M) \right)
\]

for \( (M, k_M) \) in \( C^S \). On morphisms, \( \hat{B}(m) = \overline{Bm} \). The left mate \( \lambda B B \) fulfills the following equation \( U^T \lambda B B = \psi^B \). It will be useful to make the following notation \( \Psi(\psi^B) = \lambda B B \).
Another notation for such a functor would be \( \mathcal{L}_\psi(B) := \hat{B} \), where the author is considering that \( \mathcal{L} \) stands for lifting. Also, the notation \( B^\psi \) is particularly useful. The author will use any of these notations that suits better to the context at hand.

The bar over the morphism \( m \) means that, although \( m \) is in \( \mathcal{C} \), it fulfills an additional requirement for algebras. For example, in \( U^S(m \underline{m}) = m \), this requirement is forgotten.

iii) For \( \theta : (A, \psi^A) \to (B, \psi^B) \), \( \Phi^m_E(\theta) = (\hat{\theta}, \hat{\hat{\theta}}) \), where \( \hat{\theta} : \hat{A} \to \hat{B} : \mathcal{C}^S \to \mathcal{D}^T \) and whose component, at \((M, k_M)\) in \( \mathcal{C}^S \), is

\[
\hat{\theta}(M, k_M) = \overline{\theta M}
\]

1.5 The 2-adjunction \( \Phi^m_E \dashv \Psi^m_E \)

The 2-adjunction can be completed, along the previous pair of 2-functors, by giving the following unit and counit.

i) The component of the unit \( \eta^{\Phi} : 1_{\text{Adj}^{\mathcal{R}}(\mathcal{C})} \to \Psi^m_E \Phi^m_E \), on the 0-cell \( L \dashv R \), is given by

\[
\eta^{\Phi}(L \dashv R) := (1_{\mathcal{C}}, \mathcal{R}^{RL}_E) = L \dashv R \to F^{RL} \dashv U^{RL}
\]

where \( \mathcal{R}^{RL}_E : \mathcal{D} \to \mathcal{C}^{RL} \) is the comparison functor. The notation for this functor tries to codify as much its domain as its codomain, in order to minimize possible confusions.

ii) The component of the counit \( \varepsilon^{\Phi} : \Phi^m_E \Psi^m_E \to 1_{\text{Mnd}(\mathcal{C})} \), on the 0-cell \((\mathcal{C}, S)\) is

\[
\varepsilon^{\Phi}(\mathcal{C}, S) := (1_{\mathcal{C}}, 1_S) : (\mathcal{C}, S) \to (\mathcal{C}, S)
\]

All of the previous data make the following Proposition.

**Proposition 1.5.1** There exists a 2-adjunction

\[
\begin{array}{ccc}
\text{Adj}^{\mathcal{R}}(\mathcal{C}) & \xrightarrow{\Psi^m_E} & \text{Mnd}(\mathcal{C}) \\
\Phi^m_E & \xleftarrow{\varepsilon^{\Phi}} & \text{Mnd}(\mathcal{C})
\end{array}
\]

\[\square\]

2 Adjoint Objects in 2-categories

In this section, the definitions of adjoint objects are developed as much in \( \text{Adj}^{\mathcal{R}}(\mathcal{C}) \) as in \( \text{Mnd}(\mathcal{C}) \).

2.1 Adjoint Objects in \( \text{Adj}^{\mathcal{R}}(\mathcal{C}) \)

In this subsection, a characterization of adjoint objects in the 2-category \( \text{Adj}^{\mathcal{R}}(\mathcal{C}) \) is given. These adjoint objects corresponds to the usual definition of an adjoint object in a general 2-category \( \mathcal{A} \), nevertheless the definition is reviewed in order to characterize these structures.

**Definition 2.1.1** An adjoint object in \( \text{Adj}^{\mathcal{R}}(\mathcal{C}) \) is comprised of the following.

i) A pair of 1-cells

\[
\begin{align*}
(J, V, \lambda^{JV}) & : L \dashv R \to T \dashv \overline{R}, \\
(K, W, \lambda^{KW}) & : T \dashv \overline{R} \to L \dashv R.
\end{align*}
\]
ii) A pair of 2-cells called unit and counit, respectively

\[(\eta^{KJ}, \eta^{WV}) : (1_C, 1_D) \rightarrow (KJ, WV)\]
\[(\varepsilon^{JK}, \varepsilon^{VW}) : (JK, VW) \rightarrow (1_X, 1_Y)\]

such that they fulfill the following triangular identities

\[(\varepsilon^{JK} \circ J \eta^{KJ}, \varepsilon^{VW} \circ V \eta^{WV}) = (1_J, 1_V)\]
\[(K \varepsilon^{JK} \circ K \eta^{KJ}, W \varepsilon^{VW} \circ W \eta^{WV}) = (1_K, 1_W)\]

Similar to Theorem 3.13 in [1], this type of adjoint object can be characterized by the existence of a natural transformation inverse.

**Proposition 2.1.2** Consider the following diagram in \(\mathbf{Cat}\), where \(L \dashv R\) and \(\mathcal{L} \dashv \mathcal{R}\),

\[
\begin{array}{ccc}
C & \xrightarrow{K} & D \\
\downarrow & & \downarrow \\
L & \xleftarrow{J} & \mathcal{L} \\
\downarrow & & \downarrow \\
\mathcal{X} & \xleftarrow{W} & \mathcal{Y} \\
\end{array}
\]

Consider \(J \dashv K\), \(V \dashv W\) as classical adjunctions and \((J, V, \lambda^J V)\) a morphism in \(\mathbf{Adj}_R(\mathcal{C})\). The following assertions are equivalent:

i) Exists an adjoint object in \(\mathbf{Adj}_R(\mathcal{C})\), where \((K, W)\) is extended to a 1-cell \((K, W, \lambda^K W)\),

\[(J, V, \lambda^J V) \dashv (K, W, \lambda^K W).\]

ii) \(\lambda^J V\) is invertible.

In such a case, \(\lambda^K W\) is the mate of the inverse of \(\lambda^J V\). The natural transformation \(\lambda^K W\) might be called adjoint of \(\lambda^J V\), the corresponding notation is \(\lambda^K W = \text{ad}(\lambda^J V)\)

**Proof:**

i \(\Rightarrow\) ii.

The proposed inverse, for \(\lambda^J V\), is the following

\[\gamma^J V = \varepsilon^{VW} \mathcal{L} J \circ V \lambda^K W J \circ V L \eta^{KJ}\]  \hspace{1cm} (2)

For example,

\[\gamma^J V \circ \lambda^J V = \varepsilon^{VW} \mathcal{L} J \circ V \lambda^K W J \circ V L \eta^{KJ} \circ \lambda^J V = \varepsilon^{VW} \mathcal{L} J \circ V \lambda^K W J \circ \lambda^J V J \circ \mathcal{L} \eta^{KJ} = \mathcal{L} \varepsilon^{JK} J \circ \mathcal{L} \eta^{KJ} \circ \mathcal{L}(\varepsilon^J K J \circ \eta^{KJ}) = \mathcal{L} 1_J = 1_{\mathcal{L} J}\]
In the third equality, it was used the fact that \((\varepsilon^{JK}, \varepsilon^{VW})\) is a 2-cell in \(\text{Adj}_R(\mathcal{C})\). In the fifth one, the triangular identity associated to \(J\), for the adjunction \(J \dashv K\), was applied.

In a similar way, \(\lambda^JW \circ \gamma^Jv = 1_{vL}\).

\[ii \Rightarrow i.\]

Supposing the existence of the inverse, the natural transformation \(\lambda^{KW}\) is defined as follows

\[\lambda^{KW} = W\overline{\varepsilon^{JK}} \circ W\gamma^Jv K \circ \eta^{Wv} LK\]

That is to say, \(\lambda^{KW} = _K\text{m}_w(\gamma^Jv)\).

In order for \((K,W,\lambda^{KW})\) to be a morphism in \(\text{Adj}_R(\mathcal{C})\), the mate of \(\lambda^{KW}\) must be an isomorphic natural transformation. Then, consider the mate of \(\lambda^{KW}\)

\[\rho^{KW} := RW\varepsilon^{WR} \circ RW\overline{\varepsilon^{JK}} \circ RW\gamma^Jv K \circ R\eta^{Wv} LK\overline{\varepsilon^{WR}} \circ \eta^{RL} K\overline{\varepsilon^{WR}} \circ \overline{\varepsilon^{LR}} W\varepsilon^{WR}\]

and its proposed inverse is

\[\delta^{KW} := K\overline{\varepsilon^{Wv}} \circ K\overline{\varepsilon^{WR}} \circ K\overline{\varepsilon^{Wv}} W \circ K\overline{\varepsilon^{Wv}} L\varepsilon^{WR} \circ K\overline{\varepsilon^{Wv}} JRW \circ \eta^{KJ} RW\]

The equation \(\rho^{KW} \circ \delta^{KW} = 1_{rW}\) is proved as follows.

\[\rho^{KW} \circ \delta^{KW} = RW\varepsilon^{WR} \circ RW\overline{\varepsilon^{JK}} \circ RW\gamma^Jv K \circ R\eta^{Wv} LK\overline{\varepsilon^{WR}} \circ \eta^{RL} K\overline{\varepsilon^{WR}} \circ \overline{\varepsilon^{LR}} W\varepsilon^{WR}\]

In the third equality, the triangular identity associated to \(\overline{\varepsilon^{JK}}\) of the composed adjunction \(\overline{\varepsilon^{JK}} \dashv K\overline{\varepsilon^{WR}}\) was used. In the fifth, the triangular identity associated to \(\overline{\varepsilon^{LR}}\) was applied.

In a similar manner, it can be proved that \(\delta^{KW} \circ \rho^{KW} = 1_{sW}\). Therefore, \((K,W,\lambda^{KW})\) is a morphism, or 1-cell, in \(\text{Adj}_R(\mathcal{C})\).

Remains to prove that the pair \((\eta^{KJ}, \eta^{Wv}) : (1_{\mathcal{C}}, 1_{\mathcal{X}}) \rightarrow (KJ, WV, W\lambda^Jv \circ \lambda^{KW} J) : L \dashv R \rightarrow L \dashv R\) is a 2-cell in \(\text{Adj}_R(\mathcal{C})\). In particular, it is required that

\[W\lambda^Jv \circ \lambda^{KW} J \circ L\eta^{KJ} = \eta^{Wv} L\]

Therefore,

\[W\lambda^Jv \circ \lambda^{KW} J \circ L\eta^{KJ} = W\lambda^Jv \circ W\overline{\varepsilon^{JK}} J \circ W\gamma^Jv K J \circ \eta^{Wv} LKJ \circ L\eta^{KJ}\]

\[= W\lambda^Jv \circ W\overline{\varepsilon^{JK}} (J \circ J\eta^{KJ}) \circ W\gamma^Jv \circ \eta^{Wv} L\]

\[= W\lambda^Jv \circ W\gamma^Jv \circ \eta^{Wv} L = \eta^{Wv} L\]

In the third equality, it was used the triangular identity, of \(J \dashv K\), associated to \(J\).
That the pair \((\varepsilon^{JK}, \varepsilon^{VW}) : (JK, VW, V\lambda^K \circ \lambda^V K) : L \dashv R \rightarrow L \dashv R\) is a 2-cell in \(\text{Adj}_R(\mathcal{C})\) is proved similarly. Finally, the triangular identities are fulfilled since composition, and whiskering, of 2-cells in \(\text{Adj}_R(\mathcal{C})\) are composed, and whiskered, componently as in \(\text{Cat}\).

\[\square\]

### 2.2 Adjoint Objects in \(\text{Mnd}(\mathcal{C})\)

As in the previous section, a detailed account of adjoint objects in the 2-category \(\text{Mnd}(\mathcal{C})\) is given.

**Definition 2.2.1** An adjoint object in \(\text{Mnd}(\mathcal{C})\) is comprised of the following items:

1. A pair of 1-cells,

\[(J, \psi^J) : (C, S) \rightarrow (D, T), \quad (K, \psi^K) : (D, T) \rightarrow (C, S)\]

2. A pair of 2-cells, the unit and the counit of the adjoint object

\[\eta^{KJ} : (1_C, 1_S) \rightarrow (KJ, K\psi^J \circ \psi^K J) : (C, S) \rightarrow (C, S)\]

\[\varepsilon^{JK} : (JK, J\psi^K \circ \psi^J K) \rightarrow (1_D, 1_T) : (D, T) \rightarrow (D, T)\]

such that they fulfill the triangular identities

\[\varepsilon^{JK} J \circ J \eta^{KJ} = 1_J\]

\[K \varepsilon^{JK} \circ \eta^{KJ} K = 1_K\]

This type of object can be characterised using the Theorem 3.13 in [1]. However, it is restated and proved again within the context of this article.

**Proposition 2.2.2** Consider the following adjunction \(J \dashv K : C \rightarrow D\), such that \(J\) is part of the 1-cell

\[(C, S) \xrightarrow{(J, \psi^J)} (D, T)\]

in \(\text{Mnd}(\mathcal{C})\). Then the following assertions are equivalent:

1. Exists an adjoint object in \(\text{Mnd}(\mathcal{C})\), where \(K\) is extended to a 1-cell \((K, \psi^K)\),

\[(J, \psi^J) \dashv (K, \psi^K)\]

2. \(\psi^J\) is invertible.

in such case, \(\psi^K = \text{ad}(\psi^J)\).

**Proof:**

\[i \Rightarrow ii\]

The definition of the inverse \(\zeta^J\), of \(\psi^J\), goes as follows
$$\zeta^J := \varepsilon^{JK} TJ \circ J \psi^K J \circ JS \eta^{KJ}$$

The equality $$\psi^j \circ \zeta^J = 1_{JS}$$ is proved

$$\psi^j \circ \zeta^J = \psi^j \circ \varepsilon^{JK} TJ \circ J \psi^K J \circ JS \eta^{KJ}$$
$$= \varepsilon^{JK} JS \circ JK \psi^j \circ J \psi^K J \circ JS \eta^{KJ}$$
$$= \varepsilon^{JK} JS \circ J \eta^{KJ} S$$
$$= 1_{JS}$$

In the third equality, it was used the fact that $$\eta^{KJ}$$ is a 2-cell in the 2-category $$\text{Mnd}(\mathcal{C})$$. In the fourth one, the triangular identity associated to $$J$$. The case $$\zeta^J \circ \psi^j = 1_{J\psi^j}$$ is done similarly.

$$ii \Rightarrow i.$$ Suppose that $$\psi^j$$ has an inverse, $$\zeta^J$$, then $$\psi^K$$ is defined as follows

$$\psi^K := K T \varepsilon^{JK} \circ K \zeta^J K \circ \eta^{KJ} SK$$

First, it is proved that $$(K, \psi^K) : (\mathcal{D}, T) \rightarrow (\mathcal{C}, S)$$ is a morphism in $$\text{Mnd}(\mathcal{C})$$. In order to do so, the compatibility with the products, i.e. $$\psi^K \circ \mu^S K = K \mu^T \circ \psi^K T \circ S \psi^K$$, is checked.

$$\psi^K \circ \mu^S K = K T \varepsilon^{JK} \circ K \zeta^J K \circ \eta^{KJ} SK \circ \mu^S K$$
$$= K T \varepsilon^{JK} \circ K \mu^T J K \circ K T \zeta^J K \circ K \zeta^J SK \circ \eta^{KJ} S SK$$
$$= K \mu^T \circ K T T \varepsilon^{JK} \circ K T \zeta^J K \circ K T \varepsilon^{JK} JSK \circ K T J \eta^{KJ} SK \circ K \zeta^J SK \circ \eta^{KJ} S SK$$
$$= K \mu^T \circ K T \varepsilon^{JK} T \circ K \zeta^J K \circ SK \varepsilon^{JK} SK \circ SK T \varepsilon^{JK} \circ SK \varepsilon^{JK} K \circ S \eta^{KJ} SK$$
$$= K \mu^T \circ \psi^K T \circ S \psi^K.$$ If $$(J, \psi^j) : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T)$$ is a morphism in $$\text{Mnd}(\mathcal{C})$$, then $$(J, \zeta^J) : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T)$$ is a morphism in the Kleisli dual of $$\text{Mnd}(\mathcal{C})$$. In particular, $$\zeta^J \circ J \mu^S = \mu^T J \circ \zeta^J \circ J \psi^j$$. This compatibility was used for the second equality. In the fifth equality, it was used the $$J$$ triangular identity and the definition of $$\psi^K$$.

The compatibility of the units is left to the reader.

Next, it is needed that $$\eta^{KJ} : (1_{\mathcal{C}}, 1_S) \rightarrow (K J, K \psi^j \circ \psi^K J) : (\mathcal{C}, S) \rightarrow (\mathcal{C}, S)$$, be a 2-cell in $$\text{Mnd}(\mathcal{C})$$, i.e. the following equality takes place $K \psi^j \circ \psi^K J \circ S \eta^{KJ} = \eta^{KJ} S$.

$$K \psi^j \circ \psi^K J \circ S \eta^{KJ} = K \psi^j \circ K T \varepsilon^{JK} J \circ K \zeta^J K \circ S \eta^{KJ}$$
$$= K \psi^j \circ K T \varepsilon^{JK} J \circ K T J \eta^{KJ} \circ K \zeta^J \circ \eta^{KJ} S$$
$$= K \psi^j \circ K \zeta^J \circ \eta^{KJ} S$$
$$= \eta^{KJ} S$$

In the third equality, it was used the triangular identity associated to $$J$$. In the fourth one, the fact that $$\zeta^J$$ is the inverse of $$\psi^j$$ was applied.

Likewise, $$\varepsilon^{JK} : (JK, J \psi^K \circ \psi^j K) \rightarrow (1_{\mathcal{D}}, 1_T) : (\mathcal{D}, T) \rightarrow (\mathcal{D}, T)$$ is a 2-cell in $$\text{Mnd}(\mathcal{C})$$. Since the composition of 2-cells in $$\text{Mnd}(\mathcal{C})$$, and the whiskering, is done as in the subjacent 2-category $$\text{Cat}$$, then the triangular identities are fulfilled.
In Example 3.12 in [1], the lift of an adjunction corresponds to an adjoint object in $\text{Mnd}(\mathcal{C})$. For example, conditions 3a-3d correspond to $G$ and $V$, along with $\zeta$ and $\xi$, being morphisms in $\text{Mnd}(\mathcal{C})$ and 3e-3f are the requirements for the unit and counit $(h,e)$ being 2-cells in the same 2-category.

The results on adjoint objects, using the 2-adjunction $\Phi^m_E \dashv \Psi^r_E$, can be combined. Take the 0-cells $F^S \dashv U^S$ in $\text{Adj}_R(\mathcal{C})$ and $(D,T)$ in $\text{Mnd}(\mathcal{C})$. Therefore, there exists an isomorphism of categories, natural in $F^S \dashv U^S$ and $(D,T)$

$$
\text{Hom}_{\text{Adj}_R(\mathcal{C})}(F^S \dashv U^S, \Psi^r_E(D,T)) \cong \text{Hom}_{\text{Mnd}(\mathcal{C})}(\Phi^m_E(F^S \dashv U^S), (D,T))
$$

in particular, there is a bijection between adjoint objects inside each category. If we take into account the proofs of Proposition 2.1.2 and Proposition 2.2.2 and the previous isomorphism of categories, we are left, without any need of a proof, with the following Theorem.

**Theorem 2.2.3** The following statements are equivalent

i) There exist adjoint objects in $\text{Mnd}(\mathcal{C})$ of the form

$$
(C,S) \quad \begin{array}{c}
\{(K,\psi^K)\} \\
\{(J,\psi^J)\}
\end{array} \quad (D,T)
$$

ii) There exist adjoint objects in $\text{Adj}_R(\mathcal{C})$ of the form

$$
\begin{array}{c}
C \\
\Downarrow F^S \\
C^S
\end{array} \quad \begin{array}{c}
\Downarrow \hat{K} \\
\Downarrow \hat{J}
\end{array} \quad \begin{array}{c}
\Downarrow F^T \\
\Downarrow U^T \\
D^T
\end{array} \quad \begin{array}{c}
\Downarrow \hat{D} \\
\Downarrow U^S \\
D
\end{array}
$$

iii) The natural transformation $\psi^J : TJ \rightarrow JS$ is invertible.

iv) The natural transformation $\lambda^J : F^TJ \rightarrow \hat{J}F^S$ is invertible.

3 Left Hopf 1-cells and Parametric Adjoint Objects in $\text{Adj}_R(\mathcal{C})$

Recalling that the main objective in this article is to characterize parametric adjoint objects as much in $\text{Adj}_R(\mathcal{C})$ as in $\text{Mnd}(\mathcal{C})$ and relate them through the 2-adjunction $\Phi^m_E \dashv \Psi^r_E$; therefore the definition and characterization of these structures in $\text{Cat}$ has to be given.

3.1 Preliminars

The definition of a parametric adjunction is recalled along with the corresponding theorem that characterizes it, [4].
Definition 3.1.1 Consider the following categories \( \mathcal{P}, \mathcal{C} \) and \( \mathcal{D} \). A parametric adjunction, by \( \mathcal{P} \), is a pair of functors of the form

\[
\begin{align*}
F & : \mathcal{C} \times \mathcal{P} \rightarrow \mathcal{D}, \\
G & : \mathcal{P}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{C},
\end{align*}
\]

such that for any \( P \) in \( \mathcal{P} \), there is an adjunction \( F_P : \mathcal{P} \rightarrow \mathcal{D} \) and, for \( p : P \rightarrow Q \), a conjugate morphism of adjunctions \( \tilde{p} : F_P \rightarrow G_P \rightarrow F_Q \). This parametric adjunction can be denoted as \( F \dashv P G : \mathcal{C} \rightarrow \mathcal{D} \).

Now, the corresponding characterizing theorem.

Theorem 3.1.2 Consider a functor \( F : \mathcal{C} \times \mathcal{P} \rightarrow \mathcal{D} \) such that for every \( P \) in \( \mathcal{P} \) there exists a functor \( G_P : \mathcal{D} \rightarrow \mathcal{C} \) and an adjunction

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{G_P} & \mathcal{D} \\
\xleftarrow{F_P} & & \\
\mathcal{D} & \xleftarrow{L} & \mathcal{C}
\end{array}
\]

Therefore, exists a unique \( G : \mathcal{P}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{C} \) such that for \( P \)

\[
G(P, \sim) := G_P,
\]

and for \( P' \rightarrow P \), in \( \mathcal{P}^{\text{op}} \), a natural transformation

\[
G(p^\text{op}, \sim) : G_{P'} \rightarrow G_P,
\]

further denoted as \( G_{p^{\text{op}}} \), such that

\[
G_{p^{\text{op}}} = G_{p} \varepsilon \circ G_{p} F_{P'} G_{P'} \circ \eta_{G_{P'}}.
\]

The departure from the parametric adjoint objects in \( \text{Cat} \) to the 2-category realm is given by the comonoidal adjunction, cf. [5], and the Hopf adjunction, cf. [1].

Definition 3.1.3 A comonoidal adjunction is defined as an adjunction \( L \dashv R : \mathcal{C} \rightarrow \mathcal{D} \) where \( \mathcal{C} \) and \( \mathcal{D} \) are monoidal categories and \( L \) and \( R \) are comonoidal functors and the unit and counit \( \eta_{\text{RL}} : 1_{\mathcal{C}} \rightarrow RL \) and \( \varepsilon_{\text{LR}} : LR \rightarrow 1_{\mathcal{D}} \) are natural comonoidal transformations.

In [3] there is a characterization of this comonoidal adjunctions.

Proposition 3.1.4 The following statements are equivalent

i) The adjunction \( L \dashv R : \mathcal{C} \rightarrow \mathcal{D} \) is comonoidal.

ii) The following diagram

\[
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{\otimes_{\mathcal{C}}} & \mathcal{C} \\
\mathcal{D} \times \mathcal{D} & \xrightarrow{\otimes_{\mathcal{D}}} & \mathcal{D}
\end{array}
\]

is a 1-cell, \( (\otimes_{\mathcal{C}}, \otimes_{\mathcal{D}}, \lambda_{\otimes_{\mathcal{CD}}}) \), in \( \text{Adj}_{R}(\mathcal{C}) \).

Let us remember the definition of a Hopf operator in order to start the extension of this concepts to the context of 2-categories.
Definition 3.1.5 Let \( L \dashv R : C \to D \) be a comonoidal adjunction. The left Hopf operator, \( \mathcal{H} \), is the following natural transformation

\[
\mathcal{H}(\lambda^{\otimes C_D}) : \otimes_D (L \times \varepsilon^{LR}) \circ \lambda^{\otimes C_D} (C \times R) : C \times D \to D
\]

(4)

3.2 Hopf 1-cells

The objective of this section is to extend the definition of a parametric adjunction to the 2-category of adjunctions \( \text{Adj}_R(C) \).

Consider a 1-cell in \( \text{Adj}_R(C) \) of the form \((J, V, \lambda^{JV}) : L \times \bar{L} \dashv R \times \bar{R} \to \mathcal{T} \dashv \mathcal{R}\)

Suppose that the functors \( J : C \times P \to D \) and \( V : X \times Q \to Y \) are part of classical parametric adjunctions, namely \( J \dashv_P K \) and \( V \dashv_Q W \). There is no immediate translation of a parametric adjoint object to the 2-category \( \text{Adj}_R(C) \) due to a little obstacle. The problem arises with the possible definition of the 1-cell \((K, W, \lambda^{KW})\) where the opposite adjunction, for \( \bar{L} \dashv \bar{R} \), \( \bar{R}^{op} \dashv \bar{L}^{op} : Q^{op} \to P^{op} \) change the domain and the codomain, therefore a 1-cell of the form \((K, W, \lambda^{KW})\) cannot be defined.

Hence, the objective can be changed to the study of what extension a parametric adjunction can be reasoning within the 2-category \( \text{Adj}_R(C) \). For that, the following modifications of definitions, in [1], can be given.

Definition 3.2.1 Let \((J, V, \lambda^{JV}) : L \times \bar{L} \dashv R \times \bar{R} \to \mathcal{T} \dashv \mathcal{R}\) be a 1-cell in \( \text{Adj}_R(C) \). A left Hopf operator \( \mathcal{H} \) on \((J, V, \lambda^{JV})\) is a 1-cell in \( \text{Adj}_R(C) \) of the form

\[
\mathcal{H}(J, V, \lambda^{JV}) := (J(C \times \bar{R}), V(H(\lambda^{JV}))) : L \times Q \dashv R \times Q \to \mathcal{T} \dashv \mathcal{R}
\]

where \( H(\lambda^{JV}) \) is the following natural transformation

that is to say
\[ H(\lambda^{JV}) = (L \times \varepsilon^{\text{LR}}) \cdot \rho \lambda^{JV} = V(L \times \varepsilon^{\text{LR}}) \circ \lambda^{JV}(C \times \tilde{R}) . \] (5)

**Definition 3.2.2** A left Hopf 1-cell in \( \text{Adj}_R(\mathcal{C}) \),

\[ (J, V, \lambda^{JV}) : L \times \tilde{L} \dashv R \times \tilde{R} \rightarrow \overline{T} \dashv \overline{P}, \]

is such that \( H(\lambda^{JV}) \) is invertible. In such a case, the inverse is denoted as \( N(\lambda^{JV}) \).

Consider a left Hopf 1-cell in \( \text{Adj}_R(\mathcal{C}) \), \( (J, V, \lambda^{JV}) \), therefore its left Hopf operator is invertible and so is the following natural transformation, for any \( Q \) in \( \mathcal{Q} \),

\[
\begin{array}{ccccccccc}
C & \xrightarrow{\rho^{-1}C} & C \times 1 & \xrightarrow{C \times E_Q} & C \times Q & \xrightarrow{C \times \tilde{R}} & C \times \mathcal{P} & \xrightarrow{J} & D \\
L & \downarrow & L \times 1 & \downarrow & L \times \mathcal{Q} & \xrightarrow{L \times E_Q} & L \times \tilde{L} & \xrightarrow{\lambda^{JV}} & Y \\
X & \xrightarrow{\rho^{-1}X} & X \times 1 & \downarrow & X \times \mathcal{Q} & \xrightarrow{1 \times 1 \times \mathcal{Q}} & X \times \mathcal{Q} & \xrightarrow{V} & Y \\
\end{array}
\]

The functor \( E_Q \) stands for evaluation at \( Q \) in \( \mathcal{Q} \). The previous natural transformation can be written as follows

\[
\begin{array}{cccc}
C & \xrightarrow{J_{\tilde{R}Q}} & D \\
L & \downarrow & \overline{T} \\
X & \xrightarrow{V_Q} & Y \\
\end{array}
\]

**Remark:** If \( J \dashv K \) then \( J(C \times \tilde{R}) \dashv Q K(\tilde{R}^{\text{op}} \times D) \).

Due to the previous remark, there are two parametric adjunctions on \( Q \), \( J(C \times \tilde{R}) \dashv Q K(\tilde{R}^{\text{op}} \times D) \) and \( V \dashv W \) with corresponding adjunctions \( J_{\tilde{R}Q} \dashv K_{\tilde{R}Q} \) and \( V_Q \dashv W_Q \), for any \( Q \) in \( \mathcal{Q} \), such that

\[ (J_{\tilde{R}Q}, V_Q, \lambda^{JVQ}) \]

is a 1-cell in \( \text{Adj}_R(\mathcal{C}) \) and \( \lambda^{JVQ} \) is invertible. If Proposition [2.1.2](#) is recalled for this situation, there exists an adjoint object in \( \text{Adj}_R(\mathcal{C}) \)

\[ (J_{\tilde{R}Q}, V_Q, \lambda^{JVQ}) \dashv (K_{\tilde{R}Q}, W_Q, \lambda^{KWQ}) \]

where \( \lambda^{KWQ} = ad(\lambda^{JVQ}) \).

The natural transformation \( \lambda^{KWQ} : LK_{\tilde{R}Q} \rightarrow W_Q \overline{T} : D \rightarrow X \) can be extended to a dinatural transformation of the form

\[ \lambda^{KW} : LK(\tilde{R}^{\text{op}} \times D) \rightarrow W(Q^{\text{op}} \times \overline{T}) : Q^{\text{op}} \times D \rightarrow X \]

This claim is stated as the following proposition
Proposition 3.2.3 In the previous context, there exists a dinatural transformation of the form

\[ \lambda^{K \overline{W}} : LK(\overline{R}^{op} \times D) \rightarrow W(Q^{op} \times \overline{T}) : Q^{op} \times D \rightarrow X \]

defined, on \((Q, D)\) in \(Q^{op} \times D\), as \(\lambda^{K \overline{W}}(Q, D) := \lambda^{K \overline{W}} D\) and such that for any \((q^{op}, d) : (Q', D) \rightarrow (Q, D')\) in \(Q^{op} \times D\), the following diagram commutes.

![Diagram](image)

Proof:

First, recall that

\[ K(\overline{(Rq)}^{op}, \sim) := K_{\overline{Rq}} e^{JK \overline{Rq'}} \circ K_{\overline{Rq'}} J_{\overline{Rq}} K_{\overline{Rq}} \circ \eta^{KJ \overline{Rq}} K_{\overline{Rq'}} \]

since \(J(C \times \overline{R}) \dashv Q K(\overline{R}^{op} \times D)\). There exists a similar expression for \(W(q^{op}, \sim)\). Second, the following equation takes place due to the naturality of all of the involved components

\[
W_q L d \cdot W_q e^{v W Q'} L D \cdot W_q V_q W_q' L D \cdot \eta^{W V Q} W_q' L D \cdot \lambda^{K \overline{W}} D = W_q e^{v W Q'} L D' \cdot W_q V_q W_q' L D' \cdot \eta^{W V Q} W_q' L D' \cdot \lambda^{K \overline{W}} D' \cdot L K_{\overline{Rq}} d
\]

Therefore, it is left to prove that the following equation takes place

\[
\lambda^{K \overline{W}} D \cdot L K_{\overline{Rq}} e^{JK \overline{Rq'}} \cdot L K_{\overline{Rq'}} J_{\overline{Rq}} K_{\overline{Rq}} D \cdot L \eta^{K J \overline{Rq}} K_{\overline{Rq'}} D
= W_q e^{v W Q'} \overline{T} D \cdot W_q V_q W_q' \overline{T} D \cdot \eta^{W V Q} W_q' \overline{T} D \cdot \lambda^{K \overline{W}} D
\]

This is done by the following process:

\[
\lambda^{K \overline{W}} D \cdot L K_{\overline{Rq}} e^{JK \overline{Rq'}} \cdot L K_{\overline{Rq'}} J_{\overline{Rq}} K_{\overline{Rq}} D \cdot L \eta^{K J \overline{Rq}} K_{\overline{Rq'}} D
= \lambda^{K \overline{W}} D \cdot L K_{\overline{Rq}} e^{JK \overline{Rq'}} \cdot L K_{\overline{Rq}} e^{JK \overline{Rq'}} D \cdot L \eta^{K J \overline{Rq}} D
= \lambda^{K \overline{W}} D \cdot L K_{\overline{Rq}} e^{JK \overline{Rq'}} D = W_{q^{op}} \overline{T} D \cdot \lambda^{K \overline{W}} D
= W_q e^{v W Q'} \overline{T} D \cdot W_q V_q W_q' \overline{T} D \cdot \eta^{W V Q} W_q' \overline{T} D \cdot \lambda^{K \overline{W}} D
= W_q e^{v W Q'} \overline{T} D \cdot W_q V_q W_q' \overline{T} D \cdot \eta^{W V Q} W_q' \overline{T} D \cdot \lambda^{K \overline{W}} D
\]

The first equality takes place due to the Proposition \(\text{Adj} \), where \(J_{\overline{Rq}} \) and \(K_{(\overline{Rq})^{op}} \) are conjugate morphisms. The third one, uses the triangular identity associated to \(K_{\overline{Rq}}\). The fourth one, is due to the fact that \((V_q, W_{q^{op}})\) is a 2-cell in \(\text{Adj}_{q}(C)\). The fifth one uses the triangular identity associated to \(W_q\). The seventh one is related to the fact that \(V_q\) and \(W_{q^{op}}\) are conjugate. The rest of the equalities have to do with an involved naturality and therefore the details are spare for those.
The mate of $\lambda^{KW\tilde{R}Q}$, $\rho^{KW\tilde{R}Q}$, and the inverse of this last one $\varrho^{KW\tilde{R}Q}$ can also be extended to a dinatural transformation.

**Corollary 3.2.4** The transformation defined, for $(Q,Y)$ in $Q^{op} \times Y$, as

$$\varrho^{KW\tilde{R}}(Q,Y) := \varrho^{KW\tilde{R}Q}Y$$

is dinatural, i.e. for any $(q^{op},y)$ in $Q^{op} \times Y$, the following diagram commutes

![Diagram](image.png)

The inverse of the mate of the dinatural transformation $\lambda^{KW\tilde{R}}$ is the dinatural transformation $\varrho^{KW\tilde{R}}$ then the following is a 1-cell in $\text{Adj}_R(C)$

$$(K(\tilde{R}^{op} \times D), W, \lambda^{KW\tilde{R}}) : Q^{op} \times \mathcal{T} \rightarrow Q^{op} \times \mathcal{R} \rightarrow L \rightarrow R$$

Therefore, using a left Hopf 1-cell, an object similar to a parametric adjunction could be obtained. This result is summarized, and the corresponding process, into the following statement and definition.

**Theorem 3.2.5** Consider a left Hopf 1-cell of the form

$$(J, V, \lambda^{J^V}) : L \times \tilde{L} \rightarrow R \times \tilde{R} \rightarrow \mathcal{T} \rightarrow \mathcal{T}$$

and a pair of classical parametric adjunctions $J \vdash^p K$ and $V \vdash_Q W$. Then we have

i) $(J(C \times \tilde{R}), V, \lambda^{J^V}) : L \times Q \rightarrow R \times Q \rightarrow \mathcal{T} \rightarrow \mathcal{T}$.

ii) $(K(\tilde{R}^{op} \times D), W, \lambda^{KW\tilde{R}}) : Q^{op} \times \mathcal{T} \rightarrow Q^{op} \times \mathcal{R} \rightarrow L \rightarrow R$.

as 1-cells in $\text{Adj}_R(C)$ and for each $Q$ in $Q$, an adjoint object

$$(J_{\tilde{R}Q}, V_Q, \lambda^{J^V_Q}) \vdash (K_{\tilde{R}Q}, W_Q, \lambda^{KW\tilde{R}Q})$$

Therefore, this structure might be defined as a Hopf parametric adjoint object, in $\text{Adj}_R(C)$, and denoted as

$$(J(C \times \tilde{R}), V, \lambda^{J^V}) \vdash_Q (K(\tilde{R}^{op} \times D), W, \lambda^{KW\tilde{R}})$$

□
3.3 The Antipode

Similar to the definition of an antipode in [1], the following corollary is stated in order to define this concept for a left Hopf 1-cell.

**Corollary 3.3.1** The following transformation

\[ \psi^{KR} := \theta^{KWR} p \lambda^{KWR} = \theta^{KWR} (Q^{op} \times \overline{L}) \circ R \lambda^{KWR} : RLK(\overline{R}^{op} \times D) \to K(\overline{R}^{op} \times \overline{RL}) \]

is dinatural.

According to [1], there is a certain bijection of dinatural transformations, which is now rewritten in this context for a left Hopf 1-cell in \( \text{Adj}_{\mathcal{U}}(\mathcal{C}) \).

**Proposition 3.3.2** There is a bijection between the following dinatural transformations

i) \( \psi^{KR} : RLK(\overline{R}^{op} \times D) \to K(\overline{R}^{op} \times \overline{RL}) : Q^{op} \times D \to \mathcal{C} \).

ii) \( \sigma^{KR} : RLK(\overline{R}^{op} \overline{L}^{op} \times D) \to K(\overline{P}^{op} \times \overline{RL}) : P^{op} \times D \to \mathcal{C} \).

This last dinatural transformation is called antipode.

\[ \blacksquare \]

4 Left Hopf 1-cells and Parametric Adjoint Objects in \( \text{Mnd}(\mathcal{C}) \)

In this chapter, the definitions made in the previous section are recalled but this time monads are used. The objective, in this section as in the whole article, is to give an extension of a classical parametric adjunction \( J \dashv K \) within the 2-categorical context of \( \text{Mnd}(\mathcal{C}) \).

4.1 Hopf 1-cells

Consider for this case the following 0-cells \((\mathcal{C}, S), (\mathcal{D}, T)\) and \((\mathcal{P}, E)\). For any functor \( J : \mathcal{C} \times \mathcal{P} \to \mathcal{D} \) one can think of a 1-cell, in \( \text{Mnd}(\mathcal{C}) \), of the form

\[ (J, \psi^J) : (\mathcal{C} \times \mathcal{P}, S \times E) \to (\mathcal{D}, T) \]

If one wishes to construct a parametric adjoint object then there must exist a functor \( K : \mathcal{P}^{op} \times \mathcal{D} \to \mathcal{C} \) that can be extended to a 1-cell in \( \text{Mnd}(\mathcal{C}) \), but such an extension presents a problem. Since \( E \) is a monad on \( \mathcal{P} \), \( E^{op} \) is a comonad on \( \mathcal{P}^{op} \), therefore it cannot be proposed a 1-cell \( (K, \psi^K) : (\mathcal{P}^{op} \times \mathcal{D}, E^{op} \times T) \to (\mathcal{C}, S) \), neither in \( \text{Mnd}(\mathcal{C}) \) or in the comonad dual of \( \text{Mnd}(\mathcal{C}) \) in order to complete a possible parametric adjunction. In the same way as before, a modification of the functors \( J \) and \( K \) has to be made in order to achive the proposed objective.

**Definition 4.1.1** Consider a 1-cell, in \( \text{Mnd}(\mathcal{C}) \), of the form

\[ (J, \psi^J) : (\mathcal{C} \times \mathcal{P}, S \times E) \to (\mathcal{D}, T) \]

The left Hopf operator on \( (J, \psi^J) \) is the following 1-cell in \( \text{Mnd}(\mathcal{C}) \)

\[ S_J(J, \psi^J) := (J(C \times U^E), H(\psi^J)) = (J(C \times U^E), S \times \mathcal{P}^E) \to (\mathcal{D}, T) \]

where \( H(\psi^J) \) is the following natural transformation.
that is to say,

\[ H(\psi^J) = (S \times U^E \varepsilon^F_{UE}) \cdot p \psi^J = J(S \times U^E \varepsilon^F_{UE}) \circ \psi^J(C \times U^E) \]

**Definition 4.1.2** Consider a 1-cell \((J, \psi^J) : (C \times P, S \times E) \to (D, T)\) in \(\text{Mnd}(C)\). The left fusion operator is the following 1-cell in \(\text{Mnd}(C)\)

\[ F(J, \psi^J) := (J(C \times E), F(\psi^J)) = (J(C \times E), S \times P) \to (D, T) \]

where \(F(\psi^J)\) is the following natural transformation

that is to say

\[ F(\psi^J) = (S \times \mu^E) \cdot p \psi^J = J(S \times \mu^E) \circ \psi^J(C \times E) \]

**Definition 4.1.3** A left Hopf 1-cell, in \(\text{Mnd}(C)\), is of the form \((J, \psi^J) : (C \times P, S \times E) \to (D, T)\) such that \(H(\psi^J)\) is invertible. In such case, the inverse is denoted as \(N(\psi^J)\).

**Definition 4.1.4** A left fusion 1-cell, in \(\text{Mnd}(C)\), \((J, \psi^J) : (C \times P, S \times E) \to (D, T)\) is such that \(F(\psi^J)\) is invertible. In such case, the inverse is denoted as \(G(\psi^J)\).

**Remark:** Later on, it will be checked that the Hopf and fusion 1-cell will be equivalent, which in turn will ease the difference with the Hopf monad definition on \([1]\).

Consider a classical parametric adjunction \(J \dashv P K\) and a left Hopf 1-cell \((J, \psi^J) : (C \times P, S \times E) \to (D, T)\). Since \(H(\psi^J)\) is invertible so is the following natural transformation for any \((M, k_M)\).

The previous invertible natural transformation is denoted as \(\psi^{J,UE}(M, k_M)\) or

\[ \psi^{JM} : T_{JM} \to J_{M}S : C \to D, \]
Remark: If \( J \dashv_P K \) then \( J(\mathcal{C} \times U^E) \dashv_{P_E} K(U^{E_{op}} \times \mathcal{D}) \).

For any Eilenberg-Moore algebra \((M, k_M)\), there is an adjunction \( J_M \dashv K_M \) such that the following is a 1-cell in \( \mathbf{Mnd}(\mathcal{C}) \)

\[
(J_M, \psi^{JM}) : (\mathcal{C}, S) \longrightarrow (\mathcal{D}, T)
\]

and \( \psi^{JM} \) is invertible. If the Proposition 2.2.2 is applied, an adjoint object in \( \mathbf{Mnd}(\mathcal{C}) \) is obtained

\[
(J_M, \psi^{JM}) \dashv (K_M, \psi^{KM})
\]

where \( \psi^{KM} = ad(\psi^{JM}) \). The last natural transformation can be further extended.

**Proposition 4.1.5** The transformation \( \psi^{KM} \), on \((M, k_M)\), can be extended to the following dinatural transformation

\[
\psi^{KUE} : SK(U^{E_{op}} \times \mathcal{D}) \longrightarrow K(U^{E_{op}} \times \mathcal{D})(\mathcal{P}^{E_{op}} \times T) : (\mathcal{P}^{E_{op}} \times \mathcal{D}) \longrightarrow \mathcal{C}.
\]

**Proof:**

Define \( \psi^{KUE} \) for \(((M, k_M), D)\), in \((\mathcal{P}^{E_{op}} \times \mathcal{D})\), as

\[
\psi^{KUE}((M, k_M), D) := \psi^{KM} D.
\]

The proof of the commutativity for the corresponding morphism \( \overline{p}^{op} : (M', k_{M'}) \longrightarrow (M, k_M) \) is left to the reader.

\[\square\]

In the context of the previous Proposition, the dinatural transformation can be denoted as \( \psi^{KUE} := ad(H(\psi^{J})) \) or \( \psi^{KUE} := H^2(\psi^{J}) \).

The previous process can be summarized into the following Theorem.

**Theorem 4.1.6** Consider a classical parametric adjunction \( J \dashv_P K \) and a left Hopf 1-cell in \( \mathbf{Mnd}(\mathcal{C}) \) of the form

\[
(J, \psi^J) : (\mathcal{C} \times \mathcal{P}, S \times E) \longrightarrow (\mathcal{D}, T)
\]

Then the following

i) \( (J(\mathcal{C} \times U^E), \psi^{JUE}) : (\mathcal{C} \times \mathcal{P}^{E_{op}}, S \times \mathcal{P}^{E_{op}}) \longrightarrow (\mathcal{D}, T), \)

ii) \( (K(U^{E_{op}} \times \mathcal{D}), \psi^{KUE}) : ((\mathcal{P}^{E_{op}} \times \mathcal{D}, (\mathcal{P}^{E_{op}} \times T) \longrightarrow (\mathcal{C}, S) \)

are 1-cells in \( \mathbf{Mnd}(\mathcal{C}) \) and for each \((M, k_M)\) in \( \mathcal{P}^{E} \) there is an adjoint object

\[
(J_M, \psi^{JM}) \dashv (K_M, \psi^{KM})
\]

Therefore, this structure might be defined as a Hopf parametric adjoint object, in \( \mathbf{Mnd}(\mathcal{C}) \), and denoted as

\[
(J(\mathcal{C} \times U^E), \psi^{JUE}) \dashv_{P_E} (K(U^{E_{op}} \times \mathcal{D}), \psi^{KUE})
\]

\[\square\]
4.2 Antipode

Analogous to the bijection in Proposition 3.3.2, consider the dinatural transformation

$$\psi^{KUE} : SK(U^{op} \times 1_D) \rightarrow K(U^{op} \times 1_D)(P^{op} \times T)$$

whisker it with the functor $F^{U^{op} \times D}$ and compose it with the natural transformation $K(\varepsilon^{U^{op} \times T})$ to get

$$\sigma := K(\varepsilon^{U^{op} \times T}) \circ \psi^{KUE}(F^{U^{op} \times D}) : SK(E^{op} \times D) \rightarrow K(P^{op} \times T) : P^{op} \times D \rightarrow C$$

whose component at $(P, D)$ in $P^{op} \times D$, noting that $\varepsilon^{U^{op} \times T} = (\eta^{P^{op}})^{op} = (\eta^P)^{op}$, is

$$\sigma^K(P, D) = K((\eta^P)^{op}, T) \cdot \psi^{KUE}((EP, (\mu^P)^{op}), D)$$

therefore, $\sigma^K(P, D) : SK(EP, D) \rightarrow K(P, TD)$.

In [1], A. Brugieres et. al. called this natural transformation (left) antipode. As pointed out by them, there is a bijection between the dinatural transformations $\psi^{KUE}$ and $\sigma^K$, where the inverse of the bijection acts on $\sigma^K$ as follows

$$\iota := \sigma^K(U^{op} \times D) \circ SK(\eta^{FUE^{op}} \times D) : SK(U^{op} \times D) \rightarrow K(U^{op} \times D)$$

whose component at the object $(M, k_M, D)$, noting that $\eta^{U^{op}}(M, k_M) = (\varepsilon^{FUE}(M, k_M))^{op} = (k_M)^{op}$, is

$$\iota((M, k_M, D)) = \sigma^K(M, D) \cdot SK(k_M^{op}, D) : SK(M, D) \rightarrow K(M, TD)$$

reminiscent of the properties for $\psi^{KUE}$, as a 1-cell in $\text{Mnd}(C)$, the equations that fulfills this antipode are the following

$$\sigma^K \circ \mu^S K(E^{op} \times D) = K(1_\ast \times \mu^T \circ \sigma^K(1_\ast \times T) \circ S \sigma^K(E^{op} \times D) \circ S SK((\mu^P)^{op}D)$$

$$\sigma^K \circ \eta^S K(E^{op} \times D) = K(\varepsilon^{FUE^{op}} \times \eta^T) = K((\eta^P)^{op} \times \eta^T)$$

Compare these equations with those equivalent as in Proposition 3.8.b, [1].

5 Left Hopf 1-cells through the 2-adjunction $\Phi^{\text{DR}}_E \dashv \Psi^{\text{DR}}_E$

5.1 Comparing Hopf 1-cells

Consider the 1-cell $(J, V, \lambda^{IV}) : L \times \tilde{L} \dashv R \times \tilde{R} \rightarrow \overline{T} \vdash \overline{R}$ in $\text{Adj}_{\lambda\pi}(C)$. This induces a 1-cell in $\text{Mnd}(C)$ of the form $\Phi^m_B(J, V, \lambda^{IV}) = (J, \Phi(\lambda^{IV})) = (J, \varrho^{IV}(L \times \tilde{L}) \circ \overline{R}\lambda^{IV})$, where $\varrho^{IV}$ is the inverse of the mate $\rho^{IV} = R \times \tilde{R} \text{m}_\pi(\lambda^{IV})$. Therefore

$$H(\Phi(\lambda^{IV})) = H(\varrho^{IV}(L \times \tilde{L}) \circ \overline{R}\lambda^{IV})$$

$$= J(RL \times \tilde{R} \overline{R}) \circ \varrho^{IV}(L \times \tilde{L})(C \times \tilde{R}) \circ \overline{R}\lambda^{IV}(C \times \tilde{R})$$

$$= \varrho^{IV}(L \times Q) \circ \overline{R}(V(L \times \varepsilon \tilde{R}) \circ \lambda^{IV}(C \times \tilde{R}))$$

$$= \varrho^{IV}(L \times Q) \circ \overline{R}H(\lambda^{IV}) = \Phi(H(\lambda^{IV}))$$

where the last equality takes place since $R \times \tilde{R} \text{m}_\pi(\lambda^{IV}) = R \times \tilde{R} \text{m}_\pi(H(\lambda^{IV}))$. Then, the following proposition can be stated.

**Proposition 5.1.1** Consider the 1-cell $(J, V, \lambda^{IV}) : L \times \tilde{L} \dashv R \times \tilde{R} \rightarrow \overline{T} \vdash \overline{R}$ in $\text{Adj}_{\lambda\pi}(C)$, such that $\overline{R}$ reflects isomorphisms, then the following statements are equivalent:
i) $H(\lambda^{J V})$ is invertible, i.e. the 1-cell is left Hopf in $\text{Adj}_R(\mathcal{C})$.

ii) $H(\Phi(\lambda^{J V}))$ is invertible, i.e. the induced 1-cell $\Phi(\lambda^{J V})$ is left Hopf in $\text{Mnd}(\mathcal{C})$.

Proof:

i) $\Rightarrow$ ii)

If $H(\lambda^{J V})$ is invertible, so is $\rho^{J V}(L \times Q) \circ \overline{R}H(\lambda^{J V}) = \Phi(H(\lambda^{J V}))$ and the conclusion follows from the previous equality.

ii) $\Rightarrow$ i)

If $H(\Phi(\lambda^{J V}))$ is invertible so is $\rho^{J V}(L \times Q) \circ H(\Phi(\lambda^{J V})) = \overline{R}H(\lambda^{J V})$, since $\overline{R}$ reflects isomorphisms $H(\lambda^{J V})$ is invertible.

The inverses are related as follows

$$N(\Phi(\lambda^{J V})) = \overline{R}N(\lambda^{J V}) \circ \rho^{J V}(L \times Q)$$

5.2 Hopf Parametric Adjunctions through the 2-adjunction $\Phi_E^{mt} \dashv \Psi_E^{mt}$

Using the unit of the 2-adjunction $\Phi_E^{mt} \dashv \Psi_E^{mt}$ for the 1-cell $(J(C \times \tilde{R}), V, H(\lambda^{J V}))$ the following proposition can be stated.

Proposition 5.2.1 Consider the following list:

i) $L \dashv R : C \rightarrow X$ and the induced monad $(C, S)$.

ii) $\overline{L} \dashv \overline{R} : D \rightarrow Y$ and the induced monad $(D, T)$.

iii) $\tilde{L} \dashv \tilde{R} : P \rightarrow Q$ and the induced monad $(P, E)$.

and suppose that $(J, V, \lambda^{J V})$, is a Hopf 1-cell. Therefore, there exists the following pair of commuting diagrams in $\text{Adj}_R(\mathcal{C})$

Consider the 2-adjunction $\Phi_E^{mt} \dashv \Psi_E^{mt}$, this structure gives particular classes of isomorphisms of categories. Certain 0-cells are chosen in order to get an adequate isomorphism, for example, consider the following monads, 0-cells in $\text{Mnd}(\mathcal{C})$, $(C, S)$, $(P, E)$ and $(D, T)$ and construct the 0-cells $F^S \times F^E \dashv U^S \times U^E$ and $F^T \dashv U^T$, in $\text{Adj}_R(\mathcal{C})$, therefore exists the following isomorphism of categories
\[ \text{Hom}_{\text{Adj}_H(C)}(F^S \times P^E, F^T \times (P^E, F^T)) \cong \text{Hom}_{\text{Mnd}(C)}(\langle C \times P^E, S \times P^E \rangle, (D, T)) \]

Similar isomorphisms exist for combinations of the 0-cells \((P^E)^{op} \times F^T \to (P^E)^{op} \times U^T, F^T \to U^T\) and \(F^S \to U^S\).

The following theorem can come forth which combines, through the 2-adjunction \(\Phi^m \vdash \Psi^m\) and the corresponding isomorphisms, the parametric adjoint objects in Theorem 3.2.5 and Theorem 4.1.6.

**Theorem 5.2.2** Consider a left Hopf 1-cell \((J, \psi^J) : (C \times P, S \times E) \to (D, T)\) in \(\text{Mnd}(C)\) whose functor is part of a classical parametric adjunction \(J \dashv_P K\). Therefore, there exists a bijection between the following structures

i) Hopf parametric adjunctions, in \(\text{Adj}_H(C)\), of the form

\[
\left( J(C \times U^E), [J(C \times U^E)]^{H \psi, \lambda^{H \psi}} \right) \dashv_P \left( K(U^{E^\text{op}} \times D), [K(U^{E^\text{op}} \times D)]^{H^{J \psi}, \lambda^{K H \psi}} \right)
\]

ii) Hopf parametric adjunctions, in \(\text{Mnd}(C)\), of the form

\[
(J(C \times U^E), H(\psi^J)) \dashv_P (K(U^{E^\text{op}} \times D), H^J(\psi^J))
\]

6 Lifting parametric adjunctions

In order to lift a classical parametric adjunction, to some Eilenberg-Moore categories of algebras, there are some further discussion and calculations to be done.

6.1 Hopf and Fusion 1-cells

Consider the 1-cell \((J, V, \lambda^{JV}) : L \times \tilde{L} \vdash R \times \tilde{R} \to \tilde{L} \vdash \tilde{R}\). Similar to the relation of the Hopf operators, there is the following relation between the fusion and Hopf operators

\[
F(\Phi(\lambda^{JV})) = (q^{JV},_P H(\lambda^{JV}))(C \times \tilde{L})
\] (6)

The following lemma is required.

**Lemma 6.1.1** Given an adjunction of the form \(L \vdash R : P \to Q\) and a natural transformation \(\alpha : A \tilde{R} \to B \tilde{R}\), where \(A\) and \(B\) are arbitrary parallel functors, with domain \(Q\). Therefore, \(\alpha\) is invertible if \(\alpha L\) is so. In this case the inverse of the component \(\alpha Q\) is the following

\[
\alpha^{-1}Q = A \tilde{R} \tilde{L}Q \cdot (\alpha L)^{-1} \tilde{R}Q \cdot B \eta \tilde{L} \tilde{R}Q
\] (7)

**Proof**:

The proof is similar to the Lemma 2.19 given in [1] but this time the following split fork is used

\[
\begin{array}{ccc}
\tilde{R}L \tilde{R}LQ & \xrightarrow{\tilde{R}L \tilde{R}LQ} & \tilde{R}L \tilde{R}LQ \\
\xrightarrow{\tilde{R}L \tilde{R}LQ} & & \xrightarrow{\tilde{R}L \tilde{R}LQ} \\
\end{array}
\]

The following proposition can be written.

**Proposition 6.1.2** Consider the 1-cell \((J, V, \lambda^{JV}) : L \times \tilde{L} \vdash R \times \tilde{R} \to \tilde{L} \vdash \tilde{R}\), such that \(\tilde{R}\) reflects isomorphisms, then the following statements are equivalent

i) \(H(\lambda^{JV})\) is invertible, i.e. the 1-cell is left Hopf in \(\text{Adj}_H(C)\).

ii) \(F(\Phi(\lambda^{JV}))\) is invertible, i.e. the 1-cell is left fusion in \(\text{Mnd}(C)\).
Proof:

i) \( \Rightarrow \) ii) Clear by taking into consideration (6).

ii) \( \Rightarrow \) i) In order to use the previous lemma, take \( C \) in \( \mathcal{C} \) and the natural transformation \( \alpha \) is given by

\[
(\rho^i \cdot p H(\lambda^j))((C, \sim)) : \overline{R}LJ(C, \overline{R}) \rightarrow JRLC, \overline{R},
\]

therefore \( \rho^i \cdot p H(\lambda^j) \) is invertible and so is \( \overline{R}H(\lambda^j) \), since \( \overline{R} \) reflects isomorphisms \( H(\lambda^j) \) is also invertible.

\[\Box\]

The inverses are related as follows

\[
\overline{R}N(\lambda^j) \circ \rho^i (L \times Q) = \overline{R}LJ(C \times \overline{R}^E) \circ G(\Phi(\lambda^j))(C \times \overline{R}) \circ JRL(RL \times \eta^E \overline{R})
\]

The reader is compelled to check the same expression in Lemma 2.18 [1].

Corollary 6.1.3 Consider a 1-cell \((J, \psi) : (C \times P, S \times E) \rightarrow (D, T)\). Therefore \( F(\psi) \) is invertible iff \( H(\Psi(\psi)) \) is invertible, i.e. the 1-cell is Hopf iff is fusionable.

This corollary allows the author to keep using the adjective Hopf without losing the generality of the results.

6.2 Hopf Parametric Liftings

Without any further ado, the main Theorem of the article is stated and proved.

Theorem 6.2.1 Consider a parametric adjunction \( J \dashv_P K \), and \( 0 \)-cells in \( \text{Mnd}(\mathcal{C}) \) of the form \((C, S), (D, T) \) and \((P, E)\). There is a bijection between the following structures

1) Parametric Adjoint Liftings \( \hat{J} \dashv_P \hat{K} \), where \((J, \hat{J}, \lambda^j) \) is a Hopf 1-cell in \( \text{Adj}_H(\mathcal{C}) \). This lifted parametric adjunction makes the following diagrams commutative

\[
\begin{array}{ccc}
C \times P & \xrightarrow{J} & D \\
U^S \times U^E & \downarrow & U^T \\
C^S \times P^E & \xleftarrow{\hat{J}} & D^T
\end{array}
\quad
\begin{array}{ccc}
P^op \times D & \xrightarrow{K} & C \\
U^{Eop} \times U^T & \downarrow & U^S \\
(P^E)^op \times D^T & \xrightarrow{\hat{K}} & C^S
\end{array}
\]

ii) Hopf parametric adjunctions of the form

\[
(J(C \times U^E), H(\psi^j)) \dashv_P (K(U^{Eop} \times D), \psi^{KUE})
\]

where \( \psi^{KUE} = \text{ad}(H(\psi^j)) \)

Proof:

Induce the following left Hopf 1-cell \((J, \psi) : (C \times P, S \times E) \rightarrow (D, T)\) in \( \text{Mnd}(\mathcal{C}) \), where \( \psi^j := \Phi(\lambda^j) \). According to Theorem 5.2.2 there is a bijection between

4) Hopf parametric adjunctions in \( \text{Mnd}(\mathcal{C}) \) of the form

\[
(J(C \times U^E), H(\psi^j)) \dashv_P (K(U^{Eop} \times D), H^j(\psi^j))
\]
Hopf parametric adjunctions in $\text{Adj}_R(\mathcal{C})$ of the form

$$(J(C \times U^E), [J(C \times U^E)]^{H\psi}, \lambda^{H\psi}) \dashv_{P_E} (K(U^{E^p} \times D), [K(U^{E^p} \times D)]^{H\psi}, \lambda^{KH\psi})$$

Taking into account this result, the bijection can be given as follows.

The first lifting diagram can be seen as a 1-cell $(J, \hat{J}, \lambda J)$ in $\text{Adj}_R(C)$ and the following 1-cell can be constructed $(J(C \times U^E), \hat{J}, H(J)) : F^S \times P^E \dashv U^S \times P^E \longrightarrow F^T \dashv U^T$. Using the Proposition [5.2.1] with this last 1-cell the commutative diagram can be obtained

$$\begin{array}{ccc}
C^S \times P^E & \xrightarrow{\hat{J}} & D^T \\
\downarrow 1_{C^S \times F^1} & & \downarrow 1_{D^T} \\
C^S \times (P^E)^1 & \xrightarrow{[J(C \times U^E)]^{H\psi}} & D^T
\end{array}$$

A similar argument, for the 1-cell $(K(U^{E^p} \times D), \hat{K})$, gives the following commutative diagram

$$\begin{array}{ccc}
(P^E)^{op} \times D^T & \xrightarrow{\hat{K}} & C^S \\
\downarrow F^{1^*} \times 1_{D^T} & & \downarrow 1_{C^S} \\
(P^E)^{1^*} \times C^S & \xrightarrow{[K(U^{E^p} \times D)]^{H\psi}} & C^S
\end{array}$$

These two last diagrams give the bijection of the forgetful diagrams with the components of the Hopf parametric adjunction in $\text{Adj}_R(\mathcal{C})$.

\[\square\]

### 6.3 Hopf monads on monoidal categories

Consider the case a closed monoidal category $(\mathcal{C}, \otimes, \Box, I)$. In [3], there is a bijection between monoidal liftings, 1-cells in $(\otimes, \hat{\otimes}, \lambda^{\otimes}) : (F^S \times F^S, U^S \times U^S) \longrightarrow (F^S \dashv U^S)$ in $\text{Adj}_M(\mathcal{C})$, and opmonoidal monads, 1-cells $(\otimes, \psi^\otimes) : (C \times C, S \times S) \longrightarrow S$ in $\text{Mnd}(\mathcal{C})$.

If the closure functor is to be lifted when $(\otimes, \hat{\otimes}, \lambda^{\otimes})$ is a Hopf 1-cell, the previous calculations show that

$$(J(C \times U^E), [J(C \times U^E)]^{H\psi}, \lambda^{H\psi}) = (\otimes(C \times U^S), [\otimes(C \times U^S)]^{H\psi}, \lambda^{\otimes H\psi}) \cong (\otimes, \hat{\otimes}, \lambda^{\otimes})$$

$$(K(U^{E^p} \times D), [K(U^{E^p} \times D)]^{H\psi}, \lambda^{KH\psi}) = (\Box(U^{E^p} \times C), [\Box(U^{E^p} \times C)]^{H\psi}, \lambda^{\Box H\psi})$$

are the corresponding liftings, where $\psi = \Phi(\lambda^{\otimes})$.

### 7 Conclusions and further work

This article was intended to explore more examples in classic monad theory in order to prove, what the author considers, the relevance of the 2-adjunctions of the type $\text{Adj-Mnd}$. This relevance will be significant if the recollection of a numerous quantity of useful examples is done.
The development of the article only used the left definition, nevertheless, the author hopes that the right and the left-right case can be completed without any complication whatsoever.

As far as further work is concerned, there is a pair of possible connections. The first one, is to take the framework of multivariable adjunctions in [2] for further analysis using the 2-adjunction and the parametric objects already defined.

Second, there might be a further development on categorical duality provided by this parametric objects.

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