THE HEAT TRACE EXPANSION ON COMPACT LIE GROUPS AND THE VOLUME FORMULA

SEUNGHUN HONG

ABSTRACT. Let G be a compact connected Lie group with a bi-invariant metric. Let T be a maximal torus of G. Using the representation theory of compact Lie groups and Weyl’s law for the heat trace, we verify Harish-Chandra’s formula for \( \frac{\text{vol}(G)}{\text{vol}(T)} \). The main calculation tools used are the Euler-Maclaurin formula and the Weyl integration formula.

CONTENTS

1. Introduction 1
2. Review of Basic Notions 3
3. Harish-Chandra’s Formula for \( \frac{\text{vol}(G)}{\text{vol}(T)} \) 10
References 15

1. INTRODUCTION

(1.1) From the sound waves of a drum to a quantum particle trapped in a box, important physical systems can be studied by solving the eigenvalue problem of the laplacian under Dirichlet boundary conditions:

\[
\begin{cases}
-\Delta u = \lambda u, & \text{on } \Omega; \\
 u = 0, & \text{on } \partial \Omega.
\end{cases}
\]

Here \( \Omega \) is a bounded open subset of a euclidean space. It is a standard result of elliptic differential operator theory that there is an orthonormal basis for \( L^2(\Omega) \) consisting of solutions to (1.2). Moreover, the basis elements can be ordered in such a way that their corresponding eigenvalues \( \{\lambda_k\}_{k=1}^\infty \) form an unbounded sequence of nondecreasing positive real numbers.

An important question is “how fast does \( \lambda_k \) grow?” This simple question is related to one of the most fascinating chapters in the history of science—blackbody radiation and the birth of quantum physics. Consider a cube of side length \( L \) with hollow interior, and walls that completely block any light. The intensity of the electromagnetic waves inside the cube under thermodynamic equilibrium was studied in the late 19th century, and it led to the discovery of Planck’s constant \( h \). To calculate the intensity, one needs to calculate the total number of standing waves that have frequency less than \( f \); call that number \( N(f) \). The standing waves are the eigenfunctions of equation (1.2) where \( \Omega \) is the interior of the cube. One finds (using the method of separation of variables) that the eigenvalues are of the form

\[
\lambda = \frac{\pi^2}{L^2} (l^2 + m^2 + n^2)
\]

2010 Mathematics Subject Classification. Primary 58J05, 58J37, 58J50, 58J60. 
Key words and phrases. compact Lie groups, Laplace-Beltrami operator, heat trace expansion, Weyl’s law, Harish-Chandra’s volume formula. 

This article is based on the author’s work that was presented for his oral comprehensive exam in partial fulfillment to the requirements for the degree of Ph.D. in Mathematics at the Pennsylvania State University. The author wishes to express his sincere gratitude to his advisor, Professor N. Higson, for the kind guidance and advice. The author also sends thanks to Professor J. Roe, for pointing out errors in the initial draft, Professor M. Bojowald and Professor P. Xu for reading the preliminary manuscript.
where ℓ, m, n are positive integers. Physically, λ is related to the frequency f of the standing wave by
\[ \lambda = (2\pi f)^2. \]
So \( N(f) \) is equal to the number of lattice points in the positive octant of \( \mathbb{R}^3 \) that are bound by radius \( R = 2fL \). For large R, the number \( N(f) \) can be approximated by 1/8 of the volume of the ball with radius R. The error of this estimate vanishes as R or, for that matter, f tends to infinity. Therefore,
\[ N(f) = \frac{4\pi}{3} f^3 L^3 + o(f^3). \]
From physical reasoning and experiments, this relation is expected to hold even when \( \Omega \) is not a cube and is an arbitrarily shaped cavity. In other words, we expect
\[ \lim_{f \to \infty} \frac{N(f)}{f^3} = \frac{4\pi}{3} \text{vol}(\Omega) \]
to hold; here \( \text{vol}(\Omega) \) denotes the volume of \( \Omega \).
Indeed, Weyl [14, 15] proved the asymptotic law
\[ \lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\text{vol}(\Omega)}{(4\pi)^{n/2} \Gamma(\frac{n+1}{2})} \]
where \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \) with \( n = 2 \) or \( 3 \). Gårding [5] proved the higher-dimensional case (for generic elliptic operators). On closed riemannian manifolds, the same law was proved for the laplacian by Duistermaat and Guillemin [3], and for generic elliptic operators by Minakshisundaram and Pleijel [10].
Weyl's law tells us, in particular, that if we know the spectrum of the laplacian completely then from it we can calculate the volume of the underlying space \( \Omega \). This can be viewed as the “first theorem of noncommutative geometry” [8].

Weyl's law can be reformulated as an asymptotic behavior of the function
\[ Z(t) = \sum_{k=1}^{\infty} e^{-t\lambda_k} \]
which resembles the “partition function” in physics—a function that is often invariant under the symmetry of the physical system it describes. (Let us not worry about the convergence of the sum at this point.) The relation between \( Z(t) \) and the number of eigenfunctions becomes evident as we consider the limit \( t \to 0^+ \); in that limit, the partial sum
\[ \sum_{k=1}^{K} e^{-t\lambda_k} \]
converges to \( K \), the number of the eigenvalues from \(-\lambda_1\) to \(-\lambda_K\). Weyl's law then can be shown to be equivalent to the asymptotic law
\[ t^{n/2} Z(t) \sim \frac{\text{vol}(\Omega)}{(4\pi)^{n/2} + O(t)} \]
as \( t \to 0^+ \).
In the case where \( \Omega \) is a compact connected Lie group \( G \) with a bi-invariant metric, the spectrum of the laplacian is determined by the representation theory of \( G \). Hence, the asymptotic behavior of \( Z(t) \) is also determined from the representation theory of \( G \). Our goal is to exploit this and verify Harish-Chandra's formula [6, p.203, Lem.4] for \( \text{vol}(G)/\text{vol}(T) \), where \( T \) is a maximal torus of \( G \). Our main tools for calculation are the Euler-Maclaurin formula and the Weyl integration formula.

(1.6) Before we launch into the calculations, we shall review in the next section some basic analytic and algebraic notions related to the laplacian on a compact Lie group. Throughout this article \( G \) denotes a compact connected Lie group, and \( T \) a maximal torus in \( G \). We denote by \( g \) the Lie algebra of \( G \), namely, the tangent space \( T_eG \) at the identity element \( e \in G \). We denote by \( \tilde{X} \) the left-invariant vector field on \( G \) generated by \( X \in g \).
2. Review of Basic Notions

Analytic Aspects.

(2.1) For \( g \in G \), let \( c(g) \) be the conjugation map \( G \to G, h \mapsto ghg^{-1} \). Mapping \( g \) to the differential of \( c(g) \) at the identity gives the adjoint representation of \( G \),
\[
\text{Ad} : G \to \text{Aut}(g)
\]
\[
g \mapsto c(g)_{*},c.
\]
The differential of \( \text{Ad} \) at the identity gives the adjoint representation of \( g \),
\[
\text{ad} : g \to \text{End}(g)
\]
\[
X \mapsto \text{Ad}_{g}(X).
\]
We shall often write \( \text{Ad}(g) \) as \( \text{Ad}_{g} \), and \( \text{ad}(X) \) as \( \text{ad}_{X} \).

(2.2) Bi-invariant Metrics Let \( \langle \cdot , \cdot \rangle \) denote a bi-invariant metric on \( G \). This is equivalent to an \( \text{Ad}(G) \)-invariant inner product on \( g \). Such an inner product always exists for compact Lie groups. (One can simply average over \( G \) if the inner product is not already \( \text{Ad}(G) \)-invariant.) Note that \( \text{Ad}(G) \)-invariance implies that \( \text{ad}(g) \)-action is skew-symmetric.

If \( G \) is compact and semisimple, the negative of the Killing form is an \( \text{Ad}(G) \)-invariant inner product on \( g \). (This can be checked using the fact that \( \text{ad}_{\text{Ad}_{g}(X)} = \text{Ad}_{g} \circ \text{ad}_{X} \circ \text{Ad}_{g}^{-1} \).) The Killing form of a Lie group is the bilinear form on its Lie algebra defined by
\[
\kappa : g \times g \to \mathbb{C}
\]
\[
\langle X, Y \rangle \mapsto \text{tr}(\text{ad}_{X} \circ \text{ad}_{Y}).
\]
Compact semisimple Lie groups are precisely the Lie groups whose Killing form is negative-definite [7, Prop.6.6]. Owing to the fact that \( \kappa \), we have the symmetry \( \kappa(\text{Ad}_{g} X, Y) = \kappa(X, \text{Ad}_{g^{-1}} Y) \) and \( \kappa(\text{ad}_{Z} X, Y) = -\kappa(X, \text{ad}_{Z} Y) \).

(2.3) The Laplacian Let \( \langle \cdot , \cdot \rangle \) be a bi-invariant metric on \( G \). The Laplacian is defined as follows. Let \( C_{\infty}(G) \) denote the space of smooth functions on \( G \), and let \( \mathfrak{X}(G) \) denote the space of smooth vector fields on \( G \). We define the gradient operator \( \text{grad} : C_{\infty}(G) \to \mathfrak{X}(G) \) and the divergence operator \( \text{div} : \mathfrak{X}(G) \to C_{\infty}(G) \) by
\[
\langle \text{grad} f, X \rangle = Xf,
\]
\[
\text{(div}X\text{) vol} = L_{X} \text{vol},
\]
where \text{vol} is the volume form induced by the metric, \( X \) is a vector field on \( G \), and \( L_{X} \) is the Lie derivative with respect to \( X \). Then the laplacian (or the Laplace-Beltrami operator) \( \Delta_{G} : C_{\infty}(G) \to C_{\infty}(G) \) is defined as
\[
\Delta_{G} f := \text{div}(\text{grad} f).
\]
The definition of \( \Delta_{G} \) is independent of local coordinates and depends only on the metric. Because our metric \( \langle \cdot , \cdot \rangle \) is bi-invariant, so is the laplacian.

An expression for \( \Delta_{G} \) in terms of local coordinates \( (x_{1}, \ldots, x_{n}) \) can be given as follows. Let \( g \) be the matrix defined by \( g_{ij} = \langle \partial_{i}, \partial_{j} \rangle \) where \( \partial_{i} = \partial / \partial x_{i} \). Let \( g^{-1} \) denote the \( (i, j) \)-entry of \( g^{-1} \). Then,
\[
\Delta_{G} f = \frac{1}{\sqrt{\det g}} \sum_{i,j} \partial_{i}(\sqrt{\det gg^{-1}} \partial_{j} f).
\]

(2.4) The Spectrum of the Laplacian So far the laplacian is an unbounded operator whose domain \( C_{\infty}(G) \) is a dense subspace of \( L^{2}(G) \). The domain can be extended to the Sobolev space \( H^{2}(G) \), that is, the space of measurable functions \( u \) on \( G \) such that the norm \( \| u \|_{H^{2}} = \| u \|_{L^{2}} + \sum_{i} \| \partial_{i} u \|_{L^{2}} + \sum_{i,j} \| \partial_{i} \partial_{j} u \|_{L^{2}} \) is finite. The Sobolev embedding theorem tells us that \( H^{2}(G) \) is a subspace of \( L^{2}(G) \) and that the inclusion map is compact. The extension
\[
\Delta_{G} : H^{2}(G) \to L^{2}(G)
\]
is the unique self-adjoint extension of the laplacian whose domain was originally \( C^\infty(G) \). In the language of the theory of unbounded operators, the laplacian is essentially self-adjoint on \( C^\infty(G) \). For details on these matters, see Taylor [13, §8.2].

It turns out that \((1 - \triangle_G)\), where 1 denotes the identity operator, admits an inverse that is compact [12, §5.1]. Then the spectral theorem implies that the eigenfunctions \( \{u_k\}_{k=1}^\infty \) of \((1 - \triangle_G)^{-1}\) form an orthonormal basis for \( L^2(G) \). Owing to the regularity of elliptic differential operators, the eigenfunctions are of \( C^\infty \). We can also conclude from the spectral theorem that the eigenfunctions \( u_k \) can be ordered in such a way that the corresponding eigenvalues \(-\lambda_k\) of \( \triangle_G \) give a nonincreasing unbounded sequence of negative real numbers,

\[
0 > -\lambda_1 \geq -\lambda_2 \geq -\lambda_3 \geq \cdots.
\]

(2.7) The Heat Diffusion Operator The heat diffusion operator of \( \triangle_G \) is defined by the matrix

\[
e^{t\triangle_G} = \begin{pmatrix}
e^{t\lambda_1} & 0 & \cdots & 0 \\
0 & e^{t\lambda_2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & e^{t\lambda_n}
\end{pmatrix}
\]

with respect to the basis consisting of eigenfunctions of the laplacian. Its trace is finite, which follows from the fact that (i) \( e^{t\triangle_G} \) can be represented as an integral operator on \( L^2(G) \) with a \( C^\infty \)-kernel and (ii) any smooth integral operator is of trace-class. For details, we refer to Taylor [13, §7.13]. The trace of \( e^{t\triangle_G} \) is called the heat-trace of \( \triangle_G \).

Algebraic Aspects.

(2.8) Let \( D(G) \) be the space of all differential operators that are invariant under the left-translation by \( G \). We pointed out that the laplacian is bi-invariant, that is, invariant under both the left and right translations. Algebraically, this implies that \( \triangle_G \) is in the center of \( D(G) \). Before we review such algebraic properties, we briefly go over the terminologies and notations of the representation theory we will use.

(2.9) A representation of a Lie group \( G \) is a continuous\(^1\) group homomorphism \( u : G \to \text{Aut}(V) \) where \( V \) is a topological vector space. We say that \( V \) is a \( G \)-vector space. We define the dimension of the representation to be the dimension of \( V \). We say that the representation is real or complex according to the field of scalars of \( V \). For our purposes, we shall only consider the complex representations. For \( g \in G \) and \( v \in V \), we write \( u(g)(v) \) as \( u_g(v) \), or \( g \cdot v \) if there is no risk of confusion.

A linear map \( \varphi : V \to W \) between two \( G \)-vector spaces is a \( G \)-map if it is \( G \)-equivariant, that is, \( g \cdot \varphi(v) = \varphi(g \cdot v) \) holds for all \( v \in V \). If the \( G \)-map \( \varphi : V \to W \) is homeomorphic, then we say that \( V \) and \( W \) are \( G \)-isomorphic and that the representations on \( V \) and \( W \) are equivalent.

If \( u : G \to \text{Aut}(V) \) is a finite-dimensional representation, then the induce Lie algebra representation is the differential at the identity, \( u_* : g \to \text{End}(V) \). This means that, for \( X \in \mathfrak{g} \),

\[
u_*(X)(v) = \frac{d}{dt} u(\exp tX)(v) \bigg|_{t=0}
\]

(2.10) The Universal Enveloping Algebra There are two natural \( G \)-actions on \( f \in C^\infty(G) \), namely, the left and right-regular representation:

\[
(L(h)f)(g) = f(h^{-1}g), \quad (R(h)f)(g) = f(gh).
\]

The induced Lie algebra representation of the right-regular representation coincides with the action of \( \mathfrak{g} \) as differential operators;

\[
(R(X)f)(g) = \frac{d}{dt} f(g \exp(tX)) \bigg|_{t=0} = [Xf](g).
\]

\(^1\)If \( V \) is finite-dimensional, then a continuous representation is automatically of \( C^\infty \).
This action can be extended to the universal enveloping algebra \( \mathfrak{u}(\mathfrak{g}) \), the algebra constructed by first taking the tensor algebra \( T(\mathfrak{g}) \) of \( \mathfrak{g} \) and then taking the quotient by the ideal \( \mathfrak{I}(\mathfrak{g}) \) generated by the elements of the form \( X \otimes Y - Y \otimes X - [X,Y] \):

\[
\mathfrak{u}(\mathfrak{g}) = T(\mathfrak{g})/\mathfrak{I}(\mathfrak{g}).
\]

A simple tensor \( X_1 \cdots X_k \) in \( \mathfrak{u}(\mathfrak{g}) \) defines a differential operator by

\[
X_1 \cdots X_k f = \tilde{X}_1 \cdots \tilde{X}_k f,
\]

where \( f \in C^\infty(\mathfrak{g}) \) and \( \tilde{X} \) denotes the left-invariant vector field on \( \mathfrak{g} \) generated by \( X \in \mathfrak{g} \). The correspondence

\[
\mathfrak{u}(\mathfrak{g}) \to D(\mathfrak{g}) \ni \quad X_1 \cdots X_k \mapsto \tilde{X}_1 \cdots \tilde{X}_k,
\]

is an algebra isomorphism (see [7, Ch.II, Prop.1.9]).

(2.12) **The Casimir Element** Under the identification of \( D(\mathfrak{g}) \) with \( \mathfrak{u}(\mathfrak{g}) \), what is the element in \( \mathfrak{u}(\mathfrak{g}) \) that corresponds to the laplacian? We claim that it is the (quadratic) Casimir element \( \text{Cas} \), which is defined as follows. Let \( \text{End}(\mathfrak{g}) \) denote the space of linear maps from \( \mathfrak{g} \) into itself. Let \( \mathfrak{g}^* \) be the dual space of \( \mathfrak{g} \). Using the inner product \( \langle , \rangle \) on \( \mathfrak{g} \), we can construct the isomorphisms

\[
\text{End}(\mathfrak{g}) \simeq \mathfrak{g} \otimes \mathfrak{g}^* \simeq \mathfrak{g} \otimes \mathfrak{g}.
\]

Since \( \mathfrak{g} \otimes \mathfrak{g} \) is a subspace of the tensor algebra \( T(\mathfrak{g}) \), we have a map

\[
(2.13) \quad \text{End}(\mathfrak{g}) \hookrightarrow T(\mathfrak{g}) \xrightarrow{\langle \cdot, \cdot \rangle} \mathfrak{u}(\mathfrak{g}),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the quotient map with respect to the ideal \( \mathfrak{I}(\mathfrak{g}) \) mentioned in (2.10). The image of the identity map \( \text{id}_\mathfrak{g} \in \text{End}(\mathfrak{g}) \) under the above composition is the Casimir element \( \text{Cas} \). Because the Casimir element originated from the identity map, it is in the center\(^2\) \( Z(\mathfrak{g}) \) of the universal enveloping algebra.

The definition of \( \text{Cas} \) does not depend on the particular basis for \( \mathfrak{g} \). But it has a simple expression in terms of an orthonormal basis \( X_1, \ldots, X_n \) for \( \mathfrak{g} \). Let \( \theta_1, \ldots, \theta_n \) be the dual basis for \( \mathfrak{g}^* \). Then the identity map in \( \text{End}(\mathfrak{g}) \) corresponds to \( \sum_{i=1}^n \theta_i \otimes X_i \) in \( \mathfrak{g}^* \otimes \mathfrak{g} \). So the Casimir element is

\[
\text{Cas} = \sum_{i=1}^n \theta_i X_i.
\]

Why is this the laplacian under the identification of \( \mathfrak{u}(\mathfrak{g}) \) with \( D(\mathfrak{g}) \)? Recall that the exponential map \( \exp : \mathfrak{g} \to \mathfrak{g} \) is a diffeomorphism over some neighborhood \( \mathcal{U} \) of \( 0 \in \mathfrak{g} \). This provides us the exponential chart near the identity \( e \in \mathfrak{g} \); the coordinates \( (y_1, \ldots, y_n) \) for \( g \in \exp(\mathcal{U}) \) are the components of \( X = \exp^{-1}(g) \) with respect to the basis \( X_1, \ldots, X_n \); in other words, \( g = \exp\left( \sum_{i=1}^n y_i X_i \right) \).

The exponential map we use here is the Lie-theoretic exponential map. There is also the riemannian exponential map \( \text{Exp} : \mathfrak{g} \to \mathfrak{g} \) coming from differential geometry, which is also a local diffeomorphism near \( 0 \in \mathfrak{g} \) (see [7, Ch.1, §6] for details). The exponential map depends on the metric. The Lie-theoretic exponential map \( \exp \) has nothing to do with the metric. But the two exponential maps do agree if the metric is bi-invariant, which can be seen as follows. Let \( \nabla \) be the riemannian connection so that it satisfies \( \nabla_X Y = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \). Using the identity \( 2 \langle \nabla_X Y, Z \rangle = \langle X, \nabla_Y Z \rangle + \langle Y, \nabla_X Z \rangle - \langle Z, \nabla_X Y \rangle - \langle Y, \nabla_Z X \rangle + \langle Z, \nabla_Y X \rangle \) and the skew-symmetry of the \( \text{ad}(\mathfrak{g}) \)-action, one can check that \( \nabla_X Y = \frac{1}{2} [X, Y] \) holds for all \( X, Y \in \mathfrak{g} \). In

\(^2\text{The Lie group } G \text{ acts on } \mathfrak{g} \text{ by the adjoint action (2.2). This extends to the tensor algebra } T(\mathfrak{g}) \text{ by defining } g \cdot (Y_1 \otimes \cdots \otimes Y_k) = (gY_1) \otimes \cdots \otimes (gY_k) \text{ for } g \in G \text{ and vectors } Y_1, \ldots, Y_k \text{ in } \mathfrak{g}. \text{ This leads to a } G \text{-action on } \mathfrak{u}(\mathfrak{g}). \text{ The action of } G \text{ on } A \in \text{End}(\mathfrak{g}) \text{ is given by } (g \cdot A)(X) = g \cdot A(g^{-1} \cdot X). \text{ The composition (2.13) is } G \text{-equivariant under these actions, owing mostly to the fact that the inner product } \langle , \rangle \text{ is } G \text{-invariant. The identity map in } \text{End}(\mathfrak{g}) \text{ is obviously invariant under the } G \text{-action. So the Casimir element } \text{Cas} \text{ is a } G \text{-invariant element of } \mathfrak{u}(\mathfrak{g}), \text{ and this implies that } \text{Cas} \text{ is in the center of } \mathfrak{u}(\mathfrak{g}).
particular, \( \nabla_X \tilde{X} = 0 \). It follows [7, Ch.2, Prop.1.4] that the geodesic \( \gamma_X(t) \), such that \( \gamma_X(0) = e \) and \( \gamma_X'(0) = X \), is a group homomorphism \( \mathbb{R} \to G \). By the uniqueness of 1-parameter subgroups, we have \( \gamma_X(t) = \exp(tX) \). This implies that the riemannian exponential map is identical to the Lie-theoretic exponential map. Hence, the matrix \( [g_{ij}] \) of the metric under the exponential chart satisfies \( g_{ij}(e) = \delta_{ij} \) (Kronecker delta) and \( \partial_t g_{ij}(e) = 0 \). Therefore, the laplacian at \( e \in G \) takes the form \( \sum \partial_i^2 = \sum X_i X_i \). Since the laplacian is left-invariant, we have \( \Delta_G = \sum \tilde{X}_i \tilde{X}_i \) on \( G \), and this is the differential operator corresponding to the Casimir element under the identification (2.11).

(2.14) **The Peter-Weyl Theorem** A smooth function \( f \) on \( G \) is said to be \( G \)-finite if its orbit under the left-regular action of \( G \) spans a finite-dimensional subspace of \( C^\infty(G) \). The Peter-Weyl theorem [11] says that the space \( C^\infty_\text{fin}(G) \) of all \( G \)-finite functions is dense in \( L^2(G) \):

\[
L^2(G) = \frac{C^\infty_\text{fin}(G)}{\| \cdot \|_{L^2}}.
\]

An important source of \( G \)-finite functions are finite-dimensional unitary representations. Suppose \( u : G \to \text{Aut}(V) \) is such a representation. A representative function of \( u \) is a function of the form

\[
G \to \mathbb{C},
\]

\[
g \mapsto \langle w, g^{-1} \cdot v \rangle_V,
\]

where \( w, v \) are vectors in \( V \) and \( \langle \cdot, \cdot \rangle_V \) is the inner product on \( V \). Let \( M_u \) be the space of all representative functions of \( u \). Then we have a linear isomorphism

\[
(2.15) \quad \Phi_u : M_u \ni f(g) = \langle v_1, g^{-1} \cdot v_2 \rangle_V \mapsto v_1^i \otimes v_2 \in V_u^i \otimes V_u,
\]

where \( v_1^i \) is the linear functional defined by \( w \mapsto \langle v_1, w \rangle \). This is in fact an isomorphism of \( (G \times G) \)-spaces, where \( (g, h) \in G \times G \) acts on \( f \in M_u \) by \( (g, h) \cdot f = L_g R_h f \), and on \( v_1^i \otimes v_2 \in V_u^i \otimes V_u \) by \( (g, h) \cdot (v_1^i \otimes v_2) = \tilde{u}_g v_1^i \otimes u_h v_2 \) where \( \tilde{u}_g v_1^i \{ v \} = v_1^i \{ u_g^{-1} v \} \) for \( v \in V_u \). So we have \( M_u \subseteq C^\infty_\text{fin}(G) \). But every \( G \)-finite function must arise in this way for the following reason. If \( f \) is a \( G \)-finite function, then \( f \) is an element of the finite-dimensional vector space \( V \) spanned by the orbit of \( f \) under the left-regular action of \( G \). Let \( \ell \) be the linear functional on \( V \) defined by \( \phi \mapsto \phi(e) \). Then

\[
f(g) = \ell(g^{-1} \cdot f).
\]

This shows that \( f \) is a representative function.

Hence, we have

\[
C^\infty_\text{fin}(G) = \bigoplus_{u \in \hat{G}} M_u \simeq \bigoplus_{u \in \hat{G}} V_u^i \otimes V_u,
\]

where \( \hat{G} \) denotes a complete set of distinct (non-equivalent) finite-dimensional irreducible unitary representations\(^3\). Moreover, by Schur orthogonality, the above direct sum is an orthogonal direct sum under the \( L^2 \)-inner product.

(2.16) As we have mentioned earlier, a representation \( u \) of \( G \) induces a Lie algebra representation by its differential \( u_* \) at the identity. And \( u \) is irreducible if and only if \( u_* \) is irreducible, provided that \( G \) is connected. The Lie algebra representation \( u_* \) can be extended to \( \mathfrak{u}(g) \) owing to the universal property. So the action of the Casimir element \( \text{Cas} \) on

\[
L^2(G) \simeq \bigoplus_{u \in \hat{G}} V_u^i \otimes V_u
\]

\(^3\)The qualifiers “finite-dimensional” and “unitary” are redundant. An irreducible representation of a compact Lie group has to be finite-dimensional [4, Thm.5.2]. By introducing a \( G \)-invariant inner product on a finite-dimensional representation space—which can be done by averaging the inner product over \( G \) if necessary—the original representation can be replaced by an equivalent, unitary representation.
is well-defined. (The direct sum above is a Hilbert space direct sum—the $L^2$-closure of the algebraic direct sum.) Because $\operatorname{Cas}$ is in the center of $\mathfrak{u}(g)$, Schur’s lemma tells us that it acts as a scalar—which we write $\operatorname{Cas}(u)$—on each summand. Therefore, there is an orthonormal basis for $L^2(G)$ consisting of the eigenfunctions of $\operatorname{Cas}$. And the heat-trace of the Laplacian on $G$ is given by

\begin{equation}
\operatorname{tr}(e^{it\Delta_G}) = \operatorname{tr}(e^{t \operatorname{Cas}}) \sum_{u \in \hat{G}} \dim(u) e^{t \operatorname{Cas}(u)}.
\end{equation}

**The Theory of Highest Weights.**

(2.18) The theory of highest weights allows us to parametrize $\hat{G}$ and, hence, the eigenvalues of $\operatorname{Cas}$. In short, the induced Lie algebra representation of $u \in \hat{G}$, when restricted to the maximal abelian subalgebra $t$ of $g$, becomes a sum of linear functionals on $t$; among them we can pick out the “highest” one to characterize the representation $u$. Proofs for most of the statements made in this subsection can be found in [2].

(2.19) **Weights** First, consider the case when $G$ is abelian. An abelian compact Lie group is diffeomorphic to a torus $T \cong \prod_{r=1}^{r} U(1)$. Any finite-dimensional representation $u$ of $T$ is completely reducible to 1-dimensional invariant subspaces. The 1-dimensional representations of $T$ are called the **irreducible characters of** $T$.

An irreducible character of an $r$-dimensional torus can be expressed as

\begin{equation}
\theta(z) = \exp(s \theta(H) z) = \prod_{i=1}^{r} \exp(i t_i \theta(H) z),
\end{equation}

for some integers $n_1, \ldots, n_r$. An element $t \in T$ acts on $z \in \mathbb{C}$ by $\theta(t)z$. The induced Lie algebra representation of $H \in t$ is obtained from the differential at the identity:

\begin{equation}
\theta_*(t) \equiv \frac{\partial}{\partial s} \theta_{\exp(s H) z} \bigg|_{s=0}.
\end{equation}

We identify $t$ with $\mathbb{R}^r$; so the exponential map takes the form

\begin{equation}
\exp : \mathbb{R}^r \to T = \prod_{i=1}^{r} U(1), \quad t = (e^{it_1}, \ldots, e^{it_r}) \mapsto (e^{it_1}, \ldots, e^{it_r}).
\end{equation}

Under this explicit form we have, for $H = (t_1, \ldots, t_r)$,

\begin{equation}
\theta_*(H) = \theta_*(t_1, \ldots, t_r) = i(n_1 t_1 + \cdots + n_r t_r).
\end{equation}

This is a linear function whose value is purely imaginary. It is called a **complex weight** of $T$. We define the corresponding **real weight** as the linear functional

\begin{equation}
\mu = -i \theta_*.
\end{equation}

In terms of $\mu$, the action of $H \in t$ on $z \in \mathbb{C}$ satisfies

\begin{equation}
H \cdot z = i\mu(H)z.
\end{equation}

We will be dealing with real weights most of the time, so we will call them simply as “weights”.

We denote the set of all weights of $T$ by $\Lambda_T$. It is clear from equation (2.21) that there is a group isomorphism $\Lambda_T \cong \mathbb{Z}^d$. For this reason we call $\Lambda_T$ the **weight lattice** of $T$. There is a bijection between $\hat{T}$ and the set of all irreducible characters of $T$. Hence, the weight lattice $\Lambda_T$ is isomorphic to $\hat{T}$ as well.

Moving on to the general case, let $G$ be nonabelian. Suppose we have a finite-dimensional complex representation $u : G \to \operatorname{Aut}(V)$. By restricting $u$ to a maximal torus $T$ in $G$, the representation space $V$ can decomposed into 1-dimensional invariant subspaces, each yielding a weight of $T$. Grouping the 1-dimensional invariant subspaces according to their weights, we can write $V$ as a direct sum

\begin{equation}
V = V_{\mu_1} \oplus \cdots \oplus V_{\mu_l}.
\end{equation}
where $\mu_k$, $1 \leq k \leq \ell$, are distinct weights. Put in another way, this is the eigenspace decomposition under the $t$-action induced from $u$; an element $H \in t$ acts on $v \in V_{\mu_k}$ as

$$H \cdot v = i\mu_k[H]v.$$ 

The linear functionals $\mu_1, \ldots, \mu_\ell$ are called the weights of $V$, and the subspaces $V_{\mu_k}$ are called the weight spaces of $V$. The decomposition (2.22) is called the weight space decomposition of $V$.

In the case where the representation $u$ is a real-representation, then we first complexify it by extending the action to $V_C = V \oplus iV$ by $\mathbb{C}$-linearity; and we define the weights of this complexified representation as the weights of $u$.

**2.23 Roots** The most important example of weights coming from of a real-representation are the weights of the adjoint representation $Ad : G \to \text{Aut}(g)$. To find the weights, we restrict $Ad$ to $T$ and extend its action to the complexification $g_C$ of $g$. Decomposing $g_C$ into weight spaces, we get

$$g_C = g_{C,0} \oplus g_{C,\alpha_1} \oplus \cdots \oplus g_{C,\alpha_k},$$

where $g_{C,0}$ is the zero-weight space, and $\alpha_1, \ldots, \alpha_k$ are the nonzero weights. The nonzero weights are called the roots of $G$. We will denote the set of all roots of $G$ by $\Phi$.

Note that the zero weight space is the centralizer of $g$ extend its action to the complexification $g_C$ of $g$, we must have $g_{C,0} = t_C$. Therefore, the decomposition (2.24) can be written as

$$g_C = t_C \oplus g_{C,\alpha_1} \oplus \cdots \oplus g_{C,\alpha_k}.$$ 

This is called the root space decomposition of $g_C$.

Using the inner product $\langle \cdot, \cdot \rangle$ on $g$, one can find, for each linear functional $\mu \in t^*$, a unique vector $X_\mu \in t$ such that

$$\mu(H) = \langle X_\mu, H \rangle, \quad \forall H \in t.$$ 

This gives a one-to-one correspondence between $t$ and $t^*$; and the inner product $\langle \cdot, \cdot \rangle$ can be transferred to $t^*$ by setting

$$\langle \mu, \nu \rangle := \langle X_\mu, X_\nu \rangle.$$ 

This inner product on $t^*$ is $Ad$-invariant. We also write

$$X_\mu(v) := v(X_\mu).$$ 

If $G$ is compact and semisimple then the set $\Phi$ of roots of $G$ satisfies the following properties:

(i) $\Phi$ spans $t^*$.

(ii) If $\alpha \in \Phi$, then $\lambda \alpha \in \Phi$ if and only if $\lambda = \pm 1$.

(iii) For any $\alpha, \beta \in \Phi$,

$$\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$

In other words, for each root $\alpha$ there is a vector $\hat{\alpha} \in t$ (called the coroot of $\alpha$) such that $\hat{\alpha}(\beta)$ is an integer for any root $\beta$ and, in particular, $\hat{\alpha}(\alpha) = 2$.

(iv) For $\alpha \in \Phi$, the linear transformation $s_\alpha(\mu) := \mu - \hat{\alpha}(\mu)\alpha$ on $t^*$ is a reflection that sends $\alpha$ to $-\alpha$. Moreover, $s_\alpha$ permutes the roots.

(v) The root spaces $g_{C,\alpha}$ are 1-dimensional.

The reflections $s_\alpha$ are called the Weyl reflections. They generate the group $W$ called the Weyl group of $G$. It is a subgroup of the isometry group of $t^*$.

Recall that the roots come in pairs $(\alpha, -\alpha)$. It is helpful to distinguish one from each pair and call them “positive”. To this end, divide $t^*$ into disconnected regions by excluding from $t^*$ the hyperplanes $H_\alpha := \ker(1 - s_\alpha)$, $\alpha \in \Phi$. Choose a connected component $K^\circ$ of $t^* \setminus (\bigcup_{\alpha \in \Phi} H_\alpha)$; its closure $K$ is called the fundamental Weyl chamber. Once we have made a choice for $K$, we define the set of positive roots as

$$\Phi^+ := \{ \alpha \in \Phi : \langle \alpha, \nu \rangle > 0, \forall \nu \in K^\circ \}.$$ 

Note that $\alpha \in \Phi^+$ implies $-\alpha \in \Phi \setminus \Phi^+$. We can decompose $\Phi$ as a disjoint union

$$\Phi = \Phi^+ \sqcup \Phi^-$$
where \( \Phi^- := \Phi \setminus \Phi^+ \). We call \( \Phi^- \) the set of negative roots. The decomposition of \( \Phi \) into positive and negative roots depends on our choice for \( K \). From now on, we assume that such a choice has been made.

(2.25) Partial Ordering on the Weight Lattice We have put some effort in introducing the roots. This is because they can be used to give a partial ordering on the weight lattice \( \Lambda_T \) and, thus, allow us to talk about “highest weights”. For \( \mu, \lambda \in \Lambda_T \) we say that \( \lambda \) is higher than \( \mu \) and write \( \mu \preceq \lambda \) if and only if \( \lambda \) can be written as

\[
\lambda = \mu + \sum_{\alpha \in \Phi^+} n_\alpha \alpha
\]

for some nonnegative integers \( n_\alpha \). Now suppose we have a finite-dimensional representation \( u \) of \( G \). Let \( A = \{ \mu_1, \ldots, \mu_k \} \) be the set of weights of this representation. We say that \( \mu_i \in A \) is a highest weight of this representation if it is a maximal element of \( A \) with respect to the partial ordering \( \preceq \). For reducible representations, there is a unique highest weight.

(2.26) Parametrization of \( \widehat{G} \) Let \( \Lambda_T \) be the weight lattice of a maximal torus \( T \) in a compact connected Lie group \( G \). Let \( K \) be the selected fundamental Weyl chamber in \( t^* \). Weyl proved that mapping \( u \in \widehat{G} \) to its highest weight gives a one-to-one correspondence between \( \widehat{G} \) and \( \Lambda_T \cap K \).

\[
\widehat{G} \leftrightarrow \Lambda_T \cap K,
\]

\[
u \mapsto \text{highest weight of } u.
\]

If \( u \in \widehat{G} \) has highest weight \( \lambda \) then it is customary to call the representation space of \( u \) as a highest weight module with highest weight \( \lambda \). The dimension of a highest weight module \( V(\lambda) \) with highest weight \( \lambda \) can be calculated from the Weyl dimension formula:

(2.27)
\[
\dim V(\lambda) = \frac{\prod_{\alpha \in \Phi^+} \langle \alpha, \lambda + \rho \rangle}{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle}.
\]

(2.28) The Value of the Casimir Finally we introduce a formula for the value of the Casimir element \( \text{Cas} \) on an irreducible representation of \( G \). Let \( V(\lambda) \) be a highest weight module with highest weight \( \lambda \). The value of the Casimir on \( V(\lambda) \) is equal to

(2.29)
\[
-\langle \lambda + \rho, \lambda + \rho \rangle + \langle \rho, \rho \rangle,
\]

where \( \rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \).

(2.30) Fundamental Weights Before we end this section, we mention the relation between the lattice \( \Lambda_\Phi \) spanned by the roots of \( G \) and the weight lattice \( \Lambda_T \) of a maximal torus \( T \) of \( G \).

First, let us choose a special basis for \( \Lambda_\Phi \). We do this by collecting the positive roots \( \alpha \in \Phi^+ \) that cannot be written as a sum of other positive roots. Such roots are said to be simple. Denote the set of all simple roots by \( \Sigma \). It is a fact that \( \Sigma \) is a linearly independent set that spans \( \Lambda_\Phi \).

Suppose \( \Sigma = \{ \alpha_1, \ldots, \alpha_\ell \} \). Then we define a “dual basis” \( \Sigma^\vee = \{ \lambda_1, \ldots, \lambda_\ell \} \) by

\[
\frac{\langle \lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}.
\]

In the special case when \( G \) is compact and semisimple, the linear functionals \( \lambda_i \in \Sigma^\vee \) satisfy the following properties:

(i) They form a basis for \( t^* \).
(ii) They lie in the fundamental Weyl chamber \( K \).
(iii) An element \( \mu = \sum_{i=1}^\ell x_i \lambda_i \) in \( t^* \) lies in \( K \) if and only if \( x_i \geq 0 \) for all \( i \).
(iv) \( \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \) satisfies

\[
\rho = \lambda_1 + \cdots + \lambda_\ell.
\]
(v) The lattice $\Lambda_G$ spanned by $\{\lambda_i\}_{i=1}^l$ contains the weight lattice $\Lambda_T$; the index is equal to the order of the fundamental group of $G$:
$$|\pi_1(G)| = [\Lambda_G : \Lambda_T].$$

The weights in $\Lambda_T \cap K$ are said to be dominant. Because of property (iii) and (v), we call the $\Sigma^\vee$ as the set of fundamental dominant weights.

3. Harish-Chandra’s Formula for $\text{vol}(G)/\text{vol}(T)$

(3.1) Our goal is to verify Harish-Chandra’s formula (3.32) using Weyl’s law and the representation theory of compact Lie groups. We will use the Euler-Maclaurin formula and the Weyl integration formula as our main tool for calculation. Before we launch into general arguments, we take $SU(2)$ and carry out some explicit calculations.

**An Example: SU(2).**

(3.2) Let $G = SU(2)$. This is a compact connected semisimple Lie group. So we take the metric generated by the negative of the Killing form $\kappa$ on $g$. A maximal torus in $G$ is given by
$$T = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} : t \in \mathbb{R} \right\} \simeq U(1).$$

We take the following matrices as the basis for the Lie algebra $g = su(2)$:
$$X = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$  

The matrix $H$ lies in the Lie algebra $\mathfrak{t}$ of $T$. Calculating the corresponding matrices under the adjoint representation, we have
$$\text{ad}_X = \begin{pmatrix} \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot \end{pmatrix}, \quad \text{ad}_Y = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 \\ \cdot & 2 & \cdot \end{pmatrix}, \quad \text{ad}_H = \begin{pmatrix} \cdot & 2 & \cdot \\ 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix},$$

where the dots represent $0$. The matrix for the metric $g = -\kappa$ is
$$g = \begin{pmatrix} 8 & \cdot & \cdot \\ \cdot & 8 & \cdot \\ \cdot & \cdot & 8 \end{pmatrix}.$$

(3.3) Next we calculate the roots. Upon complexifying the Lie algebra $su(2)$ we get $sl(2, \mathbb{C})$. The standard triple of $sl(2, \mathbb{C})$ are
$$J^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J^- = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad J^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Their Lie bracket relations are

(3.4) $[J^+, J^-] = J^0,$

(3.5) $[J^0, J^\pm] = \pm 2J^\pm.$

Equation (3.5) shows that there are two complex roots $\pm \mu$, where
$$\pm \mu(H) = \pm \mu(J^0) = \pm 2i.$$

Denoting the corresponding real roots as $\pm \alpha$, we have
$$\pm \alpha(H) = \pm 2.$$

The Weyl group is the cyclic group $Z_2$ generated by $\sigma : \alpha \mapsto -\alpha$. If we take $\alpha$ as the basis for the dual space $\mathfrak{t}^*$ of the Lie algebra of $T$, thereby identifying $\mathfrak{t}^*$ with the real line $\mathbb{R}$, then $\sigma$ is the reflection with respect to the point $0 \in \mathbb{R}$. We may choose the closed half-line $[0, \infty]$ as the fundamental Weyl chamber $K$. This amounts to choosing $\alpha$ as the positive root.
(3.6) The dual vector $X_\alpha$ of $\alpha$ is characterized by
\[ \alpha(H) = \langle X_\alpha, H \rangle = 2. \]
This implies $X_\alpha = H/4$. Since $\alpha(X_\alpha) = \langle X_\alpha, X_\alpha \rangle = 1/2$, we have
\[ \langle \alpha, \alpha \rangle = \langle X_\alpha, X_\alpha \rangle = 1/2. \]
There is only one fundamental dominant weight, and it is
\[ \rho := \frac{1}{2} \alpha. \]
Since SU(2) is simply connected, the weight lattice $\Lambda_T$ is spanned by $\rho$. The set of dominant weights is
\[ \Lambda_T \cap K = \{ \ell \rho \mid \ell = 0, 1, 2, \ldots \}. \]

(3.7) Let $V(\ell)$ denote the highest weight module with highest weight $\ell \rho$. Then,
\[ C^\infty_{\text{fin}}(G) = \bigoplus_{\ell=0}^\infty V(\ell) \otimes V(\ell)^*. \]
The Weyl dimension formula gives
\[ \dim V(\ell) = \frac{\langle \alpha, (\ell + 1)\rho \rangle}{\langle \alpha, \rho \rangle} = \ell + 1. \]
The value of the Casimir on $V(\ell)$ is
\[ \text{Cas}(\ell) = -(\ell + 1)^2 \langle \rho, \rho \rangle + \langle \rho, \rho \rangle = -\frac{(\ell + 1)^2}{8} + \frac{1}{8}. \]
Therefore, the heat-trace is
\[ \text{tr}(e^{t\Delta_G}) = \sum_{\ell=0}^\infty \dim V(\ell) e^{t\text{Cas}(\ell)} = e^{t\|\rho\|^2} \sum_{\ell=0}^\infty (\ell + 1)^2 e^{-t(\ell + 1)^2\|\rho\|^2}. \]

(3.10) For the volume of $G$, we are only interested in the leading term in the asymptotic expansion of the heat-trace. Thus, we only need to calculate
\[ Z(t) := \sum_{\ell=0}^\infty (\ell + 1)^2 e^{-t(\ell + 1)^2\|\rho\|^2}. \]
Define the polynomial
\[ d(\ell) := \ell. \]
Then,
\[ Z(t) = \sum_{n=0}^\infty d(n)^2 e^{-tn^2\|\rho\|^2} = \sum_{n=0}^\infty n^2 e^{-tn^2/8}. \]

(3.11) **The Euler-MacLaurin Formula** Set
\[ f(x) := x^2 e^{-tx^2/8}. \]
(We have suppressed the dependence of $f$ on $t$ in the notation.) Then
\[ Z(t) = \sum_{n=0}^\infty f(n). \]
The Euler-MacLaurin formula says that
\[ Z(t) = \int_0^\infty f(x) \, dx + \frac{1}{2} f(0) - \sum_{p=1}^{N} \frac{B_{p+1}}{(p + 1)!} f^{(p)}(0) + R, \]
where the coefficients $B_p$ are the Bernoulli numbers and $R$ is the remainder term that can be estimated by
\[
|R| \leq \frac{2}{(2\pi)^{2(N+1)}} \int_0^\infty |f^{(N)}(x)| \, dx.
\]
If we define the “Todd function” by the power series
\[
td(x) := \frac{x}{1 - e^{-x}} = \sum_{p=0}^\infty (-1)^n \frac{B_p}{p!} x^p,
\]
then the Euler-Maclaurin formula can be stated as
\[
Z(t) = td\left( \frac{\partial}{\partial h} \right|_0 \int_{-h}^\infty f(x) \, dx + R'.
\]
(3.13) We are only interested in the leading term of the asymptotic expansion of $Z(t)$. The leading term comes from the integral $\int_{-h}^\infty f(x) \, dx$ in equation (3.12). Hence,
\[
\sqrt{t}^3 Z(t) \sim \sqrt{t}^3 \int_0^\infty f(x) \, dx + O(t)
\]
as $t \to 0^+$. (3.14)

(3.15) The symmetry under the Weyl reflections now comes into play. The function $f(x)$ is invariant under the reflection $x \mapsto -x$. So
\[
Z(t) = \frac{1}{2} \sum_{x \in \mathbb{Z}} f(x).
\]
Then (3.14) can be rewritten as
\[
\sqrt{t}^3 Z(t) \sim \sqrt{t}^3 I(t) + O(t),
\]
where
\[
I(t) := \frac{1}{2} \int_{-\infty}^\infty f(x) \, dx
\]
(3.17)
This integral is elementary:
\[
I(t) = \frac{4\sqrt{2\pi}}{\sqrt{t}^3}.
\]
So (3.16) gives
\[
\sqrt{t}^3 Z(t) \sim 4\sqrt{2\pi} + O(t).
\]
Then, by Weyl’s law (1.5), we have
\[
\text{vol}(G) = 32\sqrt{2\pi^2}.
\]
(3.18)
The volume of $T$ can be calculated using the length of the coroot $\hat{\alpha}$. The coroot $\hat{\alpha} \in \mathfrak{t}$ is characterized by
\[
\lambda(\hat{\alpha}) = \frac{1}{2} \alpha(\hat{\alpha}) = 1.
\]
So we have $\hat{\alpha} = H$. And the volume of $T$ is
\[
\text{vol}(T) = \int_0^{2\pi} \sqrt{\langle H, H \rangle} \, dt = 2\pi \sqrt{\langle \hat{\alpha}, \hat{\alpha} \rangle} = 4\sqrt{2\pi}.
\]
Combining with (3.18), we have
\[
\text{vol}(G) = 8\pi \text{vol}(T).
\]
(3.19)
The General Case.

(3.20) We now consider the general case. Our approach is to follow in outline the calculation done above for the case of $SU(2)$. The only new ingredient that will appear is the Weyl integration formula.

As we shall shortly see, to the end of calculating $\frac{\text{vol}(G)}{\text{vol}(T)}$, we may assume that $G$ is simply connected and semisimple. By the general theory of compact connected Lie groups, every compact connected Lie group $G$ is of the form $K/F$ where $K$ is a product of a torus with a compact, connected, simply connected, semisimple Lie group, and $F$ is a finite abelian subgroup of $K$. Moreover, $F$ is contained in a maximal torus $T_K$ of $K$. Consequently, $T := T_K/F$ is a maximal torus of $K/F \simeq G$.

If the metric on $K/F$ is induced from $K$, then $\text{vol}(K)/\text{vol}(T_K) = \text{vol}(K/F)/\text{vol}(T) = \text{vol}(G)/\text{vol}(T)$. Hence, we may assume that $F$ is trivial and $G$ is a product $S \times R$ where $S$ is a compact, connected, simply connected, semisimple Lie group and $R$ is a torus. But the maximal torus of $S \times R$ is of the form $T_S \times R$ where $T_S$ is a maximal torus of $S$, and $\text{vol}(S \times R)/\text{vol}(T_S \times R) = \text{vol}(S)/\text{vol}(T_S)$. Therefore, we may as well assume that $G = S$.

We adopt the following notation for the dimension of $G$, $T$, and $G/T$:

$$ n := \dim G, \quad r := \dim T, \quad 2m := \dim G - \dim T = n - r. $$

(3.21) Since $G$ is compact, connected, simply connected, and semisimple, there is a one-to-one correspondence between $G$ and $\Lambda_T \cap K$. For each $\lambda \in \Lambda_T \cap K$, let $V(\lambda)$ denote the corresponding highest weight module. The heat-trace is

$$ \text{tr}(e^{t\Delta_G}) = \sum_{\lambda \in \Lambda_T \cap K} \dim V(\lambda)^2 e^{t\text{Cas}(\lambda)} $$

where $\text{Cas}(\lambda)$ is given by (2.29). The dimension of $V(\lambda)$ is given by the Weyl dimension formula (2.27). It is convenient to define

$$ d(\lambda) := \prod_{\alpha \in \Phi^+} \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \rho \rangle}. $$

Then the dimension formula takes the form

$$ \dim V(\lambda) = d(\lambda + \rho). $$

And the heat-trace is

$$ \text{tr}(e^{t\Delta_G}) = e^{t\|\rho\|^2} \sum_{\lambda \in \Lambda_T \cap K} d(\lambda + \rho)^2 e^{-t\|\lambda + \rho\|^2}. $$

Since we are interested in the leading term of the asymptotic expansion, we focus on

$$ Z(t) := \sum_{\lambda \in \Lambda_T \cap K} d(\lambda + \rho)^2 e^{-t\|\lambda + \rho\|^2} = \sum_{\lambda \in (\Lambda_T \cap K) + \rho} d(\lambda)^2 e^{-t\|\lambda\|^2}. $$

The shifted set $(\Lambda_T \cap K) + \rho$ is the set of the weights that lie in the interior of the Weyl chamber. (This can be deduced from the properties of the fundamental weights listed in (2.30).) Hence,

$$ Z(t) = \sum_{\lambda \in \Lambda_T \cap K^o} d(\lambda)^2 e^{-t\|\lambda\|^2}. $$

But since the restriction of the function $d$ on the boundary of the Weyl chamber is zero, we have

$$ Z(t) = \sum_{\lambda \in \Lambda_T \cap K} d(\lambda)^2 e^{-t\|\lambda\|^2}. $$

(3.25) Lemma Let $d\lambda$ be a translation invariant measure that assigns unit volume to the fundamental parallelepiped formed by the fundamental weights of $G$ in $\mathfrak{t}^*$. We have

$$ \sqrt{t^n} Z(t) \sim \sqrt{t^n} I(t) + O(t) $$

where $I(t)$ is the integral of $\sqrt{t^n} d\lambda$.
as \( t \to 0^+ \), where

\begin{equation}
(3.27) \quad I(t) = \int_{\mathbb{K}} d(\lambda)^2 e^{-t\|\lambda\|^2} \, d\lambda.
\end{equation}

**Proof.** Let \( \{\lambda_1, \ldots, \lambda_r\} \) be the set of fundamental dominant weights. Write \( \lambda = \sum x_i \lambda_i \). Then \( Z(t) \) is of the form

\[ Z(t) = \sum_{x_1=0}^{\infty} \cdots \sum_{x_r=0}^{\infty} d(x_1, \ldots, x_r)^2 e^{-t q(x_1, \ldots, x_r)} \]

where \( d \) and \( q \) are homogeneous polynomials of degree \( m \) and \( 2 \), respectively. We are comparing the leading term of the asymptotic expansion of \( Z(t) \) with that of the integral

\[ I(t) = \int_{0}^{\infty} \int_{0}^{\infty} d(x_1, \ldots, x_r)^2 e^{-t q(x_1, \ldots, x_r)} \, dx_1 \cdots dx_r. \]

The Euler-Maclaurin formula yields the following 1-dimensional result:

\[ \sqrt{\frac{1}{t}} \sum_{x=0}^{\infty} A x^2 e^{-t (B x^2 + C x)} \, dx \sim \sqrt{\frac{1}{t}} \sum_{x=0}^{\infty} A x^2 e^{-t (B x^2 + C x)} \, dx + O(t), \]

where \( A, B, C \) are real numbers, \( B > 0 \), and \( p = 0, 1, 2, \ldots \). Applying this repeatedly to \( Z(t) \) proves the lemma. \( \square \)

(3.28) The set of roots and the inner product \( \langle \cdot, \cdot \rangle \) are preserved under the action of the Weyl group \( W \). This implies that \( |d(\lambda)| \) and \( \|\lambda\| \) are preserved under the \( W \)-action. Hence, the integral (3.27) can be rewritten as

\begin{equation}
(3.29) \quad I(t) = \frac{1}{|W|} \int_{\mathbb{K}} d(\lambda)^2 e^{-t\|\lambda\|^2} \, d\lambda.
\end{equation}

We wish to apply the Weyl integration formula. To that end, let \( d\kappa \) and \( d\kappa_T \) denote the translation invariant measure on \( g \) induced from the metric. The Weyl integration formula [1, Ch.9, §6, Prop.4] says that, for any \( \mathrm{ad}(g) \)-invariant function \( f \) on \( g \),

\[ \frac{1}{\text{vol}(G)} \int_{g} f(X) \, d\kappa(X) = \frac{1}{\text{vol}(T) |W|} \int_{t} f(X) |\delta(X)|^2 \, d\kappa_T(X), \]

where the Jacobian factor \( |\delta(X)|^2 \) is given by

\begin{equation}
(3.30) \quad |\delta(X)|^2 = \det(\mathrm{ad} |_{g/T}(X)) = \prod_{\alpha \in \Phi^+} \alpha(X)^2.
\end{equation}

(3.31) **Theorem** Let \( G \) be a compact connected Lie group, and \( T \) its maximal torus. Let \( g \) and \( t \) be their Lie algebra, respectively. Let \( \kappa \) denote a bi-invariant metric on \( G \) and let \( \langle \cdot, \cdot \rangle \) denote the induced inner product on \( g \) and \( g^* \). The riemannian volume of \( G \) and \( T \) are related by

\begin{equation}
(3.32) \quad \text{vol}(G) = \frac{(2\pi)^m \text{vol}(T)}{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle}.
\end{equation}

**Proof.** We pointed out in (3.20) that we may assume that \( G \) is simply connected. From (3.22) and (3.30), we have

\[ d(\lambda)^2 = \prod_{\alpha \in \Phi^+} \frac{\langle \alpha, \lambda \rangle^2}{\langle \alpha, \rho \rangle^2} = \left| \delta(X_\lambda) \right|^2, \]

where \( X_\lambda \) is the vector in \( t \) dual to \( \lambda \) with respect to the inner product. So (3.29) can be written as

\begin{equation}
(3.33) \quad I(t) = \frac{1}{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^2} \frac{1}{|W|} \int_{t} e^{-t\|X_\lambda\|^2} |\delta(X_\lambda)|^2 \, d\lambda.
\end{equation}
The measure $d\lambda$ is a $W$-invariant measure on $t$. It is related to $d\kappa_T$ by $d\lambda = d\kappa_T / \text{vol}(P)$ where $\text{vol}(P)$ is the volume of the parallelepiped formed by the fundamental weights under the measure $d\kappa_T$. Hence,

$$I(t) = \frac{1}{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^2 |W|} \int_{\Gamma} e^{-t\|X\|^2} |\delta(X)|^2 \, d\kappa_T(X) \frac{1}{\text{vol}(P)}.
$$

Applying the Weyl integration formula, we get

$$I(t) = \frac{1}{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^2 \text{vol}(G) \text{vol}(P)} \int_{\mathfrak{g}} e^{-t\|X\|^2} \, d\kappa(X).
$$

Finally, we note the following.

(i) Let $Q$ be the parallelepiped formed by the coroots $\{\hat{\alpha}\}_i$. Its volume $\text{vol}(Q)$ under the measure $d\kappa_T$ is related to $\text{vol}(T)$ by

$$\text{vol}(T) = (2\pi)^r \text{vol}(Q).
$$

(ii) The two volumes $\text{vol}(Q)$ and $\text{vol}(P)$ are reciprocal to each other.

(iii) Choose any basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. Let $(x_1, \ldots, x_n)$ be the coordinates for the vectors in $\mathfrak{g}$ relative to this basis. Then

$$\int_{\mathfrak{g}} e^{-t\|X\|^2} \, d\kappa(X) = \int_{\mathbb{R}^n} e^{-t\|X\|^2} \sqrt{|\det \kappa|} \, dx_1 \cdots dx_n = \sqrt{\pi^n}.
$$

With these facts, we deduce from equation (3.35) that

$$I(t) = \frac{1}{(2\pi)^r} \prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^2 \frac{(\text{vol}(T))^2}{\text{vol}(G)} \sqrt{\frac{\pi^n}{t}}.
$$

Combining the equations (1.5), (3.26), and (3.37) gives

$$\frac{(\text{vol}(G))^2}{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^2} = \frac{(2\pi)^2 m^2 (\text{vol}(T))^2}{\prod_{\alpha \in \Phi^+} \langle \alpha, \rho \rangle^2}.
$$

This proves the formula (3.32).
[12] M. E. Taylor, Partial differential equations. I, Applied Mathematical Sciences, vol. 115, Springer-Verlag, New York, 1996. Basic theory. MR1395148 (98b:35002b)
[13] ______, Partial differential equations. II, Applied Mathematical Sciences, vol. 116, Springer-Verlag, New York, 1996. Qualitative studies of linear equations. MR1395149 (98b:35003)
[14] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung), Math. Ann. 71 (1912), no. 4, 441–479, DOI 10.1007/BF01456804 (German). MR1511670
[15] ______, Über die Abhängigkeit der Eigenschwingungen einer Membran und deren Begrenzung, J. Reine Angew. Math. 141 (1912), 1–11, DOI 10.1515/CRLL.1912.141.1 (German).

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802 U.S.A.
E-mail address: hong@math.psu.edu
URL: http://www.math.psu.edu/hong