ON THE INTEGRAL REPRESENTATION OF BINARY QUADRATIC FORMS
AND THE ARTIN CONDITION

CHANG LV, JUNCHAO SHENTU, AND YINGPU DENG

Abstract. For diophantine equations of the form \( ax^2 + bxy + cy^2 + g = 0 \) over \( \mathbb{Z} \) whose coefficients satisfy some assumptions, we show that a condition with respect to Artin reciprocity map, which we call the Artin condition, is the only obstruction to the local-global principle for integral solutions of the equation. Some concrete examples are presented.

1. Introduction

The main theorem of a book by Cox [1] is a beautiful criterion of the solvability of the diophantine equation \( p = x^2 + ny^2 \). The specific statement is

**Theorem.** Let \( n \) be a positive integer. Then there is a monic irreducible polynomial \( f_n(x) \in \mathbb{Z}[x] \) of degree \( h(-4n) \) such that if an odd prime \( p \) divides neither \( n \) nor the discriminant of \( f_n(x) \), then \( p = x^2 + ny^2 \) is solvable over \( \mathbb{Z} \) if and only if \( \left( \frac{-n}{p} \right) = 1 \) and \( f_n(x) = 0 \) is solvable over \( \mathbb{Z}/p\mathbb{Z} \). Here \( h(-4n) \) is the class number of primitive positive definite binary forms of discriminant \(-4n\). Furthermore, \( f_n(x) \) may be taken to be the minimal polynomial of a real algebraic integer \( \alpha \) for which \( L = K(\alpha) \) is the ring class field of the order \( \mathbb{Z}[\sqrt{-n}] \) in the imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-n}) \).

There are some generalizations considering the problem over quadratic fields. By using classical results in the class field theory, the first and third author [3] gave the criterion of the integral solvability of the equation \( p = x^2 + ny^2 \) for some \( n \) over a class of imaginary quadratic fields, where \( p \) is a prime element.

Recently, Harari [2] showed that the Brauer-Manin obstruction is the only obstruction for the existence of integral points of a scheme over the ring of integers of a number field, whose generic fiber is a principal homogeneous space (torsor) of a torus. After then Wei and Xu [9, 10] construct the idele groups which are the so-called \( \mathbf{X} \)-admissible subgroups for determining the integral points for multi-norm tori, and interpret the \( \mathbf{X} \)-admissible subgroup in terms of finite Brauer-Manin obstruction. In [9, Section 3] Wei and Xu also showed how to apply this method to binary quadratic diophantine equations. As applications, they gave some explicit criteria of the solvability of equations of the form \( x^2 \pm dy^2 = a \) over \( \mathbb{Z} \) in [9, Sections 4 and 5].

Later Wei [7] applied the method in [9] to give some additional criteria of the solvability of the diophantine equation \( x^2 - dy^2 = a \) over \( \mathbb{Z} \) for some \( d \). He also determined which integers can be written as a sum of two integral squares for some of the quadratic fields \( \mathbb{Q}(\sqrt{\mp p}) \) (in [6]), \( \mathbb{Q}(\sqrt{-2p}) \) (in [8]) and so on.

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In this article, we apply the method in [9] to diophantine equations of the form
\[ ax^2 + bxy + cy^2 + g = 0 \]
over \( \mathbb{Z} \), a binary quadratic form representing an integer. With some additional assumptions, by choosing \( X \)-admissible subgroups for (1.1) the same as in [9, Sections 4, 5] and [6], we obtain criteria of the solvability of (1.1), as a variant of [9, Proposition 4.1] and [9, Proposition 5.1]. In the case \( b = 0 \), the first and second author [4] also gave some corresponding results.

Specifically, the main results of this article are (for notation one can see Section 2.1):

**Theorem.** Let \( a, b, c \) and \( g \) be integers such that \( d = 4ac - b^2 < 0 \). Suppose \( -d = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r} \) where \( r > 0, m_k \geq 1 \) are not all even and \( p_k \) are distinct odd primes such that one of the following assumptions holds:

1. \( p_i \equiv 3 \pmod{4} \) for some \( i \).
2. \( r = 2 \) or \( r > 3 \) is odd, \( p_i \equiv 1 \pmod{4} \), \( m_i = 1 \) for all \( i \) and \( (p_i/p_j) = -1 \) for all \( i \neq j \).

Set \( E = \mathbb{Q}(\sqrt{-d}) \), \( L = \mathbb{Z} + \mathbb{Z}\sqrt{-d} \) and \( H_L \) the ring class field corresponding to \( L \). Let \( X = \text{Spec}(\mathbb{Z}[x,y]/(ax^2 + bxy + cy^2 + g)) \). Then \( X(\mathbb{Z}) \neq \emptyset \) if and only if there exists
\[
\prod_{p \leq \infty} (x_p, y_p) \in \prod_{p \leq \infty} X(\mathbb{Z}_p)
\]
such that
\[
\psi_{H_L/E}(\tilde{f}_E(\prod_{p} (x_p, y_p))) = 1.
\]

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\]
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\]

In Section 2 we introduce from [9] notation and the general result we mainly use in this paper, but in a modified way which focus on our goal. Then we give our results on the equation (1.1) in Section 3. If the discriminant \( d \) is positive we need no additional assumption. But if \( d \) is negative, we add some assumptions on it (as it is done in [9, Section 5]). The results state that the integral local solvability and the Artin condition (see Remark 2.12) completely describe the global integral solvability. We also give some examples showing the explicit criteria of the solvability.
2. Solvability by the Artin Condition

2.1. Notation. Let $F$ be a number field, $\mathfrak{o}_F$ the ring of integers of $F$, $\Omega_F$ the set of all places in $F$ and $\infty_F$ the set of all infinite places in $F$. Let $\mathfrak{p}$ be the completion of $F$ at $\mathfrak{p}$ and $\mathfrak{o}_F$ be the valuation ring of $F$ for each $\mathfrak{p} \in \Omega_F \setminus \infty_F$. We also write $\mathfrak{o}_F = F_\mathfrak{p}$ for $\mathfrak{p} \in \infty_F$. The adele ring (resp. idele group) of $F$ is denoted by $\mathbb{A}_F$ (resp. $\mathbb{I}_F$).

Let $a, b, c$ and $g$ be elements in $\mathfrak{o}_F$ and suppose that $-d = b^2 - 4ac$ is not a square in $F$. Let $E = F(\sqrt{-d})$ and $X = \text{Spec}(\mathfrak{o}_F[x, y]/(ax^2 + bxy + cy^2 + g))$ be the affine scheme defined by the equation $ax^2 + bxy + cy^2 + g = 0$ over $\mathfrak{o}_F$. The equation

$$ax^2 + bxy + cy^2 + g = 0$$

is solvable over $\mathfrak{o}_F$ if and only if $X(\mathfrak{o}_F) \neq \emptyset$.

Now we denote

$$\tilde{x} := 2ax + by,$$
$$\tilde{y} := y,$$
$$n := -4ag.$$

Then we can write (2.1) as

$$\tilde{x}^2 + d\tilde{y}^2 = n.$$  

Denote $R_{E/F}(\mathbb{G}_m)$ the Weil restriction of $\mathbb{G}_{m, E}$ to $F$. Let

$$\varphi : R_{E/F}(\mathbb{G}_m) \to \mathbb{G}_m$$

be the homomorphism of algebraic groups which represents

$$x \mapsto N_{E/F}(x) : (E \otimes_F A)^	imes \to A^	imes$$

for any $F$-algebra $A$. Define the torus $T := \ker \varphi$. Let $X_F$ be the generic fiber of $X$. We can identify elements in $T(A)$ (resp. $(X_F(A)$) as $u + \sqrt{-d}v$ (resp. $\tilde{x} + \sqrt{-d}\tilde{y}$). Then $X_F$ is naturally a $T$-torsor by the action:

$$T(A) \times X_F(A) \to X_F(A)$$

$$(u + \sqrt{-d}v, \tilde{x} + \sqrt{-d}\tilde{y}) \mapsto (u + \sqrt{-d}v)(\tilde{x} + \sqrt{-d}\tilde{y}).$$

Note that $T$ has an integral model $T = \text{Spec}(\mathfrak{o}_F[x, y]/(x^2 + dy^2 - 1))$ and we can view $T(\mathfrak{o}_F)$ as a subgroup of $T(\mathfrak{p}_F)$.

Denote by $\lambda$ the embedding of $T$ into $R_{E/F}(\mathbb{G}_m)$. Clearly $\lambda$ induces a natural injective group homomorphism

$$\lambda_E : T(\mathbb{A}_F) \to \mathbb{I}_E.$$

Let $L = \mathfrak{o}_F + \sqrt{-d}$ in $E$ and $L_\mathfrak{p} = L \otimes_{\mathfrak{o}_F} \mathfrak{o}_F$ in $E_{\mathfrak{p}} = E \otimes_F F_{\mathfrak{p}}$. Then

$$T(\mathfrak{o}_F) = \{ \beta \in L_\mathfrak{p}^\times \mid N_{E_{\mathfrak{p}}/F_{\mathfrak{p}}}(\beta) = 1 \}.$$ 

It follows that $\lambda_E(T(\mathfrak{o}_F)) \subseteq L_\mathfrak{p}^\times$. Note that $\lambda_E(T(F)) \subseteq E^\times$ in $\mathbb{I}_E$. Let $L := \prod_{\mathfrak{p} \in \Omega_F} L_\mathfrak{p}^\times$ which is an open subgroup of $\mathbb{I}_E$. Then the following map induced by $\lambda_E$ is well-defined:

$$\tilde{\lambda}_E : T(\mathbb{A}_F)/T(F) \prod_{\mathfrak{p} \in \Omega_F} T(\mathfrak{o}_F) \to \mathbb{I}_E/E^\times \prod_{\mathfrak{p} \in \Omega_F} L_\mathfrak{p}^\times.$$ 

Now we assume that

$$X(F) \neq \emptyset,$$ 

where
i.e. $X_F$ is a trivial $T$-torsor. Fixing a rational point $P \in X_F(F)$, for any $F$-algebra $A$, we have an isomorphism

$$\phi_P : X_F(A) \cong T(A)$$

induced by $P$. Since we can view $\prod_{p \in \Omega_F} X(o_{F_p})$ as a subset of $X_F(A_F)$, the composition $f_E := \lambda_E \phi_P : \prod_{p} X(o_{F_p}) \rightarrow \mathbb{I}_E$ makes sense, mapping $x$ to $P^{-1}x$ in $\mathbb{I}_E$. Note that $P$ is in $E^\times \subset \mathbb{I}_E$ since it is a rational point over $F$. It follows that we can define the map $\tilde{f}_E$ to be the composition

$$\prod_{p} X(o_{F_p}) \xrightarrow{f_E} \mathbb{I}_E \xrightarrow{\times P} \mathbb{I}_E$$

$$x \mapsto P^{-1}x \mapsto x.$$ 

It can be seen that the restriction to $X(o_{F_p})$ of $\tilde{f}_E$ is defined by

$$\tilde{f}_E[(x_p, y_p)] = \begin{cases} (\tilde{x}_p + \sqrt{-d}\tilde{y}_p, \tilde{x}_p - \sqrt{-d}\tilde{y}_p) & \text{if } p \text{ splits in } F/E, \\ (\tilde{x}_p + \sqrt{-d}\tilde{y}_p) & \text{otherwise,} \end{cases}$$

where $\mathfrak{P}_1$ and $\mathfrak{P}_2$ (resp. $\mathfrak{P}$) are places of $E$ above $p$.

Recall that $L = \mathfrak{o}_F + \mathfrak{o}_F \sqrt{-d}$, $L_p = L \otimes_{\mathfrak{o}_F} \mathfrak{o}_{F_p}$ and $\Xi_L = \prod_p L_p^\times$ is an open subgroup of $\mathbb{I}_E$. By the ring class field corresponding to $L$ we mean the class field $H_L$ corresponding to $\Xi_L$ under the class field theory, such that the Artin map gives the isomorphism $\psi_{H_L/E} : \mathbb{I}_E/E^\times \Xi_L \cong \text{Gal}(H_L/E)$. For any $\prod_{p \in \Omega_F} (x_p, y_p) \in \prod_{p \in \Omega_F} X(o_{F_p})$, noting that $P$ is in $E$, we have

$$\psi_{H_L/E}(\tilde{f}_E(\prod_p (x_p, y_p))) = 1 \text{ if and only if } \psi_{H_L/E}(\tilde{f}_E(\prod_p (x_p, y_p))) = 1.$$ 

**Remark 2.6.** If $\prod_{p \in \Omega_F} X(o_{F_p}) \neq \emptyset$, then the assumption (2.3) holds automatically by the Hasse-Minkowski theorem on quadratic equations. Hence we can pick an $F$-point $P$ of $X_F$ and obtain $\phi_P$.

### 2.2. A general result

In the previous section, we choose the subgroup to be $\Xi_L = \prod_p L_p^\times$ where $L = \mathfrak{o}_F + \mathfrak{o}_F \sqrt{-d}$ and $L_p = L \otimes_{\mathfrak{o}_F} \mathfrak{o}_{F_p}$. By some additional assumptions, we prove that $\Xi_L$ can be viewed as an admissible subgroup for $X = \text{Spec}(\mathfrak{o}_F[x, y]/(ax^2 + bxy + cy^2 + g))$.

**Lemma 2.7.** Let $U$ be a complete system of representatives of $\mathfrak{o}_F^\times/(\mathfrak{o}_F^\times)^2$. Suppose for every $u \in U$, the equation $x^2 + dy^2 = u$ is solvable over $\mathfrak{o}_F$ or is not solvable over $\mathfrak{o}_{F_p}$ for some place $p$. Then the map $\lambda_E$ is injective.

**Proof.** Recall that $T = \ker(R_E/F(G_m) \rightarrow G_m)$ and $T$ is the group scheme defined by the equation $x^2 + dy^2 = 1$ over $\mathfrak{o}_F$. Therefore we have

$$T(F) = \{ \beta \in E^\times \mid N_{E/F}(\beta) = 1 \}$$

and

$$T(o_{F_p}) = \{ \beta \in L_p^\times \mid N_{E/F_p}(\beta) = 1 \}.$$ 

Suppose $t \in T(A_F)$ such that $\lambda_E(t) = 1$. Write $t = \beta i$ with $\beta \in E^\times$ and $i \in \prod_p L_p^\times$. Since $t \in T(A_F)$ we have

$$N_{E/F}(\beta)N_{E/F}(i) = N_{E/F}(\beta i) = 1.$$
If \( N_{E/F}(i) = N_{E/F}(\beta^{-1}) \in F^\times \cap \prod_p \mathfrak{o}_F^\times = \mathfrak{o}_F^\times \).

So by the definition of \( \mathcal{U} \), we have \( N_{E/F}(i) = uv^2 \) for some \( u \in \mathcal{U} \) and \( v \in \mathfrak{o}_F^\times \). Then

\[
N_{E/F}(iu^{-1}) = u,
\]

from which we know that the equation \( x^2 + dy^2 = u \) is solvable over \( \mathfrak{o}_{F_p} \) for every place \( p \) of \( F \), since \( v^{-1} \in \mathfrak{o}_F \). Thus the assumption tells us that \( x^2 + dy^2 = u \) is solvable over \( \mathfrak{o}_F \). Let \( (x_0, y_0) \in \mathfrak{o}_F^2 \) be such a solution and let

\[
\begin{align*}
\zeta &= x_0 + y_0\sqrt{-d}, \\
\gamma &= \beta \nu \zeta, \\
and \ q &= iu^{-1} \zeta^{-1}.
\end{align*}
\]

Then \( N_{E/F}(\gamma) = N_{E/F}(j) = 1 \). Note that \( \zeta \in L^\times \), and we have \( \gamma \in T(F) \) and \( j \in \prod_p T(\mathfrak{o}_{F_p}) \). It follows that \( t = \beta \nu = \gamma j \in T(F) \prod_p T(\mathfrak{o}_{F_p}) \). This finishes the proof. \( \square \)

As a result, we can obtain criteria of the solvability in a more explicit way. We state it in the following proposition, which is a Corollary to [9, Corollary 1.6].

**Proposition 2.8.** Let symbols be as before and \( \mathcal{U} \) satisfy the assumption in Lemma 2.7. Then \( X(\mathfrak{o}_F) \neq \emptyset \) if and only if there exists

\[
\prod_{p \in \Omega_F} (x_p, y_p) \in \prod_{p \in \Omega_F} X(\mathfrak{o}_{F_p})
\]

such that

\[
\psi_{H_L/E}(\tilde{f}_E(\prod_p (x_p, y_p))) = 1.
\]

**(2.9)**

**Proof.** By the assumption we know from Lemma 2.7 that

\[
(2.10)
\tilde{\lambda}_E : T(\mathfrak{A}_F)/T(F) \prod_p T(\mathfrak{o}_{F_p}) \rightarrow \mathfrak{I}_{E/F} \times \prod_p L^\times_p
\]

is injective.

If \( X(\mathfrak{o}_F) \neq \emptyset \), then

\[
\tilde{f}_E \left( \prod_p X(\mathfrak{o}_{F_p}) \right) \cap E^\times \prod_p L^\times_p \supseteq \tilde{f}_E(X(\mathfrak{o}_F)) \cap E^\times \neq \emptyset
\]

Hence there exists \( x \in \prod_{p \in \Omega_F} X(\mathfrak{o}_{F_p}) \) such that \( \psi_{H_L/E}\tilde{f}_E(x) = 1 \).

Conversely, suppose there exists \( x \in \prod_p X(\mathfrak{o}_{F_p}) \) such that \( \psi_{H_L/E}\tilde{f}_E(x) = 1 \) (here \( \tilde{f}_E \) makes sense by Remark 2.10, i.e. \( \lambda_E \phi_F(x) = \tilde{f}_E(x) \in \Xi_L = E^\times \prod_p L^\times_p \)). Since \( \tilde{\lambda}_E \), i.e. \( 2.10 \), is injective, there are \( \tau \in T(F) \) and \( \sigma \in \prod_p T(\mathfrak{o}_{F_p}) \) such that \( \tau \sigma = \phi_F(x) = P^{-1}x \), i.e. \( \tau \sigma(P) = x \). Since \( P \in X_F(F) \) and

\[
(2.11)
gX(\mathfrak{o}_{F_p}) = X(\mathfrak{o}_{F_p}) \text{ for all } g \in T(\mathfrak{o}_{F_p}),
\]

it follows that

\[
\tau(P) = \sigma^{-1}(x) \in X(F) \cap \prod_p X(\mathfrak{o}_{F_p}) = X(\mathfrak{o}_F).
\]
Then the proof is done. \qed

Remark 2.12. The condition (2.9) is called the Artin condition in, for example, Wei’s [7, 6, 8]. It interprets the fact that the Brauer-Manin obstruction is the only obstruction for existence of the integral points by conditions in terms of the class field theory. Consequently, if the assumption in the proposition holds, the integral local solvability and the Artin condition completely describe the global integral solvability. As a result, in cases where the ring class fields are known it is possible to calculate the Artin condition, giving explicit criteria for the solvability.

3. The Integral Representation of Binary Quadratic Forms over \( \mathbb{Z} \)

Now we consider the case where \( F = \mathbb{Q} \) which is our focus. We now distinguish the sign of the discriminant \( d \).

3.1. The case where the discriminant \( d > 0 \).

**Theorem 3.1.** Let \( a, b, c \) and \( g \) be integers and suppose that \( d = 4ac - b^2 > 0 \). Set \( E = \mathbb{Q}(\sqrt{-d}) \), \( L = \mathbb{Z} + \mathbb{Z}\sqrt{-d} \) and \( H_L \) the ring class field corresponding to \( L \). Let \( X = \text{Spec}(\mathbb{Z}[x, y]/(ax^2 + bxy + cy^2 + g)) \). Then \( X(\mathbb{Z}) \neq \emptyset \) if and only if there exists

\[
\prod_{p \leq \infty} (x_p, y_p) \in \prod_{p \leq \infty} X(\mathbb{Z}_p)
\]

such that

\[
\psi_{H_L/E}(\tilde{f}_E(\prod_p(x_p, y_p))) = 1.
\]

**Proof.** Since \( d > 0 \) it is clear that \( x^2 + dy^2 = -1 \) is not solvable over \( \mathbb{R} \), which is to say the assumption in Proposition 2.8 holds since the only units of \( \mathbb{Z} \) are \( \{\pm 1\} \). Then the result follows from Proposition 2.8. \qed

We now give an example where the explicit criterion is obtained using this result.

**Example 3.2.** Let \( g \) be a negative integer and \( l(x) = x^4 - x^3 + x + 1 \in \mathbb{Z}[x] \). Write \( g = -2^{s_1} \times 7^{s_2} \times \prod_{k=1}^r p_k^{m_k} \), where \( s_1, s_2, k \geq 0, m_k \geq 1, p_1, p_2, \ldots, p_r \neq 2, 7 \) are distinct primes. Define \( C = \{3, p_1, p_2, \ldots, p_r\} \) and

\[
D = \{p \in C \mid (\frac{-14}{p}) = 1 \text{ and } l(x) \mod p \text{ irreducible}\}.
\]

Then the diophantine equation \( 3x^2 + 2xy + 5y^2 + g = 0 \) is solvable over \( \mathbb{Z} \) if and only if

1. \( 3g \times 2^{-s_1} \equiv \pm 1 \pmod{8} \),
2. \( (2 \times 7^{-s_2}) = 1 \),
3. for all \( p \nmid 2 \times 3 \times 7 \) with odd \( m_p := v_p(g), (\frac{-14}{p}) = 1 \),
4. and \( \sum_{p \in D} v_p(3g) \equiv 0 \pmod{2} \).

**Proof.** In this example, we have \( a = 3, b = 2, c = 5, d = 4ac - b^2 = 4 \times 14 \). Let \( E = \mathbb{Q}(\sqrt{-d}) \). Since \( b = 2 \), we can simplify the equation (2.2) by canceling 4 in both sides. Thus we set

\[
\begin{align*}
n &= -4ag/4 = -3g, \\
\tilde{x} &= (2ax + by)/2 = 3x + y, \\
\tilde{y} &= y.
\end{align*}
\]
In fact we may assume $d = 14$ and Theorem 3.1 still applies. Because if $d = 14$, we still have $E = \mathbb{Q}(\sqrt{-d})$, $\tilde{x}^2 + 2\tilde{y}^2 = n$ and also (2.11) holds. It follows that $L = \mathbb{Z} + \mathbb{Z}\sqrt{-14} = \mathfrak{o}_E$ and $H_L = H_E = E(\alpha)$ the Hilbert field of $E$ where the minimal polynomial of $\alpha$ is $l(x)$. The Galois group

$$
\text{Gal}(H_L/E) = \langle \sqrt{-1} \rangle \cong \mathbb{Z}/4\mathbb{Z}.
$$

Let $X = \text{Spec}(\mathbb{Z}[x,y]/(3x^2 + 2xy + 5y^2 + g))$ and

$$
\tilde{f}_E([x_p, y_p]) = \begin{cases} 
(\tilde{x}_p + \sqrt{-14}\tilde{y}_p, \tilde{x}_p - \sqrt{-14}\tilde{y}_p) & \text{if } p \text{ splits in } E/\mathbb{Q}, \\
\tilde{x}_p + \sqrt{-14}\tilde{y}_p & \text{otherwise},
\end{cases}
$$

Then by Theorem 3.1 $X(\mathbb{Z}) \neq \emptyset$ if and only if there exists

$$
\prod_{p \leq \infty} (x_p, y_p) \in \prod_{p \leq \infty} X(\mathbb{Z}_p)
$$

such that

$$
\psi_{H_L/E}(\tilde{f}_E(\prod_{p} (x_p, y_p))) = 1.
$$

Next we calculate these conditions in details. Recall that $n = -3q$. By a simple calculation we know the local condition

$$
\prod_{p \leq \infty} X(\mathbb{Z}_p) \neq \emptyset
$$

is equivalent to

$$
\begin{cases} 
 n \times 2^{-s_1} \equiv \pm 1 \pmod{8}, \\
 \left( \frac{a x^2 + b}{p} \right) = 1, \\
 \text{for all } p \mid 2 \times 3 \times 7 \text{ with odd } m_p = v_p(n), \left( \frac{-14}{p} \right) = 1.
\end{cases}
$$

(3.3)

For the Artin condition, let $(x_p, y_p)_p \in \prod_p X(\mathbb{Z}_p)$. Then

$$
(\tilde{x}_p + \sqrt{-14}\tilde{y}_p)(\tilde{x}_p - \sqrt{-14}\tilde{y}_p) = n \text{ in } E_\mathfrak{P} \text{ with } \mathfrak{P} | p,
$$

and since $H_L/E$ is unramified, for any $p \neq \infty$ we have

$$
1 = \begin{cases} 
\psi_{H_L/E}(p_\mathfrak{P})\psi_{H_L/E}(\bar{p}_\mathfrak{P}), & \text{if } p = \mathfrak{P}\bar{\mathfrak{P}} \text{ splits in } E/\mathbb{Q}, \\
\psi_{H_L/E}(p_{\mathfrak{P}}), & \text{if } p = \mathfrak{P} \text{ is inert in } E/\mathbb{Q},
\end{cases}
$$

(3.5)

where $p_{\mathfrak{P}}$ (resp. $p_{\bar{\mathfrak{P}}}$) is in $\mathfrak{P}_E$ such that its $\mathfrak{P}$ (resp. $\bar{\mathfrak{P}}$) component is $p$ and the other components are 1. We calculate $\psi_{H_L/E}(\tilde{f}_E([x_p, y_p]))$ separately:

1. If $p = 2, 2 = \mathfrak{P}_2^2$ in $E/\mathbb{Q}$. Suppose $\mathfrak{P}_2 = \pi_2\mathfrak{o}_{E_{\mathfrak{P}_2}}$ for $\pi_2 \in \mathfrak{o}_{E_{\mathfrak{P}_2}}$. Noting that $H_L/E$ is unramified, since $\mathfrak{P}_2^2$ is principal in $E$ but $\mathfrak{P}_2$ is not, we have $\psi_{H_L/E}(\pi_2\mathfrak{P}_2) = -1$. By (3.4) we have

$$
\nu_{\mathfrak{P}_2}(\tilde{x}_2 + \sqrt{-14}\tilde{y}_2) = \nu_{\mathfrak{P}_2}(\tilde{x}_2 - \sqrt{-14}\tilde{y}_2) = \frac{1}{2} \nu_{\mathfrak{P}_2}(n) = \nu_{2}(n) = s_1.
$$

It follows that

$$
\psi_{H_L/E}(\tilde{f}_E([x_2, y_2])) = \psi_{H_L/E}(\tilde{x}_2 + \sqrt{-14}\tilde{y}_2) = (-1)^{s_1},
$$

and

$$
\psi_{H_L/E}(\tilde{f}_E([x_p, y_p])) = \psi_{H_L/E}(\tilde{x}_p + \sqrt{-14}\tilde{y}_p).
$$

(3.4)
where \( \tilde{f}_E[(x_2, y_2)] \) is also regarded as an element in \( I_E \) such that the component above 2 is given by the value of \( \tilde{f}_E[(x_2, y_2)] \) and 1 otherwise.

(2) If \( p = 7 \), a similar argument shows that \( \psi_{H_L/E}(\tilde{f}_E[(x_7, y_7)]) = (-1)^{s_2} \).

(3) If \( \left( \frac{-14}{p} \right) = 1 \) then by (3.5) we can distinguish the following cases:

(i) \( l(x) \mod p \) splits into linear factors. Then \( \psi_{H_L/E}(p\psi) = \psi_{H_L/E}(p\bar{\psi}) = 1 \) and

\[
\psi_{H_L/E}(\tilde{f}_E[(x_p, y_p)]) = (-1)^{v_p(x_p + \sqrt{-14}y_p)}\psi_{H_L/E}(\tilde{f}_E[(\bar{x}_p - \sqrt{-14}y_p)\bar{\psi}])
\]

where \( m = v_p(n) \) since

\[
v_p(x_p + \sqrt{-14}y_p) + v_p(\bar{x}_p - \sqrt{-14}y_p)
\]

(ii) \( l(x) \mod p \) splits into two irreducible factors. Then \( \psi_{H_L/E}(p\psi) = \psi_{H_L/E}(p\bar{\psi}) = -1 \).

It follows that

\[
\psi_{H_L/E}(\tilde{f}_E[(x_p, y_p)]) = \psi_{H_L/E}(\tilde{f}_E[(\bar{x}_p - \sqrt{-14}y_p)\bar{\psi}])
\]

(3.6) \( \sum_{p \in D} v_p(3g) \equiv 0 \mod 2 \).

Putting the above argument together, and noting that \( D \neq \emptyset \) since \( 3 \in D \) and that \( n = -3g \), we know the Artin condition is

(3.6) \( \sum_{p \in D} v_p(3g) \equiv 0 \mod 2 \).

The proof is done if we put the local condition (3.3) and the Artin condition (3.6) together. \( \Box \)

3.2. The case where the discriminant \( d < 0 \). In this case, \( x^2 + dy^2 = -1 \) is solvable over \( \mathbb{R} \), so we must look for place \( p \) of \( \mathbb{Q} \) such that \( x^2 + dy^2 = -1 \) is not solvable over \( \mathbb{Z}_p \). For a rational prime \( p \) that divides \( d \), we observe that, by Hensel’s Lemma, \( x^2 + dy^2 = -1 \) is solvable over \( \mathbb{Z}_p \) if and only if it is solvable over \( \mathbb{Z}/p\mathbb{Z} \), i.e. \( (\frac{-14}{p}) = 1 \). So if \( d \) is divisible by some rational prime \( p \) where \( p \equiv 3 \mod 4 \) then \( x^2 + dy^2 = -1 \) is not solvable over \( \mathbb{Z}_p \). Otherwise if none of the prime divisors of \( d \) are congruent to 3 modulo 4, we hope that \( x^2 + dy^2 = -1 \) is solvable over \( \mathbb{Z} \), in order to make the assumption true in Proposition 2.8. We need the following result by Morris Newman [5].
Theorem 3.7. Let \( r > 1 \) be 2 or odd, \( p_1, p_2, \ldots, p_r \) be distinct primes such that
\[
p_i \equiv 1 \pmod{4}, \quad 1 \leq i \leq r,
\]
\[
\left( \frac{p_i}{p_j} \right) = -1, \quad 1 \leq i \neq j \leq r.
\]
Then the diophantine equation \( x^2 - p_1 p_2 \cdots p_r y^2 = -1 \) has a solution in \( \mathbb{Z} \).

Now we have the criterion for certain \( d < 0 \).

Theorem 3.8. Let \( a, b, c \) and \( g \) be integers such that \( d = 4ac - b^2 < 0 \). Suppose \( -d = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r} \)
where \( r > 0, m_k \geq 1 \) are not all even and \( p_k \) are distinct odd primes such that one of the following assumptions holds:

1. \( p_i \equiv 3 \pmod{4} \) for some \( i \).
2. \( r = 2 \) or \( r > 3 \) is odd, \( p_i \equiv 1 \pmod{4}, m_i = 1 \) for all \( i \) and \( (p_i/p_j) = -1 \) for all \( i \neq j \).

Set \( E = \mathbb{Q}(\sqrt{-d}) \), \( L = \mathbb{Z} + \mathbb{Z}\sqrt{-d} \) and \( H_L \) the ring class field corresponding to \( L \). Let \( X = \text{Spec}(\mathbb{Z}[x,y]/(ax^2 + bxy + cy^2 + g)) \). Then \( X(\mathbb{Z}) \neq \emptyset \) if and only if there exists
\[
\prod_{p \leq \infty} (x_p, y_p) \in \prod_{p \leq \infty} X(\mathbb{Z}_p)
\]
such that
\[
\psi_{H_L/E}(f_E(\prod_{p} (x_p, y_p))) = 1.
\]

Proof. The units of \( \mathbb{Z} \) are \( \{ \pm 1 \} \) so we only need to consider the unit \(-1\). If (1) holds, i.e. \( p_i \equiv 3 \pmod{4} \) for some \( i \), one can see immediately that \( x^2 + dy^2 = -1 \) is not solvable over \( \mathbb{Z}_p \). Otherwise (2) holds and then \( x^2 + dy^2 = -1 \) is solvable over \( \mathbb{Z} \) by Theorem 3.7. Hence the assumption in Proposition 3.8 holds and we complete the proof by Proposition 3.8.

We now give an example for this case.

Example 3.9. Let \( g \) be a nonzero integer and \( l(x) = x^3 - x^2 - 4x + 2 \in \mathbb{Z}[x] \). Write \( g = \pm 2^{s_1} \times 79^{s_2} \times \prod_{k=1}^{r} p_k^{m_k} \), where \( s_1, s_2, k \geq 0, m_k \geq 1, p_1, p_2, \ldots, p_r \neq 2, 79 \) are distinct primes.

Define \( C = \{ 5, p_1, p_2, \ldots, p_r \} \) and
\[
D = \{ p \in C \mid \left( \frac{79}{p} \right) = 1 \text{ and } l(x) \mod p \text{ irreducible} \}.
\]

Then the diophantine equation \( 5x^2 + 14xy - 6y^2 + g = 0 \) is solvable over \( \mathbb{Z} \) if and only if
\[
(1) \quad \left( \frac{2g(79)^{-2}}{79} \right) = -1,
(2) \quad \text{for all } p \mid 2 \times 5 \times 79 \text{ with odd } m_p \colonequals v_p(g), \left( \frac{79}{p} \right) = 1,
(3) \quad \text{and if } \{ p \in D \mid v_p(5g) = 1 \} \neq \emptyset \text{ then } r > 1.
\]

Proof. In this example, we have \( a = 5, b = 14, c = -6, d = 4ac - b^2 = -4 \times 79 \). Let \( E = \mathbb{Q}(\sqrt{-d}) \).

Since \( 2 \mid b \), we may cancel 4 in both sides and assume \( d = -79 \) as we do in the previous example. Since \( 79 \equiv 3 \pmod{4} \) the assumption (1) in Theorem 3.8 is correct. It follows that we can apply the theorem for \( d = -79 \). Thus we set
\[
\begin{align*}
n &= -4ag/4 = -5g, \\
\tilde{x} &= (2ax + by)/2 = 5x + 7y, \\
\tilde{y} &= y.
\end{align*}
\]
Now $E = \mathbb{Q}(\sqrt[7]{79})$, $x^2 - 79y^2 = n$ and $L = \mathbb{Z} + \mathbb{Z}\sqrt[7]{79} = \mathfrak{o}_E$ and $H_L = H_E = E(\alpha)$ the Hilbert field of $E$ where the minimal polynomial of $\alpha$ is $l(x)$. The Galois group
\[
\text{Gal}(H_L/E) = \langle \omega \rangle \cong \mathbb{Z}/3\mathbb{Z}.
\]
Let $X = \text{Spec}(\mathbb{Z}[x,y]/(5x^2 + 14xy - 6y^2 + g))$ and
\[
\tilde{f}_E[(x_p,y_p)] = \begin{cases} 
(\tilde{x}_p + \sqrt[7]{79}\tilde{y}_p, \tilde{x}_p - \sqrt[7]{79}\tilde{y}_p) & \text{if } p \text{ splits in } E/\mathbb{Q}, \\
(\tilde{x}_p + \sqrt[7]{79}\tilde{y}_p) & \text{otherwise,}
\end{cases}
\]
Then by Theorem 3.1 $X(\mathbb{Z}) \neq \emptyset$ if and only if there exists
\[
\prod_{p \leq \infty} (x_p,y_p) \in \prod_{p \leq \infty} X(\mathbb{Z}_p)
\]
such that
\[
\psi_{H_L/E}(\tilde{f}_E(\prod_p (x_p,y_p))) = 1.
\]
By a simple computation the local condition
\[
\prod_{p \leq \infty} X(\mathbb{Z}_p) \neq \emptyset
\]
is equivalent to the first two condition (1) and (2) above. For the Artin condition, let $(x_p,y_p)_p \in \prod_p X(\mathbb{Z}_p)$. Then
\[
(\tilde{x}_p + \sqrt[7]{79}\tilde{y}_p)(\tilde{x}_p - \sqrt[7]{79}\tilde{y}_p) = n \text{ in } E_\mathfrak{P} \text{ with } \mathfrak{P} | p,
\]
and since $H_L/E$ is unramified, for any $p \neq \infty$ we have
\[
(3.10) \quad 1 = \begin{cases} 
\psi_{H_L/E}(p\mathfrak{P})\psi_{H_L/E}(p\bar{\mathfrak{P}}), & \text{if } p = \mathfrak{P}\bar{\mathfrak{P}} \text{ splits in } E/\mathbb{Q}, \\
\psi_{H_L/E}(p\mathfrak{P}), & \text{if } p = \mathfrak{P} \text{ is inert in } E/\mathbb{Q}.
\end{cases}
\]
We calculate $\psi_{H_L/E}(\tilde{f}_E([x_p,y_p]))$ separately:
(1) If $p = 2, 2 = \mathfrak{P}_2^2$ in $E/\mathbb{Q}$. Suppose $\mathfrak{P}_2 = \pi_2\mathfrak{o}_{E_{\mathfrak{P}_2}}$ for $\pi_2 \in \mathfrak{o}_{E_{\mathfrak{P}_2}}$. Noting that $H_L/E$ is unramified, since $\mathfrak{P}_2$ is principal in $E$, we have $\psi_{H_L/E}(\pi_2\mathfrak{P}_2) = 1$. Hence $\psi_{H_L/E}(\tilde{f}_E([x_2,y_2])) = 1$.
(2) If $p = 79$, a similar argument shows that $\psi_{H_L/E}(\tilde{f}_E([x_{79},y_{79}])) = 1$.
(3) If (2p) = 1 then by (3.10) we can distinguish the following two cases:
(i) $l(x) \mod p$ splits into linear factors. Then $\psi_{H_L/E}(p\mathfrak{P}) = \psi_{H_L/E}(p\bar{\mathfrak{P}}) = 1$ and
\[
\psi_{H_L/E}(\tilde{f}_E([x_p,y_p])) = 1.
\]
(ii) $l(x) \mod p$ is irreducible. Then $\psi_{H_L/E}(p\mathfrak{P}) = (\psi_{H_L/E}(p\bar{\mathfrak{P}}))^{-1} = \omega^{-1}$. It follows that
\[
\psi_{H_L/E}(\tilde{f}_E([x_p,y_p])) = \psi_{H_L/E}((\tilde{x}_p + \sqrt[7]{79}\tilde{y}_p)p)\psi_{H_L/E}((\tilde{x}_p - \sqrt[7]{79}\tilde{y}_p)p)
\]
\[
= \omega^\pm (v_p(\tilde{x}_p + \sqrt[7]{79}\tilde{y}_p) + v_p(\tilde{x}_p - \sqrt[7]{79}\tilde{y}_p))/\omega^\pm 2v_p(\tilde{x}_p) \sqrt[7]{79}\tilde{y}_p
\]
\[
= \omega^\pm (m-2u)
\]
where $m = v_p(n)$ and $u = v_p(\tilde{x}_p - \sqrt[7]{79}\tilde{y}_p)$ (in $\mathbb{Q}_p$, $0 \leq u \leq m$). By Hensel’s lemma, we can choose a local solution $(x_p,y_p)$ suitably, such that $u$ riches any value between
0 and $m$. Hence

$$
\psi_{H_L/E}(\tilde{f}_E([x_p, y_p])) = \begin{cases} 
1 & \text{if } m = 0, \\
\omega^{\pm 1} & \text{if } m = 1, \\
1 \text{ or } \omega^{\pm 1} & \text{if } m \geq 2,
\end{cases}
$$

where the values are chosen freely in each case.

(4) If $(\frac{29}{p}) = -1$ then $p$ is inert in $E/\mathbb{Q}$. By (3.10) we have $\psi_{H_L/E}(\tilde{f}_E([x_p, y_p])) = 1$.

(5) At last if $p = \infty$, since $H_L/E$ is unramified, we have $\psi_{H_L/E}(\tilde{f}_E([x_\infty, y_\infty])) = 1$.

Putting the above argument together, and noting that $5 \in D$ and $n = -5g$, we know the Artin condition is exactly the last condition (3) in the example. This completes the proof. \qed

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