Research Article
Normality of the p-Harmonic and Log-p-Harmonic Mappings

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In this paper, the concepts of p-harmonic mappings and log-p-harmonic mappings in the unit disk have been introduced and
studied by many researchers. We proved the normality of the p-harmonic mappings and log-p-harmonic mappings, which
extend the related results of harmonic mappings of earlier authors.

1. Introduction and Preliminaries
For real-valued harmonic functions defined in \( D \), Lappan [1] established that \( \phi \) is normal if
\[
\sup_{z \in D} \left( 1 - |z|^2 \right) \left| \frac{\text{grad} \phi(z)}{1 + \phi^2(z)} \right| < \infty,
\]
where \( \text{grad} \phi \) is the gradient vector of \( \phi \). In [2], the authors also proved geometric properties of real-valued harmonic
normal functions. Namely, a real-valued harmonic function \( \phi \) with the property
\[
\int_{\Omega} \left( \left| \frac{\text{grad} \phi(z)}{1 + \phi^2(z)} \right| \right)^2 d\Omega < \infty,
\]
is normal.

Recently, many authors considered the properties of the complex-valued harmonic mappings and harmonic quasi-
conformal mappings in [3–13]. We are motivated to establish the topic of normality for complex-valued p-harmonic
mappings and log-p-harmonic mappings defined in the unit disk. An important concept related with normal harmonic
functions is the Bloch function, which was studied by Colonna in [14]. It is a classical result of Lewy [15] that a harmonic
mapping is locally univalent in a domain \( \Omega \) if and only if its Jacobian does not vanish. In terms of the canonical
decomposition, the Jacobian of harmonic mappings \( f = h + ig \) is given by \( J_f = |h'|^2 - |g'|^2 \), and thus, a locally univalent har-
monic mapping in a simply connected domain \( \Omega \) will be sense-preserving if \( |h'| > |g'| \).

Following the above ideas, particularly the definition of Bloch harmonic function given by Colonna [14], we will prove that the polyharmonic mapping \( F \) and log-p-harmonic mapping \( f \) defined in the unit disk \( D \) are normal if they satisfy a Lipschitz type condition. Further, for the complex-valued polyharmonic mappings and log-p-
harmonic mappings, we give out some additional conditions for which are normal. These conditions cannot be omitted. A
2p-times continuously differentiable complex-valued function \( F(z) = u(z) + iv(z) \) in a domain \( D \subseteq \mathbb{C} \) is polyharmonic
mapping or p-harmonic if \( F(z) \) satisfies the p-harmonic equation
\[
\Delta_p F = \Delta (\Delta^{p-1}) F = 0,
\]
where the Laplacian operator
\[
\Delta F = 4 F_{zz} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}.
\]

As we see in Proposition 1 in [16], we know that a mapping \( F \) is polyharmonic in a simply connected domain \( D \subseteq \mathbb{C} \) if and only if \( F \) has the following representation
\[
F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z),
\]
where each $G_{p-k+1}$ is harmonic for $k \in \{1, \ldots, p\}$. When $p = 1$, the mapping $F$ is called harmonic. When $p = 2$, the mapping $F$ is called biharmonic. If $F$ is called log-$p$-harmonic mapping if $\log F$ is $p$-harmonic mapping. When $p = 1$, the mapping $f$ is called log-harmonic. When $p = 2$, the mapping $f$ is called log-biharmonic, which can be regarded as generalizations of holomorphic functions. So we say that $f$ is called log-$p$-harmonic mapping in a simply connected domain $D \subseteq \mathbb{C}$ if and only if $f$ has the form

$$f(z) = \sum_{k=1}^{p} g_{p-k+1}(z) \left|z^{2(k-1)}\right|,$$

where each $g_{p-k+1}$ is log-harmonic for $k \in \{1, \ldots, p\}$.

For a continuously differentiable mapping $f$ in $D$, we define

$$\Lambda_f(z) = \max_{0 \leq |\theta| \leq \pi} |f'(z)| = \max_{0 \leq |\theta| \leq \pi} \left|\frac{d}{d\theta} f(z)\right| = \max_{0 \leq |\theta| \leq \pi} \left|f'(z)\right|,$$

$$\lambda_f(z) = \min_{0 \leq |\theta| \leq \pi} |f'(z)| = \min_{0 \leq |\theta| \leq \pi} \left|\frac{d}{d\theta} f(z)\right| = \min_{0 \leq |\theta| \leq \pi} \left|f'(z)\right|$$

(7)

Recently, many authors considered Landau-type theorems for harmonic mappings, biharmonic mappings, and p-harmonic mappings [16–23]. Li and Wang [24] introduced the log-p-harmonic mappings and derived two versions of Landau-type theorems. However, in virtue of being inspired by these results, we establish the normality of polyharmonic Landau-type theorems. However, in virtue of being inspired by these results, we establish the normality of polyharmonic Landau-type theorems.

2. Necessary Lemmas

In order to derive our main results, we need the following lemmas.

**Lemma 1.** [14] Suppose that $f(z) = h(z) + g(z)$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on $\mathbb{D}$. If $|f(z)| < M$ for all $z \in \mathbb{D}$, then

$$\Lambda_f(z) \leq \frac{4M}{\pi (1 - |z|^2)},$$

(8)

**Lemma 2.** [22] Suppose that $f(z) = h(z) + g(z)$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on $\mathbb{D}$. If $|f(z)| < M$ for all $z \in \mathbb{D}$, then for $|z| = r < 1$, we have

$$|f(z)| \leq \frac{4Mr}{\pi (1 - r)}.$$

(9)

**Lemma 3.** [25] Suppose that $f(z) = h(z) + g(z)$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on $\mathbb{D}$ and $\lambda_f(0) = 1$. If $\Lambda_f(z) \leq \Lambda$ for all $z \in \mathbb{D}$, then

$$|a_n| + |b_n| \leq \frac{\Lambda^2 - 1}{n\Lambda}, \quad n = 2, 3, \ldots.$$ 

(10)

When $\Lambda > 1$, the above estimates are sharp for all $n = 2, 3, \ldots$, with the extremal functions $f_n(z)$ as follows

$$f_n(z) = \Lambda^2 z - \left(\Lambda^3 - \Lambda\right) \int_0^z \frac{1}{1 + z^{n-1}} dz.$$ 

(11)

When $\Lambda = 1$, then $f(z) = a_1 z + b_1 z$ with $|a_1| - |b_1| = 1$.

**Lemma 4.** [19] Suppose that $f(z) = h(z) + g(z)$ is a harmonic mapping of the unit disk $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on $\mathbb{D}$. If $\Lambda_f(z) \leq \Lambda$ for all $z \in \mathbb{D}$, then for each $z \in \mathbb{D}$,

$$|f(z)| \leq \Lambda z.$$ 

(12)

We recall that the chordal distance on the generalized complex plane $\hat{\mathbb{C}}$, which is defined by

$$\chi(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2} \sqrt{1 + |z_2|^2}}; \quad z_1, z_2 \in \mathbb{C},$$

$$\chi(z, \infty) = \frac{1}{\sqrt{1 + |z|^2}}.$$ 

(13)

If $P_{z_1}, P_{z_2}$ are the two points on the Riemann sphere, under stereographic projection, corresponding to $z_1$ and $z_2$, respectively, we have

$$|P_{z_1} - P_{z_2}| = \chi(z_1, z_2).$$

(14)

Therefore,

$$L(\Gamma) \geq \rho(z_1, z_2) \geq \chi(z_1, z_2),$$

(15)

where $\rho(z_1, z_2)$ is the spherical distance of $z_1$ and $z_2$. $\Gamma$ is any rectifiable curve in $\mathbb{C}$ with endpoints $z_1, z_2$, and

$$L(\Gamma) = \int_\Gamma \frac{|du|}{I + |u|^2}.$$

(16)

is the spherical length of $\Gamma$. On the basis of the paper, given $z_1, z_2 \in \mathbb{D}$, $\rho(z_1, z_2)$ denotes the hyperbolic distance between $z_1, z_2$. Therefore, if $\tau$ denotes the hyperbolic geodesic joining $z_1$ to $z_2$, then

$$\rho(z_1, z_2) = \int_\tau \frac{|dv|}{I - |v|^2}.$$ 

(17)

More explicitly,

$$\rho(z_1, z_2) = \frac{1}{2} \log \frac{1 + \lambda}{1 - \lambda}.$$ 

(18)
where
\[
\lambda = \left[ \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right].
\]  
(19)

With these notations, a polyharmonic mapping or log-p-harmonic mapping \( f : \mathbb{D} \to \mathbb{C} \) is called a normal polyharmonic mapping or normal log-p-harmonic mapping, if
\[
\sup_{z_1 \neq z_2} \frac{\chi(f(z_1), f(z_2))}{\rho(z_1, z_2)} < \infty.
\]  
(20)

The following lemma provides an alternative method for deciding when a polyharmonic mapping or log-p-harmonic mapping is normal.

**Lemma 5.** Let \( f(z) \) be a polyharmonic mapping or log-p-harmonic mapping in the unit disk \( \mathbb{D} \), then \( f \) is normal if
\[
\|f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2) \frac{|f(z)| + |f'(z)|}{1 + |f'(z)|} < \infty.
\]  
(21)

**Proof.** Suppose that \( \|f\| < \infty \) and let \( z_1, z_2 \in \mathbb{D} \). If \( \tau : [0, 1] \to \mathbb{D} \) is the hyperbolic geodesic with endpoints \( z_1 \) and \( z_2 \),
\[
\chi(f(z_1), f(z_2)) \leq \int_0^1 \frac{|df(\tau(t))\tau'(t)|}{1 + |f'(\tau(t))|^2} \, dt,
\]  
(22)

where \( df \) stands for the differential of \( f \). From here and (21), we have
\[
\chi(f(z_1), f(z_2)) \leq \int_0^1 \frac{|df(\tau(t))\tau'(t)|}{1 + |f'(\tau(t))|^2} \, dt,
\]  
(23)

Hence, we obtain
\[
\sup_{z_1 \neq z_2} \frac{\chi(f(z_1), f(z_2))}{\rho(z_1, z_2)} < \infty.
\]  
(24)

So it implies that \( f \) is normal.

**3. Main Results and Their Proofs**

In this section, we prove the normality of the polyharmonic mappings and log-p-harmonic mappings as follows.

**Theorem 6.** Let \( F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) \) be a polyharmonic mapping in the unit disk \( \mathbb{D} \) satisfying \( F(0) = \lambda_F(0) - I = 0 \). Suppose that for each \( k \in \{1, \cdots, p\} \), we have
\[
G_{p-k+1}(z) \text{ is harmonic in } \mathbb{D}, \text{ and } G_{p-k+1}(0) = 0; \\
|G_{p-k+1}(z)| \leq M_{p-k+1}, \text{ and } |G_{p-k+1}(z)| \leq \Lambda_p, \text{ where } M_{p-k+1} \geq 0, \Lambda_p \geq 1.
\]  
(25)

Then, \( F \) is normal polyharmonic mapping of the unit disk \( \mathbb{D} \).

**Proof.** We may represent the harmonic functions \( G_{p-k+1}(z) \) in series form as
\[
G_{p-k+1}(z) = \sum_{j=1}^{\infty} a_{j-p-k+1} z^j + \sum_{j=1}^{\infty} b_{j-p-k+1} \bar{z}^j, k \in \{1, 2, \cdots, p\}.
\]

Firstly, we calculate the boundedness of the derivative of \( F \).
\[
|F_z(z) + F_z(z)| \leq |F_z(z)| + |F_z(z)|
\]
\[
= |(G_p)_z(z) + \sum_{k=1}^{p-1} \left[ |z|^{2k} (G_{p-k})(z) + k G_{p-k}(z) z^{k-1} \right]|
\]
\[
+ |(G_p)_z(z) + \sum_{k=1}^{p-1} \left[ |z|^{2k} (G_{p-k})(z) + k G_{p-k}(z) z^{k-1} \right]|
\]
\[
\leq A_1 + A_2 + A_3 + A_4.
\]  
(27)

where
\[
A_1 = \left| (G_p)_z(z) + (G_p)_z(z) \right|,
\]
\[
A_2 = \sum_{k=1}^{p-1} |z|^{2k} \left| (G_{p-k})(z) + (G_{p-k})(z) \right|,
\]
\[
A_3 = \sum_{k=1}^{p-1} k G_{p-k}(z) \left( z^{k-1} \right) \left( z^{k-1} \right),
\]
\[
A_4 = \left| (G_p)_z(z) - (G_p)_z(z) + (G_p)_z(z) - (G_p)_z(z) \right|.
\]  
(28)

By a simple calculation, we have
\[
A_1 \leq \Lambda G_p(0) \leq A_p.
\]  
(29)
Using Lemma 1, we have
\[
A_2 \leq \sum_{k=1}^{p-1} |z|^{2k} \left( |(G_{p-k})_z(z)| + |(G_{p-k})_{\bar{z}}(z)| \right) \leq \sum_{k=1}^{p-1} \frac{2^k 4M_{p-k}}{\pi(1-r^2)}.
\]

By Lemma 2, we have
\[
A_3 \leq \sum_{k=1}^{p-1} |kG_{p-k}(z)| \left( |z^{k-1}z| + |z^{k-1}\bar{z}| \right) \leq \frac{8}{\pi(1-r)} \sum_{k=1}^{p-1} kM_{p-k}r^{2k}.
\]

Using Lemma 3, we have
\[
A_4 \leq \left| (G_{p})_z(z) - (G_{p})_{\bar{z}}(z) \right| + \left| (G_{p})_{\bar{z}}(z) - (G_{p})_z(0) \right| \leq \sum_{n=2}^{\infty} \left( |a_{np}z^n + b_{np}\bar{z}^n| \right)n^{r-1}
\]
\[
\leq \sum_{n=2}^{\infty} \left( \frac{A_p^2 - 1}{A_p} \right) \frac{r^{n-1}}{n^{r-1}} = \frac{A_p^2 - 1}{A_p} \frac{r}{1-r}.
\]

By the above estimates, we obtain the following result
\[
|F_2(z) + F_2(z)| \leq S_1(r),
\]
where
\[
S_1(r) = \frac{4}{\pi(1-r^2)} \sum_{k=1}^{p-1} M_{p-k}r^{2k} + \frac{8}{\pi(1-r)} \sum_{k=1}^{p-1} kM_{p-k}r^{2k}
\]
\[
+ \left( \frac{A_p^2 - 1}{A_p} \right) \frac{r}{1-r} + A_p.
\]

Now, differentiating \( S_1(r) \), we have
\[
S_1'(r) = \sum_{k=1}^{p-1} \frac{4M_{p-k} [2kr^{2k-1}(1-r^2) + 2r^{2k+1}]}{\pi(1-r)^2} + \sum_{k=1}^{p-1} \frac{8kM_{p-k} [2kr^{2k+1} - r^{2k}]}{\pi(1-r)^2} + \frac{A_p^2 - 1}{A_p} \frac{r^2}{(1-r)^2}.
\]

In view of \( A_p \geq 1 \) and \( r \in (0,1) \), after a simple calculation, it shows that \( S_1'(r) > 0 \). It is simple to verify that \( S_1(r) \) is strictly increasing in \( (0,1) \).

\[
\lim_{r \to 0^+} S_1(r) = A_p, \quad \lim_{r \to 0^+} S_1(r) = +\infty.
\]

Obviously, \( S_1(r) \) has only one pole \( r = 1(\{|z| = r\}) \). In other words, \( S_1(r) \) is bounded in the interval \((0,1)\).

Finally, we consider the boundedness of \( F \) for any \( |z| = r_1 \), then we have
\[
|F(z)| = \sum_{n=1}^{\infty} \left( a_{np}z^n + b_{np}\bar{z}^n \right) + \sum_{k=1}^{p-1} |z|^{2k}G_{p-k}(z)
\]
\[
\leq |a_{1p}z + b_{1p}\bar{z}| + \sum_{n=2}^{\infty} \left( a_{np}z^n + b_{np}\bar{z}^n \right) + \frac{A_p^2 - 1}{A_p} \frac{r^n}{1-r_1^n}
\]
\[
+ \frac{4r_1}{\pi(1-r_1)} \sum_{k=1}^{p-1} M_{p-k}r_1^{2k}.
\]

So \( |F(z)| \leq S_2(r_1) \), where
\[
S_2(r_1) = A_p r_1 - \frac{A_p^2 - 1}{A_p} \frac{r_1 + \ln (1-r_1)}{1-r_1} + \frac{4r_1}{\pi(1-r_1)} \sum_{k=1}^{p-1} M_{p-k}r_1^{2k}.
\]

By the similar approach for differentiating \( S_2(r_1) \), we have the following one
\[
S_2'(r_1) = \sum_{k=1}^{p-1} \frac{4M_{p-k} [2(k+1)r^{2k+1}(1-r_1) + r^{2k+1}]}{\pi(1-r_1)^2} + \frac{A_p^2 - 1}{A_p} \frac{r^{2k+1}}{(1-r_1)^2}.
\]

By elementary calculations, we get \( S_2'(r_1) > 0 \). It implies that \( S_2(r_1) \) is increasing for \( A_p \geq 1 \) and \( r_1 \in (0,1) \). It is simple to verify that \( S_2(r_1) \) is bounded in \((0,1)\). Combined with Lemma 5 and Estimation (33), we conclude ultimately that \( F \) is normal polyharmonic mapping in the unit disk \( \mathbb{D} \). The proof of this theorem is complete.

**Theorem 7.** Let \( F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) \), be a polyharmonic mapping of \( D \) satisfying \( F(0) = \lambda_p(0) - 1 = 0 \). Suppose that for \( k \in \{1,2,\cdots, p\} \), we have
\[
G_{p-k+1}(z) \text{ is harmonic in } \mathbb{D}, \text{ and } G_{p-k+1}(0) = 0; \quad \lambda_{G_{p-k+1}}(z) \leq \lambda_{p-k+1} \text{ for all } z \in \mathbb{D}, \text{ where } \lambda_{p-k+1} \geq 0, k = 2,3,\cdots, p, \text{ and } A_p \geq 1.
\]

Then, \( F \) is normal polyharmonic mapping in the unit disk \( \mathbb{D} \).
Proof. We represent the harmonic functions $G_{p-k+1}(z)$ in series form as

$$G_{p-k+1}(z) = \sum_{j=1}^{\infty} a_{j,p-k+1} z^j + \sum_{j=1}^{\infty} b_{j,p-k+1} \bar{z}^j,$$

for each $k \in \{1, 2, \cdots, p\}$, and

$$|a_{1,p}| + |b_{1,p}| = \Lambda_F(0) = |(G_p)_z(0)| + |(G_{p-1})_z(0)| = \Lambda_{G_p}(0) \leq \Lambda_p,$$

$$\|a_{1,p}\| = |b_{1,p}| = \lambda_F(0) = \|(G_p)_z(0)\| - \|(G_{p-1})_z(0)\| = \lambda_{G_p}(0) = 1$$

To prove the normality of $F$, we first determine that the derivative of $F$ is bounded in $D$. Then, as in the proof of Theorem 6, we have

$$|F_z(z) + F_{\bar{z}}(z)| \leq B_1 + B_2 + B_3 + B_4,$$

where

$$B_1 \leq |(G_p)_z(0)| + |(G_{p-1})_z(0)| = \Lambda_{G_p}(0) \leq \Lambda_p,$$

$$B_2 \leq \sum_{k=0}^{p-1} 2^k \Lambda_{G_p-k}(z) \leq \sum_{k=0}^{p-1} 2^k \Lambda_{p-k} r^{2k}.$$

By the condition (2) of Theorem 7 and Lemma 4, we have

$$B_3 \leq \sum_{k=1}^{p-1} |G_{p-k}(z)| (|z^k z^{k-1}| + |z^{k-1} z^k|) \leq \sum_{k=1}^{p-1} 2k \Lambda_{p-k} r^{2k}.$$

Applying Lemma 3, we have

$$B_4 \leq \sum_{n=2}^{\infty} n |a_{n,p}| + |b_{n,p}| \leq \frac{\Lambda_p^2 - 1}{\Lambda_p} \sum_{n=2}^{\infty} r^{n-1}$$

By above estimates, we obtain that

$$|F_z(z) + F_{\bar{z}}(z)| \leq \sum_{k=1}^{p-1} \Lambda_{p-k} r^{2k} + \sum_{k=1}^{p-1} 2k \Lambda_{p-k} r^{2k}$$

$$+ \frac{\Lambda_p^2 - 1}{\Lambda_p} \frac{r}{1-r} + \Lambda_p$$

$$= \sum_{k=1}^{p-1} (2k + 1) \Lambda_{p-k} r^{2k} + \frac{\Lambda_p^2 - 1}{\Lambda_p} \frac{r}{1-r} + \Lambda_p.$$

Set

$$S_3(r) = \sum_{k=1}^{p-1} (2k + 1) \Lambda_{p-k} r^{2k} + \frac{\Lambda_p^2 - 1}{\Lambda_p} \frac{r}{1-r} + \Lambda_p.$$

For differentiating $S_3(r)$, we have

$$S_3'(r) = \sum_{k=1}^{p-1} 2k(2k + 1) \Lambda_{p-k} r^{2k-1} + \frac{\Lambda_p^2 - 1}{\Lambda_p} \frac{r}{(1-r)^2}.$$

By a simple calculation, we obtain $S_3'(r) > 0$. It shows that $S_3(r)$ is strictly increasing in $r \in (0, 1)$, and

$$\lim_{r \to 0} S_3(r) = \Lambda_p, \lim_{r \to 1} S_3(r) = +\infty.$$

This shows that $S_3(r)$ has only one pole $r = 1$, and it says that $S_3(r)$ is bounded in $(0, 1)$. So $|F_z(z) + F_{\bar{z}}(z)|$ is a finite value in the unit disk $D$.

Now, we consider any $z$ with $|z| = r_2$. Then, we have

$$|F(z)| = \sum_{n=1}^{\infty} (a_{n,p} z^n + b_{n,p} \bar{z}^n) + \sum_{n=2}^{\infty} |z|^{2k} G_{p-k}(z)$$

$$\leq |a_{1,p} z + b_{1,p} \bar{z}| + \sum_{n=2}^{\infty} (a_{n,p} z^n + b_{n,p} \bar{z}^n)$$

$$+ \sum_{n=2}^{\infty} |z|^{2k} G_{p-k}(z) + A_p r_2 + \frac{\Lambda_p^2 - 1}{\Lambda_p} \sum_{n=2}^{\infty} \frac{r_2^n}{n}$$

$$+ \sum_{k=1}^{p-1} \Lambda_{p-k} r_2^{2k+1} = \sum_{k=1}^{p-1} \Lambda_{p-k} r_2^{2k+1}$$

$$- \frac{\Lambda_p^2 - 1}{\Lambda_p} [r_2 + \ln (1 - r_2)] + A_p r_2 \cdot S_4(r_2)$$

Differentiating $S_4(r)$, we have

$$S_4'(r_2) = \sum_{k=1}^{p-1} (2k + 1) \Lambda_{p-k} r_2^{2k} + \frac{\Lambda_p^2 - 1}{\Lambda_p} \frac{r}{(1-r)^2} + 1$$

After elementary calculations, we have that $S_4'(r_2) > 0$. It implies that $S_4(r_2)$ is strictly increasing. It is simple to verify that $S_4(r_2)$ is finite for $r_2 \in (0, 1)$. It says that $F$ is bounded in $D$. Using these above estimates and Condition (21), we
conclude that $F$ is normal polyharmonic mapping in $\mathbb{D}$. This proof of Theorem 7 is complete.

Finally, we establish the normality of log-p-harmonic mappings as follows.

**Theorem 8.** Let $f(z) = \prod_{k=1}^{p} [g_{p-k+1}(z)]^{z_{k+1}}$ be a log-p-harmonic mapping in the unit disk $\mathbb{D}$ satisfying $f(0) = g_0(0) = \lambda_f(0) = 1$. Suppose that for each $k \in \{1, \cdots, p\}$ we have

$$g_{p-k+1}(z) \text{ is log-harmonic in } \mathbb{D},$$

$$|g_{p-k+1}(z)| \leq M_{p-k+1}, \quad \text{let } G_p = \log g_p, \quad A_{G_p} \leq A_p,$$  

$$\text{where } M_{p-k+1} \geq 1, \quad A_p \geq 1.$$  

Then, $f$ is a normal log-p-harmonic mapping in the unit disk $\mathbb{D}$.

*Proof. Let $F(z) = \sum_{k=1}^{p} z^{k-1}G_{p-k+1}(z)$, for each $k \in \{1, \cdots, p\}$. We may represent the harmonic functions $G_{p-k+1}(z) = \log g_{p-k+1}$ in series form as

$$G_{p-k+1}(z) = \sum_{n=1}^{\infty} a_{jk} z^n + \sum_{n=1}^{\infty} b_{jk} z^n.$$  

Then, $F = \log f$ is a polyharmonic mapping in $\mathbb{D}$. We know that

$$\lambda_f(0) = ||f(z)|| - |f(0)|| = |f(0)||[F(z)] - |F(z)||,$$  

and $f(0) = 1$, so it follows from $g_p(0) = \lambda_f(0) = 1$, we have $G_p(0) = A_p(0) = 1 = 0$.

Obviously,

$$|G_{p-k+1}| = |\log g_{p-k+1}| = |\log |g_{p-k+1}| + \pi, \quad \text{so we have}$$

$$|G_{p-k+1}| \leq \log M_{p-k+1} + \pi = M_{p-k+1}.$$  

In order to prove the normality of $f$, it follows from Theorem 6 that we have

$$|F_z(z) + F_z(z)| \leq S_6(r).$$  

Finally, we consider any $z$ with $|z| = r_3$; then, we have

$$\log f(z) = |F(z)| = \sum_{n=1}^{\infty} (a_{np} z^n + b_{np} z^n) + \sum_{k=1}^{p-1} |F_{z}(z)|,$$  

$$\leq |a_{np} z + b_{np} z| + \sum_{n=1}^{\infty} (a_{np} z^n + b_{np} z^n), \quad \text{so we obtain}$$

$$|f_z(z) + f_z(z)| \leq S_6(r).$$  

Differentiating $S_6(r)$, we have

$$S_6'(r) = \sum_{k=1}^{p} \frac{4M_{p-k}[2k r^{k-1}(1-r^2) + 2 r^{k+1}]}{\pi(1-r)^2} + \frac{8}{\pi(1-r)} \sum_{k=1}^{p-1} k M_{p-k} r^{2k}$$

$$+ \frac{A_p^2 - 1}{A_p} r + A_p.$$  

By elementary calculations, we get $S_6'(r) > 0$. It follows that $S_6(r)$ is strictly increasing in $(0, 1)$, and

$$\lim_{r \to 0} S_6(r) = A_p, \quad \lim_{r \to 1} S_6(r) = +\infty.$$  

It implies that $S_6(r)$ is bounded in $(0, 1)$. Hence, there exists a finite value $m_1$ such that

$$|f_z(z) + f_z(z)| \leq m_1.$$  

Finally, we consider any $z$ with $|z| = r_3$; then, we have

$$\log f(z) = |F(z)| = \sum_{n=1}^{\infty} \left( a_{np} z^n + b_{np} z^n \right) + \sum_{k=1}^{p-1} |F_{z}(z)|,$$  

$$\leq |a_{np} z + b_{np} z| + \sum_{n=1}^{\infty} \left( a_{np} z^n + b_{np} z^n \right), \quad \text{so we obtain}$$

$$|f_z(z) + f_z(z)| \leq m_1.$$  

Differentiating $S_6(r_3)$, we have the following result

$$S_6'(r_3) = \sum_{k=1}^{p} \frac{4M_{p-k}[2k r^{k-1}(1-r^2) + 2 r^{k+1}]}{\pi(1-r)^2} + \frac{8}{\pi(1-r)} \sum_{k=1}^{p-1} k M_{p-k} r^{2k}$$

$$+ \frac{A_p^2 - 1}{A_p} r + A_p.$$  

It is not difficult to verify that $S_6'(r_3) > 0$, which means that $S_6(r_3)$ is strictly increasing. It is also easily seen that $S_6(r_3)$ is bounded in $(0, 1)$. Hence, there is a finite value $m_2$ such that
\[ | \log f(z) | \leq m_2. \]  

(66)

Applying Lemma 5 and (63), (66), we obtain \( f \) is normal in \( \mathbb{D} \). The proof of Theorem 8 is complete.

**Data Availability**

The data used to support the findings of this study are included with the article.

**Conflicts of Interest**

We declare that we have no competing interests.

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**References**

[1] P. Lappan, “Some results on harmonic normal functions,” *Mathematische Zeitschrift*, vol. 90, no. 2, pp. 155–159, 1965.
[2] R. Aulaskari and P. Lappan, “An integral condition for harmonic normal functions,” *Complex Variables, Theory and Application: An International Journal*, vol. 23, no. 3-4, pp. 213–219, 1993.
[3] X. Chen and A. Fang, “A schwarcz–pick inequality for harmonic quasiconformal mappings and its applications,” *Journal of Mathematical Analysis and Applications*, vol. 369, no. 1, pp. 22–28, 2010.
[4] X. Chen and T. Qian, “Estimation of hyperbolically partial derivatives of \( \rho \)-harmonic quasiconformal mappings and its applications,” *Complex Variables and Elliptic Equations*, vol. 60, no. 6, pp. 875–892, 2015.
[5] X. Chen and Y. Que, “Quasiconformal extensions of harmonic mappings with a complex parameter,” *Journal of the Australian Mathematical Society*, vol. 102, no. 3, pp. 307–315, 2017.
[6] S. Chen and S. Ponnusamy, “John Disks and \( k \)-quasiconformal harmonic mappings,” *The Journal of Geometric Analysis*, vol. 27, no. 2, pp. 1468–1488, 2017.
[7] S. Chen, S. Ponnusamy, and X. Wang, “Recent results on harmonic and \( p \)-harmonic mappings,” *Journal of Analysis*, vol. 18, pp. 99–128, 2011.
[8] S. Chen and J. Zhu, “Schwarz type Lemmas and a Landau type theorem of functions satisfying the biharmonic equation,” *Bulletin des Sciences Mathématiques*, vol. 154, pp. 36–63, 2019.
[9] Z. Liu and S. Ponnusamy, “Bohr radius for subordination and \( k \)-quasiconformal harmonic mappings,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 5, pp. 2151–2168, 2019.
[10] Z. Liu and S. Ponnusamy, “Some properties of univalent log-harmonic mappings,” *Filomat*, vol. 32, no. 15, pp. 5275–5288, 2018.
[11] R. Hernández and M. J. Martin, “Stable geometric properties of analytic and harmonic functions,” *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 155, no. 2, pp. 343–359, 2013.
[12] S. M. S. Kanas, S. Maharana, and J. K. Prajapat, “Norm of the pre-Schwarzian derivative, Bloch’s constant and coefficient bounds in some classes of harmonic mappings,” *Journal of Mathematical Analysis and Applications*, vol. 474, no. 2, pp. 931–943, 2019.
[13] J. K. P. S. Ponnusamy and A. S. Kaliraj, “Uniformly starlike and uniformly convex harmonic, mappings,” *Journal of Analysis*, vol. 23, pp. 121–129, 2015.
[14] F. Colonna, “The Bloch constant of bounded harmonic mappings,” *Indiana University Mathematics Journal*, vol. 38, no. 4, pp. 829–840, 1989.
[15] H. Lewy, “On the non-vanishing of the Jacobian in certain one-to-one mappings,” *Bulletin of the American Mathematical Society*, vol. 42, no. 10, pp. 689–693, 1936.
[16] S. Chen, S. Ponnusamy, and X. Wang, “Bloch constant and Landau’s theorem for planar \( p \)-harmonic mappings,” *Journal of Mathematical Analysis and Applications*, vol. 373, no. 1, pp. 102–110, 2011.
[17] M. Dorff and M. Nowak, “Landau’s theorem for planar harmonic mappings,” *Computational Methods and Function Theory*, vol. 4, no. 1, pp. 151–158, 2004.
[18] M.-S. Liu, “Landau’s theorems for biharmonic mappings,” *Complex Variables and Elliptic Equations*, vol. 53, no. 9, pp. 843–855, 2008.
[19] X. Bai and M. Liu, “Landau-type theorems of polyharmonic mappings and log-\( p \)-harmonic mappings,” *Complex Analysis and Operator Theory*, vol. 13, no. 2, pp. 321–340, 2019.
[20] Y. Jiang, Z. Liu, and S. Ponnusamy, “Univalent harmonic mappings and lift to the minimal surfaces,” *Lobachevskii Journal of Mathematics*, vol. 40, no. 9, pp. 1295–1312, 2019.
[21] M.-S. Liu, “Landau’s theorem for planar harmonic mappings,” *Computers & Mathematics with Applications*, vol. 57, no. 7, pp. 1142–1146, 2009.
[22] M. Liu and Z. Liu, “Landau-type theorems for \( p \)-harmonic mappings or log-\( p \)-harmonic mappings,” *Applicable Analysis*, vol. 93, no. 11, pp. 2462–2477, 2014.
[23] Z. Mao, S. Ponnusamy, and X. Wang, “Schwarzian derivative and Landau’s theorem for logharmonic mappings,” *Complex Variables and Elliptic Equations*, vol. 58, no. 8, pp. 1093–1107, 2013.
[24] P. Li and X. Wang, “Landau’s theorem for log-\( p \)-harmonic mappings,” *Applied Mathematics and Computation*, vol. 218, pp. 4806–4812, 2012.
[25] M. Liu, “Estimates on Bloch constants for planar harmonic mappings,” *Science in China Series A: Mathematics*, vol. 52, no. 1, pp. 87–93, 2009.