Energy spectra and quantized Hall conductance in a 2D lattice subjected to light irradiation

Sukriti Sarkar¹, Moumita Dey¹ and Santanu K. Maiti²
¹Department of Physics, Adamas University, Barasat-Barrackpore Road, Kolkata-700 126, West Bengal, India
²Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 Barrackpore Trunk Road, Kolkata-700 108, India
sukritisarkar62@gmail.com; moumita.dey@adamasuniversity.ac.in; sanatnu.maiti@isical.ac.in

Abstract. We make an in-depth analysis of the phenomenon of integer quantum Hall effect, considering a two-dimensional (2D) square lattice in presence of light irradiation. Following the well known Landauer-Buttiker multi-probe formalism we compute transverse conductance in a four-probe setup where the irradiation effect is incorporated through a minimal coupling scheme based on the Floquet-Bloch approach. The energy spectrum gets significantly modified with the irradiation which thus directly affects the quantum Hall plateaus. Moreover, we also put an emphasis on how to get an effective one-dimensional Harper equation from the 2D Hall system. Several characteristic features emerge from our analysis that might be interesting and important as well.

1. Introduction

The phenomenon of Integer Quantum Hall Effect (IQHE) was one of the greatest discoveries of twentieth century. The Hall conductance, measured by Klaus von Klitzing [1] in a Si based MOSFET at very high magnetic field shows ‘sharp’ quantized behavior in units of \( \frac{e^2}{h} \), where \( e \) and \( h \) are the fundamental constants. For this remarkable discovery, Klitzing was awarded Nobel prize in Physics in 1985. It is well known that for a ‘ballistic conductor’ the conductance is quantized in units of \( \frac{e^2}{h} \), where electrons transfer without momentum relaxation, but for the Hall system, the unprecedented quantized nature is obtained even in the presence of impurity [2]. The accuracy level is specified to nearly one part per billion and because of this exact quantization it allows us to define the practical standard of electrical resistance [3].

The underlying physical mechanism of this phenomenon relies on the complete suppression of backscattering, and thus momentum relaxation processes when the Fermi energy resides within the Landau levels. At high magnetic fields, the states carrying currents in opposite directions are spatially separated in a large extent towards the two edges of the sample which thus hardly allow any overlap between these two types of states. Therefore, a complete elimination of momentum relaxation processes occurs in the presence of impurity [2,4].
An enormous amount of work has been done to explain the phenomenon of IQHE, and almost all these works are involved in the *continuum model* [5-8] based on free electron theory, since electron-electron interaction is no longer required to analyze it. But, the description of this quantum phenomenon in a *lattice model* is highly limited which sometimes provide a deeper insight to the phenomenon. Moreover, utilizing the robust Landauer-Buttiker prescription [2] we can easily determine the Hall conductance (or resistance) in a 4-probe setup (see Fig. 1), even in presence of impurities. Solving a quantum mechanical problem with impurities in the continuum model is always difficult, whereas a tight-binding (TB) prescription can easily manage it. In one of our early attempts [4], we have analyzed IQHE in a lattice model where Hall and Longitudinal resistances were measured using the four probe setup and, in the present work we extend it in presence of light irradiation which might explore several interesting features as it directly modifies the phase of electrons. The irradiation significantly affects the band structure i.e., Landau levels and thus Hall plateaus between these levels.

Our motivation behind this work is two-fold: (i) spectral analysis in presence of irradiation and (ii) quantized Hall conductances associated with it. A TB framework is given to illustrate the four-probe setup where the effect of light is incorporated through the minimal coupling scheme following the standard Floquet-Bloch approach [9-11]. The Hall conductance is computed using the Landauer-Buttiker prescription based on Green’s function technique [2,4].

The rest part of the paper is organized as follows. In Sec. 2, we present the quantum Hall system and give a detailed theoretical description, for the benefit of general readers to calculate the Hall conductance. All the results are presented and thoroughly analyzed in Sec. 3, and finally, we conclude in Sec. 4.

2. Quantum Hall system and theoretical framework

2.1. Quantum Hall system and tight-binding Hamiltonian

Let us start with the Hall setup shown in Fig.1 where a 2D lattice is attached to four contact leads. Applying a suitable bias between the leads 1 and 4, a finite current is allowed to pass through the sample along the X direction. Now, as long as a magnetic field is applied directed perpendicular to the sample, a non-zero voltage, called the Hall voltage is developed along the Y direction. This Hall

![Figure 1: Schematic illustration of the quantum Hall setup in presence of irradiation.](image)
voltage is measured by the other two leads connected at the two opposite sides of the system. The 2D lattice is finally irradiated and in our work, we essentially focus on its role in (i) band structure and (ii) quantized conductance.

The Hamiltonian of the full system is expressed as a sum

\[ H = H_{\text{sample}} + H_{\text{leads}} + H_{\text{coupling}} \]  

where \( H_{\text{sample}} \), \( H_{\text{leads}} \) and \( H_{\text{coupling}} \) are sub-Hamiltonians associated with the 2D conductor, side attached leads, and the coupling between the conductor and the leads. All these Hamiltonians are described within a TB framework in non-interacting electron picture.

In the absence of any light irradiation the TB Hamiltonian of the square lattice subjected to the uniform transverse magnetic field is written as,

\[ H_{\text{sample}} = \sum_{m,n} \epsilon_{m,n} c_{m,n}^\dagger c_{m,n} + \sum_{m,n} t \left( c_{m+1,n}^\dagger c_{m,n} + c_{m,n+1}^\dagger c_{m,n} + \text{h.c.} \right) \]

where, \( t \) is the nearest neighbour hopping (NNH) integral, which is assumed to be isotropic, both along the X and Y directions. \( \epsilon_{m,n} \) denotes the site energy of an electron at the lattice site \((m,n)\), and \( c_{m,n}^\dagger \), \( c_{m,n} \) are the conventional Fermionic operators. \( \theta_m = 2\pi m B a^2 \) \( (a \text{ being the lattice spacing}) \) is the phase factor which appears in the Hamiltonian due to the hopping of electrons along Y direction in presence of the B field. It is commonly known as the Aharonov-Bohm (AB) phase. The occurrence of the AB phase in the hopping term depends on the specific choice of the gauge associated with the vector potential, though the results are independent of the choice of the gauge.

The effect of irradiation is incorporated through the vector potential \( A(\tau) \) by a minimal coupling scheme following the Floquet-Bloch approximation. For a homogeneous electric field we have

\[ A(\tau) = \left( A_s \sin(\omega \tau), A_s \sin(\omega \tau + \phi) \right) \]

where \( A_s \) and \( A_s \) are the amplitudes, and \( \phi \) is the phase factor and \( \omega \) represents the frequency. Due to this irradiation, NNH integrals get modified as [9-11]

\[ t \frac{1}{3} \int_{0}^{3} e^{i \omega \tau} e^{i \vec{A} \cdot \vec{d}_{j}} d\tau \]

where, \( \vec{d}_{j} \) is the unit vector joining the neighbouring sites \( j \) to \( l \), and \( \Xi \) is the time period.

The side attached leads are assumed to be reflectionless, semi-infinite square lattice ribbons. The widths of these leads are considered to be half of the conductor. A similar kind of TB Hamiltonian is used for these contact leads, apart from the phase factors. We refer the site energy and NNH integrals by the parameters \( \epsilon_{0} \) and \( t_{0} \) respectively. The coupling of the leads to the sample is described by the physical parameter \( t_{q} \) \( (q = 1, 2, 3, 4) \).

2.2. Theoretical framework

To calculate Hall resistance and thus Hall conductance, we use Landauer-Buttiker multi-probe formalism [2,4], where all the leads (current and voltage leads) are considered in identical footing. In this prescription, the current through any lead \( p \) (say) is written as

\[ I_{p} = \sum_{q=1}^{n} G_{pq} (V_{p} - V_{q}) \]
where, \( G_{pq} \) represents the electrical conductance. \( V_p \) and \( V_q \) are the applied biases in the leads \( p \) and \( q \) respectively. In terms of the transmission probability \( T_{pq} \) of an electron through the conductor, \( G_{pq} \) is expressed as [2]

\[
G_{pq} = \frac{e^2}{h} T_{pq}.
\]  

(5)

Combining the current expressions of four distinct leads we get the following matrix equation

\[
\begin{pmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4
\end{pmatrix} =
\begin{pmatrix}
G_{12} + G_{13} + G_{14} & -G_{12} & -G_{13} & -G_{14} \\
-G_{21} & G_{21} + G_{23} + G_{24} & -G_{23} & -G_{24} \\
-G_{31} & -G_{32} & G_{31} + G_{32} + G_{34} & -G_{34} \\
-G_{41} & -G_{42} & -G_{43} & G_{41} + G_{42} + G_{43}
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{pmatrix}
\]  

(6)

Without loss of any generality, we set \( V_4 = 0 \), as the current essentially depends on the bias difference. Thus the above matrix equation reduces to the following form

\[
\begin{pmatrix}
I_1 \\
I_2 \\
I_3
\end{pmatrix} =
\begin{pmatrix}
G_{12} + G_{13} + G_{14} & -G_{12} & -G_{13} & -G_{14} \\
-G_{21} & G_{21} + G_{23} + G_{24} & -G_{23} & -G_{24} \\
-G_{31} & -G_{32} & G_{31} + G_{32} + G_{34} & -G_{34}
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
V_3
\end{pmatrix}
\]  

(7)

Inverting Eq. 7 we get the voltages in terms of the current as

\[
\begin{pmatrix}
V_1 \\
V_2 \\
V_3
\end{pmatrix} =
\begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix}
\begin{pmatrix}
I_1 \\
I_2 \\
I_3
\end{pmatrix}
\]  

(8)

As, lead 2 and lead 3 are the voltage probes no current will pass through them, i.e., \( I_2 = I_3 = 0 \), which yields, \( V_2 = I_1 R_{21} \) and \( V_3 = I_1 R_{31} \). From these relations we obtain the expressions of Hall resistance as

\[
R_H = \frac{V_2 - V_3}{I_1} = R_{21} - R_{31}.
\]  

(9)

Knowing \( R_H \), we can easily determine the Hall conductance \( G_H \).

**Brief description for the calculation of** \( T_{pq} \): We calculate transmission probability \( T_{pq} \) by using Green’s function formulation [2], the most robust approach compared to the other available methods. Here the effects of the contact leads are incorporated through self-energy corrections. In terms of the self-energies \( \Sigma_p \) (\( p = 1, 2, 3, 4 \)), the effective Green’s function of the sample reads as [2]

\[
G' = G_a^+ = \left( E - H_{\text{sample}} - \Sigma_1 - \Sigma_2 - \Sigma_3 - \Sigma_4 \right)^{-1}
\]  

(10)

Using \( G' \) and \( G_a \) we compute \( T_{pq} \) from the relation given below [2]

\[
T_{pq} = \text{Tr}\left[ \Gamma_p G' \Gamma_q G_a \right]
\]  

(11)

where \( \Gamma_p \)’s are the coupling matrices and they are defined by [2]

\[
\Gamma_p = -2 \text{Im}\left[ \Sigma_p \right].
\]  

(12)

3. Results and discussions
For all the numerical calculations we set the site energies in the 2D lattice and side attached electrodes to 0, for the sake of simplification and all kinds of hopping strengths to 1 eV. The results are performed for a square lattice having size $20 \times 20$, keeping the system temperature to absolute zero. The effect of irradiation is incorporated in the high frequency limit i.e., $\hbar\omega >> t$. Under this condition, the lowest order term of Eq. 3 eventually contributes. The central focus of our analysis is to examine the critical roles played by the irradiation on (i) the density of states (DOS) and (ii) the quantized Hall conductance.

Figure 2: DOS as a function of energy $E$ in the (a) absence and (b) presence ($A_x = A_y = 1.5$) of light irradiation. Here we set, $\phi = 0$ and $B = 0.1$.

Let us start our discussion with DOS spectrum. We determine the DOS using the following relation

$$\rho(E) = -\frac{1}{N^2 \pi} \text{Im} \left[ \text{Tr} \left( G' \right) \right]$$

The results are presented in Fig. 2, where two different cases are shown depending on the parameters $A_x$ and $A_y$. In (a) we set $A_x = A_y = 0$ (no irradiation), while in (b) we take $A_x = A_y = 1.5$. Comparing the spectra, a strong dependence of irradiation is clearly observed. Now, we analyze the appearance of distinct bands in the DOS spectra. For the chosen gauge of the vector potential associated with the transverse magnetic field, the translational invariance is lost along the $X$ direction. Therefore, we can write the wave amplitude at any arbitrary lattice site as

$$\psi_{m,n} = u_m e^{i k_y n a}$$

where, $k_y$ is the wave vector along $Y$ axis. Using this form, we get an effective difference equation doing some straightforward calculation as

$$\left[ E - \left\{ \varepsilon + 2t \cos \left( \frac{2\pi m \phi}{\phi_0} + k_y a \right) \right\} \right] u_m = t (u_{m+1} + u_{m-1})$$

This is the well-known Harper equation [12]. Setting $b = \frac{\phi}{\phi_0}$ in the above relation, we get

$$\left[ E - \left\{ \varepsilon + 2t \cos \left( 2\pi b m + k_y a \right) \right\} \right] u_m = t (u_{m+1} + u_{m-1})$$

It looks like that we eventually reach to an effective 1D system. The wave vector $k_y$ has some quantized values and that can be obtained by knowing the lattice number along $Y$. For each value of $k_y$, we have one discrete 1D chain, and therefore the energy eigenvalues of the parent 2D lattice can easily be evaluated from these individual decoupled chains.
Depending on $b$ (rational or irrational), we get a periodic or deterministic disordered system. Here we focus only on the rational values of $b$. To make the system periodic, we need to choose $b (= 1/q, \text{say})$ in such a way that the number of lattice sites along X, which is $N$ in our analysis, is exactly divisible by the factor $q$. Under this condition, $q$ distinct bands appear, providing identical degeneracy (as expected for the Landau bands in the continuum model). The degeneracy factor is given by $D = \frac{N^2}{q}$.

Since we set, $B = 0.1$ i.e., $q = 10$, we get ten distinct bands in the DOS spectrum. For each of these bands $D = 40$.

With the inclusion of irradiation, the NNH integrals are modified and therefore the band structures get changed. The allowed energy window decreases in large extent providing smaller gaps between the bands. But the fact is that, the number of such bands and the degeneracy factor remain unchanged, compared to the irradiation free sample.

Figure 3 displays the behavior of Hall conductance $G_H$ as a function of Fermi energy $E_F$ for both the two cases depending on the irradiation parameters $A_x$ and $A_y$. In each of these two cases, Hall conductance shows sharp quantized values in units of $\frac{e^2}{h}$. Whenever the Fermi energy crosses one Landau band, the contribution towards the conductance becomes exactly equals to $\frac{e^2}{h}$ and hence from $\nu$ such bands the net contribution is $\nu \frac{e^2}{h}$. For the irradiated system, the widths of the Hall plateaus are relatively small than the irradiation free system as in the later case the Landau levels are closely spaced due to the modification of NNH integrals.

4. Conclusion

In this work, we have investigated the phenomenon of IQHE in a 2D lattice model, in the presence of light irradiation. The effect of light has been included in the minimal coupling scheme following the Floquet-Bloch ansatz. For the benefit of the readers we have discussed detailed theoretical prescription for the calculation of Hall conductance in a four-probe junction setup using the well known Landauer-Buttiker multi-probe methodology. All the results have also been thoroughly discussed. The pronounced effect of light has been clearly reflected. Further analysis can be done by considering
different combinations of the light parameters ($A_x$, $A_y$ and $\varphi$) and we hope various non-trivial features may appear in the Hall conductance.

Acknowledgement

The financial assistance of DST-SERB, Government of India, is thankfully acknowledged for carrying out the research works under the Project Grant Number EMR/2017/000504.

References

[1] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980).
[2] S. Datta, *Electronic transport in mesoscopic systems*, Cambridge University Press, Cambridge (1995).
[3] A. Hartland, Metrologia 29, 175 (1992).
[4] S. K. Maiti, M. Dey, and S. N. Karmakar, Phys. Lett. A 376, 1366 (2012).
[5] R. B. Laughlin, Phys. Rev. B 23, 5632 (1981).
[6] R. E. Prange, Phys. Rev. B 23, 4802 (1981).
[7] K. von Klitzing, Rev. Mod. Phys. 58, 519 (1986).
[8] J. E. Avron and R. Seiler, Phys. Rev. Lett. 54, 259 (1985).
[9] A. Gomez-Leon and G. Platero, Phys. Rev. Lett. 110, 200403 (2013).
[10] F. Gallego-Marcos and G. Platero, Phys. Rev. B 95, 075301 (2017).
[11] P. Delplace, A. Gomez-Leon, and G. Platero, Phys. Rev. B 88, 245422 (2013).
[12] Y. E. Kraus, Y. Lahini, Z. Ringel, M. Verbin, and O. Zilberberg, Phys. Rev. Lett. 109, 106402 (2012).