Angular analysis of $B$ decaying into $J/\psi$ Tensor, $J/\psi$ Vector and $J/\psi$ Scalar modes

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Abstract

The analysis of $B \to J/\psi K^*_2(1430)$ decay mode is complicated by the fact that close to the $J^{PC} = 2^{++}$ meson $K^*_2(1430)$, there lie other $J^{PC} = 1^{--}$ and $J^{PC} = 0^{++}$ resonances, $K^*(1410)$ and $K^*_0(1430)$ respectively. We show how an angular analysis can be used to isolate the contributions from the different resonances and partial waves contributing to the final state $B \to J/\psi K_X$, where $K_X$ could be any of the resonance $K^*_2(1430)$, $K^*(1410)$ or $K^*_0(1430)$. For this purpose we study the time integrated differential decay rate. We also construct a time dependent angular asymmetry that enables a clean measurement of the mixing phase $\beta$ in the mode $B \to J/\psi K^*_2(1430)$ alone, without contributions from the decay modes $B \to J/\psi K^*_0(1430)$ or $B \to J/\psi K^*(1410)$.

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I. INTRODUCTION

The large numbers of B mesons produced at the B-factories have resulted in an accurate measurement[1] of the $B^0_d - \overline{B}^0_d$ mixing phase referred to as $\beta$ (also called $\phi_1$). This measurement is done primarily using the golden mode $B \rightarrow J/\psi K_S$, but is now being done for a variety of other channels. In principle, $\beta(\phi_1)$ can be measured just as cleanly using any mode that involves the same quark level process $b \rightarrow c \bar{c} s$. These include modes that involve excitations of the $K$ meson such as $B \rightarrow J/\psi K^*(892)$, $B \rightarrow J/\psi K_0^*(1430)$, $B \rightarrow J/\psi K^*(1410)$, $B \rightarrow J/\psi K_2^*(1430)$, etc., or modes involving various excitations of the $c\bar{c}$ mesons such as $B \rightarrow J/\psi' K_S$ etc., or even a combination of any of these states.

The study presented in this paper is inspired by the recent observation of the decays $B \rightarrow J/\psi K^*_X(1430)$, where $X = 0, 2$ [2–4]. There exist two mesons, a tensor meson $K^*_2(1430)$ and a scalar meson $K^*_0(1430)$ at the same mass of 1430 MeV. One is not only interested in obtaining the $B^0_d - \overline{B}^0_d$ mixing phase precisely, but also in measuring the branching ratios for the various decay modes. The branching ratio for $B \rightarrow J/\psi K^*$ has been measured and it is well known that $\beta(\phi_1)$ can also be measured using this decay mode[5]; the only complication is the need for an angular analysis to separate the contributions from the $CP$–even and $CP$–odd partial waves. However, modes such as $B \rightarrow J/\psi K_2^*(1430)$ require much more effort. The additional complication arises due to the presence of scalar and vector meson resonances $K_0^*(1430)$ and $K^*(1410)$ that overlap with the tensor meson $K^*_2(1430)$. Since, the decay modes $B \rightarrow J/\psi K_0^*(1430)$, $B \rightarrow J/\psi K^*(1410)$ and $B \rightarrow J/\psi K_2^*(1430)$ contribute to the same final state $B \rightarrow K \pi \ell^+ \ell^-$, contributions from the various decay channels cannot be separated by cuts on the kinematics. The purpose of the paper is to show how the contributions from various resonances and partial waves can be isolated, not only to measure the branching ratios but also to study the time dependent decay to the mode $B \rightarrow J/\psi K_2^*(1430)$, leading to a measurement of $\sin 2\beta$. The study performed here finds immediate application in the analysis of data collected by BaBar and Belle collaborations at the B factories running at SLAC (U.S.A.) and KEK (Japan) respectively [5].

In section II we write out the most general effective matrix element using Lorentz invariance and current conservation, for the decay channels $B \rightarrow J/\psi S$, $B \rightarrow J/\psi V$ and $B \rightarrow J/\psi T$, where $S$, $V$ and $T$ are scalar ($J^P = 0^+$), vector ($J^P = 1^-$) and tensor ($J^P = 2^+$) mesons respectively. Our calculations are general enough in formalism to include any scalar,
vector or tensor meson, in addition to $K^*_X(1430)$ and $K_1(1410)$, as long as they decay into modes identical to the modes into which $K^*_X(1430)$ is reconstructed. The $K^*_X(1430)$ and $K_1(1410)$ are considered to decay to $K\pi$ and $J/\psi$ to $\ell^+\ell^-$. In section III we primarily investigate the decay spectrum for the final state $K\pi\ell^+\ell^-$ assuming that the $K\pi$ arise from the decay of either $S$, $V$ or $T$. We study the angular distribution of the $K$ in the $K\pi$ center of mass (c.m.) frame and $\ell^-$ in the $\ell^+\ell^-$ c.m. frame. We also study the correlation between the $K\pi$ decay plane and the $\ell^+\ell^-$ decay plane in the B rest frame. We explicitly demonstrate how one can isolate the contributions to the different final states as well to $CP$ even and $CP$ odd partial waves, thereby allowing the measurement of $\beta(\phi_1)$. We conclude in Sec. IV.

II. MATRIX ELEMENT

We consider exclusive two body decays of a B meson into states involving the $J/\psi$ or its excitations (all these states will be generically referred to as $J/\psi$). In particular, we restrict ourselves to the decays $B \to J/\psi S$, $B \to J/\psi V$ and $B \to J/\psi T$, where $S$, $V$ and $T$ are scalar ($J^P = 0^+$), vector ($J^P = 1^-$) and tensor ($J^P = 2^+$) mesons respectively. We assume that each of the resonances $S$, $V$ and $T$ decay into $K\pi$, and that $J/\psi$ is reconstructed in the $\ell^+\ell^-$ decay mode. Hence each of the three decay channels results in the same decay process $B \to K\pi\ell^+\ell^-$, allowing for interference between the three channels. This decay involving a 4 body final state, can be described in terms of 5 variables $s_{\ell}$, $s_K$, $\theta_\ell$, $\theta_K$ and $\phi$. The kinematical variables $s_{\ell}$ and $s_K$ are the invariant mass squared of the lepton pairs($\ell^+\ell^-$) and the $K\pi$ pairs respectively (it is assumed that the $\ell^+\ell^-$ momentum is along the $+\text{z axis}$), $\theta_\ell$ is the angle of $\ell^-$ in the $\ell^+\ell^-$ c.m. system with the $z$-axis, $\theta_K$ is the angle of $K$ in the $K\pi$ c.m. system with the $z$-axis, and $\phi$ is the angle between the normals to the planes defined by momenta of $\ell^+\ell^-$ and $K\pi$, in the B rest frame[6]. The 4-momenta of $S$ (or $V$ or $T$) is assumed to be $k$ and that of the $J/\psi$ to be $q$. The $K$, $\pi$, $\ell^-$ and $\ell^+$ are defined to have the 4-momenta $k_1$, $k_2$, $q_1$ and $q_2$ respectively. We further define $K^\mu = k_1^\mu - k_2^\mu$ and $Q^\mu = q_1^\mu - q_2^\mu$ and note that $s_\ell = q^2$ and $s_K = k^2$.

Let us first consider the decay $B \to J/\psi S$. The most general amplitude for this decay mode may be written using Lorentz invariance as:

$$A(B \to J/\psi (q) S (k)) = ak_\mu \epsilon_{J/\psi}^{\mu} ,$$

(1)
where $\epsilon^*_{J/\psi}$ is the polarization 4-vector of the $J/\psi$, $q$ and $k$ are the 4-momentum of the $J/\psi$ and $S$ respectively, and $a$ is Lorentz scalar. The subsequent decay of the scalar $S$ into two pseudoscalars, i.e. $S \to K\pi$, will only result in the multiplication of the above amplitude by an arbitrary function of $k_1.k_2$. Therefore, the amplitude for the process $B \to J/\psi S \to J/\psi (K\pi)_S$ may be written as:

$$A_S = A(B \to J/\psi (K\pi)_S) \propto ak_{\mu} \frac{1}{(k^2 - M_S^2 + i\epsilon)} \epsilon^*_{J/\psi}^{\mu}, \quad (2)$$

where, the subscript $S$ in $(K\pi)_S$ indicates that the $K\pi$ state results from the decay of $S$. The amplitude for $B \to J/\psi S$ consists of $P$ wave only.

The amplitude for $B \to J/\psi V$ may similarly be written as,

$$A(B \to J/\psi (q)V(k)) \propto (bg_{\mu \nu} + \frac{c}{\sqrt{slsK}}k_{\mu}q_{\nu} + i\frac{d}{\sqrt{slsK}}\epsilon_{\mu \nu \alpha \beta}k^\alpha q^\beta)\epsilon^*_{J/\psi}^{\mu} \epsilon_V^{\nu}, \quad (3)$$

The subsequent decay of the vector $V$ into $K\pi$ can itself be obtained using the same approach to have the form $K^\mu \epsilon_V^\mu$. Hence, the amplitude for $B \to J/\psi V \to J/\psi (K\pi)_V$ may be written as[7, 8]:

$$A_V = A(B \to J/\psi (K\pi)_V) \propto \frac{(bg_{\mu \nu} + \frac{c}{\sqrt{slsK}}k_{\mu}q_{\nu} + i\frac{d}{\sqrt{slsK}}\epsilon_{\mu \nu \alpha \beta}k^\alpha q^\beta)}{(k^2 - M_V^2 + i\epsilon)} \theta^{\nu \rho} K_{\mu} \epsilon^*_{J/\psi}^{\mu} \epsilon_V^{\nu}, \quad (4)$$

where,

$$\sum \epsilon_V^{\nu} \epsilon_V^{\mu} = - 2g^{\nu \rho} + \frac{k^{\nu}k^{\rho}}{k^2}. \quad \text{We clearly see that the amplitude } B \to J/\psi V \text{ consists of three partial waves - } S, P \text{ and } D.$$

Once again the amplitude for $B \to J/\psi T$ may be may written as:

$$A_T \propto \left( \frac{e}{\sqrt{slsK}}q_{\rho}g_{\mu \nu} + \frac{f}{\sqrt{slsK}}k_{\mu}q_{\rho}q_{\nu} + i\frac{g}{\sqrt{slsK}}\epsilon_{\mu \nu \alpha \beta}q_{\alpha}q^\beta \right) \epsilon^*_{J/\psi}^{\mu} \epsilon_T^{\nu \rho}, \quad (5)$$

where $\epsilon_T^{\nu \rho}$ is the polarization of tensor meson. In writing Eq.5, we used the symmetry of $\epsilon_T^{\nu \rho}$ and retain only terms that contribute. The amplitude for the subsequent decay of the tensor $T$ into $K\pi$ must be of the form $\epsilon_T^{\nu \rho} (k_\lambda k_\sigma + k_\sigma K_\lambda - k_\lambda K_\sigma - K_\lambda K_\sigma)$. Therefore, the amplitude for $B \to J/\psi T \to J/\psi (K\pi)_T$ may be written as:

$$A_T \propto \left( \frac{e}{\sqrt{slsK}}q_{\rho}g_{\mu \nu} + \frac{f}{\sqrt{slsK}}k_{\mu}q_{\rho}q_{\nu} + i\frac{g}{\sqrt{slsK}}\epsilon_{\mu \nu \alpha \beta}q_{\alpha}q^\beta \right) \Theta^{\mu \nu \lambda \sigma} \frac{1}{k^2 - M_T^2 + i\epsilon} (k_\lambda k_\sigma + k_\sigma K_\lambda - k_\lambda K_\sigma - K_\lambda K_\sigma) \epsilon^*_{J/\psi}^{\mu}, \quad (6)$$
where $\Sigma^\nu\rho\epsilon T_\nu\epsilon = \Theta^\nu\rho\lambda\sigma \equiv \frac{1}{2}(\theta^{\lambda\nu}\theta^{\sigma\rho} + \theta^{\nu\sigma}\theta^{\lambda\rho}) - \frac{1}{3}\theta^{\lambda\sigma}\theta^{\nu\rho}[9]$. From 6, the amplitude $B \to J/\psi T$ consists of three waves $-P$, $D$ and $F$ waves; The form factors, $c$ and $f$, are the mixture of $P$ and $F$ partial waves and the form factor, $g$, is related to the $CP$-odd, $D$ partial wave. It is shown in III that all these partial waves can be extracted.

We next incorporate the decay of $J/\psi \to \ell^+\ell^-$ which is common to all the three decay channels. The amplitude for $J/\psi \to \ell^+\ell^-$ is:

$$A(J/\psi \to \ell^+\ell^-) \propto \epsilon_{J/\psi}^\mu (\bar{u}(q_1)\gamma_{\mu'}v(q_2)) .$$

Hence, the full decay process $B \to K(k_1)\pi(k_2)\ell^- (q_1)\ell^+ (q_2)$ may be described by the matrix element

$$\mathcal{M} = \frac{G_F \alpha}{\sqrt{2}\pi} H^\mu \left(\frac{-g^{\mu}\epsilon + q^{\mu}q_{\mu'}}{q^2 - M_{J/\psi}^2 + i\epsilon}\right)(\bar{u}(q_1)\gamma_{\mu'}v(q_2)) ,$$

where $H^\mu$ is the hadronic part of the matrix element and is the sum of individual contributions from the scalar, vector and tensor mesons. $H^\mu$ as derived above, depends on 7 independent form factors and is given by:

$$H^\mu \propto ak^\mu_{\rho} \left(\frac{1}{k^2 - M_S^2 + i\epsilon}\right)$$

$$\left(\frac{bg_{\mu\nu} + \frac{c}{s_{\ell S}} k_{\rho} q_{\nu} + i \frac{d}{s_{\ell S}} \epsilon_{\mu\nu\rho\lambda\sigma} k^\alpha q^\beta}{(k^2 - M_V^2 + i\epsilon)}\right)$$

$$\Theta^{\nu\rho\lambda\sigma}(k_\lambda k_\sigma + k_\sigma K_\lambda - k_\lambda K_\sigma) ,$$

In what follows we assume that the mass of scalar, vector and tensor particles are approximately equal i.e. $M_S \approx M_V \approx M_T = M$ and define new form factors $A$, $B$, $C$, $D$, $E$, $F$, $G$ so as to absorb the factor

$$\frac{1}{(k^2 - M^2 + i\epsilon)(q^2 - M_{J/\psi}^2 + i\epsilon)} .$$

Defining, $H'_\mu = \frac{H^\mu}{q^2 - M_{J/\psi}^2 + i\epsilon}$, we have,

$$H'_\mu = Ak^\mu_{\rho} + \left(\frac{C}{s_{\ell S}} k_{\rho} q_{\nu} + i \frac{D}{s_{\ell S}} \epsilon_{\mu\nu\rho\lambda\sigma} k^\alpha q^\beta\right)\Theta^{\nu\rho\lambda\sigma}$$

$$+ \left(\frac{E}{s_{\ell S}} q_{\rho} g_{\mu\nu} + \frac{F}{s_{\ell S}} k_{\rho} q_{\nu} + i \frac{G}{s_{\ell S}} \epsilon_{\mu\nu\rho\lambda\sigma} q_{\lambda\sigma}\right) ,$$

The matrix element is finally expressed as:

$$\mathcal{M} = \frac{G_F \alpha}{\sqrt{2}\pi} H'_\mu \left(\frac{-g^{\mu}\epsilon + q^{\mu}q_{\mu'}}{q^2}\right)(\bar{u}(q_1)\gamma_{\mu'}v(q_2)) .$$
Having obtained the matrix element for the process $B \rightarrow K(k_1)\pi(k_2)\ell^-(q_1)\ell^+(q_2)$ it is straightforward to write the matrix element $\mathcal{M}$, for the conjugate process $\bar{B} \rightarrow \bar{K}(k_1)\pi(k_2)\ell^-(q_1)\ell^+(q_2)$ as

$$\mathcal{M} = \frac{G_F\alpha}{\sqrt{2\pi}} \overline{H}_\mu(-g^\mu\nu + q^\mu q'^\nu)\bar{u}(q_1)\gamma_\nu v(q_2),$$  \hspace{1cm} (12)

where, $\overline{H}_\mu = \overline{H}_\mu(q^2 - M^2_{J/\psi} + i\epsilon)$. Where $H_\mu$ is the hadronic part of the matrix element for the conjugate process and is the sum of individual contributions from the scalar, vector and tensor mesons. $\overline{H}_\mu'$ depends on the same 7 independent form factors and using $CPT$ may be expressed as,

$$\overline{H}_\mu' = A k_\mu + (B g^\mu{}_{\nu} + \frac{C}{s_{\ell} s_K} k_\mu q_\nu - i \frac{D}{s_{\ell} s_K} \epsilon_{\mu\nu\alpha\beta} k_\alpha q^\beta) \theta^\mu\nu K_\rho$$

$$+ (\frac{E}{s_{\ell} s_K} q_\mu q_{\nu} + \frac{F}{s_{\ell} s_K} k_\mu q_\nu - i \frac{G}{s_{\ell} s_K} \epsilon_{\mu\nu\alpha\beta} q_\alpha k^\beta)$$

$$\Theta^\mu\nu(\lambda\sigma)(k_\lambda k_\sigma + k_\lambda K_\sigma - k_\sigma K_\lambda - K_\lambda K_\sigma).$$  \hspace{1cm} (13)

The time dependent study of the process $B \rightarrow K(k_1)\pi(k_2)\ell^-(q_1)\ell^+(q_2)$ requires the knowledge of the matrix element for both the process and the conjugate process. We can thus study the complete time dependent angular decay of the process. However, in this section we will derive the time integrated decay rate and leave the time dependent study until Sec. III C.

The modulus squared of the total matrix element after summing over lepton polarizations is

$$|\mathcal{M}|^2 = \frac{G^2_F\alpha^2}{2\pi^2} H^\mu_\mu H'^\mu_\mu L^{\mu\nu},$$  \hspace{1cm} (14)

with the leptonic tensor $L^{\mu\nu}$ being given by

$$L^{\mu\nu} = q^\mu q'^\nu - Q^\mu Q'^\nu - g^{\mu\nu} q^2.$$  \hspace{1cm} (15)

Next we cast all the Lorentz scalars in terms of the kinematical variables. It is straightforward to derive that[6, 10]:

$$k.q = \frac{1}{2}(M^2_B - s_K - s_\ell) \equiv x\sqrt{s_{\ell} s_K}$$

$$k.Q = X \cos \theta_\ell$$

$$q.K = \xi q.k + \beta X \cos \theta_K.$$
\[ K.Q = \xi k.Q + \beta (k.q \cos \theta_K \cos \theta_\ell - \sqrt{s_\ell s_K} \sin \theta_K \sin \theta_\ell \cos \phi) \]

\[ \epsilon_{\mu\nu\alpha\beta} k^\mu K^\nu q^\alpha Q^\beta = -\sqrt{s_\ell s_K} \beta X \sin \theta_K \sin \theta_\ell \sin \phi \]

\[ X = (k.q^2 - s_\ell s_K)^{1/2} = \frac{1}{2} \lambda^{1/2}(M_B^2, s_\ell, s_K) \]

\[ k.K = \xi s_K \]

\[ q.Q = 0 \] (17)

where,

\[ \beta = \frac{\lambda^{1/2}(k^2, M_K^2, M_\pi^2)}{k^2} \] (18)

\[ \xi = \frac{(M_K^2 - M_\pi^2)}{s_K} \] (19)

\[ \lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca \] (20)

and we have set \( k_1^2 = M_K^2, \quad k_2^2 = M_\pi^2 \).

The matrix element modulas squared obtained in Eq. (14) may finally be written in terms of the kinematical variables \( s_K, s_\ell, \theta_K, \theta_\ell \) and \( \phi \) as follows:

\[ |M|^2 = \frac{G_F^2 \alpha^2}{2\pi^2} (f_0 + f_1 \cos \phi + f_2 \cos 2\phi + f_3 \sin \phi + f_4 \sin 2\phi) , \] (21)

where,

\[ f_p = \sum (a_p^{m_n} \cos m\theta_\ell \cos n\theta_K + b_p^{m_n} \sin m\theta_\ell \sin n\theta_K) \] (22)

We note that terms proportional to \( \sin \theta_\ell \) are also proportional to \( \sin \theta_K \). It can be seen from Eq. (16) that none of the Lorentz scalars is proportional to just one of either \( \sin \theta_\ell \) or \( \sin \theta_K \). Hence there can be no term odd in \( \theta_\ell \) and even in \( \theta_K \) or vice versa – only terms even in both \( \theta_\ell \) and \( \theta_K \) or odd in both \( \theta_\ell \) and \( \theta_K \) are possible. The form chosen for \( f_p \) in Eq. (22) is thus easily understood. All the non vanishing co-efficients \( a_p^{m_n} \) and \( b_p^{m_n} \) are listed under Tables I-VI in the Appendix. Identical results were obtained using the helicity formalism [11].

III. SOLUTION OF THE FORM FACTORS AND ANALYSIS

It was seen in Sec. II that angular analysis can be used to obtain coefficients \( a_p^{m_n} \) and \( b_p^{m_n} \). We begin this section by demonstrating the explicit solutions that enable us to obtain both the magnitudes and phases of the form factors \( a, b, c, d, e, f \) and \( g \). We show that
angular analysis for this mode will enable us to isolate the contributions from each of the partial-waves. We can thus separate the contributions of the partial waves into $CP$ even and $CP$ odd, allowing us to measure the mixing phase $\beta$ without worrying about dilution from the wrong $CP$ parity contributions. In this section we also consider both the time integrated differential decay rate and time dependent differential decay rate. We study various angular asymmetries and point out certain interesting features that the angular study should obey. We also construct a time dependent angular asymmetry that enables a clean measurement of the mixing phase $\beta$ in the mode $B \to J/\psi K_2^*(1430)$ alone without contributions from the decay modes $B \to J/\psi K_0^*(1430)$ and $B \to J/\psi K^*(1410)$.

A. Solution of the form factors

We note that the angular analysis allows one to measure a large number of coefficients $a_{mn}^p$, $b_{mn}^p$. In spite of several of these coefficient, being zero, there is still enough information to solve for both the magnitude and phase of all the form factors. In fact, the zero values act as constraints and are hence useful in experimental fits to the coefficients. We do not explicitly integrate over $s_\ell$ and $s_K$ in our discussions, however, for the purpose of experimental analysis these variables may be integrated over, before extracting the form factors. In this subsection we present explicit solutions to the magnitudes and phases of the form factors, and thereby established that an angular analysis allows us to disentangle the contribution from each of the three resonances considered. To set up our notation we define $A = |A| \exp(i\phi_A)$, $B = |B| \exp(i\phi_B)$, $C = |C| \exp(i\phi_C)$, $D = |D| \exp(i\phi_D)$, $E = |E| \exp(i\phi_E)$, $F = |F| \exp(i\phi_F)$ and $G = |G| \exp(i\phi_G)$. Using the tables I-VI, it is straightforward to verify that $|B|$, $|D|$, $|E|$, $|F|$, $|G|$ are given in terms of experimentally obtainable coefficients by,

$$|B|^2 = -\frac{2}{\beta^2 s_\ell s_K} (a_{22}^0 + a_{02}^0 - 4a_{02}^2)$$  \hspace{1cm} (23)

$$|D|^2 = -\frac{2}{\beta^2 s_\ell s_K (x^2 - 1)} (a_{22}^0 + a_{02}^0 + 4a_{02}^2)$$  \hspace{1cm} (24)

$$|E|^2 = \frac{32}{\beta^4 s_\ell s_K (x^2 - 1)} (4a_{04}^2 - (a_{04}^0 + a_{24}^0))$$  \hspace{1cm} (25)

$$|F|^2 = \frac{32}{\beta^4 s_\ell s_K (x^2 - 1)^3} (4x^2 a_{04}^2 - (x^2 + 3)a_{24}^0 - (x^2 - 1)a_{04}^0 - 4x b_{24}^1)$$  \hspace{1cm} (26)
\[ |G|^2 = \frac{-32}{\beta^4 s_\ell s_K (x^2 - 1)^2} (4a_{04}^2 + a_{04}^0 + a_{24}^0) \] (27)

Also, \( \text{Im}(BD^*), \text{Re}(BE^*), \text{Re}(DG^*), \text{Im}(EG^*), \text{Re}(FE^*) \) and \( \text{Im}(FG^*) \), can easily be written in terms of observables as:

\[
\text{Im}(EG^*) = \frac{128}{\beta^4 s_\ell s_K (x^2 - 1)^{3/2}} a_{04}^4 \quad (28)
\]

\[
\text{Im}(FG^*) = \frac{64}{\beta^4 s_\ell s_K (x^2 - 1)^{5/2}} (b_{24}^3 - 2x a_{04}^4) \quad (29)
\]

\[
\text{Im}(BD^*) = \frac{8}{\beta^2 s_\ell s_K \sqrt{(x^2 - 1)}} a_{02}^4 \quad (30)
\]

\[
\text{Re}(FE^*) = \frac{-32}{\beta^4 s_\ell s_K (x^2 - 1)^2} (2b_{24}^1 + x(a_{04}^0 + a_{24}^0) - 4x a_{04}^2) \quad (31)
\]

\[
\text{Re}(CD^*) = \frac{-4}{\beta^3 s_\ell s_K (x^2 - 1)^{5/2}} ((x^2 - 1)a_{03}^0 + (x^2 + 3)a_{23}^0 + 4x^2 a_{23}^2 + 4xb_{23}^1) \quad (32)
\]

\[
\text{Re}(BE^*) = \frac{4}{\beta^3 s_\ell s_K (x^2 - 1)^{1/2}} (a_{01}^0 + a_{21}^0 - 4a_{01}^2) \quad (33)
\]

\[
\text{Re}(DG^*) = \frac{4}{\beta^3 s_\ell s_K (x^2 - 1)^{3/2}} (a_{01}^0 + a_{21}^0 + 4a_{01}^2) \quad (34)
\]

We set \( \phi_G = 0 \) by convention. The phases \( \phi_B, \phi_D, \phi_E, \phi_F \) can be obtained using \( \text{Im}(BD^*), \text{Re}(BE^*), \text{Re}(DG^*), \text{Im}(EG^*), \text{Re}(FE^*) \) and \( \text{Im}(FG^*) \), which have already been evaluated with respect to observables using,

\[
\phi_F - \phi_E = \cos^{-1}\left(\frac{\text{Re}(FE^*)}{|E||F|}\right) \quad (35)
\]

\[
\phi_E - \phi_G = \sin^{-1}\left(\frac{\text{Im}(EG^*)}{|E||G|}\right) \quad (36)
\]

\[
\phi_F - \phi_G = \sin^{-1}\left(\frac{\text{Im}(FG^*)}{|F||G|}\right) \quad (37)
\]

\[
\phi_B - \phi_E = \cos^{-1}\left(\frac{\text{Re}(BE^*)}{|B||E|}\right) \quad (38)
\]

\[
\phi_D - \phi_G = \cos^{-1}\left(\frac{\text{Re}(DG^*)}{|D||G|}\right) \quad (39)
\]

\[
\phi_B - \phi_D = \sin^{-1}\left(\frac{\text{Im}(BD^*)}{|B||D|}\right) \quad (40)
\]

Note that ambiguities in the solution are somewhat reduced due to the constraints already obtained in the above equations. The resolution of ambiguities is not discussed in detail here. However, it may be envisaged that the approach used by the BaBar collaboration [11] can also be applied here to completely remove ambiguities in the present study. The
remaining form factors can also be solved as follows:

\[
\text{Re}(CE^*) = \frac{1}{2s_\ell s_K x(x^2 - 1)^{3/2}} \left( -32a_{23}^0 - \sqrt{x^2 - 1} \beta^3 (2x(x^2 - 1)\text{Re}(BF^*) + \right.
\]
\[
(2x^2 + 1)\text{Re}(BE^*) + (x^2 - 1)(2x^2 - 1)\text{Re}(CD^*) + \text{Re}(DG^*)) \right) s_\ell s_K \]  

(41)

Having obtained the value of |B|, |D|, |E|, |F|, |G|, \(\phi_B\), \(\phi_D\), \(\phi_E\), \(\phi_F\) and \(\phi_G\), \(\text{Re}(BE^*) = |B||E|\cos(\phi_B - \phi_E)\) and \(\text{Re}(BF^*) = |B||F|\cos(\phi_B - \phi_F)\) are in fact, also obtained in terms of observables. On putting these values in Eq. (41), \(\text{Re}(CE^*)\) is evaluated in terms of observables. \(\phi_C\) and |C| can now be solved for using Eqs. (32) and (41), to give,

\[
\phi_C = \sin^{-1} \left( \frac{\cos \phi_E - \eta \cos \phi_D}{\sqrt{1 + \eta^2 - 2\eta \cos(\phi_D - \phi_E)}} \right) \]  

(42)

\[
|C| = \frac{\text{Re}(CD^*)}{|D| \cos(\phi_C - \phi_D)} \]  

(43)

where,

\[
\eta = \frac{|D|\text{Re}(CE^*)}{|E|\text{Re}(CD^*)} \]

Using a similar procedure one can also solve for the only remaining quantities \(\phi_A\) and |A|.

We have,

\[
\text{Re}(AE^*) = \frac{1}{12\beta^2 s_\ell s_K(x^2 - 1)} \left( -96b_{21}^1 + \beta^2 s_\ell s_K(24|B|^2 + \beta^2(x^2 - 1)|E|^2 + \right.
\]
\[
24(x^2 - 1)\text{Re}(BC^*) + \beta^2(x^2 - 1)^2 \text{Re}(FE^*)) \right) \]  

(44)

\[
\text{Re}(AB^*) = \frac{1}{12\beta^2 s_\ell s_K \sqrt{x^2 - 1}} \left( -48b_{21}^1 + \beta^3 \sqrt{x^2 - 1} s_\ell s_K \right.
\]
\[
((x^2 - 1)(3\text{Re}(CE^*) - \text{Re}(BF^*)) + 2x \text{Re}(BE^*)) \) \]  

(45)

With the values of \(\phi_C\) and |C| already obtained, \(\text{Re}(BC^*)\) and \(\text{Re}(CE^*)\) are also expressed in terms of observables. We finally have after solving Eqs. (44) and (45):

\[
\phi_A = \sin^{-1} \left( \frac{\cos \phi_E - \varsigma \cos \phi_B}{\sqrt{1 + \varsigma^2 - 2\varsigma \cos(\phi_B - \phi_E)}} \right) \]  

(46)

\[
|A| = \frac{\text{Re}(AB^*)}{|B| \cos(\phi_A - \phi_B)} \]  

(47)

where,

\[
\varsigma = \frac{|B|\text{Re}(AE^*)}{|E|\text{Re}(AB^*)} \]
B. Time Integrated Differential Decay Rate

The differential decay rate is

$$d\Gamma = \frac{\beta X}{2^{15}\pi^6 M_B^2} |\mathcal{M}|^2 ds_\ell \, ds_K \, d\cos\theta_\ell \, d\cos\theta_K \, d\phi.$$  

Therefore,

$$\frac{d\Gamma}{ds_\ell \, ds_K \, d\cos\theta_\ell \, d\cos\theta_K \, d\phi} = \frac{\beta X}{2^{15}\pi^6 M_B^2} |\mathcal{M}|^2$$  

The physical regions of the angular variables are

$$0 \leq \phi \leq 2\pi, \quad -1 \leq \cos\theta_K \leq 1 \quad \text{and} \quad -1 \leq \cos\theta_\ell \leq 1,$$

and $s_\ell$ and $s_K$ are integrated over the relevant resonances. We derive the one-dimensional angular distributions $d\Gamma/(ds_\ell \, ds_K \, d\cos\theta_\ell)$, $d\Gamma/(ds_\ell \, ds_K \, d\cos\theta_K)$, and $d\Gamma/(ds_\ell \, ds_K \, d\phi)$ from the differential decay rate. These distributions as well as the other observables extracted by the angular analysis, depend on different combinations of the co-efficients $a_{m0}^p$ and $b_{m0}^p$.

1. Decay rate as a function of $\cos\theta_K$

Integrating the Eq. (21) over $\cos\theta_\ell$ and $\phi$ we obtain

$$\frac{d\Gamma}{ds_\ell \, ds_K \, d\cos\theta_K} = \frac{\beta X G_F^2 \alpha^2 4\pi}{2^{16}\pi^8 M_B^3} \frac{3}{3} \{3a_{00}^0 - a_{20}^0 + (3a_{01}^0 - a_{21}^0) \cos\theta_K + (3a_{02}^0 - a_{22}^0) \cos 2\theta_K$$

$$+ (3a_{03}^0 - a_{23}^0) \cos 3\theta_K + (3a_{04}^0 - a_{24}^0) \cos 4\theta_K\}$$  

Now we define the forward–backward(FB) asymmetry in $K\pi$ system

$$A^K_{FB} = \frac{\int_0^1 d\Gamma \, ds_\ell \, ds_K \, d\cos\theta_K - \int_{-1}^0 d\Gamma \, ds_\ell \, ds_K \, d\cos\theta_K}{\int_0^1 d\Gamma \, ds_\ell \, ds_K \, d\cos\theta_K + \int_{-1}^0 d\Gamma \, ds_\ell \, ds_K \, d\cos\theta_K}$$

$$= \frac{15}{2} \frac{a_{00}^0 - (a_{02}^0 + a_{20}^0) - 3a_{04}^0 + 5a_{22}^0 + a_{24}^0}{45a_{00}^0 - 15(a_{02}^0 + a_{20}^0) - 3a_{04}^0 + 5a_{22}^0 + a_{24}^0}$$

This is not vanishing due to the presence of $\cos\theta_K$ and $\cos 3\theta_K$. These terms are present due to interference between ‘vector and scalar’ as well as between ‘vector and tensor’ mesons contributions as intermediate states. However, the forward–backward asymmetry vanishes in each of the $B \to J/\psi S$, $B \to J/\psi V$ and $B \to J/\psi T$ decay modes.
2. Decay rate as a function of $\cos \theta_{\ell}$

Integrating the Eq. (21) over $\cos \theta_{K}$ and $\phi$ we obtain

$$\frac{d\Gamma}{ds_{\ell} ds_{K} d\cos \theta_{\ell}} = \frac{\beta XG_{F}^{2} \alpha^{2}}{2^{10} \pi^{8} M_{B}^{3}} \frac{4\pi}{15} \{15a_{00}^{0} - 5a_{02}^{0} - a_{04}^{0}
\quad + (15a_{20}^{0} - 5a_{22}^{0} - a_{24}^{0}) \cos 2\theta_{\ell}\}$$

(53)

It is easy to see that the forward-backward (FB) asymmetry in the $\ell^{-}\ell^{+}$ system vanishes, i.e. $A_{FB}^{\ell} = 0$. The absence term odd in $\cos \theta_{\ell}$ is connected with the fact that the $\ell^{-}\ell^{+}$ system is in a pure $L = 1$ state. As a consequence, the forward–backward (FB) asymmetry in the system vanishes.

3. Decay rate as a function of $\phi$

Finally, the distribution in the angle $\phi$ between the lepton and meson planes, after integration of the Eq. (21) over $\cos \theta_{\ell}$ and $\cos \theta_{k}$, we obtain

$$\frac{d\Gamma}{ds_{\ell} ds_{K} d\phi} = \frac{\beta XG_{F}^{2} \alpha^{2}}{2^{10} \pi^{8} M_{B}^{3}} \frac{4\pi}{45} \{45a_{00}^{0} - 15(a_{02}^{0} + a_{20}^{0}) + 5a_{04}^{0} - 3a_{04}^{0} + a_{24}^{0}
\quad + (45a_{20}^{2} - 15(a_{02}^{2} + a_{20}^{2}) + 5a_{22}^{2} - 3a_{04}^{2} + a_{24}^{2}) \cos 2\phi
\quad + (45a_{00}^{4} - 15(a_{02}^{4} + a_{20}^{4}) + 5a_{04}^{4} - 3a_{04}^{4} + a_{24}^{4}) \sin 2\phi\}$$

(54)

The presence of $\sin 2\phi$ term is a clean signal of CP violation in the $\phi$ distribution in the decay process.

C. Time Dependent Differential decay rate and measurement of $\beta$

In this subsection we demonstrate how a clean measurement of $\sin 2\beta$ can be performed using one of the partial wave contributing to $B \rightarrow J/\psi K_{s}^{*}(1430)$. For this purpose we use the $CP$-even partial waves contributing to $B \rightarrow J/\psi K_{s}^{*}(1430)$. In Sec. III A we showed $|G|^{2}$ can be extracted using the measurements of coefficients $a_{04}^{0}$, $a_{24}^{0}$, $a_{04}^{2}$. Here, we derive an angular and time-dependent asymmetry that cleanly measures $\sin 2\beta$, without the need for a complete solutions to all the contributing form factors. It is straightforward to see that the coefficient of time dependent $\sin(\Delta M t)$ term corresponding to $a_{04}^{0}$, $a_{24}^{0}$ and $a_{04}^{2}$ of Table I, II and III can be obtained by the replacement $|G|^{2} \rightarrow |G|^{2} \sin 2\beta$. A time dependent angular
analysis can be performed to isolate the \( \sin(\Delta M t) \) term to \( a_{04}^0, a_{24}^0, a_{04}^2 \). An asymmetry that isolates such a term is given by:

\[
A_{CP} \sin(\Delta M t) = \frac{1}{\int_{-1}^{1} d \cos \theta_K \int_{-1}^{1} d \cos \theta_\ell \int_{0}^{2\pi} d \phi \left( \frac{d(\Gamma(t) + \bar{\Gamma}(t))}{d \cos \theta_\ell d \cos \theta_K d \phi} \right)} \times \left( \int_{P} \int_{-1}^{1} d \cos \theta_K \int_{0}^{2\pi} d \phi + \int_{P} \int_{T} \int_{-1}^{1} d \cos \theta_K \int_{0}^{2\pi} d \phi + \int_{P} \int_{Q} \int_{0}^{2\pi} d \phi \right) \left( \frac{d(\Gamma(t) - \bar{\Gamma}(t))}{d \cos \theta_\ell d \cos \theta_K d \phi} \right).
\]

(55)

where,

\[
\int_{P} d \cos \theta_K = \left( \int_{\frac{\pi}{5}}^{\frac{2\pi}{5}} - \int_{\frac{3\pi}{5}}^{\frac{4\pi}{5}} + \int_{\frac{\pi}{5}}^{\frac{3\pi}{5}} - \int_{\frac{4\pi}{5}}^{\pi} \right)(- \sin \theta_K) d \theta_K
\]

(56)

\[
\int_{P} d \cos \theta_\ell = \left( \int_{\frac{\pi}{5}}^{\frac{2\pi}{5}} + \int_{\frac{3\pi}{5}}^{\frac{4\pi}{5}} \right)(- \sin \theta_\ell) d \theta_\ell
\]

(57)

\[
\int_{Q} d \phi = \left( \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} - \int_{\frac{5\pi}{4}}^{\frac{7\pi}{4}} + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} - \int_{\frac{7\pi}{4}}^{\frac{9\pi}{4}} \right) d \phi
\]

(58)

In Eq. (55) integrals over the relevant range for \( s_\ell \) and \( s_K \) are implicit. Using Eqns. (21) and (22) Table I-V, we see that

\[
A_{CP} = R \sin 2\beta,
\]

(59)

where

\[
R \propto \frac{4 a_{04}^2 + a_{04}^0 + a_{24}^0}{45 a_{00}^0 - 15(a_{02}^0 + a_{20}^0) - 3 a_{04}^2 + 5 a_{02}^2 + a_{24}^2}.
\]

(60)

\( R \) can itself be obtained directly from experimental data by an analogous asymmetry using time integrated partial decay rates as follows:

\[
R = \frac{1}{\int_{-1}^{1} d \cos \theta_K \int_{-1}^{1} d \cos \theta_\ell \int_{0}^{2\pi} d \phi \left( \frac{d\Gamma}{d \cos \theta_\ell d \cos \theta_K d \phi} \right)} \times \left( \int_{P} \int_{-1}^{1} d \cos \theta_K \int_{0}^{2\pi} d \phi + \int_{P} \int_{T} \int_{-1}^{1} d \cos \theta_K \int_{0}^{2\pi} d \phi + \int_{P} \int_{Q} \int_{0}^{2\pi} d \phi \right) \left( \frac{d\Gamma}{d \cos \theta_\ell d \cos \theta_K d \phi} \right).
\]

(61)

We can thus obtain a clean measurement of \( \sin(2\beta) \) using the \( CP \)-even part of the mode \( B \to J/\psi K_2^*(1430) \) alone, without any contributions from the decay modes \( B \to J/\psi K_0^*(1430) \) or \( B \to J/\psi K^*(1410) \).
IV. CONCLUSION

We have studied how an angular analysis can be used to isolate contributions from each of the decay modes $B \rightarrow J/\psi K^*_0(1430)$, $B \rightarrow J/\psi K^*(1410)$ and $B \rightarrow J/\psi K^*_2(1430)$, where $K^*_0(1430)$, $K^*(1410)$ and $K^*_2(1430)$ are overlapping and contribute to the same final state $B \rightarrow K\pi\ell^+\ell^-$. Angular analysis also allows us to isolate the contributions from different partial waves contributing to the each of these final states. We have studied the time integrated differential decay rate and derived explicit solutions to both the magnitudes and the phases of the form factors contributing. We showed that the forward-backward (FB) asymmetry in $\ell^+\ell^-$ system vanishes, since the $\ell^-\ell^+$ system is in a pure $L = 1$ state. We have also studied the forward-backward (FB) asymmetry in $K\pi$ system; such terms are present due to interference between contributions from vector and scalar, as well as between vector and tensor intermediate states. We also construct a time dependent angular asymmetry that enables a clean measurement of the mixing phase $\beta$ in the mode $B \rightarrow J/\psi K^*_2(1430)$ alone, without contributions from the decay modes $B \rightarrow J/\psi K^*_0(1430)$ or $B \rightarrow J/\psi K^*(1410)$. The study performed here finds immediate application in the analysis of data collected by the Belle and BaBar collaborations.

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VI. APPENDIX

| $a_{0i}$ | $\frac{1}{2304} s_{\ell} s_K \left( (576 |A|^2(x^2-1) + 144 (|B|^2(2x^2 + 3) + (x^2 - 1)(3|D|^2 + 2|C|^2(x^2 - 1)))\beta^2 \\
+ (x^2 - 1) (|E|^2(27 + 22x^2) + (x^2 - 1)(27 |G|^2 + 22 |F|^2(x^2 - 1)))\beta^4 \\
+ 4(x^2 - 1)\beta^2(-24(x^2 - 1) \text{Re}(A F^*) + x(-24 \text{Re}(A E^*) + 144 \text{Re}(B C^*) \\
+ 11(x^2 - 1)\beta^2 \text{Re}(F E^*)) \right)$ |
| $a_{01}$ | $\frac{1}{96} \beta \sqrt{x^2 - 1} s_{\ell} s_K \left( -48x \text{Re}(A B^*) - 48(x^2 - 1) \text{Re}(A C^*) + \beta^2(10x(x^2 - 1) \text{Re}(B F^*) \\
+ (10x^2 + 9) \text{Re}(B E^*) + (x^2 - 1)(10((x^2 - 1) \text{Re}(C D^*) + 10x \text{Re}(C E^*) + 9 \text{Re}(D G^*))) \right)$ |
| $a_{02}$ | $\frac{1}{96} \beta^2 s_{\ell} s_K \left( 6|B|^2(2x^2 - 3) + 4(x^2 - 1)(27|G|^2 + 12|C|^2(x^2 - 1) \\
+ (|E|^2x^2 + |F|^2(x^2 - 1))^2\beta^2 - 12(x^2 - 1)^2 \text{Re}(A F^*) + 2x(x^2 - 1)(-6 \text{Re}(A E^*) \\
+ 12 \text{Re}(B C^*) + (x^2 - 1)\beta^2 \text{Re}(F E^*)) \right)$ |
| $a_{03}$ | $\frac{1}{32} \beta^3 s_{\ell} s_K \sqrt{x^2 - 1} \left( 2x(x^2 - 1) \text{Re}(B F^*) + (2x^2 - 3) \text{Re}(B E^*) \\
+ (x^2 - 1)(2(x^2 - 1) \text{Re}(C D^*) + 2x \text{Re}(C E^*) - 3\text{Re}(D G^*)) \right)$ |
| $a_{04}$ | $\frac{1}{256} \beta^4 s_{\ell} s_K (x^2 - 1) \left( |E|^2(2x^2 - 3) + (x^2 - 1)(3|G|^2 + 2|F|^2(x^2 - 1)) \\
+ 4x(x^2 - 1) \text{Re}(F E^*) \right)$ |

TABLE I: Non-vanishing $a_{0n}$
\[ a_{20}^0 = \frac{1}{2304} s_\ell s_K \left( (-576|A|^2(x^2 - 1) - 144(|B|^2(2x^2 - 1) + (x^2 - 1)(-|D|^2 + 2|C|^2(x^2 - 1)))\beta^2 \
- (x^2 - 1)(|E|^2(-9 + 22x^2) + (x^2 - 1)(-9|G|^2 + 22|F|^2(x^2 - 1)))\beta^4 \
+ 4(x^2 - 1)\beta^2(24(x^2 - 1)\text{Re}(AF^*) + 24x(\text{Re}(AE^*) - 6\text{Re}(BC^*)) \
\quad - 11x(x^2 - 1)\beta^2\text{Re}(FE^*)) \right) \]

\[ a_{21}^0 = \frac{1}{96} \beta \sqrt{x^2 - 1} s_\ell s_K \left( 48x\text{Re}(AB^*) + 48(x^2 - 1)\text{Re}(AC^*) + \beta^2(-10x(x^2 - 1)\text{Re}(BF^*) \
\quad + (-10x^2 + 3)\text{Re}(BE^*) - (x^2 - 1)(10((x^2 - 1)\text{Re}(CD^*) + 10x\text{Re}(CE^*) - 3\text{Re}(DG^*))) \right) \]

\[ a_{22}^0 = -\frac{1}{96} \beta^2 s_\ell s_K \left( 6|B|^2(2x^2 + 1) + (x^2 - 1)(6|D|^2 + 12|C|^2(x^2 - 1) \
\quad + (|E|^2x^2 + |F|^2(x^2 - 1)^2)\beta^2) - 12(x^2 - 1)^2\text{Re}(AF^*) + 2x(x^2 - 1)(-6\text{Re}(AE^*) \
\quad + 12\text{Re}(BC^*) + (x^2 - 1)\beta^2\text{Re}(FE^*)) \right) \]

\[ a_{23}^0 = -\frac{1}{32} \beta^3 s_\ell s_K \sqrt{x^2 - 1} \left( 2x(x^2 - 1)\text{Re}(BF^*) + (2x^2 + 1)\text{Re}(BE^*) \
\quad + (x^2 - 1)(2(x^2 - 1)\text{Re}(CD^*) + 2x\text{Re}(CE^*) + \text{Re}(DG^*)) \right) \]

\[ a_{24}^0 = -\frac{1}{256} \beta^4 s_\ell s_K (x^2 - 1) \left( |E|^2(2x^2 + 1) + (x^2 - 1)(|G|^2 + 2|F|^2(x^2 - 1)) \
\quad + 4x(x^2 - 1))\text{Re}(FE^*) \right) \]

**TABLE II: Non-vanishing** \[ a_{2n}^0 \]
| $a_{00}$  | $\frac{1}{256} \beta^2 s_\ell s_K (|B|^2 + (x^2 - 1)(16|D|^2 - (|E|^2 - |G|^2(x^2 - 1)) \beta^2))$ |
| $a_{01}$  | $\frac{1}{32} \beta^3 s_\ell s_K \sqrt{(x^2 - 1)(-\text{Re}(BE^*)) + (x^2 - 1)\text{Re}(DG^*))}$ |
| $a_{02}$  | $\frac{1}{16} \beta^2 s_\ell s_K (|B|^2 - (x^2 - 1)|D|^2)$ |
| $a_{03}$  | $-\frac{1}{32} \beta^3 s_\ell s_K \sqrt{(x^2 - 1)(-\text{Re}(BE^*)) + (x^2 - 1)\text{Re}(DG^*))}$ |
| $a_{04}$  | $\frac{1}{256} \beta^4 s_\ell s_K (x^2 - 1)(|E|^2 - |G|^2(x^2 - 1))$ |
| $a_{20}$  | $-\frac{1}{256} \beta^2 s_\ell s_K (|B|^2 + (x^2 - 1)(16|D|^2 - (|E|^2 - |G|^2(x^2 - 1)) \beta^2))$ |
| $a_{21}$  | $-\frac{1}{32} \beta^3 s_\ell s_K \sqrt{(x^2 - 1)(-\text{Re}(BE^*)) + (x^2 - 1)\text{Re}(DG^*))}$ |
| $a_{22}$  | $-\frac{1}{16} \beta^2 s_\ell s_K (|B|^2 - (x^2 - 1)|D|^2)$ |
| $a_{23}$  | $\frac{1}{32} \beta^3 s_\ell s_K \sqrt{(x^2 - 1)(-\text{Re}(BE^*)) + (x^2 - 1)\text{Re}(DG^*))}$ |
| $a_{24}$  | $-\frac{1}{256} \beta^4 s_\ell s_K (x^2 - 1)(|E|^2 - |G|^2(x^2 - 1))$ |

**TABLE III:** Non-vanishing $a_{mn}^2$
| $a_{00}$ | $-\frac{1}{128} \beta^2 s_\ell s_K \sqrt{(x^2 - 1)} \left(16 \text{Im}(BD^*) + (x^2 - 1) \beta^2 \text{Im}(EG^*)\right)$ |
| $a_{01}$ | $-\frac{1}{32} \beta^3 s_\ell s_K (x^2 - 1) \left(\text{Im}(BG^*) - \text{Im}(DE^*)\right)$ |
| $a_{02}$ | $\frac{1}{8} \beta^2 s_\ell s_K \sqrt{(x^2 - 1)} \text{Im}(BD^*)$ |
| $a_{03}$ | $\frac{1}{32} \beta^3 s_\ell s_K (x^2 - 1) \left(\text{Im}(BG^*) - \text{Im}(DE^*)\right)$ |
| $a_{04}$ | $\frac{1}{128} \beta^4 s_\ell s_K (x^2 - 1)^{3/2} \text{Im}(EG^*)$ |
| $a_{20}$ | $\frac{1}{128} \beta^2 s_\ell s_K \sqrt{(x^2 - 1)} \left(16 \text{Im}(BD^*) + (x^2 - 1) \beta^2 \text{Im}(EG^*)\right)$ |
| $a_{21}$ | $\frac{1}{32} \beta^3 s_\ell s_K (x^2 - 1) \left(\text{Im}(BG^*) - \text{Im}(DE^*)\right)$ |
| $a_{22}$ | $-\frac{1}{8} \beta^2 s_\ell s_K \sqrt{(x^2 - 1)} \text{Im}(BD^*)$ |
| $a_{23}$ | $-\frac{1}{32} \beta^3 s_\ell s_K (x^2 - 1) \left(\text{Im}(BG^*) - \text{Im}(DE^*)\right)$ |
| $a_{24}$ | $-\frac{1}{128} \beta^4 s_\ell s_K (x^2 - 1)^{3/2} \text{Im}(EG^*)$ |

**TABLE IV:** Non-vanishing $a_{mn}^4$
\[
\begin{array}{l}
\begin{array}{l}
b_{21}^1 = \frac{1}{48} \beta s_\ell s_K \sqrt{(x^2 - 1)} \left( -24 \text{Re}(AB^*) - \beta^2((x^2 - 1) \text{Re}(BF^*) + 2x \text{Re}(BE^*) \\
+3(x^2 - 1) \text{Re}(CE^*)) \right)
\end{array} \\
\begin{array}{l}
b_{22}^1 = \frac{1}{96} \beta^2 s_\ell s_K \left( 24 |B|^2 x + |E|^2 x (x^2 - 1) \beta^2 - 12(x^2 - 1) \text{Re}(AE^*) \\
+24(x^2 - 1) \text{Re}(BC^*) + (x^2 - 1)^2 \beta^2 \text{Re}(FE^*) \right)
\end{array} \\
\begin{array}{l}
b_{23}^1 = \frac{1}{16} \beta^3 s_\ell s_K \sqrt{(x^2 - 1)} \left( (x^2 - 1) \text{Re}(BF^*) + 2x \text{Re}(BE^*) + (x^2 - 1) \text{Re}(CE^*) \right)
\end{array} \\
\begin{array}{l}
b_{24}^1 = \frac{1}{64} \beta^4 s_\ell s_K (x^2 - 1) \left( |E|^2 x + (x^2 - 1) \text{Re}(FE^*) \right)
\end{array}
\end{array}
\]

**TABLE V:** Non-vanishing $b_{mn}^1$

\[
\begin{array}{l}
\begin{array}{l}
b_{21}^3 = \frac{1}{48} \beta s_\ell s_K (x^2 - 1) \left( -24 \text{Im}(AD^*) + \beta^2(3x \text{Im}(BG^*) \\
+(x^2 - 1)(3 \text{Im}(CG^*) + \text{Im}(DF^*)) + x \text{Im}(DE^*)) \right)
\end{array} \\
\begin{array}{l}
b_{22}^3 = \frac{1}{96} \beta^2 s_\ell s_K \sqrt{(x^2 - 1)} \left( -12(x^2 - 1) \text{Im}(AG^*) + 24x \text{Im}(BD^*) \\
+(x^2 - 1)(24 \text{Im}(CD^*) + \beta^2((x^2 - 1) \text{Im}(FG^*) + x \text{Im}(EG^*)) \right)
\end{array} \\
\begin{array}{l}
b_{23}^3 = \frac{1}{16} \beta^3 s_\ell s_K (x^2 - 1) \left( x \text{Im}(BG^*) + (x^2 - 1)(\text{Im}(CG^*) - \text{Im}(DF^*)) - x \text{Im}(DE^*) \right)
\end{array} \\
\begin{array}{l}
b_{24}^3 = \frac{1}{64} \beta^4 s_\ell s_K (x^2 - 1)^{3/2} \left( (x^2 - 1) \text{Im}(FG^*) + x \text{Im}(EG^*) \right)
\end{array}
\end{array}
\]

**TABLE VI:** Non-vanishing $b_{mn}^3$