On maximal regularity for the Cauchy-Dirichlet mixed parabolic problem with fractional time derivative

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Abstract

We prove two maximal regularity results in spaces of continuous and Hölder continuous functions, for a mixed linear Cauchy-Dirichlet problem with a fractional time derivative $D_t^\alpha$. This derivative is intended in the sense of Caputo and $\alpha$ is taken in $(0,2)$. In case $\alpha = 1$, we obtain maximal regularity results for mixed parabolic problems already known in mathematica literature.

Keywords: Fractional time derivatives, mixed Cauchy-Dirichlet problem, maximal regularity.

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1 Introduction

The aim of this paper is the study the following mixed Cauchy-Dirichlet problem:

\begin{align*}
\left\{
\begin{array}{l}
\mathbb{D}_C^{\alpha} u(t, x) = A(x, D_x)u(t, x) + f(t, x), \quad t \in [0, T], x \in \Omega, \\
u(t, x') = g(t, x'), \quad (t, x') \in [0, T] \times \partial \Omega, \\
D_{t}^{k} u(0, x) = u_k(x), \quad x \in \Omega, k \in \mathbb{N}_0, k < \alpha,
\end{array}
\right.
\end{align*}

with $\mathbb{D}_C^{\alpha} u$ fractional time derivative in the sense of Caputo of order $\alpha$ in $(0,2)$, $A(x, D_x)$ elliptic in the bounded domain $\Omega$ and Dirichlet (not necessarily homogeneous) conditions on the boundary $\partial \Omega$ of $\Omega$ (precise assumptions will be stated in the following, see (A1)-(A4)).

Mixed boundary value problems with fractional time derivatives have attracted the attention of researchers in these latest time. An application of a nonlinear version of (1.1) to a problem in viscoelasticity is mentioned in [5] (see also the references in this paper). Explicit solutions in cases with simple geometries and various boundary conditions where found in many situations (see, for example, [22], with its bibliography). A general discussion of several mathematical models of heat diffusion (even with fractional derivatives) is contained in [9].

We prove here two maximal regularity results which are already known in the case $\alpha = 1$.

The first of these results (Theorem 1.1) prescribes necessary and sufficient conditions on the data $f$, $g$, $u_k$ ($k \in \mathbb{N}_0, k < \alpha$), in order that, given $\theta$ in $(0,2) \setminus \{1\}$ with $\alpha \theta < 2$, there exists a unique solution

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$u$ which is continuous (as a function of $t$) with values in $C^2(\Omega)$, bounded with values in $C^{2+\theta}(\Omega)$ and such that $\mathbb{D}_{C^1(\Omega)}^\alpha u$ and $A(\cdot, D_x)u$ are continuous with values in $C(\Omega)$ and bounded with values in $C^\theta(\Omega)$. The case $\alpha = 1$ was proved in [12] and generalized to general mixed parabolic problems in [13]. Related questions were discussed in [23].

The second of these results (Theorem 1.2) prescribes necessary and sufficient conditions on the data $f$, $g$, $u_k$ ($k \in \mathbb{N}_0$, $k < \alpha$), in order that, given $\theta$ in $(0, 2) \setminus \{1\}$ with $\alpha \theta < 2$, there exists a unique solution $u$ which is continuous (as a function of $t$) with values in $C^2(\Omega)$, bounded with values in $C^{2+\theta}(\Omega)$ and such that $\mathbb{D}_{C^1(\Omega)}^\alpha u$ and $A(\cdot, D_x)u$ belong to the class $C^{2+\theta}((0,T] \times \Omega)$. The case $\alpha = 1$ is classical and is completely illustrated in [17] and [18]. See also [19] for a semigroup approach. We are not aware of generalizations to the case $\alpha \neq 1$.

Our study might be the starting point to consider nonlinear problems by linearization procedures.

We quote other papers connected with the content of this one.

In [8] the Cauchy problem in $\mathbb{R}^n$ is studied in case $\alpha \in (0,1)$. A fundamental solution is constructed. The simplest case, namely the case with $n = 1$ and the elliptic operator with a constant coefficients, is studied in [20].

In [10] the authors consider the abstract Cauchy problem

$$
\begin{align*}
\mathbb{D}_{X}^\alpha u(t) &= Au(t) + f(t), \quad t > 0, \\
u(0) &= u_0,
\end{align*}
$$

(1.2)

with $\alpha \in (0,1]$ (we shall precise in Definition 2.11 the meaning of the expression $\mathbb{D}_{X}^\alpha u$). They consider the case that $A$ is the infinitesimal generator of a $\beta$—times integrated semigroup in the Banach space $X$. Their results are also applied to our problem (see their Example 8.3), with $X = C(\Omega) \times C(\partial \Omega)$, but they do not seem to be of maximal regularity.

Maximal regularity results are discussed in [3] and [4] for the general abstract system

$$
\begin{align*}
\mathbb{D}_{X}^\alpha u(t) &= Au(t) + f(t), \quad t \in [0,T], \\
D_t^\alpha u(0) &= u_k, \quad k \in \mathbb{N}_0, k < \alpha,
\end{align*}
$$

(1.3)
in case $\alpha \in (0,2)$, $-A$ is a sectorial operator of type less than $(1 - \frac{\alpha}{2})\pi$ (see Definition 2.2). Two topics are discussed:

(I) necessary and sufficient conditions on the data, in order that $\mathbb{D}_{X}^\alpha u$ and $Au$ are bounded with values in the real interpolation space $(X, D(A))_{\theta,\infty} (0 < \theta < 1)$;

(II) necessary and sufficient conditions on the data, in order that $\mathbb{D}_{X}^\alpha u$ and $Au$ are both in the space of Hölder continuous functions $C^\theta([0,T]; X)$.

In [3] the case $\alpha \in (0,1]$ is considered. In this case the results found are essentially complete. The case $\alpha \in (1, 2)$ is considered in [4]. Here only sufficient conditions are prescribed. In order to prove Theorem 1.2 we shall consider the case (II), with $\beta < \alpha$. It turns out that the sufficient conditions prescribed in [4] are also necessary. This is proved in the preprint [14]. In the following (see Theorems 2.19, 2.20) we shall come back to these results, as we shall need them. Here we mention only the fact that, given $\theta$ in (say) $(0,1)$, the operator $-A$ such that

$$
\begin{align*}
D(A) &= \{ u \in C^{2+\theta}(\Omega) : u|_{\partial \Omega} = 0 \} \\
Au &= A(\cdot, D_x)u
\end{align*}
$$

is not sectorial in the Banach space $C^\theta(\Omega)$ (see [18], Example 3.1.33): the best available estimate is [22].

So, even in the case of homogeneous boundary conditions, the results of [3] and [4] are not sufficient for our purposes.

Other results of maximal regularity for (1.3) are discussed in [5] and in [1] (see also [2]). Finally, maximal regularity for equations involving versions of the Caputo derivative in $\mathbb{R}$, in spaces of order continuous functions on the line are given in [21] and [15].

Now we introduce some notations which we are going to use in the paper.
If $\alpha \in \mathbb{R}$, $[\alpha]$ will indicate the maximum integer less or equal than $\alpha$. $\mathbb{R}^+$ will indicate the set of (strictly) positive real numbers. If $\lambda \in \mathbb{C} \setminus \{0\}$.

If $X$ is a complex Banach space with norm $\| \cdot \|$, $B([0,T]; X)$ will indicate the class of functions with values in $X$ with domain $[0,T]$; if $B$ is a (generally unbounded) linear operator from $D(A) \subseteq X$ to $X$, $\rho(B)$ will indicate the resolvent set of $B$. If $A$ is a closed operator in $X$, $A : D(A)(\subseteq X) \to X$, $D(A)$, equipped with the norm

$$\| x \|_{D(A)} := \| x \| + \| Ax \|$$

is a Banach space.

Given a function $f$ with domain $\overline{\Omega}$, with $\Omega \subseteq \mathbb{R}^n$, $\gamma f$ will indicate the trace of $f$ on the boundary $\partial \Omega$ of $\Omega$.

If $X_0, X_1$ are Banach spaces such that $X_1 \hookrightarrow X_0$, $\xi \in (0,1)$ and $p \in [1, \infty]$, we shall indicate with $(X_0, X_1)_{\xi,p}$ the corresponding real interpolation space. We shall freely use the basic facts concerning real interpolation theory (see, for example, [18], [26]). If $X_0 \hookrightarrow X \hookrightarrow X_0$, we shall write $X \in J_{\xi}(X_0, X_1)$, $X \in K_{\xi}(X_0, X_1)$ if $X \hookrightarrow (X_0, X_1)_{\xi,\infty}$.

If $\beta \in \mathbb{N}_0$ and $\Omega$ is an open, bounded subset of $\mathbb{R}^n$, we shall indicate with $C^\beta(\overline{\Omega})$ the class of complex valued functions which are continuous in $\overline{\Omega}$, together with their derivatives (extensible by continuity to $\overline{\Omega}$) of order not exceeding $\beta$. If $\beta \in \mathbb{R}_+ \setminus \mathbb{N}$, $C^\beta(\overline{\Omega})$ will indicate the class of functions in $C^{[\beta]}(\overline{\Omega})$ whose derivatives of order $[\beta]$ are H"older continuous of order $\beta - [\beta]$ in $\overline{\Omega}$. These definitions admit natural extensions to function with values in a Banach space $X$. In this case, we shall use the notation $C^\beta(\Omega;X)$ (in particular $C^\beta([a,b]; X)$, in case $\Omega = (a,b) \subseteq \mathbb{R}$). By local charts, if $\partial \Omega$ is sufficiently regular, we can consider the spaces $C^\beta(\partial \Omega)$.

All these classes will be assumed to be equipped of natural norms. We shall use the notation

$$C^\beta_0(\overline{\Omega}) := \{ f \in C^\beta(\overline{\Omega}) : \gamma f = f|_{\partial \Omega} = 0 \}.$$  

If $\alpha, \beta \in [0, \infty)$, $T \in \mathbb{R}^+$ and $\Omega$ is an open bounded subset of $\mathbb{R}^n$, we set

$$C^{\alpha,\beta}([0,T] \times \overline{\Omega}) := C^{\alpha}([0,T]; C^{\beta}(\overline{\Omega})) \cap B([0,T]; C^{\beta}(\overline{\Omega})).$$

An analogous meaning will have $C^{\alpha,\beta}([0,T] \times \partial \Omega)$. If $X$ is a Banach space, $\text{Lip}([0,T]; X)$ will indicate the class of Lispchitz continuous functions from $[0,T]$, equipped with a natural norm.

Let $\phi \in (0, \pi)$, $R \in (0, \infty)$. We shall indicate with $\Gamma(\phi, R)$ a piecewise $C^1$ path, describing

$$\{ \lambda \in \mathbb{C} : |\lambda| \geq R, |\text{Arg}(\lambda)| = \phi \} \cup \{ \lambda \in \mathbb{C} : |\lambda| = R, |\text{Arg}(\lambda)| \leq \phi \},$$

$k$ oriented from $\infty e^{-i\phi}$ to $\infty e^{i\phi}$. We shall write $\lambda \in \Gamma(\phi, R)$ to indicate that $\lambda$ belongs to the range of $\Gamma(\phi, R)$.

Finally, $C$ will indicate a positive real constant we are not interested to precise (the meaning of which may be different from time to time). In a sequence of inequalities, we shall write $C_1, C_2, \ldots$.

After these preliminaries, we list the basic assumptions we are going to work with. We assume that:

(A1) $\Omega$ is an open, bounded subset in $\mathbb{R}^n$ lying on one side of its boundary $\partial \Omega$, which is a $n-1$-submanifold of $\mathbb{R}^n$ of class $C^{2+\theta}$, with $\theta \in (0,2) \setminus \{1\}$.

(A2) $\alpha \in (0,2)$, $A(x, D_x) = \sum_{|\rho| \leq 2} a_\rho(x) D_x^\rho$, with $a_\rho \in C^0(\overline{\Omega})$, $a_\rho$ complex valued; $A(x, D_x)$ is assumed to be elliptic, in the sense that $\sum_{|\rho| = 2} a_\rho(x) \xi^\rho \neq 0 \ \forall \xi \in \mathbb{R}^n \setminus \{0\}$; we suppose, moreover, that

$$|\text{Arg}\left( \sum_{|\rho|=2} a_\rho(x) \xi^\rho \right)| < \left(1 - \frac{\alpha}{2}\right)\pi, \ \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

(A3) $D^\alpha_{C(\overline{\Omega})} u$ is the fractional Caputo derivative of $u$ with respect to $t$ with values in $C(\overline{\Omega})$ (see the following Definition 2.11).

(A4) $\alpha \theta < 2$.

We are looking for necessary and sufficient conditions in order that (1.1) has a unique solution $u$ such that:
Lemma 2.1. Let
\[
\mathbb{D}^{2+\theta}_{C(\Omega)} u \text{ exists and belongs to } C([0,T];C(\Omega)) \cap B([0,T];C^\theta(\Omega)).
\]

(B2) \( u \in C([0,T];C^{2}(\Omega)) \cap B([0,T];C^{2+\theta}(\Omega)). \)

We want to prove the following

Theorem 1.1. Suppose that the assumptions (A1)-(A4) are fulfilled. Then the following conditions are necessary and sufficient, in order that (1.1) has a unique solution \( u \) satisfying (B1)-(B2):

(I) \( f \in C([0,T];C(\Omega)) \cap B([0,T];C^{\theta}(\Omega)). \)

(II) \( u_0 \in C^{2+\theta}(\Omega) \) and, in case \( \alpha \in (1,2) \), \( u_1 \in C^{\alpha+2(1-\frac{1}{\theta})} \).

(III) \( g \in C([0,T];C^2(\partial \Omega)) \cap B([0,T];C^{2+\theta}(\partial \Omega)), \mathbb{D}^{\alpha}_{C(\partial \Omega)} g \text{ exists and belongs to } C([0,T];C(\partial \Omega)) \cap B([0,T];C^{\alpha}(\partial \Omega)). \)

We want to prove the following

Theorem 1.2. Suppose that the assumptions (A1)-(A4) are fulfilled. Then the following conditions are necessary and sufficient, in order that (1.1) has a unique solution \( u \) such that:

(D1) \( u \in C([0,T];C^2(\Omega)) \cap B([0,T];C^{2+\theta}(\Omega)). \)

(D2)

We want to prove the following

Theorem 2.1. Let \( \Omega \) be an open subset in \( \mathbb{R}^n \) as in (A1). Let \( 0 \leq \beta_0 < \beta_1 \leq 2 + \theta \). Then:

(I) if \( \xi \in (0,1), \)
\[
C^{(1-\xi)\beta_0+\xi\beta_1}(\Omega) \in J_{\xi}(C^{\beta_0}(\Omega),C^{\beta_1}(\Omega)) \cap K_{\xi}(C^{\beta_0}(\Omega),C^{\beta_1}(\Omega));
\]

(II) if \( (1-\xi)\beta_0 + \xi\beta_1 \notin \mathbb{N}, \)
\[
C^{(1-\xi)\beta_0+\xi\beta_1}(\Omega) = (C^{\beta_0}(\Omega),C^{\beta_1}(\Omega))_{\xi,\infty}
\]

with equivalent norms.

(III) If \( \beta \in (0,2+\theta] \setminus \mathbb{N} \), any bounded and closed subset of \( C^{\beta}(\Omega) \) is closed in \( C(\Omega) \).

(IV) There exists an element \( R \) if \( L(C(\partial \Omega),C(\Omega)) \) such that \( \gamma Rg = g \forall g \in C(\partial \Omega) \) and, for any \( \xi \) in \([0,2+\theta]\), \( R_{L(C(\partial \Omega))} \) belongs to \( L(C^{\xi}(\partial \Omega),C^{\xi}(\Omega)) \).

2 Preliminaries

We begin by recalling with some properties of the class of spaces \( C^\beta(\Omega) \) (\( 0 \leq \beta \leq 2+\theta \)).

Lemma 2.1. Let \( \Omega \) be an open subset in \( \mathbb{R}^n \) as in (A1). Let \( 0 \leq \beta_0 < \beta_1 \leq 2 + \theta \). Then:

(I) if \( \xi \in (0,1), \)
\[
C^{(1-\xi)\beta_0+\xi\beta_1}(\Omega) \in J_{\xi}(C^{\beta_0}(\Omega),C^{\beta_1}(\Omega)) \cap K_{\xi}(C^{\beta_0}(\Omega),C^{\beta_1}(\Omega));
\]

(II) if \( (1-\xi)\beta_0 + \xi\beta_1 \notin \mathbb{N}, \)
\[
C^{(1-\xi)\beta_0+\xi\beta_1}(\Omega) = (C^{\beta_0}(\Omega),C^{\beta_1}(\Omega))_{\xi,\infty}
\]

with equivalent norms.

(III) If \( \beta \in (0,2+\theta] \setminus \mathbb{N} \), any bounded and closed subset of \( C^{\beta}(\Omega) \) is closed in \( C(\Omega) \).

(IV) There exists an element \( R \) if \( L(C(\partial \Omega),C(\Omega)) \) such that \( \gamma Rg = g \forall g \in C(\partial \Omega) \) and, for any \( \xi \) in \([0,2+\theta]\), \( R_{L(C(\partial \Omega))} \) belongs to \( L(C^{\xi}(\partial \Omega),C^{\xi}(\Omega)) \).
Then some preliminaries. We start from the following simple operator \( B \) in case of functions in \( X \)

\[
R_0g(x',x_n) := g(x')\chi(x_n), \quad (x',x_n) \in \mathbb{R}_+^n,
\]

with \( \chi \in C^\infty([0,\infty)), \chi(t) = 1 \) if \( t \in [0,\delta_1], \chi(t) = 1 \) if \( t \geq \delta_2, \) \( 0 < \delta_1 < \delta_2. \) \( R \) can be constructed employing \( R_0, \) local charts and a partition of unity.

We introduce the definition and some properties of the Caputo derivative \( D^\alpha_X u. \) We shall consider the case of functions \( u \) defined in \([0,T]\) with values in the complex Banach space \( X. \) The definition requires some preliminaries. We start from the following simple operator \( B: \)

\[
\begin{align*}
B_X : & \{ v \in C^1([0,T];X) : v(0) = 0 \} \to C([0,T];X), \\
B_X v & := D_tv.
\end{align*}
\]

Then \( \rho(B_X) = \mathbb{C} \) and \( B_X \) is a positive operator in \( C([0,T];X) \) of type \( \frac{\pi}{2} \) in the sense of the following definition:

**Definition 2.2.** Let \( B \) be a linear operator in the complex Banach space \( Y. \) We shall say that \( B \) is positive of type \( \omega, \) with \( \omega \in (0,\pi), \) if

\[
\{ \lambda \in \mathbb{C} \setminus \{ 0 \} : |\text{Arg}(\lambda)| > \omega \} \cup \{ 0 \} \subseteq \rho(B).
\]

Moreover, \( \forall \epsilon \in (0,\pi - \omega), \) there exists \( M(\epsilon) \) positive such that

\[
\| \lambda(\lambda - A)^{-1} \|_{\mathbb{L}(Y)} \leq M(\epsilon)
\]
in case \( \lambda \in \mathbb{C} \setminus \{ 0 \}, \) \( |\text{Arg}(\lambda)| \geq \omega + \epsilon. \)

We pass to define the powers of a positive operator. For the definition of positive operator see \[25\], Definition 2.3.1, where also the condition that \( D(B) \) is dense in \( Y \) is requires. In order to describe and prove the properties if the fractional properties of \( B_X, \) we shall appeal (if possible) to corresponding results in [23], concerning fractional powers of positive operators with dense domain. If \( B \) is a positive operator in \( X \) of type \( \omega, \) and \( \alpha \in \mathbb{R}^+, \) we set

\[
B^{-\alpha} := -\frac{1}{2\pi i} \int_{\Gamma(\phi,R)} \lambda^{-\alpha}(\lambda - B)^{-1} d\lambda.
\]

with \( \phi \in (\omega,\pi) \) and \( R \) positive, such that \( \{ \lambda \in \mathbb{C} : |\lambda| \leq R \} \subseteq \rho(B). \) It turns out (applying standard computations techniques of complex integrals) that, we have, \( \forall \alpha \in \mathbb{R}^+, \forall f \in C([0,T];X), \forall t \in [0,T]: \)

\[
B^{-\alpha}_X f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f(s)ds.
\]

With arguments similar to those employed in [25], Chapter 2.3, one can show the following

**Lemma 2.3.** Let \( B_X \) be the operator defined in (2.1) and let \( \alpha, \beta \in \mathbb{R}^+. \) Then:

(a) \( B^{-\alpha}_X B^{-\beta}_X = B^{-\alpha+\beta}_X. \)
(b) \[2.3\] is consistent with the usual definition of \( B^{-\alpha}_X \) in case \( \alpha \in \mathbb{N}. \)
(c) \( B^{-\alpha}_X \) is injective.

So we can define, for any \( \alpha \in \mathbb{R}^+, \)

\[
B_X^\alpha := (B^{-\alpha}_X)^{-1}.
\]

Of course the domain \( D(B_X^\alpha) \) of \( B_X^\alpha \) is the range of \( B_X^{-\alpha}. \)
Lemma 2.4. Let $\alpha, \beta \in \mathbb{R}^+$. Then:

(a) $D(B_X^{\alpha+\beta}) = \{u \in D(B_X^\beta) : B_X^\beta u \in D(B_X^\beta)\}$; moreover, if $u \in D(B_X^{\alpha+\beta})$, $B_X^{\alpha+\beta}u = B_X^\beta(B_X^\beta u)$;

(b) if $\alpha \leq \beta$, $D(B_X^\alpha) \subseteq D(B_X^\beta)$;

(c) if $\alpha \in \mathbb{N}$,

\[ D(B_X^\alpha) = \{u \in C^\alpha([0, T]; X) : u^{(k)}(0) = 0 \quad \forall k \in \mathbb{N}_0, k < \alpha \}. \]

(d) If $\alpha \in \mathbb{R}^+$, $\rho(B_X^\alpha) = \mathbb{C}$ and $\forall \lambda \in \mathbb{C}$, $\forall f \in C([0, T]; X)$, $\forall t \in [0, T]$

\[ ((\lambda - B_X^\alpha)^{-1}f)(t) = -\frac{1}{2\pi i} \int_{\gamma(\phi, R)} (x - \mu)^{-1}(\mu - B)^{-1}d\mu = \frac{1}{2\pi i} \int_0^t \left( \int_{\gamma(\phi, R)} \frac{e^{\phi(t-s)}}{\lambda - \mu}d\mu \right) f(s)ds, \]

with $\phi \in (\frac{\pi}{2}, \pi)$, $R^\alpha > |\lambda|$.

(e) If $\alpha \in (0, 2)$, $B_X^\alpha$ is positive of type $\frac{\pi}{2}$.

Proof. Concerning (a), (b), one can follow the arguments in [25], Chapter 2.3.

(c) is trivial.

Concerning (d), let $\lambda \in \mathbb{C}$. Let $\Gamma(\lambda) = \Gamma(\phi, R)$, with $\phi$ and $R$ as in the statement. If $f \in C([0, T]; X)$, we set

\[ T(\lambda) := -\frac{1}{2\pi i} \int_{\gamma(\lambda)} (\lambda - \mu)^{-1}(\mu - B)^{-1}d\mu. \]

It is easily seen (observing that $\Gamma(\lambda)$ can be chosen locally independently of $\lambda$) that $T$ is entire with values in $L(C([0, T]; X))$. By well known facts of analytic continuation, in order to show that $T(\lambda) = (\lambda - B^\alpha)^{-1}$, it is sufficient to show that this holds if $\lambda$ belongs to some ball centred at 0. We set $R(\lambda) := (\lambda - B^\alpha)^{-1}$, with $\lambda$ sufficiently close to 0, in such a way that it belongs to $\rho(B^\alpha)$ (as $0 \in \rho(B_X^\alpha)$). We prove that $T^{(k)}(0) = R^{(k)}(0)$ for every $k \in \mathbb{N}_0$. In fact, we have

\[ R^{(k)}(0) = -k!B^{-(k+1)}\alpha. \]

On the other hand,

\[ T^{(k)}(0) = \frac{k!}{2\pi i} \int_{\gamma(0)} \mu^{-(k+1)}(\mu - B)^{-1}d\mu = -k!B^{-(k+1)}\alpha, \]

and the conclusion follows.

(e) We consider first the case $\alpha \in (0, 1)$. Employing formula (2.5), we can follow the argument in [25], Proposition 2.3.2 and get the conclusion in this case. If $\alpha \in (1, 2)$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $|\arg(\lambda)| > \frac{\pi}{2}$, then $|\arg(\pm \lambda^{1/2})| > \frac{\pi}{\alpha}$ (on account of $\frac{\pi}{\alpha} < \frac{\pi}{2}$). We deduce that

\[ \lambda - B_X^\alpha = (\lambda^{1/2} + B_X^\alpha)(\lambda^{1/2} - B_X^\alpha), \]

implying easily the conclusion. \(\square\)

Remark 2.5. Let $\alpha \in (0, 2)$. Then, as $\rho(B^\alpha) = \mathbb{C}$, it is easily seen that for every $c$ in $\mathbb{C}$ $c + B_X^\alpha$ is positive of type $\frac{\alpha\pi}{2}$.

Now we examine the domain $D(B_X^\alpha)$ of $B_X^\alpha$. We shall employ the following

Proposition 2.6. Let $X$ be a complex Banach space, $B$ a linear operator in $X$ such that, for some $\phi \in (-\pi, \pi]$, there exists $R$ positive such that

\[ \{\lambda \in \mathbb{C} : |\lambda| \geq R, \arg(\lambda) = \phi\} \subseteq \rho(B) \]

and, for some $C$ positive and $\lambda$ in this set,

\[ ||(\lambda - B)^{-1}||_{L(X)} \leq C|\lambda|^{-1}. \]

Then:

(I) if $j, k \in \mathbb{N}$ and $j < k$, $D(B^j) \subseteq K_{j/k}(X, D(B^k)) \cap J_{j/k}(X, D(B^k));$
(II) if \( j_0, k_0, j_1, k_1 \) are nonnegative integers such that \( j_0 < k_0, j_1 < k_1, \xi_0, \xi_1 \in (0,1), (1 - \xi_0)j_0 + \xi_0k_0 = (1 - \xi_1)j_1 + \xi_1k_1, \) then, for any \( p \) in \([1,\infty]\),

\[
(D(B^{j_0}), D(B^{k_0}))_{\xi_0,p} = (D(B^{j_1}), D(B^{k_1}))_{\xi_1,p}.
\]

(III) If \( \xi \in (0,1), \)

\[
(X,D(B))_{\xi,\infty} = \{f \in X : \limsup_{t \to \infty} t^{\xi} \|B(te^{i\theta} - B)^{-1}f\| < \infty\}.
\]

Proof. For (I) see [25], Chapter 1.14.3. (II) follows from (I) and the reiteration property of the real method. (III) is proved in [10], Theorem 3.1. \( \square \)

Lemma 2.7. Let \( X, X_1 \) be complex Banach spaces, such that \( X_1 \hookrightarrow X \) and closed, bounded subsets of \( X_1 \) are also closed in \( X \). Let \( \alpha \in \mathbb{R}^+, u \in D(B^\alpha_{X'}) \) and suppose that \( B^\alpha_Xu \in B([0,T];X_1) \). Then:

(I) if \( \alpha \not\in \mathbb{N} \), then \( u \in C^\alpha([0,T];X_1) \).

(II) If \( \alpha \in \mathbb{N} \), then \( u \in C^{\alpha-1}([0,T];X_1) \) and \( D^{\alpha-1}_tu \in Lip([0,T];X_1) \).

(III) If \( k \in \mathbb{N}_0 \) and \( k < \alpha \), \( D^ku(0) = 0 \).

Proof. (I) Suppose first that \( \alpha \in (0,1) \). Let \( f := B^\alpha_Xu \). Then \( f \in C([0,T];X) \cap B([0,T];X_1) \) and, if \( t \in [0,T] \),

\[
u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau)d\tau.
\]

Then, clearly, \( u(0) = 0 \). Moreover, if \( 0 \leq s < t \leq T \),

\[
u(t) - \nu(s) = \frac{1}{\Gamma(\alpha)} \int_s^t (t - \tau)^{\alpha-1} f(\tau)d\tau + \frac{1}{\Gamma(\alpha)} \int_0^s [(t - \tau)^{\alpha-1} - (s - \tau)^{\alpha-1}] f(\tau)d\tau = I_1 + I_2.
\]

We set, for \( n \in \mathbb{N}, f_n : [0,T] \to X_1, f_n(0) = 0, f_n(t) = f(kT_n) \) if \( t \in (kT_n, kT_n], 1 \leq k \leq n \). Then

\[
\|\int_s^t (t - \tau)^{\alpha-1} f_n(\tau)d\tau\|_{X_1} \leq \frac{(t - s)^\alpha}{\alpha} \|f\|_{B([0,T];X_1)}.
\]

As

\[
\lim_{n \to \infty} \int_s^t (t - \tau)^{\alpha-1} [f_n(\tau) - f(\tau)]d\tau\|_{X_1} = 0,
\]

we deduce that \( I_1 \in X_1 \) and

\[
\|I_1\|_{X_1} \leq \frac{(t - s)^\alpha}{\Gamma(\alpha + 1)} \|f\|_{B([0,T];X_1)}.
\]

Analogously, one can show that \( I_2 \in X_1 \) and

\[
\|I_2\|_{X_1} \leq \frac{1}{\Gamma(\alpha)} \int_0^s [(s - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1}] d\tau \|f\|_{B([0,T];X_1)}
\]

\[
= \frac{1}{\Gamma(\alpha + 1)} [(s - t)^\alpha - (t - s)^\alpha] \|f\|_{B([0,T];X_1)} \leq \frac{1}{\Gamma(\alpha + 1)} (t - s)^\alpha \|f\|_{B([0,T];X_1)}.
\]

We assume now that \( \alpha > 1 \). We set \( f := B^{\alpha-\alpha}u \). Then \( B^{\alpha-\alpha}f \in C([0,T];X) \cap B([0,T];X_1) \), so that \( f \in C^{\alpha-\alpha}([0,T];X_1) \). So the claim follows from the identity

\[
u(t) = \frac{1}{(\alpha - 1)!} \int_0^t (t - s)^{\alpha-1} f(s)ds, \quad t \in [0,T].
\]

Similarly, one can show (II). (III) follows from Lemma [24] (b)-(c). \( \square \)
Proposition 2.8. (I) Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then

$$D(B_X^\alpha) \subset K_{\alpha-a}(D(B_X^{[\alpha]}), D(B_X^{[\alpha]+1})) \cap J_{\alpha-[\alpha]}(D(B_X^{[\alpha]}), D(B_X^{[\alpha]+1})).$$

(II) If $0 \leq \alpha_0 < \alpha_1$, $\xi \in (0, 1)$ and $(1 - \xi)\alpha_0 + \xi\alpha_1 \notin \mathbb{N}$, then

$$(D(B_X^{[\alpha]}), D(B_X^{[\alpha]+1}));_{\xi,\infty} = \{ f \in C^{(1-\xi)\alpha_0 + \xi\alpha_1}([0,T]; X) : f^{(k)}(0) = 0 \quad \forall k \in \mathbb{N}, k < (1 - \xi)\alpha_0 + \xi\alpha_1 \},$$

with equivalent norms.

Proof. (I) Using the fact that, for any $k \in \mathbb{N}$, $0 < \alpha_0 < \alpha_1$, $B_X^k$ is an isomorphism between $D(B_X^{[\alpha]+k})$ and $D(B_X^\alpha)$, it is clear that the claim in the general case follows from the particular case $0 < \alpha < 1$. First of all, it is well known that in this case,

$$(C([0,T]; X), D(B_X));_{\alpha,\infty} = \{ f \in C^\alpha([0,T]; X) : f(0) = 0 \},$$

(2.6)

with equivalent norms. This is proved in [10], Appendix, if we replace $C([0,T]; X)$ with $C_0([0,T]; X) := \{ f \in C([0,T]; X) : f(0) \}$. If $f \in C([0,T]; X), D(B_X);_{\alpha,\infty}$, $f$ belongs to the closure of $D(B)$ in $C([0,T]; X)$, so that necessarily $f(0) = 0$.

This implies (applying the definition of $(C([0,T]; X), D(B_X));_{\alpha,\infty}$ by the K method (see [18], Chapter 1.2.1) that

$$(C([0,T]; X), D(B_X));_{\alpha,\infty} = (C_0([0,T]; X), D(B));_{\alpha,\infty}.$$ 

So the fact that $D(B_X^\alpha) \subset K_{\alpha-a}(C([0,T]; X), D(B_X))$ follows from Lemma 2.7 and 2.4. By Proposition 1.2.13 in [18], in order to prove that $D(B_X^\alpha) \subset J_{\alpha-[\alpha]}(C([0,T]; X)D(B_X))$ (again in case $0 < \alpha < 1$), it suffices to show that there exists $C$ positive such that, if $f \in D(B_X)$,

$$\|B_X^\alpha f\|_{C([0,T];X)} \leq C\|f\|_{C([0,T];X)}^\alpha \|B_X f\|_{C([0,T];X)}^\alpha.$$ 

This can be shown following the argument in [24], Proposition 2.3.3.

(II) Let $m, n \in \mathbb{N}$, with $m < \alpha_0 < \alpha_1 < n$. Then, by (I), Proposition 2.6 (I) and the reiteration property, if $j \in \{0, 1\}$, $D(B_X^{[\alpha]}) \subset J_{\alpha-[\alpha]}(D(B_X^{[\alpha]})) \cap K_{\alpha-[\alpha]}(D(B_X^{[\alpha]}), D(B_X)).$ So, again by the reiteration theorem,

$$(D(B_X^{[\alpha]}), D(B_X^{[\alpha]+1}));_{\xi,\infty} = (D(B_X^{[\alpha]}), D(B_X^{[\alpha]+1}));_{\xi,\infty}^{(1-\xi)\alpha_0 + \xi\alpha_1 - m, \infty} = (D(B_X^{[\alpha]}), D(B_X^{[\alpha]+1}));_{\xi,\infty}^{(1-\xi)\alpha_0 + \xi\alpha_1 - k, \infty}$$

if $k = [(1 - \xi)\alpha_0 + \xi\alpha_1]$. By the interpolation property, $B_X^k$ is an isomorphism of Banach spaces between $(D(B_X^{[\alpha]}), D(B_X^{[\alpha]+1}));_{\xi,\infty}^{(1-\xi)\alpha_0 + \xi\alpha_1 - k, \infty}$ and $(X, D(B_X));_{\xi,\infty}^{(1-\xi)\alpha_0 + \xi\alpha_1 - k, \infty}$. So

$$(D(B_X^{[\alpha]}), D(B_X^{[\alpha]+1}));_{\xi,\infty}^{(1-\xi)\alpha_0 + \xi\alpha_1 - k, \infty} = \{ f \in D(B_X^{[\alpha]}); f^{(k)} \in C((1-\xi)\alpha_0 + \xi\alpha_1 - k; [0,T]; X), f^{(k)}(0) \},$$

which implies (II). \qed

Now we prove that functions which are representable in a certain way belong to $D(B_X^\alpha)$:

Proposition 2.9. Let $\phi_0 \in (\frac{\pi}{2}, \pi)$, $R \in \mathbb{R}^+$, $\alpha \in [0, \infty)$. Let $F : \{ \lambda \in \mathbb{C} : |\lambda| > R, |Arg(\lambda)| < \phi_0 \} \to X$ be such that:

(a) $F$ is holomorphic;

(b) there exists $M \in \mathbb{R}^+$ such that $|F(\lambda)| \leq M|\lambda|^{-1-\alpha}$, if $\lambda \in \mathbb{C}$, $|\lambda| > R$, $|Arg(\lambda)| < \phi_0$;

(c) for some $F_0 \in X$, $\lim_{|\lambda| \to \infty} \lambda^{1+\alpha}F(\lambda) = F_0$. Let $R' > R$, $\frac{\pi}{2} < \phi_1 < \phi_0$.

We set, for $t \in (0,T]$,

$$u(t) := \begin{cases} \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} F(\lambda) d\lambda & \text{if } 0 < t \leq T, \\ 0 & \text{if } t = 0, \alpha > 0, \\ F_0 & \text{if } t = 0, \alpha = 0. \\ \end{cases}$$

Then $u \in D(B_X^\alpha)$, for $t \in (0,T]$,

$$B_X^\alpha u(t) = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} \lambda^{\alpha} F(\lambda) d\lambda$$

and

$$B_X^\alpha u(0) = F_0.$$
Proof. We begin by considering the case $\alpha = 0$. It is clear that $u \in C((0,T]; X)$. We show that
\[
\lim_{t \to 0} u(t) = F_0.
\]
By standard properties of holomorphic functions, we have, for $t \in (0, \min\{1, T\}]
\[
u(t) = \frac{1}{2\pi i} \int_{\Gamma(R', \phi_1)} e^{\lambda t} d\lambda = \frac{1}{2\pi i} \int_{\Gamma(R', \phi_1)} \frac{\lambda^{\alpha}}{X} t^{-\alpha} \lambda F(t^{-1} \lambda) d\lambda
\]
\[
= F_0 + \frac{1}{2\pi i} \int_{\Gamma(R', \phi_1)} \frac{e^{\lambda t}}{X} \frac{\lambda^{\alpha}}{t^{\alpha}} - F_0 d\lambda
\]
and the second summand vanishes as $t \to 0$, by the dominated convergence theorem.
Suppose now that $\alpha > 0$. We set, for $t \in (0, T]$,
\[
f(t) = \frac{1}{2\pi i} \int e^{\lambda t} \lambda^\alpha d\lambda.
\]
Then, employing what we have seen in case $\alpha = 0$, we deduce $f \in C([0,T]; X)$ and $f(0) = F_0$. We check that
\[
B^{-\alpha}_X f = u.
\]
In fact, if we put
\[
v(t) = B^{-\alpha}_X f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,
\]
and we consider the extensions of $u$, $f$, $v$ to $[0, \infty)$, we have that
\[
\|u(t)\| + \|f(t)\| \leq Ce^{R't}, \quad \forall t \in [0, \infty),
\]
for some $C$ positive. By the inversion formula of the Laplace transform, we have, for $Re(\lambda) > R'$,
\[
\mathcal{L}u(\lambda) = F(\lambda), \quad \mathcal{L}f(\lambda) = \lambda^\alpha F(\lambda).
\]
As
\[
\mathcal{L}(\frac{t^{\alpha-1}}{\Gamma(\alpha)}) = \lambda^{-\alpha}, \quad Re(\lambda) > 0,
\]
we deduce that
\[
\mathcal{L}v(\lambda) = F(\lambda),
\]
so that $u = v$. \qed

Remark 2.10. Suppose that $F$ fulfills the assumptions of Proposition 2.9 with $\alpha = 0$ and the (possible) exception of (c). Then $u \in B([0,T]; X)$.

Now we are able to define the Caputo derivative of order $\alpha$:

Definition 2.11. Let $\alpha \in \mathbb{R}^+$, $u \in C^{[\alpha]}([0,T]; X)$. Then the Caputo derivative of order $\alpha \mathbb{D}_X^\alpha u$ exists if $u - \sum_{k<\alpha} \frac{t^k}{k!} u^{(k)}(0)$ belongs to $D(B^\alpha_X)$ and
\[
\mathbb{D}_X^\alpha u := B^\alpha_X (u - \sum_{k<\alpha} \frac{t^k}{k!} u^{(k)}(0)).
\]

Remark 2.12. It is easy to see that, if $\alpha \in \mathbb{N}$, $\mathbb{D}_X^\alpha u$ exists if and only if $f \in C^{[\alpha]}([0,T]; X)$ and $\mathbb{D}_X^\alpha u = u^{(\alpha)}$. In any case, by Lemma 2.7, $u \in C^{[\alpha]}([0,T]; X)$.

Now we introduce the following unbounded operator $A$: let $A(x, D_x)$ the partial differential operator introduced in (A2). We set
\[
\begin{align*}
D(A) &= \{ u \in \cap_{1 \leq p < \infty} (W^{2,p} (\Omega) \cap W^{1,p}_0 (\Omega)) : A(\cdot, D_x)u \in C(\overline{\Omega}) \}, \\
A u &= A(\cdot, D_x)u.
\end{align*}
\]
**Proposition 2.13.** Suppose that (A1)-(A2) hold. Then:

(I) there exist $\omega$ in $(0, (1 - \frac{\alpha}{2})\pi)$, $R$ and $C$ positive such that

$$\{\lambda \in \mathbb{C} : |\lambda| \geq R, |\text{Arg}(\lambda)| \geq \omega\} \subseteq \rho(-A)$$

and, if $\lambda$ is in this set,

$$\| (\lambda + A)^{-1} \|_{L(\mathbb{C}^n, \mathbb{M})} \leq C|\lambda|^{-1}.$$ 

(II) As a consequence, there exists $\delta \geq 0$, such that $\delta - A$ is a positive operator in $C(\mathbb{M})$ of type $\omega$.

(III) If $\xi \in (0, 1) \setminus \{\frac{1}{2}\}$, $(C(\mathbb{M}), D(A))_{\xi, \infty} = C^{2\xi}_0(\mathbb{M})$, with equivalent norms.

(IV) If $\lambda \in \rho(-A)$ and $f \in C^\theta(\mathbb{M})$, $$(\lambda + A)^{-1}f \in C^{2+\theta}(\mathbb{M}).$$

(V) There exists $C$ positive, such that, if $|\lambda| \geq R$ and $|\text{Arg}(\lambda)| \geq \omega$,

$$\| (\lambda + A)^{-1}f \|_{C^{2+\theta}(\mathbb{M})} + |\lambda|\| (\lambda + A)^{-1}f \|_{C^\theta(\mathbb{M})} \leq C(\|f\|_{C^\theta(\mathbb{M})} + |\lambda|^2\|f\|_{C(\mathbb{M})}).$$  \hspace{1cm} (2.8)

**Proof.** (I) By compactness, there exist $\omega'$ in $(0, (1 - \frac{\alpha}{2})\pi)$, such that $|\text{Arg}(A(x, \xi))| \leq \omega'$ for any $(x, \xi)$ in $\mathbb{M} \times (\mathbb{R}^n \setminus \{0\})$. Let $\omega = (\omega', (1 - \frac{\alpha}{2})\pi)$. So we have that, if $|\psi| \in [\omega, \pi]$,

$$e^{i\psi}r^2 - \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \neq 0$$

implying that $-e^{i\psi}D_t^2 + A(x, D_x)$ is properly elliptic in $\mathbb{R} \times \mathbb{M}$ (see [25], Chapter 3.7). If we have a properly elliptic operator with Dirichlet boundary conditions, the complementing condition is satisfied. So let $p \in (1, \infty)$. We set

$$D(A_p) := W^{2,p}(\mathbb{M}) \cap W^{1,p}_0(\mathbb{M}),$$

$$A_p u = A(\cdot, D_x)u.$$ 

$A_p$ is thought as an unbounded operator in $L^p(\mathbb{M})$. By Theorem 3.8.1 in [25], there exists $R$ positive, such that, if $|\lambda| \geq R$ and $|\text{Arg}(\lambda)| \geq \omega$, then $\lambda \in \rho(-A_p)$ and, for some $C_p$ positive,

$$\| (\lambda + A_p)^{-1} \|_{L^p(\mathbb{M})} \leq C_p|\lambda|^{-1}.$$

Employing the method introduced in [24], we deduce the claim.

(II) follows immediately from (I).

(III) can be proved with the argument in [11], Theorem 3.6.

(IV) Let $\epsilon \in [0, 1]$. We set

$$A_\epsilon(x, D_x) := (1 - \epsilon)\Delta_x + \epsilon A(x, D_x).$$

We observe that

$$|\text{Arg}((1 - \epsilon)|\xi|^2 + \epsilon A(x, \xi))| < (1 - \frac{\alpha}{2})\pi \ \forall \epsilon \in [0, 1], \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$ 

Employing a well known method due to Agmon (see, for example, [25], Chapter 3.7) we can show the following a priori estimate: there exist $R', C$ positive, such that, if $|\lambda| \geq R'$, $|\text{Arg}(\lambda)| \geq \omega$, $\epsilon \in [0, 1]$, $u \in C^{2+\theta}(\mathbb{M})$,

$$\|u\|_{C^{2+\theta}(\mathbb{M})} + |\lambda|\|u\|_{C^\theta(\mathbb{M})} \leq C(\|A_\epsilon u\|_{C^\theta(\mathbb{M})} + |\lambda|^2\|u\|_{C(\mathbb{M})}).$$

If $\epsilon = 0$, it is well known that $(\lambda + A_0)^{-1}f \in C^{2+\theta}(\mathbb{M})$ in case $f \in C^\theta(\mathbb{M})$. So claim (IV) follows from the continuation method if $|\lambda| \geq R'$, $|\text{Arg}(\lambda)| \geq \omega$. The general case can be obtained fixing $\lambda_0$ in this set and recalling that, for any $\lambda$ in $\rho(A)$,

$$(\lambda + A)^{-1} = (\lambda_0 + A)^{-1} + (\lambda_0 - \lambda)(\lambda_0 + A)^{-1}(\lambda + A)^{-1}.$$ 

(V) can be obtained with the argument in [13], Theorem 1.6. \hfill \Box
Remark 2.14. By Proposition 2.13 (I), if \( \omega \in (0, (1 - \frac{\alpha}{2})\pi) \) is such that \( |\text{Arg}(A(x, \xi))| < \omega \) for any \((x, \xi)\) in \( \Omega \times (\mathbb{R}^n \setminus \{0\}) \), for some \( R \) positive

\[
\{ \lambda \in \mathbb{C} : |\lambda| \geq R, |\text{Arg}(\lambda)| \leq \pi - \omega \} \subseteq \rho(A).
\]

Moreover, if \( \lambda \) is in this set

\[
\| (\lambda - A)^{-1} \|_{\mathcal{L}(\mathcal{C}(\Omega))} \leq C|\lambda|^{-1}.
\]

We observe that

\[
\pi - \omega > \frac{\alpha \pi}{2}.
\]

Now we consider the abstract equation

\[
B_X^\alpha u(t) - Au(t) = f(t), \quad t \in [0, T],
\]
with the following general conditions:

\( (E) \) is a complex Banach space, \( \alpha \in (0, 2), A : D(A)(\subseteq X) \to X \) is an operator in \( X \), such that, for some \( \delta \geq 0 \), \( -A \) is positive of type \( \eta \) less than \((1 - \frac{\alpha}{2})\pi\).

We introduce the following

Definition 2.15. Suppose that \( (E) \) holds. A strict solution of \( \text{(2.9)} \) is an element \( u \) of \( D(B_X^\alpha) \cap C([0, T]; D(A)) \) such that \( B_X^\alpha u(t) - Au(t) = f(t) \) \( \forall t \in [0, T] \).

It is convenient to introduce in the space \( Y := C([0, T]; X) \), the operator \( A \), defined as follows:

\[
\begin{align*}
D(A) & = C([0, T]; D(A)), \\
(Au)(t) & = Au(t), \quad t \in [0, T].
\end{align*}
\]

and write \( \text{(2.9)} \) in the form

\[
(B_X^\alpha - \delta)u + (\delta - A)u = f.
\]

We observe that \( B_X^\alpha - \delta \) is positive of type \( \frac{2\alpha}{\pi} \), \( \delta - A \) is positive of type \( \eta \), \( \frac{2\alpha}{\pi} + \eta < \pi \). We observe also that \( \rho(A) = \rho(A) \) and, \( \forall \lambda \in \rho(A), f \in C([0, T]; C(\mathcal{C}(\Omega))), t \in [0, T], \)

\[
[(\lambda - A)^{-1}f](t) = (\lambda - A)^{-1}f(t).
\]

Finally, if \( \lambda \in \rho(B_X^\alpha), \mu \in \rho(A), \) then

\[
(\lambda - B_X^\alpha)^{-1}(\mu - A)^{-1} = (\mu - A)^{-1}(\lambda - B_X^\alpha)^{-1}.
\]

So we are in position to apply a slight generalization of the theory developed in [6], concerning sums of operators with commuting resolvents. This slight generalization can be found in [7], Theorem 2.2. Applying this theorem, we can deduce the following

Proposition 2.16. Suppose that \( (E) \) holds. Then:

(1) for any \( f \) in \( C([0, T]; X) \) \((\text{2.9)} \) has, at most, one strict solution \( u \).

(2) Such strict solution can be represented (if existing) in the form

\[
u = Sf = \frac{1}{2\pi i} \int_{\Gamma(\pi - \eta', R)} (B_X^\alpha - \lambda)^{-1}(\lambda - A)^{-1} f d\lambda,
\]

with \( \eta < \eta' < (1 - \frac{\alpha}{2})\pi \) and \( R \in \mathbb{R}^+ \) such that \( \{ \lambda \in \mathbb{C} : |\lambda| \geq R, |\text{Arg}(\lambda)| \leq \pi - \eta' \} \subseteq \rho(A) \).

Corollary 2.17. Suppose that \((A1)-(A4)\) hold. Consider equation \( \text{(2.9)} \), in case \( X = C(\mathcal{C}(\Omega)) \) and \( A \) is the operator defined in \( \text{(2.7)} \). Then:

(1) for any \( f \) in \( C([0, T]; C(\mathcal{C}(\Omega))) \) \( \text{(2.9)} \) has, at most, one strict solution \( u \).

(2) Such strict solution can be represented (if existing) in the form

\[
u(t) = \frac{1}{2\pi i} \int_{\phi} \left( \int_{(\phi, r)} e^{\lambda(t-s)}(\lambda^\alpha - A)^{-1} d\lambda \right) f(s) ds,
\]

with \( \frac{\pi}{2} < \phi < \frac{\pi - \omega}{\alpha} \), \( \omega \) as in Remark 2.13 \( r \) positive, such that \( \{ \lambda \in \mathbb{C} : |\lambda| \geq r^\alpha, |\text{Arg}(\lambda)| \leq \pi - \omega \} \subseteq \rho(A) \).
The second identity is justified by the estimate (1.3) has a unique strict solution
necessary and sufficient in order that (1.3) has a unique strict solution

Then it follows from Lemma 2.4 (d) that, for any

Suppose that (E) holds. Let

Proof. (1) We consider (2.12), with \( \eta' = \omega \). We fix \( \phi \) as in the statement and \( r \) positive such that \( r^\alpha > R \).
Then it follows from Lemma 2.3 (d) that, for any \( \lambda \in \Gamma(\pi - \omega, R) \), one has

\[
(\lambda - B_X^\alpha)^{-1} f(t) = \frac{1}{2\pi i} \int_0^t \left( \int_{\Gamma(\phi, \tau)} \frac{e^{\mu(t-s)}}{\lambda - \mu} d\mu \right) f(s) ds.
\]

So, from (2.12) we deduce, applying Cauchy’s integral formula, that, for any \( t \) in \([0, T]\),

\[
u(t) = \frac{1}{2\pi i} \int_{\Gamma(\phi, \tau)} e^{\mu(t-s)} \left( \frac{1}{\mu^\alpha - \lambda} \int_{\Gamma(\pi - \omega, R)} (\mu^\alpha - \lambda)^{-1} f(s) ds d\lambda \right) d\mu f(s) ds.
\]

The second identity is justified by the estimate

\[
\int_{\Gamma(\pi - \omega, R)} \left( \int_{\Gamma(\phi, \tau)} e^{\mu(t-s)} \frac{|Re(\mu)|}{|\mu^\alpha - \lambda|} d\mu \right) ||(\lambda - A)^{-1} f(s)|| ds d\lambda
\]

\[
\leq C_1 \int_{\Gamma(\pi - \omega, R)} \int_{\Gamma(\phi, \tau)} \min\{t, |Re(\mu)|^{-1}\} |\lambda|^{-1} (|\mu^\alpha + |\lambda|)^{-1} d\mu |d\lambda|
\]

\[
\leq C_2 \int_{\Gamma(\phi, \tau)} \min\{t, |Re(\mu)|^{-1}\} |\mu|^{-\alpha} ln(|\mu| + 1) d\mu < \infty.
\]

We pass to consider the abstract system (1.3).

Definition 2.18. Let \( X \) be a complex Banach space, \( \alpha \in \mathbb{R}^+ \), \( A \) a closed operator in \( X \), \( f \in C([0, T]; X) \),
\( u_k \in X \) for each \( k \in \mathbb{N} \), \( k < \alpha \). A strict solution \( u \) of (1.3) is an element of \( C([0, T]; D(A)) \), such that \( D^\alpha_X u \) is defined and all the conditions in (1.3) are satisfied pointwise.

The two following results of maximal regularity hold:

Theorem 2.19. Suppose that (E) holds. Let \( \beta \in (0, \min\{1, \alpha\}) \). Then the following conditions are necessary and sufficient in order that (1.3) has a unique strict solution \( u \), with \( D^\alpha_X u \) and \( Au \) (that is, \( Au \)) belonging to \( C^\beta([0, T]; X) \):

(a) \( f \in C^\beta([0, T]; X) \);
(b) \( u_0 \in D(A) \);
(c) \( Au_0 + f(0) \in (X, D(A))_{\beta/\alpha, \infty} \);
(d) if \( \alpha > 1 \), \( u_1 \in (X, D(A))_{1 - \frac{\beta}{\alpha}, \infty} \).

Theorem 2.20. Suppose that (E) holds. Let \( \beta \in (0, 1) \), \( \alpha \beta < 1 \). Then the following conditions are necessary and sufficient in order that (1.3) has a unique strict solution \( u \), with \( D^\alpha_X u \) and \( Au \) belonging to \( C([0, T]; X) \cap B([0, T]; (X, D(A))_{\beta, \infty}) \):

(a) \( f \in C([0, T]; X) \cap B([0, T]; (X, D(A))_{\beta, \infty}) \);
(b) \( u_0 \in D(A), Au_0 \in (X, D(A))_{\beta, \infty} \);
(c) if \( \alpha > 1 \), \( u_1 \in (X, D(A))_{\beta + 1 - \frac{\beta}{\alpha}, \infty} \).

As we already mentioned, the case \( \alpha \in (0, 1] \) is treated in [3]. Concerning the case \( \alpha \in (1, 2) \), the sufficiency of the conditions (a)-(d) and (a)-(c) to get the conclusion is proved in [4]. Their necessity is shown in [2].

Applying Theorems 2.19, 2.20 in the case that \( A \) is the operator defined in (2.7), we deduce, on account of Proposition 2.13.

Corollary 2.21. Suppose that (A1)-(A2) are fulfilled. We consider system (1.5) in the case \( X = C(\Omega) \), and \( A \) as in (2.7). Let \( \alpha \in (0, 2) \), \( \beta \in (0, \min\{1, \alpha\}) \). Then the following conditions are necessary and sufficient in order that (1.3) has a unique strict solution \( u \), with \( D^\alpha_X u \) and \( Au \) belonging to \( C^\beta([0, T]; C(\Omega)) \):

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(a) \( f \in C^\beta([0,T];C(\Omega)) \);
(b) \( u_0 \in D(A); \)
(c) \( A(\cdot, D_x)u_0 + f(0) \in (C(\Omega), D(A))_{\beta/\alpha, \infty} \) (\( A(\cdot, D_x)u_0 + f(0) \in C^{2\beta/\alpha}_0(\Omega) \) in case \( \beta \neq \frac{2}{\alpha} \));
(c) if \( \alpha > 1 \), \( u_1 \in (C(\Omega), D(A))_{1 - \frac{\beta}{\alpha}, \infty} \) (\( u_1 \in C_0^{2(1 - \frac{1 - \beta}{\alpha})}(\Omega) \) in case \( \beta \neq 1 - \frac{2}{\alpha} \)).

Corollary 2.22. Suppose that (A1)-(A2) are fulfilled. We consider system (7.5) in the case \( X = C(\Omega) \), and \( A \) as in (2.7). Let \( \alpha \in (0, 2), \alpha \theta < 2 \). Then the following conditions are necessary and sufficient in order that (3.3) has a unique strict solution \( u \), with \( D_xu \) and \( Au \) belonging to \( C([0,T]; C(\Omega)) \land B([0,T]; C_0^{\theta}(\Omega)) \):
(a) \( f \in C([0,T]; C(\Omega)) \cap B([0,T]; C_0^{\theta}(\Omega)) \);
(b) \( u_0 \in C^{2+\theta}_0(\Omega), A(\cdot, D_x)u_0 \in C_0^{\theta}(\Omega) \);
(c) if \( \alpha > 1 \), \( u_1 \in (C(\Omega), D(A))_{\frac{1}{2}+1 - \frac{\beta}{2}, \infty} \) (\( u_1 \in C_0^{\theta+2 - \frac{\beta}{2}}(\Omega) \) in case \( \alpha \neq \frac{2}{\theta+2} \)).

3 Proof of Theorem 1.1

We begin with some preliminary results.

Lemma 3.1. Suppose that (A1)-(A4) hold. Let \( u \in C([0,T]; C(\Omega)) \) satisfy (B1)-(B2). Then:
(I) if \( \alpha \in (0, 2) \setminus \{1\} \), \( u \in C^\alpha([0,T]; C^\theta(\Omega)) \);
(II) if \( \alpha = 1 \), \( u \in Lip([0,T]; C^\theta(\Omega)) \);
(III) in any case, \( u \in C^{\frac{\beta}{\alpha} + \frac{2}{\alpha}}([0,T]; C^\theta(\Omega)) \);
(IV) \( u(0) \in C^{2+\theta}(\Omega) \);
(V) if \( \alpha \in (1, 2) \), \( D_tu(0) \in C^{\theta+2(1-1/\alpha)}(\Omega) \).

Proof. (IV) is obvious.

We show (I). We set \( v(t) := u(t) - \sum_{k<\alpha} t^k D^{(k)}_t(0) \). Then, by Lemma 2.7 with \( X = C(\Omega) \), \( X_1 = C^\theta(\Omega) \), \( v \in C^\alpha([0,T]; C^\theta(\Omega)) \). By (IV), in case \( \alpha \in (0, 1) \), we obtain the assertion, because
\[
u(t) = v(t) + u(0).
\]
Assume that \( \alpha \in (1, 2) \). Then, by difference, \( t D_tu(0) = u(t) - u(0) - v(t) \in C^\theta(\Omega) \) for any \( t \in [0,T] \). We deduce that necessarily \( D_tu(0) \in C^\theta(\Omega) \). So the conclusion follows from
\[
u(t) = v(t) + u(0) + t D_tu(0).
\]
The proof of (II) is similar.

We show (V). By Theorem 3.2 in [7], from \( u \in C^\alpha([0,T]; C^\theta(\Omega)) \cap B([0,T]; C^{2+\theta}(\Omega)) \), we deduce that \( D_tu \) is bounded with values in the interpolation space
\[
(C^\theta(\Omega), C^{2+\theta}(\Omega))_{1 - \frac{\beta}{\alpha}, \infty} = C^{\theta+2(1-\frac{1}{\alpha})}(\Omega),
\]
by Lemma 2.4 (II). From this we deduce also that
\[
u \in Lip([0,T]; C^{\theta+2(1-\frac{1}{\alpha})}(\Omega)).
\]

We show (III). We recall that \( C^2(\Omega) \in J_{1-\theta/2}(C^\theta(\Omega)), C^{2+\theta}(\Omega) \) (by Lemma 2.1(II)). So, in case \( \alpha \in (0, 1) \), from (I)-(II) we deduce, \( \forall t, s \in [0,T] \), for some positive constant \( C \),
\[
\|u(t) - u(s)\|_{C^2(\Omega)} \leq C \|u(t) - u(s)\|^\theta_{C^\theta(\Omega)} \|u(t) - u(s)\|_{C^{2+\theta}(\Omega)}^{1-\theta}
\leq 2^{1-\theta/2} C \|u\|^\theta_{C^\alpha([0,T]; C^\theta(\Omega))} \|u\|_{B([0,T]; C^{2+\theta}(\Omega))}^{1-\theta/2} (t-s)^{\alpha \theta/2}.
\]
Suppose that \( \alpha \in (1, 2) \). Then, (3.1) holds. Observe that \( \theta + 2(1 - 1/\alpha) < 2 \), because \( \alpha \theta < 2 \). So the conclusion follows from the fact that
\[
C^2(\Omega) \in J_{1-\alpha \theta/2}(C^{\theta+2(1-1/\alpha)}(\Omega), C^{\theta+2}(\Omega)).
\]
\( \square \)
Lemma 3.2. Suppose that the assumptions (A1)-(A4) are satisfied. Then the conditions (I)-(VI) in the statement of Theorem 1.1 are necessary, in order that there exists a solution \( u \) fulfilling (B1)-(B2).

Proof. (I) is obviously necessary.

The necessity of (II) follows from Lemma 3.1 (IV)-(V).

The necessity of (IV) is clear.

We show that (III) is necessary. First, as \( u \in C([0, T]; C^2(\Omega)) \cap B([0, T]; C^{2+\theta}(\Omega)) \), necessarily \( g = \gamma u \in C([0, T]; C^2(\partial\Omega)) \cap B([0, T]; C^{2+\theta}(\partial\Omega)) \) Next, we set \( h := D_c^a \). Then \( h \in C([0, T]; C(\Omega)) \cap B([0, T]; C^\theta(\Omega)) \) and

\[
 u(t) = \sum_{k<\alpha} t^k u_k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \quad t \in [0, T].
\]

From (IV) we obtain

\[
 g(t) = \sum_{k<\alpha} t^k g^{(k)}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma h(s) ds, \quad t \in [0, T],
\]

implying that \( D_{c(\partial\Omega)}^\alpha g \) is defined and \( D_{c(\partial\Omega)}^\alpha g = \gamma h = \gamma D_{c(\partial\Omega)}^\alpha u \).

So \( D_{c(\partial\Omega)}^\alpha g \) has to belong to \( C([0, T]; C(\partial\Omega)) \cap B([0, T]; C^\theta(\partial\Omega)) \).

We show that (V) is necessary. We have

\[
 \gamma f - D_{c(\partial\Omega)}^\alpha g = \gamma(f - D_{c(\partial\Omega)}^\alpha u) = -\gamma(A(\cdot, D_x)u).
\]

By Lemma 3.1 (III), \( A(\cdot, D_x)u \in C^{\alpha/2}([0, T]; C(\Omega)) \), so that \( \gamma(A(\cdot, D_x)u) \in C^{\alpha/2}([0, T]; C(\partial\Omega)) \).

Finally, (VI) follows from

\[
 \gamma[A(\cdot, D_x)u_0 + f(0)] = \gamma(D_{c(\partial\Omega)}^\alpha u)(0) = D_{c(\partial\Omega)}^\alpha g(0).
\]

\( \square \)

It remains to prove that the assumptions (I)-(VI) of Theorem 1.1 are also sufficient. To this aim, we begin to consider the case \( u_0 = u_1 = 0, \ g \equiv 0 \). So we consider the equation

\[
 B_{c(\Omega)}^\alpha u(t) = Au(t) + f(t), \quad t \in [0, T],
\]

(\( A \) is the operator defined in (2.7)), with the following conditions:

(C1) \( f \in C([0, T]; C(\Omega)) \cap B([0, T]; C^{\theta}(\Omega)) \).

(C2) \( \gamma f \in C^{\alpha/2}([0, T]; C(\partial\Omega)) \).

(C3) \( \gamma[f(0)] = 0 \).

By Corollary 2.17 the unique possible solution of (3.2) is

\[
 u(t) = \int_0^t T(t-s)f(s)ds,
\]

with

\[
 T(t) = \frac{1}{2\pi i} \int_{\Gamma(\phi, r)} e^{\lambda t} (\lambda^\alpha - A)^{-1} d\lambda,
\]

with \( \phi \) in \( \mathbb{(\frac{\pi}{2} + \frac{\pi}{\alpha})} \) and \( r \geq R^{1/\alpha}, \omega \) and \( R \) as in Remark 2.1. Then we have:

Lemma 3.3. Suppose that (C1)-(C3) are satisfied. Then the function \( u \) given by (3.3) is a strict solution of (3.2).
Proof. Let $R$ be the operator introduced in Lemma \[2.1\] (IV). We observe that, for any $t$ in $[0, T]$, \[f(t) = (f(t) - R\gamma f(t)) + R\gamma f(t).\] 

$f - R\gamma f$ belongs to $C([0, T]; C(\Omega)) \cap B([0, T]; C^\alpha(\Omega))$, $R\gamma f$ belongs to $C^\alpha([0, T]; C(\Omega))$ and $R\gamma f(0) = 0$. So the conclusion follows from Corollaries \[2.21\, 2.22\]. \hfill \Box

**Lemma 3.4.** Let $\omega$ and $R$ be as in Proposition \[2.13\]. Let $\xi \in [\theta, 2 + \theta]$. Then there exists $C(\xi)$ positive such that, $\forall \lambda \in \mathbb{C}$, with $|\lambda| \geq R$, $|\text{Arg}(\lambda)| = \omega$, $\forall f \in C^\theta(\Omega)$, \[
\| (\lambda + A)^{-1} f \|_{C^\theta(\Omega)} \leq C(\xi)|\lambda|^{\frac{\omega - \theta}{2\alpha} - 1}(\| f \|_{C^\theta(\Omega)} + \| f \|_{C(\partial\Omega)}).\]

**Proof.** The case $\xi \in \{\theta, 2 + \theta\}$ follows from Proposition \[2.13\] (V). The case $\xi \in (\theta, 2 + \theta)$ follows from the foregoing and Lemma \[2.1\] (I). \hfill \Box

As a consequence, we obtain the following

**Lemma 3.5.** Let us consider the family of operators $(T(t))_{t \geq 0}$, introduced in \[3.4\]. Then $T \in C(\mathbb{R}^+; L(C^\theta(\Omega), C^{2+\theta}(\Omega)))$. Moreover, for any $\xi \in [\theta, 2 + \theta]$ there exists $C(\xi)$ positive, such that $\forall f \in C^\theta(\Omega)$, $t$ in $(0, T]$, \[
\| T(t)f \|_{C^\theta(\Omega)} \leq C(\xi)t^{\theta(1 - \frac{\omega - \theta}{2\alpha}) - 1}(\| f \|_{C^\theta(\Omega)} + t^{-\frac{\omega}{2\alpha}} \| f \|_{C(\partial\Omega)}).\]

**Proof.** Again, we fix $\phi$ in $[\frac{\pi}{2}, \frac{3\pi}{2}]$ and $r \geq R^{1/\alpha}T$, with $\omega$ and $R$ as in Remark \[2.14\]. Then we have \[
T(t)f = \frac{1}{2\pi i} \int_{\Gamma(\phi, r)} e^{\lambda t}(\lambda^\alpha - A)^{-1} f d\lambda = \frac{t^{-1}}{2\pi i} \int_{\Gamma(\phi, r)} e^{\lambda t^{-\alpha}(\lambda^\alpha - A)^{-1}} f d\lambda
\] so that \[
\| T(t)f \|_{C^\theta(\Omega)} \leq C(\xi)t^{-1} \int_{\Gamma(\phi, r)} e^{R\mu |t^{-\alpha} \mu^\alpha|^{\frac{\omega - \theta}{2\alpha} - 1}(\| f \|_{C^\theta(\Omega)} + |t^{-\alpha} \mu^\alpha|^{\frac{\omega}{2\alpha}} \| f \|_{C(\partial\Omega)})} |d\mu|,
\] which implies the statement. \hfill \Box

**Lemma 3.6.** If $t \in \mathbb{R}^+$, we set \[
T_1(t) := \int_0^t T(s) ds. \tag{3.5}
\]

Then $T_1 \in C(\mathbb{R}^+; L(C^\theta(\Omega), C^{2+\theta}(\Omega)))$. Moreover, for any $\xi \in [\theta, 2 + \theta]$ there exists $C(\xi)$ positive, such that $\forall f \in C^\theta(\Omega)$, $t$ in $(0, T]$, \[
\| T_1(t)f \|_{C^\theta(\Omega)} \leq C(\xi)t^{\theta(1 - \frac{\omega}{2\alpha})}(\| f \|_{C^\theta(\Omega)} + t^{-\frac{\omega}{2\alpha}} \| f \|_{C(\partial\Omega)}).\]

**Proof.** We start by observing that, in force of Lemma \[3.3\], the integral in \[3.5\] converges in $L(C^\theta(\Omega), C^\xi(\Omega))$, for any $\xi$ in $[\theta, 2)$. In general, it is easily seen that \[
T_1(t) = \frac{1}{2\pi i} \int_{\Gamma(\phi, r)} e^{\lambda t^{-1}(\lambda^\alpha - A)^{-1}} d\lambda.
\] In fact, the second term vanishes for $t = 0$ and has derivative $T(t)$ for $t$ positive. So the assertion can be obtained with the same method of Lemma \[3.3\]. \hfill \Box

**Lemma 3.7.** Let $\alpha \in (0, 2)$, let $\phi \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, $r, \xi$ positive, such that $r^\alpha < \xi$. Then, for any $t$ positive \[
h(t, \xi) := \frac{1}{2\pi i} \int_{\Gamma(\phi, r)} \frac{e^{\lambda t}}{\lambda^\alpha - \xi} d\lambda = \frac{1}{\pi} \int_0^\infty \frac{e^{-\frac{\tau}{2\alpha}} \sin(\alpha\pi)}{\tau^{2\alpha} - 2\xi \cos(\alpha\pi)\tau^\alpha + \xi^2} d\tau.
\]
Then we have, by Lemma 3.6,

$B$ a strict solution of (3.2). Moreover, which is valid for some $C$ positive, depending on $\alpha$, $\beta$, $\gamma$, $\delta$, $\xi$.

Suppose that (A1)-(A4) and (C1)-(C3) are fulfilled. Let Proposition 3.10.

Proof. The fact that $u$ is a strict solution has been proved in Lemma 3.3. In order to show the remaining part of the assertion, it suffices to show that $u \in C([0, T]; C^2[\Omega])$ and that $Au \in B([0, T]; C^{2+\theta}[\Omega])$, in force of the inequality

$$
\|u\|_{C^{2+\theta}[\Omega]} \leq C(\|u\|_{C^2[\Omega]} + \|Au\|_{C^\theta[\Omega]}),
$$

which is valid for some $C$ positive independent of $u$, by Proposition 2.13.

To this aim, we begin by observing that

$$
u(t) = \int_0^t T(t - s)[f(s) - f(t)]ds + T_1(t)f(t) := u_1(t) + u_2(t).
$$

Then we have, by Lemma 3.6,

$$
\|u_2(t)\|_{C^\theta[\Omega]} \leq C_3t^{\theta/\alpha}(\|f(t)\|_{C^\theta[\Omega]} + t^{-\theta/\beta}\|\gamma_f(t)\|_{C(\Omega)})) \leq C_5 t^{\alpha/\beta},
$$

so that $u_2 \in C([0, T]; C^2[\Omega])$. Moreover, again by Lemma 3.6,

$$
\|Au_2(t)\|_{C^{\theta}[\Omega]} \leq C_3t^{\theta/\alpha}(\|f(t)\|_{C^{\theta}[\Omega]} + t^{-\theta/\beta}\|\gamma_f(t)\|_{C(\Omega)})) \leq C_5.
$$

So $u_2 \in B([0, T]; C^{2+\theta}[\Omega])$.

Now we consider $u_1$. By Lemma 3.3,

$$
\|u_1(t)\|_{C^2[\Omega]} \leq C(2) \int_0^t (t - s)^{\alpha/\beta - 1}(2\|f\|_{B([0, T]; C[\Omega])} + \|\gamma f\|_{C^{\alpha/\beta}[0, T]; C[\Omega]})ds
$$

$$
\leq C_6 t^{\alpha/\beta}(\|f\|_{B([0, T]; C[\Omega])} + \|\gamma f\|_{C^{\alpha/\beta}[0, T]; C[\Omega]}).
$$

So $u_1 \in C([0, T]; C^2[\Omega])$. It remains to estimate $\|Au_1(t)\|_{C^\theta[\Omega]}$. By Proposition 2.10 (III) and Proposition 2.13 (III),

$$
C^\theta[\Omega] = \{f \in C(\Omega) : \sup_{\xi \geq 2R} \xi^\theta \|A(\xi - A)^{-1}f\|_{C(\Omega)} < \infty \}.
$$

**Remark 3.8.** In case $\alpha = 1$, we have $h(t, \xi) = 0$.
with $R$ as in Remark 2.14. Moreover, the norm
\[ f \to \max\{\|f\|_{C^0(\mathcal{M})}, \sup_{\xi \geq 2R} \xi^{\frac{\beta}{2}} \|A(\xi - A)^{-1}f\|_{C^0(\mathcal{M})}\} \]
with $f \in C^0_0(\mathcal{M})$, is equivalent to $\|\cdot\|_{C^0(\mathcal{M})}$. So, in order to complete the proof, we can show that there exists $C$ positive, such that, for any $t$ in $(0, T)$, for any $\xi$ in $[2R, \infty)$,
\[ \|A(\xi - A)^{-1}Au_1(t)\|_{C^0(\mathcal{M})} \leq C\xi^{-\frac{\beta}{2}}. \tag{3.6} \]
We put
\[ \Gamma := \Gamma(\frac{\pi - \omega}{\alpha}, R^{1/\alpha}). \tag{3.7} \]
Let $\xi \in [2R, \infty)$. Then we have, by the resolvent identity,
\[ A(\xi - A)^{-1}AT(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} A(\xi - A)^{-1}A(\lambda^\alpha - A)^{-1}d\lambda = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} A(\xi - A)^{-1}[-1 + \lambda^\alpha(\lambda^\alpha - A)^{-1}]d\lambda = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} A(\xi - A)^{-1}(\lambda^\alpha - A)^{-1}d\lambda = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} A(\xi - A)^{-1}d\lambda = K_1(t, \xi) + K_2(t, \xi), \]
so that
\[ A(\xi - A)^{-1}Au_1(t) = \int_0^t K_1(t - s, \xi)[f(s) - f(t)]ds + \int_0^t K_2(t - s, \xi)[f(s) - f(t)]ds. \]
We have
\[ \| \int_0^t K_1(t - s, \xi)[f(s) - f(t)]ds \|_{C(\mathcal{M})} \leq \frac{1}{2\pi} \int_0^t (\int_{\Gamma_1} e^{(t-s)Re(\lambda)} \frac{|\lambda^\alpha|}{|\xi - \lambda^\alpha|} \|A(\lambda^\alpha - A)^{-1}[f(s) - f(t)]\|_{C(\mathcal{M})}|d\lambda|)ds \leq \frac{C_1}{\xi^{-\frac{\beta}{2}}} \|f\|_{C([0, T]; C^0(\mathcal{M}))}, \]
\[ \int_0^t (\int_{\Gamma_2} e^{(t-s)Re(\lambda)} \frac{|\lambda^\alpha|}{|\xi - \lambda^\alpha|} \|A(\lambda^\alpha - A)^{-1}[f(s) - f(t)]\|_{C(\mathcal{M})}|d\lambda|)ds \leq \frac{C_2}{\xi^{-\frac{\beta}{2}}} \|f\|_{C([0, T]; C(\mathcal{M}))}, \]
\[ \int_0^t (\int_{\Gamma_3} e^{(t-s)Re(\lambda)} \frac{|\lambda^\alpha|}{|\xi - \lambda^\alpha|} \|A(\lambda^\alpha - A)^{-1}[f(s) - f(t)]\|_{C(\mathcal{M})}|d\lambda|)ds \leq \frac{C_3}{\xi^{-\frac{\beta}{2}}} \|f\|_{C([0, T]; C^0(\mathcal{M}))}, \]
By Lemma 3.4 we have
\[ \|A(\lambda^\alpha - A)^{-1}[f(s) - f(t)]\|_{C(\mathcal{M})} \leq \frac{C_4}{\xi^{-\frac{\beta}{2}}} \|f\|_{B([0, T]; C^0(\mathcal{M}))} + (t - s)^{\frac{\beta}{2}} \|\gamma f\|_{C^{\frac{\beta}{2}}([0, T]; C^0(\mathcal{M}))}, \]
\[ \leq \frac{C_5}{\xi^{-\frac{\beta}{2}}} \|f\|_{B([0, T]; C^0(\mathcal{M}))} + (t - s)^{\frac{\beta}{2}} \|\gamma f\|_{C^{\frac{\beta}{2}}([0, T]; C^0(\mathcal{M}))}, \]
\[ \leq \frac{C_6}{\xi^{-\frac{\beta}{2}}} \|f\|_{B([0, T]; C^0(\mathcal{M}))} + (t - s)^{\frac{\beta}{2}} \|\gamma f\|_{C^{\frac{\beta}{2}}([0, T]; C^0(\mathcal{M}))}, \]
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so that

\[ I + J \leq C_4 \left( \int_{\mathbb{R}^+} e^{-\|f\|_{C^0(\mathbb{T})}} d\tau \right) \|f\|_{B([0,T];C^0(\mathbb{T}))} + \int_{\mathbb{R}^+} e^{-\|f\|_{C^0(\mathbb{T})}} d\tau \|\gamma f\|_{C^{2,\theta}(\partial\Omega)} \]

\[ + \int_{\mathbb{R}^+} e^{-\|f\|_{C^0(\mathbb{T})}} d\tau \|\gamma f\|_{C^{2,\theta}(\partial\Omega)} \]

\[ = C_5 \xi^{-\frac{n}{2}} \left( \|f\|_{B([0,T];C^0(\mathbb{T}))} + \|\gamma f\|_{C^{2,\theta}(\partial\Omega)} \right), \]

in force of Lemma 3.9.

Finally, we observe that

\[ \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \frac{\lambda^n}{\lambda^n - \xi} d\lambda = \xi h(t, \xi), \]

so that

\[ \int_0^t K_2(t-s, \xi) [f(s) - f(t)] ds = \int_0^t h(t-s, \xi) A(\xi - A)^{-1} [f(s) - f(t)] ds. \]

So, by Lemmata 3.4, 3.7, 3.9, we have

\[ \| \int_0^t K_2(t-s, \xi) [f(s) - f(t)] ds \|_{C(\mathbb{T})} \]

\[ \leq C_6 \xi \int_0^t \left( \int_{\mathbb{R}^+} e^{-\|f\|_{C^0(\mathbb{T})}} d\tau \right) [A(\xi - A)^{-1} [f(s) - f(t)] \|_{C(\mathbb{T})} \]

\[ \leq C_7 \xi \int_{\mathbb{R}^+} e^{-\|f\|_{C^0(\mathbb{T})}} d\tau \|\gamma f\|_{C^{2,\theta}(\partial\Omega)} \]

\[ + \int_{\mathbb{R}^+} e^{-\|f\|_{C^0(\mathbb{T})}} d\tau \|\gamma f\|_{C^{2,\theta}(\partial\Omega)} \]

\[ = C_8 \xi^{-\theta/2} \left( \|f\|_{B([0,T];C^0(\mathbb{T}))} + \|\gamma f\|_{C^{2,\theta}(\partial\Omega)} \right). \]

So (3.3) holds and the assertion is completely proved.

Now we consider the case \( g \equiv 0 \). We begin with the following

**Lemma 3.11.** Suppose that (A1)-(A4) hold. Moreover,

(I) \( f \in C([0,T];C(\mathbb{T})) \cap B([0,T];C^0(\mathbb{T})) \);

(II) \( u_0 \in C^2(\mathbb{T}) \);

(III) \( \gamma \in C^{\frac{1}{2}}([0,T];C(\mathbb{T})) \);

(IV) \( \gamma (Au_0 + f(0)) = 0 \).

If \( t \in (0,T] \), we set

\[ u(t) := u_0 + \int_0^t T(t-s)[f(s) + Au_0] ds, \]

with \( T(t) \) as in (3.4). Then \( u \) satisfies (B1)-(B2) and is a solution to (1.1), with \( g \equiv 0 \) and, in case \( \alpha \in (1,2), u_1 = 0 \).

**Proof.** We set, for \( t \in (0,T], \)

\[ v(t) := \int_0^t T(t-s)[f(s) + Au_0] ds. \]

Then, by Proposition 3.10 \( v \) is a strict solution to

\[ B_{C(\mathbb{T})}^\alpha v(t) = Av(t) + f(t) + Au_0, \quad t \in [0,T], \]

and, moreover, \( B_{C(\mathbb{T})}^\alpha v \in B([0,T];C^0(\mathbb{T})) \) and \( v \in B([0,T];C^2(\mathbb{T})) \). We deduce that \( u(0) = u_0 \), in case \( \alpha \in (0,1) \), \( D_t u(0) = D_t v(0) = 0 \) (by Lemma 2.7), \( \mathbb{D}_{C(\mathbb{T})}^\alpha u \) is defined and, for \( t \in [0,T], \)

\[ \mathbb{D}_{C(\mathbb{T})}^\alpha u(t) = B_{C(\mathbb{T})}^\alpha v(t) = Av(t) + Au_0 + f(t) = Au(t) + f(t). \]

The fact that \( u \) satisfies (B1)-(B2) is clear.
Lemma 3.12. Suppose that (A1)-(A4) hold, with \( \alpha \in (1,2) \). Let \( u_1 \in C_0^{\theta + 2(1 - \frac{1}{\alpha})}(\Omega) \). We adopt again the convention (3.7) and set, for \( t \in [0,T] \)

\[
u(t) := \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-2}(\lambda^\alpha - A)^{-1} u_1 d\lambda.
\]

Then \( u \) satisfies (B1)-(B2) and is a solution to (1.7), with \( u_0 = 0, f \equiv 0, g \equiv 0 \).

Proof. We put, for \( |\lambda| \geq R^{1/\alpha}, |\text{Arg}(\lambda)| \leq \frac{\pi}{\alpha} \),

\[
F(\lambda) := \lambda^{\alpha-2}(\lambda^\alpha - A)^{-1} u_1.
\]

As \( u_1 \) belongs to the closure of \( D(A) \) in \( C(\Omega) \) (because \( 2u_1 = 0 \)), we have

\[
\lim_{|\lambda| \to \infty} \lambda^2 F(\lambda) = \lim_{|\lambda| \to \infty} \lambda^\alpha (\lambda^\alpha - A)^{-1} u_1 = u_1.
\]

We deduce from Proposition 2.9 that \( u \) belongs to \( D(B_{C(\Omega)}) \) and \( B_{C(\Omega)} u(0) = u_1 \), so that \( u(0) = 0, D_t u(0) = u_1 \). If \( t \in (0,T] \), we have

\[
u(t) - tu_1 = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} [\lambda^{\alpha-2}(\lambda^\alpha - A)^{-1} u_1 - \lambda - u_1] d\lambda = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{-2} A(\lambda^\alpha - A)^{-1} u_1 d\lambda.
\]

By Proposition 2.13 (III), we have

\[
C_0^{\theta + 2(1 - \frac{1}{\alpha})}(\Omega) = (C(\Omega), D(A))_{\frac{\alpha}{2} + 1 - \frac{1}{\alpha}, \infty},
\]

so that, by Proposition 2.6 (III), we have

\[
\|A(\lambda^\alpha - A)^{-1} u_1\|_{C(\Omega)} \leq C|\lambda|^{1-\alpha(\frac{\alpha}{2} + 1)}.
\]

We deduce that

\[
\lim_{|\lambda| \to \infty} |\lambda|^{1+\alpha} \|\lambda^{-2} A(\lambda^\alpha - A)^{-1} u_1\|_{C(\Omega)} = 0.
\]

So by Proposition 2.9 \( D_{C(\Omega)}^\alpha u \) is defined. Moreover, \( \forall t \in [0,T] \)

\[
D_{C(\Omega)}^\alpha u(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-2} A(\lambda^\alpha - A)^{-1} u_1 d\lambda = Au(t).
\]

If remains to show that \( Au \) is bounded with values in \( C^\theta(\Omega) \). To this aim, we introduce the operator \( A_\theta \), defined as follows:

\[
\begin{cases}
D(A_\theta) := \{u \in C^2(\Omega) : Au \in C_0^\theta(\Omega)\}, \\
A_\theta u = Au, \quad u \in D(A_\theta).
\end{cases}
\]

As

\[
C_0^\theta(\Omega) = (C(\Omega), D(A))_{\theta/2, \infty} = (C(\Omega), D(A^2))_{\theta/4, \infty},
\]

\( A_\theta \), as unbounded operator in \( C_0^\theta(\Omega) \), can be taken as operator \( B \) in Proposition 2.6. We have that

\[
D(A_\theta) = (D(A), D(A^2))_{\theta/2, \infty} = (C(\Omega), D(A^2))_{\frac{\alpha}{2} + 1 - \frac{1}{\alpha}, \infty},
\]

with equivalent norms. So, by Proposition 2.6 and the reiteration theorem, we deduce

\[
(C_0^\theta(\Omega), D(A_\theta))_{1-\frac{1}{\alpha}, \infty} = (C(\Omega), D(A^2))_{\frac{\alpha}{2} + 1 (1-\frac{1}{\alpha}), \infty} = (C(\Omega), D(A))_{\frac{\alpha}{2} + 1 - \frac{1}{\alpha}, \infty} = C_0^{\theta + 2(1 - \frac{1}{\alpha})}(\Omega).
\]

We deduce that, if \( \lambda \in \Gamma \),

\[
\|\lambda^{\alpha-2} A(\lambda^\alpha - A)^{-1} u_1\|_{C^\theta(\Omega)} = \|\lambda^{\alpha-2} A(\lambda^\alpha - A)^{-1} u_1\|_{C_0^{\theta + 2(1 - \frac{1}{\alpha})}(\Omega)} \leq C|\lambda|^{-1} \|u_1\|_{C_0^{\theta + 2(1 - \frac{1}{\alpha})}(\Omega)},
\]

so that, by Remark 2.10 \( Au \in B([0,T]; C_0^\theta(\Omega)) \) and \( u \in B([0,T]; C^{2+\theta}(\Omega)). \)

\[\square\]
Corollary 3.13. Suppose that (A1)-(A4) are fulfilled. Consider system (1.1) in case $g \equiv 0$. Then the following conditions are necessary and sufficient, in order that there exists a unique solution $u$ satisfying (B1)-(B2):

1. $f \in C((0, T]; C^2(\Omega)) \cap B([0, T]; C^\theta(\Omega))$.
2. $u_0 \in C^{2+\theta}_0(\Omega)$ and, in case $\alpha \in (1, 2)$, $u_1 \in C^{\alpha+2(1-\frac{1}{2})}_0(\Omega)$.
3. $\gamma f \in C^{2\theta}_0([0, T]; C(\partial\Omega))$.
4. $\gamma_1 f \in C^{2\theta}_0([0, T]; C(\partial\Omega))$.

Proof. The necessity of conditions (I)-(IV) follows from Lemma 3.2. The uniqueness of a solution follows from Proposition 2.16.

Proof of Theorem 1.1. The uniqueness of a solution follows from Proposition 2.16. We prove the existence. Let $R$ be the operator introduced in Lemma 2.1 (IV). We set

$$v(t) := R g(t), \quad t \in [0, T].$$

Then $v \in C([0, T]; C^2(\Omega)) \cap B([0, T]; C^{2+\theta}(\Omega))$. We set $h := D^\alpha_{C(\partial\Omega)}g$. Then, by (IV), if we put $u_1 = 0$ in case $\alpha \in (0, 1]$, we have

$$g(t) = \gamma u_0 + t \gamma u_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) \, ds, \quad t \in [0, T].$$

We deduce

$$v(t) = R \gamma u_0 + t R \gamma u_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R h(s) \, ds, \quad t \in [0, T].$$

so that $D^\alpha_{C(\partial\Omega)} v$ exists and coincides with $R D^\alpha_{C(\partial\Omega)} g$, implying that $D^\alpha_{C(\partial\Omega)} v$ belongs to $C([0, T]; C^2(\Omega)) \cap B([0, T]; C^{2+\theta}(\Omega))$. From this we deduce (applying Lemma 3.3 (III)) that $v \in C^{2\theta}_0([0, T]; C^2(\Omega))$.

Now we take, as new unknown, $w := u - v$. $w$ should solve the system

$$\begin{cases}
D^\alpha_{C(\partial\Omega)} w(t, x) = A(x, D_x) w(t, x) + f(t, x) - D^\alpha_{C(\partial\Omega)} v(t, x) + A(x, D_x) v(t, x), & t \in [0, T], \ x \in \Omega, \\
w(t, x') = 0, & (t, x') \in [0, T] \times \partial\Omega, \\
D^k_x w(0, x) = u_k(x) - R(\gamma u_k)(x), & x \in \Omega, k \in \mathbb{N}_0, k < \alpha.
\end{cases}$$

It is easily seen that Corollary 3.13 is applicable to system (3.9). So there exist a solution $w$ satisfying (B1)-(B2). If we put $u := v + w$, we obtain a solution of (1.1), satisfying (B1)-(B2).

4 Proof of Theorem 1.2

We begin by showing that conditions (I)-(V) in Theorem 1.2 are necessary to get the conclusion. If $u$ has the required regularity, it satisfies also (B1)-(B2). So conditions (I)-(VI) in the statement of Theorem 1.1 are all necessary. It is clear that, necessarily, $f$ should belong to $C^{2\theta}_0([0, T] \times \Omega)$. Moreover, as

$$D^\alpha_{C(\partial\Omega)} g = \gamma D^\alpha_{C(\partial\Omega)} u,$$

necessarily, $D^\alpha_{C(\partial\Omega)} g$ belongs to $C^{2\theta}_0([0, T] \times \partial\Omega)$.

Now we show that these conditions are also sufficient. Following the argument in the proof of Theorem 1.1, we define $v$ as in (3.8). Then

$$v \in C([0, T]; C^2(\Omega)) \cap B([0, T]; C^{2+\theta}(\Omega)),$$
\[ \mathbb{D}_C^{\alpha} v \text{ exists and coincides with } R \mathbb{D}_C^{\alpha} g, \text{ implying that } \mathbb{D}_C^{\alpha} v \text{ belongs to } C^{\alpha,\beta}([0,T]\times\overline{\Omega}). \] By Lemma 3.9, \( v \in C^{\alpha,\beta}([0,T]; C^2(\overline{\Omega})) \), so that \( A(\cdot, D_x)v \) belongs to \( C^{\alpha,\beta}([0,T]\times\overline{\Omega}). \) Subtracting \( v \) to \( u \), we are reduced to consider system \( (\ref{3.9}) \). Arguing as in the proof of Theorem 1.1 we see that its solution \( w \) satisfies \( (B1)-(B2) \). Moreover, \( f - \mathbb{D}_C^{\alpha} v + A(\cdot, D_x)v \) belongs to \( C^{\alpha,\beta}([0,T]; C(\overline{\Omega})) \), \( u_0 - R(\gamma u_0) \in D(A) \), if \( \alpha \in (1,2) \), \( u_1 - R(\gamma u_1) \in C^{\alpha+2(1-\frac{\beta}{\alpha})}([0,T]\times\overline{\Omega}). \)

\[
A(u_0 - R\gamma u_0) + f(0) - \mathbb{D}_C^{\alpha} v(0) + A(\cdot, D_x)v(0) = A(\cdot, D_x)u_0 + f(0) - \mathbb{D}_C^{\alpha} v(0) \in C^{\alpha,\beta},
\]

\[
\gamma[A(\cdot, D_x)u_0 + f(0) - \mathbb{D}_C^{\alpha} v(0)] = \gamma[A(\cdot, D_x)u_0 + f(0)] - \mathbb{D}_C^{\alpha}(0) = 0.
\]

We deduce from Corollary 2.22 that \( \mathbb{D}_C^{\alpha} w \) and \( Aw = A(\cdot, D_x)w \) belong to \( C^{\alpha,\beta}([0,T]; C(\overline{\Omega})) \), so that \( w \) satisfies \( (D1)-(D2) \). The conclusion is that \( u = v + w \) satisfies \( (D1)-(D2) \).

\[ \square \]

**Remark 4.1.** In case \( \alpha = 1 \), \( (D1)-(D2) \) imply that \( u \) belongs to \( C^{1+\frac{\beta}{2}}([0,T]; C(\overline{\Omega})) \), so that \( u \) belongs to \( C^{1+\frac{\beta}{2},\beta}([0,T]\times\overline{\Omega}) \). This suggest that in the general case \( u \) should belong to \( C^{\alpha+\frac{\beta}{2},\beta}([0,T]\times\overline{\Omega}) \).

In case \( \alpha \neq 1 \), \( u \) may satisfy \( (D1)-(D2) \) without belonging to any space \( C^{\alpha+\epsilon}([0,T]; C(\overline{\Omega})) \) for any \( \epsilon \) positive. Consider the following example: let \( \alpha \in (0,2) \setminus \{1\} \). Fix \( f_0 \) in \( C^{2+\theta}(\overline{\Omega}) \setminus \{0\}, \theta \in (0,2) \setminus \{1\} \), and define

\[
\left\{
\begin{array}{l}
u : [0,T] \times \overline{\Omega} \rightarrow C, \\
u(t,x) = \frac{\alpha}{1+\alpha}\frac{f_0(x)}{f_0(0)}
\end{array}\right.
\]

Then \( u \) solves \( (\ref{1.1}) \), if we take \( f(t,x) = f_0(x) - \frac{\alpha}{1+\alpha}\frac{f_0(0)}{f_0(0)} [A(\cdot, D_x)f_0](x) \), \( g \equiv 0, D^k u(0, \cdot) = 0 \) if \( k \in \mathbb{N}_0, k < \alpha \). It is easily seen that in this case the assumptions (I)-(V) of Theorem 1.2 are satisfied. However, \( u \) does not belong to any space \( C^{\alpha+\epsilon}([0,T]; C(\overline{\Omega})) \), for any \( \epsilon \) positive.

Nevertheless, let \( v \in D(B^\alpha_X) \) be such that \( B^\alpha_X v \in C^\beta([0,T]; X) \), with \( \alpha + \beta, \beta \in \mathbb{R}^+ \setminus \mathbb{N} \). Then \( v \) can be represented in the form

\[
v(t) = \sum_{k \in \mathbb{N}_0, k < \beta} t^{k+\alpha} v_k + w(t),
\]

with \( v_k \in X \) for each \( k \), \( w \in C^{\alpha+\beta}([0,T]; X) \), \( w^{(j)}(0) = 0 \), for each \( j \) in \( \mathbb{N}_0, j < \alpha + \beta \) (see \cite{14}, Proposition 12). We deduce that in the situation of Theorem 1.2 at least in case \( \alpha (1 + \frac{\beta}{2}) \notin \mathbb{N}_0 \), the solution \( u \) can be written in the form

\[
u(t) = U(t) + t^\alpha v_0,
\]

with \( v_0 \in C(\overline{\Omega}), U \in C^{\alpha+\frac{\beta}{2}}([0,T]; C(\overline{\Omega})) \).

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