Percolation and Magnetization in the Continuous Spin Ising Model

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Abstract. In the strong coupling limit the partition function of \( SU(2) \) gauge theory can be reduced to that of the continuous spin Ising model with nearest neighbour pair-interactions. The random cluster representation of the continuous spin Ising model in two dimensions is derived through a Fortuin-Kasteleyn transformation, and the properties of the corresponding cluster distribution are analyzed. It is shown that for this model, the magnetic transition is equivalent to the percolation transition of Fortuin-Kasteleyn clusters, using local bond weights. These results are also illustrated by means of numerical simulations.

1 Introduction

It has recently been proposed that the deconfinement transition in \( SU(2) \) gauge theory can be characterized as percolation of Polyakov loop clusters \cite{1}. This idea is based on the fact that \( SU(2) \) gauge theory and the Ising model belong to the same universality class \cite{2}, and that the magnetization transition in the Ising model can be specified as percolation of clusters defined through local bond weights \cite{3}. It thus seems natural that the same holds for the corresponding Polyakov loop clusters in \( SU(2) \) gauge theory, provided suitable bond weights can be defined. In \cite{1}, it was shown that in a lattice formulation effectively corresponding to the strong coupling limit, \( SU(2) \) gauge theory indeed leads to the predicted Ising critical exponents.

In the present paper, we want to consider the classical continuous spin model on \( \mathbb{Z}^2 \) introduced by Griffiths \cite{4}, an Ising model with spins taking values continuously between \(-1\) and \(+1\), and prove that in this case magnetization and percolation transitions coincide. We also present a detailed numerical study of the model with simulations on the 2d square lattice.

For \( SU(2) \) lattice gauge theory it was shown in the strong coupling limit \cite{5} that the partition function can be written in a form which, apart from a factor which depends on the group measure, is the partition function of the classical spin model, with spins varying continuously in some bounded real interval \cite{4}. These continuous spins thus are the natural counterparts of the Polyakov loops in \( SU(2) \) gauge theory. Following the pioneering work \cite{3}, the relationship between the thermodynamical features of spin models and the properties of
the corresponding geometrical clusters have in the past years received considerable attention (see [6] - [11]). We want to address here the random percolation content of the continuous spin Ising model [4] obtained through a Fortuin-Kasteleyn transformation [3] of the partition function. A specific class of geometrical clusters [12] shows a percolation transition whose critical behaviour matches the conventional Ising counterparts. In particular, it can be shown that up to finite constants, the spin magnetization equals the probability of long range cluster connectivity; moreover, the spin susceptibility becomes equal (up to the same constants) to some mean cluster size [13].

The numerical method to be used here is the Wolff algorithm [12]. It is based on the formulation of a single cluster update algorithm for spin systems with unprecedented performances regarding the problem of critical slowing down behaviour near phase transition [12]. It turns out that the Wolff clusters exhibit critical percolation behaviour at the point of the Ising phase transition. Very recently, this specific property of Wolff clusters has been put into a rigorous framework [10], leading to a concept of Wolff measures which provides the theoretical basis for the method to be used below. We shall extend this work to the 2-d continuous spin model and corroborate the results through numerical simulations.

2 The Model.

Let \( \Lambda \subset \mathbb{Z}^2 \) be a finite lattice with edge set \( E \) and vertex set \( V = \{1, \ldots, N\} \) and \( \mathbf{S} \) a random configuration of spin variables \( \mathbf{S} = \{S_i\} \subset \Sigma^V \), where the \( S_{i,i\in V} \) take values independently in \( \Sigma = [-1, +1] \). Consider the Hamiltonian

\[
H(\mathbf{S}) = - \sum_{<i,j>} J_{i,j} S_i S_j,
\]

where the sum is over all nearest neighbour pairs \((<i,j>)\) of spins and \( J_{i,j} > 0 \). Rewriting the spin variables \( S_i = l_i \sigma_i \), where \( \sigma_i = \text{sign}(S_i) = \pm 1 \) and \( l_i = |S_i| \in [0, 1] \), one has

\[
H(\mathbf{S}) = H(\mathbf{\sigma}, l) = - \sum_{i,j} J_{i,j} l_i l_j \sigma_i \sigma_j
\]

2.1 Joint Distribution over Spin Amplitudes and Bond Variables

Following Edwards and Sokal [8], the random cluster representation (RCR) [3, 4] of this model is derived in the following way. Let \( \mathbf{n} \subset \{0, 1\}^E \) be a configuration of bond variables on \( \Lambda \) where,

- \( n_{ij} = 1 \) means that there is a bond between sites \( i \) and \( j \) in \( V \),
- \( n_{ij} = 0 \) is the opposite event.

Consider the following joint probability distribution of the random variables \((\sigma, l, n)\),

\[
\mathbb{P}(\mathbf{S}, \mathbf{n}) = \mathbb{P}(\mathbf{l}, \mathbf{\sigma}, \mathbf{n}) = Z_\Lambda^{-1} \prod_{<i,j>} e^{\beta J_{i,j} l_i l_j} \left[ p_{ij} \delta_{n_{ij},1} \delta_{\sigma_i \sigma_j} + (1 - p_{ij}) \delta_{n_{ij},0} \right]
= Z_\Lambda^{-1} \prod_{<i,j>} w_{\beta,J}(\mathbf{l}, \mathbf{\sigma}, \mathbf{n}),
\]

(2.3)
where \( p_{i,j} = 1 - e^{-2\beta J_{i,j} l_{i,j}} \) is a weighting factor to be interpreted as the probability for a link \( n_{ij} \) to exist \((n_{ij} = 1) \) between two nearest neighbour spins \( S_i \) and \( S_j \) that share the same sign \((\delta_{\sigma_i\sigma_j} = 1) \), and \((1 - p_{i,j}) \) the probability of having no link \((n_{ij} = 0) \) between them. \( Z_\Lambda \) is the partition function of the model given by

\[
Z_\Lambda = \sum_{\{\sigma\}} \prod_i d f_i(l_i) \exp \left[ \beta \sum_{<i,j>} J_{i,j} l_{i,j} \sigma_i \sigma_j \right],
\]

where \( df_i \) is the probability distribution of the spin amplitudes \( l_i \) and \( \beta \) is the inverse temperature.

Starting with the joint distribution \((2.3)\), it is straightforward to express the weights \( w_{\beta,J}(l,\sigma,n) \) in the following way

\[
w_{\beta,J}(l,\sigma,n) = \prod_{<i,j>} e^{\beta J_{i,j} l_{i,j}} \prod_{n_{ij}=1} p_{i,j} \delta_{\sigma_i\sigma_j} \prod_{n_{ij}=0} (1 - p_{ij}).
\]

Now, we write \( \tilde{n} \) for a bond configuration fulfilling the compatibility condition that a bond \( \tilde{n}_{i,j} \) exists between sites \( i \) and \( j \) \((\tilde{n}_{i,j} = 1) \) iff \( \sigma_i = \sigma_j \) and call \( c(\tilde{n}) \) the number of clusters of bonds in the configuration \( \tilde{n} \). Then, the weights \( w_{\beta,J} \) take the form (FK-representation)

\[
w_{\beta,J}(l,\tilde{n}) = \prod_{<i,j>} e^{\beta J_{i,j} l_{i,j}} \left(2^{c(\tilde{n})} \prod_{<i,j>\in\tilde{n}} p_{i,j} \prod_{<i,j>\notin\tilde{n}} (1 - p_{ij})\right).
\]

These weights are called the Wolff weights of the random cluster distribution. The joint distribution on the configurations \((l,\tilde{n})\) provide the right framework to relate the critical behavior in the original spin system and its associated percolation representation.

**Remark 2.1** The Wolff weights \( w_{\beta,J}(l,\tilde{n}) \) differ from the (FK) weights of the Ising case (see \([3,7]\)) by the term \( \prod_{<i,j>} e^{\beta J_{i,j} l_{i,j}} \) that reflect the measure on the spin amplitudes in the continuous case.

### 3 Wolff Cluster Algorithm and Distributions

Non-local cluster Monte Carlo (MC) algorithms have brought significant improvements in the simulation of Ising models near criticality. Starting from the ground-breaking work of Fortuin and Kasteleyn \([3]\), which relates the partition function of spin systems with that of a correlated percolation model, Swendsen and Wang \([5]\) derived a non-local cluster algorithm which drastically reduces the critical slowing down phenomena near the transition point, with a dynamical critical exponent near 0.25, where \( z \) is defined by

\[
\tau = \xi^z.
\]

Here \( \tau \) is the correlation time in MC simulations (measured in MC steps per site) and \( \xi \) is the correlation length. For a system of size \( L \), near the critical point, \( \xi \) scales like \( L^2 \) in two dimensions. In the case of a local update algorithm like Metropolis or Heatbath, \( z \) is found
to be close to 2, so the required time to reach stable configurations at criticality in this case is of order $L^4$, reducing noticeably the possible size of samples to study.

Building on the Swendsen-Wang idea, Wolff [12] improved the method (see below) and derived a non-local update $MC$ scheme with a dynamical critical exponent $z$ smaller than or equal to Swendsen-Wang’s in any dimension and furthermore easier to implement.

3.1 The Wolff Algorithm

First let us recall briefly the main features of the Wolff method. Consider the 2-D ferromagnetic Ising Model on $\mathbb{Z}^2$ with coupling constant $J$ and inverse temperature $\beta$. Starting from a randomly chosen spin $\sigma_0$, visit all nearest neighbours and

- with probability $p_b = 1 - \exp(-2\beta J)$,
- and only if they have the same orientation as $\sigma_0$,

include them in the same cluster as $\sigma_0$; spins not satisfying both conditions are excluded. Repeat iteratively this procedure with newly added spins in the cluster until no more neighbours fulfill the above compatibility condition (III.1). Now flip all the spins in that cluster with probability 1. After that, erase all the bonds and start this procedure again. It turns out that this dynamics verifies the detailed balance condition, i.e. it samples the Gibbs distribution of the Ising model (see [12]). The distinguishing feature of the Wolff method compared to Swendsen-Wang’s is that, in the latter (following Fortuin and Kasteleyn [3]) one needs to build, with the same growing probability $p_b$ as before, all possible clusters of like spins and then, with probability 1/2, flip all the spins in those clusters. Then all the bonds are erased and one starts again from the newly created spin configuration. It can also be shown that this method verifies the detailed balance condition [6]. In summary, the following remarks can be made:

- The building procedure of an individual cluster in both methods is clearly the same, thus the Wolff cluster belongs to the set of Swendsen-Wang clusters.
- When the Wolff cluster is built, the randomly chosen spin has obviously higher probability to fall in a large Swendsen-Wang cluster than in a smaller one.

It results that the distribution of Wolff clusters is given by the distribution of Swendsen-Wang clusters (see below) modified by an additional weight that takes into account the size of the clusters.

**Remark 3.1** A (rather non economic) way of building the Wolff cluster would be to construct the set of all $N$ Swendsen-Wang bond clusters $C_{SW}^i$, $i \in \{1, \ldots, N\}$, then to pick at random a lattice spin $\sigma_\chi$ and flip with probability 1 all the spins belonging to the cluster $C_{SW}^i(\sigma_\chi)$. Obviously, the randomly chosen spin $\sigma_\chi$ has higher probability to sit in a large cluster.

**Remark 3.2** In the case of our model, the bond probability $p_b(S_i, S_j) = 1 - \exp(-2\beta J_{i,j})$ takes also into account the spin amplitudes. It turns out that, the Wolff dynamics (as described above) is no longer ergodic (because spin amplitudes are locally conserved) and must be supplemented by a local update method (like Metropolis or Heatbath) in order to respect detailed balance. In practice it is sufficient to insert a Heatbath sweep after several Wolff
sweeps have been performed. This of course slightly alters the gain brought by the non-local cluster update method concerning critical slowing down near the phase transition.

The proof that the Wolff algorithm applied to our infinite-spin model fulfills both ergodicity criteria and detailed balance conditions will be omitted here since it follows closely the derivations that can be found in [15, 18, 20, 21].

3.2 Conditional Distributions and Properties of Wolff Clusters

We state in this section our main results concerning the properties of the \textit{RC} distribution.

First, it is straightforward to see that integrating \( P(l, \sigma, n) \) over \( n \) gives the marginal distribution of the model with Hamiltonian (2.2).

**Proposition 3.1** The marginal spin distribution in (2.3) gives the Boltzmann weight associated with Hamiltonian (2.2).

Tracing over bond variables in (2.3), one gets

\[
\frac{1}{Z^\lambda} \prod_{<i,j>} \sum_{n_{ij}=0} \left\{ e^{\beta J_{i,j} l_i l_j} \left[ p_{i,j} \delta_{n_{ij},1} \delta_{\sigma_i \sigma_j} + (1 - p_{i,j}) \delta_{n_{ij},0} \right] \right\}
\]

using the expression for \( p_{i,j} \) and re-arranging terms, this can be rewritten as

\[
\frac{1}{Z^\lambda} \prod_{<i,j>} \left\{ e^{\beta J_{i,j} l_i l_j} \left[ p_{i,j} \delta_{\sigma_i \sigma_j} + (1 - p_{i,j}) \right] \right\}
\]

thus we get for this marginal distribution

\[
Z^\lambda \prod_{<i,j>} e^{\beta J_{i,j} l_i l_j \sigma_i \sigma_j} = Z^\lambda \exp \left( \sum_{<i,j>} \beta J_{i,j} l_i l_j \sigma_i \sigma_j \right) = Z^\lambda \exp \left( \sum_{<i,j>} \beta J_{i,j} S_i S_j \right) = \frac{e^{H(S)}}{Z^\lambda},
\]

which is just the Boltzmann weight for the spin configuration \( S \) of our model.

The nice features that make Wolff clusters useful reside in the following propositions:

**Proposition 3.2** The magnetization \( M \) of the spin system is equal to the probability of long range connectivity in the percolation model.

Considering the joint bond-spin distribution (2.3) with weights (2.5), one can ask: given a bond configuration, what is the conditional distribution on the spin \( \sigma \)? Let us assume (+) boundary conditions for the spins (which amounts at fixing \( l_i \sigma_i = 1 \) on the boundary). First, due to the compatibility condition (III.1), if site \( k \) is connected to the boundary through some path in the percolation model, then necessarily \( \sigma_k = +1 \). The same is true with (−) boundary conditions due to the sign symmetry of the exponential factor in the statistical weight. Second, a site that is not connected to the boundary is equally likely to be in a ±1 state. From this it follows that \( M \) is exactly given by the probability of connectivity to
the boundary, thence long range order in the spin system is the same as percolation in the random cluster model.

A more detailed argument will be found in [13]. It is actually easy to show that absence of percolation is equivalent to zero magnetization. Consider again the random cluster model (RCM) characterized by the joint distribution (2.3). First let us state the

**Definition 3.1** Let \( P_{\Lambda,i}(\beta) \) be the probability for a site \( i \) to be connected to the boundary of a finite domain \( \Lambda \subset \mathbb{Z}^2 \) through some path of active bonds in the percolation model. We say that there is no percolation in the RCM iff

\[
\forall i \in \Lambda, \lim_{\Lambda \to \mathbb{Z}^2} P_{\Lambda,i}(\beta) = P_{\infty,i}(\beta) = 0
\]

**Proposition 3.3** Consider the joint bond-spin distribution (2.3) and suppose that for the infinite-spin model at inverse temperature \( \beta \) there corresponds no percolation in the joint correlated geometrical bond representation; then the magnetization vanishes.

The argument is as follows. Let \( \sigma_n = (\sigma_{i_1}, \sigma_{i_2}, ..., \sigma_{i_n}) \) a finite set of spins and \( M^{(2)}(\sigma_n) \) the corresponding magnetization in the spin model with boundary conditions (2.3). Let (2.3) \( -\epsilon \), the boundary condition identical to (2.3) except that each spin value on the boundary has been lowered by an amount \( \epsilon \). Then one has

\[
\langle M_{\Lambda}^{(2)}(\sigma_n + \epsilon) \rangle \equiv \langle M_{\Lambda}^{(2-\epsilon)}(\sigma_n) \rangle
\]

where \( \sigma_n + \epsilon = (\sigma_{i_1} + \epsilon, \sigma_{i_2} + \epsilon, ..., \sigma_{i_n} + \epsilon) \). Clearly the geometrical clusters corresponding to these two boundary conditions are the same, thus for all bond configurations in which sites \( i_1, i_2, ..., i_n \) are not connected to the boundary, the corresponding contributions to the magnetization are the same.

Now, using the uniqueness of the limiting state at given \( \beta \), it is easy to conclude that, in the infinite volume limit of definition (3.1), absence of percolation is the same as zero magnetization.

From the proposition above follow also interesting results concerning high temperature decay of correlations for functions of the spin variables when their relative distance goes to infinity but we will not enter this question here and refer to [13].

The stronger statement of proposition (1.2) whose details of proof will also be found in the last reference (see also [10, 11]) is the following: not only the onset of percolation is the signature of the magnetic ordering but moreover, there exist a constant \( C \) depending only smoothly on \( \beta \) such that

\[
\mathbb{P}_{\infty}(\beta) \geq M(\beta) \geq C \mathbb{P}_{\infty}(\beta)
\]

Explicitly, there is spontaneous magnetization if and only if there is percolation and the critical behaviour is the same for these two quantities. Namely, considering on one hand the order parameter exponent \( \beta \), defined for the spin system by \( M \sim (T_c - T)^{\beta} \), where \( T_c \) is the critical temperature, and on the other hand the order parameter for the percolation model, i.e. the fraction of sites \( \mathbb{P}_{\infty} \) in the percolating cluster, which behaves like \( (p - p_c)^{\tilde{\beta}} \), where \( p_c \) is the percolation threshold. It follows from (3.2) that \( \beta = \tilde{\beta} \).

A similar result as (3.2) can be proven (see [13]) between the magnetic susceptibility of the continuous spin system and its geometric counterpart.

In the following we shall carry out lattice simulations to determine the critical exponents for cluster percolation, to illustrate that these indeed lead to the values of the Ising universality class.
4 Numerical results

We have performed extensive simulations of our model, choosing six different lattice sizes, namely $64^2$, $96^2$, $128^2$, $160^2$, $240^2$ and $300^2$.

Our update step consisted of one heatbath for the spin amplitudes and three Wolff flippings for the signs, which turned out to be a good compromise to reduce sensibly the correlation of the data without making the move be too much time-consuming. Every five updates we measured the quantities of interest. The physical quantities are the energy and the magnetization of the system. The percolation quantities have been evaluated in this way: for a given configuration we consider the sign of the spins corresponding to the sign of the global magnetization, say up. Then we form clusters of spins up using the local bond weights $1 - \exp(-2(J/kT)s_is_j)$. Once we embody all spins up in clusters we assign to each cluster a size $s$ which is the number of spins belonging to it. We say that a cluster percolates if it spans the lattice in both directions, that is if it touches all four sides of the lattice. This choice was made to avoid the possibility that, due to the finite lattice size, one could find more than one percolating cluster, making ambiguous the evaluation of our variables.

Finally we assign to the percolation probability $P$ the value of the cluster size of the percolating cluster divided by the number of lattice sites ($P = 0$ if there is no percolation) and to the average cluster size $S$ the value of the expression

$$S = \sum_s \left( \frac{n_s s^2}{\sum_s n_s s} \right),$$

where $n_s$ is the number of clusters of size $s$ and the sums exclude the percolating cluster.

After some preliminary scans of our program for several values of the temperature $\beta$ ($\beta = J/kT$), we focused on the beta range between 1.07 and 1.11, where the transition seems to take place. The number of iterations for each run goes from 20000 (for $\beta$ values close to the extremes of the range) to 50000 (around the center of the range). We interpolated our data using the density of states method (DSM) [22].

To locate the critical point of the physical transition we used the Binder cumulants

$$g_r = \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2} - 3.$$

Figure 1 shows $g_r$ as a function of $\beta$ for the different lattice sizes we used. The lines cross remarkably at the same point, which suggests that also in our case $g_r$ is a scaling function. As a numerical proof we replot our lines as a function of $(\beta - \beta_{\text{crit}}) L^{1/\nu}$, choosing for the exponent $\nu$ the Onsager value 1. The plot (Figure 2) shows that indeed $g_r$ is a scaling function.

To get the critical point of the percolation transition we used the method suggested in [23], which is based on the same principle. For each $\beta$ value we count the number of configurations where we found a percolating cluster and we divide it by the total number of configurations. If we plot the results as a function of $\beta$ for the different lattice sizes the corresponding lines should cross at the threshold of the geometrical transition (Fig. 3).

As the figures show, the agreement between the two thresholds we found is excellent.

For the evaluation of the exponents we used the $\chi^2$ method [24]. The results we got are reported in Table I.
Figure 1: Binder cumulants as a function of $\beta$ for our six lattice sizes.

Figure 2: Rescaled Binder cumulants. We took $\beta_{\text{crit}} = 1.0932$ and for the exponent $\nu$ the Onsager value 1.
Figure 3: Fraction of percolating samples as a function of $\beta$ for our six lattice sizes.

### Table I

| Critical point | $\beta/\nu$   | $\gamma/\nu$ |
|----------------|---------------|---------------|
| Thermal results | $1.093120^{+0.00120}_{-0.00080}$ | $0.128^{+0.005}_{-0.006}$ | $1.745^{+0.007}_{-0.007}$ |
| Percol. results | $1.093200^{+0.000080}_{-0.000080}$ | $0.130^{+0.028}_{-0.029}$ | $1.753^{+0.006}_{-0.006}$ |

The two thermal exponents' ratios are in good agreement with the Onsager values. Unfortunately we encountered some troubles in deriving the percolation $\beta/\nu$ exponents' ratio because the best fit values change rapidly in the little range of beta values for which the $\chi^2$ corresponds to the 95% confidence level. Consequently, the relative error of $\beta/\nu$ turns out to be large. For a more precise evaluation of this ratio much higher statistics or higher lattice sizes seem to be necessary.

However, the value of the other exponents' ratio $\gamma/\nu$ is in good agreement with the corresponding Onsager value 7/4 and its error seems to exclude the possibility for it to be instead the random percolation ratio 43/24.

### 5 Conclusions

For the continuous spin Ising model, the percolation transition of Fortuin-Kasteleyn clusters is indeed equivalent to the magnetic transition due to spontaneous $Z_2$ symmetry breaking. We have also determined by means of lattice simulations the critical exponents of the percolation variables and shown that they belong to the Ising universality class. The simulation method used here can also be employed in the study of percolation in $SU(N)$ gauge theories, where analytic proofs do not exist. We expect that they will eventually allow a general demonstration that the deconfinement transition in such theories can indeed be defined as Polyakov cluster percolation.
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