BOUNDARY IDEMPOTENTS AND 2-PRECLUSTER-TILTING CATEGORIES

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Abstract. The homological theory of Auslander–Platzeck–Todorov on idempotent ideals laid much of the groundwork for higher Auslander–Reiten theory, providing the key technical lemmas for both higher Auslander correspondence as well as the construction of higher Nakayama algebras, among other results. Given a finite-dimensional algebra $A$ and idempotent $e$, we expand on a criterion of Jasso-Külshammer in order to determine a correspondence between the 2-precluster-tilting subcategories of $\text{mod}(A)$ and $\text{mod}(A/\langle e \rangle)$. This is then applied in the context of generalising dimer algebras on surfaces with boundary idempotent.

1. Introduction

Higher cluster-tilting subcategories were introduced and studied by Iyama [3, 4]. They remain only partially understood, and not easy to find in general. A generalisation to this are higher precluster-tilting subcategories, as introduced by Iyama and Solberg [5]. These higher precluster-tilting subcategories are a weaker version of higher cluster-tilting subcategories, but which are of interest in their own right.

A particular class of algebras where we would expect to find higher (pre)cluster-tilting subcategories are from a higher versions of dimer algebras on a disc the sense of [2]. In recent work [8], we were able to find higher precluster-tilting subcategories, as long as the boundary was omitted. In this work we give an inductive criterion to construct higher precluster-tilting subcategories, and show how this can be applied to boundary idempotents. This criterion is motivated by the following example.

Example 1.1. Given a semisimple algebra $A$ with two vertices 1 and 3, we may add a vertex 2 together with arrows and relations, to produce a (pre)cluster-tilting subcategory in the following ways: we can form an Auslander algebra of type $A_2$:
which has a cluster-tilting subcategory given by $S_1, S_3$ and the projective-injectives. Alternatively, we can form the (self-injective) preprojective algebra $\Pi(A_3)$, obtained from $A_3$ be adding arrows $\alpha: j \to i$ for $\alpha: i \to j \in Q$ with admissible ideal $I$ generated by $\sum_{\alpha \in Q} \alpha \delta - \delta \alpha$.

1 $\longmapsto$ 2 $\longmapsto$ 3

which has a precluster-tilting subcategory given by $S_1, S_3$ and the projective-injectives.

This inductive criterion is based on a result of Jasso–Külshammer (Lemma 3.1), who needed an inductive approach to define higher Nakayama algebras, and show that they have higher cluster-tilting subcategories.

**Theorem 1.2.** Let $A$ be a finite-dimensional algebra and $e$ an idempotent of $A$ and $\tilde{C} \subseteq \mod(A)$ a 2-precluster-tilting subcategory. Let $C \cong \tilde{C} \cap \mod(A/\langle e \rangle)$. If also

(i) $\Hom_A(A/\langle e \rangle, \tilde{C}) \subseteq \tilde{C}$.
(ii) $A/\langle e \rangle \otimes_A \tilde{C} \subseteq \tilde{C}$.

Then $C \subseteq \mod(A/\langle e \rangle)$ is 2-precluster tilting.

Conversely we may (inductively) construct 2-precluster-tilting subcategories.

**Theorem 1.3.** Let $A$ be a finite-dimensional algebra, $e$ an idempotent of $A$ and $C \subseteq \mod(A/\langle e \rangle)$ a 2-precluster-tilting subcategory. If also

(i) $\Ext^1_A(DA, A) = 0$.
(ii) $Ae, D(eA)$ are projective-injective $A$-modules.
(iii) There is an equality of sets

\[
\{X \in C|\Ext^2_A(X, J) \neq 0 \forall J \in \inj(A/\langle 1-e \rangle)\} = \{(\tau_2^-)A|P|P \in \proj(A)\setminus\proj(A/\langle e \rangle)\}.
\]

(iv) There is an equality of sets

\[
\{X \in C|\Ext^2_A(A/\langle 1-e \rangle, X) \neq 0\} = \{(\tau_2)A|I|I \in \inj(A)\setminus\inj(A/\langle e \rangle)\}.
\]

Then $C \cup \proj(A) \cup \inj(A) =: \tilde{C} \subseteq \mod(A)$ is 2-precluster tilting.

The final two conditions have a combinatorial meaning in terms of the relations in the algebra. For a 2-(internally)-Calabi Yau algebras, for example the preprojective algebra above, we expect significant simplifications of the above conditions. Likewise if proj.dim$(S)$ = inj.dim$(S)$ = 1, which is often the case for a simple module over a higher Nakayama algebra [7].

2. Background and Notation

Consider a finite-dimensional algebra $A$ over a field $K$, and fix a positive integer $d$. We will assume that $A$ is of the form $KQ/I$, where $KQ$ is the path algebra over some quiver $Q$ and $I$ is an admissible ideal of $KQ$. For two arrows in $Q$, $\alpha: i \to j$
and \( \beta : j \rightarrow k \), we denote their composition as \( \beta \alpha : i \rightarrow k \). Let \( A^{\text{op}} \) denote the opposite algebra of \( A \). An \( A \)-module will mean a finitely-generated left \( A \)-module; by \( \text{mod}(A) \) we denote the category of \( A \)-modules. The functor \( D = \text{Hom}_K(\cdot, K) \) defines a duality; by \( \otimes \) we mean \( \otimes_K \) and we denote the syzygy by \( \Omega \). Denote by \( \nu := DA \otimes_A \cong D\text{Hom}_A(\cdot, A) \) the Nakayama functor in \( \text{mod}(A) \). Let \( \text{add}(M) \) be the full subcategory of \( \text{mod}(A) \) composed of all \( A \)-modules isomorphic to direct sums of finite direct sums of copies of \( M \).

2.1. Higher precluster-tilting subcategories.

**Definition 2.1.** [4, Definition 2.2] For a finite-dimensional algebra \( A \), a module \( M \in \text{mod}(A) \) is a \( d \)-cluster-tilting module if it satisfies the following conditions:

\[
\text{add}(M) = \{ X \in \text{mod}(A) | \text{Ext}_A^i(M, X) = 0 \forall 0 < i < d \}.
\]

\[
\text{add}(M) = \{ X \in \text{mod}(A) | \text{Ext}_A^i(X, M) = 0 \forall 0 < i < d \}.
\]

In this case \( \text{add}(M) \) is a \( d \)-cluster-tilting subcategory of \( \text{mod}(A) \).

Define \( \tau_d := \tau \Omega^{d-1} \) to be the \( d \)-Auslander–Reiten translation and \( \tau_d^{-1} := \tau^{-1} \Omega^{-(d-1)} \) to be the inverse \( d \)-Auslander–Reiten translation.

**Definition 2.2.** [5, Definition 3.2] For a finite-dimensional algebra \( A \), a module \( M \in \text{mod}(A) \) is \( d \)-precluster tilting if it satisfies the following conditions:

(P1) The module \( M \) is a generator-cogenerator for \( \text{mod}(A) \).

(P2) We have \( \tau_d M \in \text{add}(M) \) and \( \tau_d^{-1} M \in \text{add}(M) \).

(P3) There is an equality \( \text{Ext}_A^i(M, M) = 0 \) for all \( 0 < i < d \).

For a \( d \)-precluster-tilting module \( M \), the subcategory \( \text{add}(M) \subseteq \text{mod}(A) \) is called a \( d \)-precluster-tilting subcategory.

**Proposition 2.3.** [4, Theorem 1.5] We have the following

- If \( \text{Ext}_A^i(M, A) = 0 \) for all \( 0 < i < d \), then \( \text{Ext}_A^i(M, N) \cong D\text{Ext}_A^{d-i}(N, \tau_d M) \) for all \( M \in \text{mod}(A) \) and all \( 0 < i < d \).

- If \( \text{Ext}_A^i(DA, N) = 0 \) for all \( 0 < i < d \), then \( \text{Ext}_A^i(M, N) \cong D\text{Ext}_A^{d-i}(\tau_d^{-1} N, M) \) for all \( N \in \text{mod}(A) \) and all \( 0 < i < d \).

2.2. Homological theory of idempotent ideals. Now we review the some homological theory of idempotent ideals, as introduced by Auslander, Platzeck and Todorov. Throughout this section we will let \( F := \text{Hom}_A(A/\langle e \rangle, -) \).

**Proposition 2.4.** [1 Proposition 1.1] Let \( N \) be an \( A \)-module, and let \( 1 \leq d \leq \infty \). Then the following are equivalent:

(i) \( \text{Ext}_A^i(A/\langle e \rangle, N) = 0 \) for all \( i \) such that \( 0 < i < d \).

(ii) Let \( M \) be in \( \text{mod}(A/\langle e \rangle) \). Then there are isomorphisms:

\[
\text{Ext}_A^i(M, FN)) \rightarrow \text{Ext}_A^i(M, N)
\]

for all \( 0 < i < d \).
A third equivalent condition was incorrectly stated in the original article. The result we will need instead is the following:

**Corollary 2.5.** Let $N$ be an $A$-module, and let $0 \to N \to I_0 \to I_1 \to \cdots \to I_d$ be the beginning of a minimal injective coresolution of $N$ and let $0 < i < d$. Then each equivalent condition of Proposition 2.4 implies

$$0 \to FN \to FI_0 \to \cdots \to FI_d$$

is the beginning of an injective coresolution of $FN$ in $\text{mod}(A/\langle e \rangle)$.

**Proof.** Suppose that $\text{Ext}^i_A(A/\langle e \rangle, N) = 0$ for all $i$ such that $0 < i < d$, and let $C_j := \ker(I_{j-1} \to I_j)$. Then we have an exact sequence:

$$0 \to FN \to \cdots \to FI_{d-2} \to FI_{d-1} \to 0$$

since $\text{Ext}^i_A(A/\langle e \rangle, N) = 0$ for all $0 < i < d$. Moreover, the exact sequences

$$0 \to FC_{d-1} \to FI_{d-1} \to FC_d$$

$$0 \to FC_d \to FI_d$$

combine to give the result. \qed

We note that the resulting injective coresolution is not necessarily minimal. There is now a characterisation

**Proposition 2.6.** [1, Proposition 1.3] Let $A$ be a finite-dimensional algebra and $e$ an idempotent of $A$. Then the following are equivalent

(i) There are isomorphisms $\text{Ext}^i_A(A/\langle e \rangle, M) \to \text{Ext}^i_A(M, N)$ for all $M, N \in \text{mod}(A/\langle e \rangle)$ and all $0 \leq i \leq d$.

(ii) $\text{Ext}^i_A(A/\langle e \rangle, N) = 0$ for all $N \in \text{mod}(A/\langle e \rangle)$ for all $i$ such that $0 < i < d$.

(iii) $\text{Ext}^i_A(A/\langle e \rangle, I) = 0$ for all $I \in \text{inj}(A/\langle e \rangle)$ for all $i$ such that $0 < i < d$.

In this case, the ideal $\langle e \rangle$ is said to be $(d-1)$-idempotent. A related useful result is the following. For a positive integer $d$, we define $I_d$ to be the full subcategory of $\text{mod}(A)$ consisting of the $A$-modules $M$ having an injective resolution

$$0 \to M \to I_0 \to I_1 \to \cdots$$

with $I_j \in \text{add}(I)$ for all $0 \leq i \leq d$.

**Proposition 2.7.** [1, Proposition 2.6] Let $A$ be a finite-dimensional algebra, $e$ an idempotent of $A$ and $I = D(1-e)A$ and $1 \leq d < \infty$. Then the following are equivalent

(i) $N \in I_d$.

(ii) $\text{Ext}^i_A(M, N) = 0$ for all $M \in \text{mod}(A/\langle e \rangle)$ for all $i$ such that $0 \leq i < d$.

(iii) $\text{Ext}^i_A(A/\langle e \rangle, N) = 0$ for all $0 \leq i < d$. 
2.3. Main results.

**Theorem 2.3.** Let $A$ be a finite-dimensional algebra and $e$ an idempotent of $A$ and $\mathcal{C} \subseteq \text{mod}(A)$ a 2-precluster-tilting subcategory. If also

(i) $\text{Hom}_A(A/\langle e \rangle, \mathcal{C}) \subseteq \mathcal{C}$

(ii) $A/\langle e \rangle \otimes_A \mathcal{C} \subseteq \mathcal{C}$.

Then $\mathcal{C} := \mathcal{C} \cap \text{mod}(A/\langle e \rangle) \subseteq \text{mod}(A/\langle e \rangle)$ is 2-precluster tilting.

**Proof.** Suppose $\mathcal{C} \subseteq \text{mod}(A)$ is 2-precluster-tilting. By assumption (i), we have $\text{inj}(A/\langle e \rangle) \subseteq \mathcal{C}$ and by assumption (ii) $\text{proj}(A/\langle e \rangle) \subseteq \mathcal{C}$. So condition (H1) is satisfied. Secondly, Proposition 2.6(iii) implies that $\langle e \rangle$ is 1-idempotent, and hence that $\text{Ext}_A^1(M, N) \cong \text{Ext}_A^1(M, N) = 0$ for all $M, N \in \mathcal{C}$. Hence condition (H3) is also satisfied.

Finally, let $N \in \mathcal{C}$ be a non-injective $A$-module and let

$$0 \to N \to \hat{I}_0 \to \hat{I}_1 \to \hat{I}_2$$

be the beginning of a minimal injective coresolution of $N$ in $\text{mod}(A)$. It follows that $(\tau_2^{-1})_A N = \text{coker}(P_1 \to P_2)$. Let $I_j = \text{Hom}_A(A/\langle e \rangle, \hat{I}_j)$. Then Corollary 2.5 implies that $0 \to N \to I_0 \to I_1 \to I_2$ is the beginning of an injective coresolution of $N$ in $\text{mod}(A/\langle e \rangle)$. Since $I_0$ is necessarily minimal, the only case where non-minimality may arise is a trivial map to a summand of $I_2$. It follows $(\tau_2^{-1})_A N$ is a summand of $\text{coker}(P_1 \to P_2) = \text{Hom}_A(A/\langle e \rangle, (\tau_2^{-1})_A N) \in \mathcal{C}$ by assumption. Hence $(\tau_2^{-1})_A N \in \mathcal{C}$ and dually $\mathcal{C}$ is closed under $(\tau_2)_A$. So condition (H2) holds, and $\mathcal{C} \subseteq \text{mod}(A/\langle e \rangle)$ is a 2-precluster-tilting subcategory. \hfill $\square$

We need a technical result:

**Lemma 2.8.** Let $A$ be a finite-dimensional algebra such that $Ae, D(eA) \in \text{proj-inj}(A)$.

Then for any $M \in \text{mod}(A)$ with minimal injective resolution

$$0 \to M \to I_0 \to I_1 \to I_2,$$

then

$$I_2 \in \text{add}(D(eA)) \iff \text{Ext}_A^2(A/\langle 1-e \rangle, M) = 0.$$

**Proof.** First note $\text{Ext}_A^2(A/\langle 1-e \rangle, M) \cong \text{Hom}_A(A/\langle 1-e \rangle, \Omega^{-2}(M))$. Now any morphism $A/\langle 1-e \rangle \to \Omega^{-2}(M)$ that factors through an injective must factor through an injective summand of $\Omega^{-2}(M)$. This is impossible, since any such summand is in $\text{add}(D(eA))$ and therefore projective by assumption. Hence $\text{Hom}_A(A/\langle 1-e \rangle, \Omega^{-2}(A)) \cong \text{Hom}_A(A/\langle 1-e \rangle, \Omega^{-2}(A))$. So $\text{Ext}_A^2(A/\langle 1-e \rangle, M) \cong \text{Hom}_A(A/\langle 1-e \rangle, \Omega^{-2}(M)) \cong \text{Hom}_A(A/\langle 1-e \rangle, \Omega^{-2}(M))$ and the result follows from Proposition 2.7. \hfill $\square$

**Theorem 2.3.** Let $A$ be a finite-dimensional algebra, $e$ an idempotent of $A$ and $\mathcal{C} \subseteq \text{mod}(A/\langle e \rangle)$ a 2-precluster-tilting subcategory. If also
(i) \( \text{Ext}^1_A(DA, A) = 0. \)
(ii) \( A e, D(eA) \in \text{proj-inj}(A). \)
(iii) There is an equality of sets
\[ \{ X \in C | \text{Ext}^1_A(X, J) \neq 0 \forall J \in \text{inj}(A/\langle 1-e \rangle) \} = \{ (\tau_2^-)_A P | P \in \text{proj}(A) \setminus \text{proj}(A/\langle e \rangle) \}. \]
(iv) There is an equality of sets
\[ \{ X \in C | \text{Ext}^2_A(A/\langle 1 - e \rangle, X) \neq 0 \} = \{ (\tau_2)_AI | I \in \text{inj}(A) \setminus \text{inj}(A/\langle e \rangle) \}. \]

Then \( C \cup \text{proj}(A) \cup \text{inj}(A) =: \hat{C} \subseteq \text{mod}(A) \) is 2-precluster tilting.

Proof. Suppose that \( C \subseteq \text{mod}(A/\langle e \rangle) \) is 2-precluster-tilting. We have that \( \hat{C} \) is a generator-cogenerator by construction. Since \( \text{Ext}^1_A(DA, A) = 0, \) Proposition 2.3 implies for any \( N \in C \cup \text{proj}(A) \) the calculation
\[ \text{Ext}^1_A(DA, N) \cong D\text{Ext}^1(N, (\tau_2)_A DA) \cong 0, \]
since \( (\tau_2)_A DA \in C. \) Dually \( \text{Ext}^1_A(M, A) = 0 \) for all \( M \in C. \) By construction \( \text{Ext}^1_A(P, I) = 0 \) for any \( P \in \text{proj}(A/\langle e \rangle) \) and \( I \in \text{inj}(A/\langle e \rangle) \) (there are no arrows in the quiver of \( A \) from a sink in the quiver of \( A/\langle e \rangle \) to a source in the quiver of \( A/\langle e \rangle). \) So Proposition 2.6 implies \( 0 = \text{Ext}^1_A/M, N = \text{Ext}^1_A(M, N) \) for all \( M, N \in C. \) Hence condition (P3) is also satisfied.

Finally, we have to show closure under \( (\tau_2^-)_A, \) we do this for a given \( X \in C \) with minimal injective resolution in \( \text{mod}(A/\langle e \rangle): \) \( 0 \to X \to I_0 \to I_1 \to I_2, \) where we set \( \tilde{J}, \tilde{J}' \) to be injective \( A \)-modules such that \( \text{Hom}_A(X, \tilde{J}) = 0. \)

(a) \( 0 \to X \to \tilde{I}_0 \to \tilde{I}_1(\oplus \tilde{J}) \to \tilde{I}_2 \oplus \tilde{J}' \) is a minimal injective resolution of \( X \) in \( \text{mod}(A) \): then by assumption \( (\tau_2^-)_A X \in \text{inj}(A) \in \hat{C}. \)
(b) \( 0 \to X \to \tilde{I}_0 \to \tilde{I}_1(\oplus \tilde{J}) \to \tilde{I}_2 \) is an injective resolution of \( X \) in \( \text{mod}(A) \): then simply \( (\tau_2^-)_A X \cong (\tau_2^-)_A/\langle e \rangle X \in \hat{C}. \)
(c) \( 0 \to X \to \tilde{I}_0 \to \tilde{J} \) is an injective resolution of \( X \) in \( \text{mod}(A) \): then \( X \in \text{inj}(A/\langle e \rangle) \) and \( (\tau_2^-)_A X \in \text{proj}(A/\langle e \rangle) \in \hat{C}. \)

Hence \( \hat{C} \) is closed under \( (\tau_2^-)_A, \) and dually also under \( (\tau_2)_A. \) So \( \hat{C} \subseteq \text{mod}(A) \) is a 2-precluster-tilting subcategory. \( \square \)

3. Examples

In this section we will consider algebras with vertices labelled by subsets of \( \{1, 2, \ldots, n\}. \) There is a canonical way of constructing the algebra. Let \( Q_0 \) be a set of \( (d + 1) \)-subsets of \( \{1, 2, \ldots, n\}. \) For \( X, Y \in Q_0, \) define \( Q_1 \) by adding arrows \( \alpha_i(X) : X \to Y \) whenever \( X \setminus \{i\} = Y \setminus \{i + 1\} \) for some \( i \in X. \) Let \( I \) be the admissible ideal of \( KQ \) generated by the elements
\[ \alpha_j(\alpha_i(I)) - \alpha_i(\alpha_j(I)), \]
which range over all $X \in Q_0$. By convention, $\alpha_i(X) = 0$ whenever $X$ or $X \cup \{i + 1\} \setminus \{i\}$ is not a member of $Q_0$. Hence there are zero relations included in the ideal $I$.

3.1. Higher Nakayama algebras. One of the motivating examples comes from higher Nakayama algebras [6]. In order to define higher Nakayama algebras and define higher cluster-tilting subcategories, Jasso and Külshammer make use of the following result, which motivates our main results.

Lemma 3.1. [6 Lemma 1.20] Let $A$ be a finite-dimensional algebra and $\tilde{C}$ a $d$-cluster-tilting subcategory of mod$(A)$. Let $e \in A$ be an idempotent such that the following conditions are satisfied:

- All the projective and all the injective $A/\langle e \rangle$-modules belong to $C$.
- Every indecomposable $M \in C$ which does not lie in mod$(A/\langle e \rangle)$ is projective-injective.

Then $\langle e \rangle$ is a $(d-1)$-idempotent ideal and $C \subseteq$ mod$(A/\langle e \rangle)$ is $d$-cluster tilting.

Using this result, Jasso and Külshammer are able to inductively define higher cluster-tilting subcategories. For example, the following higher Nakayama algebra $A$ has quiver

![Quiver Diagram]

and relations indicated by the dotted arrows. We may easily apply Theorems 1.2 and 1.3 to the idempotent $e_{04}$, since $S_{04}$ has projective dimension and injective dimension 1.

3.2. Boundary idempotents. Scott found a cluster structure on the Grassmannian $\mathbb{C}[\text{Gr}(k, n)]$, with clusters given by non-crossing $k$-subsets of $[n]$. On the other hand, Oppermann and Thomas [9] generalised the cluster structure of triangulations of convex polygons to cyclic polytopes. Combinatorially, a triangulation of a cyclic polytope is given by maximal-by-size collections of non-intertwining subsets, where, given two $k$-subsets $I = \{i_1, i_2, \ldots, i_l\}$ and $J = \{j_1, j_2, \ldots, j_l\}$, then $I$ intertwines $J$ if

$$i_1 < j_1 < i_2 < \cdots < i_l < j_l.$$ 

While no cluster algebra is formed, such triangulations are related to the representation theory of higher Auslander algebras of Dynkin type $A$. Two $k$-subsets
$I$ and $J$ are said to be non-crossing if there do not exist elements $s < t < u < v$ (ordered cyclically) where $s, u \in I - J$, and $t, v \in J - I$.

Oppermann and Thomas \cite{9} were able to describe higher Auslander algebras of type $A$ by maximal collections of non-intertwining subsets, and also to triangulations of cyclic polytopes. In \cite{8}, we extended this to tensor products of higher Auslander algebras of type $A$ by introducing maximal collections of non-$l$-intertwining subsets. Critically, we were only able to find higher precluster-tilting subcategories in general.

Baur, King and Marsh \cite{2} studied dimer algebras on a disc, which are related to maximal non-crossing collections (and tensor products of type $A$ quivers). In their work, boundary idempotents (given by consecutive subsets) play a key role. The criterion in Theorem 1.3 gives us hope that we may inductively add these boundary idempotents back in, when we consider non-intertwining and non-$l$-intertwining collections. This is illustrated in the following example:

**Example 3.2.** Consider the algebra $A$, it can be checked that the idempotent $e = E_{125,236,145,367,147}$ satisfies the conditions for Theorem 1.2 and 1.3. $A/\langle e \rangle$ can be described by a maximal non-intertwining collection of 3-subsets of $\{1, 2, \ldots, 6\}$ $(135, 136, 146)$ as well as a semisimple algebra $(256, 347)$. So $\text{mod}(A/\langle e \rangle)$ contains a 2-cluster-tilting subcategory, hence $\text{mod}(A)$ also contains a (pre)cluster-tilting subcategory.

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256 236 367 347
136
125 135 146 147
145
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