Large $|U_{e3}|$ and Tri-bimaximal Mixing

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Abstract

We investigate in a model-independent way to what extent one can perturb tri-bimaximal mixing in order to generate a sizable value of $|U_{e3}|$, while at the same time keeping solar neutrino mixing near its measured value, which is close to $\sin^2 \theta_{12} = \frac{1}{3}$. Three straightforward breaking mechanisms to generate $|U_{e3}| \simeq 0.1$ are considered. For charged lepton corrections, the suppression of a sizable contribution to $\sin^2 \theta_{12}$ can be achieved if CP violation in neutrino oscillations is almost maximal. Generation of the indicated value of $|U_{e3}| \simeq 0.1$ through renormalization group corrections requires the neutrinos to be quasi-degenerate in mass. The consistency with the allowed range of $\sin^2 \theta_{12}$ together with large running of $|U_{e3}|$ forces one of the Majorana phases to be close to $\pi$. This implies large cancellations in the effective Majorana mass governing neutrino-less double beta ($\beta\beta_{0\nu}$)-decay, constraining it to lie near its minimum allowed value of $m_0 \cos 2\theta_{12}$, where $m_0 \gtrsim 0.1$ eV. Finally, explicit breaking of the neutrino mass matrix in the inverted hierarchical and quasi-degenerate neutrino mass spectrum cases is similarly correlated with the $(\beta\beta)_{0\nu}$-decay effective Majorana mass, although to a lesser extent. The implied values for the atmospheric neutrino mixing angle $\theta_{23}$ are given in all cases.

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1 Introduction

It is a remarkable achievement of experimental neutrino physics to have identified the leading form of lepton mixing, or Pontecorvo-Maki-Nakagawa-Sakata (PMNS), mixing matrix [1] \( U \):

\[
U \simeq U_{\text{TBM}} P , \quad \text{where} \quad U_{\text{TBM}} = \begin{pmatrix}
\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]  

(1)

and \( P = \text{diag}(1, e^{i\alpha_2/2}, e^{i\alpha_3/2}) \) contains the Majorana phases [2, 3]. The above matrix Eq. (1) defines tri-bimaximal mixing (TBM) [4]:

\[
\sin^2 \theta_{12} = \frac{1}{3}, \quad \sin^2 \theta_{23} = \frac{1}{2}, \quad U_{e3} = 0 .
\]

(2)

Currently, the values of \( \sin^2 \theta_{12} \) and \( \sin^2 \theta_{23} \) obtained from global fits of the neutrino oscillation data are indeed very close to those predicted by TBM [5]:

\[
\sin^2 \theta_{23} = 0.466^{+0.073, 0.178}_{-0.058, 0.135}, \\
\sin^2 \theta_{12} = 0.312^{+0.019, 0.063}_{-0.018, 0.049} .
\]

(3)

Here we have given the best-fit values as well as the 1\( \sigma \) and 3\( \sigma \) ranges (see also [6–8]). Obviously, even if Nature has chosen TBM as the lepton mixing scheme, one expects deviations from it on very general grounds. Straightforward examples are charged lepton corrections (i.e., corrections stemming from the diagonalization of the charged lepton mass matrix), renormalization effects, or explicit breaking in the neutrino mass matrix giving rise to TBM. Interestingly, in what regards the third mixing angle \( \theta_{13} \), a weak indication towards a non-vanishing value has recently emerged from a combination of two independent hints in solar, reactor and atmospheric neutrino data. Reference [5] quotes the following best-fit value and 1\( \sigma \) range:

\[
\sin^2 \theta_{13} = 0.016 \pm 0.010 ,
\]

(4)

or \( |U_{e3}| = \sin \theta_{13} = 0.126^{+0.035}_{-0.049} \), or \( \theta_{13} = (7.3^{+2.0}_{-2.8})^{\circ} \). Vanishing \( \theta_{13} \) is thus disfavored at 1.6\( \sigma \). Similar values and ranges have been found in other, independent analyses [10]. We note that the hint in the atmospheric data has been questioned [11], but that the recent MINOS data show an excess of electron events [12], which may be interpreted [13] as another hint for a non-zero \( \theta_{13} \).

The allowed ranges for the mass-squared differences from the current global fit performed in [5] are

\[
\Delta m_{31}^2 = 7.67^{+0.16, 0.52}_{-0.19, 0.53} \times 10^{-5} \text{eV}^2 , \\
|\Delta m_A^2| = 2.39^{+0.11, 0.42}_{-0.08, 0.33} \times 10^{-3} \text{eV}^2 .
\]

(5)

Note that the sign of \( |\Delta m_A^2| \) is \( \sim |\Delta m_{31}^2| \), i.e., the ordering of neutrino masses is still not known. Regarding the neutrino mass scale, there are mainly three possibilities: normal hierarchy (NH) with \( m_1 \ll m_2 \ll m_3 \), inverted hierarchy (IH) with \( m_3 \ll m_1 \approx m_2 \), or

\[ \text{The experimental results are so close to TBM that parameterizations of the PMNS matrix with TBM as the starting point have been proposed [9].] } \]
quasi-degenerate neutrinos (QD) with $m_0^2 = m_1^2 \simeq m_2^2 \simeq m_3^2 \gg \Delta m_{\odot}^2, |\Delta m_A^2|$. The latter requires that $m_{1,2,3} \gtrsim 0.10$ eV. For the QD case also one can still ask the question whether $m_1$ or $m_3$ is the lowest mass, i.e. whether $\Delta m_A^2 > 0$ or $\Delta m_A^2 < 0$.

In the present article we investigate in a model-independent way the possibility of having a sizeable value of $|U_{e3}|$ as a result of a perturbation of tri-bimaximal neutrino mixing. For concreteness, we will use the range of $|U_{e3}|$ in Eq. (4) in our analysis. In the light of expected deviations from TBM, it represents an interesting and testable benchmark scenario for various breaking mechanisms. In this respect, the problem of possible deviations of the neutrino mixing matrix from the TBM form has not been studied in detail (see e.g. Ref. [14] for some qualitative statements on the subject). Very specific perturbations to TBM in the framework of concrete models, with the goal of allowing sizable non-zero $\theta_{13} \simeq 0.1$, have recently been discussed in Refs. [15]. However, a detailed, quantitative and model-independent analysis, in particular in the light of the recent hints for a non-zero $U_{e3}$, has not been performed before and in our opinion is at the present stage both timely and useful.

More specifically, in this paper we consider values of $|U_{e3}| \simeq 0.1$ suggested by Eq. (4) and try to obtain them by starting from TBM. The main challenge is to keep at the same time $\sin^2 \theta_{23}$, and especially $\sin^2 \theta_{12}$, close to their experimentally determined and thus close to the TBM predicted values. As any breaking mechanism introduces correlations between the observables, we are able to make characteristic and testable predictions within each case. Interestingly, all predictions are connected with CP properties of the lepton sector.

The paper is organized as follows. In Section 2 we will start by deviating TBM with charged lepton corrections and find that CP violation in neutrino oscillations gets constrained to be almost maximal by the joint requirement of large $|U_{e3}|$ and small deviations from $\sin^2 \theta_{12} = \frac{1}{3}$. Atmospheric mixing deviates from maximal by order $\sin^2 \theta_{23} = \frac{1}{3} + \mathcal{O}(|U_{e3}|^2)$. Section 3 deals with quantum corrections to TBM and shows that only quasi-degenerate neutrinos in the Minimal Supersymmetric Standard Model (MSSM) can give rise to sizable $|U_{e3}|$, while it is impossible to produce the required $|U_{e3}|$ if the effective theory is the standard model (SM). Solar neutrino mixing is particularly affected by renormalization effects, but the modification of $\theta_{12}$ can be suppressed by certain values of the Majorana CP violating phases. These values in turn influence the magnitude of the effective Majorana mass in neutrino-less double beta ($\beta\beta_0\nu$-)decay, leading to large cancellations. It is worth noting that the quantum corrections, within the context of the MSSM, make $\sin^2 \theta_{12}$ increase, whereas the $1\sigma$ range obtained from global fits lies below $\frac{1}{3}$. Atmospheric neutrino mixing deviates in general from maximal stronger than in the case of charged lepton corrections, namely $\sin^2 \theta_{23} = \frac{1}{2} + \mathcal{O}(|U_{e3}|)$. A similar but weaker correlation between $|U_{e3}|$ and the effective Majorana mass in $\beta\beta_0\nu$-decay is found when we explicitly perturb a neutrino mass matrix which without perturbations would lead to TBM. This possibility is analyzed in Section 4. We also find that in this case sizable corrections to $\sin^2 \theta_{23} = \frac{1}{2}$ of order $|U_{e3}|$ are expected. We finally summarize and conclude in Section 5.

2 Breaking Tri-bimaximal Mixing with Charged Lepton Corrections

The PMNS matrix is, in general, a product of two unitary matrices,

\[ U = U_\ell^T U_\nu, \] (6)
where $U_\nu$ diagonalizes the neutrino mass matrix and $U_\ell$ is associated with the diagonalization of the charged lepton mass matrix. Several authors have discussed charged lepton corrections to various neutrino mixing scenarios [16–20]. It has been shown [18] that, after eliminating the unphysical phases, the matrix which diagonalizes the neutrino mass matrix can be written as:

$$U_\nu = P_\nu \tilde{U}_\nu Q_\nu ,$$

where $\tilde{U}_\nu$ is a “PDG-like” mixing matrix, i.e.,

$$\tilde{U}_\nu = \begin{pmatrix} c_{12}^{\nu} c_{13}^{\nu} & s_{12}^{\nu} c_{13}^{\nu} & s_{13}^{\nu} e^{-i\xi} \\ -s_{12}^{\nu} c_{23}^{\nu} - c_{12}^{\nu} s_{23}^{\nu} s_{13}^{\nu} e^{i\xi} & c_{12}^{\nu} c_{23}^{\nu} - s_{12}^{\nu} s_{23}^{\nu} s_{13}^{\nu} e^{i\xi} & s_{23}^{\nu} c_{13}^{\nu} \\ s_{12}^{\nu} s_{23}^{\nu} - c_{12}^{\nu} c_{23}^{\nu} s_{13}^{\nu} e^{i\xi} & -c_{12}^{\nu} s_{23}^{\nu} s_{13}^{\nu} e^{i\xi} - c_{12}^{\nu} c_{23}^{\nu} s_{13}^{\nu} e^{i\xi} & c_{23}^{\nu} c_{13}^{\nu} \end{pmatrix} .$$

It contains three angles, $\theta_{12}^{\nu}$, $\theta_{23}^{\nu}$, and $\theta_{13}^{\nu}$, and one phase, $\xi$. The diagonal matrices $P_\nu = \text{diag}(1, e^{i\phi}, e^{i\omega})$ and $Q_\nu = \text{diag}(1, e^{i\sigma}, e^{i\tau})$, in general, cannot be neglected. Note, however, that $Q_\nu$ does not affect neutrino oscillation observables [2, 21]. The unitary matrix $U_\ell$ which diagonalizes the charged lepton mass matrix can be written as

$$\tilde{U}_\ell = \begin{pmatrix} c_{12}^{\ell} c_{13}^{\ell} & s_{12}^{\ell} c_{13}^{\ell} & s_{13}^{\ell} e^{-i\psi} \\ -s_{12}^{\ell} c_{23}^{\ell} - c_{12}^{\ell} s_{23}^{\ell} s_{13}^{\ell} e^{i\psi} & c_{12}^{\ell} c_{23}^{\ell} - s_{12}^{\ell} s_{23}^{\ell} s_{13}^{\ell} e^{i\psi} & s_{23}^{\ell} c_{13}^{\ell} \\ s_{12}^{\ell} s_{23}^{\ell} - c_{12}^{\ell} c_{23}^{\ell} s_{13}^{\ell} e^{i\psi} & -c_{12}^{\ell} s_{23}^{\ell} s_{13}^{\ell} e^{i\psi} - c_{12}^{\ell} c_{23}^{\ell} s_{13}^{\ell} e^{i\psi} & c_{23}^{\ell} c_{13}^{\ell} \end{pmatrix} .$$

We have used in $U_{\nu,\ell}$ the obvious abbreviations $c_{ij}^{\nu} = \cos \theta_{ij}^{\nu}$ and $s_{ij}^{\nu} = \sin \theta_{ij}^{\nu}$.

Let us assume next that $U_\nu$ corresponds to TBM, i.e., $\tilde{U}_\nu$ is given by $U_{TBM}$ from Eq. (1). Assume further that the charged lepton corrections are “CKM-like”, i.e. that

$$\sin \theta_{12}^\ell = \lambda , \quad \sin \theta_{23}^\ell = A \lambda^2 , \quad \sin \theta_{13}^\ell = B \lambda^3 ,$$

with $A, B$ real and of order one. We therefore have in mind here a GUT-like scenario, in which tri-bimaximal mixing from the neutrino sector (presumably owing its origin from a see-saw mechanism) is corrected by $U_\ell$, which via some quark-lepton symmetry is related to the CKM mixing. A natural expectation for $\lambda$ is then that it is kindred to the sine of the Cabibbo angle, $\lambda \simeq \sin \theta_C \simeq 0.227$. In scenarios based on $SU(5)$ Grand Unification it often happens that a Clebsch-Gordan coefficient of $\frac{1}{2}$ occurs in between the charged lepton and down quark diagonalization, in which case $\lambda \simeq \frac{1}{3} \sin \theta_C \simeq 0.076$.

In the case of CKM-like corrections it is straightforward to calculate from $U = U_T U_\nu$ the neutrino mixing observables $\sin^2 \theta_{12} = |U_{e2}|^2/(1 - |U_{e3}|^2)$, $\sin^2 \theta_{23} = |U_{\mu 2}|^2/(1 - |U_{e3}|^2)$ and $\sin \theta_{13} = |U_{e3}|$. Moreover, it is of interest to obtain the rephasing invariant

$$J_{CP} = \text{Im} \{ U_{\mu 1}^* U_{\mu 3}^* U_{e3}^* U_{\mu 1} \} ,$$

which controls the magnitude of CP violation in neutrino oscillations [22], generated by the Dirac CP violating phase in the PMNS matrix. In the standard PDG-parametrization of the PMNS matrix we have $J_{CP} = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta$. The result for the observables is

$$\sin^2 \theta_{12} \simeq \frac{1}{3} \left( 1 - 2 \lambda \cos \phi + \frac{1}{2} \lambda^2 \right) , \quad |U_{e3}| \simeq \frac{\lambda}{\sqrt{2}} ,$$

$$\sin^2 \theta_{23} \simeq \frac{1}{2} \left( 1 - \left( \frac{1}{2} - 2 A \cos (2 \omega - \phi) \right) \lambda^2 \right) , \quad J_{CP} \simeq \frac{1}{8} \lambda \sin \phi ,$$

(12)
plus terms of order $\lambda^3$. The magnitude of $|U_{e3}|$ is in our analysis fixed by the range in Eq. (4). Therefore we can estimate the following interesting range for $\lambda$: $\lambda \simeq 0.18^{+0.05}_{-0.07}$.

We can be more general, however, and refine this analysis. To this end, we consider the exact and lengthy expression for $|U_{e3}|$ and use a random number generator to generate the values of $\lambda, A, B, \omega, \phi, \psi$. We let $\lambda$ vary between 0 and 0.3, the phases between 0 and $2\pi$, and $A, B$ within 0.2 and 5. In order to have a hierarchy in $U_{\ell}$, we take care that $\sin^2 \theta_{12}$ is at least five times as large as $\sin \theta_{23}^2$, which in turn is at least five times as large as $\sin \theta_{13}^2$. We obtain then from the requirement of reproducing the 1$\sigma$ ranges of the mixing angles given in Eqs. (3, 4) the range

$$0.104 \leq \lambda \leq 0.247.$$  \hspace{1cm} (13)

This is the range for $\lambda$ we will use for the rest of this Section. Interestingly, the sine of the Cabibbo angle is included in this range, while one third of it is not. It turns out that $\sin^2 \theta_{12}$ can lie anywhere in its currently allowed range given in Eq. (4). In contrast, as can be seen in Eq. (12), atmospheric mixing receives only small corrections of order $\lambda^2$, i.e.,

$$\sin^2 \theta_{23} = \frac{1}{4} \mp O(|U_{e3}|^2).$$

To be quantitative, we find

$$0.437 \leq \sin^2 \theta_{23} \leq 0.533.$$  \hspace{1cm} (14)

From the expressions for the mixing parameters given in Eq. (12), an interesting correlation appears [19]: a sizable value of $|U_{e3}|$, and therefore of $\lambda$, introduces a sizable contribution to $\sin^2 \theta_{12}$ of the same order [4]. To be more precise, we have:

$$\frac{1}{3} - \sin^2 \theta_{12} \simeq \frac{2\sqrt{2}}{3} |U_{e3}| \cos \phi.$$  \hspace{1cm} (15)

The observed value of $\frac{1}{3} - \sin^2 \theta_{12}$ is at 1$\sigma$ between 0.002 and 0.039. Thus, $\cos \phi$ should lie below 0.33, 0.26, or 0.53, if $|U_{e3}|^2 = 0.016, 0.026, or 0.006$. The closer $\sin^2 \theta_{12}$ is to $\frac{1}{3}$, the smaller $\cos \phi$ is. Consequently, $|\sin \phi|$ is close to one and CP violation in neutrino oscillation is “maximal”, in the sense that the invariant describing it takes (as a function of $|U_{e3}|$) almost its maximal value [3].

We illustrate the phenomenology of this framework in Fig. 1. The values of the parameters in $U_{\ell}$ are the same as the ones leading to Eqs. (13) and (14). It is easy to see that $\sin^2 \theta_{23}$ can have values in a limited interval, and that CP violation is very close to maximal, i.e., $\delta = \phi \mod \pi \simeq \pi/2$ or $3\pi/2$. The blue solid lines in Fig. 1 display the maximal value that $|J_{CP}|$ can take. The sign of $\sin \delta$ cannot be predicted, because the charged lepton corrections to the CP conserving quantities $\sin^2 \theta_{12}$ and $|U_{e3}|$ fix only $\cos \phi$, whereas CP violation depends necessarily on $\sin \phi$. Note that atmospheric neutrino mixing can be maximal.

Finally, we note an alternative second type of correction from the relation $U = U_{\ell}^T U_{\nu}$, namely when $U_{\ell}^T$ corresponds to TBM and $U_{\nu}$ is CKM-like [20]. In this case, $|U_{e3}| \simeq \sin \theta_{23}/\sqrt{3}$. The parameter $|U_{e3}|$ is therefore governed by the 23-element of $U_{\nu}$, which by the same arguments as given above for the first case, is expected to be quite small, namely of order $\lambda^2$. Even for the lowest considered value of $|U_{e3}|^2 = 0.006$ this scenario would require that $\sin \theta_{23} = 0.134$, a comparably large number, given the GUT-inspired paradigm of “small corrections” in the relation $U_{\ell}^T U_{\nu}$. We will therefore not discuss this possibility further, except for noting two

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2Similar result holds in the case of hierarchical $U_{\ell}$ and $U_{\nu}$ having bimaximal mixing form [18].

3Solar neutrino mixing then receives correction by the “NLO” term in Eq. (12): $\sin^2 \theta_{12} = \frac{1}{3} \left(1 + \frac{1}{3} \lambda^2\right)$. 

5
Figure 1: Charged lepton corrections to tri-bimaximal mixing. The left plot shows $|U_{e3}|$ against $\sin^2 \theta_{23}$ (the axes cover the whole 1\textsigma range) while the right plot gives $|U_{e3}|$ against $J_{\text{CP}}$. The blue solid lines display the maximal value that $|J_{\text{CP}}|$ can take.

things. First, sizable corrections of order $\lambda$ would arise, in general, for $\sin \theta_{12}$. Suppressing them by choosing a specific value of a CP violating phase is possible, but this phase is not related to CP violation in neutrino oscillations. Second, there would also be a very similar correlation to Eq. (15), namely

$$\sin^2 \theta_{23} - \frac{1}{2} \simeq \sqrt{2} |U_{e3}| \cos \phi.$$  

(16)

Note that in this case the atmospheric neutrino mixing angle is correlated with $|U_{e3}|$ and CP violation in neutrino oscillations. We refer to Ref. [20] for more details on this mixing scenario.

3 Breaking Tri-bimaximal Mixing with Quantum Corrections

Another straightforward breaking mechanism is the application of renormalization group (RG) corrections to TBM [19,23–25], which is essential to be considered if the tri-bimaximal scenario is assumed to have been generated at some high energy scale. In contrast to charged lepton corrections the results now depend on the neutrino mass values and their ordering. In general, in the usual PDG-parametrization of the mixing matrix, the corrections to the mixing angles can be expressed as [23, 24, 26]:

$$\theta_{ij}^{\Lambda} \simeq \theta_{ij}^{\lambda} + C \ k_{ij} \Delta_{\tau} + \mathcal{O}(\Delta_{\tau}^2),$$

(17)

where $\Lambda$ is the high scale at which TBM is implemented and $\lambda$ is the low energy scale at which measurements take place. We will indicate high scale values by a superscript $\Lambda$ in the following, and omit for simplicity the superscript $\lambda$, which would indicate low scale values. Hence we have $\theta_{12}^{\Lambda} = \sin^{-1} \sqrt{1/3}$, $\theta_{23}^{\Lambda} = \pi/4$ and $\theta_{13}^{\Lambda} = 0$. We consider the RG evolution of the neutrino masses and the mixing parameters in the effective theory and for definiteness assume the high scale to be $\Lambda = 10^{12}$ GeV. The low scale is taken to be $\lambda = 10^2$ GeV when the effective theory is the Standard Model (SM), while we take $\lambda = 10^3$ GeV when the effective
theory at low energy is the MSSM. The constant $C$ in Eq. (17) is given by $C = -3/2$ for the SM and $C = +1$ for the MSSM. The result in Eq. (17) is obtained in first order in the parameter

$$\Delta \tau \equiv \left\{ \begin{array}{ll} \frac{m_2^2}{m_1^2} (1 + \tan^2 \beta) \ln \frac{\Lambda}{\Lambda} \simeq 1.4 \cdot 10^{-5} (1 + \tan^2 \beta) & \text{(MSSM)}, \\ \frac{m_3^2}{m_2^2} \ln \frac{\Lambda}{\Lambda} \simeq 1.5 \cdot 10^{-5} & \text{(SM)}, \end{array} \right.$$  

with $\Delta_{e,\mu}$ having been neglected since $m_{e,\mu} \ll m_{\tau}$ and the vev of the Higgs is taken to be $v/\sqrt{2} = 174$ GeV.

The dependence on the neutrino mass and mixing parameters is encoded in [23, 24, 27]

$$k_{12} = \frac{\sqrt{2}}{6} \left| m_1 + m_2 e^{i\alpha_2} \right|^2,$$

$$k_{23} = -\left( \frac{1}{3} \left| m_2 + m_3 e^{i(\alpha_3 - \alpha_2)} \right|^2 + \frac{1}{6} \left| m_1 + m_3 e^{i\alpha_3} \right|^2 \right),$$  

$$k_{13} = -\frac{\sqrt{2}}{6} \left( \left| m_2 + m_3 e^{i(\delta - \alpha_3 - \alpha_2)} \right|^2 - \frac{\Delta_{m_3}^2}{\Delta_{m_3}^2} \right) - \frac{4 m_3^2 \alpha_{m_3}^2}{\Delta_{m_3}^2} \sin^2 \frac{\delta}{2},$$

where we have used $\theta_{ij} = \theta_{ij}^V$, which is correct up to $O(\Delta_\tau)$, and $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$. The running of the masses has been neglected in the above expressions for the $k_{ij}$. The masses are decreasing from high to low scale and run as

$$|m_i| = I_K \left( |m_i^V| + \mu_i \Delta_\tau \right),$$

where $I_K$ is a scalar factor that depends on the SU(2) and U(1) gauge coupling constants and the Yukawa matrix in the up quark sector [27–29] and $\mu_i$ are $O(1)$ numbers. Thus, neglecting the running of masses introduces an error $O(\Delta_\tau)$ in $k_{ij}$ and hence $O(\Delta_\tau)^2$ in $\theta_{ij}$. One also observes that for $|\Delta_\tau| \gtrsim (|m_2^V|^2 - |m_1^V|^2)/|m_0^V|^2$, the $O(\Delta_\tau)$ terms dominate over the $O(\Delta_\tau)$ terms in the evolution of $m_2^2 - m_1^2$ [27]. For such cases Eqs. (19) will no longer be cogent. Thus, for the validity of these equations, we require $(m_2^V)^2 \Delta_\tau \lesssim (m_2^V)^2 - (m_1^V)^2$, which may not be satisfied if $(m_2^V)^2 - (m_1^V)^2$ is indeed very small. We will therefore use the full running equations for the mass matrix itself for the plots and numerical values to be presented. Analytical estimates are made with the expressions of the $k_{ij}$ and, as we show, these estimates can explain the numerical results with a sufficient degree of correctness.

\[4\] Note that the masses appear in both the denominator and numerator of the $k_{ij}$.

| Model | mass ordering | $\sin^2 \theta_{12}$ | $\sin^2 \theta_{23}$ |
|-------|--------------|----------------|----------------|
| SM    | $\Delta m_{31}^2 > 0$ | ⬇️ | ⬇️ |
|       | $\Delta m_{31}^2 < 0$ | ⬆️ | ⬆️ |
| MSSM  | $\Delta m_{31}^2 > 0$ | ⬇️ | ⬇️ |
|       | $\Delta m_{31}^2 < 0$ | ⬆️ | ⬆️ |

Table 1: Direction of RG correction to the observables $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ for the SM and the MSSM and both possible neutrino mass orderings.
There is a subtle issue involved when we consider $k_{13}$ in Eq. (19). As is seen, $k_{13}$ depends on the Dirac CP phase $\delta$ which is unphysical for the case of $\theta_{13} = 0$ at the high scale $\Lambda$. However as discussed in [27,30], the value of $\delta$ at this point depends on the values of the masses and the Majorana phases and RG evolution takes care of that automatically.

For analytical estimates, it is convenient to consider the shift of the mixing angles $\theta_{ij}$ from their initial values. From the above expressions for the $k_{ij}$, and in the limit of $|k_{ij} \Delta_\tau| \ll 1$, one obtains the following expressions for the observables:

$$|\sin \theta_{13}| \simeq |C k_{13} \Delta_\tau|, \quad \sin^2 \theta_{23} \simeq \frac{1}{2} - C k_{23} \Delta_\tau, \quad \sin^2 \theta_{12} - \frac{1}{3} \simeq \frac{2\sqrt{2}}{3} C k_{12} \Delta_\tau. \quad (21)$$

In the spirit of our analysis we require (see Eqs. (11,14)) that $|C k_{13} \Delta_\tau| = 0.077 - 0.161$, while $-C k_{12} \Delta_\tau = 2.8 \cdot 10^{-3} - 4.2 \cdot 10^{-2}$. Note that for the $1\sigma$ range we are taking, $C k_{12} \Delta_\tau$ (and therefore $C$) is supposed to be negative. Hence, within the MSSM the required deviation from TBM cannot be realized. Therefore we use the $3\sigma$ mass-squared differences: $|\Delta m_{ij}^2|$. The RG evolution at low scale the parameters are consistent with the chosen ranges of the TBM scenario, while the masses and the CP phases are chosen randomly so that after starting with the SM, Fig. 2 shows the allowed region in the $m_0 - \sin^2 \theta_{13}$ plane at the low scale $\Lambda$, after performing the RG evolution, for both the normal (left panel) and inverted (right panel) mass orderings. Recall that $m_0^2 \gg |\Delta m_{ij}^2|$ is the common neutrino mass scale

Note that for the same reason any initial value of $\sin^2 \theta_{12} > \frac{1}{3}$ (including bimaximal mixing) at high scale is excluded unless of course highly model-dependent see-saw threshold effects [30,31] are taken into account.
Figure 2: The running of $|U_{e3}|^2 = \sin^2 \theta_{13}$ in SM for both the normal (left panel) and the inverted (right panel) mass orderings. The high scale values of mixing angles are kept fixed at TBM values while the masses and phases are varied randomly such that after RG evolution the parameter values are within current experimental ranges.

for quasi-degenerate neutrinos. As can be seen, to generate values of $|U_{e3}|$ within the range of interest, neutrino masses should exceed the direct limit of 2.3 eV from tritium decay [32], and hence also the more stringent but model-dependent limits from cosmology. We conclude that a high scale value of $\theta_{13} = 0$ is incompatible with the indicated range of $|U_{e3}|$. The dependence of this statement on the initial values of $\theta_{12}^A$ and $\theta_{23}^A$ is moderate and hence this statement is valid in general.

We will focus on the MSSM in what follows. As already stated above, we require the $3\sigma$ ranges of the oscillation parameters to be satisfied, because, strictly speaking, the MSSM cannot reproduce the $1\sigma$ range, due to its prediction of $\sin^2 \theta_{12} \geq \frac{1}{3}$.

Fig. 3 shows the allowed region in the $m_0 - \sin^2 \theta_{13}$ plane, when the effective theory is the MSSM, for $\tan \beta = 5, 20$ and the normal mass ordering. The left panel shows that $\sin^2 \theta_{13}$ lies in the required range when $0.8 \text{eV} \lesssim m_0 \lesssim 1.2 \text{eV}$ for $\tan \beta = 5$, while the allowed mass range becomes $0.2 \text{eV} \lesssim m_0 \lesssim 0.34 \text{eV}$ for $\tan \beta = 20$, as can be seen from the right panel. Thus, the relevant range of $m_0 \tan \beta$ is given by (see below for analytical estimates) $4.1 \lesssim (m_0/\text{eV}) \tan \beta \lesssim 6.9$. Hence the allowed mass ranges depend strongly on $\tan \beta$ and for higher values of $\tan \beta$, lower values of $m_0$ are sufficient to produce the required running of $\theta_{13}$. It has been checked that for a fixed $\tan \beta$ value, there is no significant dependence on the mass ordering, other than the direction of the correction to $\theta_{23}$. From the allowed mass ranges obtained in Fig. 3 it is seen that to have $\sin^2 \theta_{13}$ in the $1\sigma$ range under consideration, we need the neutrinos to be quasi-degenerate even for the MSSM with $\tan \beta = 20$. Fig. 4 shows scatter plots of the allowed region of the neutrino mass scale $m_0$ and the Majorana phase $\alpha_2$, which is particularly important for the running of $\theta_{12}$ [27] (see also [33]). We compare the allowed regions at high and low scale for a normal mass ordering and $\tan \beta = 5, 20$. The scattered plots obtained for the inverted mass ordering show the same characteristics. We see that $|\alpha_2|$ is restricted in a narrow region around $|\alpha_2| = \pi$ for all cases.

In order to explain the plots analytically we consider Eqs. (19) in the QD regime $m_0^2 \gg \Delta m_A^2$.
Figure 3: Scatter plots showing the running of $\sin^2 \theta_{13}$ with $m_0$, for MSSM with normal mass ordering and $\tan \beta = 5, 20$. The high scale mixing angles are fixed at TBM and the masses and phases are varied randomly such that after RG evolution the parameter values are within current experimental ranges. For a given $\tan \beta$, the allowed regions are the same as above for the inverted mass ordering.

to obtain:

$$\left(\sin^2 \theta_{12} - \frac{1}{3}\right)_{\text{QD}} \simeq \frac{4}{9} C \Delta r (1 + \cos \alpha_2) \frac{m_0^2}{\Delta m^2_{\odot}},$$  \hspace{1cm} (22)

$$|\sin \theta_{13}|_{\text{QD}} \simeq \frac{\sqrt{2}}{3} C \Delta r \frac{m_0^2}{\Delta m^2_{\odot}} |(1 + R) \cos(\delta + \alpha_3 - \alpha_2) - \cos(\delta + \alpha_3) + R \cos \delta|,$$  \hspace{1cm} (23)

where $R = \Delta m^2_{\odot}/\Delta m^2_{\odot}$. From Eq. (22) one can understand that the low energy constraint on $\sin^2 \theta_{12}$ from the current experimental data restricts $|\alpha_2|$ to remain close to $\pi$, as shown in Fig. 4 making $(1 + \cos \alpha_2)$ small so that there is less running of $\theta_{12}$ even with large neutrino masses. The plots in Fig. 4 further show that $\alpha_2^A$ is also close to $\pi$ and that $\alpha_2$ stays close to $\pi$ in the course of its RG evolution. This can be estimated from the fact that the running of $\alpha_2$ can be expressed as $\alpha_2^A \simeq \alpha_2^A + a_2 \Delta r$ [24,27] with $a_2 \simeq -2/(3 \Delta m^2_{\odot}) m_1^A m_2^A \sin \alpha_2^A$. From Eq. (22) we note that the maximum running for $\theta_{12}$ is obtained for $\alpha_2 = 0$. In absence of any lower bound on $\theta_{13}$ this value was still allowed [24]. However if we put $\alpha_2 = 0$ in Eq. (23) then the running of $\theta_{13}$ is suppressed by the factor $|R| = \Delta m^2_{\odot}/\Delta m^2_{\odot}$. Thus, the requirement of large running of $\theta_{13}$ disfavors $\alpha_2 = 0$ and further strengthens the bound in the $\alpha_2 - m_0$ plane.

In the limit of $\alpha_2 = \pi$ and quasi-degenerate neutrinos, the maximum value of $|\sin \theta_{13}|$ that can be achieved starting from $\theta_{13}^A = 0$ can be estimated from Eq. (23) as

$$|\sin \theta_{13}|_{\text{QD}} \leq \frac{2 \sqrt{2}}{3} C \Delta r \frac{m_0^2}{\Delta m^2_{\odot}} \left(1 + R\right),$$  \hspace{1cm} (24)

with $\alpha_3 = 0$, $\delta = \pm \pi$ or $\alpha_3 = \mp \pi$, $\delta = 0$. Thus, from Eq. (24) one can estimate that $\theta_{13} \geq 0.077$ requires $m_0 \gtrsim 2.66$ eV for the SM and $m_0 \gtrsim 0.72$ (0.18) eV for $\tan \beta = 5$ (20) with the MSSM. The estimates are in good agreement with the allowed mass ranges obtained from Fig. 2 and Fig. 3 respectively.
Figure 4: Scatter plots in the $m_0 - |\alpha_2|$ plane for MSSM ($\tan \beta = 5, 20$) and normal mass ordering, both for high (black circles) and low (red squares) energy scales. The high scale mixing angles are fixed at the TBM values and the masses and phases are varied randomly at the high scale so that the low energy parameters are consistent with the current experimental data. The data for the inverted mass ordering shows the same variation.

Fig. 5 shows the correlation between the low scale values of $\sin^2 \theta_{13}$ and $\sin^2 \theta_{23}$. For normal ordering $\sin^2 \theta_{23} > \frac{1}{2}$, whereas for inverted ordering $\sin^2 \theta_{23} < \frac{1}{2}$. For normal ordering $\theta_{13}$ and $\theta_{23}$ are correlated, i.e., a higher value of $\theta_{13}$ requires a higher value of $\theta_{23}$. For the inverted ordering the predicted values of the two angles are anti-correlated. The plots obtained with $\tan \beta = 20$ are identical to those shown in Fig. 5 for $\tan \beta = 5$, when the mass ordering is the same. For a different $\tan \beta$ the value of $m_0$ adjusts itself to comply with the low energy cuts on the parameters and the allowed points in the $\sin^2 \theta_{23} - \sin^2 \theta_{13}$ plane remain same.

We note here that maximal atmospheric neutrino mixing is not possible. To be more quantitative, we find that

$$0.55 \leq \sin^2 \theta_{23} \leq 0.64 \quad \text{for } \Delta m_{31}^2 > 0,$$

$$0.33 \leq \sin^2 \theta_{23} \leq 0.45 \quad \text{for } \Delta m_{31}^2 < 0,$$

independent on the value of $\tan \beta$.

In Fig. 6 we plot the effective Majorana mass

$$\langle m \rangle = \cos^2 \theta_{13} \left| m_1 \cos^2 \theta_{12} + m_2 \sin^2 \theta_{12} e^{i \alpha_2} + m_3 \tan^2 \theta_{13} e^{i (\alpha_3 + 2 \delta)} \right|,$$

which governs the rate of $(\beta \beta)_0$-decay at low energy. The scatter points show the values of $\langle m \rangle$ allowed by the low energy neutrino oscillation data after RG analysis. The solid (black) lines indicate the maximum and minimum possible values of $\langle m \rangle$ at low scale for a given $m_0$, obtained by varying the oscillation parameters in their current $3\sigma$ range and the phases between 0 to $2\pi$. The plots show that the effective mass obtained after RG analysis lies close to its minimum allowed range. As can also be seen from Fig. 6 for $\tan \beta = 5$, $\langle m \rangle$ takes values between 0.26 and 0.50 eV, to be compared with the general upper and lower limits of 0.2 eV and 1.4 eV. If $\tan \beta = 20$, then $0.07 \text{ eV} \lesssim \langle m \rangle \lesssim 0.11 \text{ eV}$, while in general the effective Majorana mass could be in between 0.05 eV and 0.34 eV. As can be estimated from Eq. (26), the maximum value that $\langle m \rangle$ can achieve for quasi-degenerate neutrinos is when
\[ \cos 2 \theta_{12} - 2 |U_{e3}|^2 / (1 + \tan^2 \theta_{12}) \]

(27)

As we have seen, RG evolution combined with low energy constraints imply QD neutrinos with \( \alpha_2 \) close to \( \pi \). In this limit

\[ \langle m \rangle_{\text{QD}}_{\alpha_2=\pi} \simeq m_0 \cos 2 \theta_{12} + \tan^2 \theta_{13} e^{i(\alpha_3 + 2\delta)} \]

(28)

and since \( \theta_{13} \) is small at all energy scales, expanding in powers of \( |U_{e3}| = \sin \theta_{13} \) we can write [34]

\[ \langle m \rangle_{\text{QD}}_{\alpha_2=\pi} \simeq m_0 \left( \cos 2 \theta_{12} + \mathcal{O}(|U_{e3}|^2) \right) \]

(29)

Thus, RG evolution constrains \( \langle m \rangle \) towards the minimum allowed value, which is confirmed by the figure.

We can give very simple forms of the neutrino mass matrix in the flavor basis satisfying the above constraints. In general the mass matrix generating TBM reads

\[
\langle m \rangle_{\text{TBM}} = U_{\text{TBM}}^\dagger \cdot \begin{pmatrix}
A & B \\
\frac{1}{2} (A + B + D) & \frac{1}{2} (A + B - D)
\end{pmatrix} U_{\text{TBM}}^\dagger.
\]

(30)

The parameters \( A, B, D \) are in general complex and functions of the neutrino masses and Majorana phases:

\[
A = \frac{1}{3} \left( 2 m_1 + m_2 e^{-i \alpha_2} \right), \quad B = \frac{1}{3} \left( m_2 e^{-i \alpha_2} - m_1 \right), \quad D = m_3 e^{-i \alpha_3}.
\]

(31)
Now to estimate the texture of $m_\nu$ at high scale let us insert $m_{1,2,3} = m_0$, TBM and $\alpha_2 = \pi$. It follows
\[
\frac{3}{m_0} m_\nu \simeq \begin{pmatrix}
1 & -2 & -2 \\
\frac{1}{2} (-1 + 3 e^{-i\alpha_3}) & 1 + 3 e^{-i\alpha_3} & \frac{1}{2} (-1 + 3 e^{-i\alpha_3}) \\
\cdot & \cdot & \cdot
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & -2 & -2 \\
1 & 1 & -2 \\
1 & -2 & 1 \\
\cdot & \cdot & -2
\end{pmatrix}
\]
for $\alpha_3 = 0$, where we have set two specific values of $\alpha_3$. Corrections to these expressions are of order $\sqrt{\Delta m_{\odot}^2/m_0}$, $\sqrt{\Delta m_{\text{atm}}^2/m_0}$ and hence small for QD neutrinos.

4 Breaking Tri-bimaximal Mixing explicitly

We can explicitly break TBM by perturbing the neutrino mass matrix. As for RG effects, we will see that there is crucial dependence on the neutrino mass ordering and values of neutrino masses. In its general form, $m_\nu$ leading to TBM is given in Eq. (30). A possible strategy to perturb TBM, outlined in detail in Ref. [35], is to modify the mass matrix in the following way:
\[
m_\nu = \begin{pmatrix}
A(1 + \epsilon_1) & B(1 + \epsilon_2) & B(1 + \epsilon_3) \\
\cdot & \frac{1}{2}(A + B + D)(1 + \epsilon_4) & \frac{1}{2}(A + B - D)(1 + \epsilon_5) \\
\cdot & \cdot & \frac{1}{2}(A + B + D)(1 + \epsilon_6)
\end{pmatrix}.
\] (32)
The complex perturbation parameters $\epsilon_i$ are taken to be $|\epsilon_i| \leq 0.2$ for $i = 1 - 6$ with their phases $\phi_i$ allowed to lie between zero and $2\pi$. In case of a normal hierarchy, one finds [35] that
$|U_{e3}|^2$ is of order $\epsilon^2 R$, where $\epsilon$ is the magnitude of one of the $\epsilon_i$, and $R = \Delta m^2_{\odot}/\Delta m^2_\Lambda$. Hence, a too small value of $|U_{e3}|^2$ is generated in this case. It turns out that at least $m_1 \simeq 0.015$ eV is required in order to generate $|U_{e3}|^2$ above 0.006. This is illustrated in Fig. 7.

Such values correspond to a scenario with a partial mass hierarchy: $m_1 \simeq m_2 \lesssim m_3$. With the increase of $m_1$ starting from 0.015 eV, the maximal value of $|U_{e3}|$ grows almost linearly with $m_1$. In contrast, in the case of inverted hierarchy (ordering), one can generate large values of $|U_{e3}|$ even for a vanishing value of the smallest neutrinos mass $m_3$. For quasi-degenerate neutrinos, obviously, sizeable values of $|U_{e3}| \simeq 0.1$ can also be generated. In addition, in the cases of neutrino mass spectrum with partial hierarchy, with inverted hierarchy and of quasi-degenerate type, there exists a correlation between the effective Majorana mass in $(\beta\beta)_{0\nu}$-decay and the value of $|U_{e3}|$ thus generated.

To illustrate the above comments, consider the following analytic estimates in the case of spectrum with inverted ordering. We first set $\alpha_2 = \pi$. In this case one has $A \simeq \sqrt{\Delta m^2_\Lambda}/3$ and $B \simeq -2\sqrt{\Delta m^2_\Lambda}/3$. Consider now a perturbation of the form

$$m_\nu = \begin{pmatrix} A & B (1 + \epsilon) & B (1 - \epsilon) \\ \cdot & \frac{1}{2}(A + B + D) & \frac{1}{2}(A + B - D) \\ \cdot & \cdot & \frac{1}{2}(A + B + D) \end{pmatrix}. \tag{33}$$

for real $\epsilon$, either negative or positive. In this case we get the largest effects on $|U_{e3}|$ and $\sin^2 \theta_{23}$ [35]:

$$|U_{e3}|^2 \simeq \epsilon^2 \left( \frac{8}{81} + \frac{16}{27} \frac{m_3}{\sqrt{\Delta m^2_\Lambda}} \right) \lesssim 10^{-2} \quad \text{and} \quad \left| \sin^2 \theta_{23} - \frac{1}{2} \right| \simeq \frac{8}{9} \epsilon^2 \gtrsim 0.18. \tag{34}$$

Note that $A \simeq \sqrt{\Delta m^2_\Lambda}/3 \simeq 0.016$ eV is the minimal possible value of the $(\beta\beta)_{0\nu}$-decay effective Majorana mass in the case of spectrum with inverted hierarchy under discussion. The cancellation arises due to the chosen CP conserving value of the Majorana phase $\alpha_2$.

In the other extreme case of $\alpha_2 = 0$, we have $B/A \simeq \frac{1}{6} \Delta m^2_\odot/\Delta m^2_\Lambda$ and we find that $|U_{e3}|^2$ is at most of order $(\epsilon B/A)^2 \simeq 10^{-6}$ and therefore completely negligible.
Figure 8: Scatter plot of $|U_{e3}|$ against $\sin^2 \theta_{23}$ as well as of $|U_{e3}|$ against $\langle m \rangle$ for an explicitly broken TBM mass matrix in case of an inverted hierarchy. The left plots show the cases $\alpha_2 = 0$ and $\alpha_2 = \pi$, the right plots have free $\alpha_2$. Indicated are also the 1$\sigma$ ranges of the oscillation parameters, and the upper and lower limits of the effective mass.

In the case of perturbed $\mu\mu$ and $\tau\tau$ entries of $m_\nu$,

$$m_\nu = \begin{pmatrix} A & B \\ \cdot & \frac{1}{2}(A + B + D)(1 - \epsilon) \\ \cdot & \frac{1}{2}(A + B - D) \\ \cdot & \frac{1}{2}(A + B + D)(1 + \epsilon) \end{pmatrix},$$

(35)

the largest possible deviation of $\theta_{23}$ from $\pi/4$ is obtained for $\alpha_2 = 0$: $|\sin^2 \theta_{23} - \frac{1}{2}| \simeq \epsilon/2 \simeq 0.1$. We conclude that [35], if initially the phase $\alpha_2$ takes a CP conserving value of $\pi$ and for an inverted hierarchy neutrino mass spectrum, perturbed TBM leads to values of the $(\beta\beta)_{00^{-}}$ decay effective Majorana mass close to the minimal one, $\langle m \rangle = c_{13}^2 \sqrt{\Delta m_{A}^2} \cos 2\theta_{12}$. These values are correlated with sizable values of $|U_{e3}|$ and relatively large deviations from maximal atmospheric neutrino mixing. The benchmark value of $|U_{e3}|$ from Eq. (4) can be reconciled with minimal allowed values of the effective Majorana mass. In contrast, if the effective Majorana mass $\langle m \rangle$ is close to its possible maximal value, $\langle m \rangle \simeq c_{13}^2 \sqrt{\Delta m_{A}^2}$, negligible values of $|U_{e3}|$ are predicted. Hence, the benchmark value of $|U_{e3}|$ from Eq. (4) cannot be reconciled with values of $\langle m \rangle$ close to its maximal value. The expected deviation from $\sin^2 \theta_{23} = \frac{1}{2}$ is also smaller than in the previous case. It turns out, however, that the case of free $\alpha_2 \neq 0$ or $\pi$ allows non-minimal values of $\langle m \rangle$ for sizeable $|U_{e3}|$ as well (see below), which means that
the correlations discussed above rely on extreme initial values of $\alpha_2$.

In Fig. 8 we show scatter plots resulting from a corresponding numerical analysis. We have diagonalized Eq. (32), where we have taken random values for the complex $\epsilon_i$, by starting with $m_2 = 0.051$ eV, $m_1 = 0.0502424$ eV and $m_3 = 0.01$ eV. We required the resulting oscillation observables to lie in their $3\sigma$ ranges. We have also chosen as initial values $\alpha_2 = 0, \pi$, but let $\alpha_2$ vary freely as well. The largest and smallest possible values of the effective mass $\langle m \rangle$ are approximately $0.059$ eV and $0.0135$ eV, respectively, and we have indicated them in the figure. The analytical estimates from above are confirmed here. Fig. 9 shows the same analysis for quasi-degenerate neutrinos, where we have started with $m_3 = 0.10$ eV, $m_2 = 0.08778$ eV and $m_1 = 0.08735$ eV. The effective Majorana mass $\langle m \rangle$ in this case lies between $0.023$ eV and $0.105$ eV. In both cases it is evident that the value of $\theta_{23}$ is not a good discriminator. In particular, maximal atmospheric neutrino mixing is always possible.

5 Conclusions and Summary

Tri-bimaximal mixing provides a very close description of neutrino mixing angles. However, the present hint of non-zero $\theta_{13}$ coming from analyses of the global neutrino oscillation data may indicate that it is broken. In this paper we consider three breaking mechanisms from exact TBM – charged lepton corrections, radiative corrections and explicit breaking. While the deviation from maximal $\sin^2 \theta_{23} = \frac{1}{2}$ is allowed by the data to be of the same order $0.1$ as the values of $|U_{e3}|$ that we study, the challenge is to simultaneously keep the deviations
Table 2: Requirements on and predictions of the three breaking scenarios in order to generate the 1σ range $0.077 \leq |U_{e3}| \leq 0.161$. IH denotes inverted hierarchy, while PD stands for a partial hierarchal and QD for a quasi-degenerate mass scheme.

| sin$^2 \theta_{23}$ | charged leptons | renormalization (MSSM) | explicit breaking |
|------------------|----------------|------------------------|------------------|
| 0.44 - 0.53      | 0.55 - 0.64 ($\Delta m^2_{A} > 0$) | $\propto m_0^2 / \Delta m^2_{A}$ (1 + tan$^2 \beta$) | $\propto \epsilon$ (IH) |
| $|U_{e3}|$        | $|U_{e3}| \simeq \frac{\lambda}{\sqrt{2}}$ | $\propto \epsilon$, $|U_{e3}| \propto \frac{m_0}{\sqrt{\Delta m^2_{A}}}$ (PD/QD) | III, PD, QD |
| mass             | QD: $m_0 \tan \beta \simeq (4 - 7)$ eV | QD: $\sqrt{m_0^2 \sin^2 \theta_{12}}$ | II, PD, QD |
| $\langle m \rangle$ | $m_0 e^{i \frac{2}{13}} \cos 2\theta_{12}$ | $\sqrt{\Delta m^2_{A} e^{i \frac{2}{13}} \cos 2\theta_{12}}$ (IH) | III, PD, QD |
| CP               | oscillations: almost maximal CP violation | $\alpha_2 \simeq \pi$ | large $|U_{e3}|$ requires suppressed $\langle m \rangle$ only when initially $\alpha_2 \simeq \pi$ |

The correction parameter $\lambda$, which is the sine of the 12-rotation in the usual parametrization of $U_{e}$, can be restricted as $0.104 \leq \lambda \leq 0.247$ from the current 1σ ranges of the mixing angles. We note that the sine of the Cabibbo angle is included in this range, but one third of it is not. In this picture $|U_{e3}| \simeq \lambda / \sqrt{2}$. A sizable value of $U_{e3}$ therefore implies a sizable $\lambda$. Suppressing the leading ($O(\lambda)$) correction to sin$^2 \theta_{12}$ is possible by choosing the Dirac CP phase in neutrino oscillations to be $\pi/2$ or $3\pi/2$ corresponding to maximal CP violation in neutrino oscillations. The charged lepton corrections to tri-bimaximal mixing do not depend on the neutrino mass values and their ordering. The atmospheric neutrino mixing parameter sin$^2 \theta_{23}$ is deviated from $1/3$ by terms of order $|U_{e3}|^2$. To be precise, it is within the range $0.44 \lesssim \sin^2 \theta_{23} \lesssim 0.53$. In particular it is allowed to be maximal.

Generating a large $|U_{e3}|$ via radiative corrections implies quasi-degenerate neutrinos, namely $m_0 \gtrsim 2.6$ eV for the SM when $U_{e3} = 0$ at the high scale. Thus, the current neutrino mass limits rule out a possible RG origin of $|U_{e3}| \simeq 0.1$ in the SM. For the MSSM one requires $m_0 \gtrsim 0.8$ (0.2) eV with tan$\beta = 5$ (20). The implied constraints of the 1σ range of $|U_{e3}|$ on $m_0$ and tan$\beta$ can be summarized as $4 \lesssim (m_0/eV) \tan \beta \lesssim 7$. The running in the MSSM predicts that sin$^2 \theta_{12}$ increases from its initial high scale value. Large running of $\theta_{13}$ to generate $|U_{e3}| \simeq 0.1$ together with the requirement that sin$^2 \theta_{12}$ is within its current 3σ range forces the Majorana phase $\alpha_2 \simeq \pi$. Interesting correlations are also obtained between sin$^2 \theta_{13}$ and sin$^2 \theta_{23}$. The latter parameter is necessarily non-maximal and lies in the range $0.55 \leq \sin^2 \theta_{23} \leq 0.64$ for a normal ordering and $0.33 \leq \sin^2 \theta_{23} \leq 0.45$ for an inverted ordering. The RG evolved effective neutrino Majorana mass observed in neutrino-less double beta decay is found to lie close to its minimum allowed value because of the constraint of $\alpha_2 \simeq \pi$. In case of tan$\beta = 5$, $\langle m \rangle$ lies between 0.26 and 0.50 eV, to be compared with its general upper and lower limits of 0.2 eV.
and 1.4 eV. If \( \tan \beta = 20 \), then \( 0.07 \, \text{eV} < \langle m \rangle < 0.11 \, \text{eV} \), while in general the effective mass could be in between 0.05 eV and 0.34 eV.

We also consider the possibility of deviating from tri-bimaximal mixing by adding explicit breaking terms to the neutrino mass matrix, i.e., every entry is multiplied with an individual factor \( 1 + \epsilon_i \). For this mechanism to generate sizable \( |U_{e3}| \) the neutrino mass spectrum has to be partially degenerate, or quasi-degenerate, or with inverted hierarchy. Atmospheric neutrino mixing is allowed to take any of its currently allowed values, including \( \sin^2 \theta_{23} = \frac{1}{2} \). In this breaking scenario the requisite sizeable \( |U_{e3}| \) value cannot be reconciled with initial maximal values of the effective Majorana mass governing neutrino-less double beta decay, corresponding to the three indicated types of neutrino mass spectrum.

To sum up, the CP violating phases in the neutrino mixing matrix play a crucial role for having only relatively small corrections to \( \theta_{12} \) when large corrections to \( U_{e3} = 0 \) are generated. This interesting fact together with the predictions for \( \theta_{23} \) may be used to distinguish breaking scenarios to tri-bimaximal mixing.

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