The Absence of Renormalization of the Specific Heat Coefficient of the Interacting Fermion Systems

Yunkyu Bang

Department of Physics, Chonnam National University, Kwangju 500-757, and Asia Pacific Center for Theoretical Physics, Pohang 790-784, Korea

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Contrary to the longtime and widely conceived belief, we proved that the specific heat coefficient $\gamma$ of the interacting fermion system is not renormalized by the wave-function renormalization factor $Z$ as far as the system remains a Fermi liquid state.

In this paper, we showed that there was an error in the proof of Luttinger and the SH coefficient $\gamma$ of the interacting fermion system is not fully renormalized so that $\gamma/\gamma_0 = m^*/m_0$ is not true. Our finding should have far reaching consequences in the study of various interacting fermion systems such as strongly correlated metals, liquid $^3$He, neutron stars, etc. In this paper, we will be focusing only on the questions of where was wrong in the Luttinger’s proof and what is the correct answer for the SH coefficient $\gamma$ of the interacting fermion systems.

SH coefficient $\gamma$ and DOS – It is well known that the SH coefficient of the non-interacting fermion system $\gamma_0$ is given by

$$\lim_{T \to 0} C(T)/T \equiv \gamma_0 = \frac{\pi^2}{3} N_0(0),$$

where $N_0(0)$ is the density of states (DOS) of the non-interacting fermion system at the chemical potential. Intuitively, the SH coefficient of the interacting fermion system $\gamma$ is expected to be given with the above equation by replacing $N_0(0)$ by the DOS of the interacting fermion system $N(0)$ such as $\gamma = \frac{\pi^2}{3} N(0)$. But this absolutely reasonable intuition falls in a serious trouble as follows. The DOS $N(0)$ of the interacting fermion system can be calculated if we know the exact one-particle Green’s function which is formally written as $G(k, \omega) = \frac{1}{\omega - \lambda(k) - \Sigma(k, \omega)}$ with the exact self-energy $\Sigma(k, \omega)$. However, we can show that $N(0)/N_0(0) \neq m^*/m_0$ and that even the inequality $N(0)/N_0(0) > 1$ is not guaranteed, as shown below. This finding is in stark contrast to the common knowledge that the SH coefficient should be enhanced by interaction such as $\gamma/\gamma_0 \approx m^*/m_0 > 1$. There are two possible options to resolve this dilemma: (1) $\gamma = \frac{\pi^2}{3} N(0)$ is not true for the interacting system; or (2) the common belief $\gamma/\gamma_0 \approx m^*/m_0$ is wrong. The main conclusion of this paper is that the option (2) is the correct answer, namely, $\gamma$ does not measure the effective mass $m^*$ of the renormalized fermionic q.p.s.

Let us begin with calculating $N(0)$. It is well known that the self-energy in the Fermi liquid state has the well defined expansion such as $\Sigma(k, \omega) = \Sigma(k, \omega) + \gamma_0 \delta(k) - \lambda(k) \omega - i\delta$, where $\gamma_0 = \frac{\partial \Sigma(k, 0)}{\partial k}$ and $\lambda_k = \frac{\partial \Sigma(k, \omega)}{\partial \omega}|_{\omega=0}$, respec-
are defined, respectively, and the wave-function renormalization factor $\lambda$ is renormalized as $\lambda > 1$ due to the causality – completely drops in the interaction effect (i.e. setting $\delta = 0$). However, the reduction of the q.p. spectral weight of the DOS should be conserved. The width of the q.p. region which is not fully displayed here but the total spectral strength $\Im G(\omega)$ comes unstable. This implies that the exact DOS defined in Eq.(5) is still renormalized by the static renormalization factor $Y$. However, although there is no general constraint to guarantee $Y > 1$ or $\omega < 1$ as in the case of $\omega > 1$, the known cases, such as the Hartree-Fock exchange correction with the Coulomb potential, indicate that $Y > 1$ is usually satisfied unless the Fermi liquid state becomes unstable. This implies that the exact DOS defined in Eq.(2) tends to be reduced by interaction, quite contrary to the common knowledge. In this paper, however, we will mainly focus on the dynamic renormalization factor $Z$, because $Z$ is the dominant renormalization effect in most of the strongly interacting fermion systems.

To demonstrate the correctness of the result of Eq.(5), we show the numerical results of $N(\omega) = \frac{1}{\pi} \sum_k \Im G(k, \omega)$ of a simple toy model in Fig.1(a) neglecting the static renormalization effect (i.e. setting $Y = 1$). In this exemplary calculation, we assumed a box like DOS for the non-interacting fermion system as $N_0(\omega) = 1.0$ for $-\Lambda < \omega < \Lambda$ and the effect of interaction is simulated by the Fermi liquid type self-energy $\Im \Sigma(\omega) = \omega \omega^2$ for $-\Lambda < \omega < \Lambda$ including the corresponding real part $\Re \Sigma(\omega)$. We chose $\Lambda = 5$. The results are self-explaining, showing $N(\omega) = N_0(\omega)$ for all interaction strength of $\alpha$. Increasing the interaction strength, the width of the q.p. DOS around $\omega = 0$ becomes progressively narrowed and the spectral weight outside of it is depleted toward the high energy region which is not fully displayed here but the total spectral weight of the DOS should be conserved. The width of the q.p. DOS around $\omega = 0$ is roughly proportional to $1/\Lambda$ and the value of $Z$ is determined by the combination of the interaction strength $\alpha$ and the band width scale $\Lambda$.

**SH coefficient $\gamma$ of Interacting Fermi Systems** – To find an exact theoretic formula to calculate the SH coefficient $\gamma$ of the interacting Fermi systems, we start with the same Hamiltonian

$$H = \sum_r \varepsilon_r c_r^+ c_r + \frac{1}{2} \sum_{r,s,\sigma,\sigma'} c_r^+ c_{\sigma'}^+ c_{\sigma'} c_{\sigma} \langle rs|v|s'\rangle'$$

where $\varepsilon_r$ is the energy measured from the chemical potential of the non-interacting single particle states with the index $r = (k, \sigma)$ for both momentum and spin. $c_r^+$, $c_r$ are the creation and annihilation fermion operators, respectively, and $(rs|v|s')$ is the general four point fermion interaction matrix.

In Ref. [2], Luttinger and Ward wrote down the celebrated free energy functional of the interacting fermion system as

$$\Omega(T) = -T \sum_{r,m} e^{i\omega_m T} \{\ln[\varepsilon_r + \Sigma_r(\omega_m) - i\omega_m] + G_r(\omega_m)\Sigma_r(\omega_m)\} + \Omega'(T)$$

for the interacting fermion system used by Luttinger and Ward [2].

![Image](image_url)
where \( \omega_n = \pi T (2n + 1) \) is Matsubara frequency. \( G_r (\omega_n) \) and \( \Sigma_r (\omega_n) \) are the full Green’s function and the full proper self-energy, respectively. The functional \( \Omega' \) is defined by LW (refers to Ref. [2]) as

\[
\Omega' = \text{(contribution of all closed-linked skeleton diagrams, but with replacing all Green’s function lines by the full Green’s functions \( G_r (\omega_n) \)).}
\]

(8)

The explicit expression of \( \Omega' \) was given in LW(50) (this notation means Eq.(50) of Ref.[2]), but for our purpose we don’t need to know the details of the structure of \( \Omega' \). The functional \( \Omega \) was ingeniously designed by Luttinger to satisfy the famous variational theorem of the total free energy functional:

\[
\frac{\partial \Omega}{\partial \Sigma_r} = 0.
\]

(9)

And this theorem can be satisfied only if the functional \( \Omega' \) satisfies the following variational property

\[
\frac{\partial \Omega'}{\partial \omega_n} = T \sum_{r,n} [G_r (\omega_n)]^2 \Sigma_r (\omega_n)
\]

(10)

which was shown in LW(51). Up to now, we have just copied the key results of Ref.[2]. For our purpose, we only need one slight generalization of Eq.(10) as follows

\[
\frac{\partial \Omega'}{\partial \omega_n} = -T \sum_{r,n} [G_r (\omega_n)]^2 \left( 1 - \frac{\partial \Sigma_r}{\partial \omega_n} \right) \Sigma_r (\omega_n).
\]

(11)

The proof of Eq.(11) is easily deduced from Eq.(10) if we note the expression of \( G_r^{-1} = i \omega_n - \epsilon_r - \Sigma_r (\omega_n) \) and the trivial relations

\[
\frac{\partial G_r (\omega_n)}{\partial \Sigma_r (\omega_n)} = [G_r (\omega_n)]^2
\]

(12)

and

\[
\frac{\partial G_r (\omega_n)}{\partial \omega_n} = -[G_r (\omega_n)]^2 \left( 1 - \frac{\partial \Sigma_r}{\partial \omega_n} \right).
\]

(13)

The Eq.(11) is the crucially important relation for our purpose and will be used later.

In order to calculate the entropy from the free energy functional Eq.(7), we need to extract the leading temperature dependent parts of it. Using a standard method of the Matsubara frequency summation, Eq.(7) is written as

\[
\Omega(T) = \oint \frac{dz}{2 \pi i n_F(z)} \sum_r \{ \ln[\epsilon_r + \Sigma_r(z) - z] + G_r(z) \Sigma_r(z) \} - \oint \frac{dz}{2 \pi i n_F(z)} \Omega'(z)
\]

(14)

where all Matsubara frequencies of Eq.(7) are analytically transformed to complex numbers as \( i \omega_n = z \) and the functional \( \Omega(z) \) is also understood as \( \Omega' (i \omega_n \to z) \) after replacing the overall Matsubara frequency summation – \( T \sum_\omega \) of the original functional \( \Omega (i \omega_n) \) by the contour integral \( \oint \frac{dz}{2 \pi i n_F(z)} \) with the Fermi-Dirac distribution function \( n_F(z) \). Now it is clear that there are only two places which contain the temperature dependence in the above free energy functional Eq.(14): \( n_F(z) \) and \( \Sigma_r(z) \). As Luttinger argued [3], the leading temperature dependence should come from the explicit summation of \( i \omega_n \) (equivalently in \( n_F(z) \)) and the temperature variation of \( \Sigma_r(T) - \Sigma_r(T = 0) \) is a higher order and should be neglected. Therefore, using \( S(T) = -d \Omega(T)/dT \), we can write down \( S(T) \) as follows

\[
S(T) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi T} \left[ \frac{\partial n_F (\omega)}{\partial \omega} \right] \cdot \sum_r \text{Im} \{ \ln G_r^{-1} (\omega) + G_r(\omega) \Sigma_r(\omega) \} - \int_{-\infty}^{\infty} \frac{d\omega}{\pi T} \left[ \frac{\partial n_F (\omega)}{\partial \omega} \right] \text{Im} \Omega'(\omega),
\]

(15)

where the contour path of \( \oint \) is deformed along the real frequency axis \( \text{à la} \) the appendix of Ref.[2]; the \( \omega \)-integration for \([-\infty, \infty]\) should be carried infinitesimally above the real axis, i.e. for \( \omega + i \eta \). \( \Sigma_r'(\omega) \) means \( \Sigma_r(\omega, T = 0) \) and it is understood that every \( \Sigma_r(T) \) implicit in the above expression is replaced by \( \Sigma_r'(\omega) \). While the above expression of \( S(T) \) is undoubtedly the exact expression, Luttinger argued in Ref.[3] that the leading temperature dependence of \( \Omega(T) \) (Eq.(7)) is contained only in

\[
\Omega_{\text{Luttinger}}(T) \approx -T \sum_{r,n} e^{i \omega_n 0^+} \ln[\epsilon_r + \Sigma_r(\omega_n) - i \omega_n]
\]

(16)

and ignored the last two terms of Eq.(7) because the leading temperature dependent parts in the remaining terms \( -T \sum_{r,n} e^{i \omega_n 0^+} [G_r(\omega_n) \Sigma_r(\omega_n)] + \Omega \) cancels each other. Hence, Luttinger has obtained the entropy from Eq.(16) as follows

\[
S_{\text{Luttinger}}(T) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi T} \left[ \frac{\partial n_F (\omega)}{\partial \omega} \right] \sum_r \text{Im} \{ \ln G_r^{-1} (\omega) \} \times \sum_r \text{Im} \{ \ln[\epsilon_r + \Sigma_r(\omega) - \omega] \},
\]

(17)

(18)

The above expression \( S_{\text{Luttinger}}(T) \) is the only the first term of the exact entropy expression \( S(T) \) of Eq.(15). Then it is obvious question how to justify using \( S_{\text{Luttinger}}(T) \) to calculate the SH instead of using the exact \( S(T) \). The only justification is that both expressions Eq.(15) and Eq.(17) give the same result or put in other words the contributions of the last two terms of Eq.(15) cancel each other as Luttinger claimed. However, below we show that the cancellation between the two terms are incomplete and an important contribution remains. Therefore we have to use the full expression of the entropy Eq.(15).

Expectedly the calculation results of the SH coefficient \( \gamma \) from \( S(T) \) and \( \gamma_{\text{Luttinger}} \) from \( S_{\text{Luttinger}}(T) \) are totally different:
the former one yields $\gamma$ unrenormalized by the wave-function renormalization factor $Z$ regardless of the strength of the interaction while the latter one yields an enhanced $\gamma_{\text{Luttinger}}$ proportional to the value of $Z$ as widely believed in the community ever since the proof of Luttinger [3].

To calculate $\gamma \equiv \lim_{T \rightarrow 0} C(T)/T = \lim_{\omega \rightarrow 0} dS(T)/dT$, we only need to extract $T$-linear contributions in $S(T)$ or $S_{\text{Luttinger}}(T)$. Utilizing Sommerfeld expansion, we then only need to extract $\omega$-linear terms in the integrand of $\text{Im}...$ in $S(T)$ or $S_{\text{Luttinger}}(T)$. Let us first calculate $\gamma_{\text{Luttinger}}$ from $S_{\text{Luttinger}}$. The leading Taylor expansion of the integrand of $S_{\text{Luttinger}}$ can be read from Eq.(18) as

\[
\text{Im}\{\ln \epsilon_r + \Sigma^0_r - \omega \} = \text{Im}G_r(0)(1 - \frac{\partial \text{Re} \Sigma^0_r}{\partial \omega} \big|_{\omega=0}) \cdot \omega + O(\omega^2)...
\]

Using the notation $(1 - \frac{\partial \text{Re} \Sigma^0_r}{\partial \omega} \big|_{\omega=0}) = 1 + \lambda_r = Z_r$, and combining it with Eqs.(2)-(5), we obtained

\[
N_{\text{Luttinger}}(0) = -\frac{1}{\pi} \sum_{\omega} \text{Im}G_r(0)(1 + \lambda_r) = N(0) \cdot Z \frac{Y}{Y}
\]

where $Z$ is a Fermi surface average of $Z_r$. In Fig.1(b), we showed the numerical calculations of $-\frac{1}{\pi} \sum_{\omega} \text{Im}G_r(0)(1 - \frac{\partial \text{Re} \Sigma^0_r}{\partial \omega})$ with the varying interaction strength. This quantity has no direct physical meaning (it becomes even negative at higher energies) but its zero frequency value $N_{\text{Luttinger}}(0)$ clearly demonstrated the result of Eq.(21) and showed what quantity was used by Luttinger for the calculation of the SH coefficient. Substituting the results of Eqs.(19)-(21) into Eq.(18), the leading temperature dependence of $S_{\text{Luttinger}}$ is the following

\[
S_{\text{Luttinger}}(T) \approx -\int_{-\infty}^{\infty} d\omega \frac{\text{Im}G_r(0)}{T \omega} \left[ \frac{\partial n_F(\omega)}{\partial \omega} \right] N_{\text{Luttinger}}(0) \cdot T
\]

and from this we can derive the same result as Luttinger had obtained [3] as

\[
\gamma_{\text{Luttinger}} = \frac{\pi^2}{3} \cdot N_{\text{Luttinger}}(0) = \frac{\pi^2}{3} N(0) \cdot Z \frac{Y}{Y}
\]

so that the SH coefficient $\gamma_{\text{Luttinger}}$ is indeed enhanced by the factor $Z$ compared to the non-interacting case. Note that $N_{\text{Luttinger}}(0)$ defined in Eq.(20) is nothing but the quasi-particle DOS $N_{qp}(0)$ which was conventionally defined by re-scaling fermion operators $c_r$ by the factor $\sqrt{Z_r}$. Hence the Luttinger’s result of Eq.(23) has firmly established that the SH coefficient $\gamma$ measures the q.p. DOS $N_{qp}(0)$.

Now let us use the exact expression $S(T)$ of Eq.(15) to derive $\gamma$. The coefficients of the $\omega$-linear terms of the integrand of $S(T)$, $\{ \ln G_r^{-1}(\omega) + G_r(\omega) \Sigma^0_r(\omega) \} - \Omega(\omega)$, are the following

\[
= G_r(1 - \frac{\partial \Sigma^0_r}{\partial \omega})
\]

\[
- [G_r]^2 \Sigma^0_r(1 - \frac{\partial \Sigma^0_r}{\partial \omega}) + G_r \frac{\partial \Sigma^0_r}{\partial \omega}
\]

\[
+ [G_r]^2 \Sigma^0_r(1 - \frac{\partial \Sigma^0_r}{\partial \omega}) = G_r.
\]

Above we have arranged the Taylor expansions of each three terms $\text{Im}G_r^{-1}(\omega)$, $G_r(\omega) \Sigma^0_r(\omega)$ and $-\Omega(\omega)$ into three separate lines for clarity. In particular, we have used the important relation of Eq.(11) for $\frac{\partial \Sigma^0_r}{\partial \omega}$ in the third line. There are lots of cancellations and the final result should be compared to Eq.(19) obtained from $S_{\text{Luttinger}}(T)$. In fact, the above cancellation is the consistent result of the Luttinger’s variational theorem of Eq.(9) which requires that all variations of $\partial \Sigma^0_r$ in the total free energy functional $\Omega$ should sum up to zero [2]. In this sense, the expression of $S_{\text{Luttinger}}$ in Eq.(18) with Eq.(19) cannot be correct since it contains $\partial \Sigma^0_r$ term.

Now it is a trivial matter to calculate the SH coefficient $\gamma$ substituting the result of Eq.(24) into Eq.(15) as

\[
S(T) \approx \int_{-\infty}^{\infty} d\omega \frac{\text{Im}G_r(0)}{T \omega} \left[ \frac{\partial n_F(\omega)}{\partial \omega} \right] \sum_r \text{Im}G_r(0) \cdot \omega
\]

and combining with Eqs.(2)-(5), we have

\[
\gamma = \frac{\pi^2}{3} \cdot N(0) = \frac{\pi^2}{3} \cdot \frac{N(0)}{Y}.
\]

The above result shows that the SH coefficient $\gamma$ of the interacting Fermi system measures the exact DOS $N(0)$ defined by Eq.(2), which is consistent with our physical intuition. However, due to the absence of the wave-function renormalization factor $Z$ in contrast to the Luttinger’s result of Eq.(23), we do not expect a strong enhancement of $\gamma$ by the interaction in a Fermi liquid state unless the static renormalization factor $Y$ becomes $0 < Y < 1$.

**Other Physical Quantities** – The renormalized q.p. mass $m^* \approx m \cdot \frac{\Pi}{4}$ due to interaction is measured by different experimental probes. Indeed the energy dispersion of the q.p. pole $E(k)$, defined by $\omega - \epsilon(k) - \Sigma(k, \omega) = 0$, is renormalized as $E(k) \approx \epsilon(k) \cdot \frac{\Sigma(k, \omega)}{\omega}$ and should be directly measured by ARPES without any interpretation or confusion. Another common tool to measure $m^*$ is the dHvA effect with the applied external field $H$. In this case, the effective mass is measured from the temperature reduction factor of the signal strength which is given by the Lifshitz-Kosevich formula $R_T \sim \exp(-T/\omega_c)\left[\frac{\omega_c}{E}\right]^2$, where $\omega_c$ is the cyclotron frequency. $\omega_c$ is determined by the q.p. energy distance between the Landau levels quantized by the field $H$ as $\Delta E = \hbar \omega_c$, and the Landau level is determined by the q.p. dispersion $E(k)$ to the first approximation,
hence $\omega_c = eH/m^*c$. Therefore, the dHvA effect measurement can provide an information of $m^*$. Lastly, the optical spectroscopy measurements need a more careful interpretation. The total spectral density near the Fermi level is not enhanced by interaction as shown in Eq.(5), but the width of the q.p. dispersion is narrowed by the factor $1/Z$ as shown in Fig.1(a). Therefore, for example, the width of the Drude spectra in the optical conductivity is expected to be reduced by the factor $1/Z$, while the absolute magnitude of the zero frequency conductivity $\sigma(\omega = 0)$ nor the total Drude spectral weight is not expected to be enhanced. However, because the optical conductivity is a transport property, it is essential also to count on the renormalized Fermi velocity $\tilde{v}_F$ and the scattering rate $1/\tau_{tr}$ due to the interaction besides the q.p. DOS. Therefore, for more complete details of the optical properties of the interacting fermion systems, we need to analyze the two particle correlation function which is beyond the scope of the current paper.

Conclusions – In summary, we have shown the following: (1) Luttinger’s calculation of $\gamma_{Luttinger}$ is not correct because it started with an approximate functional $\Omega_{Luttinger}$; (2) the SH coefficient $\gamma$ measures the exact DOS $N(0)$ defined in Eq.(2) and is not enhanced by $Z$ the wave-function renormalization factor; therefore, (3) the q.p. DOS $N_{qp}(0)$ is only a fictitious concept and not a measurable quantity. These results are in stark contrast to the longtime accepted idea of the interaction-enhanced SH coefficient since the proof of Luttinger in 1960.

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