A Public-Key Cryptosystem Using Cyclotomic Matrices

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Abstract

Confidentiality and Integrity are two paramount objectives of asymmetric key cryptography. Where two non-identical but mathematically related keys- a public key and a private key effectuate the secure transmission of messages. Moreover, the private key is non-shareable and the public key has to be shared. The messages could be secured if the amount of computation rises to very high value. In this work, we propose a public key cryptosystem using the cyclotomic numbers, where cyclotomic numbers are certain pairs of solutions \((a, b)_e\) of order \(e\) over a finite field \(\mathbb{F}_q\) with characteristic \(p\). The strategy employs cyclotomic matrices of order \(2^{l^2}\), whose entries are cyclotomic numbers of order \(2^{l^2}\), \(l\) be prime. The public key is generated by choosing a particular generator \(\gamma'\) of \(\mathbb{F}_p^*\). Secret key (private key) is accomplished by discrete logarithm problem (DLP) over a finite field \(\mathbb{F}_p\).

Keywords: Cyclotomic numbers, Finite fields, Cyclotomic matrix, Discrete logarithm problem, Public key, Secret key

1. Introduction

Apart from a rich history of Message encryption, it became more popular in the 20th century upon the evolution of information technology. In a cryptosystem, both parties (in a two-party system) have a pair of public enciphering and secret deciphering keys \([1]\). A party can send encrypted messages to a designated party using a public enciphering key. However, only the
designated party can decrypt the message using their corresponding secret deciphering key $k_2$.

Discrete logarithm problem (DLP) is a mathematical problem that occurs in many settings and it is hard to compute exponent in a known multiplicative group $[3]$. Diffie-Hellman $[4]$ and ElGamal $[5]$ cryptosystems are the schemes developed under the Discrete logarithm algorithm. Diffie-Hellman brought the new direction in the cryptosystem that introduced key exchange protocol which is based on DLP $[4]$. For the security perspective, $[5]$ cryptosystem was proposed to introduce a digital signature scheme (DSS) which is based on Diffie-Hellman DLP and key distribution scheme. Many researches had done to overcome the shortcomings of the ElGamal cryptosystem $[6]$ and to secure against mathematical and brute force attacks $[7]$. Elliptic curve cryptosystem (ECC) is another widely used crypto scheme which is based on DLP. The composite discrete logarithm problem (CDLP) is a generalization of DLP which is also used to design public key cryptosystems. McCurley $[8]$ proposed an ElGamal signature scheme that is based on CDLP. Pointcheval $[9]$ developed an efficient authentication scheme based on the CDLP which is more secured than factorization.

Cyclotomic numbers are one of the most important objects in number theory. These numbers have been extensively used in cryptography, coding theory and other branches of information theory. Thus determination of cyclotomic numbers, so called cyclotomic number problem, of different orders is one of basic problems in number theory. Complete solutions for cyclotomic number problem for $e = 2 - 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 22, l, 2l, l^2, 2l^2$ with $l$ an odd prime have been investigated by many authors see $([10, 11, 12, 13]$ and the references there in).

In 1978, McEliece $[14]$ proposed a public key cryptosystem based on Goppa codes Hamming metric. Gabidulin $[15]$ introduced the rank metric and the Gabidulin codes over a finite field with $q$ element, where $q = p^n$ i.e. $F_q^n$, as an alternative for the Hamming metric. In 2006, Delgosha and Fekri $[1]$ developed a public key cryptosystem using paraunitary matrices, whose entries are polynomials with coefficients from a finite field. Further, in 2018, Lau and Tan $[16]$ proposed new encryption with public key matrix by considering the addition of a random distortion matrix over $F_q$ of full column rank $n$. In this work, we consider two important problems in the theory of cyclotomic numbers over $F_p$. The first one deals with an algorithm for fast computation of all the cyclotomic numbers of order $2l^2$, where $l$ is prime. The second one deals with the public key cryptosystem based on cyclotomic matrices of
The paper is organized as follows: Section 2 presents the definition and notations, including some well-known properties of cyclotomic numbers of order \(2l^2\). Section 3 presents the construction of cyclotomic matrices of order \(2l^2\). Section 4 contains methods of encryption and decryption along with a numerical example. Finally, a brief conclusion is reflected in Section 5.

2. Cyclotomic numbers

One of the central problems in the study of cyclotomic numbers is the determination of all cyclotomic numbers of a specific order for a given finite field in terms of solutions of certain Diophantine systems. Complete solutions to the cyclotomy problem over a finite field \(\mathbb{F}_q\) with characteristic \(p\) have been investigated by many authors for some specific orders. The problem of cyclotomy of order \(2l^2\) concerns to formulate all \(4l^4\) cyclotomic numbers of order \(2l^2\). The section contains the definition of cyclotomic numbers of order \(e\), useful notations followed by properties of cyclotomic numbers of order \(2l^2\). These properties play a major role in determining which cyclotomic numbers of order \(2l^2\) are sufficient for the determination of all cyclotomic numbers of order \(2l^2\). The section also examines the cyclotomic matrices of order \(2l^2\).

2.1. Definition and notations

Let \(e \geq 2\) be an integer, and \(p \equiv 1 \pmod{e}\) an odd prime. One writes \(p = ek + 1\) for some positive integer \(k\). Let \(\mathbb{F}_p\) be the finite field of \(p\) elements and let \(\gamma\) be a generator of the cyclic group \(\mathbb{F}_p^*\). For \(0 \leq a, b \leq e - 1\), the cyclotomic number \((a, b)_e\) of order \(e\) is defined as the number of solutions \((s, t)\) of the following:

\[
\gamma^{es+a} + \gamma^{et+b} + 1 \equiv 0 \pmod{p}; \quad 0 \leq s, t \leq k - 1.
\] (2.1)

2.2. Properties of cyclotomic numbers of order \(2l^2\)

Let \(p \equiv 1 \pmod{2l^2}\) be a prime for an odd prime \(l\) and we write \(p = 2l^2k + 1\) for some positive integer \(k\). It is clear that \((a, b)_{2l^2} = (a', b')_{2l^2}\) whenever \(a \equiv a' \pmod{2l^2}\) and \(b \equiv b' \pmod{2l^2}\) as well as \((a, b)_{2l^2} = (2l^2 - a, b - a)_{2l^2}\). These imply the following:

\[
(a, b)_{2l^2} = \begin{cases} (b, a)_{2l^2} & \text{if } k \text{ is even}, \\ (b + l^2, a + l^2)_{2l^2} & \text{if } k \text{ is odd}. \end{cases}
\] (2.2)
Applying these facts, one can check that
\[
\sum_{a=0}^{2l^2-2} \sum_{b=0}^{2l^2-1} (a, b)_{2l^2} = q - 2 \tag{2.3}
\]
and
\[
\sum_{b=0}^{2l^2-1} (a, b)_{2l^2} = k - n_a, \tag{2.4}
\]
where \( n_a \) is given by
\[
n_a = \begin{cases} 
1 & \text{if } a = 0, 2 \mid k \text{ or if } a = l^2, 2 \nmid k; \\
0 & \text{otherwise}.
\end{cases}
\]

3. Cyclotomic Matrices

This section presents the procedure to determine cyclotomic matrices of order \( 2l^2 \) for prime \( l \). We determine the equality relation of cyclotomic numbers and discuss how few of the cyclotomic numbers are enough for the construction of whole cyclotomic matrix. Further generators for a chosen value of \( p \) will be determined followed by the generation of a cyclotomic matrix. At every step, we have included a numerical example for the convenience to understand the procedure easily.

**Definition:-** Cyclotomic matrix of order \( 2l^2 \), where \( l \) be a prime, is a square matrix of order \( 2l^2 \), whose entries are pair of solutions \((a, b)_{2l^2}; 0 \leq a, b \leq 2l^2 - 1\), of the equation \(2.1\).

| (a,b) | b   |
|-------|-----|
| a     | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
| 0     | (0,0)| (0,1)| (0,2)| (0,3)| (0,4)| (0,5)| (0,6)| (0,7)|
| 1     | (1,0)| (1,1)| (1,2)| (1,3)| (1,4)| (1,5)| (1,6)| (1,7)|
| 2     | (2,0)| (2,1)| (2,2)| (2,3)| (2,4)| (2,5)| (2,6)| (2,7)|
| 3     | (3,0)| (3,1)| (3,2)| (3,3)| (3,4)| (3,5)| (3,6)| (3,7)|
| 4     | (4,0)| (4,1)| (4,2)| (4,3)| (4,4)| (4,5)| (4,6)| (4,7)|
| 5     | (5,0)| (5,1)| (5,2)| (5,3)| (5,4)| (5,5)| (5,6)| (5,7)|
| 6     | (6,0)| (6,1)| (6,2)| (6,3)| (6,4)| (6,5)| (6,6)| (6,7)|
| 7     | (7,0)| (7,1)| (7,2)| (7,3)| (7,4)| (7,5)| (7,6)| (7,7)|

Table 1: Cyclotomic matrix of order 8

For instance Table 1 depicts a typical cyclotomic matrix of order 8 (assuming \( l=2 \)). Whose construction steps have been given in the next subsection.
3.1. Construction of cyclotomic matrix

Typically construction of a cyclotomic matrix has been subdivided into four subsequent steps. Below are those ordered steps for the construction of a cyclotomic matrix;

1. For given \( l \), choose a prime \( p \) such that \( p = 2l^2k + 1, k \in \mathbb{Z}^+ \).

The initial entries of the cyclotomic matrix are the arrangement of pair of numbers \((a, b)_{2^l}\) where \(a\) and \(b\) usually vary from 0 to \(2l^2 - 1\).

2. Determine the equality relation of pair of \((a, b)_{2^l}\), which reduces the complexity of pair of solution \((a, b)_{2^l}\) of equation \(2.1\), that is discuss in next sub-section.

3. Determine the generators of chosen \( p \) (i.e. generators of \( \mathbb{F}_p^* \)). Let \( \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n \) be generators of \( \mathbb{F}_p^* \).

4. Choose a generator (say \( \gamma_1 \)) of \( \mathbb{F}_p^* \) and put in equation \(2.1\). This will give cyclotomic matrix of order \(2l^2\) w.r.t. chosen generator \( \gamma_1 \).

The first step initializes the entries of cyclotomic matrix of order \(2l^2\). Value of \( p \) will be determined for given \( l \). Assuming \( l=2 \), an example of such initialization of matrix of order 8 has been shown in Table 1.

For the construction of cyclotomic matrix, it does not require to determine all the cyclotomic numbers of a cyclotomic matrix which is shown in Table 1. By well-known properties of cyclotomic numbers of order \(2l^2\), cyclotomic numbers are divided into various classes, therefore there are a pair of the relation between the entries of initial table of a cyclotomic matrix. Thus to avoid calculating the same solutions in multiple times, we determine the equality relation of cyclotomic numbers (i.e. equality of solutions of \((a, b)_{2^l}\)). In the next subsection, we will discuss which cyclotomic numbers are enough for the construction of the cyclotomic matrix. Thus it helps us to the faster computation of cyclotomic matrix.

3.2. Determination of equality relation of cyclotomic numbers

This subsection presents the procedure to determine the equality relation of cyclotomic numbers (i.e. the relation of pair of \((a, b)_{2^l}\)), which reduces the complexity of solutions of pair of \((a, b)_{2^l}\) (see also \(2.1\)). For the determination of cyclotomic matrices, it is not necessary to obtain all the cyclotomic numbers of order \(2l^2\). The minimum number of cyclotomic numbers required to determine all the cyclotomic numbers (i.e. required for construction of
cyclotomic matrix) depends on the value of positive integer $k$ on expressing prime $p = 2l^2k + 1$. By (2.2), if $k$ is even, then

$$
(a, b)_{2l^2} = (b, a)_{2l^2} = (a - b, -b)_{2l^2} = (b - a, -a)_{2l^2} = (-a, b - a)_{2l^2} = (-b, a - b)_{2l^2}
$$

(3.1)

otherwise

$$
(a, b)_{2l^2} = (b + l^2, a + l^2)_{2l^2} = (l^2 + a - b, -b)_{2l^2} = (l^2 + b - a, l^2 - a)_{2l^2} = (-a, b - a)_{2l^2} = (l^2 - b, a - b)_{2l^2}.
$$

(3.2)

Thus by (3.1) and (3.2), cyclotomic numbers $(a, b)_{2l^2}$ of order $2l^2$ can be divided into various classes.

- **$2 | k$ and $l \neq 3$:** In this case, (3.1) gives classes of singleton, three and six elements. $(0, 0)_{2l^2}$ form singleton class, $(-a, 0)_{2l^2}$, $(a, a)_{2l^2}$, $(0, -a)_{2l^2}$ form classes of three elements where $1 \leq a \leq 2l^2 - 1 \pmod{2l^2}$ and rest $4l^4 - 3 \times 2l^2 + 2$ of the cyclotomic numbers form classes of six elements.

- **$2 | k$ and $l = 3$:** In this case, (3.1) divide cyclotomic numbers $(a, b)_{18}$ of order 18 into classes of singleton, second, three and six elements. $(0, 0)_{18}$ form singleton class, $(-a, 0)_{18}$, $(a, a)_{18}$, $(0, -a)_{18}$ form classes of three elements where $1 \leq a \leq 17 \pmod{18}$, $(6, 12)_{18} = (12, 6)_{18}$ which is grouped into classes of two elements and rest $4l^4 - 3 \times 2l^2$ of the cyclotomic numbers form classes of six elements.

- **$2 \nmid k$ and $l \neq 3$:** Using (3.2), once again we get classes of singleton, three and six elements. $(0, l^2)_{2l^2}$ forms singleton class, $(0, a)_{2l^2}$, $(a + l^2, l^2)_{2l^2}$, $(l^2 - a, -a)_{2l^2}$ form classes of three elements, where $0 \leq a \neq l^2 \leq 2l^2 - 1 \pmod{2l^2}$ and rest $4l^4 - 3 \times 2l^2 + 2$ of the cyclotomic numbers form classes of six elements.

- **$2 \nmid k$ and $l = 3$:** In this situation, (3.2) partitions cyclotomic numbers $(a, b)_{18}$ of order 18 into classes of singleton, two, three and six elements. Here $(0, 9)_{18}$ form singleton class, $(0, a)_{18}$, $(a + 9, 9)_{18}$, $(9 - a, -a)_{18}$ form classes of three elements, where $0 \leq a \neq 9 \leq 17 \pmod{18}$, $(6, 3)_{18} = (12, 15)_{18}$ which is grouped into classes of two elements and rest $4l^4 - 3 \times 2l^2$ of the cyclotomic numbers form classes of six elements.
### Algorithm 1 Equality relation of cyclotomic numbers

1: START  
2: Declare integer variable $e, l, p, k, flag$.  
3: INPUT $l$  
4: if $l$ is not a prime then  
5: go to 3  
6: else  
7: $e = 2l^2$  
8: end if  
9: Declare an array of size $e \times e$, where each element of array is 2 tuple structure (i.e. ordered pair of $(a, b)_{2l^2}$, where $a$ and $b$ are integers).  
10: INPUT $p$, prime number greater than 2  
11: if $(p - 1)\%e == 0$ then  
12: $k = (p - 1)/e$  
13: if $k$ even then  
14: Update table (E)  
15: else  
16: Update table (O)  
17: end if  
18: end if  

Here **Update table (E)** means each entry $(a, b)_{2l^2}$ of the table will be updated by applying the relations $(a, b)_{2l^2} = (b, a)_{2l^2} = (a - b, -b)_{2l^2} = (b - a, -a)_{2l^2} = (-a, b - a)_{2l^2} = (-b, a - b)_{2l^2}$, and **Update table (O)** means each entry $(a, b)_{2l^2}$ of the table will be updated by applying the relations $(a, b)_{2l^2} = (b + l^2, a + l^2)_{2l^2} = (l^2 + a - b, -b)_{2l^2} = (l^2 + b - a, l^2 - a_{2l^2}) = (-a, b - a)_{2l^2} = (l^2 - b, a - b)_{2l^2}$.

Further, if entries of the updated table are non-negative, then each entry should be replace by \((\text{mod} \ 2l^2)\), otherwise add $2l^2$. It is clear from above exploration, cyclotomic numbers of order $2l^2$ are divided into different classes depending on the values of $k$ and $l$. For $l = 2$ and let $k$ be even, then $(0, 0)_8$ give unique solution, cyclotomic numbers of the form $(-a, 0)_8$, $(a, a)_8$, $(0, -a)_8$ where $1 \leq a \leq 7 \ (\text{mod} \ 8)$ gives the same solutions and rest of cyclotomic numbers (i.e. 42) which forms classes of six elements has maximum 7 distinct numbers of solutions. Therefore the initial table (i.e. Table[1]) of cyclotomic matrix reduces to Table[2]. Similarly, for $l = 2$ and let $k$ be odd, then $(0, 4)_8$ give unique solution, cyclotomic numbers of the form
\((0, a)_8, (a + 4, 4)_8, (4 - a, -a)_8\) where \(0 \leq a \neq 4 \leq 7 \pmod{8}\) gives the same solutions and rest of cyclotomic numbers (i.e. 42) which forms classes of six elements has maximum 7 distinct numbers of solutions. Therefore the initial table (i.e. Table 1) of cyclotomic matrix reduces to Table 3. One can observe that 64 pairs of two parameter numbers \((a, b)_8\) reduced to 15 distinct pairs (see Table 2 and Table 3).

| (a,b) | b |
|---|---|
| a  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
| 0  | (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) | (0,6) | (0,7) |
| 1  | (0,1) | (0,7) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) | (1,2) |
| 2  | (0,2) | (0,6) | (1,6) | (2,4) | (2,5) | (2,4) | (2,4) | (1,3) |
| 3  | (0,3) | (1,3) | (1,6) | (0,5) | (1,5) | (2,5) | (2,5) | (1,6) |
| 4  | (0,4) | (1,4) | (2,4) | (1,5) | (0,4) | (1,4) | (2,4) | (1,5) |
| 5  | (0,5) | (1,5) | (2,5) | (2,5) | (1,4) | (0,3) | (1,3) | (1,6) |
| 6  | (0,6) | (1,6) | (2,4) | (2,5) | (2,4) | (1,3) | (0,2) | (1,2) |
| 7  | (0,7) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) | (1,2) | (0,1) |

Table 2: Cyclotomic matrix of order 8 for even k

| (a,b) | b |
|---|---|
| a  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
| 0  | (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) | (0,6) | (0,7) |
| 1  | (1,0) | (1,1) | (1,2) | (1,3) | (0,5) | (0,3) | (1,3) | (1,7) |
| 2  | (2,0) | (2,1) | (2,0) | (1,7) | (0,6) | (1,3) | (0,2) | (1,2) |
| 3  | (1,1) | (2,1) | (2,1) | (1,0) | (0,7) | (1,7) | (1,3) | (0,1) |
| 4  | (0,0) | (1,0) | (2,0) | (1,1) | (0,0) | (1,0) | (2,0) | (1,1) |
| 5  | (1,0) | (0,7) | (1,7) | (1,2) | (0,1) | (1,1) | (2,1) | (2,1) |
| 6  | (2,0) | (1,7) | (0,6) | (1,3) | (0,2) | (1,2) | (2,0) | (2,1) |
| 7  | (1,1) | (1,2) | (1,3) | (0,5) | (0,3) | (1,3) | (1,7) | (1,0) |

Table 3: Cyclotomic matrix of order 8 for odd k

**Remark 3.0.** By Algorithm 1 to compute \(2l^2\) cyclotomic numbers, it is enough to compute \(2l^2 + \left\lceil \frac{(2l^2 - 1)(2l^2 - 2)}{6} \right\rceil\), if \((2l^2 - 1)(2l^2 - 2)|6\), otherwise \(2l^2 + \left\lceil \frac{(2l^2 - 1)(2l^2 - 2)}{6} \right\rceil + 1\). Further, when \(l\) is the least odd prime i.e. \(l = 3\), then \((2l^2 - 1)(2l^2 - 2) \nmid 6\). Therefore \(l = 3\), it is enough to calculate 64 distinct cyclotomic numbers of order \(2l^2\) and for \(l \neq 3\), it is sufficient to calculate \(2l^2 + (2l^2 - 1)(2l^2 - 2)/6\) distinct cyclotomic numbers of order \(2l^2\).
3.3. Determination of generators of $\mathbb{F}_p^*$

To determine the solutions of (2.1), we need the generator of the cyclic group $\mathbb{F}_p^*$. Let us choose finite field of order $p$ that satisfy $p = 2l^2k+1; k \in \mathbb{Z}^+$. Let $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n$ be generators of $\mathbb{F}_p^*$. We consider finite field of order 17 (i.e. $\mathbb{F}_{17}$), since the chosen value of $p = 17$ with respect to the value of $l$ take previously. Now to determine the generators of cyclic group $\mathbb{F}_{17}^*$. The detail procedure to obtain the generator of $\mathbb{F}_{17}^*$ has been depicted in Algorithm 2. If $G_{17}$ is a set that contain all the generator of $\mathbb{F}_{17}^*$, we could get elements of $G_{17}$ as $\{3, 5, 6, 7, 10, 11, 12, 14\}$. 
Algorithm 2 Determination of generators of $\mathbb{F}_p^*$

1: Declare integer variable $p$, count
2: Declare integer array $arr_{\mathbb{F}_p[p]}$, $arr_{\mathbb{F}_p[flag][p]}$
3: for $i = 1$ to $p - 1$ do
4:   $arr_{\mathbb{F}_p[i]} = i$
5:   $arr_{\mathbb{F}_p[flag][i]} = 0$
6: end for
7: Declare integer array $arr_{\mathbb{G}_p}[max]$
8: Declare integer variable $flag = 0$, $r$, $\gamma$
9: for $i = 1$ to $p - 1$ do
10:   count = 0
11:   for $f = 1$ to $p - 1$ do
12:     $arr_{\mathbb{F}_p[flag][f]} = 0$
13:   end for
14:   $\gamma = arr_{\mathbb{F}_p[i]}$
15:   for $a = 1$ to $p - 1$ do
16:     $r = \text{power}(\gamma, a) \pmod{p}$
17:     for $j = 1$ to $p - 1$ do
18:       if $r$ is equal to $arr_{\mathbb{F}_p[j]}$ then
19:         $arr_{\mathbb{F}_p[flag][j]} = 1$
20:       end if
21:     end for
22:   end for
23:   for $k = 1$ to $p - 1$ do
24:     if $arr_{\mathbb{F}_p[flag][k]}$ is equal to 1 then
25:       count++
26:     end if
27:   end for
28:   if count is equal to $p - 1$ then
29:     $\gamma$ is generator
30: end if
31: end for

3.4. Generation of cyclotomic matrices

This subsection presents an algorithm for the generation of cyclotomic matrices of order $2l^2$. Note that entries of cyclotomic matrices are solutions
of (2.1). Thus we need the generator of the cyclic group $\mathbb{F}_p^*$, which is discussed in the previous subsection. On substituting the generators of $\mathbb{F}_p^*$ in Algorithm 3 we obtain the cyclotomic matrices of order $2l^2$ corresponding to different generators of $\mathbb{F}_p^*$. The chosen value of $p = 17$ implies $k = 2$ w.r.t. assume value of $l = 2$. Therefore the cyclotomic matrix will be obtain from Table 2. Let us choose a generator (say $\gamma_1 = 3$) from set $G_{17}$. On substituting $\gamma_1 = 3$ in Algorithm 3 it will generate cyclotomic matrix of order 8 over $\mathbb{F}_{17}$. w.r.t. chosen generator $\gamma_1 = 3$. Matrix $B_0$ show the corresponding cyclotomic matrix of order 8 w.r.t. chosen generator $3 \in \mathbb{F}_{17}^*$.

### Algorithm 3 Generation of cyclotomic matrix

1: INPUT: The value of $p, l, \gamma$
2: Declare an array $arr[ROW][COL]$ (where elements are two tuple structure)
3: Declare integer variable $p, l, k, \gamma, x, y, A, s, t, a, b, count = 0, p_1, p_2$
4: for $a$ equal to 0 to number of rows do
5:   for $b$ equal to 0 to number of columns do
6:     for $x$ is equal to 0 to $k$ do
7:       for $y$ is equal to 0 to $k$ do
8:         $p_1 = 2l^2 * s + arr[a][b].l$
9:         $p_2 = 2l^2 * t + arr[a][b].m$
10:        $A = power(\gamma, p_1) + power(\gamma, p_2) + 1$
11:       if $A \pmod{p}$ is equal to 0 then
12:          count ++
13:       end if
14:     end for
15:   end for
16:   $arr[a][b].n = count$
17:   count = 0
18: end for
19: end for
Remark 3.1. If we change the generator \( \gamma \) to a new generator \( \gamma^{r_0} \) of \( \mathbb{F}_p^* \), then \((a, b)_e\) becomes \((r^0a, r^0b)_e\).

Remark 3.2. It is noted that if we change the generator of \( \mathbb{F}_p^* \), then entries of cyclotomic matrices get interchanged among themselves but their nature remains the same.

Remark 3.3. It is obvious that (by (2.4)) cyclotomic matrices of order \( 2l^2 \) is always singular if the value of \( k = 1 \).

4. The public-key cryptosystem

In this section, we present the approach for designing a public key cryptosystem using cyclotomic matrices discussed in section 3. The scheme employ matrices of order \( 2l^2 \), whose entries are cyclotomic numbers of order \( 2l^2 \). The public key is obtained by choosing a generator \( \gamma' \in \mathbb{F}_p^* \) and apply Algorithm 3. It gives a cyclotomic matrix of order \( 2l^2 \) and further check that whether the matrix is non-singular or not. If this matrix is non-singular, then it is assigned for the public key. A key expansion algorithm is employed for secret key (see Algorithm 4), to form a non-singular matrix of order \( 2l^2 \) by the value of another generator \( \gamma'' (\gamma'' \neq \gamma') \) in \( \mathbb{F}_p^* \). The complexity of anonymous decryption could be understood as; if we assume that an attacker wants to recover the secret key by using all the informations available to them. Then they need to solve the discrete logarithm problem (DLP) to find the secret key followed by a number of steps described in algorithm 6.

Let \( p \) be a prime and \( \gamma', \gamma'' \in \mathbb{F}_p^* \). We write \( \log_{\gamma'}(\gamma'') = n \) if \( n \in \mathbb{Z} \) satisfies \( \gamma'^{(n)} = \gamma'' \). The problem of finding such an integer \( n \) for a given \( \gamma', \gamma'' \in \mathbb{F}_p^* \) (with \( \gamma' \neq 1 \)) is the discrete logarithm problem (DLP). However, although most mathematicians and computer scientists believe that the DLP is unsolvable. The complexity of the DLP depends on the cyclic group.
is believed to be a hard problem for the multiplicative group of a finite field of large cardinality. Therefore even determining the very first step is nearly unsolvable.

If it is the case that somehow attacker manages to solve the DLP, then they have to determine equation (2.1) and calculate all the solutions corresponding to different pairs \((a, b)_{2l^2}\). Further, it is required to determine the relation matrix based on equality relation among the solutions of equation (2.1). Where entries of the relation matrix are two-tuple structure of \((a, b)_{2l^2}\). Finally, entries of inverse of the relation matrix are required to replace through the implication of DLP.

Here we could observe the computational complexity as it increases with the value of \(p\) and \(2l^2\). Therefore it is nearly impossible to determine the secret key for a large value of \(p\) and \(2l^2\); hence uphold the secure formulation claim of the proposed work.

**Algorithm 4** Key Expansion

1: INPUT: The value of \(p, l\) and \(\gamma''\)
2: Algorithm 1
3: Algorithm 3

**Algorithm 5** Encryption

1: Transfer the plain text (message) into its numerical value and store in matrix of order \(2l^2\)
2: INPUT: The value of \(p\) and \(2l^2\)
3: Algorithm 2
4: INPUT: The value of \(\gamma'\)
5: Algorithm 3
6: Check: Generated matrix by Algorithm 3 is non-singular
7: Choose a generator \(\gamma''\) which is different from \(\gamma'\) in \(\mathbb{F}_p^*\).
8: Determine the relation of \(\gamma'\) and \(\gamma''\) by remark 4.1 and send the value of \(\gamma', r_0, p\) and \(l\)
Algorithm 6 Decryption

1: Determine $\gamma''$ by the value of $r_0$ and $\gamma'$
2: Algorithm [1]
3: Each entries of equality of cyclotomic matrix (i.e. output matrix of Algorithm [1]) is multiply by $r_0$. The entries of the generated matrix are pair of cyclotomic number
4: Compute the inverse of generated matrix in step 3 and substitute the value of each pair of cyclotomic number from generated matrix in step 2
5: Now multiply the cipher text matrix to generated matrix in step 4, we get back to the original plain text message.

Example 1. Here is an example for our cryptosystem. Let us consider $2l^2 = 8$ and $p = 17$. Suppose we want to send a message $X$ whose numerical value store in matrix $A$ of order $2l^2$.

$$A = \begin{bmatrix}
2 & 3 & 5 & 9 & 8 & 0 & 2 & 1 \\
1 & 5 & 9 & 2 & 9 & 3 & 0 & 5 \\
2 & 1 & 3 & 2 & 5 & 6 & 8 & 7 \\
5 & 3 & 0 & 7 & 8 & 7 & 3 & 1 \\
4 & 2 & 3 & 1 & 9 & 8 & 7 & 3 \\
0 & 9 & 2 & 3 & 5 & 6 & 8 & 9 \\
1 & 0 & 2 & 9 & 6 & 7 & 9 & 8 \\
9 & 1 & 3 & 2 & 4 & 4 & 5 & 6
\end{bmatrix}$$

Let us choose value of generator $\gamma = 11$ (by Algorithm [2]) of cyclic group $\mathbb{F}_{17}^*$. Then the public key is given by Algorithm [3] which is

$$B_3 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

Determinant of $B_3$ is equal to 1, implies non-singular. Now we encrypt the message $A$ by multiplying matrix $B_3$ and $A$, which is as follows:
We choose a generator that is different from public key generator $\gamma = 11$ in $\mathbb{F}^{*}_{17}$. Let us consider $\gamma = 3$. Now, we determine the relationship between $\gamma = 3$ and $\gamma = 11$. One can write $3^7 = 11 \mod 17$. Consider that $r_0 = 7$. For the decryption, determine the value of $\gamma = 3$ by using the values of $\gamma = 11$ and $r_0 = 7$. Now by applying Algorithm 1 and 3, we get a cyclotomic matrix, which is shown by matrix $B_0$. Now each entries of equality of cyclotomic matrix (i.e. output matrix of Algorithm 1) is multiply by $r_0$. We get matrix $D$ whose entries are pair of cyclotomic numbers.

$$D = \begin{bmatrix}
(0,0) & (0,7) & (0,6) & (0,5) & (0,4) & (0,3) & (0,2) & (0,1) \\
(0,7) & (0,1) & (1,2) & (1,6) & (1,5) & (1,4) & (1,3) & (1,2) \\
(0,6) & (1,2) & (0,2) & (1,3) & (2,4) & (2,5) & (2,4) & (1,6) \\
(0,5) & (1,6) & (1,3) & (0,3) & (1,4) & (2,5) & (2,5) & (1,5) \\
(0,4) & (1,5) & (2,4) & (1,4) & (0,4) & (1,5) & (2,4) & (1,4) \\
(0,3) & (1,4) & (2,5) & (2,5) & (1,5) & (0,5) & (1,6) & (1,3) \\
(0,2) & (1,3) & (2,4) & (2,5) & (2,4) & (1,6) & (0,6) & (1,2) \\
(0,1) & (1,2) & (1,6) & (1,5) & (1,4) & (1,3) & (1,2) & (0,7)
\end{bmatrix}$$

Now compute the inverse of $D$ and substitute the value from $B_0$ to each pair of cyclotomic numbers. The matrix becomes

$$D^* = \begin{bmatrix}
-1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & -1 & 0 & 1 & -1 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\
-1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\
1 & -1 & 0 & 1 & 1 & -1 & 1 & -1
\end{bmatrix}$$

Finally we obtain $D^* \times C = A$. 

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5. Conclusion

In this paper, we have introduced a secured asymmetric key cryptography model applying the principle of cyclotomic numbers over a finite field. Procedure to generate cyclotomic matrix along with public key and private key have been presented where the relation between the public key and private key has acquired by discrete logarithm problem (DLP). Finally, a convincing argument to strengthen the claim has been presented followed by the method of encryption, decryption and a numerical example.

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