Dissipation-free and dispersion-optimized explicit time-domain finite element method for room acoustic modeling

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Abstract: This paper presents a proposal for an efficient room acoustic solver with dissipation-free and dispersion-optimized explicit time-domain FEM (TD-FEM), with investigation of its applicability for broadband room acoustic modeling from three numerical experiments. Recently, FEM-based room acoustic solvers have attracted great attention because of their strength in handling complex geometries. However, the development of higher-efficiency solvers is unavoidable to perform acoustic modeling of real-size rooms at kilohertz frequency with small discretization error: the dispersion error. The present paper first formulates a novel room acoustic solver with dissipation-free fourth-order accurate explicit TD-FEM using a three-step time integration method. A dispersion-optimized solver is further proposed in which dispersion error is minimized in the axial and diagonal directions at a specific frequency under given spatial resolution mesh or elements, by which the approximation capability at higher frequencies is enhanced without any additional computational cost. The performance of the optimized solver in broadband acoustic simulation using cubic elements is then examined in comparison with the original fourth-order accurate solver and the standard implicit TD-FEM. Finally, higher efficiency of the optimized solver is also demonstrated for acoustic simulation in a larger rectangular room discretized with rectangular and distorted hexahedral elements.

Keywords: Dispersion error, Explicit time integration, Room acoustics simulation, Time-domain FEM

1. INTRODUCTION

Developments of FEM-based room acoustic solvers in both the frequency-domain [1,2] and time-domain [3–13] are active research areas in recent years. Actually, FEM is attractive for room acoustic modeling because of its outstanding potential in modeling arbitrarily shaped rooms and various acoustical materials accurately. However, as with other numerical methods such as the finite difference time-domain method [14–16] and the finite volume time-domain method [17], the acoustic FEM includes inherent discretization error, i.e., dispersion error, limiting the computable range with respect to frequency and room size. The dispersion error is frequency-dependent and directional angle-dependent sound speed error, which engenders anisotropic sound wave propagation with slower or faster sound speed than the exact one. A simple strategy to minimize the dispersion error is reducing the element size in spatial discretization, called $h$-refinement. Although a famous rule of thumb for spatial discretization with linear finite elements (FEs) exists, i.e., at least 10 points per wavelength [18], this condition engenders huge computational demands for acoustic simulations in a real-size room at several kilohertz of frequency. Therefore, the construction of a high efficient solver that is able to reduce the dispersion error with a small computational cost is necessary for application of FEM to practical room acoustic problems.

Dispersion error can also be reduced by increasing the polynomial order of FEs ($p$-refinement), seen in high-order FEMs, e.g., the quadratic finite element [19], spline FEM [20], $p$-FEM [21], $hp$-FEM [22], spectral element method (SEM) [23], and discontinuous Galerkin finite element method (DGFEM) [24]. Among them, $p$-FEM reduces the dispersion error using only the increase of polynomial order of a hierarchical shape function with a fixed FE mesh. However, the $hp$-FEM, the SEM and the DGFEM use the
combination of \( h \)-refinement and \( p \)-refinement, known as \( hp \)-refinement, reducing the dispersion error with an exponential rate. Recent works \([7-9,13]\) have applied SEM and the nodal DGFEM to room acoustic problems, presenting their applicability in room acoustics modeling including both frequency-independent and frequency-dependent impedance boundaries. These higher-order FEMs can generally conduct acoustic simulations with fewer computational costs than the linear FEM because the use of higher-order polynomials can reduce the degrees of freedom (DOF) to achieve a similar level of accuracy. However, higher-order elements engender wider matrix bandwidth with the increase of polynomial order, requiring much higher computational cost per DOF than lower-order elements. The DGFEM can soften the shortcoming by element-wise calculation without storing matrices. However, the DGFEM presents larger dispersion error than the SEM with the same order polynomials \([24,25]\), and involves dissipation error. The use of higher-order polynomials is necessary to acquire comparable accuracy to that of the SEM. Although the benefit of using the \( hp \)-framework is apparent in terms of accuracy, its implementation is difficult compared to standard FEM code.

Another highly accurate FEM, called partition of unity FEM (PU-FEM), which uses a refinement technique: \( q \)-refinement \([26,27]\). In acoustic simulations, PU-FEM enhances the approximation capability of sound fields with incorporation of plane waves propagating in various directions at the node of FEs, by which a drastic DOF reduction is achieved while maintaining high accuracy. A recent report of the literature \([2]\) discussing its potential as a room acoustic solver showed the DOF reduction reaches at least 1/100 of the standard linear FEM. However, its efficiency in 3D room acoustic problems remains unclear. Although application of PU-FEM to a time-domain wave propagation problem was conducted in the literature \([28]\), computing a wideband response is still a challenging task. It is noteworthy that other dispersion-reducing techniques are presented in an earlier report \([18]\).

The authors proposed efficient room acoustic FEM solvers in both frequency and time domains \([1,3,6,11]\), using low-order dispersion-reduced FEs \([29,30]\), which has minimum computational costs per DOF. The outstanding attractiveness of the solvers is to achieve fourth-order accuracy in terms of dispersion error while maintaining the same bandwidth of linear FEs, meaning that the dispersion error reduction is achieved without additional computational cost, unlike the higher-order FEMs. Furthermore, the implementation is simple. Among them, an implicit TD-FEM \([3]\) offers high capability in room acoustic simulations. Its applicability on real-scale 3D room acoustic problems has been demonstrated via acoustic simulations in a concert hall and a reverberation room in earlier works \([6,31]\). Two recent studies \([11,12]\) extended its applicability further by incorporating locally reacting frequency-dependent absorbing boundary conditions and an extended reacting equivalent fluid model for porous sound absorbers.

However, the authors explore construction of highly efficient explicit TD-FEM \([4,5,10]\), which avoids solving a linear system of equations at each time step, unlike implicit TD-FEM. Consequently, we proposed a fourth-order accurate explicit TD-FEM based on a low-order spatial discretization. We demonstrated its applicability to 2D real-scale room acoustic problems, where sound wave propagation in a rectangular room with complex-shaped sound diffusers and a concert hall were calculated at kilohertz frequencies \([10]\). However, the method accompanies a shortcoming: a dissipation error is introduced by the inherent characteristics of a time integration method employed for temporal discretization. Although the magnitude of dissipation error can minimize with smaller time intervals, the strategy engenders increasing computational costs. The present paper presents a solution for inefficiency using a three-step time integration method, which is uniquely developed in the present paper, i.e., one of the novelties of the present paper, based on a general form of linear multi-step methods \([32]\), computing next step values using previous three-step results. We also present a dispersion-optimized method, as the second point of novelties of the present paper, to enhance the approximation capability of sound fields at higher frequencies.

With these two novel methods, this paper finally proposes a novel dispersion-optimized and dissipation-free explicit TD-FEM for room acoustic simulations and presents examination of performance via basic numerical experiments. We show the present method as more efficient and accurate than the previous dissipative scheme, as a contribution of present work. Although the present method can incorporate frequency-dependent impedance boundary conditions, we only show its performance on interior acoustic simulation with a rigid boundary because the most paper space is used to explain how we eliminate the dissipation error and how we optimize dispersion error theoretically with detailed descriptions.

The remainder of the paper is organized as follows. Section 2 formulates the dissipation-free fourth-order accurate explicit TD-FEM, defined hereinafter as D-4th explicit method. Section 3 proposes a dissipation-free and dispersion-optimized explicit TD-FEM, which we call D-opt explicit method, with a novel dispersion optimization strategy. Section 4 presents examination of the accuracy and efficiency of D-4th and D-opt explicit methods through three numerical experiments compared to an analytical solution and a standard implicit TD-FEM.
2. ROOM ACOUSTIC SOLVER WITH A DISSIPATION-FREE FOURTH-ORDER ACCURATE EXPLICIT TD-FEM

This section presents a description of the formulation of room acoustic solver with D-4th explicit method. The dissipation-free scheme is realized using a three-step time integration method. Additionally, fourth-order accuracy with respect to dispersion error is obtained using the Gauss–Legendre rule with modified integration points in element matrices calculations and appropriate weight coefficients in the time integration method. We first present a basic matrix equation for TD-FEM and a scheme for dissipative fourth-order explicit TD-FEM [10]. Then, an origin of dissipation error and how to remove it is explained. The modified integration points and appropriate weight coefficients for the D-4th explicit TD-FEM are derived from three-dimensional dispersion error analysis. Finally, we exemplify the theoretical dispersion error characteristics of D-4th explicit TD-FEM.

2.1. Basic Semi-discretized Equation

To simulate sound propagation in an enclosed space \( \Omega \) with boundary \( \Gamma \) second-order scalar wave equation is defined as

\[
\frac{\partial^2 p(r,t)}{\partial t^2} - c_0^2 \nabla^2 p(r,t) = \rho_0 c_0^2 \frac{\partial q(r,t)}{\partial t} \delta(r - r_s),
\]

with the following rigid boundary \( \Gamma_0 \) and vibration boundary \( \Gamma_V \) as shown below.

\[
\frac{\partial p(r,t)}{\partial n} = \begin{cases} 0 & \text{on } \Gamma_0, \\ -\rho_0 \dot{v}_n(r,t) & \text{on } \Gamma_V. \end{cases}
\]

Therein, \( p \), \( c_0 \), and \( \rho_0 \) respectively denote the sound pressure, sound speed, and density of air. Also, \( q \), \( t \) and \( \nabla^2 \) respectively represent the added fluid mass per unit volume, the time, and the Laplacian operator. \( r \) and \( r_s \) respectively denote the arbitrary coordinate vectors in the space and a point source. Furthermore, \( \delta \) is the delta function. Additionally, \( v_n \) is the vibration velocity and the symbol \( \cdot \) is the first-order time derivative. The following weak form is derived using Green’s theorem as

\[
\int_{\Omega} \left( \frac{\partial^2 p(r,t)}{\partial t^2} + c_0^2 \nabla \phi \cdot \nabla p(r,t) \right) d\Omega - c_0^2 \int_{\Gamma} \phi_t \frac{\partial p(r,t)}{\partial n} d\Gamma = \rho_0 c_0^2 \int_{\Gamma} \phi_t \frac{\partial q(r,t)}{\partial t} \delta(r - r_s) d\Omega,
\]

where \( \phi_t \) represents the arbitrary weight function. Using the Galerkin method with the two boundary conditions above engenders the second-order ordinary differential equation (ODE) as [3,6,31]

\[
M \ddot{p} + c_0^2 K p = f,
\]

with

\[
M = \sum_{i=1}^{N_e} M_{ei} = \sum_{i=1}^{N_e} \int_{\Omega_e} N^T N d\Omega_e,
\]

\[
K = \sum_{i=1}^{N_e} K_{ei} = \sum_{i=1}^{N_e} \int_{\Omega_e} \nabla N^T \nabla N d\Omega_e.
\]

Here, \( M \) and \( K \) respectively represent the global consistent mass matrix and the global stiffness matrix assembled with each element matrix \( M_e \) and \( K_e \). \( N \) denotes the shape function, and \( p \) and \( f \) respectively denote the sound pressure vector and the external force vector. \( N_e \) stands for the number of finite elements \( \Omega_e \). The symbol \( \cdot \) signifies the second-order time derivative. A standard implicit TD-FEM formulation solves Eq. (4) with direct time integration methods such as the Newmark \( \beta \) method [33]. For spatial discretization of the 3D domain, the present paper uses eight-node hexahedral linear FEIs. In the FEIs, the numerical integrations in Eqs. (5) and (6) are performed with two-point Gauss–Legendre quadrature in each direction as

\[
M_e = \sum_{i=1}^{8} N(\alpha_{m,i})^T N(\alpha_{m,i}) \det(J),
\]

\[
K_e = \sum_{i=1}^{8} \nabla N(\alpha_{e,i})^T \nabla N(\alpha_{e,i}) \det(J),
\]

where \( \alpha_{m,i} \) and \( \alpha_{e,i} \) signify local coordinates vectors assembled with integration points \( \alpha_m = \alpha_e = \pm \sqrt{1/3} \) in the computation of \( M_e \) and \( K_e \). \( J \) stands for the Jacobian matrix. The standard implicit TD-FEM has second-order accuracy in terms of dispersion error. It is used for the performance comparison in Sect. 4 as a baseline solver.

2.2. Time-marching Scheme for Dissipative Fourth-order Explicit TD-FEM

The explicit TD-FEM solves the following simultaneous first-order ODEs equivalent to the second-order system of Eq. (4) [4,5,10].

\[
D \ddot{p} = Mv,
\]

\[
D \dot{v} = f - c_0^2 Kp.
\]

Here, \( D \) is the lumped mass matrix calculated from the consistent mass matrix \( M \) using the row-sum method [34,35]. Additionally, \( v \) denotes the auxiliary vector equivalent to the first derivative of \( p \) with respect to time. In an earlier study [10], \( \dot{p} \) in Eq. (9) was discretized using a modified Adams method. Also, \( \dot{v} \) in Eq. (10) was discretized using the first-order backward difference approximation as

\[
p^{n+1} = p^n + \frac{\Delta t}{12} D^{-1}(14Mv^{n-1} - 5Mv^{n-2} + 4Mv^{n-3} - Mu^{n-4}),
\]

\[
v^{n+1} = v^{n-1} + \Delta t D^{-1}(f^n - c_0^2 Kp^n).
\]
Therein, $\Delta t$ and $n$ respectively represent the time interval and the time step. Additionally, to achieve fourth-order accuracy in space, the method uses the following modified integration points in the element mass and stiffness matrices calculations instead of conventional integration points of $\pm \sqrt{1/3}$.

$$\alpha_m = \pm \sqrt{\frac{4}{3}}, \quad \alpha_k = \pm \sqrt{\frac{2}{3}}.$$  \hspace{1cm} (13)

Applying the dispersion error analysis described in Sect. 2.3.2 yields the following evaluation equation for the numerical sound speed $c^h$ as

$$c^h \approx c_0 \left(1 - \frac{(kh)^4}{368,640} \left(14,952 \tau^4 + 1,230 + C\right) - i \frac{(\omega \Delta t)^5}{24}\right).$$  \hspace{1cm} (14)

where

$$C = 295 \cos 2\theta + 514 \cos 4\theta + 9 \cos 6\theta + 16(29 - 9 \cos 2\theta) \cos 4\phi \sin \theta.$$ \hspace{1cm} (15)

Here, $\theta$ and $\phi$ respectively stand for the elevation and azimuth in the spherical coordinate system. $\tau$ is the Courant number defined as $c_0 \Delta t/h$. In addition, $k$ and $\omega$ denote the wavenumber and the angular frequency. $i$ is the imaginary number. From Eqs. (14) and (15), the explicit TD-FEM using the modified Adams method achieves fourth-order accuracy with respect to the dispersion error. However, the sound speed becomes a complex value, which indicates that the method is inherently dissipative. The magnitude is represented by a fifth-order dissipation term. Therefore, the dissipation-free scheme can be created by eliminating the imaginary part from an equation of dispersion relation. Using Von Neumann’s stability analysis [36], the stability limit of time interval $\Delta t_{\text{limit}}$ for the explicit TD-FEM using the modified Adams method is derived as

$$\Delta t_{\text{limit}} = 0.365801 h/c_0.$$ \hspace{1cm} (16)

2.3. Time-marching Scheme for Dissipation-free Explicit TD-FEM

2.3.1. Three-step time integration method

To construct a dissipation-free high-order scheme, we discretize $\phi$ in Eq. (9) using the three-step time integration method as

$$p^n = \sum_{i=1}^{3} a_i p^{n-i} + \Delta t D^{-1} \sum_{j=1}^{3} b_j M v^{n-j}. \hspace{1cm} (17)$$

Here, $a_i$ and $b_j$ respectively represent the $i$-th and $j$-th weight coefficients that satisfy the following property:

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{3} b_j = 1.$$ \hspace{1cm} (18)

For time-marching computation of $v$ Eq. (12) is used. Both D-4th and D-opt explicit TD-FEMs use the time-marching scheme of Eqs. (17) and (12). As described later, they respectively use different numerical integration points in element mass and stiffness matrices and the weight coefficients in Eq. (17).

2.3.2. Dispersion error analysis

Dispersion error analysis presented here follows the procedure described in literature [4] and assumes plane wave propagation in a free field under isotropically discretized mesh with cubic elements of length $h$, where $\theta$ and $\phi$ respectively denote the elevation and azimuth in the spherical coordinate system.

For time-marching computation of $v$ Eq. (12) is used. Both D-4th and D-opt explicit TD-FEMs use the time-marching scheme of Eqs. (17) and (12). As described later, they respectively use different numerical integration points in element mass and stiffness matrices and the weight coefficients in Eq. (17).
construct a dissipation-free scheme by satisfying the following conditions:
\[ a_1 + 1 = a_3 - a_2, \quad a_3 = 1, \quad b_1 = b_3. \] (22)

From Eqs. (18) and (22), we obtain
\[ a_1 = 2, \quad a_2 = -2, \quad a_3 = 1, \quad b_2 = 1 - 2b_1. \] (23)

With Eqs. (23) and (21), Eq. (19) is transformed into
\[ (4 - 6 \cos \omega \Delta t + 2 \cos 2\omega \Delta t) p_{\alpha,1}^n + c_2^2 \Delta t^2 A ((1 - 2b_1) + 2b_1 \cos \omega \Delta t) p_{\alpha,1}^{n-1} = 0. \] (24)

Solving Eq. (24) with respect to \( c_0 \) engenders the dispersion relation as
\[ c_0 = \sqrt{\frac{\tilde{h}^2 (-4 + 6 \cos \tilde{h} \Delta t - 2 \cos 2\tilde{h} \Delta t)}{\Delta t^2 M_c ((1 - 2b_1) + 2b_1 \cos \tilde{h} \Delta t)}}, \] (25)

where
\[ M_c = \frac{h}{16} (3 + C_x + C_y + C_z - C_{xy} - C_{yz} - C_{xz} - 3C_{xyz}) - 2\alpha_3^2 (-3 + C_x + C_y + C_z + C_{xy} + C_{yz} + C_{xz} - 3C_{xyz}) - 3\alpha_3^2 (-1 + C_x + C_y + C_z - C_{xy} - C_{yz} - C_{xz} + 3C_{xyz}), \] (26)

\[ C_x = \cos (\tilde{h} \sin \theta \cos \phi), \quad C_y = \cos (\tilde{h} \sin \theta \sin \phi), \]
\[ C_z = \cos (\tilde{h} \cos \theta), \quad C_{xy} = C_x C_y, \quad C_{yz} = C_y C_z, \]
\[ C_{xz} = C_z C_x, \quad C_{xyz} = C_x C_y C_z. \] (28)

Here, \( \tilde{h} \) signifies the numerical wavenumber. Similarly to the explicit TD-FEM using the modified Adams method, the D-4th explicit TD-FEM uses modified integration points in Eq. (13) to maintain fourth-order accuracy in space. Taking Taylor expansion with respect to \( \tilde{h} \) in Eq. (25), \( \tilde{h} \) is evaluated as
\[ \tilde{h} = c_0 \left( 1 + \frac{(\omega \Delta t)^2}{24} (12b_1 - 13) + O(\Delta t^4) \right). \] (29)

The use of \( b_1 = 13/12 \) eliminates the second-order dispersion term. The fourth-order accuracy in time is achieved as
\[ \tilde{h} \approx c_0 \left( 1 - \frac{(\tilde{h} \Delta t)^2}{368,640} (14,952 \tau^4 + 1,230 + C) \right). \] (30)

Details of \( C \) in Eq. (30) are described in Eq. (15). The resulting numerical sound speed becomes a real value, meaning that the scheme inherently includes no dissipative characteristics. Because Eq. (25) has no imaginary part, the numerical sound speed always takes a real value irrespective of the order of Taylor expansion. Additionally, the dispersion property of D-4th explicit TD-FEM is identical to that of the explicit TD-FEM using the modified Adams method. By performing Von Neumann's stability analysis [36], the stability limit of time interval \( \Delta t_{\text{limit}} \) is given as
\[ \Delta t_{\text{limit}} = 0.490774h/c_0. \] (31)

From Eq. (31), D-4th explicit TD-FEM relaxed the stability condition compared with the explicit TD-FEM using the modified Adams method. Figure 2 shows the dispersion error characteristics of D-4th explicit TD-FEM at the stability limit for the case with \( f = 2.5 \) kHz, \( h = 0.02 \) m, and \( c_0 = 343.7 \) m/s. The spatial resolution \( R \) is 6.87, where \( R \) is defined as \( \lambda/h \) with wavelength \( \lambda \) at \( f = 2.5 \) kHz. The maximum dispersion error shows at axial directions. The values are smaller than 0.6%. Furthermore, the relations between \( R \) and dispersion error at axial direction and at diagonal direction were presented, respectively, in Fig. 3. Here, \( \Delta t \) was set to \( \Delta t_{\text{limit}} \) in Eq. (31). The D-4th explicit TD-FEM can restrict maximum dispersion error smaller than 1% with \( R \geq 5.98 \). As a reference,
standard implicit TD-FEM using linear FEs requires spatial resolution of $R = 12$ at least to obtain dispersion error of less than 1%.

3. DISPERSION-OPTIMIZED EXPLICIT TD-FEM

This section presents another dissipation-free explicit TD-FEM having dispersion-optimized property abbreviated as D-opt explicit TD-FEM. As demonstrated in subsequent numerical experiments, the present method can produce higher accuracy than D-4th explicit TD-FEM at higher frequencies under the same spatial resolution mesh. The D-opt explicit TD-FEM uses different integration points $\alpha_m$ and $\alpha_k$ in element matrices calculation, and different weight coefficients in the time integration method. Conventionally, the modified integration and higher-order time integration method had been used to increase the convergence rate of discretization error. However, the proposed optimization method, i.e., the second point of novelties of the present paper, minimizes dispersion error in the axial and diagonal directions at a specific frequency under given spatial resolution mesh or elements. We also present theoretical dispersion error characteristics of D-opt explicit TD-FEM.

3.1. Optimizing Spatial Discretization Error

Taking the Fourier transformation with time factor of $e^{i\omega \Delta t}$, Eqs. (9) and (10) are transformed into

$$
i_0 \mathbf{D} \mathbf{p} = \mathbf{M} \mathbf{v},$$

(32)

$$
i_0 \mathbf{D} \mathbf{v} = \mathbf{f} - c_0^2 \mathbf{K} \mathbf{p} - ioc_0 \mathbf{C} \mathbf{p}.$$  

(33)

Using the same procedure described in the authors’ earlier work [10], three-dimensional dispersion relation is expressed as

$$
c_0 = c_k k^h \sqrt{\frac{\nu_0^0}{M^0 K^0}}. $$

(34)

Replacing $c_k$ and $k^h$ in Eq. (34), respectively, with $c_0$ and $k$, the dispersion error from spatial discretization, $\epsilon_{\text{spatial}}$, is evaluated using the following equation.

$$
\epsilon_{\text{spatial}} = \frac{c_0 k^h \sqrt{\frac{\nu_0}{M^0 K^0}} - c_0}{c_0}.
$$

(35)

Minimizing the dispersion error so that $\epsilon_{\text{spatial}} = 0$ at axial direction, the following optimum point of $\alpha_m$ is obtained as

$$
\alpha_m = \pm \frac{4 \pi^2 + R^2 (\cos^2 \frac{2 \pi}{n} - 1) \sqrt{R^2 (\cos \frac{2 \pi}{n}) - 1}}{3 X^2 X^4 X^5 X^6},
$$

(36)

The integration point depends only on the spatial resolution $R$, i.e., the optimum $\alpha_m$ is fixed with analyzed frequency and used FE mesh or elements. Then, with the optimum $\alpha_m$, the dispersion error in Eq. (35) is minimized in the diagonal direction. We obtain the optimum $\alpha_k$ depending only $\alpha_m$ and $R$ as

$$
\alpha_k = \pm \sqrt{\frac{1}{R} \frac{X_1 + X_2}{3 X^2 X^4 X^5 X^6}},
$$

(37)

with

$$
X_1 = -X^2 X^2 X^2 h(R^2 X^2 (Y_1 - Y_2)^2 (-X_6) + 48 \pi^2 X_5),
$$

$$
X_2 = \alpha_m^2 X^3 X^3 Y_1 - X_3 X_3 Y_1 (Y_1 + 1)^2 (1 + Y_2) + \alpha_m^2 (Y_1^2 - 1) (-3 + Y_2 - 3 Y_1 Y_2) X_7 - \alpha_m^2 X^2 X_7^2,
$$

$$
X_3 = Y_1 - 1,
$$

$$
X_4 = 1 + \alpha_m^2 + Y_1 - \alpha_m^2 Y_1,
$$

$$
X_5 = Y_2 - 1,
$$

$$
X_6 = 1 + \alpha_m^2 + Y_2 - \alpha_m^2 Y_2,
$$

$$
X_7 = -3 - Y_1 - Y_2 + 3 Y_1 Y_2,
$$

$$
Y_1 = \cos \frac{\pi}{R},
$$

$$
Y_2 = \cos \frac{2 \pi}{R}.
$$

(38)

3.2. Optimizing Temporal Discretization Error

Substituting $\frac{\nu_0}{M^0 K^0} = 1/k^2$ into Eq. (25), the dispersion error from a temporal discretization error, $\epsilon_{\text{temporal}}$, is expressed as

$$
\epsilon_{\text{temporal}} = \sqrt{\frac{(-4 - 6 \cos \omega \Delta t - 2 \cos 2 \omega \Delta t)}{\omega^2 \Delta t^2 (1 - 2 \Delta t) + 2 \Delta t \cos \omega \Delta t} - c_0} c_0.
$$

(39)

Solving the equation above under condition $\epsilon_{\text{temporal}} = 0$ engenders the optimum weight coefficient $b_1$ as

$$
b_1 = \frac{1 - 2 \cos (\omega \Delta t)}{\omega^2 \Delta t^2} + \frac{1}{4 \sin^2 \left(\frac{\omega \Delta t}{2}\right)}.
$$

(40)

Optimum $b_1$ depends only on the frequency and the time interval. As a summary of D-opt explicit TD-FEM, it uses the modified integration points of Eqs. (36) and (37) and weight coefficients of Eq. (23) with the coefficient $b_1$ of Eq. (40). We show the dispersion error characteristics of D-opt explicit TD-FEM in Fig. 4 under the same condition as in Sect. 2.3.2. D-opt explicit TD-FEM shows much less dispersion error for all directions than those in D-4th explicit TD-FEM shown in Fig. 2. Also, Fig. 5 presents the relations between $R$ and dispersion error at the axial direction and at the diagonal direction in two optimizing cases, where dispersion errors were minimized, respectively, at $R = 5.5$ and $R = 6.87$ using a stability limit time interval value. In both cases, the dispersion errors were minimized at the intended spatial resolution. The dispersion error in the axial direction is greater than that in the diagonal direction under the same optimizing conditions, irrespective of $R$. Although the dispersion errors increased drastically before and after the optimized spatial resolutions, the D-opt explicit TD-FEM maintains small dispersion at broadband, where the maximum dispersion
errors were smaller than 0.4% with $R / C = 99$ and smaller than 0.2% with $R / C = 78$, respectively, when optimizing at $R = 5.5$ and $R = 6.874$. It is noteworthy that D-opt explicit TD-FEM requires no additional computational costs compared to the D-4th explicit TD-FEM while increasing the accuracy at higher frequencies. The next section shows, numerically, that the method outperforms D-4th explicit TD-FEM in broadband acoustic simulations.

4. NUMERICAL EXPERIMENTS

The present section presents examination of the performance of D-4th explicit TD-FEM and D-opt explicit TD-FEM via three numerical experiments. Because the D-opt explicit TD-FEM was derived by conditions that minimize dispersion error in axial and diagonal directions at a specific frequency with cubic elements, it will be particularly important to show how one can use D-opt explicit TD-FEM to broadband room acoustic simulations where the sound field is discretized with irregularly shaped FEs. Sound waves propagate to various directions. In this section, we examine the performance of both methods using acoustic simulation in a small cubic room and larger rectangular room with rigid boundaries where an analytical solution is available. The benefits of both methods are demonstrated in comparison with standard implicit TD-FEM, which uses Fox–Goodwin method as a direct time integration method, having second-order accuracy in space and fourth order accuracy in time [3,11]. In both problems, the upper-limit frequency was 2.5 kHz. As a source signal, we use a modulated Gaussian pulse. Its volume acceleration waveform and frequency characteristics are presented in Fig. 6.

For quantitative evaluation of accuracy, we defined the relative error, $e_r$, as

$$e_r = \frac{1}{N_{rp}} \sum_{j=1}^{N_{rp}} e(r_j) \times 100 \%,$$  \hspace{1cm} (41)

where $N_{step}$ is the total number of time steps. $N_{rp}$ is the total numbers of receiving points, which are respectively 6 and 11 for the cubic room analysis and the rectangular room analysis. Also, $p_{analytic}(i, r_j)$ and $p_{FEM}(i, r_j)$ respectively represent the analytical solution [14] and the numerical solutions calculated using TD-FEMs at $i$-th step on $j$-th receiver position $r_j$. In addition, to examine the accuracy of the proposing optimization method on frequency responses we further evaluated the absolute error in 1/96 octave band sound pressure level (SPL), $e_{abs}$, as

$$e(r_j) = \sqrt{\frac{1}{N_{step}} \sum_{i=1}^{N_{step}} \left( p_{analytic}(i, r_j) - p_{FEM}(i, r_j) \right)^2},$$ \hspace{1cm} (42)
Regarding time discretization, we used \( \frac{1}{C_1} \) kHz for D-opt explicit TD-FEM with the two meshes. Mesh M3. Additionally, dispersion error was minimized at and the standard implicit TD-FEM uses only the finest mesh M1 and M2, this numerical experiment, D-4th explicit TD-FEM and D-opt explicit TD-FEM use only coarser mesh M1 and M2. The stability limit of the implicit FEM is calculated as shown below \[31\].

\[
\Delta t_{\text{lim}} = \frac{C_0}{\sqrt{\delta}}. \tag{44}
\]

Therein, \( L_{\text{analytic}} (f_i, r_j) \) and \( L_{\text{FEM}} (f_i, r_j) \) are respectively the 1/96 octave band SPL at center frequency of \( f_i \) on \( r_j \) calculated by the analytical method and the TD-FEMs. The octave band SPLs were accessed from the time responses using Matlab function \text{poctave} after windowing process with the hann function having the same length. All numerical experiments were performed using a computer (Intel Xeon E5-2650 v4, 2.2 GHz, Precision Tower 7810; Dell) with a Fortran compiler (ver. 2019; Intel Corp.).

4.1. Cubic Room

We considered sound propagation in a simple cubic room of 1 m\(^3\) having rigid boundary surfaces. Three FE meshes M1–M3 were created with cubic elements having three element lengths 0.025, 0.02, and 0.0125 m. The resulting FE meshes have spatial resolution of \( R = 5.5 \) (M1), \( R = 6.87 \) (M2), and \( R = 11 \) (M3) at the upper-limit frequency. It is noteworthy that only M3 satisfies the rule of thumb for linear elements, i.e., ten elements per wavelength. A source point was located at \( (x, y, z) = (0, 0, 0) \), which is the room corner. Six receivers were located on the line of \( y = z = 0.5 \) with 0.2 m spacing. In this numerical experiment, D-4th explicit TD-FEM and D-opt explicit TD-FEM use only coarser mesh M1 and M2, and the standard implicit TD-FEM uses only the finest mesh M3. Additionally, dispersion error was minimized at 2.5 kHz for D-opt explicit TD-FEM with the two meshes. Regarding time discretization, we used \( \Delta t \) at the stability limit for the three methods. For explicit TD-FEM, the value is obtained from Eq. (31). The stability limit of the implicit TD-FEM is calculated as shown below \[31\].

\[
\Delta t_{\text{lim}} = \frac{C_0}{\sqrt{\delta}}. \tag{44}
\]

Figures 7(a)–7(c) respectively present comparisons of waveforms at \( (x, y, z) = (0.6, 0.5, 0.5) \) among the analytical solutions, D-4th explicit TD-FEM, D-opt explicit TD-FEM with M2 and the standard implicit TD-FEM with M3. The D-opt explicit TD-FEM shows the best fit to the analytical solution. The D-4th explicit TD-FEM also shows better agreement than the standard implicit TD-FEM, despite the use of coarser mesh. For quantitative evaluation, \( e_s \) of the three methods are, respectively, 1.27% (D-4th explicit TD-FEM), 0.73% (D-opt explicit TD-FEM) and 2.35% (standard implicit TD-FEM). In addition, for results with M1, the \( e_s \) in D-4th and D-opt explicit methods were 2.16% and 1.37%, which are lower than that in the standard implicit TD-FEM with M3. These results demonstrated clearly that both explicit TD-FEMs using coarse mesh show higher accuracy than the implicit TD-FEM.

Figure 8 presents comparisons of 1/96 octave band SPLs at \( (x, y, z) = (0.6, 0.5, 0.5) \) among the analytical solution, the D-opt and the D-4th explicit TD-FEM with M-1. The D-opt method agrees well with the analytic solution while the D-4th’s result shows discrepancies above 2.1 kHz. Figure 9 shows a comparison of \( e_{\text{abs}} \) at 1–2.5 kHz among the explicit TD-FEMs with M1 and M2, and the standard implicit TD-FEM with M3. Below 1 kHz, \( e_{\text{abs}} \) were smaller than 0.2 dB irrespective of the methods. The \( e_{\text{abs}} \) of D-4th explicit TD-FEM and the implicit TD-FEM increase as with frequency. On the other hand, the D-opt explicit TD-FEM indicates flat error characteristics at high frequencies, showing lower error magnitude than the other methods.
We can also demonstrate higher efficiency of the two explicit TD-FEMs, in which the computational times were 8.8 s for M-1 and 20.5 s for M-2, which were extremely shorter than 483.4 s for the standard implicit TD-FEM. We conclude that the D-opt explicit TD-FEM can perform much better than D-4th explicit TD-FEM for broadband interior acoustic simulations.

4.2. Rectangular Room

We analyzed sound propagation in the rectangular room of dimensions $3\text{m}\times2\text{m}\times1\text{m}$ as shown in Fig. 10, by which we demonstrate the applicability of D-opt explicit TD-FEM for the case using rectangular FEs and irregular hexahedral FEs in comparison with D-4th explicit TD-FEM. In the rectangular room, a source point was placed at $(x, y, z) = (1, 1, 0.5)$; 11 receivers were located on the grid of $1\text{m}\times1\text{m}$ on the plane of $z = 0.5$. The time response was calculated up to 0.05 s. Then, $e_{rs}$ and $e_{abs}$ were respectively evaluated using Eqs. (41) and (43) for each case.

4.2.1. Rectangular FEs

The rectangular room was spatially discretized using rectangular FEs of $0.02\text{m}\times0.017\text{m}\times0.01\text{m}$. For D-opt explicit TD-FEM, dispersion error was minimized at $R = 6.87$ for the upper-limit frequency of 2.5 kHz using the maximum edge length. Additionally, $b_1$ was configured to minimize dispersion error at 2.5 kHz with $\Delta t$ satisfying the stability condition Eq. (31) using the minimum element edge of 0.01 m.

Figure 11 presents comparisons of waveforms at $(x, y, z) = (1, 2, 0.5)$ among the analytical solution and the two explicit TD-FEM solutions. It seems readily apparent that D-opt explicit TD-FEM shows better fit to the analytical solution, whereas D-4th explicit TD-FEM shows slower sound propagation around 0.04 s, which is typical dispersion error behavior. Additionally, the figure presents that the two methods have no dissipation effect. Quantitatively, D-opt explicit TD-FEM has the $e_{rs}$ value of 0.436%, which is lower than 0.793% for D-4th explicit TD-FEM, demonstrating better performance of dispersion-optimized method in the case using rectangular FEs.

Figure 12 presents comparisons of 1/96 octave band SPLs at $(x, y, z) = (1, 2, 0.5)$ among the analytic solution, the D-opt and the D-4th explicit TD-FEMs. The D-opt method well fits to the analytic solution. In contrast, the D-4th method differs from the analytic solution at high frequencies. Figure 13 presents a comparison of $e_{abs}$ at 1–2.5 kHz between the D-opt explicit TD-FEM and the D-4th explicit TD-FEM. Here, the errors below 1 kHz were smaller than 0.14 dB for both explicit TD-FEMs. The D-opt method shows much lower $e_{abs}$ at high frequencies than that of the
D-4th method with the benefit of optimization. Although the D-opt method indicates peaks of $e_{abs}$ at some frequencies, it is an error at dips which is no problem in practice. The error reduction at high frequencies contributes the higher accuracy of D-opt method in computing waveform with broadband spectrum.

### 4.2.2 Irregular hexahedral FEs

Finally, the applicability of D-opt explicit TD-FEM in acoustic simulations using irregular hexahedral FEs was examined. The distorted elements were created by discretizing edges in Fig. 10 as follows: AB and BC with 80 elements, AD and DE with 72 elements, and AF with 100 elements. The maximum and minimum edge sizes in resulting FE mesh are, respectively, 0.02 m and 0.01 m. We used the same $\Delta t$ and $b_1$ as Sect. 4.2.1. Because elements in the mesh have different maximum edge sizes, the following two methods OPT1 and OPT2 were used to determine the integration points: OPT1 uses the same integration points in all elements, which were determined with $R$ at the upper-limit frequency calculated using the maximum edge length 0.02 m; OPT2 uses different integration points in each element with $R$ at the upper-limit frequency calculated using maximum edge length in each element. Therefore, OPT1 minimizes the dispersion error at different frequency at each element, whereas OPT2 minimizes the dispersion error at the same frequency in all elements.

As a result, values of $e_r$ for D-4th explicit TD-FEM, D-opt explicit TD-FEM (OPT1) and (OPT2) were, respectively, 0.530%, 0.470% and 0.380%. This result shows higher accuracy of D-opt explicit TD-FEM even in the case of using irregular FEs. Moreover, OPT2 is a better choice for ascertaining the integration points. The comparison of $e_{abs}$, which is shown in Fig. 14, shows the higher accuracy of D-opt explicit TD-FEMs at high frequencies. Irrespective of the methods, the errors were smaller than 0.1 dB at frequencies lower than 1 kHz. Note that the D-opt methods show relatively large $e_{abs}$ at around 2.4 kHz but it is a dip error as with shown in Sect. 4.2.1. Also, we can observe the higher accuracy of OPT2 than the OPT1. As references, Figs. 15(a)–15(c) present comparisons of waveforms at $(x, y, z) = (1, 2, 0.5)$ among the analytical solution and the numerical solutions calculated using the D-opt explicit TD-FEM (OPT1), the D-opt explicit TD-FEM (OPT2) and the D-4th explicit TD-FEM. Both D-opt explicit TD-FEM (OPT1) and (OPT2) well capture the analytic solution, whereas the discrepancy in waveform of D-4th explicit method from the analytical waveform is seen gradually over time. These results indicate higher accuracy of the optimized methods. In addition, the D-opt explicit TD-FEM (OPT2) shows the smaller value of $e_r$ in Sect. 4.2.2 than that in Sect. 4.2.1. This is because the mesh used in Sect. 4.2.2 has more elements with higher minimum $R$ than that used in Sect. 4.2.1. Thus, the D-opt explicit TD-FEM will probably work well for the case using a mesh with largely different minimum $R$ if the elements with the lowest minimum resolution, which are dominant to the accuracy, are appropriately optimized. However, a detailed investigation is a subject of future works.

### 5. CONCLUSION

The present paper proposed the two novel room acoustic solvers with dissipation-free explicit TD-FEM and showed its high capability in broadband room acoustic modeling. As an important contribution of this study, we presented the inherent mode of eliminating dissipation error that appears in the earlier dissipative fourth-order accurate explicit TD-FEM [10]. This was achieved using a three-step time integration method. Then, we respectively
formulated D-4th explicit TD-FEM and D-opt explicit TD-FEM. The D-4th explicit TD-FEM presents an additional benefit of the relaxed stability condition compared to the dissipative fourth-order method, while maintaining the same dispersion property without numerical dissipation. The D-opt explicit TD-FEM can further reduce dispersion error at higher frequencies without additional computational costs. The used optimization is based on an idea of minimizing dispersion error in axial and diagonal directions at a specific frequency under given spatial resolution mesh or elements. The theoretical dispersion error analysis revealed that the optimization can keep dispersion error low in broad frequency ranges. The performance of two dissipation-free explicit TD-FEMs were examined via three numerical examples with a particular focus on whether D-opt explicit TD-FEM performs well for broadband acoustic simulation using irregularly shaped FEs. The first experiment on a cubic room discretized with cubic FEs showed that the two explicit TD-FEMs have higher accuracy and efficiency in broadband acoustic simulation than the standard implicit TD-FEM. The second and final experiments were conducted to explore the applicability of D-opt explicit TD-FEM, respectively, using rectangular FEs and irregular hexahedral FEs. Results revealed that D-opt explicit TD-FEM offers higher accuracy than D-4th explicit TD-FEM in both cases. Furthermore, we demonstrated the effectiveness of minimizing dispersion error at the element level. The implementation of frequency-dependent absorption boundary condition and application into realistic room acoustic problems are left as subjects for future study.

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