Schwarzschild Fuzzball and Explicitly Unitary Hawking Radiations

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We provide a fuzzball picture for Schwarzschild black holes, in which matters and energy consisting the hole are not positioned on the central point exclusively but oscillate around there in a serial of eigen-modes, each of which features a special level of binding degrees and are quantum mechanically possible to be measured outside the horizon. By listing these modes explicitly for holes as large as $6M_{pl}$, we find that their number increases exponentially with the area. Basing on this picture, we present a simple but explicitly unitary derivation of hawking radiations.

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The horizon and central singularity are two key ingredients of general relativistic black holes, either from observational \cite{1} or from pure theoretical \cite{2} aspects. They are also birth-lands of many radical proposition and exciting progresses in quantum gravitation researches, typically the information missing puzzle \cite{3,7} and the Anti-de Sitter/Conformal Field Theory correspondence \cite{7} or more generally the gauge/gravity duality (AdS/CFT here after). Although initiative researchers such as L. Susskind, basing on general ideas of gauge/gravity duality and special picture of string theories \cite{8,20}, claims that the war between him and S. Hawking has finished \cite{21}, new ideas in the information puzzle’s reformulation and resolution continue to appear endlessly, ranging from the famous AMPS observation of firewall paradoxes \cite{22,23} and the ER=EPR proposition \cite{24}, to various nonlocal/entanglement \cite{25,35} revision believed being ignored in hawking’s original calculation, and to totally new mechanisms for black holes to save information \cite{36}, although in challenges \cite{37,38}.

Basic ideas The general idea of gauge/gravity duality that microstates of black holes could be explained in terms of lower dimensional gauge field theories brings us misunderstandings that, the information of the black holes is stored locally in their near horizon region. However, even in the most well understood fuzzball picture of string theories \cite{11,20}, S. D. Mathur, et al tell us that for large classes of asymptotically AdS black holes constructible from or related with special D1-D5 brane configurations, the information carriers are distributed across the whole region covered by the horizon surface. For more general black holes, especially the Schwarzschild ones, the string theory still finds no way to give the relevant information saving mechanism a concrete explanation. We considered in ref. \cite{39} a possibility that, matters inside the Schwarzschild black holes, which we call Schwarzschild contents in this paper, are not positioned on the central point of the hole statically but are experiencing periodical motion of collapsing, collapsing overdue to the other side and collapsing again during which the radial mass profile preserves continuously. We argue that it is just this radial mass profiles’ diversity, chosen at arbitrary given times $\tau = \tau_0$, with the future determined by Einstein equation, that leads to the microstates’ multiplicity of black holes. The inner metric of these holes when written in the co-moving observer’s proper time has the form

$$ds^2 = -(h^{-1}\frac{\eta^2}{m^2} + 1)d\tau^2 + h^{-1}dr^2 + r^2d\Omega_2^2$$

(1)

$$h = 1 - \frac{2Gm(\tau, r)}{r}, \quad r < r_0 \equiv 2Gm_{total}$$

(2)

By looking $m(\tau_0, r)$ as independent coordinate and introducing a wave functional $\Psi[m(\tau_0, r)]$ to denote the amplitude the hole being at profile $m(\tau_0, r)$, we establish in ref. \cite{39} a functional differential equation controlling the form of $\Psi[m(\tau_0, r)]$ through quantisations of the Hamiltonian constraint of the system, thus translate the question of black hole microstates’ definition and counting a functional eigenvalue problem. However, due to complexes of the functional differential equation, we get only rough estimations for the eigen-state of 1- and 2-$M_{pl}$ mass black holes. The purpose of this work is to provide an alternative definition for this functional problem and an almost exact thus more convincing proof of the microstate number’s exponential area law. Basing on this proof, we will also give a hamiltonian thus explicitly unitary derivation for hawking radiations.

The classic picture behind our micro-state definition and counting is shown in FIG.1 schematically. Just as was done in \cite{39}, we will still focus on black holes consisting of zero pressure dusts for simplicity. The advantage of this doing is that, we can easily prove that the co-moving observer’s geodesic motion $\{u^0 = 1, g_{\mu\nu}u^\mu u^\nu = -1\}$ follows directly as part of Einstein equation $G_{\mu\nu} = 8\pi G\eta_{\mu\nu}$, i.e. $G_{\eta_{\mu\nu}} = g_{\mu\nu} = m_{\mu\nu}$. But different from \cite{39}, in this time we will decompose the radial mass distribution profile into several concentric shells at the first beginning. We will show that the number of this shell partition, as well as their quantum state are both countable, with the latter equals to $e^{rA/4G}$. In the mostly simple layering scheme, all contents of the hole are concentrated in one shell of dusts, the equations of motion, looking from exterior observers which use time $t$ and feel a Schwarzschild geometry v.s. interior observers which use $\tau$ and feel a...
Minkowskian geometry, can be easily written as

\[
\begin{align*}
\{ h^* = \gamma < \hat{x}^0 + \Gamma^0_{\mu\nu} \hat{x}^\mu \hat{x}^\nu, \quad \hat{\gamma}^2 = \gamma^2 - \hbar, \quad h = 1 - \frac{2Gm}{r} \\
\{ \hat{q}^2 = \gamma^2 - \hbar, \quad h = 1 - \frac{2Gm}{r} \}
\end{align*}
\]  
(3)

or

\[
\{ \hat{q}^2 = \gamma^2 - \hbar, \quad h = 1 - \frac{2Gm}{r} \} \]  
(4)

where \( \gamma \) is an integration constant equaling to the value of \( h \) on \( r = r_{\text{rel}} \) where the shell is released from static and allowed to freely moving under self-gravitations. If the shell is released from outside the horizon, \( \gamma \) will be real and less than 1. But we will focus on cases where the shell is released from inside the horizon, so that \( \gamma \) is purely imaginary and \( \gamma^2 < 0 \). Talking about this motion inside horizons is meaningful because we could have observers on the central point around which the space-time is simply minkowskian before the shell arrives onto. While as the shell arrives onto, its radial speed equals to that of light \( \frac{dr^*}{dt} \rightarrow 0 \), so it cannot be stopped there but have to go across that point and making oscillations there.

It can be easily verified that, the vanishing at origin, square integrable solution to this equation of motion exists only when \( \frac{d\gamma}{dr} = 1, 2, \ldots, n \). This is almost the same as the simple hydrogen atoms. However, due to the condition that the shell is released from inside the horizon, thus \( \gamma^2 = h(r_{\text{rel}} < 0) \), we must have \( 1 < \beta \). This constrains the allowed wave function to be, \( L_{q-1}(\beta \hat{r}) \) here is the first order associated Legendre polynomial,

\[ \Psi = \Psi_{\beta}(\hat{r}) = e^{-\beta \hat{r}} \beta \hat{r} L_{q-1}(2\beta \hat{r}) \]  
(8)

\[ 1 < \beta, \quad \frac{2Gm \hbar h^{-1}}{2\beta} \equiv q = 1, 2, \ldots, q_{\text{max}} \]  
(9)

That is, for a sphere shell of given mass, the number of allowed quantum wave functions corresponding to classic oscillation modes released statically from inside the horizon is finite, equals to the maximal integer no larger than \( Gm \hbar^{-1} \) or symbolically

\[ q_{\text{max}} = \text{Floor}[Gm \hbar^{-1}] \]  
(10)

While for the more interesting case where the whole Schwarzschild contents are consisting of several layer of different mass shells \( m = \{m_1, m_2, \ldots, m_l\} \). In the case when all these shells do not cross each other, the wave function of the whole system can be written as \( \Psi^1 \otimes \Psi^2 \otimes \cdots \otimes \Psi^\ell \). While if shell crossing occurs \([40, 41]\), we will need some symmetrisation procedure on this direct product to get wave functions of the system which have similar physical interpretations. But as long as the microstate number counting is concerned, such a consideration is not needed. Denoting the total mass inside the \( i \)-th shell, including the \( i \)-th shell itself, with \( M_i = \sum_{j=1}^{l} m_j \), and repeating the calculations across \( [42, 11] \), we will find that for each mass shell \( m_i \), the corresponding wave function are simply

\[ \Psi^i_m = \Psi^i_{m_{\beta_{\ell}}}(\hat{r}) = e^{-\beta \hat{r}} \beta \hat{r} L_{q-1}(2\beta \hat{r}), \hat{r} = rm_i h^{-1} \]  
(11)

\[ \frac{2GM_i m_i}{\beta_i h} \equiv q^m_{i} = 1, 2, \ldots, q^m_{\text{max}} \equiv \text{Floor}[\frac{GM_i m_i}{h}] \]  
(12)

So the total number of microstate allowed by this layering scheme \( m \), and by the whole Schwarzschild contents equals to respectively

\[ w_m = \prod_{i=1}^{\ell} q^m_{i \text{max}}, \quad w = \sum_{m} w_m \]  
(13)

This way, the question of black hole microstates’ counting becomes a simple mass/energy layering partition and the corresponding quantum number listing. For a 2-\( M_{\text{pl}} \) mass black holes, we have only 3 methods to partition the mass/energy contents into layers that lead to distinguished quantum state. The results are presented in TABLE[4]. While TABLE[11] lists all the quantum numbers
allowed by a 3-\(M_{\text{pl}}\) mass black holes. For more large black holes up to 6-\(M_{\text{pl}}\) masses, listing out and counting up their quantum numbers one by one are also possible using computer programs. However, as the black hole becomes even larger, the number of quantum states allowed by their contents increases exponentially with their horizon area. We plot the results in FIG 2 explicitly, from which we easily see that the entropy of the system

\[
S \equiv k_{\text{pl}} \log[w] = \frac{k_{\text{pl}}}{4G} \cdot \frac{0.52 k_{\text{pl}}}{\pi^2} \cdot A = 16\pi G^2 M^2\]  

(14)

Except for a numeric factor of \(\frac{0.52}{\pi^2} \approx \frac{1}{2\pi^2}\), this yields perfectly the area law of Bekenstein-Hawking formulas. We will not distinguish \(k_{\text{pl}}\) and \(k'_{\text{pl}}\) in the following and will denote them ambiguously as \(k\). We note here that this result is only for 4-dimensional black holes. Continuations to other-dimensions are possible but nontrivial. For example, in 5-dimensions, the function \(h\) appearing in (3) should be changed to \(1 - \frac{2GM}{r}\). This change will bring us to the very trouble Calogero problem [12], which is still not understood clearly in quantum mechanics but necessary for deriving conditions like [13] and [12].

![FIG 2: The logarithmic value of the number of microstate of Schwarzschild black holes with masses less than or equal to 6-\(M_{\text{pl}}\).](image1)

So, let us come back to the physic meaning of the black hole contents’ wave function \(\Psi_{\beta \geq 1/q}[r]\) themselves. We plot in FIG 3 six typical wave functions of this type for a 2-\(M_{\text{pl}}\) mass Schwarzschild black hole. From the figure, we firstly see that either in the one or two layer partition case, we always have nonzero probabilities to find the contents being outside the horizon. The horizon, behaves only approximately as the boundary of contents distribution. That is, it only requires that maximal values of the wave function occur inside it. From this aspects, our pictures are almost a quantum mechanical version of the string theory fuzzy balls [1, 11–16, 18–20]. However, we get this picture basing on standard general relativity and simple quantum mechanics, instead of metaphysic ideas such as extra-dimension or supersymmetry et al. The second point we need to emphasise in

![FIG 3: Typical wave functions corresponding to eigen-modes of motion executed by the mass/energy contents of a 2\(M_{\text{pl}}\) mass black holes. In the left panel, contents of the hole are concentrated in one mass shell which has only four possible quantum states. While in the right panel, the contents are distributed in two layer of shells each with mass \(\frac{M_{\text{pl}}}{2}\) and \(\frac{M_{\text{pl}}}{2}\) respectively. In this latter case, the wave function of the system is the direct product of outside layer’s \(\Psi[r, 2\frac{M}{2}]\) and inside layer’s \(\Psi[r, 2\frac{M}{2}]\) or \(\Psi[r, 1\frac{M}{2}]\).](image2)

| \(\{m_i\}/M_{\text{pl}}\) | 2 | 1.1 | \((\frac{3}{2}, \frac{1}{2})\) |
| \(\{M_i\}/\sum m_j<1\} \times \{2\} | 1.2 | \((\frac{3}{2}, \frac{1}{2})\) |
| \(GM_{\text{pl}} \times \{1\} \times \{4\} \times \{1\} \times \{1\} \times \{1\} \times \{1\} \times \{1\} \times \{1\}\) | num.states | 4 | 1.2 | \((\frac{3}{2}, \frac{1}{2})\) |

**TABLE I:** The layering scheme and corresponding quantum states of a 2\(M_{\text{pl}}\) mass black holes. Such a black hole can be layered into (i) one shell of total mass 2\(M_{\text{pl}}\) or (ii) two shells of equal mass 1\(M_{\text{pl}}\) + 1\(M_{\text{pl}}\) or (iii) two shells of unequal mass \(\frac{3}{2}M_{\text{pl}}\) + \(\frac{1}{2}M_{\text{pl}}\). Other layering scheme such as \(\{M_{\text{pl}} + \epsilon, M_{\text{pl}} - \epsilon\}\) with \(0 < \epsilon < \frac{M_{\text{pl}}}{2}\) would not lead to radial quantum numbers different from listed above. While layerings such as \(\{\frac{M_{\text{pl}}}{2}, \frac{M_{\text{pl}}}{2} - \epsilon\}\) with \(0 < \epsilon < \frac{M_{\text{pl}}}{2}\) break conditions that \(\frac{GM_{\text{pl}} m_i}{(1\text{st})}\) \(\in\mathbb{Z}\). So the total number of all possible quantum state is 8.
a mass $\Delta$ hawking particle and becomes a mass $b - \Delta$ hole is proportional to $[\text{the parameter } (8\pi kGb)^{-1}]$ will be denoted by an effective temperature $kT$
\[ P_{\Delta} = \frac{e^{kGb(\Delta)}}{e^{kGb} - 1} \delta_{\Delta} \frac{e^{-4kGb^2}}{e^{8\pi kGb\Delta}} \propto e^{-\Delta/kT} \] (15)

Due to randomness of quantum decays, the average energy of a hawking mode emitted in one such spontaneous event is

for fermions: $\langle E \rangle = \langle \hbar \omega e^{-\hbar \omega/kT} + 0 \rangle = \frac{\hbar \omega}{e^{\hbar \omega/kT} + 1}$ (16)

for bosons: $\langle E \rangle = \sum_{\ell=0}^{\infty} \frac{n \hbar \omega e^{-n \hbar \omega/kT}}{\sum_{n} e^{-n \hbar \omega/kT}} = \frac{\hbar \omega}{e^{\hbar \omega/kT} - 1}$ (17)

This is nothing but the spectrum of hawking radiations. However, we get it here basing on only standard quantum mechanics instead of any semi-classic consideration of quantum field theories in the curved background of space-time.

In fact, this simple quantum mechanic picture allows us to go more further than obtaining the power spectrum of hawking radiations. Basing on it, we may even construct a Hamiltonian thus explicitly unitary formulation for the whole process as follows,

\[ H = H_{\text{BH}} + H_{\text{vac}} + H_{\text{int}} \] (18)

\[ H = \begin{pmatrix} b_n \\ b_{n-1} \\ \vdots \\ 0 \end{pmatrix} + \frac{\hbar \omega}{kT} a_k^+ a_k + \sum_{|b_n b_v|} g_{uv} b_{uv}^* a_k (19) \]

where $H_{\text{BH}}$, $H_{\text{vac}}$ and $H_{\text{int}}$ are respectively hawkingians of the black hole, the environment and interactions between them two. The concrete form of $H_{\text{BH}}$ is unknown but unimportant. We need only to know that its eigenvalues are $\{b_n, b_n, \ldots, b_1, b_0 = 0\}$ and respectively $\{w = e^{4kGb^2}, w = e^{4kGb^2}, \ldots, 1, 1\}$-times degenerating. The vacuum hawkingian $H_{\text{vac}}$ is denoted by many harmonic oscillators, bosonic ones here for simplicity. When a mode’s $\hbar \omega$ happens to be equal to the difference between two eigenvalues of $H_{\text{BH}}$, it can go on shell and radiated away from the hole. The interaction part $H_{\text{int}}$ between the holes and the environment functions to bring energies from the former to the latter and vice versa. Denoting the quantum state of a mass $b_\ell$, binding degree $u$ values in $\{1, 2, \ldots, e^{4kGb^2}\}$, black hole and its environment consisting of hawking modes as $|e^\ell u, n \ell \rangle$, then

\[ b_{uv}^* \equiv (b_{uv} a_k)|e^\ell u, n \ell \rangle = |(\ell + k)^u, n \ell \rangle \] (20)

\[ b_{uv}^* = b_{uv}, a_k^* = a_k^* \] (21)

This transition $|e^\ell u, n \ell \rangle \rightarrow |(\ell + k)^u, n \ell \rangle$ could be induced by pure gravitational, especially mass monopole interactions. The proportionality in (21) just reflects the fact that two black holes which have more similar microscopic wave functions could be more rapidly to transit to each other. If we further denote the initial state of the system as $|n^\sigma, 0 \rangle$, and the latter time state as follows

\[ |\psi(t)\rangle = \sum_{\ell=0}^{\infty} \sum_{u=1}^{\ell} e^{i \beta_{\ell u} c_{\ell u}(t)} |e^\ell u, n \ell \rangle \] (22)

then the standard Schrödinger equation $i\hbar \partial_t |\psi(t)\rangle = H |\psi(t)\rangle$ will tell us

\[ i\hbar \partial_t c_{\ell u}(t) = (b_{\ell u} - b_{\ell u})c_{\ell u}(t) + \sum_{j \neq \ell} \sum_{u=1}^{\ell} g_{uv} e^{i(h_{\ell u} - h_{\ell u})} c_{\ell u}(t) \] (23)

\[ c_{\ell u}(0) = c_{\ell u}(0) = c_{\ell u}(0) \ldots = 0 \] (24)

$^{1}$ $b_u$, $b_v$ are abbreviations for $b_{uv} = b_\ell$ and $b_{(\ell + k)u} = b_{(\ell + k)u}$ respectively.
If we know which binding state the initial black hole is at, i.e. the \( w \) value in \( n^w \), then we will be able to predict its latter time evolutions exactly, 
\[
\begin{align*}
  n^w & \xrightarrow{gw} n^0 \xrightarrow{gw} n^w & \ldots
\end{align*}
\]
Conversely, if we can precisely monitor a black hole’s evaporation process, especially the time feature of its horizon size variations, then we will be able to infer its initial state exactly. However, in semi classic discussion of hawking radiations, the initial state effects are completely averaged. All initial states are assumed to decay indistinguishably to the final state with equal chances, thus leading to the information missing puzzle.

In practical observations, initial states of black holes are almost unknowable. However, according to the general idea of probability theories, it is very natural that they have more chances to lie on positions of the phase space, binding degree space here, around which the density of microstate takes more larger values. Experiences from TABLE.I, TABLE.II and more larger black holes’ microstate listing tells us that, such positions happens to be the classically continuous distribution of matter contents inside the horizon. Comparing with the conventional black hole pictures with central singularity, the two body system in our pictures has totally different quadrupole structures, so is dis/verifiable through gravitational wave observations, such as those reported in GW150914-170814 [43–47]. However, in all these observations, the inner structure of black holes are not measured because theoretical temples [48–57] used in them to extract information from the highly noised signals adopt simple horizon boundary conditions to account for effects following from non-trivial quadrupole structures of the system. If we consider inner structure effects in theoretical temples, we expect to see that the gravitational wave forms, e.g. FIG.1 in [43] following from binary black holes will become more close to that measured in the binary neutron star events, e.g. FIG.2 of GW170817

Many new works could be done from our fuzzball pictures. Most direct but non-trivially, generalises to other dimension, and to other asymptotic background such as AdS and dS space-times would be very interesting and necessary for better understanding of the origin of Bekenstein-Hawking entropy’s area law feature. On the other hand, the most interesting work may be the experimental dis/verification of our inner structure picture for black holes through gravitation wave observation. Considering data accumulations today and the observation technique’s maturity for this exploration [43–47, 58], what we need for this dis/verification work is just a theoretical template to model the inner-structure carrying binary black hole’s merging. We wish to come back this point in near futures.

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\[
\begin{align*}
\{m_i\}/M_{\text{pl}} & \quad \{3\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
M_i \equiv \sum m_i & \quad \{3\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
GM_{m_i} & \quad \{9\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
GM_{m_i} & \quad \text{in state} & \{1,2,9\} & \{12,7\} & \{1\} \\
\#\text{state} & \quad & 9 & 1 & 5 \\
\{m_i\} & \quad \{\frac{\sqrt{2}}{3}\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
M_i & \quad \{\frac{\sqrt{2}}{3}\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
GM_{m_i} & \quad \{4,3\} & \{\frac{\sqrt{2}}{3},4\} & & \{\frac{\sqrt{2}}{3},5\} \\
\#\text{state} & \quad & 4 & 3 & 1 & 5 & 1 & 6 \\
\{m_i\} & \quad \{\frac{\sqrt{2}}{3}\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
M_i & \quad \{\frac{\sqrt{2}}{3}\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
GM_{m_i} & \quad \{3,1,1,2\} & \{2,2,2,1\} & & \{1,3,2\} \\
GM_{m_i} & \quad \text{in state} & \{123\} & \{12\} & \{123\} \\
\#\text{state} & \quad & 3 & 1 & 2 & 2 & 2 & 1 & 3 & 2 \\
\{m_i\} & \quad \{\frac{\sqrt{2}}{3}\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
M_i & \quad \{\frac{\sqrt{2}}{3}\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
GM_{m_i} & \quad \{2,1,3\} & \{1,2,3\} & & \{1,1,4\} \\
GM_{m_i} & \quad \text{in state} & \{123\} & \{12\} & \{123\} \\
\#\text{state} & \quad & 2 & 1 & 3 & 1 & 2 & 1 & 4 \\
\{m_i\} & \quad \{\frac{\sqrt{2}}{3}\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
M_i & \quad \{\frac{\sqrt{2}}{3}\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
GM_{m_i} & \quad \{3,1,1,1\} & \{2,2,1,1\} & & \{1,1,2,1\} \\
GM_{m_i} & \quad \text{in state} & \{123\} & \{12\} & \{12\} \\
\#\text{state} & \quad & 3 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\
\{m_i\} & \quad \{\frac{\sqrt{2}}{3}\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
M_i & \quad \{\frac{\sqrt{2}}{3}\} & \{\frac{\sqrt{2}}{3}\} & & \{\frac{\sqrt{2}}{3}\} \\
GM_{m_i} & \quad \{1,1,1,2\} & \{1,1,1,1\} & & \{1,1,1,1\} \\
GM_{m_i} & \quad \text{in state} & \{1\} & \{1\} & \{1\} \\
\#\text{state} & \quad & 1 & 1 & 1 & 1 \\
\end{align*}
\]

TABLE II: The same as TABLE III but for a 3M\text{pl} mass black hole. Other layering schemes such as decomposing 3M\text{pl} into \{8\frac{M_{\text{pl}}}{\text{pl}} - \epsilon, \frac{M_{\text{pl}}}{\text{pl}} + \epsilon\} with |\epsilon| < \frac{M_{\text{pl}}}{\text{pl}} will not lead to quantum numbers different from listed above. So this table lists out all possible quantum state of a 3M\text{pl} mass Schwarzschild black hole exclusively, whose total number is 126.

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