Incompressible impinging jet flow with gravity

Jianfeng Cheng²,³ · Lili Du¹,² · Zhouping Xin³

Received: 4 May 2021 / Accepted: 30 January 2023 / Published online: 17 March 2023 © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract
In this paper, we investigate steady two-dimensional free-surface flows of an inviscid and incompressible fluid emerging from a nozzle, falling under gravity and impinging onto a horizontal wall. More precisely, for any given atmosphere pressure $p_{atm}$ and any appropriate incoming total flux $Q$, we establish the existence of two-dimensional incompressible impinging jet with gravity. The two free surfaces initiate smoothly at the endpoints of the nozzle and become to be horizontal in downstream. By transforming the free boundary problem into a minimum problem, we establish the properties of the flow region and the free boundaries. Moreover, the asymptotic behavior of the impinging jet in upstream and downstream is also obtained.

Mathematics Subject Classification 76B10 · 76B03 · 35Q31 · 35J25

Contents
1 Introduction and main results ...................................... 2
   1.1 Introduction ............................................. 2
   1.2 Statement of the physical problem ................................. 3
   1.3 Main results ............................................. 5
2 Mathematical formulation of the impinging flow problem ....................... 8
   2.1 A free boundary problem ...................................... 8
   2.2 On the asymptotic widths ...................................... 9
   2.3 The variational approach ...................................... 12

Communicated by Neil S Trudinger.

Lili Du
dulili@szu.edu.cn
Jianfeng Cheng
jfcheng@scu.edu.cn
Zhouping Xin
zpxin@ims.cuhk.edu.hk

¹ College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, People’s Republic of China
² Department of Mathematics, Sichuan University, Chengdu 610064, People’s Republic of China
³ The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong
1 Introduction and main results

1.1 Introduction

The problem describing a jet emerging from a nozzle and impacting on a solid wall is important. It possesses the key ingredients of a challenging mathematical problem with numerous practical applications, such as the industrial process of fabricating glassy metals, a jet of molten metal emerging from a nozzle is directed onto a moving plate and develops a glassy structure due to the rapid cooling. The more obvious industrial applications involve a Vertical/Short Takeoff and Landing (V/STOL) aircraft, terrestrial rocket launch, the formulation of continuous fibers from molten liquid extruded through an orifice [14]. Although these practical applications are different in many respects, the one underlying feature is the fundamental fluid mechanics of the impinging jet. Consequently, an accurate but mathematically tractable model to describe the essential features of the impinging jet would be desirable.

The impinging jet problem has been studied from the computational and experimental expects, by Dias-Elcrat and Trefethen, who presented an efficient procedure for solving the impinging jet problem numerically in [11]. See also in [19] for a numerical solution to the incompressible impinging jet in the absence of gravity. King and Bloor introduced a method in [20] for determining the free streamline of a free jet of ideal weightless fluid impacting on a wall. We also would like to refer the surveys [7] by Birkhoff and Zarantonello, [18] by Jacob, [16] by Gurevich, [21] by Milne-Thomson, and [29] by Wu for the mathematical theory on jets of ideal fluid.

The first systematic existence theory on the incompressible impinging jet of ideal weight-less fluids was mentioned in the monograph [13] of Friedman (Page 365 and Page 416), and some existence results of an impinging jet emerging from a two-dimensional nozzle and impacting on an infinite wall has been established in an unpublished paper by Caffarelli and Friedman. Very recently, Cheng, Du and Wang in [9] extended the existence and non-existence results to the oblique impinging jet. The “oblique” means that the orifice of the nozzle is not parallel to the rigid wall. The main idea follows from the techniques introduced by Alt, Caffarelli and Friedman in [1, 2, 5], the original physical problem was transformed into a minimum problem, and then the existence of incompressible jet and the properties of the free boundary is established. The main purpose of this paper is to extend the existence result by Caffarelli and Friedman to the case where the gravity field is present. The analysis of jets in the presence of gravity is difficult because of the strong nonlinearity of the dynamic boundary condition that must be satisfied on both unknown free surfaces bounding the jet. In this paper, we consider the steady irrotational flow of an incompressible inviscid fluid emerging from a nozzle, falling vertically under the effects of the gravity and impinging on a rigid wall. The total incoming flux and atmospheric pressure are imposed, and some existence results on incompressible impinging jets under gravity are established.
Another motivation to investigate the impinging jet problem under gravity comes from the classical result on a falling jet under gravity without the horizontal plate in the significant work [3] by Alt, Caffarelli and Friedman. As mentioned in Page 59 in [3],

“The mathematical literature on jets with gravity is very meager. The reason for this is that the hodograph method which has been successfully used in steady 2-dimensional problems for jets and cavities without gravity cannot be extended to the case where gravity is present.”

They considered the incompressible jet under gravity emerging from a channel in axially symmetric case and two-dimensional asymmetric case, assuming the height of the channel to be infinite. In the axially symmetric case, existence and uniqueness of an incompressible jet with gravity were established, and surprisingly, some non-existence result on asymmetric jet under gravity was also obtained in Section 14 in [3]. They gave a counterexample to the existence of asymmetric jets under gravity, and showed that in general the two free streamlines cannot connect the endpoints at the same time. Nevertheless, in this paper, we consider the jet emerging from the nozzle with finite height, which does not fall into the case of their counterexamples, and try to seek a mechanism to establish the well-posedness theory of the asymmetric jet under gravity. This seems to be the first well-posedness result on two-dimensional asymmetric incompressible jets under gravity.

1.2 Statement of the physical problem

The problem we will address is that the two-dimensional steady flow of an incompressible inviscid fluid emerges from a semi-infinitely long nozzle, and impinges onto the ground. We shall take into account of the gravity but neglect all other forces, such as surface tension and air resistance.

The situation is shown in Fig. 1. $N : y = 0$ is denoted as the ground, and an open semi-infinite nozzle is bounded by the nozzle walls $N_1$ and $N_2$. $N_i$ is bounded by $x = g_i(y) \in C^{2,\alpha}(H, H_i)$ with $g_1(y) < g_2(y)$ and $H < H_1 < H_2$, $i = 1, 2$. $H$ is the distance between the orifice of the nozzle and the ground. Without loss of generality, we assume that

$$g_i(H) = (-1)^i, \quad \lim_{y \to H_i^-} g_i(y) = -\infty \text{ and the width of the orifice is } 2. $$

Denote $A_1 = (-1, H)$ and $A_2 = (1, H)$ be the endpoints of the semi-infinite nozzle. The gravity acts in the negative $y$-direction.
The resulting motion of the incompressible impinging jet is governed by the continuity equation and momentum equations in two dimensions,

\[
\begin{align*}
    u_x + v_y &= 0, \\
    uu_x + uv_y + p_x &= 0, \\
    uv_x + vv_y + p_y &= -g.
\end{align*}
\] (1.1)

Here, \((u, v)\) is the velocity field, \(p\) is the pressure of the incompressible fluid, and \(g\) is the acceleration due to the gravity. The irrotational condition is written as

\[v_x - u_y = 0.\] (1.2)

The nozzle walls \(N_1 \cup N_2\) and the ground \(N\) are assumed to be impermeable, and then the velocity satisfies the following no-flow condition,

\[(u, v) \cdot \mathbf{n} = 0 \text{ on } N \cup N_1 \cup N_2,\] (1.3)

where \(\mathbf{n}\) is the outer normal to the boundaries.

In this paper, we will seek an impinging jet acted on by gravity and with two asymmetric free streamlines, and the free streamlines initiate from the endpoints \(A_1\) and \(A_2\) of the nozzle, and extend to infinity.

Let \(\Gamma_1\) and \(\Gamma_2\) be the free streamlines connecting the nozzle walls \(N_1\) and \(N_2\), respectively. However, the location of the free streamlines is not known a priori. If the surface tension is negligible, the pressure on the free surface should be the constant atmospheric value \(p_{\text{atm}}\). \(\Omega_0\) denotes the flow region (which is unknown), bounded by the nozzle wall \(N_1 \cup N_2\), the free boundaries \(\Gamma_1 \cup \Gamma_2\) and the ground \(N\) (see Fig. 2).

Furthermore, the following notations will be used,

\[
\begin{align*}
    Q &: \text{ the total incoming flux in upstream}, \\
    Q_1 &: \text{ the effluent flux in negative } x\text{-direction}, \\
    Q_2 &: \text{ the effluent flux in positive } x\text{-direction}.
\end{align*}
\] (1.4)
It should be noted that the total incoming flux \( Q = Q_1 + Q_2 \) is imposed in our problem and the quantities \( Q_1 \) and \( Q_2 \) are unknown a priori, which can be determined by the solution itself.

Now the impinging jet flow problem can be reformulated as: For any given total mass flux \( Q \), determine an incompressible flow \((u, v, p)\) with free streamlines \( \Gamma_1 \) and \( \Gamma_2 \) connecting smoothly at the endpoints \( A_1 \) and \( A_2 \), on which the pressure is the given atmospheric pressure \( p_{\text{atm}} \).

More precisely, a solution to the incompressible impinging jet problem is defined as follows.

**Definition 1.1** (a solution to the incompressible impinging jet problem) A vector \((u, v, p, \Gamma_1, \Gamma_2)\) is called a solution to the incompressible impinging jet problem, provided that the following properties hold:

**Property 1** The free streamlines \( \Gamma_1 \) and \( \Gamma_2 \) can be described by \( C^1 \)-smooth functions \( x = k_1(y) \) and \( x = k_2(y) \), respectively, and there exist two constants \( h_1, h_2 \in (0, H) \), such that

\[
\lim_{y \to h_1^+} k_1(y) = -\infty \quad \text{and} \quad \lim_{y \to h_2^+} k_2(y) = +\infty.
\]

**Property 2** The free boundaries \( \Gamma_1 \) and \( \Gamma_2 \) are analytic, and satisfy

\[
g_1(H) = k_1(H) = -1 \quad \text{and} \quad g_2(H) = k_2(H) = 1, \tag{1.5}
\]

and

\[
g_1'(H + 0) = k_1'(H - 0) \quad \text{and} \quad g_2'(H + 0) = k_2'(H - 0). \tag{1.6}
\]

**Property 3** \((u, v, p) \in \left(C^1,\alpha(\Omega_0) \cap C^0(\overline{\Omega_0})\right)^3\) solves the steady incompressible Euler system (1.1), the irrotational condition (1.2) and the boundary condition (1.3).

**Property 4** \( p = p_{\text{atm}} \) on \( \Gamma_1 \) and \( \Gamma_2 \).

**Remark 1.1** The constants \( h_1 \) and \( h_2 \) are indeed the asymptotic widths of the impinging jet in left and right downstream, respectively. The property 1 in Definition 1.1 implies that the free boundaries \( \Gamma_1 \) and \( \Gamma_2 \) can not oscillate in downstream.

**Remark 1.2** The conditions (1.5) and (1.6) are so-called continuous fit conditions and smooth fit conditions for the impinging jet, which mean that the free boundaries connect the nozzle walls smoothly.

**Remark 1.3** The constant atmosphere pressure \( p_{\text{atm}} \) is imposed arbitrarily in advance.

### 1.3 Main results

The main results of this paper are stated as follows. The first one is on the existence.

**Theorem 1.1** For any given atmosphere pressure \( p_{\text{atm}} \) and total incoming flux \( Q > 2\sqrt{gH^3} \), there exists a solution \((u, v, p, \Gamma_1, \Gamma_2)\) to the incompressible impinging jet problem with gravity. Furthermore,

(1) the vertical velocity of the impinging jet is negative, namely, \( v < 0 \) in \( \Omega_0 \cup \Gamma_1 \cup \Gamma_2 \).
Fig. 3 The interface $\Gamma$

(2) there exists a unique smooth streamline $\Gamma : x = k(y)$, which separates the impinging jet with two different downstream, and $\Gamma$ goes to the inlet of the nozzle and intersects the ground $N$ at the unique point $S$ (see Fig. 3).

(3) The streamline $\Gamma$ is perpendicular to the ground $N$ at $S$, namely, $k'(0+0) = 0$.

Next, we will give the asymptotic behavior of the incompressible impinging jet under gravity.

**Theorem 1.2** Assume that there exists a large $R_0 > 1$, such that the nozzle wall $N_i$ can be described by $y = g_i^{-1}(x)$ for $x < -R_0$, $i = 1, 2$, then the impinging jet flow obtained in Theorem 1.1 satisfies the following asymptotic behavior in downstream,

$$(u, v, p) \rightarrow \left( -\frac{Q}{h_1}, 0, p_1(y) \right), \nabla(u, v) \rightarrow 0 \text{ and } \nabla p \rightarrow (0, -g) \quad (1.7)$$

uniformly in any compact subset of $(0, h_1)$ as $x \to -\infty$, and

$$(u, v, p) \rightarrow \left( \frac{Q - Q_1}{h_2}, 0, p_2(y) \right), \nabla(u, v) \rightarrow 0 \text{ and } \nabla p \rightarrow (0, -g) \quad (1.8)$$

uniformly in any compact subset of $(0, h_2)$ as $x \to +\infty$, where $Q_1$ is determined uniquely by

$$\frac{Q_1^2}{h_1^2} + 2gh_1 = \frac{(Q - Q_1)^2}{h_2^2} + 2gh_2.$$ 

$p_1(y) = p_{atm} + g(h_1 - y)$ for $y \in (0, h_1)$ and $p_2(y) = p_{atm} + g(h_2 - y)$ for $y \in (0, h_2)$.

Similarly, in upstream,

$$(u, v, p) \rightarrow \left( \frac{Q}{H_2 - H_1}, 0, p_0(y) \right), \nabla(u, v) \rightarrow 0 \text{ and } \nabla p \rightarrow (0, -g) \quad (1.9)$$

uniformly in any compact subset of $(H_1, H_2)$ as $x \to -\infty$, where $p_0(y) = p_{atm} + g(h_1 - y) + \frac{Q_1^2}{2h_1} - \frac{Q^2}{2(H_1 - H_2)^2}$ for $y \in (H_1, H_2)$. 

Springer
We would like to give some comments on the results as follows.

**Remark 1.4** In the significant work [3], the authors showed that in general there does not exist an asymmetric jet in gravity field issuing from a nozzle with infinite height, which satisfies the continuous fit conditions. However, the semi-infinite channel considered here possesses finite height, which seems physically reasonable and yet does not fall into the class studied in [3]. We establish the existence of the incompressible asymmetric impinging jet with continuous fit conditions under gravity for this class nozzles. This is the one of main differences to the work [3]. Meanwhile, ideas developed in this paper may provide a different way to establish the well-posedness of asymmetric jet flows under gravity without the horizontal plate \( N \), by taking the limit \( H \to +\infty \). This will be considered in the future.

**Remark 1.5** The mechanical behavior of the jet near the orifice is quite complicated because the flow has to adjust from a distribution compatible with the flow in a nozzle to the flow in a jet. Here we use the continuous and smooth fit conditions near the orifice, which is physically acceptable in the sense of Brillouin in [8] that the detachment is smooth at the fixed detached point. Yet, to ensure that such conditions are fulfilled mathematically is one of key elements to establish the existence of the impinging jet flow in this paper. The main new ideas are as follows. First, due to the Bernoulli’s law, the constant pressure condition on the free streamlines implies that
\[
\frac{1}{2}(u^2 + v^2) + gy = \text{constant} \quad \text{on} \quad \Gamma_1 \cup \Gamma_2,
\]
and this constant, denoted as \( \lambda \), is undetermined at the present stage. On the other hand, as pointed out before, the effluent flux \( Q_1 \) is also not imposed here. Thus, to solve the impinging jet problem, we can treat the two quantities \( (\lambda, Q_1) \) as a pair of free parameters which will be adjusted to guarantee the continuous and smooth fit conditions of the free boundaries. This is also the main difference from the problem of free jets without solid nozzle walls. It should be also mentioned that there are literatures on numerical and analytical results on the free jets of ideal fluids, see [10, 17, 25] and the references therein.

**Remark 1.6** Recall that in the absence of gravity, the existence of an impinging jet with two asymptotic directions and non-existence of an impinging jet with only one asymptotic direction have been shown in [9]. However, in this paper, our proof still excludes the critical cases \( Q_1 = 0 \) and \( Q_1 = Q \) and shows that \( Q_1 \in (0, Q) \) is the sufficient condition to the existence of impinging jet under gravity. This coincides with the results on impinging jets without gravity.

**Remark 1.7** One of the main differences between the impinging jets in [2–4] and general jets is the occurrence of the interface between the two fluids with different downstream. Here, we have to show the existence of the interface, and the uniqueness of the intersection point of the interface and the ground. In fact, we will show that the intersection point is a unique stagnation point in the fluid domain, furthermore, the interface \( \Gamma \) intersects the ground perpendicularly at the stagnation point \( S \) in Proposition 5.1.

**Remark 1.8** It should be noted that the assumption \( Q > 2\sqrt{gH^2} \) in Theorem 1.1 is a sufficient condition to exclude the stagnation point in the fluid domain (see Remark 2.1), especially on the free boundary. The advantage of this fact lies in exclusion the possible singularity on the free boundary of water waves with gravity. The singularity of the free surface flows with gravity at the stagnation point has been studied extensively in the elegant works [6, ...
22, 24, 26–28], which is closely related to a very interesting problem, the so-called Stokes Conjecture (Stokes [23] conjectured in 1880 that, at any stagnation point the free surface has a symmetric corner of $\frac{2\pi}{3}$). Therefore, unlike the results in [3], we do not restrict the deflection angle of the nozzle wall at the endpoints in this paper.

The remainder of this paper is organized as follows. Section 2 describes the set-up of the physical problem and formulates a free boundary problem for the stream function with two parameters $(\lambda, Q_1)$. The solvability of the free boundary problem for any parameters follows from the standard variational approach in Sect. 2. To verify the continuous fit and smooth fit conditions, we will show that there exists a pair of parameters $(\lambda, Q_1)$, such that the desired conditions hold in Sect. 3. We give the asymptotic behavior of the impinging jet in Sect. 4. Finally, we will investigate the existence, regularity and the properties of the interface $\Gamma_1$ and the branching point $S$ in Sect. 5.

2 Mathematical formulation of the impinging flow problem

In this section, we will formulate a boundary value problem with free boundaries, and furthermore, solve the free boundary problem by the variational method developed by Alt, Caffarelli and Friedman in [1–3]. For convenience to the readers, we will sketch the details and point out the differences as follows.

2.1 A free boundary problem

Due to the continuity equation, there exists a stream function $\psi(x, y)$, such that

$$\frac{\partial \psi}{\partial y} = u \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -v.$$  \hspace{1cm} (2.1)

The slip boundary conditions on the nozzle walls and the free boundaries imply that $N_1 \cup \Gamma_1$ and $N_2 \cup \Gamma_2$ are level sets of the stream function $\psi$, and then without loss of generality, one assumes that

$$\psi = 0 \quad \text{on} \quad N_1 \cup \Gamma_1, \quad \psi = Q_1 \quad \text{on} \quad N \quad \text{and} \quad \psi = Q \quad \text{on} \quad N_2 \cup \Gamma_2.$$  

As mentioned before, in this work, we will seek an impinging jet under gravity with negative vertical velocity, and then it is reasonable to consider the free boundaries below the horizontal line $y = H$. Hence, we may assume that the possible fluid region is contained in the domain $\Omega$, bounded by $N_1, N_2, L_1, L_2$ and $N$, where

$$L_1 = \{(x, H) \mid x \leq -1\} \quad \text{and} \quad L_2 = \{(x, H) \mid x \geq 1\}.$$  

Moreover, we can define the free streamlines $\Gamma_1$ and $\Gamma_2$ as

$$\Gamma_1 = \{0 < y < H\} \cap \partial \psi > 0 \quad \text{and} \quad \Gamma_2 = \{0 < y < H\} \cap \partial \psi < Q,$$  \hspace{1cm} (2.2)

and the interface $\Gamma$ as

$$\Gamma = \Omega \cap \{\psi = Q_1\}.$$  

Thus, the fluid region $\Omega_0 = \Omega \cap \{0 < \psi < Q\}$.

The irrotational condition implies

$$\Delta \psi = 0 \quad \text{in} \quad \Omega \cap \{0 < \psi < Q\},$$
and the Bernoulli’s law gives that
\[
\frac{|\nabla \psi|^2}{2} + gy + p = \text{constant in } \Omega \cap \{0 < \psi < Q\}. \tag{2.3}
\]
This, together with the constant pressure condition on the free streamlines, implies that
\[
\frac{|\nabla \psi|^2}{2} + gy \text{ remains a constant on } \Gamma_1 \cup \Gamma_2, \text{ we denote this constant as } \lambda. \text{ Hence, we formulate}
\]
the following free boundary value problem in terms of a stream function.
\[
\begin{aligned}
\Delta \psi &= 0 \text{ in } \Omega \cap \{0 < \psi < Q\}, \\
\frac{\partial \psi}{\partial \nu} &= -\sqrt{2\lambda - 2gy} \text{ on } \Gamma_1, \quad \frac{\partial \psi}{\partial \nu} = \sqrt{2\lambda - 2gy} \text{ on } \Gamma_2, \\
\psi &= 0 \text{ on } N_1 \cup \Gamma_1, \quad \psi = Q \text{ on } N_2 \cup \Gamma_2, \quad \psi = Q_1 \text{ on } N,
\end{aligned}
\tag{2.4}
\]
where \(\nu\) is the outer normal to the free boundaries.

The problem is now to find a stream function \(\psi(x, y)\) solving the free boundary problem (2.4) with the continuous fit and smooth fit conditions. It should be emphasized that there are two undetermined parameters \(\lambda\) and \(Q_1\) in the free boundary problem, which will be determined by continuous fit conditions. In other words, we will solve the free boundary problem for any \(\lambda\) and \(Q_1\) first, and then show the existence of a pair of parameters \((\lambda, Q_1)\) to guarantee the continuous fit conditions.

Once the free boundary problem (2.4) has been solved, the velocity field is given by (2.1) and the pressure is given by Bernoulli’s law, and then the free boundaries \(\Gamma_1\) and \(\Gamma_2\) are determined by (2.2).

### 2.2 On the asymptotic widths

In this subsection, we will find the relationship between the parameters \((\lambda, Q_1)\) and the asymptotic widths \(h_1\) and \(h_2\) of the impinging jet in downstream if it exists.

**Proposition 2.1** For any given \(Q > 2\sqrt{gH^3}\),

1. the parameters \(\lambda > 0\) and \(Q_1 \in [0, Q]\) can be determined uniquely by the asymptotic heights \(h_1 \in [0, H]\) and \(h_2 \in [0, H]\).

2. for any given \(Q_1 \in (0, Q)\) and \(\lambda \geq \max\left(\frac{Q_1^2(Q_1 - Q)^2}{2H^2} + gH\right)\), there exists a unique pair of asymptotic heights \((h_1, h_2)\) of the impinging jet flow with \(h_1 \in (0, H]\) and \(h_2 \in (0, H]\).

Furthermore, \(h_1\) increases with respect to \(Q_1\) while \(h_2\) decreases with respect to \(Q_1\).

**Proof** (1). It is clear that
\[
Q_1 = \sqrt{2\lambda - 2gh_1}h_1 \quad \text{and} \quad Q - Q_1 = \sqrt{2\lambda - 2gh_2}h_2. \tag{2.5}
\]

**Case 1.** \(h_1 = h_2 > 0\), one has
\[
Q_1 = \frac{Q}{2} \quad \text{and} \quad \lambda = \frac{Q^2}{8h_1^2} + gh_1 = \frac{Q^2}{8h_2^2} + gh_2.
\]

**Case 2.** \(h_1h_2 = 0\), one has
\[
Q_1 = 0 \quad \text{and} \quad \lambda = \frac{Q^2}{2h_2^2} + gh_2 \quad \text{for} \quad h_1 = 0,
\]
and
\[
Q_1 = Q \quad \text{and} \quad \lambda = \frac{Q^2}{2h_1^2} + gh_1 \quad \text{for} \quad h_2 = 0.
\]
Case 3. $h_1 \neq h_2$ and $h_1 h_2 > 0$, it follows from (2.5) that
\[ Q_1^2 = 2\lambda h_1^2 - 2gh^3 \quad \text{and} \quad (Q - Q_1)^2 = 2\lambda h_2^2 - 2gh^3, \]
which implies that
\[ (h_2^2 - h_1^2)Q_1^2 + 2h_1^2 Q_1( - (h_1^2 Q_1^2 + 2gh_1^2 h_2^2(h_2 - h_1)) = 0. \]
Then we have
\[ Q_1 = \frac{-Qh_1^2 + \sqrt{Q^2 + 2g(h_2^2 - h_1^2)(h_2 - h_1)h_1 h_2}}{h_2^2 - h_1^2}, \]
(2.6)
or
\[ Q_1 = \frac{-Qh_1^2 - \sqrt{Q^2 + 2g(h_2^2 - h_1^2)(h_2 - h_1)h_1 h_2}}{h_2^2 - h_1^2} \begin{cases} < 0, & \text{if } h_1 < h_2, \\ > Q, & \text{if } h_1 > h_2, \end{cases} \]
which can be dropped. It should be noted that
\[ \sqrt{Q^2 + 2g(h_2^2 - h_1^2)(h_2 - h_1)} > Q \quad \text{for any} \quad h_1 \neq h_2. \]
It suffices to show that $Q_1$ described in the formula (2.6) lies in $[0, Q]$. In fact, a direct calculation gives that
\[ Q_1 = \frac{h_1(Q^2 + 2gh_1^2 h_2)}{Qh_1 + \sqrt{Q^2 + 2g(h_2^2 - h_1^2)(h_2 - h_1)h_2}}, \]
and
\[ Q_1 - Q = \frac{-Qh_1^2 + \sqrt{Q^2 + 2g(h_2^2 - h_1^2)(h_2 - h_1)h_1 h_2}}{h_2^2 - h_1^2} = \frac{h_2(-Q^2 + 2gh_2 h_1^2)}{Qh_2 + \sqrt{Q^2 + 2g(h_2^2 - h_1^2)(h_2 - h_1)h_1}}. \]
Noting that $Q_1 \geq 0$ and $Q_1 - Q \leq 0$, due to the condition $Q > 2\sqrt{gH^3}$.

Hence, $Q_1 \in [0, Q]$ can be uniquely determined by $h_1$ and $h_2$. Furthermore, $\lambda = \frac{Q_1^2}{2h_1^2} + gh_1$ can be uniquely determined by $h_1$ and $Q_1$.

(2). Since $Q > 2\sqrt{gH^3}$, we have $Q_1 > \sqrt{gH^3}$ or $Q - Q_1 > \sqrt{gH^3}$. Without loss of generality, we assume that $Q_1 > \sqrt{gH^3}$.

Recall that $\lambda = \frac{Q_1^2}{2h_1^2} + gh_1 = \frac{(Q - Q_1)^2}{2h_1^2} + gh_2$. Set
\[ \Lambda_1(t) = \frac{Q_1^2}{2t^2} + gt \quad \text{and} \quad \Lambda_2(t) = \frac{(Q - Q_1)^2}{2t^2} + gt \]
for any fixed $Q_1 \in [0, Q]$ and $t > 0$. It is easy to check that
\[ \Lambda_1'(t) = \frac{gt^3 - Q_1^2}{t^3} < 0 \quad \text{for any} \quad t \in (0, H]. \]
(2.7)
Since $\lambda \geq \max[\frac{Q_1^2}{2h_1^2}, \frac{(Q - Q_1)^2}{2H^2}] + gH \geq \Lambda_1(H)$, it follows from (2.7) that $h_1 \in (0, H]$ is determined uniquely by $\lambda$ and $Q_1$. 

Springer
Similarly, we have
\[ \Lambda_2'(t) = \frac{-(Q - Q_1)^2}{t^3} + gH \Lambda_2''(t) = \frac{3(Q - Q_1)^2}{t^4} > 0 \text{ for any } t > 0. \]

Set \( t_0 = \frac{(Q - Q_1)^2}{g^3} \), such that \( \Lambda_2'(t_0) = 0 \). Then we consider the following two cases.

**Case 1.** \( t_0 \geq H \) (see Fig. 4).

Then \( \Lambda_2'(t) < 0 \) for any \( t \in (0, H] \). On another side, the fact \( \lambda \geq \frac{\max\{Q_1^2, (Q - Q_1)^2\}}{2H^2} + gH \geq \Lambda_2(H) \) implies that \( h_2 \in (0, H] \) can be determined uniquely by \( \lambda = \frac{(Q - Q_1)^2}{2h_2^2} + gh_2 \).

**Case 2.** \( t_0 < H \) (see Fig. 5), which is equivalent to \( Q - Q_1 < \sqrt{gH^3} \). This together with \( Q_1 > \sqrt{gH^3} \) gives that
\[ \Lambda_2(H) = \frac{(Q - Q_1)^2}{2H^2} + gH < \frac{Q_1^2}{2H^2} + gH \leq \lambda, \]
which implies that \( h_2 \in (0, H] \) is determined uniquely by \( \lambda \) and \( Q_1 \), namely, \( \lambda = \frac{(Q - Q_1)^2}{2h_2^2} + gh_2 \).

Furthermore, we can conclude that \( h_1 \) is increasing with respect to \( Q_1 \) and \( h_2 \) is decreasing with respect to \( Q_1 \). \( \Box \)

**Remark 2.1** Note that \( \lambda = \frac{Q_1^2}{2h_1^2} + gh_1 = \frac{(Q - Q_1)^2}{2h_2^2} + gh_2 \). It follows from the proof of Proposition 2.1 that the condition \( Q > 2\sqrt{gH^3} \) ensures that one can choose \( \lambda \) depending on \( h_1 \) and \( h_2 \) monotonically provided that
\[ \lambda \geq \frac{\max\{Q_1^2, (Q - Q_1)^2\}}{2H^2} + gH. \]
Clearly, the right hand side has the following lower bound

\[ \frac{Q^2}{8H^2} + gH, \]

which excludes the stagnation point on the free boundary \( \Gamma_1 \cup \Gamma_2 \).

In particularly, for any \((Q_1, \lambda) \in [0, Q] \times \left[ \max\{Q_1^2(Q-Q_1)^2\}/2H^2 + gH, +\infty \right] \), there exists a unique pair \((h_1, h_2)\), such that

\[ \lambda = \frac{Q_1^2}{2h_1^2} + gh_1 = \frac{(Q - Q_1)^2}{2h_2^2} + gh_2. \]

### 2.3 The variational approach

To solve the boundary value problem (2.4), we will use the variational method, which was developed by Alt, Caffarelli and Friedman in [1–3] for two-dimensional asymmetric incompressible jets and jets under gravity.

Define a functional with a parameter \(\lambda\) as follows,

\[ J_\lambda(\psi) = \int_\Omega |\nabla \psi|^2 + (2\lambda - 2gy)\chi_{[0<\psi<Q] \cap D}dx dy, \quad (2.8) \]

where \(\chi_E\) is the characteristic function of a set \(E\) and \(D = \{0 < y < H\}\). Define an admissible set with a parameter \(Q_1\),

\[ K_{Q_1} = \{\psi \in H^1_{loc}(\mathbb{R}^2) \mid \psi = Q_1 \text{ below } N, \psi = Q \text{ in the right of } N_2 \cup L_2, \psi = 0 \text{ in the left of } N_1 \cup L_1\}. \]
However, noting that the functional \( J_\lambda(\psi) = +\infty \) for any \( \psi \in K_{Q_1} \), we will consider the variational problem in a truncated domain. For any \( \mu > 1 \), denote

\[
\begin{align*}
H_{1,\mu} &= \max\{y \mid g_1(y) = -\mu\} \\
H_{2,\mu} &= \min\{y > H_{1,\mu} \mid g_2(y) = -\mu\}, \\
N_{1,\mu} &= \{(x, y) \mid x = g_1(y), H \leq y \leq H_{1,\mu}\}, \\
N_{2,\mu} &= \{(x, y) \mid x = g_2(y), H \leq y \leq H_{2,\mu}\}, \\
\sigma_{1,\mu} &= \{x = -\mu, 0 \leq y \leq H\}, \\
\sigma_{2,\mu} &= \{x = \mu, 0 \leq y \leq H\}, \\
N_\mu &= N \cap \{-\mu \leq x \leq \mu\}, \\
\sigma_\mu &= \{x = -\mu, H_{1,\mu} \leq y \leq H_{2,\mu}\},
\end{align*}
\]

and \( \Omega_\mu \) is bounded by \( N_{1,\mu}, N_{2,\mu}, L_{1,\mu}, L_{2,\mu}, \sigma_{1,\mu}, \sigma_{1,\mu}, \sigma_{2,\mu} \) and \( N_\mu \) (see Fig. 6).

Hence, one may consider the following truncated variational problem in the bounded domain \( \Omega_\mu \). Define a functional

\[
J_{\lambda,\mu}(\psi) = \int_{\Omega_\mu} |\nabla \psi|^2 + (2\lambda - 2g\gamma) \chi_{[0<\psi<Q]} \cap D_\mu \ dx \ dy,
\]

where \( D_\mu = \{-\mu < x < \mu, 0 < y < H\} \).

The truncated variational problem \((P_{\lambda,Q_1,\mu})\): For any \( \mu > 1 \), \( Q_1 \in [0, Q] \) and \( \lambda \geq \frac{\max\{Q_1^2, (Q-Q_1)^2\}}{2Hz} + gH \), find a \( \psi_{\lambda,Q_1,\mu} \in K_{\lambda,Q_1,\mu} \), such that

\[
J_{\lambda,\mu}(\psi_{\lambda,Q_1,\mu}) = \min_{\psi \in K_{\lambda,Q_1,\mu}} J_{\lambda,\mu}(\psi),
\]
where the admissible set \( K_{\lambda, Q_1, \mu} = \{ \psi \in K_{Q_1} \mid \psi = \Psi_{\lambda, Q_1, \mu}(x, y) \text{ on } \partial \Omega_\mu \} \) and the boundary function

\[
\Psi_{\lambda, Q_1, \mu}(x, y) = \begin{cases}
0, & \text{on } N_{1, \mu} \cup L_{1, \mu}, \\
Q, & \text{on } N_{2, \mu} \cup L_{2, \mu}, \\
Q_1, & \text{on } N_{\mu}, \\
\max \left\{ -\sqrt{2\lambda - 2gh_1y + Q_1, 0} \right\}, & \text{on } \sigma_{1, \mu}, \\
\min \left\{ \sqrt{2\lambda - 2gh_2y + Q_1, Q} \right\}, & \text{on } \sigma_{2, \mu}, \\
\left(\frac{y-H_1(\sigma_{\mu})}{H_2(\mu)-H_{1, \mu}}\right)Q & \text{on } \sigma_{\mu}.
\end{cases}
\]

Here, \( h_1 \) and \( h_2 \) are the asymptotic heights of the impinging jets in left and right downstream, depending uniquely on \( \lambda \) and \( Q_1 \) as in \((2.5)\).

The existence of a minimizer \( \psi_{\lambda, Q_1, \mu} \) follows directly from Theorem 1.3 in [1], so the details are omitted here.

The corresponding free boundaries \( \Gamma_{1, \lambda, Q_1, \mu} \) and \( \Gamma_{2, \lambda, Q_1, \mu} \) are defined as

\[
\Gamma_{1, \lambda, Q_1, \mu} = D_{\mu} \cap \partial \{ \psi_{\lambda, Q_1, \mu} > 0 \}\text{ and } \Gamma_{2, \lambda, Q_1, \mu} = D_{\mu} \cap \partial \{ \psi_{\lambda, Q_1, \mu} < Q \}.
\]

It follows from Lemma 1.3, Lemma 2.3 and Lemma 2.4 in [1] that the following properties hold for the minimizer \( \psi_{\lambda, Q_1, \mu} \) to the truncated variational problem \((P_{\lambda, Q_1, \mu})\).

**Proposition 2.2** The minimizer \( \psi_{\lambda, Q_1, \mu} \) satisfies that

1. \( \psi_{\lambda, Q_1, \mu} \in C^{0,1}(\Omega_\mu) \) and is harmonic in \( \Omega_\mu \cap \{ 0 < \psi_{\lambda, Q_1, \mu} < Q \} \).
2. \( 0 \leq \psi_{\lambda, Q_1, \mu} \leq Q \) in \( \Omega_\mu \) and \( 0 < \psi_{\lambda, Q_1, \mu} < Q \) in \( \Omega_\mu \cap \{ y > H \} \).
3. The free boundaries \( \Gamma_{1, \lambda, Q_1, \mu} \) and \( \Gamma_{2, \lambda, Q_1, \mu} \) are analytic.
4. \( |\nabla \psi_{\lambda, Q_1, \mu}| = \sqrt{2\lambda - 2gh_1y} \) on \( \Gamma_{1, \lambda, Q_1, \mu} \cup \Gamma_{2, \lambda, Q_1, \mu} \).

Next, we will give a uniform estimate to the gradient of the minimizer away from the endpoints of the nozzle walls. The proof follows from the similar arguments in Section 5 in [2] immediately, and then the details are omitted here.

**Lemma 2.3** The minimizer \( \psi_{\lambda, Q_1, \mu} \) is continuous in \( \tilde{\Omega}_\mu \), and for any \( \delta > 0 \), if \( X \in \tilde{\Omega}_\mu \) with \( |X - A_1| > \delta \) and \( |X - A_2| > \delta \), then there exists a constant \( C \) depending on \( \delta \) and \( \Omega_\mu \), such that

\[
|D \psi_{\lambda, Q_1, \mu}(X)| \leq C(\sqrt{\lambda} + Q).
\]

Furthermore, for any domain \( K \subseteq \Omega_\mu \) containing a free boundary point, the Lipschitz coefficient of \( \psi_{\lambda, Q_1, \mu} \) in \( K \) is estimated by \( C\sqrt{\lambda} \), where the constant \( C \) depends on \( K \) and \( \Omega_\mu \), independent of \( Q \).

### 2.4 The properties of the minimizer and the free boundaries

Next, we will establish some important properties of the minimizer and the free boundaries, such as monotonicity, uniqueness and regularity. First, we consider the monotonicity of the minimizer with respect to \( x \). To this end, we will give a priori bounds of the minimizer as follows.

**Lemma 2.4** The minimizer \( \psi_{\lambda, Q_1, \mu} \) to the truncated variational problem \((P_{\lambda, Q_1, \mu})\) satisfies

\[
\max \left\{ -\sqrt{2\lambda - 2gh_1y + Q_1, 0} \right\} \leq \psi_{\lambda, Q_1, \mu}(x, y) \leq \min \left\{ \sqrt{2\lambda - 2gh_2y + Q_1, Q} \right\} \text{ in } D_{\mu}, \tag{2.10}
\]
and
\[
\max \left\{ \frac{(y - H_{1,\mu})Q}{H_{2,\mu} - H_{1,\mu}}, 0 \right\} < \psi_{\lambda, Q_1,\mu}(x, y) < Q \quad \text{in} \quad \Omega_\mu \cap \{ y > H \},
\]
(2.11)
where \( H_{1,\mu} \) and \( H_{2,\mu} \) are defined in (2.9).

**Remark 2.2** The bound (2.10) gives that
\[
0 < -\sqrt{2\lambda - 2gh_1} y + Q_1 \leq \psi_{\lambda, Q_1,\mu}(x, y) \quad \text{for} \quad y < h_1,
\]
and
\[
\psi_{\lambda, Q_1,\mu}(x, y) \leq \sqrt{2\lambda - 2gh_2} y + Q_1 < Q \quad \text{for} \quad y < h_2,
\]
which in fact imply that the free boundary \( \Gamma_{i,\lambda, Q_1,\mu} \) lies above the asymptotic height \( h_i \) for \( i = 1, 2 \). On another side, the bounds in (2.11) imply that the free boundaries lies below the horizontal line \( y = H \).

**Proof** Consider the upper bound in (2.10) first. Since \( \psi_{\lambda, Q_1,\mu} \leq Q \) in \( \Omega_\mu \), it suffices to obtain the upper bound in (2.10) for the case \( Q_1 < Q \).

Define an auxiliary comparison function
\[
\omega_\varepsilon(y) = \min \left\{ \sqrt{2\lambda - 2g(h_2 - \varepsilon)} y + Q_1, Q \right\} \quad \text{for any} \quad \varepsilon \geq 0,
\]
and a strip
\[
D_{\mu,\varepsilon} = \Omega_\mu \cap \left\{ y < \frac{Q - Q_1}{\sqrt{2\lambda - 2g(h_2 - \varepsilon)}} \right\} \quad \text{for any} \quad \varepsilon \geq 0.
\]

Due to Lemma 2.3, \( \psi_{\lambda, Q_1,\mu} \) is Lipschitz continuous near \( N_\mu \), this implies that there is a positive distance between the right free boundary and \( N_\mu \). Thus, take a sufficiently large \( \varepsilon > 0 \), such that
\[
y = \frac{Q - Q_1}{\sqrt{2\lambda - 2g(h_2 - \varepsilon)}} \quad \text{lies below the right free boundary} \Gamma_{2,\lambda, Q_1,\mu}.
\]
It is easy to check that
\[
\omega_\varepsilon \geq \psi_{\lambda, Q_1,\mu} \quad \text{on} \quad \sigma_{1,\mu} \cup \sigma_{2,\mu}
\]
for any \( \varepsilon \geq 0 \), and it follows from Proposition 2.2 that
\[
\psi_{\lambda, Q_1,\mu} \leq Q = \omega_\varepsilon \quad \text{on} \quad y = \frac{Q - Q_1}{\sqrt{2\lambda - 2g(h_2 - \varepsilon)}},
\]
for \( Q_1 < Q \). If \( \varepsilon \) is sufficiently large, \( \psi_{\lambda, Q_1,\mu} \) and \( \omega_\varepsilon \) are harmonic in \( D_{\mu,\varepsilon} \). Applying the maximum principle in \( D_{\mu,\varepsilon} \) gives that
\[
\psi_{\lambda, Q_1,\mu} < \omega_\varepsilon \quad \text{in} \quad D_{\mu,\varepsilon},
\]
provided that \( \varepsilon \) is sufficiently large.

Let \( \varepsilon_0 \geq 0 \) be the smallest one and \( y_0 = \frac{Q - Q_1}{\sqrt{2\lambda - 2g(h_2 - \varepsilon_0)}} \) (see Fig. 7), such that
\[
\psi_{\lambda, Q_1,\mu}(x, y) < \omega_{\varepsilon_0}(y) \quad \text{in} \quad D_{\mu,\varepsilon_0}, \quad \text{and} \quad \omega_{\varepsilon_0}(y_0) = \psi_{\lambda, Q_1,\mu}(x_0, y_0) \quad \text{for some} \quad x_0 \in (-\mu, \mu).
It suffices to show that $\varepsilon_0 = 0$, and then the upper bound of the minimizer $\psi_{\lambda, Q_1, \mu}$ follows. If not, then $\varepsilon_0 > 0$, which implies that
\[
y_0 = \frac{Q - Q_1}{\sqrt{2\lambda - 2g(h_2 - \varepsilon_0)}} < h_2.
\]

Denote $X_0 = (x_0, y_0)$. Since the free boundary $\Gamma_{2, \lambda, Q_1, \mu}$ is analytic at $X_0$, it follows from Hopf’s lemma that
\[
\sqrt{2\lambda - 2g} y_0 > \frac{\partial \psi_{\lambda, Q_1, \mu}}{\partial \nu} = \sqrt{2\lambda - 2g(h_2 - \varepsilon_0)} = \frac{Q - Q_1}{y_0} = \frac{\sqrt{2\lambda - 2gh_2}}{y_0} \text{ at } X_0,
\]
where $\nu = (0, 1)$ is the outer normal vector to $\Gamma_{2, \lambda, Q_1, \mu}$ at $X_0 = (x_0, y_0)$. This is nothing but
\[
y_0 \sqrt{2\lambda - 2gh_0} > h_2 \sqrt{2\lambda - 2gh_2} \quad \text{for} \quad y_0 < h_2. \tag{2.12}
\]

On another side, it is clear that the function $t \sqrt{2\lambda - 2gt}$ is strictly increasing with respect to $t \in (0, H)$, which contradicts to (2.12).

Similarly, one can show that
\[
\psi(x, y) \geq \max \left\{ -\sqrt{2\lambda - 2gh_1} y + Q_1, 0 \right\} \text{ in } D_{\mu}.
\]

Finally, consider the estimate (2.11) in the nozzle. Denote $G_{\mu} = \Omega_{\mu} \cap \{ y > H_{1, \mu} \}$. Applying the maximum principle in $G_{\mu}$ gives that $\frac{(y-H_{1, \mu})Q}{H_{2, \mu}-H_{1, \mu}} \leq \psi_{\lambda, Q_1, \mu}$ in $G_{\mu}$. Since $0 < \psi_{\lambda, Q_1, \mu} < Q$ in $\Omega_{\mu} \cap \{ y > H \}$, it follows
\[
\max \left\{ \frac{(y-H_{1, \mu})Q}{H_{2, \mu}-H_{1, \mu}}, 0 \right\} < \psi_{\lambda, Q_1, \mu}(x, y) < Q \quad \text{in } \Omega_{\mu} \cap \{ y > H \}.
\]

With the aid of Lemma 2.4, one can obtain the uniqueness and monotonicity of the minimizer $\psi_{\lambda, Q_1, \mu}$.
Proposition 2.5  The minimizer $\psi_{\lambda, Q_1, \mu}(x, y)$ to the truncated variational problem $(P_{\lambda, Q_1, \mu})$ is increasing with respect to $x$. Furthermore, the minimizer $\psi_{\lambda, Q_1, \mu}$ is unique for any fixed $\lambda$, $Q_1$ and $\mu$.

Proof  Suppose that $\psi(x, y) = \psi_{\lambda, Q_1, \mu}(x, y)$ and $\tilde{\psi}(x, y) = \tilde{\psi}_{\lambda, Q_1, \mu}(x, y)$ are two minimizers to the truncated variational problem $(P_{\lambda, Q_1, \mu})$. Define $\tilde{\psi}(x, y) = \psi(x - \varepsilon, y)$ for any $\varepsilon \geq 0$, the corresponding functional $J_{\lambda, \mu, \varepsilon}$ with admissible set $K_{\lambda, Q_1, \mu}$ in truncated domain $\Omega_{\mu}^\varepsilon = \{(x, y) \mid (x - \varepsilon, y) \in \Omega_{\mu}\}$. Extend $\psi(x, y)$ and $\tilde{\psi}(x, y)$ to larger domain, respectively, as

$$
\psi(x, y) = \min \left\{ \sqrt{2\lambda - 2gh_2y} + Q_1, Q \right\} \text{ for } \mu \leq x \leq \mu + \varepsilon, 0 \leq y \leq H
$$

and

$$
\tilde{\psi}_\varepsilon(x, y) = \left\{ \begin{array}{ll}
\max \left\{ -\frac{\sqrt{2\lambda - 2gh_1y} + Q_1, 0} \right\} , & \text{ for } -\mu \leq x \leq -\mu + \varepsilon, 0 \leq y \leq H, \\
\max \left\{ \frac{\sqrt{y - H_{1, \mu}}}{H_{2, \mu} - H_{1, \mu}}, 0 \right\} , & \text{ for } -\mu \leq x \leq -\mu + \varepsilon, H \leq y \leq H_{2, \mu}.
\end{array} \right.
$$

Set

$$
\psi_1 = \max \{ \psi, \tilde{\psi}_\varepsilon \} \text{ and } \psi_2 = \min \{ \psi, \tilde{\psi}_\varepsilon \}.
$$

Then Lemma 2.4 gives that

$$
\psi_1 = \psi \text{ in } \Omega_{\mu}^\varepsilon \setminus \Omega_{\mu} \text{ and } \psi_2 = \tilde{\psi}_\varepsilon \text{ in } \Omega_{\mu} \setminus \Omega_{\mu}^\varepsilon, \tag{2.13}
$$

which implies that $\psi_1 \in K_{\lambda, Q_1, \mu}$ and $\psi_2 \in K_{\lambda, Q_1, \mu}^\varepsilon$. Next, we claim that

$$
J_{\lambda, \mu}(\psi) = J_{\lambda, \mu}(\psi_1) \text{ and } J_{\lambda, \mu}^\varepsilon(\tilde{\psi}_\varepsilon) = J_{\lambda, \mu}^\varepsilon(\psi_2). \tag{2.14}
$$

Indeed, since $J_{\lambda, \mu}(\psi) \leq J_{\lambda, \mu}(\psi_1)$ and $J_{\lambda, \mu}^\varepsilon(\tilde{\psi}_\varepsilon) \leq J_{\lambda, \mu}^\varepsilon(\psi_2), (2.14)$ will follow from

$$
J_{\lambda, \mu}(\psi) + J_{\lambda, \mu}^\varepsilon(\tilde{\psi}_\varepsilon) = J_{\lambda, \mu}(\psi_1) + J_{\lambda, \mu}^\varepsilon(\psi_2). \tag{2.15}
$$

In fact, (2.13) implies that $\psi \geq \tilde{\psi}_\varepsilon$ in $\left( \Omega_{\mu}^\varepsilon \setminus \Omega_{\mu} \right) \cup \left( \Omega_{\mu} \setminus \Omega_{\mu}^\varepsilon \right)$. Therefore one has

$$
\Omega_{\mu} \cap \{ \psi < \tilde{\psi}_\varepsilon \} = \Omega_{\mu}^\varepsilon \cap \{ \psi < \tilde{\psi}_\varepsilon \}. \tag{2.16}
$$

Due to (2.16), one may get

$$
\int_{\Omega_{\mu}} |\nabla \psi_1|^2 dxdy + \int_{\Omega_{\mu}^\varepsilon} |\nabla \psi_2|^2 dxdy
= \int_{\Omega_{\mu} \cap \{ \psi \geq \tilde{\psi}_\varepsilon \}} |\nabla \psi|^2 dxdy + \int_{\Omega_{\mu} \cap \{ \psi < \tilde{\psi}_\varepsilon \}} |\nabla \tilde{\psi}_\varepsilon|^2 dxdy
+ \int_{\Omega_{\mu}^\varepsilon \cap \{ \psi \geq \tilde{\psi}_\varepsilon \}} |\nabla \tilde{\psi}_\varepsilon|^2 dxdy + \int_{\Omega_{\mu}^\varepsilon \cap \{ \psi < \tilde{\psi}_\varepsilon \}} |\nabla \psi|^2 dxdy
= \int_{\Omega_{\mu} \cap \{ \psi \geq \tilde{\psi}_\varepsilon \}} |\nabla \psi|^2 dxdy + \int_{\Omega_{\mu}^\varepsilon \cap \{ \psi < \tilde{\psi}_\varepsilon \}} |\nabla \tilde{\psi}_\varepsilon|^2 dxdy
+ \int_{\Omega_{\mu}^\varepsilon \cap \{ \psi \geq \tilde{\psi}_\varepsilon \}} |\nabla \tilde{\psi}_\varepsilon|^2 dxdy + \int_{\Omega_{\mu}^\varepsilon \cap \{ \psi < \tilde{\psi}_\varepsilon \}} |\nabla \psi|^2 dxdy
= \int_{\Omega_{\mu}} |\nabla \psi|^2 dxdy + \int_{\Omega_{\mu}^\varepsilon} |\nabla \tilde{\psi}_\varepsilon|^2 dxdy. \tag{2.17}
$$
Similarly, one has
\[
\int_{D_{\mu}} (2\lambda - 2gy) \chi_{\{0 < \psi < Q\}} \, dx \, dy + \int_{D_{\mu}^c} (2\lambda - 2gy) \chi_{\{0 < \psi < \bar{Q}\}} \, dx \, dy \\
= \int_{D_{\mu}} (2\lambda - 2gy) \chi_{\{0 < \psi < Q\}} \, dx \, dy + \int_{D_{\mu}^c} (2\lambda - 2gy) \chi_{\{0 < \psi < \bar{Q}\}} \, dx \, dy,
\]
which, together with (2.17), gives (2.15). And hence the claim (2.14) is proved.

Moreover, we will show that if \(0 < \psi(X_0) = \tilde{\psi}_e(X_0) < Q\) with \(X_0 \in \Omega_\mu \cap \Omega_\mu^e\), then
\[
either \psi \geq \tilde{\psi}_e \text{ or } \psi \leq \tilde{\psi}_e \text{ in some neighborhood of } X_0.\tag{2.18}
\]
By the continuity of \(\psi\), \(0 < \psi < Q\) in \(B_r(X_0) \subset \Omega_\mu \cap \Omega_\mu^e\) for some small \(r > 0\). Define a function \(\phi\) as follows,
\[
\Delta \phi = 0 \text{ in } B_r(X_0) \text{ and } \phi = \psi_1 \text{ on } \partial B_r(X_0).\tag{2.19}
\]
Suppose that (2.18) is not true, we then claim that \(\psi_1\) is not harmonic in \(B_r(X_0)\) for any small \(r > 0\). In fact, if \(\psi_1\) solves the boundary value problem (2.19), and it is easy to check that \(\Delta \psi = 0\) in \(B_r(X_0)\) and \(\psi \leq \psi_1\) on \(\partial B_r(X_0)\), the strong maximum principle gives that \(\psi \equiv \psi_1\) in \(B_r(X_0)\), due to \(\psi(X_0) = \psi_1(X_0)\). Namely, \(\psi \geq \tilde{\psi}_e\) in \(B_r(X_0)\), which contradicts to our assumption.

Since \(\psi_1\) is not harmonic in \(B_r(X_0)\), and \(\psi_1 \neq \phi\) in \(B_r(X_0)\). Then it holds that
\[
\int_{B_r(X_0)} |\nabla \phi|^2 \, dx \, dy - \int_{B_r(X_0)} |\nabla \psi_1|^2 \, dx \, dy \\
= -\int_{B_r(X_0)} |\nabla (\phi - \psi_1)|^2 \, dx \, dy + 2 \int_{B_r(X_0)} \nabla \phi \cdot \nabla (\phi - \psi_1) \, dx \, dy \\
= -\int_{B_r(X_0)} |\nabla (\phi - \psi_1)|^2 \, dx \, dy < 0.\tag{2.20}
\]

Extend \(\phi = \psi_1\) outside \(B_r(X_0)\), such that \(\phi \in K_{\lambda, Q, 1, \mu}\). It follows from (2.20) and the fact that \(0 < \phi < Q\) in \(\bar{B}_r(X_0)\) that
\[
J_{\lambda, \mu}(\phi) < J_{\lambda, \mu}(\psi_1) = J_{\lambda, \mu}(\psi),
\]
which contradicts to the fact that \(\psi\) is a minimizer to the variational problem \((P_{\lambda, Q, 1, \mu})\). Hence, the claim (2.18) is proved.

Next, we will show that
\[
\Omega_\mu \cap \{0 < \psi < Q\} \text{ and } \Omega_\mu^e \cap \{0 < \tilde{\psi}_e < Q\} \text{ are connected}.\tag{2.21}
\]
If (2.21) fails, then without loss of generality, we assume that \(\Omega_\mu \cap \{0 < \psi < Q\}\) is not connected. Since \(0 < \psi < Q\) in \(\Omega_\mu \cap \{y > H\}\) (see (2.11)), let \(E\) be a maximal connected subset of \(\Omega_\mu \cap \{0 < \psi < Q\}\), such that \(\Omega_\mu \cap \{y > H\} \subset E\). Then there exists a point \(Y_0 \in \Omega_\mu \cap \{0 < \psi < Q\}\), such that \(Y_0 \notin E\). The continuity of \(\psi\) gives that there exists a maximal connected subset \(F\) in \(\Omega_\mu \cap \{0 < \psi < Q\}\) and \(Y_0 \in F\).

If \(\partial \Omega_\mu \cap \partial E = \emptyset\), we first show that \(\psi = 0\) or \(\psi = Q\) on \(\partial E\). If not, there exists a point \(X_0 \in \partial E\) with \(0 < \psi(X_0) < Q\). Due to the continuity of \(\psi\) again, there exists a connected subset \(\bar{F}\) of \(\Omega_\mu \cap \{0 < \psi < Q\}\), such that \(F \subseteq \bar{F}\), which contradicts to the definition of the set \(\bar{F}\). Since \(\psi\) is harmonic in \(F\) and \(0 < \psi(Y_0) < Q\) with \(Y_0 \in F\), the strong maximum principle gives that \(0 < \psi < Q\) in \(F\). Moreover, \(\psi \equiv C_0\) on the outer boundary \(\partial F\), where \(C_0 = Q\) or \(C_0 = 0\). Let \(F_0\) be a domain bounded by the outer boundary \(\partial F\), and define a function \(\psi_0\), such that
\[
\psi_0 = \psi \text{ in } \Omega_\mu \setminus F_0 \text{ and } \psi_0 = C_0 \text{ in } F_0.
\]
Obviously, \( \psi_0 \in K_{\lambda, \Omega_1, \mu} \) and \( J_{\lambda, \mu}(\psi_0) < J_{\lambda, \mu}(\psi) \). This leads to a contradiction.

If \( \partial \Omega_\mu \cap \partial F \neq \emptyset \), the definition of \( F \) implies that \( \psi \equiv 0 \) or \( \psi \equiv Q \) on \( \Omega_\mu \cap \partial F \). Without loss of generality, one may assume that \( \psi \equiv 0 \) on \( \Omega_\mu \cap \partial F \). Hence, \( \psi \equiv 0 \) on \( \partial \Omega_\mu \cap \partial F \), since \( \psi \) is continuous up to \( \partial \Omega_\mu \) and is monotone with respect to \( y \) on \( \partial \Omega_\mu \). Therefore, \( \psi \equiv 0 \) on the connected part of \( \partial F \), which together with the strong maximum principle and the definition of \( F \) gives that \( 0 < \psi < Q \) in \( F \). By using above arguments, one can obtain a contradiction to the minimality of \( \psi \).

Hence, the claim (2.21) is obtained.

By virtue of (2.18), one has that

\[
\text{there does not exist some point } X_0 \in \Omega_\mu, \text{ such that } 0 < \psi(X_0) = \tilde{\psi}_e(X_0) < Q. \tag{2.22}
\]

In fact, suppose that there exists a point \( X_0 = (x_0, y_0) \in \Omega_\mu \), such that \( 0 < \psi(X_0) = \tilde{\psi}_e(X_0) < Q \). Then there are two cases to be considered.

**Case 1.** \( X_0 \in \Omega_\mu \cap \Omega_\mu^e \). In view of the claim (2.18), we have that \( \psi \geq \tilde{\psi}_e \) or \( \psi \leq \tilde{\psi}_e \) in \( B_r(X_0) \subset (\Omega_\mu \cap \{0 < \psi < Q\}) \cap (\Omega_\mu^e \cap \{0 < \tilde{\psi}_e < Q\}) \) for some small \( r > 0 \). The strong maximum principle gives that \( \psi \equiv \tilde{\psi}_e \) in \( B_r(X_0) \), due to \( \psi(X_0) = \tilde{\psi}_e(X_0) \). By virtue of (2.21), applying strong maximum principle again, we can conclude that

\[
\psi \equiv \tilde{\psi}_e \text{ in } (\Omega_\mu \cap \{0 < \psi < Q\}) \cap (\Omega_\mu^e \cap \{0 < \tilde{\psi}_e < Q\}),
\]

which leads to a contradiction to the fact that \( \tilde{\psi}_e < Q = \psi \) on \( N_{2, \mu} \).

**Case 2.** \( X_0 \in \Omega_\mu \setminus \Omega_\mu^e \). Then there are three subcases.

**Subcase 2.1.** \( X_0 \in (\Omega_\mu \setminus \Omega_\mu^e) \cap \{y > H_1, \mu\} \). The extension of \( \tilde{\psi}_e \) implies that \( 0 < \psi(X_0) = \tilde{\psi}_e(X_0) = (y_0 - H_1, \mu)Q \frac{h_2 - h_1}{H_2, \mu - H_1, \mu} < Q \), which contradicts to the lower bound in (2.11).

**Subcase 2.2.** \( X_0 \in (\Omega_\mu \setminus \Omega_\mu^e) \cap \{h_1 \leq y \leq H_1, \mu\} \). It is easy to see that \( \tilde{\psi}_e(X_0) = 0 \), which contradicts to our assumption \( 0 < \tilde{\psi}_e(X_0) < Q \).

**Subcase 2.3.** \( X_0 \in (\Omega_\mu \setminus \Omega_\mu^e) \cap \{y < h_1\} \). Noticing the extension of \( \tilde{\psi}_e \), one has that \( 0 < \psi(X_0) = \tilde{\psi}_e(X_0) = \sqrt{2\lambda - 2gh_1} y_0 + Q_1 < Q \). The continuity of \( \psi \) gives that there exists a small \( r > 0 \), such that \( 0 < \psi < Q \) in \( B_r(X_0) \subset \Omega_\mu \) and \( y_0 + r < h_1 \). Due to the lower bound \( \psi \geq \sqrt{2\lambda - 2gh_1} y + Q_1 \) in (2.10), the strong maximum principle implies that \( \psi(X_0) \equiv \sqrt{2\lambda - 2gh_1} y_0 + Q_1 \) in \( B_r(X_0) \). In view of (2.21), by using the strong maximum principle again, one gets that \( \psi(X_0) \equiv \sqrt{2\lambda - 2gh_1} y_0 + Q_1 \) in \( \Omega_\mu \cap \{y < h_1\} \), which leads to a contradiction to the boundary value of \( \psi \) on \( \sigma_{2, \mu} \).

Since \( \tilde{\psi}_e < Q = \psi \) on \( N_{2, \mu} \), it follows from (2.21) and (2.22) that

\[
\psi(x, y) \geq \tilde{\psi}(x - \varepsilon, y) \quad \text{in } \Omega_\mu. \tag{2.23}
\]

Similarly, one can obtain that

\[
\psi(x, y) \leq \tilde{\psi}(x + \varepsilon, y) \quad \text{in } \Omega_\mu. \tag{2.24}
\]

Taking \( \varepsilon \to 0 \) in (2.23) and (2.24) implies that \( \psi = \tilde{\psi} \).

In particular, taking \( \psi = \tilde{\psi} \) in (2.23), we conclude that \( \psi(x, y) \) is monotone increasing with respect to \( x \).
2.5 The free boundaries of the minimizer

Thanks to the monotonicity of $\psi_{\lambda, Q_1, \mu}$ with respect to $x$ in Proposition 2.5, there exist two functions $k_{1, \lambda, Q_1, \mu}(y)$ and $k_{2, \lambda, Q_1, \mu}(y)$ such that

$$D_\mu \cap \{0 < \psi_{\lambda, Q_1, \mu} < Q\} = \{(x, y) \in D_\mu \mid k_{1, \lambda, Q_1, \mu}(y) < x < k_{2, \lambda, Q_1, \mu}(y) \text{ for } 0 < y < H\}. \quad (2.25)$$

In order to establish the continuity of free boundaries $x = k_{1, \lambda, Q_1, \mu}(y)$ and $x = k_{2, \lambda, Q_1, \mu}(y)$ with respect to $y$, we need the following non-oscillation lemma, which implies that the free boundary cannot oscillate near any free boundary point.

**Lemma 2.6** (non-oscillation lemma) Suppose that there exist some $\alpha_1, \alpha_2$ with $\alpha_1 < \alpha_2$ and a domain $E \subset D_\mu \cap \{0 < \psi_{\lambda, Q_1, \mu} < Q\}$, such that

1. $E$ is bounded by two disjoint arcs $\gamma_1, \gamma_2$ ($\gamma_1, \gamma_2 \subset \Gamma_{2, \lambda, Q_1, \mu}$), the lines $\{x = \alpha_1\}$ and $\{x = \alpha_2\}$ (see Fig. 8). Denote the endpoints of $\gamma_i$ as $(\alpha_i, \beta_i)$ and $(\alpha_i, \eta_i)$ for $i = 1, 2$.

2. $\text{dist}(E, A_1 A_2) > c_0$ for some $c_0 > 0$.

There exists a constant $C > 0$, depending only on $\lambda, H, c_0$ and $Q$, such that

$$|\alpha_2 - \alpha_1| \leq C \max(|\beta_1 - \beta_2|, |\eta_1 - \eta_2|).$$

A similar result holds for $\gamma_1, \gamma_2 \subset \Gamma_{1, \lambda, Q_1, \mu}$.

**Proof** Denote $h = \max(|\beta_1 - \beta_2|, |\eta_1 - \eta_2|)$ and $\psi = \psi_{\lambda, Q_1, \mu}$. Since $\psi$ is harmonic in $E$, it is easy to check that $\int_{\partial E} \frac{\partial \psi}{\partial v} dS = 0$, which yields that

$$\int_{\gamma_1 \cup \gamma_2} \sqrt{2\lambda - 2gy} dS = \int_{\gamma_1 \cup \gamma_2} \frac{\partial \psi}{\partial v} dS = -\int_{\partial E \cap \{x = \alpha_2\}} \frac{\partial \psi}{\partial x} dy + \int_{\partial E \cap \{x = \alpha_1\}} \frac{\partial \psi}{\partial x} dy,$$

where $\gamma_1$ and $\gamma_2$ are two arcs of the right free boundary $\Gamma_{2, \lambda, Q_1, \mu}$.

\(\square\) Springer
By virtue of the Lipschitz continuity of $\psi$, one has
\begin{equation}
- \int_{\partial E \cap \{ x = \alpha_2 \}} \frac{\partial \psi}{\partial x} \, dy + \int_{\partial E \cap \{ x = \alpha_1 \}} \frac{\partial \psi}{\partial x} \, dy \leq Ch. \tag{2.26}
\end{equation}
On the other hand, one gets
\begin{equation*}
\int_{\gamma_1 \cup \gamma_2} \sqrt{2\lambda - 2gy} \, dS \geq 2\sqrt{2\lambda - 2gH(\alpha_2 - \alpha_1)},
\end{equation*}
which together with (2.26) gives that
\begin{equation*}
\alpha_2 - \alpha_1 \leq Ch.
\end{equation*}
\hfill \Box

**Remark 2.3** The non-oscillation Lemma 2.6 remains true if one of the arcs, $\gamma_1$ for example, is a line segment (see Fig. 9), and
\begin{equation}
\frac{\partial \psi_{\lambda, Q_1, \mu}}{\partial \nu} \geq 0 \quad \text{on} \quad \gamma_1. \tag{2.27}
\end{equation}

With the aid of the non-oscillation Lemma 2.6, we can show the continuity of $k_{1, \lambda, Q_1, \mu}(y)$ and $k_{2, \lambda, Q_1, \mu}(y)$ with respect to $y$.

**Lemma 2.7** $k_{1, \lambda, Q_1, \mu}(y)$ is continuous in $(h_1, H)$ and $k_{2, \lambda, Q_1, \mu}(y)$ is continuous in $(h_2, H)$. Furthermore, $k_{i, \lambda, Q_1, \mu}(H) = \lim_{y \to H^-} k_{i, \lambda, Q_1, \mu}(y)$ exists for $i = 1, 2$.

**Proof** Denote $\psi = \psi_{\lambda, Q_1, \mu}$ and $k_i(y) = k_{i, \lambda, Q_1, \mu}(y)$ ($i = 1, 2$) for simplicity. We consider only the continuity of $k_2(y)$ in the following, and the continuity of $k_1(y)$ can be obtained by similar arguments.

As mentioned in Remark 2.2, the free boundary $\Gamma_{2, \lambda, Q_1, \mu}$ lies above $y = h_2$. First, we will show that the one side limit of $k_2(y)$ exists as $y \to y_0^+$ or $y \to y_0^-$ for any $y_0 \in (h_2, H)$.

Suppose that there exists a $y_0 \in (h_2, H)$, such that there are two limits $\alpha_1$ and $\alpha_2$ with $\alpha_1 < \alpha_2$ as $y \to y_0^-$, that is, the free boundary $\Gamma_{2, \lambda, Q_1, \mu}$ oscillates as $y \to y_0^-$. 

\begin{center}
\includegraphics[width=\textwidth]{fig9}
\end{center}

*Fig. 9* The domain $E$
The monotonicity $\psi(x, y)$ with respect to $x$ implies that there exist two sequences $\{y_n\}_{n=1}^{\infty}$ and $\{\tilde{y}_n\}_{n=1}^{\infty}$ with $y_n < \tilde{y}_n < y_{n+1}$ for $n \geq 1$, such that $y_n \to y_0^−$, $\tilde{y}_n \to y_0^+$ and

$$\psi(x, y_n) = Q \quad \text{and} \quad \psi(x, \tilde{y}_n) < Q,$$

(2.28)

for $\delta_1 < x < \delta_2$, where $\delta_1 = \frac{3\alpha_1 + 3\alpha_2}{4}$ and $\delta_2 = \frac{\alpha_1 + 3\alpha_2}{4}$.

Let $E_n \subset \Omega_{\mu} \cap \{\psi < Q\}$ be a domain, bounded by the arcs $x = \delta_1, x = \delta_2, y = \tilde{y}_n(x)$ and $y = y_n(x)$, where $(x, y_n(x))$ and $(x, \tilde{y}_n(x))$ are free boundary points of $\Gamma_{\delta_1}, Q_{\mu_1}, \mu_{\delta_1}$ with $y_n(x) < \tilde{y}_n(x)$. The existence of the domain $E_n$ follows from (2.28). It is easy to check that

$$\varepsilon_n = \sup_{\delta_1 < x < \delta_2} \{\tilde{y}_n(x) - y_n(x)\} \to 0 \quad \text{as} \quad n \to \infty.$$

The Lipschitz continuity of $\psi$ gives that

$$\psi > 0 \quad \text{in} \quad E_n \quad \text{for sufficiently large} \quad n.$$

Therefore, by the non-oscillation Lemma 2.6 in $E_n$ for sufficiently large $n$, one has

$$0 < \frac{\alpha_2 - \alpha_1}{2} = \delta_2 - \delta_1 \leq C\varepsilon_n \quad \text{for sufficiently large} \quad n,$$

which leads to a contradiction.

Thus, $\lim_{y \to y_0^−} k_2(y)$ exists for any $y_0 \in (h_2, H)$. Similar arguments yield the existence of $\lim_{y \to H^−} k_2(y)$, denoted by $k_2(H)$, and the existence of $\lim_{y \to y_0^+} k_2(y)$ for any $y_0 \in (h_2, H)$.

Finally, we will show that $k_2(y)$ is a continuous function in $(h_2, H)$. Denote

$$k_2(y_0 + 0) = \lim_{y \to y_0^+} k_2(y) \quad \text{and} \quad k_2(y_0 - 0) = \lim_{y \to y_0^-} k_2(y),$$

and it suffices to show that

$$k_2(y_0 + 0) = k_2(y_0 - 0) = k_2(y_0) \quad \text{for any} \quad y_0 \in (h_2, H).$$

Suppose that there exists a $y_0 \in (h_2, H)$, such that $k_2(y_0 - 0) \neq k_2(y_0)$, and without loss of generality we assume $k_2(y_0 - 0) > k_2(y_0)$. By virtue of the monotonicity of $\psi$ with respect to $x$, one has that $y_0 = \{(x, y_0) | x_1 < x < x_2\}$ is a part of the free boundary $\Gamma_{\delta_1}, Q_{\mu_1}, \mu_{\delta_1}$, where

$$x_1 = \frac{k_2(y_0 - 0) + 3k_2(y_0)}{4} \quad \text{and} \quad x_2 = \frac{3k_2(y_0 - 0) + k_2(y_0)}{4}.$$

It follows from Proposition 2.2 that

$$\frac{\partial \psi(x, y_0 - 0)}{\partial y} = \sqrt{2\lambda - 2g_{y_0}} \quad \text{and} \quad \psi = Q \quad \text{on} \quad y_0.$$

The continuity of $\psi$ implies that there exists a small $\varepsilon > 0$ such that

$$0 < \psi < Q \quad \text{in} \quad \mathcal{E}_\varepsilon,$$

where $\mathcal{E}_\varepsilon = \{(x, y) | x_1 < x < x_2, y_0 - \varepsilon < y < y_0\}$.

It follows from the Cauchy-Kovalevskaya theorem and the unique continuation that

$$\psi(x, y) = \sqrt{2\lambda - 2g_{y_0}(y - y_0)} + Q \quad \text{in} \quad \mathcal{E}_\varepsilon,$$

where $\mathcal{E}_\varepsilon = \{ -\infty < x < +\infty, y_0 - \varepsilon < y < y_0\} \cap \Omega_{\mu}$, which leads to a contradiction to the boundary condition of $\psi$. □
With the aid of Lemma 2.7, the free boundary can be denoted as
\[ \Gamma_{i,\lambda, Q_1, \mu} = \{(x, y) \in D_\mu \mid x = k_{i,\lambda, Q_1, \mu}(y) \text{ for } h_i < y < H_i, i = 1, 2. \]

Up to now, one can conclude that for any \( Q_1 \in [0, Q] \) and \( \lambda \geq \frac{\max(Q_1^2, (Q - Q_1)^2)}{2H^2} + gH \), there exists a unique minimizer \( \psi_{\lambda, Q_1, \mu} \) to the variational problem \( (P_{\lambda, Q_1, \mu}) \) with the smooth free boundaries \( \Gamma_{1,\lambda, Q_1, \mu} \) and \( \Gamma_{2,\lambda, Q_1, \mu} \).

### 2.6 Almost continuous fit conditions of the free boundaries

Next, we will verify the continuous fit conditions between the rigid boundaries and the free boundaries. Namely, there exists an appropriate pair \( (\bar{\lambda}_\mu, \bar{Q}_{1,\mu}) \) to guarantee the continuous fit conditions
\[
\begin{align*}
k_{1,\bar{\lambda}_\mu, \bar{Q}_{1,\mu}}(H) &= -1 \quad \text{and} \quad k_{2,\bar{\lambda}_\mu, \bar{Q}_{1,\mu}}(H) = 1 \quad \text{for any} \quad \mu > 1.
\end{align*}
\]

This subsection, we only give almost continuous fit conditions (in Proposition 2.12) for \( \psi_{\lambda, Q_1, \mu} \) in the truncated domain \( \Omega_{\mu} \), and the continuous fit conditions for the minimizer in the whole fluid field will be verified in the next section.

This is one of key points and the essential difference to the asymmetric incompressible jets with gravity in [3]. In [3], Alt, Caffarelli and Friedman showed that there does not exist a jet under gravity from an asymmetric nozzle with infinite height, such that the continuous fit conditions hold. In other words, in general, the asymmetric free boundaries cannot connect the endpoints of the nozzle walls at the same time. Here, a new observation is that assuming the asymmetric nozzle is of finite height, we can obtain the existence result on jets under gravity with continuous fit conditions.

The desired result will be shown by the following facts.

**Fact 1.** The minimizer \( \psi_{\lambda, Q_1, \mu} \) and the free boundaries \( x = k_{i,\lambda, Q_1, \mu}(y) \) depend continuously on \( \lambda \) and \( Q_1 \).

**Fact 2.** For any \( Q_1, Q_1' \in [0, Q] \) with \( Q_1 > Q_1' \), it holds that
\[
\psi_{\lambda, Q_1, \mu}(x, y) \geq \psi_{\lambda, Q_1', \mu}(x, y) \quad \text{for any} \quad (x, y) \in \Omega_{\mu}.
\]

This fact indeed implies that the free boundary \( x = k_{i,\lambda, Q_1, \mu}(y) \) is decreasing with respect to \( Q_1, i = 1, 2 \).

**Fact 3.** For \( Q_1 = \frac{Q}{2} \), there exists some \( \lambda > \frac{Q^2}{8H^2} + gH \), such that
\[
k_{1,\lambda, Q_1, \mu}(H) < -1 \quad \text{and} \quad k_{2,\lambda, Q_1, \mu}(H) > 1. \tag{2.29}
\]

This fact implies that the following set is non-empty:
\[
\Sigma_{\mu} = \{\lambda \mid \text{there exists a } Q_1 \in (0, Q), \text{ such that(2.29) holds}\}.
\]

**Fact 4.** The set \( \Sigma_{\mu} \) is uniformly bounded for any \( \mu > 1 \).

**Fact 5.** Define \( \bar{\lambda}_\mu = \sup_{\lambda \in \Sigma_{\mu}} \lambda. \) There exists a \( \bar{Q}_{1,\mu} \in [0, Q] \), such that the free boundaries \( \Gamma_{1,\lambda, Q_1, \mu} \) and \( \Gamma_{2,\lambda, Q_1, \mu} \) satisfy the almost continuous fit conditions (in Proposition 2.12).

First, we show that the minimizer and the free boundaries depend continuously on \( \lambda \) and \( Q_1 \), and complete the proof of Fact 1.

**Lemma 2.8** For any sequences \( \{\lambda_n\} \) and \( \{Q_{1,n}\} \) with
\[
Q_{1,n} \in [0, Q] \quad \text{and} \quad \lambda_n \geq \frac{\max(Q_{1,n}^2, (Q - Q_{1,n})^2)}{2H^2} + gH,
\]

...
If $\lambda_n \to \lambda$ and $Q_{1,n} \to Q_1$, then
\[ \psi_{\lambda_n, Q_{1,n}} \to \psi_{\lambda, Q_1, \mu} \] uniformly in any compact subset of $\Omega_\mu$.

and
\[ k_{1, \lambda_n, Q_{1,n}, \mu}(H) \to k_{1, \lambda, Q_1, \mu}(H) \] and
\[ k_{2, \lambda_n, Q_{1,n}, \mu}(H) \to k_{2, \lambda, Q_1, \mu}(H). \]

\[ \text{Proof} \] Denote $\psi_n = \psi_{\lambda_n, Q_{1,n}, \mu}$ and $k_{i,n}(y) = k_{i, \lambda_n, Q_{1,n}, \mu}(y) \ (i = 1, 2)$ for simplicity. Thanks to Lemma 2.3, one has
\[ |\nabla \psi_n(X)| \leq C(\sqrt{\lambda_n} + Q) \leq C(\sqrt{2\lambda} + Q) \] in any compact subset $K$ in $\Omega_\mu$,
where the constant $C$ depends only on $K$ and $\Omega_\mu$, provided that $n$ is sufficiently large. Then there exists a subsequence still labeled as $\{\psi_n\}$ such that
\[ \psi_n \to \omega \] weakly in $H^1_{\text{loc}}(\mathbb{R}^2)$ and $\psi_n \to \omega$ in $C^\alpha(\bar{E})$,
(2.30) for any compact subset $E$ of $\Omega_\mu$ and $0 < \alpha < 1$.

Step 1. $D_\mu \cap \partial \{0 < \psi_n < Q\} \to D_\mu \cap \partial \{0 < \omega < Q\}$ in the Hausdorff distance, where
the Hausdorff distance $d(E, F)$ between two sets $E$ and $F$ is defined as follows,
\[ d(E, F) = \inf \left\{ \varepsilon > 0 \mid E \subset \bigcup_{X \in F} B_\varepsilon(X) \text{ and } F \subset \bigcup_{X \in E} B_\varepsilon(X) \right\}. \]

For any $X_0 \in D_\mu$, if $X_0 \neq D_\mu \cap \partial \{0 < \omega < Q\}$, then there exists a small $r \geq 0$, with $0 < r < \frac{1}{2} \text{dist}(X_0, \partial \Omega_\mu)$, such that $B_r(X_0) \subset D_\mu$ and $B_r(X_0) \cap \partial \{0 < \omega < Q\} = \emptyset$. We next claim that
\[ B_{\frac{2r}{\sqrt{2}}} (X_0) \cap \partial \{0 < \psi_n < Q\} = \emptyset \] for sufficiently large $n$.
(2.31)
In fact, it follows from (2.30) that the claim (2.31) is true for $0 < \omega < Q$ in $B_r(X_0)$. On another hand, if $\omega \equiv Q$ in $B_r(X_0)$, for any small $\varepsilon > 0$, the Lipschitz continuity of $\psi_n$ in Lemma 2.3 gives that there exists a positive integer $N(\varepsilon)$, such that
\[ \psi_n > 0 \quad \text{and} \quad |Q - \psi_n(X)| < \varepsilon \text{ in } B_{\frac{2r}{\sqrt{2}}} (X_0) \text{ for any } n > N(\varepsilon). \]

Then we have
\[ \frac{2}{r} \int_{\partial B_{\frac{2r}{\sqrt{2}}} (X_0)} (Q - \psi_n) dS < \frac{2\varepsilon}{r} \leq c^* \sqrt{2\lambda_n - 2g\overline{H}} \] for sufficiently large $n$,
where the constant $c^*$ is same as in Lemma 6.2. Therefore, Lemma 6.2 implies that $\psi_n \equiv Q$ in $B_{\frac{2r}{\sqrt{2}}} (X_0)$ for sufficiently large $n$, and thus the claim (2.31) is true. Similarly, one can obtain (2.31), provided that $\omega \equiv 0$ in $B_r(X_0)$.

Reverse, for any $X_0 \in D_\mu$, if $X_0 \neq D_\mu \cap \partial \{0 < \psi_n < Q\}$ for sufficiently large $n$, then there exists a small $r \geq 0$, with $0 < r < \frac{1}{2} \text{dist}(X_0, \partial \Omega_\mu)$, such that $B_r(X_0) \cap \partial \{0 < \omega < Q\} = \emptyset$. Next, we claim that
\[ B_{\frac{2r}{\sqrt{2}}} (X_0) \cap \partial \{0 < \omega < Q\} = \emptyset. \]
(2.32)
If $0 < \psi_n < Q$ in $B_r(X_0)$ for a sequence $\{\psi_n\}$, then
\[ \Delta \psi_n = 0 \text{ in } B_r(X_0), \]
which implies that
\[ \Delta \omega = 0 \text{ and } 0 \leq \omega \leq Q \text{ in } B_{\frac{2r}{\sqrt{2}}} (X_0). \]
The strong maximum principle yields that

\[ \text{either } \omega \equiv Q \text{ or } \omega \equiv 0 \text{ or } 0 < \omega < Q \text{ in } B_{r}(X_{0}), \]

which gives the claim (2.32).

If \( \psi_{n} \equiv Q \) or \( \psi_{n} = 0 \) in \( B_{r}(X_{0}) \) for a sequence \( \{\psi_{n}\} \), it is easy to see that the claim (2.32) is true.

Hence, we have shown the convergence of the free boundary in the Hausdorff distance. Step 2. \( \chi(0 < \psi_{n} < Q) \rightarrow \chi(0 < \psi_{n} < Q) \) in \( L^{1}(D_{\mu}) \).

For any \( X_{n} \in K \subseteq D_{\mu} \) with \( X_{n} \in \partial\{0 < \psi_{n} < Q\} \), it follows from the results in Step 1 that there exists a subsequence \( \{X_{n}\} \), such that \( X_{n} \to X_{0} \in D_{\mu} \cap \partial\{0 < \psi_{n} < Q\} \).

Since \( \text{dist}(X_{n}, A_{i}) \geq \text{dist}(K, \partial\Omega_{\mu}) \) for \( i = 1, 2 \), the Lipschitz continuity of \( \psi_{n} \) in Lemma 2.3 implies that there exists a small \( r_{0} \) with \( 0 < r_{0} < \frac{1}{2} \text{dist}(K, \partial D_{\mu}) \), such that

\[ \psi_{n} > 0 \text{ in } B_{r}(X_{n}) \text{ for } X_{n} \in D_{\mu} \cap \partial\{\psi_{n} < Q\}, \tag{2.33} \]

and

\[ \psi_{n} < Q \text{ in } B_{r}(X_{n}) \text{ for } X_{n} \in D_{\mu} \cap \partial\{\psi_{n} > 0\}, \tag{2.34} \]

for any \( r \in (0, r_{0}) \). With the aid of (2.33) and (2.34), it follows from Lemma 6.1 and Lemma 6.2 for \( \psi_{n} \) that

\[ c^{*}\sqrt{2\lambda_{n} - 2gH} \leq \frac{1}{r} \int_{\partial B_{r}(X_{n})} (Q - \psi_{n})dS \leq C^{*}\sqrt{2\lambda_{n}} \text{ for } X_{n} \in D_{\mu} \cap \partial\{\psi_{n} < Q\}, \tag{2.35} \]

and

\[ c^{*}\sqrt{2\lambda_{n} - 2gH} \leq \frac{1}{r} \int_{\partial B_{r}(X_{n})} \psi_{n}dS \leq C^{*}\sqrt{2\lambda_{n}} \text{ for } X_{n} \in D_{\mu} \cap \partial\{\psi_{n} > 0\} \tag{2.36} \]

for any \( r \in (0, r_{0}) \), where the constant \( C^{*} \) and \( c^{*} \) are uniform constants as in Lemma 6.1 and Lemma 6.2, respectively.

Then taking \( n \to +\infty \) in (2.35) and in (2.36), one has

\[ c^{*}\sqrt{2\lambda - 2gH} \leq \frac{1}{r} \int_{\partial B_{r}(X_{n})} (Q - \omega)dS \leq C^{*}\sqrt{2\lambda} \text{ for } X_{0} \in D_{\mu} \cap \partial\{\omega < Q\}, \tag{2.37} \]

and

\[ c^{*}\sqrt{2\lambda - 2gH} \leq \frac{1}{r} \int_{\partial B_{r}(X_{n})} \omega dS \leq C^{*}\sqrt{2\lambda} \text{ for } X_{0} \in D_{\mu} \cap \partial\{\omega > 0\}, \tag{2.38} \]

for any \( r \in (0, r_{0}) \). Those together with Theorem 4.5 in [1] imply that

\[ \mathcal{H}^{1}(D_{\mu} \cap \partial\{0 < \omega < Q\}) < +\infty, \]

where \( \mathcal{H}^{1} \) is the one-dimensional Hausdorff measure on \( \mathbb{R}^{2} \). Consequently,

\[ \mathcal{L}^{2}(D_{\mu} \cap \partial\{0 < \omega < Q\}) = 0, \tag{2.39} \]

where \( \mathcal{L}^{2} \) is the two-dimensional Lebesgue measure on \( \mathbb{R}^{2} \).

Let \( O_{\epsilon_{n}}^{1} \) be an \( \epsilon_{n} \)-neighborhood of \( D_{\mu} \cap \partial\{\omega > 0\} \) and \( O_{\epsilon_{n}}^{2} \) be an \( \epsilon_{n} \)-neighborhood of \( D_{\mu} \cap \partial\{\omega < Q\} \), such that

\[ D_{\mu} \cap \partial\{\psi_{n} > 0\} \subset O_{\epsilon_{n}}^{1} \text{ and } D_{\mu} \cap \partial\{\psi_{n} < Q\} \subset O_{\epsilon_{n}}^{2}, \]
and
\[ \mathcal{L}^2(D_\mu \cap O_{\varepsilon_n}) \to 0 \quad \text{as } \varepsilon_n \to 0, \quad O_{\varepsilon_n} = O_{\varepsilon_1}^1 \cup O_{\varepsilon_2}^2, \] (2.40)
due to \( D_\mu \cap \partial \{0 < \psi_n < Q\} \to D_\mu \cap \partial \{0 < \omega < Q\} \) in the Hausdorff distance in Step 1. Hence, one has
\[ \int_{D_\mu} \left| \chi_{\{0 < \psi_n < Q\}} - \chi_{\{0 < \omega < Q\}} \right| \, dx \, dy \leq \int_{D_\mu \cap O_{\varepsilon_n}} 1 \, dx \, dy = \mathcal{L}^2(D_\mu \cap O_{\varepsilon_n}), \] (2.41)
for sufficiently large \( n \), which together with (2.40) gives that
\[ \chi_{\{0 < \psi_n < Q\}} \to \chi_{\{0 < \omega < Q\}} \text{ in } L^1(D_\mu). \]

**Step 3.** \( \nabla \psi_n \to \nabla \omega \) a.e. in \( \Omega_\mu \).

Let \( E \) be any compact subset of \( \Omega_\mu \cap \{0 < \omega < Q\} \). It follows from the result in Step 1 that the minimizer \( \psi_n \) solves Laplace equation in \( E \) for sufficiently large \( n \). Thanks to the standard elliptic estimates for \( \psi_n \), one has
\[ \nabla \psi_n \to \nabla \omega \text{ uniformly in } E. \] (2.42)

Next, we will show that
\[ \nabla \psi_n \to \nabla \omega \text{ a.e. in } \Omega_\mu \cap \{\omega = Q\}. \] (2.43)

Since \( \Omega_\mu \cap \{\omega = Q\} \) is \( L^2 \)-measurable and \( L^2(\Omega_\mu \cap \partial \{\omega < Q\}) = 0 \), it follows from Corollary 3 in Section 1.7 in [12] that
\[ \lim_{r \to 0} \frac{\mathcal{L}^2(B_r(X) \cap \Omega_\mu \cap \{\omega = Q\})}{\mathcal{L}^2(B_r(X))} = 1 \text{ for } L^2 \text{ a.e. } X \in \Omega_\mu \cap \{\omega = Q\}. \]

Denote
\[ S = \left\{ X \in \Omega_\mu \cap \{\omega = Q\} \mid \lim_{r \to 0} \frac{\mathcal{L}^2(B_r(X) \cap \Omega_\mu \cap \{\omega = Q\})}{\mathcal{L}^2(B_r(X))} = 1 \right\}. \]

Then we claim that
\[ \frac{Q - \omega(X_0 + X)}{|X|} \to 0 \quad \text{as } |X| \to 0 \text{ for any } X_0 \in S. \] (2.44)

In fact, suppose that there exists an \( X_0 \in S \), such that \( Q - \omega(Y) > kr \) for some \( Y \in B_r(X_0) \) with \( r \to 0 \) and \( k > 0 \). With the aid of (2.37) and (2.38), it follows from Theorem 4.3 and Remark 4.4 in [1] that \( \omega \in C^{0,1}(\Omega_\mu) \), which implies that
\[ Q - \omega > \frac{k}{2} r \text{ in } B_{\varepsilon kr}(Y) \subset B_{2r}(X_0) \text{ for some small } \varepsilon > 0, \quad 0 < r < \frac{1}{4} \text{ dist}(X_0, \partial \Omega_\mu). \]

This gives that \( \Omega_\mu \cap \{\omega < Q\} \) has positive density at \( X_0 \), namely,
\[ \lim_{r \to 0} \frac{\mathcal{L}^2(B_{2r}(X_0) \cap \Omega_\mu \cap \{\omega < Q\})}{\mathcal{L}^2(B_{2r}(X_0))} > \frac{\varepsilon^2 k^2}{4}, \]
which contradicts to the fact \( X_0 \in S \).

With the aid of (2.30) and (2.44), for any \( \varepsilon > 0 \), one has
\[ \frac{Q - \psi_n}{r} < \varepsilon \text{ and } \psi_n > 0 \text{ in } B_r(X_0) \text{ for small } r \text{ with } 0 < r < \frac{1}{4} \text{ dist}(X_0, \partial \Omega_\mu), \]
provided that \( n \) is sufficiently large, that is \( n > N(\epsilon, r) \). It follows from the Lemma 6.2 in the “Appendix” that \( \psi_n \equiv Q \) in \( B_{\frac{r}{n}}(X_0) \), which implies that \( \omega \equiv Q \) in \( B_{\frac{r}{n}}(X_0) \). Thus, \( \mathcal{S} \) is open, and furthermore,

\[
\psi_n \equiv \omega \text{ in any compact subset of } \mathcal{S} \text{ for sufficiently large } n.
\]

This completes the proof of (2.43).

Similarly, one can show that

\[
\nabla \psi_n \rightarrow \nabla \omega \text{ a.e. in } \Omega_\mu \cap \{ \omega = 0 \}.
\]

Since \( \mathcal{L}^2(\Omega_\mu \cap \partial \{ 0 < \omega < Q \}) = 0 \), it holds that \( \nabla \psi_n \rightarrow \nabla \omega \text{ a.e. in } \Omega_\mu \).

**Step 4.** \( \omega = \psi_{\lambda, Q_1, \mu} \). Denote \( \psi = \psi_{\lambda, Q_1, \mu} \) in the proof for the notational simplicity.

First, we will check that \( \omega \) is a minimizer to the truncated variational problem \((P_{\lambda, Q_1, \mu})\).

It follows from (2.30) that \( \omega \in K_{\lambda, Q_1, \mu} \), and

\[
J_{\lambda, \mu}(\psi) \leq J_{\lambda, \mu}(\omega).
\]

It suffices to show that

\[
J_{\lambda, \mu}(\psi) \geq J_{\lambda, \mu}(\omega).
\]

(2.45)

For any \( \eta \in C_0^1(\Omega_\mu) \) with \( 0 \leq \eta \leq 1 \), set

\[
\phi_n = \psi + (1 - \eta)(\psi_n - \omega).
\]

It is easy to check that \( \phi_n \in K_{\lambda_n, Q_1, \mu} \) and

\[
\chi_{\{0 < \phi_n < Q\} \cap D_\mu} \leq \chi_{\{0 < \phi_n < Q\} \cap D_\mu} + \chi_{\{\eta < 1\} \cap D_\mu}.
\]

Hence,

\[
\int_{\Omega_\mu} |\nabla \psi_n|^2 + (2\lambda_n - 2g \eta) \chi_{\{0 < \phi_n < Q\} \cap D_\mu} \, dxdy
\]

\[
\leq \int_{\Omega_\mu} |\nabla \phi_n|^2 + (2\lambda_n - 2g \eta) \chi_{\{0 < \phi_n < Q\} \cap D_\mu} \, dxdy
\]

\[
\leq \int_{\Omega_\mu} |\nabla \phi_n|^2 + (2\lambda_n - 2g \eta) \chi_{\{0 < \phi_n < Q\} \cap D_\mu} \, dxdy + 2\lambda_n \int_{D_\mu} \chi_{\{\eta < 1\}} \, dxdy.
\]

(2.46)

Since \( \mathcal{L}^2(\Omega_\mu \cap \partial \{ 0 < \psi < Q \}) = 0 \), by virtue of the results in Step 2 and Step 3 and taking \( n \rightarrow +\infty \) in (2.46), one has

\[
\int_{\Omega_\mu} |\nabla \omega|^2 + (2\lambda - 2g \eta) \chi_{\{0 < \omega < Q\} \cap D_\mu} \, dxdy
\]

\[
\leq \int_{\Omega_\mu} |\nabla \psi|^2 + (2\lambda - 2g \eta) \chi_{\{0 < \psi < Q\} \cap D_\mu} \, dxdy + 2\lambda \int_{D_\mu} \chi_{\{\eta < 1\}} \, dxdy.
\]

(2.47)

Choosing \( \eta(X) = d_\epsilon(X) = \min\left(\frac{1}{\epsilon} \text{dist}(X, \mathbb{R}^2 \setminus D_\mu), 1\right) \) for \( \epsilon > 0 \) in (2.47) and taking \( \epsilon \rightarrow 0 \) yield (2.45). Thanks to the uniqueness of the minimizer to the truncated variational problem \((P_{\lambda, Q_1, \mu})\), one gets that \( \omega = \psi = \psi_{\lambda, Q_1, \mu} \).

**Step 5.** In this step, we will show the convergence of the free boundaries at the initial points, namely, \( k_{i,n}(H) \rightarrow k_{i,\lambda, Q_1, \mu}(H) \) as \( n \rightarrow +\infty \) (\( i = 1, 2 \)).
Suppose not, without loss of generality, we assume that there exist two subsequences still labeled by \( \{\lambda_n\} \) and \( \{Q_{1,n}\} \) such that 
\[
k_{2,n}(H) \to k_{2,\lambda_n,\phi_{1,\mu}}(H) + \delta \quad \text{with} \quad \delta \neq 0.
\]
Denote \( \psi = \psi_{\lambda,\phi_{1,\mu}} \) and \( k_2(H) = k_{2,\lambda_n,\phi_{1,\mu}}(H) \) for simplicity. There are three cases to be considered.

**Case 1.** \( \delta < 0 \). We consider two subcases in the following.

**Subcase 1.1.** \( k_2(H) + \delta \geq 1 \). Denote
\[
I_0 = \left\{(x, H) \mid k_2(H) + \frac{3\delta}{4} < x < \frac{k_2(H) + \frac{\delta}{4}}{2}\right\}
\]
and
\[
U_{\delta,\varepsilon} = \left\{(x, y) \mid k_2(H) + \frac{3\delta}{4} < x < \frac{k_2(H) + \frac{\delta}{4}}{2}, H - \varepsilon < y < H + \varepsilon\right\}
\]
for small \( \varepsilon > 0 \).

It is easy to check that \( I_0 = U_{\delta,\varepsilon} \cap \partial \{\psi < Q\} \). We extend \( \psi_n = Q \) and \( \psi = Q \) in \( U_{\delta,\varepsilon} \cap \Omega_{\mu} \). Since \( \text{dist}(A_i, U_{\delta,\varepsilon}) = -\frac{\delta}{2} \) \( (i = 1, 2) \), Lemma 2.3 gives that \( \psi_n \in C^{0,1}(U_{\delta,\varepsilon}) \). Obviously, \( 0 < \psi_n \leq Q \) in \( U_{\delta,\varepsilon} \) for small \( \varepsilon > 0 \).

Next, we show that 
\[
\Gamma_{2,\lambda_n,\phi_{1,\mu}} \cap \{y = H\} = \emptyset \quad \text{and} \quad U_{\delta,\varepsilon} \cap \partial \{\psi_n < Q\} \neq \emptyset \quad \text{for sufficiently large} \quad n.
\]

Noting that \( k_{2,n}(H) \to k_{2,\lambda_n,\phi_{1,\mu}}(H) + \delta \geq 1 \) with \( \delta < 0 \), then \( k_{2,n}(H) \leq k_{2,\lambda_n,\phi_{1,\mu}}(H) + \frac{\delta}{2^n} \) for sufficiently large \( n \). The definition of \( k_{2,n}(H) \) implies that \( \Gamma_r(k_{2,n}(H), H) \cap \Gamma_{2,\lambda_n,\phi_{1,\mu}} \neq \emptyset \) for any \( r > 0 \), which yields that \( \Gamma_{2,\lambda_n,\phi_{1,\mu}} \cap \{y = H\} = \emptyset \), due to that the free boundary \( \Gamma_{2,\lambda_n,\phi_{1,\mu}} \) is a y-graph. Recalling that \( \psi_n \to \psi \) in \( U_{\delta,\varepsilon} \) and \( \psi < Q \) in \( U_{\delta,\varepsilon} \cap \{y < H\} \), it follows from \( k_{2,n}(H) \leq k_{2,\lambda_n,\phi_{1,\mu}}(H) + \frac{\delta}{2^n} \) that \( U_{\delta,\varepsilon} \cap \partial \{\psi_n < Q\} \neq \emptyset \) for sufficiently large \( n \).

Since the right free boundary of \( \psi_n \) is analytic, it holds that \( U_{\delta,\varepsilon} \cap \partial \{\psi_n < Q\} \) is \( C^{1,\alpha} \) \( (0 < \alpha < 1) \), and
\[
\Delta \psi_n = 0 \text{ in } U_{\delta,\varepsilon} \cap \{\psi_n < Q\}, \quad \frac{\partial \psi_n}{\partial v} = \sqrt{2\lambda_n - 2g\psi_n} \text{ on } U_{\delta,\varepsilon} \cap \partial \{\psi_n < Q\},
\]
where \( v \) is the outer normal. Moreover, the \( C^{1,\alpha} \)-norm of \( U_{\delta,\varepsilon} \cap \partial \{\psi_n < Q\} \) is independent of \( n \). In fact, for \( \phi_n = \frac{Q - \psi_n}{\sqrt{2g\lambda_n}} \), it is easy to check that
\[
\phi_n = 0 \text{ and } 1 - \frac{4gH}{3\lambda_n} \leq |\nabla \phi_n| = \sqrt{1 - \frac{2g\psi_n}{\lambda_n}} \leq 1 \text{ on } U_{\delta,\varepsilon} \cap \partial \{\psi_n < Q\},
\]
for sufficiently large \( n \). Applying the results in Section 8 in [1], we can conclude that the free boundary \( U_{\delta,\varepsilon} \cap \partial \{\psi_n < Q\} \) is analytic and the uniform estimate of \( |\nabla \phi_n| \) gives that the \( C^{1,\alpha} \)-norm of \( U_{\delta,\varepsilon} \cap \partial \{\psi_n < Q\} \) is independent of \( n \).

Furthermore, by virtue of the results in previous steps, one has
\[
\psi_n \to \psi \quad \text{uniformly in } U_{\delta,\varepsilon}, \quad \nabla \psi_n \to \nabla \psi \quad \text{weakly in } L^2(U_{\delta,\varepsilon}),
\]
and
\[
U_{\delta,\varepsilon} \cap \{\psi_n < Q\} \to U_{\delta,\varepsilon} \cap \{\psi < Q\} \text{ in } L^2 \text{ measure, } I_0 = U_{\delta,\varepsilon} \cap \partial \{\psi < Q\}.
\]
Therefore, we can apply the convergence of free boundaries in Lemma 6.1 in Chapter 3 in [13] to obtain

\[ \Delta \psi = 0 \text{ in } U_{\delta,e} \cap \{ \psi < Q \}, \]

\[ \psi = Q \text{ and } \frac{\partial \psi}{\partial y} = \sqrt{2\lambda - 2 g H} \text{ on } I_0 = U_{\delta,e} \cap \partial \{ \psi < Q \}, \]

which implies that

\[ \frac{\partial \psi(x, H - 0)}{\partial y} = \sqrt{2\lambda - 2 g H} \text{ on } I_0. \]

It follows from the Cauchy-Kovalevskaya theorem for \( \psi \) in \( U_{\delta,e} \cap \{ y < H \} \) that

\[ \psi = \sqrt{2\lambda - 2 g H} (y - H) + Q \text{ in } U_{\delta,e} \cap \{ y < H \}. \]

By using the unique continuation for the harmonic function, one has

\[ \psi = \sqrt{2\lambda - 2 g H} (y - H) + Q \text{ in } \{(x, y) | -\infty < x < +\infty, H - \varepsilon < y < H\} \cap \Omega_\mu, \]

which is impossible.

**Subcase 1.2.** \( k_2(H) + \delta < 1 \). The monotonicity of \( \psi(x, y) \) gives that \( \psi(x, H) = Q \) for \( k_2(H) + \delta \leq x \leq \min\{k_2(H), 1\} \). Denote \( I_0 = \{(x, H) | k_2(H) + \delta + \varepsilon < x < k_2(H) + \delta + 2\varepsilon \} \) for \( \varepsilon = \frac{\min\{1-k_2(H),0\}-\delta}{3} \). Similar to Subcase 1.1, by virtue of the convergence of the free boundary of \( \psi_n \), one has

\[ \frac{\partial \psi(x, H + 0)}{\partial y} = \sqrt{2\lambda - 2 g H} \text{ on } I_0, \]

which also leads to a contradiction by using the Cauchy-Kovalevskaya theorem.

**Case 2.** \( \delta > 0 \) and \( k_2(H) < 1 \). It follows from the arguments in Lemma 5.6 (i) in [4] that

\[ \frac{\partial \psi_n(x, H - 0)}{\partial y} \geq \sqrt{2\lambda - 2 g H} \text{ on } \left\{ k_2(H) + \frac{\delta}{4} < x < k_2(H) + \frac{3\delta}{4}, y = H \right\}, \]

for sufficiently large \( n \).

Let \( E_n \) be the domain bounded by \( y = H, x = k_{2,n}(y), x = k_2(H) + \frac{\delta}{4} \text{ and } x = k_2(H) + \frac{3\delta}{4} \). Since \( k_{2,n}(H) \rightarrow k_2(H) + \delta \) with \( \delta > 0 \), one has \( k_{2,n}(H) \geq k_2(H) + \frac{7\delta}{8} \) for sufficiently large \( n \). Denote \( y_\delta = \max \{ y | k_2(y) = k_2(H) + \frac{\delta}{8} \} \). It follows from \( D_\mu \cap \partial \{0 < \psi_n < Q\} \rightarrow D_\mu \cap \partial \{0 < \omega < Q\} \) in the Hausdorff distance in Step 1 that

\[ D_\delta \cap \partial \{\psi_n < Q\} \rightarrow D_\delta \cap \partial \{\omega < Q\} \]

in the Hausdorff distance,

where \( D_\delta = D_\mu \cap \{k_2(H) < x < k_2(H) + \delta\} \). This implies that

\[ k_{2,n}(y_\delta) \rightarrow k_{2,H,Q_1,\mu}(y_\delta) = k_2(H) + \frac{\delta}{8}. \]

Thus, one has \( \Gamma_{2,\lambda,n,Q_1,\mu} \cap \{ x = k_2(H) + \frac{\delta}{4} \} \neq \emptyset \) for sufficiently large \( n \). Therefore, the domain \( E_n \) is well-defined.
Furthermore, it follows from $\psi_n \to \psi$ and $\psi = Q$ on $L_{2,\mu}$ that
\[
y_n = \max \left\{ y \mid k_{2,n}(y) = k_2(H) + \frac{3\delta}{4} \right\} \to H \quad \text{as} \quad n \to +\infty.
\]
Thanks to the non-oscillation Lemma 2.6 and Remark 2.3 for $\psi_n$ in $E_n$, there exists a constant $C$ independent of $n$, such that
\[
0 < \frac{\delta}{2} \leq C(H - y_n),
\]
which gives a contradiction for sufficiently large $n$. \hfill \Box

Second, the monotonicity of the minimizer $\psi_{\lambda, Q_1, \mu}$ with respect to the parameter $Q_1$ is obtained as follows.

**Lemma 2.9** For any $Q_1, Q'_1 \in [0, Q]$ with $Q_1 > Q'_1$, it holds that
\[
\psi_{\lambda, Q_1, \mu}(x, y) \geq \psi_{\lambda, Q'_1, \mu}(x, y) \quad \text{in} \quad \Omega_{\mu}.
\]

**Proof** It is easy to check that
\[
\min\{\psi_{\lambda, Q_1, \mu}, \psi_{\lambda, Q'_1, \mu}\} \in K_{\lambda, Q'_1, \mu} \quad \text{and} \quad \max\{\psi_{\lambda, Q_1, \mu}, \psi_{\lambda, Q'_1, \mu}\} \in K_{\lambda, Q_1, \mu}.
\]
Since $\psi_{\lambda, Q_1, \mu} = Q'_1 < Q_1 = \psi_{\lambda, Q_1, \mu}$ on $N_{\mu}$, it follows from the similar arguments in Proposition 2.5 that
\[
\psi_{\lambda, Q_1, \mu}(x, y) \geq \psi_{\lambda, Q'_1, \mu}(x, y) \quad \text{in} \quad \Omega_{\mu}.
\]
\hfill \Box

Next, we will verify the Fact 3.

**Lemma 2.10** For $Q_1 = \frac{Q}{2}$, there exists a $\lambda > \frac{Q^2}{8H^2} + gH$, such that if $\lambda - \frac{Q^2}{8H^2} - gH$ is small, then
\[
k_{1,\lambda, Q_1, \mu}(H) < -1 \quad \text{and} \quad k_{2,\lambda, Q_1, \mu}(H) > 1.
\]

**Proof** Suppose not, without loss of generality, we assume that there exists a sequence $\{\lambda_n\}$ with $\lambda_n > \lambda_0 = \frac{Q^2}{8H^2} + gH$, such that $\lambda_n \downarrow \lambda_0$ and $k_{2,\lambda_n, Q_1, \mu}(H) \leq 1$.

Then Lemma 2.4 implies that
\[
\max \left\{ -\sqrt{2\lambda_n - 2gh_1,n + Q_1,0} \right\} \leq \psi_{\lambda_n, Q_1, \mu}(x, y) \leq \min \left\{ \sqrt{2\lambda_n - 2gh_2,n + Q_1}, Q \right\}
\]
in $D_{\mu}$, where $h_{1,n} = h_{2,n} \in (0, H]$ is determined uniquely by
\[
\lambda_n = \frac{Q^2}{8h_{1,n}^2} + gh_{1,n}.
\]
With the aid of Lemma 2.8, taking $n \to +\infty$ in above inequality gives that
\[
\frac{Q}{2} \left( 1 - \frac{y}{H} \right) = -\sqrt{2\lambda_0 - 2gHy + \frac{Q}{2}} \leq \psi_{\lambda_0, Q_1, \mu}(x, y) \leq \sqrt{2\lambda_0 - 2gHy + \frac{Q}{2}} = \frac{Q}{2} \left( 1 + \frac{y}{H} \right).
\]
in $D_\mu$, which implies that $0 < \psi_{\lambda_0, Q_1, \mu}(x, y) < Q$ in $D_\mu$, and thus both of the free boundaries $\Gamma_{1, \lambda_0, Q_1, \mu}$ and $\Gamma_{2, \lambda_0, Q_1, \mu}$ are empty in $D_\mu$. For the case $k_{2, \lambda_0, Q_1, \mu}(H) \leq 1$, we claim that the free boundary $\Gamma_{2, \lambda_0, Q_1, \mu}$ is non-empty. In fact, if $\Gamma_{2, \lambda_0, Q_1, \mu}$ is empty, namely, $\psi_{\lambda_0, Q_1, \mu} < Q$ in $D_\mu$. Since $k_{2, \lambda_0, Q_1, \mu}(H) \leq 1$, the definition of $k_{2, \lambda_0, Q_1, \mu}(H)$ gives that $B_r(k_{2, \lambda_0, Q_1, \mu}(H)) \cap \Gamma_{2, \lambda_0, Q_1, \mu} \neq \emptyset$ for any $r > 0$. Therefore, the free boundary $\Gamma_{2, \lambda_0, Q_1, \mu}$ is non-empty. This contradicts to our assumption that the free boundary $\Gamma_{2, \lambda_0, Q_1, \mu}$ is empty.

Denote $\psi = \psi_{\lambda_0, Q_1, \mu}$ and $\psi_n = \psi_{\lambda_n, Q_1, \mu}$ for simplicity. Set

$$I = \{(x, H) \mid \frac{3 + \mu}{4} < x < \frac{1 + 3\mu}{4}\}$$

and

$$U_\varepsilon = \{(x, y) \mid \frac{3 + \mu}{4} < x < \frac{1 + 3\mu}{4}, H - \varepsilon < y < H + \varepsilon\}$$

for small $\varepsilon > 0$.

Obviously, $I = U_\varepsilon \cap \partial(\psi < Q)$ and $\text{dist}(A_i, U_\varepsilon) \geq \frac{1}{4}$ $(i = 1, 2)$. Extend $\psi_n = Q$ and $\psi = Q$ in $U_\varepsilon \setminus \Omega_\mu$. It follows from Lemma 2.3 that $\psi_n \in C^{0,1}(\tilde{U}_\varepsilon)$ and $0 < \psi_n \leq Q$ in $U_\varepsilon$ for small $\varepsilon > 0$. The analyticity of $\Gamma_{2, \lambda_n, Q_1, \mu}$ implies that $U_\varepsilon \cap \partial(\psi_n < Q)$ is $C^{1, \alpha}$ ($0 < \alpha < 1$), and

$$\Delta\psi_n = 0 \text{ in } U_\varepsilon \cap \{\psi_n < Q\}, \quad \frac{\partial\psi_n}{\partial\nu} = \sqrt{2\lambda_n - 2g\nu} \text{ on } U_\varepsilon \cap \{\psi_n < Q\},$$

where $\nu$ is the outer normal. Moreover, following the similar arguments in the proof of Lemma 2.8, one has

$$\psi_n \to \psi \text{ uniformly in } U_\varepsilon, \quad \nabla\psi_n \rightharpoonup \nabla\psi \text{ weakly in } L^2(U_\varepsilon),$$

and

$$U_\varepsilon \cap \{\psi_n < Q\} \to U_\varepsilon \cap \{\psi < Q\} \text{ in } L^2 \text{ measure, } I = U_\varepsilon \cap \partial(\psi < Q).$$

Therefore, we can apply the convergence of free boundaries in Lemma 6.1 in Chapter 3 in [13] to get

$$\Delta\psi = 0 \text{ in } U_\varepsilon \cap \{\psi < Q\}, \quad \psi = Q \text{ and } \frac{\partial\psi(x, H - 0)}{\partial\nu} = \sqrt{2\lambda_n - 2g\nu} \text{ on } I = U_\varepsilon \cap \partial(\psi < Q).$$

It follows from the Cauchy-Kovalevskaya theorem for $\psi$ in $U_\varepsilon \cap \{y < H\}$ that

$$\psi = \sqrt{2\lambda_n - 2g\nu}(y - H) + Q \text{ in } U_\varepsilon \cap \{y < H\}.$$

By using the unique continuation for the harmonic function, one has

$$\psi = \sqrt{2\lambda_n - 2g\nu}(y - H) + Q \text{ in } \{(x, y) \mid -\infty < x < +\infty, H - \varepsilon < y < H\} \cap \Omega_\mu,$$

which is impossible. $\square$

The following lemma gives the Fact 4.

**Lemma 2.11** For any $Q_1 \in (0, Q)$, there exists a positive constant $C_0$ independent of $\mu$ and $Q_1$, such that

$$k_{1, \lambda_1, Q_1, \mu}(H) > -1 \text{ or } k_{2, \lambda, Q_1, \mu}(H) < 1,$$

where $k_{1, \lambda_1, Q_1, \mu}(H)$ and $k_{2, \lambda, Q_1, \mu}(H)$ are defined in (2.48).
for any $\lambda > C_0$.

**Proof** Suppose that there exist a $Q_1 \in (0, Q)$ and a large $\lambda$, such that

$$k_{1,\lambda}(H) \leq -1 \text{ and } k_{2,\lambda,Q_1}(H) \geq 1.$$  

Denote the two initial points of the free boundaries as $B_1 = (k_{1,\lambda}(H), H)$ and $B_2 = (k_{2,\lambda,Q_1}(H), H)$. Then one has

$$\frac{1}{r} \int_{\partial B_r(X_0)} \psi_{\lambda, Q_1} dS \leq \frac{Q}{r} \text{ for any } X_0 \in \Gamma_{1,\lambda, Q_1}.$$  

Let $r_0 = \frac{Q}{c^* \sqrt{\lambda - 2gH}}$ ($c^*$ as in Lemma 6.2). Then

$$\frac{1}{r_0} \int_{\partial B_{r_0}(X_0)} \psi_{\lambda, Q_1} dS \leq \frac{Q}{r_0} = c^* \sqrt{\lambda - 2gH} \text{ for any } X_0 \in \Gamma_{1,\lambda, Q_1}.$$  

We claim that

$$B_{r_0}(X_0) \cap \{\psi_{\lambda, Q_1, \mu} = Q\} \neq \emptyset \text{ for any } X_0 \in \Gamma_{1,\lambda, Q_1}.$$  

In fact, suppose that there exists an $X_0 \in \Gamma_{1,\lambda, Q_1}$, such that $B_{r_0}(X_0) \cap \{\psi_{\lambda, Q_1, \mu} = Q\} = \emptyset$, which implies that $\psi_{\lambda, Q_1, \mu} < Q$ in $B_r(X_0)$. The non-degeneracy Lemma 6.3 in the “Appendix” implies that $\psi_{\lambda, Q_1, \mu} \equiv 0$ in $B_{\bar r}(X_0)$, which contradicts to the fact $X_0 \in \Gamma_{1,\lambda, Q_1}$.

Thus it holds that

$$B_{2r_0}(B_1) \text{ intersects } \{\psi_{\lambda, Q_1, \mu} = Q\}.$$  

Similarly,

$$B_{2r_0}(B_2) \text{ intersects } \{\psi_{\lambda, Q_1, \mu} = 0\}.$$  

Without loss of generality, we assume that the free boundary $\Gamma_{2,\lambda, Q_1}$ lies above of the free boundary $\Gamma_{1,\lambda, Q_1}$ near the segment $A_1A_2$. Let $E \subset \Omega_{\mu}$ be the domain bounded by $x = \frac{1}{2}$, $x = -\frac{1}{2}$ and $\Gamma_{2,\lambda, Q_1, \mu}$ and $y = H_0$ (see Fig. 10), where $H_0 = \min \{y \mid g_2(y) = \frac{1}{2}\}$. Set $\Psi_{\lambda}(X) = \frac{Q - \psi_{\lambda, Q_1, \mu}(X)}{\sqrt{2\lambda}}$. It is easy to check that $\Psi_{\lambda}$ is harmonic in $E$ and

$$\Psi_{\lambda} = 0 \text{ and } \frac{1}{2} \leq |\nabla \Psi_{\lambda}| = \frac{\sqrt{2\lambda - 2gy}}{\sqrt{2\lambda}} \leq 1 \text{ on } \Gamma_{2,\lambda, Q_1, \mu},$$

for sufficiently large $\lambda$. By means of the results in Section 8 in [1], we can conclude that the free boundary $\Gamma_{2,\lambda, Q_1, \mu}$ of $\Psi_{\lambda}$ is analytic and the $C^3$-norm of $\partial E \cap \Gamma_{2,\lambda, Q_1, \mu}$ is independent of $\lambda$. Applying the elliptic estimate for harmonic function $\Psi_{\lambda}$ on the boundary $\partial E \cap \Gamma_{2,\lambda, Q_1, \mu}$ (see Corollary 6.7 in [15]), we have

$$\frac{1}{2} \leq |\nabla \Psi_{\lambda}| \leq C \max_{\chi \in E} |\Psi_{\lambda}(X)| \leq C \frac{Q}{\sqrt{2\lambda}} \text{ on } (\partial E \cap \Gamma_{2,\lambda, Q_1, \mu}) \cap \left\{-\frac{1}{4} \leq x \leq \frac{1}{4}\right\},$$

where the constant $C$ depends on $E$ and the $C^3$-norm of $\partial E \cap \Gamma_{2,\lambda, Q_1, \mu}$, and does not depend on $\lambda$. This leads to a contradiction for sufficiently large $\lambda$. \hfill $\Box$

Set

$$\bar \lambda_{\mu} = \sup_{\lambda \in \Sigma_{\mu}} \lambda.$$  

(2.49)
It follows from Lemma 2.11 that there exists a $C_0 > 0$ independent of $\mu$ and $Q_1$, such that
\[
\frac{\max\{Q_1^2, (Q - Q_1)^2\}}{2H^2} + gH \leq \bar{\lambda}_\mu \leq C_0.
\] (2.50)

Next, we can establish the almost continuous fit conditions to the impinging jet flow under gravity.

**Proposition 2.12** (almost continuous fit conditions) For any $\mu > 1$, there exist $\tilde{Q}_{1,\mu} \in [0, Q]$ and $\tilde{\lambda}_\mu \geq \frac{\max\{Q_1^2, (Q - Q_1)^2\}}{2H^2} + gH$, such that

1. $k_1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu,\mu}(H) \leq -1$ and $k_2, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu,\mu}(H) \geq 1$.
2. $k_1, \tilde{\lambda}_\mu, , \tilde{Q}_{1,\mu,\mu}(H) = -1$ or $k_2, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu,\mu}(H) = 1$.
3. $k_2, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu,\mu}(H) = 1$ for $\tilde{Q}_{1,\mu} < Q$.
4. $k_1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu,\mu}(H) = -1$ for $\tilde{Q}_{1,\mu} > 0$.

**Remark 2.4** Obviously, as long as the critical cases $\tilde{Q}_{1,\mu} = 0$ and $\tilde{Q}_{1,\mu} = Q$ are excluded, then the continuous fit conditions

\[
k_1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu,\mu}(H) = -1 \quad \text{and} \quad k_2, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu,\mu}(H) = 1
\]

are satisfied. This will be done in the next section.

**Proof** (1). By using the similar arguments in Lemma 2.8, taking a sequence $\{\lambda_n\}$ with $\lambda_n \in \Sigma_\mu$ and $\lambda_n \uparrow \tilde{\lambda}_\mu$, we can obtain from the definition of $\Sigma_\mu$ that there exists a sequence $\{Q_{1,n}\}$ with $Q_{1,n} \in (0, Q)$, such that
\[
k_1, \lambda_n, Q_{1,n,\mu}(H) < -1 \quad \text{and} \quad k_2, \lambda_n, Q_{1,n,\mu}(H) > 1.
\] (2.51)

It follows from the similar arguments in the proof of Lemma 2.8 that there exist two subsequences $\{\lambda_n\}$ and $\{Q_{1,n}\}$, such that
\[
\lambda_n \rightarrow \tilde{\lambda}_\mu, \quad Q_{1,n} \rightarrow \tilde{Q}_{1,\mu}.
\]
and

\[ \psi_{\lambda_n, Q_{1.n, \mu}} \rightarrow \psi_{\tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}} \] in \( H^1(\Omega_\mu) \) and uniformly in any compact subset of \( \Omega_\mu \), as \( n \rightarrow +\infty \). Furthermore,

\[ \lim_{n \rightarrow +\infty} k_1, \lambda_n, Q_{1.n, \mu}(H) = k_1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}(H) \quad \text{and} \quad \lim_{n \rightarrow +\infty} k_2, \lambda_n, Q_{1.n, \mu}(H) = k_2, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}(H). \]

Therefore, the assertion (1) follows from (2.51).

(2) Suppose that the assertion (2) is not true. Then it follows from the assertion (1) that

\[ k_1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}(H) < -1 \quad \text{and} \quad k_1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}(H) > 1. \] (2.52)

Due to the continuous dependence of \( k_{i, \lambda, Q_{1,\mu}}(H) \) on \( \lambda \) and \( Q_1 \) in Lemma 2.8, there exist \( \lambda' > \tilde{\lambda}_\mu \) and \( Q_1' \in (0, Q) \) with \( \lambda' - \tilde{\lambda}_\mu \) and \( |Q_{1,\mu} - Q_1'| \) small enough, such that

\[ k_1, \lambda', Q_1'(H) < -1 \quad \text{and} \quad k_2, \lambda', Q_1'(H) > 1. \]

Then we have that \( \lambda' \in \Sigma_{\mu} \), which contradicts to the definition of \( \tilde{\lambda}_\mu \) in (2.49).

(3) Suppose that the opposite is true. Then \( k_2, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}(H) > 1 \) for \( \tilde{Q}_{1,\mu} < Q \), and the assertion (2) implies that \( k_1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}(H) = -1 \). Due to Lemma 2.9, it holds that

\[ \psi_{\tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}}(x, y) \geq \psi_{\tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}}(x, y) \quad \text{in} \quad \Omega_\mu, \]

for any \( Q_1 \in (\tilde{Q}_{1,\mu}, Q) \). Denote

\[ \psi = \psi_{\tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}}, \quad \psi_1 = \psi_{\tilde{\lambda}_\mu, Q_{1,\mu}}(x, y), \quad k_1, \tilde{Q}_{1,\mu}(y) = k_1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}(y) \quad \text{and} \quad k_2, Q_1(y) = k_2, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}(y) \]

for \( i = 1, 2 \) in the following. Therefore, \( k_1, Q_1(H) \leq k_1, \tilde{Q}_{1,\mu}(H) = -1 \). Next, we claim that

\[ k_1, Q_1(H) < k_1, \tilde{Q}_{1,\mu}(H) = -1, \] (2.53)

for any \( Q_1 \in (\tilde{Q}_{1,\mu}, Q) \). Suppose that there exists a \( Q_1 \in (\tilde{Q}_{1,\mu}, Q) \) such that \( k_1, Q_1(H) = k_1, \tilde{Q}_{1,\mu}(H) \). It follows from the results in Section 9 in [4] and Section 11 in Chapter 3 in [13] that the continuous fit condition implies the smooth fit condition. Hence, \( N_{1, \mu} \cup \Gamma_{1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}} \) and \( N_{1, \mu} \cup \Gamma_{1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}} \) are \( C^1 \) at \( A_1 \). Furthermore, \( \nabla \psi \) is uniformly continuous in a \( \{\psi > 0\}\)-neighborhood of \( A_1 \), and \( \nabla \psi_1 \) is uniformly continuous in a \( \{\psi_1 > 0\}\)-neighborhood of \( A_1 \), namely,

\[ -\sqrt{2\lambda - 2gH} = \frac{\partial \psi}{\partial v} = \frac{\partial \psi_1}{\partial v} = -\sqrt{2\lambda - 2gH} \quad \text{at} \quad A_1, \]

where \( v \) is the outer normal vector. It should be noted that it’s difficult to verify the inner ball property at the point \( A_1 \), then one can not apply the Hopf’s lemma at \( A_1 \).

First, we show that

\[ \Gamma_{1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}} \cap \Gamma_{1, \tilde{\lambda}_\mu, Q_{1,\mu}} = \emptyset. \] (2.54)

Suppose not, there exists a \( X_0 = (x_0, y_0) \in \Gamma_{1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}} \cap \Gamma_{1, \tilde{\lambda}_\mu, Q_{1,\mu}} \). Since \( \Gamma_{1, \tilde{\lambda}_\mu, \tilde{Q}_{1,\mu, \mu}} \) and \( \Gamma_{1, \tilde{\lambda}_\mu, Q_{1,\mu}} \) are analytic at \( X_0 \), Hopf’s lemma gives that

\[ -\sqrt{2\lambda - 2g y_0} = \frac{\partial \psi_1}{\partial v} < \frac{\partial \psi}{\partial v} = -\sqrt{2\lambda - 2g y_0} \quad \text{at} \quad X_0, \]

where \( v \) is the outer normal vector, which leads to a contradiction.
The continuity of $\psi$ and $\psi_1$ implies that
\[ \psi < Q \text{ and } \psi_1 < Q \text{ in } B_r(A_1), \]
for any small $r > 0$. Let $\mathcal{M} = B_r(A_1) \cap N_{1,\mu}$. It follows from the strong maximum principle that
\[ \psi < \psi_1 \text{ in } B_r(A_1) \cap \{ \psi > 0 \}. \tag{2.55} \]
Since $N_{1,\mu}$ are $C^{2,\alpha}$ and $\psi = \psi_1 = 0$ on $\mathcal{M}$, thanks to Hopf’s lemma, one has
\[ \frac{\partial \psi_1}{\partial v} < \frac{\partial \psi}{\partial v} \text{ on } \mathcal{M}, \]
where $v$ is the outer normal vector of $\mathcal{M}$. It follows from (2.55) that there exists a small $\eta_1 > 0$, such that
\[ \psi_1 \geq (1 + \eta_1)\psi \text{ on } \partial B_r(A_1) \cap \{ \psi > 0 \}. \]
This, together with (2.54) and (2.55), implies that there exists a small $\eta \in (0, \eta_1)$ such that
\[ \psi_1 \geq (1 + \eta)\psi \text{ on } \partial (B_r(A_1) \cap \{ \psi > 0 \}). \]
It follows from the maximum principle that
\[ \psi_1 > (1 + \eta)\psi \text{ in } B_r(A_1) \cap \{ \psi > 0 \}, \]
which together with $\psi_1 = \psi = 0$ at $A_1$ gives that
\[ -\sqrt{2\lambda - 2gH} = \frac{\partial \psi_1}{\partial v} \leq (1 + \eta)\frac{\partial \psi}{\partial v} = -(1 + \eta)\sqrt{2\lambda - 2gH} \text{ at } A_1, \]
which leads to a contradiction. Thus, the claim (2.53) holds true.

Since $k_{2,\tilde{Q}_{1,\mu}}(H) > 1$, by using the continuous dependence of $k_{2,\tilde{\lambda}_{1,\mu},Q_1,\mu}(H)$ with respect to $Q_1$, we have
\[ k_{2,Q_1}(H) > 1 \text{ for } Q_1 > \tilde{Q}_{1,\mu} \text{ and } Q_1 - \tilde{Q}_{1,\mu} \text{ is small.} \tag{2.56} \]

In view of (2.53) and (2.56), we can obtain a contradiction to the definition of $\tilde{\lambda}_{1,\mu}$ by using the continuous dependence of $k_{i,Q_1}(H)$ with respect to $\lambda$ and $Q_1$.

(4) Similar to (3), one can show that
\[ \text{if } \tilde{Q}_{1,\mu} > 0, \text{ then } k_{1,\tilde{\lambda}_{1,\mu},\tilde{Q}_{1,\mu},\mu}(H) = -1. \]

\[ \square \]

3 The existence of the solution to the incompressible impinging flow problem

In this section, we will give the existence of the solution to the incompressible impinging flow problem based on the previous results.

Proposition 3.1 For any $Q > 2\sqrt{gH^3}$, there exist a pair $(\lambda, Q_1)$ and a solution $\psi_{\lambda,Q_1}$ to the free boundary problem (2.4) with $Q_1 \in [0, Q]$ and $\lambda \geq \max\{\tilde{Q}_1, (Q - Q_1)^2\} + gH$. Moreover,
(1) $\psi, Q_1(x, y)$ is increasing with respect to $x$ and the free boundary $\Gamma_{i, \lambda, Q_1}$ is analytic. Furthermore, the free boundary $\Gamma_{i, \lambda, Q_1}$ can be described by a continuous function $x = k_{i, \lambda, Q_1}(y)$ for $y \in (h_1, H)$, respectively, $i = 1, 2$.

(2) (almost continuous fit conditions) $k_{1, \lambda, Q_1}(H) \leq -1$ and $k_{2, \lambda, Q_1}(H) \geq 1$. Furthermore,

$$k_{1, \lambda, Q_1}(H) = -1 \text{ or } k_{2, \lambda, Q_1}(H) = 1,$$

and

$$k_{1, \lambda, Q_1}(H) = -1 \text{ if } Q_1 > 0, \text{ and } k_{2, \lambda, Q_1}(H) = 1 \text{ if } Q_1 < Q.$$ 

**Proof** Let $\{\mu_n\}$ be a sequence such that $\mu_n \to +\infty$. It follows from the similar arguments in the proof of Lemma 2.8 that

$$\tilde{\lambda}_{\mu_n} \to \lambda \text{ and } \tilde{Q}_{1, \mu_n} \to Q_1,$$

and

$$\psi_{\tilde{\lambda}_{\mu_n}, \tilde{Q}_{1, \mu_n}, \mu_n} \rightharpoonup \psi_{\lambda, Q_1} \text{ weakly in } H^1_{\text{loc}}(\Omega) \text{ and uniformly in any compact subset of } \Omega.$$ 

It follows from (2.50) that

$$\lambda \leq C_0. \quad (3.1)$$

Next, we will show that $\psi_{\lambda, Q_1}$ is a local minimizer to the variational problem $(P_{\lambda, Q_1})$ (see Remark 6.1). Set $\eta_n = \psi_{\tilde{\lambda}_{\mu_n}, \tilde{Q}_{1, \mu_n}, \mu_n}$. We first claim that

$$\int_E |\nabla \eta_n|^2 + (2\tilde{\lambda}_{\mu_n} - 2gy)\chi_{[0<\eta_n<Q]}|E|dxdy \leq \int_E |\nabla \tilde{\psi}|^2 + (2\tilde{\lambda}_{\mu_n} - 2gy)\chi_{[0<\tilde{\psi}<Q]}|E|dxdy, \quad (3.2)$$

for any $\tilde{\psi} \in H^1(E)$ and $\tilde{\psi} = \psi_n$ on $\partial E$, provided that $n$ is sufficiently large, where $E$ is any compact subset of $\Omega$. In fact, there exists a $N > 0$, such that $E \subset \Omega_{\mu_n}$ for any $n > N$. Extend $\tilde{\psi} = \psi_n$ outside of $E$ such that $\tilde{\psi} = \psi_{\tilde{\lambda}_{\mu_n}, \tilde{\lambda}_{\mu_n}, \mu_n}(y)$ on $\partial \Omega_{\mu_n}$, then it implies that (3.2) is valid.

With the aid of (3.2), by using the similar arguments in Step 4 in the proof of Lemma 2.8, one can conclude that $\psi_{\lambda, Q_1}$ is a local minimizer to the variational problem $(P_{\lambda, Q_1})$, and thus it follows from Proposition 2.2 that $\psi_{\lambda, Q_1}$ is harmonic in $\Omega \cap \{0 < \psi_{\lambda, Q_1} < Q\}$ and the free boundary $\Gamma_{i, \lambda, Q_1}$ is analytic with $|\nabla \psi_{\lambda, Q_1}| = \sqrt{2\lambda - 2gy}$ on $\Gamma_{i, \lambda, Q_1}$. Furthermore, Lemma 2.5 gives that the local minimizer $\psi_{\lambda, Q_1}(x, y)$ is increasing with respect to $x$. By using similar arguments as in Lemma 2.7, one can conclude that the free boundary $\Gamma_{i, \lambda, Q_1}$ of $\psi_{\lambda, Q_1}$ can be described by a generalized continuous function $x = k_{i, \lambda, Q_1}(y)$ ($i = 1, 2$), namely, $k_{i, \lambda, Q_1}(y + 0) = \lim_{t \to y+0} k_{i, \lambda, Q_1}(t)$ and $k_{i, \lambda, Q_1}(y - 0) = \lim_{t \to y-0} k_{i, \lambda, Q_1}(t)$ exist and may be infinite, and $k_{i, \lambda, Q_1}(y + 0) = k_{i, \lambda, Q_1}(y - 0) \in [-\infty, +\infty]$ for and $y \in (h_1, H)$.

Next, we will show that

$$k_{1, \lambda, Q_1}(H) = -1 \text{ for } Q_1 > 0, \text{ and } k_{2, \lambda, Q_1}(H) = 1 \text{ for } Q_1 < Q. \quad (3.3)$$

If not, without loss of generality, one may assume that $k_{1, \lambda, Q_1}(H) \neq -1$ for $Q_1 > 0$. Since $Q_1 > 0$, so $\tilde{Q}_{1, \mu_n} > 0$ for sufficiently large $n$, and it follows from Lemma 2.12 that $k_{1, \lambda, Q_1, \mu_n}(H) = -1$. Similar to the Step 5 in the proof of Lemma 2.8, one can obtain a contradiction by using the convergence of the free boundary and the Cauchy-Kovalevskaya theorem.

□
Next, \(k_{i,\lambda, Q_1}(y)\) has the following properties for \(i = 1, 2\).

**Proposition 3.2** The free boundary \(\Gamma_{i,\lambda, Q_1} : x = k_{i,\lambda, Q_1}(y)\) is a bounded continuous function for any \(x \in (h_i, H)\), where \(h_i\) is determined uniquely by (2.5) for \(i = 1, 2\). Furthermore,

\[
\lim_{y \to h_i^+} k_{1,\lambda, Q_1}(y) = -\infty \text{ if } Q_1 > 0, \quad \text{and} \quad \lim_{y \to h_i^+} k_{2,\lambda, Q_1}(y) = +\infty \text{ if } Q_1 < 0.
\]

**Proof** Consider first that \(Q_1 < Q\). In this case, \(h_2 \in (0, H)\). It follows from (2) in Proposition 3.1 that \(k_{2,\lambda, Q_1}(H) = 1\), and thus \(k_{2,\lambda, Q_1}(y)\) is finite near \(y = H\). It remains to show that

\[
k_{2,\lambda, Q_1}(y)\text{is continuous and finite in } (h_2, H) \quad \text{and} \quad \lim_{y \to h_2^+} k_{2,\lambda, Q_1}(y) = +\infty, \quad (3.4)
\]

which will be proved in three steps.

**Step 1.** Let \((\beta_i, \alpha_i)\) be the maximal intervals \((i = 1, 2, \ldots)\), such that \(k_{2,\lambda, Q_1}(y)\) is finite valued, \(\alpha_1 = H\) and \(\beta_i \geq \alpha_{i+1}\). We first claim that

the number of intervals \((\beta_i, \alpha_i)\) is finite.

If not, then \(\alpha_i - \beta_i \to 0\) and \(\beta_i - \alpha_{i+1} \to 0\) as \(i \to +\infty\). There are the following two cases to be considered.

**Case 1.** \(k_{2,\lambda, Q_1}(\alpha_i - 0) = k_{2,\lambda, Q_1}(\beta_i + 0) = -\infty\). (See Fig. 11)

Denote \(G_i \subset \{ (x, y) \in D \mid x < k_{2,\lambda, Q_1}(y), \beta_i < y < \alpha_i \}\), such that \(G_i\) satisfies

\[
\forall \, X_1 = (x_1, y_1) \in G_i, \exists \, y_0 \text{ such that } x_1 = k_{2,\lambda, Q_1}(y_0) \text{ and } X_0 = (x_1, y_0) \in \Gamma_{2,\lambda, Q_1}.
\]

Thanks to the Lipschitz continuity of \(\psi_{\lambda, Q_1}\), we have

\[
Q - \psi_{\lambda, Q_1}(X_1) = \psi_{\lambda, Q_1}(X_0) - \psi_{\lambda, Q_1}(X_1) \leq C(\alpha_i - \beta_i),
\]

which implies that

\[
\psi_{\lambda, Q_1}(X_1) > 0 \quad \text{for any} \quad X_1 \in G_i, \quad (3.5)
\]

provided that \(\beta_i - \alpha_i\) is small enough such that \(C(\alpha_i - \beta_i) < Q\).

Hence, we can derive a contradiction to the non-oscillation Lemma 2.6 in the region \(G_i \cap \{-2R < x < -R\}\) (for some \(R\) sufficiently large) provided that \(\alpha_i - \beta_i\) is small enough.

**Case 2.** \(k_{2,\lambda, Q_1}(\alpha_i - 0) = +\infty\) or \(k_{2,\lambda, Q_1}(\beta_i + 0) = +\infty\). Without loss of generality, we assume that \(k_{2,\lambda, Q_1}(\beta_i + 0) = +\infty\) and consider the following two subcases.

**Subcase 2.1.** \(k_{2,\lambda, Q_1}(\alpha_{i+1} - 0) = +\infty\) (see Fig. 12).

For sufficiently large \(R > 0\), set

\[
G_{i, R} = \left\{ (x, y) \in D \mid x > R, \frac{\alpha_{i+1} + \beta_{i+1}}{2} < y < \frac{\alpha_i + \beta_i}{2} \right\}.
\]

Similar to (3.5), we can conclude that \(\psi_{\lambda, Q_1} > 0\) in \(G_{i, R}\) for sufficiently large \(i\). This leads to a contradiction by using the non-oscillation Lemma 2.6 in the region \(G_{i, R} \cap \{ x < 2R \}\).

**Subcase 2.2.** \(k_{2,\lambda, Q_1}(\alpha_{i+1} - 0) = -\infty\) (see Fig. 13).

By using the monotonicity of \(\psi_{\lambda, Q_1}(x, y)\) with respect to \(x\), we have that \(\psi_{\lambda, Q_1} = Q\) on the horizontal line \(\{ y = \alpha_{i+1} \}\), and thus the line \(\{ y = \alpha_{i+1} \}\) is the free boundary of \(\psi_{\lambda, Q_1}\). It follows from Proposition 2.2 that

\[
\psi_{\lambda, Q_1} = Q \text{ and } \frac{\partial \psi_{\lambda, Q_1}}{\partial y} = \sqrt{2\lambda - 2g\alpha_{i+1}} \text{ on } \{ y = \alpha_{i+1} \}.
\]
which contradicts to the Cauchy-Kovalevskaya theorem.
Hence, we can conclude that the number of \((\alpha_i, \beta_i)\) is finite.

**Step 2.** In this step, we will show that

\[
\text{the number of } (\alpha_i, \beta_i) \text{ is in fact one, namely, } i = 1. \tag{3.6}
\]

If not, the number of the intervals \((\beta_i, \alpha_i)\) is at least 2. Then there are the following three cases.

**Case 1.** \(k_{2,\lambda, Q_1}(\beta_1 + 0) = k_{2,\lambda, Q_1}(\alpha_2 - 0) = +\infty\).
The non-oscillation Lemma 2.6 gives that \(\alpha_2 < \beta_1\). We first claim that

\[
k_{1,\lambda, Q_1}(y) < +\infty \text{ for any } \alpha_2 < y < \beta_1. \tag{3.7}
\]

If not, without loss of generality, one may assume that there exists a \(\beta_0 \in (\alpha_2, \beta_1)\), such that

\[
\lim_{y \to \beta_0^{-}} k_{1,\lambda, Q_1}(y) = +\infty \text{ and } k_{1,\lambda, Q_1}(y) < +\infty \text{ for } y \in (\beta_0, \beta_1).
\]

Since \(k_{2,\lambda, Q_1}(\beta_1 + 0) = k_{1,\lambda, Q_1}(\beta_0 + 0) = +\infty\), we conclude that the free boundaries \(\Gamma_{1,\lambda, Q_1}\) and \(\Gamma_{2,\lambda, Q_1}\) satisfy the flatness condition (see Section 7 in [1]) near \(y = \beta_0\) and \(y = \beta_1\). Then there exists a large \(x_0 > 0\), such that

\[
\Gamma_{1,\lambda, Q_1} \cap \{(x, y) \mid x > x_0, \beta_0 < y < \beta_1\} \text{ is described by } y = g_1(x), \text{ and } g_1(x) \downarrow \beta_0
\]
as \(x \to +\infty\), and

\[
\Gamma_{2,\lambda, Q_1} \cap \{(x, y) \mid x > x_0, \beta_1 < y < \alpha_1\} \text{ is described by } y = g_2(x), \text{ and } g_2(x) \downarrow \beta_1
\]
as \(x \to +\infty\). Furthermore, it holds that

\[
g'_i(x) \to 0 \text{ as } x \to +\infty, \quad \left| g^{(j)}_i(x) \right| \leq C \text{ for } i = 1, 2, j = 2, 3.
\]

Due to the uniform elliptic estimates, there exists a sequence \(\{\psi_{\lambda, Q_1}(x + n, y)\}\), such that

\[
\psi_{\lambda, Q_1}(x + n, y) \to \psi_0(x, y), \text{ in } C^{2,\alpha}(\mathbb{T})
\]

where \(\mathbb{T} = (-\infty < x < +\infty) \times \{\beta_0 < y < \beta_1\}\), and

\[
\begin{cases}
\Delta \psi_0 = 0 & \text{in } \mathbb{T}, \\
\frac{\partial \psi_0(x, \beta_0)}{\partial y} = \sqrt{2\lambda - 2g\beta_1} \text{ and } \frac{\partial \psi_0(x, \beta_1)}{\partial y} = \sqrt{2\lambda - 2g\beta_1}, \\
\psi_0(x, \beta_0) = 0 \text{ and } \psi_0(x, \beta_1) = Q. \tag{3.8}
\end{cases}
\]

A direct computation gives that

\[
\psi_0(y) = \sqrt{2\lambda - 2g\beta_1}(y - \beta_1) + Q \quad \text{for } \beta_0 < y < \beta_1,
\]

and

\[
\frac{\partial \psi_0(x, \beta_1)}{\partial y} = \sqrt{2\lambda - 2g\beta_1} \quad \text{for } -\infty < x < +\infty,
\]

which contradicts to \(\frac{\partial \psi_0(x, \beta_1)}{\partial y} = \sqrt{2\lambda - 2g\beta_0}, \text{ due to } \sqrt{2\lambda - 2g\beta_1} < \sqrt{2\lambda - 2g\beta_0} \).

Denote

\[
G_R = \{(x, y) \in D \mid R < x < 2R, \alpha_2 < y < H\}
\]

for sufficiently large \(R > 0\). Then it follows from the claim (3.7) that \(\psi_{\lambda, Q_1} > 0 \text{ in } G_R\), which contradicts to the non-oscillation Lemma 2.6 in \(G_R\).
Case 2. $k_{2,\lambda, Q_1}(\beta_1 + 0) = -k_{2,\lambda, Q_1}(\alpha_2 - 0) = +\infty$.

Due to the monotonicity of $\psi_{\lambda, Q_1}(x, y)$ with respect to $x$, it follows from Proposition 2.2 that
\[ \psi_{\lambda, Q_1} = Q \quad \text{and} \quad \frac{\partial \psi_{\lambda, Q_1}}{\partial y} = -\sqrt{2\lambda - 2g\alpha_2} \quad \text{on} \{y = \alpha_2\}, \]
which contradicts to the Cauchy-Kovalevskaya theorem.

Case 3. $k_{2,\lambda, Q_1}(\beta_1 + 0) = -k_{2,\lambda, Q_1}(\alpha_2 - 0) = -\infty$.

Similarly to Case 2, one can get
\[ \psi_{\lambda, Q_1} = Q \quad \text{and} \quad \frac{\partial \psi_{\lambda, Q_1}}{\partial y} = \sqrt{2\lambda - 2g\beta_1} \quad \text{on} \{y = \beta_1\}, \]
which also leads to a contradiction.

Case 4. $k_{2,\lambda, Q_1}(\beta_1 + 0) = k_{2,\lambda, Q_1}(\alpha_2 - 0) = -\infty$.

There are the following two subcases.

Subcase 4.1. $k_{2,\lambda, Q_1}(\beta_2 + 0) = -\infty$. We first claim that
\[ k_{1,\lambda, Q_1}(y) = -\infty \quad \text{for any} \quad \beta_2 < y < \alpha_2. \tag{3.9} \]
If not, without loss of generality, we assume that there exists a $\beta_0 \in (\beta_2, \alpha_2)$, such that
\[ \lim_{y \to \beta_0^+} k_{1,\lambda, Q_1}(y) = -\infty \quad \text{and} \quad k_{1,\lambda, Q_1}(y) > -\infty \quad \text{for} \quad y \in (\beta_0, \beta_0 + \varepsilon) \quad \text{and} \quad \varepsilon > 0. \]

Similar to Case 1 in Step 2, there exists a sequence $\{\psi_{\lambda, Q_1}(x - n, y)\}$, such that
\[ \psi_{\lambda, Q_1}(x - n, y) \to \psi_0(x, y), \quad \text{in} \quad C^{2,\alpha}(\{x < k_{2,\lambda, Q_1}(y), \beta_2 < y < \alpha_2\}), \]
and $\psi_0$ satisfies (3.8), which leads to a contradiction. The claim (3.9) implies that
\[ \psi_{\lambda, Q_1} > 0 \quad \text{in} \quad \{(x, y) \in D \mid x < k_{2,\lambda, Q_1}(y), \beta_2 < y < \alpha_2\}, \]
which contradicts to the non-oscillation Lemma 2.6.

Subcase 4.2. $k_{2,\lambda, Q_1}(\beta_2 + 0) = +\infty$. Similar to the claim (3.9), one can get
\[ k_{1,\lambda, Q_1}(y) = -\infty \quad \text{for any} \quad 0 < y < \alpha_2. \tag{3.10} \]
The non-oscillation Lemma 2.6 yields that
\[ k_{2,\lambda, Q_1}(y) = +\infty \quad \text{for any} \quad 0 < y < \beta_2. \tag{3.11} \]

With the aid of (3.10) and (3.11), by using the similar arguments in Case 1 in Step 2, one can get a sequence $\{\psi_{\lambda, Q_1}(x + n, y)\}$, such that
\[ \psi_{\lambda, Q_1}(x + n, y) \to \psi_1(x, y), \quad \text{in} \quad C^{2,\alpha}(\{-\infty < x < +\infty \times \{0 < y < \beta_2\}), \]
and $\psi_1$ satisfies
\[ \Delta \psi_1 = 0 \quad \text{in} \quad \{-\infty < x < +\infty \times \{0 < y < \beta_2\}, \]
\[ \frac{\partial \psi_1(x, y)}{\partial y} = \sqrt{2\lambda - 2g\beta_2}, \quad \psi_1(x, \beta_2) = Q. \]

By the uniqueness for the Cauchy-Kovalevskaya theorem, one has
\[ \psi_1(x, y) = \sqrt{2\lambda - 2g\beta_2}(y - \beta_2) + Q \quad \text{in} \quad \{-\infty < x < +\infty \times \{0 < y < \beta_2\}. \tag{3.12} \]
It follows from Lemma 2.4 that
\[ \max\{-\sqrt{2\lambda - 2gh_1} + Q_1, 0\} \leq \psi_1(x, y) \leq \min\{\sqrt{2\lambda - 2gh_2} + Q_1, Q\}. \]
in $D$, which implies that $\psi_1(x, 0) = Q_1$. Hence
\[
Q - Q_1 = \sqrt{2\lambda - 2g\beta_2\beta_2}.
\] (3.13)

Similarly, there exists a sequence $\{\psi_\lambda, Q_1(x - n, y)\}$, such that
\[
\psi_\lambda, Q_1(x - n, y) \to \psi_2(x, y), \text{ in } C^{2,\alpha}(\mathbb{T}),
\]
where $\mathbb{T} = (-\infty < x < +\infty) \times \{0 < y < \alpha_2\}$, and $\psi_2$ satisfies
\[
\begin{cases}
\Delta \psi_2 = 0 & \text{in } \mathbb{T}, \\
\frac{\partial \psi_2(x, \alpha_2)}{\partial y} = \sqrt{2\lambda - 2g\alpha_2}, & \psi_2(x, \alpha_2) = Q.
\end{cases}
\]
It implies that
\[
Q - Q_1 = \sqrt{2\lambda - 2g\alpha_2\alpha_2}.
\] (3.14)

It follows from (3.13) and (3.14) that we obtain a contradiction to $\alpha_2 > \beta_2$.

Hence, we have shown that $k_{2,\lambda, Q_1}(y)$ is continuous and finite in $(\beta_1, H)$ and infinite in $(0, \beta_1)$.

Furthermore,
\[
\lim_{y \to \beta_1^+} k_{2,\lambda, Q_1}(y) = +\infty.
\] (3.15)

Step 3. In this step, we will show that $\beta_1 = h_2$.

Due to (3.15), it follows from the similar arguments in Case 1 in Step 1 that there exists a sequence $\{\psi_\lambda, Q_1(x + n, y)\}$, such that
\[
\psi_\lambda, Q_1(x + n, y) \to \psi_0(x, y) \text{ in } C^{2,\alpha}(\mathbb{T}),
\]
where $\mathbb{T} = (-\infty < x < +\infty) \times \{0 < y < \beta_1\}$, and
\[
\begin{cases}
\Delta \psi_0 = 0 & \text{in } \mathbb{T}, \\
\frac{\partial \psi_0(x, \beta_1)}{\partial y} = \sqrt{2\lambda - 2g\beta_1} \text{ and } \psi_0(x, \beta_1) = Q \text{ for } -\infty < x < +\infty.
\end{cases}
\]

Similar arguments in the proof in Subcase 4.2 in Step 2 show
\[
Q - Q_1 = \sqrt{2\lambda - 2g\beta_1\beta_1}.
\]

This and Proposition 2.1 give $\beta_1 = h_2$.

Next we consider the case that $Q_1 = Q$. It follows from similar arguments above that $k_{1,\lambda, Q_1}(y)$ is continuous and finite in $(h_1, H)$ and $\lim_{y \to h_1^+} k_{1,\lambda, Q_1}(y) = -\infty$.

Finally, using the non-oscillation Lemma 2.6, one can show that $h_2 = 0$, $k_{2,\lambda, Q_1}(y)$ is continuous and finite in $(0, H)$.

$$
\square
$$

The previous result gives the almost continuous fit conditions. The continuous fit conditions will follow once the critical value $Q_1 = 0$ or $Q_1 = Q$ can be excluded.
Proposition 3.3  The value $Q_1$ in Proposition 3.1 lies in $(0, Q)$.

Proof  Without loss of generality, we assume that $Q_1 = 0$, which implies that $h_2 \in (0, H)$. The non-oscillation Lemma 2.6 gives that

$$k_{1,\lambda,0}(0) = \lim_{y \to 0^+} k_{1,\lambda,0}(y) \text{ exists.}$$

Next, we consider the following three cases.

Case 1. $k_{1,\lambda,0}(0) = +\infty$ (see Fig. 14).

Since $k_{2,\lambda,0}(h_2) = +\infty$, it follows from the similar arguments in Case 1 in Step 2 in the proof of Proposition 3.2 that there exists a sequence $\{\psi_{\lambda,0}(x + n, y)\}$, such that

$$\psi_{\lambda,0}(x + n, y) \to \psi_0(x, y) \quad \text{in} \ C^{2,\alpha}([-\infty < x < +\infty] \times \{0 < y < h_2\}) ,$$
and $\psi_0$ satisfies
\[
\begin{align*}
\Delta \psi_0 &= 0 \quad \text{in } \{-\infty < x < +\infty\} \times \{0 < y < h_2\}, \\
\frac{\partial \psi_0(x,0)}{\partial y} &= \sqrt{2\lambda} \quad \text{and} \quad \frac{\partial \psi_0(x,h_2-0)}{\partial y} = \sqrt{2\lambda - 2gh_2}, \\
\psi_0(x,0) &= 0 \quad \text{and} \quad \psi_0(x,h_2) = Q.
\end{align*}
\]
This is an overdetermined problem, due to $\sqrt{2\lambda - 2gh_2} < \sqrt{2\lambda}$.

**Case 2.** $k_{1,\lambda,0}(0) = -\infty$ (see Fig. 15).

The maximum principle gives that
\[
\frac{\partial \psi_0}{\partial y} \geq 0 \quad \text{on} \quad N.
\]

Similar to the Case 1 in the proof of Proposition 3.2, one can obtain a contradiction by using the non-oscillation Lemma 2.6 and Remark 2.3 for $\psi_{\lambda,0}$ in $D \cap \{-2R < x < -R\} \cap \{\psi_{\lambda,0} > 0\}$, provided that $R$ is sufficiently large.

**Case 3.** $k_{1,\lambda,0}(0) \in (-\infty, +\infty)$ (see Fig. 16).

Set $X_0 = (k_{1,\lambda,0}(0), 0)$. It follows from the results in Section 9 in [4] and Section 11 in Chapter 3 in [13] that the continuous fit conditions imply the smooth fit conditions, we have that the free boundary $\Gamma_{1,\lambda,0}$ is $C^1$-smooth at $X_0$. Furthermore, $\nabla \psi_{\lambda,0}$ is uniformly continuous in a $[\psi_{\lambda,0} > 0]$-neighborhood of $X_0$, and thus
\[
|\nabla \psi_{\lambda,0}| = \frac{\partial \psi_{\lambda,0}}{\partial y} = \sqrt{2\lambda} \quad \text{at} \quad X_0.
\]

Consider a function
\[
\omega_0(x, y) = \min\{y, y, Q\} \quad \text{for} \quad \gamma = \frac{\sqrt{2\lambda - 2gh_2} + \sqrt{2\lambda}}{2}.
\]
It follows from the Step 3 in the proof of Proposition 3.1 that
\[
\psi_{\lambda,0}(x, y) \rightarrow \min\{\sqrt{2\lambda - 2gh_2}y, Q\} \quad \text{as} \quad x \rightarrow +\infty.
\]
Since \( \gamma > \sqrt{2\lambda - 2gh_2} \), so \( \psi_{\lambda,0} \leq \omega_0 \) in the far field, which together with the maximum principle gives that

\[
\psi_{\lambda,0}(x, y) \leq \omega_0(x, y) \quad \text{in} \quad D \cap \{\omega_0 < Q\}.
\]

In view of \( \psi_{\lambda,0} = \omega_0 = 0 \) at \( X_0 \), thus we have

\[
\sqrt{2\lambda} = \frac{\partial \psi_{\lambda,0}}{\partial y} = \frac{\partial \psi_{\lambda,0}}{\partial v} \leq \frac{\partial \omega_0}{\partial v} = \frac{\partial \omega_0}{\partial y} = \gamma = \frac{\sqrt{2\lambda - 2gh_2} + \sqrt{2\lambda}}{2} \quad \text{at} \quad X_0,
\]

which leads to a contradiction, due to \( h_2 > 0 \).

\[ \square \]

**Remark 3.1** Proposition 3.3 implies that the impinging jet under gravity with continuous fit conditions must possess two asymptotic directions in far fields. This conclusion coincides the results on impinging jet in absence of gravity in [9].
As a consequence of Propositions 3.1–3.3, we can get

**Theorem 3.4** For any $Q > 2\sqrt{gH^3}$, there exist an effluent flux $Q_1 \in (0, Q)$ and a $\lambda > \frac{\max\{Q_1^2, (Q - Q_1)^2\}}{2H^2} + gH$, such that there exists a solution $(u, v, p, \Gamma_1, \Gamma_2)$ to the incompressible impinging jet flow problem, where

$$u = \frac{\partial \psi_{\lambda, Q_1}}{\partial y}, \quad v = -\frac{\partial \psi_{\lambda, Q_1}}{\partial x} \quad \text{and} \quad p = p_{\text{atm}} + \lambda - \frac{|\nabla \psi_{\lambda, Q_1}|^2}{2} - gy,$$

and

$$\Gamma_i = \Gamma_{i, \lambda, Q_1} : x = k_{i, \lambda, Q_1}(y) = k_i(y) \quad \text{for} \quad y \in (h_i, H) \quad \text{and} \quad i = 1, 2.$$

**Proof** It follows from Propositions 3.1 and 3.3 that

$$Q_1 \in (0, Q) \quad \text{and} \quad \lambda \geq \frac{\max\{Q_1^2, (Q - Q_1)^2\}}{2H^2} + gH.$$

Furthermore, Propositions 3.1 and 3.2 give that

$$k_{1, \lambda, Q_1}(H) = -1 \quad \text{and} \quad k_{2, \lambda, Q_1}(H) = 1,$$

and

$$\lim_{y \to h_1^+} k_{1, \lambda, Q_1}(y) = -\infty \quad \text{and} \quad \lim_{y \to h_2^+} k_{2, \lambda, Q_1}(y) = +\infty,$$

where $h_1$ and $h_2$ are uniquely determined by (2.5). It follows from Lemma 6.5 in the “Appendix” that the free boundary $\Gamma_{1, \lambda, Q_1}$ and $\Gamma_{2, \lambda, Q_1}$ are analytic, and the Bernoulli’s law gives that $p = p_{\text{atm}}$ on $\Gamma_{1, \lambda, Q_1} \cup \Gamma_{2, \lambda, Q_1}$.

Next, we will show that $\lambda > \frac{\max\{Q_1^2, (Q - Q_1)^2\}}{2H^2} + gH$. If not, without loss of generality, one may assume that $Q_1 \geq \frac{Q}{2} > \sqrt{gH^3}$ and $\lambda = \frac{Q_1^2}{2H^2} + gH$. Since $Q_1 = \sqrt{2\lambda - 2gh_1h_1}$, it follows from the proof of Proposition 2.1 that $h_1 = H$. This implies that the free boundary $\Gamma_{1, \lambda, Q_1}$ is empty, which leads to a contradiction.

Springer
Therefore, the continuous fit conditions are fulfilled for the free boundary $\Gamma_{i,\lambda, Q_1}$ at $A_i$ ($i = 1, 2$). It follows from the results in Section 9 in [4] and Section 11 in Chapter 3 in [13] that the continuous fit conditions imply the smooth fit conditions, hence, $N_i \cup \Gamma_{i,\lambda, Q_1}$ is $C^1$-smooth at $A_i$, that is
\[ k_{1,\lambda, Q_1}'(H - 0) = g_1'(H + 0) \quad \text{and} \quad k_{2,\lambda, Q_1}'(H - 0) = g_2'(H + 0). \]
Furthermore, $\nabla \psi_{\lambda, Q_1}$ is uniformly continuous in a $\{\psi_{\lambda, Q_1} > 0\}$-neighborhood of $A_1$, and $\nabla \psi_{\lambda, Q_1}$ is uniformly continuous in a $\{\psi_{\lambda, Q_1} < Q\}$-neighborhood of $A_2$.

It remains to show that
\[ v = -\frac{\partial \psi_{\lambda, Q_1}}{\partial x} < 0 \quad \text{in} \quad (\Omega \cap \{0 < \psi_{\lambda, Q_1} < Q\}) \cup \Gamma_{1,\lambda, Q_1} \cup \Gamma_{2,\lambda, Q_1}. \quad (3.16) \]
By virtue of the monotonicity of $\psi_{\lambda, Q_1}$ with respect to $x$, one has
\[ \psi_{\lambda, Q_1}(x_1, y) \geq \psi_{\lambda, Q_1}(x_2, y) \quad \text{for any} \quad x_1 \geq x_2, \]
which implies that
\[ -\frac{\partial \psi_{\lambda, Q_1}}{\partial x} \leq 0 \quad \text{in} \quad \Omega \cap \{0 < \psi_{\lambda, Q_1} < Q\}. \]
Consider $v = -\frac{\partial \psi_{\lambda, Q_1}}{\partial x}$ in $\Omega_0$, which solves the Laplace equation in $\Omega_0$. The strong maximum principle gives that $v < 0$ in $\Omega_0$. Finally, we claim that
\[ v < 0 \quad \text{on} \quad \Gamma_{1,\lambda, Q_1} \cup \Gamma_{2,\lambda, Q_1}. \]
If not, without loss of generality, suppose that there exists a $X_0 = (x_0, y_0) \in \Gamma_{2,\lambda, Q_1}$, such that $v(X_0) = 0$. Since the free boundary $\Gamma_{2,\lambda, Q_1}$ is analytic at $X_0$, we can assume that $v = (0, 1)$ is the outer normal vector to $\Gamma_{2,\lambda, Q_1}$ at $X_0$. Then Hopf’s lemma implies that
\[ v_y = \frac{\partial v}{\partial y} > 0 \quad \text{at} \quad X_0. \quad (3.17) \]
It follows from Proposition 2.2 that $u^2 + v^2 = |\nabla \psi_{\lambda, Q_1}|^2 = 2\lambda - 2gy$ on $\Gamma_{2,\lambda, Q_1}$, and
\[ \frac{\partial (u^2 + v^2)}{\partial s} = \frac{\partial (2\lambda - 2gy)}{\partial s} = (0, -2g) \cdot (1, 0) = 0 \quad \text{at} \quad X_0, \quad (3.18) \]
where $s = (1, 0)$ is the tangential vector of $\Gamma_{2,\lambda, Q_1}$ at $X_0$. On the other hand, it follows from (3.17) that
\[ \frac{\partial (u^2 + v^2)}{\partial s} = 2uu_x + 2vv_y = -2uv_y = -2\sqrt{2\lambda - 2gy}v_y > 0 \quad \text{at} \quad X_0, \]
which contradicts to (3.18).

Since $v < 0$ on $\Gamma_{1,\lambda, Q_1} \cup \Gamma_{2,\lambda, Q_1}$, the implicit function theorem gives that $x = k_{i,\lambda, Q_1}(y)$ is $C^1$-smooth for any $x \in (h_1, H)$, $i = 1, 2$. \qed

4 The asymptotic behaviors in the far field

Since $k_{2,\lambda, Q_1}(h_2 + 0) = +\infty$, it follows from the similar arguments in Step 3 in the proof of Proposition 3.2 that there exists a sequence $\{\psi_{\lambda, Q_1}(x + n, y)\}$, such that
\[ \psi_{\lambda, Q_1}(x + n, y) \to \sqrt{2\lambda - 2gh_2y} + Q_1 \quad \text{uniformly in} \quad C^{2,\alpha}(G), \]
for any $G \subset \subset (-\infty, +\infty) \times (0, h_2)$. This gives that
\[ (-v(x, y), u(x, y)) = \nabla \psi_{\lambda, Q_1}(x, y) \to (0, \sqrt{2\lambda - 2gh_2}) \]
uniformly in any compact subset of $(0, h_2)$, as $x \to +\infty$. It follows from Bernoulli’s law that
\[ p(x, y) \to p_{atm} + g(h_2 - y) \]
uniformly in any compact subset of $(0, h_2)$, as $x \to +\infty$.

Furthermore, it holds that
\[ \nabla(u, v) \to 0 \quad \text{and} \quad \nabla p \to (0, -g) \]
uniformly in any compact subset of $(0, h_2)$, as $x \to +\infty$.

By using the similar arguments, one gets
\[ (u, v, p) \to (Q_{H_2 - H_1}, 0, p_0(y)) \]
uniformly in any compact subset of $(0, h_1)$ as $x \to -\infty$, where $p_1(y) = p_{atm} + g(h_1 - y)$.

Next, we will obtain the asymptotic behavior of the impinging jet flow in the upstream. Define the function $\psi_n(x, y) = \psi_{\lambda, Q_1}(x - n, y)$ for $x < \frac{n}{2}$. For any compact subset $G$ of $S = (-\infty, +\infty) \times (H_1, H_2)$, with the aid of the assumptions of the nozzle walls $N_1$ and $N_2$ in the inlet, it follows from the standard elliptic estimates that we have
\[ \|\psi_n\|_{C^2,\alpha(G)} \leq C(G) \quad \text{for sufficiently large} \ n, \ 0 < \alpha < 1. \tag{4.1} \]
Arzela-Ascoli lemma gives that there exists a subsequence still labeled by $\psi_n$, such that
\[ \psi_n \to \psi_0 \quad \text{uniformly in} \ C^{2,\beta}(G) \quad \text{for some} \ 0 < \beta < \alpha. \tag{4.2} \]
Furthermore, $\psi_0$ satisfies
\[ \begin{cases} 
\Delta \psi_0 = 0 \quad \text{in} \ S, \\
\psi_0(x, H_1) = 0, \ \psi_0(x, H_2) = Q, \\
0 \leq \psi_0 \leq Q \quad \text{in} \ S.
\end{cases} \tag{4.3} \]
It is easy to check that the boundary value problem (4.3) possesses a unique solution
\[ \psi_0(y) = \frac{Q}{H_2 - H_1} y \quad \text{in} \ S. \tag{4.4} \]
Hence, this together with (4.2) yields that
\[ (u, v) \to \left( \frac{Q}{H_2 - H_1}, 0 \right), \ \nabla(u, v) \to 0 \]
uniformly in any compact subset of $(H_1, H_2)$ as $x \to -\infty$. It then follows from the Bernoulli’s law that
\[ p \to p_0(y) \quad \text{and} \quad \nabla p \to (0, -g) \]
uniformly in any compact subset of $(H_1, H_2)$ as $x \to -\infty$, where $p_0(y) = p_{atm} + \lambda - \frac{Q^2}{2(H_2 - H_1)^2} - gy$. Springer
5 The properties of the interface $\Gamma$

In this section, we will investigate the properties of the interface $\Gamma = \Omega \cap \{\psi_{\lambda, Q_1} = Q_1\}$ between two fluids with different downstreams. This is another important difference between the impinging jet flows and the general jet flows.

**Proposition 5.1** The interface $\Gamma = \Omega \cap \{\psi_{\lambda, Q_1} = Q_1\}$ can be denoted by $x = k(y)$ for $y \in (0, H_3)$, where $H_3 = \frac{Q_1(H_2 - H_1)}{Q} + H_1$. Furthermore, $\lim_{y \to 0^+} k(y)$ exists and is finite, and $k'(0+0) = 0$.

**Proof** Recall that $\Omega_0 = \Omega \cap \{0 < \psi_{\lambda, Q_1} < Q\}$. Since $v < 0$ in $\Omega_0$, the interface $\Gamma$ is a $y$-graph, and the implicit function theorem implies that $\Gamma$ can be denoted by a $C^1$-smooth function $x = k(y)$ for any $y \in (0, H_3)$, where $H_3 = \frac{Q_1(H_2 - H_1)}{Q} + H_1$ is nothing but the asymptotic height of the interface $\Gamma$ in upstream. It follows from the asymptotic behavior of $\psi_{\lambda, Q_1}$ in Section 4 that

$$\lim_{y \to H_3^-} k(y) = -\infty \text{ and } k(y) \text{ is finite for any } y \in (0, H_3).$$

Next, we will show that

$$\lim_{y \to 0^+} k(y) \text{ exists and is finite.} \quad (5.1)$$

Suppose that there exist two sequences $y_n \downarrow 0$ and $\tilde{y}_n \downarrow 0$, such that

$$\lim_{n \to +\infty} k(y_n) = x_1 \text{ and } \lim_{n \to +\infty} k(\tilde{y}_n) = x_2. \quad (5.2)$$

Without loss of generality, one may assume that $x_1 > x_2$. We claim that

$$\frac{\partial \psi_{\lambda, Q_1}}{\partial y} = 0 \text{ on the segment } I = \{(x, 0) \mid x_2 < x < x_1\}. \quad (5.3)$$

In fact, suppose that there exists a point $X_0 = (x_0, 0) \in I$, such that

$$\frac{\partial \psi_{\lambda, Q_1}}{\partial y} \neq 0 \text{ at } X_0.$$

Without loss of generality, assume that $\frac{\partial \psi_{\lambda, Q_1}(X_0)}{\partial y} > 0$. Then one has

$$\psi_{\lambda, Q_1}(x_0, y) < Q_1 \text{ for } 0 < y < \varepsilon, \quad (5.4)$$

for small $\varepsilon > 0$, due to $\psi_{\lambda, Q_1}(x_0, 0) = Q_1$.

Due to the monotonicity of $\psi(x, y)$ with respect to $x$, it follows from (5.2) that

$$\psi_{\lambda, Q_1}(x_0, \tilde{y}_n) \geq \psi_{\lambda, Q_1}(k(\tilde{y}_n), \tilde{y}_n) = Q_1 \text{ and } \psi_{\lambda, Q_1}(x_0, y_n) \leq \psi_{\lambda, Q_1}(k(y_n), y_n) = Q_1,$$

for any small $\tilde{y}_n, y_n \in (0, \varepsilon)$, which contradicts to (5.4).

Take $X_0 = (x_0, 0) \in I$. Then $x_2 < x_0 - \varepsilon < x_1$ for small $\varepsilon > 0$. Denote $\psi_{\varepsilon}(x, y) = \psi_{\lambda, Q_1}(x - \varepsilon, y)$. The monotonicity of $\psi_{\lambda, Q_1}(x, y)$ with respect to $x$ implies that

$$\psi_{\varepsilon}(x, y) \leq \psi_{\lambda, Q_1}(x, y) \text{ in } \Omega.$$

In view of $\psi_{\lambda, Q_1} = \psi_{\varepsilon} = Q_1$ at $X_0$, then Hopf’s lemma shows that

$$\frac{\partial (\psi_{\lambda, Q_1} - \psi_{\varepsilon})}{\partial y} > 0 \text{ at } X_0.$$
which contradicts to (5.3). Hence, \( \lim_{y \to 0^+} k(y) \) exists.

Next, we will show that

\[-\infty < \lim_{y \to 0^+} k(y) < +\infty.\]

If not, without loss of generality, one may assume that \( \lim_{y \to 0^+} k(y) = +\infty \). The asymptotic behavior of \( \psi_{\lambda, Q_1} \) implies that there exists a sufficiently large \( R_0 > 0 \), such that

\[
\sqrt{2\lambda - 2gH} \leq |\nabla \psi_{\lambda, Q_1}(x, y)| \leq \sqrt{2\lambda}
\]

in any subdomain of \( \{(x, y) \mid x \geq R_0, 0 < y < h_2\} \). Denote \( G = \{(x, y) \mid x < k(y), 0 < y < h_2\} \). Then one has

\[
\Delta \psi_{\lambda, Q_1} = 0 \quad \text{and} \quad \psi_{\lambda, Q_1} < Q_1 \quad \text{in} \quad G_{R_0},
\]

where \( G_{R_0} = G \cap \{R_0 < x < 2R_0\} \). The maximum principle gives that

\[
\frac{\partial \psi_{\lambda, Q_1}}{\partial v} \geq 0 \quad \text{on} \quad N, \quad \text{and} \quad \frac{\partial \psi_{\lambda, Q_1}}{\partial v} \geq \sqrt{2\lambda - 2gH} \quad \text{on} \quad \Gamma,
\]

where \( v \) is the outer normal vector.

It follows from (5.5) and (5.6) that

\[
\sqrt{2\lambda - 2gH}R_0 \leq \int_{(\Gamma \cup N) \cap \{R_0 < x < 2R_0\}} \frac{\partial \psi_{\lambda, Q_1}}{\partial v} dS
\]

\[
= \int_{G_{R_0}} \Delta \psi_{\lambda, Q_1} dxdy - \int_{\partial G_{R_0} \cap \{x = R_0\}} \frac{\partial \psi_{\lambda, Q_1}}{\partial x} dy + \int_{\partial G_{R_0} \cap \{x = 2R_0\}} \frac{\partial \psi_{\lambda, Q_1}}{\partial x} dy
\]

\[
\leq 2\sqrt{2\lambda}H,
\]

which leads to a contradiction, provided that \( R_0 \) is sufficiently large.

Finally, we will show that the interface \( \gamma \) intersects the ground \( N \) perpendicularly.

Denote \( S = (k(0), 0) \) as the stagnation point. Let \( \tilde{\psi}(x, y) = \psi_{\lambda, Q_1}(k(0) + x, y) - Q_1 \). It follows from Proposition 2.2 that \( \tilde{\psi}(x, y) \) is harmonic in \( B_r(0) \cap \{y > 0\} \) for some \( r > 0 \) and \( \tilde{\psi} \) vanishes on \( B_r(0) \cap \{y = 0\} \). Hence, \( \tilde{\psi} \) can be extended to a harmonic function in \( B_r(0) \). Then \( \{\tilde{\psi} = 0\} \) consists of arcs forming equal angles at the origin. By virtue of the previous arguments, there exists a continuous arc \( \gamma \) initiating at the point \( (0, 0) \) and \( \tilde{\psi} \) vanishes on \( \gamma \). Hence, \( \gamma \) must intersect \( \{y > 0\} \) orthogonally at the point \( (0, 0) \), which implies that

\[ k'(0 + 0) = 0. \]

\[ \square \]

6 Appendix

In this section, we will list some important lemmas for the minimizer \( \psi_{\lambda, Q_1, \mu} \), which have been established in [1, 2, 4].

It follows from Lemma 3.2 in [1] and Lemma 3.1 in [2] that the following lemma holds.
Lemma 6.1 There exists a universal constant $C^*$ such that, for any disc $B_r(X_0) \subset \Omega_\mu$, if
\[
\frac{1}{r} \int_{\partial B_r(X_0)} \psi_{\lambda, Q_1, \mu} dS \geq C^* \sqrt{2\lambda}, \quad \text{then } \psi_{\lambda, Q_1, \mu} > 0 \text{ in } B_r(X_0).
\]
Similarly, if
\[
\frac{1}{r} \int_{\partial B_r(X_0)} (Q - \psi_{\lambda, Q_1, \mu}) dS \geq C^* \sqrt{2\lambda}, \quad \text{then } \psi_{\lambda, Q_1, \mu} < Q \text{ in } B_r(X_0).
\]

By using the similar arguments for Lemma 3.4 in [1] and Lemma 2.4 in [4], one can have the following lemma.

Lemma 6.2 There exists a universal positive constant $c^*$, such that for any disc $B_r(X_0)$ with $X_0 \in D_\mu$, if
\[
\frac{1}{r} \int_{\partial B_r(X_0)} \psi_{\lambda, Q_1, \mu} dS \leq c^* \sqrt{2\lambda - 2gH}, \quad \text{and } \psi_{\lambda, Q_1, \mu} < Q \text{ in } B_r(X_0),
\]
then $\psi_{\lambda, Q_1, \mu} = 0$ in $B_{\frac{r}{8}}(X_0) \cap D_\mu$; similarly, if
\[
\frac{1}{r} \int_{\partial B_r(X_0)} (Q - \psi_{\lambda, Q_1, \mu}) dS \leq c^* \sqrt{2\lambda - 2gH}, \quad \text{and } \psi_{\lambda, Q_1, \mu} > 0 \text{ in } B_r(X_0),
\]
then $\psi_{\lambda, Q_1, \mu} = Q$ in $B_{\frac{r}{8}}(X_0) \cap D_\mu$.

Lemma 6.2 implies the following non-degeneracy lemma.

Lemma 6.3 For any $X_0 \in \{\psi_{\lambda, Q_1, \mu} > 0\} \cap D_\mu$, if $\psi_{\lambda, Q_1, \mu} < Q$ in $B_r(X_0)$ for some $r > 0$, then
\[
\frac{1}{r} \int_{\partial B_r(X_0)} \psi_{\lambda, Q_1, \mu} dS \geq c^* \sqrt{2\lambda - 2gH}. \tag{6.1}
\]
In particular,
\[
\sup_{\partial B_r(X_0)} \psi_{\lambda, Q_1, \mu} \geq c^* \sqrt{2\lambda - 2gHr}. \tag{6.2}
\]
Similarly, the result holds with $\psi_{\lambda, Q_1, \mu}$ replaced by $Q - \psi_{\lambda, Q_1, \mu}$.

It follows from Lemma 5.1 in [2] that the following bounded gradient lemma holds.

Lemma 6.4 Let $X_0 = (x_0, y_0)$ be a free boundary point in $D_\mu$ and $B_r(X_0) \subset B_R(X_0) \subset D_\mu$. Then
\[
|\nabla \psi_{\lambda, Q_1, \mu}(x, y)| \leq C \text{ in } B_r(X_0),
\]
where $C$ depends only on $\lambda$ and $(1 - \frac{e}{R})^{-1}$, but it is independent of $Q$.

Since $Q > 2\sqrt{gH^3}$, it follows from Remark 2.1 that $\lambda \geq \frac{\max(Q^2 + (Q_1)^2)}{8H^2} + gH \geq \frac{3}{2}gH$, we have that $\sqrt{2\lambda - 2gy}$ is analytic for any $y \in (0, H)$. Then the regularity of the free boundary is obtained in Theorem 8.4 in [1].

Lemma 6.5 The free boundary $D_\mu \cap \partial\{0 < \psi_{\lambda, Q_1, \mu} < Q\}$ is locally analytic.
Remark 6.1 Those lemmas still hold for the local minimizer $\psi_{\lambda, Q_1}$ to the variational problem $(P_{\lambda, Q_1})$. A local minimizer $\psi_{\lambda, Q_1}$ to the variational problem $(P_{\lambda, Q_1})$ means that

$$J_E(\psi_{\lambda, Q_1}) \leq J_E(\phi) \quad \text{for any} \quad \phi \in K_{Q_1}, \quad \phi = \psi_{\lambda, Q_1} \text{ on } \partial E,$$

for any bounded domain $E \subset \Omega$ with smooth boundary, where the functional

$$J_E(\psi) = \int_E |\nabla \psi|^2 + \sqrt{2\lambda - 2g y\chi_{\{0 < \psi < Q\}} \cap D} \, dx \, dy$$

and the admissible set $K_{Q_1}$ is defined in Sect. 2.3.

Acknowledgements Cheng and Xin are supported in part by the Zheng Ge Ru Foundation and Hong Kong RGC Grants: 14300819, 14300917, 14302819. Cheng is also supported partially by National Key R&D Program of China (Grant No. 2022YFA1007700) and by National Natural Science Foundation of China (Grant No. 12001387). Xin is also supported partially by the Key Project of National Natural Science Foundation of China (Grant No. 12131010) and by Guangdong Providence Basic and Applied Basic Research Foundation (Grant No. 2020B1515310002). Du is supported by National Nature Science Foundation of China (Grant No. 11971331, 12125102), and Sichuan Youth Science and Technology Foundation (Grant No. 2021JJDYT0024)

Funding Part of the work was done when the first author was visiting The Institute of Mathematical Sciences, The Chinese University of Hong Kong. He thanks the institute for its hospitality and support

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Alt, H.W., Caffarelli, L.A.: Existence and regularity for a minimum problem with free boundary. J. Reine Angew. Math. 325, 105–144 (1981)
2. Alt, H.W., Caffarelli, L.A., Friedman, A.: Asymmetric jet flows. Commun. Pure Appl. Math. 35, 29–68 (1982)
3. Alt, H.W., Caffarelli, L.A., Friedman, A.: Jet flows with gravity. J. Reine Angew. Math. 331, 58–103 (1982)
4. Alt, H.W., Caffarelli, L.A., Friedman, A.: Axially symmetric jet flows. Arch. Rational Mech. Anal. 81, 97–149 (1983)
5. Alt, H.W., Caffarelli, L.A., Friedman, A.: Variational problems with two phases and their free boundaries. Trans. Am. Math. Soc. 282, 431–461 (1984)
6. Amick, C.J., Fraenkel, L.E., Toland, J.F.: On the Stokes conjecture for the wave of extreme form. Acta Math. 148, 193–214 (1982)
7. Birkhoff, G., Zarantonello, E.H.: Jets, Wakes and Cavities. Academic Press, New York (1957)
8. Brilliouin, M.: Les surfaces de glissement de Helmholtz et la résistance des fluides. Ann. Chim. Phys. 23, 145–230 (1911)
9. Cheng, J.F., Du, L.L., Wang, Y.F.: On incompressible oblique impinging jet flows. J. Differ. Equ. 265, 4687–4748 (2018)
10. Donaldson, C.D., Snedeker, R.S.: A study of free jet impingement. Part 1. Mean properties of free and impinging jets. J. Fluid Mech. 45, 281–319 (1971)
11. Dias, F., Elcrat, A.R., Trefethen, L.N.: Ideal jet flow in two dimensions. J. Fluid Mech. 185, 275–288 (1987)
12. Evans, L.C.: Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics. CRC Press, Cambridge (1992)
13. Friedman, A.: Variational Principles and Free-Boundary Problems. Pure and Applied Mathematics. Wiley, New York (1982)
14. Gifford, W.A.: A finite element analysis of isothermal fiber formation. Phys. Fluid 25, 219–225 (1982)
15. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order, Classics in Mathematics. Springer, Berlin (2001)
16. Gurevich, M.I.: The Theory of Jets in an Ideal Fluid, International Series of Monographs in Pure and Applied Mathematics, vol. 93. Pergamon Press, Oxford (1966)
17. Hureau, J., Weber, R.: Impinging free jets of ideal fluid. J. Fluid Mech. 372, 357–374 (1998)
18. Jacob, C.: Introducțion Mathematésque à la Mécanque des Fluides. Gauthier-Villars, Paris (1959)
19. Jenkins, D.R., Barton, N.G.: Computation of the free-surface shape of an inviscid jet incident on a porous wall. IMA J. Appl. Math. 41, 193–206 (1988)
20. King, A.C., Bloor, M.I.G.: Free-surface flow of a stream obstructed by an arbitrary bed topography. Quart. J. Mech. Appl. Math. 43, 87–106 (1990)
21. Milne-Thomson, L.M.: Theoretical Hydrodynamics, 5th edn. Macmillan, London (1968)
22. Plotnikov, P.I.: Proof of the Stokes conjecture in the theory of surface waves. Stud. Appl. Math. 108, 217–244 (2002)
23. Stokes, G.G.: Considerations relative to the greatest height of oscillatory irrotational waves which can be propagated without change of form, in Mathematical and Physical Papers, vol. I, pp. 225–228. Cambridge University Press, Cambridge (1880)
24. Strauss, W.A.: Steady water waves. Bull. Am. Math. Soc. 47, 671–694 (2010)
25. Tuck, E.O.: The shape of free jets of water under gravity. J. Fluid Mech. 76, 625–640 (1976)
26. Varvaruca, E.: Singularities of Bernoulli free boundaries. Commun. Partial Differ. Equ. 31, 1451–1477 (2006)
27. Varvaruca, E., Weiss, G.S.: A geometric approach to generalized Stokes conjectures. Acta Math. 206, 363–403 (2011)
28. Varvaruca, E., Weiss, G.S.: The Stokes conjecture for waves with vorticity. Ann. Inst. H. Poincaré Anal. Non Linéaire 29, 861–885 (2012)
29. Wu, T.Y.: Cavity and wakes flows. Annu. Rev. Fluid Mech. 4, 243–284 (1972)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.