Renormalization and topological susceptibility on the lattice: SU(2) Yang-Mills theory.

Bartomeu Allés\textsuperscript{a}, Massimo Campostrini\textsuperscript{b}, Adriano Di Giacomo\textsuperscript{b}, Yiğit Gündüz\textsuperscript{c}, and Ettore Vicari\textsuperscript{b}

\textsuperscript{a} Departamento de Física Teórica y del Cosmos, Universidad de Granada, Spain.
\textsuperscript{b} Dipartimento di Fisica dell’Università and I.N.F.N., Pisa, Italy.
\textsuperscript{c} Hacettepe University Physics Department, Beytepe Ankara, Turkey.

The renormalization functions involved in the determination of the topological susceptibility in the SU(2) lattice gauge theory are extracted by direct measurements, without relying on perturbation theory. The determination exploits the phenomenon of critical slowing down to allow the separation of perturbative and non-perturbative effects. The results are in good agreement with perturbative computations.

\* Partially supported by MURST, by a CICYT contract, and by NATO (grant no. CRG 920028).
The topological susceptibility of the ground state of Yang-Mills theories is an important parameter for understanding the breaking of the U(1) axial symmetry. It is defined by the vacuum expectation value

\[ \chi = \int d^4x \langle 0 | T \{ q(x)q(0) \} | 0 \rangle , \]  

where

\[ q(x) = \frac{g^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x) \]

is the topological charge density. Lattice is the ideal tool to determine \( \chi \) (for a review cf. e.g. Ref. [1]).

On the lattice a topological charge density operator with the appropriate classical continuum limit can be defined as [2]

\[ q_L(x) = -\frac{1}{2^4 \times 32\pi^2} \sum_{\mu\nu\rho\sigma=\pm 1} \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left[ \Pi_{\mu\nu} \Pi_{\rho\sigma} \right] , \]

where \( \Pi_{\mu\nu} \) is the product of link variables around a plaquette. Its classical continuum limit is

\[ q_L(x) \underset{a \to 0}{\longrightarrow} a^4 q(x) + O(a^6) , \]

where \( a \) is the lattice spacing. We define the lattice bare topological susceptibility \( \chi^L \) from the correlation at zero momentum of two \( q^L(x) \) operators:

\[ \chi^L = \langle \sum_x q^L(x)q^L(0) \rangle = \frac{1}{V} \langle (\sum_x q^L(x))^2 \rangle . \]

\( \chi^L \) is connected to \( \chi \) by a non-trivial relationship: the presence of irrelevant operators of higher dimension in \( q^L(x) \) and in the lattice action induces quantum corrections. Eq. (4) must be therefore corrected by including a renormalization function \( Z(\beta) \) [3]:

\[ q_L(x) \underset{a \to 0}{\longrightarrow} a^4 Z(\beta) q(x) + O(a^6) . \]

Further contributions arise from contact terms, i.e. from the singular limit \( x \to 0 \) of Eq. (5). They appear as mixings with the trace of the energy-momentum tensor

\[ T(x) = \frac{\beta(g)}{g} F^a_{\mu\nu} F^a_{\mu\nu}(x) \]

and with the unity operator, which are the only available renormalization-group-invariant operators with dimension equal or lower than \( \chi \). Therefore the relationship between the lattice and the continuum topological susceptibility takes the form [4]
\[ \chi^L(\beta) = a^4 Z^2(\beta) \chi + a^4 A(\beta) \langle T(x) \rangle_{\text{N.P.}} + P(\beta) \langle I \rangle + O(a^6) , \] (8)

where N.P. denotes taking the non-perturbative part. \( T(x) \) is the correct operator to describe the mixing \(^5\), since it has no anomalous dimension. In order to be consistent with traditional notation, we will rescale \( T(x) \) to the gluon condensate \( G_2 \), which is defined by \(^6\)

\[ G_2 = -\frac{1}{4\pi^2 b_0} \langle T \rangle , \] (9)

where \( b_0 = 11N/48\pi^2 \) is the first coefficient of the \( \beta \)-function. In Eq. (8) the proportionality constant is fixed by requiring

\[ G_2 = \left\langle \frac{g^2}{4\pi^2} F^a_{\mu\nu} F^a_{\mu\nu}(x) \right\rangle + O(g^4) . \] (10)

We will then write Eq. (8) in the form

\[ \chi^L(\beta) = a^4 Z^2(\beta) \chi + a^4 A(\beta) G_2 + P(\beta) \langle I \rangle + O(a^6) . \] (11)

\( Z(\beta) \), \( P(\beta) \), and \( A(\beta) \) are ultraviolet effects, i.e. they have their origin in the ultraviolet cut-off-dependent modes. They can be computed in perturbation theory. In the present paper, we will show explicitly that \( Z(\beta) \), \( P(\beta) \) and \( A(\beta) \) (computed non-perturbatively) are well approximated by the first few perturbative terms. This fact is far from trivial, since in this case the perturbative series are not even Borel-summable (cf. Ref. [7]). Moreover, in principle they could be affected by non-perturbative contributions; however, there are arguments \(^8\) implying that non-perturbative effects should not appear in the first few terms of the perturbative expansion.

In Ref. [9] a “heating” method was proposed to estimate \( Z(\beta) \) and \( P(\beta) \) directly from numerical simulations, without using perturbation theory. This method has already been employed to study, on the lattice, the topological properties of 2-d \( \text{CP}^{N-1} \) models \(^9\) \(^\text{[6]}\), which are toy models enjoying many similarities with QCD. In the present paper we apply the heating method to the 4-d \( \text{SU}(2) \) Yang-Mills theory on the lattice, and we show that it can also be used to disentangle the contribution of the mixing with the operator \( T(x) \).

The main idea of the method is to exploit the fact that renormalizations are produced by short-ranged quantum fluctuations, at the scale of the cut-off \( a \), whereas physical effects like gluon condensation or topological properties involve distances of the order of the correlation length. When using local updating procedures in Monte Carlo simulations, fluctuations at distance \( l \sim a \) are soon thermalized, whereas fluctuations at the scale of the correlation length are critically slowed down when approaching the continuum limit. For a standard local algorithm, e.g. Metropolis or heat bath, the number of sweeps needed to thermalize the fluctuations at distance \( \xi \) should grow proportionally to \( \xi^z \), with \( z \simeq 2 \). Quantities like topological charge, involving changes of global properties of the configurations, are expected to be even slower to reach equilibrium, experiencing a more severe form of critical slowing down. This is suggested by the fact that in the cooling procedure \(^{13}\) the topological charge survives long after the disappearance of the string tension \(^{14}\). In large-\( N \) lattice \( \text{CP}^{N-1} \) models the autocorrelation time of the topological susceptibility grows exponentially with
respect to the correlation length \[\xi.\] A similar phenomenon has been observed in the 2-d $U(1)$ gauge model \[\xi.\]

The heating method consists in constructing on the lattice a smooth configuration carrying a definite topological charge $Q_0$, and heating it by a local updating procedure at a given value of $\beta$. Short-ranged fluctuations, contributing to $Z(\beta)$ and $P(\beta)$, are rapidly thermalized, while the initial global topological structure is preserved for a much larger number of local updatings. This allows us to obtain estimates of $Z(\beta)$ when heating instanton configurations and of the mixings when heating flat configurations.

In particular when heating a flat configuration (a configuration with zero fields, i.e. with all the link variables equal to the identity), which has zero topological charge, we expect the production of instantons at the scale of $\xi$ to take place much later than the thermalization of local quantum fluctuations. The above-mentioned considerations on critical slowing down effects lead to the expectation that, when $\xi \gg a$, the heating procedure on a flat configuration should show the following intermediate stages before reaching full equilibrium:

(a) Short-ranged fluctuations at $l \sim a$ contributing to $P(\beta)$ get thermalized in a number of updating sweeps independent of $\xi$. The signal of $\chi^L$ should show a plateau giving $P(\beta)$.

(b) When the number of heating sweeps $n$ increases up to $n \propto \xi^2$, fluctuations at $l \sim \xi$ start to be thermalized, and gluon condensate and its mixing with $\chi^L$ sets on. This should produce an increase in $\chi^L$.

(c) Since a more severe form of critical slowing down is expected to affect the topological properties, the $\chi^L$ signal should show a second plateau in which the topological charge is still zero and the whole signal is mixing to the identity operator and to the trace of the energy-momentum tensor. This can be checked by cooling back the sample of configurations and controlling that $Q = 0$.

(d) Eventually the modes responsible of the topological structure will be thermalized and $\chi^L$ will reach its equilibrium value.

Actually, in the SU(2) lattice gauge theory the situation is complicated by the fact that we are forced to work at small $\xi$ (low $\beta$): the term involving $\chi$ in Eq. (11) is exponentially suppressed with respect to the mixing with the identity operator; therefore it becomes rapidly smaller than the errors and is not detectable at larger $\xi$ (large $\beta$). Using the Wilson action, the optimal region where to investigate Eq. (11) is around $\beta = 2.5$. For these values of $\beta$, the correlation length as obtained from the square root of the string tension is $\xi_\sigma \simeq 4$, while from the lowest glueball mass one obtains $\xi_g \simeq 2$. Of course the relevant correlation length depends on the quantity we are studying. For example, since the gluon condensate is closely connected to the glueball propagation, the relevant correlation length should be $\xi_g$. In this case of small $\xi$, the first two regimes (a) and (b) in the heating procedure may become not clearly distinguishable. However, we will still have a clear intermediate plateau where the topological structure is trivial (stage (c)), allowing us to separate the pure topological contribution from the mixing terms in Eq. (11).

In Section II we present our numerical results. They include data from standard Monte Carlo simulations and from the heating procedure. In Section III we draw our conclusions.

II. NUMERICAL RESULTS
A. Monte Carlo simulations and perturbative results

We performed standard Monte Carlo simulations on a $12^4$ lattice using the Wilson action and collecting data for $\chi^L$ over an extended range of $\beta$. We employed the over-heat-bath updating procedure \[16\]. We also measured the topological susceptibility $\chi_{\text{cool}}$ by using the cooling method \[13,17\], which consists in measuring the topological susceptibility on an ensemble of configurations cooled by locally minimizing the actions (starting from equilibrium configurations). The topological content of the cooled configurations is measured by using $Q = \sum_x q^L(x)$. The topological susceptibility measured on cooled configurations is seen to gradually reach a long plateau. We estimate $\chi_{\text{cool}}$ from the plateau measurements. Data for $\chi^L$ and $\chi_{\text{cool}}$ are reported in Table I.

In Eq. (11), $Z(\beta), A(\beta)$ and $P(\beta)$ can be calculated in perturbation theory following the field theory prescriptions. The first few terms of the series are

$$Z(\beta) = 1 + \frac{z_1}{\beta} + \frac{z_2}{\beta^2} + ... , \quad (12)$$

where $z_1$ has been calculated finding $z_1 = -2.1448$ \[3\].

$$A(\beta) = \frac{b_2}{\beta^2} + \frac{b_3}{\beta^3} + ... , \quad (13)$$

where $b_2 = 1.874 \times 10^{-3}$ \[18\], and

$$P(\beta) = \frac{c_3}{\beta^3} + \frac{c_4}{\beta^4} + \frac{c_5}{\beta^5} + ... , \quad (14)$$

where $c_3 = 2.648 \times 10^{-4}$ \[2\] and $c_4 = 0.700 \times 10^{-4}$ \[19\].

While the first calculated terms of $P(\beta)$ fit the data well at large $\beta$ ($\beta \geq 4$), more terms must be included as $\beta$ decreases. Lacking an analytical calculation for these terms, one must fit them from the data. In order to estimate them we, fitted data for $\beta \geq 2.8$, where the whole non-perturbative signal is smaller than the errors. One more term proved to be sufficient for a fit with $\chi^2/\text{d.o.f} \simeq 1$. We found

$$c_5 = 3.6(4) \times 10^{-4} . \quad (15)$$

We extrapolate this result to get an estimate of $P(\beta)$ at $\beta \simeq 2.5$. An overall fit to the $\chi^L$ data of Table I gives a value of $c_5$ consistent with (13), and allows to extract the signal of dimension 4 in Eq. (11):

$$\frac{Z^2(\beta)\chi}{A^4_L} + \frac{A(\beta)G_2}{A^4_L} \simeq 0.3 \times 10^5 , \quad (16)$$

for $\beta \simeq 2.5$. There is no way within this method to separate the term proportional to $\chi$ from the mixing to $G_2$ without both a direct computation of $Z(\beta)$ and $A(\beta)$ and an independent determination of $\chi$, e.g. by cooling. In fact the term proportional to $G_2$ is indistinguishable from contributions $O(1/\beta^6)$ (or higher) in $Z^2(\beta)$.

Evidence for the existence of the mixing to $G_2$ was obtained by comparing different definitions of $\chi_L$ \[20\]. In what follows we will instead obtain the two contributions independently.
B. Heating an instanton configuration

We start from a configuration \( C_0 \) which is an approximate minimum of the lattice action and carries a definite topological charge \( Q^L_0 \) (typically \( Q^L_0 \approx \pm 1 \)). We heat it by a local updating procedure in order to introduce short-ranged fluctuations, taking care to leave the background topological structure unchanged. We construct ensembles \( C_n \) of many independent configurations obtained by heating the starting configuration \( C_0 \) for the same number \( n \) of updating steps, and average the topological charge over \( C_n \) at fixed \( n \). Fluctuations of length \( l \approx a \) should rapidly thermalize, while the topological structure of the initial configuration is left unchanged for a long time. If, for a given \( \beta \), we plot \( Q^L = \sum_x q^L(x) \) averaged over \( C_n \) as a function of \( n \), we should observe first a decrease of the signal, originated by the onset of \( Z(\beta) \) during thermalization of the short-ranged modes, followed by a plateau. The average of \( Q^L \) over plateau configurations should be approximately equal to \( Z(\beta) Q^L_0 \). Since we do not expect short-ranged fluctuations to be critically slowed down, the starting point of the plateau should be independent of \( \beta \).

In order to check that heating does not change the background topological structure of the initial configuration, after a given number \( n_c \) of heating sweeps we cool the configurations (by locally minimizing the action) and verify that the cooled configurations have topological charge equal to \( Q^L_0 \).

We remind that the size of our lattice is \( 12^4 \). As heating procedure we used the heat-bath algorithm, which is efficient in updating short-ranged fluctuations, but is severely affected by critical slowing down for larger modes. We constructed on the lattice an instanton configuration according to the method described in Ref. [21]. On a \( 12^4 \) lattice we found that the optimal size of the instanton is \( \rho = 4 \). We performed also a few cooling steps in order to smooth over the configuration at the lattice periodic boundary. After this procedure we end up with a smooth configuration with \( Q^L \approx 0.90 \) (on the lattice \( Q^L \) measured on a discrete approximation to an instanton is exactly 1 only for very large instantons and in the infinite-volume limit).

In Fig. 1 we plot \( Q^L(C_n)/Q^L_0 \), where \( Q^L(C_n) \) is the lattice topological charge \( Q^L \) averaged over the ensemble \( C_n \). The data in Fig. 1 were taken at \( \beta = 2.5 \) and \( \beta = 3.0 \). We see clearly a plateau starting from \( n = 4 \) for both values of \( \beta \). The check of the stability of the topological structure was performed at \( n_c = 5 \). According to the above-mentioned considerations, the value of \( Q^L/Q^L_0 \) at the plateau gives an estimate of \( Z(\beta) \). We repeated this procedure for other values of \( \beta \). The behavior of \( Q^L(C_n)/Q^L_0 \) is always very similar to the case reported in Fig. 1. The results are presented in Table II.

A fit of the data to a polynomial

\[
Z(\beta) = 1 - \frac{2.1448}{\beta} + \frac{z_2}{\beta^2}
\]  

(17)

gives \( z_2 = 0.48(4) \) with \( \chi^2/\text{d.o.f.} \approx 0.3 \).

In Ref. [4] \( z_2 \) was estimated by comparing \( \chi_{\text{cool}} \) with the signal of dimension 4, and it was determined to be \( \approx 1.2 \). However, it is clear from Eq. (17) that the estimate included the mixing with \( G_2 \). By using the new determination of \( Z(\beta) \), we can extract \( A(\beta) G_2 \) from the old data; approximating \( A(\beta) \) with the first term \( b_2/\beta^2 \) is consistent with the data and gives
\[ \frac{G_2}{\Lambda_L^4} \approx 0.5(2) \times 10^8. \] (18)

C. Heating a flat configuration

We now proceed to the analysis of the ensembles \( C_n \) of configurations obtained by heating the flat configuration (using the heat-bath algorithm), for several values of \( \beta \). In Figs. 2 and 3 we plot the average value of \( \chi^L \) as a function of the number \( n \) of heating steps respectively for \( \beta = 2.45 \) and \( \beta = 2.5 \). At \( n_c = 10 \) for \( \beta = 2.45 \) and \( n_c = 10, 15 \) for \( \beta = 2.5 \) we check by cooling that the topological charge is still zero.

For both values of \( \beta \) we observe long plateaus starting from \( n \approx 10 \), which are lower than the equilibrium values of \( \chi^L \) (see Table I), but also definitely higher than the estimate of \( P(\beta) \) obtained in Section II A. We repeated this procedure for other values of \( \beta \). The behavior of \( \chi^L(C_n) \) is always similar to the cases plotted in the figures. Data for the quantity measured on the plateaus \( \chi_{pl}^L \) are reported in Table III.

We believe that \( \chi_{pl}^L \) contains the mixings both to the identity operator and to the trace of the energy-momentum tensor for the following reasons:

(i) The plateaus observed are long and after cooling back we do not find any topological structure.
(ii) The plateau values of \( \chi^L \) are systematically higher then the values of \( P(\beta) \) obtained from Eqs. (14) and (15).
(iii) At \( \beta \approx 2.5 \), the correlation length relevant to the gluon condensate is small (it comes from the lowest gluebal mass: \( \xi_g \approx 2 \)), therefore at \( n \sim \xi_g^2 \) the fluctuations contributing to the gluon condensate start to be thermalized; for \( n \geq 10 \) they could be already approximately thermalized.
(iv) Data are not inconsistent with a first plateau at \( P(\beta) \), followed by a second one; they are also not inconsistent with the expected shift of the starting point of the second plateau, when \( \beta \) is increased from 2.45 to 2.5. However, since the change in correlation length is small, these phenomena can not be detected clearly within our error bars.

The quantity

\[ \chi_h(\beta) \equiv \frac{\chi^L(\beta) - \chi_{pl}^L(\beta)}{Z(\beta)} \] (19)

should then measure the physical topological susceptibility. Data for \( \chi_h \) are reported in Table IV. In order to evaluate \( \chi_h(\beta) \), we inserted into Eq. (13) the parametrization for \( Z(\beta) \) given by Eq. (14), using the fitted value of \( z_2 \).

III. CONCLUSIONS

In the previous section we obtained two independent estimates of the topological susceptibility: \( \chi_{cool} \), by cooling method, and \( \chi_h \), given by the relationship (19). The comparison is satisfactory, although \( \chi_{cool} \) seems to be systematically lower. This behavior could be explained by the fact that \( Q^L \) underestimates the topological charge content of the cooled
configurations (for the lattice size we are working with), as we found out explicitly when we constructed an instanton configuration on the lattice (cf. Section II B).

In the limit $\beta \to \infty$, the lattice spacing $a$ is given by the two-loop renormalization formula

$$a = \frac{1}{\Lambda_L} f(\beta), \quad f(\beta) = \left( \frac{6}{11} \pi^2 \beta \right)^{51/121} \exp \left( -\frac{3}{11} \pi^2 \beta \right).$$

(20)

In Fig. 4 we plot $\chi_h/\Lambda_L^4$ and $\chi_{cool}/\Lambda_L^4$ versus $\beta$. Scaling is quite good within the errors. By fitting to a constant we found:

$$\frac{\chi_h}{\Lambda_L^4} = 3.5(4) \times 10^5, \quad \frac{\chi_{cool}}{\Lambda_L^4} = 2.8(2) \times 10^5.$$  

(21)

According to the arguments given in Section II C, the difference:

$$M(\beta) \equiv \chi_{PL}^L(\beta) - P(\beta)$$

should be proportional to the gluon condensate. Assuming that the first non-trivial term of the perturbative expansion ( Eq. (13) ) gives a good approximation of the function $A(\beta)$, we can estimate $G_2$ by

$$G_2 \approx G_M \equiv \frac{M(\beta)}{b_2/\beta^2}.$$  

(23)

In Fig. 4 we plot $G_M/\Lambda_L^4$. Although the errors are large, the signal is clear. Fitting the data to a constant we obtained

$$\frac{G_M}{\Lambda_L^4} = 0.38(6) \times 10^8.$$  

(24)

This result is consistent with Eq. (18). This estimate can also be compared with an independent determination obtained by studying the plaquette operator [22,23]: $G_2/\Lambda_L^4 = 0.30(2) \times 10^8 [5]$. The comparison is again satisfactory.

We have shown that Eq. (11) is physically meaningful on the lattice, since it separates contributions having different physical origin. Mixings with the unity operator and with the trace of the energy-momentum tensor are well defined, and we have estimated them. Estimates of $\chi$ and $G_2$ coming from Eq. (11) and the heating method are consistent with those obtained independently by other methods.

Finally it is pleasant to notice that our results strongly support the feasibility of the determination of $G_2$ from the lattice topological susceptibility in full QCD, i.e. in the presence of dynamical quarks, as proposed in Ref. [24].

Acknowledgments

We would like to thank H. Panagopoulos for many useful discussions; two of us (B. A. and Y. G.) thank the theory group in Pisa for hospitality.
REFERENCES

[1] A. Di Giacomo, Proceedings of the QCD '90 conference, Montpellier, 1990, Nucl. Phys. (Proc. Suppl.) B23 (1991) 191.
[2] P. Di Vecchia, K. Fabricius, G.C. Rossi and G. Veneziano, Nucl. Phys. B192 (1981) 392.
[3] M. Campostrini, A. Di Giacomo, and H. Panagopoulos, Phys. Lett. B212 (1988) 206.
[4] M. Campostrini, A. Di Giacomo, H. Panagopoulos, and E. Vicari, Nucl. Phys. B329 (1990) 683.
[5] A. Di Giacomo, H. Panagopoulos, and E. Vicari, Phys. Lett. B240 (1990) 423.
[6] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B147 (1979) 385, 448, 519.
[7] S. David, Nucl. Phys. B209 (1982) 433; Nucl. Phys. B234 (1984) 237.
[8] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B249 (1985) 445.
[9] A. Di Giacomo and E. Vicari, Phys. Lett. B 275 (1992) 429.
[10] A. Di Giacomo, F. Farchioni, A. Papa, and E. Vicari, Phys. Rev. D 46, 4630 (1992).
[11] M. Campostrini, P. Rossi, and E. Vicari, Phys. Rev. D 46, 2647 (1992).
[12] M. Campostrini, P. Rossi, and E. Vicari, Phys. Rev. D 46, 4643 (1992).
[13] M. Teper, Phys. Lett. B171 (1986) 81,86.
[14] M. Campostrini, A. Di Giacomo, M. Maggiore, H. Panagopoulos, and E. Vicari, Phys. Lett. B225 (1989) 403.
[15] M. L. Laursen, J. Smit, and J. C. Vink, Phys. Lett. B262 (1991) 467.
[16] R. Petronzio and E. Vicari, Phys. Lett. B254 (1991) 444.
[17] E. M. Ilgenfritz, M. L. Laursen, M. Müller-Preußker, G. Schierholz, and H. Schiller, Nucl. Phys. B268 (1986) 693.
[18] B. Allés and M. Giannetti, Phys. Rev. D 44, 513 (1991).
[19] B. Allés, M. Campostrini, A. Feo, and H. Panagopoulos, Pisa preprint IFUP-TH 31-1992.
[20] B. Allés, A. Di Giacomo, and M. Giannetti, Phys. Lett. B249 (1990) 490.
[21] J. Hoek, Comp. Phys. Comm. 61 (1990) 304.
[22] A. Di Giacomo and G. Rossi, Phys. Lett. B100 (1981) 692.
[23] A. Di Giacomo, in Non-Perturbative Methods, S. Narison editor, World Scientific 1985.
[24] B. Allés and A. Di Giacomo, Phys. Lett. B294 (1992) 269.
FIGURES

FIG. 1. Determination of the multiplicative renormalization $Z(\beta)$. Dashed lines indicate the value of $Z(\beta)$ estimated by averaging data on the plateau.

FIG. 2. $\chi^L$ vs. the number of updatings when heating a flat configuration at $\beta = 2.45$. The dashed lines indicate the equilibrium value of $\chi^L$ and the dot-dashed lines the value of $P(\beta)$ (with the respective errors). The solid line shows the value of $\chi^L_{pl}$ estimated by averaging data on the plateau. The result obtained by cooling the configurations after $n_c$ heating sweeps is indicated with the symbol ■.

FIG. 3. $\chi^L$ vs. the number of updatings when heating a flat configuration at $\beta = 2.5$. The dashed lines indicate the equilibrium value of $\chi^L$ and the dot-dashed lines the value of $P(\beta)$ (with the respective errors). The solid line shows the value of $\chi^L_{pl}$ estimated by averaging data on the plateau. The results obtained by cooling the configurations after $n_c$ heating sweeps are reported with the symbol ■.

FIG. 4. Plot of $\chi_h/\Lambda^4_L$ ( ◊), $\chi_{cool}/\Lambda^4_L$ ( × ) and $G_M/\Lambda^4_L$ ( ■ ) vs. $\beta$. Data for $\chi_h$ and $\chi_{cool}$ are slightly displaced for sake of readability.
### TABLE I

$\chi^L$, $\chi_{\text{cool}}$, obtained by standard Monte Carlo simulations, and $\chi_h$ vs. $\beta$. The column “stat” reports the statistics of the Monte Carlo simulations.

| $\beta$  | stat | $10^5 \chi^L$ | $10^5 \chi_{\text{cool}}$ | $10^5 \chi_h$ |
|----------|------|---------------|-----------------|---------------|
| 2.45     | 80k  | 3.14(3)       | 7.7(5)          | 9.8(1.8)      |
| 2.475    | 80k  | 2.91(3)       | 6.5(4)          | 7.6(1.2)      |
| 2.5      | 60k  | 2.69(4)       | 4.4(3)          | 5.1(1.1)      |
| 2.525    | 80k  | 2.56(2)       | 3.0(3)          | 4.6(1.0)      |
| 2.7      | 4k   | 1.84(6)       |                 |               |
| 2.8      | 8k   | 1.60(4)       |                 |               |
| 2.9      | 4k   | 1.36(4)       |                 |               |
| 3.0      | 7k   | 1.21(3)       |                 |               |
| 3.25     | 4k   | 0.93(3)       |                 |               |
| 3.5      | 4k   | 0.69(2)       |                 |               |
| 3.75     | 4k   | 0.58(2)       |                 |               |
| 4.0      | 3k   | 0.45(2)       |                 |               |
| 4.5      | 3k   | 0.33(1)       |                 |               |
| 5.0      | 3k   | 0.227(8)      |                 |               |
| 6.0      | 3k   | 0.127(5)      |                 |               |

### TABLE II

Measure of the multiplicative renormalization of $Q^L$, starting from an instanton of size $\rho = 4$ on a lattice $12^4$ and with $Q^L_0 = 0.90$. The estimate of $Z(\beta)$ is taken by averaging the data in the range of $n$ reported in the column “plateau”. Since data on the plateau are correlated, as error we report the typical error of data in the plateau.

| $\beta$  | stat | plateau | $Z^L$ |
|----------|------|---------|-------|
| 2.45     | 2k   | 4–7     | 0.20(2) |
| 2.475    | 3k   | 4–7     | 0.22(2) |
| 2.5      | 5k   | 4–7     | 0.22(1) |
| 2.55     | 3k   | 5–7     | 0.23(1) |
| 2.6      | 2k   | 4–7     | 0.25(2) |
| 2.8      | 1k   | 5–7     | 0.32(2) |
| 3.0      | 0.6k | 4–7     | 0.33(2) |
TABLE III. $\chi_{\text{pl}}^L$ vs. $\beta$ starting from a flat configuration. $\chi_{\text{pl}}^L$ is estimated by averaging the data in the range of $n$ reported in the column “plateau”. As error we report the typical error of data in the plateau.

| $\beta$ | stat | plateau | $10^5 \chi_{\text{pl}}^L$  |
|-------|------|---------|-----------------------------|
| 2.45  | 5k   | 10–20   | 2.72(6)                     |
| 2.475 | 8k   | 10–15   | 2.56(4)                     |
| 2.5   | 10k  | 11–15   | 2.44(4)                     |
| 2.525 | 10k  | 10–15   | 2.32(4)                     |