Do spatially non-uniform phases of matter with no long-range order exist?

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In this Letter, the existence of spatially non-uniform phases with no long-range order is investigated in continuum models of first order phase transitions with quartic non-linearity. The central result of the paper is the development of a mathematical method allowing to find “disordered” solutions (infinite sets of spatially non-uniform analytical solutions with no long range order) to partial differential equations. The new method is applied for the Gaussian measure, and it has been found that the Gaussian phase is not present in the investigated model family. In addition to this exact result we show that the method is adapted to predicting disordered phases and phase transitions in general.

Prior to the discovery of topological phase transitions, spontaneous symmetry breaking was the major concept of classifying transition processes between phases of matter. When a highly symmetric phase becomes unstable against natural fluctuations, it transforms into a lower energy phase, and the transition is accompanied with the reduction of the symmetry of the initial phase. At present, a universal nature of spontaneous symmetry breaking transitions is that the final state displays order in the sense of symmetry properties. In addition to ordered states, however, spatially non-uniform stationary states with no long-range order (henceforth called “disordered” in this paper) also emerge in pattern formation models [1]. To date, however, there is no mathematical tool to confirm or disprove the existence of disordered phases (a set of infinitely many equal stationary points of the free energy) [2] and potential phase transitions involving them. In this Letter the development and test of a novel mathematical concept is reported, which extends the domain of possible analytical solutions to differential equations with disordered functions, and therefore can be used to predict disordered phases and phase transitions (involving these phases) in continuum models of first order phase transformations. Since the underlying mathematical idea is not restricted to partial differential equations, the method is expected to have major implications in all fields of condensed matter physics where disordered structures emerge.

Since physical phases represent stationary points of the free energy density of the system, the configurations forming the disordered phase must be analytical solutions to the Euler-Lagrange (EL) equation. Consequenly, to describe a spontaneous uniform to disordered transition, first we need to develop a mathematicial method which allows us to find analytical disordered solutions to partial differential equations. The essence of the method is the exact transformation of the EL equation into a PDE for the parameter of a probability measure that generates the disordered solutions of the EL equation. This transformation is based on the “Concentration of Measure Phenomenon” (CMP) [3], which states that a balanced, Lipschitz-continuous function of a large number of independent random variables (henceforth denoted by \( X \)) is nearly constant. More formally, \( P[|f(X)−\bar{f}|>\lambda]<\exp(−\alpha_N\lambda) \), i.e., the probability that the random variable \( Y := f(X) \) is outside of the \( \lambda \) vicinity of the expectation value decays exponentially with \( \lambda \), while \( \alpha_N \) is a positive, strictly increasing function of the number of the random variables \( N \). If \( \lim_{N\rightarrow\infty}\alpha_N\rightarrow\infty \), \( P[|f(X)−\bar{f}|=1 \) in this case, i.e., the random variable \( Y = f(X) \) becomes “almost surely” constant in this limit (the term “almost surely” means that the probability of that a realisation of \( Y \) is not constant is 0). A well-known manifestation of this property is the equivalence of statistical physical ensembles in the thermodynamic limit [4]. In particular, the relative amplitude of energy fluctuations vanish for an infinitely large system, i.e., \( \lim_{N\rightarrow\infty}(\langle E(X)/E−1 \rangle_2^2 \) = 0 for the Gibbs measure, and therefore the energy density \( c := E/V \) is universal: \( P[c(X) = \bar{c}] = 1 \). We note that this holds even when the random variables are correlated, and the energy is not Lipschitz continuous, though the price is that the energy is constant only for certain measures.

Let now \( x := (\phi_{k_1}, \phi_{k_2}, \ldots) \) be an ordered list of the Fourier coefficients of a periodic scalar field \( \phi(x) \) in \( d \) spatial dimensions, and let the vector-vector function \( F(x) := \{ F_{k_1}(x), F_{k_2}(x), \ldots \} \) be the Fourier representation of the partial differential equation \( \hat{O}(\phi(x)) = 0 \). Since \( \mathbb{Z}^d \) is bijective with \( \mathbb{N} \) for finite \( d \), the set of those periodic functions in \( d \) spatial dimensions whose Fourier coefficients exist is bijective with the infinite dimensional Euclidean space, and is therefore measurable in the probability sense. Consequently, if there exist a non-singular (in at least one component of \( x \)) measure \( d\mu(x) \) for which \( P[F(X) = 0] = 1 \) holds, the real-space equivalent \( \phi(r) \) of a realisation of the random vector \( X \) is almost surely a solution to the differential equation \( \hat{O}(\phi(r)) = 0 \). Finally, if the concept can be extended to infinite volume (corresponding to true no long-range order), and the solutions represent a unique minimum of the free energy density, then the set of these solutions form a thermodynamic phase. In the rest of the paper we work the above idea...
and investigate its predictions for the Gaussian measure in the Phase-Field Crystal (PFC) model.

In the PFC model, the dimensionless free energy of the inhomogeneous system is defined as \[ F[\varphi] := \int \text{d}r \left\{ \frac{1}{2} \varphi \mathcal{L} \varphi + \frac{1}{4} \varphi^4 \right\} , \] (1)

where the scalar field \( \varphi(\mathbf{r}, t) \) is related to the dimensionless coarse-grained one-particle density of the system, \( \mathcal{L} = (1 + \nabla^2)^2 - \epsilon \) is a linear operator favouring the \( |k| = 1 \) wavelength, \( \epsilon \) is a scalar parameter, while \( \int \text{d}r \ (\varphi - \bar{\varphi}) = 0 \) emerges from mass conservation. In addition to the uniform, 1D periodic, 2D hexagonal and 3D body-centred cubic (bcc) stationary points of the free energy functional, disordered minima were also found in numerical simulations [12], which suggests the existence of (a) disordered phase(s) in the theory. The stationary points of \( F[\varphi] \) can be determined by solving the Euler-Lagrange equation \( \delta_{\varphi} F = \mathcal{L} \varphi + \varphi^3 = \Lambda \), where \( \delta_{\varphi} F \) denotes the first functional derivative of \( F[\varphi] \) with respect to \( \varphi \), while \( \Lambda \in \mathbb{R} \) is a Lagrange multiplier emerging from mass conservation. Introducing \( \phi := \varphi - \bar{\varphi} \) and re-arranging the equation results in:

\[
(\mathcal{L} + 3 \bar{\varphi}^2) \phi + (3 \bar{\varphi}) \phi^2 + \phi^3 - \nu = 0 ,
\] (2)

where \( \int \text{d}r \phi = 0 \). To find non-uniform analytical solutions with no long-range order to Eq. (2), we select \( \phi(\mathbf{r}) \) randomly according to the probability measure \( d\mu(\phi) \) on the set of those periodic functions in three spatial dimensions with period \( L \), for which the Fourier coefficients exist. Accordingly, \( \phi(\mathbf{r}) \) can be represented by the vector of its Fourier coefficients (henceforth denoted by \( \Phi \)). Assuming that \( \phi(\mathbf{r}) \) is smooth enough so that \( \nabla^2 \phi \) and \( \nabla^4 \phi \) exist, the Fourier representation of Eq. (2) reads:

\[
\psi_k(\Phi) := [\mathcal{L}(k) + 3 \bar{\varphi}^2] \phi_k + (3 \bar{\varphi}) \sum_{k', k''} \phi_{k'} \phi_{k''} \delta_{k' + k'' - k} + \sum_{k', k'', k'''} \phi_{k'} \phi_{k''} \phi_{k'''} \delta_{k' + k'' + k''' - k} - \nu \delta_k = 0 ,
\] (3)

where \( \phi_k \) is the Fourier coefficient of \( \phi(\mathbf{r}) \) to the wave number \( \mathbf{k} = (2\pi/L)\mathbf{n} \) (where \( \mathbf{n} \in \mathbb{Z}^3 \)), \( \mathcal{L}(k) = (1 - k^2)^2 - \epsilon \), while \( k', k'' \) and \( k''' \) run for all possible wave numbers, and \( \nu \) is responsible for \( \phi_0 = 0 \). \( \Phi \) is a solution of the Euler-Lagrange equation if and only if \( \psi_k(\Phi) = 0 \) for any \( \mathbf{k} \). In a disordered solution the components of \( \Phi \) are randomly selected according to a probability measure, and therefore we need \( P[\psi_k(\Phi) = 0] = 0 \), which, according to the limit case of the CMP discussed above, is a valid mathematical scenario. A practical reformulation of the equation \( P[\psi_k(\Phi) = 0] = 0 \) reads:

\[
\langle |\psi_k(\Phi)|^2 \rangle = 0 \quad \text{for} \quad \forall \mathbf{k} ,
\] (4)

where \( \langle \cdot \rangle = \int \{ \cdot \} d\mu(\phi) \) stands for the expectation value, and \( |\cdot|^2 \) denotes the squared magnitude of a complex number. If Eq. (4) applies for a probability measure, any realisation of the random function (the real-space equivalent of \( X \)) is an analytical solution to the Euler-Lagrange equation.

In practice, solving Eq. (4) for \( d\mu(\phi) \) means solving the equation for the parameter of a parametrised probability measure. In this work we choose the Gaussian measure, which is indicated by time-dependent simulations in the linearly unstable liquid regime of the phase diagram near the critical point and the liquid instability limit (see Fig. 1). The amplitude of the disordered pattern develops exponentially in this regime, and (regardless of the system size) the pattern finally collapses into the lattice symmetric bcc energy minimum. Consequently, the Gaussian solution is expected to be (at least) unstable in this model, which must be recovered by our mathematical method. Accordingly, we set \( \phi_k := \alpha_k + \beta_k \), where \( \alpha_k \) and \( \beta_k \) are zero-mean Gaussian random numbers with the following correlations (emerging from the translational and rotational invariance of the solution, and from \( \phi(\mathbf{r}) \in \mathbb{R} \)): \( \langle \alpha_k \alpha_{k'} \rangle = (\sigma_k^2/2) \delta_{k-k'} - \delta_k \delta_{k'} \), \( \langle \beta_k \beta_{k'} \rangle = (\sigma_k^2/2) \delta_{k-k'} - \delta_k \delta_{k'} \), where \( \delta_k = \delta_k \delta_{k'} \delta_{k''} \) is the three-dimensional Kronecker symbol. Using Eq. (4) in Eq. (4), the emerging expectation values can be calculated by using Isserlis’ Theorem [13]. Using the discrete number.

\[
\int \text{d}r \left\{ \frac{1}{2} \varphi \mathcal{L} \varphi + \frac{1}{4} \varphi^4 \right\} ,
\]

FIG. 1. Disordered state in the conserved Swift-Hohenberg dynamics \( \partial_t \phi = \nabla^2(\Delta F) \) at \( \bar{\varphi} = -0.025 \) and \( \epsilon = 0.001922 \) in a numerical simulation on a 256 \( ^3 \) grid. The initial condition was \( \phi(\mathbf{r}, 0) = 10^{-3} \varphi + U[-1, 1] \), where \( U[-1, 1] \) stands for a uniformly distributed random number on the interval \((-1, 1)\).

(a) Scaled dimensionless free energy density \( \Delta f = 10^7 (f - f_0) \) vs scaled dimensionless time \( t/10^3 \). The dashed section of the curve indicates the exponential growth of the disordered pattern in the small amplitude regime; (b) 2D cross section of the pattern at the point indicated on panel (a); (c) Numerical realisation of a coloured Gaussian random field with correlation function \( C(r) \propto \sin(r)/r \). Note the structural similarity with panel (b).
form of the trivial correlators yields \( \langle \prod_{i=1}^{2m-1} \phi_{k_i} \rangle = 0 \) and 
\( \langle \prod_{i=1}^{2m} \phi_{k_i} \rangle = \left( \prod_{i=1}^{2m} \sigma_{k_i} \right) \sum \delta_{k-k'} \), where \( m \in \mathbb{N} \),
while the notation \( \sum \prod \) stands for summing over all distinct ways of partitioning the set \( \{ k_1, \ldots, k_{2m} \} \) into \( m \) pairs \( (k', k'') \), and each terms is the product of the Kronecker symbol over the pairs. We note that the expectation values exist since the space of the Fourier coefficients is bijective with the infinite dimensional Euclidean space, which is measurable in the probability sense (see the classical Wiener measure, for instance). After lengthy but straightforward algebraic manipulations Eq. (4) reads:
\[
[\mathcal{L}(k) + 3(\varphi^2 + c_0)]^2 \sigma_k^2 + (18 \varphi^2) \sum_{k'} \sigma_{k'}^2 \sigma_k^2 = \omega \delta_k \, ,
\]
where \( c_0 = \sum_k \sigma_k^2 \), and \( \omega \in \mathbb{R} \) is responsible for the condition \( \sigma_0^2 = 0 \) (emerging from \( \phi_0 = 0 \)). Since \( \sigma_k^2 \geq 0 \) must hold, the only exact solution of the above equation is \( \sigma_k^2 = 0 \). Since Eq. (6) is exact for Eq. (4), the pure Gaussian solution is absent in the PFC model, which corresponds to our expectations. In addition, the same result holds for arbitrary \( \mathcal{L}(k) \), and therefore there is no pure Gaussian solution emerges in any theory with free energy functional defined by Eq. (1).

The above result indicates that disordered solutions in \( \varphi \)'s scalar field theories of first order phase transformations contain further components, which might be a deterministic contribution to the solution, the presence of correlations in the Fourier space, or even a different measure. However, instead of trying to find out which of these components dominate in the PFC model, we focus on demonstrating the predictive power of the developed mathematical method in the present scenario. Since true disordered solutions display no long-range order, we take Eq. (5) in the \( L \to \infty \) (aperiodic) limit. Using \( \sigma_k^2 (k) \equiv \lim_{L \to \infty} [L/(2\pi)]^3 \sigma_k^2 < \infty \), the inverse Fourier transform of Eq. (5) reads:
\[
[\hat{\mathcal{L}} + 3(\hat{\varphi}^2 + C_0)]^2 \mathcal{C} + (18 \varphi^2) \mathcal{C} + 6 \mathcal{C} = \omega \, ,
\]
where \( \mathcal{C}(r) = \int dk \{ \sigma^2(k)e^{-i k \cdot r} \} \) is the correlation function of the coloured Gaussian random field, while \( C_0 = \mathcal{C}(0) \). Furthermore, the condition \( \sigma^2(0) = 0 \) indicates \( \mathcal{C}(r) = 0 \), while the lack of long-range order indicates \( \lim_{r \to \infty} \mathcal{C}(r) = 0 \), thus yielding \( \omega = 0 \). Any solution of Eq. (6) satisfying the above conditions and \( \sigma^2(k) \geq 0 \) is a Gaussian solution of the PFC model. An isotropic solution of Eq. (6) reads [13]:
\[
\mathcal{C}(r) = A^2 \gamma \sin(r)/r + O(\gamma^2) \, ,
\]
where \( \gamma = \epsilon - 3 \bar{\varphi}^2 \geq 0 \) is a small parameter (distance from the liquid stability line), while \( A \sqrt{\pi} \) is the characteristic amplitude of the disordered pattern. Since \( \sigma(k) \propto A^2 \gamma \sin(k-1) \geq 0 \), an approximating analytical solution exists in the leading order of \( \gamma \), meaning that the realisations of a coloured Gaussian random field with correlation function \( \mathcal{C}(r) = A^2 \gamma \sin(r)/r \) are approximating analytical solutions to the Euler-Lagrange equation for some \( A > 0 \) (to be determined later). Moreover, since \( \mathcal{C}(r) \) is smooth, \( \phi(r) \) is also smooth in the realisations of the random field, and therefore Eq. (3) is valid [15]. From the Fourier transform of Eq. (6) it is clear that the solution terminates in a higher order of \( \gamma \) (where \( \sigma^2(k) \geq 0 \) does not hold any more), but now we neglect this information and work further with the leading order solution to demonstrate the predictive power of the mathematical method.

The first step is to calculate the amplitude in Eq. (7) for which the disordered configurations represent approximate analytical solutions to EL equation in the leading order of \( \gamma \). For this we need to calculate the expectation value and the variance of the energy density for the Gaussian phase. Using \( \varphi = \bar{\varphi} + \phi \) and the Fourier representation of \( \phi \) in \( f_L = \mathcal{F}[\varphi]/L \) results in:
\[
f_L(\Phi) = f_0 + \frac{1}{2} \sum_{k,k'} [\mathcal{L}(k) + 3 \bar{\varphi}^2] \phi_k \phi_{k'} \delta_{k+k'} \\
+ \bar{\varphi} \sum_{k,k',k''} \phi_k \phi_{k'} \phi_{k''} \delta_{k+k'+k''} \\
+ \frac{1}{4} \sum_{k,k',k'',k'''} \phi_k \phi_{k'} \phi_{k''} \phi_{k'''} \delta_{k+k'+k''+k'''} 
\]
where \( f_0 = [(1-\epsilon)/2] \bar{\varphi}^2 + \bar{\varphi}^4/4 \) is the free energy density of the uniform phase of density \( \bar{\varphi} \). The average free energy density can be calculated by taking the expectation value on Eq. (8), using Iserlis’ Theorem to evaluate the terms \( \langle \prod_{i=1}^{2m} \phi_{k_i} \rangle \), then taking the \( L \to \infty \) limit and using \( \sigma^2(2) = 1/(2\pi)^3 \int dr \{ \mathcal{C}(r) \exp(-i k \cdot r) \} \), thus yielding:
\[
\bar{f} = f_0 + (1/2) \lim_{r \to 0} [\mathcal{L} + 3 \bar{\varphi}^2] \mathcal{C} + (3/4) C^2 
\]
A similar calculation can be performed to determine the variance of the energy density \( \sigma_f^2 = \lim_{L \to \infty} \langle (f_L - \langle f_L \rangle)^2 \rangle \), yielding:
\[
\sigma_f^2 = (1/2) \| \hat{\mathcal{Q}} \mathcal{C} \| + (6 \bar{\varphi}) \| \mathcal{C} \| + (3/2) \| \mathcal{C} \|^2 
\]
where \( \mathcal{Q} = \mathcal{L} + 3(\bar{\varphi}^2 + C_0) \), \( \| \| = \lim_{L \to \infty} L^{-3} \int_{\Omega_L} dr \{ \} \) stands for the spatial average in the infinite volume limit \( \Omega_L \) is any period in the 3D space. Since \( \lim_{R \to \infty} R^{-3} \int_{[1]}(\sin(r)/r)^n r^2 \, dr = 0 \) for \( n = 2, 3 \) and 4, \( \sigma_f^2 = 0 \) for \( \mathcal{C}(r) \propto \sin(r)/r \). Since the free energy density of the disordered approximate solutions is unique, they form a phase in the PFC model. Again, even if the solution of Eq. (6) eventually terminates in a higher order of \( \gamma \), the mathematical method is capable of predicting disordered phases in continuum theories, which is a major achievement. Using now Eq. (7) in Eq. (9) results in the Landau free energy \( \bar{f} - f_0 = (\gamma^2/2) A^2 [\mathcal{Q}/2, A^2 - 1] \) with a maximum at \( A = 0 \) (uniform) and two minima at \( A = \pm 1/\sqrt{3} \) (approximate Gaussian), which indicates...
a spontaneous symmetry breaking (no first order transition) from the uniform phase to the Gaussian one. Since the cubic term of the free energy density is identically zero for the Gaussian measure, this result also accords with our expectations.

To determine the stability of the individual configurations in the approximate Gaussian phase, we start from the Taylor expansion of the energy density around a solution of the Euler-Lagrange equation, which reads:

\[
\Delta f_L(\Phi, \delta \Phi) := f_L(\Phi + \delta \Phi) - f_L(\Phi) = \sum_{n=2}^{4} \frac{1}{n!} \sum_{k_1, \ldots, k_n} d_{k_1, \ldots, k_n}^{(n)} \prod_{i=1}^{n} \delta \phi_{k_i},
\]  

where \(d_{k_1, \ldots, k_n}^{(n)} = [\partial^n f_L/(\partial \phi_{k_1} \ldots \partial \phi_{k_n})] |_{\Phi}\). Similarly to the free energy density calculations, we calculate the expectation value and the variance of the energy density difference \(\Delta f_L(\Phi, \delta \Phi)\) for a fixed \(\delta \psi(r) := \sqrt{\gamma} \eta \psi(r)\) perturbation in the infinite volume limit, where the magnitude of the perturbation is measured in \(\sqrt{\gamma}\) units (the characteristic amplitude of the disordered approximate solutions to the EL), \(O(\psi(r)) = 1\), and \(|\eta| \ll 1\). Since the free energy density of the Gaussian phase is unique, the calculations yield:

\[
\Delta f = (1/2) ||\delta \phi \hat{Q} \delta \phi|| + \varphi ||\delta \phi^3|| + (1/4) ||\delta \phi^4|| \quad (12)
\]

\[
\sigma^2_{\Delta f} = (3 \varphi^2) ||\delta \phi^2(C \otimes \delta \phi^2)|| + (6 \varphi^2) ||\delta \phi^3(C \otimes \delta \phi^2)|| + (9/2) ||\delta \phi^2(C^2 \otimes \delta \phi^2)|| + ||\delta \phi^3(C \otimes \delta \phi^3)||. \quad (13)
\]

where \((f \circ g)(r) = L^{-3} \int_{\Omega} dr' \{f(r') \eta(r - r')\}\). Assuming \(0 < ||\psi^2(C \otimes \psi^2)|| < \infty\), the leading order of Eq. (13) reads \(\sigma^2_{\Delta f} \propto \varphi^2 \gamma^3 \eta^4\). The Gaussian phase is stable if (but not only if) \(\Delta f > 0\) and \(\sigma^2_{\Delta f} = 0\) for any non-trivial perturbation. Since \(Q(k) = (1 - k^2)^2\) in the leading order of \(\gamma\) for Eq. (1), \(\Delta f \propto \gamma \eta^2 > 0\) in the leading order for any perturbation containing at least one \(k \neq 1\) Fourier mode. The corresponding variance of the relative energy response \(||\Delta f|| := \Delta f / \Delta f\) is \(\sigma^2_{||\Delta f||} = \sigma^2_{\Delta f} / \Delta f^2 \propto \varphi^2 \gamma\). Assuming normal distribution for \(\Delta f\), \(\mathbb{P}(\Delta f \leq 0) = (1/2) \text{erfc}[1/(\sigma_{\Delta f} \sqrt{2})]\), and therefore there exists a finite \(\gamma_0 > 0\) for any \(\delta \phi\) so that the Gaussian phase is practically stable for \(0 < \gamma < \gamma_0\) against perturbations containing at least one \(|k| \neq 1\) Fourier mode. For perturbations consisting of only \(|k| = 1\) modes, however, the quadratic term \((1/2) ||\delta \phi \hat{Q} \delta \phi|| = (1/2) \sum_k Q(k) ||\delta \phi_{k}||^2\) cancels in Eq. (12), which directly emerges from \(Q(1) = 0\). For perturbations providing \(||\psi^3|| \neq 0\), \(\Delta f \propto \varphi^{3/2} \eta^3\), which can be negative. It is easy to see that \(||\psi^3|| > 0\) for the bcc and 2D hexagonal structures, and therefore \(\Delta f < 0\) for \(\varphi < 0\) and \(\eta > 0\), meaning that the approximating Gaussian phase is unstable against perturbations displaying the structure of the stable phase of the system. Consequently, our analytical results predict an unstable Gaussian phase in the PFC model in the leading order of \(\gamma = \epsilon - 3 \varphi^2 \geq 0\), thus confirming the absence of this phase and the collapse of the Gaussian pattern into the stable bcc phase in time-dependent numerical simulations.

To summarise the results, a novel mathematical tool has been developed and tested to investigate the existence of spatially non-uniform phases without long-range order in continuum mean-field theories of first order phase transitions. The general idea of finding “disordered” analytical solutions to time-independent deterministic partial differential equation relies on the infinite dimensional limit case of the Concentration of Measure Phenomenon. The method was implemented for coloured Gaussian random fields in continuum mean-field theories with quartic non-linearity, and it has been found that the Gaussian phase is absent in this model family. In addition to this analytically exact result, the practicality and predictive power of the general mathematical concept have also been demonstrated by calculating the properties of an approximating Gaussian phase in the leading order of a small parameter in the Phase-Field Crystal model. It has been found that our method is capable of predicting disordered phases (a disordered phase is an infinite set of spatially non-uniform analytical solutions to the Euler-Lagrange equation with no long-range order and unique free energy density), and applicable to determine the stability properties of these.

The next step is to find a disordered phase far from the critical point in the PFC, where, according to the results of numerical simulations, the disordered configurations are stable. The method will also be applied for more quantitative mean-field theories (such as the Classical Density Functional Theory of soft matter and the Fundamental Measure Theory), where conditions for the existence of the disordered phase will be given in terms of physical quantities. Since we search for disordered analytical solutions of Euler-Lagrange equations directly (instead of minimising only a parametrised free energy density functional, which might miss the actual stationary points), this step has great potential for achieving a major progress in glass theory. In addition, developing the analogue of the Kramers/Fokker-Planck equation governing the time evolution of parameters of probability measures on continuum dynamics is also vital. Finally, we note that the general mathematical idea is not restricted to continuum models, since the only requirement on the level of the mathematical model is an (at least) countable infinitely many dimensional representation of the physical state of the system. Consequently, our results are expected to have important implications in all fields of condensed matter research where static structures without long-range order emerge.

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