Extending du Bois-Reymond’s Infinitesimal and Infinitary Calculus Theory

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Abstract

The discovery of the infinite integer leads to a partition between finite and infinite numbers. Construction of an infinitesimal and infinitary number system, the Gossamer numbers. Du Bois-Reymond’s much-greater-than relations and little-o/big-O defined with the Gossamer number system, and the relations algebra is explored. A comparison of function algebra is developed. A transfer principle more general than Non-Standard-Analysis is developed, hence a two-tiered system of calculus is described. Non-reversible arithmetic is proved, and found to be the key to this calculus and other theory. Finally sequences are partitioned between finite and infinite intervals.

Introduction

The papers are an interdisciplinary collaboration. The coauthor is a retired high school and tertiary Physics/Maths teacher with over 33 years teaching experience. I produced the mathematics, considered by myself for over 20 years, and collaborated with William, over the last four years. With the goal to make calculus more accessible.

The theory is developed to support applications with comparison algebra and infinitesimals. However, it soon emerged that much larger issues needed to be considered, as the real numbers do not contain infinities. What we found was a need to fundamentally consider what numbers really are, and to find and then use algebra to describe the ‘missing’ mathematics.

While constructing a paper trail to validate and prove the comparison theory (Part 3), which is used in applications [12, Convergence sums …] which we developed before the theory, we came across du Bois-Reymond’s work via Hardy in ‘Orders of infinity’ [7]. This had extensive symbolism and theory of the magnitude relations \( \succ \) and \( \succeq \) which correspond respectively to little-o and big-O relations.

It made sense to use this, and explore another kind of calculus. We had the classic problem, does the chicken come before or after the egg? The idea of magnitude arguments to solve limits is a natural consideration from the limit problem. We believe the symbol relations greatly benefit calculus arguments, because they describe mathematics that is otherwise difficult to consider.

Indeed, the issues of calculus, not just its discovery, the very existence of infinitesimals and infinities has caused the greatest mathematical argument, arguably spanning millennia. The Greek’s rejection of the infinitesimal, and their other mistakes, we have repeated.
It is our opinion that each of the six parts are connected, but can also stand alone. The only dependencies in the Parts are that Part 1 and Part 2 use comparison algebra in their proofs. Hence, a forward reference for Parts 1 and 2 is used. In defense of this, what better way of justifying a theory than to use it, thereby demonstrating its utility.

We are producing a mathematical foundation that will allow the ‘integration’ of separate ideas in a consistent notation.

This paper develops the theory (the egg) where later papers develop our applications (the chicken).

We strongly believe that infinitesimals and infinities must be included at every stage with infinitary calculus, because like real numbers, they can no longer be ignored.
1 Gossamer numbers

The discovery of what we call the gossamer number system \( *G \), as an extension of the real numbers includes an infinitesimal and infinitary number system; by using ‘infinite integers’, an isomorphic construction to the reals by solving algebraic equations is given. We believe this is a total ordered field. This could be an equivalent construction of the hyperreals. The continuum is partitioned: \( 0 < \Phi^+ < \mathbb{R}^+ + \Phi < +\Phi^{-1} < \infty \). A one-one correspondence over an infinity of infinite intervals is interpreted differently resulting in the infinite-integers having a higher cardinality than the countable integers, and a likely consequence that \( *G \) and the hyperreals have a higher cardinality than the real numbers.

1.1 Introduction

An infinite integer is an integer at infinity, larger than any finite integer. The set of integers \( \mathbb{J} \) is a countable infinity. However, by representing this countability as an infinite integer \( n \mid n = \infty \), in Non-Standard-Analysis (NSA) \( \omega \); using evaluation at a point notation Definition [1.3] if we take the reciprocal \( \frac{1}{n} \mid n = \infty \) we have an infinite number that is positive and not rational, or even a real number, but a positive infinitesimal.

This number system is of interest to anyone working with calculus. Infinities and infinitesimals in mathematics are universal.

Despite this, infinitesimals and infinities largely are not recognised as actual numbers. That is, they are not declared, like integers or reals. Limits are used all the time, and these “are” infinitesimals, that what we daily use as shorthand is not acknowledged. Infinitesimals are so successful that they are accepted as known and are applied so often. (Since we use them, why is there any need to study them)

We have gone back in time to a mathematician who systematically considered infinitesimals and infinities. Paul du Bois-Reymond, like everyone else, was at first uncomfortable with infinity. By comparing functions (Example 3.20), he investigated the continuum and realized
the space which we would call Non-Standard-Analysis today. Our subsequent sections in this series consider this in some detail.

The purpose of this paper is to construct a number system that both contains infinitesimals and infinities, and a more natural extension to the real numbers than other alternatives.

A fact of significant importance is that the real numbers are composed of integers. Rational numbers are built from the ratio of two integers, algebraic numbers from solving an equation of rational coefficients, and so other numbers with infinite processes are constructed.

With the discovery of infinite integers \(^{17, p.1}\) and A. Robinson’s enlargement of the integers \(\mathbb{J}: *\mathbb{J}^{16, p.97}\), by a similar process to the construction of the real numbers from integers we can construct infinite and infinitesimal numbers. We introduce a number system which is comparable to, and an alternative to hyperreal and surreal numbers.

By having ‘infinity’ itself as a number, like \(i\) for complex numbers, we can follow a standard construction analogous to building the real numbers from integers.

We believe that no such simple construction has been given that provides the usability without the complexity. (i.e. avoids the complicated mathematics of logic and set theory)

With time, we would like to see the claimed misuse of infinitesimals as ‘not being rigorous’, start to be reconsidered. A number system which directly supports their use in the traditional sense is possible. In Part 4 and Part 5 we argue that rigour should not be the only goal of a number system.

We take a constructionist’s approach to the development of the infinitesimal and infinitary number system. A series of papers for a larger theory, based on du Bois-Reymond’s ideals and our extensions to them, is developed.

Working with infinitesimals and infinities, we find a need for better representation. Indeed, the more we consider calculus, the greater is the need for this.

By adding infinities and infinitesimals to the real numbers, and declaring these additions to be of these types, this primarily is an extension of the real number line. Rather than saying a number becomes infinitely small, or infinitely large, the numbers are able to be declared as infinitesimals and infinities respectively. Theorems with this characterisation naturally follow.

Why should we be interested in infinitesimals or infinities? We are indeed, using them all the time through limits, applied mathematics and theory. Why should we want to further characterise them?

In Flatland \(^{10}\), set in a world of two dimensions, where a 3D creature, a sphere, attempting to communicate, describes its world to a flatlander inhabitant, the square. Proving its
existence by removing an item from a closed 2D cupboard, and materializing it elsewhere.

If we imagine the infinitesimals and infinities as working in a higher dimension, then we project back to the reals afterwards. See Part 4 The transfer principle.

The name we give to the number system which we construct is the “gossamer numbers”, for, analogous to a spider’s web, there can be ever finer strands between the real numbers. About any real number, there are infinitely many numbers asymptotic to a given real number. Similarly about any curve, there are infinitely many curves asymptotic to it.

What this view gives is a deeper fabric of space to work with. Propositions in $\mathbb{R}$ may be better explained as propositions in the higher dimensional space.

We look at the problem from an equation’s perspective. Just as complex numbers solve the equation $x^2 = -y^2$ in terms of $x, y = xi$, we can then put forward an equation at infinity which likewise needs new ideas and new numbers.

Historically, the development of a number system preceded with an equation or problem which needed to be solved. By looking at ancient mathematics such as Diophantine equations, words were used in place of a mathematical language such as our modern algebra.

Without the language of algebra, it took a genius to solve by today’s measure routine problems. Similarly with Newton’s calculus, the English mathematicians lagged behind the continent until the adoption of Leibniz’s calculus. In short, the mathematical language can greatly impact on both the development and use of mathematics affecting the culture.

While it can be argued that infinitesimals and infinities are known and managed with other mathematics such as asymptotic notation and the hyperreal number system, by producing two fields of mathematics, we argue that there is a need for change to include new mathematical language. Similarly, the application of hyperreal numbers justified their existence.

1.2 The infinite integer

We believe that what an atom is to a physicist, an infinite integer is to a mathematician, well that is how it should be! With this belief, we go on to derive infinitesimals and infinities, building on the existence of the infinite integer $\mathbb{J}_\infty$ (Definition 1.18).

An infinite integer’s existence does not contradict Euclid’s proof that there is no largest integer because the existence of infinite integers is not a single largest integer, but a class of integers.

Certain properties become apparent. We define a partition between finite and infinite numbers, but there is no lowest or highest infinite number, as there is no lowest or highest infinite
integer.

\[ \ldots, n - 2, n - 1, n, n + 1, n + 2, \ldots |_{n \in \mathbb{J}_\infty} \]

In the construction of the extended numbers, what distinguishes these new numbers is the inclusion of an infinity within the number. While the real numbers are constructed created by infinite processes, the real numbers do not contain an infinity. We can imagine such a number with the inclusion of an infinity as ever changing, and not static. We need such a construction to explain limits.

That we are continually being asked to explain the existence of real numbers, let alone numbers with infinities is a challenge. However, these numbers provide immeasurable insight into the mechanics of calculus. Would we deny the existence of quantum physics because we felt only the need for classical physics? This is similar to the denial of the infinitesimal and infinitary numbers because we already have the real numbers.

We are accustomed to the use of complex numbers which have a separate component, the imaginary part. Similarly with the extension of the real number, an infinitesimal or infinitary component is added. However, this will generally be seen as part of the number, and not represented as separate components. Having said that, in this paper we do represent the components separately, to help prove the basic properties of the number system.

Once the infinite integer is accepted, an infinity not in \( \mathbb{R} \), the infinitesimal follows as its reciprocal \( \frac{1}{n} \notin \mathbb{R} \).

While the set of integers is a countable infinity, we need to distinguish between infinite and finite numbers.

**Definition 1.1.** We say \( \mathbb{J}_< \) to mean a finite integer, \( k \in \mathbb{J} \) and \( k \) is finite. \( \mathbb{J}_< \subset \mathbb{J} \) We similarly define \( \mathbb{N}_< \subset \mathbb{N} \) the finite natural numbers, \( \mathbb{Q}_< \subset \mathbb{Q} \) the finite rational numbers, \( \mathbb{A}_< \subset \mathbb{A} \) the finite algebraic numbers.

**Definition 1.2.** We say \( \mathbb{J}_< < \mathbb{J}_\infty \) to mean than for all finite integers \( i \in \mathbb{J} \), and all ‘infinite integers’ \( j \in \mathbb{J}_\infty \) then \( i < j \).

We note that the set of finite numbers is infinite, but any given finite number is not infinite. We can consider any finite integer as large as we please, but once we do it is in existence it is without infinity.

### 1.3 Preliminaries

This paper is one of six in this series, which we believe is establishing a new field of mathematics. The proofs and notation contain mathematics in Part 2 and Part 3; however if \( f > g \) then \( \frac{f}{g} \in \Phi \) is equivalent to \( f = o(g) \), where \( \Phi \) is an infinitesimal (Definition 1.7). The proofs can solve for a relation, which is studied in Part 3.
We introduce a notation at infinity which can express both a limit and the realization of being at infinity. We later develop and justify this choice more fully Part 2, Part 4, Part 5.

**Definition 1.3.** $f(x)|_{x=\infty}$ represents the notion of the “order” of $f(x)$ as a function “at infinity”, or a limit at infinity. See Evaluation at-a-point Definition 2.1.

**Remark: 1.1.** We will later see that formal objects can be formed with a number system that reflects the requirements of our notion of infinitesimals and infinities.

Now we give an example of an infinitary equation, which can only be solved by an infinitary number.

Solve $(x^2 + 1)y|_{x=\infty} = x^3|_{x=\infty}$. $y = \frac{x^3}{x^2 + 1}|_{x=\infty}$ is a solution, another is $y = x|_{x=\infty}$.

We already use infinitesimals and infinities with little-o and big-O notation. However, unlike real numbers, we often do not declare them. And this may be a problem, because, unlike real numbers, they can have radically different properties.

So here is the contradiction, you use these numbers ubiquitously, for example, calculating limits, but you do not declare them as such.

Now we are not entirely adverse to the utility of this approach, things need to be done and calculations performed.

However, not even relatively modern mathematicians such as Hardy are immune. While having extensive powers of computation, he never defined an infinitesimal as a number.

Examples of infinitesimal and infinitary numbers are everywhere.

Zeno’s paradoxes [2] assume a truth and argue to a paradox. Hence disproving the assumed truth. In assuming continuity as an infinity of divisibility for physical questions of motion, logical contradictions follow. Achilles, when chasing the tortoise, never catches up, as, when Achilles travels towards the tortoise, there is always some remaining distance to reach it [3].

The distance left over is of course a positive infinitesimal, a number smaller than any positive real numbers [12, Section 2: What does a sum at infinity mean?]

From [1], to counter Zeno’s arguments, Atomism, while ascribed to physical theory, the indivisibility of matter and presumably time. However, Aristotle believed in the continuity of magnitude.

Aristotle identifies continuity and discreteness as attributes applying to the category of Quantity. As examples of continuous quantities, or continua, he offers lines, planes, solids (i.e. solid bodies), extensions, movement, time and space; among discrete quantities he includes number and speech [1].
Infinities and infinitesimals are crucial in the development of calculus, before their ostracism (19th century) from mainstream calculus. We believe this is a case of ‘throwing the baby out with the bath water’.

Hence, the banishment of infinitesimals was a rejection which sent them underground. The Greeks, with their geometric methods, had followed a similar story. Where they could not describe numbers, as this had to wait for the discovery of zero and the positional number system, reasoning was replaced by other arguments (Egyptian fractions for calculations and Roman numerals for date calculations remained).

In modern times, the first person to systematically study the infinitesimals and infinities was du Bois-Reymond, whose writing around 1870 onwards, on the scales of infinities [7, pp.9–21], for example \( \ldots, x^{-2}, x^{-1}, 1, x, x^2, x^3, \ldots \big|_{x=\infty} \), and the ratios of infinities, which are instances of comparing functions.

... certain problems have ”familiarized mathematicians with the use of scales of comparison other than those of powers of a variable .. This extension goes back above all to the works of P. du Bois-Reymond who was the first to approach systematically the problems of the comparison of functions in the neighborhood of a point, ... [9, Bourbaki p.157]

Cantor had a competing theory which represented the continuum with sets and he believed infinitesimals did not fit in. So our development may have been skewed by the rise of set theory, which became dominant and is heavily present in Abraham’s NSA.

From the work of Abraham Robinson, infinitesimals have in recent years been made more rigorous; however they have not been made accessible, that is, easy to use. To address the inaccessibility, reformations of NSA have been constructed. However, NSA by its nature is very technical and used for high-end mathematics.

Note that the gossamer number system is also technical. However, the gossamer number system we present is more accessible. It has been built to be used with functions. We would have no idea how to implement many of the applications that later follow such as the rearrangement theorems with NSA, or the theory which is later developed. This is not a trivial distinction: the tools that a worker uses to do a job matter. We may not wish to deal with low level set theory logic while working with functions, particularly if it is unnecessary.

Our concerns are NSA’s debasement of meaning. That is, the meaning and use of infinitesimals and infinities is lost or “muddied” during applications and proofs.

This is not to invalidate NSA. Indeed, later we do use NSA which as a reference is invaluable. Just as we have axiomatic geometry, does not mean that we have to reason with it. NSA is specialized, and requires more knowledge to use it, if you want to use it in the first place.

Instead, we will look where possible for an alternative. From the premise of this and later
papers, it follows that Robinson’s NSA is not the only way.

The paper has two primary tasks: describe the construction of the number system, and then work towards proving $\ast G$ is an ordered field. We also believe others can forward this work with the benefits being that the number system can be used in many ways.

These are separate goals, the justification of introducing additional complexities as we have separated the components. There is likely other ways to go about this.

This is not the same as using this mathematics. The real numbers themselves are used in so many different ways, that we expect a generalisation of them to be even more diverse.

### 1.4 Infinitesimals and infinities

A more general way of considering infinity is the realization of reaching infinity.

Infinitesimals and infinities naturally occur in any description at infinity. E.g. an infinitesimal, $\frac{1}{n}|_{n=\infty}$. The inverse is an infinity. $1/(1/n) = n|_{n=\infty} = \infty$. It would be fair to say that calculus (for the continuous variable) without infinitesimals and infinities would not exist (for example, a limit or a derivative could not exist).

If $x$ is infinite and we realize $x$ to infinity, then at infinity $x$ becomes an extended real. If $x$ is an infinite integer, then it is an integer at infinity. Considering Robinson’s NSA, it becomes clear that with infinite integer $\omega$, we can have an infinity of infinite integers ($\omega, \omega + 1, \omega + 2, \ldots$). In this context, infinity is its own number system, where we have arrived at infinity, and it is a very large space. A lower case $n$ at infinity will be understood as an infinite integer, the same as $\omega$. (Though with the notation any variable can be used.) With infinitesimals the situation is similar, since given an infinity we can always construct an infinitesimal by dividing 1 by the infinity.

We can similarly construct infinite rational numbers. The numbers themselves need not all be infinite, but a composition of infinities and reals. E.g. $\frac{n^2 + 1}{n^3 - 2}|_{n=\infty}$. Similarly, there are infinite surds. E.g. $\sqrt{2n}|_{n=\infty}$. What about infinite reals? That is, a number which is as dense as a real, but at infinity. The infinite numbers will in many respects behave similarly to their finite counterparts.

Meaning can be attributed to expressions at infinity. We may consider $n|_{n=\infty}$ as the process of repeatedly adding 1. Similarly $\ln n|_{n=\infty}$ corresponds with summing the harmonic series. If divergent series, which are ubiquitous, are asymptotic to divergent functions, then the functions can have a geometric meaning. Another example, $\sin x|_{x=\infty}$ continually generates a sine curve at infinity.

The next jump is that from the reals to du Bois-Reymond’s infinitesimals and infinities at
infinity, realizing the space at infinity. To do this we separate finite and infinite numbers, therefore separating finite and infinite space. For example, integers \( \mathbb{J}_< \) and infinite integers \( \mathbb{J}_\infty \) at infinity. \( \mathbb{J}_< < \mathbb{J}_\infty \) Definition 1.2.

*Infinities and infinitesimals form their own number system. As a number, extending the reals with infinities, the infinities are a supremum for the reals, with an extended Dedekind cut, as an infinity is larger than any finite number.*

**Definition 1.4.** Define a positive infinite number to be “larger” than any number in \( \mathbb{R} \)

**Definition 1.5.** Define a negative infinite number to be “smaller” than any number in \( \mathbb{R} \)

**Definition 1.6.** Define an infinite number, ‘an infinity’ as a positive infinity or negative infinity, the set of all these being denoted \( \Phi^{-1} \). However exclude \( \pm \infty \notin \Phi^{-1} \) for reversible multiplication.

**Definition 1.7.** We say a number is an “infinitesimal” \( \Phi \) if the reciprocal of the number is an infinity. If \( \frac{1}{x} \in \Phi^{-1} \) then \( x \in \Phi \).

**Definition 1.8.** If \( x \) is a positive infinitesimal then \( x \in +\Phi \) or \( x \in \Phi^+ \).

**Definition 1.9.** If \( x \) is a negative infinitesimal then \( x \in -\Phi \) or \( x \in \Phi^- \).

**Definition 1.10.** If \( x \) is a positive infinity or negative infinity then \( x \in +\Phi^{-1} \) or \( -\Phi^{-1} \) respectively.

**Definition 1.11.** Let \( \frac{1}{0} = \infty \), 0 and \( \infty \) are mutual inverses.

**Corollary 1.1.** 0 is not an infinitesimal.

*Proof. \( \frac{1}{0} = \infty \notin \Phi^{-1} \). \( \infty \) is ‘infinity’, not ‘an’ infinity (see Definition 1.6).*

0 is finite, but not an infinitesimal. 0 is a special number, which could be called a super infinitesimal. The reason for not including 0 as an infinitesimal is to make reasoning clearer by avoiding division by zero, having \( \mathbb{R}\{0\} \cup R_\infty \) (see Definition 1.24) with respect to multiplication form an abelian group.

**Proposition 1.1.** A infinitesimal is less than any finite positive number.

*Proof. Solving for a comparison relation Part 3. A negative number is less than a positive number, then only the positive infinitesimal case remains. \( x \in \mathbb{R}^+ \); \( \delta \in +\Phi \); then \( \frac{1}{\delta} \in +\Phi^{-1} \), compare the infinitesimal and real number, \( \delta z x \), \( 1 z \frac{x}{\delta} \), \( \frac{1}{z} \frac{x}{\delta} \), \( \mathbb{R}^+ z +\Phi^{-1} \), by Definition 1.4 \( z = < \) then \( \delta < x \), \( \Phi < \mathbb{R}^+ \).*
Definition 1.12. We say $R_\infty$ is an ‘infinireal’ number if the number is an infinitesimal or an infinity. If $x \in R_\infty$ then either $x < R^-$ or $x > R^+$.

\[
x \in R_\infty \text{ then } x \in \Phi \text{ or } x \in \Phi^{-1}, R_\infty = \Phi \cup \Phi^{-1}
\]

Definition 1.13. $\overline{R}_\infty = \{-\infty, 0, \infty\}$ (A realization of $R_\infty$)

Definition 1.14. If $a \not\in b$ then no element in $a$ is in set $b$. Equivalent to \( \{b\}\backslash\{a\} \) or \( \{b\} - \{a\} \).

For example, $\Phi \not\in f$ means the variable or function contains no infinitesimals. Similarly $\Phi^{-1} \not\in f$ means $f$ contains no infinities.

\[
\not\in \ R_\infty \equiv R + \Phi[R \neq 0]
\]

Definition 1.15. Define the numbers 0 and $\infty$: $0 < |\Phi|$ and $|\Phi^{-1}| < \infty$.

Theorem 1.1. Partitioning the number line, $0 < \Phi^+ < R^+ \cup \Phi < +\Phi^{-1} < \infty$.

Proof. $\delta_1, \delta_2, \delta_3 \in \Phi; x \in R^+; z \in \mathbb{B}$ a binary relation. Consider $\delta_1 z x + \delta_2, \delta_1 - \delta_2 z x$. Since $(\Phi, +)$ is closed, let $\delta_3 = \delta_1 - \delta_2$. $\delta_3 z x$. By $\Phi < R^+$ (Proposition 1.1) $z =$, since adding and subtracting on both sides does not change the inequality.

Given the existence of the infinite integers, a constructive definition $^*G$ Definition 1.24 of an extended real number system follows. The number system at infinity is isomorphic with the real number system, defining integers, rational numbers, algebraic numbers, irrational numbers and transcendental numbers all at infinity.

Definition 1.16. Let $A_<$ be the symbol for the finite algebraic numbers.

Definition 1.17. Let $A'_<$ be the symbol for the finite transcendental numbers.

Definition 1.18. Define an “infinite integer” $J_\infty$ at infinity, larger in magnitude than any finite integer.

Definition 1.19. Define an “infinite natural number” $N_\infty$ as a positive infinite integer.

Definition 1.20. Define an “infinite rational number” $Q_\infty$ as a ratio of finite integers and “infinite integers”.

\[
\{J_\infty, J_\infty, J_\infty\} \in Q_\infty
\]

Definition 1.21. Define an “infinite algebraic number” $A_\infty$ which is the root of a non-zero finite polynomial in one variable with at least one infinite rational number coefficient.

Definition 1.22. Define an “infinite irrational number” $Q'_\infty$ as an infinity that is not an infinite rational number.
**Definition 1.23.** Define an “infinite transcendental number” $A'_\infty$ as an infinity which is not an infinite algebraic number.

**Definition 1.24.** Define the gossamer numbers, $*G$ as numbers that comprise of Definitions 1.18–1.23.

**Example 1.1.** Given a fraction of the form $\frac{J_\infty}{J_\infty}, \frac{2(n+1)}{n+1}|_{n=\infty} = \frac{2}{1}$ cancelling like $J_\infty$ terms leaves a fraction of the form $\frac{J_\infty}{J_\infty} \in J_\infty$.

As a byproduct of the $*G$ construction, $\mathbb{R}$ is embedded within Definition 1.24 as there exist $\frac{J_\infty}{J_\infty} \in J_\infty$. We could choose to define $Q_\infty$ to exclude $Q_\infty$. Then all $\{J_\infty, Q_\infty, Q', A_\infty, A'_\infty\}$ would contain an explicit infinity within the number. By definition all infinireals $R_\infty$ contain an infinity as an element within the number.

A hierarchy diagram for the number systems shows the parent number system which is used to build the child number system (see Figure 1). In this way, all reals are composed of integers and gossamer numbers are composed of integers and infinite integers. The reals are embedded within the gossamer numbers.

**Remark: 1.2.** We have constructed the real numbers not with $\mathbb{J}$ but $J_\infty$ as the real numbers cannot contain infinities. In this way, we have also provided a construction of the real numbers. Restating with an explicit construction: Define a rational number $Q_\infty$ as a ratio of $J_\infty$ without division by zero. Define an irrational number $Q'_\infty$ which is not a rational number and finite. Define the finite algebraic number $A_\infty$ as a root of a non-zero polynomial in one variable with at least one rational coefficient $Q_\infty$ and finite. Define a transcendental number $A'_\infty$ which is not algebraic $A_\infty$ and finite.
Remark: 1.3. While a finite number is not an infinity, the collection of finite numbers is an infinity, as there is no greatest finite number. Hence, the countability of the finite numbers and the finite numbers is separated. This apparent paradox is explained by any instance of a finite number begin less than the whole. (infinity is non-unique and has other possibilities)

Definition 1.25. We define extended gossamer numbers $*G$, where $*G = *G \cup \pm \infty$

![Figure 2: gossamer numbers and infinireals composition (Not a Venn diagram)](image)

The infinitary component generally is not unique. Where the intersection in Figure 2 represents adding different components, between the numbers there exist overlaps. Infinity dominates, with any combination of non-infinity numbers with infinity resulting in an infinity.

Definition 1.26. Let $\Phi^{-1}$ be a number type, the infinitreal infinity without reals or infinitesimals, under addition in the base. If $x \in \Phi^{-1}\{\Phi, \mathbb{R}\}$ then $x \in \Phi^{-1}$.

$$\Phi^{-1} = \Phi^{-1} - \mathbb{R} - \Phi$$

The following sets are disjoint and partition $\Phi^{-1}$.

$$\{\Phi^{-1} + \Phi\} \cup \{\Phi^{-1} + \mathbb{R}\{0\}\} \cup \{\Phi^{-1} + \mathbb{R}\{0\} + \Phi\} \cup \{\Phi^{-1}\} = \Phi^{-1}$$

Example 1.2. Example of numbers and their relation to Figure 2. $n + \frac{1}{n}|_{n=\infty} \in \Phi^{-1} + \Phi$ has an infinitary and infinitesimal component; $\sqrt{2} + \frac{1}{n} \in \mathbb{R} + \Phi$ has both a real and an infinitesimal component.

If we compare $\mathbb{R}_\infty$ and $\mathbb{R}$, $\mathbb{R}_\infty$ have an infinity within the number itself, an infinite variable, and the reals do not, hence the reals are constant. We could then conceive of the numbers $\mathbb{R}_\infty$ and $\mathbb{R} + \Phi$ as ever changing and containing an infinity, something of which the reals are not allowed.
Having gossamer numbers, it would logically follow to define complex gossamer numbers \((a, b \in \ast \mathbb{G}; a + bi)\). Given how useful complex numbers are, this would enable complex numbers with infinitesimals and infinities.

Converting a given element of \(\ast \mathbb{G}\) into an algebraic form can help identify what type the number is. If this is not possible, and the number is not of the other simpler types, then the number is generally an infinite transcendental number.

**Example 1.3.** Let \(k \in \mathbb{J}_<\), \(k = \frac{k}{1} \in \mathbb{Q}_<\). Let \(n \in \mathbb{J}_\infty\), \(n = \frac{n}{1} \in \mathbb{Q}_\infty\).

**Example 1.4.** \(n^2 + \frac{1}{n} |_{n=\infty} \in \mathbb{Q}_\infty\) because \(n^2 + \frac{1}{n} = \frac{n^2+1}{n} |_{n=\infty}\) of form \(\frac{1}{\mathbb{J}_\infty} \in \mathbb{Q}_\infty\).

**Example 1.5.** \(y^2 + 2y + 1 = 2n^2 |_{n=\infty}\) is of form \(\mathbb{J}_< y^2 + \mathbb{J}_< y + \mathbb{J}_< = \mathbb{J}_\infty, \mathbb{Q}_< y^2 + \mathbb{Q}_< y + \mathbb{Q}_< = \mathbb{Q}_\infty\) is in algebraic form, \(y \in \mathbb{A}_\infty\).

**Example 1.6.** Show \(\sqrt{2n} |_{n=\infty} \in \mathbb{A}_\infty\). Let \(y = 2^{\frac{1}{2}} n, y^2 = 2n^2, 2n^2 = \frac{2n^2}{1} \in \mathbb{Q}_\infty, y^2 + 0y^1 - \frac{2n^2}{1}y^0 = 0, \mathbb{Q}_< y^2 + \mathbb{Q}_< y^1 + \mathbb{Q}_\infty y^0 = 0, y \in \mathbb{A}_\infty\).

If we consider a number in \(\ast \mathbb{G}\) as composed of three components, infinitesimals, reals, and infinities while the number is unique, this composition is not. \(n + 2 |_{n=\infty} \in \Phi^{-1}\) is an infinity, but is composed of both an infinity and an integer. This is because \(\mathbb{R}^+ < n + 2 |_{n=\infty}\), and partitions the reals, hence the number is an infinity, but \(2 \in \mathbb{J}_<, 2 + n \in \mathbb{J}_< + \mathbb{J}_\infty \in \Phi^{-1}\).

As a consequence of the infinitesimals \(\Phi\) and infinities \(\Phi^{-1}\) definitions, they are not symmetrically defined. \(\frac{1}{n} + 3 |_{n=\infty} \not\in \Phi, \text{ but } n^2 + 1 |_{n=\infty} \in \Phi^{-1}\). An infinity can have an arbitrary real or infinitesimal part, but an infinitesimal cannot have either a real or an infinity part.

**Proposition 1.2.** The gossamer number system comprises of reals and infinitesimal numbers.

\[\ast \mathbb{G} = \Phi + \mathbb{R} + \Phi^{-1}\]

However, this does not mean that \(\ast \mathbb{G}\) as a number cannot be uniquely represented. A unique representation may aid with theorems and proofs. We define the following infinity to exclude infinitesimals and real numbers in an addition definition. Using \(\Phi^{-1}\), a gossamer number can be uniquely represented as three components, \(\Phi, \mathbb{R}, \Phi^{-1}\). As a vector of three independent components.

**Definition 1.27.** Uniquely represent \(\ast \mathbb{G}\) by three independent components.

\[a \in \ast \mathbb{G} \text{ then } a = (\Phi, \mathbb{R}, \Phi^{-1})\]
Any number in $\ast G$ is in one of the following seven forms: $\Phi$, $\Phi + \mathbb{R}$, $\Phi + \Phi^{-1}$, $\Phi + \mathbb{R} + \Phi^{-1}$, $\mathbb{R}$, $\mathbb{R} + \Phi^{-1}$, $\Phi^{-1}$

For proofs, with the component representation, we can define addition and multiplication between two unique numbers in $\ast G$, for the components which we know and do not know.

**Definition 1.28.** Addition $a + b$; $a,b \in \ast G$;

$$a + b = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

We know $\mathbb{R}\setminus\{0\} \cdot \Phi \in \Phi$ (Proposition 1.46), $\mathbb{R}\setminus\{0\} \cdot \Phi^{-1} \in \Phi^{-1}$ (Proposition 1.47), $\mathbb{R} \cdot \mathbb{R} \in \mathbb{R}$, $\Phi \cdot \Phi \in \Phi$ (Proposition 1.16), and $\Phi^{-1} \cdot \Phi^{-1} \in \Phi^{-1}$ (Proposition 1.17), hence for these multiplications we can determine which components they belong to.

**Definition 1.29.** Multiplication $a \cdot b$; $a,b \in \ast G$;

$$a \cdot b = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3)$$

$$a \cdot b = (a_1b_2 + a_2b_1 + a_1b_1, a_2b_2, a_3b_3 + a_3b_2 + a_2b_3) + (a_1b_3 + a_3b_1)$$

**Remark: 1.4.** We do not know the product type of $a_1b_3$ and $a_3b_1$ as the numbers could be in any or multiple categories. This is the familiar indeterminate case $0 \cdot \infty$. $\frac{5}{n+1} \cdot n |_{n=\infty} = 5 - \frac{5}{n+1} \in \mathbb{R} + \Phi$

We have constructed a number system, and from this construction believe the following properties, while not proved, are true. This would require a proof that $\ast G$ is a total order field Proposition 1.45, in perhaps a similar way to a proof of $\mathbb{R}$ as a total order field.

### 1.5 Properties

We believe the $\ast G$ construction is valid is actually a field, but we are less certain about the cardinality. This could be a consequence of the representations and definitions, or possible errors, or that by doing the mathematics in another way, some insight may have been gained.

The following may be of interest to the set theoretical mathematicians. We do not consider one-one correspondence over an infinity of infinite intervals to be correct, but over a finite number of infinite intervals to be true. A finite number of infinite sets is not an infinite number of infinite sets.

To explain why the one-one correspondence is different, we believe that by including infinities within the set we are effectively increasing the cardinality.
Definition 1.30. For determining cardinality, a one-one correspondence over an infinite interval is a one-to-one correspondence between a finite number of infinite intervals.

Theorem 1.2. The cardinality of $\mathbb{J}_\infty$ is larger than the cardinality of $\mathbb{J}$

Proof. $1 \ldots n$ cannot be put in one-one correspondence with $\mathbb{J}_\infty$, as one element of $\mathbb{J}_\infty$ alone can be put into one-one correspondence with $\mathbb{J}$. Let $k \in \mathbb{J}_\infty$, placing $k$ in one-one correspondence with $\mathbb{J}$, $(\ldots, (-2, k - 2), (-1, k - 1), (0, k), (1, k + 1), \ldots)$. Since there is an infinity of such $k$ elements, unlike the diagonalization argument with 2 infinite axes (which proves rational numbers have the same cardinality as the integers), this case has no one-one correspondence. E.g. Consider $(k) = (n, n^2, n^3, n^4, \ldots)|_{n=\infty}$. If this sequence were finite it could be put into a one-one correspondence, but it is not.

A simpler way, countable infinities describe, $1 \ldots n$, $1 \ldots n^2$, $\ldots$, $1 \ldots n^w|_{n=\infty}$ for finite $w$, but not infinite $w$.  

This may have consequences for the continuum hypothesis: that there is no cardinality between the integers and the real numbers. However, if the continuum hypothesis is true, the infinite-integers would have greater than or equal to cardinality than the real numbers.

Theorem 1.2 is a different one-one correspondence from Cantor. With non-uniqueness at infinity, there may be other mathematics here, so the truth or falsehood is relative to the mathematical system chosen, if one is chosen at all.

This also has a consequence for the cardinality of the hyperreals which by the current theory has the same cardinality as the real numbers [28]. We believe that this is incorrect, and hence the current theory is not modelling the situation. In contrast, our theory would result in the reals having a lower cardinality than that of the hyperreals.

We really believe that the hyperreals and $*G$ have the same cardinality.

Intuitively, we do not agree that $*G$ has the same cardinality as $\mathbb{R}$, particularly as we have defined $\mathbb{R}_<$ as the real numbers, which are devoid of any infinity $n$ terms.

The nature of the gossamer numbers gives reason to believe infinitesimals and infinities are much more dense than reals, for about any real number there is an infinity of infinitesimal numbers, which map back to a unique real number. (Part 4)

The hyperreal point of view is that the geometric line is capable of sustaining a much richer and more intricate set than the real line. [27 p.14]

The preceding quote does seem to contradict the current position of the hyperreals having the same cardinality as the reals. Why would this property not be reflected in the cardinality?
**Conjecture 1.1.** The cardinality of the gossamer numbers is larger than the cardinality of the reals.

Since the construction of the number system of reals and gossamer numbers is identical except with different number types, and the cardinality of the infinite integers $\mathbb{J}_\infty$ is larger than the cardinality of the finite integers $\mathbb{J}_<$, then the input having different cardinality results in an output with different cardinality.

Another approach is to consider the solution space, by comparing the transcendental solution space with the infinite transcendental solution space. Let $|\mathbb{N}_\infty|$ describe the infinite integers cardinality.

\[
\begin{align*}
    a_k \in \mathbb{Q}; \sum_{N_\infty} a_k x^k = 0 & \text{ has } N_0^{[N_\infty]} \text{ solution space.} \\
    b_k \in \mathbb{Q}_\infty; \sum_{N_\infty} b_k x^k = 0 & \text{ has } |N_\infty|^{[N_\infty]} \text{ solution space.}
\end{align*}
\]

Solving with comparison algebra shows $|N_\infty|^{[N_\infty]} \succ N_0^{[N_\infty]}$ and the gossamer numbers have a larger cardinality.

After scales of infinities (also see Part 2), another geometric example at infinity involves the visualization of infinitesimally close curves. We can have infinitely many curves, infinitesimally close to a single curve, separated by infinitely small distances, $f(x) - g(x) \in \Phi$ (see Example 3.20).

Infinitesimals are required to describe such a space, in the same way that real numbers are needed to extend rational numbers, and rational numbers to extend integers.

Infinitesimal and infinity exclusion can be compared with removing other classes of numbers, and is staggering. Not even Apostol [5] discusses infinitesimals or infinities as their own number system, but implicitly uses infinitesimals and infinities when convenient. Given the banishment of infinitesimals from calculus in general, for example, their exclusion as a number from Apostol and generally every modern calculus text (we cited Apostol as this is highly regarded, but does not describe infinitesimals as numbers). Such a choice may be understandable, but it is not complete.

What follows is a construction of an extended real number system, we call $*G$, which includes infinitesimals and infinities, which is similar to the hyperreals, but different.

We differentiate between infinitesimals and zero. Similarly we differentiate between an infinity such as $n^2|_{n=\infty}$ and the “number” $\infty$.

### 1.6 Field properties

# Unproven propositions. Other propositions assume their truth.
Proposition 1.3 \((+\Phi^{-1}, +)\) is closed

Proposition 1.4 \((+\Phi^{-1}, +)\) is closed

Proposition 1.5 \((+\Phi, +)\) is closed

Proposition 1.6 \((\Phi \cup \{0\}, +)\) is closed

Proposition 1.7 \((\Phi \cup \{0\}, +)\) is commutative

Proposition 1.8 \((\Phi \cup \{0\}, +)\) is associative

Proposition 1.9 \((\Phi^{-1} \cup \{0\}, +)\) is closed

Proposition 1.10 \((\Phi^{-1} \cup \{0\}, +)\) is commutative

Proposition 1.11 \((\Phi^{-1} \cup \{0\}, +)\) is associative

Proposition 1.12 \((\ast G, +)\) is closed

Proposition 1.13 \((\ast G, +)\) is commutative

Proposition 1.14 \((\ast G, +)\) is associative

Proposition 1.15 \((\ast G, \cdot)\) is closed

Proposition 1.16 \((\Phi, \cdot)\) is closed

Proposition 1.17 \((\Phi^{-1}, \cdot)\) is closed

Proposition 1.18 \((+\Phi^{-1}, \cdot)\) is closed

Proposition 1.19 \((\Phi^{-1}, \cdot)\) is closed

Proposition 1.20 \(\Phi \cdot \Phi^{-1} \in \ast G \setminus \{0\}\)

Proposition 1.21 \(\Phi^{-1} \cdot \Phi \in \ast G \setminus \{0\}\)

Proposition 1.22 \((\mathbb{R}_{\infty}, \cdot) \in \ast G \setminus \{0\}\)

Proposition 1.23 \((\ast G, \cdot)\) is closed

Proposition 1.24 \((\Phi^{-1}, \cdot)\) is commutative

Proposition 1.25 \((\Phi^{-1}, \cdot)\) is commutative

Proposition 1.26 \((\Phi, \cdot)\) is commutative

Proposition 1.27 \((\Phi, \Phi^{-1}, \cdot)\) is commutative

Proposition 1.28 \((\ast G \setminus \{0\}, \cdot)\) is commutative

Proposition 1.29 \((\ast G \setminus \{0\}, \cdot)\) is associative

Proposition 1.30 \((\ast G, +, \cdot)\) is distributive, \(a \cdot (b + c) = (a \cdot b) + (a \cdot c)\)

Proposition 1.31 \((\Phi \cup \{0\}, +)\) has identity 0

Proposition 1.32 \((\Phi^{-1} \cup \{0\}, +)\) has identity 0

Proposition 1.33 \((\ast G, +)\) has identity \((0, 0, 0)\)

Proposition 1.34 \((\Phi^{-1} \cup \{0\}, +)\) has an inverse

Proposition 1.35 \((\ast G, +)\) has an inverse

Proposition 1.36 \((\Phi^{-1} \cup \{0\}, +)\) has an inverse

Proposition 1.37 \((\ast G \setminus \{0\}, \cdot)\) has an inverse

Proposition 1.38 \((\Phi \cup \{0\}, +)\) has an inverse

Proposition 1.39 \((\ast G, +)\) has an inverse

Proposition 1.40 \((\ast G, +)\) is an abelian group

Proposition 1.41 \((\ast G \setminus \{0\}, \cdot)\) has an identity

Proposition 1.42 \((\ast G \setminus \{0\}, \cdot)\) has inverse

Proposition 1.43 \((\ast G \setminus \{0\}, \cdot)\) is a group

Proposition 1.44 \(\ast G\) is a field

Proposition 1.45 \(\ast G\) is a total order field. If \(a \leq b\) then \(a + c \leq b + c\). If \(0 \leq a\) and \(0 \leq b\) then \(0 \leq a \times b\).

Proposition 1.46 \(\mathbb{R} \setminus \{0\} \cdot \Phi \in \Phi\)

Proposition 1.47 \(\mathbb{R} \setminus \{0\} \cdot \Phi^{-1} \in \Phi^{-1}\)

Proposition 1.48 \(\mathbb{R} \setminus \{0\} \cdot \mathbb{R}_{\infty} \in \mathbb{R}_{\infty}\)
Proposition 1.49 Let $\theta \in \ast \mathbb{G}\setminus\{0\}$, $\theta > 0$, $z \in \{<, \leq, =, >, \geq\}$. $f(z)g \Leftrightarrow \theta \cdot f(z)\theta \cdot g$.

Proposition 1.50 Let $z \in \{<, \leq, =, >, \geq\}$. The relations $z$ invert when multiplied by a negative number in $\ast \mathbb{G}$. For the equality case the left and right sides of the relation are interchanged.

Proposition 1.3. $(+\Phi^{-1}, +)$ is closed

Proof. $a, b \in +\Phi^{-1}$; since $b > 0$, $a + b > a$, $a + b \in +\Phi^{-1}$

Proposition 1.4. $(+\Phi^{-1}_*, +)$ is closed

Proof. Since the components are separate, by definition; $a, b \in +\Phi^{-1}_*$; $a + b = (0, 0, a_3) + (0, 0, b_3) = (0, 0, a_3 + b_3) \in +\Phi^{-1}_*$ since adding two positive numbers are also positive.

Proposition 1.5. $(+\Phi, +)$ is closed

Proof. $a, b \in +\Phi^{-1}$; $\frac{1}{a}, \frac{1}{b} \in \Phi$, $\frac{1}{a} + \frac{1}{b} = \frac{b+a}{ab}$, since $ab > a + b$ then $\frac{a+b}{ab} \in \Phi$.

Proposition 1.6. $(\Phi \cup \{0\}, +)$ is closed

Proof. $a, b \in +\Phi^{-1}$; For both positive number and negative numbers are closed by Proposition 1.5. Consider case adding infinitesimals of opposite sign, $\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$. If $a = b$ then $\frac{1}{a} - \frac{1}{b} = \frac{0}{ab} = 0$ is closed. If $a \neq b$ then $b - a < ab$ and $\frac{b-a}{ab} \in \Phi$ is closed.

Proposition 1.7. $(\Phi \cup \{0\}, +)$ is commutative

Proposition 1.8. $(\Phi \cup \{0\}, +)$ is associative

Proposition 1.9. $(\Phi^{-1}_* \cup \{0\}, +)$ is closed

Proof. Since the components are separate, by definition; $a, b \in \Phi^{-1}_*$; $a + b = (0, 0, a_3) + (0, 0, b_3) = (0, 0, a_3 + b_3) \in \Phi^{-1}_*$ or $(0, 0, 0)$.

Proposition 1.10. $(\Phi^{-1}_* \cup \{0\}, +)$ is commutative

Proposition 1.11. $(\Phi^{-1}_* \cup \{0\}, +)$ is associative

Proposition 1.12. $(\ast \mathbb{G}, +)$ is closed

Proof. Since $(\Phi, +)$ is closed, $(\mathbb{R}, +)$ is closed, and $(\Phi^{-1}_*, +)$ is closed, then, as an addition in $\ast \mathbb{G}$ is the sum of three independent components are all closed.

Proposition 1.13. $(\ast \mathbb{G}, +)$ is commutative

Proof. Since the components (Propositions 1.1, 1.10 and $(\mathbb{R}, +)$) are independent, and commutative.

Proposition 1.14. $(\ast \mathbb{G}, +)$ is associative
**Proof.** Since the components (Propositions 1.8, 1.11 and \((\mathbb{R}, +)\)) are independent, and associative.

**Proposition 1.15.** 
\((+\Phi, \cdot)\) is closed

**Proof.**

\[ a, b \in +\Phi^{-1}; \frac{1}{a}, \frac{1}{b} \in +\Phi; \frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab} \in +\Phi \text{ as } ab \in +\Phi^{-1}. \]

**Proposition 1.16.** 
\((\Phi, \cdot)\) is closed

**Proof.** Since the sign can be factored out, consider the positive case only, by Proposition 1.15 this is true.

**Proposition 1.17.** 
\((\Phi^{-1}, \cdot)\) is closed

**Proof.** Since a negative sign can be factored out, only need to consider positive infinities. \(x, y \in +\Phi^{-1};\) choose \(x \leq y\), then \(x \leq y \leq xy\), as \(x \geq 1\). \(xy \in \Phi^{-1}\) by Definition 1.4.

**Proposition 1.18.** 
\((+\Phi^{-1}, \cdot)\) is closed

**Proof.**

\[ a, b \in +\Phi^{-1}; \frac{1}{a}, \frac{1}{b} \in +\Phi; \frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab} \in +\Phi \text{ as } ab \in +\Phi^{-1}. \]

**Proposition 1.19.** 
\((\Phi_{*}^{-1}, \cdot)\) is closed

**Proof.** Factor out the negative sign, since \((+\Phi_{*}^{-1}, \cdot)\) Proposition 1.18 then closed.

**Proposition 1.20.**

\[ \Phi \cdot \Phi_{*}^{-1} \in *G \setminus \{0\} \]

**Proposition 1.21.**

\[ \Phi_{*}^{-1} \cdot \Phi \in *G \setminus \{0\} \]

**Proposition 1.22.**

\[ (\mathbb{R}_{\infty}, \cdot) \in *G \setminus \{0\} \]

**Proof.** From Propositions 1.20 and 1.21.

**Proposition 1.23.** 
\((*G, \cdot)\) is closed

**Proof.** Any product with 0 results in 0. Consider non-zero products, since multiplying by the components is closed, and multiplying by an infinitesimal and infinity is in \(*G \setminus \{0\}\) by Propositions 1.20 and 1.21, then by the multiplication of components Definition 1.29 the product is closed.

**Proposition 1.24.** 
\((\Phi^{-1}, \cdot)\) is commutative

**Proposition 1.25.** 
\((\Phi_{*}^{-1}, \cdot)\) is commutative

**Proof.** By Proposition 1.24, a subset is also commutative. \(a, b \in \Phi; a', b' \in \Phi_{*};\) If \(a \cdot b = b \cdot a\), and let the unique expressions be \(a' = a\), \(b' = b\), then it follows by substitution \(a' \cdot b' = b' \cdot a'.\)

**Proposition 1.26.** 
\((\Phi, \cdot)\) is commutative
Proof. \( a, b \in \Phi^{-1}; \) By Proposition 1.24 the infinities are commutative, \( \frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab} = \frac{1}{b} \cdot \frac{1}{a} \)

Proposition 1.27. \( (\Phi, \Phi^{-1}, \cdot) \) is commutative

Proposition 1.28. \( (*G\{0\}, \cdot) \) is commutative

Proof. Since the components are commutative, and multiplication between infinitesimals and infinities is commutative, then general multiplication is commutative.

Proposition 1.29. \( (*G\{0\}, \cdot) \) is associative

Proof. Using Maxima symbolic mathematics package. Implement multiplication, with the fourth element in an unknown status.

\begin{align*}
f_4(a_1,a_2,a_3,a_4, b_1,b_2,b_3,b_4) &= \text{expand}(a_1*b_2+a_2*b_1+a_1*b_1), \text{expand}(a_2*b_2)\text{, expand}(a_3*b_3+a_3*b_2+a_2*b_4)\
&\text{ expand}(a_1*b_3+a_3*b_1+a_4*(b_1+b_2+b_3)+b_4*(a_1+a_2+a_3)+a_4*b_4))\
&f_5(a,b) := f_4(\ a[1],\ a[2],\ a[3],\ a[4],\ b[1],\ b[2],\ b[3],\ b[4]);
\end{align*}

The following gave the same output, proving the associativity.

\begin{align*}
f_5(\ f_5(\ [a_1,a_2,a_3,0],\ [b_1,b_2,b_3,0],\ [c_1,c_2,c_3,0] ));
&f_5(\ [a_1,a_2,a_3,0],\ f_5(\ [b_1,b_2,b_3,0],\ [c_1,c_2,c_3,0] ));
\end{align*}

\begin{align*}
[a_1b_2c_2+a_2b_1c_2+a_1b_1c_2+a_2b_2c_1+a_1b_2c_1+a_1b_1c_2+a_2b_1c_2+a_3b_3c_3
\text{ + a}_2b_3c_3+a_3b_2c_3+a_2b_2c_3+a_3b_3c_3+a_2b_3c_2+a_3b_2c_2+a_1b_3c_3
\text{+ a}_1b_2c_3+a_2b_3c_2+a_1b_3c_2+a_2b_3c_1+a_3b_3c_1+a_2b_3c_1+a_3b_3c_1+a_2b_3c_1+a_3b_3c_1]
\end{align*}

Proposition 1.30. \( (\Phi^{-1}\cup\{0\}, +, \cdot) \) is distributive

Proposition 1.31. \( (\Phi\cup\{0\}, +, \cdot) \) is distributive

Proposition 1.32. \( (*G, +, \cdot) \) is distributive, \( a \cdot (b + c) = (a \cdot b) + (a \cdot c) \)

Proof. \( a(b + c) = (a_1, a_2, a_3) \cdot (b_1 + c_1, b_2 + c_2, b_3 + c_3) = (a_1(b_2 + c_2) + a_2(b_1 + c_1) + a_3(b_1 + c_1), a_2(b_2 + c_2) + a_3(b_3 + c_3) + a_3(b_2 + c_2) + a_2(b_3 + c_3) + a_1(b_3 + c_3) + a_3(b_1 + c_1) = (a_1b_2 + a_1c_2 + a_2b_1 + a_2c_1 + a_1b_1 + a_1c_1, a_2b_2 + a_2c_2, a_3b_3 + a_3c_3 + a_3b_2 + a_2c_3 + a_2b_3 + a_2c_3) + a_1b_3 + a_1c_3 + a_3b_1 + a_3c_1)

\begin{align*}
ab + ac &= (a_1, a_2, a_3)(b_1, b_2, b_3) + (a_1, a_2, a_3)(c_1, c_2, c_3) \\
&= [(a_1b_2 + a_1c_2 + a_2b_1 + a_1b_1 + a_1c_1, a_2b_2 + a_3b_3 + a_3b_2 + a_2b_3) + a_1b_3 + a_1c_3 + a_3b_1 + a_3c_1] + [(a_1c_2 + a_2c_1 + a_1c_1, a_2c_2, a_3c_3 + a_3c_2 + a_2c_3) + a_1c_3 + a_3c_1 + a_3c_1] \\
&= (a_1b_2 + a_2b_1 + a_1b_1 + a_1c_2 + a_2c_1 + a_1c_1 + a_2b_2 + a_2c_2, a_3b_3 + a_3b_2 + a_2b_3 + a_2c_3 + a_3c_3 + a_3c_2 + a_2c_3) + a_1b_3 + a_1c_3 + a_3b_1 + a_3c_1
\end{align*}

Consider that the three components are distributive, Propositions 1.30 and 1.31 and \( \mathbb{R} \). They are also commutative, Propositions 1.10 and 1.7 and \( \mathbb{R} \). Then the expressions are equal.

Proposition 1.33. \( (\Phi\cup\{0\}, +) \) has identity 0

Proof. \( a \in +\Phi^{-1}; \frac{1}{a} + 0 = \frac{1}{a} + \frac{0}{a} = \frac{1+0}{a} = \frac{1}{a} \).

Proposition 1.34. \( (\Phi^{-1}\cup\{0\}, +) \) has identity 0
Proof. By Proposition 1.33 since true for any $\Phi^{-1}$.

Proposition 1.35. $(*G, +)$ has identity $(0, 0, 0)$

Proof. Solve $a + b = a$ for identity $b$. $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1, a_2, a_3)$ then by the components independence, $a_1 + b_1 = a_1$, $b_1 = 0$ by Proposition 1.14, $a_2 + b_2 = a_2$, $b_2 = 0$ as a real number, $a_3 + b_3 = a_3$, $b_3 = 0$ by Proposition 1.34.

Proposition 1.36. $(\Phi^{-1} \cup \{0\}, +)$ has an inverse

Proof. By assumption that any number in $\Phi$ can have an integer coefficient multiplier of $\pm 1$; $x \in \Phi^{-1}$; consider $x + (-x) = x(1 + -1) = x \cdot 0 = 0$

Proposition 1.37. $(\Phi^{-1} \cup \{0\}, +)$ has an inverse

Proof. By Proposition 1.36 as true for any $\Phi^{-1}$.

Proposition 1.38. $(\Phi \cup \{0\}, +)$ has an inverse

Proof. $a, b \in \Phi^{-1}$; $\frac{1}{a} + \frac{1}{b} = 0$, $\frac{b}{ba} + \frac{a}{ba} = \frac{b + a}{ba} = 0$ by Proposition 1.36 when $b = -a$, and inverse of $\frac{1}{a}$ is $-\frac{1}{a}$.

Proposition 1.39. $(*G, +)$ has an inverse

Proof. Since each component has an inverse, Propositions 1.37 and 1.38, $a + b = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (0, 0, 0)$, $b = (-a_1, -a_2, -a_3)$.

Proposition 1.40. $(*G, +)$ is an abelian group

Proof. Group properties: closure Proposition 1.12, associativity Proposition 1.14, identity Proposition 1.35, inverse Proposition 1.39, Abelian group by Proposition 1.13.

Proposition 1.41. $(*G \setminus \{0\}, \cdot)$ has an identity 1

Proof. $(a_1, a_2, a_3) \cdot (0, 1, 0) = (a_1 \cdot 1 + a_2 \cdot 0 + a_3 \cdot 0, a_2 \cdot 1 + a_3 \cdot 1 + a_2 \cdot 0, a_1 \cdot 0 + a_3 \cdot 0)$

Proposition 1.42. $(*G \setminus \{0\}, \cdot)$ has inverse

Proof. If we consider the 7 forms of $a_1 b_3$ and the 7 forms of $a_3 b_1$ there are 49 combinations, leading to 49 sets of equations of the form $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = (0, 1, 0)$. Since this is 3 linear equations with 3 unknowns, providing the equations do not contradict, there is always a unique solution.
Contradictory solutions are a consequence of the different cases. For example, \(a_2 \neq 0\) for the following set of equations. 

\[ a_1 a_3 \in \Phi; a_3 b_1 \in \Phi; a_1 b_2 + a_2 b_1 + a_1 b_1 + a_1 b_3 + a_3 b_1 = 0; a_2 b_2 = 1; a_3 b_3 + a_3 b_2 + a_2 b_3 = 0; \]

**Proposition 1.43.** \((\ast G \setminus \{0\}, \cdot)\) is a group

**Proof.** Properties: closure Proposition 1.23, associativity Proposition 1.29, identity Proposition 1.41, inverse Proposition 1.42.

**Proposition 1.44.** \(*G\) is a field

**Proof.** Properties: Proposition 1.40, \((\ast G, +)\) is a group. Proposition 1.43, \((\ast G \setminus \{0\}, \cdot)\) is a group. Proposition 1.32, \((\ast G, +, \cdot)\) is distributive.

**Proposition 1.45.** \(*G\) is a total order field \(\{18\}\). \(a, b, c \in \ast G;\) If \(a \leq b\) then \(a + c \leq b + c.\) If \(0 \leq a\) and \(0 \leq b\) then \(0 \leq a \times b.\)

Scalar multiplication by a real number except 0 is closed.

**Proposition 1.46.** \(\mathbb{R} \setminus \{0\} \cdot \Phi \in \Phi\)

**Proposition 1.47.** \(\mathbb{R} \setminus \{0\} \cdot \Phi^{-1} \in \Phi^{-1}\)

**Proposition 1.48.** \(\mathbb{R} \setminus \{0\} \cdot \mathbb{R}_\infty \in \mathbb{R}_\infty\)

**Proof.** \(\mathbb{R}_\infty\) is the above infinitesimal and infinity cases, Propositions 1.46 and 1.47.

**Proposition 1.49.** \(\theta \in \ast G \setminus \{0\};\) if \(\theta\) is positive, \(z \in \{<, \leq, =, >, \geq\}\).

\[ f(z) \cdot g \iff \theta \cdot f(z) \cdot \theta \cdot g \]

**Proposition 1.50.** \(z \in \{<, \leq, =, >, \geq\};\) The relations \(z\) invert when multiplied by a negative number in \(*G\). For the equality case the left and right sides of the relation are interchanged.

**Proof.** \(a, b \in \ast G; a \cdot b, -a (-z) - b, -a + b (-z) 0, b (-z) a.\)

### 1.7 Conclusion

It is with disbelief that so many practising mathematicians have little explicit application of infinitesimals; that the online forums have so little discussion and that only the specialized few use NSA. This must change. Without explicitly partitioning the finite and infinite, there is a world of analysis that will not see the light of day.
The gossamer number system structure is the simplest explanation (Occam’s razor) for infinitesimals and infinities. The construction is identical with the implicit equation construction of real numbers except where the integers, the building blocks of the real number system, are replaced with ‘infinite integers’.

Since \( \ast G \) (we believe) is a field, the theory is general. For example, you could perhaps plug \( \ast G \) into Fourier analysis theory, and extend the theory.

Du Bois-Reymond expressed the numbers like we do today as functions. Largely to avoid accounts of the numbers being fiction and devoid of real world meaning.

However, had he realised the number system (which we believe needed the discovery of the infinite integers), he could have expressed his theory with two number systems \( \mathbb{R} \) and \( \ast G \) or \( \ast R \). Many others, including Newton, Leibniz, Cauchy, Euler and Robinson have considered these questions. A two-tiered number system \( \mathbb{R} \) and \( \ast G \) has evolved, not as an option, but as a necessity in explaining mathematics.

We have attempted to find a rigorous formulation to “traditional non-rigorous extensions”, which we believe has not been considered. By construction, the zero divisors problem relevant to the hyperreals is avoided, as in \( \ast G \) there is only one zero.

Acknowledgment, help from a reviewer: thank you for reviewing the original paper in earlier stages, which was subsequently split into six parts (due to the scope of the investigation), and then this paper. Thank you.

In response, better communication with numerous suggestions and extensions were subsequently applied, and the construction of the gossamer number system was found.

2 The much greater than relations

An infinitesimal and infinitary number system the Gossamer numbers is fitted to du Bois-Reymond’s infinitary calculus, redefining the magnitude relations. We connect the past symbol relations much-less-than \( < \) and much-less-than or equal to \( \preceq \) with the present little-o and big-O notation, which have identical definitions. As these definitions are extended, hence we also extend little-o and big-O, which are defined in Gossamer numbers. Notation for a reformed infinitary calculus, calculation at a point is developed. We proceed with the introduction of an extended infinitary calculus.
2.1 Introduction

While the majority of mathematicians readily accepted the emancipation of analysis from geometry there were, nonetheless, powerful voices raised against the arithmetization programs. One of the sharpest critics was Paul du Bois-Reymond (1831-1889) who saw the arithmetization as a contentless attempt to destroy the necessary union between number and magnitude. [19, p.92]

The separation between geometry and number by the arithmetization of analysis has led to the dominance of set theory. However, just as we have different languages, problems can be described with functions or set theory and other ideas with infinity. Language for theories is both an evolution and also can be more of a choice, but ‘does’ have an effect on how we see the mathematics.

We believe that arguments of magnitude are essential to understand real and gossamer numbers. Without a theory from this viewpoint many things are left without explanations. Without the relations described, the symbolism and language of algebra that they describe is harder to encapsulate.

Contradictory to du Bois-Reymond, we find arithmetization in his relations that lead to a transfer principle (Part 4) and non-reversible arithmetic Part 5. So we claim that they are important.

Before becoming aware of du Bois-Reymond’s work, we defined $\gg$ equivalently to $\succ$, as during a mathematical modelling subject a lecturer had symbolically used the symbol to describe (without definition) large differences in magnitude.

The notion of the ‘order’ or the ‘rate of increase’ of a function is essentially a relative one [7, p.2]. Consider functions $f(x)$ and $\phi(x)$, we could have functions satisfying relation $f \succ \phi$. However, what about their ratio? Knowing only $>$ or $\geq$ does not give a size difference of the numbers involved.

Consider monotonic functions which over time settle down, and have properties such as their ratio is monotonic too. In examining these well behaved functions, families of ratios, scales of infinities (Section 2.4) are considered. From these investigations, the characterisation of an infinity in size difference was discovered, and defined as a relation $\succ$ (Definition 2.13) and $\succeq$ (Definition 2.12). Here, it is not the sign of the number, but the size of the number which determines the relation.

With the particular system of notation that he invented, it is, no doubt, quite possible to dispense; but it can hardly be denied that the notation is exceedingly useful, being clear, concise, and expressive in a very high degree. [7, p. (v)]
However, the notation was quickly superseded by little-o and big-O, primarily because the magnitude relation, instead of being expressed separately, could be packaged as a variable. E.g. $\sin x = x + O(x^3)$ instead of $x^3 \geq -\frac{x^4}{3!} + \frac{x^5}{5!} - \ldots \bigg|_{x=0}$

We believe du Bois-Reymond’s relations do have a critical place, where we develop an algebra for comparing functions Part 3. For many reasons, we introduce the at-a-point notation, which is used throughout our series of papers towards the development of an alternative to non-standard analysis which we refer to as infinitary calculus.

By fitting an infinitesimal number system to du Bois-Reymond’s infinitary calculus definitions (Section 2.3), the theory is better explained. Instead of defining a limit in $\mathbb{R}$, the limit is defined in $\ast\mathbb{G}$ the extended number system.

The benefits continued with the later development of a transfer principle Part 4 between $\ast\mathbb{G}$ and $\mathbb{R}$, which explains mathematics that would not make sense without infinitesimals and infinities. General limit calculations do not make sense in $\mathbb{R}$ because the number system has no infinity elements.

This theory is then used to derive a new field of mathematics ‘convergence sums’ [12], with applications to convergence or divergence of positive series. Where du Bois-Reymond’s theory of comparing functions has been forgotten, we now believe we have found useful applications.

We believe this shows value and general applicability of the mathematics. The Gossamer number system’s utility is demonstrated. So, we see this as a building paper.

The relation $\succ$, is defined as equivalent to little-o, and is general because it exists in a number system which includes infinities and infinitesimals, and not a modified or implicitly defined $\mathbb{R}$. The current practise implicitly uses infinitesimals and infinities, without declaring them as their own number type Part 1.

In this sense we have extended du-Bois Reymond and Hardy’s work. By explicitly having a number system, we can better compare functions. In a later paper on the transfer principle, we will argue that this is not just an option, but a fundamental part of calculus.

Having said the above, the objective of this paper is to introduce definitions and notations, re-state du Bois-Reymond’s infinitary calculus, and connect the past with the present little-o and big-O notation.

### 2.2 Evaluation at a point

Motivation: Approximating functions by truncation in calculation is common practice. We use infinitesimals all the time.
Example 2.1. A quick numerical check will provide evidence by approximation, where successive powers significantly reduce in magnitude, $x = 0.1$, $x^2 = 0.01$, $x^3 = 0.001$.

\[ f(x) = x + x^2 + x^3 + \ldots |_{x=0} \text{ at } x = 0 \text{ we mean } x \in \Phi \text{ (see Definition 2.3). This may be represented by } f(x) = x, f(x) = x + x^2, f(x) = x + x^2 + x^3 \text{ or any other number of first terms at } x = 0. \text{ As } x^2 \text{ is much smaller than } x, x^3 \text{ is much smaller than } x^2, \ldots \]

When we send $x \to 0$, which functions that go to zero faster matters, as these may be truncated, and we can start using infinitesimals. It is this sort of reasoning and calculation that leads to the definition of the magnitude relation (see Definitions 2.12 and 2.13), and then to little-o and big-O notation which we use today.

Some people, who find negative numbers difficult to accept, will happily add and subtract positive numbers, but are unable to do so using negative numbers. In a similar way, there may occasionally be a problem where people can reason with infinity but not zero, or the other way round. Logically 0 and $\infty$ as numbers are very similar.

Example 2.2. Similar reasoning can be done with infinities. Let $x = 1/n$ and assume a solution. Consider the series first three terms. $y = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}, yn^3 = n^2 + n + 1$ when $n = \infty$. As $n$ is much greater than 1, assume $n + 1 = n, yn^3 = n^2 + n$, reversing, $y = \frac{1}{n} + \frac{1}{n^2}$ and $\frac{1}{n^3}$ was truncated.

Truncation non-uniqueness in calculation: Let $f(x)$ be a function of infinite terms used in function $h(x) = g(f(x))$. Since a truncated $f(x)$ can solve $h(x)$ for infinitely many truncations, we say $f(x)$ is not unique, as an infinite number of solutions may give a satisfactory result. When evaluating $h(x)$, $f(x)$ is not unique, as an infinite number of truncated evaluations can occur. It is often desirable to use the minimum number of first terms of $f(x)$ to evaluate $h(x)$. In this way, asymptotic expansions as given by $f(x)$ are said to be non-unique.

Calculation is a major part of analysis, and one of the most common evaluations is the limit of a function. This evaluation can be thought of more generally by considering the behaviour at a point, with the inclusion of infinity as a number and as a point.

When the ideas of a point are extended to include such properties as continuity, infinity, existence and divergence at a point, then it becomes clear that a point, whatever it may be, is both what we interpret and how we calculate.

With a view to realising something more general than a limit, the following definition at a point is given.

*By virtue of reaching a point, we have to pass through or approach the point. The definition of evaluation at a point will also encompass approaches to the point.*

This interpretation of a point accepts non-uniqueness - two parallel lines could meet at infinity or they may never meet at infinity. In particular, asymptotic expansions are not
unique, but subject to orders of magnitude.

**Definition 2.1.** Let \( f(x) \) be an expression. Then evaluation at-a-point \( f(x)|_{x=a} \) is the evaluation of \( f(x) \) at \( x = a \). (Optionally omit variable assignment, \( f(x)|_a \))

Case 1. All possibilities or
Case 2. Context dependent evaluation

Infinity can be considered as a point.

*Case 1* concerns itself with all the different ways a point could be interpreted and calculated, and is a conceptual tool.

Given a problem either theoretical or practical, there are often different views or interpretations which may help. (For example from a programming perspective, an object orientated approach to problem modelling.)

Let \( C^j \) describe a curve continuous in the first \( j \) derivatives, then let \( C^0 \) describe a continuous curve. When building a curve that is a function, except at a point, the following possibilities may occur (see Figure 3). The curve may be discontinuous at the point, or its vector equations are continuous but the function has infinitely many values at the point, or \( C^0 \) but not \( C^1 \) continuous, or the curve is a function and also an s-curve between an interval. With a point at infinity the possibilities are endless.

Let \( q \)

- \( f(x) \notin C^0 \)
- \( (x, f(x)): \) one-one
- \( q \)

- \( f(x) \notin C^1 \)
- \( f(x) \in C^0 \)
- \( f(x) \in C^\infty \)

\( x(t), y(t) \in C^0 \)
\( (x(t), y(t)) \)
\( f'(q) = \infty \)
\( q \)

\( f(x), q \in +\Phi^{-1} \)
\( f(x) = \infty \)
\( q = \infty \)

**Figure 3**: Examples of interpretations at a point

*Case 2* is the practical aspect of calculating, where a choice of interpretation has been made, on proceeding with the “actual calculation”. The context calculation separates responsibility
for the justification from the theory to the point of use. This decoupling is important. If there is another way of calculating or using another branch of mathematics, the evaluation at-a-point is simply interpreted then. The trade is that less can be said, in that the definitions and theory are less exacting, but this is mitigated by the calculation being context specific, and more adaptable to our problem solving.

The consequences of decoupling can be non-trivial. For example, we do not believe in necessarily using a field when extending the reals. A trade-off for a different kind of generality may be a different number system, or chosen differently, not depending on what you want to do.

Evaluating a function at-a-point can often result in the evaluation of the limit. Indeed, this limit at a point, is a subset of the possibilities. In the context of a calculation, \( \lim_{x \to a} f(x) \) can be represented by \( f(x)|_{x=a} \).

As mathematics is a language, a further purpose of Definition 2.1 is to communicate to the reader that other ways of calculation might be employed. For instance, where these could be incompatible with rigorous argument, one way of distinguishing differences could be through the above notation.

The notation can also be used to make existing arguments more explicit. For example \( f(x) = O(g(x))|_{x=\infty} \) says that the function is being considered at infinity, not 0 or any other finite value.

Motivation for a separate notation is used so that different mathematics can work side by side with standard mathematics, and in a sense be contained. The limit concept is so ‘ingrained’ that doing operations that use a different paradigm, without clear communication to the reader, would be unsatisfactory.

A consequence of such flexibility is that mathematical inconsistencies can and are invariably introduced into the calculations. Where this is viable, the benefits brought to the calculation can outweigh any adverse presumptions. Designing methods to protect against inconsistencies other than narrowed definition and practice can actually make the calculus more accessible and interesting.

**Definition 2.2.** Let the “at-a-point” definition, appearing on the right-hand side, apply to all functions within the expression, unless overridden by another at-a-point definition.

**Example 2.3.** \( f(x) \; z \; g(x)|_{x=a} \) means \( f(x)|_{x=a} \; z \; g(x)|_{x=a} \) where \( z \) is any relation. Optionally we can include round brackets around the expression, \( (f(x) \; z \; g(x))|_{x=a} \).

**Definition 2.3.** We say \( f(x) \; z \; g(x)|_{x=a} \) can have a context meaning, where \( f(x) \) and \( g(x) \) are dependent, and in some way governed by operator or relation \( z \).

When forming conditions for infinitely small or infinitely large, we employ a bound, which itself is going to zero or infinity; for instance, when forming definitions.
Definition 2.4. In context, a variable \( x \) can be described at infinity \( |x=\infty, \) then \( \exists x_0, \forall x : x > x_0 \)

Definition 2.5. In context, a variable \( x \) can be described at zero \( |x=0 \) then \( \exists x_0, \forall x : |x| < x_0 \)

With the transfer principle Part 4, Definitions 2.4 and 2.5 which describe the neighborhood can be better expressed with infinitesimals Definition 2.7 and infinities Definition 2.6 are defined with sequences more generally in Part 6.

Definition 2.6. In context, a variable \( x \) can be described at infinity \( |x=\infty, \) then \( x \in +\Phi^{-1} \) an infinity.

Definition 2.7. In context, a variable \( x \) can be described at zero \( |x=0 \) then \( x \in \Phi \) an infinitesimal.

Definition 2.8. In context, we say \( f(x)|_{x=\infty} \) then \( \lim_{x \to \infty} f(x). \)

Generally the equality “=” with respect to assignment is defined with a left-to-right ordering see Definition 2.9. Essentially as reasoning, such that the right follows from the left. This can be exact, as in one form is converted to another, or as a generalization, or rather implication. Therefore the context needs to be understood.

Definition 2.9. In context, assignment has a left-to-right ordering.

In context, instance = generalization

Example 2.4. How to use the notation is open, some examples follow.
1. Limit calculations \((1 + \frac{1}{n})^n|_{n=\infty} = e\)
2. Divergent sums \(\sum_{k=1}^{n} \frac{1}{k} = \ln n|_{n=\infty}\)
3. A conversion between series and integrals, read from left-to-right.
   \(\sum_{1}^{n} a_n = \int_{1}^{n} a(n) \, dn + c|_{n=\infty}\)
4. A comparison relation. \(n! > n^2|_{n=\infty}\)
5. An infinitary calculus relation \(f_n < g_n|_{n=\infty}\) (as described in Definition 2.13).
6. Asymptotic results

Provided there are no contradictions, the expressions at infinity can be handled algebraically in the usual ways.

Example 2.5. \(n! = (2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}} e^{-n}|_{n=\infty} \) then times by \(e^n|_{n=\infty}\) gives \(n!e^n = cn^{n+\frac{1}{2}}|_{n=\infty}.\)

Example 2.6. \(\ln(n!)|_{n=\infty} = \sum_{k=1}^{n} \ln k|_{n=\infty} = \int_{1}^{n} \ln k \, dk + \gamma|_{n=\infty} = [k \ln k - k]|_{1}^{n} + \gamma|_{n=\infty} = n \ln n - n|_{n=\infty}\)

Definition 2.10. Generalize the at-a-point Definition 2.2 to include condition \(c(x)\) in relation \(f(x)|_{c(x)}\). Where \(c(x)\) can describe an interval.
Example 2.7. $f(x) \rightarrow g(x)|_{x=(0,1]}$ describes the relation $z$ (see Part 3) over the interval $(0,1]$.

Example 2.8. The vertical bar notation is more general when working across different situations. Such as when little-o and big-O notation may be cumbersome $O(x^2) + O(x^3) = O(x^2)$ becomes $x^2 + x^3 = x^2|_{x=0}$, as an alternative to the approximation symbol so $a \approx b = c$ becomes $a = b = c|_{n=\infty}$, $x^n e^{-n} \approx n^n e^{-n} e^{-\xi^2/2} |_{n=\infty}$, $f \sim g$ becomes $f = g|_{n=\infty}$, so assignment becomes consistent.

Since non-uniqueness is accepted with the notation; $\sin x = x - x^3/3!|_{x=0}$ can be understood to mean truncation - an exact happening at $x = 0$: an approximation in $\ast G$ where $\Phi \mapsto 0$ see Part 4. The notation gives you the choice. If you want to be more 'exact' or explicit, then use other or further relations. $\sin x = x - x^3/3! + O(x^5)|_{x=0}$.

If say, after $k$ or more terms, the calculation is invariant with non-reversible arithmetic, we define this as an 'exact happening'. Increasing the number of terms does not change the calculation. After a transfer, the calculations produce the same result. Such situations are common, where to few terms in the approximations give incorrect results.

The notation is also built with comparing functions in mind, where non-reversible arithmetic (see Part 5) is applied. An alternative to the limit notation, as it applies across relations and as an aid to Landau notation, can be replacing $x \rightarrow \infty$ with $|x=\infty$. The concept of approach is logically equivalent to being at the value, and the notation can say this.

A notation for chaining arguments using commas as an implication and context is used. The last proposition uses the first expression. As a free algebra, this does place responsibility on both reading expressions and writing expressions. The notation is concise. When there are errors in evaluation or proofs, the chaining arguments can be rewritten one expression per line and edited. (Later, for similar reasons but a narrower purpose, in context we have defined $=$, with a left-to-right ordering.)

Example 2.9. $x, y \in \mathbb{R}; x > 0, y > 0, x + y > 0$

Definition 2.11. Mathematical arguments can be chained with context by commas (‘,’) and semicolons (‘;’), from left-to-right order. The next statement optionally has a left-to-right implication. The semicolons have a lower precedence.

In evaluating a function at-a-point we can shift any point $x = a$ to the origin or infinity, stating the definition at $\infty$, is as general as stating the definition at any other point.

Proposition 2.1. If; $f, x \in \ast G$; then $f(x)|_{x=a} = f(x + a)|_{x=0} = f(\frac{1}{x} + a)|_{x=\infty}$

Hence the investigation at infinity is similar to the corresponding theory at zero. It is here that infinitesimals (near zero) and infinities (near infinity) of infinitary calculus operate.
2.3 Infinitary calculus definitions

The following is a summary and extension, by means of the gossamer numbers Part 1, of infinitary calculus definitions and a comparison with the derived work of little-o and big-O relations. This is a calculus of magnitudes.

Occasionally ≪ and ≫ are used to indicate much smaller or larger numbers. Correspondingly ≪ and ≫ implement the idea of much smaller than and much greater than numbers by defining infinitely smaller and infinitely larger relations.

While a finite number is not infinity, a very large number treated as infinity would model the situation and allow reasoning. These relations, at zero or infinity provide an implementation of this.

The idea of much larger numbers is extended to infinity, where it becomes obvious that there are much larger numbers than others; hence du Bois-Reymond’s development of the Definitions 2.13 and 2.12, which are equivalent to little-o and big-O respectively.

We demonstrate the connection between modern relations and du Bois-Reymond’s relations and restate some definitions of du Bois-Reymond, referred to by G. Hardy in Orders of Infinity [7, pp 2–4] with their representation in Landau notation.

Definition 2.12. We say \( f(x) \preceq g(x) \mid_{x=\infty} \) if there exists \( M \in \mathbb{R}^+ \): \( |f(x)| \leq M |g(x)| \mid_{x=\infty} \).

\( f(x) \preceq g(x) \) is the same as \( f(x) = O(g(x)) \)

Definition 2.13. We say \( f(x) \prec g(x) \mid_{x=\infty} \) then \( f(x) \leq \delta \mid_{x=\infty} \).

\( f(x) \prec g(x) \) is the same as \( f(x) = o(g(x)) \)

Definition 2.13 is equivalently defined \( |f(x) \leq M_2 |g(x)| \mid_{x=\infty} \), \( M_2 \in \Phi^+ \). We see that this is almost the same as Definition 2.12 except \( M \in \mathbb{R}^+ \). One is bounded in \( \mathbb{R} \), the other in *G.*

Proposition 2.2. If \( f(x) \prec g(x) \mid_{x=\infty} \) then \( \frac{f(x)}{g(x)} \mid_{x=\infty} = 0 \) in \( \mathbb{R} \). As a definition see [7, p.2].

Proof. Apply a transfer \( \Phi \mapsto 0 \) (see Part 4) to Definition 2.13.

Proposition 2.3. If \( f(x) \prec g(x) \mid_{x=\infty} \) then \( \frac{g(x)}{f(x)} \mid_{x=\infty} = \infty \).

Proof. By Definition 2.13 let \( \delta \in \Phi \), \( \frac{f(x)}{g(x)} \mid_{x=\infty} = \delta \), \( \frac{g(x)}{f(x)} \mid_{x=\infty} \in \Phi^{-1} = \infty \).

Proposition 2.4. \( \delta \in \Phi^+ \); if \( \frac{f(x)}{g(x)} \preceq \delta \) then \( f(x) \prec g(x) \).
Proof. Since \( f \) and \( g \) are positive, an infinitesimal is their upper bound. Since choosing any infinitesimal in \((0, \delta]\) satisfies the much-less-than relation, \( f \prec g \).

[ When applying Proposition 2.5, we will need to avoid division by zero via a transfer \( 0 \mapsto \Phi \) and \( \infty \mapsto \Phi^{-1} \), thereby treating 0 and \( \infty \) separately. See Part 4 ]

Proposition 2.5. \( 0 \prec \Phi \) and \( \Phi^{-1} \prec \infty \)

Proof. Since a magnitude relation, we need only consider the positive case.

\[ 0 \prec \Phi^+: \delta_1, \delta_2 \in \Phi^+ \text{ Consider } 0 \prec \delta_1 \text{ as there is no smallest number. } \]
\[ 0 < \delta_2 < \delta_1. \text{ Since 0 is smaller than } \delta_2 \text{ then } 0 \prec \delta_1. \]

\[ +\Phi^{-1} \prec \infty: \text{ By inverting the relation we obtain the infinite case. Choose } \delta_2 \prec \delta_1, \frac{1}{\delta_2} > \frac{1}{\delta_1}, \frac{1}{\delta_1} \prec \frac{1}{\delta_2}, \text{ however } \frac{1}{\delta_2} < \infty \text{ then } \frac{1}{\delta_1} \prec \infty. \]

Definition 2.14.

When \( f(x) \prec g(x) \) we say \( f(x) \) is much-less-than \( g(x) \)

When \( f(x) \succ g(x) \) we say \( f(x) \) is much-greater-than \( g(x) \)

Definition 2.15.

\( g(x) \prec f(x) \) is the same as \( f(x) \succ g(x) \)

\( g(x) \preceq f(x) \) is the same as \( f(x) \succeq g(x) \)

Definition 2.16.

\( f(x) \succ g(x) \) is the same as \( f(x) = \omega(g(x)) \)

\( f(x) \succeq g(x) \) is the same as \( f(x) = \Omega(g(x)) \)

Because little-o and big-O are defined on a right-hand side order for \( \prec, \preceq \), additional symbols are needed for \( \succ, \succeq \). Here, infinitary calculus has a notational advantage.

Definition 2.17. We say \( f(x) \asymp g(x)|_{x=\infty} \) if \( f(x) \succeq g(x)|_{x=\infty} \) and \( f(x) \preceq g(x)|_{x=\infty} \). (See Definition 2.22 and Proposition 2.7.)

\[ f(n) \asymp g(n) \text{ is the same as } f(n) = \Theta(g(n)) \]

Definition 2.18. We say \( a \simeq b \) then \( a \) and \( b \) are infinitesimally close, \( a - b \in \Phi \cup \{0\} \) [16, p.57]

Definition 2.19. We say \( f(x) \sim g(x)|_{x=\infty} \) then \( \frac{f(x)}{g(x)}|_{x=\infty} \simeq 1 \)
We may consider the asymptotic relation \( \sim \) as an equality with respect to the product, and the infinitesimally close relation \( \simeq \) as an equality with respect to addition.

**Definition 2.20.** We say \( f(x) \propto g(x) |_{x=\infty} \) if \( f(x)/g(x) |_{x=\infty} \simeq c \). This uses a different relation symbol from Hardy’s in [7, pp 2–4].

The functions \( f(x)/g(x) \) may not necessarily be compared, particularly if oscillating between categories at infinity occurs. [7, p.4] \( f \succ g \) and \( f \preceq g \) are not each other’s logical negations in general.

**Example 2.10.** A counter example demonstrating logical negation does not imply a much less than or equal to relation.

\[
(f_n/g_n) |_{n=\infty} = (0, \infty, 0, \infty, \ldots) \quad \text{(Sequence at infinity)}
\]

Assume \( f_n \not\simeq g_n \) implies \( f_n \preceq g_n |_{n=\infty} \) \( (f_n/g_n) \simeq (g_n/g_n) \) \( (f_n/g_n) \preceq (1, 1, 1, \ldots) \) \( (0, \infty, 0, \infty, \ldots) \preceq (1, 1, 1, \ldots) \) \( (A \text{ component-wise contradiction}) \)

**Proposition 2.6.** If the relation between \( f \) and \( g \) are \( \{ f \prec g, f \asymp g, f \succ g \} \), then the negation of one of these relations would imply one of the other two relations.

**Proof.** \( f \succ g \) and \( f \prec g \) are disjoint. Since \( \asymp \): \( f \succeq g \) or \( f \preceq g \) covers the remaining cases. Since this is given as disjoint, only one of the three cases can occur. \( \square \)

Further theorems follow: if \( f \succ g, g \succeq h \), then \( f \succ h \). This is interesting from an application perspective as the ratio \( f/g \) has settled down into one of the three relations.

In comparing relations \( \{ o(), O(), \omega(), \Omega() \} \) with \( \{ \prec, \preceq, \succ, \succeq \} \), while the variable relation and symbols are equivalent, the symbols can be easier to manage and understand in comparison.

However the relation variables of Landau’s notation have a major advantage over infinitary calculus relation symbols in that the relation is packaged as a variable in the equation. \( \frac{1}{1-x} = 1 + x + x^2 + O(x^3) \).

Consequently the definitions of infinitary calculus symbols and Landau notation can be viewed as complementary.

We add further definitions to infinitary calculus that extend its use as an infinitesimal calculus analysis.

**Definition 2.21.** We say \( c(x) \prec \infty \) when \( c(x) \neq \pm \infty \) and that \( c(x) \) is bounded.
a ≺ ∞ is not the same as a ∈ R as the bound includes infinitesimals. While the function c(x) has finite bounds, c₀ < c(x) < c₁, these functions do not need to converge. E.g. f(x) = \sin x|_{x=∞} ≺ ∞, f(x)|_{x=∞} ≺ ∞. At times such functions behave similarly to constants. However an infinitesimal when realized is 0 and not positive, hence the need to exclude infinitesimals from a finite positive bound definition.

**Definition 2.22.** A variable has a “finite positive bound” if a < x < b, \{a, b\} ∈ R⁺.

**Example 2.11.** \((\frac{1}{n}, 1)|_{n=∞}\) is not a finite positive bound as the infinitesimal \(\frac{1}{n}|_{n=∞} \notin R⁺\), \(\frac{1}{n}|_{n=∞}\) is not finite. Further when realizing the infinitesimal, the interval is not all positive, as it includes 0. \((\frac{1}{n}, 1)|_{n=∞} = [0, 1)\) Similarly \(\sin \frac{1}{n} = \frac{1}{n}|_{n=∞} = 0\) does not have finite positive bound.

**Proposition 2.7.** If \(f \succ g\) then \(\frac{f}{g}\) has a finite positive bound.

**Proof.** From Definition 2.17 if \(f ≍ g\) then \(f ≪ g\) and \(f ≳ g\). \(M, M₂ \in R⁺\); from Definition 2.22 if \(f ≳ g\) then \(Mf ≥ g\). \(f ≪ g\) then \(f ≤ M₂g\), \(\frac{f}{M₂} ≤ g ≤ Mf\), \(\frac{1}{M₂} ≤ \frac{f}{g} ≤ M\). Inverting, \(M₂ ≥ \frac{f}{g} ≥ \frac{1}{M₂}\).

Hardy in *Orders of Infinity* [7, p.4] states several theorems with the much greater than relations and their transitivity. The infinitesimal numbers developed earlier can be used as a tool to prove these theorems. When proving, without loss of generality, consider positive functions.

While Hardy states the reader will be able to prove the theorems without difficulty, here a new number system is used for that purpose.

**Proposition 2.8.** \(f > φ, φ ≳ ψ\), then \(f > ψ\)

**Proof.** \(δ ∈ Φ⁺\);

\[
\begin{align*}
f > φ & \text{ then } φ = δf & \text{(Definition 2.13)} \\
φ ≳ ψ & \text{ then } Mφ ≥ ψ & \text{(Definition 2.12)} \\
Mδf ≥ ψ & \text{ (Redefine } δ \text{ to absorb } M) \\
δf ≥ ψ & \text{ (Proposition 2.4)} \\
δ ≥ \frac{ψ}{f} & \\
f > ψ &
\end{align*}
\]

**Definition 2.23.** Let \(¬\) be the negation operation, and \(z\) the binary relation. \(¬(f z g) = (f z g)\).

\(¬(f z g) = (f (¬z) g)\)

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Example 2.12. Examples of negation in $\mathbb{R}$ or $*G$. $(\neg <) = \geq$. $(\neg ==) = \neq$.

Theorem 2.1. $a, b, c \in *G \setminus \{0\}$; If $ab \neq c$ then $a \neq cb^{-1}$

Proof. $(ab \neq c) = \neg(ab = c) = \neg(a = cb^{-1}) = (b \neq cb^{-1})$

Proposition 2.9. $f \succeq \phi$ implies the negation of $f \prec \phi$, $(\neg(f \prec \phi))$. However $(\neg(f \prec \phi))$ does not imply $f \succeq \phi$.

Proof. Without loss of generality, consider positive $f$ and $\phi$. Since $f \succeq \phi$, $\exists M : M \in \mathbb{R}^+$ then $Mf \geq \phi$, $M \geq \frac{\phi}{f}$. Since $\frac{\phi}{f}$ is positive and bounded above, and $\frac{\phi}{f} \in \mathbb{R}^+ + \Phi$.

If we consider the negation relation, $\delta \in \Phi$, $f \neq \phi$, $(\neg(f \prec \phi))$, $(\neg(\frac{f}{\phi} = \delta))$, $\frac{f}{\phi} \neq \delta$, which excludes infinitesimals, but not infinities. Since $\frac{f}{\phi}$ is positive, then expressed as an interval $\frac{\phi}{f} \in [\mathbb{R}^+ + \Phi, +\Phi^{-1}]$.

We can see the first interval is a subinterval in the second, hence implication is confirmed, but that the second interval is not contained in the first, then the ‘not implied’ confirmed.

2.4 Scales of infinity

In music a scale ordered by increasing pitch is an ascending scale, while descending scales are ordered by decreasing pitch. Indeed everyone has heard musicians going through the scales in rehearsal before a performance.

In an analogous way mathematics has its scales where families of functions ascend and descend. Because of a property of the numbers zero and infinity, the scales are defined at these points, giving a number system at zero and at infinity.

Since these scales are intimately involved with the evaluation of a function at a point (extended sense), the scales apply to any function evaluation. A simple example is that when $a \neq 0$, $x^2 |_{x=a} = (x + a)^2 |_{x=0} = x^2 + 2ax + a^2 |_{x=0}$, we also see $x^2 \prec 2ax \prec a^2 |_{x=0}$, correlates to the scale $x^2 \prec x \prec 1 |_{x=0}$

Hardy discusses in detail the rates of growth of functions, and compares two functions where different functions could be ordered. Hence I believe the title of his book is fittingly “Orders of Infinity”. A new function can always be inserted between two ordered functions. Different families of functions possess different orderings. This is similar to the real number system where we can always find a number between two other numbers.

The notion of numbers being much greater than $(\succ)$ or much smaller than $(\prec)$ other numbers makes sense for numbers that are infinitely large or infinitely small.
Consider the family of functions $x^k$ as $x \to \infty$. Moving away from the origin, each function with increasing exponent $k$, becomes steeper.

![Figure 4: powers at infinity](image)

From the considerations of infinitely large functions, relational scales can be developed. The progression of these functions form a scale of higher infinities.

Let $x \to \infty$ then $x^2/x \to \infty, x^3/x^2 \to \infty, \ldots$

By defining a measure of the magnitude as the absolute value of the ratio between two functions at infinity, the scales of infinity are more easily expressed. See the $\succ$ relation (Definition 2.13).

$$(\ldots \succ x^3 \succ x^2 \succ x^1 \succ x^0 \succ x^{-1} \succ \ldots) |_{x=\infty}$$

This relation is conveniently symmetrical such that by swapping the function’s sides, the arrow reverses in direction in the same way $3 \prec 5$ becomes $5 \succ 3$.

$$(x \prec x^2 \prec x^3 \prec \ldots) |_{x=\infty}$$

Other examples: $(e^x \prec e^{e^x} \prec e^{e^{e^x}} \prec \ldots) |_{x=\infty}$ and importantly the logarithmic scale $(n \succ \ln n \succ \ln \ln n \succ \ln \ln \ln n \succ \ldots) |_{n=\infty}$

**Definition 2.24.** Let $k$-powers of $e$ be represented by $e_k(x)$: $e_0(x) = x$, $e_k(x) = e^{e_{k-1}(x)}$

**Definition 2.25.** Let $k$-nested natural logarithms be represented by $\ln_k(x)$: $\ln_{-1} x = 1$, $\ln_0 x = x$, $\ln_k x = \ln(\ln_{k-1} x)$

**Definition 2.26.** As a convention, when $\ln_k$ has no argument, we define $\ln_k = \ln_{k \to \infty} n$.

Consider $\ln_k |_{n=\infty}$. If $n$ reaches infinity before the $k$-nested log functions, $\ln_k = \infty$ is guaranteed. Looking at this another way, let $k$ be finite. This avoids the possibility of the
logarithm becoming negative or complex. Each of these infinities belongs to a family scale.

\[(\ln x \succ \ln_2 x \succ \ln_3 x \succ \ldots)\bigr|_{x=\infty}\]

**Conjecture 2.1.** Given \(f(x)\bigr|_{x=\infty} = \infty\), \(k = \infty\), \(\ln_k f(x) = \infty\) when \(x\) reaches infinity before \(k\).

While Conjecture 2.1 is usually expressed as a definition, the possibility of an ordering of variables at infinity should be expected, and this may provide much further investigation. A variable reaching infinity before another variable could better explain partial differential equations, where other variables are held constant, and the target variable differentiated.

**Definition 2.27.** Given initial relation \(\phi_1 \succ \phi_2\), and function \(\phi : \phi_{n+1} = \phi(\phi_n)\) with the property \(\phi_n \succ \phi_{n+1}\), the relations \((\phi_1 \succ \phi_2 \succ \ldots \succ \phi_n \succ \ldots)\bigr|_{n=\infty}\), are referred to as ‘scales of infinity’ [7, p.9]. Similarly with the much-less-than relation \(<\).

With the definition of much less than and much greater than, multiplying the scale by constants has no effect.

**Proposition 2.10.** \(f, g \in \ast G; \ \alpha_1, \alpha_2 \in \mathbb{R}\backslash\{0\}\); then \(f \succ g \Leftrightarrow \alpha_1 f \succ \alpha_2 g\)

**Proof.** \(f \succ g\) then \(\frac{g}{f} = \Phi\), \(\frac{\alpha_1 g}{\alpha_1 f} = \Phi\), \(\frac{\alpha_2 g}{\alpha_2 f} = \frac{1}{\alpha_1} \Phi = \Phi\), then \(\alpha_1 f \succ \alpha_2 g\)

**Corollary 2.1.**

Given \((\phi_1 \succ \phi_2 \succ \ldots)\bigr|_{n=\infty}\)

If \(a_n \in \mathbb{R}\backslash\{0\}\), then \((a_1 \phi_1 \succ a_2 \phi_2 \succ a_3 \phi_3 \ldots)\bigr|_{n=\infty}\)

**Proof.** Apply Proposition 2.10 to each relation.

Infinitely small magnitude scales can likewise be considered. For the powers of \(x^k\) this has the effect of reversing the relation when evaluating \(x\) at 0.

\((\ldots \prec x^4 \prec x^3 \prec x^2 \prec \ldots)\bigr|_{x=0}\)
The scales of infinity are often used implicitly in calculations. For example, truncating the Taylor series, or with limit calculations by ignoring the infinitesimals which effectively sets the infinitesimals to zero.

Since scales of infinity describe infinitesimals, calculus can be constructed with these ideas. A most useful scale in algebraic simplification orders different families of curves, whereby different types of infinitesimals and infinities are compared.

\[
\begin{align*}
(c & < \ln(x) < x^p)|_{p>0} < a^x|_{a>1} < x! < x^x)|_{x=\infty} \\
(\ldots & < x^{-2} < x^{-1} < 1 < x < x^2 < \ldots)|_{x=\infty} \\
(\ldots & > x^{-2} > x^{-1} > 1 > x > x^2 > \ldots)|_{x=0} \\
(\ldots & < e^{e^{-x}} < e^{-x} < 1 < e^x < e^{e^x} < \ldots)|_{x=\infty} \\
(x & > \ln x > \ln_2 x > \ldots)|_{x=\infty} \\
(v & < \ln v > \ln_2 v > \ln_3 v > \ldots)|_{v=0^+} \\
(hf' & > \frac{h^2}{2!}f^{(2)} > \frac{h^3}{3!}f^{(3)} > \ldots)|_{h=0^+} \text{ when } f^{(k)} < \infty
\end{align*}
\]

Table 1: Summary of scales

2.5 Little-o and big-O notation

Since infinitary calculus has equivalent definitions for little-o and big-O notation, it can be used to do the same things. It can describe function growth, compare functions, and derive theorems.

Where little-o and big-O notation surpasses the infinitary calculus notation, we see both
notations as complementary. In particular, the Landau notation’s strength is that it contains
the relation as an end term to a formula. That is, a relation is packaged and managed as a
variable.

\[ e^x = 1 + x + \frac{1}{2}x^2 + O(x^3) \]

The infinitary calculus symbols are not “side dependent”, \( f(x) \succ g(x) \) is the same as \( g(x) \prec f(x) \), which can give the algebra a sense of freedom. The Landau notation introduced \( \omega(x) \) and \( \Omega(x) \) to express the relations on the “other side”, see Definition 2.16.

Before proceeding, properties of the magnitude relations \( \{\prec, \preceq, \succ, \succeq\} \) are derived using \( \ast G \), thus demonstrating its usefulness in proofs. These properties are then used to prove theorems
with little-o and big-O, demonstrating an equivalence with the magnitude relations.

To simplify the proofs, from Proposition 2.13, we can make the arguments positive. Therefore
with assumptions regarding \( a \) and \( b \), we can always transform the problem to one with
sequences positive or greater than zero, since these relations are not affected by the sign of
the elements of the sequence.

Since \( a \) and \( b \) are positive numbers, either infinitesimals, infinities, or real numbers except
0, then we can multiply and divide \( a \) and \( b \), before realizing the infinitesimal or infinity.

**Proposition 2.11.** \( b \succ a \iff \frac{1}{b} \prec \frac{1}{a} \)

**Proof.** \( a \neq 0, b \neq 0 \), let \( \delta \in \Phi, b \succ a \) then \( \frac{a}{b} = \delta, \frac{1}{b} = \frac{1}{\delta}, \frac{1}{a} = \frac{1}{\delta}, \frac{1}{b} \prec \frac{1}{a} \). Similarly if \( \frac{1}{b} \prec \frac{1}{a} \) then \( \frac{1}{a} = \delta, \frac{1}{b} = \delta, a \prec b \).

**Proposition 2.12.** \( b \succ a \iff cb \succ ca, c \in \ast G \backslash \{0\} \)

**Proof.** \( a \neq 0, b \neq 0 \), let \( \delta \in \Phi, b \succ a \) then \( \frac{a}{b} = \delta, \frac{ca}{cb} = \delta, cb \succ ca \). Similarly if \( cb \succ ca \) then \( \frac{ca}{cb} = \delta, \frac{a}{b} = \delta, b \succ a \).

**Proposition 2.13.** \( a \prec b \iff -a \prec b \)

**Proof.** \( a \neq 0, b \neq 0 \), let \( \delta \in \Phi, a \prec b \) then \( \frac{a}{b} = \delta, -\frac{a}{b} = -\delta, -a \prec b \). Similarly if \( -a \prec b \) then \( -\frac{a}{b} = \delta, \frac{a}{b} = -\delta, a \prec b \).

**Proposition 2.14.** \( a \preceq b \iff a + \lambda \preceq b + \lambda \text{ when } \lambda \prec a \text{ and } \lambda \prec b \).

**Proof.** \( a \neq 0, b \neq 0 \), let \( \delta \in \Phi, a \prec b \) then \( \frac{a}{b} = \delta, -\frac{a}{b} = -\delta, a \prec b \). Reversing the argument, \( \frac{a}{b} = \frac{a+\lambda}{b+\lambda} \in \Phi^{-1} \) then \( a + \lambda \preceq b + \lambda \).

**Proposition 2.15.** \( a \succeq b \iff \frac{1}{a} \preceq \frac{1}{b} \)

**Proof.** \( a \neq 0, b \neq 0, a \succeq b, \exists \alpha: |a| \geq |b|, \alpha \geq \frac{|b|}{|a|}, \alpha \frac{1}{|a|} \geq \frac{1}{|b|}, \frac{1}{a} \succeq \frac{1}{b}, \frac{1}{b} \succeq \frac{1}{a}, \frac{1}{a} \preceq \frac{1}{b} \).

**Proposition 2.16.** \( a \succeq b \iff ca \succeq cb \)
Proof. Let \( c \in \ast G \setminus \{0\} \). Consider \( a \geq b \), \( \exists \alpha : \alpha |a| \geq |b| \), \( \alpha |c||a| \geq |c||b| \), \( \alpha |ca| \geq |cb| \), \( ca \geq cb \) then \( a \geq b \Rightarrow ca \geq cb \), reversing the argument gives the implication in the other direction.

**Proposition 2.17.** \( a \preceq b \iff -a \succeq b \)

**Proof.** Consider \( a \succeq b \), \( \exists \alpha : \alpha |a| \geq |b| \), \( \alpha |-a| \geq |b| \), \( -a \succeq b \) then \( a \succeq b \Rightarrow -a \succeq b \), reversing the argument gives the implication in the other direction.

**Proposition 2.18.** \( \lambda < \infty \), \( a \succeq b \Rightarrow a + \lambda \succeq b + \lambda \)

**Proof.** \( a \succeq b \), \( \exists \alpha : \alpha |a| \geq |b| \), \( \alpha |a| + \lambda \geq |b| + \lambda \), Assume \( \alpha > 1 \) as we can always increase \( \alpha \). Case \( \lambda > 0 \), \( \alpha |a| + \alpha \lambda \geq |b| + \lambda \), \( \alpha |a| + \lambda \geq |b| + \lambda \), \( a + \lambda \succeq b + \lambda \). Case \( -\lambda \) then \( \alpha |a| - \lambda \geq |b| - \lambda \), \( \alpha |a| + \lambda \geq |b| + \lambda \), the above positive case. Hence \( a \succeq b \Rightarrow a + \lambda \succeq b + \lambda \)

**Example 2.13.** For proving the following big-O theorem we found infinitary calculus to be easier to reason with than the solution given in [4, Theorem 2.8].

If \( g(x) = o(1) \) then \( \frac{1}{1 + O(g(x))} = 1 + O(g(x)) \)

**Proof.** Let \( v(x) = O(g(x)) \)

\[
\frac{1 + v(x) \succeq 1}{1 + v(x)} \preceq 1 \quad \text{(from Proposition 2.15)}
\]

\[
\frac{v(x)}{1 + v(x)} \preceq v(x) \quad \text{(from Proposition 2.16)}
\]

\[
\frac{-v(x)}{1 + v(x)} \preceq v(x) \quad \text{(from Proposition 2.17)}
\]

\[
\frac{-v(x) - 1 + 1}{1 + v(x)} \preceq v(x)
\]

\[
\frac{1}{1 + v(x)} - 1 \preceq v(x)
\]

\[
\frac{1}{1 + v(x)} \preceq 1 + v(x) \quad \text{(from Proposition 2.18)}
\]

\[
\frac{1}{1 + O(g(x))} = 1 + O(g(x))
\]

**Verification:** rather than building the inequality, the inequality can be verified directly. \( \frac{1}{1 + v(x)} \preceq 1 + v(x) ; \frac{1}{1} \preceq 1 + v(x) \) is true, since \( 1 + v(x)|_{x=a} = 1 \) and \( v(x) \preceq g(x) < 1|_{x=a} \)
Example 2.14. Consider the proof of the following theorem from [4, Theorem 2.8] .

\[
\text{If } g(x) = o(1) \text{ then } \frac{1}{1 + o(g(x))} = 1 + o(g(x))
\]

Using an inequality in infinitary calculus to prove the theorem. Let \( h(x) = o(g(x)) \) \( x = a \), \( h(x) \prec g(x) \) \( x = a \), \( 1 + h(x) \geq 1 \) \( x = a \), \( \frac{1}{1 + h(x)} \leq 1 \) \( x = a \), \( \frac{h(x)}{1 + h(x)} \leq h(x) \) \( x = a \), \( -h(x) \leq h(x) \) \( x = a \), \( \frac{1}{1 + h(x)} \leq 1 \) \( x = a \), \( \frac{1}{1 + h(x)} - 1 \leq o(g(x)) \) \( x = a \), \( \frac{1}{1 + h(x)} - 1 = o(g(x)) \) \( x = a \).

From [4] the theorem is derived in the standard way by taking the limit. Applying the little-o definition directly:

\[
\lim_{x \to a} \frac{1 + h(x)}{g(x)} = \lim_{x \to a} \frac{1 - (1 + h(x))}{g(x)} = -\lim_{x \to a} \frac{h(x)}{g(x)} = -\lim_{x \to a} \frac{h(x)}{g(x)} \frac{1}{h(x)}
\]

The same calculation with infinitary calculus evaluation at the point and applying the definition:

\[
\lim_{x \to a} \frac{1 + h(x)}{g(x)} = \lim_{x \to a} \frac{1 - (1 + h(x))}{g(x)} = -\lim_{x \to a} \frac{h(x)}{g(x)}
\]

It can be shown that the scales with big-O notation, using the left side to right side definition, big-O notation is defined where \( f = O(g) \) is not the same as \( O(f) = f \). Let \( a > 0, b > 0, k > 0 \). We can write \( O(e^{-ax}) = O(x^{-b}) = O(ln(x)^{-k}) \) which has a left-to-right definition of \( O() \) and express in infinitary calculus as \( e^{-ax} \equiv x^{-b} \equiv ln(x)^{-k} \) \( x = \infty \).

3 Comparing functions

An algebra for comparing functions at infinity with infinireals, comprising of infinitesimals and infinities, is developed: where the unknown relation is solved for. Generally, we consider positive monotonic functions \( f \) and \( g \), arbitrarily small or large, with relation \( \preceq \): \( f \preceq g \). In general we require \( f, g, f - g \) and \( \frac{f}{g} \) to be ultimately monotonic.

3.1 Introduction

In extending du Bois-Reymond’s theory, we have discovered a new number system Part 1, and used this to rephrase du Bois-Reymond’s much greater than relations Part 2. However, at the heart of Reymond’s theory is the comparison of functions.

Today this may seem of little interest because there are no applications which directly require this. Even Hardy, through writing and extending du Bois-Reymond’s work [7] thought this.
Others better incorporated the theory: little-o and big-O notation have the same definitions as relational operators \( \prec, \preceq \) and similarly other relations Part 2.

Instead, du Bois-Reymond’s work became a catalyst for other higher mathematics and itself as an operational calculus was largely forgotten. In this era of immense change and other issues which they faced, this is not surprising. For just one example, the theory on divergent sums and functions was being established.

Our aim is to open the field of infinitary calculus through the development of another infinitesimal and infinitary calculus that is derived from du Bois-Reymond’s work. The method solves for relations between functions. Subsequent papers giving applications for comparing functions are found. (E.g. sum convergence [12, Convergence sums])

Comparing functions is the key discovery in the theory’s development.

With the method of comparing functions at infinity described in this paper, we believe there is a significant improvement over the method described by Hardy, which either computes the relation with a limit, which is fine, or uses logarithmico-exponential scales [7, pp.31–33].

By comparing functions at infinity, du-Bois Reymond showed the existence of curves infinitely close to each other. For example, the construction of infinitely many curves infinitesimally close to a straight line (see Example 3.20). Thereby demonstrating the curves exist at infinity.

However, the number system in which the curves reside includes infinitesimals and infinities, and hence is a non-standard analysis.

While du Bois-Reymond did not define a number system as Abraham Robinson had done with Non-standard analysis (NSA), the constructions prove that such a number system exists.

That Abraham made little reference to du Boise-Reymond’s work is unexplained. Though he did use similar notation. For example in [16, p.97]: if \( a \in \mathbb{J} \) and \( b \in \ast J \) then \( a \prec b \). Elements of \( \ast J \) he called infinite.

As this is a reference paper in the sense that it contains propositions which we have collected, while working through problems. Subsequent papers reference this paper.

We can consider comparison in terms of addition or multiplication. Where \( z \) is a relation.

| \( f z g \) | Comparison |
|---------------|-----------|
| \( f - g z 0 \) | additive sense |
| \( \frac{f}{g} z 1 \) | multiplicative sense |

Table 2: Binary relation comparison
3.2 Solving for a relation

While we are very familiar with solving for variables as values, in general we do not solve variables for relations. However there is no real reason not to do so.

In the course of devising an alternative way to compare functions at infinity, a new way of comparing functions has been developed, where the primary focus is to solve for the relation.

**Definition 3.1.** Let \( f(x) \ z g(x) \) be a comparison of the functions \( f(x) \) and \( g(x) \) where \( z \) is the variable relation.

When possible, we could then solve for a relation \( z \), for example \( z \in \{<, \leq, \leq, >, \geq, \succ, \succeq, =, \preceq, \neq, \ll, \gg \} \).

**Definition 3.2.** Given \( f(x) \ z g(x) \), with relation \( z \), then applying function or operator \( \phi \) to one or both sides of the variable relation \( z \) results in a new relation \( (\phi(z)) \).

The brackets about the relation are an aid to distinguish the relation as a variable.

\[
\begin{align*}
& f(x) \ z g(x), \ \phi(f(x)) (\phi(z)) \phi(g(x)) \\
& \text{The function is also applied to the middle as changing either } f \text{ or } g \text{ can change the relation. For example, applying an exponential function to both sides and the middle gives } e^{f(x)} (e^z) e^{g(x)}, \text{ where } (e^z) \text{ is the new relation. Applying the logarithm function to all parts, } \\
& \ln f(x) (\ln z) \ln g(x), \text{ where } (\ln z) \text{ is the new relation. Applying differentiation to all parts, } \\
& Df(x) (Dz) Dg(x), \text{ where } (Dz) \text{ is the new relation. } D \text{ is a shorthand operator for differentiation } \frac{d}{dx}. \\
\end{align*}
\]

Differentiating or integrating positive monotonic functions, with the condition that their difference is monotonic, preserves the \(<\) and \(\leq\) relations: \( Df(x) (Dz) Dg(x) \leftrightarrow f(x) \ z g(x) \) (see Part 6), and by notation, relations \((Dz) = (z)\), \((\int z) = (z)\).

**Definition 3.3.** We say \( z \in \mathbb{B} \) to mean that \( z \) is a binary relation.

**Definition 3.4.** When \( f(\phi(z)) \ g \) and \( (\phi(z)) = z_2 \), where \( z, z_2, (\phi(z)) \in \mathbb{B} \); we may propose \( \phi(z) = z_2 \), provided that there is no contradiction.

In practice, when solving for a relation, we presume such a relation exists and then proceed to solve for it. The method of brackets about the relation is a label of applied operations. If this is reversible, a solution exists to unravel the said operations. Definition 3.4 allows you to proceed with the solution process without having to formerly say so.

If \((D^n z) = \succ\) then solve \(D^n z = \succ\)

**Example 3.1.** \( 2x^2 \ z 5x|_{x=\infty}, 4x (Dz) 5|_{x=\infty} = \succ, \) removing the brackets and solving for \( z, Dz = \succ, \int Dz = \int \succ, \ z = \int \succ = \succ \) (see Table 3).
Definition 3.5. Let a “finite relation” be a relation without consideration of infinitesimal or infinitary arithmetic.

Example 3.2. The relations $\forall n > n_0 : n + 1 > n$, for positive $n$, then $\frac{1}{n} > 0$ is not finite relations, as they include infinite arithmetic. No lower bound exists. The infimum (greatest lower bound) 0 exists, but is another type of number. Positive $n$ has no upper bound either, but has an infimum $\infty$. The “bounds” do exist, but involve infinite arithmetic, and in a sense are numbers of another dimension.

Definition 3.6. Let a relation that is not finite have infinitesimal or infinite arithmetic. Any operation on the relation produces a new comparison, or new $z_k$, however such a system allows us to solve for the initial $z$, through the myriad of possibilities.

Example 3.3. Compare in a multiplicative-sense $n^2$ and $n$. See Figure 6. In solving for $\succ$, the relations $z_1$, $z_2$, $z_3$ did not change after division, $(\ldots, n^2 \succ n, n \succ 1, 1 \succ n^{-1} \ldots)|_{n=\infty}$, by $b \succ a \Leftrightarrow cb \succ ca$ Proposition [2,12]

Figure 6: Example calculation of relations connecting $*G$ and $\mathbb{R}$

Definition 3.7. Given relations $z_1$ and $z_2$,
If $f(x) \ z_1 \ g(x) \Rightarrow \phi(f(x)) \ z_2 \ \phi(g(x))$ then we say $\phi(z_1) = z_2$.

If $f(x) \ z_1 \ g(x) \Rightarrow e^{f(x)} \ z_2 \ e^{g(x)}$ then we say $e^{z_1} = z_2$.
If $f(x) \ z_1 \ g(x) \Rightarrow \ln f(x) \ z_2 \ \ln g(x)$ then we say $\ln z_1 = z_2$.

Use as an aid in calculation as a left to right operator when solving relations, where all functions in the relations are positive, $e^\succ = \succ$, $\ln \succ = \succ$, $\succ = \succ$.

The following compares the rate of increase of functions in infinitary calculus, which we find computationally easier than that described by Hardy, and at the least is an alternate way of performing such calculations.
The Caterpillar was the first to speak. “What size do you want to be?” it asked.
“Oh, I’m not particular as to size,” Alice hastily replied; “only one doesn’t like changing so often, you know.” “I don’t know,” said the Caterpillar. [15] pp 72–73

After eating the mushroom in her right hand, Alice was shrunk, and eating from the left hand she was magnified.

In an analogous way, by applying powers and logarithms we can magnify or shrink aspects of the function comparison. Combined with non-reversible arithmetic Part 5 (for example $n^2 + n = n^2|_{n=\infty}$), we can solve for the relation.

Powers and logarithms are mutual inverses. While logs of different bases can undo any powers, the natural logarithm ln and $e$ are the most useful. In solving relations, it is often convenient to apply these functions to both sides of a relation, in a similar manner to solving equations. Then apply infinitary arithmetic with non-reversible algebra to simplify the relation.

Consider raising both sides of a finite inequality to a power. E.g. $3 > 2, e^3 > e^2$ and the relation symbol did not change.

**Example 3.4.** Now consider a relation where both numbers are diverging to infinity. For example, $3x > 2x|_{x=\infty}$, then $e^{3x} > e^{2x}|_{x=\infty}$ but more importantly $e^{3x} > e^{2x}|_{x=\infty}$ as $e^{3x}/e^{2x} = e^x|_{x=\infty} = \infty$.

If two numbers are positive and one is much larger than the other, then the weaker relation that one of the numbers is greater than the other, must be true.

**Theorem 3.1.** $f = \infty, g = \infty$, if $f \succ g$ then $f > g$

**Proof.** $f \succ g$ then $f = \delta, \delta \in \Phi, g = f\delta$. Comparing, $f \succ g$, $f \succ f\delta, 1 \succ \delta, z = \succ$.

Magnifying a less than or greater than relationship magnifies the inequality, provided their difference is not finite (see Theorems 3.2 and 3.3). Demonstrating this, consider a condition with an infinitesimal difference, so the inequality exists in $*G$, but not $\mathbb{R}$.

**Example 3.5.** Show : $f < g$ does not imply $e^f < e^g$. $\delta \in \Phi; g = f + \delta, f = \infty, f < g, f < f + \delta|_{\delta=0}, e^f \succ e^{f+\delta}|_{\delta=0}, 1 z e^\delta|_{\delta=0}, 1 z 1 + \delta + \frac{1}{2}\delta^2 + \ldots|_{\delta=0}, 0 z \delta + \frac{1}{2}\delta^2 + \ldots|_{\delta=0}, 0 z \delta|_{\delta=0}, z = \prec, but \ z \ is \ not \ \prec.$

If we realize the infinitesimals $*G \leftrightarrow \mathbb{R}$ and we have equality. $1 z 1 + \delta|_{\delta=0}, \delta \mapsto 0, 1 \equiv 1$.

**Theorem 3.2.** $f = \infty, g = \infty$, If $f < g$ and $f - g < \infty \Rightarrow e^f < e^g$

**Proof.** $f < g, 0 < g - f, e^0 < e^{g-f}, 1 < e^{g-f}, e^f < e^g$
Theorem 3.3. \( f = \infty, g = \infty \), if \( f < g \) and \( g - f = \infty \) \( \Rightarrow \) \( e^f \prec e^g \)

Proof. \( e^f z e^g, e^0 z e^{g-f}, 1 z e^{\infty}, z = \prec. \)

Theorem 3.4. \( f = \infty, g = \infty \), if \( f \prec g \) then \( g - f = \infty \)

Proof. \( \delta \in \Phi; \) since \( f \prec g \), let \( f = \delta g \). Consider \( g - f = g - g\delta = g(1 - \delta) \simeq g = \infty. \)

Theorem 3.5. \( f = \infty, g = \infty \), if \( f \prec g \) then \( e^f \prec e^g \)

Proof. \( f \prec g, g - f = \infty. e^f z e^g, 1 z e^{g-f}, 1 z e^{\infty}, 1 z \infty, z = \prec \)

Proposition 3.1. \( f = \infty, g = \infty \), if \( f \succ g \) then \( \ln f - \ln g = \infty \)

Proof. \( f \succ g \) then let \( f\delta = g \) where \( \delta \in +\Phi; f = \delta^{-1} g. \) Consider \( \ln f - \ln g = \ln(\delta^{-1} g) - \ln g = \ln \delta^{-1} + \ln g - \ln g = \ln \delta^{-1} = \infty. \)

In reducing a large number, the log function applied to both sides of a relation can decrease the inequality.

Theorem 3.6. \( f = \infty, g = \infty \) If \( f < g \) then \( \ln f < \ln g \)

Proof. \( f < g, \) at any point let \( g = f^{1+\delta}, \delta > 0. \) Compare \( \ln f z \ln g, \ln f z \ln f^{1+\delta}, \ln f z (1 + \delta)\ln f, 0 z \delta \ln f, z = \prec \) then \( \ln f < \ln g. \)

Theorem 3.7. \( f = \infty, g = \infty \) If \( f \prec g \) then \( \ln f < \ln g \)

Proof. \( f \prec g \) then \( \frac{f}{g} = \delta; \delta \in \Phi^+; f = \delta g. \) Since \( g \) is not an infinitesimal, multiplying by an infinitesimal decreases the number, then \( f < g. \) Alternatively \( f z g, \delta g z g, \delta z 1, z = \prec \) as an infinitesimal is smaller than any positive number. As \( \ln \) preserves the relation, \( f < g \Rightarrow \ln f < \ln g \)

In reducing an infinite number with a log from a much less than relation does not imply another much less than relation, nor does it exclude it.

Proposition 3.2. \( f = \infty, g = \infty, f \prec g \neq \ln f \prec \ln g \)

Proof. Counter example: Example 3.6
Theorem 3.8. \( f = \infty, g = \infty \), if \( f \succ g \) then \( Df \succ Dg \).

Proof. \( Df \succ Dg, 1z \frac{Dg}{Df} \), but \( \frac{Dg}{Df} = \frac{g}{f} \) then \( 1z \frac{g}{f} \). \( g = \delta f; \delta \in \Phi^+; 1z \delta, z = \succ \). □

Example 3.6. \( e^{3x} \succ e^{2x} \mid x = \infty \rightarrow 3x \succ 2x \mid x = \infty \) as \( \frac{3x}{2x} \mid x = \infty = \frac{3}{2} \neq \infty \).

When asking what happens when we reduce an infinity, in a similarly way to magnifying the relationship, we can consider the two complete cases \( g - f < \infty \) and \( g - f = \infty \), thereby showing these conditions to be necessary and sufficient for determining what happens when reducing the relation.

Proposition 3.3. \( f = \infty, g = \infty \), \( f \propto g \iff \ln g - \ln f \prec \infty \).

Proof. \( f \propto g \then f \preceq g \) and \( f \succeq g \). Part 2. \( a, b \in \mathbb{R}^+ \); If \( f \propto g \) then \( a < \frac{f}{g} < b \). Case \( a < \frac{f}{g}, ag < f, \ln a + \ln g (\ln <) \ln f \ln a (\ln <) \ln f - \ln g. \) Similarly, \( \frac{f}{g} < b, f < bg, \ln f (\ln <) \ln b + \ln g, \ln f - \ln g (\ln <) \ln b. \) Putting the two conditions together, \( \ln a (\ln <) \ln f - \ln g (\ln <) \ln b. \) By Theorem 3.6. \( \ln \prec \infty \). \( \ln a < \ln f - \ln g < \ln b. \) Similarly ; \( a', b' \in \mathbb{R}^+ \); then \( a' < \frac{f}{g} < b' \) gives finite bounds. □

The development of solving for the unknown relation as a variable comes about through comparing functions, which includes the calculation of limits, see Part 5. After developing the theory, while investigating infinitesimals in Orders of Infinity [7], similar problems were found and some of du Bois-Reymond’s known theorems were rediscovered. At this point the alternate calculation was already useful, rather than trying to follow Hardy’s calculations.

Using the equality symbol as an operator reading from left to right, define \( z_1 = z_2 \), from Definition 3.7 a table of relation implications, the equals symbol is interpreted from left to right, generally leading to the right side being a generalization from the left. As an operator analogy; Mercedes = car; BMW = car. The right-hand side is the generalisation of the left.

When using equality operator = for generalization, place the variable relation being solved for, on the left side of the equals sign. For example, writing \( \ln z = \succ \) instead of \( \succ = \ln z \). Then the generalisation can be combined with solving the variable. \( \ln z = \succ, e^{\ln z} = e^\succ = \succ, z = \succ \). Examples of exponential and log functions:

In \( *G \), given \( f = \infty, g = \infty \), \( f \succeq g \) and \( \phi(f) (\phi(z)) \phi(g) \), then \( \phi(z) = z_2 \). The functions are continuous and monotonic in \( *G \).
| φ(\{<, <\}) | φ(\{>, >\}) | Condition | Reference |
|----------------|----------------|-----------|-----------|
| e^< = < | e^> = > | f - g < ∞ | Th. 3.2 |
| e^< = < | e^> = > | f - g = ∞ | Th. 3.3 |
| e^< = < | e^> = > | e^< = > | Th. 3.4 |
| ln^< = < | ln^> = > | f ≺ g | Th. 3.5 |
| ln^< = < | ln^> = > | Ignore integration constants | Part 5 |
| ln^< = < | ln^> = > | f ≺ g | Part 6 |
| D_< = < | D_> = > | Df - Dg is not constant | Part 6 |
| D_< = < | D_> = > | Df - Dg is not constant | Part 6 |
| D_< = ≤ | D_> = ≥ | Df - Dg is not constant | Part 6 |

Table 3: Relation simplification for positive divergent functions

Example 3.7. Decreasing/reducing the infinities

\[ 3x > 2x|_{x=∞} \]
\[ \ln(3x) (\ln >) \ln(2x)|_{x=∞} \]
\[ \ln 3 + \ln x (\ln >) \ln 2 + \ln x|_{x=∞} \]
\[ \ln 3 (\ln >) \ln 2 \]
\[ \ln 3 > \ln 2 \text{ then } \ln > = > \]

Example 3.8. Increasing/magnifying the infinities

\[ 3x > 2x|_{x=∞} \]
\[ e^{3x} (e^>) e^{2x}|_{x=∞} \]
\[ e^{3x} > e^{2x}|_{x=∞} \text{ then } e^> = > \]

To show how all this works, take an example problem from [7, p.8]. Solve for \( z \) the following, where \( \Delta \) is an arbitrarily large but fixed value.

Example 3.9.
At first, raising a relation to a power may seem silly, but it is useful when applied as a notational aid in the solution; understood as a magnification it makes sense. However, the solution is not always unique, \( z = e^\tau = > \) is true too, as \( \tau = > \) with the left-to-right reading.

The comparison at infinity can ignore added constants, that is, the comparison is with infinite elements.

**Proposition 3.5.** If \( f = \infty, g = \infty, f \succ \alpha, g \succ \beta, z \in \{>, \geq, \succ, \geq\} \) then

\[
 f + \alpha \ z \ g + \beta \ \Rightarrow \ f \ z \ g
\]

**Proof.** \( f + \alpha \ z \ g + \beta, f \ z \ g + \beta \) because \((f + \alpha = f \ as \ f \succ \alpha), \ f \ z \ g \ as \ g \succ \beta\)

**Example 3.10.** Demonstrated by example, another problem from [7, p.8]. Given \( P_m(x) = \sum_{k=0}^{m} p_k x^k, p_k \) is positive and \( Q_n(x) = \sum_{k=0}^{n} q_k x^k, q_k \) is positive. Show \( \ln \ln P_m(x) \sim \ln \ln Q_n(x) \mid x=\infty \)

\[
\ln \ln P_m(x) \ z \ln \ln Q_n(x) \mid x=\infty \\
\ln \ln \sum_{k=0}^{m} p_k x^k \ z \ln \ln \sum_{k=0}^{n} q_k x^k \mid x=\infty \quad (\text{Apply } x^k \succ x^{k-1}, p_k x^k + p_{k-1} x^{k-1} = p_k x^k \mid x=\infty)
\]

Contrast the above with an example where the highest order diverging terms are simplified (subtracting equally infinite quantities); the next highest order diverging terms determine the relation.

**Example 3.11.** Solve for \( z \), for the comparison \( n^n n \ z \ e^n n! \mid_{n=\infty} \). \( n \ln n + \ln n \ (\ln z) \ n + \sum_{k=1}^{n} \ln k \mid_{n=\infty} \). Given \( \sum_{k=1}^{n} \ln k = n \ln n - n \mid_{n=\infty} / \sum_{k=1}^{n} \ln k n = \int_1^n \ln k n \ dn = n \ln n - n \mid_{n=\infty} / \) then, \( n \ln n + \ln n \ (\ln z) \ n + n \ln n - n \mid_{n=\infty}, \ ln n \ (\ln z) \ 0 \mid_{n=\infty}, \ ln z = >, \ z = e^\tau = >, \ n^n n \succ e^n n! \mid_{n=\infty} \)

The application of the logarithm has simplified the problem from products of functions to sums of functions.
Example 3.12. Consider the following theorems from [7, pp.300–302] Theorem 7.11. If $a > 0, b > 0$ we have

$$
\lim_{x \to \infty} (\ln x)^b / x^a = 0, \quad \lim_{x \to \infty} x^b / e^{ax} = 0
$$

Proof. $\ln x < x|_{x=\infty},$ ln $x \neq x^a|_{x=\infty},$ ln $x \ln x|_{x=\infty},$ ln $x \ln x|_{x=\infty},$ $z = e^x = \infty$, ln $x < x^a|_{x=\infty},$ (ln $x)^b z_2 x^a|_{x=\infty},$ b ln $x \ln z_2 \ln x|_{x=\infty},$ b ln $x < a \ln x|_{x=\infty},$ (ln $x)^b \neq x^a|_{x=\infty},$ (ln $x)^b / x^a|_{x=\infty} = 0$

Proof. $e^x \neq x|_{x=\infty},$ $e^x \neq x^b|_{x=\infty},$ x (ln $z$) b ln $x|_{x=\infty},$ x $\neq b \ln x|_{x=\infty},$ z = $e^x = \infty$, $e^x \neq x^b|_{x=\infty},$ $e^{ax} z_2 x^b|_{x=\infty},$ a x (ln $z_2$) b ln $x|_{x=\infty},$ a x $\neq b \ln x|_{x=\infty},$ $e^{ax} \neq x^b|_{x=\infty}$. $x^b / e^{ax}|_{x=\infty} = 0$

Applying infinitary calculus to problems can result in choosing whether to use a theorem, or solving by calculating directly.

Example 3.13. Find $\lim_{x \to 0^+} x^\alpha \ln x,$ where $\alpha > 0, x^\alpha \ln x|_{x=0^+},$ $x^{-\alpha} \ln x^{-1}|_{x=\infty},$ ln $x^{-1} \neq x^\alpha|_{x=\infty},$ $x^{-\alpha} \ln x^{-1}|_{x=0^+} = 0, x^\alpha \ln x|_{x=0^+} = 0.$

Another way. Let $y = x^\alpha \ln x|_{x=0^+} = x^{-\alpha} \ln x^{-1}|_{x=\infty},$ ln $y = \ln(x^{-\alpha} \ln x^{-1})|_{x=\infty} = \ln x^{-\alpha} + \ln 2 x^{-1}|_{x=\infty} = \ln x^{-\alpha}|_{x=\infty}$ as ln $x \neq \ln 2 x|_{x=\infty},$ $y = x^{-\alpha}|_{x=\infty} = 0$

Example 3.14. [7, p.31] Compare the rate of increase of $f = (\ln x)^{(\ln x)\mu}$ and $\phi = x^{(\ln x)^{-v}}$.

$$f z \phi|_{x=\infty}$$

$$z_2 (\ln x)^\mu \ln 2 x (\ln z) (\ln x)^{-v} \ln x|_{x=\infty}$$

$$\mu \ln 2 x + \ln 3 x (\ln z) - v \ln 2 x + \ln 2 x|_{x=\infty}$$

$$(\mu + v) \ln 2 x + \ln 3 x (\ln z) \ln 2 x|_{x=\infty}$$

Case $\mu + v = 1, \ln 3 x (\ln z) 0|_{x=\infty}, f > \phi.$
Case $\mu + v < 1, (\mu + v) \ln 2 x (\ln z) \ln 2 x|_{x=\infty}, 0 < (1 - \mu - v) \ln 2 x|_{x=\infty}, f < \phi.$
Case $\mu + v > 1, (\mu + v) \ln 2 x (\ln z) \ln 2 x|_{x=\infty}, (\mu + v - 1) \ln 2 x|_{x=\infty} > 0, f > \phi.$

In solving relations of infinite magnitude, another case occasionally arises where both sides of the relation are infinite, but opposite in sign. Raise all parts to a power, with the effect of pulling the positive infinity further to infinity, and the negative infinity to zero, effectively pushing the relation further apart.

Example 3.15. $n = \infty$

$$-n (\ln z) \ln n$$

$$e^{-n} z e^{ln n}$$

$$e^{-(n+1)} z e^{ln(n+1)}$$

$$e^{-(n+2)} z e^{ln(n+2)}$$

$$0 \leftarrow z \rightarrow \infty$$

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Proposition 3.6.

If \( f = \infty, \ g = \infty, \ -f \leq g \) then \( e^{-f} \prec e^g \)

\[ \text{Proof.} \] Solving for \( z \), \( e^{-f} \leq e^g \), \( \frac{1}{e^f} \leq e^g \), \( 0 \prec \infty \), as \( e^f = +\infty \) then \( e^{-f} = 0 \). \( z = \prec \).

Example 3.16. Solve \( n \ln x (\ln z) \ln n|_{n=\infty} \) when \( x = (0,1) \). Within this interval \( \ln x \) is negative. \( n \ln x \succ \ln n|_{n=\infty} \), \( e^{n \ln x} (e^z) e^n n|_{n=\infty} \), \( x^n (e^z) n|_{n=\infty} \), \( x^n \prec n|_{n=\infty} \), \( z = e^z = \prec \).

Definition 3.8. Let \( == \) mean an equality relation.

Definition 3.9. Let \( z == \) mean equality is assigned to the variable \( z \)

We have further introduced a use of assignment as a left-to-right generalisation. As maths is a language, this decision was made to chain together implications.

Having the context of the problem being solved for is important.

Multiplying by \(-1\) reverses the orders direction, the order relations \( \{<, \leq, >, \geq\} \) are not effected by positive multiplication or addition.

Comparison in an additive sense ‘is’ effected by adding and subtracting terms. Not in the sense of the order as described, but the magnitudes can be shifted, hence the magnitude relations can change direction. If \( a \succ b \), \( a - a \prec b - a \), \( 0 \prec b - a \). Part 2

When adding the same value to both sides the much-greater-than relation can change direction, unlike inequalities \( \{<, \leq\} \) which are invariant.

Example 3.17.

\[
\begin{align*}
    x^2 & > x|_{x=\infty} \\
    x^2 - x^2 & < -x^2 + x|_{x=\infty} \\
    0 & < -x^2 + x|_{x=\infty}
\end{align*}
\]

(direction changes)

In solving a relation, say relation \( z \), if the aim in solving is to satisfy all the expressions involving \( z \), then adding to both sides can introduce contradictory solutions. If after addition, the highest terms magnitude is removed, the next highest order determines the magnitude relation.

Example 3.18. Solve \( z_1 \) and \( z_2 \) for the same relation.

\[
\begin{align*}
    n^2 + 3 z_1 n^2 + n|_{n=\infty} & \texttt{ (subtract } n^2) \\
    3 z_2 n|_{n=\infty} & \\
    z_1 = \prec, \ z_2 = \prec
\end{align*}
\]
However, for usability, we really do not wish to be this formal.

Example 3.19.

\[ n^2 + 3 \; z \; n^2 + n|_{n=\infty} \]

\[ 3 \; z \; n|_{n=\infty} \]

\[ z = < \]

(subtract \( n^2 \))

(solving \( z = < \) contradicts the first expression)

(True for both expressions)

This management of variables is part of the mathematics. At times, it is only necessary to solve with forward implications. Other times, for example in constructing proofs, reversibility, having implications in both directions is required. Or more generally, we want to solve not for several binary relation variables, when we can do so with one.

Up till now we have explored some of the mechanics for solving the relation. Of course, the premise was simple, solve for the relation as a variable.

However, infinitary calculus concerns itself with functions and curves, and continuous families of curves; we imagine curves between curves as real numbers between other real numbers. Just as the relation separated and defined different numbers, relations again separate and define different curves.

In discussing the infinity of curves near an existing curve, G. Fisher [9, pp 109–110] comments and beautifully quotes du Bois-Reymond in developing an infinity of curves close to \( y = x \), with the following relationship at infinity. \( x^{p+1} < x^{\frac{\ln 2}{\ln 2 + 1}} < x \big|_{x=\infty}, \; p \in \mathbb{N} \). That is, there is a function between \( x^{p+1} \) and \( x \), and there are an infinitely many such functions, \( x^{p+1} < \phi(x) < x \big|_{x=\infty} \). Therefore, there are an infinitely many functions close to \( y = x \) at infinity, where the space of real valued functions diverge, \( f(x) \big|_{x=\infty} = \infty \).

Example 3.20. There exists an infinity of curves infinitesimally close to the straight line \( y = x \). Show \( x^{p+1} < x^{\frac{\ln 2}{\ln 2 + 1}} < x \big|_{x=\infty} \).

We will use an indirect inequality approach, where we introduce another inequality. Undoing the base \( x \), show \( \frac{p}{p+1} < \frac{x}{x+1} < \frac{\ln 2 x}{\ln 2 (x+1)} \big|_{x=\infty} < 1 \).

\[ \frac{p}{p+1} \; z \; x \big|_{x=\infty} \]

\[ p(x + 1) \; z \; (p + 1)x \big|_{x=\infty} \]

\[ px + p \; z \; px + x \big|_{x=\infty} \]

\[ p \; z \; x \big|_{x=\infty}, \; z = < \]
\[ \frac{x}{x+1} \cdot \ln(x) \cdot \frac{\ln_2(x)}{\ln_2(x+1)} \cdot \ln x \bigg|_{x=\infty} \]

\[ \frac{x}{x+1} \cdot (\ln z) \cdot \frac{\ln_2(x)}{\ln_2(x+1)} \bigg|_{x=\infty} \]

\[ x \cdot \ln_2(x+1) \cdot (\ln z) \cdot (x+1) \ln_2 x \bigg|_{x=\infty} \]

\[ \ln x \cdot (\ln_2 z) \cdot \ln(x+1) \bigg|_{x=\infty} \]

\[ (\text{as } \ln x > \ln_3 x \bigg|_{x=\infty}) \]

\[ \ln z = <, \quad z = e^< = < \]

\[ \text{The last relation trivially follows, } \frac{\ln_2 x}{\ln_2(x+1)} \bigg|_{x=\infty} = 1, \quad \ln_2 x \cdot \ln_2(x+1) \bigg|_{x=\infty} =<. \text{ Then } \]

\[ x^{\frac{\ln_2 x}{\ln_2(x+1)}} \bigg|_{x=\infty} < x^{\frac{\ln_2 x}{\ln_2(x+1)}} \bigg|_{x=\infty} < x^1 \bigg|_{x=\infty}. \]

We did not need to introduce the additional inequality. However, like algebra in general, we may not get the minimal solution. The same calculation, without the added inequality, and using an asymptotic approximation could have been used, see [13, Example 2.5].

As a rule of thumb, think of infinity as being as large as the reals, so that you could construct any graph there, and then know that the space is larger still. However what is striking is the existence of the relations themselves, and this property, to not just partition, but develop algorithms at infinity. It is obvious then that if you can iterate in that space, have relations existing in that space, then you can construct algorithms and mathematical reasoning in that space.

In comparing sequences and functions, monotonic sequences and functions, that is sequences and functions which are either equal to or increasing, or equal to and decreasing, are of great interest.

Since a monotonic sequence can be made into a monotonic function, and a monotonic function back into a monotonic sequence, the theory of functions given is true for sequences.

Roughly, if we can determine that two functions are monotonic, we can do other things such as compare their ratio, and other mathematics.

Hence, the interest with Hardy’s L-functions, which are monotonic functions, that comprise of a finite combination of \{+,-,\div,\times,\ln,e\} operations.

Also, with monotonic functions, the association of a sequence of points to a curve would allow sequences and functions to be connected, via the same relations.

As sequences are indexed, a connection between the discrete and continuous can be made.
Proposition 3.7. Given a relation \( z \), and functions \( (f,g) \), then \( (z,(f,g)) \) is a relation if for all values in the domain, there is no contradiction. If there is a contradiction, \( (z,(f,g)) \) is not a relation, and is disproved as one.

Proof. Transforming the functions into sets, since the set generated by \( (z,(f,g)) \) is not an exact subset of the relation set generated from \( z \), by definition \( (z,(f,g)) \) is not a relation.

Example 3.21. Show \( n^2 > e^n \) contradicts. By using L’Hôpital Part 5, \( \infty > \infty \), differentiate, \( 2n > e^n \), \( 2 > e^n \), contradicts as \( 2 < e^n \).

Another approach is by solving, and showing that the symbols are contradictory.

Proposition 3.8. If \( f, g \in +\Phi \); and \( z \in \{<,\leq,=,>,\geq\} \),

\[ f \ z \ g \iff e^f \ z \ e^g \]

Proof. Assuming the partial sums of the exponential function with an infinitesimal are asymptotic to the infinite sum, \( e^\delta = 1 + \delta + \frac{\delta^2}{2} + \ldots \sim 1 + \delta + \sum_{i=0}^{\infty} \frac{f^{(i)}}{i!} \).

We have the general partial sum comparisons \( z_i, 1 + f \ z \ 1 + g, 1 + f + \frac{f^2}{2} \ z_2 1 + g + \frac{g^2}{2}, 1 + f + \frac{f^2}{2} + \frac{f^3}{3!} \ z_3 1 + g + \frac{g^2}{2} + \frac{g^3}{3!}, \ldots \)

Case \( e^f \ z \ e^g \) implies \( f \ z \ g \): \( e^f \ z \ e^g, 1 + f + \frac{f^2}{2} + \ldots \ z 1 + g + \frac{g^2}{2} + \ldots \), apply non-reversible addition Part 5 Theorem 5.1 to \( f \) and \( g \) partial sums, for example \( \frac{f^{(i)}}{i!} + \frac{f^{(i+1)}}{(i+1)!} = \frac{f^{(i)}}{i!} \) as \( f^i \succ f^{i+1} \) and \( z_{i+1} = z_i \). \( 1 + f \ z \ 1 + g, f \ z \ g \).

Reversing the process to prove the implication in the other direction. Adding the infinitesimal does not change the relation as the next term is much less than in magnitude to the previous term. \( f \ z \ g, 1 + f \ z \ 1 + g, 1 + f + \frac{f^2}{2} \ z \ 1 + g + \frac{g^2}{2}, 1 + f + \frac{f^2}{2} + \frac{f^3}{3!} \ldots \ z 1 + g + \frac{g^2}{2} + \frac{g^3}{3!} + \ldots \), recognising the expression as the exponential functions, \( e^f \ z \ e^g \).

For comparing of functions, we do need well behaved functions, hence the monotonic requirements. The classes of functions may appear to be restricted, but this can be expanded in many ways. Non-reversible arithmetic can be used to remove transient terms. For an additive comparison, we may only need \( f - g \) to be ultimately monotonic, and not the ratio.

So important is the determination of these classes of functions that they lead to the following definition and conjecture.
In accordance with redefining other infinitary calculus relations, we redefine the L-function in $\ast G$.

**Definition 3.10.** Define an L-function in $\ast G$ without implicit complex numbers as a finite combination of $\{+, -, \div, \times, \ln, e\}$ operations.

The following is given by Hardy as a theorem [7, p.24 Appendix I], but here stated as a conjecture because we re-defined the L-functions in $\ast G$ instead of $\mathbb{R}$. For example, the limit in $\ast G$ Part 6 can be a function, as it can contain infinitesimals and infinities. A transfer of the theorem in $\ast G \mapsto \mathbb{R}$ Part 4 would result in the theorem stated by Hardy.

**Conjecture 3.1.** Any L-functions at infinity is ultimately continuous and monotonic.

If $f$ and $g$ are L-functions then at $f \sim g|_{n=\infty}$, $z$ is unique and $z \in \{<, >, \sim\}$

Additionally the L-functions have the property that if $f$ and $g$ are L-functions, so is their ration $f/g$. Truncated infinite series can also be L-functions.

**Example 3.22.** $x^2 = e^{2\ln x}|_{x=\infty}$ is an L-function.

### 3.3 M-functions an extension of L-functions

Surprisingly, considering infinity as a point in the comparison theory at infinity is not enough. On occasion, it is beneficial to compare between two infinities.

This idea of comparison between two infinities was indirectly taken and adapted from NSA where convergence was determined by integrating at infinity: integrated between two infinities to determine convergence or divergence. Similarly for comparison, we can compare at infinity by comparing at infinity, over an infinite interval.

**Conjecture 3.2.** L-functions can be compared over any infinite interval. $f, g \in L$-functions; $a, b \in \Phi^{-1}; b - a = \pm\infty; \{<, >, \sim\} \in z$: $f \sim g|_{x=[a,b]} \equiv f \sim g|_{x=\infty}$

We would like to extend Conjecture 3.1 to include other functions such as $n!|_{n=\infty}$ which do not contain a finite number of multiplication operations, but is monotonic, and either ever increasing or ever decreasing. The following is an attempt to capture this.

What is the M-word? Marriage. We define a marriage of properties from Conjecture 3.1 the monotonic L-functions and ‘infinite term functions’.

By the following definition, assuming the conjecture is true, all L-functions are M-functions.
Definition 3.11. M-functions satisfy one of the following comparisons: $\propto$, $\prec$ or $\succ$ for any infinite interval.

If the functions have the same behaviour at infinity, that is $f, g \in \Phi^{-1}$; then only consider one infinite interval.

$$f(n), g(n) \in \text{M-functions}; a, b, b - a \in \Phi^{-1};$$

If $f \prec g|_{[a,b]}$ then by definition $f \prec g|_{n=\infty}$

Consider the following problem, first easily solved with Stirling’s formula, then without.

With Stirling’s formula $e^n n! = cn^{n+\frac{1}{2}}|_{n=\infty}$, the comparison is obvious.

$$n^n z e^n n!|_{n=\infty}$$
$$n^n z cn^{n+\frac{1}{2}}|_{n=\infty}$$
$$1 z cn^{n}\frac{1}{2}|_{n=\infty}$$
$$z = \prec$$

Example 3.23. Without Stirling’s formula, the problem appears more difficult. Firstly, re-organize the comparison.

$$n^n z e^n n!|_{n=\infty}$$
$$n^n z e^n \prod_{k=1}^{n} k|_{n=\infty}$$
$$n^n z \prod_{k=1}^{n} e^k|_{n=\infty}$$
Consider the comparison between two infinities. For example \((n, 2n)|_{n=\infty}\).

\[
\prod_{k=n+1}^{2n} n \quad \prod_{k=n+1}^{2n} e^k
\]
\[
\prod_{k=n+1-n}^{2n-n} n \quad \prod_{k=n+1-n}^{2n-n} e(k+n)
\]
\[
n^n \prod_{k=1}^{n}(e(n+k))|_{n=\infty}
\]
\[
n^n \prod_{k=1}^{n}(ne(1 + \frac{k}{n}))|_{n=\infty}
\]
\[
n^n z \prod_{k=1}^{n}(1 + \frac{k}{n})|_{n=\infty}
\]
\[1 \quad e^n \prod_{k=1}^{n}(1 + \frac{k}{n})|_{n=\infty}
\]
\[
z = \prec
\]

In Example 3.23 it is assumed that \(n^n, e^n n! \in M\)-functions.

The theory of functions is very important, as if we can guarantee certain properties, the analysis can be developed in powerful ways. In whatever form that the theory finally takes, identifying and restricting the functions will allow the application of comparison algebra to be more consistent and exacting. The development of the applications theory which we believe is a new field of mathematics rests on the comparison function theory. It is no longer a question of ‘will it work’, but ‘how does it work’. Getting this right potentially means being independent from NSA for solving large classes of problems, a goal worth striving for. A language of functions rather than a language of sets for solving theory with functions is required.

### 4 The transfer principle

Between gossamer numbers and the reals, an extended transfer principle founded on approximation is described, with transference between different number systems in both directions, and within the number systems themselves. Therefore an extended transfer principle with non-reversibility is established. As a great variety of transfers are possible, hence a mapping notation is given. In \(\ast G\) we find equivalence with a limit with division and comparison to a transfer \(\ast G \mapsto \mathbb{R}\) with comparison.
4.1 Introduction

The transfer principle in Non-Standard Analysis (NSA) generally translates between the hyperreals $\mathbb{R}^*$ and the reals $\mathbb{R}$. We are similarly interested in a transfer principle between the gossamer number system $G^*$ Part 1 and real numbers $\mathbb{R}$; including their variants.

For many reasons, we need to work in the more detailed number system. Any such work requires us to interpret or bring back the results into $\mathbb{R}$ or otherwise. We note that $G^*$ has $\mathbb{R}$ embedded within, making the transfer from $\mathbb{R}$ to $G^*$ possible. However, the nature of a statement in $G^*$ may not be able to be expressed in $\mathbb{R}$. While the transfer is possible, the meaning may change. (See Example 4.3)

We provide another view of the transfer principle that is based on approximation, a process with indeterminacy. This is motivated by the fact that if given a number in $G$ that is not an infinity, we can truncate successive orders of infinitesimals, and when all the infinitesimals are truncated we have only the real component remaining. Truncating all the infinitesimals is defined as the standard part $st()$ function, which results in a transfer from $G^* \mapsto \mathbb{R}$.

However, taking just one truncation can change an inequality. Hence, a more general view of a transfer from one state to another is warranted. We can also see the algebra of comparing functions Part 3 as transfers.

Then truncating a Taylor polynomial is a transfer; $G^* \mapsto G^*$, as truncating a Taylor polynomial may involve infinitesimals, which are not in $\mathbb{R}$.

More general questions can be asked. Consider the two number systems, one with infinitesimals and infinities, the other the reals. If $a > b$ in $G^*$, is this true in $\mathbb{R}$? Under what conditions is this true?

That is, can we in one number system, transfer to the other number system? So, rather than working in reals, and extending the reals which is implicitly done (for example the evaluation of a limit), you can deliberately work in one or the other number systems, and transfer between them.

Surprisingly we are applying the transfer principle all the time, for example in evaluating limits. The limits themselves, having infinitesimals or infinities do not belong in $\mathbb{R}$. By taking the limit, and truncating the infinitesimals that remain, you are effectively taking the standard part of the expression. That this is not discussed but assumed true, is part of our culture.
4.2 Transference

The transfer principle itself is a realization of the ‘Law of continuity’: a heuristic principle developed by Leibniz described in [29, p.2]

The rules of the finite succeed in the infinite and vice versa...

Leibniz rules can be explained by $\mathbb{R}$ and $\ast G$ being fields. This law (before Leibniz’s characterisation) from the beginnings of calculus was used for considering infinitesimal quantities with algebra, and then getting back tangible results. For example, Fermat integrated a power $\int a^n \, dn = \frac{a^{n+1}}{n+1}$.

To demonstrate the necessity of infinitesimals and infinities in the mathematics, from [30, p.64–65] we reconstruct the calculation in $\ast G$.

Example 4.1. Fermat integrating $y = x^n$ over $x = [0, a]$, partitioned by $ar^i$ where $i = \infty \ldots 0$. $t_i = ar^i$; $\int_0^a x^n \, dx = \sum_{i=\infty}^0 t_i(t_{i+1} - t_i)$ (a Riemann sum of unequal partitions); $t_{i+1} - t_i = ar^i(r - 1) \in \Phi$; $r \rightarrow 1$: $r \in 1 + \Phi$; for the Riemann sum to exist, else a divergent sum.

$$\sum_{i=\infty}^0 t_i(t_{i+1} - t_i) = -\sum_{i=0}^\infty a^n r^m ar^i(r - 1) |_{r=1} = (1 - r)a^{n+1} \sum_{i=0}^\infty (r^{n+1})^i |_{r=1} = (1 - r)a^{n+1}\frac{1}{1 - r} |_{r=1} = (1 - r)a^{n+1}\frac{a^{n+1}}{n+1}$$

To handle the apparent paradox, the new state is in $\ast G$ a higher dimensional calculus. The above example is done in $\ast G$ and implicitly brings the result back to real numbers. This is exactly what you do when taking a limit. The infinitesimals are approximated to 0 by $\Phi \mapsto 0$ and we are again in the standard real number system. Of course there can be no infinities here, so if you intend to increase $n$ to infinity, the expression is still in $\ast G$.

Hence, the mechanics of calculus require a higher dimensional view.

That we approximate in $\ast G$ produces a transition. But to what purpose? While the ‘rules’ require the law, the transfer itself is usually between states.

For geometric surfaces, we can easily visualise the law of continuity applying to computer generated meshes. As the mesh is refined, a smoother surface appears. For a 2D example, at infinity, a polygon of equal sides inside a circle becomes the circle.

That space with infinitesimals existing was predominant in their minds. Leibniz gives the example of two parallel lines infinitesimally close that never meet [29, p.1552]. Du Bois-Reymond constructs an infinity of curves infinitely close, therefore parallel to a straight line Part 3 Example 3.20. A transfer could be made from these curves to the straight line.

The following example is perhaps a more complicated transfer, as a radical state change occurs, but only at infinity. We describe a fixed ellipse with one focal point at the origin
and send the other focal point to infinity. The ellipse becomes a parabola, but only after the variable of the focal point is sent to infinity before the other variables. See 6.7. A variable reaching infinity before another

**Example 4.2.** [29, p.8–9] With a closed curve the ellipse becomes an open curve, the parabola, but only with the focus at infinity. Send the focus $h$ to infinity.

$$(x^2 + y^2)^\frac{1}{2} + (x^2 + (y-h)^2)^\frac{1}{2} = h + 2|h=\infty$$

$$(x^2 + y^2 + x^2 + (y-h)^2 + 2((x^2 + y^2)(x^2 + (y-h)^2))^\frac{1}{2} = (h + 2)^2|h=\infty$$

$$2x^2 + 2y^2 - 2yh + 2((x^2 + y^2)(x^2 + (y-h)^2))^\frac{1}{2} = 4h + 4|h=\infty$$

(Apply non-reversible arithmetic Part 5)

$$(2x^2 + 2y^2 - 2yh = -2yh|_{h=\infty} as \hspace{.1cm} -2yh > 2x^2 + 2y^2|_{h=\infty}, \hspace{.1cm} 4h + 4 = 4h|_{h=\infty})$$

$$((x^2 + y^2)(x^2 + (y-h)^2))^\frac{1}{2} = yh + 2h|_{h=\infty}$$

$$(x^2 + y^2)(x^2 + (y-h)^2) = (h(y + 2))^2|h=\infty$$

$$(x^2 + y^2)h^2 = h^2(y + 2)^2|h=\infty$$

$$x^2 = 4y + 4$$

That is, a closed curve is broken open. The ellipse is broken to form a parabola at infinity. For any finite values the curve is always closed, and is an ellipse.

The example highlights the directional nature of change. After applying non-reversible arithmetic to the equation, a transfer process takes place to the new state.

**Definition 4.1.** $\mathbb{R}^* = \mathbb{R} \cup \pm \infty$ the extended real numbers.

**Example 4.3.** From [26] reformed in $*G$. Let $n \in \mathbb{J}_\infty$, $w \in \mathbb{J}$ be finite then $\sum_{k=1}^{w} 1 < n|_{n=\infty}$ cannot be transferred to $\mathbb{R}$ because it lacks infinity elements in $\mathbb{R}$ then $*G \not\leftrightarrow \mathbb{R}$. However, since the extended reals $\mathbb{R}^*$ have infinity the transfer is possible; $*G \leftrightarrow \mathbb{R}$: $\sum_{k=1}^{w} 1 < \infty$, which is slightly different as the extended reals $\mathbb{R}^*$ only have two infinity elements, $\pm \infty$.

**Example 4.4.** In $*G$, $2 + \frac{1}{n} > 2|_{n=\infty}$, but $*G \not\leftrightarrow \mathbb{R}$. However, if we replace the strict inequality to include equality, the transfer is possible. $2 \geq 2$ in $\mathbb{R}$. (See Theorem 4.6)

A transfer principle states that all statements of some language that are true for some structure are true for another structure [26].

A sentence in $\varphi$ in $L(V(S))$ is true in $V(S)$ if and only if its *-transform $*\varphi$ is true in $V(*S)$ [22, p.82]

From Example [4.4] we see the transfer definitions given above are not adequate. While it is very important and most useful to take a proposition in one number system, and have the proposition true in another. For example, theorem proving where if true in one system implies the truth in the other. However, the principle as stated is not complete because a transfer can change the relation’s meaning.
We put forward a definition of the extended transfer principle, which in part, is based on approximation. Where, by realizing infinitesimals, we can truncate expressions. By seeing the continued truncation of infinitesimals as a sequence of smaller operations, we can transfer within the same number system.

We find such truncation can describe non-reversible processes, which lead to non-reversible arithmetic Part 5.

The second part of the transfer principle generalization is its directional nature. Transfers exist in both directions.

Possibilities arise from non-uniqueness, for example, a single point of discontinuity in $\mathbb{R}$ can be continuous in $\ast G$; transferring from $\ast G$ to the point discontinuity in $\mathbb{R}$ can be done in several ways. Perhaps a deeper transfer is the promotion of an infinitesimal to a small value.

**Definition 4.2.** Transfer principle: Assume an implementation of the "Law of continuity" between $\mathbb{R}$ or $\overline{\mathbb{R}}$ and $\ast G$ or $\overline{\ast G}$. For each number $x$ in the target space, $x \mapsto x'$ in the image space. If true over the domain in the target space, then it is true in the image space.

**Definition 4.3.** Extended transfer principle: Depending on context we can transfer in either direction, and in any combinations of number systems and operations. Further, dependent on the transfer, the relations may change.

Example 4.3 is an extended transfer. For further examples see Table 4 Mapping examples.

We differentiate between infinitesimals and zero. Similarly we differentiate between and an infinity such as $n^2|_{n=\infty}$ and the number $\infty$.

We will define an operation to convert from "an infinitesimal" to zero, and an operation to convert from "an infinity" to infinity. In other words, zero is a generalization of infinitesimals and its own unique number. Similarly, infinity is a generalization of infinities, and its own unique number.

**Definition 4.4.** We say "realizing an infinitesimal" is to set the infinitesimal to 0, and "realizing an infinity" is to set an infinity to $\infty$.

With these definitions an infinitesimal is not 0, but a realization of it. "An infinity" is not infinity, but an instance of it. By the 'realization' operation we convert infinitesimals and infinities respectively to 0 or $\infty$. The numbers 0 or $\infty$, while mutual inverses, have no specific inverses. After a realization, you cannot go back.

**Example 4.5.** $\infty \notin \Phi^{-1}$, but $\Phi^{-1} = \infty$ as a left-to-right generalization is true. Similarly $0 \notin \Phi$, but $\Phi = 0$. 

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Example 4.6. We may have \( n^2 |_{n=\infty} = \infty \). The left side is a specific instance of the right side generalization. Similarly for zero, \( \frac{1}{n} |_{n=\infty} = 0 \).

Example 4.7. \((\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \ldots ) |_{n=\infty} = (0, 0, 0, \ldots)\) is a null sequence, \( \frac{1}{n} |_{n=\infty} \in \Phi \).

Example 4.8. If we consider realizing an infinitesimal before dividing, \( 1/\delta = 1/0, \delta \in \Phi \), then we can interpret \( 1/0 = \infty \) as a generalization of a specific infinity which we do not know about, nor necessarily care.

Definition 4.5. Let \( rz(x) \) realize infinitesimals and infinities in an expression \( x \). Lower order of magnitude terms are deleted by non-reversible addition: if \( a \succ b \) then \( a + b = a \) Part 5.

Example 4.9. \( rz(n^2 + n + 1) |_{n=\infty} = n^2 |_{n=\infty} \) as \( n^2 \succ n + 1 |_{n=\infty} \).

Definition 4.6. Let \( rz(z) \) realize infinitesimals and infinities in an additive sense about the relation \( z \).

Example 4.10. Realize relation \( n z n^2 |_{n=\infty}, n rz(z) n^2 |_{n=\infty}, 0 z n^2 |_{n=\infty} \) because \( n z n^2, 0 z n^2 - n, 0 z n^2 |_{n=\infty} \) as \( n^2 - n = n^2 |_{n=\infty} \).

Fisher [9, p.115] comments, while following du Bois-Reymond’s infinitary calculus:

For the objects of his infinitary calculus, ..., are functions which do not form a field under the operations he considers (addition and composition), whereas his reference to ordinary mathematical quantities obeying the same rules that hold for finite quantities might be taken to refer to a field.

While *G we believe forms a field Part 1, this is the first step before applying arithmetic that has no inverse. Realizing can simplify the expression, however such an operation is non-reversible. Continued application could lead to 0 or \( \infty \), which could then be captured as a theorem. However in this case 0 and \( \infty \) have no inverses, with respect to their state before realization. While we define \( \infty \) and 0 as mutual inverses, the realization was likely done before this, perhaps through a limit.

The properties which make infinitary calculus not a field are the valuable properties. We approximate and use the field. The transfer (when realizing) is not independent of the number system, but part of it.

If after realizing a number, has the number changed type? Has the meaning of the relation changed?

Example 4.11. \( x^2 + \frac{1}{x} |_{x=\infty} \) realized to \( x^2 \) by discarding the infinitesimal, apply a transfer principle to bring back to \( \mathbb{R} \), \( y = x^2 \).

When transferring from a higher dimension number system to a lower dimension number system, in general, we need to consider the law of continuity after truncating.
The purpose of working in one number system is often to transfer the results to another. For example, we could solve a problem in integers with real numbers, and transfer the result back to working in the integer domain.

**Definition 4.7.** Let $A$ and $B$ be number systems, we say $A \mapsto B$ to mean the number system $A$ is projected onto or maps to the number system $B$.

Hence when realizing infinitesimals in $\ast G$ to $\mathbb{R}$ we approximate and simplify expressions. Let $x = a + \delta, a \in \mathbb{R}, \delta \in \Phi$. Truncating all the infinitesimals (assuming $\Phi^{-1} \notin x$) is the same as Robinson’s NSA, “taking the standard part” [16, p.57], $\text{st}(a + \delta) = a$. If we interpret this as converting a number with infinitesimals to a real number, we can see $\ast G$ as a more detailed space, where the numbers can ultimately be realized as reals. So $0$ in reals may be expressed as an infinitesimal in $\ast G$, as $0$ is a projection of an infinitesimal to the reals.

**Proposition 4.1.** $\delta \in \Phi; \ast G \mapsto \mathbb{R}: \delta \mapsto 0$

*Proof.* By approximation, repeated truncation of the infinitesimals leaves $0$. □

In the realization of infinitesimals and infinities, information can be lost. $\frac{1}{n} < 1|_{\infty}^\infty$ becomes $0 < 1$. If the relation was reordered differently, the much greater than relation could remain, but the realization would not be to the real number system, but the extended reals. $\frac{1}{n} < 1|_{\infty}^\infty, 1 < n|_{\infty}^\infty, 1 < \infty$.

**Example 4.12.** Let $\delta \in \Phi$, definition $f/g = \delta$ in $\ast G$ becomes $f/g = 0$ in $\mathbb{R}$.

A similar process exists for infinities, where the “infinities” are realized and converted to the "infinity".

**Example 4.13.** $\frac{1}{n+1} < 1|_{n=\infty}^\infty$ in $\ast G$ is true, but contradicts in $\mathbb{R}$ when we realize the infinitesimals: $0 < 0$. Similarly rearranging to compare infinities, $\frac{1}{n+1} < 1|_{\infty}^\infty, \frac{n}{n+1} < \infty$, $n < n + 1|_{n=\infty}^\infty$, realizing the infinities contradicts; $\infty < \infty$.

**Example 4.14.** $\frac{1}{n} > 1|_{n=\infty}^\infty, n^2 \frac{1}{n} > n^2 \frac{1}{n^2}|_{n=\infty}^\infty, n > 1|_{n=\infty}^\infty, \infty > 1$ is true.

$\frac{1}{n} > 1|_{n=\infty}^\infty, \text{realizing } \ast G \mapsto \mathbb{R}, 0 \not> 0$ is false, but $> \Rightarrow \geq$ is true.

Infinitesimals being smaller than any number in $\mathbb{R}$. Within an inequality, they can change to equality when removed.

**Example 4.15.** Let $\delta \in \Phi$, $(\ast G, e^f < e^{f+\delta}) \mapsto (\mathbb{R}, e^f = e^f)$

The following theorems may, if unfamiliar, seem trivial. If some proposition is true for a range, it is also true for its subrange: why would we make this into a theorem?
The very reduction of range can greatly simplify the complexity of cases involved. Hence, why construct theorems for reals and infinities if we only need to handle the infinite case?

Doing so, we believe leads to a radically different view of convergence and a new way to integrate: [12, Convergence sums ...] and rearrangement theorems with order on the infinite interval [14, Rearrangements of convergence sums at infinity].

By partitioning an interval between the infinireals and other numbers, we can separate arguments on finite numbers and infinireals to arguments on infinireals alone, and transfer when we need to go back to real or gossamer numbers.

Particularly important is the implication that partitioning by a finite bound, and including the infinity cases implies the infinity case, be it infinitely small or infinitely large.

We have kept the transfer notation \( \mapsto \) as we are losing information in the process.

**Definition 4.8.** A ‘bounded number’ is a number that is not an infinireal. All reals are bounded numbers, and so are all reals except 0 with infinitesimals. If \( x \) is a bounded number then \( x \in \ast G - \mathbb{R}_\infty \).

**Definition 4.9.** We say an ‘implicit infinite condition’ has a domain that includes both finite numbers (which can include infinitesimals) and infinireals \( \mathbb{R}_\infty \). Let \( x \) and \( x_0 \) be either real or gossamer numbers.

1. \( \forall x > x_0 \) where \( x_0 \) is finite.
2. \( \forall x : |x| < x_0 \) where \( x_0 \) is finite.

Since the finite numbers are partitioned by the infinite numbers (the infinireals), we remove the finite condition.

**Theorem 4.1.** If an implicit infinite condition at infinity determines some proposition \( P \) then we can transfer to the infinitely large domain.

\[
(\forall x : x > x_0) \mapsto (x \in +\Phi^{-1})
\]

**Proof.**

\[
[x > x_0] = [x > x_0][x < +\Phi^{-1}] + [x > x_0][x \in +\Phi^{-1}]
\]

Since choosing \( x \in +\Phi^{-1} \) in the domain always satisfies the condition, the transfer is always possible. \( \square \)

**Theorem 4.2.** If an implicit infinite condition at the infinitely small determines some proposition \( P \) then we can transfer to the infinitely small domain.

\[
(\forall x : -x_0 < x < x_0) \mapsto (x \in \Phi)
\]
Proof. 

\[ |x| < x_0 = |x| < x_0| |x| \in \Phi \] + \[ x < x_0| x \not\in +\Phi \]

Since choosing \( x \in \Phi \) always satisfies \( |x| < x_0 \) in the above, the infinitely small case is always true and the transfer is always possible.

\[ \square \]

**Definition 4.10.** In context, a variable \( x \) can be described at infinity \( |x| = \infty \) corresponds with Theorem 4.1

**Definition 4.11.** In context, a variable \( x \) can be described at zero \( |x| = 0 \) corresponds with Theorem 4.2

Theorems 4.1 and 4.2 are a common reduction within the transfer. Because of the variations of mapping involving the transfer from one domain to another, with different relations, we have developed a loose and not exact notation to communicate the mapping, and its context.

**Definition 4.12.** Let \((K, <f_b>, <x>)\) describe the number system and context, where the angle brackets indicate optional arguments. \( K \) is the number type, \( f_b \in \mathbb{B} \) a binary relation.

\((\text{number type}, <\text{relation}>, <\text{number}>)\)

A mapping between domains can be described by

\((K, <f_b>, <x>) \mapsto (K', <f'_b>, <x'>)\)

**Remark:** 4.1. A number may be input that is not of the same type as its result. For example \( f/g \) may be in number system \( K \) but neither \( f \) nor \( g \) need necessarily be in \( K \). Limit calculations happily accept input with infinities and infinitesimals, but the limit can be in \( \mathbb{R} \).

Mapping can occur in different contexts: realization, rearrangement of expression, transfer principle. The mapping can be in many different combinations. We summarize with a flexible notation; it is not at all strict.
Mapping examples

| Mapping | Comment |
|---------|---------|
| \((*G, /) \mapsto \mathbb{R}/\mathbb{R}\) | Limit \(\frac{a_n}{b_n}|_{n=\infty}\) evaluation |
| \((\mathbb{R}, /) \mapsto (*G, /)\) | Undoing an implicit limit |
| \((\forall x > x_0) \mapsto (\Phi^{-1}, |x=\infty\rangle)\) | Law of continuity from \(\mathbb{R}\) to \(*G\) Theorems 4.1, 4.2 |
| \((\forall |x| < x_0) \mapsto (\Phi, |x=0\rangle)\) | Realize infinitesimals |
| \((*G, \Phi^{-1}) \mapsto (\mathbb{R}, \infty)\) | Realize infinities |
| \((*G, \Phi) \mapsto (\mathbb{R}, 0)\) | Realize infinities, apply \(\text{st}()\) the standard part |
| If \((*G, \neq)\) then \((*G, \text{rz}(\langle\rangle)) \mapsto (\mathbb{R}/\mathbb{R}, \langle\rangle)\) | Theorem 4.4 |
| If \((*G, \neq)\) then \((*G\setminus\{\Phi^{-1}\}, \langle\rangle) \mapsto (\mathbb{R}, \langle\rangle)\) | Corollary 4.2 |
| \((*G, \langle\rangle) \mapsto (\mathbb{R}/\mathbb{R}, \leq)\) | Loses information, Theorem 4.6 |
| \((*G, \langle\rangle) \mapsto (\mathbb{R}, (0 < 0))\) | See Example 4.13 |
| \((*G, \infty) \mapsto (\mathbb{R}, \infty)\) | Infinity is not in \(\mathbb{R}\) |
| \((\mathbb{R}, f \not\in C^0) \mapsto (*G, f_2 \in C^0)\) | Adding information 11 |
| \((J_{\infty}, n) \mapsto (*G, n)\) | Discrete to continuous domain |
| \((\Phi, \delta_n) \mapsto (\mathbb{R}, \delta_n)\) | Algorithm example Part 5 Example 5.4 |
| \(\sum a_n|_{n=\infty} \mapsto \sum_{k=b_0}^\infty a_k\) | Convergences sums to sums 12, Theorem 11.1 |

Table 4: Mapping examples

Consider a limit. While the image space may be \(\mathbb{R}\), the solution space is \(*G\) as it holds infinitesimals and infinities. Hence, given \(*G \mapsto \mathbb{R}\), we can consider \(*G\) and postpone or avoid the transfer. The implicit nature of the limit can be undone.

**Example 4.16.** \(\frac{n^2+1}{n^2}|_{n=\infty} = 1 \in \mathbb{R}\), but \(n^2 + 1|_{n=\infty} \in *G\) and \(n^2|_{n=\infty} \in *G\). Hence \(\frac{n^2+1}{n^2}|_{n=\infty} \in *G\). However, the limit calculation can be described by \(*G \mapsto \mathbb{R}\).

If we consider the more general question of function evaluation, we have numbers that may be transferred between the number systems \(\mathbb{R}\) and \(*G\). While a function returns a value, by a transfer process it may not actually be calculated in that type. A transfer can occur between the calculation and the function’s returned value.

Consider now, the function return value location as holding a local variable. If the function type does not match the location type, a transfer is made.

In the evaluation, we can show the implicit transfer. \((*G, =) \mapsto (\mathbb{R}, =)\). Then \(\frac{a_n}{b_n}|_{n=\infty} = 1\) can be multiplied through to \(a_n = b_n|_{n=\infty}\) in \(*G\).

We introduce a notation to explicitly describe a relation, to help describe the transference rather than of practical use. With transference, the relation argument types are likely to be in the less detailed number system, but the evaluation in the more detailed number system \(*G\).

**Definition 4.13.** Let two arguments of a binary relation be described by their type \(T_1, T_2\).
where $z$ is the binary relation.

$$(T1 \ z \ T2)$$

**Example 4.17.** To undo an infinite operation, we need to promote the numbers to $\ast G$. The comments indicate the left and right types on either side of the equality relation.

$$\frac{n^2 + 1}{n^2} \big|_{n=\infty} = 1 \quad (\ast G = \mathbb{R} \ or \ \ast G)$$

$$\ast G \mapsto \mathbb{R}$$

$$\frac{n^2 + 1}{n^2} \big|_{n=\infty} = 1 \quad (\ast G = \ast G)$$

$$n^2 + 1 = n^2 \big|_{n=\infty} \quad (\ast G = \ast G)$$

**Example 4.18.** Promote a limit to a limit in $\ast G$. $(\mathbb{R}, 0) \mapsto (\ast G, \Phi)$

$$\frac{\sin \frac{1}{n}}{n} \big|_{n=\infty} = 0 \quad (\ast G = \mathbb{R})$$

$$\frac{\sin \frac{1}{n}}{n} \big|_{n=\infty} = \delta; \ \delta \in \Phi \quad (\ast G = \ast G)$$

$\mathbb{R} \mapsto \ast G$ is one-one as $\mathbb{R}$ is embedded in $\ast G$. However $\ast G \mapsto \mathbb{R}$ is different, as information about the infinitesimals is lost. Because $\ast G$ is more dense than $\mathbb{R}$, the transfer principle applied to the strict inequalities $\{<, >\}$ for variables/functions which are infinitesimally close, fail. Examples 4.13 and 4.14 implicitly worked in $\ast G$ and the relations changed when projected onto $\mathbb{R}$.

**Proposition 4.2.** $\delta \in \Phi; \ast G \mapsto \mathbb{R}/\mathbb{R}$: If $h \neq 0$ in $\ast G$ then $h \neq 0$ in $\mathbb{R}/\mathbb{R}$.

*Proof.* Either $\mathbb{R} \in h$ or $\Phi^{-1} \in h$, both components map to non-zero elements in $\mathbb{R}/\mathbb{R}$. □

**Proposition 4.3.** $\ast G \mapsto \mathbb{R}/\mathbb{R}$: If $h > \Phi$ in $\ast G$ then $h > 0$ in $\mathbb{R}/\mathbb{R}$.

*Proof.* If $h > \Phi$ in $\ast G$ then either $h \in +\Phi^{-1} \mapsto \infty$ or $h$ has $\mathbb{R}^+ \mapsto \mathbb{R}^+$. Neither result is 0 in $\mathbb{R}$. □

**Corollary 4.1.** If $f > 0$ in $\ast G$ and $\mathbb{R}$ then $f \neq 0$.

*Proof.* Assume true and show a contradiction. Let $f = \delta$ in $\ast G$, $\delta \in \Phi^+$ then $f > 0$ in $\ast G$. $(\ast G, \ \delta > 0) \mapsto (\mathbb{R}, \ 0 > 0)$ is contradictory. □
Example 4.19. \( *G \mapsto \mathbb{R}, \frac{1}{n+1} < \frac{1}{n} \mapsto \frac{1}{n+1} = \frac{1}{n}|_{n=\infty} \)

**Theorem 4.3.** If \((*G, \sim)\) then \((*G, <) \mapsto (\mathbb{R}/\mathbb{R}, =)\)

*Proof.* \( f < g, \frac{f}{g} < 1 \), since \( f \sim g \) then let \( \frac{f}{g} = 1 - \delta \) to preserve the less than relation, \( \delta \in \Phi^+ \). Apply transfer, \( \delta \rightarrow 0 \), \( \frac{f}{g}|_{n=\infty} = 1 \), \( f = g \). \( \square \)

Example 4.20. \((*G, n rz(<) n^2)|_{n=\infty} \mapsto (\mathbb{R}, 0 < \infty)\). \( n \ z \ n^2|_{n=\infty}, 0 \ z \ n^2|_{n=\infty}, 0 < \infty\)

**Theorem 4.4.** If \((*G, \not\sim)\) then \((*G, rz(<)) \mapsto (\mathbb{R}/\mathbb{R}, <)\)

*Proof.* Let \( f \sim g \). If either \( \{f, g\} \in \Phi^{-1} \), after realization, for the negative infinity case: \(-\Phi^{-1} < 0 \mapsto -\infty < 0\), positive infinity case: \( 0 < \Phi^{-1} \mapsto 0 < \infty \). Without infinities, let \( g = \alpha + f, \alpha > 0 \) to maintain the inequality, \( \mathbb{R} \in \alpha \) as \( f \not\sim g \) then \( \alpha \in \mathbb{R}^+ \). In \( *G \), \( f < \alpha + g \mapsto st(f) z_2 \alpha' + st(f) \) where \( \alpha' \in \mathbb{R}^+, 0 \ z_2 \alpha', z_2 = < \). \( \square \)

**Corollary 4.2.** If \((*G, \not\sim)\) then \((*G\setminus\{\Phi^{-1}\}, <) \mapsto (\mathbb{R}, <)\)

*Proof.* \((*G, \not\sim)\) without infinity becomes \((*G, \not\sim)\) at infinitely close test only. \( \square \)

**Theorem 4.5.** \((\mathbb{R}/\mathbb{R}, <) \mapsto (*G, \not\sim)\) and \((*G, <)\)

*Proof.* Since \( \mathbb{R} \) embedded in \( *G \), the \( < \) relation follows. Since elements in \( \mathbb{R}/\mathbb{R} \) are separated by an infinity or real number the elements cannot be asymptotic in \( *G \). \( \square \)

**Proposition 4.4.** \( rz(\sim) = \sim \)

*Proof.* The limit itself considers magnitudes, usually by dividing the infinities and realizing the infinitesimals. \( \square \)

**Theorem 4.6.** \((*G, <) \mapsto (\mathbb{R}/\mathbb{R}, \leq)\)

*Proof.* Since \( \sim \) and \( \not\sim \) cover all cases, combining Theorem 4.3 and Theorem 4.4 covers all cases, the union of the two images. \( \square \)

The limit calculation can be described as an evaluation in a more detailed number system with infinitesimals and infinities which is projected back to the real numbers.

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \] can be expressed as \((*G = \mathbb{R})\) or \((\mathbb{R}, =)\), even though the fraction may not be in \( \mathbb{R} \). However the result of the ratio is in \( \mathbb{R} \), hence we state it this way.
We found the limit to be a transfer as it realizes infinitesimals and infinities. Hence, we provided a number system $*G$, which contains infinireals and better describes the limit calculation.

In what follows, we are able to decouple a fraction about 1, multiplying the numerator and denominator out, while still able to simplify as a fraction, through transfers.

**Theorem 4.7.** $z_1, z_2 \in \mathbb{B}; a_n > 0, b_n > 0$.

$$\frac{a_n}{b_n}|_{n=\infty} z_1 \Leftrightarrow a_n z_2 b_n|_{n=\infty}$$

| Condition 1 | Condition 2 |
|-------------|-------------|
| $z_1$       | $z_2$       |
| $<$         | $<$         |
| $>$         | $>$         |
| $=$         | $\sim$      |

*Proof.* While we can bring the fraction into $*G$ and multiply out the denominator, to show equivalence we need to show that the transfer back from $*G$ to $\mathbb{R}/\mathbb{R}$ is equivalent to the limit.

Condition 2 can map $*G \rightarrow \mathbb{R}/\overline{\mathbb{R}}$ as the cases are disjoint and cover $*G$. If we consider the fractions equality case, by definition is $\sim$. However, this case is considered by excluding the inequality cases, partitioning $*G$ into the three disjoint cases. For the inequality, condition 2 leads to condition 1 by Theorem 4.3. All cases of $*G \rightarrow \mathbb{R}/\overline{\mathbb{R}}$ have been considered.

**Theorem 4.8.** Extend Theorem 4.7: $a_n \neq 0, b_n \neq 0$,

$$\frac{a_n}{b_n}|_{n=\infty} z_1 \Leftrightarrow a_n (\text{sgn}(b_n) z_2) b_n|_{n=\infty}$$

*Proof.* Since multiplying by a negative number inverts the inequality which is included within the theorem, the proof is identical to the proof given in Theorem 4.7.

While Theorem 4.7 loses information as $*G$ is more dense, so is not reversible, we can map back to $*G$. Consider the table relations $z_1 \rightarrow z_2$, if we consider $\mathbb{R}/\overline{\mathbb{R}} \rightarrow *G$, as $\mathbb{R}$ is embedded, the inequalities hold.

### 4.3 Overview and prelude

The significance of ratio comparison being expressed in $*G$ is that we can undo tests about a value. A ratio comparison is often given as a fixed truth set in stone, the pinnacle result.
For example, the ratio test. We need to investigate, undo or analyse tests, to modify or change them perhaps, to find out how they may work or build new ones.

The major aspect to this paper is the idea of transference, where we explore the change in relations as a transition. In this way, we can understand why a statement with infinitesimals may have a strict inequality, but when transferred to reals has an equality.

Many of the definitions are introduced as a basis for later work. The partitioning between finite and infinite domains is particularly important, as seen in Definitions 4.10 and 4.11 because they are a problem separation and an extension to classical analysis. From Fermat and other examples, these elements always existed, but they needed form.

The mappings Table 4 gives a range of applications and theory, and yet it is only a snapshot. However, we felt a need to describe the extent of the theory, and hence in a simpler way start to reveal why $*G$ would be useful in numerical calculations as well as theoretical mathematics.

While $\mathbb{R}$ is embedded in $*G$, this does not mean a translation between $\mathbb{R} \mapsto *G$ is one-to-one. For example, we could add information by inserting a curve at a singularity.

That transference explains infinitesimal and infinitary calculations is interesting. The missing mathematics of infinitesimals and infinities which have eluded analysis for centuries are the same operations that are performed on a day to day basis: primarily, approximation.

There is a paradigm shift. In particular, the exploration of possibilities is opened up. The user of this mathematics will need other tools for the larger workspace, but we believe will more likely find the mechanics of the mathematics.

It is hard to understand how infinitesimals fell. For they were the most successful aspect of calculus and are a part of it. We feel that the inability to construct a number system that explained the infinitesimals and infinities led to their demise. Infinitesimals never truly disappeared, but instead went underground, incorporated in much of todays mathematics (see [17]). Their resurrection in 1960s through NSA was a sustained correction, that even today has not been fully achieved. Two hundred years of denial are not so easily fixed.

Our view of the calculus is two tiered, with transference explaining the contradictions. It also opens worlds of possibilities. This is not the answer, only the beginning of the journey.

We are not taught to deal with contradiction well. In freeing the mathematics by acknowledging different possibilities, the ability to analyse mathematics is greatly increased.

In relation to quantum mechanics described in the language of statistics theory, Einstein said “God does not play dice”. We believe this meant that everything has a structure, whether we can see it or not. It is quite conceivable that statistics in quantum mechanics works but may not reveal many hidden structures. Any theory, after all, is only one viewpoint.
Similarly, Gödel’s incompleteness theorems [31] could be interpreted to mean that there is an infinity of mathematics to be discovered. How can we explore mathematics if we are unable to easily incorporate different points of view? How do we analyse different theories and algorithms, side by side?

So, the major obstacle to the acceptance and use of infinitary mathematics is the mathematical language itself. The notation \( |_{x=0} \) Definition [4.11] is defined in context. That is, until you use it in a calculation, it is undefined. This is different to most definitions, which are invariant and unchanging. However, in order to ‘free’ the mathematics, we need to manage cases where we cannot foresee the future. This is the responsibility of the user of the mathematics. We believe this is necessary, because of the complexity of the subject itself. This decoupling of responsibility with notation has a purpose. It can produce very flexible mathematics, essentially creating dynamic definitions.

I once asked a researcher if we could compare two algorithms with different structure that gave the same result. The answer was an affirmative no. However, if we could pull apart the algorithms, and perhaps investigate them in the infinite domain, it may just be possible to find answers and show linkages which without \(*G\) or hyperreals would be impossible to find.

The goal is not necessarily harmony or unity in the context of one set of rules or only one consistently logical mathematical framework, but instead connected transitions and freedom which can give a more holistic view.

So we find transference describes the world of mathematics which we see well, and hence its development could lead to tools that every numerical scientist or mathematician could use.

5 Non-reversible arithmetic and limits

Investigate and define non-reversible arithmetic in \(*G\) and the real numbers. That approximation of an argument of magnitude, is arithmetic. For non-reversible multiplication we define a logarithmic magnitude relation \( \succ\succ \). Apply the much-greater-than relation \( \succ \) in the evaluation of limits. Consider L'Hopital’s rule with infinitesimals and infinities, and in a comparison \( f(z)g \) form.

5.1 Introduction

Two parallel lines may meet at infinity, or they may always be apart. Infinity is non-unique. We believe different number systems can co-exist by the non-unique nature of infinity.

We focus on the infinite case where the largest magnitude dominates, \( a+b=a \) where \( b \neq 0 \).
For example, arbitrarily truncating a Taylor series. 
\[ f(x + h)|_{h=0} = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \ldots |_{h=0} = f(x) + hf'(x)|_{h=0}. \]

That an infinite sum of the discarded lower order terms have no effect on the outcome is explained by sum convergence theory. However, if such a sum was for example bounded above by an infinitesimal, then the realization of the sum (a transfer) to real numbers would discard the infinitesimal terms, \( \Phi \mapsto 0 \).

DU BOIS-REYMOND in his journal articles on the infinitary calculus is not much interested in a theory of sets as such, or even explicitly in sets on lines. He is interested in the nature of a linear continuum, but chiefly because he wants to consider a more general "continuum" of functions. He especially concerns himself with limit processes as they occur in a linear continuum, since he wants to consider limit processes among functions. [9, p.110]

In our previous sections we have identified the investigation of functions as working in a higher dimensional number system than \( \mathbb{R} \), and fitted \( *G \) to du Bois-Reymond’s work, as a representation of the continuum that he was investigating.

Since we believe \( *G \) is a field, multiplication and addition by non-zero elements are reversible prior to realization. For example truncation \( *G \mapsto *G \). After this process, information is lost and in general you cannot go back.

What we demonstrate is that arithmetic ‘is’ non-reversible; and that this is a major aspect of analysis. Ultimately we see this as a way for working with transference, with reasoning by magnitude.

For example, we prove L’Hopital’s rule Proposition [5.4] not with an equality but with an argument of magnitude. This, after reading the original translation of the rule is closer to the discovery than the formation with the mean value theorem. And this is a problem, that mathematics is being used not in an intuitive way, but as a means of proof. Newton did not publish the Principia with his calculus, but the traditional geometric arguments which are “extremely” difficult to use. Analysis is not complete without a more detailed investigation into arguments of magnitude.

5.2 Non-reversible arithmetic

The following is a complicated argument that associates finite mathematics with reversible processes, and infinitary calculus with non-reversible processes. This results in non-uniqueness in the additive operator, the consequences of which are profound.

To start, consider addition and multiplication. Given \( x + 2 \), then \((x + 2) - 2\) gives back \( x \), the \(-2\) undoing the \(+2\) operation. Similarly for multiplication, if we start at 5, \( 5 \times 2 = 10 \).
Reversing, $10/2 = 5$. Similarly with powers, where logarithms and powers are each other's inverse.

However, if the operator is not reversible you cannot undo the operator previously applied; it is as if the operation has disappeared. $n^2 + \ln n = n^2|_{n=\infty}$ leaves no evidence of adding $\ln n$, an infinity. We need a number system with infinities for this to occur. The much-greater-than relation with realization explains this.

**Theorem 5.1.** $a, b \in *G; (a, b \neq 0)$ Assume transfer between $\Phi$ and $0$.

If $a + b = a \leftrightarrow a \succ b$.

**Proof.** $\delta \in \Phi$;

If $a + b = a$, $\frac{a}{a} + \frac{b}{a} = \frac{a}{a}$, $\frac{b}{a} = 0$, $0 \mapsto \Phi$, $\frac{b}{a} = \delta$, $a \succ b$.

If $a \succ b$, $\frac{b}{a} = \delta$. Consider $a + b = y$, $\frac{a}{a} + \frac{b}{a} = \frac{y}{a}$, $1 + \delta = \frac{y}{a}$, $\delta \mapsto 0$, $1 = \frac{y}{a}$, $a = y$, $a + b = a$.

**Corollary 5.1.** $a, b \in *G; b \neq 0$

If $a + b = a$ then after the addition and realization in general, it is impossible to determine $b$.

**Proof.** By Definition 5.2.

This is a fundamental statement about our numbers. We use Theorem 5.1 whenever we approximate. We approximate whenever we use infinitesimal or infinite numbers. By considering an alternative definition to the much-greater-than relation as an infinitesimal ratio ($a \succ b$ then $\frac{b}{a} \in \Phi$) is explained.

If $a \succ b$ is alternatively defined[7, p.19]: if $a - b = \infty$ then $a \succ b$. We cannot apply the transfer $\Phi \rightarrow 0$ then $a + b = a$ is not true in general. Consider $a = e^{x+1}$, $b = -e^x$, $e^{x+1} - e^x = (e - 1)e^x|_{x=\infty} = \infty$ but $e^{x+1} \neq (e - 1)e^x|_{x=\infty}$. This lacks the ‘$a + b = a$’ property given in Theorem 5.1. Since this paper is concerned about non-reversibility we do not want to lose this property.

**Proposition 5.1.** If $a_n \sim b_n|_{n=\infty}$ then there exists $c_n$: $a_n + c_n = b_n|_{n=\infty}$ where $c_n \prec a_n$.

**Proof.** $a_n + c_n \sim b_n$, $a_n \sim b_n$ as $a_n + c_n = a_n$, $z = \sim$.

**Definition 5.1.** An Archimedean number system has no infinitesimals or infinities. A non-Archimedean number system is not an Archimedean number system.
That is, a non-Archimedean number system has non-reversible arithmetic, see Definition 5.2. A consequence of the existence of ratios being an infinitesimal or an infinity.

A consequence of the Archimedean property are unique inverses, both additive and multiplicative, which allow unique solutions to equations involving addition and multiplication.

With the infinireals, as long as we do not approximate via the “realization” operation, and assuming non-zero numbers, we have unique inverses when adding and subtracting and unique inverses when multiplying or dividing.

However the non-Archimedean property is necessary for Theorem 5.1. We need an infinity of possibilities with addition before we can say an operation is non-reversible. However Archimedean systems can have non-reversible arithmetic.

**Theorem 5.2.** Multiplication by 0 in \( \mathbb{R} \) number systems is non-reversible.

**Proof.** Let \( x \in \mathbb{R} \). Consider the equation \( x \times 0 = 0 \). If you then ask what was the value of \( x \), there is an infinity of solutions in \( \mathbb{R} \). By Definition 5.2, the multiplication is non-reversible.

The number zero collapses other numbers through multiplication. What was the original number being collapsed? For this reason 0 is considered separately from other numbers. By defining \( 0^0 = 0 \) we can extend the number system further, see Example 5.5.

The reals, with which we are so familiar, form an Archimedean number system which excludes infinity.

This exclusion of infinity in many respects is illusionary, as an arbitrary large number acts as an infinity. This is exploited in proofs by retaining the property of being a finite number. For the statement “\( \forall n > n_0 \) the following is true...” implicitly defines infinity.

Excluding that infinity implicitly exists even when ignored, the Archimedean property essentially means that there are no infinitesimals or infinities as elements.

There is fascination that both the finite perspective and infinite perspective can co-exist. Both can describe, even when they are contradictory viewpoints, that is, they are completely different views of the same event. 1. \( a + b \) for finite numbers excluding zero always changes the sum. 2. \( a + b = a \)

If we add a cent to a million dollars we have a million dollars plus a cent. Alternatively, if we treat a million dollars as the infinity, then adding a cent does ‘nothing’ to the sum, and the sum remains unchanged, as its magnitude is not changed. Hence, the accountant and the businessman may have different views on the same transaction.

Looking at infinitary calculus with infinity as a point, consider \( \ln n + n^2 = n^2|_{n=\infty} \). The
The term $\ln n|_{n=\infty} = \infty$ is infinite, but acts like a zero when added to a much larger infinity, a consequence of $\ln n \prec n^2|_{n=\infty}$. Replacing $\ln n$ with $n$ gives a similar result where $n|_{n=\infty}$ acts as a zero element, when $n + n^2 = n^2|_{n=\infty}$. The additive identity is not unique. It should be apparent that there is an infinity of additive elements.

The same can also be true with regard to multiplication, where a multiplicative identity is not unique. That is, $1$ is not the only element which, when multiplied, does not change the number.

**Example 5.1.** The following demonstrates non-unique multiplicative identities. Let $f = \infty, g = \infty, h = \infty$. Consider $f \cdot g = h, \ln(f \cdot g) = (\ln z) \ln h, \ln f + \ln g = (\ln z) \ln h$. Let $\ln f \succ \ln g$, then $\ln f + \ln g = \ln f$ and $\ln f (\ln z) \ln h$. When reversing the process is possible, $f \equiv h$ and $g$ is a multiplicative identity.

**Definition 5.2.** We say that an arithmetic is non-reversible if there is an infinite number of additive identities or there is an infinite number of multiplicative identities.

Let $g(n)$ be one of the infinitely many additive identities. Let $h(n)$ be one of the infinitely many multiplicative identities.

If one of the following is true, then we say that the arithmetic is non-reversible arithmetic.

$$f(n) + g(n) = f(n) \text{ or } f(n)h(n) = f(n)$$

Consider when lower order terms realized?

Theorem 5.1 proves non-reversible addition. From this, Theorem 5.3 proves non-reversible multiplication. These operations, particularly addition have wide applicability. However, we stress that this is only one possibility at infinity when via algebra and a transfer, $\Phi \mapsto 0$. That at infinity we can have non-uniqueness is further exploration of what constitutes the continuum.

While Theorem 5.2 multiplication by $0$ in $\mathbb{R}$ is non-reversible, it is isolated. By excluding the element $0$ in $\mathbb{R}$, multiplication and division are reversible. $\infty \times c = \infty$ when $c \neq 0$ has a similar property, but in $\mathbb{R}$, $\infty$ is excluded. $0 \times \infty$ is better considered in $\ast G$ before the infinitesimals or infinities are realized. The order in which numbers are realized matters.

Generally addition is easier for reasoning with non-reversible arithmetic than multiplication; probably a consequence of multiplication being defined as repeated addition, a more complex operation. Hence, for example, when applicable, $\prod_{k=1}^{n} a_k = e^{\sum_{k=1}^{n} \ln a_k}$ transforms a product to a power with a sum for reasoning.

Calculus has been extended in many ways to use infinity in calculations and theory because infinity is so useful. Calculus often states the numbers in $\mathbb{R}$ but then reasons in $\ast G$. Limits are a typical example. We generally agree with this as the utility of calculation is paramount.
However, just as we discuss the atom in teaching physics, because it describes fundamental properties, a teaching of the infinitesimal is warranted. Not the exclusion of its existence, which is fair to say is the current practice.

Infinitary calculus facilitates arguments with magnitude, and has the potential to reduce the use of inequalities in analysis, beyond what Non-Standard-Analysis or present analysis does. This could make problem-solving more accessible by reducing the technical difficulties associated with the use of inequalities.

For example, one may experience the problem of not knowing or not using the right inequality, and become stuck; without specific knowledge, no progress is likely to be made. This applies to specialized domains where networking may be required. If, however, the problem could be done without “networking”, time would be saved.

As well as providing alternative arguments to problems, infinitary calculus can be combined with standard mathematical arguments and inequalities, leading to them being used in new ways.

Well how can this be achieved?

Simply put, by employing non-reversible arithmetic; that is, in number systems with non-Archimedean properties, non-reversible arguments can be made. This is indeed possible with the much-less-than ($\prec$) and much-greater-than ($\succ$) relations defined by du Bois-Reymond.

We have an equation that we would like to solve hence the need for a new number system. Considering the scales of infinity, there does not exist $c \in \mathbb{R}$ with such a property to move between the infinities because $c$ is finite.

**Example 5.2.** Let $c \in \mathbb{R}$ then $c \prec \infty$. If $c = n^2|_{n=\infty}$, $c = n|_{n=\infty}$, $c \in \Phi^{-1}$. This implies $c \not\in \mathbb{R}$. If we restricted $c$ to $\mathbb{R}$ then the equation would not have been solved, as $c$ was infinite.

An Archimedean number system cannot solve this. Just as we needed $i$ to solve $x^2 = -1$, we need a non-Archimedean number system to solve the equation with infinities.

The definition of $\prec$ and little-o distinguish between infinities. $a \prec b$ if $\frac{a}{b} \in \Phi$. If two functions differ infinitely through division, then they must also differ infinitely through addition to a greater degree (See Theorem 5.1).

Depending on the context, lower-order-magnitude terms may be disregarded. $f(x) + g(x) = g(x)|_{x=\infty}$. $f(x) = \infty$, $g(x) = \infty$. Here $f(x)|_{x=\infty}$ acts as a zero identity element, even though $f(x)$ is not zero. However $f(x)$ has its magnitude dwarfed by the much larger $g(x)$, so $f(x)$ is negligible.

To avoid summing infinite collections of terms, the general restriction when applying the
simplification \( a_n + b_n = a_n|_{n=\infty} \) for infinitary or infinitesimal \( a_n \) and \( b_n \) is that the rule is only good for a finite sum of infinitesimals or infinities.

A sum of infinities or infinitesimals to infinity, can itself step up in orders. In other words, generally apply the simplification to a finite number of times. If you were to sum infinities or infinitesimals to infinity, you would need to integrate instead. Or apply truncation when sum convergence is known.

This is discussed in detail [12, Convergence sums ...]. The assumption of independence of sums may by invalid at infinity [14]. Briefly, how we view finite mathematics may be completely different to how we view mathematics “at infinity” because it is a much larger space.

The advantages of such simplification can allow classes of functions to be reduced.

**Example 5.3.** \( \{ \frac{1}{x^2 + \pi}, \frac{1}{x - 3^2}, \ldots \}|_{x=\infty} \) simplify to considering \( \frac{1}{x^2}|_{x=\infty} \).

Applications include taking the limit which applies in summing infinitesimals to zero, hence truncating a series. The arguments can apply to diverging sums as well.

Use of extended calculus in \( *G \) as a heuristic, for developing algorithms.

**Example 5.4.** Developing an algorithm for approximating \( \sqrt{2} \). \( x, x_n, \delta_n \in *G; \delta \in \Phi \).

\[
(x + \delta)^2 = 2 \\
x^2 + 2x\delta + \delta^2 = 2|_{\delta=0} \\
x^2 + 2x\delta = 2|_{\delta=0} \quad (2x\delta \succ \delta^2|_{\delta=0}) \\
x_n^2 + 2x_n\delta_n = 2|_{\delta=0} \quad (\text{Developing an iterative scheme.}) \\
\delta_n = \frac{1}{x_n} - \frac{x_n}{2} \quad (\text{Solving for } \delta_n) \\
x_{n+1} = x_n + \delta_n \quad (\text{Progressive sequence of } x_n)
\]

If \( \delta_n \to 0 \) then at infinity \( \delta_n \) becomes an infinitesimal (see [13, Example 2.12]), that is \( \delta_n|_{n=\infty} \in \Phi \), then \( (x + \delta)^2 \simeq 2 \) is solved, and \( x_{n+1} \simeq x_n + \delta_n \) has \( x_n \) converging. Starting the approximation with \( x_0 = 1.5 \), \( x_5 \) is correct to 47 places (for a numerical calculation with Maxima see [24]), where the algorithm was transferred from \( *G \) to \( \mathbb{R} \). \( (*G, \delta_n, x_n) \mapsto (\mathbb{R}, \delta_n, x_n) \), an infinitesimal was promoted to a real number.

Turning towards the number system, what is common is the operation of numbers at zero or infinity. That is where the numbers display non-Archimedean properties.

In fact, all function evaluation is at zero or infinity. Simply shift the origin. Zero and infinity form a number system, \( \mathbb{R}_\infty \). The cardinality of \( \mathbb{R}_\infty \) is infinitely larger than the cardinality of
Then the gossamer number system $\mathbb{R} \ast G$ is much larger than the real number system (the reals which are embedded within it).

In this paper, addition simplification is applied to solving relations by converting a series of relations to a sum, where lower order terms are discarded and the relations solved.

**Definition 5.3.** Using the Iverson bracket notation (see [23, p.24])

$$[f \vDash g] = \begin{cases} 1 & \text{when the relation } f \vDash g \text{ is true,} \\ 0 & \text{when then relation } f \vDash g \text{ is false.} \end{cases}$$

In a more radical approach to demonstrate addition as a basis for building relations, we can define $0^0 = 0$ as another extension to the real number system, which then allows the building of the comparison greater than function (see Example 5.5).

Canceling the 0’s is not allowed, as by definition this is now a non-reversible process, as we view either multiplication by 0 or multiplication by $\frac{0}{0}$ as collapsing the number to 0. We also get to test if a number is zero or not.

**Example 5.5.** Non reversible mathematics to build the relations. See Definition 5.3

$$[x > 0] = \frac{x + |x|}{2x}$$

When $x = 0$, $\frac{0 + |0|}{2 \times 0} = \frac{0}{0} = 0$. When $x > 0$, $\frac{x + |x|}{2x} = \frac{2x}{2x} = 1$. When $x < 0$, $\frac{x + |x|}{2x} = \frac{0}{2x} = 0$.

When $x = 0$, $0^0 = 0$. When $x \neq 0$, $x^0 = 1$.

### 5.3 Logarithmic change

We introduce a relation, for better explaining magnitudes, and their comparison. While $f \succ g$ describes an infinity in the ratio of $f$ and $g$, there could be a much larger change in the functions themselves.

If we consider the operations of addition, multiplication as repeated addition, a power as repeated multiplication, all these operations accelerate change. Conversely, subtraction, division, and logarithms of positive numbers to greater degrees decelerate change.

Consider the log function as undoing change, then applying to both sides of a relation and comparing, we can determine a much-greater than relationship, and ‘if’ one exists, we infer an infinity between the functions.
For example, in solving $f z g$ if we find a $\ln f \succ \ln g$ relationship. The undoing log operation revealed a much-greater-than relationship.

**Definition 5.4.** We describe a logarithmic magnitude.

- We say $f \gg g$ when $\ln f \succ \ln g$
- We say $f \ll g$ when $\ln f \prec \ln g$
- We say $f$ ‘log dominates’ $g$ when $f \gg g$.
- We say $f$ is ‘log dominated’ by $g$ when $f \ll g$.

The much-greater-than relation $f \succ g$ may have a log dominating relation $f \gg g$ or be log dominated by $g$: $f \ll g$ or no such relationship. That is the relations between the magnitude and the logarithmic magnitude are not necessarily in the same direction. An exception is when both positively diverge, see Proposition 5.2.

A logarithmic magnitude is like a derivative. A derivative’s sign is not necessarily the same as the function’s sign. The much greater than relation is independent in direction to the log dominating relation.

Logarithmic magnitude can describe non-reversible product arithmetic (Definition 5.2). $0 \cdot \infty$ indeterminate case arises in the calculation of the limit. We prove the non-reversible product as a consequence of non-reversible addition (see Theorem 5.3).

**Example 5.6.** Consider $x^n z n|_{n=\infty}$ where $|x| < 1$.

\[
\begin{align*}
  x^n z n|_{n=\infty} \\
  \ln(x^n) (\ln z) n|_{n=\infty} \\
  n \ln x (\ln z) n|_{n=\infty} \\
  n \ln x \succ \ln n|_{n=\infty} \\
  x^n \gg n|_{n=\infty}
\end{align*}
\]

(By Definition 5.4)

While $x^n \prec n|_{n=\infty}$ as $0 \prec \infty$, the logarithmic magnitude of $x^n$ is much greater than $n$ with $x^n \gg n|_{n=\infty}$.

When simplifying products by non-reversible arithmetic, for example in the calculation of limits, rather than solve with products, reason by exponential and logarithmic functions which are each other inverses, converting the problem of multiplication to one with addition. $f \cdot g = e^{\ln(f)g} = e^{\ln f + \ln g}$. If possible, apply non-reversible arithmetic: $\ln f + \ln g = \ln f$ or $\ln f + \ln g = \ln g$.

**Example 5.7.** When $|x| < 1$, evaluate $x^n \cdot n|_{n=\infty}$.
This is an indeterminate form 0·∞. \( x^n \cdot n = x^n|_{n=\infty} \) is harder to understand than when the problem is reformed and when simplifying, non-reversible arithmetic applied on a sum and not a product.

\[
x^n \cdot n|_{n=\infty} = e^{\ln(x^n) \cdot n}|_{n=\infty} = e^{n \ln x + \ln n}|_{n=\infty} = e^{n \ln x}|_{n=\infty} = x^n|_{n=\infty}
\]

\[ (n \succ n \text{ then apply non-reversible arithmetic}) \quad (n \ln x + \ln n = n \ln x|_{n=\infty}) \]

**Theorem 5.3.** Non-reversibility in a product. Let \( a \) and \( b \) be positive.

If \( a \gg b \) then \( a \cdot b = a \)

*Proof.* \( ab = e^{\ln(ab)} = e^{\ln a + \ln b} = e^{\ln a} = a, \) as \( a \gg b \) then \( \ln a \gg \ln b. \) □

**Example 5.8.** If we know the log magnitude relationship, we may directly calculate.

\[
x^n \cdot n|_{n=\infty} = x^n|_{n=\infty} \text{ as } x^n \gg n|_{n=\infty}
\]

**Proposition 5.2.** Let \( f = \infty, \) \( g = \infty. \) If \( f \gg g \) then \( f \gg g. \)

*Proof.* \( f \gg g \) then \( \ln f \gg \ln g. \) Since there is no smallest infinity, \( f_2 = \ln f = \infty, \) \( g_2 = \ln g = \infty. \) \( f_2 \gg g_2. \) By the following theorem: \( a = \infty, b = \infty, \) if \( a \gg b \) then \( e^a \gg e^b \) (see Table 2) then \( f_2 \gg g_2, e^{f_2} \gg e^{g_2}, f \gg g. \) □

### 5.4 Limits at infinity

**Definition 5.5.** In context, we say \( f(x)|_{x=\infty} \) then \( \sup_{x \to \infty} \lim_{x \to \infty} f(x), \) similarly \( \inf_{x \to \infty} \lim_{x \to \infty} f(x) \)

When the definition is put into a context such as a relation, since the condition is assumed to be true for all \( n \) at infinity (else the condition is false and a contradictory statement), the exact lower and upper bound language can optionally be excluded.

**Example 5.9.** Condition \( \inf_{n \to \infty} \rho_n > 1 \) becomes \( \rho_n|_{n=\infty} > 1. \)

Similarly condition \( \sup_{n \to \infty} \rho_n < 1 \) becomes \( \rho_n|_{n=\infty} < 1. \)

The following demonstrates the application of the notation and ideas discussed in this paper about limits and the more general at-a-point evaluation.
In computation of limits, infinity can be as useful in simplifying expressions as infinitesimals. So rather than dividing and forming the infinitesimals, instead apply arguments of magnitude. Let the user choose. The non-reversible arithmetic works either way.

**Example 5.10.** A simple example will show this. $\frac{2n^2}{3n^2} |_{n=\infty} = \frac{2\infty}{3\infty} |_{n=\infty} = \frac{2}{3}$ The justification being $3n + 5 = 3n |_{n=\infty}

**Example 5.11.** Apostol [5, 3.6.7], $\lim_{x \to \infty} \frac{x^2-a^2}{x^2+2ax+a^2} \neq 0$, $\lim_{x \to 0} \frac{x^2-a^2}{x^2+2ax+a^2} = -1$ as $a^2 \gg x^2 |_{x=0}$ and similarly $a^2 \gg 2ax \gg x^2 |_{x=0}$

**Example 5.12.** Apostol [5, 7.17.28], $\lim_{x \to \infty} (x^5 + 7x^4 + 2)^c - x \neq 0$ for non-zero limit $(c = 0$ may collapse to $1 - x |_{x=\infty})$. Using $x^5 + 7x^4 + 2 = x^5 |_{x=\infty}$ as $x^5 \gg x^4 \gg x^0 |_{x=\infty}$,

$(x^5 + 7x^4 + 2)^c - x |_{x=\infty} = x^5c - x |_{x=\infty} = b$ then $c = \frac{1}{5}$ as the difference reduces the power by one to a finite value. I.e. a limit.

**Example 5.13.** $(x^5(1 + \frac{7}{x} + \frac{2}{x^2}))^{\frac{1}{5}} - x |_{x=\infty} = (x^5(1 + \frac{7}{x}))^{\frac{1}{5}} - x |_{x=\infty}$ expand with the binomial theorem. $(1 + x)^w = 1 + wx + w(w-1)x^2 + \ldots$ then $x(1 + \frac{7}{x})^{\frac{1}{5}} - x |_{x=\infty} = x + \frac{7}{5} + \frac{49}{25} + \ldots - x |_{x=\infty} = \frac{7}{5} + \frac{49}{25} + \ldots |_{x=\infty}$ = $\frac{7}{5}$

Arguments of magnitude are commonly used in calculations. Apostol [5, pp 289–290] discusses polynomial approximations used in the calculation of limits, where the relation is within the little-o variable.

Computing the terms separately lead to the indeterminate form 0/0; by computing the numerator and denominator as a coupled problem, leading magnitude terms may be subtracted (for example through factorization).

When we remove little-o the calculation is not cluttered. If you need to be exact include it, but if not then it may as well be omitted.

Using the identity $\frac{1}{1-x} = 1 + x + x^2 + \ldots$, $\frac{1}{1-(\frac{4}{x}+o(x^3))} = 1 + \frac{1}{2}x^2 - o(x^3)$ as $x \to 0$ becomes $\frac{1}{1-x^2} = 1 + \frac{x^2}{2} |_{x=0}$ Truncation is part of calculations context and assumed to be the case.

Applying the at-a-point notation to some limits. Since the series expressions have terms forming a scale of infinities, often only a fixed number of terms with the expansions need be used. Taylor series, the binomial expansion, trigonometric series and others can be viewed as not unique since they have an infinity of terms.

**Example 5.14.** $\frac{e^{x} a - e^{x} b}{x} |_{x=0} = (e^{x} \ln a - e^{x} \ln b) |_{x=0}$, expanding the exponential series for the first three terms, $\frac{e^{x} b x}{a} |_{x=0} = (1 + x \ln a + (x \ln a)^2 \frac{1}{2} - (1 + x \ln b + (x \ln b)^2 \frac{1}{2} |_{x=0} = (x \ln a - x \ln b) \frac{1}{2} |_{x=0} = \ln a - \ln b = \ln \frac{a}{b}$

With known algebraic identities, such $\frac{n}{n!} |_{n=\infty} = 1$ or $e = \frac{n}{(n!)^{\frac{1}{n}}} |_{n=\infty}$ can easily be used to solve limits.
Example 5.15. p.39, 2.3.23.b] \( (\frac{n!}{n^n})^{\frac{1}{n}}\big|_{n=\infty} = e^{(\frac{n!}{n^n})^{\frac{1}{n}}} = e^{((\frac{n!}{n^n})^{\frac{1}{n}})^3} = e^{e^{-1}} = e^{-2} \)

The typical interchange between zero and infinity is useful.

\[ x|_{x=0^+} = \frac{1}{n}|_{n=\infty} \text{ then } f(x)|_{x=0^+} = f\left(\frac{1}{n}\right)|_{n=\infty} \]

Example 5.16. 7.17.18[ \lim_{x\to0^-} (1 - 2^x)^{\sin x} = \lim_{x\to0^-} (1 - 2^x)^x \text{ as } \sin(x) = x \text{ for small } x \]

Show \( y = 1 \). Let \( y \in *G : y = (1 - 2^x)|_{x=0^-}, \ln y = x \ln(1 - 2^x)|_{x=0^-}, 0 \cdot \infty \text{ form.} \)

With a log expansion, the problem can be solved. \( \ln(1 - 2^x)|_{x=0^-} = 2^x + \frac{(2^x)^2}{2} + \frac{(2^x)^3}{3} + \cdots \), \( \ln y = x \ln(1 - 2^x)|_{x=0^-} = x2^x|_{x=0^-}, y = (2^x)^2|_{x=0^-} = (e^x)^2|_{x=0^-} = 1 \)

Example 5.17. Solve \( \delta^\delta = y, \delta \in \Phi \), \( \ln(\delta^\delta) = \ln y, \delta \ln \delta = \ln y \). Noticing \( \delta \ln \delta = 0 \cdot \infty = \frac{1}{\infty} \cdot \infty \), \( 0 \cdot \infty \text{ can be expressed as } -\infty/\infty \text{ and differentiated using L'Hopital's rule, } \delta \ln \delta = \frac{\ln \delta}{\delta^\delta} = \frac{\frac{1}{\delta}}{1/\delta^2} = -\delta = 0 \text{ by } (*G, \Phi) \mapsto (\mathbb{R}, 0), 0 = \ln y \text{ then } y = 1 \).

Since the magnitude \( \{<, \leq\} \) and other relations are defined in terms of ratios, when comparing two functions in a “multiplicative sense”, we convert between the fraction and the comparison.

Proposition 5.3. When \( \frac{f}{g} \in *G \) and \( z \) is defined in a “multiplicative sense”, \( g \neq 0 \)

\[ \frac{f}{g} \mapsto f z g \]

Proof. Since no information is lost, and the operation is reversible, \( \frac{f}{g} \cdot g z 1 \cdot g, f z g \).

Consider the problem process \( \frac{f}{g} \Rightarrow f z g. \frac{f}{g} = 1, (*)G = 1 \mapsto (*)G = (*)G \). Though non-uniqueness also has advantages. \( (*)G z 1 \mapsto (*)G z \cdot (*)G \).

It is common for a problem to be phrased as if in \( \mathbb{R} \) but in actuality is in \( *G \). Then after algebraically manipulating in \( *G \), the information has to transfer back to \( \mathbb{R} \), or be phrased as such.

In fractional form \( \frac{f}{g} \) then becomes particularly convenient to apply what we know about fractions to the comparison. For example we may transform the comparison to a point where L'Hopital’s rule can be applied. With the extended number system \( *G \), the indeterminate forms \( 0/0 \) and \( \infty/\infty \) are expressed as \( \Phi/\Phi \) and \( \Phi^{-1}/\Phi^{-1} \) respectively.

We proceed with another proof of L'Hopital’s rule where we use infinitary calculus theory and work in \( *G \). L'Hopital’s argument is interpreted in \( *G \) (see Proposition 5.4) and the algebra is explained with non-reversible arithmetic, directly calculating the ratio (see [25]).
Lemma 5.1. A ratio of infinitesimals is equivalent to a ratio of infinities. \( \frac{\Phi}{\Phi} \equiv \frac{\Phi^1}{\Phi^1} \)

Proof. \( f, g \in \Phi^{-1}; \ \frac{f}{g} = \frac{1}{f'}, \) but \( \frac{1}{g} \in \Phi; \) then \( \frac{f}{g} \) of the form \( \frac{\Phi}{\Phi}. \) The implication in the other direction has a similar argument. \( a, b \in \Phi; \ \frac{a}{b} = \frac{1}{b}, \) but \( \frac{1}{a}, \frac{1}{a} \in \Phi^{-1}; \) then \( \frac{a}{b} \) of the form \( \frac{\Phi^1}{\Phi^1}. \)

Proposition 5.4. \( f, g \in \Phi; \) If \( \lim_{x \to a} \frac{f(x)}{g(x)} \) exists then \( \frac{f(a)}{g(a)} = \frac{f'(a)}{g'(a)} \)

Proof. Since \( \lim_{x \to a} \frac{f'(x)}{g'(x)} \) exists, we can vary about \( x = a \) in \( f \) and \( g. \)

\[
\frac{f(a)}{g(a)} = \frac{f(a+h)}{g(a+h)} \bigg|_{h=0} = \frac{f(a)+f'(a)h}{g(a)+g'(a)h} \bigg|_{h=0}.
\]

Choose \( h \in \Phi : f'(a)h \succeq f(a) \) and \( g'(a)h \succeq g(a). \) Then \( f(a)+f'(a)h = f'(a)h \big|_{h=0}, \) \( g(a)+g'(a)h = g'(a)h \big|_{h=0}. \)

\[
\frac{f(a)+f'(a)h}{g(a)+g'(a)h} \bigg|_{h=0} = \frac{f'(a)h}{g'(a)h} \bigg|_{h=0} = \frac{f'(a)}{g'(a)}
\]

Theorem 5.4. L’Hospital’s rule (weak) in \( \ast G. \)

Proof. The indeterminate form \( 0/0 \) is represented by \( \Phi/\Phi \) in \( \ast G. \) A transfer \( \Phi \mapsto 0 \) confirms this. The indeterminate form \( \infty/\infty \) is represented as \( \Phi^{-1}/\Phi^{-1}. \) Similarly a transfer \( \Phi^{-1} \mapsto \infty \) confirms this.

The indeterminate form \( \Phi^{-1}/\Phi^{-1} \) by Lemma 5.1 can be transformed to the indeterminate form \( \Phi/\Phi. \)

Apply Proposition 5.4 to the indeterminate form \( \Phi/\Phi. \)

Theorem 5.5. Comparison form of L’Hospital’s rule. If \( f/g \) is in indeterminate form \( \frac{\Phi}{\Phi} \) or \( \frac{\Phi^{-1}}{\Phi^{-1}}, \) when \( f'/g' \) exists then \( f \sim g \implies f' \sim g' \) where \( z \in \{<, \infty, >\}. \)

Proof. Equivalent to L’Hospital’s rule. See Theorem 5.4

Example 5.18. \( f' \succeq g' \implies f \succeq g. \) \( 1 \succeq \frac{1}{n} \big|_{n=\infty}, n \succeq \ln n \big|_{n=\infty}. \)

Example 5.19. Applying L’Hospital’s rule reaches a \( 0/\infty \) form. Hence a much greater than or much less than relationship. Solve \( \ln x \sim x^2 \big|_{x=\infty}. \) This is in indeterminate form \( \infty/\infty, \) differentiate. \( \frac{1}{x} (Dz) \ 2x \big|_{x=\infty}, \frac{1}{x} \prec 2x \big|_{x=\infty}, Dz = \prec, z = \int \prec = \prec \) then \( \ln x \prec x^2 \big|_{x=\infty}. \)

If the limit exists then the comparison and limit are solved for \( f(x) \sim g(x) \big|_{x=a} \) as a standard application of L’Hopital’s rule with the comparison notation.
Example 5.20. Computing an indeterminate form 0/0. \( \lim_{x \to 2} \frac{3x^2 + 2x - 16}{x-2} \), \( x^2 - x - 2|_{x=2} = 0 \) form then differentiate. 6x+2 (Dz) 2x-1|_{x=2}, 14 (Dz) 3, \( \frac{3x^2 + 2x - 16}{x^2 - x - 2}|_{x=2} = \frac{14}{3} \)

Example 5.21. Indeterminate form \( \infty/ - \infty \). \( \frac{u}{v} \big|_{v=0} = \frac{1}{v} \big|_{v=0} = v \big|_{v=0} = 0 \) then \( v \prec \ln v \big|_{v=0} \). Since the relation could occur with the relation notation, \( v \ln v \big|_{v=0} \), 1 Dz \( \frac{1}{v} \big|_{v=0} \), 1 Dz \( \infty \), Dz \( \prec \), \( z = \prec \).

Example 5.22. [7] p.8. Show \( P_m \succ Q_n \) when \( m > n \), given \( P_m(x) = \sum_{k=0}^{m} p_k x^k \) and \( Q_n(x) = \sum_{k=0}^{n} q_k x^k \) for positive coefficients. Let \( m = n + a \), \( a > 0 \). \( P_m \succ Q_n \big|_{x=\infty} \), \( \infty \succ \infty \), \( D^n P_m \big( D^n z \big) \), \( D^n Q_n \big|_{x=\infty} \), \( D^n P_m \big( D^n z \big) \) \( \alpha \big|_{x=\infty} \), \( \beta x^a \big( D^n z \big) \) \( \alpha \big|_{x=\infty} \), \( \beta x^a \succ \alpha \big|_{x=\infty} \), integrating \( n \) times preserves this relation and solves for \( z \).

Comparison can be in a “multiplicative sense”, or an “additive sense”. In the additive sense, we treat the expression more as a relation with addition and we may add and subtract, but drawing conclusions with much-larger-than relations may be problematic. \( 2x \prec 3x \), \( 0 \prec x \), \( 0 \prec \infty \), may then mistakenly draw the conclusion \( 2x \prec 3x \big|_{x=\infty} \). In the multiplicative sense, divide by \( x \), \( 2x \succ 3x \big|_{x=\infty} \), \( 2 \succ 3 \) is false. Both comparisons are beneficial.

Example 5.23. Consider \( x^2 \succ x \big|_{x=\infty} \). By L’Hôpital, \( x^2 \succ x \), \( 2x \succ 1 \), \( z \succ \). By multiplicative \( z \), \( x^2 \succ x \), \( \frac{x^2}{x} \succ 1 \), \( x \succ 1 \), \( z \succ \).

However a variation, divide by \( x \), \( x \succ 1 \), \( 1 \succ \frac{1}{x} \), realize the infinitesimal, \( 1 \succ 0 \), \( z \succ \) does not solve for \( \succ \). In ‘realizing’ the infinitesimal information is lost as in \( \mathbb{R} \).

Example 5.24. [20] WolframMathworld] An occasional example where L’Hôpital’s rule fails. Applying the rule swaps the arguments to opposite sides. Since the relation is equality, this is indeed true. \( \frac{u}{(u^2 + 1)^2} \big|_{u=\infty} \), \( u \succ (u^2 + 1) \big|_{u=\infty} \), \( u \big|_{u=\infty} = \frac{u}{(u^2 + 1)^2} \big|_{u=\infty} \) = \( s \| u \big|_{u=\infty} = 1 \)

6 Part 6 Sequences and calculus in \( *G \)

With the partition of positive integers and positive infinite integers, it follows naturally that sequences are also similarly partitioned, as sequences are indexed on integers. General convergence of a sequence at infinity is investigated. Monotonic sequence testing by comparison. Promotion of a ratio of infinite integers to non-rational numbers is conjectured. Primitive calculus definitions with infinitary calculus, epsilon-delta proof involving arguments of magnitude are considered.
6.1 Introduction

The discovery of infinite integers leads to the obvious existence of the infinitesimals Part 1, as dividing 1 by an infinite integer is not a real number. However, it also does so much more. For theorems, the separation of the finite and infinite is possible, rather than having a single theorem which addresses both cases.

As sequences are indexed by integers, we similarly find that sequences can be partitioned. The following investigates some of the mechanics of sequences, particularly at infinity. For example it could be that over time sequences supersede sets. Sequences can be viewed as more primitive structures.

6.2 Sequences and functions

We have extended the sequence notation to include intervals, as we deem that ‘the order’ is the most important property.

Definition 6.1. Join or concatenate two sequences. \((a) + (b) = (a, b)\) A sequence can be deconstructed. \((a, b) = (a) + (b)\) The operator + is not commutative, \((b) + (a) = (b, a)\), and in general \((b, a) \neq (a, b)\).

Definition 6.2. Compare sequences component wise on relation \(z\). \((a_1, a_2, \ldots) z (b_1, b_2, \ldots)\) then \((a_1 z b_1, a_2 z b_2, \ldots)\)

Example 6.1. While we often use functions to state a comparison of sequences, we may use sequence notation. Here, infinite positive integers are implied. \((n) \prec (n^2)_{|n=\infty}\)

Definition 6.3. If a set has a ‘less than’ relation, the sequence of the set is ordered. If \(X\) is the set, let \((X)\) be the sequence of \(X\) with the order relation.

Example 6.2. \((+\Phi)\) is the ordered sequence of positive infinitesimals, \((\mathbb{N}_\infty)\) is the ordered sequence of infinite positive integers, \((+\Phi^{-1})\) is the ordered sequence of positive infinite numbers, \((\mathbb{N})\) is the ordered sequence of natural numbers, \((\mathbb{N}_<)\) ordered sequence of finite natural numbers.

We can iterate over infinity in the following way. Consider the infinite sequence \(1, 2, 3, \ldots\). We can express this with a variable \(n\). i.e. \(1, 2, 3, \ldots, n, n + 1, n + 2, n + 3, \ldots |_{n=\infty}\).

Definition 6.4. Any integer sequence can be composed of both finite and infinite integers.

\[ (1, 2, 3, \ldots, k)_{|k<\infty} + (\ldots, n - 1, n, n + 1, n + 2, n + 3, \ldots)_{|n=\infty} \]

\[ (\mathbb{N}_<) + (\mathbb{N}_\infty) \]
With the establishment of the existence of the infinite integers, since a sequence is indexed by integers we can partition the sequence into finite and infinite parts. Further all sequences with integer indices, implicitly or explicitly are of this form. A finite sequence is deconstructed with no infinite part.

**Definition 6.5.** Define a sequence at infinity \((a_n)|_{n=\infty}\) to iterate over the whole infinite interval,

\[
(\ldots, a_{n-2}, a_{n-1}, a_n, a_{n+1}, a_{n+2}, \ldots)|_{n=\infty}
\]

or to count from a point onwards, generally in a positive direction.

\[
(a_n, a_{n+1}, a_{n+2}, \ldots)|_{n=\infty}
\]

The concept of a sequence at infinity is particularly important, as we now can separate and partition finite and infinite numbers.

**Definition 6.6.** A sequence can be deconstructed into both finite and infinite parts.

\[
(a_1, a_2, \ldots) = (a_1, a_2, \ldots, a_k)|_{k<\infty} + (\ldots, a_n, a_{n+1}, \ldots)|_{n=\infty}
\]

\[
(a_1, a_2, \ldots) = (a_k)|_{1\leq k<\infty} + (a_{N_\infty})
\]

\[
(a_1, a_2, \ldots) = (a_{N_\infty} + (a_{N_\infty})
\]

What is striking is that at infinity there is no minimum or maximum elements. If \(n = \infty\) is an infinity, so is \(n - 1, n - 2, \ldots\). Similarly for the continuous variable. If \(x = \infty\), so is \(x - 1, x - 2, \ldots\). While infinity has no lower or upper bound, we may find it useful to define the ‘ideal’ min and max elements, as these can describe an interval.

**Definition 6.7.** Ideal minimum and maximum numbers

Let \(\min(N_\infty)\) be an ideal minimum of the lowest positive infinite integer.

Let \(\max(N_\infty)\) be an ideal maximum of the highest positive infinite integer.

Let \(\min(+\Phi^{-1})\) be an ideal minimum of the lowest positive infinite number.

Let \(\max(+\Phi^{-1})\) be an ideal maximum of the highest positive infinite number.

\[
(\ldots, n - 1, n, n + 1, \ldots)|_{n=\infty} = (\min(N_\infty), \ldots, \max(N_\infty)) = (N_\infty)
\]

\[
(\ldots + [x - 1, x] + [x, x + 1] + [x + 1, x + 2] + \ldots)|_{x=\infty} = (\min(+\Phi^{-1}), \max(+\Phi^{-1})) = (+\Phi^{-1})
\]

Considering 1, 2, 3, 4, \ldots, we believe that the infinity was thought of as an open set, \((1, 2, \ldots)\), but with infinite integers, this may be better expressed with an interval notation \([1, 2, \ldots, \infty]\).

**Definition 6.8.** An interval can be deconstructed into real and infinite real parts

\[
(a, \infty] = (x)|_{a<x<\infty} + (+\Phi^{-1})
\]
That there is no infinite integer lower bound often does not matter. Once we arrive at infinity, we may iterate from a chosen point onwards.

**Example 6.3.** In describing the function $\frac{1}{n}$ as a sequence, we often say $(\frac{1}{2}, \frac{1}{3}, \ldots)$ which includes both finite and infinitesimal numbers. By considering the infinite sequence, $\frac{1}{n}|_{n=\infty}$ we now are describing the infinitesimals only.

We would like to iterate over infinity for various reasons. On occasion it is necessary to iterate not over all the infinities, but between two infinities. For example, like with NSA (Non-Standard Analysis), iterate between two infinities $\omega$ and $2\omega$. In our notation, we can start counting at infinity, till we reach the next infinity. Construct an auxiliary sequence for this purpose.

$$(a_{n+1}, a_{n+2}, \ldots a_{2n}) = (b_1, b_2, \ldots b_n) \text{ where } b_k = a_{n+k}|_{n=\infty}$$

From another perspective, we can use the same notation to iterate over all infinity. Iterating over infinity at infinity, so that the finite part is removed, $(a_{n+k})|_{k=n=\infty}$ iterates over all the infinite elements.

While there is no lower infinite bound for the infinite integers, this is not really a problem as we need not consider all infinite elements, but elements from a certain point onwards, hence Definition 6.5.

We require sequences at infinity when building other structures at infinity. The ordering property of sequences is separate to sets, which by their definition are unordered.

Sequences can be transformed and or rearranged, from one sequence to another, with an infinity of elements, in such a way to guarantee a property based on the order. Our subsequent papers [12, Convergence sums …], [13] both require sequences in the ideas and proofs.

Sequences are not restricted to discrete variables. As we can consider a function as a continuous sequence of points, we extend the sequence notation to the continuous variable. We would then consider the index which is also a continuous variable, the domain.

**Example 6.4.** Partition the interval, $[\alpha, \infty] = (x)|_{x=[\alpha, \infty]} = (x)|_{(\alpha, x<\infty)} + (x)|_{(+\Phi^{-1})}$. We say a function is “monotonically increasing” if $f(x+\delta) \geq f(x)$, “monotonically decreasing” if $f(x+\delta) \leq f(x)$, $\delta \in \Phi$.

**Definition 6.9.** We say a sequence is “monotonically increasing” if $a_{n+1} \geq a_n$, “monotonically decreasing” if $a_{n+1} \leq a_n$.

**Definition 6.10.** We say a sequence or function has “monotonicity” if the sequence or function is monotonic: monotonically increasing or monotonically decreasing.
Determine if a function is monotonic by comparing successive terms and solving for the relation. For a continuous function we can often take the derivative. However, for sequences this may not be possible.

**Conjecture 6.1.** We can determine the monoticity of sequence \( a_n |_{n=\infty} \) by solving for relation \( z \) in \( *G \), \( a_{n+1} z a_n |_{n=\infty} \), or if it exists its continuous version \( a(n+1) z a(n) \).

**Example 6.5.** Determine if the sequence \( (a_n) |_{n=\infty} \) is monotoninc. \( a_n = \frac{1}{n^2} \), compare sequential terms, \( a_{n+1} z a_n |_{n=\infty} = \frac{1}{(n+1)^2} z \frac{1}{n^2} |_{n=\infty} \), \( n^2 z (n+1)^2 |_{n=\infty} \), \( n^2 z n^2 + 2n + 1 |_{n=\infty} \), 0 \( 2n + 1 |_{n=\infty} \), \( z = <, a_{n+1} < a_n \) and the sequence is monotonically decreasing.

**Example 6.6.** Test if the sequence \( (a_n) |_{n=\infty} \) is monotoninc, \( a_n = \frac{1}{n^{2} + (-1)^n} |_{n=\infty} \). Let \( j = (-1)^n, a_n z a_{n+1} |_{n=\infty} = \frac{1}{n^{2} + j} z \frac{1}{(n+1)^2 - j} |_{n=\infty} \), \( (n + 1)^{\frac{1}{2}} |_{n=\infty} \), \( n^{\frac{1}{2}} + j |_{n=\infty} \), \( (n + 1)^{\frac{1}{2}} - n^{\frac{1}{2}} |_{n=\infty} = 0, 0 z 2j |_{n=\infty} \), \( 0 z (-1)^n |_{n=\infty} \), \( z = <, >, \ldots \) then \( (a_n) |_{n=\infty} \) is not monotonic.

**Example 6.7.** An example of when not to apply infinitary argument simplification \( a+b=a \).

Test if the sequence \( a_n = \frac{1}{n^{2} + (-1)^n} |_{n=\infty} \) is monotoninc. \( n^{\frac{1}{2}} > (-1)^n |_{n=\infty} \), if we say \( a_n = \frac{1}{n^{2} + (-1)^n} |_{n=\infty} = \frac{1}{n^{2} |_{n=\infty}} \), the sequence \( \frac{1}{n^{2}} |_{n=\infty} \) is easily shown to be monotoninc. However example 6.6 shows the sequence is not monotoninc. Even though the magnitude is infinitely small compared with the other function, the property of monotonicity by adding \( (-1)^n \) was changed.

### 6.3 Convergence

We now move on to a more theoretical use of at-a-point definition. The Cauchy convergence, Cauchy sequence and limit, can be defined at infinity instead of both a finite and infinite perspective definition. Given that a number system exists at infinity and zero, this is more justified, and since the definitions may be more primitive, may subsume the standard definitions.

The problem with both the limit existence and the Cauchy sequence convergence is that they both define convergence to the point to include the point of convergence in the same space. While this is incredibly useful it is a subset of a more general convergence.

For example Hille [6, p.17, Theorem 1.3.1] already assumes complex numbers, \( z_k \in \mathbb{C} \) and defines Cauchy convergence

\[
|z_m - z_n| < \epsilon \text{ for } m, n > M(\epsilon).
\]

Then provides the following corollary from the definition, expressed as a limit. [6, p.71 (4.1.12)]

\[
\lim_{m, n \to \infty} ||z_m - z_n|| = 0
\]
However turning this around, the corollary is the more primitive operation, that being their difference is zero. Make this the definition, defining a sequence as converging at infinity (Definition 6.14) and derive the Cauchy sequence (Definition 6.15).

**Definition 6.12.** Convergence is the negation of divergence.

**Definition 6.13.** A sequence with singularities diverges.

**Theorem 6.1.** A sequence without singularities before infinity can only diverge at infinity.

**Proof.** Every finite sequence converges because the number of terms is finite and the terms are not singularities.

**Corollary 6.1.** For a sequence without finite singularities, convergence or divergence is determined at infinity.

**Proof.** Since convergence is defined as the negation of divergence (Definition 6.12), and divergence can only happen at infinity (Theorem 6.1), then both convergence and divergence can only be completely determined at infinity. Note: this does not contradict finite sums converging, as at infinity their sum is 0.

**Definition 6.14.** A sequence \((a_n)\) converges at infinity if given \(\{m, n\} \in \mathbb{N}_{\infty}\):

\[
a_m - a_n |_{m,n=\infty} \simeq 0
\]

**Definition 6.15.** A Cauchy sequence converges if the sequence \((x_n)|_{n=\infty}\) converges (Definition 6.14) and the finite and infinite numbers are the same type of number. \(n < \infty\) then \(x_n \in W\) and \(x_m |_{m=\infty} \in W\)

In Definition 6.14 of sequence convergence, the number types can be different as \(\Phi\) is composed of the infinireals. Cauchy convergence Definition 6.15 derives from defining convergence Definition 6.14.

Similarly the limit definition changes. Define evaluation at a point, then define the limit as the evaluation at the point and in the same space. Definition 6.20 the limit derives from Definition 6.19 evaluation at a point.

The idea of a sequence not being convergent because it is not ‘complete’ is a narrow view.

Consider the computation of two integer sequences \(a_n\) an \(b_n\) where their ratio for finite values is always rational, but what they are approximating is not. The Cauchy sequence convergence does not explain the differing number types, only convergence.

The limit fails to be defined when a ratio between these two sequences is considered. This is a simple operation. The best answer that can explain the calculation is that at infinity the
ratio is promoted, a rational approximation at infinity can be promoted to a transcendental number.

**Conjecture 6.2.** There exists ratios of infinite integers of the form \( \frac{N}{N_\infty} \) which can be transferred to real numbers.

Since all irrationals including transcendental numbers are calculated by integer sequences, such a restriction on the ratio of two integer sequences not converging is absurd. Such sequences do converge at infinity.

If \( \frac{a_n}{b_n} \big|_{n=\infty} \) converges at a point not in the limit.

If \( \{a_n, b_n\} \in \mathbb{N} \) are integers, \( \frac{a_n}{b_n} \in \mathbb{Q} \), but \( \frac{a_n}{b_n} \big|_{n=\infty} \not\to \mathbb{Q}' \) then \( \lim_{n \to \infty} \frac{a_n}{b_n} \) does not exist. However \( \frac{a_n}{b_n} \big|_{n=\infty} \) does not have this restriction. \( \frac{a_n}{b_n} \big|_{n=\infty} \in \mathbb{J}_\infty \), but \( \mathbb{Q}_\infty \) for any non-rational number approximation is promoted to \( \mathbb{Q}' \), if the approximation exists in \( \mathbb{R} \).

These rational approximations are common. All calculations of numbers in \( \mathbb{R} \) are reduced to integer calculations. However such calculations need to be explained in a higher dimension number, at least with \( \mathbb{R}_\infty \) because infinite integers are involved.

**Example 6.8.** Construct an integer sequence to approximate \( \sqrt{3} \). Hence, we consider \( \sqrt{3} \) as the ratio of two infinite integers at infinity. \( (x - 1)^2 = 3 \) has a solution \( x = 1 + \sqrt{3} \). \( x^2 - 2x + 1 = 3, \ x = 2 + \frac{2}{x} \). Develop an iterative scheme. \( x_{n+1} = 2 + \frac{2}{x_n} \). Assume an integer solution, \( x_n = \frac{a_n}{b_n}, \ \frac{a_{n+1}}{b_{n+1}} = 2 + \frac{2b_n}{a_n}, \ \frac{a_{n+1}}{b_{n+1}} = \frac{2a_n + 2b_n}{a_n} \). Let \( b_{n+1} = a_n \) then \( a_{n+1} = 2a_n + 2a_{n-1} \). For two initial values, \( a_0 = 1, a_1 = 1, a_2 = 2a_1 + 2a_0 = 2 \cdot 1 + 2 \cdot 1 = 4 \), the sequence generated is \( 1, 1, 4, 10, 28, 76, 208, 568, 1552, \ldots \). \( \sqrt{3} = \frac{a_n}{b_n} - 1 = \frac{a_n + 2a_{n-1}}{a_n} \big|_{n=\infty} \).

The way around this difficulty by saying it is not important through definition is problematic. Simply promoting the two numbers being divided to the same number system (as demonstrated by Hille [9, p.17, Theorem 1.3.1]), so that by definition and only by definition they are the same number type; and therefore the ratio converges in the same space, is incomplete.

The alternative Definition 6.14 define the same concept more generally. Admittedly the problem of ‘promotion’ is not explained, but is acknowledged.

While this may be a controversial finding, it suggests either that notions of convergence have not been entirely settled, or that they are incomplete. Especially for the most basic operations.

In summary, there are two problems with the Cauchy sequence. It is derived from a more general convergence, and is better explained in a space at infinity, with infinite integers.
6.4 Limits and continuity

The standard epsilon definition of a limit Definition 6.16 can be improved by explicitly defining \( \{ \varepsilon, \delta \} \in \Phi^+ \) as, by the conditions, these numbers become infinitesimals. Therefore this is an implicit infinitesimal definition.

**Definition 6.16.** The symbol \( \lim_{x \to p} f(x) = A \) means that for every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( |f(x) - A| < \varepsilon \) whenever \( 0 < |x - p| < \delta \).

If we consider a limit definition [5, p.129] given by Apostol, we can generalize the definition in \(*G\) to include infinitesimals, thereby making the definition explicit. A statement with infinitesimals, let \( \varepsilon \in +\Phi: |f(x) - A| < \varepsilon \) can be equivalently expressed: \( f(x) - A \in \Phi \cup \{0\} \).

**Definition 6.17.** The symbol \( \lim_{x \to p} f(x) = A \) in \(*G\) means that when \( x - p \in \Phi \) then \( f(x) - A \in \Phi \cup \{0\} \).

**Example 6.9.** \( \lim_{n \to \infty} \frac{n^3 + \frac{1}{n^3}}{4n^3} = \lim_{n \to \infty} \frac{3n^2 - \frac{2}{n^2}}{2n^2} = \lim_{n \to \infty} \frac{6n + 2n^{-3}}{24n} = \lim_{n \to \infty} \frac{6 - 6n^{-4}}{24} = \frac{1}{4} - \frac{1}{4n} \mid_{n=\infty} \)

The limit \( \lim_{n \to \infty} \frac{a_n}{b_n} \) in \( \mathbb{R} \) implicitly applies a transfer \(*G \mapsto \mathbb{R} \) Part 4. A limit in \(*G\) (Definition 6.17) by default does not do a transfer, but this is easily done.

**Proposition 6.1.** If a limit exists in \( \mathbb{R} \) then \(*G \mapsto \mathbb{R} \): in \(*G\), \( \text{st}(\lim f(x)) = \lim f(x) \) in \( \mathbb{R} \).

**Proof.** \( \mathbb{R} \) is a subset of \(*G\). Then a transfer must exist, since limits are actually calculated in \(*G\). During the transfer, infinitesimals are mapped to zero, \( \Phi \mapsto 0. \)

Limits and continuity are tied together in \( \mathbb{R} \), however we will see that this is often not the case in \(*G\). For example, we can have a discontinuous staircase function in \( \mathbb{R} \) which is continuous in \(*G\).

However, we can similarly define continuity in \(*G\) with limits. Since this is before the transfer principle \(*G \mapsto \mathbb{R} \) is applied then there is no paradox. We believe that the definition of continuity via limits has the advantage of an ‘at-a-point’ perspective.

Consider the ‘principle of variation’, which for a continuous variable is the ‘law of adequality’ [17, p.5]: \( d(f(x)) = f(x + \delta) - f(x) \) leads to the derivative, as a ratio of infinitesimals. \( df(x) = f(x + dx) - f(x), \quad \frac{df(x)}{dx} = \frac{f(x + dx) - f(x)}{dx} \mid_{dx=0} \)
However, the principle of variation is also applicable to discrete change, where \( dn = (n + 1) - n = 1 \) is a change in integers, which we interpret to derive a derivative of a sequence [13].

Continuity can be defined either by the principle of variation or the limit. Continuity can be expressed as a variation; taking two points infinitesimally close, and their difference is an infinitesimal.

**Definition 6.18.** A function \( f : \ast G \to \ast G \) is continuous at \( x \); \( f(x), x, y \in \ast G ; \delta x, \delta y \in \Phi \);

\[
y = f(x) \\
y + \delta y = f(x + \delta x)
\]

**Definition 6.19.** A function \( f : \ast G \to \ast G \) is continuous at \( x = a \) and has been evaluated to \( L \): \( f(x)|_{x=a} = L \),

\[
\text{If } \forall x : x \simeq a \text{ then } f(x) \simeq L.
\]

\( x \simeq a \) is equivalent to \( x + \delta = a \) when \( \delta \in \Phi \) or \( x = a \) Definition 2.18

**Lemma 6.1.** Definition 6.18 implies Definition 6.19.

**Proof.** Consider Definition 6.18 \( y + \delta y = f(x + \delta x)|_{x=a} \), \( y \simeq f(x + \delta x)|_{x=a} \), let \( L = f(x)|_{x=a} \), \( L \simeq f(x)|_{x=a} \) \( \square \)

A definition of a limit, less general than a definition of evaluation at a point is given.

**Definition 6.20.** A function \( f : \ast G \to \ast G \) is a limit at \( x = a \) if \( f(x) \) is continuous at \( x = a \) and is the same number type \( W \).

\[
\text{If } f(x)|_{x=a} = L \text{ and } \{ f(x), L \} \in W \text{ then } f(x)|_{x=a} \text{ is a limit.}
\]

The symbol \( \lim_{x \to a} f(x) = L \) in \( \ast G \) means that when \( x \simeq a \) then \( f(x) \simeq L \).

Considering the larger picture between the two-tiered number systems. What is being claimed is that providing a finite and infinite perspective definition does not describe well what is happening, particularly when “at infinity” simplifies the explanation. These ideas of defining at infinity extend into many other definitions.

As there appear to be different kinds of arithmetics and convergence as governed by what happens at infinity, defining convergence in general at infinity makes more sense.
Example 6.10. Define \( x > 1 : \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \). Then we can have an infinite sum with positive terms that has a negative solution. Let \( \frac{1}{1-x} = -w \), solve for \( x = \frac{1+w}{w} \). Case \( w = 2, x = \frac{3}{2}, \sum_{k=0}^{\infty} (\frac{3}{2})^k = -2 \).

Not only has a positive sum of terms become negative, we needed an infinity of terms to interpret the sum, for the sum to have meaning.

This example is relevant because our applications following this series \[12, \text{Convergence sums} \ldots\] define convergence at infinity, as does Robinson’s non-standard analysis (here infinity defined as not finite is interpreted as successive orders of numerical infinities \( \omega \)).

6.5 Epsilon-delta proof

We again look at the limit in the guise of the epsilon-delta proof. \[21\] comments on the generalization in multidimensional space with the norm and open balls.

For complex proofs, NSA has been successful as an alternative to Epsilon-Delta management, and other proofs solving in the higher number system and transferring the results back into the reals. We would expect our calculus to also be useful in proving propositions and theorems, with similar purpose to NSA but in another way.

An epsilon-delta definition proof, if \(|x - x_0| < \delta\) then \(|f(x) - f(x_0)| < \epsilon\), in a minimalistic sense is not a finite inequality, but an inequality at infinity with infinitesimals, as we can derive the finite inequality. By the transfer principle, project the statement from \(G\) into \(R\). E.g. \( \delta_1 \in \Phi; \delta_2 \in R^+; (\ast G, |x - x_0|\delta_1) \mapsto (R, |x - x_0|\delta_2) \)

An abstraction, by removing the inequality relation, expressing the relation as a variable which is infinitesimal, hence a more direct reasoning.

Definition 6.21. The Epsilon-Delta Proof with \( \Phi \)

\[ \text{If } x - x_0 = \Phi \text{ then } f(x) - f(x_0) = \Phi, n = \infty. \]

Example 6.11. \[20\] Epsilon-Delta Proof] \( f(x) = ax + b; a, b \in R; a \neq 0 \). Show \( f(x) \) is continuous.

\[ x - x_0 = \Phi \]

\[ f(x) - f(x_0) = (ax + b) - (ax_0 + b) = a(x - x_0) = \Phi \tag{as a\Phi = \Phi} \]

An Epsilon-Delta definition and proof with a similar structure \[21\] could be given where the real numbers are replaced by \(G\). For example it may not be enough that the numbers
are infinitesimals (Definition 6.21), but we may require the infinitesimals to be continually approaching 0. (See Proposition 6.3)

\[
\delta_n \to 0 \text{ replaced with } \delta_n \succ \delta_{n+1}
\]

**Definition 6.22.** The Epsilon-Delta Proof in \( ^\ast G \)

\[
\delta_n = x - x_0 \in \Phi; \quad \epsilon_n = f(x) - f(x_0) \in \Phi; \quad n = \infty
\]

*If \( \delta_n \succ \delta_{n+1} \) then \( \epsilon_n \succ \epsilon_{n+1} \)*

### 6.6 A two-tiered calculus

We can work in \( ^\ast G \) and project back or transfer to \( \mathbb{R} \) or \( \overline{\mathbb{R}} \), or \( ^\ast G \). The overall reason for doing this was a separation of the finite and infinite domains, thereby separating and isolating the problem.

We introduce the following sequence definitions as a consequence of a two-tiered calculus. It is possible for a sequence to plateau in \( ^\ast G \) and project back to a convergent sequence in \( \mathbb{R} \). Similarly a divergent sequence could plateau in \( ^\ast G \) and diverge in \( \overline{\mathbb{R}} \). We need to be able to describe arbitrarily converging and diverging sequences to guarantee certain properties and avoid the plateau. Hence, additional requirements are needed to manage the sequences.

**Definition 6.23.** We say \( x_n \to 0 \) then \( x_n \in \Phi \) and is decreasing in magnitude: \( |x_{n+1}| \leq |x_n| \).

**Definition 6.24.** We say \( x_n \to \infty \) then \( x_n \in \Phi^{-1} \) and is increasing in magnitude: \( |x_{n+1}| \geq |x_n| \).

Since a variable may be expressed as a point, the sequences described can be extended to the continuous variable. An adaptable notation, given that we may need different sequences for particular problems and theory.

For example, \( x \to \infty \), \( (x) \) indefinitely increases and is positive monotonic, \( f(x)|_{x=\infty} = \ldots \)

The other type of sequences in general use are a partition. For example, for all \( x > x_0 \).

**Definition 6.25.** In context, a variable \( x \to 0 \) can be described at zero by \( x \in \Phi \) or Definition 6.25 or Definition 6.29 or other as \( |x=0| \).

**Definition 6.26.** In context, a variable \( x \to \infty \) can described at infinity by \( x \in \Phi^{-1} \) or Definition 6.24 or Definition 6.29 or other as \( |x=\infty| \).

**Definition 6.27.** A ‘subsequence’ is a sequence formed from a given sequence by deleting elements without changing the relative position of the elements.
Definition 6.28. We say a sequence is ‘indeﬁnitely decreasing’ in magnitude. $x_n \to 0$; $n$, $n_2 \in \mathbb{N}_\infty$; there exists $n_2 : n_2 > n$ and $x_{n_2} < x_n$.

Definition 6.29. We say a sequence is ‘indeﬁnitely increasing’ in magnitude. $x_n \to \infty$; $n$, $n_2 \in \mathbb{N}_\infty$; there exists $n_2 : n_2 > n$ and $x_{n_2} > x_n$.

Proposition 6.2. If $x_n \to 0$ is indeﬁnitely decreasing there exists a subsequence $(\nu_n)$: $\nu_{n+1} < \nu_n |_{n=\infty}$ and $\nu_n \to 0$

Proof. Since $x_n$ is decreasing in magnitude, we can always choose a subsequent much-less-than term.

We use inﬁnity arguments with order in the proof of Proposition 6.3. Normally we would send $h \to 0$ before $\delta \to 0$. However, if the solution is independent of the inﬁnity, consider $\delta \to 0$ before $h \to 0$. Then we reason that the derivative must be an inﬁnitesimal.

Proposition 6.3. If $\delta_n \to 0$ is indeﬁnitely decreasing and strictly positive monotonic decreasing then

$$D\delta_n |_{n=\infty} \in -\Phi$$

Proof. $h \in +\Phi$; Strictly monotonic decreasing $\delta_n$ then $\delta_{n+1} < \delta_n$, $\delta_{n+1} - \delta_n < 0$, $\frac{\delta_{n+1} - \delta_n}{h} < 0$, $D\delta_n = \frac{\delta_{n+1} - \delta_n}{h} |_{h=0}$ is negative.

Consider the inﬁnite state where $\delta_n \to 0$ before $h \to 0$. Since $\delta_n$ can be made arbitrarily small, then $\delta_{n+1} - \delta_n < h$, $\delta_{n+1} - \delta_n \in \Phi$, $\frac{\delta_{n+1} - \delta_n}{h} |_{h=0} \in \Phi$, $D\delta_n \in \Phi$.

Lemma 6.2. If $f(x)$ and $g(x)$ are positive monotonic functions, with relation $z : z \in \{<, \leq, >, \geq\}$, $f(x) z g(x)$ for all $x$ in a given domain, such a relation can be reformed to a positive inequality: $\phi > 0$ or $\phi \geq 0$

Proof. A less than relation can always be expressed as a greater than relation by swapping the arguments sides. If $f < g$ then $g > f$. If $f \leq g$ then $g \geq f$. If $f(x) > g(x)$ then $f(x) - g(x) > 0$. If $f(x) \geq g(x)$ then $f(x) - g(x) \geq 0$.

Proposition 6.4. $z \in \{>, \geq\}$: If $f(x) z 0$ and $f(x)$ is monotonically increasing then $Df(x) z 0$ where $f(x)$ is not constant.

Proof. $h \in \Phi^+$, $f(x + h) z f(x)$, $f(x + h) - f(x) z 0$, $\frac{f(x+h)-f(x)}{h} z 0$, $\frac{f(x+h)-f(x)}{h} |_{h=0} z 0$, $Df(x) z 0$

Proposition 6.5. $z \in \{>, \geq\}$: If $f(x)$ and $g(x)$ are positive monotonic functions: $f(x) z g(x)$ over a positive domain then $Df(x) z Dg(x)$, where $f(x) - g(x)$ is monotonically increasing.
Proof. Reorganise the relation to be positive, Lemma \ref{lemma6.2}. Apply Proposition \ref{proposition6.4}.

**Proposition 6.6.** If \( f \succ 0 \) and \( f \) is a positive monotonic increasing function then ignoring integration constants and integrating in a positive interval, \( \int f \succ 0 \).

Proof. Since integrating an infinitesimal or infinity does not result in 0, then integral must have a much-greater-than relationship with 0.

**Theorem 6.2.** Let \( f \) and \( g \) be positive monotonic functions: \( f \succ g \). If integrated over a positive interval ignoring integration constants then \( \int f \succ \int g \).

Proof. \( f \succ g \), \( f - g \succ 0 \), apply Proposition \ref{proposition6.6}.

This infinitesimal and infinitary analysis is more suited to a functional approach and does not explicitly use sets, compared with NSA. Hence the complexity of use would likely make this calculus more accessible. We have found, empirically, different solutions to problems and in many cases simpler reasoning than with standard calculus, such as found in \cite{8}. That is, we have constructed a new calculus of sum convergence \cite[Convergence sums ...]{12}.

The following propositions define continuity and calculus in \( \ast G \).

**Proposition 6.7.** A function \( f : \ast G \mapsto \ast G \) is uniformly continuous if \( f(x) \simeq f(y) \) when \( x, y \in \ast G \) and \( x \simeq y \).

**Proposition 6.8.** A function \( f : \ast G \mapsto \ast G \) is differentiable at \( x \in \ast G \) iff there exists \( b \in \ast G \):
\[
\frac{f(x) - f(a)}{x - a} \simeq b \text{ when } x \simeq a
\]

Many of the classical results can be proved in \( \ast G \). Assuming the Taylor series in \( \ast G \) with arbitrary truncation, that is a well-behaved function, prove Newton’s method.

**Theorem 6.3.** When \( f(x_{n+1}) \prec f(x_n) \big|_{n=\infty} \) and \( x_{n+1} \simeq x_n \big|_{n=\infty} \) then \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \big|_{n=\infty} \)

Proof. \( h, f(x_n) \in \Phi; x_n, f^{(w)}(x_n) \in \ast G \)
\[
f(x_n + h) = f(x_n) + hf'(x_n) + \frac{h^2}{2!}f''(x_n) + \ldots \big|_{n=\infty} \big|_{h=0} \quad \text{(Assume continuity)}
\]
\[
f(x_n + h) = f(x_n) + hf'(x_n) \big|_{n=\infty} \big|_{h=0} \quad \text{(Choose } h = x_{n+1} - x_n) \]
\[
f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n)f'(x_n) \big|_{n=\infty} \quad \text{(Non-reversible arithmetic)}
\]
\[
(f(x_n) - f(x_{n+1}) = f(x_n) \big|_{n=\infty} \text{ as } f(x_n) \succ f(x_{n+1}) \big|_{n=\infty})
\]
\[
0 = f(x_n) + (x_{n+1} - x_n)f'(x_n) \big|_{n=\infty} \quad \text{(Solve for } x_{n+1})
\]
\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \big|_{n=\infty} \quad \text{(Transfer principle } \ast G \mapsto \mathbb{R})
\]
6.7 A variable reaching infinity before another

With the knowledge of partial derivatives, there should be no argument against a variable reaching infinity before another. That this is not taught or seen this way, once stated should be accepted as fact.

The partial derivative is equivalent to one variable reaching infinity before the other variables.

With the finite and infinite separation, we will seek further mathematics. The order of one variable reaching infinity before another is common as demonstrated by partial differential equations. We also have other language to capture the infinite state, such as ‘the characteristic differential equation’.

This raises the possibility of combinations of variables reaching infinity in different orders. The algebra and space is amazingly complex, yet understandable.

Rather than purge mathematics of complexity for certainty, a consideration of orderings we believe has led to new rearrangement theorems and analysis (see [12], [14]). This is a necessary correction to current mathematics which rejects infinitary calculus and Euler in superficial ways.

Uniform convergence, absolute sum convergence and other concepts, which for example highlight when the problem is independent of the order, could be investigated with orderings.

Turning the problem around, if the ordering does not matter, we only need to find the solution of one ordering to determine the whole solution. (for example sum rearrangements at infinity [14])

Example 6.12. See Proposition 2.11

An example of a variable reaching infinity before another explains what others claim is ‘extraordinary reasoning’.

Example 6.13. In following and explaining Euler’s derivation of the exponential function expansion for $e^x$, Robert Goldblatt [27, p.8] describes Euler’s reasoning

$$\frac{j(j-1)(j-2)\ldots(j-n+1)}{j^n} \bigg|_{j=\infty} = 1$$

as ‘extraordinary’, which, with polite wording, is an academics way of describing something as humbug. However, the calculation is explained by one variable reaching infinity before the other.
Firstly, consider the numerical evidence. \( \frac{j}{j} \big|_{j=\infty} = 1 \); \( \frac{(j-1)}{j} \big|_{j=\infty} = 1 \); \( \frac{(j-1)(j-2)}{j} \big|_{j=\infty} = 1 \); \( \ldots \)

If we consider the general \( n \) term, we arrive at the expression \( \frac{j(j-1)(j-2)\ldots(j-n+1)}{j} \big|_{j=\infty} = 1 \). If \( j = \infty \) before \( n = \infty \), then from \( j \)'s perspective, \( n \) is a constant. This is no different from the partial derivative case. Hence, simplify the constant by non-reversible arithmetic, \( j(j-1)(j-2)\ldots(j-n+1) \big|_{j=\infty} = j^n \big|_{n=\infty} \) and the result follows.

Consequently, \( j \succ n \big|_{j,n=\infty} \). There is the possibility of developing algebra for these situations, to analyse the mathematics.

We should point out that this is not the only possibility at infinity (as noted in subsequent papers there is algebra where lower order terms prevail at infinity), and that non-uniqueness exists at infinity. However, it 'is' a valid possibility.

For the given problem, we can state Euler's reasoning in \( *G \).

\[ \omega \in \Phi; j = \frac{x}{\omega}; x \notin \mathbb{R}_{\infty}; \omega = \frac{x}{\omega^2} \text{ and } j \in \Phi^{-1}; \text{consider the Binomial expansion in } *G, \text{ which is most likely acceptable as we believe } *G \text{ is a field.} \]

\[
(1 + k\omega)^j = 1 + j \frac{k\omega}{1!} + j(j-1) \frac{(k\omega)^2}{2!} + \ldots \\
= 1 + \sum_{n=1}^{\infty} j(j-1)(j-2)\ldots(j-n+1) \frac{k^n \omega^n}{n!} \big|_{j=\infty} \\
= 1 + \sum_{n=1}^{\infty} j(j-1)(j-2)\ldots(j-n+1) \frac{k^n x^n}{n!} \big|_{j=\infty} \\
= 1 + \sum_{n=1}^{\infty} \frac{k^n x^n}{n!} \big|_{j=\infty}
\]

Hence, in this case Euler's logic is justified and explained in a more rigorous way, in the exact way Euler stated. Euler was not in error, but exactly correct.

Choosing \( k = 1 \)

\[
(1 + \omega)^{\frac{x}{\omega}} = \sum_{k=0}^{\infty} \frac{x^n}{n!}
\]

If \( x = 1 \) then \( (1 + \omega)^{\frac{x}{\omega}} = e \). \( (1 + \omega)^{\frac{x}{\omega}} = ((1 + \omega)^{\frac{1}{\omega}})^x = e^x \) then \( e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!} \).

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