Adiabatic noise-induced escape rate for nonequilibrium open systems

Suman Kumar Banik, Jyotipratim Ray Chaudhuri and Deb Shankar Ray*
Department of Physical Chemistry, Indian Association for the Cultivation of Science, Jadavpur, Calcutta 700032, India.
(January 11, 2022)

We consider the motion of an overdamped particle in a force field in presence of an external, adiabatic noise, without the restriction that the noise process is Gaussian or the stochastic process is Markovian. We examine the condition for attainment of steady state for this nonequilibrium open system and calculate the adiabatic noise-induced rate of escape of the particle over a barrier.

PACS number(s) : 05.40.-a, 02.50.Ey

I. INTRODUCTION

The motion of a Brownian particle in a fluid was first explained by Einstein in terms of fast thermal motion of fluid molecules striking the Brownian particle and causing it to undergo a random walk. One essential requirement of the theory is that the noise is of internal origin. This implies that the dissipative force which the Brownian particle experiences in course of its motion in the fluid and the stochastic force acting on the particle as a result of random impact of molecules arise from a common mechanism. From a microscopic point of view the system-reservoir Hamiltonian description developed over the last few decades suggests that the coupling of the system and the reservoir co-ordinates determines both the noise and the dissipative terms of the Langevin equation describing the motion of the particle. It is therefore not difficult to anticipate that these two entities get related through the celebrated fluctuation-dissipation theorem (Einstein’s relation for diffusion and mobility is the first of its kind). The spiritual root of fluctuation-dissipation relation lies at the dynamical balance between inward flow of energy due to fluctuation of the reservoir into the system and the outward flow of energy from the system to the reservoir due to dissipation of the system. This dynamical balance is essential for attainment of a steady state of the nonequilibrium system. In their treatise on nonequilibrium statistical mechanics Lindenberg and West have classified these systems as thermodynamically closed. However, there are quite a large number of physical situations, a comprehensive account of which has been given in, where the system is thermodynamically open, i.e., when the system is driven by an external noise which is independent of system’s characteristic damping. These are important for describing the effects of pump fluctuations on the emission of a dye laser, effects of fluctuating rate constants on a chemical reaction, effects of noise on electronic parametric oscillators etc. The dynamics is still governed by a Langevin equation. Although there exists no relationship between the fluctuation and the dissipation in such a situation it is interesting to search for the condition under which the steady state of the thermodynamically open system is attained. Physically the sytem is described by three timescales; the timescale of dissipation or relaxation unlike the closed system is independent of correlation time \( \tau_c \) of the system. In the present communication we consider the system to be driven by fluctuation which is adiabatically slow such that it is characterized by a very long correlation time \( \tau_c \) where

\[
\beta^{-1} \ll \Delta t \ll \tau_c. \tag{1}
\]

Here \( \Delta t \) refers to the timescale over which we look for the average motion of the system. The latter inequality implies that \( \beta^{-1} \), i.e., the inverse of damping constant defines the shortest timescale in the dynamics in contrast to the case of standard Brownian dynamics which obeys \( \beta^{-1} \gg \Delta t \gg \tau_c \). Secondly, we do not put the restriction that the stochastic processes is Markovian or the noise is Gaussian. The two assumptions have been discussed so much in the recent literature that it is necessary to emphasize that the present consideration is free from these assumptions. We mention, in passing, that some interesting limiting situations have been examined for linearized systems driven by Gaussian noise processes

Our aim in the present article is two-fold. First, we examine the condition for a steady state distribution of the system driven by an external noise obeying the inequality. Second, Having obtained this condition we calculate the external, adiabatic noise induced steady state escape rate over a barrier in the spirit of Kramers’-Smoluchowski theory. We show how the third-order noise plays an important role in both of these issues of stochastic dynamics.

II. THIRD ORDER NOISE AND STEADY STATE PROBABILITY DENSITY FOR OPEN SYSTEMS

To start with we consider the equation of motion of a particle of unit mass in one dimension when it is acted upon by an external field of force corresponding to a potential \( V(x) \) and an external, adiabatic stochastic force \( \xi(t) \),
\[ \dot{x} = -\frac{1}{\beta} V'(x) + \frac{1}{\beta} \xi(t) . \]  

(2)

where \( \beta \) and the correlation time \( \tau_c \) of \( \xi(t) \) satisfy the inequality \( \beta \gg \tau_c \). Also note that by virtue of this we have considered the overdamped limit. In a preceding paper \( \text{[3]} \) the equation of motion for probability density distribution function \( P(x,t) \) in phase space corresponding to the Langevin description \( \text{[2]} \) was derived. It has been shown that \( P(x,t) \) obeys the differential equation of motion which contains third order terms (beyond the usual Fokker-Planck terms) giving rise to third order noise. The appearance of these terms is generic for the stochastic process pertaining to the separation of timescales \( \text{[1]} \) we consider here. The general expression for time evolution of probability density function is given by

\[
\frac{\partial P(x,t)}{\partial t} = \frac{1}{\beta} \frac{\partial}{\partial x}[V'(x)P(x,t)] + \frac{c_01}{\beta^2} \frac{\partial^2 P(x,t)}{\partial x^2} - \frac{c_2}{\beta^2} \frac{\partial}{\partial x^2}[V'(x)P(x,t)].
\]

(3)

\( c_0, c_1 \) and \( c_2 \) in Eq.(3) are as follows:

\[
c_0 = 0,
\]

\[
c_1 = \int_0^\infty \tau \langle \xi(t) \xi(t-\tau) \rangle \, d\tau,
\]

\[
c_2 = \int_0^\infty \tau^2 \langle \xi(t) \xi(t-\tau) \rangle \, d\tau,
\]

(4)

where we have also assumed, for convenience \( \langle \xi(t) \rangle = 0 \).

The equation (3) describes the time evolution of an overdamped particle in a force field (derivable from a potential \( V(x) \)) simultaneously subjected to an external adiabatic stochastic force. \( c_0, c_1 \) and \( c_2 \) measure the strength of the noise term. While the first term in Eq.(3) can be identified as the usual deterministic dynamical term, the second and the third terms involve second and third order diffusion coefficients due to stochasticity of \( \xi(t) \). The expansion scheme associated with Eq.(3) should not be confused with Kramers-Moyal [KM] expansion (consideration of third order diffusion term of a KM expansion may lead to serious interpretative difficulties \( \text{[4]} \) because the probability distribution diffusion function often turns out to be negative) which serves as the standard starting point for analysis of stochastic processes with fast noise corresponding to the separation of timescale \( \beta^{-1} \gg \Delta t \gg \tau_c \). What is implicit in a KM expansion is that the moments are assumed to be linear in \( \tau \) and thus the validity of the coefficients of a KM expansion rests on the smallness of \( \tau \). The cumulant expansion \( \text{[1]} \) similarly relies on the smallness of correlation time \( \tau_c \). On the other hand Eq.(3) is based on “adiabatic following approximation” \( \text{[2,3]} \) which involves an expansion in \( 1/\beta \) \( \text{[3]} \), which is evident from the nature of the coefficients of the terms in the right hand side. The remarkable departure from the standard form of Fokker-Planck equation (Smoluchowski equation) is due to the presence of the third order noise. For other details we refer to \( \text{[3]} \).

In the next step we recast the third order equation \( \text{[3]} \) in the form of the familiar continuity equation and identify a steady state current \( J \) in the following equation for a steady state probability distribution function \( P(x) \)

\[
\frac{d^2}{dx^2} \{ V'(x)P \} - a \frac{dP}{dx} - b \{ V'(x)P \} = \frac{\beta^3 J}{c_2},
\]

(5)

where \( a \) and \( b \) are given by

\[
a = \frac{\beta c_01}{c_2} \quad \text{and} \quad b = \frac{\beta^2}{c_2}.
\]

(6)

We consider a Kramers type potential \( V(x) \) shown in Fig.(1) and look for the steady state current at the barrier top by linearizing the potential at \( x = 0 \). If \( \omega_0 \) refers to frequency of the inverted well and \( E_0 \) defines the potential at the barrier top, one obtains the following inhomogenous modified Bessel equation of order \( \nu \)

\[
\zeta^2 \frac{d^2 W}{d\zeta^2} + \zeta \frac{dW}{d\zeta} - (\nu^2 + c^2) W = \frac{D}{b^2(1+\nu)} \zeta^{1+\nu}.
\]

(7)

In the above derivation following transformations and abbreviations have been used

\[
P(x) = x^{(1-\gamma)/2} W(x), \quad \zeta = \sqrt{b} x,
\]

(8a)

\[
D = -\frac{\beta^3 J}{c_2 \omega_0}, \quad \gamma = \frac{2a^2 + a}{\omega_0}, \quad \nu = \frac{1}{2} \left(1 + \frac{a}{\omega_0}\right).
\]

(8b)

The homogenous counterpart corresponding to the above Eq.(3) is the standard modified Bessel equation of order \( \nu \). The general solution of Eq.(3) can be written as

\[
W(\zeta) = A I_\nu(\zeta) + B K_\nu(\zeta) + \frac{D}{b^2(1+\nu)} \int_{0}^{\zeta} \zeta^\nu K_\nu(\zeta') \, d\zeta'
\]

\[
- \frac{D}{b^2(1+\nu)} I_\nu(\zeta) \int_{0}^{\infty} \zeta^\nu I_\nu(\zeta') \, d\zeta',
\]

(9)

where \( I_\nu(\zeta) \) and \( K_\nu(\zeta) \) are modified Bessel functions of order \( \nu \); \( A \) and \( B \) are the two arbitrary constants of integration corresponding to the homogenous part of Eq.(3). The \( D \) containing term results from the particular integral corresponding to the inhomogenous contribution of Eq.(3) obtained by the method of variation of parameters. Making use of the relations \( \text{[3]} \) we revert back to the original variables \( x \) and \( P(x) \) to obtain the general solution of Eq.(3) as

\[
P(x) = A x^{-\nu} I_\nu(\sqrt{b} x) + B x^{-\nu} K_\nu(\sqrt{b} x)
\]

\[
+ D x^{-\nu} \left[I_\nu(\sqrt{b} x) \int x^{\nu} K_\nu(\sqrt{b} x') \, dx'\right]
\]

\[
- K_\nu(\sqrt{b} x) \int x^{\nu} I_\nu(\sqrt{b} x') \, dx'.
\]

(10)
We now impose two natural boundary conditions on
the solution for the probability density function; (i) \( P(x) \)
vanishes for \(|x| \to \infty \) and (ii) \( P(x) \) remains finite for all
\( x \). The first condition ascribes the essential notion of a
probability function which demands \( A = 0 \). The second
one is a necessary requirement since the relevant modified
Bessel function is singular at \( x = 0 \). This singularity at
\( x = 0 \) can be removed by constructing a relation between
the constants \( B \) and \( D \) of the form
\[
B = (-1)^n D \frac{n!}{\sqrt{\pi}} x^{(2n+1)/2} \frac{1}{b^{(2n+3)/4}},
\]
(11)
such that all powers of \( \frac{1}{x} \) vanish identically from the
exact solution \([14]\) of the modified Bessel equation \([8]\)
which satisfy both the boundary conditions simultane-
ously provided \( \nu \) is an odd half integer, i.e.,
\[
\nu = n + (1/2), \quad n = 1, 2, \ldots
\]
so that from \([5]\) and the definition of \( \nu \) in \([8b]\) we have
(note that \( n = 0 \) is not physically allowed)
\[
\beta \frac{\epsilon_0}{2 \omega_0 c_2} = n, \quad n = 1, 2, \ldots
\]
(12)
The above Eq. (12) relates the dissipation constant \( \beta \) to
the correlation of fluctuations ( \( \epsilon_0, c_2 \) ) of the external
adiabatic noise as defined in Eq.(4). This is a fluctuation-
dissipation like relation for the open systems. The in-
tegers \( n \) characterize the distinct stable steady states
in terms of the positive definite probability distribution
function as given by (for \( x > 0 \))
\[
P_{n+\frac{1}{2}}(x) = (-1)^n D \frac{2^n n!}{\sqrt{\pi} b^{(n+2)/2}} \sum_{k=0}^{n} f_k \frac{e^{-\sqrt{\pi} x}}{x^{k+n+1}}
- \frac{D}{2 \sqrt{b}} \sum_{i=0}^{n} \sum_{k=0}^{n} \sum_{j=0}^{n-k} \left\{ (-1)^i + (-1)^{j-n} \right\}
\times f_k^n f_k^i \frac{(n-k)!}{j!} \left( \frac{1}{\sqrt{b}} \right)^{k-j+1}
\]
(13)
with
\[
f_k^n = \frac{(n+k)!}{2^n k^{k/2} k!(n-k)!}.
\]
(14)
The normalization constant \( D \) is related to the steady
state current \( J \) (positive, implying a flow from left to
right) through the first of the relations \([8]\) and is given by
\[
D = b \left[ 2 \text{Ein}(\sqrt{\Delta}) + 2^{n+1} n! \sum_{k=0}^{n} \sum_{j=0}^{k+n-1} \frac{(-1)^k}{2^n k!(n-k)! (n+k-j)} \right]^{-1}
\]
(15)
where \( \Delta \) is large but finite and the function \( \text{Ein}(x) \)
is defined \([15]\) as
\[
\text{Ein}(x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k!}.
\]
(16)
For \( x < 0 \), the corresponding probability distribution
function can be calculated from the symmetry of the differen-
tial equation \([6]\).
Expression (13) reduces to the following simple form for,
\( n = 1 \), which is depicted in Fig. (2) for several
values of third order noise strength \( c_2 \)
\[
P_{3/2}(x) = -2D b^{-3/2} e^{-\sqrt{\pi} x} \left( \frac{1}{x^2} + \frac{1}{\sqrt{b x^3}} \right)
- \frac{D}{b} \left( \frac{1}{x} - \frac{2}{bx^3} \right) ; \quad x > 0
\]
(17a)
\[
= -2D b^{-3/2} e^{\sqrt{\pi} x} \left( \frac{1}{x^2} - \frac{1}{\sqrt{b x^3}} \right)
+ \frac{D}{b} \left( \frac{1}{x} - \frac{2}{bx^3} \right) ; \quad x < 0
\]
(17b)
\[
= -2 D b^{-3/2} ; \quad x = 0
\]
(17c)
Thus the steady states are physically allowed only when
one can relate the dissipation \( \beta \) to the strength of sec-
tand third order external noise for these integers \( n \).
Our analysis suggests that such a steady state condition
\([12]\) for the open systems can be realized at least in the
specific issues as in the present instance. The third order
noise is an essential ingredient for the dynamic balance
we have referred to. It is thus also apparent why the ther-
mosdynamic open systems described by a Fokker-Planck
equation for fast fluctuations which include only second
order diffusion coefficient reaches a steady state with no
fluctuation-dissipation relation.

III. DYNAMICS OF BARRIER CROSSING
INDUCED BY ADIABATIC NOISE

Once the condition for attainment of the steady state is
realized it becomes possible to consider the situation such
that a particle in a force field, e.g., originally confined in
a potential well may escape under the influence of exter-
nal adiabatic noise by maintaining a steady state current
over the barrier. It is therefore pertinent to calculate the
noise-induced escape rate in the spirit of Kramers and
Smoluchowski and to elucidate the aspect of dependence
of escape rate on dissipation. The counterpart of the lat-
ter issue in the theory of fast fluctuation is the well-known
turn-over problem in Kramers’ theory \([10,11]\).

To proceed further we again make use of a Kramers’
type potential \( V(x) \) ( Fig. (1) ) under which the particle
moves, in the third order equation \([3]\) for the probability


distribution function. The popular flux-over-population method \[10,17\] is then employed. The calculation rests on the evaluation of two quantities; (i) the steady state current \( J \) over the barrier top, (located at \( x = 0 \)) that results if the particles are continuously fed into the domain of attraction (say, in the region of left well) and are subsequently and continuously removed in the neighboring domain of attraction. (ii) steady state population \( n_a \) in the initial domain of attraction, i.e., the left well. The rate is defined by

\[
K = J / n_a .
\]  

(18)

For linearization the potential \( V(x) \) at the bottom of the left well at \( x = -\Delta \) we approximate \( V(x) \cong \frac{1}{2} \omega_b^2 (x + \Delta)^2 \), where \( \omega_b \) refers to the frequency at the bottom of the left well. The above approximation to left well and \( J = 0 \) condition reduce the third order equation of motion (Eq.\([3]\)) for probability density \( P_b(x) \) near the bottom of the well to the following form.

\[
(x + \Delta) \frac{d^2 P_b}{dx^2} + \gamma' \frac{dP_b}{dx} - b (x + \Delta) P_b = 0 .
\]  

(19)

The above equation is valid near the bottom of the left well \( (x \approx -\Delta) \). Here \( \gamma' \) is defined as

\[
\gamma' = \frac{2\omega_b^2 - a}{\omega_b^2}
\]

(20)

and \( a \) by Eq.\([6]\). Eq.\([13]\) can be written as

\[
z^2 \frac{d^2 W}{dz^2} + z \frac{dW}{dz} - [\nu'^2 + b z^2] W = 0 ,
\]

(21)

making use of the following sets of \( \nu \) transformations,

\[
y(z) = z^{\frac{1}{2}(\sigma+1)} W(z) , \quad \nu' = \frac{\sigma - 1}{2} ,
\]

(22a)

\[
2 - \gamma' = \sigma , \quad z = x + \Delta , \quad y = z P_b .
\]

(22b)

The solutions of Eq.\([21]\) are again the modified Bessel functions. Reverting back to original variables, the general solution for the steady state probability distribution near the bottom of the left well is given by

\[
P_b(x) = A' (x + \Delta)\nu' I_{\nu'}[\sqrt{b} (x + \Delta)] + B' (x + \Delta)^\nu' K_{\nu'}[\sqrt{b} (x + \Delta)] .
\]

(23)

\( A' \) and \( B' \) are the two arbitrary constants of integration. The solution \( P_b(x) \) must satisfy the following boundary conditions ; (i) \( P_b(x) \) must vanish at \( x \rightarrow \infty \) and (ii) \( P_b(-\Delta) = P_t^{n+\frac{1}{2}}(-\Delta) \), where the stationary probability \( P_t^{n+\frac{1}{2}}(-\Delta) \) corresponds to the vanishing current \( J = 0 \) along \( x \) pertaining to the homogenous version of Eq.\([7]\).

As usual, \( P_t^{n+\frac{1}{2}}(x) \) must also satisfy the boundary condition that for \( |x| \rightarrow \infty \), \( P_t^{n+\frac{1}{2}}(x) \) vanishes. The first condition leads to \( A' = 0 \) and the second one gives the value of \( B' \) in terms of \( B \) of Eq.\([10]\) and therefore

\[
P_b(x) = B' (x + \Delta)^\nu' K_{\nu'}[\sqrt{b} (x + \Delta)] .
\]  

(24)

where

\[
B' = \sqrt{\frac{\pi}{2}} \frac{b^{\nu'/2} \Gamma(\nu')}{\Gamma(2\nu' - 1)} \sum_{k=0}^{n} (-1)^{k+n+1} f_k^n e^{\sqrt{b} \Delta} \Delta^{k+n+1} .
\]

(25)

Therefore \( P_b(x) \) in Eq.\([24]\) may be expressed as

\[
P_b(x) = B \sqrt{\frac{\pi}{2}} \frac{b^{\nu'/2} \Gamma(\nu')}{\Gamma(2\nu' - 1)} \sum_{k=0}^{n} (-1)^{k+n+1} f_k^n e^{\sqrt{b} \Delta} \Delta^{k+n+1} (x + \Delta)^\nu' \times K_{\nu'}[\sqrt{b} (x + \Delta)] .
\]

(26)

The above distribution which is valid near the bottom of the left well may be used to calculate the population inside the left well as,

\[
n_a = 2 \int_{-\Delta}^{0} P_b(x) \, dx .
\]

(27)

Due to the presence of \( K_{\nu'}(x) \) the probability \( P_b(x) \) is a rapidly decreasing function. We may extend the above integration limit to infinity. This yields \( (\text{using Eq.}\([24]\) )

\[
n_a = B' \sqrt{\frac{\pi}{2}} \frac{\Gamma(\nu' + \frac{1}{2})}{(\sqrt{b})^{\nu'+1}} .
\]

(28)

Using the relations \([25]\) and \([28]\) we finally have

\[
n_a = \sqrt{\frac{\pi}{2}} \frac{B \Gamma(\nu' + \frac{1}{2})}{\Gamma(\nu')} \sum_{k=0}^{n} (-1)^{k+n+1} f_k^n e^{\sqrt{b} \Delta} \Delta^{k+n+1} b^{-3/4} .
\]

(29)

Having determined the population of the left well, \( n_a \) and the steady state current, \( J \) ( from first of the relations in \([31]\) and \([13]\) ) over the barrier we are now in a position to calculate the escape rate \( K_{n+\frac{1}{2}} \) \( (= J/n_a) \). We quote the final result:

\[
K_{n+\frac{1}{2}} = \frac{\epsilon_0 \omega_0^2}{\sqrt{\pi} b^3} \frac{1}{n!} \frac{\Gamma(n\frac{2}{\omega_0^2} - \frac{1}{2})}{\Gamma(n\frac{2}{\omega_0^2} - \frac{1}{2})} \left\{ \frac{e^{-\sqrt{b} \Delta}}{\sum_{k=0}^{n} (-1)^{k+n+1} \frac{(n+k+1)!}{k!(n-k)!} \Delta^{k+n+1} b^{-3/4}} \right\} .
\]

(30)
\( \Delta \) is approximately given by \( \Delta = (2E_0/\omega_0^2)^{1/2} \).

The above expression can be made more transparent by demonstrating a representative transition rate, say, by virtue of linearization of for some damping value \( \beta \rightarrow 0 \). This behavior is somewhat reminiscent of Kramers’ theory \([16]\), where it was noted earlier that two limiting behavior implies a maximal rate at a force field, simultaneously subjected to adiabatic fluctuations due to external non-Gaussian noise leads to a third order noise strength. With increasing friction, the rate undergoes a turnover from an increasing behavior at low friction to an inverse behavior in the high friction limit.

\[ K_{3/2} = \frac{1}{8 \sqrt{\pi}} \frac{c_{01}}{c_2^2} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})} (\Delta \beta)^2 \exp \left( -\frac{\beta \Delta}{\sqrt{c_2}} \right) \quad (31) \]

The above expression is analogous to Kramers’ formula for the rate of escape from a potential well over a finite barrier of height \( E_0 \) under the influence of an external nonthermal adiabatic noise pertaining to the timescale \( \beta \).

The escape rate expressions derived above suggest that the rate approaches zero both for \( \beta \rightarrow \infty \) and \( \beta \rightarrow 0 \). This behavior is somewhat reminiscent of Kramers’ theory \([14]\), where it was noted earlier that two limiting behavior implies a maximal rate at some damping value \( \beta \). The rate therefore undergoes a turnover in a form of a bell-shaped curve. In Fig.(3) we plot a representative variation of the escape rate versus dissipation \( \beta \) for different third order noise strength. With increasing friction, the rate undergoes a turnover from an increasing behavior at low friction to an inverse behavior in the high friction limit.

IV. CONCLUSION

In conclusion, we consider the motion of particle in a force field, simultaneously subjected to adiabatic fluctuations of external origin. The equation of motion for probability distribution function includes a third order noise term. We show that although the system is thermodynamically open, the specific interplay of the characteristic dissipation of the system and the correlation of fluctuations due to external non-Gaussian noise leads to distinct steady states for the open system. We calculate the external adiabatic noise-induced rate of escape of a particle confined in a well. A typical variation of the escape rate as a function of dissipation which is reminiscent of Kramers’ turn-over problem, has been demonstrated. In view of several experimental investigations on external noise induced transitions in the recent past \([2, 3]\), the study of thermodynamically open systems has been specially relevant in both physical and chemical sciences. Although the driving noise processes in these cases are fast, suitable extension to adiabatic noise limit may throw new light on the present issue. The population inversion in a two-level system by an adiabatically varying stochastic electromagnetic field, as considered earlier in Refs. \([3, 19]\) may also be worthwhile candidate for further studies in this context.

ACKNOWLEDGMENTS

SKB is indebted to Council of Scientific and Industrial Research (C.S.I.R.), Government of India for financial support. We express our sincerest thanks to Prof. J. K. Bhattacharjee for various discussions. DSR thanks Prof. N. Satyamurthy for his kind invitation for the present contribution.

[1] W. H. Louisell, Quantum Statistical Properties of radiation (John Wiley, New York) 1973.
[2] K. Lindenberg and B. J. West, The Nonequilibrium Statistical Mechanics of Open and Closed Systems (VCH Publishers Inc., New York) 1990.
[3] R. Roy, A. W. Yu and S. Zhu, Phys. Rev. Lett. 55 (1985) 2794.
[4] L. Arnold, W. Horsthemke and R. Lefever, Z. Phys. B 29 (1978) 367.
[5] T. Kawakubo, S. Kabashima and M. Itsumi, J. Phys. Soc. Japan 41 (1976) 699.
[6] J. M. Porrà, K. G. Wang and J. Masoliver, Phys. Rev. E 53 (1996) 5872.
[7] K. G. Wang and J. Masoliver, Physica A 231 (1996) 615.
[8] S. K. Banik, J. Ray Chaudhuri and D. S. Ray, J. Phys. A 31 (1998) 7301.
[9] C. W. Gardiner, Handbook of Stochastic Methods (Springer-Verlag, Berlin) (1983) pp.299.
[10] H. Risken and K. Vogel in Far from Equilibrium Phase Transition (Lecture Notes in Physics, Springer-Verlag, Berlin) edited by L. Garrido, 319 (1988) 237.
[11] N. G. van Kampen, Phys. Rep. 24 (1976) 171.
[12] M. D. Crisp, Phys. Rev. A 8 (1973) 2128.
[13] S. K. Banik and D. S. Ray, J. Phys. A 31 (1998) 3937.
[14] C. M. Bender and S. A. Orszag, Advanced Mathematical Methods for Scientists and Engineers (McGraw-Hill Book Company, New York) 1978.
[15] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables (Dover, New York) 1965.
[16] H. A. Kramers, Physica 7 (1940) 284.
[17] P. Hänggi, P. Talkner and M. Borkovec, Rev. Mod. Phys. 62 (1990) 251.
[18] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products (Academic Press, New York) 1980.
[19] D. Grischkowsky, E. Courtens and J. A. Armstrong, Phys. Rev. Lett. 31 (1973) 422.

Figure Captions

1. Fig.(1) : A schematic plot of the Kramers’ type potential \( V(x) \) used for the calculation of barrier crossing dynamics.
2. Fig.(2) : The normalized probability distribution function \( P_{3/2}(x) \) is plotted as a function of \( x \) for
various values of the third order noise strength $c_2$ ($\beta = 1.0$ and $\Delta = 2.5$).

3. **Fig.(3)**: Escape rate $K_{3/2}$ is plotted as a function of the characteristic dissipation $\beta$ of the system for various values of $c_2$ ($c_{01} = 7.0, \omega_b = 0.80$ and $\Delta = 2.5$).
