Multi-bump solutions for a class of quasilinear problems involving variable exponents

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Abstract We establish the existence of multi-bump solutions for the following class of quasilinear problems

$$-\Delta_{p(x)} u + (\lambda V(x) + Z(x))u^{p(x)-1} = f(x, u) \text{ in } \mathbb{R}^N, \ u \geq 0 \text{ in } \mathbb{R}^N,$$

where the nonlinearity $f: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function having a subcritical growth and potentials $V, Z: \mathbb{R}^N \to \mathbb{R}$ are continuous functions verifying some hypotheses. The main tool used is the variational method.

Keywords Variational Methods · Positive solutions · Asymptotic behavior of solutions · $p(x)$-Laplacian

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1 Introduction

In this paper, we consider the existence and multiplicity of solutions for the following class of problems

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Here, $\lambda > 0$ is a parameter, $p: \mathbb{R}^N \to \mathbb{R}$ is a Lipschitz function, $V, Z: \mathbb{R}^N \to \mathbb{R}$ are continuous functions with $V \geq 0$, and $f: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous having a subcritical growth. Furthermore, we take into account the following set of hypotheses:

1. $1 < p_- = \inf_{\mathbb{R}^N} p \leq p_+ = \sup_{\mathbb{R}^N} p < N$. 
2. $\Omega = \text{int } V^{-1}(0) \neq \emptyset$ and bounded, $\overline{\Omega} = V^{-1}(0)$ and $\Omega$ can be decomposed in $k$ connected components $\Omega_1, \ldots, \Omega_k$ with $\text{dist}(\Omega_i, \Omega_j) > 0$, $i \neq j$.
3. There exists $M > 0$ such that $\lambda V(x) + Z(x) \geq M$, $\forall x \in \mathbb{R}^N, \lambda \geq 1$.
4. There exists $K > 0$ such that $|Z(x)| \leq K$, $\forall x \in \mathbb{R}^N$.

$(f_1)$

$$\limsup_{|t| \to \infty} \frac{|f(x, t)|}{|t|^{q(x)-1}} < \infty, \text{ uniformly in } x \in \mathbb{R}^N,$$

where $q: \mathbb{R}^N \to \mathbb{R}$ is continuous with $p_+ < q_-$ and $q \ll p^* = \frac{Np}{N-p}$. Here, the notation $q \ll p^*$ means that $\inf_{\mathbb{R}^N} (p^* - q) > 0$.

$(f_2)$ $f(x, t) = o(|t|^{p+1}), t \to 0$, uniformly in $x \in \mathbb{R}^N$.

$(f_3)$ There exists $\theta > p_+$ such that

$$0 < \theta F(x, t) \leq f(x, t)t, \forall x \in \mathbb{R}^N, t > 0,$$

where $F(x, t) = \int_0^t f(x, s) \, ds$.

$(f_4)$ $\frac{f(x, t)}{t^{p+1}}$ is strictly increasing in $t \in (0, \infty)$, for each $x \in \mathbb{R}^N$.

$(f_5)$ $\forall a, b \in \mathbb{R}, a < b$, $\sup_{x \in \mathbb{R}^N} \sup_{t \in [a, b]} |f(x, t)| < \infty$.

A typical example of nonlinearity verifying $(f_1)$ -- $(f_5)$ is

$$f(x, t) = |t|^{q(x)-2}t, \forall x \in \mathbb{R}^N \text{ and } \forall t \in \mathbb{R},$$

where $p_+ < q_-$ and $q \ll p^*$.

Partial differential equations involving the $p(x)$-Laplacian arise, for instance, as a mathematical model for problems involving electrorheological fluids and image restorations, see [1,2,11–13,29]. This explains the intense research on this subject in the last decades. A lot of works, mainly treating nonlinearities with subcritical growth, are available (see [4–9,16–18,20–24,28] for interesting works). Nevertheless, to the best of the author’s knowledge, this is the first work dealing with multi-bump solutions for this class of problems.
The motivation to investigate problem $(P_\lambda)$ in the setting of variable exponents has been the papers \cite{3} and \cite{15}. In \cite{15}, inspired by del Pino and Felmer \cite{14} and Séré \cite{30}, the authors considered $(P_\lambda)$ for $p = 2$ and $f(u) = u^q$, $q \in (1, \frac{N+2}{N-2})$ if $N \geq 3$; $q \in (1, \infty)$ if $N = 1, 2$. The authors showed that $(P_\lambda)$ has at least $2^k - 1$ solutions $u_\lambda$ for large values of $\lambda$. More precisely, one solution for each non-empty subset $\Upsilon$ of $\{1, \ldots, k\}$. Moreover, fixed $\Upsilon \subset \{1, \ldots, k\}$, it was proved that, for any sequence $\lambda_n \to \infty$, we can extract a subsequence $(\lambda_{n_k})$ such that $(u_{\lambda_{n_k}})$ converges strongly in $H^1(\mathbb{R}^N)$ to a function $u$, which satisfies $u = 0$ outside $\Omega_\Upsilon = \bigcup_{j \in \Upsilon} \Omega_j$ and $u|_{\Omega_j}$, $j \in \Upsilon$, is a least energy solution for

\[
\begin{aligned}
-\Delta u + Z(x)u &= u^q, \quad \text{in } \Omega_j, \\
u &\in H^1_0(\Omega_j), \ u > 0, \quad \text{in } \Omega_j.
\end{aligned}
\]

In \cite{3}, employing some different arguments than those used in \cite{15}, Alves extended the results described above to the $p$-Laplacian operator, assuming that in $(P_\lambda)$ the nonlinearity $f$ possesses a subcritical growth and $2 \leq p < N$. In particular, fixed $\Upsilon \subset \{1, \ldots, k\}$, for any sequence $\lambda_n \to \infty$, we can extract a subsequence $(\lambda_{n_k})$ such that $(u_{\lambda_{n_k}})$ converges strongly in $W^{1,p}(\mathbb{R}^N)$ to a function $u$, which satisfies $u = 0$ outside $\Omega_\Upsilon$ and $u|_{\Omega_j}$, $j \in \Upsilon$, is a least energy solution for

\[
\begin{aligned}
-\Delta_p u + Z(x)u &= f(u), \quad \text{in } \Omega_j, \\
u &\in W^{1,p}_0(\Omega_j), \ u > 0, \quad \text{in } \Omega_j.
\end{aligned}
\]

In the present paper, we extend the results found in \cite{3} to the $p(x)$-Laplacian operator. However, we would like to emphasize that in a lot of estimates, we have used different arguments from that found in \cite{3}. The main difference is related to the fact that for equations involving the $p(x)$-Laplacian operator it is not clear that Moser’s iteration method is a good tool to get the estimates for the $L^\infty$-norm. Here, we adapt some ideas explored in \cite{18} and \cite{25} to get these estimates. For more details see Sect. 5.

Since we intend to find nonnegative solutions, throughout this paper, we replace $f$ by $f^+: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

\[
f^+(x, t) = \begin{cases} f(x, t), & \text{if } t > 0 \\ 0, & \text{if } t \leq 0. \end{cases}
\]

Nevertheless, for the sake of simplicity, we still write $f$ instead of $f^+$.

The main theorem in this paper is the following:

**Theorem 1.1** Assume that $(H_1)$ – $(H_3)$ and $(f_1)$ – $(f_3)$ hold. Then, there exist $\lambda_0 > 0$ with the following property: for any non-empty subset $\Upsilon$ of $\{1, 2, \ldots, k\}$ and $\lambda \geq \lambda_0$, problem $(P_\lambda)$ has a solution $u_\lambda$. Moreover, if we fix the subset $\Upsilon$, then for any sequence $\lambda_n \to \infty$, we can extract a subsequence $(\lambda_{n_k})$ such that $(u_{\lambda_{n_k}})$ converges strongly in $W^{1,p(x)}(\mathbb{R}^N)$ to a function $u$, which satisfies $u = 0$ outside $\Omega_\Upsilon = \bigcup_{j \in \Upsilon} \Omega_j$ and $u|_{\Omega_j}$, $j \in \Upsilon$, is a least energy solution for

\[
\begin{aligned}
-\Delta_{p(x)} u + Z(x)u &= f(x, u), \quad \text{in } \Omega_j, \\
u &\in W^{1,p(x)}_0(\Omega_j), \ u \geq 0, \quad \text{in } \Omega_j.
\end{aligned}
\]

**Notations:** The following notations will be used in the present work:

- $C$ and $C_i$ will denote generic positive constant, which may vary from line to line;
- In all the integrals, we omit the symbol $dx$. 

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• If \( u \) is a measurable function, we denote \( u^+ \) and \( u^- \) its positive and negative part, i.e., 
\[ u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}. \]

• If \( u, v \) are measurable functions, \( u_- = \text{ess inf}_{\mathbb{R}^N} u, u_+ = \text{ess sup}_{\mathbb{R}^N} u \) and the notation \( u \ll v \) means that \( \text{ess inf}_{\mathbb{R}^N} (v - u) > 0 \). Moreover, we will denote by \( u^* \) the function
\[
u^*(x) = \begin{cases} \frac{Nu(x)}{N-u(x)}, & \text{if } u(x) < N, \\ \infty, & \text{if } u(x) \geq N. \end{cases}
\]

2 Preliminaries on variable exponents Lebesgue and Sobolev spaces

In this section, we recall some results on variable exponents Lebesgue and Sobolev spaces found in [8,19,21] and their references.

Let \( h \in L^\infty(\mathbb{R}^N) \) with \( h^- = \text{ess inf}_{\mathbb{R}^N} h \geq 1. \) The variable exponent Lebesgue space \( L^h(\mathbb{R}^N) \) is defined by
\[
L^h(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R}; u \text{ is measurable and } \int_{\mathbb{R}^N} |u|^h < \infty \right\},
\]
endowed with the norm
\[
|u|_h = \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} \left( \frac{|u|^h}{\lambda} \right) \leq 1 \right\}.
\]
The variable exponent Sobolev space is defined by
\[
W^{1,h}(\mathbb{R}^N) = \left\{ u \in L^h(\mathbb{R}^N); |\nabla u| \in L^h(\mathbb{R}^N) \right\},
\]
with the norm
\[
\|u\|_{1,h} = \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} \left( \frac{|\nabla u|^h}{\lambda} + |u|^h \right) \leq 1 \right\}.
\]
If \( h^- > 1 \), the spaces \( L^h(\mathbb{R}^N) \) and \( W^{1,h}(\mathbb{R}^N) \) are separable and reflexive with these norms.

We are mainly interested in subspaces of \( W^{1,h}(\mathbb{R}^N) \) given by
\[
E_W = \left\{ u \in W^{1,h}(\mathbb{R}^N); \int_{\mathbb{R}^N} W(x)|u|^h < \infty \right\},
\]
where \( W \in C(\mathbb{R}^N) \) is such that \( W^- > 0. \) Endowing \( E_W \) with the norm
\[
\|u\|_W = \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} \left( \frac{|\nabla u|^h}{\lambda} + W(x) \frac{|u|^h}{\lambda} \right) \leq 1 \right\},
\]
$E_W$ is a Banach space. Moreover, it is easy to see that $E_W \hookrightarrow W^{1,h(x)}(\mathbb{R}^N)$ continuously. In addition, we can show that $E_W$ is reflexive. For the reader’s convenience, we recall some basic results.

**Proposition 2.1** The functional $\varphi : E_W \to \mathbb{R}$ defined by

$$
\varphi(u) = \int_{\mathbb{R}^N} \left( |\nabla u|^{h(x)} + W(x) |u|^{h(x)} \right),
$$

(2.1)

has the following properties:

(i) If $\|u\|_W \geq 1$, then $\|u\|^{\frac{h}{h^+}}_W \leq \varphi(u) \leq \|u\|^{\frac{h}{h^-}}_W$.

(ii) If $\|u\|_W \leq 1$, then $\|u\|^{\frac{h}{h^+}}_W \leq \varphi(u) \leq \|u\|^{\frac{h}{h^-}}_W$.

In particular, for a sequence $(u_n)$ in $E_W$,

$$
\|u_n\|_W \to 0 \iff \varphi(u_n) \to 0, \text{ and},
$$

$(u_n)$ is bounded in $E_W \iff \varphi(u_n)$ is bounded in $\mathbb{R}$.

**Remark 2.2** For the functional $\varphi_{h(x)} : L^{h(x)}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$
\varphi_{h(x)}(u) = \int_{\mathbb{R}^N} |u|^{h(x)},
$$

an analogous conclusion to that of Proposition 2.1 also holds.

**Proposition 2.3** Let $m \in L^\infty(\mathbb{R}^N)$ with $0 < m_- \leq m(x) \leq h(x)$ for a.e. $x \in \mathbb{R}^N$. If $u \in L^{h(x)}(\mathbb{R}^N)$, then $|u|^{m(x)} \in L^{\frac{h(x)}{m(x)}}(\mathbb{R}^N)$ and

$$
\left| |u|^{m(x)} \right|^{\frac{h(x)}{m(x)}} \leq \max \left\{ |u|^{m_-}_{h(x)}, |u|^{m_+}_{h(x)} \right\} \leq |u|^{m_-}_{h(x)} + |u|^{m_+}_{h(x)}.
$$

Related to the Lebesgue space $L^{h(x)}(\mathbb{R}^N)$, we have the following generalized Hölder’s inequality.

**Proposition 2.4** (Hölder’s inequality) If $h_- > 1$, let $h' : \mathbb{R}^N \to \mathbb{R}$ such that

$$
\frac{1}{h(x)} + \frac{1}{h'(x)} = 1 \quad \text{for a.e. } x \in \mathbb{R}^N.
$$

Then, for any $u \in L^{h(x)}(\mathbb{R}^N)$ and $v \in L^{h'(x)}(\mathbb{R}^N)$,

$$
\int_{\mathbb{R}^N} |u v| \, dx \leq \left( \frac{1}{h_-} + \frac{1}{h_-'} \right) |u|_{h(x)} |v|_{h'(x)}.
$$

We can define variable exponent Lebesgue spaces with vector values. We say $u = (u_1, \ldots, u_L) : \mathbb{R}^N \to \mathbb{R}^L \in L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)$ if, and only if, $u_i \in L^{h(x)}(\mathbb{R}^N)$, for $i = 1, \ldots, L$. On $L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)$, we consider the norm $|u|_{L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)} = \sum_{i=1}^L |u_i|_{h(x)}$.

We state below lemmas of Brezis–Lieb type. The proof of the two first results follows the same arguments explored at [26], while the proof of the latter can be found at [8].

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Proposition 2.5 (Brezis–Lieb lemma, first version) Let \((u_n)\) be a bounded sequence in 
\(L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)\) such that \(u_n(x) \rightarrow u(x)\) for a.e. \(x \in \mathbb{R}^N\). Then, \(u \in L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)\) and
\[
\int_{\mathbb{R}^N} |u_n|^{h(x)} - |u_n - u|^{h(x)} - |u|^{h(x)} \, dx = o_n(1). \tag{2.2}
\]

Proposition 2.6 (Brezis–Lieb lemma, second version) Let \((u_n)\) be a bounded sequence in 
\(L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)\) with \(h > 1\) and \(u_n(x) \rightarrow u(x)\) for a.e. \(x \in \mathbb{R}^N\). Then
\[
u_n \rightharpoonup u \quad \text{in} \quad L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L).
\]

Proposition 2.7 (Brezis–Lieb lemma, third version) Let \((u_n)\) be a bounded sequence in 
\(L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)\) with \(h > 1\) and \(u_n(x) \rightarrow u(x)\) for a.e. \(x \in \mathbb{R}^N\). Then
\[
\int_{\mathbb{R}^N} |u_n|^{h(x)-2} u_n - |u_n - u|^{h(x)-2} (u_n - u) - |u|^{h(x)-2} u \, dx = o_n(1), \tag{2.3}
\]

To finish this section, we notice that for any open subset \(\Omega \subset \mathbb{R}^N\), we can define in the
same way the spaces \(L^{h(x)}(\Omega)\) and \(W^{1,h(x)}(\Omega)\). Moreover, all the above propositions have
analogous versions for these spaces and, besides, we have the following embedding theorem
of Sobolev’s type.

Proposition 2.8 ([21, Theorems 1.1, 1.3]) Let \(\Omega \subset \mathbb{R}^N\) an open domain with the cone
property, \(h: \overline{\Omega} \rightarrow \mathbb{R}\) satisfying \(1 < h_- \leq h_+ < N\) and \(m \in L^\infty_+(\Omega)\).

(i) If \(h\) is Lipschitz continuous and \(h \leq m \leq h^*\), the embedding \(W^{1,h(x)}(\Omega) \hookrightarrow L^{m(x)}(\Omega)\)
is continuous;
(ii) If \(\Omega\) is bounded, \(h\) is continuous and \(m \ll h^*\), the embedding \(W^{1,h(x)}(\Omega) \hookrightarrow L^{m(x)}(\Omega)\)
is compact.

3 An auxiliary problem

In this section, we work with an auxiliary problem adapting the ideas explored in del Pino
and Felmer [14] (see also [3]).

We start noting that the energy functional \(I_\lambda: E_\lambda \rightarrow \mathbb{R}\) associated with \((P_\lambda)\) is given by
\[
I_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left( |\nabla u|^{p(x)} + (\lambda V(x) + Z(x)) |u|^{p(x)} \right) - \int_{\mathbb{R}^N} F(x, u),
\]
where \(E_\lambda = (E, \| \cdot \|_\lambda)\) with
\[
E = \left\{ u \in W^{1,p(x)}(\mathbb{R}^N); \int_{\mathbb{R}^N} V(x)|u|^{p(x)} < \infty \right\},
\]
and
\[
\|u\|_\lambda = \inf \{ \sigma > 0; \rho_\lambda \left( \frac{u}{\sigma} \right) \leq 1 \},
\]
being
\[ q_\lambda(u) = \int_{\mathbb{R}^N} \left( |\nabla u|^{p(x)} + (\lambda V(x) + Z(x))|u|^{p(x)} \right). \]

Thus, \( E_\lambda \hookrightarrow W^{1,p(x)}(\mathbb{R}^N) \) continuously for \( \lambda \geq 1 \) and \( E_\lambda \) is compactly embedded in \( L^{h(x)}_{loc} (\mathbb{R}^N) \), for all \( 1 \leq h < p^* \). In addition, we can show that \( E_\lambda \) is a reflexive space. Also, being \( \mathcal{O} \subset \mathbb{R}^N \) an open set, from the relation
\[ q_{\lambda, \mathcal{O}}(u) = \int_{\mathcal{O}} \left( |\nabla u|^{p(x)} + (\lambda V(x) + Z(x))|u|^{p(x)} \right) \geq M \int_{\mathcal{O}} |u|^{p(x)} = M q_{p(x), \mathcal{O}}(u), \]
for all \( u \in E_\lambda \) with \( \lambda \geq 1 \), writing \( M = (1 - \delta)^{-1} \nu \), for some \( 0 < \delta < 1 \) and \( \nu > 0 \), we derive
\[ q_{\lambda, \mathcal{O}}(u) - \nu q_{p(x), \mathcal{O}}(u) \geq \delta q_{\lambda, \mathcal{O}}(u), \quad \forall u \in E_\lambda, \lambda \geq 1. \] (3.2)

**Remark 3.1** From the above commentaries, in this work the parameter \( \lambda \) will be always bigger than or equal to 1.

We recall that for any \( \epsilon > 0 \), the hypotheses \((f_1), (f_2)\) and \((f_3)\) yield
\[ f(x, t) \leq \epsilon |t|^{p(x) - 1} + C_x |t|^{q(x) - 1}, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}, \] (3.3)
and, consequently,
\[ F(x, t) \leq \epsilon |t|^{p(x)} + C_x |t|^{q(x)}, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}, \] (3.4)
where \( C_x \) depends on \( \epsilon \). Moreover, for each \( \nu > 0 \) fixed, the assumptions \((f_2)\) and \((f_3)\) allow us considering the function \( a : \mathbb{R}^N \to \mathbb{R} \) given by
\[ a(x) = \min \left\{ a > 0 ; \frac{f(x, a)}{a^{p(x) - 1}} = \nu \right\}. \] (3.5)

From \((f_2)\), it follows that
\[ 0 < a_- = \inf_{x \in \mathbb{R}^N} a(x). \] (3.6)

Using the function \( a(x) \), we set the function \( \tilde{f} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) given by
\[ \tilde{f}(x, t) = \begin{cases} f(x, t), & t \leq a(x) \\ \nu t^{p(x) - 1}, & t \geq a(x) \end{cases}, \]
which fulfills the inequality
\[ \tilde{f}(x, t) \leq \nu |t|^{p(x) - 1}, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}. \] (3.7)

Thus
\[ \tilde{f}(x, t) \leq \nu |t|^{p(x)}, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}, \] (3.8)
and
\[ \tilde{F}(x, t) \leq \frac{\nu}{p(x)} |t|^{p(x)}, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R}, \] (3.9)
where \( \tilde{F}(x, t) = \int_0^t \tilde{f}(x, s) \, ds \).
Now, once that \( \Omega = \text{int } V^{-1}(0) \) is formed by \( k \) connected components \( \Omega_1, \ldots, \Omega_k \) with 
\( \text{dist}(\Omega_i, \Omega_j) > 0, \ i \neq j, \) then for each \( j \in \{1, \ldots, k\} \), we are able to fix a smooth bounded domain \( \Omega'_j \) such that
\[
\overline{\Omega'_j} \subset \Omega'_j \quad \text{and} \quad \overline{\Omega'_i} \cap \overline{\Omega'_j} = \emptyset, \quad \text{for } i \neq j. \tag{3.10}
\]

From now on, we fix a non-empty subset \( \Upsilon \subset \{1, \ldots, k\} \) and
\[
\Omega_{\Upsilon} = \bigcup_{j \in \Upsilon} \Omega_j, \quad \Omega'_{\Upsilon} = \bigcup_{j \in \Upsilon} \Omega'_j, \quad \chi_{\Upsilon} = \begin{cases} 1, & \text{if } x \in \Omega'_{\Upsilon} \\ 0, & \text{if } x \notin \Omega'_{\Upsilon}. \end{cases}
\]

Using the above notations, we set the functions
\[
g(x, t) = \chi_{\Upsilon}(x) f(x, t) + (1 - \chi_{\Upsilon}(x)) \tilde{f}(x, t), \ (x, t) \in \mathbb{R}^N \times \mathbb{R}
\]
and
\[
G(x, t) = \int_0^t g(x, s) \, ds, \ (x, t) \in \mathbb{R}^N \times \mathbb{R},
\]
and the auxiliary problem
\[
(A_\lambda) \begin{cases} -\Delta_{p(x)} u + (\lambda V(x) + Z(x)) |u|^{p(x)-2} u = g(x, u), & \text{in } \mathbb{R}^N, \\ u \in W^{1, p(x)}(\mathbb{R}^N). \end{cases}
\]

The problem \((A_\lambda)\) is related to \((P_\lambda)\) in the sense that, if \( u_\lambda \) is a solution for \((A_\lambda)\) verifying
\[
uu(x) \leq a(x), \ \forall x \in \mathbb{R}^N \setminus \Omega'_{\Upsilon},
\]
then it is a solution for \((P_\lambda)\).

In comparison with \((P_\lambda)\), problem \((A_\lambda)\) has the advantage that the energy functional associated with \((A_\lambda)\), namely, \( \phi_\lambda : E_\lambda \to \mathbb{R} \) given by
\[
\phi_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left( |\nabla u|^{p(x)} + (\lambda V(x) + Z(x)) |u|^{p(x)} \right) - \int_{\mathbb{R}^N} G(x, u),
\]
satisfies the \((PS)\) condition, whereas \( I_\lambda \) does not necessarily satisfy this condition. In this way, the mountain pass level (see Theorem 3.6) is a critical value for \( \phi_\lambda \).

**Proposition 3.2** \( \phi_\lambda \) satisfies the mountain pass geometry.

**Proof** From (3.4) and (3.9),
\[
\phi_\lambda(u) \geq \frac{1}{p_+} \mathcal{Q}_{\lambda}(u) - \epsilon \int_{\mathbb{R}^N} |u|^{p(x)} - C_\epsilon \int_{\mathbb{R}^N} |u|^{q(x)} - \frac{\nu}{p_-} \int_{\mathbb{R}^N} |u|^{p(x)},
\]
for \( \epsilon > 0 \) and \( C_\epsilon > 0 \) be a constant depending on \( \epsilon \). By (3.1), fixing \( \epsilon < \frac{M}{p_+} \) and \( \nu < p_- M \left( \frac{1}{p_+} - \frac{\epsilon}{M} \right) \) and assuming \( \|u\|_{\lambda} < \min \{1, 1/C_q\} \), where \( |v|_{q(x)} \leq C_q \|v\|_{\lambda}, \ \forall v \in E_\lambda, \) we derive from Proposition 2.1
\[
\phi_\lambda(u) \geq \alpha \|u\|_{\lambda}^{p_+} - C \|u\|_{\lambda}^{q_-},
\]
where \( \alpha = \left( \frac{1}{p^+} - \frac{q}{p^M} \right) - \frac{v}{p^- M} > 0 \). Once \( p^+ < q^- \), the first part of the mountain pass geometry is satisfied. Now, fixing \( v \in C_0^\infty(\Omega_T) \), we have for \( t \geq 0 \)

\[
\phi_\lambda(tv) = \int_{\mathbb{R}^N} \frac{t^{p(x)}}{p(x)} \left( |\nabla v|^{p(x)} + Z(x)|v|^{p(x)} \right) - \int_{\mathbb{R}^N} F(x, tv).
\]

If \( t > 1 \), by (3.8) and (3.9),

\[
\phi_\lambda(tv) \leq \frac{t^{p^+}}{p^-} \int_{\mathbb{R}^N} \left( |\nabla v|^{p(x)} + Z(x)|v|^{p(x)} \right) - C_1 t^\theta \int_{\mathbb{R}^N} |v|^\theta - C_2,
\]

and so,

\[
\phi_\lambda(tv) \to -\infty \text{ as } t \to +\infty.
\]

The last limit implies that \( \phi_\lambda \) verifies the second geometry of the mountain pass. \( \square \)

**Proposition 3.3** All \((PS)_d\) sequences for \( \phi_\lambda \) are bounded in \( E_\lambda \).

**Proof** Let \((u_n)\) be a \((PS)_d\) sequence for \( \phi_\lambda \). So, there is \( n_0 \in \mathbb{N} \) such that

\[
\phi_\lambda(u_n) - \frac{1}{\theta} \phi'_\lambda(u_n) u_n \leq d + 1 + \|u_n\|_\lambda, \text{ for } n \geq n_0.
\]

On the other hand, by (3.8) and (3.9)

\[
\tilde{F}(x, t) - \frac{1}{\theta} \tilde{f}(x, t)t \leq \left( \frac{1}{p(x)} - \frac{1}{\theta} \right) v|t|^{p(x)}, \quad \forall x \in \mathbb{R}^N, \quad t \in \mathbb{R},
\]

which together with (3.2) gives

\[
\phi_\lambda(u_n) - \frac{1}{\theta} \phi'_\lambda(u_n) u_n \geq \left( \frac{1}{p^+} - \frac{1}{\theta} \right) \delta \varrho_\lambda(u_n), \quad \forall n \in \mathbb{N}.
\]

Hence

\[
d + 1 + \max \{ \varrho_\lambda(u_n)^{1/p^-}, \varrho_\lambda(u_n)^{1/p^+} \} \geq \left( \frac{1}{p^+} - \frac{1}{\theta} \right) \delta \varrho_\lambda(u_n), \quad \forall n \geq n_0,
\]

from where it follows that \((u_n)\) is bounded in \( E_\lambda \). \( \square \)

**Proposition 3.4** If \((u_n)\) is a \((PS)_d\) sequence for \( \phi_\lambda \), then given \( \epsilon > 0 \), there is \( R > 0 \) such that

\[
\limsup_{n} \int_{\mathbb{R}^N \setminus B_R(0)} \left( |\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)} \right) < \epsilon.
\]

(3.11)

Hence, once that \( g \) has a subcritical growth, if \( u \in E_\lambda \) is the weak limit of \((u_n)\), then

\[
\int_{\mathbb{R}^N} g(x, u_n) u_n \, dx \to \int_{\mathbb{R}^N} g(x, u) u \, dx \quad \text{and} \quad \int_{\mathbb{R}^N} g(x, u_n) v \, dx \to \int_{\mathbb{R}^N} g(x, u) v \, dx, \quad \forall v \in E_\lambda.
\]

**Proof** Let \((u_n)\) be a \((PS)_d\) sequence for \( \phi_\lambda \), \( R > 0 \) large such that \( \Omega_{\Gamma} \subset B_{R}(0) \) and \( \eta_R \in C^\infty(\mathbb{R}^N) \) satisfying

\[
\eta_R(x) = \begin{cases} 
0, & x \in \frac{B_{R}(0)}{2} \\
1, & x \in \mathbb{R}^N \setminus B_{R}(0),
\end{cases}
\]

\( \square \) Springer
0 ≤ \eta_R ≤ 1 and |\nabla \eta_R| \leq \frac{C}{R}, where C > 0 does not depend on R. This way,

\[\int_{\mathbb{R}^N} \left( |\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)} \right) \eta_R \]

\[= \phi'_\lambda(u_n) (u_n \eta_R) - \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \eta_R + \int_{\mathbb{R}^N \setminus \Omega'_\epsilon} f(x, u_n) u_n \eta_R.\]

Denoting

\[I = \int_{\mathbb{R}^N} \left( |\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)} \right) \eta_R,\]

it follows from (3.8),

\[I \leq \phi'_\lambda(u_n) (u_n \eta_R) + \frac{C}{R} \int_{\mathbb{R}^N} |u_n| |\nabla u_n|^{p(x)-1} + \nu \int_{\mathbb{R}^N} |u_n|^{p(x)} \eta_R.\]

Using Hölder’s inequality 2.4 and Proposition 2.3, we derive

\[I \leq \phi'_\lambda(u_n) (u_n \eta_R) + \frac{C}{R} \int_{\mathbb{R}^N} |u_n| |\nabla u_n|^{p(x)-1} + \nu \int_{\mathbb{R}^N} |u_n|^{p(x)} \eta_R.\]

Since \((u_n)\) and \(|\nabla u_n|\) are bounded in \(L^{p(x)}(\mathbb{R}^N)\) and \(\frac{\nu}{M} = 1 - \delta\), we obtain

\[\int_{\mathbb{R}^N \setminus B_R(0)} \left( |\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)} \right) \leq o_n(1) + \frac{C}{R}.\]

Therefore

\[\limsup_n \int_{\mathbb{R}^N \setminus B_R(0)} \left( |\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x))|u_n|^{p(x)} \right) \leq \frac{C}{R}.\]

So, given \(\epsilon > 0\), choosing a \(R > 0\) possibly still bigger, we have that \(\frac{C}{R} < \epsilon\), which proves (3.11). Now, we will show that

\[\int_{\mathbb{R}^N} g(x, u_n) u_n \rightarrow \int_{\mathbb{R}^N} g(x, u) u.\]

Using the fact that \(g(x, u) u \in L^1(\mathbb{R}^N)\) together with (3.11) and Sobolev embeddings, given \(\epsilon > 0\), we can choose \(R > 0\) such that

\[\limsup_{n \to +\infty} \int_{\mathbb{R}^N \setminus B_R(0)} |g(x, u_n) u_n| \leq \frac{\epsilon}{4} \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_R(0)} |g(x, u) u| \leq \frac{\epsilon}{4}.\]

On the other hand, since \(g\) has a subcritical growth, we have by compact embeddings

\[\int_{B_R(0)} g(x, u_n) u_n \rightarrow \int_{B_R(0)} g(x, u) u.\]
Combining the above information, we conclude that
\[ \int_{\mathbb{R}^N} g(x, u_n) u_n \rightarrow \int_{\mathbb{R}^N} g(x, u) u. \]
The same type of arguments works to prove that
\[ \int_{\mathbb{R}^N} g(x, u_n) v \rightarrow \int_{\mathbb{R}^N} g(x, u) v \quad \forall v \in E_\lambda. \]

\[ \square \]

**Proposition 3.5** \( \phi_\lambda \) verifies the \((PS)\) condition.

**Proof** Let \((u_n)\) be a \((PS)_d\) sequence for \(\phi_\lambda\) and \(u \in E_\lambda\) such that \(u_n \rightharpoonup u\) in \(E_\lambda\). Thereby, by Proposition 3.4,
\[ \int_{\mathbb{R}^N} g(x, u_n) u_n \rightarrow \int_{\mathbb{R}^N} g(x, u) u \quad \text{and} \quad \int_{\mathbb{R}^N} g(x, u_n) v \rightarrow \int_{\mathbb{R}^N} g(x, u) v, \quad \forall v \in E_\lambda. \]
Moreover, the weak limit also gives
\[ \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \cdot (u_n - u) \rightarrow 0 \]
and
\[ \int_{\mathbb{R}^N} (\lambda V(x) + Z(x)) |u|^{p(x)-2} u (u_n - u) \rightarrow 0. \]
Now, if
\[ P_n^1(x) = \left( |\nabla u|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) \cdot (u_n - u), \]
and
\[ P_n^2(x) = \left( |u|^{p(x)-2} u - |u|^{p(x)-2} u \right) (u_n - u), \]
we derive
\[ \int_{\mathbb{R}^N} \left( P_n^1(x) + (\lambda V(x) + Z(x)) P_n^2(x) \right) = \phi_\lambda'(u_n) u_n + \int_{\mathbb{R}^N} g(x, u_n) u_n - \phi_\lambda'(u) u - \int_{\mathbb{R}^N} g(x, u_n) u \]
\[ - \int_{\mathbb{R}^N} \left( |\nabla u|^{p(x)-2} \nabla u \cdot (u_n - u) + (\lambda V(x) + Z(x)) |u|^{p(x)-2} u (u_n - u) \right). \]
Recalling that \(\phi_\lambda'(u_n) u_n = o_n(1)\) and \(\phi_\lambda'(u) u = o_n(1)\), the above limits lead to
\[ \int_{\mathbb{R}^N} \left( P_n^1(x) + (\lambda V(x) + Z(x)) P_n^2(x) \right) \rightarrow 0. \]
Now, the conclusion follows as in [8]. \( \square \)

**Theorem 3.6** The problem \((A_\lambda)\) has a (nonnegative) solution, for all \(\lambda \geq 1\).

**Proof** The proof is an immediate consequence of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [10]. \( \square \)
4 The \((PS)_\infty\) condition

A sequence \((u_n) \subset W^{1,p(x)}(\mathbb{R}^N)\) is called a \((PS)_\infty\) sequence for the family \((\phi_\lambda)_\lambda\geq 1\), if there is a sequence \((\lambda_n) \subset [1, \infty)\) with \(\lambda_n \to \infty\) as \(n \to \infty\), verifying
\[
\phi_{\lambda_n}(u_n) \to c \quad \text{and} \quad \|\phi_{\lambda_n}(u_n)\| \to 0, \quad \text{as} \quad n \to \infty.
\]

**Proposition 4.1** Let \((u_n) \subset W^{1,p(x)}(\mathbb{R}^N)\) be a \((PS)_\infty\) sequence for \((\phi_\lambda)_\lambda\geq 1\). Then, up to a subsequence, there exists \(u \in W^{1,p(x)}(\mathbb{R}^N)\) such that \(u_n \to u\) in \(W^{1,p(x)}(\mathbb{R}^N)\). Furthermore,

(i) \(\varrho_{\lambda_n}(u_n - u) \to 0\) and, consequently, \(u_n \to u\) in \(W^{1,p(x)}(\mathbb{R}^N)\);

(ii) \(u = 0\) in \(\mathbb{R}^N \setminus \Omega_\gamma\), \(u \geq 0\) and \(u|_{\Omega_j} \colon j \in \gamma\), is a solution for

\[
(P_j)\begin{cases}
-\Delta_p(x)u + Z(x)|u|^{p(x)-2}u = f(x,u), \quad \text{in} \ \Omega_j, \\
u \in W^{1,p(x)}(\Omega_j);
\end{cases}
\]

(iii) \(\int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^{p(x)} \to 0\);

(iv) \(\varrho_{\lambda_n,\Omega_j}(u_n) \to \int_{\Omega_j} \left(\left|\nabla u(x)\right|^{p(x)} + Z(x)|u|^{p(x)}\right)\), for \(j \in \gamma\);

(v) \(\varrho_{\lambda_n,\mathbb{R}^N \setminus \Omega_\gamma}(u_n) \to 0\);

(vi) \(\varphi_{\lambda_n}(u_n) \to \int_{\Omega_\gamma} \frac{1}{p(x)} \left( \left|\nabla u\right|^{p(x)} + Z(x)|u|^{p(x)} \right) - \int_{\Omega_\gamma} F(x,u)\).

**Proof** Using the same reasoning as in the proof of Proposition 3.3, we obtain that \((\varrho_{\lambda_n}(u_n))\) is bounded in \(\mathbb{R}\). Then \((\|u_n\|_{\lambda_n})\) is bounded in \(\mathbb{R}\) and \((u_n)\) is bounded in \(W^{1,p(x)}(\mathbb{R}^N)\). So, up to a subsequence, there exists \(u \in W^{1,p(x)}(\mathbb{R}^N)\) such that

\[
u_n \to u \quad \text{in} \quad W^{1,p(x)}(\mathbb{R}^N) \quad \text{and} \quad u_n(x) \to u(x) \quad \text{for} \quad \text{a.e.} \ x \in \mathbb{R}^N.
\]

Now, for each \(m \in \mathbb{N}\), we define \(C_m = \left\{ x \in \mathbb{R}^N : V(x) \geq \frac{1}{m} \right\}\). Without loss of generality, we can assume \(\lambda_n < 2(\lambda_n - 1), \ \forall n \in \mathbb{N}\). Thus

\[
\int_{C_m} |u_n|^{p(x)} \leq \frac{2m}{\lambda_n} \int_{C_m} \left( \lambda_n V(x) + Z(x) \right) |u_n|^{p(x)} \leq \frac{2m}{\lambda_n} \varrho_{\lambda_n}(u_n) \leq \frac{C}{\lambda_n}.
\]

By Fatou’s lemma, we derive

\[
\int_{C_m} |u|^{p(x)} = 0,
\]

which implies that \(u = 0\) in \(C_m\) and, consequently, \(u = 0\) in \(\mathbb{R}^N \setminus \overline{\Omega}\). From this, we are able to prove \((i) - (vi)\).

\[(i)\] Since \(u = 0\) in \(\mathbb{R}^N \setminus \overline{\Omega}\), repeating the argument explored in Proposition 3.5 we get

\[
\int_{\mathbb{R}^N} \left( P^1_n(x) + (\lambda_n V(x) + Z(x)) P^2_n(x) \right) \to 0,
\]

where

\[
P^1_n(x) = \left( \left|\nabla u_n\right|^{p(x)-2} \nabla u_n - \left|\nabla u\right|^{p(x)-2} \nabla u \right) \cdot (\nabla u_n - \nabla u).
\]
and

\[ P_n^2(x) = \left( |u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u \right)(u_n - u). \]

Therefore, \( \varrho_{\lambda_n}(u_n - u) \to 0 \), which implies \( u_n \to u \) in \( W^{1,p(x)}(\mathbb{R}^N) \).

(ii) Since \( u \in W^{1,p(x)}(\mathbb{R}^N) \) and \( u = 0 \) in \( \mathbb{R}^N \setminus \Omega \), we have \( u \in W^{1,p(x)}_0(\Omega) \) or, equivalently, \( u|_{\Omega_j} \in W^{1,p(x)}_0(\Omega_j) \), for \( j = 1, \ldots, k \). Moreover, the limit \( u_n \to u \) in \( W^{1,p(x)}(\mathbb{R}^N) \) combined with \( \phi'_{\lambda_n}(u_n) \varphi \to 0 \) for \( \varphi \in C_0^\infty(\Omega_j) \) implies that

\[
\int_{\Omega_j} \left( |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi + Z(x)|u|^{p(x)-2}u \varphi \right) - \int_{\Omega_j} g(x,u)\varphi = 0, \quad (4.1)
\]

showing that \( u|_{\Omega_j} \) is a solution for

\[
\begin{cases}
-\Delta_{p(x)}u + Z(x)|u|^{p(x)-2}u = g(x,u), & \text{in } \Omega_j, \\
u \in W^{1,p(x)}_0(\Omega_j).
\end{cases}
\]

This way, if \( j \in \Upsilon \), then \( u|_{\Omega_j} \) satisfies \( (P_j) \). On the other hand, if \( j \notin \Upsilon \), we must have

\[
\int_{\Omega_j} \left( |\nabla u|^{p(x)} + Z(x)|u|^{p(x)} \right) - \int_{\Omega_j} \tilde{f}(x,u)u = 0.
\]

The above equality combined with \((3.8)\) and \((3.2)\) gives

\[ 0 \geq \varrho_{\lambda_j}(\Omega_j)(u) - \nu_{\varphi_{p(x)},\Omega_j}(u) \geq \delta_{\varrho_{\lambda_j},\Omega_j}(u) \geq 0, \]

from where it follows \( u|_{\Omega_j} = 0 \). This proves \( u = 0 \) outside \( \Omega_{\Upsilon} \) and \( u \geq 0 \) in \( \mathbb{R}^N \).

(iii) It follows from (i), since

\[
\int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^{p(x)} = \int_{\mathbb{R}^N} \lambda_n V(x)|u_n - u|^{p(x)} \leq 2\varrho_{\lambda_n}(u_n - u).
\]

(iv) Let \( j \in \Upsilon \). From (i),

\[
\varrho_{p(x),\Omega_j'}(u_n - u), \varrho_{p(x),\Omega_j'}(\nabla u_n - \nabla u) \to 0.
\]

Then by Proposition 2.5,

\[
\int_{\Omega_j'} \left( |\nabla u_n|^{p(x)} - |\nabla u|^{p(x)} \right) \to 0 \quad \text{and} \quad \int_{\Omega_j'} Z(x)(|u_n|^{p(x)} - |u|^{p(x)}) \to 0.
\]

From (iii),

\[
\int_{\Omega_j'} \lambda_n V(x)(|u_n|^{p(x)} - |u|^{p(x)}) = \int_{\Omega_j'} \lambda_n V(x)|u_n|^{p(x)} \to 0.
\]

This way

\[
\varrho_{\lambda_n,\Omega_j'}(u_n) - \varrho_{\lambda_n,\Omega_j'}(u) \to 0.
\]
Once $u = 0$ in $\Omega_j \setminus \Omega_j$, we get
\[
\varrho_{\lambda_n, \Omega_j}(u_n) \to \int_{\Omega_j} \left( |\nabla u|^{p(x)} + Z(x)|u|^{p(x)} \right).
\]

(v) By (i), $\varrho_{\lambda_n}(u_n - u) \to 0$, and so,
\[
\varrho_{\lambda_n, \mathbb{R}^N \setminus \Omega}(u_n) \to 0.
\]

(vi) We can write the functional $\phi_{\lambda_n}$ in the following way
\[
\phi_{\lambda_n}(u_n) = \sum_{j \in \Upsilon} \int_{\Omega_j} \frac{1}{p(x)} \left( |\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)} \right)
+ \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{p(x)} \left( |\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)} \right) - \int G(x, u_n).
\]

From (i) – (v),
\[
\int_{\Omega_j} \frac{1}{p(x)} \left( |\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)} \right)
\to \int_{\Omega_j} \frac{1}{p(x)} \left( |\nabla u|^{p(x)} + Z(x)|u|^{p(x)} \right),
\]

\[
\int_{\mathbb{R}^N \setminus \Omega} \frac{1}{p(x)} \left( |\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)} \right) \to 0.
\]

and
\[
\int_{\mathbb{R}^N} G(x, u_n) \to \int_{\Omega} F(x, u).
\]

Therefore
\[
\phi_{\lambda_n}(u_n) \to \int_{\Omega} \frac{1}{p(x)} \left( |\nabla u|^{p(x)} + Z(x)|u|^{p(x)} \right) - \int F(x, u).
\]

\[
\Box
\]

5 The boundedness of the $(A_{\lambda})$ solutions

In this section, we study the boundedness outside $\Omega'_{\Upsilon}$ for some solutions of $(A_{\lambda})$. To this end, we adapt for our problem arguments found in [18] and [25].

**Proposition 5.1** Let $(u_{\lambda})$ be a family of solutions for $(A_{\lambda})$ such that $u_{\lambda} \to 0$ in $W^{1,p(x)}(\mathbb{R}^N \setminus \Omega_{\Upsilon})$, as $\lambda \to \infty$. Then, there exists $\lambda^* > 0$ with the following property:
\[
|u_{\lambda}|_{\infty, \mathbb{R}^N \setminus \Omega'_{\Upsilon}} \leq a_-, \quad \forall \lambda \geq \lambda^*.
\]

Hence, $u_{\lambda}$ is a solution for $(P_{\lambda})$ for $\lambda \geq \lambda^*$.

Before to prove the above proposition, we need to show some technical lemmas.
Lemma 5.2 There exist \( x_1, \ldots, x_l \in \partial \Omega_1' \) and corresponding \( \delta_{x_1}, \ldots, \delta_{x_l} > 0 \) such that

\[
\partial \Omega_1' \subset \mathcal{N} (\partial \Omega_1') := \bigcup_{i=1}^{l} B_{\delta_{x_i}} (x_i).
\]

Moreover,

\[
q^{x_i}_+ \leq (p^{-}_x)^{*},
\]

where

\[
q^{x_i}_+ = \sup_{B_{\delta_{x_i}} (x_i)} q, \quad p^{x_i}_+ = \inf_{B_{\delta_{x_i}} (x_i)} p \quad \text{and} \quad (p^{-}_x)^{*} = \frac{N p^{x}_+}{N - p^{x}_-}.
\]

**Proof** From (3.10), \( \Omega_1' \subset \Omega_1 \). So, there is \( \delta > 0 \) such that

\[
B_{\delta} (x) \subset \mathbb{R}^N \setminus \Omega_1, \quad \forall x \in \partial \Omega_1'.
\]

Once \( q \ll p^{*} \), there exists \( \epsilon > 0 \) such that \( \epsilon \leq p^{*} (y) - q (y) \), for all \( y \in \mathbb{R}^N \). Then, by continuity, for each \( x \in \partial \Omega_1' \), we can choose a sufficiently small \( 0 < \delta_x \leq \delta \) such that

\[
q^{x}_+ \leq (p^{-}_x)^{*},
\]

where

\[
q^{x}_+ = \sup_{B_{\delta_x} (x)} q, \quad p^{x}_+ = \inf_{B_{\delta_x} (x)} p \quad \text{and} \quad (p^{-}_x)^{*} = \frac{N p^{x}_+}{N - p^{x}_-}.
\]

Covering \( \partial \Omega_1' \) by the balls \( B_{\delta_x} (x) \), \( x \in \partial \Omega_1' \), and using its compactness, there are \( x_1, \ldots, x_l \in \partial \Omega_1' \) such that

\[
\partial \Omega_1' \subset \bigcup_{i=1}^{l} B_{\delta_{x_i}} (x_i).
\]

\( \Box \)

Lemma 5.3 If \( u_\lambda \) is a solution for \( (A_\lambda) \), in each \( B_{\delta_{x_i}} (x_i) \), \( i = 1, \ldots, l \), given by Lemma 5.2, it is fulfilled

\[
\int_{A_{k,R,x_i}} |\nabla u_\lambda|^p_{x_i} \leq C \left( (k^{q^p} + 2)|A_{k,\delta,x_i}| + (\delta - \bar{\delta})(p^{-}_x)^{*} \int_{A_{k,\delta,x_i}} (u_\lambda - k)(p^{x}_-)^{*} \right),
\]

where \( 0 < \bar{\delta} < \delta < \delta_{x_i} \), \( k \geq \frac{a_{-} - 4}{4} \), \( C = C(p_{-}, p_{+}, q_{-}, q_{+}, \nu, \delta_{x_i}) > 0 \) is a constant independent of \( k \), and for any \( R > 0 \), we denote by \( A_{k,R,x_i} \) the set

\[
A_{k,R,x_i} = B_R (x_i) \cap \left\{ x \in \mathbb{R}^N ; u_\lambda (x) > k \right\}.
\]

**Proof** We choose arbitrarily \( 0 < \bar{\delta} < \delta < \delta_{x_i} \) and \( \xi \in C^\infty (\mathbb{R}^N) \) with

\[
0 \leq \xi \leq 1, \quad \text{supp} \; \xi \subset B_{\delta} (x_i), \; \xi = 1 \text{ in } B_{\bar{\delta}} (x_i) \quad \text{and} \quad |\nabla \xi| \leq \frac{2}{\delta - \bar{\delta}}.
\]

For \( k \geq \frac{a_{-} - 4}{4} \), we define \( \eta = \xi^{p^+} (u_\lambda - k)^{+} \). We notice that

\[
\nabla \eta = p_{+} \xi^{p^+ - 1} (u_\lambda - k) \nabla \xi + \xi^{p^+} \nabla u_\lambda.
\]
on the set \( \{ u_{\lambda} > k \} \). Then, writing \( u_{\lambda} = u \) and taking \( \eta \) as a test function, we obtain

\[
p_+ \int_{A_k, \lambda, x_i} \xi^{p+1} (u - k) | \nabla u |^{p(x) - 2} \nabla u \cdot \nabla \xi + \int_{A_k, \lambda, x_i} \xi^{p+1} | \nabla u |^{p(x)} \\
+ \int_{A_k, \lambda, x_i} (\lambda V(x) + Z(x)) u^{p(x) - 1} \xi^{p+1} (u - k) = \int_{A_k, \lambda, x_i} g(x, u) \xi^{p+1} (u - k).
\]

If we set

\[
J = \int_{A_k, \lambda, x_i} \xi^{p+1} | \nabla u |^{p(x)},
\]

using that \( \nu \leq \lambda V(x) + Z(x), \forall x \in \mathbb{R}^N \), we get

\[
J \leq p_+ \int_{A_k, \lambda, x_i} \xi^{p+1} (u - k) | \nabla u |^{p(x) - 1} | \nabla \xi | \\
- \int_{A_k, \lambda, x_i} \nu u^{p(x) - 1} \xi^{p+1} (u - k) + \int_{A_k, \lambda, x_i} g(x, u) \xi^{p+1} (u - k). \tag{5.2}
\]

From (5.2), (3.3) and (3.7),

\[
J \leq p_+ \int_{A_k, \lambda, x_i} \xi^{p+1} (u - k) | \nabla u |^{p(x) - 1} | \nabla \xi | \\
- \int_{A_k, \lambda, x_i} \nu u^{p(x) - 1} \xi^{p+1} (u - k) \\
+ \int_{A_k, \lambda, x_i} (\nu u^{p(x) - 1} + C_{\nu} u^{q(x) - 1}) \xi^{p+1} (u - k),
\]

from where it follows

\[
J \leq p_+ \int_{A_k, \lambda, x_i} \xi^{p+1} (u - k) | \nabla u |^{p(x) - 1} | \nabla \xi | + C_{\nu} \int_{A_k, \lambda, x_i} u^{q(x) - 1} (u - k).
\]

Using Young’s inequality, we obtain, for \( \chi \in (0, 1) \),

\[
J \leq \frac{p_+ (p_+ - 1)}{p_-} \chi^{p_+ - 1} J + \frac{2p_+ p_+}{p_-} \chi^{-p_+} \int_{A_k, \lambda, x_i} \left( \frac{u - k}{\delta - \delta} \right)^{p(x)} \\
+ \frac{C_{\nu} (q_+ - 1)}{q_-} \int_{A_k, \lambda, x_i} u^{q(x)} + \frac{C_{\nu} (1 + \delta^{q_+}_{x_i})}{q_-} \int_{A_k, \lambda, x_i} \left( \frac{u - k}{\delta - \delta} \right)^{q(x)}.
\]

Writing

\[
Q = \int_{A_k, \lambda, x_i} \left( \frac{u - k}{\delta - \delta} \right)^{(p_+)}_x,
\]

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for \( \chi \approx 0^+ \) fixed, due to (5.1), we deduce

\[
J \leq \frac{1}{2} J + \frac{2^{p_+} p_+}{p_-} \chi^{-p_+} (|A_{k,\bar{\delta},x_i}| + Q) + \frac{C_v 2^{q_+} (q_+ - 1) (1 + \delta^{q_+}_{\bar{n}})}{q_-} (|A_{k,\bar{\delta},x_i}| + Q)
\]

\[
+ \frac{C_v 2^{q_+} (q_+ - 1) (1 + k^{q_+})}{q_-}|A_{k,\bar{\delta},x_i}| + \frac{C_v (1 + \delta^{q_+}_{\bar{n}})}{q_-} (|A_{k,\bar{\delta},x_i}| + Q).
\]

Therefore

\[
\int_{A_{k,\bar{\delta},x_i}} |\nabla u|^{p(x)} \leq J \leq C \left[(k^{q_+} + 1)|A_{k,\bar{\delta},x_i}| + Q\right] + |A_{k,\bar{\delta},x_i}|
\]

for a positive constant \( C = C(p_-, p_+, q_-, q_+, \nu, \delta_{x_i}) \) which does not depend on \( k \). Since

\[
|\nabla u|^{p_{\bar{n}}} - 1 \leq |\nabla u|^{p(x)}, \forall x \in B_{\delta_{x_i}}(x_i),
\]

we obtain

\[
\int_{A_{k,\bar{\delta},x_i}} |\nabla u|^{p_{\bar{n}}} \leq C \left[(k^{q_+} + 2)|A_{k,\bar{\delta},x_i}| + (\bar{\delta} - \delta)^{-\left(p_{\bar{n}}\right)^*} \int_{A_{k,\bar{\delta},x_i}} (u - k)^{\left(p_{\bar{n}}\right)^*}\right],
\]

for a positive constant \( C = C(p_-, p_+, q_-, q_+, \nu, \delta_{x_i}) \) which does not depend on \( k \). \( \square \)

The next lemma can be found at ([27, Lemma 4.7]).

**Lemma 5.4** Let \((J_n)\) be a sequence of nonnegative numbers satisfying

\[
J_{n+1} \leq CB^n J_n^{1+\eta}, \ n = 0, 1, 2, \ldots,
\]

where \( C, \eta > 0 \) and \( B > 1 \). If

\[
J_0 \leq C^{-\frac{1}{\eta}} B^{-\frac{1}{\eta^2}},
\]

then \( J_n \to 0, \) as \( n \to \infty. \)

**Lemma 5.5** Let \((u_\lambda)\) be a family of solutions for \((A_\lambda)\) such that \( u_\lambda \to 0 \) in \( W^{1,p(x)}(\mathbb{R}^N \setminus \Omega_T) \), as \( \lambda \to \infty. \) Then, there exists \( \lambda^* > 0 \) with the following property:

\[
|u_\lambda|_{\infty,N(\beta \Omega_T)} \leq a_-, \forall \lambda \geq \lambda^*.
\]

**Proof** It is enough to prove the inequality in each ball \( B_{\delta_{x_i}}(x_i), i = 1, \ldots, l, \) given by Lemma 5.2. We set

\[
\bar{\delta}_n = \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2n+1}, \delta_n = \frac{\bar{\delta}_n + \bar{\delta}_{n+1}}{2}, k_n = \frac{a_-}{2} \left(1 - \frac{1}{2n+1}\right), \forall n = 0, 1, 2, \ldots.
\]

Then

\[
\bar{\delta}_n \downarrow \frac{\delta_{x_i}}{2}, \delta_n + 1 < \delta_n < \bar{\delta}_n, \ k_n \uparrow \frac{a_-}{2}.
\]
From now on, we fix

\[ J_n(\lambda) = J_n = \int_{A_{\delta_x_n \cdot x}} (u_{\lambda}(x) - k_n) \left( p_{\lambda}^x \right)^* , \quad n = 0, 1, 2, \ldots \]

and \( \xi \in C^1(\mathbb{R}) \) such that

\[ 0 \leq \xi \leq 1, \quad \xi(t) = 1, \quad \text{for} \ t \leq \frac{1}{2}, \quad \text{and} \ \xi(t) = 0, \quad \text{for} \ t \geq \frac{3}{4}. \]

Setting

\[ \xi_n(x) = \xi \left( \frac{2^{n+1}}{\delta_{x_i}} \left( |x - x_i| - \frac{\delta_{x_i}}{2} \right) \right), \quad x \in \mathbb{R}^N, \quad n = 0, 1, 2, \ldots, \]

we have \( \xi_n = 1 \) in \( B_{\delta_{x_n+1}}(x_i) \) and \( \xi_n = 0 \) outside \( B_{\delta_{x_n}}(x_i) \). Writing \( u_{\lambda} = u \), we get

\[ J_{n+1} \leq \int_{A_{\delta_{x_n+1} \cdot x} - \delta_{x_i}} (u(x) - k_{n+1}) \xi_n(x) \left( p_{\lambda}^x \right)^* \]

\[ = \int_{B_{\delta_{x_n}}(x_i)} \left( (u - k_{n+1})^{+}(x) \xi_n(x) \right) \left( p_{\lambda}^x \right)^* \]

\[ \leq C(N, p_{\lambda}^x) \left( \int_{B_{\delta_{x_n}}(x_i)} |\nabla ((u - k_{n+1})^{+}\xi_n)(x)|_{p_{\lambda}^x} \right) \left( p_{\lambda}^x \right)^* \]

\[ \leq C(N, p_{\lambda}^x) \left( \int_{A_{\delta_{x_n+1} \cdot x} - \delta_{x_i}} |\nabla u|_{p_{\lambda}^x}^{p_{\lambda}^x} + \int_{A_{\delta_{x_n+1} \cdot x} - \delta_{x_i}} (u - k_{n+1})^{+} \xi_n \left( p_{\lambda}^x \right)^* \right). \]

Since

\[ |\nabla \xi_n(x)| \leq C(\delta_{x_i}) 2^{n+1}, \quad \forall x \in \mathbb{R}^N, \]

writing \( \frac{\left( p_{\lambda}^x \right)^*}{p_{\lambda}^x} = \tilde{J}_{n+1} \), we obtain

\[ \tilde{J}_{n+1} \leq C(N, p_{\lambda}^x, \delta_{x_i}) \left( \int_{A_{\delta_{x_n+1} \cdot x} - \delta_{x_i}} |\nabla u|_{p_{\lambda}^x}^{p_{\lambda}^x} + 2^{np_{\lambda}^x} \int_{A_{\delta_{x_n+1} \cdot x} - \delta_{x_i}} (u - k_{n+1})^{+} \xi_n \right). \]
Using Lemma 5.3,

\[ \tilde{J}_{n+1} \leq C \left( N, p_{x_i}^+, \delta_{x_i} \right) \left( \left( k_{n+1}^{q_+} + 2 \right) |A_{k_{n+1}, \tilde{\delta}_{x_i}}| \right) + \left( \frac{2^n+3}{\delta_{x_i}} \right) \left( p_{x_i}^- \right)^* \int_{A_{k_{n+1}, \tilde{\delta}_{x_i}}} (u - k_{n+1}) \left( p_{x_i}^- \right)^* + 2^n p_{x_i}^- \int_{A_{k_{n+1}, \tilde{\delta}_{x_i}}} (u - k_{n+1}) p_{x_i}^- \right)

\leq C \left( N, p_{x_i}^-, \delta_{x_i} \right) \left( \left( k_{n+1}^{q_+} + 2 \right) |A_{k_{n+1}, \tilde{\delta}_{x_i}}| \right) + 2^n \left( p_{x_i}^- \right)^* \int_{A_{k_{n+1}, \tilde{\delta}_{x_i}}} (u - k_{n+1}) \left( p_{x_i}^- \right)^* + 2^n p_{x_i}^- \int_{A_{k_{n+1}, \tilde{\delta}_{x_i}}} (u - k_{n+1}) p_{x_i}^- \right).

From Young’s inequality

\[ \int_{A_{k_{n+1}, \tilde{\delta}_{x_i}}} (u - k_{n+1}) p_{x_i}^- \leq C \left( p_{x_i}^- \right) \left( |A_{k_{n+1}, \tilde{\delta}_{x_i}}| \right) + \int_{A_{k_{n+1}, \tilde{\delta}_{x_i}}} (u - k_{n+1}) \left( p_{x_i}^- \right)^* \right).

Thus

\[ \tilde{J}_{n+1} \leq C \left( N, p_{x_i}^-, \delta_{x_i} \right) \left( \left( \left( \frac{a_-}{2} \right)^{q_+} + 2 \right) |A_{k_{n+1}, \tilde{\delta}_{x_i}}| + 2^n \left( p_{x_i}^- \right)^* \right) \left( J_n + 2^n p_{x_i}^- J_n \right).

Now, since

\[ J_n \geq \int_{A_{k_{n+1}, \tilde{\delta}_{x_i}}} (u - k_n) \left( p_{x_i}^- \right)^* \geq (k_{n+1} - k_n) \left( p_{x_i}^- \right)^* \left( A_{k_{n+1}, \tilde{\delta}_{x_i}} \right) \]

it follows that

\[ |A_{k_{n+1}, \tilde{\delta}_{x_i}}| \leq \left( \frac{2^n+3}{\frac{n+3}{a_-}} \right) \left( p_{x_i}^- \right)^* \]

and so,

\[ \tilde{J}_{n+1} \leq C \left( N, p_{x_i}^-, \delta_{x_i}, a_-, q_+ \right) \left( 2^n \left( p_{x_i}^- \right)^* \right) J_n + 2^n \left( p_{x_i}^- \right)^* \left( J_n + 2^n \left( p_{x_i}^- \right)^* \right) J_n + 2^n \left( p_{x_i}^- \right)^* \left( J_n + 2^n \left( p_{x_i}^- \right)^* \right) J_n. \]

Fixing \( \alpha = \left( p_{x_i}^+ + \left( p_{x_i}^- \right)^* \right) \), it follows that

\[ J_{n+1} \leq C \left( N, p_{x_i}^+, \delta_{x_i}, a_-, q_+ \right) \left( \frac{\alpha \left( p_{x_i}^+ \right)^*}{2} \right)^* \left( J_n \right) \left( \frac{\left( p_{x_i}^- \right)^*}{p_{x_i}^-} \right)^*, \]

and consequently

\[ J_{n+1} \leq C B^n J_n^{1+\eta}, \]
Proof of Proposition 5.1

We derive

\[ \int_{\Omega_1} \left( u_\lambda - \frac{a}{4} \right)^p = J_0(\lambda) = C \left( \frac{1}{4} \right)^p \lambda \geq \lambda_i. \]

From Lemma 5.4, \( J_n(\lambda) \to 0 \), \( n \to \infty \), for all \( \lambda \geq \lambda_i \), and so,

\[ u_\lambda \leq \frac{a}{2} < a_- \quad \text{in} \quad B_{\delta i}, \quad \text{for all} \quad \lambda \geq \lambda_i. \]

Now, taking \( \lambda^* = \max\{ \lambda_1, \ldots, \lambda_l \} \), we conclude that

\[ |u_\lambda|_{\infty, N(\beta \Omega')} < a_- \quad \forall \lambda \geq \lambda^*. \]

\( \square \)

Proof of Proposition 5.1

Fix \( \lambda \geq \lambda^* \), where \( \lambda^* \) is given at Lemma 5.5, and define \( \overline{u}_\lambda : \mathbb{R}^N \setminus \Omega_Y' \to \mathbb{R} \) given by

\[ \overline{u}_\lambda(x) = (u_\lambda - a_-)^+ (x). \]

From Lemma 5.5, \( \overline{u}_\lambda \in W^{1,p(x)}(\mathbb{R}^N \setminus \Omega_Y'). \) Our goal is showing that \( \overline{u}_\lambda = 0 \) in \( \mathbb{R}^N \setminus \Omega_Y' \). This implies

\[ |u_\lambda|_{\infty, \mathbb{R}^N \setminus \Omega_Y'} \leq a_- \]

In fact, extending \( \overline{u}_\lambda = 0 \) in \( \Omega_Y' \) and taking \( \overline{u}_\lambda \) as a test function, we obtain

\[ \int_{\mathbb{R}^N \setminus \Omega_Y'} \left| \nabla u_\lambda \right|^{p(x)-2} \nabla u_\lambda \cdot \nabla \overline{u}_\lambda + \int_{\mathbb{R}^N \setminus \Omega_Y'} \left( \lambda V(x) + Z(x) \right) u_\lambda^{p(x)-2} u_\lambda \overline{u}_\lambda = \int_{\mathbb{R}^N \setminus \Omega_Y'} g(x, u_\lambda) \overline{u}_\lambda. \]

Since

\[ \int_{\mathbb{R}^N \setminus \Omega_Y'} \left| \nabla u_\lambda \right|^{p(x)-2} \nabla u_\lambda \cdot \nabla \overline{u}_\lambda = \int_{\mathbb{R}^N \setminus \Omega_Y'} \left| \nabla \overline{u}_\lambda \right|^{p(x)}, \]

\[ \int_{\mathbb{R}^N \setminus \Omega_Y'} \left( \lambda V(x) + Z(x) \right) u_\lambda^{p(x)-2} u_\lambda \overline{u}_\lambda = \int_{\mathbb{R}^N \setminus \Omega_Y'} \left( \lambda V(x) + Z(x) \right) u_\lambda^{p(x)-2} (\overline{u}_\lambda + a_-) \overline{u}_\lambda \]

and

\[ \int_{\mathbb{R}^N \setminus \Omega_Y'} g(x, u_\lambda) \overline{u}_\lambda = \int_{\mathbb{R}^N \setminus \Omega_Y'} g(x, u_\lambda) \overline{u}_\lambda (\overline{u}_\lambda + a_-) \overline{u}_\lambda, \]

where

\[ \left( \mathbb{R}^N \setminus \Omega_Y' \right)_+ = \left\{ x \in \mathbb{R}^N \setminus \Omega_Y' : u_\lambda(x) > a_- \right\}, \]

we derive

\[ \int_{\mathbb{R}^N \setminus \Omega_Y'} \left| \nabla \overline{u}_\lambda \right|^{p(x)} + \int_{\mathbb{R}^N \setminus \Omega_Y'} \left( \lambda V(x) + Z(x) \right) u_\lambda^{p(x)-2} - \frac{g(x, u_\lambda)}{u_\lambda} \right) (\overline{u}_\lambda + a_-) \overline{u}_\lambda = 0, \]
Now, by (3.7),
\[(\lambda V(x) + Z(x))u_\lambda^{p(x)-2} - \frac{g(x, u_\lambda)}{u_\lambda} > v u_\lambda^{p(x)-2} - \frac{\tilde{f}(x, u_\lambda)}{u_\lambda} \geq 0 \text{ in } (\mathbb{R}^N \setminus \Omega'_Y)_+ .\]

This form, \(\tilde{u}_\lambda = 0\) in \((\mathbb{R}^N \setminus \Omega'_Y)_+\). Obviously, \(\tilde{u}_\lambda = 0\) at the points where \(u_\lambda \leq a_-\), consequently, \(\tilde{u}_\lambda = 0\) in \(\mathbb{R}^N \setminus \Omega'_Y\).

### 6 A special critical value for \(\phi_\lambda\)

For each \(j = 1, \ldots, k\), consider
\[I_j(u) = \int_{\Omega_j} \frac{1}{p(x)} \left( |\nabla u|^{p(x)} + Z(x)|u|^{p(x)} \right) - \int_{\Omega_j} F(x, u), \; u \in W_0^{1,p(x)}(\Omega_j),\]

the energy functional associated to \((P_j)\), and
\[\phi_{\lambda, j}(u) = \int_{\Omega'_j} \frac{1}{p(x)} \left( |\nabla u|^{p(x)} + (\lambda V(x) + Z(x))|u|^{p(x)} \right) - \int_{\Omega'_j} F(x, u), \; u \in W^{1,p(x)}(\Omega'_j),\]

the energy functional associated to
\[\left\{ -\Delta_{p(x)} u + (\lambda V(x) + Z(x))|u|^{p(x)-2} u = f(x, u), \quad \text{in } \Omega'_j, \right.\]
\[\left. \frac{\partial u}{\partial \eta} = 0, \quad \text{on } \partial \Omega'_j. \right\}

It is fulfilled that \(I_j\) and \(\phi_{\lambda, j}\) satisfy the mountain pass geometry and let
\[c_j = \inf_{\gamma \in \Gamma_j} \max_{t \in [0, 1]} I_j(\gamma(t)) \text{ and } c_{\lambda, j} = \inf_{\gamma \in \Gamma_{\lambda, j}} \max_{t \in [0, 1]} \phi_{\lambda, j}(\gamma(t)),\]

their respective mountain pass levels, where
\[\Gamma_j = \left\{ \gamma \in C\left([0, 1], W_0^{1,p(x)}(\Omega_j) \right); \gamma(0) = 0 \text{ and } I_j(\gamma(1)) < 0 \right\}\]
and
\[\Gamma_{\lambda, j} = \left\{ \gamma \in C\left([0, 1], W^{1,p(x)}(\Omega'_j) \right); \gamma(0) = 0 \text{ and } \phi_{\lambda, j}(\gamma(1)) < 0 \right\} .\]

Invoking the \((PS)\) condition on \(I_j\) and \(\phi_{\lambda, j}\), we ensure that there exist \(w_j \in W_0^{1,p(x)}(\Omega_j)\) and \(w_{\lambda, j} \in W^{1,p(x)}(\Omega'_j)\) such that
\[I_j(w_j) = c_j \text{ and } I'_j(w_j) = 0\]
and
\[\phi_{\lambda, j}(w_{\lambda, j}) = c_{\lambda, j} \text{ and } \phi'_{\lambda, j}(w_{\lambda, j}) = 0.\]

**Lemma 6.1** There holds that

(i) \(0 < c_{\lambda, j} \leq c_j, \forall \lambda \geq 1, \forall j \in \{1, \ldots, k\}\);

(ii) \(c_{\lambda, j} \to c_j\) as \(\lambda \to \infty, \forall j \in \{1, \ldots, k\} .\)
Proof (i) Once $W^{1,p(x)}_0(\Omega_j) \subset W^{1,p(x)}(\Omega'_j)$ and $\phi_{\lambda,j}(\gamma(1)) = I_j(\gamma(1))$ for $\gamma \in \Gamma_j$, we have $\Gamma_j \subset \Gamma_{\lambda,j}$. This way

$$c_{\lambda,j} = \inf_{\gamma \in \Gamma_{\lambda,j}} \max_{t \in [0,1]} \phi_{\lambda,j}(\gamma(t)) \leq \inf_{\gamma \in \Gamma_j} \max_{t \in [0,1]} \phi_{\lambda,j}(\gamma(t)) = \inf_{\gamma \in \Gamma_j} \max_{t \in [0,1]} I_j(\gamma(t)) = c_j.$$

(ii) It suffices to show that $c_{\lambda_n,j} \to c_j$, as $n \to \infty$, for all sequences $(\lambda_n)$ in $[1, \infty)$ with $\lambda_n \to \infty$, as $n \to \infty$. Let $(\lambda_n)$ be such a sequence and consider an arbitrary subsequence of $(c_{\lambda_n,j})$ (not relabeled). Let $w_n \in W^{1,p(x)}(\Omega'_j)$ with

$$\phi_{\lambda_n,j}(w_n) = c_{\lambda_n,j} \text{ and } \phi'_{\lambda_n,j}(w_n) = 0.$$ 

By the previous item, $(c_{\lambda_n,j})$ is bounded. Then, there exists $(w_{n_k})$ subsequence of $(w_n)$ such that $\phi_{\lambda_{n_k},j}(w_{n_k})$ converges and $\phi'_{\lambda_{n_k},j}(w_{n_k}) = 0$. Now, repeating the same type of arguments explored in the proof of Proposition 4.1, there is $w \in W^{1,p(x)}(\Omega_j) \setminus \{0\} \subset W^{1,p(x)}(\Omega'_j)$ such that

$$w_{n_k} \to w \text{ in } W^{1,p(x)}(\Omega'_j), \text{ as } k \to \infty.$$ 

Furthermore, we also can prove that

$$c_{\lambda_{n_k},j} = \phi_{\lambda_{n_k},j}(w_{n_k}) \to I_j(w)$$

and

$$0 = \phi'_{\lambda_{n_k},j}(w_{n_k}) \to I'_j(w).$$

Then, by (f4),

$$\lim_k c_{\lambda_{n_k},j} \geq c_j.$$ 

The last inequality together with item (i) implies

$$c_{\lambda_{n_k},j} \to c_j, \text{ as } k \to \infty.$$ 

This establishes the asserted result.

\[ \square \]

In the sequel, let $R > 1$ verifying

$$0 < I_j\left(\frac{1}{R}w_j\right), I_j(Rw_j) < c_j, \text{ for } j = 1, \ldots, k. \quad (6.1)$$

There holds that

$$c_j = \max_{t \in [1/R^2, 1]} I_j(tRw_j), \text{ for } j = 1, \ldots, k.$$ 

Moreover, to simplify the notation, we rename the components $\Omega_j$ of $\Omega$ in way such that $\Upsilon = \{1, 2, \ldots, l\}$ for some $1 \leq l \leq k$. Then, we define:

$$\gamma_0(t_1, \ldots, t_l)(x) = \sum_{j=1}^l t_jRw_j(x), \ \forall (t_1, \ldots, t_l) \in [1/R^2, 1]^l,$$

$$\Gamma_* = \left\{ \gamma \in C([1/R^2, 1]^l, E_\lambda \setminus [0]) \ ; \ \gamma = \gamma_0 \text{ on } \partial [1/R^2, 1]^l \right\}$$

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and
\[
b_{\lambda, \gamma} = \inf_{\gamma \in \Gamma_*} \max_{(t_1, \ldots, t_l) \in [1/R^2, 1]^l} \phi_{\lambda, j}(\gamma(t_1, \ldots, t_l)).
\]

Next, our intention is proving that \(b_{\lambda, \gamma}\) is a critical value for \(\phi_{\lambda, j}\). However, to do this, we need to some technical lemmas. The arguments used are the same found in [3]; however, for reader’s convenience, we will repeat their proofs.

**Lemma 6.2** For all \(\gamma \in \Gamma_*\), there exists \((s_1, \ldots, s_l) \in [1/R^2, 1]^l\) such that
\[
\phi_{\lambda, j}'(\gamma(s_1, \ldots, s_l)) = 0, \quad \forall j \in \mathcal{Y}.
\]

**Proof** Given \(\gamma \in \Gamma_*\), consider \(\tilde{\gamma} : [1/R^2, 1]^l \to \mathbb{R}^l\) such that
\[
\tilde{\gamma}(t) = (\phi_{\lambda, 1}'(\gamma(t)), \ldots, \phi_{\lambda, l}'(\gamma(t)))\gamma(t), \quad \text{where } t = (t_1, \ldots, t_l).
\]
For \(t \in \partial [1/R^2, 1]^l\), it holds \(\tilde{\gamma}(t) = \tilde{\gamma}_0(t)\). From this, we observe that there is no \(t \in \partial [1/R^2, 1]^l\) with \(\tilde{\gamma}(t) = 0\). Indeed, for any \(j \in \mathcal{Y}\),
\[
\phi_{\lambda, j}'(\gamma_0(t))\gamma_0(t) = I_j'(t_j R w_j)(t_j R w_j).
\]
This form, if \(t \in \partial [1/R^2, 1]^l\), then \(t_j = 1\) or \(t_j = \frac{1}{R}\), for some \(j \in \mathcal{Y}\). Consequently,
\[
\phi_{\lambda, j}'(\gamma_0(t))\gamma_0(t) = I_j'(R w_j)(R w_j) \quad \text{or} \quad \phi_{\lambda, j}'(\gamma_0(t))\gamma_0(t) = I_j'\left(\frac{1}{R} w_j\right)\left(\frac{1}{R} w_j\right).
\]
Therefore, if \(\phi_{\lambda, j}'(\gamma_0(t))\gamma_0(t) = 0\), we get \(I_j(R w_j) \geq c_{j_0}\) or \(I_j'\left(\frac{1}{R} w_j\right) \geq c_{j_0}\), which is a contradiction with (6.1).

Now, we compute the degree \(\deg(\tilde{\gamma}, (1/R^2, 1)^l, (0, \ldots, 0))\). Since
\[
\deg(\tilde{\gamma}, (1/R^2, 1)^l, (0, \ldots, 0)) = \deg(\tilde{\gamma}_0, (1/R^2, 1)^l, (0, \ldots, 0)),
\]
and, for \(t \in (1/R^2, 1)^l\),
\[
\tilde{\gamma}_0(t) = 0 \iff t = \left(\frac{1}{R}, \ldots, \frac{1}{R}\right),
\]
we derive
\[
\deg(\tilde{\gamma}, (1/R^2, 1)^l, (0, \ldots, 0)) \neq 0.
\]
This shows what was stated. \(\square\)

**Proposition 6.3** If \(c_{\lambda, \gamma} = \sum_{j=1}^{l} c_{\lambda, j}\) and \(c_{\gamma} = \sum_{j=1}^{l} c_{j}\), then

(i) \(c_{\lambda, \gamma} \leq b_{\lambda, \gamma} \leq c_{\gamma}, \forall \lambda \geq 1\);

(ii) \(b_{\lambda, \gamma} \to c_{\gamma}\), as \(\lambda \to \infty\);

(iii) \(\phi_{\lambda, j}(\gamma(t)) < c_{\gamma}, \forall \lambda \geq 1, \gamma \in \Gamma_*\) and \(t = (t_1, \ldots, t_l) \in \partial [1/R^2, 1]^l\).

**Proof** (i) Once \(\gamma_0 \in \Gamma_*\),
\[
b_{\lambda, \gamma} \leq \max_{(t_1, \ldots, t_l) \in [1/R^2, 1]^l} \phi_{\lambda, j}(\gamma_0(t_1, \ldots, t_l)) = \max_{(t_1, \ldots, t_l) \in [1/R^2, 1]^l} \sum_{j=1}^{l} I_j(t_j R w_j) = c_{\gamma}.
\]
Now, fixing \( s = (s_1, \ldots, s_l) \in [1/R^2, 1]^l \) given in Lemma 6.2 and recalling that
\[
c_{\lambda, j} = \inf \left\{ \phi_{\lambda, j}(u) : u \in W^{1, p(x)}(\Omega_j') \setminus \{0\} \text{ and } \phi_{\lambda, j}'(u)u = 0 \right\},
\]
it follows that
\[
\phi_{\lambda, j}(\gamma(s)) \geq c_{\lambda, j}, \forall j \in \Upsilon.
\]
From (3.9),
\[
\phi_{\lambda, R^N \setminus \Omega'_\Upsilon}(u) \geq 0, \forall u \in W^{1, p(x)}(R^N \setminus \Omega'_\Upsilon),
\]
which leads to
\[
\phi_{\lambda}(\gamma(t)) \geq \sum_{j=1}^l \phi_{\lambda, j}(\gamma(t)), \forall t = (t_1, \ldots, t_l) \in [1/R^2, 1]^l.
\]
Thus
\[
\max_{(t_1, \ldots, t_l) \in [1/R^2, 1]^l} \phi_{\lambda}(\gamma(t)) \geq \phi_{\lambda}(\gamma(s)) \geq c_{\lambda, \Upsilon},
\]
showing that
\[
b_{\lambda, \Upsilon} \geq c_{\lambda, \Upsilon};
\]
(ii) This limit is clear by the previous item, since we already know \( c_{\lambda, j} \to c_j \), as \( \lambda \to \infty \); (iii) For \( t = (t_1, \ldots, t_l) \in \partial [1/R^2, 1]^l \), it holds \( \gamma(t) = \gamma_0(t) \). From this,
\[
\phi_{\lambda}(\gamma(t)) = \sum_{j=1}^l I_j(t_j R w_j).
\]
Writing
\[
\phi_{\lambda}(\gamma(t)) = \sum_{j=1}^l I_j(t_j R w_j) + I_{j_0}(t_{j_0} R w_{j_0}),
\]
where \( t_{j_0} \in \left\{ \frac{1}{R^2}, 1 \right\} \), from (6.1) we derive
\[
\phi_{\lambda}(\gamma(t)) \leq c_\Upsilon - \epsilon,
\]
for some \( \epsilon > 0 \), so (iii).

\[\square\]

**Corollary 6.4** \( b_{\lambda, \Upsilon} \) is a critical value of \( \phi_{\lambda} \), for \( \lambda \) sufficiently large.

**Proof** Assume \( b_{\lambda, \Upsilon} \) is not a critical value of \( \phi_{\lambda} \) for some \( \tilde{\lambda} \). We will prove that exists \( \lambda_1 \) such that \( \tilde{\lambda} < \lambda_1 \). Indeed, by item (iii) of Proposition 6.3, we have seen that
\[
\phi_{\lambda}(\gamma_0(t)) < c_\Upsilon, \forall \lambda \geq 1, t \in \partial [1/R^2, 1]^l.
\]
This way
\[
\mathcal{M} = \max_{t \in \partial [1/R^2, 1]^l} \phi_{\lambda}(\gamma_0(t)) < c_\Upsilon.
\]
Since $b_{\lambda,\Upsilon} \to c_\Upsilon$ (item (ii) of Proposition 6.3), there exists $\lambda_1 > 1$ such that if $\lambda \geq \lambda_1$, then

$$\mathcal{M} < b_{\lambda,\Upsilon}.$$  

So, if $\lambda \geq \lambda_1$, we can find $\tau = \tau(\lambda) > 0$ small enough, with the ensuing property

$$\mathcal{M} < b_{\lambda,\Upsilon} - 2\tau. \quad (6.2)$$

From the deformation’s lemma [31, Page 38], there is $\eta: E_{\lambda} \to E_{\lambda}$ such that

$$\eta\left(\phi_{\lambda}^{b_{\lambda,\Upsilon} + \tau}\right) \subset \phi_{\lambda}^{b_{\lambda,\Upsilon} - \tau} \text{ and } \eta(u) = u, \text{ for } u \neq \phi_{\lambda}^{-1}\left([b_{\lambda,\Upsilon} - 2\tau, b_{\lambda,\Upsilon} + 2\tau]\right).$$

Then, by (6.2),

$$\eta(\gamma_0(t)) = \gamma_0(t), \quad \forall t \in \partial[1/R^2, 1]^d.$$  

Now, using the definition of $b_{\lambda,\Upsilon}$, there exists $\gamma_\star \in \Gamma_{\star}$ satisfying

$$\max_{t \in [1/R^2, 1]^d} \phi_{\lambda}^{\tau}(\gamma_\star(t)) < b_{\lambda,\Upsilon} + \tau. \quad (6.3)$$

Defining

$$\tilde{\gamma}(t) = \eta(\gamma_\star(t)), \quad t \in [1/R^2, 1]^d,$$

due to (6.3), we obtain

$$\phi_{\lambda}^{\tau}(\tilde{\gamma}(t)) \leq b_{\lambda,\Upsilon} - \tau, \quad \forall t \in [1/R^2, 1]^d.$$  

But since $\tilde{\gamma} \in \Gamma_{\star}$, we deduce

$$b_{\lambda,\Upsilon} \leq \max_{t \in [1/R^2, 1]^d} \phi_{\lambda}^{\tau}(\tilde{\gamma}(t)) \leq b_{\lambda,\Upsilon} - \tau,$$

a contradiction. So, $\tilde{\lambda} < \lambda_1$. \hfill \Box

7 The proof of the main theorem

To prove Theorem 1.1, we need to find nonnegative solutions $u_\lambda$ for large values of $\lambda$, which converges to a least energy solution in each $\Omega_j$ ($j \in \Upsilon$) and to 0 in $\Omega_c^\Upsilon$ as $\lambda \to \infty$. To this end, we will show two propositions which together with the Propositions 4.1 and 5.1 will imply that Theorem 1.1 holds.

Henceforth, we denote by

$$r = R^{p^+} \sum_{j=1}^l \left( \frac{1}{p^+} - \frac{1}{\theta} \right)^{-1} c_j, \quad B_r^j = \{ u \in E_\lambda : \varrho_{\lambda}(u) \leq r \}$$

and

$$\phi_{\lambda}^{c_j} = \{ u \in E_\lambda : \phi_{\lambda}(u) \leq c_j \}.$$  

Moreover, for small values of $\mu$,

$$\mathcal{A}_{\mu}^j = \{ u \in B_r^j : \varrho_{\lambda,\mathbb{R}^N \setminus \Omega_\Upsilon}(u) \leq \mu, \ |\phi_{\lambda,j}(u) - c_j| \leq \mu, \ \forall j \in \Upsilon \}.$$  

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We observe that
\[ w = \sum_{j=1}^{l} w_j \in A^\mu_{\lambda} \cap \phi^C_{\lambda}, \]
showing that \( A^\mu_{\lambda} \cap \phi^C_{\lambda} \neq \emptyset \). Fixing
\[ 0 < \mu < \frac{1}{4} \min_{j \in \Gamma} c_j, \quad (7.1) \]
we have the following uniform estimate of \( \| \phi'_{\lambda}(u) \| \) on the region \( (A^\lambda_{2\mu} \setminus A^\lambda_{\mu}) \cap \phi^C_{\lambda} \).

**Proposition 7.1** Let \( \mu > 0 \) satisfying (7.1). Then, there exist \( \Lambda_* \geq 1 \) and \( \sigma_0 > 0 \) independent of \( \lambda \) such that
\[ \| \phi'_{\lambda}(u) \| \geq \sigma_0, \quad \text{for } \lambda \geq \Lambda_* \text{ and all } u \in (A^\lambda_{2\mu} \setminus A^\lambda_{\mu}) \cap \phi^C_{\lambda}. \quad (7.2) \]

**Proof** We assume that there exist \( \lambda_n \to \infty \) and \( u_n \in (A^\lambda_{2\mu} \setminus A^\lambda_{\mu}) \cap \phi^C_{\lambda_n} \) such that
\[ \| \phi'_{\lambda_n}(u_n) \| \to 0. \]
Since \( u_n \in A^\lambda_{2\mu} \), this implies \( (\phi_{\lambda_n}(u_n)) \) is a bounded sequence and, consequently, it follows that \( (\phi_{\lambda_n}(u_n)) \) is also bounded. Thus, passing a subsequence if necessary, we can assume \( \phi_{\lambda_n}(u_n) \) converges. Thus, from Proposition 4.1, there exists \( 0 \leq u \in \nabla_{1,p}(\Omega_\Gamma) \) such that \( u|_{\Omega_j}, j \in \Upsilon, \) is a solution for \( (P_j), \)
\[ \phi_{\lambda_n}(u_n) \to 0 \quad \text{and} \quad \phi_{\lambda_n,j}(u_n) \to I_j(u). \]

We know that \( c_j \) is the least energy level for \( I_j \). So, if \( u|_{\Omega_j} \neq 0 \), then \( I_j(u) \geq c_j \). But since \( \phi_{\lambda_n}(u_n) \leq c_\Gamma \), we must analyze the following possibilities:

(i) \( I_j(u) = c_j, \forall j \in \Upsilon; \)
(ii) \( I_{j_0}(u) = 0, \) for some \( j_0 \in \Upsilon. \)

If (i) occurs, then for \( n \) large, it holds
\[ \phi_{\lambda_n}(u_n) \leq \mu \quad \text{and} \quad \phi_{\lambda_n,j}(u_n) - c_j \leq \mu, \forall j \in \Upsilon. \]
So \( u_n \in A^\lambda_{\mu}, \) a contradiction.

If (ii) occurs, then
\[ \phi_{\lambda_n,j_0}(u_n) - c_{j_0} \to c_{j_0} > 4\mu, \]
which is a contradiction with the fact that \( u_n \in A^\lambda_{2\mu}. \) Thus, we have completed the proof. \( \square \)

**Proposition 7.2** Let \( \mu > 0 \) satisfying (7.1) and \( \Lambda_* \geq 1 \) given in the previous proposition. Then, for \( \lambda \geq \Lambda_* \), there exists a solution \( u_{\lambda} \) of \( (A_{\lambda}) \) such that \( u_{\lambda} \in A^\lambda_{\mu} \cap \phi^C_{\lambda}. \)

**Proof** Let \( \lambda \geq \Lambda_* \). Assume that there are no critical points of \( \phi_{\lambda} \) in \( A^\lambda_{\mu} \cap \phi^C_{\lambda}. \) Since \( \phi_{\lambda} \) is a \( (PS) \) functional, there exists a constant \( d_{\lambda} > 0 \) such that
\[ \| \phi'_{\lambda}(u) \| \geq d_{\lambda}, \quad \text{for all } u \in A^\lambda_{\mu} \cap \phi^C_{\lambda}. \]
\( \square \)
From Proposition 7.1, we have
\[ \| \phi'_\lambda(u) \| \geq \sigma_0, \quad \text{for all } u \in (A_{2\mu}^\lambda \setminus A_\mu^\lambda) \cap \phi_{\lambda}^{CY}, \]
where \( \sigma_0 > 0 \) does not depend on \( \lambda \). In what follows, \( \Psi : E_\lambda \to \mathbb{R} \) is a continuous functional verifying
\[ \Psi(u) = 1, \quad \text{for } u \in A_{2\mu}^\lambda, \quad \Psi(u) = 0, \quad \text{for } u \notin A_{2\mu}^\lambda \quad \text{and} \quad 0 \leq \Psi(u) \leq 1, \quad \forall u \in E_\lambda. \]
We also consider \( H : \phi_{\lambda}^{CY} \to E_\lambda \) given by
\[ H(u) = \begin{cases} -\Psi(u)\| Y(u) \|^{-1} Y(u), & \text{for } u \in A_{2\mu}^\lambda, \\ 0, & \text{for } u \notin A_{2\mu}^\lambda, \end{cases} \]
where \( Y \) is a pseudo-gradient vector field for \( \Phi_\lambda \) on \( K = \{ u \in E_\lambda : \phi'_\lambda(u) \neq 0 \} \). Observe that \( H \) is well defined, once \( \phi'_\lambda(u) \neq 0 \), for \( u \in A_{2\mu}^\lambda \cap \phi_{\lambda}^{CY} \). The inequality
\[ \| H(u) \| \leq 1, \quad \forall \lambda \geq \Lambda_\ast \quad \text{and} \quad u \in \phi_{\lambda}^{CY}, \]
guarantees that the deformation flow \( \eta : [0, \infty) \times \phi_{\lambda}^{CY} \to \phi_{\lambda}^{CY} \) defined by
\[ \frac{d\eta}{dt} = H(\eta), \quad \eta(0, u) = u \in \phi_{\lambda}^{CY} \]
verifies
\[ \frac{d}{dt} \phi_{\lambda}(\eta(t, u)) \leq -\frac{1}{2} \Psi(\eta(t, u))\| \phi'_\lambda(\eta(t, u)) \| \leq 0, \quad (7.3) \]
\[ \left\| \frac{d\eta}{dt} \right\|_{\lambda} = \left\| H(\eta) \right\|_{\lambda} \leq 1 \quad (7.4) \]
and
\[ \eta(t, u) = u \quad \text{for all } t \geq 0 \quad \text{and} \quad u \in \phi_{\lambda}^{CY} \setminus A_{2\mu}^\lambda. \quad (7.5) \]

We study now two paths, which are relevant for what follows:

- The path \( t \mapsto \eta(t, \gamma_0(t)), \) where \( t = (t_1, \ldots, t_l) \in [1/R^2, 1]^l \).

The definition of \( \gamma_0 \) combined with the condition on \( \mu \) gives
\[ \gamma_0(t) \notin A_{2\mu}^\lambda, \quad \forall t \in \partial[1/R^2, 1]^l. \]
Since
\[ \phi_{\lambda}(\gamma_0(t)) < c_T, \quad \forall t \in \partial[1/R^2, 1]^l, \]
from (7.5), it follows that
\[ \eta(t, \gamma_0(t)) = \gamma_0(t), \quad \forall t \in \partial[1/R^2, 1]^l. \]

So, \( \eta(t, \gamma_0(t)) \in \Gamma_\ast, \) for each \( t \geq 0 \).

- The path \( t \mapsto \gamma_0(t), \) where \( t = (t_1, \ldots, t_l) \in [1/R^2, 1]^l. \)

We observe that
\[ \text{supp} (\gamma_0(t)) \subset \overline{\Omega_T} \]
and
\[ \phi_{\lambda}(\gamma_0(t)) \text{ does not depend on } \lambda \geq 1, \]
forall $t \in [1/R^2, 1]$]. Moreover,

$$\phi_\lambda (\gamma_0 (t)) \leq c_\Gamma, \ \forall t \in [1/R^2, 1]$$

and

$$\phi_\lambda (\gamma_0 (t)) = c_\Gamma \text{ if, and only if, } t_j = \frac{1}{R}, \ \forall j \in \mathcal{Y}.$$ 

Therefore

$$m_0 = \sup \left\{ \phi_\lambda (u) : u \in \gamma_0 ([1/R^2, 1]) \setminus A^\lambda_\mu \right\}$$

is independent of $\lambda$ and $m_0 < c_\Gamma$. Now, observing that there exists $K_* > 0$ such that

$$|\phi_\lambda, j (u) - \phi_\lambda, j (v)| \leq K_* \| u - v \|_{\lambda, \omega_j}, \ \forall u, v \in B^\lambda_\mu \text{ and } \forall j \in \mathcal{Y},$$

we derive

$$\max_{t \in [1/R^2, 1]} \phi_\lambda \left( \eta (T, \gamma_0 (t)) \right) \leq \max \left\{ m_0, c_\Gamma - \frac{1}{2} \sigma_0 \mu \right\}, \quad (7.6)$$

for $T > 0$ large.

In fact, writing $u = \gamma_0 (t), \ t \in [1/R^2, 1]$, if $u \notin A^\lambda_\mu$, from (7.3),

$$\phi_\lambda (\eta (t, u)) \leq \phi_\lambda (u) \leq m_0, \ \forall t \geq 0,$$

and we have nothing more to do. We assume then $u \in A^\lambda_\mu$ and set

$$\tilde{\eta} (t) = \eta (t, u), \ \tilde{d}_\lambda = \min \{ d_\lambda, \sigma_0 \} \text{ and } T = \frac{\sigma_0 \mu}{K_* \tilde{d}_\lambda}.$$ 

Now, we will analyze the ensuing cases:

**Case 1:** $\tilde{\eta} (t) \in A^{\lambda}_{2\mu} \forall t \in [0, T]$.

**Case 2:** $\tilde{\eta} (t_0) \in \partial A^{\lambda}_{2\mu}$, for some $t_0 \in [0, T]$.

**Analysis of Case 1**

In this case, we have $\Psi (\tilde{\eta} (t)) = 1$ and $\left\| \phi_\lambda (\tilde{\eta} (t)) \right\| \geq \tilde{d}_\lambda$ for all $t \in [0, T]$. Hence, from (7.3),

$$\phi_\lambda (\tilde{\eta} (T)) = \phi_\lambda (u) + \int_0^T \frac{d}{ds} \phi_\lambda (\tilde{\eta} (s)) \ ds \leq c_\Gamma - \frac{1}{2} \int_0^T \tilde{d}_\lambda \ ds,$$

that is,

$$\phi_\lambda (\tilde{\eta} (T)) \leq c_\Gamma - \frac{1}{2} \tilde{d}_\lambda T = c_\Gamma - \frac{1}{2} \sigma_0 \mu,$$

showing (7.6).

**Analysis of Case 2**

In this case, there exist $0 \leq t_1 \leq t_2 \leq T$ satisfying

$$\tilde{\eta} (t_1) \in \partial A^{\lambda}_{\mu}, \quad \tilde{\eta} (t_2) \in \partial A^{\lambda}_{2\mu},$$

$$\tilde{\eta} (t_1) \in \partial A^{\lambda}_{\mu}, \quad \tilde{\eta} (t_2) \in \partial A^{\lambda}_{2\mu},$$
and
\[ \tilde{\eta}(t) \in A^\lambda_{2^\mu} \setminus A^\lambda_{\mu}, \forall t \in (t_1, t_2]. \]

We claim that
\[ \| \tilde{\eta}(t_2) - \tilde{\eta}(t_1) \| \geq \frac{1}{2K^* \mu}. \]

Setting \( w_1 = \tilde{\eta}(t_1) \) and \( w_2 = \tilde{\eta}(t_2) \), we get
\[ \rho_{\lambda, R \setminus \Omega_\gamma}(w_2) = \frac{3}{2} \mu \quad \text{or} \quad |\varphi_{\lambda, j_0}(w_2) - c_{j_0}| = \frac{3}{2} \mu, \]
for some \( j_0 \in \Upsilon \). We analyze the latter situation, once that the other one follows the same reasoning. From the definition of \( A^\lambda_{\mu} \),
\[ |\varphi_{\lambda, j_0}(w_1) - c_{j_0}| \leq \mu, \]
consequently,
\[ \| w_2 - w_1 \| \geq \frac{1}{K^*} |\varphi_{\lambda, j_0}(w_2) - \varphi_{\lambda, j_0}(w_1)| \geq \frac{1}{2K^* \mu}. \]

Then, by mean value theorem, \( t_2 - t_1 \geq \frac{1}{2K^* \mu} \) and, this form,
\[ \varphi_{\lambda}(\tilde{\eta}(T)) \leq \varphi_{\lambda}(u) - \int_0^T \Psi(\tilde{\eta}(s)) \| \varphi'_{\lambda}(\tilde{\eta}(s)) \| ds \]
implying
\[ \varphi_{\lambda}(\tilde{\eta}(T)) \leq c_\gamma - \int_{t_1}^{t_2} \sigma_0 \, ds = c_\gamma - \sigma_0(t_2 - t_1) \leq c_\gamma - \frac{1}{2K^*} \sigma_0 \mu, \]
which proves 7.6. Fixing \( \varphi(t_1, \ldots, t_l) = \eta(T, \gamma_0(t_1, \ldots, t_l)) \), we have that \( \tilde{\eta} \in \Gamma_* \) and, hence,
\[ b_{\lambda, \gamma} \leq \max_{(t_1, \ldots, t_l) \in \{1/R^2, 1\}} \varphi_{\lambda}(\tilde{\eta}(t_1, \ldots, t_l)) \leq \max \left\{ m_0, c_\gamma - \frac{1}{2K^*} \sigma_0 \mu \right\} < c_\gamma, \]
which contradicts the fact that \( b_{\lambda, \gamma} \to c_\gamma \). \( \square \)

**Proof of Theorem 1.1** According Proposition 7.2, for \( \mu \) satisfying (7.1) and \( \Lambda_* \geq 1 \), there exists a solution \( u_{\lambda} \) for \( (A_{\lambda}) \) such that \( u_{\lambda} \in A^\lambda_{\mu} \cap \phi_{\lambda}^{C_\gamma} \), for all \( \lambda \geq \Lambda_* \).

**Claim:** There are \( \lambda_0 \geq \Lambda_* \) and \( \mu_0 > 0 \) small enough, such that \( u_{\lambda} \) is a solution for \( (P_{\lambda}) \) for \( \lambda \geq \Lambda_0 \) and \( \mu \in (0, \mu_0) \).

Indeed, assume by contradiction that there are \( \lambda_n \to \infty \) and \( \mu_n \to 0 \), such that \( (u_{\lambda_n}) \) is not a solution for \( (P_{\lambda_n}) \). From Proposition 7.2, the sequence \( (u_{\lambda_n}) \) verifies:

(a) \( \phi'_{\lambda_n}(u_{\lambda_n}) = 0, \forall n \in \mathbb{N}; \)
(b) \( \rho_{\lambda_n, R \setminus \Omega_\gamma}(u_{\lambda_n}) \to 0; \)
(c) \( \phi_{\lambda_n, j}(u_{\lambda_n}) \to c_j, \forall j \in \Upsilon. \)
The item (b) ensures we can use Proposition 5.1 to deduce $u_{\lambda_n}$ is a solution for $(P_{\lambda_n})$, for large values of $n$, which is a contradiction, showing this way the claim.

Now, our goal is to prove the second part of the theorem. To this end, let $(u_{\lambda_n})$ be a sequence verifying the above limits. Since $\Phi_{\lambda_n}(u_{\lambda_n})$ is bounded, passing a subsequence, we obtain that $\Phi_{\lambda_n}(u_{\lambda_n}) \to c$. This way, using Proposition 4.1 combined with item (c), we derive $u_{\lambda_n}$ converges in $W^{1,p(x)}(\mathbb{R}^N)$ to a function $u \in W^{1,p(x)}(\mathbb{R}^N)$, which satisfies $u = 0$ outside $\Omega$ and $u|_{\Omega_j}, \ j \in \N$, is a least energy solution for

\begin{align*}
-\Delta_{p(x)} u + Z(x)u &= f(u), \quad \text{in } \Omega_j, \\
u \in W^{1,p(x)}_0(\Omega_j), \ u \geq 0, \quad \text{in } \Omega_j.
\end{align*}

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