THE HADAMARD FORMULA AND THE RAYLEIGH-FABER-KRAHN
INEQUALITY FOR NONLOCAL EIGENVALUE PROBLEMS

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Abstract. In this paper we obtain a Hadamard type formula for simple eigenvalues and an
analog to the Rayleigh-Faber-Krahn inequality for a class of nonlocal eigenvalue problems. Such
class of equations include among others, the classical nonlocal problems with Dirichlet and Neu-
mann conditions. The Hadamard formula is computed allowing domain perturbations given by
embeddings of $n$-dimensional Riemannian manifolds (possibly with boundary) of finite volume
while the Rayleigh-Faber-Krahn inequality is shown by rearrangement techniques.

1. Introduction

There are many works in the literature which connect the shape of a region to the eigenvalues
and eigenfunctions of a given operator. For instance, the minimization of eigenvalues (or combi-
nation of them) has attracted a lot of attention since the early part of the twentieth century. As
far as we know, this issue first came out in the famous book of Rayleigh entitled The theory of
sound [35] where he conjectured that the disk should minimize the first Dirichlet eigenvalue of the
Laplacian among all open sets of given measure.

In the 1920’s, Rayleigh’s conjecture was simultaneously proved by Faber [13] and Krahn [25]
using rearrangement techniques. Such result is called Rayleigh-Faber-Krahn inequality being one
of the most famous isoperimetric inequalities. Naturally similar questions have been investigated
for other eigenvalues as well as for other operators. For instance, we can mention the Payne-Pólya-
Weinberger isoperimetric inequality for the quotient of the first two Dirichlet eigenvalues of the
Laplacian [2]; the Szegő-Weinberger inequality, which is an isoperimetric inequality for the first
nontrivial Neumann eigenvalue of the Laplacian [39, 40]; and [3] where the Rayleigh’s conjecture
has been considered for the clamped plate. For other examples and a more complete bibliography
about these issues, we refer to the following surveys [7, 20, 21].

Notice that the importance of such kinds of results in analysis, calculus of variations and
applied mathematics is self-evident. Therefore, the development of more general techniques and
approaches to deal with the optimization of functions depending on the shape of the domains and
the eigenvalues of a given operator are required. In this context, the rate of change of simple
eigenvalues play an essential role and it has been studied since the pioneering work of Hadamard
[18] who first computed the domain derivative of a simple Laplace eigenvalue under Dirichlet
boundary conditions in 1908.

Since then, the Hadamard formula has been generalized in a number of significant ways. Such
generalizations include the use of Neumann and Robin boundary conditions, multiple eigenvalues,
and second order variations for a large class of differential and integral operators. Among many
references, we cite the monographs [22, 23, 27, 34] and the recent works [16, 17, 28, 30], all of them
concerned with boundary perturbation problems to differential equations and their applications
to eigenvalue problems. In particular, we mention [32] where a proof of the Rayleigh-Faber-Krahn
inequality is obtained as a consequence of the analysis of the Hadamard formula for the first
eigenvalue.

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In this work, we study a class of nonlocal eigenvalue problems with non-singular kernels in $n$-dimensional Riemannian manifolds of finite volume. Our main goal is two fold. We obtain a Hadamard type formula for simple eigenvalues, and an analog of the Rayleigh-Faber-Krahn inequality. The Hadamard formula is computed in Section 4 using the approach developed in [36] to deal with boundary perturbation problems. The analog of the Rayleigh-Faber-Krahn inequality is obtained in Section 4 for open bounded sets in $\mathbb{R}^n$ as a direct consequence of the rearrangement techniques and the Riesz rearrangement inequality shown respectively in [7] and [6].

2. Our nonlocal eigenvalue problem

We consider an $n$-dimensional Riemannian manifold $(\mathcal{M}, g)$ (possibly with boundary) of finite volume and the following nonlocal eigenvalue problem

$$ (2.1) \quad a_{\mathcal{M}}(x)u(x) - \int_{\mathcal{M}} J(x, y)u(y)dy = \lambda(\mathcal{M}) u(x), \quad x \in \mathcal{M} $$

for some unknown value $\lambda(\mathcal{M})$ where $a_{\mathcal{M}} : \overline{\mathcal{M}} \to \mathbb{R}$ is assumed to be a continuous function, and $J$ is a non-singular kernel satisfying

$$(\mathcal{H}) \quad J \in C(\mathcal{M} \times \mathcal{M}, \mathbb{R}) \text{ is a nonnegative, symmetric function } (J(x, y) = J(y, x))$$

with $J(x, x) > 0$.

We also assume that $\int_{\mathcal{M}} J(x, y)dy < \infty$.

Remark 2.1. Here, $dy$ refers to the measure on the manifold, which in coordinates is equivalent to $\sqrt{g(x)}dx$ and $g$ is the determinant of the matrix $g_{ij}$.

Notice that analysing the spectral properties of (2.1) is equivalent to study the spectrum of the linear operator $B_{\mathcal{M}} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ given by

$$ (2.2) \quad B_{\mathcal{M}}u(x) = a_{\mathcal{M}}(x)u(x) - \int_{\mathcal{M}} J(x, y)u(y)dy, \quad x \in \mathcal{M}. $$

See that $B_{\mathcal{M}}$ is the difference of the multiplication operator $a_{\mathcal{M}}$, which maps $u(x) \mapsto a_{\mathcal{M}}(x)u(x)$, and the integral operator $J_{\mathcal{M}} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ given by

$$ J_{\mathcal{M}}u(x) = \int_{\mathcal{M}} J(x, y)u(y)dy, \quad x \in \mathcal{M} $$

which is self-adjoint and compact by [36] Propositions 3.5 and 3.7 since $\mathcal{M}$ is a measurable metric space.

The prototype of the nonlocal equation given by $B_{\mathcal{M}}$ is the Dirichlet problem which is defined by $a_{\mathcal{M}}(x) \equiv 1$. For instance, if one takes $\mathcal{M} = \Omega \subset \mathbb{R}^n$, $J(x, y) = J(|x-y|)$, and assume $u(x) \equiv 0$ in $\mathbb{R}^n \setminus \Omega$ with $\int_{\mathbb{R}^n} J(z)dz = 1$, the operator $B_\Omega$ becomes

$$ B_\Omega u(x) = \int_{\mathbb{R}^n} J(|x-y|)(u(x) - u(y))dy, \quad x \in \Omega. $$

Notice that in this case, $\Omega^c = \mathbb{R}^n \setminus \Omega$ is a hostile surrounding since the particles (whose density is set by $u$) die when they land in $\Omega^c$. As observed, for instance in [1] [13], this is a nonlocal analog to the Laplace operator with Dirichlet boundary condition in bounded open sets of $\mathbb{R}^n$.

On the other hand, if we take $a_\Omega(x) = \int_\Omega J(|x-y|)dy$, we get a nonlocal analog to the Laplacian with Neumann boundary condition

$$ B_\Omega u(x) = \int_\Omega J(|x-y|)(u(x) - u(y))dy, \quad x \in \Omega. $$

In this case the particle can just jump inside of $\Omega$ living in an isolated surrounding. As expected, under this Neumann condition, the constant function satisfies equation (2.1) whenever one takes $\lambda(\Omega) = 0$. 

R. D. BENGURIA, M. C. PEREIRA AND M. SÁEZ
According to [29], nonlocal diffusion equations such as [24] were used early in population genetics models, see for instance [11]. In Ecology, Othmer et al. [31] were the first authors to introduce a jump process to model the dispersion of individuals, which later, was generalised by Hutson et al. [24] associating $J$ to a radial probability density.

Actually, several continuous models for species and human mobility have been proposed using such nonlocal equations, in order to look for more realistic dispersion models [4, 6, 10, 28]. Besides the applied models with such kernels, the mathematical interest is mainly due to the fact that, in general, there is no regularizing effect and therefore no general compactness tools are available.

The spectrum of $B_M$. It is known from [30] Theorem 3.24 (see also [29] Theorem 2.2) for open bounded sets $\mathcal{M} = \Omega \subset \mathbb{R}^n$ that the spectrum set $\sigma(B_M)$ of $B_M$ satisfies

$$\sigma(B_M) = R(a_M I) \cup \{ \lambda_n(\mathcal{M}) \}_{n=0}^l$$

for some $l \in \{0, 1, ..., \infty\}$ where $R(a_M I)$ denotes de range of the map $a_M I$ and $\lambda_n(\mathcal{M})$ are the eigenvalues of $B_M$ with finite multiplicity. Also, the essential spectrum of $B_M$ is given by

$$\sigma_{ess}(B_M) = [m, M]$$

where

$$m = \min_{x \in \mathcal{M}} a_M(x) \quad \text{and} \quad M = \max_{x \in \mathcal{M}} a_M(x).$$

As a consequence of the characterization (2.3), we notice that the eigenfunctions of $B_M$ possess the same regularity of the functions $J$ and $a_M$. In fact, for all $x \in \mathcal{M}$, one has

$$B_M u(x) = \lambda(\mathcal{M}) u(x) \iff (a_M(x) - \lambda(\mathcal{M})) u(x) = \int_{\mathcal{M}} J(x, y) u(y) dy.$$  

On the other hand, the convolution-type operator $(J * u)(x) = \int_{\mathcal{M}} J(x, y) u(y) dy \in C^k(\mathcal{M})$ whenever $J(\cdot, y) \in C^k(\mathcal{M})$ for every $y \in \mathcal{M}$ and $u \in L^1(\mathcal{M})$. Therefore, if $\lambda(\mathcal{M})$ is an eigenvalue of $B_M$ with corresponding eigenfunction $u$, we obtain from (2.3) that $\lambda(\mathcal{M}) \in [m, M]^c$ implying that $a_M - \lambda(\mathcal{M}) \neq 0$ in $\mathcal{M}$. Consequently, we get from (2.3) that

$$u \in C^k(\mathcal{M}) \quad \text{whenever} \quad J(\cdot, y) \text{ and } a_M \text{ are } C^k\text{-functions for every fixed } y \in \mathcal{M}$$

for $k = 0, 1, 2, ...$

Under appropriate conditions, the existence of the principal eigenvalue of $B_M$ is guaranteed by [29] Theorem 2.1. Recall that the principal eigenvalue of a linear and bounded operator is the minimum of the real part of the spectrum which is simple, isolated and it is associated with a continuous and strictly positive eigenfunction.

3. Hadamard formula for simple eigenvalues

Now let us perturb simple eigenvalues of the operator $B_M$ computing derivatives with respect to several kinds of variations of the manifold $\mathcal{M}$. In the particular case of $\mathcal{M} = \Omega \subset \mathbb{R}^n$ our approach agrees with the one introduced in [29] for perturbing a fixed domain $\Omega$ by diffeomorphisms. As a consequence, we extend the expressions obtained to the domain derivative for simple eigenvalues given in [8, 14].

Let $(\mathcal{M}, g_\mathcal{M})$ and $(\mathcal{N}, g_\mathcal{N})$ be $C^1$-regular manifolds ($\mathcal{M}$, possibly with boundary). Assume in addition that $\mathcal{M}$ is compact. If $h : \mathcal{M} \mapsto \mathcal{N}$ is a $C^1$ imbedding, i.e., a diffeomorphism to its image, we set the composition map $h^*$ (also called the pull-back) by

$$h^* v(x) = (v \circ h)(x), \quad x \in \mathcal{M},$$

when $v$ is any given function defined on $h(\mathcal{M})$. The metric on $\mathcal{N}$ induces the pullback metric on $\mathcal{M}$ through $h$ as follows: for $u, v \in T_{h^{-1}(x)} \mathcal{M}$ we have $h^* g_\mathcal{N}(u, v) = g_\mathcal{M}(dh_x(u), dh_x(v))$. It is not difficult to see that $h^* : L^2(h(\mathcal{M}), g_\mathcal{N}) \mapsto L^2(\mathcal{M}, h^* g_\mathcal{N})$ is an isomorphism with inverse $(h^*)^{-1} = (h^{-1})^*$. 

We assume that $\mathcal{N}$ has a Riemannian metric $g_{\mathcal{N}}$ and we denote by $g_h = h^* g_{\mathcal{N}}$ the metric on $\mathcal{M}$ induced by the embedding $h$. For instance, if $\mathcal{M} = \Omega \subset \mathbb{R}^3 = \mathcal{N}$ then the metric $h^* g_{\mathcal{N}}$ is given by $g_{ij} = \frac{\partial h}{\partial x_i} \cdot \frac{\partial h}{\partial x_j}$ and in particular, if $h = id_{\mathcal{N}}$ in the interior of $\Omega$ the metrics in $\mathcal{M}$ and $\mathcal{N}$ agree in that set.

In general, for any embedding $h$ we can consider the operator

$$\left( B_{h(\mathcal{M})} \right)(y) = (a_{h(\mathcal{M})} \circ h)(x)(v \circ h)(x) - \left( J_{h(\mathcal{M})} v \right)(h(x))$$

(3.5)

$$= (h^* a_{h(\mathcal{M})})(x)(h^* v)(x) - (h^* J_{h(\mathcal{M})}v)(x)$$

if $y = h(x)$ for $x \in \mathcal{M}$, $a_{h(\mathcal{M})} : h(\mathcal{M}) \mapsto \mathbb{R}$ is assumed to be a continuous function in $h(\mathcal{M})$ for any isomorphism $h$. Notice that $B_{h(\mathcal{M})} : L^2(h(\mathcal{M}), g_N) \mapsto L^2(h(\mathcal{M}), g_N)$ is a self-adjoint operator for any $h$ as is the operator $J_{h(\mathcal{M})}$.

On the other hand, we can use the pull-back operator $h^*$ to consider

$$h^* B_{h(\mathcal{M})} : L^2(\mathcal{M}) \mapsto L^2(\mathcal{M})$$

defined by $h^* B_{h(\mathcal{M})} h^{-1} u(x) = B_{h(\mathcal{M})} (u \circ h)(h^{-1}(x))$. Hence,

(3.6)

$$h^* B_{h(\mathcal{M})} h^{-1} u(x) = (h^* a_{h(\mathcal{M})})(x) u(x) - \left( h^* J_{h(\mathcal{M})} h^{-1} u \right)(x), \quad \forall x \in \mathcal{M}.$$

As it is known, expressions (3.5) and (3.6) are the customary way to describe deformations or motions of regions. Equation (3.5) is called the Lagrangian description, and (3.6) the Eulerian one. The latter is written in fixed coordinates while the Lagrangian is not.

Due to (3.5) and (3.6), it is easy to see that

(3.7)

$$h^* B_{h(\mathcal{M})} h^{-1} u(x) = B_{h(\mathcal{M})} v(y), \quad \text{and} \quad h^* J_{h(\mathcal{M})} h^{-1} u(x) = J_{h(\mathcal{M})} v(y)$$

whenever $y = h(x)$ and $v(y) = (u \circ h^{-1})(y) = h^{-1} u(y)$ for $y \in h(\mathcal{M})$.

Moreover, we have $B_{h(\mathcal{M})} v(y) = \lambda v(y)$ for $y \in h(\mathcal{M})$ and some value $\lambda$, if and only if,

$$h^* B_{h(\mathcal{M})} h^{-1} u(x) = \lambda u(x), \quad \forall x \in \mathcal{M},$$

with $v(y) = h^{-1} u(y)$. Hence, as $B_{h(\mathcal{M})}$ is a self-adjoint operator for any imbedding $h$, we obtain that the spectrum of $h^* B_{h(\mathcal{M})} h^{-1}$ is also a subset of the real line. We have the following:

**Proposition 3.1.** Let $h : \mathcal{M} \mapsto \mathcal{N}$ be an imbedding.

Then, $\sigma \left( h^* B_{h(\mathcal{M})} h^{-1} \right) = \sigma \left( B_{h(\mathcal{M})} \right) \subset \mathbb{R}$ where $\sigma \left( B_{h(\mathcal{M})} \right)$ is given by (2.3). More precisely, $\lambda \in \mathbb{R}$ is an eigenvalue of $B_{h(\mathcal{M})}$, if and only if, is an eigenvalue of $h^* B_{h(\mathcal{M})} h^{-1}$. Also,

$$\sigma_{ess} \left( h^* B_{h(\mathcal{M})} h^{-1} \right) = \sigma_{ess} \left( B_{h(\mathcal{M})} \right).$$

Proof. As $B_{h(\mathcal{M})}$ is a self-adjoint operator in $L^2(h(\mathcal{M}), g_h)$, we have that $\sigma \left( B_{h(\mathcal{M})} \right) \subset \mathbb{R}$. Also, we know from relationship (3.7) that a value $\lambda$ is an eigenvalue of $B_{h(\mathcal{M})}$, if and only if, is an eigenvalue of $h^* B_{h(\mathcal{M})} h^{-1}$. Thus, $\sigma \left( h^* B_{h(\mathcal{M})} h^{-1} \right) = \sigma \left( B_{h(\mathcal{M})} \right) \subset \mathbb{R}$ with $\sigma \left( B_{h(\mathcal{M})} \right)$ given by (2.3).

Now, it follows from (2.3) and expressions (2.2) and (3.6) that

$$\sigma_{ess} \left( B_{h(\mathcal{M})} \right) = [m_h, M_h] \quad \text{and} \quad \sigma_{ess} \left( h^* B_{h(\mathcal{M})} h^{-1} \right) = [m_{h^*}, M_{h^*}]$$

where

$$m_h = \min_{y \in h(\mathcal{M})} a_{h(\mathcal{M})}(y), \quad M_h = \max_{y \in h(\mathcal{M})} a_{h(\mathcal{M})}(y)$$

and

$$m_{h^*} = \min_{x \in \mathcal{M}} a_{h(\mathcal{M})}(h(x)), \quad M_{h^*} = \max_{x \in \mathcal{M}} a_{h(\mathcal{M})}(h(x)),$$

As $m_h = m_{h^*}$ and $M_h = M_{h^*}$, the proof is completed. \[\square\]
Remark 3.1. Notice that Proposition 3.1 guarantees that the essential spectrum of $B_{h(M)}$ does not change under perturbations given by embeddings $h : M \mapsto N$.

From now on, we consider a family of embeddings $h : M \times [0, T] \mapsto N$ that depends on a parameter $t$. We denote the perturbed domain $h(M, t)$ by $M_t$ in order to simplify the notation. We study the differentiability of simple eigenvalues $\lambda(M_t)$ for $B_{M_t}$ with respect to $t$. This corresponds to the Gâteaux derivative with respect to the function $h$.

We remark that for a function $f : N \mapsto \mathbb{R}$ it holds that
\[
\frac{d}{dt} (\langle h^* f \rangle (x, t)) = \langle \frac{d}{dt} (f(h(x, t), t)) \rangle = \langle h^* \nabla f, \frac{\partial h}{\partial t} \rangle + h^* \frac{\partial f}{\partial t},
\]
where $\nabla$ denotes de tangential gradient on $N$. Then we denote
\[
(D_t) = \frac{\partial}{\partial t} - \left( \frac{\partial h}{\partial t} \cdot \nabla \right),
\]
where $\langle \cdot, \cdot \rangle$ denotes the inner product in $N$.

If $N = \mathbb{R}^n$ and $M = \Omega \subset \mathbb{R}^n$ this quantity can be written in coordinates as
\[
D_t = \frac{\partial}{\partial t} - U(t, x) \cdot \frac{\partial}{\partial x} \quad \text{with} \quad U(t, x) = \frac{\partial h^{-1} \partial h}{\partial t} \quad \text{for} \quad x \in \Omega
\]
and it is known as the anti-convective derivative $D_t$ in the reference domain $\Omega$.

We denote by $\text{Diff}^1(M) \subset C^1(M, N)$ the set of $C^1$-functions $h : M \mapsto N$ which are embeddings. We assume that $N$ has a Riemannian metric $g_N$ and we denote by $g_h = h^* g_N$ the metric on $M$ induced by the embedding $h$. For instance, if $M = \Omega \subset \mathbb{R}^n = \mathbb{R}^n$ then the metric $h^* g_N$ is given by $g_{ij} = \langle \partial_x i, h \partial_x j \rangle$ and, if the dimension of $\Omega$ is $n$, the tangent spaces of $h(\Omega)$ and $\mathbb{R}^n$ agree in the interior and the volume element is $|Dh| \, dx$.

Consider the map
\[
F : \text{Diff}^1(M) \times \mathbb{R} \times L^2(M) \mapsto L^2(M) \times \mathbb{R}
\]
\[
(h, \lambda, u) \mapsto \left( (h^* B_{h(M)} h^{-1} - \lambda) u, \int_M u^2(x) \, dv_{g_h} \right).
\]

Here $dv_{g_h}$ is the volume element of the metric on $g_h$. It is not difficult to see that $\text{Diff}^1(M)$ is an open set of $C^1(M, N)$ (which denotes the space of $C^1$-functions from $M$ into $N$ whose derivatives extend continuously to the closure $\overline{M}$ with the usual supremum norm). Hence, $F$ can be seen as a map defined between Banach spaces.

In what follows we will consider that $M \subset N$ (perhaps by identifying $M$ with its image with an initial fixed embedding).

Notice that if $\lambda_0 \in \mathbb{R}$ is an eigenvalue for $B_M$ for some $u_0 \in L^2(M)$ with $\int_M u_0^2(x) \, dx = 1$, then $F(i_M, \lambda_0, u_0) = (0, 1)$ where $i_M \in \text{Diff}^1(M)$ denotes the inclusion map of $M$ into $N$. On the other hand, whenever $F(h, \lambda, u) = (0, 1)$, we have from Proposition 3.1 that
\[
B_M \lambda v(y) = \lambda v(y), \quad y \in M_h, \quad \text{with} \quad \int_{M_h} v^2(y) \, dy = 1
\]
where $v(y) = (u \circ h^{-1})(y)$ for $y \in M_h$. In this way, we can proceed as in [23] using the map $F$ to deal with eigenvalues and eigenfunctions of $B_{M_h}$ and $h^* B_{M_h} h^{-1}$ perturbing the eigenvalue problem to the fixed manifold $M$ by diffeomorphisms $h$.

**Theorem 3.1.** Let $\lambda_0$ be a simple eigenvalue of $B_M$ with corresponding normalized eigenfunction $u_0$ and $J \in C^1(\mathbb{N} \times \mathbb{R}, \mathbb{R})$ satisfying (H). Also, let us assume that $\Phi : \text{Diff}^1(M) \mapsto C^1(M)$ given by $\Phi(h)(x) = (h^* a_{M_h})(x), \quad x \in \Omega$, is differentiable as a map defined between Banach spaces.
Then, there is a neighbourhood $O$ of the inclusion $i_M \in \text{Diff}^1(M)$, and $C^1$-functions $u_h$ and $\lambda_h$ from $O$ into $L^2(M)$ and $\mathbb{R}$ respectively satisfying for all $h \in O$ that
\begin{equation}
(3.9) \quad h^* B_{h(M)} h^{*-1} u_h (x) = \lambda_h u_h (x), \quad x \in M,
\end{equation}
with $u_h \in C^1(M)$.

Moreover, $\lambda_h$ is a simple eigenvalue with $(\lambda_{i_M}, u_{i_M}) = (\lambda_0, u_0)$ and the domain derivative
\begin{equation}
(3.10) \quad \frac{\partial \lambda}{\partial h}(i_M)V = - \int_{\partial M} (a_M(s) - \lambda_0) u_0^2(s) \langle V^T, N(s) \rangle dS + \int_M u_0^2(x) D_t^T (h^* a_{\Omega_h}) \bigg|_{t = 0} dx
\end{equation}
\begin{equation*}
+ \int_M (\lambda_0 - a_0) u_0^2(w) \langle \bar{H}, V^\perp \rangle dv_0(w) - \int_M u_0^2(w) \langle \nabla w a_0(w), V^\perp \rangle dv_0(w),
\end{equation*}
for all $V \in \mathcal{X}^1(N)$ where $\mathcal{X}^1(N)$ denotes the set of $C^1$ vector fields on $N$ and $D_t^T f = \frac{\partial f}{\partial t} - \langle V^T, \nabla f \rangle$, $V^T$ is the component of $V$ tangential to $M$. Note that at the boundary the tangent space of $M$ splits into vectors that are tangential to $\partial M$ (and to $M$) and one vector that is normal to $\partial M$ (and tangential to $M$). Then $N \in T(M)$ denotes this unitary normal vector that is normal to $\partial M$. $\bar{H}$ is the mean curvature vector associated to $M$ and $V^\perp$ is the component of $V$ normal to $M$.

**Proof.** The proof of the existence of the neighbourhood $O \subset \text{Diff}^1(M)$ and the $C^1$-functions $u_h$ and $\lambda_h$ satisfying (3.9) is similar to that one performed in [S] Lemma 4.1. As one can see, it is a consequence of the Implicit Function Theorem applied to the map $F$. Here, we compute the derivative of $\lambda_h$ at $h = i_M$. For this, it is enough to consider a curve of imbeddings $h(t, x)$ that satisfies $h(0, x) = i_M$ and $\frac{\partial h}{\partial t} = V(x)$ for a fixed vector field $V \in \mathcal{X}^1(N)$. To simplify the notation, we denote the eigenvalue and eigenfunction on $h(t, M)$ by $\lambda_t$ and $u_t$ respectively. It follows from
\begin{equation*}
h(t)^* B_{h(t,M)} h(t)^{-1} u_t (x) = \lambda_t u_t, \quad x \in M,
\end{equation*}
that
\begin{equation}
(3.11) \quad \frac{\partial}{\partial t} \left(h(t)^* B_{h(t,M)} h(t)^{-1} u_t (x) \right) \bigg|_{t = 0} = - \frac{\partial \lambda}{\partial t} \bigg|_{t = 0} u_0 + \lambda_0 \frac{\partial u_t}{\partial t} \bigg|_{t = 0} \quad \text{in } M.
\end{equation}

Now, we need to compute the derivative of the left-hand side of (3.11). Notice that
\begin{equation*}
\frac{\partial}{\partial t} \left(h(t)^* B_{h(t,M)} h(t)^{-1} u_t (x) \right) \bigg|_{t = 0} = \frac{\partial}{\partial t} \left(h^* a_{h(t,M)} u_t \right) \bigg|_{t = 0}
\end{equation*}
\begin{equation*}
- \frac{\partial}{\partial t} \left(h(t)^* J_{h(t,M)} h(t)^{-1} u_t \right) \bigg|_{t = 0} \quad \text{in } M.
\end{equation*}

Also, for any function $w : M \times [0, T) \to \mathbb{R}$ it holds that $\frac{\partial}{\partial t} (h^* w) = h^* \frac{\partial}{\partial t} w + (h^* \nabla w, \frac{\partial}{\partial t})$. Here $\nabla$ denotes de tangential gradient on $N$ and $\langle \cdot, \cdot \rangle$ the inner product in $N$. Then we have
\begin{equation}
(3.12) \quad D_t \left(h(t)^* J_{h(t,M)} h(t)^{-1} u_t \right) = h(t) \frac{\partial}{\partial t} \left(J_{h(t,M)} h(t)^{-1} u_t \right) \quad \text{in } N.
\end{equation}
In the case of domains of $\mathbb{R}^n$ this derivative is known from [S] Lemma 4.1 under the Dirichlet condition.

Hence, setting $v(t, y) = h(t)^* u_t(y) = u_t(h^{-1}(t, y))$, $y \in h(t, M)$, we get from (3.7)
\begin{equation*}
\frac{\partial}{\partial t} \left(J_{h(t,M)} h(t)^{-1} u_t \right) \bigg|_{t = 0} = \frac{\partial}{\partial t} \left(J_{h(t,M)} v \right) \bigg|_{t = 0}
\end{equation*}
\begin{equation*}
= \frac{\partial}{\partial t} \left( \int_{h(t,M)} J(v) v(t, w) dv \bigg|_{t = 0} \quad \text{for } y \in h(t, M).
\end{equation*}
To explicitly compute this derivative we recall that \( \frac{dv_y}{dt} \bigg|_{t=0} = \text{tr} \left( g^{-1} \frac{dv}{dt} (0) \right) dv_0 \). Since \( g_{ij} (t) = (\partial_{x_i}, h, \partial_{x_j}, h) \) we have that \( \frac{dv_y}{dt} = (\partial_{x_i}, h, \partial_{x_j}, V) + (\partial_{x_j}, V, \partial_{x_i}, h) \). Now we denote \( V = V^\perp + V^T \), where \( V^\perp \) is normal component of \( V \) and \( V^T \) the tangential one. Then we have

\[
\frac{dv_y}{dt} \bigg|_{t=0} = (\text{div}_M V^T + \langle \vec{H}, V^\perp \rangle) dv_0,
\]

where \( \vec{H} \) is the mean curvature vector associated to \( M \). To keep in mind the variable that we are using in the computation, we will add a subscript to \( \nabla \) (e.g. \( \nabla_w J(w, x) \) or \( \nabla_x J(w, x) \)). Then

\[
\frac{\partial}{\partial t} \left( \int_{h(t, M)} J(y, w)v(t, w) dv_y (w) \right) \bigg|_{t=0} = \int_M \text{div}_M (J(y, w)v(0, w)V^T) dv_0 (w) + \int_M J(y, w) \frac{dv}{dt} v(0, w) dw + \int_M J(y, w)v(t, w) \langle \vec{H}, V^\perp \rangle dv_0 (w) + \int_M \langle \nabla_w (J(y, w)v(0, w)), V^\perp \rangle dv_0 (w) = \int_{\partial M} J(y, w) u_0 (w) \langle V^T, N \rangle dS + \int_M J(y, w) D_t u_0 dv_0 (w) \]

where \( N \in T(M) \cap (T(\partial M))^\perp \) is the unitary normal vector to \( \partial M \). Since \( J \) is \( C^1 \), the eigenfunctions \( u_t \) and their derivatives can be continuously extended to the border \( \partial M \). Hence, \( u_t \) possesses trace and the expression above is well defined. Since \( u_0 \) is a function defined on \( M \) we have \( \nabla_N u_0 (w) = \nabla_M u_0 (w) \) and, \( \nabla_M u_0 (w) \) is tangential to \( M \), then \( \langle \nabla_N u_0 (w), V^\perp \rangle = 0 \). We will also denote by \( a_0 = a_{h(0, M)} = a_{M} \).

Consequently, (3.8) and (3.12) imply (3.13)

\[
\frac{\partial}{\partial t} \left( h(t)^* J_{h(t, \Omega)} h(t)^{-1} u_t \right) \bigg|_{t=0} = \langle V, \nabla (J_{\Omega} u_0) \rangle + J_{\Omega} (D_t u) \bigg|_{t=0} + \int_{\partial M} J(y, w) u_0 (w) \langle V^T, N \rangle dS + \int_M J(y, w) u_0 (w) \langle \vec{H}, V^\perp \rangle dv_0 (w) \]

We get from (3.11) and (3.13) that

\[
\frac{\partial \lambda_t}{\partial t} u_0 + \lambda_0 \frac{\partial u_t}{\partial t} \bigg|_{t=0} = \frac{\partial}{\partial t} \left( h^* a_{h(t, M)} \right) \bigg|_{t=0} = u_0 + a_0 \frac{\partial u_t}{\partial t} - \langle V, \nabla (J_M u_0) \rangle - J_M (D_t u) \bigg|_{t=0} - \int_{\partial M} J(y, w) u_0 (w) \langle V^T, N \rangle dS - \int_M J(y, w) u_0 \langle \vec{H}, V^\perp \rangle dv_0 (w) - \int_M u_0 (w) \langle \nabla_w (J(y, w)), V^\perp \rangle dv_0 (w).
\]
Thus, multiplying (3.14) by the normalized eigenfunction $u_0$ and integrating on $\mathcal{M}$, we obtain

$$
\frac{\partial \lambda}{\partial t} + \lambda_0 \int_{\mathcal{M}} \frac{\partial u}{\partial t} u_0(x) d\nu_0(x) = \int_{\mathcal{M}} \frac{\partial}{\partial t} (h^* a_{b(t, \mathcal{M})}) \left|_{t=0} \right. u_0^2(x) d\nu_0(x) + \int_{\mathcal{M}} \mathcal{J}_M u_0 \frac{\partial u}{\partial t} d\nu_0(x)
$$

$$
+ \int_{\mathcal{M}} (a_0 - \mathcal{J}_M u_0) \frac{\partial u}{\partial t} d\nu_0(x) - \int_{\mathcal{M}} \langle V, \nabla (\mathcal{J}_M u_0) \rangle u_0(x) d\nu_0(x)
$$

$$
- \int_{\mathcal{M}} J(x, w) u_0(x) u_0(w) \langle \mathcal{H}, V \rangle d\nu_0(x)
$$

$$
- \int_{\mathcal{M}} \int_{\partial\mathcal{M}} J(x, z) u_0(z) \langle V^T, N \rangle dS(z) d\nu_0(x)
$$

which in turn implies

$$
\frac{\partial \lambda}{\partial t} \bigg|_{t=0} = \int_{\mathcal{M}} \frac{\partial}{\partial t} (h^* a_{b(t, \mathcal{M})}) \left|_{t=0} \right. u_0^2(x) d\nu_0(x)
$$

$$
+ \int_{\mathcal{M}} \mathcal{J}_M u_0 \frac{\partial u}{\partial t} d\nu_0(x) - \int_{\mathcal{M}} u_0 \langle V, \nabla (\mathcal{J}_M u_0) \rangle d\nu_0(x)
$$

(3.15)

$$
- \int_{\mathcal{M}} u_0(x) \left[ \mathcal{J}_M(D_t u|_{t=0}) + \int_{\partial\mathcal{M}} J(x, y) u_0(z) \langle V^T, N \rangle dS(z) \right] d\nu_0(x)
$$

$$
- \int_{\mathcal{M}} \int_{\mathcal{M}} J(x, w) u_0(x) u_0(w) \langle \mathcal{H}, V \rangle d\nu_0(x)
$$

since $(a_0 - \mathcal{J}_M) u_0 = \lambda_0 u_0$ in $\mathcal{M}$. The last two integrals are obtained from the symmetry $J(x, w) = J(w, x)$, which also implies

$$
\int_{\mathcal{M}} u_0 \left[ \mathcal{J}_M(D_t u|_{t=0}) + \langle V, \nabla (\mathcal{J}_M u_0) \rangle \right] d\nu_0(x)
$$

$$
= \int_{\mathcal{M}} \int_{\mathcal{M}} J(x, w) u_0(x) \left( \frac{\partial u}{\partial t}(w) - \langle V(w), \nabla u_0(w) \rangle \right) d\nu_0(x) d\nu_0(x)
$$

$$
+ \int_{\mathcal{M}} u_0 \langle V, \nabla (a_0 u_0 - \lambda_0 u_0) \rangle d\nu_0(x)
$$

$$
= \int_{\mathcal{M}} \frac{\partial u}{\partial t} \mathcal{J}_M u_0 d\nu_0(x) - \int_{\mathcal{M}} u_0 \langle V, \nabla (a_0 u_0 - \lambda_0 u_0) \rangle d\nu_0(x)
$$

$$
= \int_{\mathcal{M}} \frac{\partial u}{\partial t} \mathcal{J}_M u_0 d\nu_0(x) + \int_{\mathcal{M}} u_0^2 \langle V, \nabla u_0 \rangle d\nu_0(x).
$$

Finally we observe

$$
\int_{\mathcal{M}} u_0(w) \langle \nabla w, (\mathcal{J}_M u_0(w)), V \rangle d\nu_0(w) = \int_{\mathcal{M}} u_0^2(w) \langle \nabla w, (a_0(w)), V \rangle d\nu_0(w).
$$

Here we used that $\nabla u_0$ is tangential to $\mathcal{M}$.

Consequently, we get from (3.15) that

$$
\frac{\partial \lambda}{\partial t}(0) = \int_{\mathcal{M}} \frac{\partial}{\partial t} (h^* a_{b(t, \mathcal{M})}) \left|_{t=0} \right. u_0^2(x) d\nu_0(x) - \int_{\mathcal{M}} u_0^2 \langle V, \nabla a_0 \rangle d\nu_0(x)
$$

$$
- \int_{\mathcal{M}} u_0(x) \left[ \mathcal{J}_M(D_t u|_{t=0}) + \int_{\partial\mathcal{M}} J(x, y) u_0(z) \langle V^T, N \rangle dS(z) \right] d\nu_0(x)
$$

$$
- \int_{\mathcal{M}} \int_{\mathcal{M}} J(x, w) u_0(x) u_0(w) \langle \mathcal{H}, V \rangle d\nu_0(x)
$$

$$
= \int_{\mathcal{M}} \frac{\partial u}{\partial t} \mathcal{J}_M u_0 d\nu_0(x) - \int_{\partial\mathcal{M}} u_0^2 \langle V, N \rangle dS
$$

$$
- \int_{\mathcal{M}} \langle a_0 - \lambda_0 \rangle u_0^2(w) \langle \mathcal{H}, V \rangle d\nu_0(w)
$$

$$
= \int_{\mathcal{M}} \frac{\partial u}{\partial t} \mathcal{J}_M u_0 d\nu_0(x) - \int_{\partial\mathcal{M}} u_0^2 \langle V, N \rangle dS
$$

$$
- \int_{\mathcal{M}} \langle a_0 - \lambda_0 \rangle u_0^2(w) \langle \mathcal{H}, V \rangle d\nu_0(w)
$$

where $D_T f = \frac{\partial f}{\partial t} - \langle V^T, \nabla f \rangle$. Observing that $\mathcal{J}_M u_0 = (a_0 - \lambda_0) u_0$ we complete the proof.  □
Remark 3.2. In the case that $\mathcal{M}$ is a open domain of $\mathbb{R}^{n+1}$ (with co-dimension 0). The formula becomes
\[
\frac{\partial \lambda}{\partial h}(i_{\Omega})V = - \int_{\partial \Omega} (a_{\Omega}(s) - \lambda_0) u_0^2(s) \langle V, N(s) \rangle dS + \int_{\Omega} u_0^2(x) D_t(h^* a_{\Omega}) (1)_{t=0} dx.
\]

Next, we give some preliminary examples setting suitable nonlocal operators computing their Hadamard formula.

Example 3.1 (The sphere). Consider $\mathcal{M} = S^n$ and $\mathcal{N} = \mathbb{R}^{n+1}$. If we take $a \equiv 0$, then $\tilde{H}(p) = \frac{1}{H} p$ and
\[
\frac{\partial \lambda}{\partial h}(i_{\Omega})V = \frac{n \lambda_0}{R} \int_{S^n} u_0^2(w) \langle w, V^\perp \rangle dv_0(w).
\]

Example 3.2 (The Dirichlet problem on the upper hemisphere). Consider $\mathcal{M} = S^3_+$ (that is $p \in S^n$ with $x_n+1 \geq 0$) and $\mathcal{N} = \mathbb{R}^{n+1}$. If we take $a \equiv 1$, then $\tilde{H}(p) = \frac{1}{R} p$ and $N(p) = e_{n+1}$
\[
\frac{\partial \lambda}{\partial h}(i_{\Omega})V = - (1 - \lambda_0) \int_{S^3_+} u_0^2 V_{n+1} dS - \frac{n(1 - \lambda_0)}{R} \int_{S^3} u_0^2 (w) \langle w, V^\perp \rangle dv_0(w).
\]

Example 3.3 (One parameter family of functions $a$). Consider $\Omega \subset \mathbb{R}^n$, $\mathcal{M} = \Omega \times [0, 1]$ and $\mathcal{N} = \mathbb{R}^{n+1}$. In this case $H = 0$, but assume that $a$ depends on the variable $x_n+1$ then
\[
\frac{\partial \lambda}{\partial h}(i_{\Omega})V = - \int_{\partial \Omega} (a_{\Omega}(s) - \lambda_0) u_0^2(s) \langle V \cdot N(s) \rangle dS + \int_{\Omega} u_0^2(x) D_t(h^* a_{\Omega}) (1)_{t=0} dx
\]
\[
- 2 \int_{\Omega \times [0, 1]} u_0^2(w) \langle \nabla_w a_0(w), V^\perp \rangle dv_0(w).
\]

Domain derivative of eigenfunctions. Let us now determine the domain derivative of the function $u_h$ introduced by Theorem 3.1 at the reference manifold $\mathcal{M}$.

Due to (3.11), we have for all $V \in \mathcal{X}'(N)$ that
\[
\frac{\partial \lambda_{i \mathcal{M}}}{\partial t} u_0 + \lambda_0 \frac{\partial u_{i \mathcal{M}}}{\partial t} = \frac{\partial}{\partial t} (h^* a_{(t, \mathcal{M})}) \bigg|_{t=0} u_0 + B_{\mathcal{M}} \left( \frac{\partial u_{i \mathcal{M}}}{\partial t} \right)
\]
\[
+ [\mathcal{J}_{\mathcal{M}}, \langle V, \nabla(\cdot) \rangle] u_0 - \int_{\partial \mathcal{M}} J(y, w) u_0 (w) (V, N) dS(w)
\]
\[
- \int_{\mathcal{M}} J(y, w) u_0 (\tilde{H}, V^\perp) dv_0(w) - \int_{\mathcal{M}} u_0 (w) \langle \nabla_w (J(y, w)), V^\perp \rangle dv_0(w).
\]

where $[A, B] u := ABu - B Au$, and then, $[\mathcal{J}_{\mathcal{M}}, \langle V, \nabla(\cdot) \rangle] u_0 = \mathcal{J}_{\mathcal{M}}(\langle V, \nabla u_0 \rangle) - \langle V, \nabla(\mathcal{J}_{\mathcal{M}} u_0) \rangle$.

Hence,
\[
(\lambda_0 - B_{\mathcal{M}}) \frac{\partial u_{i \mathcal{M}}}{\partial t} = - \frac{\partial \lambda_{i \mathcal{M}}}{\partial t} u_0 + \frac{\partial}{\partial t} (h^* a_{(t, \mathcal{M})}) \bigg|_{t=0} u_0 + [\mathcal{J}_{\mathcal{M}}, \langle V, \nabla(\cdot) \rangle] u_0
\]
\[
- \int_{\partial \mathcal{M}} J(y, w) u_0 (w) (V, N) dS(w) - \int_{\mathcal{M}} J(y, w) u_0 (\tilde{H}, V^\perp) dv_0(w)
\]
\[
- \int_{\mathcal{M}} u_0 (w) \langle \nabla_w (J(y, w)), V^\perp \rangle dv_0(w).
\]

Thus, we can conclude that the derivative of $u_h$ at $h = i_{\mathcal{M}}$ in $V \in C^1(N, \mathcal{N})$ is the solution of
\[
(\lambda_0 - B_{\mathcal{M}}) w = f_V
\]
where $f_V \in L^2(\mathcal{M})$ is the function given by the right side of (3.10) which is well defined since $u_0$, $\lambda_0$, $\frac{\partial \lambda_{i \mathcal{M}}}{\partial t}$ and $\frac{\partial}{\partial t} (h^* a_{(t, \mathcal{M})}) \bigg|_{t=0}$ are known.

Notice that $\lambda_0$ is a simple eigenvalue of $B_{\mathcal{M}}$, and then, we have
\[
L^2(\mathcal{M}) = R(\lambda_0 - B_{\mathcal{M}}) \oplus [u_0].
\]
Therefore, (3.16) possesses unique solution, if and only if, \( \int_\mathcal{M} u_0 f_V dx = 0 \) for each \( V \in \mathcal{X}^1(\mathcal{N}) \).

Indeed, it follows from (3.10) and the assumption \( B_\mathcal{M} u_0 = \lambda_0 u_0 \) in \( \mathcal{M} \) that
\[
\int_\mathcal{M} u_0 f_V dx = -\frac{\partial \lambda}{\partial t} t_0 + \int_\mathcal{M} D_1(h^* a_{h(t, \cdot)}) u_0^2 dx + \int_\mathcal{M} u_0(\nabla(\cdot)) a_\mathcal{M} + [J_\mathcal{M}, \langle V, \nabla(\cdot) \rangle] u_0 dx
\]
\[
- \int_\mathcal{M} \int_{\partial \mathcal{M}} J(x, z) u_0(x) \frac{u_0(z)}{V, N} dS(z) dx
\]
\[
- \int_\mathcal{M} \int_{\mathcal{M}} J(x, y) u_0(x) \frac{u_0(z)}{V, V^\perp} dz dx
\]
\[
- \int_\mathcal{M} \int_{\mathcal{M}} u_0(x) \frac{u_0(z)}{(\nabla, V^\perp)} dz dx
\]
\[
= \int_\mathcal{M} u_0 \left( J_\mathcal{M}(V^T, \nabla u_0) \right) - a_\mathcal{M}(V^T, \nabla u_0) + \lambda_0 \langle V^T, \nabla u_0 \rangle dx
\]
\[
= \int_\mathcal{M} u_0 (\lambda_0 - B_\mathcal{M}) \left( V^T, \nabla u_0 \right) dx.
\]

Thus, since \( B_\mathcal{M} \) is a self-adjoint operator, \( u_0 \) is a \( C^1 \)-function and \( \langle V, \nabla(\cdot) \rangle u_0 \in L^2(\mathcal{M}) \), one has
\[
\int_\mathcal{M} u_0 f_V dx = \int_\mathcal{M} \langle V, \nabla u_0 \rangle (\lambda_0 - B_\mathcal{M}) u_0 dx = 0
\]
for all \( V \in \mathcal{X}^1(\mathcal{N}) \) which proves the following result.

**Corollary 3.1.** Let \( u_h \) be the family of eigenfunctions associated with the operator \( B_{h(\mathcal{M})} \) and eigenvalues \( \lambda_h \) given by Theorem 3.1.

Then, the derivative of \( u_h \) at \( h = i_{\mathcal{M}} \) and \( V \in \mathcal{X}^1(\mathcal{N}) \) is the unique solution of
\[
(\lambda_0 - B_\mathcal{M}) w = f_V
\]
where \( f_V \in L^2(\mathcal{M}) \) is the function given by
\[
f_V = -\frac{\partial \lambda}{\partial h}(i_{\mathcal{M}}) V u_0 + \frac{\partial h}{\partial t}(h^* a_{h(t, \cdot)}) \bigg|_{t=0} u_0 + [J_\mathcal{M}, \langle V, \nabla(\cdot) \rangle] u_0
\]
\[
- \int_{\partial \mathcal{M}} J(y, w) u_0(w) (V, N) dS(w)
\]
\[
- \int_{\mathcal{M}} J(y, w) u_0(\vec{H}, V^\perp) dv_0(w)
\]
\[
- \int_{\mathcal{M}} u_0(w) (\nabla w(J(y, w)), V^\perp) dv_0(w)
\]
with \( \frac{\partial \lambda}{\partial h}(i_{\mathcal{M}}) V \) given by (3.10).

**Some examples in Euclidean spaces.** In the sequel we compute some examples assuming \( \mathcal{M} = \Omega \) is an open set in \( \mathbb{R}^n \), \( J(x, y) = J(|x - y|) \) with \( \int_{\mathbb{R}^n} J(z) dz = 1 \). It is worth noting that such examples often appear in the literature associated with nonlocal equations. Below we give appropriate references for each example considered.

**Example 3.4** (Dirichlet problem). If we take \( a_{\Omega}(x) \equiv 1 \) in (2.2), we have what is called the Dirichlet nonlocal problem. In this case, the Hadamard formula is known and it was first obtained in [15] for the first eigenvalue. In [8], we have proved that the same formula still holds for any simple eigenvalue. Since \( a_{\Omega} \) is constant, \( D_1(h^* a_{\Omega}) \big|_{t=0} = 0 \) and, from Theorem 3.1, we get
\[
\frac{\partial \lambda}{\partial h}(\Omega) V = - (1 - \lambda_0) \int_{\partial \Omega} u_0^2 V \cdot N dS \quad \forall V \in C^1(\Omega, \mathbb{R}^n)
\]
with \( \cdot \) denoting the scalar product in \( \mathbb{R}^n \).
Example 3.5 (Neumann problem). In the literature, see for instance [1] [14] [24], the nonlocal Neumann problem is established taking

$$a_{\Omega}(x) = \int_{\Omega} J(|x - y|)dy, \quad x \in \mathbb{R}^n.$$ 

As expected, zero is its first eigenvalue for any measurable open set \(\Omega\) which is simple and it is associated with a constant eigenfunction. Clearly, the rate of the first eigenvalue with respect to the domain must be null. Let us take its rate for any other simple eigenvalue. For this, we first compute the anti-convective derivative of \(a_{\Omega}\) to the domain must be null. Let us take its rate for any other simple eigenvalue. For this, we

$$\int_{\partial D} J(|x - s|)(V \cdot N)(s) dS, \quad x \in \Omega.$$ 

Hence, we obtain from Theorem 3.1 that

$$D_t [h^s(t)a_{h(t,\Omega)}] \big|_{t=0} = h^s(t)\frac{\partial}{\partial t} \left[ \int_{h(t,\Omega)} J(|\cdot - w|)dw \right] \big|_{t=0}$$ 

Due to

$$\frac{\partial}{\partial t} \int_{\partial \Omega} J(|x - s|)(V \cdot N)(s) dS, \quad x \in \Omega.$$ 

Notice in the last integral the term \(J_{\Omega} u_0^2\) which is the operator \(J_{\Omega}\) applied to the square of the normalized eigenfunction \(u_0\).

Example 3.6. Let \(D \subset \mathbb{R}^n\) be a bounded open set and take \(A \subset D\), another open bounded set strictly contained in \(D\) in such way that \(\partial A \cap \partial D = \emptyset\). Next, consider \(\Omega = D \setminus A\) defining

$$a_{\Omega}(x) = \int_{\mathbb{R}^n \setminus A} J(|x - y|) dy, \quad x \in \mathbb{R}^n.$$ 

The nonlocal operator \(B_{\Omega}\) given for such function \(a_{\Omega}\) is a kind of Dirichlet/Neumann problem. It takes Dirichlet boundary condition side out of \(D\) setting Neumann condition on the hole \(A\). Such operator is given by

$$B_{\Omega}(x) = \int_{\mathbb{R}^n \setminus A} J(|x - y|)(u(x) - u(y)) dy, \quad x \in \Omega,$$ 

assuming \(u \equiv 0\) in \(\mathbb{R}^n \setminus D\) and has been studied for instance in [33]. Let us compute its Hadamard formula. Due to

$$a_{\Omega}(x) = \int_{\mathbb{R}^n} J(|x - y|) dy - \int_{\partial A} J(|x - y|) dy$$ 

one gets again from [23] Lemma 2.1 and [23] Theorem 1.1 that

$$D_t [h^s(t)a_{h(t,\Omega)}] \big|_{t=0} = h^s(t)\frac{\partial}{\partial t} \left[ 1 - \int_{h(t,D)} J(|\cdot - y|) dy + \int_{h(t,\Omega)} J(|\cdot - y|) dy \right] \big|_{t=0}$$ 

Due to

$$\int_{\partial D} J(|x - s|)(V \cdot N)(s) dS + \int_{\partial \Omega} J(|x - s|)(V \cdot N)(s) dS$$ 

one gets again from [23] Lemma 2.1 and [23] Theorem 1.1 that

$$\int_{\partial A} J(|x - s|)(V \cdot N)(s) dS, \quad x \in \Omega.$$
since \( \partial \Omega = \partial D \cup \partial A \) with \( \partial D \cup \partial A = \emptyset \). Hence,
\[
\frac{\partial \lambda}{\partial h}(\Omega)V = -\int_{\partial \Omega} (a_\Omega(s) - \lambda_0) u^2_\Omega(s) (V \cdot N)(s) \, dS + \int_{\partial A} (\mathcal{J}_\Omega u^2_\Omega)(s)(V \cdot N)(s) \, dS.
\]

4. ISOPERIMETRIC INEQUALITIES FOR EIGENVALUES

In this section, we obtain an analogue of the Rayleigh-Faber-Krahn inequality for the operator \( B_M \) assuming \( M = \Omega \) is an open set in \( \mathbb{R}^n \) and the function \( J \) satisfies
\[
J \in C(\mathbb{R}^n, \mathbb{R}) \text{ is a nonnegative function, spherically symmetric and radially decreasing}
\]
\[\text{(H)}\]
\[
\text{with } J(0) > 0 \text{ and } \int_{\mathbb{R}^n} J(x) \, dx = 1.
\]

As we will see, it is a direct consequence of rearrangements (or Schwarz symmetrization) first introduced by Hardy and Littlewood [19]. In the following we recall some basic definitions and properties concerning spherically symmetric rearrangements. We mention [7], as well as [9] [21], for more detailed discussions and proofs concerning this subject.

Let \( \Omega \subset \mathbb{R}^n \) be a measurable set and \( |\Omega| \) its Lebesgue measure. If \( |\Omega| \) is finite, we denote by \( \Omega^* \) an open ball with the same measure as \( \Omega \), otherwise, we write \( \Omega^* = \mathbb{R}^n \). We consider \( u : \Omega \to \mathbb{R} \) a measurable function assuming either that \( |\Omega| \) is finite or that \( u \) decays at infinity, i.e., the set \( \{x \in \Omega : |u(x)| > t\} \) is finite for all \( t > 0 \).

The function \( \mu(t) = |\{x \in \Omega : |u(x)| > t\}| \) defined for \( t \geq 0 \) is called the distribution function of \( u \). It is non-increasing, right-continuous with \( \mu(0) = |\text{supp}(u)| \) and \( \text{supp}(\mu) = [0, \|u\|_{L^\infty(\Omega)}] \).

The decreasing rearrangement \( u^\# : \mathbb{R}^+ \to \mathbb{R}^+ \) of \( u \) is the distributional function of \( \mu \), and it can be used to set the decreasing symmetric rearrangement \( u^* : \Omega \to \mathbb{R}^+ \) of \( u \) which is defined by
\[
u^*(x) = u^\#(c_n|x|^n)
\]
where the constant \( c_n = \pi^{n/2} (\Gamma(n/2 + 1))^{-1} \) is the measure of the \( n \)-dimensional unit ball.

It follows from [7] Lemma 3.4 that \( u^* \) is spherically symmetric and radially decreasing. Also, the measure of the level set \( \{x \in \Omega^* : u^*(x) > t\} \) is the same as the measure of \( \{x \in \Omega : |u(x)| > t\} \) for any \( t \geq 0 \).

Quite analogous to the decreasing rearrangements are the definitions of increasing ones. If the measure of \( \Omega \) is finite, we set by \( u^\#(s) = u^\#(|\Omega| - s) \) the increasing rearrangement of \( u \). Hence, the symmetric increasing rearrangement \( u_* : \Omega^* \to \mathbb{R}^+ \) of \( u \) is defined by
\[
u_*(x) = u^\#(c_n|x|^n).
\]

Due to the symmetry condition imposed on the kernel \( J \), we can show an analogue of the Rayleigh-Faber-Krahn inequality for the operator \( B_\Omega \) assuming \( a_\Omega \) is a non-negative function. In this way, we improve previous results obtained in [3] [37] for the Dirichlet nonlocal problem and the compact operator \( J_\Omega \). We show that the first eigenvalue of \( B_\Omega \) possesses as a lower bound, the first eigenvalue of the following self-adjoint operator: \( B^*_\Omega : L^2(\Omega^*) \to L^2(\Omega^*) \) given by
\[
B^*_\Omega u(x) = a_\Omega^*(x)u(x) - \int_{\Omega^*} J(x-y)u(y) \, dy, \quad x \in \Omega^*
\]
where the function \( a_\Omega^* \) is the symmetric increasing rearrangement of \( a_\Omega \). It is a consequence of the Riesz rearrangement inequality proved in [8] Symmetrization Lemma. It is known that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)g(y-x)h(x) \, dx \, dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f^*(y)g^*(y-x)h^*(x) \, dx \, dy
\]
for any nonnegative measurable functions \( f, g \) and \( h \) defined in \( \mathbb{R}^n \).
Theorem 4.1. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $a_\Omega : \overline{\Omega} \mapsto \mathbb{R}^+$ a non-negative continuous function and $\Omega^*$ an open ball with the same measure as $\Omega$. Assume that there exist the first eigenvalues of $B_\Omega$ and $B_{\Omega^*}$ denoted respectively by $\lambda_1(\Omega)$ and $\lambda_1^*(\Omega^*)$.

Then, under conditions (H), we have that

$$\lambda_1(\Omega) \geq \lambda_1^*(\Omega^*).$$

Proof. First, let us recall that [7, Theorem 3.8] implies that

\begin{equation}
\int_{\Omega} \phi(x) \varphi(x) \, dx \geq \int_{\Omega^*} \phi^*(x) \varphi^*(x) \, dx
\end{equation}

for any non-negative functions $\phi$ and $\varphi$ defined on $\Omega \subset \mathbb{R}^n$. Consequently, it follows from (4.19) and the Riesz rearrangement inequality (4.18) that

\begin{align*}
\int_{\Omega} u(x) (B_\Omega u)(x) \, dx &= \int_{\Omega} a_\Omega(x) u^2(x) \, dx - \int_{\Omega} \int_{\Omega} J(x-y) u(y) u(x) \, dy \, dx \\
&\geq \int_{\Omega} \int_{\Omega^*} a_{\Omega^*}(x) u^2(x) \, dx - \int_{\Omega^*} \int_{\Omega^*} J(x-y) u^*(y) u^*(x) \, dy \, dx \\
&\geq \int_{\Omega^*} u^*(x) (B_{\Omega^*} u^*)(x) \, dx
\end{align*}

since $J$ is nonnegative, spherically symmetric and radially decreasing.

Now, due to [7, Theorem 3.6], we know that $\|u^*\|_{L^2(\Omega^*)} = \|u\|_{L^2(\Omega)}$ for any nonnegative function $u$. Thus, if $u_1$ is the corresponding eigenfunction of $\lambda_1(\Omega)$, one has

$$\lambda_1(\Omega) \geq \int_{\Omega^*} u_1^*(x) (B_{\Omega^*} u_1^*)(x) \, dx \geq \lambda_1^*(\Omega^*)$$

completing the proof. \hfill \Box

Remark 4.1. We recall that, under appropriate conditions, the existence of the first eigenvalue $\lambda_1(\Omega)$ of $B_\Omega$ is guaranteed by [29, Theorem 2.1]. In particular, $\lambda_1(\Omega)$ exists if $a_\Omega$ satisfies

\begin{equation}
\int_{\Omega} \frac{dx}{a_\Omega(x) - m} = \infty
\end{equation}

with $m = \min_{x \in \overline{\Omega}} a_\Omega(x)$. Now, we known from Proposition 5.1 (which is a consequence of the layer-cake formula) that

\begin{equation}
\int_{\Omega} \Phi(u(x)) \, dx = \int_{\Omega^*} \Phi(u_*(x)) \, dx
\end{equation}

for any non-negative measurable function $u$ and any decreasing function $\Phi$ satisfying

\begin{equation}
\lim_{t \to \infty} \Phi(t) = 0 \quad \text{and} \quad \lim_{t \to a^+} \Phi(t) = \infty.
\end{equation}

Therefore, since $\Phi(x) = (x - m)^{-1}$ is a non-negative decreasing function on $(m, +\infty)$ satisfying (4.22), we obtain from (4.20) and (4.21) that

$$\int_{\Omega^*} \frac{dx}{a_{\Omega^*}(x) - m} = \infty.$$

Thus, it follows from [29, Theorem 2.1] that the first eigenvalue $\lambda_{1\Omega}^*$ of (4.17) also exists ensuring the application of the isoperimetric inequality given by Theorem 4.1 to a large class of nonlocal operators $B_\Omega$. 
Notice that the ball is not the unique minimizer of $\lambda_1(\Omega)$ even up to displacements. Indeed, since $L^2(\Omega)$ does not change if we remove from $\Omega$ a set of zero measure, any kind of open sets as $\Omega^* \setminus A$ with $|A| = 0$ gives a minimizer for $\lambda_1(\Omega)$.

As we have already mentioned, the operators $B_\Omega$ and $B^*_{\Omega^*}$ can be introduced by a jump process used to model dispersion of individuals in a given habitat. In fact, if $u(x, t)$ is thought of as a population density at a point $x$ and a time $t$, and $J(x - y)$ is the probability distribution of jumping from a location $y$ to the position $x$, the amount $\int_\Omega J(x - y) u(y, t) dy$ gives the rate in which individuals are arriving at location $x$ from all the other places $y \in \Omega$. On the other hand, $-a_{\Omega}(x)u(x, t)$ can be thought of as the rate in which individuals are leaving position $x$ to the others sites in the habitat. Therefore, in the absence of external or internal sources, we have that the density $u$ satisfies the evolution equation

$$u_t(x, t) = -B_\Omega u(x, t), \quad x \in \Omega.$$ 

Hence, it follows from Theorem 4.1 that the minimum decay rate of the population density $u(x, t)$ is attained when the habitat is a ball.

Under a Neumann condition, i.e., assuming $a_{\Omega}(x) = \int_\Omega J(x - y) dy$ in the definition of $B_\Omega$, it is clear that zero is the lower bound for the first eigenvalues since it is the principal eigenvalue of $B_\Omega$ for any bounded open set $\Omega \subset \mathbb{R}^n$. Anyway, as we have $a_{\Omega^*}(x) = a_{\Omega}(x)$ in $\Omega^*$, we also recover this obvious property using Theorem 4.1.

Finally, we notice that in general, the first eigenvalue $\lambda_1(\Omega)$ of (2.1) does not have a maximizer among open bounded sets with constant measure. For the Dirichlet problem, i.e., under the assumption $a_{\Omega}(x) \equiv 1$ for all $x \in \Omega$, this has been pointed out in [8, Remark 4.2]. Other examples can be obtained in a very similar way.

5. Appendix

Here, we see that the integral of the absolute value of functions is invariant under rearrangement. We have:

**Proposition 5.1.** Let $\Phi : (a, +\infty) \subset \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a continuous increasing map satisfying

$$\lim_{t \to \infty} \Phi(t) = 0 \quad \text{and} \quad \lim_{t \to a^+} \Phi(t) = \infty.$$

Then,

$$\int_{\Omega^*} \Phi(u^*) dx = \int_{\Omega} \Phi(|u|) dx = \int_{\Omega^*} \Phi(u_+) dx.$$

**Proof.** It is a direct consequence of the layer-cake formula given for instance at [7, Theorem 10.1]. We choose $m(dx) = dx$ setting $\Phi(t) = \nu([0, t]^c)$.

**Remark 5.1.** An analogous result holds if we assume that $\Phi$ is increasing and satisfies $\Phi(0) = 0$. See [7, Theorem 3.6].

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