CURTIS-TITS GROUPS OF SIMPLY-LACED TYPE

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Abstract. The classification of Curtis-Tits amalgams with connected, triangle free, simply-laced diagram over a field of size at least 4 was completed in [4]. Orientable amalgams are those arising from applying the Curtis-Tits theorem to groups of Kac-Moody type, and indeed, their universal completions are central extensions of those groups of Kac-Moody type. The paper [3] exhibits concrete (matrix) groups as completions for all Curtis-Tits amalgams with diagram $A_{n-1}$. For non-orientable amalgams these groups are symmetry groups of certain unitary forms over a ring of skew Laurent polynomials. In the present paper we generalize this to all amalgams arising from the classification above and, under some additional conditions, exhibit their universal completions as central extensions of twisted groups of Kac-Moody type.

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1. Introduction

A celebrated theorem of Curtis and Tits [7, 21] (later extended by Timmesfeld (see [17, 18, 19, 20] for spherical groups), by P. Abramenko and B. Mühlherr [1] and Caprace [5] to 2-spherical groups of Kac-Moody type) on groups with finite BN-pair states that these groups are central quotients of the universal completion of the concrete amalgam of the Levi components of the parabolic subgroups with respect to a given (twin-) BN-pair. Following Tits [24] a group of Kac-Moody type is by definition a group with RGD system such that a central quotient is the subgroup of $\text{Aut}(\Delta)$ generated by the root groups of an apartment in a Moufang twin-building $\Delta$. This central quotient will be called the associated adjoint group of Kac-Moody type.

1.1. Classification of Curtis-Tits amalgams

In [4], for a given connected diagram, we consider all possible rank-2 amalgams which are locally isomorphic to an amalgam arising from the Curtis-Tits theorem in the simply-laced case. Then, extending the techniques from [3], we classify Curtis-Tits amalgams as defined in Definition 2.5 (called Curtis-Tits structures in loc. cit.) with property (D) (see Definition 2.6) and connected simply-laced diagram. We show that any Curtis-Tits amalgam with connected simply-laced diagram without triangles that has a non-trivial universal completion must have property (D). Although many of the results in the present and preceding papers can be proved under the more general assumption of property (D), some rely on 3-sphericity, for instance the simple connectedness of $\Delta^0$ in Section 4 and Lemma 2.9.
Assumption 1. Throughout the paper we will make the following assumptions:

1. $k$ is a field with at least 4 elements.
2. $\Gamma = (I, E)$ is a connected simply-laced Dynkin diagram of rank $n = |I|$ without triangles.
3. All Curtis-Tits amalgams have property (D).

In this context, we quote Theorem 1 from [4]:

**Classification Theorem** Let $\Gamma$ be a connected simply laced Dynkin diagram without triangles and $k$ a field with at least 4 elements. There is a natural bijection between isomorphism classes of universal Curtis-Tits amalgams with property (D) over the field $k$ on the graph $\Gamma$ and group homomorphisms $\omega: \pi(\Gamma, 0) \rightarrow \langle \tau \rangle \times \text{Aut}(k)$.

We’d like to point out that the classification theorem generalizes similar classification results of sound Moufang foundations in [12, 24]. Here, $\pi(\Gamma, 0)$ denotes the (first) fundamental group of the graph $\Gamma$ with base point $0 \in I$, and $\langle \tau \rangle \times \text{Aut}(k)$ arises as a group of automorphisms of $\text{SL}_2(k)$ stabilizing a given torus (see Subsection 2.4). It is clear from their definition that any Curtis-Tits amalgam is a homomorphic image of a universal Curtis-Tits amalgam. Note that the standard Curtis-Tits amalgam of type $\Gamma_\omega(k)$ in Definition 2.25 is universal. We shall denote this amalgam by $G_\omega(k)$. We say that a Curtis-Tits amalgam has type $\Gamma_\omega(k)$ if it is a homomorphic image of $G_\omega(k)$. The Classification Theorem can thus be reformulated as follows.

**Corollary** Every Curtis-Tits amalgam with property (D) with connected simply-laced triangle free Dynkin diagram over a field $k$ of size at least 4 has type $\Gamma_\omega(k)$ for some $\omega$.

1.2. Curtis-Tits amalgams yielding Curtis-Tits groups

For the remainder of the paper $\Gamma$, $\omega$, and $k$ will be as in the Classification Theorem. We wish to decide which general Curtis-Tits amalgam of type $\Gamma_\omega(k)$ have non-trivial universal completion.

**Definition 1.1.** Let $\sim_0$ be the equivalence relation on $I$ so that for distinct $i, j \in I$ we have $i \sim_0 j$ if and only if

- (B1) $i$ and $j$ are not adjacent, but have a common neighbor in $\Gamma$, and
- (B2) the neighbors of $i$ in $\Gamma$ coincide with the neighbors of $j$ in $\Gamma$.

Now consider two equivalence relations $\sim$ and $\sim'$ on $I$. We say that $\sim$ refines $\sim'$ if and only if $i \sim j$ implies $i \sim' j$ for all $i, j \in I$; that is, each equivalence classes of $\sim$ is contained in an equivalence class of $\sim'$.

**Definition 1.2.** For any Curtis-Tits amalgam $\mathcal{A} = \{A_i, A_{i,j}, a_{i,j} | i, j \in I(\Gamma)\}$ of type $\Gamma_\omega(k)$, let $\sim_{\mathcal{A}}$ be the equivalence relation generated by the pairs $(i, j) \in I \times I$ such that for distinct $i, j \in I$ we have $i \sim_{\mathcal{A}} j$ if and only if

- (B0) $A_{i,j} \cong \text{SL}_2(k) \times \text{SL}_2(k)/\langle (z, z) \rangle$. 

where \( \langle z \rangle = Z(\text{SL}_2(k)) \).

**Definition 1.3.** Next, suppose that the Curtis-Tits amalgam \( \mathcal{A} = \{ A_i, A_{i,j}, a_{i,j} \mid i, j \in I(\Gamma) \} \) of type \( \Gamma_\omega(k) \) has a non-trivial completion \((A, \alpha_\bullet)\). Let \( \sim_A \) be the relation of the pairs \((i, j) \in I \times I\) such that for distinct \(i, j \in I\) we have \(i \sim_A j\) if and only if

\[
\alpha_{i,j}(A_{i,j}) \cong \text{SL}_2(k) \times \text{SL}_2(k)/\langle (z, z) \rangle,
\]

where \( \langle z \rangle = Z(\text{SL}_2(k)) \).

We shall prove in Proposition 2.18 that \( \sim_A \) is in fact an equivalence relation, and that \( \sim_\alpha \) refines \( \sim_A \).

Our first result allows one to decide which Curtis-Tits amalgams of type \( \Gamma_\omega(k) \) yield non-trivial Curtis-Tits groups. The proof is obtained in Subsection 3.3.

**Theorem A.** Let \( \mathcal{A} \) be a Curtis-Tits amalgam with property (D) of type \( \Gamma_\omega(k) \). Then, \( \mathcal{A} \) has a non-trivial universal completion if and only if \( \sim_\alpha \) refines \( \sim_0 \).

Theorem A says that \( \sim_\alpha \) must refine \( \sim_0 \). For the standard Curtis-Tits amalgam \( \mathcal{G}_\omega(k) \) of type \( \Gamma_\omega(k) \) the relation \( \sim_{\mathcal{G}_\omega(k)} \) is the equality relation. Hence we obtain

**Corollary B.** Any standard Curtis-Tits amalgam arising from the Classification Theorem has a non-trivial universal completion.

**Definition 1.4.** We shall denote \((\tilde{G}_\omega(k), \tilde{\alpha}_\omega(k)_\bullet)\) as the universal completion of the standard Curtis-Tits amalgam \( \mathcal{G}_\omega(k) \) of type \( \Gamma_\omega(k) \) and call \( \tilde{G}_\omega(k) \) the *simply-connected Curtis-Tits group of type \( \Gamma_\omega(k) \)*. Since \( \omega \) and \( k \) are fixed we often omit them from the notation. The quotient of \( \tilde{G}_\omega(k) \) over its center will be called the *adjoint Curtis-Tits group of type \( \Gamma_\omega(k) \)*. From the proof of Theorem A it will emerge that this group corresponds to the \( \sim_0 \) relation.

### 1.3. Curtis-Tits amalgams injecting into their Curtis-Tits groups

Curtis-Tits amalgams with a non-trivial completion do not necessarily inject into their universal completion. We now determine when they do.

Write \((\tilde{G}, \tilde{\alpha}_\bullet) = (\tilde{G}_\omega(k), \tilde{\alpha}_\omega(k)_\bullet)\).

**Definition 1.5.** For any amalgam \( \mathcal{A} \) of type \( \Gamma_\omega(k) \) with completion \((A, \alpha_\bullet)\) consider the map \( d^A: (k^*)^I \to A \) of Definition 2.26. Define

\[
z_{ij} = (a_k)_{k \in I} \in (k^*)^I \text{ where } a_k = \begin{cases} -1 & \text{if } k \in \{i, j\}, \\ 1 & \text{else}. \end{cases}
\]

\[
Z^0 = \langle z_{ij} : i \sim_0 j \text{ with } i, j \in I \text{ distinct} \rangle \leq (k^*)^I.
\]

\[
Z^A = \ker d^A \cap Z^0 \leq (k^*)^I.
\]

Let \( \mathcal{G}_1 \) be the image of \( \mathcal{G}_\omega(k) \) in \( \tilde{G} \), let \( \sim_1 = \sim_{\mathcal{G}_1} \), and let \( Z^1 = Z^{\tilde{G}} \). Using these groups, we define a map of equivalence relations on \( I \) sending \( \sim \) to the equivalence relation \( \equiv \) generated by the pairs \((i, j)\) of distinct elements of \( I \) such that

\[
z_{ij} \in \langle Z^1, z_{kl} : k \sim l \text{ with } k, l \in I \text{ distinct} \rangle.
\]
Clearly \( \bar{\sim} = \sim \) so \( \bar{\sim} \) is a closure operator with respect to the refinement order.

We now have the following result. The proof is obtained in Subsection 3.3.

**Theorem C.** Let \( \mathcal{A} \) be a Curtis-Tits amalgam with property (D) of type \( \Gamma_\omega(k) \) such that \( \sim_{\mathcal{A}} \) refines \( \sim_0 \). Let \( \mathcal{A} \) be the image of \( \mathcal{A} \) in its universal completion. Then, \( \sim_{\mathcal{A}} = \bar{\sim}_{\mathcal{A}} \) hence \( \mathcal{A} \) injects into its universal completion if and only if \( \sim_{\mathcal{A}} = \bar{\sim}_{\mathcal{A}} \).

To summarize Theorems A and C, using properties (B1) and (B2) of Definition 1.1 one can compute \( \sim_0 \) directly from \( \Gamma \). From this, one can determine which Curtis-Tits amalgams of type \( \Gamma_\omega(k) \) have a non-trivial universal completion. To find \( \sim_1 \) and the closure operator of Definition 1.5, one needs group theoretic properties of \( \bar{\tilde{\Gamma}} \). Once these are known we can find the Curtis-Tits amalgams of type \( \Gamma_\omega(k) \) which inject into their universal completion as those corresponding to the elements closed with respect to \( \bar{\sim} \) in the interval between \( \sim_0 \) and \( \sim_1 \) ordered by refinement.

### 1.4. Construction of Curtis-Tits groups

Let \( \mathcal{H} \) be the endomorphism of \( (k^*)^I \) described in Definition 2.27.

In the non-orientable case, there exists a canonical orientable Curtis-Tits amalgam \( \mathcal{L} \) over a 2-sheeted covering \( \Lambda = (I(\Lambda), E(\Lambda)) \) of \( \Gamma \). Let \( L \) be the quotient of the universal completion of \( \mathcal{L} \) over its center. Then, the monodromy group of the covering of graphs \( \Lambda \to \Gamma \) induces an involution \( \theta \) of \( L \). Let \( L^\theta \) be the centralizer of \( \theta \) in \( L \) and let \( G \) be its commutator subgroup. Let \( \tilde{D} = \langle \tilde{\gamma}_i(D_i) : i \in I \rangle_{\tilde{\Gamma}} \), with \( D_i \) as in Definition 2.6.

**Theorem D.** First assume that \( \Gamma_\omega(k) \) is orientable, (that is \( \text{im} \omega \leq \text{Aut}(k) \)) and \( |k| \geq 4 \).

1. The group \( \tilde{G} \) is a central extension of a group of Kac-Moody type over \( k \) with diagram \( \Gamma \). Moreover, \( Z(\tilde{G}) = d(\ker \mathcal{H}) \).

Next, assume that \( \Gamma_\omega(k) \) is not orientable, (that is, \( \text{im} \omega \not\leq \text{Aut}(k) \)) and that \( |k| \geq 7 \). With the notation above we have

2. \( (G, \gamma_\bullet) \) is a non-trivial completion of \( \mathcal{A} \), for some \( \gamma_\bullet \).

3. \( L^\theta/G \) is an elementary abelian 2-group which can be explicitly computed using the operator \( \mathcal{H} \).

4. Suppose \( L^\theta = G \) and \( \pi: \tilde{G} \to L^\theta \) is the canonical map. Then, \( \ker \pi = Z(\tilde{G}) \cap \tilde{D} = d(\ker \mathcal{H}) \).

Note that the first half of part 1 of Theorem D, which was already accomplished in [4], is similar to, but stronger than the Example following Theorem A in [5]. Indeed, we do not presuppose that the amalgam is isomorphic to the Curtis-Tits amalgam of a group of Kac-Moody type. The proof of this part is reviewed in Subsection 3.1 The second half, on the other hand is new. The proofs of Part 2, Part 3, and Part 4 are obtained in Subsections 3.2, 5.2, and 5.3 respectively.

We also record the following geometric result, arising from the proof of Theorem D, which is of independent interest from a geometric group theory point of view. The proof of this theorem is obtained in Subsection 5.2.
Theorem E. Under the assumptions of part 2. of Theorem D, $\tilde{G}$ acts flag-transitively on a connected and simply 2-connected geometry $\Delta^\theta$.

The simple 2-connectedness of the geometry $\Delta^\theta$ follows in a similar manner as in [3] (see Section 4). The flag-transitivity of $\tilde{G}$ follows from Corollary 5.10.

Finally, in Section 6 we apply the methods developed in this paper to some concrete examples of orientable and non-orientable Curtis-Tits groups.

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2. Definitions and properties of Curtis-Tits amalgams

2.1. Amalgams and diagrams

In this section we briefly recall the notion of a Curtis-Tits amalgam over a commutative field $k$ from [4] and prove that up to finite central extension one can restrict to universal Curtis-Tits amalgams.

Definition 2.1. An amalgam over a poset $(\mathcal{P}, \prec)$ is a collection $\mathcal{A} = \{A_x \mid x \in \mathcal{P}\}$ of groups, together with a collection $a_{\cdot} = \{a_{xy} \mid x \prec y, x, y \in \mathcal{P}\}$ of monomorphisms $a_{xy}: A_x \hookrightarrow A_y$, called inclusion maps such that whenever $x \prec y \prec z$, we have $a_{yz} = a_{zy} \circ a_{xy}$. A completion of $\mathcal{A}$ is a group $A$ together with a collection $\alpha_{\cdot} = \{\alpha_x \mid x \in \mathcal{P}\}$ of homomorphisms $\alpha_x: A_x \rightarrow A$, whose images generate $A$, such that for any $x, y \in \mathcal{P}$ with $x \prec y$ we have $\alpha_y \circ a_{xy} = \alpha_x$. The amalgam $\mathcal{A}$ is non-collapsing if it has a non-trivial completion. A completion $(\tilde{A}, \tilde{\alpha}_{\cdot})$ is called universal if for any completion $(A, \alpha_{\cdot})$ there is a unique surjective group homomorphism $\pi: \tilde{A} \rightarrow A$ such that $\alpha_{\cdot} = \pi \circ \tilde{\alpha}_{\cdot}$.

Remark 2.2. Throughout the paper we will adhere to the font conventions in Definition 2.1, that is, amalgams are in calligraphic font, their groups and connecting maps are in boldface roman, their completion maps are in greek and their image is denoted in regular math font. Finally, universal completions use the notation of ordinary completions with a tilde on top.

We record the following lemma, which will be applied in the proof of Lemma 5.14 (for another example of its use see the proof of Proposition 3.12 of [5]).

Lemma 2.3. Let $\eta: A \rightarrow B$ be a surjective group homomorphism. Suppose $\mathcal{A} = \{A_x \mid x \in \mathcal{P}\}$ is an amalgam of subgroups of $A$, with connecting maps given by inclusion, generating $A$. Let $\mathcal{B} = \{B_x = \eta(A_x) \mid x \in \mathcal{P}\}$ be the amalgam corresponding to $\mathcal{A}$ via $\eta$ and suppose that $B$ is the universal completion of $\mathcal{B}$. For each $x \in \mathcal{P}$, let $K_x = \ker \eta|_{A_x}$, then, $\ker \eta$ is the normal closure of the subgroup generated by all $K_x$. 

Proof Let $K$ be the normal closure of the subgroup generated by all $K_x (x \in P)$. Then, by definition $K \leq \ker \eta$. Moreover, for any $x \in P$ we have $K_x \leq K \cap A_x \leq \ker \eta \cap A_x = K_x$. Therefore, the canonical map $\eta': A \to A/K$ restricts to $A_x \to A_x/K_x \cong B_x$. By universality of $B$ there exists a map $B \to A/K$ extending this isomorphism. On the other hand, since $K \leq \ker \eta$, there is also a homomorphism $A/K \to B$ induced by $\eta$. The composition $B \to A/K \to B$ fixes the elements of $B$ elementwise so it must be the identity mapping. \hfill \Box

Definition 2.4. For the purposes of this paper, a simply-laced diagram is an undirected graph $(I, E)$ with finite vertex set $I \subseteq \mathbb{N}$ and edge set $E$ without circuits of length 1 or 2. All simply-laced diagrams in this paper are connected and triangle-free, that is, they also have no circuits of length 3.

Let $\Gamma = (I, E)$ denote a connected triangle-free simply-laced diagram with $3 \leq n = |I| < \infty$ nodes. Also fix a field $k$ of order at least 4.

Indexing convention. Throughout the paper we shall adopt the following indexing conventions. Indices from $I$ shall be taken modulo $n = |I|$. For any $i \in I$, we set $(i) = I \setminus \{i\}$. Also subsets of $I$ of cardinality 1 or 2 appearing in subscripts are written without set-brackets.

Definition 2.5. Let $P = \{J \mid \emptyset \neq J \subseteq I \text{ with } |J| \leq 2\}$ and $\prec$ denoting inclusion. A Curtis-Tits amalgam with diagram $\Gamma$ over $k$ is an amalgam $\mathcal{G} = \{G_i, G_{i,j}, g_{i,j} \mid i, j \in I\}$ over $P$, where, for every $i, j \in I$, we write $g_{i,j} = g_{i,j}^{(i,j)}$. Note that, due to our subscript conventions, we write $G_i = G_{\{i\}}$ and $G_{i,j} = G_{\{i,j\}}$, where

(SCT1) for any vertex $i$, we set $G_i = \text{SL}_2(k)$ and for each pair $i, j \in I$,

$$G_{i,j} \cong \begin{cases} 
\text{SL}_3(k) & \text{if } \{i, j\} \in E \\
G_i \circ G_j & \text{else}
\end{cases};$$

(SCT2) For $\{i, j\} \in E$, $g_{i,j}(G_i)$ and $g_{j,i}(G_j)$ form a standard pair for $G_{i,j}$ in the sense of [4], whereas for all other pairs $(i, j)$, $g_{i,j}$ is the natural inclusion of $G_i$ in $G_i \circ G_j$.

Here, $\circ$ denotes central product. Note that there are two cases:

$$G_i \circ G_j \cong \begin{cases} 
G_i \times G_{j}, & \text{if } \{i, j\} \not\in E \text{ and } \{i, j\} \not\in E \text{ or } \{i, j\} \not\in E \text{ and } \{i, j\} \not\in E \\
G_i \times G_{j}/\langle z, z \rangle, & \text{else}
\end{cases},$$

where $\langle z \rangle = \text{Z} (\text{SL}_2(k))$. We call the amalgam $\mathcal{G}$ universal if the first case occurs for any $\{i, j\} \not\in E$. This happens for instance if $\text{Char}(k) = 2$.universal amalgam.

We recall Property (D) from [4].

Definition 2.6. (property (D)) For any $i, j \in I$ with $\{i, j\} \in E$, let

$$D_i^j = N_{G_{i,j}}(g_{j,i}(G_j)) \cap g_{i,j}(G_i)$$

Once checks that $D_i^j$ is the only torus in $g_{i,j}(G_i)$ normalized by $D_j^i$ (see [4]). We now say that $\mathcal{G}$ has property (D) if for any pair of edges $\{i, j\}$ and $\{i, k\}$ in $E$ the map $g_{i,k} \circ
\(g_{i,j}^{-1} : D_i^j \to D_k^i\) is an isomorphism. That is, we can select tori \(D_i \leq G_i\) such that for all edges \(\{i, j\}\) we have \(g_{i,j}(D_i) = D_i^j\).

2.2. Spherical Curtis-Tits amalgams

In the spherical case we have the following corollary to a version of the Curtis-Tits theorem due to Timmesfeld [17, Theorem 1]:

**Theorem 2.7.** Let \(\mathcal{G}\) be a Curtis-Tits amalgam (universal or not) over a commutative field \(k\), having spherical diagram \(\Gamma\). Then any non-trivial completion \(G\) of \(\mathcal{G}\) is a perfect central extension of the adjoint Chevalley group of type \(\Gamma\) and a central quotient of the universal Chevalley group of type \(\Gamma\).

**Proof** The condition (H1) preceding Theorem 1 of [17] is satisfied. For each \(i, j \in I\), let \(X_i = G_i\), and \(H_i = D_i\) (the standard torus \(D_i\) arising from Property (D)). It follows that, for each \(i, j \in I\) we have \(X_{i,j} = G_{i,j}\).

We verify the conditions in (H1): (1) We have \(X_i \cong SL_2(k)\), which is a central extension of \(PSL_2(k)\).

(3) Since the images of \(X_i\) and \(X_j\) form a standard pair in \(X_{i,j} \cong SL_3(k)\), whenever \(\{i, j\}\) is an edge of \(\Gamma\), and \(X_{i,j}\) is the central product of the images of \(X_i\) and \(X_j\) whenever \(\{i, j\}\) is not an edge in \(\Gamma\), this condition is satisfied.

(2) Property (D) ensures that this condition is satisfied as well. We also check the rest of the conditions of Theorem 1 of loc. cit.. Namely, if \(|k| = 4\) and \(\{i, j\}\) is an edge of \(\Gamma\), then \(G_{i,j} = SL_3(4)\), which has \(|Z(SL_3(k))| = 3\). The result follows. \(\Box\)

**Remark 2.8.** From [9] we have the following identifications.

1. The universal and adjoint Chevalley groups of type \(A_n\) are \(SL_{n+1}(k)\) and \(PSL_{n+1}(k)\).
2. The universal and adjoint Chevalley groups of type \(D_n\) over \(k\) are the spin group \(Spin^{+}_{2n}(k)\) and \(P\Omega^{+}_{2n}(k)\), the derived subgroup of the group of linear isometries of a non-degenerate quadratic form of Witt index \(n\), modulo the center of this derived subgroup.
3. We have \(Spin^{+}_{2n}(k) \cong 2^2 P\Omega^{+}_{2n}(k)\) if \(n\) is even and \(Spin^{+}_{2n}(k) \cong 4 P\Omega^{+}_{2n}(k)\) if \(n\) is odd [25].

2.3. General and Universal Curtis-Tits amalgams

Note that every Curtis-Tits amalgam is a quotient of a unique universal one that we call \(\mathcal{G}\). This means in particular that any completion of a Curtis-Tits amalgam is a completion of the corresponding universal amalgam. We shall now investigate the relation between the universal completions of universal and other Curtis-Tits amalgams.

Let \(\mathcal{G}\) be a universal Curtis-Tits amalgam with connected simply-laced triangle-free diagram \(\Gamma\) over \(k\). We shall only be interested in amalgams that admit a non-trivial completion \((A, \alpha_\bullet)\). From [4] it then follows that \(\mathcal{G}\) has property (D). Using connectedness of \(\Gamma\) and the fact that conjugates of the vertex groups generate the edge groups, one shows that in fact all maps \(\alpha_J (J \subseteq I \text{ with } 0 < |J| \leq 2)\) are non-trivial.
Notation. Let \((A, \alpha_s)\) be a completion of \(G\) such that for each \(i, j \in I\), the maps \(\alpha_i\) and \(\alpha_{i,j}\) are non-trivial. We discuss the structure of the image of \(G\) in \(A\). We show that the possibilities are only limited by the structure of \(\Gamma\).

Given a subset \(S \subseteq I\), let \(\Gamma_S\) be the full subgraph of \(\Gamma\) supported by the node set \(S\). We define the subamalgam \(G_S = \{G_J | J \subseteq S\text{ with }0 < |J| \leq 2\}\) of \(G\), with connecting maps induced by \(g_s\). Let \(A_S\) be the subgroup of \(A\) generated by the images of the elements of \(G_S\) under \(\alpha_s\). Let \((\tilde{G}_S, \tilde{\gamma}_{S,i})\) be the universal completion of \(G_S\), and let \(\pi_S : \tilde{G}_S \to A_S\) be the universal map. For \(S = I\), we set \(\tilde{G} = \tilde{G}_S\) and then \(\pi = \pi_S\).

The maps \(\alpha_i\) and \(\alpha_{i,j}\).

**Lemma 2.9.** The map \(\alpha_i\) is injective for all \(i \in I\) and \(\alpha_{i,j}\) is injective for all \(\{i, j\} \in E\).

**Proof** Since the diagram \(\Gamma\) is connected and has at least three nodes, but no triangles, every \(i \in I\) lies on an edge and any edge is part of a subdiagram of type \(A_3\). Without loss of generality let us assume that \(i, j, k \in I\) are such that \(j\) is adjacent to both \(i\) and \(k\) and let \(S = \{i, j, k\}\). By Theorem 2.7 and Remark 2.8 we see that the universal completion \(\tilde{G}_S\) of the amalgam \(G_S\) is a central extension of the group \(A_S\) isomorphic to \(\text{SL}_4(k)\). For each \(s \in S\), let \(\tilde{U}_s^{\pm}\) be the root groups mapping isomorphically to the standard root groups \(U_s^{\pm}\) of \(A_s\) under \(\pi_S\). Then, \(\text{SL}_2(k) \cong \langle \tilde{U}_s^{\pm}, \tilde{U}_j^{\pm} \rangle\) does not intersect the center of \(\tilde{G}_s\) and hence maps isomorphically to \(A_s\). For the same reason \(\text{SL}_3(k) \cong \langle \tilde{U}_i^{\pm}, \tilde{U}_j^{\pm} \rangle\) maps isomorphically to \(A_{i,j}\) and the same holds for \(j\) and \(k\). \(\square\)

Next we consider the image of \(\alpha_{i,j}\), when \(\{i, j\} \notin E(\Gamma)\).

**Definition 2.10.** If \(\text{Char} \; k\) is odd, then by \(z_i\) we denote the unique central involution of \(G_i \cong \text{SL}_2(k)\). By Lemma 2.9, the image of \(z_i\) is non-trivial in any completion of \(G_S\), where \(\Gamma_S\) is connected of size at least 2. We denote \(\tilde{z}_i = \tilde{\gamma}_{S,i}(z_i) \in \tilde{G}_S\).

**Lemma 2.11.** Let \(\{i, j\} \notin E\). If \(\text{Char} \; k\) is even then \(\alpha_{i,j}\) is injective. Otherwise \(\ker \alpha_{i,j} \leq \langle (z_i, z_j) \rangle \leq G_i \times G_j\).

**Proof** By Lemma 2.9, \(\alpha_i\) and \(\alpha_j\) are injective and their images commute. As \(\alpha_{i,j}\) extends both maps, the kernel must be central in \(G_i \times G_j\). \(\square\)

The equivalence relation \(\sim\) of bad pairs.

**Definition 2.12.** With the notation of Lemma 2.11 we call \(\{i, j\}\) a **good pair** if \(\ker \alpha_{i,j}\) is trivial and a **bad pair** otherwise. In case \(\{i, j\}\) is a bad pair we write \(i \sim_A j\).

The following is verified easily inside \(\text{SL}_4(k)\).

**Lemma 2.13.** If \(S = \{i, k, j\}\) is such that \(\Gamma_S\) has diagram \(A_3\), and \(k\) is a neighbor of \(i\) and \(j\), then \(\alpha_{i,j}(z_i z_j) \in Z(A_S)\).

**Lemma 2.14.** If \(\Gamma_S\) has diagram \(A_1 \times A_2\), then \((S, \sim_A)\) has no edges.

**Proof** Let \(S = \{i, j, k\}\) with \(i\) having no neighbor in the edge \(\{j, k\}\). If \(\{i, j\}\) is a bad pair, then \(\alpha_{i,j}(z_i, z_j) = e \in G\). This means that the non-trivial elements \(\alpha_i(z_i)\) and \(\alpha_j(z_j)\) are equal. However, in \(G\), we see that \(z_i\) commutes with all of \(G_k\), but \(z_j\) does not. Note
that by Lemma 2.9 the \( \alpha_{i,k} \) is injective so \( \alpha_j(z_j) \) does not commute with all of \( \alpha_k(G_k) \), whereas \( \alpha_i(z_i) \) does, a contradiction.

**Lemma 2.15.** Let \( S = \{i, j, k, l\} \) be such that \( \Gamma_S \) has diagram \( D_4 \) with central node \( k \). Then, \( \{\{i, j, l\}, \sim_A\} \) has one or three edges.

**Proof** For each \( s \in \{i, j, k, l\} \) let \( \tilde{z}_s = \tilde{\gamma}_{S,s}(z_s) \in \tilde{G}_S \). This is non-trivial by Lemma 2.9.

From Theorem 2.7 and Remark 2.8 we know that \( \tilde{G}_S \) is the universal Chevalley group \( \text{Spin}_8^-(k) \) having \( A_S \) as a central quotient. Moreover, from that remark we also have that \( C_2 \times C_2 \cong \{1, \tilde{z}_i\tilde{z}_j, \tilde{z}_j\tilde{z}_i, \tilde{z}_i\tilde{z}_l\} = Z(\tilde{G}_S) \). The conclusion follows. \( \square \)

**Lemma 2.16.** The graph \( (I, \sim_A) \) is a disjoint union of complete graphs.

**Proof** Suppose \( i \sim_A j \). By Lemma 2.14, and connectedness of \( \Gamma \), \( i \) and \( j \) have a common neighbor \( k \) in \( \Gamma \). Now if \( i \sim_A l \), then by Lemma 2.14, \( l \) is a neighbor of \( k \) also. By Lemma 2.15 since \( j \sim_A i \sim_A l \), also \( j \sim_A l \).

Let \( B = \{B_1, \ldots, B_m\} \) be the set of connected components of \( (I, \sim_A) \). Lemma 2.14 implies the following.

**Corollary 2.17.** If \( i \in B_s \) and \( j \in B_t \) are adjacent in \( \Gamma \), then \( \Gamma_{B_s \cup B_t} \) is a complete bipartite graph.

**Proposition 2.18.** Suppose that the Curtis-Tits amalgam \( \mathcal{A} \) of type \( \Gamma_\omega(k) \) has a non-trivial completion \( (A, \alpha_\bullet) \). Then, \( \sim_A \) is an equivalence relation. Moreover, \( \sim_\mathcal{A} \) refines \( \sim_A \) and \( \sim_A \) refines \( \sim_0 \).

**Proof** The fact that \( \sim_A \) is an equivalence relation is Lemma 2.16. Since the map \( \alpha_\bullet \) can only introduce more bad pairs, clearly \( \sim_\mathcal{A} \) refines \( \sim_A \). It follows from Corollary 2.17 that \( \sim_A \) refines \( \sim_0 \).

Isogeny of Curtis-Tits amalgams. Define a graph \( (\mathcal{B}, \sim_\mathcal{B}) \), where \( B_s \sim_\mathcal{B} B_t \) if \( \Gamma_{B_s \cup B_t} \) is a complete bipartite graph.

**Lemma 2.19.** The graph \( (\mathcal{B}, \sim_\mathcal{B}) \) is connected.

**Proof** Connectedness follows from Corollary 2.17 and connectedness of \( \Gamma \). \( \square \)

**Corollary 2.20.** Suppose \( \mathcal{B} = \{B_i \mid i = 1, \ldots, m\} \). Then, \( I = \bigcup_{i=1}^m B_i \) and for any selection \( x_i \in B_i \) \((i = 1, \ldots, m)\) we have that \( \{x_i, x_j\} \in E \) if and only if \( B_i \sim_\mathcal{B} B_j \).

**Corollary 2.21.** There exists a central elementary abelian 2-subgroup \( \tilde{Z} \) of \( \tilde{G} \) of size at most \( 2^{|I| - |\mathcal{B}|} \) such that \( \tilde{G}/\tilde{Z} \) contains a copy of the image of \( \mathcal{G} \) in \( A \). In fact \( \tilde{Z} \) is a central subgroup contained in the diagonal group \( \tilde{D} \).

**Proof** For each \( B \in \mathcal{B} \), let \( \tilde{Z}_B \) be the subgroup generated by all products \( \tilde{z}_i\tilde{z}_j \) in \( \tilde{G} = \tilde{G}_I \), where \( i, j \in B \) (so in case \( |B| = 1 \) \( \tilde{Z}_B \) is trivial). Thus \( \tilde{Z}_B \) is an elementary abelian group of size at most \( 2^{|B|-1} \). By Lemma 2.13 and since \( z_s \in D_s \), for all \( s \in I \), we have \( \tilde{Z}_B \leq Z(\tilde{G}) \cap \tilde{D} \). Take \( \tilde{Z} = \langle \tilde{Z}_B \mid B \in \mathcal{B} \rangle \). Then \( \tilde{Z} \leq Z(\tilde{G}) \cap \tilde{D} \) is an elementary abelian 2-group of size at most \( 2^{|I| - |\mathcal{B}|} \). Consider the map \( \pi : \tilde{G} \to A \). From Lemma 2.9 we know
that $\ker \pi$ intersects none of the groups $\tilde{G}_i$ and $\tilde{G}_{i,j}$ for $i \in I$ and $\{i, j\} \in E$. Moreover, $i \sim_A j$ if and only if $\tilde{z}_i \tilde{z}_j \in \ker \pi$. Thus, $\tilde{Z} \leq \ker \pi$ so that $\pi$ factors as $\tilde{G} \to \tilde{G}/\tilde{Z} \cong_A A$. Now consider distinct $i, j \in I$ such that $\{i, j\} \not\in E$. Then by Lemma 2.14 and since $\Gamma$ is triangle free and connected, $\alpha_{i,j}(z_iz_j) = 1$ implies $i \sim_A j$ and conversely. It follows that the map $\tilde{\pi}$ is the identity on the image of $\mathcal{G}$.

\[\square\]

Remark 2.22. Note that $\ker \pi$ in the proof of Corollary 2.21 does not have to be trivial.

Given $(A, \alpha_\bullet)$, we shall denote the central elementary abelian 2-group of Corollary 2.21 by $\tilde{Z}_A$.

Definition 2.23. We call two groups $A$ and $B$ isogenic if there is a group $C$ which is a finite central quotient of both $A$ and $B$.

Theorem F. Given a Curtis-Tits amalgam $\mathcal{A}$ of type $\Gamma_\omega(k)$ with non-trivial completion, then its universal completion $(\tilde{A}, \tilde{\alpha}_\bullet)$ is isogenic to the universal completion of the corresponding universal Curtis-Tits amalgam of type $\Gamma_\omega(k)$. More precisely, we have $\tilde{A} = \tilde{G}/\tilde{Z}_A$.

Proof Let $\mathcal{G}$ be the universal Curtis-Tits amalgam corresponding to $\mathcal{A}$. This means that there is a map of amalgams $\phi_\bullet: \mathcal{G} \to \mathcal{A}$. Let $(\tilde{G}, \tilde{\gamma}_\bullet)$ be the universal completion of $\mathcal{G}$ and let $(\tilde{A}, \tilde{\alpha}_\bullet)$ be the universal completion of $\mathcal{A}$. Then $\phi_\bullet$ induces a map $\tilde{\phi}: \tilde{G} \to \tilde{A}$. Thus, $(\tilde{A}, \tilde{\phi} \circ \tilde{\gamma}_\bullet)$ is a completion of $\mathcal{G}$. We now apply Corollary 2.21. First note that the image of $\mathcal{G}$ in $\tilde{A}$ is also the image of $\mathcal{A}$ in $\tilde{A}$. Since $\tilde{G}/\tilde{Z}_A$ contains a copy of $\mathcal{A}$, $(\tilde{G}/\tilde{Z}_A, \tilde{\alpha}_\bullet)$ is a completion of $\mathcal{A}$ for some map $\tilde{\alpha}_\bullet$. The proof also establishes a map $\tilde{\phi}: \tilde{G}/\tilde{Z}_A \to \tilde{A}$ (called $\tilde{\pi}$ there). By universality of $(\tilde{A}, \tilde{\alpha}_\bullet)$ as a completion of $\mathcal{A}$, we must have $\tilde{A} = \tilde{G}/\tilde{Z}_A$. Since $\tilde{Z}_A$ is finite, we are done.

Remark 2.24. Note that in [5, Proposition 3.12] it is proved that given a Kac-Moody group $G$ of 2-spherical type $\Gamma$, and given the groups $G^{sc}$, which is the simply-connected Kac-Moody group of the same type, and $\tilde{G}$ the universal completion of the Curtis-Tits amalgam contained in $G$, then, the canonical map $\pi: G^{sc} \to \tilde{G}$ has finite central kernel. In the proof of Corollary 2.21, $\tilde{G}$ and $A$ play the roles of $G^{sc}$ and $\tilde{G}$ respectively, so in fact the kernel of $\pi$ is a central 2-group of size at most $2^{\mid I \mid - |E|}$.

2.4. Universal Curtis-Tits amalgams; standard Curtis-Tits amalgams

Because of Theorem F we shall now restrict our attention to universal Curtis-Tits amalgams and their completions. These universal Curtis-Tits amalgams were classified in [4] (see Classification Theorem in the Introduction).

In this subsection we will make this correspondence precise and define one standard universal Curtis-Tits amalgam for each $\omega: \pi(\Gamma, 0) \to \Aut(k) \times \langle \tau \rangle$. Before we do this, we specify an action of the group $\Aut(k) \times \langle \tau \rangle$ (with $\tau$ of order 2) on $\SL_2(k)$. We let $\alpha \in \Aut(k)$ act entry-wise on the matrix $A \in \SL_2(k)$ and let $\tau$ act by sending each $A \in \SL_2(k)$ to its transpose inverse $A^{-1}$ with respect to the standard basis. Note that $\tau$ acts as an inner automorphism.
CURTIS-TITS GROUPS OF SIMPLY-LACED TYPE

Fix a spanning tree $T = (I, E(T))$ for $\Gamma$ rooted at $0 \in I$ and suppose that for any $e \in E(T)$, the vertex of $e$ nearest to $0$ (in $T$) has the smallest label. Given $e = \{i, j\} \in E - E(T)$ with $i < j$, let $\gamma(e)$ be the (the homotopy class of) the unique minimal loop of the graph $(I, E(T) \cup \{e\})$ directed such that $e$ is traversed from $i$ to $j$. This establishes an isomorphism of free groups $F(E - E(T)) \to \pi(\Gamma, 0)$. Thus the homomorphisms $\omega$ of the Classification Theorem correspond bijectively to set maps $E - E(T) \to \text{Aut}(k) \times \langle \tau \rangle$ given by $e \mapsto \omega(\gamma(e))$; we shall denote this set map also by $\omega$.

Definition 2.25. Let $\mathcal{P} = \{J \mid \emptyset \neq J \subseteq I \text{ with } |J| \leq 2\}$ and $\prec$ denoting inclusion. Given a set-map $\omega : E - E(T) \to \text{Aut}(k) \times \langle \tau \rangle$, the standard Curtis-Tits amalgam of type $\Gamma_\omega(k)$ is the amalgam $\mathcal{G}_\omega(k) = \{G_i, G_{i,j}, g_{i,j} \mid i, j \in I\}$ over $\mathcal{P}$, where, for every $i, j \in I$, we write

$$G_i = G_{\{i\}}, \quad G_{i,j} = G_{\{i,j\}}, \quad \text{and } g_{i,j} = g_{\{i,j\}}$$

where

(SCT1) for any vertex $i$, we set $G_i = \text{SL}_2(k)$ and for each pair $i, j \in I$,

$$G_{i,j} \cong \begin{cases} \text{SL}_3(k) & \text{if } \{i, j\} \in E \\ G_i \times G_j & \text{else} \end{cases};$$

(SCT2) For $\{i, j\} \in E$ with $i < j$ we have

$$g_{i,j} : G_i \to G_{i,j} \quad g_{j,i} : G_j \to G_{j,i}$$

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \quad A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A^{\omega_{j,i}} \end{pmatrix},$$

whereas for all other pairs $(i, j)$, $g_{i,j}$ is the natural inclusion of $G_i$ in $G_i \times G_j$.

Here

$$\omega_{j,i} = \begin{cases} \omega(\{j, i\}) & \text{if } \{j, i\} \in E - E(T) \text{ and } i < j \\ 1 & \text{else.} \end{cases}$$

$\omega_{\{i,j\}}$ We call $\mathcal{G}_\omega(k)$ orientable if $\text{im } \omega \leq \text{Aut}(k)$.

This notation directly generalizes that of [3]. By [4] every Curtis-Tits amalgam over $k$ with connected diagram $\Gamma$ is of the form $\mathcal{G}_\omega(k)$ for some $\omega$.

2.5. Combinatorics of centers in Curtis-Tits groups

Let $(G, \gamma_{\bullet})$ be a completion of a universal Curtis-Tits amalgam $\mathcal{G} = \{G_i, G_{i,j}, g_{i,j} \mid i, j \in I\}$ of type $\Gamma_\omega(k)$ for some $\omega$.

Definition 2.26. For any $i \in I$ and $a \in k^*$, let

$$d_i^G(a) = \gamma_i \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

$$s_i = \gamma_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
Let \( D \) be the subgroup of \( G \) generated by the tori \( D_i = \text{im} \, d_i \). Then \( d^G : (k^*)^I \rightarrow D \) given by \( d^G((a_i)_{i \in I}) = \prod d_i^G(a_i) \) is a surjective homomorphism.

Note that by Lemma 2.9, \( \gamma_i \) is injective for all \( i \in I \) and, if \( \{ i, j \} \in E \), then also \( \gamma_{i,j} \) is injective. Moreover, note that in this case \( \gamma_i = \gamma_{i,j} \circ g_{i,j} \). Therefore any computation in the subgroup \( \langle d_i^G(x), s_i, d_j(y), s_j \mid x, y \in k^* \rangle \) can be done entirely inside of \( G_{i,j} \). If \( G \) is clear from the context, we shall often omit the superscript \( G \).

We will investigate the group \( Z(G) \cap D \). To this end we have the following

**Definition 2.27.** For any \( i, j \in I \) with \( i \sim j \), let \( \rho_{j,i} = \omega_{i,j}^{-1} \omega_{j,i} \in \text{Aut}(k) \times \langle \tau \rangle \leq \text{Aut}(k^*) \). Here \( \tau \) embeds as the map \( x \mapsto x^{-1} \). We define the map \( \mathcal{K} = \mathcal{K}_\gamma = (\mathcal{K}_i)_{i \in I} : (k^*)^I \rightarrow (k^*)^I \mathcal{K} \), by setting, for \( a = (a_i)_{i \in I} \in (k^*)^I \)

\[
\mathcal{K}_i(a) = a_i^{-2} \prod_{j \succ i} \rho_{j,i}(a_j).
\]

We can now describe the action of \( d(a) \) on \( G_i \).

**Lemma 2.28.** Suppose that \( d(a) = d^G(a) = \prod_{j \in I} d_j(a_j) \) and that

\[
g = \gamma_i \begin{pmatrix} x & y \\ w & z \end{pmatrix} \in \gamma_i(G_i).
\]

Then,

\[
(d(a))^{-1} gd(a) = \gamma_i \begin{pmatrix} x & yk \\ k^{-1}w & z \end{pmatrix},
\]

where \( k = \mathcal{K}_i((a_j)_{j \in I}) \).

**Proof** First note \( d_j(a_j) = \gamma_j \begin{pmatrix} a_j & 0 \\ 0 & a_j^{-1} \end{pmatrix} = \gamma_{j,i} \begin{pmatrix} a_j & 0 \\ 0 & a_j^{-1} \end{pmatrix} \). This equals

\[
\gamma_{j,i} \begin{pmatrix} \omega_{j,i}(a_j) & 0 & 0 \\ 0 & \omega_{j,i}(a_j)^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

if \( j < i \),

\[
\gamma_{j,i} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{j,i}(a_j) & 0 \\ 0 & 0 & \omega_{j,i}(a_j)^{-1} \end{pmatrix}
\]

if \( j > i \).

In both cases conjugation gives

\[
(d_j(a_j))^{-1} gd_j(a_j) = \gamma_i \begin{pmatrix} x & y \rho_{j,i}(a_j) \\ \rho_{j,i}(a_j)^{-1}w & z \end{pmatrix}.
\]

Finally, note that

\[
(d_i(a_i))^{-1} gd_i(a_i) = \gamma_i \begin{pmatrix} x & ya_i^{-2} \\ a_i^{-2}w & z \end{pmatrix}.
\]

The result follows since the actions of all \( d_j(a_j) \) \( (j \in I) \) commute. \( \square \)

The following is an immediate consequence of Lemma 2.28.
Theorem 2.29. Let $a = (a_i)_{i \in I} \in (k^*)^I$. Then, $d^G(a) \in Z(G)$ if and only if $a \in \ker \mathcal{X}$.

Remark 2.30. Note that $\mathcal{X}$ does not depend on $G$, but $d^G$ does.

Theorem 2.31. Suppose that $G$ is a group of Kac-Moody type and $\mathcal{G}$ is the Curtis-Tits amalgam arising from the action on the twin-building $\Delta$ associated to its twin BN-pair. Then the following are equivalent for $a \in (k^*)^I$.

1. $d^G(a) \in Z(G)$,
2. $\mathcal{X}(a) = (1)^I$,
3. $d^G(a)$ acts trivially on $\Delta$.

Proof. By Theorem 2.29, 1. and 2. are equivalent and by Lemma 3.4 of [5] 3. and 1. are equivalent. □

Remark 2.32. Note that in the case when $G$ is a group of Lie type, this result is equivalent to the one of Steinberg (see for example section 12.1 of [6]) but is more elementary as it does not make use of weight lattices. Moreover, this method applies to the more general case of Curtis-Tits amalgams.

3. Construction of the amalgam $\mathcal{G}^\omega(k)$ inside a group of Kac-Moody type

In this section we shall prove Corollary B.

3.1. Orientable Curtis-Tits amalgams

Proof of Theorem D.1. The second statement follows from Theorem 2.31. The main statement follows directly from Corollary 1.2 of [4], which we quote here.

Corollary 3.1. Let $\Gamma$ be a connected simply laced Dynkin diagram with no triangles and $k$ a field with at least 4 elements. The universal completion of the standard Curtis-Tits amalgam over a commutative field $k$ and diagram $\Gamma$ is a group of Kac-Moody type over $k$ with Dynkin diagram $\Gamma$ (and $\mathcal{G}$ is the Curtis-Tits amalgam for this group) if and only if $\mathcal{G}$ is orientable.

Recall from Definition 1.4 that the universal completion of this standard amalgam $\mathcal{G}$ of type $\Gamma^\omega(k)$ is called the simply-connected Curtis-Tits group of type $\Gamma^\omega(k)$. In order to fix notation for future reference we recall some relevant details from its proof here. Fix $\Gamma$, $I$, $k$, and $\omega$, as well as $\mathcal{G}$ as in the Classification Theorem and see Definition 2.25 for a precise definition of a standard Curtist-Tits amalgam.

For each $i \in I$, using the identification $G_i \cong \text{SL}_2(k)$, for each $t \in k$ define

$$U_i^+(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U_i^-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

and let $U_i^\varepsilon = \{U_i^\varepsilon(t) : t \in k^\varepsilon\} \leq G_i$, for $\varepsilon = +, -$. Since $\text{im} \omega \leq \text{Aut}(k)$ (i.e. $\mathcal{G}$ is orientable) we have that $g_{i,j}(U_j^\varepsilon)$ and $g_{i,j}(U_j^\varepsilon)$ belong to the same Borel subgroup of $G_{i,j}$ for all $i, j \in I$. This allows us to create a sound Moufang Foundation whose rank 1 and rank 2 residues are buildings associated to split algebraic groups. Then we use the
integration results by Mühlherr [12] to prove that there exists a Moufang twin-building \( \Delta = (\Delta_+, \Delta_-, \delta_i) \) with diagram \( \Gamma \) with this foundation. This building is constructed from a combinatorial building over the covering of \( \Gamma \) associated to \( \pi_1(\Gamma;i_0) \) by taking certain fixed points under the action of \( \pi_1(\Gamma;i_0) \). Note that Mühlherr’s theorem allows infinite type sets and as such the image of \( \omega \) does not have to be finite. This building has a pair of opposite chambers \( c_+ \in \Delta_+, c_- \in \Delta_- \) such that \( U^\varepsilon_i \) can be identified with the root group \( U_{\varepsilon \alpha_i} \leq \text{Aut}(\Delta) \) \((\varepsilon = +, -, i \in I(\Gamma))\). Here \( \Pi = \{\alpha_i : i \in I\} \) is a fundamental system of a system \( \Phi \) of (real) roots of type \( \Gamma \). That is, if \( \Sigma = (\Sigma_+, \Sigma_-) \) is the twin-apartment of \( \Delta \) determined by \( (c_+, c_-) \), then \( \Pi \) is the set of roots in \( \Sigma_+ \) containing \( c_+ \) and determined by the panels on \( c_+ \). Moreover, let \( G = \langle U_\alpha : \alpha \in \Phi \rangle \leq \text{Aut}(\Delta) \) \((\text{In fact since } \Gamma \text{ is simply-laced we have } G = \langle U_\alpha, U_{-\alpha} : \alpha \in \Pi \rangle \) ). Then, it is known that \( (G, \{U_\alpha\}_{\alpha \in \Phi}) \) is a group with twin root group datum. For \( \alpha, \beta \in \Pi \), let
\[
G_\alpha = \langle U_\alpha, U_{-\alpha} \rangle,
G_{\alpha, \beta} = \langle G_\alpha, G_\beta \rangle.
\]

The proof of Corollary 3.1 furthermore asserts that the identification isomorphisms \( U^\varepsilon_i \to U_{\varepsilon \alpha_i} \) extend to surjective homomorphisms
\[
\gamma_i : G_i \xrightarrow{\sim} G_{\alpha_i},
\gamma_{i,j} : G_{i,j} \to G_{\alpha_i, \alpha_j}
\]
where the latter is an isomorphism whenever \( \{i, j\} \) is an edge of \( \Gamma \), provided one assumes that \( |I| \geq 4 \). Universality of \( (\tilde{G}, \tilde{\gamma}_\bullet) \) gives that the collection \( \gamma_\bullet = \{\gamma_i, \gamma_{i,j} : i, j \in I\} \) induces a surjective homomorphism \( \tilde{G} \to G \). Thus, \( (G, \gamma_\bullet) \) is a completion of \( \mathcal{G} \). Combined with Theorem 2.31, this completes the proof of part 1 in Theorem D.

For future reference also recall [14] that, setting
\[
U^\varepsilon = \langle U_\alpha : \alpha \in \Phi^\varepsilon \rangle \quad (\varepsilon = +, -)
N = \langle m(u) : u \in U_\alpha - \{1\}, \alpha \in \Pi \rangle,
D = \bigcap_{\alpha \in \Phi} N_c(U_\alpha)
\]
we have a twin- BN-pair \((B^+, N), (B^-, N))\), where \( B^\varepsilon = DU^\varepsilon \). In fact this is the twin BN-pair giving rise to \( \Delta = (\Delta_+, \Delta_-, \delta_\varepsilon) \). \( U^\varepsilon \cdot NDB^\varepsilon \)

3.2. Non-orientable Curtis-Tits amalgams and the proof of Theorem D.2.

From now on we fix non-orientable \( \delta \) and drop it as a superscript.

Covering graphs, fibers and deck-transformations. Let \( K \) be the kernel of the composition of homomorphisms \( \pi(\Gamma, 0) \xrightarrow{\sim} \text{Aut}(k) \times \langle \tau \rangle \to \langle \tau \rangle \). Recall that there is a covering graph \( \Lambda = (I(\Lambda), E(\Lambda)) \) such that if \( p : \Lambda \to \Gamma \) is the projection map and \( \hat{0} \) is a vertex mapping to \( 0 \), the induced homomorphism is an isomorphism \( p_* : \pi_1(\Lambda, \hat{0}) \to K \) and as \( K \) is normal, \( \pi(\Gamma, 0)/K \) acts as a group of deck transformations commuting with \( p \). Let \( \theta \) be the generator of \( \pi(\Gamma, 0)/K \cong \mathbb{Z}_2 \).
Let $i, j \in I$ be distinct, and let $\Lambda_{i,j}$ denote the subgraph induced on the $p$ fiber over $\{i, j\}$. Then, since $\Gamma$ has no circuits of length $\leq 2$, $\Lambda_{i,j}$ consists of pairwise disjoint edges if $\{i, j\} \in E(\Gamma)$ and has no edges otherwise. Since $K < \pi(\Gamma, 0)$, in fact $\theta$ restricts to an automorphism of $\Lambda_{i,j}$.

**Definition 3.2.** Let $\mathcal{L} = \{L_i, L_{i,j}, l_{i,j} \mid i, j \in I(\Lambda)\}$ be the amalgam such that, for all $i, j \in I(\Lambda)$,

1. $(L_1)$ $L_i$ is a copy of $G_{p(i)}$;
2. $(L_2)$ $L_{i,j}$ is a copy of $\begin{cases} G_{p(i), p(j)} & \text{if } \{i, j\} \in E(\Lambda), \\ G_{p(i)} \times G_{p(j)} & \text{else.} \end{cases}$
3. $(L_3)$ $l_{i,j} = \begin{cases} g_{p(i), p(j)} & \text{if } \{i, j\} \in E(\Lambda), \\ \text{canonical inclusion} & \text{else.} \end{cases}$

Given any $J \subseteq I(\Lambda)$ with $1 \leq |J| \leq 2$, and denoting by $\pi_\bullet : \mathcal{L} \to \mathcal{G}$ the homomorphism of amalgams induced by $p$, then we have a commuting diagram of isomorphisms

\[
\begin{array}{ccc}
x \in L_J & \xrightarrow{\theta_J} & L_{\theta(J)} \ni x \\
\pi_J \downarrow & & \downarrow \pi_{\theta(J)} \\
x \in G_{p(J)} & & \\
\end{array}
\]

That is, identifying $G_{p(J)}$ with its copies $L_J$ and $L_{\theta(J)}$, the maps $\pi_\bullet$ and $\theta_\bullet$ are given by the identity mapping. Then, by $(L_3)$ $\theta_\bullet$ is an automorphism of $\mathcal{L}$.

Let $\\hat{\omega} = \omega \circ p_\bullet$.

**Definition 3.3.** We first apply the construction of Subsection 3.1 replacing $\Gamma, \omega, \mathcal{G}, G_\bullet$, and $(G, \gamma)_\bullet$ by $\Lambda, \hat{\omega}, \mathcal{L}, L_{sc}$, and $(L_{sc}, \lambda_{sc})$ respectively, and leaving all other notation as it is. Thus $L_{sc}$ is the simply-connected Curtis-Tits group of type $\Lambda_{sc}(k)$. Next, we will pass to the adjoint setting via the canonical map $L_{sc} \to L_{sc}/Z(L_{sc})$ replacing $L_{sc}$ and $(L_{sc}, \lambda_{sc})$ by $L_\circ$ and $(L, \lambda_\circ)$, not changing the notation otherwise (so for instance $U_\alpha$ now denotes the (isomorphic) image of the corresponding root group in $L_{sc}$). Thus, $(L, \{U_\alpha\}_{\alpha \in \Phi})$ is a twin-root group datum, where $\Phi$ is the root system of type $\Lambda$ with fundamental system $\Pi = \{\alpha_i \mid i \in I(\Lambda)\}$ and for each $\alpha, \beta \in \Phi$ we have $L_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$ and $L_{\alpha, \beta} = \langle L_\alpha, L_\beta \rangle$. It follows that, not only $(L_{sc}, \lambda_{sc})$, but also $(L, \lambda_\circ)$ is a non-trivial completion of $\mathcal{L}$. Now let

\[
D_\alpha = N_{L_\alpha}(U_\alpha) \cap N_{L_\alpha}(U_{-\alpha}),
\]

\[
D = \prod_{i \in I(\Lambda)} D_{\alpha_i}.
\]

The group $L$ acts on the twin-building $\Delta = (\Delta_+, \Delta_-, \delta_\circ)$ associated to the twin-root datum, alternatively described as in [4, §5]. The automorphism $\theta_\bullet$ induces a diagram automorphism of the root group data of $L$. Indeed, if $(\tilde{L}, \tilde{\lambda})$ is the universal completion
of $\mathcal{L}$, then $\theta_\bullet$ induces an automorphism of $\tilde{L}$. Hence $\theta_\bullet$ induces an automorphism of $L \cong \tilde{L}/Z(\tilde{L})$. More precisely, the following diagrams commute

\[ \begin{array}{ccc}
\lambda_i & \xrightarrow{\lambda} & L \\
\downarrow & & \downarrow \theta \\
\theta_i^{-1} \lambda_{\theta(i)} & \xrightarrow{\theta} & \theta_i^{-1} \lambda_{\theta(i)} \\
\end{array} \]

\[ \begin{array}{ccc}
\lambda_{i,j} & \xrightarrow{\lambda_{i,j}} & L \\
\downarrow & & \downarrow \theta \\
\theta_{i,j} \lambda_{\theta(i),\theta(j)} & \xrightarrow{\theta} & \theta_{i,j} \lambda_{\theta(i),\theta(j)} \\
\end{array} \]

Note that $\theta(U_{\alpha_i}) \in \{U_{\alpha_i}, U_{-\alpha_i}\}$. Moreover, if $\{i, j\} \in E(\Gamma)$, then because of the Chevalley relations, if $\theta(U_{\alpha_i}) = U_{\epsilon\alpha_i}$, then also $\theta(U_{\alpha_j}) = U_{\epsilon\alpha_j}$, for some $\epsilon = \pm$. Since $\mathcal{G}$ is a non-orientable amalgam, there exists some $i$ for which $\theta(U_{\alpha_i}) = U_{-\alpha_i}$, and so by connectivity of $\Lambda$, we must have $\theta(U_{\alpha_i}) = U_{-\alpha_i}$, for all $i \in I(\Lambda)$. This shows that $\theta$ interchanges the two halves of $\Delta$ and there exists a chamber $c_+ \in \Delta_+$, such that $c_+ \text{ opp } c_\alpha$. Namely, $c_+$ is the chamber corresponding to the Borel subgroup corresponding to the fundamental system $\Pi$.

Let $L^\theta$ denote the group of fixed points of $L$ under $\theta$. Our next aim is to show that $L^\theta$ contains a non-trivial completion of $\mathcal{G}$.

We consider the following setup. Let $i, j \in I(\Gamma)$ and $k \in p^{-1}(i)$ and $l \in p^{-1}(j)$ so that if $\{i, j\} \in E(\Gamma)$, then $\{k, l\} \in E(\Lambda_{i,j})$.

**Lemma 3.4.**

1. $\lambda(L_{k,\theta(k)}) \cong \lambda(L_k) \times \lambda(L_{\theta(k)}) \cong \text{SL}_2(k) \times \text{SL}_2(k)$,
2. $\langle \lambda(L_{k,l}), \lambda(L_{\theta(k),\theta(l)}) \rangle \cong \lambda(L_{k,l}) \circ \lambda(L_{\theta(k),\theta(l)})$.

**Proof** By Lemma 2.9 $\lambda(L_k) \cong \text{SL}_2(k) \cong \lambda(L_{\theta(k)})$. Moreover, since $k$ and $\theta(k)$ are at distance more than 2 in $\Lambda$, and $\Gamma$ is triangle free and connected, by Lemma 2.14, $\lambda(L_{k,\theta(k)}) \cong \lambda(L_k) \times \lambda(L_{\theta(k)})$.

Similarly, it follows from the structure of $\Lambda_{i,j}$ that $\lambda(L_{k,l})$ and $\lambda(L_{\theta(k),\theta(l)})$ commute, so they can only intersect in their center. \hfill $\square$

**Definition 3.5.** Next, define the amalgam $\mathcal{L}^\theta = \{L_{i,j}^\theta, L_{i,j}^\theta, L_{i,j}^\theta | i, j \in I\}$, where for $i, j \in I$ and $k, l \in I(\Lambda)$ as before, we define the following subgroups of $L$:

$\mathcal{L}_{i,j}^\theta = \{\lambda_k(y)\theta(\lambda_k(y)) | y \in L_k\}$ and $\mathcal{L}_{i,j}^\theta = \{\lambda_k(y)\theta(\lambda_k(y)) | y \in L_{k,l}\}$,

where the connecting maps $L_{i,j}$ are inclusions of subgroups of $L$. Also define a map of amalgams $\mu_\bullet : \mathcal{G} \to \mathcal{L}^\theta$ setting

$\mu_i : y \mapsto \lambda_k(y)\theta(\lambda_k(y))$ \quad \text{for all } y \in G_i,$

$\mu_{i,j} : y \mapsto \lambda_k(y)\theta(\lambda_k(y))$ \quad \text{for all } y \in G_{i,j}.$

Note that we are implicitly using the map $\pi_\bullet$.

**Proposition 3.6.**

1. $\mathcal{L}^\theta$ is an amalgam of subgroups in $L^\theta$.
2. $\mu_\bullet : \mathcal{G} \to \mathcal{L}^\theta$ is a non-trivial morphism of amalgams.
Proof Part 1. In the notation of Definition 3.5, for any \( y \in G_J \), we have \( \theta(\lambda_J(y)\theta(\lambda_J(y))) = \theta(\lambda_J(y))\lambda_J(y) = \lambda_J(y)\theta(\lambda_J(y)) \) for \( J = \{k\} \) and \( J = \{k, l\} \) due to the central and direct products appearing in Lemma 3.4.

Part 2. To see that \( \mu_i \) is non-trivial, we show that ker \( \mu_i \) is central in \( G_i \). Namely, if \( y \in G_i \) is such that \( \mu_i(y) = 1 \), then apparently \( \lambda_i(y) \in L_k \cap L_{\theta(k)} \) which is central in \( L_k \) (and \( L_{\theta(k)} \)). Since the map \( y \mapsto y \) identifies \( G_i \) with \( L_k \) the claim follows.

Since the products appearing in Lemma 3.4 are direct or central, \( \mu_i \) and \( \mu_{i,j} \) are group homomorphisms. Hence it suffices to show that \( \mu_{i,j} \circ g_{i,j} = l_{i,j}^0 \circ \mu_i \). To this end note that the connecting maps of \( \mathcal{L} \) are compatible with the identification in (3.2) of \( L_k \) with \( G_i \) (resp. \( L_{k,l} \) with \( G_{i,j} \)) and of Equation (3.3). 

Proof of Theorem D.2. By Proposition 3.6, the non-orientable amalgam \( \mathcal{G} \) has a non-trivial completion in the group \( L^0 \). This proves part 2. of Theorem D.

From now on we shall denote by \((G, \gamma_\bullet)\) the completion of \( \mathcal{G} \) generated by \( \mathcal{L}^0 \) in \( L^0 \).

3.3. Proof of Theorems A and Theorem C

Proof of Theorem A. Let \( \mathcal{A} \) be any Curtis-Tits amalgam of type \( \Gamma_\omega(k) \) and let \((\hat{A}, \tilde{\alpha}_\bullet)\) be its universal completion. Since \( \mathcal{G}_\omega(k) \) is a universal amalgam, we have a map \( \rho_\bullet : \mathcal{G}_\omega(k) \rightarrow \mathcal{A} \) and \((\hat{A}, \rho_\bullet)\) is a completion of \( \mathcal{G}_\omega(k) \). By Proposition 2.18, \( \sim_\mathcal{A} \) refines \( \sim_0 \).

For the converse, let \( \mathcal{G}_\omega(k) \) be the standard Curtis-Tits amalgam of type \( \Gamma_\omega(k) \). Then, by Corollary 3.1 and Proposition 3.6 it has a non-trivial universal completion \((\hat{G}, \hat{\gamma}_\bullet)\).

Consider the completion \((G_0, \gamma_{0,\bullet})\), where \( G_0 = \hat{G}/Z(\hat{G}) \) and its relation \( \sim_{G_0} \). Then, by the preceding paragraph \( \sim_{G_0} \) refines \( \sim_0 \). We now claim that we have equality. Namely, let \( i, j \in I \) be such that \( i \sim_0 j \) and let \( z \) be the central involution of \( G_{i,j} = G_i \times G_j \). Then, by Definition 1.1, for any \( k \in I - \{i, j\} \), either \( k \) is not connected to either \( i \) or \( j \) in \( \Gamma \) or it is connected to both. In the former case, clearly \( \gamma_{0;i,j}(z) \) commutes with \( \gamma_{0;k}(G_k) \) and in the latter, it follows from Lemma 2.13 that \( \gamma_{0;i,j}(z) \) commutes with \( \gamma_{0;k}(G_k) \). That is, apparently \( \gamma_{0;i,j}(z) \in Z(G_0) = \{1\} \) and we must have \( i \sim_{G_0} j \) already.

It now follows that if \( \sim_\mathcal{A} \) refines \( \sim_0 \), then we have a homomorphism of amalgams \( \mathcal{A} \rightarrow \mathcal{G}_0 \), where \( \mathcal{G}_0 \) is the image of \( \mathcal{G}_\omega(k) \) under \( \gamma_{0,\bullet} \) in \( G_0 \). In particular, \( \mathcal{A} \) has \( G_0 \) as a non-trivial completion and hence \( \mathcal{A} \) also has a non-trivial universal completion. 

Proof of Theorem C. Suppose \( \mathcal{A} \) is a Curtis-Tits amalgam with property (D) of type \( \Gamma_\omega(k) \) such that \( \sim_\mathcal{A} \) refines \( \sim_0 \). By Theorem A this is equivalent to the assumption that \( \mathcal{A} \) has a non-trivial universal completion, which we shall denote \((\hat{A}, \tilde{\alpha}_\bullet)\). Since \( \mathcal{G}_\omega(k) \) is universal, there is a map \( \mathcal{G}_\omega(k) \rightarrow \mathcal{A} \). By universality of the completion \((\hat{G}, \hat{\gamma}_\bullet)\) of \( \mathcal{G}_\omega(k) \), we have a surjective homomorphism \( \pi : \hat{G} \rightarrow \hat{A} \) taking \( \hat{\gamma}_\bullet(\mathcal{G}_\omega(k)) = \mathcal{G}_1 \) to \( \mathcal{A} \). Hence, we have \( d\hat{A} = \pi \circ d\hat{G} \).

Define \( Z_{\hat{A}} = \langle Z^1, z_{kl} : k \sim_\mathcal{A} l \text{ with } k, l \in I \text{ distinct} \rangle \leq (k^*)^I \). Note that for any \( i, j \in I \),

\[
\begin{align*}
    i \sim_\mathcal{A} j & \iff z_{ij} \in Z_{\hat{A}}, \\
    i \sim_{\mathcal{A}} j & \iff z_{ij} \in Z_{\hat{A}}.
\end{align*}
\]
Note that we have $Z\tilde{A} \leq Z\tilde{A}$. Consequently, we have $d\tilde{G}(Z\tilde{A}) \leq d\tilde{G}(Z\tilde{A}) \leq \ker \pi$. Consider the quotient $\tilde{A} = \tilde{G}/d\tilde{G}(Z\tilde{A})$. Then, we have a canonical map $\bar{\pi}: \tilde{A} = \tilde{G}/d\tilde{G}(Z\tilde{A}) \rightarrow \tilde{G}/d\tilde{G}(Z\tilde{A}) \rightarrow \tilde{G}/d\tilde{G}(Z\tilde{A}) \rightarrow G/d\tilde{G}(Z\tilde{A}) \rightarrow \ker \pi = \tilde{A}$.

Since $z_{ij} \in Z\tilde{A}$ for all distinct $i, j \in I$ with $i \sim_{\mathcal{A}} j$, there is a completion $(\tilde{A}, \tilde{\alpha})$ of $\mathcal{A}$. Note that we have $(\tilde{A}, \tilde{\alpha})$ to $\mathcal{A}$. By universality of $(\tilde{A}, \tilde{\alpha})$ we now must have $\tilde{A} = \tilde{A}$ and we obtain $Z\tilde{A} = Z\tilde{A}$ and $\sim_{\mathcal{A}} = \sim_{\mathcal{A}}$. The second claim of Theorem C follows immediately from the first. □

4. The geometry $(\Delta^\theta, \approx)$ for non-orientable Curtis-Tits amalgams

4.1. Definition of $(\Delta^\theta, \approx)$

The construction of $\Delta^\theta$, and the proof of its simple-connectedness follows the pattern of [3].

Let $\Delta = (\Delta_+, \Delta_-, \delta_\ast)$ be the twin-building associated to $L$ as in Section 3.2 and let opp denote opposition. Note that $\Delta$ has diagram $\Lambda$ over the index set $I(\Lambda)$.

Recall that the action of $\theta$ on $L$ induces an action of $\theta$ on $\Delta$ that interchanges $\Delta_+$ and $\Delta_-$, and such that $c^\pm \text{opp } c^\pm \approx c_\pm$, where, for $\varepsilon = +, -, c_\varepsilon$ is the chamber corresponding to the Borel group $B_\varepsilon$, containing $U_{\varepsilon, \alpha}$ for all $\alpha \in \Pi$.

Definition 4.1. We define a relaxed incidence relation on $\Delta_\varepsilon$ as follows. We say that $d_\varepsilon$ and $c_\varepsilon$ are $i$-adjacent if and only if $d_\varepsilon$ and $e_\varepsilon$ are in a common $p^{-1}(i)$-residue. In this case we write

$$d_\varepsilon \approx_i e_\varepsilon,$$

where we let $i \in I$. Note that the residues in this chamber system are $J$-residues of $\Delta_\varepsilon$ where $J^\theta = J \subseteq I(\Lambda)$.

Define the fixed subgeometry $\Delta^\theta = \{d \in \Delta_+ \mid d \text{ opp } d^\theta\}$ endowed with the adjacency relation $\approx$. Note that $c_\varepsilon \approx \Delta^\theta$.

It is easy to see that residues of $\Delta^\theta$ are the intersections of certain residues of $(\Delta_+, \approx)$ with the set $\Delta^\theta$.

Remark 4.2. The motivation for defining $\Delta^\theta$ is that the action of $L$ on $\Delta$ induces an action of $L^\theta$ on $\Delta^\theta$. We shall prove in Section 5.3 that this action is flag-transitive and use Tits’ Lemma [22] to show that under certain conditions, the universal completion of $\mathcal{G}$ is a central extension of $L^\theta$.

The following results from [3] and their proofs remain valid for general connected simply-laced diagrams without triangles and we mention them here for their value in this more general context: Lemma 4.24 (using that $\theta$ has no fixed vertices or edges on $\Lambda$), 4.26, 4.27, Lemma 4.28, Lemma 4.30, Proposition 4.31, Proposition 4.32 and Theorem 4.45. As in loc. cit. we obtain the following theorem.

Theorem 4.3. Let $k$ be a field of size at least 7. Then, $\Delta^\theta$ is connected and simply $2$-connected.

The following corollary is of general interest.

Corollary 4.4. Let $d \in \Delta_\varepsilon$. Then, there exists $(c, c^\theta) \in \Delta^\theta$ such that $d \in \Sigma_+(c, c^\theta)$. 
5. The action of the universal completion $\tilde{G}$ on $\Delta^\theta$

5.1. The torus of $L^\theta$

Definition 5.1. Let $L$ and $L^\theta$, and $\mathcal{L}^\theta$ be as in Subsection 3.2 and let $(\tilde{G}, \tilde{\gamma})$ be the universal completion of $G$ and recall from Proposition 3.6 that we have a non-trivial morphism of amalgams $\mu_*: \mathcal{G} \to \mathcal{L}^\theta$. By the universal property we have a map

$$\pi: \tilde{G} \to L^\theta$$

that restricts to a non-trivial homomorphism of amalgams on $\mathcal{L}^\theta$. Recall that $G = \pi(\tilde{G})$ by definition. For $i \in I$, let $\tilde{D}_i = \tilde{\gamma}_i(D_i) \leq \tilde{G}$, $\tilde{D} = \langle \tilde{D}_i: i \in I(\Gamma) \rangle \leq \tilde{G}$, and $D^\theta = \mu_i(D_i) = \pi(\tilde{D}_i)$. Finally, let $D_\theta = \prod_{i \in I} D^\theta_i \leq D \cap L^\theta$ (where $D$ is as in Definition 3.3).

Remark 5.2. All arguments up until Theorem 5.20 do not require the assumption that $Z(L) = 1$ and are valid for any quotient of $L^\alpha$ that admits $\theta$ as an automorphism.

The aim of this subsection is to investigate the difference between $D_\theta$ and $D \cap L^\theta$.

Lemma 3.3 of [5] has the following consequence.

Lemma 5.3. We have $\text{Stab}_L((c_+, c_-)) = \bigcap_{\alpha \in \Phi} N_L(U_\alpha) = \prod_{i \in I(\Lambda)} D_{\alpha_i} = D$. Therefore, $\text{Stab}_{L^\theta}((c_+, c_-)) = D \cap L^\theta$ and $\text{Stab}_G((c_+, c_-)) = D \cap G$.

Definition 5.4. We now define the following maps. $d: (k^*)^{I(\Lambda)} \to D$, and $\theta, \nu: (k^*)^{I(\Lambda)} \to (k^*)^{I(\Lambda)}$ by

$$d(x_i)_{i \in I(\Lambda)} = \prod_{i \in I(\Lambda)} d_i(x_i); \quad \theta((x_i)_{i \in I(\Lambda)}) = (x_{\theta(i)})_{i \in I(\Lambda)}; \quad \nu(x) = x\theta(x)^{-1};$$

Note that $\theta$ and $\nu$ on $(k^*)^{I(\Lambda)}$ induce $\theta$ and $\nu$ on $D$ via $d$ (see (3.3)). We now set

$$F = \ker \nu = \{ x \in (k^*)^{I(\Lambda)} \mid x = \theta(x) \},$$

$$M = \text{im} \nu,$$

$$K = \ker d.$$

Now note that $d(F) = D_\theta = \langle d_i(a) d_{\theta(i)}(a) \mid i \in I(\Lambda), a \in k^* \rangle \leq D \cap L^\theta$. However, this inclusion may be proper, as we will now investigate. The diagram in Figure 1 arises from applying the snake lemma to the commuting diagram involving the two rows connected via $d$, which are exact. The columns are all exact.

The snake lemma gives the following result.

Lemma 5.5. We have an isomorphism $\partial: (M \cap K)/\nu(K) \cong (D \cap L^\theta)/D_\theta$. 
We shall show in Corollary 5.10 that \( G = L^θ \) if and only if the following condition holds:

\[
(D) \quad (D ∩ L^θ)/D_θ = 0
\]

By Lemma 5.5, this is equivalent to the fact that \( M ∩ K = \nu(K) \).

**Lemma 5.6.** \((M ∩ K)/\nu(K) \cong (D ∩ L^θ)/D_θ\) is an elementary abelian 2-group. If \( k \cong \mathbb{F}_q \), then it has rank at most \( n \).

**Proof** Let \( x = (x_i)_{i \in I(\Lambda)} \in M \cap K \). This means that \( x_i = a_i a_i^{-1}_θ = \nu(a) \) for some \( a = (a_i)_{i \in I(\Lambda)} \) and \( d(x) = 1 \in D \). Note that \( x^2 = xθ(x)^{-1} = \nu(x) \in \nu(K) \). This means that \((M ∩ K)/\nu(K)\) is an elementary abelian 2-group. It is a quotient of a subgroup of \( M \) which, if \( k = \mathbb{F}_q \), is a product of \( n \) cyclic groups isomorphic to \( \mathbb{F}_q^* \), and the second claim follows. □

**Remark 5.7.** If we define \( L \) so that \( Z(L) = 1 \) (see Definition 3.3), we have \( K = \ker \mathcal{X}_2 \). In this case we can verify condition (D) combinatorially (see Example 6.3).

### 5.2. The amalgam of parabolics

For any subset \( J \subseteq I \), let \( \overline{J} = p^{-1}(J) \subseteq I(\Lambda) \), where \( p: \Lambda \to \Gamma \) is the projection map.

**Definition 5.8.** Note that \( G \) is the subgroup of \( L^θ \) generated by the \( \pi \)-images of the groups in \( \mathcal{L}^θ \). Let \( \mathcal{B} = \{B_J \mid J \subseteq I\} \) denote the amalgam of all parabolic subgroups of \( G \) with respect to its action on \( \Delta^θ \). That is, for any \( J \subseteq I \), we let \( B_J = \text{Stab}_G(R_{\overline{J}}) \), where \( R_{\overline{J}} \) is the \( \overline{J} \)-residue of \( \Delta_+ \) on \( c_+ \) such that \( R^θ_J \) opp \( R_{\overline{J}} \). In particular, \( B_θ = D ∩ G \). Also, for each \( J \subseteq I \), let \( G_J = \langle \pi(L^θ_j) \mid j \in J \rangle_G \) (see Definition 3.5).

**Lemma 5.9.** The stabilizer \( B_{\{i\}} \) in \( G \) of the \( \{i, θ(i)\} \)-residue on \( c_+ \) of \( \Delta_+ \) is transitive on its intersection with \( \Delta^θ \).
The stabilizer in $\Delta^\theta$ has type $A_1 \times A_1$. Therefore by Lemma 3.4 the stabilizer in $L$ of $R$ acts as $G_i \times G_{\theta(i)} \cong \text{SL}_2(k) \times \text{SL}_2(\mathcal{O})$ on $R$ and the stabilizer in $G$ of $R$ contains $\pi(L^\theta_i) = \{ \pi(gg^\theta) \mid g \in L_i \}$. Let $R_i, R_{\theta(i)}$ be the $i$ and $\theta(i)$-panels on $c^+$. Identify $R$ with $\pi(R_i(d)) \pi(R_{\theta(i)}(d))$. For any of these chambers $d$ we have $\delta_s(d, d^\theta) \in \{1, s_{\theta(i)} \}$.

Let $S = \{ d \in R \mid \delta_s(d, d^\theta) = s_{\theta(i)} \}$. Note that every panel in $R$ meets $S$ in exactly one chamber. Also $\pi(L^\theta_i)$ acts 2-transitively on $S$. Any chamber $c' \in R - S = R \cap \Delta^\theta$, corresponds to a unique ordered pair of distinct chambers of $S$, that is $(R_i \cap S, R_{\theta(i)} \cap S)$, where $R_i$ is the $i$-panel on $c'$ and $R_{\theta(i)}$ is the $\theta(i)$-panel on $c'$. By 2-transitivity $\pi(L^\theta_i)$ is transitive on such pairs, hence is transitive on $R \cap \Delta^\theta$.

**Corollary 5.10.** Suppose that $\Delta^\theta$ is connected. For any $J \subseteq I$, the group $B_J$ is flag-transitive on $R_J \cap \Delta^\theta$. In particular, $G$ itself is flag-transitive on $\Delta^\theta$, and as a consequence, so is $L^\theta$. Hence, for any $J \subseteq I$, we have $B_J = G_J : (D \cap G)$ and $L^\theta = G \cdot \text{Stab}_{L^\theta}(\{ c_+, c_- \}) = G \cdot (D \cap L^\theta)$ (product of groups).

**Proof** Lemma 5.9 combined with the connectedness of $\Delta^\theta$ in fact shows that already $G_J$ is transitive on the chambers of $R_J \cap \Delta^\theta$ and the conclusion follows.

Proof of Theorem E. By assumption of Part 2. of Theorem D we have $|k| \geq 7$ and so by Theorem 4.3, $\Delta^\theta$ is connected and simply 2-connected. That $G$ and hence $\tilde{G}$ is flag-transitive on $\Delta^\theta$ follows from Corollary 5.10.

**Proposition 5.11.** Suppose that the conclusion of Theorem 4.3 holds. For any $J \subseteq I$ with $|J| \geq 3$, $B_J$ is the universal completion of the amalgam $\mathcal{B}_J = \{ B_K \mid K \subseteq J \}$. In particular, $G$ is the universal completion of the amalgam $\mathcal{B}$.

**Proof** This follows from Tits’ Lemma [22, Corollaire 1] combined with Corollary 5.10.

**Proposition 5.12.** $G = [L^\theta, L^\theta]$ and $L^\theta / G$ is an elementary abelian 2-group. If $k = \mathbb{F}_q$, then it has rank at most $n = |I|$. 

**Proof** As $G$ is generated by perfect groups it is perfect itself, so $G \leq [L^\theta, L^\theta]$. By Corollary 5.10 and Lemma 5.3 $L^\theta = G(D \cap L^\theta)$ and the latter group normalizes $G$; namely $D$ normalizes $\pi(L_{i, \theta(i)})$ and since $L^\theta \cap \pi(L_{i, \theta(i)}) = \pi(L^\theta_i)$ we see that $D \cap L^\theta$ normalizes $\pi(L^\theta_i)$ for every $i \in I$. Therefore, $L^\theta / G$ is isomorphic to a quotient of $(D \cap L^\theta) / D_{\theta}$, which is abelian, so that $G = [L^\theta, L^\theta]$. The result follows from Lemma 5.6.

Proof of Theorem D.3. This follows from Proposition 5.12 and Theorem 2.31. For a concrete example see Subsection 6.3.

5.3. $\tilde{G}$ is a central extension of $L^\theta$

From now on we shall assume that $(M \cap K) = \nu(K)$. This happens frequently, as $K$ itself is generally already very small. By Lemma 5.5 and Corollary 5.10 we have $G = L^\theta$ and $B_{\theta} = D \cap G = D_{\theta}$. 

GOLDING-TITS GROUPS OF SIMPLY-LACED TYPE
This condition is necessary to ensure that the groups \( B_J \) are entirely obtainable from the amalgam \( \mathcal{L}^\theta \) via Corollary 5.10.

We also assume throughout this section that the conclusion of Theorem 4.3 holds, that is, \((\Delta^\theta, \simeq)\) is simply-connected. Recall that \((\hat{G}, \hat{\gamma})\) is the universal completion of \(\mathcal{G}\). We denote proper subgroups of \(\hat{G}\) and \(I^\theta\) by \(\hat{H}\) and \(H^\theta\) respectively, with the assumption that \(\pi(\hat{H}) = H^\theta\). In particular, for \(G_i \in \mathcal{G}\) we set \(\hat{\gamma}(G_i) = \hat{G}_i\).

**Definition 5.13.** Define the following amalgam in \(\hat{G}\). \(\hat{\mathcal{B}} = \{\hat{B}_J \mid J \subseteq I\}\), where \(\hat{B}_J = \langle \hat{G}_j, \hat{D}_i \mid j \in J, i \in I \rangle_{\hat{G}}\). The connecting maps of \(\hat{\mathcal{B}}\) are given by inclusion of subgroups in \(\hat{G}\). Recall that we have a surjective homomorphism \(\pi: \hat{G} \to G\) given by universality of \(\hat{G}\). For each \(J \subseteq I\), we also define

\[
\hat{K}_J = \ker \pi \cap \hat{B}_J \quad \text{and} \quad \hat{K} = \langle \{\hat{K}_J \mid J \subseteq I\} \rangle,
\]

that is, the smallest normal subgroup of \(\hat{G}\), containing all \(\hat{K}_J\).

**Lemma 5.14.** For any \(J \subseteq I\), of rank at least 3, \(\hat{K}_J\) is the normal closure in \(\hat{B}_J\) of all \(\hat{K}_{J'}\) with \(J' \subsetneq J\). In particular \(\hat{K} = \ker \pi\).

**Proof** By Corollary 5.10, since \(B_J = G_{J'}(D \cap G) = G_{J'}D\), the restriction \(\pi|_{\hat{B}_J}: \hat{B}_J \to B_J\) takes \(\hat{B}_{J'}\) onto \(B_{J'}\) for any \(J' \subsetneq J\). By Proposition 5.11, \(B_J\) is the universal completion of the amalgam \(\{B_{J'} \mid J' \subsetneq J\}\). Therefore by Lemma 2.3, \(\hat{K}_J\) is the normal closure in \(\hat{B}_J\) of all \(\hat{K}_{J'}\) with \(J' \subsetneq J\). In particular \(\hat{K} = \ker \pi\).

Recall that \(\hat{D} = \langle \gamma_i(D_i) \mid i \in I \rangle\). Our next aim is to show that \(\hat{K} \subseteq Z(\hat{G}) \cap \hat{D}\). To this end it suffices to show that \(\ker \pi \cap \hat{B}_J \subseteq Z(\hat{G}) \cap \hat{D}\). We start at rank 0.

**Lemma 5.15.** We have \(\hat{D} \cap \pi^{-1}(Z(G)) \subseteq Z(\hat{G}) \cap \hat{D}\). In particular, \(\hat{K}_\theta = \hat{D} \cap \ker \pi \subseteq Z(\hat{G}) \cap \hat{D}\).

**Proof** Note that \(\hat{D}\) normalises every \(\hat{G}_i\). Thus, if \(x \in \hat{D} \cap \pi^{-1}(Z(G))\), then for every \(g \in \hat{G}_i\), also \(g^x \in \hat{G}_i\) and \(\pi(g) = \pi(g^x)\). As \(\pi|_{\hat{G}_i}\) is injective we get \(g^x = g\), so \(x \in C_{\hat{G}}(\hat{G}_i)\). This holds for all \(i\) so \(x \in Z(\hat{G}) \cap \hat{D}\).

**Lemma 5.16.** For every \(J \subseteq I\) with \(|J| \leq 2\) we have \(\hat{K}_J \subseteq Z(\hat{G}) \cap \hat{D}\).

**Proof** Let \(x \in \hat{K}_J\), then \(x = gd^{-1}\) for some \(g \in \hat{G}_J\) and \(d \in \hat{D}\). We now have \(\pi(g) = \pi(d)\). Moreover, for any \(y \in \hat{G}_J\), we have \(\pi(y^g) = \pi(y^d)\), where \(y^g, y^d \in \hat{G}_J\), as the latter is normalized by both \(g\) and \(d\). By Lemma 2.9, if \(J\) is a vertex or an edge, then \(\pi|_{\hat{G}_J}\) is injective and so we have \(y^g = y^d\), so \(g \in \hat{G}_J\) acts as a diagonal, necessarily inner, automorphism of \(\hat{G}_J\). Therefore \(g \in \hat{D}_J\) so that \(x = \hat{D} \cap \ker \pi\), which is contained in \(Z(\hat{G})\) by Lemma 5.15. On the other hand, if \(J = \{i, j\}\) is a non-edge pair, then consider \(g = g_ig_j\) with \(g_i \in \hat{G}_i\), \(g_j \in \hat{G}_j\). For any \(y \in \hat{G}_i\), since \(\hat{G}_i\) is normal in \(\hat{B}_J\), \(y^d, y^g \in \hat{G}_i\), and since \(\pi|_{\hat{G}_i}\) is injective, we obtain \(y^d = y^g = y^g_1\). As before we conclude that \(g_i \in \hat{D}_i\). Likewise \(g_j \in \hat{D}_j\) so that \(g \in \hat{D}\). The conclusion that \(x \in Z(\hat{G}) \cap \hat{D}\) follows as above.
Corollary 5.17. For every $J \subseteq I$, we have $\bar{K}_J \leq Z(\tilde{G}) \cap \tilde{D}$. In particular, $\ker \pi \leq Z(\tilde{G}) \cap \tilde{D}$.

Proof. This follows from an inductive application of Lemma 5.14 combined with Lemma 5.16. □

Lemma 5.18. 
\[ Z(\tilde{G}) \cap \tilde{D} = \pi^{-1}(Z(G) \cap D). \]

Proof. The inclusion ‘$\subseteq$’ follows easily as $\pi(Z(\tilde{G})) \subseteq Z(G)$, since $\pi$ is onto, and $\pi(\tilde{D}) \subseteq D$ by definition. To see the converse inclusion, suppose that $d \in Z(G) \cap D$. Then, since $Z(G) \cap D = Z(G) \cap D_\theta$ by assumption, there exists $\tilde{d} \in \tilde{D}$ with $\pi(\tilde{d}) = d$. By Lemma 5.15 $\tilde{d} \in Z(\tilde{G}) \cap \tilde{D}$. By Corollary 5.17 $\pi^{-1}(d) = \tilde{d} \cdot \ker \pi \subseteq Z(\tilde{G}) \cap \tilde{D}$ as well. □

Lemma 5.19. $Z(G) \cap D = Z(L) \cap G$.

Proof. We first claim that $Z(G) \cap D \leq Z(L) \cap G$. Recall $D = \text{Stab}_L(c, e^\theta)$, where $(c, e^\theta) \in \Delta^\theta$. Let $g \in Z(G) \cap D$, then clearly $gc = c$ and, for any $h \in G$ we have $ghc = hgc = hc$. By Corollary 5.10, $g$ fixes all of $\Delta^\theta$.

We now claim that $g$ fixes all of $\Delta$. Let $\pi$ be any panel on $c$. Then, \[ \{d \in \pi \mid (d, d^\theta) \in \Delta^\theta\} = \pi - \{\text{proj}^*(e^\theta)\}. \] It follows that $g$ fixes all chambers of $\pi$ and hence $g$ fixes $E_1(c) \cup \{e^\theta\}$. By Theorem 1 of [24, §4.2], $g$ acts as the identity on $\Delta$. Therefore, by Lemma 3.4 of [5], $g \in Z(L)$, proving the claim.

Finally, note that we have inclusions 
\[ Z(L) \cap G = Z(L) \cap D \cap G \subseteq Z(G) \cap D \subseteq Z(L) \cap G. \]
The first equality follows from Theorem 2.31. As for the second equality follows from the inequality just obtained. □

Theorem 5.20. Suppose that $L$ is the adjoint Kac-Moody group of type $\Gamma$, that is, $Z(L) = 1$. Then, $\ker \pi = Z(\tilde{G}) \cap \tilde{D}$.

Proof. The inclusion $\subseteq$ is Corollary 5.17. To establish the converse inclusion, note that by Lemmas 5.18 and 5.19, we have $\pi(Z(G) \cap \tilde{D}) \subseteq Z(G) \cap D = Z(L) \cap D = 1$. □

Proof of Theorem D.4. The fact that $\ker \pi = Z(\tilde{G}) \cap \tilde{D}$ is Theorem D. The equality $\ker \pi = d^\theta(\ker \mathcal{K})$ is Theorem 2.29.

We finish this section with an observation showing that, although it isn’t a group of Kac-Moody type, the twisted group has the notion of a Weyl group. Recall $W = N/D$, where $N = \text{Stab}_L(\Sigma_\pm)$ and $D = \text{Stab}_L((c, e^\theta))$. Since $\Sigma_\pm = \Sigma(c, e^\theta)$, $\theta$ acts naturally on $W$, and we let 
\[ W^\theta = \{w \in W \mid w^\theta = w\}, \]
\[ S^\theta = \{ss^\theta \mid s \in S\}. \]

Lemma 5.21. (1) $\pi(W^\theta, S^\theta)$ is a Coxeter system whose diagram is the diagram induced on the $\theta$-orbits in $I(\Lambda)$. 
Define \( N^\theta = \text{Stab}_{L^\theta}(\Sigma_{\pm}) \). Then, we also have \( W^\theta = N^\theta D / D \cong N^\theta / (N^\theta \cap D) = N^\theta / D^\theta \).

**Proof** The orbits of \( \theta \) on \( I(\Lambda) \) are pairs \((i, i^\theta)\) with \( i \in I(\Lambda) \) such that \( s_i \) and \( s_i^\theta \) commute. Thus \( S^\theta \) is the set of longest words corresponding to the orbits of \( \theta \) on \( I(\Lambda) \). Therefore, by the main result of [11], \((W^\theta, S^\theta)\) is a Coxeter system.

Consider the map \( N \to W \) sending \( n \mapsto nD \). For each \( s \in S \) select \( n_s \in N \) so that \( n_s D \) represents \( s \). Then, \( \langle n_s n_s^\theta | s \in S \rangle \leq N^\theta \). By the first part of this lemma, this subgroup already maps surjectively to \( W^\theta \). Since \( N^\theta \) is fixed by \( \theta \) also the image is fixed by \( \theta \). Now we have \( N^\theta / (N^\theta \cap D) \cong W^\theta \). Note that \( N^\theta \cap D = D^\theta \). \( \square \)

6. Examples

In this section we use the methods developed in the preceding sections to compute \( Z(\tilde{G}) \cap \tilde{D} \) for some Curtis-Tits groups \( \tilde{G} \) and we verify condition (D) for another group.

6.1. The generalized Cartan matrix

In this subsection we consider the situation of Subsection 2.5.

**Definition 6.1.** Let \( E \) be the subring with identity \( 1_E \) of \( \text{End}(k^*) \) generated by all \( \{ \rho_{j,i} | i, j \in I, i \sim j \} \) with the natural operations given by

\[
(\alpha \oplus \beta)(x) = \alpha(x) \cdot \beta(x)
\]

\[
(\alpha \otimes \beta)(x) = (\alpha \circ \beta)(x)
\]

Here by \( 0_E \) we denote the map \( x \mapsto 1_k \) for all \( x \in k^* \) and \( 1_E = \text{id}_k \) and the \( \rho_{i,j} \) are those defined in Definition 2.27. Henceforth, we shall impose the restriction that \( \langle \rho_{j,i} | i \sim j, i, j \in I \rangle \leq \text{Aut}(k^*) \) is finite and commutative, so that in particular, the ring \( E \) is commutative. This happens for instance in the case where \( k \) is finite. The generalized Cartan matrix is the matrix \( \mathcal{K} = (\mathcal{K}_{i,j})_{i,j \in I} \in M_n(E) \leq \text{End}((k^*)^I) \) given by

\[
\mathcal{K}_{i,j} = \begin{cases} \text{id}^{-2} & \text{if } i = j \\ \rho_{j,i} & \text{if } j \sim i \\ 0 & \text{if } i \not\sim j. \end{cases}
\]

Theorem 2.29 now says that \( \ker \mathcal{K} \) consists of all elements in \((k^*)^I\) mapping to central elements in \( G \) under the map \( d \).

**Remark 6.2.** Note that \( \text{id}^{-2} \) is the additive inverse of \( \text{id} \oplus \text{id} \). In particular, if all \( \rho_{i,j} \) are the identity map, then the matrix representing \( \mathcal{K} \) is the negative of the Cartan matrix \((A_{i,j})_{i,j \in I}\), making the connection to the characterization of the center due to Steinberg concrete.

6.2. Computation of centers

We will compute the centers of the Curtis-Tits groups \( G \) with the diagrams shown below.
Example 1. Here we consider a group of Lie type with diagram $E_6$. By uniqueness of the amalgam we can assume that all $\rho_{i,j}$ are trivial.

In the case of $E_6$, we find that $d = \prod_{i=1}^{6} d_i(a_i) \in Z(G)$ if and only if the following conditions hold: Setting $a_1 = a$, $a_6 = b$, $a_2 = c$, we get $a_3 = a^2$, $a_5 = b^2$, $a_4 = a^3 = b^3 = c^2$, $a_4^2 = a^2 b^2 c$. It then follows that $a^3 = 1 = ab = c$, so the center is of order 3 if and only if $k$ has a primitive third root of 1 and trivial otherwise.

Example 2. In this case we assume that $\rho_{j,i} = 1$ for all $1 \leq i \neq j \leq 6$. This means that $\mathcal{G}$ is orientable and $\tilde{G}$ is a split group of Kac-Moody type. Assuming the field $k$ is finite, we can use a discrete logarithm to view the matrix $K$ as having coefficients in $\mathbb{Z}/(q-1)\mathbb{Z}$ and one computes that this matrix has determinant $-13$. Therefore if 13 is invertible modulo $q-1$, the center is trivial. More precisely, we find that $d = \prod_{i=1}^{6} d_i(a_i) \in Z(G)$ if and only if the following conditions hold: Setting $a_1 = a$, $a_2 = c$, we get $a_3 = a^2$, $a_4 = c^2$, $a_5 = c^2 a^{-2}$, $a_6 = a^3 c^{-2}$, $a_2^2 = c^2 a_6 = a^3$ and similarly $a_6^2 = c^3$. It follows that $c^6 = a^7$ and $a^6 = c^7$ so that $ac = 1 = a^{13}$. Therefore this $G$ has a non-trivial center of order 13 if and only if $k$ contains a primitive 13-th root of 1.

Example 3. Now suppose the amalgam has the same diagram as in Example 2, but is given by

$$\omega_{j,i} = \begin{cases} \tau & \text{if } (j, i) = (5, 4) \\ \text{id} & \text{otherwise} \end{cases}.$$ 

Then, $G$ is not orientable and $\rho_{5,4} = \rho_{4,5}: x \mapsto x^{-1}$. The matrix of the corresponding $\mathcal{K}$ is that of the previous example replacing $\mathcal{K}_{5,4} = \mathcal{K}_{4,5} = -1$. Now the determinant is 3 and a computation similar to the previous one shows that $G$ now has a non-trivial center of order 3 if and only if $k$ has a primitive 3-rd root of 1. More precisely, $(a_1, \ldots, a_6) = (a, a^2, a^2, a, a^2, a^2)$, where $a^3 = 1$. 

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) [circle, draw] {1};
\node (2) at (1,0) [circle, draw] {2};
\node (3) at (2,0) [circle, draw] {3};
\node (4) at (3,0) [circle, draw] {4};
\node (5) at (4,0) [circle, draw] {5};
\node (6) at (5,0) [circle, draw] {6};
\draw (1) -- (2);
\draw (2) -- (3);
\draw (3) -- (4);
\draw (4) -- (5);
\draw (5) -- (6);
\end{tikzpicture}
\end{center}
In case $k = \mathbb{F}_q$ is finite of characteristic $p$, it may happen that $\omega_{j,i} : x \mapsto x^{p^s}$ for some $s$. In this case $\rho_{j,i}$ and $\rho_{i,j}$ are represented additively by $\mathcal{H}_{j,i} = p^s$ and $\mathcal{H}_{i,j} = -p^s$. The determinant now should be computed modulo $q - 1$, and in principle the center of $G$ can be computed as above.

### 6.3. Verifying Condition (D)

We consider the non-orientable amalgam $\mathcal{G}$ of Example 3. The corresponding amalgam $\mathcal{L}$ has the diagram shown below. The diagram is numbered using $I(\Lambda) = \{i, i' \mid i \in \{1, 2, 3, 4, 5, 6\} = I\}$, where $i' = \theta(i)$. Note that $\mathcal{L}$ corresponds to the map $\omega : I(\Lambda) \to \text{Aut}(k) \times \langle \tau \rangle$ given by

$$
\omega_{j,i} = \begin{cases} 
\tau & \text{if } (j, i) \in \{(5, 4'), (5', 4)\} \\
\text{id} & \text{otherwise}
\end{cases}.
$$

A computation similar to the one in Example 3 reveals the following. Any element $a(\zeta) \in K = \ker \mathcal{H}$ is a sequence $(a_1, \ldots, a_6, a_1', \ldots, a_6') \in (k^*)^{I(\Lambda)}$, where $a_i$ is the label of the node $i$ in the diagram and $\zeta^{39} = 1$ in $k^*$. For example if we choose $a_1 = \zeta$, then $a_6 = \zeta^{11}$.

By definition $a(\zeta)$ belongs to $M \cap K$ if and only if $\theta(a(\zeta)) = a(\zeta)^{-1}$, that is $\zeta^{-1} = \zeta^{-14}$, which means that $\zeta^{13} = 1$ in $k^*$.

Now, note that if $\mu_n$ is the set of $n$-th roots of 1 in $k^*$, then the map $\mu_{39} \to \mu_{13}$ given by $x \mapsto x^{15}$ is surjective. Noting that $\nu(a(\xi)) = a(\xi)^{15}$ for any $\xi$ a 39-th root of 1, we conclude that $(M \cap K)/\nu(K) = 0$, that is, condition (D) is satisfied and $L^\theta = G$ in this case.

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