Higher dimensional charged $f(R)$ black holes

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We construct a new class of higher dimensional black hole solutions of $f(R)$ theory coupled to a nonlinear Maxwell field. In deriving these solutions the traceless property of the energy-momentum tensor of the matter field plays a crucial role. In $n$-dimensional spacetime the energy-momentum tensor of conformally invariant Maxwell field is traceless provided we take $n = 4p$, where $p$ is the power of conformally invariant Maxwell lagrangian. These black hole solutions are similar to higher dimensional Reissner-Nordstrom AdS black holes but only exist for dimensions which are multiples of four. We calculate the conserved and thermodynamic quantities of these black holes and check the validity of the first law of black hole thermodynamics by computing a Smarr-type formula for the total mass of the solutions. Finally, we study the local stability of the solutions and find that there is indeed a phase transition for higher dimensional $f(R)$ black holes with conformally invariant Maxwell source.

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I. INTRODUCTION

A large volume of observational evidences indicate that our universe is now experiencing a phase of accelerated expansion. There are two main approaches for explanation of this cosmic acceleration. The first approach which try to explain the problem in the framework of general relativity, requires the existence of a strange type of energy called “dark energy” whose gravity is repulsive and consist an un-clustered component through the universe. The second approach is to modify general relativity by adding higher powers of the scalar curvature $R$, the Riemann and Ricci tensors, or their derivatives in the lagrangian formulation. Among the latter attempts are Lovelock gravity, braneworld cosmology, scalar-tensor theories like Brans-Dicke one and also $f(R)$ theories. A fairly comprehensive review on dark energy models can be seen in [1]. $f(R)$ theories were proved to be able to mimic the whole cosmological history, from inflation to the actual accelerated expansion era [2, 3] (see also [4] for a recent review on $f(R)$ theories). Diverse applications of $f(R)$ theories on gravitation and cosmology have been also widely studied [4], as well as multiple ways to observationally and experimentally distinguish them from general relativity.

When one considers $f(R)$ theory as a modification of general relativity, it is quite natural to ask about black hole existence and its features in this theory. It is expected that some signatures of black holes in $f(R)$ theory may differ from the expected physical results in Einstein’s gravity. Therefore, the investigation on $f(R)$ black holes is of particular interest. Some attempts have been done to construct black hole solutions in $f(R)$ theories (see [5–8] and references therein). In general, in the presence of a matter field, the field equations of $f(R)$ gravity are complicated and it is not easy to find exact analytical solutions. However, if one considers the traceless energy-momentum tensor for the matter field, one can extract exact analytical solutions from $R + f(R)$ theory coupled to a matter field [9]. Since the energy-momentum tensor of Maxwell and Yang-Mills fields are traceless only in four dimensions, therefore black hole solutions from $R + f(R)$ theory coupled to the matter field were derived only in four dimensions [9]. The studies were also generalized to the four dimensional charged rotating black holes [10], charged rotating black string [11] and magnetic string solutions [12] in $R + f(R)$-Maxwell theory. However, since the standard Maxwell energy-momentum tensor is not traceless in higher dimensions, they failed to derive higher dimensional black hole/string solutions from $R + f(R)$ gravity coupled to standard Maxwell field. A natural question then arises: Is there an extension of Maxwell action in arbitrary dimensions that is traceless and hence possesses the conformal invariance? The answer is positive and the
conformally invariant Maxwell action was presented as \[ S_m = -\int d^n x \sqrt{-g} (F_{\mu\nu} F^{\mu\nu})^p, \] (1)

where \( p \) is a positive integer, i.e., \( p \in \mathbb{N} \). The associated energy-momentum tensor of the above conformally invariant Maxwell action is given by

\[ T_{\mu\nu} = 2 \left( p F_{\mu\eta} F_{\nu}^{\eta} F^{p-1} - \frac{1}{4} g_{\mu\nu} F^p \right), \] (2)

where \( F = F_{\alpha\beta} F^{\alpha\beta} \) is the Maxwell invariant. One can easily check that the above energy momentum tensor is traceless for \( n = 4p \). The theory of conformally invariant Maxwell field is considerably richer than that of the linear standard Maxwell field and in the special case \( (p = 1) \) it recovers the Maxwell action. It is worthwhile to investigate the effects of exponent \( p \) on the behavior of the solutions and the laws of black hole mechanics. The motivation is to take advantage of the conformal symmetry to construct the analogues of the four dimensional Reissner-Nordstrom (RN) solutions, in higher dimensions. Recently, the studies on the black object solutions with a nonlinear Maxwell source in Einstein \[ \text{[13–15]} \] and Gauss-Bonnet \[ \text{[16]} \] gravity have got a lot of attentions.

In this work we would like to extend the investigation on the conformally invariant Maxwell field to \( f(R) \) gravity. We will consider the action \[ \text{[11]} \] as the matter source of the field equations in \( R + f(R) \) theory with constant curvature scalar. Our purpose is to find the analogues of the four-dimensional charged black hole solutions of \( R + f(R) \)-Maxwell theory \[ \text{[9]} \] in higher dimensional spacetime. In contrast to the higher dimensional black holes of Einstein gravity with a conformally invariant Maxwell source presented in \[ \text{[13]} \] which has vanishing scalar curvature \( R = 0 \), the spacetime we construct here in \( R + f(R) \) gravity coupled to a nonlinear Maxwell field has a constant scalar curvature \( R = R_0 \). Our solutions also differ from higher dimensional RN solutions in that the electric charge term in the metric coefficient goes as \( r^{-(n-2)} \) while in the standard RN case is \( r^{-2(n-3)} \). Also, the electric field in higher dimensions does not depend on \( n \) and goes as electric field in four dimensions.

This paper is outlined as follows. In Sec. \[ \text{II} \] we construct exact spherically symmetric black hole solutions of \( R + f(R) \) theory coupled to a nonlinear Maxwell field in \( n = 4p \) dimensions and investigate their properties. In Sec. \[ \text{III} \] we obtain the conserved and thermodynamic quantities of the solutions and verify the validity of the first law of black hole thermodynamics. We also study local stability of the solutions in this section. We summarize our results in Sec. \[ \text{IV} \]
II. FIELD EQUATIONS AND SOLUTIONS

We consider the action of $R + f(R)$ gravity in $n$-dimensional spacetime coupled to a conformally invariant Maxwell field

$$S = \int_{\mathcal{M}} d^n x \sqrt{-g} [R + f(R) - (F_{\mu\nu} F^{\mu\nu})^p],$$  \hspace{1cm} (3)

where $R$ is the scalar curvature, $f(R)$ is an arbitrary function of scalar curvature, $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ is the electromagnetic field tensor and $A_\mu$ is the electromagnetic potential. The equations of motion can be obtained by varying action (3) with respect to the gravitational field $g_{\mu\nu}$ and the gauge field $A_\mu$,

$$R_{\mu\nu} (1 + f'(R)) - \frac{1}{2} g_{\mu\nu} (R + f(R)) + (g_{\mu\nu} \nabla^2 - \nabla_\mu \nabla_\nu) f'(R) = T_{\mu\nu},$$ \hspace{1cm} (4)

$$\partial_\mu (\sqrt{-g} F^{\mu\nu} F^{\nu-1}) = 0,$$ \hspace{1cm} (5)

where $F = F_{\alpha\beta} F^{\alpha\beta}$ is the Maxwell invariant and the “prime” denotes differentiation with respect to $R$. In order to obtain the constant curvature black hole solution in $f(R)$ gravity theory coupled to a matter field, the trace of stress-energy tensor $T_{\mu\nu}$ should be zero \[9\]. Hence, two candidates for the matter field in four dimensions are Maxwell and Yang-Mills fields. Since the assumption of traceless energy-momentum tensor is essential for deriving exact black hole solutions in $f(R)$ gravity coupled to the matter field, therefore the solutions exist only for $n = 4p$ dimensions.

Assuming the constant scalar curvature $R = R_0 = \text{const.}$, then the trace of Eq. (4) yields

$$R_0 \left(1 + f'(R_0)\right) - \frac{n}{2} (R_0 + f(R_0)) = 0.$$ \hspace{1cm} (6)

Solving the above equation for $R_0$, gives

$$R_0 = \frac{nf(R_0)}{2f'(R_0) + 2 - n} = \frac{2n}{n - 2} \Lambda_f < 0.$$ \hspace{1cm} (7)

Substituting the above relation into Eq. (4), we obtain the following equation for Ricci tensor

$$R_{\mu\nu} \left(1 + f'(R_0)\right) - \frac{g_{\mu\nu} R_0}{n} \left(1 + f'(R_0)\right) = T_{\mu\nu}.$$ \hspace{1cm} (8)

We are looking for the $n$-dimensional static spherically symmetric solutions. Motivated by the metric of higher dimensional charged black holes in Einstein gravity, we assume the metric has the following form

$$ds^2 = -N(r) dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega_{n-2}^2,$$ \hspace{1cm} (9)
where $d\Omega^2_{n-2}$ denotes the metric of an unit $(n-2)$-sphere and $N(r)$ is a functions of $r$ which should be determined. We are seeking for a purely radial electric solution which means that the only non-vanishing component of the Maxwell tensor is $F_{tr}$. In this case, the Maxwell equations (5) can be integrated immediately, where, for the spherically symmetric spacetime (9), all the components of $F_{\mu\nu}$ are zero except $F_{tr}$:

$$F_{tr} = \frac{q}{r^{\frac{n-2}{2}}},$$

where $q$ is an integration constant. Substituting $n = 4p$ in the above relation, the Maxwell field becomes

$$F_{tr} = \frac{q}{r^2}.$$  

(11)

It is important to note that the electric field in higher dimensions does not depend on $n$ and its value coincides with the RN solution in four dimensions. Using metric (9) and the Maxwell field (11), one can show that Eq. (8) has a solution of the form

$$N(r) = 1 - \frac{2m}{r^{n-3}} + \frac{q^2}{r^{n-2}} \times \frac{(-2q^2)^{(n-4)/4}}{(1 + f'(R_0))} - \frac{R_0}{n(n-1)}r^2,$$

(12)

where $m$ is an integration constant which is related to the mass of the solution. In four dimension $(n = 4)$ the solution recovers the result of [9]. In order to have a real solution we should restrict ourself to the dimensions which are multiples of four, i.e., $n = 4, 8, 12, \ldots$, which means that $p$ should be only positive integer, as we mentioned already.

Next we study the physical properties of the solutions. To do this, we first look for the curvature singularities. A simple calculation shows that the Kretschmann scalar $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ diverges at $r = 0$, it is finite for $r \neq 0$ and is proportional to $R_0^2$ as $r \to \infty$. Therefore, there is a curvature singularity located at $r = 0$. As one can see from Eq. (12), the solution is ill-defined for $f'(R_0) = -1$. Let us consider the cases with $f'(R_0) > -1$ and $f'(R_0) < -1$ separately. In the first case where $f'(R_0) > -1$, we have a black hole solution (see Fig. 1). Indeed, for $1 + f'(R_0) > 0$ and $R_0 < 0$ black hole can have two inner and outer horizons, an extreme black hole or a naked singularity provided the parameters of the solutions are chosen suitably (see Fig. 2). In the latter case where $f'(R_0) < -1$, we encounter a cosmological horizon for $R_0 < 0$. Indeed, in this case the signature of the spacetime changes and the conserved quantities such as mass become negative, as we will see in the next section, thus this is not a physical case and we rule it out from our consideration.

It is apparent that the spacetime described by solution (12) is asymptotically AdS provided we define $R_0 = -n(n-1)/l^2$. However, the solution presented here differ from the standard
FIG. 1: The function $N(r)$ versus $r$ for $m = 2, n = 4, q = 1$ and $R_0 = -12$. $f'(R_0) = -0.5$ (bold line) and $f'(R_0) = -2$ (dashed line).

FIG. 2: The function $N(r)$ versus $r$ for $m = 2, q = 1, n = 4$ and $R_0 = -12$. $f'(R_0) = -0.2$ (bold line), $f'(R_0) = -0.54$ (continuous line) and $f'(R_0) = -0.8$ (dashed line).

higher dimensional Reissner-Nordstrom AdS (RNAdS) solutions since the electric charge term in the metric coefficient goes as $r^{-(n-2)}$ while in the standard RNAdS case is $r^{-2(n-3)}$. In four dimensions, with the following replacement

$$\frac{\alpha q^2}{(1 + f'(R_0))} \rightarrow Q^2$$

$$R_0 \rightarrow 4\Lambda$$

the solution reduces to standard RNAdS black holes for $\Lambda = -3/l^2$.

We use the subtraction method of Brown and York (BY) [17] to calculate the quasilocal mass of the charged $f(R)$ black hole. Such a procedure causes the resulting physical quantities to depend on the choice of reference background. In order to use the BY method one should write the metric
in the following form

\[ ds^2 = -W(r)dt^2 + \frac{dr^2}{V(r)} + r^2d\Omega_{n-2}^2. \]  

(15)

Since metric (11) has the above form, it is sufficient to choose the background metric to be the metric (15) with

\[ W_0(r) = V_0(r) = N_0(r) = 1 + \frac{r^2}{l^2} \]

(16)

Where we have defined \( R_0 = -n(n-1)/l^2 \) which show that the solutions are asymptotically AdS as we mentioned. It is well-known that the Ricci scalar for AdS spacetime should have this value (see e.g. [18]). To compute the conserved mass of the spacetime, we choose a timelike Killing vector field \( \xi \) on the boundary surface \( \mathcal{B} \) of the spacetime (15). Then the quasilocal conserved mass can be written as

\[ \mathcal{M} = \frac{1}{8\pi} \int_{\mathcal{B}} d^{n-2}x \sqrt{\sigma} \left\{ (K_{ab} - K h_{ab}) - \left( K^0_{ab} - K^0 h^0_{ab} \right) \right\} n^a \xi^b, \]

(17)

where \( \sigma \) is the determinant of the metric of the boundary \( \mathcal{B} \), \( K^0_{ab} \) is the extrinsic curvature of the background metric and \( n^a \) is the timelike unit normal vector to the boundary \( \mathcal{B} \). In the context of counterterm method, the limit in which the boundary \( \mathcal{B} \) becomes infinite \( (\mathcal{B}_\infty) \) is taken, and the counterterm prescription ensures that the action and conserved charges are finite. Thus, we obtain the mass through the use of the above subtraction method of BY as

\[ M = \frac{(n-2)\Omega_{n-2}}{8\pi} m [1 + f'(R_0)], \]

(18)

where \( \Omega_{n-2} \) is the volume of the unit \( (n-2) \)-sphere. In the limiting case \( (f'(R_0) = 0) \), this expression for the mass reduces to the mass of the \( n \)-dimensional AdS black hole.

### III. THERMODYNAMICS OF CHARGED \( f(R) \) BLACK HOLES

In this section we are going to explore thermodynamics of higher dimensional charged \( f(R) \) black holes. The Hawking temperature of the black holes can be easily obtained by requiring the absence of conical singularity at the horizon in the Euclidean sector of the black hole solutions. One obtains the associated temperature with the outer event horizon \( r = r_+ \) as

\[ T = \frac{1}{4\pi} \left( \frac{dN(r)}{dr} \right)_{r=r_+} = \frac{[1 + f'(R_0)] (2r^2(n-1) + 2l^2(n-3)) + (-2q^2)^{n/4}r^{2-n}l^2}{\pi l^2 [1 + f'(R_0)]}. \]

(19)

where we have used equation \( N(r_+) = 0 \) for omitting the mass parameter \( m \) from temperature expression. Next, we calculate the entropy of the black hole. Let us first give a brief discussion
regarding the entropy of the black hole in $f(R)$ gravity. To this aim, we follow the arguments presented in [19]. If one use the Noether charge method for evaluating the entropy associated with black hole solutions in $f(R)$ theory with constant curvature, one finds

$$S = \frac{A}{4G} f'(R_0),$$

(20)

where $A = 4\pi r_+^2$ is the horizon area. As a result, in $f(R)$ gravity, the entropy does not obey the area law and one obtains a modification of the area law. Inspired by the above argument, for the $n$-dimensional charged black hole solutions in $R + f(R)$ gravity, we find the entropy as

$$S = \frac{r_+^{n-2} \Omega_{n-2}}{4} [1 + f'(R_0)].$$

(21)

The charge of conformally invariant $f(R)$ black holes can be found by calculating the flux of the electric field at infinity, yielding

$$Q = \frac{n(-2)^{(n-4)/4} \Omega_{n-2}^{(n-2)/2}}{16\pi \sqrt{1 + f'(R_0)}}.$$  

(22)

The electric potential $\Phi$, measured at infinity with respect to the horizon, is defined by

$$\Phi = A_\mu \chi^\mu |_{r\to\infty} - A_\mu \chi^\mu |_{r=r_+},$$

(23)

where $\chi = \partial_t$ is the null generator of the horizon. We find

$$\Phi = \frac{q}{r_+} \sqrt{1 + f'(R_0)}.$$  

(24)

Then, we investigate the validity of the first law of thermodynamics for higher dimensional charged $f(R)$ black hole. For this purpose, we obtain a Smarr-type formula, namely the mass $M$ as a function of extensive quantities $S$, and $Q$. Using the expression for the mass, the entropy and the charge given in Eqs. (18), (21) and (22) and the fact that $N(r_+) = 0$, we find

$$M(S,Q) = \frac{n-2}{16\pi l^2} \left( \frac{4S}{1 + f'(R_0)} \right)^{-1/(n-2)} \left[ [1 + f'(R_0)] \left( \frac{4S}{1 + f'(R_0)} \right) \times \left( l^2 + \left( \frac{4S}{1 + f'(R_0)} \right)^{2/(n-2)} \right) + l^2 q^2 (-2q^2)^{(n-4)/4} \right].$$

(25)

where $q$ is a function of $Q$ according to Eq. (22). One may then regard the parameters $S$, and $Q$ as a complete set of extensive parameters for the mass $M(S,Q)$ and define the intensive parameters conjugate to $S$ and $Q$. These quantities are the temperature and the electric potential

$$T = \left( \frac{\partial M}{\partial S} \right)_Q, \quad \Phi = \left( \frac{\partial M}{\partial Q} \right)_S.$$  

(26)
Numerical calculations show that the intensive quantities calculated by Eq. (26) coincide with Eqs. (19) and (24). Thus, the thermodynamics quantities we obtained in this section satisfy the first law of black hole thermodynamics

\[ dM = TdS + \Phi dQ. \]

In this way we constructed all conserved and thermodynamic quantities of charged \( f(R) \) black holes and verified the validity of the first law of thermodynamics on the event horizon.

Finally, we study the local stability of the charged \( R + f(R) \) black holes in the presence of conformally invariant Maxwell source. In the canonical ensemble, the positivity of the heat capacity \( C_Q = T/(\partial^2 M/\partial S^2)_Q \) and therefore the positivity of \( (\partial^2 M/\partial S^2)_Q \) is sufficient to ensure the local stability. We have shown the behavior of \( (\partial^2 M/\partial S^2)_Q \) as a function \( q \) and \( r_+ \) for different value of \( n, q \) and \( f'(R_0) \) in figures 3-7 where we have taken \( R_0 = -n(n-1)/l^2 \). From figures 3 and 4 we see that the system is always thermally stable in four dimensions for different value of the parameters. However, in higher dimensions the system has an unstable phase as one can see from Fig. 5. As an example, we find that in 8-dimensions black holes are always unstable and there is no phase transition (see Fig. 6), while in 12-dimensions the system has a transition from unstable phase to stable phase (see Fig. 7).

IV. SUMMARY AND DISCUSSION

In general the field equations of \( f(R) \) gravity coupled to a matter field are complicated and it is not easy to construct exact analytical solutions. Recently, it was shown \[9\] that by assuming

![Figure 3](image-url): The function \( (\partial^2 M/\partial S^2)_Q \) versus \( q \) for \( l = 1, n = 4, r_+ = 0.8. \) \( f'(R_0) = -0.5 \) (bold line), \( f'(R_0) = 0 \) (continuous line) and \( f'(R_0) = 0.5 \) (dashed line).
FIG. 4: The function \( (\partial^2 M/\partial S^2)_Q \) versus \( r_+ \) for \( l = 1, \; f'(R_0) = 1 \) and \( n = 4 \). \( q = 0.4 \) (bold line), \( q = 1 \) (continuous line) and \( q = 2 \) (dashed line).

FIG. 5: The function \( (\partial^2 M/\partial S^2)_Q \) versus \( q \) for \( l = 1, \; r_+ = 0.8 \) and \( f'(R_0) = 1 \). \( n = 4 \) (bold line), \( n = 8 \) (continuous line) and \( n = 12 \) (dashed line).

FIG. 6: The function \( (\partial^2 M/\partial S^2)_Q \) versus \( q \) for \( l = 1, \; n = 8 \) and \( r_+ = 0.8 \). \( f'(R_0) = -0.5 \) (bold line), \( f'(R_0) = 0 \) (continuous line) and \( f'(R_0) = 0.5 \) (dashed line).
a traceless energy-momentum tensor for the matter field as well as the constant curvature scalar, one can extract some analytical black hole solutions in $R + f(R)$ theory coupled to a matter field. Two examples for the traceless $T_{\mu\nu}$ in four dimensions are Maxwell and Yang-Mills fields [9]. However, the energy-momentum tensor of Maxwell field is not traceless in higher dimensions. Seeking for a traceless energy-momentum tensor in arbitrary dimensions, the authors of [13] found a conformally invariant nonlinear Maxwell action which its energy-momentum tensor is traceless in $n = 4p$ dimensions where $p$ is a positive integer. They also studied the black hole solutions in Einstein gravity with nonlinear Maxwell field [13].

In this paper we obtained a class of higher dimensional black holes from $R + f(R)$ gravity with conformally invariant Maxwell source. The two key assumptions in finding these solutions are: (i) the constant scalar curvature $R = R_0$ and (ii) the traceless energy momentum tensor. These solutions are similar to higher dimensional RNAdS black holes with appropriate replacement of the parameters, but only exist in dimensions which are multiples of four. Besides, the solutions presented here differ from higher dimensional RNAdS black holes in two features. First, the electric charge term in the metric coefficient goes as $r^{-(n-2)}$ while in the standard RNAdS case is $r^{-2(n-3)}$. Second, the electric field in higher dimensions does not depend on $n$ and goes as electric field in four dimensional RNAdS black holes. Our solutions also differ from the higher dimensional black holes of Einstein gravity with a conformally invariant Maxwell source [13] in that they have vanishing scalar curvature $R = 0$, while the obtained solutions here in $R + f(R)$ gravity coupled to a nonlinear Maxwell field have a constant curvature scalar $R = R_0$. In addition, the conserved and thermodynamic quantities computed here depend on function $f'(R_0)$ and differ completely from those of Einstein theory in AdS spaces. Clearly the presence of the general function $f'(R_0)$
changes the physical values of conserved and thermodynamic quantities. Furthermore, unlike Einstein gravity, for the black hole solutions obtained here in $f(R)$ gravity, the entropy does not obey the area law.

After studying the physical properties of the solutions, we computed the mass, charge, electric potential and temperature of the black holes. We also found the entropy expression which does not obey the area law for the $f(R)$ black holes. We obtained a Smarr-type formula for the mass, and verified that the conserved and thermodynamics quantities satisfy the first law of black hole thermodynamics. We also studied the phase behavior of the higher dimensional charged $f(R)$ black holes. We found that the system is always thermally stable in four dimensions, while in higher dimensions, there is a phase transition in the presence of the conformally invariant Maxwell field in $R + f(R)$ theory with constant curvature.

Finally, we would like to mention that the $n$-dimensional charged $f(R)$ black hole solutions obtained here are static. Thus, it would be interesting if one can construct charged rotating black brane/hole solutions from $R + f(R)$ theory in the presence of conformally invariant Maxwell source. These issues are now under investigation and will be appeared elsewhere.

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