A NEW KIND OF SLANT HELIX IN PSEUDO-RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we define a new kind of slant helix called $f$-eikonal $V_n$-slant helix in Pseudo-Riemannian manifolds and give the definition of harmonic curvature functions related to the $f$-eikonal $V_n$-slant helix in Pseudo-Riemannian manifolds. Moreover, we give some characterizations of $f$-eikonal $V_n$-slant helix by making use of the harmonic curvature functions.

1. Introduction

Curves theory is an important framework in the differential geometry studies. Helix is one of the most fascinating curves because we see helical structure in nature, science and mechanical tools. Helices arise in the field of computer aided design, computer graphics, the simulation of kinematic motion or design of highways, the shape of DNA and carbon nanotubes. Also, we can see the helical structure in fractal geometry, for instance hyperhelices (9, 15, 20).

Izumiya and Takeuchi defined a new kind of helix (slant helix) and they gave a characterization of inclined curves and slant helices in $E^3$ (8). For example, the Riemannian condition (5) is very important subject.

A curve of constant slope or general helix in Euclidean 3-space $E^3$, is defined by the property that its tangent vector field makes a constant angle with a fixed straight line (the axis of general helix). A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (11 and 17) is: A curve $\alpha$ of constant slope or general helix in $E^3$ makes its tangent vector field makes a constant angle with a fixed straight line (the axis of general helix). A curve is an inclined curve if and only if $\sum_{i=1}^{n} H_i^2 = \text{constant}$ (1.1)

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In 2000, Önder et al. defined a new kind of slant helix in Euclidean 4-space $E^4$ which is called $B_2$-slant helix and they gave some characterizations of these slant helices in Euclidean 4-space $E^4$ (12). And then in 2009, Gök et al. generalized $B_2$-slant helix in $E^4$ to $E^n$, $n > 3$, called $V_n$-slant helix in Euclidean and Minkowski $n$-space (6, 7). Lots of authors in their papers have investigated inclined curves and slant helices using the harmonic curvature functions in Euclidean and Minkowski $n$-space (11, 12, 14, 15). But, Şenol et al. (18) see for the first time that the characterization of inclined curves and slant helices in (1.1) is true only for the case necessity but not true for the case sufficiency in Euclian $n$-space. Then, they consider the pre-characterizations about inclined curves and slant helices and restructure them with the necessary and sufficient condition.

Let $M$ be a Riemannian manifold, where $\langle \cdot, \cdot \rangle$ is the metric. Let $f : M \to \mathbb{R}$ be a function and let $\nabla f$ be its gradient, i.e., $df(X) = \langle \nabla f, X \rangle$. We say that $f$ is eikonal if it satisfies: $\|\nabla f\|$ is constant (4). $\nabla f$ is used many areas of science such as mathematical physics and geometry. So, $\nabla f$ is very important subject. For example, the Riemannian condition $\|\nabla f\|^2 = \text{constant}$ (for non-constant $f$ on connected $M$) is precisely the eikonal equation of geometrical optics. Thus on a connected $M$, a non-constant real valued $f$ is Riemannian if $f$ satisfies this eikonal equation.

In the geometrical optical interpretation, the level sets of $f$ are interpreted as light rays orthogonal to the wave fronts. These geodesics can be interpreted as light rays orthogonal to the wave fronts.

In this paper, we define $f$-eikonal $V_n$-slant helix in $n$-dimensional pseudo-Riemannian manifolds and give the definition of harmonic curvature functions related to $f$-eikonal $V_n$-slant helix in $n$-dimensional pseudo-Riemannian manifolds.
pseudo-Riemannian manifolds. Moreover, we give some characterizations of $f$-eikonal $V_n$-slant helix by making use of the harmonic curvature functions.

2. Preliminaries

In this section, we give some basic definitions from differential geometry.

**Definition 2.1.** A metric tensor $g$ on a smooth manifold $M$ is a symmetric non-degenerate $(0,2)$ tensor field on $M$. In other words, $g(X,Y) = g(Y,X)$ for all $X,Y \in \mathcal{T}M$ (tangent bundle) and at the each point $p \in M$ if $g(X_p, Y_p) = 0$ for all $Y_p \in T_p(M)$, then $X_p = 0$ (non-degenerate), where $T_p(M)$ is the tangent space of $M$ at the point $p$ and $g : T_p(M) \times T_p(M) \to \mathbb{R}$.

**Definition 2.2.** A pseudo-Riemannian manifold (or semi-Riemannian manifold) is a smooth manifold $M$ with a metric tensor $g$. That is, a pseudo-Riemannian manifold is an ordered pair $(M, g)$. We shall recall the notion of a proper curve of order $n$ in an $n$-dimensional pseudo-Riemannian manifold $M$ with the metric tensor $g$. Let $\alpha : I \to M$ be a non-null curve in $M$ parametrized by the arclength $s$, where $I$ is an open interval of the real line $\mathbb{R}$. We denote the tangent vector field of $\alpha$ by $V_1$. We assume that $\alpha$ satisfies the following Frenet formula:

$$
\nabla V_1 V_1 = k_1 V_2, \\
\nabla V_i V_i = -\varepsilon_{i-1} k_{i-1} V_{i-1} + k_i V_{i+1}, \quad 1 < i < n \\
\nabla V_n V_n = -\varepsilon_n k_{n-1} V_{n-1},
$$

where

$$
k_1 = \|\nabla V_1 V_1\| > 0, \quad k_i = \|\nabla V_i V_i + \varepsilon_{i-1} k_{i-1} V_{i-1}\| > 0, \quad 2 \leq i \leq n-1, \quad \varepsilon_{i-1} = g(V_j, V_j) \quad (= \pm 1), \quad 1 \leq j \leq n, \text{on } I,
$$

and $\nabla$ is Levi-Civita connection of $M$.

We call such a curve a proper curve of order $n$, $k_i$ $(1 \leq i \leq n-1)$ its $i$-th curvature and $V_1, ..., V_n$ its Frenet Frame field.

Moreover, let us recall that $\|X\| = \sqrt{g(X,X)}$ for $X \in \mathcal{T}M$, where $\mathcal{T}M$ is the tangent bundle of $M$.

3. $f$-eikonal $V_n$-slant helix curves in pseudo-Riemannian manifolds

In this section, we define $f$-eikonal $V_n$-slant helix curves and we give characterizations for a $f$-eikonal $V_n$-slant helix curve in $n$-dimensional pseudo-Riemannian manifold $M^n$ by using harmonic curvature functions in terms of $V_n$ of the curve.

**Definition 3.1.** Let $M$ be an $n$-dimensional pseudo-Riemannian manifold and let $\alpha(s)$ be a proper curve of order $n$ (non-null) with the curvatures $k_i$ $(i = 1, ..., n-1)$ in $M$. Then, harmonic curvature functions of $\alpha$ are defined by

$$
H_i^* : I \subset \mathbb{R} \to \mathbb{R}
$$

along $\alpha$ in $M$, where

$$
H_0^* = 0, \\
H_1^* = \varepsilon_{n-3} k_{n-2}^{-1} k_{n-1}^{-1}, \\
H_i^* = (k_{n-1} H_{i-2}^* - \nabla V_i H_{i-1}^*) \frac{\varepsilon_{n-(i+2)} \varepsilon_{n-(i+1)} k_{n-(i+1)}}{k_{n-(i+1)}}, \quad 2 \leq i \leq n-2.
$$

Note that $\nabla V_i H_{i-1}^* = V_i (H_{i-1}^*) = H_{i-1}^*.$

**Definition 3.2.** Let $(M, g)$ be an $n$-dimensional pseudo-Riemannian manifold. Let $f \in \mathcal{F}(M)$ and $\nabla f$ be its gradient, i.e.

$$
g(\nabla f, X) = df(X) = X(f)
$$
for all \( X \in TM \), where \( f \) is the set of all smooth real-valued functions on \( M \). Then, we say that \( f \) is eikonal function if \( f \) satisfies the eikonal equation

\[
g(\nabla f, \nabla f) = \text{constant}.
\]

**Lemma 3.1.** The Hessian \( H^f \) of \( f \in f \) is the symmetric \((0,2)\) tensor field such that

\[
H^f(X,Y) = g(\nabla_X (\text{grad} f), Y),
\]

where \((M,g)\) is a pseudo-Riemannian manifold and \( \nabla \) is Levi-Civita connection of \( M \).

The above Lemma has the following corollary.

**Corollary 3.1.** \( H^f = 0 \) if \( \nabla f = \text{grad} f \) is parallel in \( M \).

**Proof.** We assume that \( H^f = 0 \). Since \( g \) is non-degenerate metric, \( \nabla_X (\text{grad} f) = 0 \) for all \( X,Y \in TM \).

In other words, \( \nabla f \) is parallel in \( M \).

Conversely, if \( \nabla f \) is parallel in \( M \), then \( \nabla_X (\text{grad} f) = 0 \) for all \( X \in TM \). Hence, \( H^f = 0 \). This completes the proof. \( \square \)

**Definition 3.3.** Let \((M,g)\) be a \( n \)-dimensional pseudo-Riemannian manifold and let \( \alpha(s) \) be a proper curve of order \( n \) (non-null) in \( M \). Let \( f \in f \) be a eikonal function along curve \( \alpha \), i.e. \( g(\nabla f, \nabla f) \) is constant along curve \( \alpha \). If the function

\[
g(\nabla f, \nabla f) = \text{constant}.
\]

is non-zero constant function along \( \alpha \), then \( \alpha \) is called a \( e \)-eikonal \( V_n \)-slant helix curve, where \( V_n \) is \( n \)-th Frenet Frame field. And, \( \nabla f \) is called the axis of the \( e \)-eikonal \( V_n \)-slant helix curve \( \alpha \).

**Theorem 3.1.** Let \((M,g)\) be a \( n \)-dimensional pseudo-Riemannian manifold and let \( \alpha(s) \) be a proper curve of order \( n \) (non-null) in \( M \). Let us assume that \( f \in f \) be a eikonal function along curve \( \alpha \), i.e. \( g(\nabla f, \nabla f) = \text{constant} \) along curve \( \alpha \) and the Hessian \( H^f = 0 \). If \( \alpha \) is a \( e \)-eikonal \( V_n \)-slant helix curve with the axis \( \nabla f \), then the system

\[
g(V_{n-(i+1)}, \nabla f) = H^*_i g(V_n, \nabla f), \quad i = 1, \ldots, n-2
\]

holds, where \( \{V_1, \ldots, V_n\} \) and \( \{H^*_1, \ldots, H^*_n\} \) are the Frenet frame and the harmonic curvature functions of \( \alpha \), respectively.

**Proof.** Since \( \{V_1, \ldots, V_n\} \) is the orthonormal frame of the curve \( \alpha \) in \( M \), \( \nabla f \) can be expressed in the form

\[
\nabla f = \lambda_1 V_1 + \cdots + \lambda_n V_n.
\]

By using the definition of \( e \)-eikonal \( V_n \)-slant helix curve and (3.2), we get

\[
g(V_n, \nabla f) = \lambda_n \varepsilon_{n-1} = \text{constant}.
\]

If we take the derivative in each part of (3.3) in the direction \( V_1 \) in \( M \), then we have

\[
g(\nabla V_1 \nabla f, V_n) + g(\nabla f, \nabla V_1 V_n) = 0.
\]

On the other hand, from Corollary 3.1, \( \nabla f \) is parallel in \( M \). That is, \( \nabla_X \nabla f = 0 \) for all \( X \in TM \). So, \( \nabla V_1 \nabla f = 0 \) for \( X = V_1 \). Hence, by using (3.4) and Frenet formulas, we obtain

\[
-\varepsilon_{n-2} \varepsilon_{n-1} k_{n-1} (\nabla f, V_{n-1}) = 0.
\]

And, since \( \varepsilon_{n-2} \varepsilon_{n-1} = 0 \) is different from zero, from (3.5), we get

\[
g(\nabla f, V_{n-1}) = 0.
\]

By taking the derivative in each part of (3.6) in the direction \( V_1 \) in \( M \), we can write the equality

\[
g(\nabla V_1 \nabla f, V_{n-1}) + g(\nabla f, \nabla V_1 V_{n-1}) = 0.
\]

And, since \( \nabla V_1 \nabla f = 0 \), by using (3.7) and Frenet formulas, we obtain

\[
-\varepsilon_{n-3} \varepsilon_{n-2} k_{n-2} g(\nabla f, V_{n-2}) + k_{n-1} g(\nabla f, V_n) = 0.
\]

Therefore, from (3.8), we have

\[
\begin{align*}
g(\nabla f, V_{n-2}) &= \frac{k_{n-1}}{\varepsilon_{n-3} \varepsilon_{n-2} k_{n-2}} g(\nabla f, V_n) \\
g(\nabla f, V_{n-2}) &= \frac{\varepsilon_{n-3} \varepsilon_{n-2}}{(\varepsilon_{n-3})^2 (\varepsilon_{n-2})^2} \frac{k_{n-1}}{k_{n-2}} g(\nabla f, V_n) \\
g(\nabla f, V_{n-2}) &= \frac{k_{n-1}}{k_{n-2}} g(\nabla f, V_n).
\end{align*}
\]
Moreover, since $H_i^* = \varepsilon_{n-3}\varepsilon_{n-2}\frac{k_{n-1}}{k_{n-2}}$, from (3.9), we can write
\[ g(\nabla f, V_{n-2}) = H_i^* g(\nabla f, V_n). \]
It follows that the equality (3.1) is true for $i = 1$. According to the induction theory, let us assume that the equality (3.1) is true for all $k$, where $1 \leq k \leq i$ for some positive integers $i$. Then, we will prove that the equality (3.1) is true for $i + 1$. Since the equality (3.1) is true for some positive integers $i$, we can write
\[ g(V_{n-(i+1)}, \nabla f) = H_i^* g(\nabla f, V_n) \quad (3.10) \]
for some positive integers $i$. If we take derivative in each part of (3.10) in the direction $V_1$ in $M$, we have
\[ g(\nabla V_1 V_{n-(i+1)}, \nabla f) + g(V_{n-(i+1)}, \nabla V_1 \nabla f) = V_1 (H_i^* g(\nabla f, V_n)). \quad (3.11) \]

And, by using (3.11) and Frenet formulas, we get the equality
\[ V_1 (H_i^* g(\nabla f, V_n)) = -\varepsilon_{n-(i+3)}\varepsilon_{n-(i+2)} k_{n-(i+2)} g(V_{n-(i+2)}, \nabla f) + k_{n-(i+1)} g(V_{n-i}, \nabla f) + g(V_{n-(i+1)}, \nabla V_1 \nabla f). \quad (3.12) \]

Moreover, $\nabla V_1 \nabla f = 0$. Hence, from (3.12), we can write
\[ -\varepsilon_{n-(i+3)}\varepsilon_{n-(i+2)} k_{n-(i+2)} g(V_{n-(i+2)}, \nabla f) + k_{n-(i+1)} g(V_{n-i}, \nabla f) = V_1 (H_i^* g(\nabla f, V_n)). \quad (3.13) \]

And, from (3.13), we obtain
\[ g(V_{n-(i+2)}, \nabla f) = \left\{ -V_1 (H_i^* g(\nabla f, V_n)) + k_{n-(i+1)} g(V_{n-i}, \nabla f) \right\}. \quad (3.14) \]
\[ = \left\{ -V_1 (H_i^* g(\nabla f, V_n)) + k_{n-(i+1)} g(V_{n-i}, \nabla f) \right\} \varepsilon_{n-(i+3)}\varepsilon_{n-(i+2)} \frac{1}{k_{n-(i+2)}}. \]

On the other hand, since the equality (3.1) is true for $i - 1$ according to the induction hypothesis, we have
\[ g(V_{n-i}, \nabla f) = H_{i-1}^* g(\nabla f, V_n). \quad (3.15) \]

Therefore, by using (3.3), (3.14) and (3.15), we get
\[ g(V_{n-(i+2)}, \nabla f) = \left\{ -V_1 (H_i^*) + k_{n-(i+1)} H_{i-1}^* \right\} \varepsilon_{n-(i+3)}\varepsilon_{n-(i+2)} \frac{1}{k_{n-(i+2)}} g(\nabla f, V_n) \quad (3.16) \]

Moreover, we obtain
\[ H_{i+1}^* = \left\{ k_{n-(i+1)} H_{i-1}^* - V_1 (H_i^*) \right\} \varepsilon_{n-(i+3)}\varepsilon_{n-(i+2)} \frac{1}{k_{n-(i+2)}}. \quad (3.17) \]

for $i + 1$ in the Definition 3.1. So, we have
\[ g(V_{n-(i+2)}, \nabla f) = H_{i+1}^* g(\nabla f, V_n) \]
by using (3.16) and (3.17). It follows that the equality (3.1) is true for $i + 1$. Consequently, we get
\[ g(V_{n-(i+1)}, \nabla f) = H_i^* g(\nabla f, V_n) \]
for all $i$ according to induction theory. This completes the proof. \[ \Box \]

**Theorem 3.2.** Let $(M, g)$ be a $n$-dimensional pseudo-Riemannian manifold and let $\alpha(s)$ be a proper curve of order $n$ (non-null) in $M$. Let us assume that $f \in C(M)$ be a eikonal function along curve $\alpha$, i.e. $g(\nabla f, \nabla f) =$ constant along curve $\alpha$ and the Hessian $H^f = 0$. If $\alpha$ is a $f$-eikonal $V_n$-slant helix curve with the axis $\nabla f$, then the axis of the curve $\alpha$
\[ \nabla f = \{ \varepsilon_0 H_{n-2}^* V_1 + \ldots + \varepsilon_{n-3} H_1^* V_{n-2} + \varepsilon_{n-1} V_n \} g(\nabla f, V_n), \]
where $\{V_1, V_2, \ldots, V_n\}$ and $\{H_1^*, \ldots, H_{n-2}^*\}$ are the Frenet frame and the harmonic curvatures of $\alpha$, respectively.
Theorem 3.3. Let curve of order $n$ be a proper curve of order $n$ (non-null) in $M$. Let us assume that $f \in F(M)$ be a eikonal function along curve $\alpha$, i.e. $g(\nabla f, \nabla f) = \text{constant}$. Moreover, by the definition of metric tensor, we have

$$g(\nabla_V \nabla f, V_n) = \text{constant}.$$  \hspace{1cm} (3.18)

If we take the derivative in each part of (3.18) in the direction $V_1$ in $M$, then we have

$$g(\nabla_{V_1} \nabla f, V_n) + g(\nabla f, \nabla_{V_1} V_n) = 0.$$  \hspace{1cm} (3.19)

On the other hand, from Corollary 3.1, $\nabla f$ is parallel in $M$. That’s why, $\nabla_{V_1} \nabla f = 0$. Then, we obtain

$$- \varepsilon_n \epsilon_n k_{n-1} (\nabla f, V_{n-1}) = 0$$  \hspace{1cm} (3.20)

by using (3.19) and Frenet formulas. Since $\varepsilon_n \epsilon_n k_{n-1}$ is positive function, (3.20) implies that

$$g(\nabla f, V_{n-1}) = 0.$$  

Hence, we can write the axis of $\alpha$ as

$$\nabla f = \lambda_1 V_1 + \lambda_2 V_2 + \ldots + \lambda_{n-1} V_{n-2} + \lambda_n V_n.$$  \hspace{1cm} (3.21)

Moreover, from (3.21), we get

$$\varepsilon_0 \lambda_1 = g(\nabla f, V_1)$$
$$\varepsilon_1 \lambda_2 = g(\nabla f, V_2)$$
$$\quad 
$$
$$\quad 
$$
$$\varepsilon_{n-3} \lambda_{n-2} = g(\nabla f, V_{n-2})$$
$$\varepsilon_{n-1} \lambda_n = g(\nabla f, V_n)$$

by using the metric $g$. On the other hand, from Theorem 3.1, we know that

$$\lambda_1 = g(\nabla f, V_1) = \varepsilon_0 H_{n-2}^* g(\nabla f, V_n)$$
$$\lambda_2 = g(\nabla f, V_2) = \varepsilon_1 H_{n-3}^* g(\nabla f, V_n)$$
$$\quad 
$$
$$\quad 
$$
$$\lambda_{n-2} = g(\nabla f, V_{n-2}) = \varepsilon_{n-3} H_{n-2}^* g(\nabla f, V_n)$$
$$\lambda_n = \varepsilon_{n-1} g(\nabla f, V_n).$$  \hspace{1cm} (3.22)

Thus, it can be easily obtained the axis of the curve $\alpha$ as

$$\nabla f = \{\varepsilon_0 H_{n-2}^* V_1 + \ldots + \varepsilon_{n-3} H_{n-2}^* V_{n-2} + \varepsilon_{n-1} V_n\} g(\nabla f, V_n),$$

by making use of the equality (3.21) and the system (3.22). This completes the proof. \hfill \Box

Theorem 3.3. Let $(M, g)$ be a $n$-dimensional pseudo-Riemannian manifold and let $\alpha (s)$ be a proper curve of order $n$ (non-null) in $M$. Let us assume that $f \in F(M)$ be a eikonal function along curve $\alpha$, i.e. $g(\nabla f, \nabla f) = \text{constant}$. Moreover, from (3.21), we get

$$\nabla f = \{\varepsilon_0 H_{n-2}^* V_1 + \ldots + \varepsilon_{n-3} H_{n-2}^* V_{n-2} + \varepsilon_{n-1} V_n\} g(\nabla f, V_n).$$  \hspace{1cm} (3.23)

Therefore, from (3.23), we can write

$$g(\nabla f, \nabla f) = (g(\nabla f, V_n))^2 (\varepsilon_0 H_{n-2}^* + \ldots + \varepsilon_{n-3} H_{n-2}^* + \varepsilon_{n-1} V_n).$$  \hspace{1cm} (3.24)

Moreover, by the definition of metric tensor, we have

$$|g(\nabla f, \nabla f)| = \|\nabla f\|^2.$$
According to this Theorem, \( \alpha \) is a \( f \)-eikonal \( V_n \)-slant helix curve. So, \( \| \nabla f \| = \text{constant} \) and \( g (\nabla f, V_n) = \text{non-zero constant along } \alpha \). Hence, from (3.24), we obtain
\[
\varepsilon_n^3 H_{n-2}^2 + \ldots + \varepsilon_n^3 H_{n-1}^2 + \varepsilon_{n-1}^3 = \text{constant}.
\]
In other words,
\[
\varepsilon_n H_{n-2}^2 + \ldots + \varepsilon_n H_{n-1}^2 = \text{constant}.
\]
Now, we will show that \( H_{n-2}^* \neq 0 \). We assume that \( H_{n-2}^* = 0 \). Then, for \( i = n - 2 \) in (3.1),
\[
g (V_1, \nabla f) = H_{n-2}^* g (\nabla f, V_n) = 0. \tag{3.25}
\]
If we take derivative in each part of (3.25) in the direction \( V_1 \) on \( M \), then we have
\[
g (\nabla V_1, V_1, \nabla f) + g (V_1, \nabla V_1, \nabla f) = 0. \tag{3.26}
\]
On the other hand, from Corollary 3.1, \( \nabla f \) is parallel in \( M \). That’s why \( \nabla V_1 \nabla f = 0 \). Then, from (3.26), we have \( g (V_1, V_1, \nabla f) = k_1 g (V_2, \nabla f) = 0 \) by using the Frenet formulas. Since \( k_1 \) is positive, \( g (V_2, \nabla f) = 0 \). Now, for \( i = n - 3 \) in (3.1),
\[
g (V_2, \nabla f) = H_{n-3}^* g (V_n, \nabla f).
\]
And, since \( g (V_2, \nabla f) = 0 \), \( H_{n-3}^* = 0 \). Continuing this process, we get \( H_i^* = 0 \). Let us recall that \( H_i^* = \varepsilon_n H_{n-3}^2 + \varepsilon_n H_{n-1}^2 + \ldots + \varepsilon_0 H_{n-2}^2 \) is non-zero constant if and only if \( V_1 (H_{n-2}^*) = H_{n-2}^* = k_1 H_{n-3}^*, \) where \( V_1 \) and \( \{ H_1^*, \ldots, H_{n-2}^* \} \) are the unit tangent vector field and the harmonic curvatures of \( \alpha \), respectively.

Lemma 3.2. Let \((M, g)\) be a \( n \)-dimensional pseudo-Riemannian manifold and let \( \alpha (s) \) be a proper curve of order \( n \) (non-null) in \( M \). Let us assume that \( H_{n-2}^* \neq 0 \) for \( i = n - 2 \). Then, \( \varepsilon_n H_{n-2}^2 + \varepsilon_n H_{n-1}^2 + \ldots + \varepsilon_0 H_{n-2}^2 \) is non-zero constant if and only if \( V_1 (H_{n-2}^*) = H_{n-2}^* = k_1 H_{n-3}^* \), where \( V_1 \) and \( \{ H_1^*, \ldots, H_{n-2}^* \} \) are the unit tangent vector field and the harmonic curvatures of \( \alpha \), respectively.

Proof. First, we assume that \( \varepsilon_n \neq 0 \). Consider the following functions given in Definition 3.1
\[
H_i^* = (k_{n-i} H_{n-2}^* - H_{n-1}^*) \frac{\varepsilon_n (i+2) \varepsilon_n (i+1)}{k_{n-(i+1)}},
\]
for \( 3 \leq i \leq n - 2 \). So, from the equality, we can write
\[
k_{n-(i+1)} H_i^* = \varepsilon_n (i+2) \varepsilon_n (i+1) (k_{n-i} H_{n-2}^* - H_{n-1}^*). \tag{3.27}
\]
Hence, in (3.27), if we take \( i + 1 \) instead of \( i \), we get
\[
\varepsilon_n (i+3) \varepsilon_n (i+2) H_i^* = \varepsilon_n (i+3) \varepsilon_n (i+2) k_{n-(i+1)} H_{n-1}^* - k_{n-(i+2)} H_{n-1}^* = 2 \leq i \leq n - 3
\]
together with
\[
H_i^* = \frac{1}{\varepsilon_n (i+2) \varepsilon_n (i+1)}
\]
or
\[
H_i^* = -\varepsilon_n (i+3) \varepsilon_n (i+2) H_{n-2}^*. \tag{3.29}
\]
On the other hand, since \( \varepsilon_n H_{n-2}^2 + \varepsilon_n H_{n-1}^2 + \ldots + \varepsilon_0 H_{n-2}^2 \) is constant, we have
\[
\varepsilon_n H_{n-2}^2 H_{n-1}^* + \varepsilon_n H_{n-1}^2 H_{n-2}^* + \ldots + \varepsilon_0 H_{n-2}^2 H_{n-2}^* = 0
\]
and so,
\[
\varepsilon_0 H_{n-2}^2 H_{n-2}^* = -\varepsilon_n H_{n-2}^2 H_{n-1}^* - \varepsilon_n H_{n-1}^2 H_{n-2}^* - \ldots - \varepsilon_1 H_{n-3}^2 H_{n-3}^*.
\]
By using (3.28) and (3.29), we obtain
\[
H_1 H_i^* = -\varepsilon_n (i+2) \varepsilon_n (i+1) H_{n-2}^* \tag{3.31}
\]
and
\[
\varepsilon_n (i+3) \varepsilon_n (i+2) H_i^* H_{n-2}^* = \varepsilon_n (i+3) \varepsilon_n (i+2) k_{n-(i+1)} H_{n-1}^* - k_{n-(i+2)} H_{n-1}^* = 2 \leq i \leq n - 3.
\]
Therefore, by using (3.30), (3.31) and (3.32), an algebraic calculus shows that
\[
\varepsilon_0 H_{n-2}^2 H_{n-2}^* = \varepsilon_0 k_1 H_{n-3}^* H_{n-2}^* \tag{3.33}
\]
or
\[
H_{n-2}^* H_{n-2}^* = k_1 H_{n-3}^* H_{n-2}^*.
\]
Since \( H_{n-2}^* \neq 0 \), we get the relation
\[
H_{n-2}^* = k_1 H_{n-3}^*.
\]
Conversely, we assume that
\[ H_{n-2}^* = k_1 H_{n-3}^*. \] (3.33)
By using (3.33) and \( H_{n-2}^* \neq 0 \), we can write
\[ H_{n-2}^* H_{n-2}^* = k_1 H_{n-2}^* H_{n-3}^*. \] (3.34)
From (3.32), we have the following equation system:

\[
\begin{align*}
&\text{for } i = n-3, \quad \varepsilon_1 H_{n-3}^* H_{n-3}^* = \varepsilon_1 k_2 H_{n-4}^* H_{n-3}^* - \varepsilon_0 k_1 H_{n-3}^* - k_{n-2}^* H_{n-2}^*, \\
&\text{for } i = n-4, \quad \varepsilon_2 H_{n-4}^* H_{n-4}^* = \varepsilon_2 k_3 H_{n-5}^* H_{n-4}^* - \varepsilon_2 k_2 H_{n-4}^* H_{n-3}^*, \\
&\text{for } i = n-5, \quad \varepsilon_3 H_{n-5}^* H_{n-5}^* = \varepsilon_3 k_4 H_{n-6}^* H_{n-5}^* - \varepsilon_2 k_3 H_{n-5}^* H_{n-4}^*, \\
&\quad \quad \quad \quad \quad \vdots \\
&\text{for } i = 2, \quad \varepsilon_{n-4} H_2^* H_2^* = \varepsilon_{n-4} k_{n-3} H_1^* H_2^* - \varepsilon_{n-5} k_{n-4} H_2^* H_3^*. 
\end{align*}
\]
Moreover, from (3.31) and (3.34), we obtain
\[ \varepsilon_{n-3} H_1^* H_1^* = -\varepsilon_{n-4} k_{n-3} H_1^* H_2^* \] (3.35)
and
\[ \varepsilon_0 H_{n-2}^* H_{n-2}^* = \varepsilon_0 k_1 H_{n-2}^* H_{n-3}^*. \] (3.36)
So, by using the above equation system, (3.35) and (3.36), an algebraic calculus shows that
\[ \varepsilon_{n-3} H_1^* H_1^* + \varepsilon_{n-4} H_2^* H_2^* + \ldots + \varepsilon_0 H_{n-2}^* H_{n-2}^* = 0. \] (3.37)
And, by integrating (3.37), we can easily say that
\[ \varepsilon_{n-3} H_1^* H_1^* + \varepsilon_{n-4} H_2^* H_2^* + \ldots + \varepsilon_0 H_{n-2}^* H_{n-2}^* \]
is a non-zero constant. This completes the proof. \( \square \)

**Corollary 3.2.** Let \((M,g)\) be a \(n\)-dimensional pseudo-Riemannian manifold and let \(\alpha(s)\) be a proper curve of order \(n\) (non-null) in \(M\). Let us assume that \(f \in \mathcal{F}(M)\) be a eikonal function along curve \(\alpha\), i.e. \(g(\nabla f, \nabla f) = \text{constant}\) along curve \(\alpha\) and the Hessian \(H^f = 0\). If \(\alpha\) is a \(f\)-eikonal \(V_{n}\)-slant helix curve, 
\[ V_1 (H_{n-2}^*) = H_{n-2}^* = k_1 H_{n-3}^*. \]

**Proof.** It is obvious by using Theorem 3.3 and Lemma 3.2. \( \square \)

### 4. Conclusions

In this work, it is defined \(f\)-eikonal \(V_{n}\)-slant helix by the gradient vector field \(\nabla f\) and \(\nabla f\) is called as the axis of the eikonal slant helix. Besides, it gives new characterizations on eikonal slant helices by using the harmonic curvature functions in \(n\)-dimensional pseudo-Riemannian manifolds.

On the other hand, we want to emphasize an important point. The axis \(\nabla f\) defined in this work is non-constant. If the axis \(\nabla f\) is considered as a constant vector field, then the eikonal slant helix defined in this paper coincides with \(V_{n}\)-slant helix which is introduced in \([7]\). Also, if \(\nabla f\) is a Levi-Civita parallel vector field, then eikonal slant helix is a LC-slant helix defined by in \([13,14]\).

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