Particle decays and stability on the de Sitter universe

Jacques Bros¹, Henri Epstein² and Ugo Moschella³
¹Service de Physique théorique - CEA. Saclay. 91191 Gif-sur Yvette.
²Institut des Hautes Études Scientifiques, 91440 Bures-sur-Yvette.
³Università dell’Insubria, Como and INFN Milano

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Abstract

We study particle decay in de Sitter space-time as given by first order perturbation theory in a Lagrangian interacting quantum field theory. We study in detail the adiabatic limit of the perturbative amplitude and compute the “phase space” coefficient exactly in the case of two equal particles produced in the disintegration. We show that for fields with masses above a critical mass \( m_c \) there is no such thing as particle stability, so that decays forbidden in flat space-time do occur here. The lifetime of such a particle also turns out to be independent of its velocity when that lifetime is comparable with de Sitter radius. Particles with mass lower than critical have a completely different behavior: the masses of their decay products must obey quantification rules, and their lifetime is zero.

1 Introduction

Some important progress in the astronomical observations of the last ten years [1, 2] have led in a progressively convincing way to the surprising conclusion that the recent universe is dominated by an almost spatially homogeneous exotic form of energy density to which there corresponds an effective negative pressure. Such negative pressure acts repulsively at large scales, opposing itself to the gravitational attraction. It has become customary to characterize such energy density by the term "dark".

The simplest and best known candidate for the "dark energy" is the cosmological constant. As of today, the ΛCDM (Cold Dark Matter) model, which is obtained by adding a cosmological constant to the standard model, is the one which is in better agreement with the cosmological observations, the latter being progressively more precise. Recent data show that dark energy behaves as a cosmological constant within a few percent error. In addition, if the description provided by the ΛCDM model is correct, Friedmann’s equation shows that the remaining energy components must in the future progressively thin out and eventually vanish thus letting the cosmological constant term alone survive.

In the above scenario the de Sitter geometry [3, 4], which is the homogeneous and isotropic solution of the vacuum Einstein equations with cosmological term, appears to take the double role of reference geometry of the universe, namely the geometry of spacetime deprived of its matter and radiation content and of geometry that the universe approaches asymptotically. On the other hand, it seems reasonable to imagine that the presence of a small cosmological constant, while having a huge impact on our understanding of the universe as a whole, would not influence microphysics in its quantum aspects. However this conclusion may have to be reassessed, because in the presence of a cosmological constant, however small, it is the notion of elementary particle itself which has to be reconsidered: indeed, the usual asymptotic theory is based on concepts which refer closely to the global structure of Minkowski spacetime and to its Fourier representation, and do not apply to the de Sitter universe.
which is not asymptotically flat. Secondly, even if one may think that interactions between elementary particles happen in a "laboratory" so that "infinity" is a distance of the order of meters, our present understanding of perturbative quantum field theory is also based on global concepts; in particular, the calculation of perturbative amplitudes involves integrations over the whole spacetime manifold and it should be expected that different topological global structures result in different physical properties in the "small".

The literature about de Sitter quantum field theory is very extensive, but there is no comparison with the understanding one has of Minkowskian field theory as regards both general and structural results as well as its operative and computational possibilities. This second point is particularly doleful: calculations of perturbative amplitudes which in the Minkowskian case would be simple or even trivial become rapidly prohibitive or impossible in the case of de Sitter or anti de Sitter universes: this in spite of the fact that one is dealing with maximally symmetric manifolds which have invariance groups of the same dimension as the Poincaré group. The technical, but also the physical, difference lies precisely in the above mentioned fact that much of the usual quantum field theory is based on concepts which are characteristic of the global structure of Minkowski spacetime and which do not persist in the presence of curvature, already in the presence of a mere cosmological constant, where Minkowskian spacetime is replaced by the de Sitter or by the anti de Sitter one.

In this paper we give a full description of how to solve the problem of calculating the mean lifetime of unstable scalar particles on de Sitter spacetime at first order in perturbation theory. This interesting physical problem provides also an example of a concrete perturbative calculation in presence of the cosmological constant. The task already presents considerable mathematical difficulties.

To our knowledge this calculation was first taken up by O. Nachtmann [5] in 1968. He showed, in a very special case, that while a Minkowskian particle can never decay into heavier products, a dS-particle can, although this effect is exponentially small in the dS-radius.

The subject has acquired a greater physical interest with the advent of inflationary cosmology. In particular, the idea that particle decays during the (quasi-)de Sitter phase may have important consequences on the physics of the early universe has been suggested recently [6, 7, 8]. The mathematical and physical difficulties related to the lack of time-translation symmetry of the de Sitter universe, and more generally of non-static cosmological backgrounds, have been tackled [5, 7, 8] by using the Schwinger-Keldysh formalism, which is suitable for studying certain aspects of the quantum dynamics of systems out of equilibrium. An important ingredient of this approach is the so called Dynamic Renormalization Group [9] which allows a kind of resummation of an infinite series of infrared diverging quantities. That method is however based on the introduction of a practical notion of lifetime of an unstable particle which is quite different from the definition commonly used in quantum physics. Also, the hard technical difficulties of the concrete calculation involved in solving a complicated integro-differential equation have only been faced in the favorably special conformal and minimally coupled massless cases although in principle the method can be used to deal with particles of generic mass [6, 7, 8].

In this paper we perform a computation which is similar to the one outlined by Nachtmann and follows the conventional quantum field theoretical perturbative approach for computing probability amplitudes. Our work gives significantly wider results w.r.t. [5], e.g. regarding the so-called adiabatic limit, complementary-series-particles, and explicit expressions of the relevant Källén-Lehmann weights. On the other side comparing our result with those of [6, 7, 8] is not easy because of the non standard (but interesting) definition of lifetime chosen in [6, 7, 8].

These findings have been summarized in a recent short communication [10]. The results exhibit significant differences compared to the Minkowski case, and decay processes which are normally forbidden become possible and, vice-versa, processes that are normally possible are now forbidden. The maximal symmetry of the de Sitter universe implies the existence of a global square-mass operator, one of the two Casimir operators of the de Sitter group $SO_0(1, d)$ (see e.g. [11]): this quantity is conserved for de Sitter invariant field theories. However, in contrast with the Poincaré group case, the tensor product
of two unitary irreducible representations of masses \( m_1 \) and \( m_2 \) decomposes into a direct integral of representations whose masses \( m \) do not satisfy the ‘subadditivity condition’ \( m \geq m_1 + m_2 \). All representations of mass larger than a certain critical value (principal series) appear in the decomposition. This fact was shown in \([5]\) for the two-dimensional case and will be established here in general. This means that the de Sitter symmetry does not prevent a particle with mass in the principal series from decaying into e.g. pairs of heavier particles. This phenomenon also implies that there can be nothing like a mass gap in that range. This is a major obstruction to attempts at constructing a de Sitter S-matrix; the Minkowskian asymptotic theory makes essential use of an isolated point in the spectrum of the mass operator, and this will generally not occur in the de Sitter case. We will also show that the tensor product of two representations of sufficiently small mass below the critical value (complementary series) contains an additional finite sum of discrete terms in the complementary series itself (at most one term in dimension 4). This implies a form of particle stability, but the new phenomenon is that a particle of this kind cannot disintegrate unless the masses of the decay products have certain quantized values. Stability for the same range of masses has also been recently found \([8]\) in a completely different context. Other remarks about the physical meaning and applicability of our results will be presented in the concluding section.

### 1.1 Notation

We denote \( \mathbb{C}_+ = -\mathbb{C}_- \) the open upper complex half-plane. Let \( \Delta = \mathbb{C} \setminus [-1, 1], \Delta_1 = \mathbb{C} \setminus (-\infty, 1] \). The function \( \log \) is defined as holomorphic on \( \mathbb{C} \setminus (-\infty, 0] \) and real on \( (0, +\infty) \) and \( \zeta \mapsto \zeta^n \) as \( \exp(\mu \log(\zeta)) \). It is entire in \( \mu \). If \( \zeta \in \Delta_1 \) and \( \rho > 0 \), then \( (\rho \zeta)^\mu = \rho^\mu \zeta^\mu \). If \( \zeta \in \mathbb{C}_+ \) and \( s \in \mathbb{C}_- \), then \( (s \zeta)^\mu = s^\mu \zeta^\mu \). We define \( z \mapsto (z^2 - 1)^{1/2} \) as holomorphic on \( \Delta \) and asymptotic to \( z \) at large \( |z| \). It is Herglotz, negative on \( (-\infty, -1) \) and positive on \( (1, +\infty) \).

### 2 Free fields in Minkowski and de Sitter spacetimes

In this section we give a short summary of the theory of free and generalized free quantum fields on de Sitter spacetime. Since there are infinitely many inequivalent representations of the field algebra, a (mathematical) choice has to be made on physical grounds. Ours is based on the analyticity properties of the vacuum expectation values: see the condition \((W2)\) below. In the Minkowski space, this is equivalent to the positivity of the energy. In the de Sitter case, it admits a thermal interpretation \([12, 13, 14, 15]\). The reader can find in \([13, 14, 15]\) a general approach to de Sitter QFT based on such analytic properties. It includes the so called Bunch-Davies, also called Euclidean vacuum of de Sitter scalar Klein-Gordon fields as a basic example.

The real (resp. complex) \( d \)-dimensional Minkowski spacetime \( M_d \) (resp. \( M_d^{(c)} \)) is \( \mathbb{R}^d \) (resp. \( \mathbb{C}^d \)) equipped with the Lorentzian inner product

\[
x \cdot x' = x^0 x'^0 - x^1 x'^1 - \cdots - x^{d-1} x'^{d-1} = x^0 x'^0 - \vec{x} \cdot \vec{x}'
\]

\( (2.1) \)

w.r.t. an arbitrarily chosen Lorentz frame \( \{ e_\sigma, \sigma = 0, \ldots, d-1 \} \). When no ambiguity arises, \( x^2 \overset{\text{def}}{=} x \cdot x \). The real (resp. complex) de Sitter spacetime \( X_d \) (resp. \( X_d^{(c)} \)) with radius \( R > 0 \) are the hyperboloids

\[
X_d = \{ x \in M_{d+1} : x \cdot x + R^2 = 0 \}, \quad X_d^{(c)} = \{ x \in M_{d+1}^{(c)} : x \cdot x + R^2 = 0 \},
\]

\( (2.2) \)

equipped with the pseudo-riemannian metric induced by \((2.1)\). \( L_+^1(d) = \text{SO}(1, d-1; \mathbb{R}) \) is the connected Lorentz group acting on \( M_d \), and \( L_+^c(\mathbb{C}; d) = \text{SO}(1, d-1; \mathbb{C}) \) is the connected complex Lorentz group acting on \( M_d^{(c)} \). The connected group of displacements on \( X_d \) (resp. \( X_d^{(c)} \)) is \( L_+^1(d+1) \).
The following properties:

- The future and past tubes in the complex Minkowski spacetime $M_d$ are given by:
  \[ V_+ = \{ x \in M_d : x \cdot x > 0, \quad x^0 > 0 \} = -V_-, \]
  \[ C_+ = \{ x \in M_d : x \cdot x = 0, \quad x^0 \geq 0 \} = -C_- . \]

The future and past tuboids in $X_d$ are the intersections of the future and past tubes in $M_d$ with the complex de Sitter manifold $X_d$:

\[ T_\pm = T_\pm \cap X_d . \]

We will use the letter $\mathcal{X}$ to denote either $M_d$ or $X_d$ when the same discussion applies to both, $\mathcal{X}(\mathbb{C})$ denoting the complexified object. $dx$ will denote the standard invariant measure on $\mathcal{X}$, i.e. using the frame $(e_0, \ldots, e_n)$, $dx = dx^0 \ldots dx^{d-1}$ in the case of $M_d$, and $dx = 2\delta(x^2 + R^2) dx^0 \ldots dx^d$ for $X_d$.

A (neutral scalar) generalized free field $\phi$ on $\mathcal{X}$ is entirely specified by its 2-point function. This is a tempered distribution $\mathcal{W}$ on $\mathcal{X} \times \mathcal{X}$ (we denote $\mathcal{W}(x, x') = \mathcal{W}(x', x)$), which we require to have the following properties:

- **(W1) Hermiticity:**
  \[ \mathcal{W}(x, x') = \mathcal{W}(x', x) . \]

- **(W2) Analyticity and invariance:** there is a function $w$ of one complex variable, holomorphic in the cut plane $\mathbb{C} \setminus \mathbb{R}_+$, with tempered behavior at infinity and at the boundaries, such that, in the sense of tempered distributions,
  \[ \mathcal{W}(x, x') = \lim_{z \to x, \quad z' \to x'} w((z - z')^2) . \]
  \[ \mathcal{W}(x, x') = \lim_{z \to x, \quad z' \to x'} w((z - z')^2) . \]

For complex $z, z' \in X_d$ such that $(z - z')^2 \in \mathbb{C} \setminus \mathbb{R}_+$ we will denote $W(z, z') = w((z - z')^2)$.

Note that this implies

\[ W(z, z') = W(z', z) = W(-z', -z) , \]

and

\[ W(x, x') = W(-x', -x) . \]

**(W1) and (W2) also imply**

\[ W(z, z') = \overline{W(z, z')} . \]

**(W3) Positivity:** For every $f \in \mathcal{S}(\mathcal{X})$,

\[ \langle \mathcal{W}, f \rangle = \int_{\mathcal{X} \times \mathcal{X}} \overline{f(x)} \mathcal{W}(x, x') f(x') \, dx \, dx' \geq 0 . \]
Conversely, given \( \mathcal{W} \) and \( w \) having these properties, after having identified the kernel \( \mathcal{N}_1 = \{ f \in \mathcal{S}(\mathcal{X}) : \langle \mathcal{W}, \mathcal{F} \otimes f \rangle = 0 \} \) we can construct a Hilbert space \( \mathcal{F}_1 \) by completing \( \mathcal{S}(\mathcal{X})/\mathcal{N}_1 \) equipped with the scalar product \( \langle f, g \rangle = \langle \mathcal{W}, \mathcal{F} \otimes g \rangle \), and then exponentiate \( \mathcal{F}_1 \) into a Fock space \( \mathcal{F} \)

\[
\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n, \quad \mathcal{F}_0 = \mathbb{C}, \quad \mathcal{F}_n = S\mathcal{F}_1^\otimes n \quad \text{for} \quad n \geq 1.
\] (2.13)

The vacuum \( \Omega \) is the unit vector \( 1 \in \mathcal{F}_0 = \mathbb{C} \). There is a continuous unitary representation \( U \) of the Poincaré or de Sitter group acting on \( \mathcal{F} \) and preserving the \( \mathcal{F}_n \), with \( U\Omega = \Omega \). The generalized free field \( \phi \) is defined on a dense domain in \( \mathcal{F} \) and \( (\Omega, \phi(x)\phi(x')\Omega) = \mathcal{W}(x, x') \).

As a result of the analyticity property (W2), the Wick powers of a generalized free field are well-defined local fields operating in the same Fock space. Their vacuum expectation values are obtained by the standard Wick formulae as sums of products of \( \mathcal{W} \).

We note that a function \( \mathcal{W} \) on \( X_d \times X_d \) possessing the properties (W1) and (W2) automatically extends (through (2.7)) to a function with the same properties on \( M_{d+1} \times M_{d+1} \), so that a generalized free field on \( X_d \) has an extension as a generalized free field on \( M_{d+1} \). However the extension of \( \mathcal{W} \) need not satisfy (W3) on \( M_{d+1} \times M_{d+1} \) even if it does on \( X_d \times X_d \).

A free field \( \phi \) of mass \( m > 0 \) on \( \mathcal{X} \) is a generalized free field such that \( \mathcal{W} \) is a solution of the Klein-Gordon equation with mass \( m \) in both arguments, and is normalized so as to obey the canonical commutation relations. In that case \( \mathcal{W} \) is uniquely determined by \( m \) and will be denoted \( \mathcal{W}_m \). In the Minkowskian case, the representation \( U|\mathcal{F}_1 \) is irreducible and equivalent to the representation \([m, 0]\) of the Poincaré group.

As usual, the representation \( U \) provides a representation of the Lie algebra of the (Poincaré or de Sitter) group and its enveloping algebra by self-adjoint (or \( i \times \) self-adjoint) operators on \( \mathcal{F} \). In particular the square-mass operator \( M^2 \) is given by \( M^2 = P^\mu P_\mu \) in the Minkowskian case, and by \( M^2 = M_{\mu\nu} M_{\mu\nu}/2R^2 \) in the de Sitter case. In both cases, \( M^2 \Psi = m^2 \Psi \) for every \( \Psi \in \mathcal{F}_1 \). See e.g. [11].

2.1 Special features of free fields in de Sitter space-time

In the de Sitter case the mass \( m \) can be related to a dimensionless parameter \( \nu \) as follows

\[
m^2 R^2 = \left( \frac{d-1}{2} \right)^2 + \nu^2,
\] (2.14)

\[
\nu = \pm \left[ m^2 R^2 - \left( \frac{d-1}{2} \right)^2 \right]^{1/2} = \pm R(m^2 - m_c^2)^{1/2}, \quad m_c = \frac{d-1}{2R}.
\] (2.15)

In this case, if no ambiguity arises, we shall often denote \( \mathcal{W}_\nu = \mathcal{W}_{-\nu} \) to mean \( \mathcal{W}_m \), and similarly \( \mathcal{W}_\nu \) and \( w_\nu \). Explicitly, if \( z, z' \in X^{(c)}_d \), \((z - z')^2 \notin \mathbb{R}_+ \), and hence \( \zeta = z \cdot z'/R^2 \) does not belong to the real interval \([-\infty, -1] \),

\[
\mathcal{W}_\nu(z, z') = \frac{\Gamma \left( \frac{d-1}{2} + i\nu \right) \Gamma \left( \frac{d-1}{2} - i\nu \right)}{2(2\pi)^{d/2} R^{d-2}} \left( \zeta^2 - 1 \right)^{-\frac{d-2}{2}} \mathcal{F} \left( \frac{d-1}{2} + i\nu, \frac{d-1}{2} - i\nu, \frac{d-1}{2}; \frac{1-\zeta}{2} \right).
\] (2.16)

Since \( \Gamma(\zeta)^{-1} \mathcal{F}(a, b; c; z) \) is entire in \( a, b, \) and \( c \) the rhs of (2.17) is meromorphic in \( \nu \) with simple poles at \( \nu = \pm i((d-1)/2 + n), n \geq 0 \) an integer. In other words \( \mathcal{W}_\nu(z, z') \) extends to a holomorphic function of \( \nu, z \) and \( z' \) in the domain \( \{ \nu \in \mathbb{C}, z \in X^{(c)}_d, z' \in X^{(c)}_d : \nu \notin \pm i((d-1)/2 + \mathbb{Z}_+), (z - z')^2 \notin \mathbb{R}_+ \} \). However \( w_\nu \) possesses the positivity property (W3) (see (2.12)) only if either
(1) \( \nu \) is real, i.e. \( m \geq m_c = (d-1)/2R \). In this case \( U|\mathcal{F}_1 \) is an irreducible unitary representation of the “principal series”.

or

(2) \( \nu \) is pure imaginary with \( i\nu \in (-d-1)/2, (d-1)/2 \), i.e. \( 0 < m \leq (d-1)/2R \). In this case \( U|\mathcal{F}_1 \) is an irreducible unitary representation of the “complementary series”.

We shall need a small part of the harmonic analysis on the de Sitter space-time as developed in [14]. If \( z \in T_\pm \subset X^d_+ \) and \( \xi \in C_+ \setminus \{0\} \subset M_{d+1} \), then \( \pm \text{Im}(z : \xi) > 0 \), so that \( (z : \xi) \) is well-defined and holomorphic in \((z, \lambda) \in (T_+ \cup T_-) \times \mathbb{C} \). The role of plane waves on \( X_d \) is played by the distributions

\[
\psi^\pm_\lambda(x, \xi) = \lim_{y \to 0} (x \pm iy) \cdot \xi) = \psi^\pm_\lambda(x, \xi).
\]

(2.18)

An important formula expressing the de Sitter case two-point \( W_\nu \) as a Fourier superposition of plane-waves is the following (see [14]):

\[
W_\nu(z, z') = R_{d,\nu} \int_\gamma (z \cdot \xi)^{-\frac{d-1}{2} + i\nu} (\xi \cdot z')^{-\frac{d-1}{2} - i\nu} \alpha(\xi),
\]

where \( z_1 \in T_-, z_2 \in T_+ \), and

\[
c_{d,\nu} = \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)e^{-\pi\nu}}{2^{d+1}\pi^d}.
\]

(2.20)

In (2.19), \( \gamma \) is a \((d-1)\)-cycle in \( C_+ \setminus \{0\} \) homologous to the sphere \( S_0 = C_+ \cap \{\xi \ : \ \xi^0 = 1\} \). The \((d-1)\)-form \( \alpha \) is given, in the standard coordinates, by

\[
\alpha = (\xi^0)^{-1} \sum_{j=1}^d (-1)^{j+1} \xi^j \, d\xi_1 \cdots \hat{d}\xi_j \cdots d\xi^d.
\]

(2.21)

If a smooth function \( f \) on \( C_+ \setminus \{0\} \) is homogeneous of degree \((1-d)\), the form \( f\alpha \) is closed, so that the linear functional

\[
f \mapsto I_0(f) = \int_\gamma f(\xi) \alpha(\xi)
\]

(2.22)

is independent of \( \gamma \). This implies that it is Lorentz-invariant. We often denote \( d\mu_\gamma \) the measure defined on \( \gamma \) by the restriction of \( \alpha \) by the restriction of \( \alpha \) to the \((d-1)\)-form \( S_0 \) is the standard volume form on that sphere, normalized by \( \int_{S_0} d\mu_\gamma = 2\pi^{d/2}/\Gamma(d/2) \). It is possible to take the limit of (2.19), in the sense of distributions, when \( z_1 \) and \( z_2 \) tend to the reals:

\[
W_\nu(x, x') = R_{d,\nu} \int_\gamma (z \cdot \xi)^{-\frac{d-1}{2} + i\nu} (\xi \cdot z')^{-\frac{d-1}{2} - i\nu} d\mu_\gamma(x). 
\]

(2.23)

Comparing (2.17) with (2.19) and (2.20) gives

\[
\int_\gamma (z \cdot \xi)^{-\frac{d-1}{2} + i\nu} (\xi \cdot z')^{-\frac{d-1}{2} - i\nu} d\mu_\gamma(\xi) = \frac{e^{\pi\nu^2 2\pi^{d/2}}}{R^{d-1} \Gamma\left(\frac{d}{2}\right)} F\left(\frac{d-1}{2} + i\nu, \frac{d-1}{2} - i\nu ; \frac{1}{2} - \frac{\xi^0 \pm 1}{2}\right)
\]

(2.24)

Both sides of this equation are holomorphic in \( z_1, z_2, \nu \) in the domain \( T_- \times T_+ \times \mathbb{C} \), hence the equation (2.24) holds in this domain.

**Remark 2.1** If \( T \) is a homogeneous distribution of degree \( \beta \) on \( C_+ \setminus \{0\} \), it can be restricted to any \( C^\infty \) submanifold of dimension \( d-1 \) which is transversal to the generators of \( C_+ \), in particular to hyperplanar sections such as \( S_0 = \{\xi \in C_+ : \xi^0 = 1\} \) and \( V_0 = \{\xi \in C_+ : \xi^0 + \xi^d = 1\} \). If \( \gamma \) is of this type and compact, \( \int_\gamma T(\xi) \alpha(\xi) \) is well-defined and, if \( \beta = 1 - d \), it is independent of \( \gamma \).
Remark 2.2 For any complex \( \alpha, (z, \xi) \mapsto (z \cdot \xi)^\alpha \) is \( C^\infty \) in \( \xi \) and holomorphic in \( z \) on \( T_\mathbb{K} \times (C_+ \setminus \{0\}) \) and it is an entire function of \( \alpha \). For each \( \xi \) it has a limit in the sense of tempered distributions on \( X_\mathbb{D} \) as \( z \) tends to the reals, and this has been denoted \( \psi_\alpha^\pm(x, \xi) \). It is an entire function in \( \alpha \). Furthermore its invariance under \( G_0 \) implies that, if \( \varphi \in \mathcal{S}(C_+ \setminus \{0\}) \), \( \int_{C_+} \psi_\alpha^\pm(x, \xi) \varphi(\xi) \, d\xi \) is \( C^\infty \) in \( x \). Indeed any small displacement of \( x \) can be effected by a group transformation close to the identity, which can be transferred to \( \xi \) and thence to \( \varphi \). In the same way, \( \psi_\alpha^\mp(x, \xi) \) is \( C^\infty \) in \( \xi \) (as well as homogeneous) when integrated with a smooth test-function in \( x \). This explains the meaning of formulae such as (2.28).

Note that the integral in this formula is entire in \( \nu \). For similar reasons, for any \( \varphi \in \mathcal{S}(X_\mathbb{D}) \), \( \int_{X_\mathbb{D}} \varphi(x) \mathcal{W}_\nu(x, x') \, dx \) is \( C^\infty \) in \( x' \) and meromorphic in \( \nu \).

2.2 More features common to Minkowski and de Sitter space-time

An important formula, which holds in Minkowski as well as in de Sitter space-time (but in this case only if \( m, m' \geq m_c \)), is the projector identity:

\[
\int_X \mathcal{W}_m(z, x) \mathcal{W}_{m'}(x, y) \, dx = C_1(m, d) \delta(m^2 - m'^2) \mathcal{W}_m(z, y). \tag{2.25}
\]

Here

\[
C_1(m, d) = 2\pi \text{ for Minkowski space-time,} \tag{2.26}
\]

\[
C_1(m, d) = C_0(\nu) = 2\pi |\coth(\pi \nu)| \text{ for de Sitter space-time.} \tag{2.27}
\]

The proof of the above identity is trivial in the Minkowskian case. For the de Sitter case it will be provided in Appendix [19]. Note that \( C_0(mR) \) tends to \( 2\pi \) as \( R \to +\infty \) for a fixed \( m > 0 \).

The Källén-Lehmann decomposition theorem exists in both \( M_\mathbb{D} \) and \( X_\mathbb{D} \). In the case of \( M_\mathbb{D} \), (see [16], p. 360), it asserts that, for every \( W \) having the properties (W1) and (W2) there is a tempered \( \rho \) such that

\[
W(z, z') = \int_{\mathbb{R}_+} \rho(m^2) \mathcal{W}_m(z, z') \, dm^2. \tag{2.28}
\]

If \( \mathcal{W} \) satisfies (W3), then \( \rho \) is a tempered positive measure. The same holds in the dS case provided \( \mathcal{W} \) satisfies some decrease property. In this case, the integral runs on masses of the principal series, i.e. \( m > m_c = (d - 1)/2R \). For proofs and details, see [14], [17]. In particular if \( m_j \geq 0 \) and, in the dS case, \( m_j > m_c \) for \( 1 \leq j \leq N \),

\[
\prod_{j=1}^N \mathcal{W}_{m_j}(x, x') = \int_{a \geq b} \rho(a^2; m_1, \ldots, m_N) \mathcal{W}_a(x, x') \, da. \tag{2.29}
\]

Here \( b = m_c \) in the de Sitter case, \( b = \sum_j m_j \) in the Minkowski case.

3 Particle decays: general formalism

There is at the moment nothing like the Haag-Ruelle asymptotic theory (HRT) (see [18], [19], [16]) for the de Sitter universe. Indeed all the ingredients of that theory are missing in the de Sitter case. For example, as it will be shown in this paper, even in a free field theory of mass \( m > m_c \), the mass \( m \) is not an isolated point in the mass spectrum. Moreover the solutions of the Klein-Gordon equation do not have the kind of localization at infinity which plays an essential role in the HRT. The concept of a particle is therefore not obvious in de Sitter space-time, except for localized observations. Here we adopt Wigner’s point of view: a one-particle vector state is a state belonging to an invariant subspace of the Hilbert space.
in which the representation of the invariance group reduces to an irreducible representation. In the dS case, we also require that this irreducible representation belong to the principal or complementary series, i.e. it should be equivalent to one of the representations which occur in the $F_1$ of a free field.

We shall study the decay of a particle using first-order perturbation theory. The initial framework and calculations are the same for the Minkowski and de Sitter cases: its ingredients are the projector identity and the Källén-Lehmann representation. (It can also be extended to the Minkowskian thermal case (20) although there is no Källén-Lehmann representation there). Let

$$
\phi_0, \phi_1, \ldots, \phi_N
$$

be $1 + N$ independent free scalar fields with masses $m_0 > 0$, $m_1 > 0, \ldots, m_N > 0$, acting in a common Fock space $\mathcal{H}$, the tensor product of the individual Fock spaces for the $\phi_k$:

$$
\mathcal{H} = \bigotimes_{k=0}^{N} F^{(k)} ,
$$

(3.2)

$$
(\Omega, \phi_j(\phi_k)(\Omega)) = \delta_{jk} \mathcal{W}_{m_j}(x, y) .
$$

(3.3)

We denote

$$
\mathcal{H}_{j_0,\ldots,j_N} = F_{j_0}^{(0)} \otimes \ldots \otimes F_{j_N}^{(N)} .
$$

(3.4)

This is the subspace of states in $\mathcal{H}$ containing $j_k$ k-particles. $E_{j_0,\ldots,j_N}$ denotes the hermitian projector onto this subspace. We now switch on an interaction term

$$
\int_{X} \gamma g(x) \mathcal{L}(x) dx , \quad \mathcal{L}(x) : = \phi_0(x)\phi_1(x)\gamma_1 \ldots \phi_N(x)\gamma_N : .
$$

(3.5)

Here the $q_j$ are non-negative integers, and we denote $q! = \prod_{j=1}^{N} q_j!$. $\gamma$ is a small constant. $g$ is a smooth, rapidly decreasing function over $X$. In the end, $g$ should be made to tend to 1 (adiabatic limit). According to perturbation theory, the transition amplitude between two normalized states $\psi_0$ and $\psi_1$ in $\mathcal{H}$ is given by $(\psi_0, S(\gamma g)\psi_1)$, where $S(\gamma g)$ is the formal series in $\gamma g$

$$
S(\gamma g) = \sum_{n=0}^{\infty} \frac{i^n \gamma^n}{n!} \int_{X^n} g(x_1) dx_1 \ldots g(x_n) dx_n T(\mathcal{L}(x_1) \ldots \mathcal{L}(x_n))
$$

(3.6)

In (3.6), $T(\mathcal{L}(x_1) \ldots \mathcal{L}(x_n))$ denotes the (renormalized) time-ordered product of $\mathcal{L}(x_1), \ldots, \mathcal{L}(x_n)$. In the first order in $\gamma g$, the transition amplitude between two orthogonal states $\psi_0$ and $\psi_1$ is

$$
(\psi_0, iT_1(\gamma g)\psi_1) , \quad T_1(\gamma g) = \int_{X} \gamma g(x) \mathcal{L}(x) dx .
$$

(3.7)

We take

$$
\psi_0 = \int f_0(x) \phi_0(x) \Omega dx ,
$$

(3.8)

$$
\psi_1 = \int f_1(x_{1}, \ldots, x_{1q_1}, \ldots, x_{N1}, \ldots, x_{Nq_N}) : \prod_{j=1}^{N} \prod_{k=1}^{q_j} \phi_j(x_{jk}) dx_{jk} : \Omega ,
$$

(3.9)

where $f_0$ and $f_1$ are smooth rapidly decreasing functions. The states of the form (3.8) generate $\mathcal{H}_{0,\ldots,0}$ and the states of the form (3.9) generate $\mathcal{H}_{0,q_1,\ldots,q_N}$. The probability of transition from $\psi_0$ to any state in $\mathcal{H}_{0,q_1,\ldots,q_N}$ is:

$$
\Gamma = \frac{(\psi_0, T_1(\gamma g)E_{0,q_1,\ldots,q_N}T_1(\gamma g)^* \psi_0)}{(\psi_0, \psi_0)} = \frac{q! \gamma^2}{(\psi_0, \psi_0)} \int f_0(x) f_0(y) g(u) g(v) \times
$$
\begin{equation}
\times W_{m_0}(x, u) \left\{ \prod_{j=1}^{N} W_{m_j}(u, v)^{q_j} \right\} W_{m_0}(v, y) dx du dv dy . \tag{3.10}
\end{equation}

From now on, we suppose, in the dS case, that $m_k > m_c$, $0 \leq k \leq N$, i.e all particles belong to the principal series. We may then replace the central two-point function in $u$ and $v$ by its Källén-Lehmann decomposition:

\begin{equation}
\prod_{j=1}^{N} W_{m_j}(u, v)^{q_j} = \int \rho(a^2; m_1, \ldots, m_1, \ldots, m_N, \ldots, m_N) W_a(u, v) da^2 . \tag{3.11}
\end{equation}

Here $m_j$ occurs $q_j$ times as an argument of $\rho$. This gives

\begin{equation}
\Gamma = \frac{q! \gamma^2}{(\Psi_0, \Psi_0)} \int f_0(x) f_0(y) g(u) g(v) \rho(a^2; m_1, \ldots, m_1, \ldots, m_N, \ldots, m_N) \times \\
\times W_{m_0}(x, u) W_a(u, v) W_{m_0}(v, y) dx du dv dy da^2 . \tag{3.12}
\end{equation}

The next step would be the so-called adiabatic limit, and should consist in letting the cut-off $g$ tend to 1 in this formula. It is however easier to set first only one of the $g$'s equal to 1, say $g(u) = 1$ in (3.12). It then becomes possible to perform the integration over $u$ by using the projector identity (2.25) and we find for the transition probability:

\begin{equation}
\Gamma = L_1(f_0, g) \times q! \rho(m_0^2; m_1, \ldots, m_1, \ldots, m_N, \ldots, m_N) , \tag{3.13}
\end{equation}

where

\begin{equation}
L_1(f_0, g) = \frac{\gamma^2 C_1(m_0, d) \int g(v) f_0(x) W_{m_0}(x, v) W_{m_0}(v, y) f_0(y) dx dy dv}{\int f_0(x) W_{m_0}(x, y) f_0(y) dx dy} . \tag{3.14}
\end{equation}

This formula exhibits an interesting factorization: the first factor depends only on the wavepacket $f_0$, the mass $m_0$ of the incoming particle and the switching-off factor $\gamma^2 g$; the adiabatic limit still remains to be done there; the second factor contains all the information about the decay products.

If we now attempt to set $g(v) = 1$ in (3.13) and to integrate over $v$ using again (2.25), the result is proportional to $\delta(m_0^2 - m_c^2)$, i.e. the integral diverges. This difficulty was resolved in the 1930’s by aiming at the average transition probability per unit time (see e.g. [21], pp. 60-62). We first review the well-known Minkowski case, in a form which can serve as a model for the de Sitter case. In fact even this famous old case deserves some re-examination on its own right and it is possible, in this case, to allow the two $g$ in (3.12) to tend to 1 simultaneously, or even at different rates. This is done in Appendix A

It is found that, if both $g$ are taken as in (3.13), the result is the same as found above. But this is not necessarily the case for other $g$. Nevertheless the procedure announced above (i.e. setting the first $g$ in (3.12) to equal to 1, then discussing the time average of the limit as the second $g$ tends to 1) will be used in the de Sitter case, since it gives good results in the Minkowski case, and since calculations in the dS case would become much more difficult otherwise. Note that in the de Sitter case (3.13) and (3.14) are applicable only when $m_0 > m_c$ and the range of integration over $a^2$ in (3.12) contains only values $a^2 > m_c^2$ ($m_c = (d - 1)/2R$). In the case of the decay into two particles of mass $m_1$, it will be seen below that this includes the case $m_1 > m_c$, but also the case $m_c > m_1 > m_c\sqrt{3}/2$.

4 Minkowski case

4.1 Adiabatic limit: the Fermi golden rule

The simplicity of the Minkowskian case arises from being able to use of the Fourier representations:

\begin{equation}
f_0(x) = \int e^{-ipx} f_0(p) dp, \quad g(x) = \int e^{-ipx} g(p) dp, \quad w_m(x, y) = (2\pi)^{1-d} \int e^{ip(y-x)} \delta(p^2 - m^2) \theta(p^0) dp . \tag{4.1}
\end{equation}

Then the factor in (3.14) becomes

\[ L_1(f_0, g) = \frac{(2\pi)^2 \gamma^2}{\tau(f_0)} = \frac{2 \pi \int \bar{f}_0(p) \delta(p^2 - m_0^2) \theta(p^0) \bar{f}_0(q) \delta(q^2 - m_0^2) \theta(q^0) \bar{g}(p - q) \, dp \, dq}{\int |f_0(p)|^2 \delta(p^2 - m_0^2) \theta(p^0) \, dp}. \]  

(4.2)

We now specialize the cut-off \( g \) to depend only on the time coordinate of the chosen frame \( g(v) = h(v^0) = h(t) \), i.e. we think of the interaction as smoothly switched on and then turned off. The Fourier representation is then \( \bar{g}(p) = \bar{h}(p^0) \delta(p) \) and eq. (4.2) becomes

\[ L_1(f_0, g) = \frac{(2\pi)^2 \gamma^2}{\tau(f_0)} \frac{\int |f_0(p)|^2 \delta(p^2 - m_0^2) \theta(p^0) \, dp}{\int |f_0(p)|^2 \delta(p^2 - m_0^2) \theta(p^0) \, dp}. \]  

(4.3)

If we choose for \( g \) the indicator function of a time-slice of thickness \( T \), i.e. \( h(t) = \theta(t + T/2) \theta(T/2 - t) \), \( \bar{h}(0) = T/2\pi \), we get

\[ L_1(f_0, g) = T \times \left( \frac{(2\pi)^2 \gamma^2}{\tau(f_0)} \frac{\int |f_0(p)|^2 \delta(p^2 - m_0^2) \theta(p^0) \, dp}{\int |f_0(p)|^2 \delta(p^2 - m_0^2) \theta(p^0) \, dp} \right). \]  

(4.4)

Therefore, as noted above, removing the cut-off produces infinity. However, according to the Fermi golden rule, what is physically meaningful is not the amplitude but the amplitude per unit time. Therefore, dividing this by \( T \) and taking the limit as \( T \to \infty \) (a particularly trivial operation in this case) we finally get the following expression for the transition probability per unit time:

\[ \frac{1}{\tau(f_0)} = \frac{(2\pi)^2}{\pi} \frac{\int |f_0(p)|^2 \delta(p^2 - m_0^2) \theta(p^0) \, dp}{\int |f_0(p)|^2 \delta(p^2 - m_0^2) \theta(p^0) \, dp} q! \rho(m_0^2; m_1, \ldots, m_1, \ldots, m_N, \ldots, m_N). \]  

(4.5)

The reciprocal of this expression is the lifetime of the 0-particle in the state \( f_0 \). The dependence on the wavepacket \( f_0 \) is a crucial feature of the special relativistic Minkowski case as it will be readily recognized. For instance to compute the lifetime \( \tau_0 \) of a particle at rest in the chosen frame we may let \( |f_0(p)|^2 \) tend to \( \delta(p) \), e.g. by taking

\[ \bar{f}_0(p) = \varepsilon^{(1-d)/2} \bar{\varphi}(\varepsilon) \}, \quad f_0(x) = 2 \pi \delta(x_0) \varepsilon^{(d-1)/2} \varphi(\varepsilon x), \quad \varepsilon > 0, \]  

(4.6)

with \( \varphi \in S(R^{d-1}) \), and letting \( \varepsilon \to 0 \). Then (4.5) tends to

\[ \frac{1}{\tau_0} = \frac{\pi \gamma^2}{m_0} q! \rho(m_0^2; m_1, \ldots, m_1, \ldots, m_N, \ldots, m_N). \]  

(4.7)

We may act with a Lorentz boost on the same particle by replacing in (4.5) the wavepacket \( f_0 \) by \( f_0^\Lambda(x) = f_0(\Lambda^{-1} x) \), \( \Lambda \in L_1 \); the amplitude is modified as follows

\[ \frac{1}{\tau(f_0^\Lambda)} = \frac{(2\pi)^2}{\pi} \frac{\int |f_0(p)|^2 \delta(p^2 - m_0^2) \theta(p^0) \, dp}{\int |f_0(p)|^2 \delta(p^2 - m_0^2) \theta(p^0) \, dp} q! \rho(m_0^2; m_1, \ldots, m_1, \ldots, m_N, \ldots, m_N). \]  

(4.8)

If again \( |f_0(p)|^2 \to \delta(p) \), the final result is the expression in (4.7) multiplied by \( 1/\Lambda_{00} \). If \( \Lambda = \exp(sM_1,0) \), i.e. the particle is moving with velocity \( v = th \), \( \Lambda_{00} = \cosh s = (1 - v^2)^{-1/2} \) gives the usual correction to the lifetime:

\[ \tau_v = \tau_0/\sqrt{1 - v^2}. \]  

(4.9)

**Remark 4.1** It is worthwhile to stress once more that this effect, which expresses the behavior of the life-time of a moving particle in special relativity, crucially depends on the peculiar way in which the wavepacket enters in the transition amplitude per unit time (4.5).
4.2 Källén-Lehmann weights

The weight $\rho$ can be explicitly computed only in the case of one particle decaying into two particles. For a particle of mass $m_0 > 0$ decaying into two identical particles of mass $m_1 > 0$, i.e. the case $N = 1$, $q_1 = 2$, the well-known formula is

$$\rho(m_0^2; m_1, m_1) = \frac{(m_0^2 - 4m_1^2)^{\frac{d-3}{2}}}{(4\pi)^{\frac{d-1}{2}} 2^{d-2} \Gamma \left(\frac{d-1}{2}\right)} \frac{\theta(m_0^2 - 4m_1^2)}{m_0}, \quad (4.10)$$

and (4.7) becomes

$$\frac{1}{\tau_0} = \text{life time of } m_0 \text{)}^{-1} = \frac{\pi \gamma^2 (m_0^2 - 4m_1^2)^{\frac{d-3}{2}}}{(4\pi)^{\frac{d-1}{2}} 2^{d-3} \Gamma \left(\frac{d-1}{2}\right)} \frac{\theta(m_0^2 - 4m_1^2)}{m_0^2}. \quad (4.11)$$

For $d = 4$ this is

$$\frac{1}{\tau_0} = \frac{\gamma^2 (m_0^2 - 4m_1^2)^{\frac{1}{2}}}{8\pi m_0^2} \theta(m_0^2 - 4m_1^2), \quad (4.12)$$

in agreement with the computation in e.g. [21].
5 de Sitter case

5.1 Adiabatic limit in the de Sitter case

The discussion of the adiabatic limit is more complicated in the de Sitter case. Taking the adiabatic limit is of course technically much more involved than in the Minkowski case (and we will relegate all the technical details to the appendices). But the really intricate and maybe perplexing issue is the physical interpretation of the whole procedure and, even more, of the somewhat surprising results.

Having in mind the Minkowskian case that we have just discussed, the first question that should be asked is what is "time" in the de Sitter universe and what does it means that an interaction lasts for a certain time. In the Minkowski case we have the solid foundation of special relativity and a privileged class of frames, the inertial frames, each of them having an inherent precise notion of time.

In the de Sitter case (and the situation is even worse in a general curved spacetime) we have no such thing. Instead we have many possible coordinate systems, that may or may not cover the whole manifold, and many possible choices of temporal coordinates that have no special relation to each other.

For example, the de Sitter universe is the only known spacetime manifold admitting three different inequivalent choices of cosmic time so that the de Sitter metric takes the appearance of a, respectively, closed, flat, or open Friedmann-Robertson-Walker universe. But there are also other possibilities. The choice of time coordinate made in 1917 by de Sitter in his original papers [3, 4] describes a wedge-like region of the de Sitter manifold as a static spacetime with bifurcate Killing horizons [22].

We choose to proceed heuristically in analogy with the Minkowskian case. Concretely, we will work out the adiabatic limit using two of the three possible cosmological coordinate systems, namely the closed and the flat systems. Starting again from eq. (3.14) we take the cutoff $g$ appearing in there as the indicator (or characteristic) function of some “cosmic time-slice” of thickness $T$ w.r.t. to the relevant choice of cosmic time.

We will see that in both the closed and flat case the amplitude diverges linearly in $T$ precisely as in the Minkowskian case. Therefore, to extract a finite limit we are entitled (and have no other choice than) to use the Fermi golden rule and compute in the above two frames the probability per unit time by dividing by $T$; there is at this point a small difference w.r.t. the flat case: the amplitude per unit time at finite $T$ depends on $T$. However, letting $T \to \infty$ gives a well-defined limit which exhibits a much more disturbing difference with the Minkowskian case.

Closed FRW model: The relevant coordinate system is the following:

$$x(t, \vec{u}) = \begin{cases} x^0 &= R \sinh(t/R), \\ x^i &= R \cosh(t/R) \ (-\vec{u}), \quad \vec{u} \in S^{d-1}, \end{cases}$$

(5.1)

(the minus sign at rhs is for further convenience). In this coordinate system the constant time slices are hyperspheres. These coordinates have the advantage to globally cover the de Sitter manifold; they gives to the metric the form of a closed FRW model with scale factor $a(t) = \cosh(t/R)$:

$$ds^2 = dt^2 - R^2 \cosh^2(t/R) d\sigma^2(\vec{u}) = dt^2 - R^2 \cosh^2(t/R) (d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)) \quad (d = 4).$$

(5.2)

In (5.2) $d\sigma^2(\vec{u})$ is the square line element on $S^{d-1}$ at $\vec{u}$, and (5.3) includes its expression in Euler angles. In these coordinates, we choose

$$g(x) = g_T(x) = \theta(t + T/2) \theta(T/2 - t).$$

(5.4)
**Flat FRW model:** These are the coordinates currently used in the context of inflationary models. Hypersurfaces of constant time are flat:

\[
x(t, y) = \begin{cases} 
  x^0 = R \sinh \frac{t}{R} + \frac{1}{2R} e^{\pi y^2}, \\
  x^j = e^{\frac{t}{R}} y^j, \quad 1 \leq j \leq d - 1, \\
  x^d = R \cosh \frac{t}{R} - \frac{1}{2R} e^{\pi y^2}, 
\end{cases}
\]  
\[y \in \mathbb{R}^{d-1}, \quad (5.5)\]
\[
ds^2 = dt^2 - e^{2t/R} \left( d y_1^2 + \ldots + d y_{d-1}^2 \right), \quad (5.6)
\]

In these coordinates we choose

\[
g(x) = g_T(x) = \theta(t + T/2) \theta(T/2 - t). \quad (5.7)
\]

But the coordinates (5.5) only cover one half of \(X_d\), the region where \(x^0 + x^d > 0\), and the adiabatic limit will have to include the contribution of the other half not covered by the coordinate system.

It turns out that the limit

\[
\frac{1}{\tau} = \left( \lim_{T \to \infty} \frac{L_1(f_0, g)}{T} \right) \times q! \rho(m_0^2; m_1, \ldots, m_1, \ldots, m_N, \ldots, m_N) \quad (5.8)
\]

exists and is the same for both kinds of slices. The calculations are tedious and not quite straightforward, and will be given in Appendices B and C. For the spherical slices of the closed FRW system, only the calculations for \(d = 2, 3, 4\) have been carried out. The method for the flat FRW coordinates works for all \(d\).

The inverse lifetime that results is

\[
\Gamma_{1, q_1, \ldots, q_N}(f_0) = \frac{\gamma^2 \pi \coth(\pi \kappa)^2 R}{|\kappa|} \times q! \rho(m_0^2; m_1, \ldots, m_1, \ldots, m_N, \ldots, m_N) \quad (5.9)
\]

where we denoted \(\kappa = R(m_0^2 - m_c^2)^{1/2}\). Note the similarity of this formula with (4.7) and in fact the first factor in (5.9) tends to the corresponding factor in (4.7) when \(R \to \infty\) at fixed \(m_0\). However there is a most striking difference with (4.5): the rhs of (5.9) does not depend on \(f_0\), the initial wave function of the decaying particle. Therefore in particular the lifetime of a particle does not depend on its velocity. We will comment on this feature, at first sight embarrassing, in the conclusions.
6 Källén-Lehmann weights

As in the flat case, an explicit computation of the Källén-Lehmann weight is only possible for decays of one particle into two. Here the discussion will be restricted to the case of a particle of mass \( m_0 > 0 \) decaying into two particles of equal masses \( m_1 = m_2 > 0 \) and we suppose at the beginning \( m_1 > m_0 \).

The more difficult case \( m_1 \neq m_2 \) will be treated in a paper in preparation \[23\]. We shall find an explicit \( \rho(a^2; m_1, m_1) \) such that

\[
W_{\rho}(z, z') = \int \rho(a^2; m_1, m_1)W_a(z, z') da^2 . \tag{6.1}
\]

We change to variables \( \kappa = [a^2 R^2 - (d-1)/2]^1/2 \), \( \nu = [m_1^2 R^2 - (d-1)/2]^1/2 \), and (by abuse of notation) seek a function \( \rho(\kappa; \nu, \nu) \) (mostly abbreviated as \( \rho(\kappa) \)) such that

\[
W_{\nu}(z, z') = \int_0^\infty 2\kappa \rho(\kappa; \nu, \nu) W_{\kappa}(z, z') d\kappa . \tag{6.2}
\]

By \[2.16\], this is equivalent to

\[
C_{d,\nu}^2 (x^2 - 1)^{-d-2} P_{-\frac{d-2}{2} + i\nu}(x)^2 = \int_0^\infty 2C_{d,\kappa} \rho(\kappa) P_{-\frac{d-2}{2} + i\kappa}(x) d\kappa , \tag{6.3}
\]

with

\[
C_{d,\nu} = \frac{\Gamma \left( \frac{d-1}{2} + i\nu \right) \Gamma \left( \frac{d-1}{2} - i\nu \right)}{2(2\pi)^{\frac{d}{2}} R^{d-2}} . \tag{6.4}
\]

The generalized Mehler-Fock theorem \[24\], p. 398 asserts that

\[
g(x) = \int_0^\infty P_{-\frac{d-2}{2} + i\kappa}(x)f(\kappa) d\kappa \iff f(\kappa) = \frac{\kappa}{\pi} \sinh(\pi\kappa) \Gamma \left( \frac{1}{2} - \sigma + i\kappa \right) \Gamma \left( \frac{1}{2} - \sigma - i\kappa \right) \int_1^\infty P_{-\frac{d-2}{2} + i\nu}(x)g(x) dx . \tag{6.5}
\]

Therefore \[6.3\] implies

\[
\rho(\kappa) = \frac{C_{d,\nu}^2 \kappa \sinh(\pi\kappa) \Gamma \left( \frac{d-1}{2} + i\kappa \right) \Gamma \left( \frac{d-1}{2} - i\kappa \right)}{2C_{d,\kappa} \pi} \times h_d(\kappa, \nu, \nu) . \tag{6.6}
\]

\[
h_d(\kappa, \nu, \nu) \overset{\text{def}}{=} \int_1^\infty (x^2 - 1)^{-d-2} \left[ P_{-\frac{d-2}{2} + i\kappa}(x) \right]^2 P_{-\frac{d-2}{2} + i\nu}(x) dx . \tag{6.7}
\]

It is possible to obtain an explicit expression of \( h_d(\kappa, \nu, \nu) \) by using Mellin transform techniques (see \[25\]) and a lemma of Barnes (see \[26\]). Recall that if \( \varphi \in D((0, \infty)) \), its Mellin transform \( \hat{\varphi} \) is given by

\[
\hat{\varphi}(s) = \int_0^\infty \zeta^{s-1} \varphi(\zeta) d\zeta , \tag{6.8}
\]

It is entire in \( s = \sigma + i\tau \), decreasing faster than any negative power of \( \tau \) for fixed \( \sigma \), and

\[
\varphi(\zeta) = \frac{1}{2i\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} \zeta^{-s} \hat{\varphi}(s) ds \quad \forall \sigma \in \mathbb{R} , \tag{6.9}
\]

\[
\psi(\zeta) = \varphi(1/\zeta) \iff \hat{\psi}(s) = \hat{\varphi}(-s) . \tag{6.10}
\]

If \( \varphi, \varphi_1, \varphi_2 \) are in \( D((0, \infty)) \),

\[
\varphi(\zeta) = \int_0^\infty \varphi_1(\zeta/u) \varphi_2(u) \frac{du}{u} \iff \hat{\varphi}(s) = \hat{\varphi}_1(s)\hat{\varphi}_2(s) . \tag{6.11}
\]
In particular (Mellin-Plancherel identity)
\[
\int_0^\infty \varphi_1(u) \varphi_2(u) \frac{du}{u} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \hat{\varphi}_1(-s) \hat{\varphi}_2(s) \, ds .
\]
These properties can be extended to other functions and generalized functions (see [23]), and, in many interesting cases, although the Mellin transforms are no longer entire, the above formulae survive provided the integration in (6.9) or (6.12) is performed on a suitable contour.

By the change of variable \( x = \sqrt{1+\zeta} \) in (6.7) we find
\[
h_d(\kappa, \nu, \nu) = \int_0^\infty G_1(\zeta) G_2(\zeta) \frac{d\zeta}{\zeta} ,
\]
with
\[
G_1(\zeta) = P^\mu(\sqrt{1+\zeta}) \quad \text{and} \quad G_2(\zeta) = \zeta^\frac{d}{2} \left[ P^\mu(\sqrt{1+\zeta}) \right]^2 .
\]
and
\[
\alpha = -\frac{d}{2} + i\kappa, \quad \beta = -\frac{d}{2} + i\nu, \quad \mu = 1 - \frac{d}{2} .
\]
The Mellin transforms of \( G_1 \) and \( G_2 \) are known (see [25], 17(1) p. 257, and 28(1) p. 263.)
\[
\hat{G}_1(s) = \frac{2^\mu-1}{\Gamma(1+\mu)\Gamma(\alpha-\mu/2)} \Gamma\left[ \frac{s}{2} - \frac{\kappa}{2}, 1 + \frac{\alpha}{2} - s, 1 - \frac{\alpha}{2} - s \right] \quad \text{provided} \quad \text{Re} \mu < 2 \text{Re} \alpha < \min\{2 + \text{Re} \alpha, 1 - \text{Re} \alpha\} ,
\]
\[
\hat{G}_2(s) = \frac{1}{\pi^s \Gamma(1+\beta-\mu)\Gamma(-\beta-\mu)} \Gamma\left[ 1 - \frac{\beta}{2} + s, 1 - \frac{\beta}{2} - s, -\frac{\beta}{2} - s \right] \quad \text{provided} \quad \text{Re} \mu < \text{Re}(s+1-\mu/2) < \min\{-\text{Re} \beta, \frac{d}{2}\} , \quad \mu \notin \mathbb{N} .
\]

Remark 6.1 We have actually checked the above formulae using the methods described in [25]. However some other formulae appearing in that extremely useful reference have misprints.

By the Mellin-Plancherel theorem
\[
h_d(\kappa, \nu, \nu) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \hat{G}_1(s) \hat{G}_2(-s) \, ds ,
\]
and the preceding formulae give
\[
\hat{G}_1(s) \hat{G}_2(-s) = \frac{1}{2^\mu \sqrt{\pi} \Gamma(\mu-\frac{d}{2})} \Gamma\left[ \frac{d-4}{3d-6} + s, \frac{d-4}{3d-6} - i\nu + s, \frac{d-4}{3d-6} + i\nu + s, \frac{d-4}{3d-6} + s, \frac{d-4}{3d-6} - s \right] \times \Gamma\left[ \frac{d-4}{3d-6} + s, \frac{d-4}{3d-6} - i\nu + s, \frac{d-4}{3d-6} + i\nu + s, \frac{d-4}{3d-6} + s, \frac{d-4}{3d-6} - s \right] .
\]

It is now possible to use Barnes’ Second Lemma [26] p. 112) :

Lemma 6.1 (Barnes)
\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma\left[ a_1 + s, a_2 + s, a_3 + s, b_1 - s, b_2 - s \right] \, ds = \Gamma\left[ a_1 + b_1, a_2 + b_1, a_3 + b_1, a_1 + b_2, a_2 + b_2, a_3 + b_2 \right] ,
\]
provided
\[
a_1 + a_2 + a_3 + b_1 + b_2 - c = 0
\]
and that the contour of integration in (6.20) separates the increasing and decreasing series of poles.
As a function of $s$, $\hat{G}_1(s)\hat{G}_2(-s)$, as given by (6.19), is, up to a factor, of the form of the integrand of (6.20) if we take

$$a_1 = \frac{d-4}{4} + iq, \quad a_2 = \frac{d-4}{4} - iq, \quad a_3 = \frac{d-4}{4}, \quad b_1 = \frac{3}{4} + i\kappa, \quad b_2 = \frac{3}{4} - i\kappa, \quad c = \frac{3d-6}{4}. \quad (6.22)$$

This choice satisfies the condition (6.21). Therefore

$$h_d(\kappa; \nu, \nu) = \frac{1}{2^{d-1} \sqrt{\pi}} \frac{1}{\Gamma \left[ \frac{d+1}{2} + i\kappa, \frac{d+1}{2} - i\kappa, \frac{d-1}{2} + i\nu, \frac{d-1}{2} - i\nu \right]} \times \Gamma \left[ \frac{d+1}{2} - i\nu, \frac{d+1}{2} + i\nu, \frac{d-1}{2} - i\nu, \frac{d-1}{2} + i\nu \right]. \quad (6.23)$$

From here on, we will use the notation $\mu = (d-1)/4$.

We can recast the above expression for $h_d(\kappa; \nu, \nu)$ by using Legendre’s duplication formula:

$$h_d(\kappa; \nu, \nu) = \frac{1}{2^{d-1} \sqrt{\pi}} \frac{\kappa \sinh(\pi\kappa)}{\Gamma \left( \mu + i\kappa \right) \Gamma \left( \mu - i\kappa \right) \Gamma \left( \mu + i\nu \right) \Gamma \left( \mu - i\nu \right) \Gamma \left( \mu + i\nu \right)^2 \Gamma \left( \mu - i\nu \right)^2} \times \Gamma \left( \mu - i\kappa \right) \times \Gamma \left( \mu - i\kappa \right)^2, \quad \mu = \frac{d-1}{4}. \quad (6.24)$$

In this form the formula is a special case of the formula for two unequal masses which will appear in (6.23). We note that in the derivation of (6.23) or (6.24) with $h_d$ defined in (6.7), $d$ is not restricted to be an integer. These formulæ hold wherever both sides are defined. Eqs. (6.23) and (6.24) give

$$\kappa \rho(\kappa; \nu, \nu) = \frac{\kappa \sinh(\pi\kappa)}{2^{d+2} \pi^{d-2} \Gamma \left( \frac{1}{2} + \mu + i\kappa \right) \Gamma \left( \frac{1}{2} - \mu + i\kappa \right) \Gamma \left( \frac{1}{2} + \mu - i\kappa \right) \Gamma \left( \frac{1}{2} - \mu - i\kappa \right) \Gamma \left( 2\mu \right) \times \Gamma \left( \mu + i\kappa \right) \Gamma \left( \mu - i\kappa \right) \Gamma \left( \mu + i\nu \right) \Gamma \left( \mu - i\nu \right) \Gamma \left( \mu + i\nu \right)^2 \Gamma \left( \mu - i\nu \right)^2} \times \prod_{\epsilon = \pm 1} \Gamma \left( \mu + i\kappa \right) \prod_{\epsilon' = \pm 1} \Gamma \left( \mu + i\kappa \right)^2 \times (6.25)$$

or, using $\kappa \sinh(\pi\kappa) = \pi \Gamma \left( i\kappa \right) \Gamma \left( -i\kappa \right)^{-1}$,

$$\kappa \rho(\kappa; \nu, \nu) = \frac{1}{2^{d+2} \pi^{d-2} \Gamma \left( \frac{1}{2} + \mu + i\kappa \right) \Gamma \left( \frac{1}{2} + \mu - i\kappa \right) \Gamma \left( \frac{1}{2} - \mu + i\kappa \right) \Gamma \left( \frac{1}{2} - \mu - i\kappa \right) \Gamma \left( 2\mu \right) \times \prod_{\epsilon = \pm 1} \Gamma \left( \mu + i\kappa \right) \prod_{\epsilon' = \pm 1} \Gamma \left( \mu + i\kappa \right)^2 \Gamma \left( \mu + i\nu \right) \Gamma \left( \mu - i\nu \right) \Gamma \left( \mu + i\nu \right)^2 \Gamma \left( \mu - i\nu \right)^2} \times (6.26)$$

This obviously extends to an even analytic function of $\kappa$, hence

$$w_\nu(z, z')^2 = \int_{-\infty}^{\infty} \kappa \rho(\kappa; \nu, \nu) w_\kappa(z, z') \, d\kappa. \quad (6.27)$$

Moreover, for real $\nu$ and $\kappa \neq 0$, $\kappa \rho(\kappa; \nu, \nu)$ is strictly positive. This shows that, in the presence of a suitable interaction term (see (3.5)), any “principal” particle can decay into any pair of equal-mass “principal” particles.

## 6.1 Minkowskian limit

Setting $\kappa = MR > 0$ and $\nu = MR > 0$ in (6.25) gives

$$\rho(MR; MR, MR) = \frac{\sinh(\pi MR)}{2^{d+2} \pi^{d-2} \Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d+1}{2} \right)} \left| \prod_{j=1}^{3} \Gamma(x_j + iRu_j) \right|^2, \quad (6.28)$$
with
\[ x_1 = x_2 = x_3 = \frac{d - 1}{4}, \quad x_4 = \frac{d + 1}{4}, \]
\[ u_1 = \frac{M}{2} + m, \quad u_2 = \frac{M}{2} - m, \quad u_3 = \frac{M}{2}, \quad u_4 = \frac{M}{2}. \]  
(6.29)

Recall Stirling’s formula \( [27], \text{p. } 47): \)
\[ \Gamma(z) = (2\pi)^{\frac{1}{2}}e^{-\frac{1}{2}z} \left( 1 + \frac{1}{12}z + O(z^{-3}) \right), \quad a_1 = 1/12, \quad a_2 = 1/288, \]  
(6.30)
valid for \( z \not\in \mathbb{R}_- \). By a straightforward calculation it follows that if \( z = x + iy \) and \( |x| \) remains bounded while \( |y| \to +\infty, \)
\[ |\Gamma(x + iy)|^2 \sim 2\pi e^{-\pi|y|} |y|^{2x-1} \left( 1 + \frac{(x - \frac{i}{2})^2 + 2(a_1 x - a_2) + a_1^2}{y^2} \right). \]  
(6.31)

Using this in (6.28), we find that as \( R \to \infty, \)
\[ R^2 \rho(RM; Rm, Rm) \sim \frac{\exp \pi R (\frac{M}{2} - m - |\frac{M}{2} - m|)}{2^{d-1} \pi^{\frac{d+1}{2}} \Gamma \left( \frac{d-1}{2} \right) M^d} \left( \frac{M^2 - 4m^2}{4} \right)^{\frac{d-3}{2}} (1 + AR^{-2}), \]  
(6.32)
where
\[ A = \sum_{j=1}^{3} \frac{(x_j - \frac{i}{2})^2 + 2(a_1 x_j - a_2) + a_1^2}{u_j^2} - \frac{(x_4 - \frac{i}{2})^2 + 2(a_1 x_4 - a_2) + a_1^2}{u_4^2} = \frac{17}{16} \left( \frac{1}{(M + 2m)^2} + \frac{1}{(M - 2m)^2} \right) - \frac{107}{24M^2} \text{ for } d = 4. \]  
(6.33)

Note that the argument of the exponential in (6.32) is 0 if \( M - 2m \geq 0 \), otherwise \( -\pi R(2m - M) \) and, in this case, \( R^2 \rho(RM; Rm, Rm) \) tends rapidly to 0. In all cases, (6.32) shows that \( R^2 \rho(RM; Rm, Rm) \) tends to \( \rho^{\text{Minkowski}}(M^2; m, m) \) (see \( [10] \)).

### 6.2 Complementary particles

One benefit of having the explicit formula (6.25) is being able to examine the case of “complementary” particles. The integrand of (6.27) is meromorphic in \( \kappa \) and \( \nu \). We can rewrite (6.27) as
\[ w_\nu(z, z')^2 = \int_R \frac{\kappa}{2^{d+5} \pi^{d+4} \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d-1}{2} \right) R^{2d-4}} F \left( \frac{d - 1}{2} + i\kappa, \frac{d - 1}{2} - i\kappa; \frac{d}{2}; \frac{1 - i}{2} \right) \times \Gamma \left( \mu + \frac{i\kappa}{2} \right)^2 \Gamma \left( \mu - \frac{i\kappa}{2} \right)^2 \prod_{\epsilon, \epsilon' = \pm 1} \Gamma \left( \mu + \frac{i\kappa}{2} + \frac{i\epsilon}{2} \right) d\kappa, \quad \mu = \frac{d - 1}{4}. \]  
(6.34)

The integrand is meromorphic in \( \kappa \) and \( \nu \). It has no singularity when both are real. The lhs is holomorphic in \( (\nu : \nu \not\in \pm i((d - 1)/2 + \mathbb{Z}_+)) \). We analytically continue the integral in the variable \( \nu \): choose \( \nu \) complex with \( \text{Re } \nu > 0 \) and \( \alpha = \text{Im } \nu > 0 \). Recall that, for integer \( n \geq 0, \)
\[ z + n \sim 0 \quad \Rightarrow \quad \Gamma(z) \sim \frac{(-1)^n}{n!(z + n)}. \]  
(6.35)

The poles of the functions \( \kappa \mapsto \Gamma(\mu \pm i\kappa/2) \) are at \( \kappa = \pm 2 \nu (\mu \pm n) \) \( (n \geq 0 \text{ integer}) \), and are independent of \( \nu \). The other poles of the integrand are as follows \( (n \geq 0 \text{ integer}):\)
\[ \frac{i\kappa}{2} + \mu \pm i\nu + n \sim 0 \quad \Rightarrow \quad \Gamma \left( \frac{i\kappa}{2} + \mu \pm i\nu \right) \sim \frac{(-1)^n}{2^n n!(\kappa - 2i(\mu \pm i\nu + n))}. \]  
(6.36)
\[ \frac{-i\kappa}{2} + \mu \pm iv + n \sim 0 \Rightarrow \Gamma \left( -\frac{i\kappa}{2} + \mu \pm iv \right) \sim \frac{(-1)^n}{-\pi n!(\kappa + 2i(\mu \pm iv + n))}. \] (6.37)

The poles \( \kappa - 2i(\mu + iv + n) = 0 \) (see (6.33)) and the poles \( \kappa + 2i(\mu - iv + n) = 0 \) (see (6.37)) are on the line \(-2\Re\nu + i\Re\). Their mutual distances do not change as \( \nu \) varies, and they all move down as \( \Im\nu \) increases. The poles \( \kappa + 2i(\mu + iv + n) = 0 \) (see (6.37)) and \( \kappa - 2i(\mu - iv + n) = 0 \) (see (6.36)) are the opposites of those described before. They lie on \( 2\Re\nu + i\Re \) and move up as \( \Im\nu \) increases.

If \( \Im\nu \) increases from 0 but \( 0 < \Im\nu < \mu \), no pole reaches the real axis and the formula (6.27) continues to hold. This is true in particular if \( \mu = i\alpha \) with \( 0 < \alpha < (d - 1)/4 = m_c/2 \), corresponding to \( m_c > m_1 > m_c\sqrt{3}/2 \). If this condition is satisfied and \( m_0 > m_c \), eqs. (6.12) and (6.14) hold and the adiabatic limit exists just as in the case \( m_1 > m_c \).

When \( \Im\nu \) reaches \( \mu \) we have
\[ w_\nu(z, z')^2 = \int_C \kappa \rho(\kappa, \nu, \nu) w_\kappa(z, z') d\kappa, \] (6.38)
where the contour \( C \) is obtained from \( \Re \) by a small downward excursion to avoid the pole at \(-2\Re\nu\), and another small upward excursion to avoid the pole at \( 2\Re\nu\). Once \( \mu < \Im\nu < \mu + 1 \), we can extract the residues of the poles at \( \kappa = \pm 2i(\mu + iv + n) \). A similar situation occurs when the successive poles \( \kappa = \pm 2i(\mu + iv + n) \) cross the real axis, so that, for \( \Re\nu > 0, \Im\nu \geq 0, \Im\nu - \mu \notin \mathbb{Z}, N = \max \{j \in \mathbb{Z} : j < \Im\nu - \mu\} \),
\[ w_\nu(z, z')^2 = \int_{\Re} \kappa \rho(\kappa, \nu, \nu) w_\kappa(z, z') d\kappa + \sum_{n=0}^{N} \left[ A_n(\nu) \frac{w_{2i(\mu + iv + n)}(z, z')}{2} + \frac{A_n(\nu)}{2} w_{-2i(\mu + iv + n)}(z, z') \right]. \] (6.39)

Note that if \( \Im\nu < \mu \) (\( N < 0 \)), the discrete sum is not present. It turns out that \( A'_n(\nu) = A_n(\nu) \), which is consistent with \( \kappa \rho \) being even in \( \kappa \). Recall that \( w_\nu = w_{-\nu} \) for any \( \nu \). Thus, for \( N = \max \{j \in \mathbb{Z} : j < \Im\nu - \mu\} \), (always supposing \( \Re\nu > 0 \)),
\[ w_\nu(z, z')^2 = \int_{\Re} \kappa \rho(\kappa, \nu, \nu) w_\kappa(z, z') d\kappa + \sum_{n=0}^{N} A_n(\nu) w_{2i(\mu + iv + n)}(z, z'). \] (6.40)

We find, for integer \( n \geq 0 \),
\[ A_n(\nu) = \frac{(-1)^n}{n!2^{d-1}\pi^{\frac{d-2}{2}}R^{d-2}\Gamma(2\mu)} \times \frac{\Gamma(2\mu + 2iv + n)\Gamma(-2iv - n)\Gamma(2\mu + n)\Gamma(-iv - n)\Gamma(2\mu + iv + n)}{\Gamma(-2\mu - 2iv - 2n)\Gamma(2\mu + 2iv + 2n)\Gamma(\frac{1}{2} - iv - n)\Gamma(\frac{1}{2} + 2\mu + iv + n)}. \] (6.41)

If now we let \( \nu \) tend to \( i\alpha \) (\( \Re\nu \) tends to 0), (6.41) will continue to hold provided both parts of the rhs remain meaningful. Therefore, if \( 0 < \alpha < (d - 1)/2, \alpha - \mu \notin \mathbb{Z} \), and \( N = \max \{j \in \mathbb{Z} : j < \alpha - \mu\} \), \( \mu = (d - 1)/4 \),
\[ w_{i\alpha}(z, z')^2 = \int_{\Re} \kappa \rho(\kappa; i\alpha, i\alpha) w_\kappa(z, z') d\kappa + \sum_{n=0}^{N} A_n(i\alpha) w_{2i(\mu - \alpha - n)}(z, z'). \] (6.42)

\[ \kappa \rho(\kappa; i\alpha, i\alpha) = \frac{1}{2^{d+2}\pi^{\frac{d+2}{2}}R^{d-2}\Gamma(2\mu)} \times \frac{\Gamma(\mu + \frac{d}{2} - \alpha)\Gamma(\mu - \frac{d}{2} - \alpha)\Gamma(\mu + \frac{d}{2} + \alpha)\Gamma(\mu - \frac{d}{2} + \alpha)\Gamma(\mu + i\alpha)\Gamma(\mu - i\alpha)}{\Gamma(i\kappa)\Gamma(-i\kappa)\Gamma(\mu + \frac{i\alpha}{2})\Gamma(\mu + \frac{i\alpha}{2})}. \] (6.43)
This is obviously positive. For $A_n(i\alpha)$ we find

$$A_n(i\alpha) = \frac{1}{n!} \frac{\Gamma(2\alpha - n)\Gamma(2\mu + n)\Gamma(\alpha - n)\Gamma(2\mu - \alpha + n)}{R^{d-2}\Gamma(2\mu)\Gamma(2\alpha - 2\mu - 2n)\Gamma(\frac{d}{2} - \alpha - n)\Gamma(\frac{d}{2} + 2\mu - \alpha + n)} \times (-1)^n \frac{\Gamma(2\mu - 2\alpha + n)}{\Gamma(2\mu - 2\alpha - 2n)} \times$$

$$\times (-1)^n \frac{\Gamma(n + x)}{\Gamma(2n + x)} = (-1)^n \prod_{q} (q + x)^{-1}. \quad (6.45)$$

The last product contains $n$ negative factors and the result is positive, so that $A_n(i\alpha) \geq 0$. Thus the Hilbert space with scalar product given by the lhs of (6.42) appears as a direct integral of Hilbert spaces associated with unitary irreducible representations of $G_0$.

We conclude that

1. Any particle from the principal series can decay into two particles (of equal masses) of any series.
2. A particle of the complementary series with parameter $\kappa = i\beta$, with $0 < \beta < 2\mu$ can decay into two particles with parameter $i\alpha$, $\alpha = \frac{1}{2}\beta + \mu + n$, where $n$ is any integer such that $0 \leq n$ and $\alpha < 2\mu$, i.e. $n < \mu - \beta/2$. This relation can also be written as

$$(2\mu - \beta) = 2(2\mu - \alpha) + 2n < 2\mu. \quad (6.46)$$

This implies a form of particle stability, but the new phenomenon is that a particle of this kind cannot disintegrate unless the masses of the decay products have certain quantized values. Stability for the same range of masses has also been recently found [28] in a completely different context.

7 Concluding remarks

In trying to interpret the results concerning the lack of mass subadditivity in the de Sitter universe, one can wonder whether they might be due to the thermodynamical properties [12-15] of the fundamental state we have been using. We have tested this possibility against a similar computation in flat thermal field theory that however does not exhibit this phenomenon in two-particle decays. Another issue has to do with energy conservation and the relation mass/energy. dS invariant field theories admit ten conserved quantities (in $d = 4$). The identification of a conserved energy among these quantities has proven to be useful in classical field theory [29]. The same quantity remains exactly conserved also at the quantum level although it becomes an operator whose spectrum is not positive [13-15] even when restricted to the region where the corresponding classical expression is positive [29]; the thermodynamical properties of dS fields arise precisely in this restriction [12-15]. Energy is conserved also in the decay processes that violate mass subadditivity, once the adiabatic limit has been performed. The breakdown of the subadditivity property of masses in dS spacetime just reflects the nonexistence of an Abelian translation group and thereby of a linear energy-momentum space.

When we consider the adiabatic limit problem and its meaning in the de Sitter context a first complication is the existence of several choices of cosmic time, having different physical implications and the result might depend on one’s preferred choice. We have studied the closed and the flat cosmological and found that in both models the first factor in (3.13) diverges like $T$; thus it has to be divided by $T$ to extract a finite result which is the same in both models.

Here the second (unforeseen) result comes in: in contrast to the Minkowskian case the limiting probability per unit of time does not depend on the wavepacket! This result seems to contradict what we see
everyday in laboratory experiments, a well known effect of special relativity (Eq. [4.9]). Furthermore, in contrast with the violation of particle stability that is exponentially small in the de Sitter radius, this phenomenon does not depend on how small is the cosmological constant. How can we solve this paradox and reconcile the result with everyday experience? The point is that the idea of probability per unit time (Fermi’s golden rule) has no scale-invariant meaning in de Sitter: if we use the limiting probability to evaluate amplitudes of processes that take place in a short time we get a grossly wrong result. This is in strong disagreement with what happens in the Minkowski case where the limiting probability is attained almost immediately (i.e. already for finite $T$). Therefore to describe what we are really doing in a laboratory we should not take the limit $T \to \infty$ and rather use the probability per unit of time relative to a laboratory consistent scale of time. In that case we will recover all the standard wisdom even in presence of a cosmological constant. But, if an unstable particle lives a very long time ($\gg R$) and we can accumulate observations then a nonvanishing cosmological constant would radically modify the Minkowski result and de Sitter invariant result will emerge. This result should not be shocking: after all erasing any inhomogeneity is precisely what the quasi de Sitter phase is supposed to do at the epoch of inflation; in the same way, from the viewpoint of an accelerating universe all the long-lived particles look as if they were at rest and so their lifetime would not depend on their peculiar motion.

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A Appendix. More details in the Minkowski case

In this appendix we study in more detail the adiabatic limit in the Minkowski case: it is possible to let the two occurrences of $g$ in (A.12) tend to 1 together, or even at different rates. Let

$$ U(f_0, \varphi_1, \varphi_2, \rho) = \int f_0(x) f_0(y) \varphi_1(u) \varphi_2(v) \rho(\sigma^2) w_m(x, u) w_n(u, v) w_m(v, y) \, dx \, dv \, dy \, d\sigma^2 . $$

We will assume that $\rho$ is $C^\infty$ and has support in $c^2 + R_+$, with $0 < c < m_0$, and that, for each integer $n \geq 0$, there are constants $C_n \geq 0$ and $L_n \geq 0$, such that, for all real $t \geq c^2$,

$$ |\rho^{(n)}(t)| \leq C_n (1 + |t|)^{L_n} . $$

We take

$$ \varphi_j(x) = \int_{\mathbb{R}^d} e^{-ip x} \tilde{\varphi}_j(p) \, dp, \quad j = 1, 2, $$

$$ \tilde{\varphi}_j(p) = \varepsilon_j^{-1} \tilde{g}_j(p_0/\varepsilon_j) \tilde{\psi}(p_0) \delta(p), \quad \varphi_j(x) = \frac{1}{2\pi} \int_{\mathbb{R}} g_j(\varepsilon_j(x - t)) \psi(t) \, dt, \quad (j = 1, 2), $$

$$ g_j(t) = \int_{\mathbb{R}} e^{-itw} \tilde{g}_j(w) \, dw, \quad \psi(t) = \int_{\mathbb{R}} e^{-itw} \tilde{\psi}(w) \, dw. $$

Here $\varepsilon_j = T_j^{-1} > 0$. The function $\psi$ belongs to $\mathcal{S}(\mathbb{R})$ with $\tilde{\psi}(0) = 1$. The function $g_j$ is $L^\infty$ with compact support. (The cases of real interest are $g_j(t) = \theta(1/2 - |t|)$ or $g_j(t) = \theta(t)\theta(1 - t)$.) We find, after using the various delta-functions,

$$ U(f_0, \varphi_1, \varphi_2, \rho) = (2\pi)^{d+3} \varepsilon_1^{-1} \varepsilon_2^{-1} \int_{\mathbb{R}^d} (2p_0 - 1) \tilde{f}_0(p)^2 \tilde{g}_1 \left( \frac{p_0 - \varepsilon_1}{\varepsilon_1} \right) \tilde{g}_2 \left( \frac{\varepsilon_1}{\varepsilon_2} \right) \delta(p_0 - h_0) \tilde{\psi}(h_0 - p_0) \delta(p_0 - \varepsilon_1) \tilde{\psi}(\varepsilon_1 - p_0) \, dp_0 \, dh_0 . $$

We now change from the variable $h_0$ to the variable $w$ such that $h_0 = p_0 + w$:

$$ \varepsilon_2 U(f_0, \varphi_1, \varphi_2, \rho) = \varepsilon_1^{-1} \int H(w) \tilde{g}_1 \left( -\frac{w}{\varepsilon_1} \right) \tilde{g}_2 \left( \frac{w}{\varepsilon_2} \right) \tilde{g}_1 \left( -\frac{\varepsilon_1 r}{\varepsilon_2} \right) \tilde{g}_2 \left( \frac{\varepsilon_1 r}{\varepsilon_2} \right) \, dr. $$

Here

$$ H(w) = (2\pi)^{d+3} |\tilde{\psi}(w)|^2 \int (2p_0 - 1) \tilde{f}_0(p)^2 \delta(p_0 - m_0^2) \theta(p_0) \rho(m_0^2 + w(2p_0 + w)) \theta(p_0 + w) \, dp_0, $$

and we set

$$ H(t) = \int_{\mathbb{R}} e^{-itw} H(w) \, dw. $$

Then

$$ \varepsilon_2 U(f_0, \varphi_1, \varphi_2, \rho) = (2\pi)^{-2} \int_{\mathbb{R}^2} H(x) g_1(\varepsilon_1 x + \frac{\varepsilon_1}{\varepsilon_2} y) g_2(y) \, dx \, dy. $$

With our assumptions on $\rho$, $H \in \mathcal{S}(\mathbb{R})$. Since $g_j$ is $L^\infty$ with compact support and $H \in \mathcal{S}(\mathbb{R})$, the above integral (A.10) is absolutely convergent, uniformly in $\varepsilon_1$ and $\varepsilon_2$. Hence

$$ \varepsilon_2 U(f_0, \varphi_1, \varphi_2, \rho) = (2\pi)^{-2} \int \lim_{\varepsilon_1, \varepsilon_2 \to 0} \lim_{\varepsilon_1, \varepsilon_2 \to 0} H(x) g_1(\varepsilon_1 x + \frac{\varepsilon_1}{\varepsilon_2} y) g_2(y) \, dx \, dy. $$

We find

$$ G(x, \varepsilon_1, \varepsilon_2) = \int g_1 \left( \varepsilon_1 x + \frac{\varepsilon_1}{\varepsilon_2} y \right) g_2(y) \, dy = \int g_1 \left( \frac{\varepsilon_1}{\varepsilon_2} y \right) g_2(y - \varepsilon_2 x) \, dy. $$
We assume from now on $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$. Since $g_j \in L^\infty \cap L^1$ and translation is continuous on $L^1$, $G$ is continuous in $x$.

**Example 1.** We suppose that $g_1, g_2$ are $C^\infty$ with compact support. In this case the limits when $\varepsilon_j$ tend to 0 can be taken under the integral sign in (A.12).

(1.1) if $\varepsilon_1$ tends to 0 at fixed $\varepsilon_2$, $G$ tends to the constant $g_1(0) \int g_2(y) dy$, independent of $\varepsilon_2$.

(1.2) if both $\varepsilon_1$ and $\varepsilon_2$ tend to 0 and $\varepsilon_1 / \varepsilon_2 \to 0$, $G$ also tends to $g_1(0) \int g_2(y) dy$.

(1.3) if both $\varepsilon_1$ and $\varepsilon_2$ tend to 0 and $\varepsilon_1 / \varepsilon_2 \to \lambda \in (0, 1)$, then $G$ tends to the constant $\int g_1(\lambda y)g_2(y) dy$, and

$$
\varepsilon_2 \mathcal{U}(f_0, \varphi_1, \varphi_2, \rho) \to (2\pi)^{-2} \int H(x) dx \int g_1(\lambda y)g_2(y) dy. \quad (A.13)
$$

This holds in particular if $\varepsilon_1$ and $\varepsilon_2$ are kept equal so that $\lambda = 1$. The constant $\int g_1(\lambda y)g_2(y) dy$ may be equal to the preceding constant $g_1(0) \int g_2(y) dy$, for example if $g_1(\lambda y) = g_1(0)$ on the support of $g_2$.

**Example 2.** We consider the case when $g_j(x) = \theta(x)\theta(1 - x)$, i.e. $g_j$ is the indicator function of $[0, 1]$. Then (see Fig. 2)

$$
G(x, \varepsilon_1, \varepsilon_2) = (1 + \varepsilon_2 x)\theta(1 + \varepsilon_2 x)\theta(-x) + \theta(x)\theta(\varepsilon_1^{-1} - \varepsilon_2^{-1} - x) + 
\varepsilon_2(\varepsilon_1^{-1} - x)\theta(x - \varepsilon_1^{-1} + \varepsilon_2^{-1})\theta(\varepsilon_1^{-1} - x). \quad (A.14)
$$

$G(x, \varepsilon_1, \varepsilon_2)$ tends to 1 when both $\varepsilon_j \to 0$ (with $\varepsilon_1 \leq \varepsilon_2$). $G$ also tends to 1 if $\varepsilon_1 \to 0$ at fixed $\varepsilon_2$ and $\varepsilon_2 \to 0$. Since $H \in \mathcal{S}(\mathbb{R})$, the integral $\mathcal{A.11}$ tends to $(2\pi)^{-2} \int H(x) dx = (2\pi)^{-1} \tilde{H}(0)$.

**Example 3.** Consider now the case $g_j(x) = \theta(1/2 - |x|)$, i.e. $g_j$ is the indicator function of $[-1/2, 1/2]$. In that case $G(x, \varepsilon_1, \varepsilon_2)$ is even in $x$ (see Fig. 3):

$$
G(x, \varepsilon_1, \varepsilon_2) = \theta(\varepsilon_1^{-1} - \varepsilon_2^{-1} - 2|x|) + 
\left( \frac{1}{2} \left( 1 + \frac{\varepsilon_2}{\varepsilon_1} \right) - \varepsilon_2 |x| \right) \theta(2|x| - \varepsilon_1^{-1} + \varepsilon_2^{-1})\theta(\varepsilon_1^{-1} + \varepsilon_2^{-1} - 2|x|). \quad (A.15)
$$

$G(x, \varepsilon_1, \varepsilon_2)$ tends to 1 either if $\varepsilon_1$ tends to 0 at fixed $\varepsilon_2$, or if both $\varepsilon_j \to 0$ (with $\varepsilon_1 \leq \varepsilon_2$), and the integral $\mathcal{A.11}$ tends to $(2\pi)^{-2} \int H(x) dx = (2\pi)^{-1} \tilde{H}(0)$.

**Conclusion** With the two last choices of $g_j$ just described,

$$
\varepsilon_2 \mathcal{U}(f_0, \varphi_1, \varphi_2, \rho) \to (2\pi)^{d+2} \rho(m_0^2) \int (2p^0)^{-1} |\tilde{f}_0(p)|^2 \delta(p^2 - m_0^2)\theta(p^0) dp. \quad (A.16)
$$
If we relax the conditions set on $\rho$. Horizontal slices have been described in subsect. 5.1. In this appendix, we study lim $\epsilon$ 

\[ \lim_{\epsilon \to 0} g_1(x) = g_2(x) = \theta(1/2 - |x|). \]

For other choices of $g_j$, the limit as $\epsilon_1 \to 0$, then $\epsilon_2 \to 0$ need not be the same as when $\epsilon_1 = \epsilon_2 \to 0$.

If we relax the conditions set on $\rho$, the same conclusions hold if e.g. $\rho(s) = \rho(s; m_1, m_2)$ and $d \geq 4$.

**B Appendix. Adiabatic limit (dS): horizontal slices**

Horizontal slices have been described in subsect. 5.1. In this appendix, we study $\lim_{T \to +\infty} T^{-1} L_1(f_0, g)$ where $g(x)$ is given by (5.4) in the coordinates (5.1). We denote $\kappa = \sqrt{\frac{m_0^2}{d} - (d - 1)^2/4}$ and recall that $m_0 > (d - 1)/2$ (hence $\kappa > 0$), and $C_L(m_0, d) = C_0(\kappa)$ (see (2.27)). Inserting the representation (2.23) for the three occurrences of $w_{m_0}^\kappa$ (denoted also $w_{\kappa^+}$) in the formula (5.14) for $L_1(f_0, g)$ gives

\[ L_1(f_0, g) = \frac{\gamma^2 C_0(\kappa) c_{d, \kappa} \int_{\gamma} h_0(\xi, \xi', g) h_0(\xi') d\mu_\gamma(\xi) d\mu_\gamma(\xi')} {\int h_0(\xi) h_0(\xi) d\mu_\gamma(\xi)} , \]  

(B.1)

\[ h_0(\xi) = \int_{X_d} \psi^+_{\xi_1 - \xi_2} \psi(-\xi_1 - \alpha_2)(x, \xi) f_0(x) dx , \]  

(B.2)

\[ K_{\kappa}(\xi, \xi', g) = \int_{X_d} \psi^+_{\xi_1 - \alpha_2}(x, \xi) \psi(-\xi_1 + \alpha_2)(x, \xi') g(x) dx . \]  

(B.3)

We take $\gamma = S_0 = \{ \xi \in C_+ : \xi^0 = 1 \} \simeq S^{d-1}$, the unit sphere in $\mathbb{R}^d$. In this appendix, we also set $R = 1$: a general $R$ can be reinstated in the results by homogeneity. Note that, for any $\varphi \in C^\infty(S_0 \times S_0)$,

\[ \int \varphi(\xi, \xi') \psi^+_{\xi_1 - \alpha_2}(x, \xi) \psi(-\xi_1 + \alpha_2)(x, \xi') d\mu_{S_0}(\xi) d\mu_{S_0}(\xi') \]  

is $C^\infty$ in $\xi$. Hence, for any bounded $g$ with bounded support, $K_{\kappa}$ is a distribution on $S_0 \times S_0$ in the variables $(\xi, \xi')$. We will take $g$ invariant under the rotation group in $d$ dimensions (leaving $\epsilon_0$ invariant), hence $K_{\kappa}(\xi, \xi', g) = K_{\kappa}(L\xi, L\xi', g)$ for every such rotation $L$. Hence $K_{\kappa}$ is $C^\infty$ in $\xi$ when smeared with a test-function in $\xi'$. Studying the limit of $T^{-1} L_1(f_0, g)$ for $g$ as in (5.4), is therefore equivalent to studying the limit of $T^{-1} K_{\kappa}(\xi, \xi', g)$ as a distribution in $\xi'$ for fixed $\xi$. In this appendix the cases $d = 2$ and $d = 4$ will be treated. The case $d = 3$, more straightforward than $d = 4$ (no need to use $d = 2$), will be omitted. The result in these three cases is the same (see (3.33) and (3.33)).

**B.1 Case $d = 2$**

We use the following parametrizations

\[
\begin{align*}
x^0 &= \sinh t \\
x^1 &= \cosh t \sin \theta \\
x^2 &= \cosh t \cos \theta
\end{align*}
\]

\[
\begin{align*}
\xi^0 &= 1 \\
\xi^1 &= 0 \\
\xi^2 &= -1
\end{align*}
\]

\[
\begin{align*}
x^0 &= 1 \\
x^1 &= -\sin \phi \\
x^2 &= -\cos \phi
\end{align*}
\]

(B.4)
has a non-zero imaginary part of the same sign as \( s \) and remains analytic in \( t \) in the variable \( \theta \).

In these variables, the measure \( dx \) takes the form \( \cosh t \, dt \, d\theta \). For small \( \varepsilon > 0 \), changing \( t \) into \( t + i\varepsilon \) pushes \( x \) into \( T_\varepsilon \). If \( g(x) = g_0(t) \),

\[
K_\kappa(\xi, \xi', g) = \int_{t \in \mathbb{R}, -\pi \leq \theta \leq \pi} g_0(t)
\]

\[
[\sinh(t + i0) + \cosh(t + i0) \cos(\theta)]^{-\frac{1}{2} - i\kappa} [\sinh(t - i0) + \cosh(t - i0) \cos(\theta)]^{-\frac{1}{2} + i\kappa} \cosh t \, dt \, d\theta. \quad (B.6)
\]

For real \( s \) with \( 0 < |s| < \pi/2 \) and real \( \alpha \),

\[
\sinh(t + is) + \cosh(t + is) \cos(\alpha) = \cos(s)(\sinh t + \cosh t \cos(\alpha)) + i \sin(s)(\cosh t + \sinh t \cos(\alpha)) \quad (B.7)
\]

has a non-zero imaginary part of the same sign as \( s \), so its power \( \mu \) can be taken for any complex \( \mu \) and remains analytic in \( t \) for all real \( t \), smooth and periodic with period \( 2\pi \) in \( \alpha \). The integral over \( \theta \) in (B.6) will be performed, using Plancherel’s formula, by first computing the discrete Fourier transform, in the variable \( \theta \), of the two last factors in the integrand, i.e.

\[
K_\kappa(\xi, \xi', g) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \int_{t \in \mathbb{R}} e^{im\phi} F_m(t + i0) \overline{F_m(t + i0)} \, g_0(t) \, \cosh(t) \, dt, \quad (B.8)
\]

with

\[
F_m(t + is) = \int_{-\pi}^{\pi} [\sinh(t + is) + \cosh(t + is) \cos(\theta)]^{-\frac{1}{2} - i\kappa} e^{im\phi} \, d\theta. \quad (0 < |s| < \pi/2). \quad (B.9)
\]

By changing \( \theta \) to \(-\theta\) in the integration, we get:

\[
F_m(t + is) = F_{-m}(t + is). \quad (B.10)
\]

We use the formula \((27, (15 \text{ p.}157))\)

\[
P^m_\mu(z) = \frac{\Gamma(\mu + m + 1)}{2\pi i \Gamma(\mu + 1)} \int_{-\pi}^{\pi} [z + (z^2 - 1)^{1/2} \cos \phi]^\mu e^{im\phi} \, d\phi, \quad (z \in \Delta_1, \ \text{Re} \, z > 0). \quad (B.11)
\]

This is stated for \( m \) integer and \( \geq 0 \), but by using (see \(27, (7 \text{ p.}140))\)

\[
\frac{P^{-m}_\lambda(z)}{\Gamma(\lambda - m + 1)} = \frac{P^m_\lambda(z)}{\Gamma(\lambda + m + 1)}, \quad m \in \mathbb{Z}, \quad (B.12)
\]

it is seen to hold for all \( m \in \mathbb{Z} \). This gives, for \( 0 < s < \pi/2 \),

\[
F_m(t + is) = e^{-i\pi/4 + \pi \kappa/2} \frac{2\pi i (\frac{1}{2} - i)}{\Gamma(\frac{1}{2} + 1)} P_m^{-\frac{1}{2} - i}(\sinh(t + is)) \quad \text{for } t > 0, \quad (B.13)
\]
\[
F_m(t + is) = (-1)^m e^{-i\pi/4 + \pi \kappa/2} \frac{2\pi i (\frac{1}{2} - i)}{\Gamma(\frac{1}{2} + 1)} P_m^{-\frac{1}{2} + i}(\sinh(t + is)) \quad \text{for } t < 0, \quad (B.14)
\]
\[
\overline{F_m(t + is)} = e^{i\pi/4 + \pi \kappa/2} \frac{2\pi i (\frac{1}{2} + i)}{\Gamma(-m + \frac{1}{2} + i)} P_m^{-\frac{1}{2} + i}(i \sinh(t - is)) \quad \text{for } t > 0, \quad (B.15)
\]
\[
= (-1)^m e^{i\pi/4 + \pi \kappa/2} \frac{2\pi i (\frac{1}{2} + i)}{\Gamma(-m + \frac{1}{2} + i)} P_m^{-\frac{1}{2} - i}(i \sinh(t - is)) \quad \text{for } t < 0. \quad (B.16)
\]

Taking \( g_0(t) = \theta(T/2 - t) \theta(t + T/2), \) we rewrite \( (B.8) \) as

\[
K_\kappa(\xi, \xi', g) = \sum_{m \in \mathbb{Z}} e^{im\phi} (I_m^+ + I_m^-), \quad (B.17)
\]
where

\[
I_m^+ = \frac{1}{2\pi} \int_0^{T/2} F_m(t + i0) \overline{F_m(t + i0)} \cosh(t) dt \tag{B.18}
\]

\[
= 2\pi e^{\kappa}(-1)^m \int_0^{\pi/T} P_{-\frac{i}{z} - x}(i\mu + \varepsilon) P_{\frac{i}{z} + x}(-i\mu + \varepsilon) du. \tag{B.19}
\]

In the last expression we have used \(\Gamma(\frac{1}{2} + x)\Gamma(\frac{1}{2} - x) = \pi/\cos(\pi x)\), and changed to the variable \(u = \sinh t\). Similarly (with now \(u = -\sinh t\)),

\[
I_m^- = \frac{1}{2\pi} \int_{-T/2}^0 F_m(t + i0) \overline{F_m(t + i0)} \cosh(t) dt \tag{B.20}
\]

\[
= 2\pi e^{\kappa}(-1)^m \int_0^{\pi/T} P_{-\frac{i}{z} - x}(i\mu + \varepsilon) P_{\frac{i}{z} + x}(-i\mu + \varepsilon) du. \tag{B.21}
\]

Remark B.1 Using (ah.10) and \(\Gamma(\frac{1}{2} + z)\Gamma(\frac{1}{2} - z) = \pi/\cos(\pi z)\) shows that the rhs of \(\text{(B.1)}\) can be obtained from the rhs of \(\text{(B.19)}\) by changing in the integrand (but not outside the integral) \(\kappa\) to \(-\kappa\). Note also that \(I_m^+ = I_m^-\) by \(\text{(B.10)}\).

The meaning of \(\text{(B.17)}\) is that \(K_\xi(\xi', g)\) is a distribution in \(\xi'\) as expressed in the coordinate \(\phi\), and that \(m \to I_m^+ + I_m^-\) is its discrete Fourier transform. It is tempered, i.e. \(|I_m^+ + I_m^-|\) does not increase faster than a power of \(|m|\) as \(|m| \to \infty\). To prove that \(T^{-1} K_\xi\) tends to a limit (also a distribution in \(\phi\)) as \(T \to \infty\) is equivalent to proving that

1. For each \(m\), \(T^{-1}(I_m^+ + I_m^-)\) tends to a limit \(U_m\) as \(T \to \infty\),
2. there are two positive constants \(P\) and \(Q\) such that \(T^{-1}|I_m^+ + I_m^-| \leq P(1 + |m|^Q)\) for all \(m\) and \(T\).

If both conditions are satisfied, \(U_m\) is the \(m^{th}\) Fourier coefficient of the limit, i.e. \(\lim T^{-1}K_\xi = \sum_m U_m e^{im\phi}\).

B.1.1 Condition (1)

We need the asymptotic behavior of \(P^m_\lambda(z)\) as \(|z| \to \infty\), as described in \(\text{[27]}, \text{pp. 123, 124, 126, and 164. For } z \in \Delta_1 \text{ and } \zeta = z^{-2},

\[
P^0_\lambda(z) = \frac{2^{-\lambda-1} \Gamma(-\frac{1}{2} - \lambda) \zeta^{-\lambda-1+\mu} (z^2 - 1)^{-\mu/2}}{\Gamma(-\lambda - \mu)} F(\frac{1}{2} + \lambda/2 - \mu/2, \frac{1}{2} + \lambda/2 - \mu/2; \ \frac{1}{2} + \lambda/2 - \mu/2; \ \zeta) + \frac{2^{\lambda-1} \Gamma(\frac{1}{2} - \lambda) \zeta^{\lambda+\mu} (z^2 - 1)^{-\mu/2}}{\Gamma(1 + \lambda - \mu)} F(-\lambda/2 - \mu/2, \frac{1}{2} - \lambda/2 - \mu/2; \ \frac{1}{2} - \lambda - \mu). \tag{B.22}
\]

If \(\lambda + \frac{1}{2} \notin \mathbb{Z}\), the two hypergeometric functions can be expanded into convergent power series for \(|\zeta| < 1\). For \(z \in \Delta_1 \text{ and } |z| \to \infty\), we find

\[
P^0_\lambda(z) \sim \frac{2^{-\lambda-1} \Gamma(-\frac{1}{2} - \lambda) \zeta^{-\lambda-1}}{\Gamma(-\lambda - \mu)} + \frac{2^{\lambda-1} \Gamma(\frac{1}{2} + \lambda) \zeta^\lambda}{\Gamma(1 + \lambda - \mu)}. \tag{B.23}
\]

Hence, as \(u \to +\infty\),

\[
P^m_{\frac{i}{z} - x}(iu) P^m_{\frac{i}{z} + x}(iu) \sim
\]

\[
\left[ 2^{-\frac{i}{z} + \kappa} \pi^{-1/2} \Gamma(i\kappa) e^{i\pi/4 + i\kappa/2} u^{-\frac{i}{z} + i\kappa} \frac{2^{-\frac{i}{z} + \kappa} \pi^{-1/2} \Gamma(-i\kappa) e^{-i\pi/4 - i\kappa/2} u^{-\frac{i}{z} - i\kappa}}{\Gamma(\frac{1}{2} + i\kappa - m)} \right] \times \]

\[
\left[ 2^{-\frac{i}{z} - \kappa} \pi^{-1/2} \Gamma(-i\kappa) e^{-i\pi/4 - i\kappa/2} u^{-\frac{i}{z} - i\kappa} \frac{2^{-\frac{i}{z} - \kappa} \pi^{-1/2} \Gamma(i\kappa) e^{i\pi/4 - i\kappa/2} u^{-\frac{i}{z} + i\kappa}}{\Gamma(\frac{1}{2} + i\kappa + m)} \right]. \tag{B.24}
\]
We first consider the ‘off-diagonal terms’ of this product:
\[
\frac{2^{-1+2i\kappa} \pi^{-1} \Gamma(i\kappa)^2 u^{-1+2i\kappa}}{\Gamma(\frac{i}{2} + i\kappa - m) \Gamma(\frac{i}{2} + i\kappa + m)} + \frac{2^{-1-2i\kappa} \pi^{-1} \Gamma(-i\kappa)^2 u^{-1-2i\kappa}}{\Gamma(\frac{i}{2} - i\kappa - m) \Gamma(\frac{i}{2} - i\kappa + m)}.
\]  
(B.25)

These two terms are exchanged by changing \(\kappa\) to \(-\kappa\). The contribution of the first to \(I_m^+ / T\) is of the form
\[
\text{Const.}\frac{1}{T} \int_{1}^{\sinh(T/2)} u^{-1+2i\kappa} du = \text{Const.}\frac{1}{2i\kappa T}(\sinh(T/2)^{2i\kappa} - 1).
\]  
(B.26)

This tends to zero as \(T \to +\infty\). The same happens for the second term. The ‘diagonal terms’ are
\[
\frac{2^{-1}\pi^{-1} \Gamma(i\kappa) \Gamma(-i\kappa) e^{\pi\kappa} u^{-1}}{\Gamma(\frac{i}{2} + i\kappa - m) \Gamma(\frac{i}{2} - i\kappa + m)} + \frac{2^{-1}\pi^{-1} \Gamma(-i\kappa) \Gamma(+i\kappa) e^{-\pi\kappa} u^{-1}}{\Gamma(\frac{i}{2} - i\kappa - m) \Gamma(\frac{i}{2} + i\kappa + m)}.
\]  
(B.27)

Again these two terms are exchanged by changing \(\kappa\) to \(-\kappa\). Their sum can be reexpressed as
\[
\frac{(-1)^m \cosh(\pi\kappa)^2 u^{-1}}{\pi\kappa \sinh(\pi\kappa)}.
\]  
(B.28)

Since
\[
\int_{1}^{\sinh(T/2)} u^{-1} du = \log(\sinh(T/2)) \sim T/2,
\]  
(B.29)

\[
\frac{1}{T} I_m^+ \sim \frac{e^{\pi\kappa} \cosh(\pi\kappa)^2}{\pi\kappa \sinh(\pi\kappa)}.
\]  
(B.30)

Because of Remark 3.1, \(I_m^+ / T\) has the same limit as \(I_m^+ / T\) and
\[
U_m = \lim_{T \to +\infty} \frac{1}{T} (I_m^+ + I_m^-) = \frac{2e^{\pi\kappa} \cosh(\pi\kappa)^2}{\kappa \sinh(\pi\kappa)}.
\]  
(B.31)

\(U_m\) is independent of \(m\), so that if Condition (2) is satisfied,
\[
\lim_{T \to +\infty} \frac{1}{T} K_{\alpha}(\xi, \xi', g_T) = \frac{4\pi e^{\pi\kappa} \cosh(\pi\kappa)^2}{\kappa \sinh(\pi\kappa)} \delta(\phi) = \frac{4\pi e^{\pi\kappa} \cosh(\pi\kappa)^2}{\kappa \sinh(\pi\kappa)} \delta_{\Sigma}(\xi, \xi'),
\]  
(B.32)

and (see (B.1))
\[
\lim_{T \to +\infty} T^{-1} L_1(f_0, g_T) = \gamma^2 C_0(\kappa) c_2, \kappa \frac{4\pi e^{\pi\kappa} \cosh(\pi\kappa)^2}{\kappa \sinh(\pi\kappa)} = \frac{\gamma^2 \pi \coth(\pi\kappa)^2}{|\kappa|}.
\]  
(B.33)

We note that, owing to the delta function in (B.32), the dependence on \(h_0\) (i.e. on \(f_0\)) has completely disappeared from the limit. This result agrees with (5.9).

### B.1.2 Condition (2)

In this subsection \(\lambda\) always denotes \(-\frac{i}{2} - i\kappa\) with \(\kappa \in \mathbb{R}\) and \(\kappa \neq 0\). We first return to the first step of the preceding subsection in the case \(m = 0\). From the identity (B.22) and the analyticity of \(\zeta \mapsto F(a, b, c; \zeta)\) in the unit disk, it follows that there is a \(M_0(\kappa) > 0\) such that
\[
|P_\lambda(z) - \frac{2^{-\frac{i}{2} + i\kappa} \pi^{-1/2} \Gamma(i\kappa) z^{-\frac{i}{2} + i\kappa}}{\Gamma(\frac{i}{2} + i\kappa)} - \frac{2^{-\frac{i}{2} - i\kappa} \pi^{-1/2} \Gamma(-i\kappa) z^{-\frac{i}{2} - i\kappa}}{\Gamma(\frac{i}{2} - i\kappa)}| < M_0(\kappa) |z|^{-5/2} \quad z \in \Delta_1, \quad |z| > 2.
\]  
(B.34)

By (B.13, B.14), there is also an \(M_1(\kappa) > 0\) such that, for \(0 < s < \pi/2\),
\[
|F_0(t + is)| \leq M_1(\kappa) |\sinh(t + is)|^{-1/2}.
\]  
(B.35)
We now obtain crude bounds for $|F_m|$. For $t \geq 0$ and $0 < |s| < \pi/2$, changing $\theta$ to $\theta + \pi$ in (B.39), we get

$$F_m(t + is) = (-1)^m \cosh(t + is)^{\lambda} \int_{-\pi}^{\pi} ((1 - \cos(\theta)) - (1 - \th(t + is)))^{\lambda} e^{im\theta} d\theta. \quad (B.36)$$

Changing to the variable $\varphi = \theta/2$,

$$F_m(t + is) = (-1)^m 2(2 \cosh(t + is))^{\lambda} A_m(z), \quad (B.37)$$

$$A_m(z) = \int_{-\pi/2}^{\pi/2} (\sin^2(\varphi) - z^2)^{\lambda} e^{2im\varphi} d\varphi, \quad (B.38)$$

$$z^2 = \frac{1}{4}(1 - \th(t + is)) = \frac{1}{4}(1 - t)(1 - i t \sin s) . \quad (B.39)$$

We now suppose $0 < \tan s < 1/4$. It follows, after some calculations:

$$\tan s \leq \frac{\Im z^2}{\Re z^2} \leq \tan(2s), \quad \tan(s/2) \leq \frac{\Im z}{\Re z} \leq \tan s, \quad (B.40)$$

We define $z = x - iy$ with $x > 0$. Then

$$0 < x < |z| < 3/4, \quad 0 < y \leq x \tan s < 3/16, \quad \frac{e^{-i}}{\sqrt{2}} \leq |z| \leq x \sqrt{1 + \tan^2 s} \leq x \sqrt{17/16}. \quad (B.41)$$

Recall that for $\rho > 0$, $-\pi < \theta < \pi$, $\zeta \in \mathbb{C}$,

$$|\langle \rho e^{i\theta} \zeta \rangle| = \rho \Re \zeta \, e^{-\theta} \Im \zeta \leq \rho \Re \zeta \, e^{|\Im \zeta|}. \quad (B.42)$$

Thus

$$e^{-|\zeta|} A_m(z) \leq H(z) = \int_{-\pi/2}^{\pi/2} |\sin^2 \varphi - z^2|^{-1/2} \, d\varphi = 2 \int_{0}^{1} \frac{dt}{\sqrt{1 - t^2 \sqrt{|t^2 - z^2|}}} . \quad (B.43)$$

After splitting the integration interval as $[0, 1] = [0, x] \cup [x, \sqrt{3}/2] \cup [\sqrt{3}/2, 1]$, straightforward estimates give

$$e^{-|\zeta|} A_m(z) \leq H(z) \leq \frac{4\pi}{3\sqrt{3}} + 8 + 4 \log(\sqrt{3}/x) \leq 27 + 4t . \quad (B.44)$$

We now consider

$$A_0(z) - A_m(z) = \int_{-\pi/2}^{\pi/2} (\sin^2(\varphi) - z^2)^{\lambda} (1 - e^{2im\varphi}) \, d\varphi = \int_{-\pi/2}^{\pi/2} (\sin^2(\varphi) - z^2)^{\lambda} (1 - e^{2im\varphi} + 2im\varphi) \, d\varphi. \quad (B.45)$$

Using

$$|1 - e^{2im\varphi} + 2im\varphi| \leq 2m^2 \varphi^2 \leq \frac{m^2\pi^2}{2} \sin^2(\varphi) \quad \forall \varphi \in [-\pi/2, \pi/2] \quad (B.46)$$

we get

$$e^{-|\zeta|} |A_0(z) - A_m(z)| \leq m^2 \pi^2 \int_{0}^{\pi/2} |\sin^2(\varphi) - z^2|^{1/2} \sin^2(\varphi) \, d\varphi
= \frac{m^2 \pi^2}{2} \sqrt{H(z)} + 2m^2 \pi^2 \int_{0}^{\pi/2} |\sin^2(\varphi) - z^2|^{1/2} \, d\varphi
\leq \frac{m^2 \pi^2}{2} \sqrt{H(z)} + \frac{5\pi^3 m^2}{8}. \quad (B.47)$$
Since \( |z|^2 \leq 17x^2/16 \) and \( 2x^2 \log(1/x) < 1/e \), there is a constant \( M_2 > 0 \) such that, for all \( m \),
\[
|A_0(z) - A_m(z)| \leq e^{\pi |\kappa|} M_2 m^2,
\]
and hence
\[
|F_0(t + is) - ( -1)^m F_m(t + is) | \leq \sqrt{2} e^{2\pi |\kappa|} \cosh(t + is)^{-1/2 - i\kappa} |M_2 m^2|
\]
(B.49)
With \( 0 < \tan s < 1/4 \), as we have chosen, \( |\cosh(t + is)^{-1/2 - i\kappa}| \leq (\cosh(t)^{-1/2}(17/16)^{1/4} e^{4|\kappa|/4} \), so that
\[
|F_0(t + is) - ( -1)^m F_m(t + is) | \leq \sqrt{2} e^{(2\pi + 1/4)|\kappa|/4} (17/16)^{1/4} |\cosh(t)|^{-1/2} M_2 m^2,
\]
and, by (B.33), there is an \( M_3(\kappa) > 0 \) such that
\[
|F_m(t + is) | \leq M_3(\kappa)(1 + m^2) |\sinh(t)|^{-1/2}.
\]
(B.51)
Therefore, using the bound (B.43), independent of \( m \), for \( 0 \leq t \leq t_1 \), and the bound (B.51) for \( t_1 \leq t \leq T/2 \), we find that there is a constant \( M_4(\kappa) > 0 \) such that
\[
|T^{-1} I_m^+ | \leq M_4^2(\kappa)(1 + m^2)^2 \quad \forall \ m \in \mathbb{Z}.
\]
(B.52)
The same holds for \( |T^{-1} I_m^- | \). This proves that Condition (2) is satisfied.

### B.2 Other dimensions

In this subsection the dimension of the de Sitter space-time \( X \) is \( n = d > 2 \), i.e. the ambient Minkowski space-time is \( \mathbb{R}^{d+1} \). The notation \( n = d \) is used to stay close to [30], Chap. IX, p 448 ff, which is constantly used in this section. As in Subsect. B.1.1 we wish to compute
\[
K_\nu(\xi, \xi', g) = \int_X \psi^+_{\lambda}(x, \xi) g(x) \psi^-_{\lambda}(x, \xi') \, dx
\]
(B.53)
where \( \xi, \xi' \in \partial V_+ \subset \mathbb{R}^{d+1} \) and \( \xi^0 = \xi'^0 = 1 \), \( \lambda = -(n - 1)/2 - i\nu \), \( \psi^\pm_{\lambda}(x, \xi) \) are as defined in (2.18). We use the following parametrization for \( x = (x^0, \vec{x}) \in X, \xi = (1, \xi) \in \partial V_+, \xi' = (1, \xi') \in \partial V_+ \) (see [30], p. 448).

\[
\begin{align*}
x^0 &= \sinh t, \quad \vec{x} = -\cosh t \vec{u} \\
u^1 &= \sin \theta_{n-1} \ldots \sin \theta_2 \sin \theta_1 \\
u^2 &= \sin \theta_{n-1} \ldots \sin \theta_2 \cos \theta_1 \\
u^3 &= \sin \theta_{n-1} \ldots \cos \theta_2 \\
\vdots \\
u^{n-1} &= \sin \theta_{n-1} \cos \theta_{n-2} \\
u^n &= \cos \theta_{n-1}
\end{align*}
\]
\[
\begin{align*}
\xi^0 &= 1 \\
\xi^1 &= 0 \\
\xi^2 &= \sin \phi_{n-1} \ldots \sin \phi_2 \sin \phi_1 \\
\xi^3 &= \sin \phi_{n-1} \ldots \sin \phi_2 \cos \phi_1 \\
\vdots \\
\xi^{n-1} &= \sin \phi_{n-1} \cos \phi_{n-2} \\
\xi^n &= \cos \phi_{n-1}
\end{align*}
\]
(B.54)
Here \( t \in \mathbb{R}, 0 \leq \theta_1 < 2\pi, 0 \leq \phi_1 < 2\pi, 0 \leq \theta_k < \pi \) for \( k > 1 \), \( 0 \leq \phi_k < \pi \) for \( k > 1 \). With these notations
\[
dx = \cosh^{-1} t \, dt \, d\vec{u}, \quad d\vec{u} = \sin^{-1} \theta_{n-1} \, d\theta_{n-1} \ldots \sin \theta_2 \, d\theta_2 \, d\theta_1.
\]
(B.55)
We also use the normalized measure \( d\sigma(\vec{u}) \) on \( S^{n-1} \),
\[
d\sigma(\vec{u}) = \Omega_{n-1}^{-1} \, d\vec{u}, \quad \Omega_n = \int_{S^{n-1}} d\vec{u} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.
\]
We restrict \( g \) to be of the form \( g(x) = g_T(x) = g_0(t) = \theta(T/2 - t)\theta(t + T/2), T > 0 \). The integral (B.53) takes the form
\[
K_\nu(\xi, \xi', g) = \int_{\mathbb{R}} g_0(t) (\cosh t)^{-n} \, dt \int F(t, \vec{u}) G(t, \vec{u}) \, d\vec{u},
\]
(B.57)
\[ F(t, \bar{u}) = (\sinh(t + i0) + \cosh(t + i0) \cos(\theta_{n-1}))^\lambda. \]  
(B.58)

For \( G \), we have \( G(t, \bar{u}) = (x_+ \cdot \xi^\prime)^\lambda \). Note that \( \xi^\prime = R\xi \), where \( R \) is the rotation in \( \mathbb{R}^n \)

\[ R = e^{\phi_1 M_{21}} \cdots e^{\phi_{n-1} M_{nn-1}}, \quad M_{jk} = e_j \wedge e_k. \]  
(B.59)

For example

\[ e^{\phi_{n-1} M_{nn-1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \phi_{n-1} & \sin \phi_{n-1} \\ 0 & 0 & -\sin \phi_{n-1} & \cos \phi_{n-1} \end{pmatrix}. \]  
(B.60)

Therefore

\[ G(t, \bar{u}) = F(t, R^{-1}\bar{u}). \]  
(B.61)

As in the case \( d = 2 \), we reexpress the integral over \( \bar{u} \) in \((B.57)\) using harmonic analysis on the sphere.

Harmonic analysis on \( S^{n-1} \) uses an orthonormal basis \( \{ \Xi^\ell_K \} \) of functions on the sphere (\( \ell = 0, 1, 2, \ldots, K \) is a multiindex). This is fully described in \([30, \text{Chap IX}]\):

\[ \int_{S^{n-1}} \Xi^\ell_K(\bar{u}) \overline{\Xi^{\ell^\prime}_{K^\prime}(\bar{u})} d\sigma(\bar{u}) = \delta_{\ell\ell^\prime}\delta_{KK^\prime}. \]  
(B.62)

For fixed \( \ell \) the functions \( \{ \Xi^\ell_K \} \) generate a finite-dimensional subspace \( \mathcal{H}^{\ell\ell} \) of \( L^2(S^{n-1}) \) in which the regular representation of \( SO(n) \) reduces to an irreducible unitary representation, characterized by its matrix elements in the basis \( \{ \Xi^\ell_K \} \): for any \( g \in SO(n) \),

\[ \Xi^\ell_K(g^{-1}\bar{u}) = \sum_M t^\ell_{MK}(g) \Xi^\ell_M(\bar{u}). \]  
(B.63)

The functions \( \{ \Xi^\ell_K \} \) and \( \{ t^\ell_{MK} \} \) are analytic on \( S^{n-1} \) and \( SO(n) \) respectively. Given two arbitrary \( L^2 \) functions \( h_1, h_2 \) on \( S^{n-1} \) we have (for \( j = 1, 2 \))

\[ h_j(\bar{u}) = \sum_{\ell, K} h^\ell_{jK} \Xi^\ell_K(\bar{u}), \]  
(B.64)

\[ h^\ell_{jK} = \int_{S^{n-1}} h_j(\bar{u}) \Xi^{\ell^\prime}_{K^\prime}(\bar{u}) d\sigma(\bar{u}), \]  
(B.65)

\[ \int_{S^{n-1}} h_1(\bar{u}) \overline{h_2(\bar{u})} d\sigma(\bar{u}) = \sum_{\ell, K} h^\ell_{1K} \overline{h^\ell_{2K}}. \]  
(B.66)

These formulae imply that

\[ \sum_{\ell, K} \Xi^\ell_K(\bar{u}) \Xi^{\ell^\prime}_{K^\prime}(\bar{v}) = \Omega_n \delta_{S^{n-1}}(\bar{u}, \bar{v}) \]  
(B.67)

where \( \delta_{S^{n-1}}(\bar{u}, \bar{v}) \) denotes the distribution (actually measure) on \( S^{n-1} \times S^{n-1} \) defined by

\[ \int_{S^{n-1} \times S^{n-1}} \delta_{S^{n-1}}(\bar{u}, \bar{v}) \varphi(\bar{u}, \bar{v}) d\bar{u} d\bar{v} = \int_{S^{n-1}} \varphi(\bar{u}, \bar{u}) d\bar{u}. \]  
(B.68)

Actually, as is the case for all invariant distributions on \( S^{n-1} \times S^{n-1} \), smearing \( \delta_{S^{n-1}}(\bar{u}, \bar{v}) \) only in \( \bar{v} \) with a \( \mathcal{C}^\infty \) function produces a \( \mathcal{C}^\infty \) function of \( \bar{u} \):

\[ \int_{S^{n-1}} \delta_{S^{n-1}}(\bar{u}, \bar{v}) \psi(\bar{v}) d\bar{v} = \psi(\bar{u}). \]  
(B.69)
Choosing in particular \( \vec{u} = \vec{e}_n \), we can use the formula (30, IX 4.1 (1-4) and text there)

\[
\Xi^\ell_K(\vec{e}_n) = \delta_{K0} \sqrt{\frac{\Gamma(\ell + n - 2)(2\ell + n - 2)}{\ell! \Gamma(n-1)}}.
\] (B.70)

Inserting this in (B.67) gives

\[
\delta_{S_n-1}(\vec{e}_n, \vec{v}) = \Omega_n^{-1} \sum_{\ell} \frac{\Gamma(\ell + n - 2)(2\ell + n - 2)}{\ell! \Gamma(n-1)} \Xi^\ell_0(\vec{v}).
\] (B.71)

Taking \( \vec{v} = \vec{\xi}^\ell \) with \( \vec{\xi}^\ell \) given by (B.54) and using (30, IX 4.1 (3-4),

\[
\Xi_M^\ell(g e^n_{\vec{u}}) = \sqrt{\frac{\Gamma(\ell + n - 2)(2\ell + n - 2)}{\ell! \Gamma(n-1)}} t^\ell_{M0}(g)
\] (B.72)

we get

\[
\delta_{S_n-1}(\vec{e}_n, \vec{\xi}^\ell) = \Omega_n^{-1} \sum_{\ell} \frac{\Gamma(\ell + n - 2)(2\ell + n - 2)}{\ell! \Gamma(n-1)} t^\ell_{00}(R),
\] (B.73)

with \( R \) given by (B.59). Harmonic analysis extends to distributions on the sphere, as it does on \( S^1 \).

We apply (B.64)-B.66 to the case \( h_1(\vec{u}) = F(t, \vec{u}), h_2(\vec{u}) = G(t, \vec{u}) \). Because \( F(t, \vec{u}) \) depends only on \( \cos \theta_{n-1} \),

\[
f^\ell_K(t) = \int_{S^{n-1}} F(t, \vec{u}) \Xi^\ell_K(\vec{u}) d\Omega(\vec{u}) = \delta_{K0} t^\ell_{00}(t)
\] (B.74)

Note that \( t \) can be complexified in (B.73), i.e. \( t \) can be replaced by \( t + is \) with \( 0 < |s| < \pi/2 \). In the sequel we omit the \( t \)-dependence of \( f^\ell_K(t) \), writing simply \( f^\ell_K \) unless the \( t \)-dependence becomes significant. We have

\[
\Xi^\ell_0(\vec{u}) = A^0_0 C_{\ell}^{\frac{n-2}{2}}(\cos \theta_{n-1}), 
A^0_0 = \sqrt{\frac{\ell! \Gamma(n-2)(2\ell + n - 2)}{\Gamma(\ell + n - 2)(n-2)}}
\] (B.75)

\[
G(t, \vec{u}) = \sum_{\ell} f^\ell_0 \Xi^\ell_0(R^{-1} \vec{u}) = \sum_{\ell, K} (f^\ell_0 t^\ell_{K0}(R)) \Xi^\ell_K(\vec{u})
\] (B.76)

Therefore

\[
\int_{S^{n-1}} F(t, \vec{u}) G(t, \vec{u}) d\Omega = \Omega_n \sum_{\ell} t^\ell_{00}(R) |f^\ell_0|^2,
\] (B.77)

\[
K_{\nu}(\xi, \xi', g r) = \Omega_n \sum_{\ell} t^\ell_{00}(R) \int_{-T/2}^{T/2} |f^\ell_0(t)|^2 (\cosh t)^{n-1} dt
\] (B.78)

Also

\[
t^\ell_{00}(R) = \frac{\ell! \Gamma(n-2)}{\Gamma(\ell + n - 2)} C_{\ell}^{\frac{n-2}{2}}(\cos \phi_{n-1})
\] (B.79)

For (B.79) see (30, IX 3.6 (6,7) p. 480. For (B.79) see (30, IX 4.2 (8) p. 484. We thus have

\[
f^\ell_0 = \Omega_n^{-1} A^0_0 \int [\sinh(t+i0) + \cosh(t+i0) \cos \theta_{n-1}]^\lambda C_{\ell}^{\frac{n-2}{2}}(\cos \theta_{n-1}) \sin^{n-2} \theta_{n-1} d\theta_{n-1} \ldots \sin \theta_2 d\theta_2 \sin \theta_1
\]

\[
= \Omega_n^{-1} \Omega_{n-1}^{-1} A^0_0 \int_0^\pi [\sinh(t+i0) + \cosh(t+i0) \cos \theta]^\lambda \sin^{n-2} \theta C_{\ell}^{\frac{n-2}{2}}(\cos \theta) d\theta
\] (B.80)

In these formulae \( C_{\ell}^\mu \) is a Gegenbauer polynomial: see (27) p. 175 for the definition. The formulae (27) p. 176 (9), and (30, IX 3.1 (3), giving the explicit coefficients of \( C_{\ell}^\mu \) coincide, so we are dealing with the same objects.
B.3 The case $d = n = 4$

We now restrict our attention to the case $d = 4$, keeping the notations of the preceding subsection. In this case $\lambda = -3/2 - i\kappa$, $\Omega_4 = 2\pi^2$, $A_0 = 1$. We exclude the case $\kappa = 0$. Since $(n - 2)/2 = 1$, the formula (B.30) gives:

$$f^*_\ell(t + is) = \frac{2}{\pi} \int_0^\pi (\sinh(t + i0) + \cosh(t + i0) \cos \theta)^\lambda C^1_\ell(\cos \theta) \sin^2 \theta \, d\theta.$$  

(B.81)

We have (27, 3.15.1 (15) p. 177)

$$C^1_\ell(\cos \theta) = \frac{\sin(\ell + 1)\theta}{\sin \theta}.$$  

(B.82)

Therefore, for sufficiently small $s > 0$,

$$f^*_\ell(t + is) = \frac{2}{\pi} \int_0^\pi (\sinh(t + is) + \cosh(t + is) \cos \theta)^\lambda \sin(\ell + 1)\theta \sin \theta \, d\theta$$

(B.83)

$$= \frac{1}{\pi} \cosh(t + is)^\lambda \int_{-\pi}^\pi [\cosh(t + is) + \cos \theta]^\lambda \sin(\ell + 1)\theta \sin \theta \, d\theta$$

(B.84)

$$= \frac{(\ell + 1)}{\pi(\lambda + 1)} \cosh(t + is)^\lambda \int_{-\pi}^\pi [\cosh(t + is) + \cos \theta]^{\lambda + 1} \cos(\ell + 1)\theta \, d\theta$$

(B.85)

$$= \frac{(\ell + 1)}{\pi(\lambda + 1)} \cosh(t + is)^{-1} \int_{-\pi}^\pi (\sinh(t + is) + \cosh(t + is) \cos \theta)^{\lambda + 1} \cos(\ell + 1)\theta \, d\theta.$$  

(B.86)

Recall that $\lambda + 1 = -1/2 - i\kappa$. Therefore, comparing (B.86) with (B.9), we find, for $0 < |s| < \pi/2$,

$$f^*_\ell(t + is) = \frac{(\ell + 1)}{2\pi(-2 - i\kappa) \cosh(t + is)} (F_{\ell+1}(t + is) + F_{-(\ell+1)}(t + is))$$

(B.87)

$$= \frac{(\ell + 1)}{\pi(-2 - i\kappa) \cosh(t + is)} F_{\ell+1}(t + is),$$

using (B.10). Hence

$$K_\kappa(\xi, \xi', g_T) = \Omega_4 \sum_\ell \int_0^{\ell^*_0(R)} t^{\ell^*_0(R)} \frac{(\ell + 1)^2}{\pi^2(\kappa^2 + 1/4)} \int_{-T/2}^{T/2} F_{\ell+1}(t + i0) F_{\ell+1}(t + i0) \cosh t \, dt$$

(B.88)

$$= \sum_\ell \int_0^{\ell^*_0(R)} \frac{4\pi(\ell + 1)^2}{(\kappa^2 + 1/4)} (I^+_{\ell+1} + I^-_{\ell+1})$$

(B.89)

with the notations of (B.18). Therefore, by (B.31) and the proof of Condition (2) for $d = 2$ (subsect. B.1.2),

$$\lim_{T \to +\infty} \frac{1}{T} K_\kappa(\xi, \xi', g_T) = \frac{8\pi e^{\pi\kappa} \cosh(\pi\kappa)^2}{(\kappa^2 + 1/4)\kappa \sinh(\pi\kappa)} \sum_\ell t^{\ell^*_0(R)} (\ell + 1)^2.$$  

(B.90)

In the case $n = 4$, (B.73) becomes

$$\sum_\ell (\ell + 1)^2 t^{\ell^*_0(R)} = \Omega_4 \delta_{S^3}(\xi^1, \xi^3) = 2\pi^2 \delta_{S^3}(\xi, \xi').$$  

(B.91)

Thus

$$\lim_{T \to +\infty} \frac{1}{T} K_\kappa(\xi, \xi', g_T) = \frac{16\pi^3 e^{\pi\kappa} \cosh(\pi\kappa)^2}{(\kappa^2 + 1/4)\kappa \sinh(\pi\kappa)} \delta_{S^3}(\xi, \xi').$$  

(B.92)

It follows (see (B.1) that

$$\lim_{T \to +\infty} T^{-1} L_1(f_0, g_T) = \frac{\gamma^2 \pi \coth(\pi\kappa)^2}{|\kappa|}.$$  

(B.93)

This is the same as in the case $d = 2$. 
Appendix. Adiabatic limit (dS): parabolic slices

We again take $R = 1$. We again start from the formulae (B.1-B.3) of Appendix B but we only require $k \in \mathbb{R} \setminus \{0\}$. The function $g$ will be chosen as announced in subsect 5.1 The map $(t, y) \mapsto x(t, y)$ defined in (C.2) is a diffeomorphism of $\mathbb{R}^d$ onto the “upper half” $X_d^{up} = \{x \in X_d : x^0 + x^d > 0\}$, and $(t, y) \mapsto -x(t, y)$ is a diffeomorphism of $\mathbb{R}^d$ onto the “lower half” $X_d^{down} = -X_d^{up}$. The cycle $\gamma$ appearing in (B.1) will be chosen as $V_0 = C_+ \cap \{\xi \in M_{d+1} : \xi^0 + \xi^d = 1\}$. It can be parametrized by the diffeomorphism $\eta \mapsto \xi(\eta)$ of $\mathbb{R}^{d-1}$ onto $V_0$:

$$
\xi(\eta) = \begin{cases} 
\xi^0 = \frac{1}{2}(1 + \eta^2), \\
\xi^j = \eta_j, \quad (1 \leq j \leq d - 1), \\
\xi^d = \frac{1}{2}(1 - \eta^2),
\end{cases} \quad \eta^2 = \sum_{j=1}^{d-1} \eta_j^2. \quad (C.1)
$$

Thus $V_0$ is a Euclidean space with $(d\xi \cdot d\xi) = -d\eta^2$ on $V_0$. The stability group of the vector $e_0 - e_d$ in $G_0$ leaves $V_0$ invariant and acts as the group of Euclidean displacements there. As noted in Remarks 2.1 and 2.2, the $G_0$ invariance and homogeneity of $\psi^\pm_\omega(x, \xi)$ imply that it can be regarded as a distribution in $\xi$ on $V_0$, $C^\infty$ in $x$ on $X_d$. For a real $g \in S(X_d)$, if we denote $\tilde{g}(x) = g(-x)$, we find

$$
K_\kappa(\xi, \xi', \tilde{g}) = e^{2\pi \kappa^2 \int \overline{K_{\kappa}}(\xi, \xi', g)}. \quad (C.2)
$$

It will turn out that $g$ can be chosen invariant under the stability group of $e_0 - e_d$. Then $K_\kappa$ is an invariant distribution on $V_0 \times V_0$. For our purposes it will suffice (and be possible) to study the limit of $T^{-1}K_\kappa(\xi, \xi', \tilde{g})$ with

$$
g_{\tilde{r}}(x) = \theta(t + T/2)\theta(T/2 - t), \quad t = \log(x^0 + x^d), \quad (C.3)
$$

and to add in the end the limit of $T^{-1}K_\kappa(\xi, \xi', \tilde{g})$ obtained from (C.2).

With $x$ parametrized as in (5.5) and $\xi$ as in (C.1), we have

$$
x(t, s) : \xi = \frac{1}{2}e^t(y - \eta)^2 - e^{-t} = \frac{(y - \eta)^2}{2s} - \frac{s}{2}, \quad s = e^{-t}. \quad (C.4)
$$

For $k \in \mathbb{R}^{d-1}$, we find

$$
\psi^{\pm}_{-\frac{d-1}{2} + iv}(k, s, \eta) \overset{\text{def}}{=} \int_{\mathbb{R}^{d-1}} \psi^{\pm}_{-\frac{d-1}{2} + iv}(x(t, s), \xi)e^{iky} \, dy \\
= 2^{d-1} - iv e^{iky} \int_{\mathbb{R}^{d-1}} \left[ \frac{y^2}{s + i\epsilon} - (s + i\epsilon) \right]^{-\frac{d-1}{2} + iv} e^{iky} \, dy \\
= 2^{d-1} - iv e^{iky} \int_{0}^{\infty} \left[ \frac{a^2}{s + i\epsilon} - (s + i\epsilon) \right]^{-\frac{d-1}{2} + iv} da \int_{\mathbb{R}^{d-1}} \delta(y^2 - a^2) e^{iky} \, dy \\
= 2^{d-1} - iv \frac{d-1}{2} k^{\frac{d-3}{2}} e^{iky} \int_{0}^{\infty} \left[ \frac{y^2}{s + i\epsilon} - (s + i\epsilon) \right]^{-\frac{d-1}{2} + iv} y^{d-3} J_{d-3}(ky) \, dy, \quad (C.5)
$$

where $k = |k|$ and $s = e^{-t}$. We use the following formula (31) (51) p. 95 with some notational changes)

$$
\int_{0}^{\infty} (y^2 + z^2)^{-\lambda + i\alpha} y^{\lambda - 1} J_{\lambda-1}(ky) \, dy = \left( \frac{k}{2} \right)^{\lambda - i\alpha - 1} z^{i\alpha} K_{-\lambda}(kz) \frac{\Gamma(\lambda - i\alpha)}{\Gamma(\lambda - i\alpha)}. \quad (C.6)
$$

This is valid provided $k > 0$, $Re z > 0$, $Re \lambda > 0$, and $Re(\lambda - 2i\alpha + 1/2) > 0$. Note that none of these parameters except $k$ needs to be real. In our application, $\lambda = (d - 1)/2$ and $\alpha = \nu \in \mathbb{R} \setminus \{0\}$. We take $z = -is + \varepsilon$, $s > 0$, $\varepsilon > 0$ arbitrarily small, $\alpha = \nu$. Since $(y^2 + z^2)$ then has a small negative imaginary part, this will correspond to the case of $\psi^{\pm}_{-\lambda + iv}$ in (C.5). In (C.6) $K_{-\lambda}$ denotes the Macdonald...
function. This will introduce no lasting ambiguity since we will use the identities ([31], (5), (6) p. 4, (15) p. 5)

\[ K_{-i\nu}(-iks) = \frac{i\pi}{2}e^{-\pi\nu/2}H^{(1)}_{i\nu}(ks) = \frac{i\pi e^{-\pi\nu/2}}{2\sinh(\pi\nu)}(J_{i\nu}(ks) - e^{-\pi\nu}J_{-i\nu}(ks)). \]  

(C.7)

This yields

\[ \tilde{\psi}_{-\lambda+i\nu}(k, s, \eta) = e^{ik\eta} \frac{2^d \pi^{d+1}k^{d-1}s^d}{\Gamma(\lambda - i\nu)\sinh(\pi\nu)}(e^{\pi\nu}J_{i\nu}(ks) - J_{-i\nu}(ks)), \quad \lambda = \frac{d-1}{2}. \]  

(C.8)

\[ \tilde{\psi}_{-\lambda-\nu}(k, s, \eta) \] can be obtained from this since, for real \( \nu \), it is the complex conjugate of \( \tilde{\psi}_{-\lambda+i\nu}(-k, s, \eta) \):

\[ \tilde{\psi}_{-\lambda-\nu}(k, s, \eta) = e^{ik\eta} \frac{(-i)2^d \pi^{d+1}k^{d-1}s^d}{\Gamma(\lambda + i\nu)\sinh(\pi\nu)}(e^{\pi\nu}J_{-i\nu}(ks) - J_{i\nu}(ks)), \quad \lambda = \frac{d-1}{2}. \]  

(C.9)

**Remark C.1**  By the preceding remarks, if \( x \in V_0 \) is expressed in terms of \( \eta \in \mathbb{R}^{d-1} \) as in (C.11), \( \psi_\alpha^+(x, \xi) \) is a tempered distribution in \( \eta \), a \( C^\infty \) function of \( x \), and an entire function in \( \alpha \). If \( x \) is expressed as in (5.5), its Fourier transform with respect to the variable \( y \) is also a tempered distribution in the variable \( k \) conjugated to \( y \) and in \( \eta \); \( C^\infty \) in \( s \) and holomorphic in \( \alpha \), and, in this sense, the formulae (C.8) and (C.9) can be continued to all \( \nu \). If \( \nu \) is taken real in these formulae, their rhs becomes locally bounded in \( k \) in particular locally \( L^2 \).

Supposing \( g(x) = G((x^0 + x^d)^{-1}) \) (for example if \( g(x) = g_T(x) = G_T(s) = \theta(s - e^{-T/2})\theta(e^{T/2} - s) \)), Plancherel’s formula gives

\[ K_\alpha(\xi, \xi', g) = (2\pi)^{-d} \int_{s > 0, \ k \in \mathbb{R}^{d-1}} s^{-d}G(s) \psi_{-\alpha+in}(k, s, \eta)\tilde{\psi}_{-\alpha-in}(-k, s, \eta') \, ds \, dk. \]  

(C.10)

Inserting (C.8), we find

\[ K_\alpha(\xi, \xi', g) = (2\pi)^{-d} \int_{\mathbb{R}^{d-1}} e^{ik(\eta - \eta')} \tilde{K}_\alpha(k, g) \, dk, \]  

(C.11)

\[ \tilde{K}_\alpha(k, g) = \int_0^\infty s^{-1}G(s) \left[ A J_{in}(ks)J_{-in}(ks) + B J^2_{in}(ks) + C J^2_{-in}(ks) \right] \, ds, \]  

(C.12)

where

\[ A = \frac{2d\pi^{d+1}e^{\pi\kappa}\cosh(\pi\kappa)}{\Gamma(\frac{d-1}{2} + ik)\Gamma(\frac{d-1}{2} - i\kappa)\sinh^2(\pi\kappa)}, \]  

(C.13)

\[ B = C = \frac{-2d\pi^{d+1}e^{\pi\kappa}}{\Gamma(\frac{d-1}{2} + ik)\Gamma(\frac{d-1}{2} - i\kappa)\sinh^2(\pi\kappa)}. \]  

(C.14)

Going back to eq. (C.12), we divide the integration range into the intervals [0, 1] and [1, \( \infty \]). After dividing by \( T \), the contribution of the second interval is bounded by

\[ \text{Const.} \frac{1}{T} \int_1^{\infty} k^{-1}s^{-2} \, ds = \text{ Const.} \frac{1}{kT}. \]  

(C.15)

This is because \( |J_\alpha(x)| \leq \text{ Const.} x^{-1/2} \) as \( x \to +\infty \) (see [31], p. 85). Hence the contribution of the second interval tends to 0 as \( T \) tends to \( +\infty \). The function \( J_\alpha \) can be written as

\[ J_\alpha(z) = (z/2)^\alpha \sum_{m=0}^\infty \frac{(-1)^m(z/2)^{2m}}{m!\Gamma(m + \alpha + 1)} = (z/2)^\alpha \left( \frac{1}{\Gamma(1 + \alpha)} + O(z^2) \right). \]  

(C.16)
Thus as $T$ tends to $+\infty$, 
\[
\frac{1}{T} \tilde{K}_\kappa(k, g_T) \sim \frac{1}{T} \int_{e^{-T/2}}^{1} s^{-1} \left[ A J_{i\kappa}(ks) J_{-i\kappa}(ks) + B J_{i\kappa}^2(ks) + C F_{i\kappa}^2(ks) \right] ds \sim \frac{1}{T} \int_{e^{-T/2}}^{1} s^{-1} \left[ A + B (ks/2) e^{2i\kappa} + C (ks/2)^2 e^{-2i\kappa} + \text{Const.} k^2 s^2 \right] ds
\]

\[
= \frac{A}{2\Gamma(1+i\kappa)\Gamma(1-i\kappa)} + \frac{B(ks/2) e^{2i\kappa}}{2it\kappa\Gamma(1+i\kappa)^2} + \frac{C(ks/2)^2 e^{-2i\kappa}}{-2it\kappa\Gamma(1-i\kappa)^2} + \text{Const.} k^2 (1-e^{-T})
\]

Hence

\[
\lim_{T \to +\infty} \frac{1}{T} \tilde{K}_\kappa(k, g_T) = \frac{A}{2\Gamma(1+i\kappa)\Gamma(1-i\kappa)} = \frac{\sinh(\pi\kappa)}{2\pi\kappa}.
\]

This gives

\[
\lim_{T \to +\infty} \frac{1}{T} K_{\kappa}(\xi, \xi', g_T) = \frac{A \sin(\pi\kappa)}{2\pi\kappa} \delta(\eta - \eta'),
\]

and (see (B.1))

\[
\lim_{T \to +\infty} \frac{1}{T} L_1(f_0, g_T) = \frac{\gamma^2 C_0(\kappa) c_{d.4} A \sin(\pi\kappa)}{2\pi\kappa} = \frac{\gamma^2 \pi \coth(\pi\kappa)}{2|\kappa|}.
\]

This is half of the result in (B.33) or (B.93), but it is doubled by the addition of the contribution of $\tilde{g}_T$.

### D Proof of the projector identity

In this appendix, we give a proof of the formula (2.23) in the de Sitter case, with masses $m$ and $m'$ in the principal series, i.e. $m^2 = \mu^2 + (d - 1)^2/4$, $m'^2 = \nu^2 + (d - 1)^2/4$, with real $\mu \neq 0$ and $\nu \neq 0$. We set $R$ equal to $1$. The meaning of (2.23) is

\[
\lim_{g \to g_T} \frac{1}{S(X_d)} \int_X W_m(z, x) W_{m'}(x, y) g(x) dx = C_4(m, d) \delta(m^2 - m'^2) W_m(z, y).
\]

For $g \in S(X_d)$, the integral in this formula is well defined (see Remark 2.2). The same method as in Appendix C will be used. Using Eq. (2.23) reduces the problem to the study of $g$ tends to $1$, of

\[
K_{\mu,\nu}(\xi, \xi', g) = \int_X \psi^+ \frac{1}{2} - i\psi^+ (x, \xi) \psi^+ \frac{1}{2} + iv (x, \xi') g(x) dx.
\]

Recalling Remarks 2.3, 2.2 and C.1 and parametrizing $\xi$ and $\xi'$ in terms of $\eta$ and $\eta'$ as in (C.1), we see that, for a general smooth fast decreasing $g$, this is well defined as a distribution in $\eta$ and $\eta'$, and an entire function in $\mu$ and $\nu$, and, denoting $\tilde{g}(x) = g(-x)$, it satisfies

\[
K_{\mu,\nu}(\xi, \xi', \tilde{g}) = e^{\pi(\mu + \nu)} K_{-\mu, -\nu}(\tilde{\gamma} \xi', \tilde{\gamma} \xi).
\]

(It is sufficient to verify this formula for real $\mu$ and $\nu$). We will use the same coordinates (5.3) and many of the formulae of Appendix C. We wish to take $g$ as $g_\mu(x) = \theta(x_0 + x^d)$, or $g_\mu = \tilde{g}_\mu$. Thus $g_\mu (\mu$ stands for “upper”) is the indicator function of the domain covered by the coordinates (5.3). We denote $K_{\mu,\nu}^u(\xi, \xi', \tilde{g}) = K_{\mu,\nu}(\xi, \xi', \tilde{g}_\mu)$ and $K_{\mu,\nu}^d(\xi, \xi', \tilde{g}) = K_{\mu,\nu}(\xi, \xi', \tilde{g}_\mu)$. To make the integral converge, we first replace $g_\mu$ by a better behaved $g_\mu^c$ of the form $g_\mu^c(x(t, y)) = G_\epsilon(e^{-i}) g_\mu(x)$ which will tend to $g_\mu(x)$ as $\epsilon \to 0$. We thus consider

\[
K_{\mu,\nu}^u(\xi, \xi', \tilde{g}_\mu^c) = K_{\mu,\nu}(\xi, \xi', \tilde{g}_\mu^c) = \int_{X_d^c} \psi^+ \frac{1}{2} - i(\mu) (x, \xi) \psi^+ \frac{1}{2} + iv (x, \xi') g_\mu^c(x) dx,
\]

\[
K_{\mu,\nu}^d(\xi, \xi', \tilde{g}_\mu^c) = K_{\mu,\nu}(\xi, \xi', \tilde{g}_\mu^c) = e^{\pi(\mu + \nu)} K_{-\mu, -\nu}(\tilde{\gamma} \xi', \tilde{\gamma} \xi).
\]
We now take \( \mu \) and \( \nu \) real and furthermore require \( \mu \nu > 0 \). Using the coordinates (5.5) and parametrizing \( \xi \) and \( \xi' \) as in Appendix C (see (C.1)), we may use the Plancherel formula as was done there. We obtain

\[
K_{\mu,\nu}^u(\xi, \xi') = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i k \cdot (\eta - \eta')} \widetilde{K}_{\mu,\nu}^u(k) \, dk, \\
K_{\mu,\nu}^d(\xi, \xi') = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i k \cdot (\eta - \eta')} \widetilde{K}_{\mu,\nu}^d(k) \, dk, \\
K_{\mu,\nu}^{d,\varepsilon}(\xi, \xi') = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i k \cdot (\eta - \eta')} \widetilde{K}_{\mu,\nu}^{d,\varepsilon}(k) \, dk, \\
K_{\mu,\nu}^{d,\varepsilon}(\xi, \xi') = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i k \cdot (\eta - \eta')} \widetilde{K}_{\mu,\nu}^{d,\varepsilon}(k) \, dk, \\
\widetilde{K}_{\mu,\nu}^{d,\varepsilon}(\xi, \xi') = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i k \cdot (\eta - \eta')} \widetilde{K}_{\mu,\nu}^{d,\varepsilon}(k) \, dk, 
\]

where \( k = |k| \). We can use the following formula (31, 7.7.4 (30), p. 51):

\[
\int_0^\infty J_\alpha (as) J_\beta (as) s^{-\rho} \, ds = \frac{(a/2)^{\rho-1} \Gamma(\rho) \Gamma((\alpha + \beta + 1 - \rho)/2)}{2 \Gamma((1 + \alpha + \beta + \rho)/2) \Gamma((1 - \alpha + \beta + \rho)/2) \Gamma((1 + \alpha - \beta + \rho)/2)}, \quad \text{Re}(\alpha + \beta + 1) > \text{Re} \rho > 0, \quad a > 0. 
\]

Choosing \( G_\varepsilon(s) = s^\varepsilon \) with \( 0 < \varepsilon < 1 \), and using (D.7) to obtain \( \widetilde{K}_{\mu,\nu}^{u,\varepsilon}(k) \) from \( \widetilde{K}_{\mu,\nu}^{u,\varepsilon}(k) \), we obtain

\[
\widetilde{K}_{\mu,\nu}^{u,\varepsilon}(k) + \widetilde{K}_{\mu,\nu}^{d,\varepsilon}(k) = \frac{2^{d-2} \pi^{d+1} k^d \varepsilon (\mu - \nu) / 2 - \varepsilon} {\sinh(\pi \mu) \sinh(\pi \nu) \Gamma \left( \frac{d+1}{2} - i \mu \right) \Gamma \left( \frac{d+1}{2} + i \nu \right) \times} \\
\left( \frac{1}{\Gamma((2 - i\mu + i\nu - \varepsilon)/2) \Gamma((2 + i\mu + i\nu - \varepsilon)/2) \Gamma((2 - i\mu - i\nu - \varepsilon)/2) \Gamma((2 + i\mu - i\nu - \varepsilon)/2)} \right) \\
\frac{1}{\Gamma((2 - i\mu - i\nu - \varepsilon)/2) \Gamma((2 + i\mu - i\nu - \varepsilon)/2) \Gamma((2 - i\mu + i\nu + \varepsilon)/2) \Gamma((2 + i\mu + i\nu + \varepsilon)/2)} \\
\frac{1}{\Gamma((2 + i\mu + i\nu - \varepsilon)/2) \Gamma((2 + i\mu + i\nu - \varepsilon)/2) \Gamma((2 + i\mu + i\nu + \varepsilon)/2) \Gamma((2 + i\mu + i\nu + \varepsilon)/2)} \\
\frac{1}{\Gamma((2 + i\mu - i\nu + \varepsilon)/2) \Gamma((2 + i\mu - i\nu + \varepsilon)/2) \Gamma((2 + i\mu - i\nu + \varepsilon)/2) \Gamma((2 + i\mu - i\nu + \varepsilon)/2)}. \]

These expressions have well-defined limits in the sense of distributions in \( \mu \) and \( \nu \). In the numerator of each term inside the square brackets we make the substitution \( \Gamma(z) = \Gamma(1 + z)/z \). As \( \varepsilon \to 0 \), we find

\[
\widetilde{K}_{\mu,\nu}^{u,\varepsilon}(k) + \widetilde{K}_{\mu,\nu}^{d,\varepsilon}(k) \sim \frac{2^{d-1} \pi^{d+1} k^d \varepsilon (\mu - \nu)} {\sinh(\pi \mu) \sinh(\pi \nu) \Gamma \left( \frac{d+1}{2} - i \mu \right) \Gamma \left( \frac{d+1}{2} + i \nu \right) \times} \\
\left( \frac{e^{\pi \mu + \varepsilon}} {\Gamma \left( 1 + \frac{i \mu + i \nu}{2} \right) \Gamma \left( 1 - \frac{i \mu + i \nu}{2} \right)} \right) \\
\left( \frac{1}{\Gamma \left( \frac{1}{2} (\mu + \nu) - \varepsilon \right) \Gamma \left( \frac{1}{2} (\mu - \nu) + \varepsilon \right)} \right) \\
\left( \frac{1}{\Gamma \left( \frac{1}{2} (\mu + \nu) + \varepsilon \right) \Gamma \left( \frac{1}{2} (\mu - \nu) - \varepsilon \right)} \right). \]

Using \((it + \varepsilon)^{-1} + (-it + \varepsilon)^{-1} \approx 2\pi / \sinh(\pi t)\), and \( \Gamma(1 + iz) \Gamma(1 - iz) = \pi z / \sinh(\pi z) \), this gives

\[
\widetilde{K}_{\mu,\nu}^{u,\varepsilon}(k) + \widetilde{K}_{\mu,\nu}^{d,\varepsilon}(k) \sim \frac{2^{d-1} \pi^{d+1} k^d \varepsilon (\mu - \nu)} {\sinh(\pi \mu) \Gamma \left( \frac{d+1}{2} - i \mu \right) \Gamma \left( \frac{d+1}{2} + i \nu \right) \times} \\
\left( \frac{e^{\pi \mu + \varepsilon}} {\Gamma \left( 1 + \frac{i \mu + i \nu}{2} \right) \Gamma \left( 1 - \frac{i \mu + i \nu}{2} \right)} \right) \\
\left( \frac{1}{\Gamma \left( \frac{1}{2} (\mu + \nu) - \varepsilon \right) \Gamma \left( \frac{1}{2} (\mu - \nu) + \varepsilon \right)} \right) \\
\left( \frac{1}{\Gamma \left( \frac{1}{2} (\mu + \nu) + \varepsilon \right) \Gamma \left( \frac{1}{2} (\mu - \nu) - \varepsilon \right)} \right). \]
although we have assumed $\mu \neq 0$ and $\nu \neq 0$ have the same sign. In this case $\delta(\mu + \nu) = 0$, and $|\mu|^{-1}\delta(\mu - \nu) = 2\delta(\mu^2 - \nu^2)$. Thus, in this case,

$$
K_{\mu,\nu}(\xi, \xi', g = 1) = K_{\mu,\nu}^u(\xi, \xi') + K_{\mu,\nu}^d(\xi, \xi') = \frac{2^{d+2}\pi^{d+1}e^{\pi\mu}x^\mu|x|}{\Gamma\left(\frac{d+1}{2} - i\mu\right)\Gamma\left(\frac{d+1}{2} + i\mu\right)}\delta(\mu^2 - \nu^2)\delta(\eta - \eta'),
$$

(D.14)

Recall that this holds when $\mu$ and $\nu$ are both non-zero and have the same sign. Still in the same case, using (2.23) we have

$$
\int_{X_d} W_{\mu}(x, y)W_{\nu}(y, x')\, dy = c_{d,\mu}c_{d,\nu}\int_{\gamma \times \gamma} \psi_{\frac{d+1}{2} + i\mu, x, \xi}K_{\mu,\nu}(\xi, \xi', 1)\psi_{\frac{d+1}{2} - i\mu, x', \xi'}\, d\mu(\xi)\, d\gamma(\xi)\, d\mu(\xi')\, d\gamma(\xi') .
$$

(D.15)

We choose $\gamma = V_0$ as described at the beginning of Appendix C and of this Appendix. With the parametrization (C.1), this is a $(d-1)$-Euclidean space and $d\mu(\xi) = d^{d-1}\eta$. Therefore, by (D.14), the rhs of (D.15) is given by

$$(c_{d,\mu})^2\delta(\mu^2 - \nu^2)\frac{2^{d+2}\pi^{d+1}e^{\pi\mu}|x|}{\Gamma\left(\frac{d+1}{2} - i\mu\right)\Gamma\left(\frac{d+1}{2} + i\mu\right)}\int_{\gamma} \psi_{\frac{d+1}{2} + i\mu, x, \xi}\psi_{\frac{d+1}{2} - i\mu, x', \xi'}\, d\mu(\xi)\, d\gamma(\xi),
$$

(D.16)

and finally

$$
\int_{X_d} W_{\mu}(x, y)W_{\nu}(y, x')\, dy = 2\pi|x|\coth(\mu)|\delta(\mu^2 - \nu^2)|W_{\mu}(x, x') .
$$

(D.17)

Although we have assumed $\mu$ and $\nu$ to have the same sign in the derivation, it follows from $W_{\mu} = W_{-\mu}$ and the form of the formula above that it holds for all possible relative signs, provided $\mu \neq 0$ and $\nu \neq 0$.

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