THE WEB OF REFLEXIVE POLYGONS IS CONNECTED

MAKOTO MIURA

Abstract. We discuss the problem on the connectedness of various webs of lattice polytopes by introducing a geometric point of view from the toric Mori theory. To this end, we provide a combinatorial description of toric Sarkisov links in terms of certain sets of lattice points, which we call primitive generating sets. In two dimensions, the description is further translated into the language of lattice polygons. As an application, we prove in two ways (constructive and non-constructive) that reflexive or terminal polygons form a single connected web via inclusion relations even without taking modulo unimodular equivalences.

1. Introduction

As a byproduct of their famous works on the classification of reflexive polytopes, Kreuzer and Skarke have shown that the web of $d$-dimensional reflexive polytopes is connected modulo unimodular equivalences if $d \leq 4$ [KS02]. This is important in relation to the unsolved mathematical problem, so-called Reid’s fantasy [Rei87], which asks whether the web of smooth Calabi–Yau 3-folds is connected via geometric transitions. In fact, their result implies that the web of smooth Calabi–Yau 3-folds described as anticanonical hypersurfaces in toric varieties (in the sense of Batyrev [Bat94] and Fredrickson [Fre15]) is connected via geometric transitions and flops. Since their argument is purely combinatorial and relies on the results of computer-aided classification of reflexive polytopes, it is still an interesting problem to analyze the web of reflexive polytopes from a more geometric point of view.

The present paper provides an argument based on the perspectives from the Mori theory in birational geometry. Let $Z$ be a smooth or mildly singular projective variety over the field of complex numbers $\mathbb{C}$. By running the minimal model program (MMP, for short), one conjecturally obtains a sequence of elementary birational maps directed by the canonical divisor $K_Z$,

$$\varphi : Z = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_m = X,$$  \hspace{1cm} (1.1)

where $X$ is either a minimal model or a Mori fiber space $p : X \rightarrow S$. Although the output $X$ is not unique, it is known that different birational minimal models are related by a sequence of flops [Kaw08] and different birational Mori fiber spaces are related by a sequence of Sarkisov links [Cor95], [HM13]. Thus, in either case, one obtains a single web connecting all birational models via elementary birational maps if the MMP works. For projective toric varieties, the Mori theory has been established by the seminal paper of Reid [Rei83] and developed through a number of studies; for example, [Mat02 Chapter 14], [FS04], [CLS11 Chapters 14–15]. By running MMP starting from a $\mathbb{Q}$-factorial projective toric variety $Z$, one always ends up with a toric Mori fiber space (we do not assume that it has only terminal singularities). The resulting web of birational models can be coarse-grained into the web of Fano polytopes via inclusion relations. Here a Fano polytope (also known as a $\mathbb{Q}$-Fano polytope) is a lattice polytope containing the origin as an interior point whose vertices are all primitive lattice points. This coarse-graining is done by taking the convex hull $\text{Conv} \ G(\Sigma)$ of the set of primitive ray generators $G(\Sigma)$ of each projective
fan $\Sigma$. For an elementary birational map $X_\Sigma \dashrightarrow X_{\Sigma'}$ in (1.1), we have inclusions $G(\Sigma) \supset G(\Sigma')$ and $\text{Conv}(G(\Sigma)) \supset \text{Conv}(G(\Sigma'))$.

The web of Fano polytopes is connected since there is a sequence $\nabla \subset \text{Conv}(\nabla \cup \nabla') \supset \nabla'$ for any pair of Fano polytopes $\nabla$ and $\nabla'$. Hence, a natural problem is whether the web of a certain restricted class of Fano polytopes is connected. If the answer to the following Problem 1.1 is yes, we say that the class of lattice polytopes is \textit{globally connected}.

\textbf{Problem 1.1.} For a given class of lattice polytopes, do they form a single connected web via inclusion relations?

We introduce five known classes of lattice polytopes including reflexive polytopes, for which one could naturally ask Problem 1.1. Let $N \cong \mathbb{Z}^d$ and $M = \text{Hom}(N, \mathbb{Z})$ be the dual pair of free abelian groups of rank $d$ and $N_\mathbb{R} := N \otimes \mathbb{R}$. The \textit{polar dual} and the \textit{Mavlyutov dual} of a lattice polytope $\nabla$ in $N_\mathbb{R}$ is defined as

$$\nabla^* := \{ u \in M_\mathbb{R} \mid \langle u, v \rangle \geq -1 \text{ for all } v \in \nabla \} \quad \text{and} \quad \lfloor \nabla^* \rfloor := \text{Conv}(\nabla^* \cap M),$$

respectively. A reflexive (resp. pseudoreflexive) polytope is defined such as the polar duality (resp. the Mavlyutov duality) holds among the same class of lattice polytopes. Namely, $\nabla$ is \textit{reflexive} if $\nabla^*$ is also a lattice polytope, and \textit{pseudoreflexive} if $\lfloor \lfloor \nabla^* \rfloor \rfloor = \nabla$ holds. While these polytopes are well suited to mirror symmetry as shown by [Bat94] and [Mav11], the following polytopes are more fitted to the arguments in the toric Mori theory. A lattice polytope $\nabla$ is called \textit{canonical} if the origin is the unique interior lattice point, \textit{terminal} if $\nabla \cap N_\mathbb{Z}$ consists of the vertices of $\nabla$ and the origin, and \textit{almost pseudoreflexive} if the Mavlyutov dual $\lfloor \nabla^* \rfloor$ contains the origin as an interior point. We shortly remark the geometric meaning of these three classes of lattice polytopes. A canonical (resp. terminal) polytope $\nabla$ corresponds to a toric Fano variety with at worst canonical (resp. terminal) singularities [Rei83], which is the anticanonical model of the toric variety defined by a projective simplicial fan $\Sigma$ satisfying $\text{Conv} G(\Sigma) = \nabla$. An almost pseudoreflexive polytope $\nabla$ corresponds to a toric Fano variety whose general elephant (i.e., a member of the anticanonical linear system $|\sim K|$) is a Calabi–Yau variety. In fact, the latter is equivalent to that the Newton polytope $\lfloor \nabla^* \rfloor$ is almost pseudoreflexive (and hence, pseudoreflexive in this case) as proved by [Bat17, Theorem 2.23] (and by [ACG16, Theorem 1] for one direction). There are the following implications among these five classes of polytopes:

$$\text{terminal} \Rightarrow \text{canonical} \iff \text{almost pseudoreflexive} \iff \text{pseudoreflexive} \iff \text{reflexive}. \quad (1.3)$$

Only the second implication is nontrivial, which follows because $\lfloor \nabla^* \rfloor \supset \nabla$ contains only the origin as an internal lattice point by [ACG16, Lemma 1.5]. A Fano polytope contained in an almost pseudoreflexive (resp. canonical, terminal) polytope is again an almost pseudoreflexive (resp. canonical, terminal) polytope. Hence, by running MMP, Problem 1.1 for these three classes of Fano polytopes comes down to Problem 1.2 below.

\textbf{Problem 1.2.} For any pair of almost pseudoreflexive (resp. canonical, terminal) polytopes represented by toric Mori fiber spaces, is there a pair of representatives connected by a sequence of Sarkisov links consisting only of birational models whose associated polytopes are almost pseudoreflexive (resp. canonical, terminal)?

Note that the global connectedness of $d$-dimensional almost pseudoreflexive polytopes implies that of $d$-dimensional pseudoreflexive polytopes. In fact, this is clear by taking the pseudoreflexive hull $\lfloor \lfloor \nabla^* \rfloor \rfloor$ for each almost pseudoreflexive polytope $\nabla$ in a sequence connecting a pair of pseudoreflexive polytopes. Together with the fact that a pseudoreflexive polytope is always
reflexive when \( d \leq 4 \) [Ska96], Problem [1.2] for almost pseudoreflexive polytopes possibly re-proves
the connectedness of the web of four-dimensional reflexive polytopes due to [KS02] without taking
modulo unimodular equivalences nor using the classification.

In order to make Problem [1.2] more precise, we explore the combinatorial description of Sarkisov
links for toric Mori fiber spaces. However, it turns out that there is an obstacle in the description
of Sarkisov links only in terms of Fano polytopes (Example 3.8). On the other hand, we obtain a
satisfactory description of Sarkisov links in terms of primitive generating sets in arbitrary dimen-
sions, which are the combinatorial objects (corresponding to \( G(\Sigma) \) above) that appear between
projective fans and Fano polytopes. In Section 2, we introduce and develop the notion of primitive
generating sets. In Section 3, we prove Theorem 3.1 which gives a combinatorial description of
Sarkisov links in terms of primitive generating sets.

In two dimensions, things becomes very simple. Especially, the description in Theorem 3.1
can be further translated into the language of Fano polygons without any additional conditions
(Proposition 3.9). Using this simplicity, we give two proofs (constructive and non-constructive)
for the following theorem in Section 4 by solving Problem [1.2] in two dimensions:

**Theorem 1.3.** The web of reflexive polygons are connected via inclusion relations without taking
modulo unimodular equivalences. The same is true for the web of terminal polygons.

The latter proof also implies that elliptic elephants in smooth projective rational surfaces can be
connected via smooth transitions associated with blow-ups and blow-downs decomposing a given
birational map between ambient spaces (Corollary 4.5). This could be regarded as a toy version
of Reid’s fantasy in one dimension.

Note that, for a lattice polygons, reflexive is equivalent to canonical, so that Theorem 1.3 yields
a complete answer to Problem [1.1] for all the five classes of polytopes in two dimensions. On the
other hand, the proof is still based on the classification of smooth toric Mori fiber surfaces and the
hand-picked choice of sequences of Sarkisov links. Hopefully, more detailed study of the Sarkisov
program could improve the argument and provide a way to solve Problem [1.1] in higher dimensions
and also a new perspective to Reid’s fantasy for Calabi–Yau 3-folds.

**Acknowledgements.** This paper is dedicated to Professor Shinobu Hosono on the occasion of his
sixtieth birthday. The author thanks Atsushi Ito for helpful discussions. He was supported by
Grants-in-Aid for Scientific Research (21K03156).

2. PRIMITIVE GENERATING SETS

**Definition 2.1.** We call a finite set \( A \subset N \) a primitive generating set of \( N_\mathbb{R} \) if \( A \) consists of
primitive lattice points and generates \( N_\mathbb{R} \) as a cone, i.e., \( N_\mathbb{R} = \{ \sum_{v \in A} \lambda_v v \mid \lambda_v \geq 0 \text{ for all } v \in A \} \).

**Definition 2.2.** An inclusion \( A \supset A' \) of primitive generating sets is said to be a reduction
of \( A \) if \( |A| = |A'| + 1 \). We denote by \( A \succ A' \) a reduction of \( A \).

**Definition 2.3.** For a primitive generating set \( A \), an inclusion \( A_f \subset A \) is said to be a fiber
structure on \( A \) if \( A_f \neq \emptyset \) is a primitive generating set of the linear span \( L \) of \( A_f \) and \( A_f = L \cap A \).
A fiber structure \( A_f \subset A \) defines the associated exact sequence

\[
0 \longrightarrow N_f \longrightarrow N \longrightarrow^{\pi} N_b \longrightarrow 0, \tag{2.1}
\]

where \( N_f := N \cap L \) and \( N_b := N/N_f \). For any \( v \in A \setminus A_f \), the ray \( \mathbb{R}_+ \pi(v) \) in \( (N_b)_\mathbb{R} \) has the unique
primitive generator \( \pi(v) \in N_b \). Thus there is a map

\[
\pi : A \setminus A_f \to A_b := \{ \pi(v) \mid v \in A \setminus A_f \} \subset N_b. \tag{2.2}
\]
We call $A_b$ the base of the fiber structure $A_f \subset A$, which turns out to be a primitive generating set of $(N_b)_{\mathbb{R}}$. The map (2.2) is also denoted by $\pi : A \rightarrow A_b$.

A fiber structure $A_f \subset A$ is called irreducible if $|A| = |A_f| + |A_b|$ holds. A Mori fiber structure is an irreducible fiber structure satisfying $|A_f| = \dim L + 1$. We call a primitive generating set $A$ that admits a Mori fiber structure a Mori fiber primitive generating set. A Mori fiber structure $A_f \subset A$ on a Mori fiber primitive generating set $A$ is denoted by $A_f \in A$.

By abuse of notation, we use all the notions in Definition 2.2 and Definition 2.3 also for Fano polytopes by replacing a primitive generating set $\nabla \cap N_{\text{prim}}$ with a Fano polytope $\nabla$, where $N_{\text{prim}}$ is the set of primitive lattice points in $N$. Thus, for a reduction $\nabla > \nabla'$ of Fano polytopes, $\nabla'$ is a Fano polytope defined as the convex hull of all primitive lattice points in $\nabla \setminus \{v\}$ for a vertex $v \in \nabla$. Similarly, for a Mori fiber structure $\nabla_f \subset \nabla$ of Mori fiber Fano polytopes (or Mori fiber polytopes, for short), $\nabla_f$ is a terminal simplex and $|\nabla \cap N_{\text{prim}}| = |A_b| + \dim L + 1$ holds, where $A_b$ is the base of $\nabla_f \subset \nabla$. Notice that, even for a fiber structure $\nabla_f \subset \nabla$ of Fano polytopes, the base $A_b$ needs not to be a Fano polytope.

Let $X_\Sigma$ be a $Q$-factorial projective toric variety defined by a simplicial projective fan $\Sigma$. Denote by $\overline{\text{NE}}(X_\Sigma)$ the closed convex cone generated by numerical equivalence classes of curves. The following is evident from the description of extremal contractions, e.g., [CLS11, Proposition 15.4.5].

**Lemma 2.4.** Let $\varphi_R : X_\Sigma \rightarrow X_\Sigma'$ be an extremal contraction of a $Q$-factorial projective toric variety $X_\Sigma$ with respect to an extremal ray $R \subset \overline{\text{NE}}(X_\Sigma)$. If $\varphi_R$ is a small contraction, it preserves $G(\Sigma) = G(\Sigma')$. If $\varphi_R$ is a divisorial contraction, it gives a reduction $G(\Sigma) > G(\Sigma')$. If $\varphi_R$ is a fibering contraction, it gives a Mori fiber structure $G(\Sigma_f) \subset G(\Sigma)$ with the base $G(\Sigma')$ for a naturally defined subfan $\Sigma_f$ of $\Sigma$.

**Remark 2.5.** Note that, if $\varphi_R$ is a fibering contraction (i.e., the last case in Lemma 2.4), $X_\Sigma$ is automatically a toric Mori fiber space with the fibers isomorphic to $X_{\Sigma_f}$, that is, $K_{X_\Sigma}.R < 0$. In fact, $R$ is the outer normal of a facet of the pseudoeffective cone $\overline{\text{Eff}}(X_\Sigma)$, and $-K_{X_\Sigma}$ is big so that its numerical class is contained in the interior of $\overline{\text{Eff}}(X_\Sigma)$. Since $\overline{\text{Eff}}(X_\Sigma)$ is strongly convex, $K_{X_\Sigma}.R < 0$ always holds.

For any primitive generating set $A$, there exists a simplicial projective fan $\Sigma$ such that $G(\Sigma) = A$, which we call a projective A-maximal fan, following the case where $A$ is (the set of primitive lattice points in) a reflexive polytope [Fre13, Definition 2.1]. The following is a key lemma which shows the converse of Lemma 2.4 also holds in a sense:

**Lemma 2.6.** For any primitive generating set $A$ and any pair of projective A-maximal fans $\Sigma$ and $\Sigma'$, the natural birational map $\varphi : X_\Sigma \rightarrow X_{\Sigma'}$ (induced by the identity map of $N$) is decomposed into a sequence of flips, flops and inverse flips. For any reduction $A > A'$ and any projective $A'$-maximal fan $\Sigma'$, there exists an extremal divisorial contraction $X_\Sigma \rightarrow X_{\Sigma'}$ such that $\Sigma$ is a projective $A$-maximal fan. For any Mori fiber structure $A_f \subset A$ with the base $A_b$ and any projective $A_b$-maximal fan $\Sigma_b$, there exists a toric Mori fiber space $X_\Sigma \rightarrow X_{\Sigma_b}$ with fibers isomorphic to $X_{\Sigma_f}$ such that $\Sigma$ is a projective $A$-maximal fan and $\Sigma_f \subset \Sigma$ is a projective $A_f$-maximal fan.

**Proof.** All the statements in Lemma 2.6 follow from the description of the movable cone $\overline{\text{Mov}}(X_\Sigma)$ of a $Q$-factorial projective toric variety $X_\Sigma$ as the corresponding cone $\text{Mov}_{\text{GKZ}}$ in the secondary fan $\Sigma_{\text{GKZ}}$ defined by $A = G(\Sigma)$ (see (15.1.5), Proposition 15.1.4 and Theorem 15.1.10(c) in [CLS11]). The first statement follows because $\text{Mov}_{\text{GKZ}}$ is a full-dimensional convex rational polyhedral cone decomposed into finite chambers corresponding to projective $A$-maximal fans. In fact, along a general path between the two chambers corresponding to $\Sigma$ and $\Sigma'$, one obtains a sequence of
flips, flops and inverse flips, which decomposes $\varphi : X_\Sigma \rightarrow X_{\Sigma'}$ (see [CLSIII Theorem 15.3.13]). The second statement is a simple extension of [Pre15, Lemma 6.1] and the same proof works. That is, since $\text{Mov}_{\text{GKZ}}$ is full-dimensional and convex, there exists a chamber containing the cone corresponding to $\Sigma'$ as a face, which gives an extremal divisorial contraction (refer to [CLSIII Lemma 14.4.6] for the face structure of the secondary fan). Exactly the same proof works for the third statement if one starts from a generalized fan $\pi^{-1}\Sigma_b := \{ \pi^{-1}_R(\sigma) \mid \sigma \in \Sigma_b \}$ instead of $\Sigma'$, where $\pi_R : N_R \rightarrow (N_b)_R$ is the natural projection defined by (2.1).

**Remark 2.7.** The shed of a fan $\Sigma$ is defined as $\text{shed}(\Sigma) = \bigcup_{\sigma \in \Sigma} \text{Conv} \left( \{ 0 \} \cup \sigma(\pi) \right) \subset \text{Conv} G(\Sigma)$. As observed originally by [Rei83, Proposition 4.3], for a birational contraction $\varphi_R : X_\Sigma \rightarrow X_{\Sigma'}$ of an extremal ray $R \subset \tilde{\text{NE}}(X_\Sigma)$, $\text{shed}(\Sigma') \subset \text{shed}(\Sigma)$ (resp. $\text{shed}(\Sigma') = \text{shed}(\Sigma)$, $\text{shed}(\Sigma') \supset \text{shed}(\Sigma)$) if and only if $K_{X_\Sigma}.R < 0$ (resp. $K_{X_\Sigma}.R = 0$, $K_{X_\Sigma}.R > 0$). Be aware that the sign of $K_{\Sigma}.R$ for a divisorial contraction $\varphi_R : X_\Sigma \rightarrow X_{\Sigma'}$ actually depends not only on the primitive generating sets $G(\Sigma)$ and $G(\Sigma')$ but also on $\Sigma'$ in general. However, for a reduction $\nabla \supset \nabla'$ of Fano polytopes, any divisorial contraction $\varphi_R : X_\Sigma \rightarrow X_{\Sigma'}$ with $\text{Conv} G(\Sigma) = \nabla$ and $\text{Conv} G(\Sigma') = \nabla'$ satisfies $K_{\Sigma}.R < 0$ since $\text{shed}(\Sigma') \subset \nabla'$ does not contain $v \in \nabla \setminus \nabla'$.

### 3. Description of toric Sarkisov links

A **Sarkisov link** is one of the following four types of diagrams of normal projective varieties that connects two Mori fiber spaces $p : X \rightarrow S$ and $p' : X' \rightarrow S'$:

\[
\begin{align*}
\text{type I} & : X \xrightarrow{p} S \xleftarrow{p'} X' \xrightarrow{p'} S' \\
\text{type II} & : X \xleftarrow{p} S \xrightarrow{p'} X' \xrightarrow{p'} S' \\
\text{type III} & : X \xrightarrow{p} S \xrightarrow{p'} S' \xrightarrow{p'} X' \\
\text{type IV} & : X \xrightarrow{p} S \xrightarrow{p'} S' \xrightarrow{p'} X'
\end{align*}
\]

(3.1)

Here all the varieties except $R$ are $\mathbb{Q}$-factorial. Every dashed arrow denotes a sequence of flips, flops and inverse flips, and every solid arrow denotes an extremal contraction. In the latter case, each solid arrow between $X$’s is a divisorial contraction of a $K$-negative extremal ray, each arrow between $S$’s is either divisorial or fibering contraction, and each arrow toward $R$ is either small or fibering contraction.

**Theorem 3.1.** Let $A_I \subset A$ and $A'_I \subset A'$ be Mori fiber primitive generating sets associated with toric Mori fiber spaces $p : X \rightarrow S$ and $p' : X' \rightarrow S'$, respectively. Suppose that these are connected by a single Sarkisov link in (3.1). Then $A_I \subset A$ and $A'_I \subset A'$ coincide or are related by one of the following diagrams and their inverses (i.e., the diagrams switching $A_I \subset A$ and $A'_I \subset A'$):

\[
\begin{align*}
\text{type I}_d & : A \supset A' \quad A = A'' \supset A' \\
\text{type I}_m & : A \cup A' \\
\text{type II}_{irr} & : A \cup A' \\
\text{type II}_{ni} & : A \cup A' \\
\text{type IV}_m & : A = A'' = A'
\end{align*}
\]

(3.2)

We call one of the diagrams in (3.2) and their inverses an **elementary link** for Mori fiber primitive generating sets. The inverse of an elementary link of type $I_d$ (resp. type $I_m$) is referred to as that of type $III_d$ (resp. type $III_m$). It is natural to add the trivial link, $A = A$ with $A_I = A'_I$, as
an elementary link of type IV_s. The scripts “s”, “d”, “m” indicate that the contraction of S’s in the corresponding Sarkisov link is a small contraction, a divisorial contraction, a Mori fiber space, respectively. Also, “irr” and “ni” for type II links indicate that all fibers of the composite contractions X'' → S and X' → S' are irreducible, and that the fibers are not irreducible over a common invariant prime divisor of the base S = S', respectively.

**Proof.** We begin with describing the composition of two extremal contractions q and q',

\[ Y \xrightarrow{q} Y' \xrightarrow{q'} Y'' \]  

(3.3)

appeared in one of the Sarkisov links in (3.1) for \( \mathbb{Q} \)-factorial projective toric varieties. There are four types of such compositions:

1. q is a fibering contraction and q' is a small contraction,
2. q is a fibering contraction and q' is a divisorial contraction,
3. q is a divisorial contraction and q' is a fibering contraction,
4. both q and q' are fibering contractions.

By Lemma 2.4, the contractions q and q' are described by primitive generating sets as

1. \( B_t \subsetneq B \) and \( B_b = B'_b \),
2. \( B_t \subsetneq B \) and \( B_b > B'_b \),
3. \( B > B' \) and \( B'_b \subsetneq B' \),
4. \( B_t \subsetneq B \) and \( B_{b,t} \subsetneq B_b \),

for each type of composition, respectively. We shall combine two of such descriptions into an elementary link for Mori fiber primitive generating sets. Hereafter, we denote by \( A_t \subsetneq A \) and \( A'_t \subsetneq A' \) the two Mori fiber structures corresponding to p and p' in each Sarkisov link, respectively.

A composition of type (1) only appears in the Sarkisov link of type IV_s. In this Sarkisov link, both compositions are of type (1) so that \( A = A' = B \) and \( A_b = A'_b = B_b \) by Lemma 2.4 and hence, \( A_t = A'_t \) as well. This yields the trivial elementary link of type IV_s.

A composition of type (2) only appears in the Sarkisov link of type I_d (or type III_d as its inverse). In this case, we have a reduction \( B > B' := \pi^{-1}(B'_b \cup \{0\}) \cap B \) since the fiber structure \( B_t \subsetneq B \) is irreducible so that \( B \setminus B' = \pi^{-1}(B_b \setminus B'_b) \cap B \) consists of exactly one point. Thus we obtain the elementary link of type I_d,

\[ \begin{align*}
B > B' \\
\cup \\
B_t = B'_t
\end{align*} \]

(3.4)

with \( B_b > B'_b \) and \( B'_t := B' \cap L \), where L is the linear span of \( B_t \). Clearly, the fiber structure \( B'_t \subsetneq B' \) is irreducible and its base coincides with \( B'_b \) given in advance. We may put \( A = B \) and \( A' = B' \) (resp. \( A = B' \) and \( A' = B \)) for the Sarkisov link of type I_d (resp. type III_d). In either case, the opponent composition is of type (3) below.

For a type (3) composition, we have a sequence \( B > B' \supseteq B'_t \). Let v be the unique element in \( B \setminus B' \) and \( L' \) be the linear span of \( B'_t \). Then, depending on whether \( v \notin L' \) or \( v \in L' \), we can make the following diagrams, respectively:

\[ \begin{align*}
& B > B', \\
& \cup \quad \cup \\
B_t = B'_t \\
& B_t > B'_t
\end{align*} \]

(3.5)

In fact, in the case of \( v \notin L' \), it is clear that \( B_t := B'_t \subsetneq B \) defines a fiber structure. Furthermore, if the fiber structure is irreducible, the left diagram in (3.5) coincides with the elementary link
of type I_d (3.4), which is already discussed. If the fiber structure is not irreducible, we call the diagram a non-irreducible half-link, which should be coupled with another half-link. In the case of \( v \in L' \), we have a reduction \( B_t := B \cap L' > B'_t \) as in the right diagram in (3.5), and \( B_t \subset B \) becomes an irreducible fiber structure. We call the diagram an irreducible half-link I.

Finally, in the case of a type II, we denote by \( B'_t \) the base of the fiber structure \( B_{h,t} \equiv B_h \). Accordingly, we set a diagram of lattices and fit all relevant sets of lattice points together:

\[
\begin{align*}
N & \xrightarrow{\pi} N_h \xrightarrow{\pi_h} N'_h \quad B \quad \xrightarrow{\pi} B_h \xrightarrow{\pi_h} B'_h \\
\cup & \quad \cup & & \cup & \quad \cup \\
N'_h & \longrightarrow N_{h,t} & B'_t & \longrightarrow B_{h,t} & \\
\cup & \quad \cup & & \cup & \quad \cup \\
N_t & \longrightarrow \quad B_t
\end{align*}
\]

where \( N'_h = \ker(\pi_h \circ \pi) \) and \( B'_t = B \cap N'_h \). Since \( B_{h,t} \subset N_{h,t} \) is a primitive generating set, so is \( B'_t \subset N'_h \), and hence, \( B'_t \subset B \) and \( B_t \subset B'_t \) are fiber structures. By definition, Mori fiber structures \( B_t \subset B \) and \( B_{h,t} \subset B_h \) are irreducible, and hence, the equalities \( |B| = |B_t| + |B_h| = |B'_t| + |B_{h,t}| \) hold. Then the inequalities \( |B| \geq |B'_t| + |B'_h| \) and \( |B'_t| \geq |B_t| + |B_{h,t}| \) hold. Hence, both \( B'_t \subset B \) and \( B_t \subset B'_t \) are also irreducible. In particular, \( B_t \subset B'_t \) is a Mori fiber structure. Thus we obtain another diagram:

\[
B = B' \\
\cup & \quad \cup \\
B_t \subset B'_t
\]

We call the diagram an irreducible half-link II.

Now we combine two of half-links to compose remaining elementary links between \( A_t \subseteq A \) and \( A'_t \subseteq A' \), which is very easy. If the fiber structure in the middle is not irreducible, there is only one way to combine them, resulting the elementary link of type II_m in (3.2). If the fiber structure in the middle is irreducible, one has the remaining three ways, I-I, I-II, and II-II, except switching how to set \( A_t \subseteq A \) and \( A'_t \subseteq A' \), which results the remaining elementary links of type II_{irr}, type I_m, and type IV_m in (3.2), respectively. This completes the proof.

Corollary 3.2. For any pair of Mori fiber primitive generating sets \( A_t \subseteq A \) and \( A'_t \subseteq A' \) in \( N_R \), there exists a sequence of elementary links connecting them.

Proof. By Lemma 2.6 there exists a pair of toric Mori fiber spaces \( p : X \to S \) and \( p' : X' \to S' \) which represent \( A_t \subseteq A \) and \( A'_t \subseteq A' \), respectively. The \( p \) and \( p' \) are connected by a sequence of toric Sarkisov links by the Sarkisov program for toric varieties [Mat02 Section 14.5], [HM13]. By coarse-graining the resulting sequence of Sarkisov links, one obtains a sequence of elementary links connecting \( A_t \subseteq A \) and \( A'_t \subseteq A' \) by Theorem 3.1. We define a log Sarkisov link as one of the four types of diagrams in (3.1) but each divisorial contraction of \( X \)'s is not imposed to be K-negative.

Corollary 3.3. For any sequence of elementary links connecting two Mori fiber primitive generating sets \( A_t \subseteq A \) and \( A'_t \subseteq A' \), there exists a sequence of log Sarkisov links connecting two toric Mori fiber spaces \( p : X \to S \) and \( p' : X' \to S' \) which represent \( A_t \subseteq A \) and \( A'_t \subseteq A' \), respectively.

Proof. Assume \( A_t = A'_t \) and \( A = A' \) first. There is a natural birational map \( \varphi : S \dashrightarrow S' \) induced by the identity map of \( N_h \). By the first statement of Lemma 2.6, \( \varphi_b \) is decomposed into flips, flops, and inverse flips. Since each intermediate birational model \( S'' \) also represents the base \( A_h \)

7
of \( A_f \subset A \), there exists a toric Mori fiber space \( p'' : X'' \to S'' \) which represent \( A_f \subset A \) by the third statement of Lemma 2.6. Hence, by replacing \( p' \) with \( p'' \), one may assume \( \varphi_b = \) either a single flip, flop, or inverse flip. In either case, it fits a log Sarkisov link of type IV, because the natural map \( \varphi : X' \to X'' \) is decomposed into flips, flops, and inverse flips and \( \varphi_b \circ p = p' \circ \varphi \) holds.

Next, assume that \( A_f \subset A \) and \( A'_f \subset A' \) are related by a single elementary link. As in the first case, one can make a log Sarkisov link from the bottom of its diagram for each elementary link by applying Lemma 2.6 repeatedly. For this purpose, it is helpful to rewrite the elementary links in (3.2) into alternative descriptions by using the bases of fiber structures:

\[
\begin{array}{cccc}
type I_d & type I_m & type II_{m'/n_l} & type IV_m \\
A > A' & A = A'' > A' & A < A'' > A' & A = A'' = A' \\
\downarrow & \downarrow & \downarrow & \downarrow \\
A_b > A'_b & A_b \rightarrow A''_b = A'_b & A_b = A''_b = A'_b & A_b \rightarrow A''_b \leftarrow A'_b \\
\end{array}
\]

(3.8)

Finally, in general case, the log Sarkisov links obtained from each elementary link are glued together by the sequence of log Sarkisov links of type IV in the first case, so that one obtains a sequence of log Sarkisov links connecting \( p : X \to S \) and \( p' : X' \to S' \).

**Corollary 3.4.** For any sequence of elementary links connecting two Mori fiber polytopes \( \nabla_f \subset \nabla \) and \( \nabla'_f \subset \nabla' \) consisting of only Fano polytopes, there exists a sequence of Sarkisov links connecting two toric Mori fiber spaces \( p : X \to S \) and \( p' : X' \to S' \) which represent \( \nabla_f \subset \nabla \) and \( \nabla'_f \subset \nabla' \), respectively.

**Proof.** This is an immediate consequence of Corollary 3.3 and the last sentence of Remark 2.7. \( \square \)

In order to establish the whole picture only in terms of Fano polytopes, the following two problems could be keys, the latter of which is regarded as the Fano polytope version of Corollary 3.2.

**Problem 3.5.** For any minimal Fano polytope \( \nabla \), is there a Mori fiber structure \( \nabla_f \subset \nabla \)?

**Problem 3.6.** For any pair of Mori fiber polytopes \( \nabla_f \subset \nabla \) and \( \nabla'_f \subset \nabla' \) in \( N_R \), is there a sequence of elementary links consisting only of Fano polytopes which connects \( \nabla_f \subset \nabla \) and \( \nabla'_f \subset \nabla' \)?

**Remark 3.7.** Problem 3.3 is required to guarantee that the MMP works only inside the category of Fano polytopes. Namely, it assures that one ends up with a Mori fiber polytope after finite times of reductions of a given Fano polytope. Note that, by triangulating a minimal Fano polytope \( \nabla \) using only its vertices, one can always find a fiber structure \( \nabla_f \subset \nabla \) with a minimal Fano simplex \( \nabla_f \) (see [Kas10] Proposition 3.2, and also [KS02] Lemma 1) for the structure of minimal primitive generating sets). Hence, one part of the problem is whether a minimal Fano simplex \( \nabla_f \) allows a fiber structure \( \nabla_f \subset \nabla_f \) such that \( \nabla'_f \) is a terminal simplex.

The following example shows the subtleties of Problem 3.6 in three or higher dimensions. The obstacle stems from the fact that, for a relevant Fano polytope \( \nabla \), a projective simplicial terminal fan \( \Sigma \) satisfying \( \text{Conv} G(\Sigma) = \nabla \) needs not to be a projective \( \nabla \)-maximal fan.

**Example 3.8.** Set \( v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (-1,-1,0), v_4 = (-1,0,0), v_5 = (0,0,1), v_6 = (-1,0,-1), \) and \( v_7 = (-2,0,-1) \) in \( N \approx \mathbb{Z}^3 \). As an abbreviation, we write Fano polytopes \( \nabla_{i_1 \ldots i_r} = \text{Conv}(v_{i_1}, \ldots, v_{i_r}) \) and primitive generating sets \( A_{i_1 \ldots i_r} = \{v_{i_1}, \ldots, v_{i_r}\} \) with concatenated subscripts for any indices \( i_1, \ldots, i_r \). In the following, the latter symbol is used if and only if
\[ A_{1i,\ldots,i_r} \neq \nabla_{1i,\ldots,i_r} \cap N_{\text{prim}}. \] Then we have a sequence of elementary links connecting a pair of Mori fiber polytopes \( \nabla_{1i} \subset \nabla_{123457} \) and \( \nabla_{123} \subset \nabla_{1236}. \)

\[
\begin{align*}
\nabla_{123457} &= \nabla_{123457} > A_{12357} < \nabla_{12356} > \nabla_{123} \\
\nabla_{14} &= \nabla_{1234} > \nabla_{123} = \nabla_{123} = \nabla_{123}.
\end{align*}
\] (3.9)

The sequence (3.9) corresponds to that of Sarkisov links between toric Mori fiber spaces, which are all projective bundles (cf. [CLS11, Proposition 7.3.3]),

\[
\begin{array}{c}
\mathbb{P}^1 \times \mathbb{P}^1 ((\mathcal{O} \oplus \mathcal{O}(-1,-2)) \xrightarrow{\text{I}_{a}} \mathbb{P}^1 (\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2)) \xrightarrow{\text{II}_a} \mathbb{P}^1 (\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)),
\end{array}
\] (3.10)

where we put the types of the Sarkisov links on dashed arrows. Note that the coarse-graining of the sequence (3.9) into Fano polytopes does not attain a sequence of elementary links. On the other hand, one can also take another sequence of elementary links connecting the same pair of Mori fiber polytopes which consists only of Fano polytopes:

\[
\begin{align*}
\nabla_{123457} &< \nabla_{1234567} > \nabla_{123456} = \nabla_{12356} > \nabla_{123} \\
\nabla_{14} &= \nabla_{1234} > \nabla_{123} = \nabla_{123} = \nabla_{123}.
\end{align*}
\] (3.11)

The sequence (3.11) corresponds to the following sequence of Sarkisov links:

\[
\begin{array}{c}
\mathbb{P}^1 \times \mathbb{P}^1 ((\mathcal{O} \oplus \mathcal{O}(-1,-2)) \xrightarrow{\text{II}_a} \mathbb{P}^1 (\mathcal{O} \oplus \mathcal{O}(-1,-1)) \xrightarrow{\text{I}_{a}} \mathbb{P}^1 (\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)).
\end{array}
\] (3.12)

Now, let us see that the situation becomes better in two dimensions once we use the classification of the Sarkisov links for smooth rational Mori fiber surfaces:

\[
\begin{array}{cccc}
type I_m & type II_{ni} & type III_m & type IV_m \\
F_{m} & S_x & F_{m} & F_{m+1} \\
\mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^2 & \mathbb{P}^2 \\
\mathbb{P}^1 & \mathbb{P}^1 & \mathbb{P}^1 & \mathbb{P}^1 \\
pt & pt & pt & pt
\end{array}
\] (3.13)

Here \( F_m = \mathbb{P}^1 (\mathcal{O} \oplus \mathcal{O}(-m)) \) is the Hirzebruch surface for \( m \geq 0 \), \( p_i : F_0 = \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) is the \( i \)-th projection \( (i = 1, 2) \), and \( \text{bl}_x : S_x \rightarrow F_m \) is a blow-up at a point \( x \in F_m \). The Sarkisov link of type \( II_{ni} \) in (3.13) is classically known as an elementary transform, that is, a blow-up at a point \( x \in F_m \) followed by the contraction of the strict transform of the fiber passing through \( x \) with respect to a ruling \( F_m \rightarrow \mathbb{P}^1 \) [Har77, Example 5.7.1]. We denote it by

\[
eq \text{el}_x : F_m \rightarrow F_{m+1}
\] (3.14)

in the following. Note that, for \( m \geq 1 \), the elementary transform at \( x \in F_m \) results \( F_{m+1} \) or \( F_{m-1} \) depending on whether \( x \) lies on the section \( D \subset F_m \) with negative self-intersection, \( D^2 = -m \).

**Proposition 3.9.** Problem 3.5 and Problem 3.6 are affirmative in two dimensions.

**Proof.** It is easy to verify that all smooth toric Mori fiber surfaces \( \mathbb{P}^2 \) and \( F_m \) \( (m \geq 0) \) are described by Mori fiber polygons, and all toric Sarkisov links in (3.13) are described by elementary links consisting only of Fano polygons (see Section 4 for concrete forms). \qed
4. Two Proofs of Theorem 1.3

In this section, we give a constructive proof of Theorem 1.3 first, and then a non-constructive proof of it. The latter proof uses a stronger claim (Lemma 4.2), which also implies the connectedness of the web of elliptic elephants in smooth projective rational surfaces (Corollary 4.5).

Constructive proof of Theorem 1.3. Recall that reflexive is equivalent to canonical for lattice polygons. By repeating reductions for a canonical polygon, one ends up with a canonical Mori fiber polygon by Proposition 3.9. Then it suffices to show that any pair of canonical Mori fiber polygons is connected via a sequence of elementary links whose intermediate polygons are all canonical. The same is true for terminal polygons. Up to unimodular equivalences, there are only three terminal Mori fiber polygons \( \nabla_{-\infty}, \nabla_0, \nabla_1 \) and four canonical Mori fiber polygons \( \nabla_{-\infty}, \nabla_0, \nabla_1, \nabla_2 \). Here, for any integer \( m \geq 0 \) and a fixed lattice basis \( \{e_1, e_2\} \) of \( N \cong \mathbb{Z}^2 \), we set

\[
\nabla_{-\infty} := \text{Conv}(e_1, e_2, -e_1 - e_2) \quad \text{and} \quad \nabla_m := \text{Conv}(e_1, e_2, -e_1, -me_1 - e_2) \quad (4.1)
\]

corresponding to \( \mathbb{P}^2 \) and \( \mathbb{F}_m \), respectively. There is an elementary link of type II connecting \( \nabla_m \) and \( \nabla_{m+1} \),

\[
\nabla_m \prec \text{Conv}(\nabla_m \cup \nabla_{m+1}) \succ \nabla_{m+1}
\]

(4.2)

The elementary link (4.2) and its inverse are denoted by \( \ell^+_m \) and \( \ell^-_m \), respectively. It is easy to observe that all polygons in \( \ell^+_0 \) and all polygons in \( \ell^-_1 \) are canonical. By Lemma 4.1 below, we may assume each Mori fiber polygon is moved to standard form (4.1) in advance. By using \( \ell^+_m \) and another elementary link \( \ell^-_{-\infty} \) of type III,

\[
\nabla_{-\infty} \prec \text{Conv}(\pm e_1) \succ \nabla_{-\infty} \prec \text{Conv}(\pm e_1)
\]

(4.3)

with its inverse \( \ell^-_{-\infty} \), each pair of terminal Mori fiber polygons from \( \{\nabla_{-\infty}, \nabla_0, \nabla_1\} \) (resp. canonical Mori fiber polygons from \( \{\nabla_{-\infty}, \nabla_0, \nabla_1, \nabla_2\} \)) can be connected via a sequence of elementary links whose intermediate polygons are all terminal (resp. canonical). This completes the proof. □

Lemma 4.1. With the same notation as above, any unimodular transformation of \( \nabla_{-\infty} \) (resp. \( \nabla_m \) for \( m \geq 0 \)) is decomposed into a sequence of elementary links unimodular equivalent to \( \ell^+_0 \), \( \ell^+_{-\infty} \) and \( \ell^-_j \) (resp. \( \ell^+_0, \ell^-_j \) with \( 0 \leq j \leq \max\{1, m - 1\} \)), where \( \ell^+ \) is the following elementary link of type IV,

\[
\nabla_0 \prec \text{Conv}(\pm e_1) \succ \nabla_0 \prec \text{Conv}(\pm e_2)
\]

(4.4)

and \( \ell^- \) is its inverse. In particular, any terminal (resp. canonical) Mori fiber polygons is connected to that in standard form (4.1) through a sequence of elementary links whose intermediate polygons are all terminal (resp. canonical).

Proof. It is enough to show the claim only for the pairs of polygons, either \( \nabla_{-\infty} \) or \( \nabla_m \) in standard form (4.1) and one of those moved by the action of a generator of \( GL(N) \cong GL(2, \mathbb{Z}) \), say,

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(4.5)
Indeed, any other pair of transformed polygons is related by a finite composition of some conjugates of the above moves, and, for such a conjugate move, it can be decomposed into a sequence of the corresponding conjugates of the elementary links. Thus the claim can be verified by direct computation. For example, the pair $\nabla_{-\infty}$ and $S\nabla_{-\infty}$ is connected by a sequence,

$$
\begin{array}{c}
\begin{array}{c}
\text{Mfp}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
<
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
=
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
<
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
>
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
<
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
>
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
=
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
>
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Mfp}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Mfp}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Mfp}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Mfp}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Mfp}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Mfp}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Mfp}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Mfp}
\end{array}
\end{array}
\end{array},
\tag{4.6}
\end{array}
$$

where we encode the fiber structure $\nabla_f \subset \nabla$ as the thick interval overlaid in gray if $\dim \nabla_f = 1$ and as the polytope filled in gray if $\nabla_f = \nabla$. Also, the marker “Mfp” indicates that $\nabla_f \in \nabla$ is a Mori fiber polygon. If we write the conjugate and the composition of elementary links as in ordinary maps, the sequence \([4.6]\) is expressed as $S\nabla = (U_{-\infty} l^{-1}) \circ (U l^0 U^{-1}) \circ l^0 \circ t^+ (\nabla)$. Similarly, as another example, the pair $\nabla_m$ and $S\nabla_m$ is connected by a sequence $S\nabla_m = (S l^+_m t^- S^{-1}) \circ \cdots \circ (S l^+_m S^{-1}) \circ t^+ \circ l^+_1 \circ \cdots \circ l^+_{m-1} (\nabla_m)$. Remaining four pairs are left to the reader.

Non-constructive proof of Theorem 1.3. Let us set $\mathbb{F}_\infty = \mathbb{F}^2$ for convenience. For any sequence $s$ of Sarkisov links in \([3.13]\), we write $M(s)$ as the maximal $m$ such that $\mathbb{F}_m$ appears in the sequence $s$, and $M(\mathbb{F}_m) = m$ for all $m \in \{-\infty, 0, 1, \ldots\}$.

**Lemma 4.2.** Let $p : X \to S$ and $p' : X' \to S'$ be smooth rational Mori fiber surfaces and $\varphi : X \dashrightarrow X'$ be any birational map. Then $\varphi$ is decomposed into a sequence $s$ of Sarkisov links such that $M(s) \leq \max \{1, M(X), M(X')\}$.

**Proof.** By the Sarkisov program in two dimensions, any birational map $\varphi : X \dashrightarrow X'$ can be decomposed into a sequence of Sarkisov links. Suppose $M(s) > \max \{1, M(X), M(X')\}$ for a sequence $s$ decomposing $\varphi$. Then $s$ should be locally in the following form:

$$
\begin{array}{c}
\begin{array}{c}
s : \cdots \longrightarrow \mathbb{F}_{M(s) - 1} \longrightarrow \mathbb{F}_{M(s)} \longrightarrow \mathbb{F}_{M(s) - 1} \longrightarrow \cdots,
\end{array}
\end{array}
\tag{4.7}
$$

where $x \in \mathbb{F}_{M(s) - 1}$ is on the curve of negative self-intersection, $y \in \mathbb{F}_{M(s)}$ is away from the curve of negative self-intersection, and $e_l$ and $e_y$ are elementary transforms \([3.11]\). If $y$ is not an infinitely near point to $x$ (i.e., if $y$ does not lie on the strict transform of the exceptional curve of the blow-up of $\mathbb{F}_{M(s) - 1}$ at $x$), one can replace $s$ with the following sequence $s'$ by switching the order of taking elementary transforms:

$$
\begin{array}{c}
\begin{array}{c}
s' : \cdots \longrightarrow \mathbb{F}_{M(s)} \longrightarrow \mathbb{F}_{M(s) - 2} \longrightarrow \mathbb{F}_{M(s) - 1} \longrightarrow \cdots,
\end{array}
\end{array}
\tag{4.8}
$$

where we denote the corresponding points $x \in \mathbb{F}_{M(s) - 2}$, $y \in \mathbb{F}_{M(s) - 1}$ as the same symbols. Similarly, if $y$ is an infinitely near point to $x$ (which is only redundant in toric case), by taking another elementary transform at $z \in \mathbb{F}_{M(s) - 1}$ in advance, one can replace $s$ with the following sequence $s'$ (by using elementary relations in the sense of \([Kal13]\) twice):

$$
\begin{array}{c}
\begin{array}{c}
s' : \cdots \longrightarrow \mathbb{F}_{M(s)} \longrightarrow \mathbb{F}_{M(s) - 2} \longrightarrow \mathbb{F}_{M(s) - 1} \longrightarrow \mathbb{F}_{M(s) - 2} \longrightarrow \mathbb{F}_{M(s) - 1} \longrightarrow \cdots,
\end{array}
\end{array}
\tag{4.9}
$$

where $z$ is chosen away from neither the curve of negative self-intersection nor the fiber passing through $x$, and $e_l^{-1}$ denotes the inverse of $e_l$ away from the fiber passing through $x$. In either case, the number of times that $\mathbb{F}_{M(s)}$ appears is strictly decreased in $s'$. Thus, after a finite number of replacing, one may have $M(s') < M(s)$. Repeating the same process, one obtains a sequence $s$ that satisfies $M(s) \leq \max \{1, M(X), M(X')\}$ in the end. \qed
Let $\nabla$ and $\nabla'$ be any pair of reflexive, and hence, canonical polygons. By Proposition 3.9, one may assume that those are canonical Mori fiber polygons by repeating reductions in advance. By Lemma 2.6 for any pair of canonical Mori fiber polygons $\nabla_f \oplus \nabla$ and $\nabla'_f \oplus \nabla'$, there is a pair of smooth toric Mori fiber surfaces $p : X \to S$ and $p' : X' \to S'$ which respectively represent $\nabla_f \oplus \nabla$ and $\nabla'_f \oplus \nabla'$ and the natural birational map $\varphi : X \dashrightarrow X'$ (all of which are uniquely determined in two dimensions). Note that $X$ and $X'$ are either $F_m$ ($m \leq 2$) in this case. Clearly from the proof, the sequence in Lemma 4.2 can be chosen as a sequence of toric Sarkisov links for the equivariant birational map $\varphi : X \dashrightarrow X'$. Therefore, the sequence of Sarkisov links can be coarse-grained into a sequence of elementary links consisting only of Fano polygons, which connects $\nabla_f \oplus \nabla$ and $\nabla'_f \oplus \nabla'$ again by Proposition 3.9. From the concrete description of elementary links, all the intermediate Fano polygons turn out to be canonical. The same argument also works for terminal Mori fiber polygons. This completes the proof. 

Remark 4.3. Lemma 4.2 is a variant of the classical Noether–Castelnuovo theorem, which states that the Cremona group $\text{Bir}(\mathbb{P}^2)$ is generated by automorphisms in $\text{Aut}(\mathbb{P}^2) \simeq \text{PGL}(3, \mathbb{C})$ and the standard quadratic transformation $\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ written as $[a : b : c] \mapsto [\frac{a}{b} : \frac{b}{c} : \frac{c}{a}]$. In fact, for a given birational automorphism of $\mathbb{P}^2$, one can easily rewrite the sequence obtained in Lemma 4.2 into the composition of the transformations described as

$$h^{-1} \circ \sigma \circ g : \mathbb{P}^2 \xrightarrow{\text{bl}_{x}} \mathbb{F}_1 \xrightarrow{\text{el}_y} \mathbb{F}_0 \xrightarrow{\text{el}_z} \mathbb{F}_1 \xrightarrow{\text{bl}_{x}} \mathbb{P}^2$$

(4.10)

by inserting redundant Sarkisov links. Here in (4.10), $x, y, z \in \mathbb{P}^2$ are any three points not on a line, $g \in \text{Aut}(\mathbb{P}^2)$ is an automorphism which moves $x, y, z$ to the torus fixed points, and $h \in \text{Aut}(\mathbb{P}^2)$ is defined to produce $\sigma$ by fixing torus action depending on the choice of $g$.

To clarify the relationship between Lemma 4.2 and Reid’s fantasy, we introduce the notion of geometric transitions for Calabi–Yau elephants in arbitrary dimensions.

Definition 4.4. Let $X$ and $X'$ be normal projective varieties whose general elephants $D$ and $D'$ are Calabi–Yau varieties, and $\varphi : X \to X'$ be a birational contraction. We say that the Calabi–Yau varieties $D$ and $D'$ are related by a geometric transition associated with $\varphi$ if $\varphi|_D : D \to \varphi_* D$ is a birational contraction to a normal variety, and $\varphi_* D$ and $D'$ are related by a flat deformation inside $X'$. The opposite operation is also called a geometric transition associated with $\varphi^{-1}$. Furthermore, if $\varphi|_D$ is an isomorphism, and $\varphi_* D$ and $D'$ are related by a smooth deformation inside $X'$, we say that $D$ and $D'$ are related by a smooth transition.

Corollary 4.5. Let $X$ and $X'$ be smooth projective rational surfaces whose general elephants $E$ and $E'$ are smooth elliptic curves. Then any birational map $\varphi : X \dashrightarrow X'$ is decomposed into a sequence of blow-ups and blow-downs such that $E$ and $E'$ are connected via a sequence of smooth transitions associated with them.

Proof. Let $X$ be a smooth projective rational surface whose general elephant $E$ is a smooth elliptic curve. Then any $(-1)$-curve $C$ on $X$ intersects with $E$ at one point by the adjunction formula $C.(C + K) = -2$. Hence, a contraction of $(-1)$-curves sends $E$ to its isomorphic image without changing the property of being an elephant. By contracting all $(-1)$-curves in advance, one may assume that $X$ and $X'$ are smooth rational Mori fiber surfaces whose elephants are elliptic curves, i.e., $F_m$ ($m \leq 2$). Now, the birational map $\varphi : X \dashrightarrow X'$ is decomposed into a sequence of Sarkisov links obtained by Lemma 4.2. In each Sarkisov link in the sequence, all birational models allow elliptic elephants since they are described by reflexive polygons in its toric description. Thus, the elliptic elephants are related via a smooth transition at each step of the Sarkisov links. More
precisely, for a blow-up at a point, one can smoothly specialize \( E \) to another elephant passing through the center of the blow-up and send it to its isomorphic strict transform that is again an elephant in a new ambient space. \( \square \)

**Remark 4.6.** Corollary \[4.5\] is compared with the Sarkisov program for Mori fibered Calabi–Yau pairs established in arbitrary dimensions by [CK16], which shows, in the setting of Corollary \[4.5\], \( E \) and \( E' \) are connected only by a sequence of strict transforms if \( \varphi : (X, E) \dashrightarrow (X', E') \) is a volume-preserving birational map of Calabi–Yau pairs, i.e., \( p^*(K_X + E) = q^*(K_{X'} + E') \) for a common log resolution \((p, q) : W \to X \times X'\).

### References

[ACG16] Michela Artebani, Paola Comparin, and Robin Guilbot, *Families of Calabi-Yau hypersurfaces in \(Q\)-Fano toric varieties*, J. Math. Pures Appl. (9) **106** (2016), no. 2, 319–341. MR 3515305

[Bat94] Victor V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), no. 3, 493–535. MR MR1269718 (95c:14046)

[Bat17] Victor Batyrev, *The stringy Euler number of Calabi-Yau hypersurfaces in toric varieties and the Mavlyutov duality*, Pure Appl. Math. Q. **13** (2017), no. 1, 1–47. MR 3585013

[CK16] Alessio Corti and Anne-Sophie Kaloghiros, *The Sarkisov program for Mori fibred Calabi-Yau pairs*, Algebr. Geom. **3** (2016), no. 3, 370–384. MR 3504536

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322

[Cor95] Alessio Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** (1995), no. 2, 223–254. MR 1311348

[Fre15] Karl Fredrickson, *Generalized compactifications of Batyrev hypersurface families*, 2015, arXiv/1410.8287.

[FS04] Osamu Fujino and Hiroshi Sato, *Introduction to the toric Mori theory*, Michigan Math. J. **52** (2004), no. 3, 649–665. MR 2097403

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

[HM13] Christopher D. Hacon and James McKernan, *The Sarkisov program*, J. Algebraic Geom. **22** (2013), no. 2, 389–405. MR 3019454

[Kal13] Anne-Sophie Kaloghiros, *Relations in the Sarkisov program*, Compos. Math. **149** (2013), no. 10, 1685–1709. MR 3123306

[Kas10] Alexander M. Kasprzyk, *Canonical toric Fano threefolds*, Canad. J. Math. **62** (2010), no. 6, 1293–1309. MR 2760660

[Kaw08] Yujiro Kawamata, *Flops connect minimal models*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 419–423. MR 2426353

[KS02] Maximilian Kreuzer and Harald Skarke, *Reflexive polyhedra, weights and toric Calabi-Yau fibrations*, Rev. Math. Phys. **14** (2002), no. 4, 343–374. MR 1901220 (2003b:14062)

[Mat02] Kenji Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002. MR 1875410

[Mav11] Anvar R. Mavlyutov, *Mirror Symmetry for Calabi–Yau complete intersections in Fano toric varieties*, 2011, arXiv/1103.2093.

[Rei83] Miles Reid, *Decomposition of toric morphisms*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 395–418. MR 717617

[Rei87] _______, *The moduli space of 3-folds with \( K = 0 \) may nevertheless be irreducible*, Math. Ann. **278** (1987), no. 1-4, 329–334. MR 909231 (88h:32016)

[Ska96] Harald Skarke, *Weight systems for toric Calabi-Yau varieties and reflexivity of Newton polyhedra*, Modern Phys. Lett. A **11** (1996), no. 20, 1637–1652. MR 1397227

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

Email address: miirror.jp@gmail.com