Finding Eigenvalues the Rupert Way

Colin Grudzien and Christopher K.R.T. Jones

March 28, 2014
1 Abstract

The goal of this paper is to verify the numerical method developed by Rupert Way and Tom Bridges in the PhD Thesis “Dynamics in the Hopf bundle, the geometric phase and implications for dynamical systems” [9] for $\mathbb{C}^2$, to generalize the method for higher dimensional systems, and to extend the method to boundary value problems. Implementing the numerical methods for calculating the geometric phase in the Hopf bundle developed in Way’s thesis, we will conclude with open questions introduced by the computational methods developed by Rupert Way.

Way conjectured that the parallel transport generated in the Hopf bundle through the embedding of $S^{2n-1} \subset \mathbb{C}^n$ could measure the multiplicity of of eigenvalues contained within a contour in the spectral plane. The spectral parameter is introduced through the eigenvalue problem $(L - \lambda)u = 0$ where $u$ is a steady state for a PDE and $L$ is the associated linear operator about the state. For the conjecture, Way assumed that the flow generated on $\mathbb{C}^n$ through $(L - \lambda)$ is non-autonomous with asymptotically autonomous systems at $\pm\infty$.

Way proposed using a solution which approximates the stable manifold for the asymptotic system at $+\infty$ and projecting the solution onto the Hopf bundle $S^{2n-1}$ to measure the parallel transport along the spectral path. Provided the approximate solution is integrated “close” to the asymptotic system at $-\infty$, Way conjectured that the phase along the spectral path should agree with the multiplicity of the eigenvalues contained within the contour. A full development of the method for $n = 2$ is in section 2 and 3, and the adaptations the authors have made to this method for $n \geq 2$ will be developed in section 4. Adaptations to this method for boundary value problems are treated in section 5, and results with Way’s numerical method adapted for the authors’ construction are presented in section 6.
2 Introduction

2.1 Motivation of Techniques

The motivation for considering the dynamics on the Hopf bundle comes from the analogous methods for detecting eigenvalues of linear operators with the Evans function. This problem arises when considering the stability of travelling wave solutions as steady states for PDE’s, where often the question of stability is reduced to the existence of eigenvalues of positive real part for the linearized operator corresponding to the PDE [5].

A motivating example from the stability analysis of travelling waves is the system of ODE’s derived from systems of reaction diffusion equations of the form:

\[ U_t = U_{xx} + F(U) \quad U(x,0) = U_0(x) \in \mathbb{R}^n \]

where \( F \) is a non-linearity. A well studied example of this type of system is the case where \( u \in \mathbb{R} \) is a single equation, and the non-linear system is given:

\[
\begin{align*}
  u_t &= u_{xx} + f(u) \\
  f(u) &= u(u-a)(1-u)b \\
  -1 &\leq a \leq \frac{1}{2} \quad b \neq 0 \\
  u(x,0) &= u_0(x)
\end{align*}
\]

(1)

The value \( x \) is assumed to be in \( \mathbb{R} \) so that we can find a travelling wave solution, ie: a solution of the single variable \( \xi = x - ct \), so \( U(\xi) \) satisfies:

\[-cU'' = U''' + f(U)\]

(\( ' = \frac{d}{d\xi} \))

The system (1) is re-written as

\[ u_t = u_{\xi\xi} + cu_{\xi} + f(u) \]

(2)

for which the travelling wave is a a time independent solution. Linearizing (2) about the steady state, travelling wave solution we obtain the \( \xi \) dependent operator \( \mathcal{L} \) such that:

\[ \mathcal{L}(p) = p_{\xi\xi} + cp_{\xi} + f'(u)p \]

with \( p \) in the appropriate function space. For \( \lambda \in \mathbb{C} \) we consider the equation

\[ (\mathcal{L} - \lambda I)(p) = 0 \]

that has the equivalent formulation as the system

\[
\begin{align*}
  p' &= q \\
  q' &= -cq + (\lambda - f'(u))p
\end{align*}
\]

If we let \( z = (p, q) \in \mathbb{C}^2 \), we can write the above as \( z' = A(\lambda, \xi)z \) where

\[
A(\lambda, \xi) = \begin{pmatrix} 0 & 1 \\ \lambda - f'(u) & -c \end{pmatrix}
\]

(3)
The system \( z' = A(\lambda, \xi)z \) is a non-autonomous with dependence on \( u(\xi) \); the travelling wave solution \( u(\xi) \) must be bounded as \( \xi \to \pm \infty \), and we assume \( \lim_{\xi \to \pm \infty} u(\xi) = u(\pm \infty) \). There are thus asymptotic autonomous systems \( z' = A_{\pm \infty}(\lambda)z \) defined by

\[
A_{\pm \infty}(\lambda) := \lim_{\xi \to \pm \infty} A(\lambda, \xi) = \begin{pmatrix} 0 & 1 \\ \lambda - f'(u(\pm \infty)) & -c \end{pmatrix}
\]

The values of \( \lambda \) that satisfy \( L(p) = \lambda p \) are called \textit{temporal eigenvalues} while the eigenvalues of the \( A_{\pm \infty} \) systems are denoted the \textit{spatial eigenvalues}. Eigenfunctions for \( L \) must be bounded, and the spatial eigenvalues determine the growth and decay rates of solutions to the operator. We are thus motivated to investigate the spatial eigenvalues and eigenvectors of \( A_{\pm \infty} \).

The un/stable eigenvectors of the system at \( \pm \infty \) will help us determine the behaviour of solutions which lie in the un/stable manifolds of the critical points of the asymptotic systems; these solutions will allow us to demonstrate the existence of eigenfunctions and temporal eigenvalues for the operator \( L \) by constructing solutions which decay at \( \pm \infty \). This method is the Evans function construction outlined in [2].

The Evans function itself is a \( \mathbb{C} \)-analytic function whose zeros, including multiplicity, correspond to temporal eigenvalues \( \lambda \) satisfying \( (L - \lambda)v = 0 \). Therefore, the strategy for determining stability of the wave with the Evans function is as follows:

1. Through analytic arguments, bound the possible temporal eigenvalues of \( L \) in the right half of the complex plane to some domain \( D \)
2. Choose a contour \( K \) which encloses the domain, but does not intersect the spectrum of \( L \).
3. Use the argument principle with the Evans function to count the zeros enclosed by \( K \), and the multiplicity of the eigenvalues with positive real part.

An elegant setting for constructing the Evans function is the unstable bundle developed in [2], where we can take advantage of the unique classification of complex vector bundles over spheres with their Chern class. It is in this setting we will frame the discussion of the eigenvalue problem so that we can establish
the link between the existing Evans function method and the method conjectured by Rupert Way.

2.2 The Unstable Bundle

Let $L$ be a linear operator with spectrum $\sigma(L)$, and suppose we derive a flow on $\mathbb{C}^n$ from the equation $(L - \lambda)u = 0$ where $\lambda \in \Omega \subset \mathbb{C}$:

$$Y' = A(\lambda, \xi)Y \quad A_{\pm \infty}(\lambda) := \lim_{\xi \to \pm \infty} A(\lambda, \xi)$$

Assume $\Omega$ is open, simply connected and contains only discrete eigenvalues of $L$. We ask that the asymptotic systems are always hyperbolic and “split” in $\Omega$, ie: $A_{\pm \infty}$ each have $k$ unstable and $n-k$ stable complex directions in $\mathbb{C}^n$ respectively for every $\lambda \in \Omega$. Let $K$ be a smooth, simple closed curve in $\Omega \subset \mathbb{C}$ such that there is no spectrum of $L$ in $K$, and let $K$ be parametrized by $\lambda(s) : [0, 1] \mapsto K$.

The asymptotic systems $A_{\pm \infty}(\lambda)$ allow us to introduce a dependent variable which is morally a compactification of time. Let us define

$$\xi =: \frac{1}{2\kappa} \ln \left( \frac{1+\tau}{1-\tau} \right)$$

We append the dependent variable to obtain the system

$$Y' = A(\lambda, \tau)Y \quad \tau' = \frac{d}{d\xi}$$

where the operator $A(\lambda, \tau)$ is defined by

$$A(\lambda, \tau) = \begin{cases} 
A(\lambda, \xi(\tau)) & \text{for } \tau \neq \pm 1 \\
A_{\pm \infty}(\lambda) & \text{for } \tau = \pm 1 
\end{cases}$$

On finite time scales the flow is smooth by linearity, but [2] shows there are generic conditions on $\kappa$ related to the travelling wave, from which we derive $L$, that guarantee the flow to be $C^1$ on the entire compact interval. This compactification of time is somewhat natural in this setting, as the asymptotic conditions allow us to “cap” the phase space with a natural boundary system. Within the invariant planes $\{\tau = \pm 1\}$, the dynamics are governed entirely by the linear equations

$$X' = A_{\pm \infty}X \quad \tau' = 0 \quad (\tau = \pm 1)$$

so that solutions in the planes are determined entirely by the stable and unstable directions of the asymptotic system. Solutions in $\{\tau \in (-1, +1)\}$ will reach a limit in the invariant planes in infinite forward and backward time. The alpha and omega limit sets of such a solution are invariant, so the limit of a solution as it reaches the invariant planes must be unbounded or zero; this is because the only fixed points of the asymptotic systems are $(0, \pm 1) \in \mathbb{C}^n \times \{\tau = \pm 1\}$ by the linearity of the flows.

The advantage of framing the dynamics in the asymptotically capped system is twofold. Firstly, with the appended time parameter we can exploit the
geometry induced by the flow of time as a spacial variable. Secondly and most importantly, we now have a set up to use invariant manifold theory for critical points. Specifically we will consider the un/stable manifolds of the critical points \((0, \pm 1) \in \mathbb{C}^n \times \{ \tau = \pm 1 \}\) in the flow on \(\mathbb{C}^n \times \{ \tau \in [-1, +1] \}\). As mentioned above, the dynamics in the invariant planes is linear with \(k\) unstable directions and \(n-k\) stable directions; with the appended \(\tau\) equation, the system gains one real unstable/stable direction at \(\tau = \pm 1\) respectively. Standard invariant manifold theory dictates that there is a \(2k + 1\) (real) dimensional local unstable manifold in some neighborhood of \((0, -1)\) that can be extended globally by taking its flow forward for all time. In the invariant plane \(\tau = -1\), the unstable manifold is just the collection of unstable eigen directions, but for \(\tau > -1\), this becomes a \(\tau\) dependent subspace of \(\mathbb{C}^n\). A solution is bounded as \(\xi \to \pm \infty\) if and only if it is in the un/stable manifold of \((0, \pm 1)\) respectively, and \([2]\) shows that \(\lambda \in \Omega\) is an eigenvalue if and only if there is a non-trivial intersection of these two manifolds for \(\lambda\).

At its heart, we are constructing a topological winding number which matches the multiplicity of the temporal eigenvalues contained in the region bounded by \(K\), and we want to describe the winding in terms of vector bundles on the sphere. We will use the curve \(K\) and the time interval to construct a “parameter sphere”, above which we can view solutions to the flow as paths in an appended trivial \(\mathbb{C}^n\) bundle. Firstly consider, \(K \times \{ \tau \in [-1, +1] \}\) defines a topological cylinder as \(K\) is topologically equivalent to \(S^1\); if we cap this cylinder with \(K^\circ\), the region enclosed by \(K\) in the complex plane, we obtain a topological 2-sphere, \(M \equiv K \times \{ \tau \in [-1, +1] \} \cup K^\circ \times \{ \tau = \pm 1 \}\). The trivial \(\mathbb{C}^n\) bundle where we view solutions parameterized by \(M\) is thus defined \(M \times \mathbb{C}^n\). It is in this space we would like to construct the winding number which measures the multiplicity of the eigenvalues in \(K^\circ\).
However, the ambient \( \mathbb{C}^n \) bundle doesn’t acquire any winding as \( \tau \) moves between \( \pm 1 \), and we need to choose instead a subspace that “picks up” information from the flow. The natural candidates in this situation are the un/stable manifolds of the critical points in the invariant planes. Indeed, \([2]\) shows that for fixed \( \lambda \in K \) the unstable manifold of \((0, -1)\) will converge to the unstable space at \( \tau = 1 \) as a subspace, ie: in Grassmann norm. We may extend the unstable manifold to \( \{ \lambda \in K^o \} \times \{ \tau = 1 \} \) by considering the unstable directions for the linear flow of \( A_{+\infty}(\lambda) \). This unstable manifold will be a non-trivial vector bundle contained in the trivial, ambient vector bundle \( M \times \mathbb{C}^n \) and forms the “unstable bundle”. This vector bundle has a unique Chern class equal to the number of temporal eigenvalues in \( K^o \) including multiplicity. We will recast the method of Rupert Way into the framework of the unstable bundle to exploit the similarities of the methods. For a full discussion of the unstable bundle the reader is referred to \([2]\).
2.3 The Method of Rupert Way

The Hopf bundle is a principle fibre bundle with full space \( P \equiv S^{2n-1} \), base space \( M \equiv \mathbb{C}P^{n-1} \), and fibre \( G \equiv S^1 \); \( S^1 \) acts naturally on \( S^{2n-1} \) by scalar multiplication, and with respect to this action, the quotient is exactly \( \mathbb{C}P^{n-1} \).

The standard coordinate atlas for the Hopf bundle \( S^{2n-1} \subset \mathbb{C}^n \) is defined by \( U_j = \{(z_1, \ldots, z_j, \ldots, z_n) \in S^{2n-1} : z_j \neq 0\} \); these neighbourhoods yield an open covering of \( S^{2n-1} \) which define an associated open covering, \( \{\pi(U_j) = V_j\} \), for \( \mathbb{C}P^{n-1} \). Trivializations are thus defined by \( \psi_j : U_i \rightarrow V_j \times S^1 \) \( \psi_j^{-1} : V_j \times S^1 \rightarrow U_i \) so that their composition gives a self diffeomorphism [8].

There exists an intuitive choice of connection between fibres for all real dimensions \( 2n-1 \), and the choice is unique when the dimension is 3 [9]. Consider, \( S^{2n-1} \) is always naturally embedded in \( \mathbb{C}^n \); we will use this embedding to define the connection.

**Definition 1. The Natural Connection** If \( P = S^{2n-1} \) is the Hopf bundle, viewed in coordinates embedded in \( \mathbb{C}^n \), we define the connection 1-form \( \omega \) pointwise for \( p \in P \) as a mapping of the embedded tangent space \( T_p(S^{2n-1}) \subset T_p(\mathbb{C}^n) \)

\[
\omega_p : T_p(S^{2n-1}) \rightarrow \mathbb{R} \cong \mathfrak{g}
\]

(9)

where \( \mathfrak{g} \) is the Lie algebra of the fibre \( G = S^1 \). Define \( \omega \) to be the natural connection on the Hopf bundle.

By the choice of a connection we decompose the tangent space \( T S^{2n-1} \) of the Hopf bundle into horizontal and vertical subspaces, which allows us to introduce parallel translation in the fibres. The vertical subspace is always canonically defined by the kernel of the projection operator and the horizontal subspace is isomorphic to the tangent space of the base space, but is defined by the choice of the connection. In particular, the horizontal subspace is defined as the kernel of the connection 1-form.

We may always choose a corresponding “horizontal lift” to a given path which has the same projection onto the base manifold, but that always has tangent vector in the horizontal space. A path having its tangent vector always in the horizontal subspace is equivalent to saying that it experiences no motion in the fibre. Parallel translation is the displacement in the fibres between a given path and the horizontal lift of the path. The fibre of the Hopf bundle is the circle, so parallel translation induces a natural winding number for loops in \( P \) through the displacement in the circle.
In the above Figure 2.3, \( v_t \) is a path in the Hopf bundle with non-trivial motion in the fibre, whereas \( u_t \) represents the horizontal lift. Both of these paths have the same projection onto the base manifold in the neighbourhood \( U \), and parallel transport is defined through the path of corrections \( a_t \) induced in the fibre. Thus the path \( a_t \) is the path in \( S^1 \) which, dependent on the path parameter \( t \), returns \( v_t \) to its horizontal lift via the group action.

If we are given \( v \in S^{2n-1} \), for some \( \theta \in \mathbb{R} \), let \( v = e^{i\theta}w \). We write \( v \) in the Hopf bundle locally as \( (e^{i\theta},[w]) \), \([w] \in \mathbb{C}P^{m-1}\) and the vertical part of the tangent space is represented by \( \text{span}(\frac{\partial}{\partial \theta}) \equiv i\mathbb{R} \equiv G \). We give the Hopf bundle the connection \( \Gamma \) with associated connection 1-form \( \omega \). If \( v(\lambda) \) is a path in \( S^{2n-1} \), we let \( v(\lambda) = e^{i\theta(\lambda)}w(\lambda) \), where \( w(\lambda) \) is horizontal. Equivalently \( v(\lambda)e^{-i\theta(\lambda)} = w(\lambda) \), so \( e^{-i\theta(\lambda)} \) is the path of corrections in the group \( G \) that make \( v(\lambda) \) a horizontal path. The ODE which defines parallel translation is derived in [7], and for the path described above the equation becomes:

\[
\left( \frac{\partial}{\partial \lambda} e^{-i\theta(\lambda)} \right) e^{i\theta(\lambda)} = \omega(v'(\lambda)) \iff -i\theta'(\lambda) = -\omega(v(\lambda))
\]

\[
\theta'(\lambda) = -i\omega(v'(\lambda))
\]

This equation defines the correction in the fibre to make \( v(\lambda) \) horizontal. We call the quantity \( \theta \) the phase and we call \( \theta(\lambda) \) the phase curve with respect to \( v(\lambda) \). By integrating the connection 1-form, pulling back by some curve, we get \( i \) times the value \( \theta(\lambda(b)) - \theta(\lambda(a)) \), \( \lambda \in [a,b] \). Moreover, assuming \( v(\lambda(a)) = w(\lambda(a)) \) we may take \( \theta(\lambda(a)) = 0 \). Moreover, Way demonstrates in [9] that the ODE (10) becomes:

\[
\theta'(\lambda) = -iv^H(\lambda)v(\lambda) = \text{Im}(v^H(\lambda)v'(\lambda))
\]

where the \( H \) represents the conjugate transpose.
If we’re given a path on $S^{2n-1}$ by a projection of a path in $\mathbb{C}^n$ the right hand side of equation (10) is equivalent to $\text{Im}(u^H u') / (u^H u)$; thus pulling the form back along a path $\gamma$ we get

$$\int_{[a,b]} \gamma^* \omega = \int_a^b i \frac{\text{Im}(\langle u'(s), u(s) \rangle)}{\langle u(s), u(s) \rangle} ds$$

(12)

This means the winding of the phase is given by

$$\theta(b) = i \int_a^b \frac{\text{Im}(\langle u'(s), u(s) \rangle)}{\langle u(s), u(s) \rangle} ds$$

(13)

For a complete derivation of the above equations, and the proof that the natural connection is indeed a connection, the reader is referred to section 3 in [9].

Suppose now we are given a system on $\mathbb{C}^n$ with the same assumptions on the flow as in system (4); any non-zero solution to the system will have a projection onto the Hopf bundle $S^{2n-1}$ by normalization. Rupert Way in [9] developed a numerical method to link the winding in the geometric phase induced by the projection of a solution in the $\mathbb{C}^n$ system to the multiplicity of eigenvalues for $L$. The method Way originally proposed is in some ways dual to the unstable bundle, and we will instead describe a variant of his numerical method more suited to the unstable bundle construction. Way’s conjecture follows several steps:

- Begin by choosing a contour $K$ in $\mathbb{C}$ which does not intersect the spectrum of the operator $L$. For any fixed $\lambda \in K$ we may choose an eigenvector $X^+(\lambda)$ for $A_{-\infty}(\lambda)$ of most positive real part.

- As we vary $\lambda \in K$ we may thus define a loop of eigenvectors for the $A_{-\infty}(\lambda)$ system, parameterized in $\lambda$

- We choose $\xi_0$ sufficiently “close” to $-\infty$ and let $X^+(\lambda, \xi_0)$ be a loop of initial conditions for the dynamics on $\mathbb{C}^n$
• Proceed by integrating the loop of eigenvectors according to the $A(\lambda, \xi)$ system to some forward state $\xi_1$, for $\xi_1$ “close” to $+\infty$.
• The forward integrated loop $X^+(\lambda, \xi_1)$ may then be projected onto the Hopf bundle $S^{2n-1} \subset \mathbb{C}^n$ by normalizing the loop at every point, i.e:

$$\hat{X}^+(\lambda, \xi_1) \equiv \frac{X^+(\lambda, \xi_1)}{\|X^+(\lambda, \xi_1)\|}$$

Figure 2.7

• We thus measure the geometric phase of the path $\hat{X}^+(\lambda, \xi_1)$ in the Hopf bundle, according to the natural connection.

Way’s conjecture was that the geometric phase of $\hat{X}^+$ should equal the multiplicity of the temporal eigenvalues in $K^c$ when $\xi_1$ is taken sufficiently large. We will proceed by reformulating this idea into the machinery developed with the unstable bundle; this is to take advantage of the existing computation of the multiplicity of temporal eigenvalues through the Chern class.

In order to formulate our procedure rigorously we will develop the techniques first in the case where the phase space is $\mathbb{C}^2$. The low dimension will allow us to use some geometric intuition and will simplify the problem by reducing the unstable bundle to a complex line bundle on the parameter sphere.

Note: Way originally proposed a dual method to the one presented above, integrating a loop of eigenvectors corresponding to the eigenvalue of most negative real part close to $+\infty$ backwards toward $-\infty$. This doesn’t make a significant difference in the formulation of the method, but this method will not work in general for systems defined on $\mathbb{C}^n$ where $n > 2$. For this method to work in higher dimensions, we will make further adaptations to obtain the eigenvalue count.
3 The Two Dimensional Case

In this section we will restrict our consideration of the unstable bundle construction to the case where the asymptotic system is symmetric, by which we mean \( \lim_{\xi \to \pm \infty} A(\lambda, \xi) = A_{\infty}(\lambda) \). This restriction on the boundary conditions will flesh out the geometric intuition of the method, but the restriction is not necessary in general. We will adapt the method presented in section 2 to the general construction of the unstable bundle for \( n \) dimensions, \( k \) unstable directions, and non-symmetric boundary conditions in the section 3.

3.1 Set Up of the System on \( \mathbb{C}^2 \)

Suppose now we are given a system on \( \mathbb{C}^2 \) derived from the equation \((\mathcal{L} - \lambda)v = 0\) where \( \lambda \in \Omega \subset \mathbb{C} \):

\[
\begin{align*}
Y' &= A(\lambda, \tau)Y \\
\tau' &= \kappa(1 - \tau^2)
\end{align*}
\]

\[\begin{align*}
A(\lambda, \tau) &= \begin{cases} 
A(\lambda, \xi(\tau)) & \text{for } \tau \neq \pm 1 \\
A_{\infty}(\lambda) & \text{for } \tau = \pm 1
\end{cases}
\end{align*}\] (14)

where \( \Omega \) is open and simply connected, and the system at infinity determined by \( A_{\infty} \) has one stable and one unstable complex direction in \( \mathbb{C}^2 \) respectively for every \( \lambda \in \Omega \). Let \( K \) be a smooth, simple closed curve in \( \Omega \subset \mathbb{C} \) which contains no spectrum of \( \mathcal{L} \), let the enclosed region be denoted \( K^o \), and let \( K \) be parameterized by \( \lambda(s) : [0, 1] \to K \). We will denote the spectrum of \( \mathcal{L} \) to be \( \sigma(\mathcal{L}) \).

Let the unstable manifold of the point \((0, -1) \in \mathbb{C}^2 \times [-1, 1] \) be given by \( W^u_0 \). For each fixed \((\lambda, \tau)\) this will be a subspace of \( \mathbb{C}^2 \) that approaches \((0, -1) \in \mathbb{C}^2 \times [-1, 1] \) exponentially decaying as \( \xi \to -\infty \). For a fixed \((\lambda, \tau)\), \( W^u_0(\lambda, \tau) \) will denote the unstable subspace of \((0, -1) \) at \((\lambda, \tau)\). In particular, if the eigenvalues of \( A_{\infty}(\lambda) \) are given by \( \mu_1(\lambda), \mu_2(\lambda) \) such that

\[\text{Re}(\mu_1) < 0 < \text{Re}(\mu_2)\]

for each \( \lambda \), then the vector

\[X := e^{-\mu_2(\lambda)\xi}Y\]

is in \( W^u_0(\lambda, \tau) \) provided \( Y \in W^u_0(\lambda, \tau) \), because \( W^u_0(\lambda, \tau) \) is a subspace. Let \( \frac{d}{d\xi} = \cdot' \), then

\[
\begin{align*}
X' &= -\mu_2(\lambda)e^{-\mu_2(\lambda)\xi}Y + e^{-\mu_2(\lambda)\xi}Y' \\
&= (A - \mu_2I)X
\end{align*}\]

(15)

This motivates us considering the following system on \( \mathbb{C}^2 \):

\[
\begin{align*}
X' &= BX \\
\tau' &= \kappa(1 - \tau^2) \\
B(\lambda, \tau) &:= (A(\lambda, \tau) - \mu_2(\lambda)I) \\
B_{\infty}(\lambda) &:= \lim_{\xi \to \pm \infty} B(\lambda, \tau)
\end{align*}\] (16)
The $B$ system (16) should be considered a dual system to $A$ on $\mathbb{C}^2$, in the sense that solutions to the $A$, respectively $B$, system can be transformed into a solution of the $B$, respectively $A$, system by a $\xi$ dependent rescaling. Solutions to the $A$ system will be denoted with a $\tilde{z}$; if $Z^\sharp \in W^u_1(\lambda, \tau_0)$, then there is a $\xi_0$ such that $Z \equiv e^{-\mu_2(\lambda)(\xi-\xi_0)}Z^\sharp$ is the unique solution to the $B$ system (16) which agrees with $Z^\sharp$ at $(\lambda, \tau_0)$. Similarly if $Z$ is a solution to (16), then $Z^\sharp \equiv e^{\mu_2(\lambda)(\xi-\xi_0)}Z$ is the unique solution to (14) which agrees with $Z$ at $\xi_0$.

We know from [2] Lemma 3.7 that a solution to (14) that is in $W^u_1$, is unbounded and converges to the unstable subspace of $(0, +1)$ in the Grassmann norm as $\xi \to +\infty$. Thus we say

$$\| Z^\sharp - C_0 e^{\mu_2(\lambda)\xi} X^\sharp(\lambda) \| \to 0 \quad \xi \to +\infty \quad (17)$$

where $X^\sharp(\lambda)$ is the unstable direction of $A_\infty(\lambda)$.

This is because the flow on $\mathbb{C}^2$ is dominated by the flow induced by $A_\infty$ as $\xi \to \infty$, and knowing that the solution $Z^\sharp$ is unbounded converging onto the unstable space of the $A_\infty$ system at $(0, +1)$, the component of $Z^\sharp$ in the stable eigen direction of $A_\infty$ must be going to zero while the component of $Z^\sharp$ in the unstable eigen direction experiences $\mu_2(\lambda)$ exponential-like growth.

Let $\lambda \in K$ and $X^\sharp(\lambda)$ be an eigenvector associated to $\mu_2(\lambda)$. In the $B$ system (16), let $\lambda$ be fixed, if $Z \in \text{span}_\mathbb{C}\{X^\sharp(\lambda)\}$ then $(Z, \pm 1)$ is a fixed point. By the construction of $B_\infty$, in $\tau = -1$, there is exactly one complex stable direction, one complex center direction corresponding to the line of fixed points, and the real unstable $\tau$ direction. We may thus construct the center-unstable manifold of a non-zero path of eigenvectors corresponding to the zero eigenvalue in the $B(\lambda)$ system (16).

### 3.2 The Induced “Flow” on $S^3$

Suppose we choose unstable eigenvectors for $A_\infty(\lambda)$, $X^\sharp(\lambda)$, smoothly dependent on $\lambda \in \Omega$ - this may be done because the region $\Omega \subset \mathbb{C}$ is simply connected [2]. Under spherical projection we may lose $\mathbb{C}$ differentiability, but we will retain the differentiability in $s$, where $A(s) : [0, 1] \to K$.

Recall, $(0, \pm 1)$ is a fixed point of system (16), and so any non-zero solution to (14) may be viewed in hyper-spherical coordinates in $\mathbb{R} \times S^3 \times \{\tau \in (-1, 1)\}$ because no solution reaches zero in finite time. We claim that if $Z$ is a solution to the $B$ system (16) which is in the center-unstable manifold of a point within the line of critical points at $\tau = -1$, then $Z$ has a hyper-spherical projection which can be extended to $S^3 \times [-1, 1]$; it will suffice to show the projection of the solution is non-singular as $\xi \to \pm \infty$. Having a solution with spherical projection defined for all $\tau \in [-1 + 1]$, we will use the projection of the loop of eigenvectors for $A_\infty$ in the invariant hyperplane $\{\tau = +1\}$ as reference path to measure how much winding was induced in the unstable bundle.

As in section 4 of [2], the center-unstable manifold of the path $X^\sharp(\lambda)$ viewed in the $B$ system is given as a function of $(\lambda, \tau)$

$$Z(\lambda, \tau) \quad Z(\lambda, -1) \equiv X^\sharp(\lambda)$$

and is $\mathbb{C}$ differentiable in $\lambda$ for $\tau \in [-1, 1]$ fixed. The dual, $\xi$ dependent scaling of $Z$

$$Z^\sharp(\lambda, \tau) = e^{\mu_2(\lambda)\xi} Z(\lambda, \tau)$$

and thus

15
yields a solution to the $A$ system which is necessarily in $W^u_\lambda$, by the exponential decay condition as $\xi \to -\infty$. Therefore $Z(\lambda, \tau)$ spans $W^u_{\lambda}\xi(\lambda, \tau)$ for each $\tau \in [-1, +1)$. This dual relationship between the $A$ and $B$ systems allows us to describe sections of the unstable bundle with solutions to the $B$ system that behave better at $\tau = \pm 1$. We extend the $Z$ solution to the closed interval $[-1, 1]$ as equation (13) gives the following estimate

$$\| Z e^{\mu_2(\lambda)\xi} - C_2 e^{\mu_2(\lambda)\xi} X^+(\lambda) \| \to 0 \quad \xi \to \infty$$

$$\Rightarrow \| Z e^{\mu_2(\lambda)\xi} - C_2 e^{\mu_2(\lambda)\xi} X^+(\lambda) \| \leq 1 \quad \text{for } \xi \text{ sufficiently large} \quad (18)$$

$$\Rightarrow \| Z - C_2 X^+(\lambda) \| \leq e^{-Re(\mu_2(\lambda)\xi)} \quad \text{for } \xi \text{ sufficiently large}$$

ie: $Z$ converges to a non-zero vector in $\mathbb{C}^2$ as $\xi \to \infty$, so that it spans the unstable bundle for $\tau \in [-1, 1]$, and has a non-singular projection on to $S^3$ for all time.

### 3.3 The Induced Phase on the Hopf Bundle

Let $\hat{Z}, \hat{X}^+(\lambda)$ be the projections of $Z, X^+(\lambda)$ onto $S^3$ respectively, then $\hat{Z}$ defines a path on $S^3$ for which the following hold:

- $\hat{Z}(\lambda, \tau) \to \hat{X}^+(\lambda)$ as $\xi \to -\infty$
- $\hat{Z}(\lambda, \tau) \to \zeta(\lambda) \hat{X}^+(\lambda)$ as $\xi \to +\infty$ for some $\zeta(\lambda) \in \mathbb{C}$
- $\text{span}_\mathbb{C} \{\hat{Z}(\lambda, \tau)\} \equiv W^u_{\lambda}(\lambda, \tau)$

Notice that $\zeta$ as given above is a complex scalar which takes $\hat{X}^+(\lambda)$ to $\hat{Z}(\lambda, +1)$, but by our construction they are both unit vectors, ie: $\zeta(\lambda) \in S^1$. Recall, $\hat{X}^+(\lambda)$ is the projection of $Z(\lambda, -1)$ so we may consider the set of values $\zeta(\lambda)$ as a measure of the winding picked up in the $S^1$ fibre in the Hopf bundle as the loop $Z(\lambda, -1)$ was flowed to infinity. Firstly we want to prove that as a function of $s$, $\zeta$ is differentiable. Having this condition, we will explore the connection between $\zeta(s)$, the multiplicity of the eigenvalue in $K^0$, and the geometric phase.

**Proposition 1.** For each $\lambda \in K$, let us define $\zeta(\lambda)$ such that $\hat{Z}(\lambda, +1) = \zeta(\lambda) \hat{X}^+(\lambda)$.

We claim that if $\lambda(s)$ is a smooth parametrization of $K$ such that $\zeta(\lambda(s))$ is a differentiable function.

$$\zeta(\lambda(s)) : [0, 1] \to S^1 \quad (19)$$

**Proof.** Firstly consider that the solutions $Z(\lambda, \tau)$ will converge locally uniform to their limits $Z(\lambda, +1)$ for $\lambda \in \Omega \setminus \sigma(L)$. This must be the case as the $A$ and $B$ systems both have $\lambda$ dependence through the operator $(L - \lambda)$, and on a compact set bounded from $\sigma(L)$, the convergence of $(L - \lambda)$ to its $\xi \to \pm \infty$ limit will be uniform.

In section 4 of [2] it is demonstrated that the solutions $Z(\lambda, \tau)$ are defined for $\tau \in [-1, 1)$, and for fixed $\tau$, analytic in $\lambda \in \Omega$. We notice then, the limiting
function of $\lambda$, $Z(\lambda, +1)$, is also analytic in $\lambda$ by the uniform convergence on $K$. The spherical projection is not $\mathbb{C}$ analytic, but it will be real differentiable as a map from $\mathbb{R}^4 \to S^3$. This means the composition function $\hat{Z}(\lambda(s), +1)$ is differentiable with respect to the real parameter $s \in [0, 1]$. The quantity $\zeta(\lambda)$ is given as the ratio of components of $\hat{Z}(\lambda, 1)$ and $\hat{X}^+$, and one component of $\hat{Z}(\lambda, 1)$ is always non-zero as it is a unit vector; the proposition is thus proven.

**Remark:** We may like to compare $\hat{Z}(\lambda, +1)$ to its unique horizontal lift in the Hopf bundle; in this way we could directly measure the parallel translation of $\hat{Z}(\lambda, \xi)$ as $\xi$ approaches $+\infty$. However, because of technical considerations for creating trivializations of the unstable bundle, we must avoid this. The horizontal lift in general **will not be a loop**, and for reasons which will become clear later, it cannot be for all $\xi$.

Using $\hat{X}^+(\lambda)$ as the reference path is thus a matter of convenience to more easily construct trivializations of the unstable bundle. For more discussion on this choice of reference path, versus the horizontal lift of $\hat{Z}(\lambda, \tau)$, see the concluding remarks in section 7.

### 3.4 The Trivializations and the Transition Map

We recall now the construction of the parameter sphere, defined $M \equiv K^+ \times \{\tau = \pm 1\} \cup K \times \{\tau \in (-1, +1)\}$. The unstable bundle is a non-trivial complex line bundle contained in the ambient trivial $\mathbb{C}^2$ vector bundle above the sphere; for fixed $\lambda$, as $\tau$ moves between $\pm 1$, the parameters in the sphere are the values $(\lambda, \tau)$ which induce the motion of solutions $Z(\lambda, \tau)$. Taking a trivialization of this line bundle amounts to finding a linear isomorphism

$$\phi_\alpha : U_\alpha \times \mathbb{C} \leftrightarrow U_\alpha \times \mathbb{C}^2$$  \hspace{1cm} (20)

where $U_\alpha$ is a neighborhood in $M$, and the image of $\phi_\alpha$ is the unstable bundle over $U_\alpha$. 
We make the following definitions:

- Let $H_-$ be the “lower hemisphere” of $M$, given by $K^0 \times \{\tau = -1\} \cup K \times \{\tau \in [-1,1]\} \cup V \times \{\tau = +1\}$ where $V$ is an open neighborhood in $K^0$ homotopy equivalent to $S^1$ with $K$ in the closure of $V$. Assume no eigenvalue of $L$ is contained in $V$.

- Let $H_+$ be the “upper hemisphere” of $M$, given by $K^0 \times \{\tau = +1\} \cup K \times \{\tau \in (-1,1]\}$

- Let $Z, \hat{Z}$ be as given in section 3.3, and let $Z$ be an extension of $Z$ into $V \times \{\tau = +1\}$ so that for $\lambda \in V$, $Z(\lambda, +1)$ is smoothly compatible with the values $Z(\lambda, +1)$, $\lambda \in K$. Abusing notation, let $\hat{Z}(\lambda, +1)$ be the spherical projection of $Z$ for $\lambda \in V$.

- Let $W(\lambda, \tau)$ be a solution to the $B$ system for which $W(\lambda, \tau) \to \hat{X}^+(\lambda)$ as $\xi \to +\infty$, ie: $W(\lambda, \tau)$ is in the stable manifold of the path $\hat{X}^+(\lambda)$ in the invariant plane $\tau = +1$, where $\hat{X}^+(\lambda)$ are critical points on $S^3$ with respect to the $B$ system. Extend $W$ into $K^0 \times \{\tau = +1\}$ so that for $\lambda \in K^0$, $W(\lambda, +1)$ is an eigenvector for the unstable direction of $A^\infty(\lambda)$, smoothly compatible with the values on the boundary $K$. We define the spherical projection of $W$ to be $\hat{W}$.

By construction, for fixed $(\lambda, \tau)$, where they are defined $\hat{Z}, \hat{W}$ each span the unstable bundle. $\hat{Z}$ is defined over $H_-$ and $\hat{W}$ is defined over $H_+$, so that for any point $p$ in the unstable bundle we may choose a unique $z \in \mathbb{C}$ for which $p \equiv (\lambda, \tau, z\hat{Z})$ if $p$ is is over $H_-$, or choose a unique $w \in \mathbb{C}$ for which $p \equiv (\lambda, \tau, w\hat{W})$ if $p$ is over $H_+$. But then the solutions $\hat{Z}, \hat{W}$ give a choice of trivializations for the unstable bundle over $H_-, H_+$ respectively; we define the following maps as such:

$$\phi_- : H_- \times \mathbb{C} \leftrightarrow H_- \times \mathbb{C}^2$$

$$(\lambda, \tau, z) \mapsto (\lambda, \tau, z\hat{Z}(\lambda, \tau))$$

$$\phi_+ : H_+ \times \mathbb{C} \leftrightarrow H_+ \times \mathbb{C}^2$$

$$(\lambda, \tau, w) \mapsto (\lambda, \tau, w\hat{W}(\lambda, \tau))$$

These maps are trivializations of the unstable bundle, as they are linear vector bundle isomorphisms.
Fixing \( \tau \) such that \((\lambda, \tau) \in H_- \cap H_+ \ \forall \lambda \in K\), the transition map can be seen as a mapping from \( S^1 \) to \( GL(1, \mathbb{C}) \), i.e., we take the restriction of the composition of trivializations to the \( \lambda \) parameter

\[
\phi_+^{-1} \circ \phi_-(\lambda, \tau, -) : \ K \cong S^1 \rightarrow GL(1, \mathbb{C}) \\
(\lambda, -) \mapsto \hat{\phi} \\
\hat{\phi} : \ \mathbb{C} \rightarrow \mathbb{C} \\
z \mapsto w
\]

where \( z(\lambda, \tau) \hat{Z}(\lambda, \tau) = w(\lambda, \tau)\hat{W}(\lambda, \tau) \). But for a fixed \( z \in \mathbb{C}^* \),

\[
[\phi_+^{-1} \circ \phi_-(\lambda, \tau, z)] \in \pi_1(GL(1, \mathbb{C})) \cong \pi_1(S^1).
\]

That is for \( z \) fixed, we may associate \( \hat{\phi}_z(\lambda) \), equal to \( m \), is the Chern class of the unstable bundle.

Notice that in \( \{ \tau = +1 \} \cap H_- \times \mathbb{C} \) the transition map can be described through

\[
z \mapsto z\hat{Z}(\lambda(s), +1) \equiv z\zeta(\lambda(s))\hat{X}^+(\lambda(s)) \equiv z\zeta(\lambda(s))\hat{W}(\lambda(s), +1) \rightarrow z\zeta(\lambda(s))
\]

so that the transition map \( \hat{\phi} \) is exactly given by \( z \mapsto \zeta(\lambda(s))z \).

But the number of windings \( \zeta(\lambda(s)) \) takes around the circle is given by

\[
m = \frac{1}{2\pi} \int_K \frac{1}{z}dz
\]

\[
= \frac{1}{2\pi} \int_0^1 \frac{1}{\zeta(\lambda(s))}\zeta'(\lambda(s))\lambda'(s)ds
\]

provided \( \lambda(s) \) has standard orientation. That is

\[
[\zeta(\lambda(s))] \equiv [e^{i\frac{1}{4\pi} \int_{\lambda(0)}^s \zeta'(t)\lambda'(t)dt}]
\]

where \( s \in [0, 2\pi] \).

### 3.5 The Geometric Phase and the Transition Map

We have now found the relationship between the induced phase \( \zeta(s) \) and the Chern class of the unstable bundle over \( M \), but we still need to interpret this in terms of the geometric phase in the Hopf bundle. We will now reformulate Way’s central conjecture.

Let \( Z, \hat{Z} \) be defined as in section 3. \( Z, \hat{Z} \in W^u_A \) for all \( \xi \) and

\[
Z^\sharp := e^{i\mu_2(\lambda(s))\xi}Z
\]

is the corresponding solution to the \( A \) system (1) at \( \xi \). We want to show that the geometric phase of the two solutions agree for each \( \xi \), and that this phase converges to the winding of the transition map for the unstable bundle as \( \xi \rightarrow +\infty \). We will begin by verifying the geometric phase of the \( B \) system solution agrees with the winding of the transition map.
The natural connection on the Hopf bundle is given by the 1-form
\[
\omega_p(Q) \equiv \langle Q, p \rangle_{\mathbb{C}^2}, \quad Q \in T_p(S^3) \subset T_p(\mathbb{C}^2)
\] (26)
so that to calculate the geometric phase of \( \hat{Z}(\lambda(s),+1) \), we consider
\[
\hat{Z}(\lambda(s), +1) = \zeta(\lambda(s))\hat{X}^+(\lambda(s))
\]
\[
\Rightarrow \quad \frac{d}{ds} \hat{Z}(\lambda(s), +1) = \zeta'(\lambda(s))\lambda'(s)\hat{X}^+(\lambda(s)) + \zeta(\lambda(s)) \frac{d}{ds} \hat{X}^+(\lambda(s)) \quad (27)
\]
\[
\Rightarrow \quad \omega(\frac{d}{ds} \hat{Z}(\lambda(s), +1)) = \frac{\zeta(s)}{\zeta(\lambda(s))} \zeta'(\lambda(s))\lambda'(s) + \omega(\frac{d}{ds} \hat{X}^+(\lambda(s)))
\]
because \( \hat{X}^+(\lambda(s)) \) is a unit vector, \( \zeta(s) \in S^1 \), and because \( \omega(V_p) \equiv \langle V_p, p \rangle_{\mathbb{C}^2} \).

But by \([9]\) 3.32-3.35 we see the geometric phase of \( \hat{Z}(\lambda, +1) \) is given by
\[
\frac{\Delta \theta}{2\pi} = \frac{i}{2\pi} \int_K \omega(\hat{Z}'(\lambda(s), +1)))d\lambda = \frac{1}{2\pi} \int_0^1 \omega(\hat{Z}(\lambda(s)))\lambda'(s)ds \quad (28)
\]
so that the geometric phase \( \frac{\Delta \theta}{2\pi} \) of \( \hat{Z}(\lambda(s), +1) \) equals the Chern class of the unstable bundle over \( M \) plus the geometric phase of the reference path.

We will now introduce two lemmas to show the geometric phase of \( \hat{Z}(\lambda, +1) \) will be independent of the choice of reference path, for a path which can be extended to the bundle construction.

**Lemma 1.** Consider every loop of unstable eigenvectors \( V(\lambda) \) for \( A_{+\infty} \), parametrized by \( K \) and smooth in \( \lambda \), for which \( V(\lambda) \) can be extended over \( K^c \), as to define the reference path at \( \tau = +1 \) compatible with the unstable bundle; the geometric phase of all such paths must agree.

**Proof.** Suppose \( \sigma(\lambda) \) is a smooth scaling \( \sigma : K \rightarrow \mathbb{C}^* \) such that \( V(\lambda) \equiv \sigma(\lambda)X^+(\lambda) \) is also a loop of unstable eigenvectors for \( A_{+\infty} \) which can be extended to define the unstable bundle. Then let us define \( \hat{\sigma}(\lambda) \) to be the mapping to \( S^1 \), for which \( \hat{V}(\lambda) \equiv \hat{\sigma}(\lambda)\hat{X}^+(\lambda) \), note that \( \hat{\sigma} \) and \( \sigma \) have the same winding.

Following the constructions as in the previous steps, let us define \( \rho(\lambda(s)) \) to be the induced phase for the reference path \( \hat{V}(\lambda) \), i.e. \( \hat{Z}(\lambda(s), 1) = \rho(\lambda(s))\hat{V}(\lambda(s)) \).

Then the winding of \( \rho(\lambda(s)) \) as before agrees with the Chern class of the unstable bundle.

But notice:
\[
\hat{Z}(\lambda(s), 1) = \rho(\lambda(s))\hat{V}(\lambda(s))
\]
\[
\Rightarrow \quad \hat{Z}(\lambda(s), 1) = \rho(\lambda(s))\hat{\sigma}(s)\hat{X}^+(\lambda(s))
\]
\[
\Rightarrow \quad \omega(\frac{d}{ds} \hat{Z}(\lambda(s), 1)) = \rho'(\lambda(s))\hat{\sigma}(s)\hat{X}^+(\lambda(s)) + \omega(\frac{d}{ds} \hat{X}^+(\lambda(s)))
\]
\[
\Rightarrow \quad \omega(\frac{d}{ds} \hat{Z}(\lambda(s), 1)) = \rho'(\lambda(s)) + \omega(\frac{d}{ds} \hat{V}(\lambda(s)))
\]
that is, the geometric phase of \( \hat{Z}(\lambda, 1) \) equals the winding of \( \hat{\sigma}(s) \) plus the Chern class plus the geometric phase of \( \hat{X}^+(\lambda) \), and also equals the Chern class plus the geometric phase of \( \hat{V}(\lambda) \). Therefore, the geometric phase of \( \hat{V}(\lambda) \) is equal to that of \( \hat{X}^+(\lambda) \) plus the winding of \( \hat{\sigma}(s) \)
If \( \sigma(\lambda) \) has non-trivial winding there is immediately a contradiction because if:

- the winding number is positive, any smooth extension of \( \sigma(\lambda) \) into \( K^\circ \) must have a zero of non-trivial order in \( K^\circ \) contradicting the construction of the unstable bundle, scaling a fibre by zero.

- the winding number is negative, any smooth extension of \( \sigma(\lambda) \) into \( K^\circ \) must have a pole of non-trivial order in \( K^\circ \) contradicting the construction of the unstable bundle, scaling a fibre by infinity.

Thus the calculation is independent of the choice of the reference path, provided that the reference path can be used in the unstable bundle construction.

We introduce the above lemma to guide some of the discussion of the numerical results later on - indeed the choices of other reference paths which are not compatible with the bundle construction introduce some interesting behaviour which we will discuss in examples in section 6.

**Lemma 2.** For a choice of reference path which can be used to define the unstable bundle, the geometric phase is always zero.

**Proof.** Given the above lemma 1, let us consider \( \hat{X}^+ \) for our reference path. We want to show that integrating the tangent vector to \( \hat{X}^+(\lambda) \) is always zero, and to do so we will consider the connection in a differential form view.

The natural connection can be written \( \omega \equiv VHdV \) where \( H \) will denote the hermitian transpose of the vector \( V \). Written as a 1-form on \( \mathbb{R}^{2n} \) \( \omega \) then becomes \( a_jda^j - b_jdb^j \), where \( v_j = a_j + ib_j \). Notice, if we take the exterior derivative \( d\omega \) this will always equal zero

\[
\begin{align*}
\omega &=(\partial_{a_k}da^k + \partial_{b_k}db^k) \land (a_jda^j - b_jdb^j) \\
\Leftrightarrow d\omega &= 0
\end{align*}
\]

For this reason it will be natural to consider Stokes' theorem. Indeed, suppose \( \hat{X}^+(\lambda) \) is extended over a neighbourhood \( U \) which contains \( K \) and over \( U \) \( \hat{X}^+(\lambda) \) is locally an injection; we consider here \( \lambda \equiv (a, b) \), so that \( \hat{X}^+(\lambda) \) is viewed this as a map from \( U \subset \mathbb{R}^+ \mathbb{R}^{2n} \). Then for each point \( \lambda \in K \) we may find a closed neighbourhood \( U \) of \( \lambda \) for which \( \hat{X}^+(\lambda) \) defines a diffeomorphism onto its image. The image of \( U \), \( V \) is thus a compact manifold with boundary of dimension 2 for which \( \omega \) defines a 1-form. Stokes' theorem implies that the integral of \( \omega \) about \( \partial V \), or any simple loop in \( V \), must equal zero. Thus the integral of \( U \cap K \) only depends on the endpoints. \( K \) is compact, so we make take a disjoint finite cover of such local neighbourhoods; the integral \( \int_K \omega(\partial_{a}\hat{X}^+) \equiv \int_{\hat{X}^+(K)} \omega \) thus decomposes into the local integrals, the sum of which vanishes.

To prove our lemma, it will suffice thus suffice that \( \hat{X}^+(\lambda) \) is locally injective. Thus let’s assume it is not locally injective - we will prove when this is the case the phase must still be zero.
If $\dot{X}^+(\lambda)$ is not locally injective, then there must exist some $\lambda_n \to \lambda_0$ for which $\dot{X}^+(\lambda_n) \equiv X^+(\lambda_0)$. Consider then the limit:

$$
\partial_\lambda |_{\lambda_0} X^+(\lambda) = \lim_{n \to \infty} \frac{X^+(\lambda_n) - X^+(\lambda_0)}{\lambda_n - \lambda_0} = \lim_{n \to \infty} \frac{\dot{X}^+(\lambda_n)[||X^+(\lambda_n)|| - ||X^+(\lambda_0)||]}{\lambda_n - \lambda_0} = \lim_{n \to \infty} \frac{\dot{X}^+(\lambda_0)[||X^+(\lambda_0)|| - ||X^+(\lambda_0)||]}{\lambda_n - \lambda_0} = \dot{X}^+(\lambda_0) \lim_{n \to \infty} \frac{||X^+(\lambda_n)|| - ||X^+(\lambda_0)||}{\lambda_n - \lambda_0}
$$

So that the limit

$$
\lim_{n \to \infty} \frac{||X^+(\lambda_n)|| - ||X^+(\lambda_0)||}{\lambda_n - \lambda_0}
$$

must converge to a value in $C$. This means in particular

$$
\partial_\lambda |_{\lambda_0} X^+(\lambda) = \alpha X^+(\lambda_0)
$$

where the derivative in $\lambda$ is component-wise. Thus we see that the unique solution to this equation is $X^+(\lambda_0)e^{\alpha \lambda}$, and $X^+(\lambda) = \dot{X}^+(\lambda_0)e^{i\text{Im}(\alpha \lambda)}$. Let $\lambda(s)$ be a parameterization of $K$, then

$$
\int_{\dot{X}^+(K)} \omega = i \int_0^1 \text{Im}(\alpha \lambda'(s)) ds = 0 \quad (29)
$$

as $\lambda(s)$ defines a closed loop. Therefore, if $\dot{X}^+(\lambda)$ is locally injective the phase is zero, and if not, the loop is actually somewhat trivial and the phase is again zero. \qed

**Corollary 1.** The geometric phase of the projected solution $\tilde{Z}(\lambda, +1)$ equals the Chern class of the unstable bundle, and therefore the multiplicity of the eigenvalues enclosed by the contour $K$.

Finally we will explore the relationship between the solutions to the $B$ system where we calculate the phase in the proof, and the solutions to the $A$ system. Suppose $\mu_2(\lambda) \equiv \alpha(\lambda) + i\beta(\lambda)$, and recall the solution to the $A$ system given by $Z^2(\lambda, \xi) = e^{\mu_2(\lambda)\xi}Z(\lambda, \xi)$. Note that $Z^2$ is the unique solution to the $A$ system which agrees with $Z$ at $\xi = 0$. The projection of $Z^2$ onto the Hopf bundle is given by $\tilde{Z}^2(\lambda, \xi) \equiv e^{i\beta(\lambda)\xi}Z(\lambda, \xi)$, so that calculating the phase:

$$
\tilde{Z}^2(\lambda, \xi) = e^{i\beta(\lambda)\xi}Z(\lambda, \xi)
$$

$$
\Rightarrow \frac{\partial}{\partial \lambda} \tilde{Z}^2(\lambda, \xi) = i\beta'(\lambda)\xi e^{i\beta(\lambda)\xi}Z(\lambda, \xi) + e^{i\beta(\lambda)\xi} \frac{\partial}{\partial \lambda} Z(\lambda, \xi) = i\beta'(\lambda)\xi + \omega(\frac{\partial}{\partial \lambda} \tilde{Z}(\lambda, \xi)) \quad (30)
$$

But $\mu_2(\lambda), \mu'_2(\lambda)$ are each holomorphic by construction so that integrating $\mu'_2(\lambda) \equiv \alpha'(\lambda) + i\beta'(\lambda)$ about a closed curve, the real and imaginary parts both must equal zero; the $i\beta'(\lambda)\xi$ term thus vanishes from the above expression. Thus the geometric phase of the $A$ system at $\xi$ corresponds to the phase of the solution to the $B$ system for arbitrary $\xi$. 

22
Knowing that the multiplicity is an integer, and that for $\xi_0$ arbitrarily large the geometric phase of a solution to the $A$ system at $\tau(\xi_0)$ is arbitrarily close to the geometric phase of $\hat{Z}(\lambda(s), +1)$, we may obtain the multiplicity of the isolated eigenvalue with this adaptation of Way’s method.

However, in practice it is not actually necessary to take $\xi_0$ arbitrarily close to $+\infty$ as described above; rather $\xi_0$ must only be greater than a threshold value where there is a rapid transition from zero phase to the multiplicity of the eigenvalues. This behaviour is readily seen in the numerical examples, and has theoretical justification - however there are technical considerations which are currently unclear, and are the subject of ongoing research. For more discussion of this, the reader is again referred to the concluding remarks in section 7.
4 Extending the Two Dimensional Method

In this section we will adapt the techniques we developed in the previous section to take advantage of the full generality in which the unstable bundle can be constructed. Firstly we will extend the techniques to the case when there are \( k > 1 \) unstable directions, and once we have a general method we will loosen the conditions on the asymptotic systems \( A_{\pm \infty} \).

4.1 The n Dimensional Case and Determinant Bundle

What if we are now given an \( A \) system and \( B \) system as in the two dimensional case, but we instead assume \( A \) defines a flow on \( \mathbb{C}^n \)? For all \( \lambda \in \Omega \), suppose \( A_{\infty} \) has one unstable direction and \( n-1 \) stable directions. In this case we can use essentially the same techniques as when there were two dimensions; although the ambient complex dimension has increased, the unstable bundle is still one dimensional. If we proceed through the construction as in two dimensions, we again obtain the induced parallel translation \( \zeta(\lambda) \) from a solution to the \( B \) system. The natural connection on the Hopf bundle \( S^{2n-1} \) is the same for every \( n \), so the calculations of the geometric phase is no different than in the two dimensional case.

Suppose more generally that there are \( k > 1 \) unstable directions for the system \( A_{\infty} \); the \( k \) dimensional unstable bundle \( E \) is again formed from the unstable manifold of the critical point \( (0, -1) \) in the \( A \) system, and the Chern class of this vector bundle equals the multiplicity of the eigenvalues contained in \( K^0 \). We would like to write the transition map of \( E \) as a value in \( S^1 \), so we can determine the windings of the mapping as in the two dimensional case - making a reduction of this sort requires us introducing the determinant bundle constructed from a \( k \) dimensional vector bundle. Morally, we can view this technique as reducing the dimension of the unstable bundle back down to one, while raising the ambient complex dimension in which the bundle lives, above the parameter sphere. The goal is thus to apply the same construction of the induced parallel translation to the determinant bundle of the \( k \) dimensional unstable space.

Consider the \( k \)th exterior power \( \Lambda^k(\mathbb{C}^n) \equiv \mathbb{C}^{n\choose k} \), where a natural flow is generated by
\[
Z = Z_1 \wedge \cdots \wedge Z_k
\]
\[
\Rightarrow Z' = AZ_1 \wedge \cdots \wedge Z_k + \cdots + Z_1 \wedge \cdots \wedge AZ_k
\]
\[
A^{(k)} Z := AZ_1 \wedge Z_2 \wedge \cdots \wedge Z_k + \cdots + Z_1 \wedge \cdots \wedge AZ_k
\] (31)

and \( ' = \frac{d}{dt} \) as before. By (29) it is clear that the eigenvalues for the corresponding system at infinity \( A_{\infty}^{(k)} \) are the sums of all \( k \)-tuples of eigenvalues for \( A_{\infty} \). Thus, there is a unique eigenvalue of largest positive real part given by the sum of all eigenvalues with positive real part, including geometric multiplicity. If \( \{\mu_1^+, \cdots, \mu_k^+\} \) are all eigenvalues of positive real part for \( A_{\infty} \), denote \( \mu^+ := \sum_{i=1}^k \mu_i^+ \).
There is an explicit algorithm to compute the $A^{(k)}$ system on the exterior power $\Lambda^k(\mathbb{C}^n)$ which can be automated by a computer algebra system; the coefficients of the $A^{(k)}$ are calculated through the inner product induced on $\Lambda^k(\mathbb{C}^n)$, by the inner product on $\mathbb{C}^n[4]$. Let $e_1, \cdots, e_n$ denote the standard basis on $\mathbb{C}^n$ and $\omega_1, \cdots, \omega_d$ be the standard orthonormal basis for $\Lambda^k(\mathbb{C}^n)$ generated from the $\{ e_i \}$. If $x := x_1 \wedge \cdots \wedge x_k$ and $y := y_1 \wedge \cdots \wedge y_k$ are decomposable $k$ forms, their inner product is defined

$$\langle x, y \rangle_k := \det \begin{pmatrix} \langle x_1, y_1 \rangle_{\mathbb{C}^n} & \cdots & \langle x_1, y_k \rangle_{\mathbb{C}^n} \\ \vdots & \ddots & \vdots \\ \langle x_k, y_1 \rangle_{\mathbb{C}^n} & \cdots & \langle x_k, y_k \rangle_{\mathbb{C}^n} \end{pmatrix}$$

and this extend bi-linearly to any $k$ form.

If we define $Ax = \sum_{j=1}^k x_1 \wedge \cdots \wedge Ax_j \wedge \cdots \wedge x_k$, then the coefficients of $A^{(k)}$ are calculated

$$\{ A^{(k)} \}_{i,j} = \langle \omega_i, A\omega_j \rangle_k \qquad i, j = 1, \cdots, d = \binom{n}{k}$$

An explicit calculation of the $A^{(2)}$ system on $\Lambda^2(\mathbb{C}^4)$ is performed in [1] for an example which we will return to in the section on numerics; generally for a $4 \times 4$ matrix $\{ A \}_{i,j}$ this algorithm generates the following $A^{(2)}$ system:

$$A^{(2)} = \begin{pmatrix} a_{11} + a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11} + a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11} + a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22} + a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22} + a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33} + a_{44} \end{pmatrix}$$

Computing the flow on the exterior power, we generate the corresponding appended $A$ and $B$ systems with $\tau$ defined as as before:

$$Y' = A^{(k)}(\lambda, \tau) Y \quad A^{(k)}_{\infty}(\lambda) = \lim_{\xi \to \pm \infty} A^{(k)}(\lambda, \tau)$$

$$\tau' = \kappa(1 - \tau^2)$$

$$B^{(k)}(\lambda, \tau) := (A^{(k)}(\lambda, \tau) - \mu^+(\lambda)) \quad X' = B^{(k)} X$$

$$B^{(k)}_{\infty}(\lambda) := \lim_{\xi \to \pm \infty} B^{(k)}(\lambda, \tau) \quad \tau' = \kappa(1 - \tau^2)$$

The system $B^{(k)}_{\infty}$, at $\tau = -1$, has a center direction of critical points, an unstable real direction, and all other directions are stable; the line of critical points is given by the span of the wedge of the eigenvectors corresponding to $\{ \mu^+_1, \cdots, \mu^+_k \}$, with the eigenvectors linearly independent.
Let the transition map of the unstable bundle \( E \) be denoted \( \hat{\phi}_E \). The determinant bundle \( \Lambda^k(E) \) acquires its namesake from the construction of its transition map \( \hat{\phi}_E^k \). The transition map of the k-dimensional unstable bundle is a \( \lambda \) dependent, non-singular mapping of k-frames of n dimensional complex vectors. Restricting to the equator of \( M \), we thus interpret the transition map \( \hat{\phi}_E^k : S^1 \to GL(C,k) \) \( \lambda \mapsto \Psi(\lambda) \) (36)

so that it defines an element of \( \pi_1(GL(C,k)) \). But notice, \( \det(\hat{\phi}_E(\lambda)) \in GL(C,1) \) for all \( \lambda \in K \), so that the determinant induces a homomorphism of fundamental groups

\[
\text{det} : \pi_1(GL(C,k)) \to \pi(GL(C,1))
\]

(37)

The mapping \( \det \circ \hat{\phi}_E(\lambda) \equiv \hat{\phi}_E^k \), the transition map of the determinant bundle. In [2] moreover it is verified that the Chern class of the determinant bundle agrees with the Chern class of the unstable bundle - we thus wish to calculate the induced parallel translation of the determinant bundle in \( \Lambda^k(E) \) as a complex line bundle over \( M \).

### 4.2 Solving the k Unstable Direction Case

Recall now the \( B^{(k)} \) system (27), where at infinity there is a line of critical points equal to the span of the wedge of the unstable directions of \( A_\infty \). We choose a smooth path of critical points \( X^+(\lambda) \) and let \( Z(\lambda, \tau(\xi)) \) be the center-unstable manifold of these points as before. Like the two dimensional case, \( Z^\sharp(\lambda, \tau(\xi)) := e^{\mu+\xi}Z(\lambda, \tau(\xi)) \) (38)

will be a solution to the corresponding \( A^{(k)} \) system. In particular it is in the determinant bundle, constructed from the unstable bundle over \( M \). By [2] section 4, \( Z \) forms a \( \mathbb{C} \) analytic section of the determinant bundle for \( \tau \in [-1,1] \), and we may furthermore extend this solution to a non-zero limit as \( \xi \to \infty \) as in [2] section 6. The convergence will again be uniform for \( \lambda \in K \) so that the extension of the \( B^{(k)} \) system solution \( Z(\lambda, +1) \) is \( \mathbb{C} \) analytic.

Like in the two dimensional case, we may take a hyper-spherical projection of \( Z \) onto the \( S^{(\xi+1)} \subset \mathbb{C}^{(\xi)} \cong \Lambda^k(C^n) \); we will construct \( \hat{Z}, \hat{X}^+, \hat{X}^+, W, \hat{W}, \zeta(\lambda) \) to all have the analogous meanings. Denote the projection \( \hat{X}^+(\lambda) \equiv \hat{Z}(\lambda, -1) \); as before, \( W(\lambda, \tau) \) will be solutions to the \( B^{(k)} \) system that are in the stable manifold of the path \( \hat{X}^+(\lambda) \) at \( \tau = +1 \), \( \hat{W} \) will be the projection of these solutions; the trivializations of the determinant bundle can be expressed in terms of \( \hat{Z} \) and \( \hat{W} \), which yields the familiar transition map \( \zeta(\lambda) \) at \( \tau = +1 \).

This definition of \( \zeta(\lambda) \) as the function for which

\[
\hat{Z}(\lambda, +1) \equiv \zeta(\lambda)\hat{X}^+(\lambda) \equiv \zeta(\lambda)\hat{W}(\lambda, +1)
\]

is possible because \( \text{span}_C\{Z(\lambda, +1)\} \equiv \text{span}_C\{\hat{X}^+(\lambda)\} \) (lemma 6.1 [2]), so that \( \hat{X}^+(\lambda) \) and \( \hat{Z}(\lambda, +1) \) must differ again by a scalar in \( S^1 \). The winding of \( \zeta(\lambda) \) is
thus equal to the Chern class of the determinant bundle, and is related to the geometric phase by the same formulation as in the two dimensional case.

From the above discussion, in the case of $k$ unstable directions we may calculate the multiplicity of the eigenvalues contained in the region $K^\circ$ by an adaptation of the method described by Way applied to the determinant bundle of the unstable bundle. In particular we construct the $A^{(k)}$ system and choosing a solution in the unstable manifold corresponding to the eigen direction of largest real part for $A^{(k)}_{-\infty}$, and flowing the solution sufficiently close to the system at $\tau = +1$, we may approximate the multiplicity of the eigenvalues through the geometric phase of our solution on the Hopf bundle $S^{2(n-k)-1}$.

4.3 The General Case and Non-symmetric Asymptotic Systems

In the preceding sections we developed a method for finding the multiplicity of eigenvalues for $L$ in the region $K^\circ$, but the method was restricted to the case for which $\lim_{\xi \to -\infty} A(\lambda, \xi) \equiv \lim_{\xi \to +\infty} A(\lambda, \xi)$. The unstable bundle construction, however, was valid for systems $A_{\pm\infty}$ that “split” in $\Omega$, i.e.: each have exactly $k$ unstable, and $n-k$ stable directions for every $\lambda \in \Omega$. The final modification we will make to the preceding method will be to account for non-symmetric asymptotic systems.

Suppose we have the determinant bundle system

\[
Y' = A^{(k)}(\lambda, \tau)Y \quad A^{(k)}_{-\infty}(\lambda) = \lim_{\xi \to +\infty} A^{(k)}(\lambda, \tau)
\]

\[
\tau' = \kappa(1 - \tau^2)
\]

derived from the flow $V' = AV$ on $\mathbb{C}^n$. Note, the determinant bundle reduces to the regular unstable bundle in the case where there $k = 1$, so this construction will be defined for every $k \in \mathbb{N}$.

If we were to define the $B^{(k)}$ system as in the previous section, we could construct the unstable manifold of the direction of critical points at $\tau = -1$ to obtain a section for the unstable bundle, but the behaviour of such a solution will differ from the previous constructions when $\tau \to +1$. The dominating unstable eigenvalue for the system at $\tau = +1$ in general will differ from the value at $-1$, so we need to find a new way to guarantee the hyper-spherical projection of solutions as $\xi \to \infty$.

Let us begin by defining $X^+(\lambda), \hat{X}^+(\lambda)$ for the system $A^{(k)}_{-\infty}(\lambda)$ as in the previous section - we let $Y^+(\lambda), \hat{Y}^+(\lambda)$ be a loop of eigenvectors, and the projection, for the eigenvalue of most positive real part for the system $A^{(k)}_{-\infty}$. If $Z(\lambda, \tau)$ is a solution to the $B^{(k)}$ system for which $Z(\lambda, -1) \equiv X^+(\lambda)$, and $\mu^-\lambda$ is the eigenvalue of largest positive real part for $A^{(k)}_{-\infty}(\lambda)$, we define

\[
Z^\sharp(\lambda, \tau) := e^{\mu^-(\lambda)\xi}Z(\lambda, \tau)
\]

so that $Z^\sharp$ is a section of the unstable bundle on $\{\tau \in (-1, +1)\}$ which satisfies the flow defined by (31). Suppose $\mu^+(\lambda)$ is the eigenvalue of largest positive real part for $A^{(k)}_{\infty}$; by using the $\xi$ dependent rescaling trick twice, we will produce a spanning vector for the determinant bundle on the whole parameter sphere.
Proposition 2. With \( Z, Z^\sharp, \mu^\pm \) defined as above let

\[
\Gamma(\lambda, \tau) := \begin{cases} 
  e^{(-\mu^- \xi)} Z^\sharp(\lambda, \tau) & \text{for } \tau \in [-1, 0) \\
  e^{(-\mu^+ \xi)} Z^\sharp(\lambda, \tau) & \text{for } \tau \in [0, +1) \\
  \lim_{\xi \to \infty} e^{(-\mu^+ \xi)} Z^\sharp(\lambda, \tau) & \text{for } \tau = +1
\end{cases}
\]  

Then \( \Gamma(\lambda, \tau) \) is non-zero and spans the determinant bundle \( \gamma(\lambda, \tau) \in H_- \), and is analytic in \( \lambda \) for fixed \( \tau \). Note, \( \tau = 0 \Leftrightarrow \xi = 0 \).

Proof. Firstly we notice that \( \Gamma(\lambda, \tau) \equiv Z(\lambda, \tau) \) for all \( (\lambda, \tau) \in K \times [-1, 0] \), as for these values, the scaling above is simply the inverse of the map which took \( Z \mapsto Z^\sharp \). We thus gain analyticity of \( \Gamma \) for \( \tau \in [-1, +1) \) from the analyticity of \( Z \). The scaling taking \( Z^\sharp \mapsto \Gamma \) for \( \tau \in [0, +1) \) can be seen as changing the eigen direction corresponding to the eigenvalue of largest positive real part at \( A^{(k)}_{+\infty} \) into a line of critical points, like was done at for the system \( A^{(k)}_{-\infty} \). What we would like to prove then is under the flow defined by

\[
X' = (A^{(k)}(\lambda, \tau) - \mu^+(\lambda) I)X
\]

the solution \( \Gamma \) converges uniformly in \( \lambda \) to a non-zero critical point in the zero eigen direction. The uniform convergence is the content of the proof [2] lemma 6.1, so that \( \Gamma \) indeed defines a section of the determinant bundle over \( H_- \).

To adapt the determinant bundle method from here, we need only define \( W, \hat{W} \) appropriately so they converge to the path of critical points for \( A^{(k)}_{+\infty}, \hat{Y}^+(\lambda) \). The construction of the induced parallel translation will follow, and we can again compute the multiplicity of eigenvalues in \( K^0 \) with the geometric phase of a solution to the \( A^{(k)} \). The reference path which we use to compare the Chern class with the geometric phase of \( \Gamma(\lambda, +1) \) in this case is the loop \( \hat{Y}(\lambda) \) of unstable eigenvectors for \( A^{(k)}_{+\infty} \); once again we may invoke the lemmas on the reference paths to show that the phase of \( \hat{Y}^+(\lambda) \) is unique for any path compatible with the bundle construction, and must equal zero. The equivalence of phase for \( B^{(k)} \) and \( A^{(k)} \) solutions will follow similarly, as the calculations in (30) didn’t depend on the which eigenvalue scaling, \( \mu^\pm(\lambda) \) was used.
5 Boundary Value Problems

Gardner and Jones further developed the bundle construction for the Evans Function to study boundary value problems with parabolic boundary conditions, i.e., problems of the form:

\[
\begin{align*}
    u_t &= D_{xx}u + f(x,u,u_x) && (0 < x < 1) \\
    u(x,0) &= u_0 && B_0 u = 0 \\
    u(0,t) &= B_0 u = 0 \\
    u(1,t) &= B_1 u = 0
\end{align*}
\]

where \( u \in \mathbb{R}^n \), \( f : \mathbb{R}^{2n+1} \to \mathbb{R}^n \) is \( C^2 \). The matrix \( D \) is a positive diagonal matrix and the boundary operators are defined

\[
B_0 u = D_0 u(0,t) + N_0 u_x(0,t) \\
B_1 u = D_1 u(0,t) + N_1 u_x(0,t)
\]

such that \( D_j, N_j \) are diagonal with entries \( \alpha_j^i, \beta_j^i \) respectively which satisfy

\[
(\alpha_j^i)^2 + (\beta_j^i)^2 = 1 \quad \text{for} 1 \leq i \leq n \quad \text{and} \quad i = 1, 2
\]

Austin and Bridges built upon and generalized these bundle methods into a vector bundle construction for boundary value problems in which the boundary conditions can depend on \( \lambda \), and allow for general splitting of the boundary conditions. In this section we will consider how the method of Rupert Way can be adapted to boundary value problems, using the techniques Austin and Bridges developed for the general boundary conditions.

5.1 Constructing the Boundary Bundle for \( \mathbb{C}^n \)

Suppose for \( n \geq 2 \) we are given a system of ODE’s defining a flow on \( \mathbb{C}^n \), derived from the linearization of an operator \( \mathcal{L} \) about a steady state; as in the previous constructions, assume \( \Omega \subset \mathbb{C} \) is simply connected and let \( K \subset \Omega \) be a smooth loop which contains no eigenvalue of \( \mathcal{L} \). Assume the system is of the form

\[
\begin{align*}
    u_x &= A(\lambda,x)u && 0 < x < 1 \\
    \lambda \in \Omega \subset \mathbb{C}
\end{align*}
\]

\[
\begin{align*}
    a^*_i(\lambda) : \mathbb{C} \to \mathbb{C}^n && i = 1,\ldots,n-k \\
    b^*_i(\lambda) : \mathbb{C} \to \mathbb{C}^n && i = 1,\ldots,k
\end{align*}
\]

where \( A(\lambda,x) \) depends analytically on \( \lambda \), and the \( a^*_i, b^*_i \) are holomorphic functions of \( \lambda \).

The ambient trivial bundle is once again constructed from the product of a parameter sphere and the phase space. Define

\[
M = K^\circ \times \{0\} \cup K \times \{x \in (0,1)\} \cup K^\circ \times \{1\}
\]

where \( K^\circ \) is the region bounded by \( K \) in \( \mathbb{C} \); the trivial bundle over the parameter sphere is thus \( M \times \mathbb{C}^n \). The vectors \( (\lambda, x, a^*_i), (\lambda, x, b^*_i) \) for each \( (\lambda, x) \in M \) are anti-holomorphic sections of the trivial bundle, motivating the above dual notation. For a pair \( \nu(\lambda,x), \eta(\lambda,x) \) where \( \nu \) is a holomorphic section and \( \eta \) is an anti-holomorphic section of the trivial bundle, their product is defined as:

\[
\langle \eta, \nu \rangle_\lambda = \sum_{j=1}^{n_k} \eta_j(\lambda)\overline{\nu_j(\lambda)}
\]
where \( \eta_i, \nu_j \) are components of the vector valued function. This scalar product is holomorphic for all \( \lambda \in \Omega \), and the boundary value problem is formulated in this setting as follows:

\[
\langle a_i^*(\lambda), u(\lambda, x) \rangle = 0 \quad i = 1, ..., n - k
\]

\[
\langle b_i^*(\lambda), u(\lambda, 1) \rangle = 0 \quad i = 1, ..., k
\]

Significantly different in this construction from the previous is that there are no dynamics to consider on the caps of the sphere, and we will not be concerned with the eigenvalues of a limiting system. What is needed then is an analogue to the unstable bundle which will trace the dynamics from one hemisphere to the next, and pick up the winding induced by the dynamics to describe the multiplicity of the eigenvalues contained in \( K^0 \). The most obvious choice in this situation is to look at the \( k \) dimensional subspaces of \( \mathbb{C}^n \), given by the orthogonal compliments to \( \text{span}(V_0^*(\lambda)) \), where \( V_0^*(\lambda) := \{a_i^*(\lambda) : i = 1, ..., n\} \) for \( \lambda \in \Omega \). We would like these subspaces, and their images under the flow, to vary holomorphically the with respect \( \lambda \in \Omega \) so that we can construct a non-trivial vector bundle over \( M \) through which we can calculate the geometric phase. Let us define pointwise, for \( \lambda \in \Omega \), \( \{a_i^*(\lambda) : i = k + 1, ..., n\} \) such that the set \( \{a_i^*(\lambda)\}_i^n \) is a basis for \( \mathbb{C}^n \) for each such \( \lambda \).

**Proposition 3.** Bases may be chosen \( U_0(\lambda) = \{\xi_i(\lambda) \in \mathbb{C}^n : i = 1, ..., n - k\} \), \( V_0(\lambda) = \{\nu_i(\lambda) \in \mathbb{C}^n : i = 1, ..., k\} \) which vary holomorphically for \( \lambda \in \Omega \) such that \( \text{span}(V_0(\lambda)) = \text{span}(V_0^*(\lambda)) \) for \( \lambda \in K^0 \cup K \) and \( \forall \lambda \in \Omega \), \( U_0(\lambda) \oplus V_0(\lambda) = \mathbb{C}^n \).

Moreover

\[
\langle a_i^*(\lambda), \xi_j(\lambda) \rangle = \begin{cases} 
0 & \text{if } i = 1, ..., n \text{ and } i - n + k \neq j, \\
1 & \text{if } i = n + j - k, \text{ and } j = 1, ..., n \end{cases}
\]

and

\[
\langle a_i^*(\lambda), \nu_j(\lambda) \rangle = \begin{cases} 
0 & \text{if } i = 1, ..., n, \text{ } j = 1, ..., n - k \text{ and } i \neq j, \\
1 & \text{if } i = j = 1, ..., n - k \end{cases}
\]

Austin and Bridges prove these results in lemmas 3.1 through 3.3 in [3] and the reader is referred there for a full discussion. In addition the analogous results hold for the boundary conditions at \( x = 1 \), that is if \( U_1^*(\lambda) = \{b_i^*(\lambda) : i = 1, ..., k\} \), we can choose bases \( U_1(\lambda), V_1(\lambda) \) which vary holomorphically with respect to \( \lambda \in \Omega \) such that \( \text{span}(U_1(\lambda)) = \text{span}(U_1^*(\lambda)) \), and \( U_1(\lambda) \oplus V_1(\lambda) = \mathbb{C}^n \), with the dual orthonormality results to those above holding for these bases. Note, the bases \( U_1(\lambda), U_0(\lambda) \) and \( V_1(\lambda), V_0(\lambda) \) are dimension \( k \) and \( n-k \) respectively.
Reformulating the problem in this context, we say:

\[ u(\lambda, x) \text{ is an eigen function of } \mathcal{L} \text{ with eigenvalue } \lambda \text{ if and only if } u \text{ satisfies } u_x = Au \text{ and } u(\lambda, 0) \in \text{span}(U_0(\lambda)) \text{ and } u(\lambda, 1) \in \text{span}(V_1(\lambda)) \]

In this context, we are motivated to define the boundary bundle over \( M \) by prescribing the subspaces \( U_0(\lambda), U_1(\lambda) \subset \mathbb{C}^n \) on the caps of \( M \), and find subspaces which connect these over the sides of \( M \) which pick up the information of the flow. We choose these values for the fibres of the boundary bundle above the caps firstly because the dimensions need to agree; secondly we choose these because if \( \lambda \) which is not an eigenvalue, a solution to the system \( u' = Au \) cannot be in the span \( U_0 \) and \( V_1 \) at \( x = 0 \) and \( x = 1 \) respectively.

Any collection of solutions \( \{\gamma_1(\lambda, x), ..., \gamma_k(\lambda, x)\} \) that satisfy the boundary conditions at \( x = 1 \), and are linearly independent for \( (\lambda, 0) \), will be linearly independent for \( (\lambda, x) \) where \( x \in [0, 1) \). In particular when \( \lambda \) is not an eigenvalue of \( \mathcal{L} \), then \( \{\gamma_1(\lambda, 1), ..., \gamma_k(\lambda, 1)\} \) are linearly independent and must span some compliment of \( V_1(\lambda) \); in general this need not be the subspace \( U_1(\lambda) \), but it is possible to smoothly deform the solutions into \( U_1(\lambda) \) with the projection operator.

Specifically, if

\[ Q_\lambda : \mathbb{C}^n \rightarrow U_1(\lambda) \quad (46) \]

is the projection operator, let

\[ P_\lambda : \mathbb{C}^n \rightarrow V_1(\lambda) \quad (47) \]

be the projection operator onto \( V_1(\lambda) \) which is complimentary to \( Q_\lambda \), ie: \( P_\lambda = (I - Q_\lambda) \).

**Proposition 4.** Let \( u_i(\lambda, x) \) be solutions to the flow on \( \mathbb{C}^n \) such that \( u_i(\lambda, 0) = \xi(\lambda) \) for each \( i = 1, ..., k \), and let \( \{\eta_i(\lambda) : i = 1, ..., k \text{ and } \lambda \in \Omega\} \) be a holomorphic basis for \( U_1(\lambda) \). Define

\[ 
\mu_i(\lambda, x) := (I - xP_\lambda)(u_i(\lambda, x)) \quad (\lambda, x) \in K \times [0, 1] \\
\mu_i(\lambda, 0) := \xi(\lambda) \quad \lambda \in K^\circ \quad (48)
\]

Then \( \{\mu_i(\lambda, x)\} \) are linearly independent and holomorphic for all \( (\lambda, x) \in M \setminus (K^\circ \times \{1\}) \). Moreover, \( \text{span}(\{\mu_i(\lambda, 1)\}) = U_1(\lambda) \) for \( \lambda \in K \) and we can write

\[ 
\mu_i(\lambda, 1) := \sum_{j=1}^k \alpha_{i,j}(\lambda)\eta_j(\lambda) \quad (49)
\]

for holomorphic functions \( \{\alpha_{i,j}(\lambda)\}, \lambda \in K \).

Indeed, this proposition follows immediately from the results of section 4 in \[3\], and we may now use the above construction to describe the boundary bundle over \( M \).

**Definition 2.** Define \( \mathcal{E}_{\lambda,x} \subset \mathbb{C}^n \) to be the \( k \) dimensional subspace spanned by \( \{\mu_i(\lambda, x) : i = 1, ..., k\} \) for \( (\lambda, x) \in K \times [0, 1] \), over \( K^\circ \times \{0\} \) define \( \mathcal{E}_{\lambda,0} = \text{span}(U_0(\lambda)) \) and over \( K^\circ \times \{1\} \) define \( \mathcal{E}_{\lambda,1} = \text{span}(V_1(\lambda)) \); then \( E = \{(\lambda, x, \mathcal{E}_{\lambda,x}) : (\lambda, x) \in M\} \) is defined to be the boundary bundle with respect to (36) over \( M \).
It follows from proposition 3 that $E$ is a holomorphic vector bundle for which we can use \( \{ \mu_i(\lambda, x) : i = 1, ..., k \} \) and \( \{ \eta_i(\lambda) : i = 1, ..., k \} \) to construct local trivializations over the upper and lower hemispheres of $M$, as defined in section 3. In particular define \( \{ \eta_i(\lambda, x) : i = 1, ..., k \} \), for \( (\lambda, x) \in K \times (0, 1] \), as the solutions to (36) which converge to \( \{ \eta_i(\lambda) \} \) as $x \to 1$, and suppose we extend the \( \{ \mu_i(\lambda, x) \} \) holomorphically into an open set in $K^\circ$ homotopy equivalent to $S^1$. With these extensions, we can then define the trivializations of $E$ over open sets in $M$, by

\[
\phi_- : H_- \times \mathbb{C}^k \leftrightarrow H_- \times \mathbb{C}^n \\
(\lambda, x, z e_i) \mapsto (\lambda, x, z \mu_i(\lambda, x))
\]

\[
\phi_+ : H_+ \times \mathbb{C}^k \leftrightarrow H_+ \times \mathbb{C}^n \\
(\lambda, x, z e_i) \mapsto (\lambda, x, z \eta_i(\lambda, x))
\]

whereby the transition map at \( \{1\} \times K \) is defined by the matrix with coefficients \( \{ \alpha_{i,j}(\lambda) \} \). The Chern number of this vector bundle is unique up to its isomorphism class, and Austin and Bridges show in [3] that the determinant of this matrix as a map

\[
det(\alpha_{i,j}(\lambda)) : K \to \mathbb{C} \setminus \{0\}
\]

has winding equal to the multiplicity of eigenvalues of $L$ contained within $K^\circ$, and the Chern number.

### 5.2 Adapting the Method of Rupert Way

With above construction, we are now once again in the position to utilize the method of Rupert Way to calculate the multiplicity of eigenvalues contained within $K^\circ$. As in section 3, we know that the determinant bundle, $\Lambda^k(E)$, of the vector bundle $E$ has the same Chern number as $E$ and the wedge product of the solutions \( \{ \mu_i(\lambda, x) : i = 1, ..., k \} \) forms a solution to the system on $\Lambda^k(\mathbb{C}^n)$

\[
Z = Z_1 \wedge ... \wedge Z_k
\]

\[
Z' = A Z_1 \wedge ... \wedge Z_k + ... + Z_1 \wedge ... \wedge A Z_k
\]

\[
\iff Z' = A^{(k)} Z
\]

Thus we define $U(\lambda, x) := \mu_1(\lambda, x) \wedge ... \wedge \mu_k(\lambda, x)$, and denote $\hat{U}(\lambda, x)$ to be the hyperspherical projection of $U(\lambda, x)$; $\hat{U}$ may not be holomorphic, but as in previous sections it inherits infinite differentiability in the parameter $s$, where $\lambda(s) : [0, 1] \to K$. We may thus calculate the geometric phase of the vector $\hat{U}(\lambda(s), x)$ on the Hopf bundle $S^{(2\binom{n}{k} - 1)}$.

Let $\eta(\lambda, x) := \eta_1(\lambda, x) \wedge ... \wedge \eta_k(\lambda, x)$, then we may define trivializations of the determinant bundle in the obvious way through $U(\lambda, x)$ over $H_-$ and $\eta(\lambda, x)$ over $H_+$. The Chern number of this vector bundle is equal to the winding of the transition function given exactly by the winding of the determinant of $\{ \alpha_{i,j} \}$, and while the transition function depends on the choice of the trivialization, the winding does not. Thus, we may adapt these trivializations to ones more
suitable for calculating the geometric phase. We may normalize \( \eta(\lambda, x) \), and find the path horizontal to the normalization with respect to the natural connection on the Hopf bundle. In particular, let \( \tilde{\eta}(\lambda, x) \) be the normalized, horizontal path corresponding to \( \eta(\lambda, x) \) - this will be the trivialization we will use over \( H_+ \) and we will use \( \tilde{U}(\lambda, x) \) over \( H_- \).

With respect to the above trivializations, the calculations of the winding of the transition function and the geometric phase are identical to the calculations performed in section 3. Thus calculating the geometric phase of the solution \( \tilde{U}(\lambda, x) \) at \( x = 1 \) we may once again ascertain the multiplicity of the eigenvalues contained in \( K^\circ \), proving the method of Rupert Way for boundary value problems of the form described in [3].

5.3 Implementing the Method

In practice the many of the theoretical considerations in the above discussion will drop into the background, and the method itself is simple to describe; for this reason we will outline the steps to the method here. Suppose we are given a system of the form of (42), we will begin by finding the holomorphic vectors \( \{\xi_j(\lambda)\} \) as described in Proposition 3 and \( \{\eta_i(\lambda)\} \) as described in proposition 4. The \( \{\eta_i(\lambda)\} \) will allow us to define the necessary projection operator \( Q_\lambda = (I - P_\lambda) \), with which we may compute the \( \mu_i(\lambda, 1) \).

Here we break from the proof, as it is not necessary to compute the \( \mu_i(\lambda, x) \) for any \( x \neq 1 \); the \( \mu_i(\lambda, x) := (I - xP_\lambda)u(\lambda, x) \) so that the operator \((I - xP_\lambda)\) is always applied to the independently evolved \( u_i(\lambda, x) \). Thus it suffices to find the forward integrated solutions \( u_i(\lambda, 1) \) and apply the projection at the opposite endpoint of the boundary. Taking the wedge of the \( \mu_i(\lambda, 1) \), we may normalize the corresponding loop in \( \wedge^k(C^n) \) and calculate the geometric phase of this normalized loop.
6 Numerical Results

In this section we present new numerical results obtained with modifications to Rupert Way’s Matlab code for computing the geometric phase on the Hopf bundle. Way’s method was roughly an ODE45 shooting argument in which:

- A discretization of the contour $K$ is chosen

- For each time step on the contour $\lambda_t$, the initial condition $X^+(\lambda_t, \xi_0)$ is integrated forward according to $A(\lambda_t, \xi)$.

- Similarly $X^+(\lambda_{t\pm1}, \xi_0)$ are integrated forward according to $A(\lambda_{t\pm1}, \xi)$ to the forward state

- The forward integrated vectors are normalized onto the sphere

- An approximation of the tangent vector at $\hat{X}^+(\lambda_t, \xi_1)$ is given by:
  
  $$\frac{\hat{X}^+(\lambda_{t+1}, \xi_1) - \hat{X}^+(\lambda_{t-1}, \xi_1)}{\delta_t}$$

- The connection is applied to the tangent vector, and the result is integrated along all time steps.

These results suggest future directions for research in stability analysis and the connection between the existence of eigenvalues and the underlying structure of the travelling wave. The Matlab code Rupert Way developed was modified in the following examples to fit the method described in this paper - that is, the eigenvector corresponding to the eigenvalue of greatest positive real part is integrated forward in time from a time which approximates the system at $\xi = -\infty$. In each of the following examples, there is a clear dependence on the length of the integration in the $\xi$ direction, where the phase experiences a rapid transition between zero and, when stable, the count of the multiplicity of the eigenvalue.

Of interest as well are examples with degenerate construction - that is, when choosing eigenvector initial conditions which could not be used to define the unstable bundle construction. These examples exhibit similar behaviour to the non-degenerate cases, but demonstrate a dependence on the reference path that doesn’t exist in the other case. This is especially interesting, because the proof given in this work doesn’t work for the degenerate case - it required the equivalence of the Chern class of the unstable bundle and the geometric phase to verify the result. This suggests future work to understand how far the method of Rupert Way can be extended.

6.1 Reaction Diffusion Equations

We return now to the example presented in the introduction of the general reaction diffusion equation with non-linearity of the form $f(u) = u(u-a)(u-1)b$, $0 < a \leq \frac{1}{2}$, and $b \neq 0$. In particular we will consider the cases when $a = -1, \frac{1}{2}$, $c = 0$, and $\xi = x$ where $u(\xi)$ is a steady state which can be found explicitly. In each of these cases 0 is an eigenvalue of multiplicity one for the linear operator $L$ derived from the linearization about the steady state.
In the case where \( a = -1 \), \( u(\xi) = \sqrt{2}sech(\xi) \) is a time independent solution to the equation \( u_t = u_{\xi\xi} + u(u^2 - 1) \). About this steady state we derive the system
\[
Y' = A(\lambda, \xi)Y \quad \xi \in (-\infty, \infty)
\]
\[
A = \begin{pmatrix} \lambda + 1 - 6sech^2(\xi) & 1 \\ 0 & 0 \end{pmatrix} \quad A_\infty(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda + 1 & 0 \end{pmatrix}
\]
which is equivalent to the operator \( \mathcal{L}_\xi(p) = p_{\xi\xi} + f'(u(\xi))p \). The eigenvalues/vectors for the asymptotic system are of the form
\[
\begin{cases} 
{+\sqrt{\lambda + 1}, \begin{pmatrix} 1 \\ \sqrt{\lambda + 1} \end{pmatrix}} \\
{-\sqrt{\lambda + 1}, \begin{pmatrix} 1 \\ -\sqrt{\lambda + 1} \end{pmatrix}} 
\end{cases}
\]
(54)

Modifying Rupert Way’s “Integration at \( L'' \) Matlab code, we plot the building of the geometric phase against the length of the \( \xi \) integration interval. Define the spectral path \( \lambda(s) = 1.1e^{2\pi s} \) about the eigenvalue at zero, with the \( s \) integration time step \( \frac{2\pi}{10000} \). The \( \lambda \) path of unstable eigenvectors for the asymptotic system at \( -\infty \) is integrated in the \( \xi \) direction from \([-11, -11 + j]\), with \( j = .1 : .1 : 22 \), and the geometric phase calculated for each \( j \) - that is we calculate the integral of the natural connection about the spectral path at each of the forward time steps \(-11 + j\).

We see in the plot that the geometric phase remains close to zero, with minor fluctuations, until the \( \lambda \) dependent path of eigenvectors is integrated past \( \xi = 0 \); immediately the calculation sees a rapid transition to the phase being close to one, the multiplicity of the eigenvalue contained in the spectral path.
Suppose we instead scale our initial condition

\[
\begin{pmatrix}
1 \\
\sqrt{\lambda + 1}
\end{pmatrix}
\]

by the factor \( \frac{1}{\lambda} \); over the contour given above this will \textbf{not} be compatible with the construction in [2] as there is a pole enclosed at 0. Trying to extend this loop of eigenvectors over \( K^\circ \) would be tantamount to scaling a fibre by \( \infty \).

Interestingly the pattern of rapid transition persists, but the whole phase plot is translated by the count of the pole.

Should we instead scale our initial condition

\[
\begin{pmatrix}
1 \\
\sqrt{\lambda + 1}
\end{pmatrix}
\]

by the factor \( \lambda \), again this will \textbf{not} be compatible with the construction in [2] as there is a zero enclosed by the contour at 0. Again this would be a degenerate construction, defining only a zero fibre above \( \lambda = 0 \)
Interestingly once again we see the behaviour persist despite the degeneracy in the construction, though in this case translated by the count of the zero.

Similarly now in the case where \( a = \frac{1}{3} \) with \( f(u) = u(u - \frac{1}{2})(u - 1)^8 \), \( u(x) = \frac{1}{2} - \frac{1}{2} \tanh^2(\xi) \) is a time independent solution to the equation \( u_t = u_{\xi\xi} + u(u - \frac{1}{2})(u - 1)^8 \), with \( c = 0 \). We derive the following system from the linearization about the steady state:

\[
Y' = A(\lambda, \xi)Y \quad \xi \in (-\infty, \infty)
\]

\[
A = \begin{pmatrix}
0 & 0 \\
\lambda - 2 + 6 \tanh^2(\xi) & 1
\end{pmatrix} \quad A_\infty(\lambda) = \begin{pmatrix}
0 & 1 \\
\lambda + 4 & 0
\end{pmatrix}
\]

which is equivalent to the operator \( \mathcal{L}_\xi(p) = p_{\xi\xi} + f'(u(\xi))p \). The eigenvalues/vectors for the asymptotic system are of the form

\[
\begin{align*}
\{ &+\sqrt{\lambda + 4}, \begin{pmatrix} 1 \\ \sqrt{\lambda + 4} \end{pmatrix} \} \\
&-\sqrt{\lambda + 4}, \begin{pmatrix} 1 \\ -\sqrt{\lambda + 4} \end{pmatrix} \}
\end{align*}
\]

Following the same setup as in the case for \( a = -1 \), we plot the build up of the geometric phase versus the length of the integration in the \( \xi \) direction \([-11, -11 + j], j = .1 : .1 : 22\).
Again the geometric phase sees the rapid transition from zero to the multiplicity of the eigenvalue, approximately when the $\lambda$ dependent path of eigenvectors is integrated past $\xi = 0$.

For a further demonstration of the degenerate constructions, we plot the initial conditions scaled by a pole of order 5 and zero of order 5 below.
This behaviour is of great interest because it suggests there is further generality in which the method can be used, which is yet to be determined.

6.2 Rupert’s Test Case

In [9] Rupert Way draws a motivating example from quantum mechanics, which he refers to as the “Test Case”. The Test Case is a system of the form:

\[ Y' = A(\lambda, \xi)Y \quad \lambda \in \mathbb{C} \setminus \{(-\infty, -1]\} \quad x \in (-\infty, \infty) \]

\[ A(\lambda, \xi) = \begin{pmatrix} 0 & 1 \\ \lambda + 1 + 3 \text{sech}^2(\frac{\xi}{2}) & 0 \end{pmatrix} \quad A_{\infty}(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda + 1 & 0 \end{pmatrix} \]

(57)

The operator from which this system is derived has continuous spectrum on the interval \((-\infty, -1] \subset \mathbb{C}\) and discrete eigenvalues at \(\lambda = -\frac{3}{4}, 0, \frac{5}{4}\). The eigenvalues/vectors of the asymptotic system are given by

\[ \{ +\sqrt{1 + \lambda}, \begin{pmatrix} 1 \\ \sqrt{1 + \lambda} \end{pmatrix} \} \quad \{ -\sqrt{1 + \lambda}, \begin{pmatrix} 1 \\ -\sqrt{1 + \lambda} \end{pmatrix} \} \]

(58)

For each of the eigenvalues \(\lambda = 0, \frac{5}{4}\), we define the spectral path to be the circle of radius \(r = .2\) centred at \(\lambda\), but in the case \(\lambda = -.75\) the radius is instead chosen to be \(r = .01\) due to the sensitivity to the nearby spectrum. Following the same procedure as in the reaction diffusion examples, we plot the geometric phase versus the length of the \(\xi\) integration.
With the Test Case, we see the familiar pattern of rapid transition between geometric phase close to zero moving to the multiplicity of the eigenvalue, though with some delay in the case $\lambda = -.75$.

Consider the spectral path given by the circle with center at $-0.37$ and radius equal to $0.5$; in this case the path encloses the eigenvalues at $-0.75, 0 \in \mathbb{C}$ each with multiplicity one. With the same step $\frac{2\pi}{10000}$ for the integration of the connection about the spectral path, we plot the build up of the geometric phase versus the length of integration in the $\xi$ direction $[-11, -11 + j]$ for $j = .1 : .1 : 22$. 
Up to minor fluctuations, the rapid transition between phase close to zero and phase close to multiplicity is still clear, with the phase reaching a value close to 2 with the delay seen in the integration about $\lambda = -.75$.

If we take the spectral path to be the circle with center at 1.25 and radius $r = 2.1$, this loop encloses the three eigenvalues and proceeding as before we plot the geometric phase versus the length of integration in the $\xi$ direction $[-11, -11 + j]$ for $j = .1 : .1 : 22$. 
The pattern of rapid transition is again seen, but this time with greater fluctuation and time between states.

### 6.3 Hocking Stewart Pulse

Way also wrote code for determining eigenvalues for the linearization of the scaled Ginzburg-Landau equation, where he studied a marked sensitivity of the phase to the choice of the contour. The scaled Ginzburg-Landau equation is given

$$\rho e^{i\omega} Y_t = Y_{\xi\xi} - (1 + i\omega)^2 Y + (1 + i\omega)(2 + i\omega) |Y|^2 Y$$  \hspace{1cm} (59)

and is linearized about the Hocking-Stewart pulse

$$Y(x, t) = (\cosh \xi)^{-1 - i\omega}$$  \hspace{1cm} (60)

The ODE system he considers is constructed for Evans function analysis using the second exterior power on $\mathbb{C}^4$, yielding the asymptotically autonomous system on $\mathbb{C}^6$ defined

$$u_\xi = A(\lambda, \xi) \quad \xi \in \mathbb{R} \quad \lambda \in \mathbb{C} \quad u \in \mathbb{C}^6$$

$$A(\lambda) = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ a_{32} & 0 & 0 & 0 & 0 & 0 \\ a_{42} & 0 & 0 & 0 & 0 & 1 \\ -a_{31} & 0 & 0 & 0 & 0 & -1 \\ -a_{41} & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_{41} & -a_{31} & -a_{42} & a_{32} & 0 \end{pmatrix}$$  \hspace{1cm} (61)

with the asymptotic system

$$A_\infty(\lambda) = \lim_{\xi \to \pm \infty} A(\lambda, \xi) = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ p(\lambda) & 0 & 0 & 0 & 0 & 0 \\ \tau(\lambda) & 0 & 0 & 0 & 0 & 1 \\ -\tau(\lambda) & 0 & 0 & 0 & 0 & -1 \\ p(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & -p(\lambda) & -\tau(\lambda) & -\tau(\lambda) & -p(\lambda) & 0 \end{pmatrix}$$

where the $a_{ij}$ depend on the parameters $\rho, \omega, \psi, \lambda$ and on $\xi$. The asymptotic system has a unique eigenvalue of most positive and most negative real part given by $\sigma^-, \sigma^+$ respectively, and associated eigenvectors

$$\{\sigma^- = \sqrt{2} \sqrt{\tau + \sqrt{\tau^2 + p^2}}, \quad \chi^-(\lambda) = \begin{pmatrix} 2\sigma^- \\ -2p \\ (\sigma^-)^2 \\ -2p \\ (\sigma^-)^2 - 2\tau \end{pmatrix} \}$$  \hspace{1cm} (62)

$$\{\sigma^+ = -\sqrt{2} \sqrt{\tau + \sqrt{\tau^2 + p^2}}, \quad \chi^+(\lambda) = \begin{pmatrix} 2\sigma^+ \\ -2p \\ (\sigma^+)^2 \\ -2p \\ (\sigma^+)^2 - 2\tau \end{pmatrix} \}$$
Way calculates the geometric phase of the unique eigenvector corresponding to the eigenvalue of most negative real part, integrated from $x = 10$ to $x = -10$. This analysis for the system on $\mathbb{C}^6$, although dual to the construction we have outlined, falls within the general framework by taking advantage of the calculation of phase on the path which corresponds to the determinant bundle and thus essentially agrees with the method outlined in this paper\cite{9}.

For the parameter values $\omega = 3$, $\rho = \frac{1}{\sqrt{5}}$, and $\psi = \arctan(2)$ there is a known double eigenvalue at $\lambda = 0$, and simple eigenvalues at approximately $\lambda = -6.6357$ and $\lambda = 15$ calculated in \cite{11}. Way noticed sensitivity in this case to the choice of his contours about the eigenvalues. Particularly Way observed a marked decay of the phase away from integer values by choosing circles of decreasing radii about the eigenvalues. The phase dropped to order $10^{-3}$ when he took the radius of the contour about $\lambda = -6.6357$ as low as $r = .0003$, and for $\lambda = 0$, the phase dropped below 1 when the radius was as low as $r = .01$. Way also suggested a discrepancy in results for the precise location of the eigenvalue $\lambda \approx -6.6357$ calculated in \cite{11}. Phase reached a local maximum for a mesh of circles with radius $r = .0003$ and centres spaced .0005 at $\lambda = -6.641$; all of the phases were of reduced order but the peak did not occur at $\lambda = -6.6357$ as would have been expected.

Following the analysis as in previous examples, we plot the development of the geometric phase versus integration in the $\xi$ direction for contours centred at each of the $\lambda = 0, 15, -6.641, -6.6357$ for radii varying between $r = 1$ and $r = 0.0003$. We will plot, as in previous examples, the development of phase over the $\xi$ interval for contours of descending radius to illustrate the sensitivity Way described and propose new considerations for the sensitivity analysis.

To begin we choose the contours to be the circles of radius one descending centred at $\lambda = 0$. The eigenvector corresponding to the unique largest eigenvalue is integrated from $[-11, -11 + j]$ where $j = .1 : .1 : 22$, where in this case the single trajectory is an approximation of the determinant bundle of the unstable bundle in the space $\Lambda^2(\mathbb{C}^4) \equiv \mathbb{C}^6$. 

\section*{44}
In the case of $\lambda = 15$ we see a rapid transition of phase building again in a prototypical fashion with little variation to this trend until the radius $r = .001$. When the contour is chosen to be the circle $r = .001$ about $\lambda = 15$ there is a sudden degeneracy of the method where only one spike is seen which agrees with the multiplicity of the eigenvalue. The stability of the calculation up until the sudden transition of this example suggest that $\lambda = 15$ is an approximation of the eigenvalue good to $O(10^{-2})$, but differs by at least by .001.

This first example differs significantly from the analysis at each of the other
two eigenvalues; in each of these cases we see much greater variability in the phase calculation and gradual degeneracy introduced through the reduction of the radii for the contours.

In the case where $\lambda = 0$, Way saw a complete degeneracy of the method for all $r \leq .01$. However, plotting the phase versus the integration in the $x$ direction, there is a more complicated picture. The phase calculation becomes increasingly unstable and dependent on the length of $\xi$ integration with each reduction of the radius - total degeneracy only occurs though at $O(10^3)$. Up to this point there is increasingly chaotic behaviour of the phase, but there is some loose structure of the phase about the multiplicity equal to 2.
Here we will look at the plots for descending radii for the two approximations for the eigenvalue near $\lambda = -6.6357$ and $\lambda = -6.641$. Beginning with the plot for $\lambda = -6.6357$ and radius $r = 1$, we see an immediate contrast with the previous results in the initial fluctuations of phase before the rapid transition of modes. The reducing the radius once again introduces instability to the phase calculation and large fluctuations of the phase over time. The calculation of phase remains relatively consistent, besides the isolated jumps, up to $r = .006$. When $r = .003$, the first contour which excludes the approximation by Way $\lambda = -6.641$ there is a sudden change of behaviour. The peaks of the phase plot do not exceed .7, and while the plot maintains some structure, the drop in the phase is marked. Total degeneracy occurs for $r = .001$, and increasingly chaotic behaviour dominates the phase plot after the point of transition.
For $\lambda = -6.641$ we see a similar plot to the ones for $\lambda = -6.6537$, but with increasingly degenerate behaviour for small radii. The oscillations become more dramatic for $r = .06$, and for the $r = .03$ where $\lambda = -6.6357$ is excluded from the contour, there is a greater concentration of phase around zero after the point of rapid transition. Total degeneracy occurs for $r = .01$ though the pattern of large fluctuation continues for $r = .0003$. 
The plot of phase against the length of the $x$ interval yields surprising results in terms of the variability of the phase beyond the transition point. The greater degeneracy of the plot for $\lambda = -6.641$ versus the plot for $\lambda = -6.6357$ suggests the greater accuracy of the approximation in [II], though any future study of this eigenvalue must clearly take into account the variability of the calculation with respect to the length of the $x$ interval. Way’s choice of integrating from $x = 10$ to $x = -10$ found the peak of the phases attained at $\lambda = -6.641$, though from the results it should be clear that this maximum is subject to the length of the integration.

In each of the above examples there is a rapid transition between the phase equal to zero and non-trivial phase plots. Where the calculation is the most stable, the transition is displayed clearly, though sometimes with some fluctua-
tions; where the calculation becomes less stable there is still a rapid transition between trivial and chaotic behaviour and the significance of this transition for all of the above examples is a subject for future analysis.
7 Summary of Results and Ongoing Questions

In this paper we have adapted and verified the method of Rupert Way for the determination of the multiplicity of eigenvalues for asymptotically stable and boundary value problems. Building from the groundwork laid in [9] we proved a dual form of Way’s central conjecture using the unstable bundle as in the Evans function construction in [2], and using the technique of the determinant bundle, generalized the method for higher dimensional systems. Building on the work of [3] and [5], Way’s method was adapted to fit $\lambda$ dependant boundary value problems. In addition, we have produced new numerical results which further the study of the method of Rupert Way in applications.

The numerics developed by Way exhibit a clear dependence on the length of the forward integration in the $\xi$ direction. The results suggest firstly that it is not in general necessary to forward integrate the $\lambda$ dependent loop of eigenvectors to a value “close to $\infty$”, but rather, just past some critical point at which there is a rapid transition between the geometric phase being close to zero and the geometric phase being close to the multiplicity of the eigenvalues enclosed by the spectral path. The significance of the critical value is currently unclear, and the connection to the underlying structure of the steady state about which the system is derived is an open question.

We return here to the remarks at the end of sections 3.4 and 3.5 in regards to the reference path and the calculation of geometric phase. For simplicity, consider the general symmetric-asymptotic system as in section 4, given by

$$Y' = A(\lambda, \tau)Y \quad \tau \in (-1, 1)$$

$$\lim_{\tau \to \pm 1} A(\lambda, \tau) = A_{\pm \infty}$$

(63)

Let $Z$ denote the solution to the corresponding $B^{(k)}$ system for which $Z(\lambda, -1)$ is the loop of eigenvectors corresponding to the eigenvalue of largest positive real part and suppose $\lambda(s)$ is a parametrization of $K$; for each $\tau \in [-1, 1]$ we will define $Z(\lambda(s), \tau)$ to be the horizontal lift of the hyperspherical projection $\hat{Z}(\lambda(s), \tau)$. Supposing we could use $Z(\lambda(s), \tau)$ as a trivialization for the unstable bundle, it is easily verified that computing the geometric phase of $\hat{Z}(\lambda(s), \tau)$ is equivalent to finding the winding of the $\lambda(s)$ dependent change of basis matrix from $\hat{Z}(\lambda(s), \tau)$ to $Z(\lambda(s), \tau)$. However, a horizontal lift of a loop in general will not be a loop, and when it is not a loop, it cannot define a trivialization of the unstable bundle. Consider, $Z(\lambda(s), \tau)$ cannot be for all $\tau$ because whenever it is a loop this implies $\hat{Z}(\lambda(s), \tau)$ has exactly integer geometric phase. By the smoothness of the calculation of phase in $\tau$, it requires that the geometric phase take intermediary values between the modes shown in the numerics. Moreover, if the geometric phase of our trivialization begins at zero, as demonstrated in the proof, then for $Z(\lambda, \tau)$ to always be a loop would require the phase to be everywhere zero.

The numerics, however, do suggest an interesting geometric investigation: in the above cases there is a clear transition between hemispheres where $Z(\lambda, \tau)$ is a loop and can be used as a trivialization for the unstable bundle. Indeed, while the phase is integer and agrees with the Chern class, $Z(\lambda, \tau)$ will suffice as a trivialization because it is a loop, and the winding of the transition map does agree with the Chern class. One of the open questions in this construction is thus...
to find the relationship between the “hemispheres” of the unstable bundle and the geometric phase. Yet another question to pursue is how far the method can be extended beyond the unstable bundle construction, in light of the persisting behaviour in the degenerate constructions.

Many thanks go to Rupert Way and Tom Bridges for opening this line of research with their work in 9, opening the way for what will hopefully be many useful new techniques in stability analysis.
References

[1] Andrei L. Afendikov and Thomas J. Bridges. Instability of the hocking-stewartson pulse and its implications for three-dimensional poiseuille flow. Proceedings: Mathematical, Physical and Engineering Sciences, 457(2006):257 – 272, 2001.

[2] J. Alexander, R. Gardner, and C. Jones. A topological invariant arising in the stability analysis of traveling waves. JOURNAL FUR DIE REINE UND ANGEWANDTE MATHEMATIK, 410:167 – 212, 1990.

[3] Francis R. Austin and Thomas J. Bridges. A bundle view of boundary-value problems: generalizing the Gardner-Jones bundle. JOURNAL OF DIFFERENTIAL EQUATIONS, 189:412 – 439, 2003.

[4] Thomas J. Bridges. The orr-sommerfeld equation on a manifold. Proceedings: Mathematical, Physical and Engineering Sciences, 455(1988):319 – 340, 1999.

[5] R. Gardner and C. K. R. T Jones. A stability index for steady state solutions of boundary value problems for parabolic systems. JOURNAL OF DIFFERENTIAL EQUATIONS, 91:181 – 203, 1991.

[6] Christopher K. R. T. Jones. Stability of the travelling wave solution of the Fitzhugh-Nagumo system. Transactions of the American Mathematical Society, 286(2):431–469, 1984.

[7] Shoshichi Kobayashi. Foundations of differential geometry. Wiley, New York, 1996.

[8] Gerard Walschap. Metric Structures in Differential Geometry. Springer, New York, 2004.

[9] R. Way. Dynamics in the Hopf bundle, the geometric phase and implications for dynamical systems. PhD thesis, University of Surrey, 2009.