A null model for Dunbar’s circles

Manuel Jiménez-Martín,1 Ignacio Tamarit,2 Javier Rodríguez-Laguna,1 and Elka Korutcheva1

1Dpto. Física Fundamental, UNED, C/ Senda del Rey 9, 28040, Madrid, Spain
2Universidad Carlos III de Madrid, Grupo Interdisciplinar de Sistemas Complejos (GISC), Departamento de Matemáticas, 28911 Leganés, Madrid, Spain

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An individual’s social group may be represented by their ego-network, formed by the links between the individual and their acquaintances. Ego-networks present an internal structure of increasingly large nested layers of decreasing relationship intensity, whose size exhibits a precise scaling ratio. Starting from the notion of limited social bandwidth, and assuming fixed costs for the links in each layer, we propose a grand-canonical ensemble that generates the observed hierarchical social structure. This result suggests that, if we assume the existence of layers demanding different amounts of resources, the observed internal structure of ego-networks is indeed a natural outcome to expect. In the thermodynamic limit, realized when the number of ego-network copies is large, the specific layer degrees reduce to Poisson variables. We also find that, under certain conditions, equispaced layer costs are necessary to obtain a constant group size scaling. Finally, we fit and compare the model with an empirical social network.

I. INTRODUCTION

The computational capacity to store and manage an ever-changing social network is thought to depend roughly on neocortical size, which evolved driven by the need of managing increasingly large social groups [1]. This is the statement of the far reaching social brain hypothesis [2], which links brain volume in humans, primates and other mammals with the size of their social groups. For humans, the Dunbar’s number constitutes an upper limit of ∼150 for the social group size. That is, the total number of active relationships that we can maintain at any given time; a cognitive constraint that seems to operate also in virtual environments, such as Twitter [3]. Certainly, to monitor and handle social ties comes at a cost, it takes time to cultivate these relationships [4], and the process is limited by cognitive constraints, such as memory and mentalising skills [5]. Current sociological studies rely on large digital datasets in order to build weighted networks of human relationships, where link weights encoding actor-to-actor interaction frequency are used as a proxy for emotional closeness [6, 7]. In this framework, an individual’s social group is equivalent to its set of neighbors, which is often called its ego-network.

Ego-networks are internally highly structured and their links can be sorted by their weights [5]. Moreover, links can be clustered into groups of increasing number of links and decreasing emotional closeness [3]. These layers form a nested hierarchy, where the cumulative sizes of consecutive groups follow a preferred scaling ratio of approximately 1/3, forming a sequence of typical group sizes of ∼5, ∼15, ∼50 and ∼150, which are sometimes called Dunbar’s circles. A smaller inner layer of size ∼1.5 [10], and two larger groups of sizes ∼500 and ∼1500 [11], have also been reported. In each case, the scaling relationship between consecutive groups holds. This hierarchical structure appears to be a fundamental organizational principle of human groups, and has been confirmed in online games [12], online social networks [13, 14] and telephone call detail records [15].

While several null models suitable for weighted social networks have been proposed recently [16–18], no such models have yet been proposed to analyze the observed ego-network hierarchical structure. In this paper, we propose a grand-canonical ego-network statistical ensemble that reproduces qualitatively the hierarchical structure, and is able to fit experimental data successfully. We simplify the problem by assigning the ties in each layer constant costs and postulate an abstract social capital [19], or resource, that is spent in placing the links into the different layers. Then, by fixing the actors average total degree and resource, the hierarchical structure emerges spontaneously. This ensemble is an unbiased null model for ego-networks which offers a parsimonious explanation for the nested structure. It can also be used to generate synthetic data with the desired layer scaling, as well as for hypothesis testing against more complex models. The ensemble probability distribution, as well as its thermodynamic limit, is presented in section II. In section III we examine the hierarchical structure and prove that, in our setting, equispaced costs are a condition for a constant group size scaling in the outer layers. We fit and compare the model to an empirical dataset in section IV. Finally, section V is dedicated to the discussion.

II. GRAND-CANONICAL ENSEMBLE FOR DUNBAR’S CIRCLES

Let us assume the existence of a social resource, s, that an individual can employ to establish ties or relationships of different emotional intensity with their k acquaintances. We consider r = 1, . . . , K different relationship layers with respective costs, s_r ∈ R, with 0 ≤ s_r ≤ s.
and sorted in order of decreasing emotional intensity, such that \( s_r > s_{r+1} \). We define the layer-degree, \( k_r \), as the number of ties of cost \( s_r \). In sum, a given individual identified by index \( j \) will have a total degree and social resource verifying

\[
k(j) = \sum_r k_r(j), \quad s(j) = \sum_r s_j k_r(j).
\]

(1a)

(1b)

The Dunbar’s circles are inclusive groupings of decreasing emotional closeness [9], hence the group at level \( r \) includes all layers of cost higher or equal to \( s_r \). In our setting, the variables corresponding to the size of Dunbar’s circles are the cumulative group-sizes, defined as \( n_r = \sum_{l=1}^r k_l \).

Thus, an individual’s ego-network in our setting is completely described by the configuration variables \( k_r \). This system can take any state, \( \{k_r\}_{r=1,\ldots,K} \) that verifies the constraints Eqs. (1). The problem is equivalent to that of distributing \( k \) particles among \( K \) energy levels \( \{s_r\} \) in a quantum micro-canonical ensemble [20, 21]. Without making any assumption \textit{a priori} about the probabilities of any given configuration, we could study the average layer structure in a micro-canonical ensemble. The micro-canonical ensemble assigns an homogeneous probability distribution on the configuration space, hence the problem would solved if we could compute the total number of allowed configurations, i.e. the partition function. However, as in classical statistical mechanics, it is simpler to formulate instead a generalized or grand-canonical ensemble, consisting on a large number of identical copies of the system, for which the constraints Eqs. (1) are verified only on average. Consider a group of \( N \) individuals or egos, each of which having different degree, \( k(j) \) and resource \( s(j) \); with given averages

\[
\langle k \rangle = \sum_{j=1}^N k(j) P(\{k_r(j)\}), \quad \langle s \rangle = \sum_{j=1}^N s(j) P(\{k_r(j)\}).
\]

(2a)

(2b)

The least biased distribution \( P(\{k_r\}) \) verifying the constraints Eqs. (2) can be calculated following a maximum entropy principle [22]. The distribution entropy is \( S = -\sum_{k_r} P(\{k_r\}) \ln P(\{k_r\}) \), where the sum runs over all the allowed configurations. Maximizing \( S \) subject to the constraints Eqs. (2) plus normalization, we obtain a Gibbs distribution

\[
P(\{k_r\}) = \frac{1}{Z} D(\{k_r\}) e^{H(\{k_r\})},
\]

(3)

where \( D(\{k_r\}) = \binom{N}{k} k! / \prod_r k_r! \) is the degeneracy of the configuration \( \{k_r\} \), which counts all possible ways of selecting \( k = \sum_r k_r \) links out of \( N \) actors, and all ways of assigning \( k \) distinguishable links into layers of degree \( k_r \). Finally, \( Z \) and \( H(\{k_r\}) \) are the partition function and cost function, respectively.

\[
Z = \sum_{\{k_r\}} D(\{k_r\}) e^{H(\{k_r\})},
\]

(4)

\[
H(\{k_r\}) = \lambda k + \mu s + \sum_r h_r k_r,
\]

(5)

where \( \lambda \) and \( \mu \) are the Lagrange multipliers and will act as fitting parameters in order to enforce the constraints Eqs. (2). We have also included the auxiliary fields \( h_r \) for convenience. The maximum entropy method has been applied to formulate a large number of complex network models with prescribed features, known generally as exponential random graphs [16, 15, 23, 25]. Our approach here seeks instead a probability distribution for ego-network configurations \( \{k_r\} \). All the information of our system, including the cumulants of the layer degrees and group sizes, as well as their marginal distributions, are recovered from the partition function, which can be calculated analytically.

\[
Z = \sum_{\{k_r\}} \left( \frac{N}{k} \right)^k \prod_k k_r! e^{\sum_r (\lambda s_r + h_r) k_r},
\]

(6a)

\[
= \sum_{k=0}^N \left( \frac{N}{k} \right)^k \prod k_r! e^{\sum_r (\lambda s_r + h_r) k_r}
\]

(6b)

\[
= \sum_{k=0}^N \left( \frac{N}{k} \right)^k \prod r e^{\lambda s_r + h_r}
\]

(6c)

where the symbol \( \sum_{\{k_r\}} \) on the second line denotes sums over configurations \( \{k_r\} \) with total degree \( k \), and we have used the multinomial and binomial sums on the second and third lines, respectively. The cumulants of a single layer degree \( k_r \) can be computed by taking derivatives with respect to the respective auxiliary field \( h_r \). For instance, the average layer degree, variance, as well as the correlograms between different layers are given by

\[
\langle k_r \rangle = \frac{\partial \ln Z}{\partial h_r} \bigg|_{h_r=0} = \frac{N x y^{s_r}}{1 + \sum_r x y^{s_r}}, \quad \sigma_{k_r}^2 = \frac{1}{(\langle k_r \rangle)^2} \frac{\partial^2 \ln Z}{\partial h_r^2} \bigg|_{h_r=0} = \frac{1 + \sum_r x y^{s_r} \langle k_r \rangle}{1 + \sum_r x y^{s_r}},
\]

(7)

(8)

where we have defined \( x = e^\lambda \) and \( y = e^{\mu} \). We will prove later on that the layer degree marginal distributions become uncorrelated Poisson distributions in the thermodynamic limit \( N \to \infty \). Let us first write, however, the saddle point equations, used to fix the \( k \) and \( s \) ensemble averages.

\[
\langle k \rangle = \frac{\partial \ln Z}{\partial \lambda} \bigg|_{\lambda=0} = N \frac{\sum_r x y^{s_r}}{1 + \sum_r x y^{s_r}}, \quad \langle s \rangle = \frac{\partial \ln Z}{\partial \mu} \bigg|_{\mu=0} = N \frac{\sum_r s_r x y^{s_r}}{1 + \sum_r x y^{s_r}}.
\]

(9)

(10)

(11)
Notice that the average degree and resource verify \( \langle k \rangle = \sum_r \langle k_r \rangle \) and \( \langle s \rangle = \sum_r s_r \langle k_r \rangle \), respectively. In our maximum entropy setting, Eqs. \( \text{(10)} \) and \( \text{(11)} \) are solved for \( x \) and \( y \) in order to obtain the parameter values that fix the desired \( \langle k \rangle \) and \( \langle s \rangle \).

Further derivatives recover the subsequent \( k \) and \( s \) cumulants. For instance, the variances are

\[
\sigma_k^2 = \partial_x^2 \ln Z \mid _{k_r=0} = \frac{\langle k \rangle}{1 + \sum_r xy^{r_r}}.
\]

\[
\sigma_s^2 = \partial_s^2 \ln Z \mid _{s_r=0} = \sum_r s_r^2 \langle k_r \rangle - \frac{\langle s \rangle^2}{N}.
\]

Finally, we can write the configuration probability function, which may be sampled with Monte Carlo methods.

\[
P(\{k_r\}) = \left(\frac{N}{k}\right)^k \frac{k!}{\prod_r k_r!} \prod_r (xy^{r_r})^{k_r} \frac{1}{1 + \sum_r xy^{r_r}}.
\]

**Thermodynamic limit**

Let us study the thermodynamic limit for the grandcanonical ensemble, that is when the number of ego-networks \( N \to \infty \) while keeping \( \langle k \rangle \) and \( \langle s \rangle \) constant. From Eq. \( \text{(10)} \), we can write \( \sum_r xy^{r_r} = \langle k \rangle / (N - \langle k \rangle) \). This in turn, allows us to rewrite the partition function as

\[
Z = \left(1 + \frac{\langle k \rangle / N}{1 - \langle k \rangle / N}\right)^N \rightarrow e^{\langle k \rangle} \prod_r e^{\langle k_r \rangle}.
\]

The expected layer degrees can be expressed as \( \langle k_r \rangle = xy^{r_r}(N - \langle k \rangle) \approx Nxy^{r_r} \), for which it is needed that \( xy^{r_r} \to 0 \) as \( N \to \infty \). Then, the configuration probability distribution Eq. \( \text{(14)} \), reduces to

\[
P(\{k_r\}) = \frac{N^{-k} N!}{(N - k)!} \prod_r \frac{(k_r)^{k_r}}{k_r!} e^{-\langle k_r \rangle} \frac{1}{N} \rightarrow \prod_r p_r(k_r),
\]

where \( p_r(k_r) \) are the layer degree marginal distributions, and the prefactor tends to 1 as \( N \to \infty \), for finite \( k \). Indeed, using Stirling’s approximation and the exponential limit

\[
\frac{N^{-k} N!}{(N - k)!} \approx e^{-k} \left( \frac{N}{N - k} \right)^{N-k} \rightarrow 1.
\]

Thus, the resulting layer degree marginal distributions are Poisson distributions.

\[
p_r(k_r) = P(k_r; N, x, y) = \frac{e^{Nxy^{r_r}}}{k_r!} (Nxy^{r_r})^{k_r}.
\]

Noticing that \( \sum_r xy^{r_r} \to 0 \), we can see that Eqs. \( \text{(17)} \) and \( \text{(18)} \) reduce to

\[
\langle k_r \rangle = Nxy^{r_r},
\]

\[
\sigma_{k_r}^2 = \langle k_r \rangle,
\]

\[
\sigma_{k_r, k_i} = 0.
\]

Moreover, the saddle point equations, Eqs. \( \text{(10)} \) and \( \text{(11)} \), become

\[
k = \sum_r Nxy^{r_r},
\]

\[
s = \sum_r s_r Nxy^{r_r}.
\]

Thus, as we anticipated, the ego-network grandcanonical ensemble generates an uncorrelated layer structure in the thermodynamic limit. Hence, sampling corresponds to drawing independent random Poisson variables from the layer degree marginal distributions.

### III. HIERARCHICAL STRUCTURE

Before proceeding to study the constant group size scaling condition, let us first discuss the meaning of the ensemble parameters, \( x \) and \( y \). Both \( x \) and \( y \) are positive, since they are defined as exponentials of the real Lagrange multipliers \( \lambda \) and \( \mu \). We can identify \( y^{r_r} \) as the relative weight of each layer when writing the average link weight

\[
\bar{s} = \frac{s}{k} = \frac{\sum_r s_r y^{r_r}}{\sum_r y^{r_r}}.
\]

Moreover, the parameter \( y \) relates the expected layer-degree scaling with the difference of the link costs:

\[
\frac{\langle k_r+1 \rangle}{\langle k_r \rangle} = y^{s_r+1-s_r}.
\]

On the other hand, \( x \) acts a volumetric parameter which fixes the total degree through the constant product \( Nx = k / \sum_r y^{r_r} \).

From equation \( \text{(25)} \), the hierarchical structure is made apparent, that is \( \langle k_r \rangle < \langle k_{r+1} \rangle \), as long as \( y < 1 \). Let us check that this is indeed the case. Notice that the partition function \( Z \) measures the number of allowed configurations and that the layer costs \( s_r \) are arbitrary positive numbers. Consider that we increase the cost of one of the layers, \( s_c \), while keeping the imposed average values, \( \langle k \rangle \) and \( \langle s \rangle \) constant. For large enough \( s_c \) and fixed \( \langle s \rangle \), an overwhelming majority of ego-networks will not have any links placed in layer \( s_c \), and the number of allowed configurations must decrease. In the limit \( s_c \to \infty \), a well-behaved partition function demands \( \mu < 0 \). Consequently, \( y \in [0, 1] \) and the layer degrees \( \langle k_r \rangle \) are monotonically decreasing with the layer cost \( s_r \).

In sum, in a maximum entropy setting corresponding to the least unbiased guess about the ego-network configurations, a hierarchical structure arises naturally from the constraints Eqs. \( \text{(2)} \). Next, let us consider the condition of a constant group size scaling, as it is observed on empirical ego-networks.
Constant group size scaling condition

In human social groups, a constant scaling is found on average between the cumulative sizes of consecutive layers. In the grand-canonical ensemble, this is expressed by the expected group-size scaling, \( \langle n_r/n_{r+1} \rangle \). This quantity is the quotient of two functions of the configuration variables and cannot be calculated analytically. Instead, we must approximate it in terms of \( \langle k_r \rangle \), \( \langle k_r^2 \rangle \) and \( \langle k_r k_s \rangle \), as explained in the appendix A. The ensemble average is given by

\[
\langle n_r/n_{r+1} \rangle = \langle n_r \rangle \left( 1 + \epsilon_{r+1} \right), \tag{26}
\]

where \( \epsilon_{r+1} \) is a second order correction term, which can be expressed as

\[
\epsilon_{r+1} = \frac{\langle n_r \rangle \langle k_r^2 \rangle - \langle k_r \rangle^2 \langle n_{r+1} \rangle}{\langle n_{r+1} \rangle^2} - \frac{\langle n_r \rangle^2 - \langle k_r \rangle \langle n_{r+1} \rangle}{\langle n_{r+1} \rangle^2}. \tag{27}
\]

The rightmost expression was obtained by using the identity for Poisson variables: \( \langle k_r^2 \rangle = \langle k_r \rangle + \langle k_r \rangle^2 \).

Let us now consider the scaling of two consecutive group pairings \( n_{r+1}/n_r \) and \( n_r/n_{r-1} \). We will consider the implications of having a constant group-size scaling, such as observed in empirical social networks. Imposing \( n_{r+1}/n_r = n_r/n_{r-1} \) we get

\[
\langle n_r^2 \rangle = \langle n_{r+1} \rangle \langle n_{r-1} \rangle R_r. \tag{28}
\]

The correction factor, \( R_r = (1 + \epsilon_r)/(1 + \epsilon_{r+1}) \), tends to 1 provided that \( \langle n_r \rangle \gg 1 \). Indeed, this is a good approximation for the outer layers, as shown in the inset of Fig. 2. Considering \( R_r \approx 1 \), simple manipulations lead to

\[
\langle n_{r-1}/n_r \rangle \approx \frac{\langle k_r \rangle}{\langle k_{r+1} \rangle} = y^{s_r-s_{r+1}}. \tag{29}
\]

This result states that a constant group-size scaling in the outer layers is possible only if the cost difference between them is constant.

\[
s_r - s_{r+1} = \Delta, \quad \text{(for } r \text{ s.t. } n_r \gg 1). \tag{30}
\]

IV. FIT TO AN EMPIRICAL SOCIAL NETWORK

The grand-canonical ensemble presented above may function as a null model for ego-networks, providing a benchmark against which to test more complicated features. In order to accept the ensemble as a good model for social structure, the model should meet two demands: (i) It should generate ego-network instances with \( k \) and \( s \) values similar to the empirical ones (similar macrostate); and (ii) Those instances should present a nested layer structure. The assumption of constant layer costs makes the ensemble specially suited to model data from surveys, where the ties weights are chosen from predefined discrete scores or categories. We have fitted the grand-canonical ensemble to the Reciprocity Survey dataset network \([26]\). In this experiment, a total of \( N = 84 \) undergraduate students were asked to score their relationship with each of the other participants in a scale from 0 to 5, where 0 meant no relationship, and 1 to 5 represented increasing degree of friendship. We have considered the zero weight as a no-link. Thus, the allowed layer costs are \( \{s_1, s_2, s_3, s_4, s_5\} = \{5, 4, 3, 2, 1\} \), which verify the equispaced layer cost condition, Eq. (30). The global and hierarchical structure of the empirical network is summarized in Table I. The individual ego-networks show the expected Dunbar’s structure, albeit with some remarks: By design, the total active network is incomplete as the maximum possible degree is limited by the total number of experiment participants, \( k_{\text{max}} = N - 1 \). Consequently, the outer layer degree, \( \langle k_s \rangle \), departs from the expected scaling. However, all the participants belong to the same course and live in the same campus, hence we can expect that a significant fraction of their actual social network is captured by the experiment. Indeed, the inner groups show an almost constant ratio of approximately 0.4, which is consistent with the ratio of \( \sim 1/3 \) reported by larger scale studies \([10, 13–15]\). The degree distribution is peaked close to the number of participants, with low variance. The weights distribution, however, is more spread-out.

We have fitted the grand-canonical ensemble to the observed data, by substituting the mean values of \( \langle k \rangle = 73.63 \) and \( \langle s \rangle = 145.45 \), along with \( N = 84 \); into Eqs. (10) and (11). Solving the saddle point equations numerically, we obtained the parameter values \( x = 0.74 \) and \( y = 0.56 \). The resulting distribution, along with the data are shown in Figure 1. We have employed a Wang-Landau algorithm \([27, 28]\) in order to compute the joint density of states in the \( k-s \) space (the macrostate space). This function is defined as \( P(k,s) = \sum(k_r) P\{k_r\} \delta(k-\sum_k k_r) \delta(s-\sum_r k_r s_r) \), where \( P\{k_r\} \) is the grand-canonical probability distribution from Eq. (14). We have made available online a python implementation of the algorithm \([31]\). As it can be seen on the figure, the presence of various outliers with very low \( k \) and \( s \) displaces the averages from the bulk of the distribution. Other than that, the probability distribution
is a well behaved unimodal function, hence fulfilling our first demand, (i).

Next, let us compare the RS layer structure with the layer structure generated by the grand-canonical ensemble. Figure 2 shows the empirical layer degree and layer group sizes distributions along with the ensemble averages. Observing the RS layer distributions we find that the empirical averages of the three first layers lie within one standard deviation of the ensemble expected values, both for degrees, $k_r$, and group sizes, $n_r$. On the other hand, the outermost layers degrees suffer from finite size effects. The discrepancy between ensemble and data is however corrected for the accumulated variables, $n_r$, which follow closely the ensemble scaling trend. Thus, condition (ii) is verified as well. Remarkably, the obtained value of $y = 0.56$ is comparable with the empirical group size scaling for the outer layers, $\langle n_4/n_5 \rangle = 0.61$. Indeed, the approximation from Eq. (29) is better for the larger outer layers, where the correction factor, $R_r$, tends to 1, as shown on the inset of Fig. 2. We have also fitted the ensemble thermodynamic limit to the empirical data. Interestingly, despite the small size of the RS network, the grand-canonical ensemble results are barely distinguishable from its thermodynamic limit. Both ensembles average layer degrees and group sizes are equivalent, and only the variances of the outer layer degrees are slightly larger in the thermodynamic limit.

V. DISCUSSION

We have proposed a grand-canonical ensemble as a null model for the hierarchical structure of social networks. The ensemble generates the observed nested structure of increasingly large layers of decreasing link weight. More-over we show that, at least for the outer layers, a constant group size scaling between consecutive group pairings is possible only if the difference between costs of consecutive layers is constant. In the thermodynamic limit, that is, when the number of actors is large, the layer degrees are uncorrelated Poisson variables, which are related through the group size scaling, $y$. Interestingly, a recent paper providing more evidence on Dunbar’s theory shows the good fit of Poisson distributions to the layer-specific degree distributions 20. In the case of the dataset analyzed, after fitting the average values of the social bandwidth or resource $(s)$ and degree $(k)$, we find that typical samples of the ensemble are similar to the empirical ego-networks.

The proposed null model succeeds at modeling survey social data, where the available categories could be directly used as representations of layers. However, in larger scale studies of ego-network social structure, layers are not given a priori, but rather inferred from the interaction patterns. The link weights usually represent frequency of interaction which acts as a proxy for emotional closeness, and ties are then clustered into discrete groups according to these weights 10 13 14. It is impor-
tant to assert that we do not identify our link weights, or costs, with interaction frequency. Rather, we introduce the abstract notion of social resource, which can be spent in assigning discrete weights to the social ties. In order to justify this construction it can be argued that a limited social bandwidth may arise both from biological or temporal constraints, since maintaining a social relationship is costly \[1\,3\]. Nevertheless our model remains uninformative about the psychological or sociological nature of that cost. The other main assumption of our model is the discrete nature of the layers. It is important to stress that we do not intend to neither justify nor provide any sort of explanation to their existence. We rather rely on the existing literature, where these layers have been consistently identified and even given specific names: the support clique of size \(\sim 5\), the sympathy group of size \(\sim 15\), the affinity group of size \(\sim 50\), and the total active network whose size equals the Dunbar’s number, \(\sim 150\) \[9\]. Assigning the layers a constant cost is not only a convenient simplification, but has also a rather natural interpretation. Indeed the very existence of the layers would implicitly define different types of relationships for the ego. We simply consider that relationships within a given layer are similar precisely because they have the same cost. Then, the problem we focused on was to compute in how many different ways a given number of actors can be distributed in an ego’s network with the above mentioned restrictions. In our setting, once the layers, and the total average degree and social resource are fixed, the hierarchical structure emerges in a natural way, as the number of possible configurations with few strong and many weak links is much higher than configurations made of only strong or weak links. This result suggests that the observed hierarchical structure could be a consequence of the existence of an internal discrete categorization in which individuals organize different types of relationships. However, the reason why those categories or layers would exist in the first place remains unknown and shall be further explored.

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Appendix A: Cumulants of a quotient function

The cumulants of a quotient function \(r(x, y) = x/y\) cannot be obtained directly from the partition function. However, they can be approximated by expanding \(r(x, y)\) around the expected values, \(\langle x \rangle\) and \(\langle y \rangle\), as done in reference \[30\]. The mean and variance up to second order are given by

\[
\langle r \rangle = \frac{\langle x \rangle}{\langle y \rangle} \left[1 + \frac{\langle y^2 \rangle}{\langle y \rangle^2} - \frac{\langle x y \rangle}{\langle x \rangle \langle y \rangle}\right],
\]

\[
\sigma_r^2 = \frac{\langle x^2 \rangle}{\langle y \rangle^2} \left[\frac{\langle x^2 \rangle}{\langle x \rangle^2} + \frac{\langle y^2 \rangle}{\langle y \rangle^2} - \frac{2 \langle x y \rangle}{\langle x \rangle \langle y \rangle}\right].
\]

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[31] It is straightforward to generalize a Wang-Landau algorithm to compute a joint bivariate density of states in an integer valued configuration space. The code in python is available at [https://manu-jimenez.github.io/2017/04/19/Wang-Landau-for-joint-DOS.html](https://manu-jimenez.github.io/2017/04/19/Wang-Landau-for-joint-DOS.html)