Since January 2020 Elsevier has created a COVID-19 resource centre with free information in English and Mandarin on the novel coronavirus COVID-19. The COVID-19 resource centre is hosted on Elsevier Connect, the company's public news and information website.

Elsevier hereby grants permission to make all its COVID-19-related research that is available on the COVID-19 resource centre - including this research content - immediately available in PubMed Central and other publicly funded repositories, such as the WHO COVID database with rights for unrestricted research re-use and analyses in any form or by any means with acknowledgement of the original source. These permissions are granted for free by Elsevier for as long as the COVID-19 resource centre remains active.
The attribute reductions of three-way concept lattices

Ruisi Ren, Ling Wei

School of Mathematics, Northwest University, Xi’an, Shaanxi 710069, PR China

ARTICLE INFO

Article history:
Received 25 January 2015
Revised 8 January 2016
Accepted 30 January 2016
Available online 18 February 2016

Keywords:
Three-way concept
Attribute reduction
Irreducible element
Object-induced three-way object concept
Attribute-induced three-way attribute concept
Discernibility attribute set

ABSTRACT

Three-way concept analysis is a newly proposed area of formal concept analysis from which one can obtain both the inclusion decision and the exclusion decision. In general, given a context, some attributes may not be essential in three-way concept analysis, such as forming three-way concept lattice. So in this paper, we study the attribute reductions of three-way concept lattices in order to make the data easily be understood. Firstly, based on different criteria generated from object-induced three-way concept (OE-concept), four kinds of attribute reductions are proposed. The four reductions together embody different characteristics of a formal context and can be used in different occasions. Secondly, we discuss their relationships, including their advantages and disadvantages and the relationships among consistent sets and among the cores. Thirdly, based on attribute-induced three-way concept (AE-concept), we also give four attribute-induced three-way attribute reductions and discuss their relationships. Finally, the approaches to computing these attribute reductions are presented and the obtained results are demonstrated and verified by an empirical case. In this paper, we systematically investigate the attribute reductions of three-way concept lattices which enriches the study of formal concept analysis.

© 2016 Elsevier B.V. All rights reserved.

1. Introduction

The three-way decision (3WD), an extension of the two-way decision model with an added third option, is a common human practice in problem solving and is widely used in our daily life’s decision making process [1]. Because of its extensive usages, Yao [2] outlined the theory of three-way decision. The basic idea of 3WD is to divide the universe of discourse into three disjoint subsets by given criteria. These three disjoint parts are called positive, negative and boundary regions from which one can construct different rules to make three-way decision. One can construct rules of acceptance from positive region, rules of rejection from negative region and rules of non-commitment from boundary region.

In the past few years, more and more attentions have been paid to this newly proposed theory and the 3WD has been studied from the different views. Actually, 3WD plays a key role in knowledge-based systems and has been widely used in many fields and disciplines, including computer science, information science, management science, engineering, social science, medical decision-making, etc.[1,3–13].

Formal concept analysis (FCA), as an efficient tool for decision making and knowledge discovery, was proposed by Wille in 1982 [14], and has subsequently been extended by Wille and Ganter [15] and other scholars. As an effective method for data analysis and knowledge processing, FCA has been used in various areas, such as data mining, information retrieval, and software engineering [16–22].

However, in FCA we can only get the two-way decision rather than the three-way decision. That is, based on a formal concept, one can only determine whether an object (attribute) certainly possesses (is shared by) all elements in the intent (extent) and such decision is called inclusion method. Actually, the formal context also offers us the information about whether an object (attribute) does not possesses (is not shared by) any attribute (object), which is called exclusion method, but this cannot be reflected in the formal concept. Since this kind of exclusion method is commonly used in our daily life, Qi et al. [23] applied 3WD to FCA and proposed three-way concept analysis (3WCA). Two key components in 3WCA are three-way concepts and three-way concept lattices. Similar with the formal concept in FCA, a three-way concept is also determined by two parts named extent and intent. However, the extent or intent of a three-way concept should be an orthopair studied by Ciucci [24]. That is to say, in 3WCA, “the extent (or the intent) of a three way concept is equipped with two parts: positive one and negative one. These two parts are used to express the semantics ‘jointly possessed’ and ‘jointly not possessed’ in a formal context.” [25] This newly proposed three-way concept combines inclusion method (positive region) with exclusion method (negative region). Thus based on a three-way concept, one can divide
the object (attribute) universe into three regions to make three-way decision.

Being an important issue in knowledge discovery, attribute reduction has been extensively studied in different fields of soft computing since it can decrease the dimension and make the data easily be understood [26–35]. It is also an interesting topic in FCA. Many significant results in attribute reduction have been obtained based on a formal context. For example, Zhang et al. [36] discussed the lattice reduction theory where the notion of “reducing the context” in [15] was developed. Within such framework, Zhang et al. [36] constructed judgment theorems of consistent sets, and developed approaches to attribute reduction based on discernibility matrix, which was given a further simplification by Qi [37] from the viewpoint of parent-child concepts. Based on the lattice reduction theory of formal contexts, Wei et al. [38] also studied the reduction theory of formal decision contexts under two different meanings of consistency. And Li et al. [39] have proposed the attribute reduction in order to preserve the rules in the formal decision context. On the other hand, Wang and Ma [40] proposed another attribute reduction in order to preserve the extents of meet-irreducible elements in the original formal concept lattice. Based on the irreducible elements, Li et al. [41] defined the join-irreducible attribute reduction in order to preserve the extents of join-irreducible elements in the original formal concept lattice. From the viewpoint of granular computing, granular reduction was proposed by Wu et al. [42] Additionally, Liu et al. [43] have studied the attribute reduction of object oriented formal concept lattices and property oriented formal concept lattices, and furthermore Wang et al. [44] discussed relations of attribute reductions between object and property oriented formal concept lattices. Because of the extensive usages of 3WD and FCA, 3WCA will become a very useful tool for knowledge discovery and data analysis. In order to make the discovery and representation of implicit knowledge in three-way concept lattices easier and simpler, based on the different criteria generated from three-way concepts, we firstly propose different attribute reductions. Since the generalization and the specialization of knowledge contained in the context are all reflected in the three-way concept lattices, if our reduction can preserve the lattice structure, then the basic knowledge reflected in the three-way concept lattices is preserved. Also, because every three-way concept is the join (meet) of join (meet)-irreducible elements, if our reduction can preserve the irreducible elements, we actually preserve the basic elements in lattice construction. Thus this kind of reduction is very important in lattice construction. From the perspective of granular computation, a three-way object or attribute concept can be regarded as a kind of granule, so a kind of attribute reduction which can preserve the extents of the two-way object (attribute) concept is proposed. All these attribute reductions together embody different characteristics of a context. The basic ideas of these attribute reductions are similar with the classical attribute reductions mentioned in the previous paragraph. However, different from the attribute reductions in FCA, all the reductions in this paper depend on the three-way concept and the three-way concept can reflect more information than the classical formal concept, so the results and meaning of these attribute reductions in 3WCA are different with the classical ones in FCA. Then, we analyze the relationships among these three-way reductions. Finally, we give some approaches to these attribute reductions. Since the study we make is based on the three-way concept analysis, our work can be regarded as a three-way extension of the attribute reductions mentioned in previous paragraph.

This paper mainly discusses the different attribute reductions of three-way concept lattices and the relationships among them. The content is organized as follows. In Section 2, the basic knowledge of classical and three-way concept analysis is reviewed. Section 3 gives the definitions of four different OE-attribute reductions and discusses their relationships, including the relationships among the corresponding consistent sets and cores. Analogously, in Section 4, we define four different AE-attribute reductions and discuss the relationships among them. Then, the approaches to computing these attribute reductions are presented in Section 5. Finally, an empirical case is shown in Section 6 and this paper is concluded with a summary in Section 7.

2. Preliminaries

In order to make this paper self-contained, we firstly review some basic notions in formal concept analysis and three-way concept analysis.

2.1. Formal concept analysis

Definition 1 [15]. A formal context $(G, M, I)$ consists of two sets $G$ and $M$, and a relation $I$ between $G$ and $M$. The elements of $G$ are called the objects and the elements of $M$ are called the attributes of the context. In order to express that an object $x$ is in a relation $I$ with an attribute $m$, we write $xIm$ or $(x, m) \in I$ and read it as “the object $x$ has the attribute $m$”.

A pair of operators are defined on $X \subseteq G$ and $A \subseteq M$ respectively in $(G, M, I)$ [15]:

$$X^* = \{m | m \in M, \forall x \in X, xIm\}$$

$$A^* = \{x | x \in G, \forall m \in A, xIm\}$$

The operator $*$ is same with the modal-style operator $R^4$ in [45], which expresses the meaning of jointly possessing. The properties of this pair of operators are shown in [15], so we do not describe here.

Definition 2 [15]. A formal concept of the context $(G, M, I)$ is a pair $(X, A)$ with $X^2 = A$ and $X = A^* (X \subseteq G, A \subseteq M)$. We call $X$ the extent and $A$ the intent of the formal concept $(X, A)$.

The formal concepts of a formal context $(G, M, I)$ are ordered by

$$(X_1, A_1) \leq (X_2, A_2) \iff X_1 \subseteq X_2 (\iff A_1 \supseteq A_2).$$

All formal concepts of $(G, M, I)$ can form a complete lattice called the formal concept lattice of $(G, M, I)$ and denoted by $L(G, M, I)$. The infimum and supremum are given by

$$(X_1, A_1) \cap (X_2, A_2) = (X_1 \cap X_2, (A_1 \cup A_2)^*)$$

$$(X_1, A_1) \cup (X_2, A_2) = ((X_1 \cup X_2)^*, A_1 \cap A_2)$$

In order to distinguish from the following three-way concept analysis, the formal concept mentioned above will be called classical concept and other notions in the FCA also be named in the same way in the following parts of this paper.

Now, we give the definition of reduct as follows.

Definition 3 [36]. Let $(G, M, I)$ be a formal context. If there exists an attribute set $D \subseteq M$ such that $I(D, D, I_D) \equiv I(G, M, I)$, then $D$ is called a consistent set of $(G, M, I)$. And further, if $\forall d \in D, L(G, D – \{d\}, I_D) \sqsubseteq L(G, M, I)$, the set $D$ is called a reduct of $(G, M, I)$. The intersection of all the reducts is called the core of $(G, M, I)$. Here $I_D = I \cap (G \times D)$.

Definition 4 [46]. Let $L$ be a lattice. An element $x \in L$ is join-irreducible if

1. $x \neq 0$ (in case $L$ has a zero)
2. $x = a \lor b$ implies $x = a$ or $x = b$ for all $a, b \in L$.

A meet-irreducible element is dually defined.
2.2. Three-way concept analysis

Applying the idea of three-way decision into formal concept analysis, Qi et al. [23] recently proposed three-way concept analysis. Firstly, based on the formal concept in Definition 2 and the operators defined on it, called positive operators here, another pair of two-way operators are given, named negative operators, as follows:

for \( X \subseteq G \) and \( A \subseteq M \), we have

\[
X^\ast = \{ m \in M \mid \forall x \in X(-x \land m) \} \\
A^\ast = \{ x \in G \mid \forall m \in A(-x \land m) \}
\]

Combining the operators \( * \) and \( \neg \), a pair of three-way operators are defined on \( X \subseteq G \) and \( A \subseteq M \) respectively:

\[
X^\leq = (X^\ast, X^\top) \\
A^\leq = (A^\ast, A^\top)
\]

In relation to the above three-way operators, their inverse can be defined as follows [23]: for \( X, Y \subseteq G \) and \( A, B \subseteq M \),

\[
(X, Y)^> = \{ m \in M \mid m \in X^\ast \text{ and } m \in Y^\top \} \\
(A, B)^> = \{ x \in G \mid x \in A^\ast \text{ and } x \in B^\top \}
\]

Then the three-way concept is given.

**Definition 5** [23]. Let \((G, M, I)\) be a formal context. A pair \((X, (A, B))\) of an object subset \(X \subseteq G\) and two attribute subsets \(A, B \subseteq M\) is called an object-induced three-way concept, for short, an OE-concept, of \((G, M, I)\), if and only if \(X^\leq = (A, B)\) and \((A, B)^> = X. X\) is called the extent and \((A, B)\) is called the intent of the OE-concept \((X, (A, B))\).

In particular, \((X^\top, X^\leq)\) is an OE-concept for all \(X \subseteq G\), which is called OE-object concept. Here, for convenience, we write \(X^\leq\) instead of \([X^\leq]\) for any \(X \subseteq G\).

If \((X, (A, B))\) and \((Y, (C, D))\) are OE-concepts, then they can be ordered by

\[
(X, (A, B)) \subseteq (Y, (C, D)) \iff X \subseteq Y \text{ and } (A, B) \subseteq (C, D).
\]

Here \((C, D) \subseteq (A, B)\) if and only if \((X, (C, D))\) is a sub-concept of \((Y, (A, B))\), and \((Y, (A, B))\) is called a super-concept of \((X, (C, D))\), which is denoted by \((X, (A, B)) \triangleright (Y, (C, D))\).

All the OE-concepts form a complete lattice, which is called the object-induced three-way concept lattice of \((G, M, I)\) and written as \(OE(G, M, I)\) in the classical formal concept analysis, the reduction is defined related to the extents of the formal concepts. In order to give and analyze the attribute reductions of an OE-concept lattice, we firstly give the necessary and sufficient condition for an object set to be an extent of an OE-concept.

**Theorem 1.** Let \((G, M, I)\) be a formal context.

\[
Ext_{OE}(G, M, I) = \{ X \subseteq G \mid X^\ast \cap X^\top = X \}
\]

**Proof.** Suppose \((X, (A, B))\) is an OE-concept of \((G, M, I)\). From Definition 5, we know that \(X^\leq = (A, B)\) which means \((X^\ast, X^\top) = (A, B)\). Also from the Definition 5, we can get \((A, B)^> = X\). That is, \(A^\ast \cap B^\top = X\). Thus, \(X^\ast \cap X^\top = X\). On the contrary, if \(X^\ast \cap X^\top = X\) it's obvious that \((X, (X^\ast, X^\top))\) is an OE-concept. \(\square\)

The three-way concept lattices are the core structures in 3WCA, because the hierarchical knowledge of three-way concepts is all reflected in these three-way concept lattices. Thus based on the OE-concept lattice, we propose the OE-lattice reduction of \((G, M, I)\).

**Definition 6** [23]. Let \((G, M, I)\) be a formal context. A pair \((X, (Y, A))\) of two object subsets \(X, Y \subseteq G\) and an attribute subset \(A \subseteq M\) is called an attribute-induced three-way concept, for short, an AE-concept, of \((G, M, I)\), if and only if \((X, (Y, A))\) is a sub-concept of \((X, (Y, A))\), and \((X, (Y, A))\) is called a parent-concept of \((X, (Y, A))\), which is denoted by \((X, (Y, A)) \leq (X, (Y, A))\).

All AE-concepts also form a complete lattice, which is called the attribute-induced three-way concept lattice of \((G, M, I)\) and written as \(AE(G, M, I)\). The infimum and supremum are given by

\[
((X, (Y, A)) \land ((Z, W), B)) = ((X, (Y, A)) \cap (Z, W), B) \\
((X, (Y, A)) \lor ((Z, W), B)) = ((X, (Y, A)) \cup (Z, W), B)
\]

**3. Four types of attribute reductions of an OE-concept lattice**

In the classical formal concept analysis, the reduction is defined related to the extents of the formal concepts. In order to give and analyze the attribute reductions of an OE-concept lattice, we firstly give the necessary and sufficient condition for an object set to be an extent of an OE-concept.

**Definition 7**. Let \((G, M, I)\) be a formal context. An attribute set \(D \subseteq M\) is called an object-induced three-way concept lattice (OE-lattice) consistent set of \((G, M, I)\) if \(Ext_{OE}(G, M, I) = Ext_{OE}(G, D, I_D)\). Here, \(I_D = I \cap (G \times D)\). Furthermore, if \(Ext_{OE}(G, D - \{d\}, I_D-\{d\}) \neq Ext_{OE}(G, M, I)\) for any \(d \in D\), then set \(D\) is called an OE-lattice reducible of \((G, M, I)\).}

From [15], we know that in a finite lattice, the set of meet-irreducible elements is infimum-dense and the set of join-irreducible elements is supremum-dense. So the meet-irreducible elements and join-irreducible elements are important in the structure of the lattice. Thus from this point of view, we define the object-induced three-way irreducible elements preserving reductions as follows.

**Definition 8**. Let \((G, M, I)\) be a formal context. An attribute set \(D \subseteq M\) is called an object-induced three-way meet-irreducible elements preserving (OE-MIE-preserving) consistent set of \((G, M, I)\) if \(Ext_{OE}(G, M, I) = Ext_{OE}(G, D, I_D)\). If set \(D\) is an OE-MIE-preserving consistent set of \((G, M, I)\), and there is no proper subset \(E \subset D\) such that \(E\) is an OE-MIE-preserving consistent set of \((G, M, I)\), then set \(D\) is called an OE-MIE-preserving reducible of \((G, M, I)\).

Here, \(Ext_{OE}(G, M, I)\) is the extent set of all the meet-irreducible elements in the OE-concept lattice \(OE(G, M, I)\).

**Definition 9**. Let \((G, M, I)\) be a formal context. An attribute set \(D \subseteq M\) is called an object-induced three-way join-irreducible elements preserving (OE-JIE-preserving) consistent set of \((G, M, I)\) if \(Ext_{OE}(G, M, I) = Ext_{OE}(G, D, I_D)\). If set \(D\) is an OE-JIE-preserving consistent set of \((G, M, I)\), and there is no proper subset \(E \subset D\) such that \(E\) is an OE-JIE-preserving consistent set of \((G, M, I)\), then set \(D\) is called an OE-JIE-preserving reducible of \((G, M, I)\).

Here, \(Ext_{OE}(G, M, I)\) is the extent set of all the join-irreducible elements in the OE-concept lattice \(OE(G, M, I)\).
Table 1
The comparison of four OE-attribute reductions.

| Reduction Type                      | Completeness of Knowledge | Computation          | Applicable Occasion                  |
|------------------------------------|---------------------------|----------------------|--------------------------------------|
| OE-lattice reduction               | No knowledge loss         | Complicate           | No knowledge loss simplification     |
| OE-MIE-preserving reduction        | No knowledge loss         | Simple               | No knowledge loss simplification     |
| OE-JIE-preserving reduction        | Has knowledge loss        | Simple               | Lattice construction                 |
| OE-granular reduction              | Has knowledge loss        | Simple               | Granular computing                  |

Theorem 2. Let \( (G, M, I) \) be a formal context, then \( CS(OEL) = CS(OLE) \), \( Red(OLE) = Red(OEM) \) and \( Core(OEL) = Core(OEM) \).

Proof. Firstly, we give the proof of \( CS(OEL) = CS(OLE) \).

If \( D \subseteq CS(OLE) \), then \( Ext_{OLE}(G, M, I) \subseteq Ext_{OLE}(G, D, I_p) \) holds by Definition 7. For any \( X \in Ext_{OLE}(G, M, I) \), from the definition of meet-irreducible element, we can get \( X \neq Y \cap Z \) (\( Y, Z \notin Ext_{OLE}(G, M, I) \) and \( Y \neq X, Z \neq X \)). And because of \( Ext_{OLE}(G, M, I) = Ext_{OLE}(G, D, I_p) \), \( X \in Ext_{OLE}(G, D, I_p) \) and \( X \neq Y \cap Z \) (\( Y, Z \notin Ext_{OLE}(G, D, I_p) \) and \( Y \neq X, Z \neq X \)). So, according to the definition of meet-irreducible elements, \( X \in Ext_{OLE}(G, D, I_p) \). Thus \( Ext_{OLE}(G, M, I) \subseteq Ext_{OLE}(G, D, I_p) \), and vice versa. Therefore \( Ext_{OLE}(G, M, I) = Ext_{OLE}(G, D, I_p) \), that is \( D \subseteq CS(OLE) \). Thus \( CS(OEL) \subseteq CS(OLE) \).

Suppose \( D \subseteq CS(OLE) \), we need to show \( Ext_{OLE}(G, M, I) = Ext_{OLE}(G, D, I_p) \). Since \( Ext_{OLE}(G, D, I_p) \subseteq Ext_{OLE}(G, M, I) \) holds for any \( D \subseteq M \), we only need to prove \( Ext_{OLE}(G, M, I) \subseteq Ext_{OLE}(G, D, I_p) \). For any \( X \in Ext_{OLE}(G, M, I) \), there exist \( X \in Ext_{OLE}(G, M, I) \) that \( X \neq Y \cap Z \) due to Lemma 1. Since \( Ext_{OLE}(G, M, I) = Ext_{OLE}(G, D, I_p) \), and \( Ext_{OLE}(G, D, I_p) \subseteq Ext_{OLE}(G, D, I_p) \), we obtain \( X = \cap_{i \in \tau} X_i \) Ext \( G, D, I_p \). Thus, \( Ext_{OLE}(G, M, I) \subseteq Ext_{OLE}(G, D, I_p) \). Therefore, \( Ext_{OLE}(G, M, I) = Ext_{OLE}(G, D, I_p) \) which means \( D \subseteq CS(OLE) \). Hence, \( CS(OEL) \subseteq CS(OLE) \). So, we obtain \( CS(OEL) = CS(OLE) \).

The first equation of this theorem shows that \( D \) is an OE-lattice consistent set if and only if \( D \) is an OE-MIE-preserving consistent set. Therefore, the following parts \( Red(OEL) = Red(OEM) \), \( Core(OEL) = Core(OEM) \) are naturally obtained.

Lemma 2. Let \( (G, M, I) \) be a formal context, \( D \subseteq M \), and \( X, Y \subseteq G \). We have \( X \cap D \cap X \cap M = X \cap M \). Consequently, \( X \cup Y \subseteq G \). If \( D \subseteq CS(OLE) \), then the two equations hold.

Proof. \( X \subseteq G \), since \( D \subseteq M \), we have \( X \cap D \subseteq X \cap M \). Then, \( X \cap D \cap X \cap M \subseteq X + X \cap M \). Let \( X = \{ x \} \), \( X \cap D \cap X \cap M \subseteq X + X \cap M \) holds naturally. Suppose \( D \) is an OE-lattice consistent set. \( X \subseteq G \), since \( X \cap D \subseteq Ext_{OLE}(G, D, I_p) \), \( X \cap M \subseteq Ext_{OLE}(G, M, I) \), and \( Ext_{OLE}(G, M, I) \), \( D \subseteq Ext_{OLE}(G, D, I_p) \). Therefore, \( X \subseteq Ext_{OLE}(G, D, I_p) \). Similarly, let \( X = \{ x \} \), \( X \cap D \subseteq X \cap M \) also holds.

Lemma 2 says that if \( D \) is an OE-lattice consistent set, then \( X \in G \), \( X \cap D \subseteq X \cap M \) holds. Then, \( D \) is also an OE-granular consistent set according to Definition 10. Therefore, we have the following results.

Theorem 3. Let \( (G, M, I) \) be a formal context, if \( D \subseteq M \) is an OE-lattice consistent set, then \( D \) is an OE-granular consistent set. That is, \( CS(OEL) \subseteq CS(OEG) \). Also, we have \( Red(OEL) \subseteq CS(OEG) \) and \( Core(OEL) \subseteq Core(OEG) \).

Proof. Based on Lemma 2, \( CS(OEL) \subseteq CS(OEG) \) is easy to be obtained. Because \( Red(OEL) \subseteq CS(OEL) \), we have \( Red(OEL) \subseteq CS(OEG) \). That is, an OE-lattice reduce must be an OE-granular consistent set. Now we give the proof of \( Core(OEL) \subseteq Core(OEG) \).

From the definition of core, we know that \( Core(OEG) = \cap_{i \in \tau} Core(D, I_p) \) and \( Core(OEL) = \cap_{i \in \tau} CS(OEL) \). By the first part of Theorem 3, we have \( CS(OEL) \subseteq CS(OEG) \). Thus, \( \cap_{i \in \tau} CS(OEL) \subseteq \cap_{i \in \tau} CS(OEG) \). That is, \( Core(OEG) \subseteq Core(OEL) \).
\[
\begin{align*}
\text{Red}(OEL) & \subseteq \text{CS}(OEL) \subseteq \text{CS}(OEG) = \text{CS}(OEJ) \\
\text{Red}(OEM) & \subseteq \text{CS}(OEM)
\end{align*}
\]

Core(OEJ) = Core(OEG) \subseteq Core(OEL) = Core(OEM)

Fig. 1. The relationships among four kinds of OE-reductions.

In order to analyze the relationship between OE-granular reduction and OE-JIE-preserving reduction, we firstly discuss the relationship between OE-object concept and the join-irreducible element of OEI(G, M, I) in the following proposition.

**Lemma 3.** Let (G, M, I) be a formal context, \( X \subseteq G, A \subseteq M \) and \( B \subseteq M \). Then \((X, (A, B))\) is a join-irreducible element of OEI(G, M, I), if and only if \((X, (A, B))\) is an OE-object concept.

**Proof.** Necessity. Let \((X, (A, B)) \in \text{OEI}(G, M, I)\). Then

\[
\text{Red}(x_i^\ast) \subseteq \text{Red}(x_i^\ast) = \text{Red}(x_i^\ast) \subseteq \text{Red}(x_i^\ast) = \text{Red}(x_i^\ast) = \text{Red}(x_i^\ast)
\]

By the properties of \( \ast \) and \( \pi \), we can obtain

\[
\text{Red}(x_i^\ast) = (U_{x_i \in X} x_i^\ast) = X^\ast = A
\]

Thus,

\[
(X, (A, B)) = \text{Red}(x_i^\ast) = (U_{x_i \in X} x_i^\ast) = X^\ast = A
\]

Sufficiency. Assume \((X, (x_i^\ast, x_i^\ast))\) is an OE-object concept induced by \( x_i \). From Lemma 1 we know that \((X, (x_i^\ast, x_i^\ast))\) is a join of join-irreducible elements, and the necessity of Lemma 3 shows that the join-irreducible element of OEI(G, M, I) must be an OE-object concept.

Similarly, we can give four attribute reductions of an AE-concept lattice. Analogously to the proofs of theorems in Section 3, the proofs in this section are omitted.

**Definition 11.** Let \((G, M, I)\) be a formal context. An attribute set \( D \subseteq M \) is called an attribute-induced three-way concept lattice (attribute-induced three-way meet-irreducible elements preserving; attribute-induced three-way join-irreducible elements preserving) consistent set of \((G, M, I)\) if \( \text{Ext}_{AE}(G, M, I) = \text{Ext}_{AE}(G, D, I_2) \). Furthermore, if there is no proper subset \( E \subseteq D \) such that \( E \) is an AE-lattice (AE-MIE-preserving; AE-JIE-preserving) consistent set of \((G, M, I)\), then \( D \) is called an AE-lattice (AE-MIE-preserving; AE-JIE-preserving) reduct of \((G, M, I)\).

In AE-concept lattice, the granule is different from the one in OE-concept lattice. Here, the AE-attribute concept can be regarded as granule, since every AE-concept can be represented by the meet of AE-attribute concepts. Then the AE-granular reduction is given as follows.

**Definition 12.** Let \((G, M, I)\) be a formal context. An attribute set \( D \subseteq M \) is called an attribute-induced three-way granular (AE-granular) consistent set of \((G, M, I)\) if \( m^{\text{AE}} = m^{\text{AE}} \) for all \( m \in M \). If set \( D \) is an AE-granular consistent set of \((G, M, I)\), and there is no proper subset \( E \subseteq D \) such that \( E \) is an AE-granular consistent set of \((G, M, I)\), then \( D \) is called an AE-granular reduct of \((G, M, I)\).

Then, we give the relationships among four AE-attribute reductions, which are similar to the last section.

**Theorem 5.** Let \((G, M, I)\) be a formal context and set \( D \subseteq M \). Then \( \text{CS}(AEI) = \text{CS}(AEM) \). Red(AEI) = Red(AEM) and Core(AEI) = Core(AEM).

**Lemma 4.** Let \((G, M, I)\) be a formal context, \( X \subseteq G, Y \subseteq G \) and \( A \subseteq M \). Then \((X, (Y, A))\) is a meet-irreducible element of AEL(G, M, I), if and only if \((X, (Y, A))\) is an AE-attributive concept.

**Theorem 6.** Let \((G, M, I)\) be a formal context, then \( \text{CS}(OEG) = \text{CS}(OEJ) \). Red(OEG) = Red(OEJ) and Core(OEG) = Core(OEJ).

Based on the above analysis, we can sum up the relationships among these four kinds of OE-reductions in Fig. 1, and its explanation is shown in Fig. 2.
\(Red(AEL) \subseteq CS(AEL) \subseteq CS(AEJ)\)

\[
\begin{align*}
Red(AEM) & \subseteq CS(AEM) \\
Red(AEG) & \subseteq CS(AEG)
\end{align*}
\]

Core(AEL) \subseteq Core(AEL) = Core(AEM) = Core(AEG)

Fig. 3. The relationships among four kinds of AE-reductions.

\(D\) is an AE-lattice consistent set
\(\Downarrow\)

\(D\) is an AE-MIE-preserving consistent set
\(\Downarrow\)

\(D\) is an AE-granular-preserving consistent set
\(\Downarrow\)

\(D\) is an AE-JIE-preserving consistent set

Fig. 4. The explanation of Fig. 3.

**Theorem 6.** Let (\(G, M, I\)) be a formal context and set \(D \subseteq M\). Then \(CS(AEG) = CS(AEM)\), \(Red(AEG) = Red(AEM)\) and Core(AEG) = Core(AEM).

Moreover, from Theorem 5, we know that the AE-lattice consistent set and AE-MIE-preserving consistent set are same, so the following theorem is easy to get:

**Theorem 7.** Let (\(G, M, I\)) be a formal context and set \(D \subseteq M\). Then, \(CS(AEL) = CS(AEM) = CS(AEG)\), \(Red(AEL) = Red(AEM) = Red(AEG)\) and Core(AEL) = Core(AEM) = Core(AEG).

Finally, we present the relationship between AE-lattice reduction and AE-JIE-preserving reduction. Because \(Ext\_AE(G, M, I) \subseteq Ext\_AE(G, M, I)\), it is easy to see that if set \(D \subseteq M\) can preserve the extent set of all AE-concept, then it must preserve the extent set of join-irreducible elements of \(AEL(G, M, I)\). Thus the following proposition is obtained.

**Theorem 8.** Let (\(G, M, I\)) be a formal context and set \(D \subseteq M\). Then, \(CS(AEL) \subseteq CS(AEJ)\) and Core(AEJ) \(\subseteq Core(AEL)\).

Based on above analysis, we can sum up the relationships among the four kinds of AE-reductions in Fig. 3, and its explanation is shown in Fig. 4.

5. Approaches to attribute reductions

5.1. Approaches to attribute reductions of an OE-concept lattice

Based on the similar idea of discernibility matrix proposed by Zhang et al. [36], we introduce the discernibility matrix and discernibility functions which can be used to find the four different OE-reductions.

**Definition 13.** Let (\(G, M, I\)) be a formal context, and \((X, (A, B)), (Y, (C, D)) \in OE\_L(G, M, I)\). Then

\[
DIS\_OE\_L((X, (A, B)), (Y, (C, D))) = \begin{cases} 
(A \setminus C, B \setminus D), & (X, (A, B)) < (Y, (C, D)) \\
\emptyset, & \text{otherwise}
\end{cases}
\]

is called the OE-discernibility attributes set between \((X, (A, B))\) and \((Y, (C, D))\).

\(\Lambda\_OE\_L = (DIS\_OE\_L((X, (A, B)), (Y, (C, D))))((X, (A, B)), (Y, (C, D)) \in OE\_L(G, M, I))\) is called the OE-discernibility matrix of \((G, M, I)\).

\(\Lambda\_OE\_L = (DIS\_OE\_L((X, (A, B)), (Y, (C, D))))\), where \((X, (A, B))\) is the join-irreducible element of \(OE\_L(G, M, I)\).

Because we only need to use the non-empty sets in the attribute reduction, we also let \(\Lambda\) denote the set of the non-empty OE-discernibility attributes sets.

**Theorem 9.** Let \((G, M, I)\) be a formal context, \(B \subseteq M\) and \(X \in Ext\_OE\_L(G, M, I)\). Then \(\chi(B)X = X\) if and only if \(B \cap DIS\_OE\_L((X, (A, B)), (Y, (C, D)) = \emptyset\) for any \((Y, (C, D)) \in PC(X, X^{M})\). Here \(PC(X, X^{M})\) is a set of all the parent-OE-concepts of \((X, X^{M})\).

**Proof.** For simplicity, we firstly define \((A, B) = (C, D) := A \cap C\) and \(B \subseteq D\), or \(A \subseteq C\) and \(B \subseteq C\). And \(B \cap (C, D) = (B \cap C, B \cap D) = (A, B, C, D) \subseteq \emptyset\).

Necessity. Suppose \(X^{C \setminus B} \subseteq Y\) and \((Y, (C, D)) \in PC(X, X^{M})\).

(i) If \((Y, (C \cap B, D \cap B)) \in OE\_L(G, M, I)\), then \(Y \cap (X^{C \setminus B}) = (X^{C \setminus B}, X^{M \setminus (C \cap B)} \cap Y, X^{C \setminus B}, X^{M \setminus (C \cap B)})\), equivalently, \(B \cap (X^{C \setminus B} \cap (Y, (C \cap B), D \cap B)) \neq \emptyset\), obviously, \(B \cap DIS\_OE\_L((X, X^{M}), (Y, (C, D))) \neq \emptyset\).

(ii) If \((Y, (C \cap B, D \cap B)) \notin OE\_L(G, M, I)\), then \(Y \subseteq (X^{C \setminus B} \cap (Y, (C \cap B), D \cap B)) \subseteq (X^{C \setminus B} \cup (X^{M \setminus (C \cap B)} \cap Y), X^{M \setminus (C \cap B)} \cup (X^{M \setminus (C \cap B)}))\). Moreover, it's sure that \((C \cap B, D \cap B) \subseteq X^{C \setminus B} \cup (X^{M \setminus (C \cap B)} \cap Y)\). Because if \(X^{C \setminus B} \cup (X^{M \setminus (C \cap B)} \cap Y) \neq (C \cap B, D \cap B)\), then \(X^{C \setminus B} \cup (X^{M \setminus (C \cap B)} \cap Y) = (C \cap B, D \cap B)\), which is a contradiction. Thus \(B \cap DIS\_OE\_L((X, X^{M}), (Y, (C, D))) \neq \emptyset\).

Sufficiency. Assume \(B \cap DIS\_OE\_L((X, X^{M}), (Y, (C, D))) \neq \emptyset\) for any \((Y, (C, D)) \in PC(X, X^{M})\) and \(X^{B} \subseteq X\). Then we have \(X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X\). Since \((X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X) \in OE\_L(G, M, I)\), we obtain \(X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X\). Moreover, it's sure that \((X, X^{M}) = (X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X)\). If there is \((E, F) \in OE\_L(G, M, I)\) such that \((X, X^{M}) = (X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X)\), then \(X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X\), which implies \(X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X\), thus \(X^{A} \subseteq X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X\), which is impossible. Thus \((X, X^{M}) = (X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X)\), \(X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X^{B} \subseteq X\), which is impossible. Hence, \(X^{B} \subseteq X\).

**Theorem 9** gives us a necessary and sufficient condition of an OE-concept to hold its extent unchanged. Considering the definitions of the four different OE-attribute reductions and their relationships, we give the following methods to computing these reductions.

In order to give a more concise description of the following methods, we also use \(DIS\_OE\_L((X, (A, B)), (Y, (C, D)))\) to denote the union of its components in the following definition. For instance, if \(DIS\_OE\_L((X, (A, B)), (Y, (C, D))) = (E, F)\), then \(E \cup F\) is also denoted as \(DIS\_OE\_L((X, (A, B)), (Y, (C, D)))\). And we also use \(\Lambda\_OE\_L\) to represent the OE-discernibility matrix of \((G, M, I)\).

**Definition 14.** Let (\(G, M, I\)) be a formal context. The OE-discernibility function, OEM-discernibility function, OJE-discernibility function and OEC-discernibility function are defined as follows:

\[
f(\Lambda\_OE\_L) = f(\Lambda\_OE\_M) = \bigcup_{H \subseteq \Lambda\_OE\_L} \{ h \}
\]

\[
f(\Lambda\_OE\_J) = f(\Lambda\_OE\_A) = \bigcup_{H \subseteq \Lambda\_OE\_J} \{ h \}
\]

By absorption law and distributive law, the OE-discernibility function \(f\) can be transformed to a minimal disjunctive normal form, whose items can form all the OE-reductions of \((G, M, I)\).

**Example 1.** Table 2 shows a formal context \((G_1, M_1, I_1)\), in which, the object set is \(G_1 = \{1, 2, 3, 4\}\), and the attribute set is \(M_1 = \)
There are 11 OE-concepts: \((G_1, (\varnothing, \varnothing)), (13, (d, c)), (23, (\varnothing, e)), (124, (ab, \varnothing)), (14, (abe, \varnothing)), (24, (abc, d)), (1, (abde, c)), (3, (dabce), (2, (abc, de)), (4, (abce, d)), (OEL, (M_1, M_1)), \) which are labeled as \(T_2^i (i = 1, 2, \ldots, 11)\). According to Definition 13, the \(\Lambda_{OEL}\) of \(OE\) is shown in Table 3.

More specifically, if \(T_2\) is a child-concept of \(T_2\), then the OE-discriminability attributes set is \(\Lambda_{OEL}(T_2, T_2)\) at the \(i\)th row and \(j\)th column of Table 3. Now we can calculate the discriminability functions.

\[
f(A_{OEL}) = f(A_{OEM})
= \bigwedge_{h \in \Lambda_{OEL}} (\bigvee_h)
= (d \vee e) \land (a \lor b) \land (a \lor e
\land (d \lor c) \land (a \lor b)
\lor (a \land c \land e) \lor (b \land c \land e)
\lor (a \land d \land e) \lor (b \land d \land e)
\]

So the OE-lattice (OE-MIE-preserving) reduces are \(\{a, c, e\}, \{b, c, e\}, \{a, d, e\}\) and \(\{b, d, e\}\). And the OE-JIE-preserving (OE-granular) reduces are \(\{c, e\}\) and \(\{d, e\}\).

Considering the space limitation, we only present the OE-concept lattices of \((G_1, M_2, L_{M_2})\) and \((G_1, M_3, L_{M_3})\) in Figs. 6 and 7, respectively. Here \(M_2\) is OE-lattice (OE-MIE-preserving) reduce \(\{a, c, e\}\) and \(M_3\) is OE-JIE-preserving (OE-granular) reduce \(\{c, e\}\).

From Figs. 5–7, we can see that OE-lattice (OE-MIE-preserving) reduce must be OE-JIE-preserving (OE-granular) reduce, but OE-JIE-preserving (OE-granular) reduce may not be OE-lattice (OE-MIE-preserving) reduce.
Table 3
The OEI-discernibility matrix of the OEI-concept lattice in Fig. 5.

| TC_1 | TC_2 | TC_3 | TC_4 | TC_5 | TC_6 | TC_7 | TC_8 | TC_9 | TC_10 | TC_11 |
|------|------|------|------|------|------|------|------|------|-------|-------|
| TC_1 | (d, c) | | | | | | | | | |
| TC_2 | (c, d) | | | | | | | | | |
| TC_3 | (c, e) | | | | | | | | | |
| TC_4 | (ab, σ) | | | | | | | | | |
| TC_5 | (e, σ) | | | | | | | | | |
| TC_6 | (c, d) | | | | | | | | | |
| TC_7 | (d, c) | | | | | | | | | |
| TC_8 | (d, abc) | (ab, d) | | | | | | | | |
| TC_9 | (e, σ) | (c, e) | | | | | | | | |
| TC_10 | (c, d) | (e, σ) | | | | | | | | |
| TC_11 | (c, abde) | (abc, d) | (de, abc) | (d, abce) | | | | | | |

Table 4
The AEI-discernibility matrix of the AEI-concept lattice in Fig. 8.

| TC_1 | TC_2 | TC_3 | TC_4 | TC_5 | TC_6 | TC_7 | TC_8 | TC_9 | TC_10 | TC_11 |
|------|------|------|------|------|------|------|------|------|-------|-------|
| TC_1 | c | | | | | | | | | |
| TC_2 | ab | | | | | | | | | |
| TC_3 | e | | | | | | | | | |
| TC_4 | d | | | | | | | | | |
| TC_5 | ab | c | | | | | | | | |
| TC_6 | e | ab | | | | | | | | |
| TC_7 | d | e | | | | | | | | |
| TC_8 | e | d | ab | | | | | | | |
| TC_9 | d | ab | | | | | | | | |
| TC_10 | d | c | | | | | | | | |
| TC_11 | | | | | | | | | | |

Fig. 8. The AEI-concept lattice of \( (G_1, M_1, I_1) \).

Fig. 9. The AEI-concept lattice of \( (G_1, M_4, I_4) \).

AEG-discernibility function are defined as follows:

\[
f(\Lambda_{AEI}) = f(\Lambda_{AEM}) = f(\Lambda_{AEG})
\]

\[
= \bigwedge_{H \in \Lambda_{AEI}} \bigvee_{b \in H} h
\]

or

\[
= \bigwedge_{H \in \Lambda_{AEG}} \bigvee_{b \in H} h.
\]

By absorption law and distributive law, the AEG-discernibility function \( f \) can be transformed to a minimal disjunctive normal form, whose items can form all the AE-reducts of \( (G, M, I) \).

Example 2 (Continued with Example 1). The AEI-concept lattice of \( (G_1, M_1, I_1) \) is shown in Fig. 8.

There are 11 AEI-concepts: \( (G_1, G_1, \varnothing) \), \( ([24, 13], c) \), \( ([124, 3], \varnothing) \), \( ([124, 3], ab) \), \( ([14, 23], e) \), \( ([13, 24], d) \), \( ([24, 3], abc) \), \( ([14, 3], abe) \), \( ([1, 2], de) \), \( ([4, 3], abce) \), \( ([1, \varnothing], abde) \), \( ([\varnothing, \varnothing], M_1) \), which are labeled as TC_i (i = 1, 2, ..., 11). According to Definition 15, the \( \Lambda_{AEI} \) of AEI(\( G_1, M_1, I_1 \)) is shown in Table 4.

Then we calculate the discernibility functions.

\[
f(\Lambda_{AEI}) = f(\Lambda_{AEM}) = f(\Lambda_{AEG})
\]

\[
= \bigwedge_{H \in \Lambda_{AEG}} \bigvee_{b \in H} h
\]

\[
= c \land e \land d \land (a \lor b)
\]

\[
= (a \land c \land e \land d) \lor (b \land c \land e \land d)
\]

So the AEI-lattice (AEI-MIE-preserving and AEI-granular) reducts are \{a, c, e, d\} and \{b, c, e, d\}.

Considering the space limitation, we only present the AEI-concept lattice of \( (G_1, M_4, I_4) \) in Fig. 9. Here \( M_4 \) is AEI-lattice (AEI-MIE-preserving and AEI-granular) reduce \{a, c, e, d\}.
6. An empirical case

In this section, a real-life database is analyzed to demonstrate the application of the proposed attribute reduction methods. The chosen database is about the patients who suffer from severe acute respiratory syndrome (SARS) and their symptoms. Here we only use OE-reductions as examples and the AE-reductions can be studied similarly. The details are as follows:

Example 3. Table 5 depicts a dataset of four patients who suffer from severe acute respiratory syndrome (SARS) [47]. Let G be the set of four patients and M be the set of five symptoms. For convenience, we denote the four patients by 1, 2, 3, 4, respectively, and the five symptoms (Fever, Cough, Headache, Difficulty Breathing and Diarrhea) by a, b, c, d, e, respectively. That is G = \{1, 2, 3, 4\} and M = \{a, b, c, d, e\}.

| Patient | Fever | Cough | Headache | Difficulty breathing | Diarrhea |
|---------|-------|-------|----------|----------------------|----------|
| 1       | Yes   | No    | Yes      | No                   | Yes      |
| 2       | Yes   | No    | Yes      | No                   | No       |
| 3       | No    | Yes   | No       | Yes                  | No       |
| 4       | No    | No    | Yes      | No                   | Yes      |

Firstly, we can get all the formal concepts of Table 5 and the corresponding formal concept lattice is shown in Fig. 10. There are 7 formal concepts: (G, ∅), (14, ce), (12, a), (1, ace), (2, ad), (3, b), (2, ad). Then, we can get 12 OE-concepts: (G, (∅, ∅)), (134, (∅, d)), (124, (∅, b)), (34, (∅, ad)), (14, (ce, bd)), (12, (a, b)), (23, (ce, ce)), (4, (ce, ad)), (1, (ace, bd)), (2, (ad, bce)), (3, (b, acce)), (2, (M, M)).)

Also that the objects in the extent don't have the attributes in the second component of intent (exclusion method). For example, the formal concept (14, ce) only tells us that patients 1 and 4 belong to one class because they have same symptoms Headache and Diarrhea, but the OE-concept (14, (ce, bd)) tells us that patients 1 and 4 belong to one class only not because they have the symptoms Headache and Diarrhea but also because they don’t have the symptoms Cough and Difficulty Breathing. Thus three-way concepts can reflect more knowledge than the classical formal concepts.

However, in order to obtain all OE-concepts and the corresponding OE-concept lattice, one needs to consider all attributes. In this case, there are only 5 attributes, but from Fig. 11 we can see that the representations of all OE-concepts are a little bit complicated. Actually, in real life, we even meet more complicated data. Thus the attribute reductions are necessary to be considered.

Since the ordered hierarchical structure of all the OE-concepts is reflected by the OE-concept lattice, OE-concept lattice is a core structure in 3WCA. Hence, we show the OE-lattice attribute reduction firstly. According to Definition 13, the \( \Lambda_{OE} \) of OE(G, M, I) is shown in Table 6.

Now we calculate the discernibility functions.

\[
f(\Lambda_{OE}) = f(\Lambda_{DEM})
\]

\[
= \bigwedge_{H \in \Lambda_{OE}} (\bigvee H)
\]

\[
= d \land b \land a \land (b \lor
\]

\[
\land (c \lor e) \land (c \lor e \lor d)
\]

\[
\land (a \lor b \lor c \lor d \lor e)
\]

\[
= (a \lor b \lor d \lor c) \land (a \lor b \lor d \lor e)
\]

So the OE-lattice (OE-MIE-preserving) reducts are \{a, b, d, c\} and \{a, b, d, e\}. We only present the OE-concept lattice of \((G, M_5, I_{M_5})\) in Fig. 12. Here \(M_5\) is the OE-lattice reduct \{a, b, d, c\}.

From Fig. 12, we can see that the OE-lattice of \((G, M_5, I_{M_5})\) is isomorphic to the OE-lattice of \((G, M, I)\). Thus the classification of patients and the hierarchy of the three-way concepts don’t change, but we only need to consider smaller amount of attributes. For instance, only based on 4 symptoms Fever, Cough, Headache and Difficulty Breathing, patients 1 and 4 can still be classified into one class. So after OE-lattice reduction, the knowledge is represented in an easier way.

Then, we calculate the OE-JIE-preserving (OE-granular) reducts.

\[
f(\Lambda_{OE}) = f(\Lambda_{JE})
\]

\[
= \bigwedge_{H \in \Lambda_{JE}} (\bigvee H)
\]

\[
= a \land (b \lor c \lor e) \land (c \lor e \lor d)
\]
Table 6
The OEL-discriminability matrix of the OEL-concept lattice in Fig. 11.

|   | TC₁ | TC₂ | TC₃ | TC₄ | TC₅ | TC₆ | TC₇ | TC₈ | TC₉ | TC₁₀ | TC₁₁ | TC₁₂ |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|
| TC₁ | (∅, d) |   |   |   |   |   |   |   |   |     |     |     |
| TC₂ | (∅, a) | (c, d) |   |   |   |   |   |   |   |     |     |     |
| TC₃ | (a, c) |   | (d, e) |   |   |   |   |   |   |     |     |     |
| TC₄ | (a, b) |   |   | (c, d) |   |   |   |   |   |     |     |     |
| TC₅ | (b, c) |   |   |   | (d, e) |   |   |   |   |     |     |     |
| TC₆ | (b, ad) |   |   |   |   | (d, ce) |   |   |   |     |     |     |
| TC₇ | (d, ce) | (ad, b) |   |   |   |   | (d, ce) |   |   |     |     |     |
| TC₈ | (ad, ce) | (db, ace) | (bce, ad) | (ac, de) |   |   |   |   |   |     |     |     |

(F, (∅, ∅))

Fig. 13. The OEL-concept lattice of (G, M₈, I₈).

\[ \land (a \lor d \land b) = (a \lor c) \land (a \lor b) \land (a \lor b \lor d) \]

The OEL-JIE-preserving (OEL-granular) reducts are \{a, c\}, \{a, e\} and \{a, b, d\}. The OEL-concept lattice of (G, M₈, I₈) is presented in Fig. 13. Here M₈ is the OEL-JIE-preserving (OEL-granular) reduct \{a, c\}.

From Fig. 13, we can see that the OEL-granular reduct cannot preserve the structure of the OEL-lattice. That means some knowledge may be lost after reduction. For example, now in Fig. 13, the patients 1, 2 and 4 can not belong to one class because the attribute Difficulty Breathing has been removed after reduction. But the extents of the OEL-object concepts and the join-irreducible elements of OEL-lattice in Fig. 11 don’t change. Since the join-irrelevant elements are important in the lattice construction and the OEL-object concepts are important in granular computing, these two reductions are important in lattice construction and granular computing.

7. Conclusion

In this paper, the attribute reductions of three-way concept lattices have been systematically studied. We have defined eight kinds of attribute reductions of three-way concept lattices, four OEL-attribute reductions and four AE-attribute reductions, and discussed their advantages and disadvantages. Then we investigated the relationships among these newly proposed attribute reductions, the relationships among their consistent sets, as well as the relationships among their cores. Also, we have presented the approaches to computing most of these reductions. Finally, an empirical case is shown to illustrate our analysis. Since three-way concept analysis can be regarded as an extension of formal concept analysis and the three-way concept lattice can reflect more knowledge than the classical one, the study of the three-way concept lattice is a newly proposed but meaningful part of formal concept analysis. Also because the attribute reductions of three-way concept lattices are necessary in three-way concept analysis, the attribute reductions we discussed in this paper enrich the study of formal concept analysis.

The current approach to AE-JIE-preserving reduction is complicated and cannot be unified under the frame of the discernibility matrices and discernibility functions, so, in this paper, we omitted the discussion of it and focused on the relationships among different attribute reductions. Also, since the three-way concept analysis can be regarded as a generalization of the classical formal concept analysis, there must exist some relationships between these two theories. Hence, the relationship among the attribute reductions on these two theories should be studied in the future. Actually, for any kind of attribute reduction mentioned in this paper, the attributes can be classified into three types according to their roles in the reduction [36]. Let \((G, M, I)\) be a formal context. For a certain kind of reduction, suppose the set \(\{I_i \mid i \in \tau\}\) includes all associated reducts of \((G, M, I)\). Then attribute \(M\) can be classified into three types: absolutely necessary attribute, also called core attribute (attribute in \(\bigcap_{i \in \tau} I_i\)), relatively necessary attribute (attribute in \(\bigcup_{i \in \tau} I_i - \bigcap_{i \in \tau} I_i\)), and absolutely unnecessary attribute (attribute in \(M - \bigcup_{i \in \tau} I_i\)). Of course, different attributes play different roles in different reductions. For further research, it is of significance to investigate their relationships based on different reductions. Also, in order to apply the theories we discussed in this paper in the real world, in the future, we will propose the corresponding algorithms of these eight different attribute reductions, which may help us to deal with the big data problems in our daily lives more conveniently. And the utility value will also be introduced and added to every concept and the new attribute reductions based on the utility value will be studied, which will make our study more suitable in practices.

Acknowledgement

The authors of this paper really appreciate the anonymous reviewers. This work was partially supported by the National Natural Science Foundation of China (Grant Nos. 11371014, 11071281 and 61202206).

References

[1] H. Fujita, T.R. Li, Y.Y. Yao, Advances in three-way decisions and granular computing, Knowl.-Based Syst. 91 (2016) 1–3.
[2] Y.Y. Yao, An outline of a theory of three-way decisions, in: J. Yao, Y. Yang, R. Slowinski, S. Greco, H. Li, S. Mitra, L. Polkowski (Eds.), Rough Set and Knowledge Technology, Lecture Notes in Computer Science, vol. 7413, Springer, Heidelberg, 2012, pp. 1–17.
[3] X.F. Deng, Y.Y. Yao, Decision-theoretic three-way approximations of fuzzy sets, Inf. Sci. 279 (2014) 702–715.
[4] Y.Y. Yao, Granular computing and sequential three-way decisions, in: P. Lingras, M. Wolksi, C. Cornelis, S. Mitra, P. Wasielski (Eds.), Rough Set and Knowledge Technology, Lecture Notes in Computer Science, vol. 8171, Springer, Heidelberg, 2013, pp. 16–27.
