SPECIALIZING WIDE ARONSZAJN TREES WITHOUT ADDING REALS

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Abstract. We show that under certain circumstances wide Aronszajn trees can be specialized iteratively without adding reals. We then use this fact to study forcing axioms compatible with CH and list some open problems.

1. Introduction

The purpose of this short note is to prove a technical strengthening of a theorem of Shelah’s on specializing Aronszajn trees and connect it to some open problems in iterated forcing and the continuum hypothesis. Specializing Aronszajn trees iteratively without adding reals goes back to the work of Jensen separating CH from ♢, see [6]. Reworking this result, in [8, Chapter V] Shelah introduced the class of dee-complete and ℵ₀-proper forcing notions, an iterable class which does not add reals and showed that there is a forcing notion in this class that specializes Aronszajn trees. However the countability of the levels of the trees is essential in Shelah’s proof in contrast to the ccc specializing forcing introduced in [4], which adds reals but where the width of the tree plays no role. Therefore it remains unclear when one can specialize wider trees without adding reals. In this note we provide a partial solution to this problem by proving that under certain circumstances there are dee-complete and ℵ₀-proper posets to specialize wide trees. Specifically we show the following.

Theorem 1.1. Suppose T is an ℵ₀-tree (countable levels) and S ⊆ T is a (potentially wide) Aronszajn tree with the induced suborder. Then there is a forcing notion PT,S which specializes S and is dee-complete and ℵ₀-proper.

Using this poset we give an application to forcing axioms compatible with CH.

Theorem 1.2. Under the forcing axiom for dee-complete and ℵ₀-proper forcing notions, all ℵ₀-trees are essentially special and therefore there are no Kurepa trees.

This latter theorem was shown by Shelah under the additional assumption that CH and 2^ℵ₀ = ℵ₂ holds. What’s new here is that using the forcing notion from Theorem [6] we can remove the cardinal arithmetic assumption.

The general question of when one can specialize a wide tree without adding reals turns out to be very interesting and there are many open questions still. The note finishes with some brief further observations and open problems. In particular, a connection to cardinal characteristics is observed.
2. Preliminaries: Dee-complete Forcing, < ω₁-Properness and Trees

Given a ZFC⁻ model $N$, a forcing notion $\mathbb{P} \in N$ and a condition $p \in \mathbb{P}$ let $\text{Gen}(N, \mathbb{P}, p)$ denote the set of $\mathbb{P}$-generic filters over $N$ containing $p$. The following definitions come from [8, Chapter V] and a particularly good exposition is also given in [1]. What we call a completeness system here is called a “countably complete” completeness system in [8]. However, every completeness system considered in this note is countably complete so we omit the additional notation.

**Definition 2.1.** A completeness system is a function $\mathbb{D}$ defined on triples $(N, \mathbb{P}, p)$ such that $N \models ZFC^-$, $\mathbb{P} \in N$ is a forcing notion and $p \in \mathbb{P} \cap N$ is a condition and the following hold:

1. $\mathbb{D}(N, \mathbb{P}, p)$ is a family of sets, $A$, such that each $A \subseteq \text{Gen}(N, \mathbb{P}, p)$.
2. If $A_i \in \mathbb{D}(N, \mathbb{P}, p)$ for each $i < \omega$ then the intersection $\bigcap_{i < \omega} A_i$ is non-empty.

If $\theta$ is a cardinal and $\mathbb{D}$ is defined on the set of countable elementary substructures of $N \prec H_\theta$ then we call $\mathbb{D}$ a completeness system on $\theta$.

Completeness systems in general are quite easy to construct, which leads one to question their utility. In general we will only be interested therefore in ones which are “nicely defined” a notion Shelah refers to as simple.

**Definition 2.2.** A completeness system $\mathbb{D}$ is simple if there is a formula $\phi$ and a parameter $s \in H_{\omega_1}$ such that $\mathbb{D}(N, \mathbb{P}, p) = \{ A_u \mid u \in H_{\omega_1} \}$ where $A_u = \{ G \mid G \in \text{Gen}(N, \mathbb{P}, p) \text{ and } (H_{\omega_1}, \in, \bar{N}, \bar{\mathbb{P}}, \bar{p}, s) \models \phi(\bar{N}, \bar{\mathbb{P}}, \bar{p}, s) \}$ where $\bar{N}, \bar{\mathbb{P}}, \bar{p}, ...$ are the Mostowski collapsed versions of $N, \mathbb{P}, p$, etc.

Using this, we can define dee-completeness.

**Definition 2.3.** We say that $\mathbb{P}$ is dee-complete if for every sufficiently large $\theta$ there is a simple completeness system $\mathbb{D}$ on $\theta$ such that whenever $\mathbb{P} \in N \prec H_\theta$, with $N$ countable and $p \in \mathbb{P} \cap N$ there is an $A \in \mathbb{D}(N, \mathbb{P}, p)$ such that for all $G \in A$ there is a condition $q \in \mathbb{P}$ so that $q \leq r$ for all $r \in G$.

Given a poset $\mathbb{P}$ we say that a (not necessarily simple) completeness system $\mathbb{D}$ is a completeness system for $\mathbb{P}$ if it satisfies the requirements of the definition of dee-completeness. Observe that the existence of a completeness system for $\mathbb{P}$ implies that $\mathbb{P}$ is proper and adds no new reals (or indeed $\omega$ sequences of ordinals) since the condition $q$ as in the definition of dee-completeness is an $(N, \mathbb{P})$-generic condition and witnesses that any countable intersection of dense open sets is non-empty. Note also that somewhat trivially all $\sigma$-closed forcing notions are dee-complete since every generic for any $N \prec H_\theta$ has a lower bound. In order to iterate dee-complete forcing notions we need a slight strengthening of properness however.

**Definition 2.4 (α-Properness).** Let $\theta$ be a cardinal and $\alpha < \omega_1$. An $\alpha$-tower for $H_\theta$ is a sequence $\bar{N} = \langle N_i \mid i < \alpha \rangle$ of countable elementary substructures of $H_\theta$ so that for each $\beta < \alpha$ $\langle N_i \mid i \leq \beta \rangle \in N_{\beta+1}$ and if $\lambda < \alpha$ is a limit ordinal then $N_\lambda = \bigcup_{i<\lambda} N_i$. We say that $\mathbb{P}$ is $\alpha$-proper if for all sufficiently large $\theta$, all $p \in \mathbb{P}$
and all $\alpha$-towers $\vec{N}$ so that $p, P \in N_0$ there is a $q \leq p$ which is simultaneously $(N_i, P)$-generic for all $i < \alpha$. We say that $P$ is $< \omega_1$-proper if it is $\alpha$-proper for all $\alpha < \omega_1$.

Note that properness is 1-properness. The point is the following theorem due to Shelah, [8, Chapter V, Theorem 7.1].

**Theorem 2.5.** Countable support iterations of dee-complete and $< \omega_1$-proper forcing notions are dee-complete and $< \omega_1$-proper. In particular such iterations do not add reals.

As an immediate consequence, we obtain, relative to a supercompact cardinal, the consistency of DCFA, the forcing axiom for dee-complete and $< \omega_1$-proper forcing notions and even its consistency with CH simply by running the standard Baumgartner argument. Of course DCFA does not imply CH since PFA implies DCFA. Very little attention has gone into DCFA as an axiom in its own right outside of [7]. However one notable exception is [3] where it is shown (though not remarked in this language) that DCFA implies the P-Ideal Dichotomy.

The main purpose of this note is to look at applications of dee-complete forcing to trees. Let me review some notation and terminology related to this here for reference. Recall that a tree $T = \langle T, \leq_T \rangle$ is a partially ordered set so that for each $t \in T$ the set of $s \leq_T t$ is well ordered. A branch through a tree is a maximal linearly ordered subset.

**Definition 2.6.** Let $T$ be a tree, $\alpha$ an ordinal and $\kappa$ and $\lambda$ cardinals.

1. The $\alpha$th-level of $T$, denoted $T_\alpha$ is the set of all $t \in T$ so that $\{s \mid s \leq_T t\}$ has order type $\alpha$. Also let $T_{\leq \alpha} = \bigcup_{i \leq \alpha} T_i$ and $T_{< \alpha} = \bigcup_{i < \alpha} T_i$.
2. The height of $T$ is the least $\alpha$ with $T_\alpha = \emptyset$.
3. We say that $T$ is a $\kappa$-tree if it has height $\kappa$ and each level has size $< \kappa$.
4. $T$ is a $\kappa$-Aronszajn tree if it is a $\kappa$-tree with no branch of size $\kappa$. If $\kappa = \aleph_1$ we just say $T$ is an Aronszajn tree.
5. $T$ is a $(\kappa, \leq)\text{-Aronszajn tree}$ if it is a tree of height $\kappa$ with each level of size $\leq \lambda$ and no branch of size $\kappa$. An $(\aleph_1, \leq)\text{-Aronszajn tree}$ is called a fat Aronszajn tree if $\lambda$ is uncountable and the bound is realized at some level.
6. A (fat) Aronszajn tree is special if it can be decomposed into countably many antichains. Equivalently if there is a specializing function $f : T \to \mathbb{Q}^+$, the set of positive rationals so that $f$ is strictly increasing on linearly ordered subsets of $T$.
7. An $\omega_1$-tree is Kurepa if it has more than $\aleph_1$ many uncountable branches. A tree is a weak Kurepa tree if it is a tree of height and cardinality $\aleph_1$ with more than $\aleph_1$-many uncountable branches.
8. A tree of height $\omega_1$ is essentially special if there is an $f : T \to \mathbb{Q}$ which is (weakly) increasing on chains and for all $s, t, u \in T$ if $s \leq_T t, u$ and $f(s) = f(t) = f(u)$ then $t$ and $u$ are comparable.
Note that, as observed in Theorem 7.4 of [5] coupled with the remarks preceding its statement on page 949 of the same article, every essentially special tree has at most $\aleph_1$-many uncountable branches and hence is not weakly Kurepa.

3. Specializing a Wide Tree Without Adding Reals

In this section I work towards proving Theorem 1.1. The forcing notion used is inspired by the forcing from [2] and the exposition here mirrors the one in that article closely. Throughout, fix an $\omega_1$-tree $T$ (possibly with branches) and let $S \subseteq T$ be an $(\omega_1, \leq \omega_1)$-Aronszajn tree with the induced suborder. Without loss we may assume that $T \subseteq H_{\omega_1}$. The first step is to define the forcing $P$. The idea is to force with partial specializing functions $f : S \rightarrow Q$ but use the structure of $T$ to extend. The fact that $S$ is Aronszajn will be used to ensure properness while the countable levels of $T$ will be used to ensure the forcing can be controlled. I begin by defining the objects that will build up the conditions.

**Definition 3.1.** Recall that $S \subseteq T$ are trees, $T$ has countable levels but potentially cofinal branches and $S$ may have uncountable levels but no cofinal branches.

1. A **partial specializing function** of height $\alpha$ is a function $f : T_{\leq \alpha} \cap S \rightarrow Q^+$ which is strictly increasing on linearly ordered chains. We write $ht(f)$ to denote the height of $f$.

2. A (possibly partial) function $h : T_{\beta} \rightarrow Q^+$ **bounds** a partial specializing function if $\beta \geq \alpha + 1$ and for all $t$ in the domain of $h$ which projects into $S$ we have that $h(t) > f(t \upharpoonright \gamma)$ for all $\gamma \leq \alpha$ for which $t \upharpoonright \gamma \in S$.

3. A **requirement** $H$ of height $\beta$ is a relation whose domain is a subset of $T_{\beta}$ which has infinitely many projections into $S$ and whose range is $Q^+$. If $s \in T_{\beta}$ is in the domain of the relation, we say say that $H$ **bounds** a partial specializing function $f$ at $s$ if there is a rational $x$ so that $sHx$ and $f(s \upharpoonright \gamma) < x$ for all $\gamma \leq ht(f)$ for which $s \upharpoonright \gamma \in S$.

4. A partial specializing function $f$ **fulfills** a requirement $H$ if the height of $f$ is at most the height of $H$ and for every finite $\tau \subseteq T_{\beta}$, $\beta$ the height of $H$, either $\tau$ has no projections in $S$ or else there is an $s$ in the domain of $H$ so that $H$ bounds $f$ at $s$ and $s \notin \tau$.

5. A **promise** is a function $\Gamma$ defined on a tail set of countable ordinals, the first of which we denote $\beta = \beta(\Gamma)$ so that for each $\gamma \geq \beta$, $\Gamma(\gamma)$ is a countable set of requirements of height $\gamma$ and if $\gamma' \geq \gamma$ then $\Gamma(\gamma) = \Gamma(\gamma') \upharpoonright \gamma$ i.e. every $H \in \Gamma(\gamma)$ is the projection of some $H' \in \Gamma(\gamma')$ and this projection is always well defined.

6. A partial specializing function $f$ **keeps** a promise $\Gamma$ if $\beta(\Gamma) \geq ht(f)$ and $f$ fulfills every $H \in \Gamma(\gamma)$ for all $\gamma \geq \beta$. Note that by the projection property given in the definition of a promise, to keep a promise it suffices to fulfill the requirements at the first level.
The proof is by induction on something bounding \( f \). Then there is a \( q \) function of height \( \leq q \) if \( f_p \geq f_q \) and for all \( \gamma \geq ht(q) \) \( \Gamma_p(\gamma) \geq \Gamma_q(\gamma) \).

The proof of Theorem \( \[\text{(7)} \] \) is broken up into several lemmas. I begin by showing that we can always extend to arbitrarily high levels.

**Lemma 3.2** (The Extension Lemma). Suppose \( p \in \mathbb{P} \) of height \( \alpha \) and let \( \beta \geq \alpha \). Then there is a \( q \leq p \) of height \( \beta \). Moreover, if \( g : T_{\beta} \to \mathbb{Q}^+ \) is a (potentially partial) function that bounds \( f_p \) then \( q \) can be found so that \( h \) bounds \( q \) as well.

Let me fix some terminology before I prove this lemma. If \( f \) is a partial specializing function of height \( \alpha \) and \( H \) is a requirement of height \( \beta \), with \( \beta \geq \alpha \) and \( \tau \subseteq T_{\beta} \) is finite then let's say that some extension of \( f \) of height \( \beta \) fulfills \( H \) with respect to \( \tau \) at \( s \in T_{\beta} \) if \( s \notin \tau \) and the extension of \( f \) is bounded by \( H \) at \( s \). In other words, \( s \) witnesses that \( f \) fulfills the requirement \( H \) with respect to the fact that there is something bounding \( f \) and disjoint from \( \tau \).

**Proof.** The proof is by induction on \( \beta \). There are two cases.

**Case I:** \( \beta \) is a successor ordinal. By induction assume that \( \beta = \alpha + 1 \). By our inductive hypothesis therefore we may assume that there is a function \( f_p : T_\alpha \cap S \to \mathbb{Q}^+ \). We need to find an \( \tilde{f} : T_{\beta} \cap S \to \mathbb{Q}^+ \) so that \( (f_p \cup \tilde{f}, \Gamma_p \setminus \Gamma_p(\alpha)) \in \mathbb{P} \). I will define such an \( \tilde{f} \) in countably many stages, noting that there are only countably many things that we need to account for. Indeed I will define finite functions \( f_n \) for \( n < \omega \) so that \( f_n \subseteq f_{n+1} \) and the union will be \( \tilde{f} \). Let \( \{t_n \mid n < \omega \} \) enumerate the elements of \( T_{\beta} \cap S \) and let \( \{(H_n, \tau_n) \mid n < \omega \} \) enumerate all possibly pairs of requirements from \( \Gamma_p(\beta) \) and finite subsets \( \tau \subseteq T_{\beta} \). Without loss we may assume that \( T_{\beta} \cap S \) is infinite since if it is finite (or even empty) then one easily checks that any extension of \( f_p \) to those finitely many levels that is bounded by \( g \) will witness the lemma. Now let \( f_0 \) be defined as follows: by the definition of \( \mathbb{P} \), there is a \( t = t_{k_0} \in T_{\beta} \cap S \) and a rational number \( h_t \in \text{range}(H_{k_0}) \) so that \( t \notin \tau_0 \) and \( H \) bounds \( f_p \) at \( t \) or \( t \notin S \). Let \( f_0(t_{k_0}) \) be any rational above \( f_p(t_{k_0} \uparrow \alpha) \) less than the value of \( h_t \) and \( g(t_{k_0}) \) if the latter is defined. Then, if \( t_{k_0} \neq t_0 \), define \( f_0(t_0) \) to be any value less than \( g(t_0) \), again assuming this value is defined (again above \( f_p \)). These are the only two points in the domain. Suppose now that we have defined \( f_i \) so that for all \( i \leq n \) we have that \( f_i \subseteq f_{i+1} \) and there is a \( t_{k_i} \in \text{dom}(f_i) \) witnessing that \( F_i \) bounds \( f_i \) at \( t_i \) and \( t_0, \ldots, t_i \in \text{dom}(f_i) \). Moreover assume that \( f_i \) is bounded by \( g \). Now I define \( f_{n+1} \) by performing the same procedure as described for \( f_0 \), except that \( \tau_{n+1} \) is replaced by \( \tau_{n+1} \cup \text{dom}(f_n) \). Note that this set is still finite so we can find a good \( t_{k_{n+1}} \) by the definition of a requirement. Now let \( \tilde{f} = \bigcup_{n<\omega} f_n \). Clearly this function is defined on all of \( T_{\beta} \cap S \) and keeps the promise \( \Gamma(\beta) \) so we are done.

**Case II:** \( \beta \) is a limit ordinal. Fix a strictly increasing sequence \( \langle \beta_n \mid n < \omega \rangle \) so that \( \beta_0 = \alpha \) and \( \sup_n \beta_n = \beta \). Via the same procedure described in case 1, we can define a function \( \tilde{f} : T_{\beta} \cap S \to \mathbb{Q}^+ \) so that \( f_p \cup \tilde{f} \) is a partial specialization on \( (T_{\leq \alpha} \cup T_{\beta}) \cap S \) which is bounded by \( g \). Now, using the inductive assumption, we can
define $f_{i+1} \supseteq f_i \supseteq f_\sim$ so that $f_i$ is defined on $T_\beta \cap S$ and is bounded by $f_\sim$. Finally let $f_q = \bigcup_{n<\omega} f_n \cup f$. This function is then defined on all of $T_\beta \cap S$ and keeps all requisite promises and is bounded by $g$ so we’re done.

Next I show how to add promises. Given two promises $\Gamma$ and $\Psi$ I write $\Psi \subseteq \Gamma$ to mean that $\beta(\Gamma) \geq \beta(\Psi)$ and for all $\gamma \geq \beta(\Gamma)$ we have that $\Psi(\gamma) \subseteq \Gamma(\gamma)$. Also, I will write $\Gamma \cup \Psi$ to mean the promise $\Delta$ so that for all $\gamma \geq \max(\beta(\Gamma), \beta(\Psi))$ $\Delta(\gamma) = \Gamma(\gamma) \cup \Psi(\gamma)$.

**Lemma 3.3 (Adding Promises).** Suppose $p \in P$ is of height $\alpha$, $\beta \geq \alpha$ and $g : T_\beta \to \mathbb{Q}^+$ is a (potentially partial) function bounding $f_p$. Let $\Psi_g$ be a promise so that $\beta(\Psi_g) \geq \beta$ and for all $H \in \Psi_g(\beta(\Psi_g))$ if $(t, x) \in H$ then $t \upharpoonright \beta$ is in the domain of $g$ and $\geq g(t \upharpoonright \beta)$. Then there is an extension $q \leq p$ so that $\Gamma_q \supseteq \Psi_g$.

**Proof.** Note that the conditions on $g$ imply that already $(f_p, \Gamma_p \cup \Psi_g) \in P$. But then any extension of this condition to level $\beta$ (using the previous lemma) is as required.

Intuitively the previously lemma states that if $g$ bounds some condition, it’s not dense to insist that extensions are not bounded by $g$. In particular, we can always avoid growing above $g$. This is key for proof of properness and it’s here that the need for promises stems from. To prove that $P$ is proper I will need the following lemma.

**Lemma 3.4 (The Submodel Lemma).** Let $\theta$ be sufficiently large and let $M \prec H_\theta$ be countable containing $T, S, P, \omega_1$. Let $p \in P \cap M$ and let $\delta = M \cap \omega_1$. Note that $M \cap T = T_\delta$. Let $D \in M$ be a dense open subset of $P$ and let $h : T_\delta \to \mathbb{Q}^+$ be a finite function bounding $p$. Then there is an extension $q \in D \cap M$ so that $q$ is also bounded by $h$.

**Proof.** Suppose the statement of the lemma is false and let $M, T, S, p, D, h$ etc be a counter example. Let me fix that $ht(p) = \alpha$. Note that if $q \leq p$ and $q \in D \cap M$, then $q$ is not bounded by $h$. I will reach a contradiction by showing how to add a promise to $p$ as in Lemma 3.3 which ensures that any further extension is bounded by $h$. Let us enumerate the domain of $h$ by $t_0^h, ..., t_{n-1}^h$. Since $T$ is normal, there is a least level $\gamma > \alpha$ so that for all $i < j < n$ the projections $t_i^h \upharpoonright \beta$ in the sense of $S$ (!!) and $t_j^h \upharpoonright \beta$ are distinct. Moreover, without loss, assume that all such projections are in $S$ since otherwise they do not matter. Let $h_{\gamma}$ be the projection on $h$ to this level. Note that $h_{\gamma} \in M$ (since it’s finite), bounds $p$ and if $q \leq p$ is in $M \cap D$ then $q$ must not be bounded by $h_{\gamma}$ since otherwise we would contradict our assumption. Thus we obtain that $M \models \forall q \leq p \text{ if } q \in D \text{ then } q \text{ is not bounded by } h_{\gamma}$”. Note by elementarity this is also true in $V$. Note also, that since the fact that $\gamma$ was least was not used, we could have chosen any $\gamma' \geq \gamma$ and the statement above would have held with $\gamma$ replaced by $\gamma'$. Hence $M$ thinks there are cofinally many $\gamma'$ so that $h_{\gamma'}$ is as in the statement $M$ models above.

Now, I want to use the property described of $h_{\gamma}$ to define a collection of $n$-tuples of $T$. Let’s say that an $n$-tuple $\bar{s} = (s_0, ..., s_{n-1})$ of elements of $S$ all of the same
hieght is bad if its projection to level $\gamma$ is $\text{dom}(h_\gamma)$ (say $s_i \upharpoonright \gamma = \rho_i \upharpoonright \gamma$) and the function $h_\gamma$ whose domain is $\vec{s}$ and, to each $s_i$ assigns the rational $h(t^i_\gamma)$ bounds $p$ but is such that there is no $q \leq p$ so that $q \in D$ and $q$ is bounded by $h_\vec{s}$. In particular the domain of $h_{\gamma'}$ is bad for each $\gamma' \geq \gamma$, but other sets, which are not the projection of the domain of $h$ may also be bad. Let $B \subseteq S^\alpha$ be the collection of all bad tuples. For simplicity we let $B(i)$ be the collection of all bad tuples on level $i < \omega_1$. Note that since $M \models B$ is unbounded this is true in $V$. Also $B$ is closed downwards above $\gamma$ in the sense that if $\vec{s} \in B(j)$ and $\gamma < i < j$ then $\vec{s} \upharpoonright i \in B(j)$. If $\vec{s}_0, \vec{s}_1 \subseteq S_\alpha$ let’s write $\vec{s}_0 \leq \vec{s}_1$ if the elements of $\vec{s}_0$ are pairwise below the elements of $\vec{s}_1$ on the tree. We define recursively $B_0 = B$, $B_{i+1} = \{s \in B_i \mid \text{for uncountably many levels } j \exists \vec{u} \in B(j) \vec{s} \subseteq \vec{u}\}$, and $B_\lambda = \bigcap_{i<\lambda} B_i$ for $\lambda$ limit. Note that $\text{dom}(h_{\gamma}) \subseteq B_i$ for every $i$. Let $B_\infty = B_\rho$ where $\rho$ is the least so that $B_\rho = B_{\rho+1}$.

Claim 3.5. Every $\vec{s} \in B_\infty$ has two disjoint extensions under $\leq$ in $B_\infty$.

Proof of Claim. Suppose not and let $\vec{s} \in B_\infty$ be a counter example. I will use $\vec{s}$ to define a branch through $S$ contradicting the fact that $S$ is Aronszajn. Let $W \subseteq B_\infty$ be the collection of all $\vec{u}$ extending $\vec{s}$. Since every element of $B_\infty$ has extensions on confinally many levels, this set in particular has extensions on all levels above the height of $\vec{s}$. Let’s denote this height by $s_\vec{s}$. Finally note that, given any $\gamma_s < i < j < \omega_1$ we have that if $\vec{s}^i \in W(i)$ and $\vec{s}^j \in W(j)$ then it must be that these two tuples have complementary elements by the assumption of $\vec{s}$ being a counter example to the claim.

Let $U$ be an ultrafilter on $W$ all of whose elements are uncountable. For any $x \in S$ and $k < n$ let $Y_{x,k}$ be the collection of all elements $\vec{z} \subseteq W$ so that $x$ is comparable with the $k$th element of $\vec{z}$. Notice by the above assumption, we get that $W = \bigcup_{i<n} \bigcup_{k<n} Y_{z^i,k}$ where $z^i \subseteq W(i)$ for any $i \in (\gamma_s, \omega_1)$. But for any such $i$ we must have that there is an $l_i < n$ and a $k_i < n$ so that $Y_{z^i,l_i,k_i} \in U$. But then for some $k$ the set $\{i \in (\gamma_s, \omega_1) \mid k_i = k\}$ is uncountable and the corresponding $z^i_k$'s must generate a cofinal branch in $W$, contradiction.

Moving on, by bootstrapping the above argument, there is a level $i$ so that $B_{\infty}(i)$ has infinitely many disjoint bad tuples. Let $i$ be such a level. But now such a level generates a promise $\Psi$: we simply take the projections of the nodes (which are all in the same level in $S$) to the least level in $T$ in which all such projections are defined and on each such level let $H$ be the requirement so that $s_i H(t^i_\gamma)$. Such a promise is in $M$, by running the argument above in $M$, and moreover, $p$ keeps this promise so we can add it to $p$ (in $M$). Therefore by Lemma 3.3 there is a $p' \leq (f_p, \Gamma_p \cup \Psi)$ in $M$. But now let $q \leq p'$ be any element in $D \cap M$. Then $q$ keeps the promise $\Psi$ but this contradicts the definition of a bad function.

This lemma is the key ingredient in the proof that $\mathbb{P}$ is proper.

Lemma 3.6. $\mathbb{P}$ is proper. In fact, $\mathbb{P}$ is dee-complete for some simple completeness system $\mathbb{D}$. 

Proof. Work in the setting of Lemma 3.4. I want to prove the existence of a master condition for $M$. Let $\langle D_n \mid n < \omega \rangle$ be an enumeration of the dense open subsets of $\mathbb{P}$ in $M$. Let $p \in P \cap M$ and let $\langle t_i \mid i < \omega \rangle$ enumerate the elements of $T_\delta$ with projections into $S$ and let $\langle \tau_k \mid k < \omega \rangle$ enumerate all the finite subsets of $T_\delta$. I want to define a sequence $p \geq p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots$ so that $p_i \in D_i \cap M$ for all $M$ and there is a condition $q$ extending the union of the $p_i$’s. Such a $q$ defines a generic over $M$. The idea is to use Lemma 3.4 $\omega$-many times to make sure that the union of an $M$ generic filter is bounded and hence can be extended into a further condition. I will then extract from the proof a definition of the generics bounded by such a $q$ and this will be used to define a simple completeness system as needed.

Fix an enumeration in order type $\omega$ of all triples $e_i = (m_i, n_i, k_i)$ so that $m_i, n_i, k_i \in \omega$ and the first occurrence of $m$ in the first coordinate is after the $m^\text{th}$ element of the enumeration and each such triple appears infinitely often. Now, using Lemma 3.4 recursively define conditions $p_{i+1}$ and functions $h_i$ satisfying the following conditions:

i) $p_{i+1} \leq p_i$ and $p_{i+1} \in D_{i+1}$ so that we ensure that $p_{i+1}$ fulfills the $n_i^\text{th}$ requirement of $p_{m_i}$ with respect to $\tau_{k_i}$ (this uses Lemma 3.3). Let $t_{j_{i+1}}$ be the node of the tree that we bound $p_{i+1}$ by in this step. We can assume inductively that $t_{j_{i+1}}$ is not in the domain of $h_i$.

ii) $h_i$ has a finite domain consisting of $t_0, \ldots, t_i$ and $t_{j_0}, \ldots, t_{j_i}$, bounds $p_i$ and is at most $(f_{p_i}(t_{j_i} \upharpoonright ht(p_i)) + h_{t_{j_i}})/2$ where $h_{t_{j_i}}$ is the rational in the range of the $n_i^\text{th}$ requirement of $\Gamma_{p_i}(\delta)$ which corresponds to $t_{j_i}$ and was chosen in the $i^\text{th}$ step of the process.

iii) $h_{i+1}$ is a finite function from $T_\delta$ to $\mathbb{Q}^+$ bounding $f_{p_{i+1}}$ which extends $h_i$ to include in its domain $t_{i+1}$ and $t_{j_{i+1}}$ (potentially these are the same).

It’s clear by what we have done that such a sequence can be constructed and generates a generic filter on $M$. I need to show that there is a lower bound, $q$. Note that $\bigcup_{n<\omega} f_{p_n}$ is a partial specializing function defined on $T_{<\delta} \cap S$. I claim that we can extend it to a function defined on $T_\delta \cap S$ which fulfills the promises $\bigcup_{n<\omega} \Gamma_{p_n}$. Indeed, let $q(t_i) = h_i(t_i)$ for $t_i \in S$. This is defined, since we insisted that $t_i \in \text{dom}(h_i)$. Also, since $h_i$ bounded all $p_i$, $q(t_i)$ is at least the supremum of the values of $f_n(t_i \upharpoonright \beta)$ for all $\beta < \delta$. What needs to be checked is that $f_q$ actually fulfills all the promises in the $p_i$’s. This is what was planned for though. If $H \in \Gamma_q(\delta)$ then $H \in \Gamma_{p_i}(\delta)$ for some $i$ and for any $\tau \subseteq T_\delta$ finite with projections into $S$, there was a stage where we ensured that $f_q$ was bounded by some $h_n$ which included being bounded by some $H$ on a node disjoint from $\tau$. Then, from that stage on, since all $n_j$’s were bounded by this $h$, we get that $f_q$ fulfills that instance of the promise.

Thus we have shown that $q$ is an $(M, \mathbb{P})$-master condition so $\mathbb{P}$ is proper. It remains to show that it is in fact dee-complete. This amounts to showing that we can define the collection of filters of the form $G_q = \{p \in \mathbb{P} \cap M \mid q \leq p\}$. Clearly such filters can be recovered from the choice of the enumeration of the nodes of $T_\delta$, the requirements, $\langle e_l \mid l < \omega \rangle$, and the choice of the $h_i$’s. This set of parameters is hereditarily countable so it’s coded by a single element of $H_{\omega_1}$. Fix such a code $s$ and let $\phi(G, x, N, \mathbb{P}, p, s)$ be the formula which says that “$G$ is $\mathbb{P}$-generic over $N$.
(in fact the Mostowski collapsed versions of these), \( p \in G \) and either \( x \) codes the same enumerations as \( s \), except maybe the functions \( h_i \) coded by \( x \) are larger than those coded by \( s \) or else we take all generics". Then clearly this is a well defined simple completeness system. To check that countable intersections are non-empty, note that for any \( x \), the set \( A_x \) contains the generic bounded by the \( h_i \)'s coded by \( s \).

Finally I prove that \( P \) is \( \alpha \)-proper for all \( \alpha < \omega_1 \).

**Lemma 3.7.** Let \( \alpha < \omega_1 \) and let \( \bar{N} = \langle N_i \mid i \leq \alpha \rangle \) be a tower of length \( \alpha \) for \( N_i \prec H_\theta, \theta \) sufficiently large with \( P \in N_0 \). Then for any \( p \in N_0 \cap P \) there is a \( q \leq p \) which is \( (N_i, P) \)-generic simultaneously for every \( i \leq \alpha \).

**Proof.** We induct on \( \alpha \). If \( \alpha \) is a successor ordinal, this is just the proof of properness given above so assume that \( \alpha \) is a limit ordinal. Pick an increasing sequence \( \langle \alpha_n \mid n < \omega \rangle \) with \( \sup_{n<\omega} \alpha_n = \alpha \). Let \( \delta = \omega_1 \cap N_\alpha \). One can perform the same proof as when it was proved that \( P \) was proper, except now we insist (via the inductive assumption) that \( p_i \) be \( (N_j, P) \)-generic for all \( j < i \) and \( p_i \in N_i \) as opposed to \( p_i \) being in some specified dense open. Since, by the definition of a tower \( \langle N_j \mid j < i \rangle \in N_i \) this is possible (given the sequence, by elementarity, \( N_i \) can find a master condition). Moreover, since, again by definition of a tower, the sequence of models is continuous and in particular, \( N_\alpha = \bigcup_{n<\omega} N_{\alpha_n} \), the set \( \{ r \in N_\alpha \cap P \mid \exists i p_i \leq r \} \) is \( (N_\alpha, P) \)-generic. The only thing to be careful about is that the union of the \( p_i \)'s can be extended to some \( q \) of height \( \delta \). However, by iteratively applying Lemma 3.4 as in the previous proof this is easily accounted for.

**Corollary 3.8.** Assume DCFA. Every (potentially wide) Aronszajn tree which embeds into an \( \omega_1 \)-tree is special.

I will give a concrete application of such a tree in the next section. Let me note first that the condition is not trivial: there are wide Aronszajn trees in ZFC which cannot be embedded into \( \omega_1 \) trees.

**Lemma 3.9.** (Essentially Todočević, see \cite[Definition 3.2]{[9]}) There is an \( (\omega_1, \leq 2^{\aleph_0}) \)-Aronszajn tree which is ZFC-provably non-special, and cannot be specialized by any forcing not adding reals.

**Proof.** Let \( E \subseteq \omega_1 \) be stationary co-stationary and let \( T(E) \) be the tree of attempts to shoot a club through \( E \). In other words, elements of \( T \) are closed, bounded, countable initial segments of \( E \) ordered by end extension. This poset is well known to be \( \sigma \)-distributive, hence the tree has height \( \aleph_1 \). Also, every element is a countable set of ordinals hence coded by a real so it has width \( 2^{\aleph_0} \). Therefore \( T(E) \) is a \( (\omega_1, \leq 2^{\aleph_0}) \)-Aronszajn tree. However, it can't be special, since, as mentioned before, forcing with this tree does not add reals, so in particular, \( \omega_1 \) is preserved. For the last part, note that, if \( P \) does not add reals then the reinterpretation of \( T(E) \) in \( V^P \) is just \( T(E) \) so it's still \( \sigma \)-distributive and hence it must still not be special. \qed
Putting together Lemma 3.9 and Theorem 1.1 we conclude the following odd result which may be of independent interest. Note that below is a ZFC theorem.

**Theorem 3.10.** For any stationary set $E \subseteq \omega_1$ the tree $T(E)$ cannot be embedded into any $\omega_1$-tree.

*Proof.* Suppose $T(E)$ could be embedded into an $\omega_1$-tree. Then, by forcing with the forcing from Theorem 1.1 we could make $T(E)$ special without adding reals. But this contradicts Lemma 3.9. □

**Corollary 3.11.** DCFA is consistent with the existence of non-special trees of size $\aleph_1$.

*Proof.* If CH holds, which it does in the natural model of DCFA, then the tree $T(E)$ witnesses the corollary. □

### 4. Kurepa Trees

I now use the forcing from the previous section to provide an application of DCFA.

**Theorem 4.1.** Under DCFA all $\omega_1$ trees are essentially special and hence there are no Kurepa Trees.

Note that [8, Chapter VII, Application G] proves the same thing under the additional assumptions that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. What is new is that the cardinal arithmetic is unnecessary. The proof also gives more information since it shows that certain wide trees are special under DCFA. The proof of Theorem 4.1 follows Baumgardner’s original proof from PFA, however using the poset from Theorem 1.1.

The argument is sketched with reader referred to [5, Section 7] for more details.

*Proof.* Assume DCFA and let $T$ be an $\omega_1$-tree. Let $\lambda \geq \aleph_2$ be the number of branches through $T$. First, force with $Col(\lambda, \aleph_1)$, the $\sigma$-closed forcing to collapse $\lambda$ to $\aleph_1$. Note that, being $\sigma$-closed, this is $\omega_1$-complete and $< \omega_1$-proper. Work in the collapse extension. As noted in Lemma 7.11 of [5] $\sigma$-closed forcing won’t add uncountable branches to a tree of width $< 2^{\aleph_0}$ hence, in particular, there are no new branches added to to $T$ in the extension so there are now at most $\aleph_1$ many branches.

I use the following claim, due to Baumgardner, see [5, Lemma 7.7].

**Claim 4.2.** There is an uncountable subtree $S \subseteq T$ with no uncountable branches and, by specializing it, we obtain that $T$ is essentially special i.e. there is an $f : T \to \mathbb{Q}$ which is (weakly) increasing on chains and for all $s, t, u \in T$ if $s \leq_T t, u$ and $f(s) = f(t) = f(u)$ then $t$ and $u$ are comparable.

Thus, by applying the specializing forcing $\mathbb{P}_{T,S}$ from Theorem 1.1 to $S$ and working in that extension we have that $S$ is special and so $T$ is essentially special. Now, finally applying DCFA we can pull back to $V$ and find an $f : T \to \mathbb{Q}$ witnessing that $T$ is essentially special, so we’re done. □

In contrast with the case of PFA, note that Corollary 3.11 this result cannot be improved to trees of width $\omega_1$. 

5. Cardinal Characteristics and Open Problems

The previous sections suggest some new directions for studying wide trees, particularly in connection with cardinal characteristics. While I leave an in-depth investigation of these ideas for future research I want to finish this chapter by recording some easy observations and connecting them back to what has been shown.

The main observation is that the behavior of trees is as much connected to their width and cardinality as to their height. This is obscured by the fact that the ccc forcing to specialize a tree works equally well regardless of the width of the tree. However, the trees of the form $T(E)$ suggest that there is something more subtle going on with regards to specializing wider trees. The obvious question is whether Theorem 1.1 can be improved.

**Question 1.** Under what conditions can a wide tree be specialized without adding reals? Are there (in ZFC) trees which can be specialized without adding reals but are not embeddible into $\omega_1$-trees?

Note that if all trees of size $< 2^{\aleph_0}$ are special (as happens, e.g. under $\text{MA} + \neg \text{CH}$) then, trivially all wide trees of width $< 2^{\aleph_0}$ can be specialized without adding reals. However, as far as I can tell the following is open.

**Question 2.** Suppose that there is a non-special tree of width $\aleph_1$. Does it follow that there is an Aronszajn tree of width $\omega_1$ which cannot be specialized without adding reals?

Note that this is true under $\text{CH}$ as witnessed by the tree $T(E)$. More generally, we can think of this problem in terms of cardinal characteristics.

**Definition 5.1.**

1. $\text{st}$, the special tree number, is the least cardinal $\lambda$ such that there is a non-special $(\omega_1, \leq \lambda)$-Aronszajn tree of cardinality $\lambda$.
2. $\text{no}$, the no new reals number, is the least cardinal $\lambda$ of an $(\omega_1, \leq \lambda)$-Aronszajn tree of cardinality $\lambda$ which cannot be forced to be special without adding reals.
3. $\text{et}$, the embeddible tree number, is the least cardinal $\lambda$ such that there is an $(\omega_1, \leq \lambda)$-Aronszajn tree of cardinality $\lambda$ which cannot be embedded into an $\omega_1$-tree (possibly with branches).

Let’s make some easy observations.

**Observation 5.2.** All three numbers above are uncountable, $\text{st}, \text{et} \leq \text{no} \leq 2^{\aleph_0}$, and $\text{et} \leq \aleph_2$.

**Proof.** That all three numbers are uncountable is essentially by definition. To see that $\text{st} \leq \text{no}$ it suffices to note that any special tree is obviously specializable without adding reals (by trivial forcing). The inequality $\text{et} \leq \text{no}$ is the content of Theorem 1.1. For the upperbound, Todorcevic’s tree $T(E)$ defined above witnesses that there is always a tree of size continuum that cannot be specialized without adding reals. The fact that $\text{et} \leq \aleph_2$ is a cardinality argument. □
Question 3. What provable relations exist between the cardinals introduced above and other cardinal characteristics of the continuum?

Question 4. Are there (in ZFC) trees which can be specialized without adding reals but are not embeddible into $\omega_1$-trees?

The use of forcing notions which specialize wide trees is key in several important applications of PFA including failure of various square principles, and the tree property on $\omega_2$. Therefore a natural question is whether the forcing $P_{T,S}$ can be substituted in these arguments.

Question 5. What other consequences of DCFA (possibly with some additional cardinal arithmetic assumption) can be obtained using $P_{T,S}$? Does DCFA + $\neg$CH imply the tree property on $\omega_2$? Does it imply the failure of weak square on $\omega_1$?

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