Cosmological solutions for a two-branes system in a vacuum bulk

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Abstract. We study the cosmology for a two branes model in a space-time of five dimensions where the extra coordinate is compactified on an orbifold. The hidden brane is filled with a real scalar field endowed with a quadratic potential that behaves as primordial dark matter field. This case is analyzed when the radion effects are negligible in comparison with the density energy; all possible solutions are found by means of a dynamical system approach.

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INTRODUCTION.

Recent research in string theory and its generalization M-theory [1, 2, 3, 4] have suggested that 11 dimensions are necessary to have a consistent quantum string theory. Inherited in these models are p-branes (0 < p < 9) and D-branes, which would be the fundamental constituents of the Universe, as they open the possibility that our visible universe can be a very large D-brane extending over 3 spatial dimensions (see [5] for a pedagogical review).

The main idea of the string theory is that the Standard Model of Particles Physics (SMPP) is made of open strings that are confined on a D-brane, while gravity and other exotic fields, such as the dilaton field, can freely propagate in the extra dimensions (the bulk). This scenario is called brane cosmology, or brane-world cosmology, for which the reduction to 5D of M-theory was first suggested by Horava and Witten [6, 7, 8, 9], providing then the basis for many brane models: Arkani-Dimopoulou-Dvali (ADD) [10], Randall-Sundrum type 1 (RS1), type 2 (RS2)[11, 12], and Dvali-Gabadadze-Porrati (DGP) brane models of 5D gravity [13], among others.

In this paper, using the assumptions of RS1 models and some previous results [14, 15], we focus our attention in generalizations of the solutions found in [16] for a vacuum 5D bulk. This formalism generates a dynamical equation for the Hubble parameter in our brane, \( H_c \), which is closely related with the dynamics of the Hubble parameter in hidden brane, \( H_0 \) [17]; i.e., the fields immersed in the hidden brane have a gravitational influence on the dynamics in the visible brane.

We investigate the behavior of quadratic scalar field at early times with the use of the dynamical brane equations. The main question we want to address is: can the scalar field provide the source of inflation at high energies through brane world dynamics? The behaviour of the field is analyzed by means of dynamical system theory, under the assumption that the radion has negligible effects on the brane equations of motion. In the following we use units in which \( c = \hbar = 1 \).

TWO BRANES EMBEDDED IN A 5D SPACE-TIME.

To start with, the universe will be modeled as a two brane system embedded in a 5-dimensional manifold. The fifth extra dimension is represented by the coordinate \( y \), and then the branes are located at \( y = 0 \) (visible), and \( y = y_c \) (hidden), respectively, and their corresponding matter fields are confined into 4-dimensional hypersurfaces. We write the most general metric in the form

\[
ds^2 = -n^2(t,|y|)dt^2 + a^2(t,|y|)g_{ij}dx^idx^j + b^2(t,|y|)dy^2,
\] (1)
where we have assumed that the two branes are dominated by perfect fluid components which satisfy the following equations of state: $p_0 = \omega_0 \rho_0$, and $p_c = \omega_c \rho_c$, respectively. The energy momentum tensor is written as

$$\tilde{T}_{AB}^b = \tilde{T}_{AB}^a + \frac{\delta(y)}{b_0} \mathrm{diag}(-\rho_0, p_0, p_0, 0) + \frac{\delta(y-y_0)}{b_c} \mathrm{diag}(-\rho_c, p_c, p_c, 0),$$

where the first term corresponds to the bulk contribution and the second and third term corresponds to the branes embedded in the 5D manifold. As usual, the term $\tilde{T}_{AB}^b$ is in the form of a five dimensional cosmological constant, namely

$$\tilde{T}_{AB} = -\frac{\Lambda_5}{\kappa_5} g_{AB},$$

We have shown previously [15] that the general metric that satisfies the 5-dim Einstein’s equations ($G_{AB} = \kappa_5^2 T_{AB}$) in a vacuum bulk is

$$a(t, y) = a_0 \left[ 1 + (m-1) \frac{\kappa_5^2}{6} \rho_0 b_0 y \right]^{1/(1-m)}, \quad (4)$$

$$n(t, y) = n_0 \left[ 1 + \left( \frac{m}{2} + 3 \omega_0 \right) \frac{\kappa_5^2}{6} \rho_0 b_0 y \right] \left[ 1 + (m-1) \frac{\kappa_5^2}{6} \rho_0 b_0 y \right]^{m/(2-2m)}, \quad (5)$$

$$b(t, y) = b_0 \left[ 1 + (m-1) \frac{\kappa_5^2}{6} \rho_0 b_0 y \right]^{m/(2-2m)}, \quad (6)$$

where $\kappa_5^2$ is the 5-dim gravitational constant related with the brane tension $\lambda_0$, and to the 4-dim gravitational constant as $\kappa_4^2 = 6 \kappa_5^2 / \lambda_0$. The functions $a_0, n_0$ and $b_0$ correspond to the time-dependent values of the metric coefficients in the brane at $y = 0$, and $m$ is a parameter which determine the bulk geometry. Substituting Eqs. (4), (5), and (6), into the boundary conditions,

$$\left[ \frac{a'}{a b_0} \right]_0 = -\frac{\kappa_5^2}{3} \rho_0, \quad \left[ \frac{a'}{a b_c} \right]_c = -\frac{\kappa_5^2}{3} \rho_c,$$

we can show that there exists a relationship among the evolution of energy densities, $\rho_c$ and $\rho_0$, in the two branes given by

$$\rho_c = -\rho_0 \left[ 1 + (m-1) \frac{\kappa_5^2}{6} \rho_0 b_0 y c \right]^{(m-2)/(2-2m)}, \quad (8)$$

where we have assumed that the metric coefficients satisfy the mirror symmetry, $[F']_0 = 2 F'|_{y=0}$ and $[F']_c = -2 F'|_{y=y_c}$, and a prime means derivative with respect to $y$. Similarly, using the boundary conditions,

$$\left[ \frac{n'}{n b_0} \right]_0 = \frac{\kappa_5^2}{3} (3 p_0 + 2 \rho_0), \quad \left[ \frac{n'}{n b_c} \right]_c = \frac{\kappa_5^2}{3} (3 p_c + 2 \rho_c),$$

in Eqs. (4), (5), and (6), we obtain that the equations of state (EoS), $\omega_0$ and $\omega_c$, are both related by the expresion

$$\omega_c = \frac{\omega_0 + (\frac{m}{2} + 3 \omega_0) (\frac{m}{2} - 1) \frac{\kappa_5^2}{6} \rho_0 b_0 y c}{1 + (\frac{m}{2} + 3 \omega_0) \frac{\kappa_5^2}{6} \rho_0 b_0 y c}, \quad (10)$$
Assuming a Friedmann-Robertson-Walker (FRW) metric on the visible brane with \( n_c = 1 \), we can write the Hubble parameters, \( H_0 = -\varepsilon \frac{\kappa_0^2}{6} n_0 \rho_0 \) and \( H_c = \varepsilon \frac{\kappa_0^2}{6} n_c \rho_c \) in the two branes as [15]

\[
H_0 = -\varepsilon \frac{\kappa_0^2}{6} \rho_0 \left[ 1 + (m - 1) \frac{\kappa_5^2}{6} \rho_0 b_0 y_c \right]^{-m/(2-2m)},
\]

\[
H_c = -\varepsilon \frac{\kappa_0^2}{6} \rho_0 \left[ 1 + (m - 1) \frac{\kappa_5^2}{6} \rho_0 b_0 y_c \right]^{(m-2)/(2-2m)},
\]

where \( m \neq 1 \) is a parameter which defines the global geometry, and \( \varepsilon = \pm 1 \). A complementary equation to solve the whole system is the conservation equation in the hidden brane

\[
\rho_0 = -3(\rho_0 + \rho_0)H_0 = -3\rho_0(1 + \omega_0)H_0,
\]

where the sign of \( \varepsilon \) is chosen such that we obtain an expanding universe within the visible brane.

In the following sections, we consider the particular case \( m = 0 \). In this scenario, \( b_0 y_c = R \) is a constant and corresponds to a stabilized radius of compactification; this is the most simple case in the family of solutions found in [15].

**SCALAR FIELD DARK MATTER COMPONENT ON THE HIDDEN BRANE.**

Scalar field dark matter (SFDM) is an interesting alternative model to the dark matter problem, that may be able to resolve many conflicting behavior of the standard model Λ-CDM in the formation of structure at different scales. A very useful potential in SFDM cosmology is the quadratic potential, \( V(\phi) = m_\phi^2 \phi^2 / 2 \), where \( m_\phi \) is the scalar field mass, whose value could be as low as \( m_\phi \sim 10^{-22} \text{eV} \) [20, 21]. Our scenario then considers a SFDM dynamics at high energies using the braneworld context, and assuming the topology shown in Eqs. (11)-(12). We investigate what kind of conditions could make the SFDM be the source of inflation and the possible consequences of it in physical observables, i.e. the SFDM coupled with brane gravity can modify the physical observables in inflation (e.g. spectral index, power spectrum, etc.) generating evidence of modified gravity and likely that the SFDM could generate inflation in early universe ages.

Based in the results of the previous section, it is possible to compute that in the limit \( \left| \kappa_0^2 \rho_0 R \right| \ll 1 \), Eqs. (11), and (12), can be written as

\[
H_0 = H_c = -\varepsilon \frac{\kappa_0^2}{6} \rho_0.
\]

Physically, this limit is consistent with an epoch in which the Hubble radius is much larger than radius of compactification, namely \( H_0^{-1} \gg R \). The above equation implies that both branes evolves closely related. If we assume that \( \rho_0 > 0 \), then \( \varepsilon = -1 \) in order to have positive solutions for \( H_\ell \). However, from Eq. (8), we can see that the energy densities in the branes have opposite signs due to the \( Z_2 \) symmetry imposed upon them, i.e. the negative effective energy density is a topological effect due to the mirror symmetry as in the RS case.

Eq. (14) is consistent with the high energy limit in brane-world models, in which, traditionally the energy density is decomposed in two parts: \( \rho_0 = \rho_{m,0} + \Lambda_0 \). Substituting this last ansatz in Eq. (14) and squaring, we have [18, 19]

\[
H_0^2 = \frac{\kappa_0^2}{3} \rho_{0,m} \left( 1 + \frac{\rho_{0,m}}{2\Lambda_0} \right) + \frac{\kappa_0^4}{36} \Lambda_0^2, \tag{15}
\]

where, in the Randall-Sundrum approximation [11, 12], \( \kappa_0^2 \Lambda_0 + \Lambda_5 = 0 \), and for our analysis, \( \Lambda_5 \approx 0 \). In the high energy limit, \( \rho_{0,m} \ll \Lambda_0 \), Eq. (16) becomes

\[
H_0^2 \approx \frac{\kappa_0^2}{6} \rho_{0,m} = \frac{\kappa_0^4}{36} \rho_{0,m}^2. \tag{16}
\]
Now, we consider $\rho_0 = \rho_{0,m}$ is composed only for a real scalar field living in the hidden brane, with
\[
\rho_0 = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m_\phi^2 \phi^2, \quad p_0 = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} m_\phi^2 \phi^2,
\]
and the equation of motion is the Klein-Gordon equation
\[
\ddot{\phi} + 3H_0 \dot{\phi} + m_\phi^2 \phi = 0.
\]

In order to analyze the dynamics of the system, it is convenient to rewrite the equations of motion in terms of the new dimensionless variables:
\[
x^2 = \frac{\kappa_5^2}{12H_0} \dot{\phi}^2, \quad y^2 = \frac{\kappa_5^2 m_\phi^2}{12H_0} \phi^2, \quad s = \frac{m_\phi}{H_0},
\]
where $s = m_\phi/H_0 = m_\phi R_H$ is also directly proportional to the Hubble radius. Then, the dynamical system of our system reads
\[
x' = -3\varepsilon x^3 - 3x - sy, \\
y' = -3\varepsilon x^2 y + sx, \\
s' = -6\varepsilon s x^2,
\]
where a prime indicates derivative with respect to the $e$-foldings number $N \equiv \ln a_0$. In order to have an expanding universe in our brane, we choose $\varepsilon = -1$.

Variables $x$ and $y$ are subjected to the Friedmann constraint $x^2 + y^2 = 1$, and then more appropriate variables are
\[
x = \cos \theta, \quad y = \sin \theta.
\]

Then, the equations of motion (20) can be written as
\[
\theta' = 3\cos \theta \sin \theta + s, \\
s' = 6\varepsilon \sin^2 \theta.
\]

The critical points, together with their stability, of this dynamical system (22), are listed in Table 1. The stability of the critical points was determined by calculating the eigenvalues and eigenvectors of the Jacobian matrix
\[
\mathcal{M}_{(\theta,s)} = \begin{bmatrix}
3 \left( \cos^2 \theta - \sin^2 \theta \right) & 1 \\
-12s \cos \theta \sin \theta & 6\varepsilon \sin^2 \theta
\end{bmatrix}.
\]

**TABLE 1.** Properties of the critical points of the dynamical system (22). All critical points are unstable, but the saddle points are the source for inflationary solutions.

| Critical point $\{\theta,s\}$, $n \in \mathbb{Z}$ | Eigenvalue | Stability |
|-----------------------------------------------|------------|-----------|
| $\{n\pi,0\}$ | $\{6.3\}$ | Unstable |
| $\{(n + \frac{1}{2})\pi,0\}$ | $\{-3.0\}$ | Saddle |

Typical solutions of the dynamical system on the $(\theta,s)$ plane are shown in Fig. 1. We observe that the points $\{\theta = n\pi, s = 0\}$ (where the kinetic energy is dominant) are unstable. However, it is possible to see that the system eventually evolves towards the critical point $\{\theta = \pi/2, s \neq 0\}$ (where the potential energy is dominant). Afterwards, the scalar field shows an oscillatory behaviour at late times, this can be clearly seen in Fig. 2.

For sufficiently small values of variable $s$, the solutions move closely to the saddle points, for which exists a natural exponential expansion in our (visible) brane that satisfies the conditions for inflation, see Fig. 2 and[22]. It is possible to observe that SFDM could be a good candidate not only for dark matter, but the addition of brane world can likewise make it a strong candidate for inflation.
FIGURE 1. Analytical solutions of the dynamical system (22). Note that the point \( \{n\pi,0\} \) are unstable, whereas the points \( \{(n+\frac{1}{2})\pi,0\} \) are saddle. We can see the expansion of the Universe starts in a kinetic dominated epoch, and goes to a potential dominated epoch, and the scalar field eventually ends up in an oscillatory behavior, see also Fig. 2.

CONCLUSIONS

Using previous results, for which the branes are connected by imposing topological constraints and assuming a vacuum bulk, (with the aim of obtain an exact solution for the metric coefficients), we study the behavior of SFDM in the hidden brane and their visible effects in our brane in high energies. Under the consideration that the radion effects are negligible, we analyse the general solutions of the equations of motion. The main results can be enumerated in the following way:

• The universe goes towards different epochs: the first one is dominated by the kinetic energy of the field, the second one is dominated by the potential energy, and the last one is oscillatory.
• The solutions that drive out the inflationary epoch, and that also satisfy the slow roll conditions, are located near the saddle critical points. This is always the case if \( m_\phi \ll H \), which generates an exponential expansion that is typical of inflationary models.
• It is important to remark that it is possible to obtain an expanding solution in the visible brane, as long as the effective (or apparently) energy density \( \rho_c \) is negative. But, it is possible to argue that this is a consequence of the \( Z_2 \) symmetry, as is also the case of RS models.

A more general study should include an analysis of the evolution of the Universe for both the high and low energy regimes. This is work currently under preparation that will be published elsewhere.

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FIGURE 2. Evolution of the system (22) for different values of $\theta$ near the origin of coordinates (color lines); the universe begins at a kinetic dominated point. The inflationary solutions (dashed lines), which satisfy the slow-roll conditions, are located near of the saddle points. The corresponding evolutions of the kinetic $x$ and potential $y$ energies, and of EoS $\omega_0$, with respect to $s$ are also shown. We can notice the oscillatory behavior of all variables at late times.

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