EFFECTS OF RANDOMNESS ON THE ANTIFERROMAGNETIC SPIN-1 CHAIN

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We investigate the effect of weak randomness on the antiferromagnetic anisotropic spin-1 chain. We use Abelian bosonization to construct the low-energy effective theory. A renormalization group calculation up to second order in the strength of the disorder is performed on this effective theory. We observe in this framework the destruction of the antiferromagnetic ordered phase à la Imry-Ma. We predict the effects of a random magnetic field along z axis, a random field in the XY plane as well as random exchange with and without XY symmetry. Instabilities of massless phases appear in general by mechanisms different from the case of the 2-leg spin ladder.

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I. INTRODUCTION

Quantum one-dimensional spin systems are characterized by a variety of behavior much richer than their higher-dimensional counterpart. For example, antiferromagnetic spin chains have different physical properties according to whether the spins are integer or half-integer. Let us consider the prototypical Heisenberg spin chain:

\[ H = J \sum_i S_i \cdot S_{i+1}, \]  

(1.1)

where \( i \) are the sites along the chain and \( S_i \) quantum spin operators with \( S_i^2 = S(S+1) \). The exchange coupling \( J \) is positive, i.e. antiferromagnetic. Then the \( S=1/2 \) chain is gapless and has algebraic decay of the spin correlations \( \langle S_0 \cdot S_n \rangle \simeq (-1)^n/n \) for \( n \to \infty \).

The \( S=1 \) chain has a quite different physics: the isotropic antiferromagnetic chain is in the Haldane phase. There is a gap for spin excitations and the correlations decay exponentially. When adding anisotropy, this Haldane phase survives in a region of parameters around the isotropic point. When anisotropies become too large, then the system is eventually driven towards more conventional ferro or antiferro phases.

This spin-1 chain is closely related to the two-leg spin-1/2 ladder. In the ladder system, one is dealing with two \( S=1/2 \) spin chains coupled by rungs perpendicular to the chains. The ladder has a gap for all values of the perpendicular coupling. In fact, it is known that the \( S=1 \) chain and the two-leg spin ladder share the same low-energy effective field theory.

Understanding the influence of disorder on such spin systems is an important issue. In all real materials there is of course some amount of disorder. One of the probe of the physics of the system is to introduce artificially defects or impurities. The study of disorder in one-dimensional quantum systems is also of considerable theoretical interest. Recently it has been realized that it is possible to obtain exact results in such disordered systems, while this is generally impossible in the classical realm.

In the case of the spin-1/2 chain, one simple kind of disorder is a distribution of quenched bond disorder. The real-space renormalization group study shows that the ground state is a collection of singlet bonds over arbitrarily large distances in a random pattern according to the initial distribution of randomness. This is the so-called random singlet phase. The influence of weak disorder on the spin-1/2 chain has been also studied by means of a renormalization group (RG) study on the effective field theory. This method is a perturbative RG calculation but it is easy to study widely different kinds of disorder. The real-space RG method has also been applied to systems with a distribution of ferro- and antiferromagnetic bonds as well as dimerized systems. There are also direct numerical studies.

In the case of the spin-1 chain we are dealing with a gapped system that has also some hidden long-range order as exhibited by the approximate valence-bond-solid (VBS) ground state wavefunction. Certainly one expects the influence of disorder to be quite different. A real-space RG has been performed on the isotropic \( S=1 \) chain. It shows that there is destruction of the Haldane gap beyond a nonzero disorder strength, then there is a gapless phase.

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which sustains the hidden long-range order and then for even stronger disorder one finds the featureless random-singlet phase. This is analogous to the doping of conventional superconductors with magnetic impurities[3]: here the hidden long-range order plays the role of the superconducting condensate.

In this paper we study the S=1 spin chain with exchange and single-ion anisotropies. This model has a rich phase diagram and the influence of disorder is extremely sensitive to the nature of the ground state of the unperturbed system. We use a perturbative RG treatment of the disorder on the effective field theory. The average over the disorder is performed by use of the replica trick as pioneered by Giamarchi and Schulz[4] in the context of the one-dimensional electron gas. Here we perform the calculation up to second order in the disorder strength. This is important as shown by Fujimoto and Kawakami[5] because, at this order of perturbation, one captures the interplay between interactions and disorder when they are both relevant. In the spin-1/2 case they have shown that it is possible to explicitly obtain from the RG treatment the destruction of the antiferromagnetic phase under a random z-axis magnetic field for an arbitrarily small amount of randomness as predicted by the argument of Imry and Ma[6]. The case of the spin ladder[7] is closely related to our work but the relevant operators are not always the same. In the spin-1 chain there are operators generated by disorder that involve products of the two spin-1/2 operators used to describe the spin-1 degrees of freedom. They do not exist in the ladder case.

In section II, we give the construction of the effective bosonic theory for anisotropic S=1 chain without randomness. In section III, we translate in the bosonic language, the operators induced by the disorder. Section IV shows why gapped phases are in general stable under weak disorder. Section V contains the derivation of the renormalization group equations. Section VI gives the results of our study on each phase of the S=1 chain. Finally section VII contains our conclusions.

II. THE PURE S=1 SPIN CHAIN

In this section we give the construction of the effective Bosonic low-energy theory for the S=1 spin chain. We will use this continuum theory as a starting point for the treatment of randomness in sect. III. We concentrate on the anisotropic Heisenberg S=1 antiferromagnetic spin chain with the following Hamiltonian:

\[ H = \sum_i S^x_i S^x_{i+1} + S^y_i S^y_{i+1} + \Delta S^z_i S^z_{i+1} + D(S^z_i)^2. \]  

(2.1)

The \( S^\alpha_i \) are S=1 quantum spin operators, \( \Delta \) is the exchange anisotropy and \( D \) is the single-ion anisotropy. All energies are measured in units of \( J \). The phase diagram of this model is well-known in the absence of disorder[8]. There are two phases with conventional long-range magnetic order: a ferromagnetic (F) phase for \( \Delta \) large and negative and an antiferromagnetic (AF) phase for \( \Delta \) large and positive. In the intermediate regime, there is the Haldane phase (H) which is gapped and has no obvious long-range order. In particular it does not break translational or rotational symmetry. There are two gapless XY-like phases that differ only in the behaviour of the spin correlations. Finally, when \( D \) is very large and positive there is a non-magnetic phase which is trivially obtained by perturbation theory from the strong-coupling limit where the ground state has all spins set to \( S^z_i = 0 : |000\ldots00\rangle \), called the large-D phase (D) in what follows.

The starting point is to write each spin-1 operator as the sum of two spin-1/2 operators \( \sigma^\alpha_i = \sigma^\alpha_i + \tau^\alpha_i \), an approximation suggested by Luther and Peschel[9]. It has been shown by Schulz that Abelian bosonization[10] is able to reproduce most of the phase diagram. One first performs a Jordan-Wigner transformation on each spin-1/2 \( \sigma \) and \( \tau \) by introducing two species of spinless fermions. Explicitly we write:

\[ \sigma^+_n = (-)^n c^+_n e^{i\pi \sum_{m<n} c^+_m c_m} , \]  

(2.2a)

\[ \sigma^-_n = (-)^n c_n e^{i\pi \sum_{m<n} c^+_m c_m} , \]  

(2.2b)

\[ \sigma^z_n = c^+_n c_n - \frac{1}{2} , \]  

(2.2c)

where \( c \) is a fermion operator and similarly one introduces a \( d \) fermion for the other set of spins-1/2 \( \tau^\alpha_i \). Since one is interested in the long-wavelength, low-energy behaviour, it is then convenient to take the continuum limit by introducing two continuous Fermi fields \( \psi_c(x) \) and \( \psi_d(x) \). The Hamiltonian (2.1) leads then to a theory of two interacting Fermi fields in one space dimension. The physical content of this Fermi system is obtained by translation into the language of interacting Bosons[11]. In fact, in one space dimension a theory involving a Fermi field \( \psi(x) \) can be translated in a Bose field theory by the following relations:
where $a_L$ and $a_R$ are the chiral components of a Bose field: $a_{L,R} = 1/2(\phi \mp \int x \Pi(x') dx')$. Thus there are two Bose fields $\phi_a$ and $\phi_d$ corresponding to the two fermions $\psi_c$ and $\psi_d$. The effective theory for the spin-1 chain can be simplified by use of the natural “acoustic” and “optic” linear combinations:

$$\phi_a = \frac{1}{\sqrt{2}} (\phi_c + \phi_d) \quad \text{and} \quad \phi_o = \frac{1}{\sqrt{2}} (\phi_c - \phi_d).$$

The Bose theory can then be written as:

$$\mathcal{H} = \mathcal{H}_a + \mathcal{H}_o,$$

where the a-sector is a sine-Gordon theory:

$$\mathcal{H}_a = 2\left[ K_a (\Pi_a^x) + \frac{1}{K_a} (\nabla \phi_a)^2 \right] + \frac{g_1}{(\pi \alpha)^2} \cos(\sqrt{8\pi \phi_a}),$$

and the o-sector is a generalized sine-Gordon theory:

$$\mathcal{H}_o = 2\left[ K_o (\Pi_o^x) + \frac{1}{K_o} (\nabla \phi_o)^2 \right] + \frac{g_2}{(\pi \alpha)^2} \cos(\sqrt{8\pi \phi_o}) + \frac{g_3}{(\pi \alpha)^2} \cos(\sqrt{2\pi \phi_o}).$$

Here $\phi_o = \phi_o^L - \phi_o^R$ is the dual field of $\phi_o$, and the couplings are given by $g_1 = g_2 = D - \Delta$, $g_3 = -1$ at first order in $D, \Delta$. The velocities $v_a, v_o$ and the two parameters $K_a, K_o$ are also functions of the initial parameters of the problem and can be computed to the first order in $D$ and $\Delta$:

$$v_a = 1 + \frac{3\Delta + D}{\pi}, \quad v_o = 1 + \frac{\Delta - D}{\pi}, \quad K_a = 1 - \frac{3\Delta + D}{\pi}, \quad K_o = 1 - \frac{\Delta - D}{\pi}.$$
III. DISORDER IN THE BOSONIC LANGUAGE

In this section we describe an approximate treatment of quenched randomness in the Bosonized theory obtained in sect. II. We give the operators that should be added to the Bosonic field theory (2.5) to describe randomness. We first list the various kinds of randomness that we have been able to treat. We will consider the effects of a random field along the z-axis. We add to the Hamiltonian a term:

$$H_{ZF} = \sum_i h_i S_i^z,$$

where the local fields $h_i$ are Gaussian random variables with zero mean, uncorrelated from site to site:

$$\overline{h_i h_j} = \mathcal{D}_{ZF} \delta_{ij}.$$  \hspace{1cm} (3.2)

The bar means average over the random distribution and the variance of the distribution is $\mathcal{D}_{ZF}$, characterizing the strength of the disorder. This kind of disorder preserves the rotational symmetry around the z-axis which a symmetry of the spin hamiltonian. It is known from the Imry-Ma argument\cite{20} that it has a dramatic effect on the phases with long-range magnetic order: an arbitrarily small amount of random z-field destroy the ordering. We consider also the case of a random planar field, i.e. lying in the XY plane:

$$H_{PF} = \sum_i h_x S_x i + h_y S_y i,$$

with $h_\alpha i h_\beta j = \mathcal{D}_{PF} \delta_{ij} \delta_{\alpha\beta}$. The rotational symmetry is destroyed in this case. We treat random planar exchange :

$$H_{PE} = \sum_i J_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y),$$

with $\overline{J_i J_j} = \mathcal{D}_{PE} \delta_{ij}$. Finally randomness may break XY-symmetry by random planar anisotropy :

$$H_{PA} = \sum_i \gamma_i (S_i^x S_{i+1}^x - S_i^y S_{i+1}^y),$$

with $\overline{\gamma_i \gamma_j} = \mathcal{D}_{PA} \delta_{ij}$.

When bosonizing the spin operators in Eqs. (3.1,3.3,3.4,3.5), there is one part with small momenta $q \approx 0$ and one part with $q \approx \pi$. For example, in the ZF case we have:

$$S^z(x,t) = -\sqrt{\frac{2}{\pi \alpha}} \partial_x \phi_a + \frac{(-)^x}{\pi \alpha} \cos(\sqrt{2\pi \phi_a}) \cos(\sqrt{2\pi \phi_a}),$$

in the continuum limit. Inserted in Eq. (3.1) this formula will pick the $q \approx 0$ part of the fluctuating field $h_i$ as well as the $q \approx \pi$ part :

$$H_{ZF} = \int dx h^{q=0}(x) O_1(x,t) + h^{q=\pi}(x) O_2(x,t),$$

with:

$$O_1 = -\sqrt{\frac{2}{\pi \alpha}} \partial_x \phi_a(x,t) \quad \text{and} \quad O_2 = \cos(\sqrt{2\pi \phi_a}) \cos(\sqrt{2\pi \phi_a}).$$

In fact this formula is generic for all kinds of disorder. We have listed in Table II the operators $O_1$ and $O_2$ corresponding to all cases cited above. We have kept only the most relevant operators in the renormalization group sense. The continuous random fields $h^{q=0}(x)$ and $h^{q=\pi}(x)$ are also Gaussian. To treat the quenched disorder we follow the replica trick: we introduce $n$ copies of the system and compute the free energy as $F = \lim_{n \to 0} \{Z^n - 1\}/n$. Integrating over the Gaussian disorder $h^{q=0}(x)$ leads in the Euclidean effective action to the following operator:

$$S_{eff} = -\mathcal{D}_1 \int dx dt d\tau' \sum_{i,j} O_1^{(i)}(x,\tau) O_1^{(j)}(x,\tau'),$$

\hspace{1cm} (3.9)
where $D_i$ is the variance of $h^{q=0}(x)$, $i, j$ are replica indices. It is important to note that this operator is not local: since the random field $h^{q=0}(x)$ is quenched i.e. does not depend on time, the Gaussian integration leads to the two-time integral in Eq.(3.9). We obtain exactly similar formulas for all operators given in table II. In each case there is a variance $D$ corresponding to the Gaussian random field coupled to the operator considered. We will use the obvious notation $ZF(0), ZF(\pi), PF(0), \ldots$. The $ZF(0)$ operator corresponds to a forward scattering process in the language of the Jordan-Wigner fermions. When written as a Bosonic operator it may be absorbed by a change of variable in the path integral over the field $\phi_a$ which leads to simple leading-order RG calculations when interactions in the $a$-sector are irrelevant. Here we will retain this $ZF(0)$ operator explicitly in our RG treatment because, when its strength diverges, it leads to a very important phenomenon: the vanishing of the critical disorder strength for massive phases.

It is important to note that the most relevant $q = \pi$ operators are, in all cases over study, always given by $ZF(\pi)$. This is due to the fact that even when $ZF(\pi)$ does not appear at the bare level, it is generated by renormalization: this has been pointed out first by Doty and Fisher in the study of the $S=1/2$ chain. In our case a similar reasoning leads immediately to the appearance of $ZF(\pi)$ in the PF case as well as in the PA case (in the ZF and PE case, $ZF(\pi)$ appear at the bare level).

The operators $PA(0)$ and $PE(0)$ are specific to the $S=1$ spin chain as opposed to the spin ladder: they come from the cross coupling $\sigma_i \cdot \tau_{i+1}$ that occurs when expressing $S=1$ spin operators into $S=1/2$ entities. In the spin ladder it would be replaced by $\sigma_i \cdot \tau_i$ across the rung.

The operators of the type $(\ref{eq:4.7})$ break the Lorentz invariance which is present in the effective field theory of the pure system. This means that in a renormalization group calculation many new operators will be generated in addition to those introduced at the bare level. In principle it is thus extremely difficult to keep the RG flow under control. In the RG calculation at second order that we performed, we have to add for internal RG consistency only three new operators:

\begin{align}
S_{a1} &= -D_a \int dx \, dt \, d\tau' \sum_{i,j} \partial_\tau \phi_a^i(x, \tau) \partial_\tau \phi_a^j(x, \tau'), \\
S_{a2} &= -D_a \int dx \, dt \, d\tau' \sum_{i,j} \partial_\omega \phi_a^i(x, \tau) \partial_\omega \phi_a^j(x, \tau'), \\
S_o &= -D_o \int dx \, dt \, d\tau' \sum_{i,j} \partial_\omega \phi_a^i(x, \tau) \partial_\tau \phi_a^j(x, \tau'),
\end{align}

where $z = x + v_0 \tau, \tau = x - v_0 \tau, w = x + v_0 \tau$ and $\bar{w} = x - v_0 \tau$. They are called $a1, a2$ and $o$ in what follows.

**IV. REINORMALIZATION GROUP TREATMENT**

In this section we perform a second-order RG treatment of the Bosonic theory given by the field theory Eq. (2.3) perturbed by the operators of the type (3.9). There are five distinct bare operators ($ZF(0), ZF(\pi), PF(0), PE(0), PA(0)$) and three generated by renormalization ($a1, a2, o$). The perturbation theory can start only from the massless theory with no cosine operators $g_1 = g_2 = g_3 = 0$ in Eq. (2.3). We write the effective theory including randomness as:

$$S = S^* + \sum_i a^{x_i-2} g_i \int dx \, dt \, O_i,$$

where $S^*$ is the massless theory characterized by $K_a$ and $K_o$, $a$ is the cut-off and $x_i$ is the scaling dimension of the operator $O_i$. The beta functions of the couplings $g_i$ are then given by the following formula:

$$\frac{dg_i}{dt} = (2 - x_k) g_k - \pi \sum_{i,j} C_{ijk} g_i g_j + O(g^3),$$

where the Wilson coefficients $C_{ijk}$ are obtained from the Operator Product Expansion:

$$O_i(x, \omega) \sim \sum_k \frac{1}{|x - x_i|^z} C_{ijk} O_k(\omega).$$

This procedure gives the beta function for an arbitrary number of replicas and is very useful in order to find which operators are generated by renormalization. While this technique can be applied straightforwardly to local operators
like those appearing in the sine-Gordon or generalized sine-Gordon theories, this is not so when dealing with the non-local operators \([3,9]\) generated by randomness. We will take into account only the strongest singularities (which are in \(1/|\text{Re} z|\)) in the OPE containing nonlocal operators, and assume that the OPE is still valid.

We now sum up the renormalization group equations of the system. In the replica \(n \to 0\) limit, we obtain the flow for the cosine couplings:

\[
\frac{dy_1}{dl} = 2(1 - K_o)y_1 - \frac{1}{8\pi^2}K_o^2y_1(2D_{ZF}^{(0)} + 2D_{a1} + D_{a2}),
\]

\[
\frac{dy_2}{dl} = 2(1 - K_o)y_2 - \frac{1}{8\pi^2}K_o^2D_o y_2,
\]

\[
\frac{dy_3}{dl} = (2 - \frac{1}{2K_o})y_3 - \frac{1}{32\pi^2}D_o y_3,
\]

where \(y_1 = g_1/v_o, y_{2,3} = g_{2,3}/v_o\). In the first equation (4.4a), in the absence of disorder the renormalization eigenvalue \(2(1 - K_o)\) leads to a massive phase when \(K_o < 1\) : the cosine is relevant since \(y_1\) scales to infinity in this case. This picture may be altered since the randomness leads to an additional second-order coupling \(-y_1 \times D_{ZF}^{(0)}\) between the strength of the interaction and the forward scattering term \(ZF^{(0)}\). If \(D_{ZF}^{(0)}(1)\) scales fast enough to infinity then it can revert the flow of \(y_1\) and leads to the destruction of the massive phase. This behaviour has been obtained in the case of the Hubbard model by Fujimoto and Kawakami.\(^{11}\) We will show later that this happens also in the case of the spin chain under study. Of course to capture such phenomena, it is important to deal explicitly with \(ZF^{(0)}\) instead of eliminating it by a change of variables in the functional integral.

There are also two flow equations for the stiffness constants:

\[
\frac{dK_o}{dl} = -(D_{ZF}^{(\pi)}) - \frac{2}{\pi^2}g_1^2K_o^2 + \frac{1}{4}D_{PF}^{(0)} + D_{PA}^{(0)},
\]

\[
\frac{dK_o}{dl} = -(D_{ZF}^{(\pi)}) - \frac{2}{\pi^2}g_2^2K_o^2 + \frac{1}{4}D_{PF}^{(0)} + D_{PA}^{(0)} + \frac{1}{\pi^2}g_4^2,
\]

where \(u = v_o/v_o\). The operators that do not involve gradients have simple renormalization properties. In fact they scale according to the dimension that can be found immediately from the massless theory:

\[
\frac{dD_{ZF}^{(\pi)}}{dl} = (3 - K_o - K_o)D_{ZF}^{(\pi)},
\]

\[
\frac{dD_{PF}^{(0)}}{dl} = (3 - \frac{1}{4K_o} - \frac{1}{4K_o})D_{PF}^{(0)},
\]

\[
\frac{dD_{PF}^{(0)}}{dl} = (3 - \frac{1}{K_o})D_{PF}^{(0)},
\]

\[
\frac{dD_{PA}^{(0)}}{dl} = (3 - \frac{1}{K_o})D_{PA}^{(0)}.
\]

These equations allow immediate statements about the stability of the massless phases according to the relevance or irrelevance of the random operators. This simplicity however does not extend to the operators \(ZF^{(0)}, a1, a2\) and \(o\) because they appear in the OPE of various combinations of the other operators. The RG equations thus couple the various random perturbations:

\[
\frac{dD_{ZF}^{(0)}}{dl} = D_{ZF}^{(0)} + \pi(\frac{1}{4}f(u)D_{ZF}^{(\pi)} - \frac{1}{8}\delta(u)D_{PF}^{(0)} - \frac{1}{2}hD_{PA}^{(0)}),
\]
Defining $y$, $D$ and $K$ will appear in the study of the Ising-ordered phase. 

up to some critical disorder which, strictly speaking, is outside of reach of perturbation theory. A formal proof of the system under weak disorder at this order of perturbation theory. It is clear that the massive phase will persist arbitrarily weak random $z$-field, as predicted by the Imry-Ma argument.

will show in the following section that these divergences lead to the disappearance of the Ising-ordered phase for an

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region

K

disorder will not affect this behavior. The value of $K_o$ pure case corresponds to a regime where interactions are relevant, the presence of infinitesimally small (but finite) small initial disorder, the disorder operator will become irrelevant, whereas the interaction operator is relevant. If the

improved by use of the renormalization group then we can’t say much about the flow to strong coupling. The phases that are unstable cannot be characterized by our tools. We can only find whether or not the randomness will have an immediate effect on the system. Even in the stable case, we of course expect that for strong enough disorder the physics of the system will change through a phase transition of some kind. This strategy has been applied with some success in the $S=1/2$ chain in a random $z$-field ans in the related problem of 1D fermions with attractive interactions submitted to a random site potential. Bosonization predicts stability of the XY phase of the $S=1/2$ chain in the region $-1 < \Delta < -1/2$ and there is ample evidence for the correctness of this result from direct numerical studies. Our general study should be followed by attacks by similar methods for an independent consistency check.

\begin{align*}
\frac{dD_o}{dl} &= D_o + \pi \left\{ \frac{1}{4u} f(u) D^{(2)}_{ZF} - \frac{1}{8u} g(u) D^{(0)}_{P F} - \frac{1}{2u} k(u) D^{(2)}_{P F} \right\}, \tag{4.7b}
\end{align*}

Equations similar to (4.7) hold also for the couplings $D_{a1}$ and $D_{a2}$. The second-order terms in the flow equations above involve the following functions:

\begin{align*}
&f(u) = \left[ \int_{-\infty}^{+\infty} \frac{dy}{(1 + y^2) K_{a1} / 2} \right]^2, \quad g(u) = \left[ \int_{-\infty}^{+\infty} \frac{dy}{(1 + y^2)^{1/8} K_{a1} / 2} \right]^2, \tag{4.8a} \\
&h = \left[ \int_{-\infty}^{+\infty} \frac{dy}{(1 + y^2) K_{a2} / 2} \right]^2, \quad k(u) = \left[ \int_{-\infty}^{+\infty} \frac{dy}{(1 + y^2)^{1/2} K_{a2}} \right]^2. \tag{4.8b}
\end{align*}

\begin{align*}
S_{\text{eff}} &= \frac{1}{2K} \int dx \, d\tau (\partial_{\phi} \phi)^2 + \frac{g}{(\pi \alpha)^2} \int dx \, d\tau \cos(\sqrt{4\pi a^2} \phi(x, \tau)) - \frac{d}{(\pi \alpha)^2} \int dx \, d\tau \, d\tau' \cos(\sqrt{4\pi b^2} \phi(x, \tau)) \cos(\sqrt{4\pi b^2} \phi(x, \tau')). \tag{4.9}
\end{align*}

Defining $y = g/\pi u$, and $D = d\alpha/2\pi u^2$, where $u$ is the velocity, we derive the following renormalization group equations for $y$, $D$ and $K$:

\begin{align*}
\frac{dy}{dl} &= (2 - a^2 K) y, \tag{4.11a} \\
\frac{dD}{dl} &= (3 - b^2 / K) D, \tag{4.11b} \\
\frac{dK}{dl} &= -y^2 K^2 + D. \tag{4.11c}
\end{align*}

The corresponding RG flow is given in Figure 2 when $(ab)^2 < 6$. Here, the crucial parameter is the stiffness constant $K$, because it governs the relative relevance of interaction and disorder operators. For instance, if $K$ flows to zero for small initial disorder, the disorder operator will become irrelevant, whereas the interaction operator is relevant. If the pure case corresponds to a regime where interactions are relevant, the presence of infinitesimally small (but finite) disorder will not affect this behavior. The value of $K^*$ in the pure case is enough to predict the qualitative behavior of the system under weak disorder at this order of perturbation theory. It is clear that the massive phase will persist up to some critical disorder which, strictly speaking, is outside of reach of perturbation theory. A formal proof of stability is given in appendix A. The flow equations (4.11) are only first-order. The interest of our second-order study will appear in the study of the Ising-ordered phase.

Finally, we comment on the correctness of the bosonization approach. Since the scheme we use is perturbation improved by use of the renormalization group then we can’t say much about the flow to strong coupling. The phases that are unstable cannot be characterized by our tools. We can only find whether or not the randomness will have an immediate effect on the system. Even in the stable case, we of course expect that for strong enough disorder the physics of the system will change through a phase transition of some kind. This strategy has been applied with some success in the $S=1/2$ chain in a random $z$-field ans in the related problem of 1D fermions with attractive interactions submitted to a random site potential. Bosonization predicts stability of the XY phase of the $S=1/2$ chain in the region $-1 < \Delta < -1/2$ and there is ample evidence for the correctness of this result from direct numerical studies. Our general study should be followed by attacks by similar methods for an independent consistency check.

\section*{V. PHASE DIAGRAM IN THE PRESENCE OF DISORDER}

In this section, we exploit the RG flow equations obtained previously to discuss the influence of randomness on the phase diagram of the $S=1$ spin chain phase diagram pictured in Fig. (1). We begin our discussion by the massive phases, with and without long-range spin order. All our findings are summarized in Table III. The key element of the stability is the operator $ZF^{(\pi)}$ since it is present in all kinds of randomness.
A. AF phase

This phase is characterized by the following IR stiffness constants in the pure case: \( K_a^* = K_o^* = 0 \). For such a fixed point, the functions appearing at second-order in Eqs. (4.7) take the values \( g(u) = h = k(u) = 0 \) while the function \( f \) is singular. The \( Z^F(\pi) \) operator is relevant in this phase. Since the cosine interaction is relevant in the a-sector, it seems that we are in a case with interaction and disorder that are relevant. However, the second-order beta functions allow a more precise conclusion: when \( K_a^* \) and \( K_o^* \) flow to a small enough value then \( f \) diverges. When approaching this singular point, \( D_{ZF}^{(0)} \) becomes arbitrarily large even for infinitesimally small values of \( D_{ZF}^{(0)}(l = 0) \) and \( D_{ZF}^{(0)}(l = 0) \). In Eq. (4.4a), the coupling \(-y_1 \times D_{ZF}^{(0)} \) becomes so large that the interaction \( y_1 \) is driven to zero. We infer thus that the Mott-like gap of the Ising phase is immediately destroyed by disorder, i.e. by infinitesimal disorder.

In the case of a random \( z \)-field, this is really what we predict from the Imry-Ma reasoning. In the case of the random planar field and random planar anisotropy, the random \( z \)-field is generated by the RG process so this is why there is also destruction of the Ising phase. The only trouble with the bosonization treatment is that it also leads to instability under random planar exchange, a fact which is not expected from the Imry-Ma argument. Since stability in this case is expected we see this as a problem of our approach (if we apply the same treatment to the \( S=1/2 \) case then one finds also instability under random planar exchange of the corresponding Ising phase). However this approach correctly reproduces the immediate destruction of the gap in the Ising phase.

B. Haldane and large-D phases

In the renormalization group equations, nothing distinguishes the H and D phases. So we expect that they behave in the same way under disorder. These phases are characterized by \( K_a^s = 0, K_o^s = \infty \), and \( \phi_a \) has a vacuum expectation value. The function \( f \) is now well defined when we reach the unperturbed fixed point, and near this fixed point, \( f(u) = 0 \). The operators \( ZF(\pi) \), \( PF(0) \) and \( PA(0) \) are irrelevant. This implies immediately stability under random XY symmetry-breaking exchange. Concerning the random fields, we note that the random forward scattering \( D_{ZF}^{(0)} \) scales as \( e^l \) while \( y_1 \) and \( y_3 \) scale faster as \( e^{2l} \) according to Eqs. (4.7a,4.4a,4.4c). Thus there is no possibility of destruction of the role of interactions contrary to the case of the AF phase. Thus there is stability under random \( z \)-field as well as under random XY-field.

The only special case is the random XY-symmetric coupling. The \( PE(0) \) operator is relevant and the function \( k(u) \) is singular. This drives the \( o \)-operator to large negative values of \( D_o \). This goes outside the reach of perturbation theory. But for such large negative values, \( y_3 \) scales much faster than the disorder like \( e^{D_o l} \). We take this as an indication of the robustness of the gap against disorder. So we expect stability also in this case. The two gapped phases without long-range spin order are thus stable, up to some critical strength of the disorder presumably, which is beyond the reach of the methods we employ here.

C. XY1 phase

In this phase, the a-sector is gapless, and the \( ZF(\pi) \) operator is irrelevant near the unperturbed fixed point in the whole phase so the spectrum can appear only through the other operators induced by randomness.

- Random \( z \)-field: the forward scattering \( ZF(0) \) operator can be absorbed by a field redefinition. Thus the excitation spectrum is unchanged and the correlation functions are affected in a simple way.

- Random XY field: the \( PF(0) \) operator is relevant; so the phase is unstable. In this case, we note that the function \( g \) is singular, so the \( ZF(0) \) operator is singular too and scale to \(+\infty\).

- Random XY symmetric coupling: the \( PE(0) \) operator is relevant, as in the Haldane/large-D phase. The same discussion apply: the stability of the \( o \)-sector lead us to conclude that the XY1 phase remains stable under this kind of disorder.

- Random XY symmetry-breaking coupling: The \( PA(0) \) operator is relevant and the a-sector is not gapped; so the phase is unstable. The operator leading to the instability is different from the ladder case because it involves \( \sigma\tau \) couplings that are typical of the S=1 case.
D. Frontier line between Haldane and large-D phases

This line is characterized by \( y_1 = 0 \), so that the field \( \phi_0 \) is a free massless field. There is no gap along this line in the \( a \)-sector. It can be seen as an extension of the XY1 phase. Nevertheless, \( K_a \) is renormalized by disorder and \( K_o \) flows to infinity. Thus, the function \( g \) is equal to zero, whereas \( h \) is either finite or singular according to the value of \( K_a \). The stability discussion is exactly the same as that of the XY1 phase apart from the random XY field. The novelty is that \( K_a \) may be less than 1 but the \( a \)-sector is still massless so there is a transition line in the \( D_{ZF}^{(0)} - K_a \) plane, corresponding to a critical value of \( (K_a)_c = 1/12 \). For \( K_a > (K_a)_c \), the line we consider is unstable under weak PF disorder. The line starts from the XY1 boundary for which \( K_a = 1 \) and crosses the frontier line between AF and H phases. The stiffness \( K_a \) decreases along the line but we do not know if it reaches the value 1/12 before arriving at the tricritical point. We conclude that the line is unstable under a random XY field at least in the neighborhood of the XY phase.

E. XY2 phase

This phase is characterized by the following IR values in the pure case : \( K_a^* > 1 \) and \( K_o^* = 0 \). We are in a most interesting case in which the operator \( ZF^{(\sigma)} \) may be either relevant or irrelevant within the bulk of this phase according to the scaling equation \((4.6a)\). We first consider the influence of the random \( z \)-field : we just need to consider the \( a \)-sector and the operator \( ZF^{(\sigma)} \), which simplifies the calculation and does not change the result (the random forward scattering \( ZF^{(0)} \) does not change the global picture). The effective renormalization group equations are then at lowest order (since the function \( g \) is not singular near the IR pure fixed point, nothing new happens at the next order) :

\[
\frac{dy_1}{dl} = 2(1 - K_a)y_1, \tag{5.1a}
\]

\[
\frac{dD_{ZF}^{(\sigma)}}{dl} = (3 - K_a)D_{ZF}^{(\sigma)}; \tag{5.1b}
\]

\[
\frac{dK_a}{dl} = -(D_{ZF}^{(\sigma)} + 2\pi^2 y_1^2)K_a^2. \tag{5.1c}
\]

In each plane \( y_1 = 0 \) and \( D_{ZF}^{(\sigma)} = 0 \), the flow has a simple form. We can draw the corresponding renormalization flow in the three dimensional space : it is given in Figure 3. We observe that the \( ZF^{(\sigma)} \) operator breaks the XY2 phase into two phases. The first one (for large \( K \)) is a massless phase stable under small disorder, and the interaction term also flows to zero; the other phase is unstable under weak disorder. Thus the XY2 phase is only partially stable under small disorder, and this is consistent with physical intuition. Indeed, when the single-ion \( D(S_i^z)^2 \) term becomes large and negative, we expect that the \( S=1 \) spin chain will behave as the \( S=1/2 \) spin chain. As shown by Doty and Fisher for the \( S=1/2 \) spin chain, there is a region of exchange parameter \( \Delta \) stable under weak disorder (it is also a superfluid phase arising for the disordered boson gas; see Giamarchi and Schulz). In the \( S=1/2 \) spin chain, this stability region is located in the interval \(-1 < \Delta < -1/2\). In the \( S=1 \) case, we find that such a stable phase also exists for \( K \) large enough, and this is consistent with the fact that we should recover the \( S=1/2 \) behavior.

In the case of a random \( XY \) field, since there is a random \( z \)-field generated we are in the same situation as above. For the random \( XY \) symmetric coupling, the \( PE^{(0)} \) operator is irrelevant and the above discussion is again valid.

In the random \( XY \) symmetry-breaking case, the \( PA^{(0)} \) operator is always relevant whereas the \( a \)-sector is ungapped. Thus, the chain is unstable. This is again consistent in the limit of large negative \( D \) with the Doty and Fisher results for the \( S=1/2 \) chain.

Finally, we briefly comment the case of random \( z \)-exchange that we have not considered yet in this paper. In terms of the \( S=1 \) spins, it is given by : \( \mathcal{H}_{ZF} = \sum J_{ij} S_i^z S_{i+1}^z \). When written with the two kinds of spins \( S=1/2 \), terms of the form \( \sigma \tau \) appear. They lead to \( \cos(\sqrt{8\pi} \phi_0) + \cos(\sqrt{8\pi} \phi_0) \) when expressed in the boson language, a coupling that has not been studied in the context of spin ladders. The \( \cos(\sqrt{8\pi} \phi_0) \) leads to an operator which is relevant in Haldane phase, and make the forward scattering \( ZF^{(0)} \) singular and infinite with the consequence of vanishing interactions in the \( a \)-sector. The AF phase is also unstable : the \( ZF^{(\sigma)} \) operator is also contained in the boson expression of this kind of disorder, as if a random \( z \)-field was generated. These instabilities clearly deserve more studies. This problem appears also in the \( S=1/2 \) case. The XY1 phase is stable while XY2 is unstable.
VI. CONCLUSION

We have performed a renormalization group study of the influence of disorder on the effective theory which describes the S=1 chain. The RG equations allow a discussion of the stability of the various phases of the system. Our discussion includes the effects of the second order of the renormalization group calculation. This allows us to capture the Imry-Ma destruction of the AF phase.

The gapped Haldane and large-D phases are stable under all kinds of disorder we have studied: random z-field, random XY field and random XY symmetric and antisymmetric exchange. On the contrary, the AF phase which is also gapped is less stable. In fact, our RG calculation that goes up to the second order in the disorder strength shows that the phase is unstable to an arbitrarily weak random z-field, in agreement with the Imry-Ma argument. It is also unstable under random XY-field and random planar anisotropy since these perturbations do generate a random z-field.

Concerning the gapless XY phases, then again the situation is quite rich. The XY1 phase is stable under random z-field and random planar exchange while it is immediately unstable under perturbations breaking the planar symmetry: random XY-field and planar anisotropy. The XY2 phase is only partially stable under random fields and random planar exchange. In this case the phase breaks up into two parts: there is a stable massless domain with irrelevant disorder and another domain with relevant disorder. Finally XY2 is totally unstable under random planar anisotropy.

Of course it remains to characterize more completely the phases in which the disorder is relevant. In our calculation, we can simply observe that the system flows to strong coupling but the methods we use are not informative on its fate. An outstanding problem is to describe in a unified manner the weak-coupling regime which we observe here with the regime probed by the real-space RG calculations. The gapless phase with hidden long-range order of the random spin-1 chain appears presumably only beyond some critical strength and it seems to be out of reach of the methods we have used in this paper.

Note added: A recent work by Y. Nishiyama on the effect of a random z-field on the Haldane gap and the XY1 phase is in agreement with our findings (see e-print cond-mat/9805110).

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APPENDIX A: STABILITY OF A GAPPED PHASE

The pure SG system is gapped when $K \to 0$. We call $K^0$ and $y^0$ a solution of the pure differential system. We set $K = K^0 + \delta K$ and $y = y^0 + \delta y$. A straightforward calculation leads to:

\begin{align*}
\delta K' &= D - 2y^0(K^0)^2 \delta y - 2(y^0)^2 K^0 \delta K \\
\delta y' &= (2 - a^2 K^0) \delta y - a^2 y^0 \delta K
\end{align*}

(A1) (A2)

From the flow equations, we have $D(l) < D_0 e^{3l}$, $\delta K > 0$ and $\delta y > 0$. Furthermore, the initial values of the functions $\delta K$ and $\delta y$ are 0. From this statement, we can solve exactly the differential system for $\delta K$ and $\delta y$ if we replace $D(l)$ by $D_0 e^{3l}$. We get:

\begin{align*}
\delta K(l) &= D_0 F(l) \\
\delta y(l) &= D_0 G(l)
\end{align*}

(A3) (A4)

where the functions $F$ and $G$ obey the following differential system:

\begin{align*}
F'(x) &= e^{3x} - 2y^0(x)^2 K^0(x) F(x) - 2y^0(x) K^0(x)^2 \\
G'(x) &= (2 - a^2 K^0(x)) G(x) - a^2 y^0(x) F(x) \\
F(0) &= G(0) = 0.
\end{align*}

(A5) (A6) (A7)
If we choose a small $\epsilon > 0$, it is possible to find $l = l_0$ so that $K(0) < \epsilon/2$. Moreover, we know that the real function $\delta K$ is inferior to $D_0F(x)$ where $F(x)$ is a known function; so we can choose the initial value of $D(l)$ so that $\delta K(l_0) < \epsilon/2$. In this case we have $K(l_0) < \epsilon$. Because of the positivity of the functions $\delta K$ and $\delta y$, we know that $y^2(l)K^2(l) > (y^0)^2(l)(K^0)^2(l)$ for all $l$, and so $dK/dl < -(y^0)^2(l)(K^0)^2(l) + D_0e^{3l}$. This inequality proves that for sufficiently small $D_0$, we can also have $K'(l_0) < 0$. This argument shows that, for all $\epsilon > 0$, there is a positive $l_0$ and a small enough $D_0$ so that $K(l_0) < \epsilon$ and $K'(l_0) < 0$. For $l = l_0$, $3 - b^2/K(l_0) \sim -b^2/K(l_0)$. As a result, the disorder contribution in $K'$ will decrease to 0 as fast as $e^{-b^2/K(l_0)}$, whereas the other term will decrease slower than $K^2(l)$. Thus this last term dominates the value of $K'$ which remains negative for all $l > l_0$. It follows that for $l \gg l_0$, we have $K' \sim -y^2K^2$, so that $K(l) \to 0$ for a sufficiently small but finite value of $D_0$.

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FIG. 1. Spin-1 chain phase diagram without randomness

FIG. 2. renormalization flow in the case $(ab)^2 < 6$
FIG. 3. Effective renormalization flow for AF and XY2 phases

|       | AF | Haldane | large-D | XY1 | XY2 |
|-------|----|---------|---------|-----|-----|
| $< \phi_a >$ | 0 | 0 | $\pi/\sqrt{8}$ | dis | dis |
| $< \phi_o >$ | 0 | dis | dis | dis | 0 |
| $K_a^*$ | dis | 0 | 0 | 0 | dis |
| $K_a^*$ | 0 | $\infty$ | $\infty$ | $> 1$ | $> 1$ |

**TABLE I.** Values of the different parameters characterizing the phases of the pure S=1 antiferromagnetic chain.

|       | $q = 0$ | $q = \pi$ |
|-------|---------|----------|
| ZF    | $\partial_x \phi_a$ | $\cos(\sqrt{2}\pi\phi_a) \cos(\sqrt{2}\pi\phi_o)$ |
| PF    | $\cos(\sqrt{\pi/2}\phi_a) \cos(\sqrt{\pi/2}\phi_o)$ | $\cos(\sqrt{2}\pi\phi_a) \cos(\sqrt{2}\pi\phi_o)$ |
| PE    | $\cos(\sqrt{2}\pi\phi_o)$ | $\cos(\sqrt{2}\pi\phi_a) \cos(\sqrt{2}\pi\phi_o)$ |
| PA    | $\cos(\sqrt{2}\pi\phi_o)$ | $\cos(\sqrt{2}\pi\phi_a) \cos(\sqrt{2}\pi\phi_o)$ |

**TABLE II.** Bosonized formulas of the quenched disorder.

|       | AF    | Haldane | large-D | XY1   | XY2   |
|-------|-------|---------|---------|-------|-------|
| ZF    | Unstable | Stable  | Stable  | Stable | Partially stable |
| PF    | Unstable | Stable  | Stable  | Unstable | Partially stable |
| PE    | Unstable | Stable  | Stable  | Partially stable | Unstable |
| PA    | Unstable | Stable  | Unstable | Unstable | Unstable |

**TABLE III.** Stability of the S=1 phases with respect to quenched disorder.