Tidal Disruption Event Disks around Supermassive Black Holes: Disk Warp and Inclination Evolution

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ABSTRACT
After the Tidal Disruption Event (TDE) of a star around a SuperMassive Black Hole (SMBH), the bound stellar debris rapidly forms an accretion disk. If the accretion disk is not aligned with the spinning SMBH's equatorial plane, the disk will be driven into Lense-Thirring precession around the SMBH's spin axis, possibly affecting the TDE's light curve. We carry out an eigenmode analysis of such a disk to understand how the disk's warp structure, precession, and inclination evolution are influenced by the disk's accretion rate and viscosity, as well as the SMBH's mass and spin. We find the TDE disk has a warp profile that is generally more tilted out of the SMBH's equatorial plane at the inner radius than the outer radius, and an oscillatory warp may develop as a result of strong non-Keplarian motion near the SMBH. The global disk precession frequency matches the Lense-Thirring precession frequency of a rigid disk around a spinning black hole within a factor of a few when the disk's accretion rate is high, but deviates significantly at low accretion rates. Viscosity aligns the disk with the SMBH's equatorial plane within a few years for high accretion rates, but within a few days to months for low accretion rates. We also examine the effect of fall-back material on the warp evolution of TDE disks, and find that the fall-back torque aligns the TDE disk with the SMBH’s equatorial plane in a few to tens of days for the parameter space investigated. Our results place constraints on models of TDE emission which rely on the changing disk orientation with respect to the line of sight to explain observations.

Key words: accretion, accretion disks; black hole physics; relativistic processes; stars: black holes; X-rays: bursts

1 INTRODUCTION
When a star wanders too close to a SuperMassive Black Hole (SMBH) at the center of a galaxy, the tidal force exerted on the star by the SMBH overcomes the star’s self-gravity, and the star tidally disrupts. Such tidal disruption events (TDEs) are expected to produce distinct electromagnetic flares (Rees 1988); half of the stellar debris escapes from the SMBH on an unbound orbit, while the other half remains gravitationally bound to the SMBH. This bound material rains down onto the SMBH at a characteristic accretion rate $\dot{M} \propto t^{-5/3}$, and forms an accretion disk after eccentric fluid streams collide with one another (Rees 1988; Evans & Kochanek 1989; Cannizzo, Lee & Goodman 1990). This TDE disk proceeds to accrete rapidly onto the SMBH, producing a luminous flare over a few months to years proportional to the fall-back material accreted onto the disk.

Over the last few decades, dozens of TDEs or TDE candidates have been discovered in various spectral bands, ranging from soft X-rays (e.g. Bade, Komossa & Dahlem 1996; Komossa & Bade 1999; Greiner et al. 2000; Maksym, Ulmer & Fracheous 2010; Donato et al. 2014; Maksym, Lin & Irwin 2014; Khaliullin & Sazonov 2014; Lin et al. 2015), hard X-rays (Bloom et al. 2011; Burrows et al. 2011; Levan et al. 2011; Zauderer et al. 2011; Chenko et al. 2012; Brown et al. 2015), to UV (e.g. Gezari et al. 2006, 2008, 2009) and optical (e.g. Komossa et al. 2008; van Velzen et al. 2011; Wang et al. 2011, 2012; Arcavi et al. 2014; Chornock et al. 2014). Ongoing and future transient surveys like ASAS-SN, PTF, Pan-STARRS, ZTF, and LSST are poised to discover and characterize many more TDEs in the coming decade.

Various models attribute TDE emission arising from inefficient circularization of tidal disruption debris (Guillochon, Manukian & Ramirez-Ruiz 2014; Piran et al. 2015; Shokaw et al. 2015; Krolik, et al. 2016), or outflows supported by radiation pressure (Loeb & Ulmer 1997; Bogdanović et al. 2004; Strubbe & Quataert 2009; Curd & Narayan 2018). If the outflow absorbs the inner accretion disk’s X-ray and

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ultraviolet emission (Metzger & Stone 2016), the full range of observed emission from TDEs may be explained by the observer’s viewing geometry (Dai et al. 2018). It is often assumed that the TDE disk is parallel to the SMBH’s equatorial plane. But if the star disrupts on an orbit misaligned with the SMBH’s spin axis, the disk will be driven into precession from Lense-Thirring torques, and potentially align with the SMBH’s spin over longer timescales.

Some models explaining a TDE’s hard X-ray emission invoke a misaligned accretion disk as an essential component. The spectra of the TDEs Swift J164449.3+573451 (Bloom et al. 2011, Burrows et al. 2011, Levan et al. 2011, Zauderer et al. 2011), Swift J2058.4+0516 (Cenko et al. 2012a), and Swift J1112.2-8238 (Brown et al. 2015) were highly non-thermal, implying a jet was contributing to the tidal disruption flare’s emission. Moreover, the light curve of Swift J164449.3+573451 displayed order-of-magnitude quasi-periodic variations in the hard X-ray, with a period of order ~ 2.7 days (Burrows et al. 2011, Saxton et al. 2012). If the TDE disk is misaligned with the spinning SMBH’s equatorial plane, the Lense-Thirring effect drives the disk to precess around the SMBH’s spin axis (Stone & Loeb 2012, Tchekhovskoy et al. 2014, Franchini, Lodato & Facchini 2016). The jet axis would vary with respect to the observer’s line of sight, causing variations in the hard X-ray’s light curve. Some studies assume that the inner edge of the accretion disk is nearly aligned with the SMBH’s equatorial plane (Lei, Zhang & Gao 2013), while others argue that the entire disk is nearly flat and precesses like a rigid plate around the SMBH (Stone & Loeb 2012, Shen & Matzner 2014, Franchini, Lodato & Facchini 2016). Most studies of misaligned TDE disks assume the only torque aligning the disk with the SMBH’s equatorial plane is from the disk’s viscosity (Stone & Loeb 2012, Lei, Zhang & Gao 2013, Franchini, Lodato & Facchini 2016). Recently, Ivanov, Zhuravlev & Papaloizou (2018) included the torque acting on the disk from the stellar debris’ fall-back material, and showed that typical TDE disks cannot complete one full precession period before aligning with the SMBH’s equatorial plane.

In this work, we attempt to clarify these theoretical issues on the warp profile, precession and inclination dynamics of TDE disks around SMBHs. Section 2 examines the warp structure and dynamical evolution of a thick \(H/r \gtrsim \alpha\) disk with a power-law surface density and constant aspect ratio. Section 3 introduces our simple model TDE disks. Section 4 contains our results for the precession and damping rates of a viscous TDE disk around a SMBH. Section 5 investigates how the fall-back material influences the alignment of the TDE disk with the SMBH’s equatorial plane. Section 6 discusses theoretical uncertainties and observational implications of our work. Section 7 summarizes our key results.

2 WARPED DISK UNDERGOING
LENSE-THIRRING PRECESSION

Before considering more detailed models of TDE disks (Sec. 3), in this section we study the warp and dynamical evolution for a simple model of an accretion disk orbiting a SuperMassive Black Hole (SMBH) of mass \(M_\bullet\) and dimensionless spin parameter \(a_\bullet\). We denote the black hole’s gravitational radius by \(R_g = GM_\bullet/c^2\). We take the disk’s surface density \(\Sigma(r, t) \propto r^{-1/2}\), and the disk aspect ratio \(H/r\) to be constant across the disk’s annular extent. The inner truncation radius of the disk is taken to be the Innermost Stable Circular Orbit (ISCO) of a test particle orbiting prograde around a spinning black hole \(r_{\text{ISCO}}\) (Bardeen, Press & Teukolsky 1972)

\[
r_{\text{in}} = r_{\text{ISCO}} = \left[3 + Z_2 - \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}\right] R_g,
\]

where

\[
Z_1 = 1 + \left(1 - a_\bullet^2\right)^{1/3} \left[1 + \left(\frac{a_\bullet^2}{3}\right)^{1/3} + \left(1 - a_\bullet^2\right)^{1/3}\right],
\]

\[
Z_2 = \sqrt{3a_\bullet^2 + Z_1^2}.
\]

The outer truncation radius of the disk is set to be \(r_{\text{out}} = 94.2 R_g\).

We assume the misalignment between the orbital angular momentum unit vector of the disk \(\hat{l} = \hat{l}(r, t)\) and the SMBH’s spin vector \(\hat{\kappa}\) is small everywhere. Defining the complex warp amplitude \(W(r, t) = \hat{l}(\hat{x} + i\hat{y})\) in a Cartesian coordinate system with \(\hat{x} = \hat{l} + \Omega(W)\hat{y}\), the disk warp evolves in time according to (Ogilvie 1999, Lubow & Ogilvie 2000)

\[
\Sigma r^2 \frac{\partial W}{\partial t} = \frac{1}{r^2} \frac{\partial G}{\partial r} + T,
\]

where \(G(r, t) = G_\nu(r, t) + iG_\rho(r, t)\) is the disk’s complex internal torque per unit area, while \(T\) is the complex external torque acting on the disk.

Equation (4) is closed with an equation for the disk’s internal torque, written in terms of the disk warp. This closure expression depends on whether or not the disk lies in the so-called resonant regime (Papaloizou & Pringle 1983, Papaloizou & Lin 1995, Ivanov & Illarionov 1997, Ogilvie 1999, Lubow & Ogilvie 2000), which occurs when bending waves are able to travel at approximately half the disk’s sound speed. In the appendix, we derive the dispersion relation for inertial-density waves in viscous disks with non-Keplerian epicyclic frequencies \(\kappa\). We show the approximate condition for bending waves to globally propagate across the disk with a velocity \(v_{\text{bw}} \approx c_\nu/2\) is

\[
\frac{H}{r} \gtrsim \alpha \quad \text{and} \quad \frac{H}{r} \gtrsim \frac{\Omega^2 - \kappa^2}{2\Omega^2} \equiv \tilde{\kappa},
\]

where \(H/c_\Omega = \kappa\) is the disk scale-height, \(\alpha\) is the dimensionless Shakura-Sunyaev viscosity parameter, and \(\Omega\) is the orbital frequency. When condition (5) is satisfied, the disk lies in the resonant regime, with \(G(r, t)\) given by (Ogilvie 1999)

\[
\frac{\partial G}{\partial t} = i\tilde{\kappa} \Omega G - \alpha \Omega G + \frac{\Sigma H^2 r^3 \Omega^2}{4} \frac{\partial W}{\partial r}.
\]

When condition (5) is violated, the disk lies in the diffusive regime, and \(G(r, t)\) is given by (Ogilvie 1999)

\[
G = \Sigma H^2 r^3 \Omega^2 \left(Q_\nu \frac{\partial W}{\partial r} + iQ_\rho \frac{\partial W}{\partial r}\right),
\]

assuming a steady-state disk (radial velocity \(v_r = \frac{1}{2} \Omega H^2 \Omega/r\)). The viscous and pressure coefficients \(Q_\nu\) and \(Q_\rho\) differ depending on whether or not the disk lies in the resonant regime.
$Q_p$ are given by \cite{Ogilvie1999} Appendix A5
\begin{align}
Q_p &= \frac{2\tilde{k} + \alpha^2[3 + 2(3 + \tilde{k})(2\tilde{k} - \alpha^2)]}{2[(2\tilde{k} - \alpha^2)^2 + 4\alpha^2]} + O(|W|^2), \\
Q_v &= \frac{\alpha[1 + 2\tilde{k} + \alpha^2(7 + 2\tilde{k})]}{(2\tilde{k} - \alpha^2)^2 + 4\alpha^2} + O(|W|^2).
\end{align}
Although the dependence of $Q_v$ and $Q_p$ on $\alpha$ and $\tilde{k}$ is complicated, two limiting cases are particularly relevant for astrophysical disks. The first is the high viscosity limit ($|\tilde{k}| \ll \alpha^2 \ll 1$), where $Q_v$ and $Q_p$ reduce to (assuming $|W| \ll 1$)
\begin{align}
Q_p &\approx \frac{3}{8} \quad \text{and} \quad Q_v \approx \frac{1}{4\alpha}.
\end{align}
Clearly for $\alpha \leq 1$, viscosity is the main internal torque working to maintain the disk's coplanarity ($Q_v \gg Q_p$). The opposite limit is the low viscosity regime ($\alpha^2 \ll |\tilde{k}| \ll 1$), where $Q_p$ and $Q_v$ reduce to (assuming $|W| \ll 1$)
\begin{align}
Q_p &\approx \frac{1}{4\tilde{k}} \quad \text{and} \quad Q_v \approx \frac{\alpha}{4\tilde{k}}.
\end{align}
In the low viscosity limit, pressure can be the main internal torque working to maintain the disk's coplanarity ($Q_p \gg Q_v$).

The dimensionless function $\tilde{k}$ [Eq. (5)] measures the amount of apsidal precession for a slightly eccentric fluid particle's orbit. Around a Kerr black hole, it is given by (e.g. \cite{Kato1990})
\begin{align}
\tilde{k}(r) = \frac{3R_k}{r} - \frac{4\alpha R_k^{3/2}}{r^{3/2}} + \frac{3\alpha^2 R_k^2}{2r^2}.
\end{align}
Note that $\tilde{k} \sim 1$ in the inner region of the disk.

Most previous studies of disk warps in the diffusive regime around spinning black holes have assumed the disk lies in the high viscosity limit and used Equation (10), thus essentially neglecting $\tilde{k}$ (see e.g. \cite{Kumar&Pringle1985, Scheuer&Feiler1996, Lodato&Pringle2000, Martin, Pringle&Tout2009, Tremaine&Davis2014, Chakraborty&Bhattacharyya2017}). Such studies find the inner disk to be closely aligned with the spinning black hole's equatorial plane (the Bardeen-Peterson effect; \cite{Bardeen&Petterson1975}). Hydrodynamical simulations have reproduced this result in the $\alpha \gg H/r$ regime, some by neglecting apsidal precession induced by the black hole (e.g. \cite{Nealon2000, Sorathiya2013, Krolik2015, Hawley2018, Liska2018}). However, ignoring apsidal precession may neglect important features of the disk's warp profile, since warps induce radial pressure gradients in the disk (e.g. \cite{Ivanov1997, Lodato&Pringle2007}), and the epicyclic frequency can become highly non-Keplerian ($\tilde{k} \sim 1$) near the ISCO. Some analytic works (e.g. \cite{Ivanov1997, Enbouw, Ogilvie&Pringle2002, King2005, Zhuravlev2014}) and hydrodynamical simulations (e.g. \cite{Franklin2007, Zhuravlev2014, MoralesTeixeira2014, Nealon2010, Liska2018}) of disks in the $\alpha \lesssim H/r$ limit which include apsidal precession find a very different picture, with the inner disk highly misaligned with respect to the black hole's equatorial plane. As we show below, this behavior arises from the non-negligible influence of pressure torques when the non-Keplerian epicyclic frequency near the black hole is properly taken into account.

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These effects are relevant for TDE disks in particular since they are likely to be in the $H/r \lesssim \alpha$ regime.

Since $\tilde{k}$ is non-negligible around a black hole, thin ($H/r \ll 1$) disks have some radial extent where the resonant condition (5) is violated. We define the radius $r_\kappa$ via
\begin{align}
\left[ \frac{H}{r} \right]_{r=r_\kappa} = \tilde{k}(r_\kappa).
\end{align}
Because of the non-monotonic behavior of $\tilde{k}$, in this section we assume $r_\kappa$ has a single value, taken as a free parameter.

The internal torque $G$ is given by Equation (6) when $r > r_\kappa$, and Equation (7) when $r \leq r_\kappa$. We assume the external torque on the disk is the Lense-Thirring torque (\cite{Kato1990})
\begin{align}
T = i\Sigma r^2 \Omega \omega_\bullet W, \quad (14)
\end{align}
where
\begin{align}
\Omega(r) &= \frac{c}{R_k} \left( \frac{r^{3/2}}{R_k^{3/2}} - a_\bullet \right)^{-1},
\end{align}
is the orbital frequency, and
\begin{align}
\omega_\bullet(r) &= \Omega \left( \frac{2\alpha R_k^{3/2}}{r^{3/2}} - \frac{3\alpha^2 R_k^2}{2r^2} \right),
\end{align}
is the Lense-Thirring precession frequency.

To solve Equations (4), (9), and (7), we look for solutions of the form
\begin{align}
W(r,t) &= \tilde{W}(r)e^{i\lambda t}dr, \quad (17)
G(r,t) &= \tilde{G}(r)e^{i\lambda t}dr,
\end{align}
where in general the complex eigenfrequency $\lambda = \lambda(t)$ can change in time with the disk properties and external torque. In this section however, $\lambda$ is constant. We assume a zero-torque boundary condition
\begin{align}
\tilde{G}(r_{\text{in}}) = \tilde{G}(r_{\text{out}}) = 0. \quad (19)
\end{align}
Because the solutions are linear in $G$ and $W$, we are free to choose a normalization condition for $\tilde{W}$ and $\tilde{G}$, which we take to be the disk warp at the outer truncation radius $W(r_{\text{out}})$.

Equation (4) can be integrated over $W^*dr$ ($W^*$ is the complex conjugate of $W$) to obtain (after integration by parts) an integral expression for the complex eigenfrequency $\lambda = \gamma + i\omega$, with
\begin{align}
\omega = \omega_\bullet + \omega_\kappa, \quad \gamma = \tilde{\gamma}_v, \quad (20)
\end{align}
where
\begin{align}
\omega_\bullet &= \left[ \frac{1}{L_+} \int_{r_\kappa}^{r_{\text{out}}} \Sigma r^3 \Omega \omega_\bullet |W|^2 dr \right. + \left. \frac{2}{L_-} \int_{r_{\text{in}}}^{r_\kappa} \tilde{Q}_p |\tilde{G}|^2 \frac{4\tilde{k}}{\Sigma H^2 r^4} dr \right],
\omega_\kappa &= \left[ \frac{\alpha}{L_-} \int_{r_{\text{in}}}^{r_\kappa} \tilde{Q}_p |\tilde{G}|^2 \frac{4\tilde{k}}{\Sigma H^2 r^4} dr \right. + \left. \frac{\alpha}{L_+} \int_{r_{\text{out}}}^{r_\kappa} \tilde{Q}_p |\tilde{G}|^2 \frac{4\tilde{k}}{\Sigma H^2 r^4} dr \right],
\tilde{\gamma}_v &= \left[ \frac{1}{L_+} \int_{r_{\text{in}}}^{r_\kappa} \Sigma r^3 |W|^2 |\tilde{G}|^2 dr \right. + \left. \frac{L_+}{L_-} \int_{r_{\text{out}}}^{r_\kappa} \Sigma r^3 \Omega |W|^2 |\tilde{G}|^2 \right] d\Omega, \quad (23)
L_{\pm} &= \left[ \int_{r_{\text{in}}}^{r_\kappa} \Sigma r^3 \Omega |W|^2 dr \right. + \left. \int_{r_{\text{in}}}^{r_\kappa} \Sigma r^3 \Omega \omega_\bullet |W|^2 dr \right] \frac{4\tilde{k}}{\Sigma H^2 r^4} dr, \quad (24)
\end{align}
where $r_\kappa = \max(r_{\text{in}}, r_\kappa)$, and
\begin{align}
\tilde{Q}_p &= \left\{ \begin{array}{ll}
\frac{4\tilde{k}}{Q_p/(Q_p^2 + Q_\kappa^2)} & r \geq r_\kappa, \\
Q_p/(Q_p^2 + Q_\kappa^2) & r < r_\kappa,
\end{array} \right. \quad (25)
\end{align}
\[
\tilde{Q}_v = \begin{cases} 
\frac{4\alpha}{Q_v/(Q_p^2 + Q_v^2)} & r \geq r_\kappa \\
6 & r < r_\kappa 
\end{cases}
\] (26)

When the disk is rigidly precessing around the black hole (|\tilde{W}| = constant, |\tilde{G}| = 0), the global precession rate reduces to the rigid-body precession frequency \( \omega = \bar{\omega}_{\text{rigid}} \), where

\[
\bar{\omega}_{\text{rigid}} = \frac{\int_{r_{\text{in}}}^{r_{\text{out}}} \rho r^2 \Omega d\tau}{\int_{r_{\text{in}}}^{r_{\text{out}}} \Sigma r^2 d\tau}
\] (27)

Equation (27) is often used to estimate the precession rate of a disk around a spinning black hole (e.g. Fragile et al. 2007; Stone & Loeb 2012). In addition, the simple estimate (Bate et al. 2000)

\[
\gamma_{\text{rate}} = \alpha \left( \frac{r}{H} \right)^2 \frac{\bar{\omega}_{\text{rigid}}}{\Omega} \bigg|_{r = r_{\text{out}}}
\] (28)

is often used as an approximation to the disk's viscous damping rate (e.g. Foucart & Lai 2014; Franchini, Lodato & Facchini 2016). We will investigate the validity of this approximation in later sections.

Although the eigenfrequency is given by the integral expressions above [Eqs. (21)-(24)], in practice it must be determined numerically alongside the eigenfunctions \( \tilde{W} \) and \( \tilde{G} \). We use a shooting algorithm [Press et al. 2002] written in C++ to calculate the warped disk's eigenmodes.

### 2.1 Disk Warp Profile

Figure 1 plots the disk warp \( \tilde{\beta}(r) \) (top panel) and twist \( \tilde{\varphi}(r) \) (bottom panel) profile for the complex warp amplitude \( \tilde{W} = \beta e^{i\varphi} \) around a black hole (BH) with a moderate spin (\( a_* = 0.5 \)). The disk warp is close to flat over most of the disk’s radial extent, but there is an increase in the warp amplitude \( \beta \) near the disk’s inner edge. This increase is due to the influence of pressure torques, and persists even when the inner disk is in the dissipative regime (dashed line solution with \( r_\kappa = 10 R_g \)). A small twist \( \tilde{\varphi} \) develops due to viscous torques. Notice there are only minor differences between the disk warp profile when the entire disk is in the resonant regime (solid lines), and when the inner disk lies in the dissipative regime (dashed lines), because the dependence of the pressure coefficient \( Q_p \) on the dimensionless non-Keplerian epicyclic frequency \( \kappa \) has correctly been accounted for.

Figure 2 is the same as Figure 1 except the BH is spinning maximally (\( a_* = 1.0 \)). Because the disk is much more extended around such a BH (\( r_{\text{out}} \approx 100 r_{\text{in}} \) when \( a_* = 1 \)), the disk tilt \( \tilde{\beta} \) oscillations become more pronounced, so that \( \tilde{\beta}(r_{\text{in}}) > 2 \beta(r_{\text{out}}) \). The differences between the \( r_\kappa = 0 \) and \( r_\kappa = 10 R_g \) cases are minor, indicating that tilt oscillations occur around maximally spinning BHs when pressure torques are properly taken into account, no matter what regime (resonant or dissipative) the disk lies in (assuming \( \alpha \lesssim H/r \)). The disk twist \( \tilde{\varphi} \) increases rapidly when the disk warp \( \tilde{\beta} \) is small, which can be shown to be due to a coordinate singularity when \( \beta = 0 \). The gradual increase in disk tilt \( \tilde{\varphi} \) before and after the dip in disk warp \( \tilde{\beta} \) is due to viscous torques.

In contrast to Figure 2, Figure 3 shows that when \( \kappa \) is neglected, the efficacy of pressure torques to drive tilt oscillations is reduced. We see that the disk warp \( \tilde{\beta} \) decreases smoothly and monotonically to the inner disk edge. This suggests the Bardeen-Peterson effect [\( \tilde{\beta}(r_{\text{in}}) \ll \tilde{\beta}(r_{\text{out}}) \)] is only possible when pressure torques are negligible (\( Q_p \ll Q_v \)).

### 2.2 Precession/Damping Rates

Figure 4 plots the precession frequency \( \omega \) and damping rate \( \gamma \) for a maximally spinning black hole (\( a_* = 1 \)), as a function of the disk scale-height \( H/r \). Sharp dips in the precession frequency \( \omega \) occur when the Lense-Thirring part of the precession frequency \( \bar{\omega}_{\kappa} \) [Eq. (21)] becomes nearly equal to the non-Keplerian part of the precession frequency \( \bar{\omega}_{i} \) [Eq. (22)]. During these dips, the damping rate \( \gamma \) becomes larger than the precession frequency \( \omega \), implying that an initially misaligned disk would rapidly align with the black hole’s equatorial plane. Note that the precession frequency \( \omega \)
Figure 2. Same as Fig. 1 except $a_\ast = 1.0$. The Eigenvalues are $\omega = -0.0054 \Omega(r_{\text{out}})$, $\gamma = -0.0444 \Omega(r_{\text{out}})$ for $r_\kappa = 0$, and $\omega = 0.0015 \Omega(r_{\text{out}})$, $\gamma = -0.080 \Omega(r_{\text{out}})$ for $r_\kappa = 10 R_g$. See Eq. (13) and discussion thereafter for definition of $r_\kappa$.

Figure 3. Same as Fig. 2 except that we set $\tilde{\kappa} = 0$. The Eigenvalues are $\omega = 0.10 \Omega(r_{\text{out}})$, $\gamma = -0.045 \Omega(r_{\text{out}})$ for $r_\kappa = 0$, and $\omega = 0.10 \Omega(r_{\text{out}})$, $\gamma = -0.048 \Omega(r_{\text{out}})$ for $r_\kappa = 10 R_g$.

Many qualitative features of the $\omega$ value’s erratic dependence on $H/r$ may be understood by examining the WKB limit of the disk. As discussed in Appendix B, the WKB approximation of the disk warp

$$\frac{d^2 W}{dr^2} + \frac{V}{r^2} W = 0,$$  \hspace{1cm} (29)

depends non-trivially on the disk aspect ratio $H/r$, and can differ many orders of magnitude from the naive rigid-body estimate $\omega_{\text{rigid}}$ [Eq. (27)].

Figure 4 plots the warp $\tilde{\beta}$ (top panel) and twist $\tilde{\varphi}$ (bottom panel) profiles of the complex warp amplitude $\tilde{W} = \beta e^{i \varphi}$ at several values of $H/r$, denoted by the vertical black lines of Fig. 4. We see that the spikes in the precession frequency $\omega$ coincide with an increase in the number of nodes [where $\beta(r) \approx 0$] in $\tilde{\beta}$. Sharp increases in $\tilde{\varphi}$ occur near warp nodes.

Many qualitative features of the $\omega$ value’s erratic dependence on $H/r$ may be understood by examining the WKB limit of the disk. As discussed in Appendix B, in the WKB approximation the disk warp

$$\frac{d^2 W}{dr^2} + \frac{V}{r^2} W \simeq 0,$$  \hspace{1cm} (29)

where

$$V(r) = t_{bw}^2 (\omega - \omega_\ast)(\omega - \tilde{\kappa} \Omega),$$  \hspace{1cm} (30)

and $t_{bw} = 2r/c_s$ is the bending wave travel time. Notice the similarity between Eq. (29) and the equation for non-radial stellar oscillations in the WKB limit [e.g. Fuller & Lai 2012, Eq. (16)]. Low-frequency $|\omega| < |\omega_\ast|, |\tilde{\kappa} \Omega|$ disk warp modes would correspond to g-modes of stellar oscillations. Non-trivial propagating and evanescent zones have been shown to create complex g-mode responses for white dwarf binary stars [Fuller & Lai 2012, 2013]. Therefore, a complex response is expected of the disk precession frequency to the disk parameters as the disk scale-height (or equivalently $t_{bw}$) is varied.

3 TIDAL DISRUPTION EVENT DISK MODEL

This section introduces our model for the disk structure formed after a Tidal Disruption Event (TDE) of a star. We begin by reviewing the physics of TDEs, then follow with our model of TDE disks.
3.1 TDE review

A TDE occurs when a star of mass $M_\star$ and radius $R_\star$ approaches a SMBH of mass $M_\bullet$ on a nearly parabolic orbit, with pericenter distance

$$r_p \lesssim R_\star = R_\star \left( \frac{M_\bullet}{M_\star} \right)^{1/3}. \quad (31)$$

The energy spread of the star’s debris $\Delta E$ after the TDE is (assuming $R_\star \ll r_p$)

$$\Delta E \simeq \frac{GM_\bullet R_\star}{r_p^2}. \quad (32)$$

Since the star is on a nearly parabolic orbit, the mean energy of the debris $E \approx 0$, so the shortest period of all debris streams is (assuming $r_p \approx R_\star$)

$$t_\chi = \frac{G M_\star}{(\Delta E)^{3/2}} = \frac{\pi}{2} \frac{r_p^3}{\sqrt{GM_\bullet R_\star}}$$

$$= 41 \left( \frac{M_\star}{10^6 M_\odot} \right)^{1/2} \left( \frac{1 M_\odot}{M_\star} \right) \left( \frac{R_\star}{1 R_\odot} \right)^{3/2} \text{ days}. \quad (33)$$

After $t_\chi$, the streams begin to self-intersect and circularize [Rees 1988; see Sec. 6.4 for discussion on circularization efficiency], and an accretion disk forms. The remaining bound stellar debris then rains down onto the newly formed accretion disk at a rate $\dot{M}_{\text{fb}}/dt$. Since Kepler’s laws give $|E| \propto P^{-2/3}$ ($P$ is the orbital period of the debris stream), the fall-back rate of the bound debris onto the disk is

$$\dot{M}_{\text{fb}} = \frac{dM_{\text{fb}}}{dt} = \frac{dM_{\text{fb}}}{dE} \frac{dE}{dt} = \frac{M_\star}{3\ell_\chi} \left( \frac{t_\chi}{t} \right)^{5/3}. \quad (34)$$

Here, we have assumed the spread in energy of the fall-back material $\Delta E$ is uniform in $E$ ($dM_{\text{fb}}/dE \approx \text{constant}$). Analytic arguments and simulations suggest that this approximation breaks down when $t \sim t_\chi$, but is excellent when $t \gg t_\chi$ [Lodato, King & Pringle 2009; Guillochon & Ramirez-Ruiz 2013].

Normalizing Eq. (34) to the Eddington accretion rate $\dot{M}_{\text{Edd}} = \eta L_{\text{Edd}}/c^2$, where $\eta$ is an efficiency factor and $L_{\text{Edd}}$ is the Eddington Luminosity, we have

$$\dot{M}_{\text{fb}}/\dot{M}_{\text{Edd}} = 133.8 \left( \frac{\eta}{0.1} \right) \left( \frac{10^6 M_\odot}{M_\star} \right)^{3/2} \left( \frac{M_\star}{1 M_\odot} \right) \left( \frac{t_\chi}{t} \right)^{5/3}. \quad (35)$$

We see almost all TDEs have super-Eddington accretion rates over some portion of their lifetime.
3.2 Disk Model

Our disk model is motivated by Cannizzo, Lee & Goodman (1990), and does not include effects which are important soon after the formation of the TDE disk (e.g. viscous spreading) discussed in Montesinos Armijo & de Freitas Pacheco (2011) and Shen & Matzner (2014). We assume the disk forms rapidly after $t \geq t_i$. The inner truncation radius $r_{in}$ of the disk is given by the ISCO [Eq. (1)]. The outer truncation radius of the disk $r_{out}$ is given by the circularization radius of the nearly parabolic debris stream:

$$ r_{out} \approx 2R_t = 0.94 \text{ au} \left( \frac{M_*}{10^6 M_\odot} \right)^{1/3} \left( \frac{R_*}{R_\odot} \right)^{1/3} \left( \frac{1 M_\odot}{M_*} \right)^{1/3} \approx 94.2 \left( \frac{R_*}{1 R_\odot} \right)^{2/3} \left( \frac{10^6 M_\odot}{M_*} \right)^{1/3} \frac{1}{R_\odot}. \quad (36) $$

We assume the disk is in a steady state, with an accretion rate $\dot{M} = -2\pi r v_s \Sigma \approx -3\pi r^2 \Sigma$ that is radially constant ($v_s$ is the radial velocity). Parameterizing the disk viscosity through the Shakura-Sunyaev prescription ($\nu = \alpha H^2 \Omega, \alpha = \text{constant}$), the viscous heating rate (per unit area) of the disk is

$$ Q^+_{visc} = \nu \Sigma r^2 \left( \frac{d\Omega}{dr} \right)^2 \approx \frac{9}{4} \alpha \Sigma H^2 \Omega^4, \quad (37) $$

where the disk scale-height $H$ is related to the isothermal sound-speed $c_s$ via $H = c_s/\Omega$. The disk is cooled by advection and radiation. The advective cooling rate is (Abramowicz et al. 1988, 1995)

$$ Q_{adv} = \Sigma v_s T \frac{d\Omega}{dr} \approx \frac{9}{4} \alpha \Sigma c_s^2 / 4r^2, \quad (38) $$

where $s$ is the disk entropy, $T$ is the temperature, and we have assumed the constant $\xi$ in Abramowicz et al. (1995) is $\xi \approx 3/2$. The radiative cooling rate of the disk is

$$ Q^-_{rad} = \frac{4\alpha c^4 T^4}{3\pi \Sigma}. \quad (39) $$

We focus on the early phase of the TDE disk, when the disk is supported primarily by radiation pressure ($p \approx \rho c_s^2 = aT^4/3$). Since the disk’s surface density $\Sigma$ is related to the density $\rho$ via $\Sigma = 2H \rho$, the sound-speed $c_s$ is given by

$$ c_s^2 = \frac{p}{\rho} \approx \frac{2\alpha HT^4}{3\Sigma}. \quad (40) $$

Using Equations (37)–(40), energy balance ($Q^+_{visc} = Q_{adv} + Q^-_{rad}$) gives the disk’s aspect ratio:

$$ \frac{H}{r} = \sqrt{\frac{1}{\mathcal{H}^2 + 1} - \mathcal{H}}, \quad (41) $$

where

$$ \mathcal{H}(r,t) = \frac{4\pi cr}{3m} = \frac{3.89 \times 10^{-2}}{m} \left( \frac{\eta}{0.1} \right) \left( \frac{r}{R_\odot} \right) \quad (42) $$

parameterizes the relative importance of advective to radiative cooling in the disk, $\kappa = 0.34 \text{cm}^2/\text{g}$ is the electron scattering opacity, and

$$ \dot{m} = \frac{\dot{M}}{M_{\text{Edd}}}. \quad (43) $$

When the disk is advective ($\mathcal{H} \ll 1$), the disk aspect ratio reduces to

$$ \frac{H}{r} \approx 1 \quad (44) $$

while when the disk is radiative ($\mathcal{H} \gg 1$),

$$ \frac{H}{r} \approx \frac{1}{(2\mathcal{H})}. \quad (45) $$

In a steady state,

$$ \Sigma(r,t) \approx \frac{M}{3\pi \alpha H^2 \Omega^4}, \quad (46) $$

so $\Sigma \propto M^{-1/2}$ when $\mathcal{H} \ll 1$, while $\Sigma \propto M^{-1} r^{-3/2}$ when $\mathcal{H} \gg 1$. Notice the disk surface density increases with $r$ when the accretion rate becomes sufficiently small.

The scale-height given by Equation (41) is essentially that derived by Strubbe & Quataert (2009), except we do not include the factor $f = 1 - \sqrt{r/r_{in}}$ to force the viscous torque to be zero at $r = r_{in}$. The late time behavior of TDEs is better modeled by an accretion disk without the zero-torque boundary condition at the ISCO radius (Balbus & Mummery 2018), and MHD simulations of accretion onto BHs show that the viscous torques do not necessarily vanish at the ISCO radius (e.g. Hawley 2000, Hawley, Guan & Krolik 2011).

We will have to consider the gas-pressure dominated regime ($p \approx \rho g = \rho k T/\mu m_p$) when the disk’s accretion rate falls below (e.g. Shen & Matzner 2014)

$$ \dot{m}_{\text{gas}} = \frac{M}{M_{\text{Edd}}} \left|_{r_{\text{rad}}=p_{\text{gas}}, r=r_{out}} \right. $$

$$ = 1.14 \times 10^{-2} \left( \frac{\eta}{0.1} \right) \left( \frac{0.01}{\alpha} \right) \left( \frac{10^6 M_\odot}{M_*} \right)^{1/8} \times \left( \frac{R_*}{1 R_\odot} \right)^{21/16} \left( \frac{1 M_\odot}{M_*} \right)^{7/16}. \quad (47) $$

When $\dot{m} \lesssim \dot{m}_{\text{gas}}$, the disk scale-height falls from Eq. (45) to

$$ \frac{H}{r} = 5.56 \times 10^{-3} \dot{m}_{\text{gas}}^{1/5} \left( \frac{0.1}{\eta} \right)^{1/5} \times \left( \frac{10^6 M_\odot}{M_*} \right) \left( \frac{0.01}{\alpha} \right)^{1/10} \left( \frac{r}{R_\odot} \right)^{1/20}. \quad (48) $$

In reality, the disk scale-height $H$ does not transition smoothly from Equation (45) to (48) for our prescription for the disk’s viscosity. Rather, when $p \approx p_{\text{rad}}$ and $Q^+_{\text{visc}} \approx Q^-_{\text{rad}}$, the disk is susceptible to a thermal instability, and oscillates between the two states given by Equations (44) and (48) (Lightman & Eardley 1974, Abramowicz et al. 1988, Shen & Matzner 2014, Xiang-Gruess, Ivanov & Papaloizou 2016). We therefore take Equation (41) to be a conservative upper limit to the disk aspect ratio when the disk is radiative. Observations support the lack of the Lightman & Eardley (1974) instability occurring in TDE accretion disks, since a disk with scale-height (48) would have emission in the soft X-ray and ultraviolet much fainter than observed (van Velzen et al. 2018a).

Our model for a TDE disk assumes a steady-state accretion rate. This assumption is valid as long as the viscous time $t_v = r^2/\nu$ is everywhere much less than the timescale over which the TDE disk evolves [$M_* / 2M \sim t_i$, see Eq. (33)].
This section examines how the TDE disk model described in Section 2 evolves its radial warp profile, precession and damping rates in time, due to the Lense-Thirring torque from the SMBH. As discussed in Section 2, we expect a non-trivial response of the disk’s precession frequency \( \omega \) to the evolving disk aspect ratio \( H/r \). We solve Equations (1), (6), and (7) numerically using the shooting method written in C++ (Press et al. 2002) for the eigenfunction \( \tilde{W} \) and eigenfrequency \( \lambda = \gamma + i\omega \).

Figure 6 plots the precession frequency \( \omega \) and damping rate \( \gamma \) as a function of the disk’s accretion rate \( \dot{m} = \dot{M}/\dot{M}_{\text{Edd}} \). When the accretion rate is high (\( \dot{m} \gtrsim 0.4 \)), \( \omega \) is close to (within a factor of 2) the rigid-body Lense-Thirring precession frequency \( \omega_{\text{rigid}} \) (see Eq. (27)), and \( \omega \) deviates the most from \( \omega_{\text{rigid}} \) when the black hole is spinning rapidly (\( a_* \gtrapprox 1 \)). At these high accretion rates, \( \omega \) does not depend strongly on the disk’s viscosity parameter \( \alpha \). When the accretion rate is low (\( \dot{m} \lesssim 0.4 \)), the evolution of \( \omega \) depends heavily on \( \alpha \). For disks with a high viscosity (\( \alpha = 0.1 \)), \( \omega \) slowly decreases with \( \dot{m} \) to values significantly below \( \omega_{\text{rigid}} \). Disks with low viscosities (\( \alpha = 0.01 \)) have a very different dependence on \( \dot{m} \), and undergo the large variations in \( \omega \) as \( \dot{m} \) varies (see Sec. 2.2).

A disk with a high viscosity (\( \alpha = 0.1 \)) suppresses large variations in \( \omega \) as \( \dot{m} \) decreases. The disk’s damping rate \( \gamma \) usually tracks the dependence of \( \omega \) on \( \dot{m} \), and \( \gamma \) is typically smaller in magnitude than \( \omega \). However, when the accretion rate and disk viscosity are low (\( \dot{m} \lesssim 0.4, \alpha \sim 0.01 \)), \( \gamma \) has its own dependence on \( \dot{m} \) and can exceed \( \omega \) in magnitude.

Note a disk with a low viscosity (\( \alpha = 0.01 \)) can have a viscous damping rate greater than a disk with a high viscosity (\( \alpha = 0.1 \)) when the accretion rate is low (\( \dot{m} \lesssim 0.4 \)), due to the non-linear effects from disk warping. The evolution of \( \omega_{\text{rigid}} \) is due to the disk’s surface density evolution (see Fig. 6).

Figure 7 is identical to Figure 6 except we increase the mass of the SMBH to \( M_* = 10^7 M_\odot \). Like Figure 6, when the accretion rate is high (\( \dot{m} \gtrsim 0.2 \)), the disk’s precession rate \( \omega \) closely tracks the rigid-body precession frequency \( \omega_{\text{rigid}} \). Unlike Figure 6, \( \omega \) of a disk around a slowly spinning \( (a_* = 0.4) \) high-mass \( (M_* = 10^7 M_\odot) \) SMBH does not depend strongly on the viscosity \( \alpha \), and increases above \( \omega_{\text{rigid}} \) as \( \dot{m} \) decreases. A disk around a moderately spinning \( (a_* = 0.7) \) high-mass \( (M_* = 10^7 M_\odot) \) SMBH also has an \( \omega \) which can exceed the rigid-body precession frequency \( \omega_{\text{rigid}} \), but oscillates as the accretion rate \( \dot{m} \) decreases, and has a much stronger dependence on \( \alpha \). A disk around a maximally spinning \( (a_* = 1.0) \) high-mass \( (M_* = 10^7 M_\odot) \) SMBH has an \( \omega \) that matches the qualitative dependence of the low-mass \( (M_* = 10^6 M_\odot) \) SMBH on \( \dot{m} \). When the viscosity is high (\( \alpha = 0.1 \)), \( \omega \) decreases below \( \omega_{\text{rigid}} \) as \( \dot{m} \) decreases. When the viscosity is low (\( \alpha = 0.01 \)), the precession rate \( \omega \) undergoes large variations as the accretion rate \( \dot{m} \) decreases. The damping rate \( \gamma \) of a disk around a high-mass \( (M_* = 10^7 M_\odot) \) SMBH can exceed \( \omega \), and a low viscosity disk \( (\alpha = 0.01) \) can have a \( \gamma \) larger than a high viscosity disk \( (\alpha = 0.1) \) when the accretion rate is low (\( \dot{m} \lesssim 0.2 \)) and SMBH spin high (\( a_* \gtrapprox 0.7 \)).

Figure 8 plots the warp \( \tilde{\beta}(r) \) and twist \( \tilde{\varphi}(r) \) radial

\[
\frac{\Sigma(r_m, \phi)}{\Sigma(r_m, 0)} \propto \frac{\dot{m}}{\dot{M}_{\text{Edd}}} \left( 1 + \frac{r_m}{R_g} \right) \left( \frac{R_g}{R} \right)^2 \left( \frac{M_*}{M_\odot} \right) \times \left( \frac{R_g}{R} \right)^{3/2} \left( \frac{1 M_\odot}{M_*} \right)^{1/2} \text{days},
\]

this is a reasonable assumption early in the disk’s lifetime when the disk is hot (\( H/r \sim 1 \)), but will break down when the disk has cooled significantly (\( H/r \approx r_{\text{out}} \approx 0.3 \)).

Figure 6 shows the surface density profile \( \Sigma \) and aspect ratio \( H/r \) of a TDE disk at different times during its evolution. At early times, \( \Sigma \propto r^{-1/2} \) and decreases with the accretion rate \( \dot{M} \). At later times, the disk begins to cool and the scale-height decreases at a rate proportional to \( \dot{M} \) and becomes \( H/r \propto r^{-1} \), while the surface density increases at the disk’s outer edges. The radial profile of the disk’s surface density \( \Sigma \) switches from \( \Sigma \propto r^{-1/2} \) to \( \Sigma \propto r^{3/2} \) at these late radiation-cooled stages.
Figure 7. Precession ($\omega$, green) and damping ($\gamma$, magenta) rates as functions of the disk’s accretion rate $\dot{m} = \dot{M}/\dot{M}_{Edd}$, for a viscosity parameter of $\alpha = 0.1$ (solid) and $\alpha = 0.01$ (dotted), with dimensionless black hole spin parameter $a_\bullet$ as indicated. The black dashed line denotes the rigid-body Lense-Thirring precession frequency $\omega_{\bullet, \text{rigid}}$ [Eq. (27)], while red lines denote the disk warp estimate $\gamma_{\text{Bate}}$ [Eq. (28)], for $\alpha = 0.1$ (dashed) and $\alpha = 0.01$ (dotted). Here, $M_\bullet = 10^6 M_\odot$, $M_\star = 7 M_\odot$, and $R_\star = 1 R_\odot$.

Figure 8. Same as Figure 7 except $M_\bullet = 10^7 M_\odot$.

profiles for high viscosity ($\alpha = 0.1$) eigenfunctions, while Figure 10 plots $\beta(r)$ and $\tilde{\varphi}(r)$ for low viscosity ($\alpha = 0.01$) eigenfunctions, for select accretion rates. Over most of the TDE disk’s evolution, a decreasing accretion rate $\dot{m}$ and decreasing $H/r$ gives rise to an increasing inner disk warp $\beta(r_{in})/\beta(r_{out})$ and increasing inner disk twist $\tilde{\varphi}(r_{in}) - \tilde{\varphi}(r_{out})$. This is because the internal torque maintaining disk rigidity is proportional to the disk scaleheight ($G \propto H$), and becomes less effective at enforcing rigidity for low $\dot{m}$ values. However, when $\dot{m}$ sufficiently low ($\dot{m} \lesssim 0.2$), the $\beta$ eigenfunctions undergo a qualitative change which cannot be described by simply scaling the solutions to an increased $\beta(r_{in})/\beta(r_{out})$ and $\tilde{\varphi}(r_{in}) - \tilde{\varphi}(r_{out})$. Instead of $\beta$ increasing as the radius $r$ decreases, $\beta$ generally decreases in magnitude, but oscillates as the inner truncation radius $r_{in}$ is approached. The low viscosity $\beta$ solutions develop some nodes [when $\beta(r) \approx 0$], while the high viscosity $\beta$ solutions do not. This indicates high viscosities are able to maintain a smooth non-oscillatory warp profile. The high viscosity $\tilde{\varphi}$ profile increases steadily as the inner disk is approached, while the low viscosity $\tilde{\varphi}$ increases rapidly when near a warp node [when $\beta(r) \approx 0$], but only modestly otherwise.

Figure 11 shows how the precession and damping rates of the disk depend on the SMBH spin $a_\bullet$. At the relatively high accretion rate ($\dot{M} = \dot{M}_{Edd}$), the rigid-body Lense-Thirring precession frequency $\omega_{\bullet, \text{rigid}}$ [Eq. (27)] is an excellent approximation to the disk’s precession frequency $\omega$, and deviates at most by a factor of $\sim 2 - 3$ only for near-
maximally spinning BHs ($a_\bullet \gtrsim 0.9$) due to effects from disk warping. The viscous damping rate $\gamma$ is at least an order of magnitude below $\omega$ for the entire range of viscous parameters $\alpha$ considered, unless the SMBH spin is near maximal ($a_\bullet \gtrsim 0.9$). The Bate damping rate $\gamma_{\text{bate}}$ [Eq. (25)] is comparable to $\gamma$ (within a factor of 10) unless the SMBH is spinning rapidly ($a_\bullet \gtrsim 0.9$)

5 EFFECT OF FALL-BACK MATERIAL

After the star tidally disrupts around the SMBH, the stellar debris rains down on the TDE accretion disk at a rate given by Equation (34). The mass from the fall-back material also deposits angular momentum to the disk, exerting a torque. This section parameterizes the fall-back torque and examines how the combined influence of Lense-Thirring and fall-back torques affect the disk structure, precession and inclination evolution, using the TDE disk model of Section 3.

Consider a star which disrupts on a parabolic orbit with the orbital angular momentum axis $\mathbf{l}_*$. We parameterize the torque per unit area acting on the TDE disk by

$$T_l = \Sigma r^2 \Omega \gamma \delta \mathbf{l}_* ,$$

where

$$\gamma_l = \frac{M_{fb}}{\Sigma r_{out}} \delta (r - r_{out})$$

is the rate of fall-back material accreting onto the outer disk, and $\delta(x)$ is the delta function. We assume $\mathbf{l}_*$ is fixed in time. For simplicity, we consider the case where $\mathbf{l}_*$ does not deviate much from the SMBH spin $\hat{s}$ (the z-axis), and define $W_\star \equiv \mathbf{l}_* (\hat{s} + i \hat{\gamma})$. Thus the complex fall-back torque (per unit area) may be written as

$$T_l = \Sigma r^2 \Omega \gamma (W_\star - W) ,$$

and the total torque (per unit area) acting on the disk is then

$$T = i \Sigma r^2 \Omega_{\bullet} W + T_l .$$

Because of the delta function in Equation (51), the fall-back torque $T_l$ must be handled with care when included in the warp equations (4), (6), and (7). Integrating equation (4) over $dr$ using the total torque (53) from $r = r_{out} - \epsilon$ to $r = r_{out} + \epsilon$, we see $T_l$ causes a discontinuity in the internal torque of

$$[G]_{r = r_{out} - \epsilon} = \lim_{\epsilon \to 0+} \left( G|_{r = r_{out} + \epsilon} - G|_{r = r_{out} - \epsilon} \right)$$

$$= \frac{M_{fb} r^2 \Omega}{2\pi} (W_\star - W) ,$$

(54)

Requiring $G|_{r = r_{out} + \epsilon} = 0$, we see equation (44) can be solved with the total torque (53) by taking

$$T = i \Sigma r^2 \Omega \omega W$$

(55)

when $r < r_{out}$, and forcing $G$ to satisfy the boundary con-
To solve equations (57)-(58), we look for solutions of the form

\[ W(r, t) = W(r)e^{f(t)r} + \tilde{W}_s(r), \]

\[ \tilde{G}(r, t) = \tilde{G}_s(r) e^{f(t)r} + \tilde{G}_a(r). \]

Inserting equations (57)-(58) into equations (4), (6), and (7), we see the functions \( W \) and \( \tilde{G} \) satisfy the homogeneous equations

\[ \frac{d\tilde{G}}{dr} = \Sigma r^3 \Omega(\lambda + i\omega_*) \tilde{W}, \]

\[ \frac{d\tilde{W}}{dr} = \frac{(\lambda - i\kappa\Omega + \alpha\Omega)\tilde{G}}{\Sigma H^2 r^2 \Omega^3} \text{ when } r \geq r_\kappa, \]

\[ \frac{d\tilde{W}}{dr} = \frac{(2\kappa_0 - Q_0)\tilde{G}}{(Q_0^2 + Q_0^2)/\Sigma H^2 r^2 \Omega^3} \text{ when } r < r_\kappa, \]

with the boundary conditions (assuming \( \dot{M}_0 = \dot{M} \))

\[ \tilde{G}(r_{in}) = 0, \quad \tilde{G}(r_{out}) = -\frac{3}{2} \alpha \Sigma H^2 r^2 \Omega^2 \tilde{W} \bigg|_{r=r_{out}}, \]

while the functions \( \tilde{W}_s \) and \( \tilde{G}_s \) satisfy the equations

\[ \frac{d\tilde{W}_s}{dr} = \frac{4(\alpha - i\kappa)\tilde{G}_s}{\Sigma H^2 r^3 \Omega^4}, \]

\[ \frac{d\tilde{G}_s}{dr} = (Q_s + iQ_p)\tilde{G}_s \text{ when } r < r_\kappa, \]

\[ \frac{d\tilde{G}_s}{dr} = \frac{(Q_s^2 + Q_p^2)\Sigma H^2 r^2 \Omega^3}{(Q_s^2 + Q_p^2)/\Sigma H^2 r^2 \Omega^3} \text{ when } r \geq r_\kappa, \]

with the boundary conditions (assuming \( \dot{M}_0 = \dot{M} \))

\[ \tilde{G}_s(r_{in}) = 0, \quad \tilde{G}_s(r_{out}) = -\frac{3}{2} \alpha \Sigma H^2 r^2 \Omega^2 |\tilde{W}|^2 \bigg|_{r=r_{out}}. \]

Since \( \omega_* \rightarrow 0 \) as \( r \rightarrow \infty \), the boundary condition (66) is equivalent to \( W_s(r_{out}) \approx W_s \) when \( r_{out} \gg r_{in} \). The solutions \( W_s \) and \( \tilde{G}_s \) do not evolve in time, and correspond to the disk’ steady-state profile. The simulations of Xiang-Gruess, Ivanov & Papaloizou (2016) looked solely at the steady-state warp profile \( W_s \), while Ivanov, Zhuravlev & Papaloizou (2018) simulated the steady-state profile \( W_s \) and the precessing/damping solution \( \tilde{W} \) simultaneously.

Decomposing the complex eigenfrequency into its real and imaginary parts \( \lambda = \gamma + i\omega \), integrating equation (4) over \( W^2 r^dr \) gives (after integration by parts)

\[ \omega = \omega_* + \omega_\gamma, \quad \gamma = \gamma_\gamma + \gamma_\Omega, \]

where \( \omega_* \) is given by equation (21), \( \omega_\gamma \) by equation (22), \( \gamma_\gamma \) by equation (23), and \( \gamma_\Omega \) by equation (24),

\[ \gamma_\Omega = -\frac{3}{2L_e} \alpha H^2 r^2 \Omega^2 |\tilde{W}|^2 \bigg|_{r=r_{out}} \]

The modified disk angular momentum \( L_+ \) is given in equation (24).

Including the fall-back torque \( T_f \) causes the dynam-


5.2 Steady-State Warp Profiles

Figures 14 and 15 plot the disk’s complex steady-state warp profile \( W_* = \beta_* e^{i \varphi_*} \) for select accretion rate values \( \dot{m} \) and SMBH spin parameters \( \alpha_* \), as indicated. In many ways, the steady-state warp profiles are similar to their precessing warp profile counterparts \( W = \beta e^{i \varphi} \). When the accretion rate is high (\( \dot{m} \gtrsim 1 \)), only nearly-extremal SMBHs (\( \alpha_* \approx 1 \)) have warp profiles \( \beta_* \) which vary substantially across the disk. At these high \( \dot{m} \) rates, the variation in the disk’s twist \( \varphi_* \) is negligible, unless the SMBH is near extremal and disk viscosity high (\( \alpha_1 \approx 0.1 \)). When the accretion rate is low (\( \dot{m} \lesssim 1 \)), \( \beta_* \) becomes non-trivial. Low viscosity disks (\( \alpha_0 \approx 0.01 \)) have much more oscillatory \( \beta_* \), in comparison to their high viscosity (\( \alpha_0 \approx 0.1 \)) counterparts. High viscosity disks with low accretion rates have \( \varphi_* \), which increase steadily across the disk’s radial extent, while low viscosity \( \varphi_* \) variations are negligible unless near a warp node when \( \beta_1(r) \approx 0 \).

The main difference between the steady-state warp profiles \( W_* = \beta_* e^{i \varphi_*} \) (Figs. 14, 15) and the precession profiles \( W = \beta e^{i \varphi} \) (Figs. 10, 11) are the normalization conditions, which cause the profiles to evolve differently as the disk’s accretion rate \( \dot{m} \) drops. The \( W_* \) normalization cannot be freely chosen, and is determined by the tidally-disrupted star’s orbital angular momentum \( W_* = \beta_* e^{i \varphi_*} \). As \( \dot{m} \) decreases, so does the fall-back torque’s magnitude, and it becomes more difficult to tilt the outer disk in opposition to the SMBH’s
Lense-Thirring torque. As a result, \( \dot{\beta}_k \) decreases in magnitude across the entire disk when \( \dot{m} \) is lowered.

6 DISCUSSION

6.1 Theoretical Uncertainties

A major uncertainty in our work is how efficiently the fall-back material influences the disk warp [Eq. (36)]. We have adopted a simple prescription, and fixed the location of the fall-back angular momentum deposition to be at the outer truncation radius of the disk \( r_{\text{out}} \) [Eq. (36)]. Letting the angular momentum be deposited at locations \( r_{\text{dep}} \approx r_{\text{out}} \) will change the damping rates \( \dot{\gamma}_t \) in Figures 12 and 13 by factors of order unity (see also Shen & Matzner 2014 [Xiang-Gruess Ivanov & Papaloizou 2016 Ivanov, Zhuravlev & Papaloizou 2018]. Another uncertainty is our assumption \( M = M_\text{fb} \) when computing our damping rates in Figures 12 and 13. As the disk cools, the viscous time \( t_v \) [Eq. (40)] will become longer than the timescale over which the fall-back torque decreases [\( M_\ast/2M_\text{fb} \sim t_v \), Eq. (53)]. However, the fall-back accretion rate at these times is typically small, and fall-back damping \( \dot{\gamma}_t \) will be negligible compared to viscous damping \( \dot{\gamma}_v \). If the disk is Eddington limited at early times \( (M \lesssim M_{\text{Edd}}) \) as suggested by some (e.g. Metzger & Stone 2016), then \( M \approx M_\text{fb} \approx M_{\text{Edd}} = \text{constant during the Eddington limited phase} \), and our results remain valid.

There are both theoretical justification and observa-
tional evidence that TDE disks may be eccentric. Depending on the pericenter distance of the star’s orbit and the SMBH spin, the eccentric debris streams can take anywhere from $t \sim 1 - 10 \, t_\text{f}$ to completely circularize, with long circularization times ($\sim 4 - 10 \, t_\text{f}$) the most common (e.g. Rees [1988], Guillochon, Mansukian & Ramirez-Ruiz [2013], Piran et al. [2015], Guillochon & Ramirez-Ruiz [2015], Shiokawa et al. [2015], Bonnerot et al. [2016], Hayasaki, Stone & Loeb [2016], Krolik, et al. 2016). Moreover, some emission lines from TDE debris are fit much better by modeling the accretion disk with an order-unity eccentricity (Liu et al. 2017; Cao et al. 2015; Guillochon & Ramirez-Ruiz 2015; Shiokawa et al. 2015; Bonnerot et al. 2016; Hayasaki, Stone & Loeb 2016; Krolik, et al. 2016). In order to understand how such eccentric disks are twisted under the competing influences of relativistic apsidal precession and internal pressure torques, the non-linear eccentric disk theory of Ogilvie (2001) must be used (see also Ogilvie & Lynch 2019), and will also affect the energy dissipation rate in the disk (Barker & Ogilvie 2014; Chan, Krolik & Piran 2018; Wienkers & Ogilvie 2018). A theory of eccentric and warped disks has yet to be developed, making it unclear how relaxing our assumption of circular disks will affect our results.

To model the warped TDE accretion disk, we have used fully relativistic expressions for the apsidal and nodal precession frequencies around the spinning SMBH (e.g. Kato 1990), but neglected changes in length scales due to relativity (affecting $\partial/\partial r$) and time dilation (affecting $\partial/\partial t$) in the SMBH’s accretion disk. Although relativistic theories of warped accretion disks around spinning black holes have been developed (Ivanov & Illarionov 1997; Demianski & Ivanov 1997; Zhuravlev & Ivanov 2011), we chose to use the formalism of Ogilvie (1999) and Lubow & Ogilvie (2000) with the fully relativistic apsidal and nodal precession frequencies for simplicity. We note these other warped disk theories are not fully relativistic: Ivanov & Illarionov (1997), Demianski & Ivanov (1997) include only the leading order post-Newtonian corrections to changes in length scales and time dilation, while Zhuravlev & Ivanov (2011) assume a slowly spinning black hole ($a_\bullet \ll 1$). Since the ISCO radius [Eq. (1)] is approximately equal to the black hole’s event horizon when the black hole is near extremal ($a_\bullet \approx 1$), order unity black hole spins must be included in a relativistic theory to fully understand how time dilation modifies the TDE disk’s precession frequency.

6.2 Observational Implications

Figure 14. Disk warp $\beta_\Lambda(r)$ (top panels) and twist $\tilde{\varphi}_\star(r)$ (bottom panels) radial profiles for the steady-state solution $\dot{W}_\star = \beta_\Lambda e^{i\tilde{\varphi}_\star}$, for $\dot{m} = 10$ (solid), $\dot{m} = 1$ (dashed), and $\dot{m} = 0.1$ (dot-dashed), with dimensionless SMBH spin parameters $a_\bullet$ as indicated. We normalize the solutions to the complex orbital angular momenta of the tidally disrupted star $W_\star = \beta_\star e^{i\varphi_\star}$. Here, $\alpha = 0.1, M_\star = 10^6 M_\odot, M_\bullet = 1 M_\odot$, and $R_\star = 1 R_\odot$. We assume $M = M_\odot$.

When the accretion rate is high ($\dot{M} \gtrsim 0.4 \dot{M}_{\text{Edd}}$), we have shown the rigid-body Lense-Thirring precession frequency $\dot{\omega}_{\text{rigid}}$ [Eq. (27)] is a good approximation to the lowest-order precession frequency $\omega$ of the disk (Figs. 7<sup>C</sup> & 11), differing by factors of $\sim 2 - 3$ only when the SMBH spin is sufficiently high ($a_\bullet \gtrsim 0.9$, Fig. 11). But when the TDE disk’s accretion rate is low ($\dot{M} \lesssim 0.4 \dot{M}_{\text{Edd}}$), the disk’s precession frequency $\omega$ can differ from $\dot{\omega}_{\text{rigid}}$ substantially. Figures 7<sup>C</sup> and 11 show the deviation of $\omega$ from $\dot{\omega}_{\text{rigid}}$ can be a factor of a few for disks with high viscosities ($\alpha = 0.1$),
and be more than an order of magnitude for disks with low viscosities ($\alpha = 0.01$). The precession frequency of TDE disks with low accretion rates and viscosities can vary with $M$ in such a dramatic manner, that it is nearly impossible to get any information on the SMBH by analyzing the TDE disk’s precession rate (see Sec. 2.2 for discussion of the reason behind a highly variable $\omega$ with $M$). Any stable detected quasi-periodic oscillations in TDEs from precessing accretion disks (e.g. Burrows et al. 2011; Saxton et al. 2012) must therefore be either in a high accretion phase ($\dot{M} \gtrsim 0.4 M_{\text{Edd}}$), or have high viscosities ($\alpha \sim 0.1$).

When the fall-back material has a negligible influence on the precessing disk’s evolution, we find a wide range of viscous alignment timescale of the TDE disk with the SMBH’s equatorial plane. The viscous alignment timescales range anywhere between a few years to a few days, depending on the disk’s accretion rate and viscosity (Figs. 7-8), as well as the SMBH spin (Fig. 11). The viscous damping rate is typically at least an order of magnitude below the precession frequency, unless the SMBH spin is high ($\alpha \gtrsim 0.9$) and viscosity parameter is large ($\alpha \sim 0.1$). The TDE disk should therefore stably precess around the SMBH spin vector for many precession periods unless the SMBH is near extremal and disk viscosity high. When the accretion rate is sufficiently low ($\dot{M} \sim 0.1 M_{\text{Edd}}$), the viscous damping rates $\gamma_\nu$ are comparable to the precession frequencies $\omega$ for high viscosity disks ($\alpha = 0.1$), and can exceed the $\omega$ values of low viscosity disks ($\alpha = 0.01$). Therefore, when the accretion rate is low, coherent precession is unlikely to be detectable due to the rapid alignment of the TDE disk with the SMBH’s equatorial plane.

In contrast, the fall-back alignment rates are typically comparable to or exceeding the disk’s precession frequency when the disk’s accretion rate is high ($\dot{M} \gtrsim M_{\text{Edd}}$; Figs. 12-13). The inclusion of the fall-back torque causes the initially misaligned and precessing TDE disk to rapidly evolve into its steady-state warp profile. For the TDE parameters investigated in this work, the disk warp evolves to its steady-state profile in a few to tens of days. If quasi-periodic oscillations in the hard X-ray from tidal disruption flares are emitted by a precessing accretion disk (e.g. Burrows et al. 2011; Saxton et al. 2012), then the fall-back material must deposit far less angular momentum to the accretion disk than we assumed with our prescription (50).

For both the rigidly precessing (Figs. 9-10) and steady-state (Figs. 14-15) warp profiles of the TDE disk, the inner disk always has a higher tilt to the SMBH’s equitorial plane than the outer disk [$\dot{\beta}(r_{\text{in}}) > \dot{\beta}(r_{\text{out}})$ and $\dot{\beta}_*(r_{\text{in}}) > \dot{\beta}_*(r_{\text{out}})$], in sharp contrast to the “standard” picture dating back to Bardeen & Petterson (1975). Previous models of warped TDE disks obtained different results because they neglected the dominant internal torque (pressure rather than viscosity) acting the disk (e.g. Lei, Zhang & Gao 2013). Including the highly tilted inner edge of a TDE accretion disk will further constrain models explaining the variability in the hard X-ray of jetted TDEs with Lense-Thirring precession (Stone & Loeb 2012), since the TDE jet is likely to be tightly coupled to the inner edge of the accretion disk (Liska et al. 2018a).

Figure 15. Same as Figure 14 except $\alpha = 0.01$. 

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**Figure 15.** Same as Figure 14 except $\alpha = 0.01$. 

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7 CONCLUSIONS

We have carried out a systematic analysis of the dynamics and evolution of warped accretion disks that are misaligned with the equatorial plane of the central SMBH. Such disks are naturally produced in TDEs when the stellar orbital angular momentum axis is misaligned with the BH spin axis. Even with our somewhat idealized model of the TDE disks, our work clarifies many disagreements in the literature, and uncovers several new dynamical behaviors of the TDE disk evolution.

Section 2 examines the warp profile, precession and viscous damping rates of simple disk models (powerlaw $\Sigma$, constant $H/r$) around a black hole. We find that to properly calculate the warp profile, it is important to include pressure torques, which dominate viscous torques because of relativistic apsidal precession. The inner disk is generally more tilted than the outer disk (Figs. 12-13). The global disk precession frequency and viscous damping rate vary by more than an order of magnitude as the disk scaleheight is varied (Fig. 4), due to the sensitive dependence of the effective potential on the disk scaleheight [Eq. (30)].

Section 3 constructs a simple model for the TDE disk soon after the star tidally disrupts. We obtain analytic prescriptions for the disk’s surface density and aspect ratio (Fig. 5), which depend on the accretion rate.

Section 4 uses our analytic disk model (Sec. 3) to calculate a TDE disk’s tilt profile, as well as the precession and damping rates of the disk with respect to the SMBH’s equatorial plane. Like the simple disk model studied in Section 2, we find the inner disk to be more tilted than the outer disk, and the tilt profile to become more oscillatory with lower accretion rates and viscosities (Figs. 9-10). Disks with high accretion rates have global precession frequencies which closely match the disk’s rigid-body Lens-Thirring precession frequency, but disks with low accretion rates can have precession frequencies which depart from the rigid-body Lens-Thirring precession frequency by orders of magnitude (Figs. 7-8).

Section 5 examines how angular momentum deposition by fall-back material affects the warp structure and inclination evolution of the TDE disk. The main effect of the fall-back material is to cause the disk tilt to rapidly evolve to its steady-state profile, over a timescale shorter than the disk’s global precession period (Figs. 2-3). The steady-state warp profiles have a similar structure as the precessing warp profiles, except the steady-state warp amplitude decreases with the disk’s accretion rate (Figs. 14-15).

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APPENDIX A: DENSITY WAVE DISPERSION RELATION IN A VISCOUS, NON-KEPLERIAN DISK

As discussed in Section 2, the behavior of bending waves depends critically on the disk aspect ratio $H/r$ in comparison to the Shakura-Sunyaev viscosity parameter $\alpha$ and dimensionless non-Keplerian epicyclic frequency $\kappa = (\Omega^2 - \kappa^2)/2\Omega^2$. This section shows this condition may be understood using WKB theory for inertial-density waves.

Consider an accretion disk with vertically isothermal sound-speed $c_s = H\Omega$, unperturbed density $\rho(r,z) = \rho(r)e^{-r^2/2H^2}$, pressure $p(r,z) = c_s^2\rho(r,z)$, and azimuthal fluid velocity $v_\varphi(r) = r\Omega$, and with radial $v_r$ and vertical $v_z$ velocity components equal to zero. We perturb each equilibrium state quantity $X$ by a perturbation $\delta X$ which satisfies

$$\frac{\partial \delta X}{\partial r} - \frac{\partial \delta X}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\delta X}{r} \right).$$

Moreover, we assume the disk is thin ($H/r \ll 1$), so the equilibrium quantities $X(r,z)$ satisfy

$$\frac{\partial X}{\partial z} \gg \frac{\partial X}{\partial r} \sim X.$$  \hspace{1cm} (A2)

Decomposing the azimuthal and time dependences of the perturbations $\delta X$ as $\delta X(r,z,\varphi,t) = \delta X(r,z)e^{i(m\varphi - \omega t)}$, we have (e.g. Fu & Lai 2009)

$$- i\omega \delta \varphi + \rho \frac{\partial}{\partial r} \delta v_r + \frac{\partial}{\partial z} (\rho \delta v_z) = 0,$$  \hspace{1cm} (A3)

$$- i\omega \delta v_r - 2\Omega \delta v_\varphi = - \frac{1}{\rho} \frac{\partial}{\partial r} (\rho \delta v_r) + (f_\varphi)_r,$$  \hspace{1cm} (A4)

$$- i\omega \delta v_\varphi + \frac{\kappa^2}{2\Omega} \delta v_z = (f_z)_\varphi,$$  \hspace{1cm} (A5)

$$- i\omega \delta v_z = - \frac{1}{\rho} \frac{\partial}{\partial r} \delta p + \frac{1}{\rho^2} \frac{\partial}{\partial z} (\rho \delta p) + (f_z)_z,$$  \hspace{1cm} (A6)

where

$$f_\varphi = \nu \left( \frac{4}{3} \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \left( \frac{\partial}{\partial z} \right) + \frac{\partial^2}{\partial z^2} \right) \delta v_\varphi,$$  \hspace{1cm} (A7)

$$f_\varphi = \nu \left( \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \left( \frac{\partial}{\partial z} \right) + \frac{\partial^2}{\partial z^2} \right) \delta v_\varphi,$$ \hspace{1cm} (A8)

$$f_z = \nu \left( - \frac{2}{3} \frac{\partial}{\partial z} + \frac{1}{3} \frac{\partial^2}{\partial z^2} \right) \delta v_z,$$ \hspace{1cm} (A9)

are the viscous force terms, and

$$\omega = \omega - m\Omega.$$ \hspace{1cm} (A10)

We assume $m = O(1)$ throughout this section.

A1 High Vertical Wavenumber Limit

When the vertical gradients of the fluid perturbations satisfy $\partial \delta X/\partial z \gg \delta X/H$, we may assume $\delta X \propto e^{i(k_r r + k_z z)}$, and equations (A3)-(A6) become

$$- i\omega \delta \varphi + ik_r r \Omega \delta v_r + ik_z r \Omega \delta v_z = 0,$$ \hspace{1cm} (A11)

$$- i\omega r \Omega \delta v_r - 2r\Omega^2 \delta v_\varphi = -ik_r c_s^2 \delta \varphi - \alpha \kappa^2 k_z^2 \delta v_z,$$ \hspace{1cm} (A12)
Specifically, writing

\[ -i\omega \rho \Omega \delta \tilde{v}_r + \frac{r n^2}{2} \delta \tilde{v}_r = 0 \]  
\[ -i\omega \rho \Omega \delta \tilde{v}_z = -ic_2^2 k_z \delta \rho - \alpha r c_2^2 k_z^2 \delta \tilde{v}_r, \]

where \( \delta \rho = \delta \rho / \rho, \delta \tilde{v} = \delta \tilde{v} / r \Omega, \)

\[ k_z^2 = \frac{4}{3} k_z^2 + k_z^2, \]
\[ k_{2z}^2 = k_z^2 + \frac{4}{3} k_z^2, \]
\[ k_{2z}^2 = -\frac{2}{3} k_z k_r + k_z k_z = \frac{1}{3} k_z k_z, \]

and

\[ \tilde{\varphi}_r = \varphi + i\alpha r c_2^2 \tilde{\varphi}_r, \]
\[ \tilde{\varphi}_z = \varphi + i\alpha r c_2^2 \tilde{\varphi}_z, \]
\[ \tilde{\varphi}_z = \varphi + i\alpha r c_2^2 \tilde{\varphi}_z. \]

Equations (A11)-(A14) may be solved for the dispersion relation

\[ (\tilde{\varphi}_r, \tilde{\varphi}_z - k^2)(\Omega^2 - k_{2z}^2 \Omega^2) + c_2^2 \tilde{\varphi}_z \Omega^2. \]

From Equation (A23), we see the cross terms \( k_{rz} \) and \( k_{rz} \) are negligible when \( \alpha c_2^2 k_z^2 \ll \Omega^2 \). Assuming \( k_z = \sqrt{n} / H \), where \( n \) is the order of the wave and measures the number of vertical nodes, we see the cross terms are negligible when \( n \alpha \ll 1 \). The next section derives the dispersion relation for these low-order density waves \( n = O(1) \).

## A2 Low-Order Inertial-Density Waves

This section derives the dispersion relation for low-wavenumber density waves \( n \ll \alpha^{-1} \). Using the fact that the cross viscous force terms are negligible when \( n \) is sufficiently low, the viscous force components reduce to

\[ \langle f_r \rangle \propto \alpha H^2 \Omega \left( \frac{4}{3} \frac{\partial^2}{\partial z^2} + \frac{\partial \ln \rho}{\partial z} \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2} \right) \delta v_r, \]
\[ \langle f_\varphi \rangle \propto \alpha H^2 \Omega \left( \frac{\partial^2}{\partial r^2} + \frac{4}{3} \frac{\partial \ln \rho}{\partial z} \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2} \right) \delta v_\varphi, \]
\[ \langle f_z \rangle \propto \alpha H^2 \Omega \left( \frac{\partial^2}{\partial r^2} + \frac{4}{3} \frac{\partial \ln \rho}{\partial z} \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2} \right) \delta v_z. \]

Since \( \partial \ln \rho / \partial z = -z / H^2 \), one may decompose the vertical dependence of the fluid perturbations in terms of Hankel Functions \( H_n(Z) \):

\[ H_n(Z) = (-1)^n e^{z^2/2} \left( \frac{d}{dz} \right)^n e^{-z^2/2}. \]

Specifically, writing

\[ \delta \rho = \rho \delta \rho H_n \left( \frac{z}{H} \right) e^{ik_r r}, \]
\[ \delta v_r = r \Omega \delta \tilde{v}_r H_n \left( \frac{z}{H} \right) e^{ik_r r}, \]
\[ \delta v_\varphi = r \Omega \delta \tilde{v}_\varphi H_n \left( \frac{z}{H} \right) e^{ik_r r}. \]

### A3 Limiting Case: Bending Waves

For a bending wave \( m = n = 1 \), dispersion relation (A30) for a low-frequency \( (\omega \ll \Omega) \), long radial wavelength \( (k_r \ll H^{-1}) \) disk with \( \alpha \ll 1 \) reduces to

\[ \omega^2 + (i\alpha + \tilde{k}) \Omega - \frac{1}{4} k_r^2 c_2^2 \approx 0. \]

Equation (A40) may be solved for the group velocity of the bending wave \( v_{bw} = d\omega / dk_r \):

\[ v_{bw} \approx \pm \frac{k_r H_c}{2 \sqrt{(i\alpha + \tilde{k})^2 + k_r^2 H^2}}. \]

The group velocity (A41) expresses the efficiency of angular momentum exchange by bending waves.

When \( |i\alpha + \tilde{k}| \leq k_r \), Equation (A41) reduces to

\[ v_{bw} \approx \pm \frac{k_r H_c}{2(i\alpha + \tilde{k})}. \]

In other words, when the disk viscosity parameter \( \alpha \) and dimensionless non-Keplerian epicyclic frequency \( \tilde{k} \) are sufficiently low, bending waves travel at half the sound-speed. When \( |i\alpha + \tilde{k}| \gg k_r \), Equation (A41) reduces to a different expression:

\[ v_{bw} \approx \pm \frac{k_r H_c}{2 \sqrt{(i\alpha + \tilde{k})^2 + k_r^2 H^2}}. \]

Thus, when \( \alpha \) becomes sufficiently large, the bending waves becomes diffusive, while when \( |\tilde{k}| \) becomes sufficiently large, bending waves travel at speeds significantly less than \( c_s / 2 \).

For the long radial wavelength bending waves of interest for this work \( (k_r \sim r^{-1}) \), the condition for bending waves to travel at \( |v_{bw}| \gtrless |c_s / 2| \) is

\[ \alpha \lesssim \frac{H}{r} \quad \text{and} \quad \tilde{k} \lesssim \frac{H}{r}. \]
We begin with the warped disk equations [see Eqs. (4) & (6)].

\[ \frac{\partial G}{\partial t} = i\kappa G + \frac{\Sigma c_s^2 r^3 \Omega}{4} \frac{\partial W}{\partial r}, \]  

(B2)

where \( \dot{\Omega}_\perp = (\Omega^2 - \Omega^2_\perp)/2G^2 \) is the dimensionless non-Keplerian nodal precession rate, \( \Omega_\perp \) is the nodal precession rate, and all other quantities are the same as usual. We assume the disk is inviscid (\( \alpha = 0 \)). Looking for eigenmode solutions of the form \( W, G \propto e^{i\omega t} \), the warp equations may be rearranged to give

\[ \frac{\partial}{\partial r} \left[ \frac{\Sigma c_s^2 r^3 \Omega}{4(\omega - \kappa \Omega)} \frac{\partial W}{\partial r} \right] + \Sigma r^3 \Omega (\omega - \dot{\Omega}_\perp \Omega) W = 0. \]  

(B3)

When \( c_s^2 \ll r^2 (\omega - \dot{\Omega}_\perp \Omega)/(\omega - \kappa \Omega) \), we may assume \( \partial^2 W/\partial r^2 \gg \partial W/\partial r \), and the above equation simplifies to

\[ \frac{\partial^2 W}{\partial r^2} + \frac{4(\omega - \dot{\Omega}_\perp \Omega)/(\omega - \kappa \Omega)}{c_s^2} W \approx 0. \]  

(B4)

For the rest of this section, we assume \( c_s = \text{constant} \). Letting \( t_{bw} = 2r/c_s \) and \( x = \ln(r/R_g) \), the above equation may be rearranged to give

\[ \frac{\partial^2 W}{\partial x^2} + t_{bw}^2 (\omega - \dot{\Omega}_\perp \Omega)/(\omega - \kappa \Omega) W = 0. \]  

(B5)

Let

\[ V(x) = t_{bw}^2 (\omega - \dot{\Omega}_\perp \Omega)/(\omega - \kappa \Omega), \]

and defining

\[ k = \sqrt{|V|}, \]  

(B7)

the WKB solution of Equation (B5) is

\[ W(x) = \frac{A_+}{\sqrt{k}} e^{i\int \kappa \, dkx'} + \frac{A_-}{\sqrt{k}} e^{-i\int \kappa \, dkx'}. \]  

(B8)

when \( V(x) > 0 \), and

\[ W(x) = \frac{B_+}{\sqrt{k}} e^{\int \kappa \, dkx'} + \frac{B_-}{\sqrt{k}} e^{-\int \kappa \, dkx'} \]

(B9)

when \( V(x) < 0 \), where \( A_\pm, B_\pm \) are constants to be determined by the boundary conditions.

For simplicity, we assume the disk is truncated at \( x_{in} = 0 \) (\( r_{in} = R_g \)) and leave \( x_{out} \) (\( r_{out} = e^{x_{out}} R_g \)) to be a free parameter. We examine the disk's eigenmodes for a toy model:

\[ \dot{\Omega}_\perp = \kappa = x\omega_0/\Omega. \]  

(B10)

We assume the usual torque-free boundary conditions:

\[ \frac{\partial W}{\partial x} \bigg|_{x=x_{out}} = 0, \]  

(B11)

with a normalization condition \( W(x_{out}) = 1 \). We look for low-frequency solutions \( (|\omega| \leq \omega_0) \). We define \( x_c = \omega/\omega_0 \) as the critical radius where \( V(x_c) = 0 \). Notice with our toy model, \( V(x) \geq 0 \) everywhere.

The outer boundary condition gives [assuming \( k(x_{out}) \gg 1 \)]

\[ W(x) \approx \sqrt{\frac{k(x_{out})}{k}} \cos \left( \int_x^{x_{out}} k dx' \right), \]  

(B12)

while the inner boundary condition is satisfied when [assum-
ing $k(0) \gg 1$
\[
\int_0^{x_{\text{out}}} |x - x_c| \, dx \cong \frac{\pi n}{t_{\text{bw}} \omega_0}, \tag{B13}
\]
This equation has solutions
\[
\frac{\omega}{\omega_0} = \frac{x_{\text{out}}}{2} \left( 1 \pm \sqrt{\frac{4\pi n}{x_{\text{out}} t_{\text{bw}} \omega_0} - 1} \right), \tag{B14}
\]
where $n$ is an integer. Requiring the square root in expression to be positive forces
\[
n > \frac{x_{\text{out}} t_{\text{bw}} \omega_0}{4\pi}. \tag{B15}
\]
Therefore, the lowest order eigenvalue $\omega$ is given by
\[
\omega = \frac{\omega_0 x_{\text{out}}}{2} \left( 1 - \sqrt{\frac{4\pi n_{\text{min}}}{x_{\text{out}} t_{\text{bw}} \omega_0} - 1} \right), \tag{B16}
\]
where
\[
n_{\text{min}} = \left\lceil \frac{x_{\text{out}} t_{\text{bw}} \omega_0}{4\pi} \right\rceil, \tag{B17}
\]
and $\lceil \cdot \rceil$ is the ceiling function. Note that $n$ counts the number of nodes in the disk’s warp amplitude [when $W(x) = 0$].

Figure [31] plots the numerically and analytically computed eigenfrequencies $\omega$ as a function of $t_{\text{bw}} \omega_0$. Eigenvalues computed numerically solve Equation (B13) using a shooting algorithm, while the analytic eigenvalues are given by Equation (B16). Although these eigenfrequencies quantitative values differ by a factor of $\sim \omega_0$ due to the crudeness of the WKB approximation, both display oscillations in the rigid-body precession frequency $\omega$ as $t_{\text{bw}} \omega_0$ increases. This is because when $t_{\text{bw}} \omega_0$ increases, the $n_{\text{min}}$ value of the disk’s lowest-order eigenmode changes. This causes an increase in the disk’s eigenfrequency $\omega$. A characteristic of the disk’s eigenfunction $W(x)$ when $n_{\text{min}}$ changes values is the number of nodes [when $W(x) = 0$] increases.

The fact that the number of nodes changes at each of the eigenfrequency peaks is shown clearly in Figure [52]. Displayed are the eigenfunctions for the $t_{\text{bw}} \omega_0$ values marked by vertical black lines in Fig. [31] As $t_{\text{bw}} \omega_0$ increases, so does the number of nodes in the disk’s warp amplitude for the lowest-order eigenmode.

This toy model is analogous to a disk around a spinning black hole, since a black hole disk’s effective potential $V(r) > 0$ over most of the disk’s radial extent [see Eqs. [12], [16], & (30)]. The precession frequency of a disk around a spinning black hole also has a sensitive and non-monotonic dependence on the bending wave crossing timescale $t_{\text{bw}}$, or equivalently the disk scaleheight $H/r$.

**REFERENCES**

Abramowicz M. A., Chen X., Kato S., Lasota J.-P., Regev O., 1995, ApJ, 438, L37

Abramowicz M. A., Czerny B., Lasota J. P., Szušskiewicz E., 1988, ApJ, 332, 646

Arcavi I., et al., 2014, ApJ, 793, 38

Bade N., Komossa S., Dahlem M., 1996, A&A, 309, L35

Ballbus S. A., Munnery M. A., 2018, MNRAS, 2356

Bardeen J. M., Petterson J. A., 1975, ApJ, 195, L65

Bardeen J. M., Press W. H., Teukolsky S. A., 1972, ApJ, 178, 347

Barker A. J., Ogilvie G. I., 2014, MNRAS, 445, 2637

Bate M. R., Bonnell I. A., Clarke C. J., Lubow S. H., Ogilvie G. I., Pringle J. E., Tout C. A., 2000, MNRAS, 317, 773

Bloom J. S., et al., 2011, Sci, 333, 203

Bogdanović T., Eracleous M., Mahadevan S., Sigurdsson S., Laguna P., 2004, ApJ, 610, 707

Bonnerot C., Rossi E. M., Lodato G., Price D. J., 2016, MNRAS, 455, 2253

Brown G. C., et al., 2015, MNRAS, 452, 4297

Burrows D. N., et al., 2011, Natur, 476, 421

Cao R., Liu F. K., Zhou Z. Q., Komossa S., Ho L. C., 2018, MNRAS, 480, 2929

Cannizzo J. K., Lee H. M., Goodman J., 1990, ApJ, 351, 38

Cenko S. B., et al., 2012, ApJ, 753, 77

Cenko S. B., et al., 2012, MNRAS, 420, 2684

Chakraborty C., Bhattacharyya S., 2017, MNRAS, 469, 3062

Chen C.-H., Krolik J. H., Piran T., 2018, ApJ, 856, 12

Chornock R., et al., 2014, ApJ, 780, 44

Curd B., Narayan R., 2018, ArXiv e-prints, [arXiv:1811.06971]

Dai L., McKinney J. C., Roth N., Ramirez-Ruiz E., Miller M. C., 2018, ApJ, 859, L20

Demianski M., Ivanov P. B., 1997, A&A, 324, 829

Donato D., et al., 2014, ApJ, 781, 59

Evans C. R., Kochanek C. S., 1989, ApJ, 346, L13

Foucart F., Lai D., 2014, MNRAS, 445, 1731

Fragile P. C., Blaes O. M., Anninos P., Salmonson J. D., 2007, ApJ, 668, 417

Fragile P. C., Anninos P., 2005, ApJ, 623, 347

Fu W., Lai D., 2009, ApJ, 690, 1386

Fuller J., Lai D., 2013, MNRAS, 430, 274

Fuller J., Lai D., 2012, MNRAS, 421, 426

Franchini A., Lodato G., Facchini S., 2016, MNRAS, 455, 1946

Gezari S., et al., 2006, ApJ, 653, L25

Gezari S., et al., 2008, ApJ, 676, 944

Gezari S., et al., 2009, ApJ, 698, 1367

Greiner J., Schwartz R., Zharkov S., Orlo M., 2000, A&A, 362, L25

Guillochon J., Manukian H., Ramirez-Ruiz E., 2014, ApJ, 783, 23

Guillochon J., Ramirez-Ruiz E., 2013, ApJ, 767, 25

Guillochon J., Ramirez-Ruiz E., 2015, ApJ, 809, 166

Hawley J. F., 2000, ApJ, 528, 462

Hawley J. F., Guan X., Krolik J. H., 2011, ApJ, 738, 84

Hawley J. F., Krolik J. H., 2018, ArXiv e-prints, [arXiv:1809.01979]

Hayasaki K., Stone N., Loeb A., 2016, MNRAS, 461, 3760

Ivanov P. B., Illarionov A. F., 1997, MNRAS, 285, 394

Ivanov P. B., Zhuravlev V. V., Papaloizou J. C. B., 2018, MNRAS, 2379

Kato S., 1990, PASJ, 42, 99

Khalilullah I., Sazonov S., 2014, MNRAS, 444, 1041

King A. R., Lubow S. H., Ogilvie G. I., Pringle J. E., Tout C. A., 2000, MNRAS, 317, 773

Komossa S., Bade N., 1999, A&A, 343, 775

Komossa S., et al., 2008, ApJ, 678, L13

TDE Disk Warp and Inclination Evolution 19
Calculate the warp and alignment timescale for a SMBH misaligned with the AGN disk. Neglect pressure torques.