Spectral Properties of the $k$–Body Embedded Gaussian Ensembles of Random Matrices for Bosons

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October 30, 2018

Abstract

We consider $m$ spinless Bosons distributed over $l$ degenerate single–particle states and interacting through a $k$–body random interaction with Gaussian probability distribution (the Bosonic embedded $k$–body ensembles). We address the cases of orthogonal and unitary symmetry in the limit of infinite matrix dimension, attained either as $l \to \infty$ or as $m \to \infty$. We derive an eigenvalue expansion for the second moment of the many–body matrix elements of these ensembles. Using properties of this expansion, the supersymmetry technique, and the binary correlation method, we show that in the limit $l \to \infty$ the ensembles have nearly the same spectral properties as the corresponding Fermionic embedded ensembles. Novel features specific for Bosons arise in the dense limit defined as $m \to \infty$ with both $k$ and $l$ fixed. Here we show that the ensemble is not ergodic, and that the spectral fluctuations are not of Wigner–Dyson type. We present numerical results for the dense limit using both ensemble unfolding and spectral unfolding. These differ strongly, demonstrating the lack of ergodicity of the ensemble. Spectral unfolding shows a strong tendency towards

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picket–fence type spectra. Certain eigenfunctions of individual realizations of the ensemble display Fock–space localization.

PACS numbers: 02.50.Ey, 05.30.Jp, 05.45.-a, 21.10.-k, 24.60.Lz
1 Introduction

Random–matrix theory (RMT) was introduced by Wigner [1] to describe the complexity of the spectra and wave functions of atomic nuclei. It was soon realized that RMT could be applied successfully to a large variety of systems [2, 3]. Examples include atoms, molecules, atomic nuclei and quantum dots. For such systems, the RMT modeling is not completely realistic since all many–body systems are effectively governed by one– and two–body forces. This fact motivated work on the two–body random ensembles for Fermions (see Refs. [4] to [10]) and inspired the pioneering work of French and collaborators on the subject. Especially, it led to the introduction of the $k$–body embedded ensembles by Mon and French [11] (see also Ref. [12]). In the embedded ensembles (EE), stochasticity is generated at the $k$–body level. The matrix of the $k$–body interaction is then taken between $m$–particle states with $k \leq m$. The latter are obtained by distributing $m$ particles over $l$ degenerate single–particle levels. For $k < m$, the matrix elements of the $k$–body interaction in the space of $m$–body states are correlated. The number of independent random variables is much smaller than in RMT. The question arose whether these more realistic ensembles yield the same results as RMT. Early numerical experiments performed for interacting Fermion systems [4, 5, 6] of rather small matrix dimension showed that spectral fluctuation properties of the EE agreed with those of RMT. For Bosonic systems, Manfredi [13] argued that the spectral fluctuations should coincide with the predictions of RMT. More recently Patel et al. [14] performed simulations for a Bosonic case with $m > l$ and concluded that the agreement with RMT carries over into the so–called dense limit. As far as we know, there are no results on the ergodic behavior of the EE for Boson systems.

Interest in model Hamiltonians with random two–body interactions has resurged in recent years in several areas of many–body physics (see Refs. [15] to [30]), and the question of possible further differences between such models and RMT has resurfaced. For instance, Altshuler et al. [17] suggested localization in Fock space as a realistic possibility in a system of interacting electrons in a random one–body potential. In a different context, calculations using random two–body interactions within the nuclear shell–model [21] and in the framework of the Interacting Boson Model [23] led to the observation of pairing properties and band structures in the spectra. These “regular” features in the low–lying part of the spectrum are robust and seem to contradict the expectations based on the canonical ensembles of RMT.
Recently, three of the present authors found a novel analytical approach to the Fermionic embedded ensembles \cite{31, 32} in the limit of infinite matrix dimension \((l \to \infty)\). The main results of this approach are: (i) Within a certain parameter range \((2k > m)\), the average spectrum has semicircle shape, and the spectral fluctuation properties coincide with those of RMT. (ii) The spectral density changes shape at or near \(2k = m\) and becomes Gaussian. (iii) In the dilute limit \((k \ll m \ll l)\) the spectral fluctuations are completely uncorrelated (Poissonian). (iv) The spectral fluctuations change gradually from Wigner–Dyson to Poissonian.

In this paper, we extend our analytical treatment to the case of the \(k\)-body embedded Gaussian ensembles of random matrices for Bosonic systems. Our motivation for doing so is twofold. First, it is instructive to understand possible differences arising from symmetry (fully symmetric versus fully antisymmetric states). To this theoretical argument we add a second and more practical consideration: In an atomic trap, Bosonic atoms occupy partly degenerate single-particle states. The interaction will lift the degeneracy. A random-matrix approach should reveal the generic features of the resulting spectrum.

Our approach is largely patterned after the one for Fermionic systems which in turn is based on the invariance properties of the EE. We focus attention on the second moment \(B^{(k)}\) of the random \(k\)-body interaction in the Hilbert space of many-body states. We derive an eigenvector expansion formally equivalent to the Fermionic one. We exploit this formal equivalence and obtain results for the shape of the spectral density and for the spectral fluctuations in terms of the parameters \(k, m,\) and \(l\) of the problem. All our results are obtained in the limit of infinite Hilbert-space dimension. For Bosons this limit is realized either by letting \(l \to \infty\) (the same limit as for Fermions), or by letting \(m \to \infty\) with \(k\) and \(l\) fixed. This second case is novel and has no analogue in the Fermionic case. We refer to this case as to the dense limit. The tools at our disposal are the supersymmetry method \cite{33, 34} and the binary correlation method. Neither of these methods really works in the dense limit except to show that standard Wigner–Dyson statistics will not apply. Therefore, we resort here to numerical simulations.

The Bosonic embedded ensembles are defined in such a way that for \(k = m\), they coincide with one of the standard ensembles of RMT. Deviations can occur only for \(k < m\). In the dilute limit \((k \ll m \ll l)\) there is no difference between Bosons and Fermions. Therefore, we expect that the embedded ensembles for Fermions and Bosons are qualitatively similar as long as \(k <
This is indeed what we find. The physically most interesting and novel case is, therefore, the dense limit. Here, we prove analytically that the ensemble is not ergodic. Numerical results for the spectral correlations are obtained both by ensemble unfolding and by spectral unfolding. In the latter case we find a strong tendency of the spectra towards picket–fence behavior. We also show that certain eigenfunctions display Fock–space localization.

The paper is organized as follows. The Bosonic embedded ensemble are defined in Section 2. The second moment $B^{(k)}$ is introduced, and an important identity is derived, in Section 3. The eigenvector expansion derived in Section 4 is a central analytical result of this work. This expansion is formally identical to the one found for the Fermionic ensembles. It enables us to calculate the low moments of the interaction (Section 5) and to apply the supersymmetry method (Section 6) in a straightforward way. In Section 7 we present numerical results. Section 8 contains our conclusions. A preliminary account of some of our results was given in Ref. [35].

2 Definitions and Elementary Facts

We consider $m$ spinless Bosons in $l$ degenerate single–particle states with associated creation and annihilation operators $b_j^\dagger$ and $b_j$ where $j = 1, \ldots, l$. The Bosons are coupled through a random $k$–body interaction $V_k(\beta)$, with $k \leq m$, given by

$$V_k(\beta) = \sum_{1 \leq j_1 \leq \ldots \leq j_k \leq l} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq l} v_{j_1, \ldots, j_k; i_1, \ldots, i_k} b_{j_1}^\dagger \cdots b_{j_k}^\dagger b_{i_k} \cdots b_{i_1} \frac{\mathcal{N}(j_1, \ldots, j_k)\mathcal{N}(i_1, \ldots, i_k)}{N(j_1, \ldots, j_k)}.$$

(1)

We refer to $k$ as to the rank of the interaction. As in the canonical case, we use the labels $\beta = 1$ and $\beta = 2$ for the orthogonal and the unitary ensemble, respectively. The matrix element $v_{j_1, \ldots, j_k; i_1, \ldots, i_k}$ of the $k$–body interaction taken between the single–particle states $j_1, \ldots, j_k$ and $i_1, \ldots, i_k$ is totally symmetric with respect to $j_1, \ldots, j_k$ and $i_1, \ldots, i_k$. The elements differing in the sequence of indices $\{j_1 \ldots j_k; i_1 \ldots i_k\}$ (except for permutations of $\{j_1 \ldots j_k\}$ and of $\{i_1 \ldots i_k\}$ and for symmetries specified by $\beta$) are uncorrelated Gaussian–distributed random variables with zero mean and a common second moment $v_0^2$. Thus,

$$v_{j_1, \ldots, j_k; i_1, \ldots, i_k} = v_0^2[\delta_{j_1 i_1} \cdots \delta_{j_k i_k}\delta_{i_1 j_1} \cdots \delta_{i_k j_k} + \delta_{j_1 i_1} \cdots \delta_{j_k i_k}\delta_{i_1 j_1} \cdots \delta_{i_k j_k}].$$

(2)
The overbar denotes the average over the ensemble. Without loss of generality we put \( v_0^2 = 1 \) in the sequel. The normalization coefficients \( \mathcal{N}(j_1, \ldots, j_k) \) and \( \mathcal{N}(i_1, \ldots, i_k) \) in Eq. (1) are defined below and are introduced in order to obtain for \( k = m \) the canonical ensembles of random–matrix theory. The number of independent random variables is given by \( K_\beta = \frac{\sum_{\alpha=1}^r n_\alpha}{l+m-1} \), which defines the Bosonic \( k \)–body embedded Gaussian orthogonal (unitary) ensemble of random matrices, respectively, in short BEGOE\((k)\) and BEGUE\((k)\).

Hilbert space is spanned by the orthonormal \( m \)–particle states \(|\mu\rangle\) given by

\[
|\mu\rangle = \frac{b_{j_1} \cdots b_{j_m}^\dagger}{\mathcal{N}(j_1, \ldots, j_m)} |0\rangle, \tag{3}
\]

with \( j_1 \leq j_2 \leq \ldots \leq j_m \) indicating the single–particle levels occupied by the \( m \) Bosons. Another equivalent definition of the states \(|\mu\rangle\) is obtained by choosing partitions of \( m \), i.e., sequences of positive integers \( \{n_1, n_2, \ldots, n_r\} \) with \( r \leq l \) and \( \sum_{\alpha=1}^r n_\alpha = m \), and sets \( j'_1 < j'_2 < \ldots < j'_r \) of single–particle levels. For every such choice, each of the levels \( j'_s \), \( s = 1, \ldots, r \) is occupied by \( n_s \) Bosons, and the corresponding \( m \)–particle state has the form

\[
|\mu\rangle = \frac{(b_{j'_1}^\dagger)^{n_1} \cdots (b_{j'_r}^\dagger)^{n_r}}{\sqrt{n_1! \cdots n_r!}} |0\rangle. \tag{4}
\]

The states \(|\mu\rangle\) are normalized. States belonging to different partitions or, for the same partition, to different sequences \( j'_1 < j'_2 < \ldots < j'_r \) of single–particle levels, are mutually orthogonal. Comparison of Eqs. (3) and (4) yields an implicit definition of the coefficients \( \mathcal{N}(j_1, \ldots, j_m) \). These same coefficients appear also in Eq. (1). The dimension of Hilbert space is

\[
N = \binom{l+m-1}{m}. \tag{5}
\]

Eq. (4) can be used to define classes of states. All states in a given class have in common the same set \( \{n_1, \ldots, n_r\} \) of positive integers appearing in Eq. (4), irrespective of the choice of single–particle states \( j'_1, \ldots, j'_r \). By definition, permutations of the \( l \) single–particle levels leave the class invariant. We note that for Fermions there is only one class of states. The number \( N_{cl} \) of classes is given by the number of partitions of \( m = \sum n_\alpha \) subject to the constraint \( n_1 \geq n_2 \geq \ldots \geq n_r (r \leq l) \). The number of non–zero
matrix elements coupling the state \(|\mu\rangle\) to other states depends on the class to which \(|\mu\rangle\) belongs. Therefore, a graphical representation using vertices for the Hilbert space vectors and links for the non–zero non–diagonal matrix elements (cf. Ref. [32]) leads to a non–regular graph in general. However, the subgraph defined only among states of the same class is a regular graph, i.e., the number of links emanating from each vertex is the same for all vertices in the subgraph.

3 The Second Moment of the Interaction

As in the Fermionic case, the second moment of the matrix \(\langle \nu | V_k(\beta) | \mu \rangle\) is of central importance in our approach. We show that the duality relation found for Fermions also holds for the Bosonic ensembles.

3.1 The Second Moment

By virtue of the randomness of \(V_k(\beta)\), the elements of the matrix \(\langle \nu | V_k(\beta) | \mu \rangle\) are random variables with a Gaussian probability distribution and zero mean value. The distribution is completely specified in terms of the second moment \(B^{(k)}_{\mu\nu,\rho\sigma}(\beta)\) defined by

\[
B^{(k)}_{\mu\nu,\rho\sigma}(\beta) = \langle \mu | V_k(\beta) | \sigma \rangle \langle \rho | V_k(\beta) | \nu \rangle .
\]  

(6)

To simplify the notation, we define the operators

\[
B^\dagger_{\alpha(r)} = \frac{b^\dagger_{j_1} \cdots b^\dagger_{j_r}}{N(j_1, \ldots, j_r)},
\]  

(7)

and likewise for the adjoint \(B_{\alpha(r)}\). The index \(\alpha(r)\) is a short–hand notation for the rank \(r\) and for the sequence of indices \(\{j_1 \ldots j_r\}\). The matrix elements \(\langle \mu | B^\dagger_{\alpha(r)} B_{\alpha(r)} | \nu \rangle\) are real. Using this fact, Eqs. (6) and (8), and the Hermitecity of \(V_k(\beta)\), we obtain

\[
B^{(k)}_{\mu\nu,\rho\sigma}(\beta) = \sum_{\alpha(k),\gamma(k)} \langle \mu | B^\dagger_{\alpha(k)} B_{\gamma(k)} | \sigma \rangle
\]

\[
\times \left[ \langle \rho | B^\dagger_{\gamma(k)} B_{\alpha(k)} | \nu \rangle + \delta_{\beta 1} \langle \nu | B^\dagger_{\gamma(k)} B_{\alpha(k)} | \rho \rangle \right] .
\]  

(8)
The second moment, Eq. (8), has the same central importance for the theory developed here as its counterpart, Eq. (14) of Ref. [32], for the Fermionic ensemble. For \( k = m \) and \( \beta = 2 \), Eq. (8) reduces to \( \delta_{\mu\nu}\delta_{\rho\sigma} \), and correspondingly for \( \beta = 1 \). This shows that for \( k = m \), our ensembles reduce to the GUE or GOE, respectively, with the radius of the semicircle growing as \( \sqrt{N} \).

3.2 Duality

The duality relation establishes a connection between the second moment of the \( k \)-body interaction and that of the \((m-k)\)-body interaction. The proof presented here is somewhat more general than the one given in Ref. [32].

We consider Eq. (8) for the unitary case \( \beta = 2 \). We observe that the bra-state \( \langle \mu | B^{\dagger}_{\alpha(k)}(k) \rangle \) contains \((m-k)\) Bosons and yields the vacuum state upon the action of a uniquely defined operator \( B^{\dagger}_{\beta(m-k)} \). For the matrix element in Eq. (8) not to vanish, the ket-state \( |B^{\dagger}_{\gamma(k)}| \sigma \rangle \) must contain the same \((m-k)\) Bosons as \( \langle \mu | B^{\dagger}_{\alpha(k)}(k) \rangle \). In other words, this state yields the vacuum state by the action of the operator \( B^{\dagger}_{\gamma(k)} \) with the same index \( \beta(m-k) \). The \((m-k)\)-Boson states \( B^{\dagger}_{\beta(m-k)}|0\rangle \) are orthonormal and complete, i.e., we have

\[
\sum_{\beta(m-k), \delta(m-k)} \langle \mu | B^{\dagger}_{\alpha(k)}(k) B^{\dagger}_{\beta(m-k)}|0\rangle \langle 0| B^{\dagger}_{\beta(m-k)} B^{\dagger}_{\gamma(k)}|\sigma \rangle \times \langle \nu | B^{\dagger}_{\gamma(k)} B^{\dagger}_{\delta(m-k)}|0\rangle \langle 0| B^{\dagger}_{\delta(m-k)} B^{\dagger}_{\alpha(k)}|\nu \rangle = \sum_{\beta(m-k), \delta(m-k)} \langle \mu | B^{\dagger}_{\beta(m-k)} B^{\dagger}_{\gamma(k)}|0\rangle \langle 0| B^{\dagger}_{\gamma(k)} B^{\dagger}_{\beta(m-k)}|\sigma \rangle .
\]

The first equality in Eq. (9) follows from the definition of the second moment and the completeness of the \((m-k)\)-Boson states. In the second one,
we first rearrange the order of the factors conveniently and then use the
commutation properties of the Bosonic operators. The last line follows from
the completeness of the $k$–Boson states. Using the definition of the second
moment, Eq. (8), we obtain

$$B_{\mu\nu,\rho\sigma}^{(k)}(2) = B_{\mu\sigma,\rho\nu}^{(m-k)}(2).$$

This is the duality relation. The duality relation is different from and has
nothing to do with the particle–hole symmetry that is used for treating
Fermions in more than half–filled shells.

4 Eigenvector Expansion of the Second
Moment

In the Fermionic case, results for the shape of the spectrum and spectral
fluctuations could be derived with the help of the eigenvector decomposition
of the second moment. A similar decomposition exists in the case of Bosons
and is derived below. A central role in the derivation is played by the identity

$$\sum_{j_1 \leq \ldots \leq j_k} b_{j_1} \ldots b_{j_k} \overline{b}_{j_1} \ldots \overline{b}_{j_k} = \frac{1}{k!} \sum_{j_1, \ldots, j_k} b_{j_1} \ldots b_{j_k} \overline{b}_{j_1} \ldots \overline{b}_{j_k}.$$  

This identity relates a sum over the indices $j_1, \ldots, j_k$ restricted by the con-
dition $j_1 \leq \ldots \leq j_k$ containing the appropriate weights to the unrestricted
sum over the same indices. The weight factor $k!/[N(j_1, \ldots, j_k)]^2$ is related
to the number of permutations with repetitions of $k$ indices. A proof of this
identity is given in the Appendix.

We consider first the unitary ensemble. Proceeding as in the case of Fer-
monics, we solve the eigenvalue equation

$$\sum_{\rho\sigma} B_{\mu\nu,\rho\sigma}^{(k)} C_{\sigma\rho}^{(sa)} = \Lambda_{B}^{(s)}(k) C_{\sigma\rho}^{(sa)}$$

and show that the eigenvectors have the form

$$C_{\sigma\rho}^{(sa)} = \langle \sigma | B_{\beta(s)}^{\dagger} B_{\delta(s)} | \rho \rangle.$$

Here, $s$ is the rank of $B_{\beta(s)}^{\dagger}$ and $B_{\delta(s)}$, and $a$ labels the pair $(\beta(s), \delta(s))$. We
assume first that no index in $\beta(s)$ equals any of the indices in $\delta(s)$. Using
Eqs. (8) and (12) in the eigenvalue equation, we obtain

$$\sum_{\rho\sigma} B_{\mu\nu,\rho\sigma}^{(k)}(2) C_{\sigma\rho}^{(sa)} = \sum_{\alpha(k)\gamma(k)} \langle \mu | B_{\alpha(k)}^{\dagger} B_{\delta(s)} B_{\gamma(k)} B_{\beta(s)}^{\dagger} B_{\alpha(k)} | \nu \rangle.$$  

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We have used that there are no common indices in $\beta(s)$ and $\delta(s)$, so $B_{\beta(s)}^\dagger$ and $B_{\delta(s)}$ commute. To perform the summation over $\gamma(k)$ we use the identity (11) and the fact that $\sum_j b_j b_j^\dagger = l + \sum_j b_j b_j$. This yields $\binom{l+m+s-1}{k}$. For the sum over $\alpha(k)$ we proceed similarly, rearranging first the order of the operators to bring together the operators involving $\alpha(k)$. We use an identity similar to (11) to compute the summation. We obtain the factor $\binom{m-s}{k}$ multiplying $C_{\alpha\rho}^{(sa)}$. This shows that the matrix $C_{\alpha\rho}^{(sa)}$ defined in Eq. (12) is indeed an eigenvector with eigenvalue

$$\Lambda_{\beta}^{(s)}(k) = \binom{m-s}{k} \binom{l+m+s-1}{k}.$$  

In Eq. (14) and below, the label B stands for Bosons; we shall use the label F for Fermions. The eigenvalues $\Lambda_{\beta}^{(s)}(k)$ decrease monotonically with increasing $s$ and terminate at $s = m - k$, as for the case of Fermions. Again, this is consistent with the GUE result for $k = m$, where only the term $s = 0$ occurs and where $\Lambda_{\beta}^{(0)}(m) = N$.

Eigenvectors of the form (12) with no common index in $\beta(s)$ and $\delta(s)$ are orthogonal. In the case where there is a common index, a construction identical to the Fermionic case is required. The eigenvectors can be “lifted” from a lower rank $s'$ to a higher one with $s > s'$ by insertion of powers of the number operator. The construction of linearly independent eigenvectors $C_{\alpha\rho}^{(sa)}$ proceeds as for Fermions (see Ref. [32] for details), and one finds that these eigenvectors posses the same eigenvalues (14). As in the Fermionic case, the eigenvectors do not depend upon the rank $k$ of the interaction, only the eigenvalues do. The construction of the eigenvectors is identical in both cases. The dimension $D_B^{(s)}$ of the subspace spanned by degenerate eigenvectors characterized by $s$ is given by $D_B^{(0)} = 1$ and for $s \geq 1$ by

$$D_B^{(s)} = \binom{l+s-1}{s}^2 - \binom{l+s-2}{s-1}^2.$$  

It follows that $\sum_{s=0}^m D_B^{(s)} = N^2$. Therefore, the eigenvectors form a complete basis. We impose Hermitecity on the eigenvectors and normalize them according to

$$\sum_{\mu\nu} C_{\mu\nu}^{(sa)} C_{\nu\mu}^{(tb)} = N \delta_{st} \delta_{ab}.$$  

It is instructive to compare the eigenvalues for Fermions and for Bosons. We recall that for Fermions, the eigenvalues have a form similar to Eq. (14),

$$\Lambda_{\alpha}^{(s)}(k) = \binom{m-s}{k} \binom{l+m+s-1}{k}.$$  

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with the second factor on the right-hand side replaced by \( \ell_{i}^{(s)}(k) \). Since the dimensions of the Hilbert spaces for Bosons and for Fermions differ, and since the eigenvalues depend on the dimension, a fair comparison is obtained by normalizing the eigenvalues to the value \( \Lambda_{i}^{(0)}(k) \). For Bosons and Fermions, the ratios \( \ell_{i}^{(s)}(k) = \Lambda_{i}^{(s)}(k)/\Lambda_{i}^{(0)}(k) \) with \( i = F, B \) are given by

\[
\ell_{B}^{(s)}(k) = \prod_{i=1}^{k} \left[ \frac{m+1-i-s}{m+1-i} \right] \left[ \frac{l+m-i+s}{l+m-i} \right], \quad (17)
\]

\[
\ell_{F}^{(s)}(k) = \prod_{i=1}^{k} \left[ \frac{m+1-i-s}{m+1-i} \right] \left[ \frac{l-m+i-s}{l-m+i} \right]. \quad (18)
\]

It is easy to see from Eqs. (17) and (18) that \( \ell_{F}^{(s)}(k) < \ell_{B}^{(s)}(k) \) while \( \ell_{F}^{(0)}(k) = \ell_{B}^{(0)}(k) = 1 \) by definition. In the dilute limit \( k \ll m \ll l \) where the distinction between Fermions and Bosons is irrelevant, one finds the expected result \( \ell_{F}^{(s)}(k) \sim \ell_{B}^{(s)}(k) \). On the other hand, for fixed \( k \) and \( l \), the difference between the Fermionic and the Bosonic case increases with increasing particle number. For Bosons, the eigenvalues with larger \( s \) become more important. This fact suggests stronger deviations from Wigner–Dyson statistics for Bosons than for Fermions. This is particularly obvious for values of \( m \) beyond half-filling, \( 2m > l \), and \( m-k > l-m \), where \( \ell_{F}^{(s)}(k) = 0 \) for \( s > l-m \) while \( \ell_{B}^{(s)}(k) \) is different from zero. This is demonstrated in Fig. [4] which shows \( \ell^{(s)}(k) \) for Bosons and for Fermions as a function of \( s \), for \( k = 2, l = 40 \) and for several values of \( m \).

For the orthogonal case we observe that the matrix \( B^{(k)}(1) \) in Eq. (8) consists of two terms, each one of the form of \( B^{(k)}(2) \). Combining the expressions for \( \beta = 1 \) and for \( \beta = 2 \), we find for the second moment the eigenvector expansion

\[
B_{\mu\nu,\rho\sigma}^{(k)}(\beta) = \frac{1}{N} \sum_{sa} \Lambda_{B}^{(s)}(k) \left[ C^{(sa)}_{\mu\nu} C^{(sa)}_{\rho\sigma} + \delta_{\beta1} C^{(sa)}_{\mu\rho} C^{(sa)}_{\nu\sigma} \right]. \quad (19)
\]

This expression shows a formal analogy to the Fermionic case. However, the dimension of Hilbert space, the values of the \( \Lambda_{B}^{(s)}(k) \), and the definitions of the eigenvectors, are all different.
Figure 1: The ratio $\ell^{(s)}(k) = \Lambda^{(s)}(k)/\Lambda^{(0)}(k)$ as a function of $s$ for (a) Bosons and (b) Fermions, for $k = 2$, $l = 40$, and for several values of $m$. Only non-zero eigenvalues are shown. We note the difference in horizontal scales in the two plots.

5 Low Moments of $V_k$

We apply the results obtained in Sections 3 and 4 to calculate the dependence on $k$, $m$, and $l$ of measures of the shape of the average spectrum in the limit $N \to \infty$. Only in this limit can we expect to obtain generic results.

In contradistinction to the Fermionic case, where the limit $N \to \infty$ implies $l \to \infty$, the limit $N \to \infty$ can for Bosons also be obtained by letting $m \to \infty$. In particular, we will be interested in this case for fixed values of $l$ and $k$. We refer to this case as to the dense limit. Having chosen the Bosonic ensembles in such a way that they agree with GUE or GOE for $k = m$, and observing that for $k \ll m \ll l$ Bosonic and Fermionic ensembles must agree, we do not expect a qualitatively different behavior for Bosonic and Fermionic systems as long as $m < l$. Thus, the dense limit is the really interesting and novel case for Bosons. It has no analogue for Fermions.

We consider three ratios that yield information about the shape of the average spectrum. The ratio $S$ measures the fluctuations of the center of the spectrum in units of the average width of the spectrum. The ratio $R$ measures the relative fluctuation of the width of the spectrum. These two quantities yield information about finite–size effects in numerical calculations [36]. The third ratio is the kurtosis $\kappa$, written as $\kappa = Q + 2$. The quantity $Q$ marks the difference between the semicircular ($Q = 0$) and the Gaussian shape ($Q = 1$) of the average spectrum. The definitions of $S$, $R$, and $Q$ are given in
Ref. \[32\]. Since the eigenvectors and the duality relation are formally similar to the Fermionic case, the formal expressions for the three quantities \( S \), \( R \), and \( Q \) are identical, too. Differences are due to the changed form of the eigenvalues, to different ranges of the parameters \( k \), \( m \), and \( l \), and to the following fact.

The creation and annihilation operators for Bosons are not nilpotent like those for Fermions. Therefore, contributions of the form \( \sum_{\mu\nu} C^{(sa)\mu\nu} C^{(sa)\mu\nu} \) that for Fermions are of relative order \( O(1/l^2) \) and vanish in the large \( N \) limit, may become important for the Bosonic ensembles. Such terms can be worked out explicitly exploiting the symmetry or antisymmetry of the Hermitean eigenvectors \( C^{(sa)}_{\mu\nu} \) of the second moment. We define the “parity” \( \pi_a(s) \) as zero (one) when \( C^{(sa)}_{\mu\nu} \) is symmetric (antisymmetric) in the indices \( \mu, \nu \), and have \( C^{(sa)}_{\mu\nu} = (-1)^{\pi_a(s)} C^{(sa)}_{\nu\mu} \). Using duality and the orthogonality relations, Eqs. (10) and (16), in \( \sum_{sa} \Lambda(s)(m) \sum_{\mu\nu} C^{(sa)}_{\mu\nu} C^{(sa)}_{\mu\nu} \), we obtain the sum rule \( \sum_{sa} (-1)^{\pi_a(s)} = N \). From this sum rule, it is easy to derive the following important identity for the partial sum \( d^{(s)}_B = \sum_a (-1)^{\pi_a(s)} \),

\[
d^{(s)}_B = \left( \frac{l + s - 1}{s} \right) - \left( \frac{l + s - 2}{s - 1} \right) = \left( \frac{l + s - 2}{s} \right).
\]

Using Eq. (20) and the result for the Fermionic ensembles, we obtain

\[
S(k, m, l) = (1 + \delta_{\beta 1}) \Lambda_B^{(0)}(m - k) \left[ \text{tr}\left[ V_k(\beta) \right] \right]^{-1},
\]

\[
R(k, m, l) = 2 \left\{ (1 + \delta_{\beta 1}) \sum_s (\Lambda_B^{(s)})^2 D_B^{(s)} + 2\delta_{\beta 1} \sum_{s=0}^k (\Lambda_B^{(s)}(m - k))^2 d_B^{(s)} \right\}
\times \left[ \text{tr}\left[ V_k(\beta) \right] \right]^{-2},
\]

and

\[
Q(k, m, l) = \frac{1}{N} \left\{ \sum_{s=0}^{\min(m-k,k)} \Lambda_B^{(s)}(k)\Lambda_B^{(s)}(m - k)[D_B^{(s)} + 2\delta_{\beta 1} d_B^{(s)}] \\
+ \delta_{\beta 1} \left[ \sum_{s=0}^{m-k} (\Lambda_B^{(s)}(k))^2 d_B^{(s)} - \frac{2}{N} \left( \sum_s \Lambda_B^{(s)} d_B^{(s)} \right)^2 \\
+ \frac{2}{N^2} \sum_{(sa), (tb)} (-1)^{\pi_a(s) + \pi_b(t)} \Lambda_B^{(t)} \text{tr}(C^{(sa)} C^{(sa)} C^{(tb)} C^{(tb)}) \right\} \}
\]
The width of the spectrum is given by
\[
\frac{1}{N} \text{tr}[V_{k}(\beta)]^2 = \Lambda^{(0)}_{B}(k) + \frac{\delta \beta}{N} \sum_{s} \Lambda^{(s)}_{B} d^{(s)}_{B} .
\] (24)

In Eqs. (21)–(24), traces are always taken in the Hilbert space of the many-body states $|\mu\rangle$. We have simplified the notation by omitting the dependence on $k$ or $m - k$ of the eigenvalues $\Lambda^{(s)}_{B}$ in the terms which are invariant under the exchange $k \leftrightarrow m - k$ because of the duality relation.

We consider the large $N$ limit, first realized by letting $l \rightarrow \infty$, with $m/l$ fixed. The results are similar to the Fermionic case: For $k \geq 1$ both $S \rightarrow 0$ and $R \rightarrow 0$. The convergence to the limit is particularly slow when both $k$ and the filling factor $m/l$ are fixed. In this case, $S \propto l^{-k} \propto (\ln N)^{-k}$ and $R \propto l^{-2k} \propto (\ln N)^{-2k}$ while for the canonical ensembles ($k = m$) the limit is approached as $S(m, m, l) \propto 1/N^2$ and $R(m, m, l) \propto 1/N^2$. The behavior of $Q$ suggests that the shape of the spectrum coincides with the semicircle for $2k > m$. The semicircle undergoes a smooth transition to Gaussian shape for $2k \leq m$. The critical value of $k$ where this transition takes place is associated, as in the case for Fermions, to duality.

We turn to the dense limit, letting $m \rightarrow \infty$ with both $k$ and $l$ fixed. We find

\[
\lim_{m \rightarrow \infty} S(k, m, l) = \frac{(1 + \delta \beta)(2k)}{2k} \left[ \sum_{s=0}^{k} \left( \frac{2k}{k+s} \right)^{l+k+s-1-1} d^{(s)}_{B} \right],
\] (25)

\[
\lim_{m \rightarrow \infty} R(k, m, l) = \frac{2 \sum_{s=0}^{k} \left[ \frac{2k}{k+s} \right]^{l+k+s-1-1} \left[ D^{(s)}_{B} + \delta \beta \sum_{s=0}^{k} \left( \frac{2k}{k+s} \right)^{l+k+s-1-1} d^{(s)}_{B} \right]}{\left[ \frac{2k}{k} + \delta \beta \sum_{s=0}^{k} \left( \frac{2k}{k+s} \right)^{l+k+s-1-1} d^{(s)}_{B} \right]^2},
\] (26)

\[
\lim_{m \rightarrow \infty} Q(k, m, l) = \sum_{s=0}^{k} \left( \frac{2k}{k+s} \right)^{l+k+s-1-1} D^{(s)}_{B}
= 1 \quad (\beta = 2).
\] (27)
The identity used to obtain the second equality in Eq. (27) is derived in Ref. [37]. Eq. (27) applies only in the unitary case ($\beta = 2$) and implies that in the dense limit, the average spectrum has Gaussian shape. We have been unable to extend this result to the orthogonal case ($\beta = 1$). This is because we did not succeed in calculating the term $\text{tr}(C^{(sa)}C^{(sa)}C^{(tb)}C^{(tb)})$ appearing in Eq. (23). On physical grounds, however, we expect an equation analogous to Eq. (27) to be valid also for the BEGOE($k$). We conclude that the shape of the average spectrum has Gaussian form. This is in keeping with the results of Ref. [38].

A more important and surprising result lies in the fact that the right-hand sides of Eqs. (25) and (26) do not vanish: The fluctuations of the centroids and of the variances of individual spectra do not vanish asymptotically. This feature differs from the behavior both of canonical random-matrix theory and of the embedded Fermionic ensembles. We are led to the important conclusion that the Bosonic ensembles are not ergodic in the dense limit $m \to \infty$ with $k$ and $l$ fixed: Unfolding the spectra by taking the ensemble average, or the spectral average over a single member of the ensemble, will yield different results. We attribute this non-ergodic behavior of the ensemble to the slow rate at which the eigenvalues $\Lambda^{(s)}_B(k)$ decrease as $s$ increases. Physically, it is a consequence of the fact that the number of independent random variables $K_\beta$ in the ensemble does not grow with $m$ but stays fixed with $k$ and $l$. We return to this point in Section 7.

6 Supersymmetry Approach

As in the case of Fermions, the eigenvector expansion (19) permits us to use the supersymmetry approach. We shall not give a self-contained presentation of this technique in the present context. There are review papers which describe it extensively. We mention for instance, Refs. [33] and [34]. Moreover, the supersymmetry technique was also described in some detail in our preceding paper on Fermions [32]. Suffice it to say that the resulting expressions are formally the same as in the Fermionic case. Because of specific properties of the eigenvalues $\Lambda^{(s)}_B(k)$, in particular in the dense limit, new contributions appear in the lowest-order terms in the loop expansion. These contributions may be finite and, thus, imply departures from RMT behavior. In the dense limit, such behavior must be expected because of the non-ergodic features found in Section 5.
6.1 Saddle Point

We summarize the main steps of the supersymmetric approach for the one- and two-point functions; details may be found in Ref. [32]. After averaging over the ensemble, the integrand of the generating functional contains an exponential whose argument is a sum of squares of bilinear forms in the integration variables involving the matrix $B^{(k)}$. We use the eigenvector expansion (19) and perform the Hubbard–Stratonovich transformation. For each eigenvector $C^{(sa)}$, this introduces a supermatrix $\sigma^{(sa)}$ of composite variables. We use the saddle-point approximation. We define $E = (E_1 + E_2)/2$ and $\epsilon = E_2 - E_1$, where $E_1$ and $E_2$ are the energy arguments of the advanced and retarded Green functions, respectively. Omitting small terms of order $1/N$, we find the coupled saddle-point equations

$$\sigma^{(sa)} = \frac{1}{N} \text{tr}_\mu \left( [E - \sum_{tb} \lambda^{(t)}_B(k) \sigma^{(tb)} C^{(tb)}]^{-1} \lambda^{(s)}_B(k) C^{(sa)} \right),$$

(28)

where $\lambda^{(s)}_B(k)$ is the positive square root of $\Lambda^{(s)}_B(k)$. As in the Fermionic case, this set of equations is solved iteratively by introducing the variable $X_{\rho\sigma} = \sum_{sa} \lambda^{(s)}_B(k) \sigma^{(sa)} C^{(sa)}$ and using the identity $G(E) = [E - X]^{-1}$ for the averaged Green function. We find that $X$ obeys the saddle–point equation $X = \Lambda^{(0)}_B(k) [E - X]^{-1}$. Hence, the solution diagonal in the superindices has the form $X^{\text{diag}}_{\mu\nu} = \delta_{\mu\nu} \Lambda^{(0)}_B(k) \tau^{(0)}$, with

$$\tau^{(0)} = \frac{E}{2\lambda^{(0)}_B(k)} \pm i \sqrt{1 - \left( \frac{E}{2\lambda^{(0)}_B(k)} \right)^2},$$

(29)

for $|E| \leq 2\lambda^{(0)}_B$. The $\pm$ signs refers to the retarded ($G^{-}(E)$) and advanced ($G^{+}(E)$) case, respectively. This implies that the solutions of Eq. (28) reduce to $\tau^{(0)}$ for $s = 0$, while $\sigma^{(sa)} = 0$ for $s \geq 1$. For the one–point function this shows that, within the range of validity of the saddle–point approximation, the semicircle describes the shape of the average spectrum.

As in the case of the canonical ensembles, the invariance of the effective Lagrangean under general pseudounitary transformations implies that the two–point function possesses not a single saddle point but a saddle–point manifold which is generated by $T^{-1}\sigma^{(0)}T$. The matrices $T$ parameterize the coset space $\text{UOSP}(1, 1/1, 1)/[\text{UOSP}(1, 1) \otimes \text{UOSP}(1, 1)]$, and $\sigma^{(0)}$ is a diagonal supermatrix of dimension four with entries given by Eq. (29) in the usual
way \cite{34}. Using this and the saddle–point solution in the expression of the
effective Lagrangean, the first–order term in \( \epsilon \) takes the canonical form

\[ -\frac{i\pi \epsilon}{d} \text{trg}(LT^{-1}LT) . \] (30)

Here \( d \) is the average level spacing and the diagonal supermatrix \( L \) (defined in
Ref. \cite{34}) distinguishes the retarded and advanced cases. The argument also
carries through for higher–point correlation functions. Eq. (30) shows that
within the range of validity of the saddle–point approximation, the spectral
fluctuations of the BEGUE(\( k \)) are identical to those of the GUE.

6.2 Corrections of the Loop Expansion

To determine the range of validity of the saddle–point solution in the limit
\( N \to \infty \), we use the loop expansion, a power–series expansion of the effec-
tive Lagrangean around the saddle–point, followed by an expansion of the
resulting exponential. Again, the formal equivalence between the Fermionic
and Bosonic eigenvector expansions permits a straightforward tr anscription
of the results for Fermions.

For the one–point function, the first correction term in the loop expansion
yields

\[ |G^+(0)| \leq \frac{N|\text{trg}J|}{\lambda_B^{(0)}(k)} \left( 1 + Q(k, m, l) \right) . \] (31)

From Eq. (31) we conclude that for \( l \to \infty \), the shape of the spectrum is the
semicircle for \( 2k > m \), and finite corrections arise for \( 2k \leq m \) (see Section 5).
These results coincide with those for the Fermionic ensembles. In particular,
in the dilute limit a Gaussian shape of the spectrum is obtained. This holds
also true in the dense limit \( m \to \infty \) with \( k \) and \( l \) fixed as given by Eq. (27).

For the two–point function we drop terms that vanish as \( 1/N \) for \( N \to \infty \).
The resulting expression can be written as an exponential which contains two
non–trivial terms. The first one contains the expression

\[ -\frac{i\pi \epsilon}{d_1} \text{trg}(LT^{-1}LT) , \] (32)

where \( d_1 \) is a modified level spacing which includes the terms that lead to the
bound given in Eq. (31) for the average Green function. More interesting is
the second term which, as in the Fermionic case, is proportional to

\[
R(k, m, l) \left( \frac{1}{d_1} \text{tr} \left[ \frac{1}{2} \epsilon LTL^{-1} + LTJ^{-1} \right] \right)^2.
\]  

(33)

Kravtsov and Mirlin [39] have shown that terms of this form produce deviations from universal behavior. The strength of the fluctuations which arise from this term is proportional to the ratio \( R(k, m, l) \) given in Eq. (22). We recall that \( R \) measures the relative fluctuations of the width of the spectrum. While in the dilute limit this quantity tends slowly to zero as \( N \to \infty \), in the dense limit it attains a constant value. Such finite corrections imply that the saddle–point solution is no longer appropriate in the dense limit, and deviations from Wigner–Dyson statistics must occur.

We conclude that in the dense limit, \( \text{BEGUE}(k) \) possesses spectral fluctuations which deviate significantly from the predictions of RMT. As mentioned above, we attribute this feature to the fact that the number of independent random variables of the ensemble remains the same as \( m \) increases, see Section 7. Alternatively, we attribute this result to the rather slow decrease of the eigenvalues \( \Lambda_B^{(s)} \) with increasing \( s \). Contributions other than the canonical term \( s = 0 \) are relevant and give rise to important corrections for the spectral fluctuations. These features are similar to those characterizing the dilute limit. Here, the spectral correlations become Poissonian. In that case, however, the function \( \tilde{R}(k, m, l) \) vanishes asymptotically, albeit very slowly. Therefore, the first loop correction of the supersymmetry method is not able to provide solid evidence for non–RMT spectral fluctuations in the dilute limit. However, in this limit another method is available: The extension of the binary correlation approximation of Mon and French [11] developed in Ref. [32]. This method works equally for Bosons and for Fermions (in the dilute limit, the distinction between the two cases disappears) and, thus, yields identical results: The spectral density acquires Gaussian shape, and the spectral fluctuations become Poissonian.

7 Numerical Results: the Dense Limit

In the dense limit, there are no analytical techniques to predict spectral fluctuations. Moreover, the Bosonic ensembles are not ergodic. Therefore, unfolding procedures based upon averaging over the ensemble carry a question mark and spectral unfolding is the preferred procedure to study the
spectral fluctuations. In this Section, we present numerical results on the spectral fluctuation properties of BEGOE(\(k\)) for \(k = 2\) in the dense limit. We exhibit the difference between spectral unfolding and unfolding by using the average over the ensemble. For brevity, the latter method is called ensemble unfolding.

Numerically, the dense limit is conveniently modeled by taking \(l = 2\). The dimension of Hilbert space is then given by \(N = m + 1\) (cf. Eq. (5)), and the situation \(m \gg l\) can be easily attained numerically. Furthermore, for \(l = 2\) and \(k = 2\) the number of independent random variables of the ensemble is \(K_1 = 6\). We emphasize that \(K_1\) is independent of the number \(m\) of Bosons. The many–particle states \(|\mu_n\rangle\) are written as \((m - n, n)\) where \(m - n\) and \(n\) indicate the number of Bosons occupying the first and the second single–particle state, respectively. The many–particle states \(|\mu_n\rangle\) are arranged in a sequence \((m, 0), (m - 1, 1), \ldots, (0, m)\) of increasing values of \(n\). Then, the Hamiltonian matrix attains a band structure, with non–zero matrix elements on the main diagonal and on the \(k\) closest diagonals. We restrict ourselves to the physically most relevant case \(k = 2\).

### 7.1 Ensemble Unfolding versus Spectral Unfolding

The evaluation of measures of spectral fluctuations requires that individual spectra be unfolded: The average level spacing must be constant. Spectra can be unfolded in two ways. Spectral unfolding consists in handling each spectrum independently. One looks for a transformation which yields the value unity for the average level spacing. This transformation need not be the same for different spectra. Ensemble unfolding uses the average spectral density (obtained by averaging all spectra in the ensemble) to find a single transformation which is then applied to all spectra of the ensemble.

Spectral unfolding is typically done by fitting a polynomial to the staircase function of each realization of the ensemble. In the results presented below, we have determined the degree of the polynomial for each realization by taking the value which gives the first minimum of the associated \(\chi^2\) when increasing the degree starting from 1. The maximum degree considered was 20. Typically the first minimum was found around 11.

Ensemble unfolding is usually preceded by removing the fluctuations of the center and of the width \([13, 36]\). All spectra are recentered and rescaled so as to have a common center and width. This procedure is essentially equivalent to the more sophisticated method used in Refs. \([2, 14, 40]\) which
Figure 2: Spectral density of the BEGOE(2) for \( l = 2 \) and \( m = 3000 \), normalized to the dimension \( N = 3001 \) of Hilbert space. The curves (a) and (b) show the level densities of two individual realizations of the ensemble; curve (c) shows the ensemble-averaged spectral density. The dashed curve corresponds to a Gaussian shape of the spectrum.

corrects higher moments of the distribution and relies strongly on the Gaussian shape of the average spectrum. The recentered and rescaled individual spectra are then superposed to obtain the ensemble-averaged spectral density and the staircase function for the ensemble. The latter is fitted with a polynomial, and each spectrum of the ensemble is unfolded using this polynomial. However, for the Bosonic ensembles in the dense limit, this procedure yields a non-Gaussian average level density as a consequence of the non-ergodic behavior of the ensemble. Therefore, we did not recenter and rescale the individual spectra for constructing the ensemble-averaged spectral density and the corresponding staircase function. We have used a polynomial of degree 11 for fitting the average staircase function. The stability of the results has been checked by varying the degree of the polynomial and the size of the bins used to construct the average spectral density.

Fig. 2 shows the spectral densities of two realizations of the ensemble and the ensemble-averaged spectral density obtained from 1512 realizations of the ensemble for \( m = 3000 \). The integral over each histogram is normalized to 3001, the dimension of Hilbert space. In the Figure, we have also plotted a Gaussian density (dashed curve), whose width is given by the the-
oretical prediction Eq. (24). Comparison of the ensemble–averaged density with this curve confirms the prediction of a Gaussian shape of the average spectrum. The striking differences between each of the two spectral densities and between those and the ensemble–averaged spectral density illustrate the non–ergodic character of the Bosonic ensemble for \(m \gg l\). The Gaussian form of the average spectrum arises as a consequence of the Central Limit Theorem. It has no bearing on individual spectral shapes.

### 7.2 Spectral Statistics

We turn to the spectral fluctuation properties of BEGOE(2). We have analyzed 1512 members of the ensemble for \(m = 3000\). After fitting the polynomials, including all levels of each realization for both spectral and ensemble unfolding, we only considered 60% of all levels, i.e., 1800 levels for each realization. These were the levels closest to the center of the spectrum.

In Fig. 3 we present the nearest–neighbor spacing distribution \(P(s)\) obtained by ensemble unfolding and by spectral unfolding. In both cases the distribution \(P(s)\) corresponds neither to the Wigner surmise nor to a Poisson distribution. In the case of ensemble unfolding the level repulsion characteristic of the GOE is clearly lost. On the other hand, the tail of the distribution decays rather rapidly in comparison to the exponential decay of the Poisson distribution. For spectral unfolding the distribution \(P(s)\) is dominated by a prominent peak centered at \(s = 1\). This suggests that individual spectra have an almost constant level spacing, i.e., are nearly of picket–fence type (one–dimensional harmonic oscillator).

The two–point correlations of the spectra are characterized by the \(\Delta_3\)–statistics of Dyson and Mehta measured at the centers of the spectra and shown in Fig. 4. We have computed \(\Delta_3(L)\) directly from the spectra using the expression derived by Bohigas and Giannoni. For comparison we have also plotted the results for a GOE spectrum \(\Delta_3(L) \sim [\ln(2\pi L) + \gamma - \pi^2/8 - 5/4]/\pi^2\), an uncorrelated (Poisson) spectrum \(\Delta_3(L) = L/15\), and a picket–fence spectrum \(\Delta_3(L) = 1/12\) (see Ref. [41] for details). We have checked that the increase of \(\Delta_3(L)\) with \(L\) for small values of \(L\) is approximately linear in both panels of Fig. 4. Ensemble unfolding (Fig. 4(a)) yields a \(\Delta_3\)–statistics which deviates rapidly from GOE behavior, and increases almost linearly. This is consistent with the expected deviations from RMT predictions found in Section 3. In the case of spectral unfolding, \(\Delta_3(L)\) is almost constant up to \(L \sim 20\). This result again suggests a tendency of individual spectra to
Figure 3: Nearest–neighbor spacing distribution $P(s)$ obtained by (a) ensemble unfolding and (b) spectral unfolding. The dotted curves correspond to the Wigner surmise and the dashed ones to the Poisson distribution. We note the different vertical scales used in the frames.

have a picket–fence–like behavior. Beyond this point, the $\Delta_3(L)$ grows albeit less rapidly than after ensemble unfolding.

7.3 Structure of Individual Spectra and Wave Functions

The results presented above show the deviations from RMT behavior expected after ensemble unfolding. For spectral unfolding the results suggest that spectra of individual realizations have a very rigid structure with an almost constant spacing in some regions of the spectrum. We now describe in more detail the structure of individual spectra. While we do not have a
Figure 4: The $\Delta_3$-statistics (solid lines) measured at the centers of the spectra after (a) ensemble unfolding and (b) spectral unfolding. For comparison we have plotted the results for the GOE (dotted curve), for a Poissonian spectrum (dashed line), and for a picket-fence spectrum (dotted-dashed line).

Full analytical understanding for the tendency towards picket-fence behavior, we believe that the following observations are related to the small number of independent random variables of the Bosonic ensemble for $m \gg l$, and to the associated graph structure.

Fig. 5 shows the staircase function $\eta(E)$ for a typical realization of the ensemble $(k = 2, l = 2)$ for $m = 1000$. The spectrum is dominated by levels with almost constant spacing. The staircase function typically displays one or more points where an abrupt change in the density of states takes place: The spectrum has (almost) constant spacing up to the level 619. This behavior suddenly changes at the level 620. While the spacing between neighboring levels is no longer constant, the spacing between next-to-nearest-neighbors
Figure 5: Detail of the staircase function $\eta(E)$ for a typical realization of BEGOE(2) with $m = 1000$ and $l = 2$. The vertical lines help to show how two (almost) equidistant spectra overlap around the level 620. We note the change in the density of states after this level.

is and is almost the same as it was before level 620 appeared. This is illustrated by the vertical lines plotted above and below the staircase function. We remark that these observations imply non–stationary properties of the spectra.

These results suggest that the spectrum of an individual realization consists of pieces of overlapping segments of spectra with almost constant level spacings. The kink at level 620 in Fig. 5 marks the left edge of the overlap region. Such spectra would lead to results like those in Figs. 3(b) and 4(b). We notice that if we were to use for spectral unfolding a piecewise polynomial fit instead of a polynomial fit covering the whole spectrum, the rigid features of the spectrum would appear in an even more convincing form in the spectral statistics. We have not pursued this point further.

If the spectra do possess the structure proposed above, eigenfunctions belonging to different segments of the spectrum should differ qualitatively. To verify this hypothesis, we write the eigenvectors $|i\rangle$ as linear combinations of the ordered many–body basis states $|\mu_n\rangle$,

$$ |i\rangle = \sum_{n=0}^{m} c_{in} |\mu_n\rangle .$$

(34)
Figure 6: Probabilities $|c_{in}|^2$ for eigenvectors belonging to eigenvalues in the vicinity of the kink shown in Fig. 5. The two overlapping segments of nearly equidistant levels are easily distinguished by the structure of the eigenfunctions.
In Fig. 6 we plot the probabilities $|c_{in}|^2$ of eigenvectors $|i\rangle$ belonging to eigenvalues in the vicinity of the kink shown in Fig. 5. Eigenvectors up to $i = 619$ behave similarly and are somewhat extended, although the distribution of the intensities clearly deviates from the Porter–Thomas distribution expected from RMT. This behavior of the eigenfunctions changes abruptly at $i = 620$ where the eigenvector is a rather “coherent” superposition of close-lying basis states. We emphasize the difference in the horizontal scales used in Fig. 6 for $i = 619$ and $i = 620$, for instance. The eigenvectors with $i = 621, 623$ display the same behavior as the eigenvectors up to $i = 619$, and thus correspond to the first segment of the equidistant spectra. The eigenvectors with $i = 620, 622, 624$ are much more localized in Fock space. They differ in the number of intensity oscillations. As $i$ increases, so does this number, and the levels on this sequence become more and more delocalized. At some point the spread of the eigenfunctions in the second segment is indistinguishable from that in the first one.

8 Conclusions

We have studied the shape of the average spectrum and the spectral fluctuations of the Bosonic embedded ensembles BEGOE($k$) and BEGUE($k$) in the limit of infinite matrix dimension, attained by letting either the number $l$ of degenerate single-particle states go to infinity, or by letting the number $m$ of Bosons go to infinity. In the first case, the results are qualitatively the same as for the Fermionic embedded ensembles $[32]$: For sufficiently high rank $k$ of the random interaction ($2k > m$), these ensembles behave generically. The average spectrum has semicircle shape, and the spectral fluctuations obey Wigner–Dyson statistics. A smooth transition to a different regime takes place at or near $2k = m$. This new regime is characterized by a Gaussian spectral shape. The spectral fluctuations also change and are not of Wigner–Dyson type for $2k \lesssim m$; in the dilute limit $k \ll m \ll l$ the spectral fluctuations become Poissonian. We expect that the transition away from Wigner–Dyson statistics is more rapid in the Bosonic than in the Fermionic case.

In the second case ($m > l$), and in particular in the dense limit, where $l$ and the rank of the interaction $k$ remain fixed as $m \to \infty$, we have shown that the Bosonic ensembles are not ergodic. To the best of our knowledge, this is the first case that a random–matrix model has been shown to be non–ergodic.
in the limit of infinite matrix dimension. Moreover, we have shown that the spectral fluctuations deviate strongly from RMT results. This finding disagrees with the conclusions based on numerical simulations of Refs. [13, 14]. We ascribe this to the fact that the ratio $m/l$ for the parameter settings investigated both in Ref. [13] ($m = 11$, $l = 4$) and Ref. [14] ($m = 10$, $l = 5$) were still too small to encounter deviations from Wigner–Dyson statistics. We have studied numerically the structure of the spectra of individual realizations of the ensemble in the dense limit using the case $m = 3000$, $l = 2$ and $k = 2$ as example. In both short– and long–range correlations we found a tendency towards a picket–fence type of behavior. We presented evidence pointing to a non–stationary behavior of the spectra. The spectrum of an individual realization seems generically to consist of segments of spectra of picket–fence type. These segments overlap. In the overlap region, the eigenvectors can be attributed to the different segments by their structure which is characteristic for each segment. In particular, we have shown that some of the eigenfunctions display strong localization in Fock space. We cannot offer an analytical explanation for these properties. We believe that they are caused by the small and constant (independent of $m$) number of independent random variables $K_\beta$ and by the specific graph structure associated with the Hamiltonian matrix of the Bosonic ensembles. This causes the Hamiltonian matrix to attain a particular structure with highly correlated matrix elements. In particular, we do not know how the spectra change as $l$ increases for $m/l \gg 1$.

Acknowledgment We are grateful to O. Bohigas and T.H. Seligman for stimulating discussions and useful suggestions. T.A. acknowledges support from the Japan Society for the Promotion of Science.

Appendix: Proof of the Identity Eq. (11)

We claim that

$$S_k = \sum_{j_1 \leq j_2 \leq \ldots \leq j_k} \frac{b_{j_1} \ldots b_{j_k} b_{j_k}^\dagger \ldots b_{j_1}^\dagger}{[N(j_1, \ldots, j_k)]^2}$$

is equal to

$$T_k = \frac{1}{k!} \sum_{j_1, j_2, \ldots, j_k} b_{j_1} \ldots b_{j_k} b_{j_k}^\dagger \ldots b_{j_1}^\dagger.$$
We prove the claim by induction. The claim is trivial for $k = 1$ and $k = 2$. We assume that the assertion holds for $k$ and prove it for $k + 1$. On the right–hand side of Eq. (35), written for $k + 1$, we use the definition of the coefficients $N(j_1, . . . , j_{k+1})$. We write

$$S_{k+1} = \sum_{\{n_i\}} \frac{(b_{j_1})^{n_1} \ldots (b_{j_r})^{n_r} (b_{j_1}')^{n_1} \ldots (b_{j_1}')^{n_1}}{n_1! \ldots n_r!},$$

where the sum over the non–zero integers $n_i$ is restricted by $\sum_i n_i = k + 1$. Moreover, $j_1' < . . . < j_r'$. We arrange the terms in the sum in Eq. (37) in increasing powers of $b_{j_r}'$ (and $b_{j_1}'$) and apply the induction hypothesis to the remaining sums involving at most $k$ factors $b_{j_r}'$. Hence,

$$S_{k+1} = \sum_{n=1}^{k+1} \frac{1}{(k+1-n)! n!} \times \sum_{j_1, . . . , j_{k+1} < j_{k+2}} b_{j_1} \ldots b_{j_{k+1} - n} (b_{j_{k+2} - n})^n (b_{j_{k+2} - n}', b_{j_{k+2} - n}^)' \ldots b_{j_1}'$$

$$= \frac{1}{(k+1)!} \sum_{n=1}^{k+1} \binom{k+1}{n} \times \sum_{j_1, . . . , j_{k+1} < j_{k+2}} b_{j_1} \ldots b_{j_{k+1} - n} (b_{j_{k+2} - n})^n (b_{j_{k+2} - n}', b_{j_{k+2} - n}^)' \ldots b_{j_1}' .$$

(38)

Consider now the sum on the right–hand side of Eq. (36) for $k + 1$. For each term, we identify the largest of the indices $j_1, . . . , j_{k+1}$. We call it $j_{\text{max}}$. The $b_{j_{\text{max}}}$’s and $b_{j_{\text{max}}}'$’s come in powers $n_{j_{\text{max}}} = 1, . . . , k + 1$. For a fixed value $n_{j_{\text{max}}} = n$, the number of terms in the sum is $\binom{k+1}{n}$. Relabeling the indices in the sum, so that $j_{\text{max}} \rightarrow j_{k+2-n}$, the sum can be rewritten in the form of Eq. (38).

**References**

[1] E.P. Wigner, Contribution to the Conference on neutron physics by time–of–flight, Gatlinburg, Tennessee (1956). Reprinted in C.E. Porter (ed.), Statistical theory of spectra: fluctuations, Academic Press, 1965.
[2] T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. 53 (1981) 385.

[3] T. Guhr, A. Müller–Groeling, and H.A. Weidenmüller, Phys. Rep. 299 (1998) 189.

[4] J.B. French and S.S.M. Wong, Phys. Lett. B 35 (1971) 5.

[5] O. Bohigas and J. Flores, Phys. Lett. B 34 (1971) 261.

[6] O. Bohigas and J. Flores, Phys. Lett. B 35 (1971) 383.

[7] A. Gervois, Nucl. Phys. A 184 (1972) 507.

[8] T.A. Brody, Lett. Nuovo Cim. 7 (1973) 482.

[9] J.B. French, Revista Mexicana de Fisica 22 (1973) 221.

[10] T.A. Brody, E. Cota, J. Flores, and P.A. Mello, Nucl. Phys. A 259 (1976) 87.

[11] K.K. Mon and J.B. French, Ann. Phys. (N.Y.) 95 (1975) 90.

[12] J.J.M. Verbaarschot and M.R. Zirnbauer, Ann. Phys. (N.Y.) 158 (1984) 78.

[13] V.R. Manfredi, Lett. Nuovo Cimento 40 (1984) 135.

[14] K. Patel, M.S. Desai, V. Potbhare and V.K.B. Kota, Phys. Lett. A 275 (2000) 329.

[15] V.V. Flambaum, G.F. Gribakin, and F.M. Izrailev, Phys. Rev. E 53 (1996) 5729.

[16] V. Zelevinsky, B.A. Brown, N. Frazier, and M. Horoi, Phys. Rep. 276 (1996) 85.

[17] B. Altshuler, Y. Gefen, A. Kamenev, and L.S. Levitov, Phys. Rev. Lett. 78 (1997) 2803.

[18] V.V. Flambaum and F.M. Izrailev, Phys. Rev. E 56 (1997) 5144.

[19] P. Jacquod and D.L. Shepelyansky, Phys. Rev. Lett. 79 (1997) 1837.
[20] B. Georgeot and D. Shepelyansky, Phys. Rev. Lett. 81 (1998) 5129.

[21] C.W. Johnson, G.F. Bertsch, and D.J. Dean, Phys. Rev. Lett. 80 (1998) 2749.

[22] C. Mejia–Monasterio, J. Richert, T. Rupp, and H.A. Weidenmüller, Phys. Rev. Lett. 81 (1998) 5189.

[23] R. Bijker, A. Frank, and S. Pittel, Phys. Rev. C 60 (1999) 021302R.

[24] C.W. Johnson, G.F. Bertsch, D.J. Dean, and I. Talmi, Phys. Rev. C 61 (1999) 014311.

[25] X. Leyronas, J. Tworzydlo, and C.W.J. Beenakker, Phys. Rev. Lett. 82 (1999) 4894.

[26] R. Bijker, and A. Frank., Phys. Rev. Lett. 84 (2000) 420.

[27] X. Leyronas, P.G. Silvestrov, and C.W.J. Beenakker, Phys. Rev. Lett. 84 (2000) 3414.

[28] T. Rupp, H.A. Weidenmüller and J. Richert, Phys. Lett. B 483 (2000) 331 - 336; Erratum p. 376.

[29] Y. Alhassid, Ph. Jacquod, and A. Wobst, Phys. Rev. B 61 (2000) R13357.

[30] V.K.B. Kota, Physics Reports 347 (2001) 223.

[31] L. Benet, T. Rupp, and H.A. Weidenmüller, Phys. Rev. Lett. 87 (2001) 010601.

[32] L. Benet, T. Rupp, and H.A. Weidenmüller, Ann. Phys. (N.Y.) 292 (2001) 67.

[33] K.B. Efetov, Adv. Phys. 32 (1983) 53.

[34] J.J.M. Verbaarschot, H.A. Weidenmüller, and M.R. Zirnbauer, Phys. Rep. 129 (1985) 367.

[35] T. Asaga, L. Benet, T. Rupp, and H.A. Weidenmüller, Eur. Phys. Lett. 56 (2001) 340.
[36] J. Flores, M. Horoi, M. Müller, and T.H. Seligman, Phys. Rev. E 63 (2001) 026204.

[37] T. Rupp, Ph.D. Thesis, Univ. of Heidelberg, 2001 (unpublished).

[38] V.K.B. Kota and V. Potbhare, Phys. Rev. C 21 (1980) 2637.

[39] V.E. Kravtsov and A.D. Mirlin, Pis’ma Zh. Eksp. Teor. Fiz. 60 (1994) 645 [JETP Lett. 60 (1994) 656].

[40] G.J.H. Laberge and R.U. Haq, Can. J. Phys. 68 (1990) 301.

[41] O. Bohigas and M.J. Giannoni, Ann. Phys. (N.Y.) 89 (1975) 393.