FOUR DIMENSIONAL EINSTEIN SPACES ON SIX DIMENSIONAL RICCI FLAT BASE SPACES

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Abstract

Some examples of ten-dimensional vacuum Einstein spaces \( (10 R_{ij} = 0) \) composed of four-dimensional Ricci-flat \( (4 R_{ij} = 0) \) Einstein spaces and six-dimensional Ricci-flat base spaces \( (6 R_{ij} = 0) \) defined by the solutions of the classical Korteweg-de Vries equation are constructed.

The properties of geodesics of such type of the spaces are studied.

1 Introduction

The properties of classical four dimensional Einstein spaces are determined by the energy-momentum tensor of matter \( T_{ik} \)

\[
R_{ij} = \frac{8\pi\kappa}{c^4} \left( T_{ik} - \frac{1}{2} g_{ik} T \right). \tag{1}
\]

Tensor \( T_{ik} \) is the self-dependent object in the Einstein theory of gravitation and in general does not has geometric description.

The most popular approach to the geometric description of the tensor \( T_{ik} \) takes place within the bounds of the Kaluza-Klein theories using the string theory on the Calaby-Yau manifolds as the component.

We present here a new possibilities to such type of considerations.

2 Three-dimensional space of zero curvature

We start from three dimensional space with the metrics in form

\[
ds^2 = y^2 dx^2 + 2 \left( l(x, z)y^2 + m(x, z) \right) dx \, dz + 2 \, dy \, dz + \\
+ \left( (l(x, z))^2 y^2 - 2 \left( \frac{\partial}{\partial x} l(x, z) \right) y + 2 l(x, z) m(x, z) + 2 l(x, z) \right) dz^2 \tag{2}
\]

with arbitrary functions \( l(x, z) \) and \( m(x, z) \) ([1]).

The condition on the curvature tensor

\( R_{ijkl} = 0 \)

for the metric (2) lead to the relations

\[
R_{1313} = \left( \frac{\partial^3}{\partial x^3} l(x, z) - 3 l(x, z) \frac{\partial}{\partial x} l(x, z) + \frac{\partial}{\partial z} l(x, z) \right) y + \\
+ \left( -l(x, z) \frac{\partial^2}{\partial x^2} m(x, z) + \frac{\partial^2}{\partial x \partial z} m(x, z) - 3 \left( \frac{\partial}{\partial x} l(x, z) \right) \frac{\partial}{\partial x} m(x, z) - 2 m(x, z) \frac{\partial^2}{\partial x^2} l(x, z) - \frac{\partial^2}{\partial x^2} l(x, z) \right) -
\]
\[-m(x, z) \frac{\partial}{\partial z} m(x, z) + m(x, z) l(x, z) \frac{\partial}{\partial x} m(x, z) + \left( \frac{\partial}{\partial x} l(x, z) \right) m(x, z) + 2 \left( \frac{\partial}{\partial x} l(x, z) \right) (m(x, z))^2 \right) / y = 0,

and

\[ R_{1323} = \left( \frac{\partial}{\partial x} l(x, z) - \frac{\partial}{\partial z} m(x, z) + 2 \left( \frac{\partial}{\partial x} l(x, z) \right) m(x, z) + l(x, z) \frac{\partial}{\partial x} m(x, z) \right) / y = 0, \]

which are equivalent the system of equations for the functions \( l(x, z) \) and \( m(x, z) \)

\[ \frac{\partial^3}{\partial x^3} l(x, z) - 3 l(x, z) \frac{\partial}{\partial x} l(x, z) + \frac{\partial}{\partial z} l(x, z) = 0, \quad (3) \]

\[ \frac{\partial}{\partial x} l(x, z) - \frac{\partial}{\partial z} m(x, z) + 2 \left( \frac{\partial}{\partial x} l(x, z) \right) m(x, z) + l(x, z) \frac{\partial}{\partial x} m(x, z) = 0. \quad (4) \]

So the following theorem is valid

**Theorem 1** There exists a class of 3-dimensional flat metrics defined by the solutions of the system of equations (3-4).

Remark that the first equation of the system is the famous KdV-equation. Its solutions are very well known and may be used in particular to the description of three orthogonal coordinates systems ([2]).

Let us consider some examples.

1. \( l(x, z) = -\frac{x}{3z}, \quad m(x, z) = -1/2 + F_1(\frac{z}{x^3})x^{-2}. \)

2. \( l(x, z) = -4 (\cosh(x - 4 z))^{-2}, \quad m(x, z) = -\frac{1}{2} \)

**Remark 1** In a most simplest case of 3-dim metrics of zero curvature look as

\[ ds^2 = y^2 dx^2 + 2 (l(x, z)y^2 - 1/2) dx dz + 2 dy dz + (l(x, z))^2 y^2 - 2 \left( \frac{\partial}{\partial x} l(x, z) \right) y + l(x, z) \right) dz^2 \]

where the function \( l(x, z) \) is solution of classical KdV-equation

\[ \frac{\partial}{\partial z} l(x, z) - 3 l(x, z) \frac{\partial}{\partial x} l(x, z) + \frac{\partial^3}{\partial x^3} l(x, z) = 0. \]

In particular the functions

\[ l(x, z) = -\frac{x}{3z}, \quad l(x, z) = -4 (\cosh(x - 4 z))^{-2}, \]

and

\[ l(x, z) = -24 \quad \frac{4 \cosh(2x - 8z) + \cosh(4x - 64z) + 3}{(3 \cosh(x - 28z) + \cosh(3x - 36z))^2} \]

give us the examples of such type of metrics.

In spite of the fact that the determinant of the metric does not depends from the function \( l(x, z) \) it is possible to distinguish the properties of the metrics with help of the eigenvalue equation for the Laplace operator defined on the 1-form

\[ A(x, y, z) = A_i dx^i. \]

It has the form

\[ g^{ij} \nabla_i \nabla_j A_k - R_k^i A_l = -\lambda A_k. \]
In particular case \( l(x, z) = 0 \) and \( A_t(x, y, z) = [h(y), q(y), f(y)] \) this equations take the form

\[
-\frac{1}{4} \frac{d}{dy} h(y) - 4 \left( \frac{d}{dy} f(y) \right) y^2 - \left( \frac{d^2}{dy^2} h(y) \right) y + 4 \lambda h(y) y^3 = 0,
\]

\[
-\frac{1}{4} \frac{d}{dy} h(y) - 3 \frac{d}{dy} q(y) + 3 \left( \frac{d}{dy} h(y) \right) y + 4 \left( \frac{d}{dy} f(y) \right) y^2 - 4 f(y) y^2 + 4 \lambda q(y) y^4 = 0,
\]

\[
-\frac{1}{4} \frac{d}{dy} f(y) - \left( \frac{d^2}{dy^2} f(y) \right) y + 4 \lambda f(y) y^3 = 0.
\]

The solutions of such type system depends from the eigenvalues \( \lambda \) and characterize the properties of corresponding flat metric.

**Remark 2** In 3-dimensional geometry the density of Chern-Simons invariant defined by

\[
CS(\Gamma) = \epsilon^{ijk}(\Gamma^p_{iq} \Gamma^q_{kp}; j + 2 \Gamma^p_{iq} \Gamma^q_{kp})
\]

(5)

has an important role ([4]).

For the metric (2) one get the expression

\[
CS(\Gamma) = \frac{10}{9} x csgn(y) z^2 y - 10/3 F_1(\frac{z}{x^3}) csgn(y) z^{-1} x^{-2} y^{-3}.
\]

In a second example this quantity is

\[
CS(\Gamma) = 160 \frac{\cosh(x - 4 z) (\sinh(x - 4 z))^3 csgn(y)}{y}.
\]

**Remark 3** By analogy the properties of zero curvature metrics can be investigated with help of solutions of the MKdF-equation.

Our construction of ten dimensional space consists from a few steps.

The first of them is the creation of six dimensional basic space with necessary properties.

With this aim we use the notion of the Riemann extension of a given three dimensional space of zero curvature.

### 3 Six dimensional Riemann extensions of the metrics of zero curvature

The Riemann extension of riemannian or nonriemannian spaces can be constructed with the help of the Christoffel coefficients \( \Gamma^i_{jk} \) of corresponding Riemann space or with connection coefficients \( \Pi^i_{jk} \) in the case of the space of affine connection.
The metrics of the Riemann extension of any n-dimensional riemannian space looks as

\[ 2^n ds^2 = -2\Gamma_{ij}^k dx^idx^j\psi_k + 2dx^i d\psi_i, \]  

where \( \psi_i \) are an additional coordinates ((5)).

The properties of the spaces equipped with the metric (6) and their applications was first studied in the works of author ([6]-[15]).

In the case of the metric (2) we get the following expressions for nonzero Christoffel coefficients

\[
\Gamma_{11}^{1} = \frac{1}{2} \frac{2(l(x,z)y^2 - 1)}{y}, \quad \Gamma_{11}^{2} = \frac{1}{4} \frac{-4y^3 \frac{\partial}{\partial z}l(x,z) + 8l(x,z)y^2 - 1}{y}, \quad \Gamma_{11}^{3} = -y, \\
\Gamma_{12}^{1} = y^{-1}, \quad \Gamma_{12}^{2} = 1/2 y^{-1}, \quad \Gamma_{13}^{1} = 1/2 \frac{(2l(x,z)y^2 - 1)l(x,z)}{y}, \\
\Gamma_{13}^{2} = \frac{1}{4} \frac{-4l(x,z)y^3 \frac{\partial}{\partial z}l(x,z) + 8(l(x,z))^2 y^2 - l(x,z) - 4 \left( \frac{\partial^2}{\partial z^2}l(x,z) \right) y^2 + 2 \left( \frac{\partial}{\partial z}l(x,z) \right) y}{y}, \\
\Gamma_{13}^{3} = -l(x,z)y, \quad \Gamma_{23}^{1} = \frac{l(x,z)}{y}, \quad \Gamma_{23}^{2} = 1/2 \frac{l(x,z) - 2 \left( \frac{\partial}{\partial z}l(x,z) \right) y}{y}, \\
\Gamma_{33}^{1} = 1/2 \frac{2 \left( \frac{\partial}{\partial z}l(x,z) \right) y - 4 l(x,z)y \frac{\partial}{\partial x}l(x,z) + 2 \frac{\partial^2}{\partial x^2}l(x,z) + 2 \left( l(x,z) \right)^3 y^2 - \left( l(x,z) \right)^2}{y}, \\
4y\Gamma_{33}^{2} = -4 \left( l(x,z) \right)^2 y^3 \frac{\partial}{\partial z}l(x,z) - 4l(x,z)y^2 \frac{\partial^2}{\partial x^2}l(x,z) + 4 \left( \frac{\partial}{\partial z}l(x,z) \right) y + 2 \frac{\partial^2}{\partial x^2}l(x,z) + \\
+8y^2 \left( \frac{\partial}{\partial x}l(x,z) \right)^2 + 8 \left( l(x,z) \right)^3 y^2 - 8l(x,z)y \frac{\partial}{\partial x}l(x,z) - \left( l(x,z) \right)^2 - 4 \left( \frac{\partial^2}{\partial x \partial z}l(x,z) \right) y^2, \\
\Gamma_{33}^{3} = -(l(x,z))^2 y + \frac{\partial}{\partial x}l(x,z).
\]

Using these expressions and (5) we get the metric of extended six dimensional space with coordinates

\( (x, y, z, u, v, w) \)

\[ 6 ds^2 = 4y \left( \frac{\partial}{\partial x}l(x,z) \right) dy dv + \\
y/8 \left( 16v \frac{\partial^2}{\partial x \partial y}l(x,z) - 16v \left( \frac{\partial^2}{\partial x^2}l(x,z) \right) l(x,z) + 32v \left( \frac{\partial}{\partial x}l(x,z) \right)^2 \right) dy^2 + \\
+(-4 l(x,z) dy + 2 dz - 2 l(x,z) dx) dv + \\
+1/8 \left( -16u \frac{\partial}{\partial z}l(x,z) - 16v \frac{\partial}{\partial z}l(x,z) - 16u \frac{\partial}{\partial z}l(x,z) + 32u \left( \frac{\partial}{\partial x}l(x,z) \right) l(x,z) - 16v \left( \frac{\partial}{\partial x}l(x,z) \right) l(x,z) \right) dy^2 + \\
+4v \left( \frac{\partial}{\partial x}l(x,z) \right) dy dz + 2 dy dw - 2 l(x,z) dy du - 4v \left( \frac{\partial}{\partial x}l(x,z) \right) l(x,z) dx dy + \\
+1/8 \left( 8v \frac{\partial^2}{\partial x \partial z}l(x,z) - 16u \frac{\partial^2}{\partial x \partial z}l(x,z) \right) dy^2 + 4v \left( \frac{\partial^2}{\partial x^2}l(x,z) \right) dx dy + \\
+2v \left( \frac{\partial}{\partial x}l(x,z) \right) dx^2 + 1/8 \left( -16v \frac{\partial}{\partial x}l(x,z) - 64 u \frac{\partial}{\partial x}l(x,z) \right) dx dy + (1/2 dy + dx) dv + \\
+1/8 \left( -32 u \frac{\partial^2}{\partial x^2}l(x,z) - 12 v \frac{\partial^2}{\partial x^2}l(x,z) \right) dy^2 + (dy + 2 dx) du + \\
\]
3 SIX DIMENSIONAL RIEMANN EXTENSIONS OF THE METRICS OF ZERO CURVATURE

The space with the metric (9) is a Ricci flat on the solutions of the KdF-equation, but its curvature tensor depends from the function $H(x,z)$ and it is also equal to zero on the solutions of the KdF-equation.

In result after the Riemann extension of three dimensional flat space we get six dimensional flat space.

Next step of our construction is receiving of the Ricci flat $^6R_{ijkl} = 0$ but not a flat $^6R_{ijkl} \neq 0$ space.

With this aim we can insert into the expression for the metric (7) additional terms as a result of which can be obtained six dimensional Ricci flat but non a flat space.

Remark that there are a lot possibilities to realize that.

In a simplest case we item an additional term

$$2H(x,z)dx dz$$

with unknown function $H(x,z)$ into the (7) and in result we get the metric of perturbed space in form

$$ ^6ds^2 = \left(2y^2v \frac{\partial}{\partial x}l(x,z) - 4yvl(x,z) - 2yul(x,z) + 1/2 \frac{v}{y} + \frac{u}{y} + 2yw \right) dx^2 + 2 \left(\frac{v}{y} - 2 \frac{u}{y} \right) dx dy +$$

$$+ 2 \left( H(x,z) + 1/2 \frac{vl(x,z)}{y} + 2yvl(x,z) \right) dz dx +$$

$$+ \left( 2 yv \frac{\partial^2}{\partial x^2}l(x,z) - 4 yv (l(x,z))^2 + 2 yv^2 l(x,z) \frac{\partial}{\partial x}l(x,z) - v \frac{\partial}{\partial x}l(x,z) + \frac{vl(x,z)u}{y} - 2 y (l(x,z))^2 u \right) dx dz +$$

$$+ 2 dx du + 2 \left( - \frac{vl(x,z)}{y} - 2 \frac{l(x,z)u}{y} + 2v \frac{\partial}{\partial x}l(x,z) \right) dy dz + 2 dy dv +$$

$$+ \left( 1/2 \frac{v (l(x,z))^2}{y} - 2 v \frac{\partial}{\partial z}l(x,z) - 4 yv \left( \frac{\partial}{\partial x}l(x,z) \right)^2 + 2 yw (l(x,z))^2 - 2 w \frac{\partial}{\partial z}l(x,z) \right) dz^2 +$$

$$+ \left( 2 yvl(x,z) \frac{\partial^2}{\partial x^2}l(x,z) - 2 yu (l(x,z))^3 \right) dz^2 +$$

$$+ \left( u \frac{\partial}{\partial z}l(x,z) \right)^2 y - 2 v \frac{\partial}{\partial x}l(x,z) - 4 yv \left( \frac{\partial}{\partial x}l(x,z) \right)^2 + 2 yw \frac{\partial}{\partial x}l(x,z) - 4 yv (l(x,z))^3 \right) dz^2 +$$

$$+ \left( 2 yv^2 (l(x,z))^2 \frac{\partial}{\partial x}l(x,z) + 4 v(l(x,z)) \frac{\partial}{\partial x}l(x,z) + 4 ul(x,z) \frac{\partial}{\partial x}l(x,z) - 2 w \frac{\partial}{\partial x}l(x,z) \right) dz^2 + 2 dz dw.$$
The linear system for the coordinates \((u, v, w)\)

\[
\frac{d^2}{ds^2}u(s) + 2 \left( \frac{d}{ds}x(s) \right) y \frac{d}{ds}w(s) + 1/2 \left( \left( \frac{d}{ds}x(s) \right) y - 2 \left( \frac{d}{ds}y(s) \right) y \frac{d}{ds}v(s) \right) +
\]

\[
+1/2 \left( \frac{d}{ds}x(s) \right) y - 4 \left( \frac{d}{ds}y(s) \right) y \frac{d}{ds}u(s) + 1/2 \left( 2 \left( \frac{d}{ds}x(s) \right) y^2 - 4 \left( \frac{d}{ds}y(s) \right) \left( \frac{d}{ds}x(s) \right) y^2 \right) w(s) +
\]

\[
+1/2 \left( 6 \left( \frac{d}{ds}y(s) \right) y^2 - 3 \left( \frac{d}{ds}y(s) \right) \frac{d}{ds}x(s) \right) u(s) + 1/2 \left( \frac{d}{ds}y(s) \right) \frac{d}{ds}v(s) = 0,
\]

\[
\frac{d^2}{ds^2}v(s) - 2 \left( \frac{d}{ds}x(s) \right) \frac{d}{ds}u(s) - \left( \frac{d}{ds}x(s) \right)^2 w(s) - 1/4 \left( \frac{d}{ds}x(s) \right) \left( -24 \frac{d}{ds}y(s) + 3 \frac{d}{ds}x(s) \right) u(s) -
\]

\[
- \left( \frac{d}{ds}x(s) \right) \frac{d}{ds}v(s) = 0,
\]

\[
\frac{d^2}{ds^2}w(s) - 2 \left( \frac{d}{ds}x(s) \right) \frac{d}{ds}u(s) - \left( \frac{d}{ds}x(s) \right)^2 w(s) - 1/4 \left( \frac{d}{ds}x(s) \right) \left( -24 \frac{d}{ds}y(s) + 3 \frac{d}{ds}x(s) \right) u(s) -
\]

\[
- \left( \frac{d}{ds}x(s) \right) \frac{d}{ds}v(s) = 0.
\]

And nonlinear system for the coordinates \((x, y, z)\)

\[
2 \left( \frac{d^2}{ds^2}x(s) \right) y - \left( \frac{d}{ds}x(s) \right)^2 + 4 \left( \frac{d}{ds}y(s) \right) \frac{d}{ds}x(s) = 0,
\]

\[
4 \left( \frac{d^2}{ds^2}y(s) \right) y - \left( \frac{d}{ds}x(s) \right)^2 + 4 \left( \frac{d}{ds}y(s) \right) \frac{d}{ds}x(s) = 0,
\]

\[
\frac{d^2}{ds^2}z(s) - y \left( \frac{d}{ds}x(s) \right)^2 = 0.
\]

In the case \(l(x, z) \neq 0\) the geodesics of the metric (9) are more complicated.

Testing with the GRTensorII package show that six-dim spaces with the metric (9) have vanishing scalar curvature invariants and from this point they are same with the spaces of gravitational waves of the classical Einstein theory.

## 4 Four-dim Schwarzschild space-time on six-dim Ricci flat background

Here we present the construction of ten dimensional space made up on basis of six dimensional Ricci flat space (9) and some four dimensional Einstein space-time.

In the role of the Einstein space-time as a case in point the Schwarzschild space-time will be considered.

The metric can be choice in the form

\[
10ds^2 = \left( 2yw + \frac{u}{y} + 1/2 \frac{v}{y} + 2y^2 v \frac{\partial l(x, z)}{\partial x} - 2yl(x, z) - 4yl(x, z) \right) dx^2 + 2 \left( -2 \frac{u}{y} + \frac{v}{y} \right) dx dy +
\]

\[
+2 \left( 2yl(x, z)w - 2y(l(x, z))^2 u + \frac{l(x, z)u}{y} + 2yl(x, z)^2 l(x, z) + 2y^2 l(x, z)^2 \frac{\partial l(x, z)}{\partial x} \right) dx dz +
\]
where \( H(x, z, r) \) is some arbitrary function.

The condition \( 10^2 R_{ik} = 0 \) on the Ricci tensor of the metric (10) lead to the equations

\[-1/2 \left( \frac{\partial}{\partial r} H(x, z, r) \right) r - \left( \frac{\partial^2}{\partial r^2} H(x, z, r) \right) r^2 + \left( \frac{\partial}{\partial r} H(x, z, r) \right) M + \left( \frac{\partial^2}{\partial r^2} H(x, z, r) \right) rM = 0\]

and

\[-2 \frac{\partial^2}{\partial x^2} l(x, z) + 3 l(x, z) \frac{\partial}{\partial x} l(x, z) - \frac{\partial^3}{\partial x^3} l(x, z) = 0.\]

The solution of the first equation is defined by

\[ H(x, z, r) = F_1(x, z) + F_2(x, z) (-\ln(r) + \ln(M - r)) \]

where \( F_i(x, z) \) are an arbitrary functions and the second equation is the famous KdF-equation.

As result the family of the metrics of ten dimensional Einstein spaces with nonzero curvature made up on basis of Schwarzschild space and of some six dimensional space depended on the solutions of KdF-equation has been constructed.

So the properties of the classical Schwarzschild space and corresponding six dimensional basic space depending from the solutions of KdF-equation are interconnected.

In particular we get new types of trajectories in the Schwarzschild space-time which are defined by these solutions.

This statement follows from the properties of equations of geodesics for coordinates \( r, x, z \) of the metric (10)
where the function \( l(x, z) \) is solution of the KdF-equation.

Remark that full ten dimensional space has nonzero scalar curvature invariants. The simplest of them is

\[
C_{iklm} C^{iklm} = 12 \frac{M^2}{r^6}
\]

where \( C_{iklm} \) is the Weyl tensor of the metric (10).

In conclusion it may be said that suggested method of investigation of the properties of classical Einstein spaces admits a large generalization from the various points of view.

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