An invariance principle for a class of non-ballistic random walks in random environment

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Abstract

We are concerned with random walks on $\mathbb{Z}^d$, $d \geq 3$, in an i.i.d. random environment with transition probabilities $\varepsilon$-close to those of simple random walk. We assume that the environment is balanced in one fixed coordinate direction, and invariant under reflection in the coordinate hyperplanes. The invariance condition was used in [1] as a weaker replacement of isotropy to study exit distributions. We obtain precise results on mean sojourn times in large balls and prove a quenched invariance principle, showing that for almost all environments, the random walk converges under diffusive rescaling to a Brownian motion with a deterministic (diagonal) diffusion matrix. We also give a concrete description of the diffusion matrix. Our work extends the results of Lawler [9], where it is assumed that the environment is balanced in all coordinate directions.

1 Introduction and main results

1.1 The model

Denote by $e_i$ the $i$th unit vector of $\mathbb{Z}^d$. We let $\mathcal{P}$ be the set of probability distributions on $\{\pm e_i : i = 1, \ldots, d\}$ and put $\Omega = \mathcal{P}^{\mathbb{Z}^d}$. Denote by $\mathcal{F}$ the natural product $\sigma$-field on $\Omega$ and by $\mathbb{P} = \mu^{\otimes \mathbb{Z}^d}$, the product probability measure on $(\Omega, \mathcal{F})$.

Given an element (or environment) $\omega \in \Omega$, we denote by $(X_n)_{n \geq 0}$ the canonical nearest neighbor Markov chain on $\mathbb{Z}^d$ with transition probabilities

$$p_{\omega}(x, x + e) = \omega_x(e), \quad e \in \{\pm e_i : i = 1, \ldots, d\},$$

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the random walk in random environment (RWRE for short). We write $P_{x,\omega}$ for the “quenched” law of $(X_n)_{n \geq 0}$ starting at $x \in \mathbb{Z}^d$.

We are concerned with RWRE in dimensions $d \geq 3$ which is an $\varepsilon$-perturbation of simple random walk. To fix a perturbative regime, we shall assume the following condition.

- Let $0 < \varepsilon < 1/(2d)$. We say that $A_0(\varepsilon)$ holds if $\mu(P_{\varepsilon}) = 1$, where

$$P_{\varepsilon} = \{q \in \mathcal{P} : |q(\pm e_i) - 1/(2d)| \leq \varepsilon \text{ for all } i = 1, \ldots, d\}.$$ 

Furthermore, we work under two centering conditions on the measure $\mu$. The first rules out ballistic behavior, while the second guarantees that the RWRE is balanced in direction $e_1$.

- We say that $A_1$ holds if $\mu$ is invariant under all $d$ reflections $O_i : \mathbb{R}^d \to \mathbb{R}^d$ mapping the unit vector $e_i$ to its inverse, i.e. $O_i e_i = -e_i$ and $O_i e_j = e_j$ for $j \neq i$. In other words, the laws of $(\omega_0(O_i e_i))_{|e_i|=1}$ and $(\omega_0(e_i))_{|e_i|=1}$ coincide, for each $i = 1, \ldots, d$.

- We say that $B$ holds if $\mu(P_{s,1}) = 1$, where

$$P_{s,1} = \{p \in \mathcal{P} : p(e_1) = p(-e_1)\}.$$ 

We now state our results. Then we discuss them together with our conditions in the context of known results from the literature.

### 1.2 Our main results

Our first statement shows $\mathbb{P}$-almost sure convergence of the (normalized) RWRE mean sojourn time in a ball when its radius gets larger and larger. Let $V_L = \{y \in \mathbb{Z}^d : |y| \leq L\}$ denote the discrete ball of radius $L$, and $V_L(x) = x + V_L$. Denote by $\tau_{V_L(x)} = \inf\{n \geq 0 : X_n \notin V_L(x)\}$ the first exit time of the RWRE from $V_L(x)$. We write $E_{x,\omega}$ for the expectation with respect to $P_{x,\omega}$. We always assume $d \geq 3$.

**Theorem 1.1** (Quenched mean sojourn times, $d \geq 3$). Assume $A_1$ and $B$. Given $0 < \eta < 1$, one can find $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that if $A_0(\varepsilon)$ is satisfied for some $\varepsilon \leq \varepsilon_0$, then the following holds: There exists $D \in [1-\eta, 1+\eta]$ such that for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$\lim_{L \to \infty} \left( E_{0,\omega} [\tau_{V_L}] / L^2 \right) = D.$$ 

Moreover, one has for each $k \in \mathbb{N}$, for $\mathbb{P}$-almost all $\omega$,

$$\lim_{L \to \infty} \left( \inf_{x:|x| \leq L^k} E_{x,\omega} [\tau_{V_L(x)}] / L^2 \right) = \lim_{L \to \infty} \left( \sup_{x:|x| \leq L^k} E_{x,\omega} [\tau_{V_L(x)}] / L^2 \right) = D.$$ 

Standard arguments then imply the following bound on the moments.
**Corollary 1.1** (Quenched moments). *In the setting of Theorem 1.1, for each* \( k, m \in \mathbb{N} \) *and* \( \mathbb{P} \)-*almost all* \( \omega \),

\[
\limsup_{L \to \infty} \left( \sup_{x : |x| \leq L^k} \mathbb{E}_{x, \omega} \left[ \tau_{V_{L}(x)}^m \right] / L^{2m} \right) \leq 2^m m! .
\]

Combined with results on the spatial behavior of the RWRE from [1], we prove a functional central limit theorem under the quenched measure. In [1], it was shown that under \( A_1 \) and \( A_0(\varepsilon) \) for \( \varepsilon \) small, the limit

\[
2p_\infty(\pm e_i) = \lim_{L \to \infty} \sum_{y \in \mathbb{Z}^d} \mathbb{E} \left[ P_{0, \omega} \left( X_{\tau_{V_L}} = y \right) \right] \frac{y_i^2}{|y|^2}
\]

exists for \( i = 1, \ldots, d \), and \( |p_\infty(e_i) - 1/(2d)| \to 0 \) as \( \varepsilon \downarrow 0 \). Let

\[
\Lambda = (2p_\infty(e_i)\delta_j(j))_{i,j=1}^d \in \mathbb{R}^{d \times d} ,
\]

and define the linear interpolation

\[
X_t^n = X_{[tn]} + (tn - [tn]) \left( X_{[tn]+1} - X_{[tn]} \right) , \quad t \geq 0 .
\]

The sequence \((X_t^n, t \geq 0)\) takes values in the space \( C(\mathbb{R}_+, \mathbb{R}^d) \) of \( \mathbb{R}^d \)-valued continuous functions on \( \mathbb{R}_+ \). The set \( C(\mathbb{R}_+, \mathbb{R}^d) \) is tacitly endowed with the uniform topology and its Borel \( \sigma \)-field.

**Theorem 1.2** (Quenched invariance principle, \( d \geq 3 \)). *Assume \( A_0(\varepsilon) \) for small* \( \varepsilon > 0 \), \( A_1 \) *and* \( B \). *Then for* \( \mathbb{P} \)-*a.e. \( \omega \in \Omega \), *under* \( P_{0, \omega} \),

\[
X_t^n / \sqrt{n} \text{ converges in law to a } d \text{-dimensional Brownian motion with diffusion matrix } D^{-1}\Lambda, \text{ where } D \text{ is the constant from Theorem 1.1 and } \Lambda \text{ is given by (2).}
\]

Under Conditions \( A_0(\varepsilon) \) and \( A_1 \), a local limit law for RWRE exit measures was proved in [1] for dimensions three and higher. Before that, similar results were obtained by Bolthausen and Zeitouni [6] for the case of isotropic perturbative RWRE. While the results of [6] and [1] do already imply transience for the random walks under consideration, they do not prove diffusive behavior, since there is no control over time. This was already mentioned in [6]: “In future work we hope to combine our exit law approach with suitable exit time estimates in order to deduce a (quenched) CLT for the RWRE.”

Under the additional Condition \( B \) which shall be discussed below, we fulfill here their hope.

In dimensions \( d > 1 \), the RWRE under the quenched measure is an irreversible (inhomogeneous) Markov chain. A major difficulty in its analysis comes from the presence of so-called traps, i.e. regions where the random walk can hardly escape and therefore spends a lot of time. In the ballistic regime where the limit velocity \( \lim_{n \to \infty} X_n/n \) is an
almost sure constant different from zero, powerful methods leading to law of large numbers and limit theorems have been established, see e.g. Sznitman [11, 12, 13], Berger [2] or the lecture notes of Sznitman [14] with further references. They involve the construction of certain regeneration times, where, roughly speaking, the walker does not move “backward” anymore.

In the non-ballistic case, different techniques based on renormalization schemes are required. In the small disorder regime, results can be found under the classical isotropy condition on $\mu$, which is stronger than our condition $A1$. It requires that for any orthogonal map $O$ acting on $\mathbb{R}^d$ which fixes the lattice $\mathbb{Z}^d$, the law of $(\omega_0(Oe))_{|e|=1}$ and $(\omega_0(e))_{|e|=1}$ coincide. Under this condition, Bricmont and Kupiainen [7] provide a functional central limit theorem under the quenched measure for dimensions $d \geq 3$.

However, it is of a certain interest to find a new self-contained proof of their result. A continuous counterpart is studied in Sznitman and Zeitouni [15]. For $d \geq 3$, they prove a quenched invariance principle for diffusions in a random environment which are small isotropic perturbations of Brownian motion. Invariance under all lattice isometries is also assumed in the aforementioned work of Bolthausen and Zeitouni [6].

In the non-perturbative setting, Bolthausen et al. [5] use so-called cut times as a replacement of regeneration times. At such times, past and future of the path do not intersect. However, in order to ensure that there are infinitely many cut times, it is assumed in [5] that the projection of the RWRE onto at least $d_1 \geq 5$ components behaves as a standard random walk. Among other things, a quenched invariance principle is proved when $d_1 \geq 7$ and the law of the environment is invariant under the antipodal transformation sending the unit vectors to their inverses.

Our Condition $B$ requires only that the environment is balanced in one fixed coordinate direction ($e_1$ for definiteness). Then the projection of the RWRE onto the $e_1$-axis is a martingale under the quenched measure, which implies a priori bounds on the sojourn times, see the discussion in Section 4. Clearly, assuming just Conditions $A0(\varepsilon)$ and $B$ could still result in ballistic behavior; but the combination of $A0(\varepsilon)$, $A1$ and $B$ provides a natural framework to investigate non-ballistic behavior of “partly-balanced” RWRE in the perturbative regime.

To our knowledge, we are the first who study random walks in random environment which is balanced in only one coordinate direction. The study of fully balanced RWRE when $P(\omega_0(e_i) = \omega_0(-e_i) \text{ for all } i = 1, \ldots, d) = 1$ goes back to Lawler [9]. He proves a quenched invariance principle for ergodic and elliptic environments in all dimensions. Extensions within the i.i.d. setting to the mere elliptic case were obtained by Guo and Zeitouni [8], and recently to the non-elliptic case by Berger and Deuschel [3].

Since the results of [11] do also provide local estimates, we believe that with some more effort, Theorem 1.2 could be improved to a local central limit theorem. Furthermore, we expect that our results remain true without assuming Condition $B$. Getting rid of this condition would however require a complete control over large sojourn times, which remains a major open problem.
Organization of the paper and rough strategy of the proofs

We first introduce the most important notation. For ease of readability, we recapitulate in Section 2 those concepts and results from [1] which play a major role here. In Section 3 we provide the necessary control over Green’s functions. To a large extend, we can rely on the results from [1], but we need additional difference estimates for our results on mean sojourn times.

In Section 4 we prove Theorem 1.1. This will allow us to deduce Theorem 1.1. A strategy of proof of this statement can be found at the beginning of Section 5.

In the last part of this paper starting with Section 5, we show how Theorem 1.1 can be combined with the results on exit laws from [1] to obtain Theorem 1.2. A strategy of proof of this statement can be found at the beginning of Section 5.

1.3 Some notation

We collect here some notation that is frequently used in this text.

Sets and distances

We put \( \mathbb{N} = \mathbb{N}_0 = \{0, 1, 2, 3, \ldots \} \). For \( x \in \mathbb{R}^d \), \( |x| \) is the Euclidean norm of \( x \). The distance between \( A, B \subset \mathbb{R}^d \) is denoted \( d(A, B) = \inf \{|x - y| : x \in A, \ y \in B\} \). Given \( L > 0 \), we let \( V_L = \{x \in \mathbb{R}^d : |x| \leq L\} \), and for \( x \in \mathbb{Z}^d \), \( V_L(x) = x + V_L \). Similarly, put \( C_L = \{x \in \mathbb{R}^d : |x| < L\} \). The outer boundary of \( V \subset \mathbb{Z}^d \) is given by \( \partial V = \{x \in \mathbb{Z}^d \setminus V : d(\{x\}, V) = 1\} \). For \( x \in C_L \), we let \( d_L(x) = L - |x| \). Note for \( x \in V_L \), \( d_L(x) \leq d(\{x\}, \partial V_L) \). Finally, for \( 0 \leq a < b \leq L \), put \( Sh_L(a, b) = \{x \in V_L : a \leq d_L(x) < b\} \), \( Sh_L(b) = Sh_L(0, b) \).

Functions

We use the usual notation \( a \wedge b = \min\{a, b\} \) for reals \( a, b \). We further write log for the logarithm to the base e. Given two functions \( F, G : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R} \), we write \( FG \) for the (matrix) product \( FG(x, y) = \sum_{u \in \mathbb{Z}^d} F(x, u)G(u, y) \), provided the right hand side is absolutely summable. \( F^k \) is the \( k \)-th power defined in this way, and \( F^0(x, y) = \delta_x(y) \). \( F \) can also operate on functions \( f : \mathbb{Z}^d \to \mathbb{R} \) from the left via \( Ff(x) = \sum_{y \in \mathbb{Z}^d} F(x, y)f(y) \).

As usual, \( 1_W \) stands for the indicator function of the set \( W \), but we will also write \( 1_W \) for the kernel \( (x, y) \mapsto 1_W(x)\delta_y(y) \), where the Delta function \( \delta_x(y) \) is equal to one if \( y = x \) and zero otherwise. If \( f : \mathbb{Z}^d \to \mathbb{R} \), \( \|f\|_1 = \sum_{x \in \mathbb{Z}^d} |f(x)| \in [0, \infty] \) denotes its \( L^1 \)-norm. For a (signed) measure \( \nu : \mathbb{Z}^d \to \mathbb{R} \), we write \( \|\nu\|_1 \) for its total variation norm.
Transition kernels, exit times and exit measures

Denote by $\mathcal{G}$ the $\sigma$-algebra on $(\mathbb{Z}^d)^\mathbb{N}$ generated by the cylinder functions. If $p = (p(x, y))_{x, y \in \mathbb{Z}^d}$ is a family of (not necessarily nearest neighbor) transition probabilities, we write $P_{x,p}$ for the law of the canonical random walk $(X_n)_{n \geq 0}$ on $((\mathbb{Z}^d)^\mathbb{N}, \mathcal{G})$ started from $X_0 = x \ P_{x,p}$-a.s. and evolving according to the kernel $p$.

The simple random walk kernel is denoted $p_\omega(x, x \pm e_i) = 1/(2d)$, and we write $P_x$ instead of $P_{x,p_\omega}$. For transition probabilities $p_\omega$ defined in terms of an environment $\omega$, we use the notation $P_{x,\omega}$. The corresponding expectation operators are denoted by $E_x$, $E_x$ and $E_{x,\omega}$, respectively. Every $p \in \mathcal{P}$ gives in an obvious way rise to a homogeneous nearest neighbor transition kernel on $\mathbb{Z}^d$, which we again denote by $p$.

For a subset $V \subset \mathbb{Z}^d$, we let $\tau_V = \inf\{n \geq 0 : X_n \notin V\}$ be the first exit time from $V$, with $\inf\emptyset = \infty$.

Given $x, z \in \mathbb{Z}^d, p \in \mathcal{P}$ and a subset $V \subset \mathbb{Z}^d$, we define

$$\pi_V^{(p)}(x, z) = P_{x,p}(X_{\tau_V} = z).$$

For an environment $\omega \in \Omega$, we set

$$\Pi_{V,\omega}(x, z) = P_{x,\omega}(X_{\tau_V} = z).$$

We mostly drop $\omega$ in the notation and interpret $\Pi_V(x, \cdot)$ as a random measure.

Recall the definitions of the sets $\mathcal{P}$, $\mathcal{P}_\kappa, \mathcal{P}_\varepsilon$ from the introduction. For $0 < \kappa < 1/(2d)$, let

$$\mathcal{P}_\kappa = \{p \in \mathcal{P}_{\kappa} : p(e_i) = p(-e_i), i = 1, \ldots, d\},$$

i.e. $\mathcal{P}_\kappa$ is the subset of $\mathcal{P}_\kappa$ which contains all symmetric probability distributions on $\{\pm e_i : i = 1, \ldots, d\}$. The parameter $\kappa$ was introduced in [1] to bound the range of the symmetric transition kernels under consideration. We can think of $\kappa$ as a fixed but arbitrarily small number (the perturbation parameter $\varepsilon$ is chosen afterward).

Miscellaneous comments about notation

Our constants are positive and depend only on the dimension $d \geq 3$ unless stated otherwise. In particular, they do not depend on $L$, $p \in \mathcal{P}_\kappa$, $\omega$ or on any point $x \in \mathbb{Z}^d$.

By $C$ and $c$ we denote generic positive constants whose values can change even in the same line. For constants whose values are fixed throughout a proof we often use the symbols $K, C_1, c_1$.

Many of our quantities, e.g. the transition kernels $\hat{\Pi}_L$, $\hat{\pi}_L$ or the kernel $\Gamma_L$, are indexed by $L$. We normally drop the index in the proofs. In contrast to [1], we do here not work with an additional parameter $r$.

We often drop the superscript $(p)$ from notation and write $\pi_V$ for $\pi_V^{(p)}$. If $V = V_L$ is the ball around zero of radius $L$, we write $\pi_L$ instead of $\pi_V$, $\Pi_L$ for $\Pi_V$ and $\tau_L$ for $\tau_V$.

By $P$ we denote sometimes a generic probability measure, and by $E$ its corresponding expectation. If $A$ and $B$ are two events, we often write $P(A; B)$ for $P(A \cap B)$. 
If we write that a statement holds for “$L$ large (enough)”, we implicitly mean that there exists some $L_0 > 0$ depending only on the dimension such that the statement is true for all $L \geq L_0$. This applies also to phrases like “$\delta$ (or $\varepsilon$, or $\kappa$) small (enough)”.

Some of our statements are only valid for large $L$ and $\varepsilon$ (or $\delta$, or $\kappa$) sufficiently small, but we do not mention this every time.

2 Results and concepts from the study of exit laws

Our approach uses results and constructions from [1], where exit measures from large balls under $A_0$ and $A_1(\varepsilon)$ are studied. We adapt in this section those parts which will be frequently used in this paper. Some auxiliary statements from [1], which play here only a minor role, will simply be cited when they are applied.

The overall idea of [1] is to transport estimates on exit measures inductively from one scale to the next, via a perturbation expansion for the Green’s function, which we recall first.

2.1 A perturbation expansion

Let $p = (p(x, y))_{x,y \in \mathbb{Z}^d}$ be a family of finite range transition probabilities on $\mathbb{Z}^d$, and let $V \subset \mathbb{Z}^d$ be a finite set. The corresponding Green’s kernel or Green’s function for $V$ is defined by

$$g_V(p)(x, y) = \sum_{k=0}^{\infty} (1_V p)^k (x, y).$$  \hspace{1cm} (3)

Now write $g$ for $g_V(p)$ and let $P$ be another transition kernel with corresponding Green’s function $G$ for $V$. With $\Delta = 1_V (P - p)$, the resolvent equation gives

$$G - g = g\Delta G = G\Delta g.$$  \hspace{1cm} (4)

An iteration of (4) leads to further expansions. Namely, first one has

$$G - g = \sum_{k=1}^{\infty} (g\Delta)^k g,$$  \hspace{1cm} (5)

and then, with $R = \sum_{k=1}^{\infty} \Delta^k p$, one arrives at

$$G = g \sum_{m=0}^{\infty} (Rg)^m \sum_{k=0}^{\infty} \Delta^k.$$  \hspace{1cm} (6)

We refer to [1] for more details.
2.2 Coarse grained transition kernels

We fix once for all a probability density $\varphi \in C^\infty(\mathbb{R}_+, \mathbb{R}_+)$ with compact support in $(1, 2)$. Given a transition kernel $p \in \mathcal{P}$ and a strictly positive function $\psi = (m_x)_{x \in W}$, where $W \subset \mathbb{R}^d$, we define coarse grained transition kernels on $W \cap \mathbb{Z}^d$ associated to $(\psi, p)$,

$$\hat{\pi}_\psi^{(p)}(x, \cdot) = \frac{1}{m_x} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{m_x} \right) \pi_{V_t(x)}^{(p)}(x, \cdot) dt, \quad x \in W \cap \mathbb{Z}^d. \quad (7)$$

Often $\psi \equiv m > 0$ will be a constant, and then (7) makes sense for all $x \in \mathbb{Z}^d$ and therefore gives coarse grained transition kernels on the whole grid $\mathbb{Z}^d$.

We now introduce particular coarse grained transition kernels for walking inside the ball $V_L$, for both symmetric random walk and RWRE. We set up a coarse graining scheme which will make the link between different scales and allows us to transport estimates on mean sojourn times from one level to the next. Our scheme is similar to that in [1], but does not depend on an additional parameter $r$.

Let $s_L = \frac{L}{(\log L)^3}$ and $r_L = \frac{L}{(\log L)^{15}}$.

We fix a smooth function $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$h(x) = \begin{cases} x & \text{for } x \leq 1/2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

such that $h$ is concave and increasing on $(1/2, 2)$. Define $h_L : \overline{C}_L \to \mathbb{R}_+$ by

$$h_L(x) = \frac{1}{20} \max \left\{ s_L h \left( \frac{d_L(x)}{s_L} \right), r_L \right\}. \quad (8)$$

Note that in the setting of [1], this means that we always work with the choice $r = r_L$, and there is no need keep $r$ in the notation.

We write $\hat{\Pi}_L = \hat{\Pi}_{L, \omega}$ for the coarse grained RWRE transition kernel inside $V_L$ associated to $(\psi = (h_L(x))_{x \in V_L}, p_\omega)$,

$$\hat{\Pi}_L(x, \cdot) = \frac{1}{h_L(x)} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{h_L(x)} \right) \Pi_{V_t(x) \cap V_L}(x, \cdot) dt, \quad (9)$$

and $\hat{\pi}_L^{(p)}$ for the corresponding kernel coming from symmetric random walk with transition kernel $p \in \mathcal{P}$, where in the definition (9) the random RWRE exit measure $\Pi$ is replaced by $\pi^{(p)}$. For points $x \in \mathbb{Z}^d \setminus V_L$, we set $\hat{\Pi}_L(x, \cdot) = \hat{\pi}_L^{(p)}(x, \cdot) = \delta_x(\cdot)$.

Note our small abuse of notation: $\hat{\pi}_L^{(p)}$ is always defined in this way and does never denote the coarse grained kernel $\hat{\Pi}_L$ associated to the constant function $\psi \equiv L$. Also note that $\hat{\Pi}_L$ was denoted $\hat{\Pi}_{L, r_L}$ in [1], and similarly $\hat{\pi}_L$ was denoted $\hat{\pi}_{L, r_L}$. The kernel $\hat{\Pi}_L$ is a random transition kernel depending on $\omega$. However, when we consider $\hat{\Pi}_L$ under $P_{x, \omega}$, then $\omega$ is fixed, but even in this case we usually write $\hat{\Pi}_L$ instead of $\hat{\Pi}_{L, \omega}$.

Two Green’s function will play a crucial role (cf. (3)).
2.3 Propagation of Condition C1

- \( \hat{G}_L \) denotes the (coarse grained) RWRE Green’s function corresponding to \( \hat{\Pi}_L \).
- \( \hat{g}_L^{(p)} \) denotes the Green’s function corresponding to \( \hat{\pi}_L^{(p)} \).

The “goodified” version \( \hat{G}_L^q \) of \( \hat{G}_L \) will be introduced in Section 2.5.

2.3 Propagation of Condition C1

We recapitulate in this part the technical Condition C1(\( \delta, L_0, L_1 \)), which is propagated from one level to the next in [1].

Assignment of transition kernels

Let \( L_0 > 0 \) (\( L_0 \) shall play the role of a large constant). Define \( L \)-dependent symmetric transition kernels by

\[
 p_L(\pm e_i) = \begin{cases} 
 1/(2d) & \text{for } 0 < L \leq L_0 \\
 \frac{1}{2} \sum_{y \in \mathbb{Z}^d} \mathbb{E} \left[ \hat{\Pi}_L(0, y) \right] \frac{y^2}{|y|^d} & \text{for } L > L_0
\end{cases}
\]  

To be in position to formulate Condition C1, we recall some notation from [1]. Let \( \mathcal{M}_t \) be the set of smooth functions \( \psi : \mathbb{R}^d \to \mathbb{R}_+ \) with first four derivatives bounded uniformly by 10 and

\[
 \psi \left( \{ x \in \mathbb{R}^d : t/2 < |x| < 2t \} \right) \subset (t/10, 5t).
\]

For \( p, q \in \mathcal{P} \) and \( \psi \in \mathcal{M}_t \), define

\[
 D_{t,p,\psi,q}^* = \sup_{x \in V_t/5} \left\| \left( \Pi_{V_t} - \hat{\pi}_{V_t}^{(p)} \right) \hat{\pi}_{V_t}^{(q)}(x, \cdot) \right\|_1,
\]

\[
 D_{t,p}^* = \sup_{x \in V_t/5} \left\| \left( \Pi_{V_t} - \hat{\pi}_{V_t}^{(p)} \right)(x, \cdot) \right\|_1.
\]

With \( \delta > 0 \), define for \( i = 1, 2, 3 \)

\[
 b_i(L, p, \psi, q, \delta) = \mathbb{P} \left( \{ (\log L)^{-9+9(i-1)/4} < D_{L,p,\psi,q}^* \leq (\log L)^{-9+9i/4} \} \cap \{ D_{L,p}^* \leq \delta \} \right),
\]

and

\[
 b_4(L, p, \psi, q, \delta) = \mathbb{P} \left( \{ D_{L,p,\psi,q}^* > (\log L)^{-3+3/4} \} \cup \{ D_{L,p}^* > \delta \} \right).
\]

Let \( t = (\log L_0)^{-7} \). Then Condition C1 is given as follows.

**Condition C1**

Let \( \delta > 0 \) and \( L_1 \geq L_0 \geq 3 \). We say that C1(\( \delta, L_0, L_1 \)) holds if

- For all \( 3 \leq L \leq 2L_0 \), all \( \psi \in \mathcal{M}_L \) and all \( q \in \mathcal{P}_t^p \),

\[
 \mathbb{P} \left( \{ D_{L,p,\psi,q}^* > (\log L)^{-9} \} \cup \{ D_{L,p}^* > \delta \} \right) \leq \exp \left( -((2L_0)^2) \right).
\]

- For all \( L_0 < L \leq L_1 \), \( L' \in [L, 2L] \), \( \psi \in \mathcal{M}_{L'} \) and \( q \in \mathcal{P}_{t'}^p \),

\[
 b_i(L', p_{L'}, \psi, q, \delta) \leq \frac{1}{4} \exp \left( -((3 + i)/4) (\log L')^2 \right) \quad \text{for } i = 1, 2, 3, 4.
\]
The main statement for C1

The first part of [1, Proposition 1.1] implies the following statement.

**Proposition 2.1.** Assume A1. For \( \delta > 0 \) small enough, there exist \( L_0 = L_0(\delta) \) large and \( \varepsilon_0 = \varepsilon_0(\delta) > 0 \) small with the following property: If \( \varepsilon \leq \varepsilon_0 \) and A0(\( \varepsilon \)) is satisfied, then C1(\( \delta, L_0, L \)) holds for all \( L \geq L_0 \).

For us, the important implication is that if C1(\( \delta, L_0, L_1 \)) is satisfied, then for any \( 3 \leq L \leq L_1 \) and for all \( L' \in [L, 2L] \), all \( \psi \in \mathcal{M}_{L'} \) and all \( q \in \mathcal{P}_s \),

\[
\mathbb{P} \left( \{ D_{L',p,L,\psi,q}^* > (\log L')^{-9} \} \cup \{ D_{L',p,L}^* > \delta \} \right) \leq \exp\left( -\left( \log (L')^2 \right) \right). \tag{11}
\]

In [1, Lemma 2.2] it is shown that the transition kernels \( p_L \) form a Cauchy sequence. Their limit is given by the kernel \( p_\infty \) defined in (1), i.e. \( \lim_{L \to \infty} p_L(e_i) = p_\infty(e_i) \) for \( i = 1, \ldots, d \). From this fact and the last display one can deduce that the difference in total variation of the exit laws \( \Pi_L \) and \( \pi(p_\infty) \) is small when \( L \) is large, in both a smoothed and non-smoothed way. See Theorems 0.1 and 0.2 of [1] for precise statements. For us, it will be sufficient to keep in mind (11) and the fact that \( \lim_{L \to \infty} p_L = p_\infty \).

We follow the convention of [1] and write “assume C1(\( \delta, L_0, L_1 \))”, if we assume C1(\( \delta, L_0, L_1 \)) for some \( \delta > 0 \) and some \( L_1 \geq L_0 \), where \( \delta \) can be chosen arbitrarily small and \( L_0 \) arbitrarily large.

### 2.4 Good and bad points

In [1, Section 2.2], the concept of good and bad points inside \( V_L \) is introduced. It turns out that for controlling mean sojourn times, we need a stronger notion of “goodness”, see Section 4.3. It is however more convenient to first recall the original classification.

Recall the assignment (10). A point \( x \in V_L \) is **good** (with respect to \( L \) and \( \delta > 0 \)), if

- For all \( t \in [h_L(x), 2h_L(x)] \), with \( q = p_{h_L(x)} \),
  \[
  \left\| \left( \Pi_{V_t(x)} - \pi_{V_t(x)}^{(q)} \right)(x, \cdot) \right\|_1 \leq \delta.
  \]

- If \( d_L(x) > 2r \), then additionally
  \[
  \left\| \left( \hat{\Pi}_L - \pi_L^{(q)} \right) \hat{p}_L^{(q)}(x, \cdot) \right\|_1 \leq (\log h_L(x))^{-9}.
  \]

The set of environments where all points \( x \in V_L \) are good is denoted \( \text{Good}_L \). A point \( x \in V_L \) which is not good is called **bad**, and the set of all bad points inside \( V_L \) is denoted by \( \mathcal{B}_L = \mathcal{B}_L(\omega) \).
2.5 Goodified transition kernels and Green’s function

By replacing the coarse grained RWRE transition kernel at bad points \( x \in V_L \) by that of a homogeneous symmetric random walk, we obtain what we call “goodified” transition kernels inside \( V_L \).

More specifically, write \( p \) for \( p_{sL/20} \) stemming from the assignment (10). The goodified transition kernels are then defined as follows.

\[
\hat{\Pi}^g_L(x, \cdot) = \begin{cases} 
\hat{\Pi}_L(x, \cdot) & \text{for } x \in V_L \setminus B_L \\
\hat{\pi}^{(p)}_L(x, \cdot) & \text{for } x \in B_L
\end{cases}
\]  

(12)

We write \( \hat{G}^g_L \) for the corresponding (random) Green’s function (denoted \( \hat{G}^g_{L,r_L} \) in [1]).

Proposition 2.1 will allow us to concentrate on environments \( \omega \in \text{Good}^L \), where \( \hat{\Pi}_L = \hat{\Pi}^g_L \) and therefore also \( \hat{G}_L = \hat{G}^g_L \). In the next section, we provide the necessary estimates for the “goodified” coarse grained RWRE Green’s function \( \hat{G}^g_L \).

3 Control on Green’s functions

We first recapitulate estimates on Green’s functions from [1]. Then we establish difference estimates for these functions, which will be used to control differences of (quenched) mean sojourn times from balls \( V_t(x) \) and \( V_t(y) \) that have a sufficiently large intersection.

3.1 Bounds on Green’s functions

Recall that \( P^s_\kappa \) denotes the set of kernels which are symmetric in every coordinate direction and \( \kappa \)-perturbations of the simple random walk kernel. The statements in this section are valid for small \( \kappa \), in the sense that there exists \( 0 < \kappa_0 < 1/(2d) \) such that for \( 0 < \kappa \leq \kappa_0 \), the statements hold true for \( p \in P^s_\kappa \), with constants that are uniform in \( p \in P^s_\kappa \).

Let \( p \in P^s_\kappa \) and \( m \geq 1 \). Denote by \( \hat{\pi}_{\psi_m} = \hat{\pi}^{(p)}_{\psi_m} \) the coarse grained transition probabilities on \( \mathbb{Z}^d \) associated to \( \psi_m = (m_x)_{x \in \mathbb{Z}^d} \), where \( m_x = m \) is chosen constant in \( x \), cf. (7). We mostly drop \( p \) from notation. The kernel \( \hat{\pi}_{\psi_m} \) is centered, with covariances

\[
\sum_{y \in \mathbb{Z}^d} (y_i - x_i)(y_j - x_j)\hat{\pi}_{\psi_m}(x, y) = \lambda_{m,i}\delta_i(j),
\]

where for large \( m \), \( C^{-1} < \lambda_{m,i}/m^2 < C \) for some \( C > 0 \). We set

\[
\mathbf{A}_m = (\lambda_{m,i}\delta_i(j))_{i,j=1}^d, \quad \mathcal{J}_m(x) = |\mathbf{A}_m^{-1/2}x| \quad \text{for } x \in \mathbb{Z}^d,
\]

and denote by

\[
\hat{g}_{m,\mathbb{Z}^d}(x, y) = \sum_{n=0}^{\infty} (\hat{\pi}_{\psi_m})^n(x, y)
\]
the Green’s function corresponding to $\hat{\pi}_{\psi_m}$. In [Π], the following behavior of $\hat{g}_{m,\mathbb{Z}^d}$ was established. The proof is based on a local central limit theorem for $\hat{\pi}_{\psi_m}$, which we do not restate here.

**Proposition 3.1.** Let $p \in \mathcal{P}_{\kappa}^s$. Let $x, y \in \mathbb{Z}^d$, and assume $m \geq m_0 > 0$ large enough.

(i) For $|x - y| < 3m$,

$$\hat{g}_{m,\mathbb{Z}^d}(x, y) = \delta_{x}(y) + O(m^{-d}).$$

(ii) For $|x - y| \geq 3m$, there exists a constant $c(d) > 0$ such that

$$\hat{g}_{m,\mathbb{Z}^d}(x, y) = \frac{c(d) \det \Lambda_m^{-1/2}}{f_m(x-y)^{d-2}} + O\left(\frac{1}{|x-y|^d} \left(\log \frac{|x-y|}{m}\right)^d\right).$$

Recall that $\tau_L = \tau_{V_L}$ denotes the first exit time from $V_L$. The proposition can be used to estimate the corresponding Green’s function for $V_L$.

$$\hat{g}_{m,V_L}(x, y) = \sum_{n=0}^{\infty} (1_{V_L} \hat{\pi}_{\psi_m})^n(x, y).$$

Indeed, $\hat{g}_{m,V_L}$ is bounded from above by $\hat{g}_{m,\mathbb{Z}^d}$, and more precisely, the strong Markov property shows

$$\hat{g}_{m,V_L}(x, y) = \mathbb{E}_{x, \hat{\pi}_{\psi_m}} \left[ \sum_{k=0}^{\tau_{L}-1} 1_{\{X_k=y\}} \right] = \hat{g}_{m,\mathbb{Z}^d}(x, y) - \mathbb{E}_{x, \hat{\pi}_{\psi_m}} \left[ \hat{g}_{m,\mathbb{Z}^d}(X_{\tau_L}, y) \right]. \quad (13)$$

Here, according to our notational convention, $\mathbb{E}_{x, \hat{\pi}_{\psi_m}}$ is the expectation with respect to $P_{x, \hat{\pi}_{\psi_m}}$, the law of a random walk started at $x$ and running with kernel $\hat{\pi}_{\psi_m}$.

We next recall the definition of the (deterministic) kernel $\Gamma_L$, which was introduced in [Π] to dominate coarse-grained Green’s functions from above.

Since we always work with $r = r_L$, we write $\Gamma_L$ instead of $\Gamma_{L,r}$ as in [Π], and in the proofs, the index $L$ is dropped as well. We formulate our definitions and results in terms of the larger ball $V_{L+r_L}$, so that we can refer to the proofs in [Π]. For $x \in V_{L+r_L}$, let

$$\tilde{d}(x) = \max \left(\frac{d_{L+r_L}(x)}{2}, 3r_L\right), \quad a(x) = \min \left(\tilde{d}(x), s_L\right).$$

For $x, y \in V_{L+r_L}$, the kernel $\Gamma_L$ is now defined by

$$\Gamma_L(x, y) = \min \left\{ \frac{\tilde{d}(x)d(y)}{a(y)^2(a(y) + |x-y|)^d}, \frac{1}{a(y)^2(a(y) + |x-y|)^{d-2}} \right\}. \quad (14)$$

For $x \in V_{L+r_L}$, we write $U(x) = V_{a(x)}(x) \cap V_{L+r_L}$ for the $a(x)$-neighborhood around $x$. Given two positive functions $F, G : V_{L+r_L} \times V_{L+r_L} \to \mathbb{R}_+$, we write $F \preceq G$ if for all $x, y \in V_{L+r_L}$,

$$F(x, U(y)) \leq G(x, U(y)).$$
where $F(x,U)$ stands for $\sum_{y \in U \cap \mathbb{Z}^d} F(x,y)$. We write $F \asymp 1$, if there is a constant $C > 0$ such that for all $x, y \in V_{L+rL}$,

$$\frac{1}{C} F(x,y) \leq F(\cdot,\cdot) \leq C F(x,y) \quad \text{on } U(x) \times U(y).$$

We shall repeatedly need some properties of $\Gamma_L$, which form part of [1, Lemma 4.4].

**Lemma 3.1 (Properties of $\Gamma_L$).**

(i) $\Gamma_L \asymp 1$.

(ii) For $1 \leq j \leq \frac{1}{3} s_L$, with $E_j = \{ y \in V_{L+rL} : \tilde{d}(y) \leq 3jr_L \}$,

$$\sup_{x \in V_{L+rL}} \Gamma_L(x, E_j) \leq C \log(j + 1),$$

and for $0 \leq \alpha < 3$,

$$\sup_{x \in V_{L+rL}} \Gamma_L(x, Sh_L(s_L, L/(\log L)^\alpha)) \leq C (\log \log L)(\log L)^{6-2\alpha}.$$

(iii) For $x \in V_{L+r}$,

$$\Gamma_L(x, V_L) \leq C \max \left\{ \frac{\tilde{d}(x)}{L}(\log L)^6, \left( \frac{\tilde{d}(x)}{r_L} \wedge \log \log L \right) \right\}.$$

We now formulate the key estimate, which shows how both $\hat{g}_L^{(q)}$ and $\hat{G}_L^q$ can be dominated from above by the deterministic kernel $\Gamma_L$. See [1, Lemma 4.2] for a proof.

**Lemma 3.2.**

(i) There exists a constant $C > 0$ such that for all $q \in P^n$,

$$\hat{g}_L^{(q)} \preceq CT_L.$$

(ii) Assume $C1(\delta, L_0, L_1)$, and let $L_1 \leq L \leq L_1(\log L_1)^2$. There exists a constant $C > 0$ such that for $\delta > 0$ small,

$$\hat{G}_L^q \preceq CT_L.$$

### 3.2 Difference estimates

For controlling mean sojourn times, we will need difference estimates for the coarse grained Green’s functions $\hat{g}_L^{(q)}$ and $\hat{G}_L^q$. We first recall our notation:

- For $q \in P^n$, $\hat{g}_L^{(q)}$ is the Green’s function in $V_L$ corresponding to $\hat{\pi}_L^{(q)}$. 

• $\hat{G}_L^q$ is the Green’s function in $V_L$ corresponding to $\hat{\Pi}_L^q$, cf. (12).

• For $m > 0$, $\hat{g}_{m,V_L}^{(q)}$ is the Green’s function in $V_L$ corresponding to $1_{V_L} \hat{\pi}_m^{(q)}$, where $\psi_m \equiv m$.

• $\hat{g}_{m,Z^d}^{(q)}$ is the Green’s function on $Z^d$ corresponding to $\hat{\pi}_m^{(q)}$, where $\psi_m \equiv m$.

Lemma 3.3. There exists a constant $C > 0$ such that for all $q \in \mathcal{P}_s$,

(i) \[ \sup_{x,x' \in V_L : |x-x'| \leq s_L} \sum_{y \in V_L} |\hat{g}_L^{(q)}(x,y) - \hat{g}_L^{(q)}(x',y)| \leq C(\log \log L)(\log L)^3. \]

(ii) Assume $C1(\delta, L_0, L_1)$, and let $L_1 \leq L \leq L_1(\log L_1)^2$. There exists a constant $C > 0$ such that for $\delta > 0$ small,

\[ \sup_{x,x' \in V_L : |x-x'| \leq s_L} \sum_{y \in V_L} |\hat{G}_L^q(x,y) - \hat{G}_L^q(x',y)| \leq C(\log \log L)(\log L)^3. \]

Proof: (i) The underlying one-step transition kernel is always given by $q \in \mathcal{P}_s$, which we constantly omit in this proof, i.e. $\hat{\pi}_m = \hat{\pi}_m^{(q)}$, $\hat{g}_{m,V_L} = \hat{g}_{m,V_L}^{(q)}$, $\hat{g}_{m,Z^d} = \hat{g}_{m,Z^d}^{(q)}$, or $P_x = P_{x,q}$ and so on. Also, we suppress the subscript $L$, i.e. $\hat{g} = \hat{g}_L$. Set $m = s_L/20$. We write

\begin{equation}
\sum_{y \in V_L} |\hat{g}(x,y) - \hat{g}(x',y)| \leq \sum_{y \in V_L} |(\hat{g} - \hat{g}_{m,V_L})(x,y)| + \sum_{y \in V_L} |\hat{g}_{m,V_L}(x,y) - \hat{g}_{m,V_L}(x',y)| + \sum_{y \in V_L} |(\hat{g}_{m,V_L} - \hat{g})(x',y)|.
\end{equation}

If $x \in V_L \setminus Sh_L(2s_L)$, we have $\hat{\pi}(x,\cdot) = \hat{\pi}_m^{(q)}(x,\cdot)$. Clearly, $\sup_{x \in V_L} \hat{g}_{m,V_L}(x,Sh_L(2s_L)) \leq C$. Thus, with $\Delta = 1_{V_L}(\hat{\pi}_m^{(q)} - \hat{\pi})$, the perturbation expansion (13) and Lemma 3.1 yield (remember $\hat{g} \leq CT$ by Lemma 3.2)

\begin{equation}
\sum_{y \in V_L} |(\hat{g}_{m,V_L} - \hat{g})(x,y)| = \sum_{y \in V_L} |\hat{g}_{m,V_L}(x,y)|
\leq 2\hat{g}_{m,V_L}(x,Sh_L(2s_L)) \sup_{v \in Sh_L(3s_L)} \hat{g}(v,V_L) \leq C(\log L)^3.
\end{equation}

It remains to handle the middle term of (15). By (13),

\begin{equation}
\hat{g}_{m,V_L}(x,y) - \hat{g}_{m,V_L}(x',y) = \hat{g}_{m,Z^d}(x,y) - \hat{g}_{m,Z^d}(x',y) + E_{x',\pi_m} \left[ \hat{g}_{m,Z^d}(X_{x,L},y) \right] - E_{x,\hat{\pi}_m} \left[ \hat{g}_{m,Z^d}(X_{x,L},y) \right].
\end{equation}

Using Proposition 3.1, it follows that for $|x - x'| \leq s_L$,

\begin{equation}
\sum_{y \in V_L} |\hat{g}_{m,Z^d}(x,y) - \hat{g}_{m,Z^d}(x',y)| \leq C(\log L)^3.
\end{equation}
At last, we claim that
\[
\sum_{y \in V_L} |E_{x', \tilde{\psi}_m}[\hat{g}_{m, Z'}(X_{\tau_L}, y)] - E_{x, \tilde{\psi}_m}[\hat{g}_{m, Z'}(X_{\tau_L}, y)]| \leq C(\log \log L)(\log L)^3. \tag{16}
\]
Since \( |x - x'| \leq m \), we can define on the same probability space, whose probability measure we denote by \( \mathbb{Q} \), a random walk \((Y_n)_{n \geq 0}\) starting from \( x \) and a random walk \((\tilde{Y}_n)_{n \geq 0}\) starting from \( x' \), both moving according to \( \tilde{\psi}_m \) on \( \mathbb{Z}^d \), such that for all times \( n \), \( |Y_n - \tilde{Y}_n| \leq s_L \). However, with \( \tau = \inf\{n \geq 0 : Y_n \not\in V_L\} \), \( \tilde{\tau} \) the same for \( \tilde{Y}_n \), we cannot deduce that \( |Y_\tau - \tilde{Y}_\tau| \leq s_L \), since it is possible that one of the walks, say \( Y_n \), exits \( V_L \) first and then moves far away from the exit point, while the other walk \( \tilde{Y}_n \) might still be inside \( V_L \). In order to show that such an event has a small probability, we argue similarly to [10] Proposition 7.7.1. Define
\[
\sigma(s_L) = \inf\{n \geq 0 : Y_n \in \text{Sh}_L(s_L)\},
\]
and analogously \( \tilde{\sigma}(s_L) \). Let \( \vartheta = \sigma(s_L) \wedge \tilde{\sigma}(s_L) \). Since \( |Y_\vartheta - \tilde{Y}_\vartheta| \leq s_L \),
\[
\max\{\sigma(2s_L), \tilde{\sigma}(2s_L)\} \leq \vartheta.
\]
For \( k \geq 1 \), we introduce the events
\[
B_k = \{|Y_i - Y_{\sigma(2s_L)}| > ks_L \text{ for all } i = \sigma(2s_L), \ldots, \tau\},
\]
\[
\tilde{B}_k = \{\tilde{Y}_i - \tilde{Y}_{\tilde{\sigma}(2s_L)}| > ks_L \text{ for all } i = \tilde{\sigma}(2s_L), \ldots, \tilde{\tau}\}.
\]
By the strong Markov property and the gambler’s ruin estimate of [10], p. 223 (7.26), there exists a constant \( C_1 > 0 \) independent of \( k \) such that
\[
\mathbb{Q}\left(B_k \cup \tilde{B}_k\right) \leq C_1/k
\]
for some \( C_1 > 0 \) independent of \( k \). Applying the triangle inequality to
\[
Y_\tau - \tilde{Y}_\tau = (Y_\tau - Y_\vartheta) + (Y_\vartheta - \tilde{Y}_\vartheta) + (\tilde{Y}_\vartheta - \tilde{Y}_\tau),
\]
we deduce, for \( k \geq 3 \),
\[
\mathbb{Q}\left(|Y_\tau - \tilde{Y}_\tau| \geq ks_L\right) \leq 2C_1/(k - 1).
\]
Since \( |Y_\tau - \tilde{Y}_\tau| \leq 2(L + s_L) \leq 3L \), it follows that
\[
\mathbb{E}_{\mathbb{Q}}\left(|Y_\tau - \tilde{Y}_\tau|\right) \leq \sum_{k=1}^{3L} \mathbb{Q}\left(|Y_\tau - \tilde{Y}_\tau| \geq k\right) \leq C(\log \log L)s_L.
\]
Also, for \( v, w \) outside and \( y \) inside \( V_L \),
\[
\left|\frac{1}{|v - y|^{d-2}} - \frac{1}{|w - y|^{d-2}}\right| \leq C \frac{|v - w|}{(L + 1 - |y|)^{d-1}}.
\]
Applying Proposition 3.1, (16) now follows from the last two displays and a summation over \( y \in V_L \).

(ii) We take \( p = p_{s_L/20} \) stemming from the assignment \( \Pi = \Pi(p) \) and work with \( p \) as the underlying one-step transition kernel, which will be suppressed from the notation, i.e. \( \hat{\pi} = \hat{\pi}(p) \) and \( \hat{g} = \hat{g}(p) \).

Let \( x, x' \in V_L \) with \( |x - x'| \leq s_L \) and set \( \Delta = 1_{V_L}(\Pi g - \hat{\pi}) \). With \( B = V_L \setminus Sh_L(2r_L) \),

\[
\hat{G}^g = \hat{g}1_B \Delta G^g + \hat{g}1_{V_L \setminus B} \Delta \hat{G}^g + \hat{g}.
\]

Replacing successively \( \hat{G}^g \) in the first summand on the right-hand side,

\[
\hat{G}^g = \sum_{k=0}^{\infty} (\hat{g}1_B \Delta)^k \hat{g} + \sum_{k=0}^{\infty} (\hat{g}1_B \Delta)^k \hat{g}1_{V_L \setminus B} \Delta \hat{G}^g = F + F1_{V_L \setminus B} \Delta \hat{G}^g,
\]

where we have set \( F = \sum_{k=0}^{\infty} (\hat{g}1_B \Delta)^k \hat{g} \). With \( R = \sum_{k=1}^{\infty} (1_B \Delta)^k \hat{\pi} \), expansion (6) gives

\[
F = \hat{g} \sum_{m=0}^{\infty} (R\hat{g})^m \sum_{k=0}^{\infty} (1_B \Delta)^k = \hat{g} \sum_{k=0}^{\infty} (1_B \Delta)^k + \hat{g}RF.
\]

By the arguments given in the proof of Lemma 3.2 (ii) in [1] (note \( \|1_B \Delta\|_1 \leq \delta \), and \( \|1_B \Delta \hat{\pi}\|_1 \leq C(\log L)^{-9} \)), one deduces \( |F| \leq CT \). By Lemma 3.1 (ii) and (iii), we then see that for large \( L \), uniformly in \( x \in V_L \),

\[
\|F1_{V_L \setminus B} \Delta \hat{G}^g(x, \cdot)\|_1 \leq CT(x, Sh_L(2r_L)) \sup_{v \in Sh_L(3r_L)} \Gamma(v, V_L) \leq C \log \log L.
\]

Therefore,

\[
\sum_{y \in V_L} |\hat{G}^g(x, y) - \hat{G}^g(x', y)| \leq C \log \log L + \sum_{y \in V_L} |F(x, y) - F(x', y)|.
\]

Using (17) and twice part (i),

\[
\sum_{y \in V_L} |F(x, y) - F(x', y)| \leq \sum_{y \in V_L} |\hat{g} \sum_{k=0}^{\infty} (1_B \Delta)^k (x, y) - \hat{g} \sum_{k=0}^{\infty} (1_B \Delta)^k (x', y)| + \sum_{y \in V_L} |\hat{g}RF(x, y) - \hat{g}RF(x', y)|.
\]

The first expression on the right is estimated by

\[
\sum_{y \in V_L} \sum_{w \in V_L} |\hat{g}(x, w) - \hat{g}(x', w)| \sum_{k=0}^{\infty} (1_B \Delta)^k (w, y) \leq C(\log \log L)(\log L)^3,
\]
where we have used part (i) and $\|1_B \Delta(w, \cdot)\|_1 \leq \delta$. The second factor of (18) is again bounded by (i) and the fact that for $u \in V_L$, 

$$
\sum_{y \in V_L} |RF(u, y)| = \sum_{y \in V_L} \left| \sum_{k=1}^{\infty} (1_B \Delta)^k \hat{\pi} F(u, y) \right| 
\leq \sum_{k=0}^{\infty} \|1_B \Delta(u, \cdot)\|_1^k \sup_{v \in B} \|1_B \Delta \hat{\pi}(v, \cdot)\|_1 \sup_{w \in V_L} \sum_{y \in V_L} |F(w, y)| 
\leq C (\log L)^{-9+6} = C (\log L)^{-3}.
$$

Altogether, this proves part (ii).

\[ \Box \]

## 4 Mean sojourn times

Using the results about the exit measures from Proposition 2.1 and the estimates for the Green’s functions from Section 3, we proof in this part our main results on mean sojourn times in large balls.

### 4.1 Condition C2 and the main technical statement

Similarly to Condition C1($\delta, L_0, L_1$), cf. Section 2.3, we formulate a condition on the mean sojourn times which we propagate from one level to the next.

We first introduce a monotone increasing function which will upper and lower bound the normalized mean sojourn time in the ball. Let $0 < \eta < 1$, and define $f_\eta : \mathbb{R}^+ \to \mathbb{R}^+$ by setting

$$
f_\eta(L) = \frac{\eta}{3} \sum_{k=1}^{[\log L]} k^{-3/2}.
$$

Note $\eta/3 \leq f_\eta(L) < \eta$ and therefore $\lim_{\eta \downarrow 0} \lim_{L \to \infty} f_\eta(L) = 0$.

Recall that $E_x = E_{x, p_0}$ is the expectation with respect to simple random walk starting at $x \in \mathbb{Z}^d$, and $\tau_L$ is the first exit time from $V_L$.

**Condition C2**

We say that C2($\eta, L_1$) holds, if for all $3 \leq L \leq L_1$,

$$
P (E_{0, \omega} [\tau_L] \notin [1 - f_\eta(L), 1 + f_\eta(L)] \cdot E_{0} [\tau_L]) \leq \exp \left( -(1/2)(\log L)^{-2} \right).
$$

Our main technical result for the mean sojourn times is

**Proposition 4.1.** Assume A1 and B, and let $0 < \eta < 1$. There exists $\varepsilon_0 = \varepsilon_0(\eta) > 0$ with the following property: If $\varepsilon \leq \varepsilon_0$ and A0($\varepsilon$) holds, then there exists $L_0 = L_0(\eta) > 0$ such that for $L_1 \geq L_0$,

$$
C2(\eta, L_1) \Rightarrow C2(\eta, L_1 (\log L_1)^2).
$$
Remark 4.1. Given \( \eta \) and \( L_0 \), we can always guarantee (by making \( \varepsilon \) smaller if necessary) that \( A_0(\varepsilon) \) implies \( C_2(\eta, L_0) \).

The proof of this statement is deferred to Section 4.4.

4.2 Some preliminary results

We begin with an elementary statement about the mean time a symmetric random walk with kernel \( p \in \mathcal{P}_s^\kappa \) spends in the ball \( V_L \).

Lemma 4.1. Let \( p \in \mathcal{P}_s^\kappa \), and let \( x \in V_L \). Then

\[
L^2 - |x|^2 \leq E_{x,p}[\tau_L] \leq (L + 1)^2 - |x|^2.
\]

The proof of this standard lemma (see e.g. [10, Proposition 6.2.6]) uses the fact that \( |X_n \wedge \tau_L|^2 - n \wedge \tau_L \) is a martingale, which leads by optional stopping to \( E_{x,p}[\tau_L] = E_{x,p}[|X_{\tau_L}|^2] - |x|^2 \). In particular, for different \( p, q \in \mathcal{P}_s^\kappa \), the corresponding mean sojourn times satisfy

\[
E_{0,p}[\tau_L] = E_{0,q}[\tau_L] (1 + O(L^{-1})).
\]

We will compare the RWRE sojourn times on all scales with \( E_0[\tau_L] \), the corresponding quantity for simple random walk. This is somewhat in contrast to our comparison of the exit measure in [1], where we use the scale-dependent kernels \( p_L \) given by [10].

Using that \( \mu \) is supported on transition probabilities which are balanced in the first coordinate direction, we obtain a similar upper bound for the mean sojourn time of the RWRE.

Lemma 4.2. For \( \omega \in \mathcal{P}^\epsilon \cap \mathcal{P}^{s,1} \),

\[
E_{x,\omega}[\tau_L] \leq \frac{d}{1 - 2\varepsilon d} (L + 1)^2 - (x \cdot e_1)^2.
\]

Proof: For \( \omega \in \mathcal{P}^{s,1} \), \( \omega_x(e_1) = \omega_x(-e_1) \) for each \( x \in \mathbb{Z}^d \). Then

\[
M_n = (X_n \cdot e_1)^2 - \sum_{k=0}^{n-1} (\omega_{X_k}(e_1) + \omega_{X_k}(-e_1))
\]

is a \( P_{x,\omega} \)-martingale with respect to the filtration generated by the walk \((X_n)_{n \geq 0}\). By the optional stopping theorem, \( E_{x,\omega}[M_n \wedge \tau_L] = (x \cdot e_1)^2 \). Since for \( \omega \in \mathcal{P}^\epsilon \),

\[
\omega_{X_k}(e_1) + \omega_{X_k}(-e_1) \geq 1/d - 2\varepsilon,
\]

it follows that

\[
E_{x,\omega}[n \wedge \tau_L] \leq (1/d - 2\varepsilon)^{-1} E_{x,\omega}[(X_{n \wedge \tau_L} \cdot e_1)^2] - (x \cdot e_1)^2.
\]

Letting \( n \to \infty \) proves the statement. \( \square \)
The kernel

Figure 1: We express the mean sojourn time $E_{0,\omega}[\tau_L]$ as a convolution of the coarse grained RWRE Green’s function $\hat{G}_L$ with mean sojourn times in smaller balls $V_t(x) \cap V_L$, where $t \in [h_L(x), 2h_L(x)]$ (see Lemma 4.3). Inductive control over the sojourn times on smaller scales $\leq s_L$ and over the Green’s function then allow us to obtain the right estimate for $V_L$.

Remark 4.2. Conditions A0(ε) and B guarantee that the event $(P_ε)^{2d} \cap (P^{s,1})^{2d}$ has full $\mathbb{P}$-measure. The a priori fact that $E_{0,\omega}[\tau_L] \leq CL^2$ for almost all environments will be crucial to obtain more precise bounds on these times.

We will now express the mean sojourn time of the RWRE in $V_L$ in terms of mean sojourn times in smaller balls $V_t(x) \subset V_L$, for $t \in [h_L(x),2h_L(x)]$. Recall the definition of $h_L$ and the corresponding coarse graining scheme inside $V_L$. As in Section 2.2 we put

$$s_t = \frac{t}{(\log t)^3} \quad \text{and} \quad r_t = \frac{t}{(\log t)^{15}}.$$ 

Let $h^x_t(\cdot) = h_t(\cdot - x)$, where $h_t(\cdot - x)$ is defined in (8) (with $L$ replaced by $t$). By translating the origin into $x$, we transfer the coarse graining schemes on $V_L$ in the obvious way to $V_t(x)$, using $h^x_t$ instead of $h_t$. We write $\hat{\Pi}^x_t$ for the coarse grained transition probabilities in $V_t(x)$ associated to $((h^x_t)(y))_{y \in V_t(x),\omega}$, cf. [9]. Given $p \in \mathcal{P}_\kappa$, the kernel $\hat{\pi}^{(p)x}_t$ is defined similarly, with $p_\omega$ replaced by $p$.

For the corresponding Green’s functions we use the expressions $\hat{G}^x$ and $\hat{g}^{(p)x}$. If we do not keep $x$ as an index, we always mean $x = 0$ as before. If it is clear with which $p$ we are working, we drop the superscript $(p)$. Notice that for $y,z \in V_t(x)$ and $p \in \mathcal{P}_\kappa$, we have $\hat{\pi}^{(p)x}_t(y,z) = \hat{\pi}^{(p)}_t(y-x,z-x)$ and $\hat{g}^{(p)}_t(y,z) = \hat{g}^{(p)}_t(y-x,z-x)$. Since $p_\omega$ is in general not homogeneous in space, this is not true for $\hat{\Pi}^x_t$ and $\hat{G}^x_t$.

Define the “coarse grained” RWRE sojourn time

$$\Lambda_L(x) = 1_{V_L(x)} \frac{1}{h_L(x)} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{h_L(x)} \right) E_{x,\omega} \left[ \tau_{V_t(x)\cap V_L} \right] dt,$$

and the analog for random walk with kernel $p \in \mathcal{P}_\kappa$,

$$\lambda^{(p)}_L(x) = 1_{V_L(x)} \frac{1}{h_L(x)} \int_{\mathbb{R}_+} \varphi \left( \frac{t}{h_L(x)} \right) E_{x,p} \left[ \tau_{V_t(x)\cap V_L} \right] dt.$$
We also consider the corresponding quantities $\Lambda^x_T$ and $\lambda^{(p),x}_T$ for balls $V_i(x)$. For example,

$$\Lambda^x_T(y) = 1_{V_i(x)}(y) \frac{1}{h^x_T(y)} \int_{\mathbb{R}_+} \varphi \left( \frac{s}{h^x_T(y)} \right) E_{y,\omega} \left[ \tau_{V_i(y) - V_i(x)} \right] \, ds.$$ 

Note that we should rather write $\Lambda_L,\omega$ and $\Lambda^x_T,\omega$, but we again suppress $\omega$ in the notation. In the rest of this part, we often let operate kernels on mean sojourn times from the left. As an example,

$$\hat{G}_L \Lambda_L(x) = \sum_{y \in V_L} \hat{G}_L(x,y) \Lambda_L(y).$$

Both $\Lambda_L$ and $\hat{G}_L$ should be understood as random functions, but sometimes (for example in the proof of the next statement) the environment $\omega$ is fixed.

The basis for our inductive scheme is now established by

**Lemma 4.3.** For environments $\omega \in (\mathcal{P}_\varepsilon)^{\mathbb{Z}^d}$, $x \in \mathbb{Z}^d$,

$$E_{x,\omega}[\tau_L] = \hat{G}_L \Lambda_L(x).$$

In particular, for $\omega$ the homogeneous environment with transition probabilities given by $p \in \mathcal{P}^n$, 

$$E_{x,p}[\tau_L] = \hat{y}_L^{(p)} \lambda_L^{(p)}(x).$$

**Proof:** We construct a probability space where we can observe in $V_L$ both the random walk running with transition kernel $p_\omega$ and its coarse grained version running with kernel $\hat{\Pi}_L(\omega)$. In this direction, we take a probability space $(\Xi, \mathcal{A}, \mathbb{P})$ carrying a family of i.i.d. $[1,2]$-valued random variables $(\xi_n : n \in \mathbb{N})$ distributed according to $\varphi(t)\, dt$. We now consider the probability space $((\mathbb{Z}^d)^N \times \Xi, \mathcal{G} \otimes \mathcal{A}, \mathbb{P}_{x,\omega} \otimes \mathbb{Q})$. By a small abuse of notation, we denote here by $X_n$ the projection on the $n$th component of the first factor of $(\mathbb{Z}^d)^N \times \Xi$, so that under $\mathbb{P}_{x,\omega} \otimes \mathbb{Q}$, $(X_n)_{n \geq 0}$ evolves as the canonical Markov chain under $\mathbb{P}_{x,\omega}$.

Set $T_0 = 0$ and define the “randomized” stopping times 

$$T_{n+1} = \inf \left\{ m > T_n : X_m \notin V_{\xi_{T_n},h_L(X_{T_n})}(X_{T_n}) \right\} \wedge \tau_L.$$

Then the coarse grained Markov chain in $V_L$ running with transition kernel $\hat{\Pi}_{L,\omega}$ can be obtained by observing $X_n$ at the times $T_n$, that is by considering $(X_{T_n})_{n \geq 0}$. Moreover, the Markov property of $X_n$ and the i.i.d. property of the $\xi_n$ ensure that under $\hat{\mathbb{P}}_{x,\omega}$, conditionally on $X_{T_n}$, the random vector $((X_{T_n}, X_{T_n+1}, \ldots), T_{n+1} - T_n)$ is distributed as $((X_0, X_1, \ldots), T_1)$ under $\hat{\mathbb{P}}_{X_{T_n},\omega}$. Indeed, formally one may define the filtration $\mathcal{G}_n = \sigma(X_0, \ldots, X_n, \xi_0, \ldots, \xi_{n-1})$. Then $(X_n)_{n \geq 0}$ is also a $\mathcal{G}_n$-Markov chain. By induction, one sees that $T_n$ is a $\mathcal{G}_n$-stopping time, and the strong Markov property gives the stated equality in law. Writing $E_{x,\omega}$ for the expectation with respect to $\hat{\mathbb{P}}_{x,\omega} = \mathbb{P}_{x,\omega} \otimes \mathbb{Q}$, we
obtain
\[
E_{x,\omega}[T_L] = \sum_{z \in V_L} \tilde{E}_{x,\omega} \left[ \sum_{n=0}^{\infty} 1_{\{z\}}(X_n) 1_{\{n < \tau_L\}} \right] \\
= \sum_{z \in V_L} \tilde{E}_{x,\omega} \left[ \sum_{n=0}^{\infty} \sum_{k=T_n}^{T_{n+1}-1} 1_{\{z\}}(X_k) \right] \\
= \sum_{z \in V_L} \tilde{E}_{x,\omega} \left[ \sum_{n=0}^{\infty} \left( \sum_{y \in V_L} 1_{\{y\}}(X_{T_n}) \right) \sum_{k=T_n}^{T_{n+1}-1} 1_{\{z\}}(X_k) \right] \\
= \sum_{y \in V_L} \sum_{n=0}^{\infty} \tilde{E}_{x,\omega} \left[ 1_{\{y\}}(X_{T_n}) \tilde{E}_{x,\omega} \left[ \sum_{z \in V_L} \sum_{k=T_n}^{T_{n+1}-1} 1_{\{z\}}(X_k) \right] \right] \\
= \sum_{y \in V_L} \sum_{n=0}^{\infty} \tilde{E}_{x,\omega} \left[ 1_{\{y\}}(X_{T_n}) \tilde{E}_{x,\omega} \left[ \sum_{z \in V_L} \sum_{k=0}^{T_1-1} 1_{\{z\}}(X_k) \right] \right] \\
= \sum_{y \in V_L} \sum_{n=0}^{\infty} \tilde{E}_{x,\omega} \left[ 1_{\{y\}}(X_{T_n}) \Lambda_L(y) = \sum_{y \in V_L} \hat{G}_L(x,y)\Lambda_L(y) = \hat{G}_L\Lambda_L(x) \right].
\]

\[\Box\]

4.3 Space-good/bad and time-good/bad points

We classify the grid points inside $V_L$ into good and bad points, with respect to both space and time. We start by defining space-good and space-bad points. Unlike in [11], we need simultaneous control over two scales. This suggests the following stronger notion of “goodness”.

Space-good and space-bad points

Recall the assignment [10]. We say that $x \in V_L$ is space-good (with respect to $L$ and $\delta > 0$), if

- $x \in V_L \setminus B_L$, that is $x$ is good in the sense of Section 2.4
- If $d_L(x) > 2s_L$, then additionally for all $t \in [h_L(x), 2h_L(x)]$ and for all $y \in V_t(x)$,
  - For all $t' \in [h_t^x(y), 2h_t^x(y)]$, with $\tilde{q} = p\delta^x_t(y)$, \[ \| (\Pi V_{t'}(y) - \pi_{V_{t'}(y)})(y, \cdot) \|_1 \leq \delta. \]
  - If $t - |y - x| > 2r_t$, then additionally (with the same $\tilde{q}$)
    \[ \left\| \left( \hat{\Pi}_t^x - \frac{\tilde{\pi}_t^{(\tilde{q}),x}}{\pi_t^{(\tilde{q}),x}} \right)(y, \cdot) \right\|_1 \leq (\log h_t^x(y))^{-9}. \]

In other words, for a point $x \in V_L$ with $d_L(x) > 2s_L$ to be space-good, we do not only require that $x$ is good in the sense of Section 2.4 but also that all points $y \in V_t(x)$
Lemma 4.5. Therefore, also the kernel again from Lemma 3.2.

Next Lemma 4.6), using repeatedly the estimate (11) under Proof:

(i) \( \hat{G}_L \leq C T_L. \)

(ii) If \( x \in V_L \) with \( d(x) > 2s_L \), then for all \( t \in [h_L(x), 2h_L(x)] \),

\[
\hat{G}_t^x \leq CT_1(-x, \cdot - x).
\]

Proof: (i) Since \( \text{Good}^p_L \subset \text{Good}_L \), we have \( \hat{G} = \hat{G}^p \) on \( \text{Good}^p_L \), and Lemma 3.2 applies. For (ii), with \( x \) and \( t \) as in the statement, there are no bad points within \( V_t(x) \) on \( \text{Good}^p_L \). Therefore, also the kernel \( \hat{G}_t^x \) coincides with its goodified version, and the claim follows again from Lemma 3.2.

Lemma 4.4. There exist \( \delta > 0 \) small and \( L_0 \) large such that if \( \text{C1}(\delta, L_0, L_1) \) is satisfied for some \( L_1 \geq L_0 \), then we have for \( L_1 \leq L \leq L_1(\log L)^2 \) on \( \text{Good}^p_L \),

(i) \( \hat{G}_L \leq CT_L. \)

Proof: One can argue as in the proof of [11, Lemma 2.3] (or as in the proof of the next Lemma 4.6), using repeatedly the estimate (11) under \( \text{C1}(\delta, L_0, L_1) \). We omit the details.

Time-good and time-bad points

We also classify points inside \( V_L \) according to the mean time the RWRE spends in surrounding balls. Remember Condition \( \text{C2}(\eta, L_1) \) and the function \( f_\eta \) introduced above. We now fix \( 0 < \eta < 1 \).

For points in the bulk of \( V_L \), we need control over two scales, as above. We say that a point \( x \in V_L \) is time-good if the following holds:

- For all \( x \in V_L, t \in [h_L(x), 2h_L(x)] \),
  \[
  \mathbb{E}_{x, \omega} [\tau_{V_t(x)}] \in [1 - f_\eta(s_L), 1 + f_\eta(s_L)] \cdot \mathbb{E}_x [\tau_{V_t(x)}].
  \]

- If \( d_L(x) > 2s_L \), then additionally for all \( t \in [h_L(x), 2h_L(x)] \), \( y \in V_t(x) \) and for all \( t' \in [h^\circ_t(y), 2h^\circ_t(y)] \),
  \[
  \mathbb{E}_{y, \omega} [\tau_{V_t'(y)}] \in [1 - f_\eta(s_t), 1 + f_\eta(s_t)] \cdot \mathbb{E}_y [\tau_{V_t'(y)}].
  \]
A point \( x \in V_L \) which is not time-good is called time-bad. We denote by \( \mathcal{B}^m_L = \mathcal{B}^m_L(\omega) \) the set of all time-bad points inside \( V_L \). With
\[
\mathcal{D}_L = \left\{ V_{\delta h_L(x)}(x) : x \in V_L \right\},
\]
we let OneBad\(_L^m = \{ \mathcal{B}^m_L \subset D \text{ for some } D \in \mathcal{D}_L \}, \) ManyBad\(_L^m = (\text{OneBad}\_L^m)^c, \) and Good\(_L^m = \{ \mathcal{B}^m_L = \emptyset \} \subset \text{OneBad}\_L^m.\)

The next lemma ensures that for propagating Condition C2, we can forget about environments with space-bad points or widely spread time-bad points.

**Lemma 4.6.** Assume C1(\( \delta, L_0, L_1 \)), and C2(\( \eta, L_1 \)). Then for \( L_1 \leq L \leq L_1(\log L_1)^2 \),
\[
\mathbb{P}(\text{Bad}\_L^m \cup \text{ManyBad}\_L^m) \leq \exp \left( -\left(1/2\right)(\log L)^2 \right).
\]

**Proof:** We have \( \mathbb{P}(\text{Bad}\_L^m \cup \text{ManyBad}\_L^m) \leq \mathbb{P}(\text{Bad}\_L^m) + \mathbb{P}(\text{ManyBad}\_L^m). \) Lemma 4.5 bounds the first summand. Now if \( x \in \mathcal{B}_L^m \), then either
\[
\mathbb{E}_{x,\omega} [\tau_{\mathcal{V}_i(x)}] \notin [1 - f_\eta(t), 1 + f_\eta(t)] \cdot \mathbb{E}_x [\tau_{\mathcal{V}_i(x)}]
\]
for some \( t \in [h_L(x), 2h_L(x)] \) (recall that \( f_\eta \) is increasing), or, if \( d_L(x) > 2s_L \),
\[
\mathbb{E}_{y,\omega} [\tau_{\mathcal{V}_i(y)}] \notin [1 - f_\eta(t'), 1 + f_\eta(t')] \cdot \mathbb{E}_y [\tau_{\mathcal{V}_i(y)}]
\]
for some \( y \in V_{2h_L(x)}(x), t' \in [h_\eta_L(y), 2h_\eta_L(y)]. \)

For all \( x \in V_L \), we have \( h_L(x) \geq r_L/20. \) Moreover, if \( d_L(x) > 2s_L \), then \( h_L(x) \geq r_{s_L/20}/20. \) Therefore, under C2(\( \eta, L_1 \)),
\[
\mathbb{P}(x \in \mathcal{B}_L^m) \leq s_L^d \exp \left( -\left(1/2\right)(\log (r_L/20))^2 \right) + C s_L^d s_L^d \exp \left( -\left(1/2\right)(\log (r_{s_L/20}/20))^2 \right).
\]

We now observe that if \( \omega \in \text{ManyBad}\_L^m \), then there are at least two time-bad points \( x, y \) inside \( V_L \) with \( |x - y| > 2h_L(x) + 2h_L(y). \) For such \( x, y, \) the events \( \{ x \in \mathcal{B}_L^m \} \) and \( \{ y \in \mathcal{B}_L^m \} \) are independent. With the last display, we therefore conclude that
\[
\mathbb{P}(\text{ManyBad}\_L^m) \leq CL^{6d} \left( \exp \left( -\left(1/2\right)(\log (r_{s_L/20}/20))^2 \right) \right)^2 \leq \exp \left( -\left(2/3\right)(\log L)^2 \right).
\]

\[\square\]

### 4.4 Proof of the main technical statement

In this part, we prove Proposition 4.1. We will always assume that \( \delta \) and \( L \) are such that Lemma 4.4 can be applied. We start with two auxiliary statements: Lemma 4.7 proves a difference estimate for mean sojourn times. Here the difference estimates for the coarse grained Green’s functions from Section 3.2 play a crucial role. Lemma 4.8 then provides the key estimate for proving the main propagation step.

Note that due to Lemma 4.2, we have for \( \omega \in (\mathcal{P}_e)^{\mathbb{Z}^d} \cap (\mathcal{P}^{n,1})^{\mathbb{Z}^d}, \) that is for \( \mathbb{P}\)-almost all environments,
\[
\Lambda_L(x) \leq C s_L^2 \leq C(\log L)^{-6} L^2 \quad \text{for all } x \in V_L.
\]

(19)
Lemma 4.7. Assume $\mathbf{A0}(\varepsilon)$, $\mathbf{B}$, $\mathbf{C1}(\delta, L_0, L_1)$, and let $L_1 \leq L \leq L_1(\log L_1)^2$. Let $0 \leq \alpha < 3$ and $x, y \in V_{L-2s_L}$ with $|x - y| \leq (\log s_L)^{-\alpha} s_L$. Then for $\mathbb{P}$-almost all $\omega \in \text{Good}^\alpha_T$, 

$$|\Lambda_L(x) - \Lambda_L(y)| \leq C(\log \log s_L)(\log s_L)^{-\alpha}s_L^2.$$ 

Proof: We let $\omega \in (\mathcal{P}_c)^{2d} \cap (\mathcal{P}^0, 1)^{2d} \cap \text{Good}^\alpha_T$. The statement follows if we show that for all $t \in [(1/20)s_L, (1/10)s_L]$, 

$$|E_{x,\omega}[\tau_{\hat{V}_i(x)}] - E_{y,\omega}[\tau_{\hat{V}_i(y)}]| \leq C(\log t)(\log t)^{-\alpha}t^2.$$ 

We fix such a $t$ and set $t' = (1 - 20(\log t)^{-\alpha})t$. Then $V_{t'}(x) \subset V_t(x) \cap V_t(y)$. Now put $B = V_{t'-2s_L}(x)$. By Lemma 4.3, we have the representation 

$$E_{x,\omega}[\tau_{\hat{V}_i(x)}] = \hat{G}^y_{t'} 1_B \Lambda^y_{t'}(x) + \hat{G}^x_{t'} 1_{V_t(x) \cap B} \Lambda^x_{t'}(x). \quad (20)$$ 

By (19), $\Lambda^x_t(z) \leq C(\log t)^{-6}t^2$, for all $z \in V_t(x)$. Moreover, since $\omega \in \text{Good}^\alpha_T$, we have by Lemma 4.4 $\hat{G}^x_t \leq CT_t(\cdot, - \cdot, \cdot, - \cdot)$. Applying Lemma 3.1 (ii) gives 

$$\hat{G}^x_{t'} 1_{V_t(x) \cap B} \Lambda^x_{t'}(x) \leq CT_t(0, V_t \setminus V_{t'-2s_L}) (\log t)^{-6}t^2 \leq (\log t)^{-\alpha}t^2$$ 

for $L$ (and therefore also $t$) sufficiently large. Concerning $E_{y,\omega}[\tau_{\hat{V}_i(y)}]$, we write again 

$$E_{y,\omega}[\tau_{\hat{V}_i(y)}] = \hat{G}^y_{t'} 1_B \Lambda^y_{t'}(y) + \hat{G}^y_{t'} 1_{V_t(y) \cap B} \Lambda^y_{t'}(y).$$ 

The second summand is bounded by $(\log t)^{-\alpha}t^2$, as in the display above. For $z \in B$, we have $h^y_t(z) = h^y_t(z) = (1/20)s_L$. In particular, $\hat{\Pi}_t^y(z, \cdot) = \hat{\Pi}_t^y(z, \cdot)$, and also $\Lambda^y_t(z) = \Lambda^y_t(z)$. Since both $x$ and $y$ are contained in $B \subset V_t(x) \cap V_t(y)$, the strong Markov property gives 

$$\hat{G}^y_{t'}(y, z) = \hat{G}^x_{t'}(y, z) + b(y, z),$$ 

where 

$$b(y, z) = E_{y,\hat{\Pi}_t^y(\omega)}[\hat{G}^y_{t'}(\tau_B, z); \tau_B < \infty] - E_{y,\hat{\Pi}_t^y(\omega)}[\hat{G}^x_{t'}(\tau_B, z); \tau_B < \infty].$$ 

Therefore, 

$$|E_{x,\omega}[\tau_{\hat{V}_i(x)}] - E_{y,\omega}[\tau_{\hat{V}_i(y)}]| \leq 2(\log t)^{-\alpha}t^2 + \sum_{z \in B} \left( |\hat{G}^x_{t'}(x, z) - \hat{G}^y_{t'}(y, z)| + |b(y, z)| \right) \Lambda^y_t(z).$$ 

The quantity $\Lambda^y_t(z)$ is again bounded by $C(\log t)^{-6}t^2$. For the part of the sum involving $|b(y, z)|$, we notice that if $w \in V_t(y) \setminus B$, then $t - |w - y| \leq C(\log t)^{-\alpha}t$ and similarly for $v \in V_t(x)$. We can use twice Lemma 3.1 (ii) to get 

$$\sum_{z \in B} |b(y, z)| \leq \sup_{v \in V_t(x) \setminus B} \hat{G}^x_{t'}(v, B) + \sup_{w \in V_t(y) \setminus B} \hat{G}^y_{t'}(w, B) \leq C(\log t)^{6-\alpha}. $$
Finally, for the sum over the Green’s function difference, we recall that $\hat{G}_l^x$ coincides with its goodified version. Applying Lemma 3.3 $O((\log t)^{3-\alpha})$ times gives

$$\sum_{z \in B} |\hat{G}_l^x(x, z) - \hat{G}_l^x(y, z)| \leq C(\log t)(\log t)^{5-\alpha}.$$  

This proves the statement. \hfill $\square$

**Lemma 4.8.** Assume $A0(\varepsilon)$, B, C1(δ, L_0, L_1), and let $L_1 \leq L \leq L_1(\log L_1)^2$. Let $p = p_{sL/20}$, cf. (10), and set $\Delta = 1_{V_L}(\hat{\Pi}_L - \hat{\pi}^{(p)}_L)$. For $\mathbb{P}$-almost all $\omega \in \text{Good}^p_L$,

$$\sup_{x \in V_L} |\hat{G}_l\Delta\hat{g}_L^{(p)}\Lambda_L(x)| \leq C(\log L)^{-\alpha/3}L^2.$$  

**Proof:** Again, we consider $\omega \in (\mathcal{P}_e)^{Z^d} \cap (\mathcal{P}^{sL})^{Z^d} \cap \text{Good}^p_L$. Write $\hat{g} = \hat{g}^{(p)}$, $\hat{\pi} = \hat{\pi}^{(p)}$.

First, $\hat{G}\Delta\hat{g}\Lambda_L(x) = \hat{G}\Delta\hat{\pi}\hat{g}\Lambda_L(x) + \hat{G}\Delta\Lambda_L(x) = A_1 + A_2$.

By Lemma 4.4 $\hat{G} = C^p \leq CT$. Therefore, with $B_1 = V_{L-2sL}$, we bound $A_1$ by

$$|A_1| \leq |\hat{G}_1B_1\Delta\hat{\pi}\hat{g}\Lambda_L(x)| + |\hat{G}_1V_L\setminus B_1\Delta\hat{\pi}\hat{g}\Lambda_L(x)|$$

$$\leq \sum_{v \in B_1, w \in V_L} \hat{G}(x, v)\Delta\hat{\pi}(v, w)\sum_{y \in V_L} (g(w, y) - \hat{g}(v, y))\Lambda_L(y) + C(\log L)^{-2}L^2$$

$$\leq C(\log L)^{-5/3}L^2,$$

where in the next to last inequality we have used the bound on $\Lambda_L(y)$ given by (19) and Lemma 3.1 (ii), (iii) for $\Gamma$, and in the last inequality additionally Lemma 3.3. For the term $A_2$, we let $B = V_{L-5sL}$ and split into

$$A_2 = \hat{G}_1B\Delta\Lambda_L(x) + \hat{G}_1V_L\setminus B\Delta\Lambda_L(x).$$

Lemma 3.1 (ii) yields $\hat{G}(x, Sh_L(5sL)) \leq C\log L$.

Since $\Lambda_L(y) \leq (\log L)^{-2}L^2$ by (19), this is good enough for the second summand of $A_2$. For the first one,

$$\hat{G}_1B\Delta\Lambda_L(x) \leq CT(x, B)\sup_{v \in B} |\Delta\Lambda_L(v)|.$$  

Since $\Gamma(x, B) \leq C(\log L)^6$, the claim follows we show that for $v \in B$,

$$|\Delta\Lambda_L(v)| \leq C(\log L)^{-8}L^2,$$  

which, by definition of $\Delta$, in turn follows if for all $t \in [h_L(v), 2h_L(v)]$,

$$\left|\left(\Pi_{V_t(v)} - \pi_{V_t(v)}\right)\Lambda_L(v)\right| \leq C(\log L)^{-8}L^2.$$  

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where we have set \( \pi_{V_i(v)} = \pi^{(p)}_{V_i(v)} \). Notice that on \( B, \ h_L(\cdot) = (1/20)s_L \). We now fix \( v \in B \) and \( t \in [(1/20)s_L, (1/10)s_L] \). Set \( \Delta' = 1_{V_i(v)}(\hat{\Pi}_t^v - \hat{\pi}^{(p)}_{t,v}) \) and \( B' = V_i - 2r_t(v) \). By expansion [4],

\[
(\Pi_{V_i(v)} - \pi_{V_i(v)}) \Lambda_L(v) = \hat{G}_t^v 1_{B'} \Delta' \pi_{V_i(v)} \Lambda_L(v) + \hat{G}_t^v 1_{V_i(v) \setminus B'} \Delta' \pi_{V_i(v)} \Lambda_L(v). \tag{22}
\]

Since \( \pi_{V_i(v)} = \hat{\pi}^{(p)}_{t,v} \pi_{V_i(v)}, \) we get

\[
|\hat{G}_t^v 1_{B'} \Delta' \pi_{V_i(v)} \Lambda_L(v)| \leq \hat{G}_t^v (v, B') \sup_{w \in B'} |\Delta' \hat{\pi}^{(p)}_{t,v}(w, \cdot)| \sup_{y \in \partial V_i(v)} \Lambda_L(y) 
\leq C(\log s_L)^6 (\log L)^{-6} L^2 \sup_{w \in B'} |\Delta' \hat{\pi}^{(p),v}_{t,v}(w, \cdot)|. 
\]

Here, in the last inequality we have used [19] and Lemma 3.1 (iii). In order to bound \( \|\Delta' \hat{\pi}^{(p),v}_{t,v}(w, \cdot)\|_1 \) for \( w \in B' \), we use the fact that \( v \) is space-good and \( d_L(v) > 2s_L \), which gives also control over the exit distributions from smaller balls inside \( V_i(v) \). Indeed, by definition we first have for \( w \in B' \), with \( \hat{q} = p_{\hat{\pi}^{(p)}_{t,v}} \),

\[
\|1_{V_i(v)}(\hat{\Pi}_t^v - \hat{\pi}^{(p)}_{t,v}) \hat{\pi}^{(p),v}_{t,v}(w, \cdot)\|_1 \leq \log h^{(v)}_{\hat{\pi}^{(p)}_{t,v}}(w) - 9 \leq C(\log L)^{-6}.
\]

The last inequality follows from the bound \( h^{(v)}_{\hat{\pi}^{(p)}_{t,v}}(w) \geq (1/20)r_{s_L/20}. \) Furthermore, under \( C(\delta, L_0, L_1) \), the kernel \( \hat{q} \) is close to \( p \): in fact, one has \( \|\hat{q} - p\|_1 \leq C(\log L)^{-8} \), see [1] Lemma 2.2 and the arguments in the proof there. This bound transfers to the exit measures, so that, arguing as in [1] Lemma 4.1,

\[
\|\Delta' \hat{\pi}^{(p),v}_{t,v}(w, \cdot)\|_1 = \|\hat{G}_t^v 1_{B'} \Delta' \pi_{V_i(v)} \Lambda_L(w)| \leq C(\log L)^{-8} L^2. 
\]

Putting the estimates together, we obtain as desired

\[
|\hat{G}_t^v 1_{B'} \Delta' \pi_{V_i(v)} \Lambda_L(v)| \leq C(\log L)^{-8} L^2. 
\]

For the second summand of [22], Lemma 3.1 (ii) gives \( \hat{G}_t^v (v, V_i(v) \setminus B') \leq C \), whence

\[
|\hat{G}_t^v 1_{V_i(v) \setminus B'} \Delta' \pi_{V_i(v)} \Lambda_L(v)| \leq C \sup_{w \in V_i(v) \setminus B'} |\Delta' \pi_{V_i(v)} \Lambda_L(w)|. 
\]

Fix \( w \in V_i(v) \setminus B' \). Set \( \eta = d(w, \partial V_i(v)) \leq 2r_t + \sqrt{d} \) and choose \( y_w \in \partial V_i(v) \) such that \( |w - y_w| = \eta \). With

\[
I(y_w) = \{ y \in \partial V_i(v) : |y - y_w| \leq (\log L)^{-5/2}s_L \},
\]

we write

\[
\Delta' \pi_{V_i(v)} \Lambda_L(w) = \sum_{y \in \partial V_i(v)} \Delta' \pi_{V_i(v)}(w, y) (\Lambda_L(y) - \Lambda_L(y_w)) 
= \sum_{y \in I(y_w)} \Delta' \pi_{V_i(v)}(w, y) (\Lambda_L(y) - \Lambda_L(y_w)) 
+ \sum_{y \in \partial V_i(v) \setminus I(y_w)} \Delta' \pi_{V_i(v)}(w, y) (\Lambda_L(y) - \Lambda_L(y_w)). \tag{23}
\]
For \( y \in I(y_w) \), Lemma 4.7 yields \( |\Lambda_L(y) - \Lambda_L(y_w)| \leq C (\log L)^{-7/3} s_L^2 \). Therefore,
\[
\sum_{y \in I(y_w)} |\Delta' \pi_{V_t(v)}(w, y)| |\Lambda_L(y) - \Lambda_L(y_w)| \leq C (\log L)^{-8} L^2.
\]

It remains to handle the second term of (23). To this end, let \( U(w) = \{ u \in V_t(v) : |\Delta(w, u)| > 0 \} \). Using for \( y \in \partial V_t(v) \setminus I(y_w) \) the trivial bound
\[
|\Lambda_L(y) - \Lambda_L(y_w)| \leq \Lambda_L(y) + \Lambda_L(y_w) \leq C (\log L)^{-6} L^2,
\]
see [19], we obtain
\[
\sum_{y \in \partial V_t(v) \setminus I(y_w)} |\Delta' \pi_{V_t(v)}(w, y) (\Lambda_L(y) - \Lambda_L(y_w))| \leq C (\log L)^{-6} L^2 \sup_{u \in U(w)} \pi_{V_t(v)}(u, \partial V_t(v) \setminus I(y_w)).
\]

If \( u \in U(w) \) and \( y \in \partial V_t(v) \setminus I(y_w) \), then
\[
|u - y| \geq |y - y_w| - |y_w - u| \geq (\log L)^{-5/2} s_L - 3r_t \geq (1/2)(\log L)^{-5/2} s_L.
\]

For such \( u \), we get by standard hitting estimates, see e.g. [1] Lemma 3.2 (ii),
\[
\pi_{V_t(v)}(u, \partial V_t(v) \setminus I(y_w)) \leq C r_t \sum_{y \in \partial V_t(v) \setminus I(y_w)} \frac{1}{|u - y|^d}
\leq C r_t (\log L)^{5/2} (s_L)^{-1} \leq C (\log L)^{-9}.
\]

The estimate on the sum can be obtained from [1] Lemma 3.6. This bounds the second term of (23). We have proven (21) and hence the lemma. \( \square \)

Now it is easy to prove the main propagation step.

**Lemma 4.9.** Assume A0(\( \varepsilon \)) and B. There exists \( L_0 = L_0(\eta) \) such that if \( L_1 \geq L_0 \) and C1(\( \delta, L_0, L_1 \)) holds, then for \( L_1 \leq L \leq L_1 (\log L_1)^2 \) and \( \mathbb{P} \)-almost all \( \omega \in \text{Good}_L^p \cap \text{OneBad}_L^m \),
\[
E_{0, \omega} [\tau_L] \in [1 - f_\eta(L), 1 + f_\eta(L)] \cdot E_0 [\tau_L].
\]

**Proof:** We let \( \omega \in (\mathcal{P}_L)^2 \cap (\mathcal{P}_1)^2 \cap \text{Good}_L^p \cap \text{OneBad}_L^m \). Put \( p = p_{s_L/20} \). In this proof, we keep the superscript \( (p) \) in \( \hat{g}^{(p)} \). By Lemma 4.3 and the perturbation expansion (4), with \( \Delta = |V_L| (\hat{\Pi} - \hat{\tau}^{(p)}) \),
\[
E_{0, \omega} [\tau_L] = \hat{G} \Lambda_L(0) = \hat{g}^{(p)} \Lambda_L(0) + \hat{G} \Delta \hat{g}^{(p)} \Lambda_L(0) = A_1 + A_2.
\]
Set \( B = V_L \setminus (B_L^m \cup \text{Sh}_L(L/(\log L)^2)) \). The term \( A_1 \) we split into
\[
A_1 = \hat{g}^{(p)} 1_B \Lambda_L(0) + \hat{g}^{(p)} 1_{V_L \setminus B} \Lambda_L(0).
\]
Since \( \tilde{g}(p)(0, V_L \setminus B) \leq C(\log L)^3 \) by Lemma 3.1 (ii) and \( \Lambda_L(x) \leq (\log L)^{-6} L^2 \), the second summand of \( A_1 \) can be bounded by \( O((\log L)^{-3}) E_0[\tau_L] \). The main contribution comes from the first summand. First notice that

\[
g'(p) 1_B \lambda_L^p(0) = g'(p) 1_B \lambda_L(0) \left( 1 + O \left( s_L^{-1} \right) \right) = E_0 \left[ \tau_L \right] \left( 1 + O \left( (\log L)^{-6} \right) \right).
\]

Furthermore, we have for \( x \in B \) by definition

\[
\Lambda_L(x) \in \left[ 1 - f_\delta \left( (\log L)^{-3} L \right), 1 + f_\delta \left( (\log L)^{-3} L \right) \right] \cdot \lambda_L(x).
\]

Collecting all terms, we conclude that

\[
A_1 \in \left[ 1 - O \left( (\log L)^{-3} \right) - f_\delta \left( (\log L)^{-3} L \right), 1 + O \left( (\log L)^{-3} \right) + f_\delta \left( (\log L)^{-3} L \right) \right] \times E_0 \left[ \tau_L \right].
\]

Lemma 4.8 bounds \( A_2 \) by \( O((\log L)^{-5/3}) E_0[\tau_L] \). Since for \( L \) sufficiently large,

\[
f_\delta(L) > f_\delta \left( (\log L)^{-5/3} L \right) + C(\log L)^{-5/3},
\]

we arrive at

\[
E_{0,\omega}[\tau_L] = A_1 + A_2 \in \left[ 1 - f_\delta(L), 1 + f_\delta(L) \right] \cdot E_0 \left[ \tau_L \right],
\]

as claimed. \( \square \)

Proposition 4.1 follows now as an immediate consequence of our estimates.

**Proof of Proposition 4.1** (i) From Lemmata 4.6 and 4.9 we deduce that for large \( L_0 \), if \( L_1 \geq L_0 \) and \( L_1 \leq L \leq L_1(\log L_1)^2 \), we have under \( \mathbf{C1}(\delta, L_0, L_1) \) and \( \mathbf{C2}(\eta, L_1) \)

\[
\mathbb{P}(E_{0,\omega}[\tau_L] \notin [1 - f(L), 1 + f(L)] \cdot E_0[\tau_L]) \leq \mathbb{P}(\text{Bad}^\delta \cup \text{ManyBad}^\delta) + \mathbb{P}(E_{0,\omega}[\tau_L] \notin [1 - f(L), 1 + f(L)] \cdot E_0[\tau_L]; \text{Good}^\delta \cap \text{OneBad}^\delta) \leq \exp \left( -1/2(\log L)^2 \right).
\]

By Proposition 2.1 if \( \delta > 0 \) is small and \( L_0 \) is sufficiently large, \( \mathbf{C1}(\delta, L_0, L) \) holds under \( \mathbf{A0}(\varepsilon) \) for all \( L \geq L_0 \), provided \( \varepsilon \leq \varepsilon_0(\delta) \). This proves the proposition. \( \square \)

### 4.5 Proof of the main theorem on sojourn times

We shall now prove convergence of the (non-random) sequence \( \mathbb{E}[E_{0,\omega}[\tau_L]]/L^2 \) towards a constant \( D \) that lies in a small interval around 1. Note that Proposition 4.1 together with Lemma 4.2 already tells us that for any \( 0 < \eta < 1 \), under \( \mathbf{A0}, \mathbf{B} \) and \( \mathbf{A1}(\varepsilon) \) for \( \varepsilon(\eta) \) small,

\[
\mathbb{E}[E_{0,\omega}[\tau_L]]/L^2 \in [1 - \eta, 1 + \eta] \quad \text{for large } L.
\]

**Proposition 4.2.** Assume \( \mathbf{A1} \) and \( \mathbf{B} \). Given \( 0 < \eta < 1 \), one can find \( \varepsilon_0 = \varepsilon_0(\eta) > 0 \) such that if \( \mathbf{A0}(\varepsilon) \) is satisfied for some \( \varepsilon \leq \varepsilon_0 \), then there exists \( D \in [1 - \eta, 1 + \eta] \) such that

\[
\lim_{L \to \infty} \left( \mathbb{E}[E_{0,\omega}[\tau_L]]/L^2 \right) = D.
\]
4.5 Proof of the main theorem on sojourn times

Proof: Let $0 < \eta < 1$. By choosing first $\delta$, then $L_0$ and then $\varepsilon_0$ small respectively large enough, we know from Propositions 2.1 and 4.1 that under $\text{A}1$ and $\text{B}$, whenever $\text{A}0(\varepsilon)$ is satisfied for some $\varepsilon \leq \varepsilon_0$, $\text{C}1(\delta, L_0, L)$ and $\text{C}2(\eta/2, L)$ hold true for all $L \geq L_0$. We can therefore assume both conditions. We obtain from Lemma 4.5

$$\mathbb{E}[E_{0, \omega} \tau_L] = \mathbb{E}[E_{0, \omega} \tau_L] + O \left( L^2 \exp \left( -(2/3)(\log L)^2 \right) \right).$$

Thus it suffices to look at $E_{0, \omega} \tau_L$ on the event $(\mathcal{P}_{\varepsilon})_{\leq \varepsilon} \cap (\mathcal{P}_{\varepsilon})_{\leq \varepsilon} \cap \text{Good}_L$. Setting $B = V_L \setminus (\text{Sh}_L(L/(\log L)^2))$, we see from the proof of Lemma 4.9 that on this this event,

$$E_{0, \omega} \tau_L = (g^{(p)}_1 B \Lambda_L)(0) + O \left( (\log L)^{-5/3} L^2 \right),$$

where the constant in the error term does only depend on $d$ (and not on $L$ or the environment). For $x \in B$, $h_L(x) = s_L/20$. In particular, this implies on the set $B$

$$\mathbb{E}[\Lambda_L(\cdot)] = \mathbb{E}[\Lambda_L(0)] \quad \text{and} \quad \lambda^{(p)}(\cdot) \equiv \lambda^{(p)}(0).$$

Now put $c_L = \mathbb{E}[\Lambda_L(0)] / \lambda^{(p)}(0)$. We have

$$\mathbb{E}[E_{0, \omega} \tau_L] = g^{(p)}_0(0, B) \cdot \mathbb{E}[\Lambda_L(0)] + O \left( (\log L)^{-5/3} L^2 \right)$$

$$= c_L g^{(p)}_0(0, B) \cdot \lambda^{(p)}(0) + O \left( (\log L)^{-5/3} L^2 \right)$$

$$= c_L E_{0, p} \tau_L + O \left( (\log L)^{-5/3} L^2 \right).$$

Since $E_{0, p} \tau_L / L^2$ converges to 1 by Lemma 4.1 convergence of $\mathbb{E}[E_{0, \omega} \tau_L] / L^2$ follows if we show that $\lim_{L \to \infty} c_L$ exists. Let $L' \in (L, 2L]$. As before,

$$\mathbb{E}[E_{0, \omega} \tau_{L'}] = c_{L'} E_{0, p} \tau_{L'} + O \left( (\log L)^{-5/3} L^2 \right). \tag{24}$$

On the other hand, we claim that (24) also holds with $c_{L'}$ replaced by $c_L$. To see this, we slightly change the coarse graining scheme inside $V_L$, as in the proof of [11, Proposition 1.1]. More specifically, we define for $L' \in (L, 2L]$ the coarse graining function $\tilde{h}_{L'} : \mathcal{C}_{L'} \to \mathbb{R}_+$ by setting

$$\tilde{h}_{L'}(x) = \frac{1}{20} \max \left\{ s_L h \left( \frac{d_{L'}(x)}{s_{L'}} \right), r_L \right\}.$$

Then $\tilde{h}_{L'}(x) = h_L(x) = s_L/20$ for $x \in V_L$ with $d_{L'}(x) \geq 2s_L$. We consider the analogous definition of space-good/bad and time-good/bad points within $V_L$, which uses the coarse graining function $\tilde{h}_{L'}$ instead of $h_{L'}$ and the coarse grained transition kernels $\tilde{\pi}$ and $\tilde{\pi}$ in $V_{L'}$ defined in terms of $\tilde{h}_{L', r}$, cf. [9]. Clearly, all the above statements of this section remain true (at most the constants change), and we can work with the same kernel $p = p_{sL}/20$. Denoting by $\tilde{g}^{(p)}$ the Green’s function corresponding to $\tilde{\pi}^{(p)}$ and by $\tilde{B}$ the set $V_L \setminus \text{Sh}_L(L'/(\log L)^2)$, we obtain as above

$$\mathbb{E}[E_{0, \omega} \tau_{L'}] = \tilde{g}^{(p)}(0, \tilde{B}) \mathbb{E}[\Lambda_L(0)] + O \left( (\log L)^{-5/3} L^2 \right)$$

$$= c_L \tilde{g}^{(p)}(0, B) \lambda^{(p)}(0) + O \left( (\log L)^{-5/3} L^2 \right)$$

$$= c_L E_{0, p} \tau_{L'} + O \left( (\log L)^{-5/3} L^2 \right).$$
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Note that since \( \tilde{h}_{L'}(\cdot) \equiv h_L(0) \) on \( B' \), the quantities \( c_L, \mathbb{E}[A_L(0)] \) and \( \lambda^{(p)}(0) \) do indeed appear in the above display. Comparing with (24), this shows that for some constant \( C > 0 \)

\[ |c_L - c_L'| \leq C(\log L)^{-5/3}, \]

which readily implies that \( c_L \) is a Cauchy sequence and thus \( \lim_{L \to \infty} c_L = D \) exists. From Proposition 4.1 we already know that \( D \in [1 - \eta, 1 + \eta] \). This finishes the proof. \( \square \)

We shall now employ Hoeffding’s inequality to show that \( E_{0,\omega}[\tau_L] \) is close to its mean.

**Lemma 4.10.** Assume A1 and B. There exists \( \varepsilon_0 > 0 \) such that if A0(\( \varepsilon \)) holds for some \( \varepsilon \leq \varepsilon_0 \), then

\[
\mathbb{P} \left( \frac{1}{L^2} \left| E_{0,\omega}[\tau_L] - \mathbb{E}[E_{0,\omega}[\tau_L]] \right| > (\log L)^{-4/3} \right) \leq \exp \left( -\frac{1}{3}(\log L)^2 \right).
\]

Let us first show how to prove Theorem 1.1 from this result. **Proof of Theorem 1.1:** We know from Proposition 4.2, \( D \) the constant from there,

\[
\left| \frac{1}{L^2} E_{0,\omega}[\tau_L] - D \right| \leq \frac{1}{L^2} \left| E_{0,\omega}[\tau_L] - \mathbb{E}[E_{0,\omega}[\tau_L]] \right| + \alpha(L)
\]

for some (deterministic) sequence \( \alpha(L) \to 0 \) as \( L \to \infty \). Putting

\[
\alpha'(L) = \max \left\{ (\log L)^{-4/3}, \alpha(L) \right\},
\]

we deduce from Lemma 4.10 that

\[
\mathbb{P} \left( \left| \frac{1}{L^2} E_{0,\omega}[\tau_L] - D \right| \geq 2\alpha'(L) \right) \leq \exp \left( -\frac{1}{3}(\log L)^2 \right).
\]

This implies the first statement of the theorem. For the second, we have

\[
\mathbb{P} \left( \sup_{x:|x| \leq L^k} E_{x,\omega}[\tau_{V_L(x)}]/L^2 - D \geq 2\alpha'(L) \right) \leq C L^{kd} \mathbb{P} \left( \left| E_{0,\omega}[\tau_L]/L^2 - D \right| \geq 2\alpha'(L) \right) \leq \exp \left( -\frac{1}{4}(\log L)^2 \right)
\]

for large \( L \), and the same bound holds with the supremum over \( x \) with \( |x| \leq L^k \) replaced by the infimum. The second claim of the theorem follows now from Borel-Cantelli. \( \square \)

It remains to prove Lemma 4.10. **Proof of Lemma 4.10:** By Proposition 2.1 and Lemma 4.5, we can find \( \varepsilon_0 > 0 \) such that under A0 and A1(\( \varepsilon \)) for \( \varepsilon \leq \varepsilon_0 \),

\[
\mathbb{P}(\text{Bad}_L^p) \leq \exp \left( -(2/3)(\log L)^2 \right) \quad \text{for } L \text{ large}.
\]
As in the proof of Proposition 4.2 (or Lemma 4.9), we have for \( \omega \in (\mathcal{P}_x)^{2d} \cap (\mathcal{P}^{s,1})^{2d} \) in the complement of \( \text{Bad}_L^p \), that is \( \mathbb{P} \)-almost surely on the event \( \text{Good}_L^p \),

\[
E_{0,\omega} [\tau_L] = (g^{(p)}_1 B_L)(0) + O \left( (\log L)^{-5/3} L^2 \right),
\]

where \( B = V_L \setminus (\text{Sh}_L(L/(\log L))^2) \). In the proof of Proposition 4.2 we have also seen that

\[
\mathbb{E} [E_{0,\omega} [\tau_L]] = g^{(p)}(0, B) \mathbb{E} [\Lambda_L(0)] + O \left( (\log L)^{-5/3} L^2 \right).
\]

Therefore, on \( \text{Good}_L^p \),

\[
E_{0,\omega} [\tau_L] = g^{(p)}(0, B) \mathbb{E} [\Lambda_L(0)] + \sum_{y \in B} g^{(p)}(0, y) (\Lambda_L(y) - \mathbb{E} [\Lambda_L(0)]) + O \left( (\log L)^{-5/3} L^2 \right)
\]

\[
= \mathbb{E} [E_{0,\omega} [\tau_L]] + \sum_{y \in B} g^{(p)}(0, y) (\Lambda_L(y) - \mathbb{E} [\Lambda_L(0)]) + O \left( (\log L)^{-5/3} L^2 \right).
\]

The statement of the lemma will thus follow if we show that

\[
\mathbb{P} \left( \left| \sum_{y \in B} g^{(p)}(0, y) (\Lambda_L(y) - \mathbb{E} [\Lambda_L(0)]) \right| > (\log L)^{-3/2} L^2 \right) \leq \exp \left( -\frac{(\log L)^2}{2} \right). \tag{25}
\]

We use a similar strategy as in the proof of [11, Lemma 5.4]. First, define for \( j \in \mathbb{Z} \) the interval \( I_j = (j s_L, (j + 1) s_L \). Now divide \( B \) into subsets \( W_j = B \cap (I_{j_1} \times \cdots \times I_{j_d}) \), where \( j = (j_1, \ldots, j_d) \in \mathbb{Z}^d \). Let \( J \) be the set of those \( j \) for which \( W_j \neq \emptyset \). Then there exists a constant \( K = K(d) \) and a disjoint partition of \( J \) into sets \( J_1, \ldots, J_K \), such that for any \( 1 \leq \ell \leq K \),

\[
j, j' \in J_\ell, j \neq j' \implies d(W_j, W_{j'}) > s_L. \tag{26}
\]

We set

\[
\xi_j = \sum_{y \in W_j} g^{(p)}(0, y) (\Lambda_L(y) - \mathbb{E} [\Lambda_L(0)])
\]

and \( t = t(d, L) = (\log L)^{-3/2} L^2 \). From (26) we see that the random variables \( \xi_j, j \in J_\ell \), are independent and centered (we recall again that \( \mathbb{E} [\Lambda_L(y)] = \mathbb{E} [\Lambda_L(0)] \) for \( y \in B \)). Put \( \Omega' = (\mathcal{P}_x)^{2d} \cap (\mathcal{P}^{s,1})^{2d} \). Applying Hoeffding’s inequality, we obtain with \( \|\xi_j\|_\infty = \sup_{\omega \in \Omega'} |\xi_j(\omega)| \), for some constant \( c > 0 \),

\[
\mathbb{P} \left( \left| \sum_{j \in J_\ell} \xi_j \right| > t \right) \leq K \max_{1 \leq \ell \leq K} \mathbb{P} \left( \left| \sum_{j \in J_\ell} \xi_j \right| > \frac{t}{K} \right) \leq 2 \exp \left( -c \frac{(\log L)^{-3} L^4}{\sum_{j \in J_\ell} \|\xi_j\|_\infty^2} \right). \tag{27}
\]

It remains to estimate the sup-norm of the \( \xi_j \). We have, by Lemmata 3.2 and 3.1

\[
\hat{g}(x, W_j) \leq \frac{C s_L^d}{s_L^2 (s_L + d(x, W_j))^{d-2}} = C \left( 1 + \frac{d(x, W_j)}{s_L} \right)^{2-d}.
\]

By (19),

\[
|\Lambda_L(y) - \mathbb{E} [\Lambda_L(0)]| \leq C \log L^{-6} L^2.
\]
Altogether, recalling that \( d \geq 3 \),
\[
\sum_{j \in J} \| \xi_j \|_\infty^2 \leq C \sum_{r=1}^{C(\log L)^3} r^{-d+3} L^4 (\log L)^{-12} \leq C(\log L)^{-9} L^4.
\]

Going back to (27), this shows
\[
P \left( \left| \sum_{y \in B} \hat{g}^{(p)}(0, y) (\Lambda_L(y) - \mathbb{E} [\Lambda_L(0)]) \right| \geq (\log L)^{-3/2} L^2 \right) \leq 2 \exp \left( -c(\log L)^6 \right),
\]
which is more than we need, cf. (25). This completes the proof of the lemma.

**Proof of Corollary 1.1:** Let \( k \in \mathbb{N} \), and let first \( m = 1 \). By Proposition 4.1, we obtain under our conditions (for \( \epsilon \) small)
\[
P \left( \sup_{x: |x| \leq L^k} \mathbb{E}_{y, \omega} [\tau_{V_L(x)}] / L^2 \geq 2 \right) \leq C L^{k \cdot d} \mathbb{P} \left( \sup_{y \in V_L} \mathbb{E}_{y, \omega} [\tau_L] / L^2 \geq 2 \right) \leq C L^{(k+1) \cdot d} \mathbb{P} \left( \mathbb{E}_{0, \omega} [\tau_L] / L^2 \geq 2 \right) \leq \exp \left( - (1/3) (\log L)^2 \right).
\]
This implies by Borel-Cantelli that
\[
\lim \sup_{L \to \infty} \sup_{x: |x| \leq L^k} \mathbb{E}_{y, \omega} [\tau_{V_L(x)}] / L^2 \leq 2 \mathbb{P}-\text{almost surely.} \quad (28)
\]
For the rest of the proof, take an environment \( \omega \) that satisfies (28). Assume \( m \geq 2 \). Then
\[
\mathbb{E}_{x, \omega} [\tau_{V_L(x)}^m] = \sum_{\ell_1, \ldots, \ell_m \geq 0} \mathbb{P}_{x, \omega} (\tau_{V_L(x)} > \ell_1, \ldots, \tau_{V_L(x)} > \ell_m) \\
\leq m! \sum_{0 \leq \ell_1 \leq \cdots \leq \ell_m} \mathbb{P}_{x, \omega} (\tau_{V_L(x)} > \ell_m).
\]
By the Markov property, using the case \( m = 1 \) and induction in the last step,
\[
\sum_{0 \leq \ell_1 \leq \cdots \leq \ell_m} \mathbb{P}_{x, \omega} (\tau_{V_L(x)} > \ell_m) \\
= \sum_{0 \leq \ell_1 \leq \cdots \leq \ell_{m-1}} \mathbb{E}_{x, \omega} \left[ \sum_{\ell=0}^{\infty} \mathbb{P}_{x, \omega} (\tau_{V_L(x)} > \ell; \tau_{V_L(x)} > \ell_{m-1}) \right] \\
\leq \sup_{\omega \in V_L(x)} \mathbb{E}_{x, \omega} [\tau_{V_L(x)}] \sum_{0 \leq \ell_1 \leq \cdots \leq \ell_{m-1}} \mathbb{E}_{x, \omega} [\tau_{V_L(x)} > \ell_{m-1}] \leq 2^m L^{2m},
\]
if \( L = L(\omega) \) is sufficiently large. \( \square \)
5 A quenched invariance principle

Here we combine the results on the exit distributions from $[1]$ and our results on the mean sojourn times to prove Theorem 1.2, which provides a functional central limit theorem for the RWRE under the quenched measure. Let us recall the precise statement.

Assume $A_0(\varepsilon)$ for small $\varepsilon > 0$, $A_1$ and $B$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$, under $P_{0,\omega}$, the $C(\mathbb{R}_+,\mathbb{R}^d)$-valued sequence $X^n_t/\sqrt{n}$, $t \geq 0$, converges in law to a $d$-dimensional Brownian motion with diffusion matrix $D^{-1}\Lambda$, where $D$ is the constant from Theorem 1.1, $\Lambda$ is given by (2), and $X^n_t$ is the linear interpolation $X^n_t = X_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor)(X_{\lfloor tn \rfloor + 1} - X_{\lfloor tn \rfloor})$.

The statement follows if we show that for each real $T > 0$, weak convergence occurs in $C([0, T], \mathbb{R}^d)$. In order to simplify notation, we will restrict ourselves to $T = 1$, the general case being a simple generalization of this case.

Let us first give a rough (simplified) idea of our proof. We define the step size $L_n = (\log n)^{-1}\sqrt{n}$. From Theorem 1.1 we infer that the RWRE should have left about $(\log n)^2/\sqrt{n}$ balls of radius $L_n$ in the first $n$ steps. Proposition 2.1 tells us that for sufficiently large $n$, the exit law from each of those balls is close to that of a symmetric random walk with nearest neighbor kernel $p_{L_n}$. For our limit theorem, this will imply that we can replace the coarse grained RWRE taking steps of size $L_n$, i.e. the RWRE observed at the successive exit times from balls of radius $L_n$, by the analogous coarse grained random walk with kernel $p_{L_n}$. For the latter, we apply the multidimensional Lindeberg-Feller limit theorem. Since we know that the kernels $p_{L_n}$ converge to $p_\infty$ (see (1) and the comments below Proposition 2.1), we obtain in this way the stated convergence of the one-dimensional distributions. Since our estimates on exit measures and exit times are sufficiently uniform in the starting point, multidimensional convergence and tightness then follow from standard arguments.

Figure 2: The coarse grained RWRE $\hat{X}_{n,i}$, $i \in \mathbb{N}$, which is obtained from observing the RWRE at the successive exit times from balls of radius $L_n$. Here $k_n$ denotes the maximal number of such balls which are left by the RWRE in the first $n$ steps.

5.1 Construction of coarse grained random walks on $\mathbb{Z}^d$

We start with a precise description of the coarse grained random walks. Let

$$L_n = (\log n)^{-1}\sqrt{n}.$$
Similarly to the proof of Lemma 4.3, given an environment $\omega \in \Omega$, we introduce a probability space where we can observe both the random walk with kernel $p_\omega$ and a coarse grained version of it taking steps of a size between $L_n$ and $2L_n$.

More specifically, we take a probability space $((\mathbb{Z}^d)^N \times \Xi, \mathcal{G} \otimes \mathcal{A}, \tilde{P}_{x,\omega})$, where $\tilde{P}_{x,\omega} = P_{x,\omega} \otimes Q$. On this space, $X_k$ denotes again the projection on the $k$th component of $(\mathbb{Z}^d)^N$, so that under $\tilde{P}_{x,\omega}$, $X_k$ has the law of a random walk started from $x$ with transition kernel $p_\omega$.

Set $T_{n,0} = 0$, and recursively for integers $i \in \mathbb{N}$,

$$T_{n,i+1} = \inf\{m > T_{n,i} : X_m \notin V_{\xi,T_{n,i},L_n}(X_{T_{n,i}})\}$$

$\hat{X}_{n,i} = X_{T_{n,i}}$.

Under $\tilde{P}_{x,\omega}$, for fixed $n$, $\hat{X}_{n,i}$ is the coarse grained Markov chain running with transition probabilities

$$Q_{n,\omega}(y, \cdot) = \frac{1}{L_n} \int_{\mathbb{R}_+} \varphi\left(\frac{t}{L_n}\right) \Pi_{V_t,\omega}(y, \cdot) dt$$

and started from $x$, i.e. $\tilde{P}_{x,\omega}(\hat{X}_{n,0} = x) = 1$. Note that in contrast to Lemma 4.3, the step size of the coarse grained walk takes values between $L_n$ and $2L_n$ and does not depend on the current location. We shall suppress the environment $\omega$ in the notation and write $Q_n$ instead of $Q_{n,\omega}$.

We will compare $Q_n$ with the coarse grained (non-random) kernel

$$q_n(y, \cdot) = \frac{1}{L_n} \int_{\mathbb{R}_+} \varphi\left(\frac{t}{L_n}\right) \pi_V^{(p_{L_n})}(0, \cdot - y) dt,$$

where the kernel $p_{L_n}$ stems from the assignment \[10\].

### 5.2 Good events

#### Good behavior in space

We shall now introduce an event $A_1$ with $\mathbb{P}(A_1) = 1$ on which the RWRE has a “good” behavior in terms of exit distributions. Let

$$D_{L,p,\psi,q}(x) = \left\| \left( \Pi_{V_{L}(x)} - \pi^{(p)}_{V_{L}(x)} \right) \pi^{(q)}_{\psi}(x, \cdot) \right\|_1.$$ 

We require that all smoothed differences of exit measures $D_{L,pL_n,\psi_n,pL_n}(x)$, where $x \in \mathbb{Z}^d$ with $|x| \leq n^4$, $L \in [L_n, 2L_n]$ and $\psi_n \equiv L_n$, are small when $n$ is large.

In this regard, note that Proposition 2.1 implies for large $n$

$$\mathbb{P}\left( \sup_{x:|x| \leq n^4} \sup_{L_n \leq L \leq 2L_n} D_{L,pL_n,\psi_n,pL_n}(x) > \log L_n \right) = 0$$

$$\leq Cn^{4d} \sup_{L_n \leq L \leq 2L_n} \mathbb{P}\left( D_{L_n}^z, \psi_n, pL_n > \log L_n \right) \leq \exp\left(-\frac{1}{5}\right)(\log n)^2.$$
5.3 A law of large numbers

An application of Borel-Cantelli then shows that on a set $A_1$ of full $\mathbb{P}$-measure,

$$\limsup_{L \to \infty} \sup_{x:|x| \leq n^3} \sup_{L \leq L_n \leq 2L_n} D_{L,PL_n,\psi_n,PL_n}(x) \leq (\log L_n)^{-9}. \quad (29)$$

**Good behavior in time**

We next specify an event $A_2$ with $\mathbb{P}(A_2) = 1$ on which we have uniform control over mean sojourn times. Let

$$c_{\varphi} = \int_1^2 t^2 \varphi(t) \, dt.$$

Under our usual conditions, we obtain by Theorem 1.1 and dominated convergence, for $\mathbb{P}$-almost all $\omega \in \Omega$, $D$ the constant from the theorem,

$$\lim_{n \to \infty} \left( \inf_{x:|x| \leq n^3} \tilde{E}_{x,\omega}[T_{n,1}]/L_n^2 \right) = \lim_{n \to \infty} \left( \sup_{x:|x| \leq n^3} \tilde{E}_{x,\omega}[T_{n,1}]/L_n^2 \right) = c_{\varphi} D \quad (30)$$

Moreover, by Corollary 1.1 for $\mathbb{P}$-almost all $\omega$,

$$\limsup_{n \to \infty} \left( \sup_{x:|x| \leq n^3} \tilde{E}_{x,\omega}[T_{n,1}^2]/L_n^4 \right) \leq 8. \quad (31)$$

We denote by $A_2$ the set of environments of full $\mathbb{P}$-measure on which both (30) and (31) hold true.

### 5.3 A law of large numbers

Recall Figure 2. We shall not merely consider $k_n = k_{n,1}$, but more generally for $t \in [0, 1]$

$$k_{n,t} = k_{n,t}(\omega) = \max \{ i \in \mathbb{N} : T_{n,i} \leq tn \}.$$

We shall need a (weak) law of large numbers for $k_{n,t}$ under $\tilde{\mathbb{P}}_{x,\omega}$, uniformly in $|x| \leq n^2$. In view of (30), it is natural to expect that $k_{n,t}$ has the same asymptotic behavior as $t\beta_n$, where

$$\beta_n = \left\lfloor \frac{n}{c_{\varphi}DL_n^2} \right\rfloor = \left\lfloor \frac{(\log n)^2}{c_{\varphi}D} \right\rfloor.$$

We first establish a bound on the variance of $T_{n,t}$.

**Lemma 5.1.** For $\mathbb{P}$-almost all environments,

$$\sup_{t \leq 2\beta_n} \sup_{|x| \leq n^2} \frac{\text{Var}_{\tilde{\mathbb{P}}_{x,\omega}}(T_{n,t})}{n^2} \to 0 \quad \text{as} \quad n \to \infty,$$

where $\text{Var}_{\tilde{\mathbb{P}}_{x,\omega}}$ denotes the variance with respect to $\tilde{\mathbb{P}}_{x,\omega}$.
Proof: We can restrict ourselves to \( \omega \in A_2 \). Define the successive sojourn times \( \tau_{n,i} = (T_{n,i} - T_{n,i-1}) \). Then \( T_{n,\ell} = \tau_{n,1} + \cdots + \tau_{n,\ell} \). Unlike for random walk in a homogeneous environment, the variables \( \tau_{n,i}, i = 1, \ldots, 2\beta_n \), are in general not independent under \( \tilde{P}_{0,\omega} \). However, for \( i < j \), \( \tau_{n,j} \) is conditionally independent from \( \tau_{n,i} \) given \( \tilde{X}_{n,j-1} \). By the strong Markov property (with the same justification as in the proof of Lemma 4.3), we obtain for \( i < j \leq 2\beta_n \) and \( x \in \mathbb{Z}^d \) with \( |x| \leq n^2 \),

\[
\tilde{E}_{x,\omega}[\tau_{n,i} \mid \tau_{n,j}] = \tilde{E}_{x,\omega}[\tau_{n,i} \mid \tilde{X}_{n,j-1}, \tau_{n,i}]
= \tilde{E}_{x,\omega}[\tau_{n,i} \mid \tilde{X}_{n,j-1}, \omega] \leq \sup_{|y| \leq 2n^2} \tilde{E}_{y,\omega}[T_{n,1}]^2.
\]

In the last step we used that the coarse grained random can bridge in \( 2\beta_n \) steps a distance of at most \( 4\beta_n L_n < n \) and is therefore well inside \( V_{2n^2} \) when started from \( V_{n^2} \). Similarly, we see that

\[
\tilde{E}_{x,\omega}[\tau_{n,i} \mid \tau_{n,j}] \geq \inf_{|y| \leq 2n^2} \tilde{E}_{y,\omega}[T_{n,1}]^2.
\]

For \( x \) with \( |x| \leq n^2 \) and \( i, j \leq 2\beta_n \), it also holds that

\[
\inf_{|y| \leq 2n^2} \tilde{E}_{y,\omega}[T_{n,1}]^2 \leq \tilde{E}_{x,\omega}[\tau_{n,i}] \tilde{E}_{x,\omega}[\tau_{n,j}] \leq \sup_{|y| \leq 2n^2} \tilde{E}_{y,\omega}[T_{n,1}]^2.
\]

Since by definition of the event \( A_2 \), we have for \( \omega \in A_2 \)

\[
\lim_{n \to \infty} \left( \inf_{|y| \leq 2n^2} \tilde{E}_{y,\omega}[T_{n,1}]^2 / L_n^4 \right) = \lim_{n \to \infty} \left( \sup_{|y| \leq 2n^2} \tilde{E}_{y,\omega}[T_{n,1}]^2 / L_n^4 \right),
\]

we obtain for \( i, j \leq 2\beta_n \) and \( x \) with \( |x| \leq n^2 \),

\[
\left| \tilde{E}_{x,\omega}[\tau_{n,i} \mid \tau_{n,j}] - \tilde{E}_{x,\omega}[\tau_{n,i}] \tilde{E}_{x,\omega}[\tau_{n,j}] \right|
\leq \sup_{|y| \leq 2n^2} \tilde{E}_{y,\omega}[T_{n,1}]^2 - \inf_{|y| \leq 2n^2} \tilde{E}_{y,\omega}[T_{n,1}]^2 \overset{\text{def}}{=} \alpha(n) = o(L_n^4) \quad \text{for} \quad n \to \infty.
\]

Using this for \( i \neq j \) and \( \sigma \), \( \sigma \) for the terms with \( i = j \), we conclude that for \( n \geq n(\omega), \ell \leq 2\beta_n \),

\[
\sup_{|x| \leq n^2} \text{Var}_{\tilde{P}_{x,\omega}}(T_{n,\ell}) \leq C\beta_n L_n^4 + C\beta_n^2 \alpha(n) = o(n^2).
\]

This finishes the proof. \( \square \)

We are now in position to prove a weak law of large numbers for \( k_{n,t} \).

Lemma 5.2. For \( \mathbb{P} \)-almost all environments, for every \( t \in [0,1] \) and every \( \epsilon > 0 \),

\[
\sup_{|x| \leq n^2} \tilde{P}_{x,\omega} \left( \left| k_{n,t} / \beta_n - t \right| > \epsilon \right) \to 0 \quad \text{as} \quad n \to \infty.
\]
Proof: We take $\omega \in A_2$ as in the previous lemma. There is nothing to show for $t = 0$, so assume $t \in (0, 1]$. If the statement would not hold, then we could find $\epsilon, \epsilon' > 0$ such that

$$\sup_{|x| \leq n^2} \tilde{P}_{x,\omega} (k_{n,t} < (t - \epsilon)\beta_n) > \epsilon'$$  infinitely often, or \hspace{2cm} (32)

$$\sup_{|x| \leq n^2} \tilde{P}_{x,\omega} (k_{n,t} > (t + \epsilon)\beta_n) > \epsilon'$$  infinitely often. \hspace{2cm} (33)

Let us first assume (32). Then, with $i_n = \lceil (t - \epsilon)\beta_n \rceil$, by definition

$$\sup_{|x| \leq n^2} \tilde{P}_{x,\omega} (T_{n,i_n} > tn) > \epsilon'$$  infinitely often.

Next note that by (30), by linearity of the expectation and the fact that $2i_nL_n < n$,$$
0 \leq \frac{\sup_{|x| \leq n^2} \tilde{E}_{x,\omega} [T_{n,i_n}]}{tn} \leq \frac{i_n}{tn} \sup_{|y| \leq 2n^2} \tilde{E}_{y,\omega} [T_{n,1}] \leq 1 - \epsilon/2 \quad \text{for } n \geq n_0(\omega).
$$

Chebycheff’s inequality then shows that if $n \geq n_0(\omega)$ and $x$ with $|x| \leq n^2$,

$$\tilde{P}_{x,\omega} (T_{n,i_n} > tn) = \tilde{P}_{x,\omega} \left( T_{n,i_n} - \tilde{E}_{x,\omega} [T_{n,i_n}] > tn - \tilde{E}_{x,\omega} [T_{n,i_n}] \right)$$

$$\leq \frac{1}{(tn)^2} \left( 1 - \tilde{E}_{x,\omega} [T_{n,i_n}] / (tn) \right)^{-2} \text{Var} \tilde{P}_{x,\omega} (T_{n,i_n})$$

$$\leq \frac{4}{(\epsilon tn)^2} \sup_{|y| \leq n^2} \text{Var} \tilde{P}_{y,\omega} (T_{n,i_n}).$$

The right hand side converges to zero by Lemma 5.1. This contradicts (32).

Now assume (33). We argue similarly. First, with $i_n = \lfloor (t + \epsilon)\beta_n \rfloor$ by definition

$$\sup_{|x| \leq n^2} \tilde{P}_{x,\omega} (T_{n,i_n} < tn) > \epsilon'$$  infinitely often.

Since for $\omega \in A_2$ and large $n \geq n_0(\omega)$,

$$\frac{\inf_{|x| \leq n^2} \tilde{E}_{x,\omega} [T_{n,i_n}]}{tn} \geq \frac{i_n}{tn} \inf_{|y| \leq 2n^2} \tilde{E}_{y,\omega} [T_{n,1}] \geq 1 + \epsilon/2,$$

we obtain for large $n \geq n_0(\omega)$ and $x$ with $|x| \leq n^2$,

$$\tilde{P}_{x,\omega} (T_{n,i_n} < tn) = \tilde{P}_{x,\omega} \left( \tilde{E}_{x,\omega} [T_{n,i_n}] - T_{n,i_n} > \tilde{E}_{x,\omega} [T_{n,i_n}] - tn \right)$$

$$\leq \frac{4}{(\epsilon tn)^2} \sup_{|y| \leq n^2} \text{Var} \tilde{P}_{y,\omega} (T_{n,i_n}) / n^2 \to 0 \quad \text{as } n \to \infty.$$

Therefore, neither (32) nor (33) can hold, and the proof of the lemma is complete. \qed
5.4 Proof of Theorem 1.2

We turn to the proof of Theorem 1.2. Recall our notation introduced above. Since the subscript $n$ already appears in both $k_{n,t}$ and $\beta_n$, we may safely write

$$\hat{X}_{k_{n,t}}$$

instead of $\hat{X}_{n,k_{n,t}}$, 

$$\hat{X}_{\lfloor t \beta_n \rfloor}$$

instead of $\hat{X}_{n,\lfloor t \beta_n \rfloor}$.

Since both $A_1$ and $A_2$ have full $\mathbb{P}$-measure, we can restrict ourselves to $\omega \in A_1 \cap A_2$.

We first prove one-dimensional convergence, uniformly in the starting point $x$ with $|x| \leq n^{-2}$. This will easily imply multidimensional convergence and tightness.

One-dimensional convergence

**Proposition 5.1.** For $\mathbb{P}$-almost all environments, for each $t \in [0,1]$ and $u \in \mathbb{R}$,

$$\sup_{|x| \leq n^{-2}} |\mathbb{P}_{x,\omega} \left( \left( X^n_t - x \right) / \sqrt{n} > u \right) - \mathbb{P} \left( N(0,tD^{-1}\Lambda) > u \right) | \to 0 \quad \text{as } n \to \infty,$$

where $N(0, A)$ denotes a $d$-dimensional centered Gaussian with covariance matrix $A$.

**Proof:** Let $t \in [0,1]$. We write

$$X^n_t = \hat{X}_{\lfloor t \beta_n \rfloor} + \left( X^n_t - \hat{X}_{k_{n,t}} \right) + \left( \hat{X}_{k_{n,t}} - \hat{X}_{\lfloor t \beta_n \rfloor} \right).$$

Since by definition of the random sequence $k_{n,t}$, one has

$$|X^n_t - \hat{X}_{k_{n,t}}| \leq 1 + |X_{\lfloor tn \rfloor} - \hat{X}_{k_{n,t}}| \leq 3L_n = o \left( \sqrt{n} \right),$$

our claim follows from the following two convergences when $n \to \infty$.

(i) For each $u \in \mathbb{R}$,

$$\sup_{|x| \leq n^{-2}} \left| \mathbb{P}_{x,\omega} \left( \left( \hat{X}_{\lfloor t \beta_n \rfloor} - x \right) / \sqrt{n} > u \right) - \mathbb{P} \left( N(0,tD^{-1}\Lambda) > u \right) \right| \to 0.$$

(ii) For each $\epsilon > 0$, $\sup_{|x| \leq n^{-2}} \mathbb{P}_{x,\omega} \left( \left| \hat{X}_{\lfloor t \beta_n \rfloor} - \hat{X}_{\lfloor t \beta_n \rfloor} \right| / \sqrt{n} > \epsilon \right) \to 0$.

We first prove (i). For notational simplicity, we restrict ourselves to the case $t = 1$; the general case $t \in [0,1]$ follows exactly the same lines, with $\beta_n$ replaced everywhere by $\lfloor t \beta_n \rfloor$. For later use, it will be helpful to consider here the supremum over $x$ bounded by $2n^2$ instead of $n^2$. We let $(Z_{n,i})_{i=0,\ldots,n}$ be an i.i.d. sequence of random vectors distributed according to $q_n(0, \cdot)$, independently of the RWRE. Since $|Z_{n,i}| \leq 2L_n = o(\sqrt{n})$, it suffices to show the statement for $\hat{X}_{\beta_n}$ inside the probability replaced by $\hat{X}_{\beta_n} + Z_{n,0}$ (tacitly assuming that $\hat{X}_{\beta_n}$ under $\mathbb{P}_{x,\omega}$ and $Z_{n,0}$ are defined on the same probability space, whose
probability measure we again denote by \( \tilde{P}_{x,\omega} \). Now let \( \tilde{Y}_i = Z_{n,1} + \cdots + Z_{n,i} \). Since \( \tilde{X}_{\beta_n} + Z_{n,0} \) has law \( (Q_n)^{\beta_n} q_n(x, \cdot) \) under \( \tilde{P}_{x,\omega} \), and \( x + \tilde{Y}_i \) has law \( (q_n)^i(x, \cdot) \), we get

\[
\sup_{|x| \leq 2n^2} |\tilde{P}_{x,\omega} \left( (\tilde{X}_{\beta_n} + Z_{n,0} - x) / \sqrt{n} > u \right) - P \left( (x + \tilde{Y}_{\beta_n+1} - x) / \sqrt{n} > u \right)| \leq \sup_{|x| \leq 2n^2} \| (Q_n)^{\beta_n} - (q_n)^{\beta_n} q_n(x, \cdot) \|_1.
\]

For \( \omega \in A_1 \), we obtain by iteration, uniformly in \( x \) with \( |x| \leq 2n^2 \),

\[
\| (Q_n)^{\beta_n} - (q_n)^{\beta_n} q_n(x, \cdot) \|_1 \\
\leq \| (Q_n)^{\beta_n-1} (Q_n - q_n) q_n(x, \cdot) \|_1 + \| (Q_n)^{\beta_n-1} - (q_n)^{\beta_n-1} q_n^2(x, \cdot) \|_1 \\
\leq \sup_{|x| \leq 3n^2} \| (Q_n - q_n) q_n(x, \cdot) \|_1 + \sup_{|x| \leq 2n^2} \| (Q_n)^{\beta_n-1} - (q_n)^{\beta_n-1} q_n(x, \cdot) \|_1 \\
\leq \beta_n (\log L_n)^{-9} \to 0 \quad \text{as } n \to \infty.
\]

It remains to show that \( \tilde{Y}_{\beta_n} / \sqrt{n} \) converges in distribution to a \( d \)-dimensional centered Gaussian vector with covariance matrix \( D^{-1} \Lambda \). This will be a consequence of the following multidimensional version of the Lindeberg-Feller theorem.

**Proposition 5.2.** Let \( W_{m,\ell} \), \( 1 \leq \ell \leq m \), be centered and independent \( \mathbb{R}^d \)-valued random vectors. Put \( \Sigma_{m,\ell} = (\sigma_{m,\ell}^{(ij)})_{i,j=1,\ldots,d} \), where \( \sigma_{m,\ell}^{(ij)} = E \left[ W_{m,\ell}^{(i)} W_{m,\ell}^{(j)} \right] \) and \( W_{m,\ell}^{(i)} \) is the \( i \)th component of \( W_{m,\ell} \). If \( m \to \infty \),

(a) \( \sum_{\ell=1}^m \Sigma_{m,\ell} \to \Sigma \),

(b) for each \( \mathbf{v} \in \mathbb{R}^d \) and each \( \epsilon > 0 \),

\[
\sum_{\ell=1}^m E \left[ |\mathbf{v} \cdot W_{m,\ell}|^2 ; |\mathbf{v} \cdot W_{m,\ell}| > \epsilon \right] \to 0,
\]

then \( W_{m,1} + \cdots + W_{m,m} \) converges in distribution as \( m \to \infty \) to a \( d \)-dimensional Gaussian random vector with mean zero and covariance matrix \( \Sigma \).

**Proof of Proposition 5.2** By the Cramér-Wold device, it suffices to show that for fixed \( \mathbf{v} \in \mathbb{R}^d \), \( \mathbf{v} \cdot (W_{m,1} + \cdots + W_{m,m}) \) converges in distribution to a Gaussian random variable with mean zero and variance \( \mathbf{v}^T \Sigma \mathbf{v} \). Under (a) and (b), this follows immediately from the classical one-dimensional Lindeberg-Feller theorem.

We now finish the proof of (i). Recall that \( \tilde{Y}_{\beta_n} = Z_{n,1} + \cdots + Z_{n,\beta_n} \), where the \( Z_{n,\ell} \) are independent random vectors with law \( q_n(0, \cdot) \). Since the underlying one-step transition kernel \( p_{\ell,n} \) is symmetric, the \( Z_{n,\ell} \) are centered. Moreover, denoting by \( Z_{n,\ell}^{(i)} \) the \( i \)th component of \( Z_{n,\ell} \),

\[
E \left[ Z_{n,\ell}^{(i)} Z_{n,\ell}^{(j)} \right] = 0 \quad \text{for } i \neq j, \quad i, j = 1, \ldots, d.
\]
For $i = j$, we obtain by definition
\[
\frac{1}{n} \left( \mathbb{E} \left[ \left( Z^{(i)}_{n,1} \right)^2 \right] + \cdots + \mathbb{E} \left[ \left( Z^{(i)}_{n,\beta_n} \right)^2 \right] \right)
\]
\[
= \frac{\beta_n}{n} \sum_{y \in \mathbb{Z}^d} q_n(0, y) y_i^2 = \frac{\beta_n}{n} \int_0^2 \varphi(s) \sum_{y \in \mathbb{Z}^d} \pi^{(p_{L_n})}_{y, L_n}(0, y) y_i^2 \, ds
\]
\[
= \frac{\beta_n}{n} \int_0^2 (L_n s^2) \varphi(s) \sum_{y \in \mathbb{Z}^d} \pi^{(p_{L_n})}_{y, L_n}(0, y) (y_i/s L_n)^2 \, ds.
\] (34)

We next recall that [1, Lemma 2.1] shows how to recover the kernel $p_{L_n}$ out of the exit measure $\pi^{(p_{L_n})}_{y, L_n}$, namely
\[
2p_{L_n} = \sum_{y \in \mathbb{Z}^d} \pi^{(p_{L_n})}_{y, L_n}(0, y) (y_i/s L_n)^2 + O(L_n^{-1}).
\]

Replacing $\beta_n$ by its value, we therefore deduce from (34) that
\[
\frac{1}{n} \left( \mathbb{E} \left[ \left( Z^{(i)}_{n,1} \right)^2 \right] + \cdots + \mathbb{E} \left[ \left( Z^{(i)}_{n,\beta_n} \right)^2 \right] \right)
\]
\[
= L_n^2 \beta_n c \varphi_{2p_{L_n}}(e_i) + O \left( \frac{2p_{L_n}}{D} \right) + O \left( \left( \log n \right)^{-2} \right).
\]

Since $p_{L_n}(e_i) \to p_{\infty}(e_i)$ as $n \to \infty$, we obtain with $W_{\beta_n, \ell} = Z_{n, \ell}/\sqrt{n}$, $\ell = 1, \ldots, \beta_n$, in the notation of Proposition 5.2
\[
\sum_{\ell=1}^{\beta_n} \mathbb{E}[\delta_{\beta_n, \ell}] \to D^{-1} \left( 2p_{\infty}(e_i) \delta_i(j) \right)_{i,j=1}^d = D^{-1} \mathbf{A} \quad \text{as } n \to \infty.
\]

Since $W_{\beta_n, \ell} \leq 2L_n/\sqrt{n} \leq 2(\log n)^{-1}$, point (b) of Proposition 5.2 is trivially fulfilled. Applying this proposition finally shows that $\tilde{Y}_{\beta_n}/\sqrt{n} = W_{\beta_n, 1} + \cdots + W_{\beta_n, \beta_n}$ converges in distribution to a $d$-dimensional centered Gaussian random vector with covariance matrix $D^{-1} \mathbf{A}$. This finishes the proof of (i).

It remains to prove (ii). In view of Lemma 5.2 it suffices to show that for each $\epsilon > 0$,
\[
limit_{\theta \to 0} \sup_{n \to \infty} \mathbb{P}_{x, \omega} \left( |\tilde{X}_{k_n, t} - \tilde{X}_{t, \beta_n}| > \epsilon \sqrt{n}; |k_n, t - [t, \beta_n]| < \theta \beta_n \right) = 0.
\]

Fix $\epsilon > 0$, $\theta > 0$. Define the set of integers
\[
A_n = \{ [t, \beta_n] - [\theta \beta_n], \ldots, [t, \beta_n] + [\theta \beta_n] \},
\]
and let $\ell_n = [t, \beta_n] - [\theta \beta_n]$. Then
\[
\mathbb{P}_{x, \omega} \left( |\tilde{X}_{k_n, t} - \tilde{X}_{t, \beta_n}| > \epsilon \sqrt{n}; |k_n, t - [t, \beta_n]| < \theta \beta_n \right)
\]
\[
\leq \mathbb{P}_{x, \omega} \left( \max_{\ell \in A_n} |\tilde{X}_{k_n, \ell} - \tilde{X}_{t, \beta_n}| > \epsilon \sqrt{n} \right)
\]
\[
\leq \mathbb{P}_{x, \omega} \left( \max_{\ell \in A_n} |\tilde{X}_{k_n, \ell} - \tilde{X}_{n, \ell_n}| > \epsilon/2 \sqrt{n} \right) + \mathbb{P}_{x, \omega} \left( |\tilde{X}_{n, \ell_n} - \tilde{X}_{t, \beta_n}| > \epsilon/2 \sqrt{n} \right).
\]
We only consider the first probability in the last display; the second one is treated in a similar (but simpler) way. We first remark that after $\ell_n$ steps, the coarse grained RWRE with transition kernel $Q_n$ starting in $V_n$ is still within $V_{2n}$. Therefore, by the Markov property, for $x$ with $|x| \leq n^2$,

$$
\tilde{P}_{x,\omega} \left( \max_{\ell \in A_n} |\hat{X}_{n,\ell} - \hat{X}_{n,\ell_n}| > (\epsilon/2)\sqrt{n} \right)
$$

$$
\leq \sup_{|y| \leq 2n^2} \tilde{P}_{y,\omega} \left( \max_{\ell \leq 2[\theta \beta_n]} |\hat{X}_{n,\ell} - y| > (\epsilon/2)\sqrt{n} \right). \quad (35)
$$

For estimating (35), we follow a strategy similar to Billingsley [3, Theorem 9.1]. Put

$$
E_\ell = \left\{ \max_{j \leq \ell} |\hat{X}_{n,j} - \hat{X}_{n,0}| < (\epsilon/2)\sqrt{n} \leq |\hat{X}_{n,\ell} - \hat{X}_{n,0}| \right\}.
$$

Then

$$
\tilde{P}_{y,\omega} \left( \max_{\ell \leq 2[\theta \beta_n]} |\hat{X}_{n,\ell} - y| > (\epsilon/2)\sqrt{n} \right) \leq \tilde{P}_{y,\omega} \left( |\hat{X}_{n,2[\theta \beta_n]} - y| \geq (\epsilon/4)\sqrt{n} \right)
$$

$$
+ \sum_{\ell=1}^{2[\theta \beta_n]-1} \tilde{P}_{y,\omega} \left( |\hat{X}_{n,2[\theta \beta_n]} - \hat{X}_{n,\ell}| \geq (\epsilon/4)\sqrt{n}; E_\ell \right).
$$

Concerning the first probability on the right, we already know from (i) that for $\theta < 1/2$,

$$
\sup_{|y| \leq 2n^2} \tilde{P}_{y,\omega} \left( |\hat{X}_{n,2[\theta \beta_n]} - y| \geq (\epsilon/4)\sqrt{n} \right) \to P \left( |\mathcal{N}(0, 2\theta D^{-1} \Lambda)| \geq \epsilon/4 \right) \quad \text{as } n \to \infty.
$$

For fixed $\epsilon$, the right side converges to zero as $\theta \downarrow 0$ by Chebychev’s inequality. For the sum over the probabilities in the above display, we stress that the increments of the coarse grained walk $\hat{X}_{n,\ell}$ are neither independent nor stationary under $\tilde{P}_{y,\omega}$. But we have by the Markov property at time $\ell$, for $|y| \leq 2n^2$,

$$
\sum_{\ell=1}^{2[\theta \beta_n]-1} \tilde{P}_{y,\omega} \left( |\hat{X}_{n,2[\theta \beta_n]} - \hat{X}_{n,\ell}| \geq (\epsilon/4)\sqrt{n}; E_\ell \right)
$$

$$
\leq \sum_{\ell=1}^{2[\theta \beta_n]-1} \tilde{P}_{y,\omega}(E_\ell) \sup_{|z| \leq 3n^2} \tilde{P}_{z,\omega} \left( |\hat{X}_{n,2[\theta \beta_n]} - z| \geq (\epsilon/4)\sqrt{n} \right). \quad (36)
$$

Similar to the proof of (i), we estimate for $\ell = 1, \ldots, 2[\theta \beta_n] - 1$

$$
\sup_{|z| \leq 3n^2} \tilde{P}_{z,\omega} \left( |\hat{X}_{n,\ell} - z| \geq (\epsilon/4)\sqrt{n} \right) \leq \sup_{|z| \leq 3n^2} (Q_n)^\ell q_n \left( z, Z^d \setminus V_{(\epsilon/8)\sqrt{n}}(z) \right)
$$

$$
\leq \sup_{|z| \leq 3n^2} \| ((Q_n)^\ell - (q_n)^\ell) q_n(z, \cdot) \|_1 + (q_n)^{\ell+1} \left( 0, Z^d \setminus V_{(\epsilon/8)\sqrt{n}} \right).$$
For environments \( \omega \in A_1 \), the first summand is estimated by \( \ell (\log L_n)^{-9} \) as in the proof of (i). For the expression involving \( q_n \), we use the following standard large deviation estimate (a proof is for example given in [1, Proof of Lemma 7.5]): There exist constants \( C_1, c_1 \) depending only on the dimension such that

\[
(q_n)^\ell (0, \mathbb{Z}^d \setminus V_\ell) \leq C_1 \exp \left(-c_1 r^2 / (\ell L_n^2)\right), \quad r > 0, \ \ell \in \mathbb{N}.
\]

In our setting, we obtain

\[
(q_n)^\ell (0, \mathbb{Z}^d \setminus V_{(\ell/8)\sqrt{n}}) \leq C \exp \left(-\ell c_2^2 / \theta \right) \quad \text{uniformly in } 1 \leq \ell \leq 2[\theta \beta_n].
\]

Back to (36), the fact that the \( E_i \)'s are disjoint leads to

\[
\sum_{\ell=1}^{2[\theta \beta_n]} \tilde{P}_{y,\omega} (|\hat{X}_{n,2[\theta \beta_n]} - \hat{X}_{n,\ell}| \geq (\epsilon/4) \sqrt{n}; E_\ell) \leq \beta_n^2 (\log L_n)^{-9} + \sum_{\ell=1}^{2[\theta \beta_n]} \tilde{P}_{y,\omega} (E_\ell) (q_n)^{2[\theta \beta_n]+1-\ell} (0, \mathbb{Z}^d \setminus V_{(\ell/8)\sqrt{n}}) \leq o(1) + C \exp \left(-\ell c_2^2 / \theta \right),
\]

everything uniformly in \(|y| \leq 2n^2\). The last expression converges to zero as \( \theta \downarrow 0 \). This concludes the proof of (ii) and hence of the one-dimensional convergence. \( \square \)

### Convergence of finite-dimensional distributions

In order to prove convergence of the two-dimensional distributions under \( P_{0,\omega} \), we have to show that for \( 0 \leq t_1 < t_2 \leq 1 \) and \( u_1, u_2 \in \mathbb{R} \), as \( n \to \infty \),

\[
\left| P_{0,\omega} \left( X_{t_1}^n / \sqrt{n} > u_1, \ X_{t_2}^n - X_{t_1}^n / \sqrt{n} > u_2 \right) - P \left( \mathcal{N}(0, t_1 D^{-1} \Lambda) > u_1 \right) P \left( \mathcal{N}(0, (t_2 - t_1) D^{-1} \Lambda) > u_2 \right) \right| \to 0. \tag{37}
\]

This follows easily from our uniform one-dimensional convergence. First, we may replace \( X_{t_1}^n \) by \( X_{[t_1]n} \) and \( X_{t_2}^n \) by \( X_{[t_2]n} \), since their difference is bounded by one. Then, by the Markov property

\[
P_{0,\omega} \left( X_{[t_1]n} / \sqrt{n} > u_1, \ X_{[t_2]n} - X_{[t_1]n} / \sqrt{n} > u_2 \right) = P_{0,\omega} \left( X_{[t_1]n} / \sqrt{n} > u_1, \ P_{X_{[t_1]n}\omega} \left( (X_{[t_2]n} - [t_1]n) - X_0) / \sqrt{n} > u_2 \right) \right) \leq P_{0,\omega} \left( X_{[t_1]n} / \sqrt{n} > u_1 \right) \sup_{|x| \leq n} P_{x,\omega} \left( (X_{[t_2]n} - [t_1]n) - x) / \sqrt{n} > u_2 \right).
\]

The product of the two probabilities converges by Proposition 5.1 towards

\[
P \left( \mathcal{N}(0, t_1 D^{-1} \Lambda) > u_1 \right) P \left( \mathcal{N}(0, (t_2 - t_1) D^{-1} \Lambda) > u_2 \right). \tag{38}
\]
5.4 Proof of Theorem 1.2

For the lower bound,
\[
P_{0,\omega} \left( \frac{X_{\lfloor t_1 n \rfloor}}{\sqrt{n}} > u_1, \frac{X_{\lfloor t_2 n \rfloor} - X_{\lfloor t_1 n \rfloor}}{\sqrt{n}} > u_2 \right)
\geq P_{0,\omega} \left( \frac{X_{\lfloor t_1 n \rfloor}}{\sqrt{n}} > u_1 \right) \inf_{|x| \leq n} P_{x,\omega} \left( \frac{X_{\lfloor t_2 n \rfloor} - \lfloor t_1 n \rfloor - x}{\sqrt{n}} > u_2 \right),
\]
and the right hand side converges again towards the product in (38). This proves convergence of the two-dimensional distributions under \( P_{0,\omega} \). The general case of finite-dimensional convergence is obtained similarly.

**Tightness**

The sequence of \( P_{0,\omega} \)-laws of \( (X^n_t/\sqrt{n} : 0 \leq t \leq 1) \) is tight, if the following Condition \( T \) holds true.

For each \( \epsilon > 0 \) there exist a \( \lambda > 1 \) and an integer \( n_0 \) such that, if \( n \geq n_0 \),
\[
P_{0,\omega} \left( \max_{\ell \leq n} |X_{k+\ell} - X_k| \geq \lambda \sqrt{n} \right) \leq \frac{\epsilon}{\lambda^2}
\text{ for all } k \leq n \lambda^2 / \epsilon.
\]

See [4, Theorem 8.4] for a proof of this standard criterion.

Let us now show that Condition \( T \) is indeed satisfied in our setting. First, by the Markov property at time \( n \),
\[
P_{0,\omega} \left( \max_{\ell \leq n} |X_{k+\ell} - X_k| \geq \lambda \sqrt{n} \right) \leq \sup_{|x| \leq k} P_{x,\omega} \left( \max_{\ell \leq n} |X_\ell - x| \geq \lambda \sqrt{n} \right).
\]
The random walk \( X_k \) under \( P_{x,\omega} \) has the same law as the first coordinate process on \( (\mathbb{Z}^d)^N \times \Xi \) under \( \tilde{P}_{x,\omega} \), which we also denote by \( X_k \) (see the beginning of Section 5).

We shall now consider the latter under \( \tilde{P}_{x,\omega} \). We recall that \( k_{n,1} = k_{n,1}(\omega) \) counts the number of steps the coarse grained walk performs up to time \( n \). Now we have
\[
P_{x,\omega} \left( \max_{\ell \leq n} |X_\ell - x| \geq \lambda \sqrt{n} \right)
\leq \tilde{P}_{x,\omega} \left( \max_{\ell \leq n} |X_\ell - x| \geq \lambda \sqrt{n}; k_{n,1} \leq 2 \beta_n \right) + \tilde{P}_{x,\omega} (k_{n,1} > 2 \beta_n).
\]
The second probability on the right converges to zero as \( n \) tends to infinity by Lemma 5.2 uniformly in starting points \( x \) with \( |x| \leq n^2 \). For the first probability, we find on the event \( \{k_{n,1} \leq 2 \beta_n\} \) for each \( j \leq n \) an \( \ell \leq 2 \beta_n \) such that \( |X_j - \hat{X}_{n,\ell}| \leq 2 L_n \). We therefore obtain for large \( n \)
\[
\tilde{P}_{x,\omega} \left( \max_{\ell \leq n} |X_\ell - x| \geq \lambda \sqrt{n}; k_{n,1} \leq 2 \beta_n \right) \leq \tilde{P}_{x,\omega} \left( \max_{\ell \leq 2 \beta_n} |\hat{X}_{n,\ell} - x| \geq (\lambda/2) \sqrt{n} \right).
\]
For bounding this last probability, we can follow the same steps as for estimating (35). Leaving out the details, we arrive at
\[
\sup_{|x| \leq n^2} \tilde{P}_{x,\omega} \left( \max_{\ell \leq 2 \beta_n} |\hat{X}_{n,\ell} - x| \geq (\lambda/2) \sqrt{n} \right) \leq \frac{C}{\lambda^3} + C \exp \left( -c \lambda^2 \right) \leq \frac{\epsilon}{\lambda^2},
\]
provided $\lambda = \lambda(d, \epsilon)$ is large enough. This proves that Condition $T$ is satisfied. Therefore, the sequence of $P_{0,\omega}$-laws of $(X^a_t/\sqrt{n} : 0 \leq t \leq 1)$ is tight, which concludes also the proof of Theorem 1.2.

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