ON A SHARP INEQUALITY OF ADIMURTHI-DRUET TYPE AND EXTREMAL FUNCTIONS

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ABSTRACT. Our main purpose in this paper is to establish the existence and nonexistence of extremal functions for sharp inequality of Adimurthi-Drue t type for fractional dimensions on the entire space. Precisely, we extend the sharp Trudinger-Mos er type inequality in (Calc.Var.Partial Differential Equations, 52 (2015) 125-163) for the entire space. In addition, we perform the two-step strategy of Carleson-Chang together blow up analysis method to ensure the existence of maximizers for the associated extremal problems for both subcritical and critical regimes. We also present a nonexistence result under subcritical regime for some special cases.

1. INTRODUCTION

The classical Trudinger-Moser inequality [17, 20, 23, 24] states that

\[
\sup_{u \in C^1_0(\Omega), \int_\Omega |\nabla u|^N dx \leq 1} \int_\Omega e^{\mu |u|^N} dx \begin{cases} < \infty, & \text{if } \mu \leq \mu_N \\ = \infty, & \text{if } \mu > \mu_N \end{cases}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain, \( \mu_N = N \omega_{N-1}^{\frac{1}{N-1}} \), and \( \omega_{N-1} \) is the measure of the unit sphere in \( \mathbb{R}^N \). By using symmetrization techniques J. Moser [17] was able to reduce (1.1) to the following

\[
\sup_{u \in C^1_{0,rad}(B_R), \int_{B_R} |\nabla u|^N dx \leq 1} \int_{B_R} e^{\mu |u|^N} dx \begin{cases} < \infty, & \text{if } \mu \leq \mu_N \\ = \infty, & \text{if } \mu > \mu_N \end{cases}
\]

where \( |B_R| = |\Omega| \) and \( C^1_{0,rad}(B_R) \) represents the set of the radially symmetric functions in \( C^1_0(B_R) \).

On the other hand, according to the formalism in [21, 26], the integration of radially symmetric function \( f(r) \) on a \( \theta \)-dimensional fractional space is given by

\[
\int f(r(x_0, x_1)) dx_0 = \omega_\theta \int_0^\infty r^\theta f(r) dr,
\]

where \( r(x_0, x_1) \) is the distance between two points \( x_0 \) and \( x_1 \), and \( \omega_\theta \) given by

\[
\omega_\theta = \frac{2\pi^\theta}{\Gamma(\frac{\theta}{2})}, \quad \text{with} \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.
\]
In the case that \( \theta \) is a positive integer number \( \omega_{\theta} \) agrees precisely with the measure of the unit sphere in for Euclidean space \( \mathbb{R}^{\theta+1} \). Integration over fractional dimensional spaces is often used in the dimensional regularization method as a powerful tool to obtain results in statistical mechanics and quantum field theory \([7, 19, 25, 26]\). Motivated by the above discussion, in \([9]\) the authors were able to establish a sharp Trudinger-Moser type inequality which extends the classical \((1.1)\) for fractional dimensions. Indeed, for \( 0 < R < \infty, \alpha \geq 1, \sigma \geq 0 \) and \( \alpha - p + 1 = 0 \), it is proven that

\[
\sup_{u \in AC_{loc}(0,R), u(R)=0, \int_0^R |u|^p \sigma \lambda \leq 1} \int_0^R e^{\mu |u|^\frac{\sigma}{p}} \sigma \lambda \left\{ \begin{array}{ll}
< \infty & \text{if } \mu \leq \mu_{\alpha,\sigma} \\
= \infty & \text{if } \mu > \mu_{\alpha,\sigma}
\end{array} \right.
\]

where \( \mu_{\alpha,\sigma} = (\sigma + 1) \omega_{\frac{1}{\alpha}}^\frac{1}{\alpha} \), \( AC_{loc}(0,R) \) denotes set of all locally absolutely continuous functions on the interval \( (0, R] \) and we are denoting the \( \theta \)-fractional measure (cf. \((1.3)\)) by

\[
\int_0^R f(r) d\omega_{\theta} = \omega_{\theta} \int_0^R r^\theta f(r) dr, \quad \theta \geq 0, \quad \text{and} \quad 0 < R \leq \infty.
\]

Despite its simple form, the inequality \((1.5)\) hides surprises and some interesting points have been drawing attention. Firstly, in the particular case that \( \alpha = \sigma = N - 1 \), the fractional inequality \((1.5)\) implies that the Moser’s reduction \((1.2)\) and then \((1.1)\) holds. In this sense we say that \((1.5)\) extends \((1.1)\) to weighted Sobolev spaces including fractional dimensions. Secondly, for \( \alpha = N - k \) and \( \sigma = N - 1 \) we can recover the Trudinger-Moser type inequality for the \( k \)-Hessian equation obtained by Tian and Wang \([22]\). Further, based on \((1.5)\), in \([12]\) the authors were able to investigate the existence of maximizers for that inequality obtained in \([22]\) and the existence of radially symmetric solutions for the \( k \)-Hessian equation was obtained in \([13]\). Also, for arbitrary real choices of \( \alpha \) and \( \sigma \), the estimate \((1.5)\) can be employed to study a general class of quasi-linear elliptic operators \([6, 15]\). Thirdly, in \([15]\) was proved that the exponential growth is optimal in \((1.5)\) which leads to loss of compactness in the sense of the embedding into the Orlicz-type spaces and makes the extremal problem associated to \((1.5)\) interesting. Inspired by the results due to Adimurthi and O. Druet \([1]\), in \([10]\) the authors have obtained a sharp form of \((1.5)\) and investigated the associated extremal problem.

In this paper we are mainly interested in extends the results in \([10]\) for the unbounded case, when \( \bar{R} = \infty \). In order to state our results, let us present briefly the related weighted Sobolev space introduced by P. Clément et al. \([6]\). For \( 0 < \bar{R} \leq \infty, \theta \geq 0 \) and \( q \geq 1 \), set \( L^p_\theta = L^p_\theta(0, \bar{R}) \) the Lebesgue space associated with the \( \theta \)-fractional measure \((1.6)\) on the interval \((0, \bar{R})\). Then, we denote by \( X^{1,p}_R(\alpha, \theta) \) the completion of the set of all functions \( u \in AC_{loc}(0, \bar{R}) \) such that \( \lim_{r \to R} u(r) = 0, u \in L^p_\theta \) and \( u' \in L^p_\theta \) with the norm

\[
\|u\| = (\|u\|^p_{L^p_\theta} + \|u'\|^p_{L^p_\theta})^{\frac{1}{p}}.
\]

If \( \alpha - p + 1 = 0 \) and \( \bar{R} = \infty \), we have the continuous embedding (See Section 2 below)

\[
X^{1,p}_R(\alpha, \theta) \hookrightarrow L^q_\theta \quad \text{for all } q \in [p, \infty).
\]
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Set

\[(1.9) \quad \varphi_p(t) = e^t - \sum_{k=0}^{k_0-1} \frac{t^k}{k!} = \sum_{j \geq n : j \geq p - 1} \frac{t^j}{j!}, \quad t \geq 0,\]

with \(k_0 = \min \{ j \in \mathbb{N} : p - 1 \leq j \} \). In view of the (1.8), for each term of the series expansion of \(\varphi_p(|u|^{p/(p-1)})\) belongs to \(L^1_\theta\) for all \(u \in X^{1,p}_\infty(\alpha, \theta)\). It motivates us to investigate the supremum

\[(1.10) \quad \text{AD}(\eta, \mu, \alpha, \theta) = \sup_{u \in X^{1,p}_\infty(\alpha, \theta), \|u\|=1} \int_0^\infty \varphi_p(\mu(1 + \eta\|u\|_{L^p_\theta})^{1-p}) |u|^{\frac{p}{p-1}} \, d\lambda_\theta.\]

Actually, we are able to establish the following sharp result:

**Theorem 1.1.** Let \(p \geq 2\) and \(\alpha = p - 1\) and \(\theta \geq 0\). Then,

1. \(\text{AD}(\eta, \mu, \alpha, \theta) < \infty\), for any \(\mu \leq \mu_{\alpha, \theta}\) and \(0 \leq \eta < 1\)
2. \(\text{AD}(1, \mu_{\alpha, \theta}, \alpha, \theta) = \infty\).

If \(\eta = 0\), Theorem 1.1 recovers the result in [3, Theorem 1.1] and part of the [11, Theorem 1.2]. In addition, Theorem 1.1 extends [10, Theorem 2] for the entire space \(\bar{R} = \infty\).

Maximizers for the Trudinger-Moser type inequality (1.5) and its extensions were investigated in [3,9–11]. In this paper, we will also investigate the existence and nonexistence of extremal function for the supremum (1.10). Our first existence results reads as follow:

**Theorem 1.2 (Subcritical case).** Let \(p \geq 2\) and \(\alpha = p - 1\) and \(\theta \geq \alpha\). Then the supremum \(\text{AD}(\eta, \mu, \alpha, \theta)\) is attained in the following cases:

1. \(p > 2, 0 \leq \eta < 1\) and \(0 < \mu < \mu_{\alpha, \theta}\).
2. \(p = 2, 0 \leq \eta < 1\) and \(\frac{2(1+2\eta)}{(1+\eta)^2} B_{2, \theta} < \mu < \mu_{1, \theta}\), where

\[(1.11) \quad \frac{1}{B_{2, \theta}} = \inf_{0 \neq u \in X^{1,2}_\infty(1, \theta)} \frac{\|u\|_{L^2_1}^2 \|u\|_{L^2_\theta}^2}{\|u\|_{L^4_\theta}^4}.\]

If we choose \(\eta = 0\), Theorem 1.2 recovers precisely [3, Theorem 1.2]. In addition, for the critical case \(\mu = \mu_{\alpha, \theta}\) we are able to obtain the following:

**Theorem 1.3 (Critical case).** Assume \(p, \alpha\) and \(\theta\) under the assumption of Theorem 1.2. Then, there exists \(\eta_0 \in (0, 1)\) such that \(\text{AD}(\eta, \mu_{\alpha, \theta}, \alpha, \theta)\) is attained for any \(0 \leq \eta < \eta_0\).

We note that Theorem 1.3 is new even for \(\eta = 0\). Indeed, the existence of maximizers for \(\text{AD}(0, \mu, \alpha, \theta)\) was recently ensured in [3] and [11] only for the strict case \(\mu < \mu_{\alpha, \theta}\).

On the non-existence we provide the following result which extends [3, Theorem 1.3] for \(\eta > 0\) and complements the classical non-existence results [16,18] to include non-integer dimensions.

**Theorem 1.4.** Let \(p = 2, \alpha = 1\) and \(\theta \geq 0\). Then there exists \(\mu_0 > 0\) such that \(\text{AD}(\eta, \mu, 1, \theta)\) is not attained for any \(0 \leq \eta < 1\) and \(0 < \mu < \mu_0\).
This paper is organized as follows. In the Section 2 we present some preliminary results on the weighted Sobolev space $X_{R}^{1,p}(\alpha, \theta)$. The Section 3 is devoted to prove of the sharp estimate given by Theorem 1.1. In the Section 4 we prove the attainability of $AD(\eta, \mu, \alpha, \theta)$ such as stated in Theorem 1.2 and Theorem 1.3. Finally, in the Section 5 we prove Theorem 1.4.

2. Notations and preliminary results

In this section we present briefly some notations and preliminary results on $X_{R}^{1,p}(\alpha, \theta)$. For a deeper discussion on this subject we recommend [6, 8, 11, 14] and the references therein.

According to the relation between the parameters $\alpha$ and $p$, we can distinguish two cases for $X_{R}^{1,p}(\alpha, \theta)$: the Sobolev case when $\alpha - p + 1 > 0$ and the Trudinger-Moser case for $\alpha - p + 1 = 0$. Supposing $\alpha - p + 1 > 0$, we have the following continuous embedding

$$(2.1) \quad X_{R}^{1,p}(\alpha, \theta) \hookrightarrow L_{\theta}^{q} \quad \text{if} \quad q \in [p, p^{*}] \quad \text{and} \quad \theta \geq \alpha - p$$

where the critical exponent $p^{*}$ is given by

$$p^{*} = p^{*}(\alpha, \theta, p) = \frac{(\theta + 1)p}{\alpha - p + 1}.$$ 

Also, the embeddings (2.1) are compact for the strict conditions $\theta > \alpha - p$ and $p < q < p^{*}$. On the other hand, for the Trudinger-Moser case we have the continuous embeddings

$$(2.2) \quad X_{R}^{1,p}(\alpha, \theta) \hookrightarrow L_{\theta}^{q} \quad \text{for all} \quad q \in [p, \infty)$$

which are compact for the strict case $q > p$. Of course, if $0 < R \neq \infty$ the embedding (2.1) and (2.2) can be extend to $1 \leq q \leq p^{*}$ and $1 \leq q < \infty$, respectively.

**Remark 2.1.** Let us denote $W_{R}^{1,p}(\alpha, \theta)$ the set of all functions $u \in AC_{loc}(0, R)$ such that $u \in L_{\theta}^{p}$ and $u' \in L_{\alpha}^{p}$. We recall $W_{R}^{1,p}(\alpha, \theta)$ is a Banach space endowed with the norm (1.7). In addition, according to [8, Lemma 2.2], the embeddings (2.1) and (2.2) also hold for $W_{R}^{1,p}(\alpha, \theta)$.

From [11, Lemma 4.1], for each $u \in X_{\infty}^{1,p}(\alpha, \theta)$, $p \geq 2$, we have the point-wise estimate

$$(2.3) \quad |u(r)|^{p} \leq \frac{C}{r^{(\alpha +\theta p -1)/p}} \|u\|^{p}, \quad \forall \ r > 0$$

where $C > 0$ depends only on $\alpha, p$ and $\theta$. For any $q \geq 1$, the following elementary inequality holds

$$(2.4) \quad (x + y)^{q} \leq (1 + \epsilon)^{\frac{q-1}{\theta}} x^{q} + \left(1 - (1 + \epsilon)^{-\frac{1}{\theta}}\right)^{-\frac{1}{\theta}} y^{q}, \quad x, y \geq 0$$

for all $\epsilon > 0$.

Henceforth we suppose $\alpha, \theta$ and $p$ such as in Theorem 1.1 and $\varphi_{\rho}$ as defined in (1.9).

**Lemma 2.1.** Let $R > 0$ and $\mu, \sigma \geq 0$ be arbitrary real numbers.

(i) For any $u \in X_{\infty}^{1,p}(\alpha, \theta)$ we have $\exp(\mu|u|^{1-\frac{1}{p}}) \in L_{\sigma}^{1}(0, R)$. In addition, if $\mu < \mu_{\alpha, \sigma}$ then

$$\sup_{\|u\|_{L_{\theta}^{p}} \leq 1, \|u\|_{L_{\theta}^{p}} \leq M} \int_{0}^{R} \exp(\mu|u|^{1-\frac{1}{p}}) d\lambda_{\sigma} \leq c$$
for some constant $c = c(\alpha, \sigma, \mu, M, R) > 0$.

(ii) For any $u \in X^{1,p}_\infty(\alpha, \theta)$ we have $\varphi_p(\mu |u|^{\frac{p}{p-1}}) \in L^1_\sigma([0, \infty))$. Also,

$$\sup_{\|u\|_{L^p(\infty)} \leq M, \|u\|_{L^\sigma(\infty)} \leq 1} \int_R^\infty \varphi_p(\mu |u|^{\frac{p}{p-1}}) d\lambda_\theta \leq c$$

for some constant $c = c(\alpha, \theta, \mu, M, R) > 0$.

**Proof:** For each $u \in X^{1,p}_\infty(\alpha, \theta)$, by setting $v = u - u(R)$ on $(0, R)$ we have $v \in X^{1,p}_R(\alpha, \theta)$. Thus, from (2.3) and (2.4) we have

$$|u|^{\frac{p}{p-1}} \leq \left(1 + \epsilon\right)^{\frac{1}{p}} |v|^{\frac{p}{p-1}} + \frac{c_1}{R^{\theta+1}} \|u\|^{\frac{p}{p-1}},$$

where $c_1$ depends only on $\alpha, \theta$ and $\epsilon$. Hence,

$$\int_0^R e^{\mu|u|^{\frac{p}{p-1}}} d\lambda_\sigma \leq e^{c_1 \frac{p}{p-1} \|u\|^{\frac{p}{p-1}}} \int_0^R e^{(1+\epsilon)^{\frac{p}{p-1}}} \mu |v|^{\frac{p}{p-1}} d\lambda_\sigma.$$

Hence, by choosing $\epsilon > 0$ such that $(1 + \epsilon)^{\frac{1}{p}} \mu < \mu_{\alpha, \sigma}$ the above inequality and (1.5) imply (i).

Further, the continuous embedding (2.2) and the monotone convergence theorem yield

$$\int_R^\infty \varphi_p(\mu |u|^{\frac{p}{p-1}}) d\lambda_\theta = \sum_{j \in \mathbb{N}} \int_R^\infty \mu^j |u|^{\frac{p}{p-1}} d\lambda_\theta$$

$$\leq \frac{\mu_k^0}{k_0!} (c_1 \|u\|^{\frac{p}{p-1}} + \frac{\mu_{k_0+1}}{(k_0 + 1)!} (c_1 \|u\|^{\frac{(k_0+1)p}{p-1}} + \sum_{j \in \mathbb{N}} \frac{\mu^j}{j!} \int_R^\infty |u|^{\frac{p}{p-1}} d\lambda_\theta$$

for some $c_1 > 0$ depending only on $\alpha$ and $\theta$. Also, $j \geq k_0 + 2$, from (2.3) we have

$$\int_R^\infty |u|^{\frac{p}{p-1}} d\lambda_\theta \leq (C \|u\|^p)^{\frac{1}{p-1}} \int_R^\infty \frac{1}{\theta+1} d\lambda_\theta \leq (C \|u\|^p)^{\frac{1}{p-1}} \frac{c_2}{R^{\theta+1}},$$

where $c_2$ depends only on $p, \theta$ and $R$. Using (2.7)

$$\int_R^\infty \varphi_p(\mu |u|^{\frac{p}{p-1}}) d\lambda_\theta \leq \max\{1, c_2\} e^{\max\{\mu(c_1 \|u\|^{\frac{p}{p-1}}, \mu(C \|u\|^p)^{\frac{1}{p-1}} R^{-\frac{\theta+1}{p}}\}}$$

which proves (ii).

**Remark 2.2.** By setting $\sigma = \theta$ in Lemma 2.1-(i) and combining this with (ii), we can see that $\varphi_p(\mu |u|^{\frac{p}{p-1}}) \in L^1_\sigma([0, \infty))$, for any $\mu > 0$ and $u \in X^{1,p}_\infty(\alpha, \theta)$. Moreover, in the uniform estimate (2.5) we are not supposing $\mu \leq \mu_{\alpha, \theta}$.

We finish this section with some elementary properties of the fractional integral in (1.3). First, the change of variables $s = \tau \rho$ yields

$$\int_0^\infty f(\tau \rho) d\lambda_\theta = \frac{1}{\tau^{\theta+1}} \int_0^\infty f(s) d\lambda_\theta, \quad \tau > 0.$$
Thus, by setting \( u_\tau(r) = \zeta u(\tau r) \), with \( \zeta, \tau > 0 \) and \( u \in X^{1,p}_\infty(\alpha, \theta) \) we can write
\[
\|u_\tau\|^p_{L^p_\theta} = \frac{(\zeta \tau)^p}{\tau^{\alpha+1}} \|u\|^p_{L^p_\theta}
\]
(2.10)
\[
\|u_\tau\|^q_{L^q_\theta} = \frac{\zeta^q}{\tau^{\theta+1}} \|u\|^q_{L^q_\theta}, \quad q \geq p.
\]

3. **Sharp Trudinger-Moser Inequality of Adimurthi-Druet Type**

This section is devoted to prove Theorem 1.1. We split the proof into two steps.

**Step 1:** Boundedness. We follow the argument of V.H. Nguyen [18]. Let \( u \in X^{1,p}_\infty \) be arbitrary. Set
\[
u_\tau(r) = u_\tau(\tau^{-1}r), \quad \tau > 0.
\]
Then, (2.10) yields
\[
\|u\|^p_{L^p_\theta} = \tau \|u_\tau\|^p_{L^p_\theta} \quad \text{and} \quad \|u_\tau\|^p_{L^p_\theta} = \|u_\tau\|^p_{L^p_\theta}.
\]
Consequently
\[
\sup_{\|u\|^p_{L^p_\theta} + \tau \|u\|^p_{L^p_\theta} = 1} \int_0^\infty \varphi_p(\mu_{\alpha,\theta}|u|^{p_\tau}) d\lambda_\theta = \frac{1}{\tau} \sup_{\|u\|^p_{L^p_\theta} = 1} \int_0^\infty \varphi_p(\mu_{\alpha,\theta}|u|^{p_\tau}) d\lambda_\theta
\]
which is finite due to [3, Theorem 1.1] (see also [11, Theorem 1.2]). Now, for \( \tau = 1-\eta \) and \( u \in X^{1,p}_\infty \) with \( \|u\| = 1 \), we set
\[
w = \frac{u}{(\|u\|^p_{L^p_\theta} + \tau \|u\|^p_{L^p_\theta})^{\frac{1}{p}}}
\]
Then one has \( \|w\|^p_{L^p_\theta} + \tau \|w\|^p_{L^p_\theta} = 1 \), and from above observation it follows
\[
\int_0^\infty \varphi_p(\mu_{\alpha,\theta}|w|^{p_\tau}) d\lambda_\theta \leq \frac{1}{1-\eta} \sup_{\|u\|^p_{L^p_\theta} = 1} \int_0^\infty \varphi_p(\mu_{\alpha,\theta}|u|^{p_\tau}) d\lambda_\theta.
\]
(3.1)

Since \( |u|^{p_\tau} \leq (1-\eta)\|u\|^p_{L^p_\theta}|w|^{p_\tau} \leq |w|^{p_\tau} \), it follows that \( (1+\eta)\|u\|^p_{L^p_\theta}|w|^{p_\tau} \leq |w|^{p_\tau} \) which together with (3.1) and [3, Theorem 1.1] completes the proof of the item (1).

**Step 2:** Sharpness. We will employ the Moser type sequence \( (v_n) \) given by
\[
v_n(r) = \begin{cases} \left( \frac{n}{\theta + 1} \right)^{\frac{1}{p}}  & \text{if} \quad 0 \leq r \leq e^{-\frac{n}{\theta+1}}, \\ \left( \frac{n}{\theta + 1} \right)^{-\frac{1}{p}} \ln \frac{1}{r} & \text{if} \quad e^{-\frac{n}{\theta+1}} < r < 1, \\ 0 & \text{if} \quad r \geq 1. \end{cases}
\]
(3.2)

It follows that
\[
\|v_n\|^p_{L^p_\theta} = 1 \quad \text{and} \quad \|v_n\|^p_{L^p_\theta} = \frac{\omega_{\theta}}{n(\theta + 1)p\omega_\alpha} \int_0^n s^p e^{-s} ds + O(n^{p-1}e^{-n}).
\]
Proposition 4.1. Let
\[ (4.1) \]
where
\[ R \]
Since
\[ \Gamma(p + 1) = \int_0^\infty s^p e^{-s} ds \quad \text{and} \quad \lim_{x \to \infty} \frac{e^x}{x^p} \int_x^\infty s^p e^{-s} ds = 1 \]
we can also write
\[ \left\| v_n \right\|_{L^p_\theta}^p = \frac{\omega_\theta \Gamma(p + 1)}{\omega_\alpha n(\theta + 1)^p} + O(n^{p-1}e^{-n}). \]
For \( \rho > 0 \), define \( v_{n,\rho}(r) = v_n(r/\rho) \) and \( w_{n,\rho} = v_{n,\rho}/\|v_{n,\rho}\| \). From (2.10) with \( \alpha = p - 1 \), we get
\[ (3.3) \quad \left\| v'_{n,\rho} \right\|_{L^p_\alpha} = 1 \quad \text{and} \quad \left\| v_{n,\rho} \right\|_{L^p_\theta}^p = \frac{\rho^{p+1}}{n} \left[ \frac{\omega_\theta \Gamma(p + 1)}{\omega_\alpha (\theta + 1)^p} + O(n^{p-1}e^{-n}) \right]. \]
Then
\[ \frac{1 + \|w_{n,\rho}\|_{L^p_\theta}^p}{\|v_{n,\rho}\|_{L^p_\theta}^p} = 1 + \frac{\|v_{n,\rho}\|_{L^p_\theta}^p}{\|w_{n,\rho}\|_{L^p_\theta}^p} = 1 + 2\|w_{n,\rho}\|_{L^p_\theta}^p = 1 - R_n, \]
where \( R_n = \|w_{n,\rho}\|_{L^p_\theta}^{2p}/(1 + \|v_{n,\rho}\|_{L^p_\theta}^p)^2 \). Hence, since \( \|w_{n,\rho}\|_{L^p_\theta} = 1 \) for \( n \) large enough we have
\[ AD(1, \mu, \alpha, \theta, \rho) \geq \int_0^\infty \varphi_p(\mu, \theta, 1 + \|w_{n,\rho}\|_{L^p_\theta}^p)^{1/p} \|w_{n,\rho}\|_{L^p_\theta}^p d\lambda_\theta \]
\[ \geq \rho^{\theta+1} \int_0^1 e^{-\frac{n}{\theta+1}} \varphi_p(1 - R_n) \mu, \theta, \|w_{n,\rho}\|_{L^p_\theta}^p d\lambda_\theta \]
\[ = \frac{\omega_\theta \rho^{\theta+1}}{\theta + 1} \left[ e^{n-nR_n} - \sum_{j=0}^N \frac{(1 - R_n)^j n^j}{j!} \right] e^{-n}. \]
From (3.3), we obtain \( nR_n \to 0 \) as \( n \to \infty \). Then, letting \( n \to \infty \) we obtain \( AD(1, \mu, \alpha, \theta, \rho) \geq \omega_\theta \rho^{\theta+1}/(\theta + 1) \) for any \( \rho > 0 \). Hence \( AD(1, \mu, \alpha, \theta) = \infty \).

4. Extremal function for Adimurthi-Druet type inequality

In this section we will show Theorem 1.2. Here, we are assuming the assumptions \( \alpha = p - 1 \geq 1 \) and \( \theta \geq \alpha \). We first prove a lower bound for \( AD(\eta, \mu, \alpha, \theta) \).

**Proposition 4.1.** Let \( p \geq 2 \) be an integer number and \( \eta \in [0, 1) \). Then
\[ AD(\eta, \mu, \alpha, \theta) > \begin{cases} \frac{\mu^{p-1}}{(p-1)!} (\eta + 1), & \text{if } p > 2 \text{ and } \mu \in (0, \mu_{1,\theta}] \\
\frac{\mu^{p-1}}{(p-1)!} (\eta + 1), & \text{if } p = 2 \text{ and } \mu \in \left( \frac{2(1 + 2\eta)}{(1 + \eta)^2 B_{2,\theta}}, \mu_{1,\theta} \right], 
\end{cases} \]
where
\[ \inf_{0 \neq u \in X^2_{\infty,\theta}(1,\theta)} \frac{\|u'\|_{L^2_\theta}}{\|u\|_{L^2_\theta}} = \frac{1}{B_{2,\theta}} = \frac{\inf_{0 \neq u \in X^2_{\infty,\theta}(1,\theta)} \|u'\|_{L^2_\theta}}{\|u\|_{L^2_\theta}}. \]
Proof: We follow the argument of Ishiwata [16]. Let \( u \in X_{1,p}^1(\alpha, \theta) \) such that \( \|u\|_t = 1 \), and set
\[
u_t(r) = r^{1/p}u(t^{1/p}r).
\]
From (2.10), we can easily show that
\[
\|u_t\|_{L^p_0} = t\|u\|_{L^p_0},
\]
\[
\|u_t\|_{L^p_0}^q = t^{q-p}\|u\|^q_{L^p_0}, \quad \forall q \geq p.
\]
In particular, if \( v_t = \xi_t \nu_t \) with \( \xi_t = (t + (1 - t)\|u\|_{L^p_0}^{-1/p})^{-1/p} \) with \( t > 0 \) small enough, we have
\[
\|v_t\| = 1 \quad \text{and} \quad \|v_t\|_{L^p_0}^q = \xi_t^q t^{q-p}\|u\|^q_{L^p_0}, \quad q \geq p.
\]
Noticing that \( \varphi_p(s) \geq \frac{k_0}{k_0+1} + \frac{k_0+1}{k_0+1}! \), \( s \geq 0 \) we obtain
\[
AD(\eta, \mu, \alpha, \theta) \geq \frac{\mu k_0}{k_0!} \left(1 + \eta \xi_t^p \|u\|_{L^p_0}^p \right)^{k_0+1} \|u\|^{k_0}_{L^p_0} \|u\|^{k_0+1}_{L^p_0} \xi_t^{k_0+1}
\]
\[
+ \frac{\mu k_0+1}{(k_0+1)!} \left(1 + \eta \xi_t^p \|u\|_{L^p_0}^p \right)^{k_0+1} \|u\|^{k_0+1}_{L^p_0} \xi_t^{k_0+1}.
\]
Since we are supposing that \( p \geq 2 \) is an integer number we have \( k_0 = p - 1 \). Thus,
\[
AD(\eta, \mu, \alpha, \theta) \geq \frac{\mu^{p-1}}{(p-1)!} h(t),
\]
where
\[
h(t) = \left(\xi_t^p + \eta \xi_t^{2p} \|u\|_{L^p_0}^p \right)^{p-1} \|u\|^{p-1}_{L^p_0} \xi_t^{p-1}.
\]
Since \( \xi_t \to 1/\|u\|_{L^p_0} \) as \( t \to 0^+ \), we have \( h(0) = 1 + \eta \). Thus, it remains to show that \( h'(t) > 0 \), if \( 0 < t \ll 1 \). In order to see this, note that for any \( q \geq p \) (recall \( \|u\|_{L^p_0}^p \))
\[
(4.2) \quad \frac{d}{dt} (\xi_t^q) = -\frac{2p}{p} \left(1 + (1 - t)\|u\|_{L^p_0}^p\right)^{-2+p} \left(1 - \|u\|_{L^p_0}^p\right) \to -\frac{p}{p} \|u\|_{L^p_0}^{(p+q)} \|u\|_{L^p_0}^{p-1} \text{ as } t \to 0.
\]
Therefore
\[
h'(t) = \frac{\mu}{p(p-1)} \left(\xi_t^p + \eta \xi_t^{2p} \|u\|_{L^p_0}^p \right)^{p-1} \|u\|^{p-1}_{L^p_0} \xi_t^{p-1} + f(t),
\]
where
\[
f(t) = \left\|u\right\|_{L^p_0}^p \left(\xi_t^p + \eta \xi_t^{2p} \|u\|_{L^p_0}^p \right)^{p-1} \|u\|^{p-1}_{L^p_0} \xi_t^{p-1}
\]
\[
\to -(1 + 2\eta) \frac{\left\|u\right\|_{L^p_0}^p}{\|u\|_{L^p_0}^p} \text{ as } t \to 0.
\]
Hence, if $p > 2$ we have
\[
\lim_{t \to 0} h'(t) = \infty
\]
and, thus $h'(t) > 0$, if $t > 0$ is small enough. If $p = 2$, also from (4.3), we have
\[
\lim_{t \to 0} h'(t) = \frac{\mu (1 + \eta)^2}{2} \frac{\|u\|_{L^2_0}^4}{\|u\|_{L^4_0}^4} - (1 + 2\eta) \frac{\|u'\|_{L^2_0}^2}{\|u\|_{L^4_0}^2}.
\]
Hence,
\[
\lim_{t \to 0} h'(t) > 0, \text{ if } \mu > \frac{2(1 + 2\eta)}{(1 + \eta)^2} \frac{\|u'\|_{L^2_0}^2}{\|u\|_{L^4_0}^2}.
\]
Since $1/B_{2,\theta}$ is attained (cf. [3, Proposition 7.1]) for some $u \in X^{1,2}_\infty$, with $\|u\| = 1$ our result is proved.

Let $(u_n) \subset X^{1,p}_\infty(\alpha, \theta)$ be a maximizing sequence for the supremum $AD(\eta, \mu, \alpha, \theta)$. From the Pólya-Szegö type principle in [3] (see also [4]), we can assume that each $u_n$ is a non-increasing function. Moreover, from [8, Lemma 2.2]) (see Remark 2.1), for any $R > 0$ and $q \in (1, \infty)$ we can assume that
\[
(4.4) \quad u_n \rightharpoonup u_0 \quad \text{in} \quad X^{1,p}_\infty, \quad u_n \to u_0 \quad \text{in} \quad L^q_\theta(0, R) \quad \text{and} \quad u_n(r) \to u_0(r) \quad \text{a.e in} \quad (0, \infty).
\]
In addition, since $0 < \|u_n\|_{L^p_\theta}^p \leq 1$, up to a subsequence, we can take $a \in [0, 1]$ such that
\[
(4.5) \quad \|u_n\|_{L^p_\theta}^p \to a.
\]

4.1. Proof of Theorem 1.2: Maximizers for the subcritical case. For $R > 1$, we set $v_n(r) = u_n(r) - u_n(R)$, with $r \in (0, R]$. Then $v_n \in X^{1,p}_R(\alpha, \theta)$ and since $\|u_n\| = 1$
\[
(4.6) \quad \|v_n'\|_{L^p_\theta}^p \leq 1 - \|u_n\|_{L^p_\theta}^p.
\]
For any $\epsilon > 0$, combining (2.3) with (2.4) we have
\[
(4.7) \quad \|u_n\|_{L^p_\theta}^p \leq (1 + \epsilon)^\frac{p}{p - 1} \|u_n\|_{L^p_\theta}^p + \frac{c_\epsilon}{R^{\frac{p}{p - 1}}},
\]
for some $c_\epsilon > 0$ depending only on $\epsilon, p$ and $\theta$. If $w_n = v_n/\|v_n\|_{L^p_\theta}$, then (4.6) and (4.7) imply
\[
\varphi_p(\mu(1 + \eta)\|u_n\|_{L^p_\theta}^{p - 1} |u_n|^{\frac{p - 1}{p}}) \leq e^{\mu(1+\eta)c_\epsilon R^{\frac{p}{p - 1}}} e^{\mu(1+\epsilon)^\frac{p}{p - 1} |w_n|_{L^p_\theta}^{p - 1}}.
\]
Hence, if $\epsilon > 0$ is small enough and $q > 1$ is close to 1 the inequality (1.5) ensures
\[
\sup_n \int_0^R \left[ \varphi_p(\mu(1 + \eta)\|u_n\|_{L^p_\theta}^{p - 1} |u_n|^{\frac{p - 1}{p}}) \right]^q \mathrm{d}\lambda_\theta < \infty.
\]
From Vitali’s convergence theorem
\[
\lim_{n \to \infty} \int_{0}^{R} \varphi_p \left( \mu (1 + \eta \|u_n\|^{p}_{L^p}) \frac{1}{p-1} |u_n|^{\frac{p}{p-1}} \right) d\lambda_{\theta} = \int_{0}^{R} \varphi_p \left( \mu (1 + \eta \alpha \frac{1}{p-1} |u_0|^{\frac{p}{p-1}} \right) d\lambda_{\theta}.
\]
(4.8)

Also, form (4.4), for any \( R > 0 \) we get
\[
\lim_{n \to \infty} \int_{0}^{R} (1 + \eta \|u_n\|^{p}_{L^p}) |u_n|^p d\lambda_{\theta} = (1 + \eta \alpha) \int_{0}^{R} |u_0|^p d\lambda_{\theta}.
\]
(4.9)

Now, we claim that
\[
AD(\eta, \mu, \alpha, \theta) = \begin{cases} 
\int_{0}^{\infty} \varphi_p \left( \mu (1 + \eta \alpha \frac{1}{p-1} |u_0|^{\frac{p}{p-1}} \right) d\lambda_{\theta}, & \text{if } p \notin \mathbb{N}, \\
\int_{0}^{\infty} \varphi_p \left( \mu (1 + \eta \alpha \frac{1}{p-1} |u_0|^{\frac{p}{p-1}} \right) d\lambda_{\theta} + \frac{\mu^{p-1}}{(p-1)!} (1 + \eta \alpha) \left( a - \|u_0\|^{p}_{L^p} \right) & \text{if } p \in \mathbb{N}.
\end{cases}
\]
(4.10)

Indeed, for any real number \( p \geq 2, \alpha = p - 1 \) and \( r \geq R \geq 1 \), (2.3) yields
\[
\varphi_p \left( \mu (1 + \eta \|u_n\|^{p}_{L^p}) \frac{1}{p-1} |u_n|^{\frac{p}{p-1}} \right) - \frac{\mu^{k_0}}{k_0!} \left( 1 + \eta \|u_n\|^{p}_{L^p} \right)^{k_0} |u_n|^{\frac{p k_0}{p - 1}} \\
\leq \frac{|u_n|^{p}_{L^p}}{R^{\frac{p}{p+1}}} \sum_{j=k_0+1}^{\infty} \frac{\mu^j}{j!} (1 + \eta)^{\frac{j}{p-1}} C^{\frac{j}{p-1}} \\
\leq C' \frac{|u_n|^{p}_{L^p}}{R^{\frac{p}{p+1}}},
\]
where \( C' \) does not dependent of \( n \) and \( R \). If \( p \in \mathbb{N} \) the above estimate yields
\[
\lim_{R \to \infty} \lim_{n \to \infty} \int_{R}^{\infty} \left[ \varphi_p \left( \mu (1 + \eta \|u_n\|^{p}_{L^p}) \frac{1}{p-1} |u_n|^{\frac{p}{p-1}} \right) - \frac{\mu^{p-1}}{(p-1)!} (1 + \eta \|u_n\|^{p}_{L^p}) |u_n|^p \right] d\lambda_{\theta} = 0.
\]

Hence, by splitting the integral on \((0, R)\) and \((R, \infty)\) and letting \( n \to \infty \) and then \( R \to \infty \), from (4.8) and (4.9) we obtain (4.10). If \( p \notin \mathbb{N} \), we must have \( k_0 > p - 1 \). Since \( u_n \) is a non-increasing function
\[
\|u_n\|^{p}_{L^p} \geq \int_{0}^{r} |u_n|^p d\lambda_{\theta} \geq |u_n(r)|^{p} \int_{0}^{r} d\lambda_{\theta}, \quad r > 0.
\]
Thus, there is \( c > 0 \) (independent of \( n \)) such that
\[
|u_n|^{\frac{k_0 p}{p - 1}} \leq \frac{c}{r^{(p-1)(k_0 - 1)}}, \quad r > 0.
\]
Hence, arguing as in (4.11), for \( r > R \) we can write

\[
(4.12) \quad \varphi_p \left( \mu (1 + \eta) \| u_n \|_{L^p_0}^{-\frac{1}{p-1}} \right) \leq \frac{c_2}{r^{(\theta+1)\frac{1}{p-1}}} + c_1 \| u_n \|^p R^{\frac{p-1}{p}},
\]

where \( c_1 \) and \( c_2 \) are independent of \( n \) and \( R \). By using \( k_0 > p - 1 \), from (4.12)

\[
\lim_{R \to \infty} \lim_{n \to \infty} \int_R^{\infty} \varphi_p \left( \mu (1 + \eta) \| u_n \|_{L^p_0}^{-\frac{1}{p-1}} \right) d\lambda_\theta \leq 0.
\]

Hence, by splitting the integral on \((0, R)\) and \((R, \infty)\) and letting \( n \to \infty \) and then \( R \to \infty \), from (4.8) we complete the proof of (4.10).

Now, if \( u_0 \equiv 0 \) then (4.10) yields

\[
0 < AD(\eta, \mu, \alpha, \theta) = \begin{cases} 
0, & \text{if } p \not\in \mathbb{N} \\
\frac{\mu^{p-1}}{(p-1)!} (1 + \eta) a \leq \frac{\mu^{p-1}}{(p-1)!} (1 + \eta), & \text{if } p \in \mathbb{N},
\end{cases}
\]

which contradicts the Proposition 4.1. Hence \( u_0 \not\equiv 0 \). If \( \tau = (a/\| u_0 \|_{L^p_0})^{1/(\theta+1)} \geq 1 \) and \( v_0(r) = u_0(r/\tau) \) then

\[
(4.13) \quad \| v_0 \|_{L^p_0} = \| u_0 \|_{L^p_0}^{-\theta+1} = a, \quad \text{and} \quad \| v_0' \|_{L^p_0} = \| u_0' \|_{L^p_0}.
\]

It follows that \( 1 \geq \lim \inf_n \| u_n \|^p \geq \| v_0 \|^p \). Therefore, if \( p \in \mathbb{N} \), by using (4.10)

\[
AD(\eta, \mu, \alpha, \theta) \geq \tau^{\theta+1} \int_0^\infty \varphi_p \left( \mu (1 + \eta) \| u_0 \|_{L^p_0}^{-\frac{1}{p-1}} \right) d\lambda_\theta = AD(\eta, \mu, \alpha, \theta) + (\tau^{\theta+1} - 1) \int_0^\infty \left[ \varphi_p \left( \mu (1 + \eta) \| u_0 \|_{L^p_0}^{-\frac{1}{p-1}} \right) \right]
\]

\[
- \frac{\mu^{p-1}}{(p-1)!} (1 + \eta) \| u_0 \|^p] d\lambda_\theta.
\]

Since \( u_0 \not\equiv 0 \), we obtain \( \tau = 1 \) which gives \( a = \| u_0 \|_{L^p_0} \). On the other hand, for \( p \not\in \mathbb{N} \) (4.10) yields

\[
AD(\eta, \mu, \alpha, \theta) \geq \int_0^\infty \varphi_p \left( \mu (1 + \eta) \| v_0 \|_{L^p_0}^{-\frac{1}{p-1}} \right) d\lambda_\theta = \tau^{\theta+1} \int_0^\infty \varphi_p \left( \mu (1 + \eta) \| u_0 \|_{L^p_0}^{-\frac{1}{p-1}} \right) d\lambda_\theta = \tau^{\theta+1} AD(\eta, \mu, \alpha, \theta),
\]

which gives \( \tau = 1 \) once again. Hence, for any real number \( p \geq 2 \) (if \( p = 2 \), according to Proposition 4.1, we must assume \( \mu > 2(1 + 2\eta)/(1 + \eta)^2B_{2,\theta} \)), we have the following

\[
AD(\eta, \mu, \alpha, \theta) = \int_0^\infty \varphi_p \left( \mu (1 + \eta) \| u_0 \|_{L^p_0}^{-\frac{1}{p-1}} \right) d\lambda_\theta.
\]

This together with the fact that \( \| u_0 \| \leq 1 \), implies \( \| u_0 \| = 1 \) and completes the proof.
4.2. Proof of Theorem 1.3: Maximizers for the critical case. From Theorem 1.2, for each $\epsilon > 0$ (small) there is a non-increasing function $u_\epsilon \in X^{1,p}_\infty$ with $\|u_\epsilon\| = 1$ such that

$$AD(\eta, \mu_{\alpha, \theta} - \epsilon, \alpha, \theta) = \int_0^\infty \varphi_p((\mu_{\alpha, \theta} - \epsilon)(1 + \eta \|u_\epsilon\|_L^p)^{\frac{1}{p-1}} |u_\epsilon|^{\frac{p}{p-1}}) d\lambda_\theta. \tag{4.14}$$

It is easy to see that the Lagrange multipliers theorem implies

$$\int_0^\infty |u_\epsilon'|^{p-2} u_\epsilon' \, d\lambda_\alpha = \frac{b_\epsilon}{d_\epsilon} \int_0^\infty \varphi_p'(\eta \|u_\epsilon\|_L^p) |u_\epsilon|^{\frac{p}{p-1}} v \, d\lambda_\theta$$

$$+ (c_\epsilon - 1) \int_0^\infty |u_\epsilon|^{p-1} v \, d\lambda_\theta \tag{4.15}$$

for all $v \in X^{1,p}_\infty$, where

$$\begin{aligned}
\eta_\epsilon &= \mu_\epsilon (1 + \eta \|u_\epsilon\|_L^p)^{\frac{1}{p-1}}, \text{ with } \mu_\epsilon = \mu_{\alpha, \theta} - \epsilon \\
b_\epsilon &= (1 + \eta \|u_\epsilon\|_L^p)/(1 + 2\eta \|u_\epsilon\|_L^p) \\
c_\epsilon &= \eta/(1 + 2\eta \|u_\epsilon\|_L^p) \\
d_\epsilon &= \int_0^\infty |u_\epsilon|^{\frac{p}{p-1}} \varphi_p'(\eta \|u_\epsilon\|_L^p) \, d\lambda_\theta.
\end{aligned} \tag{4.16}$$

In order to perform the blow up analysis method, we need to show that each maximizers $u_\epsilon$ belongs to $C^1[0, \infty)$. Indeed, we have the following:

Lemma 4.1. For each $\epsilon > 0$, we have $u_\epsilon \in C^1[0, \infty) \cap C^2(0, \infty)$.

Proof: The proof of this result proceeds along the same lines of [12, Lemma 6], and we limit ourselves to sketching a few differences. Firstly, given $\sigma > 0$ we will prove that

$$\lim_{r \to 0^+} r^\sigma [\varphi_p'(\eta \|u_\epsilon\|_L^p)|u_\epsilon|^{\frac{p}{p-1}} + (c_\epsilon - 1)|u_\epsilon|^{p-1}] = 0. \tag{4.17}$$

Since $0 \leq \varphi_p'(t) \leq e^t$, $t \geq 0$ it is sufficient to show that

$$\lim_{r \to 0^+} r^\sigma \varphi(r) = 0, \text{ with } \varphi =: |u_\epsilon|^{\frac{p}{p-1}} e^{\eta \|u_\epsilon\|_L^p} + (c_\epsilon - 1)|u_\epsilon|^{p-1}. \tag{4.18}$$

Let $1 < q < p$ such that $(\sigma + 1)q > p$ and $m = p/q$. Then the Hölder inequality and $\alpha = p - 1$ yield

$$\int_0^1 r^\sigma |u_\epsilon'|^m \, dr \leq C\|u_\epsilon\|^m, \tag{4.19}$$

for some $C > 0$ depending only on $\alpha$, $\sigma$ and $m$. Note that

$$\varphi' = \frac{1}{p-1} |u_\epsilon|^{\frac{2-p}{p-1}} u_\epsilon' e^{\eta \|u_\epsilon\|_L^p} + \frac{\eta \epsilon p}{p-1} |u_\epsilon|^{\frac{2-p}{p-1}} u_\epsilon' e^{\eta \|u_\epsilon\|_L^p}$$

$$+ (c_\epsilon - 1)(p-1)|u_\epsilon|^{p-2} u_\epsilon'.$$
In addition, without loss of generality we can assume $u_\varepsilon > 0$ in $(0, 1)$ and $\lim_{r \to 0} u_\varepsilon(r) = +\infty$. Hence, there exists $C > 0$ such that

$$|\varphi'| \leq C \left[ |u'_\varepsilon|e^{\bar{\eta}_\varepsilon |u_\varepsilon|^p} + |u_\varepsilon|^{\frac{2}{p-m}}|u'_\varepsilon|e^{\bar{\eta}_\varepsilon |u_\varepsilon|^p} + |u_\varepsilon|^{p-2}|u'_\varepsilon| \right],$$

on $(0, 1/2)$. Now, for $m > 1$ given by (4.19) and $q_1, q_2 > 1$ such that

$$\frac{2}{q_1(p-1)} + \frac{1}{m} + \frac{1}{q_2} = 1$$

the Hölder inequality, Remark 2.1 (recall $u_\varepsilon \in L^{q_i}_\delta(0, 1)$), (4.19) and Lemma 2.1-(i) imply

$$\int_0^{1/2} r^\sigma |u_\varepsilon|^{\frac{2}{p-1}} |u'_\varepsilon| e^{\bar{\eta}_\varepsilon |u_\varepsilon|^p} dr$$

$$\leq \left( \int_0^1 r^\sigma |u_\varepsilon|^{q_1} dr \right)^{\frac{2}{q_1(p-1)}} \left( \left( \int_0^1 r^\sigma |u'_\varepsilon|^m dr \right)^{-\frac{1}{m}} \left( \left( \int_0^1 r^\sigma e^{q_2 \bar{\eta}_\varepsilon |u_\varepsilon|^p} dr \right)^{\frac{1}{q_2}} \right)^{\frac{1}{q_2}} \right)^{\frac{1}{q_2}} \leq c,$$

for some $c > 0$ depending only on $\alpha, \sigma$ and $m$. Analogously,

$$\int_0^{1/2} r^\sigma |u'_\varepsilon| e^{\bar{\eta}_\varepsilon |u_\varepsilon|^p} dr \leq C \left( \int_0^1 r^\sigma e^{\frac{m-1}{m} |u_\varepsilon|^p} dr \right)^{-\frac{m-1}{m}} \leq c_1$$

and

$$\int_0^{1/2} r^\sigma |u_\varepsilon|^{p-2} |u'_\varepsilon| dr \leq C \left( \int_0^1 r^\sigma e^{\frac{m-1}{m} |u_\varepsilon|^p} dr \right)^{-\frac{m-1}{m}} \leq c_2.$$

Thus, from (4.20), (4.21) and (4.22) it follows that

$$\varphi' \in L^1_\sigma(0, 1/2).$$

Hence, in the same line of [12, Lemma 6], combining (4.19) and (4.23) we conclude that (4.18) holds.

Following [6], for each $r, \delta > 0$ we consider the test function $v_\delta \in X^1_\infty$ given by

$$v_\delta(s) = \begin{cases} 1 & \text{if } 0 < s \leq r, \\ 1 + \frac{1}{\delta}(r - s) & \text{if } r \leq s \leq r + \delta, \\ 0 & \text{if } s \geq r + \delta. \end{cases}$$

By using $v_\delta$ in (4.15) and letting $\delta \to 0$, we get the integral equation

$$(-u'_\varepsilon(r))^{p-1} = \frac{1}{\omega_\alpha r^\alpha} \int_0^r \frac{b_\varepsilon}{d_\varepsilon} \varphi'_{\varepsilon p}(\bar{\eta}_\varepsilon |u_\varepsilon(s)|^{p-1}) |u_\varepsilon|^{\frac{1}{p-1}} + (c_\varepsilon - 1) |u_\varepsilon(s)|^{p-1} \right) d\lambda_\theta.$$

It follows that $u_\varepsilon \in C^2(0, \infty)$. In addition, since we are supposing $\theta \geq \alpha$, the L'Hospital rule and (4.17) imply $u'_\varepsilon(0) = 0$, and thus $u_\varepsilon \in C^1[0, \infty)$. \hfill \blacksquare

**Lemma 4.2.** We have $AD(\eta, \mu_\varepsilon, \alpha, \theta) \to AD(\eta, \mu_{\alpha, \theta}, \alpha, \theta)$, as $\varepsilon \to 0$. In particular, $\liminf_{\varepsilon \to 0} d_\varepsilon > 0$. 
Proof: For any \( u \in X_{\infty}^{1,p} \) with \( \|u\| \leq 1 \), the Fatou’s Lemma yields
\[
\liminf_{\epsilon \to 0} AD(\eta, \mu_\epsilon, \alpha, \theta) \geq \int_0^\infty \varphi_p(\mu_\alpha, \theta (1 + \eta \|u\|_{L_{\theta}^p})^\frac{1}{p-1} |u|^{\frac{p}{p-1}}) d\lambda_\theta.
\]
Since \( u \) is taken arbitrary, we get
\[
\liminf_{\epsilon \to 0} AD(\eta, \mu_\epsilon, \alpha, \theta) \geq AD(\eta, \mu_\alpha, \alpha, \theta).
\]
Also, we clearly have
\[
\limsup_{\epsilon \to 0} AD(\eta, \mu_\epsilon, \alpha, \theta) \leq AD(\eta, \mu_\alpha, \alpha, \theta).
\]
Then we can write
\[
(4.26) \quad \lim_{\epsilon \to 0} AD(\eta, \mu_\epsilon, \alpha, \theta) = AD(\eta, \mu_\alpha, \alpha, \theta).
\]
Noticing that
\[
t \varphi'_p(t) = \sum_{j=k_0}^{\infty} \frac{t^j}{(j-1)!} \geq \sum_{j=k_0}^{\infty} \frac{t^j}{j!} = \varphi_p(t), \quad t \geq 0
\]
we can see that
\[
d_\epsilon \geq \frac{1}{\eta_\epsilon} AD(\eta, \mu_\epsilon, \alpha, \theta) \geq \frac{1}{\mu_\epsilon (1 + \eta)^\frac{1}{p-1}} AD(\eta, \mu_\epsilon, \alpha, \theta).
\]
Thus, using (4.26), we get
\[
(4.27) \quad \liminf_{\epsilon \to 0} d_\epsilon \geq \frac{1}{\mu_\alpha (1 + \eta)^\frac{1}{p-1}} AD(\eta, \mu_\alpha, \alpha, \theta) > 0.
\]

In the sequel, we do not distinguish sequence and subsequence. As well as in (4.4) and (4.5) we have
\[
(4.28) \quad u_\epsilon \rightharpoonup u_0 \text{ in } X_{\infty}^{1,p}(\alpha, \theta), \quad u_\epsilon \to u_0 \text{ in } L_{\theta}^p(0, R) \text{ and } u_\epsilon(r) \to u_0(r) \text{ a.e in } (0, \infty),
\]
for any \( R > 0 \) and \( q \in (1, \infty) \). In addition, we can pick \( a \in [0, 1] \) such that
\[
(4.29) \quad \|u_\epsilon\|_{L_{\theta}^p} \to a.
\]
Moreover, since \( u_\epsilon \in C^1[0, \infty) \) is a non-increasing function we can define
\[
(4.30) \quad a_\epsilon := u_\epsilon(0) = \max_{r \in (0, \infty)} u_\epsilon(r).
\]

Lemma 4.3. If \((a_\epsilon)\) is bounded, then \( AD(\eta, \mu_\alpha, \alpha, \theta) \) is attained.

Proof: Assume that there is \( c > 0 \) such that \( a_\epsilon \leq c \), for any \( \epsilon > 0 \). Hence, \( u_\epsilon(r) \leq c \), for \( r \in (0, \infty) \) and uniformly on \( \epsilon \). Consequently, for any \( R > 0 \) there is \( q > 1 \) such that
\[
\sup_{\epsilon > 0} \int_0^R \left[ \varphi_p(\eta_\epsilon |u_\epsilon|^{\frac{p}{p-1}}) \right]^q d\lambda_\theta < \infty.
\]
From (4.28) and the Vitali’s convergence theorem

\[
(4.31) \quad \lim_{\epsilon \to 0} \int_0^R \varphi_p(\eta_k | u_\epsilon |^{p-1}) \, d\lambda_\theta = \int_0^R \varphi_p(\mu_{\alpha, \theta}(1 + \eta a)^{p-1} | u_0 |^{p-1}) \, d\lambda_\theta.
\]

Also,

\[
(4.32) \quad \lim_{\epsilon \to 0} \int_0^R (1 + \eta \| u_\epsilon \|_{L^p_\theta}) | u_\epsilon |^p \, d\lambda_\theta = (1 + \eta a) \int_0^R | u_0 |^p \, d\lambda_\theta.
\]

For \( u_0 \) and \( a \) are given by (4.28) and (4.29), as in (4.10) we will prove that

\[
(4.33) \quad AD(\eta, \mu_{\alpha, \theta}, \alpha, \theta) = \left\{ \begin{array}{ll}
\int_0^\infty \varphi_p(\mu_{\alpha, \theta}(1 + \eta a)^{p-1} | u_0 |^{p-1}) \, d\lambda_\theta, & \text{if } p \not\in \mathbb{N} \\
\int_0^\infty \varphi_p(\mu_{\alpha, \theta}(1 + \eta a)^{p-1} | u_0 |^{p-1}) \, d\lambda_\theta + \mu_{\alpha, \theta}^{p-1} (1 + \eta a)(a - \| u_0 \|_{L^p_\theta}^p) & \text{if } p \in \mathbb{N}.
\end{array} \right.
\]

Firstly, (2.3) yields

\[
(4.34) \quad \varphi_p(\eta_k | u_\epsilon |^{p-1}) - \eta_k^{k_0} u_\epsilon^{k_0} | u_\epsilon |^{p-1} \leq C' | u_\epsilon |^p \frac{R}{R + k_0^{1/p}}, \quad r \geq R \geq 1
\]

where \( C' \) does not dependent of \( \epsilon \) and \( R \). Hence, if \( p \in \mathbb{N} \), i.e. \( k_0 = p - 1 \) we obtain

\[
(4.35) \quad \lim_{R \to \infty} \lim_{\epsilon \to 0} \int_0^\infty \left[ \varphi_p(\eta_k | u_\epsilon |^{p-1}) - \eta_k^{p-1} (p-1)! | u_\epsilon |^p \right] \, d\lambda_\theta = 0.
\]

Combining (4.26), (4.31), (4.32) and (4.35) we get (4.33) if \( p \in \mathbb{N} \). Now, suppose that \( p > 2 \) is not an integer number, i.e. \( k_0 > p - 1 \). Then, arguing as in (4.12), for \( r > R \) we can write

\[
(4.36) \quad \varphi_p(\eta_k | u_\epsilon |^{p-1}) \leq \frac{C_2}{r^{(\theta+1)}} + \frac{C_1 | u_\epsilon |^p}{R^{p+1}}.
\]

Hence,

\[
(4.37) \quad \lim_{R \to \infty} \lim_{\epsilon \to 0} \int_0^\infty \varphi_p(\eta_k | u_\epsilon |^{p-1}) \, d\lambda_\theta = 0.
\]

Combining (4.26), (4.31) and (4.37) we get (4.33) for the case \( p \not\in \mathbb{N} \). If \( u_0 \equiv 0 \) then (4.33) implies

\[
0 < AD(\eta, \mu_{\alpha, \theta}, \alpha, \theta) = \left\{ \begin{array}{ll}
0, & \text{if } p \not\in \mathbb{N} \\
\mu_{\alpha, \theta}^{p-1} (1 + \eta a)(1 + \eta), & \text{if } p \in \mathbb{N},
\end{array} \right.
\]

which contradicts the Proposition 4.1. Thus, \( u_0 \not\equiv 0 \) and by setting \( v_0(r) = u_0(r/\tau) \), with \( \tau = (a/\| u_0 \|_{L^p_\theta}^{1/(\theta+1)})^{1/\theta} \) we can argue as in the proof of Theorem 1.2 to conclude the result. ■
In view of Lemma 4.3, without loss of generality, in the sequel we are supposing the condition
\[(4.38) \quad \lim_{\epsilon \to \infty} a_\epsilon = +\infty.\]

To complete the proof of Theorem 1.3, we shall apply the two-step strategy of Carleson-Chang [5], namely: Supposing the condition \[(4.38), \] we exhibit an explicit constant \(U(\eta, \alpha, \theta)\) so that

**Step 1:** (Lemma 4.12)
\[(4.39) \quad AD(\eta, \mu_{\alpha, \theta}, \alpha, \theta) \leq U(\eta, \alpha, \theta), \quad \text{for any } 0 \leq \eta < 1.\]

**Step 2:** (Lemma 4.14) For each \(\eta\) small enough, there is \(v_\eta \in X_{1, p}^\infty\) with \(\|v_\eta\| = 1\) so that
\[(4.40) \quad AD(\eta, \mu_{\alpha, \theta}, \alpha, \theta) \geq \int_0^\infty \varphi_p(\mu_{\alpha, \theta}(1 + \eta\|v_\eta\|)^{\frac{p}{p-1}|v_\eta|^{\frac{p}{p-1}}}) \, d\lambda_\alpha > U(\eta, \alpha, \theta).\]

The contradiction given by \((4.39)\) and \((4.40)\) excludes \((4.38)\) and Theorem 1.3 follows from Lemma 4.3. The proof of both Step 1 and Step 2 will be divided into a series of lemmas.

**Lemma 4.4.** The sequence \((u_\epsilon)\) is concentrating at the origin, i.e.,
\[(4.41) \quad \|u_\epsilon'\|_{L_p^\infty} \leq 1, \quad u_\epsilon \to 0 \quad \text{and} \quad \lim_{\epsilon \to 0} \int_0^r |u_\epsilon'|^p \, d\lambda_\alpha = 0, \quad \text{for any } r_0 > 0.\]

**Proof:** Fix \(r_0 > 0\). We claim that
\[(4.42) \quad \lim_{\epsilon \to 0} \int_0^{r_0} |u_\epsilon'|^p \, d\lambda_\alpha = 1.\]

Of course we have \(\|u_\epsilon'\|_{L_p^\infty(0, r_0)} \leq 1\). By contradiction, suppose that there is \(0 \leq \delta < 1\) such that \(\|u_\epsilon'\|_{L_p^\infty(0, r_0)} \to \delta\), as \(\epsilon \to 0\). By setting \(v_\epsilon = u_\epsilon - u_\epsilon(r_0)\) on \((0, r_0]\), we obtain \(v_\epsilon \in X_{r_0}^{1, p}(\alpha, \theta)\) with
\[(4.43) \quad \lim_{\epsilon \to 0} \int_0^{r_0} |v_\epsilon'|^p \, d\lambda_\alpha = \lim_{\epsilon \to 0} \int_0^{r_0} |u_\epsilon'|^p \, d\lambda_\alpha = \delta.\]

For any \(\sigma > 0\), from \((2.3)\) and \((2.4)\)
\[|u_\epsilon|^\frac{p}{p-1} \leq (1 + \sigma)^\frac{1}{p} |u_\epsilon|^\frac{p}{p-1} + c_\sigma r_0^{\frac{p-1}{p}}, \quad \text{in } (0, r_0]\]
for some \(c_\sigma > 0\) depending only on \(\sigma, p\) and \(\theta\). Define \(w_\epsilon = v_\epsilon / \|v_\epsilon\|_{L_p^\infty}\). Then
\[\varphi_p(\eta, |u_\epsilon|^\frac{p}{p-1}) \leq c \mu_{(1+\sigma)^\frac{1}{p}}[(1+\eta|u_\epsilon|_{L_p^\infty})|v_\epsilon'|_{L_p^\infty}]^{\frac{p}{p-1}} |w_\epsilon|^{\frac{p}{p-1}}, \quad \text{for some } c = c(\alpha, \theta, r_0, \sigma) > 0.\]

Note that \(\|u_\epsilon\| = 1\) yields
\[|(1 + \eta|u_\epsilon|_{L_p^\infty})|v_\epsilon'|_{L_p^\infty} \leq (1 + \eta - \eta|v_\epsilon'|_{L_p^\infty})|v_\epsilon'|_{L_p^\infty}.\]

Hence, from \((4.43)\)
\[
\lim_{\epsilon \to 0} \mu_{\epsilon}(1 + \sigma)^\frac{1}{p} [(1 + \eta|u_\epsilon|_{L_p^\infty})|v_\epsilon'|_{L_p^\infty}]^{\frac{p}{p-1}} \leq \mu_{\alpha, \theta}(1 + \sigma)^\frac{1}{p} [U_{\alpha, \theta} + \eta \delta]^{\frac{1}{p-1}} < \mu_{\alpha, \theta}
\]
for $\sigma > 0$ small enough (since $\eta, \delta < 1$). Hence, we can choose $q > 1$ such that

$$q^2 \mu_\alpha(1 + \sigma)^{\frac{1}{p}}[(1 + \eta\|u_\epsilon\|_{L^p_\theta})\|u_\epsilon\|_{L^{p-1}_{L^p_\theta}}]^{\frac{1}{p-1}} < \mu_{\alpha, \theta}$$

for all $\epsilon > 0$ small enough. Thus, the above estimates and (1.5) imply

$$\int_0^{r_0} \left[ \varphi'_p(\eta_\epsilon|u_\epsilon|^{\frac{p}{q}}) \right]^q d\lambda_\theta \leq c^q \int_0^{r_0} \epsilon^{\mu_{\alpha, \theta}|u_\epsilon|^{q-1}} d\lambda_\theta < c_1$$

where $c_1 > 0$ does not depend on $\epsilon$. Set

$$f_\epsilon(r) = \frac{1}{\omega_{\alpha}} \frac{b_\epsilon}{d_\epsilon} \varphi'_p(\eta_\epsilon|u_\epsilon(r)|^{\frac{p}{q}})|u_\epsilon(r)|^{\frac{1}{p-1}} + (c_\epsilon - 1)|u_\epsilon(r)|^{p-1}.$$  

From Lemma 4.2, there is $c_2 > 0$ such that

$$\frac{1}{c_2} \int_0^{r_0} |f_\epsilon|^q d\lambda_\theta \leq \left( \int_0^{r_0} \left[ \varphi'_p(\eta_\epsilon|u_\epsilon|^{\frac{p}{q}}) \right]^q |u_\epsilon|^{q-1} d\lambda_\theta \right)^\frac{1}{q} \left( \int_0^{r_0} |u_\epsilon|^{q-1} d\lambda_\theta \right)^\frac{q-1}{q}$$

$$+ \left( \int_0^{r_0} |u_\epsilon|^{(p-1)} \frac{q^{q-1}}{q^q} d\lambda_\theta \right)^\frac{q-1}{q} \left( \int_0^{r_0} d\lambda_\theta \right)^\frac{1}{q} \leq c,$$

where we have chosen $q > 1$ (close 1) such that $q^2/(q-1) > (p-1)p$ and (4.44) hold. In particular,

$$\int_0^{r_0} |f_\epsilon|^q d\lambda_\theta \leq c,$$

for all $\epsilon > 0$ small enough. Now, from (4.25), for any $0 \leq r < r_0$ we can write

$$u_\epsilon(r) \leq u_\epsilon(r_0) + \left( \frac{\omega_{\theta}}{\theta + 1} \right)^{\frac{1}{p-1}} \left( \int_0^{r_0} |f_\epsilon|^q d\lambda_\theta \right)^\frac{1}{q} \left( \int_0^{r_0} s^{(\theta+1)q^{-2}(p-1)-1} ds \right)^\frac{q-1}{q}.$$

Hence, from (2.3) and (4.46) we obtain $a_\epsilon = u_\epsilon(0) \leq c$. This, contradicts (4.38) and proves (4.42).

Next we will prove (4.41). By contradiction, suppose that there are $0 < A < 1$ and $r_0 > 0$ such that

$$\lim_{\epsilon \to 0} \int_{r_0}^{\infty} |u'_\epsilon|^p d\lambda_\alpha > A.$$  

Thus,

$$1 = \|u_\epsilon\|_{L^p_\theta}^p + \|u'_\epsilon\|_{L^p_\theta}^p > \|u_\epsilon\|_{L^p_\theta}^p + \int_0^{r_0} |u'_\epsilon|^p d\lambda_\alpha + A,$$

and consequently

$$\int_0^{r_0} |u'_\epsilon|^p d\lambda_\alpha < 1 - A.$$

Hence, we get

$$\lim_{\epsilon \to 0} \int_0^{r_0} |u'_\epsilon|^p d\lambda_\alpha = \delta \leq 1 - A < 1,$$
which contradicts (4.42). So, (4.41) holds. Finally, from (4.41) and (4.42)
\[
1 = \lim_{\epsilon \to 0} \left[ \| u_\epsilon \|_{L^p_\theta}^p + \int_0^1 |u_\epsilon'|^p \, \text{d} \lambda_\alpha + \int_1^{\infty} |u_\epsilon'|^p \, \text{d} \lambda_\alpha \right] \geq \lim_{\epsilon \to 0} \| u_\epsilon \|_{L^p_\theta}^p + 1.
\]
Consequently \( u_\epsilon \to 0 \) in \( L^p_\theta \) and, from (4.28), we get \( u_\epsilon \rightharpoonup u_0 \equiv 0 \).

In order to investigate the behavior of the sequence \((u_\epsilon)\) around of blowing up point \( r = 0 \), we consider the following auxiliary functions

\[
\begin{align*}
\eta(r) = \frac{u_\epsilon(r \epsilon)}{a_\epsilon}, & \quad r \in (0, \infty) \\
w_\epsilon(r) = \frac{d_\epsilon}{b_\epsilon} (u_\epsilon(r \epsilon) - a_\epsilon)
\end{align*}
\]

where
\[
r_\epsilon^{\theta+1} = \frac{d_\epsilon}{b_\epsilon} a_\epsilon e^{-\frac{\mu}{\eta} - \frac{\theta}{\alpha} a_\epsilon}.
\]

**Lemma 4.5.** For any \( 0 < \mu < \mu_{\alpha, \theta} \), we have
\[
\lim_{\epsilon \to 0} r_\epsilon^{\theta+1} a_\epsilon^{-\frac{\mu}{\eta}} e^{\frac{\mu}{\eta}} = 0.
\]

*In particular, \( \lim_{\epsilon \to 0} r_\epsilon^{\theta+1} = 0 \).*

**Proof:** For any \( R > 0 \)
\[
\begin{align*}
r_\epsilon^{\theta+1} a_\epsilon^{-\frac{\mu}{\eta}} e^{\frac{\mu}{\eta}} &= \frac{e^{(\mu - \eta) a_\epsilon^{\frac{\mu}{\eta}}} + \int_0^R |u_\epsilon| \, \text{d} \lambda_\theta \\
&\quad + \int_{R}^{\infty} |u_\epsilon| \, \text{d} \lambda_\theta \\
&= I_1 + I_2.
\end{align*}
\]

Since \( \eta_\epsilon \to \mu_{\alpha, \theta} > \mu \), for \( \epsilon \) sufficiently small we obtain
\[
(\eta_\epsilon - \mu)(|u_\epsilon(r)| e^{\frac{\mu}{\eta}} - a_\epsilon e^{\frac{\mu}{\eta}}) \leq 0, \quad r > 0.
\]

Then, using that \( \varphi'_p(t) \leq e^t, \ t \geq 0 \), we can write
\[
I_1 \leq \frac{1}{b_\epsilon} \int_0^R |u_\epsilon| e^{\mu a_\epsilon^{\frac{\mu}{\eta}}} \, \text{d} \lambda_\theta.
\]

Now, by choosing \( q > 1 \) such that \( q \mu < \mu_{\alpha, \theta} \) and since \( b_\epsilon \to 1 \), from Lemma 2.1-(i) and the convergence in (4.28) with \( u_0 \equiv 0 \), we obtain
\[
I_1 \leq \frac{1}{b_\epsilon} \left( \int_0^R |u_\epsilon| e^{\frac{\mu}{\eta} (q b_\epsilon - 1)} \, \text{d} \lambda_\theta \right)^{\frac{q-1}{q}} \left( \int_0^R e^{q \mu a_\epsilon^{\frac{\mu}{\eta}}} \, \text{d} \lambda_\theta \right)^{\frac{1}{q}} \to 0, \ \text{as} \ \epsilon \to 0.
\]
From (2.3), for any \( r \geq R \) we have \(|u_\epsilon(r)|^{\frac{p}{p-1}} \leq c/R^{\frac{\theta+1}{p}}\) for some \( c > 0 \) which is independent of \( \epsilon \) and \( R \). Hence, since \( \eta \|u_\epsilon\|_{L^p_\theta} \leq 1 \) and \( \mu_\epsilon \leq \mu_{\alpha,\theta} \) we can write

\[
\eta_\epsilon |u_\epsilon|^\frac{p}{p-1} \leq \mu_{\alpha,\theta} 2^{\frac{1}{p-1}} C R^{-\frac{\theta+1}{p}}, \quad \forall r \geq R.
\]

Hence, noticing that for any \( T > 0 \) we have \( t\varphi'_p(t) \leq t^{k_0}e^t \leq t^{k_0}e^T \), for \( 0 \leq t \leq T \) we can write (use \( k_0 \geq p - 1 \))

\[
|u_\epsilon|^\frac{p}{p-1} \varphi'_p(\eta_\epsilon |u_\epsilon|^\frac{p}{p-1}) \leq \frac{e^{\mu_{\alpha,\theta} 2^{\frac{1}{p-1}} C R^{-\frac{\theta+1}{p}}} \eta_\epsilon^{k_0} |u_\epsilon|^{k_0 p}}{\eta_\epsilon} \leq c(\alpha, \eta, \theta, R) |u_\epsilon|^p, \quad \forall r \geq R
\]

where \( R > 0 \) is large such that \(|u_\epsilon(r)| \leq 1\) for \( r \geq R \). Hence, since \( u_\epsilon \to 0 \) in \( L^p_\theta \) we obtain

\[
I_2 \leq c(\alpha, \eta, \theta, R) e^{(\mu - \eta_\epsilon)\frac{p}{p-1} \int_0^\infty |u_\epsilon|^p d\lambda_\theta} \to 0, \quad \text{as} \quad \epsilon \to 0.
\]

Combining (4.49), (4.50) and (4.52) we get the result.

**Lemma 4.6.** Let \( v_\epsilon \) and \( w_\epsilon \) given by (4.47). Set

\[
w(r) = -\frac{p-1}{\mu_{\alpha,\theta}} \ln \left( 1 + c_{\alpha,\theta} r^{\frac{\theta+1}{p-1}} \right), \quad \text{with} \quad c_{\alpha,\theta} = \left( \frac{\omega_\theta}{\theta + 1} \right)^{\frac{1}{p-1}}.
\]

Then \( v_\epsilon \to 1 \) in \( C^1_{loc}[0, \infty) \) and \( w_\epsilon \to w \) in \( C^0_{loc}[0, \infty) \). In addition,

\[
\int_0^\infty e^{\frac{p}{p-1} \mu_{\alpha,\theta} w(r)} d\lambda_\theta = 1.
\]

**Proof:** By using integral equation (4.25), from the definition in (4.47) and (4.48), it is easy to see that

\[
\omega_\alpha |v'_\epsilon(r)|^{p-1} = \frac{1}{r^{\alpha}} e^{-\eta_\epsilon r^{\frac{p}{p-1}}} \int_0^r \left[ \varphi'_p(\eta_\epsilon |v_\epsilon(s)|^{\frac{p}{p-1}}) |v_\epsilon(s)|^{\frac{1}{p-1}} \right] d\lambda_\theta + \frac{(c_\epsilon - 1) r^{\frac{\theta+1}{p-1}}}{r^{\alpha}} \int_0^r |v_\epsilon(s)|^{p-1} d\lambda_\theta.
\]

Note that

\[
\varphi'_p(t) = \begin{cases} 
  e^t, & \text{if } p = 2 \\
  e^t - \sum_{j=0}^{k_0-2} \frac{t^j}{j!}, & \text{if } p > 2, \quad (t \geq 0).
\end{cases}
\]
Thus, we can write
\[
\omega_\alpha |v'_\varepsilon(r)|^{p-1} = \frac{1}{a_\varepsilon^p} \frac{1}{r^\alpha} \int_0^r e^{\eta_\alpha a_\varepsilon^{\frac{p}{p-1}} (|v'|_p^{p-1}) - 1} |v'|_{p-1}^{\frac{1}{p-1}} d\lambda_\theta \\
- \frac{e^{-\eta_\alpha a_\varepsilon^{\frac{p}{p-1}}} a_\varepsilon^{\frac{p}{p-1}}}{a_\varepsilon^p} \sum_{j=0}^{k_0-2} \frac{\eta_\alpha^{j} a_\varepsilon^{\frac{p}{p-1}}}{j!} \frac{1}{r^\alpha} \int_0^r |v'|_p^{p-1} d\lambda_\theta \\
+ \frac{(c_\varepsilon - 1)r_\varepsilon^{\theta+1}}{r^\alpha} \int_0^r |v'|_p^{p-1} d\lambda_\theta,
\]
(4.56)
where the second term on the right hand side does not appear if \( p = 2 \). Fix \( r_0 > 0 \). Since \( \theta \geq \alpha \), \( |v_\varepsilon| \leq 1 \), \( a_\varepsilon \to \infty \), \( c_\varepsilon \to \eta \) and \( r_\varepsilon^{\theta+1} \to 0 \), we conclude from (4.56) that \( v'_\varepsilon \to 0 \) uniformly on \([0, r_0]\). Since \( v_\varepsilon(0) = 1 \), we obtain \( v_\varepsilon \to 1 \) in \( C^1_{loc}[0, \infty) \). In addition, since \( w'_\varepsilon = a_\varepsilon^{\frac{p}{p-1}} v'_\varepsilon \), from (4.56) we also have
\[
\omega_\alpha |w'_\varepsilon(r)|^{p-1} = \frac{1}{a_\varepsilon^p} \frac{1}{r^\alpha} \int_0^r e^{\eta_\alpha a_\varepsilon^{\frac{p}{p-1}} (|v'|_p^{p-1}) - 1} |v'|_{p-1}^{\frac{1}{p-1}} d\lambda_\theta \\
- \frac{e^{-\eta_\alpha a_\varepsilon^{\frac{p}{p-1}}} a_\varepsilon^{\frac{p}{p-1}}}{a_\varepsilon^p} \sum_{j=0}^{k_0-2} \frac{\eta_\alpha^{j} a_\varepsilon^{\frac{p}{p-1}}}{j!} \frac{1}{r^\alpha} \int_0^r |v'|_p^{p-1} d\lambda_\theta \\
+ \frac{(c_\varepsilon - 1)a_\varepsilon^{p_r^{\theta+1}}}{r^\alpha} \int_0^r |v'|_p^{p-1} d\lambda_\theta.
\]
(4.57)
Analogously, since \( \theta \geq \alpha \), \( |v_\varepsilon| \leq 1 \) and \((c_\varepsilon - 1)a_\varepsilon^{p_r^{\theta+1}} \to 0 \) we have from (4.57) that \( w'_\varepsilon \) is bounded in \( C^0[0, r_0] \). Since \( w_\varepsilon(0) = 0 \), we have that \((w_\varepsilon)\) is uniformly bounded equicontinuous sequence in \( C[0, r_0] \). Thus, the Ascoli-Arzelà theorem gives \( w_\varepsilon \to w \) uniformly for some \( w \in C[0, r_0] \). Next, we will show that \( w \) has the expression given by (4.53). To get this, we observe
\[
a_\varepsilon^{\frac{p}{p-1}} (|v_\varepsilon(s)|^{\frac{p}{p-1}} - 1) = \frac{p}{p-1} w_\varepsilon(s) [1 + O_\varepsilon(|v_\varepsilon| - 1)].
\]
(4.58)
By integrating in (4.57) on \((0, r)\) we obtain
\[
w_\varepsilon(r) = - \int_0^r \left( \frac{1}{\omega_\alpha t^\alpha} \int_0^t g_\varepsilon(s) d\lambda_\theta \right)^{\frac{1}{p-1}} dt,
\]
where
\[
g_\varepsilon(s) = e^{\eta_\alpha a_\varepsilon^{\frac{p}{p-1}} (|v'|_p^{p-1}) - 1} |v'|_{p-1}^{\frac{1}{p-1}} + (c_\varepsilon - 1)a_\varepsilon^{p_r^{\theta+1}} |v'|_p^{p-1} \\
- \frac{e^{-\eta_\alpha a_\varepsilon^{\frac{p}{p-1}}} a_\varepsilon^{\frac{p}{p-1}}}{a_\varepsilon^p} \sum_{j=0}^{k_0-2} \frac{\eta_\alpha^{j} a_\varepsilon^{\frac{p}{p-1}}}{j!} |v'|_{p-1}^{\frac{p}{p-1}}.
\]
Now, fixed \( t > 0 \) arbitrary, from (4.58) we obtain
\[
\lim_{\varepsilon \to 0} g_\varepsilon(s) = e^{\mu_\alpha t^{\frac{p}{p-1}} w(s)}, \quad \forall s \in (0, t).
\]
Hence,
\[ w(r) = - \int_0^r \left( \frac{1}{\omega_\alpha r^\alpha} \int_0^t e^{\mu_{\alpha, \theta} \frac{p}{p-1} w(s)} \, d\lambda_\theta \right) \frac{1}{r^\alpha} \, dt. \]

It is easy to show that \( w \) must satisfy the equation
\[
\begin{align*}
- \omega_\alpha (r^\alpha |w'|^{p-2} w')' &= \omega_\theta r^\theta e^{|w(r)|} \mu_{\alpha, \theta} w(r) \quad \text{on} \quad [0, \infty) \\
w(0) &= w'(0) = 0.
\end{align*}
\]

Noticing that the unique solution for the above ODE is the function given in (4.53) we get the desired expression for \( w \). Performing the change of variable \( s = c_\alpha \theta r^{(\theta+1)/(p-1)} \) and using the identities (see [2])
\[
\Gamma(1) = 1, \quad \Gamma(x+1) = x\Gamma(x) \quad \text{and} \quad \int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}} ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0
\]
we obtain (4.54). \( \square \)

**Lemma 4.7.** For each \( c > 1 \), set \( u_{\epsilon, c} = \min \{ u_\epsilon, \frac{a_\epsilon}{c} \} \). Then
\[
\lim_{\epsilon \to 0} \int_0^\infty |u'_{\epsilon, c}|^p \, d\lambda_\alpha = \frac{1}{c}.
\]

**Proof:** We have \( v_\epsilon \to 1 \) in \( C^1_{\text{loc}}[0, \infty) \) and then \( u_\epsilon = a_\epsilon (1 + o_\epsilon(R)) \) uniformly on \( (0, r_\epsilon R) \). Then
\[
\frac{\int_0^{r_\epsilon R} |u_\epsilon| q \, d\lambda_\theta}{\int_0^{r_\epsilon R} e^{\eta_\epsilon |u_\epsilon|^{p-1}} \, d\lambda_\theta} = o_\epsilon(R), \quad \forall q > 1
\]
where \( o_\epsilon(R) \) means that \( \lim_{\epsilon \to 0} o_\epsilon(R) = 0 \) if \( R \) is fixed. Since \( \|u_\epsilon\|_{L^p_\theta} = o_\epsilon(1) \), from (4.15), we can write
\[
\int_0^\infty |u'_{\epsilon, c}|^p \, d\lambda_\alpha = \frac{b_\epsilon}{d_\epsilon} \int_0^\infty \left| u_\epsilon \right|^\frac{1}{p-1} \varphi_p'(\eta_\epsilon \left| u_\epsilon \right|^\frac{p}{p-1}) u_{\epsilon, c} \, d\lambda_\theta + o_\epsilon(1).
\]

In view of (4.55) and (4.59), for \( \epsilon > 0 \) small enough such that \( (0, r_\epsilon R) \subset \{ u_\epsilon \geq \frac{a_\epsilon}{c} \} \), we have
\[
\int_0^\infty |u'_{\epsilon, c}|^p \, d\lambda_\alpha \geq \frac{b_\epsilon a_\epsilon^{\frac{1}{p-1}}}{d_\epsilon} \frac{1}{c} (1 + o_\epsilon(R)) \int_0^{r_\epsilon R} e^{\eta_\epsilon |u_\epsilon|^{p-1}} \, d\lambda_\theta + o_\epsilon(1).
\]

By using the change of variable \( r = r_\epsilon s \), from (4.48) we get
\[
\int_0^\infty |u'_{\epsilon, c}|^p \, d\lambda_\alpha \geq \frac{1}{c} (1 + o_\epsilon(R)) \int_0^R e^{\eta_\epsilon a_\epsilon^{\frac{1}{p-1}} (|u_\epsilon|^{p-1} - 1)} \, d\lambda_\theta + o_\epsilon(1).
\]

Using (4.54) and (4.58), setting \( \epsilon \to 0 \) and then \( R \to \infty \) we obtain
\[
\int_0^\infty |u'_{\epsilon, c}|^p \, d\lambda_\alpha \geq \frac{1}{c}.
\]

Similarly, we obtain
\[
\int_0^\infty \left| \left( u_\epsilon - \frac{a_\epsilon}{c} \right)^+ \right|^p \, d\lambda_\alpha \geq \frac{c - 1}{c},
\]
where \( u^+ = \max \{ u, 0 \} \). Also, since \( \| u_\epsilon \| = 1 \) and \( \| u_\epsilon \|_{L_0^p}^p = o_\epsilon(1) \) we have

\[
(4.62) \quad \int_0^\infty |u_{\epsilon,c}'|^p d\lambda_\alpha + \int_0^\infty |((u_\epsilon - \frac{\alpha_\epsilon}{c})')|^p d\lambda_\alpha = \int_0^\infty |u_\epsilon'|^p d\lambda_\alpha = 1 + o_\epsilon(1).
\]

Combining (4.60) (4.61) and (4.62), we conclude the proof.

Lemma 4.8. It holds

\[
AD(\eta, \mu_\alpha, \alpha, \theta) = \lim_{\epsilon \to 0} \frac{d_\epsilon}{\epsilon^{\frac{p}{p-1}}}. 
\]

In particular, \( d_\epsilon / a_\epsilon^\sigma \to \infty \), for any \( \sigma < p/(p-1) \). Also, \( a_\epsilon^{p/(p-1)}/d_\epsilon \) is bounded.

Proof: Let \( u_{\epsilon,c} = \min \{ u_\epsilon, a_\epsilon/c \} \), \( c > 1 \) be given by Lemma 4.7. Since \( \varphi_p'(t) \geq \varphi_p(t), t \geq 0 \) we obtain

\[
AD(\eta, \mu_\epsilon, \alpha, \theta) \leq \int_0^\infty \varphi_p(\eta_\epsilon |u_{\epsilon,c}|^{\frac{p}{p-1}}) d\lambda_\theta + \frac{c_\epsilon^{\frac{p}{p-1}}}{a_\epsilon^{\frac{p}{p-1}}} \int_0^\infty |u_\epsilon|^{\frac{p}{p-1}} \varphi_p(\eta_\epsilon |u_{\epsilon,c}|^{\frac{p}{p-1}}) d\lambda_\theta 
\]

\[
\leq \int_0^\infty \varphi_p(\eta_\epsilon |u_{\epsilon,c}|^{\frac{p}{p-1}}) d\lambda_\theta + \frac{c_\epsilon^{\frac{p}{p-1}}}{a_\epsilon^{\frac{p}{p-1}}} 
\]

From Lemma 4.7, \( \| u_{\epsilon,c} \|_p \to 1/c < 1 \) as \( \epsilon \to 0 \). Hence, using (1.5) and arguing as in (4.8), (4.9) we get

\[
(4.63) \quad \lim_{\epsilon \to 0} \int_0^\infty \varphi_p(\eta_\epsilon |u_{\epsilon,c}|^{\frac{p}{p-1}}) d\lambda_\theta = 0.
\]

Letting \( \epsilon \to 0 \) and then \( c \searrow 1 \), we obtain

\[
(4.64) \quad AD(\eta, \mu_\alpha, \alpha, \theta) \leq \liminf_{\epsilon \to 0} AD(\eta, \mu_\epsilon, \alpha, \theta) \leq \liminf_{\epsilon \to 0} \frac{d_\epsilon}{a_\epsilon^{\frac{p}{p-1}}}. 
\]

Noticing that \( \varphi_p'(t) = \varphi_p(t) + t^{k_0-1}/((k_0 - 1)! \) we can see that

\[
d_\epsilon \leq a_\epsilon^{\frac{p}{p-1}} AD(\eta, \mu_\epsilon, \alpha, \theta) + \frac{\nu_0^{k_0-1}}{(k_0 - 1)!} \| u_\epsilon \|_{L_0^p}. 
\]

Hence, using that \( \| u_\epsilon \|_{L_0^p} \to 0, q \geq p \) and (4.64) and the above inequality we complete the proof.

Lemma 4.9. For any \( v \in C_0(0, \infty) \), we have

\[
(4.65) \quad \lim_{\epsilon \to 0} \int_0^\infty \frac{a_\epsilon b_\epsilon}{d_\epsilon} |u_\epsilon|^{\frac{1}{p-1}} \varphi_p'(\eta_\epsilon |u_{\epsilon,c}|^{\frac{p}{p-1}}) v d\lambda_\theta = v(0).
\]

In particular, the sequence \( \{ F_\epsilon \} \) given by

\[
(4.66) \quad F_\epsilon = \frac{a_\epsilon b_\epsilon}{d_\epsilon} |u_\epsilon|^{\frac{1}{p-1}} \varphi_p'(\eta_\epsilon |u_{\epsilon,c}|^{\frac{p}{p-1}}), \ \epsilon > 0
\]
satisfies

\[ \lim_{\epsilon \to 0} \int_0^\rho F_\epsilon \, d\lambda_\theta = 1, \quad \rho > 0. \]

**Proof:** For \( R > 0 \) and \( c > 1 \) we can divide \([0, \infty) = A_1 \cup A_2 \cup A_3\) into the three disjoints sets

\[ A_1 = \{ u_\epsilon > a_\epsilon/c \} \cap [r_\epsilon R, \infty), \quad A_2 = \{ u_\epsilon \leq a_\epsilon/c \} \cap [r_\epsilon R, \infty) \quad \text{and} \quad A_3 = (0, r_\epsilon R). \]

We split the integral in (4.65) into three integrals over these sets and denote by \( I^1_\epsilon \), \( I^2_\epsilon \) and \( I^3_\epsilon \), respectively. Setting \( m = \max_{r \in [0, \infty)} |v(r)| \) and using (4.55) we can write

\[ I^1_\epsilon \leq mb_\epsilon c \left[ 1 - \frac{1}{b_\epsilon} \int_0^R |v_\epsilon|^{p-1} e^{\eta_\epsilon a_\epsilon^{\frac{p}{p-1}} (|v_\epsilon|^{p-1} - 1)} \, d\lambda_\theta + o_\epsilon(R) \right], \]

where we use \( a_\epsilon \to \infty \). From Lemma 4.6, letting \( \epsilon \to 0 \) and then \( R \to \infty \), we obtain \( I^1_\epsilon \to 0 \) as \( \epsilon \to 0 \). Also, from Lemma 4.8 we have \( a_\epsilon/d_\epsilon \to 0 \), then if \( L > 0 \) is chosen such that \( \supp v \subset (0, L) \), we obtain

\[ I^2_\epsilon \leq \frac{a_\epsilon b_\epsilon m}{d_\epsilon} \int_{A_2} |u_\epsilon|^{\frac{1}{p-1}} \varphi_p (\eta_\epsilon |u_\epsilon|^{\frac{p}{p-1}}) \, d\lambda_\theta + \frac{\eta_\epsilon^{k_0 - 1}}{(k_0 - 1)!} \int_0^L |u_\epsilon|^{\frac{(k_0 - 1)p + 1}{p-1}} \, d\lambda_\theta \]
\[ \leq \frac{a_\epsilon^{p-1} b_\epsilon m}{d_\epsilon} \frac{1}{c^{p-1}} \int_0^\infty \varphi_p (\eta_\epsilon |u_\epsilon|^{\frac{p}{p-1}}) \, d\lambda_\theta + o_\epsilon(1). \]

From Lemma 4.8, \( a_\epsilon^{p/(p-1)}/d_\epsilon \) is bounded, then using (4.63) we get \( I^2_\epsilon \to 0 \). Finally, for some \( \tau \in [0, R] \)

\[ I^3_\epsilon = \frac{a_\epsilon^{p-1} b_\epsilon \theta^{p+1}}{d_\epsilon} v(r_\epsilon, \tau) \int_0^R |v_\epsilon|^{\frac{1}{p-1}} \varphi_p' (\eta_\epsilon a_\epsilon^{\frac{p}{p-1}} |v_\epsilon|^{\frac{p}{p-1}}) \, d\lambda_\theta \]
\[ = v(r_\epsilon, \tau) \left[ \int_0^R |v_\epsilon|^{\frac{1}{p-1}} e^{\eta_\epsilon a_\epsilon^{\frac{p}{p-1}} (|v_\epsilon|^{\frac{p}{p-1}} - 1)} \, d\lambda_\theta \right] \]
\[ - e^{-\eta_\epsilon a_\epsilon^{\frac{p}{p-1}}} \sum_{j=0}^{k_0 - 2} \frac{\eta_\epsilon^{j+1} a_\epsilon^{\frac{j+1}{p-1}}}{j!} \int_0^R |v_\epsilon|^{\frac{p+1}{p-1}} \, d\lambda_\theta \].

By (4.54) and (4.58), letting \( \epsilon \to 0 \) and then \( R \to \infty \) we obtain \( I^3_\epsilon \to v(0) \) which proves (4.65). To get (4.67), we choose \( v \in C^0_c(0, \infty) \) such that \( v \geq 0 \) and \( v \equiv 1 \) in \([0, \rho]\). For \( \epsilon > 0 \) small enough we have \( r_\epsilon R < \rho \) and thus \( (\rho, \infty) \subset A_1 \cup A_2 \). So, (4.67) follows from \( I^1_\epsilon \to 0 \) and \( I^3_\epsilon \to 0 \) and (4.65).

Next, we employ [10, Lemma 9] to ensure the existence of the Green-type functions to equation (4.15).
Lemma 4.10. Let $p \geq 2$, $1 < q < p$, $0 \leq \eta < 1$ and $R > 0$. Set $g_\epsilon = \frac{1}{\epsilon^{1-\frac{1}{q}}} u_\epsilon$, $\epsilon > 0$. Then there is $g_\eta$ such that $(g_\epsilon)$ converges weakly to $g_\eta$ in $W^{1,p}_R(\alpha, \theta)$ and $g_\eta$ satisfies the equation

\begin{equation}
\omega_\alpha r^\alpha |g_\eta(r)|^{p-1} + \int_0^r |g_\eta|^p \, d\lambda_\theta = 1 + \eta \int_0^r |g_\eta|^{p-1} \, d\lambda_\theta, \quad r > 0.
\end{equation}

In addition, $g_\epsilon \rightarrow g_\eta$ in $C^0_{loc}(0, \infty)$ and $g_\epsilon' \rightarrow g_\eta'$ in $L^p_\alpha(r_1, \infty)$, for all $r_1 > 0$.

**Proof:** By using (4.15), for any $v \in X^{1,p}_R$ we obtain

\begin{equation}
\int_0^\infty g_\epsilon' |g_\epsilon|^p |g_\epsilon'|^{p-2} \, d\lambda_\alpha = \int_0^\infty F_\epsilon v \, d\lambda_\theta + (c_\epsilon - 1) \int_0^\infty |g_\epsilon|^{p-1} \, d\lambda_\theta,
\end{equation}

where $F_\epsilon$ is given by (4.66). Firstly, for any $R > 0$ we claim that

\begin{equation}
\sup_{\epsilon > 0} \int_0^R |g_\epsilon|^{p-1} \, d\lambda_\theta \leq c.
\end{equation}

By contradiction, assume that there exists $R > 0$ such that $\lim_{\epsilon \rightarrow 0} \|g_\epsilon\|_{L^{p-1}_\alpha(0,R)} = \infty$. Setting $h_\epsilon = g_\epsilon/\|g_\epsilon\|_{L^{p-1}_\alpha(0,R)}$ on $(0,R)$, we have $\|h_\epsilon\|_{L^{p-1}_\alpha(0,R)} = 1$ and $h_\epsilon' = g_\epsilon'/\|g_\epsilon\|_{L^{p-1}_\alpha(0,R)}$. Also, from (4.69)

\begin{equation}
\int_0^\infty |h_\epsilon'|^p |h_\epsilon'|^{p-2} h_\epsilon' \, d\lambda_\alpha = \int_0^\infty \tilde{F}_\epsilon v \, d\lambda_\theta
\end{equation}

where

\begin{equation}
\tilde{F}_\epsilon = \frac{F_\epsilon}{\|g_\epsilon\|_{L^{p-1}_\alpha(0,R)}} + (c_\epsilon - 1)\|g_\epsilon\|^{p-1}.
\end{equation}

From (4.67), we can see that $(\tilde{F}_\epsilon)$ is bounded in $L^1_\theta(0,R)$. Let $\tilde{h}_\epsilon = h_\epsilon - h_\epsilon$ on $(0,R)$. Then each $\tilde{h}_\epsilon \in X^{1,p}_R(\alpha, \theta)$ and satisfies

\begin{equation}
\int_0^R |\tilde{h}_\epsilon'|^p |\tilde{h}_\epsilon'|^{p-2} \, d\lambda_\alpha = \int_0^R |h_\epsilon'|^p |h_\epsilon'|^{p-2} h_\epsilon' \, d\lambda_\alpha = \int_0^R \tilde{F}_\epsilon v \, d\lambda_\theta, \forall v \in X^{1,p}_R(\alpha, \theta).
\end{equation}

Applying [10, Lemma 9], for any $1 < q < p$ we get

\begin{equation}
\|h_\epsilon\|_{L^q_\alpha(0,R)} = \|\tilde{h}_\epsilon\|_{L^q_\alpha(0,R)} \leq C(\alpha, q, R, c_0)
\end{equation}

where where $c_0$ is an upper bound of $(\tilde{F}_\epsilon)$ in $L^1_\theta(0,R)$. Then, if $\bar{q} = p - 1$ (cf. Ramark 2.1), we have

\begin{equation}
\|h_\epsilon\|_{W^{1,\bar{q}}_R} = (\|h_\epsilon\|_{L^{\bar{q}}_\alpha(0,R)} + \|h_\epsilon\|_{L^{\bar{q}}_\theta(0,R)})^{1/\bar{q}} \leq (C(\alpha, \bar{q}, R, c_0) + 1)^{1/\bar{q}}.
\end{equation}

Hence, $(h_\epsilon)$ bounded in $W^{1,\bar{q}}_R(\alpha, \theta)$. Also, since $\theta \geq \alpha$ and $\alpha = p - 1$ we have $\alpha - \bar{q} + 1 = 1 > 0$ and $\theta \geq \alpha - \bar{q} = 0$. Further,

\begin{equation}
\bar{q}' = \frac{\theta + 1}{\alpha - \bar{q} + 1} = (\theta + 1)(p - 1) \geq p, \forall p \geq 2.
\end{equation}

Hence, from (2.1), it follows that the continuous embedding

\begin{equation}
W^{1,\bar{q}}_R(\alpha, \theta) \hookrightarrow L^q_\theta(0,R), \quad 1 < q \leq \bar{q}'.
\end{equation}
Note that \( p \leq q^* \) and, from (4.73), \((h_\epsilon)\) is bounded in \( W^{1,\pi}_R(\alpha, \theta) \). Then (4.74) yields

\[
(4.75) \quad \sup_{\epsilon > 0} \|h_\epsilon\|_{L^{q}_\theta(0,R)} \leq C_0, \quad \forall 1 < q < p.
\]

Combining (4.72) and (4.75), we conclude that \((h_\epsilon)\) bounded in \( W^{1,q}_R(\alpha, \theta) \), for any \( 1 < q < p \). Therefore,

\[
(4.76) \quad h_\epsilon \rightharpoonup h \quad \text{weakly in} \quad W^{1,q}_R(\alpha, \theta), \quad 1 < q < p.
\]

But, we have \( \alpha - q + 1 > \alpha - p + 1 = 0 \) and \( \theta \geq \alpha > \alpha - q \), for all \( 1 < q < p \). According with (2.1) (cf. Remark 2.1) we have the compact embedding

\[
W^{1,q}_R(\alpha, \theta) \hookrightarrow L^s(0,R), \quad 1 < s < q^*(\alpha, \theta, q) = \frac{(\theta + 1)q}{\alpha - q + 1}.
\]

Since \( \alpha - p + 1 = 0 \), we have \( q^*(\alpha, \theta, q) > p \) if \( q \) is sufficiently near \( p \). In particular, using that \((h_\epsilon)\) is bounded in \( W^{1,q}_R(\alpha, \theta) \) and (4.76) we obtain

\[
(4.77) \quad h_\epsilon \rightarrow h \quad \text{in} \quad L^s(0,R), \quad 1 < s < p.
\]

Combining (4.71), (4.76) and (4.77), we get

\[
(4.78) \quad \int_0^R |h'|^{p-2}h'v' \, d\lambda_\alpha = (\eta - 1) \int_0^R |h|^{p-1}v \, d\lambda_\theta, \quad \forall v \in X^{1,p}_R(\alpha, \theta).
\]

Analogous to (4.25), by using the test function \( v_\delta \) given by (4.24) and noticing that \( h \) is a non-increasing function, we can write

\[
(4.79) \quad |h'(r)|^{p-1} = (-h'(r))^{p-1} = \frac{\eta - 1}{\omega_\alpha r^\alpha} \int_0^r |h|^{p-1} \, d\lambda_\theta, \quad r \in (0, R].
\]

Since \( \|h\|_{L^{p-1}_\theta} = \lim_{\epsilon \to 0} \|h_\epsilon\|_{L^{p-1}_\theta} = 1 \), from (4.79), it follows that

\[
|h'(R)|^{p-1} = \frac{\eta - 1}{\omega_\alpha R^\alpha} < 0
\]

which is a contradiction. Hence, (4.70) holds. Thus, from Lemma 4.9 and (4.70), by setting \( f_\epsilon = F_\epsilon + (c_\epsilon - 1)|g_\epsilon|^{p-1} \) we get that the sequence \((f_\epsilon)\) is bounded in \( L^1(0,R) \) for all \( R > 0 \). From (4.69) (setting \( v \equiv 0 \) on \( [R, \infty) \)) we obtain

\[
(4.80) \quad \int_0^R |g_\epsilon|^{p-2}g_\epsilon v' \, d\lambda_\alpha = \int_0^R f_\epsilon v \, d\lambda_\theta, \quad \forall v \in X^{1,p}_R(\alpha, \theta).
\]

Using [10, Lemma 9], we get

\[
(4.81) \quad \|g_\epsilon\|_{L^q_\theta(0,R)} \leq C(\alpha, q, R, c_0), \quad 1 < q < p,
\]

where \( c_0 \) is an upper bound of \( (f_\epsilon) \) in \( L^1(0,R) \). Now, from (4.70), we can argue such as in (4.73), (4.74) and (4.75) to conclude that \((g_\epsilon)\) is bounded in \( W^{1,q}_R(\alpha, \theta) \), \( 1 < q < p \). Hence, there exists \( g_\eta \in W^{1,q}_R(\alpha, \theta) \) such that

\[
(4.82) \quad g_\epsilon \rightharpoonup g_\eta \quad \text{weakly in} \quad W^{1,q}_R(\alpha, \theta), \quad 1 < q < p.
\]
Further, from the compact embedding (4.76), for any $R > 0$

\begin{equation}
(4.83) 
\quad g_\varepsilon \rightarrow g_\eta \text{ in } L_0^{p-1}(0, R) \text{ and } g_\varepsilon(r) \rightarrow g_\eta(r) \text{ a.e in } (0, R).
\end{equation}

We proceed to show that $g_\eta$ satisfies the equation (4.68). From (4.25), we obtain

\begin{equation}
(4.84) 
\quad -g_\varepsilon'(r) = \frac{1}{\omega_\alpha(r)} \left[ \int_r^\infty (F_\varepsilon + (c_\varepsilon - 1)|g_\varepsilon|^{p-1})d\lambda_\theta \right]^{\frac{1}{p-1}}, \quad r > 0.
\end{equation}

Fixed $\rho > 0$ arbitrarily, by integrating (4.84) on $(r, \rho)$ we obtain

\begin{equation}
(4.85) 
\quad g_\varepsilon(r) = g_\varepsilon(\rho) + \frac{1}{\omega_\alpha(r)} \int_r^\rho \frac{1}{t} \left[ \int_t^\infty (F_\varepsilon + (c_\varepsilon - 1)|g_\varepsilon|^{p-1})d\lambda_\theta \right]^{\frac{1}{p-1}} dt.
\end{equation}

Using Lemma 4.9 and (4.83),

\begin{equation}
(4.86) 
\quad g_\eta(r) = g_\eta(\rho) + \frac{1}{\omega_\alpha(r)} \int_r^\rho \frac{1}{t} \left[ 1 + (\eta - 1) \int_t^\rho |g_\eta|^{p-1}d\lambda_\theta \right]^{\frac{1}{p-1}} dt.
\end{equation}

By differentiating this equation we obtain (4.68). Now, let $[r_1, R] \subset (0, \infty)$ be an arbitrary compact interval. From (4.65) and (4.84) we obtain $|g'_\varepsilon| \leq c/r_1$ on $[r_1, R]$, where $c > 0$ does not depend on $\varepsilon$. In addition, by (4.83) we can pick $\rho_0 > R$ such that $g_\varepsilon(\rho_0) \rightarrow g_\eta(\rho_0)$ as $\varepsilon \rightarrow 0$. This fact combined with (4.85) and (4.65) ensure $|g_\varepsilon(r)| \leq c_1 + c_2 \ln(\rho_0/r_1)$ on $[r_1, R]$, with $c_1, c_2 > 0$ do not depend on $\varepsilon$. Hence, the Ascoli-Arzellà theorem and (4.83) imply that $(g_\varepsilon)$ converges to $g_\eta$ in $C^0[r_1, R]$. Also, from (4.65) and (4.84) we have $|g'_\varepsilon(r)| \leq c/r_1$ on $[r_1, \infty)$. Also, by using (4.84) and (4.86) we conclude that $g'_\varepsilon(r) \rightarrow g'_\eta(r)$ a.e in $(r_1, \infty)$. Hence, from the Lebesgue dominated convergence theorem we have $g'_\varepsilon \rightarrow g'_\eta$ in $L_0^p(r_1, \infty)$. \hfill \blacksquare

**Remark 4.1.** From (4.68), we can see that $g_\eta$ satisfies the equation

\begin{equation}
(4.87) 
\quad \int_0^\infty |g'_\eta|^p - 2g'_\eta v'd\lambda_\alpha + \int_0^\infty |g_\eta|^{p-1}v d\lambda_\theta = \delta_0(v) + \eta \int_0^\infty |g_\eta|^{p-1}v d\lambda_\theta,
\end{equation}

for any $v \in X^p \cap C[0, \infty)$, where $\delta_0$ is the Dirac measure concentrated at origin $r = 0$.

**Lemma 4.11.** Let $g_\eta$ be given by Lemma 4.10. Then there exists

\[ \mathcal{A}_\eta = \lim_{r \rightarrow 0} \left[ g_\eta(r) + \frac{\theta + 1}{\mu_{\alpha, \theta}} \ln r \right]. \]

Also, for some $z \in C^1(0, \infty)$ and $z(r) = O(r^{\theta+1} \ln r^{p-1})$ as $r \rightarrow 0$, $g_\eta$ takes the form

\begin{equation}
(4.88) 
\quad g_\eta(r) = -\frac{\theta + 1}{\mu_{\alpha, \theta}} \ln r + \mathcal{A}_\eta + z(r), \quad 0 < r \leq 1.
\end{equation}

**Proof:** For $t > 0$, we set $h_{g_\eta}(t) = (1 - \eta) \int_0^t |g_\eta|^{p-1}d\lambda_\theta$. We claim that

\begin{equation}
(4.89) 
\quad \lim_{t \rightarrow 0} \frac{h_{g_\eta}(t)}{t} = 0.
\end{equation}
Indeed, by (4.86) we have $t^\sigma g_\eta(t) \to 0$ as $t \to 0$, for $\sigma > 0$. Then, since $g_\eta$ belongs to $L^{p-1}_\theta(0, R)$, with $R > 0$, the L’Hospital rule yields (4.89). Now, we note that

\begin{equation}
[1 - h_{g_\eta}(t)]^{\frac{1}{p-1}} = 1 - \frac{1}{p-1} h_{g_\eta}(t) + E_2(t), \quad t > 0
\end{equation}

where

\[ E_2(t) = - \frac{p-2}{2!(p-1)^2} (1 - \tau_0 h_{g_\eta}(t))^{-\frac{2p-3}{p-1}} h_{g_\eta}^2(t), \]

for some $\tau_0 = \tau_0(t) \in (0, 1)$. In view of (4.89) we also have

\begin{equation}
\lim_{t \to 0} E_2(t) = 0.
\end{equation}

From (4.86) and (4.90), for any $0 < r \leq 1$

\begin{equation}
g_\eta(r) = g_\eta(1) - \frac{\theta + 1}{\mu_{\alpha, \theta}} \ln r - \frac{\theta + 1}{\mu_{\alpha, \theta}} \frac{1}{p-1} \int_r^1 \frac{h_{g_\eta}(t)}{t} dt + \frac{\theta + 1}{\mu_{\alpha, \theta}} \int_r^1 \frac{E_2(t)}{t} dt.
\end{equation}

Hence, from (4.89) and (4.91), there exists

\begin{equation}
\mathcal{A}_\eta = \lim_{r \to 0} \left[ g_\eta(r) + \frac{\theta + 1}{\mu_{\alpha, \theta}} \ln r \right]
\end{equation}

\begin{equation}
= g_\eta(1) - \frac{\theta + 1}{\mu_{\alpha, \theta}} \frac{1}{p-1} \int_0^1 \frac{h_{g_\eta}(t)}{t} dt + \frac{\theta + 1}{\mu_{\alpha, \theta}} \int_0^1 \frac{E_2(t)}{t} dt.
\end{equation}

From (4.92) we also can write

\begin{equation}
g_\eta(r) = - \frac{\theta + 1}{\mu_{\alpha, \theta}} \ln r + \mathcal{A}_\eta + z(r)
\end{equation}

with

\[ z(r) = \frac{\theta + 1}{\mu_{\alpha, \theta}} \frac{1}{p-1} \int_0^r \frac{h_{g_\eta}(t)}{t} dt - \frac{\theta + 1}{\mu_{\alpha, \theta}} \int_0^r \frac{E_2(t)}{t} dt. \]

We observe that $z(0) = 0$. In addition, the L’Hospital rule and (4.94) yield

\begin{equation}
\lim_{r \to 0} \frac{h_{g_\eta}(r)}{r^{\theta+1} |\ln r|^{p-1}} = \frac{\omega_\theta}{\omega_\alpha} \frac{1 - \eta}{\theta + 1}.
\end{equation}

Thus, we obtain

\begin{equation}
\lim_{r \to 0} \frac{\int_0^r \frac{h_{g_\eta}(t)}{t} dt}{r^{\theta+1} |\ln r|^{p-1}} = \frac{\omega_\theta}{\omega_\alpha} \frac{1 - \eta}{(\theta + 1)^2}.
\end{equation}

Since $E_2(t) = O(h_{g_\eta}^2(t))$ as $t \to 0$, from the above argument we can also write

\[ \lim_{r \to 0} \frac{\int_0^r \frac{E_2(t)}{t} dt}{r^{\theta+1} |\ln r|^{p-1}} = 0. \]

Then we get

\[ z(r) = O(r^{\theta+1} |\ln r|^{p-1}) + o(r^{\theta+1} |\ln r|^{p-1}), \quad \text{as} \quad r \to 0. \]
Lemma 4.12. For $A_0$ is given by Lemma 4.11, we have
\[ AD(\eta, \mu_{\alpha, \theta}, \alpha, \theta) \leq \frac{\omega_\theta}{\theta + 1} e^{\mu_{\alpha, \theta} A_0 + \gamma + \Psi(p)}, \]
where $\Psi(x) = \frac{d}{dx} (\ln \Gamma(x))$ is the digamma function and $\gamma = -\Psi(1)$ is the Euler-Mascheroni constant.

Proof: Fix $\rho > 0$. From the same argument in (4.51), for $r \geq \rho$ we can write
\[ |u_\epsilon(r)|^{\frac{p}{p-1}} \varphi_p(\eta_\epsilon |u_\epsilon(r)|^{\frac{p}{p-1}}) \leq C(\alpha, \eta, \theta) e^{\mu_{\alpha, \theta} C^2} \rho^{-\frac{d+1}{p-1}} |u_\epsilon(r)|^{\frac{b_\rho}{p-1}}. \]
Using Lemma 4.8, the convergence in (4.28) and the Fatou’s lemma we get
\[ \int_{\rho}^{\infty} g_\epsilon F_\epsilon d\lambda_\theta = \frac{a_{p-1} b_\epsilon}{\epsilon} \int_{\rho}^{\infty} |u_\epsilon|^{\frac{p}{p-1}} \varphi_p(\eta_\epsilon |u_\epsilon|^{\frac{p}{p-1}}) d\lambda_\theta = o_\epsilon(\rho) \]
where $o_\epsilon(\rho) \to 0$ as $\epsilon \to 0$. In addition, (4.84) yields
\[ \int_{\rho}^{\infty} |g_\epsilon'|^p d\lambda_\alpha = - \int_{\rho}^{\infty} g_\epsilon'(s) \left[ \int_{0}^{s} (F_\epsilon + (c_\epsilon - 1)|g_\epsilon|^{p-1}) d\lambda_\theta \right] ds \]
\[ = \omega_\alpha \rho^\alpha |g_\epsilon'(\rho)|^{p-1} g_\epsilon(\rho) + \int_{\rho}^{\infty} (g_\epsilon F_\epsilon + (c_\epsilon - 1)|g_\epsilon|^p) d\lambda_\theta. \]
Thus,
\[ \int_{\rho}^{\infty} |g_\epsilon'|^p d\lambda_\alpha + (1 - c_\epsilon) \int_{\rho}^{\infty} |g_\epsilon|^p d\lambda_\theta = \omega_\alpha \rho^\alpha |g_\epsilon'(\rho)|^{p-1} g_\epsilon(\rho) + o_\epsilon(\rho). \]
Combining (4.67), (4.83) and (4.84), we can see that $\omega_\alpha \rho^\alpha |g_\epsilon'(\rho)|^{p-1} \leq c$, for some $c > 0$ which does not depend on $\epsilon$. Since $c_\epsilon \to \eta$, from (4.96)
\[ \int_{\rho}^{\infty} |g_\epsilon|^p d\lambda_\theta \leq c_1 g_\epsilon(\rho) + o_\epsilon(\rho). \]
Lemma 4.10 together with Fatou’s lemma yields
\[ \int_{\rho}^{\infty} |g_\eta|^p d\lambda_\theta \leq \lim_{\epsilon \to 0} \int_{\rho}^{\infty} |g_\epsilon|^p d\lambda_\theta \leq c_1 g_\eta(\rho). \]
In particular, $g_\eta \in L^p_0(\rho, \infty)$, for $\rho > 0$ and we obtain $g_\eta(r) \to 0$ as $r \to \infty$. Thus, from (4.97) we obtain
\[ \lim_{\rho \to \infty} \lim_{\epsilon \to 0} \int_{\rho}^{\infty} |g_\epsilon|^p d\lambda_\theta = \lim_{\rho \to \infty} \int_{\rho}^{\infty} |g_\eta|^p d\lambda_\theta = 0. \]
Using Lemma 4.10 we conclude that $g_\epsilon \to g_\eta$ in $L^p_0(0, \infty)$. Thus, from (4.68) we have
\[ \omega_\alpha \rho^\alpha |g_\eta'(\rho)|^{p-1} g_\eta(\rho) = g_\eta(\rho) + (\eta - 1) g_\eta(\rho) \int_{0}^{\rho} |g_\eta|^{p-1} d\lambda_\theta = g_\eta(\rho) + o_\rho(1), \text{ as } \rho \to 0. \]
where we have used $0 \leq g_\eta(\rho) \int_0^\rho |g_\eta|^{p-1} d\lambda_\theta \leq \int_0^\rho |g_\eta|^p d\lambda_\theta \to 0$ as $\rho \to 0$. Hence, from (4.96) we have

$$\int_\rho^\infty |u_\epsilon'|^p d\lambda_\alpha + \int_\rho^\infty |u_\epsilon|^p d\lambda_\theta = \frac{1}{a_\epsilon^{p-1}} [g_\eta(\rho) + \eta \int_\rho^\infty |g_\eta|^p d\lambda_\theta + o_\rho(1) + o_\epsilon(\rho)]$$

$$= \frac{1}{a_\epsilon^{p-1}} [g_\eta(\rho) + \eta \|g_\eta\|^p_{L_\theta^p} + o_\epsilon(\rho) + o_\rho(1)].$$

We also have

$$\int_0^\rho |u_\epsilon|^p d\lambda_\theta = \frac{1}{a_\epsilon^{p-1}} (\int_0^\rho |g_\eta|^p d\lambda_\theta + o_\epsilon(\rho)) = \frac{1}{a_\epsilon^{p-1}} (o_\rho(1) + o_\epsilon(\rho)) .$$

Since $\|u_\epsilon\| = 1$ the two previous estimates yield

$$\int_0^\rho |u_\epsilon'|^p d\lambda_\alpha = 1 - \frac{1}{a_\epsilon^{p-1}} [g_\eta(\rho) + \eta \|g_\eta\|^p_{L_\theta^p} + o_\epsilon(\rho) + o_\rho(1)].$$

Set $\tau_{\epsilon, \rho} = \|u_\epsilon'|_{L_\theta^p(0, \rho)}$. Define $v_{\epsilon, \rho} = u_\epsilon - u_\epsilon(\rho)$ on $(0, \rho]$. We have $v_{\epsilon, \rho} \in X_{\rho}^{1,p}$. Also, by using Lemma 4.4 we have that $(v_{\epsilon, \rho}/\tau_{\epsilon, \rho}^{1/p})$ is concentrating at the origin. Hence, by [10, Lemma A] (see also [5]) we obtain

$$\limsup_{\epsilon \to 0} \int_0^\rho (e^{\mu_{\alpha, \theta} |v_{\epsilon, \rho}|/\tau_{\epsilon, \rho}^{1/p})^{\frac{p}{\mu_{\alpha, \theta} + 1}} - 1) d\lambda_\theta \leq \frac{\omega_{\beta} \rho^{\theta + 1}}{\gamma + \Psi(p)} .$$

Fix $R > 0$. We have $0 < r_\epsilon R \leq \rho$, for $\epsilon > 0$ small enough. From Lemma 4.6 we can write $u_\epsilon = a_\epsilon (1 + o_\epsilon(R))$ uniformly on $[0, r_\epsilon R]$. Thus,

$$\eta_\epsilon |u_\epsilon|^{\frac{p}{p-1}} \leq \mu_{\alpha, \theta} |v_{\epsilon, \rho}|^{\frac{p}{p-1}} + \frac{\mu_{\alpha, \theta} \eta}{p - 1} \|g_\eta\|^p_{L_\theta^p} + o_\epsilon(R).$$

Analogously

$$\|u_\epsilon\|^\frac{p}{p-1} = (v_{\epsilon, \rho} + u_\epsilon(\rho))^\frac{p}{p-1} = |v_{\epsilon, \rho}|^{\frac{p}{p-1}} + \frac{p}{p-1} |v_{\epsilon, \rho}|^{\frac{p}{p-1}} u_\epsilon(\rho) + o_\epsilon(\rho).$$

So, it follows that

$$\eta_\epsilon |u_\epsilon|^{\frac{p}{p-1}} \leq \mu_{\alpha, \theta} |v_{\epsilon, \rho}|^{\frac{p}{p-1}} + \frac{p \mu_{\alpha, \theta}}{p - 1} |v_{\epsilon, \rho}|^{\frac{1}{p-1}} u_\epsilon(\rho) + \frac{\mu_{\alpha, \theta} \eta}{p - 1} \|g_\eta\|^p_{L_\theta^p} + o_\epsilon(R) + o_\rho(1),$$

on $[0, r_\epsilon R]$. From Lemma 4.10, $v_{\epsilon, \rho}^{\frac{1}{p-1}} u_\epsilon(\rho) = g_\eta(\rho) + o_\epsilon(R)$ and by definition of $\tau_{\epsilon, \rho}$ we can see that

$$\tau_{\epsilon, \rho}^{\frac{1}{p-1}} = 1 - \frac{1}{p - 1} \frac{1}{a_\epsilon^{p-1}} \left( g_\eta(\rho) + \eta \|g_\eta\|^p_{L_\theta^p} + o_\epsilon(\rho) + o_\rho(1) \right).$$

Hence, by using $v_{\epsilon, \rho} = a_\epsilon [1 + o_\epsilon(R) + o_\epsilon(\rho)]$ uniformly on $[0, r_\epsilon R]$ and $\tau_{\epsilon, \rho} = 1 + o_\epsilon(\rho)$, we can write

$$\eta_\epsilon |u_\epsilon|^{\frac{p}{p-1}} \leq \mu_{\alpha, \theta} |v_{\epsilon, \rho}/\tau_{\epsilon, \rho}^{1/p}|^{\frac{p}{p-1}} + \mu_{\alpha, \theta} g_\eta(\rho) + o_\epsilon(\rho) + o_\epsilon(R) + o_\rho(1).$$
Since $0 < r_c R \leq \rho$ and $\varphi_p(t) \leq e^t$, for $t \geq 0$
\[ \int_0^{r_c R} \varphi_p(\eta_c|u_c|^{\frac{p}{p-1}})d\lambda_\theta \leq \left[ e^{\mu_{\alpha,\theta}g_\theta(\rho) + o_\rho(\rho) + o_\rho(1)} \right] \times \int_0^{r_c R} (e^{\mu_{\alpha,\theta}|\omega_{\alpha,\theta}|^{1/p} R^{p-1} - 1) d\lambda_\theta + o_\rho(\rho). \]

Thus, from (4.99) we obtain
\[ \limsup_{\epsilon \to 0} \int_0^{r_c R} \varphi_p(\eta_c|u_c|^{\frac{p}{p-1}})d\lambda_\theta \leq \frac{\omega_\theta \rho^{\theta+1}}{\theta + 1} e^{\mu_{\alpha,\theta}g_\theta(\rho) + o_\rho(1)} e^{\gamma + \Psi(p)}. \]

Recalling (4.59) we can see that
\[ \int_0^{r_c R} \varphi_p(\eta_c|u_c|^{\frac{p}{p-1}})d\lambda_\theta = (1 + o_\rho(1)) \frac{d_c}{a_\rho^{p-1}} \int_0^R e^{(1+o_\rho(R))\mu_{\alpha,\theta} v_c^{\frac{p}{p-1}}} d\lambda_\theta. \]

Letting $\epsilon \to 0$ and $R \to \infty$ and using Lemma 4.8 we obtain
\[ \lim_{R \to \infty} \lim_{\epsilon \to 0} \int_0^{r_c R} \varphi_p(\eta_c|u_c|^{\frac{p}{p-1}})d\lambda_\theta = AD(\eta, \mu_{\alpha,\theta}, \alpha, \theta). \]

Then
\[ AD(\eta, \mu_{\alpha,\theta}, \alpha, \theta) \leq \frac{\omega_\theta \rho^{\theta+1}}{\theta + 1} e^{\mu_{\alpha,\theta}g_\theta(\rho) + o_\rho(1)} e^{\gamma + \Psi(p)}. \]

Letting $\rho \to 0$ and using Lemma 4.11 we conclude the proof. $\blacksquare$

**Lemma 4.13.** For any $z > 0$ and $p \geq 2$ we have
\[ \int_0^z \frac{s^{p-1}}{(1 + s)^p} ds = \ln(1 + z) - [\gamma + \Psi(p)] + \int_1^1 \frac{1 - s^{p-1}}{1 - s} ds \]
and
\[ \int_0^z \frac{s^{p-2}}{(1 + s)^p} ds = p - 1 - \int_1^\infty \frac{1 - s^{p-2}}{1 - s} ds. \]

In addition, we have
\[ \frac{\Gamma(p)\Gamma(1 + x)}{\Gamma(p + x)} = 1 - [\Psi(p) + \gamma]x + O(x^2), \quad \text{as} \quad x \to 0. \]

**Proof:** We recall (see [2] for instance) the following
\[ \Gamma(1) = 1, \quad \Gamma(x + 1) = x\Gamma(x) \quad \text{and} \quad \int_0^\infty \frac{s^{x-1}}{(1 + s)^{x+y}} ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \quad x, y > 0 \]
and the integral representation of $\Psi(x)$ due to Dirichlet
\[ \Psi(x) = \int_0^\infty \frac{1}{z} \left( 1 - \frac{1}{(1 + z)^x} \right) dz, \quad x > 0. \]
ON A SHARP INEQUALITY OF ADIMURTHI-DRUET TYPE

Since $\Psi(1) = -\gamma$, by using (4.104) and the change of variables $s = 1/(1 + z)$ it follows that

$$
\Psi(p) + \gamma = \int_0^\infty \frac{1}{z} \left[ \frac{1}{(1 + z)^p} - \frac{1}{1 + z} \right] dz = \int_0^1 \frac{1 - s^{p-1}}{1 - s} ds.
$$

Thus, setting $1/\tau = 1 + s$ we can write

$$
\int_0^\tau \frac{s^{p-1}}{(1 + s)^p} ds = \int_0^1 \left[ \frac{1}{\tau} + \frac{1}{\tau} ((1 - \tau)^{p-1} - 1) \right] d\tau
= \ln(1 + z) - [\Psi(p) + \gamma] + \int_0^1 \frac{1 - s^{p-1}}{1 - s} ds
$$

which proves (4.100). Analogously, with help of (4.103) we obtain (4.101). Finally, the Taylor’s expansion at $x = 0$ of the function $x \mapsto \Gamma(p)\Gamma(1 + x)/\Gamma(p + x)$ yields (4.102).

**Lemma 4.14.** There exists $\eta_0 \in (0, 1)$ such that

$$AD(\eta, \mu_{\alpha, \theta}, \alpha, \theta) > \frac{\omega_\theta}{\theta + 1} C^{\mu_{\alpha, \theta} + \gamma + \Psi(p)},$$

for any $0 < \eta < \eta_0$.

**Proof:** For any $\epsilon > 0$ small enough, let us define

$$v_\epsilon(r) = \begin{cases}
  c + \frac{1}{C^{p-1}} \left[ - \frac{p - 1}{\mu_{\alpha, \theta}} \ln \left( 1 + c_{\alpha, \theta} \left( \frac{r}{\epsilon} \right)^{\frac{\theta + 1}{\theta - 1}} \right) + b \right], & \text{if } r \leq \epsilon L \\
  \frac{1}{C^{p-1}} g_\eta(r), & \text{if } r > \epsilon L
\end{cases}
$$

(4.105)

where $g_\eta$ is given by Lemma 4.10, $c_{\alpha, \theta} = (\omega_\theta/(\theta + 1))^{1/(p-1)}$, $L = -\ln \epsilon$ and $b, c$ are constants determined later. To ensure $v_\epsilon \in X^{1,p}_{\infty}$, we assume the identity

$$c + \frac{1}{C^{p-1}} \left[ - \frac{p - 1}{\mu_{\alpha, \theta}} \ln \left( 1 + c_{\alpha, \theta} L^{\frac{\theta + 1}{\theta - 1}} \right) + b \right] = \frac{1}{C^{p-1}} g_\eta(\epsilon L).
$$

(4.106)

From (4.88) and the choice $L = -\ln \epsilon$, we can write

$$\frac{\epsilon}{C^{p-1}} = \frac{1}{\mu_{\alpha, \theta}} \ln \frac{\omega_\theta}{\theta + 1} - \frac{\theta + 1}{\mu_{\alpha, \theta}} \ln \epsilon + A_\eta - b + O(L^{-\frac{\theta + 1}{\theta - 1}}).
$$

(4.107)

By simplicity, let us define $C_\eta(q) = \int_0^r |g_\eta|^q d\lambda_\theta$, $q \geq 1$, $r > 0$. In the same way of (4.95), we have

$$\lim_{r \to 0} \frac{C_\eta(q)}{r^{\theta+1} \ln r^q} = \frac{\omega_\theta}{\theta + 1} \frac{1}{\omega_\alpha}.
$$

(4.108)

Using (4.68), we have

$$\int_{\epsilon L}^{\infty} |g_\eta'|^p d\lambda_\alpha = \omega_\alpha (\epsilon L)\alpha |g_\eta'(\epsilon L)|^{p-1} g_\eta(\epsilon L) + (\eta - 1) \int_{\epsilon L}^{\infty} |g_\eta|^p d\lambda_\theta.$$
From this and by using (4.68) again we can write
\[
\int_{\epsilon L}^{\infty} |v'_e|^p d\lambda_e + \int_{\epsilon L}^{\infty} |v_e|^p d\lambda_e
= \frac{1}{c_{p+1}} \left[ \eta \|g_\eta\|_{L^p_0}^p + g_\eta(\epsilon L) + (\eta - 1)g_\eta(\epsilon L)C_\eta^{p-1}(\epsilon L) - \eta C_\eta(\epsilon L) \right].
\]
Therefore, from (4.108) and Lemma 4.11
\[
\int_{\epsilon L}^{\infty} |v'_e|^p d\lambda_e + \int_{\epsilon L}^{\infty} |v_e|^p d\lambda_e
(4.109)
= \frac{1}{c_{p+1}} \left[ \eta \|g_\eta\|_{L^p_0}^p - \frac{\theta + 1}{\mu_{\alpha,\theta}} \ln \epsilon - \frac{\theta + 1}{\mu_{\alpha,\theta}} \ln L + A_\eta + O((\epsilon L)^{\theta+1} |\ln(\epsilon L)|^p) \right].
\]
By simplicity, we denote \( z_L = c_{\alpha,\theta} L^{\theta+1}. \) Hence, from Lemma 4.13
\[
\int_{0}^{\epsilon L} |v'_e|^p d\lambda_e = \frac{p - 1}{c_{p+1}} \left[ \ln (1 + z_L) - [\gamma + \Psi(p)] + \int_{1}^{\epsilon L} \frac{1 - s^{p-1}}{1 - s} ds \right].
\]
Noticing that
\[
\lim_{z \to \infty} (1 + z) \int_{1}^{1 + z} \frac{1 - s^{p-1}}{1 - s} ds = p - 1 \quad \text{\scriptsize (by L’Hospital rule)}
\]
and
\[
\frac{p - 1}{\mu_{\alpha,\theta}} \ln (1 + z_L) = \frac{\theta + 1}{\mu_{\alpha,\theta}} \ln L + \frac{1}{\mu_{\alpha,\theta}} \ln \left( \frac{\omega_L}{\theta + 1} \right) + O \left( L^{-\frac{\theta+1}{p-1}} \right)
\]
we can see that
\[
(4.110)
\int_{0}^{\epsilon L} |v'_e|^p d\lambda_e = \frac{1}{c_{p+1}} \left[ \frac{\theta + 1}{\mu_{\alpha,\theta}} \ln L + \frac{1}{\mu_{\alpha,\theta}} \ln \left( \frac{\gamma + \Psi(p)}{\omega_L} \right) + O \left( L^{-\frac{\theta+1}{p-1}} \right) \right].
\]
On the other hand, by using \( L = -\ln \epsilon, \) (4.88) and (4.106) we obtain for any \( q \geq 1 \)
\[
(4.111)
\int_{0}^{\epsilon L} |v_e|^q d\lambda_e = \frac{1}{c_{p+1}} \left[ \int_{0}^{\epsilon L} \left| g_\eta(\epsilon L) + \frac{p - 1}{\mu_{\alpha,\theta}} \ln \left( \frac{1 + c_{\alpha,\theta}^{L^{\theta+1}}}{1 + c_{\alpha,\theta}^{L^{\theta+1}}} \right) \right|^q d\lambda_e \right]
\]
Also, for any \( q \geq p, \) directly from (4.105) and (4.108) we have
\[
(4.112)
\int_{\epsilon L}^{\infty} |v_e|^q d\lambda_e = \frac{1}{c_{p+1}} \left[ \|g_\eta\|_{L^p_0}^q + O((\epsilon L)^{\theta+1} |\ln(\epsilon L)|^q) \right].
\]
Combining (4.111) and (4.112) we have
\[
(4.113)
\|v_e\|_{L^p_0}^q = \frac{1}{c_{p+1}} \left[ \|g_\eta\|_{L^p_0}^q + O((\epsilon L)^{\theta+1} |\ln(\epsilon L)|^q) \right], \quad q \geq p.
\]
In addition, from (4.109), (4.110) and (4.111)

$$
\|v_\epsilon\|_p^p = \frac{1}{c_{p-1}^p} \left[ \eta \|g_\eta\|_{L_0^p}^p + \frac{1}{\mu_{\alpha, \theta}} \ln \frac{\omega}{\theta + 1} - \frac{1}{\mu_{\alpha, \theta}} \ln \epsilon \right] + A_\eta - \frac{p - 1}{\mu_{\alpha, \theta}} [\gamma + \Psi(p)] + O \left( L^{-\frac{\theta + 1}{p-1}} \right).
$$

For $\epsilon > 0$ small enough, we can choose $b$ in (4.107) of the form

(4.114) $$b = -\eta \|g_\eta\|_{L_0^p}^p + \frac{p - 1}{\mu_{\alpha, \theta}} [\gamma + \Psi(p)] + O \left( L^{-\frac{\theta + 1}{p-1}} \right)$$

such that $c_{p-1}^p$ satisfies the equation (4.107) and $\|v_\epsilon\| = 1$. In addition, $c$ satisfies

(4.115) $$c_{p-1}^p = \eta \|g_\eta\|_{L_0^p}^p + \frac{1}{\mu_{\alpha, \theta}} \ln \frac{\omega}{\theta + 1} - \frac{1}{\mu_{\alpha, \theta}} \ln \epsilon + A_\eta - \frac{p - 1}{\mu_{\alpha, \theta}} [\gamma + \Psi(p)] + O \left( L^{-\frac{\theta + 1}{p-1}} \right).$$

Now, for any $q > 0$

$$
\left(1 + \eta \|v_\epsilon\|_{L_0^p}^p\right)^q = 1 + q\eta \|v_\epsilon\|_{L_0^p}^p + \frac{q(q - 1)}{2!} (\eta \|v_\epsilon\|_{L_0^p}^p)^2 + R(\|v_\epsilon\|_{L_0^p}^p)
$$

where, for some $0 < \tau_\epsilon < 1$,

$$R(\|v_\epsilon\|_{L_0^p}^p) = \frac{q(q - 1)(q - 2)}{3!} (1 + \tau_\epsilon \eta \|v_\epsilon\|_{L_0^p}^p)^{q - 3} (\eta \|v_\epsilon\|_{L_0^p}^p)^3.$$

Thus, taking into account (4.113), (4.115) and $L = -\ln \epsilon$ we can see that

(4.116) $$\left(1 + \eta \|v_\epsilon\|_{L_0^p}^p\right)^{\frac{1}{p-1}} = 1 + \frac{1}{p - 1} \frac{\eta \|g_\eta\|_{L_0^p}^p}{c_{p-1}^p} - \frac{p - 2}{2(p - 1)^2} \eta^2 \|g_\eta\|_{L_0^p}^{2p} + O \left( c^{-\frac{3p}{p-1}} \right).$$

Since $\varphi_p(t) \geq \frac{t^{k_0}}{k_0!}$, for $t \geq 0$, by using (4.112) we can write

$$
\int_{\epsilon L}^{\infty} \varphi_p(\mu_{\alpha, \theta})(1 + \eta \|v_\epsilon\|_{L_0^p}^p)^{\frac{1}{p-1}} |v_\epsilon|^{\varphi_p} \ d\lambda_\theta
\geq \left[ \frac{\mu_{\alpha, \theta}}{k_0!} \right] \left[ \|g_\eta\|_{L_0^p}^{\frac{k_0 p}{p^{p-1}}} + O \left( (\epsilon L)^{\theta + 1} |\ln(\epsilon L)|^{\frac{k_0 p}{p^{p-1}}} \right) \right].
$$

From (4.113), $\|v_\epsilon\|_{L_0^p}^p = O(c^{-\frac{p}{p-1}})$ and consequently

(4.117) $$\int_{\epsilon L}^{\infty} \varphi_p(\mu_{\alpha, \theta})(1 + \eta \|v_\epsilon\|_{L_0^p}^p)^{\frac{1}{p-1}} |v_\epsilon|^{\varphi_p} \ d\lambda_\theta
\geq \left[ \frac{\mu_{\alpha, \theta}}{k_0!} \right] \left[ \|g_\eta\|_{L_0^p}^{\frac{k_0 p}{p^{p-1}}} + O \left( (\epsilon L)^{\theta + 1} |\ln(\epsilon L)|^{\frac{k_0 p}{p^{p-1}}} \right) + O \left( c^{-\frac{p}{p-1}} \right) \right].$$

Note that for $\epsilon > 0$ small enough, we have

$$c^{-\frac{p}{p-1}} \left[ -\frac{p - 1}{\mu_{\alpha, \theta}} \ln \left( 1 + c_{\alpha, \beta} \left( \frac{\theta + 1}{\epsilon} \right)^{\frac{1}{p-1}} \right) + b \right] > -1,$$
for any \( r \in (0, \epsilon L] \). Hence, the inequality \((1 + t)^d \geq 1 + dt\) for \( t > -1 \) and \( d \in (1, 2] \) yields
\[
\left| \frac{v_r}{c} \right|^{\frac{p}{p-1}} \geq 1 + \frac{p}{p-1} \frac{1}{c^{p-1}} \left[ w \left( \frac{r}{\epsilon} \right) + b \right],
\]
where \( w \) is given by (4.53). Using (4.114)

\[
|v_\epsilon|^{\frac{p}{p-1}} \geq c^{\frac{p}{p-1}} - \frac{p\eta}{p-1} \|g_n\|_{L^p_2}^p + \frac{p}{\mu_{\alpha, \theta}} [\gamma + \Psi(p)] + \frac{p}{p-1} w(r/\epsilon) + O \left( L^{-\frac{\theta+1}{p-1}} \right).
\]

From (4.116), we have for any \( r \in (0, \epsilon L] \)
\[
\left( 1 + \eta \|v_\epsilon\|^p_{L^p_2} \right)^{\frac{p}{p-1}} |v_\epsilon|^{\frac{p}{p-1}} = \left( 1 + \frac{\eta \|g_n\|^p_{L^p_2}}{(p-1)c^{p-1}} \right) |v_\epsilon|^{\frac{p}{p-1}}
\]
(4.119)
\[
+ \left[ \frac{2-p}{2(p-1)^2} \frac{\eta^2 \|g_n\|^{2p}_{L^p_2}}{c^{3p-1}} + O \left( c^{-\frac{3p}{p-1}} \right) \right] |v_\epsilon|^{\frac{p}{p-1}}.
\]

Next, we shall estimate each term on the right hand side of (4.119). Firstly, from (4.118)
\[
\left( 1 + \frac{\eta \|g_n\|^p_{L^p_2}}{(p-1)c^{p-1}} \right) |v_\epsilon|^p_{p-1} \geq \left( 1 + \frac{\eta \|g_n\|^p_{L^p_2}}{(p-1)c^{p-1}} \right) \frac{p}{p-1} w(r/\epsilon)
\]
\[
+ c^{\frac{p}{p-1}} - \eta \|g_n\|^p_{L^p_2} - \frac{p\eta^2 \|g_n\|^{2p}_{L^p_2}}{(p-1)^2 c^{p-1}} + \frac{p}{\mu_{\alpha, \theta}} [\gamma + \Psi(p)]
\]
\[
+ \frac{p}{\mu_{\alpha, \theta}} [\gamma + \Psi(p)] \frac{\eta \|g_n\|^p_{L^p_2}}{(p-1)c^{p-1}} + O \left( L^{-\frac{\theta+1}{p-1}} \right).
\]

Using the identity in (4.115), we can also write
\[
\left( 1 + \frac{\eta \|g_n\|^p_{L^p_2}}{(p-1)c^{p-1}} \right) |v_\epsilon|^p_{p-1} \geq \left( 1 + \frac{\eta \|g_n\|^p_{L^p_2}}{(p-1)c^{p-1}} \right) \frac{p}{p-1} w(r/\epsilon)
\]
(4.120)
\[
+ \frac{1}{\mu_{\alpha, \theta}} \ln \frac{\omega_n}{\theta + 1} - \frac{\theta + 1}{\mu_{\alpha, \theta}} \ln \epsilon + A_n + \frac{1}{\mu_{\alpha, \theta}} [\gamma + \Psi(p)] - \frac{p\eta^2 \|g_n\|^{2p}_{L^p_2}}{(p-1)^2 c^{p-1}}
\]
\[
+ \frac{p}{\mu_{\alpha, \theta}} [\gamma + \Psi(p)] \frac{\eta \|g_n\|^p_{L^p_2}}{(p-1)c^{p-1}} + O \left( L^{-\frac{\theta+1}{p-1}} \right).
\]

On the other hand, from (4.118) it is easy to see that
\[
\left( \frac{2-p}{2(p-1)^2} \frac{\eta^2 \|g_n\|^{2p}_{L^p_2}}{c^{3p-1}} + O \left( c^{-\frac{3p}{p-1}} \right) \right) |v_\epsilon|^{\frac{p}{p-1}} = \frac{1}{c^{p-1}} \Phi_\epsilon
\]
(4.121)
where
\[ \Phi_\epsilon = \left[ \frac{2 - p}{2(p - 1)^2} \eta^2 \| g_\eta \|_{L^p_\theta}^{2p} + O \left( \epsilon^{-\frac{p}{p-1}} \right) \right] \times \left[ 1 - \frac{p\eta \| g_\eta \|_{L^p_\theta}^{p\eta}}{(p - 1)c_{p-1}} + \frac{p\gamma + \Psi(p)}{\mu_{\alpha,\theta}c_{p-1}} + \frac{pw(r/\epsilon)}{(p - 1)c_{p-1}} + O \left( L^{\frac{\theta+1}{p-1}} \right) \right]. \]

Note that (recall \( \alpha = p - 1 \))
\[ \lim_{\epsilon \to 0} \left[ \Phi_\epsilon + \frac{p^2 - p + 2}{2(p - 1)^2} \eta^2 \| g_\eta \|_{L^p_\theta}^{2p} + O \left( \frac{c_{p-1}}{L^{\frac{\theta+1}{p-1}}} \right) \right] \]
(4.122)
\[ = \begin{cases} (p - 2)^2 \eta^2 \| g_\eta \|_{L^p_\theta}^{2p} > 0, & \text{if } p > 2 \\ 2\eta^2 \| g_\eta \|_{L^p_\theta}^4 > 0, & \text{if } p = 2. \end{cases} \]

For \( \epsilon > 0 \) small enough, combining (4.119), (4.120), (4.121) and (4.122), we have
\[ \left( 1 + \eta \| v_\epsilon \|_{L^p_\theta}^p \right)^{\frac{1}{p-1}} |v_\epsilon|^\frac{1}{p-1} \geq \left( 1 + \frac{\eta \| g_\eta \|_{L^p_\theta}^p}{(p - 1)c_{p-1}} \right)^{\frac{p}{p-1}} \frac{w(r/\epsilon)}{p - 1} \]
(4.123)
\[ + \frac{1}{\mu_{\alpha,\theta}} \ln \frac{\omega_\theta}{\theta + 1} - \frac{\theta + 1}{\mu_{\alpha,\theta}} \ln \epsilon + A_\eta + \frac{1}{\mu_{\alpha,\theta}} \left[ \gamma + \Psi(p) \right] \]
\[ + \frac{1}{\mu_{\alpha,\theta}} \left[ \gamma + \Psi(p) \right] \frac{p\eta \| g_\eta \|_{L^p_\theta}^p}{(p - 1)c_{p-1}} - \frac{p^2 - p + 2 \eta^2 \| g_\eta \|_{L^p_\theta}^{2p}}{2(p - 1)^2} \frac{c_{p-1}}{L^{\frac{\theta+1}{p-1}}}. \]

From (1.9), (4.111), (4.113), (4.115) and \( L = -\ln \epsilon \), we have
\[ \int_0^{\epsilon L} \varphi_p \left( \mu_{\alpha,\theta} \left( 1 + \eta \| v_\epsilon \|_{L^p_\theta}^p \right)^{\frac{1}{p-1}} |v_\epsilon|^\frac{1}{p-1} \right) d\lambda_\theta \]
(4.124)
\[ = \int_0^{\epsilon L} \mu_{\alpha,\theta} \left( 1 + \eta \| v_\epsilon \|_{L^p_\theta}^p \right)^{\frac{1}{p-1}} |v_\epsilon|^\frac{1}{p-1} d\lambda_\theta + O \left( L^{\frac{\theta+1}{p-1}} \right). \]

By simplicity, set
\[ Y_{p,\eta} = \frac{1}{c_{p-1}} \frac{p\eta}{p - 1} \| g_\eta \|_{L^p_\theta}^p. \]

With this notation, from (4.102) and (4.103) we have
\[ \frac{1}{\epsilon^{\theta+1}} \int_0^{\epsilon L} \mu_{\alpha,\theta} \left( 1 + \frac{\eta \| g_\eta \|_{L^p_\theta}^p}{(p - 1)c_{p-1}} \right)^{\frac{p}{p-1}} w(r/\epsilon) d\lambda_\theta = (p - 1) \int_0^{\epsilon L} \frac{s^{p-2}}{(1 + s)^{p + Y_{p,\eta}}} ds \]
\[ = \frac{\Gamma(p) \Gamma(1 + Y_{p,\eta})}{\Gamma(p + Y_{p,\eta})} - (p - 1) \int_{z_L}^{\infty} \frac{s^{p-2}}{(1 + s)^{p + Y_{p,\eta}}} ds \]
\[ = 1 - [\Psi(p) + \gamma] Y_{p,\eta} + O \left( \epsilon^{-\frac{2}{p-1}} \right) + O \left( L^{\frac{\theta+1}{p-1}} \right). \]
This together with (4.123) and the inequality \( e^x \geq 1 + x \) yields

\[
\int_0^{\epsilon L} e^{\mu_{\alpha, \theta} \left(1 + \eta \|v_\epsilon\|^p_{L^p_\theta}\right)} \frac{1}{\|v_\epsilon\|^p_{L^p_\theta}} |v_\epsilon|^p_{L^p_\theta} d\lambda_\theta \\
\geq \frac{\omega_\theta}{\theta + 1} e^{\mu_{\alpha, \theta} A_\eta + \gamma + \Psi(p)} \left[ 1 - \frac{p^2 - p + 2 \mu_{\alpha, \theta} \eta^2 \|g_\eta\|^p_{L^p_\theta}}{2(p - 1)^2} \right] + O \left( c^{-\frac{2p}{p-1}} \right) + O \left( L^{-\frac{\theta+1}{p-1}} \right)
\]

\[
+ \left[ \gamma + \Psi(p) \right] \frac{p^2 - p + 2 \mu_{\alpha, \theta} \eta^2 \|g_\eta\|^p_{L^p_\theta}}{2(p - 1)^2} Y_{p, \eta} \right].
\]

Hence,

\[
\int_0^{\epsilon L} e^{\mu_{\alpha, \theta} \left(1 + \eta \|v_\epsilon\|^p_{L^p_\theta}\right)} \frac{1}{\|v_\epsilon\|^p_{L^p_\theta}} |v_\epsilon|^p_{L^p_\theta} d\lambda_\theta \\
\geq \frac{\omega_\theta}{\theta + 1} e^{\mu_{\alpha, \theta} A_\eta + \gamma + \Psi(p)} \\
+ \frac{\omega_\theta \eta \mu_{\alpha, \theta} \eta \|g_\eta\|^p_{L^p_\theta}}{\theta + 1} e^{\mu_{\alpha, \theta} A_\eta + \gamma + \Psi(p)} \left[ \|g_\eta\|^p_{L^p_\theta} \right] + O \left( c^{-\frac{2p}{p-1}} \right) + O \left( L^{-\frac{\theta+1}{p-1}} \right)
\]

\[
(4.125)
\]

Now, by using (4.117), (4.124), (4.125) we obtain

\[
(4.126)
\]

where

\[
H(\epsilon, \eta) = \frac{\mu_{\alpha, \theta} \|g_\eta\|^p_{L^p_\theta} \ln \frac{k_{\alpha, \theta}}{k_{0, \theta}}}{\eta \|g_\eta\|^p_{L^p_\theta}} \ln \frac{k_{\alpha, \theta} L^p_\theta}{k_{0, \theta} L^p_\theta} \left[ O \left( c^{-\frac{p}{p-1}} (\epsilon L)^{\theta+1} \ln \frac{k_{\alpha, \theta} L^p_\theta}{k_{0, \theta} L^p_\theta} \right) + O(1) \right]
\]

\[
+ \frac{\omega_\theta \eta \mu_{\alpha, \theta} \eta \|g_\eta\|^p_{L^p_\theta}}{\theta + 1} e^{\mu_{\alpha, \theta} A_\eta + \gamma + \Psi(p)} \left[ \|g_\eta\|^p_{L^p_\theta} \right] \left[ \gamma + \Psi(p) \right] + O \left( c^{-\frac{2p}{p-1}} \right) + O \left( L^{-\frac{\theta+1}{p-1}} \right).
\]

Now, it is sufficient to show that \( H(\epsilon, \eta) > 0 \), for \( \epsilon > 0, \eta \geq 0 \) small.

Case 1: \( 2 \leq p \in \mathbb{R}, \eta = 0 \) and \( \epsilon > 0 \) small. In this case we have

\[
H(\epsilon, 0) = \frac{\mu_{\alpha, \theta} \|g_0\|^p_{L^p_\theta} \ln \frac{k_{\alpha, \theta}}{k_{0, \theta}}}{\eta \|g_0\|^p_{L^p_\theta}} \ln \frac{k_{\alpha, \theta} L^p_\theta}{k_{0, \theta} L^p_\theta} \left[ O \left( c^{-\frac{p}{p-1}} (\epsilon L)^{\theta+1} \ln \frac{k_{\alpha, \theta} L^p_\theta}{k_{0, \theta} L^p_\theta} \right) + O(1) \right]
\]

\[
+ O \left( c^{-\frac{2p}{p-1}} \right) + O \left( L^{-\frac{\theta+1}{p-1}} \right).
\]
Noting that \( p - 1 \leq k_0 < p \) and by using (4.115) with \( L = -\ln \epsilon \) we can see that

\[
(4.127) \quad c^{\frac{p}{p-1}} \to \infty, \quad c^{\frac{k_0 p}{p-1}} \to \infty, \quad c^{\frac{k_0 p}{p-1}} L^{\frac{\theta+1}{p-1}} \to 0, \quad c^{\frac{p}{p-1}} (\epsilon L)^{\theta+1} \ln \frac{k_0 p}{p-1} (\epsilon L) \to 0,
\]

as \( \epsilon \to 0 \). Thus, we obtain \( H(\epsilon, 0) > 0 \) for \( \epsilon > 0 \) small enough.

**Case 2:** \( 2 \leq p \in \mathbb{N}, \epsilon > 0 \) and \( \eta > 0 \) small. Let \( g_0 \) be the solution of the equation (4.68), that is,

\[
\omega_\alpha r^\alpha |g_0(r)|^{p-1} + \int_0^r |g_0|^{p-1}d\lambda_\theta = 1, \quad r > 0.
\]

A simple scaling argument shows that \( g_\eta(r) = g_0((1-\eta)^{1/(\theta+1)} r) \). Hence, we have

\[
(4.128) \quad A_\eta = A_0 - \frac{1}{\mu_{\alpha, \theta}} \ln(1-\eta) \quad \text{and} \quad \|g_\eta\|_{L_p}^q = (1-\eta)^{-1} \|g_0\|_{L_p^q}, \quad q \geq p.
\]

Now, since \( p \in \mathbb{N} \), we have \( k_0 = p - 1 \). Thus, by using (4.128)

\[
H(\epsilon, \eta) = \frac{\mu_{\alpha, \theta}}{(p-1)!} + \frac{1}{c^{\frac{p}{p-1}}} |g_0|_{L_p}^p \left[ O \left( c^{\frac{p}{p-1}} (\epsilon L)^{\theta+1} \ln(\epsilon L) \right)^p \right] + O(1)
\]

\[
+ \left[ \frac{\eta}{1-\eta} \omega_\theta \mu_{\alpha, \theta} \mu_{\alpha, \theta} A_0 + \gamma + \Psi(p) \left( \frac{p}{p-1} \gamma + \Psi(p) \right) \frac{\eta}{1-\eta} \|g_0\|_{L_p^p} - 1 \right] \times
\]

\[
\left[ \frac{p^2 - p + 2}{2(p-1)^2} \frac{\eta}{1-\eta} \|g_0\|_{L_p^p}^p \right] + O \left( c^{-\frac{p}{p-1}} \right) + O \left( c^{\frac{p}{p-1}} L^{\frac{\theta+1}{p-1}} \right).
\]

From this, we conclude that \( H(\epsilon, \eta) > 0 \), if \( \epsilon, \eta > 0 \) is small enough.

**Case 3:** \( 2 \leq p \not\in \mathbb{N}, \epsilon > 0 \) and \( \eta > 0 \) small. Here \( p - 1 < k_0 < p \) and, from (4.115) and (4.128)

\[
\eta c^{\frac{p}{p-1}} \left( L^{\frac{k_0}{p-1} - 1} \right) = \left[ \frac{1}{\mu_{\alpha, \theta}} \frac{\eta}{1-\eta} \|g_0\|_{L_p^p} + \frac{1}{\mu_{\alpha, \theta}} \|g_0\|_{L_p^p} \right] - \frac{\theta + 1}{\mu_{\alpha, \theta}} \ln(1-\eta) + O \left( L^{\frac{\theta+1}{p-1}} \right) \left( \frac{1}{\eta^{\theta+1}} \right) - \frac{1}{\mu_{\alpha, \theta}} \frac{1}{\eta^{\theta+1}} \|g_0\|_{L_p^p} + O \left( L^{\frac{\theta+1}{p-1}} \right) \left( \frac{1}{\eta^{\theta+1}} \right).
\]

Hence, by choosing \( \epsilon = \eta > 0 \) we have

\[
(4.129) \quad \eta c^{\frac{p}{p-1}} \left( L^{\frac{k_0}{p-1} - 1} \right) \to 0, \quad \text{as} \quad \epsilon = \eta \to 0.
\]
Moreover, using (4.128)
\[
H(\epsilon, \eta) = \frac{k_0^p}{\kappa_0} \frac{\Vert g_0 \Vert^{p-1}_{L^p_0}}{\kappa_0^{p-1}_{L^p_0}} + \frac{1}{\kappa_0^{p-1}_{L^p_0}} \left[ O\left( c_{p-1}^{\frac{p}{\theta}} (\epsilon L)^{\frac{p}{\theta} + 1} \right) + O(1) \right]
\]
\[
+ \left[ \left( \frac{p(\gamma + \Psi(p))}{p-1} - 1 \right) \frac{\kappa_0^{p-1}_{L^p_0}}{(1-\eta)^2} \omega_\eta \mu_{\alpha, \theta, \epsilon} \frac{\omega_{\eta, \mu, \theta} A_0 + \eta + \Psi(p)}{\theta + 1} \right] \times \left[ \left( \frac{p(\gamma + \Psi(p))}{p-1} - 1 \right) \frac{\kappa_0^{p-1}_{L^p_0}}{(1-\eta)^2} \right] \left[ \frac{p^2 - p + 2}{2(p-1)^2} \| g_0 \|^p_{L^p_0} \right]
\]
\[
+ O\left( \kappa_0^{(p-1)2-2p-1}_{L^p_0} \right) + O\left( \kappa_0^{(p-1)2} L^{-\theta+1}_{\theta-1} \right).
\]
Thus, using the convergences in (4.127) and (4.129) we have \( H(\epsilon, \eta) > 0 \), if \( \epsilon = \eta > 0 \) is small enough. 

5. Proof of Theorem 1.4: Non-existence of maximizers

We follow the argument of Ishiwata of [16]. For \( p = 2, [9, Theorem 1.4] \) yields
\[
C_{\mu, \theta} = \sup_{0 \neq u \in X_{1, \infty}^2} \frac{\| u \|^2_{L^2_0}}{\| u \|^2_{L^2_0}} \int_0^\infty \left( \frac{\mu \| u \|_{L^2_0}^2}{\| u \|_{L^2_0}^2} - 1 \right) d\lambda_\theta < \infty,
\]
for any \( 0 < \mu < \mu_{1, \theta} = 2\pi(\theta + 1) \). Thus, the series expansion of \( x \mapsto e^x - 1 \) yields
\[
\| u \|^2_{L^2_0} \leq C_{\mu, \theta} \frac{1}{\mu^j} \| u \|^{2(j-1)}_{L^2_1} \| u \|^{2j}_{L^2_0}, \quad j \geq 1.
\]
Let \( M = \{ u \in X_{1, \infty}^2 \mid \| u \| = 1 \} \). For any \( u \in M \), we set
\[
u_r = u \frac{u}{\| u \|}, \quad \nu_r = \frac{\nu_r}{\| \nu_r \|}, \quad \tau > 0
\]
Hence \( v_r \in M \). Define
\[
J(u) = \int_0^\infty \left( e^{\mu(1+\eta)\| u \|_{L^2_0}^2 u^2} - 1 \right) d\lambda_\theta.
\]
If \( u \) is a maximizer for \( AD(\eta, \mu, 1, \theta) \) then \( u \in M \). Since \( v_r \) is a curve in \( M \) with \( v_1 = u \) we must have
\[
\frac{d}{d\tau} J(v_r) \bigg|_{\tau=1} = 0.
\]
From (2.10), we can write
\[
J(v_r) = \sum_{j=1}^\infty \frac{\mu^j}{j!} \left( 1 + \eta \frac{\| u \|^2_{L^2_0}}{\tau \| u \|^2_{L^2_0} + \| u \|^2_{L^2_0}} \right)^{\tau^{-1} j} \left( \frac{\tau \| u \|^2_{L^2_0} + \| u \|^2_{L^2_0}}{\tau \| u \|^2_{L^2_0} + \| u \|^2_{L^2_0}} \right)^{\tau^{-1} j}.
\]
Since \( \|u\| = 1 \), it follows that
\[
\frac{d}{d\tau} J(v_\tau) \bigg|_{\tau=1} = -\mu \|u\|_{L^2_\theta}^2 \|u'\|_{L^2_1}^2 + \sum_{j=2}^\infty \frac{\mu^j}{j!} \left( 1 + \eta \|u\|_{L^2_\theta}^2 \right)^{j-1} \|u\|_{L^2_\theta}^{2j} \left[ -j \eta \|u\|_{L^2_\theta}^2 \|u'\|_{L^2_1}^2 \right] + (1 + \eta \|u\|_{L^2_\theta}^2) (j - 1 - j \|u'\|_{L^2_1}^2) \right),
\]
\[
\leq -\mu \|u\|_{L^2_\theta}^2 \|u'\|_{L^2_1}^2 + \sum_{j=2}^\infty \frac{(2\mu)^j}{(j-1)!} \|u\|_{L^2_\theta}^{2j}.
\]

For any \( 0 < \gamma < \mu_{1,\theta} \), (5.1) yields
\[
\frac{d}{d\tau} J(v_\tau) \bigg|_{\tau=1} \leq \|u\|_{L^2_\theta}^2 \|u'\|_{L^2_1}^2 \left( -\mu + \frac{4\mu^2}{\gamma^2 C_{\gamma,\theta}} \sum_{j=2}^\infty \frac{j(2\mu)^{j-2}}{\gamma^{j-2}} \|u'\|_{L^2_1}^{2(j-2)} \right)
\]
\[
\leq \mu \|u\|_{L^2_\theta}^2 \|u'\|_{L^2_1}^2 \left( -1 + \frac{4\mu}{\gamma^2 C_{\gamma,\theta}} \sum_{j=0}^\infty \frac{(j+2)(2\mu)^j}{\gamma^j} \right).
\]

Thus, for \( \mu < \mu_{1,\theta}/4 \) and by choosing \( \gamma = 3\mu_{1,\theta}/4 \), we get
\[
\frac{d}{d\tau} J(v_\tau) \bigg|_{\tau=1} \leq \mu \|u\|_{L^2_\theta}^2 \|u'\|_{L^2_1}^2 \left( -1 + \mu C \right), \quad \text{with} \quad C = \frac{4}{\gamma^2 C_{\gamma,\theta}} \sum_{j=0}^\infty \frac{(j+2)}{3^j}.
\]

But, for any \( \mu < \min \{ \mu_{1,\theta}/4, 1/C \} \), this contradicts (5.2).

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