Using FFT to reduce the computational complexity of sub-Nyquist sampling based wideband spectrum sensing

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Abstract. Sub-Nyquist sampling based wideband spectrum sensing can usually be summarized as two steps. The first step estimates the signal power spectrum. The second step uses the energy detection method to make a decision. The traditional method usually adopts the least square algorithm to estimate the power spectrum. In order to reduce the computational complexity of the power spectrum estimation, a novel algorithm is proposed to replace the least square algorithm. We find the system matrix of Multi-Coset Sampling (MCS) based power spectrum estimation has a special structure. Its row vectors are all selected from the row vectors of the Discrete Fourier Transform (DFT) matrix. We exploit this special structure and use Fast Fourier Transform (FFT) to reduce the computational complexity of power spectrum estimation. Simulation shows the proposed algorithm can achieve the spectrum sensing performance similar to the traditional method at much lower computational complexity.

1. Introduction
Sub-Nyquist Sampling based wideband spectrum sensing has been widely used in cognitive radio and spectrum monitoring [1-2]. Spectrum sensing is usually carried out by first estimating the signal power spectrum and then using the energy detection method to determine whether the primary users are active [3-5]. The signal power spectrum is usually estimated by least square algorithm. Unfortunately, the least square algorithm requires a large amount of computation when the dimension of unknowns is high. Although high performance Central Processing Unit (CPU) can be used, the cost is usually extremely high in terms of price and power consumption.

In this paper, we aim at reducing the computational complexity of Multi-Coset Sampling (MCS) based power spectrum estimation. The structure of the system matrix of power spectrum estimation is first analyzed. Interestingly, we find that this matrix has a special structure. Its rows are all selected from the rows of the Discrete Fourier Transform (DFT) matrix. The proposed algorithm firstly processes the measured cross-spectrum vector so as to transform the original tall system matrix of power spectrum estimation into a square DFT matrix. Then Fast Fourier Transform (FFT) [6] is used to estimate the signal power spectrum. Simulation shows that the proposed power spectrum estimation algorithm can achieve the spectrum sensing performance similar to the traditional method at much lower computational complexity.

2. Power Spectrum Estimation Model
The model of MCS based power spectrum estimation is first briefly introduced. For more detail, we refer the reader to [4]. Denote $x(t)$ as the input signal and $T$ as its Nyquist sampling interval. Define
\( X(f) \) as the Fourier Transform (FT) of \( x(t) \). \( q \) samples are selected from \( L \) consecutive Nyquist samples in MCS. The MCS system can be written as

\[
Y(e^{j2\pi ft/LT}) = Ax(f), \quad f \in \mathcal{F}_0, \quad \mathcal{F}_0 = [0,1/LT),
\]

where \( Y(e^{j2\pi ft/LT}) \) is a \( q \times 1 \) vector, and its \( i \)-th element corresponds to the Discrete Time Fourier Transform (DTFT) of the \( i \)-th uniform sampling sequence multiplying a constant phasor. \( A \) in Equation (1) is the \( q \times L \) MCS system matrix, and its element at the \( i \)-th row and \( l \)-th column is

\[
A_{il} = e^{-j2\pi ilr/LT}, \quad 1 \leq i \leq q, \quad -L/2 \leq l \leq L/2 - 1.
\]

\( x(f) \) in Equation (1) is a \( L \times 1 \) vector and its \( l \)-th element is

\[
x(f)_l = X(f)_{l} = X(f - \frac{j}{LT}).
\]

\( x(t) \) comprises of signal component \( s(t) \) and noise component \( n(t) \). Let \( S(f) \) and \( N(f) \) denote the FT of \( s(t) \) and \( n(t) \), respectively. Similar to the definition of \( x(f) \), define two \( L \times 1 \) vectors \( s(f)_l = S(f)_{l} \) and \( n(f)_l = N(f)_{l} \). The covariance matrix of \( Y(e^{j2\pi ft/LT}) \) can be written as

\[
\begin{align*}
R_Y(e^{j2\pi ft/LT}) &= E[Y(e^{j2\pi ft/LT})Y^H(e^{j2\pi ft/LT})]. \tag{2}
\end{align*}
\]

Likewise, the covariance matrix of \( x(f) \), \( s(f) \) and \( n(f) \) can be expressed as \( R_x(f) = E[x(f)x^H(f)] \), \( R_s(f) = E[s(f)s^H(f)] \) and \( R_n(f) = E[n(f)n^H(f)] \). Using Equation (1), Equation (2) can be further written as

\[
R_Y(e^{j2\pi ft/LT}) = AR_x(f)A^H = A[R_s(f) + R_n(f)]A^H. \tag{3}
\]

The covariance matrix \( R_x(f) \) is proved to be a diagonal matrix \([4]\), and its \( i, l \)-th element can be written as

\[
[R_x(f)]_{i,l} = E[|s(f)_i|^2]\delta(i-l) + \sigma^2\delta(i-l). \tag{4}
\]

This property is used to further develop the power spectrum estimation method. Let \( \text{Vec}(\cdot) \) and \( \otimes \) represent the matrix vectorization operator and Kronecker product, respectively. Define \( r_Y(e^{j2\pi ft/LT}) = \text{Vec}[R_Y(e^{j2\pi ft/LT})] \). Equation (3) can be rewritten as

\[
r_Y(e^{j2\pi ft/LT}) = (A^* \otimes A) \text{Vec}[R_x(f)], \tag{5}
\]

where the matrix identity \( \text{Vec}(AXB) = (B^T \otimes A) \text{Vec}(X) \) is used. The vector form of the diagonal matrix \( R_x(f) \) can be written as \( \text{Vec}[R_x(f)] = [P_1(f)e_1^T, P_2(f)e_2^T, ... , P_L(f)e_L^T]^T \), where \( P_i(f) \) is the \( i \)-th diagonal element and \( e_i \) is selected from the \( i \)-th column of a \( L \times L \) identity matrix. Further define \( \tilde{r}_x(f) = [P_1(f), P_2(f), ..., P_L(f)]^T \) and a \( L^2 \times L \) selection matrix \( B \) which has 1 at the \([(j-1)L + \tilde{j}] \)-th row and the \( \tilde{j} \)-th column. The vector form of \( R_x(f) \) can be written as \( \text{Vec}[R_x(f)] = B\tilde{r}_x(f) \). Equation (5) can be further expressed as

\[
r_{\tilde{j}}(e^{j2\pi ft/LT}) = \left( A^* \otimes A \right) B\tilde{r}_x(f) = \Phi \tilde{r}_x(f), \tag{6}
\]

where \( \Phi = (A^* \otimes A)B \) is a \( q^2 \times L \) matrix. If \( \Phi \) has full column rank, then least square algorithm can be used to estimate the power spectrum

\[
\hat{r}_x(f) = \Phi^+ r_Y(e^{j2\pi ft/LT}), \tag{7}
\]

where \( \Phi^+ \) is the Moore-Penrose inverse of \( \Phi \).

3. The Proposed Power Spectrum Estimation Algorithm

To begin with, let us explore the structure of matrix \( \Phi = (A^* \otimes A)B \). Let \( a_i^j \) and \( a_{ij} \), \( 1 \leq i \leq q \), denote the \( i \)-th row of matrix \( A^* \) and \( A \), respectively. \( A^* \) and \( A \) can be further written as \( A^* = (a_1^1, a_2^2, ..., a_q^q)^T \) and \( A = (a_1^T, a_2^T, ..., a_q^T)^T \), respectively. The structure of \( q^2 \times L^2 \) matrix \( A^* \otimes A \) is firstly explored and it can be written as
\[
A^* \otimes A = \begin{bmatrix}
(a_i^* \otimes a_j)^T, (a_i^* \otimes a_2)^T, \ldots, (a_i^* \otimes a_q)^T, \\
(a_2^* \otimes a_1)^T, (a_2^* \otimes a_2)^T, \ldots, (a_2^* \otimes a_q)^T, \\
\vdots \\
(a_q^* \otimes a_1)^T, (a_q^* \otimes a_2)^T, \ldots, (a_q^* \otimes a_q)^T
\end{bmatrix}
\]

where \(a_i^* \otimes a_k, 1 \leq i, k \leq q\) is the \((i-1)q + k\)-th row vector of matrix \(A^* \otimes A\). The \(L^2 \times L\) selection matrix \(B\) can be written as

\[
B = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1
\end{pmatrix}^T.
\]

Based on the structure of matrix \(B\), it is not difficult to find that multiplying a \(1 \times L^2\) vector \(a_i^* \otimes a_k\) with a \(L^2 \times L\) selection matrix \(B\) is equivalent to \(a_i^*.a_k\), where \(.*\) denotes the matrix dot product. So matrix \(\Phi = (A^* \otimes A)B\) can be rewritten as

\[
\Phi = \begin{pmatrix}
(a_1^*.a_1)^T, (a_1^*.a_2)^T, \ldots, (a_1^*.a_q)^T, \\
(a_2^*.a_1)^T, (a_2^*.a_2)^T, \ldots, (a_2^*.a_q)^T, \\
\vdots \\
(a_q^*.a_1)^T, (a_q^*.a_2)^T, \ldots, (a_q^*.a_q)^T
\end{pmatrix}
\]

Since the row vector \(a_i\) of matrix \(A\) can be written as \(a_i = \begin{bmatrix} e^{-j2\pi c_1 i} & e^{-j2\pi c_2 i} & \ldots & e^{-j2\pi c_L i} \end{bmatrix}\), the \((i-1)q + k\)-th row vector of matrix \(\Phi = (A^* \otimes A)B\) can be further written as

\[
a_i^*.a_k = \begin{bmatrix} e^{-j2\pi (c_k - c_l) i} & e^{-j2\pi (c_k - c_l) i} & \ldots & e^{-j2\pi (c_k - c_l) i} \end{bmatrix}.
\]

Note that \(a_i^*.a_k\), the \((i-1)q + k\)-th row of matrix \(\Phi\), has some special structure. It can be viewed as the \(c_k - c_l + 1\)-th row of the \(L \times L\) DFT matrix when \(c_k - c_l\) is greater than or equal to 0.

**Algorithm 1: Proposed Power Spectrum Estimation Algorithm**

**Step 1:**
Rearrange the vector \(r_Y(e^{j2\pi f/L})\) according to \(c_k - c_l + 1\) from small to large, and a new vector \(\tilde{r}_Y(e^{j2\pi f/L})\) can be obtained.

**Step 2:**
Average the elements corresponding to the same \(c_k - c_l + 1\) in \(\tilde{r}_Y(e^{j2\pi f/L})\) and only keep their average. At this point, the new matrix \(\Phi\) is a DFT matrix.

**Step 3:**
Estimate the power spectrum using the FFT.

The proposed power spectrum estimation algorithm is presented in Algorithm 1. It can be summarized as the following three steps. First, since the vector \(r_Y(e^{j2\pi f/L})\) is the result of vectorizing the matrix \(R_Y(e^{j2\pi f/L})\), the order of elements in vector \(r_Y(e^{j2\pi f/L})\) is independent of \(c_k - c_l + 1\). In order to take advantage of the property of DFT matrix, the vector \(r_Y(e^{j2\pi f/L})\) is rearranged according to \(c_k - c_l + 1\). We can then obtain a new vector \(\tilde{r}_Y(e^{j2\pi f/L})\). Secondly, elements in \(\tilde{r}_Y(e^{j2\pi f/L})\) corresponding to the same \(c_k - c_l + 1\) are averaged. Only the averaged results are kept and these original elements are deleted. Since we assume that the matrix \(\Phi\) has full column rank, we can then obtain a \(L \times L\) DFT matrix \(\Phi\). Finally, FFT is used to estimate the signal power spectrum.
Compared with the traditional least square algorithm, the proposed algorithm can save a large amount of computation. Table 1 gives the comparison of the computational complexity between the traditional algorithm and the proposed algorithm for a single frequency bin. ‘ADD’ and ‘MULT’ denote complex addition and complex multiplication, respectively. From Equation (7), we can find that the number of complex multiplication and addition required by the traditional algorithm are \( L \times [q(q-1)+1] \) and \( L \times q(q-1) \), respectively.

Since the rearrangement does not require addition and multiplication, we assume that the first step of the proposed algorithm does not increase computational complexity. The second step of the proposed algorithm averages the elements corresponding to the same \( c_k - c_i + 1 \). The number of complex addition required is \( q(q-1)+1-L \). Since the number of divisions in the second step is much smaller than its following step, and the specific number of divisions depends on the sampling pattern of MCS, the division in the second step is not taken into account in the total computational complexity. The third step uses the FFT to estimate the power spectrum. The computational complexity of this step depends on \( L \). According to the computational complexity of FFT [6], the number of complex multiplication and addition required are \( L/2 \times \log(L) \) and \( L \times \log(L) \), respectively. Therefore, the total number of complex multiplication and addition required by the proposed algorithm are \( L/2 \times \log(L) \) and \( L \times \log(L) + q(q-1)+1-L \), respectively.

### Table 1: Computational complexity comparison.

| MCS Parameters | Traditional | Proposed | Saving |
|----------------|-------------|----------|--------|
|                | ADD  | MULT  | ADD  | MULT  | ADD  | MULT  |
| \( L=64, q=10 \) | 5760 | 5824 | 539  | 256   |  90.7% | 95.7% |
| \( L=32, q=7 \)   | 1344 | 1376 | 171  | 80    |  87.3% | 94.2% |
| \( L=16, q=5 \)   | 480  | 496  | 79   | 32    |  83.6% | 93.6% |

4. Simulation Results

Quadrature Phase-Shift Keying (QPSK) signal is used as the test signal, and it is generated by the following model:

\[
\begin{align*}
    x(t) &= \sum_{k=1}^{M} \sum_{i=1}^{N_k} l_k[i] g_k(t-IT_k) \cos(2\pi f_k t) \\
    &\quad + \sum_{j=1}^{N_q} Q_k[j] g_k(t-JT_k) \sin(2\pi f_k t) + n(t), 
\end{align*}
\]

where \( M \) is the number of primary users, \( N_k \) is the number of random bits, \( l_k[i] \) and \( Q_k[j] \) are random bit streams, the pulse-shaping function \( g_k(t) \) is root-raised cosine with roll-over factor 0.1, \( T_k \) is the symbol duration, \( f_k \) is the carrier frequency of the \( k \)-th primary user, and \( n(t) \) is the additive white Gaussian noise. The bandwidth of the primary user is 8MHz. We consider several settings of MCS parameters. The first setting is \( L=16, q=5 \), and \( M=4 \). \( f_k \) is selected from \([0,128MHz]\) randomly. The second setting is \( L=32, q=7 \) and \( M=8 \). \( f_k \) is selected from \([0,256MHz]\) randomly. The third setting is \( L=64, q=10 \) and \( M=16 \). \( f_k \) is selected from \([0,512MHz]\) randomly. For each simulation, 200 trials are performed.

Fig. 1 shows the spectrum sensing performance at different Signal-to-Noise Ratios (SNR). Fig. 1(a) is the detection probability. Fig. 1(b) is the false alarm probability. The curves corresponding to ‘Setting 1(Traditional)’ and ‘Setting 1(proposed)’ are the spectrum sensing performance using the traditional least square algorithm and the proposed algorithm, respectively. It can be seen from Fig. 1 that the proposed algorithm can achieve the spectrum sensing performance similar to the traditional least square algorithm at much lower computational complexity.
5. Conclusion

The proposed power spectrum estimation algorithm greatly reduces the computational complexity of MCS based wideband spectrum sensing. The traditional least square algorithm is replaced with a computationally efficient algorithm which is based on FFT. Simulation shows that the proposed algorithm can achieve the spectrum sensing performance similar to the traditional method at much lower computational complexity. Moreover, the proposed algorithm is easy to implement in practice since current mainstream Field Programmable Gate Array (FPGA) manufacturers usually provide FFT Intellectual Property (IP) cores.

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References

[1] Yucek, T., and Huseyin, A.: ‘A survey of spectrum sensing algorithms for cognitive radio applications’, IEEE Commun. Surv. Tutor., 2009, 11, (1), pp. 116-130, doi: 10.1109/SURV.2009.090109

[2] Ma, Y., Gao, Y., Liang, Y.C., and Cui, S.G.: ‘Reliable and efficient sub-Nyquist wideband spectrum sensing in cooperative cognitive radio networks’, IEEE J. Sel. Areas Commun., 2016, 34, (10), pp. 2750-2762, doi: 10.1109/JSAC.2016.2605998

[3] Sun, W.C., Huang, Z.T., Wang, F.H., and Wang, X.: ‘Compressive wideband spectrum sensing based on single channel’, IET Electron. Lett., 2015, 51, (9), pp. 693-694, doi: 10.1049/el.2014.4223

[4] Yen, C.P., Tsai, Y.M., and Wang, X.D.: ‘Wideband spectrum sensing based on sub-Nyquist sampling’, IEEE Trans. Signal Process., 2013, 61, (12), pp. 3028-3040, doi: 10.1109/TSP.2013.2251342

[5] Liu, C.J., Wang, H.J., Zhang, J., and He, Z.M.: ‘Wideband Spectrum Sensing Based on Single-Channel Sub-Nyquist Sampling for Cognitive Radio’, Sensors, 2018, 18, (7): 2222, doi: 10.3390/s18072222

[6] Oppenheim, A.V., and Schafer, R.W.: ‘Digital Signal Processing’ (Prentice-Hall, New Jersey, NJ, USA, 1975)