An $H_1$-BMO duality theory in the framework of semigroups of operators

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Abstract

Let $(M, \mu)$ be a sigma-finite measure space. Let $(T_t)$ be a semigroup of positive preserving maps on $(M, \mu)$ with standard assumptions. We prove a $H_1$-BMO duality theory with assumptions only $(T_t)$ itself. The BMO is defined as spaces of functions $f$ such that $\sup_t \| T_t f - T_t T_{-t} f \|^2 < \infty$. The $H_1$ is defined by square functions of P. A. Meyer’s gradient form. Our argument does not rely on the geometric/metric structure of $M$ nor on the kernel of the semigroups of operators. This allows our main results extend to the noncommutative setting as well, e.g. the case where $L_\infty(M, \mu)$ is replaced by von Neumann algebras with a semifinite trace. We also prove a Carleson embedding theorem for semigroups of operators.

Key words: tent space, BMO space, Hardy space, Carleson measure, semigroup of positive operators, von Neumann algebra.

0 Introduction

E. Stein ([St70]) studied a “universal” $H^p$ theory for $1 < p < \infty$ in the framework of semigroup of operators. After Stein’s work, many other mathematicians (e.g. M. Cowling, P. A. Meyer, N. Varopoulos, Doung/Yan, Auscher/ McIntosh and their coauthors) have been working on Fourier multipliers and Hardy/BMO spaces associated with semigroups of operators (see [Cow83], [FS82], [Mey74], [Var80], [DY05], [ADM04], [HM09], [HLMY] etc.). In particular, Doung/Yan proved an $H_1$-BMO duality for semigroups of operators with heat kernel bounds in their remarkable article [DY05]. S. Hofmann and S. Mayboroda proved an $H_1$-BMO duality for semigroups of operators generated by divergence form elliptic operators. A novelty of Doung/Yan’s definitions of BMO

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and $H_1$ is that they are according to the growth of the kernel of the underlying semigroups of operators.

One aim of those work is to establish a Hardy spaces theory which relies on less geometric/metric properties of Euclidean spaces. This article continues the effort to this aim and provides an abstract approach to an $H_1$-BMO duality with assumptions only on the semigroups of operators. By “abstract”, we mean that the formulation of the duality is unified and the corresponding constants are absolute, as in Stein’s work for Littlewood-Paley theory in the case of $1 < p < \infty$.

Let $(M, \mu)$ be a sigma-finite measure space. Let $(T_t)$ be a standard semigroup of operators on $L^p(M)$. Under standard assumptions (see Def 1.1), $T_t$ can be viewed as a natural alternative to the classical mean value operator. This led to two natural definition of BMO-norms,

$$\|f\|_{BMO(T)} = \sup_t \|T_t|f - T_t f|^2\|_\infty^{\frac{1}{2}}.$$  

$$\|f\|_{bmo(T)} = \sup_t \|T_t|f|^2 - |T_t f|^2\|_\infty^{\frac{1}{2}}.$$  

These two BMO norms are equivalent to the usual interval BMO norms if $(T_t)$ are the heat semigroups on $\mathbb{R}^n$. But $\| \cdot \|_{BMO(T)}$ and $\| \cdot \|_{bmo(T)}$ may not be equivalent to each other in general. Let $BMO(T)$ and $bmo(T)$ be Banach spaces characterized by the corresponding BMO norms. In [JM12], Junge and the author of this paper prove that an interpolation result holds between $BMO(T)$ and $L_1(M)$ and, the interpolation theory for $bmo(T)$ and $L_1(M)$ also hold with an additional continuity assumption (see Lemma 1.6).

Let us consider the following analogues of Lusin area integral and Littlewood-Paley $G$ function.

$$S_T(f) = (\int_0^\infty T_s \Gamma(T_s f) ds)^{\frac{1}{2}}.$$  

$$G_T(f) = (\int_0^\infty \Gamma(T_s f) ds)^{\frac{1}{2}}.$$  

Here $\Gamma(\cdot)$ is P. A. Meyer’s gradient form (Carré du Champ) for semigroups of operators (see Def 1.3). If $(T_t)$ are the classical heat semigroups (or Ornstein-Uhlenbeck) semigroups on $\mathbb{R}^n$ (with the Gaussian measure), then

$$\Gamma(f) = |\partial_x f|^2.$$  

This is also the case, if $T_t$ is the Laplace-Beltrami operator on a Riemannian manifold. We use $\int_0^\infty T_s(\cdot) ds$ in the definition of the S-function as an alternative
to the integration on the cones.

Let us define $H^S_1(T)$ (resp. $H^G_1(T)$) as the space of all $f \in L^1(M)$ such that $\|f\|_{H^S_1(T)} = \|S_T(f)\|_1 < \infty$ (resp. $\|f\|_{H^G_1(T)} = \|G_T(f)\|_1 < \infty$).

Denote by $\tau(\cdot)$ the integration operator $f \cdot d\mu$ on $M$. Our first result is the following duality inequality between $H_1$ and BMO.

**Theorem 0.1** Let $(T_t)$ be a standard semigroup of operators satisfying the $\Gamma_2 \geq 0$ condition (1.5). Then $bmo(T) \subset (H^S_1(T))^*$ and

$$\tau(fg) \leq c_1 \|f\|_{H^S_1(T)} \|g\|_{bmo(T)} \leq c_2 \|f\|_{H^G_1(T)} \|g\|_{bmo(T)},$$

with absolute constants $c_1, c_2$.

Our attention then turns to the following questions.

- When does the other direction $bmo(T) \supseteq (H^S_1(T))^*$ hold?
- When does $bmo(T) = BMO(T)$?
- When does $H^S_1(T) = H^G_1(T)$?

The answers to them are all “yes” if the semigroup $(T_t)$ satisfies two more conditions.

**Theorem 0.2** Let $T_t$ be a standard semigroup of operators satisfying the $\Gamma_2 \geq 0$ condition (1.5). Assume, in addition, that there exist constants $c_3, c_4, r > 0$ such that

(i) $\|(T_{t+\varepsilon} - T_t)f\|_1 \leq c_3 \varepsilon r \|f\|_1$, for all $\varepsilon > 0, t > 0$ and $f \in L^1(M)$.

(ii) Denote $M_t = \frac{1}{t} \int_0^t T_s ds$.

$$\|(M_t f)^2\|_{L^1(M)} \leq c_4 \|f\|_{L^1(M)}, \quad (0.1)$$

for all $t > 0$ and $f \in L^1_+(M)$.

Then every linear functional $\ell$ on $H^S_1(T)$ can be represented as $\tau(g \cdot)$ with $\|g\|_{bmo(T)} \simeq c \|\ell\|_{(H^S_1(T))^*}$. Moreover,

$$bmo(T) = BMO(T) \quad \text{and} \quad H^G_1(T) = H^S_1(T).$$

The equivalent constants only depend on $c_3, c_4, r$.

Our argument does not rely on any geometric/metric structure of $M$ nor on the kernel of the semigroups of operators. In fact, our argument only needs an
abstract $L_\infty$ space and a standard semigroup of operators on this space satisfying the assumptions in Theorem 0.1 and 0.2. We consider these assumptions as an reflection of the geometric properties of $M$, as what in the mind of many other researchers, e.g. D. Bakry, etc. This abstract argument allows to extend our main results to the noncommutative setting, which is our final goal. The drawback is that we require the semigroups of operators are positive preserving, while Doung/Yan and Hofmann/Mayboroda’s theorems go beyond those type of semigroups of operators.

From the point of view of functional analysis, every $L_\infty$ space on a sigma-finite measure spaces is a commutative von Neumann algebra. This led to defining noncommutative $L^p$ spaces as von Neumann algebras $\mathcal{M}$ with “nice” linear functionals, called traces and denoted by $\tau$, which play the role of integration with respect to $\mu$.

The importance of analyzing semigroups of operators on von Neumann algebras has been impressively demonstrated by the recent work of Popa and Ozawa [OP10] and also occurs in the work of Shlyahktenko/Connes [CS05] on Betti numbers for von Neumann algebras. M. Junge and the author of this article build up a connection between semigroups of operators on von Neumann algebras and M. Rieffel’s quantum metric spaces (see [Rie], [JM10]).

Noncommutative analogues of analytic Hardy spaces have been developed mainly by W. Arveson (see [A67]). Pisier/Xu and their collaborators have established the noncommutative theory of martingale Hardy spaces (see [PX97]). Junge-Le Merdy-Xu studied noncommutative real Hardy spaces for $1 < p < \infty$ in [JLX06]. In particular, an $H_1$-BMO duality for noncommutative martingales is proved in [PX97] and [JX03]. [M07] proves an analogue of the classical real variable $H_1$-BMO duality in the semi-commutative case. [M08] is a first try on a real variable $H_1$-BMO duality in the general noncommutative setting. [JM12] established an interpolation result between semigroup BMO spaces and noncommutative $L^p$ spaces. Noncommutative fourier multiplier theories are further developed in [JM10] and [JMP] by using BMO spaces associated with semigroups of operators defined above. This article improves the method developed in [M08] and extends Theorem 0.1 and 0.2 to the noncommutative case in Section 3.
1 Preliminaries

1.1 Semigroups of operators

Let \((M, \sigma, \mu)\) be a sigma-finite measure space. Let \(L^p(M)\) be the space of all complex valued \(p\)-integrable functions on \(M\). Denote by \(f^*\) the pointwise complex conjugate of a function \(f\) on \(M\).

**Definition 1.1** A family of operators \((T_y)_y\) is a standard semigroup of operators, if \(T_y T_{y_2} = T_{y_1 + y_2}, T_0 = \text{id}\) and

(i) \(T_y\) are contractions on \(L^p(M)\) for all \(1 \leq p \leq \infty\).

(ii) \(T_y\) are symmetric, i.e. \(T_y = T_y^*\) on \(L^2(M)\).

(iii) \(T_y(1) = 1\)

(iv) \(T_y(f) \to f\) in \(L^2\) as \(y \to 0^+\) for \(f \in L^2\).

The conditions (i), (iii) above imply \(T_y\) is positivity preserving for each \(y\), i.e. \(T_y(f) \geq 0\) if \(f \geq 0\). We will need the following Kadison-Schwarz inequality for unital (completely) positive contraction \(T\) on \(L^p(M)\),

\[
|T(f)|^2 \leq T(|f|^2), \quad \forall f \in L^p(M).
\] (1.2)

A standard semigroup \((T_y)\) always admits an infinitesimal generator \(L = \lim_{y \to 0} \frac{T_y - \text{id}}{y}\). \(L\) is a unbounded operator densely defined on \(L^2(M)\). We will write \(T_y = e^{yL}\). Some of the conditions (i)-(iv) may be weaken but that is beyond the main interests of this article.

**Definition 1.2** P. A Meyer’s gradient form \(\Gamma\) (also called “Carré du Champ”) associated with \(T_t\) is defined as,

\[
2\Gamma(f, g) = L(f^*g) - (L(f^*)g) - f^*(L(g)),
\] (1.3)

for \(f, g\) with \(f^*, g, f^*g \in D(L)\).

When \(f = g\), we simply write \(\Gamma(f) = \Gamma(f, g)\).

For convenience, we assume that there exists a \(^*\)-algebra \(\mathcal{A}\) which is weak* dense in \(L_\infty(M)\) such that \(T_s(\mathcal{A}) \subset \mathcal{A} \subset D(L)\). This assumption is to guarantee that \(\Gamma(T_s f, T_s g)\) make senses for \(f, g \in \mathcal{A}\), which is not easy to verify in general, although the other form \(T_t \Gamma(T_s f, T_s g)\) is what we need essentially in
this article and can be read as \( LT_t(Tsf^*T sg) - T_t((LTsf^*)T sg) - T_t(Tsf^*(LTsg)) \) for any \( f, g \in L_p(M), 1 \leq p \leq \infty, s, t > 0. \)

It is easy to verify that for \( L = \Delta = \frac{\partial^2}{\partial x^2}, \Gamma(f, g) = \frac{\partial f^*}{\partial x} \cdot \frac{\partial g}{\partial x}. \) It is well known that the positive-preserving property of a standard semigroup of operators implies that \( \Gamma(f) \geq 0 \) for all \( f. \)

**Definition 1.3** Bakry-Émery’s iterated gradient form \( \Gamma_2 \) is defined as

\[
\Gamma_2(f, g) = L\Gamma(f, g) - \Gamma(f^*L(g)) - \Gamma((Lf^*)g),
\]

for \( f, g \in A. \)

When \( f = g, \) we simply write \( \Gamma_2(f) = \Gamma_2(f, g). \) For \( L = \Delta = \frac{\partial^2}{\partial x^2}, \Gamma_2(f, g) = \frac{\partial^2 f^*}{\partial x^2} \cdot \frac{\partial g}{\partial x}. \) However, \( \Gamma_2(f, f) \geq 0 \) is not always true. For \( L \) being the Laplace-Beltrami operator on a complete manifold, \( \Gamma_2(f) \geq 0 \) is equivalent to the positivity of the Ricci curvature of the manifold. See [BBG12] and references therein for more details on \( \Gamma_2 \) and examples of semigroups of operators satisfying the “\( \Gamma_2 \geq 0 \)” condition, which is the so-called curvature-dimension criterion \( CD(0, \infty) \) in [BBG12]. We still use the relative old notation “\( \Gamma_2 \geq 0 \)” because it makes sense even in the noncommutative setting.

It is easy to check that, for a standard semigroup of operator \( (T_t)_t, \Gamma_2(f) \geq 0 \) if

\[
\Gamma(Tv f) \leq T_s \Gamma(f)
\]

for all \( v > 0, f \in A. \)

We will need the following Lemma due to P.A. Meyer. We add a short proof for the convenience of the reader.

**Lemma 1.3** For any \( f \in L_p(M), 1 \leq p \leq \infty, s > 0, \) we have

\[
T_s|f|^2 - |T_s f|^2 = 2 \int_0^s T_{s-t} \Gamma(T_t f) dt.
\]

**Proof.** For \( s \) fixed, let

\[ F_t = T_{s-t}(|T_t f|^2). \]

Then

\[
\frac{\partial F_t}{\partial t} = \frac{\partial T_{s-t}(|T_t f|^2)}{\partial t} + T_{s-t}[(\frac{\partial T_t}{\partial t} f^*) f] + T_{s-t}[f^* (\frac{\partial T_t}{\partial t} f)]
\]

\[ = -T_{s-t} \Gamma(T_t f). \]
Therefore

\[ T_s|f|^2 - |T_s f|^2 = -F_s + F_0 = \int_0^s T_{s-t} \Gamma(T_t f) dt. \]

Definition 1.4 Given a standard semigroup of operators \((T_y)_y\) with an infinitesimal generator \(L\), the semigroup \((P_y)_y\) defined as

\[ P_y = e^{-y \sqrt{-L}} \]

is again a standard semigroup of operators. We call it the subordinated Poisson semigroup of \((T_y)_y\).

Note \(P_y\) is chosen such that

\[ \left( \frac{\partial^2}{\partial s^2} + L \right) P_s = 0. \] (1.6)

It is well known that (see [St2])

\[ P_y = \frac{1}{2\sqrt{\pi}} \int_0^\infty ye^{-\frac{y^2}{4u}} u^{-\frac{3}{2}} T_u du. \] (1.7)

Apply \(\Gamma(f) \geq 0\) and (1.7), it is easy to deduce that \(\Gamma_2 \geq 0\) also implies \(\Gamma(P_v f, P_v f) \leq P_v \Gamma(f)\) for any \(v > 0\).

(1.7) also implies that

\[ \frac{P_y}{y}(f) \leq \frac{P_t}{t}(f) \text{ and } |(P_y - P_{y+t}) f| \leq \frac{8t}{y} P_{\frac{y}{2}} f, \] (1.8)

for any \(0 \leq t \leq y, f \geq 0\), since \(T_u\) is positive and \(e^{-\frac{y^2}{4u}} u^{-\frac{3}{2}}\) is a function decreasing with respect to \(y\).

The classical heat semigroup and Ornstein-Uhlenbeck semigroup on \(\mathbb{R}^n\) are typical examples of standard semigroups of operators. They can be presented as

\[ T_t = e^{t\Delta} \]
\[ O_t = e^{t(\frac{1}{2} \Delta + x \cdot \frac{\partial}{\partial x})} \]

with \(\Delta = \frac{\partial^2}{\partial x^2} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\), the Laplacian operator on \(\mathbb{R}^n\).
It is easy to check that for these two semigroups,

$$\Gamma(f, g) = \frac{\partial}{\partial x} f^* \cdot \frac{\partial}{\partial x} g,$$

and they both satisfy the $\Gamma_2 \geq 0$ condition (1.5).

1.2 BMO spaces associated with semigroups of operators

Recall we set

$$\|f\|_{\text{bmo}(T)} = \sup_{0 < t < \infty} \|T_t|f|^2 - |T_t f|^2\|_{L^1_{\infty}}^{\frac{1}{2}}.$$  

(1.12)

$$\|f\|_{\text{BMO}(T)} = \sup_{0 < t < \infty} \|T_t|f - T_t f|^2\|_{L^1_{\infty}}^{\frac{1}{2}}.$$  

(1.13)

It is easy to see by Kadison-Schwarz inequality (1.2) that for any positive sequence $t_n$ which converges to 0,

$$\|f\|_{\text{bmo}(T)} \simeq \sup_{n \in \mathbb{N}} \|T_{t_n}|f|^2 - |T_{t_n} f|^2\|_{L^1_{\infty}}^{\frac{1}{2}}.$$  

(1.14)

$$\|f\|_{\text{BMO}(T)} \simeq \sup_{n \in \mathbb{N}} \|T_{t_n}|f - T_{t_n} f|^2\|_{L^1_{\infty}}^{\frac{1}{2}}.$$  

(1.15)

**Lemma 1.4** ([JM12]) Let $(T_t)$ be a standard semigroup of operators. Then (i) $\|f\|_{\text{bmo}(T)} = 0$ iff $\|f\|_{\text{BMO}(T)} = 0$ iff $f \in \ker(L) = \{f \in D(L)Lf = 0\}$. (ii) If in addition $(T_t)$ satisfies the $\Gamma_2 \geq 0$ condition (1.5), then

$$\|f\|_{\text{BMO}(T)} \simeq \|f\|_{\text{bmo}(T)} + \sup_t \|T_t f - T_{2t} f\|.$$  

**Lemma 1.5** Suppose $(T_t)$ is a standard semigroup of operators. Then

$$\sup_t \|(T_t|f - T_t f(\cdot))^p(\cdot)\|_{L^1_{\infty}}^{\frac{1}{p}} \simeq^p \|f\|_{\text{bmo}(T)},$$  

(1.16)

for any $0 < p < \infty$. And, if $(T_t)$ satisfies the $\Gamma_2 \geq$ condition (1.5), then

$$\sup_t \|T_t|f - T_t f|^p\|_{L^1_{\infty}}^{\frac{1}{p}} \simeq^p \|f\|_{\text{BMO}(T)},$$  

(1.17)

for any $0 < p < \infty$.  

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Proof. Let $W_t$ be a Markov process, so that

$$E_s f(W_t) = T_{t-s} f(W_s), \quad (1.18)$$

for any $0 \leq s < t < \infty$. The Markov process $(W_t)$ can be constructed by setting its covariance function $C_{s,t} = K_{t-s}$, the kernel of $(T_t)$. Fix a $t > 0$, we consider the martingale $E_s(f(W_t))$, $s = 2^{-n} t$, $n \in \mathbb{N}$. By the John-Nirenberg inequality for the little BMO space of martingales, we have,

$$\sup_{s=2^{-n} t} \| E_s(|f(W_t) - E_s f(W_t)|^p) \|_\infty \simeq_p \sup_{s=2^{-n} t} \| E_s|f(W_t) - E_s f(W_t)|\|_p. \quad (1.19)$$

On the other hand, by the Markov property, we have that, on the set \{ $W_s = v$ \},

$$E_s(|f(W_t) - E_s f(W_t)|^p) = E_s(|f(W_t) - T_{t-s} f(v)|^p) = (T_{t-s}|f - T_{t-s} f(v)|^p)(v). \quad (1.20)$$

Therefore,

$$\| E_s(|f(W_t) - E_s f(W_t)|^p) \|_\infty = \| (T_{t-s}|f - T_{t-s} f(\cdot)|^p)(\cdot) \|_\infty. \quad (1.21)$$

Combining (1.19) and (1.21) we obtain the John-Nirenberg inequality for $bmo(T)$,

$$\sup_t \| (T_t|f - T_t f(\cdot)|^p)(\cdot) \|^\frac{1}{p} \simeq_p \| f \|_{bmo(T)}, \quad (1.22)$$

for any $0 < p < \infty$.

Now, for any $p > 2$,

$$\begin{align*}
& (T_t|f - T_t f(\cdot)|^p(\cdot))^{\frac{1}{p}} \\
& \leq (T_t|f - T_t f(\cdot)|^p(\cdot))^{\frac{1}{p}} + (T_t|T_t f - T_t f(\cdot)|^p(\cdot))^{\frac{1}{p}} \\
& \leq [T_t(|f - T_t f(\cdot)|^p(\cdot))]^{\frac{1}{p}} + [T_{2t}|f - T_{2t} f(\cdot)|^p(\cdot)]^{\frac{1}{p}} + ([T_t(f) - T_{2t} f]|(\cdot)).
\end{align*}$$

Taking supremum on both sides, we get by (1.16) and Lemma 1.4 (ii) that

$$\begin{align*}
\sup_t \| T_t(|f - T_t f|^p) \|^\frac{1}{p} \leq & \| f \|_{bmo(T)} + \sup_t \| T_t(f) - T_{2t} f \|_\infty \\
& \leq c_p \| f \|_{BMO(T)}.
\end{align*} \quad (1.23)$$
For $0 < q < 2$, let $p = 4 - q$, by Hölder’s inequality,

$$T_t(|f - T_t f|^2)(\cdot) = T_t(|f - T_t f|^q |f - T_t f|^{\frac{4-q}{2}})(\cdot) \leq [T_t(|f - T_t f|^q)(\cdot)]^{\frac{1}{2}} \cdot [T_t(|f - T_t f|^p)(\cdot)]^{\frac{1}{2}}.$$  

Taking supremum on both sides and applying (1.23), we get

$$\|f\|_{BMO(T)} \leq c_q \sup_t \|T_t(|f - T_t f|^q)\|^{\frac{1}{q}} \|f\|_{BMO(T)}.$$ 

Therefore,

$$\|f\|_{BMO(T)} \leq c_q \sup_t \|T_t(|f - T_t f|^q)\|^{\frac{1}{q}}.$$ 

\[\square\]

**Lemma 1.6** ([JM12]) Assume that $(T_t)$ is a standard semigroup of operators. Then

$$[BMO(T), L^0_1(M)]^{\frac{1}{p}} = L^0_p(M)$$

for $1 < p < \infty$. If, in addition, $(T_t)$ admits a Markov dilation which has a. u. continuous path, then

$$[bmo(T), L^0_1(M)]^{\frac{1}{p}} = L^0_p(M)$$

for $1 < p < \infty$.

Here $L^0_p(M) = L_p(M)/\ker L$.

**Remark 1.1** Lemma 1.4 could include here a proof of Lemma 1.6 in a few lines, if the little BMO space of martingales works as an interpolation-end point for $L_p$ martingale spaces. But that is not the case in general.

## 2 Proof of Theorem 0.1, 0.2

Recall, for $f \in L_1(M)$, we set

$$S_{\Gamma}(f) = (\int_0^\infty T_s \Gamma(T_s f) ds)^{\frac{1}{2}},$$

$$G_{\Gamma}(f) = (\int_0^\infty \Gamma(T_s f) ds)^{\frac{1}{2}}.$$
Set
\[
\|f\|_{\mathcal{H}_1^\Gamma(T)} = \|S_T(f)\|_{L^1}, \\
\|f\|_{\mathcal{H}_1^G(T)} = \|G_T(f)\|_{L^1}.
\]

It is easy to see that
\[
\|f\|_{\mathcal{H}_1^G} \leq 2\|f\|_{\mathcal{H}_1^\Gamma},
\]
by \(\Gamma_2 \geq 0\). Let \(\mathcal{H}_1^\Gamma, \mathcal{H}_1^G, \text{bmo}(T)\) and \(\text{BMO}(T)\) be the corresponding Banach spaces.

Set truncated square functions \(S_s, G_s\) as follows:
\[
S_s = \left( \int_s^\infty T_{y-\frac{s}{2}}(\Gamma(T_{y+\frac{s}{2}}f)dy) \right)^{\frac{1}{2}},
\]
\[
G_s = \left( \int_s^\infty \Gamma(T_{2y}f)dy \right)^{\frac{1}{2}}.
\]

\(S_s, G_s\) are constructed to satisfy our key Lemma.

**Lemma 2.1**

\[
G_s \leq S_s;
\]
\[
\frac{dT_{(a+b)s}(S_s)}{ds} \geq \frac{a+b}{(a+b)s}T_{as}(\frac{dT_{bs}(S_s)}{ds}),
\]
\[
\frac{dT_{\frac{a+b}{2}s}(S_s)}{ds} \leq 0,
\]
for any \(a, b > 0\).

**Proof.** (2.4) is true because of the fact
\[
\Gamma(T_{2y}f) \leq T_{y-\frac{s}{2}}\Gamma(T_{y+\frac{s}{2}}f),
\]
which follows from the \(\Gamma_2 \geq 0\) condition (1.5).

Apply (1.5) again, we get \(S_s \geq S_t\) for any \(s \leq t\), then
\[
T_{(a+b)s+(a+b)\Delta s}(S_{s+\Delta s}) - T_{(a+b)s}(S_s)
\]
\[
= T_{as}[T_{bs+(a+b)\Delta s}(S_{s+\Delta s}) - T_{bs}(S_s)]
\]
\[
\geq T_{as}[T_{bs+(a+b)\Delta s}(S_{s+\frac{a+b}{b}\Delta s}) - T_{bs}(S_s)].
\]
Divide by $\Delta s$ both sides, we get (2.5).

We go to prove (2.6). By (1.2) and $\Gamma_2 \geq 0$ condition (1.5), we get

\[
T_{\Delta s} S_{s+2\Delta s} = T_{\Delta s} \left( \int_{s+2\Delta s}^{\infty} T_y - \frac{s}{2} - \Delta s \Gamma(T_y + \frac{s}{2} + \Delta s f)dy \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int_{s+2\Delta s}^{\infty} T_y - \frac{s}{2} \Gamma(T_y + \frac{s}{2} + \Delta s f)dy \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int_{s+2\Delta s}^{\infty} T_y - \frac{s}{2} + \frac{\Delta s}{2} \Gamma(T_y + \frac{s}{2} + \frac{\Delta s}{2} f)dy \right)^{\frac{1}{2}}
\]

\[
= \left( \int_{s+2\Delta s}^{\infty} T_u - \frac{s}{2 + \Delta s} \Gamma(T_u + \frac{s}{2 + \Delta s} f)du \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int_{s}^{\infty} T_y - \frac{s}{2} \Gamma(T_y + \frac{s}{2} f)dy \right)^{\frac{1}{2}} = S_s.
\]

Then

\[
T_{\frac{s+2\Delta s}{2}} S_{s+2\Delta s} - T_{\frac{s}{2}} S_s \leq 0.
\]

Taking $\Delta s \to 0$ we obtain (2.6).

**Lemma 2.2** We have

\[
|\tau \int_{0}^{\infty} \Gamma(T_{2s} f, T_{(2+v)s} \varphi_s) ds |
\]

\[
\leq (8 + 4v)^{\frac{1}{2}} \|G_{\Gamma}(f)\|_{1}^{\frac{1}{2}} \left( -\tau \int_{0}^{\infty} (T_{\frac{s}{2} + v}) y \int_{0}^{y} \Gamma(\varphi_s) ds \frac{\partial T_y}{\partial y} (S_y) dy \right)^{\frac{1}{2}}
\]

for any $v > 0$ and $f \in \mathcal{A}$ and any $\varphi_s \in \mathcal{A}$.

**Proof.** We can assume $G_s, S_{v,s}$ are invertible by approximation. By (1.2), (2.1) and Cauchy-Schwarz inequality, we get

\[
|\tau \int_{0}^{\infty} \Gamma(T_{2s} f, T_{(2+v)s} \varphi_s) ds |
\]

\[
\leq (\tau \int_{0}^{\infty} \Gamma(T_{2s} f) G_{s}^{-1} ds)^{\frac{1}{2}} (\tau \int_{0}^{\infty} \Gamma(T_{(2+v)s} \varphi_s) G_{s} ds)^{\frac{1}{2}}
\]

\[
\overset{def}{=} I^\frac{1}{2} II^\frac{1}{2}.
\]

Note that $-\frac{\partial G_s^2}{\partial s} = \Gamma(T_{2s} f)$. For $I$, we have

\[
I = \tau \int_{0}^{\infty} -\frac{\partial G_s^2}{\partial s} G_{s}^{-1} ds = 2\tau \int_{0}^{\infty} -\frac{\partial G_s}{\partial s} ds = 2\|G_0\|_1.
\]
We estimate $II$. By (2.4) and $\Gamma_2 \geq 0$ we have

\[
II \leq \tau \int_0^{\infty} \Gamma(T_{(2+v)s} \varphi_s)S_s ds \\
\leq \tau \int_0^{\infty} \int_0^{\infty} \Gamma(\varphi_s)\int_s^\infty - \frac{\partial T_{(2+v)s}(S_y)}{\partial y} dy ds \\
= -\tau \int_0^{\infty} \int_0^{y} \Gamma(\varphi_s)ds \frac{\partial T_{(2+v)s}(S_y)}{\partial y} dy.
\]

(2.7)

Applying (2.6), with $a, b = \frac{3}{2} + v, \frac{1}{2}$ to (2.8), we get

\[
II \leq -(4 + 2v)\tau \int_0^{\infty} \int_0^{y} \Gamma(\varphi_s)ds T_{(\frac{3}{2} + v)y}(\frac{\partial T_{\frac{3}{2}}(S_y)}{\partial y}) dy \\
= -(4 + 2v)\tau \int_0^{\infty} (T_{(\frac{3}{2} + v)y}) \int_0^{y} \Gamma(\varphi_s)ds \frac{\partial T_{\frac{3}{2}}(S_y)}{\partial y} dy.
\]

(2.8)

Combining the estimates of $I$ and $II$, we get the desired inequality.

**Proof of Theorem 0.1.** Note $\tau Lf = 0, \tau (Lf)g = \tau f(Lg)$ for $f, g \in \mathcal{A}$. By the definition of $\Gamma$ (1.3),

\[
\tau f \varphi^* = -\tau \int_0^{\infty} \frac{\partial}{\partial s}(T_{2s}fT_{(3+v)s} \varphi^*) ds = (5 + v)\tau \int_0^{\infty} \Gamma(T_{2s}f, T_{(3+v)s} \varphi) ds
\]

Applying Lemma 2.2 to $\varphi_s = T_s \varphi$, we get

\[
|\tau f \varphi^*| \leq 12\sqrt{3}\|G_{\Gamma}(f)\|_1 \left(-\tau \int_0^{\infty} \int_0^{y} T_{(\frac{3}{2} + v)y} \Gamma(T_s \varphi) ds \frac{\partial T_{\frac{3}{2}}(S_y)}{\partial y} dy\right)^{\frac{1}{2}},
\]

for any $0 < v < 1$.

Denote $M_y = \frac{1}{y} \int_0^{y} T_s ds$. Integrate on $v$ for $0 < v < 1$, we have

\[
|\tau f \varphi^*|^2 \leq c\|G_{\Gamma}(f)\|_1 \left(-\tau \int_0^{\infty} \int_0^{y} M_y T_{(\frac{3}{2})y} \Gamma(T_s \varphi) ds \frac{\partial T_{\frac{3}{2}}(S_y)}{\partial y} dy\right),
\]

Then, by (2.6), we have
\[ |τfϕ^*|^2 \leq c\|G_Γ(f)\|_1(\sup_y \| \int_0^y M_y T_γ(y) Γ(T_γϕ)ds \|_{∞}) \int_0^∞ - \frac{∂T_y(S_y)}{∂y} dy \]

\[ \leq c\|G_Γ(f)\|_1(\sup_y \| \int_0^y M_y T_γ(y) Γ(T_γϕ)ds \|_{∞}) \|S_Γ(f)\|_1. \]

On the other hand, note \( M_y T_{\frac{y}{2}+s} \leq 3M_y \) for \( 0 < s < y \), we have

\[ \int_0^y M_y T_γ(y) Γ(T_γϕ, T_γϕ)ds = \int_0^y M_y T_{\frac{y}{2}+s} T_{y-s} Γ(T_γϕ)ds \]

\[ \leq 3M_y \int_0^y T_{y-s} Γ(T_γϕ)ds \]

(Lemma 1.3) = \( 3M_y (T_y |ϕ|^2 - |T_yϕ|^2) \).

Therefore,

\[ \sup_y \| \int_0^y M_y T_γ(y) Γ(T_γϕ)ds \|_{∞} \leq 3\|ϕ\|_{bmo(Γ)}^2, \]

because \( M_{3y} \) is a bounded operator on \( L_∞ \).

We concluded that

\[ |τ(fϕ^*)|^2 \leq c\|G_Γ(f)\|_1\|S_Γ(f)\|_1\|ϕ\|_{bmo(Γ)}^2. \]

\[ \square \]

**Remark 2.2** Known examples of semigroups operators satisfying the \( Γ_2 ≥ 0 \) conditions include all standard semigroups of operators on group von Neumann algebras (see Example 3), the Ornstein-Uhlenbeck semigroups on \( \mathbb{R}^n \), and the heat semigroups generated by the Laplace-Beltrami operator on a compact manifold with positive curvature.

**Remark 2.3** The proofs of Lemma 2.1 and 2.2 work for any bilinear form \( B(\cdot, \cdot) \) satisfying (i) \( B(f, f) ≥ 0 \); (ii) \( B(T_γf, T_γf) ≤ T_γB(f, f) \). In particular, the proofs work for \( B(f_s, g_s) = s^2 \frac{∂f_s}{∂s} \frac{∂g_s}{∂s} \) and will give the estimation

\[ |τfg^*| \leq c(\int_0^{∞} \| T_γf \|_{L_1(Γ)}^2 sds) \frac{1}{2} \| T_γf \|_{L_1(Γ)} \| T_γg \|_{L_1(Γ)} \sup_t \| T_t \| \| T_s \|_{L_∞(Γ)} \int_0^{∞} \| T_s \|_{L_1(Γ)}^2 sds \|_{L_∞(Γ)} \].

The author shows in [M08] that,

\[ \sup_t \| T_t \| \int_0^{t} \| T_s \|_{L_∞(Γ)}^2 sds \|_{L_1(Γ)} \leq c\|g\|_{bmo(Γ)}, \]

\[ (2.9) \]
if $T_s$ is a subordinated Poisson semigroup. It would be nice if there is a less strict assumption on $T_t$ which implies (2.9).

Proof of Theorem 0.2. Note by (1.17),

$$\|\varphi\|_{BMO(T)} = \sup_t \|T_t|\varphi - T_t\varphi\|_{\infty} = \sup_f \tau(\varphi f), \quad (2.10)$$

Here the supremum takes for all $f = hT_t(g) - T_t(hT_tg)$ with $g \geq 0$, $\|g\|_1, \|h\|_\infty \leq 1$. We need to show that such $f$'s are in $H^S_1(T)$ with norm $\leq c$. The inclusion $(H^S_1)^* \subset BMO(T)$ then follows from a density argument. The equivalence $(H^S_1)^* = bmo(T) = BMO(T)$ and $H^G_1(T) = H^S_1(T)$ follow from inequality (2.1) and Lemma 1.4.

Fix such a $f$. Recall $M_t = \frac{1}{t} \int_0^t T_s ds$.

$$\tau(\int_0^t T_s \Gamma(T_s f) ds)^{\frac{1}{2}} \leq \tau(\int_0^t M_t T_s \Gamma(T_s f) ds)^{\frac{1}{2}}$$

$$\leq \tau(\int_0^t 3M_t T_{s-t} \Gamma(T_s f) ds)^{\frac{1}{2}}$$

(Lemma 1.3) $= \tau(3M_t T_t |f|^2 - 3M_t |T_t f|^2)^{\frac{1}{2}}$

$$\leq \tau(3M_t T_t |f|^2)^{\frac{1}{2}}$$

$$\leq 2\sqrt{3} \tau(M_t T_t |T_t g|^2)^{\frac{1}{2}}$$

(apply assumption $(ii)$) $\leq 4\sqrt{2} \tau(M_{2t} |T_t g|^2)^{\frac{1}{2}} \leq c$. (2.11)

For the other part,

$$\tau(\int_t^\infty T_s \Gamma(T_s f) ds)^{\frac{1}{2}} = \tau(\sum_{n=0}^\infty \int_{2^n t}^{2^{n+1} t} T_s \Gamma(T_s f) ds)^{\frac{1}{2}}$$

$$\leq \sum_{n=0}^\infty \tau(\int_{2^n t}^{2^{n+1} t} T_s \Gamma(T_s f) ds)^{\frac{1}{2}}$$

(let $v = s - 2^n t$) $\leq \sum_{n=0}^\infty \tau(\int_0^{2^n t} T_{2^n t+v} \Gamma(T_v T_{2^n t} f) dv)^{\frac{1}{2}}$

(apply (2.11)) $\leq \sum_{n=0}^\infty \tau(3M_{3:2^n t} T_{2^n t} |T_{2^n t} T_{2^n - t} f|^2)^{\frac{1}{2}}$

$$\leq \sum_{n=0}^\infty \tau(4M_{8:2^n - 1 t} |T_{2^n - t} T_{2^n - 1 t} f|^2)^{\frac{1}{2}}$$

(apply assumption $(ii)$) $\leq c \sum_{n=0}^\infty \|T_{2^n - 1 t} f\|_1$. 

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Note by assumption (i) we have
\[ \|T_{2^n-1}f\|_1 = \|T_{2^n-1}(T_t hT_t g) - hT_t g\|_1 \leq \frac{c}{2^{rn}}. \]

Therefore,
\[ \tau(\int_0^\infty T_s \Gamma(T_s f) ds)^{\frac{1}{2}} \leq c + \sum_{n=0}^\infty \frac{c}{2^{rn}} \leq c. \]

\[ \square \]

**Example 1** Let us illustrate the assumptions of Theorem 0.2 for the heat semigroups \((T_t)\) on a weighted Riemannian manifold with a doubling measure. Saloff-Coste’s survey (see [SC10]) gives a lot of examples of \(T_t\) with kernels satisfying the following upper Gaussian bounds
\[ p(t, x, y) \leq \frac{1}{V(x, \sqrt{t})} \exp -\frac{d^2(x, y)}{ct}, \] (2.12)
\[ \left| \frac{\partial p(t, x, y)}{\partial t} \right| \leq \frac{1}{tV(x, \sqrt{t})} \exp -\frac{d^2(x, y)}{ct}. \] (2.13)

The assumptions (i), (ii) are easily verified for such \(T_t\)’s. (i) is obvious by (2.13). For (ii), it is enough to check the extreme points \(f = \delta_{x_0}\). Then \(T_tf = p(t, x_0, y)\) and
\[ \left( \frac{1}{t} \int_{c_1 t}^{ct} T_s ds |T_t f|^2 \right)^{\frac{1}{2}} \leq \frac{1}{V^2(x, \sqrt{t})} \exp -\frac{d^2(x, y)}{ct} \]
which belongs to \(L^\frac{1}{2}\) uniformly in \(t\).

**Example 2** Let \((O_t)\) be the Ornstein-Uhlenbeck semigroups on \(\mathbb{R}^n\) with the gaussian measure \(d\mu = e^{-x^2}dx\). The infinitesimal generator of \((O_t)\) is \(L = \frac{1}{2} \Delta - x \cdot \partial x = \sum_{i=1}^n \frac{1}{2} \frac{\partial^2}{\partial x_i^2} - x_i \cdot \frac{\partial}{\partial x_i}\). \(O_t\) satisfies the \(\Gamma_2 \geq 0\) condition and the assumption (i) of Theorem 0.2. This can be verified by the kernel of \(O_t\). So Theorem 0.1 applies to \(O_t\). It is a pity that \(O_t\) does not satisfy the assumption (ii) of Theorem 0.2 although the following inequality holds
\[- \int_{\mathbb{R}^n} \left( \frac{1}{t} \int_t^{2t} \tilde{O}_s ds |O_tf|^2 \right)^{\frac{1}{2}} d\mu \leq c \int_{\mathbb{R}^n} f d\mu, \] (2.14)
for any \(f \geq 0\). Here \(\tilde{O}_t\) is the semigroup generated by \(\tilde{L} = \frac{1}{2} \Delta - 2x \cdot \partial x\).
3 Extension to the noncommutative setting

We refer the readers to [PX03] for an introduction of noncommuative $L^p$ spaces and to [JLX06] Chapter 10 for noncommutative semigroups of operators. Given a semigroups of operators on a semifinite von Neumann algebras $M$, all the definitions and Lemmas in Section 1 still work in the noncommutative setting except Lemma 1.5. The noncommutative generalization of Theorem 0.1 and its proof are straightforward (we kept the noncommutative version in mind when writing the proof of Theorem 0.1). Let us state it as follows without a proof.

**Theorem 3.1** Let $(T_s)_s$ be a semigroups of operators on a semifinite von Neumann algebra $(M, \tau)$ satisfying $\Gamma_2 \geq 0$. Then $\text{bmo}(T) \subset (H^S_1(T))^*$ and

$$ |\tau(fg^*)| \leq c_1 \|f\|_{H^S_1(T)} \|g\|_{\text{bmo}(T)} \leq c_2 \|f\|_{H^S_1(T)} \|g\|_{\text{bmo}(T)}, $$

for all $f, g \in A$ with absolute constants $c_1, c_2$.

The generalization of Theorem 0.2 takes a little more effort. Because the John-Nirenberg inequality for noncommutative martingales are not very “nice”, and the noncommutative version of Lemma 1.5 is not available to us (at least by now).

**Theorem 3.2** Let $(T_s)_s$ be as in theorem 3.1. Assume that, in addition, there exist constants $c_3, c_4, r > 0$ such that,

(i) $\|\langle T_{t+\varepsilon} - T_t \rangle f \|_1 \leq c_3 \varepsilon \|f\|_1$, for all $\varepsilon > 0, t > 0$ and $f \in L_1(M)$.

(ii) For all $t > 0$ and $g \in L^1(M), h \in L^2(M)$,

$$ \tau[(M_{st}|h(T_t g)^{\frac{1}{2}}|^2)^{\frac{1}{2}}] \leq c_3 \tau(g)^{\frac{1}{2}} \tau(h^2)^{\frac{1}{2}}. \quad (3.1) $$

Then $(H^S_1)^* = \text{bmo}(T) = \text{BMO}(T)$ and $H^G_1 = H^S_1$.

**Proof.** Note, for the $\text{BMO}(T)$ norm,

$$ \|T_t|\varphi - T_t\varphi|^2\|_{\text{BMO}}^\frac{1}{2} = \sup_f |\tau(\varphi^*f)|, \quad (3.2) $$

Here the supremum takes over for all $f = h(T_t g)^{\frac{1}{2}} - T_t(h(T_t g)^{\frac{1}{2}})$ with $g \geq 0, \|g\|_1, \|h\|_2 \leq 1$. This is easily verified as follows.
\[
\|T_t|\varphi - T_1\varphi|^2\|_\infty = \sup_{g \geq 0, \tau g \leq 1} \tau[(T_t|\varphi - T_1\varphi|^2)g] \\
= \sup_g \tau[|\varphi - T_1\varphi|^2(T_tg)] \\
= \sup \tau[|(\varphi - T_1\varphi)(T_tg)\frac{1}{2}|^2] \\
= \sup_{g, \tau h^2 \leq 1} |\tau[h(T_tg)\frac{1}{2}(\varphi - T_1\varphi)^*]| \\
= \sup_{g, h} |\tau\varphi^*[h(T_tg)\frac{1}{2} - T_1(h(T_tg)\frac{1}{2})]| 
\]

We need to show such \( f \)'s are in \( H_1^S(\mathcal{T}) \) with norm \( \leq c \). Then the inclusion \((H_1^S)^* \subset BMO(\mathcal{T})\) follows from a density argument. And the equivalence \((H_1^S)^* = bmo(\mathcal{T}) = BMO(\mathcal{T})\) and \( H_1^G(\mathcal{T}) = H_1^S(\mathcal{T}) \) follow from inequality (2.1) and Lemma 1.4.

Fix such a \( f \). Recall \( M_t = \frac{1}{T} \int_0^t T_s ds \).

\[
\tau(\int_0^t T_s\Gamma(T_sf)ds)^{\frac{1}{2}} \leq \tau(\int_0^t M_tT_s\Gamma(T_sf)ds)^{\frac{1}{2}} \\
\leq \tau(\int_0^t 3M_{3t}T_{t-s}\Gamma(T_sf)ds)^{\frac{1}{2}} \\
(\text{Lemma 1.3}) = \tau(3M_{3t}T_{t}|f|^2 - 3M_{3t}|T_tf|^2)^{\frac{1}{2}} \\
= \tau(3M_{3t}T_{t}|f|^2)^{\frac{1}{2}} \\
(\text{Lemma 1.3}) \leq c\tau[(M_{3t}|h(T_tg)\frac{1}{2}|^2)^{\frac{1}{2}}] + \tau([3M_{3t}T_2|h(T_tg)\frac{1}{2}|^2)^{\frac{1}{2}}] \\
= \tau(A\frac{1}{2}) + \tau(B\frac{1}{2}) \leq c.
\]

The rest part of the proof remains the same as that for Theorem 0.2. Note that the noncommutative \( L^2 \)-quasi norm is 2-convex, we still have \( \tau[(A + B)^{\frac{1}{2}}] \leq \tau[A^{\frac{1}{2}}] + \tau[B^{\frac{1}{2}}] \) for \( A, B \geq 0 \).

**Example 3** Let \( G \) be a discrete group. Let \( \lambda_g, g \in G \) be the translation-operator on \( \ell_2(G) \) defined as

\[
\lambda_g(m)(h) = m(g^{-1}h).
\]

\( g \mapsto \lambda_g, g \in G \) is called the left regular representation of \( G \). The so called group von Neumann algebras \( \mathcal{M}_G \) of \( G \) is the weak* closure of the linear span of the \( \lambda_g \)'s in \( B(\ell_2(G)) \). The canonical trace \( \tau \) on \( \mathcal{M}_G \) is defined as \( \tau\lambda_e = 1 \) and \( \tau(\lambda_g) = 0 \) if \( g \neq e \). If \( G \) is abelian, then \( L^p(\mathcal{M}_G) \) is the canonical \( L^p \) space of functions on the dual group \( \hat{G} \) of \( G \). In particular, if \( G = \mathbb{Z} \), the integer group, then \( \lambda_k = e^{ikt}, k \in \mathbb{Z} \) and \( L^p(\mathcal{M}_G) = L^p(\mathbb{T}) \), the function space on the unit circle.
Let $\phi$ be a scalar valued function on $G$. We say $\phi$ is conditionally negative if
\[ \sum_{g,h} a_g a_h \phi(g^{-1}h) \leq 0 \]
for any finitely many coefficients $a_g \in \mathbb{C}$ with $\sum a_g = 0$. Schoenberg’s theorem claims that all standard semigroups of operators on the group von Neumann algebras $\mathcal{M}_G$ are in the form of $T_t(\lambda_g) = e^{-\phi(t)\lambda_g}$ with $\phi$ a real valued conditionally negative function and $\phi(e) = 0$, $\phi(g) = \phi(g^{-1})$.

Let $K_{\phi}(g, h) = \frac{1}{2}(\phi(g) + \phi(h) - \phi(g^{-1}h))$, the Gromov form associated with $\phi$. Then $K_{\phi}$ is a positive definite function on $G \times G$. So is $K^2_{\phi}$. It is easy to compute by the definition that
\begin{align*}
\Gamma(\sum_g a_g \lambda_g) &= \sum_{g,h} \bar{a}_g a_h K_{\phi}(g, h) \lambda_g^{-1}h, \\
\Gamma_2(\sum_g a_g \lambda_g) &= \sum_{g,h} \bar{a}_g a_h K^2_{\phi}(g, h) \lambda_g^{-1}h,
\end{align*}
(3.4) (3.5)

Therefore the $\Gamma_2 \geq 0$ condition is automatically hold for such $T_t$. So Theorem 3.1 applies to all such $(T_t)_t$’s.

Let $\mathbb{R}[G]$ be the algebra of all real valued bounded functions on $G$. Then
\[ \langle \sum_g a_g \delta_g, \sum_h b_h \delta_h \rangle_{\phi} = \sum_{g,h} a_g a_h K_{\phi}(g, h) \]
defines a semi-inner product on $\mathbb{R}[G]$. After quotient the null space
\[ N_{\phi} = \{ x \in \mathbb{R}[G], \langle x, x \rangle_{\phi} = 0 \}, \]
$\mathbb{R}[G]/N_{\phi}$ becomes a Hilbert space. In a forthcoming article, we are going to show that $T_t$ satisfies the assumptions of Theorem 3.2 if $\mathbb{R}[G]/N_{\phi}$ is finite dimensional.

Appendix—A Carleson embedding theorem

Let $(M, \mu)$ be a sigma-finite measure space. Assume $(T_t)$ is a standard semigroup of operators on $L_p(M)$ with infinitesimal generator $L$. Let $\mathcal{P} = (P_t)$ be the subordinated Poisson semigroup $P_t = e^{-t\sqrt{-L}}$. Given a $f \in L_p(M)$, then $F(t) = \mathcal{P} f = P_t f$ is $L$-harmonic on $M \times (0, \infty)$ in the sense that $(\partial^2_t + L) F = 0$. Let $\nu = \nu_t d\mu$ be a measure on $M \times (0, \infty)$ with $\nu_t$ an integrable function on $M$. Viewing $P_t$’s as analogues of the mean value operators, we say that $\nu$ is a Carleson measure with respect to $L$ if
\[ ||\nu||_{L,\alpha} = \sup_t \| P_t \int_0^t \nu_s ds \|_\infty < \infty. \]

**Theorem.** Suppose \((T_t)\) is a standard semigroup of operators on a sigma-finite measure space \((M, \mu)\). Assume \(T_t\) satisfies the \(\Gamma_2 \geq 0\) condition (1.5). Let \(1 < p < \infty\). Then

\[ \| P f \|_{L^p(M \times (0, \infty), \nu)} \leq c_p \| f \|_{L^p(M)} \]

for all \(f \in L^p(M)\) if \(\| \nu \|_{P,4} \leq c\).

**Proof.** It is clear that \(\| P f \|_{L^\infty(\nu)} \leq \| f \|_{L^\infty}\). Let

\[ H_1(M) = \{ f \in L_1(M); \sup_{\|g\|_{\text{BMO}(P)} \leq 1} \| \tau f g \| < \infty \}. \]

By Lemma 1.3, we have

\[ (L_\infty(M), H_1(M))_{\frac{1}{p}} = L_p(M). \]

It is then enough to show that \(\| P f \|_{L^1(\nu)} \leq c \| f \|_{H_1(M)}\). Note

\[ \int_{M \times (0, \infty)} |P f| d\nu \leq \sup_{\|g\|_\infty \leq 1} \int_M g(\int_0^\infty (P_t f) \nu_t dt) d\mu \]

\[ = \sup_{\|g\|_\infty \leq 1} \int_M f(\int_0^\infty P_t (g \nu_t) dt) d\mu \]

\[ \leq \| f \|_{H_1(M)} \sup_{\|g\|_\infty \leq 1} \| \int_0^\infty P_t (g \nu_t) dt \|_{\text{BMO}(P)}. \]

We go to show that \(\| (f_0^\infty P_t (g \nu_t) dt) \|_{\text{BMO}(P)} \leq c \| \nu \|_{P,4}\). In fact,

\[ P_s | \int_0^\infty P_t (g \nu_t) dt - P_s (\int_0^\infty P_t (g \nu_t) dt) | \]

\[ \leq P_s | \int_0^s (P_t - P_{t+s}) (g \nu_t) dt + \int_s^\infty (P_t - P_{t+s}) (g \nu_t) dt | \]

(apply (1.8)) \(\leq 5 P_s (\int_0^s |g \nu_t| dt) + 8 s \int_s^\infty \frac{1}{t} P_\frac{t}{2} (|g \nu_t|) dt \]

\( (\|g\|_\infty \leq 1 \) \(\leq 5 \| \nu \|_{P,1} + 8 \sum_k \int_0^{2k+1} \frac{1}{2k} P_\frac{t}{2} (|\nu_t|) dt \]

\( \leq 5 \| \nu \|_{P,1} + \frac{8}{2k} \sum_k \int_0^{2k+1} P_\frac{t}{2} (|\nu_t|) dt \]

(apply (1.8)) \(\leq 5 \| \nu \|_{P,1} + \frac{16}{2k} \sum_k \int_0^{2k+1} P_{2k-1} (|\nu_t|) dt \]

\[ \leq 5 \| \nu \|_{P,1} + \frac{16}{2k} \sum_k \int_0^{2k+1} P_{2k-1} (|\nu_t|) dt \]

\[ \leq 5 \| \nu \|_{P,1} + \sum_k \int_0^{2k+1} P_{2k-1} (|\nu_t|) dt \]

\[ \leq 5 \| \nu \|_{P,1} + \sum_k \int_0^{2k+1} P_{2k-1} (|\nu_t|) dt \]

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Taking the supremum over $s$ we get, by Lemma 1.5, that

$$\left\| \int_0^\infty P_t(g\nu_t) \, dt \right\|_{\text{BMO}(\mathcal{P})} \leq 37\|\nu\|_{\mathcal{P}, 4} \leq c.$$  

By interpolation, we get

$$\|\mathcal{P}f\|_{L^p(M \times (0, \infty), \nu)} \leq c_p\|f\|_{L^p(M, \mu)}.$$  

\[\blacksquare\]

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