On the nonadiabatic geometric quantum gates

Wang Xiang-Bin* and Matsumoto Keiji†
Imai Quantum Computation and Information Project,
ERATO, Japan Sci. and Tech. Corp.
Dani Hongo White Bldg. 201, 5-28-3, Hongo Bunkyo, Tokyo 113-0033, Japan

Abstract

Motivated for the fault tolerant quantum computation, a quantum gate by adiabatic geometric phase shift is extensively investigated recently. Due to the enviromental noise, the decoherence time for a quantum state can be very short. In the adiabatic quantum computation, if we change the Hamiltonian slowly, the decoherence error is increased, if we change the Hamiltonian faster, the adiabatic condition is broken and the error caused by the nonadiabaticity is increased. In this paper, we give a nonadiabatic scheme for the geometric phase shift and conditional geometric phase shift with the nuclear magnetic resonance system(NMR). Essentially, the new scheme is simply to add an appropriate additional field. With this additional field, the state evolution can be controlled exactly on a dynamical phase free path. Geometric quantum gates for single qubit and the controlled NOT gate for two qubits are given. In our non-adiabatic scheme, the geometric quantum gates can run as fast as the usual quantum gates.

*correspondence author, email: wang@qci.jst.go.jp
†email:keiji@qci.jst.go.jp
I. INTRODUCTION

It has been shown that, a quantum computer, if available, can perform certain tasks much more efficiently than a classical Turing machine. A fault-tolerant quantum logic gate \cite{1} is the central issue in realizing the basic constituents of a quantum information processor. Quantum computation via the controlled geometric phase shift \cite{2} provides a nice scenario to this purpose. Due to its geometric property, a geometric phase \cite{3,4} shift can be robust with respect to certain types of operational errors. In particular, suppose that the spin(qubit) undergoes a random fluctuation about its path in the evolution. The final value of the geometric phase shift will not be affected provided that the random fluctuation does not change the total area on Bloch sphere \cite{2}.

Recently, it was reported \cite{2,5–9} that the conditional Berry phase(adiabatic cyclic geometric phase) shift can be used in quantum computation. In particular, in ref \cite{2}, an experiment was done with nuclear magnetic resonance(NMR) \cite{10–14} under the adiabatic condition.

Demand on both running speed and precision of every gate in a quantum computer is quite high. By the available technology at present( and at near future), due to the enviromental noise, distortion to a quantum state increases fast with time. This requires all quantum gates complete the task as fast as possible. However, the schemes raised previously \cite{4,5} required the adiabatic condition. That is to say, if we run the gate too fast(i.e. change the Hamiltonian rapidly), the distortion from the nonadiabaticity must be significant. It has been reported in the recent experiment that \cite{2}, due to the requirement of adiabatic condition, faster running speed causes severe distortions to the results \cite{2}. The adiabatic condition makes the fast speed and high precision conflict each other in the conditional Berry phase shift gate. Increasing the running speed of the logic gate can exponentially increase the power of a quantum computation such as quantum factorization \cite{15}. It seems that the geometric phase shift gate will only be practical if it can run at a speed comparable to that of the usual quantum gate. Therefore one is tempted to find an easy way to make the conditional geometrical phase shift through nonadiabatic state evolution. In this paper, we present an easy scheme to make the geometric
phase shift nonadiabatically with NMR. In section II we study the state evolution on a cone. We give a simple way to nonadiabatically control the state evolution on the cone. With an appropriate setting, the state evolves on a dynamical phase-free path. In section III we implement the result of section II to the NMR system. A scheme for nonadiabatic conditional geometric shift is given. In section IV we investigate the nonadiabatic geometric realization of various types of quantum logic gates, including the phase shift gate, the Hadamard gate and the NOT gate for single qubit, and the C-NOT (controlled-NOT) gate for two qubits. We give a concluding remark in section V.

II. EXACT STATE EVOLUTION ON THE CONE.

It is well known that a spin half nucleus can gain a geometric phase shift in the conical evolution. We now assume that the initial spin state is on the cone and we will demonstrate a method to make it evolve nonadiabatically. The Hamiltonian for a spin in a constantly rotating magnetic field \[ H(t) = \left[ \omega_0 \sigma_z + \omega_1 \sigma_x(t) \right] / 2. \] (1) Here \( \sigma_x(t) = \begin{pmatrix} 0 & e^{-i\gamma t} \\ e^{i\gamma t} & 0 \end{pmatrix} \), \( \omega_0 \) is the amplitude of vertical field and \( \omega_1 \) is the amplitude of the horizontal field, which is rotating around \( z \)-axis in the constant angular speed \( \gamma \). The initial state \( |\psi_0\rangle \) is an eigenstate of \( H_0 = H(0) \). Explicitly,

\[ |\psi_0\rangle = \cos \frac{\theta}{2} |\uparrow\rangle + \sin \frac{\theta}{2} |\downarrow\rangle \] (2)

and

\[ \cos \theta = \frac{\omega_0}{\sqrt{\omega_0^2 + \omega_1^2}}. \] (3)

We shall extend it to the case of arbitrary time-dependent rotating speed in the end of this section.
To know the cause of the state distortion, we study the time evolution first. (The similar idea has
been used for the adiabatic rotational splitting with NQR previously [16].) The time evolution
operator $U$ generated by the Hamiltonian $H_0$ is determined by the time-dependent Schrödinger
equation

$$i\frac{\partial}{\partial t} U = H(t)U.$$  (4)

Solving this equation we obtain

$$U(t) = e^{-i\gamma \sigma_z t} e^{-i(H_0 - \gamma \sigma_z/2)t}.$$  (5)

In particular, at time $t = \tau = 2\pi/\gamma$, when the external field completes a $2\pi$ rotation, the state
is evolved to

$$\psi(\tau) = e^{-i\pi - i(H_0 - \gamma \sigma_z/2)\tau} |\psi_0\rangle.$$  (6)

The adiabatic approximation assumes that at time $\tau$, the state completes a cyclic evolution
provided that $\gamma$ is small. However, due to the decoherence time limitation, $\gamma$ cannot be too
small. The non-zero $\gamma$ distorts the state in the evolution and makes it noncyclic at time $\tau$. This
non-zero $\gamma$ in the adiabatic approximation causes two types of errors. One is that the noncyclic
completion of the evolution at time $\tau$ will cause further errors in the succeeding operations in
the sequence. (A sequence of operations is used to remove the dynamic phase in realizing the
geometric quantum gate [2].) The other is that the geometric phase acquired over period $\tau$ is
not $-\pi(1 - \cos \theta)$, as for the ideal adiabatic cyclic evolution.

Now we give an easy way to exactly control the state evolution on the cone. Here the
external field can be rotated around $z$-axis arbitrarily fast. We use $|\psi_0\rangle$, the eigenstate of $H_0$
for the initial state. We switch on a static vertical magnetic field $\omega_z = \gamma$ while the external
field is rotated around $z$-axis. In this way, the new time-dependent Hamiltonian is

$$H_W(t) = \frac{1}{2} [((\omega_0 + \gamma)\sigma_z + \omega_1 \sigma_x(t)].$$  (7)

The time evolution operator $U_W(t)$ generated by this Hamiltonian satisfies the Schrödinger
equation
\begin{equation}
    i \frac{\partial}{\partial t} U_W(t) = H_W(t) U_W(t)
\end{equation}

with the boundary condition \( U_W(0) = 1 \). The state at time \( t \) is related to the state at time 0 by \( |\psi(t)\rangle = U_W(t) |\psi_0\rangle \), \( \psi_0 = \psi(0) \). Denoting \( R = e^{i\gamma t \sigma_z/2} \) we obtain the following equivalent equation

\begin{equation}
    i \partial (RU_W) / \partial t = H_0(RU_W).
\end{equation}

Now that \( H_0 \) is time-independent we have

\begin{equation}
    RU_W(t) = e^{-iH_0t}.
\end{equation}

This is equivalent to

\begin{equation}
    U_W(t) = e^{-i\gamma t \sigma_z/2} e^{-iH_0t}.
\end{equation}

Consequently, at any time \( t \), state

\begin{equation}
    |\psi(t)\rangle = e^{-i\lambda t} e^{-i\gamma t \sigma_z/2} |\psi_0\rangle
\end{equation}

is exactly the instantaneous eigenstate of Hamiltonian \( H(t) \), where \( \lambda \) is the eigenvalue of \( H_0 \) for eigenstate \( |\psi_0\rangle \),

\begin{equation}
    \lambda = \pm \sqrt{\omega_0^2 + \omega_1^2}.
\end{equation}

In particular, at time \( \tau \),

\begin{equation}
    |\psi(\tau)\rangle = e^{-i\pi - i\lambda \tau} |\psi_0\rangle.
\end{equation}

It only differs to \( |\psi_0\rangle \) by a phase factor. Note that the time evolution operator \( U_W(t) \) here is generated by the Hamiltonian \( H_W(t) \) instead of \( H(t) \).

Here the additional vertical field \( \gamma \) plays an important role. Without this field, the time evolution operator generated by \( H(t) \) is

\begin{equation}
    U(t) = e^{-i\gamma t \sigma_z/2} e^{-iH_1t}
\end{equation}
and $H_1 = H_0 - \gamma \sigma_z / 2$. Obviously, this time evolution operator $U(t)$ will distort a qubit with the initial state $|\psi_0\rangle$ in the evolution. But this $U(t)$ can exactly control the qubit with the initial state $|\psi_1\rangle$, the eigenstate of $H_1$. If initially we set the spin state to $|\psi_1\rangle$, then the spin will evolve exactly on its cone without the additional field $\gamma$. So, we have two ways to control the spin evolution exactly. We can use the additional field $\gamma$, if the initial state is set to be $|\psi_0\rangle$. Alternatively, we can also set the initial state to be $|\psi_1\rangle$ and then we need not add any additional field when the external field is rotated. In this letter, we adopt the former one, i.e. we set the initial spin state to $|\psi_0\rangle$.

The additional magnetic field $\gamma$ can make the state $|\psi_0\rangle$ keep up the rotating field exactly in the evolution. So the state completes a cyclic evolution when the field completes a $2\pi$ rotation. However, after a cyclic evolution, the total phase shift for the state in general includes both dynamic and geometric contribution \[17\]. It is possible to restrict the state evolution on the dynamic phase path if we choose certain specific value of the rotating speed $\gamma$ (and also the additional field $\gamma$). For this purpose, we require

$$\gamma \cos \theta = -\sqrt{\omega_0^2 + \omega_1^2}. \quad (16)$$

The solution is

$$\gamma = -\frac{\omega_1^2 + \omega_0^2}{\omega_0}. \quad (17)$$

The negative sign in the right hand side indicates that the additional field $\gamma$ is anti-parallel with the field $\omega_0$. By this setting, the total magnetic field in $H_W$ is always "perpendicular" to the state vector expressed in the Bloch ball. Details of this are shown in Fig. 1. One can easily show that the instantaneous dynamical phase

$$< \psi_0 | U_W^\dagger(t) H_W(t) U_W(t) | \psi_0 >= 0. \quad (18)$$

We have assumed above that the field rotates at a constant speed. Actually, we can easily modify the above scheme for arbitrary time-dependent rotating speed $\gamma(t)$, $\int_0^T \gamma(t) dt = 2\pi$. In this case, we need only change the term $\gamma t$ in $H_W(t)$, $U_W(t)$, $R$ and $\psi(t)$ into $\int_0^t \gamma(t') dt'$.\[6\]}
accordingly. Consequently, the additional vertical field is now a time-dependent field $\omega_z(t) = \gamma_a(t)$ instead of a static field. To remove the dynamical phase, we can use the time-dependent magnetic field $\omega_1(t)$ and $\omega_0(t)$. We require

$$
\gamma(t) = -\frac{\omega_1^2(t) + \omega_0^2(t)}{\omega_1(t)} \tag{19}
$$

and

$$
\omega_1(t)/\omega_0(t) = \omega_1(0)/\omega_0(0) = \tan \theta \tag{20}
$$

This extension to the time-dependent case can be important in case it is difficult to rotate the field (or the fictitious field [5]) in a constant angular speed. Punctual results can be obtained here through the exact feedback system where the value of additional vertical field is always instantaneously equal to the angular velocity of the rotating field.

Thus we see, by adding an additional magnetic field that is equal to the rotating frequency of the external field, we do get the exact result. Obviously if we rotate the field inversely, additional vertical field should be in the inverse direction ($-z$) accordingly.

### III. THE NONADIABATIC CONDITIONAL GEOMETRIC PHASE SHIFT.

#### A. NMR system and the rotational framework.

Consider the interacting nucleus spin pair (spin $a$ and spin $b$) in the NMR quantum computation [10–13]. If there is no horizontal field the Hamiltonian for the two qubit system is

$$
H_i = \frac{1}{2}(\omega_a \sigma_{za} + \omega_b \sigma_{zb} + J \sigma_{za} \cdot \sigma_{zb}) \tag{21}
$$

, where $\omega_a (\omega_b)$ is the resonance frequency for spin $a (b)$ in a very strong static magnetic field (e.g. $\omega_a$ can be 500MHz [2]), $J$ is the interacting constant between nuclei and $\sigma_{za} = \sigma_{zb} = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. After adding a circularly polarized RF field in horizontal plane, the Hamiltonian for spin $a$ in static framework is
\[ H' = \frac{1}{2}\omega_0\sigma_z + \frac{1}{2}\omega_a'\sigma_z + \frac{1}{2}\omega_1 \begin{pmatrix} 0 & e^{-i\omega_a't} \\ e^{i\omega_a't} & 0 \end{pmatrix}. \]  

(22)

Here \( \omega_a' \) and \( \omega_1 \) are the angular frequency and amplitude of the RF field respectively,

\[ \omega_0 = \omega_a - \omega_a' \pm J \]  

(23)

and the "±" sign in front of \( J \) is dependent on the specific state of spin \( b \), up or down respectively. To selectively manipulate spin \( a \), we shall use the rotational framework that is rotating around \( z \)-axis in speed \( \omega_a' \). We assume \( \omega_a' \) close to \( \omega_a \) but quite different from \( \omega_b \). The Hamiltonian for spin \( a \) in the rotational framework is

\[ H_a = R'H'\dot{R}'^{-1} + i(\partial R'/\partial t)R'^{-1} = \frac{1}{2}\omega_0\sigma_z + \frac{1}{2}\omega_1\sigma_z(t) = H(t) \]  

(24)

and \( R' = e^{i\omega_a'\sigma_zt/2} \). Note here \( H_0 \) is dependent on state of spin \( b \) through \( \omega_0 \). In the rotational framework, if the horizontal field is rotated in the angular speed \( \gamma \), the Hamiltonian for spin \( a \) is just \( H(t) \), as defined in eq(1). In the NMR system, we require \( |\psi_0\rangle \) to be the eigenstate of \( H_0 \) in rotational the framework. Previously [2], state \( |\psi_0\rangle \) for spin \( a \) was produced adiabatically. In the following we demonstrate how to nonadiabatically produce the state \( |\psi_0\rangle \).

B. Creating the conditional initial state with NMR.

In rotating the state around \( z \)-axis, we have started from state \( |\psi_0\rangle \), which is the eigenstate of \( H_0 \). State \( |\psi_0\rangle \) can be created from the spin state \( |\uparrow\rangle \) or \( |\downarrow\rangle \). We have a nonadiabatic way to create state \( |\psi_0\rangle \) for spin \( a \). Note that we are working in the rotational framework. We denote \( \delta = \omega_a - \omega_a' \). We can use the following sequence of operations to create the conditional angle \( \theta \). (For simplicity we will call the following sequence as \( S \) operation later on.) We have

\[ S = \left[ \frac{\pi}{2} \right]^y \rightarrow J'(\varphi_\pm(t_c)) \rightarrow [-\delta \cdot t_c]^z \rightarrow \left[ \frac{\pi}{2} \right]^x \rightarrow [-\varphi']^y \]  

(25)

Here all terms inside the [\(\cdots\)] represent the Bloch sphere rotation angles caused by RF pulses. The superscripts indicate the axis the Bloch sphere is rotated around. \( J'(\varphi_\pm(t_c)) \) is the time
evolution over period $t_c$ by the Hamiltonian $\frac{1}{2}(\delta \pm J)\sigma_z$. This evolution rotates spin $a$ around $z-$axis for an angle $\varphi_{\pm} = (\delta \pm J)t_c$. (Or $\varphi_{\pm} + \pi$, if spin $a$ is down initially. For clarity we omit this case and always assume spin $a$ is initially up.) After this $S$ operation, the angle between spin $a$ and the $z$-axis is $\theta_{\pm} = \frac{\pi}{2} - (\varphi' + \varphi_{\pm})$ (see Fig. 2). To ensure the state to be eigenstate of Hamiltonian $H_0$ after $S$ operation we require $\tan(\varphi' + Jt_c) = \frac{\delta + J}{\omega_1}$ and $\tan(\varphi' - Jt_c/2) = \frac{\delta - J}{\omega_1}$ simultaneously. This is equivalent to

$$
\begin{cases}
Jt_c = \frac{\arctan \frac{\delta + J}{\omega_1} - \arctan \frac{\delta - J}{\omega_1}}{2} \\
\varphi' = \frac{\arctan \frac{\delta + J}{\omega_1} + \arctan \frac{\delta - J}{\omega_1}}{2}
\end{cases}
$$

(26)

From this constraint, given specific values of $\delta$, $J$ and $\omega_1$, we can easily obtain the scaled time $J \cdot t_c$ (Fig. 3) and the angle $\varphi'$ (Fig. 4) in controlling the $S$ operation. In particular, for $\delta/J = 1.058$, the value adopted in the recent NMR experiment, the relation curves for $Jt_c$ vs $\omega_1$ and $\varphi'$ vs $\omega_1$ are shown in Fig. 5.

C. The nonadiabatic conditional geometric phase shift.

After the $S$ operation, the state of spin $a$ is an eigenstate of Hamiltonian $H_0$ no matter whether spin $b$ is up or down. After this $S$ operation we can use the additional magnetic field to nonadiabatically control the state evolution on the cone. However, here we need again remove the dynamical phase contribution. Note the situation here is different from the single qubit case. Qubit $b$ could be either up or down. We need choose the appropriate $\gamma$ value so that the instantaneous dynamic phase for qubit $a$ is always zero no matter qubit $b$ is up or down. So in this case the following simultaneous conditions are required.

$$
\gamma \cos \theta_{\pm} = -\sqrt{(\delta \pm J)^2 + \omega_1^2}.
$$

(27)

Setting

$$
\omega_1 = \sqrt{\delta^2 - J^2}.
$$

(28)

will make the simultaneous conditions hold. With this setting we propose the following scheme to make the nonadiabatic conditional geometric phase shift.
\[ S \to \begin{pmatrix} \gamma \\ C \end{pmatrix} \to S^{-1}. \] (29)

The term \( \begin{pmatrix} \gamma \\ C \end{pmatrix} \) represents doing operation \( C \) with an additional vertical field \( \gamma \). \( C \) represents rotating the external field around \( z \)-axis for \( 2\pi \) in a uniform speed \( \gamma \). We should choose the rotating direction so that additional field \( \gamma \) is anti-parallel the field \( \delta \pm J \), \( (\delta > J) \). Since qubit \( a \) evolves on the dynamical phase free path, the scheme raised here can remove the dynamic phase and retain only the geometric phase. It can be shown that the geometric phase acquired after the operation is \( \Gamma_+ \), \( -\Gamma_+ \), \( \Gamma_- \) and \( -\Gamma_- \) respectively for the four different initial state \( (|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle) \),

\[ \Gamma_{\pm} = -\pi - \pi \cos \theta_{\pm} \] (30)

and

\[ \cos \theta_{\pm} = \frac{\delta \pm J}{\sqrt{\left(\delta \pm J\right)^2 + \omega^2_1}} = \sqrt{\frac{\delta \pm J}{2\delta}} \] (31)

IV. GEOMETRIC QUANTUM LOGIC GATES

A. Geometric gates for single qubit

To make a quantum computation, we need both single qubit gates and the two qubit C-NOT gate. It is impossible to make any single qubit geometric gate with NMR in the adiabatic condition because the interaction among different qubits cannot be switched off. Though the interaction is small, the effects from the interaction may accumulate significantly after a long term. Strictly speaking, the adiabatic geometric quantum computation raised previously [2] is only a partial geometric quantum computation. However, in the our non-adiabatic scheme, we can easily make single qubit gate. In this nonadiabatic case, the effect of the interaction among qubits is negligible because the operation time is very short.
In section II, we have required the initial state of the qubit be the eigenstate of the initial Hamiltonian, $H_0$. However, this requirement is not necessary in making the geometric quantum gates, instead of making geometric phase shift to the qubit directly. If we are only interested to the geometric quantum gates instead of making the phase shift to the qubit itself, we can directly use arbitrary initial state. We now demonstrate this by some examples. Without any loss of generality, we suppose the initial state of the single qubit is $|\uparrow\rangle$ or $|\downarrow\rangle$. We start from the phase shift gate. We set the initial magnetic field $\omega$ in the $z$ direction. We then rotate $\omega$ anti-clockwise around $z'$ axis. The rotating angular velocity is $\gamma$. The angle between $z'$ and $z$ axis is $\theta_0$ (see Fig. 6). Meanwhile we add the additional field $\gamma$. The rotating speed and additional field should satisfy eq.(16). Note that in this case, the additional field is in the $-z'$ direction. After a $2\pi$ angle rotation the initial states will gain phase shifts according to the following equation

$$
\begin{pmatrix}
|\uparrow\rangle \\
|\downarrow\rangle
\end{pmatrix} = W_p(\theta_0) \begin{pmatrix}
|\uparrow\rangle \\
|\downarrow\rangle
\end{pmatrix}
$$

(32)

and

$$W_p(\theta_0) = \begin{pmatrix}
e^{-2\pi i \cos \theta_0} & 0 \\
0 & e^{2\pi i \cos \theta_0}
\end{pmatrix}.
$$

(33)

The phase shift here is determined by area enclosed by the state evolution loop on Bloch sphere, i.e. by angle $\theta_0$. Since $\theta_0$ can take arbitrary value, we can make arbitrary phase shift gate through the above equation.

Now we consider the Hadamard gate, which is another fundamentally important gate in quantum computation. We come back to $z$ axis. Now $\theta_0$ is the angle between magnetic field $\omega$ and $z$ axis. With the constraint of eq.(16), suppose the additional magnetic field and rotating speed of $\omega$ is $\gamma$. State $|\uparrow\rangle$ or $|\downarrow\rangle$ can be written in the following linear superposed forms respectively

$$|\uparrow\rangle = \cos \frac{\theta_0}{2} |+\rangle - \sin \frac{\theta_0}{2} |-\rangle$$

(34)
and
\[
|\downarrow> = \sin \frac{\theta_0}{2} |+> + \cos \frac{\theta_0}{2} |->
\]
where the states $|\pm> = \sin \theta_0 |+> + \cos \theta_0 |->$ are the spin states parallel or anti-parallel to the direction of $\omega = \sqrt{\omega_0^2 + \omega_1^2}$. After the field rotates one loop around $z$ axis, the initial state $|\uparrow>$ or $|\downarrow>$ changes by the following formula
\[
\begin{pmatrix}
|\uparrow>
|\downarrow>
\end{pmatrix} = W_s \begin{pmatrix}
|\uparrow>
|\downarrow>
\end{pmatrix} = W_s
\]
where
\[
W_s = \begin{pmatrix}
\cos \frac{\theta_0}{2} - \sin \frac{\theta_0}{2} & e^{i\Gamma} 0 \\
\sin \frac{\theta_0}{2} \cos \frac{\theta_0}{2} & 0 e^{-i\Gamma}
\end{pmatrix} \begin{pmatrix}
\cos \frac{\theta_0}{2} \sin \frac{\theta_0}{2} \\
-\sin \frac{\theta_0}{2} \cos \frac{\theta_0}{2}
\end{pmatrix}
\]
i.e.
\[
W_s = \begin{pmatrix}
\cos \gamma \cos \theta_0 + i \sin \gamma \sin \theta_0 & i \sin \gamma \sin \theta_0 \\
i \sin \gamma \sin \theta_0 & \cos \gamma - i \sin \gamma \sin \theta_0
\end{pmatrix}
\]
Here $\Gamma$ is the geometric phase gained by state $|+>$ after it completes the loop around $z$ axis. If we take $|\sin \gamma \sin \theta_0| = \sqrt{2}/2$ (root $\theta_0$ for this equation obviously exists), this $W_s$ is
\[
W_{s0} = \frac{\sqrt{2}}{2} \begin{pmatrix}
e^{i\phi} & i \\
i & -e^{-i\phi}
\end{pmatrix}
\]
Here $\phi = \tan^{-1} \frac{\sin \gamma \sin \theta_0}{\cos \gamma}$. Combining this $W_{s0}$ gate with an appropriate phase shift gate we can obtain the Hadamard gate. Explicitly we have
\[
\begin{pmatrix}
e^{-i(\phi-\pi/2)} 0 \\
0 1
\end{pmatrix} W_{s0} \begin{pmatrix}
e^{-i(\phi-\pi/2)} 0 \\
0 1
\end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\
1 & -1
\end{pmatrix} = W_H.
\]
Obviously, by combining the gate $W_H$ and the $W_p$, we can also make the NOT gate. It is
\[
W_H W_p(\theta_0 = \cos^{-1} \frac{1}{4}) W_H = i \begin{pmatrix} 0 & 1 \\
1 & 0
\end{pmatrix}.
\]
The NOT gate of single qubit played an important role in the adiabatic scheme to cancel the dynamical phase \[2,5\].

So far we have demonstrated the geometric realization for several types of logic gates in single qubit operation. Here geometric phase is the only contribution in the time evolution. Actually, all single qubit gate can be realized in a similar way. The running speed of the above geometric gates can be in principle arbitrarily fast.

**B. Geometric C-NOT gate**

With the conditional geometric phase shift, it is very easy to construct the C-NOT gate. To do so, we take \( \delta = \frac{4\sqrt{7}}{7} J \), then we have

\[
\Gamma_+ = \Gamma_- - \frac{\pi}{2}
\]

(42)

and

\[
S \begin{pmatrix} \gamma \\ C \end{pmatrix} S^{-1} = \begin{pmatrix} e^{i(\Gamma_- - \pi/2)} & 0 \\ 0 & e^{-i(\Gamma_- - \pi/2)} \end{pmatrix}
\]

(43)

if qubit \( b \) is up. And

\[
S \begin{pmatrix} \gamma \\ C \end{pmatrix} S^{-1} = \begin{pmatrix} e^{i\Gamma_-} & 0 \\ 0 & e^{-i\Gamma_-} \end{pmatrix}
\]

(44)

if qubit \( b \) is down. We can use a specific phase shift gate \( W_p \) on qubit \( a \) to convert the above form of conditional phase shift gate into

\[
S \begin{pmatrix} \gamma \\ C \end{pmatrix} S^{-1} = \begin{pmatrix} e^{-i\Gamma_-} & 0 & 0 & 0 \\ 0 & e^{i\Gamma_-} & 0 & 0 \\ 0 & 0 & e^{-i\Gamma_-} & 0 \\ 0 & 0 & 0 & e^{i\Gamma_-} \end{pmatrix}
\]

(45)

\[
S \begin{pmatrix} \gamma \\ C \end{pmatrix} S^{-1} = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

Sandwitching it by two Hadamad gates to the single qubit \( a \) we obtain the C-NOT gate, i.e.
\[
\frac{\sqrt{2}}{2} \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1
\end{pmatrix} \begin{pmatrix}
-i & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1
\end{pmatrix} = \begin{pmatrix}
0 & -i & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (46)

V. CONCLUDING REMARK.

To make the conditional geometric phase shift gate we need be able to control two elementary operations. One is to exactly control the cyclic state evolution on a cone; the other is to produce the initial state on the cone with an angle \( \theta_\pm \) conditional on the other bit. Previously \(^{[4]}\), these two elementary operations were done adiabatically. We have shown that both of these two tasks can be done nonadiabatically. We have demonstrated the non-adiabatic geometric phase shift and quantum logic gates for both single qubit and two qubit case. The single qubit geometric gate can run arbitrarily fast in principle. The running speed of two qubit gate is is only limited by the \( J \), the value of the qubits interaction. This is the case for all normal conditional quantum gates. So it possible to run the geometric quattm gate in a speed comparable to that of the normal quantum gate. The idea on nonadiabatic geometric phase shift gate demonstrated by the NMR system here should in principle also work for the other two level systems, such as Josephson Junction system \(^{[3]}\) and the harmonic oscillator system \(^{[2, 18]}\), where the decoherence time can be much shorter. It should be interesting to study the nonadiabatic scheme for the nonabelian geometric quantum computation \(^{[19–22]}\).

Acknowledgment We thank Prof Imai Hiroshi for supporting, Dr Hachimori M., Mr Tokunaga Y and Miss Moriyama S for various helps. W.X.B thanks Prof. Ekert A(Oxford) for suggestions. W.X.B. also thanks Dr. Pan J.W.(U. Vienna), Dr Kwek LC(NIE) for discussions and Prof Oh CH(NUS) for pointing out ref \(^{[3]}\).
FIG. 1. Control the conical evolution exactly. The initial state is the eigen state of $H_0$. $\omega$ is the field in $H_0$. $\omega$ is rotating around $z$ axis in the direction indicated by the arrow. Additional field $\gamma$ is equal to the rotating speed of $\omega$. With this additional field, the state will evolve keep up the field rotating $\omega$ exactly. We can see that, choosing an appropriate $\gamma$, the total field is instattaneously perpendicular to the field $\omega$, thus eigen state of $H_0$ evolves in a dynamic phase free path.
FIG. 2. Using $S$ operation to produce the eigenstate of $H_0$ nonadiabatically. Picture $a$ shows the possible initial state for spin $a$. The up(down) arrow on the solid line represents the up(down) state for spin $a$ while spin $b$ is up. The up(down) arrow on the dashed line represents the up(down) state for spin $a$ while spin $b$ is down. In picture $f$, angle $GOA$ is $\theta_+ = \arctan \frac{\omega_1}{\delta + J}$ and angle $GOB$ is $\theta_- = \arctan \frac{\omega_1}{\delta - J}$.

FIG. 3. The time control in $S$ operation. Parameter $\omega_1/J$ varies from 0 to 10, $\delta/J$ varies from 0 to 5. Vertical axis is for the scaled time $Jt_c$. 
FIG. 4. **The rotating angle control in S operation.** Parameter $\omega_1/J$ varies from 0 to 10, $\delta/J$ varies from 0 to 5. Vertical axis is for the angle $\varphi'$. 

FIG. 5. **The time and rotating angle control in S operation for specific $\delta/J$ value.** Horizontal axis represents $\omega_1/J$. Vertical axis is for the scaled control $Jt_c$(the solid line) or the angle $\varphi'$(dashed line). Here $\delta/J = 1.058$, as for the experimental condition[2]. By this figure, given the specific values of $\omega_1/J$, we can always find the corresponding point in the two curves thus we can take the suitable time control(for $t_c$) and rotation control(for $\varphi'$) in the S operation.
FIG. 6. Rotating the field around $z'$ axis. Here the field $\omega$ is in $+z$ direction, rotates around $z'$ axis. Additional field $\gamma$ is in $-z'$ direction.
REFERENCES

[1] E. Knill, R. Laflamme and W. H. Zurek, Science 279, 342(1998)

[2] J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Nature 403, 869(2000)

[3] M. V. Berry, Proc. R. Soc. Lond. A 392, 45(1984)

[4] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58, 1593(1987)

[5] G. Falci, R. Fazio, G. M. Palma, J. Siewert, & V. Vedral, V. Nature 407, 355(2000)

[6] I. Fuentes-Guridi, S. Bose, & V. Vedral, Phys. Rev. Lett. 85, 5018(2000).

[7] T. Pellizzari, S. A. Gardiner, J. I. Cirac, and & P. Zoller, Phys. Rev. Lett. 75, 3788-3791(1995)

[8] D. V. Averin, Solid State Commun. 105, 659-664(1998)

[9] A. Ekert et al. J. Mod. Optic. 47, 2501(2000).

[10] D. G. Cory, A. F. Fahmy, & T. F. Havel, Proc. Nat. Acad. Sci. USA, 94, 1634-1639(1997).

[11] N. A. Gershenfeld & I. L. Chung, science 275, 350-356(1997)

[12] J. A. Jones & M. Mosca, J. Chem. Phys. 109, 1648-1653(1998)

[13] J. A. Jones, R. H. Hansen & M. Mosca, J. Magn. Reson. 135, 353-360(1998)

[14] M. A. Nielsen and I. L. Chuang Quantum Computation and Quantum Information, Cambridge, 2000.

[15] A. Ekert and R. Jozsa, Rev. of Mod. Phys. 68, 733(1996)

[16] Robert Tycko, Phys. Rev. Lett. 58, 2281(1987).

[17] Tim Piller et al, Private communication.

[18] X. B. Wang, L. C. Kwek and C. H. Oh Phys. Rev. A 62 032105(2000)
[19] Duan L.M., Cirac J. I. and Zoller P., Science, 292, 1695(2001).

[20] Pachos J., Phys. Rev. A62, 052318(2000) and A61, 01305(2000).

[21] Zannardi P., Phys. Lett. A264, 94(1999).

[22] Llyds S, Science, 292, 1669(2001).