Nonstandard Mathematics and New Zeta and L-Functions

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Abstract

I define new zeta functions in a nonstandard setting and examine some of their properties. I further develop $p$-adic interpolation in the nonstandard setting and define the concept of interpolation with respect to two primes. The final section of the dissertation examines the work of M. J. Shai Haran and makes initial attempts of viewing it from a nonstandard perspective.
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CHAPTER 1

Introduction

There are many examples of modern work in number theory being developed by fusing it with another field of mathematics. Examples include quantum field theory, with work by D. Broadhurst and D. Kreimer, and the more well known field of geometry, with the work by A. Connes ([Co1] and [Co2]). This enables studying old problems from a new direction. The other alternatives in tackling number theory include proving a known theorem in a new way and trying to prove an old problem directly (which is becoming less popular due to the difficulty of such problems - Riemann hypothesis, Birch Swinnerton-Dyer conjecture, . . .).

The approach in this work is via number-theory-fusion with the area of nonstandard mathematics. The work splits into four related parts with the underlying theme being nonstandard mathematics. This chapter and chapter 2 (first part) are introductory; setting out the work and introducing nonstandard mathematics from a model theoretic basis. A review of the main results of Robinson on nonstandard algebraic number theory is provided as the work in following chapters further develops his work and approaches. Chapters 3 and 4 (second part) introduce nonstandard versions of the Riemann and Dedekind zeta functions. They also prove some nonstandard analytical properties.

The third part consists of chapters 5 and 6. In the first of these chapters the ideas of $p$-adic Mahler interpolation are interpreted in a nonstandard setting using shadow (standard part) maps (which enables nonstandard objects to be viewed in a standard setting). These ideas are pursued in the next chapter by viewing the Morita gamma function and Kubota-Leopoldt zeta function in a nonstandard way. One advantage of using nonstandard spaces (*$\mathbb{Q}$ for example) is the property of being able to consider finite primes in a symmetrical way (for example subquotients of *$\mathbb{Q}$ include $\mathbb{Q}_{p_1}$ and $\mathbb{Q}_{p_2}$ resulting in functions on this nonstandard space being able to be interpreted as as $p_1$-adic and $p_2$-adic functions). This enables double interpolation with respect to two distinct finite rational primes to be defined, and more generally interpolation with respect to a finite set of rational primes. An
example given is the double interpolation of the Riemann zeta function. The final chapter (the final part) is on the work of Shai Haran. A review of some of his work is given in conjunction with a review of other authors’ work which relates to his. A small section is dedicated to the beginning of a nonstandard interpretation of his work.

1.1 Chapter 3 Overview

The first application of nonstandard analysis in this work is in relation to the Riemann zeta function. A nonstandard version of the Riemann zeta function, the hyper Riemann zeta function, is defined. In particular some of the first analytical properties are examined.

The framework developed from the model theory provides a foundation for most of the work. Further (analytical) tools are needed and in particular the shadow maps. They exist in a general setting on hyper topological spaces with a shadow map corresponding to a non trivial absolute value. Essentially they link (part of) an object on a nonstandard space to a related object on the standard space. On $^\ast \mathbb{R}$ the shadow map of interest is the real shadow map corresponding to the real prime $\eta$ and the usual absolute value (definition 3.1.1). This map can also be extended to a shadow map on $^\ast \mathbb{C}$ (definition 3.4.16). Further it is shown how the shadow map can act on functions (definition 3.1.7) which is of most use in this chapter. Other tools introduced include the hyperfinite version of sums, products and integrals along with hyper sequences.

The central object in this chapter is the new function, the hyper Riemann zeta function, $\zeta_{^\ast \mathbb{Q}}$. The definition follows from defining the internal ideals of $^\ast \mathbb{Z}$ and their corresponding norm. These internal ideals are in bijection with the elements of $^\ast \mathbb{N}$ (lemma ??). Also in order to define $\zeta_{^\ast \mathbb{Q}}$ the definitions of $^\ast \exp$ (the hyper exponential function) and $^\ast \log$ (the hyper logarithm function) are needed in order to define the power function $z^s$ for general $z, s \in ^\ast \mathbb{C}$. The approach taken with these hyper functions is different to those in the literature, for example [S-L], section 2.4. In this work most objects are defined from first principles for completeness.

Combining the ideas from the previous paragraph gives definition 3.3.1

$$\zeta_{^\ast \mathbb{Q}}(s) = \sum_{n \in ^\ast \mathbb{N} \setminus \{0\}} \frac{1}{n^s}.$$

The first analytical properties are proved in the $\mathbb{Q}$-topology which is the finer topology. The rest of section 2.3 develops these results along with the first parts of section 2.4. Examples include the region of $\mathbb{Q}$-convergence and continuity in the hyper half plane $^\ast \Re s > 1$. Like the classical function
there is also a product decomposition into "Euler factors" via proposition 3.3.5

\[ \zeta_{\mathbb{Q}}(s) = \prod_p (1 - p^{-s})^{-1}. \]

The product above is taken over all positive primes in \(*\mathbb{Z}*, \(*\mathbb{Z}\) is a ring so primes (or prime ideals) can be defined. Some of the \(Q\)-analytical results require some hyper complex analysis which is taken from the work of Robinson, for example the hyper Cauchy theorem.

The rest of the chapter is spent proving the main result in this chapter (theorem 3.4.12 and corollaries 3.4.13 and 3.4.14), the functional equation and \(Q\)-analytic continuation of \(\zeta_{\mathbb{Q}}\).

**Theorem 3.4.12**  \(\zeta_{\mathbb{A}}(s) = \zeta_{\mathbb{A}}(1 - s)\).

Here \(\zeta_{\mathbb{A}}\) is the completed hyper Riemann zeta function, that is the Euler product of all the local zeta functions including the contribution from the prime at infinity \(\pi^{-(s/2)}\Gamma(s/2)\).

**Corollary 3.4.13** \(\zeta_{\mathbb{A}}(s)\) can be \(Q\)-analytically continued on \(*\mathbb{C}\).

**Corollary 3.4.14** \(\zeta_{\mathbb{Q}}(s)\) can be \(Q\)-analytically continued on \(*\mathbb{C}\),

\[ \zeta_{\mathbb{Q}}(1 - s) = \frac{\pi^{1/2-s} \Gamma(s/2) \zeta_{\mathbb{Q}}(s)}{\Gamma(1-s/2)}, \]

with a pole at \(s=1\) and trivial zeros at \(s = -2*\mathbb{Z}_{\geq 0}\).

Contained in this expression and in the proof are nonstandard versions of classical functions, many of which introduced are new objects.

The most basic is the **hyper gamma function** which is a (hyperfinite) product for the hyper naturals (the hyper factorial),

\[ *\Gamma(n) = \prod_{1 \leq j \leq n} j := n!, \]

for \(n \in *\mathbb{N}\) with \(*\Gamma(0) = 0! = 1\) (definition 3.2.13). The definition of the hyper gamma function for general hyper complex numbers follows by interpolation of the hyper factorial (definition 3.2.22)

\[ *\Gamma(s) = \int_{*\mathbb{R}} \exp(-y)y^{s-1}dy. \]

The **functional equation** for this function is given by theorem 3.2.25

\[ *\Gamma(s + 1) = s*\Gamma(s). \]
The method of proof of the functional equation follows the classical version involving theta functions. A hyper theta function is defined (via definition 3.3.7)

\[ *\Theta(s) = \sum_{n \in *\mathbb{Z}} *\exp(\pi in^2 s), \]

for \( *\Im(s) > 0 \). In order to obtain this, hyper Fourier transforms and hyper Poisson summation are needed. Results in this area require evaluation of definite hyper integrals and the interchanging of hyper products and hyper integrals. As a result the functional equation is given by

\[ *\Theta(-1/s) = (s/i)^{1/2} *\Theta(s). \]

The functional equation of \( \zeta_{*Q} \) follows from this as \( \zeta_{*Q} \) can be expressed as a hyper integral involving \( *\Theta \).

The main methods of proof are a combination of some analysis and a transfer of results from the classical setting for \( \zeta_Q \). It is emphasized that results do not transfer directly and so some analytical methods have to be used to bind the proofs together. A common example of proof is when one has a function which is a sum over \( *\mathbb{N} \) and has a classical counterpart which is a sum over \( \mathbb{N} \). Usually only properties of "an approximation" to the hyper function can be deduced from the classical counterpart which can then be extended to the hyper function using analysis. From the basic definitions of the hyper functions it is seen that they have a similar definition to their classical counterparts and it is unsurprising that proofs of nonstandard results follow a similar structure to the classical case.

1.2 Chapter 4 Overview

This chapter can be considered as a further application of the nonstandard and analytical tools from the previous chapter. Just as the Dedekind zeta function generalises the Riemann zeta function so the hyper Dedekind zeta function generalises the hyper Riemann zeta function by being a zeta function for hyper number fields. The obvious question is why the first chapter is included since it can be deduced from the work of this chapter. The main reason is that the first chapter serves to introduce some nonstandard tools not covered in the introduction to model theory and techniques of proof used in later chapters. Also some of the nonstandard versions of classical functions are used in this chapter (for example the hyper gamma function) and in other chapters. The above reasons could have yielded a chapter focussing on nonstandard tools and some classical functions in a nonstandard setting. With some extra work the hyper Riemann zeta function and some of its properties have been proven.
In order to define the hyper Dedekind zeta function a small amount of nonstandard analytic number theory is needed to give a solid meaning to hyper (algebraic) number fields and their properties. An important characteristic of any hyper number field is the ring of hyperintegers, section 3.2.3. This naturally leads into ideals which have been studied to an extent by Robinson (detailed in chapter 2). Finally combining all the above the hyper Dedekind zeta function for a given hyper number field $^*K$ can be defined (definition 4.3.1)

$$\zeta^{^*K}(s) = \sum_{a} \frac{1}{(N(a))^s},$$

where the sum runs over the set of internal ideals of the ring of hyperintegers of $^*K$ and $N$ is the norm of such an ideal. The first properties in the $Q$-topology are given including convergence and a product formula.

The second half of the chapter is devoted to proving the main result of the chapter; the functional equation $Q$-analytic continuation of $\zeta^{^*K}$ to $^*\mathbb{C}$ (corollary 4.4.24).

**Corollary 4.4.24** $\zeta^{^*K}(s)$ has a $Q$-analytic continuation to $^*\mathbb{C} \setminus \{1\}$. It has a simple pole at $s = 1$ with residue

$$\frac{2^{r_1}(2\pi)^{r_2}}{w|d^{^*K}|^{1/2}}h^{^*K}\Gamma(s).$$

It also satisfies the functional equation

$$\zeta^{^*K}(1 - s) = ^*A(s)\zeta^{^*K}(s).$$

Here

$$^*A(s) = 2^n \times (2\pi)^{-ns}|d^{^*K}|^{1/2-s}(^*\cos(\pi s/2))^{r_1+r_2}(^*\sin(\pi s/2))^{r_2}(\Gamma(s))^n.$$  

The proof follows a similar approach to that of the chapter but in the higher dimensional case, for example introducing a higher dimensional hyper gamma function and a higher dimensional hyper theta function (definition 4.4.5) which can be defined for each complete lattice $(L)$ in the hyper Minkowski space

$$^*\Theta_L(z) = \sum_{g \in L} ^*\exp(\pi i \langle gz, z \rangle),$$

where $\langle , \rangle$ is a hermitian scalar defined in section 3.2.

The functional equation (theorem 4.4.12) follows from a Poisson summation type formula,

$$^*\Theta_L(-1/z) = \frac{\sqrt{(z/i)}}{\text{vol}(L)^{1/2}}^*\Theta_{L'}(z),$$

where $L'$ is the dual lattice defined in theorem 4.4.11.
To finish off the proof a relationship between $\zeta_{*K}$ and $*\Theta_L$ (for some lattice) is needed in order to use the functional equation of the hyper theta function. This is provided essentially by a hyper Mellin transform (section 3.1).

These first two chapters do follow, and rely upon, the classical theory for the reasons given in the previous section. The reason for including them are that several nonstandard objects are defined rigorously with the hope they may have uses outside of the work in later chapters (which include in chapter 5 the hyper Riemann zeta function and hyper gamma function being $p$-adically interpolated in a nonstandard setting. It should also be noted that for a concise and non-repetitive presentation some of the arguments are not detailed in depth, for example the evaluation of definite hyper integrals. This is because in the first chapter rigorous proofs have been given and the method is the same each time, though clear referencing is given to enable the result to be proven in depth if so required.

1.3 Chapter 5 Overview

The concept of $p$-adic interpolation is wide in its application and in the actual method of obtaining a $p$-adic object from a real object. This chapter takes the concept of $p$-adic interpolation by the method of Mahler and interprets it in a nonstandard way via $p$-adic shadow maps. There are two main theorems in this chapter 5.3.1 and 5.4.1 with the latter strengthening of the former.

**Theorem 5.3.1** Let $f : \mathbb{N} \to \mathbb{Q}_p$ be a uniformly continuous function, with respect to the $p$-adic metric, on $\mathbb{N}$ and let $*f : *\mathbb{N} \to *\mathbb{Q}_p$ be the extension to its hyper function. Then $\text{sh}_p(*f) : \mathbb{Z}_p \to \mathbb{Q}_p$ is the the unique $p$-adic function obtained by Mahler interpolation.

Here $\text{sh}_p$ is defined in definition 3.1.7 as it is not defined on all of $*\mathbb{Q}_p$ since it contains unlimited elements.

**Theorem 5.4.1** Let $f : \mathbb{N} \to \mathbb{Q}_p$ be a uniformly continuous function, with respect to the $p$-adic metric, on $\mathbb{N}$. Then there exists a hyper function $*g : *\mathbb{N} \to *\mathbb{Q}_{\text{lim}p}$ such that $\text{sh}_p(*g) : \mathbb{Z}_p \to \mathbb{Q}_p$ is the the unique $p$-adic function obtained by Mahler interpolation.

Mahler interpolation (theorems 5.1.1 and 5.1.2) can be stated in two stages. Firstly any uniformly continuous function $f : \mathbb{N} \to \mathbb{Q}_p$ can be extended to a uniformly and continuous function $F : \mathbb{Z}_p \to \mathbb{Q}_p$ such that if $x \in \mathbb{N}$ then $F(x) = f(x)$. The second part shows that any continuous function $h : \mathbb{Z}_p \to \mathbb{Q}_p$ can be written as a series over $\mathbb{N}$. The first theorem 5.3.1 shows how this interpolation can be viewed in a nonstandard way. Indeed, any function $f : \mathbb{N} \to \mathbb{Q}_p$ can
be extended to a hyperfunction \( *f : *\mathbb{N} \to *\mathbb{Q}_p \) (using the tools of the introduction by regarding \( *f(\ast n) = (f(n_1), f(n_2), \ldots) \) \((\ast n = (n_1, n_2, \ldots))\) modulo the equivalence relation) with \( *f(n) = f(n) \) for \( n \in \mathbb{N} \). Applying the \( p \)-adic shadow map to this function does not give the original function \( f \) because \( \text{sh}_p(\ast \mathbb{N}) \neq \mathbb{N} \). As shown in chapter 2 this result is stronger as \( \text{sh}_p(\ast \mathbb{N}) = \mathbb{Z}_p \). So actually a function \( g : \mathbb{Z}_p \to \mathbb{Q}_p \) with \( g(n) = f(n) \) for \( n \in \mathbb{N} \) is obtained. The main work in the theorem is showing that the resulting function \( g \) from the \( p \)-adic shadow map is indeed the function which would have been obtained by Mahler interpolation of \( f \).

The proof shows that the hyper function \( (*f) \) can be written as a sum over \( \ast \mathbb{N} \) which is basically the sum over \( \mathbb{N} \) for \( f \) with added nonstandard terms (proposition 5.2.6). The properties of \( *f \) can now be partially deduced from the properties of \( f \) and some more analysis, in particular showing the continuity and convergence properties. Some care is needed when taking the shadow map. Although the properties of convergence and continuity carry through actually taking the shadow map on a sum over \( \ast \mathbb{N} \) is not trivial unlike taking it over a hyperfinite set.

The second main theorem uses the fact that \( \text{sh}_p(\ast \mathbb{Q}^{\lim_p}) = \mathbb{Q}_p \). So instead of considering \( p \)-adic uniform \( \mathbb{Q} \)-continuous hyper functions with values lying in \( \ast \mathbb{Q}_p \) consider the subset of it \( \ast \mathbb{Q}^{\lim_p} \) and in particular uniformly \( \mathbb{Q} \)-continuous hyper functions of the form \( g : \ast \mathbb{N} \to \ast \mathbb{Q}^{\lim_p} \in \mathcal{C}_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p}) \), which is the set of all \( p \)-adically uniformly \( \mathbb{Q} \)-continuous functions from \( \ast \mathbb{N} \) to \( \ast \mathbb{Q}^{\lim_p}_p \). This theorem shows that the shadow map is a surjective homomorphism from \( \mathcal{C}_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p}) \) to \( \mathcal{C}_p(\mathbb{Z}_p, \mathbb{Q}_p) \), the space of \( p \)-adically uniformly continuous functions from \( \mathbb{Z}_p \) to \( \mathbb{Q}_p \). Showing the map is a homomorphism and that the map lies in \( \mathcal{C}_p(\mathbb{Z}_p, \mathbb{Q}_p) \) follows from the definition of the \( p \)-adic shadow map. The hardest part is to show that the map is surjective. To show this a hyper sequence of hyper functions \((\ast f_m : \ast \mathbb{N} \to \ast \mathbb{Q}^{\lim_p} (m \in \ast \mathbb{N}))\) is constructed from the original function. It is eventually shown that there exist hyper functions in this hyper sequence \((\ast f_m, m \in \ast \mathbb{N} \setminus \mathbb{N})\) which are infinitely close to the original function by placing a hyper norm on \( \mathcal{C}_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p}) \). This hyper norm is based on the classical norm acting on functions in \( \mathcal{C}_p(\mathbb{Z}_p, \mathbb{Q}_p) \). In particular it is shown that functions of \( \mathcal{C}_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p}) \) have a monad (hyper functions of \( \mathcal{C}_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p}) \) which are infinitely close to a given function under the hyper norm). In each monad there is only one standard function infinitely close to the original function. This can also be interpreted as the hyper norm is defined for functions in \( \mathcal{C}_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p}) \) and in \( \mathcal{C}_p(\mathbb{Z}_p, \mathbb{Q}_p) \) and for each \( f \in \mathcal{C}_p(\mathbb{Z}_p, \mathbb{Q}_p) \) there exists at least one hyper function in \( \mathcal{C}_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p}) \) such that the \( p \)-adic shadow map of this function is \( f \).

A potential advantage of considering Mahler interpolation in this form is that on one level the notion of \( p \)-adic spaces almost "disappears" as the interpolated function is essentially a hyper function \( *f : *\mathbb{N} \to *\mathbb{Q} \) which has many of the same properties as functions from \( \mathbb{N} \) to \( \mathbb{Q} \). In future this
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could make work studying Mahler interpolation simpler.

1.4 Chapter Overview

The theme of this chapter is $p$-adic interpolation but in a more general setting than the previous chapter. There are many more methods of $p$-adic interpolation such as those used to interpolate the Riemann zeta function or the gamma function. There is no reason why these cannot be viewed from a nonstandard viewpoint as well.

The gamma function provides an ideal candidate for interpolation and the study of the Morita gamma function is in some ways easier than Mahler interpolation. In a lot of literature on nonstandard analysis calculus and certain elements of analysis are presented and defined entirely in a nonstandard way. This philosophy can be carried through to $p$-adically interpolating the gamma function. The aim of the interpolation is to find a uniformly $Q$-continuous hyper function from $^*\mathbb{N}$ to $^*\mathbb{Q}^\lim_p$ which interpolates the gamma function. Simply defining the hyper factorial (definition 3.2.13) does not work due to lack of continuity from the powers of $p$. By defining a restricted factorial (as in section 5.1) such a hyper function can be found, for all $n \in ^*\mathbb{N}$ with $n \geq 2$

$^*\Gamma_p : ^*\mathbb{N} \rightarrow ^*\mathbb{Z}, \quad ^*\Gamma_p(n) = (-1)^n \prod_{1 \leq j < n, p \nmid n} j,$

with $^*\Gamma(0) = -^*\Gamma(1) = 1$. It is shown in section 5.1 that it has a functional equation. One can now study this object as the interpolation of the gamma function. Naturally the link to the standard world is provided by the $p$-adic shadow map and results in the Morita gamma function.

The next section searches for a version of the Kubota-Leopoldt zeta function in the nonstandard world. A key point is that the hyper Riemann zeta function is used to provide values which combine to give the nonstandard function which interpolates the Riemann zeta function. By proving continuity properties the Kubota-Leopoldt function is obtained under the $p$-adic shadow maps. One observation is the role of the hyper Riemann zeta function. In section 2.4.1 it was shown that the shadow map onto the complex numbers takes $\zeta_{^*\mathbb{Q}}$ to $\zeta_{\mathbb{Q}}$ and similarly in this section $\zeta_{^*\mathbb{Q}}$ is mapped to the Kubota-Leopoldt zeta functions via the $p$-adic shadow maps by

Theorem 6.2.6 For a fixed $\sigma_0 \in \{-1, 1, 3, \ldots p - 3\}$

$\text{sh}_p((^*f_{\sigma_0}(\sigma)) = \zeta_{p, \sigma_0+1}(\text{sh}_p(\sigma))$.

(Here $\zeta_{p, \sigma_0}$ are the branches from the classical Kubota-Leopoldt function and $^*f_{\sigma_0}$ is the nonstandard
function interpolating the Kubota-Leopoldt function defined in section 6.2.1. In this way \( \zeta_{*Q} \) can be seen as a source of these zeta functions.

Probably one of the most important concepts contained in this work is that of double interpolation, that is interpolating a real object with respect to two distinct finite primes. Such a concept in the standard world is seemingly not possible, apart from the trivial case when the values lie in \( \mathbb{Q} \), because such a double interpolated function would have to take values in \( \mathbb{Q}_{p_1} \) and \( \mathbb{Q}_{p_2} \) which are non-isomorphic. By treating interpolation from a nonstandard perspective a solution can be found. An interpolating set of hyper values are needed with the requirement that they are uniformly \( \mathbb{Q} \)-continuous with respect to both prime valuations. This is not too restrictive and they form a subset of hyper functions which are interpolated with respect to a single prime. The nonstandard space of \( *\mathbb{Q} \) provides an ideal sanctuary for such hyper functions since the only requirement is that the values lie in \( *\mathbb{Q}_{\lim p_1} \cap *\mathbb{Q}_{\lim p_1} \). So a function for double interpolation is going to be of the form

\[
*f : *\mathbb{N} \to *\mathbb{Q}_{\lim p_1} \cap *\mathbb{Q}_{\lim p_2}.
\]

By taking the respective shadow maps standard functions are obtained. The aim of double interpolation is the same as that of single interpolation in that by looking at a real problem from a different perspective new information about the object can be found. As a note the process of double interpolation extends to interpolation with respect to a finite set of primes and in a special case to all finite primes.

An explicit example of double interpolation is given of the Riemann zeta function. The ideal choice of numbers to interpolate would be the set \( \{ (1 - p^m)(1 - q^m)\zeta_{*Q}(-m) \}_{m \in *\mathbb{N}} \). This follows from the work on the Kubota-Leopoldt zeta function. Firstly these need to be made continuous and by use of the Kummer congruences the double interpolation can take place. The next part of this looks at the resulting \( p \)-adic function (one only needs to consider one of the primes by symmetry) coming from the shadow map of the double Riemann zeta function. By extending some results of Katz on \( p \)-adic measures (in particular theorem 6.4.9) it is shown that a \( p \)-adic measure exists (lemma 6.4.11) which corresponds to the \( p \)-adic function and enables it to be written as an integral over \( \mathbb{Z}_p^\times \) (lemma 6.4.18).

**Theorem 6.4.9** Let \( a \in \mathbb{N} \) with \( a \geq 2 \) and \( (a, p) = 1 \). Also let \( r \in \mathbb{N} \), \( (r, p) = 1 \). Then for all \( m \in \mathbb{N} \)

\[
(1 - a^{m+1})r^m \zeta_{*Q}(-m) = \left( \frac{d}{dt} \right)^m \Psi_r(t) \big|_{t=1}.
\]

Here \( \Psi_r(t) = (1 - t^a)^{-1} \sum_{b=1}^{a} \xi_r(br)t^{br} \) and
\(\xi_r : \mathbb{Z} \rightarrow \mathbb{Z},\)

\[\begin{align*}
    n &\mapsto \begin{cases} 
    0 & r \nmid n, \\
    1 & r \mid n, ra \nmid n, \\
    1 - a & r \mid n, ra \mid n.
  \end{cases}
\end{align*}\] (1.4.1)

Although the Morita gamma function was simple to interpolate using nonstandard methods the "double gamma" function is the trivial value 1. The actual problem in determining its trivial existence is from elementary number theory. The problem is as follows. Let \(M_n = \{x : 1 \leq x \leq n, p \nmid x, q \nmid x\}\) where \(p\) and \(q\) are fixed primes. Then does there exist a \(j \in \mathbb{N} \setminus \{0, 1\}\) with \(p, q \nmid j\) such that \(j\) has an inverse in \(M_{pr}\) and \(M_{qs}\) for all \(r, s \in \mathbb{N}\) (where \(M_n\) can be considered as a multiplicative set of elements modulo \(n\))? The proof is given in theorem 6.6.1.

As stated above there is a special case of interpolation with respect to all finite primes (which would not be possible for the Riemann zeta function because it would be the interpolation of the trivial function 1). It is the function \(n \rightarrow n^x\) for some fixed \(n \equiv 1 \pmod{p}\). This section shows that there is a hyper function which is continuous with respect to each finite prime and by taking the shadow map for any prime \(p\) the original \(n^x\) function is obtained.

The final section gives a method of constructing a nonstandard Teichmüller character. It has the feel of an artificial method (section 5.8.1) as the hyperfunction is constructed based on a set of properties it is expected to have. Using this a definition is given for a double Hurwitz zeta function but none of the properties are explored. Finally a brief mention is given of the difficulty in constructing double \(L\)-functions.

1.5 Chapter Overview

This final chapter gives a conceptual overview of the work of Shai Haran. The central aim of this chapter is to translate my extensive studies of his work into a simplified version. Much of his work is contained in his book, Mysteries of the Real Prime ([Hall]). In some extended lectures on his work he gave several years after its publication he described his own book as "very condensed and hard to read" and as a result very few people understand this and hence his research. The hope is that this chapter may be useful on its own to help people who may wish to study his book (perhaps as an overview of his work before tackling his book) enabling the reader to have clear ideas of what his work involves. Naturally the aim of any research is to try and produce some original work. The
final section of this chapter contains some very basic attempts to view some parts of his work in a nonstandard way.

Shai Haran’s work is centred around the dictionary between arithmetic and geometry. Although this dictionary is very powerful there are two main problems with it. The first problem relates the geometric property of adding the point at infinity to the affine line which produces projective geometry. The analogue of \( \infty \) in the arithmetic picture is the real prime \( \eta \). This is not an unfamiliar object for example the completion of \( \mathbb{Q} \) with respect to \( \eta \) is \( \mathbb{R} \). What is unfamiliar are the real integers, corresponding to the \( p \)-adic integers. What is \( \mathbb{Z}_\eta \)?

The second problem is the lack of, what he calls, an arithmetical surface. For example in geometry the product of two affine lines is a plane yet the corresponding entities in the arithmetic picture is trivial as the product in the category of commutative rings of \( \mathbb{Z} \otimes \mathbb{Z} \) is simply \( \mathbb{Z} \). The hunt is then for a category where this does not happen.

Much of his work could be defined as ultimately trying to find a new language in which the dictionary between arithmetic and geometry is more complete. His way of doing this is to try and view all primes of \( \mathbb{Q} \) (finite \( p \) and the real prime \( \eta \)) on an equal footing. Many of his methods in his book are examples of this technique.

His book mainly covers work in the direction of the first problem. By taking the view that the real integers should be in some way comparable to the \( p \)-adic integers a \( p \)-adic approach can be made to the real integers since plenty of information is known about \( \mathbb{Z}_p \). This approach is refreshing given that for the past century real results have been applied in the \( p \)-adic world yet the \( p \)-adic world in some respects is simpler. His "tool of comparison" is the \( q \)(uantum)-world. He shows that this world interpolates between the real and \( p \)-adic worlds of several objects. By looking at limits of when \( q \to 0 \) and \( q \to 1 \) he obtains \( p \)-adic objects and real objects respectively. By considering a certain \( q \) version of well studied \( p \)-adic Markov chains he gives an interpretation of some aspects of the real integers by looking at the chain when \( q \to 1 \). It should be stressed that it his work is an interpretation and future work will show the validity of this work. Also, in one sense quite separate from this, though related to the \( q \)-world, is the second half of his book where he works on the Riemann zeta function which is closely related to the work of Connes ([Co1] and [Co2]).

The second problem is in its infancy and his work takes quite a different direction to the work which is already being developed in this area. The main object is the "field of one element" (\( \mathbb{F} \)). Most other work examines objects related to this such as schemes, varieties and zeta functions over this "field". Haran differs in his approach and tries to develop a new geometric language based on, what he calls \( \mathbb{F} \)-rings. These are certainly related to the "field of one element" but in a less than obvious
way. Also in this part I make a small review of the other, more traditional, work related to the "field of one element".
The Nonstandard Algebraic Number Theory of Robinson

2.1 Nonstandard Mathematics

Throughout the history of mathematics infinitely large numbers and, more commonly, infinitely small numbers (infinitesimals) have caused problems in establishing their existence. Infinitesimals initially appeared in the mathematical work of the Greek atomist philosopher Democritus when he put forward the question of whether it is possible to generate a cone by piling up circular plane surfaces with decreasing diameter and in his argument for the existence of atoms. (This problem and similar ones related to the concept of the continuum. The concepts of discreteness and continuity are still of interest today for example in philosophy and in the nature of time.) Archimedes also used infinitesimals to give results regarding areas and centres of gravity but did not state them as proofs because he did not believe in them. Most well known is the work of Leibniz in the development of calculus and (the differential notation $dx$) though this was later replaced by the $\epsilon - \delta$ method of the nineteenth century as the transfer principle of Leibniz (that results in the reals can be extended to the infinitesimals) could not be rigorously justified.

The concept of infinitely large numbers has a similar history beginning with the Indians in the ancient Yajur Veda (c. 1200–900 BC) which states that "if you remove a part from infinity or add a part to infinity, still what remains is infinity". It continued with Greeks (for example the paradoxes of Zeno) and to the modern work in axiomatic set theory.

The rigorous logical framework for infinitely large and small numbers was pioneered by Robinson in the 1960s via nonstandard mathematics, a branch of mathematical logic - model theory. A basic
Chapter 2: The Nonstandard Algebraic Number Theory of Robinson

Fact in model theory is that every infinite mathematical structure has nonstandard models (this basically means that there are non-isomorphic structures which satisfy the same elementary properties). The existence of nonstandard models has been known since the 1920s from the work of Thoralf Skolem. More interest began in the fifties but it was not until Robinson applied this model theoretic machinery to analysis that nonstandard analysis was founded.

Robinson’s original presentation ([Ro1]) was considered by most mathematicians to be unnecessarily complicated because of the logical formalism needed. Several other approaches appeared and there exists at least eight different (simpler) presentations of the methods of nonstandard analysis. They fall into two categories: semantic, or model-theoretic, approach and the syntactic approach. The most used presentation is the via superstructures (which is model theoretic like Robinson’s original work) introduced by Robinson and Zakon (it was also the first one to be purely set-theoretic in nature).

The syntactic approach was created by Nelson in the 1970s using less model theory by introducing an axiomatic formulation of non-standard analysis, called internal set theory.

2.1.1 Overview of Nonstandard Analysis

This section introduces some of the basic notions of nonstandard analysis without focussing on the detailed and specific formulation which follows later. In approaching it in this generality some logical rigour is sacrificed but an initial feel can be obtained. The fundamental object in defining nonstandard analysis is a universe.

Definition 2.1.1. A universe, \( \mathbb{U} \), is a non empty collection of "mathematical objects" that is closed under subsets and closed under basic mathematical operations. These operations are union of sets, intersection of sets, set difference, ordered pair, Cartesian product, powerset and function set. A universe is also assumed to contain (copies of) \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \). Further \( \mathbb{U} \) is assumed to be transitive (\( a \in A \in \mathbb{U} \Rightarrow a \in \mathbb{U} \)).

One point of interest is the notion of "mathematical objects". These are taken to include all the objects of mathematics (numbers, sets, functions, relations, ordered tuples, Cartesian products, etc.). In fact sets are actually enough and all the objects can be formalized in the foundational framework of Zermelo-Fraenkel axiomatic set theory. (For example a function \( f : A \to B \) can be identified with the set of pairs \( \{ (a, b) : b = f(a) \} \) which is a subset of the Cartesian product \( A \times B \). Conversely a function from \( A \) to \( B \) can be defined set theoretically.)

The nonstandard universe is obtained via a star map. This is a one-to-one map \( \ast : \mathbb{U} \to \mathbb{V} \) between
two universes that maps object \( A \in \mathbb{U} \) to its hyper-extension (or sometimes termed nonstandard extension) \(*A \in \mathbb{V} \). Further it is assumed that for all \( n \in \mathbb{N} \), \(*n = n \) and \(*\mathbb{N} \neq \mathbb{N} \).

The star map has a powerful property in that it preserves a large class of properties of the standard universe. This is the transfer principle (or often called the Leibniz principle). In a non-rigorous way let \( P(a_1, \ldots, a_n) \) be a property of standard objects \( a_1, \ldots, a_n \) which has a bounded formalization in a language is true iff it is true about the corresponding hyper-extensions \(*a_1, \ldots, *a_n \). This will be made more precise by using mathematical logic.

Combining the previous two paragraphs leads to the following definition.

**Definition 2.1.2.** A model of nonstandard analysis is a triple \((*, \mathbb{U}, \mathbb{V}) \) where \(* : \mathbb{U} \rightarrow \mathbb{V} \) is a star-map satisfying the transfer principle.

The other fundamental principle of nonstandard analysis is the saturation property. Saturation has a precise definition is model theory, which will be detailed later. However in terms of set theory saturation can be given an elementary formulation as an intersection property.

**Definition 2.1.3.** An internal set is any \( x \) with \( x \in *A \) for some standard \( A \). An external set is an element of the nonstandard model that is not internal.

**Definition 2.1.4.** Let \( X \) be a set with \( A = (A_i)_{i \in I} \) a family of subsets of \( X \). Then the collection \( A \) has the finite intersection property, if any subcollection \( J \subset I \) has non-empty intersection \( \bigcap_{i \in J} A_i \neq \emptyset \).

**Definition 2.1.5** (\( \kappa \)-Saturation). Let \( \kappa \) be an infinite cardinal. Then the \( \kappa \)-saturation principle states that if \( I \) is an index set with cardinality \( |I| < \kappa \) and \( (A_i)_{i \in I} \) is a family of internal sets of an internal set \( A \) having the finite intersection property, then \( \bigcap_{i \in I} A_i \neq \emptyset \).

Robinson’s original presentation contained a weaker form of saturation where he used concurrent relations. It was Luxemburg who introduced \( \kappa \)-saturation as a fundamental tool in nonstandard analysis and in particular for the nonstandard study of topological spaces. The final point relates to the existence of a star map which can be constructed using ultraproducts.

### 2.1.2 Model Theory

Mathematical logic, like many areas of mathematics, has various branches. One branch is the study of mathematical structures by considering the first order sentences true in these structures and sets definable by first order formulae: *model theory*. The two main, but connected, reasons for studying
model theory are finding out more about a mathematical structure using model theoretic techniques and given theories proving general theorems about their models. Some recent results from model theory have been related to number theoretic problems. The two main results are the Mordell-Lang conjecture\(^1\) (for function fields in positive characteristic) and another proof of the Manin-Mumford conjecture\(^2\), both were proved by Hrushovski. For a concise overview of model theory see \cite{Mar} and for a more detailed introduction the textbook of \cite{C-Kei} provides a solid grounding in this area of logic.

The fundamental objects are structures and the components of a language.

**Definition 2.1.6** (Language). A language is a collection of symbols of three types:

- set of function symbols \((\mathcal{F})\) and a \(n_f \in \mathbb{N}\) for each \(f \in \mathcal{F}\);
- set of relation symbols \((\mathcal{R})\) and \(n_R \in \mathbb{N}\) for each \(R \in \mathcal{R}\);
- set of constant symbols \((\mathcal{C})\).

The numbers \(n_f\) and \(n_R\) are the arities of the function \(f\) and relation \(R\) respectively.

A simple example is given by the language of rings \(\mathcal{L}_r = \{+,-,\cdot,0,1\}\) where \(+, -\) and \(\cdot\) are binary function symbols and 0 and 1 are constants.

**Definition 2.1.7** (\(\mathcal{L}\)-Structure). A \(\mathcal{L}\)-structure \(\mathcal{M}\) consists of the following:

- a non empty set \(M\) (the universe of \(\mathcal{M}\));
- for each function symbol \(f \in \mathcal{F}\) a function \(f^\mathcal{M} : M^{n_f} \rightarrow M\);
- for each relation symbol \(R \in \mathcal{R}\) a set \(R^\mathcal{M} \subset M^{n_R}\);
- for each constant symbol \(c \in \mathcal{C}\) an element \(c^\mathcal{M} \in M\).

\(f^\mathcal{M}, R^\mathcal{M}\) and \(c^\mathcal{M}\) are the interpretation of the symbols \(f\), \(R\) and \(c\) respectively.

When there is no confusion the superscript \(\mathcal{M}\) is often dropped. Naturally maps can be considered between \(\mathcal{L}\)-structures and the ones of interest are those which preserve the interpretation of \(\mathcal{L}\).

\(^1\)The generalised Mordell-Lang conjecture states that the irreducible components of the Zariski closure of a subset of a group of finite rank inside a semi-abelian variety are translates of closed algebraic subgroups.

\(^2\)One way of stating the Manin-Mumford conjecture is that a curve \(C\) in its Jacobian variety \(J\) can only contain a finite number of points that are of finite order in \(J\), unless \(C = J\).
Definition 2.1.8. Suppose \( \mathcal{M} \) and \( \mathcal{N} \) are \( \mathcal{L} \)-structures with universes \( M \) and \( N \) respectively. Then an \( \mathcal{L} \)-embedding \( \theta : \mathcal{M} \to \mathcal{N} \) is a one-to-one map \( \theta : M \to N \) such that:

- for all \( f \in \mathcal{F} \) and \( a_1, \ldots, a_{n_f} \in M \), \( \theta(f^M(a_1, \ldots, a_{n_f})) = f^N(\theta(a_1), \ldots, \theta(a_{n_f})) \);
- for all \( R \in \mathcal{R} \) and \( a_1, \ldots, a_{m_R} \in \mathcal{M} \), \( (a_1, \ldots, a_{m_R}) \in R^M \) iff \( (\theta(a_1), \ldots, \theta(a_{m_R})) \in R^N \);
- for \( c \in \mathcal{C} \), \( \theta(c^M) = c^N \).

Further \( \mathcal{M} \) is a substructure of \( \mathcal{N} \) (or \( \mathcal{N} \) is an extension of \( \mathcal{M} \)) if \( M \subset N \) and the inclusion map is an \( \mathcal{L} \)-embedding. A bijective \( \mathcal{L} \)-embedding is called a \( \mathcal{L} \)-isomorphism and in the case \( M = N \) it is called a \( \mathcal{L} \)-automorphism. Finally the cardinality of \( \mathcal{M} \) is \( |M| \).

An example of a structure is \( (\mathbb{R}, +, 0) \) which is an \( \mathcal{L}_g \)-structure where \( \mathcal{L}_g = \{+, 0\} \) with + is a binary function and 0 a constant. The structure \( (\mathbb{Z}, +, 0) \) is a substructure of \( (\mathbb{R}, +, 0) \).

In any language (mathematical or linguistical) one aim is to use it to use it to convey ideas (often via sentences). In logic the first step is to create formulae to describe properties of \( \mathcal{L} \)-structures. Formulae are strings built using the symbols of \( \mathcal{L} \) and the (assumed disjoint) set of logical symbols (which consists of logical connectives (\( \land, \lor, \neg, \rightarrow \) and \( \leftrightarrow \)), parentheses (\( [\, ] \)), quantifiers (\( \forall \) and \( \exists \)), the equality symbol (\( = \)) and variable symbols). Simplistically terms are expressions obtained from constants and variables by applying functions.

Definition 2.1.9. The set of \( \mathcal{L} \)-terms is the smallest set \( T \) such that

- for each \( c \in \mathcal{C} \), \( c \in T \),
- each variable symbol is an element of \( T \), and
- if \( f \in \mathcal{F} \) and \( t_1, \ldots, t_{n_f} \in T \) then \( f(t_1, \ldots, t_{n_f}) \in T \).

A \( \mathcal{L} \)-term (\( t \)) has a unique interpretation in a \( \mathcal{L} \)-structure \( \mathcal{M} \) as a function \( t^M : M^m \to M \). For a subterm \( s \) of a term \( t \) and \( \bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M \), \( s^M(\bar{a}) \) can be defined inductively

- if \( s \) is the constant symbol \( c \) then \( s^M(\bar{a}) = c^M \),
- if \( s \) is the variable \( v_{i_j} \) then \( s^M(\bar{a}) = a_{i_j} \),
- if \( s \) is the term \( f(t_1, \ldots, t_{n_f}) \) (where \( f \) is a function symbol of \( \mathcal{L} \) and \( t_i \) are terms) then \( s^M(\bar{a}) = f^M(t_1^M(\bar{a}), \ldots, t_{n_f}^M(\bar{a})) \).
Finally the definition of \( L \)-formulae can be given.

**Definition 2.1.10.** \( L \)-formulae are defined via atomic \( L \)-formulae. An object \( \phi \) is said to be an atomic \( L \)-formula if \( \phi \) is either \( t_1 = t_2 \) (for terms \( t_1, t_2 \)) or \( R(t_1, \ldots, t_n) \) (for \( R \in \mathcal{R} \) and \( t_i \) terms). Then the set of \( L \)-formulae is the smallest set \( \mathcal{W} \) containing the atomic formulae such that

- if \( \phi \) is in \( \mathcal{W} \) then so is \( \neg \phi \),
- if \( \phi \) and \( \psi \) are in \( \mathcal{W} \) then so are \((\phi \wedge \psi)\) and \((\phi \vee \psi)\), and
- if \( \phi \) is in \( \mathcal{W} \) then so are \( \forall v_i \phi \) and \( \exists v_i \phi \) (where \( v_i \) is a variable).

The set of \( L \)-formulae split into two depending on the variables of the formula. Indeed a variable is said to be bound in a quantifier if it occurs inside a \( \exists \) or \( \forall \) quantifier, otherwise it is said to be free. A formula is called a sentence if it has no free variables otherwise it is called a predicate.

The next concept is *satisfaction*. That is given an \( L \)-formula \( (\phi(x), \bar{x} = (x_1, \ldots, x_n) \) are free variables) and a \( L \)-structure \( \mathcal{M} \) the notion of \( \phi(\bar{a}) \) being true in \( \mathcal{M} \), \( \bar{a} = (a_1, \ldots, a_n) \) is an \( n \)-tuple of elements in \( \mathcal{M} \). This is denoted by \( \mathcal{M} \models \phi(\bar{a}) \) with the negation denoted by \( \mathcal{M} \not\models \phi(\bar{a}) \).

**Definition 2.1.11.** The notion of \( \mathcal{M} \models \phi(\bar{a}) \) is defined inductively by

- If \( \phi \) is \( t_1 = t_2 \) then \( \mathcal{M} \models \phi(\bar{a}) \) if \( t_1^\mathcal{M}(\bar{a}) = t_2^\mathcal{M}(\bar{a}) \);
- If \( \phi \) is \( R(t_1, \ldots, t_n) \) (a \( n \)-ary relation) then \( \mathcal{M} \models \phi(\bar{a}) \) if \( (t_1^\mathcal{M}(\bar{a}), \ldots, t_n^\mathcal{M}(\bar{a})) \in \mathcal{R}^\mathcal{M} \);
- If \( \phi \) is \( \phi_1 \wedge \phi_2 \) then \( \mathcal{M} \models \phi(\bar{a}) \) if \( \mathcal{M} \models \phi_1(\bar{a}) \) and \( \mathcal{M} \models \phi_2(\bar{a}) \);
- If \( \phi \) is \( \phi_1 \vee \phi_2 \) then \( \mathcal{M} \models \phi(\bar{a}) \) if \( \mathcal{M} \models \phi_1(\bar{a}) \) or \( \mathcal{M} \models \phi_2(\bar{a}) \);
- If \( \phi \) is \( \neg \phi_1 \) then \( \mathcal{M} \models \phi(\bar{a}) \) if \( \mathcal{M} \not\models \phi(\bar{a}) \);
- If \( \phi \) is \( \exists v \psi(\bar{v}, x) \) (where the free variables of \( \psi \) are among \( \bar{v}, x \)) then \( \mathcal{M} \models \phi(\bar{a}) \) if there is a \( b \in \mathcal{M} \) such that \( \mathcal{M} \models \psi(\bar{a}, b) \);
- If \( \phi \) is \( \forall x \psi(\bar{v}, x) \) then \( \mathcal{M} \models \phi(\bar{a}) \) if \( \mathcal{M} \models \psi(\bar{a}, b) \) for all \( b \in \mathcal{M} \).

**Definition 2.1.12.** A \( L \)-theory is a set of sentences of the language \( L \). A model of a theory \( T \) is a \( L \)-structure \( \mathcal{M} \) which satisfies all the sentences of \( T \), denoted \( \mathcal{M} \models T \). Further a \( L \)-theory, \( T \), is satisfiable iff there exists a model of \( T \) and it is consistent iff a formal contradiction can be derived from \( T \).
(It can be shown as a corollary from the completeness theorem below that $T$ is satisfiable iff $T$ is consistent.)

**Definition 2.1.13.** Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures with $M \subset N$. $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ (or $\mathcal{N}$ is an elementary extension of $\mathcal{M}$), denoted $\mathcal{M} \prec \mathcal{N}$, iff for any formula $\varphi(\overline{x})$ and tuple $\overline{a}$ from $M$

$$\mathcal{M} \models \varphi(\overline{a}) \iff \mathcal{N} \models \varphi(\overline{a}).$$

Further a map $f : \mathcal{M} \to \mathcal{N}$ is called an elementary embedding iff it is an embedding and $f(M) \prec \mathcal{N}$.

**Definition 2.1.14.** Let $\phi$ be an $\mathcal{L}$-sentence and $T$ an $\mathcal{L}$-theory. A proof of $\phi$ from $T$ is a finite sequence of $\mathcal{L}$-formulae $\psi_1, \ldots, \psi_m$ such that $\psi_m = \phi$ and $\psi_i \in T$ or $\psi_i$ follows from $\psi_1, \ldots, \psi_{i-1}$ by a simple logical rule for each $i$. ($T \vdash \phi$ if there is a proof of $\phi$ from $T$.)

These definitions lead to some important theorems including the very important compactness theorem, one of the crucial tools of model theorists.

**Theorem 2.1.15 (Completeness Theorem).** Let $T$ be an $\mathcal{L}$-theory and $\phi$ an $\mathcal{L}$-sentence, then $T \models \phi$ if and only if $T \vdash \phi$. Moreover, if $T$ has infinite models then $T$ has a model where the model has cardinality $\kappa$, for all $\kappa \geq |\mathcal{L}| + \aleph_0$.

**Theorem 2.1.16 (Compactness Theorem).** A $\mathcal{L}$-theory $T$ has a model iff every finite subset of $T$ has a model.

A proof of the compactness theorem can be given via ultraproducts though it is also a consequence of the completeness theorem.

It is often useful to work in a very rich model of a theory, for example it is often easier to prove things in an algebraically closed field of infinite transcendence degree and in the context of this work the nonstandard methods in assuming there are infinite elements when dealing with the reals. This is made precise by the use of types and the property of saturation.

For an $\mathcal{L}$-structure $\mathcal{A}$ let $\mathcal{L}_A = \mathcal{L} \cup \{c_a : a \in A\}$ be the expansion of the language by adjoining constant symbols. This leads onto the method of diagrams but this is not needed in this introduction. Instead suppose that for some $\mathcal{L}$-structure $\mathcal{M}$, $A \subset M$ then let $\text{Th}_A(M)$ be the set of all $\mathcal{L}_A$-sentences, $\varphi$, such that $\mathcal{M} \models \varphi$.

**Definition 2.1.17 (Types).** An $n$-type over $A$ is a set of $\mathcal{L}_A$-formulas in free variables $x_1 \ldots x_n$ that is consistent with $\text{Th}_A(M)$. A complete $n$-type is a maximal $n$-type. Let $S_n(A)$ be the set of complete $n$-types over $A$. 
A formula $\phi(x_1, \ldots, x_n)$ is said to be consistent with a $\mathcal{L}$-theory $T$ iff there exists a model $\mathcal{U}$ which realises $\phi$ (iff for some $n$-tuple of elements satisfies $\phi$ in $\mathcal{U}$). A more expansive definition is to say that a complete $n$-type is a set $q$ of $\mathcal{L}$-formulae consistent with $\text{Th}_A(\mathcal{M})$ in the free variables $x_1, \ldots, x_n$ such that for any $\mathcal{L}$-formula, $\varphi(\bar{x})$, either $\varphi(\bar{x}) \in q$ or $\neg\varphi(\bar{x}) \in q$.

**Definition 2.1.18.** Let $\kappa$ be an infinite cardinal. A structure $\mathcal{M}$ is $\kappa$-saturated if for every $A \subset M$, $|A| < \kappa$ and $p \in S_1(A)$ then $p$ is realized in $\mathcal{M}$. Induction shows that in this case every $n$-type over $A$ is also realized in $\mathcal{M}$. $\mathcal{M}$ is said to be saturated if it is $|\mathcal{M}|$-saturated.

Finally a nonstandard model can be defined.

**Definition 2.1.19.** A nonstandard model of, a $\mathcal{L}$ structure, $\mathcal{M}$ is a saturated elementary extension of $\mathcal{M}$ which is usually denoted by

$$^*: \mathcal{M} \to {}^*\mathcal{M}.$$ 

As mentioned at the end of section 2.1.1 an important point is the existence of a nonstandard model. One method of existence is via ultraproducts. In fact this is a basic method of constructing models in general and originated in the work of Skolem in the 1930s and has been used extensively since the work of Los in 1955.

**Definition 2.1.20** (Filters and Ultrafilters). Let $I$ be a set. A filter on $I$ is a subset $\mathcal{F}$ of (the power set of $I$) $\mathcal{P}(I)$ satisfying the following properties:

1. $I \in \mathcal{F}, \emptyset \notin \mathcal{F}$;
2. if $U \in \mathcal{F}$ and $U \subset V$ then $V \in \mathcal{F}$;
3. if $U, V \in \mathcal{F}$ then $U \cup V \in \mathcal{F}$.

An ultrafilter on $I$ is a filter on $I$ which such that for any $U \in \mathcal{P}(I)$ either $U \in \mathcal{F}$ or $I \setminus U \in \mathcal{F}$.

An ultrafilter ($\mathcal{F}$) on a set $I$ is principal if there is $i \in I$ such that $\{i\} \in \mathcal{F}$ (so $U \in \mathcal{F} \iff i \in U$). An ultrafilter is non-principal if it is not principal. The existence of ultrafilters is provided by following theorem and corollary

**Theorem 2.1.21** (Ultrafilter Theorem). If $E \subset \mathcal{P}(I)$ and $E$ has the finite intersection property then there exists an ultrafilter $\mathcal{F}$ of $I$ such that $E \subset \mathcal{F}$.

(A proof can be found in proposition 4.1.4 of [C-Kei].)
Corollary 2.1.22. Any proper filter of \( I \) can be extended to an ultrafilter over \( I \).

Definition 2.1.23 (Cartesian Products of \( \mathcal{L} \)-Structures). Let \( I \) be an index set, \( \mathcal{L} \) a fixed language and \( (\mathcal{M}_i)_{i \in I} \) a family of \( \mathcal{L} \)-structures. Then the \( \mathcal{L} \)-structure \( \mathcal{M} = \prod_{i \in I} \mathcal{M}_i \) is defined as follows:

- universe - the cartesian product of \( \mathcal{M}_i \)s (the set of sequences \((a_i)_{i \in I}\) such that \( a_i \in \mathcal{M}_i \) for each \( i \in I \).
- constant symbol - for each \( c \) (constant symbol) of \( \mathcal{L} \) define \( c^\mathcal{M} = (c^\mathcal{M}_i)_{i \in I} \).
- relation symbol - for \( R \) an \( n_R \)-ary relation symbol define \( R^\mathcal{M} = \prod_{i \in I} R^\mathcal{M}_i \).
- function symbol - for \( f \) a \( n_f \)-ary function symbol and \((a_{1,i}), ..., (a_{n,i})_{i \in I} \) then \( f^\mathcal{M}((a_{1,i}), ..., (a_{n,i})_{i \in I}) = (f^\mathcal{M}_i(a_{1,i}), ..., a_{n,i})_{i \in I} \).

Let \( I \) be a set and \( I \) a filter on this set. Let \( \mathcal{M} \) be a cartesian product of \( \mathcal{L} \)-structures with \( (\mathcal{M}_i)_{i \in I} \) the related family of \( \mathcal{L} \)-structures. An equivalence relation \((\equiv_F)\) can be put on \( \mathcal{M} = \prod_{i \in I} \mathcal{M}_i \) by

\[
(a_i)_{i \in I} \equiv_F (b_i)_{i \in I} \iff \{ i \in I : a_i = b_i \} \in \mathcal{F}.
\]

The equivalence class of the element \((a_i)_{i \in I}\) is denoted by \((a_i)_\mathcal{F}\). This is used to define another \( \mathcal{L} \)-structure.

Definition 2.1.24 (Reduced Products of \( \mathcal{L} \)-Structures). The reduced product of the \( \mathcal{M}_i \)s over \( \mathcal{F} \) is denoted by \( \prod_{i \in I} \mathcal{M}_i \setminus \mathcal{F} \). It is the quotient structure defined by:

- universe - the quotient of \( \prod_{i \in I} \mathcal{M}_i \) by \( \equiv_F \).
- the interpretation of a constant symbol \( c \) of \( \mathcal{L} \) is \((c^\mathcal{M}_i)_{\mathcal{F}}\).
- for \( R \) an \( n_R \)-ary relation symbol, \( f \) an \( n_f \)-ary function symbol in \( \mathcal{L} \) and \((a_{1,i}), ..., (a_{n,i})_{i \in I} \) then

\[
\prod_{i \in I} \mathcal{M} \setminus \mathcal{F} \models R(a_1, ..., a_n) \iff \{ i \in I : (a_{1,i}, ..., a_{n,i}) \in R^\mathcal{M}_i \} \in \mathcal{F},
\]

and

\[
f^\mathcal{M}(a_1, ..., a_n) = (f^\mathcal{M}_i(a_{1,i}, ..., a_{n,i}))_{\mathcal{F}}.
\]

This quotient structure is well-defined by the properties of filters. In the special case when \( \mathcal{F} \) is an ultrafilter then \( \prod_{i \in I} \mathcal{M}_i \setminus \mathcal{F} \) is called the ultraproduct of the \( \mathcal{M}_i \)s with respect to \( \mathcal{F} \).

The key result is Łos’ theorem which basically connects what formulae are satisfied in the ultraproduct and in the original structure. The proof is by structural induction on the complexity of the formulae.
Theorem 2.1.25 (Łoś Theorem). Let $I$ be a set, $\mathcal{F}$ an ultrafilter on $I$ and $(\mathcal{M}_i)\ (i \in I)$ a family of $\mathcal{L}$-structures. Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula, and let $a_1, \ldots, a_n \in \prod_{i \in I} M_i \setminus \mathcal{F}$ be represented by $(a_1, i), \ldots, (a_n, i) \in \prod_{i \in I} M_i$. Then
\[
\prod_{i \in I} M_i \setminus \mathcal{F} \models \varphi(a_1, \ldots, a_n) \iff \{i \in I : M_i \models \varphi(a_1, \ldots, a_n, i)\} \in \mathcal{F}.
\]

Corollary 2.1.26. Let $I$ be a set, $\mathcal{F}$ an ultrafilter on $I$ and $\mathcal{M}$ an $\mathcal{L}$-structure. Then the natural map $\mathcal{M} \to \mathcal{M}' \setminus \mathcal{F}$, $a \mapsto (a)_{\mathcal{F}}$, is an elementary embedding. (Here $(a)_{\mathcal{F}}$ is the equivalence class of the sequence with all terms equal to $a$.)

2.1.3 The Hyperreals

Putting all the work of the previous section together enables the hyperreals to be defined. Let $I = \mathbb{N}$ and let $\mathcal{F}$ be a nonprincipal ultrafilter (such ultrafilters exist on $\mathbb{N}$ by the axiom of choice, see [Go] corollary 2.6.2 for a proof). Let $\mathcal{L} = \{+, -, \leq, 0\}$ be the language of abelian ordered groups (with $+$ and $-$ binary function symbols, $\leq$ a binary relation symbol and $0$ a constant) and endow $\mathbb{R}$ with its natural $\mathcal{L}$-structure to get a $\mathcal{L}$-structure $R$. Let $\mathcal{M}_i = R$ for all $i \in \mathcal{N}$ Let $\mathcal{R} = \prod_{i \in \mathcal{N}} \mathcal{M}_i \setminus \mathcal{F} = \mathbb{R}^\mathcal{N} \setminus \mathcal{F}$. Then by the corollary 2.1.26 $\mathcal{R}$ is an elementary extension of $R$ and further it is saturated (see chapter 6 of [C-Kei]) - hence it is a nonstandard model of $R$, therefore let $^*\mathcal{R} = \mathbb{R}^\mathcal{N} \setminus \mathcal{F}$. The map of the corollary is the $^*$-map desired with the transfer principle provided by the map being an elementary embedding. By applying the transfer principle (to the corresponding statement for $\{\mathbb{R}, +, -, <\}$) the structure $\{^*\mathbb{R}, +, -, <\}$ is a complete ordered field. One could also check this by going through the axioms for a field and checking they hold for $^*\mathbb{R}$.

Corollary 2.1.26 implies that $\mathbb{R}$ can be considered as embedded in $^*\mathbb{R}$ but are these the only elements? (In fact a general nonstandard model $^*\mathcal{M}$ does contain elements distinct from those in the original structure because of saturation.) For the existence of such elements in $^*\mathbb{R}$ consider the sets $X_n = \{x \in \mathbb{R} : n < x\}$ for each $n \in \mathbb{N}$. Then $(X_n)$ is a countable family of sets satisfying the finite intersection property. Therefore the intersection of their image $(\cap^* X_n)$ is non-empty in $^*\mathbb{R}$. Then any such element ($\omega$) of the intersection satisfies $\omega >^* x = x$ for all $x \in \mathbb{R}$. The set of infinite numbers is the set $\{s \in ^*\mathbb{R} : n < |s| \forall n \in \mathbb{N}\}$. The set of infinite integers are often called hyperfinite integers. A number $s \in ^*\mathbb{R}$ is said to be limited (or finite) if $|s| < n$ for some $n \in \mathbb{N}$.

A similar construction can be used to show that infinitesimal elements exist by considering the sets $Y_n = \{y \in \mathbb{R} : 0 < x < 1/n\}$ for $n \in \mathbb{N} \setminus \{0\}$. The set of infinitesimals are denoted by $\mu_\eta(0) = \{x \in ^*\mathbb{R} : |x| < 1/n \forall n \in \mathbb{N}\}$.

An equivalence relation ($\simeq_\eta$) can be defined for $x, y \in ^*\mathbb{R}$ by $x \simeq_\eta y$ iff $x - y \in \mu_\eta(0)$. This leads
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to the real shadow map from limited elements of $^*\mathbb{R}$ to $\mathbb{R}$. It is shown in full details in definition 3.1.1 that if $x \in ^*\mathbb{R}$ is limited then there exists a unique $\rho \in \mathbb{R}$ such that $\rho \simeq_\eta x$ enabling the definition of the shadow/standard part of $x$ to be $\text{sh}_\eta(x) = \rho$.

A very important concept is to be able to descend from the nonstandard model to the standard model. This can be done for a general topological space with a valuation (for example see [Mac1]). For the hyperreals there is the real shadow map, denoted $\text{sh}_\eta$, briefly described in the previous paragraph and developed in full detail in chapter 3.

In the construction of $^*\mathbb{R}$ via ultrafilters, one essentially views elements of $^*\mathbb{N}$ as infinite sequences of real numbers. This construction can also be used to define functions and subsets on $^*\mathbb{R}$. For example the important concept of an internal set can be defined. Suppose there is a given sequence of subsets $(A_n)_\mathbb{N}$ of $\mathbb{R}$ then a subset $[A_n]$ of $^*\mathbb{R}$ can be defined by specifying that for each $r \in ^*\mathbb{R}$,

$$r \in [A_n] \leftrightarrow \{n \in \mathbb{N} : r_n \in A_n\} \in \mathcal{F},$$

where $(r_n)$ is the equivalence class of $r$ modulo the ultrafilter $\mathcal{F}$. This can be shown to be well-defined ([Go], 11.1). A set which is not internal is said to be external. Examples of internal sets include $^*\mathbb{N}$, $^*\mathbb{Q}$, $^*\mathbb{Z}$ and $^*\mathbb{R}$. Full details of internal sets of $^*\mathbb{R}$ can be found in [Go], chapter 11. Similar constructions using sequences of functions can be used to create hyper functions, and internal functions can also be defined in an analogous way to internal sets.

A problem with $^*\mathbb{R}$, and more generally for a nonstandard enlargement of a topological space, is the lack of a canonical topology. Given a topological space $T$, with topology $\tau$, and enlargement $^*T$ there are two main topologies which can be put on this hyper space - the S(standard)-topology and the finer Q-topology.

- The basis of fundamental neighbourhoods for the S-topology are generated by $^*U$ (where $U$ runs through the open subsets of $T$).
- The basis of fundamental neighbourhoods for the Q-topology are $^*V$ (where $V$ is the set of fundamental neighbourhoods in $\tau$).

In the case of the hyperreals the S-neighbourhoods are of the form $((r - \epsilon, r + \epsilon)) = \{x \in ^*\mathbb{R} : \text{sh}_\eta |r - x| < \epsilon\}$ where $r \in ^*\mathbb{R}$ and $\epsilon \in \mathbb{R}_{>0}$. The S-open sets are the union of S-neighbourhoods and the S-open sets form the S-topology on $^*\mathbb{R}$. The Q-neighbourhoods are of the form $(s - \delta, s + \delta)$ for $s \in ^*\mathbb{R}$ and $\delta \in ^*\mathbb{R}_{>0}$. Further details can be found in chapter 4 of [Ro1].

All the analytical results in this work are proven in the Q-topology as it is the finer topology.
To summarize the hyperreals can be "explicitly" considered as certain elements of $\mathbb{R}^N$ or just $\mathbb{R}$ with added elements which are "infinitely close" to each element of $\mathbb{R}$ and "infinitely large". Most of the properties of $\mathbb{R}$ hold in the hyperreals by using the transfer principle. It should be noted that the construction of the hyperreals is not unique because of the choice of ultrafilter. In fact if the continuum hypothesis is assumed it can be shown that all quotients of $\mathbb{R}^N$ with respect to nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic as ordered fields ([Go], 3.16). A guide to the hyperreals can be found in the form of [Go].

A slightly more general construction is instead of considering a structure with universe $\mathbb{R}$ is to consider a superstructure. Indeed let $X$ be a non empty set of atoms (where atoms are objects that can be elements of sets but are not themselves and are "empty" with respect to $\in$). Usually it is assumed that (a copy of) the natural numbers $\mathbb{N} \subset X$.

**Definition 2.1.27.** The superstructure over $X$ is defined to be

$$U(X) = \bigcup_{n \in \mathbb{N}} X_n,$$

where $X_0 = X$ and by induction $X_{n+1} = X_n \cup P(X_n)$ ($P(A)$ is the powerset of $A$).

Note that

$$X = X_0 \subset X_1 \subset X_2 \subset \ldots .$$

It is a simple exercise to check that the conditions for a universe are satisfied by the superstructure. Superstructures enable a mathematical object $Z$ to be investigated by ensuring that (a copy of) $Z$ is contained in $X$. The set elements of $U(X)$ are called the entities of $U(X)$ and the individuals of $U(X)$ are the elements of $X_0 = X$.

**Definition 2.1.28.** A superstructure based on a set of atoms $X$ is the set $U(X)$ together with the notions of equality and membership on the elements of $U(X)$: $(U(X), \in, =)$.

(The notion of equality is assumed given for individuals and equality between entities (no atom equals any entity) is when they have the same elements.)

As a solid example consider the superstructure $(\mathcal{N})$ based on the natural numbers as the atoms. The basic algebra of $\mathbb{N}$ is part of $\mathcal{N}$. For example addition can be taken to be the following entity

$$S = \{(a, b, c) : a, b, c \in \mathbb{N}, a + b = c\}.$$

Further number systems can be obtained as entities of $\mathcal{N}$ by taking pairs of entities. For example $\mathbb{Z}$ is formed from ordered pairs of $\mathbb{N}$, $\mathbb{Q}$ is formed from pairs of $\mathbb{Z}$ and $\mathbb{R}$ from Dedekind cuts. Virtually anything occurring in classical analysis is an entity of $\mathcal{N}$.
Using the model theory above it has a nonstandard extension along with the transfer principle. From a formal view an advantage of a superstructure is that all the "useful" objects such as functions, metrics,... are already extended to the nonstandard setting. The definition of internal is clearer as an object \( A \) is internal if \( A \in *B \) for some \( B \in \mathbb{U}(X) \). Internal objects are important as from the definition of the transfer principle; transfer takes place from standard objects to internal objects. For example induction only takes place on internal subsets of \(*\mathbb{N}\. For a full introduction to nonstandard analysis via superstructures see [S-L].

In this work there is some abuse of notation, for example an element of a nonstandard space \(*X\) should be written as \(*x\) but generally the * is dropped without confusion.

### 2.1.4 Applications and Criticisms

As a final point it should be mentioned about the uses of nonstandard analysis and the potential criticism. So far nonstandard analysis has been successfully applied to many areas such as probability theory (for example certain products of infinitely many independent, equally weighted random variables), mathematical economics (for example the behaviour of large economies) and mathematical physics.

There was initial expectation that nonstandard analysis might revolutionize the way mathematicians reasoned with the real numbers but it never happened. Due to the construction of nonstandard analysis any nonstandard proof can be reinterpreted using standard techniques. This does not reduce nonstandard analysis to a mere redundant method of proof since the methods can actually be quite powerful not only in simplifying standard proofs, proving/refuting conjectures but also in giving precise meaning to many informal notions/concepts which do not make sense classically (like infinitesimals). Despite these points there is still skepticism about just how much nonstandard methods add to mathematics. A (well-)known critic is Alain Connes, as he mentioned in his famous book Non Commutative Geometry ([Co3]). The reader is left to make their own opinions.

(Further it cannot be overlooked that the 300 year old problem of formalizing infinitesimals was solved by Robinson using the power of 20th century logic in the form of nonstandard analysis. In [Kei] he shows how integral and differential calculus can be developed entirely using hyperreal numbers.)

Future progress and applications of nonstandard analysis, and of model theory, can be found in [Fe1] and [Fe2].
2.2 Nonstandard Number Theory

One of the first applications of nonstandard mathematics was to algebraic number theory. Unsurprisingly Robinson produced much work in this area and there have been important contributions from Roquette and from MacIntyre. This section gives a brief summary of some of the work in this area both to give an application of nonstandard mathematics and to form a basis for which this thesis is an extension.

The main work of Robinson in this area is contained in the papers [Ro2]–[Ro5] and the joint paper with Roquette ([R-R]). The first few papers make use of a certain external nonstandard ideal in a nonstandard extension of a Dedekind ring in order to look at properties of ideals in the standard Dedekind ring.

2.2.1 The Quotient Ring \( \Delta \)

Let \( D \) be a Dedekind domain which possesses at least one proper ideal (where a proper ideals is any ideal other than \( D \) or the zero ideal). \( D \) can be enlarged to give an integral domain \(*D\). (To obtain this extension \( D \) and \( \mathbb{N} \) are embedded in a structure \( M \) and then \( M \) is enlarged. From the work above this is done so as to be consistent in making references to hyperfinite integers and other nonstandard objects.) Further let \( \Omega \) be the set of ideals of \( D \) and this enlarges to the set of internal ideals of \( D, *\Omega \).

**Definition 2.2.1.** Define the monad \( \mu \) to be the intersection of all proper standard ideals in \(*D\). (An internal ideal \( A \in *\Omega \) is standard if there exists an ideal \( B \) in \( D \) such that \( A = *B \).)

The following properties of \( \mu \) are easy to establish and full proofs can be found in section 3 of the first paper in [Ro2].

**Proposition 2.2.2.**

1. \( \mu \) is an external ideal in \(*D\).

2. The only element of \( D \) in \( \mu \) is the zero element.

3. For any ideal, \( B \), in \( D \) then \( a \in D \cap *B \iff a \in B \).

4. There exists an internal proper ideal \( J \) in \(*D\) such that \( J \subset \mu \).

**Definition 2.2.3.** Let \( \Delta \) be the quotient ring of \(*D\) with respect to \( \mu \), \( \Delta = *D/\mu \).

The main result from this paper is the following theorem.

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Theorem 2.2.4. There exists a one-to-one multiplicative mapping from the proper ideals of $D$ into the classes of associated elements of $\Delta$.

The second paper of [Ro2] continues the above development. In particular the factorization laws of internal ideals in $\Delta$ are examined and the introduction of prime ideals. There are several theorems in the paper but the most important is in section 7 regarding the factorization of elements of $D$ in terms of prime elements of $^*D$.

Theorem 2.2.5. Let $a$ be an element of $D$, which can be regarded as a subset of $\Delta$. Further decompose $(a)$ in $D$ into its prime ideals as $(a) = P_1^{n_1} \cdots P_j^{n_j}$. Then there exists representative primes $\pi_1, \ldots \pi_j$ of $\Delta$ and a unit, $\epsilon$, such that the following decomposition is unique $a = \epsilon^{n_1} \pi_1 \cdots \pi_j^{n_j}$.

Details of nonstandard finite factorization can be found in section 7 of [Ro2].

The final paper in this series is [Ro3]. This paper relates $\Delta$ to the theory of $p$-adic numbers and adeles. In the previous papers the enlargements were constructed by the use of concurrent relations. In this one ultrapowers are used in a condensed way, similar to the previous chapter. This construction is required to prove the following theorem.

Theorem 2.2.6. Let $D$ be a countable Dedekind ring (not a field) such that the quotient rings $D \setminus P$ are finite. (Here $P$ is a non-trivial prime ideal in $D$.) Let $^*D$ be a comprehensive enlargement then the ring $\Delta$ is isomorphic to the strong direct sum of $P$-adic completions of $D$.

In section 5 of [Ro4] the results regarding $p$-adic completions and adeles are extended to algebraic extensions. This is also summarized in section 2 of [Mac1].

The penultimate paper moves on from the work of the properties of $\Delta$ and to some other topics in algebraic number theory. These areas move into, for example, infinite Galois theory and class field theory. This work is not pursued in this thesis.

The work in [Ro5] continues with work on ideals. In particular the theory of entire ideals in an infinite algebraic extension of $\mathbb{Q}$ (and also some class field theory which is not mentioned here because this thesis does not develop this area). In the previous works the quotient ring $\Delta$ has been studied for Dedekind rings and finite algebraic number fields. This work looks at the quotient ring in infinite algebraic extensions of $\mathbb{Q}$.

Indeed let $F$ be an infinite algebraic extension of $\mathbb{Q}$ with rings of integers $F^i$ and $\mathbb{Q}^i$ respectively. The nonstandard enlargements can be considered, with $^*F$ an enlargement of $F$ and $^*F^i$ an enlargement of $F^i$. Further let $\Phi = \{F_n\}$ be a tower subfields of $F$ such that $F_0 = \mathbb{Q}$, each $F_n$ a finite extension of $\mathbb{Q}$ and $F = \bigcup_n F_n$. In the enlargement $^*\Phi$ is a mapping from $^*\mathbb{N}$ into the subfields of $F$.
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* with \( *\Phi = \{H_n\} \) and \( H_n = F_n \) for finite \( n \). In particular for any infinite \( n = \omega \) (set \( H = H_\omega \)) \( F \subset H \subset *F \).

**Definition 2.2.7.** Let \( \mu \) be the subset of \(*F^i\) defined as \( \mu = \{x : x \in *F^i \text{ and } x \text{ is divisible by all non zero standard rational integers} \} \).

This naturally leads to the quotient ring \( \Delta = *F^i/\mu \) and the canonical mapping \( \delta : *F^i \to \Delta \). Since \( \mu \) does not contain any standard elements it follows that \( \delta \) injects \( F^i \) into \( \Delta \). Further let \( \mu_H = \mu \cap H \) then \( \Delta_H = H^i/\mu_H \) can be identified with a subring of \( \Delta \). Let \( \delta_H \) be the restriction of \( \delta \) to \( H \) so that it maps \( H \) on \( \Delta_H \) and injects \( F^i \) into \( \Delta_H \).

**Theorem 2.2.8.** Let \( S_H \) be the set of internal ideals in \( H^i \) and \( a \in F^i \) \((a \neq 0)\). Then \( \delta_H(a) \) is invertible in \( \Delta_H \) iff all prime ideals \( P_j \in S_H \) which divide the ideal \( (a)_H \) generated by \( a \) in \( H \) have norms \( NP_j \) that are powers of nonstandard primes.

The final papers of interest are [R-R] and [Mac1]. The paper by [R-R] is extremely important in nonstandard number theory. It gives a new and simplified proof of the finiteness theorem of Siegel-Mahler theorem concerning Diophantine equations by the use of nonstandard methods. The main idea is to relate the algebraic geometry of a number field \( K \) and the nonstandard arithmetic of a nonstandard enlargement of \( K, *K \). So an algebraic function field in one variable over a algebraic number field \( K \) can be viewed as a field of functions over \( K \) and as a subfield of \( *K \). The Siegel-Mahler theorem can then be restated as

**Theorem 2.2.9.** If \( F \) is an algebraic function field in one variable over some fixed number field \( K \) such that \( K \subset F \subset *K \), and if \( F \) has genus \( g > 0 \), then every non constant element \( x \) of \( F \) admits at least one nonstandard prime divisor of \( *K \) in its denominator.

The method of proof is based on a transfer principle which can symmetrically translate arguments of \( F \) into equivalent functional properties of \(*F\). A key facet of the proof is that it does not use the Mordell-Weil theorem \( ^3 \) (although there are areas of the proof which are similar to areas of the Mordell-Weil theorem). This enables the revelation of an effective bound relative to effective bounds in Roth’s theorem \( ^4 \) see section 7 in [Mac1].

[Mac1] builds on the work in [R-R]. The work was originally given as a talk intended to outline

\(^3\)The Mordell-Weil theorem states that for an abelian variety \( A \) over a number field \( K \), the group \( A(K) \) of \( K \)-rational points of \( A \) is a finitely-generated abelian group.

\(^4\)Given an algebraic number \( \alpha \) and a given \( \epsilon > 0 \) the inequality \( |\alpha - p/q| < q^{-\epsilon/2} \) has only a finite set of solutions for coprime integers \( p \) and \( q \).
model-theoretic methods in Diophantine geometry. The main result relates to another formulation of Weil’s theory of distributions via a "covering theorem" relating geometric and arithmetical ideles.
The Hyper Riemann Zeta Function

3.1 The Shadow Maps of $^\ast \mathbb{Q}$

The shadow (or sometimes termed standard) maps enable a standard entity to be taken from a nonstandard entity. In this work they are extremely important. In $^\ast \mathbb{Q}$ the shadow maps used in this work correspond directly to the valuations of $\mathbb{Q}$. The simplest one to develop is the real shadow map.

**Definition 3.1.1** (Real Shadow Map). Let $\left| \cdot \right|_{\eta}$ be the standard archimedean valuations on $\mathbb{R}$ extended to a hyper valuation on $^\ast \mathbb{R}$.

1. Two elements $x, y \in ^\ast \mathbb{R}$ are said to be infinitesimally close with respect to $\left| \cdot \right|_{\eta}$ (denoted $x \simeq_{\eta} y$) if $\left| x - y \right|_{\eta}$ is an infinitesimal element of $^\ast \mathbb{R}^+ = \{ y \in ^\ast \mathbb{R} : y \geq 0 \}$. Define $^\ast \mathbb{R}^{\inf \rho} = \{ x \in ^\ast \mathbb{R} : x \simeq_{\eta} 0 \}$.

2. Define the monad of an element $x \in ^\ast \mathbb{R}$ to be the set of elements which are infinitely close with respect to $\left| \cdot \right|_{\eta}$, $\mu_{\eta}(x) = \{ y \in ^\ast \mathbb{R} : x \simeq_{\eta} y \}$.

3. $x \in ^\ast \mathbb{R}$ is said to be limited if there exists $a, b \in \mathbb{R}$ such that $a < \left| x \right|_{\eta} < b$. Define $^\ast \mathbb{R}^{\lim \rho} = \{ x \in ^\ast \mathbb{R} : x \text{ is limited} \}$.

4. For $x \in ^\ast \mathbb{R}^{\lim \rho}$ define $\text{sh}_{\eta}(x)$ to be the unique element of $\mathbb{R}$ to which $x$ is infinitesimally close.

This definition and the justification for the final statement can be found in [Go]. He further goes onto define (in chapter 18) the $p$-adic shadow maps but in quite an explicit way unlike the concise way for the real shadow map. Indeed define the natural map.
\[ \theta_p : \ast \mathbb{Z} \rightarrow \mathbb{Z}_p, \]
\[ x \mapsto \langle x \pmod{p}, x \pmod{p^2}, \ldots, x \pmod{p^n}, \ldots \rangle. \]

This is a surjective homomorphism with kernel:

\[ \ast \mathbb{Z}^{\inf_p} = \{ x \in \ast \mathbb{Z} : \theta_p(x) = 0 \}, \]
\[ = \{ p^N q : N \text{ is unlimited and } q \in \ast \mathbb{Z} \}. \]

This extends to \( \ast \mathbb{Q} \) by setting \( \theta_p(x y) = \theta_p(x) \theta_p(y) \), (with \( x/y \) in lowest form). For this to be defined it is required that \( \theta_p(y) \neq 0 \). In fact this map is well defined for \( x \in \ast \mathbb{Q}^{\lim_p} = \{ q \in \ast \mathbb{Q} : |q|_p \text{ is limited} \} \). This is also a ring homomorphism and has kernel \( \ast \mathbb{Q}^{\inf_p} = \{ x \in \ast \mathbb{Q} : |x|_p \simeq 0 \text{ in } \ast \mathbb{R} \}. \)

So I define the \( p \)-adic shadow maps on \( \ast \mathbb{Q}_p \) using the \( p \)-adic valuation directly.

**Definition 3.1.2 (\( p \)-adic Shadow Map).**

- Let \( x, y \in \ast \mathbb{Q}_p \), \( x \) is \( p \)-adically infinitely close to \( y \) (denoted \( x \simeq_p y \)) if there exists \( N \in \ast \mathbb{N} - \mathbb{N} \) such that \( |x - y|_p < p^{-N} \).
- The \( p \)-adic monad of \( x \in \ast \mathbb{Q}_p \) is defined to be \( \mu_p(x) = \{ y \in \ast \mathbb{Q}_p : x \simeq_p y \} \).
- For \( x \in \ast \mathbb{Q}_p^{\lim_p} = \{ x \in \ast \mathbb{Q}_p : |x|_p \text{ is limited} \} \), \( \text{sh}_p(x) \) is defined to be the unique element of \( \mathbb{Q}_p \) which is infinitely close in the \( p \)-adic valuation to \( x \).

The last statement is proved by the following theorem.

**Theorem 3.1.3.** Every \( x \in \ast \mathbb{Q}_p^{\lim_p} \) is infinitely close to exactly one number in \( \mathbb{Q}_p \).

**Proof:** Let \( x \in \ast \mathbb{Q}_p^{\lim_p} \). Dealing with uniqueness first. Assume \( x \) is infinitely close to two elements in \( \mathbb{Q}_p \), \( x \simeq_p a \) and \( x \simeq_p b \) with \( a \neq b \) and \( a, b \in \mathbb{Q}_p \). Then \( |a - b|_p = |(a - x) + (x - b)|_p \leq \max\{ |a - x|_p, |x - b|_p \} \). Both of these are infinitesimal by definition and hence \( a \simeq_p b \). This is a contradiction since \( a, b \in \mathbb{Q}_p \) and therefore \( a = b \).

Now for existence. As \( x \in \ast \mathbb{Q}_p^{\lim_p} \) there exists an expansion,

\[ x = a_n p^n + \ldots + a_0 + a_1 p + \ldots + a_N p^N + \ldots, \]

where \( n \) is not negative unlimited (\( n \notin \ast (\mathbb{N} - \mathbb{N}) \)), otherwise \( x \) would not be in \( \ast \mathbb{Q}_p^{\lim_p} \). Also \( 0 \leq a_r < p \). Then let

\[ \bar{x} = \sum_{r \geq n, r \in \mathbb{N}} a_r p^r \in \mathbb{Q}_p, \]
this is the standard part of \( x \). Let
\[
\hat{x} = \sum_{r \geq n, r \in \ast \mathbb{N} - \mathbb{N}} a_r p^r,
\]
This is the nonstandard part of \( x \). Thus, \( x = \bar{x} + \hat{x} \), equivalently \( \hat{x} \simeq_p 0 \). From basic \( p \)-adic analysis, if the sum \( \sum_{n \in \mathbb{N}} c_n \) is absolutely \( p \)-adically convergent for some \( c_n \in \mathbb{Q}_p \) then \( |\sum_{n \in \mathbb{N}} c_n|_p \leq \max_{n \in \mathbb{N}} \{|c_n|_p\} \). By transfer this can be applied to finding \( |ar{x}|_p \),
\[
|\bar{x}|_p \leq \max_{r \in \ast \mathbb{N} - \mathbb{N}} \{|a_r p^r|_p\}.
\]
As \( |a_r|_p < 1 \) the result follows since \( r \) is nonstandard.

\[\square\]

**Theorem 3.1.4** (Properties of \( \text{sh}_p \)). Let \( x, y \in \ast \mathbb{Q}_p^{\lim_p} \) then

1. \( \text{sh}_p(x \pm y) = \text{sh}_p(x) \pm \text{sh}_p(y) \),
2. \( \text{sh}_p(xy) = \text{sh}_p(x) \cdot \text{sh}_p(y) \),
3. \( \text{sh}_p(x/y) = \text{sh}_p(x)/\text{sh}_p(y) \), if \( \text{sh}_p(y) \neq 0 \).
4. \( \text{sh}_p(|x|_p) = |\text{sh}_p(x)|_p \).

**Proof:** The three statements are basic exercises in manipulation of the \( p \)-adic expansions of \( x \) and \( y \) then use the definition of theorem 3.1.3. The final property is very similar in nature and relies on a proof by cases.

Firstly suppose \( x \in \ast \mathbb{Q}_p^{\inf_p} \). Then \( x = \sum_{r \geq n, r \in \ast \mathbb{N} - \mathbb{N}} a_r p^r \) where \( N \in \ast \mathbb{N} - \mathbb{N} \) thus \( |x|_p = p^{-N} \) and \( \text{sh}_p(|x|_p) = 0 \). However \( \text{sh}_p(x) = 0 \) and so \( |\text{sh}_p(x)|_p = 0 \).

In the other case \( x \in \ast \mathbb{Q}_p^{\lim_p} \) such that \( x = \sum_{r \geq n, r \in \ast \mathbb{N}} a_r p^r \) with \( n \in \mathbb{N} \). Thus \( |x|_p = p^{-n} \) and \( \text{sh}_p(|x|_p) = p^{-n} \). By the definition of the \( p \)-adic shadow map \( \text{sh}_p(x) = \sum_{r \geq n, r \in \ast \mathbb{N}} a_r p^r \) and \( |\text{sh}_p(x)|_p = p^{-n} \).

\[\square\]

**Lemma 3.1.5.** \( \text{sh}_p \) is a surjective map with kernel \( \ast \mathbb{Q}_p^{\inf_p} \).

**Proof:** The map is surjective because \( \mathbb{Q}_p \subset \ast \mathbb{Q}_p^{\lim_p} \) and \( \text{sh}_p \) acts trivially on \( \mathbb{Q}_p \).

The kernel is \( \text{ker}(\text{sh}_p) = \{x \in \ast \mathbb{Q}_p^{\lim_p} : \text{sh}_p(x) = 0\} \). By the definition of the \( p \)-adic shadow map this happens precisely when \( x \in \ast \mathbb{Q}_p^{\inf_p} = \{x \in \ast \mathbb{Q}_p^{\lim_p} : |x|_p \simeq_p 0\} \).
Corollary 3.1.6.

\[ \mathbb{Q}_p^{\text{lim}} / \mathbb{Q}_p^{\text{inf}} \cong \mathbb{Q}_p. \]

In both the infinite and the finite prime cases the associated shadow maps are surjective ring homomorphisms such that \( \mathbb{Q}_p^{\text{lim}} / \mathbb{Q}_p^{\text{inf}} \cong \mathbb{R} \) and \( \mathbb{Q}_p^{\text{lim}} / \mathbb{Q}_p^{\text{inf}} \cong \mathbb{Q}_p. \)

The shadow maps can be extended from acting on just the above spaces to the functions on them.

Definition 3.1.7 (Shadow Image). For a function \( \ast h : \ast X \to \ast Y \) (where \( \ast X \) and \( \ast Y \) are sets upon which the hyper valuation \( |.| \) is defined) the shadow image of \( \ast h \) (with respect to the valuation \( |.| \)) is denoted by \( \text{sh}_{|.|}(\ast h) \). This is a function with domain consisting of the standard parts \( \text{sh}_{|.|}(x) \) (\( x \in \ast X \)) such that \( \ast h(x) \) is infinitely close with respect to the valuation to a standard element in \( \mathbb{R} \). This gives the image consisting of these \( \text{sh}_{|.|}(\ast h(x)) \) for \( x \) in the domain.

Further details of the real shadow map acting on functions can be found in [Go].

3.2 Nonstandard Tools

In order to develop the analytical theory one needs some basic notions in nonstandard analysis.

3.2.1 Properties of \( \ast \mathbb{Z} \)

To define the hyper Riemann zeta function one first needs the idea notion of ideals for \( \ast \mathbb{Z} \).

Let \( \Lambda \) and \( \ast \Lambda \) be the sets of all prime ideals in \( \mathbb{Z} \) and all internal prime ideals in \( \ast \mathbb{Z} \) respectively. Also, let \( \Pi \) and \( \ast \Pi \) be the sets of primes numbers in \( \mathbb{Z} \) and \( \ast \mathbb{Z} \) respectively.

Lemma 3.2.1. I is an internal prime ideal in \( \ast \mathbb{Z} \iff I = p \ast \mathbb{Z} \) for a unique \( p \in \Pi \).

Proof:

(\( \forall \ I \in \Lambda \) (\( \exists! \ p \in \Pi \) (I=p\mathbb{Z})), \( (3.2.1) \))

↓ \( \ast \)-transform,

(\( \forall \ I \in \ast \Lambda \) (\( \exists! \ n \in \ast \Pi \) (I=p\ast\mathbb{Z})), \( (3.2.2) \))

Conversely,

(\( \forall \ p \in \Pi \) (\( \exists! \ I \in \Lambda \) (I=p\mathbb{Z})), \( (3.2.3) \))
For $I \in \Lambda$ define the norm of the ideal to be $N(I) = [Z : I]$. This definition can be transferred to $I \in \Lambda$, $N(I) = [^*Z : I] \in ^*Z_{\geq 0}$.

Further properties of $^*Z$ can be found in [Go], [Ro1] and in the papers by Robinson on algebraic integers and Dedekind rings in [Ro2], [Ro3] and [Ro4].

### 3.2.2 Hyperfinite Sums, Products, Sequences and Integrals

Internal functions can be constructed out of internal sets and the full definition can be found in chapter 12 of [Go].

In order to define hyperfinite summation the symbol $'\sum'$ has to be defined for nonstandard integers. This can be found in chapter 19 of [Go] enabling $\sum_{n=1}^{M} f(n)$ to be defined for all internal $^*f : ^*\mathbb{N} \to ^*\mathbb{C}$ and unlimited $M$. This enables the following definition:

**Definition 3.2.2.** Let $^*f : ^*\mathbb{N} \to ^*\mathbb{C}$ be an internal function then the infinite sum exists, $\sum_{n \in ^*\mathbb{Z}_{>0}} f(n) = S \in ^*\mathbb{C}$, if

$$(\forall \epsilon \in ^*\mathbb{R}_{>0})(\exists v \in ^*\mathbb{Z}_{>0})(\forall k \in ^*\mathbb{Z}_{>0})(^*d(\sum_{0 < n \leq k} f(n), S) < \epsilon).$$

In an analogous way the hyperfinite product can also be defined. Let $f : \mathbb{N} \to \mathbb{C}$ be a complex function then a finite product can be defined for $m \in \mathbb{N}$: $\prod_{n=1}^{m} f(n) = f(1) \ldots f(m)$. $'\prod'$ can be regarded as a function from finite sequences to the complex numbers - where the finite sequence is $\{f(n)\}_{n=1,\ldots,m}$. By transfer (see chapter 19 of [Go]) $'\prod'$ extends to act on all internal hyperfinite sequences. So $'\prod'$ is then defined for any $M \in ^*\mathbb{N}$ and internal hyper complex functions.

**Definition 3.2.3.** Let $^*f : ^*\mathbb{N} \to ^*\mathbb{C}$ be an internal function then the infinite product exists, $\prod_{n \in ^*\mathbb{Z}_{>0}} f(n) = P \in ^*\mathbb{C}$, if

$$(\forall \epsilon \in ^*\mathbb{R}_{>0})(\exists v \in ^*\mathbb{Z}_{>0})(\forall k \in ^*\mathbb{Z}_{>0})(^*d(\prod_{0 < n \leq k} f(n), P) < \epsilon).$$

As already seen above chapter 19 of [Go] enables hyper finite internal sequences and internal sequences to be defined.

Indeed let $^*f : ^*\mathbb{N} \to ^*\mathbb{C}$ be an internal function. Then a hyper internal sequence is defined to be $\{z_{m}\}$ where $z_{m} = ^*f(m)$.
**Definition 3.2.4.** 1. A hyper internal sequence \( \{ z_m \} \) Q-converges to a limit \( S \in \ast \mathbb{C} \) if \( (\forall \epsilon \in \ast \mathbb{R}_{>0})(\exists N \in \ast \mathbb{Z}_{>0})(\forall n \in \ast \mathbb{Z}_{\geq N})(\ast d(z_n, S) < \epsilon) \).

2. Let \( \{ z_m \} \) be a hyper internal sequence in \( \ast \mathbb{C} \) then \( \{ z_m \} \) is a hyper internal Cauchy sequence in \( \ast \mathbb{C} \) if \( (\forall \epsilon \in \ast \mathbb{R}_{>0})(\exists N \in \ast \mathbb{Z}_{>0})(\forall m, n \in \ast \mathbb{Z}_{\geq N})(\ast d(z_n, z_m) < \epsilon) \).

**Proposition 3.2.5.** If a hyper internal sequence \( \{ z_m \} \) Q-converges then it is a hyper Cauchy internal sequence.

**Proof:** Suppose \( \{ z_m \} \) Q-converges to \( S \in \ast \mathbb{C} \). By definition of Q-convergence,

\[
(\forall \epsilon/2 \in \ast \mathbb{R}_{>0})(\exists N \in \ast \mathbb{Z}_{>0})(\forall m, n \in \ast \mathbb{Z}_{\geq N})(\ast d(z_n, S) < \epsilon/2 \land \ast d(z_m, S) < \epsilon/2).
\]

Under these conditions and by using the triangle inequality,

\[
\ast d(z_n, z_m) \leq \ast d(z_n, S) + \ast d(z_m, S),
\]

\[
< \epsilon/2 + \epsilon/2 = \epsilon.
\]

\[\square\]

**Proposition 3.2.6.** If \( \{ z_m \} \) is a hyper internal Cauchy sequence then it Q-converges to a limit in \( \ast \mathbb{C} \).

**Proof:** Using the transfer principle and the definition of a hyper internal Cauchy sequence,

\[
(\forall \epsilon \in \ast \mathbb{R}_{>0})(\exists N \in \ast \mathbb{Z}_{>0})(\forall m, n \in \ast \mathbb{Z}_{\geq N})(\ast d(z_n, z_m) < \epsilon),
\]

\[\downarrow \ast\text{-transform},\]

\[
(\forall \epsilon \in \mathbb{R}_{>0})(\exists N \in \mathbb{Z}_{>0})(\forall m, n \in \mathbb{Z}_{\geq N})(d(z_n, z_m) < \epsilon)
\]

This implies \( \{ z_m \} \) for \( m \in \ast \mathbb{Z}_{>0} \) is a Cauchy sequence in \( \mathbb{C} \). By the standard theorem this converges to a limit, \( A \), say, in \( \mathbb{C} \). By the transfer principle

\[
(\forall \epsilon \in \ast \mathbb{R}_{>0})(\exists N \in \mathbb{Z}_{>0})(\forall n \in \mathbb{Z}_{\geq N})(d(z_n, A) < \epsilon)
\]

\[\downarrow \ast\text{-transform},\]

\[
(\forall \epsilon \in \ast \mathbb{R}_{>0})(\exists N \in \ast \mathbb{Z}_{>0})(\forall n \in \ast \mathbb{Z}_{\geq N})(\ast d(z_n, A) < \epsilon),
\]

\[\square\]

The work on defining hyperfinite integrals can be found beginning in chapter 5 of [Ro1].
3.2.3 Q-Topology

Consider a nonstandard set $X$ with a hypermetric $\ast D : X \times X \to \ast \mathbb{R}_{>0}$.

**Definition 3.2.7.**
1. A $Q$-ball is the set $\ast B(y, r) = \{ x \in X : \ast D(x, y) < r \}$ for $y \in X$ and $r \in \ast \mathbb{R}_{>0}$.
2. A set $W \subset X$ is called a $Q$-neighbourhood of $y \in X$ if it contains a ball $\ast B(y, r)$.
3. A set is $Q$-open if it is a $Q$-neighbourhood of each of its elements.
4. The complement of a $Q$-open set with respect to $X$ is termed to be $Q$-closed with respect to $X$.

Using the triangle inequality it follows that every $Q$-ball is $Q$-open. Naturally many more properties can be defined and results developed via the transfer principle from the standard case but these are not relevant for this work. The only property needed is the analogue of compactness in the nonstandard case.

**Definition 3.2.8.**
1. Consider an internal collection of $Q$-open sets. These form a $Q$-opencovering for $X$ if $X$ is contained in the union of these sets. A $Q$-subcovering is a subcollection with the same property.
2. A hyperfinite $Q$-covering of $X$ is a $Q$-open covering of $X$ consisting of a hyperfinite number of sets.

These lead to the definition of compactness in the nonstandard case.

**Definition 3.2.9.** A set $X$ is hypercompact with respect to the $Q$-topology iff every $Q$-open covering of $X$ contains a hyperfinite $Q$-subcovering.

3.2.4 Hyper Exponential and Logarithm Functions

On $\mathbb{C}$ take the usual metric and when extended to $\ast \mathbb{C}$ it makes $\ast \mathbb{C}$ into a hyper metric space where the metric takes values in $\ast \mathbb{R}_{>0}$.

**Definition 3.2.10.** Let $s \in \ast \mathbb{C}$ with $s = u + iv$ ($u, v \in \ast \mathbb{R}$). Define the hyperreal part to be $\ast \Re(s) = u \in \ast \mathbb{R}$ and the hyper imaginary part to be $\ast \Im(s) = v \in \ast \mathbb{R}$.

**Definition 3.2.11.** Let $f(z)$ be a hypercomplex function. $f(z)$ is Q-continuous if

$$(\forall \epsilon \in \ast \mathbb{R}_{>0})(\exists \delta \in \ast \mathbb{R}_{>0})(\forall z, w \in \ast \mathbb{C}, \ast d(z, w) < \delta)(\ast d(f(z), f(w)) < \epsilon).$$
Lemma 3.2.12. The limit function of a uniformly $Q$-convergent internal sequence of $Q$-continuous functions is itself $Q$-continuous.

Proof: Suppose that the functions $f_n(x)$ are $Q$-continuous and uniformly $Q$-converge to $f(z)$ on a set $E$. For any $\epsilon \in ^*\mathbb{R}_{>0}$, $n \in ^*\mathbb{N}$ can be found such that $^*d(f_n(z), f(z)) < \epsilon/3 \forall z \in E$. Let $z_0$ be a point in $E$. As $f_n(z)$ is $Q$-continuous at $z_0$, $\exists \delta \in ^*\mathbb{R}_{>0}$ such that $^*d(f_n(z), f_n(z_0)) < \epsilon/3 \forall z \in E$ with $^*d(z, z_0) < \delta$. Under the same conditions on $z$,

$$^*d(f(z), f(z_0)) \leq ^*d(f(z), f_n(z)) + ^*d(f_n(z), f_n(z_0)) + ^*d(f_n(z_0), f(z_0)) < \epsilon.$$ (3.2.5)

Definition 3.2.13. Let $N \in ^*\mathbb{Z}_{>0}$ and define the hyper factorial inductively by $0! = 1$ and $N! = N \times (N - 1)!$. So $N! = \prod_{1 \leq n \leq N} n$. This function is interpolated by the hyper gamma function to be defined below.

Definition 3.2.14. Suppose $f(z)$ is an internal hypercomplex function then $f'(z) \in ^*\mathbb{C}$ is the $Q$-derivative of $f(z)$ at $z = z_0$ if

$$\forall \epsilon \in ^*\mathbb{R}_{>0} (\exists \delta \in ^*\mathbb{R}_{>0}) (\forall h \in ^*\mathbb{C}, ^*d(h, 0) < \delta) (^*d(f(z_0 + h) - f(z_0), f'(z_0)) < \epsilon).$$

Definition 3.2.15. Let $B$ be a set of points in $^*\mathbb{C}$ ($f(z)$ as above). Then $f(z)$ is $Q$-analytic in $B$ if $f(z)$ is infinitely $Q$-differentiable at all points of $B$.

Definition 3.2.16. Let $s \in ^*\mathbb{C}$ and define the hyper exponential as

$$^*\exp(s) = ^*e^s = \sum_{n \in ^*\mathbb{Z}_{>0}} \frac{s^n}{n!}.$$

Lemma 3.2.17. Properties of $^*e^s$.

1. $^*e^s$ uniformly $Q$-converges $\forall s \in ^*\mathbb{C}$, so $^*e^s : ^*\mathbb{C} \to ^*\mathbb{C}^\times$.

2. $^*e^s$ is $Q$-analytic $\forall s \in ^*\mathbb{C}$.

3. $^*\exp(z)' = ^*\exp(z)$.

4. $^*e^{s+t} = ^*e^s e^t$ ($s, t \in ^*\mathbb{C}$). In particular $^*e^0 = 1$.

5. $^*\exp : ^*\mathbb{C}/2\pi i^*\mathbb{Z} \to ^*\mathbb{C}^\times$ is a group is a group isomorphism.

Proof:
1. Let $N \in \mathbb{Z}_{>0}$ and define $(N)e^s = \sum_{n=1}^{N} \frac{n^s}{n!}$. In the classical case $(N)e^s$ converges uniformly $\forall s \in \mathbb{C}$.

$$\phi_N(\epsilon) \in \mathbb{R}_{>0} \Rightarrow (\forall m, n \in \mathbb{Z}_{>0})(\forall s \in \mathbb{C})(d((n)e^s, (m)e^s) < \epsilon),$$

\[ \text{\textdownarrow \text {-transform},} \]

$$(\forall s \in \mathbb{R}_{>0})(\exists N \in \mathbb{Z}_{>0})(\forall m, n \in \mathbb{Z}_{>0})(\forall s \in \mathbb{C})(*d((n)e^s, (m)e^s) < \epsilon).$$

2. The arguments of 3.4.1, 3.4.5 and 3.4.6 see below, will give the results.

3. Since, by 2, * exp is Q-analytic it remains to find the derivative. In the standard case $\exp(s) = \exp(s)$.

$$\phi_N(\epsilon) \in \mathbb{R}_{>0} \Rightarrow (\forall m, n \in \mathbb{Z}_{>0})(\forall s \in \mathbb{C})(d((n)\exp(s), (m)\exp(s)) < \epsilon),$$

\[ \text{\textdownarrow \text {-transform},} \]

$$(\forall s \in \mathbb{R}_{>0})(\exists N \in \mathbb{Z}_{>0})(\forall m, n \in \mathbb{Z}_{>0})(\forall s \in \mathbb{C})(*d((n)\exp(s), (m)\exp(s)) < \epsilon).$$

4. By 3: $\frac{d}{ds}(*e^z) = *e^z$. Using this and the product rule gives for some constant $c \in \mathbb{C}$:

$$\frac{d}{dz}(*e^ze^{-z}) = *e^ze^{-z} + *e^z(-*e^{-z}) = 0.$$ So $*e^ze^{-z} = K, \text{constant (K} \in \mathbb{C})$. Let $z = 0$ then $K = *e^c$. Now let $z = s, c = s + t$ and the result follows. In particular $*e^s e^{-s} = 1$ which implies $*e^s$ is never zero.

5. In the standard case $\exp(s) = 1 \iff c \in 2\pi i \mathbb{Z}$.

$$\phi_N(\epsilon) \in \mathbb{R}_{>0} \Rightarrow (\forall n \in \mathbb{Z})(\forall N \in \mathbb{Z}_{>0})(\exists \epsilon \in \mathbb{R}_{>0})(d((N)e^{2\pi in}, 1) < \epsilon),$$

\[ \text{\textdownarrow \text {-transform},} \]

$$(\forall n \in \mathbb{Z})(\forall N \in \mathbb{Z}_{>0})(\exists \epsilon \in \mathbb{R}_{>0})(*d((N)e^{2\pi in}, 1) < \epsilon).$$

Conversely,

$$(\forall s \in \mathbb{C} - 2\pi i \mathbb{Z})(\forall N \in \mathbb{Z}_{>0})(\exists \epsilon \in \mathbb{R}_{>0})(d((N)e^s, 1) > \epsilon),$$

\[ \text{\textdownarrow \text {-transform},} \]

$$(\forall s \in \mathbb{C} - 2\pi i \mathbb{Z})(\forall N \in \mathbb{Z}_{>0})(\exists \epsilon \in \mathbb{R}_{>0})(*d((N)e^s, 1) > \epsilon).$$

\[ \square \]
**Definition 3.2.18.** \( ^* \log : ^* C \setminus (-^* R > 0) \to ^* C. \) It is the inverse to \( ^* \exp \) \( z = ^* \log(w) \) is a root of the equation \( ^* \exp(z) = w. \) This equation has infinitely many solutions so the hyper logarithm is multivalued. So define the principal value of the hyper logarithm to be \( ^* \log(w) = ^* \log |z| + i \text{Arg}(z) \) where \( |z| > 0 \) and \( -\pi < \text{Arg}(z) \leq \pi. \) Also define \( (n) \log(w) = z \) to be a root of the equation \( (n) \exp(z) = w \forall n \in ^* Z > 0 \) provided \( -\pi < \text{Arg}(z) \leq \pi. \)

**Lemma 3.2.19.** 1. \( ^* \log(st) = ^* \log(s) + ^* \log(t) \) iff \( -\pi < \text{Arg}(s) + \text{Arg}(t) \leq \pi. \)

2. \( ^* \log(s) \) is \( Q \)-analytic \( \forall s \in ^* C \setminus (-^* R > 0). \) Also in this region, \( \frac{d}{ds} ^* \log(s) = \frac{1}{s}. \)

**Proof:**

1. Let \( u = ^* \log(s) \) and \( V = ^* \log(t) \) then \( u \) and \( v \) satisfy \( ^* \exp(u) = s \) and \( ^* \exp(v) = t. \)

By \( 3.2.17(4) \) \( ^* \exp(u)^* \exp(v) = ^* \exp(u + v) = st \) and the result follows since from the definition \( 3.2.18 \) and the proof in the classical case.

2. For the first part use the arguments of \( 3.4.1 \) \( 3.4.5 \) and \( 3.4.6 \) below with \( (n) \log(s) \). For the second part use the arguments of \( 3.2.17(3) \).

**Definition 3.2.20.** For \( z \in ^* C \setminus i^* R \) and \( s \in ^* C \) define \( z^s = ^* \exp(s^* \log(z)) \), where the principal value of the hyper logarithm is used.

**Lemma 3.2.21.** Let \( z_1, z_2 \in ^* C \setminus i^* R \) and \( s, t \in ^* C. \)

1. \( (z_1 z_2)^s = z_1^s z_2^s \) is not true in general. It does hold when lemma \( 3.2.19 \) \( 1 \) holds.

2. \( z^{s+t} = z^s z^t. \)

3. \( \frac{d}{dz} z^s = sz^{s-1}. \)

**Proof:**

1. Suppose that \( z_1 \) and \( z_2 \) satisfy the properties of lemma \( 3.2.19 \) then

\[
(z_1 z_2)^s = ^* \exp(s^* \log(z_1 z_2)),
\]

\[
= ^* \exp(s^* \log(z_1) + s^* \log(z_2)),
\]

\[
= ^* \exp(s^* \log(z_1)) \cdot ^* \exp(s^* \log(z_2)) = z_1^s z_2^s.
\]
2. $$z^{s+t} = \ast \exp((s+t)\ast \log(z)),$$
    
    $$= \ast \exp(s\ast \log(z) + t\ast \log(z)),$$
    
    $$= z^s z^t.$$

3. Using the chain rule,

$$\frac{d}{dz} z^s = \frac{d}{dz}(\ast \exp(s \ast \log(z))),$$

$$= sz^s \frac{d}{dz}(\ast \log(z)),$$

$$= sz^{s-1}.$$

□

3.2.5 Hyper Gamma Function

**Definition 3.2.22.** Hyper gamma function. For $$s \in \ast \mathbb{C}$$ and $$\ast \Re(s) \in [\epsilon, r], \epsilon \in \ast \mathbb{R}_{>0}$$ and $$r > \epsilon$$ define

$$\ast \Gamma(s) = \int_{\ast \mathbb{R}^+} \ast e^{-y}s^{-1}dy.$$

**Proposition 3.2.23.** $$\ast \Gamma(s)$$ is absolutely $$Q$$-convergent for $$\ast \Re(s) > 0.$$ 

**Proof:** In the standard case

$$\Gamma(s) = \int_{\mathbb{R}^+} e^{-y}s^{-1}dy,$$

is absolutely convergent for $$\Re(s) \in [\epsilon, r]$$ ($$r, \epsilon \in \mathbb{R}_{>0}$$ and $$r > \epsilon$$). Let

$$I_{n,r}(s) = \int_{0}^{r} (n) e^{-y}s^{-1}dy, \text{ where } (n)e^{s} = \sum_{j=1}^{n} \frac{s^j}{j!},$$

$$(\forall s \in \mathbb{C}, \Re(s) \in [\epsilon, r]) (\forall \delta \in \mathbb{R}_{>0})(\exists m, n \in \mathbb{Z}_{>0})(\exists r, t \in \mathbb{R}_{>0}) (d(I_{n,r}(s), I_{m,t}(s)) < \delta),$$

(3.2.6)

↓ $$\ast$$-transform,

$$(\forall s \in \ast \mathbb{C}, \ast \Re(s) \in [\epsilon, r]) (\forall \delta \in \ast \mathbb{R}_{>0})(\exists m, n \in \ast \mathbb{Z}_{>0})(\exists r, t \in \ast \mathbb{R}_{>0}) (d(I_{n,r}(s), I_{m,t}(s)) < \delta),$$

(3.2.7)

So $$\{I_{n,r}\}$$ form a hyper internal Cauchy sequence with limit function $$\ast \Gamma(s).$$

□
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Lemma 3.2.24. For $\Re(s) > 0$, $\int_{\mathbb{R}^+}^* \frac{d}{dy} (y^s e^{-y}) dy = 0$.

Proof: The classical result for $\Re(s) > 0$ is $\int_{\mathbb{R}^+} \frac{d}{dy} (y^s e^{-y}) dy = 0$. Using a similar method as the previous proposition. Let $I_{n,r}(s) = \int_0^r (\frac{d}{dy} (y^n e^{-y}) dy)$. Then,

$$(\forall \epsilon \in \mathbb{R}_{>0}) (\exists n \in \mathbb{Z}_{>0}) (\exists r \in \mathbb{R}_{>0}) (d(I_{n,r}(s), 0) < \epsilon),$$

$\downarrow$ *-transform,

$$(\forall \epsilon \in \mathbb{R}^*_{>0}) (\exists n \in \mathbb{Z}^*_{>0}) (\exists r \in \mathbb{R}^*_{>0}) (*d(I_{n,r}(s), 0) < \epsilon).$$

□

Theorem 3.2.25. Functional equation of $\Gamma(s)$: $\Gamma(s+1) = s\Gamma(s)$.

Proof: $\Gamma(s+1) = \int_{\mathbb{R}^+}^* e^{-y} y^s dy$. Integrating by parts and then using 3.2.24 gives,

$$\Gamma(s+1) = \int_{\mathbb{R}^+}^* \frac{d}{dy} (y^s e^{-y}) dy + s \int_{\mathbb{R}^+}^* e^{-y} y^s dy,$$

$$= s\Gamma(s).$$

□

Corollary 3.2.26. $\Gamma(1) = \Gamma(2) = 1$ and for $N \in \mathbb{Z}_{>0}$ $\Gamma(N) = (N-1)(N-2)\ldots 1 = (N-1)!$.

Proof: Using the classical result $\Gamma(1) = 1$ and transfer using the same method as 3.2.24 but with $I_{n,r} = \int_0^r (n) e^{-y} dy$ gives $\Gamma(1) = 1$. Similarly for $\Gamma(2)$.

Let $N \in \mathbb{Z}_{>0}$ and using 3.2.25 repeatedly gives the result.

□

Proposition 3.2.27. $\Gamma(s)$ has simple poles at $\mathbb{Z}_{\leq 0}$ and is non-zero.

Proof: The standard gamma function, $\Gamma(s)$ has a simple pole at $s = 0$, since the integral diverges. Using the functional equation the only other poles are simple and are at $-1, -2, -3, \ldots$. Let

$$I_{n,r} = \int_{[0,\epsilon]}^* (n) e^{-y} dy.$$

Then,

$$(\forall \alpha \in \mathbb{R}_{>0}) (\exists n \in \mathbb{Z}_{>0}) (\exists r \in \mathbb{R}_{>0}) (d(I_{n,r}, 0) > \alpha),$$

$\downarrow$ *-transform,
(∀α ∈ *R>0)(∃n ∈ *Z>0)(∃r ∈ *R>0)(∗d(I, r, 0) > α).

So there is a pole at 0. Using 3.2.25 and as in the classical case, the other poles are at the negative hyperintegers.

The function being non-zero again follows from the functional equation and by transfer principle as Γ(s) is non zero.

3.3 Hyper Riemann Zeta Function

Definition 3.3.1. Following the definition of the classical Riemann zeta function, naturally define,

\[ ζ_{∗Q}(s) = \sum_{\text{proper internal non-zero ideals } I \text{ of } *\mathbb{Z}} \frac{1}{N(I)^s} = \sum_{n ∈ *\mathbb{Z}>0} \frac{1}{n^s}. \]

Let \( ζ_N(s) = \sum_{0<n≤N} \frac{1}{n^s} \) \((N ∈ *\mathbb{Z}, s ∈ *\mathbb{C})\). From classical results, \( ζ_0(s) \) is absolutely convergent for \( \Re(s) > 1 + δ \) \((δ ∈ R>0)\) and the functions \( ζ_N(s) \) \((N ∈ \mathbb{Z}, s ∈ \mathbb{C})\) converge uniformly for \( \Re(s) > 1 + δ \). The aim of this section is to develop similar results for \( ζ_{∗Q}(s) \).

Lemma 3.3.2. \{ζ_N(s)\} are Q-continuous functions \( ∀n ∈ *\mathbb{N}, \Re(s) > 1 \).

Proof:

(∀n ∈ Z>0)(∀ε ∈ R>0)(∃δ ∈ R>0)(∀s, t ∈ R(s) > 1, d(s, t) < δ)(d(ζ_N(s), ζ_N(t))) < ε).

(3.3.1)

↓ ∗-transform,

(∀n ∈ *Z>0)(∀ε ∈ *R>0)(∃δ ∈ *R>0)(∀s, t ∈ *R(s) > 1, *d(s, t) < δ)(*d(ζ_N(s), ζ_N(t))) < ε).

(3.3.2)

□

Proposition 3.3.3. \( ζ_N(s) \) \((N ∈ *\mathbb{Z}>0)\) Q-converges uniformly to \( ζ_{∗Q}(s) \) for \( *\Re(s) > 1 + δ \), for every \( δ ∈ *\mathbb{R}>0 \).

Proof:

(∀ε ∈ R>0)(∃N ∈ Z>0)(∃δ ∈ R>0)(∀m, n ∈ N≥N)(∀s ∈ C, \Re(s) > 1 + δ)(d(ζ_m(s), ζ_n(s))) < ε).

(3.3.3)
CHAPTER 3: THE HYPER RIEMANN ZETA FUNCTION

\[ \downarrow \ast\text{-transform}, \]
\[ (\forall \epsilon \in \ast\mathbb{R}_{>0}) (\exists N \in \ast\mathbb{Z}_{>0}) (\exists \delta \in \ast\mathbb{R}_{>0}) (\forall m, n \in \ast\mathbb{N}_{\geq M}) (\forall s \in \ast\mathbb{C}, \ast\Re(s) > 1 + \delta) (\ast d(\zeta_m(s), \zeta_n(s)) < \epsilon). \]

(3.3.4)

The \( \{\zeta_N(s)\} \) form a hyper internal Cauchy sequence and a limit function exists. Fixing \( n \) and letting \( m \to \infty \) shows that the limit function is \( \zeta_{\ast\mathbb{Q}}(s) \). The uniform \( \mathbb{Q} \)-convergence also implies absolute \( \mathbb{Q} \)-convergence of \( \zeta_{\ast\mathbb{Q}}(s) \) for \( \Re(s) > 1 + \delta \).

\[ \square \]

**Corollary 3.3.4.** \( \zeta_{\ast\mathbb{Q}}(s) \) is \( \mathbb{Q} \)-continuous for \( \Re(s) > 1 \).

**Proposition 3.3.5.** For \( \Re(s) > 1 \),

\[ \zeta_{\ast\mathbb{Q}}(s) = \prod_{\text{Non-zero internal prime ideals } p \text{ of } \ast\mathbb{Z}} (1 - [\ast\mathbb{Z} : p]^{-s})^{-1} = \prod_{p \in \ast\mathbb{P}} (1 - p^{-s})^{-1} \]

**Proof:** Let \( M \in \ast\mathbb{Z} \) and let \( \Omega_M \) be the set of all primes \( \leq M \). So,

\[ \prod_{p \in \Omega_M} (1 - p^{-s})^{-1} = \prod_{p \in \Omega_M} (1 + p^{-s} + p^{-2s} + \ldots) = \sum_{m \in \Omega_M} \frac{1}{m^s}. \]

(3.3.5)

In the above sum \( \mathcal{M} \) is the set of all \( m \in \ast\mathbb{Z}_{>0} \) such that the prime factors of \( m \) are elements of \( \Omega_M \). Let \( \xi_M(s) = \sum_{m \in \mathcal{M}} \frac{1}{m^s} \). Classically \( \xi_M(s) \) converge absolutely for \( \Re(s) > 1 + \delta \) (\( \delta \in \mathbb{R}_0 \)) to \( \zeta_{\ast\mathbb{Q}}(s) \). So by the transfer principle,

\[ (\forall \delta \in \mathbb{R}_{>0}) (\forall M, N \in \mathbb{Z}_{>0}) (\exists \epsilon \in \mathbb{R}_{>0}) (\forall s \in \mathbb{C}, \Re(s) > 1 + \delta) (d(\xi_M(s), \xi_N(s)) < \epsilon/2) \]

\[ \downarrow \ast\text{-transform}, \]
\[ (\forall \delta \in \ast\mathbb{R}_{>0}) (\forall M, N \in \ast\mathbb{Z}_{>0}) (\exists \epsilon \in \ast\mathbb{R}_{>0}) (\forall s \in \ast\mathbb{C}, \ast\Re(s) > 1 + \delta) (\ast d(\xi_M(s), \xi_N(s)) < \epsilon/2) \]

Hence \( (\forall M' \in \ast\mathbb{Z}_{>0}) (\exists \epsilon' \in \ast\mathbb{R}_{>0}) (\forall s \in \ast\mathbb{C}, \ast\Re(s) > 1 + \delta) (\ast d(\xi_M'(s), \zeta_{\ast\mathbb{Q}}(s)) < \epsilon'/2) \) and by 3.3.3 \( (\forall M \in \ast\mathbb{Z}_{>0}) (\exists \epsilon \in \ast\mathbb{R}_{>0}) (\forall s \in \ast\mathbb{C}, \ast\Re(s) > 1 + \delta) (\ast d(\zeta_M(s), \zeta_M(s)) < \epsilon/2) \). Let \( \delta/2 = \max\{\epsilon, \epsilon'\} \) and \( N = \max\{M, M'\} \), then \( \forall n > N \) and using the triangle inequality,

\[ \ast d \left( \sum_{n \in \ast\mathbb{Z}_{>0}} \frac{1}{n^s}, \sum_{n \leq N} \frac{1}{n^s} \right) = \ast d(\xi_{N'}(s), \xi_N(s)), \]

\[ < \ast d(\xi_{N'}(s), \zeta_{\ast\mathbb{Q}}(s)) + d(\zeta_{\ast\mathbb{Q}}(s), \xi_N(s)), \]

\[ < \delta/2 + \delta/2, \]

\[ = \delta. \]

Hence the result as the two sequences converge to the same limit function.
Lemma 3.3.6. \( \zeta_Q(s) \) has no zeros for \( \Re(s) > 1 \).

**Proof:** In the standard case \( \zeta_Q(s) \) for \( \Re(s) > 1 \).

\[
(\forall N \in \mathbb{Z}_{>0})(\exists e \in \mathbb{R}_{>0})(\forall s \in \mathbb{C}, \Re(s) > 1)(d(\xi_N(s), 0) > \epsilon),
\]

↓ *-transform,

\[
(\forall N \in \mathbb{Z}_{>0})(\exists e \in \mathbb{R}_{>0})(\forall s \in \mathbb{C}, \Re(s) > 1)(d(\xi_N(s), 0) > \epsilon),
\]

\[\Box\]

3.3.1 A Hyper Theta Function

**Definition 3.3.7. Hyper Theta Function:** For \( \Im(s) > 0 \),

\[
*\Theta(s) = \sum_{n \in \mathbb{Z}} *e^{\pi in^2s} = 1 + 2 \sum_{n \in \mathbb{Z}_{>0}} *e^{\pi in^2s}.
\]

Classically the theta function, \( \Theta(s) = \sum_{n \in \mathbb{Z}} e^{\pi in^2s} \), is absolutely convergent \( \forall s \in \mathbb{C} \) with \( \Im(s) > 0 \).

**Proposition 3.3.8.** \( *\Theta(s) \) is absolutely \( Q \)-convergent \( \forall s \in *\mathbb{C} \) with \( *\Im(s) > 0 \).

**Proof:** Let \( N \in \mathbb{Z}_{>0} \) and define \( ^{(N)}\Theta(s) = 1 + 2 \sum_{n=1}^{N} *e^{\pi in^2s} \) for \( *\Im(s) > 0 \).

\[
(\forall m, n \in \mathbb{Z}_{>0})(\exists e \in \mathbb{R}_{>0})(\forall s \in \mathbb{C}, \Im(s) > 0)(d(^{(m)}\Theta(s), ^{(m)}\Theta(s)) < \epsilon),
\]

↓ *-transform,

\[
(\forall m, n \in \mathbb{Z}_{>0})(\exists e \in \mathbb{R}_{>0})(\forall s \in *\mathbb{C}, *\Im(s) > 0)(*d(^{(m)}\Theta(s), ^{(m)}\Theta(s)) < \epsilon),
\]

\[\Box\]

Hyper Fourier Transform

**Definition 3.3.9. Hyper Schwartz Space:** Let \( C^\infty(*\mathbb{R}) \) be the space of internal \( Q \)-smooth functions of \( *\mathbb{R} \) and define the hyper Schwartz space to be:

\[
S = \left\{ f \in C^\infty(*\mathbb{R}) : s \in *\mathbb{R}, \lim_{d(s,0) \to 0} s^n \frac{d^m f}{ds^m} = 0 \forall n, m \in \mathbb{Z}_{>0} \right\}.
\]
Definition 3.3.10. Hyper Fourier Transform: Let \( f \in S \) and define its Fourier transform to be,

\[
\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i xy} dx.
\]  (3.3.6)

In order to examine properties of this transform the idea of hyper distributions need to be introduced.

Definition 3.3.11. Hyper Dirac Distribution:

\[ *\delta(x) = \lim_{\epsilon \to 0} \Delta(x; \epsilon), \]

where \( x, \epsilon \in \mathbb{R} \) and \( \Delta(x; \epsilon) = 0(\epsilon < \epsilon), \frac{1}{2\epsilon}(\epsilon > \epsilon) \).

Using this definition the following properties can be derived.

Proposition 3.3.12. 1.

\[
\int_{\mathbb{R}} *\delta(x - x_0) dx = \lim_{\epsilon \to 0} \int_{[x_0 - \epsilon, x_0 + \epsilon]} *\delta(x - x_0) dx = 1.
\]

2. Let \( f \) be a \( \mathcal{Q} \)-continuous hyper real function. Then

\[
\int_{\mathbb{R}} *\delta(x - x_0) f(x) dx = f(x_0).
\]

3.

\[
*\hat{\delta}(y) = \int_{\mathbb{R}} *\delta(x) e^{-2\pi i xy} dx = 1.
\]

4.

\[
*\delta(x) = \int_{\mathbb{R}} *\delta(x) e^{2\pi i xy} dy.
\]

Proposition 3.3.13.

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} f(x) e^{-2\pi i (x+y)} dz dx = \int_{\mathbb{R}} dx f(x) \left( \int_{\mathbb{R}} e^{-2\pi i (x+y)} dz \right).
\]

Proof: In the standard case this result is true. So let \( I(n, r) = \int_{[\epsilon, 1]} \int_{[\epsilon, 1]} f(x)^{(n)} e^{-2\pi i (x+y)} dz dx \)
and let \( J(n, r) = \int_{[\epsilon, 1]} dx f(x) \left( dz \int_{[\epsilon, 1]} e^{-2\pi i (x+y)} \right) \) for \( r, n \in \mathbb{Z} > 0 \).

\[
(\forall \epsilon \in \mathbb{R}_0)(\exists N, R \in \mathbb{Z}_0)(\forall n, m \in \mathbb{Z}_N)(\forall s, t \in \mathbb{Z}_R)(d(I(n, s), J(m, t)) < \epsilon), \]  

\[ \downarrow *\text{-transform}, \]

\[
(\forall \epsilon \in \mathbb{R}_0)(\exists N, R \in \mathbb{Z}_0)(\forall n, m \in \mathbb{Z}_N)(\forall s, t \in \mathbb{Z}_R)(*d(I(n, s), J(m, t)) < \epsilon). \]
Proposition 3.3.14. Let $f \in S$ then $\hat{f}(y) = f(-y)$.

Proof: Using the results of the previous two propositions:

$$\hat{f}(y) = \int_{\mathbb{R}}^* \int_{\mathbb{R}}^* f(x)^* e^{-2\pi iz(x+y)}dzdx,$$

$$= \int_{\mathbb{R}}^* df(x) \left( \int_{\mathbb{R}}^* e^{-2\pi iz(x+y)} \right),$$

$$= \int_{\mathbb{R}}^* df(x)^* \delta(x - (-y)),$$

$$= f(-y).$$

Hyper Poisson Summation

Recall Fejér’s fundamental theorem concerning Fourier series. Let $f(x)$ be a bounded, measurable and periodic of period 1. Then the Fourier coefficients of $f$ are given by,

$$c_n = \int_0^1 f(x) e^{-2\pi i nx} dx,$$

for each $n \in \mathbb{Z}$. The partial sums are defined as

$$S_N(x) = \sum_{d(n,0) \leq (N)} c_n e^{2\pi inx}.$$

When $f(x)$ is continuous and $\sum_{n \in \mathbb{Z}} |c_n| < \infty$ then the function is represented by the absolutely convergent Fourier series

$$f(x) = \sum_{\mathbb{Z}} c_n e^{2\pi inx}.$$

Let $g \in S$ and periodic of period 1 ($g(x+1) = g(x), x \in \mathbb{R}$). $f$ is a real valued function and since $g \in S$, $f$ is a bounded, measurable function of period 1. By the transfer principle the Fourier coefficients $c_n$ and partial sums $S_N$, defined above, can be defined $\forall n \in \mathbb{Z}$. So,

$$(\forall m, n \in \mathbb{Z})(\exists \epsilon \in \mathbb{R}_{>0})(d(S_N, S_M) < \epsilon),$$

↓ $\ast$-transform,

$$(\forall m, n \in \mathbb{Z}^\ast)(\exists \epsilon \in \mathbb{R}_{>0})(\ast d(S_N, S_M) < \epsilon).$$

Hence $g(x)$ is represented by the absolutely $Q$-convergent hyper Fourier series $g(x) = \sum_{n \in \mathbb{Z}^\ast} c_n^\ast e^{2\pi inx}$. 

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Lemma 3.3.15. \( \int_{[0,1]}^* \sum_{k \in \mathbb{Z}} f(x + k)^* e^{-2\pi i n x} dx = \sum_{k \in \mathbb{Z}} f_{[0,1]}^* f(x + k)^* e^{-2\pi i n x} dx. \)

**Proof:** Let \( I(n, r) = \int_{[1/r, 1-1/r]}^* \sum_{k \in \mathbb{Z}} f(x + k)^{(n)} e^{-2\pi i n x} dx \)
and
\( J(n, r) = \sum_{k \in \mathbb{Z}} \int_{[1/r, 1-1/r]}^* f(x + k)^{(n)} e^{-2\pi i n x} dx, \)
for \( n, r \in \mathbb{Z}^+. \) Then use the argument of 3.3.13.

\[ \square \]

Lemma 3.3.16. *Hyper Poisson Summation:* Let \( f \in \mathcal{S} \) then
\[ \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n). \]

**Proof:** Let \( g(x) = \sum_{k \in \mathbb{Z}} f(k) \) then \( g(x) = g(x + 1) \) where \( f \in \mathcal{S}. \) This periodicity enables \( g(x) \) to be written as \( g(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x} \) where \( a_n = \int_{[0,1]}^* g(x)^* e^{-2\pi i n x} dx. \) Using the result of the previous lemma.
\[ a_n = \int_{[0,1]}^* \sum_{k \in \mathbb{Z}} f(x + k)^* e^{-2\pi i n x} dx, \]
\[ = \sum_{k \in \mathbb{Z}} \int_{[0,1]}^* f(x + k)^* e^{-2\pi i n x} dx, \]
\[ = \hat{f}(n). \]
\[ \sum_{n \in \mathbb{Z}} f(n) = g(0) = \sum_{n \in \mathbb{Z}} a_n = \sum_{n \in \mathbb{Z}} \hat{f}(n). \]

\[ \square \]

Lemma 3.3.17. \[ \int_{\mathbb{R}}^* \left( \frac{d}{dx} e^{-\pi x^2 - 2\pi i xy} \right) dx = 0. \]

**Proof:** Standard result: \( \int_{\mathbb{R}} \frac{d}{dx} (e^{-\pi x^2 - 2\pi i xy}) dx = 0. \) Using a similar method to 3.2.23 let,
\[ I_{n,r} = \int_{[-r,r]}^* \left( \frac{d}{dx} \right)^{(n)} e^{-\pi x^2 - 2\pi i xy} dx. \]
\[ (\forall \epsilon \in \mathbb{R}^+) (\exists r \in \mathbb{R}^+) (\exists n \in \mathbb{N}) (d(I_{n,r}, 0) < \epsilon), \]
\[ \downarrow \text{*-transform}, \]
\[ (\forall \epsilon \in \mathbb{R}^+) (\exists r \in \mathbb{R}^+) (\exists n \in \mathbb{N}) (\star d(I_{n,r}, 0) < \epsilon), \]
\[ 47 \]
Lemma 3.3.18. \[ \int_{\mathbb{R}} e^{-\pi x^2} dx = 1. \]

**Proof:** In the standard case \( \int_{\mathbb{R}} e^{-\pi x^2} dx = 1 \). Let \( I_{n,r} = \int_{[-r,r]} e^{-\pi x^2} dx \).

\[ (\forall \epsilon \in \mathbb{R}_{>0})(\exists r \in \mathbb{R}_{>0})(\exists n \in \mathbb{N})(d(I_{n,r}, 0) < \epsilon), \quad (3.3.9) \]

\[ \downarrow \text{*-transform}, \]

\[ (\forall \epsilon \in \mathbb{R}_{>0})(\exists r \in \mathbb{R}_{>0})(\exists n \in \mathbb{N})(d(I_{n,r}, 1) < \epsilon), \quad (3.3.10) \]

Lemma 3.3.19. Let \( h(x) = e^{-\pi x^2} \) then \( h(x) \) is its own Fourier transform.

**Proof:** \( \hat{h}(y) = \int_{\mathbb{R}} h(x) e^{-2\pi ixy} dx \). Differentiating,

\[ \frac{d\hat{h}(y)}{dy} = -2\pi i \int_{\mathbb{R}} xh(x) e^{-2\pi ixy} dx, \]

\[ = -2\pi i \hat{h}(y), \]

Integrating by parts and using \( \hat{h}(y) = C e^{-\pi y^2} \) for some constant \( C \). Letting \( y = 0 \) gives \( C = 1 \) (using \( 3.3.18 \)) and the result.

\[ \square \]

Theorem 3.3.20. Let \( s \in \mathbb{R}_{>0} \) then,

\[ \sum_{n \in \mathbb{Z}} e^{-\pi n^2/s} = s^{1/2} \sum_{n \in \mathbb{Z}} e^{-n^2/s}. \]

**Proof:** Using hyper Poisson summation, \( 3.3.18 \) and \( 3.3.19 \)

\[ \sum_{n \in \mathbb{Z}} e^{-\pi n^2/s} = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} e^{-\pi int} e^{-\pi n^2/s} dt, \]

\[ = s \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} e^{-2\pi insu - \pi sn^2} du, \]

\[ = s \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} e^{-\pi s[(u+in)^2+n^2]} du, \]

\[ = s \sum_{n \in \mathbb{Z}} e^{-\pi sn^2} \int_{\mathbb{R}} e^{-\pi su^2} du, \]

\[ = s^{1/2} \sum_{n \in \mathbb{Z}} e^{-\pi sn^2}. \]
Corollary 3.3.21. Functional equation for the hyper theta function:

$$^*\Theta(-\frac{1}{z}) = (z/i)^{1/2}^*\Theta(z).$$

Proof: Let $z = is$ and set $\omega(s) = ^*\Theta(is)$. Then 3.3.20 gives the functional equation $\omega(1/x) = x^{1/2}\omega(x)$.

3.4 Analytic Properties

Classically $\zeta_Q(s)$ is an analytic function of $s$ with $\Re(s) > 1$. It has analytic continuation to the whole of $\mathbb{C}$ with a simple pole at $s = 1$.

Lemma 3.4.1. $\zeta_N(s)$ is $Q$-analytic $\forall N \in ^*\mathbb{N}$ and $^*\Re(s) > 1$.

Proof:

$$(\forall \epsilon \in \mathbb{R}_{>0})(\exists \delta \in \mathbb{R}_{>0})(\forall s \in \mathbb{C}, \Re(s) > 1)(\exists h, g \in \mathbb{C})$$

$$d\left(\frac{\zeta_N(s + h) - \zeta_N(s)}{h}, \frac{\zeta_N(s + g) - \zeta_N(s)}{g}\right) < \epsilon \land d(h, 0) < \delta \land d(g, 0) < \delta),$$

$\downarrow$ $^*$-transform,

$$(\forall \epsilon \in ^*\mathbb{R}_{>0})(\exists \delta \in ^*\mathbb{R}_{>0})(\forall s \in ^*\mathbb{C}, \Re(s) > 1)(\exists h, g \in ^*\mathbb{C})$$

$$(^*d\left(\frac{\zeta_N(s + h) - \zeta_N(s)}{h}, \frac{\zeta_N(s + g) - \zeta_N(s)}{g}\right) < \epsilon \land ^*d(h, 0) < \delta \land ^*d(g, 0) < \delta).$$

From standard complex analysis a theorem of Weierstrass’ enables the deduction that the limit function of a uniformly convergent sequence of analytic functions is analytic. The proof of this relies on Cauchy’s theorem.

Theorem 3.4.2. $f(z)$ is $Q$-analytic in a region $\Omega \subseteq ^*\mathbb{C}$ iff

$$\int_\gamma^* f(z)dz = 0,$$  \hspace{1cm} (3.4.1)

for every cycle, $\gamma$, which is homologous to zero in $\Omega$. 

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Proof: See section 6.2.3 of [Ro2].

\[\text{\textbf{Theorem 3.4.3.}} \quad \text{1. Let } f \text{ be } \mathbb{Q}\text{-analytic on and inside and on a cycle } \gamma. \text{ Then, if } a \text{ is inside } \gamma, \]
\[
f(a) = \frac{1}{2\pi i} \int_{\gamma} f(z) \frac{dz}{z - a}. \]

2. With the same conditions,
\[
f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} f(z) \frac{dz}{(z - a)^{n+1}}. \]

Proof: See section 6.2.3 of [Ro1].

\[\text{\textbf{Theorem 3.4.4.}} \quad \text{Suppose } f_n(z) \text{ is analytic in } \Omega_n \subseteq \ast \mathbb{C} \text{ and } f_n(z) \text{ \(\mathbb{Q}\)-converges to a limit function } f(z) \text{ in a region } \Omega \subseteq \ast \mathbb{C}, \text{ \(\mathbb{Q}\)-uniformly on every hypercompact set (with respect to the \(\mathbb{Q}\)-topology) of } \Omega. \text{ Then } f(z) \text{ is } \mathbb{Q}\text{-analytic in } \Omega. \]

Proof: By hyper Cauchy’s theorem:
\[
f_n(z) = \frac{1}{2\pi i} \int_{C} f_n(w) \frac{dw}{w - z}, \tag{3.4.2}\]
where \(C\) is a hyper disc \(\ast d(w, a) \leq r\) contained in \(\Omega\). In the limit \(n \to \infty\) and by uniform \(\mathbb{Q}\)-convergence,
\[
f(z) = \frac{1}{2\pi i} \int_{C} f(w) \frac{dw}{w - z}, \tag{3.4.3}\]
and so \(f(z)\) is \(\mathbb{Q}\)-analytic in the disc. Any hypercompact (with respect to the \(\mathbb{Q}\)-topology) subset of \(\Omega\) can be covered by a hyperfinite number of such closed discs and therefore convergence is \(\mathbb{Q}\)-uniform on every hypercompact (with respect to the \(\mathbb{Q}\)-topology) subset.

\[\text{\textbf{Corollary 3.4.5.}} \quad \zeta_{\ast \mathbb{Q}}(s) \text{ is a } \mathbb{Q}\text{-analytic function for } \ast \Re(s) > 1. \]

\[\text{\textbf{Lemma 3.4.6.}} \quad \zeta'_{\mathbb{N}}(s) \text{ \(\mathbb{Q}\)-converges uniformly to } \zeta'_{\ast \mathbb{Q}}(s) \text{ for } \ast \Re(s) > 1. \]

Proof: Identical statement as in 3.3.3 with \(\zeta_{\mathbb{N}}(s)\) replaced by \(\zeta'_{\mathbb{N}}(s)\).
Definition 3.4.7.  1. $\ast \zeta_{\eta}(s) = \pi^{-s/2} \Gamma(\frac{s}{2})$.

2. $\ast \zeta_{\ast \lambda}(s) = \ast \zeta_{\eta}(s) \ast \zeta_{\ast Q}(s)$.

Proposition 3.4.8.  1. $\ast \zeta_{\eta}(s)$ is a $Q$-analytic function with poles at $-2\ast \mathbb{Z}_{\geq 0}$.

2. $\ast \zeta_{\ast \lambda}(s)$ $Q$-converges for $\ast \Re(s) > 1$.

Proof:

1. The only poles of $\ast \zeta_{\eta}(s)$ are those of $\ast \Gamma(s/2)$.

2. By composition of functions the $Q$-convergence is dependent on the $Q$-convergence of $\ast \zeta_{\ast Q}(s)$ since $\ast \zeta_{\eta}(s)$ is $Q$-convergent on $\ast \mathbb{C}$ apart from at its poles.

Lemma 3.4.9.

$$\sum_{n \in \ast \mathbb{Z}_{\geq 0}} \int_{\ast \mathbb{R}^+} \ast e^{-\pi n^2 y} y^{s-1} dy = \int_{\ast \mathbb{R}^+} \left( \sum_{n \in \ast \mathbb{Z}_{\geq 0}} \ast e^{-\pi n^2 y} \right) y^{s-1} dy.$$ 

Proof: Use the argument of 3.3.13.

Lemma 3.4.10.

$$\zeta_{\ast \lambda}(s) = \frac{1}{2} \int_{\ast \mathbb{R}^+} \left( \ast \Theta(iy) - 1 \right) y^{s/2-1} dy \quad (\ast \Re(s) > 0).$$

Proof:

$$\ast \Gamma(s) = \int_{\ast \mathbb{R}^+} \ast e^{-y} y^{s-1} dy,$$

$$\pi^{-s} \Gamma(s) \frac{1}{n^{2s}} = \int_{\ast \mathbb{R}^+} \ast e^{-\pi n^2 y} y^{s-1} dy \quad \text{(by letting } y \mapsto \pi n^2 y),$$

$$\pi^{-s} \Gamma(s) \ast \zeta_{\ast Q}(s) = \sum_{n \in \ast \mathbb{Z}_{\geq 0}} \int_{\ast \mathbb{R}^+} \ast e^{-\pi n^2 y} y^{s-1} dy,$$

$$\zeta_{\ast \lambda}(2s) = \sum_{n \in \ast \mathbb{Z}_{\geq 0}} \int_{\ast \mathbb{R}^+} \ast e^{-\pi n^2 y} y^{s-1} dy,$$

$$= \int_{\ast \mathbb{R}^+} \left( \sum_{n \in \ast \mathbb{Z}_{\geq 0}} \ast e^{-\pi n^2 y} \right) y^{s-1} dy, \quad (\ast \Re(s) > 0),$$

$$= \frac{1}{2} \int_{\ast \mathbb{R}^+} \left( \ast \Theta(iy) - 1 \right) y^{s-1} dy.$$
Lemma 3.4.11. For $s \neq 0$,
\[ \int_{[0,1]}^{\ast} y^{s/2-1} dy = \frac{2}{s}. \]
Also, for $s \neq 1/2$,
\[ \int_{\mathbb{R} \setminus [0,1]}^{\ast} y^{-1/2-s/2} dy = 1/(s - 1/2). \]

**Proof:** Using the methods of [Lemma 3.4.10] and the corresponding standard results but with $I_n = \int_{[0,1-1/n]}^{\ast} y^{s/2-1} dy$ ($n \in \mathbb{Z}_{>0}$) and $J_n = \int_{[1,n]}^{\ast} y^{-1/2-s/2} dy$ ($n \in \mathbb{Z}_{>1}$) gives the results.

\[ \]

Theorem 3.4.12. $\zeta_{(s)} = \zeta_{(1-s)}$.

**Proof:**
\[
\begin{align*}
\zeta_{(s)} &= \frac{1}{2} \int_{\mathbb{R}^+}^{\ast} (\ast \Theta(iy) - 1) y^{s/2-1} dy, \\
&= \frac{1}{2} \int_{\mathbb{R}^+ \setminus [0,1]}^{\ast} (\ast \Theta(iy) - 1) y^{s/2-1} dy + \frac{1}{2} \int_{[0,1]}^{\ast} (\ast \Theta(iy) - 1) y^{s/2-1} dy, \\
&= \frac{1}{2} \int_{\mathbb{R}^+ \setminus [0,1]}^{\ast} (\ast \Theta(iy) - 1) y^{s/2-1} dy + \frac{1}{2} \int_{\mathbb{R}^+ \setminus [0,1]}^{\ast} (\ast \Theta(-iy)) y^{-s/2-1} dy - \frac{1}{2} \int_{[0,1]}^{\ast} y^{s/2-1} dy, \\
&= \frac{1}{2} \int_{\mathbb{R}^+ \setminus [0,1]}^{\ast} (\ast \Theta(iy) - 1) y^{s/2-1} dy + \frac{1}{2} \int_{\mathbb{R}^+ \setminus [0,1]}^{\ast} (y^{1/2} \ast \Theta(iy)) y^{-s/2-1} dy - \frac{1}{s}, \\
&= -\frac{1}{s} + \frac{1}{2s-1} + \frac{1}{2} \int_{\mathbb{R}^+ \setminus [0,1]}^{\ast} [\ast \Theta(iy) - 1](y^{s/2-1} + y^{-1/2-s/2}) dy.
\end{align*}
\]
This is true for $\mathfrak{R}(s) > 1$ however the integral of the last line converges and absolutely $\forall s \in \ast \mathbb{C}$.

The standard integral in last line converges absolutely and uniformly $\forall s \in \mathbb{C}$. Using the transfer principle this last integral does Q-converge absolutely and uniformly for $s \in \ast \mathbb{C}$. Indeed, for $n \in \mathbb{Z}_{>0}$ and $r \in \mathbb{R}_{>1}$ let
\[ I_{n,r} = -\frac{1}{s} + \frac{1}{2s-1} + \frac{1}{2} \int_{[r,s]}^{\ast} [2 \sum_{n \in \mathbb{Z}_{>0}} (n \pi^2 y)](y^{s/2-1} + y^{-1/2-s/2}) dy. \]

Apply the transfer principle as in [3.2.23] Also the last integral is invariant under $s \mapsto 1 - s$ by substitution. The result follows.

\[ \]

Corollary 3.4.13. $\zeta_{(s)}$ can be Q-analytically continued on $\ast \mathbb{C}$. 

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Corollary 3.4.14. \( \zeta_{*Q}(s) \) can be \( \ast \)-analytically continued on \( \ast \mathbb{C} \) with a pole at \( s=1 \) and trivial zeros at \( s = -2 \ast \mathbb{Z} > 0 \).

**Proof:** From the definition of \( \zeta_{*Q}(s) \),

\[
\zeta_{*Q}(1 - s) = \frac{\pi^{1/2 - s} \Gamma\left(\frac{s}{2}\right) \zeta_{*Q}(s)}{\Gamma\left(\frac{1-s}{2}\right)}.
\]

Using \[3.2.27\] the only poles of \( \zeta_{*Q}(1 - s) \) in the numerator are at \( s = 0, -1, -2, \ldots \) and in the denominator \( s = 1, 3, 5, \ldots \). Evaluating at \( s = 0 \) in the equation corresponds to the pole \( \zeta_{*Q}(1) \) since by the transfer principle \( \zeta_{Q}(s) \) has a pole at \( s = 1 \). For \( s = -1, -2, -3, \ldots \) in the equation leads to zeros at \( s = -2, -4, -6, \ldots \) since the left hand side is \( \ast \)-analytic and non-zero at these values and the poles of the hyper gamma function cancel with the zeros of the zeta function. The other values above lead to the same conclusion. These are termed the trivial zeros.

\( \square \)

It has been shown that apart from the trivial zeros above any other zero \( (\gamma) \) must lie in the region \( 0 \leq \ast \Re(\gamma) \leq 1 \). Following the standard case,

**Conjecture 3.4.15.** If \( \zeta_{*Q}(s) = 0 \) and \( s \) is not a trivial zero then \( \ast \Re(s) = 1/2 \).

### 3.4.1 The Complex Shadow Map

The real shadow map can be extended to act on the hyper complex numbers. Recall the standard norm on \( \mathbb{C} \) given for \( z \in \mathbb{C} \) by \( |z| = ((\Re(z))^2 + (\Im(z))^2)^{1/2} \). This then extends to \( \ast \mathbb{C} \) by transfer. Let

\[
\ast \mathbb{C}^{\lim} = \{ z \in \ast \mathbb{C} : \exists r \in \mathbb{R} \text{ such that } |z| \ast < r \}.
\]

Then \( \ast \mathbb{C}^{\lim} = \ast \mathbb{R}^{\lim} + i \ast \mathbb{C}^{\lim} \), where \( i^2 = -1 \).

**Definition 3.4.16** (Complex Shadow Map). Let \( z \in \ast \mathbb{C}^{\lim} \) with \( x = \ast \Re(z) \) and \( y = \ast \Im(z) \).

\[
\text{sh}_{\ast \mathbb{C}} : \ast \mathbb{C}^{\lim} \to \mathbb{C},
\]

\[
x + iy \mapsto \text{sh}_{\eta}(x) + i \text{sh}_{\eta}(y).
\]

(3.4.4)

All the properties of the real shadow map carry through. As a simple application of this map consider the action on \( \zeta_{*Q} \).
Lemma 3.4.17. For all $s \in \ast \mathbb{C}^\lim$ with $\ast \Re(s) > 1$,

$$\zeta_{\ast \mathbb{C}}(s) \in \ast \mathbb{C}^\lim.$$ 

Proof: Let $s \in \ast \mathbb{C}^\lim$ with $s = x + iy$. Then as $x \in \ast \mathbb{R}^\lim$ there exist positive $a, b \in \mathbb{R}$ such that $a < x < b$.

$$|\zeta_{\ast \mathbb{C}}(s)| \leq \sum_{n \in \ast \mathbb{N}} n^{-x} \leq \sum_{n \in \ast \mathbb{N}} n^{-a} = \zeta_{\ast \mathbb{C}}(a).$$

For a fixed $a$ then $\zeta_{\ast \mathbb{C}}(a)$ is a hyperreal series which is an extension of the real series $\zeta_{\mathbb{C}}(a)$. Using the work in [Go] (6.10) one finds that $\zeta_{\mathbb{N}}(a) \simeq \zeta_{\ast \mathbb{C}}(a)$ for any $N \in \ast \mathbb{N} / \mathbb{N}$. Using the work above there exists an $N \in \ast \mathbb{N} / \mathbb{N}$ such that $|\zeta_{\ast \mathbb{C}}(a) - \zeta_{\mathbb{N}}(a)| < \epsilon/2$ and $|\zeta_{\mathbb{N}}(a) - \zeta_{\ast \mathbb{C}}(a)| < \epsilon/2$ for some fixed $\epsilon \in \ast \mathbb{R}^{\inf \eta}$. Then by a simple application of the triangle inequality

$$|\zeta_{\ast \mathbb{C}}(a) - \zeta_{\ast \mathbb{C}}(a)| \leq |(\zeta_{\ast \mathbb{C}}(a) - \zeta_{\mathbb{N}}(a)) + (\zeta_{\mathbb{N}}(a) - \zeta_{\ast \mathbb{C}}(a))|,$$

$$< \epsilon.$$ 

Therefore $\zeta_{\ast \mathbb{C}}(a) \simeq \zeta_{\ast \mathbb{C}}(a)$.

\[ \square \]

Corollary 3.4.18. For $s \in \ast \mathbb{R}^\lim$ and $\ast \Re(s) > 1$ then

$$\text{sh}_\eta(\zeta_{\ast \mathbb{C}}(s)) = \zeta_{\mathbb{C}}(\text{sh}_\eta(s)).$$

By using very basic complex analysis a complex sequence can be considered in terms of its real and imaginary parts and a direct application of the above leads to the following.

Corollary 3.4.19. For $s \in \mathbb{C}$ and $\Re(s) > 1$ then

$$\text{sh}_{\mathbb{C}}(\zeta_{\ast \mathbb{C}}(s)) = \zeta_{\mathbb{C}}(s).$$

For general $s \in \ast \mathbb{C}^\lim$ and $\ast \Re(s) > 1$ one uses the properties of the shadow map. Indeed since each term in $\zeta_{\mathbb{N}}(s)$ is limited,

$$\text{sh}_{\mathbb{C}}(\zeta_{\mathbb{N}}(s)) = \sum_{n=1}^{N} \text{sh}_{\mathbb{C}}(n^{-s}) = \zeta_{\mathbb{C}}(\text{sh}_{\mathbb{C}}(s)).$$

This last equality follows from the fact that for all $n \in \ast \mathbb{N} \setminus \mathbb{N} \ |n^{-s}|_{\mathbb{C}} \simeq_{\mathbb{C}} 0$. By the absolute Q-convergence of $\zeta_{\ast \mathbb{C}}$, $\zeta_{\ast \mathbb{C}}(s) \simeq_{\mathbb{C}} \zeta_{\mathbb{N}}(s)$ and so the following is proven.

Theorem 3.4.20. For $s \in \ast \mathbb{C}$ and $\ast \Re(s) > 1$ then

$$\text{sh}_{\mathbb{C}}(\zeta_{\ast \mathbb{C}}(s)) = \zeta_{\mathbb{C}}(\text{sh}_{\mathbb{C}}(s)).$$
The Hyper Dedekind Zeta Function

4.1 Hyper Mellin Transform

Definition 4.1.1. Let \( *f : \ast \mathbb{R}_+^\times \to \ast \mathbb{C} \) be an internal Q-continuous function with \( *f(\infty) = \lim_{y \to \infty} *f(y) \) existing. Define the set of all such functions to be \( \ast \mathcal{M} \). Then define the hyper Mellin transform to be the integral

\[
* L(*f, s) = \int_{* \mathbb{R}_+^\times} (*f(y) - *f(\infty)) y^s \frac{dy}{y}
\]

provided this integral exists.

Definition 4.1.2. Suppose \( f \) and \( g \) are internal hyper complex functions. Define the notation \( f(x) = O(g(x)) \) as \( |x| \to \infty \) iff

\[
(\exists x_0 \in \ast \mathbb{R}_+)(\exists M \in \ast \mathbb{N})(|f(x)| \leq M|g(x)| \text{ for } |x| > x_0).
\]

Theorem 4.1.3 (Mellin). Let \( *f, *g \in \ast \mathcal{M} \) with

\[
* f(y) = a_0 + O(*\exp(-cy^\alpha)), \quad * g(y) = b_0 + O(*\exp(-cy^\alpha)),
\]

for \( y \to \infty \) and positive constants \( c, \alpha \). Suppose these hyper functions satisfy

\[
* f(1/y) = C y^k * g(y),
\]

for some hyperreal number \( k > 0 \) and some hyper complex number \( C \neq 0 \). Then

1. The integrals \( * L(*f, s) \) and \( * L(*g, s) \) Q-converge absolutely and uniformly in \( \{ s \in \ast \mathbb{C} : \ast \Re(s) > k \} \). They are therefore Q-holomorphic on this space and admit Q-holomorphic continuations to \( \ast \mathbb{C} \setminus \{0, k\} \).
2. The hyper Mellin transforms have simple poles at $s = 0$ and $s = k$ with residues $\text{Res}_{s=0} \ast L(\ast f, s) = -a_0$, $\text{Res}_{s=k} \ast L(\ast f, s) = Cb_0$, $\text{Res}_{s=0} \ast L(\ast g, s) = -b_0$ and $\text{Res}_{s=k} \ast L(\ast g, s) = C^{-1}a_0$.

3. They satisfy the functional equation

\[ \ast L(\ast f, s) = C^{\ast} L(\ast g, k - s). \]

**Proof:** The Q-convergence follows from the methods used in the first chapter by looking at partial Mellin transforms

\[ \ast L(\ast f, s)_{N} = \int_{(0,N)}^{\ast} (\ast f(y) - \ast f(\infty))y^{s} \frac{dy}{y}, \]

and transfer.

Now let $\ast \Re(s) > k$ then the hyper Mellin transform can be rewritten by splitting up the interval of integration

\[ \ast L(\ast f, s) = \int_{(1,\infty)}^{\ast} (\ast f(y) - a_0)y^{s} \frac{dy}{y} + \int_{(0,1]}^{\ast} (\ast f(y) - a_0)y^{s} \frac{dy}{y}. \]

In the second integral make the substitution $y \mapsto 1/y$ and use $\ast f(1/y) = Cy^{k}g(y)$.

\[ \int_{(0,1]}^{\ast} (\ast f(y) - a_0)y^{s} \frac{dy}{y} = -\frac{a_0}{s} + C \int_{(1,\infty)}^{\ast} (\ast g(y) - b_0)y^{k-s-1} \frac{dy}{y} - \frac{Cb_0}{k-s}. \]

This Q-converges absolutely and uniformly for $\ast \Re(s) > k$ by using an identical method to the above. Hence

\[ \ast L(\ast f, s) = -\frac{a_0}{s} + \frac{Cb_0}{s-k} + \ast F(s), \]

where

\[ \ast F(s) = \int_{(1,\infty)}^{\ast} [(\ast f(y) - a_0)y^{s} + C\frac{dy}{y}(\ast g(y) - b_0)y^{k-s}] \frac{dy}{y}. \]

Then similarly

\[ \ast L(\ast g, s) = -\frac{b_0}{s} + \frac{C^{-1}a_0}{s-k} + \ast G(s), \]

where

\[ \ast G(s) = \int_{(1,\infty)}^{\ast} [(\ast g(y) - b_0)y^{s} + C^{-1}\frac{dy}{y}(\ast f(y) - a_0)y^{k-s}] \frac{dy}{y}. \]

These integrals Q-converge absolutely and locally uniformly on the whole complex plane, so they represent Q-holomorphic functions. Moreover it is clear that $\ast F(s) = C^{\ast} G(k - s)$ and $\ast L(\ast f, s) = C^{\ast} L(\ast g, k - s)$.
4.2 Nonstandard Algebraic Number Theory

In order to define the hyper Dedekind zeta function one needs to develop the notions of algebraic numbers and integers in a nonstandard setting.

4.2.1 Hyper Polynomials and Hyper Algebraic Numbers

Consider a ring commutative ring $R$ and the set of polynomials $R[x]$. Both of these sets can be enlarged in an nonstandard framework to give a hyper commutative ring $^*R$ and the set of internal hyperpolynomials $^*R[x]$ (which is different from the set of finite polynomials with coefficients from $^*R$, $(^*R)[x]$). By transfer the notion of the degree of a hyperpolynomial carries through.

**Definition 4.2.1.** A number $\alpha \in ^*\mathbb{C}$ is a hyper algebraic number if there exists a $f(x) = a_N x^N + \ldots + a_0 \in ^*\mathbb{Q}[x]$ with $a_i$ not all zero and $f(\alpha) = 0$. Further if $\alpha$ is the root of a monic $g \in ^*\mathbb{Z}[x]$ then it is said to be a hyper algebraic integer.

**Theorem 4.2.2.** Let $\alpha$ be a hyper algebraic number. Then there exists a unique hyperpolynomial $p(x) \in ^*\mathbb{Q}[x]$ which is monic, irreducible and of smallest degree such that $p(\alpha) = 0$. Moreover, if $f(x) \in ^*\mathbb{Q}[x]$ and $f(\alpha) = 0$ then $p(x)|f(x)$.

**Proof:**

Let $\alpha \in ^*\mathcal{A}$ be the set of hyper algebraic numbers and $^*S_\alpha = \{g \in ^*\mathbb{Q}[x] : g(\alpha) = 0\}$. Then by transfer

$$(\forall \alpha \in \mathcal{A})(\exists p(x) = a_{\deg(p(x))}x^{\deg(p(x))} + \ldots + a_0 \in S_\alpha)$$

$$(\forall f(x) \in S_\alpha)(\deg(p(x)) \leq \deg(f(x)) \land a_{\deg(p(x))} = 1 \land p(x)|f(x)),$$

$$\downarrow \, ^*\text{-transform},$$

$$(\forall \alpha \in ^*\mathcal{A})(\exists p(x) = a_{\deg(p(x))}x^{\deg(p(x))} + \ldots + a_0 \in ^* S_\alpha)(\forall f(x) \in ^* S_\alpha)$$

$$(\deg(p(x)) \leq \deg(f(x)) \land a_{\deg(p(x))} = 1 \land p(x)|f(x)).$$

Suppose this $p(x)$ is not irreducible then it can be written as a product of two lower degree polynomials in $^*\mathbb{Q}[x]$, say $p(x) = g(x)h(x)$. So $p(\alpha) = g(\alpha)h(\alpha) = 0$. By transfer $^*\mathbb{C}$ is a hyper integral domain and so either $g(\alpha) = 0$ or $h(\alpha) = 0$ which is a contradiction.
For uniqueness suppose there are two such polynomials \( p(x) \) and \( q(x) \) satisfying the above then \( p(x)|q(x) \) and \( q(x)|p(x) \) which implies \( p(x) = q(x) \).

\[ \square \]

**Definition 4.2.3.** Let the degree of \( \alpha \) be given by the degree of \( p(x) \), the minimal internal hyper-polynomial of \( \alpha \).

**Definition 4.2.4.** A hyper field \( \ast K \subset \ast \mathbb{C} \) is a hyper algebraic number field if its dimension over \( \ast \mathbb{Q} \) is hyper finite. The dimension of \( \ast K \) over \( \ast \mathbb{Q} \) is called the degree of \( \ast K \).

**Theorem 4.2.5.** Let \( \alpha, \beta \) be hyper algebraic numbers then there exists a hyper algebraic number \( \gamma \) such that \( \ast \mathbb{Q}(\alpha, \beta) = \ast \mathbb{Q}(\gamma) \).

**Proof:**

Using the transfer principle

\[
(\forall \alpha, \beta \in \mathcal{A})(\exists \gamma \in \mathcal{A})(\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\gamma)).
\]

\[
\downarrow \, \ast\text{-transform},
\]

\[
(\forall \alpha, \beta \in \ast \mathcal{A})(\exists \gamma \in \ast \mathcal{A})(\ast \mathbb{Q}(\alpha, \beta) = \ast \mathbb{Q}(\gamma)).
\]

\[ \square \]

**Corollary 4.2.6.** For \( n \in \ast \mathbb{N} \) let \( \alpha_1, \ldots, \alpha_n \) be a set of hyper algebraic numbers then there exists \( \gamma \in \mathcal{A} \) such that \( \ast \mathbb{Q}(\alpha_1, \ldots, \alpha_n) = \ast \mathbb{Q}(\gamma) \).

For a given hyper algebraic number, \( \theta \), let \( p(x) \) be its minimal internal hyperpolynomial of degree \( n \). Let \( \theta^{(1)} = \theta \) and \( \theta^{(2)}, \ldots, \theta^{(n)} \) be the conjugates of \( \theta \). Then \( \ast \mathbb{Q}(\theta^{(i)}) \) \( (i = 2, \ldots, n) \) is a conjugate field to \( \ast \mathbb{Q}(\theta) \). Moreover the maps \( \theta \rightarrow \theta^{(i)} \) are embeddings of \( \ast K = \ast \mathbb{Q}(\theta) \) into \( \ast \mathbb{C} \). There are \( n \) embeddings which can be split into hyperreal and hyper complex embeddings depending on whether or not the conjugate root is hyperreal or hyper complex. The hyper complex ones then split into pairs because of hyper complex conjugation. Let \( r_1 \) be the number of hyperreal embeddings and \( 2r_2 \) the number of hyper complex embeddings. Then \( n = r_1 = 2r_2 \).
Minkowski’s ideas, on a simplistic level, interpreted an algebraic number field $K$ over $\mathbb{Q}$ in terms of points in $n$-dimensional space. The following extends this in the natural way to the nonstandard algebraic number fields $^*K$ over $^*\mathbb{K}$ of hyperfinite degree $n$. There is a canonical mapping resulting from the $n$ hyper complex embeddings ($\tau$)

$$j : ^*K \to ^*K_{^*\mathbb{C}} := \prod_{\tau} ^*\mathbb{C},$$

$$\alpha \mapsto (\tau(\alpha)).$$

This $^*\mathbb{C}$-vector space is equipped with a hermitian scalar product

$$\langle x, y \rangle = \sum_{\tau} x_{\tau} \overline{y}_{\tau},$$

where represents hyper complex conjugation. Also related to the embeddings is a $^*\mathbb{R}$-vector space (the hyper Minkowski space $^*\mathbb{R}$)

$$^*K_{^*\mathbb{R}} = [\prod_{\tau} ^*\mathbb{C}]^+,,$$

which consists of points of $^*K_{^*\mathbb{C}}$ which are invariant under the involution of hyper complex conjugation. These are the points $(z_{\tau})$ such that $z_{\tau} = \overline{z}_{\tau}$. The restriction of the hermitian scalar product from $^*K_{^*\mathbb{C}}$ to $^*K_{^*\mathbb{C}}$ yields a scalar product. There is also the mapping $j : ^*K \to ^*K_{^*\mathbb{R}}$ since for all $\alpha \in ^*K$, $\overline{\tau}(\alpha) = \overline{\tau}(\alpha)$. This mapping can be used to give lattices in the hyper Minkowski space, see the following section on lattices.

Minkowski’s theory also exists in a multiplicative form and the $j$ canonical mapping can be restricted to a homomorphism

$$j : ^*K^\times \to ^*K_{^*\mathbb{C}}^\times = \prod_{\tau} ^*\mathbb{C}.$$  

In a similar way there is the space $^*K_{^*\mathbb{R}}^\times$.

There is a homomorphism on $^*K_{^*\mathbb{C}}^\times$ given by the product of the coordinates

$$N : ^*K_{^*\mathbb{K}}^\times \to ^*\mathbb{C}^\times.$$  

The usual norm defined below on $^*K$ is related to this norm by $N_{^*K\setminus\mathbb{Q}}(\alpha) = N(j(\alpha))$. The restriction leads to the norm $N$ being defined on the hyper Minkowski space.
4.2.2 Lattices

Let \( m, n \in \mathbb{N} \) (\( m \leq n \)) and let \( \{e_1, \ldots, e_m\} \) be a set of linearly independent vectors in \( \mathbb{R}^n \). The additive subgroup of \( (\mathbb{R}^n, +) \) generated by this set is called a lattice of dimension \( m \). A lattice is called complete if \( m = n \).

A hyper metric can be placed on \( \mathbb{R}^n \). Indeed let \( x, y \in \mathbb{R}^n \) and define \( \ast d_n(x, y) = (x_1 - y_1)^2 + \ldots + (x_n - y_n)^2)^{1/2} \). With this hyper metric closed balls can be introduced. Let \( x \in \mathbb{R}^n \) and \( r \in \mathbb{R}^\ast > 0 \) then \( \ast B_r[x] = \{y \in \mathbb{R}^n : \ast d_n(x, y) \leq r\} \). A subset \( X \subset \mathbb{R}^n \) is bounded if \( X \subset \ast B_r[0] \) for some \( r \). A subset of \( \mathbb{R}^n \) is discrete if and only if it intersects every \( \ast B_r[0] \) in a hyperfinite set.

Let \( \ast A_n \) be the set of additive subgroups of \( \mathbb{R}^n \), let \( \ast D_n \) be the set of discrete subgroups of \( \mathbb{R}^n \) and let \( \ast L_n \) be the set of lattices of \( \mathbb{R}^n \).

Theorem 4.2.7. An additive subgroup of \( \mathbb{R}^n \) is a lattice iff it is discrete.

Proof:

\[
(\forall n \in \mathbb{N})(\forall X \in A_n \cap L_n)(\exists Y \in D_n)(X = Y),
\]

\[\downarrow \ast\text{-transform,}\]

\[
(\forall n \in \mathbb{N})(\forall X \in \ast A_n \cap \ast L_n)(\exists Y \in \ast D_n)(X = Y),
\]

The converse is proved in an identical manner.

\[\square\]

For each lattice (generated by \( \{e_1, \ldots, e_n\} \)) a fundamental domain \( T \) can be defined which consists of all elements \( \sum_{r=1}^{n} a_r e_r \) with \( 0 \leq a_i < 1 \).

The next important concept to introduce is the notion of volume. To generalise a little the spaces being dealt with consider \( \ast V \), a euclidean vector space (a \( \ast \mathbb{R} \)-vector space of hyperfinite dimension \( n \) with a symmetric, positive definite bilinear form \( \langle \cdot, \cdot \rangle : \ast V \times \ast V \to \ast \mathbb{R} \)). On \( \ast V \) a notion of volume exists with the cube spanned by an orthonormal basis \( e_1, \ldots, e_n \) has volume 1 while the general parallelepiped, \( \phi \) spanned by \( n \) linearly independent vectors \( v_1, \ldots, v_n \) has volume

\[\text{vol}(\phi) = |\det(A)|,\]
where $A$ is the matrix of the base change. Now let $L$ be a lattice and $T$ be the associated fundamental domain then

$$\text{vol}(L) = \text{vol}(T).$$

### 4.2.3 Hyper Algebraic Integers

Let $^*K$ be a hyper algebraic number field of degree $n$ over $^*\mathbb{Q}$ and define $O_{^*K}$ to be the set of algebraic hyperintegers. A simple check shows that this is a ring. Using the transfer principle and \[\text{an \ the \ proof \ of \ 4.2.2}\] gives the following lemma.

**Lemma 4.2.8.** For $\alpha \in O_{^*K}$, its minimal hyperpolynomial is monic and an element of $^*\mathbb{Z}[x]$.

**Lemma 4.2.9.** Let $\alpha \in ^*A$ then there exists $m \in ^*\mathbb{Z}$ such that $m\alpha \in O_{^*K}$.

**Proof:**

$$(\forall \alpha \in A)(\exists m \in \mathbb{Z})(m\alpha \in O_K),$$

\[\downarrow \ast\text{-transform},\]

$$(\forall \alpha \in ^*A)(\exists m \in ^*\mathbb{Z})(m\alpha \in O_{^*K}),$$

\[\square\]

Since $^*K$ is a vector space over $^*\mathbb{Q}$ there exists a $^*\mathbb{Q}$-basis, $\omega_1, \ldots, \omega_n$ for $^*K$. Let $B_{^*K}$ be the set of all bases for $^*K$ over $^*\mathbb{Q}$.

**Definition 4.2.10.** Let $\{\omega_i\} \in B_{^*K}$ then it is said to be an integral basis if $w_i \in O_{^*K}$ for all $i$ and $O_{^*K} = ^*\mathbb{Z}\omega_1 + \ldots + ^*\mathbb{Z}\omega_n$. Let $IB_{^*K}$ be the set of integral bases for $^*K$ over $^*\mathbb{Q}$.

**Lemma 4.2.11.** For all $^*K$ there exists an integral basis.

**Proof:**

$$(\forall K)(\exists \{\omega_i\} \in B_K)(\{\omega_i\} \in IB_K),$$

\[\downarrow \ast\text{-transform},\]

$$(\forall ^*K)(\exists \{\omega_i\} \in B_{^*K})(\{\omega_i\} \in IB_{^*K}),$$

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In analogue to the standard case the norm and trace can be defined. As \(^*K\) is a hyperfinite vector space over \(^*\mathbb{Q}\) and linear maps can be defined. For any \(\alpha \in \mathbb{K}\) define a map

\[
\Phi_\alpha : \mathbb{K} \to \mathbb{K},
\]

\[
x \mapsto \alpha x.
\]

Then the trace can be defined by \(\text{Tr}_{\mathbb{K}}(\alpha) = \text{Tr}(\Phi_\alpha)\) where \(\text{Tr}\) is the usual trace of a linear map. Similarly the norm can be defined by \(\mathcal{N}_{\mathbb{Q}}(\alpha) = \det(\Phi_\alpha)\) where \(\det\) is the determinant of a linear map. The discriminant of \(^*\mathbb{K}\) can also be defined,

\[
d_{\mathbb{K}} = \det(\omega^{(j)}_i)^2,
\]

where \(\{\omega_i\}\) is an integral basis for \(^*\mathbb{K}\) and \(\omega_i^{(j)}\) is a conjugate of \(\omega_i\).

**Definition 4.2.12.** For a module \(\mathfrak{m}\) with submodule \(\mathfrak{n}\) define the index of \(\mathfrak{n}\) in \(\mathfrak{m}\) (denoted \([\mathfrak{m} : \mathfrak{n}]\)) by the number of elements in \(\mathfrak{m} \setminus \mathfrak{n}\).

### 4.2.4 Ideals

Some aspects of ideals in hyper algebraic number fields have been examined by Robinson (reference). These works have mainly looked at purely nonstandard ideals. Let \(\Omega_{\mathbb{Q}}\) be the set of proper internal ideals of \(\mathbb{Q}\). The norm of an internal ideal \(\mathfrak{a}\), denoted by \(N(\mathfrak{a})\), is its index in \(\mathbb{Q}\). Let \(\Omega_{\mathbb{Q},p}\) be the set of internal primes ideals of \(\mathbb{Q}\).

These ideals could be called integral internal ideals as there are also internal fractional ideals of \(\mathbb{Q}\). These can be defined as an \(\mathbb{Q}\)-module contained in \(\mathbb{Q}\), \(\mathcal{A}\), such that there exists \(t \in \mathbb{Z}\) with \(t\mathcal{A} \subset \mathbb{Q}\). By taking \(t = 1\) any internal integral ideal is necessarily an internal fractional ideal. Let \(\Omega_{\mathbb{Q},F}\) be the set of internal proper fractional ideals of \(\mathbb{Q}\).

**Lemma 4.2.13.** For each internal prime ideal \(\mathfrak{p} \in \Omega_{\mathbb{Q},p}\) there is an internal fractional ideal \(\mathfrak{p}^{-1}\) such that \(\mathfrak{p} \mathfrak{p}^{-1} = \mathbb{Q}\).

**Proof:**

\[
(\forall \mathfrak{p} \in \Omega_{\mathbb{Q},p})(\exists \mathfrak{p}^{-1} \in \Omega_{\mathbb{Q},F})(\mathfrak{p} \mathfrak{p}^{-1} = \mathbb{Q}),
\]

\[\downarrow \text{*-transform},\]

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\[
(\forall \wp \in \Omega_{*K,p})(\exists \wp^{-1} \in \Omega_{*K,F})(\wp \wp^{-1} = \mathcal{O}_{*K}),
\]

\[\square\]

Lemma 4.2.14. For any \(a \in \Omega_{*K}\), it can be written uniquely as a product of prime ideals.

Proof:

\[
(\forall a \in \Omega_{K})(\exists ! M \in \mathbb{N})(\exists ! \wp_1, \ldots, \wp_M \in \Omega_{K,p})(a = \wp_1 \ldots \wp_M \land N(a) = N(\wp_1) \ldots N(\wp_M)),
\]

\[\downarrow \ast\text{-transform,}\]

\[
(\forall a \in \Omega_{*K})(\exists ! M \in \ast \mathbb{N})(\exists ! \wp_1, \ldots, \wp_M \in \Omega_{*K,p})(a = \wp_1 \ldots \wp_M) \land N(a) = N(\wp_1) \ldots N(\wp_M),
\]

\[\square\]

Using an identical method of proof this result can be extended to internal fractional ideals.

Lemma 4.2.15. For any \(a \in \Omega_{*K,F}\), it can be written uniquely as a quotient of products of prime ideals.

Two internal fractional ideals \(A\) and \(B\) can be defined to be equivalent (written \(A \sim B\)) if there exists \(\alpha, \beta \in \mathcal{O}_{*K}\) such that \((\alpha)A \sim (\beta)B\). It can be simply checked that this relation is an equivalence relation. Let \(Cl_{*K}\) be the set of equivalence classes of internal ideals of \(\ast K\). A binary operation can be placed on this set by defining a product of \(I_1\) and \(I_2\) in \(Cl_{*K}\) to be the equivalence class of \(AB\) where \(A\) and \(B\) are two representatives of \(I_1\) and \(I_2\) respectively. It can be easily checked that this product is well defined and a group is formed - the internal ideal class group - with the equivalence class containing the internal principle ideals as the identity.

Definition 4.2.16. Denote by \(h_{*K}\) the cardinality of the internal ideal class group, the class number.

Theorem 4.2.17. hypclass For all hyper algebraic number fields \(\ast K\), \(h_{*K}\) is hyperfinite.

Proof:

\[
(\forall K)(\exists M \in \mathbb{N})(h_K < M),
\]


\[ \downarrow \ast\text{-transform}, \]

\[ (\forall \ast K)(\exists M \in \ast \mathbb{N})(h_{\ast K} < M), \]

Ideals of hyper algebraic integers lead to complete lattices in the Minkowski space.

**Proposition 4.2.18.** Let \( a \in \Omega_{\ast K} \) then \( I = j() \) is a complete lattice in \( \mathbb{R} \) with \( \text{vol}(I) = \sqrt{|d_{\ast K}|} N(\alpha) \).

The proof is almost identical to that in \([N]\), page 31.

4.3 Hyper Dedekind Zeta Function and its First Properties

**Definition 4.3.1.** Let \( s \in \ast \mathbb{C} \) and define the hyper Dedekind zeta function,

\[ \zeta_{\ast K} = \sum_{a \in \Omega_{\ast K}} \frac{1}{(N(a))^s}. \]

The partial hyper Dedekind zeta function can be defined for any \( M \in \ast \mathbb{N} \) by

\[ \zeta_M = \sum_{N(a) \leq M} \frac{1}{(M(a))^s}. \]

**Lemma 4.3.2.** \( \zeta_M(s) \) \((M \in \ast \mathbb{N})\) \( \ast \text{-converges uniformly to } \zeta_{\ast K}(s) \) for \( \ast \mathbb{R}(s) > 1 + \delta \) for every \( \delta > 0 \).

**Proof:**

\[ (\forall \epsilon \in \mathbb{R}, \epsilon > 0)(\exists M \in \mathbb{N})(\exists \delta \in \mathbb{R}, \delta > 0)(\forall m, n \in \mathbb{N}, m, n \geq M) \]

\[ (\forall s \in \mathbb{C}, \mathbb{R}(s) > 1 + \delta)(d(\zeta_m(s), \zeta_n(s)) < \epsilon), \]

\[ \downarrow \ast\text{-transform}, \]

\[ (\forall \epsilon \in \ast \mathbb{R}, \epsilon > 0)(\exists M \in \ast \mathbb{N})(\exists \delta \in \ast \mathbb{R}, \delta > 0)(\forall m, n \in \ast \mathbb{N}, m, n \geq M) \]

\[ (\forall s \in \ast \mathbb{C}, \ast \mathbb{R}(s) > 1 + \delta)(*d(\zeta_m(s), \zeta_n(s)) < \epsilon), \]

Thus \( \{\zeta_M(s)\} \) form a hyper internal Cauchy sequence and a limit function exists and is \( \zeta_{\ast K} \).
Lemma 4.3.3. The hyper Dedekind zeta function is absolutely and uniformly \( \mathbb{Q} \)-convergent for \( \Re(s) \geq 1 + \delta \) for every \( \delta > 0 \).

Proof: The uniform \( \mathbb{Q} \)-convergence follows from the previous lemma. By letting

\[
\zeta_A(s) = \sum_{N(a) \leq M} \frac{1}{d((N(a))^s, 0)},
\]

for all \( M \in \mathbb{N}^* \) the absolute \( \mathbb{Q} \)-convergence can be established by transfer almost identical to the previous lemma.

Lemma 4.3.4. \( \zeta_M(s) \) is a \( \mathbb{Q} \)-continuous function for all \( \mathbb{N}^* \), \( \Re(s) \geq 1 + \delta \) (\( \delta > 0 \)).

Corollary 4.3.5. \( \zeta^{*}_{K}(s) \) is \( \mathbb{Q} \)-continuous for \( \Re(s) \geq 1 + \delta \) for any \( \delta > 0 \).

The proof of the lemma is identical to that of lemma 3.3.2 and the corollary follows from this result and lemma 3.2.12

Theorem 4.3.6. For \( \Re(s) \geq 1 + \delta \) (for any \( \delta > 0 \)),

\[
\zeta^{*}_{K} = \prod_{\wp} (1 - (N(\wp))^{-s})^{-1},
\]

here \( \wp \) runs through all prime ideals of \( \mathbb{O}^{*}_{K} \).

Proof: Let \( M \in \mathbb{N}^* \) and let \( \Omega^{*}_{K,M} \) be the set of prime ideals with \( N(\wp) \leq M \). So,\n
\[
\prod_{\wp \in \Omega^{*}_{K,M}} (1 - (N(\wp))^{-s})^{-1} = \prod_{\wp \in \Omega^{*}_{K,M}} (1 + (N(\wp))^{-s} + (N(\wp))^{-2s} + \ldots) = \sum_{a \in M} \frac{1}{N(a)^{-s}},
\]

where \( M \) is the set of ideals such that all its prime ideal factors are elements of \( \Omega^{*}_{K,M} \). By letting

\[
\xi_{M}(s) = \sum_{a \in M} \frac{1}{(N(a))^{-s}}
\]

one can proceed in an identical manner to proposition 3.3.5.

4.4 Functional Equation

Like the Riemann zeta function the Dedekind zeta function can also be extended to a meromorphic function on \( \mathbb{C} \) but with a simple pole at \( s = 1 \). In the nonstandard setting a similar result is expected.
**Definition 4.4.1** (Partial Zeta Function). Let $\mathcal{I} \in Cl_K$ and define
\[
\zeta^*_{K}(\mathcal{I}, s) = \sum_{a \in \mathcal{I}} \frac{1}{(N(a))^{s}},
\]
where the sum is only over internal integral ideals. Then
\[
\zeta^*_{K}(s) = \sum_{\mathcal{I} \in Cl_K} \zeta^*_{K}(\mathcal{I}, s).
\]

This expression can be simplified by the following lemma. For any fractional ideal $\lfloor \mathcal{O}_{K} \rfloor$ acts on the set $b^x = \mathcal{O}_{K} \setminus \{0\}$. The set of orbits is denoted by $b^x \setminus \mathcal{O}_{K}^x$.

**Lemma 4.4.2.** Let $a \in \Omega^*_{K}$ and $\mathcal{I}$ the class of $a^{-1}$. There is a bijection
\[
b^x \setminus \mathcal{O}_{K}^x \to \{b \in \mathcal{I} : b \in \Omega^*_{K}\}, \quad \pi \to b = (a)a^{-1}.
\]

**Proof:** Let $a \in a^x$, then $(a)a^{-1}$ is an integral ideal and lies in $\mathcal{I}$. Suppose $(a)a^{-1} = ba^{-1}$ then $(a) = (b)$ so $ab^{-1} \in \mathcal{O}_{K}^x$ and hence injectivity. The map is also surjective. Indeed let $b \in \mathcal{I}$ then $b = (a)a^{-1}$ with $a \in ab \subset a$.

In order to rewrite the partial zeta functions one uses the hyper Minkowski space $(\mathbb{R}^d)$ and the norm on it. In the special case of a principal internal ideal $a = (a)$ of $\mathcal{O}_{K}$. Let $\omega_1, \ldots, \omega_n$ be an integral basis of $\Omega^*_{K}$ then $a\omega_1, \ldots, a\omega_n$ is an integral basis of $a$. Let $A = (a_{ij})$ be the transition matrix $a\omega_i = \sum a_{ij}\omega_j$.

Therefore for any $a \in \mathbb{K}^x$
\[
N((a)) = |N_{\mathbb{K}^x \setminus \mathbb{Q}}(a)| = |N(a)|
\]
and so
\[
\zeta^*_{K}(\mathcal{I}, s) = N(a)^s \sum_{\pi \in b^x \setminus \mathcal{O}_{K}^x} \frac{1}{|N(\pi)|^{s}}.
\]

It has been shown in proposition 4.2.18e that $a$ forms a complete lattice in the hyper Minkowski space with volume $\sqrt{d}_a (d_a = N(a)^2|d_{-K}|)$.

### 4.4.1 A Higher Dimensional Hyper Gamma Function

In order to define a higher dimensional hyper gamma function some analogues of the one dimensional case are needed, for example the Haar measure and the integration space. Let $\mathbb{R}^d$ be the
hyper Minkowski space introduced in the above section. Recall \( *R = \{ z \in *K \subset \mathbb{C} : z = \bar{z} \} \) where \( z = (z_\tau) \) and \( \bar{z} = (\bar{z}_\tau) \). Now define analogues of \( *\mathbb{R} \setminus \{ 0 \} \) and \( *\mathbb{R}_+^\times \),

\[
*R_\pm = \{ x \in *R : x = \bar{x} \} \quad \text{and} \quad *R_+^\times = \{ x \in *R_\pm : x > 0 \}.
\]

The analogue of the upper half plane can also be defined as \( *H = *R_+ + i*R_+^\times \). Two functions can be defined, \( || : *\mathbb{R}^\times + \to *R_+^\times \) and \( \log : *R_+^\times \to *R_\pm \) which act on each component as their hyperreal analogues. Using hyper complex exponentiation \( z^p \) can be defined for \( z \) not a negative hyperreal or zero by

\[
z^p = (z^{p_\tau}).
\]

A Haar measure exists on \( *R_+^\times \). One is fixed by the following. Clearly

\[
*R_+^\times = \prod_p *R_{+p}^\times,
\]

where the product is over the hyper conjugation classes \( \{ \tau, \overline{\tau} \} \) and \( *R_{+p}^\times = *[\mathbb{R}_+^\times] \) when \( p \) is hyperreal and \( *R_{+p}^\times = \{(y, y) : y \in *[\mathbb{R}_+^\times]\} \) when hyper complex. Define the isomorphism \( *R_{+p}^\times \to *[\mathbb{R}_+^\times] \) by \( y \mapsto y \) when \( p \) is hyperreal and \( y \mapsto y^2 \) when \( p \) is hyper complex. Together these result in an isomorphism

\[
\Phi : *R_+^\times \to \prod_p *R_{+p}^\times.
\]

The usual Haar measure on \( *[\mathbb{R}_+^\times] \), \( dt/t \) can give a product measure on \( *R_+^\times \) which will be called the canonical measure, denoted \( dy/y \).

**Definition 4.4.3.** For \( s = (s_\tau) \in *K \subset \mathbb{C} \) such that \( *\mathbb{R}(s_\tau) > 0 \) define the gamma function associated to \( *K \) by

\[
\Gamma_*K(s) = \int_{*R_+^\times} N(*\exp(-y)y^s) \frac{dy}{y}.
\]

The study of this function reduces to to the hyper gamma function using the isomorphism defined above. Indeed by the product decomposition

\[
\Gamma_*K(s) = \prod_p \Gamma_p(s_p).
\]

**Proposition 4.4.4.** For \( p \) hyperreal, \( s_p = s_\tau \) and

\[
\Gamma_p(s_p) = *\Gamma(s_p).
\]

For \( p \) hyper complex, \( s_p = (s_\tau, \overline{s_\tau}) \) and

\[
\Gamma_p(s_p) = 2^{1-s_\tau - \bar{s_\tau}} \Gamma(s_\tau + \bar{s_\tau}).
\]
Theorem 4.4.6: For the hyperreal case, \( \Gamma_p(s_p) = \int_{\mathbb{R}^*_+}^{\mathcal{N}} \left( *\exp(-t)t^{s_\tau} \right) dt/t \) which is just the usual gamma integral because the norm is simply the identity in this case. 

In the hyper complex case
\[
\Gamma_p(s_p) = \int_{\mathbb{R}^*_+}^{\mathcal{N}} \left( *\exp\left(-2\sqrt{t}\sqrt{t^{s_\tau+s_\tau}}\right) \right) dt/t,
\]
\[
= 2^{1-s_\tau-s_\tau}\Gamma(s_\tau + s_\tau), \quad t \mapsto (t/2)^2.
\]

Proof: From the definition of a lattice \( L = *\mathbb{Z}r_1 + \ldots + *\mathbb{Z}r_n \) where \( (r_1, \ldots, r_n) \) is a basis of \( L \) with \( r_i \) in \( \mathbb{R} \). Define a partial theta function by
\[
*\Theta_{L_M}(z) = \sum_{g \in L} \exp(\pi i \langle g, z \rangle),
\]
where \( M \in *\mathbb{M} \) and \( L_M = (*\mathbb{Z} \setminus M^*\mathbb{Z})r_1 + \ldots + (*\mathbb{Z} \setminus M^*\mathbb{Z})r_n \). Then \( *\Theta_{L_M} \) Q-converges to \( *\Theta_L \) by transfer from the classical case. Using an identical method of proof by transfer from proposition 11.2 gives uniform Q-convergence. Absolute Q-convergence follows from applying the proof to the function
\[
*\Theta_{L_M,[]}(z) = \sum_{g \in L_M} |*\exp(\pi i \langle g, z \rangle)|.
\]

4.4.2 Hyper Theta Functions

A hyper theta function was introduced in the second chapter and naturally there are many generalizations of these functions.

Definition 4.4.5. For every complete lattice \( L \) of \( *\mathbb{R} \) define the theta series for \( z \in *\mathbb{H} \)
\[
*\Theta_L(z) = \sum_{g \in L} \exp(\pi i \langle g, z \rangle).
\]

Proposition 4.4.6. The above theta series Q-converges absolutely and uniformly for all \( z \in *\mathbb{K}_{-\mathbb{C}} \) with \( *\Im(z) > \delta \) for \( \delta \in *\mathbb{R} \) and \( \delta > 0 \).

Proof: From the definition of a lattice \( L = *\mathbb{Z}r_1 + \ldots + *\mathbb{Z}r_n \) where \( (r_1, \ldots, r_n) \) is a basis of \( L \) with \( r_i \) in \( \mathbb{R} \). Define a partial theta function by
\[
*\Theta_{L_M}(z) = \sum_{g \in L} \exp(\pi i \langle g, z \rangle),
\]
where \( M \in *\mathbb{M} \) and \( L_M = (*\mathbb{Z} \setminus M^*\mathbb{Z})r_1 + \ldots + (*\mathbb{Z} \setminus M^*\mathbb{Z})r_n \). Then \( *\Theta_{L_M} \) Q-converges to \( *\Theta_L \) by transfer from the classical case. Using an identical method of proof by transfer from proposition 11.2 gives uniform Q-convergence. Absolute Q-convergence follows from applying the proof to the function
\[
*\Theta_{L_M,[]}(z) = \sum_{g \in L_M} |*\exp(\pi i \langle g, z \rangle)|.
\]
Hyper Schwartz functions can be defined on $^*\mathbb{R}$. Let $^*f \in C^\infty(^*\mathbb{R})$ be the space of Q-smooth functions $^*f : ^*\mathbb{R} \to ^*\mathbb{C}$.

**Definition 4.4.7.** Define the hyper Schwartz space

$$S = \{^*f \in C^\infty(^*\mathbb{R}) : x \in ^*\mathbb{R}, \lim_{x \to \infty} |x|^m \frac{d^k^*f}{dx^m} = 0, \forall k, m \in ^*\mathbb{N} \}$$

**Definition 4.4.8.** Let $^*f \in S$ and define its Fourier transform to be,

$$\widehat{^*f}(y) = \int_{^*\mathbb{R}^n} ^*f(x)^* \exp(-2\pi i \langle x, y \rangle) dx.$$ 

**Proposition 4.4.9.** The function $^*h(x) = ^*\exp(-\pi \langle x, x \rangle)$ is its own Fourier transform.

**Proof:** To perform the hyper integration an isometry can be used to identify the euclidean vector space $^*\mathbb{R}$ with $^*\mathbb{R}^n$. The Haar measure $dx$ becomes the Lebesgue measure $dx_1 \ldots dx_n$. As $^*h(x)$ decomposes as $h(x) = \prod_{i=1}^n ^*\exp(-\pi x_i^2)$ the Fourier transform is then $\widehat{h}(y) = \prod_{i=1}^n (^*\exp(-\pi x_i^2))$. From the case $n = 1$ (dealt with above) the result follows. 

**Proposition 4.4.10.** Let $A$ be a linear transformation of $^*\mathbb{R}$ and define the function $^*f_A(x) = ^*f(Ax)$. Then

$$\widehat{^*f_A}(y) = \frac{1}{|\det A|} \widehat{^*f(t^A^{-1}y)},$$

where $t^A$ is the adjoint transformation of $A$.

**Proof:** From the definition of the Fourier transform,

$$\widehat{f_A}(y) = \int_{^*\mathbb{R}^n} ^*f(Ax)^* \exp(-2\pi i \langle x, y \rangle) dx.$$ 

Making a change of variable $x \mapsto Ax$,

$$\widehat{f_A}(y) = \int_{^*\mathbb{R}^n} ^*f(x)^* \exp(-2\pi i \langle A^{-1}x, y \rangle) |\det A|^{-1} dx,$n

$$= |\det A|^{-1} \int_{^*\mathbb{R}^n} ^*f(x)^* \exp(-2\pi i \langle t^A^{-1}x, y \rangle) dx.$$ 

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Using this proposition the Poisson summation formula can then be proven.

**Theorem 4.4.11.** Let $L$ be a complete lattice in $^*\mathbb{R}$ and define its dual lattice

$$L' = \{ g' \in ^*\mathbb{R} : \langle g, g' \rangle \in ^*\mathbb{Z} \forall g \in L \}.$$ 

Then for any hyper Schwartz function $^*f$:

$$\sum_{g \in L} ^*f(g) = \frac{1}{\text{vol}(L)} \sum_{g' \in L'} \hat{^*f}(g').$$

**Proof:** Consider the complete lattice $^*\mathbb{Z}^n$ then there exists an invertible map, $A$, to $L$. Thus $L = A^*\mathbb{Z}^n$ and $\text{vol} = |\det A|$. Clearly the lattice $^*\mathbb{Z}^n$ is self dual and for all $n \in ^*\mathbb{Z}^n$ then

$$g' \in L \iff ^t(An)g' = ^t n^t A g' \in ^*\mathbb{Z},$$

$$\iff ^t A g' \in ^*\mathbb{Z}^n,$$

$$\iff g' \in ^t A^{-1}^*\mathbb{Z}^n.$$ 

Therefore $L' = A^*^*\mathbb{Z}^n$ with $a^* = ^t A^{-1}$. Then the statement to be proved becomes

$$\sum_{n \in ^*\mathbb{Z}^n} ^*f_A(n) = \sum_{n \in ^*\mathbb{Z}^n} \hat{^*f}_A(n).$$

Let $^*g(x) = \sum_{k \in ^*\mathbb{Z}^n} ^*f_A(x + ^* K)$. Via transfer this function is absolutely and uniformly Q-convergent. It is also clearly periodic as for all $n \in ^*\mathbb{Z}^n$

$$^*g(x + n) = ^*g(x).$$

$^*g$ is also a hyper Schwartz function. Indeed as $^*f$ is a hyper Schwartz function

$$\| ^*f(x + k) \||k||^{n+1} \leq C$$

for almost all $k \in ^*\mathbb{Z}^n$ and $x$ varying in a Q-compact domain. Therefore $^*g(x)$ is majorized by a constant multiple of the convergent series $\sum_{k \neq 0} \frac{1}{||k||^{n+1}}$. This argument can be repeated for the partial derivatives of $^*f$ to show that $^*g(x)$ is a $C^\infty$ function (where a $C^\infty$ function is defined in the Hyper Fourier Transform section of chapter 2). Combining these observations suggests that it should have some form of hyper Fourier expansion. One can just extend the results of the one dimensional case to obtain an expansion 

$$^*g(x) = \sum_{n \in ^*\mathbb{Z}^n} a_n^* \exp(2\pi i^t n x),$$

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with \( a_n = \hat{f}(n) \). Therefore

\[
\sum_{n \in \mathbb{Z}^*} f(n) = g(0),
\]

\[
= \sum_{n \in \mathbb{Z}^*} a_n,
\]

\[
= \sum_{n \in \mathbb{Z}^*} \hat{f}(n).
\]

The reason for the developing this Poisson summation formula is to prove a transformation equation for the hyper theta function.

**Theorem 4.4.12.**

\[ *\Theta_L(-1/z) = \sqrt{(z/i)} \text{vol}(L) *\Theta_L'(z). \]

**Proof:** As both sides of the transformation are Q-holomorphic in \( z \) it is sufficient to check the identity for \( z = iy \) with \( y \in \mathbb{R}^+ \). Let \( t = y^{-1/2} \) then

\[
z = \frac{i}{t^2} \quad \text{and} \quad -1/z = it^2.
\]

Then

\[ *\Theta_L(-1/z) = \sum_{g \in L} *\exp(-\pi \langle gt, gt \rangle). \]

Let \( *f(x) = *\exp(-\pi \langle g, g \rangle) \) then

\[ *\Theta_L(-1/z) = \sum_{g \in L} *f(tg) \quad \text{and} \quad *\Theta_L'(z) = \sum_{g' \in L'} *f(t^{-1}g). \]

Let \( A \) be the self-adjoint transformation of \( \mathbb{R}, x \mapsto tx \), clearly this has determinant \( T \). Then using propositions [4.4.10] and [4.4.9] with the function \( *f(x) \) in combination with Poisson summation gives the result.

\[ \square \]

**4.4.3 Integral Representation**

A theta series can be associated with the partial zeta function,

\[
\theta(a, s) = \theta_a(s/d_a^{1/n}) = \sum_{a \in a} \exp(\pi i (as/d_a^{1/n}, a)).
\]

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**Theorem 4.4.13.**

\[ |d_{*K}|^s \pi^{-ns} \Gamma_{*K}(s) \zeta_{*K}(I, 2s) = \int_{R^*_+}^* g(y)N(y)^s \frac{dy}{y}, \]

with the series

\[ g(y) = \sum_{a \in I} \exp(-\pi \langle ay/d_1^n, a \rangle). \]

**Proof:** In the integral of the hyper higher dimensional gamma function substitute \( y \mapsto \pi|a|^{2/y}d_1^{1/n}. \)

Then

\[ |d_{*K}|^s \pi^{-ns} \Gamma_{*K}(s) \frac{N(a)^{2s}}{|N(a)|^{2s}} = \int_{R^*_+}^* \exp(-\pi \langle ay/d_1^n, a \rangle)N(y)^s \frac{dy}{y}. \]

This can be summed over a full system of representatives of \( \alpha^\times \backslash O_{*K}^\times \) and gives the result. The only point which needed to be checked was the interchange of summation and integration. This is just an extension of the result in one dimension.

\[ \square \]

The Euler factor at infinity can be defined as

\[ Z_{\infty}(s) = |d_{*K}|^{s/2} \pi^{-ns/2} \Gamma_{*K}(s/2), \]

and define

\[ Z(I, s) = Z_{\infty}(s) \zeta_{*K}(I, s). \]

The aim is to relate this to the hyper theta function introduced above. The problem is that the series \( g \) is not over all \( \alpha \) like the hyper theta series. To overcome this problem a new measure is introduced.

### 4.4.4 Hyper Dirichlet Theorem

Recall the \( j \) map which embeds \( K^\times \) into \( K_{*R}. \) There is then the map from this space to \( \prod_{\tau}^* [R]^+ \) given by the hyper logarithm acting on acting on the absolute value of each component. This can be represented diagrammatically

\[ K^\times \rightarrow j K_{*R} \rightarrow l \prod_{\tau}^* [R]^+. \]

Subgroups of these groups can be considered.
In particular let \( S = \{ x \in \mathbb{R}_+^2 : \mathcal{N}(x) = 1 \} \) - this is called the norm one hypersurface. Let \( |.| : \mathbb{R}_+ \rightarrow \mathbb{R}_+^x \) with \( (x_\tau \rightarrow (|x_\tau|) \). Then \( \mathcal{O}_K^x \) is contained in this set. Every \( y \in \mathbb{R}_+^x \) can be written in the form

\[
y = xt^{1/n}
\]

with \( x = y/1^{1/n} \) and \( t = \mathcal{N}(y) \). This gives a decomposition

\[
\mathbb{R}_+^x = S \times \mathcal{O}_K^x.
\]

By transfer there exists a unique Haar measure on \( S \) such that the Haar measure \( dy/y \) on \( \mathbb{R}_+^x \) becomes the product measure \( dy/y = d^*x \times dt/t \).

The group \( |\mathcal{O}_K^x|^2 \) acts quite naturally on \( S \). Recall the logarithm map \( *\log : \mathbb{R}_+^x \rightarrow \mathbb{R}_+^* \) given by \( (x_\tau \mapsto (\log x_\tau) \). Clearly this takes \( S \) to the trace zero space \(*H = \{ x \in \mathbb{R}_+^x : \text{Tr}(x) = 0 \} \).

Let \(*\lambda \) be the composite map of the \( j \) map, \(|.| \) map and the \(*\log \) map; from \( \mathcal{O}_K^x \) to \( \mathbb{R}_+^* \). By restriction \( |\mathcal{O}_K^x| \) lies in \(*H \) and be transfer

**Lemma 4.4.14.** Under the \(*\log \) map \( |\mathcal{O}_K^x| \) is taken to a complete lattice \(*G \) in \(*H \).

A fundamental domain, \(*F \), for the above action can be taken to be the preimage of an arbitrary fundamental mesh of the lattice \( 2*G \).

**Proposition 4.4.15.** The function \(*Z(I, 2s) \) is the Mellin transform, \(*Z(I, 2s)(L(f, s)) \), of the hyper function

\[
* f(t) =* f_F(a, t) = \frac{1}{w} \int_* F \theta(a, ixt^{1/n})d^*x,
\]

with \( w = \#\mu_\mathcal{K} \) the number of roots of unity of \(*K \).

**Proof:** From the decomposition of \( \mathbb{R}_+^x \) and the definition of \(*Z(I, 2s) \)

\[
*Z(I, 2s) = \int_{*\mathbb{R}_+} \int_S \sum_{a \in a} \exp(-\pi(axt^t', a))d^*x dt^t,
\]

with \( t' = (t/d_\alpha)^{1/n} \). By the definition of the fundamental domain \(*F \),

\[
S = \bigcup_{\eta \in |\mathcal{O}_K^x|} \eta^{2*F}.
\]
CHAPTER 4: THE HYPER DEDEKIND ZETA FUNCTION

This is a disjoint union. Consider a transformation of \( S, x \mapsto \eta^2 x \). This leaves the Haar measure invariant, by definition, and maps \( F \) to \( \eta^2 F \). Therefore

\[
\int_S \sum_{a \in \mathcal{A}} \exp(-\pi \langle axt', a \rangle) d^n x = \sum_{\eta \in \mathcal{O}^\times_K} \int_{\eta^2 F} \sum_{a \in \mathcal{A}} \exp(-\pi \langle axt', a \rangle) d^n x,
\]

\[
= \frac{1}{w} \int_S \sum_{\epsilon \in \mathcal{O}^\times_K} \exp(-\pi \langle a\epsilon xt', a\epsilon \rangle) d^n x,
\]

\[
= \frac{1}{w} \int_F \left( \ast \theta(a, ixt^{1/n}) - 1 \right) d^n x,
\]

\[
= \ast f(t) - \ast f(0).
\]

The factor \( 1/w \) appears as the kernel of the map \( \mathcal{O}^\times_K \rightarrow |\mathcal{O}^\times_K| \). Indeed let \( \gamma \in \mu^\times_K \) and let \( \tau : \ast K \rightarrow \ast \mathbb{C} \) be an embedding. Then \( \ast \log |\tau(\gamma)| = \log 1 = 0 \) and thus \( \mu^\times_K \subset \ker(\ast \lambda) \). For the converse let \( \epsilon \in \ker(\ast \lambda) \) which implies \( \ast \lambda(\epsilon) = l(j(\epsilon)) = 0 \) Therefore for each embedding \( \tau : \ast K \rightarrow \ast \mathbb{C}, |\tau(\epsilon)| = 1 \). This then means that \( j(\epsilon) = (\tau(\epsilon)) \) lies in a bounded domain of \( \ast R \). Recall that the \( j \) map of ideals leads to complete lattices in \( \ast R \) and in particular \( j(\epsilon) \) is a point of the lattice \( j\mathcal{O}^\times_K \). Hence \( \ker(\lambda) \) can contain only a hyperfinite number of elements. To finish this off the following lemma is needed.

**Lemma 4.4.16.** Let \( \ast H \) be a hyperfinite subgroup of \( \ast K^\times \). Then \( \ast H \) consists of roots of unity.

This follows directly from the transfer principle.

The functional equation for the completed partial zeta function derives from the hyper theta transformation formula. To achieve this a dual lattice is needed for \( \mathcal{A} \) and the volume of \( F \) with respect to \( d^n x \) is needed.

**Definition 4.4.17.** Let \( \ast \mathcal{D}^{-1} = \{ x \in \ast K : \text{Tr}(x\mathcal{O}^\times_K) \subset \ast \mathbb{Z} \} \). Then define the different of \( \ast K \setminus \ast \mathbb{Q} \) to be the fractional inverse of \( \ast \mathcal{D}^{-1} \).

**Lemma 4.4.18.** The lattice \( L' \) which is dual to the lattice \( L = \mathcal{A} \) is given by \( \ast L' = (\mathcal{A} \mathcal{D})^{-1} \). Here the \( \ast \) represents the involution \( (x_{\text{tau}}) \mapsto (\bar{x}_\tau) \) on \( \ast R \).

**Proof:** Using the definition of \( \langle , \rangle \) and of the dual lattice

\[
\ast L' = \{ \ast g \in \ast R : \langle g, a \rangle \in \ast \mathbb{Z} \forall a \in \mathcal{A} \},
\]

\[
= \{ x \in \ast R : \text{Tr}(x\mathcal{A}) \subset \ast \mathbb{Z} \},
\]

\[
= \{ x \in \ast K : \text{Tr}(x\mathcal{A}) \subset \ast \mathbb{Z} \}.
\]

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The final condition in the middle set (Tr(xa) ⊂ ^*Z) implies x ∈ ^*K. Using the definition of the inverse different x ∈ ^*L' \leftrightarrow Tr(xaO\_K) ⊂ ^*Z for all a ∈ a ↔ xa ⊂ ^*D^{-1} \leftrightarrow x ∈ (a^*D)^{-1}.

□

Definition 4.4.19. Let t = r₁ + r₂ - 1 and ε₁, ..., εₜ be a set of fundamental units of ^*K. Define the regulator, ^*R, of ^*K to be the absolute value of

\[ \det(δ_i^* \log |τ_j(ε_j)|) \] for 1 ≤ i, j ≤ r.

Here δᵢ = 1 if τᵢ is real and δⱼ = 2 if τⱼ is complex. The embeddings are written as τ₁, ..., τᵣ, τᵣ₊₁, ..., τᵣ₊₁, τᵣ₊₁, ..., τᵣ₊₁.

Lemma 4.4.20. Let ^*F be a fundamental domain of ^*S then with respect to d^*x

\[ \text{vol}(^*F) = 2^{r-1}*R, \]

where r is the number of infinite places.

Proof: Recall the decomposition ^*R₊ = ^*S × ^*R₊. This isomorphism is given by α : (x, t) ↦ xt^{1/n}. It transforms the canonical measure into the product measure d^*x × dt/t. With respect to dt/t the set ^*I = \{t ∈ ^*R₊ : 1 ≤ t ≤ ^*exp(1)\} so vol(^*F) is also the volume of F × I with respect to d^*x × dt/t and the volume of α(^*F × ^*I) with respect to dy/y. From above there are isomorphisms

\[ ^*R₊ × ^*\log R₊ → \prod_p ^*\mathbb{R} \]

Let ψ be the composite of these. Then dy/y is transformed into the Lebesgue measure on ^*R as already mentioned above resulting in

\[ \text{vol}(^*F) = \text{vol}_{^*R}(ψα(^*F × ^*I)). \]

Let 1 = (1, ..., 1) ∈ ^*S. Then

\[ ϕ\alpha((1, t)) = e^* \log(t^{1/n}) = \frac{1}{n} e^* \log(t). \]

Here e = (e₁, ..., eᵣ) ∈ ^*R with eᵢ = 1 or 2 depending on whether eᵢ is real or complex respectively. Using the definition of ^*F

\[ ϕ\alpha(^*F × \{1\}) = 2^*T. \]
Here $^*T$ is the fundamental mesh of the unit lattice $G$ in the trace zero space $^*H = \{(x_i) \in {}^*\mathbb{R}^r : \sum x_i = 0\}$. Then

$$\Phi_\alpha(^*F \times ^*I) = 2^*T + [0,1/n]^*e.$$ 

This is the parallelepiped spanned by the vectors $2e_1, \ldots, 2e_{r-1}, 1/n$ with $e_1, \ldots, e_{r-1}$ span the fundamental mesh. This volume is $\frac{1}{n^{2r-1}}$ times the absolute value of the determinant

$$\det \begin{pmatrix} e_{11} & \ldots & e_{r-1,1} & e_{p_1} \\ \vdots & & \vdots & \vdots \\ e_{1r} & \ldots & e_{r-1,r} & e_{p_r} \end{pmatrix}.$$ 

The usual matrix operations, which carry through by transfer, can be used. In particular adding the first $r - 1$ lines to the last one makes all entries zero apart from the last one which is $n - \sum e_{p_i}$. The matrix above these zeros leads to the regulator when the determinant is taken.

\[ \square \]

**Proposition 4.4.21.** The hyper function $^*f_{^*F}(\alpha, 1/t) = t^{1/2}^*f_{^*F\mathcal{D}}((\alpha^*\mathcal{D})^{-1}, t)$ and

$$^*f_{^*F}(\alpha, t) = \frac{2^{r-1}}{w} R + O(^*\exp(-ct^{1/n})) \quad \text{for } t \to \infty, c > 0.$$ 

**Proof:** Let $L = \alpha$ be a lattice in $^*\mathbb{R}$ then by above this has volume $\text{vol}(L) = N(\alpha)|d_{*\mathcal{K}}|^{1/2}$. The lattice dual to this is given by $^*L' = (\alpha^*\mathcal{D})^{-1}$. $\langle *gz, *g \rangle = (g z, g)$ and so $^*\theta_L(z) = ^*\theta_{^*L}(z)$. Moreover $d_{(\alpha^*\mathcal{D})^{-1}} = 1/d_{\alpha}$. The transformation $x \mapsto x^{-1}$ leaves $^*d x$ unchanged and maps $^*F$ to the fundamental domain $^*F^{-1}$. Also observe that $\mathcal{N}_{\alpha}(x(t\alpha)^{1/n}) = t\alpha_x$ for $x \in \mathcal{S}$. Using these observations and the transformation formula for the hyper theta function.

$$^*f_{^*F}(\alpha, 1/t) = \frac{1}{w} \int_{^*F} ^*\theta_{\alpha}(ix(t\alpha)^{-1/n}) d^x x,$$

$$= \frac{1}{w} \int_{^*F^{-1}} ^*\theta_{\alpha}(-(ix)^{-1}(t\alpha)^{-1/n}) d^x x,$$

$$= \frac{1}{w} \frac{(t\alpha)^{1/2}}{w} \int_{^*F^{-1}} ^*\theta_{(\alpha^*\mathcal{D})^{-1}}(ix(t\alpha)^{-1/n}) d^x x,$$

$$= \frac{t^{1/2}}{w} \int_{^*F^{-1}} ^*\theta_{(\alpha^*\mathcal{D})^{-1}}(ix(t/d_{(\alpha^*\mathcal{D})^{-1}})^{-1/n}) d^x x,$$

$$= t^{1/2}^*f_{^*F\mathcal{D}}((\alpha^*\mathcal{D})^{-1}, t).$$

In order to prove the second part

$$^*f_{^*F}(\alpha, t) = \frac{1}{w} \int_{^*F} ^*d^x x + \frac{1}{w} \int_{^*F} ^*\theta_{\alpha}(ixt^{1/n}) d^x x = \frac{\text{vol}(F)}{w} + ^*r(t).$$

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The summands of the integrand are of the form \(* \exp(-\pi(ax,a)(t')^{1/n})\) with \(a \in \mathfrak{a}\) and \(t' = t/d_\mathfrak{a}\). Also \(x_\tau \geq \delta > 0\) for all \(\tau\) which results in \(\langle ax,a \rangle \geq \delta \langle a,a \rangle\) and so

\[ \ast r(t) \leq \frac{\text{vol}(F)}{w}(\ast \theta_{a}(i\delta t^{1/n}) - 1). \]

Let \(m = \min\{\langle a,a \rangle : a \in \mathfrak{a}\ \backslash \{0\}\}\) and \(M = \#\{a \in \mathfrak{a} : \langle a,a \rangle = m\}\). Thus

\[ \ast \theta_{a}(i\delta t^{1/n}) - 1 = \ast \exp(-\pi \delta mt^{1/n})(M + \sum ) \langle a,a \rangle > m \ast \exp(-\pi \delta (\langle a,a \rangle - m)t^{1/n}) = O(\ast \exp(-ct^{1/n})). \]

\[ \square \]

**Theorem 4.4.22.** The function \(* Z(\mathcal{I}, s)\) admits a \(Q\)-analytic continuation to \(* \mathbb{C} \backslash \{0,1\}\) and satisfies the functional equation

\[ \ast Z(\mathcal{I}, s) = \ast Z(\mathcal{I}', 1 - s), \]

where \(\mathcal{I}' = [\ast \mathcal{D}]\), the ideal class of \(\ast \mathcal{D}\). It has simple poles at \(s = 0\) and \(s = 1\) with residues \(-2^{r} \ast R/w\) and \(2^{r} \ast R/w\) respectively.

**Proof:** Let \(* f(t) = f_{*F}(a,t)\) and \(* g(t) = f_{*F^{-1}}((a*\mathcal{D})^{-1}, t)\) then the previous proposition implies that

\[ \ast f(t) = t^{1/2} \ast g(t), \]

and \(* f(t) = a_0 + O(\ast \exp(-ct^{1/n}))\), \(* g(t) = a_0 + O(\ast \exp(-ct^{1/n}))\), with \(a_0 = 2^{r-1} \ast R/w\). Using the section on the hyper Mellin transform enables the \(Q\)-analytic continuation of the hyper Mellin transforms of \(* f\) and \(* g\). Also the functional equation is obtained

\[ \ast L(\ast f, s) = \ast L(\ast g, 1/2 - s) \]

with the simple poles of \(* L(\ast f, s)\) at \(s = 0\) and \(s = 1/2\) with residues \(-a_0\) and \(a_0\) respectively. By the above proposition \(* Z(\mathcal{I}, s) = \ast L(\ast f, s/2)\) and so \(* Z(\mathcal{I}, s)\) admits a \(Q\)-analytic continuation to \(* \mathbb{C} / \{0,1\}\) with simple poles at \(s = 0\) and \(s = 1\) with residues \(-2^{r} \ast R/w\) and \(2^{r} \ast R/w\) respectively. Moreover it satisfies the functional equation

\[ \ast Z(\mathcal{I}, s) = \ast L(\ast f, s/2) = \ast L(\ast g, (1-s)/2) = \ast Z(\mathcal{I}', 1-s). \]

\[ \square \]

This theorem about the partial zeta functions then enables similar results to be obtained for the completed zeta function and the hyper Dedekind zeta function of \(* K\).
Chapter 4: The Hyper Dedekind Zeta Function

Corollary 4.4.23. $^*Z_K(s) = |d_K|^{s/2}n^{-s/2}\Gamma_K(s/2)\zeta_K(s) = \sum_{\mathcal{I}}^*Z(\mathcal{I}, s)$ admits a $Q$-analytic continuation to $^*\mathbb{C}/\{0, 1\}$ and satisfies the functional equation

$$^*Z_K(s) = ^*Z_K(s).$$

It has simple poles at $s = 0$ and $s = 1$ with residues $-2^r h_K^* R/w$ and $-2^r h_K^* R/w$ respectively.

Using $\zeta_K(s) = (|d_K|^{s/2}n^{-s/2}\Gamma_K(s/2))^{-1}Z_K(s)$ it is seen that the bracketed term is only zero at $s = 0$ so it cancels the pole of $^*Z_K(s)$ at this point.

Corollary 4.4.24. $^*\zeta_K(s)$ has a $Q$-analytic continuation to $^*\mathbb{C}/\{1\}$. It has a simple pole at $s = 1$ with residue

$$\frac{2^r_1 (2\pi)^{r_2}}{w|d_K|^{1/2}h_K^* R}.$$

It also satisfies the functional equation

$$^*\zeta_K(1-s) = ^*A(s)\zeta_K(s).$$

Here

$$^*A(s) = 2^n \times (2\pi)^{-ns}|d_K|^{1/2-s}(*\cos(\pi s/2))^{r_1+r_2}(*\sin(\pi s/2))^{r_2}(*\Gamma(s))^n.$$
Nonstandard Interpretation of \( p \)-adic Interpolation

5.1 Mahler’s Theorem

The classical problem of interpolation is well known. For example, interpolating the factorials, \( n! \) \((n \in \mathbb{N})\), is solved by noting that

\[
\int_{0}^{\infty} \exp(-x)x^n \, dx = n!.
\]

Hence finding a continuous function \( f(s) \) taking the value of \( n! \) for \( s = n \) results in

\[
f(s) = \int_{0}^{\infty} \exp(-x)x^s \, dx = \Gamma(s + 1),
\]

where \( \Gamma(s) \) is the familiar gamma function.

Fix a finite prime \( p \) then the \( p \)-adic analogue (of interpolation) has no direct solution since \( |n!|_p \to 0 \) as \( |n| \to \infty \), where \(|.|\) is the standard archimedean metric on \( \mathbb{R} \). In many ways it is more natural to consider

\[
\prod_{k < n, (k, p) = 1} k,
\]

as a \( p \)-adic factorial.

\( p \)-adic interpolation is not one fixed method. It consists of many ideas with the aim of obtaining a \( p \)-adic object from a real object. There are many ways of achieving this aim.

- Mahler interpolation uses the topological property of taking a function on a dense subset of \( \mathbb{Z}_p \) (for example \( \mathbb{Z} \) or \( \mathbb{N} \)) and using the natural extension to give a function on \( \mathbb{Z}_p \).
• As an extension of the previous idea the function can be \(p\)-adically analytically continued to a larger domain using ideas of Washington. [Example: \(n^s\) for \(n \in 1 + p\mathbb{Z}_p\).]

• \(p\)-adic integration as used for the \(p\)-adic version of L-functions, the Riemann zeta function and Eisenstein series among others.

• Interpolation via a twist of the original function. [Example: \(p\)-adic Hurwitz zeta function.]

• Replacing a function in a variable with a \(p\)-adic variable where this makes sense. [Example: \(p\)-adic L-functions.]

In many cases interpolation is a combination of the above ideas and more techniques as well. The natural question is why interpolate? In many ways this question is specific to the actual problem being considered. Though a clear answer is to learn more about a function from a \(p\)-adic perspective by creating a \(p\)-adic function. Often this \(p\)-adic function is a simpler function to work with compared to the original function. This enables a problem to be tackled from various aspects. By studying interpolation from a \(p\)-adic perspective one hopes to obtain new interpretations of this standard problem.

In more detail the problem of \(p\)-adic Mahler interpolation is derived as follows. Suppose there is a sequence \(\{c_k\}_{k=1}^\infty\) with \(c_k \in \mathbb{Q}_p\) (where \(p\) is a fixed finite prime). This sequence can be encoded as a function \(g : \mathbb{N} \rightarrow \mathbb{Q}_p\) with \(g(n) = c_n\). Then when does there exist a continuous function \(f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p\) such that \(f(n) = g(n) \forall n \in \mathbb{N}\)? Since \(\mathbb{Z}_p\) is compact, \(f\) must be uniformly continuous and bounded (\([\mathbb{M}]\)). This can be stated as,

\[
(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})(\forall x, y \in \mathbb{Z}_p)(|x - y|_p \leq p^{-n} \Rightarrow |f(x) - f(y)|_p \leq p^{-m}).
\]

In particular this is true \(\forall x, y \in \mathbb{N}\) as \(\mathbb{N} \subset \mathbb{Z}_p\). Then as \(f(n) = g(n) \forall n \in \mathbb{N}\),

\[
(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})(\forall x, y \in \mathbb{N})(|x - y|_p \leq p^{-n} \Rightarrow |g(x) - g(y)|_p \leq p^{-m}).
\]

Thus necessary conditions are that \(g\) is uniformly continuous and bounded \(p\)-adically.

Conversely suppose that \(g\) is uniformly continuous and bounded. Then a continuous function \(f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p\) can be constructed. For \(x \in \mathbb{Z}_p\) let \(\{x_i\}\) be a sequence of integers tending to \(x\) and define

\[
f(x) = \lim_{i \rightarrow \infty} f(x_i).
\]

A simple exercise shows that this function is well-defined because of the uniform continuity of \(g\).

This can be stated in the following theorem.
Chapter 5: Nonstandard Interpretation of $p$-adic Interpolation

Theorem 5.1.1. \([M, 112–113]\) Let \(f : \mathbb{N} \to \mathbb{Q}_p\) be uniformly continuous on \(\mathbb{N}\). Then there exists a unique function \(F : \mathbb{Z}_p \to \mathbb{Q}_p\) which is uniformly continuous and bounded on \(\mathbb{Z}_p\), and \(F(x) = f(x)\) if \(x \in \mathbb{N}\).

In a certain sense $p$-adic interpolation is trivial because when a function is defined on a dense subset of \(\mathbb{Z}_p\) (for example \(\mathbb{Z}\) or \(\mathbb{N}\)) it has a continuous extension to all of \(\mathbb{Z}_p\).

Returning to the $p$-adic gamma function. Let \(a_n = \prod_{k<n, (k,p)=1} k\) then it is easily checked (essentially Wilson’s theorem) that \(a_{n+p^s} \equiv -a_n \pmod{p^s}\). So by making a slight sign adjustment in \(a_n\) they give a uniformly continuous function on \(\mathbb{N}\) and can be $p$-adically interpolated using the above theorem to give the $p$-adic gamma function \((p \neq 2)\):

\[
\Gamma_p(n) = (-1)^n \prod_{k<n, (k,p)=1} k,
\]

and define \(\Gamma_p(0) = 1\). Thus \(\Gamma(x)\) is a uniformly continuous function for \(x \in \mathbb{Z}_p\). This construction is due to Morita \([M]\).

Mahler’s theorem can be interpreted as a $p$-adic analogue of a classical theorem of Weierstrass which states that a continuous function on a closed interval can be uniformly approximated by polynomials.

Theorem 5.1.2 (Mahler: \([M]\), chapters 9 and 10). Suppose \(f : \mathbb{Z}_p \to \mathbb{Q}_p\) is continuous and let

\[
a_n(f) = \sum_{n \in \mathbb{N}} (-1)^{n-k} \binom{n}{k} f(k).
\]

Then \(|a_n(f)|_p \to 0\) as \(|n| \to \infty\) and the series

\[
\sum_{n \in \mathbb{N}} \binom{x}{n} a_n(f)
\]

converges uniformly in \(\mathbb{Z}_p\). Moreover,

\[
f(x) = \sum_{n \in \mathbb{N}} \binom{x}{n} a_n(f).
\]

The aim of this chapter is to view this interpolation in terms of the nonstandard shadow map. It begins with some definitions in nonstandard mathematics. Then it proceeds to demonstrate that the natural extension of a continuous function to its hyper function can have an interpretation (with some extra conditions) via the $p$-adic shadow map to classical Mahler interpolation.
5.2 Interpolation Series

In order to investigate Mahler interpolation from a nonstandard perspective it seems reasonable to consider functions of the form \( f : \mathbb{N} \to \mathbb{Q}_p \). As has been previously described functions can be extended their hyper functions \( \ast f : \ast \mathbb{N} \to \ast \mathbb{Q}_p \). To investigate these functions further the necessary constraint of \( p \)-adic \( \mathbb{Q} \)-uniform continuity will be considered in order to write the corresponding hyper function as a series and for interpolation.

**Definition 5.2.1.**

- For \( n, k \in \mathbb{N} \) the classical definition of the binomial symbol gives
  \[
  \binom{n}{k} = \begin{cases} 
  \frac{n!}{k!(n-k)!} & 0 \leq k \leq n, \\
  0 & \text{otherwise}.
  \end{cases}
  \]

- This can be extended to all \( x \in \mathbb{R} \) and \( x \in \mathbb{Z}_p \) by
  \[
  \binom{x}{k} = \begin{cases} 
  x(x-1)(x-2)\ldots(x-k+1)/k! & k \geq 1, \\
  1 & k = 0.
  \end{cases}
  \]

Naturally in the latter expression for \( x \in \mathbb{N} \) this agrees with the former definition. It takes integer values for \( x \in \mathbb{Z} \) and has properties such as \( |\binom{x}{k}_{p}| \leq 1 \forall x \in \mathbb{Z}_p \) and for \( x \in \mathbb{R} \) (with \( x > 0 \)) \( (-1)^k \binom{x+k-1}{k} \). By the transfer principle, the above definition of the binomial symbol extends to \( \ast \mathbb{R} \). By using the identity for \( \binom{x}{k} \) one only needs to consider the transfer principle applied to the classical definition of the binomial symbol to obtain a definition for \( x \in \ast \mathbb{Z} \). Also the following result is true.

**Lemma 5.2.2.** \( |\binom{x}{k}_{p}| \leq 1 \forall x \in \ast \mathbb{Z} \).

In order to develop interpolation series some basic results are needed about the binomial symbol.

**Lemma 5.2.3** (Binomial Inversion Formula). Let \( \{a_k\} \) be a set of values in \( \ast \mathbb{Q}_p \) then
\[
b_n = \sum_{k \in \ast \mathbb{N}} \binom{n}{k} a_k \iff a_n = \sum_{k \in \ast \mathbb{N}} \binom{n}{k} (-1)^{n-k} b_k.
\]

**Corollary 5.2.4** (Orthogonality).
\[
\sum_{k \in \ast \mathbb{N}} (-1)^k \binom{n}{k} \binom{k}{m} = \begin{cases} 
  (-1)^m & \text{if } n = m, \\
  0 & \text{otherwise}.
  \end{cases}
\]

Both of these results are simple exercises which are easy to verify directly by the same proofs as the classical cases.
Definition 5.2.5. For \( n \in \ast \mathbb{N} \) and \( \ast f : \ast \mathbb{N} \rightarrow \ast \mathbb{Q}_p \) define
\[
a_n(\ast f) = \sum_{k \in \ast \mathbb{N}} \binom{n}{k} (-1)^{n-k} \ast f(k).
\]

This hypersum is well defined since the binomial symbol is defined in the previous section and \((-1)^{n-k}\) extends to \(\ast \mathbb{Z}\) by the transfer principle. Also the sum is hyperfinite because \(\binom{n}{k} = 0\) for \(k > n\). In the case when \(f\) is a standard function then the coefficient is the same as that found in any text on \(p\)-adic Mahler interpolation.

Proposition 5.2.6. Let \( \ast f : \ast \mathbb{N} \rightarrow \ast \mathbb{Q}_p \) be a function. Then for all \( n \in \ast \mathbb{N} \),
\[
\ast f(n) = \sum_{k \in \ast \mathbb{N}} \binom{n}{k} a_k(\ast f).
\]

Proof: This result follows from the binomial inversion formula. In the formula in lemma 7 let \( a_n = a_n(f) \) and \( b_k = \ast f(k) \). The result follows.

\[\square\]

5.3 Interpretation of Interpolation

Theorem 5.3.1. Let \( f : \mathbb{N} \rightarrow \mathbb{Q}_p \) be a uniformly continuous function, with respect to the \(p\)-adic metric, on \(\mathbb{N}\) and let \( \ast f : \ast \mathbb{N} \rightarrow \ast \mathbb{Q}_p \) be the extension to its hyper function. Then \( \text{sh}_p(\ast f) : \mathbb{Z}_p \rightarrow \mathbb{Q}_p \) is the unique \(p\)-adic function obtained by Mahler interpolation.

This gives a natural interpretation of \(p\)-adic interpolation in a nonstandard setting.

Proof: As discussed in the previous section \(\ast f\) can be written as a hyper finite sum for all \( n \in \ast \mathbb{N} \),
\[
\ast f(n) = \sum_{k \in \ast \mathbb{N}} \binom{n}{k} a_k(\ast f).
\]

This function can be extended to all hyper integers by using this sum and noting that for \( x > 0 \),
\[
\binom{-x}{k} = (-1)^k \binom{x+k-1}{k},
\]
and defining for all \( n \in \ast \mathbb{Z} \) with \( n < 0 \),
\[
\ast f(n) = \sum_{k \in \ast \mathbb{N}} (-1)^k \binom{-n+k-1}{k} a_k(\ast f).
\]

To check that \(\ast f\) is \(\mathbb{Q}\)-convergent it is required to show the nonstandard case of convergence in the \(p\)-adic metric. Classically,
Theorem 5.3.2. For \( c_n \in \mathbb{Q}_p \), a \( p \)-adic series

\[
\sum_{n \in \mathbb{N}} c_n,
\]

converges if and only if \( |c_n|_p \to 0 \) as \( |n| \to \infty \).

This result extends by transfer to \( \ast \mathbb{Q}_p \) since it is \( \ast \)-complete with respect to the extended \( p \)-adic metric.

Proposition 5.3.3. Let \( \ast f \) be as in theorem 5.3.1 and \( a_n(\ast f) \) as defined in definition 5.2.5 then

\[
|a_n(\ast f(v))|_p \to 0 \text{ as } |v|_\eta \to \infty.
\]

Proof: This is a slightly adapted proof given in [RM, 61–63] and is based on a proof of Bojanic. As \( \ast f \) is uniformly \( \mathbb{Q} \)-continuous,

\[
(\forall s \in \ast \mathbb{N})(\exists t \in \ast \mathbb{N})(\forall x, y \in \ast \mathbb{N})(|x - y|_p \leq p^{-t} \Rightarrow |\ast f(x) - \ast f(y)|_p \leq p^{-s}).
\]

Let \( x = k \in \ast \mathbb{N} \) and \( y = k + p^t \) then \( |\ast f(k + p^t) - \ast f(k)|_p \leq p^{-s} \). \( \ast \mathbb{N} \) is bounded in the \( p \)-adic metric and \( \ast f \) is uniformly continuous on \( \ast \mathbb{N} \) it is also bounded there. Let \( p^u = \max_{n \in \ast \mathbb{N}} |\ast f(n)|_p \).

One can replace \( \ast f \) by \( \ast g = p^u \ast f \) so that \( |\ast g(n)|_p \leq 1 \) for all \( n \in \ast \mathbb{N} \). So without loss of generality assume that \( |\ast f(n)|_p \leq 1 \) for all \( n \in \ast \mathbb{N} \).

From the definition of \( a_n(\ast f) \), \( |(\binom{n}{k})|_p \leq 1 \) and properties of the metric,

\[
|a_n(\ast f)|_p \leq | \sum_{k \in \ast \mathbb{N}} (-1)^{n-k} (\binom{n}{k}) |\ast f(k)|_p |
\]

\[
\leq \max_{k \in \ast \mathbb{N}} |\ast f(k)|_p |
\]

\[
\leq 1.
\]

To continue several lemmas are needed to enable a more precise bound to be put on values of \( a_n(\ast f) \).

Definition 5.3.4. Let \( \ast f \) be as above and \( n \in \ast \mathbb{N} \) then define the difference operators:

\[
D^n(f(x)) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x + k).
\]

Lemma 5.3.5. For \( m \in \ast \mathbb{N} \),

\[
D^n(f(x)) = \sum_{j=0}^{m} \binom{m}{j} D^{n+j}(f(x - m)).
\]
Proof of lemma: Using the definition of $D$,
\[
\sum_{j=0}^{m} \binom{m}{j} D^j(f(x - m)) = \sum_{j=0}^{m} \binom{m}{j} \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} f(x - m + k),
\]
\[
= \sum_{k=0}^{m} (-1)^k f(x - m + k) \sum_{j=0}^{m} \binom{m}{j} \binom{j}{k} (-1)^j.
\]
Using corollary 8 the inner sum is zero unless $k = m$ and in this case it is equal to $(-1)^m$. Then by applying $D^n$ to each side gives the result.

\[\square\]

Lemma 5.3.6. Let $^*f : {}^*\mathbb{N} \to {}^*\mathbb{Q}_p$ be any function and define,
\[
a_n(^*f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} ^*f(k).
\]
Then,
\[
\sum_{j=0}^{m} \binom{m}{j} a_{n+j}(^*f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} ^*f(k + m).
\]

Proof of lemma: (Note that this definition of $a_n$ agrees with the previous but is a little more general as there are no conditions on the function.) For $m = 0$ it is the definition of $a_n$. By the previous lemma,
\[
D^n(^*f(m)) = \sum_{j=0}^{m} \binom{m}{j} D^{n+j}(^*f(0)).
\]
However from the definition of the difference operators,
\[
D^n(^*f(m)) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} ^*f(m + k).
\]
Then letting $m = 0$ in the previous equation gives,
\[
D^n(^*f(0)) = a_n(^*f).
\]
Putting these three equations together gives the result.

\[\square\]

This lemma is now used in conjunction with the formula for $a_n$ to give,
\[
a_{n+p'}(^*f) = -\sum_{j=1}^{p'-1} \binom{p'}{j} a_{n+j}(^*f) + \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (^*f(k + p') - ^*f(k)).
\]
From elementary number theory $p$ divides $\binom{p^j}{j}$ for each $j$ in the first sum. Also using the estimates on the uniform continuity of $^*f$ at the beginning of the proof one finds,

$$|a_{n+p^t}(^*f)|_p \leq \max_{1 \leq j < p^t} \{p^{-1}|a_{n+j}(^*f)|_p, p^{-s}\},$$

$$\leq p^{-1} \text{ for } n \geq p^t,$$

since $|a_n(^*f)|_p \leq 1$. The above argument can be repeated by $n \to n + p^t$ in the penultimate inequality to get,

$$|a_n(^*f)|_p \leq p^{-2} \text{ for } n \geq 2p^t.$$

Repeating this argument $(s-1)$ times gives

$$|a_n(^*f)|_p \leq p^{-s} \text{ for } n \geq sp^t.$$

Therefore, $|a_n(^*f)|_p \to 0$ as $|n|_\eta \to \infty$. \hfill \square

This proposition gives that $|a_n(^*f)|_p \to 0$ as $|n| \to \infty$. Using theorem 5.3.2 the sum for the function $^*f(n)$ converges for all $n \in ^*\mathbb{Z}$.

**Proposition 5.3.7.** $\forall n \in ^*\mathbb{Z} \hspace{0.5em} ^*f(n) \in \lim_{p}$.

**Proof of proposition:** Returning to the original function $f$ the binomial inversion formula can be used in the classical setting to give the classical analogue of proposition 5.2.6

$$f(n) = \sum_{k \in \mathbb{N}} \binom{n}{k} a_k(f),$$

where $a_k(f)$ are defined in theorem 2. Similarly it can be extended to give function on $\mathbb{Z}$. From the definition of $a_k(^*f)$ it is seen that $a_k(f) = a_k(^*f)$ since $^*f(n) = f(n) \forall n \in \mathbb{N}$ (from the definition of extending a function to its hyper function). To show the result it is required to demonstrate that $|^*f(n)|_p$ is limited for all $n \in ^*\mathbb{Z}$. Let $f_N(n) = \sum_{n \leq N} \binom{n}{k} a_k(f)$ ($N \in \mathbb{N}$) and also $^*f_N(n) = \sum_{n \leq N} \binom{n}{k} a_k(^*f)$ ($N \in ^*\mathbb{N}$). Classicaly $\{f_N\}$ converges absolutely to $f$ ([RM, chapter 5]). By the transfer principle $\{^*f_N\}$ Q-converges absolutely to $^*f$. (Note that for $N \in \mathbb{N}$, $^*f_N = f_N$ and $^*f_N$ is the extension of $f_N$ to nonstandard $N$.) Hence,

$$\forall u \in ^*\mathbb{R}^+ \hspace{0.5em} \exists M \in ^*\mathbb{N} \forall N \geq M (|^*f - ^*f_N|_p < u). \hspace{0.5em} (5.3.1)$$
A bound now needs to be put on \( \ast f_N \). Classically the \( p \)-adic valuation satisfies for all \( x, y \in \mathbb{Q}_p \) \( |x + y|_p \leq \max\{|x|_p, |y|_p\} \). By induction, for all \( n \in \mathbb{N} \) and \( x_i \in \mathbb{Q}_p \) (\( 1 \leq i \leq n \)):

\[
\left| \sum_{1 \leq i \leq n} x_i \right|_p \leq \max_{1 \leq i \leq n} |x_i|_p.
\]

By the transfer principle this result extends to all \( n \in \ast \mathbb{N} \) and \( x_i \in \ast \mathbb{Q}_p \). This is now applied below.

\[
|\ast f_N(n)|_p = \left| \sum_{k \leq N} \binom{n}{k} a_k(\ast f)|_p, \right.
\]

\[
\leq \max_{k \leq N} \{|\binom{n}{k} a_k(\ast f)|_p\},
\]

\[
\leq \max_{k \leq N} \{|\binom{n}{k}|_p a_k(\ast f)|_p\},
\]

\[
\leq \max_{k \leq N} \{|a_k(\ast f)|_p\} \text{ (lemma 6).} \tag{5.3.2}
\]

The following lemma examines the nonstandard values of \( a_n(\ast f) \). It looks at the nonstandard part of a \( p \)-adically \( Q \)-convergent sequence and is the analogue of the real case found in chapter 5 of \([G0]\).

**Lemma 5.3.8.** Let \( \{c_n\} \) be a convergent of \( p \)-adic numbers in \( \mathbb{Q}_p \). This sequence naturally extends to a \( p \)-adically \( Q \)-convergent sequence \( \{c_n\} \) in \( \ast \mathbb{Q}_p \) for all \( n \in \ast \mathbb{N} \). (See for example chapter 5 of \([G3]\).) Then for all \( n \in \ast \mathbb{N} - \mathbb{N} \), \( c_n \in \ast \mathbb{Q}_p^{\inf} \).

**Proof:** Suppose there exists an \( N \in \ast \mathbb{N} - \mathbb{N} \) and an \( \epsilon(N) \in \mathbb{R}^+ \) such that \( |c_N|_p > \epsilon(N) \).

As the sequence is convergent, by definition \( |c_n|_p \to 0 \) as \( n \in \mathbb{N} \to \infty \):

\[
(\forall \delta \in \mathbb{R})(\exists m(\delta) \in \mathbb{N})(\forall v > m)(|c_v|_p < \delta).
\]

In particular choosing \( \delta = \epsilon(N) \in \mathbb{R}^+ \) gives that for all \( n \in \mathbb{N} \), with \( n > m(\epsilon(N)) \), \( |c_n|_p < \epsilon(N) \).

Using the transfer principle on the above logical statement gives

\[
(\forall \delta \in \ast \mathbb{R})(\exists m(\delta) \in \ast \mathbb{N})(\forall v > m)(|c_v|_p < \delta).
\]

Again choose \( \delta = \epsilon(N) \in \mathbb{R}^+ \) but this this time note the transfer principle enables the deduction that for all \( n \in \ast \mathbb{N} \), with \( n > m(\epsilon(N)) \), \( |c_n|_p < \epsilon(N) \). This is a contradiction as \( m(\epsilon(N)) \in \mathbb{N} \).

\[\square\]
This lemma is applied to the sequence $a_n(*f)$ to deduce that $a_n(*f) \in \ast\mathbb{Q}_p^{\text{inf}}$.

From proposition 5.3.3 $|a_n(*f)|_p \to 0$ as $n \to \infty$. The previous lemma shows that for nonstandard $n$, $a_n(*f) \in \ast\mathbb{Q}_p^{\text{inf}}$. As already stated $a_n(*f) = a_n(f)$ for all $n \in \mathbb{N}$ so $a_n(*f) \in \ast\mathbb{Q}_p^{\text{inf}}$ (for $n \in \ast\mathbb{N} - \mathbb{N}$) or $a_n(*f) \in \mathbb{Q}_p$ (for $n \in \mathbb{N}$). In both cases $a_n(*f) \in \ast\mathbb{Q}_p^{\text{lim}}$. Therefore using equation 5.3.2

$$|\ast f_N(n)|_p \leq p^C, \quad (5.3.3)$$

here $p^C = \max_{n \in \ast\mathbb{N}} \{|a_n(*f)|_p\}$.

Finally for some $N$ to be chosen below:

$$|\ast f|_p = |(\ast f - \ast f_N) + \ast f_N|_p,
\leq \max\{|\ast f - \ast f_N|_p, |\ast f_N|_p\}.$$  

Equation 5.4.1 enables $N$ to be chosen such that $|\ast f - \ast f_N|_p < p^C$. Thus $|\ast f|_p < p^C$ and therefore $\ast f \in \ast\mathbb{Q}_p^{\text{lim}}$.

Applying the definition of the $p$-adic shadow map to the function $\ast f$ and using the previous proposition gives a domain consisting of $\mathbb{Z}_p$. To determine the exact nature of the function the values $\text{sh}_p(\ast f)$ need to be investigated.

Let $z \in \ast\mathbb{Z}$ then $\text{sh}_p$ can be taken of $\ast f(z)$ by the previous proposition.

$$\text{sh}_p(\ast f(z)) = \text{sh}_p \left( \sum_{n \in \ast\mathbb{N}} \binom{z}{n} a_k(\ast f) \right).$$

$\text{sh}_p$ is a ring homomorphism and as the sum is absolutely $\mathbb{Q}$-convergent,

$$\text{sh}_p(\ast f(z)) = \text{sh}_p \left( \sum_{n \in \mathbb{N}} \binom{z}{n} a_n(\ast f) \right) + \text{sh}_p \left( \sum_{n \in \ast\mathbb{N} - \mathbb{N}} \binom{z}{n} a_n(\ast f) \right). \quad (5.3.4)$$

The second term vanishes. Indeed, in the classical case ([Gv, corollary 4.1.2]) consider a convergent infinite series $\sum_{n \in \mathbb{N}} b_n$ ($b_n \in \mathbb{Q}_p$) then

$$|\sum_{n \in \mathbb{N}} b_n|_p \leq \max_{n \in \mathbb{N}} \{|b_n|_p\}.$$ 

By the transfer principle the following lemma will hold.

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**Lemma 5.3.9.** Let \( \sum_{n \in \mathbb{N}^*} b_n (b_n \in \mathbb{Q}_p) \) be a \( \mathbb{Q} \)-convergent infinite series then

\[
| \sum_{n \in \mathbb{N}^*} b_n |_p \leq \max_{n \in \mathbb{N}^*} \{|b_n|_p\}.
\]

This lemma is applied to the infinite sum \( b_n = 0 (n \in \mathbb{N}) \) and \( b_n = \left( \frac{z}{n} \right) a_n \left( \star f \right) (n \in \mathbb{N}^* - \mathbb{N}) \). It is \( \mathbb{Q} \)-convergent by proposition 5.3.3 and theorem 5.3.2. Then

\[
| \sum_{n \in \mathbb{N}^*} b_n |_p \leq \max_{n \in \mathbb{N}^*} \{|b_n|_p\} \leq \max_{n \in \mathbb{N}^* - \mathbb{N}} \{|b_n|_p\},
\]

since \( b_n = 0 \) for \( n \in \mathbb{N} \). However,

\[
|b_n|_p = |(\frac{z}{n})|_p |a_n \left( \star f \right)|_p \leq |a_n \left( \star f \right)|_p,
\]

because \( |(\frac{z}{n})|_p \leq 1 \). Proposition 5.3.7 implies \( |b_n|_p \in \mathbb{Q}_p^{\inf} \) for all \( n \in \mathbb{N}^* - \mathbb{N} \). Therefore

\[
\sum_{n \in \mathbb{N}^* - \mathbb{N}} \left( \frac{z}{n} \right) a_n \left( \star f \right) \in \mathbb{Q}_p^{\inf},
\]

and the the second term does vanish in equation (5.3.4) leaving

\[
\sh_p \left( \left( \frac{z}{n} \right) a_n \left( \star f \right) \right) = \sh_p \left( \sum_{n \in \mathbb{N}} \left( \frac{z}{n} \right) a_n \left( \star f \right) \right). \tag{5.3.5}
\]

As already stated in theorem 3.1.4 \( \sh_p \) is a ring homomorphism. By using induction the following is true for all \( N \in \mathbb{N} \) and \( b_n \in \mathbb{Q}_p^{\lim} \),

\[
\sh_p \left( \sum_{n \leq N} b_n \right) = \sum_{n \leq N} \sh_p (b_n).
\]

Using the transfer principle this statement becomes true for all \( N \in \mathbb{N}^* \). In particular taking \( N \) nonstandard implies that

\[
\sh_p \left( \sum_{n \in \mathbb{N}} b_n \right) = \sum_{n \in \mathbb{N}} \sh_p (b_n).
\]

This is true because one could define \( b_n = 0 \) for \( n \in \mathbb{N}^* - \mathbb{N} \). Or alternatively note that \( \mathbb{N} \subset \{ n \in \mathbb{N}^* : n \leq N \} \). Thus equation (5.3.5) becomes

\[
\sh_p \left( \left( \frac{z}{n} \right) a_n \left( \star f \right) \right) = \sh_p \left( \sum_{n \in \mathbb{N}} \left( \frac{z}{n} \right) a_n \left( \star f \right) \right). \tag{5.3.6}
\]

Again using the ring homomorphism property,

\[
\sh_p \left( \left( \frac{z}{n} \right) a_n \left( \star f \right) \right) = \sh_p \left( \left( \frac{z}{n} \right) \right) \sh_p (a_n \left( \star f \right)).
\]
In the proof of proposition 17 it was showed that \( a_n(*) = a_n(f) \) for \( n \in \mathbb{N} \). Therefore \( a_n(*) \in \ast \mathbb{Q}_p \) and \( \text{sh}_p(a_n(*f)) = a_n(*f) \). This results in

\[
\text{sh}_p(*f(z)) = \sum_{n \in \mathbb{N}} \text{sh}_p \left( \binom{z}{n} \right) a_n(*f).
\]  

(5.3.7)

Finally for all \( z \in \ast \mathbb{Z} \) and \( n \in \mathbb{N} \), \( \binom{z}{n} = \frac{z(z-1)(z-2)\ldots(z-n+1)}{n!} \). The numerator has a finite number of terms and the denominator is standard. Therefore with a final application of the ring homomorphism property:

\[
\text{sh}_p \left( \binom{z}{n} \right) = \frac{\text{sh}_p(z)(\text{sh}_p(z) - 1)(\text{sh}_p(z) - 2)\ldots(\text{sh}_p(z) - n + 1)}{n!},
\]

\[
= \binom{\text{sh}_p(z)}{n}.
\]

Putting all this together gives

\[
\text{sh}_p(*f(z)) = \sum_{n \in \mathbb{N}} \binom{\text{sh}_p(z)}{n} a_n(f).
\]

It has already been proved that the domain of this function is \( \mathbb{Z}_p \) and the values lie in \( \mathbb{Q}_p \). Further this is identical to the function that would have been obtained via Mahler \( p \)-adic interpolation ([R, chapter 4(2.3)]):

\[
f(x) = \sum_{n \in \mathbb{N}} \binom{x}{n} a_n(f).
\]

The identification \( x = \text{sh}_p(z) \) is made and the theorem is proven.

\[\square\]

Graphically the above process can be displayed as follows:

\[
f : \mathbb{N} \to \mathbb{Q}_p, \quad \downarrow \quad (\text{hyper extension}), \quad \downarrow \quad *f : *\mathbb{N} \to *\mathbb{Q}_p, \quad \downarrow
\]
5.4 Extension

In the previous section I have obtained a nonstandard interpretation of Mahler’s theorem. This work strengthens that result by showing that all continuous $p$-adic functions $f : \mathbb{Z}_p \to \mathbb{Q}_p$ can be viewed as functions $^*g : ^*\mathbb{N} \to ^*\mathbb{Q}^\text{lim}_p$.

**Theorem 5.4.1.** Let $f : \mathbb{N} \to \mathbb{Q}_p$ be a uniformly continuous function, with respect to the $p$-adic metric, on $\mathbb{N}$. Then there exists a hyper function $^*g : ^*\mathbb{N} \to ^*\mathbb{Q}^\text{lim}_p$ such that $\text{sh}_p(^*g) : \mathbb{Z}_p \to \mathbb{Q}_p$ is the the unique $p$-adic function obtained by Mahler interpolation.

5.4.1 $p$-adic Functions

**Definition 5.4.2.** Let the set of $p$-adically uniformly continuous functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ be denoted by $C_p(\mathbb{Z}_p, \mathbb{Q}_p)$.

By restriction of a function in $C_p(\mathbb{Z}_p, \mathbb{Q}_p)$ to $\mathbb{N}$ gives a map to $C_p(\mathbb{N}, \mathbb{Q}_p)$. By Mahler’s theorem this map is an isomorphism.

Consider the nonstandard space consisting of $p$-adically uniformly $\mathbb{Q}$-continuous functions $C_p(\mathbb{N}, \mathbb{Q}^\text{lim}_p)$. There exists a map from this space to $C_p(\mathbb{Z}_p, \mathbb{Q}_p)$, the $p$-adic shadow map (sh$_p$) acting on functions. The definition of this map has been given above. It is defined for all $^*g : ^*\mathbb{N} \to ^*\mathbb{Q}^\text{lim}_p$ because sh$_p(^*\mathbb{N}) = \mathbb{Z}_p$ and sh$_p(^*\mathbb{Q}^\text{lim}_p) = \mathbb{Q}_p$ and are also surjective maps. This suggests that the $p$-adic shadow map acting on functions should also be surjective. By the definition of this map the image of sh$_p(C_p(\mathbb{N}, \mathbb{Q}^\text{lim}_p))$ is contained in $C_p(\mathbb{Z}_p, \mathbb{Q}_p)$.

By Mahler’s theorem any $f \in C_p(\mathbb{Z}_p, \mathbb{Q}_p)$ can be written as

$$f(x) = \sum_{n\in\mathbb{N}} a_n(f)\binom{x}{n}, \forall x \in \mathbb{Z}_p.$$
Here \( a_n(f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k) \) and \( |a_n|_p \to 0 \) as \( n \to \infty \). This theorem extends to the nonstandard space \( C_p(\ast \mathbb{Z}_p, \ast \mathbb{Q}_p) \) by the previous section. In particular as \( C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{lim}p) \subset C_p(\ast \mathbb{Z}_p, \ast \mathbb{Q}_p) \) any \( g \in C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{lim}p) \) can be written as

\[
^*g(m) = \sum_{n \in ^*\mathbb{N}} b_n(^*g) \binom{m}{n}, \forall m \in ^*\mathbb{N}.
\]

Here \( b_n(^*g) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} g(k) \) and \( |b_n|_p \to 0 \) as \( n \to \infty \). The \( p \)-adic shadow map acts on this hyperfinite sum since for any \( m \in ^*\mathbb{N} \), \( \binom{m}{n} = 0 \) for \( n > m \). Properties of the \( p \)-adic shadow map are used from above.

\[
\text{sh}_p(^*g(m)) = \sum_{n \in ^*\mathbb{N}} \text{sh}_p(b_n(^*g)) \text{sh}_p \left( \binom{m}{n} \right). \tag{5.4.1}
\]

However from the definition of the the image of the \( p \)-adic shadow map

\[
f(t) = \text{sh}_p(^*g(m)) = \sum_{n \in \mathbb{N}} a_n(f) \binom{\text{sh}_p(m)}{n}, \tag{5.4.2}
\]

Here \( t = \text{sh}_p(m) \). By equating these two equations at values of \( t = m \in \mathbb{N} \) it is easily shown that \( a_n(f) = \text{sh}_p(b_n(^*g)) \) for all \( n \in \mathbb{N} \) and also \( f(t) = \text{sh}_p(^*g(m)) \) for all \( m = t \in \mathbb{N} \).

In order for the two equalities to hold in general and by the previous paragraph \( \text{sh}_p(b_n(^*g)) = 0 \) for all \( n \in ^*\mathbb{N} \setminus \mathbb{N} \). This implies that \( b_n(^*g) \in ^*\mathbb{Q}^{inf}p \) for these values of \( n \).

One question which can be asked is what is the kernel of the shadow map. \( \ker(\text{sh}_p) = \{ ^*g \in C_p(^*\mathbb{N}, ^*\mathbb{Q}^{lim}p) : \text{sh}_p(^*g) = 0 \} \). By the previous paragraphs this kernel is equal to \( C_p(^*\mathbb{N}, ^*\mathbb{Q}^{inf}p) \).

For the surjectivity consider an \( f \in C_p(\mathbb{N}, \mathbb{Q}_p) \). Then this can be written as

\[
f(n) = \sum_{k \in \mathbb{N}} a_k(f) \binom{n}{k}.
\]

In particular for each \( n \in \mathbb{N} \), \( f(n) \in \mathbb{Q}_p \) and let \( q_n = f(n) \). Then by the series expansion for \( p \)-adic numbers

\[
q_n = p^n q_{n-r_n} + p^{n+1} q_{n-r_n+1} + \ldots.
\]

Let

\[
[q_n]_m = p^n q_{n-r_n} + \ldots + p^{n+m} q_{n-r_n+m} \in \mathbb{Q}.
\]

Then \( [q_n]_m \to q_n \) as \( n \to \infty \) and \( [q_n]_m |_p = |q_n|_p \forall m \in \mathbb{N} \). Now define a function \( f_m : \mathbb{N} \to \mathbb{Q} \) by \( f_m(n) = [q_n]_m \). Since \( [q_n]_m |_p = |q_n|_p f_m \in C_p(\mathbb{N}, \mathbb{Q}) \). Using the classical norm on \( C_p(\mathbb{N}, \mathbb{Q}_p) \) with \( |f|_p = \sup_{n \in \mathbb{N}} |a_n(f)|_p \) then \( f_m \to f \) as \( m \to \infty \). Now consider the nonstandard extension.
of these functions $f_m$ to $m \in \ast \mathbb{N}$. These are going to be functions of the form $\ast f_m : \ast \mathbb{N} \to \ast \mathbb{Q} \in C_p(\ast \mathbb{N}, \ast \mathbb{Q})$. Using the previous work on Mahler interpolation it can be shown that in fact $\ast f_m \in C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p})$. Moreover the classical norm on functions can also be extended to the nonstandard functions.

For $\ast g \in C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p}) \setminus C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\inf_p})$ define $|\ast g|_p = \sup_{n \in \mathbb{N}} |a_n(\ast g)|_p$. This is well defined because it has already been shown that for $n \in \ast \mathbb{N} \setminus \mathbb{N}$ that $a_n(\ast g) \in \ast \mathbb{Q}^{\inf_p}$. Since $\ast g \not\in C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\inf_p})$ there exists at least one $a_n(\ast g)$ (for some $n \in \mathbb{N}$) with $a_n(\ast g) \in \ast \mathbb{Q}^{\lim_p} \setminus \ast \mathbb{Q}^{\inf_p}$. Thus

$$\sup_{n \in \ast \mathbb{N}} |a_n(\ast g)|_p = \sup_{n \in \mathbb{N}} |a_n(\ast g)|_p.$$

It can be easily shown that this norm satisfies the triangle inequality.

For each $\ast g \in C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p}) \setminus C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\inf_p})$ define the monad of $\ast g$ to be

$$\mu(\ast g) = \{ \ast h \in C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p}) : \ast h \sim_p \ast g \} \cup \{ j \in C_p(\mathbb{N}, \mathbb{Q}_p) : j \approx_p g \}.$$

Here $\approx_p$ is the notion of two elements being infinitesimally close with respect to the norm. That is $a \approx_p b$ iff $|a - b|_p = p^{-N}$ where $N \in \ast \mathbb{N} \setminus \mathbb{N}$. It is clear that this is an equivalence relation. Moreover

**Lemma 5.4.3.** The only element of $\mu(\ast g)$ which is an element of $C_p(\mathbb{N}, \mathbb{Q}_p)$ is $\text{sh}_p(\ast g)$.

**Proof:** Dealing with uniqueness first. Suppose there are two distinct elements ($h_1$ and $h_2$) of $C_p(\mathbb{N}, \mathbb{Q}_p)$ in $\mu(\ast g)$. Then by the transitive property of $\approx_p$, $h_1 \approx_p h_2$ which is a contradiction since both functions are standard. Hence $h_1 = h_2$.

The existence requires examination of the shadow map of $\ast g$. Using the above work

$$|\ast g - \text{sh}_p(\ast g)|_p = \sup_{n \in \mathbb{N}} |a_n(\ast g) - \text{sh}_p(a_n(\ast g))|_p.$$

By the definition of the $p$-adic shadow map each of the above values is infinitesimal and hence so is the value of the norm.

□

Similar definitions hold for elements of $C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\inf_p})$ with the shadow map of any element being zero and the monad of any element being $C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\lim_p})$ and the 0 map in $C_p(\mathbb{N}, \mathbb{Q}_p)$.

Returning to any $f \in C_p(\mathbb{N}, \mathbb{Q}_p)$ there exists a sequence of functions $\ast f_m : \ast \mathbb{N} \to \ast \mathbb{Q}^{\lim_p}$ converging to $f$. In particular for $m \in \ast \mathbb{N} \setminus \mathbb{N}$, $f \approx_p \ast f_m$. Hence $f = \text{sh}_p(\ast f_m)$ and the $p$-adic shadow map on functions is surjective.
In conclusion $\text{sh}_p : C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\text{lim}_p}) \to C_p(\mathbb{N}, \mathbb{Q}_p)$ is surjective and this map provides an isomorphism

$$C_p(\mathbb{N}, \mathbb{Q}_p) \cong C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\text{lim}_p}) \setminus C_p(\ast \mathbb{N}, \ast \mathbb{Q}^{\text{inf}_p}).$$

As a final remark to this section this result extends my nonstandard work on Mahler interpolation. This can be displayed graphically

$$f : \mathbb{N} \to \mathbb{Q}_p,$$
$$\downarrow$$
(hyper extension),
$$\downarrow$$
$$*f : *\mathbb{N} \to *\mathbb{Q}^{\text{lim}_p},$$
$$\downarrow$$
($p$-adic shadow map),
$$\downarrow$$
$$f : \mathbb{Z}_p \to \mathbb{Q}_p,$$
($p$-adic interpolated function).

The reason for looking at Mahler interpolation in this way is two fold. Firstly a nonstandard perspective can often provide a different view point and in this case could also simplify the work since it deals with functions on $\ast \mathbb{N}$ which are potentially simpler than those on $\mathbb{Z}_p$. Secondly the possibility of working with more than one prime at the same time would provide different approach to $p$-adic analysis. The development of this begins in the next chapter.
Applications and Double Interpolation

The previous chapter showed how one form of $p$-adic interpolation, Mahler interpolation, could be viewed as the $p$-adic shadow map of a certain hyper function. In the introduction it was stated that there are a variety of methods which can be used for $p$-adic interpolation. Some of these are now developed from a nonstandard perspective culminating in interpolation with respect to two or more primes.

6.1 Morita Gamma Function

The gamma function has been studied since the 1700s with its importance realised by Euler and Gauss. It occurs at one solution to the problem of finding a continuous function of real or complex variable that agrees with the factorial function at the integers.

**Definition 6.1.1 (Gamma Function).** For $z \in \mathbb{C}$ with $\Re(z) > 0$

$$\Gamma(z) = \int_{0}^{\infty} t^{z-1} \exp(-t) \, dt,$$

and functional equation for $z \notin -\mathbb{N}$

$$\Gamma(z + 1) = z\Gamma(z).$$

From the $p$-adic perspective the aim is to find a $p$-adically continuous function extending the factor- rial function. One solution was found by Morita which extended the restricted factorial

$$n!_p := \prod_{1 \leq j < n, p \nmid n} j.$$

The interpolated function is the Morita gamma function.
**Definition 6.1.2** (Morita Gamma Function). *This is the p-adically continuous function*

\[ \Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p \]

*that extends \( f(n) := (-1)^n n!_p \) for \( n \geq 2 \).*

This is quite an attractive function to try and view from a nonstandard perspective. Indeed, define for all \( n \in \mathbb{N}^* \) with \( n \geq 2 \)

\[ \ast \Gamma_p : \mathbb{N}^* \to \mathbb{Z}, \quad \ast \Gamma_p(n) = (-1)^n \prod_{1 \leq j < n, p \nmid n} j, \]

with \( \ast \Gamma(0) = -\ast \Gamma(1) = 1 \). By applying Wilson’s theorem in a nonstandard setting this hyper function is \( p \)-adically \( Q \)-continuous. It also satisfies the functional equation

\[ \ast \Gamma_p(n + 1) = \ast h_p(n) \ast \Gamma_p(n), \]

where

\[ \ast h_p(n) = \begin{cases} -n & p \nmid n, \\ -1 & p \nmid n. \end{cases} \]

As the function lies in \( \ast \mathbb{Z} \) the \( p \)-adic shadow map can be taken leading to

\[ \text{sh}_p(\ast \Gamma_p) : \mathbb{Z}_p \to \mathbb{Z}_p, \]

and it satisfies \( \text{sh}_p(\ast \Gamma_p(n)) = \Gamma_p(n) \) for all \( n \in \mathbb{N} \). By the properties of the shadow map \( \text{sh}_p(\ast \Gamma_p) \) is \( p \)-adically continuous. Thus as these two \( p \)-adically continuous functions agree on a dense subset of \( \mathbb{Z}_p \) they must be equal. Alternatively one can take this construction as the definition of the Morita gamma function. So one sees that the set \( \mathbb{N}^* \setminus \mathbb{N} \) determines the non natural \( p \)-adic values of the Morita gamma function.

**Lemma 6.1.3.**

\[ \text{sh}_p(\ast \Gamma_p) = \Gamma_p. \]

It can easily be checked that the \( p \)-adic shadow map preserves all the properties of \( \text{sh}_p(\ast \Gamma_p) \). For example taking the \( p \)-adic shadow map of \( \ast h_p(x) \) leads to \( h_p(x) \) where

\[ h_p(x) = \begin{cases} -x & x \in \mathbb{Z}_p^\times, \\ -1 & x \in p\mathbb{Z}_p, \end{cases} \]

and \( \Gamma_p(x + 1) = h_p(x) \Gamma(x) \) for all \( x \in \mathbb{Z}_p \). In many ways it is a lot easier to work with a function in the integers or hyper integers than in \( \ast \mathbb{Z}_p \).
6.2 The Kubota-Leopoldt Zeta Function

For many years the value of the Riemann zeta function at integer points has attracted many mathematicians. In particular the value at negative integers lead to work in the $p$-adic area and the discovery of the $p$-adic Riemann zeta function of Kubota and Leopoldt in 1964 ([Kub-L]).

The value of $\zeta_Q(s)$ at negative integers was originally derived by Euler using divergent series. These values are related to the Bernoulli numbers ($B_n$) which are given by

$$\frac{t}{\exp(t) - 1} = \sum_{k \in \mathbb{N}} B_k \frac{t^k}{k!}.$$

**Lemma 6.2.1.** For all $k \in \mathbb{N}$,

$$\zeta_Q(-k) = (-1)^k \frac{B_{k+1}}{k+1}.$$ 

In particular as $B_k = 0$ for $k > 1$ and odd, ($k \in \mathbb{N}$)

$$\zeta_Q(-1 - k) = -\frac{B_{k+2}}{k+2}.$$

The idea of $p$-adic interpolation is obtain a $p$-adic function which has similar properties to the original real or complex function. The aim is to learn more about the original function by recasting the problem in the $p$-adic world which can often be simpler. In the case of the Riemann zeta function the interpolation is based on its values at negative integers. In order for the resulting $p$-adic function to be $p$-adically continuous the Riemann zeta function is modified by removing the $p$-Euler factor. Let

$$\zeta_{Q,p}(s) = (1 - p^{-s})\zeta_Q(s),$$

then for all $k \in \mathbb{N}$

$$\zeta_{Q,p}(-k) = -(1 - p^k) \frac{B_{k+1}}{k+1}.$$ 

It should be noted than the factor $(-1)^k$ is not needed because when $k = 0$ the $p$-Euler factor is equal to zero and so the modified zeta function is also zero.

There are now several ways to develop the $p$-adic Riemann zeta function; Kubota-Leopoldt, Iwasawa theory, and Mazur measures and $p$-adic integration. Each $p$-adic zeta function has $p - 1$ branches. Initially define for $n \in \mathbb{N} \setminus \{0\}$

$$\zeta_{p}(1 - n) = (1 - p^{n-1})\zeta_{Q,p}(1 - n) = -(1 - p^{n-1}) \frac{B_n}{n}.$$

The $p$-adic zeta function interpolates these numbers.
**Definition 6.2.2.** Let \( p \geq 5 \) be a prime and fix \( s_0 \in \{0, 1, \ldots, p - 2\} \). Then define the \( p \)-adic zeta function by

\[
\zeta_{p,s_0} : \mathbb{Z}_p \to \mathbb{Q}_p,
\]

\[
s \mapsto \lim_{t_\alpha} \zeta_p(1 - (s_0 + (p - 1)t_\alpha)).
\]

Here \( s \) is a \( p \)-adic integer with \( \{t_\alpha\}_{\alpha \geq 1} \) a sequence of natural numbers which \( p \)-adically converges to \( s \). The case \( s_0 = 0 = s \) is excluded and dealt with below.

It should be remarked that the \( p \)-adic zeta function \( \zeta_{p,s_0}(s) \) interpolates the zeta function \( \zeta_{\mathbb{Q},p}(s) \) at negative integer values \( s \) by

\[
\zeta_{p,s_0}(s) = \zeta_p(1 - n),
\]

where \( n \equiv s \pmod{p - 1} \) and \( n = s_0 + s(p - 1) \). The continuity properties are deduced from the Kummer congruences.

**Theorem 6.2.3.** Suppose \( i \in \mathbb{N}, m \geq 2 \) and \( p \) a prime with \( (p - 1) \nmid j \). if \( i \equiv j \pmod{p^m(p - 1)} \) then

\[
(1 - p^{i-1}) \frac{B_i}{i} \equiv (1 - p^{j-1}) \frac{B_j}{j} \pmod{p^{N+1}}.
\]

For some fixed \( s_0 \) let \( s, t \in \mathbb{N} \setminus \{0\} \) with \( s \equiv t \pmod{p^N} \) then for some \( k \in \mathbb{N} \) \( s = t + kp^N \). Now let \( i = s_0 + s(p - 1) \) and \( j = s_0 + t(p - 1) = s_0 + s(p - 1) + kp^N(p - 1) \) then \( i \equiv j \pmod{p^N(p - 1)} \) and by the Kummer congruences

\[
(1 - p^{i-1}) \frac{B_i}{i} \equiv (1 - p^{j-1}) \frac{B_j}{j} \pmod{p^{N+1}}.
\]

Hence,

\[
\zeta_{p,s_0}(s) \equiv \zeta_{p,s_0}(t) \pmod{p^{N+1}},
\]

and \( \zeta_{p,s_0} \) is uniformly continuous on \( \mathbb{Z}_p \) since \( \mathbb{N} \setminus \{0\} \) is dense in \( \mathbb{Z}_p \). This also proves the uniqueness of the \( p \)-adic zeta function by the interpolation property.

One should note that in the case when \( s_0 \in \{1, 3, \ldots, p - 2\} \) (the odd congruence classes) gives the zero function since for such \( s_0, B_{s_0+(p-1)k} = 0 \). This only leaves the even congruence classes and the case \( s_0 = 0 \). The latter case will enable the zeta function to be defined for \( p = 2 \) and \( p = 3 \) since the only congruence class is \( s_0 = 0 \).

Let \( s_0 = 0 \) then for \( s \neq 0 \) definition 6.2.2 can be applied. For \( s = 0 \)

\[
\zeta_{p,0} : \mathbb{Z}_p \to \mathbb{Q}_p, \quad \zeta_{p,0}(0) = \lim_{t_\alpha \to 0} (\zeta_p(1 - (p - 1)t_\alpha) = \zeta_p(1).
\]

This can be viewed as a pole of the \( p \)-adic zeta function as in the case of the Riemann zeta function. This can be seen more clearly in terms of \( p \)-adic integration.
6.2.1 A Nonstandard Construction

**Definition 6.2.4.** \( \zeta_{\mathbb{Q}}(s) = \sum_{n \in \mathbb{N}\setminus\{0\}} n^{-s}. \)

1. The hyper Bernoulli numbers are defined by \( \frac{t}{\exp(t)-1} = \sum_{k \in \mathbb{N}^*} \frac{B_k}{k!} t^k. \)

**Lemma 6.2.5.** For \( k \in \mathbb{N}^* \) and \( k > 1 \)

\[ \zeta_{\mathbb{Q}}(1-k) = \frac{B_k}{k}. \]

The proof of this result can be derived in an identical manner to the classical case. In a similar manner to the classical construction of the previous section the \( p \)-Euler factor can be removed for some fixed standard prime, \( p \). Define

\[ \zeta_{\mathbb{Q},p}(s) = (1 - p^{-s}) \zeta_{\mathbb{Q}}, \]

and

\[ \zeta_{\mathbb{Q},p}(1-k) = -(1 - p^{k-1}) \frac{B_k}{k}. \]

Now define a nonstandard function.

\[ *f : *\mathbb{N} \to *\mathbb{Q}, \]

\[ k \mapsto -(1 - p^k) \frac{B_{k+1}}{k+1} = \zeta_{\mathbb{Q},p}(-1 - k). \]

This function can be used for the \( p \)-adic continuation of the Riemann zeta function as in the standard case. Indeed let \( \sigma_0 \in \{-1,0,1,2,\ldots,p-3\} \) and \( \sigma \in *\mathbb{N} \). Let \( p \geq 5 \) and \( \sigma_0 \neq -1 \)

\[ *f_{\sigma_0} : *\mathbb{N} \to *\mathbb{Q}, \]

\[ \sigma \mapsto -(1 - p^{\sigma_0+\sigma(p-1)}) \frac{B_{\sigma_0+\sigma(p-1)+1}}{\sigma_0 + \sigma(p-1) + 1} = \zeta_{\mathbb{Q},p}(-1 - \sigma_0 - \sigma(p-1)). \]

For \( \sigma_0 \) even the value of the Bernoulli number is zero for all \( \sigma \in *\mathbb{N} \). So only odd \( \sigma_0 \) needs to be considered. This function is \( p \)-adically continuous for a fixed \( \sigma_0 \). Indeed let \( \sigma \equiv \tau \pmod{p^N} \), \( \tau = \sigma + kp^N \) (for some \( k,N \in *\mathbb{N} \).) The Kummer congruences also carry through in the nonstandard setting and imply \( *f_{\sigma_0}(\sigma) \equiv *f_{\sigma_0}(\tau) \pmod{p^{N+1}} \). The Kummer congruences also imply \( |*B_k/k|_p \leq 1 \) for \( p-1 \mid k \) and so \( *f_{\sigma_0}(\sigma) \in *\mathbb{Q}_{\text{limp}} \) for all \( \sigma \in *\mathbb{N} \). This enables the \( p \)-adic shadow map to be taken. Using work in Goldblatt ([Go], chapter 18)

\[ \text{sh}_p(*\mathbb{N}) = \mathbb{Z}_p. \]

Thus

\[ \text{sh}_p(*f_{\sigma_0}(\sigma)) = f_{p,\sigma_0}(\text{sh}_p(\sigma)) : \mathbb{Z}_p \to \mathbb{Q}_p, \]

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with
\[ f_{p,\sigma_0}(n) = -(1 - p^{\sigma_0+n(p-1)}) \frac{B_{\sigma_0+n(p-1)+1}}{\sigma_0 + n(p - 1) + 1}, \forall n \in \mathbb{N}. \]

The \( p \)-adic shadow map preserves continuity so \( f_{p,\sigma_0} \) is \( p \)-adically continuous on \( \mathbb{Z}_p \). Moreover
\[ f_{p,\sigma_0}(n) = \zeta_{p,\sigma_0+1}(n), \forall n \in \mathbb{N}. \]

As these continuous functions agree on a dense set in \( \mathbb{Z}_p \)
\[ f_{p,\sigma_0} = \zeta_{p,s_0} \]
where \( s_0 = \sigma_0 + 1 \).

This leaves the case \( \sigma_0 = -1 \). For \( \sigma \neq 0 \) the same definition of the nonstandard function above can be used to define \( *f_{-1} \) for all primes. So let \( r = n - 1 \) then for all \( r \in \ast \mathbb{N} \), \( *f_{-1}(r) = -(1 - p^{-1+(r+1)(p-1)})B_{(r+1)(p-1)}(r + 1)(p - 1) \). From classical results transferred to the nonstandard setting this function is \( p \)-adically continuous and \( |*f_{-1}(r)|_p \leq 1 \) for all \( r \in \ast \mathbb{N} \) thus the shadow map can be taken and a function is obtained
\[ \text{sh}_p(*f_{-1}(\sigma)) = f_{p,-1}(\text{sh}_p(\sigma)): \mathbb{Z}_p \to \mathbb{Q}_p, \]

with
\[ f_{p,-1}(n) = -(1 - p^{-1+n(p-1)}) \frac{B_{n(p-1)}}{n(p - 1)}, \forall n \in \mathbb{N} \setminus \{0\}. \]

Then \( f_{p,-1}(n) = \zeta_{p,0}(n) \) for all \( n \in \mathbb{N} \setminus \{0\} \). As these functions agree are on a dense set in \( \mathbb{Z}_p \) and are continuous then they are equal, proving the following theorem.

**Theorem 6.2.6.** For a fixed \( \sigma_0 \in \{-1, 1, 3, \ldots p-3\} \)
\[ \text{sh}_p(*f_{\sigma_0}(\sigma)) = \zeta_{p,\sigma_0+1}(\text{sh}_p(\sigma)). \]

### 6.3 Double Interpolation

The ideas and techniques behind \( p \)-adic interpolation have been discussed in various parts throughout this work. The previous sections show that some forms of \( p \)-adic interpolation can be viewed from a nonstandard perspective. This interpretation lends itself to extending interpolation. In particular the possibility of interpolating a function or set of numbers with certain properties with respect to more than one prime. Initially the case of interpolating with respect to two distinct primes \( p \)
and \( q \) will be considered. The range of functions which can be interpolated is reduced because of conditions required with respect to two primes rather than a single prime.

In the standard world double interpolation does not seem possible. The main reason being that an interpolated function would have to have a domain consisting of \( \mathbb{Z}_p \) and \( \mathbb{Z}_q \) and possibly a range of values in \( \mathbb{Q}_p \) and \( \mathbb{Q}_q \). It appears very difficult to achieve this because the fields \( \mathbb{Q}_p \) and \( \mathbb{Q}_q \) are not isomorphic. In a nonstandard setting these problems are removed since one can just consider the spaces \( {}^*\mathbb{N} \) (or \( {}^*\mathbb{Z} \)) and \( {}^*\mathbb{Q} \). This is the first restriction on functions which can be interpolated, one has functions of the form \( f : \mathbb{N} \to \mathbb{Q} \) and not with values in \( \mathbb{Q}_p \) or \( \mathbb{Q}_q \) like standard interpolation. The next condition relates to the continuity. For any hope of double interpolation \( f \) must be uniformly continuous and bounded when considered in the \( p \)-adic and \( q \)-adic valuation. This ensures any interpolated function is a continuous extension of \( f \). Given these conditions one constructs the nonstandard extension of \( f \), \( {}^*f : {}^*\mathbb{N} \to {}^*\mathbb{Q} \). Since the original function was assumed to be bounded in both valuations the actual function is

\[
{}^*f : {}^*\mathbb{N} \to {}^*\mathbb{Q}^{\lim_p} \cap {}^*\mathbb{Q}^{\lim_q}.
\]

This also enables the shadow maps to be taken leading to a \( p \)-adic function \( f_p : \mathbb{Z}_p \to \mathbb{Q}_p \) and a \( q \)-adic function \( f_q : \mathbb{Z}_q \to \mathbb{Q}_q \) These correspond to the single interpolation of \( f \). This can be displayed graphically

\[
\begin{array}{ccc}
& f & \\
f_p & \downarrow & f_q \\
& {}^*f & \nearrow
\end{array}
\]

Why may double interpolation be of any use? The main reason is the same as for single prime interpolation which is often specific to the problem being considered. In general the aim is to learn more about a function from the \( p \)-adic function which share some special values (though sometimes they are twisted). Often the \( p \)-adic function is easier to examine and so enables the original problem from various aspects. By using double interpolation one hopes that another aspect is added to the methods of looking at a problem.
CHAPTER 6: APPLICATIONS AND DOUBLE INTERPOLATION

6.4 Riemann Zeta Function

In this section attempts are made to double interpolate the Riemann zeta function. Ideally one would like to choose the set

\[ \{ (1 - p^m)(1 - q^m)\zeta_{Q(-m)} \}_{m \in \ast \mathbb{N}}. \]

This seems a sensible choice based on the nonstandard work on the Kubota-Leopoldt zeta function, by removing the \( p \) and \( q \) Euler factors. This also is symmetric with respect to the prime factors.

In the single prime case the continuity was deduced from the Kummer congruences. These can be extended slightly, in a symmetrical way to take account of the extra Euler factor. Recall the Kummer congruences

**Theorem 6.4.1.** If \( (p - 1) \nmid i \) and \( i \equiv j \pmod{p^n(p - 1)} \) then

\[ (1 - p^{i-1})B_i \equiv (1 - p^{j-1})B_j \pmod{p^{n+1}}. \]

As \( q \) is distinct from \( p \) and still assume the conditions in the theorem

\( p \nmid (q^{i-1} - 1) \) and so by Euler’s theorem

\[ q^{i-1} - 1 \equiv q^{j-1} - 1 \pmod{p^{n+1}}. \]

The one case when this is not true is when \( (p - 1)|(i - 1) \) and the congruence holds trivially.

Combining this observation with the Kummer congruences gives

**Corollary 6.4.2.** If \( (q, p) = 1, (p - 1) \nmid i \)

and \( i \equiv j \pmod{p^n(p - 1)} \) then

\[ (1 - q^{i-1})(1 - p^{i-1})B_i \equiv (1 - q^{j-1})(1 - p^{j-1})B_j \pmod{p^{n+1}}. \]

This can be made symmetrical by including further conditions on \( i \) and \( j \).

**Corollary 6.4.3.** If \( (q, p) = 1, (p - 1) \nmid i, (q - 1) \nmid i \)

, \( i \equiv j \pmod{q^n(q - 1)} \) and \( i \equiv j \pmod{p^n(p - 1)} \) then

\[ (1 - q^{i-1})(1 - p^{i-1})B_i \equiv (1 - q^{j-1})(1 - p^{j-1})B_j \pmod{p^{n+1}}, \]

and

\[ (1 - q^{i-1})(1 - p^{i-1})B_i \equiv (1 - q^{j-1})(1 - p^{j-1})B_j \pmod{q^{n+1}}, \]
As in the single prime case branches are considered. Firstly define

\[ f_{p,q} : \ast \mathbb{N} \to \ast \mathbb{Q}, \]

\[ k \mapsto -(1 - p^{n-1})(1 - q^{n-1}) \frac{B_n}{n}. \]

This function is not \( \mathbb{Q} \)-continuous with respect to either prime. Let \( p, q > 5 \) and without loss of generality assume \( p > q \) and let \( \sigma_0 \in \{-1, 0, 1, 2, \ldots (p - 1)(q - 1) - 2\} \). Due to the symmetry only the \( p \) case will be considered. Let \( \sigma_0 \notin \{-1, p - 1, 2(p - 1), \ldots (q - 2)(p - 1)\} \) then define

\[ f_{p,q,\sigma_0} : \ast \mathbb{N} \to \ast \mathbb{Q}, \]

\[ \sigma \mapsto -(1 - p^{\sigma_0 + \sigma(p-1)(q-1)})(1 - q^{\sigma_0 + \sigma(p-1)(q-1)}) \frac{B_{\sigma_0 + \sigma(p-1)(q-1) + 1}}{\sigma_0 + \sigma(p-1)(q-1) + 1}. \]

For fixed \( \sigma_0 \) as above this function is \( p \)-adically \( \mathbb{Q} \)-continuous. The extended Kummer congruences then give this result. The Kummer congruences also give \( \left| B_k/k \right|_p \leq 1 \) for \( (p - 1) \nmid k \) which shows the function actually lies in \( \ast \mathbb{Q}^{\lim_p} \). The shadow map leads to a \( p \)-adic function.

For \( \sigma_0 \in \{-1, 0, 1, 2, \ldots (p - 1)(q - 1) - 2\} \) with \( \sigma \neq 0 \) the same definition of the nonstandard function can be given. The case \( \sigma = 0 \) corresponds to a pole.

So with the double interpolated zeta function defined one also has new \( p \)-adic functions defined. The next section tries to gain information about these functions by using \( p \)-adic measures. As a final note on this section it is clear that this method generalises to a finite set of primes for interpolation.

### 6.4.1 Katz’s Theorem

Katz’s theorem on \( p \)-adic measures follows directly from the work on \( p \)-adic interpolation by Mahler. A proof can be found in chapter 3 of [11].

**Definition 6.4.4.** A \( \mathbb{Q}_p \)-linear map \( \phi : C_p(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p \) (where \( C_p(\mathbb{Z}_p, \mathbb{Q}_p) \) is defined in 5.4.2) is called a bounded \( p \)-adic measure if there exists a constant \( B \geq 0 \) such that \( |\phi(f)|_p \leq B|f|_p \) for all \( f \in C_p(\mathbb{Z}_p, \mathbb{Q}_p) \).

**Theorem 6.4.5.** Consider a bounded sequence of numbers \( \{b_n\} \) in \( \mathbb{Q}_p \). Let \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) be a \( p \)-adically continuous function and \( a_n(f) \) the Mahler coefficients. Then a uniquely bounded \( p \)-adic measure \( \phi \) can be defined by

\[ \int_{\mathbb{Z}_p} f d\phi = \sum_{n \in \mathbb{N}} b_n a_n(f). \]
Moreover the measure satisfies
\[ \int_{\mathbb{Z}_p} \left( \frac{x}{n} \right) d\phi = b_n, \]
for all \( n \in \mathbb{N} \). All bounded measures on \( \mathbb{Z}_p \) are obtained in this way.

**Corollary 6.4.6.** Each bounded \( p \)-adic measure having values in \( \mathbb{Q}_p \) is uniquely determined by its values at all monomials \( x^m \) \((m \in \mathbb{N})\).

One of the applications of this theorem is to develop the \( p \)-adic measure of the Riemann zeta function and again [Hi] is the reference.

**Theorem 6.4.7.** Let \( \zeta \) be the Riemann zeta function. Also let \( a \in \mathbb{N} \) with \( a \geq 2 \) and \((a,p) = 1\).
Then for all \( m \in \mathbb{N} \)
\[ (1 - a^{m+1})\zeta(-m) = \left( t \frac{d}{dt} \right)^m \Psi(t) \big|_{t=1}. \]
where \( \Psi(t) = (1 - t^a)^{-1} \sum_{b=1}^{\alpha} \xi(b)t^b \) and

\[ \xi : \mathbb{Z} \to \mathbb{Z}, \]
\[ n \mapsto \begin{cases} 1 & a \nmid n, \\ 1 - a & a \mid n. \end{cases} \tag{6.4.1} \]

**Corollary 6.4.8.** Let \( a \in \mathbb{N} \) with \( a \geq 2 \) and \((a,p) = 1\). Then there exists a unique \( p \)-adic measure \( \zeta_a \) on \( \mathbb{Z}_p \) such that
\[ \int_{\mathbb{Z}_p} x^m d\zeta_a = (1 - a^{m+1})\zeta(-m), \]
for all \( m \in \mathbb{N} \).

The central result in this section is generalizing theorem 6.4.7 and corollary 6.4.8 with the following results.

**Theorem 6.4.9.** Let \( a \in \mathbb{N} \) with \( a \geq 2 \) and \((a,p) = 1\). Also let \( r \in \mathbb{N} \), \((r,p) = 1\). Then for all \( m \in \mathbb{N} \)
\[ (1 - a^{m+1})r^m \zeta(-m) = \left( t \frac{d}{dt} \right)^m \Psi_r(t) \big|_{t=1}. \]
Here \( \Psi_r(t) = (1 - t^a)^{-1} \sum_{b=1}^{\alpha} \xi_r(b^r)t^{br} \) and
\[ \xi_r : \mathbb{Z} \rightarrow \mathbb{Z}, \]
\[ n \mapsto \begin{cases} 
0 & r \nmid n, \\
1 & r \mid n, ra \nmid n, \\
1 - a & r \mid n, ra \mid n.
\end{cases} \quad (6.4.2) \]

From this theorem 6.4.7 is a special case of the above when \( r = 1 \). To begin the proof the following short lemma is needed.

**Lemma 6.4.10.**
\[ \sum_{b=0}^{a-1} \xi_r(br) = \sum_{b=1}^{a} \xi_r(br) = 0. \]

**Proof:** From the definition, \( \xi_r(b) \neq 0 \) only for the multiples of \( r \). Hence the sum will consist of only \( a \) terms.
\[ \sum_{b=1}^{a} \xi_r(rb) = \xi_r(r) + \xi_r(2r) + \ldots + \xi_r((a-1)r) + \xi_r(ar). \]
Only the final term is divisible by \( ar \) thus the other \( a - 1 \) terms have value 1.
\[ \sum_{b=1}^{a} \xi_r(br) = 1 + 1 + \ldots + 1 + 1 + (a - 1) = 0. \]
The same method can be used to show the second equality as \( ra \mid 0 \).

**Proof of Theorem 6.4.9:** Consider the polynomial
\[ \Phi_r(t) = \frac{t^r + 1}{t^r - 1} - a \frac{t^{ar} + 1}{t^{ar} - 1}. \]
Using classical results the cotangent function satisfies
\[ \pi \cot(\pi z) = \frac{1}{z} - 2 \sum_{k \in \mathbb{N}} \zeta_Q(2k)z^{2k-1}. \]
Letting \( e(z) = \exp(2\pi iz) \) then
\[ \Phi_r(e(z)) = -(i\pi)^{-1} \sum_{k=1}^{\infty} 2(1 - a^{2k})\zeta_Q(2k)z^{2k-1}r^{2k-1}. \quad (6.4.3) \]
The expression for \( \Phi_r(t) \) can be rewritten using the \( \xi_r \) function.
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\[ \Phi_r(t) = \frac{(t^r + 1) \left( \frac{\bar{t}^{a-1}}{t^{a-1}} \right) - a(t^{a-1} + 1)}{t^{a-1}}, \]

\[ = \frac{(t^r + 1)(1 + t^r + t^{2r} + \ldots + t^{r(a-1)}) - a(t^{a-1} + 1)}{t^{a-1}}, \]

\[ = \frac{(1 - a) + t^{a-1}(1 - a) + 2(t^r + \ldots + t^{r(a-1)})}{t^{a-1}}, \]

Now let

\[ \Psi_r(t) = -2\sum_{b=1}^{a} \xi_r(br)(1 + t^r + \ldots + t^{r(b-1)}) = \frac{\sum_{b=1}^{a} \xi_r(br)t^{rb}}{1 - t^{a-1}}. \]

The last equality follows from lemma 6.4.10

\[ -2\Psi_r(t) = \Phi_r(t) - (a - 1). \quad (6.4.4) \]

The proof continues in two separate parts depending on whether \( m \) is odd or even.

For the even case use equation 6.4.3 and let \( v \in \mathbb{N} \) and \( v \geq 1 \).

\[ \left( \frac{d}{dz} \right)^{2v} \Psi_r(e(z)) = -\frac{1}{2} \left( \frac{d}{dz} \right)^{2v} \Phi_r(e(z)), \]

\[ = (2i\pi)^{-1} \sum_{k=1}^{\infty} 2(1 - a^{2k})\zeta_Q(2k)r^{2k-1}(2k - 1) \ldots (2k - 2v)z^{2k-2v-1}. \]

Therefore

\[ \left( \frac{d}{dz} \right)^{2v} \Psi_r(e(z)) |_{z=0} = 0. \]

By letting \( t = \exp(2\pi iz) \) gives

\[ \left( \frac{d}{dt} \right)^{2v} \Psi_r(t) |_{t=1} = 0. \]

Since \( \zeta_Q(s) \) has trivial zeros at \( s \in -2\mathbb{N}(s \neq 0) \) the theorem is proved for even \( m \).
For the odd case use equation 6.4.3 and let \( v \in \mathbb{N} \) and \( v \geq 1 \).

\[
-2 \left( \frac{d}{dz} \right)^{2v-1} \Psi_r(e(z)) = \left( \frac{d}{dz} \right)^{2v-1} \Phi_r(e(z)),
\]

\[
= -(i\pi)^{-1} \sum_{k=1}^{\infty} 2(1 - a^{2k}) \zeta_Q(2k)r^{2k-1}(2k-1) \cdots (2k-2v)z^{2k-2v}.
\]

Therefore

\[
-2 \left( \frac{d}{dz} \right)^{2v-1} \Psi_r(e(z)) \bigg|_{z=0} = -(i\pi)^{-1}(2v-1)2(1 - a^{2v}) \zeta_Q(2v)r^{2v-1}.
\]

Using the functional equation for \( \zeta_Q \)

\[
(1 - a^{2v}) \zeta_Q(1 - 2v) = (2\pi i)^{-2v}(2v-1)!2(1 - a^{2v}) \zeta_Q(2v),
\]

and substituting \( t = \exp(2\pi iz) \) gives

\[
\left( t \frac{d}{dt} \right)^{2v-1} \Psi_r(t) \bigg|_{t=1} = (1 - a^{2v})r^{2v-1}\zeta_Q(-(2v-1)).
\]

The theorem is then proved.

\[\square\]

An application of this theorem in combination with theorem 6.4.5 leads to a slightly modified \( p \)-adic measure, comparable to corollary 6.4.8 for the Riemann zeta function.

Lemma 6.4.11. Let \( a \in \mathbb{N} \) with \( a \geq 2 \) and \( (a, p) = 1 \). Then there exists a unique \( p \)-adic measure \( \zeta_{a,p} \) on \( \mathbb{Z}_p \) having values in \( \mathbb{Z}_p \) such that for all \( m \in \mathbb{N} \)

\[
\int_{\mathbb{Z}_p} x^m d\zeta_{a,p,q} = (1 - a^{m+1})(1 - q^m)\zeta_Q(-m).
\]

Proof: In the proof of theorem 6.4.7 Katz showed that

\[
(1 - a^{m+1})\zeta_Q(-m) = \left( t \frac{d}{dt} \right)^{m} \Psi(t) \bigg|_{t=1}.
\]

Let \( \binom{x}{n} = \sum_{k=0}^{n} c_{n,k}x^k \) with \( c_{n,k} \in \mathbb{Q} \). (These numbers are the Stirling numbers of the second kind divided by \( n! \).) The uniqueness of such a modified \( p \)-adic Riemann zeta function measure is guaranteed by corollary 6.4.6. The sequence of numbers \( \{(1 - a^{m+1})(1 - q^m)\zeta_Q(-m)\} \) is bounded in \( \mathbb{Q} \) because the classical case by Katz gives that \( |(1 - a^{m+1})\zeta_Q(-m)|_p \leq 1 \) and as
(1 − q^n) \in \mathbb{Z}, |(1 − q^n)|_p \leq 1. In order to show the existence of the measure it is required to show that \( \int_{\mathbb{Z}_p} \frac{x^n}{n} \, d\zeta_{a,p,q} \) is bounded.

\[
\left| \int_{\mathbb{Z}_p} \frac{x^n}{n} \, d\zeta_{a,p,q} \right|_p = \left| \sum_{m=0}^{n} c_{n,m} (1 - a^{m+1})(1 - q^m)\zeta_q(-m) \right|_p,
\]

\[
= \left| \sum_{m=0}^{n} c_{n,m} (1 - a^{m+1})\zeta_q(-m) \right|_p - \left| \sum_{m=0}^{n} c_{n,m} (1 - a^{m+1})q^m\zeta_q(-m) \right|_p,
\]

\[
\leq \max \left( \left| \sum_{m=0}^{n} c_{n,m} (1 - a^{m+1})\zeta_q(-m) \right|_p, \left| \sum_{m=0}^{n} c_{n,m} (1 - a^{m+1})q^m\zeta_q(-m) \right|_p \right),
\]

This last inequality follows from the classical result in theorem 6.4.7. So the proof now only requires the last term to be bounded. Let \( r \in \mathbb{N} \) and \((p, r) = 1\).

\[
\sum_{m=0}^{n} c_{n,m} (1 - a^{m+1})r^m\zeta_q(-m) = \sum_{m=0}^{n} c_{n,m} \left( \frac{t}{n} \right)^m \Psi_r(t) \big|_{t=1},
\]

\[
= \left( \frac{t}{n} \right)^m \Psi_r(t) \big|_{t=1},
\]

\[
:= \delta_n \Psi_r(t) \big|_{t=1}.
\]

Properties of the differential operator \( \delta_n \) are known via the classical results of Katz. The proofs can be found in [Hi].

**Lemma 6.4.12.**

\[
\delta_n = \frac{t^n}{n!} \frac{d^n}{dt^n}.
\]

**Lemma 6.4.13.** Let \( R' = \{ P(t)/Q(t) : P(t), Q(t) \in \mathbb{Z}_p[t], |Q(1)|_p = 1 \} \). Then \( R' \) is a ring and stable under the action of \( \delta_n \) for all \( n \in \mathbb{N} \).

From the definition of \( \Psi_r(t) \) it is apparent that \( \Psi_r(t) \in R' \) and by the previous lemma \( \delta_n \Psi_r(t) \in R' \). Therefore

\[
\delta_n \Psi_r = \frac{P_r}{Q_r},
\]

for \( P_r(t), Q_r(t) \in \mathbb{Z}_p[t] \) with \( |Q_r(1)|_p = 1 \). Thus \( P_r(1) \in \mathbb{Z}_p \) and for all \( r, n \in \mathbb{N} \)

\[
|\delta_n \Psi_r(t)|_{t=1} = \left| \frac{P_r(1)}{Q_r(1)} \right|_p \leq 1.
\]
Hence in the case \( r = q \)
\[
| \int_{\mathbb{Z}_p} \left( \frac{x}{n} \right) d\zeta_{a,p,q} |_p \leq 1.
\]

It should be noted that for the prime \( q \) the conditions in the corollary also prove that a \( q \)-adic measure \( \zeta_{a,q,p} \) exists.

\[
\int_{\mathbb{Z}_q} x^m d\zeta_{a,q,p} = (1 - a^{m+1})(1 - p^m)\zeta_{\mathbb{Q}}(-m).
\]  

(6.4.5)

**Single Interpolation**

Without loss of generality let the following analysis be completed with the prime \( p \) (as the \( q \) case is identical). The reason that the new \( p \)-adic measure has been constructed above is to try and find measures which corresponds to the single interpolation of the double interpolated set of numbers.

\[
\int_{\mathbb{Z}_p} x^m d\zeta_{a,p} = (1 - a^{m+1})(1 - p^m)\zeta_{\mathbb{Q}}(-m).
\]

**Lemma 6.4.14.**

\[
\int_{\mathbb{Z}_p^\times} x^m d\zeta_{a,p,q} = (1 - a^{m+1})(1 - p^m)(1 - q^m)\zeta_{\mathbb{Q}}(-m).
\]

In fact a slightly more general version will be proven.

**Theorem 6.4.15.** Let \( \phi \) be a locally constant function on \( \mathbb{Z}_p \) then

\[
\int_{\mathbb{Z}_p^\times} \phi(x) x^m d\zeta_{a,p,q} = \sum_{n > 1} ((\phi(n) - a^{m+1}\phi(na))n^m - (\phi(nq) - a^{m+1}\phi(naq))n^m q^m).
\]

**Proof:** As \( \phi \) is locally constant it is going to be constant on the classes modulo \( p^k \) for some \( k \in \mathbb{N} \).

Let \( F \in R' \) and define \( [\phi](F) \) by the Fourier inversion formula

\[
[\phi](F)(t) = \frac{1}{p^k} \sum_{b \pmod{p^k}} \phi(b) \sum_{\zeta p^k = 1} \zeta^{-b} F(\zeta t).
\]

Then by \([K\alpha], \) page 85, \( [\phi](F) \in R' \) and

\[
\int_{\mathbb{Z}_p} f(x) d\mu_F = \int_{\mathbb{Z}_p} f(x) d\mu_{[\phi]} F,
\]

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where for $F \in R'$, $d\mu_F$ is the measure associated to $F$ via the work above. In particular it has been shown above that associated to the function $\Psi_r(t) \in R'$ is the measure $\zeta_{a,r}$ satisfying

$$\int_{Z_p} x^m d\zeta_{a,r} = (1 - a^{m+1})r^m \zeta(-m).$$

$$\int_{Z_p} \phi(x)x^m d\zeta_{a,r} = \int_{Z_p} \phi(x)x^m d\mu_{\Psi_r},$$

$$= \int_{Z_p} x^m d\mu_{[\phi]\Psi_r},$$

$$= \left( t \frac{d}{dt} \right)^m \left( [\phi]\Psi_r \right)_{t=1}.$$  

Recall that $\Psi_r(t) = (\sum_{n=1}^r \xi_r(nr)t^{rn})/(1 - ta^r) = \sum_{n=1}^{ar} \xi_r(n)t^n/(1 - t^{an})$ which can be rewritten as

$$\Psi_r(t) = \frac{\sum_{n=1}^{ar} \xi_r(n)t^n}{1 - t^{an}}.$$  

Then by simple calculation and using the definition of $\Psi_r$

$$[\phi]\Psi_r(t) = \frac{\sum_{n=1}^{ar} \phi(n)\xi_r(n)t^n}{1 - t^{an}},$$

$$= \sum_{n\geq 1} \phi(n)\xi_r t^n,$$

$$= \sum_{n\geq 1} (\phi(nr)t^{nr} - a\phi(na^r)t^{nar}).$$  

(6.4.6)

Then

$$\left( t \frac{d}{dt} \right)^m ([\phi]\Psi_r)_{t=1} = \sum_{n>1} \left( \phi(nr) - a^{m+1}\phi(na^r) \right) n^m r^m.$$  

Recall that

$$\int_{Z_p} x^m d\zeta_{a,p,q} = \left( t \frac{d}{dt} \right)^m (\Psi_1(t) - \Psi_q(t))_{t=1},$$

then

$$\int_{Z_p} \phi(x)x^m d\zeta_{a,p,q} = \left( t \frac{d}{dt} \right)^m ([\phi]\Psi_1(t) - [\phi]\Psi_q(t))_{t=1},$$

$$= \sum_{n>1} \left( (\phi(n) - a^{m+1}\phi(na))n^m - (\phi(nq) - a^{m+1}\phi(naq))n^mq^m \right).$$
In particular choosing $\phi = \chi_{\mathbb{Z}_p^\times}$ (the characteristic function of $\mathbb{Z}_p^\times$). Then as $(a, p) = (q, p) = 1$, $\phi(n) = \phi(naq) = \phi(na)$ and the above theorem reduces to

$$\int_{\mathbb{Z}_p^\times} x^m \chi_{\mathbb{Z}_p^\times} x^m d\zeta_{a,p,q} = \int_{\mathbb{Z}_p^\times} x^m d\zeta_{a,p,q},$$

$$= (1 - p^m)(1 - q^m)(1 - a^{m+1})\zeta_Q(-m).$$

**Definition 6.4.16.** Let $p, q$ be finite distinct primes in $\mathbb{Q}$ and for all $k \in \mathbb{N} \setminus \{1\}$

$$\zeta_{p,q}(1 - k) = (1 - p^{k-1})(1 - q^{k-1})\zeta_Q(1 - k).$$

By the corollary above

$$\zeta_{p,q}(1 - k) = \frac{1}{1 - a^k} \int_{\mathbb{Z}_p^\times} x^{k-1} d\zeta_{a,p,q}.$$

It should be noted that the expression on the right does not depend on $a$. Suppose $b \neq a$ with $(b, p) = (b, q) = 1$ and $b \geq 2$. Then by the definition

$$\frac{1}{1 - b^k} \int_{\mathbb{Z}_p^\times} x^{k-1} d\zeta_{b,p,q} = \frac{1}{1 - a^k} \int_{\mathbb{Z}_p^\times} x^{k-1} d\zeta_{a,p,q},$$

since both equal $(1 - p^{k-1})(1 - q^{k-1})\zeta_Q(1 - k)$.

**Definition 6.4.17.** Fix $s_0 \in \{0, 1, 2, \ldots, p - 2\}$. For $s \in \mathbb{Z}_p$ ($s \neq 0$ if $s_0 = 0$) define

$$\zeta_{p,q,s_0}(s) = \frac{1}{1 - a^{s_0 + (p-1)s}} \int_{\mathbb{Z}_p^\times} x^{s_0 + (p-1)s-1} d\zeta_{a,p,q}.$$

If $k \in \mathbb{N}$ congruent to $s_0$ (mod $p - 1$) ($k = s_0 + (p-1)k_0$) then $\zeta_{p,q}(1 - k) = \zeta_{p,q,s_0}(k_0)$. In the case $s = 0$ when $s_0 = 0$ the denominator vanished. Using the above equality $\zeta_{p,q}(1 - k) = \zeta_{p,q,s_0}(k_0)$ it follows that the excluded case corresponds to $\zeta_{p,q}(1)$ and so has a pole at $s = 1$.

**Lemma 6.4.18.** For $p, q$ fixed and $s_0$ as above the function $\zeta_{p,q,s_0}$ is a continuous function of $s$ that does not depend upon the choice of $a \in \mathbb{N}$ with $a \geq 2$ and $(a, p) = (a, q) = 1$.

**Proof:** The continuity is clear from standard $p$-adic analysis. The independence of $a$ is established as follows. Let $b \in \mathbb{N}$ with $b \geq 2$ and $(b, p) = (b, q) = 1$ then the two functions
\[
\frac{1}{1 - a^{s_0 + (p-1)s}} \int_{\mathbb{Z}_p} x^{s_0 + (p-1)s - 1} d\zeta_{a,p,q}
\]

and

\[
\frac{1}{1 - b^{s_0 + (p-1)s}} \int_{\mathbb{Z}_p} x^{s_0 + (p-1)s - 1} d\zeta_{b,p,q}
\]

agree whenever \(s_0 + (p-1)s = k\) is an integer greater than zero since in both cases the value of the function is \((1 - p^{k-1})(1 - q^{k-1})\zeta_Q(1 - k)\). So both functions agree on the set of non-negative integers which is dense in \(\mathbb{Z}_p\) and hence are equal.

\[\square\]

This is of course the function obtained by the shadow map of the double interpolation function.

Questions

Initial attempts at proving theorem 6.4.11 tried to use the proof of theorem 6.4.5 directly. Instead of introducing the new power series \(\Psi_r(t)\) the power series \(\Psi_1(t)\) was used as in the work of Katz but introducing a different differential operator. So in trying to prove the boundedness of \(\int_{\mathbb{Z}_p} \frac{f(x)}{n} d\zeta_{a,r}\)

the integral was rewritten

\[
\int_{\mathbb{Z}_p} \frac{f(x)}{n} d\zeta_{a,r} = \sum_{m=0}^{n} c_{n,m} r^m (1 - a^{m+1}) \zeta_Q(-m),
\]

\[
= \sum_{m=0}^{n} c_{n,m} \left( rt \frac{d}{dt} \right)^m \Psi_1(t)|_{t=1},
\]

\[:= \Delta_n \Psi_1(t)|_{t=1}.\]

By the proof of lemma 6.4.11 it is known that \(|\Delta_n \Psi_1(t)|_{t=1}|_p \leq 1\).

Questions:

- Does there exist a proof of lemma 6.4.11 more \(p\)-adic in nature? That is, following directly from the proof of theorem 6.4.5 instead of having to work via power series evaluated at \(t = 1\).

- Does there exist a closed form for \(\Delta_n\) for all \(n \in \mathbb{N}\)?

- Is \(R'\) stable under \(\Delta_n\) for all \(n \in \mathbb{N}\)?
The Measure on Sets

There are several ways to view a \( p \)-adic measure. It has been seen above that one way to define a measure is via a bounded sequence. Two other methods include a definition by power series in \( \mathbb{Z}_p[[t]] \) and a definition by the open sets of \( \mathbb{Z}_p \). The power series can be easily deduced from the bounded sequence method (for the details see section 3.5 of [Hi]). Of potentially more interest is the definition via the open sets. For example corollary 6.4.8 establishes the existence of a \( p \)-adic measure for the Riemann zeta function. Work by Mazur establishes the existence of this measure from the definition of open sets via the Bernoulli distributions.

Recall that in the \( p \)-adic topology all sets of the form \( a + p^N \mathbb{Z}_p \) are both open and closed. An integration theory can be developed over such sets, see chapter 2 of [K]. An equivalent definition of a \( p \)-adic measure can be established via sets and distributions.

**Definition 6.4.19.** Suppose \( X \) is a compact-open subset of \( \mathbb{Q}_p \). A \( p \)-adic distribution, \( \mu \), on \( X \) is an additive map from the set of compact-open sets in \( X \) to \( \mathbb{Q}_p \). Thus, if \( U \subset X \) is a disjoint union of compact-open sets \( U_1, \ldots, U_n \), then

\[
\mu(U) = \mu(U_1) + \cdots + \mu(U_n).
\]

**Definition 6.4.20.** A distribution \( \mu \) of \( X \) is called a measure if there is a constant \( B \) such that

\[
|\mu(U)|_p \leq B,
\]

for all compact-open \( U \subset X \).

\( \mathbb{Q}_p \) has a basis of open sets consisting of all sets of the form \( a + p^N \mathbb{Z}_p \) for \( a \in \mathbb{Q}_p \) and \( N \in \mathbb{Z} \). Thus any open set of \( \mathbb{Q}_p \) is a union of open subsets of this type. Hence any distribution, and measure, is determined by the values on such sets, see chapter 7 of [RM] or chapter 2 of [K] for information on this and \( p \)-adic integration.

The aim of this section is to find a method which establishes the action of a \( p \)-adic measure on open sets of \( \mathbb{Z}_p \) directly from a measure defined by a bounded sequence. This action is unique from classical results on \( p \)-adic measures. I have such a method but it is entirely numerical in nature and although gives a solution it does so only one set at a time rather than give a general form of the measure.

From above the open sets of \( \mathbb{Z}_p \) are of the form \( b + p^n \mathbb{Z}_p \) with \( n \in \mathbb{N} \) and \( 0 \leq b < p^n \). Define a characteristic function of an open set.
Definition 6.4.21. Let \( b + p^n \mathbb{Z}_p \) be an open set of \( \mathbb{Z}_p \) then its characteristic function is given by

\[
\chi_{b+p^n\mathbb{Z}_p}(x) = \begin{cases} 
1 & x \in b + \mathbb{Z}_p, \\
0 & x \notin b + \mathbb{Z}_p.
\end{cases}
\]

Then by the definition of the \( p \)-adic integral with respect to a \( p \)-adic measure \( \mu \)

\[
\mu(b + p^n \mathbb{Z}_p) = \int_{b+p^n\mathbb{Z}_p} d\mu = \int_{\mathbb{Z}_p} \chi_{b+p^n\mathbb{Z}_p}(x) d\mu.
\]

By defining a \( p \)-adic measure by an the integral of \( x^k \) for \( k \in \mathbb{N} \) enables the integral of \( p \)-adic polynomials and power series using classical results. Indeed let \( f(x) = \sum_{k \in \mathbb{N}} a_k(f) \binom{x}{k} \) be a continuous function on \( \mathbb{Z}_p \) in terms of its Mahler series. Let \( \{d_k\} \) be a bounded sequence used to define a \( p \)-adic measure \( \mu \) with \( \int_{\mathbb{Z}_p} x^k d\mu = d_k \) for all \( k \in \mathbb{N} \). As in the first section let \( \binom{x}{k} = \sum_{m=0}^{k} c_{k,m} x^m \).

\[
\int_{\mathbb{Z}_p} f(x) d\mu = \int_{\mathbb{Z}_p} \sum_{k \in \mathbb{N}} a_k(f) \binom{x}{k} d\mu,
\]  
\[
= \sum_{k \in \mathbb{N}} a_k(f) \left( \int_{\mathbb{Z}_p} \binom{x}{k} d\mu \right),
\]  
\[
= \sum_{k \in \mathbb{N}} a_k(f) d_k.
\] (6.4.7)

The middle equality follows from classical results due to the Mahler series converging (section 5, [R]). Here \( d_k \) is also a bounded sequence by theorem 6.4.5.

\[
d_k = \int_{\mathbb{Z}_p} \binom{x}{k} d\mu = \sum_{m=0}^{k} c_{k,m} d_k.
\]

It should be noted that equation 6.4.7 converges since by Mahler’s theorem \( |a_k(f)|_p \to 0 \) as \( k \to \infty \) and \( d_k \) is bounded.

Returning to the characteristic function it is continuous and has a Mahler series

\[
\chi_{b+p^n\mathbb{Z}_p}(x) = \sum_{k \in \mathbb{N}} a_k(b, n) \binom{x}{k}.
\]

Therefore

\[
\mu(b + p^n \mathbb{Z}_p) = \int_{\mathbb{Z}_p} \chi_{b+p^n\mathbb{Z}_p}(x) d\mu = \sum_{k \in \mathbb{N}} a_k(b, n) d_k.
\] (6.4.8)
This expression enables \( \mu(b + p^n\mathbb{Z}_p) \) to be numerically calculated for all open sets of \( \mathbb{Z}_p \). By some initial calculations it appears that in general extensive calculations are needed to calculate the sequence \( d_k \) and then the coefficients \( a_k(b, n) \). Unless either one or both of these sets of numbers take a closed form or particularly simple form then this method does not appear greatly practical.

As an example suppose one only knew that the Riemann zeta function measure was defined by

\[
\int_{\mathbb{Z}_p} x^k d\zeta_a = (1 - a^{k+1})\zeta_Q(-k).
\]

Could it then be deduced that using the method outlined above results in

\[
\zeta_a(b + p^n\mathbb{Z}_p) = \frac{1}{a} \left[ \frac{ab}{p^n} \right] + \frac{(1/a) - 1}{2}.
\]

Here \([\cdot]\) is the integral part function. This now leaves a search for another method which, like the closed form for the Riemann zeta function measure, may specific to a problem rather than the general approach considered above.

The reason why such a process may be necessary is that it may be interesting to see if a closed form of \( \zeta_{a,p,q}(b + p^n\mathbb{Z}_p) \) and related measures exists and in particular if they involves any Bernoulli polynomials.

### 6.5 A Double Morita Gamma Function

An extension of the Morita gamma function is the double Morita gamma function which is a \( p \)-adically and \( q \)-adically \( Q \)-continuous function in \( *\mathbb{Z} \). A naive attempt would be to define for odd distinct primes

\[
\Gamma_{p,q}(n) = \prod_{1 \leq j < n, p \nmid j, q \nmid j} j \quad (j \geq 2).
\]

The problem is that this is not continuous in either valuation. The reason is due to the generalized Wilson’s theorem not being applicable. Focussing on the prime \( p \); by removing elements divisible by \( q \) one does not always have an inverse for elements in \( (\mathbb{Z} \setminus p^n\mathbb{Z})^\times \) so the product of elements is not a unit in this group. So the search is for the double Morita gamma function of the form (for \( n \geq 2 \))

\[
\Gamma_{p,q}(n) = C(n) \prod_{1 \leq j < n, A(p,q,j)} j.
\]

Here \( A(p, q, j) \) is a set of conditions on \( j \) depending on \( p \) and \( q \). (As an example the naive attempt had \( A(p, q, j) = \{p \nmid j, q \nmid j\} \).) \( A(p, q, j) \) must contain the conditions \( p \nmid j \) and \( q \nmid j \) otherwise there can be no form of continuity.
Initial attempts at finding an appropriate $A(p, q, j)$ revolve around at trying to get some inverse residues for each $j$. So fix a representation of $\mathbb{Z}/n\mathbb{Z}$ to be $\{1, 2, \ldots, n\}$. To explain, as $p, q \nmid j$ each such $j$ is invertible in $\mathbb{Z}/p^r\mathbb{Z}$ ($r \in *\mathbb{N}$) and $\mathbb{Z}/q^s\mathbb{Z}$ ($s \in *\mathbb{N}$). The problem is that the inverse in $\mathbb{Z}/p^r\mathbb{Z}$ may be divisible by $q$ and so will not be in the product (similarly for the $p$-divisible case).

So really one wants to include only those $j$ such that for each $r, s \in *\mathbb{N}$, the inverse residue of $j$ in $\mathbb{Z}/p^r\mathbb{Z}$ is not divisible by $q$ and in $\mathbb{Z}/q^s\mathbb{Z}$ is not divisible by $p$. Let $S(p, q) \subset *\mathbb{N}$ be the set of such $j$. Then for all $r$ and $s$

$$\prod_{1 \leq j < p^r, j \in S(p, q)} j \equiv 1 \pmod{p^r} \quad \text{and} \quad \prod_{1 \leq j < q^s, j \in S(p, q)} j \equiv 1 \pmod{q^s}.$$

The immediate problem regards the elements in $S(p, q)$. Clearly $1 \in S(p, q)$ but are there any other?

It is also known that if there is another element then there are an infinite number of elements because of residue inverses for $j$ in all the rings $\mathbb{Z}/p^r\mathbb{Z}$ and $\mathbb{Z}/q^s\mathbb{Z}$. If it could be shown that $S(p, q) \equiv \{1\}$ for all $p$ and $q$ then the double Morita gamma function (and all finite sets of primes version) is trivial as the minimum conditions have been imposed to gain continuity. If the sets are not trivial then this would be a pleasant solution to an elementary number theory problem but would not lead to a solution of the double Morita gamma function directly because the resulting products may not be continuous.

### 6.6 Triviality

**Theorem 6.6.1.** For all distinct primes $p$ and $q$, $S(p, q) \equiv \{1\}$.

**Corollary 6.6.2.** The double Morita gamma function is identical to $1$.

**Proof:** Let $j$ be an integer greater than $1$ such that for every positive integer $r$ and $s$ the remainder of the inverse of $j \pmod{p^r}$ is prime to $p$ and $r$ and the remainder of the inverse of $j \pmod{q^s}$ is prime to $q$. Clearly $p \nmid j$ and $q \nmid j$. Then

$$1/j = a_0 + a_1p + a_2p^2 + \ldots \in \mathbb{Z}_p,$$

with $a_0 \in \{1, 2, \ldots, p - 1\}$ and for all $i > 0$, $a_0 \in \{0, 1, 2, \ldots, p - 1\}$.

It is well known that the sequence $a_0, a_1, \ldots$ is periodic so there exists an $n \in \mathbb{N}$ such that for all $i \geq 0$, $a_{i+n} = a_i$.

Define

$$A_{mn} = \sum_{r=0}^{m-1} a_rp^r.$$
Then by periodicity,

\[ A_{mn} = (1 + p^n + p^{2n} + \ldots + p^{n(m-1)})(a_0 + a_1 p + \ldots + a_{n-1} p^{n-1}), \]

and

\[ 1/j \equiv A_{mn} \pmod{p^{mn}}. \]

Let \( b \equiv p^n \pmod{q} \), then

\( (1 + p^n + \ldots + p^{n(m-1)}) \equiv 1 + b + b^2 + \ldots + b^{m-1} \pmod{q} \).

Suppose now that \( b = 1 \) then in the case \( m = q \),

\[ q|1 + p^n + \ldots p^{n(m-1)}, \]

and so is \( A_{qn} \), a contradiction.

In the remaining cases, \( b > 1 \), so for \( m = q - 1 \),

\[ 1 + b + b^2 + \ldots + b^{q-1} = \frac{b^q - 1}{b - 1}, \]

which is divisible by \( q \) by Euler’s theorem and hence so is \( A_{n(q-1)} \), a contradiction.

\[ \square \]

6.7 The Set \( S(p, q) \)

Before proving the result in theorem 6.6.1 I looked at the problem from a numerical perspective and I present some results below which although elementary I have not seen in the literature.

As an initial attempt to tackle the problem in theorem 6.6.1 consider the integer 2. The inverses of 2 in \( \mathbb{Z}/p^r\mathbb{Z} \) and \( \mathbb{Z}/q^s\mathbb{Z} \) are \( (p^r + 1)/2 \) and \( (q^s + 1)/2 \) respectively. The conditions for 2 not to lie in \( S(p, q) \) are \( q \mid (p^r + 1)/2 \) and \( p \mid (q^s + 1)/2 \) for some \( r, s \in \mathbb{N} \). As \( p \) and \( q \) are odd it follows that the conditions become \( p^r \equiv -1 \pmod{q} \) and \( q^s \equiv -1 \pmod{p} \) with \( r < q \) and \( s < p \). So for any odd primes \( p \) and \( q \) if one of the two congruences is soluble then \( 2 \notin S(p, q) \) - this is a necessary condition. Immediately one knows this is true when one of the primes has a primitive root of the other. This leaves the case when neither prime has the other as a primitive root.

To begin with choose a primitive root \( g \) for \( p \). Then there exists integers \( 1 \geq j(q), j(-1) < p \) such that \( g^{j(q)} \equiv q \pmod{p} \) and \( g^{j(-1)} \equiv -1 \pmod{p} \). Putting these together

\[ g^{j(q)s} \equiv g^{j(-1)} \pmod{p}, \]

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This final equation is just a linear congruence which can be solved for \( s \). Indeed let the order of \( q \) in \( p \) be \( 2u \) (the order has to be even for a solution to the necessary condition). Then by basic congruences \( q^u \equiv -1 \pmod{p} \) giving \( s = u \). An example of where it does not work is given by \( p = 5 \) and \( q = 27 \). Then \( 5^{27} \equiv 1 \pmod{109} \).

Considering just the classes modulo \( p^s \); if the necessary condition \( (p^s + 1)/2 \) is not satisfied then further conditions can be investigated. For example the inverse of \( (p^s + 1)/2 \) in \( \mathbb{Z}/p^s\mathbb{Z} \) also must not be divisible by \( q \) for \( s > r \) (the case \( s \leq r \) is already covered by the original conditions since \( (p^s + 1)/2 \equiv (p^{s′} + 1)/2 \pmod{p^s} \)). The problem of finding these inverses is given by the following lemma.

**Lemma 6.7.1.** Let \( s > r \) and let \( n \in \mathbb{N} \) be maximal in satisfying \( nr < s \) (that is \( (n + 1)r \geq s \)). Then the inverse, \( x \), of \( (p^r + 1)/2 \) in \( (\mathbb{Z}/p^s\mathbb{Z})^\times \) is given by

\[
x = 2((-1)^n p^{nr} + (-1)^{n-1} p^{(n-1)r} + \ldots - p^r + 1) + (1 - (-1)^n)(p^{s}/2),
\]

(if the leading term is negative \( n \) is odd) then \( p^s \) is added to put the inverse in the range \( 0 < x < p^s \).

**Proof:** As \( (p^r + 1)/2 \in (\mathbb{Z}/p^s\mathbb{Z})^\times \) it has an inverse \( x \) satisfying \( 0 < x < p^s \) and

\[
\frac{p^r + 1}{2} x \equiv 1 \pmod{p^s}.
\]

Let \( x = 2x_1 \) then

\[
(p^r + 1)x_1 \equiv 1 \pmod{p^s}.
\]

Now let \( x_1 = x_2 + 1 \) then

\[
p^r x_2 + x_2 + p^r \equiv 0 \pmod{p^s},
\]

which implies the existence of a \( k \in \mathbb{N} \) such that

\[
p^r x_2 + x_2 + p^r = kp^s. \tag{6.7.1}
\]

Viewing this modulo \( p^r \) shows that \( x_2 \) can be written as \( x_2 = p^r x_3 \) satisfying

\[
p^r x_3 + x_3 + 1 = kp^{s-r}. \tag{6.7.2}
\]

Reducing this equation modulo \( p^r \) leads to \( x_3 = -1 + x_4 p^r \) satisfying

\[
p^r x_4 + x_4 - 1 = kp^{s-2r}. \tag{6.7.3}
\]
This can be reduced modulo $p^r$ again to give $x_4 = 1 + x_5 p^r$ satisfying
\[ p^r x_5 + x_5 + 1 = k p^{s-3r}. \] (6.7.4)

This is of the same form as equation [6.7.2]. In total $n$ reductions modulo $p^r$ can be taken reducing to the equation
\[ p^r x_{n+2} + x_{n+2} + (-1)^{n-1} = k p^{s-nr}. \] (6.7.5)

and the solution is currently in the form
\[ x = 2(1 - p^r + p^{2r} + \ldots + (-1)^{mr} p^{mr} + \ldots + x^n p^{nr}). \]

As $s - nr < r$ the equation can only be reduced modulo $p^{s-nr}$. This gives a solution of the form
\[ x_n = (-1)^{nr} + x_{n+1} p^{s-nr}. \]

Therefore the solution is
\[ x = 2(1 - p^r + p^{2r} + \ldots + (-1)^{mr} p^{mr} + \ldots + (-1)^{n-1} p^{nr} x_{n+1} p^s), \]
\[ \equiv 2(1 - p^r + p^{2r} + \ldots + (-1)^{mr} p^{mr} + \ldots + (-1)^{n-1} p^{nr} p^s) \pmod{p^s}. \]

Induction can be trivially used to show that for any $n \in \mathbb{N}$,
\[ | \sum_{m=0}^{n} (-1)^{mr} p^{mr} | < p^{nr}. \]

As $p$ is an odd prime $2p^{mr} < p^{nr+1} \leq p^s$ and hence $p^{mr} < p^{sr}/2$. Therefore $0 < |x| < ps$ and it is clear that for even $n$, $x > 0$ and for odd $n$, $x < 0$ so the solution is given by $x + p^s$.

The general method of solving a linear congruence of the form $uv \equiv 1 \pmod{k}$ is by Euclid’s algorithm. One of the implications from the above lemma is that if one tried to calculate an inverse of $(p^r + 1)/2$ in $(\mathbb{Z}/p^s \mathbb{Z})^\times$ for specific $r$ and $p$ using Euclid’s algorithm then one knows the parity of the number of steps in the algorithm before the algorithm is even carried out, it simply is the parity of $n$.

One now has more necessary conditions for $2 \neq S(p, q)$. One looks at the $q$ integrality of the inverse found in the lemma (and similarly for the $q$ case and $p$-integrality). Of course these inverses have inverses in $(\mathbb{Z} \setminus p^r \mathbb{Z})^\times$ and so on. (For example more lemmas can be given to find these inverses such as the inverse of $2 - 2 - p^2$ is $(p^2 - 2p - 3)/4 + p^2 + (p + 1)/2 \pmod{p^3}$. Of course this process can be carried out for any integer not just 2. Hopefully one can now see how a chain of elements of $S(p, q)$ can be found given an element (not 1) in $S(p, q)$.)
An example can be given for \( p = 3 \). One then proceeds to find the necessary inverses in \((\mathbb{Z}/3^s\mathbb{Z})^\times\). Hence

\[
2 \rightarrow \{2\}_3 \rightarrow \{5\}_9 \rightarrow \{11, 14\}_{27} \rightarrow \{29, 41, 59, 65\}_{81} \rightarrow \{83, 86, 122, 146, 173, 176, 191, 221\}_{243} \rightarrow \ldots
\]

One immediately sees that the primes \( q = 5, 7, 11, 13, 17, 29, 41, 43, 59, 61, 73, 173, 183, 191 \) already occur in the inverses as a divisor thus implying that \( 2 \notin S(3, q) \) for those \( q \) above.

The above lemma \[6.7.1\] can be extended to other residue classes.

**Corollary 6.7.2.** Let \( l \mid (p^r + 1) \). Also let \( s > r \) and \( n \in \mathbb{N} \) be maximal in satisfying \( nr < s \) (that is \( (n + 1)r \geq s \)). Then the inverse, \( x \), of \( (p^r + 1)/l \) in \((\mathbb{Z} \setminus p^s\mathbb{Z})^\times\) is given by

\[
x = l((-1)^n p^{nr} + (-1)^{n-1} p^{(n-1)r} + \ldots - p^r + 1) + (1 - (-1)^n)(p^s/2),
\]

(if the leading term is negative \( n \) is odd then \( p^s \) is added to put the inverse in the range \( 0 < x < p^s \)).

**Proof:** Let \( x = n(x_1 + 1) \) then the condition for finding the inverse becomes

\[
px_1 + x_1 + p = kp^s,
\]

which is the same as equation \[6.7.1\] \( \Box \)

The most general form of the lemma is given by the following.

**Corollary 6.7.3.** Let \( m, r, t, v, n, s \in \mathbb{N} \) with \( v \mid (mp^r + t) \). Let \( s > r \) and \( n \) be maximal in satisfying \( nr < s \). Then the inverse, \( x \), of \( (mp^r + t)/v \) in \((\mathbb{Z} \setminus p^s\mathbb{Z})^\times\) is given by

\[
x = v(t_s - t_s^2 mp^r + \ldots (-1)^{t_s^{l+1}}(mp^r)^l + \ldots + (-1)^n t_s^{n+1}(mp^r)^n) + (1 - (-1)^n)(p^s/2),
\]

where \( t_s \in \{1, 2, \ldots, p^s - 1\} \) satisfies \( tt_s \equiv 1 \mod p^s \).

The proof is identical in nature to the above. The only problem with this corollary is the almost circular argument used for finding the inverse of \( t \) in \((\mathbb{Z} \setminus p^s\mathbb{Z})^\times\). In the lemma and first corollary this inverse is simple to find and so the result is useful. The advantage is that for fixed \( s \) and \( t \) the inverse \( t_s \) can be found using Euclid’s algorithm and the second corollary provides a useful result for finding other inverses having only used Euclid’s algorithm once.

One can even investigate ways of finding \( t_s \) simply or more generally the inverse of a \( j \in (\mathbb{Z}/n\mathbb{Z})^\times \). The inverse of \( j \) is going to be of the form \((yn + 1)/j\). For example if \( n \equiv -1 \mod j \) then \( y = 1 \).
or if \( n \equiv 1 \pmod{j} \) then \( y = j - 1 \). The value of \( y \) depends on \( n \pmod{j} \). Indeed one needs \( yn \equiv -1 \pmod{j} \).

For the more general integer \( j \neq 1, 2 \) it is more difficult because in general an inverse residue to \( j \) cannot just be written down like it was done for the case \( j = 2 \) above, different cases modulo \( j \) have to be examined.

### 6.8 A Universal \( p \)-adic Function

The Riemann zeta function provided the first example of double interpolation and in fact of interpolation with respect to a finite set of primes. This next example shows that interpolation can take place with respect to all primes. For the Riemann zeta function this was not possible because removing all the Euler factors would result in something trivial.

#### 6.8.1 Translated Ideals

The open sets of \( \mathbb{Z}_p \) are an essential part of \( p \)-adic analysis (for example in some forms of \( p \)-adic measures and integration). These open sets are translated ideals of \( \mathbb{Z}_p \); \( p^n \mathbb{Z}_p \) for \( n \in \mathbb{N} \). Let

\[
\mathcal{S}_p = \{ u + p^v \mathbb{Z}_p : v \in \mathbb{N} \cup \{ \infty \}, 0 \leq u < p^v \}.
\]

(The point at infinity accounts for the trivial ideal.) To find a nonstandard version of these ideals in \( \ast \mathbb{Z} \) one studies its translated ideals. The ideals of \( \ast \mathbb{Z} \) are of the form \( m \ast \mathbb{Z} \) for \( m \in \ast \mathbb{N} \). In analogy with \( \mathcal{S}_p \) define

\[
\mathcal{S} = \{ a + m \ast \mathbb{Z} : m \in \ast \mathbb{N}, 0 \leq a < m \}.
\]

There is a natural connection between these two sets and that is via the \( p \)-adic shadow map restricted to \( \ast \mathbb{Z} \) (in this case it is the same as the map described by [Gv], chapter 16). It is defined by

\[
\text{sh}_p : \ast \mathbb{Z} \to \mathbb{Z}_p, \quad x \mapsto (x \mod p, x \mod p^2, \ldots).
\]

It is a homomorphism and surjective, along with other properties.

**Lemma 6.8.1.** Let \( m \ast \mathbb{Z} \) be an ideal of \( \ast \mathbb{Z} \) then \( \text{sh}_p(m \ast \mathbb{Z}) \) is an ideal of \( \mathbb{Z}_p \).

**Proof:** Using properties of the \( p \)-adic shadow map \( \ast \mathbb{Z}^{\lim p} = \ast \mathbb{Z} \) which means \( \text{sh}_p \) is defined for all \( \ast \mathbb{Z} \). Therefore it is defined on all sets of \( \ast \mathbb{Z} \). Now the basic definition of an ideal is used. Let \( I_m = \text{sh}_p(m \ast \mathbb{Z}) \subset \mathbb{Z}_p \).
Firstly, \( \forall r, s \in m^*\mathbb{Z}, r + s \in m^*\mathbb{Z} \). Let \( a = \sh_p(r) \in I_m \) and \( b = \sh_p(s) \in I_m \). Also let \( t = r + s \) and \( c = \sh_p(t) \in I_m \). Then using the homomorphism property of the \( p \)-adic shadow map,

\[
a + b = \sh_p(r) + \sh_p(s) = \sh_p(r + s) = c \in I_m
\]

Secondly, \( \forall r \in m^*\mathbb{Z}, \forall w \in m^*\mathbb{Z}, wr \in m^*\mathbb{Z} \). Let \( a = \sh_p(r) \in I_m \) and \( l = \sh_p(w) \in \mathbb{Z}_p \). Also let \( t = wr \) and \( h = \sh_p(t) \in I_m \). Then using the homomorphism property of the \( p \)-adic shadow map,

\[
la = \sh_p(l) \sh_p(r) = \sh_p(lr) = h \in I_m
\]

Therefore \( I_m \) is an ideal of \( \mathbb{Z}_p \) for all \( m \in \mathbb{N}^* \). So for some \( n(m) \in \mathbb{N} \),

\[
\sh_p(m^*\mathbb{Z}) = p^{n(m)}\mathbb{Z}_p.
\]

\[\Box\]

**Corollary 6.8.2.** Let \( a + m^*\mathbb{Z} \in S \) then \( \sh_p(a + m^*\mathbb{Z}) \in S_p \).

**Proof:** As \( a \in \mathbb{N}^* \), \( \sh_p(a) \in \mathbb{Z}_p \). So one can write

\[
\sh_p(a) = \sum_{k \in \mathbb{N}} a_k p^k.
\]

Then

\[
\sh_p(a + m^*\mathbb{Z}) = \sh_p(a) + \sh_p(m^*\mathbb{Z}) = \sh_p(a) + p^{n(m)}\mathbb{Z}_p, = (a_0 + a_1 p + \ldots + a_{n(m)} - 1 p^{n(m) - 1} + p^{n(m)}(a_{n(m)} + \ldots)) + p^{n(m)}\mathbb{Z}_p, = (a_0 + a_1 p + \ldots + a_{n(m)} - 1 p^{n(m) - 1}) + p^{n(m)}\mathbb{Z}_p, = b + p^{n(m)}\mathbb{Z}_p \in S_p,
\]

where \( b = a_0 + a_1 p + \ldots + a_{n(m)} - 1 p^{n(m) - 1} \in \mathbb{N} \), in particular \( 0 \leq b < p^{n(m)} \).

\[\Box\]

**Lemma 6.8.3.** Let \( m^*\mathbb{Z} \) be an ideal of \( \mathbb{N}^* \). Let \( m = p^r e \) with \( r \in \mathbb{N}, e \in \mathbb{N}^* \) and \( (e, p) = 1 \). Then,

\[
\sh_p(m^*\mathbb{Z}) = \begin{cases} 
p^r\mathbb{Z}_p & m \in \mathbb{N}, r \in \mathbb{N}, 
p^\infty\mathbb{Z}_p = 0 & m \in \mathbb{N}, r \in \mathbb{N} \setminus \mathbb{N}.
\end{cases}
\]
Proof: Let \( \text{sh}_p(m^*\mathbb{Z}) = p^{n(m)}\mathbb{Z}_p \). Firstly suppose \( r \in \mathbb{N} \setminus \mathbb{N} \). Then for \( \forall r \in \mathbb{N}, p^r | m \). Then for \( x \in m^*\mathbb{Z}, \)

\[
\text{sh}_p : x \mapsto (0 \mod p, 0 \mod p^2, \ldots) = 0.
\]

Thus \( n(m) = \infty \).

In the second case suppose \( r \in \mathbb{N} \). This is then split into two further cases. Suppose \( r = 0 \) then \((m, p) = 1\) and there exists solutions to equations of the form \( km + p^t s = 1 \) with \( t \in \mathbb{N} \setminus 0 \) and \( s, m \in \mathbb{Z} \). This implies there exist solutions to \( m \equiv u \mod p^t \) with \( 1 \leq u < p^t \). So for \( x \in m^*\mathbb{Z} \) it can be written as \( x = my \) with \( y \in m^*\mathbb{Z} \). Under the shadow map \( y \) maps to an element in \( \mathbb{Z}_p \). However as \( m \equiv v \mod p^t \) \( (v \neq 0) \) it follows that \( \text{sh}_p(m) \in \mathbb{Z}_p^\times \). Hence \( \text{sh}_p(m^*\mathbb{Z}) \in \mathbb{Z}_p \).

In the second subcase suppose \((r > 0)\). Then \( x \equiv 0 \mod p^s \) for \( 0 < s \leq r \). So for \( x \in m^*\mathbb{Z}, \)

\[
\text{sh}_p(x) = (0 \mod p, \ldots, 0 \mod p^r, x \mod p^{r+1}, \ldots).
\]

Therefore \( \text{sh}_p(x) \in p^r\mathbb{Z}_p \).

\[ \square \]

Corollary 6.8.4. Consider \( a + m^*\mathbb{Z} \in S \) then

\[
\text{sh}_p(a + m^*\mathbb{Z}) = \begin{cases} \ b + p^{\text{ord}_p(m)}\mathbb{Z}_p & \text{ord}_p(m) \in \mathbb{N}, \\ \text{sh}_p(a) & \text{ord}_p(m) \in \mathbb{N} \setminus \mathbb{N}. \end{cases}
\]

Here let \( \text{sh}_p(a) = \sum_{k \in \mathbb{N}} a_k p^k \) then \( b = \sum_{k=0}^{\text{ord}_p(m)-1} a_k p^k \).

6.8.2 Characteristic Functions

Choose a \( N \in \mathbb{N} \setminus \mathbb{N} \) and set \( \mathcal{P} = \prod_{p \leq N, p \text{ prime}} p \). Then this number in some ways acts as a generic finite rational prime. For example it has already been seen above that \( \text{sh}_p(\mathcal{P}^m\mathbb{Z}) = p^n\mathbb{Z}_p \) for all finite primes \( p \) and \( n \in \mathbb{N} \). By looking at translated ideals \( a + \mathcal{P}^m\mathbb{Z} \) under the shadow maps translated ideals in \( \mathbb{Z}_p \) are obtained. These ideas can be extended to look at characteristic functions. For example set \( \phi : \mathbb{Z} \to \{0, 1\} \) to be the characteristic function of \( \mathbb{Z} \). Taking the \( p \)-adic shadow map of this function leads to \( \phi_{\mathbb{Z}_p} \). Similarly defining a characteristic function on the ideal \( \mathcal{P}^m\mathbb{Z} \) leads to \( \phi_{\mathbb{Z}_p} \) and in the same manner for translated ideals.

6.8.3 The Universal Function

Classically the \( p \)-adic interpolation of the function \( n^s \) is performed to give a \( p \)-adic continuous function \( n \mapsto n^s \) with \( n \in 1 + p\mathbb{Z}_p \) and \( s \in \mathbb{Z} \). This is proved (in [Gv], 127–133) using the
binomial theorem. This function can be looked at in a nonstandard setting to obtain a nonstandard function for a fixed $n \in 1 + p^*\mathbb{Z}$

$$f : *\mathbb{Z} \to *\mathbb{Q}^{\text{lim}_p},$$

$$s \mapsto n^s.$$ 

This function is given explicitly by the $p$-adic convergent sum

$$n^s = (1 + (n - 1))^s = \sum_{k \in *\mathbb{N}} \binom{s}{k} (n - 1)^k.$$ 

By taking the $p$-adic shadow map one obtains the classical $p$-adic function.

At this point one wonders if double interpolation can take place. That is interpolation with respect to two finite distinct primes ($p$ and $q$). In the previous work this was done. An $n \equiv 1 \mod pq$ was fixed and the same binomial expansion gave a function which was $p$-adically and $q$-adically convergent. Moreover taking the respective shadow maps lead to the classical functions.

The next step is to consider interpolation with respect to all finite primes. This has to be carried out in a nonstandard space so one has the relevant congruence. For interpolation of $n^s$ with respect to each prime ($p$) it is required that

$$n \equiv 1 \mod p.$$ 

This means one needs to consider $\mathcal{P} \in *\mathbb{N} \setminus \mathbb{N}$ with $\mathcal{P} = \prod_{q \text{ prime}, q \leq M} q, (M \in *\mathbb{N} \setminus \mathbb{N})$. (It will become clear that one could consider any nonstandard $M$ as all that matters is that $\mathcal{P}$ is divisible by all standard primes.)

The 'local' information can then be gathered into a 'global' equivalence

$$n \equiv 1 \mod \mathcal{P}.$$ 

Equivalently

$$n \in 1 + \mathcal{P}^*\mathbb{Z}.$$ 

For such an $n$ define a function on $*\mathbb{Z}$ with values in $*\mathbb{Q}$ by

$$g_n(s) = n^s = (1 + (n - 1))^s = \sum_{k \in *\mathbb{N}} \binom{s}{k} (n - 1)^k.$$ 

**Lemma 6.8.5.** The function $g_n$ is $p$-adically uniformly continuous and convergent with respect to every finite prime $p$ in $\mathbb{Q}$. 

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**Proof: Continuity:** It is the case that \( n \equiv 1 \mod \mathcal{P} \) or equivalently there exists \( t \in \ast \mathbb{Z} \) such that 
\( n = 1 + \mathcal{P}t \). Using the binomial theorem on this last expression gives 
\( n^{m+1} \equiv 1 \mod \mathcal{P}^{m+1} \). Therefore \( n^{k+p^m} \equiv n^k \mod \mathcal{P}^{m+1} \). Continuity is then established since the last expression implies 
\( n^{k+p^m} \equiv n^k \mod \mathcal{P}^{m+1} \).

**Convergence:** It has to be shown that the series \( b_k = \binom{s}{k} (n - 1)^k \) is a null sequence in each \( p \)-adic norm. Since \( |(s)_{p}|_p \leq 1 \) for all \( s, k \in \ast \mathbb{Z} \) and \( p \) prime one needs only consider the coefficients \( a_k = (n - 1)^k \). For a prime \( p \),
\[
|a_k|_p = |n - 1|_p^k,
= |\mathcal{P}t|_p^k \text{ (for some } t \in \ast \mathbb{Z}),
= |\mathcal{P}/p|_p^k |pt|_p^k,
\leq p^{-k}.
\]
Thus \( a_k \) is a null sequence with respect to each prime.

**Lemma 6.8.6.** For all \( s \in \ast \mathbb{Z} \), \( g_n(s) \in \ast \mathbb{Q}^{\lim_p} \).

**Proof:** As \( n \in 1 + \mathcal{P} \ast \mathbb{Z} \), \( |n|_p \leq 1 \) for all \( p \). Raising to the power \( s \in \ast \mathbb{Z} \), \( |n|^s_p \leq 1^s = 1 \). So \( n^s \in \ast \mathbb{Q}^{\lim_p} \) for all \( p \).

This last lemma enables the \( p \)-adic shadow map to be taken for each prime. This is done by my work on nonstandard interpolation.

\[
\text{sh}_p(g_n(s)) = \text{sh}_p(n)^{\text{sh}_p(s)},
= \sum_{k \in \mathbb{N}} \binom{\text{sh}_p(s)}{k} (\text{sh}_p(n) - 1)^k.
\]
This is the classical \( p \)-adic function because the function is defined on all of \( \text{sh}_p(s) \in \mathbb{Z}_p \). Moreover using the results of section 3, the resulting function is defined on all of \( 1 + p\mathbb{Z}_p \) because \( \text{sh}_p(1 + \mathcal{P} \ast \mathbb{Z}) = 1 + p\mathbb{Z}_p \). At first glance this is quite surprising because in the nonstandard set \( (1 + \mathcal{P} \ast \mathbb{Z}) \) the only standard integer, in fact only standard number, is 1. Yet the respective shadow maps produce the required integers \( \mod p \) and \( p \)-adic integers to produce the ideals \( 1 + p\mathbb{Z}_p \).
Returning to the choice of $\mathcal{P}$. In the above proof the only point which matters is that $\mathcal{P} = e \times \prod_{q \text{ prime}, q \leq M} q$ with $(e, p) = 1$ for all finite primes. The fact that $e$ is $p$-integral for all $p$ enables the shadow map to map $\mathcal{P}^* \mathbb{Z}$ to $p\mathbb{Z}_p$ rather than $p^r \mathbb{Z}_p$ if some power of $p$ divided $e$. As was seen in section 3 all the shadow map ‘cares’ about is the $p$-part and with $e$ $p$-integral there is no effect on the final translated ideal. Similarly with the choice of $M$ it does not affect the shadow map.

In conclusion the nonstandard function $g_n$ acts as a source of the $p$-adic function $n^s$ for all finite prime $p$.

### 6.9 Double Hurwitz Zeta Function

This problem is more difficult for the Hurwitz zeta function because interpolation is based on a twisted Hurwitz zeta function. The actual numbers interpolated are explicitly dependent on the prime $p$ in the $p$-adic interpolation. So to find the correct numbers for double interpolation they have to depend explicitly on $p$ and $q$ which can be achieved by two potential methods.

- Define an analogue of the Teichmüller character which explicitly depends on $p$ and $q$.
- View the Teichmüller character as a Dirichlet character.

The second approach potentially leads onto the ideas of a double $L$-function but are not considered here.

#### 6.9.1 Nonstandard Teichmüller Character

Before trying to find a double version it would be prudent to find a nonstandard version. Let $^*\omega_p : (\mathbb{Z}/p^* \mathbb{Z})^\times \to ^*\mathbb{N}$ be defined for standard prime $p$ by

- $^*\omega_p(n) \equiv n \pmod{p}$,
- $^*\omega_p(n)^{p-1} \equiv p \pmod{1}$.

The first condition implies that there exists $t_n \in ^*\mathbb{N}$ such that $^*\omega_p(n) = n + t_n p$. Now the second condition can be used to determine the value(s) of $t_n$ or it will show that such a function does not exist. Using the binomial theorem,
A version of Hensel’s lemma in *Z cannot be used to obtain a value of \( t_n \) because of lack of completeness. Instead one looks at properties of the surjective \( p \)-adic shadow map (\( \text{sh}_p : *\mathbb{N} \to \mathbb{Z}_p \)). Then for each \( x \in \mathbb{Z}_p \) there exists a \( n_x \in *\mathbb{N} \) such that \( \text{sh}_p(n_x) = \omega_p(x) \). Moreover by the properties of the shadow map \( n_x \equiv x \pmod{p} \) and \( n_x^{p-1} \simeq_p 1 \). Therefore it satisfies the defining properties of the Teichmüller character. Using these observations one can define a (non-unique) nonstandard Teichmüller character \( *\omega_p : *\mathbb{N} \to *\mathbb{N} \) such that the shadow map is the standard Teichmüller character. In the case of \( p | n \) define \( *\omega_p(n) = 0 \). There is no unique nonstandard character because for any two such nonstandard characters the value taken by both for a given \( n \in *\mathbb{N} \) lie in the same monad.

### 6.9.2 Double Teichmüller Character

In searching for an initial interpretation of this function the following conditions should be included in the definition. Let the double Teichmüller character initially be a function defined

\[
*\omega_{p,q} : (*\mathbb{Z}/pq\mathbb{Z})^\times \to *\mathbb{N}.
\]

The obvious extension to \( *\mathbb{N} \) is given by \( *\omega_{p,q}(n) = 0 \) for \( p \nmid n \) or \( q \nmid n \) and \( *\omega_{p,q}(n) = *\omega_{p,q}(n \pmod{pq}) \) otherwise. From the definition of the Teichmüller character in its natural \( p \)-adic setting

\[
*\omega_{p,q} \text{ should satisfy for all } n \in (*\mathbb{Z}/pq\mathbb{Z})^\times
\]

- \( *\omega_{p,q}(n) \equiv n \pmod{p}, \)
- \( *\omega_{p,q}(n) \equiv n \pmod{q}, \)
- \( *\omega_{p,q}(n)^{p-1} \simeq_p 1, \)
- \( *\omega_{p,q}(n)^{q-1} \simeq_q 1. \)

Suppose such a function exists then the shadow map with respect to one of the primes will lead to a function which agrees with the respective Teichmüller character but is defined on a slightly smaller set due to the \( q \) part in the original definition.
A function satisfying the first two conditions is trivial to find, just set \( \omega_{p,q}(n) = n + kpq \) for some hyper natural number \( k \). The last two conditions are equivalent to \( \omega_{p,q}(n)^{p-1} = 1 + u_{p,n}p^{N_{p,n}} \) and \( \omega_{p,q}(n)^{q-1} = 1 + u_{q,n}q^{N_{q,n}} \) where \( N_{p,n}, N_{q,n} \in *\mathbb{N} \setminus \mathbb{N} \) and \( u_{p,n}, u_{q,n} \in *\mathbb{N} \). In order to apply the methods of finding a nonstandard character as in the previous section one needs to determine whether or not for each \( n \in *\mathbb{N} \) the following subset of \( *\mathbb{N} \) is empty
\[
\mu_p(\omega_p(n)) \cap \mu_q(\omega_q(n)).
\]
In the case that it is non-empty then a (non-)unique double nonstandard character could be defined by setting \( ^*\omega_{p,q}(n) \) to be equal to an element in the above subset. All the properties desired are then satisfied.

In solving this problem one could consider it in a slightly more general setting. Given any two distinct primes \( p \) and \( q \) let \( x_p \in \mathbb{Q}_p \) and let \( y_q \in \mathbb{Q}_q \). The problem is to determine whether or not the following set is empty
\[
S(x_p, y_q) = \mu_p(x_p) \cap \mu_q(y_q).
\]
The first case to consider is when \( x_p, y_q \in \mathbb{N} \) and are distinct. Elements \( N \in S(x_p, y_q) \) can be written in the form \( N = x_p + k_p p^{M_p} \) and \( N = y_q + k_q q^{M_q} \) for some \( k_p, k_q \in *\mathbb{N} \) and \( M_p, M_q \in *\mathbb{N} \setminus \mathbb{N} \). Without loss of generality suppose \( x_p > y_q \) and fix nonstandard values of \( M_p \) and \( M_q \). Then the problem reduces to solving the linear equation
\[
t = x_p - y_q = k_p p^{M_p} - k_q q^{M_q}.
\]
This is soluble because \( (p, q) = 1 | t \).

This process can be applied to the non-natural elements of \( \mathbb{Z}_p \) and \( \mathbb{Z}_q \). Indeed let \( x_p \in \mathbb{Z}_p \setminus \mathbb{N} \) and \( y_q \in \mathbb{Z}_q \setminus \mathbb{N} \). Then let \( U \in \mu_p(x_p) \) and \( V \in \mu_q(y_q) \), clearly \( U, V \in *\mathbb{N} \setminus \mathbb{N} \). If there exists some \( N \in S(x_p, y_q) \) then it can be written in the form \( N = S + k_p p^{M_p} \) and \( N = T + k_q q^{M_q} \) for some \( k_p, k_q \in *\mathbb{N} \) and \( M_p, M_q \in *\mathbb{N} \setminus \mathbb{N} \). As above a linear equation results and as \( S - T \in \mathbb{Z} \) one has basically the same problem as above and is soluble as \( (p, q) = 1 \).

This can then be used to find a double version of the \( p \)-adic function defined for \( x \in \mathbb{Z}_p^\times \): \( x >_p := x/\omega_p(x) \in 1 + \mathbb{Z}_p \). Then for \( n \in * \) with \( p, q \nmid n \)
\[
< n >_{p,q} := n/^*\omega_{p,q}(n).
\]
Then taking the shadow map for a given \( n \) leads to the \( p \)-adic or \( q \)-adic \( < \).>.
6.9.3 Hurwitz Zeta Function

Pick a nonstandard double Teichmüller character \( \ast \omega_{p,q} \) such that \( \ast \omega_{p,q}(1) = 1 \). The problem is that the nonstandard version cannot be viewed as a hyper Dirichlet character as it is not a multiplicative homomorphism. This is due to the condition on it not being an exact \( p - 1 \) root of unity just infinitely close to being one (similarly in the \( q \) case). It is the same for the single nonstandard Teichmüller character. So instead of considering a Dirichlet \( L \)-series consider the following \( L \)-function

\[
\ast L(s, \ast \omega_{p,q}) = \sum_{n \in \ast \mathbb{N}} \frac{\ast \omega_{p,q}(n)}{n^s}.
\]

Using the comparison test this \( Q \)-converges absolutely for at least \( \ast \Re(s) > 1 \). Using the hyper Hurwitz zeta function the above can be written as

\[
\ast L(s, \ast \omega_{p,q}) = (pq)^{-s} \sum_{r=1}^{pq} \ast \omega_{p,q}(r) \zeta_\ast \mathbb{Q}(s, r/pq).
\]

Q-analytic continuation of this \( L \)-function is then established via the Q-analytic continuation of the hyper Hurwitz zeta function. Therefore for \( n \in \ast \mathbb{N} \)

\[
\ast L(1 - n, \ast \omega_{p,q}) = -\frac{(pq)^{n-1}}{n} \sum_{r=1}^{pq} \ast \omega_{p,q}(r) \ast B_{n}(r/pq),
\]

where \( \ast B_{n}(x) \) is the \( n \)-th hyper Bernoulli polynomial. Thus

\[
\ast L(1 - n, \ast \omega_{p,q}) = -\frac{(pq)^{n-1}}{n} \sum_{r=1}^{pq} \ast \omega_{p,q}(r) \sum_{k \in \ast \mathbb{N}} \binom{n}{k} (r/pq)^{n-k} \ast B_k.
\]

Of particular importance is the Hurwitz zeta function and finding a double interpolation of it. As a first attempt consider the following function for \( n \in \ast \mathbb{N} \ \backslash \ \{0\} \), \( b, F \in \mathbb{N} \ \backslash \ \{0\} \), \( b < F \) and \( p, q \mid F \)

\[
\ast H_{p,q}(1 - n, b, F) = -\frac{1}{n} \frac{1}{F} \ast < b >_{p,q} \sum_{k \in \ast \mathbb{N}} \binom{n}{k} (F/b)^{k} \ast B_k.
\]

Since the sum is hyperfinite the actual values of this function lie in \( \ast \mathbb{Q} \). Moreover for all \( n \) the sum lies in \( \ast \mathbb{Q}^{\lim_{p}} \backslash \ast \mathbb{Q}^{\lim_{q}} \) by the von Staudt-Clausen theorem. By the properties developed above the shadow maps can be taken of \( < . >_{p,q} \) for \( p, q \mid b \). Since \( 1/F \in \mathbb{Q} \) the shadow map is trivial and so one is just left to deal with \( 1/n \). For the shadow map to be defined on this term it is required that \( |n|_{p} \) and \( |n|_{q} \) are not infinitesimal. (Under the shadow maps this last requirement corresponds to the pole of the \( p \)-adic or \( q \)-adic Hurwitz zeta function.) This suggests defining the \( p - q \)-adic Hurwitz zeta function for \( n \in -\ast \mathbb{N} \) as

\[
\ast \zeta_{p,q}(n, b, F) = -\frac{1}{1 - n} \frac{1}{F} \ast < b >_{p,q} \sum_{k \in \ast \mathbb{N}} \binom{1 - n}{k} (F/b)^{k} \ast B_k.
\]
Chapter 7

The Work of Shai Haran

7.1 Overview

The dictionary between arithmetic and geometry is a fascinating area of mathematics having been
developed by many great mathematicians from Kummer and Kronecker to Artin and Weil. The
analogy begins with $\mathbb{Z}$ in arithmetic and with $k[x]$ (the ring of polynomials in one variable over a
field $k$) in geometry. By choosing a prime $p$ of $\mathbb{Z}$ the $p$-adic integers $\mathbb{Z}_p = \lim \leftarrow \mathbb{Z}/p^n$ and field
of fractions $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$ are obtained. Similarly for a prime $f$ of $k[x]$ the geometric analogues
are $k_f[[f]] = \lim \leftarrow k[x]/f^n$ and the field of Laurent series $k_f((f)) = k_f[[f]][1/f]$. This dictionary
extends a lot further but there are two anomalies, that is two constructions in the geometric picture
which have no "obvious" analogue in the arithmetic picture.

1. To produce theorems in geometry the change is made from affine to projective geometry.
To the affine line the point at infinity is added which corresponds to the ring $k[[1/x]]$ and
its field of fractions $k((1/x))$. The analogue of $\infty$ for $\mathbb{Q}$ is the real prime (denoted by $\eta$).
The associated field is $\mathbb{Q}_\eta = \mathbb{R}$ but there is no analogue $\mathbb{Z}_\eta$ of $k[[1/x]]$. By following the
definition for finite primes $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$, $\mathbb{Z}_\eta' = [-1, 1]$ but this is not closed
under addition.

2. The second problem refers to tensor products. By taking the product of given geometrical
objects a new geometrical object is obtained. For example the affine plane $(\mathbb{A}^2)$ is the product
of the affine line $(\mathbb{A}^1)$ with itself. This corresponds to the tensor product of two polynomial
rings $k[x_1]$ and $k[x_2]$, in the category of $k$-algebras, and is equal to the ring of polynomials
in two variables, $k[x_1, x_2]$. Trying to find the corresponding arithmetical surface it is found
that in the category of commutative rings $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$ (the surface reduces to the diagonal).
In fact in any geometry based on rings the arithmetical surface reduces to the diagonal since \( \text{Spec}(\mathbb{Z}) \otimes \text{Spec}(\mathbb{Z}) = \text{Spec}(\mathbb{Z}) \). So does there exist a category in which absolute Decartes powers \( \text{Spec } \mathbb{Z} \ldots \times \text{Spec } \mathbb{Z} \) do not reduce to the diagonal?

Much of Shai Haran’s work has been trying to resolve these two issues. The bulk of this chapter will review Haran’s attempts to find an interpretation of the first point. Work in the direction of resolving the second point is still very much in its infancy and so only a small amount of space will be dedicated to the theory of non-additive geometry and a new language between arithmetic and geometry.

### 7.2 The Real Integers

Almost all Haran’s work in the search of the real integers is found in his book Mysteries of the Real Prime which in his own words is "very condense and hard to read". Only very recently have some lecture notes become available which expand on parts of his book. So the references for this work are the chapters 1–9 of his book [Ha1] and chapters 0–4 and 7 of these lecture notes.

- **Aims:** For the \( p \)-adic integers there are three important inverse limit expressions which rely upon reduction modulo \( p \).

\[
\mathbb{P}^1 = \lim_{\leftarrow N} \mathbb{P}^1(\mathbb{Z}/p^N\mathbb{Z}), \quad \mathbb{Z}_p = \lim_{\leftarrow N} \mathbb{Z}/p^N\mathbb{Z}, \quad \mathbb{Z}_p^* = \lim_{\leftarrow N} (\mathbb{Z}/p^N\mathbb{Z})^*.
\]

The aim is to understand the real analogue of these not by reduction of points but by "reduction" of certain complex valued functions.

- **Techniques:** Initial studies of Markov chains enable the above \( p \)-adic limits to be viewed as boundaries of certain trees. The \( q \)-world is used to construct chains which generalise the \( p \)-adic chains but in the limit \( q \to 0 \) the \( p \)-adic chains are recovered. Moreover in the limit \( q \to 1 \) a real chain is obtained. By interpreting the real chain as an analogue of the \( p \)-adic chain certain deductions are made regarding properties of the real prime.

- **Philosophy:** The historical tendency is to develop results in the \( p \)-adic setting from the corresponding ones in the real setting. This is based on the human perception of macroscopic 'reality' as many aspects of everyday life are based, to a good approximation, on the real numbers and measuring distances using the absolute value. Given early mathematics, engineering, physics, . . . were influenced by the physical world it is not surprising that the real
results far outnumber those of the $p$-adic numbers in the Platonic mathematical world. Often results in the $p$-adic setting are simpler and more natural than those in the real setting so by studying the reals from the $p$-adic perspective it is hoped to find new results.

### 7.2.1 Markov Chains

**Definition 7.2.1.** A Markov chain consists of a state space, a set $X$, and transition probabilities, a function $P_X : X \times X \to [0, 1]$ satisfying $P_X(x, X) = 1$ for all $x \in X$.

The source of $p$-adic chains for his work is derived from graph theory and to begin with trees. Indeed let $X$ be a tree with root $x_0 \in X$. For $n \in \mathbb{N}$ let $X_n = \{x \in X : d(x_0, x) = n\}$ then $X = \bigsqcup_{n \in \mathbb{N}} X_n$, a disjoint union. The boundary, $\delta X$, is defined as the inverse limit of the sets $X_n$. The important result relating to this is the following theorem.

**Theorem 7.2.2.** Let $M_1(\delta X)$ be the set of probability measures on the boundary. Then there is a one-to-one correspondence between $\tau \in M_1(\delta X)$ and Markov chains on $X$.

In the constructions used in the proof it is shown that there exists a special probability measure, the harmonic measure. From this probability measure it can be shown that it leads to a probability measure on $X_n$ and an associated Hilbert space $H_n = l_2(X_n, \tau_n)$. The key observation is that there is a unitary embedding $H_n \hookrightarrow H_{n+1}$ and an orthogonal projection from $H_{n+1}$ onto the subspace $H_n$. Similarly on the boundary there is the Hilbert space $H = l_2(\delta X, \tau)$, unitary embedding $H_n \hookrightarrow H$ and orthogonal projection $H$ onto $H_n$. These Hilbert spaces (and in particular the orthogonal bases) are probably the most important aspect of the work.

In order to deal with the real and $q$-chains the above theory has to be extended to chains which are not trees. One of the characterizations of a chain which is a tree is that the boundary is totally disconnected. Therefore non-tree chains also include ones which have continuous boundaries. The theory is based around harmonic functions. Given a chain with state space $X = \bigsqcup X_n$, $X_0 = \{x_0\}$ and transition probabilities $P_X$ then $P_X$ can be regarded as a matrix over $X \times X$ with entry 0 if two points are not connected. Further $P_X$ can be regarded as an operator on $l_\infty(X)$ by $P_X f(x) = \sum_{x' \in X} P_X(x, x') f(x')$.

**Definition 7.2.3.** A function $f : X \to [0, \infty)$ is called harmonic if $P_X f = f$ and $f(x_0) = 1$.

The collection of all harmonic functions is a convex set, denote this by Harm$(X)$. This decomposes in the standard way of a convex set: Harm$(X) = $ Harm$(X)_{\text{non-ext}} \cup$ Harm$(X)_{\text{ext}}$ where
CHAPTER 7: THE WORK OF SHAI HARAN

\[ \text{Harm}(X)_{\text{non-ext}} = \{ \lambda_0 f_0 + \lambda_1 f_1 : f_0, f_1 \in \text{Harm}(X), \lambda_0, \lambda_1 > 0, \lambda_0 + \lambda_1 = 1 \} \] and \( \text{Harm}(X)_{\text{ext}} \) be those functions which are not non-extreme. This leads to a decomposition of the boundary, \( \delta X = \delta X_{\text{ext}} \cup \delta X_{\text{non-ext}} \). Indeed the boundary is the compactification of \( X \) with respect to the Martin metric which itself is derived from the Martin kernel and the Green kernel. Here the Green kernel is an operator on \( X \times X \) given by \( G(x, y) = \sum_{m \in \mathbb{N}} (P_X)^m(x, y) \) and the Martin kernel is given by \( K(x, y) = G(x, y)/G(x_0, y) \).

**Theorem 7.2.4.** For the general Markov chain there is a one-to-one correspondence between the harmonic functions on \( X \) and the probability measures on \( \delta X_{\text{ext}} \).

In particular the constant function is clearly harmonic and the corresponding unique measure is called the harmonic measure.

**A \( p \)-adic Beta Chain**

The actual chain of considerable use is the non-symmetric \( p \)-adic \( \beta \) chain. The next stage is calculations. Given the chain the harmonic measure, boundary, Hilbert spaces, \ldots can all be calculated. They are lengthy calculations and so do not appear in an explicit way in his book though really to most readers they can just be accepted. To others they form an extensive set of exercises which would probably double the length of his book.

The symmetric \( \beta \) chain on \( \mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \) is very complicated so a chain is considered on the tree \( \mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p \) where \( \ltimes \) is a semidirect product.

In summary this chain has the state space \( X_n = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i + j = n\} \). The state space can be identified with \( \mathbb{N} \times \mathbb{N} \) by the following parametrization

\[ X_n \ni (i, j) \mapsto (1 : p^{n-j}) = (1 : p^i) \in \mathbb{P}^1(\mathbb{Z}/p^n)/(\mathbb{Z}/p^n)^* \ltimes (\mathbb{Z}/p^n). \]

The explicit identification with \( \mathbb{N} \times \mathbb{N} \) can be found in chapter 4 of [Ha1]. The transition probabilities are given for \( \alpha, \beta > 0 \) by

\[
P_X((i, j), (u, v)) = \begin{cases} 
\frac{1 - p^{-\beta}}{1 - p^{-\beta - \alpha}} & \text{if } (i, j) = (0, 0) \text{ and } (u, v) = (0, 1), \\
\frac{1 - p^{-\alpha - \beta}}{1 - p^{-\alpha - \beta}} & \text{if } (i, j) = (0, 0) \text{ and } (u, v) = (1, 0), \\
p^{-\beta} & j = v = 0, i \geq 1 \text{ and } u = i + 1, \\
1 - p^{-\beta} & j = 0, v = 1, i \geq 0 \text{ and } u = i, \\
1 & i \geq 0, u = i, j \geq 1 \text{ and } v = j + 1, \\
0 & \text{otherwise.}
\end{cases}
\]
The boundary is \( \mathbb{P}^1(\mathbb{Z}_p)/\mathbb{Z}_p^* \cong \mathbb{P}^1 \cup \{0\} \) with harmonic measure the gamma measure. This is defined as \( r_{\mathbb{Z}_p}^\beta = \phi_{\mathbb{Z}_p}(x)|x|_p^\beta \, dx / \zeta_p(\beta) \) where \( \zeta_p \) is the local component of the completed Riemann zeta function.

The key calculation is that of the orthogonal bases. On the boundary the basis for the associated Hilbert space, \( H = \bigoplus_{m \in \mathbb{N}} \mathbb{C}\phi_{p,m} \), is called the \( p \)-adic Jacobi basis (\( \{\phi_{p,m}\} \)). On the finite dimensional Hilbert spaces the basis of \( H_N = \bigoplus_{0 \leq m \leq N} \mathbb{C}\phi_{p,N,m} \) is given by the \( p \)-adic Hahn basis (\( \{\phi_{p,N,m}\} \)). The most important observation is reduction mod \( p^N \). This reduction takes place between the two bases using integration against the Martin kernel since the embeddings and projections mentioned above can be given in terms of the Martin kernel. In particular the projection, or what can be referred to as reduction mod \( p^N \) from \( H \) to \( H_N \) is given by

\[
K_{p,N}\phi_{p,m} = \begin{cases} 
\phi_{p,N,m} & 0 \leq m \leq N, \\
0 & N < m.
\end{cases}
\]

It is this interpretation which is sought initially in the \( q \) case and then the real case to give reduction of certain complex polynomials mod \( \eta^N \).

The initial problem encountered by Haran is finding the "correct" real chain as there are many candidates ranging from expanding a real number with some fixed base to the use of continued fractions. Haran uses the \( q \)-chains to justify his choice of real chain as the correct one.

A \( q \)-chain is a \( q \)-interpolation between a \( p \)-adic chain and the real chain. The \( q \) world is extensive in interpolating between the \( p \)-adic and real numbers. In this world the continuum \( \mathbb{Q}^*_\eta/\mathbb{Z}^*_\eta = \mathbb{R}^+ \) is approximated by \( q^\mathbb{Z} \) which resembles the \( p \)-adic \( \mathbb{Q}^*_p/\mathbb{Z}^*_p = p^\mathbb{Z} \). Recent work can be found in the papers [Ki1]-[Ki9], [KKuW], [KuW], [KuWY] and [KuOW]. Chapter 6 of [Ha1] provides a concise introduction to the \( q \)-world.

**The \( q \)-Beta Chain**

Let the state space be \( X_q = \mathbb{N} \times \mathbb{N} \) and define the transition probabilities for \( \alpha, \beta > 0 \) as

\[
P_{X_q}((i,j),(u,v)) = \begin{dcases} 
\frac{(1-q^{\beta+j})}{(1-q^{\alpha+\beta+j})} & u = i \text{ and } v = j + 1, \\
\frac{(1-q^{\beta+j})}{(1-q^{\alpha+\beta+j})} & u = i + 1 \text{ and } v = j.
\end{dcases}
\]

Clearly this is not a tree. As it is a \( q \)-chain (interpolating between the \( p \)-adic \( \beta \) chain and the real \textit{beta} chain) it has the special property that taking the \( p \)-adic limit \( (q = p^{-N}, \alpha := \alpha/N, \beta := \beta/N, N \to \infty) \) converges to the \( p \)-adic \( \beta \) chain introduced above. Moreover in the real limit \( (q := q^{2/N}, \alpha := \alpha/2, \beta := \beta/2, N \to \infty) \) a new chain is obtained and this is the real \( \beta \) chain.
The Real Beta Chain

The state space remains $\mathbb{N} \times \mathbb{N}$, $X_\eta = \sqcup_{n \in \mathbb{N}} X_{\eta(n)}$ with $X_{\eta(n)} = \{(i,j) : i + j = n\}$ and the transition probabilities become

$$P_{X_\eta}((i,j),(u,v)) = \begin{cases} \frac{\beta+2j}{\alpha+\beta+2(i+j)} & u = i \text{ and } v = j + 1, \\ \frac{\alpha+2i}{\alpha+\beta+2(i+j)} & u = i + 1 \text{ and } v = j. \end{cases}$$

The relevant calculations can be carried out and the resulting boundary is $\delta X_\eta = \mathbb{P}^1(\mathbb{R})/\{\pm 1\} = [0, \infty]$ and the harmonic measure on the boundary is the real beta measure with finite approximation $\zeta_\eta^{\alpha,\beta}(i,j) = (n!/(i!j!))(\zeta_\eta(\alpha + 2i, \beta + 2j)/\zeta_\eta(\alpha, \beta)\ (n = i + j \text{ and } \zeta_\eta(,) \text{ is the beta function}).$

The beta measure is defined as the product of two gamma measures on the real plane which is then projected onto $\mathbb{P}^1(\mathbb{R})$. Using these calculations the Hilbert spaces can be examined but the most important aspect is to find the relations (the ladder structure) between them analogous to the structure discussed above on trees.

Let the finite Hilbert spaces be denoted by $H^{\alpha,\beta}_{\eta(n)}$ and on the boundary by $H^{\alpha,\beta}_{\eta}$. Recall in the $p$-adic case the orthogonal bases were found by using the relations between the spaces. In this case integration against the Martin kernel does not lead to embeddings or projections. The theory relies on difference operators, which in his work are developed in relation to the $q$-beta chain and follow in the real case by taking the real limit. The difference operator $D_n : H^{\alpha,\beta}_{\eta(n)} \to H^{\alpha+2,\beta+2}_{\eta(n-1)}$ and its adjoint $D^{+}_n : H^{\alpha+2,\beta+2}_{\eta(n-1)} \to H^{\alpha,\beta}_{\eta(n)}$ are defined as

$$D_n \phi(i,j) = \left(\frac{\alpha+\beta}{2} + n\right)(\phi(i,j+1) - \phi(i+1,j)),$$

$$D^{+}_n \phi(i,j) = \left(\frac{\alpha+\beta}{2} + n\right)^{-1}(\phi(i,j+1) - \phi(i+1,j)) \phi(i,j-1) - i(\beta/2 + j)\phi(i-1,j)).$$

These operators satisfy the Heisenberg relation

$$D_n D^{+}_n - D^{+}_n D_n = (\frac{\alpha+\beta}{2}) (id)^{\alpha+2,\beta+2}_{H^{\alpha+2,\beta+2}_{\eta(n-1)}}.$$

This shows that a constant multiple of the identity operator is obtained for the difference in going down and up this "ladder". Moreover the constant function is characterised by the equation $D_n 1 = 0$ which means that $D_n$ can be regarded as an annihilation operator, $D^+$ a creation operation and the constant function as the vacuum. Using these observations the orthogonal basis $\varphi^{\alpha,\beta}_{\eta(n),m} = (-1)^m/m!(D^+)^m 1_{H^{\alpha+2m,\beta+2m}_{\eta(n-m)}}$, the real Hahn basis. This process is repeated for the boundary with difference operators, satisfying the Heisenberg relation, creating the ladder structure. Ultimately they lead to the real Jacobi basis $\varphi^{\alpha,\beta}_{\eta,m} = (-1)^m/m!(D^+)^m 1_{H^{\alpha+2m,\beta+2m}_{\eta}}$. The link between the
Hilbert space on the boundary and those on the finite sets is also defined via the Martin kernel as

$$K_{\eta(n)}^{\alpha,\beta}(i,j) = \int_{\delta X} K((i,j), x)\phi(x)\tau_{\eta}^{\alpha,\beta}(x),$$

where $K$ is the Martin kernel on the boundary. Then

$$K_{\eta(n)}^{\alpha,\beta}(\varphi) = \begin{cases} \varphi_{\eta}^{\alpha,\beta}, & \text{if } 0 \leq m \leq n, \\ 0, & \text{if } m > n. \end{cases}$$

This relates the real Jacobi and Hahn basis. The interpretation is "reduction mod $\eta^n"$, analogous to the $p$-adic case. So this gives an interpretation to the inverse limit for $\mathbb{P}^1(\mathbb{R})$ in terms of complex polynomials.

Hopefully the basic method can be seen in how an interpretation of complex polynomials can be given. In order to "view" the inverse limits for the other two problems suitable chains with boundary $\mathbb{Z}_p$ and $\mathbb{Z}_p^*$ are needed. In fact both chains can be deduced from the beta chain.

**Gamma Chain**

For the beta chain taking the limit $\alpha \to \infty$ leads to the gamma chain. Unfortunately in the real case this reduces to the unit shift walking from the origin to $0 \in \delta X_\eta$. However taking the appropriate normalized limit $\alpha \to \infty$ of the above operators $D$ and $D^+$ and of the real Jacobi polynomials leads to the Laguerre polynomials. This is an orthogonal basis for the boundary space $H^\beta_\eta = L^2(\mathbb{R}/\{\pm 1\}, \tau^\beta_\eta)$ where $\tau^\beta_\eta$ is the real gamma measure defined by $\tau^\beta_\eta/(x) = \exp(-\pi x^2)|x|^{\beta/2}d^*x/\Gamma(\beta/2)$. This can be also be derived from the $q$-beta chain by taking appropriate limits. Haran does not make clear the interpretation of this in terms of the real limit of the additive problem initially stated. A problem is that he does state that the real Laguerre basis should be considered as the interpretation in the introduction but this is not developed in the book at any stage. I believe part of the problem is the lack of a real chain, there are no non-trivial finite Hilbert spaces and hence no reduction type operator. Further what is the real chain with boundary $\mathbb{Z}_\eta$?

These thoughts were confirmed when reading the appendix in the lecture notes where as a problem he asks what is the real gamma chain and what is the finite real Laguerre basis?

**The Real Units**

In the $p$-adic case a non-trivial chain can be obtained with boundary $\mathbb{Z}_p^*$ by taking the limit $\alpha, \beta \to \infty$ of the $p$-adic beta chain. The harmonic measure on the boundary is just the multiplicative Haar measure on $\mathbb{Z}_p^*$ normalized by $d^*(\mathbb{Z}_p^*) = 1$. Since a real beta chain exists taking limits $\alpha, \beta \to \infty$
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should lead to a similar chain. Indeed it does with transition probabilities of 1/2 on the same state space as for the real beta chain. On this chain the Hilbert spaces exist with difference operators which lead to orthogonal bases. In this case the basis on the boundary is given by the real Hermite polynomials

\[ \varphi_{n,m}(w) = \exp(\pi w^2) \frac{(-1)^m}{m!} \left( \frac{\delta}{\delta w} \right)^m \exp(-\pi w^2). \]

On the finite layers the basis is essentially given by the elementary symmetric functions

\[ \varphi_{\eta,n,m} = \frac{(2\pi)^{m/2}}{(N(N-1)\ldots(N-m+1))^{1/2}} \sigma_{N,m}(1,\ldots,1,-1\ldots,-1), \]

where \( \sigma_{N,m} \) is the \( m \)-th elementary symmetric function of \( N \) variables. Again it is implied that the interpretation for \( \mathbb{Z}_\eta^n \) is the Hermite polynomials while for the approximations to \( (\mathbb{Z}/\eta^n\mathbb{Z})^* \) are given by the symmetric functions above. What is not made clear is the process of reduction \( \text{mod } \eta^n \) and I think this is another problem in need of resolution.

Higher Dimensional Theory

The theory developed above has a natural extension to global chains. By considering restricted direct products of chains one can obtain chains over the finite adeles. So far there is little to comment on in this area because all the properties of these global chains follow from the local chains. Of more interest is the theory of higher dimensions. In the one dimensional theory all the \( q \) chains of interest have been constructed from the products and semi-direct products of the basic chain. This has state space \( \mathbb{N} \) and transition probabilities given by

\[
P_{\beta}^{(i,j)} = \begin{cases} 
q^{\beta+i} & i = j, \\
1 - q^{\beta+i} & i + 1 = j, \\
0 & \text{otherwise.}
\end{cases}
\]

For example the \( q \)-gamma chain is constructed as the product of the basic chain \( P_{\beta}^{(q)} \) and the unit shift chain \( P_{\beta}^{(\infty)} \). The \( q \)-beta chains are also constructed from the non-stop (semi-direct) products of just two basic chains. The next step is to consider non-stop (semi-direct) products of \( r > 2 \) basic chains. This construction leads to higher dimensional beta chains.

7.2.2 Remarks

In many areas of mathematics there is often a deeper theory underlying a set of results. Following Tate’s thesis and the subsequent Langland’s program arithmetic problems are translated into
representation theory of adelic groups. This takes place in an elegant way in which although the computations are done in representation theory the flavour is algebraic geometry. At this stage the usual consequence is to study the local objects, whatever they are, and glue the results to the adeles (the method of Langlands). What Haran proposes is the alternative of whether the local objects can be put on an equal footing.

Using this approach Haran tries to attack the Riemann hypothesis by using the interpolating \( q \)-objects. (See the next section for full details.) The rest of his work searches for the "correct" local object which can capture representation theories of all local fields - a non-trivial question. Where does one begin this search? The fragments of information Haran uses are special functions which are only glimpses of representation theory. Haran’s process of using chains is one way of building these interpolating functions. The functions can be studied without this process and one can check properties and limits to show that they are matrix elements in representations of \( GL_n(F) \) for all local fields. An advantage of Haran’s process is that in defining some of the objects above he is able to give an interpretation of \( \text{mod} \eta^n \). Future work will show whether or not these interpretations are the correct ones.

### 7.3 Applications of The Quantum World

As mentioned in the previous section one of the main areas of Haran’s work has been in attacking the Riemann hypothesis using \( q \)-interpolating objects.

- **Aims:** To prove the Riemann hypothesis.
- **Techniques:** The work is based on \( q \)-interpolating objects. The particular objects of interest are the explicit sums of Weil and the Reisz potential.

#### 7.3.1 The Riemann Zeta Function

The completed zeta function is defined via the product of the local factors to give a global zeta function which is related to certain operators acting on the adeles, full details can be found in the opening of chapter 13 of [Ha1].

\[
\zeta_A(s) = \prod_{p \geq \eta} \zeta_p(s), \quad \Re(s) > 1.
\]

Here the product is over all primes including the real prime. The local factors are defined as \( \zeta_p(s) = (1 - p^{-s})^{-1} \) for the finite primes and \( \zeta_\eta(s) = \pi^{-s/2} \Gamma(s/2) \). Although the functional equation is
well known Haran views it from the perspective of the Heisenberg group to obtain

$$\zeta_A(1/2 + s) = \zeta_A(1/2 - s),$$

with simple poles at $\pm 1/2$ and residues $\pm 1$. There are numerous reformulations of the Riemann hypothesis and the main one Haran develops are the explicit sums.

For any $f \in C_c^\infty(\mathbb{R}^+)$ there exists a Mellin transform of it $\hat{f}(s) = \int_0^\infty f(a)a^sda$. Then using the residue theorem and properties of the completed zeta function

$$\sum_{\zeta_A(1/2+s)=0} \hat{f}(s) - \hat{f}(1/2) - \hat{f}(-1/2) = - \sum_{p \geq \eta} W_p(f),$$

where $W_p(f)$ is the Weil distribution defined by

$$W_p(f) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \hat{f}(s) d\log \zeta_p(1/2 + s) \zeta_p(1/2 - s).$$

For the finite primes there is a closed expression for the Weil distribution

$$W_p(f) = \log(p) \sum_{n \neq 0} p^{-|n|/2} f(p^n).$$

So essentially the sum of $\hat{f}(s)$ over the zeros of $\zeta_A(1/2 + s)$ is "equal" to the weighted sum of $f$ over the prime powers. Moreover for finite $p$, $f \geq 0$ implies $W_p(f) \geq 0$. The contribution of the real prime cannot be ignored. There are several ways to write the real Weil distribution in a finite closed form, for example

$$W_\eta(f) = \int_0^\infty \frac{f(a) - f(1)}{1 - \min(a, a^{-1})^2} \min(a, a^{-1})^{1/2}da + (\gamma + \pi/2 + \log(8\pi))f(1),$$

where $\gamma$ is the Euler constant. This formula is obtained by Haran by introducing $q$-Weil distributions and taking the real limit. The $p$-adic limit leads to the $p$-Weil distribution as expected. The importance of the Weil distributions is in relation to the Riemann hypothesis. The Mellin transform takes multiplicative convolutions of functions to multiplication of functions. Applying this to the explicit sums

$$\sum_{\zeta_A(1/2+s)=0} \hat{f}(s)\overline{\hat{f}(-\overline{s})} - 2\Re(\hat{f}(1/2)\overline{\hat{f}(-1/2)}) = - \sum_{p \geq \eta} W_p(f \circ f^*) ,$$

where $\circ$ is the multiplicative convolution and $^*$ is the adjoint function. Without loss of generality it can be assumed that $\hat{f}(\pm 1/2) = 0$. Then the Riemann hypothesis is $\zeta_A(1/2 + s) = 0$ implies $s = -\overline{s}$ which is equivalent to

$$\sum_{\zeta_A(1/2+s)=0} \hat{f}(s)\overline{\hat{f}(-\overline{s})} \geq 0, \quad \text{for all } f.$$
Therefore the Riemann hypothesis is equivalent to the positivity of 

\[- \sum_{p \geq \eta} \mathcal{W}_p(f \circ f^*) \geq 0.\]

### 7.3.2 Riesz Potentials

Haran’s next step is finding a connection between the Weil distributions and the Riesz potentials. Let \( \mathcal{F}_p \) be the Fourier transform then the Riesz potential \( R^s_p \) can be formulated for all primes acting on functions on \( \mathbb{Q}_p \) by

\[ R^s_p \phi|_{x_p} = R^s_p|_{x_p}^{-1/2} f|_{x_p}. \]

**Theorem 7.3.1 (Local Formula).**

\[ \mathcal{W}_p(f) = (\mathcal{F}_p \log|_{x_p^{-1}} \mathcal{F}_p^{-1} \mathcal{F}^{-1} f)|_{x_p}(1), \]

\[ = \frac{\delta}{\delta s}igg|_{s=0} (R^s_p|_{x_p}^{-1/2} f)|_{x_p}(1). \]

The Taylor expansion is given by

\[ \frac{1}{\zeta_p(s)} \text{tr}(R^s_p \pi(f|_{x_p}) \phi_{\mathbb{Z}_p}|_{L_2(\mathbb{Q}_p)}) = f(1) + \mathcal{W}_p(f) s + O(s^2), \text{ as } s \to \infty. \]

The main advantage of this theorem is that it enables the Weil distribution to be written in the form of the trace of an operator in at least one way. One operator to consider is the operator on \( L_2(\mathbb{Q}_p) \) given by \( R^s_p \pi(f|_{x_p}) \phi_{\mathbb{Z}_p} \) where \( \pi \) is the unitary action of the multiplicative group \( (\pi(a) f(x) = |a|^{-1/2} f(a^{-1} x)) \) and \( \phi_{\mathbb{Z}_p} \) is the operator of multiplication by the characteristic function of \( \mathbb{Z}_p \). Then

\[ \mathcal{W}_p(f) = \frac{\delta}{\delta s}igg|_{s=0} \frac{1}{\zeta_p(s)} \text{tr}(R^s_p \pi(f|_{x_p}) \phi_{\mathbb{Z}_p}|_{L_2(\mathbb{Q}_p)}). \]

There are other versions of writing the Weil distributions locally including work by Connes using cut off operators [Co1].

Indeed, let \( B_c(x) = 1 \) if \( |x|_p \leq c \) and 0 otherwise. This can be viewed as a projection on \( L_2(\mathbb{Q}_p) \).

The dual projection is given by \( \tilde{B}_c = \mathcal{F}_p B_c \mathcal{F}_p^{-1} \). Further let \( \pi^0(f|_{x_p}) \phi(x) = f d^0 a f(|a|_p)|a|^{-1/2} \phi(ax) \) where the normalized multiplicative measure is given by \( d^0 a = n_p da/|a|_p \) with \( n_p = \log(p)/(1 - p^{-1}) \) for finite primes and 1/2 for the real prime.

**Theorem 7.3.2.** For all \( N \) and as \( c \to \infty \)

\[ \text{tr}(\tilde{B}_c B_c \pi^0(f|_{x_p})) = (2 \log(c)) f(1) + \mathcal{W}_p(f) + O(c^{-N}). \]
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The next step is to provide global formulae. These provide reformulations of the Riemann hypothesis. Let \( R_\eta = \sum_{p \geq \eta} R_p \) where \( R_p = \delta \delta s \mid s = 0 \). A function \( f \in \mathcal{C}_c^\infty(\mathbb{R}^+) \) a function on \( \mathbb{A}^* \) can be associated to it \( \tilde{f} = \phi_{\mathbb{P}^*} \otimes f \), further let \( \tilde{f}_q(x) = \tilde{f}(qx) \) for \( q \in \mathbb{Q}^* \) and \( x \in \mathbb{A}^* \).

**Theorem 7.3.3** (Global Formula).

\[
\sum_{p \geq \eta} \mathcal{W}_p(f) = \hat{f}(1/2) + \hat{f}(-1/2) - \sum_{\zeta \mathbb{A}(1/2+s)=0} \hat{f}(s) = \sum_{q \in \mathbb{Q}^*} \mathcal{R}_\eta(x^{-1/2}f)(q).
\]

There also exist global trace formulae whose positivity implies the Riemann hypothesis. In a simplistic one way of stating this is

\[
\text{Riemann Hypothesis} \iff \sum_{q \in \mathbb{Q}^*} \text{CT}_{s=0} \text{tr}(\otimes_{p \geq \eta} \mathcal{R}_p^* \pi(\hat{f})^* \pi(\hat{f})) \geq 0.
\]

Here \( \text{CT}_{s=0} \) is the constant term in the Taylor series about \( s = 0 \).

This is very similar to the work of Connes, in fact he has rewritten the global trace formulas using the tools of non-commutative geometry.

In Haran’s work he claims that his work involving global trace formula provides a proof of the Riemann hypothesis for the function field case but this is not the case as he conceded himself. Whether or not this method can be used to prove the Riemann hypothesis in either the function field case or the number field case is yet to be resolved.

### 7.4 Non-Additive Geometry

This section gives a very narrow introduction of what little theory exists regarding the second problem of the arithmetic-geometry dictionary - the arithmetic surface. In many ways this is closely related to the first problem. Indeed in trying to consider \( \mathbb{Z}_\eta \) as the interval \([-1, 1]\) the problem was it not being closed under addition. So why not abandon addition? This process has led to the consideration of the “field with one element”. A detailed overview can be found in [Du].

#### 7.4.1 The Field Of One Element

The conception of the "field of one element" (denoted \( \mathbb{F}_1 \)) was in group theory in the work of Jacques Tits. Without going into detail the "the field of one element" was thought of/defined as \( G(\mathbb{F}_1) = W \) where \( G \) is a Chevalley group scheme over \( \mathbb{Z} \) and \( W \) is its Weyl group. On the face of it the idea is absurd since fields by definition must have at least two elements. However there is plenty of
evidence that something like it exists, essentially using the ideas of Tits. With finite fields, $\mathbb{F}_q$ ($q$ a power of a prime), there are numerous formula for counting structures on projective spaces over $\mathbb{F}_q$. In these formula taking $q = 1$ gives results for finite sets. As a very basic example the number of maximal flags in an $n$-dimensional vector space over $\mathbb{F}_q$ is the $q$-factorial $[n]! = [1][2] \cdots [n]$, where $[n] = (q^n - 1)/(q - 1)$. Taking $q \to 1$ gives the value $n!$, which is the number of ways to order a set with $n$ elements. This analogy extends a lot deeper and is considered so powerful that the interpretation of finite sets is as projective spaces over the "field with one element", $\mathbb{F}_1$. With this interpretation and "existence" of a "field with one element" further interpretations and definitions are made. This was, and perhaps still is, a set of suggestions and results based on these in order to find the "correct" approach to dealing with such an entity.

One must not forget that the main aim from a number theory point of view is to be able to use geometrical methods for solving number theoretical problems. Central to this is to view $\text{Spec} \, \mathbb{Z}$ as a curve over some field. In this case some of the well known conjectures of arithmetic (Riemann hypothesis, ABC, ...) become easy theorems in the geometric analogue of a curve $C$ over a finite field since the surface $C \times C$ can be formed.

This speculation began several decades after Tits' work with Manin suggesting possible zeta functions over $\mathbb{F}_1$ ([Ma1]). The possible geometrical theorems associated to $\mathbb{F}_1$ continued with a category of varieties over $\mathbb{F}_1$ in [So], derivations over $\mathbb{F}_1$ in [KuOW] and more recently work by Deitmar on schemes over $\mathbb{F}_1$, the related cohomology and zeta functions.

Part of Soulé’s work seems to have been inspired by Manin as he gives a definition of zeta functions over $\mathbb{F}_1$. Indeed a starting point is to view $\text{Spec} \, \mathbb{Z}$ as the base field of $\text{Spec} \, \mathbb{F}_1$. This means that any variety over $\mathbb{F}_1$ must have a base change over $\mathbb{Z}$, which is an algebraic variety over $\mathbb{Z}$. This base change is found in several of the works above. In order to define a variety over $\mathbb{F}_1$ it has been suggested, in unpublished work by Kapranov and Smirnov, that $\mathbb{F}_1$ should have an extension $\mathbb{F}_{1^n}$ given by adjoining roots of unity suggesting that

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[T]/(T^n - 1),$$

which is $R_n$, the ring of functions on the affine group scheme of $n$-th roots of unity.

**Definition 7.4.1.** A variety over $\mathbb{F}_1$ is a covariant functor $X$ from the category $\mathcal{R}$ (with objects $R_n$ and their finite tensor products) to finite sets. For $R \in \text{Ob} \, \mathcal{R}$ there is the natural inclusion $X(R) \subset X_{\mathbb{Z}}(R)$ where $X_{\mathbb{Z}}$ is a variety over $\mathbb{Z}$. There is also a further condition (universal property) which relates to functors.

Unfortunately this category of varieties over $\text{Spec} \, \mathbb{F}_1$ does not contain $\text{Spec} \, \mathbb{Z}$.  

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From Weil’s work a local zeta function can be defined for a scheme of finite type over \(\mathbb{Z}, X\).

\[
Z_X(p, T) = \exp \left( \sum_{n=1}^{\infty} \frac{T^n}{n} \#X(\mathbb{F}_{p^n}) \right),
\]

where \(p\) is a prime number. Soulé considered the condition that there exists a polynomial \(N(x)\) with integer coefficients for every prime \(p\) and \(N \in \mathbb{Z}\) with \(\#X(\mathbb{F}_{p^n}) = N(p^n)\). He then defines

\[
\zeta_{X|\mathbb{F}_1}(s) = \lim_{p \to 1} \frac{Z_X(p, p^{-s})^{-1}}{(p - 1)^{N(1)}} = s^{a_0}(s - 1)^{a_1} \cdots (s - n)^{a_n}.
\]

The second equality follows by setting \(N(x) = a_0 + \cdots + a_n x^n\). For example \(\zeta_{\text{Spec} \mathbb{F}_1}(s) = s\).

Zeta functions over arbitrary \(\mathbb{F}_1\)-schemes were first defined in Deitmar’s work on schemes. He shows that the condition on requiring \(\#X(\mathbb{F}_q) = N(q)\) can be made slightly weaker, by the following theorem, which enables the zeta functions to be defined.

**Theorem 7.4.2.** Let \(X\) be a \(\mathbb{Z}\)-scheme defined over \(\mathbb{F}_1\). Then there exists \(e \in \mathbb{N}\) and \(N(x) \in \mathbb{Z}[x]\) such that for every prime power \(q\)

\[
(q - 1, e) = 1 \implies \#X(\mathbb{F}_q) = N(q).
\]

The condition determines the zeta polynomial of \(X, N\), uniquely.

The zeta function of an arbitrary \(\mathbb{F}_1\)-scheme, \(X\), can then be defined via the zeta polynomial \((N_X(x) = a_0 + \cdots + a_n x^n)\) as

\[
\zeta_{X|\mathbb{F}_1}(s) = s^{a_0} \cdots (s - n)^{a_n}.
\]

Explicit calculations of zeta functions are found in [K] who also defines the Euler characteristic as

\[
\#X(\mathbb{F}_1) = \sum_{k=0}^{n} a_k.
\]

### 7.4.2 Haran

An important aspect arising from [KuOW] is the building the theory of \(\mathbb{F}_1\) on multiplication only - losing addition. Supposing that \(\text{Spec} \mathbb{Z}\) is a "curve" over \(\mathbb{F}_1\) then derivations should exist. If the concept of \(\mathbb{Z}\)-linearity is removed and one only considers the Leibniz rule then the result is derivations. This idea of losing additivity/\(\mathbb{Z}\)-linearity is applied to other \(\mathbb{F}_1\) objects in the form of a (forgetful) functor from objects over \(\mathbb{Z}\). This idea is used by Deitmar to define schemes and is a starting point in Haran’s work.
Haran’s input in the study of the "field with one element" area is different from the other work in this area. He uses his work on the real prime as guidance in particular the concept that by abandoning addition the real integers become a real object.

The initial seed for the work comes from the work of [KuOW] as they interpret vector spaces over $\mathbb{F}_1$ as finite pointed sets. This combined with a hint of the real prime leads to the definition of a category $\mathcal{F}$ which has objects finite sets and arrows as partial bijections. He chooses to use this as a model for the "field with one element". The actual connection with the previous work in this area is not made clear. Clearly there is a form of connection because the objects can also be taken to be vector spaces over $\mathbb{F}_1$ with certain maps related to the real integers. So unlike the previous work he has taken the two problems, initially stated about the arithmetic-geometric dictionary, and combined them to search for some answers/interpretations. Again this is an interpretation as the theory in his paper can be taken for a category $\mathcal{F}$ with two symmetric monoidal structures $\oplus$, $\otimes$, the unit element $(0)$ for $\otimes$ is the initial and final object of $\mathcal{F}$, $\otimes$ is distributive over $\oplus$ and it respects $X \otimes [0] = [0]$ ($X \in |\mathcal{F}|$).

From this definition various structures are imposed on $\mathbb{F}$. One of the most important is the $\mathcal{F}$-ring.

**Definition 7.4.3.** A $\mathcal{F}$-ring is a category $\mathcal{A}$ with objects $|\mathcal{F}|$ and arrows $A_{Y,X} = \text{Hom}_A(X,Y)$ containing the arrows of $\mathcal{F}$. This means there is a faithful functor $\mathbb{F} \rightarrow \mathcal{A}$ which is the identity on objects.

As examples of these he shows that there is a functor from commutative rings to $\mathcal{F}$-rings. The $\mathcal{F}$-ring associated to a commutative ring $R$ has morphisms which are matrices with values in $R$. The most important example is that relating to the real integers. Let $X \in |\mathbb{F}|$ and let $\mathbb{R} \cdot X$ denote the real vector space with inner product having $X$ as an orthonormal basis. Then for $a = (a_x) \in \mathbb{R} \cdot X$ there is a norm $|a|_\eta = (\sum_{x \in X} |a_x|^2)^{1/2}$. There is also the related operator norm for a real linear map $f \in \text{Hom}(\mathbb{R} \cdot X, \mathbb{R} \cdot Y)$, $|f|_\eta = \sup_{|a|_\eta \leq 1} |f(a)|_\eta$. Then the $\mathcal{F}$-ring, $\mathcal{O}_\mathbb{R}$, is defined to have morphisms $$(\mathcal{O}_\mathbb{R})_{Y,X} = \{ f \in \text{Hom}_\mathbb{R}(\mathbb{R} \cdot X, \mathbb{R} \cdot Y), |f|_\eta \leq 1 \}.$$ Haran also defines the residue field of $\mathcal{O}_\mathbb{R}$ to be the $\mathcal{F}$-ring of partial isometries $(\mathcal{F}_\eta)$ $$(\mathcal{F}_\eta)_{Y,X} = \{ f : V \xrightarrow{\sim} W : V \subset \mathbb{R} \cdot X, W \subset \mathbb{R} \cdot Y, \text{real sub-vector spaces and } f \text{ is a real linear isometry} \}.$$ These can be generalised to any number field, $k$, and $\eta : k \rightarrow \mathbb{C}$ a real or complex prime. In the real case he simply mentions that these constructions are analogous to the $p$-adic integers. In future work I think these $\mathcal{F}$-rings may play an important role.
The rest of the paper sets about developing the language of geometry using \( \mathbb{F} \)-rings. This naturally starts with the definition of modules, submodules and ideals of an \( \mathbb{F} \)-ring. An \( A \)-module of a \( \mathbb{F} \)-ring \( A \) is basically a collection of sets with maps which are compatible with \( A \). An \( A \)-submodule \( M' \) of \( M \) is a collection of subsets of \( M \) which are closed under the maps which are used to define \( M \). In the special case of \( M = A \) then the \( A \)-submodule is called an ideal. A clear example following on from the \( \mathbb{F} \)-rings are the modules of a commutative ring. Under the functor \( \mathbb{F} \) the modules become modules of an \( \mathbb{F} \)-ring. The main emphasis is placed on ideals in order to develop prime ideals, \( \text{Spec} \) and schemes.

The ideals which he uses are called \( H \)-(omogeneous)-ideals. A normal ideal \( a \) is a collection of subsets \( \{ a_{Y,X} \subset A_{Y,X} \} \) which are closed under the functors \( \otimes \), \( \oplus \) and under composition \( \circ \) of maps. The \( H \)-ideals are a subset of ideals with the property that an ideal \( a \) is generated by \( a_{[1],[1]} \) where \( [n] = \{0, 1, \ldots n\} \). An application of Zorn’s lemma gives

**Theorem 7.4.4.** Every \( \mathbb{F} \)-ring contains a maximal (proper) \( H \)-ideal.

Naturally there is a notion of a prime \( H \)-ideal. A \( H \)-ideal \( p \in A_{[1],[1]} \) is called prime if \( A_{[1],[1]} p \) is multiplicative closed: \( f, g \in A_{[1],[1]} \) \( p \rightarrow f.g \notin p \). The set of prime ideals of \( A \) is denoted by \( \text{Spec}(A) \).

As an example he takes the \( \mathbb{F} \)-ring of real "integers", \( \mathcal{O}_\mathbb{R} \) and states that \( m_\eta = \{ x \in \mathbb{R} : |x|_\eta < 1 \} \) is the unique maximal \( H \)-ideal of \( \mathcal{O}_\mathbb{R} \). With these ideas in place he can then define a Zariski topology on \( \text{Spec} A \). The closed sets are given for a set \( \mathcal{U} \subset A_{[1],[1]} \), \( V_A(\mathcal{U}) = \{ p \in \text{Spec} A : p \supseteq \mathcal{U} \} \) and for a \( H \)-ideal, \( a \), generated by \( \mathcal{U} \), \( V_A(\mathcal{U}) = V_A(a) \). The closed sets of the Zariski topology on \( \text{Spec} A \) are \( \{ V_A(a) : a \in H - \text{id}(A) \} \).

The theory of localization of a \( \mathbb{F} \)-ring is almost identical to that of commutative rings due the multiplicative theory. This enables \( \mathbb{F} \)-ringed spaces to be defined and a category of Zariski-\( \mathbb{F} \)-schemes. So far this work is in setting the stage for actual applications and progress towards finding the arithmetical surface. Based on the other work I would also expect zeta functions to be developed for the schemes introduced by Haran.

### 7.5 A Nonstandard Beginning

The q-world is vast and varied in nature but in this work some of the aspects which Shai Haran chose to examine are looked at through a nonstandard view point.
7.5.1 $q$-Integers

**Definition 7.5.1.** Let $q$ be an indeterminate which can be considered in the real field. For any integer $s$ define the $q$-integer $[s]_q = \frac{1-q^s}{1-q}$. This definition in fact holds for any $s \in \mathbb{C}$, the non-symmetric $q$-numbers.

From this the nonstandard non-symmetric $q$-numbers can be defined.

**Definition 7.5.2.** Let $q \in \ast \mathbb{R}$ and $s \in \ast \mathbb{C}$ then define $\ast [s]_q = \frac{1-q^s}{1-q}$. This is a function $\ast [\cdot]_q : \ast \mathbb{C} \times \ast \mathbb{R} \rightarrow \ast \mathbb{C}$. Of particular interest are the values of the function within $\mu_\eta(1)$.

Consider $q \in \mu_\eta(1)$ then for some fixed infinitesimal $\delta$, $q = 1 + \delta$ and take $s$ to lie in $\ast \mathbb{C}_\lim_\eta$ so the $\eta$-shadow map can be taken. In particular for $q$ as above and $s \in \ast \mathbb{C}_\lim_\eta$, $\ast [s]_{1+\delta} \in \ast \mathbb{C}_\lim_\eta$.

$$\ast [q]_{1+\delta} = \frac{1-(1+\delta)^s}{1-(1+\delta)} = \frac{1-\ast \exp(s^* \log(1+\delta))}{-\delta} = \frac{s^* \log(1+\delta) + (1/2)s^2 \log^2(1+\delta) + \ldots}{\delta} = s + \delta(s^2 - s/2) + \delta^2(s/3 - s^2 + s^3) + \ldots + \delta^n S_n(s) + \ldots.$$

Here for all $n \in \ast \mathbb{N}$, $S_n(s)$ is a monic hyperpolynomial of degree $n+1$. This sum is absolutely Q-convergent (by the properties of the functions). From the convergence properties of the standard function it can be deduced that $|\delta^n S_n(s)| \simeq_\eta 0$ for all $n \in \ast \mathbb{N} \setminus \mathbb{N}$ and $s \in \ast \mathbb{C}_\lim_\eta$.

$$\text{sh}_\eta\left(\frac{1-(1+\delta)^s}{1-(1+\delta)}\right) = \text{sh}_\eta\left(\sum_{n \in \ast \mathbb{N}} \delta^n S_n(s)\right),$$

$$= \sum_{n \in \ast \mathbb{N}} \text{sh}_\eta(\delta^n S_n(s)),
= s.$$

Since the nonstandard terms are zero by above and the standard terms, apart from the leading one, are zero since the delta part is infinitesimal while $S_n(s)$ is standard leading to an infinitesimal term. Therefore the only term remaining is the first one and

$$\text{sh}_\eta\left(\frac{1-q^s}{1-q}\right) = s,$$
for \( s \in \ast \mathbb{C}^{\lim_\eta} \) and \( q \in \mu_\eta(1) \). This corresponds to the standard case

\[
\lim_{q \to 1} [s]_q = s.
\]

There also exists other special values. For example let \( q \) be an infinitesimal of the form \( r^{-1/\delta} \) where \( r \in \ast \mathbb{R} \) and \( \delta \in \mu_\eta(0) \) (an infinitesimal) Let \( s \in \ast \mathbb{C}^{\lim_\eta} \) and define \( t = s\delta \in \mu_\eta(0) \) (also an infinitesimal). Then by an analogous method as above

\[
\text{sh}_\eta(*)_{\ast,[s\delta]_{r^{-1/\delta}}} = 1 - \text{sh}_\eta(r)^{-\text{sh}_\eta(s)}.
\]

This example shows that the function is not \( Q \)-continuous. This is due to terms of the form \( \epsilon \delta \), where both are infinitesimals. This makes the nonstandard function quite interesting because given two values of the function given by infinitesimals \((s_1, q_1)\) and \((s_2, q_2)\) it does not follow that \( \ast[s_1]_{q_1} \) and \( \ast[s_2]_{q_2} \) are infinitely close.

Naturally for other basic functions, such as the non-symmetric \( q \)-numbers, special values exist and the real shadow map of these corresponds to the standard limits.

### 7.5.2 \( q \)-Zeta

One definition for the \( q \) zeta function is given for \( s \in \mathbb{C} \) and \( q \in \mathbb{R}^+ \) by

\[
\zeta_q(s) = \prod_{n \in \mathbb{N}} (1 - q^{s+n})^{-1}.
\]

One important limit of this function is when \( q = p^{-N} \) (\( p \) prime) and \( s := s/N \) with \( N \to \infty \) then

\[
\zeta_{p^{-N}}(s/N) \to_{N \to \infty} \zeta_p(s) = (1 - p^{-s})^{-1}.
\]

To consider this from a nonstandard perspective one considers a nonstandard function \( \ast \zeta_q(s) = \prod_{n \in \mathbb{N}} (1 - q^{s+n})^{-1} \) with nonstandard arguments. To find an analogy to the limit consider an infinitesimal \( \delta \in \mu_\eta(0) \) and a prime \( p \) setting \( q = p^{-1/\delta} \) and for \( t \in \ast \mathbb{C}^{\lim_\eta} \) set \( s = \sigma t \). Then

\[
\ast \zeta_{p^{-1/\delta}}(s) = \prod_{n \in \mathbb{N}} (1 - p^{-t}p^{-n/\delta})^{-1}.
\]

Using classical analysis this product is absolutely \( Q \)-convergent because clearly \( \sum_{n \in \mathbb{N}} p^{-n/\delta} \) is absolutely \( Q \)-convergent.

\[
\text{sh}_\eta(\ast \zeta_{p^{-1/\delta}}(s)) = (1 - p^{-\text{sh}_\eta(s)})^{-1} \text{sh}_\eta(\prod_{n \in \mathbb{N} \setminus \{0\}} (1 - p^{-t}p^{-n/\delta})^{-1}.
\]
From the basic properties of the real shadow map and for all \( N \in \mathbb{N} \) with \( a_n \in \mathbb{C} \),

\[
\text{sh}_\eta\left( \prod_{n=1}^{N} a_n \right) = \prod_{n=1}^{N} \text{sh}_\eta(a_n).
\]

In particular choosing \( a_n = (1 - p^{-t}p^{-n/\delta})^{-1} \) for \( n \in \mathbb{N} \) and \( a_n = 1 \) for \( n \in \mathbb{N} \setminus \mathbb{N} \) then

\[
\text{sh}_\eta(*\zeta_{p^{-1/\delta}}(s)) = (1 - p^{-\text{sh}_\eta(s)})^{-1}.
\]

### 7.5.3 Chains

One of the main approaches in investigating the real prime is via Markov chains. To interpolate between the real gamma chains and \( p \)-adic gamma chains, \( q \)-chains are used.

#### \( q \)-Gamma Chain

**Definition 7.5.3.** Let \( (X_q^0, P_{X_q}^\beta) \) be the \( q \)-gamma chain where the state space is

\[
X_q^0 = \prod_{n \in \mathbb{N}} X_{q(n)}^0, \quad X_{q(n)}^0 = \{(n, j) | 0 \leq j \leq n\},
\]

and the transition probabilities are given by

\[
P_{X_q}^\beta((i, j), (u, v)) = \begin{cases} 
q^{\beta + j} & u = i + 1, v = j, \\
1 - q^{\beta + j} & u = i + 1, v = j + 1, \\
0 & \text{otherwise}.
\end{cases}
\]

There are two approaches to considering this in the nonstandard setting. The methods both use shadow maps to obtain the standard objects.

#### Real Shadow Map

Consider the following extension to the \( q \)-gamma chain given by \( (*X_q^0, *P_{X_q}^\beta) \) with state space

\[
*X_q^0 = \prod_{n \in *\mathbb{N}} X_{q(n)}^0 \quad (*X_{q(n)}^0 = \{(n, j) | 0 \leq j \leq n\})
\]

and transition probabilities as above but defined for all nonstandard natural numbers and \( q \in *(0, 1) \) and \( \beta \in *\mathbb{R}^+ \). Special values of this chain exist for a prime \( p \) and let \( q + p^{-1/\delta} \) for some \( \delta \in \mu_0 \) and consider the chain \( (*X_{p^{-1/\delta}}^0, *P_{*X_{p^{-1/\delta}}}^{\beta}) \) with \( \beta \in *\mathbb{R}^+ \). The shadow map can be taken for all \( \beta \in (*\mathbb{R}^+)\lim_0 \) leading to transition probabilities \( *\text{sh}_{p^\beta}(*X_{p}) \). The state space is \( X_{p}^0 \). This is the \( p \)-adic gamma chain.
As a note consider the case when $p$ is not necessarily prime but $q = r^{-1/\delta}$ for some $r \in (\ast \mathbb{R}^+)^{\lim_{\eta}}$. The above holds as there was no use made of $r$ being prime. Taking the shadow map results in a chain $\ast \mathbb{P}^{\text{sh}_r}(\beta)$.

In particular for $r \in \mathbb{N}$ then the resulting chain under the shadow map has a harmonic measure consisting of harmonic measures relating to the primes divisors of $r$. (This follows from the basic results on $g$-adic numbers ($g \in \mathbb{N}$) given by Mahler. These $r$-gamma chains are almost "handmade" rather than resulting from some natural structure as in the $p$-adic gamma chains ([Ha1], chapter 4).

If instead one considers $n \in \mathbb{N}$ and considers a tree

$$Y_n^0 = \prod_{N \in \mathbb{N}} Y_{n(N)}^0, \quad Y_{n(N)}^0 = \mathbb{Z}/n^N \mathbb{Z}, \quad Y_{n(0)}^0 = \{0\}.$$ 

One can think of this chain as a $n$-adic expansion with an edge $y \in Y_{n(N)}^0$ going to its $n$ pre-images in $Y_{n(N+1)}^0$. The boundary is given by

$$\partial Y_n^0 = \lim_{\rightarrow N} Y_{n(N)}^0 = \mathbb{Z}_n.$$ 

Here $\mathbb{Z}_n$ are the $n$-adic integers as described in Mahler. ([M], chapter 5). These numbers have a decomposition as a direct sum of the $p$-adic integers with $p|n$. On this boundary there is a harmonic measure $\tau_{\mathbb{Z}_n}^\beta$ which can be decomposed as a direct sum of $\tau_{\mathbb{Z}_p}^\beta$ for $p|n$. This measure induces a unique chain on this tree. Moreover the measure is invariant under the $\mathbb{Z}_n^*$ action and a quotient chain is obtained. The structure of the chain is determined by the number of distinct prime factors of $n$. For example when $n$ is prime the chain is the $p$-adic gamma chain as in Haran. Explicit examples can be easily given. I have yet to find any use for these chains in the same way there is little use for general $n$-adic numbers compared to $p$-adic numbers. These chains are different from the ones arising from the shadow map, hence the term "handmade" for the shadow map chains.

$p$-adic Shadow Maps

The second approach to the gamma chains relies on the other shadow maps of $\ast \mathbb{Q}$. Choose a non-standard integer $N \in \ast \mathbb{N} \setminus \mathbb{N}$ and consider the set $\mathcal{P} = \{p \in \ast \mathbb{N} : p \text{ prime}, p \leq N\}$ (a hyperfinite set of primes) and the product of primes

$$P = \prod_{p \in \mathcal{P}} p.$$ 

This is a hyperfinite product.

**Lemma 7.5.4.** For all $p \in \mathcal{P}$ there exists a non-unique $u \in \ast \mathbb{Q}$ such that for some non-unique $k_p, r_p \in \ast \mathbb{N}$, $u = p + k_p p^{r_p}$ which in particular is equivalent for standard primes to $\text{sh}_p(u) = p$. 

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**Proof:** Fix a set of \( \{ r_p \} \) simply by choosing a \( r_p \in \ast \mathbb{N} \setminus \mathbb{N} \) for each \( p \in \mathcal{P} \). Then it is required to solve the system of equations \( \{ u \equiv p \text{ (mod } p^{r_p}) \} \). Since \( (p^{r_p}, q^{r_q}) = 1 \) for all distinct primes the Chinese remainder theorem in a nonstandard setting can be used to obtain a solution \( u \). Any two solutions of \( u \) for a particular set of \( \{ r_p \} \) differ by a multiple of \( \prod_{p \in \mathcal{P}} p^{r_p} \). In particular for all standard primes \( u \simeq_p p \) and by taking the \( p \)-adic shadow map \( \text{sh}_p(u) = p \).

\[ \Box \]

The non-uniqueness needs to be considered and perhaps some form of equivalence relation could be used but this will be dealt with later. For the moment suppose some set \( \{ r_p \} \) has been fixed along with a solution \( u \). For such a \( u \) a hyper chain will be constructed based on the \( p \)-gamma chain. Take a state space

\[ X_u^0 = \prod_{N \in \mathbb{N}} X_u^0(N), \quad X_u^0(N) = \{ (N, j) : 0 \leq j \leq N \}. \]

For the transition probabilities let

\[ *\mathbb{P}^\beta_{X_u}((i, j), (c, d)) = \begin{cases} u^{-\beta} & c = i + 1, d = j = 0, \\ 1 - u^{-\beta} & c = i + 1, d = 1, j = 0, \\ 1 & c = i + 1, d = j + 1, j \geq 1, d \geq 2, \\ 0 & \text{otherwise.} \end{cases} \]

In order for the shadow maps to be taken \( \beta \) has to be an element of \( *\mathbb{C}^{\lim_p} \) for all standard \( p \) and if the real shadow map is to be taken then also for \( p \) the real prime. As an initial investigation just let \( \beta \in \mathbb{N} \). For these values of \( \beta \) the \( p \)-adic shadow map acting on the chain leads to the \( p \)-adic gamma chain with value \( \beta \) for all finite \( p \), since \( \text{sh}_p(u^{-\beta}) = p^{-\beta} \). Moreover for these values of \( \beta \), \( \text{sh}_q(u^{-\beta}) = 0 \) and a chain is obtained which is the real gamma chain (the unit shift chain to the right). So the initial nonstandard chain leads to all the \( p \)-gamma chains (for all standard primes and the real prime) by an application of the respective shadow map.

The problem is, of course, the restrictive nature of the values of \( \beta \). Suppose the set of values for \( \beta \) is enlarged to contain \( \mathbb{Q} \) then the in general the \( p \)-adic shadow maps cannot be taken because \( u^{-\beta} \) is irrational. Suppose instead it is taken to be \( *\mathbb{N} \) then one obtains a chain based on a \( p \)-adic number which is a "handmade" chain for a \( p \)-adic number instead of just real numbers considered above. Even so their use appears limited meaning the "maximal" set for beta is \( \mathbb{N} \). Can a topology be put on chains in order to look at sequences of chains?

The next option to consider is the transition probabilities as above but with the enlargement of the
state space

\[ *X_u^0 = \prod_{N \in \ast \mathbb{N}} *X_{u(N)}^0, \quad *X_{u(N)}^0 = \{(N, j) : 0 \leq j \leq N\}. \]

My problem with this state space is taking the shadow map. For all nonstandard \( N \in \ast \mathbb{N} \setminus \mathbb{N} \),

\[ \mathbb{N} \subset \text{sh}_p(*X_{u(N)}^0) \subset \mathbb{Z}_p. \]

The standard chain which results from the \( p \)-adic shadow map contains \( X_p^0 \) but also subsets of \( \mathbb{Z}_p \).
At the moment I am unsure of how this "fits in" with this chain.
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