A systematic study of finite BRST-BFV transformations in generalized Hamiltonian formalism

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Wroclaw, Max Born 33 Symposium, 6-10 July 2014
Based on

I.A. Batalin, P.M.L., I.V. Tyutin, arXiv:1404.4154[hep-th]
Contents

- Introduction
- BRST-BFV quantization
- Finite BRST-BFV transformations
- Compensation equation
- Functional BRST-BFV transformations for trajectories
- Ward identities
- Discussions
The BRST symmetry (C. Becchi, A. Rouet, R. Stora (1974), I.V. Tyutin (1975)) is a powerful tool to study general properties of gauge field systems (E.S. Fradkin, G.A. Vilkovisky (1975); I.A. Batalin, G.A. Vilkovisky (1977); E.S. Fradkin, T.E. Fradkina (1978); I.A. Batalin, G.A. Vilkovisky (1981); B.L. Voronov, P.M.L., I.V. Tyutin (1982); P.M.L. I.V. Tyutin (1982); I.A. Batalin, E.S. Fradkin (1983); M. Henneaux (1985)). Parameters of that symmetry are constant Fermions, although they are allowed to be functionals of fields. Usually, the symmetry is introduced infinitesimally, which means that its Fermionic parameters are considered formally as infinitely-small quantities. Usual strategy is to show that the Jacobian of BRST transformation does generate arbitrary variation of gauge-fixing functions in the path integral. This can be done by choosing necessary functional dependence of BRST parameters on fields.
For Yang-Mills theories in Lagrangian formalism based on the Faddeev-Popov method a study of finite field-dependent BRST transformations was initiated in (S.D. Joglekar, B.P. Mandal (1995)) where a differential equation for the Jacobian of such change of variables in vacuum functional has been proposed. But a solution to this equation has not been found in this paper and numerous further studies of this problem. Recently (P.M.L., O. Lechtenfeld (2013)) it was proved that the problem of finding an explicit form of the Jacobian is pure algebraic and can be solved in terms of the BRST variation of field-dependent parameter. Any finite field dependent BRST transformation of variables in the generating functional of Green functions is related to modification of gauge fixing functional.
By following the ideas of paper (S.D. Joglekar, B.P. Mandal (1995)), recently in (S.K. Rai, B.P. Mandal (2013)) an attempt was made to study finite field-dependent BRST transformations in Hamiltonian formalism within BRST-BFV quantization method of dynamical systems with constraints (E.S. Fradkin, G.A. Vilkovisky (1975); I.A. Batalin, G.A. Vilkovisky (1977)). The main result was formulated here as a differential equation for the Jacobian of these transformations. Again, an explicit solution to the differential equation was not found in this approach.
Recently (I.A. Batalin, P.M.L., I.V. Tyutin, arXiv:1404.4154[hep-th]), we developed systematically the concept of finite BRST-BFV transformations in the generalized Hamiltonian formalism, which means actually that we consider formally BRST-BFV parameters as finite Fermionic quantities. Historically, there were several authors (see E.S. Fradkin, G.A. Vilkovisky (1975); I.A. Batalin, G.A. Vilkovisky (1977); E.S. Fradkin, T.E. Fradkina (1978); I.A. Batalin, G.A. Vilkovisky (1981); B.L. Voronov, P.M.L., I.V. Tyutin (1982); P.M.L. I.V. Tyutin (1982); I.A. Batalin, E.S. Fradkin (1983); M. Henneaux (1985) and references herein) who worked sporadically with finite BRST-BFV transformations. But the final results were formulated infinitesimally even in these special cases.
Now we have a unique consistent approach. Thereby, our new strategy is to show that the Jacobian of these finite transformations does generate arbitrary finite change of gauge-fixing functions in the path integral. In order to do this, we formulate the compensation equation determining the necessary functional field dependence for finite Fermionic parameters. Then we present the explicit solution to that compensation equation. We find functional formulation of BRST-BFV transformations and derive the Ward identities as well as a relation connecting generating functionals of Green functions written in two different gauges.
The method is applied to a dynamical system that in the phase space of initial canonical variables \((p_a, q^a), \ a = 1, 2, ..., n\) is described by the Hamiltonian \(H_0 = H_0(p, q)\) and by the set of first-class constraints \(T_\alpha = T_\alpha(p, q), \ \alpha = 1, 2, ..., m\) whose Grassmann parities are \(\varepsilon(H_0) = 0, \ \varepsilon(T_\alpha) = \varepsilon_\alpha\) and the involution relations hold true

\[
\{T_\alpha, T_\beta\} = T_\gamma U^\gamma_{\alpha\beta}, \quad \{H_0, T_\alpha\} = T_\beta V^\beta_\alpha,
\]

where the structure functions \(U^\gamma_{\alpha\beta}\) possess the properties of generalized antisymmetry

\[
U^\gamma_{\alpha\beta} = -(-1)^{\varepsilon_\alpha\varepsilon_\beta} U^\gamma_{\beta\alpha}
\]

The extended phase space \(z\) is introduced

\[
z^i = (p_a, q^a; P_\alpha, c^\alpha; \bar{P}_\alpha, \bar{c}^\alpha; ...), \quad \varepsilon(z^i) = \varepsilon_i, \quad \varepsilon(c^\alpha) = \varepsilon(P_\alpha) = \varepsilon_\alpha + 1
\]
The key role in the BRST-BFV quantization is played by the generating functions $\Omega$ and $\mathcal{H}$. The fermion function $\Omega$ is the solution of equation

$$\{\Omega, \Omega\} = 0, \quad \Omega = T_\alpha c^\alpha + \cdots$$

In its turn, the boson function $\mathcal{H}$ satisfies generating equation of the form

$$\{\mathcal{H}, \Omega\} = 0,$$

with the condition

$$\mathcal{H} = H_0 + \cdots.$$
The partition function reads

\[ Z_\psi = \int Dz \exp\left[ (i/\hbar)W_\psi \right], \]

where the action \( W_\psi \) is defined as

\[ W_\psi = \int \left[ (1/2)z^i(t)\omega_{ik}\dot{z}^k(t) - H(t) \right] dt, \]

\[ H(t) = \mathcal{H}(t) + \{\Omega, \psi\}_t, \]

Here, \( z^i(t) \) are functions of time (trajectories), \( \dot{z}^k(t) = dz^k(t)/dt \), \( H(t), \mathcal{H}(t), \Omega(t), \psi(t) \) are local functions of time: \( H(t) = H(z)|_{z \rightarrow z(t)} \) and so on. \( \{,\}_t \) means the Poisson superbracket for fixed time \( t \):

\[ \{\Omega, \psi\}_t = \{\Omega(z), \psi(z)\}|_{z \rightarrow z(t)}. \]
BRST-BFV quantization

\[ \{z^i, z^k\} = \omega^{ik} = \text{const} = -\omega^{ki}(-1)^{\varepsilon_i\varepsilon_k} \]
is an invertible even supermatrix; \(\omega^{ik}\) \((\omega^{ik} = (-1)^{(\varepsilon_i+1)(\varepsilon_k+1)}\omega_{ki})\) stands for an inverse to \(\omega^{ik}\), \(H\) is a Boson with ghost number zero, while \(\Omega\) and \(\psi\) is a Fermion with ghost number +1 and −1, respectively, they are called BRST-BFV generator and gauge Fermion.

It follows from generating equations for \(\Omega\) and \(H\) and the definition of \(H\) that

\[ \{\Omega, H\} = 0. \]

Remarkable fact is the invariance of main quantities \((\Omega, H, H)\) of the formalism under the following global supersymmetry transformations

\[ z^i \rightarrow z^i + \{z^i, \Omega\}\mu \]

known as the BRST-BFV transformations. Here \(\mu\) is a constant Grassmann parameter.
Finite BRST-BFV transformations

Finite BRST-BFV transformations of phase (canonical) variables

$$\tilde{z}^k = z^k + \{z^k, \Omega\} \mu,$$

Finite BRST-BFV transformations of the trajectories

$$\tilde{z}^k(t) \equiv \tilde{z}^k \bigg|_{z \to z(t)} = z^k(t) + \{z^k, \Omega\}_t \mu.$$ 

Here $\mu$ is a finite Fermionic parameter. In general, $\mu = \mu[z]$ is a functional of the whole trajectory $z^k(t), -\infty < t < +\infty$. However, $\mu$ itself is independent of current time $t$ and phase variables $z^k$,

$$d_t \mu = 0, \quad \partial_k \mu = 0, \quad d_t = d/dt, \quad \partial_k = \partial/\partial z^k.$$ 

Thus, only a functional derivative $\delta/\delta z(t)$ is capable to differentiate $\mu[z]$ nontrivially.
Now, let us consider the functional Jacobian,

\[ J = \text{sDet} \left\{ \frac{\delta}{\delta z^j(t')} z^i(t) \right\} = \exp \left\{ \text{sTr} \ln \left[ \frac{\delta}{\delta z^j(t')} z^i(t) \right] \right\} = \]

\[ = \exp \left\{ \text{sTr} \ln \left[ \delta^i_j \delta(t - t') + \{ z^i, \Omega \}_t \left( \mu[z] \frac{\delta}{\delta z^j(t')} \right) + \right. \right. \]

\[ + \left. \left. (\{ z^i, \Omega \} \delta_j) (-1)^{\varepsilon_j} \mu[z] \delta(t - t') \right] \right\}. \]

Omitting details of calculation the final result for the Jacobian reads

\[ J = \exp \left\{ - \ln (1 + \kappa) \right\} = (1 + \kappa)^{-1}, \]

\[ \kappa = \int dt \mu[z] \frac{\delta}{\delta z^i(t)} \{ z^i, \Omega \}_t = \mu[z] \int dt \frac{\delta}{\delta z^i(t)} \{ z^i, \Omega \}_t. \]
Compensation equation

Now, we would like to use the Jacobian to generate arbitrary finite change $\delta \psi$ of the gauge Fermion $\psi$ in the action,

$$\psi \to \psi_1 = \psi + \delta \psi.$$

Let us make the transformation of the trajectories in the path integral for partition function.
First of all, the action in the new variables reads

$$W_\psi = \int \left[ \frac{1}{2} \bar{z}^i(t) \omega_{ik} d\bar{z}^k(t)/dt - \overline{H}(t) \right] dt = W_\psi$$

where we have used the relation

$$\int [\bar{z}^i(t) \omega_{ik} d\bar{z}^k(t)/dt] dt = \int [z^i(t) \omega_{ik} dz^k(t)/dt] dt + (z^k \partial_k \Omega - 2\Omega) t\mu \bigg|_{-\infty}^{+\infty},$$

and $\overline{H} = H(\bar{z}) = H(z)$. 
Now, we have for the partition function in the new variables,

\[ Z_\psi = \int D\bar{z} \exp\left[\frac{i}{\hbar}W_\psi\right] = \int DzJ \exp\left[\frac{i}{\hbar}W_\psi\right] = \]

\[ = \int Dz \exp\left\{\left(\frac{i}{\hbar}\right)\left[W_{\psi_1} - \left( - \int dt \{\Omega, \delta\psi\}_t + (\hbar/i) \ln (1 + \kappa)\right)\right]\right\}. \]

Let us require the following condition to hold

\[ J = \exp\left[\frac{i}{\hbar} \int dt \{\Omega, \delta\psi\}_t\right]. \]

\[ Z_{\psi_1} = Z_\psi, \]

for arbitrary finite \( \delta\psi \)!
The compensation equation is rewritten as

$$\int dt \mu[z] \frac{\delta}{\delta z^i(t)} \{z^i, \Omega}\rangle_t = \exp[(i/\hbar) \int dt \{\Omega, \delta \psi\}] - 1.$$  

That is a functional equation to find $\mu[z]$. Introducing a functional $x,$

$$x = (i/\hbar) \int dt \{\Omega, \delta \psi\}_t = (i/\hbar) \delta \Psi \int dt \frac{\delta}{\delta z^i(t)} \{z^i, \Omega\}t, \quad \delta \Psi = \int dt \delta \psi(t),$$

we can rewrite in the form

$$\mu[z] \int dt \frac{\delta}{\delta z^i(t)} \{z^i, \Omega\}t = f(x)x = (i/\hbar)[f(x)\delta \Psi] \int dt \frac{\delta}{\delta z^i(t)} \{z^i, \Omega\}t,$$

where

$$f(x) = (\exp(x) - 1)x^{-1}.$$
There is an obvious explicit solution to that equation

\[ \mu[\delta \psi] = \mu[z; \delta \psi] = (i/\hbar) f(x) \delta \Psi. \]

Thus we have confirmed explicitly the compensation equation to hold. In the first order in \( \delta \psi \), explicit solution takes the usual form

\[ \mu[\delta \psi] = (i/\hbar) \delta \Psi + O((\delta \psi)^2). \]
Functional BRST-BFV transformations for trajectories

It appears quite natural to make our considerations above more transparent by introducing a concept of functional BRST-BFV transformations. Namely, let us define a functional operator (differential) $\overleftarrow{d}$ of the form

$$
\overleftarrow{d} = \int dt \frac{\delta}{\delta z^i(t)} \{z^i, \Omega\}_t \quad \varepsilon(\overleftarrow{d}) = 1.
$$

Due to the property $\{\Omega, \Omega\} = 0$ the Fermionic operator $\overleftarrow{d}$ is nilpotent,

$$
\overleftarrow{d}^2 = (1/2)[\overleftarrow{d}, \overleftarrow{d}] = 0.
$$

The BRST-BFV transformation can be rewritten in terms of $\overleftarrow{d}$

$$
\overline{z}(t) = z(t)(1 + \overleftarrow{d} \mu).
$$

Thus, the operator $\overleftarrow{d}$ is a functional BRST-BFV generator.
Now, let us define the transformed action,

\[ \bar{W}_\psi = W_\psi (1 + \langle d \mu \rangle) \]

Then we get exactly the formula,

\[ \bar{W}_\psi = W_\psi + \left( \frac{1}{2} (z^k \partial_k \Omega - 2\Omega)_{t \mu} \right) \bigg|_{-\infty}^{+\infty} = W_\psi. \]

As \( \langle d \rangle \) is linear, and \( \mu \) is nilpotent, applying the operator \( 1 + \langle d \mu \rangle \) to arbitrary functional \( F(z) \), \( \bar{F}(z) = F(z)(1 + \langle d \mu \rangle) \), yields the result \( \bar{F}(z) = F(\bar{z}) \).
Functional BRST-BFV transformations for trajectories

Functional Jacobian is rewritten in terms of the generator as

\[ J = [1 + (\mu \delta \bar{d})]^{-1}. \]

The \( x \) can be represented as

\[ x = (i/\hbar)\delta \Psi \delta \bar{d}, \]

and the compensation equation takes the form

\[ \mu \delta \bar{d} = \exp[(i/\hbar)(\delta \Psi \delta \bar{d})] - 1 = (i/\hbar)[f(x)\delta \Psi] \delta \bar{d}. \]

Thus, we conclude that all the main objects in our considerations can be expressed naturally in terms of a single quantity that is the functional BRST-BFV generator.
Note that the introduced transformations form a group. Indeed, let us rewrite the transformation of variables in the form

\[ \overline{z} = z \overleftarrow{T}(\mu), \quad \overleftarrow{T}(\mu) = 1 + \overleftarrow{d} \mu. \]

then the composition law of transformations reads

\[ \overleftarrow{T}(\mu_1) \overleftarrow{T}(\mu_2) = \overleftarrow{T}(\mu_{12}), \quad \mu_{12} = \mu_1 + J_{\mu_1}^{-1} \mu_2, \]

where \( J_{\mu_1} \) is the Jacobian of the transformation with \( \mu_1 \) standing for \( \mu \). Indeed, due to the nilpotency we have

\[ \overleftarrow{T}(\mu_1) \overleftarrow{T}(\mu_2) = 1 + \overleftarrow{d} \mu_1 + \overleftarrow{d} \mu_2 + \overleftarrow{d} \mu_1 \overleftarrow{d} \mu_2 = \]

\[ = 1 + \overleftarrow{d} \mu_1 + \overleftarrow{d} \mu_2 + \overleftarrow{d} (\mu_1 \overleftarrow{d}) \mu_2. \]

By substituting here

\[ \mu_1 \overleftarrow{d} = J_{\mu_1}^{-1} - 1, \]

we arrive at the statement. Moreover, it follows from the relation

\[ [\overleftarrow{d} \mu_1, \overleftarrow{d} \mu_2] = \overleftarrow{d} \mu_{[12]}, \quad \mu_{[12]} = \mu_{12} - \mu_{21} = -(\mu_1 \mu_2) \overleftarrow{d}. \]
Ward identities

As we have defined finite BRST- BFV transformations, it appears quite natural to apply them immediately to deduce the corresponding modified version of the Ward identity. We will do that just in terms of functional BRST- BFV generator.

As usual for that matter, let us proceed with the external-source dependent generating functional,

\[
Z_\psi(\zeta) = \int Dz \exp \left[ \frac{i}{\hbar} W_\psi(\zeta) \right], \quad W_\psi(\zeta) = W_\psi + \int dt \zeta_k(t) z^k(t),
\]

where \( \zeta_k(t) \ (\epsilon(\zeta_k) = \epsilon_k) \) is an arbitrary external source. Of course, in the presence of non-zero external source, the path integral is in general actually dependent of gauge Fermion \( \psi \). However, due to the equivalence theorem (R.E. Kallosh, I.V. Tyutin (1973)) this dependence has a special form so that physical quantities do not depend on gauge. In its turn the Ward identity measures the deviation of the path integral from being gauge-independent.
Let us perform the change $z^i \rightarrow \bar{z}^i$ with arbitrary $\mu[z]$. Then, by using the gauge invariance of partition function as well as the expression for the Jacobian, we get what we call a "modified Ward identity",

$$< [1 + (i/\hbar) \int dt \zeta_k(t) (\bar{z}^k(t) \mu)] [1 + (\mu \mu)]^{-1} >_{\psi, \zeta} = 1,$$

where we have denoted the source dependent mean value

$$< (...) >_{\psi, \zeta} = [Z_{\psi}(\zeta)]^{-1} \int Dz(...) \exp [(i/\hbar) W_{\psi}(\zeta)], \quad < 1 >_{\psi, \zeta} = 1,$$

related to the source dependent action. By construction, here both $\zeta_i(t)$ and $\mu[z]$ are arbitrary. The presence of arbitrary $\mu[z]$ reveals the implicit dependence of the generating functional on the gauge-fixing Fermion $\Psi$ for nonzero external source $\zeta_i$. 

P.M. Lavrov (Tomsk)
For a constant $\mu$, $\mu = \text{const}$, the latter does drop-out completely, and we get

$$< \int dt \zeta_k(t) (z^k(t) \frac{d}{d} \mu) >_{\psi,\zeta} = 0,$$

which is exactly the standard form of a Ward identity. By identifying the $\mu$ in the "modified Ward identity" with the solution of the compensation equation, it follows according to our result,

**New relation in BRST-BFV theory**

$$Z_{\psi_1}(\zeta) = Z_\psi(\zeta) \left[ 1 + < \left( i/\hbar \right) \int dt \zeta_k(t) (z^k(t) \frac{d}{d} \mu) [-\delta \psi] >_{\psi,\zeta} \right].$$
Finally, let us notice the following. If we introduce the so-called "antifields" $\bar{z}_i^*(t)$, whose statistics is opposite to that of $z^i(t)$, by adding the term $\int dt \bar{z}_k^*(t)(\overset{\leftarrow}{z^k(t)}d)$ to the $W_\psi(\zeta)$, we get the new generating functional $Z_\psi(\zeta, z^*)$.

$$Z_\psi(\zeta, z^*) = \int Dz \exp \left[ \frac{i}{\hbar} W_\psi(\zeta, z^*) \right],$$

$$W_\psi(\zeta, z^*) = W_\psi(\zeta) + \int dt \bar{z}_k^*(t)(\overset{\leftarrow}{z^k(t)}d).$$

Now, let us perform here the change $z^i \rightarrow \bar{z}^i$ with arbitrary $\mu[z]$, we get

$$< [1 + (i/\hbar) \int dt \zeta_k(t)(\overset{\leftarrow}{z^k(t)}d)\mu] [1 + (\mu d)]^{-1} >_{\psi,\zeta,\bar{z}^*} = 1,$$

where

$$< (\ldots) >_{\psi,\zeta,\bar{z}^*} = [Z_\psi(\zeta, z^*)]^{-1} \int Dz (\ldots) \exp \left[ \frac{i}{\hbar} W_\psi(\zeta, z^*) \right],$$

$$< 1 >_{\psi,\zeta,\bar{z}^*} = 1.$$
Ward identities

For a \( \mu = \text{const} \), we get from the above identity

\[
< \int dt \zeta_k(t) (z^k(t) \overset{\to}{d}) >_{\psi, \zeta, z^*} = 0.
\]

The latter is rewritten in a variation-derivative form

\[
\int dt \zeta_k(t) \frac{\overset{\to}{\delta}}{\delta z^*_k(t)} \ln Z_{\psi}(\zeta, z^*) = 0.
\]
Now, let $S(z, z^*)$ be a functional Legendre transform to $(\hbar/i) \ln Z_\psi(\zeta, z^*)$ with respect to the external source $\zeta_i$,

$$z^k = (\hbar/i) \ln Z_\psi(\zeta, z^*) \frac{\leftarrow \delta}{\delta \zeta_k} (-1)^{\varepsilon_k} = (\hbar/i) \frac{\rightarrow \delta}{\delta \zeta_k} \ln Z_\psi(\zeta, z^*),$$

$$S(z, z^*) = (\hbar/i) \ln Z_\psi(\zeta, z^*) - \int dt \zeta_k(t) z^k(t),$$

$$S(z, z^*) \frac{\leftarrow \delta}{\delta z^j(t)} = -\zeta_j(t).$$
Then the master equation,

$$(S, S) = 0,$$

holds for $S$, where the notation $(,)\,$ in l. h. s. means the so-called ”antibracket” well-known in the BV formalism for covariant quantization of gauge field systems,

$$(f, g) = \int dt f\left( \frac{\delta}{\delta z^i(t)} \frac{\delta}{\delta z_i^*(t)} - \frac{\delta}{\delta z_i^*(t)} \frac{\delta}{\delta z^i(t)} \right) g.$$
We have introduced the conception of finite BRST-BFV transformations in the generalized Hamiltonian formalism for dynamical systems with constraints.

It was shown that the Jacobian of finite BRST-BFV transformations, being the main ingredient of the approach, can be calculated explicitly in terms of the corresponding generator acting on finite field-dependent functional parameter of these transformations.

We have introduced the compensation equation providing for a connection between the generating functionals of Green functions formulated for a given dynamical system in two different gauges.

We have found an explicit solution to the compensation equation.
We have proposed the functional approach to BRST-BFV transformations, and then reproduced all the results obtained above, in functional terms.

The functional formulation of finite BRST-BFV transformations provided for deriving in a simple way the Ward identity.

For the first time in the BRST-BFV theory we have derived connection between the generating functional of Green functions written in different gauges.
At the present we have extended our consideration to

- the generalized $Sp(2)$ Hamiltonian formalism (I.A. Batalin, P.M.L., I.V. Tyutin (1990)),
  
  I.A. Batalin, P.M.L., I.V. Tyutin, arXiv:1405.7218[hep-th]

- the field - antifield (BV) formalism (I.A. Batalin, G.A. Vilkovisky (1981)),
  
  I.A. Batalin, P.M.L., I.V. Tyutin, arXiv:1405.2621[hep-th]

- the $Sp(2)$ covariant formalism (I.A. Batalin, P.M.L., I.V. Tyutin (1990))
  
  I.A. Batalin, K. Bering, P.M.L., I.V. Tyutin, arXiv:1406.4695[hep-th]
Thank you for attention!