Perfect imaging: they do not do it with mirrors

Ulf Leonhardt and Sahar Sahebdivan

School of Physics and Astronomy, University of St Andrews, North Haugh, St Andrews KY16 9SS, UK

Received 1 July 2010, accepted for publication 29 November 2010
Published 21 January 2011
Online at stacks.iop.org/JOpt/13/024016

Abstract
Imaging with a spherical mirror in empty space is compared with the case when the mirror is filled with Maxwell’s fish eye medium. Exact time-dependent solutions of Maxwell’s equations show that perfect imaging is not achievable with an electrical ideal mirror on its own, but it is with Maxwell’s fish eye in the regime where it implements a curved geometry for full electromagnetic waves.

Keywords: perfect imaging without negative refraction, non-Euclidean transformation optics

1. Introduction
The prospect of perfect imaging with negative refraction [1] has initiated the entire research area of metamaterials that, in turn, inspired the development [2, 3] of transformation optics [4–7], the subject of this Special Issue. Yet, ironically, negatively refracting lenses have never worked perfectly in practice; only ‘poor man’s lenses’, that are substantially thinner than the wavelength, have shown subwavelength imaging [8]. The reason is that negative refraction is only possible in highly dispersive and hence highly dissipative materials [9]; here absorption not only reduces the intensity but severely limits the resolution of the, theoretically, perfect lens. Alternatives are hyperlenses [10, 11] that rely on materials with indefinite metric. These are anisotropic materials where one of the eigenvalues of the electric permittivity is negative; these materials thus implement a hyperbolic geometry [11] (hence the name hyperlens). Hyperlenses are able to funnel out light from near-fields without losing subwavelength detail, but their resolution is given by their geometric dimensions, and is not unlimited.

Another example of perfect imaging has been known, as a theoretical idea, since an 1854 paper by Maxwell [12]. This device, called Maxwell’s fish eye, because it reminded Maxwell of the eye of a fish, uses positive refraction. Maxwell’s fish eye focuses all light rays emitted from any point in an exact image point; it makes a perfect lens for light rays. Luneburg [13] discovered that it maps light rays in physical space to rays on a virtual sphere, a curved space; Maxwell’s fish eye thus is an early example (and inspiration) of non-Euclidean transformation optics [14]. However, it has been known since Abbe’s theory of imaging that the resolution of an optical instrument is limited by the wave nature of light [15]. Ray optics is not sufficient here, especially in curved geometries, unlike in Euclidean transformation optics [3] where geometrical optics [15] is exact [5, 7]. Analytic solutions of Maxwell’s equations for Maxwell’s fish eye [16, 17] proved theoretically that perfect imaging is possible with positive refraction, and recent experiments [18] indicate that it also works in practice.

Yet perfect imaging with positive refraction [16, 17, 19] challenges [20–23] some of the accepted wisdom [1] of subwavelength imaging: it does not appear to perform an amplification of evanescent waves [1] and requires a drain at the image in the stationary regime [16]. To illuminate some of the perceived paradoxes of perfect imaging, it is instructive to consider the simplest possible case [23]: imaging with a spherical mirror. In this paper, the imaging with such a mirror is compared with the imaging in Maxwell’s fish eye [17]. Exact time-dependent solutions of Maxwell’s equations reveal the similarities and characteristic differences between the two cases; the mirror cannot perfectly image, but Maxwell’s fish eye can.

Imagine a point dipole placed at the centre of a spherically symmetric mirror that completely surrounds it (figure 1). The dipole emits electromagnetic radiation that reaches the mirror, is reflected there, travels back and focuses at the centre where the dipole sits—the dipole is imaged to itself. Due to the symmetry of the sphere, this is the simplest conceivable imaging problem. Within the regime of geometrical optics [15], the focusing with the mirror would be perfect, but the highly concentrated field in the centre violates
Figure 1. Spherical mirror. In this paper we consider the electromagnetic wave emitted by a point dipole (double arrow) in the centre of a perfect spherical mirror (half sphere—only half of the mirror is shown in order to be able to see the interior). We compare this case with imaging in Maxwell’s fish eye in three dimensions [17].

The validity condition of geometrical optics [15]. Therefore, exact calculations of the complete Maxwell equations are required for elucidating the imaging performance of the mirror. The same applies to Maxwell’s fish eye [12] where we fill the space enclosed by the mirror with a dielectric material that has a certain radially symmetric graded refractive index \( n(r) \).

In both cases, the propagation of electromagnetic waves is governed by a length scale, the size of the spherical mirror, and an associated timescale, the travel time of light in the mirror. Our results will only depend on these scales. Therefore, to simplify the algebra, it is wise to measure space in units of the mirror radius; in these units the mirror surrounds the dipole at radius

\[ r = 1 \quad \text{(mirror)}. \tag{1} \]

It is also advantageous to measure the time \( t \) in units of the distance travelled in free space; in these units we obtain for the speed of light in vacuum

\[ c = 1. \tag{2} \]

Equations (1)–(5), together with Maxwell’s equations of electromagnetism, set the scene for the problem we investigate in this paper: the radiation of a dipole placed at \( r = 0 \). But before we address this problem a few (well-known) ideas on causality are needed.

2. Causality

Maxwell’s electromagnetism is completely time-reversible; a dipole may emit an electromagnetic wave, but it may also absorb the same wave run in reverse—by carefully choosing the initial conditions at the boundary an electromagnetic wave may focus at the dipole and be completely absorbed. The latter solutions of Maxwell’s equations, where the interaction of the wave with the dipole lies in the future, are called advanced, whereas the former solutions, where the dipole causes the emission of the wave, are called retarded (figure 2). Clearly, in our case retarded solutions are required; they are causal: no radiation is present prior to the event of emission at, say, time \( t = 0 \). We thus require for all field quantities \( G \):

\[ G(t) = 0 \quad \text{for } t < 0. \tag{6} \]

Suppose we represent the quantity \( G \) in terms of its Fourier components \( \tilde{G} \):

\[ G(t) = \int_{-\infty}^{+\infty} \tilde{G}(\omega)e^{-i\omega t} d\omega. \tag{7} \]

The causality condition (6) is met if \( \tilde{G}(\omega) \) is analytic on the upper half complex \( \omega \) plane and decays sufficiently fast, such that we can close the integration contour of the Fourier
Figure 3. Causality on the complex frequency plane. The Fourier transform of causal waves (figure 2) must be analytic and decaying on the upper half complex plane. The singularities (dots) of the Fourier transform lie on the lower half plane or, as in this picture, on the real axis. In such a case, the contour (grey line) of the Fourier integral (7) can be closed on the upper half plane for \( t < 0 \) and the integration gives zero, as required by causality (6). The picture shows the first 11 singularities (28) of the Fourier-transformed vector potential of dipole radiation in the spherical mirror (figure 1).

The unity vector \( \mathbf{e}_z \) only depends on the radius \( r \).

Consider a radially symmetric impedance-matched medium (3) that only depends on the radius \( r \) (our calculations will show that this assumption is consistent with Maxwell’s equations). The unity vector \( \mathbf{e}_z \) in \( z \)-direction reads in terms of the basis vectors in spherical polars [5, 7]

\[
e_z = \cos \theta \mathbf{e}_r - \frac{\sin \theta}{r} \mathbf{e}_\theta.
\]

We thus require

\[
A' = A(r, t) \left( \cos \theta, -\frac{\sin \theta}{r}, 0 \right).
\]

However, the vector potential is rather a one-form \( A_j \) [7] (also known as a covariant vector) and not the (contravariant) vector \( A' \). We thus lower the index with the metric of the spherical polar coordinates [5] and write down the ansatz for the vector potential as

\[
A_j = A(r, t)(\cos \theta, -r \sin \theta, 0).
\]

We obtain the magnetic field \( \mathbf{B} \) from \( \mathbf{B} = \nabla \times \mathbf{A} \) where for our ansatz (11) we calculate the curl in spherical polars [5] and lower the index [5]. We arrive at

\[
B_r = 0, \quad B_\theta = 0, \quad B_\phi = B \sin^2 \theta
\]

with the magnetic-field amplitude in the plane of emission (for \( \theta = \pi/2 \)) being

\[
B = -r \partial_r A.
\]

We use the symbol \( \partial_r \) to abbreviate the partial derivative \( \partial / \partial r \) and we are going to use \( \partial_\phi \) for \( \partial / \partial \phi \). The electric field \( \mathbf{E} \) we obtain from Maxwell’s equation in the medium (3)

\[
\nabla \times \mathbf{B} = -\partial_t \mathbf{E}
\]

with our units (2) and for a point dipole at the centre pointing in the \( z \)-direction that emits a flash of light at \( t = 0 \) with normalized intensity. We thus find for the non-vanishing electric-field components

\[
\partial_r E_r = \left( \frac{2B}{n^2 r^2} - \frac{\delta(r)\delta(t)}{n} \right) \cos \theta,
\]

\[
\partial_\phi E_\phi = \left( -\frac{1}{n} \partial_\phi B + \frac{r \delta(r)\delta(t)}{n} \right) \sin \theta
\]

while the azimuthal component \( \partial_\phi E_\phi \) is zero. Furthermore, we use Faraday’s law \( \nabla \times \mathbf{E} = -\partial_t \mathbf{B} \) expressed in spherical polars [5] with indices lowered [5] and our result (12) for the magnetic field, to obtain the wave equation

\[
\partial_r \frac{1}{n} \partial_r B - \frac{2B}{n^2 r^2} - \partial_\phi^2 B = r \partial_r \frac{\delta(r)\delta(t)}{n}.
\]

All other components of \( \nabla \times \partial_r \mathbf{E} \) vanish for the electric field (15) in spherical polars, which reduces the problem to the wave equation (16) and justifies our ansatz (11). Finally, we require that the mirror acts as an ideal electrical mirror where the electric-field components in the mirror are put to zero, in our case (1)

\[
E_\phi |_{r=1} = 0,
\]
which implies from expression (15) that
\[ \partial_r \left( \frac{B}{r} \right) \bigg|_{r=1} = 0. \] (18)

We see that the magnetic field must not change at the mirror. In the following we seek causal solutions of the wave equation (16) with boundary condition (18) for the magnetic field \( B \) expressed in terms (13) of the vector potential. First we consider the dipole radiation in empty space surrounded by the spherical mirror [23] and then we compare this case with Maxwell’s fish eye [17].

4. Mirror

In empty space (4) the left-hand side of the wave equation (16) for the magnetic field (13) reduces to
\[ \partial_r^2 B - \frac{2B}{r^2} - \partial_t^2 B = -r \partial_r \left( \partial_r^2 A + \frac{2}{r} \partial_r A - \partial_t^2 A \right) \] (19)
such that the wave equation (16) is satisfied if
\[ \partial_r^2 A + \frac{2}{r} \partial_r A - \partial_t^2 A = -\delta(r) \delta(t). \] (20)

We thus require that the Fourier components \( \tilde{A} \) of the vector potential, with the convention (7) for the Fourier transform, obey the inhomogeneous wave equation
\[ \partial_r^2 \tilde{A} + \frac{2}{r} \partial_r \tilde{A} + \omega^2 \tilde{A} = \frac{\delta(r)}{2\pi}. \] (21)

For \( r > 0 \) the general solution of the—then homogeneous—wave equation (21) is
\[ \tilde{A} = \frac{A_+}{r} e^{i\omega r} + \frac{A_-}{r} e^{-i\omega r}. \] (22)

Note again that in our units (2) the frequency \( \omega \) is equal to the wavenumber that normally appears in the spherical waves (22). The first term of expression (22) describes an outgoing wave and the second term an ingoing wave. The spherical mirror turns outgoing into ingoing waves and so we expect that the coefficients \( A_\pm \) are not independent. Indeed, we obtain from condition (18) for the magnetic field (13) of the wave (22)
\[ A_- = -A_+ e^{i\omega r} \frac{\omega^2 + i\omega - 1}{\omega^2 - i\omega - 1} = -A_+ e^{2i\omega r - 2i\delta} \] (23)
where \( \delta \) denotes the phase
\[ \delta = \arctan \left( \frac{\omega}{\omega^2 - 1} \right). \] (24)

At the mirror (1) the outgoing component \( A_+ \exp(i\omega r) \) thus produces the ingoing component \( A_- \exp(-i\omega r) \) and vice versa, apart from the extra phase (24). This phase vanishes for high frequencies \( \omega \rightarrow \infty \) where the wavelength approaches zero and imaging becomes perfect, regardless whether it is subwavelength-limited or not. For finite \( \omega \) the phase (24) is responsible for limiting the resolution of the mirror, as we see next.

So far we have considered the general solution of the homogeneous radial wave equation with the boundary condition (18) at the mirror. Now we turn to the solution that satisfies the inhomogeneous equation (21) and therefore describes the field generated by the point dipole. For this we write down the following combination of outgoing and ingoing waves (22):
\[ \tilde{A} = \frac{\sin(\omega - \omega r + \delta)}{8\pi^2 r \sin(\omega + \delta)} \] (25)

One verifies that this \( \tilde{A} \) obeys the condition (23) of spherical waves reflected by the mirror. We also see that
\[ \tilde{A} \sim \frac{1}{8\pi^2 r} \quad \text{for } r \rightarrow 0, \] (26)
where \((4\pi r)^{-1}\) is the Green function of the Poisson equation,
\[ \left( \partial_r^2 + \frac{2}{r} \partial_r \right) \frac{1}{4\pi r} = \nabla^2 \frac{1}{4\pi r} = -\delta(r). \] (27)

In the vicinity of the origin we can ignore features of \( \tilde{A} \) on the scale of the wavelength that depend on the frequency \( \omega \) and so the inhomogeneous wave equation (21) reduces to the Poisson equation here. Therefore the wave (25) with the asymptotics (26) satisfies the wave equation (21), and we already know that it also obeys the boundary condition (18). The Fourier components (25) thus constitute the electromagnetic field of a light flash emitted by a point dipole, provided they are causal, i.e. analytic and decaying on the upper half \( \omega \) plane.

Causality is the final point we need to consider for the field (25). Representing the sine functions in expression (25) in terms of exponentials we see that \( \tilde{A} \rightarrow 0 \) for \( \text{Im} \ \omega \rightarrow \infty \). Furthermore, \( \tilde{A} \) is analytic, apart from poles \( \omega_m \) on the real axis (figure 3) where
\[ \sin(\omega_m + \delta_m) = 0, \quad \delta_m = \delta(\omega_m). \] (28)

The Fourier transform \( \tilde{A} \) also has a quadratic singularity at \( \omega = 0 \). In order to obtain a causal solution we must move the poles slightly below the real axis, by providing \( \omega \) with an infinitesimal, positive imaginary part. Alternatively, we move the integration contour up from the real axis by an infinitesimal distance (figure 3). In both cases, we can close the contour of the Fourier integral (7) on the upper half plane for \( t < 0 \) and get zero, as required for causal waves (6). For \( t > 0 \) we close the contour on the lower half plane and obtain from Cauchy’s theorem
\[ A = \frac{1}{2\pi i r} \sum_{n=1}^{\infty} \eta_n \sin(\omega_m r) \sin(\omega_m t) \] (29)
with the coefficients
\[ \eta_m = \frac{\omega_m^4 - \omega_m^2 + 1}{\omega_m^4 - 2\omega_m^2}. \] (30)

For the light flash emitted by the dipole in the centre of the mirror, we obtained the exact solution (29) with the pole
frequencies (28) and the coefficients (30). Let us investigate the extent to which this wave images the dipole onto itself.

For perfect imaging we require that an outgoing wave is perfectly converted into an ingoing wave. The wave is focused on the spatial derivative of  

As the magnetic field \( (13) \) and the electric field \( (15) \) depend \(<\Delta_1(\delta)\) periodic delta function at the centre and bounces back and forth with a period of twice perfectly converted into an ingoing wave. The wave is focused the extent to which this wave images the dipole onto itself.

\[
\Delta(t) = \sum_{m=-\infty}^{+\infty} \delta(t-2m) = \frac{1}{2} \sum_{m=-\infty}^{+\infty} e^{im\pi t} = \frac{1}{2} + \sum_{m=1}^{\infty} \cos(m\pi t).
\]

(31)

At the mirror \( (1) \), the wave changes sign and propagates backwards. We thus require for the radial wave \( A_0 \) in the case of perfect imaging of the dipole onto itself

\[
A_0 = \frac{\Delta(t-r)}{4\pi^2r} - \frac{\Delta(t+r-2)}{4\pi^2r} = \frac{\Delta(t-r)}{4\pi^2r} - \frac{\Delta(t+r)}{4\pi^2r}.
\]

(32)

Writing \( A_0 \) in terms of the periodic delta function \( \Delta \) guarantees that \( A_0 \) is periodic with period two. We obtain from expression (31)

\[
A_0 = \frac{1}{2\pi^2} \sum_{m=1}^{\infty} \sin(m\pi r) \sin(m\pi t).
\]

(33)

Formula (33) strongly resembles our result (29) for the spherical mirror, but not perfectly. There, instead of the eigenfrequencies \( m\pi \) the pole frequencies \( \omega_m \) appear and the trigonometric series (29) contains the coefficients \( \eta_m \). The poles (28) depend on the phase (24) that, for large \( \omega \), approaches \( \omega^{-1} \). In this limit we obtain for the poles (28)

\[
\omega_m = \pm \left( \frac{m\pi}{2} + \sqrt{\frac{m^2\pi^2}{4} - 1} \right) \quad \text{(integer } m) \quad (34)
\]

that approaches \( m\pi \) for \( m \to \infty \). In this limit also \( \eta_m \to 1 \), as we see from expression (30). Consequently, whenever large- \( m \) components dominate the series (33) of the perfect wave, they also dominate the wave (29) in the spherical mirror. We see from definition (32) that this is the case at the infinitely peaked front of the light flash bouncing back and forth in the mirror. Therefore, the perfectly imaging wave makes the dominant contribution to the electromagnetic wave in the spherical mirror.

Figure 4 shows the difference \( A - A_0 \) between the vector potential (29) of the light reflected in the spherical mirror and perfectly reflected light flashes (32). Relativistic causality implies that the outgoing wave \( A \) must remain a perfect pulse until it hits the mirror. Therefore the deviation from the ideal vector potential \( A_0 \) cannot correspond to any physical field. Figure 4 indicates that, during this stage, \( A - A_0 \) is constant in space but grows in time. This growth must be linear in order for \( A \) to obey the homogeneous wave equation (20). As the magnetic field (13) and the electric field (15) depend on the spatial derivative of \( A \) in the gauge relevant to our ansatz (11), the electromagnetic fields do not deviate from free dipole radiation until the wave hits the mirror. After the first reflection the wave is no longer a perfect pulse: the vector potential carries an additional discontinuity at the position of the pulse and develops a field in its wake. Note that the mirror is not assumed to be dispersive or otherwise imperfect; these features quantify the imperfections of imaging with an ideal spherical mirror.

5. Perfection

Now imagine we fill the space enclosed by the mirror with the impedance-matched (3) medium (5) of Maxwell’s fish eye [12]. This device implements the geometry of a curved space. In particular, it creates the illusion that light propagates on the three-dimensional surface of the four-dimensional hypersphere [13]. It turns out [17] that the geometry of waves in Maxwell’s fish eye appears in the clearest possible form if we write the vector potential \( A \) as

\[
A = 2 \int (\partial_r D) n r dr
\]

(35)

or, equivalently, \( \partial_r A = 2n \partial_r D \)

We show next that the function \( D \) is the scalar Green function on the hypersphere [17]. For this we inspect the left-hand side of the wave equation (16) where

\[
\partial^2_{n} \frac{1}{n} \partial_{n} B = - \frac{2B}{n^2 r^2} - \partial^2_{n} B
\]

\[
= -2nr \partial_r \left( \frac{1}{n^2 r^2} \partial_r n r^2 \partial_r D - D - \partial^2 D \right). \quad (37)
\]
Therefore the wave equation (16) for the refractive-index profile (5) with \( n(0) = 2 \) is satisfied if \( \tilde{D} \) obeys
\[
\frac{1}{n^2 r^2} \partial_r n^2 \partial_r \tilde{D} - \partial_t^2 \tilde{D} = \frac{\delta(r)\delta(t)}{n^3},
\]
(38)
or, in Cartesian coordinates,
\[
\frac{1}{n^2} \nabla \cdot n \nabla \tilde{D} - \partial_t^2 \tilde{D} = \frac{\delta(r)\delta(t)}{n^3}.
\]
(39)
This is the conformally coupled radial scalar wave equation on the virtual space implemented by Maxwell’s fish eye, the surface of the hypersphere [17].

We see from Euler’s formula \( \exp(i \omega t) = \cos \omega t + i \sin \omega t \) and obtain for the Fourier-transformed radial wave equation the surface of the hypersphere [17]. If \( \tilde{D} \) satisfies the boundary condition (18) at the mirror in addition to the wave equation (38) \( \tilde{D} \) is the required scalar Green function.

Let us construct the Green function \( D \). For this we Fourier-transform the wave equation (39),
\[
\frac{1}{n^2} \nabla \cdot n \nabla \tilde{D} + (\omega^2 - 1)\tilde{D} = \frac{\delta(r)}{2\pi n^3},
\]
(40)
and obtain for the Fourier-transformed radial wave equation
\[
\frac{1}{n^2 r^2} \partial_r n^2 \partial_r \tilde{D} + (\omega^2 - 1)\tilde{D} = \frac{\delta(r)}{2\pi n^3}.
\]
(41)
We consider the limit \( r \to 0 \) where the far-field terms \((\omega^2 - 1)\tilde{D} \) do not influence the near field of \( \tilde{D} \) anymore, such that we can replace the left-hand side of the Fourier-transformed radial wave equation (41) by
\[
\frac{1}{n^2 r^2} \partial_r n^2 \partial_r \tilde{D} = \frac{1}{n^2} \left( \partial_r^2 + \frac{2}{n^2} \right) \tilde{D} = \frac{\nabla^2 \tilde{D}}{n^2} = -\frac{\delta(r)}{2\pi n^2}
\]
(42)
if \( \tilde{D} \) obeys the asymptotics
\[
\tilde{D} \sim \frac{1}{(4\pi)^2 r}.
\]
(43)
One easily verifies the following expression for the two fundamental solutions of the Fourier-transformed wave equation (41) with asymptotics (43):
\[
\tilde{D}_\pm = \frac{1}{(4\pi)^2} \left( r + \frac{1}{r} \right) \exp(\pm 2i\omega \arctan r).
\]
(44)
Let us write down the expression
\[
\tilde{D} = \left( r + \frac{1}{r} \right) \frac{\sin(2\omega \arctan r) - \sin(2\omega \arctan r)}{(4\pi)^2 \sin(\omega \arctan r)}.
\]
(45)
We see from Euler’s formula \( \exp(i \omega t) = \cos \omega t + i \sin \omega t \) and \( 2 \arctan r = \pi - 2 \arctan r \) that expression (45) is a linear combination of the two fundamental solutions (44) and hence a solution of the homogeneous wave equation (41) for \( r \neq 0 \). Additionally, since \( 2 \arctan r \to \pi \) for \( r \to \infty \) expression (45) obeys the required asymptotics (43) for \( r \to 0 \). Furthermore, one verifies that \( \tilde{D} \) conforms to the boundary condition (18) at the mirror,
\[
\partial_r r \partial_r \tilde{D}|_{r=1} = 0.
\]
(46)
Formula (45) thus qualifies as the Fourier transform of a Green function. Finally, \( \tilde{D} \) decays for \( \Im \omega \to \pm \infty \) and is analytic, apart from single poles at
\[
\omega_m = m \quad (\text{non-zero integer } m)
\]
(47)
and a double pole at \( \omega = 0 \). Similar to the case of the empty spherical mirror (section 4), we move the poles slightly below the real axis or move the contour of the Fourier integral (7) above the real axis (figure 3) in order to obtain the causal Green function \( D \).

The causal Green function describes the time evolution of a flash of light emitted at the point source in the centre of the fish eye mirror. We obtain from Cauchy’s theorem for the Fourier transform of expression (45)
\[
D = \frac{\Theta(t)}{4\pi^2} \left( r + \frac{1}{r} \right) \sum_{m=0}^{\infty} [\sin(2m \arctan r)
- \sin(2m \arccot r)] \sin(mt)
\]
(48)
and write this result in terms of the periodic delta functions (31) similar to formula (32) for the ideal imaging wave:
\[
D = \frac{\Theta(t)}{8\pi} \left( r + \frac{1}{r} \right) \left[ \Delta \left( \frac{t - 2 \arctan r}{\pi} \right) + \Delta \left( \frac{t - 2 \arccot r}{\pi} \right) \right].
\]
(49)
We write \( \Delta \) in terms of the defining series (31) of delta functions, extract a factor of \( \pi \) and combine in the resulting series the numbers \( 2m \) and \( 2m + 1 \) from the \( \Delta \) series (31) in one summation index \( m' \). We obtain the result
\[
D = \frac{\Theta(t)}{8\pi} \left( r + \frac{1}{r} \right) \sum_{m'=-\infty}^{\infty} [\delta(t - 2 \arctan r - m'\pi)
- \delta(t - 2 \arccot r - m'\pi)].
\]
(50)
Formula (50) shows that the light flash emitted by the point dipole propagates through Maxwell’s fish eye as a perfect flash. It is reflected at the mirror where the amplitude changes sign, it focuses at the centre and bounces back and forth without disintegrating in time: in contrast to the spherical mirror, Maxwell’s fish eye perfectly images in time (figure 5).

6. Imaging

We have seen that Maxwell’s fish eye faithfully transmits time-dependent signals, it perfectly images in time, but how does it image in space? The spherical mirror focuses only one point, the centre, to itself; an elliptical mirror images one focal point to the other focal point. Maxwell’s fish eye is a so-called perfect optical instrument [15] where all light rays from any source point \( r_0 \) focus at the corresponding image point \( r'_0 \). The fish eye is a perfect instrument for light rays, but the optical resolution of imaging is normally limited by the wave nature of light. Remarkably, one can derive exact solutions of Maxwell’s equations for electromagnetic waves in Maxwell’s fish eye [17] (that needs to be impedance-matched in 3D) but
the technicalities [7] involved in justifying and explaining them go beyond the scope of this paper. The principal technical problem is the tensor nature of the electromagnetic field. Fortunately, we can liberate ourselves from the technicalities of tensor analysis and understand the essence of perfect imaging with positive refraction by discussing the scalar Green function $D$; the electromagnetic field can be derived from $D$ by certain differential forms [7, 17] that reduce to expressions (36) and (15) for the special case of the source being at the centre of the Fourier-transformed wave equation (40) with the correct asymptotics at the source point.

\[
\tilde{D} = \left( r' + \frac{1}{r'} \right) \frac{\sin(2\omega \arccot r')}{(4\pi)^2 \sin(\pi \omega)}, \tag{51}
\]

and replace the radius $r$ by the Möbius-transformed radius $[7, 17]$

\[
r' = \frac{|r - r_0|}{\sqrt{1 + 2r \cdot r_0 + |r|^2 |r_0|^2}}. \tag{52}
\]

One can show that formulae (51) and (52) describe solutions of the Fourier-transformed wave equation (40) with the correct asymptotics at the source point,

\[
\tilde{D} \sim \frac{1}{(4\pi)^2 |r - r_0|}, \tag{53}
\]

In the ray optics of Maxwell’s fish eye [7], source and image correspond to the 0 and the $\infty$ of $r'$, the Möbius-transformed radius (52):

\[
\begin{align*}
r' = 0 & \iff r = r_0 \quad \text{(source),} \\
r' = \infty & \iff r = r'_0 = -\frac{r_0}{|r_0|^2} \quad \text{(image).} 
\end{align*} \tag{54}
\]

However, the wave $\tilde{D}$ is not singular at the image, because we obtain from formula (51)

\[
\tilde{D} \sim \frac{\omega \sin(2\omega \arccot r')}{8\pi^2 \sin(\pi \omega)} \quad \text{with} \quad \sin x = \frac{\sin x}{x}. \tag{55}
\]

We can interpret $\tilde{D}$ in two ways, as the Fourier transform of the time-dependent Green function $D$ that describes a flash of light or, alternatively, as the amplitude of the continuous wave generated by a stationary point source that emits and absorbs radiation in a stationary state. Our result (55) shows that the electromagnetic wave of a stationary source develops a diffraction-limited image, in agreement with experiment [18]. However, let us adopt the alternative interpretation of $\tilde{D}$ as the Fourier transform of a light flash $D$. We obtain in complete analogy to the calculations in section 5:

\[
D = \frac{\Theta(t)}{8\pi} \left( r' + \frac{1}{r'} \right) \sum_{m = -\infty}^{\infty} \left[ \delta(t - 2\arctan r') - 2m\pi \right]. \tag{56}
\]

The flash perfectly focuses at the image point, is reflected there and bounces back to the source etc. In Maxwell’s fish eye the

---

**Figure 5. Mirror versus Maxwell’s fish eye.** Space–time diagrams of the vector potential of light flashes evolving in time (left) for imaging in a spherical mirror and (right) in Maxwell’s fish eye supplemented by a mirror. The mirror is placed at the vertical boundaries of each diagram where the grey vertical line indicates the world line of the centre. The black curves correspond to the infinite peaks of the flashes, the contours describe the finite values of the vector potential. As the right picture contains no finite contours, just infinite peaks, the flash is not disintegrating and bounces back and forth between the mirror in Maxwell’s fish eye: imaging is perfect there. In contrast, the left picture shows the imperfections arising from reflection in an ideal electrical mirror: after the flash has bounced off the mirror the vector potential disintegrates and bounces back and forth between the mirror in Maxwell’s fish eye: imaging is perfect there. After that time the entire interior of the mirror is filled with a spatially non-uniform vector potential that corresponds to a non-vanishing electromagnetic field distorting the flash: perfect imaging is not done with mirrors.
image point thus acts like a perfect mirror. In the stationary regime we can imagine the wave as a continuous stream of light flashes. Each of the elementary flashes is reflected at the image point where its amplitude changes sign. In the stationary regime we average over the stream of flashes and so the sign change upon reflection causes the image to get blurred. Maxwell’s fish eye has the potential of perfect imaging, but this potential is not realized yet.

What is missing is a crucial ingredient of imaging: a detector. A detector extracts the field at the image point. An ideal point detector acts as a completely passive outlet with point-like resolution (infinitely small cross section). Such a detector could be part of a CCD array or represent the photosensitive molecule of a photographic material that happens to be at the image point. Suppose the detector is positioned at the point \( r_0 \) and extracts the radiation incident at that point. In this case the series (56) of flashes bouncing back and forth reduces to one flash that is emitted at the source and disappears at the image at time \( t = \pi \) in our units:

\[
D = \frac{\Theta(t)}{8\pi} \left( r' + \frac{1}{r'} \right) \delta(t - 2 \arctan r') \Theta(\pi - t). \tag{57}
\]

For this expression we obtain the Fourier transform

\[
\tilde{D} = \frac{1}{(4\pi)^2} \left( r' + \frac{1}{r'} \right) \exp(2i\omega \arctan r'). \tag{58}
\]

Note that formula (58) describes a running wave with complex wave function propagating from the source to the image where the wave disappears, in contrast to the standing wave (51) with real wave function that is reflected at the image. The spatial singularity of \( \tilde{D} \) for \( r' \to \infty \) corresponds to a supplementary source at the image point \( r_0' \), a drain. As \( 2 \arctan r' \to \pi \) for \( r' \to \infty \) the wave carries a phase delay of \( \pi \omega \) at the image [17]. The detector, acting as a drain, creates a perfect spatial image of the source point. So far, we assumed that the detector is positioned at the correct place. Now imagine the detector is elsewhere. As the detector has point-like resolution it can only detect a field that is infinitely concentrated at that point. Therefore, an ideal point detector placed at the wrong position will not detect anything; the perfectly focused field will only appear at the perfectly positioned point detector\(^2\) in agreement with experiments [18]. Of course, the detector may have a non-perfect efficiency such that it does not extract the entire field. In this case, part of the field is reflected, but the transmitted part is perfectly focused; the detected image is infinitely sharp. As any continuous distribution of sources can be thought of as a distribution of source points, any source is perfectly imaged, as long as it is detected by a continuous array of point detectors. Only the detected field is imaged with point-like precision, but detection is the very point of imaging.

However, one may still argue that the drain at the image is an artefact that perturbs the field, creating an infinitely focused field entirely on its own, the illusion of perfect imaging but not a true image [20, 22, 23]. Let us consider a counter example where the introduction of a drain turns out not to improve imaging in time: the spherical mirror. Returning to the starting point of this paper, we consider empty space enclosed by the spherical mirror. We form the following linear combination of the two fundamental solutions (22) with the reflection condition (23):

\[
\tilde{A} = \frac{1}{8\pi r'} (e^{i\omega r'} - e^{2i\omega + 2i\delta - i\omega r'}). \tag{59}
\]

This expression creates the singularity of the source at the centre and a drain with phase delay \( 2\omega + 2\delta \) in our units for a wave reflected at the mirror with phase shift \( 2\delta \) back to the source in time \( t = 2 \). However, expression (59) also develops poles in the complex frequency plane at the zeros of \( \omega^2 - i\omega - 1 \), i.e. at

\[
\omega = \frac{i \pm \sqrt{3}}{2}. \tag{60}
\]

on the upper half plane! Therefore, the solution (59) is not causal, the drain is not causally connected to the source, but rather acts as an independent source of radiation. However, as we have seen, Maxwell’s fish eye makes all the difference here; in this case the drain is consistent with causality.

7. Credo

An ideal spherical mirror cannot perfectly image on its own, the mirror inevitably distorts light pulses (left of figure 5). However, if the space enclosed by the mirror is filled with the impedance-matched Maxwell fish eye medium, pulses are no longer distorted and thus the imaging is perfect in time (right of figure 5). We have seen that the introduction of a drain at the image is allowed by causality and leads to perfect imaging in space in the stationary regime (similar to the perfect focusing of ultrasound waves by time reversal [24]). On the other hand, in imperfect imaging the drain would be in conflict with causality. One of the puzzles of perfect imaging with positive refraction seems to be resolved.

Yet one may still wonder how Maxwell’s fish eye is able to restore evanescent waves in a similar way to how negative refraction does [1]. But note that the amplification of evanescent waves is not the only physical picture explaining the performance of the negatively refracting perfect lens; there is a simple geometrical argument [25]: the lens performs a
folding transformation of space [6]. The negative-index lens appears to fold the space perceived by electromagnetic waves such that three regions are identical, one region in front of the device, the lens itself, and the region behind it (figure 6). As these regions are identical for the field, the field strength of an electromagnetic wave must be exactly the same at all triples of connected points: negative refraction makes a perfect lens by spatial transformation [25]. In this geometrical picture of imaging we do not need to discriminate between evanescent and propagating waves, because the electromagnetic field is transformed in its entirety.

Geometry also explains the perfect imaging with Maxwell’s fish eye that has a positive refractive-index profile. Like in the case of the perfect lens, this device changes the geometry of space for the electromagnetic field. It creates the effect that electromagnetic waves propagate in the three-dimensional surface of the four-dimensional hypersphere. Imagine an ordinary three-dimensional sphere with two-dimensional surface of the four-dimensional hypersphere. The figure illustrates the 2D virtual geometry of light created by the material is valid for the approximation of geometrical optics. In such cases, the virtual geometry of light created by the material is valid for full electromagnetic waves. Perfect imaging is not done with mirrors [26], but with geometry.

Acknowledgments

We thank Roberto Merlin for inspiring this work, Aaron Danner, Susanne Kehr and Tomas Tyc for discussions, and the Royal Society for support.

References

[1] Pendry J B 2000 Phys. Rev. Lett. 85 3966
[2] Leonhardt U 2006 Science 312 1777
[3] Pendry J B, Schurig D and Smith D R 2006 Science 312 1780
[4] Shalaev V M 2008 Science 322 384
[5] Leonhardt U and Philbin T G 2009 Prog. Opt. 53 69
[6] Chen H, Chan C T and Sheng P 2010 Nat. Mater. 9 387
[7] Leonhardt U and Philbin T G 2010 Geometry and Light: the Science of Invisibility (Mineola, Dover)
[8] Fang N, Lee H, Sun C and Zhang X 2005 Science 308 534
[9] Stockman M I 2007 Phys. Rev. Lett. 98 177404
[10] Jacob Z, Alekseyev L V and Narimanov E 2006 Opt. Express 14 8247
[11] Liu Z, Lee H, Xiong Y, Sun C and Zhang X 2007 Science 315 1686
[12] Maxwell J C 1854 Cam. Dublin Math. J. 8 188
[13] Luneburg R K 1964 Mathematical Theory of Optics (Cambridge: Cambridge University Press)
[14] Leonhardt U and Tyc T 2009 Science 323 110
[15] Born M and Wolf E 1999 Principles of Optics (Cambridge: Cambridge University Press)
[16] Leonhardt U 2009 New J. Phys. 11 093040
[17] Leonhardt U and Philbin T G 2010 Phys. Rev. A 81 011804
[18] Ma Y G, Ong C K, Sahebdivan S, Tyc T and Leonhardt U 2010 arXiv:1007.2530
[19] Benitez P, Miñano J C and González J C 2010 Opt. Express 18 7650
[20] Blaikie R J 2010 New J. Phys. 12 058001
[21] Leonhardt U 2010 New J. Phys. 12 058002
[22] Guenneau S, Diatta A and McPhedran R C 2010 J. Mod. Opt. 57 511
[23] Merlin R 2010 Phys. Rev. A 82 057801
[24] de Rosny L and Fink M 2002 Phys. Rev. Lett. 89 124301
[25] Leonhardt U and Philbin T G 2006 New J. Phys. 8 247
[26] Christie A 2000 They Do It With Mirrors (Harmondsworth: Penguin)