A MATRIX MODEL SOLUTION OF HIROTA EQUATION

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ABSTRACT

We present a hermitian matrix chain representation of the general solution of the Hirota bilinear difference equation of three variables. In the large N limit this matrix model provides some explicit particular solutions of continuous differential Hirota equation of three variables. A relation of this representation to the eigenvalues of transfer matrices of 2D quantum integrable models is discussed.
1 INTRODUCTION

The Hirota bilinear equations (HE) \([3]\) provide, may be, the most general view on the world of exactly solvable models, from integrable hierarchies of differential and difference equations, like KdV equations or Toda chains, to, as it was recently shown, the transfer matrices of 2D models of statistical mechanics and quantum field theory \([4, 5, 1]\) integrable by the Bethe ansatz (BA) techniques. Indeed, in the last case, it was shown that the eigenvalues of transfer matrices must obey the general HE with 3 discrete variables corresponding to the rapidity, rank and level of a representation with rectangular Young tableaux, used in the fusion procedure. The Thermodynamical Bethe Ansatz (TBA) equations discovered by Lee and Yang and widely used in the last years for the description of thermal properties and finite size effects of 2D integrable models follow almost directly from this HE \([3]\).

HE has its own history in connection with the matrix models (MM). The general solitonic solutions of HE \([7]\) bear a striking resemblance with the matrix models with logarithmic potentials. The discovery of double scaling limit in the matrix models \([8, 9, 10]\) (corresponding to the big size of matrices and a special tuning of potentials) showed that the matrix models are closely related to KdV and KP hierarchies of integrable differential equations \([11]\). The study following it provided even more general examples of this correspondence: the multi matrix models before taking any large N limit appeared to be related the classical Toda chains \([12]\). Another manifestation of these connections are the Schwinger-Dyson equation for MM which can be written in terms of Virasoro constraints \([13, 14]\) (see \([15]\) for a modern account of this approach).

We would also recall one rather mysterious coincidence: in \([16]\) an open string amplitude for the (1+1)D string with mixed (Dirichlet-Neumann) boundary conditions appeared to have the same form as the Sine-Gordon S-matrix.

All this suggests that matrix models could have something to do with the quantum 2D integrable models and the way from one to another might go through HE.

In this paper we will propose a matrix chain representation of the general solution of Hirota difference equation. We will use for that the analogy between the so called Bazhanov-Reshetikhin determinant representation \([17]\) of the transfer-matrix eigenvalues of integrable models (which obeys the HE) and the determinant representation of matrix chain, in terms of eigenvalues of the matrices. The potentials acting at every site of the chain will depend on the eigenvalues of the matrices and on the coordinate of the site. Hence the arbitrary potential is a function of 2 variables which is in general enough to parameterize any solution of a difference equation of 3 variables.

The solutions of HE relevant to TBA must obey very special boundary conditions. It can include, for example, the condition on the maximal possible size of the fusion representation (reflecting the invariance of the initial model with respect to some continuous symmetry), and Lorentz invariance of the spectrum of physical particles emerging on the top of physical (dressed) vacuum.

It is not easy to extract the physical information from HE or the corresponding TBA
equations as it is not easy to solve the nonlinear difference or integral equations with specific boundary conditions. The results obtained on this way are quite limited: they mostly concern the calculations of central charges and dimensions of operators in the conformal (ultraviolet) limits and various asymptotic expansions corresponding to high energies (see for example \cite{17, 18, 19}). It even appears to be difficult to reproduce the first loop calculation for the asymptotically free models in finite temperature, apart from some simplified models \cite{20}, although numerically the TBA equations work quite well. Another challenge is to find the planar (large N) limit of such an interesting 2D quantum field theory as the principal chiral field. The model was formally integrated by the BA approach in \cite{21, 22}. It has been solved rather explicitly in the large N limit in case of zero temperature and arbitrarily big external field \cite{23, 24}, but the attempts to generalize it to finite temperatures were not successful.

On the other hand, the general solution of HE follows from its integrability and can be represented in terms of a $\tau$-function. In case of a general difference HE it coincides with the determinant representation of Bazhanov-Reshetikhin. It might be a good idea to use this representation, and hence our matrix chain representation, to get some hand on HE and TBA. Of course, for a finite rank N of matrices (and hence finite dimensional groups of symmetry of corresponding models) our representation hardly could offer some breakthrough. But for big N we can try to use the machinery of the matrix models (like orthogonal polynomials, character expansions and various saddle point techniques) to calculate the corresponding infinite determinant.

We were not able to find any solutions of HE satisfying correct TBA-like boundary conditions. A natural way to impose these boundary conditions is the most important drawback of our representation. Leaving it to future studies we propose here some particular solutions of continuous (differential) Hirota equations which describe our matrix chain in the large N limit. They correspond to a particular choice of parameters (potentials) of the chain.

In the next section we will briefly review how the Hirota equation is connected with the integrable models of the 2D quantum field theory. The continuous differential version of them will be presented.

In section 3 we will propose the matrix chain representation of the difference Hirota equation based on the Bazhanov-Reshetikhin determinant representation of the fusion rules for transfer-matrices.

In section 4 we will present some particular examples of solution of the continuous differential HE, given by the one matrix model and the matrix oscillator with the specific boundary conditions.

In section 5 we will sketch out a general solution of the differential HE for an arbitrary time dependent matrix chain potential and consider a more explicit solution for the particular case of time independent potential.

The last section will be devoted to conclusions and prospects.
2 TBA, FUSION RULES 
AND HIROTA DIFFERENCE EQUATION

To set a more physical background for our construction let us briefly review how the HE appears from the Bethe ansatz. We will mostly follow in this section the framework and the notations of \cite{1,2}.

The transfer-matrix of an integrable 2D model with periodic boundary conditions depends (apart from a number of fixed parameters, like volume, temperature or the anisotropy $q$) on three variables: rapidity $u = i\theta$, rank (“color”) $a$ and level (“string” length) $s$. The variables $a$ and $s$ have the meaning of a representation of elementary spins filling the bare vacuum of the model, given by the rectangular Young tableau of the size $a \times s$. The corresponding transfer matrix is called $\hat{T}_a^s(u)$.

The integrability imposes the commutativity of transfer-matrices for different values of all three variables playing thus the role of spectral parameters:

\[ [\hat{T}_a^s(u), \hat{T}_{a'}^{s'}(u')] = 0 \]  

(1)

It follows from (1) that we can always work with the eigenvalues $T_a^s(u)$ instead of the transfer-matrix itself and view them as usual functions.

The transfer-matrices, as well as their eigenvalues, obey a set of relations known as fusion rules, originally found as the relations between S-matrices for particles with different spins in integrable QFT. They can be summarized in the so called Bazhanov-Reshetikhin formula \cite{17} (BR) presenting the function of three variables $T_a^s(u)$ in terms of the function of only two variables $T_1^s(u)$:

\[ T_a^s(u) = \det_{1 \leq i,j \leq a} T_{s+i-j}^1(u + i + j + a) \]  

(2)

Actually, there exists a more general BR formula, expressing the transfer-matrix eigenvalue of any skew representation $h/h'$ through $T_1^s(u)$:

\[ T_{h/h'}^s(u) = \det_{1 \leq i,j \leq a} T_{h_i-h_j}^1(u + h_i + h_j') \]  

(3)

where $h_i = m_i + a - i$ and $h'_i = m'_i + a - i$ are the so called shifted highest weight components of two representations $R$ and $R'$ of GL(N) characterized by the usual highest weight components $R = (m_1, \ldots, m_a)$ and $R' = (m'_1, \ldots, m'_a)$, so that they obey the inequalities $h_i < h_{i-1}$, $h'_i < h'_j$ and also $h'_i < h_i$. But the transfer-matrices with rectangular Young tableaux play an exceptional role since they obey a closed set of fusion rules given by the difference Hirota equation:

\[ T_a^s(u + 1)T_a^s(u - 1) - T_{s+1}^a(u)T_{s-1}^a(u) = T_{a+1}^s(u)T_{a-1}^s(u) \]  

(4)

It follows directly from (4) in virtue of the Jacobi identity for determinants. It contains little information since it is true for any function of two variables $T_1^1(u)$ in (3). To
specify it further to some particular integrable model we have to impose some boundary conditions on solutions of the eq. (4).

One of these conditions specifies the group of symmetry of the model. To make it, say, $SU(N)$ (or $A_{N-1}$ in terms of underlying algebra) we put:

$$T^a_s(u) = 0, \quad \text{for} \quad a < 0 \quad \text{and} \quad a > N$$

(5)

It is not enough since it leaves us with an infinite discrete set of possible solutions (like in quantum mechanics, fixing the boundary conditions on a wave function we are still left with infinitely many wave functions corresponding to different energy levels). We have to specify some analytical properties of solutions.

There are two ways to do it in the case of BA.

One is related to the so called bare BA where one specifies $T^0_s(u)$ and $T^N_s(u)$ to be some given polynomials in the variable $u$ whose zeroes specify completely a model, where as the functions $T^a_s(u), \quad \text{for} \quad 1 \leq a \leq N - 1$ are polynomials whose zeros we have to find. The details of the analyticity conditions for the bare BA can be found for example in (1).

Another way to fix analytical properties corresponds to the dressed BA where the elementary excitations are already the real physical particles. To precise them let us derive from (4) the TBA equations. For that we introduce the function:

$$Y^a_s(u) = \frac{T^a_{s+1}(u)T^a_{s-1}(u)}{T^a_{s+1}(u)T^a_{s-1}(u)}$$

(6)

which, in virtue of (4), satisfies the equation sometimes called Y-system [1]:

$$\frac{Y^a_{s+1}(u + 1)Y^a_s(u - 1)}{Y^a_{s+1}(u)Y^a_s(u - 1)} = \frac{[1 + Y^a_{s+1}(u)][1 + Y^a_{s-1}(u)]}{[1 + Y^a_{s+1}(u)][1 + Y^a_{s-1}(u)]}$$

(7)

Note that this system is symmetric under the change: $Y \rightarrow Y^{-1}, \quad a \rightarrow s, \quad s \rightarrow a$ (rank-level duality).

To make it a little bit more symmetric let us introduce the functions:

$$U^a_s(u) = 1 + Y^a_s(u), \quad \tilde{U}^a_s(u) = 1 + \frac{1}{Y^a_s(u)}$$

(8)

Then the eq. (7) can be rewritten as

$$\frac{U^a_s(u + 1)U^a_s(u - 1)}{U^a_{s+1}(u)U^a_{s+1}(u)} = \frac{\tilde{U}^a_s(u + 1)\tilde{U}^a_s(u - 1)}{\tilde{U}^a_{s+1}(u)\tilde{U}^a_{s+1}(u)}$$

(9)

Taking logarithm of both sides of this equation and applying the operator

$$\int_{-\infty}^{\infty} d\theta \frac{1}{\cosh \frac{\sqrt{2}}{2}(\theta - \theta'')} \cdot \cdot \cdot$$

(10)
where $\theta = iu$ we obtain:

$$C_{ss'}(\theta) \ast \ln (1 + Y_{s'}^a(\theta)) = C^{aa'}(\theta) \ast \ln (1 + \frac{1}{Y_{s'}^a(\theta)})$$  \tag{11}

where we introduced the so called “baxterized” Cartan matrices:

$$C^{aa'}(\theta) = \delta_{aa'} - \frac{1}{2 \cosh \frac{\pi}{2}(\theta)}(\delta_{a,a'+1} + \delta_{a+1,a'})$$  \tag{12}

and similarly for $C_{ss'}$. The $\ast$ sign defines the usual convolution operation: $f(\theta) \ast g(\theta) = \int_{-\infty}^{\infty} d\theta' f(\theta - \theta')g(\theta')$.

Let us now act on both sides of eq. (11) by the operator inverse to (12):

$$A_{aa'}^N = C_{aa'}^{-1}(\theta) = \frac{2}{\pi} \int_{0}^{\infty} dp \cos(p\theta) \coth(p) \frac{\sinh[(N - \text{max}(aa'))p] \sinh[\text{min}(aa')p]}{\sinh(pN)}$$  \tag{13}

Note that this inverse is respecting the boundary conditions restricting the values of $a, a'$ to $1 \leq a, a' \leq N - 1$.

Note also that the operator $C^{aa'}$ has zero modes:

$$C^{aa'}(\theta) \ast \sigma(s)m_0 \sin(a'\pi k/N) \cosh(\theta\pi k/N) = 0$$  \tag{14}

for any integer $k$ and any function $\sigma(s)$. So in acting by (13) on both sides of (11) we might be obliged to add one of zero modes. The choice of zero mode and the function $\sigma(s)$ defines completely the boundary conditions and hence the model. If we want to respect the 1+1 dimensional relativistic invariance we can add the zero mode with $k = 1$, since only it will lead to the relativistic spectrum of energies elementary excitations (which are described by this zero mode) of a type:

$$\sigma(s)m_0 \sin(a\pi/N) \cosh(\theta\pi/N)$$  \tag{15}

It gives a typical mass and energy spectrum of physical particles for integrable relativistic models of 2D QFT. The choice of $\sigma(s)$ and the range of $s$ define a particular relativistic model. For example, for $\sigma(s) = \delta_{s,1}$, $s \geq 1$ corresponds to the chiral Gross-Neveu model, whereas $\sigma(s) = \delta_{s,0}$, $-\infty \leq s \leq \infty$ corresponds to the principal chiral field (PCF) with the SU(N) symmetry.

With all these settings the final TBA equation (or similar equations for the ground state of the finite length system with the periodic boundary conditions) takes a familiar form (say, for the PCF):

$$A^{aa'}(\theta) \ast C_{ss'}(\theta) \ast \ln (1 + Y_{s'}^a(u)) - \ln (1 + \frac{1}{Y_{s'}^a(u)}) = \delta_{s,0}m_0 \sin(a\pi/N) \cosh(\theta\pi/N)$$  \tag{16}

where $\epsilon(x, y, \tau) = \log Y_{s'}^a(u)$ plays the role of the energy density of the excitations characterized by the rank $a$ and level $s$. 

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At the end of this section let us comment on the large N limit of the TBA equations. It is not obvious to us how to simplify the eq. (16) in this limit, but the difference equation equation (7) after introducing the rescaled continuous variables

\[ \tau = u/N, \quad \eta = s/N, \quad \nu = a/N \]  

becomes a second order differential equation for the quantity \( \epsilon(\eta, \nu, \tau) = \log Y_s^a(u) \)

\[ (\partial^2_{\nu} - \partial^2_{\tau}) \epsilon = (\partial^2_{\nu} - \partial^2_{\eta}) \ln (1 + \exp \epsilon) \]  

This is an integrable classical equation, as it is a consequence of the general HE and the determinant representation (2) for its solution, although the determinant becomes functional in the large N limit.

Let us give also another form of this equation in terms of new variables \( l, m \) and \( a \) (and the corresponding rescaled variables) \( \lambda, \mu \) and \( \nu \) defined through the original variables \( u, s \) and \( a \) as follows:

\[ \lambda = \frac{l}{N} = \frac{u + s + a}{2N}, \quad \mu = \frac{m}{N} = \frac{u - s + a}{2N}, \quad \nu = \frac{a}{N} \]  

Note that \( \lambda \) and \( \mu \) play the role of “light-cone” variables.

In their terms we can represent the function \( \epsilon = \ln Y \) in the large N limit as

\[ \epsilon = -\ln \frac{1}{\exp F_{\lambda \mu} - 1} \]  

where

\[ F = \lim_{N \to \infty} \frac{1}{N^2} \log T. \]  

The continuous HE (18) for this function looks in the new variables as

\[ \partial_{\lambda} \partial_{\mu} \ln(\exp[-F_{\lambda \mu} - 1]) = \partial_{\nu}(\partial_{\nu} \partial_{\lambda} - \partial_{\lambda} \partial_{\mu})F_{\lambda \mu} \]  

This form of HE will be useful for our matrix model representation of its solution.

## 3 Matrix Model Chain as Solution of the Bilinear Difference Hirota Equation

In this section we propose to parametrize the general solution of HE (4) by means of so called matrix chain integral - a matrix model widely used and investigated in the literature (see [23] for the details).

\[ I \text{ thank P. Zinn-Justin for this comment} \]
Let us define the following Green’s function:

\[ K_a(T, \phi(0), \phi(T)) = \int_{T-1}^{T} \prod d^2 \phi(k) \exp \left[ \frac{1}{2} \sum_{k=0}^{T-1} \phi(k) \phi(k+1) - \sum_{k=0}^{T} V(k, \phi_k) \right] \] (23)

where \( \phi(k) \) are \( a \times a \) hermitian matrices, each corresponding to its site \( k \) of the chain, with matrix elements \( \phi_{ij}(k) \), \( i, j = 1, 2, \cdots, N \), \( k = 0, 1, 2, \cdots, T \). Two matrices at the ends of the chain are fixed.

The continuous analogue of the chain would be just the matrix quantum mechanical Green’s function on the interval of time \((0, T)\):

\[ K_a(T, \phi(0), \Phi(T)) = \int D^2 \phi(t) \exp -tr \int_0^T dt [\frac{1}{2} \dot{\phi}^2 + V(t, \phi)] \] (24)

with the end point values of the matrix fields also fixed. The matrix quantum mechanics was introduced and solved in ([36]).

Let us now define a new quantity:

\[ Z_a(l, m) = \int d^2 \phi(0) \int d^2 \phi(T) (\det \phi(0))^l K_a(T, \phi(0), \phi(T))(\det \phi(T))^m \] (25)

where \( K \) could be any of both Green’s functions \((23)\) or \((24)\). We have added two logarithmic potentials at the ends of the chain to describe the dependence on \( l \) and \( m \) introduced by \((19)\).

We claim that the function of three discrete variables

\[ T_s^a(u) = Z_a\left( \frac{u + s + a}{2}, \frac{u - s + a}{2} \right) \] (26)

obeys the HE \((3)\). More than that, it gives the most general solution of HE parameterized by the function of 2 variables - the potential \( V(t, x) \).

The proof goes as follows. If we start, say, from the discrete version we can diagonalize each of the matrices in the chain by the unitary rotation:

\[ (\phi_a)_{ij} = \sum_k (\Omega_a^+)_{ik}(z_a)_k(\Omega_a)_{kj} \] (27)

and integrate over the relative “angles” between two consecutive matrices in the chain by means of the Itzykson-Zuber-Harish-Chandra formula. This concerns only the first term in the exponent in the r.h.s of \((23)\). The potentials, including two determinants at the ends, depend only on the eigenvalues. The overall result after the angular integration will be (see \((25)\) for the details of this calculation):

\[ Z_a(l, m) = \det_{1 \leq i,j \leq a} \int dp \int dq K(T, p, q) p^{l+i-1} q^{m+j-1} \] (28)

where

\[ K(T, p, q) = \int_{T-1}^{T} \prod dz(k) \exp \left\{ \sum_{k=0}^{T-1} z(k) z(k+1) - \sum_{k=0}^{T} V(k, z(k)) \right\} \] (29)
for the discrete chain, or

\[ K(T, p, q) = \int Dz(t) \exp \left\{ \int_0^T dt \left[ \frac{\dot{z}^2}{2} + V(t, z) \right] \right\} \]  

(30)

for the continuous quantum mechanics, where \( z(0) = p, z(T) = q \).

Now we see that due to the determinant representation (28) the function \( T_s^a(u) \) defined by the eq. (26) has the same determinant form as the BR formula (2) and hence it obeys the HE (4), if we identify

\[ T_s^1(u) = \int dp \int dq K(T, p, q) p^{u+1} q^{u-1} \]  

(31)

The formula (28) generally defines an arbitrary function of two variables \( s \) and \( u \). It is clear from the fact that it is just the Mellin transform of an arbitrary function (30) (or (29)) in two variables, which is in our case the Greens function of a quantum mechanical particle in an arbitrary time and space dependent potential. This potential obviously gives enough of freedom to define \( K(T, p, q) \) as an arbitrary function of two variables \( p \) and \( q \). So we proved (or at least made rather obvious) our statement about the generality of this representation of solution of the HE.

We can provide a more general matrix integral giving the parameterization of the most general BR formula (3):

\[ T_{h/h'} = \int d^2 \phi_0 \int d^2 \phi_T \chi[h](\phi_T) K_a(T, \phi_0, \phi_T) \chi[h'](\phi_T) \]  

(32)

where \( \chi[h](\phi) \) is the GL(a) character of the representation characterized by the highest weight \( [h] \). It can be easily proved when written in terms of eigenvalues with the use of the Weyl formula for characters.

For any finite \( N \) all this seems to be on the edge of triviality: we just defined in a sophisticated way an arbitrary function of two variables and built from it the necessary determinant. Naturally, we don’t expect this representation to be of big use for a finite \( N \). The boundary conditions of TBA will be as difficult to satisfy as before. What is our major hope is the large \( N \) limit of this matrix model which should correspond to the large \( N \) limit of the integrable models of the type of principal chiral field. In this case the determinant in the BR formula is essentially functional, and the matrix models give a rich variety of methods for the calculation of such determinants. That why our strategy will be the following: we investigate the matrix integrals of the type (25) in the large \( N \) limit for various potentials and look for physically interesting regimes. Things might become much more universal in the large \( N \) limit, and it could exist a classification of interesting regimes, like it was done for the multi-critical points in the matrix models. This paper represents of course only a few modest steps in this direction.

Let us make an important remark concerning the large \( N \) limit of the representation (28): the \( Y \) variable introduced in the previous section and presented by the formula (21)
in the large N limit (with the matrix chain partition function $Z$ instead of the transfer-matrix $T$) obeys the differential equation (18) or, in new variables (19), (22). It is almost clear from our definitions and it will be demonstrated in the following sections.

Another more formal but interesting application of this method could be the search for new solutions in the integrable equations of the type (18). In the next section we will show some particular examples of it.

4 EXAMPLES OF SOLUTIONS OF THE CONTINUOUS HIROTA EQUATION

We will demonstrate here on some examples limited to particular choices of the matrix chain potentials how this relation between the HE and MM works.

4.1 One matrix model and $GL(n)$ character

Let us start from the one matrix model partition function with an extra logarithmic potential

$$Z_a(l) = \int d^a \phi (\det \phi)^l \exp -N \text{tr} V(\phi)$$

which, after going to the eigenvalue representation, becomes

$$Z_a(l) = \det_{1 \leq i, j \leq a} \int dp p^{i+j-2} \exp -NV(x)$$

It is well know that if one chooses:

$$\exp -V(x) = \prod_{k=1}^a (b_k - x)^{-1}$$

and performs the integral in (34) along the contour encircling all these poles one will identify this partition function with the $GL(a)$ character $\chi_{a}(b)$ of the $a \times l$ rectangular Young tableau given by the Weyl determinant formula. So, much of our next formulas is valid for the characters as well.

It is easy to see from the Jacobi identity for determinants that the function

$$t^a(l) = Z_a(a - l)$$

satisfies a simplified version of the general HE (4):

$$t^a(l + 1)t^a(l - 1) - t^{a+1}(l)t^{a-1}(l) = [t^a(l)]^2$$

Introducing the variable

$$Y^a(l) = \frac{t^{a+1}(l)t^{a-1}(l)}{[t^a(l)]^2}$$

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we obtain from (37) a simplified version of the general Y-system (7):
\[
\frac{Y^a(l + 1)Y^a(l - 1)}{[Y_a(l)]^2} = \frac{[1 + Y^{a+1}(l)][1 + Y^{a-1}(l)]}{[1 + Y^a(l)]^2}
\]  
(39)

If we go to the large N limit and introduce the continuous variables \( \nu, \lambda \) we obtain from the previous equation a simplified version of the differential HE (18):

\[
\frac{\partial^2 \lambda}{\partial \log Y} = \frac{\partial^2 \nu}{\partial \log (1 + Y)}
\]  
(40)

which is again integrable, as it is obvious from the above determinant formulas. In the last equations we changed a bit the definition of the function \( Y \) rescaling the variables by \( 1/N \).

Take first a simple example of the potential \( V(x) = |x| \). The direct calculation of (34) gives:

\[
t_a(l) = \prod_{k=1}^{a} (l - a + k)! (k - 1)!
\]  
(41)

which yields \( Y_a(l) \) as

\[
Y_a(l) = \frac{a}{l - a + 1}
\]  
(42)

or, in the large N limit:

\[
Y_\nu(\lambda) = \frac{\nu}{\lambda - \nu}
\]  
(43)

which perfectly satisfies the eq. (40).

To find a general (up to some comments which will follow) solution of the eq. (40) we just have to apply the well known formulas for the saddle point approximation in the one matrix model with the potential which is now \( V(x) + (\lambda - \nu) \log x \). Omitting the standard calculations (see for example [9]) we give the result: the function \( Y_\nu(\lambda) \) obeys the following system of equations on \( Y \) and an intermediate variable \( S \):

\[
\int_{-1}^{1} \frac{du}{\pi} \frac{V'(\sqrt{Y}u + S)}{\sqrt{1 - u^2}} = \frac{\lambda - \nu}{\sqrt{S^2 - 4Y}}
\]  
(44)

\[
\int_{-1}^{1} \frac{du}{\pi} \frac{(\sqrt{Y}u + S)V'(\sqrt{Y}u + S)}{\sqrt{1 - u^2}} = \lambda + \nu
\]  
(45)

So we have obtained the solution of the eq. (40) in terms of a system of ordinary equations. For example, for a polynomial potential the equations will become algebraic. In general they are functional.

Not every potential is compatible with this solution. We restricted ourselves to the so called one cut solution implying the existence of one classically stable well in the potential. This restricts our solution to some parametrically general but still limited class of solutions of (40). The generalization to the multi-cut solution which is straightforward should in principle yield the most general solution of (40).

The equations (44-45) look like the characteristics method of solution of the eq. (40). In the next sections we shall see to what extent we can generalize it to the full differential HE (4).
4.2 Gaussian Chain Solution of HE

Now we shall consider another particular example of solution of the HE (4) by restricting all the potentials in the matrix chain to be Gaussian. Then all the integrals over \( \phi(1), ..., \phi(T) \) in (23) can be easily performed and we are left in (25) with the following two matrix integral over the endpoint variables:

\[
Z_a(l, m) = \int d\phi \int d\phi' (\det \phi)^l \exp \left( -\text{tr} \left( \frac{1}{2} \phi^2 + \frac{1}{2} \phi'^2 - c\phi\phi' \right) \right) (\det \phi')^m
\]

If we do the same with the continuous quantum mechanical integral (24) we arrive (up to a trivial coefficient typical for the Green’s function of the harmonic oscillator) at the same two matrix integral, with \( c = 1/\cosh(\omega T) \), where \( \omega \) is the frequency of the corresponding oscillator.

An important comment is in order. Although this partition function satisfies the HE for three variables it should a little bit modified for finite \( N \): note that \( Z_a(l, m) \) in (46) after passing to eigenvalues splits into the product of two determinants corresponding to \( i, j \) both even or both odd (the matrix elements with different parities of \( i, j \) are zero, see the formula (28) with the gaussian kernel). Each of these determinants satisfies the same difference HE with the shifts of discrete variables by \( \pm 2 \) and not by \( \pm 1 \). The continuous (large \( N \)) version of HE will be the same as before.

This two-matrix model has the only complication with respect to the ordinary one, containing usually only polynomial potentials: its potentials contain logarithmic parts, like in the well know one matrix Penner model. To solve it the ordinary method of orthogonal polynomials does not look convenient. We propose here another, rather powerful method worked out in a series of papers \([26, 27, 28, 29, 30, 31]\) and capable to solve some even more sophisticated models than the present one.

First we perform the integral over the relative “angles” of two matrices in (46) by means of the character expansion \([32]\):

\[
\int (d\Omega) U(N) \exp[c \text{tr}(\phi \Omega^+ \phi' \Omega)] = \sum_{0 \leq h_a < \cdots < h_1 < \infty} \frac{c^{\sum_h h_k-a(a-1)/2}}{\prod_k h_k!} \chi_h(\phi) \chi_h(\phi')
\]

We dropped here some unessential overall coefficient.

Plugging this formula into the eq. (46) we encounter two identical independent Gaussian integrals over \( \phi \) and \( \phi' \) with the characters as pre-exponentials. These integrals can be calculated (they slightly generalize the similar integrals appearing in \([33, 27, 28, 29]\) to the case of \( l, m \neq 0 \)). The result (again up to some unessential factor) is:

\[
\int d^2 \phi e^{-\frac{1}{2} \text{Tr} \phi^2} \chi_h(\phi)(\det \phi)^l = \frac{\prod_i (h^e_i + l - 1)!! \prod_j (h^o_j + l)!!}{h^e_i!} \Delta(h^e) \Delta(h^o)
\]

where we denote by \( h^e(h^o) \) the even(odd) highest weights whose numbers should be equal. \( \Delta(h) \) is the Van-der-Monde determinant of \( h \)'s. We chose here \( l \) to be even; for \( l \) odd one only has to exchange \( h^e \) and \( h^o \) in (48).
Putting all this together we obtain for (46) a representation in terms of the multiple sum over h’s (we dropped a h-independent coefficient):

\[ Z_a(l, m) = \sum_{\{h^e, h^o\}} \Delta^2(h^e)\Delta^2(h^o) c^{\sum_k (h^e_k + h^o_k)} \times \]

\[ \prod_i (h^e_i + l - 1)!! \prod_j (h^o_j + l)!! \prod_i (h^e_i + m - 1)!! \prod_j (h^o_j + m)!! \]

\[ \prod_n h_n! \]

We chose here l, m to be both even. Different parity for them is forbidden by the symmetry \( \phi \rightarrow -\phi \) of (46). This will not be important in the large N limit. As we see, the sums over h^e and h^o are decoupled and can be calculated independently.

The method of calculation of these multiple sums in the large N limit was proposed in [26] and further elaborated in [13, 27, 28, 29] and is based on the saddle point approximation of this sum. One introduces the resolvent function of shifted highest weights:

\[ H(h) = \sum_{k=1}^{a} \frac{1}{h - h_k} \] (50)

In what follows we change h by h/N (since the highest weights are supposed to be of the order N in the large N limit). So, in the large N limit:

\[ H(h) = \int_0^d dh' \frac{\tilde{\rho}(h')}{h - h'} = H_+(h) \pm i\pi\tilde{\rho}(h) \] (51)

where \( H_+(h) \) is the symmetric part of the function \( H(h) \) on the cut defined by the distribution of h’s and \( \tilde{\rho}(h) \) is the density of h’s along this cut. In the large N limit we can calculate the multiple sum by the saddle point method. The saddle point condition defines the most probable Young tableau shaped by the density \( \tilde{\rho}(h) \):

\[ \int_0^d dh' \frac{\tilde{\rho}(h')}{h - h'} = -\frac{1}{2} \ln \left( \frac{c^2(h + \lambda)(h + \mu)}{h^2} \right) \] (52)

One has to remember that a part of the most probable Young tableau is in general empty (some of the highest weight components \( m_k \) are equal to zero, see [23, 27] for the details). So, the function \( \tilde{\rho}(h) \) is equal to one on the interval (0, b) and to some nontrivial function \( \rho(h) \) on the interval (b, d). This yields, instead of (52), the equation:

\[ \int_b^d dh' \frac{\rho(h')}{h - h'} = -\frac{1}{2} \ln \left( \frac{c^2(h + \lambda)(h + \mu)}{(h - b)^2} \right) \] (53)

This linear integral equation has a one-cut solution:

\[ H(h) = \ln \left[ c^{-1}(d - b)h \right] \] (54)

\[ -\frac{1}{2} \ln \left[ (b + d + 2\lambda)h - (b + d)\lambda - 2db + 2\sqrt{(d + \lambda)(b + \lambda)(h - d)(h - b)} \right] \]

\[ -\frac{1}{2} \ln \left[ (b + d + 2\mu)h - (b + d)\mu - 2db + 2\sqrt{(d + \mu)(b + \mu)(h - d)(h - b)} \right] \]
To fix $d$ and $b$ we should recall the asymptotic of $H(h)$ with respect to large $h$:

$$H(h) = \nu/h + <h>/h^2 + 0(1/h^3)$$  \hspace{1cm} (55)$$

following from (50). Here $<h>$ is the average shifted highest weight in the most probable Young tableau. Expanding $H(h)$ in $1/h$ up to the terms $O(1/h^2)$ we obtain a system of equations defining $d$ and $b$:

$$[2\lambda + d + b + 2\sqrt{(d + \lambda)(b + \lambda)}][2\mu + d + b + 2\sqrt{(d + \mu)(b + \mu)}] = c^{-2}(d - b)^2$$  \hspace{1cm} (56)$$

$$\sqrt{(d + \lambda)(b + \lambda)} + \sqrt{(d + \mu)(b + \mu)} = 2\nu + \lambda + \mu$$  \hspace{1cm} (57)$$

From (50) we deduce the following formula for the solution of the continuous HE (22):

$$F''_{\lambda\mu} = \frac{1}{4} << \ln(h + \lambda), \ln(h + \mu) >>$$  \hspace{1cm} (58)$$

where by $<< A, B >>$ we denoted the connected average of any two $h$-dependent functions $A$ and $B$. Note that this average has a finite large $N$ limit, as it should be.

This solution can be brought into a more explicit form: the explicit formula for such correlators in the one matrix models was given in [34, 35] for $W(z, z') = << 1/z - h, 1/(z' - h) >>$:

$$W(z, z') = \frac{1}{2} \left[ \frac{(z - d)(z' - b)}{(z' - d)(z - b)} + \frac{(z' - d)(z - b)}{(z - d)(z' - b)} - \frac{1}{(z - z')^2} \right]$$  \hspace{1cm} (59)$$

Integrating it in $z$ and $z'$ and putting $z = -\lambda, z' = -\mu$ we obtain a rather explicit solution of differential HE (22)

$$F''_{\lambda\mu} = \frac{1}{8}[g(\lambda, b, d)g(\mu, d, b) + g(\lambda, d, b)g(\mu, b, d) + \ln(\lambda + \mu)]$$  \hspace{1cm} (60)$$

where

$$g(z, b, d) = \sqrt{(z + d)(z + b)} + \frac{d - b}{2}\sqrt{\frac{z + d}{z + b}}\ln[2\sqrt{(z + d)(z + b) - 2z - d - b}]$$  \hspace{1cm} (61)$$

and $d, b$ are defined by the eqs. (57).

In the next section we will give the solution of (18) in the case of a general time-independent potential $V(x)$. We will also reduce the search for the most general solution of continuous Hirota equation (18) defined by the time dependent potential $V(x, t)$ to a simpler problem.
5 COLLECTIVE FIELD METHOD FOR THE DIFFERENTIAL HE

Now let us briefly describe how our method works in the case of a the general potential in the matrix quantum mechanics defined by (24). In this case we can apply the collective coordinate method of Jevicki and Sakita [[44]] which is valid in the large N limit and can be applied to the non-stationary saddle point solutions which are needed in our case. The details of this approach can be found in [[44, 42, 45, 37]]. We will use the results of it and apply them to our case.

In terms of this method the effective action for (24) can be written for the density \( \rho(x, t) \) of eigenvalues \( x_k \) and its conjugate momentum \( P(x, t) \) developing in time as:

\[
S_{\text{eff}}[\rho, P] = \int dx \int_0^T dt [\dot{\rho}P - \frac{1}{2} \rho P'^2 + \frac{\pi^2}{6} \rho^3 + \rho V(x, t)]
\]

(62)

reflecting our specific boundary conditions at the ends of the interval \((0, T)\) following from (25).

The equations of motion corresponding to the action (62) are:

\[
\dot{\rho} + \partial_x (P'\rho) = 0 \tag{63}
\]

\[
\dot{P} + \frac{1}{2} P'^2 = \frac{\pi^2}{2} \rho^2 + V(x, t) \tag{64}
\]

Differentiating the second one in \( x \) and defining the function \( f(x, t) = P' + i\pi \rho \) we rewrite this system as only one forced Hopf equation on this complex function:

\[
\partial_t f + f \partial_x f = -V'(x, t) \tag{65}
\]

We have to impose at any moment, say, at \( t = 0 \) the normalization condition

\[
\int dx \rho(x, 0) = \nu \tag{66}
\]

Then it will be true at any \( t \) due to the condition (63).

We have excluded the end points of the interval \((0, T)\) in the last equation. The logarithmic potentials and the two Van-der-Monde determinants left at the ends of the interval can be taken into account as the boundary conditions:

\[
Re f(x, 0) = \frac{\lambda}{x} + \int_{b(0)}^{d(0)} dx' \frac{\rho(x', 0)}{x - x'}, \quad b(0) \leq x \leq d(0) \tag{67}
\]

\[
Re f(x, T) = -\frac{\mu}{x} - \int_{b(T)}^{d(T)} dx' \frac{\rho(x', T)}{x - x'}, \quad b(T) \leq x \leq d(T) \tag{68}
\]

or, introducing the resolvent: \( R_{\pm}(x, t) = \int_{b(t)}^{d(t)} dx' \frac{\rho(x', t)}{x - x'} \pm i\pi \rho(x, t) \),

\[
f(x, 0) = \frac{\lambda}{x} + R_{\pm}(x, 0) \tag{69}
\]
\[ f(x, T) = -\frac{\mu}{x} - R_-(x, T) \] (70)

Hence we reduced the problem of the virtually general (since it is parameterized by a general potential depending on 2 variables) solution of the continuous HE \((\text{HE})\) on the function of 3 variables \(\nu, \lambda \text{ et } \mu\) to the solution of the differential equation of the first order \((\text{DE})\) (assuming that the complex function \(f\) is analytical in their variables) supplemented by the boundary conditions \((\text{BC})\). The variables \(\nu, \lambda \text{ et } \mu\) appear here only as fixed parameters.

It is still quite complicated, although simpler than the original problem and, may be, physically more transparent. We don’t know how we could simplify it further in general case. So let us consider an interesting particular example of the time independent potential \(V(x)\). In that case the forced Hopf equation

\[ \partial_T f + f \partial_x f = -V'(x) \] (71)

becomes completely integrable by the characteristics method \(\text{CM}\).

We will choose \(\lambda = \bar{\mu}\), which does not look as restriction if we assume the analyticity in \(\lambda\) and \(\mu\). Then due to the time reversal symmetry we have \(f(x, T) = -\bar{f}(x, 0)\). So we can say that also \(\rho(x, T) = \rho(x, 0)\). Hence only one of these two boundary conditions is independent.

The first part of the problem is to find the solution of eq. \((\text{DE})\) with fixed endpoint density \(\rho(x, T) = \rho(x, 0)\). The result can be formulated as the following equation on the functions already at the endpoints

\[ T = \int_x^{G(x)} \frac{dy}{\sqrt{2(g_0(x) - V(y))}} \] (72)

where \(G(x) = x_T(g_T(x))\) and \(x_T(g)\) is the function to be found by solution of

\[ g(x_T) = \frac{1}{2} f^2(x_T, T) + V(x_T) \] (73)

Note that the function \(\bar{G}(x) = x_0(g_0(x))\) defined by the solution of

\[ g = \frac{1}{2} f^2(x_0, 0) + V(x_0) \] (74)

is the functional inverse of the function \(G(x)\) itself which gives the equation of A. Matytsin \((\text{MA})\):

\[ G(\bar{G}(x)) = x \] (75)

Although the dynamics of the forced Hopf equation is summarized by the relation \((\text{int})\) the last equation leads to a strong constraint on the analytical structure of the function \(G(x)\).

\(^4\text{I am grateful to A. Matytsin for the explanation of this method and its application to the forced Hopf equation}\)
Once we found the functional \( f_{\rho(x,0)}(x,0) \) as the solution of eqs. (72-75) we have to much it with our boundary conditions:

\[
\text{Re} f_{\rho(x,0)}(x,0) = \frac{\lambda}{x} + \int_{b(0)}^{d(0)} d' \rho(x',0) \frac{\rho(x',0)}{x-x'}, \quad b(0) \leq x \leq d(0)
\] (76)

With a given \( V(x) \) this defines the end-point density \( \rho(x,0) \) which is the only non-trivial information we need to find the quantity (24). Probably it is convenient to represent this boundary condition as a condition on the large \( x \) asymptotic of \( f(x,0) \):

\[
f(x,0) \to_{x \to \infty} \frac{\nu + \lambda}{x}
\] (77)

We thus reduced the solution of the continuous HE (22) to some simpler functional problem in a particular but rather representative case of the time independent potential. The solution with the harmonic oscillator potential obtained in the previous section should be also reproducible by this method.

We can also use these equation to produce more explicit solutions of the HE (22) by the method proposed in [37]: one chooses two conjugated roots \( G(x) \) and \( \bar{G}(x) \) of an algebraic equation \( x(G) = 0 \), where \( x(G) \) is some polynomial. These two roots satisfy by construction the eq. (75). Then one has to plug them into (72) and solve it as an integral equation for \( V(x) \). Of course the choice should be limited by the boundary conditions (76) or (77).

It would be interesting to analyze the case of the inverted oscillator potential corresponding to the 1+1 dimensional non-critical string theory. But this question lies beyond the scope of this paper.

6  CONCLUSIONS AND PROSPECTS

In this paper we proposed a matrix model representation for the solution of the general difference Hirota equation. For its continuous analog of the differential HE of second order on three variables the solution is represented by the large N limit of the corresponding matrix chain. It gives an effective framework for solving the continuous HE in a rather explicit way, at least for some particular cases. In particular, the problem can be reduced to the forced Hopf equation with specific boundary conditions.

Many things remain to be understood. First of all it is not clear how to find the solutions satisfying the boundary conditions of various 1+1 dimensional quantum field theories solvable by Bethe ansatz. Especially how to choose the matrix potentials to get the relativistic spectrum for the physical particles and to fix the symmetries of original models. Another question: is there some physical interpretation in terms of these integrable theories of the time variable \( t \) and of the eigenvalue variable \( x \) in our matrix representation similar to the non-critical (1+1) dimensional string theory (where these variables describe the target space of the string [42, 41])? It might be for example that
the time can be considered as the physical space dimension of the corresponding integrable theory. But it remains to be proved. We hope that the formalism proposed here at least sets a convenient framework for the attempts to find the physically interesting solutions.

A more formal use of our method might be the search for new solitonic solutions for the well known integrable equations. For example, the Toda equation \( \partial^2_\nu F = \partial_\lambda \partial_\mu \exp F \) is just a particular limit of the continuous HE (18). To our knowledge, the solitonic solutions to this equation are not yet found.

Another interesting question is how the double scaling limit in matrix models is related to HE? For example, how to find the corresponding solution for the inverted harmonic oscillator giving the description of the (1+1) dimensional string field theory. To answer this question as well as many others we have to learn how to deal with the non-stationary forced Hopf equation with our specific boundary conditions. The methods worked out in the papers [37, 38, 39, 40] could be useful for that.

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