A Generalised Exactness Structure for Sets

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Abstract

Two adjoint functors can be seen as generalisations of the two functions within a Galois connection. If instead the adjoints are not generalised from functions, but from relations, then analogously the object of study becomes a more general notion of an adjunction. A suitable method to express such functor-level relations is to consider functors into categories of families. This structure is then used to show that the central exactness structure in self-dual group theory, consisting of a chain of adjunctions, holds also for the category of sets when seen in this general form.

EDIT: Please see the note about the empty set on page 4.

1 Chains of Adjunctions as an Exactness Structure

1.1 Background

In this first section we revisit some recent developments in the study of groups with an emphasis on the study of self-dual properties. These developments are then related to older work that has been done; and subsequently, the underlying structure that forms the basis of the framework is laid out. A main objective is forming a structural mathematical conformation that captures properties of the category of groups, but that additionally holds in the context of the category of sets.

In Section 2 we elaborate on the abstract framework for which both the context of sets and groups will be examples. The general version of the structure is exposted and axioms which ensure existence of such a structure are given.

Definition 1.1. In this text, the word form is used to refer to the notion of a form in the sense of Z Janelidze [8]; that is, a faithful functor. Although a preorder determines a unique poset, in order for the notion of a form to be equivalent to a functor with posets as preimage categories, one also assumes that the functor is amnestic.

Notation 1.2. For an object X in the bottom category of a form, if they exist, we use the notation \(0 \in \Sigma X\) and \(1 \in \Sigma X\) for the smallest and largest elements of the poset associated with X.
1.2 The Chain of Adjunctions for Groups

The central focus of this paper has its origins in group theory, where an exactness structure consisting of a chain of adjunctions occurs and which gives a useful higher level perspective to self-dual group theory.

However, this exactness structure appears in the mathematical literature earlier, albeit in a different guise, in a paper by G. Janelidze and Márki [6]. In that paper the objective is to provide a framework in which radical theory, as laid out there, can be generalised to include nonassociative rings as well. Indeed, mentioned there is that (the categorical version of) their exactness structure can be used for obtaining isomorphism theorems in various areas of algebra. It is this categorical version that forms the underlying structure that manifests for groups in [4, 9, 10, 11, 18] as a self-dual framework. Furthermore, a general version of this chain of adjunctions underpins the self-dual isomorphism theorems for sets in [17]. One of the main objectives of the present paper is to study this latter (categorical) version of the structure.

**Definition 1.3.** When we refer to a chain of adjunctions or equally, the exactness structure, we refer to the functors and adjunctions as in the diagram below:

![Diagram of functors and adjunctions]

The arrows represent functors that are sequentially adjoint: \( C \dashv M_0 \dashv F \dashv M_1 \dashv D \). From here and onwards, we shall use the term exactness structure throughout in an attempt at consistency.

For \( \text{Grp} \), we define the functors in the exactness structure as follows. We let the category \( \mathcal{C} = \text{Grp} \). The top category \( \mathcal{E} \) is the category of pairs of groups \((G, S)\) where \( S \) is a subgroup of \( G \) and morphisms \( \tilde{f} : (G, S) \to (G', S') \) are homomorphisms \( f : G \to G' \) such that \( f(S) \leq S' \). We use the symbol \( \leq \), since we refer to the relation in the subgroup lattice.

The functor \( F \) is the form that is the central structure in projective group theory, sending \((G, S)\) to \( G \). The functors \( M_0 \) and \( M_1 \) select for each group \( G \) the pairs \((G, 0)\) and \((G, 1)\), respectively, where \( 0 \) is the trivial subgroup and \( 1 \) is the improper subgroup. A homomorphism \( f : G \to H \) gets sent to the morphisms that retain \( f \) as the underlying function \( M_0(f) : (G, 0) \to (H, 0) \) and \( M_1(f) : (G, 1) \to (H, 1) \), respectively, since the direct image \( f(0) \leq 0 \) and \( f(1) \leq 1 \). The functor \( D \) selects for each pair \((G, S)\) the group \( S \) and the functor \( C \) selects the group \( G/\overline{S} \) where \( \overline{S} \) is the smallest normal subgroup containing \( S \).

It is known that this structure forms adjunctions with \( C \dashv M_0 \dashv F \dashv M_1 \dashv D \) [4, 18].

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1.3 Exactness Properties

The structure that we have defined can be thought of as portraying exactness properties within the abstract framework. An object in $\mathcal{E}_0$ can be thought of as a prototypical short exact sequence $S \hookrightarrow X \twoheadrightarrow X/\Sigma$. For groups, this is the inclusion of a subgroup $S$, followed by the quotient with the smallest normal subgroup containing $S$. Hence, the functor $F$ maps short exact sequences of this particular kind to the middle object in the sequence. The other two simple possibilities, i.e., mapping the sequence down to $S$ or mapping it down to $X/\Sigma$ turn out to be what is used for the other functors going down ($D$ and $S$, respectively). As we shall see later, for the case when we study the category of sets, instead of a prototypical short exact sequence, the exactness is captured by two families of functions $\{f_k : S \hookrightarrow X \mid k \in K\}$ and $\{g_j : X \twoheadrightarrow T \mid j \in J\}$ capturing in a self-dual way properties of an equivalence relation $R$. The first family selects the equivalence classes of $R$, while the second family consists of the single surjection (up to isomorphism) which has kernel relation $R$.

1.4 The Concrete Set Theoretic Case

When moving from $\text{Grp}$ to $\text{Set}$, the contextual translation of the exactness structure does not immediately follow. Firstly, one needs to choose a suitable notion of an abstract subobject and the suitable corresponding conormal and normal abstract subobjects (with associated embeddings and projections) into which the functors $C$ and $D$ send a pair $(X, S)$ to. It turns out [17] that a suitable setting to study $\text{Set}$ from is to use as $A$-subobjects the bounded lattice $\Sigma X$ of equivalence relations on a set $X$. We likewise use the term $\text{immorphisms}$ (of $S$) to refer to the family of injections that select each equivalence class of $S$.

Then, $D$ is in fact not a functor, but a relation at the functor level: $D$ relates $(X, S)$ to the domains of injective functions which select the equivalence classes of $S$. Moreover, $D$ and $C$ will not be defined on what we will term $\mathcal{Z}$-empty functions. To formalise this weakening of a functor to the level of a functor relation and to form a notion of an adjunction by analogy of the group case, it is useful to express the functor relation as a functor into the category of families.

**Definition 1.4.** An $R$-functor $F$ between categories $A$ and $X$ is a functor, $F_A : A \rightarrow \text{Fam}X$.

**Notation 1.5.** We will write $\{0\} \simeq \{\emptyset\}$ to represent any chosen singleton set.

**Remark 1.6.** For a category $A$ and its corresponding category of families $\text{Fam}A$, there is always the identity $R$-functor: $\text{Fam}_A : A \rightarrow \text{Fam}A$ which sends $A$ to the set $\{A_i \mid i \in \{0\}\}$ where $A_0 = A$. Any functor $F : A \rightarrow X$ gives rise to an $R$-functor by composition with $\text{Fam}_X$ as $\text{Fam}_X \circ F : A \rightarrow X \rightarrow \text{Fam}X$. 

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Definition 1.7. An $\mathcal{R}$-adjunction between a pair of $\mathcal{R}$-functors $F$ and $G$ such as in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{Fam}_A} & F \text{m}_X \\
\downarrow \alpha_{X,A} & & \downarrow \alpha_{X,B} \\
\text{Fam}_A & \xleftarrow{G} & A
\end{array}
\]

is a bijection

\[
\alpha_{X,A} : \bigsqcup_{i \in I(X)} \text{hom}(F(X)_i, A) \simeq \bigsqcup_{j \in I(G(A)_j)} \text{hom}(X, G(A)_j)
\]

which is natural in $X$ and $A$.

We need to define the $\mathcal{R}$-functor $D$ in the exactness structure, applied to concrete sets. On objects $D$ is applied to a pair $(X, S)$, where $X$ is a set with an equivalence relation $S$ on it. We define $D((X, S))$ to be the family of sets $\{L_k | k \in K\}$ which are the domains of the family of inmorphisms $\{l_k : L_k \to X \mid k \in K\}$ of $S$. These are exactly the injections that select the equivalence classes of $S$; each $l_k$ is considered up to isomorphism, so that isomorphic injections can be used interchangeably.

For a morphism $f : (X, S) \to (A, T)$, $D$ sends $f$ to the family of functions $\{q : L_k \to L_j \mid k \in K, j \in J\}$ and $f l_k = l_j q$. Note that for each $l_k$ it will correspond to and factorise through a unique $l_j$. In the category of families, this correspondence gives the required function $g = D(f)_{\text{index}} : I \to J$ that simplifies the family of functions to $\{q_k : L_k \to L_{g(k)} \mid k \in K\}$ and $f l_k = l_{g(k)} q_k$.

Later, in the abstract setting, $C$ may for some instances of the general theory behave similarly to $D$ in the sense that it may need to be not just a functor but an $\mathcal{R}$-functor. However, in the concrete set theoretic case, $C$ need not be an $\mathcal{R}$-functor, since it is suitably described as a functor: it sends $(X, S)$ to the codomain of the surjective function (up to isomorphism) $r_S : X \to R_S$ with kernel relation $S$. A morphism $f : (X, S) \to (A, T)$ gets sent to the unique $p : R_S \to R_T$ such that $r_T f = p r_S$.

We leave out the details that $C \dashv M_0$, since both $C$ and $M_0$ are functors in the usual sense. Note, however, that we have not shown that $D$ is functorial.

EDIT: $D$ is functorial. The empty set is mapped to the empty family, not to a singleton with the empty set inside (the family of its inmorphism). I'll fix everything going down when I have time, by removing the star notations.

Indeed, consider the following example:

\[
\begin{array}{ccc}
\emptyset & \xrightarrow{f} & B = \{0\} \\
\downarrow \circ & & \downarrow \circ \\
\emptyset & & C = \{0, 1\}
\end{array}
\]

Let us suppose that we choose the smallest equivalence relation $T$ on $C$, and the only equivalence relations on $\emptyset$ and $B$. Suitable inmorphisms (as inclusions) for $T$ are thus
\{0\} \hookrightarrow \{0,1\} and \{1\} \hookrightarrow \{0,1\}. A suitable inclusion as an inclusion on B’s equivalence relation is \{0\} \hookrightarrow \{0\}.

The equivalence relations together with \(f\) and \(g\) correspondingly define morphisms \(\tilde{f}\) and \(\tilde{g}\) in SetRel. D applied the composite \(\tilde{g} \circ \tilde{f}\) gives \(D(\tilde{g} \circ \tilde{f}) = \{q_1: \emptyset \rightarrow \{0\}, q_2: \emptyset \rightarrow \{1\}\}\), whereas \(D(\tilde{g}) \circ D(\tilde{f}) = \{p_1: \emptyset \rightarrow \{g(0)\}\}\). These sets are not bijective, containing two elements and one element, respectively. Hence, D is not functorial. We can attain both functoriality and an \(R\)-adjunction \(M_1 \dashv D\) by making one modification to our setting.

We observe that whenever \(S\) has more than one equivalence class in \(X\):

\[
\{0\} \simeq \text{hom}(((\emptyset, \emptyset), (X, S))) \not\cong \bigsqcup_{i \in I_D((X, S))} \text{hom}(\emptyset, D((X, S)))_i \simeq \{D((X, S))_i \mid i \in I_D((X, S))\}
\]

Hence, we need to remove functions \(f\) of the type \(f: \emptyset \rightarrow X\) where \(X \notin \{\emptyset, \{0\}\}\). In the case where \(X = \{0\}\), the above bijection does hold, but leaving in the function \(\emptyset \rightarrow \{0\}\) will include as well its composites, such as the composite \(\emptyset \rightarrow \{0\} \hookrightarrow \{0,1\}\). This latter composite does not satisfy the bijection (since the discrete equivalence relation on \{0,1\} has more than one equivalence class) and so we need to exclude from our category the function \(\emptyset \rightarrow \{0\}\). Hence, we are left with the simplified requirement that \(X \neq \emptyset\).

In the language of projective set theory [17], we are removing all the \(Z\)-empty functions. In effect, if we want to draw the exactness structure diagram to reflect this, we need to replace Set with Set*, the category of sets with non-\(Z\) functions.

(Equivalently, we could define the homsets relative to a class excluding exactly all \(Z\)-empty morphisms rather than the homset over all functions \(\text{Set}_1\) of Set.) Note in the diagram, if we want to write out explicitly what we mean by the right-most \(R\)-adjunction, we need to replace \(M_1\) with its associated \(R\)-functor in the sense of Remark 1.6.

**Definition 1.8.** For a category \(C\) on which we define a class \(Z\) of \(Z\)-empty morphisms, we write \(C^*\) for the category consisting of all objects \(C_0\), and consisting of those morphisms in \(C_1\) which are non-\(Z\).

By analogy, in the abstract setting, we will write the top category in the modified exactness structure as \(E^*\). That is, \(E^*_0 = E_0\) and \(E^*_1\) consists of all morphisms in \(E_1\) which do not map to \(Z\)-empty morphisms in \(C\). In the concrete set theoretic case, this top category is written as SetRel*.
Theorem 1.9. There is an \( \mathcal{R} \)-adjunction \( M'_1 \dashv D \) between \( \textbf{Set}^* \) and \( \textbf{SetRel}^* \) (where \( M'_1 \) is the \( \mathcal{R} \)-functor corresponding to the functor \( M_1 \)):

\[
\alpha_{X,\tilde{A}}: \text{hom}(M_1(X), \tilde{A}) \simeq \bigsqcup_{i \in I_{M'_1(X) \sim \{0\}}} \text{hom}(M'_1(X)_i, \tilde{A}) \simeq \bigsqcup_{j \in I_{D(\tilde{A})}} \text{hom}(X, D(\tilde{A}))
\]

where \( \tilde{A} = (A, T) \).

Proof. A rigorous proof in the more general setting of the abstract framework for projective set theory is given in Section 2, which includes the current theorem as a specialisation.

To get an idea of how the bijections work, for a morphism \( \tilde{f}: (X, 1) \to (A, T) \) with underlying function \( f: X \to A \) we have that the image \( f1 \) is contained completely in an equivalence class of \( T \). Then, \( \tilde{f} \) is sent under \( \alpha_{X,\tilde{A}} \) to the function \( f_{\text{restr}}: X \to L_k \), where \( L_k \) is the domain of an injection which selects the equivalence class containing \( f1 \). The converse process is similar, extending the codomain of some \( g: X \to L_k \) to \( g_{\text{extnd}}: X \to A \). After this, one needs to show naturality.

\[\Box\]

2 The Abstract Setting

2.1 Overview

In this section we show that the generalised exactness structure that was constructed in Section 1.1 for the category of sets can be shown to hold in any projective set theory in the sense of [17]. For the interested reader, the full abstract axiomatic framework is defined and explained therein.

For the purposes of the present paper, we will only state the axioms that are necessary to define and prove the generalised exactness structure for sets. Our setting has extra structure that is not captured by just having a chain of four adjunctions (such as the inmorphisms and outmorphisms) and has extra properties (such as the central functor being required to be faithful).

However, the group theoretic setting that we are generalising from does exhibit these characteristics and correspondingly so does the set theoretic context. In other words, the general exactness structure by itself does not capture the full framework, although it may be worthwhile to see whether the exactness structure can be used to derive some of our structure. For example, \( C \) and \( D \) map down an object in the poset to the codomains of outmorphisms and domains of inmorphisms, respectively. Moreover, the way we defined our exactness structure for \( \textbf{Set} \), one can recover inmorphisms and outmorphisms of a concrete equivalence relation \( S \) on a concrete set \( X \) by taking \( C \) applied to \( (X, 0) \to (X, S) \) and \( D \) applied to \( (X, S) \to (X, 1) \), respectively. In the first case, the function is mapped under \( C \) to the outmorphisms of \( S \) (which are all isomorphic for concrete sets) and in the second case, the function is mapped under \( D \) to the inmorphisms of \( S \).

Thus, in this way one can recover candidates for inmorphisms and outmorphisms for an exactness structure by applying \( C \) and \( D \) to chosen morphisms. In the abstract setting
one can use this method, because later we define \( C \) and \( D \) by a direct analogue of the concrete set case.

Turning our attention back to our exposition of the general exactness structure, the left-most functor in the chain does not require using an \( \mathcal{R} \)-adjunction for concrete sets, but simply an adjunction in the usual sense. However, the abstract theory is self-dual. In particular, outmorphisms are dual to inmorphisms. This means that we need to define the starting adjunction dually to the final \( \mathcal{R} \)-adjunction in the chain.

The chain is thus, for an abstract projective set theory framework with an underlying form \( F : \mathcal{E} \rightarrow \mathcal{C} \):

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{C} & \mathcal{C} \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\mathcal{C} & \xleftarrow{D} & \mathcal{Fam} \mathcal{C}^* \\
\end{array}
\]

Note that for \( \mathcal{C} = \text{Grp} \), we have that \( \text{Grp}^* = \text{Grp} \) since there are no \( Z \)-empty homomorphisms (in short, because there is no empty group). To prove that this chain holds in any projective set theory, the first requirement is to show that in the abstract setting \( C \) and \( D \) are in fact \( \mathcal{R} \)-functors.

The factorisation axioms in [17] for inmorphisms are restated here, for which the duals hold for outmorphisms, and are sufficient to prove that the chain of functors form adjunctions (with \( \mathcal{R} \)-adjunctions at the ends).

### 2.2 The Abstract Framework

We work in a setting with the following data:

- A functor \( F : \mathcal{E} \rightarrow \mathcal{C} \), which is a form in the sense of [8], makes up the central functor for the exactness structure. Each morphism \( f : A \rightarrow B \) in \( \mathcal{C} \) gives rise to direct and inverse image maps in the sense of [4]. That is, the direct image and inverse image form a Galois connection between bounded lattices.

- For each element \( S \) in the poset \( \Sigma X \) associated with an abstract set \( X \), we have a family of inmorphisms \( \{ f_k \mid K \in I \} \) and a family of outmorphisms \( \{ g_j \mid j \in J \} \) in the sense of [17]. The requirements on inmorphisms and outmorphisms are laid out in the axioms below.

- A class \( \mathcal{N} \) of null morphisms and a class \( \mathcal{Z} \) of empty morphisms.

**Remark 2.1.** In our motivating examples, \( \text{Grp} \) and \( \text{Set} \), the central functor \( F \) is a bifibration, which from the outfunctor constructions (i.e., as two pseudofunctors) give the Galois connection for a morphism \( f : X \rightarrow Y \).
Remark 2.2. One can specify a set of axioms on the class $\mathcal{N}$ and from there define the class $\mathcal{Z}$. However, for the purposes of this paper it suffices to say that $\mathcal{N}$ is a subclass of the class $\{f \in C_1 \mid f_1 = 0\}$ and that $\mathcal{Z}$ is again a subclass of this. We described earlier what $\mathcal{Z}$ is for $\text{Set}$ (all empty functions $\emptyset \to X$ except the identity empty function). The class $\mathcal{N}$ is then all constant functions and all empty functions (including $\emptyset \to \emptyset$). For $\text{Grp}$, $\mathcal{N}$ is empty and hence so is $\mathcal{Z}$. The idea behind why $\mathcal{N}$ is necessary is that $\text{Set}$ is not pointed like $\text{Grp}$, as there is no candidate for a zero object. We use a different notion of pointedness, which is captured by the null morphisms.

The necessary axioms (of which the duals are also assumed) for our current objective are those that appear in [17] as the third main axiom. We require for any $S \in \Sigma X$ with inmorphisms $\{f_k \mid k \in K\}$, and for any morphism $f$, that:

Axiom A1.

$K \neq \emptyset$.

Axiom A2.

Each $f_k$ is a monomorphism.

Axiom A3.

$f_1 \leq S \Rightarrow \exists f_k \exists u (f = f_k u)$.

Axiom A4.

$f_k 1 \leq S$ for all $k \in K$.

Axiom A5. For any $f_k$ and $f_j$,

$\exists u \exists v (f_k u = f_j v \wedge (f_k u \text{ not } \mathcal{Z}\text{-empty})) \Rightarrow f_k \simeq f_j$.

Remark 2.3. This axiom is stronger than that which appears in the thesis on projective set theory [17]. The reason why this stronger condition is required is that we need to be able to prove that any morphism $f \not\in \mathcal{Z}$ factors through a unique inmorphism of $f_1$. This is necessary to prove the bijection later in the $\mathcal{R}$-adjunctions at the start of the chain and at the end of the chain. For the purposes of [17] it seems that it is only required that all $f \not\in \mathcal{N}$ factors through a unique inmorphism of $f_1$. Regardless, however, for $\text{Set}$ the stronger condition above does in fact hold.

Axiom A6. Fix $f_j$. Then, $\exists f_k (f_k = f_j u) \iff u \text{ is iso}$.

Axiom A7. $(f_1 \leq f_1 1 \text{ and } f \not\in \mathcal{N}) \Rightarrow \exists u (f = f_k u)$.

The duals of these axioms are required, respectively, for the outmorphisms.
**Proposition 2.4.** The previous list of axioms holds for the concrete set theoretic case, that is, where $\mathcal{C} = \text{Set}$. 

**Proof.** The proof, and the background to the theory, can be found in [17].

**Definition 2.5.** For an object $(X, S)$ in $\mathcal{E}^*$, we define:

$$D(X, S) = \{L_k \mid k \in K\}$$

where the $L_k$'s are the domains of inmorphisms of $S$.

For a morphism $f: (X, S) \rightarrow (Y, T)$, $D$ applied to $f$ is defined as:

$$D(f) = \{q: L_k \rightarrow L_j \mid k \in K \text{ and } j \in J \text{ and } f l_k = l_j q\}$$

where inmorphisms of $S$ are indexed by $K$ and inmorphisms of $T$ are indexed by $J$. The functor $C$ is defined dually.

### 2.3 The Functors in the Exactness Structure

**Theorem 2.6.** The constructions $C, D: \mathcal{E}^* \rightarrow \text{Fam}\mathcal{C}^*$ in the chain are functors and hence are $\mathcal{R}$-functors from $\mathcal{E}^*$ to $\mathcal{C}^*$.

**Proof.** We prove the theorem for $D$, from which $C$ follows dually. We need to show that $D$ is functorial. Suppose that we have morphisms $\tilde{f}: (X, S) \rightarrow (Y, T)$ and $\tilde{g}: (Y, T) \rightarrow (Z, U)$ in $\mathcal{E}^*$. From Definition 2.5, we have that:

$$D(\tilde{f}) = \{q: L_k \rightarrow L_j \mid k \in K, j \in J \text{ and } f l_k = l_j q\}$$

and

$$D(\tilde{g}) = \{p: L_j \rightarrow L_h \mid j \in J, h \in H \text{ and } gl_j = l_h p\}$$

Now, we can compose these families as morphisms in $\text{Fam}\text{Set}^*$. For each $k$ there is exactly one morphism $q: L_k \rightarrow L_j$ in the top set; the same holds for the bottom set. Moreover, each factorisation through an $l_j$ is through a unique such $l_j$. The corresponding index functions would be from $K \rightarrow J$ and then from $J \rightarrow H$ and both are injective. With this in mind, the composition $D(\tilde{g}) \circ D(\tilde{f})$ is the result of all possible compositions of members of the second argument with members of the first argument. Hence,

$$D(\tilde{g}) \circ D(\tilde{f}) = \{p \circ q: L_k \rightarrow L_j \rightarrow L_h \mid k \in K, j \in J, h \in H, f l_k = l_j q \text{ and } gl_j = l_h p\}$$

Now, applying $D$ directly, we have:

$$D(\tilde{g} \circ \tilde{f}) = \{r: L_k \rightarrow L_h \mid k \in K, h \in H \text{ and } g f l_k = l_h r\}$$

In order to show that $D(\tilde{g} \circ \tilde{f}) = D(\tilde{g}) \circ D(\tilde{f})$, we can show that their elements are equal. Suppose that $p \circ q \in D(\tilde{g}) \circ D(\tilde{f})$. Then $g f l_k = gl_j q = l_h p q$ and hence $p \circ q$ is an element of $D(\tilde{g} \circ \tilde{f})$. Suppose conversely that $r \in D(\tilde{g} \circ \tilde{f})$. Then $f l_k 1 \leq f S \leq T$ by Axiom A4
and hence there is a factorisation $fl_k = ljq$ by Axiom $A3$. Furthermore, $g1 \leq gT \leq U$ by Axiom $A4$ and hence there is a factorisation $gl_j = lh_p$, again by Axiom $A3$ and we have that $gf_l = lh_pq$. Hence, there is a decomposition $r = pq$ and $r \in D(\overline{g}) \circ D(\overline{f})$.

\[
\begin{array}{ccc}
L_k & \xleftarrow{l_k} & X \\
\downarrow{q} & & \downarrow{f} \\
L_j & \xrightarrow{l_j} & Y \\
\downarrow{p} & & \downarrow{g} \\
L_h & \xrightarrow{l_h} & Z \\
\end{array}
\]

\[r = pq \]

2.4 The Adjunctions Comprising the Exactness Structure

In general, homsets involving a functor in the usual sense and those of its corresponding $R$-functor will be bijective. (The $R$-functor corresponding to a functor in the sense of Remark $1.6$ always sends objects and morphisms to singleton families.)

The proof below that $M'_1 \dashv D$ will turn out to resemble the expected usual structure for the proof of an adjunction, that is, proving that there is a bijection as well as proving naturality when varying the variables. It is important here to note that our $R$-adjunctions have a useful property which allows us to stay close to this proof layout without being burdened by additional arguments about coproducts of homsets. The explanation for this is as follows. An inmorphism $l_k: L_k \rightarrow X$ of some $S$ composed with $f: X \rightarrow Y$, when it factors through an inmorphism $l_j: L_j \rightarrow Y$ of some $T$, will factor uniquely. This allows us for $f$, $S$ and $T$ to determine an injective correspondence between the underlying disjoint inmorphisms of $S$ into those of $T$. In other words, we can argue about the disjoint components of the coproduct relating to $S$ and automatically correspond each component to the corresponding component in the coproduct relating to $T$.

For the below argument the situation is a further simplification of this, since the left coproduct in the bijection (I) below at the start of the proof has a single constituent homset. Then, as stated in the previous paragraph, an $f: X \rightarrow A$ where $f1 \leq S$ for $\overline{A} = (A, S)$ specifies which one of the disjoint homset components of the right coproduct to map to. This will be the homset of all morphisms $X \rightarrow L_j$. For a concrete set, recall that this will be the $L_j$ that specifies the equivalence class of $S$ containing the image $f1$; and furthermore, the element of the homset we want to map to is the function $f$ but restricting the codomain from $A$ to $L_j$.

This means that the natural bijection that we are constructing preserves disjointness of the homsets that make up the two coproducts. The $f$ in question in the proof below is $f = F(\overline{m})$ where the morphism $\overline{m}: M_1(X) \rightarrow \overline{A}$ is chased along the diagram.

**Theorem 2.7.** There is an $R$-adjunction $M'_1 \dashv D$. Dually, it follows that there is an $R$-adjunction $C \dashv M'_0$. 

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Proof. $M_1$ and $M_0$ are functors and hence can be made into $\mathcal{R}$-functors. In the diagram, we use $M'_1$ for the $\mathcal{R}$-functor corresponding to $M_1$.

\[
\begin{array}{ccc}
\text{Fam}_{\mathcal{E}^*} & \xrightarrow{\mathcal{E}^*} & \text{FamC}^* \\
\downarrow & & \downarrow \\
\text{Fam}_{\mathcal{E}^*} & \xrightarrow{\mathcal{C}^*} & \text{Fam}_{\mathcal{E}^*}
\end{array}
\]

where the function $\mathcal{E}^*$ is the inmorphism with underlying $F(\mathcal{E}^*) = m$.

\[
a_{X,\bar{A}}: \text{hom}(M_1(X), \bar{A}) \simeq \bigsqcup_{i \in I_{M'_1(X)}} \text{hom}((M'_1(X)_i, \bar{A}) \simeq \bigsqcup_{j \in I_{D(A)}} \text{hom}(X, D(\bar{A}))_j)
\]  

(1)

In order to show this bijection, we need to show that $a_{X,\bar{A}}$ is a bijection which is natural as $X$ and $\bar{A} = (A, T)$ varies. Suppose that we have a morphism $\bar{m}: M_1(X) \to (A, T)$. Then, since $M_1(X) = (X, 1)$, we have $\bar{m}: (X, 1) \to (A, T)$ with underlying $F(\bar{m}) = m$ and $m1 \leq T$. Hence, $m: X \to A$ factors through an inmorphism $l_T: L_T \to A$ as $m \equiv l_Tq$ by Axiom A3. This factorisation is unique by Axiom A5. We assign $\bar{m}$.

The inverse $a^{-1}_{X,\bar{A}}(r): X \to L_T$ assigns $a^{-1}_{X,\bar{A}}(r) = l_Tr$ where $l_T$ is the inmorphism with domain $L_T$. Observe that $l_Tr: (X, 1) \to (A, T)$ is the corresponding morphism in $\mathcal{E}^*$ to $l_Tr$ since $l_T1 \leq l_T1 \leq T$. Hence, $a^{-1}_{X,\bar{A}}a_{X,\bar{A}}(\bar{m}) = a^{-1}_{X,\bar{A}}(q) = l_Tq = \bar{m}$.

Now, let us compute $a_{X,\bar{A}}a_{X,\bar{A}}(r)$. Let $r: X \to L_T$ for some domain $L_T$ of an inmorphism $l_T$. At the first step, $a^{-1}_{X,\bar{A}}(r) = l_T^*r$, where $l_T^*r: (X, 1) \to (A, T)$ is a morphism in $\mathcal{E}^*$ since $l_T1 \leq l_T1 \leq T$. Then, $a_{X,\bar{A}}(l_T^*r) = r$ since $l_T^*r$ factors through the inmorphism $l_T$ and such factorisation is through a unique inmorphism by Axiom A5.

Now we need to show naturality. We need to show that the following diagram commutes for any $f: X' \to X$ in $\mathcal{C}^*$ and $g: (A, T) \to (A', T') \in \mathcal{E}^*_1$.

\[
\begin{array}{ccc}
\text{hom}(M_1(X), \bar{A}) & \xrightarrow{a_{X,\bar{A}}} & \bigsqcup_{j \in I_{D(\bar{A})}} \text{hom}(X, D(\bar{A}))) \\
\downarrow & & \downarrow \\
\text{hom}(M_1(f)_X, \bar{A}') & \xrightarrow{a_{X',\bar{A}'}} & \bigsqcup_{j \in I_{D(\bar{A})}} \text{hom}(X', D(\bar{A}))) \\
\end{array}
\]

The function $\bigsqcup_{j \in I_{D(\bar{A})}} \text{hom}(f, D(\bar{A}))_j$ sends a morphism $t(h)$ for $h: X \to L_T$ to the family

\[
\{X' \xrightarrow{f} X \xrightarrow{h} L_T \xrightarrow{D(\bar{A})} L_{T'} | i \in I_{D(\bar{A})}\}
\]  

(2)

where $I_{D(\bar{A})}$ indexes the family of morphisms $D(\bar{A})$. 

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Here \( i \) is the insertion \( i: \text{hom}(\langle X, L_T \rangle) \to \bigsqcup_{j \in \mathcal{D}(A)} \text{hom}(X, D(\tilde{A})) \) is the usual embedding into a coproduct. But we may just as well associate \( i(h) \) with \( h \), since we can uniquely identify every morphism in a coproduct of (hom)sets \( \bigsqcup_{i \in 1} H_i \) with the constituent morphism in a specific (hom)set \( H_n \).

Now, there is only one morphism \( D(\tilde{g}) \) in the family \( D(\tilde{g}) \) that can compose with \( \alpha_{X,\tilde{A}}(\tilde{m}) \circ f = hf \), i.e., the family in \( \mathcal{B} \) is a singleton. This is because, in the way \( D \) is defined, we require that \( gl_T = l_T \circ D(\tilde{g}) \) and such choice of factorisation through an inmorphism \( l_T \) is unique by Axiom \( \mathcal{A}_5 \).

Chasing for a given \( \tilde{m}: \langle X, 1 \rangle \to \langle A, T \rangle \) with \( F(\tilde{m}) = m: X \to A \), we have the following diagram:

\[
\begin{array}{c}
(X, 1) \xrightarrow{\tilde{m}} (A, T) & \xrightarrow{\alpha_{X,\tilde{A}}(\tilde{m})} X \xrightarrow{L_T} \\
| & | \\
(X', 1) \xrightarrow{M_1(f)} (X, 1) \xrightarrow{\tilde{m}} (A, T) \xrightarrow{\tilde{g}} (A', T') \xleftarrow{\alpha_{X',\tilde{A}'}(\tilde{g} \circ \tilde{m} \circ M_1(f))} X \xrightarrow{L_T'} \\
| & | \\
& X' \xrightarrow{f} X \xrightarrow{\alpha_{X,\tilde{A}}(\tilde{m})} L_T \xrightarrow{D(\tilde{g})} L_T'
\end{array}
\]

To show the equality, first of all let \( \alpha_{X',\tilde{A}'}(\tilde{g} \circ \tilde{m} \circ M_1(f)) = h' \) where \( l_T h' = m' \) and \( m' = F(\tilde{g} \circ \tilde{m} \circ M_1(f)) = g \circ m \circ f \). Then, for \( \alpha_{X,\tilde{A}}(\tilde{m}) = h \), we have

\[
l_T h' = g \circ m \circ f \quad \text{from the definition of } h'
\]

\[
= g \circ l_T \circ h \circ f \quad \text{since } m1 \leq T, \text{ it factors through an } l_T
\]

\[
= l_T \circ D(\tilde{g}) \circ h \circ f \quad \text{by } gl_T1 \leq gT \leq T'; \text{definition of } D(\tilde{g})
\]

Hence, since \( l_T \) is a monomorphism, we have \( \alpha_{X',\tilde{A}'}(\tilde{g} \circ \tilde{m} \circ M_1(f)) = h' = D(\tilde{g}) \circ h \circ f \), which proves the equality in the diagram. \( \square \)

### 2.5 Further Observations

The generalised exactness structure elaborated in this text holds for all the examples of structures satisfying the projective group theory axioms, being a specialisation now of projective set theory. Indeed, the purpose of this paper is mostly to show that from this
general perspective, the category of sets and that of groups are both examples of the same exactness structure.

There are different avenues that can be explored from this suggested starting point for sets. For example, one may be interested to capture more aspects of set theory in a self-dual context. The approach in this paper is limited in this sense: in spirit it approaches set theory from self-dual categorical properties of groups, giving for example isomorphism theorems, but not capturing for example topos theory (which at least would require a self-dual notion of a subobject classifier). It also does not capture set theory from the more classical point of view of the axiomatic basis of set theory, for example, what would an element be of an abstract set $X$? If it is a morphism $1 \to X$, then indeed we would have a co-element $X \to \beta$, forming a self-dual counterpart.

Another avenue to pursue, for example, emerges from applications to quantum physics. From the perspective of the authors in [2,3,5] the interest is to incorporate specific properties of categorical logic in the context of quantum physics. Their corresponding notion of an abstract subobject lattice is what they term to be an “effectus”. Predicate morphisms, of the form $X \to 1 + 1$, should be dual to states, of the form $1 \to X$. In this manner, many of the concepts within quantum physics can be phrased in familiar categorical notions (such as a map being factorisable). One should also ask what the converse counterparts would be when starting with mathematical objects. An example would be to extract inmorphisms and outmorphisms from an exactness structure as explained in Section 2.1 and to describe what their corresponding meaning would be in the quantum physical setting.

In conclusion, what has been shown in this paper is that the central encapsulating structure in self-dual group theory at the same time holds for sets when studied from a suitably general perspective. The notion of categorical duality is replaced with a functorial notion of duality that has previously been applied in the work of Z Janelidze on faithful functors. This approach to set theory is does not isolate points within sets, in the sense that singleton subsets are indistinguishable as equivalence relations, and it captures an abstraction of exact sequences through the central exactness structure of a sequence of adjunctions.
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