The Landau-Ginzburg-Wilson approach to critical phenomena in the presence of gauge symmetries

Andrea Pelissetto
Dipartimento di Fisica dell’Università di Roma “La Sapienza” and INFN, Sezione di Roma I, I-00185 Roma, Italy

Antonio Tripodo, Ettore Vicari
Dipartimento di Fisica dell’Università di Pisa and INFN, Sezione di Pisa, I-56127 Pisa, Italy

We critically reconsider the Landau-Ginzburg-Wilson (LGW) approach to critical phenomena in the presence of gauge symmetries. In the LGW framework, to obtain the universal features of a continuous transition, one identifies the order parameter $\Phi$ and considers the corresponding most general $\Phi^4$ field theory that has the same symmetries as the original model. In the presence of gauge symmetries, one usually considers a gauge-invariant order parameter and a LGW field theory that is invariant under the global symmetries of the original model. We show that this approach, in which the gauge dynamics is effectively integrated out, may sometimes lead to erroneous conclusions on the nature of the critical behavior. As an explicit example, we show that the above-described LGW approach generally fails for the three-dimensional ferromagnetic and antiferromagnetic CP$^{N-1}$ models, which are invariant under global U($N$) and local U(1) transformations. We point out possible implications for the finite-temperature chiral transition of nuclear matter.

PACS numbers: 05.70.Fh, 05.70.Jk, 05.10.Cc, 25.75.Nq

I. INTRODUCTION

In the renormalization-group (RG) approach to critical phenomena, the universal properties of continuous phase transitions can be obtained by using the Landau-Ginzburg-Wilson (LGW) field-theoretical approach [1–6]. In this framework the critical features are uniquely specified by the nature of the order parameter associated with the critical modes, by the symmetries of the model, and by the symmetries of the phases coexisting at the transition, the so-called symmetry-breaking pattern. In this paper we consider models which are also characterized by gauge symmetries. In this case, the traditional LGW approach starts by considering a gauge-invariant order parameter, and a LGW field theory that is invariant under the global symmetries of the original model. We will show that in some cases this LGW approach leads to erroneous conclusions on the nature of the critical behavior.

For this purpose, we consider the three-dimensional (3D) CP$^{N-1}$ model defined by the Hamiltonian

$$H_{CP} = J \sum_{\langle xy \rangle} |\bar{z}_x \cdot z_y|^2,$$

where the sum is over the nearest-neighbor sites $\langle xy \rangle$ of a cubic lattice, and $z_x$ are $N$-component complex vectors satisfying $\bar{z}_x \cdot z_x = 1$. The model is ferromagnetic for $J < 0$ and antiferromagnetic for $J > 0$. CP$^{N-1}$ models have a global U($N$) symmetry $|z_i \rightarrow U z_i$, with a space-independent $U \in U(N)$, and a local U(1) gauge symmetry $|z_x \rightarrow e^{i\alpha x} z_x$. The thermodynamics can be derived from the standard partition function

$$Z = \sum_{\{z_x\}} \exp(-\beta H_{CP}).$$

3D CP$^{N-1}$ models are expected to undergo a finite-temperature transition between the high- and low-temperature phases. In the ferromagnetic case, the order parameter may be identified as the gauge-invariant site variable

$$Q_x^{ab} = z_x^a z_x^b - \frac{1}{N} \delta^{ab},$$

which is a hermitian and traceless $N \times N$ matrix. It transforms as

$$Q_x \rightarrow U^\dagger Q_x U,$$

under global U($N$) transformations. The order-parameter field in the corresponding LGW theory is therefore a traceless hermitian matrix field $\Phi^{ab}(x)$, which can be formally defined as the average of $Q_x^{ab}$ over a large but finite lattice domain. The LGW field theory is obtained by considering the most general fourth-order polynomial in $\Phi$ consistent with the U($N$) symmetry [4],

$$\mathcal{H} = \text{Tr}(\partial_\mu \Phi)^2 + r \text{Tr} \Phi^2 + w \text{tr} \Phi^3 + u (\text{Tr} \Phi^2)^2 + v \text{Tr} \Phi^4.$$

For $N = 2$, the cubic term vanishes and the two quartic terms are equivalent. Therefore, one recovers the O(3)-symmetric LGW theory, consistently with the equivalence between the CP$^1$ and the Heisenberg model. For $N \geq 3$, the cubic term is generically expected to be present. This is usually considered as the indication that phase transitions of systems sharing the same global
properties are of first order, as one can easily infer using mean-field arguments. However, in the large-$N$ limit, a different argument allows one to show that the critical behavior of the ferromagnetic CP$^{N-1}$ model is the same as that of an effective abelian Higgs model for an $N$-component complex scalar field coupled to a dynamical U(1) gauge field $\tilde{A}$. This equivalence is conjectured to extend to finite $N$ at the critical point $\frac{\lambda}{\nu}$. The RG flow of the abelian Higgs model presents a stable fixed point for a sufficiently large number of components $N$. Thus, for large values of $N$, 3D CP$^{N-1}$ models may undergo a continuous transition, in the same universality class as that occurring in the abelian Higgs model, contradicting the LGW results. The predictions derived from the LGW Hamiltonian are also contradicted by recent numerical studies, which provide evidence of continuous transitions in models that are expected to be in the same universality class as that of the 3D CP$^2$ model. All these results suggest that the critical modes at the transition are not exclusively associated with the gauge-invariant order parameter $Q$ defined in Eq. (3). Other features, for instance the gauge degrees of freedom, become relevant, requiring an effective description different from that of the LGW theory.

In this paper, we show that the LGW approach, in which the gauge degrees of freedom are somehow integrated out, also fails for the antiferromagnetic CP$^{N-1}$ model (ACP$^{N-1}$), i.e., in model (1) with $J > 0$, for $N = 4$. In the antiferromagnetic case the same argument used for $J < 0$ allows one to identity the order parameter with a staggered version of the site variable $Q_{x}^{ab}$. In the LGW approach, the fundamental field is therefore a hermitean traceless matrix $\Phi_{x}^{ab}(x)$ and the corresponding field theory is given by Eq. (5). At variance with the ferromagnetic case, in the ACP$^{N-1}$ case there is also a discrete $\mathbb{Z}_2$ symmetry that forces $w = 0$. For $N = 2$ and $N = 3$ this approach predicts a transition in the same universality class as that of the $O(n)$ vector model, with $n = 3$ and 8, respectively. The result for $N = 2$ is consistent with the exact mapping between CP$^1$ and vector O(3) models. The prediction for $N = 3$ has been verified numerically: for this value of $N$ the standard LGW approach provides the correct description of the critical modes. For $N > 4$, the analysis of the RG flow in the effective LGW theory does not identify stable fixed points that can be associated with continuous transitions. Therefore, the effective theory predicts that possibly present transitions are of first order. The numerical results we report here contradict this prediction. For the ACP$^3$ model, we have numerical evidence of a continuous transition. Again, the effective theory for a gauge-invariant order parameter does not provide a correct description of the critical behavior.

Our considerations may be relevant for the finite-temperature transition of nuclear matter between the low-temperature hadronic phase, in which chiral symmetry is broken, and the high-temperature quark-gluon phase, in which chiral symmetry is restored in the limit of massless quarks. The nature of this transition is still controversial, in particular in the case of two light flavors, in spite of several Monte Carlo studies using different lattice QCD formulations. Some studies favor a continuous transition, but are not sufficiently accurate to clearly identify the corresponding universality class, while others report evidence of a first-order transition.

Our understanding of the QCD finite-temperature phase transition in the presence of $N_f$ light quarks is based on the analysis of the LGW effective theory. The relevant order-parameter field $\Phi_{x}^{ab} = (N_f \times N_f)$ complex matrix field $\Phi_{ij}$, related to the bilinear quark operators $\bar{\psi}_{Li}\psi_{Rj}$, which is the analogue of the bilinear $Q_{x}^{ab}$ of the CP$^{N-1}$ models. The corresponding LGW theories have been analyzed in detail and the corresponding predictions for the nature of the transitions have been extensively discussed. Note that they are all based on the assumption that the relevant critical modes are only associated with the local, gauge-invariant bilinear fermion operators. In particular, it is implicitly assumed that gauge modes play no role at the transition, a hypothesis that should not be taken for granted, as the examples discussed in this paper indicate.

The paper is organized as follows. In Sec. II we define the LGW $\Phi^4$ theory which is supposed to describe the universal properties of the critical transition in the ACP$^{N-1}$ model and briefly review its predictions. In Sec. III we discuss the results of Monte Carlo simulations of the ACP$^3$ lattice model, which provide evidence of a continuous transition. Finally, in Sec. IV we summarize our conclusions.

II. THE LGW $\Phi^4$ THEORY FOR THE ACP$^{N-1}$ MODELS

In this section we review the derivation of the LGW theory for the ACP$^{N-1}$ model, emphasizing the main assumptions. Similar arguments also hold for the antiferromagnetic Rp$^{N-1}$ model, in which spins are real.

As a first step we should identify the order parameter. In the antiferromagnetic case we expect a breaking of translational invariance in the low-temperature phase. Therefore, a good order parameter is the staggered quantity

$$A^{ab} = \sum_{x} p_x Q_{x}^{ab} = \sum_{x} p_x x_{a,b},$$

where $p_x$ is the parity of the site $x \equiv (x_1, x_2, x_3)$ defined by $p_x = (-1)^{\sum_k x_k}$. The matrix $A^{ab}$ is hermitian and traceless. Moreover, it changes sign under translations of one site, which exchange the even- and odd-parity sublattices. In order to construct the LGW model, we replace $A$ with a local variable $\Phi(x)$ which is taken as fundamental variable (essentially, one imagines that $\Phi$ is defined as $A$, but now the summation extends only over a large, but finite, sublattice). Then, the corresponding LGW theory is
obtained by writing down the most general fourth-order polynomial that is invariant under $U(N)$ transformations and under the global $\mathbb{Z}_2$ transformation $\Phi \rightarrow -\Phi$, i.e.

$$\mathcal{H}_a = \operatorname{Tr}(\partial_\mu \Phi)^2 + r \operatorname{Tr} \Phi^2 + \frac{u_0}{4} (\operatorname{Tr} \Phi^2)^2 + \frac{v_0}{4} \operatorname{Tr} \Phi^4.$$  \hspace{1cm} (7)

Note that the symmetry $\Phi \rightarrow -\Phi$ does not hold in ferromagnetic CP$^{N-1}$ models, so that the corresponding LGW theory \cite{11} contains also a cubic term.

Since any $2 \times 2$ and $3 \times 3$ traceless matrix $\Phi$ satisfies

$$\operatorname{Tr} \Phi^4 = \frac{1}{2} (\operatorname{Tr} \Phi^2)^2,$$  \hspace{1cm} (8)

the two quartic terms of the Hamiltonian \cite{17} are equivalent for $N = 2$ and $N = 3$. Therefore, the $N = 2$ and $N = 3$ $\Phi^4$ theories \cite{17} can be exactly mapped onto the $O(3)$ and $O(8)$ symmetric $\Phi^4$ vector theories, respectively. This implies that the continuous transitions of the ACP$^3$ and ACP$^4$ models belong to the $O(3)$ and $O(8)$ vector universality classes, respectively. For $N = 3$ this is a highly nontrivial prediction, as it implies a dynamical enlargement of the symmetry at the critical point. The $O(8)$ nature of the transition has been confirmed \cite{11} by a detailed Monte Carlo (MC) study of the ACP$^2$ model. Therefore, for this value of $N$, the LGW theory provides the correct effective description of the critical behavior.

The nature of the transitions in ACP$^{N-1}$ models for $N \geq 4$ has been investigated by analyzing the RG flow of the LGW theory \cite{11} in the space of the two quartic parameters, using two different perturbative methods \cite{11}. The analysis of the five-loop series in the $\overline{\text{MS}}$ renormalization scheme \cite{33} and of the six-loop series in the massive zero-momentum scheme \cite{3, 0, 36} both indicate the absence of stable fixed points for $N = 4$ and $N = 6$.

As a consequence, the LGW approach predicts the absence of continuous transitions for these two values of $N$. However, as we shall see, this is contradicted by the numerical results for $N = 4$ presented in the next section, which provide a robust evidence of a continuous transition.

**III. NUMERICAL RESULTS FOR THE ACP$^3$ LATTICE MODEL**

In this section we present a numerical investigation of the ACP$^3$ lattice model \cite{11}. We set $J = 1$ and perform MC simulations of cubic systems of linear size $L$ with periodic boundary conditions. Because of the antiferromagnetic nature of the model we only consider even values of $L$, up to $L = 48$. We use a standard Metropolis algorithm \cite{37}.

In our MC simulations we compute correlations of the gauge invariant operator $Q^a_{xy}$ defined in Eq. \cite{12}. Its two-point correlation function is defined as

$$G(x - y) = \langle \operatorname{Tr} Q^a_{x} Q_y \rangle$$  \hspace{1cm} (9)

 Due to the staggered nature of the ordered parameter, we only consider correlations between points that have the same parity. The susceptibility and the correlation length are defined as

$$\chi = \sum_{\text{even } x} G(x) = \tilde{G}(0),$$  \hspace{1cm} (10)

$$\xi^2 = \frac{1}{4 \sin^2(p_{\text{min}}/2)} \frac{\tilde{G}(0) - \tilde{G}(p) \rangle}{G(p)},$$  \hspace{1cm} (11)

where $\tilde{G}(p) = \sum_{\text{even } x} e^{ipx} G(x)$ is the Fourier transform of $G(x)$ over the even-parity sublattice, $p = (p_{\text{min}}, 0, 0)$, and $p_{\text{min}} \equiv 2\pi/L$. Finally, we consider the Binder parameter

$$U = \frac{\langle \sum_{\text{even } x} \operatorname{Tr} Q^a_{x} Q_x \rangle^2}{\langle \sum_{\text{even } x} \operatorname{Tr} Q^a_{x} Q_x \rangle^2}.$$  \hspace{1cm} (12)

To determine the critical behavior we study the finite-size behavior. The finite-size scaling (FSS) limit is obtained by taking $\beta \rightarrow \beta_c$ and $L \rightarrow \infty$ keeping

$$X \equiv (\beta - \beta_c) L^{1/\nu}$$  \hspace{1cm} (13)

fixed, where $\beta_c$ is the inverse critical temperature and $\nu$ is the correlation-length exponent. Any RG invariant quantity $R$, such as $R_\xi \equiv \xi / L$ and $U$, is expected to asymptotically behave as

$$R(\beta, L) = f_R(X) + L^{-\omega} g_R(X) + \ldots$$  \hspace{1cm} (14)

where $f_R(X)$ is a universal function apart from a trivial normalization of the argument. In particular, the quantity $R^* = f_R(0)$ is universal within the given universality class. The approach to the asymptotic behavior is controlled by the universal exponent $\omega > 0$, which is associated with the leading irrelevant RG operator.

To identify a transition point, we check whether the estimates of $R_\xi \equiv \xi / L$ and of $U$ for different values of

![FIG. 1: MC data of $R_\xi$ for the ACP$^3$ lattice model and several lattice sizes $L$ up to $L = 48$. The data sets corresponding to different values of $L$ cross for $\beta \approx 12.2$. The dotted lines are only meant to guide the eye.](Image)
We first use the simple Ansatz

\[ R = R^* + c_1 L^{1/\nu} (\beta - \beta_c), \]

for \( R = R_\xi \) and \( U \). This Ansatz apparently describes well the data in a relatively large interval around the transition point, essentially when \( \Delta = |R_\xi - R_\xi^*| \lesssim 0.10 \). We have also performed fits considering a second-order and a third-order polynomial in \( X \), i.e., fitting \( R \) to

\[ R = R^* + \sum_{k=1}^{n} \epsilon_k X^k, \]

with \( n = 2 \) and \( n = 3 \). Finally, we also performed combined fits of \( R_\xi \) and \( U \). The data are not sufficiently precise to allow us to include scaling corrections in the fit. Therefore, to estimate their relevance, we have repeated all fits several times, each time only including data satisfying \( L \geq L_{\text{min}} \), varying \( L_{\text{min}} \) between 8 and 32. Some results are reported in Table I. The analyses of the Binder parameter give estimates with somewhat large errors. Moreover, the estimates of \( \nu \) show a significant scatter as \( L_{\text{min}} \) and \( \Delta \) are varied. Fits of \( R_\xi \) are more stable. To obtain the final estimates, we consider the combined fits, that give reasonably accurate and stable results. We finally estimate

\[ \beta_c = 12.23(6), \quad \nu = 0.77(5), \]

and \( R_\xi^* = 0.50(1), \ U^* = 1.025(1) \), where the errors take into account the dependence of the results on the range \( \Delta \), on \( L_{\text{min}} \) and on the order \( n \) of the polynomial. In Fig. 2 we show a plot of \( R_\xi \) versus \( X \), using the MC estimates of \( \beta_c \) and \( \nu \). As \( L \) increases, data collapse onto a single scaling curve.

Since both \( R_\xi \) and \( U \) satisfy Eq. (14), we must have

\[ R_\xi = F(U) + O(L^{-\omega}), \]

where \( F(U) \) is a universal function. In Fig. 3 we plot \( R_\xi \) versus \( U \). The data collapse onto a single curve without the need of tuning any parameter, confirming the scaling behavior (18). Scaling corrections are smaller for \( \beta < \beta_c \) than for \( \beta > \beta_c \).

In order to estimate the exponent \( \eta \), we analyze the FSS behavior of the susceptibility given by

\[ \chi \sim L^{2-\eta} \left[ f_\chi(X) + O(L^{-\omega}) \right]. \]

Fitting the susceptibility to Eq. (19) [we use a linear approximation for the scaling function \( f_\chi(X) \)], we obtain the estimate \( \eta = 0.07(4) \), where the error takes also into

| \( \Delta \) | \( L_{\text{min}} \) | \( n \) | \( \chi^2/\text{DOF} \) | \( \beta_c \) | \( \nu \) |
|---|---|---|---|---|---|
| 0.05 | 24 | 1 | 7.5 | 12.224(4) | 0.78(1) |
| 0.20 | 24 | 2 | 2.4 | 12.229(4) | 0.78(2) |
| 0.20 | 24 | 3 | 2.3 | 12.229(5) | 0.79(2) |
| 0.20 | 32 | 1 | 5.2 | 12.249(8) | 0.87(4) |
| 0.20 | 32 | 2 | 1.7 | 12.246(8) | 0.79(4) |
| 0.20 | 32 | 3 | 1.7 | 12.247(8) | 0.78(4) |
| 0.10 | 24 | 1 | 3.9 | 12.225(4) | 0.78(2) |
| 0.10 | 24 | 2 | 2.6 | 12.229(5) | 0.79(3) |
| 0.10 | 32 | 2 | 2.2 | 12.247(8) | 0.85(6) |
| 0.05 | 24 | 1 | 3.4 | 12.230(6) | 0.80(7) |
| 0.05 | 24 | 2 | 3.0 | 12.229(6) | 0.75(7) |
| 0.05 | 32 | 1 | 1.6 | 12.262(16) | 0.81(16) |
| 0.05 | 32 | 2 | 1.5 | 12.248(16) | 0.60(14) |
| 0.10 | 24 | 1 | 2.2 | 12.34(2) | 0.99(11) |
| 0.10 | 32 | 1 | 1.7 | 12.31(2) | 0.62(8) |
| 0.05 | 24 | 1 | 1.1 | 12.31(2) | 0.68(13) |
| 0.05 | 32 | 1 | 1.4 | 12.38(10) | 0.89(48) |
| 0.10 | 24 | 1 | 2.4 | 12.211(4) | 0.74(2) |
| 0.10 | 32 | 1 | 1.7 | 12.24(1) | 0.86(6) |
| 0.05 | 24 | 1 | 1.6 | 12.216(6) | 0.77(7) |
| 0.05 | 32 | 1 | 1.4 | 12.25(1) | 0.77(16) |
account the uncertainty on $\beta_c$ and $\nu$. The corresponding scaling plot is reported in Fig. 3.

The numerical study of the ACP$^3$ lattice model provides therefore a robust evidence that it undergoes a transition at a finite value of $\beta$. We can exclude that the transition is of first order. Indeed, at a first order transition FSS holds with $\nu = 1/d = 1/3$ \[38, 10\]. The estimate \[17\] of $\nu$ is definitely larger than $1/3$, ruling out a discontinuous transition. Therefore, the transition is continuous, contradicting the predictions of the LGW theory \[11\].

IV. CONCLUSIONS

In this paper we have critically reconsidered the LGW approach, which is used to determine the universal features of critical transitions. In this framework, one first identifies the order parameter $\Phi$, then considers the most general $\Phi^4$ theory with the same symmetries as the original model, and finally determines the stable fixed points of the RG flow. If they correspond to a bare theory with the correct symmetry-breaking pattern, they completely characterize the possibly present continuous transitions.

In the presence of gauge symmetries—the case of interest here—the method is usually applied by considering a gauge-invariant order parameter and a LGW field theory that is invariant under the global symmetries of the original model. In the effective field theory there is no remnant of the gauge invariance: the gauge degrees of freedom have been implicitly integrated out. In this work we point out that in some cases this LGW approach may lead to erroneous conclusions on the nature of the critical behavior.

As an explicit example we consider the ferromagnetic and antiferromagnetic CP$^{N-1}$ model, which presents a global U($N$) and a local U(1) gauge symmetry. The corresponding LGW theory is constructed using a hermitian traceless $N \times N$ matrix field associated with the local gauge-invariant operator $Q_{ab} = x_a x_b - \delta_{ab}/N$.

In the ferromagnetic case, for any $N \geq 3$ the LGW theory \[5\] contains a cubic term. Its presence is generally considered as an indication of first-order transitions. Therefore, one predicts that any generic CP$^{N-1}$ model should only undergo discontinuous transitions. However, this is in contradiction with analytical large-N results. In this limit the ferromagnetic CP$^{N-1}$ model is equivalent to an effective abelian Higgs model for an $N$-component complex scalar field coupled with a dynamical U(1) field \[2\]. For a sufficiently large number of components, the abelian Higgs model has a stable fixed point \[8\]. Therefore, for $N \to \infty$ CP$^{N-1}$ models may undergo a continuous transition. The LGW predictions are also contradicted by recent numerical studies of the universality class of the 3D ferromagnetic CP$^2$ model \[8, 10\].

In the antiferromagnetic case, the LGW field theory is constructed using the staggered gauge-invariant composite operator, defined as in Eq. \[6\]. It does not present cubic terms due to the antiferromagnetic nearest-neighbor coupling which gives rise to an additional global Z$_2$ symmetry. For $N = 3$, the LGW approach nicely works: it predicts a symmetry enlargement at the critical point—the leading critical behavior is O(8) invariant—which has been accurately verified numerically \[11\]. In this work we consider the model for $N = 4$. In this case the LGW predictions are in striking contradiction with the numerical results. The analyses of the RG flow presented in Ref. \[11\] using high-order perturbative series (five-loop series in the MS renormalization scheme \[33\] and six-loop series in the massive zero-momentum scheme \[3, 6, 36\]) do not find any evidence of stable fixed points. This implies that any transition should be of first order. On the other hand, the numerical results we present here provide a robust evidence of a continuous transition in the ACP$^3$ model.

The failure of the LGW approach indicates that the effective local $\Phi^4$ theory of the gauge-invariant order parameter may not always capture all the relevant modes.
at the critical point. The gauge degrees of freedom may be relevant at the transition and should therefore be included in the effective theory. This is clearly the case for the large-$N$ ferromagnetic model, which is described by the abelian Higgs model, in which the $U(1)$ gauge field plays a crucial role.

A second possibility is that some degrees of freedom decouple giving rise to continuous transitions associated with different symmetry-breaking patterns. For example this occurs in the 2D frustrated XY models \[11\], where the disordered high-temperature phase and the ordered low-temperature phase are separated by two transitions instead of one, with different critical modes and symmetry-breaking patterns at each transition (belonging to the Ising and XY universality classes, respectively).

The above considerations may be relevant for the finite-temperature transition of quantum chromodynamics (QCD). In the limit of $N_f$ massless quarks, the finite-temperature transition of QCD is related to the restoring of the chiral symmetry. The nature of the phase transition has been investigated within the LGW framework \[12\], assuming that the relevant order parameter field is an $N_f \times N_f$ complex-matrix field $\Phi_{ij}$, related to the bilinear gauge-invariant quark operators $\bar{\psi}_L \psi_{Rj}$. To define the corresponding LGW theory, one must also specify the fate of the $U(1)_A$ symmetry at the transition, something which is not clear yet. Numerical studies of lattice QCD suggest a strong suppression of $U(1)_A$ symmetry-breaking effects at $T_c \approx 1.2 - 1.3$ as predicted by the dilute instanton gas approximation \[10\]. In the LGW approach the role of the axial $U(1)$ symmetry defines the symmetry of the LGW theory and the relevant symmetry breaking pattern. If the symmetry is broken, one should consider a $\Phi^4$ model invariant under $SU(N_f)_L \otimes SU(N_f)_R$ transformations and the relevant symmetry breaking pattern is $SU(N_f)_L \otimes SU(N_f)_R \to SU(N_f)_V$; in the opposite case the symmetry is $U(N_f)_L \otimes U(N_f)_R$ and the symmetry breaking pattern is $U(N_f)_L \otimes U(N_f)_R \to U(N_f)_V$. For the particular case of two light flavors, the two different LGW theories predict two different critical behaviors, belonging to the O(4) and U(2) universality classes, respectively \[33\]. In any case, whatever the role of the axial $U(1)$ symmetry is, in the effective LGW theory the gauge symmetry does not play any role: one is essentially assuming that the gauge degrees of freedom are irrelevant at the transition. The results presented in this paper show that this assumption should not be taken for granted. It is therefore possible that the problematics consistency and interpretation of the numerical results \[18–29\] is due to the failure of the LGW framework in describing all critical modes at the transition. This point calls for further deeper investigations.

---

[1] L. D. Landau and E. M. Lifshitz, *Statistical Physics. Part I*, 3rd edition (Elsevier Butterworth-Heinemann, Oxford, 1980).
[2] K. G. Wilson and J. Kogut, The renormalization group and the $\epsilon$ expansion, Phys. Rep. 12, 77 (1974).
[3] M. E. Fisher, The renormalization group in the theory of critical behavior, Rev. Mod. Phys. 47, 543 (1975).
[4] S.-k. Ma, *Modern Theory of Critical Phenomena* (W.A. Benjamin, Reading, MA, 1976).
[5] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, fourth edition (Clarendon Press, Oxford, 2002).
[6] A. Pelissetto and E. Vicari, Critical Phenomena and Renormalization Group Theory, Phys. Rep. 368, 549 (2002).
[7] M. Moshe and J. Zinn-Justin, Quantum field theory in the large $N$ limit: A review, Phys. Rep. 385, 69 (2003).
[8] B.I. Halperin, T.C. Lubensky, and S.K. Ma, First-Order Phase Transitions in Superconductors and Smectic-A Liquid Crystals, Phys. Rev. Lett. 32, 292 (1974).
[9] A. Nahum, J. T. Chalker, P. Serna, M. Ortuno, and A. M. Somoza, 3D Loop Models and the CP$^{N-1}$ Sigma Model, Phys. Rev. Lett. 107, 110601 (2011).
[10] A. Nahum, J. T. Chalker, P. Serna, M. Ortuno, and A. M. Somoza, Phase transitions in three-dimensional loop models and the CP$^{N-1}$ sigma model, Phys. Rev. B 88, 134411 (2013).
[11] F. Delín, A. Pelissetto, and E. Vicari, Three-dimensional antiferromagnetic CP$^{N-1}$ models, Phys. Rev. E 91, 052109 (2015).
[12] R. D. Pisarski and F. Wilczek, Remarks on the chiral phase transition in chromodynamics, Phys. Rev. D 29, 338 (1984).
[13] F. Wilczek, Application of the renormalization group to a second-order QCD phase transition, Int. J. Mod. Phys. A 7, 3911 (1992).
[14] K. Rajagopal and F. Wilczek, Static and dynamic critical phenomena at a second order QCD phase transition, Nucl. Phys. B 399, 395 (1993).
[15] S. Gavin, A. Gocksch, and R. D. Pisarski, QCD and the chiral critical point, Phys. Rev. D 49, R3079 (1994).
[16] F. Wilczek, QCD In Extreme Conditions, arXiv:hep-ph/0003183.
[17] F. Karsch, Lattice QCD at High Temperature and Density, Lect. Notes Phys. 583, 209 (2002).
[18] A. Ali Khan, S. Aoki, R. Burkhalter, S. Ejiri, M. Fukugita, S. Hashimoto, N. Ishizuka, Y. Iwasaki, K. Kanaya, T. Kaneko, Y. Kuramashi, T. Manke, K. Nagai, M. Okamoto, M. Okawa, A. Ukawa, and T. Yoshie (CP-PACS Collaboration), Phase structure and critical temperature of two-flavor QCD with a renormalization group improved gauge action and clover improved Wilson quark action, Phys. Rev. D 63, 034502 (2000).
[19] C. Bernard, T. Burch, T. A. DeGrand, C. E. De Tar, S. Gottlieb, U. M. Heller, J.E. Hetrick, K. Orginos, R. L. Sugar, and D. Toussaint (MILC Collaboration), Scaling tests of the improved Kogut-Susskind quark action, Phys. Rev. D 61, 111502 (2000).
[20] F. Karsch, E. Laermann, and A. Peikert, Quark Mass and Flavour Dependence of the QCD Phase Transition, Nucl. Phys. B 605, 579 (2001).
[21] J. B. Kogut and D. K. Sinclair, Scaling behavior at the $N_f = 6$ phase transition for 2-flavor lattice QCD with massless staggered quarks, and an irrelevant 4-fermion interaction, Phys. Rev. D 64, 034508 (2001).

[22] J. Engels, S. Holtmann, T. Mendes, and T. Schulze, Finite-size-scaling functions for 3d O(4) and O(2) spin models and QCD, Phys. Lett. B 514, 299 (2001).

[23] M. D’Elia, A. Di Giacomo, and C. Pica, Two flavor QCD and confinement, Phys. Rev. D 72, 114510 (2005).

[24] J. B. Kogut and D. K. Sinclair, Evidence for O(2) universality at the finite temperature transition for lattice QCD with 2 flavors of massless staggered quarks, Phys. Rev. D 73, 074512 (2006).

[25] P. de Forcrand and O. Philipsen, The chiral critical line of $N_f = 2 + 1$ QCD at zero and non-zero baryon density, J. High Energy Phys. 01, 077 (2007).

[26] A. Bazavov, T. Bhattacharya, M. Cheng, C. DeTar, H.-T. Ding, Steven Gottlieb, R. Gupta, P. Hegde, U. M. Heller, F. Karsch, E. Laermann, L. Levkova, S. Mukherjee, P. Petreczky, C. Schmidt, R. A. Soltz, W. Soeldner, R. Sugar, D. Toussaint, W. Unger, and P. Vranas, The chiral and deconfinement aspects of the QCD transition, Phys. Rev. D 85, 054503 (2012).

[27] F. Burger, E.-M. Ilgenfritz, M. Kirchner, M. P. Lombardo, M. Müller-Preussker, O. Philipsen, C. Pinke, C. Urbach, and L. Zeidlerwich, Thermal QCD transition with two flavors of twisted mass fermions, Phys. Rev. D 87, 074508 (2013).

[28] C. Bonati, P. de Forcrand, M. D’Elia, O. Philipsen, and F. Sanfilippo, Chiral phase transition in two-flavor QCD from an imaginary chemical potential, Phys. Rev. D 96, 074030 (2014).

[29] P. de Forcrand and M. D’Elia, Continuum limit and universality of the Columbia plot, arXiv:1702.00330.

[30] A. Butti, A. Pelissetto, and E. Vicari, On the nature of the finite-temperature chiral transition in QCD, J. High Energy Phys. 08, 029 (2003).

[31] F. Basile, A. Pelissetto, and E. Vicari, The finite-temperature chiral transition in QCD with adjoint fermions, J. High Energy Phys. 02, 044 (2005).

[32] E. Vicari, Critical phenomena and renormalization-group flow of multi-parameter $\Phi^4$ field theories, PoS (LAT2007) 023; arXiv:0709.1014.

[33] A. Pelissetto and E. Vicari, Relevance of the axial anomaly at the finite-temperature chiral transition in QCD, Phys. Rev. D 88, 105018 (2013).

[34] L. A. Fernandez, V. Martín-Mayor, D. Sciretti, A. Tarancón, and J. L. Velasco, Numerical study of the enlarged O(5) symmetry of the 3D antiferromagnetic RP$^2$ spin model, Phys. Lett. B 628, 281 (2005).

[35] G. ’t Hooft and M. J. G. Veltman, Regularization and renormalization of gauge fields, Nucl. Phys. B 44, 189 (1972).

[36] G. Parisi, Cargèse Lectures (1973); Field-theoretic approach to second-order phase transitions in two- and three-dimensional systems, J. Stat. Phys. 23, 49 (1980).

[37] If we set $Z = (\text{Re}z, \text{Im}z)$, the update consists in proposing a new vector $RZ$, where $R$ is an O(2) rotation matrix acting on two randomly chosen components of $Z$. The rotation angle is chosen randomly in an interval $[-\theta_{\text{max}}, \theta_{\text{max}}]$, where $\theta_{\text{max}}$ is fixed to have an acceptance probability of approximately 30%.

[38] B. Nienhuis and M. Nauenberg, First-Order Phase Transitions in Renormalization-Group Theory, Phys. Rev. Lett. 35, 477 (1975).

[39] M. E. Fisher and A. N. Berker, Scaling for first-order phase transitions in thermodynamic and finite systems, Phys. Rev. B 26, 2507 (1982).

[40] V. Privman and M. E. Fisher, Finite-size effects at first-order transitions, J. Stat. Phys. 33, 385 (1983).

[41] M. Hasenbusch, A. Pelissetto, and E. Vicari, Multicritical behavior in the fully frustrated XY model and related systems, J. Stat. Mech. (2005) P12002; Transitions and crossover phenomena in fully frustrated XY systems, Phys. Rev. B 72, 184502 (2005); P. Minnhagen, B. J. Kim, S. Bernhardsson, and G. Cristofano, Symmetry-allowed phase transitions realized by the two-dimensional fully frustrated XY class, Phys. Rev. B 78, 184432 (2008).

[42] M. I. Buchoff, M. Cheng, N. H. Christ, H.-T. Ding, C. Jung, F. Karsch, R. D. Mawhinney, S. Mukherjee, P. Petreczky, D. Renfrew, C. Schroeder, P. M. Vranas, H. Yin, and Z. Lin, The QCD chiral transition, U(1)$_A$ symmetry and the Dirac spectrum using domain wall fermions, Phys. Rev. D 89, 054514 (2014).

[43] G. Cossu, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko, H. Matsufuru, and J.-I. Noaki, Finite temperature study of the axial U(1) symmetry on the lattice with overlap fermion formulation, Phys. Rev. D 87, 114514 (2013).

[44] S. Aoki, H. Fukaya, and Y. Taniguchi, Chiral symmetry restoration, eigenvalue density of Dirac operator and axial U(1) anomaly at finite temperature, Phys. Rev. D 86, 114512 (2012).

[45] C. Bonati, M. D’Elia, H. Panagopoulos, and E. Vicari, Change of the $\theta$ dependence of 4D SU(N) gauge theories across the deconfinement transition, Phys. Rev. Lett. 110, 252003 (2013).

[46] J. D. Gross, R. D. Pisarski, and L. G. Yaffe, QCD and instantons at finite temperature, Rev. Mod. Phys. 53, 43 (1981).