Measuremant, Trace, Information Erasure and Entropy

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We show that both information erasure process and trace process can be realized by projective measurement. And a partial trace operation consists to a projective measurement on a subsystem. We show that a nonunitary operation will destroy the wave-behavior of a particle. We give a quantum manifestation of Maxwell’s demon and give a quantum manifestation of the second law of therodynamics. We show that, considering the law of momentum-energy conversation, the evolution of a closed system should be unitary and the von Neumann entropy of the closed quantum system should be least.

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I. INTRODUCTION

When we attempt to apply classical mechanics and electrodynamics to explain atomic phenomena, they lead to results which are obviously conflict with experiment. The mechanics which governs microworld is quantum mechanics. It is the most accurate and complete description of the world known. Thus quantum mechanics occupies a very unusual place among physical theories: it contains classical mechanics as a limiting case, at the same time, it requires this limiting case for its own formulation.

Quantum mechanics tells us the evolution of a closed quantum system is described by a unitary transformation. To observe a closed quantum system to find out what is going on inside the system, one needs to interact the system with an external physical system, which makes the system no longer closed, and the unitary evolution is not necessary. Structurally, quantum mechanics has two parts, one part concerned with quantum states, the other with quantum dynamics. In this paper, we will show some properties about quantum evolution.

To find out what is going on inside a quantum system, one must perform a quantum measurement. And a partial trace operation can give the correct description of observable quantities for subsystems of a composite system. The relation between physical entropy and information may have been mentioned first by Szilard [1]. Landauer’s principle states that in erasing one bit of information, on average, at least $k_B T \ln(2)$ energy is dissipated into the environment (where $k_B$ is Boltzmann’s constant and $T$ is the temperature of the environment at which one erases) [2]. Piechocinska derived Landauer’s principle from microscopic considerations [3].

In this paper, we first give a definition of the wave behavior of a quantum system. Suppose a quantum system is in state

$$|\varphi> = \sum_i a_i |i>,$$

where $\sum_i |a_i|^2 = 1$, $\{|i>\}$ is a set of orthogonal vectors. If $i \geq 2$, we will say that this quantum system has wave behavior. Else, there is only particle behavior in this quantum system. The quantum operations formalism is a general tool for describing the evolution process of quantum systems, such process include unitary evolution, quantum measurement, and even more general processes [4]. A general quantum dynamical process is described by a quantum operation. We will use quantum operations formalism to described the quantum evolution processes in this paper. We use density operators formalism $\rho$ or Dirac’s bracket notation $|\psi>$ to described the state of quantum system. In this paper, we will use a word: momentum-energy conservation. The momentum-energy conservation is a word for short. In microword, the conversation quantities would include momentum, energy, spin and other quantity. All these quantities we call them momentum-energy conservation in this paper. The entropy of a classical system is defaulted as Shannon entropy. And the entropy of a quantum system is considered as the von Neumann entropy default in this paper.

A. Trace

The trace of a matrix $A$ is defined to be the sum of its diagonal elements,
The trace is cyclic and linear. Suppose we have physical systems A and B, whose state is described by a density operator $\rho^{AB}$. The reduced density operator for system A is defined by

$$\rho^A \equiv \text{tr}_B(\rho^{AB}),$$

where $\text{tr}_B$ is a map of operators known as the partial trace over system B. The partial trace is defined by

$$\text{tr}_B(|a_1><a_1| \otimes |b_1><b_1|) \equiv |a_1><a_1|\text{tr}(|b_1><b_1|).$$

The partial trace operator is the operation which gives description of observable quantities for subsystems of a composite system.

### B. Measurement

Distinguishing quantum states needs quantum measurement. Quantum measurements are described by a collection $\{M_m\}$ of measurement operators [5]. These are operators acting on the state space of the system being measured. The index $m$ refers to the measurement outcomes that may occur in the experiment,

$$p(m) = <\psi|M_m^\dagger M_m|\psi >,$$

where $|\psi >$ is the state of the quantum system immediately before the measurement. The state of the system after the measurement is

$$\frac{M_m|\psi >}{\sqrt{<\psi|M_m^\dagger M_m|\psi >}}$$

And the completeness equation expresses the fact that probabilities sum to one:

$$1 = \sum_m p(m) = \sum_m <\psi|M_m^\dagger M_m|\psi >.$$

The quantum measurements postulate gives a rule describing the measurement statistics and a rule describing the postmeasurement state of the system. It is familiar that quantum mechanics describe projective measurements, ‘Positive Operator-Valued Measure’ measurements (POVMs), and a general measurements. A projective measurement is described by a Hermitian operator, $M$, on the state space of the system being observed

$$M = \sum_m mP_m,$$

where $P_m$ is the projector onto the eigenspace of $M$ with eigenvalue $m$. The possible outcomes of measurement correspond to the eigenvalues, $m$, of the observable. Upon measuring the state $|\psi >$, the probability of getting result $m$ is given by

$$p(m) = <\psi|P_m|\psi >.$$  

### C. Information Erasure

The bit is a fundamental unit of information. We will assume to have a large number of bits but they will be erased individually, one by one. Landauer argues that since information erasure is a logical function which does not have a single-valued inverse, it must be associated with physical irreversibility and require heat dissipation. Suppose we have a quantum system which is in the unknown state $\rho$. A information erasure process is that we prepare this system in a standard state,

$$\epsilon : \rho \rightarrow |0><0|,$$

where $\rho$ is an arbitrary state and $|0><0|$ is a standard state. This information erasure process is nonunitary generally. If the system is in a known state, this information erasure process can be realized by a unitary operation. But in the case we have a large number of bits to erase, the information erasure operation must be irreversible. In this paper, information erasure will be considered as the quantum state erasure of a quantum system.
II. RELATION BETWEEN MEASUREMENT, TRACE, AND ERASURE

Let $H_A$ be any Hilbert space, spanned by an orthonormal basis $|1\rangle \ldots |d\rangle$. Then the trace map $\rho \rightarrow \text{tr}(\rho)$ can be represented as

$$\text{tr}(\rho) = \sum_{i=1}^{d} <i|\rho|i>.$$  \hfill (11)

If $H_A$ is both the input and output space, a quantum measurement can be described:

$$M_m(\rho) = \sum_{i=1}^{d} |m><i|\rho|><i|M_m^\dagger,$$  \hfill (12)

where $M_m$ is a quantum measurement operator.

Let $H_Q$ be any input Hilbert space, spanned by an orthonormal basis $|1\rangle \ldots |d\rangle$, and let $H_Q'$ be a one dimensional output space, spanned by the state $|0\rangle$. Then, information erasure can be described like this

$$R(\rho) = \sum_{i=1}^{d} |0><i|\rho|><i|0,$$  \hfill (13)

where $R$ is an erasure operator. No matter what the state $\rho$ is, the state of output space is $|0><0|$. A trace operation can be realized by a projective measurement when we never learn the result of the measurement. Suppose the a quantum in the state $|\psi\rangle$, the state after the measurement is given by

$$\rho' = \sum_i M_i|\psi><\psi|M_i^\dagger.$$  \hfill (14)

Relation between trace operation and the measurement can be described like this,

$$1 = \text{tr}(|\psi><\psi|) = \sum_i <\psi|M_i^\dagger M_i|\psi>.$$  \hfill (15)

Suppose we have a joint system $AQ$ in the state

$$\rho^{AQ} = \sum_{i,j} p_{ij}\rho_i^A \otimes |j_Q><j_Q|,$$  \hfill (16)

and wish to trace out the subsystem $Q$. Then the state of the subsystem $A$ becomes

$$\rho^A = \text{tr}_Q(\rho^{AQ}) = \sum_{j'} <j_Q|\rho^{AQ}|j_Q> = \sum_{i,j,j'} p_{ij} \rho_i^A \delta_{j,j'}.$$  \hfill (17)

Suppose that a projective measurement described by $P_j^Q$ is performed on the quantum system $Q$, but we never learn the result of the measurement. The state of the system $A$ after a projective measurement on the system $Q$ is given by

$$\rho^A = \sum_j P_j^Q \rho^{AQ} P_j^Q = \sum_{i,j,j'} p_{ij} \rho_i^A \delta_{j,j'}.$$  \hfill (18)

We can see that a project measurement on a subsystem is the same as a partial trace. To see it clearly, let us consider a GHZ state [6]

$$|\text{GHZ} > = \frac{1}{\sqrt{2}} (|0_A0_B0_C> + |1_A1_B1_C>)$$  \hfill (19)

of a three qubit system $ABC$. Then trace one qubit (suppose of system $A$) out of the three qubit system. It has

$$\text{tr}_A(|\text{GHZ}><\text{GHZ}|) = \frac{1}{2} \rho_{00}^{BC} + \frac{1}{2} \rho_{11}^{BC},$$  \hfill (20)
where $\rho_{00}^{BC} = |0_B0_C \rangle \langle 0_B0_C|$, $\rho_{11}^{BC} = |1_B1_C \rangle \langle 1_B1_C|$. From Eq.(20), we know that after a partial trace on system $A$, the system of $BC$ has a probability $p = \frac{1}{2}$ in state $\rho_{00}^{BC}$ or in the state $\rho_{11}^{BC}$. Obviously, the quantum measurement postulate tells us that we perform a project measurement in basis $\{|0>, |1>\}$ on the system $A$, the measurement result is in state $|0> \ (\text{consists to } \rho_{00}^{BC})$ or in state $|1> \ (\text{consists to } \rho_{11}^{BC})$ with a probability $p = \frac{1}{2}$. Consider the case the three qubit in the state $|w> [7]$

$$|w> = \frac{1}{\sqrt{3}}(|1_A0_B0_C > + |0_A1_B0_C > + |0_A0_B1_C >).$$

(21)

Let us suppose we trace out the qubit of system $A$, then we have

$$tr_A(|w><w|) = \frac{1}{3} \rho_{00}^{BC} + \frac{2}{3} |\psi_{BC}^+> <\psi_{BC}^+|,$$

(22)

where $|\psi_{BC}^+> = \frac{1}{\sqrt{2}}(|0_B1_C > + |1_B0_C >)$ is a Bell state. Clearly, after a project measurement on system $A$ in basis $\{|0>, |1>\}$, the system $A$ has a probability $p = \frac{1}{2}$ in the state $|1> \ (\text{consists to } \rho_{11}^{BC})$ and a probability $p = \frac{1}{2}$ in the state $|0> \ (\text{consists to } |\psi_{BC}^+>)$.

Information erasure operation can be realized by a quantum projective measurement and a unitary operation. First, we perform a projective measurement. Then the quantum system will be in a known state (suppose in the state $|j><j|$) after the quantum measurement. We can perform a unitary operation on the state $|j><j|$ to prepare the system in state $|0><0|$. Suppose we want to erase the state information of quantum system $A$, which is in an known state $\rho$. First, we can perform a projective measurement described by projectors $P_j$. When we gain the measurement result (suppose in state $|j>$), then we can perform a unitary operation $U$ to prepare the system in a state $|0>$,

$$M : \rho \rightarrow |j><j|,$$

$$U : |j><j| \rightarrow |0><0|.$$

This is a information erasure process.

III. WAVE-PARTICLE DUALITY: UNITARY AND NONUNITARY OPERATIONS

The evolution of the state of a closed quantum system is described by the Schrödinger's equation. A general quantum dynamical process is described by a quantum operation. Complementarity principle tells us the microscopic word has the behavior, wave-particle duality. One can not draw pictures of individual quantum processes [8]. To gain information of a quantum system, one has to perform a quantum measurement. A unitary operation on a quantum system will keep the wave behavior of this system. But, a nonunitary operation, will destroy the wave behavior of the system.

A. Nonunitary operations and particle behavior

In the quantum case, the completeness relation requires that trace of $\rho$ equal to one, $tr(\rho) = 1$. We can see it consists to the quantum measurement, Eq.(7). A trace process can be treated as a notion that we can find out the particle in the whole space to a certain. After a trace process on a quantum system, the wave behavior disappeared completely. If the quantum system is trace-preserving ( $tr(\rho) = 1$ ), which means that the trace of this quantum system is unit, the probability of find out the particle is 1.

Consider the case of quantum measurement. When we perform a projective measurement on a quantum system, the result of the measurement is in a state with the probability give by Eq.(5). Since the state is in an orthogonal state after a projective measurement, according to the definition of Eq. (1), the behavior of this system is particle. The wave behavior disappeared. But, POVMs maybe show some wave behavior. Suppose a particle is a two dimensions $\{|0>,|1>\}$ quantum system. Consider a POVM containing three elements [5],

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}}|1><1|,$$

$$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(0>-|1>)(<0|<-1|)}{2}$$

(26)
\[ E_3 = I - E_1 - E_2 \] (27)

We can see that \( E_1, E_2, \) and \( E_3 \) are not orthogonal to each other. Then after a POVM, the quantum system will keep some wave behavior and appear some particle behavior.

A information erasure operation will destroy the wave behavior completely. After a information erasure operation, the system is in the state \(|0><0|\). There is no wave behavior in this quantum system. Since the information erasure operation is physical irreversible, the wave behavior of the quantum system can not be recurred.

### B. Quantum operation and von Neumann entropy

Entropy is a key concept of quantum information theory. It measures how much uncertainty there is in the state of a physical system. Von Neumann defined the entropy of a quantum state by the formula [9]

\[ S(\rho) \equiv -\text{tr}(\rho \log \rho). \] (28)

If \( \lambda_x \) are the eigenvalues of \( \rho \) then von Neumann’s definition can be re-expressed

\[ S(\rho) = -\sum_x \lambda_x \log \lambda_x, \] (29)

where \( 0 \log 0 \equiv 0 \). The entropy of a system measures the amount of uncertainty about the system before we learn its value. It is a measure of the “amount of chaos” or of the lack of information about a system. If one has complete information, i.e., if one is concerned with a pure state, \( S(\rho) = 0 \) [10].

How does the entropy of a quantum system behave when we perform an operation on that system? Obviously, a unitary operation does not change the entropy of a system because a unitary transformation does not change the eigenvalues of \( \rho \). In general, a nonunitary transformation would change the eigenvalues of \( \rho \). So a nonunitary operation would change the von Neumann entropy of a quantum system.

Suppose \( P_i \) is a complete set of orthogonal projectors and \( \rho \) is a density operator. Then the entropy of the state \( \rho' = \sum_i P_i \rho P_i \) of the system after the measurement is at least as great as the original entropy,

\[ S(\rho') \geq S(\rho), \] (30)

with equality if and only if \( \rho = \rho' \) [5]. From Eq.(9), we know the measurement outcomes \( P_m \) is gained randomly with the probability \( m \). So the uncertainty of the system increases after under the projective measurement if we never learn the result of the measurement. The entropy of the system would increase under a projective measurement. But how does the entropy behave depends on the type of measurement which we perform. A projective measurement increases entropy of a quantum system. But, a generalized measurements can decrease entropy of a quantum system.

Consider a trace operation. The information of a quantum system disappeared completely after a trace operation. So the uncertainty of the quantum system would increase. The uncertainty of the system would increase. In another view, a trace process can be realized by a projective measurement if we do not know the measurement result. So a trace operation would increase the entropy of the system.

Information erasure process will induce the entropy of the environment increase. Suppose a quantum system is in a state \( \rho \). The von Neumann entropy of the system is

\[ S(\rho) \geq 0, \] (31)

where \( S(\rho) = 0 \) if and only if \( \rho \) is a pure state. After the information erasure operation, the system is in the known state \(|0><0|\). The entropy of the system is zero. The entropy of the system is nonincreasing. Since entropy does not decrease, the entropy of the environment must increase. To realize the information erasure process, there must be interaction between system and environment. A information erasure process has to exchange momentum-energy with environment randomly. In another view, a information erasure process can be realized by a projective measurement and a unitary operation. A unitary operation does not change the entropy of the system. The change of entropy is derived by the measurement. A measurement will induce exchange of momentum-energy between system and environment. Here, the projective measurement will decrease the entropy of a system because we have to know the result of the measurement if we want to realize the information erasure operation.
IV. QUANTUM MEASUREMENT AND THE SECOND LAW OF THERMODYNAMICS

The second law of thermodynamics states that the entropy in a closed system can never decrease. In 1871, J. C. Maxwell proposed the existence of a machine that apparently violated this law. Suppose we have a vessel in which full of air at unitary temperature are moving with velocities by no means unitary. And such a vessel is divided into two portions, A and B, by a division in which there is a small hole. Let suppose that a demon, who can see the individual molecules, opens and closes this hole, so as to allow only the swifter molecules to pass from A to B, and only the slow from B to A. He will thus, without expenditure of the work, raise the temperature of B and lower that of A, in contraction to the second law of thermodynamics [11].

The resolution to the Maxwell’s demon paradox lies in the fact that the demon must perform measurement on the molecules moving between the partitions, in order to determine their velocities. The result of this measurement must be stored in the demon’s memory. Because any physical memory is finite, the demon must erase information from its memory, in order to have space for new measurement results. By Landauer’s principle, erasing information increases the total entropy of the combined system which includes demon, gas vessel, and their environments. The second law of thermodynamics is obeyed.

We give a quantum manifestation of Maxwell’s demon. The demon and molecules in the vessel are all considered as quantum systems. To determine the velocities of molecules, the demon has to perform a quantum projective measurement. The average velocities of the molecules is

$$\bar{M} = \sum_m m P(m) = \langle \psi | M | \psi \rangle, \quad (32)$$

where $M$ is a velocity operator. The standard deviation is

$$\Delta(M) = \langle (M - \langle M \rangle)^2 \rangle = \langle M^2 \rangle - \langle M \rangle^2. \quad (33)$$

It is a measure of the typical spread of the observed values upon measurement of $M$. In the vessel, the spread of the velocity of molecules is a Boltzmann’s spread. So the standard deviation is nonzero. Every demon’s measurement on the molecules will exchange some of momentum-energy between demon and molecules. We can calculate the least value of the momentum exchange under every measurement. Suppose the volume of the vessel is $L^3$. From the Heisenberg uncertainty principle, we know that

$$\Delta x \Delta p \sim \hbar. \quad (34)$$

The uncertainty of momentum is at least $\hbar/L$ in every measurement in one axes. Every time, when demon want to determine the velocity of molecules, there is some uncertainty momentum, at least $\hbar/L$, exchanged between demon and molecules. The system of vessel is not a closed. So the entropy of the vessel can decrease. Considering the whole system including molecules, demon, and environment, the entropy of combined system does not decrease.

Let us see a laser cooling of free atoms case. The general method of cooling is to reduce the kinetic energy of an atom after it was loaded into the trap. Assume we have an unbound gas of atoms. After a laser cooling process, the atoms can be in a Bose-Einstein condensation state [12]. The entropy of the atoms decreases all long in this laser cooling process. But the entropy of the whole system including atoms and photon field increases. This laser cooling process is very like the maxwell’s demon paradox.

Consider a closed system including a large number of molecules. The average velocity of molecules is $\bar{v}$. Every molecule has a nonzero velocity in this system. Every molecule collides with others randomly all the time. There is momentum-energy exchange under every collision. The collision is a irreversible process in natural state. In a closed system, a collision process is nonreversible. Since every collision is random, the chaos of the system would increase after a long time. The entropy of this closed system increases.

V. THE PRINCIPLE OF LEAST ACTION IN MICROWORD

There is only one physical process in reality. It is well known there is the principle of least action in the classical word. It corresponds to the Fermat’s principle in optics. The principle of least action tells us that every one in classical word will has a certain position and momentum forever. There is only one path is realizable in every possible path.
Quantum evolution of a closed system is a unitary evolution which can be described by Schrödinger’s equation. A state of a closed system is determined. As we have known, microworld is a linear word. The linearity of the microword can be considered as the requirement of momentum-energy conservation and quantization. Suppose we have a two-level atom in the state

$$|\Phi> = \alpha |0> + \beta |1>,$$  \hspace{1cm} (35)

where $|0>$ is the ground state and $|1>$ is the excited state, $|\alpha|^2 + |\beta|^2 = 1$. The energy level of the ground state is $E_0$, and the excited state is $E_1$. Suppose there is a photon with energy $\Delta E = \frac{1}{2}(E_0 - E_1)$ which was absorbed into the system. If the system is in the state $|0>$ at first, then it should be in the state

$$|\Phi> = \frac{1}{\sqrt{2}}(|0> + |1>)$$  \hspace{1cm} (36)

after it absorbed the photon. But a classical probability view is that this system is either in state $|0>$ or in state $|1>$ with certain. It is easy to see that a classical probability view will induce the energy not conservation. If the system is in state $|1>$ after absorbed a photon with energy $\Delta E$, there is another $\Delta E$ required of this combined system. Then the energy of this combined system is not conservation any longer. If the system is still in the state $|0>$ after it absorbed this photon, there is some energy $\Delta E$ unwanted. The energy of the combined system is not conservation.

Another example is the double-slit interference. Reasonably, we can suppose the transverse momentum of the particle is zero. The state of particles exiting an interferometer is

$$|\Psi> = \frac{1}{\sqrt{2}}[|\psi_1(r)> + |\psi_2(r)>],$$  \hspace{1cm} (37)

where $|\psi_1(r)>$ and $|\psi_2(r)>$ represent the possibility for the particles to take path 1 or 2. If the particle only go through path 1 or path 2 every time, the transverse momentum is not conservation yet.

Comparing to the principle of least action in classical, we can see that entropy is least in the microword. The von Neumann entropy is least of a closed system, which give a limitation that the evolution of a closed system is unitary and the a microword is linear. In this case, we can say that the least of von Neumann entropy is an elementary requirement in the microword comparing to the principle of least action in classical word if and only if the momentum-conservation of energy and the quantization is needed in the microword. There is only one physics in reality. In classical word, it can be manifestation by the principle of least action. In the microword, it can be described as that the von Neumann entropy of a closed system is least.

VI. CONCLUSION AND SUGGESTIONS FOR FUTURE RESEARCH

We have presented that both a trace operation and a information erasure process can be realized by a projective measurement in physics. A partial trace can be realized by a projective measurement on a subsystem if we never know the result of the measurement. We have discussed the relation between quantum operations and the wave-particle duality behavior in microword. We have given a quantum manifestation of Maxwell’s demon. We have shown that the quantum measurement would increase the entropy of the environment. We have given a manifestation of the principle of least action in microword. And we have presented that the quantization and the conservation of momentum-energy requires the evolution of a closed system is a unitary and the entropy of this system is least.

The information erasure process is very important in quantum computation. The memory of the quantum computer has to be erased before a new job started since deleting an unknown state is impossible [13]. The relation between information erasure and heat dissipation in a quantum system should be studied deeply in the future. And the quantum manifestation of the Maxwell’s demon maybe be practical to calculate the relation between decrease of entropy of the vessel and the exchange of energy between demon and molecules.

VII. ACKNOWLEDGMENTS

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VIII. REFERENCES

[1] L. Szilard, Z. Phys. 53, 840 (1929).
[2] R. Landauer, IBM J. Res. Dev. 3, 183 (1961).
[3] B. Piechocinska, Phys. Rev. A 61, 062314
[4] M. A. Nielsen, C. M. Dawson, J. L. Dodd, A. Gilchrist, D. Mortimer, T. J. Osborne, M. J. Bremner, A. W. Harrow, and A. Hines, Phys. Rev. A 67, 052310 (2003)
[5] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, UK, 2000).
[6] D. M. Green, M. Horne, and A. Zeilinger, in Bell’s Theorem, Quantum Theory, and Conceptions of the Universe, edited by M. Kafatos (Kluwer, Dordrecht, 1989); D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. 58, 1131 (1990); N. D. Mermin, Phys. Today 43 (6), 9 (1990).
[7] A. Zeilinger, M. A. Horne, and D. M. Greenberger, NASA Conf. Publ. No. 3135 (National Aeronautics and Space Administration, Code NTT, Washington, DC, 1997).
[8] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. 74, 145 (2002).
[9] von Neumann, Göt. Nachr. 273; A. Einstein, 1914, Verh. Dtsch. Phys. Ges. 12, 820.
[10] A. Wehrl, Rev. Mod. Phys. 50 (2), 221 (1978).
[11] J. C. Maxwell, Theory of Heat. Longmans, Green, and Co., London, 1871; L. Szilard. Z. Phys., 53, 840 (1929); C. H. Bennet, Int. J. Theor. Phys., 21, 905 (1982); C. H. Bennett, Sci. Am., 295(5), 108 (1987).
[12] D. J. Wineland and Wayne M. Itano, Phys. Rev. A 20, 1521 (1979); Anthony J. Leggett, Rev Mod. Phys., 73, 307 (2001).
[13] Arun Kumar Pati and Samuel L. Braunstein, 404, 164 (2000).