INVERSE THEOREMS IN THE THEORY OF APPROXIMATION OF VECTORS IN A BANACH SPACE WITH EXPONENTIAL TYPE ENTIRE VECTORS

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Abstract. Arbitrary operator \( A \) on a Banach space \( X \) which is the generator of \( C_0 \)-group with certain growth condition at infinity is considered. The relationship between its exponential type entire vectors and its spectral subspaces is found. Inverse theorems on connection between the degree of smoothness of vector \( x \in X \) with respect to operator \( A \), the rate of convergence to zero of the best approximation of \( x \) by exponential type entire vectors for operator \( A \), and the \( k \)-module of continuity are established. Also, a generalization of the Bernstein-type inequality is obtained. The results allow to obtain Bernstein-type inequalities in weighted \( L_p \) spaces.

1. Introduction

Direct and inverse theorems which establish the relationship between the degree of smoothness of a function with respect to a differentiation operator and the rate of convergence to zero of its best approximation by trigonometric polynomials are well known in the theory of approximation of periodic functions. Bernstein’s and Jackson’s inequalities are ones among such results.

N. P. Kuptsov proposed a generalized notion of the module of continuity, expanded onto \( C_0 \)-groups in a Banach space \([1]\). Using this notion, A. P. Terekhin \([2]\) proved the generalized Bernstein-type inequalities for the cases of bounded group and \( s \)-regular group. Remind that group \( \{U(t)\}_{t \in \mathbb{R}} \) is called \( s \)-regular if resolvent of its generator \( A \) satisfies condition \( \exists \theta \in \mathbb{R} : \|R_A(e^{\theta A})\| \leq \frac{C}{\Im \lambda} \).

G. V. Radzievsky studied direct and inverse theorems \([3, 4]\), using notion of \( K \)-functional instead of module of continuity, but it should be noted that \( K \)-functional has two-sided estimates with regard to the module of continuity at least for bounded \( C_0 \)-groups.

In the papers \([5, 6]\) and \([7]\) authors investigated the case of a group of unitary operators in Hilbert space and established Bernstein-type and Jackson-type inequalities in Hilbert spaces and their rigs. These inequalities are used to estimate the rate of convergence to zero of the best approximation of both finite and infinite smoothness vectors for operator \( A \) by exponential type entire vectors.

We consider the \( C_0 \)-groups in the Banach space, generated by the so-called non-quasianalytic operators \([8]\), i.e. the groups satisfying

\[
\int_{-\infty}^{\infty} \frac{\ln \|U(t)\|}{1 + t^2} dt < \infty.
\]
We recall that the belonging of group to the \( C_0 \) class means that for every \( x \in \mathfrak{X} \) vector-function \( U(t)x \) is continuous on \( \mathbb{R} \) with respect to the norm of the space \( \mathfrak{X} \).

As it was shown in [5], the set of exponential type entire vectors for the non-quasianalytic operator \( A \) is dense in \( \mathfrak{X} \), so the problem of approximation by exponential type entire vectors is correct. On the other hand, it was shown in [9] that condition (1.1) is close to the necessary one, so in the case when (1.1) doesn’t hold, the class of entire vectors isn’t necessary dense in \( \mathfrak{X} \), and the corresponding approximation problem loses its meaning.

In [10] the generalized Jackson-type inequalities for approximation by entire vectors of exponential type of non-quasianalytic operators are established. The purpose of this work is to obtain Bernstein-type inequalities and the analogue of inverse theorem for such approximations, and to give some applications of these results to weighted \( L_p \) spaces.

In order to do this, it is proved that the set of exponential type entire vectors of type, not exceeding some \( \sigma > 0 \), coincides with some spectral subspace of non-quasianalytic operator (constructed in [5]), and the well-developed technique for spectral subspaces is used. The last result (coincidence of the two sets of vectors) improves the embedding, established in [5].

2. Preliminaries

Let \( A \) be a closed linear operator with dense domain of definition \( \mathcal{D}(A) \) in Banach space \( (\mathfrak{X}, \| \cdot \|) \) over the field of complex numbers.

Let \( C^\infty(A) \) denotes the set of all infinitely differentiable vectors of operator \( A \), i.e.

\[
C^\infty(A) = \bigcap_{n \in \mathbb{N}_0} \mathcal{D}(A^n), \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.
\]

For a number \( \alpha > 0 \) we set

\[
\mathcal{E}^\alpha(A) = \{ x \in C^\infty(A) \mid \exists c = c(x) > 0 \forall k \in \mathbb{N}_0 \| A^k x \| \leq c \alpha^k \}.
\]

The set \( \mathcal{E}^\alpha(A) \) is a Banach space with respect to the norm

\[
\| x \|_{\mathcal{E}^\alpha(A)} = \sup_{n \in \mathbb{N}_0} \frac{\| A^n x \|}{\alpha^n}.
\]

Then \( \mathcal{E}(A) = \bigcup_{\alpha > 0} \mathcal{E}^\alpha(A) \) is a linear locally convex space with respect to the topology of inductive limit of the Banach spaces \( \mathcal{E}^\alpha(A) \):

\[
\mathcal{E}(A) = \lim_{\alpha \to \infty} \mathcal{E}^\alpha(A).
\]

Elements of the space \( \mathcal{E}(A) \) are called \( 11 \) exponential type entire vectors of the operator \( A \). The type \( \sigma(x, A) \) of vector \( x \in \mathcal{E}(A) \) is defined as the number

\[
\sigma(x, A) = \inf \{ \alpha > 0 : x \in \mathcal{E}^\alpha(A) \} = \lim_{n \to \infty} \| A^n x \|^{\frac{1}{n}}.
\]

Denote by \( \Xi^\alpha(A) \) the following set

\[
(2.1) \quad \Xi^\alpha(A) = \{ x \in \mathcal{E}(A) \mid \sigma(x) \leq \alpha \}.
\]

It is easy to see that

\[
(2.2) \quad \mathcal{E}^\alpha(A) \subset \Xi^\alpha(A) = \bigcap_{\epsilon > 0} \Xi^{\alpha + \epsilon}(A).
\]

Example 1. Let \( \mathfrak{X} \) is one of \( L_p(2\pi) \) \((1 \leq p < \infty)\) – spaces of integrable in \( p \)-th degree over \([0, 2\pi] \), \( 2\pi \)-periodical functions or the space \( C(2\pi) \) of continuous \( 2\pi \)-periodical functions (the norm in \( \mathfrak{X} \) is defined in a standard way), and let \( A \) is the differentiation operator in the space \( \mathfrak{X} \) \((\mathcal{D}(A) = \{ x \in \mathfrak{X} \cap AC(\mathbb{R}) : x' \in \mathfrak{X} \} ; (Ax)(t) = \frac{d}{dt}x(t)\), where \( AC(\mathbb{R}) \) denotes the space of absolutely continuous functions over \( \mathbb{R} \)). It can be proved that in such case the space \( \mathcal{E}(A) \) coincides with the space of all trigonometric polynomials, and for
\( y \in \mathcal{E}(A) \), \( \sigma(y, A) = \deg(y) \), where \( \deg(y) \) is the degree of the trigonometric polynomial \( y \).

Note that all previous definitions do not change if we replace the operator \( A \) by any operator of the form \( e^{i\vartheta}A \), \( \vartheta \in \mathbb{R} \). Moreover, main results of this article — the theorems do not depend on which operator generates the group \( U(t) \) — either \( A \) or \( iA \). So, in what follows, we always assume that the operator \( iA \) is the generator of group of linear continuous operators \( \{U(t) : t \in \mathbb{R}\} \) of class \( C_0 \) on \( X \). Moreover, we suppose that the operator \( A \) is non-quasianalytic.

For \( t \in \mathbb{R}_+ \), we set
\[
M_U(t) := \sup_{\tau \in \mathbb{R}, |\tau| \leq t} \|U(\tau)\|.
\]

The estimation \( \|U(t)\| \leq Me^{\omega t} \) for some \( M, \omega \in \mathbb{R} \) implies \( M_U(t) < \infty \) (for all \( t \in \mathbb{R}_+ \)). It is easy to see that the function \( M_U(\cdot) \) has the following properties:

1. \( M_U(t) \geq 1, t \in \mathbb{R}_+; \)
2. \( M_U(\cdot) \) is monotonically non-decreasing on \( \mathbb{R}_+; \)
3. \( M_U(t_1 + t_2) \leq M_U(t_1)M_U(t_2), t_1, t_2 \in \mathbb{R}_+. \)

According to [1], for \( x \in \mathcal{X}, t \in \mathbb{R}_+ \) and \( k \in \mathbb{N} \) we set as a generalization of module of smoothness,
\[
\omega_k(t, x, A) = \sup_{0 \leq \tau \leq t} \|\Delta^k \tau x\|, \quad \text{where}
\]
\[
\Delta^k_h = (U(h) - I)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} U(jh), \quad k \in \mathbb{N}_0, \, h \in \mathbb{R} \quad (\Delta^0_0 \equiv 1).
\]

For arbitrary \( x \in \mathcal{X} \) we set, according to [6, 7],
\[
\mathcal{E}_r(x, A) = \inf_{y \in \mathcal{E}(A)} \|x - y\|, \quad r > 0,
\]
i.e. \( \mathcal{E}_r(x, A) \) is the best approximation of element \( x \) by exponential type entire vectors \( y \) of operator \( A \) for which \( \sigma(y, A) \leq r \). For fixed \( x \) \( \mathcal{E}_r(x, A) \) does not increase and \( \mathcal{E}_r(x, A) \to 0, \, r \to \infty \) for every \( x \in \mathcal{X} \) if and only if the set \( \mathcal{E}(A) \) of exponential type entire vectors is dense in \( \mathcal{X} \). Particularly, as indicated above, the set \( \mathcal{E}(A) \) is dense in \( \mathcal{X} \) if operator \( A \) generates the \( C_0 \)-group \( \{U(t) : t \in \mathbb{R}\} \) and this group belongs to non-quasianalytic class (that is, it satisfies (\text{\cite{[12]}})).

3. Spectral subspaces of non-quasianalytic operators

The main instrument for proving generalized Bernstein inequality is the theory of spectral subspaces of non-quasianalytic operator \( A \), constructed in \text{\cite{[5-7]}}. Recall that spectral subspaces (denoted by \( \mathcal{L}(\Delta) \)) are defined for all segments \( \Delta \subset \mathbb{R} \) and are characterized by the following properties \text{\cite{[8]}} p.446):

1. The operator \( A \) is defined on whole \( \mathcal{L}(\Delta) \) and is bounded on it;
2. \( \mathcal{L}(\Delta) \) is invariant with respect to \( A; \)
3. the spectrum of part \( A_\Delta \) of operator \( A \), induced in \( \mathcal{L}(\Delta) \), consists of intersection of spectrum of \( A \) with the interior of segment \( \Delta \) and, perhaps, the endpoints of segment \( \Delta \). And at that, if the endpoint of segment \( \Delta \) does not belong to the spectrum of \( A \), it does not belong to the spectrum of \( A_\Delta \) either;
4. if there is some subspace \( \mathcal{L} \) on which the operator \( A \) is defined everywhere and is bounded, and this subspace is invariant with respect to \( A \), and at the same time the spectrum of the \( \mathcal{L} \)-induced part of \( A \) is included in \( \Delta \), then \( \mathcal{L} \subset \mathcal{L}(\Delta) \).
Now we describe the construction of spectral subspaces and their main properties, and later prove the relationship with the entire vectors of exponential type. Let \( \theta(t) \) \((-\infty < t < \infty)\) is the entire function of order 1 with zeroes on the positive imaginary ray:

\[
\theta(t) = C \prod_{k=1}^{\infty} \left(1 - \frac{t}{it_k}\right), \quad \text{where } 0 < t_1 \leq t_2 \leq \ldots, \sum_{k=1}^{\infty} \frac{1}{t_k} < \infty,
\]

\( C \) is a constant. Note that \(|\theta(t)|\) satisfies the conditions \(|\theta(t_1+t_2)| \leq |\theta(t_1)||\theta(t_2)|, t_1, t_2 \in \mathbb{R}\) and \(\int_{-\infty}^{\infty} \frac{|\ln(\alpha(t))|}{1+\alpha^2} dt < \infty\), i.e. it belongs to \( \Omega \) (for definition of class \( \Omega \) see \[10\]).

Define by \( E_\theta^{(\infty)} \) the class of entire functions \( \phi(t) \) of finite type and order 1 which satisfies for all \( m = 0, 1, \ldots \) and for all \( a > 0 \) the condition

\[
M_{\theta}^{(m,a)}(\phi) := \int_{-\infty}^{\infty} |t^m \theta(at) \phi(t)| dt < \infty.
\]

As shown in \[8, Lemma 1.1.1\], the Fourier transform of the functions from \( E_\theta^{(\infty)} \) is non-quasianalytic, that is the following property takes place:

**Proposition 1.** For any segment \( \Delta \) of real axis and for any open finite interval \( I \supset \Delta \) there exists \( \phi(t) \in E_\theta^{(\infty)} \) such that its Fourier transform equals one in \( \Delta \) and equals zero outside \( I \).

Moreover, the class \( E_\theta^{(\infty)} \) is linear and is closed under convolutions and differentiation.

Next step is the construction of finite functions of operator \( A \). For the \( C_0 \)-group with non-quasianalytic generator there exists \[13\] such entire function \( \theta(t) \) of order 1 with zeroes on the positive imaginary ray that

\[
\|U(t)\| \leq |\theta(t)| \quad \forall t \in \mathbb{R}.
\]

Let’s consider arbitrary \( \phi(t) \in E_\theta^{(\infty)} \) and construct linear operator

\[
P_\phi = \int_{-\infty}^{\infty} \phi(t) U(t) \, dt.
\]

The operator, defined by (3.3), is bounded due to (3.2). Next, consider arbitrary segment \( \Delta \) of the real axis and denote by \( E_\theta^{(\infty)}(\Delta) \) the set of such functions \( \phi(t) \in E_\theta^{(\infty)} \) that the Fourier transform \( \hat{\phi}(\lambda) = 1 \) in some interval containing \( \Delta \). Denote by \( \mathcal{L}(\Delta) \) the subspace of vectors \( x \) such that

\[
P_\phi x = x
\]

for all \( \phi(t) \in E_\theta^{(\infty)}(\Delta) \).

Operators \( P_\phi \) are useful for studying vectors \( A^n x \) and for proving of Bernstein-type inequality because of the properties (3.3), (3.4) and the property \[8\, p.445\]

\[
AP_\phi = P_\phi A = P_{-i\phi'},
\]

which allows to deal with derivatives of some entire functions instead of Banach-space operators and vectors.

The following theorem shows the close relationship between spectral subspaces and the entire vectors of exponential type.

**Theorem 1.** For all \( \alpha > 0 \)

\[
\mathcal{E}^\alpha(A) \subset \Xi^\alpha(A) = \mathcal{L}(-\alpha, \alpha],
\]

moreover, \( \Xi^\alpha(A) \) is the closed subspace of \( \mathcal{X} \).
Proof. First we will prove the embedding $\Xi^\alpha(A) \subset \mathcal{L}([-\alpha, \alpha])$. To do this, the forth property of spectral subspaces (mentioned at the beginning of this section) will be used.

Obviously, $\mathcal{E}^\alpha(A)$ is an invariant subspace of $A$, and so is $\Xi^\alpha(A)$. Denote the $\Xi^\alpha$-part of $A$ as $A_\alpha$:

$$A_\alpha = A \upharpoonright \Xi^\alpha(A).$$

By the mentioned property of spectral subspaces, to finish the proof, it is enough to show that $\sigma(A_\alpha) \subset [-\alpha, \alpha]$ and that $A$ is bounded on $\Xi^\alpha(A)$.

Lets show $\sigma(A_\alpha) \subset [-\alpha, \alpha]$. For that we check that all points from $\mathbb{C}\setminus [-\alpha, \alpha]$ are regular.

Let $\lambda \in \mathbb{R}\setminus [-\alpha, \alpha]$. $\lambda$ cannot be an eigenvalue, otherwise for some $x \in \Xi^\alpha(A)$ and for all $n \in \mathbb{N}$ $\|A^n_x\| = |\lambda|^n\|x\|$, which implies $x \not\in \mathcal{E}^{\alpha+\epsilon}(A)$ for some $\epsilon > 0$, a contradiction with \((2.2)\). That is, $\lambda$ is not an eigenvalue of $A_\alpha$.

The equation

$$(3.6)\quad Ax - \lambda x = y$$

has a solution

$$x = -\sum_{n=0}^{\infty} \frac{A^n y}{\lambda^{n+1}}.$$

for any $\lambda \in \mathbb{R}\setminus [-\alpha, \alpha]$ and $y \in \Xi^\alpha(A)$, and this solution belongs to $\Xi^\alpha(A)$, so such $\lambda \in \rho(A_\alpha)$.

Let $\text{Im} \lambda \neq 0$. Then, as shown in [8, p.442], $\lambda$ is not an eigenvalue of $A$ (as well as $A_\alpha$) and the resolvent $R_\lambda(A)$ is defined. We set for all $y \in \Xi^\alpha(A)$ $x = R_\lambda y$. Then

$$\|A^n x\| = \|A^n R_\lambda y\| = \|R_\lambda A^n y\| \leq \|R_\lambda\| \cdot \|A^n y\|,$$

hence $x \in \Xi^\alpha(A)$ and (by definition of resolvent) $x, y$ satisfy the equation \((3.6)\). So again $\lambda \in \rho(A_\alpha)$.

Thus it is shown that $\{\lambda \in \mathbb{R} \mid |\lambda| > \alpha\} \subset \rho(A_\alpha)$ and $\{\lambda \in \mathbb{C} \mid \text{Im} \lambda \neq 0\} \subset \rho(A_\alpha)$ therefore $\sigma(A_\alpha) \subset [-\alpha, \alpha]$.

To prove the boundedness of $A$ on $\Xi^\alpha(A)$ consider the notion of $S$-operators [8, p.452]. It results from the following facts (see [8, pp.462-465 and Theorem 6.1]):

- If the operator $A$ is non-quasianalytic, then it is an $S$-operator.
- $K_\Delta$ is a spectral subspace $\mathcal{L}(\Delta)$ of operator $A$.
- There exists such bounded linear operator $\Phi_\Delta(A)$, defined on the whole $X$, that $K_\Delta = \ker \Phi_\Delta(A)$.
- Operator $A$ is defined and is bounded on whole $K_\Delta$.
- If $\mathcal{L}$ is an invariant subspace of $A$ and if the spectrum of $\mathcal{L}$-induced part $A_\mathcal{L}$ of operator $A$ is included into segment $\Delta$, then $\mathcal{L} \subset K_\Delta$.

Moreover, from these facts it follows that $\mathcal{L}(\Delta)$ is closed subspace. This means that closedness of $\Xi^\alpha(A)$ would result from the first statement of theorem ($\Xi^\alpha(A) = \mathcal{L}([-\alpha, \alpha])$).

Let’s prove the embedding $\mathcal{L}([-\alpha, \alpha]) \subset \bigcap_{r>0} \mathcal{E}^{\alpha+\epsilon}(A) = \Xi^\alpha(A)$.

According to [10, Lemma 3.1], there exists such entire function $K_\theta(t)$ for $|\theta(t)|$ that for all $r > 0$ exists a constant $c_r = c_r(\theta) > 0$ such that for all $z \in \mathbb{C}$

$$(3.7)\quad |K_\theta(rz)| \leq c_r \frac{e^{r|\text{Im} z|}}{|\theta(|z|)|}.$$
Returning to the function \( I = \{\alpha, \alpha + 4\} \supset \Delta \). According with the proof of \([8]\) Lemma 1.1.1, the Fourier transform of a function

\[
(3.8) \quad \phi(t) = \frac{K^2_0(-ct)e^{-(\alpha+2\epsilon)it} - K^2_0(et)e^{(\alpha+2\epsilon)it}}{-2\pi it} = \frac{K^2_0(et)e^{(\alpha+2\epsilon)it} - e^{(\alpha+2\epsilon)it}}{-2\pi it} = \frac{\alpha + 2\epsilon}{\pi} K^2_0(et) \sin \left(\frac{(\alpha + 2\epsilon)t}{\alpha + 2\epsilon}\right)
\]
equals one in \( \Delta \) and equals zero outside \( I \). Denote by

\[
\phi_{\epsilon,x}(z) := K^2_0(\epsilon z) \frac{\sin rz}{r z}, \quad z \in \mathbb{C}, \quad r > 0, \quad \epsilon > 0
\]
and estimate the derivatives \( \phi_{\epsilon,x}^{(n)}(t) \), \( t \in \mathbb{R} \). Using inequality

\[
\left| \frac{\sin z}{z} \right| \leq \frac{\min(1, |z|)}{|z|} e^{\text{Im} z} \leq e^{\text{Im} z}
\]
and \((3.7)\), one can find

\[
(3.9) \quad |\phi_{\epsilon,x}(z)| \leq \frac{c^2 e^{2|\text{Im} z|}}{|\theta^2(|z|)|} \cdot e^{\text{Im} z} = \frac{c^2 e^{(r + 2\epsilon)|\text{Im} z|}}{|\theta^2(|z|)|}.
\]

Similarly to the proof of \([10]\) Lemma 3.2], Cauchy integral formula for \( \gamma_{n,r}(t) := \left\{ \zeta \in \mathbb{C} : |\zeta - t| = \frac{n}{r + 2\epsilon} \right\} \) and inequality \((3.9)\) allow to obtain for \( t \in \mathbb{R} \) and \( n \in \mathbb{N} \)

\[
|\phi_{\epsilon,x}^{(n)}(t)| \leq \frac{n!}{2\pi} \int_{\gamma_{n,r}(t)} \frac{\phi_{\epsilon,x}(\xi)}{|\xi - t|^{n+1}} |d\xi| = \frac{n!}{2\pi} \int_{\gamma_{n,r}(t)} |\phi_{\epsilon,x}(\xi)||d\xi| \leq \frac{c^{(n)} e^{-n(r + 2\epsilon)n+1}}{\sqrt{2\pi n}} \int_{\gamma_{n,r}(t)} \frac{e^{(r + 2\epsilon)|\text{Im} \xi - t|}}{|\theta^2(|\xi|)|} |d\xi|, \text{ where } c^{(n)} = \sup_{k \in \mathbb{N}} \frac{k!}{\sqrt{2\pi k}} \left( \frac{e}{k} \right)^k < e^{1/12}.
\]

Using \(|\theta(t + s)| = |\theta(t)| \cdot |\theta(s)|\), it follows from the last inequality

\[
|\phi_{\epsilon,x}^{(n)}(t)| \leq \frac{c^{(n)} e^{-n(r + 2\epsilon)n+1}}{\sqrt{2\pi n} \theta^2(t)} \int_{\gamma_{n,r}(t)} \frac{e^{(r + 2\epsilon)|\text{Im} \xi - t|}}{|\theta^2(|\xi|)|} |d\xi| \leq \frac{c^{(n)} e^{-n(r + 2\epsilon)n+1}}{\sqrt{2\pi n} \theta^2(t)} \int_{\gamma_{n,r}(t)} e^{(r + 2\epsilon)|\text{Im} \xi - t|} \theta^2(|t - \xi|) |d\xi| \leq \frac{c^{(n)} e^{(r + 2\epsilon)n}}{\sqrt{2\pi n} \theta(t)} \left( \frac{n}{r + 2\epsilon} \right)^2.
\]

Returning to the function \( \phi(t) \) one can get

\[
(3.10) \quad |\phi^{(n)}(t)| = \frac{\alpha + 2\epsilon}{\pi} |\phi^{(n+2\epsilon,\epsilon)}(t)| \leq \frac{c^{(n)} e^{2\pi n}}{\pi} (\alpha + 2\epsilon)(\alpha + 4\epsilon)^n \left( \frac{n}{\alpha + 4\epsilon} \right)^2.
\]

Let \( x \in \mathcal{L}([-\alpha, \alpha]) \). By the construction \( \phi \in E_{\theta}^{(\infty)}(\Delta) \), thus \( P_{\phi}x = x \) and, accordingly to \((3.5)\),

\[
\|A^n x\| = \|A^n P_{\phi}x\| = \|P_{(-i)^n\phi^{(n)}}x\|.
\]

Using \((3.3)\) and \((3.10)\), the following estimate for the latter expression can be found

\[
(3.11) \quad \|P_{(-i)^n\phi^{(n)}}x\| \leq \int_{-\infty}^{\infty} |\phi^{(n)}(t)\theta(t)| \, dt \cdot \|x\| \leq \frac{c^{(n)} e^{2\pi n}}{\pi} (\alpha + 2\epsilon)(\alpha + 4\epsilon)^n \|x\| \theta^2 \left( \frac{n}{\alpha + 4\epsilon} \right) \int_{-\infty}^{\infty} \frac{dt}{|\theta(t)|}.
\]
It follows from (3.1) that
\[ \int_{-\infty}^{\infty} \frac{dt}{|\theta(t)|} = c_\theta < \infty, \]
so there exists such \( c > 0 \) that
\[ (3.12) \quad \|A^nx\| \leq c\sqrt{n}(\alpha + 2\epsilon)(\alpha + 4\epsilon)^n \left| \theta^2 \left( \frac{n}{\alpha + 4\epsilon} \right) \right| \|x\|, \quad \alpha > 0, \ \epsilon > 0, \ n \in \mathbb{N}. \]

The following relation holds
\[ (3.13) \quad \lim_{n \to \infty} \left( c\sqrt{n}(\alpha + 2\epsilon)(\alpha + 4\epsilon)^n \right)^{1/n} = \alpha + 4\epsilon, \quad \alpha, \epsilon \in \mathbb{R}_+. \]

As noted in the proof of [10, Theorem 3.1], for the function \(|\theta(t)|\) it holds
\[ (3.14) \quad \lim_{n \to \infty} \left( \theta^2 \left( \frac{n}{\alpha + 4\epsilon} \right) \right)^{1/n} = 1, \quad \alpha \in \mathbb{R}_+, \]
therefore from (3.12), (3.13) and (3.14) one can get
\[ \sigma(x, A) = \limsup_{n \to \infty} \|A^n x\|^{\frac{1}{n}} \leq \alpha + 4\epsilon, \]
that is \( \forall \epsilon' > 0 \ x \in \mathcal{E}^{\alpha + 4\epsilon + \epsilon'}(A) \). Due to arbitrariness of \( \epsilon \),
\[ x \in \bigcap_{\epsilon > 0} \mathcal{E}^{\alpha + \epsilon}(A), \]
which was to be proved. \( \square \)

4. Generalized Bernstein-type inequality

One of the well-known inequalities in approximation theory is the Bernstein inequality.
If \( f(x) \) is an entire function of exponential type \( \sigma > 0 \), and
\[ |f(x)| \leq M, \quad -\infty < x < \infty, \]
then
\[ (4.1) \quad |f'(x)| \leq \sigma M, \quad -\infty < x < \infty. \]

In this section some generalization of Bernstein inequality for exponential type entire vectors is proved.

Note that more detail view on (3.12) allows to obtain Bernstein-type inequality. Consider the relation (3.14). Note that it holds uniformly for all \( \alpha \geq \alpha_0 > 0 \). Therefore for all \( \epsilon > 0 \) there exists \( c_\epsilon > 0 \) such that
\[ (4.2) \quad c\sqrt{n} \left| \theta^2 \left( \frac{n}{\alpha + 4\epsilon} \right) \right| \leq c_\epsilon (1 + \epsilon)^n, \quad \forall n \in \mathbb{N}, \ \forall \alpha \in \mathbb{R}_+. \]

Inequalities (3.12) and (4.2) allow to prove

Proposition 2. For every \( \epsilon > 0 \) there exists \( c_\epsilon > 0 \), independent of \( \alpha \) and of \( n \), such that for all \( \alpha > 0 \)
\[ (4.3) \quad \|A^n x\| \leq c_\epsilon (1 + \epsilon)^n(\alpha + 2\epsilon)(\alpha + 4\epsilon)^n\|x\|, \quad x \in \Xi^\alpha(A) \text{ or } x \in \mathcal{E}^\alpha(A). \]

But in contrast with the classic Bernstein inequality, the type \( \alpha \) of vector appears in (4.3) in the degree \( n + 1 \). Lets show that the analogous inequality with the degree \( n \) holds.

Theorem 2 (Generalized Bernstein-type inequality). For all vectors \( x \in \mathcal{E}(A) \), of type, not exceeding some \( \alpha \geq 1 \), the following inequality holds
\[ (4.4) \quad \|A^n x\| \leq c_n \alpha^n\|x\|, \quad \text{where the constants } c_n > 0 \text{ do not depend on } x \text{ and on } \alpha. \]
Proof. Let's consider majorant \( \theta(t) \) for the function \( ||U(t)|| \), constructed in \[3.3\]. Remark that \( \theta(t) \) is of the form \( 3.1 \). Similarly to the proof of theorem \[1\] and as in \[10\] Lemma 3.1 by the function \( \theta(t) \) one can construct the entire function \( K(t) \) of exponential type.

Let's consider such function \( \phi_\alpha(t) \) that its Fourier transform equals 1 in \([-\alpha, \alpha]\) and equals 0 outside \((-3\alpha, 3\alpha)\). According to \[8\] Lemma 1.1.1, one can use as \( \phi_\alpha(t) \) the function

\[
(4.5) \quad \phi_\alpha(t) = \frac{K^2(\frac{t}{\pi}) \sin 2\alpha t}{\pi t}.
\]

Denote by

\[
(4.6) \quad \phi(t) := \frac{K^2(\frac{t}{\pi}) \sin 2t}{\pi t}.
\]

Then \( \phi_\alpha(t) = \alpha \phi(\alpha t) \). As it follows from \(8.3\) and \(8.4\), it is enough to estimate the quantity

\[
\int_{-\infty}^{\infty} |\phi_\alpha^{(n)}(t)\theta(t)| \, dt
\]

to prove the theorem. For \( \alpha \geq 1 \) we have \( |\theta(t)| \leq |\theta(\alpha t)| \) and

\[
\int_{-\infty}^{\infty} |\phi_\alpha^{(n)}(t)\theta(t)| \, dt \leq \int_{-\infty}^{\infty} |\phi^{(n)}(\alpha t)\theta(\alpha t)| \, d\alpha t.
\]

The change of variables \( \tau = \alpha \cdot t \) gives

\[
\frac{d^n \phi(\alpha t)}{dt^n} = \frac{d^n \phi(\tau)}{d\tau^n} \cdot \alpha^n,
\]

thus

\[
\int_{-\infty}^{\infty} |\phi^{(n)}(\alpha t)\theta(\alpha t)| \, d\alpha t = \alpha^n \cdot \int_{-\infty}^{\infty} |\phi^{(n)}(\tau)\theta(\tau)| \, d\tau.
\]

It is easy to see that the last integral exists and does not depend on \( \alpha \). Let it equals \( c_n > 0 \). Then

\[
\|A^n x\| = \|P_{(-1)^n \phi^{(n)}} x\| \leq c_n \alpha^n \|x\|,
\]

which was to be proved. \( \square \)

As the consequence of theorem \[2\] we get the following estimate for an operator \( \Delta_h^k \):

**Corollary 1.** Let \( x \in C(A) \) and \( \sigma(x) \leq \alpha, \ \alpha \geq 1. \) Then for all \( k \in \mathbb{N} \)

\[
(4.7) \quad \| \Delta_h^k x \| \leq c_k (h\alpha)^k M_U(kh) \|x\|,
\]

where the constant \( c_k \) is the same as in the theorem \[2\] and the function \( M_U(t) \) is defined by \(2.3\).

**Proof.** It holds for \( \Delta_h^k \):

\[
\Delta_h^k x = (U(t) - I)^k x = \int_0^t \cdots \int_0^t U(\xi_1 + \ldots + \xi_k) A^k x \, d\xi_1 \ldots d\xi_k.
\]

By the theorem \[2\]

\[
\|A^k x\| \leq c_k \alpha^k \|x\|,
\]

and \( \|U(\xi_1 + \ldots + \xi_k)\| \leq M_U(mt) \) by the definition. Therefore,

\[
\|\Delta_h^k x\| \leq \int_0^t \cdots \int_0^t \|U(\xi_1 + \ldots + \xi_k)\| \cdot \|A^k x\| \, d\xi_1 \ldots d\xi_k \leq c_k h^k M_U(kh) \alpha^k \|x\|. \quad \square
\]

\footnote{The majorant is named in \[13\] as \( \omega(t) \), but in this article it is denoted as \( \theta(t) \) in order not to confuse it with the module of continuity.}
5. Inverse theorem of approximation

The following results generalize classical Bernstein theorem (also known as inverse theorem).

**Theorem 3.** Let $\omega(t)$ is the function of type of module of continuity for which the following conditions are satisfied:

1. $\omega(t)$ is continuous and nondecreasing for $t \in \mathbb{R}_+$.
2. $\omega(0) = 0$.
3. $\exists c > 0 \forall t \in [0, 1] \quad \omega(2t) \leq c\omega(t)$.
4. $\int_0^1 \frac{\omega(t)}{t} dt < \infty$.

If, for $x \in X$, there exist $n \in \mathbb{N}$ and $m > 0$ such that

$$E_r(x, A) \leq \frac{m}{r^n} \omega\left(\frac{1}{r}\right), \quad r \geq 1,$$

then $x \in \mathcal{D}(A^n)$ and for every $k \in \mathbb{N}$ there exists a constant $m_k > 0$ such that

$$\omega_k(t, A^n x, A) \leq m_k \left( t^k \int_0^1 \frac{\omega(u)}{u^{k+1}} du + \int_0^t \frac{\omega(u)}{u} du \right), \quad 0 < t \leq 1/2.$$

The following lemma is used for the proof of theorem.

**Lemma 1.** Suppose that the function $\omega(t)$ satisfies conditions 1–3 of theorem 3. If, for $x \in X$, there exists $m > 0$ such that

$$E_r(x, A) \leq m \omega\left(\frac{1}{r}\right), \quad r \geq 1,$$

then, for every $k \in \mathbb{N}$ there exists a constant $\tilde{c}_k > 0$ such that

$$\omega_k(t, x, A) \leq \tilde{c}_k t^k \int_0^1 \frac{\omega(\tau)}{\tau^{k+1}} d\tau, \quad 0 < t \leq 1/2.$$

**Remark 1.** As would follow from the proof, the lemma remains true under somewhat weaker conditions than those formulated in the theorem, namely, it is sufficient that for an element $x \in X$ there exist at least one sequence $\{u_j\}^\infty_{j=1} \subset \mathcal{E}(A)$ such that $\sigma(u_j, A) \leq 2^j$ and for all $j \in \mathbb{N}$

$$\|x - u_j\| \leq m \cdot \omega\left(\frac{1}{2^j}\right).$$

**Proof of theorem.** As shown in the theorem [11], the subspaces $\Xi^r(A)$ are closed, therefore it follows from the definition and from [5.1] that there exists a sequence of vectors $\{u_j\}^\infty_{j=0} \subset \mathcal{E}(A)$ such that $\sigma(u_j, A) \leq 2^j$ and

$$\|x - u_j\| \leq m \cdot \omega\left(\frac{1}{2^j}\right).$$

From the inequality (5.5) and conditions 1, 2 one can get $\|x - u_j\| \to 0, \quad j \to \infty$, and so the vector $x$ has the representation

$$x = u_0 + \sum_{j=1}^\infty (u_j - u_{j-1}).$$
Due to $\sigma(u_j - u_{j-1}, A) \leq 2^j$, $j \in \mathbb{N}$, one can find from (4.3)

$$||A^n u_j - A^n u_{j-1}|| \leq c_n 2^j ||u_j - u_{j-1}|| \leq c_n 2^{jn} (||x - u_j|| + ||x - u_{j-1}||) \leq$$

$$\leq c_n 2^{jn} \left( \frac{m}{2^n j} \cdot \omega \left( \frac{1}{2^n j} \right) + \frac{m}{2^n (j-1)} \cdot \omega \left( \frac{1}{2^n (j-1)} \right) \right) \leq 2^{n+1} c_n m \cdot \omega \left( \frac{1}{2^n (j-1)} \right) \leq$$

$$\leq 2^{n+1} c_n m \cdot \omega \left( \frac{1}{2^n (j-1)} \right) \leq \frac{2^{n+1} c_n m}{\ln 2} \int_{2^{-j}}^{2^{-j+1}} \frac{\omega(u)}{u} du.$$

Hence, $\sum_{j=1}^{\infty} (A^n u_j - A^n u_{j-1})$ is convergent. By virtue of closedness of operator $A^n$, $x \in \mathcal{D}(A^n)$ and

$$A^n = A^n u_0 + \sum_{j=1}^{\infty} (A^n u_j - A^n u_{j-1}),$$

therefore

$$||A^n x - A^n u_{j_0}|| \leq \sum_{j=j_0+1}^{\infty} ||A^n u_j - A^n u_{j-1}|| \leq \frac{2^{n+1} c_n m}{\ln 2} \sum_{j=j_0+1}^{\infty} \int_{2^{-j}}^{2^{-j+1}} \frac{\omega(u)}{u} du =$$

$$= \frac{2^{n+1} c_n m}{\ln 2} \int_{0}^{2^{-j_0}} \frac{\omega(u)}{u} du =: \tilde{c} \Omega(2^{-j_0}), \quad j_0 \in \mathbb{N},$$

where $\tilde{c} = \frac{2^{n+1} c_n m}{\ln 2}$.

$$\Omega(t) := \int_{0}^{t} \frac{\omega(u)}{u} du.$$

It is easy to see that the function $\Omega(t)$ has the following properties:

(1) $\Omega(t)$ is continuous and monotonically nondecreasing;

(2) $\Omega(0) = 0$;

(3) for $t \in [0, 1]$, the following relation is true:

$$\Omega(2t) = \int_{0}^{2t} \frac{\omega(u)}{u} du = \int_{0}^{t} \frac{\omega(2u)}{u} du \leq c \int_{0}^{t} \frac{\omega(u)}{u} du = c \Omega(t).$$

Therefore, setting $\omega(t) = \Omega(t)$ in lemma [1] and taking remark into account, we get

$$\omega_k(t, A^n x, A) \leq \tilde{c} k \int_{0}^{1} \frac{\Omega(u)}{u^{k+1}} du = \frac{\tilde{c} k}{k} \left( \Omega(u) \frac{1}{u^{k+1}} \right) \leq$$

$$\leq m_k \left( \int_{0}^{1} \frac{\omega(u)}{u^{k+1}} du + \int_{0}^{1} \frac{\omega(u)}{u^{k+1}} du \right).$$

The theorem is proved.

\[ \square \]

**Proof of lemma [7]** By the analogy with the proof of theorem [3] it follows from (5.1) that there exists a sequence of vectors $\{u_j\}_{j=0}^{\infty} \subset \mathcal{E}(A)$ such that $\sigma(u_j, A) \leq 2^j$ and

(5.6)

$$||x - u_j|| \leq m \omega \left( \frac{1}{2^j} \right).$$

Let us take arbitrary $h \in (0, 1/2]$ and choose a number $N$ in such a way that $\frac{1}{2^{N+1}} < h \leq \frac{1}{2^N}$. Inequality (5.6) yields

(5.7)

$$||u_j - u_{j-1}|| \leq ||u_j - x|| + ||x - u_{j-1}|| \leq$$

$$\leq m \omega(2^{-j}) + m \omega(2^{-j+1}) \leq 2 m \omega(2^{-j+1}) \leq 2 cm \omega(2^{-j}).$$
By virtue of the monotonicity of \( \omega(t) \)

\[
2^k \int_{2^{-j}}^{2^{1-j}} \frac{\omega(u)}{u^{k+1}} du \geq 2^k \omega(2^{-j}) \int_{2^{-j}}^{2^{1-j}} \frac{1}{u^{k+1}} du = \frac{2^{kj}(2^k - 1)}{k} \omega(2^{-j}) \geq 2^{kj} \omega(2^{-j}).
\]

Since \( \sigma(u_j - u_{j-1}, A) \leq 2^j \) and \( \sigma(u_0, A) \leq 1 \), according to corollary \([1]\)

\[
\| \Delta^k_h u_0 \| \leq c_k h^k M_U(kh) \| u_0 \|
\]

\[
\| \Delta^k_h (u_j - u_{j-1}) \| \leq c_k h^k (2^j) M_U(kh) \| u_j - u_{j-1} \|, \quad j \geq 1.
\]

Relations \([5.6] - [5.8]\) yield

\[
\| \Delta^k_h (u_j - u_{j-1}) \| \leq 2^{\tilde{c}k} (2^j)^k \omega(2^{-j}) \leq 2^{\tilde{c}k} \int_{2^{-j}}^{2^{1-j}} \frac{\omega(u)}{u^{k+1}} du,
\]

where \( \tilde{c} = c c_k m M_U(kh) \), and

\[
\| \Delta(x - u_N) \| \leq \| (U(h) - I)^k \| \| x - u_N \| \leq (M_U(h) + 1)^k \| x - u_N \| \leq (M_U(h) + 1)^k m \omega(2^{-N}).
\]

Using these inequalities, we obtain

\[
\| \Delta^k_h x \| = \left\| \Delta^k_h u_0 + \sum_{j=1}^{N} \Delta^k_h (u_j - u_{j-1}) + \Delta^k_h (x - u_N) \right\| \leq \\
\leq c_k M_U(kh) h^k \| u_0 \| + 2^{\tilde{c}k} \int_{2^{-j}}^{2^{1-j}} \frac{\omega(u)}{u^{k+1}} du + (M_U(h) + 1)^k m \omega(2^{-N}) \leq \\
\leq c_k M_U(kh) h^k \| u_0 \| + 2^{\tilde{c}k} \int_{2^{-j}}^{2^{1-j}} \frac{\omega(u)}{u^{k+1}} du + (M_U(h) + 1)^k c m \omega(h) \leq \\
\leq c_k M_U(kh) h^k \| u_0 \| + 2^{\tilde{c}k} \int_{2^{-j}}^{2^{1-j}} \frac{\omega(u)}{u^{k+1}} du + (M_U(h) + 1)^k c m \omega(h) = \\
= h^k \left[ c_k M_U(kh) \| u_0 \| + 2^{\tilde{c}k} \int_{2^{-j}}^{2^{1-j}} \frac{\omega(u)}{u^{k+1}} du + (M_U(h) + 1)^k c m \frac{k}{1 - h^k} \int_{2^{-j}}^{2^{1-j}} \frac{\omega(u)}{u^{k+1}} du \right] \leq \\
\leq \tilde{c}_k h^k \int_{2^{-j}}^{2^{1-j}} \frac{\omega(u)}{u^{k+1}} du,
\]

where

\[
\tilde{c}_k := \frac{\| u_0 \| c_k M_U(k/2)}{\int_{2^{-j}}^{2^{1-j}} \frac{\omega(u)}{u^{k+1}} du} + 2^{\tilde{c}k} c c_k m M_U(k/2) + (M_U(1/2) + 1)^k \frac{c m k}{1 - (1/2)^k}.
\]

The last inequality holds for all \( 0 < h \leq 1/2 \). Taking into account the definition of module of continuity \([2,4]\), this inequality finishes the proof. \( \square \)

6. Examples of Application of Abstract Direct and Inverse Theorems in Particular Spaces

In this section we discuss an application of the presented theory — the approximation of continuous functions by entire functions in the weighted \( L_p(\mathbb{R}, \mu^p) \) space with growing at the infinity weight (for example, \( L_1(\mathbb{R}, x^n) \) spaces). Similar problems studied in several papers (see the review \([14]\)).

Let consider the real-valued function \( \mu(t) \) satisfying the following conditions:

1) \( \mu(t) \geq 1 \), \( t \in \mathbb{R} \);
2) \( \mu(t) \) is even, monotonically non-decreasing when \( t > 0 \);
3) \( \mu(t) \) satisfies the condition \( \mu(t + s) \leq \mu(t) \cdot \mu(s) \), \( s, t \in \mathbb{R} \);
4) \( \int_{-\infty}^{\infty} \frac{\mu(t)}{1 + t^2} dt < \infty \),
or alternatively, instead of 4), the equivalent condition holds:

\[ 4') \sum_{k=1}^{\infty} \frac{\ln \mu(k)}{k^2} < \infty. \]

Below are several important classes of functions satisfying conditions 1)–4) (see [10] for details).

1. Constant function \( \mu(t) \equiv 1, \ t \in \mathbb{R} \).
2. Functions with polynomial order of growth at infinity. For such functions the following estimate holds: \( \exists \mu \in \mathbb{N}, \exists M \geq 1 \)

\[ \mu(t) \leq M(1 + |t|)^k, \ t \in \mathbb{R}. \]

3. Functions of the form

\[ \mu(t) = e^{t^\beta}, \ 0 < \beta < 1, \ t \in \mathbb{R}. \]

4. \( \mu(t) \) represented as a power series for \( t > 0 \). I.e.,

\[ \mu(t) = \sum_{n=0}^{\infty} \frac{|t|^n}{m_n}, \]

where \( \{m_n\}_{n \in \mathbb{N}} \) is the sequence of positive real numbers satisfying three conditions:

- \( m_0 = 1, m_2 \leq m_{n-1} \cdot m_{n+1}, \ n \in \mathbb{N}; \)
- for all \( k, l \in \mathbb{N} \)

\[ \frac{(k+l)!}{m_{k+l}} \leq \frac{k! \cdot l!}{m_k \cdot m_l}. \]
- \( \sum_{n=1}^{\infty} \left( \frac{1}{m_n} \right)^{1/n} < \infty; \)

5. \( \mu(t) \) as a module of an entire function with zeroes on the imaginary axis. Lets consider

\[ \omega(t) = C \prod_{k=1}^{\infty} \left( 1 - \frac{t}{it_k} \right), \ t \in \mathbb{R}, \]

where \( C \geq 1, 0 < t_1 \leq t_2 \leq \ldots, \sum_{k=1}^{\infty} \frac{1}{t_k} < \infty, \) and set \( \mu(t) := |\omega(t)|. \)

Ls consider the space \( L_p(\mathbb{R}, \mu^p) \) of the functions \( x(s), s \in \mathbb{R} \), integrable in \( p \)-th degree with the weight \( \mu^p \):

\[ \|x\|_{L_p(\mathbb{R}, \mu^p)} = \int_{-\infty}^{\infty} |x(s)|^p \mu^p(s) \, ds. \]

\( L_p(\mathbb{R}, \mu^p) \) is the Banach space. The differential operator

\[ (Ax)(t) = \frac{dx}{dt}, \quad D(A) = \{ x \in L_p(\mathbb{R}, \mu^p) \cap AC(\mathbb{R}) : x' \in L_p(\mathbb{R}, \mu^p) \}. \]

generates the group of shifts \( \{U(t)\}_{t \in \mathbb{R}} \) in the space \( L_p(\mathbb{R}, \mu^p) \). This group isn’t bounded. As shown in [10],

\[ \|U(t)\|_{L_p(\mathbb{R}, \mu^p)} \leq \mu(|t|), \ t \in \mathbb{R}. \]

To apply the constructed theory, we need to determine how the space \( \Xi(A) \) and the space of exponential type entire functions are connected. Denote by \( B_\sigma \) the set of exponential functions of entire type \( \sigma \). We show that the following embedding holds

\[ (6.1) \ 
\Xi^\sigma(A) \subset B_\sigma \cap L_p(\mathbb{R}, \mu^p). \]

Let \( f \in \Xi^\sigma(A) \). Obviously, \( f \in L_p(\mathbb{R}, \mu^p) \). We prove that \( f \in B_\sigma \). Due to \( \mu(t) \geq 1 \) we have

\[ \|f\|_{L_p(\mathbb{R})} \leq \|f\|_{L_p(\mathbb{R}, \mu^p)}, \]

thus for all \( n \in \mathbb{N} \) and for any \( \epsilon > 0 \)

\[ (6.2) \ 
\|A^n f\|_{L_p(\mathbb{R})} \leq \|A^n f\|_{L_p(\mathbb{R}, \mu^p)} \leq c_\epsilon(f)(\sigma + \epsilon)^n, \]
and so we can construct a continuation of \( U(t) \) onto \( \mathbb{C} \) by

\[
U(z) = \sum_{n=0}^{\infty} \frac{A_n f}{n!} z^n, \quad z \in \mathbb{C}.
\]

Moreover, (6.2) ensures for all \( \epsilon > 0 \)

\[
\|f(x + z)\|_{L_p(\mathbb{R})} = \left\| \sum_{n=0}^{\infty} \frac{A_n f}{n!} z^n \right\| \leq c_\epsilon(f) \cdot \|f\|_{\epsilon(|\sigma + \epsilon)|z|}},
\]

which means \( f \in B_\sigma \), which required.

By virtue of the classical Bernstein inequality the reverse embedding to (6.1) holds for all bounded weights \( \mu(t) \). We show that it holds for all functions \( \mu(t) \), satisfying

\[
\mu(t) \geq 1 + R|t|\]

for some \( R > 0 \) and for all \( t > t_0 \geq 0 \). The condition on \( \mu(t) \) gives us \( f \in L_1(\mathbb{R}) \), \( f \in B_\sigma \), thus it is infinitely differentiable and by the Paley-Wiener theorem the support of its Fourier transform is contained in \([-\sigma, \sigma]\). Let us prove that \( f \in \mathcal{E}^\sigma(A) \) by using theorem \( \mathbf{11} \). To do this we need to show that for all \( \phi \in E^\sigma_\theta([\sigma, \sigma]) \)

\[
f = P_\phi f = \int_{-\infty}^{\infty} \phi(t) U(t) f dt.
\]

Since \( \phi \) is arbitrary, we can consider \( \phi_1(t) = \phi(-t) \in E^\sigma_\theta([-\sigma, \sigma]) \). Note that

\[
\int_{-\infty}^{\infty} \phi_1(t) U(t) f(x) dt = \int_{-\infty}^{\infty} \phi(t) f(x - t) dt = \phi * f.
\]

The Fourier transform of \( \phi * f \) equals to

\[
\hat{\phi} * \hat{f} = \hat{\phi} \cdot \hat{f} = \hat{f},
\]

because supp \( f \subset [-\sigma, \sigma] \), and by the definition of \( E^\sigma_\theta([-\sigma, \sigma]) \) we have \( \hat{\phi} = 1 \) on \([-\sigma, \sigma] \). Thus,

\[
P_\phi f = f \quad \forall \phi \in E^\sigma_\theta([-\sigma, \sigma]),
\]

so \( f \in \mathcal{L}([\sigma, \sigma]) \) and by means of theorem \( \mathbf{11} \) \( f \in \mathcal{E}^\sigma(A) \).

We have shown that \( \mathcal{E}^\sigma(A) \) coincides with \( B_\sigma \cap L_p(\mathbb{R}, \mu^p) \). Note that \( \|f - g_\sigma\|_{L_p(\mathbb{R}, \mu^p)} \) is defined only for those functions that belongs to \( L_p(\mathbb{R}, \mu^p) \) (because of \( \|g_\sigma\|_{L_p(\mathbb{R}, \mu^p)} \leq \|f - g_\sigma\|_{L_p(\mathbb{R}, \mu^p)} + \|f\|_{L_p(\mathbb{R}, \mu^p)} \)), thus the best approximation by exponential type entire vectors is the same as the best approximation by entire functions of exponential type.

By applying theorems \( \mathbf{2} \) and \( \mathbf{3} \) we get several results for the approximation theory in \( L_p(\mathbb{R}, \mu^p) \) spaces. First two results are the direct theorems (from \( \mathbf{10} \)) for spaces \( L_p(\mathbb{R}, \mu^p) \).

**Corollary 2** (\( \mathbf{10} \)). For every \( k \in \mathbb{N} \) there exists constant \( m_k(p, \mu) > 0 \) such that for all \( f \in L_p(\mathbb{R}, \mu^p) \)

\[
\mathcal{E}_r(f) \leq m_k \cdot \omega_k \left( \frac{1}{r}, f \right), \quad r \geq 1.
\]

**Corollary 3** (\( \mathbf{10} \)). Let \( f \in W^m_p(\mathbb{R}, \mu^p) \), \( m \in \mathbb{N}_0 \). Then for all \( k \in \mathbb{N}_0 \)

\[
\mathcal{E}_r(f) \leq m_{k+m} \frac{\mu}{r^{m}} \omega_k \left( \frac{1}{r}, f^{(m)} \right), \quad r \geq 1,
\]

where constants \( m_n \) (\( n \in \mathbb{N} \)) are the same as in the corollary \( \mathbf{2} \).
Corollary 4. Let $f \in L_p(\mathbb{R}, \mu^p) \cap B_\sigma$, $\sigma \geq 1$. Then for all $n \in \mathbb{N}$ there exist such constants $c_n > 0$, not depending on $\sigma$ and on $f$, that

$$||f^{(n)}||_{L_p(\mathbb{R}, \mu^p)} \leq c_n \sigma^n ||f||_{L_p(\mathbb{R}, \mu^p)}.$$ 

Corollary 5. Let $\omega(t)$ be a function of type of module of continuity for which the following conditions are satisfied:

1. $\omega(t)$ is continuous and nondecreasing for $t \in \mathbb{R}_+$.
2. $\omega(0) = 0$.
3. $\exists \varepsilon > 0 \forall t \in [0, 1]$ $\omega(2t) \leq c\omega(t)$.
4. $\int_0^1 \frac{\omega(t)}{t} dt < \infty$.

If, for $f \in L_p(\mathbb{R}, \mu^p)$ there exist such $n \in \mathbb{N}$ and $m > 0$ that

$$E_r(f) \leq \frac{m}{r^n \omega \left( \frac{1}{r} \right)}, \quad r \geq 1,$$

then $f \in W_p^n(\mathbb{R}, \mu^p)$ and for every $k \in \mathbb{N}$ there exists such $m_k > 0$ that

$$\omega_k(t, f^{(n)}) \leq m_k \left( t^k \int_t^{1+t} \frac{\omega(u)}{u^{k+1}} du + \int_0^t \frac{\omega(u)}{u} du \right), \quad 0 < t \leq 1/2.$$

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