A Relativistic Description of Gentry’s New Redshift Interpretation

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Abstract

We obtain a new expression of the Friedmann-Robertson-Walker metric, which is an analogue of a static chart of the de Sitter space-time. The reduced metric contains two functions, \( M(T, R) \) and \( \Psi(T, R) \), which are interpreted as, respectively, the mass function and the gravitational potential. We find that, near the coordinate origin, the reduced metric can be approximated in a static form and that the approximated metric function, \( \Psi(R) \), satisfies the Poisson equation. Moreover, when the model parameters of the Friedmann-Robertson-Walker metric are suitably chosen, the approximated metric coincides with exact solutions of the Einstein equation with the perfect fluid matter. We then solve the radial geodesics on the approximated space-time to obtain the distance-redshift relation of geodesic sources observed by the comoving observer at the origin. We find that the redshift is expressed in terms of a peculiar velocity of the source and the metric function, \( \Psi(R) \), evaluated at the source position, and one may think that this is a new interpretation of Gentry’s new redshift interpretation.

1 Introduction

Recently, Gentry proposed his new interpretation of the cosmological redshift and asserted that its origin might be the gravitational potential rather than the cosmic expansion. Though one immediately finds several shortcomings of his model, we will show that Gentry’s idea can be partially applied to some class of cosmological models, e.g., de Sitter space-time, in a general relativistic manner to obtain the distance-redshift relation. The distance-redshift relation is coordinate-independent (gauge invariant), however, depends on both the observer and sources. Hereafter, let us assume that the observer and sources are described by timelike geodesics. For example, the de Sitter space-time has at least 4 natural charts (coordinate systems), namely an open chart, a flat chart, a closed chart and a static chart. Since there is a family (a vector field) of the comoving geodesics in the former 3 charts, we identify the comoving geodesics with the observer and sources associated with each chart. Though the comoving geodesic family does not exist in the static chart, there is a unique family of the geodesics in which the comoving geodesic observer is included. In this way, we have the unique correspondence of a chart to the observer and sources. In the former 3 charts, the redshift is naturally interpreted as a consequence of the cosmic expansion in contrast to the case in the static chart in which the redshift appears as the gravitational Doppler effect. The later gravitational Doppler effect may be interpreted as Gentry’s new redshift interpretation.

In this paper, motivated by the above example in the de Sitter space-time, we find a new expression of the Friedmann-Robertson-Walker (FRW) metric, which is an analogue of a
static chart of the de Sitter space-time. The reduced metric contains two functions, $M(T, R)$ and $\Psi(T, R)$, which are interpreted as, respectively, the mass function and the gravitational potential. After giving the precise reformulation of Gentry’s new redshift interpretation, we try to apply our method to the FRW metric in the following way. First, because of a complicated form of the reduced metric, we approximate the metric near the coordinate origin, $R=T=0$, and find that the approximated metric becomes the static one and that the approximated metric function, $\Psi(R)$, satisfies the Poisson equation. For the reason of that, we will refer to both $\Psi(T, R)$ and its approximated form, $\Psi(R)$, as the gravitational potential. We then solve the radial geodesics on the approximated space-time to obtain the distance-redshift relation. We find that the redshift is expressed in terms of a peculiar velocity of the source and the gravitational potential, $\Psi(R)$, evaluated at the source position, and one may think that this is a correct, general relativistic description of Gentry’s new redshift interpretation.

Throughout this paper, we adopt the unit such that $c = G = 1$.

2 The Newtonian chart of the Friedmann-Robertson-Walker metric

We consider the FRW metric whose matter contents are perfect fluid and vacuum energy. We assume the equation of state, $P = (\gamma - 1)\rho$, for the perfect fluid matter, where $\gamma$ is a constant. For the vacuum energy, the relation, $P_V = -\rho_V$, holds. The FRW metric is then written in the following form,

$$ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right], \quad d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2,$$

(2.1)

where the suffix, 0, denotes quantities evaluated at the present time, $t_0$. The parameters, $H_0, \Omega_0$ and $\Omega_V$, are usually referred to as, respectively, the Hubble parameter, the density parameter (of matter) and the (normalized) cosmological constant. Another useful cosmological parameter is the deceleration parameter, $q_0$, defined by

$$q_0 = -\frac{\ddot{a}_0}{a_0 H_0^2} = \left( \frac{3}{2}\gamma - 1 \right) \Omega_0 - \Omega_V.$$

(2.2)

Note that any spherically symmetric metric is locally written in the following form,

$$ds^2 = -[1 + 2\Psi(T, R)] \, dt^2 + \left[ 1 - \frac{2M(T, R)}{R} \right]^{-1} \, dR^2 + R^2 d\Omega^2.$$

(2.3)

We now try to express the FRW metric in the above form, because, as is the case of the Schwarzschild space-time, the metric functions, $M(T, R)$ and $\Psi(T, R)$, are interpreted as, respectively, the mass function and the gravitational potential. We first introduce new coordi-
nates, $T'$ and $R$, related to $t$ and $r$ by

$$
T' = \frac{1}{H_0} \left[ F(y) + \frac{1}{(-2k)} \log(1 - K r^2) \right], \quad R = ar,
$$

$$
y = \frac{a}{a_0}, \quad \frac{dF}{dy} = \left[ \Omega_0 y^{3(1-\gamma)} - ky + \Omega_V y^3 \right]^{-1}.
$$

The metric (2.1) is then expressed in terms of $T'$ and $R$ as

$$
 ds^2 = -\left[ 1 - k \left( \frac{a_0}{a} \right)^2 H_0^2 R^2 \right] \left[ \Omega_0 \left( \frac{a}{a_0} \right)^{4-3\gamma} - k \left( \frac{a}{a_0} \right)^2 + \Omega_V \left( \frac{a}{a_0} \right)^4 \right] dT'^2 + dR^2 + R^2 d\Omega^2.
$$

One finds that the comoving geodesic observer, i.e., $r = 0$ in the Robertson-Walker (RW) chart (2.1), is mapped to $R = 0$ and that a proper time, $T$, of the observer is given by

$$
 dT^2 = \left[ \Omega_0 \left( \frac{a}{a_0} \right)^{4-3\gamma} - k \left( \frac{a}{a_0} \right)^2 + \Omega_V \left( \frac{a}{a_0} \right)^4 \right]_{R=0} dT'^2.
$$

When one adopts the new time coordinate, $T$, defined above, the metric is reduced to the standard form (2.3), which is referred to as the Newtonian chart, hereafter. We omit an explicit expression of the gravitational potential, $\Psi(T, R)$, because of its complicated form. On the other hand, the mass function, $M(T, R)$, takes the following simple form,

$$
 M(T, R) = \frac{1}{2} \left[ \Omega_0 \left( \frac{a}{a_0} \right)^{3\gamma} + \Omega_V \right] H_0^2 R^3.
$$

It is important to note that the reduced metric (2.5) has a coordinate singularity, $R = 2M(T, R)$, and we find that it coincides with the cosmological apparent horizon. That is, the expansion of the radial ingoing null geodesic congruences vanishes along the trajectory, $R = 2M(R, T)$. Moreover, we have other coordinate singularities,

$$
 1 - k \left( \frac{a_0}{a} \right)^2 H_0^2 R^2 = 0 \leftrightarrow r = \frac{1}{\sqrt{K}},
$$

$$
 \Omega_0 \left( \frac{a_0}{a} \right)^{3\gamma} - k \left( \frac{a_0}{a} \right)^2 + \Omega_V = 0 \leftrightarrow \dot{a} = 0,
$$

which indicate the maximum values of $r$ and $a(t)$ for the spatially closed ($K > 0$) cosmological models.

### 3 The chart-associated observer and the redshift

Though the distance-redshift relation is chart-independent, it crucially depends on both the observer and sources. As a simple example, the distance-redshift relation in the Minkowski
space-time and that in the Milne universe are compared in the following. In the Minkowski space-time, the metric is given by

\[ ds^2 = -dT^2 + dR^2 + R^2 d\Omega^2, \]

and the radial geodesic tangent is given by

**Timelike:** \( U^a = (\sqrt{1 + v^2}, v, 0, 0) \)

**Ingoing null:** \( K^a = (1, -1, 0, 0) \)

where \( v \) is a constant along each timelike geodesic and denotes a peculiar velocity of the source in the Newtonian chart \((3.1)\). The redshift, \( z \), measured by the comoving observer is given by

\[ 1 + z = v + \sqrt{1 + v^2}, \]

and \( z = 0 \) for the comoving sources, \( v = 0 \). In the Milne universe, the metric is given by

\[ ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 + r^2} + r^2 d\Omega^2 \right), \quad a(t) = t > 0, \]

and the radial geodesic tangent is given by

**Timelike:** \( u^a = \left( \sqrt{1 + \beta^2 \left( \frac{a_0}{a} \right)^2}, \frac{a_0 \beta}{a^2} \sqrt{1 + r^2}, 0, 0 \right) \)

**Ingoing null:** \( k^a = \left( \frac{1}{a}, -\frac{\sqrt{1 + r^2}}{a^2}, 0, 0 \right) \)

where \( \beta \) is a constant along each timelike geodesic and denotes a peculiar velocity of the source in the RW chart \((3.4)\). The redshift, \( z \), measured by the comoving observer is given by

\[ 1 + z = \frac{a_0}{a} \left[ \beta \frac{a_0}{a} + \sqrt{1 + \beta^2 \left( \frac{a_0}{a} \right)^2} \right]. \]

Note that, even when the sources are comoving, i.e., \( \beta = 0 \), the redshift does not vanish. The Milne universe consists of all radial timelike geodesics, \( R(T) \), with the following boundary conditions,

\[ T > 0, \quad \lim_{T \to 0} R(T) = 0, \quad \lim_{T \to 0} \frac{dR}{dT}(T) = \frac{v}{\sqrt{1 + v^2}}, \quad 0 \leq v < \infty, \]

in the Newtonian chart. One will find that the peculiar velocity, \( v \), in the Newtonian chart is related to the scale factor, \( a(t) \), as

\[ \frac{a_0}{a} = v + \sqrt{1 + v^2}. \]

That is, the redshift due to the peculiar velocity in the Minkowski space-time is interpreted as the redshift due to the cosmic expansion in the Milne universe. This simple example demonstrates that the comoving geodesics associated with different charts do not necessarily coincide each other.

Another interesting demonstration can be done in the de Sitter space-time\([4]\). In the Newtonian chart, the metric is given by

\[ ds^2 = -(1 - H^2 R^2) dT^2 + (1 - H^2 R^2)^{-1} dR^2 + R^2 d\Omega^2, \quad H = \sqrt{\frac{8\pi \rho V}{3}}, \]

\[ \rightarrow \quad \Psi(T, R) = -\frac{1}{2} H^2 R^2, \quad M(T, R) = \frac{1}{2} H^2 R^3, \]

(3.9)
and the radial geodesic tangent is given by

\[ U^a = \left( [1 - H^2 R^2]^{-1}, \pm H R, 0, 0 \right), \quad \text{Ingoing null: } K^a = \left( [1 - H^2 R^2]^{-1}, -1, 0, 0 \right), \]  

where we chose an integration constant such that \( U^a \) describes a family of the timelike geodesics in which the comoving geodesic observer, \( R = 0 \), is included. Up to the sign of \( U^1 \), the above choice determines the unique family of geodesics, which are interpreted as the observer and sources. We take a positive sign of \( U^1 \) and obtain the following angular diameter distance-redshift relation

\[ R = \frac{1}{H} \frac{z}{1 + z} \]  

(3.11)

Hereafter, the angular diameter distance is referred to as the distance for simplicity. Accordingly, one has

\[ z = \frac{\sqrt{-2\Psi(T, R)}}{1 - \sqrt{-2\Psi(T, R)}}. \]  

(3.12)

When \( z \ll 1 \), one has Hubble’s law, \( z \sim \sqrt{-2\Psi(R)} = HR \), and \( H \) can be identified with the Hubble parameter measured by the comoving observer in the Newtonian chart. When we take a negative sign of \( U^1 \), we have the distance-blueshift relation. Similarly, one will find the following distance-redshift relation of the comoving sources in the RW chart parametrized by \( q_0 \),

\[ R = \frac{1}{H} \left[ 1 - \frac{1}{1 + z} \sqrt{-q_0 + (1 + q_0)(1 + z)^2} \right] \approx \frac{\sqrt{-q_0}}{H} = \frac{z}{H_0} \text{ for } z \ll 1. \]  

(3.13)

Note that the observed Hubble parameter, \( H_0 \), is chart-dependent and that, when \( q_0 = -1 \), the distance-redshift relation coincides with that in the Newtonian chart. Therefore, when \( q_0 = -1 \), the same distance-redshift relation is interpreted in two different manners, namely the redshift in the RW chart due to the cosmic expansion, \( a(t) \), and the redshift in the Newtonian chart due to the gravitational potential, \( \Psi(T, R) \).

### 4 Revising of Gentry’s new redshift interpretation

We now try to interpret the cosmological redshift in terms of the gravitational potential, \( \Psi(T, R) \). In doing that, we have a problem that we cannot obtain an explicit form of the function, \( F(y) \), appearing in (2.4). However, since the Newtonian interpretation of the metric functions, \( M(T, R) \) and \( \Psi(T, R) \), is useful only when the gravitational potential, \( \Psi(T, R) \), is small. We therefore approximate the metric functions, \( M(T, R) \) and \( \Psi(T, R) \), near the coordinate origin, \( R=T=0 \), corresponding to \( r=0 \), \( t = t_0 \) in the RW chart, and will find that the approximated metric is in a static form and that the approximated gravitational potential, \( \Psi(R) \), satisfies the Poisson equation. We will then interpret the cosmological redshift (Hubble’s law) in terms of the gravitational potential, \( \Psi(T, R) \).

Assuming that \( H_0 |t_0 - t| \) and \( H_0 a_0 \Delta r \) are equally small quantities, we drop their cubic terms. One immediately finds that

\[ y = \frac{a(t)}{a_0} = 1 - H_0 \Delta t - \frac{1}{2} q_0 (H_0 \Delta t)^2 \equiv 1 - \Delta y, \]  

(4.1)
where $\Delta t = t_0 - t$. Then we have

\[
H_0 T' = F(y) + \frac{1}{(-2k)} \log(1 - K r^2)
\]

\[
= F(1) - \frac{dF}{dy}(1) \Delta y + \frac{1}{2} \frac{d^2 F}{dy^2}(\Delta y)^2 + \frac{1}{2} H_0^2 R^2
\]

\[
= F(1) - H_0 \Delta t + \frac{q_0 - 1}{2} (H_0 \Delta t)^2 + \frac{1}{2} H_0^2 R^2,
\]

which is solved for $\Delta t$ as

\[
H_0 \Delta t = H_0 \Delta T' + \frac{q_0 - 1}{2} (H_0 \Delta T')^2 + \frac{1}{2} H_0^2 R^2,
\]

where $H_0 \Delta T' = F(1) - H_0 T'$. Moreover, $y$ is expressed in terms of $\Delta T'$ and $R$ as

\[
y = 1 - H_0 \Delta T' - \frac{q_0 - 1}{2} (H_0 \Delta T')^2 - \frac{1}{2} H_0^2 R^2.
\]

With (4.4), we now have an approximate of the metric (2.5) as

\[
ds^2 = -[1 + 2\Psi(R)] (d\Delta T')^2 + \left[1 - \frac{2M(R)}{R}\right]^{-1} dR^2 + R^2 d\Omega^2,
\]

where

\[
\Psi(R) = \frac{1}{2} q_0 H_0^2 R^2, \quad M(R) = \frac{1}{2} (\Omega_0 + \Omega_V) H_0^2 R^3,
\]

\[
Q(\Delta T') = 2(q_0 - 1) H_0 \Delta T' + \left[2 + \frac{(3\gamma - 8)(3\gamma - 2)}{2} \Omega_0 + 8\Omega_V + \frac{(3\gamma - 2)^2}{2} \Omega_0^2 + (2 - 3\gamma) \Omega_0 \Omega_V + 2\Omega_V^2\right] (H_0 \Delta T')^2.
\]

A proper time, $T$, of the comoving observer, $R = 0$, is given by

\[
T = - \int_0^{\Delta T'} \sqrt{1 + Q(\tau)} \, d\tau,
\]

and we finally obtain the approximated metric as

\[
ds^2 = -[1 + 2\Psi(R)] dT^2 + \left[1 - \frac{2M(R)}{R}\right]^{-1} dR^2 + R^2 d\Omega^2.
\]

We find that the gravitational potential, $\Psi(R)$, satisfies the following Poisson equation,

\[
\nabla^2 \Psi = 3q_0 H_0^2 = 4\pi (\rho_0 + 3P_0 + \rho_V + 3P_V),
\]

which can be identified with the Newtonian approximation of the $(0,0)$-component of the Einstein equation,

\[
R_{00} = 8\pi \left(T_{00} - \frac{T^a_a}{2} g_{00}\right).
\]
The metric function, \( M(R) \), can be identified with the mass function as follows,

\[
\frac{dM}{dR} = \frac{3}{2}(\Omega_0 + \Omega_V)H_0^2R^2 = 4\pi R^2(\rho_0 + \rho_V). \tag{4.11}
\]

Moreover, we find a further physical role of \( \Psi(R) \) and \( M(R) \) as follows. A tangent vector, \( T^a \), of the apparent horizon trajectory (see Section 2), \( x_{\text{AH}}^a(t) \), is given by \( T^a = \partial x_{\text{AH}}^a/\partial t \) in the RW chart. A norm of \( \Psi(R) \) is given by

\[
\sqrt{g_{ab}T^aT^b} = -1 + \frac{q_0^2}{(\Omega_0 + \Omega_V)^2} = -1 + \left[ \frac{\Psi(R)}{M(R)} \right]^2,
\]

without any approximation. That is, if the present time is a mass-dominant era, \( |M(R)| > |R\Psi(R)| \), the apparent horizon trajectory is timelike.

It may be interesting to note that, though the metric (4.8) is derived as an approximated form of the FRW metric, it coincides with exact solutions of the Einstein equation with perfect fluid matter in the following particular cases.

- \( \gamma\Omega_0 = 0 \)

In this case, we have the de Sitter space-time for \( q_0 < 0 \) and the anti-de Sitter space-time for \( q_0 > 0 \).

- \( q_0 = 0 \)

In this case, we have Einstein’s static universe with \( \rho_V = (3\gamma/2 - 1)\rho_0 \). The same metric is also interpreted as a solution of the Tolman-Oppenheimer-Volkoff equation with the equation of state,

\[
P_{\text{eff}} = -\frac{1}{3}\rho_{\text{eff}}, \quad \rho_{\text{eff}} = \rho_0 + \rho_V = \text{constant}. \tag{4.13}
\]

Note that the perfect fluid with \( (\rho_{\text{eff}}, P_{\text{eff}}) \) satisfies all the energy conditions, i.e., the weak energy condition, the strong energy condition and the dominant energy condition.

Let us obtain the distance-redshift relation on the approximated space-time (4.8). We will show below that the redshift is independent of the mass function, \( M(R) \). The radial geodesic tangent is given by

\[
\begin{align*}
\text{Timelike : } U^a &= \left( \sqrt{\frac{1 + v^2}{1 + q_0H_0^2R^2}}, \pm \sqrt{\frac{1 - (\Omega_0 + \Omega_V)H_0^2R^2}{1 + q_0H_0^2R^2}}, 0, 0 \right), \\
\text{Ingoing null : } K^a &= \left( \frac{1}{1 + q_0H_0^2R^2}, -\sqrt{\frac{1 - (\Omega_0 + \Omega_V)H_0^2R^2}{1 + q_0H_0^2R^2}}, 0, 0 \right),
\end{align*}
\tag{4.14}
\]

where \( v \) is a constant along the geodesic and denotes a peculiar velocity of the source.

When \( q_0 \leq 0 \) and \( v = 0 \), the distance-redshift relation of the sources becomes

\[
\begin{align*}
z &= \frac{-q_0H_0R}{1 - \sqrt{-q_0H_0^2R^2}} \\
&= \frac{\sqrt{-2\Psi(R)}}{1 - \sqrt{-2\Psi(R)}} \approx \sqrt{-2\Psi(R)} \quad \text{for } |\Psi(R)| << 1, \tag{4.15}
\end{align*}
\]

\[
R = \frac{1}{\sqrt{-q_0H_0}} \frac{z}{1 + z} \approx \frac{z}{\sqrt{-q_0H_0}} \quad \text{for } z << 1,
\]

\[7\]
and the comoving observer in the approximated Newtonian chart (4.18) measures an effective Hubble parameter, \( H_{\text{eff}} = \sqrt{-q_0} H_0 \). In the de Sitter space-time, the above results hold exactly.

On the other hand, when \( q_0 > 0 \), no family of the comoving geodesics \((v = 0)\) exists. In this case, the redshift, \( z \), measured by the comoving observer, \( R = 0 \), is given by

\[
1 + z = \frac{\sqrt{1 + v^2 + \sqrt{v^2 - q_0 H_0^2 R^2}}}{1 + q_0 H_0^2 R^2} = \frac{\sqrt{1 + v^2 + \sqrt{v^2 - 2\Psi(R)}}}{1 + 2\Psi(R)},
\]

(4.16)

and the distance-redshift relation formally becomes

\[
R = \frac{1}{H_0 \sqrt{q_0} (1 + z)} \sqrt{-1 + 2(1 + v^2)^{1/2}(1 + z) - (1 + z)^2}
\]

\[
\approx \frac{1}{H_0 \sqrt{q_0}} \left(1 - \frac{z}{2}\right) \sqrt{2(1 + v^2)^{1/2} - 2} \text{ for } z << 1.
\]

(4.17)

It should be noted that \( z \) becomes negative when \( v \) is smaller than the \( R \)-dependent critical value, \( v_c = \sqrt{2\Psi(R) + \Psi(R)^2} \approx \sqrt{2\Psi(R)} \), and that, in contrast to the previous negative \( q_0 \) case, we formally have the following tilted Hubble law,

\[
R = R_0(v) + \dot{H}_0(v)z, \quad R_0(v) = \frac{\sqrt{2(1 + v^2)^{1/2} - 2}}{H_0 \sqrt{q_0}}, \quad \dot{H}_0(v) = -\frac{\sqrt{1 + v^2}^{1/2} - 1}{H_0 \sqrt{2q_0}}.
\]

(4.18)

One may think that this tilted Hubble law is inconsistent with Hubble’s law, \( z = H_0 R \). However, it is not necessarily the case because the integration constant, \( v \), is constant along each geodesic and is not necessarily a global constant. For example, the comoving geodesic, \( u^a = \delta^a_0 \), in the closed RW chart of the de Sitter space-time has its components, \( U^a \), in the Newtonian chart parametrized by

\[
v(T, R) = \frac{HR}{\sqrt{\cosh^2 HT - H^2 R^2 \sinh^2 HT}}, \quad U^a \partial_a v = 0.
\]

(4.19)

That is, the “constant” depends on both \( T \) and \( R \). Similarly, the comoving geodesic, \( u^a = \delta^a_0 \), in the open RW chart of the anti-de Sitter space-time,

\[
ds^2 = -dt^2 + \sin^2 Ht \left(\frac{dv^2}{1 + H^2 r^2} + r^2 d\Omega^2\right), \quad H^2 = -\frac{8}{3} \pi \rho_V > 0,
\]

(4.20)

has its components, \( U^a \), in the Newtonian chart,

\[
ds^2 = -(1 + H^2 R^2) dT^2 + (1 + H^2 R^2)^{-1} dR^2 + R^2 d\Omega^2,
\]

(4.21)

parametrized by the “constant”

\[
v(T, R) = \frac{HR}{\sqrt{\sin^2 HT - H^2 R^2 \cos^2 HT}} \quad U^a \partial_a v = 0.
\]

(4.22)

For generic geodesics in the Newtonian chart, one can adopt different values of \( v \), however, the choice of globally vanishing \( v \) is forbidden when \( q_0 > 0 \).
Finally, we briefly discuss the Newtonian interpretation of the integration constant, $v$. In the slow motion limit, the radial timelike geodesic is approximated as

$$\frac{d^2 R}{dT^2} = -\frac{\partial \Psi}{\partial R} = -q_0 H_0^2 R,$$

$$\rightarrow \left(\frac{dR}{dT}\right)^2 + q_0 H_0^2 R^2 = v^2, \quad v = \text{integration constant},$$

$$\rightarrow \frac{dR}{dT} = \pm \sqrt{v^2 - q_0^2 H_0^2 R^2},$$

which is compared with (4.14). One finds that when $q_0 > 0$, $v^2$ plays a role of a conserved total energy of the harmonic oscillator and, therefore, does not vanish unless $R = dR/dT = 0$.

5 Summary and discussion

In this paper, we have obtained a new expression of the FRW metric in the Newtonian chart and have found that the apparent horizon appears as a coordinate singularity. The appearance of this coordinate singularity may be understood in the following way. Let $u$ and $v$ be null coordinates such that ingoing null geodesics are represented by the equation, $u =$-constant, and that $v$ is an affine parameter of the ingoing null geodesic. Since the geodesically incompleteness indicates the initial singularity in the FRW metric, $v$ does not have any coordinate singularity. When the further coordinate transformation, $v \rightarrow R$, is done, we have a coordinate singularity if the coordinate transformation becomes singular, i.e., $|\partial R/\partial v| \rightarrow 0$ or $\infty$, and the previous condition, $\partial R/\partial v = 0$, is equivalent with the existence of the apparent horizon. After interpreting the redshift in the Milne universe and the de Sitter space-time in Gentry’s manner, we have tried to obtain the distance-redshift relation in the approximated Newtonian chart in terms of the Newtonian potential, $\Psi(R) = q_0 H_0^2 R^2/2$, and the peculiar velocity, $v$, of the source. We have found that the sign of the deceleration parameter, $q_0$, significantly affects the redshift interpretation in the Newtonian chart. That is, when $q_0$ is positive, we have no natural family of the timelike geodesics in the approximated Newtonian chart in contrast to the negative $q_0$ cases.

Now we discuss the implication of our results in the numerical relativity. Note that one often adopts particular coordinates in the numerical relativity, e.g., the maximal slice, in order to avoid the numerical divergence due to the singularities. In such numerically adopted charts, even the well-known metric will take an unfamiliar form, and, moreover, one may have unexpected numerical coordinate singularities. It is therefore useful to study how the well-known space-time is expressed in the unfamiliar charts. As far as an empty or approximately empty space-time is concerned, one may have comparatively few difficulties in interpreting the numerical results. For example, the appearance of the null apparent horizon will be interpreted as a strong evidence of the occurrence of the de Sitter expansion, e.g., the inflationary era, in numerically studying the dynamical and global evolution of the early universe. However, when the existence of the matter cannot be negligible, it is difficult to distinguish the isotropic, homogeneous space-time from generic isotropic, inhomogeneous space-time in the Newtonian chart, as has been shown in this paper.

In order to examine further properties of the apparent horizon, we calculate a trace of the
extrinsic curvature and find that

\[ K_a^a = -3H_0 \sqrt{\Omega_0 \left( \frac{a_0}{a} \right)^{3\gamma} - k \left( \frac{a_0}{a} \right)^2 + \Omega_V}, \]  

(5.1)
in the FW chart and that

\[ K_a^a = \frac{3}{2} \gamma \Omega_0 H_0 \left( \frac{a_0}{a} \right)^{3\gamma} H_0^2 R^2 \sqrt{\frac{\Omega_0 \left( \frac{a_0}{a} \right)^{3\gamma} - k \left( \frac{a_0}{a} \right)^2 + \Omega_V \left[ 1 - k \left( \frac{a_0}{a} \right)^2 H_0^2 R^2 \right] - \left[ \Omega_0 \left( \frac{a_0}{a} \right)^{3\gamma} + \Omega_V \right] H_0^2 R^2}{1 - \left[ \Omega_0 \left( \frac{a_0}{a} \right)^{3\gamma} + \Omega_V \right] H_0^2 R^2}}, \]  

(5.2)
in the Newtonian chart. One finds that \( K_a^a \) is homogeneous in the RW chart and speculates that the space-time itself is homogeneous in contrast to the case in the Newtonian chart in which \( K_a^a \) is inhomogeneous. Eardley and Smarr\[10]\] have defined a crushing singularity as

\[ |K_a^a| \rightarrow \infty \text{ uniformly.} \]  

(5.3)

In the RW chart, one immediately finds that \( a = 0 \) is a crushing singularity generically. On the other side, the generic singularity, \( a = 0 \), is outside the coordinate singularity, \( R = 2M(T,R) \), in the Newtonian chart, and the divergence of \( K_a^a \) appears only at the apparent horizon, \( R = 2M(T,R) \), unless \( \gamma \Omega_0 = 0 \). This divergence does not occur on the hypersurface, \( T = \text{constant} \), in general and is not a crushing singularity. However, one should remember that the motivation of defining the crushing singularity is the conjecture that the singular behavior of the adopted time function such that

\[ |K_i^j| \sim \frac{1}{2} \left| g^{ij} \frac{1}{\sqrt{-g_{00}}} \partial_0 g_{ij} \right| \rightarrow \infty, \]  

(5.4)
may correspond to the appearance of a singularity. In this sense, the divergence of \( K_a^a \) at the apparent horizon, \( R = 2M(T,R) \), may be interpreted as the appearance of the physical singularity by the observer in the Newtonian chart though it is not a true singularity. Moreover, we calculate a 3-dim scalar curvature, \( \mathcal{R} \), induced on the spacelike hypersurface, \( T = \text{constant} \), in the Newtonian chart and find that

\[ \mathcal{R} = 6H_0^2 \left\{ \Omega_0 \left( \frac{a_0}{a} \right)^{3\gamma} + \Omega_V + \frac{\gamma \Omega_0 \left( \frac{a_0}{a} \right)^{3\gamma} - k \left( \frac{a_0}{a} \right)^2 + \Omega_V \left[ 1 - \left[ \Omega_0 \left( \frac{a_0}{a} \right)^{3\gamma} + \Omega_V \right] H_0^2 R^2 \right]}{1 - \left[ \Omega_0 \left( \frac{a_0}{a} \right)^{3\gamma} + \Omega_V \right] H_0^2 R^2} \right\}, \]  

(5.5)
which diverges at the apparent horizon, \( R = 2M(T,R) \), unless \( \gamma \Omega_0 = 0 \). That is, the apparent horizon in the Newtonian chart almost always appears not only as a coordinate singularity but also as an apparent singularity such that the extrinsic curvature and the 3-dim curvature simultaneously diverge unless \( \gamma \Omega_0 = 0 \).

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