Pullback attractors for stochastic Young differential delay equations

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in memory of Russell Johnson

Abstract

We study the asymptotic dynamics of stochastic Young differential delay equations under the regular assumptions on Lipschitz continuity of the coefficient functions. Our main results show that, if there is a linear part in the drift term which has no delay factor and has eigenvalues of negative real parts, then the generated random dynamical system possesses a random pullback attractor provided that the Lipschitz coefficients of the remaining parts are small.

Keywords: stochastic differential equations (SDE), Young integral, random dynamical systems, random attractors, exponential stability.

1 Introduction

Consider the stochastic differential delay equation of the form

\[ dy(t) = [Ay(t) + f(y_t)]dt + g(y_t)dZ(t), \quad y_0 = \eta \in C^{0,\beta_0}([-r, 0], \mathbb{R}^d) \subset C_r := C([-r, 0], \mathbb{R}^d), \]

where \( t \in \mathbb{R}^+ \), \( y_t \) is defined by \( y_t : [-r, 0] \to \mathbb{R}^d \), \( y_t(s) = y(t+s) \) for \( s \in [-r, 0] \), \( A \in \mathbb{R}^{d \times d} \) is a matrix, \( r \) is a constant delay, \( C_r := C([-r, 0], \mathbb{R}^d) \) is the space of continuous functions on \([-r, 0]\) valued in \( \mathbb{R}^d \), \( f \) and \( g \) are functions defined on \( C_r \) valued in \( \mathbb{R}^d \) and \( \mathbb{R}^{d \times m} \) respectively, and \( Z \) is a \( \mathbb{R}^m \)-valued stationary stochastic process on a probability space \((\Omega, F, P)\) which has almost sure all the realizations in the Hölder space \( C^{0,\nu} \) for \( \frac{1}{2} < \nu \leq 1 \), the initial condition belongs to the Hölder space \( C^{0,\beta_0}([-r, 0], \mathbb{R}^d) \). Equation (1.1) is understood in the path-wise sense using Young integration [22] for the stochastic term \( g(y_t)dZ(t) \), whereas the term \( [Ay(t) + f(y_t)]dt \) is defined by the classical Riemann-Stieltjes integration. For the notion of Young integral and its properties, as well as notions and properties of spaces of Hölder continuous functions and Hölder norms the reader is referred to Section 5 Appendix.

In this paper, we investigate the asymptotic behavior of solution of the delay system (1.1) under regular assumptions. Namely,

- **H1**: \( A \) has all eigenvalues of negative real parts;
- **H2**: \( f \) is globally Lipschitz continuous and thus has linear growth, i.e there exists constants \( C_f \) such that for all \( \xi, \eta \in C_r \)

\[ \|f(\xi) - f(\eta)\| \leq C_f\|\xi - \eta\|_\infty, [-r,0]; \]

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• $\textbf{H}_3$: $g$ is $C^1$ such that its Fréchet derivative is bounded and locally Lipschitz continuous, i.e. there exists $C_g$ such that for all $\xi, \eta \in C_r$

\[ \|Dg(\xi)\|_{L(C_r, \mathbb{R}^d)} \leq C_g, \]

and for each $M > 0$, there exists $L_M$ such that for all $\xi, \eta \in C_r$ satisfying

\[ \|\xi\|_{\infty, [-r, 0]}, \|\eta\|_{\infty, [-r, 0]} \leq M \]

one has

\[ \|Dg(\xi) - Dg(\eta)\|_{L(C_r, \mathbb{R}^d)} \leq L_M \|\xi - \eta\|_{\infty, [-r, 0]}. \tag{1.2} \]

Notice that the same question for non-delay Young differential equations is well-studied in \[19\], \[12\], \[11\], \[13\], where one can prove that the system generates a random dynamical system which possesses a random attractor. For the delay system (1.1), the existence and uniqueness of the solution and the generation of a random dynamical system is affirmed in \[14\] and \[10\], but the question on asymptotic stability is still open.

Our aim in this paper is to show that under the assumptions $\textbf{H}_1, \textbf{H}_2, \textbf{H}_3$, the system (1.1) will generate a random dynamical system by means of its solution flow, and furthermore it possesses a random pullback attractor if the nonlinear term and stochastic term are small. Specifically, Theorem 4.5 states that if all the eigenvalues of $A$ have negative real parts ($\textbf{H}_1$ holds) then, provided that the Lipschitz coefficients $C_f, C_g$ of the (perturbation) terms $f$ and $g$ are small, the random dynamical system generated by the equation (1.1) possesses a random pullback attractor. Although the result seems natural, its proof is rather technical which employs recently developed methods on semigroups and greedy times. In addition, we prove in Theorem 4.7 that, in case $g$ is bounded the assumption on the parameter $C_g$ as well as on the supremum norm of $g$ can be neglected in proving the existence of attractor. Moreover, Theorem 4.8 asserts that, in case $g$ is linear the attractor is a singleton which is simultaneously a random pullback and random forward attractor.

This paper is organized as follows. We present in section 2 a recurrence formula for the solution of deterministic delayed equation, hence a formula for estimating growth rate of solutions to the equation (2.1). Section 3 presents the generation of a random dynamical system from the delay equation (1.1). In Section 4 we present our main results on existence of a random pullback attractor for the generated random dynamical system. In Section 5 for convenience of the reader we present some notions and notations used throughout the paper, namely the notions of Young integrals, Hölder spaces, Hölder norms; two versions of Gronwall inequalities—discrete and continuous are also presented.

2 A recurrence formula for solutions of deterministic delay equation

In this section we consider the deterministic equation

\[ dy(t) = [Ay(t) + f(y_t)]dt + g(y_t)dx(t), \quad y_0 = \eta \in C^{0, \beta_0}([-r, 0], \mathbb{R}^d), \tag{2.1} \]

for some $1 - \nu < \beta_0 < \nu$, and $x$ belongs to the $C^{0,\nu}([0, T], \mathbb{R}^m)$ for all $T > 0$. By assumption, almost all realizations of $Z$ belong to $C^{0,\nu}$, hence (2.1) is a representative path-wise equation of the stochastic equation (1.1).

Due to \[10\], under the assumptions $\textbf{H}_1, \textbf{H}_2, \textbf{H}_3$, the system (2.1) has unique solution which belongs to $C^{\beta_0}([-r, T], \mathbb{R}^d) \cap C^{\beta}([0, T], \mathbb{R}^d)$ for all $T > 0$, for all $\beta_0 < \beta \leq \nu$.
From now on, we fix $\beta_0 \in (1 - \nu, \nu)$, $\beta \in (\beta_0, \nu)$ and put

$$K := \frac{1}{1 - 2^{1-(\beta+\nu)}} ,$$

$$K_0 := \frac{1}{1 - 2^{1-(\beta_0+\nu)}}$$

(see details of the constants in the appendix). The following proposition is recalled from [10] Lemmas 17.1, 17.2.

**Proposition 2.1** Let $h$ be a Lipschitz continuous function on $C_r$ with Lipschitz coefficient $L$ then for each $y \in C^\alpha([a-r, b], \mathbb{R}^d)$, $0 < \alpha \leq 1$, $0 \leq a < b$, we have

(i) $\|h(y)\|_{\infty, [a,b]} \leq \|h(0)\| + L\|y\|_{\infty, [a-r, b]}$, here $0$ denotes the zero element of $C_r$,

(ii) $\|h(y)\|_{\alpha, [a,b]} \leq L \|y\|_{\alpha, [a-r, b]}$.

Denote by $\Delta_n$ and $\Delta'_n$ the intervals $[nr, (n+1)r]$ and $[(n-1)r, (n+1)r]$, respectively. For each $0 < \alpha < 1$, we introduce the notation

$$\|h\|_{\alpha, [a,b]} := \|h\|_{\infty, [a,b]} + (b - a)^\alpha \|h\|_{\alpha, [a,b]} .$$

It is obvious that $\| \cdot \|_{\alpha, [a,b]}$ and $\| \cdot \|_{\infty, \alpha, [a,b]}$ are equivalent norms on $C^\alpha([a,b], \mathbb{R}^d)$. We also introduce the following notations:

- For real numbers $a_1, \ldots, a_n$ put $a_1 \wedge \ldots \wedge a_n := \min\{a_1, \ldots, a_n\}$, and $a_1 \vee \ldots \vee a_n := \max\{a_1, \ldots, a_n\}$;
- $L_f := \|A\| + C_f$ with $\|A\|$ being the norm of operator $A$, i.e $\|A\| := \sup_{\|x\| = 1} \frac{\|Ax\|}{\|x\|}$;
- $\kappa := 4L_f r + 2$.

In Proposition 2.2 below we prove a recurrence formula for the norm of the solution of (2.1) by using the continuous Gronwall lemma and the technique of greedy sequence of times like those in [13] with a modification for $\beta$-Hölder norm which is an appropriate norm to deal with the delay system as explained in [10].

**Proposition 2.2** The solution $y$ of the equation (2.1),

$$dy(t) = [Ay(t) + f(y_t)]dt + g(y_t)dx(t), \quad y_0 = \eta \in C^{\beta_0}([-r, 0], \mathbb{R}^d),$$

satisfies

$$\|y\|_{\beta, \Delta_n} \leq e^{4L_f r + \kappa N_n(x)} \left[\|y\|_{\beta, \Delta_{n-1}} + 4r \|f(0)\| \|g(0)\| C_g \right] - 4r \|f(0)\| \|g(0)\| C_g \right] (2.2)$$

for all $n \geq 1$, and $N_n(x)$ is estimated by

$$N_n(x) \leq 1 + [2(K + 1)C_g r^\nu - \beta]^{-1} \left[\|x\|_{\nu, \Delta_n}^{\frac{1}{\beta}} \right] (2.3)$$

Proof: Fix $r \leq a \leq b$. For $a \leq s < t \leq b$, using (2.2) and Proposition 2.1 provided $y \in C^{\beta}([a-r, b], \mathbb{R}^d)$ we have

$$\|y(t) - y(s)\|$$
\[
\begin{align*}
= & \left\| \int_s^t \left[ Ay(u) + f(y_u) \right] du + \int_s^t g(y_u) dx(u) \right\| \\
\leq & \int_s^t \left( L_f \| y_u \|_{\infty,-r,0} + \| f(0) \| \right) du + \left\| \int_s^t g(y_u) dx(u) \right\| \\
\leq & \int_s^t \left( L_f \| y_s \|_{\infty,-r,0} + L_f (u - s)^\beta \| y \|_{\beta,[s,u]} + \| f(0) \| \right) du \\
& + (t - s)^\beta \| y \|_{\nu,[s,t]} \left[ C_f \| y_s \|_{\infty,-r,0} + \| g(0) \| + KC_f (t - s)^\beta \| y \|_{\beta,[s-r,t]} \right] \\
\leq & (t - s)^\beta \left( L_f \| y_s \|_{\infty,-r,0} + L_f (s - a)^\beta \| y \|_{\beta,[a,s]} + \| f(0) \| \right) + L_f \int_s^t \| y \|_{\beta,[s,u]} du \\
& + (t - s)^\beta \| y \|_{\nu,[s,t]} \left[ C_f \| y_a \|_{\infty,-r,0} + \| g(0) \| + C_f (s - a)^\beta \| y \|_{\beta,[a,s]} + KC_f (t - s)^\beta \| y \|_{\beta,[s-r,t]} \right] \\
\leq & L_f \int_a^b \| y \|_{\beta,[a,u]} du + L_f (b - a) \| y \|_{\beta,[a,s]} + (b - a)^{1-\beta} \left( L_f \| y_a \|_{\infty,-r,0} + \| f(0) \| \right) \\
& + (b - a)^{1-\beta} \| y \|_{\nu,[a,b]} \times \\
& \times \left[ C_f \| y_a \|_{\infty,-r,0} + \| g(0) \| + C_f (b - a)^\beta \| y \|_{\beta,[a,b]} + KC_f (b - a)^\beta \left( \| y \|_{\beta,[a-r,a]} + \| y \|_{\beta,[a,b]} \right) \right] \\
\leq & L_f \int_a^b \| y \|_{\beta,[a,u]} du + L_f \int_a^b \| y \|_{\beta,[a,u]} du + (b - a)^{1-\beta} \left( L_f \| y_a \|_{\infty,-r,0} + \| f(0) \| \right) \\
& + (b - a)^{1-\beta} \| y \|_{\nu,[a,b]} \times \\
& \times \left[ C_f \| y_a \|_{\infty,-r,0} + \| g(0) \| + KC_f (b - a)^\beta \| y \|_{\beta,[a-r,a]} + (K + 1)C_f (b - a)^\beta \| y \|_{\beta,[a,b]} \right] \\
\leq & 2L_f \int_a^b \| y \|_{\beta,[a,u]} du + (b - a)^{1-\beta} \left( L_f \| y_a \|_{\infty,-r,0} + \| f(0) \| \right) \\
& + (b - a)^{1-\beta} \| y \|_{\nu,[a,b]} \times \\
& \times \left[ C_f \| y_a \|_{\infty,-r,0} + \| g(0) \| + KC_f (b - a)^\beta \| y \|_{\beta,[a-r,a]} + (K + 1)C_f (b - a)^\beta \| y \|_{\beta,[a,b]} \right]
\end{align*}
\]

Here we used the inequality \( \| y_u \|_{\infty,-r,0} \leq \| y_v \|_{\infty,-r,0} + (u - v)^\beta \| y \|_{\beta,[v,u]}, v \leq u \). Hence,

\[
\| y(t) - y(s) \| \leq (t - s)^{1-\beta} \left( L_f \| y_a \|_{\infty,-r,0} + L_f (s - a)^\beta \| y \|_{\beta,[a,s]} + \| f(0) \| \right) + L_f \int_s^t \| y \|_{\beta,[s,u]} du \\
+ (t - s)^\beta \| y \|_{\nu,[s,t]} \left[ C_f \| y_s \|_{\infty,-r,0} + \| g(0) \| + C_f (s - a)^\beta \| y \|_{\beta,[a,s]} + KC_f (t - s)^\beta \| y \|_{\beta,[s-r,t]} \right].
\]

Take the supremum on \([a, b]\), we get

\[
\| y \|_{\beta,[a,b]} \leq 2L_f \int_a^b \| y \|_{\beta,[a,u]} du + (b - a)^{1-\beta} \left( L_f \| y_a \|_{\infty,-r,0} + \| f(0) \| \right)
\]

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We fix \( n \geq 1 \) and note that (2.5) holds for all \([a, b] \subset \Delta_n = [nr, (n+1)r]\). Now, for \( \mu = \frac{1}{2(K+1)Cg} \), fixed, we construct on \( \Delta_n \) a greedy sequence of time \( t_i \) satisfies

\[
t_0 = nr, \quad t_{i+1} = \sup\{ t > t_i | (t-t_i)^{\nu-\beta} \| x \|_{\nu, [t_i, t]} \leq \mu \} \wedge (n+1)r.
\]

Since \( x \in C^{0, \nu-H^\alpha([0, T], \mathbb{R}^m)}, \)

\[
\left| \| x \|_{\nu, [0, \tau]} - \| x \|_{\nu, [0, \tau+h]} \right| \leq \max \left\{ \| x \|_{\nu, [\tau, \tau+h]}, \| x \|_{\nu, [\tau-h, \tau]} \right\} \rightarrow 0 \text{ as } h \rightarrow 0^+,
\]

the function \( \tau^{\nu-\beta} \| x \|_{\nu, [0, \tau]} \) is then continuous due to the continuity of each component in \( \tau \). Hence

\[
(t_{i+1} - t_i)^{\nu-\beta} \| x \|_{\nu, [t_i, t_{i+1}]} = \mu, \quad \forall 0 \leq i \leq N_n(x) - 2,
\]

(2.6)

\[
(t_{i+1} - t_i)^{\nu-\beta} \| x \|_{\nu, [t_i, t_{i+1}]} \leq \mu, \quad \text{for } i = N_n(x) - 1,
\]

(2.7)

where

\[
N_n(x) = N(\Delta_n, x) := 1 + \max\{ i : t_i < (n+1)r \}.
\]

We show that this counting function \( N_n(x) \) is the function furnishing the statements of the proposition, i.e. we show that for this \( N_n(x) \) the inequalities (2.2) and (2.3) are satisfied. First, we prove that \( N_n(x) \) is bounded and find a bound for it. Choose \( m = \frac{1}{\nu-\beta} > 1 \), one has

\[
[N_n(x) - 1] \mu^m = \sum_{i=0}^{N_n(x)-2} (t_{i+1} - t_i)^{\nu-\beta} \| x \|_{\nu, [t_i, t_{i+1}]}^m
\]

\[
\leq \sum_{i=0}^{N_n(x)-2} (t_{i+1} - t_i)^{m(\nu-\beta)} \| x \|_{\nu, [t_i, t_{i+1}]}^m
\]

\[
\leq \sum_{i=0}^{N_n(x)-2} (t_{i+1} - t_i) \| x \|_{\nu, \Delta_n}^m
\]

\[
\leq r \| x \|_{\nu, \Delta_n}^{\frac{1}{\nu-\beta}} < \infty.
\]

Hence,

\[
N_n(x) \leq 1 + \frac{r}{\mu^{\nu-\beta}} \| x \|_{\nu, \Delta_n}^{\frac{1}{\nu-\beta}} = 1 + [2(K+1)Cg r^\nu \mu^{\nu-\beta}] \| x \|_{\nu, \Delta_n}^{\frac{1}{\nu-\beta}}.
\]

Thus \( N_n(x) \) is bounded and the inequality (2.3) is proved.

By the construction, \( t_{i+1} - t_i \leq r \) for \( 0 \leq i \leq N_n(x) - 1 \), hence for all \([a, b] \subset [t_i, t_{i+1}]\) the inequality (2.5) leads to

\[
\| y \|_{\beta, [a, b]} \leq 2L_f \int_a^b \| y \|_{\beta, [a, u]} du + (b-a)^{1-\beta} \left( L_f \| y_{a} \|_{\infty, [-r, 0]} + \| f(0) \| \right)
\]

\[
+ \frac{1}{2(K+1)Cg} \left[ C_g \| y_{a} \|_{\infty, [-r, 0]} + \| g(0) \| \right] + \frac{1}{2r^\beta} (b-a)^{\beta} \| y \|_{\beta, [a-r, a]} + \frac{1}{2} \| y \|_{\beta, [a, b]}.
\]
Hence, for all \([a, b] \subset [t_i, t_{i+1}]\) we have

\[
\|y\|_{\beta, [a, b]} \leq 4L_f \int_a^b \|y\|_{\beta, [a, u]} \, du + 2(b - a)^{1-\beta} \left( L_f \|y\|_{\infty, [-r, 0]} + \|f(0)\| \right) \\
+ \frac{1}{(K + 1)C_g r^{\beta}} \left( C_g \|y\|_{\infty, [-r, 0]} + \|g(0)\| \right) + \|y\|_{\beta, [a-r, a]}.
\]

Consequently, let \(t \in [t_i, t_{i+1}]\) be arbitrary, we have

\[
\|y\|_{\beta, [t_i, t]} \leq 4L_f \int_{t_i}^t \|y\|_{\beta, [t_i, u]} \, du + 2(t - t_i)^{1-\beta} \left( L_f \|y\|_{\infty, [-r, 0]} + \|f(0)\| \right) \\
+ \frac{1}{(K + 1)C_g r^{\beta}} \left( C_g \|y\|_{\infty, [-r, 0]} + \|g(0)\| \right) + \|y\|_{\beta, [t_i-r, t_i]}.
\]

Using the Continuous Gronwall lemma \([5.3]\) we get

\[
\|y\|_{\beta, [t_i, t]} \leq \left[ 2(t - t_i)^{1-\beta} \left( L_f \|y\|_{\infty, [-r, 0]} + \|f(0)\| \right) \\
+ \frac{1}{(K + 1)C_g r^{\beta}} \right] + \|y\|_{\beta, [t_i-r, t_i]} \times \left[ 1 + 4L_f \int_{t_i}^t e^{4L_f(t-u)} \, du \right]
\]

\[
\leq e^{4L_f(t-t_i)} \left[ 2(t - t_i)^{1-\beta} \left( L_f \|y\|_{\infty, [-r, 0]} + \|f(0)\| \right) + \frac{1}{(K + 1)C_g r^{\beta}} \right] + \|y\|_{\beta, [t_i-r, t_i]}
\]

\[
\leq e^{4L_f(t-t_i)} \left[ 2L_f(t - t_i)^{1-\beta} \|y\|_{\infty, [-r, 0]} + \|g\|_{\beta, [t_i-r, t_i]} + \frac{1}{(K + 1)C_g r^{\beta}} \right]
\]

\[
+ e^{4L_f(t-t_i)} \left( 2(t - t_i)^{1-\beta} \|f(0)\| + \frac{1}{(K + 1)C_g r^{\beta}} \right) \cdot \|g(0)\|
\]

Therefore, by substituting \(t = t_{i+1}\) we get

\[
\|y\|_{\beta, [t_i, t_{i+1}]} \leq e^{4L_f(t_{i+1} - t_i)} \left[ 2L_f(t_{i+1} - t_i)^{1-\beta} \|y\|_{\infty, [-r, 0]} + \|g\|_{\beta, [t_i-r, t_i]} + \frac{1}{(K + 1)C_g r^{\beta}} \right]
\]

\[
+ e^{4L_f(t_{i+1} - t_i)} \left( 2(t_{i+1} - t_i)^{1-\beta} \|f(0)\| + \frac{1}{(K + 1)C_g r^{\beta}} \right) \cdot \|g(0)\|
\]

Consequently, taking into account the equality \(\|y\|_{\infty, [-r, 0]} + r^\beta \|y\|_{\beta, [t_i-r, t_i]} = \|y\|_{\beta, [t_i-r, t_i]}\) and the inequality \(t_{i+1} - t_i \leq r\), we get

\[
2r^\beta \|y\|_{\beta, [t_i, t_{i+1}]}
\]

\[
\leq e^{4L_f(t_{i+1} - t_i)} \left[ \left( 4L_f r^\beta (t_{i+1} - t_i)^{1-\beta} + \frac{2}{(K + 1)} \right) \|y\|_{\beta, [t_i-r, t_i]}
\]

\[
+ e^{4L_f(t_{i+1} - t_i)} \left( 4r^\beta (t_{i+1} - t_i)^{1-\beta} \|f(0)\| + \frac{2\|g(0)\|}{(K + 1)C_g} \right) \right] \cdot \|g(0)\|
\]

\[
\leq e^{4L_f(t_{i+1} - t_i)} (4L_f r + 2) \|y\|_{\beta, [t_i-r, t_i]} + e^{4L_f(t_{i+1} - t_i)} \left( 4r \|f(0)\| + \frac{\|g(0)\|}{C_g} \right)
\]

\[6\]
\[ e^{4L_f(t_{i+1}-t_i)} \left( e^{4L_f r + 2} - 1 \right) \|y\|_{\beta, [t_i, r, t_{i+1}]} + e^{4L_f(t_{i+1}-t_i)} \left( 4r \|f(0)\| + \|g(0)\| + C_g \right) \]

\[ = e^{4L_f(t_{i+1}-t_i)} (e^\kappa - 1) \|y\|_{\beta, [t_i, r, t_{i+1}]} + e^{4L_f(t_{i+1}-t_i)} \left( 4r \|f(0)\| + \|g(0)\| + C_g \right) \]

\[ \leq e^{4L_f(t_{i+1}-t_i) + \kappa} \|y\|_{\beta, [t_i, r, t_{i+1}]} - \|y\|_{\beta, [t_i, r, t_{i+1}]} + e^{4L_f(t_{i+1}-t_i)} \left( 4r \|f(0)\| + \|g(0)\| + C_g \right). \]

Now we evaluate norm of \( y \) on \([t_i + 1 - r, t_{i+1}]\) as follows

\[ \|y\|_{\beta, [t_i + 1 - r, t_{i+1}]} = \|y\|_{\infty, [t_i + 1 - r, t_{i+1}]} + r^\beta \|y\|_{\beta, [t_i + 1 - r, t_{i+1}]} \]

\[ \leq \|y\|_{\infty, [t_i - r, t_i]} + (t_{i+1} - t_i)^\beta \|y\|_{\beta, [t_i, t_{i+1}]} + r^\beta (\|y\|_{\beta, [t_i - r, t_i]} + \|y\|_{\beta, [t_i, t_{i+1}]}). \]

By induction we obtain, for any \( i = 0, \ldots, N_n(x) - 1, \)

\[ \|y\|_{\beta, [t_i + 1 - r, t_{i+1}]} \leq e^{4L_f(t_{i+1}-t_i) + \kappa} \|y\|_{\beta, [t_i - r, t_i]} + (t_{i+1} - t_i)^\beta \|y\|_{\beta, [t_i, t_{i+1}]} \]

Choose \( i = N_n(x) - 1; \) note that \([t_0 - r, t_0] = \Delta_n - 1 \) and \([t_{N_n(x)} - r, t_{N_n(x)}] = \Delta_n, \) we arrive at

\[ \|y\|_{\beta, \Delta_n} \leq e^{4L_f r + \kappa N_n(x)} \left[ \|y\|_{\beta, \Delta_{n-1}} + \left( 4r \|f(0)\| + \|g(0)\| + C_g \right) \right] - \left( 4r \|f(0)\| + \|g(0)\| + C_g \right). \]

Thus \([2.2]\) is proved and so is the proposition.

**Remark 2.3** We notice that while the solution of \([2.1]\) belongs to \( C^\beta \) on \([0, T], \) it only belongs to \( C^{\beta_0} \) but not necessarily belongs to \( C^\beta \) on \([-r, 0]. \) Therefore we have to make separate estimations for solutions of \([2.1]\) on the first interval \([0, r]. \) By a slight modification of the proof of Proposition \(2.2\) we get the following estimates.

1. It is evident that if we replace \( \beta \) by \( \beta_0 \) and \( K \) by \( K_0, \) then \([2.2]\) and \([2.3]\) hold for all \( n \geq 0. \) In particular, letting \( n = 0 \) we get an estimate in the \( \| \cdot \|_{\beta_0, [0, r]} \) norm for the solution of \([2.1]\) on \([0, r]\) as follows

\[ \|y\|_{\beta_0, [0, r]} \leq e^{4L_f r + \kappa N_0(x)} \left[ \|y\|_{\beta_0, [-r, 0]} + \left( 4r \|f(0)\| + \|g(0)\| + C_g \right) \right] - \left( 4r \|f(0)\| + \|g(0)\| + C_g \right), \] (2.8)

where

\[ N_0(x) \leq 1 + \frac{1}{2(K_0 + 1)C_0 r^\nu} \|x\|_{\nu, [0, r]}^{\frac{1}{\nu - \beta_0}}. \] (2.9)
2. Similar to (2.4), for \(0 \leq s < t \leq r\) we have

\[
\|y(t) - y(s)\| = \left\| \int_s^t [Ay(u) + f(y(u))] \, du + \int_s^t g(y(u)) \, dx(u) \right\|
\leq \int_s^t (L_f \|y_u\|_{\infty,[-r,0]} + \|f(0)\|) \, du + \left\| \int_s^t g(y_u) \, dx(u) \right\|
\leq (t - s) (L_f \|y\|_{\infty,[-r,r]} + \|f(0)\|)
+ (t - s)\nu \|x\|_{\nu,[s,t]} \left[ C_g \|y\|_{\infty,[-r,r]} + \|g(0)\| + K_0 C_g r^{\beta_0} \|y\|_{\beta_0,[-r,r]} \right]
\leq D \left( t - s \right) + \left( t - s \right)^\nu \|x\|_{\nu,[s,t]} \left[ 1 + \|y\|_{\infty,[-r,r]} + r^{\beta_0} \|y\|_{\beta_0,[-r,r]} \right]
\leq D (t - s)^\beta \left( r^{1 - \beta} + r^{\nu - \beta} \|x\|_{\nu,0,r} \right) \left( 1 + \|y\|_{\beta_0,[-r,0]} + \|y\|_{\beta_0,0,r} \right)
\]

for some positive constants \(D\). Combining this with (2.8) and changing the constant \(D\) to a bigger one if necessary, we obtain the following estimate in the \(\|\cdot\|_{\beta,[0,r]}\) norm for the solution of (2.1) on \([0, r]\)

\[
\|y\|_{\beta,[0,r]} \leq D \left( 1 + \|x\|_{\nu,0,r} \right) \left[ 1 + \|y\|_{\beta_0,[-r,0]} + \|y\|_{\beta_0,0,r} \right]
\leq D \left( 1 + \|x\|_{\nu,0,r} \right) \left( 1 + \|y\|_{\beta_0,[-r,0]} \right) e^{KN_0(x)}
\leq D \left( 1 + \|x\|_{\nu,0,r} \right) \left( 1 + \|y\|_{\beta_0,[-r,0]} \right) e^{D \|x\|_{\nu,0,r}^\nu}. \tag{2.10}
\]

3 Generation of random dynamical systems

In this section, we present the generation of random dynamical systems for equation (1.1) under general noise \(Z\), a stationary stochastic process with almost sure all the realizations in \(C^0,\nu\). Namely, similar to (1.1) but for simplicity of presentation we consider the equation

\[
dy(t) = F(y_t) \, dt + g(y_t) \, dZ(t), \quad y_0 = \eta \in C^{0,\beta_0}([-r,0], \mathbb{R}^d), \tag{3.1}
\]

where \(\beta_0 > 1 - \nu\) is an arbitrary fixed constant. Note that (1.1) is a special case of (3.1) with the coefficient \(F(y_t)\) changed to \(Ay(t) + f(y_t)\). The initial condition is considered in the separable space \(C^{0,\beta_0}([-r,0], \mathbb{R}^d)\), the condition \(\beta_0 > 1 - \nu\) is needed to assure existence and uniqueness of solution to (3.1) in the \(C^0,\beta_0\) space (see [10]). First, we recall the definition of random dynamical system \(\theta\), which is a measurable mapping \(\theta: \mathbb{R} \times \Omega \to \Omega\) such that \(\theta_t: \Omega \to \Omega\) is \(\mathbb{P}\)-preserving, i.e. \(\mathbb{P}(B) = \mathbb{P}(\theta_t^{-1}(B))\) for all \(B \in \mathcal{F}, t \in \mathbb{R}\), and \(\theta_{t+s} = \theta_t \circ \theta_s\) for all \(t, s \in \mathbb{R}\). Let \(S\) be a Polish space, a continuous random dynamical system

\[
\varphi: \mathbb{R} \times \Omega \times S \to S, \quad (t, \omega, y_0) \mapsto \varphi(t, \omega, y_0)
\]

is then defined as a measurable mapping which is also continuous in \(t\) and \(y_0\) such that the cocycle property

\[
\varphi(t + s, \omega) = \varphi(t, \theta_{s\omega}) \circ \varphi(s, \omega), \quad \forall t, s \in \mathbb{R}, \forall \omega \in \Omega \tag{3.2}
\]

\[
\varphi(0, \omega, \cdot) = \text{Id}_S \tag{3.3}
\]

is satisfied (see Arnold [3]).
To study the existence of the random pullback attractor of the system (1.1) in the next section, we need to construct a canonical space for $Z$ which is equipped by a metric dynamical system $\theta$. In the following, we follow [3 Theorem 5] to state a similar result for stochastic valued in $C^\alpha$ for some $\alpha \in (0,1]$. Recall that $C^{0,\alpha}([a,b],\mathbb{R}^m)$ is the closure of smooth path from $[a,b]$ to $\mathbb{R}^m$ in $\alpha$-Hölder norm. It is known that $C^{0,\alpha}([a,b],\mathbb{R}^m)$ is a separable Banach space, see [16]. Denote by $C^{0,\alpha}(\mathbb{R},\mathbb{R}^m)$ the space of all $x : \mathbb{R} \to \mathbb{R}^m$ such that $x|_I \in C^{0,\alpha}([a,b],\mathbb{R}^m)$ for each compact interval $I \subset \mathbb{R}$, equipped with the compact open topology given by the $\alpha$-Hölder norm, i.e. topology generated by the metric:

$$d(x,y) := \sum_{n \geq 1} \frac{1}{2^n} \|x - y\|_{\infty,\alpha,[-n,n]} \wedge 1.$$ 

Then $C^{0,\alpha}(\mathbb{R},\mathbb{R}^m)$ is a separable metric space. Denote by $C^{0,\alpha}_0(\mathbb{R},\mathbb{R}^m)$ the subspace of $C^{0,\alpha}(\mathbb{R},\mathbb{R}^m)$ containing paths which vanish at 0. It is evident that for $x \in C^{0,\alpha}_0(\mathbb{R},\mathbb{R}^m)$

$$\|x\|_{\alpha,[-n,n]} \leq \|x\|_{\infty,\alpha,[-n,n]} \leq (1 + n^\alpha) \|x\|_{\alpha,[-n,n]}$$

for all $n$, and $C^{0,\alpha}_0(\mathbb{R},\mathbb{R}^m)$ is closed in $C^{0,\alpha}(\mathbb{R},\mathbb{R}^m)$. The following Theorem is due to [4].

**Theorem 3.1** Assume that we have a process $\tilde{X}$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and valued in $(C^{0,\alpha}_0(\mathbb{R},\mathbb{R}^m), \mathcal{B})$ with $\mathcal{B}$ being Borel $\sigma$-algebra. Assume further that $\tilde{X}$ has stationary increment. Then there exist a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ and a process $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (C^{0,\alpha}_0(\mathbb{R},\mathbb{R}^m))$ which has the same law as $\tilde{X}$ and satisfies the property:

$$X(t+s, \omega) = X(s, \omega) + X(t, \theta_s \omega), \quad \forall \omega \in \Omega, \quad t, s \in \mathbb{R}.$$ 

**Proof:** We denote by $(\Omega, \mathcal{F})$ the space $(C^{0,\alpha}_0(\mathbb{R},\mathbb{R}^d), \mathcal{B})$ and by $\mathbb{P}$ the distribution of $\tilde{X}$ on $\Omega$. On $(\Omega, \mathcal{F}, \mathbb{P})$ we set

$$\theta : \mathbb{R} \times \Omega \to \Omega, \quad \theta(t, \omega)(s) = \theta_s \omega(s) := \omega(t+s) - \omega(t),$$

and define the process $X : X(\omega)(t) = \omega(t)$ for all $\omega \in \Omega$. The properties of $(\Omega, \mathcal{F}, \mathbb{P})$ and $X$ are obtained by arguments similar to that of [4 Theorem 5, p. 8].

Now we consider the systems (1.1) and (3.1) with $Z$ defined on the canonical space constructed as above. Moreover, we assume that $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is ergodic and

$$\Gamma(\beta) := \left( E \left\| Z \right\|_{\beta,[-r,0]}^{1-\beta} \right)^{\nu-\beta} < \infty. \quad (3.4)$$

Next, we are going to study the generation of random dynamical system from the system (3.1). Note that the Young integral satisfies the shift property with respect to $\theta$ (see for instance [2]), i.e.

$$\int_a^b x(u) d\omega(u) = \int_{a-r}^{b-r} x(u + r) d\theta_r \omega(u).$$

and due to [10] the equation (3.1) possesses a unique solution $y(t, x, \eta)$ in $C^{0,\beta_0}([-r, \infty), \mathbb{R}^d)$. Moreover, the solution is continuous w.r.t $\eta$ and belongs to $C^{\beta}([0, \infty), \mathbb{R}^d)$ for $\beta_0 < \beta < \nu$. The following conclusion is followed from [14].

**Theorem 3.2** Under assumption (H2), (H3) the system (3.1) generates a random dynamical system defined by

$$\varphi : \mathbb{R}^+ \times \Omega \times C^{0,\beta_0}([-r,0], \mathbb{R}^d) \to C^{0,\beta_0}([-r,0], \mathbb{R}^d), \quad \varphi(t, x, \eta)(s) := y(t + s, x, \eta).$$

Moreover, $\varphi$ is continuous.

**Corollary 3.3** Under assumption (H1), (H2), (H3) the stochastic delay equation (1.1) generates a continuous random dynamical system with the phase space $C^{0,\beta_0}([-r,0], \mathbb{R}^d)$. 

9
4 Random pullback attractors

This section is devoted to the main result of our paper. We will show that under some natural conditions the random dynamical system generated by the stochastic Young differential delay equation (1.1) possesses a random pullback attractor. Note that for the classical theory of dynamical systems one usually studies forward attractor, but in the framework of the theory of random dynamical systems the notion of random pullback attractor seems more appropriate (see e.g. [8] and the references therein). The relation between concepts of attractors is studied in [5], [6], [15]. Particularly in relation to the nonautonomous setting with compact topological parameter space, there is a work by [18] which proves that the (nonautonomous) pullback attractor of nonautonomous dynamical systems (in terms of skew product flows) coincides with their so-called Lyapunov attractors.

First we recall the classical notion of random pullback attractors for a general random dynamical system. Let $S$ be a Polish space, i.e. a separable topological space whose topology is metrizable with a complete metric $d$. Denote by $\mathcal{B}$ the Borel-$\sigma$ algebra on $S$. For each $y \in S$, $E \subset S$, we define $d(y, E) = \inf \{d(y, z) | z \in E\}$. The Hausdorff distance between two nonempty subsets $E, F$ of $S$ is defined by $d(E|F) = \sup \{\inf \{d(y, z) | z \in F\} | y \in E\}$. Recall that a set $\hat{M} = \{M(x)\}_{x \in \Omega}$ is called a random set if it belongs to $\mathcal{F} \times \mathcal{B}$. In the case that $M(x)$ is closed or compact for each $x \in \Omega$, that the mapping $x \mapsto d(y, M(x))$ is measurable for every $y \in S$ ensures the measurability of $M$. $M$ is then said to be closed or compact random set. Given a random dynamical system $\varphi$ on $(\Omega, \mathcal{F}, \mathbb{P})$, valued on $S$. We recall the following definition from [15], [6].

**Definition 4.1** Suppose that $\varphi$ is a RDS on a Polish space $S$ and $\mathcal{D}$ is a non-empty family of subsets of $\Omega \times S$. Then a set $\mathcal{A} \subset \Omega \times S$ is a random pullback attractor for $\mathcal{D}$ if

(i) $\mathcal{A}$ is a compact random set,

(ii) $\mathcal{A}$ is strictly $\varphi-$invariant, i.e. $\varphi(t, x)A(x) = A(\theta_t x) \mathbb{P}$-almost surely for every $t \geq 0$,

(iii) $\mathcal{A}$ attracts $\mathcal{D}$ in the pullback sense, i.e for every $\hat{D} \in \mathcal{D},$

$$\lim_{t \to \infty} d(\varphi(t, \theta_{-t} x)\hat{D}(\theta_{-t} x)|\mathcal{A}(x)) = 0, \quad \mathbb{P} - a.s.$$ (4.1)

It is known that the existence of the random pullback attractor follows from the existence of the random pullback absorbing set (see [15] Theorem 3.5], [19] Theorem 2.4]), i.e. a compact random set $B$ such that $\mathbb{P}$-almost all $x$, for each $\hat{D} \in \mathcal{D}$ there exists a time $t_0(x, \hat{D})$ such that for all $t > t_0(x, \hat{D}),$

$$\varphi(t, \theta_{-t} x)\hat{D}(\theta_{-t} x) \subset B(x).$$

An universe $\mathcal{D}$ is a family of random sets which is closed w.r.t. inclusions (i.e. if $\hat{D}_1 \in \mathcal{D}$ and $\hat{D}_2 \subset \hat{D}_1$ then $\hat{D}_2 \in \mathcal{D}$). Given a universe $\mathcal{D}$ and a compact random pullback absorbing set $B \in \mathcal{D}$, there exists a unique random pullback attractor in $\mathcal{D}$, given by

$$\mathcal{A}(x) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \varphi(t, \theta_{-t} x)\overline{B(\theta_{-t} x)}.$$ (4.2)

In our setting, the problem of generation of random dynamical system by a stochastic Young differential delay equation is treated in Section 3 and our equation (1.1) generated a random dynamical system with the phase space being the function space $C^{0, \beta_0}([-r, 0], \mathbb{R}^d)$. We define the universe $\mathcal{D}$ to be a family of tempered random sets $D(x)$ which is contained in a ball $B(0, \rho(x))$ a.s., where the radius $\rho(x)$ is a tempered random variable (see Appendix).

Here we notice that while the definition of pullback attractor is formulated for a general universe $\mathcal{D}$, or even for the case $\mathcal{D}$ being an abstract collection of subsets of the product of the phase space and the probability space of the random dynamical system (see [6] Definition 9], [8] Definition 15], in practical concrete problems one needs to impose additional conditions on the growth rate of the
size of the random sets $\hat{D}(\cdot)$. Thus one may consider the universe of tempered compact random sets (see \cite[Theorem 5.10]{15}, \cite[Theorem 2.4]{19}), or the universe of deterministic bounded sets (see \cite[Definition 1.3]{21}). In this paper we follow \cite{15,19} in imposing temperedness condition on the universe $D$ as above.

Now, to understand the dynamics of the random dynamical system generated by the stochastic Young differential delay equation (1.1) we need to study the path-wise deterministic equation of (1.1). Let us look back at the system (2.1)

\[ dy(t) = [Ay(t) + f(y_t)]dt + g(y_t)dx(t), \quad y_0 = \eta \in C^{0,\beta}([-r, 0], \mathbb{R}^d), \]

with the assumptions $H_1, H_2, H_3$. Put $\Phi(t) := e^{At}$. By the variation of constants formula, the solution $y(t)$ of (2.1) satisfies

\[ y(t) = \Phi(t - t_0)y(t_0) + \int_{t_0}^{t} \Phi(t - s)f(y_s)ds + \int_{t_0}^{t} \Phi(t - s)g(y_s)dx(s), \quad \forall t \geq t_0 \geq 0. \] (4.3)

Indeed, put $z(t) = \Phi^{-1}(t)y(t)$ then

\[
\begin{align*}
dz(t) &= d\Phi^{-1}(t)y(t) + \Phi^{-1}(t)dy(t) \\
&= -\Phi^{-1}(t)d\Phi(t)\Phi^{-1}(t)y(t) + \Phi^{-1}(t)\left[(Ay(t) + f(y_t))dt + g(y_t)dx(t)\right] \\
&= -\Phi^{-1}(t)Ay(t)dt + \Phi^{-1}(t)\left[(Ay(t) + f(y_t))dt + g(y_t)dx(t)\right] \\
&= \Phi^{-1}(t)\left[f(y_t)dt + g(y_t)dx(t)\right],
\end{align*}
\]

hence $y(t) = \Phi(t)z(t) = \Phi(t)\left(z(t_0) + \int_{t_0}^{t} \Phi^{-1}(s)f(y_s)ds + \int_{t_0}^{t} \Phi^{-1}(s)g(y_s)dx(s)\right)$, from which (1.3) follows.

By the assumption $H_1$ on $A$, there exist positive constants $C_A, \lambda > 0$ (see \cite[Chapter 1, §3]{11}) such that

\[
\begin{align*}
\|\Phi\|_{\infty,[a,b]} &\leq C_A e^{-\lambda a}, \\
\|\Phi\|_{\alpha,[a,b]} &\leq \|A\|C_A e^{-\lambda a}(b - a)^{1 - \alpha}, \quad \forall \ 0 < a < b,
\end{align*}
\] (4.4) (4.5)

where $0 < \alpha \leq 1$ is arbitrary and fixed, and $\|A\|$ is the norm of operator $A$.

We introduce the following notations

\[
\begin{align*}
\lambda_0 &:= \lambda - L, \text{ where } L := C_AC_f e^{\lambda r}, \\
M_1 &:= KC_A e^{4\lambda r}r'(1 + \|A\|r).
\end{align*}
\] (4.6) (4.7)

From now on, we will assume that

\[ \lambda_0 = \lambda - L > 0. \] (4.8)

The following lemma gives us an estimate for the uniform norm of solutions to the deterministic Young equation (2.1).

**Lemma 4.2** Let $n \in \mathbb{N}$, $n \geq 1$, and $t \in \Delta_n$ be arbitrary. Then, there exists a positive constant $M_2$ independent of $t$ and $n$ such that the solution to (2.1) satisfies

\[
\begin{align*}
\|y\|_{\infty,[t-r,t]} &\leq M_2 e^{-\lambda_0 n r}\|y\|_{\infty,[0,r]} + (\|f(0)\| \vee \|g(0)\|)M_2 \sum_{k=0}^{n-1} (1 + \|x\|_{\nu,\Delta_k+1})e^{-\lambda_0 (n-k)r} \\
&\quad + C_gM_1 \sum_{k=0}^{n-1} \|x\|_{\nu,\Delta_k+1} e^{-\lambda_0 (n-k)r} \left(\|y\|_{\beta,\Delta_k} + \|y\|_{\beta,\Delta_{k+1}}\right),
\end{align*}
\] (4.9)
where $M_1$ is defined by the formula (4.7). As a sequence,

$$
\|y\|_{\infty, \Delta_n} \leq M_2 e^{-\lambda_0 n r} \|y\|_{\infty, [0, r]} + M_2 (\|f(0)\| \vee \|g(0)\|) \sum_{k=0}^{n-1} (1 + \|x\|_{\nu, \Delta_{k+1}}) e^{-\lambda_0 (n-k) r} + C_g M_1 \sum_{k=0}^{n-1} \|x\|_{\nu, \Delta_{k+1}} e^{-\lambda_0 (n-k) r} (\|y\|_{\beta, \Delta_k} + \|y\|_{\beta, \Delta_{k+1}}).
$$

(4.10)

**Proof:** First, for any $t \geq r$, by virtue of (4.3), (4.4) and the assumption $H_2$ on $f$, the following inequalities hold

$$
\|y(t)\| \leq \|\Phi(t-r)g(y)\| + \int_r^t \|\Phi(t-s)f(y_s)\| ds + \int_r^t \|\Phi(t-s)g(y_s)\| ds \leq C_A e^{-\lambda (t-r)} \|y(r)\| + \int_r^t C_A e^{-\lambda (t-s)} \left( C_f \|y_s\| + \|f(0)\| \right) ds + \int_r^t \|\Phi(t-s)g(y_s)\| ds \leq C_A e^{-\lambda (t-r)} \|y(r)\| + \frac{C_A}{\lambda} \|f(0)\| (1 - e^{-\lambda (t-r)}) + C_A C_f \int_r^t e^{-\lambda (t-s)} \|y\|_{\infty, [s-r, t]} ds + \beta(t),
$$

where

$$
\beta(t) := \begin{cases} \int_r^t \|\Phi(t-s)g(y_s)\| ds, & t \geq r \\ 0, & 0 \leq t < r. \end{cases}
$$

Now we put $\beta^*(t) := \sup_{t \leq t \leq r} \|\beta(s)\|$, $t \geq r$. It is easy to see that for $t \geq r$ we have

$$
\|y\|_{\infty, [t-r, t]} = \sup_{s \in [t-r, t]} \|y(s)\| \leq C_A e^{-\lambda (t-2r)} \|y\|_{\infty, [0, r]} + \frac{C_A}{\lambda} \|f(0)\| (1 - e^{-\lambda (t-r)}) + C_A C_f e^{\lambda r} \int_r^t e^{-\lambda (t-s)} \|y\|_{\infty, [s-r, s]} ds + \beta^*(t).
$$

Consequently,

$$
e^{\lambda (t-r)} \|y\|_{\infty, [t-r, t]} \leq C_A e^{\lambda r} \|y\|_{\infty, [0, r]} + \frac{C_A}{\lambda} \|f(0)\| (e^{\lambda (t-r)} - 1) + e^{\lambda (t-r)} \beta^*(t) + C_A C_f e^{\lambda r} \int_r^t e^{(\lambda-s) (t-s)} \|y\|_{\infty, [s-r, s]} ds.
$$

Recall from (4.6) that $L := C_A C_f e^{\lambda r}$. Applying the continuous Gronwall inequality (see Lemma 5.3 below) for the function $e^{\lambda (t-r)} \|y\|_{\infty, [t-r, t]}$ and performing some computation we get

$$
e^{(\lambda-L)(t-r)} \|y\|_{\infty, [t-r, t]} \leq C_A e^{\lambda r} \|y\|_{\infty, [0, r]} + \frac{C_A \|f(0)\|}{\lambda - L} (e^{(\lambda-L)(t-r)} - 1) + e^{(\lambda-L)(t-r)} \beta^*(t) + L \int_r^t e^{(\lambda-L)(s-r)} \beta^*(s) ds.
$$

Therefore, recall from (4.6) that $\lambda_0 = \lambda - L$,

$$
e^{\lambda_0 (t-r)} \|y\|_{\infty, [t-r, t]} \leq C_A e^{\lambda r} \|y\|_{\infty, [0, r]} + \frac{C_A \|f(0)\|}{\lambda_0} \left[ e^{\lambda_0 (t-r)} - 1 \right] + e^{\lambda_0 (t-r)} \beta^*(t) + L \int_r^t e^{\lambda_0 (s-r)} \beta^*(s) ds.
$$

(4.11)
Now, to use this inequality to prove the lemma we need to estimate the function $\beta^*$. To this end we need to make an estimate for $\beta$. First, assume that $s \in \mathbb{R}$, $s \geq r$ and $s$ is not an integer multiple of $r$. Put $n := \lfloor s/r \rfloor$, the greatest integer which is less than or equal to $s/r$. Due to the definition of $\beta(s)$, the inequality (5.2), the estimate (4.4), (4.5) and Proposition 2.4 we have

$$
\beta(s) \leq \sum_{k=1}^{n-1} \left| \int_{kr}^{(k+1)r} \Phi(s-u)g(y_u)dx(u) \right| + \left| \int_{nr}^{s} \Phi(s-u)g(y_u)dx(u) \right|
$$

$$
\leq \sum_{k=1}^{n-1} r^\nu \| x \|_{\nu, \Delta_k} \left[ \| \Phi(s-kr)g(y_{kr}) \| + Kr^{\beta} \| \Phi(s-r)g(y_r) \|_{\beta, \Delta_k} \right]
$$

$$
+ r^\nu \| x \|_{\nu, [nr, s]} \left[ \| \Phi(s-nr)g(y_{nr}) \| + Kr^{\beta} \| \Phi(s-r)g(y_r) \|_{\beta, [nr, s]} \right]
$$

$$
\leq \sum_{k=1}^{n} r^\nu \| x \|_{\nu, \Delta_k} \left[ n^\nu C_A e^{-\lambda(s-kr)} (C_g \| y_{kr} \| + \| g(0) \|) + K r^{\beta} \left( \| \Phi(s-r) \|_{\beta, \Delta_k} \| g(y_r) \|_{\infty, \Delta_k} + \| \Phi(s-r) \|_{\infty, \Delta_k} \| g(y_r) \|_{\beta, \Delta_k} \right) \right]
$$

$$
+ r^\nu \| x \|_{\nu, \Delta_n} \left[ C_A e^{-\lambda(s-nr)} (C_g \| y_{nr} \| + \| g(0) \|) + K r^{\beta} \left( \| A \| \| C_A e^{-\lambda(s-nr-r)} r^{1-\beta} \| g(y_r) \|_{\infty, \Delta_k} + C_A e^{-\lambda(s-kr-r)} \| g(y_r) \|_{\beta, \Delta_k} \right) \right]
$$

$$
\leq \sum_{k=1}^{n} r^\nu \| x \|_{\nu, \Delta_k} \left[ n^\nu C_A e^{-\lambda(s-kr)} (C_g \| y \|_{\infty, \Delta_k} + \| g(0) \|) + K r^{\beta} \left( \| A \| \| C_A e^{-\lambda(s-kr-r)} r^{1-\beta} \| y \|_{\infty, \Delta_k} + \| g(0) \|) + C_A e^{-\lambda(s-kr-r)} C_g \| y \|_{\beta, \Delta_k} \right) \right]
$$

$$
\leq \sum_{k=1}^{n} r^\nu \| x \|_{\nu, \Delta_k} K C_A (1 + \| A \| r) e^{-\lambda(s-kr-r)} \left[ C_g \left( \| y \|_{\infty, \Delta_k} + r^{\beta} \| y \|_{\beta, \Delta_k} \right) + \| g(0) \| \right]
$$

$$
\leq \sum_{k=1}^{n} r^\nu \| x \|_{\nu, \Delta_k} K C_A (1 + \| A \| r) e^{-\lambda(s-kr-r)} \left[ C_g \left( \| y \|_{\beta, \Delta_{k-1}} + \| y \|_{\beta, \Delta_k} + \| g(0) \| \right) \right].
$$

Take the supremum on $[s-r, s]$ we have

$$
\beta^*(s) \leq \sum_{k=1}^{n} e^{2\lambda r} r^\nu \| x \|_{\nu, \Delta_k} K C_A (1 + \| A \| r) e^{-\lambda(s-kr)} \left[ C_g \left( \| y \|_{\beta, \Delta_{k-1}} + \| y \|_{\beta, \Delta_k} + \| g(0) \| \right) \right].
$$

(4.12)

Therefore, since $\lambda_0 = \lambda - L > 0$ we have

$$
\beta^*(s) e^{\lambda_0 (s-r)} \leq \sum_{k=1}^{n} e^{2\lambda r} r^\nu K C_A (1 + \| A \| r) \| x \|_{\nu, \Delta_k} e^{-L s + \lambda kr} \left[ C_g \left( \| y \|_{\beta, \Delta_{k-1}} + \| y \|_{\beta, \Delta_k} + \| g(0) \| \right) \right].
$$

(4.13)
Combining this with (4.11) and use arguments similar to that of (13), for $t \in [nr, (n+1)r]$ with $n \geq 1$ being a fixed integer we obtain

$$
e^{-\lambda_0(t-r)}\|y\|_{\infty,[t-r,t]} \leq C_A e^{\lambda r} \|y\|_{\infty,[0,r]} + \frac{C_A \|f(0)\|}{\lambda_0} \left(e^{\lambda_0(t-r)} - 1\right)$$

$$+ \sum_{k=1}^{n} KC_A e^{2\lambda r} r^\nu (1 + \|A\|r) \|x\|_{\nu,\Delta_k} e^{\lambda_0 kr} e^{-L(t-kr)} \left[C_g \left(\|y\|_{\beta,\Delta_k-1} + \|y\|_{\beta,\Delta_k}\right) + \|g(0)\|\right]$$

$$+ L \int_r^{t} \sum_{k=1}^{\lfloor s/r \rfloor} KC_A e^{2\lambda r} r^\nu (1 + \|A\|r) \|x\|_{\nu,\Delta_k} e^{\lambda_0 kr} e^{-L(s-kr)} \left[C_g \left(\|y\|_{\beta,\Delta_k-1} + \|y\|_{\beta,\Delta_k}\right) + \|g(0)\|\right] ds$$

$$\leq C_A e^{\lambda r} \|y\|_{\infty,[0,r]} + \frac{C_A \|f(0)\|}{\lambda_0} (e^{\lambda_0 t} - 1)$$

$$+ \sum_{k=1}^{n} KC_A e^{2\lambda r} r^\nu (1 + \|A\|r) \|x\|_{\nu,\Delta_k} e^{\lambda_0 kr} \left[C_g \left(\|y\|_{\beta,\Delta_k-1} + \|y\|_{\beta,\Delta_k}\right) + \|g(0)\|\right] \times \left(e^{-L(t-kr)} + L \int_{kr}^{t} e^{-L(s-kr)} ds\right)$$

$$\leq C_A e^{\lambda r} \|y\|_{\infty,[0,r]} + \frac{C_A \|f(0)\|}{\lambda_0} (e^{\lambda_0 t} - 1)$$

$$+ \sum_{k=1}^{n} KC_A e^{2\lambda r} r^\nu (1 + \|A\|r) \|x\|_{\nu,\Delta_k} e^{\lambda_0 kr} \left[C_g \left(\|y\|_{\beta,\Delta_k-1} + \|y\|_{\beta,\Delta_k}\right) + \|g(0)\|\right].$$

Consequently, we have

$$\|y\|_{\infty,[t-r,t]} e^{\lambda_0(t-r)} \leq C_A e^{\lambda r} \|y\|_{\infty,[0,r]} + \frac{C_A \|f(0)\|}{\lambda_0} \left[e^{\lambda_0(t-r)} - 1\right]$$

$$+ KC_A e^{2\lambda r} r^\nu (1 + \|A\|r) \|g(0)\| \sum_{k=1}^{n} \|x\|_{\nu,\Delta_k} e^{\lambda_0 kr}$$

$$+ C_g KC_A e^{2\lambda r} r^\nu (1 + \|A\|r) \sum_{k=1}^{n} \|x\|_{\nu,\Delta_k} e^{\lambda_0 kr} \left(\|y\|_{\beta,\Delta_k-1} + \|y\|_{\beta,\Delta_k}\right).$$

This implies that there exists a positive number $M_2$ such that for all $t \in [nr, (n+1)r)$, $n \in \mathbb{N}$ we have

$$\|y\|_{\infty,[t-r,t]} \leq M_2 e^{-\lambda_0 nr} \|y\|_{\infty,[0,r]} + (\|f(0)\| \lor \|g(0)\|) M_2 \sum_{k=0}^{n-1} (1 + \|x\|_{\nu,\Delta_{k+1}}) e^{-\lambda_0 (n-k)r}$$

$$+ C_g M_1 \sum_{k=0}^{n-1} \|x\|_{\nu,\Delta_{k+1}} e^{-\lambda_0 (n-k)r} \left(\|y\|_{\beta,\Delta_k} + \|y\|_{\beta,\Delta_{k+1}}\right).$$

Now, due to continuity, this inequality also holds true for $t = (n + 1)r$, hence for all $t \in \Delta_n$. Thus (4.9) is proved. Moreover, for $t = (n + 1)r$ the estimate (4.9) turns to (4.10).

$\square$

**Remark 4.3** Inequalities (4.10) and (2.2) show that the supremum norm of the solution on $\Delta_n$ depends on itself (up to a coefficient dependent on $x$) and also on the Hölder norm of the solution.
on previous intervals. This is different from the non-delay case (see [13]) and is very challenging to deal with. We therefore need to estimate the $\beta$–Hölder norm of $y$ in the similar form to (4.10) in the following Lemma.

Assign

$$M_3 := K r^\nu e^{(L_f + 4\lambda)r} \left[ 1 + CA L_f r(1 + \|A\| r) \right].$$

\textbf{Lemma 4.4} For any $n \in \mathbb{N}, n \geq 1$, there exists a positive constant $M_4$ independent of $n$, such that the $\beta$–Hölder norm of the solution of (2.1) on $\Delta_n$, can be estimated as follows

$$r^\beta \|y\|_\beta,\Delta_n \leq M_4 e^{-\lambda_0 nr}\|y\|_\infty,[0,r] + M_4(\|f(0)\| \vee \|g(0)\|) \sum_{k=0}^{n-1} (1 + \|x\|_{\nu,\Delta_{k+1}}) e^{-\lambda_0(n-k)r}$$

$$+ C_g M_3 \sum_{k=0}^{n-1} \|x\|_{\nu,\Delta_{k+1}} e^{-\lambda_0(n-k)r} \left( \|y\|_\beta,\Delta_k + \|y\|_{\beta,\Delta_{k+1}} \right),$$

(4.15)

where the constant $M_3$ is defined by the formula (4.14).

\textbf{Proof:} We fix $v \in \Delta_n$, and consider $s, t \in [nr, v], s < t$. We have

$$\|y(t) - y(s)\| = \left\| \int_s^t [Ay(u) + f(y_u)] du + \int_s^t g(y_u) dx(u) \right\|$$

$$\leq \|f(0)\| (t-s) + (\|A\| + C_f) \int_s^t \|y_u\| du + \left\| \int_s^t g(y_u) dx(u) \right\|. $$

Since $u \in [s, t] \subset [nr, v]$ we have $\|y_u\| \leq \|y\|_\infty,[s-r,s] + (u-s)^{\beta} \|y\|_{\beta,[s,u]}$, and, furthermore, by virtue of (5.2), Proposition 2.1 and the definition of the norm $\cdot \| \cdot _{\beta,[a,b]}$,

$$\left\| \int_s^t g(y_u) dx(u) \right\| \leq (t-s)^{\nu} \|x\|_{\nu,\Delta_n} K \left[ C_g \left( \|y\|_{\beta,\Delta_{n-1}} + \|y\|_{\beta,\Delta_n} \right) + \|g(0)\| \right]. $$

Therefore, recall that $L_f := \|A\| + C_f$, we get

$$\frac{\|y(t) - y(s)\|}{(t-s)^{\beta}} \leq \|f(0)\| r^{1-\beta} + L_f r^{1-\beta} \|y\|_\infty,[s-r,s] + L_f \int_s^t \frac{(u-s)}{(t-s)^{\beta}} \|y\|_{\beta,[s,u]} du$$

$$+ Kr^{\nu-\beta} \|x\|_{\nu,\Delta_n} \left[ C_g \left( \|y\|_{\beta,\Delta_{n-1}} + \|y\|_{\beta,\Delta_n} \right) + \|g(0)\| \right]$$

$$\leq \|f(0)\| r^{1-\beta} + L_f r^{1-\beta} \|y\|_\infty,[s-r,s] + L_f \int_s^t \|y\|_{\beta,[s,u]} du$$

$$+ Kr^{\nu-\beta} \|x\|_{\nu,\Delta_n} \left[ C_g \left( \|y\|_{\beta,\Delta_{n-1}} + \|y\|_{\beta,\Delta_n} \right) + \|g(0)\| \right]$$

$$\leq (\|f(0)\| \vee \|g(0)\|) \max \left\{ r^{1-\beta}, Kr^{\nu-\beta} \right\} (1 + \|x\|_{\nu,\Delta_n}) + L_f \int_s^t \|y\|_{\beta,[s,u]} du. $$

(4.16)

Combining this with (4.9), with the notation $M_2' := L_f r M_2 + (r \vee Kr^{\nu}) e^{\lambda_0 r} > 0$ the following estimate holds for all $[s, t] \subset [nr, v]$

$$r^\beta \|y(t) - y(s)\| \leq (t-s)^{\beta}$$
Consequently,\[\begin{align*}
&\leq (\|f(0)\| \vee \|g(0)\|) \max \{r, Kr^\nu\} (1 + \|x\|_{\nu, \Delta_n}) \\
&+ L_f r M_2 e^{-\lambda_0 n r} \|y\|_{\infty, [0, r]} + (\|f(0)\| \vee \|g(0)\|) L_f r M_2 \sum_{k=0}^{n-1} (1 + \|x\|_{\nu, \Delta_{k+1}}) e^{-\lambda_0 (n-k)r} \\
&+ C_g L_f r M_1 \sum_{k=0}^{n-1} \|x\|_{\nu, \Delta_{k+1}} e^{-\lambda_0 (n-k)r} (\|y\|_{\beta, \Delta_k} + \|y\|_{\beta, \Delta_{k+1}}) \\
&+ C_g K r^\nu \|x\|_{\nu, \Delta_n} (\|y\|_{\beta, \Delta_{n-1}} + \|y\|_{\beta, \Delta_n}) + L_f r^\beta \int_s^t \|y\|_{\beta, [s, u]} \, du
\end{align*}\]

This implies
\[\begin{align*}
r^\beta \|y\|_{\beta, [nr, v]} &\leq M_2 e^{-\lambda_0 n r} \|y\|_{\infty, [0, r]} + M_2 (\|f(0)\| \vee \|g(0)\|) \sum_{k=0}^{n-1} (1 + \|x\|_{\nu, \Delta_{k+1}}) e^{-\lambda_0 (n-k)r} \\
&+ \left(C_g L_f r M_1 + C_g K r^\nu e^{\lambda_0 r}\right) \sum_{k=0}^{n-1} \|x\|_{\nu, \Delta_{k+1}} e^{-\lambda_0 (n-k)r} (\|y\|_{\beta, \Delta_k} + \|y\|_{\beta, \Delta_{k+1}}) \\
&+ L_f \int_{nr}^v r^\beta \|y\|_{\beta, [nr, u]} \, du.
\end{align*}\]

By applying the Gronwall Lemma 5.3 to the function \(r^\beta \|y\|_{\beta, [nr, \cdot]}\), we get
\[\begin{align*}
r^\beta \|y\|_{\beta, [nr, v]} &\leq \left[M_2 e^{-\lambda_0 n r} \|y\|_{\infty, [0, r]} + M_2 (\|f(0)\| \vee \|g(0)\|) \sum_{k=0}^{n-1} (1 + \|x\|_{\nu, \Delta_{k+1}}) e^{-\lambda_0 (n-k)r} + \right. \\
&\left. + \left(C_g L_f r M_1 + C_g K r^\nu e^{\lambda_0 r}\right) \sum_{k=0}^{n-1} \|x\|_{\nu, \Delta_{k+1}} e^{-\lambda_0 (n-k)r} (\|y\|_{\beta, \Delta_k} + \|y\|_{\beta, \Delta_{k+1}}) \right] \times \\
&\times \left(1 + L_f \int_{nr}^v e^{L_f (v-u)} \, du\right).
\end{align*}\]

Consequently,
\[\begin{align*}
r^\beta \|y\|_{\beta, \Delta_n} &\leq \left[M_2 e^{-\lambda_0 n r} \|y\|_{\infty, [0, r]} + M_2 (\|f(0)\| \vee \|g(0)\|) \sum_{k=0}^{n-1} (1 + \|x\|_{\nu, \Delta_{k+1}}) e^{-\lambda_0 (n-k)r} + \right. \\
&\left. + C_g K r^\nu \left(e^{\lambda_0 r} + L_f r^{1-\nu} M_1\right) \sum_{k=0}^{n-1} \|x\|_{\nu, \Delta_{k+1}} e^{-\lambda_0 (n-k)r} (\|y\|_{\beta, \Delta_k} + \|y\|_{\beta, \Delta_{k+1}}) \right] e^{L_f r}.
\end{align*}\]

Set \(M_4 := e^{L_f r} M_2\) and taking into account (4.14), (4.15) we get the lemma proved. \(\square\)
Now we are in a position to prove the main result of the paper. Namely, we return to the stochastic Young differential delay equation \((1.1)\) and prove the following theorem on existence of a random pullback attractor.

**Theorem 4.5** Consider the system \((1.1)\)

\[
dy(t) = [Ay(t) + f(y_t)]dt + g(y_t)dZ(t), \quad y_0 = \eta \in C^{0, \beta_0}([-r, 0], \mathbb{R}^d).
\]

Assume that the conditions \(H_1, H_2, H_3\) hold, and assume additionally that

\[
C_A C_f < \lambda e^{-\lambda r}.
\]

Then there exists \(\varepsilon > 0\) such that if \(C_g < \varepsilon\), then the generated random dynamical system of \((1.1)\) possesses a random pullback attractor \(\mathcal{A}(x)\) which is in \(C^\beta([-r, 0], \mathbb{R}^d) \subset C^{0, \beta_0}([-r, 0], \mathbb{R}^d)\).

**Proof:** As noticed before, the equation \((1.1)\) is understood in the path-wise sense with Riemann-Stieltjes integration and Young integration. We consider the deterministic equation \((2.1)\)

\[
dy(t) = [Ay(t) + f(y_t)]dt + g(y_t)dx(t), \quad y_0 = \eta \in C^{0, \beta_0}([-r, 0], \mathbb{R}^d),
\]

which is a representative path-wise equation of the stochastic equation \((1.1)\). With an ambiguity of notations we will denote by \(y(\cdot)\) both the solution to \((1.1)\) and the solution to \((2.1)\).

Notice that the condition \(C_A C_f < \lambda e^{-\lambda r}\) is equivalent to the condition \(\lambda_0 = \lambda - L > 0\). Put \(M_5 := M_1 + M_3\) and \(M_6 := M_2 + M_4\). Due to \((4.10)\) and \((4.15)\), for any \(n \geq 1\) we have

\[
\|y\|_{\beta, \Delta_n} \leq M_6 e^{-\lambda_0 nr} \|y\|_{\infty, [0, r]} + M_6 (\|f(0)\| \vee \|g(0)\|) \sum_{k=0}^{n-1} (1 + \|x\|_{\nu, \Delta_{k+1}}) e^{-\lambda_0 (n-k)r} \\
+ C_g M_5 \sum_{k=0}^{n-1} \|x\|_{\nu, \Delta_{k+1}} e^{-\lambda_0 (n-k)r} (\|y\|_{\beta, \Delta_k} + \|y\|_{\beta, \Delta_{k+1}}).
\]  

(4.17)

Now, we make use of Proposition 2.2. Put for \([a, b] \subset \mathbb{R}\),

\[
F(x, [a, b]) := 1 + [2(K + 1) C_g (b - a)^r]^{1/r} \|x\|_{\nu^{1/r}, [a, b]}.
\]

The estimate \((2.2)\) of Proposition 2.2 reads

\[
\|y\|_{\beta, \Delta_k} \leq e^{4C_f r + \kappa N_k(x)} \left[\|y\|_{\beta, \Delta_{k-1}} + (4r\|f(0)\| + \|g(0)\|) \frac{C_g}{C_g} \right] - (4r\|f(0)\| + \|g(0)\|),
\]

where \(\kappa = 4L f_r + 2, \) \(N_k(x)\) is the counting function of greedy times on \(\Delta_k\) as described in the proof of Proposition 2.2 which satisfies the inequality \(N_k(x) \leq F(x, \Delta_k)\) by \((2.3)\). Hence

\[
\|y\|_{\beta, \Delta_k} + \|y\|_{\beta, \Delta_{k+1}} \leq \left(1 + e^{4C_f r + \kappa N_{k+1}(x)}\right) \|y\|_{\beta, \Delta_k} + e^{4C_f r + \kappa N_{k+1}(x)} \left(4r\|f(0)\| + \|g(0)\|\right).
\]

Combining this with \((4.17)\) we get

\[
e^{\lambda_0 nr} \|y\|_{\beta, \Delta_n} \leq M_6 \|y\|_{\infty, [0, r]} + M_6 (\|f(0)\| \vee \|g(0)\|) \sum_{k=0}^{n-1} (1 + \|x\|_{\nu, \Delta_{k+1}}) e^{\lambda_0 kr}.
\]

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\[ + C_2 M_5 \sum_{k=0}^{n-1} \|x\|_{\nu, \Delta_{k+1}} e^{\lambda_0 k r} \left[ \left( 1 + e^{4C_f r + \kappa N_k + 1(x)} \right) \|y\|_{\beta, \Delta_k} + e^{4C_f r + \kappa N_k + 1(x)} \left( 4r \|f(0)\| + \frac{\|g(0)\|}{C_y} \right) \right] \]

\[ \leq M_6 \|y\|_{\infty, [0, r]} + M_8 (\|f(0)\| \lor \|g(0)\|) \sum_{k=0}^{n-1} e^{\lambda_0 k r} \left( 1 + \|x\|_{\nu, \Delta_{k+1}} \right) e^{\kappa F(x, \Delta_{k+1})} \]

\[ + C_g M_7 \sum_{k=0}^{n-1} \|x\|_{\nu, \Delta_{k+1}} \left( 1 + e^{\kappa F(x, \Delta_{k+1})} \right) e^{\lambda_0 k r} \|y\|_{\beta, \Delta_k} \]

\[ \leq M_8 \|y\|_{\infty, [0, r]} + M_8 (\|f(0)\| \lor \|g(0)\|) \sum_{k=0}^{n-1} e^{\lambda_0 k r} H(x, \Delta_{k+1}) \]

\[ + C_g M_7 \sum_{k=0}^{n-1} G(x, \Delta_{k+1}) e^{\lambda_0 k r} \|y\|_{\beta, \Delta_k}, \quad (4.18) \]

where we used the notations

\[ M_7 := M_5 e^{4C_f r}, \quad (4.19) \]

\[ M_8 := 1 + M_6 + M_5 e^{4C_f r} (4C_g r + 1), \quad (4.20) \]

\[ G(x, [a, b]) := \|x\|_{[a, b]} \left( 1 + e^{\kappa F(x, [a, b])} \right), \quad (4.21) \]

\[ H(x, [a, b]) := (1 + \|x\|_{[a, b]}) e^{\kappa F(x, [a, b])}. \quad (4.22) \]

Due to the Discrete Gronwall Lemma 5.4 from (4.18) we derive

\[ e^{\lambda_0 n r} \|y\|_{\beta, \Delta_n} \leq M_8 \|y\|_{\beta, \Delta_0} \prod_{k=0}^{n-1} \left[ 1 + C_g M_7 G(x, \Delta_{k+1}) \right] \]

\[ + M_8 (\|f(0)\| \lor \|g(0)\|) \sum_{k=0}^{n-1} e^{\lambda_0 k r} H(x, \Delta_{k+1}) \prod_{j=k+1}^{n-1} \left[ 1 + C_g M_7 G(x, \Delta_{j+1}) \right]. \quad (4.23) \]

By the construction of the random dynamical system generated by (1.11) presented in Section 8, we have

\[ G(x, \Delta_k) = G(\theta_{k r}, x, [0, r]) \] and \[ H(x, \Delta_k) = H(\theta_{k r}, x, [0, r]) \], hence, writing the solution of (1.11) in full form \( y(\cdot, x, \eta) \) indicating the dependence on the driver \( x \) and the initial condition \( \eta \) instead of short notation \( y \) we have

\[ e^{\lambda_0 n r} \|y(\cdot, x, \eta)\|_{\beta, \Delta_n} \leq M_8 \|y\|_{\beta, \Delta_0} \prod_{k=0}^{n-1} \left[ 1 + C_g M_7 G(\theta_{(k+1)r}, x, [0, r]) \right] \]

\[ + M_8 (\|f(0)\| \lor \|g(0)\|) \sum_{k=0}^{n-1} e^{\lambda_0 k r} H(\theta_{(k+1)r}, x, [0, r]) \prod_{j=k+1}^{n-1} \left[ 1 + C_g M_7 G(\theta_{(j+1)r}, x, [0, r]) \right]. \quad (4.24) \]

Notice that using (2.10) we can find a positive constant \( M_9 \) independent of \( n \) such that

\[ M_8 \|y\|_{\beta, \Delta_0} \leq M_9 \left( 1 + \|y\|_{\nu, [0, r]} \right) \left( 1 + \|\eta\|_{\beta_0, [-r, 0]} \right) e^{\kappa N_0(x)}. \]

Put \( F_0(x, [0, r]) := 1 + 2(K_0 + 1) e^{4C_f r} \|y\|_{\nu, [0, r]}. \) By (2.9) we have \( N_0(x) \leq F_0(x, [0, r]). \) Now, make a change from \( x \) to \( \theta_{-(n+1)r}x \) in (4.24), and we get

\[ \|y(\cdot, \theta_{-(n+1)r}x, \eta)\|_{\beta, \Delta_n} \leq M_9 \left( 1 + \|\eta\|_{\beta_0, [-r, 0]} \right) \left( 1 + \|\theta_{-(n+1)r}x\|_{\nu, [0, r]} \right) e^{\kappa F_0(\theta_{-(n+1)r}x, [0, r])} e^{-\lambda_0 n r} \times \]

(Continued on next page)
Consequently, from (4.25) it follows that for all 
and 

\[0 < \beta< \delta < \lambda \to 0\] 

Similarly, one can show that \(\log \) is integrable. Furthermore, \(\tilde{F}_0(x, [0, r])\) is integrable due to the fact that \(\beta_0 < \beta\), hence \(\log \tilde{F}_0(x, [0, r])\) is integrable, where \(\tilde{F}_0(x, [0, r]) := \left(1 + \| x \|_{\nu, [0, r]} \right) e^{\kappa F_0(x, [0, r])}\). Therefore, by the temperedness of integrable random variables (see Arnold \(3\), Proposition 4.1.3, p. 165) the following equalities hold almost surely

\[
\limsup_{n \to \infty} \frac{1}{n} \log \prod_{k=1}^{n} \left[ 1 + C_g M_7 G(\theta_{-kr} x, [0, r]) \right] = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log \left[ 1 + C_g M_7 G(\theta_{kr} x, [0, r]) \right] = \tilde{G}.
\]

Similarly, one can show that \(\log H\) is integrable. Furthermore, \(F_0\) is integrable due to the fact that \(\beta_0 < \beta\), hence \(\log \tilde{F}_0(x, [0, r])\) is integrable, where \(\tilde{F}_0(x, [0, r]) := \left(1 + \| x \|_{\nu, [0, r]} \right) e^{\kappa F_0(x, [0, r])}\). Therefore, by the temperedness of integrable random variables (see Arnold \(3\), Proposition 4.1.3, p. 165) the following equalities hold almost surely

\[
\limsup_{n \to \infty} \frac{\log H(\theta_{nr} x, [0, r])}{n} = \limsup_{n \to \infty} \frac{\log H(\theta_{-nr} x, [0, r])}{n} = 0.
\]

Observe that \(\log (1 + C_g M_7 G(x, [0, r]))\), as a function of \(C_g\), converges pointwise to zero as \(C_g\) tends to zero. Due to (4.26) and Lebesgue’s dominated convergence theorem, the value \(\tilde{G}\) also converges to zero as \(C_g\) tends to zero. Therefore there exists \(\varepsilon > 0\) such that if \(C_g < \varepsilon\) then \(\tilde{G} \leq \lambda_0 r\). Fix \(0 < 2\delta < \lambda_0 r - \tilde{G}\); there exists \(n_0 = n_0(\delta, x)\) such that for all \(n \geq n_0\),

\[e^{(-\delta+\tilde{G})n} \leq \prod_{k=1}^{n} \left[ 1 + C_g M_7 G(\theta_{-kr} x, [0, r]) \right] \prod_{k=1}^{n} \left[ 1 + C_g M_7 G(\theta_{kr} x, [0, r]) \right] \leq e^{(\delta+\tilde{G})n}\]

and

\[e^{-\delta n} \leq \tilde{F}_0(\theta_{-nr} x, [0, r]), H(\theta_{-nr} x, [0, r]), H(\theta_{nr} x, [0, r]) \leq e^{\delta n}.
\]

Consequently, from (4.25) it follows that for all \(n \geq n_0\) we have

\[
\| y(\cdot, \theta_{-(n+1)r} x, \eta) \|_{\beta, \Delta n} \leq M_9 \left(1 + \| \eta \|_{3\eta, [-r, 0]} \right) e^{-\lambda_0 nr} e^{(2\delta+\tilde{G})n}
\]
Namely, since $0 < \lambda_0 < \lambda_0$ and all $\eta \in D(\theta, x)$ we have

$$\|y(\cdot, \theta, t, \eta)\|_{\beta, [t-r, t]} \leq b(x).$$

In fact, one may choose

$$b(x) := 1 + M_8(\|f(0)\| \vee \|g(0)\|) \sum_{k=1}^{\infty} e^{-\lambda_0 k r} H(\theta_{-k r} x, [0, r]) \prod_{i=1}^{k} \left(1 + C_9 M_7 G(\theta_{-i r} x, [0, r])\right).$$

and the temperedness of $b(\cdot)$ is proved in [13]. For convenience of the reader we give an improved proof of temperedness of $b(\cdot)$ which is based on Lemmas 5.1 and 5.2 and is shorter than that of [13]. Namely, since $0 < 2 \delta < \lambda_0 r - \tilde{G}$ we have

$$b(x) - 1 \leq \left[M_8(\|f(0)\| \vee \|g(0)\|) \sum_{k=1}^{\infty} e^{-\delta k H(\theta_{-k r} x, [0, r])} \right] \left[\sum_{k=1}^{\infty} e^{-(\tilde{G} + \delta) k} \prod_{i=1}^{k} \left(1 + C_9 M_7 G(\theta_{-i r} x, [0, r])\right)\right].$$

The first multiplier in the right-hand side is tempered due to Lemma 5.2 (ii); the second multiplier is tempered due to Lemma 5.1 (i). This implies that $b$ is tempered because of Lemma 5.1 (ii).

Note that, for each $x \in C_0^{\mu}(\mathbb{R}, \mathbb{R}^d)$ in the canonical representation space of $Z$, the closed ball $B(x) = \{\eta \in C^{\beta, \beta_0}([-r, 0], \mathbb{R}^d) \mid \|\eta\|_{\beta, [-r, 0]} \leq b(x)\}$ is compact in $C^{\beta, \beta_0}([-r, 0], \mathbb{R}^d)$. Thus we proved that there exists a compact absorbing random set $B(x)$ with respect to the universe of tempered compact random sets. Moreover, $B(x)$ is a subset of $C^{\beta}([-r, 0], \mathbb{R}^d)$. Therefore, $\varphi$ possesses a random pullback attractor $\mathcal{A}(x) \subset B(x)$ (see [19] Theorem 2.4, [15] Theorem 3.5). Clearly, $\mathcal{A}(x) \subset C^{\beta}([-r, 0], \mathbb{R}^d) \subset C^{\beta, \beta_0}([-r, 0], \mathbb{R}^d)$.

The inequality (4.28) provides us with a tool to make further conclusions on the dynamics of the random system generated by (1.1). In case we have some additional information on the coefficient functions $f$ and $g$ as the following corollary shows. Recall from [6] that a random forward attractor is defined in a similar manner as the random pullback attractor given in Definition 4.1; namely we replace the pullback attraction condition (ii) of Definition 4.1 by the forward attraction one, i.e. for every $D \in \mathcal{D}$, $\lim_{t \to \infty} d(\varphi(t, x), \tilde{D}(x)|\mathcal{A}(\theta t x)) = 0$, $\mathbb{P}$-a.s.

**Corollary 4.6** Assume that the conditions in Theorem 4.5 are satisfied and, in addition, $f(0) = g(0) = 0$. Then there exists $\epsilon > 0$ such that for $C_9 < \epsilon$ the random pullback attractor of the system (1.1) provided by Theorem 4.5 is the set $\mathcal{A}(x) = \{0\}$ which is both the random pullback and random forward attractor of the system (1.1).

**Proof:** Clearly the origin is a fixed point of the system (1.1), hence an invariant compact random invariant set of the system (1.1). By (4.28) and the assumptions of Theorem 4.5 any solution of (1.1) tends to the origin exponentially in the pullback sense, hence the set $\mathcal{A}(x) = \{0\}$ is the random pullback attractor of (1.1) provided by Theorem 4.5. Similarly, by (4.23) and the assumptions of Theorem 4.5 any solution of (1.1) tends to the origin exponentially in the forward sense, hence $\mathcal{A}(x) = \{0\}$ attracts tempered compact random sets in the forward sense. Thus $\mathcal{A}(x) = \{0\}$ is also a random forward attractor of (1.1).
The following theorem asserts that, in case $g$ is bounded, the existence of the random pullback attractor is ensured without further assumption on $C_g$.

**Theorem 4.7** Consider the system (1.1)\[ dy(t) = [Ay(t) + f(y_t)]dt + g(y_t)dZ(t), \quad y_0 = \eta \in C^{0,\beta_0}([-r,0], \mathbb{R}^d). \]

Assume that the conditions $H_1, H_2, H_3$ hold, and, additionally,

$$C_A f < \lambda e^{-\lambda r}.$$  
Assume furthermore that $g$ is bounded, i.e. $\sup_{\eta \in C^{0,\beta_0}([-r,0], \mathbb{R}^d)} \|g(\eta)\| < \infty$. Then the generated random dynamical system of (1.1) possesses a random pullback attractor $A(x)$ which is in $C^\beta([-r,0], \mathbb{R}^d) \subset C^{0,\beta_0}([-r,0], \mathbb{R}^d)$.

**Proof:** First we notice that in this theorem we do not assume smallness of $C_g$, hence we have to employ the boundedness of $g$ instead to prove the existence of the random pullback attractor of (1.1). We will make some significant modification of the proof of Theorem 4.5 for our need here.

Recall from the proof of Theorem 4.5 that we consider the deterministic equation (2.1)\[ dy(t) = [Ay(t) + f(y_t)]dt + g(y_t)dx(t), \quad y_0 = \eta \in C^{0,\beta_0}([-r,0], \mathbb{R}^d), \]

which is a representative path-wise equation of the stochastic equation (1.1), and with an ambiguity of notations we will denote by $y(\cdot)$ both the solution to (1.1) and the solution to (2.1).

Put $\|g\|_\infty := \sup_{\eta \in C^{0,\beta_0}([-r,0], \mathbb{R}^d)} \|g(\eta)\| < \infty$. We fix $r = k_0 r$, where $k_0 \in \mathbb{N}$ will be chosen later. Let $\mu_t$ be the solution of the ordinary differential equation\[ d\mu(t) = [A\mu(t) + f(\mu_t)]dt, \quad t \geq 0, \]

with the initial condition $\mu(t) = y(t), \quad t \in [0,r]$. Following Lemma 4.2 applied to this equation, namely similar to (4.11) we get that, for all $t \geq r$

$$\|\mu\|_{\infty,[t-r,t]} \leq C_A e^{\lambda r} e^{-\lambda_0 (t-r)} \|\mu\|_{\infty,[0,r]} + \frac{C_A \|f(0)\|}{\lambda_0}. \quad (4.29)$$

This implies that

$$\|\mu\|_{\infty,[t-r,t]} \leq C_A e^{\lambda r} \|\mu\|_{\infty,[0,r]} + \frac{C_A \|f(0)\|}{\lambda_0}, \quad t \geq r.$$ 

Therefore, for all $2r \leq s < t$ and $v \in [-r,0]$ we have

$$\|\mu_t(v) - \mu_s(v)\| \leq \int_{s+v}^{t+v} (L_f ||\mu_u|| + \|f(0)\|)du = \int_{s+v}^{t+v} (L_f \sup_{-r \leq m \leq 0} ||\mu(u+m)|| + \|f(0)\|)du$$

$$\leq \int_{s+v}^{t+v} \left[ L_f \left( C_A e^{\lambda r} ||\mu||_{\infty,[0,r]} + \frac{C_A \|f(0)\|}{\lambda_0} \right) + \|f(0)\| \right]du$$

$$= (t-s) \left( C_A L_f e^{\lambda r} ||\mu||_{\infty,[0,r]} + C_A L_f \frac{\|f(0)\|}{\lambda_0} + \|f(0)\| \right)$$

$$\leq (t-s) (C_A L_f e^{\lambda r} + \lambda_0) \left( ||\mu||_{\infty,[0,r]} + \frac{\|f(0)\|}{\lambda_0} \right).$$
Consequently, for all $2r \leq s < t$,
\[
\|\mu_t - \mu_s\| = \|\mu_t(\cdot) - \mu_s(\cdot)\|_{\infty,[-r,0]} \leq (t-s) (CA_L e^{\lambda s} + \lambda_0) \left( \|\mu\|_{\infty,[0,r]} + \frac{\|f(0)\|}{\lambda_0} \right).
\] (4.30)

Put $h(t) := y(t) - \mu(t)$, then $h$ satisfies the equation
\[
dh(t) = [Ah(t) + f(y_t) - f(\mu_t)]dt + g(y_t)dx(t).
\]
From this, using $H_2$ and (5.2), we have for all $2r \leq s < t$
\[
\|h(t) - h(s)\| \leq \int_s^t L_f\|h_u\|du + (t-s)^\nu \|x\|_{\nu,[s,t]} \left[ \|g\|_{\infty} + K(t-s)^\beta \|g(y)\|_{\beta,[s,t]} \right].
\] (4.31)

We estimate $\|g(y)\|_{\beta,[s,t]}$ as follows.
\[
\|g(y_u) - g(y_v)\| \leq C_g \|y_u - y_v\| \leq C_g \left( \|h_u - h_v\| + \|\mu_u - \mu_v\| \right).
\]
Note that if $\|\mu_u - \mu_v\| \geq 1$ then $\|g(y_u) - g(y_v)\| \leq 2\|g\|_{\infty} \|\mu_u - \mu_v\|^\beta$, whereas if $\|\mu_u - \mu_v\| < 1$ then $\|\mu_u - \mu_v\| \leq \|\mu_u - \mu_v\|^\beta$. Therefore, we have
\[
\|g(y_u) - g(y_v)\| \leq C_g \|h_u - h_v\| + (2\|g\|_{\infty} \vee C_g) \|\mu_u - \mu_v\|^\beta.
\]

Combining this with (4.30), we obtain
\[
\|g(y)\|_{\beta,[s,t]} \leq C_g \|h\|_{\beta,[s-r,t]} + (2\|g\|_{\infty} \vee C_g)(CA_L e^{\lambda s} + \lambda_0)^\beta \left( \|\mu\|_{\infty,[0,r]} + \frac{\|f(0)\|^\beta}{\lambda_0^\beta} \right).
\]

Therefore (4.31) leads to
\[
\|h(t) - h(s)\|
\leq \int_s^t L_f\|h_u\|du + (t-s)^\nu \|x\|_{\nu,[s,t]} \left[ L_1 + L_2 \|\mu\|_{\infty,[0,r]} + K C_g (t-s)^\beta \|h\|_{\beta,[s-r,t]} \right],
\] (4.32)
for all $kr \leq s < t \leq (k+1)r$, $k \geq 2$, with $L_1 = \|g\|_{\infty} + K r^\beta(2\|g\|_{\infty} \vee C_g)(CA_L e^{\lambda s} + \lambda_0)^\beta \frac{\|f(0)\|^\beta}{\lambda_0^\beta}$, $L_2 = K r^\beta(2\|g\|_{\infty} \vee C_g)(CA_L e^{\lambda s} + \lambda_0)^\beta$. Note that (4.32) has the form of (2.4) but somehow simpler (we may look at (2.4) with $\|g(0)\|$ replaced by $L_1 + L_2 \|\mu\|_{\infty,[0,r]}$, and $f(0)$ and two further items in (2.4) replaced by 0). Now we repeat the arguments in Proposition 2.2 on the interval $\Delta_k$, similar to (2.2), we have the following estimate
\[
\|h\|_{\beta,\Delta_k} \leq \exp \left\{ 4L_f r + (4L_f r \vee 2) \left[ 1 + (2KC_g r^\nu)^{\frac{1}{\nu-\beta}} \|x\|_{\nu,\Delta_k}^{\frac{1}{\nu-\beta}} \right] \right\}
\times \left( \|h\|_{\beta,\Delta_{k-1}} + \frac{L_1 + L_2 \|\mu\|_{\infty,[0,r]}^\beta}{C_g} \right) - \frac{L_1 + L_2 \|\mu\|_{\infty,[0,r]}^\beta}{C_g}
\leq \exp \left\{ (4L_f r \vee 2) \left[ 2 + (2KC_g r^\nu)^{\frac{1}{\nu-\beta}} \|x\|_{\nu,\Delta_k}^{\frac{1}{\nu-\beta}} \right] \right\}
\times \left( \|h\|_{\beta,\Delta_{k-1}} + \frac{L_1 + L_2 \|\mu\|_{\infty,[0,r]}^\beta}{C_g} \right) - \frac{L_1 + L_2 \|\mu\|_{\infty,[0,r]}^\beta}{C_g}.
\] (4.33)
By induction, we obtain, for all \( k \geq 2 \),

\[
\|h\|_{\mu, \Delta_k} \leq \exp \left\{ 2(k-1)(4L_T r \vee 2) + (4L_T r \vee 2)(2KC_g r^\nu) \sum_{i=2}^{k} \|x\|_{\nu, \Delta_i} \right\} \times \left( \|h\|_{\nu, [r, 2r]} + \frac{L_1 + L_2 \|\mu\|_{\infty, [0, r]}}{C_g} \right) - \frac{L_1 + L_2 \|\mu\|_{\infty, [0, r]}}{C_g}.
\] (4.34)

Now, in a similar manner as in the proof of Proposition 2.2 we can estimate \( \|h\|_{\beta, [r, 2r]} \) as follow

\[
\|h\|_{\beta, [r, 2r]} \leq D \left( 1 + \|h\|_{\beta, [0, r]} \right) e^{D \|x\|_{\nu, [r, 2r]}},
\]

where \( D \) is some positive constant independent of \( k \) and \( x \). Since \( \|h\|_{\beta, [0, r]} = 0 \) we can write (4.34) in the form

\[
\|h\|_{\beta, \Delta_k} \leq \|\mu\|_{\infty, [0, r]} \xi_1(k, x) + \xi_2(k, x), \quad \forall k \geq 2,
\] (4.35)

in which \( \xi_1, \xi_2 \) have form \( \exp \left\{ D \left( k + \sum_{i=1}^{k} \|x\|_{\nu, \Delta_i} \right) \right\} \) for some generic positive constant \( D \) independent of \( k \) and \( x \). Applying Young’s inequality \( ab \leq \beta a^\frac{1}{\beta} + (1 - \beta)b^\frac{1}{1-\beta} \), \( \forall a, b \geq 0 \), to (4.35) we get

\[
\|h\|_{\beta, \Delta_k} \leq \epsilon \beta\|\mu\|_{\infty, [0, r]} + \left( \frac{\xi_1(k, x)}{\epsilon} \right)^{1-\frac{1}{1-\beta}} + \xi_2(k, x), \quad \forall k \geq 2, \quad \epsilon > 0.
\] (4.36)

Choose and fix \( \epsilon > 0 \) small enough such that \( \epsilon \beta < 1/2 \).

Next, to estimate \( \|\mu\|_{\beta, \Delta_k} \) we use the argument as in (4.16), and by virtue of (4.29) we get for all \( kr \leq s < t \leq (k + 1)r \)

\[
r^\beta \frac{\|\mu(t) - \mu(s)\|}{(t-s)^\beta} \leq \|f(0)\| r + L_T r \|\mu\|_{\infty, [s-r, s]} + L_T \int_s^t r^\beta \|\mu\|_{\beta, [s, u]} du \leq \|f(0)\| r + CA L_T e^{\lambda r} r e^{-\lambda_0 (k-1)r} \|\mu\|_{\infty, [0, r]} + L_T \int_s^t r^\beta \|\mu\|_{\beta, [s, u]} du.
\]

By applying the Gronwall Lemma 5.3 to the function \( r^\beta \|\mu\|_{\beta, [kr, \cdot]} \), again like in the proof of Lemma 4.4 we get

\[
r^\beta \|\mu\|_{\beta, \Delta_k} \leq \|f(0)\| e^{L_T r} + CA L_T e^{(2\lambda + L_T) r} e^{-\lambda_0 kr} \|\mu\|_{\infty, [0, r]}.
\]

Combining this with (4.29) we get

\[
\|\mu\|_{\beta, \Delta_k} \leq D \|f(0)\| + De^{-\lambda_0 kr} \|\mu\|_{\infty, [0, r]},
\] (4.37)

where \( D \) is a positive constant independent of \( k \) and \( x \). From (4.36) and (4.37) we derive

\[
\|y\|_{\beta, \Delta_k} \leq \|h\|_{\beta, \Delta_k} + \|\mu\|_{\beta, \Delta_k} \\
\leq \left( \epsilon \beta + De^{-\lambda_0 kr} \right) \|\mu\|_{\infty, [0, r]} + \xi(k, x) \\
\leq \left( \epsilon \beta + De^{-\lambda_0 kr} \right) \|y\|_{\beta, [0, r]} + \xi(k, x),
\] (4.38)
where $\xi(k,x)$ has the form similar to that of $\xi_1, \xi_2$ above. We choose and fix $k_0$ large enough so that $De^{-\lambda ak_0} < 1/2$, then, by the choice of $\epsilon$, we have $(\epsilon \beta + De^{-\lambda ak_0}) =: \gamma < 1$, and thus

$$\|y\|_{\beta, \Delta_{k_0}} \leq \gamma \|y\|_{\beta, [0, r]} + \xi(k_0, x).$$

Consequently, since our equation is autonomous we can apply the above arguments to the shifted equation and get for all $n \in \mathbb{N}$, $n \geq 2$,

$$\|y\|_{\beta, \Delta_{n k_0}} \leq \gamma^n \|y\|_{\beta, [0, r]} + \sum_{i=0}^{n-1} \gamma^i \xi(k_0, \theta_{(n-i)k_0} x).$$

Changing from $x$ to $\theta_{-nk_0} x$, we arrive at

$$\|y(\cdot, \theta_{-nk_0} x, \eta)\|_{\beta, \Delta_{nk_0}} \leq \gamma^n \|y(\cdot, \theta_{-nk_0} x, \eta)\|_{\beta, [0, r]} + \sum_{i=0}^{n-1} \gamma^i \xi(k_0, \theta_{-ik_0} x), \quad n \geq 2. \quad (4.41)$$

Note that $\xi$ is tempered under the assumption (3.3). Using the same arguments as at the end of the proof of Theorem 4.3, taking into account that $\gamma < 1$, we can find a tempered random variable $\hat{b}(x)$ such that for any tempered compact random set $\hat{D}(\cdot) \in \mathcal{D}$, there exits an integer time moment $n(x, \hat{D}) > 0$ such that for all $n \geq n(x, \hat{D})$ and all $\eta \in \hat{D}(\theta_{-nk_0} x)$ we have

$$\|y(\cdot, \theta_{-nk_0} x, \eta)\|_{\beta, \Delta_{nk_0}} \leq \hat{b}(x).$$

Here we only estimate the norm of $y$ on $\Delta_{nk_0} x$, $n \geq 1$, but one can easily get similar estimate for the norm of $y$ in the interval $[t-r, t]$ for $t \geq k_0 r$. Thus we find a compact absorbing set

$$\mathcal{B}(x) := \{\eta \in \mathcal{C}^{0, \beta_0}([-r, 0], \mathbb{R}^d) \|\eta\|_{\beta, [-r, 0]} \leq \hat{b}(x)\}$$

for the random dynamical system generated by the equation (4.1). Consequently, the random dynamical system generated by the equation (4.1) possesses a random pullback attractor $\mathcal{A}(x) \subset \mathcal{B}(x)$ (see [13, Theorem 3.5]). Clearly, $\mathcal{A}(x) \subset \mathcal{C}^{\beta}([-r, 0], \mathbb{R}^d) \subset \mathcal{C}^{0, \beta_0}([-r, 0], \mathbb{R}^d). \quad \Box$

**Theorem 4.8** Assume that the conditions in Theorem 4.3 are satisfied and, in addition, $g$ is a linear form on $C_r$. Then there exists $\epsilon > 0$ such that for $C_g < \epsilon$ the random pullback attractor of the system (4.1) provided by Theorem 4.3 is a singleton.

**Proof:** Suppose that there exist two distinct points $a_1(x), a_2(x) \in \mathcal{A}(x) \subset \mathcal{C}^{\beta}([-r, 0], \mathbb{R}^d)$, where $\mathcal{A}(x)$ is the random pullback attractor provided by Theorem 4.3. We show that this will lead to a contradiction.

Fix $n \in \mathbb{N}$, $n \geq 2$. Put $x^* := \theta_{-nr} x$ and consider the equation

$$dy(t) = [Ay(t) + f(y_t)]dt + g(y_t)dx^*(t). \quad (4.42)$$

By the invariance principle there exist two diferent points $b_1 = b_1(x^*), b_2 = b_2(x^*) \in \mathcal{A}(x^*)$ such that

$$a_i(x) = y_{nr}(\cdot, x^*, b_i), \quad i = 1, 2,$$

where, $y(\cdot, x^*, b_i)$ denotes the solution of (4.1) with the driver $x$ replaced by $x^*$ and the initial condition $\eta$ replaced by $b_i$, and $y_{nr}(\cdot, x^*, b_i)$ denotes the shifted function $y(\cdot + nr, x^*, b_i)$ considered
as a function on \([-r,0]\). Put 
\[ y^1(t) := y(t,x^*,b_1), \quad y^2(t) := y(t,x^*,b_2), \quad y(t) := y^1(t) - y^2(t) = y(t,x^*,b_1) - y(t,x^*,b_2). \]
Then we have 
\[ dy(t) = [Ay(t) + f(y_t + y^2_t) - f(y^1_t)]dt + [g(y^1_t) - g(y^2_t)]dx^*(t) \]
and we get 
\[ dy(t) = [Ay(t) + f(y_t + y^2_t) - f(y^1_t)]dt + g(y^1_t)dx^*(t) \]
(4.43)
Now we estimate the norm of \( y(\cdot) \) using (4.43) and the method of the proof of Theorem 4.5. Notice that the results of Theorem 4.5 are not applicable directly to (4.43) because (4.43) is non-autonomous. However, a careful look at the proof of Theorem 4.5 assures us that, due to the specific construction of \( f^* \) from \( f \), this proof can be modified to the case of a non-autonomous system (4.43) as well to get some useful intermediate estimates. Namely, taking into account that \( g \) is a linear function, we repeat the calculation in the proof of Theorem 4.5 in which \( f \) is replaced by \( x^* \), \( f \) is replaced by the nonautonomous function \( f^* \) (notice that the constants \( C_f, L_f \) are not changed due to the construction of \( f^* \) from \( f \)). Since \( f^*(t,0) \equiv 0, \quad g(0) = 0 \), similar to (4.23) we obtain 
\[ ||y||_{\beta,\Delta_n} \leq M_8 e^{-\lambda_0 \theta_0} \prod_{k=0}^{n-1} \left[ 1 + C_g M_7 G(\theta_{(k+1)} \theta_0, [0,r]) \right] \]
\[ \leq M_8 e^{-\lambda_0 \theta_0} \prod_{k=0}^{n-1} \left[ 1 + C_g M_7 G(\theta_{-kr}, [0,r]) \right]. \]
Therefore,
\[ ||a_1(x) - a_2(x)||_{\beta,[-r,0]} = ||y_{nr}(\cdot)||_{\beta,[-r,0]} = ||y||_{\beta,\Delta_n-1} \]
\[ \leq M_8 e^{-\lambda_0 \theta_0} \prod_{k=0}^{n-1} \left[ 1 + C_g M_7 G(\theta_{-kr}, [0,r]) \right]. \]
(4.44)
We estimate the terms in the right-hand side of the inequality in (4.44). Using (2.10) with \( x \) replaced by \( x^* \) and the fact that \( A \subset B \), where \( B \) is determined at the end of the proof of Theorem 4.5, taking into account that we have 
\[ ||y^1 - y^2||_{\beta,[0,r]} \leq ||y^1||_{\beta,[0,r]} + ||y^2||_{\beta,[0,r]} \]
\[ \leq D \left( 1 + ||x^*||_{\nu,[0,r]} \right) \left( 2 + ||y^1||_{\beta_0,[-r,0]} + ||y^2||_{\beta_0,[-r,0]} \right) e^{D ||x^*||_{\nu,[0,r]} \theta_0} \]
\[ \leq 2(1 + b(x^*)) \xi(||x^*||_{\nu,[0,r]}) \]
where \( b(\cdot) \) is the diameter of \( B \) which is tempered, and \( \xi(\cdot) \) is a tempered function similar to that of the function \( \bar{F}(\cdot) \) in the proof of Theorem 4.5. Now, since we supposed \( a_1(x) \neq a_2(x) \) hence 
\[ ||a_1(x) - a_2(x)||_{\beta,[-r,0]} \] is a positive constant (for the fixed driver \( x \) implying ||\( a_1(x) - a_2(x) \)||_{\beta,[-r,0]} > 0 \), therefore by letting \( n \to \infty \), from (4.44) we obtain 
\[ 0 = \lim_{n \to \infty} \frac{1}{n} \log ||a_1(x) - a_2(x)||_{\beta,[-r,0]} \]
\[ \leq -\lambda_0 r + \lim_{n \to \infty} \frac{1}{n} \log \left( 1 + b(\theta_{-nr} x) \right) \xi(\theta_{-nr} x)^\nu_{\nu,[0,r]} \right] + \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \left[ 1 + C_g M_7 G(\theta_{-kr}, [0,r]) \right] \]
\[ \leq -\lambda_0 r + \hat{G} < 0 \]
if $C_g$ is small enough, where $\hat{G}$ is defined by (4.27). We arrive at a contradiction provided $C_g$ is small enough. The theorem is proved. □

5 Appendix

Young integrals

For $[a, b] \subset \mathbb{R}$, denote by $C([a, b], \mathbb{R}^d)$ the space of all continuous functions $y : [a, b] \to \mathbb{R}^d$, equipped with the sup norm

$$\|y\|_{\infty,[a,b]} = \sup_{t \in [a,b]} \|y(t)\|,$$

in which $\| \cdot \|$ is the Euclidean norm of a vector in $\mathbb{R}^d$. Also, for $0 < \beta \leq 1$ denote by $C^\beta ([a, b], \mathbb{R}^d)$ the Banach space of all Hölder continuous paths $y : [a, b] \to \mathbb{R}^d$ with exponential $\beta$, equipped with the norm

$$\|y\|_{\infty,\beta,[a,b]} := \|y\|_{\infty,[a,b]} + \|y\|_{\beta,[a,b]},$$

where

$$\|y\|_{\beta,[a,b]} := \sup_{a \leq s < t \leq b} \frac{\|y(t) - y(s)\|}{(t-s)^\beta} < \infty. \quad (5.1)$$

One can easily prove that for any $a \leq s \leq t \leq u \leq b$ we have

$$\|y\|_{\beta,[s,u]} \leq \|y\|_{\beta,[s,t]} + \|y\|_{\beta,[t,u]}.$$

Note that the space $C^\beta ([a, b], \mathbb{R}^d)$ is not separable. However, the closure of $C^\infty([a, b], \mathbb{R}^d)$ denoted by $C^\alpha([a, b], \mathbb{R}^d)$ is a separable space (see [16] Theorem 5.31, p. 96), which can be defined as

$$C^{\alpha,\beta}([a, b], \mathbb{R}^d) := \left\{ x \in C^{\beta}([a, b], \mathbb{R}^d) \mid \lim_{h \to 0} \sup_{a \leq s < t < b, |t-s| \leq h} \frac{|x(t) - x(s)|}{(t-s)^\beta} = 0 \right\}. $$

It is worth to note that for $\beta < \alpha$, $C^\alpha([a, b], \mathbb{R}^d)$ is a subspace of $C^{\beta,\beta}([a, b], \mathbb{R}^d)$. Moreover, the embedding operator

$$id : C^\alpha([a, b], \mathbb{R}^d) \to C^{\beta}([a, b], \mathbb{R}^d)$$

is compact (see [16] Proposition 5.28, p. 94).

Now we recall that for $y \in C^{\gamma}([a, b], \mathbb{R}^{d \times k})$ and $x \in C^\nu([a, b], \mathbb{R}^k)$ with $\beta + \nu > 1$. Then the Young integral $\int_a^b y(t)dx(t)$ exists (see [22]) and moreover, for all $s \leq t$ in $[a, b]$,

$$\left\| \int_s^t y(u)dx(u) - y(s)[x(t) - x(s)] \right\| \leq K(t-s)^{\beta+\nu} \|y\|_{\nu,[s,t]} \|x\|_{\beta,[s,t]},$$

where $K := \frac{1}{1-2^{1-(\beta+\nu)}}$ (see Young-Loeve estimate [16] Theorem 6.8, p. 116). Hence

$$\left\| \int_s^t y(u)dx(u) \right\| \leq (t-s)^{\nu} \|x\|_{\nu,[s,t]} \left( \|y(s)\| + K(t-s)^{\beta} \|y\|_{\beta,[s,t]} \right). \quad (5.2)$$

Tempered variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with an ergodic metric dynamical system $\theta$, which is a $\mathbb{P}$ measurable mapping $\theta : \mathbb{T} \times \Omega \to \Omega$, $\mathbb{T}$ is either $\mathbb{R}$ or $\mathbb{Z}$, and $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{T}$. Recall that a random variable $\rho : \Omega \to [0, \infty)$ is called tempered if
\[
\lim_{t \to \pm \infty} \frac{1}{t} \log^+ \rho(\theta_t x) = 0, \quad \text{a.s.}
\]  
(5.3)

which, as shown in [17, p. 220], [19], is equivalent to the sub-exponential growth
\[
\lim_{t \to \pm \infty} e^{-c|t|} \rho(\theta_t x) = 0 \quad \text{a.s.} \quad \forall c > 0.
\]

Note that our definition of temperedness corresponds to the notion of temperedness from above given in [3, Definition 4.1.1(ii)].

**Lemma 5.1**

(i) If \(h_1, h_2 \geq 0\) are tempered random variables then \(h_1 + h_2\) and \(h_1 h_2\) are tempered random variables.

(ii) If \(h_1 \geq 0\) is a tempered random variable, \(h_2 \geq 0\) is a measurable random variable and \(h_2 \leq h_1\) almost surely, then \(h_2\) is a tempered random variable.

(iii) Let \(h_1\) be a nonnegative measurable function. If \(\log h_1 \in L^1\) then \(h_1\) is tempered.

**Proof:**

(i) See [3, Lemma 4.1.2, p. 164].

(ii) Immediate from the definition of tempered random variable, formula (5.3).

(iii) See [3, Proposition 4.1.3, p. 165].

**Lemma 5.2**

(i) Let \(a : \Omega \to [0, \infty)\) be a random variable, \(\log(1 + a(\cdot)) \in L^1\) and \(\hat{a} := E\log(1 + a(\cdot)) = \int_{\Omega} \log(1 + a(\cdot)) d\mathbb{P}\). Let \(\lambda > \hat{a}\) be an arbitrary fixed positive number. Put

\[
b(x) := \sum_{k=1}^{\infty} e^{-\lambda k} \prod_{i=0}^{k-1} (1 + a(\theta_{-i} x)).
\]

Then \(b(\cdot)\) is a nonnegative almost everywhere finite and tempered random variable.

(ii) Let \(c : \Omega \to [0, \infty)\) be a tempered random variable, and \(\delta > 0\) be an arbitrary fixed positive number. Put

\[
d(x) := \sum_{k=1}^{\infty} e^{-\delta k} c(\theta_{-k} x).
\]

Then \(d(\cdot)\) is a nonnegative almost everywhere finite and tempered random variable.

**Proof:**

(i) Put \(b_n(x) := \sum_{k=1}^{n} e^{-\lambda k} \prod_{i=0}^{k-1} (1 + a(\theta_{-i} x))\). Then \(b_n(\cdot)\), \(n \in \mathbb{N}\), is an increasing sequence of nonnegative random variable, hence converges to the nonnegative random variable \(b(\cdot)\). Since \(\log(1 + a(\cdot)) \in L^1\), by Birkhoff ergodic theorem there exists a \(\theta\)-invariant set \(\Omega' \subset \Omega\) of full measure such that for all \(x \in \Omega'\) we have \(\lim_{n \to \pm \infty} (\sum_{i=0}^{n-1} \log(1 + a(\theta_{-i} x))/n = \hat{a}\). Hence given any fixed \(\delta > 0\) for all \(n\) big enough \(\prod_{i=0}^{n-1} (1 + a(\theta_{-i} x)) < \exp(\hat{a} + \delta)n\). Consequently, since \(\lambda > \hat{a}\) the sequence \(b_n(\cdot), n \in \mathbb{N}\) tends to limit \(b(\cdot)\) which is finite almost everywhere.

Now we show that \(b(\cdot)\) is tempered. For \(m \in \mathbb{N}\) and \(x \in \Omega'\) we have

\[
b(\theta_{-m} x) = \sum_{k=1}^{\infty} e^{-\lambda k} \prod_{i=0}^{k-1} (1 + a(\theta_{-i} \theta_{-m} x)) = \sum_{k=1}^{\infty} e^{-\lambda k} \prod_{j=m}^{k+m-1} (1 + a(\theta_{-j} x))
\]

\[
\leq e^{\lambda m} \sum_{l=1+m}^{\infty} e^{-\lambda l} \prod_{j=0}^{l-1} (1 + a(\theta_{-j} x)) \leq e^{\lambda m} \sum_{l=1}^{\infty} e^{-\lambda l} \prod_{j=0}^{l-1} (1 + a(\theta_{-j} x)) = e^{\lambda m} b(x).
\]
This implies that \( \limsup_{m \to \infty} \frac{1}{m} \log^+ b(\theta_m x) \leq \lambda \). By virtue of [3] Proposition 4.1.3(i), p. 165] and [20] Lemma 4, Corollary 4], for all \( x \in \Omega' \) we have
\[
\limsup_{m \to \infty} \frac{1}{m} \log^+ b(\theta_m x) = \limsup_{m \to -\infty} \frac{1}{m} \log^+ b(\theta_m x) = 0,
\]
what implies that \( b(\cdot) \) is tempered.

(ii) Put \( d_n(x) := \sum_{k=1}^{n} e^{-\delta_k} c(\theta_k x) \). Then \( d_n(\cdot), n \in \mathbb{N}, \) is an increasing sequence of nonnegative random variable, hence converges to the nonnegative random variable \( d(\cdot) \). By temperedness of \( c(\cdot) \) we can find a measurable set \( \tilde{\Omega} \subset \Omega \) of full measure such that for all \( x \in \tilde{\Omega} \) there exists \( n_0(x) > 0 \) such that for all \( n \geq n_0(x) \) we have \( c(\theta_n x) \leq e^{\delta n/2} \). Hence \( d_n(x), n \in \mathbb{N}, \) is an increasing sequence of positive numbers tending to finite value \( d(x) \). Thus \( d(\cdot) \) is finite almost everywhere. Furthermore, for \( m \in \mathbb{N} \) and \( x \in \tilde{\Omega} \) we have
\[
d(\theta_m x) = \sum_{k=1}^{\infty} e^{-\delta_k} c(\theta_k \theta_m x) = \sum_{l=m+1}^{\infty} e^{-\delta(k-l)} a(\theta_l x) \leq e^{\delta m} \sum_{l=1}^{\infty} e^{-\delta l} c(\theta_l x) = e^{\delta m} d(x).
\]
This implies that \( \limsup_{m \to \infty} \frac{1}{m} \log^+ d(\theta_m x) \leq \delta \). Similar to (i) above, \( d(\cdot) \) is tempered. \( \square \)

**Gronwall lemma**

**Lemma 5.3 (Continuous Gronwall Lemma)** Let \([t_0, T]\) be an interval on \(\mathbb{R}\). Assume that \(u(\cdot), a(\cdot) : [t_0, T] \to \mathbb{R}^+\) are positive continuous functions and \(\beta > 0\) is a positive number, such that
\[
\int_{t_0}^{t} \beta u(s) ds, \quad \forall t \in [t_0, T].
\]
Then the following inequality holds
\[
u(t) \leq a(t) + \int_{t_0}^{t} \beta u(s) ds, \quad \forall t \in [t_0, T].
\]

**Proof:** See [2] Lemma 6.1, p 89]. \( \square \)

**Lemma 5.4 (Discrete Gronwall Lemma)** Let \(a\) be a non negative constant and \(u_n, a_n, \beta_n\) be nonnegative sequences satisfying for all \(n \in \mathbb{N}, n \geq 0\), the equalities
\[
u_n \leq a + \sum_{k=0}^{n-1} a_k u_k + \sum_{k=0}^{n-1} \beta_k.
\]
Then for all \(n \in \mathbb{N}, n \geq 1\), the following inequalities hold
\[
u_n \leq \max\{a, u_0\} \prod_{k=0}^{n-1} (1 + a_k) + \sum_{k=0}^{n-1} a_k \prod_{j=k+1}^{n-1} (1 + a_j). \tag{5.4}
\]

**Proof:** See [13]. \( \square \)

**Acknowledgments**

This work is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.03-2019.310. P.T. Hong would like to thank the IMU Breakout Graduate Fellowship Program for the financial support.
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