PROJECTIVE LIMIT OF A SEQUENCE OF COMPATIBLE WEAK SYMPLECTIC FORMS ON A SEQUENCE OF BANACH BUNDLES AND DARBOUX THEOREM

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Abstract. Given a projective sequence of Banach bundles, each one provided with a weak symplectic form, we look for conditions under which the corresponding sequence of weak symplectic forms gives rise to a weak symplectic form on the projective limit bundle. Then we apply this results to the tangent bundle of a projective limit of Banach manifolds. This naturally leads to ask about conditions under which the Darboux Theorem is also true on the projective limit of Banach manifolds. We will give some necessary and some sufficient conditions so that such a result is true. Then we discuss why, in general, the Moser’s method can not work on projective limit of Banach weak symplectic Banach manifolds without very strong conditions like Kumar’s results ([17]). In particular we give an example of a projective sequence of weak symplectic Banach manifolds on which the Darboux Theorem is true on each manifold, but is not true on the projective limit of these manifolds.

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1. Introduction

In the Banach context, it is well known that a symplectic form can be strong or weak (see Definition 1). The Darboux Theorem was firstly proved for strong symplectic Banach manifolds by Weinstein ([27]). But Marsden ([20]) showed that the Darboux theorem fails for a weak symplectic Banach manifold. However Bambusi ([3]) found necessary and sufficient conditions for the validity of Darboux theorem for a weak symplectic Banach manifold (Darboux-Bambusi Theorem). The proofs of all these versions of Darboux Theorem were all established by Moser’s method.

In a wider context like Fréchet or convenient manifolds, a symplectic form is always weak. Recently, a new approach to differential geometry in Fréchet context was initiated and developed by G. Galanis, C. T. J. Dodson, E. Vassiliou and their collaborators in terms of projective limits of Banach manifolds (see [7] for a panorama of these results). In this situation, P. Kumar, in [17], proves a version of Darboux Theorem, by Moser method, for a projective sequence of weak symplectic manifolds which satisfy the assumption of the Darboux-Bambusi Theorem but under very strong added conditions on this sequence. On the other hand, a metric approach of differential geometry on Fréchet manifold was firstly introduced by Muller. This concept gives rise to Keller-differentiable calculus as exposed in details by Glockner in [13]. In this way we can consider the so called bounded Fréchet framework (cf. [23]) in which a classical implicit function Theorem is true and a Theorem of existence of local flow can be proved (cf. [8]). In this context Eftekharinasab in [9], proves a version of Darboux Theorem using Moser’s method also under very strong assumptions. In fact when such a Fréchet manifold is also a projective limit of Banach manifolds this result seems to recover Kumar’s result.

More generally we can look for conditions under which a family of weak symplectic forms on a projective sequence of Banach bundles gives rise to a weak symplectic form on the projective limit bundle: this is the essential purpose of this paper.
Of course this naturally leads to application for projective limit of weak Banach manifolds and the problem of the existence of a Darboux theorem on the projective limit of weak symplectic Banach manifolds under the assumption of Darboux-Bambusi Theorem.

More precisely let \((E_i, \ell_i^1)_{j \geq i}\) be a reductive\(^1\) projective sequence Banach spaces and \((\omega_i)_{i \in \mathbb{N}}\) be a sequence of (linear) weak symplectic forms \(\omega_i\) on \(E_i\). We say that \((\omega_i)_{i \in \mathbb{N}}\) is a sequence of compatible symplectic forms if each \(\ell_i^{1+i}\) satisfies
\[
\ker\ell_i^{1+i} \cap (\ker\ell_i^{1+i})^\perp = \{0\}\] and \((\ell_i^{1+i})^* \omega_i = \omega_{i+1}\) in restriction to \((\ker\ell_i^{1+i})^\perp\) where \((\ker\ell_i^{1+i})^\perp\) is the orthogonal of \(\ker\ell_i^{1+i}\) relatively to \(\omega_{i+1}\) (cf. Definition\(^2\)).

Now consider reductive projective sequence of Banach bundles \((E_i, \lambda_i^j)_{j \geq i}\) over a projective sequence \((M_i, \delta_i^j)_{j \geq i}\) of manifolds and let \((\omega_i)_{i \in \mathbb{N}}\) be a sequence of weak symplectic forms \(\omega_i\) on \(E_i\). We say that \((\omega_i)_{i \in \mathbb{N}}\) is a sequence of compatible symplectic forms if; for any \(x_i \in M_i\) and \(i \in \mathbb{N}\), the sequence \(((\omega_i)_{x_i})_{i \in \mathbb{N}}\) is a sequence of compatible (linear) weak symplectic forms on the projective sequence \((\pi_i^{-1}(x_i), (\lambda_i^j)_{x_i})_{j \geq i}\) of Banach spaces (cf. Definition\(^3\)).

In this context, we have (cf. Theorem\(^4\) and Corollary\(^5\)):

**Theorem 1.** Consider a reductive\(^1\) projective sequence \((E_i, \lambda_i^j)_{j \geq i}\) of Banach bundles over a projective sequence of Banach manifolds \((M_i, \delta_i^j)_{j \geq i}\).

1. Let \((\omega_i)_{i \in \mathbb{N}}\) be a sequence of compatible weak symplectic forms on \((E_i, \lambda_i^j)_{j \geq i}\). Then \(\omega = \lim\omega_i\) is a well defined weak symplectic form on the Fréchet bundle \(E = \varprojlim E_i\) over \(M = \varprojlim M_i\).

2. Conversely, let \(\omega\) be a weak symplectic form on a projective limit bundles \((E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)\) of a submersive\(^4\) projective sequence of Banach bundles \((E_i, \lambda_i^j)_{j \geq i}\). Assume that for each \(x = \varprojlim x_i\), the map \((\lambda_i)_x : \pi_i^{-1}(x) \to \pi_i^{-1}(x_i)\) is a symplectic submersion\(^4\). Then \(\omega\) induces a weak symplectic 2-form \(\omega_i\) on \(E_i\) which gives rise to a family of compatible weak symplectic forms. Moreover, the 2-form on \(E\) defined by this sequence \((\omega_i)\) is precisely the given 2-form \(\omega\).

3. A 2-form \(\omega\) on a Fréchet bundle, projective limit \((E = \varprojlim E_i, \pi = \varprojlim \pi_i, M = \varprojlim M_i)\) of a submersive sequence of Banach fibre bundles \((E_i, \lambda_i^j)_{j \geq i}\) is a weak symplectic form if and only if there exists a sequence of coherent weak symplectic forms \((\omega_i)_{i \in \mathbb{N}}\) on \(E_i\) such that \(\omega = \varprojlim \omega_i\).

As corollary we obtain (cf. Theorem\(^6\)):

**Theorem 2.**

1. Let \((M_i, \lambda_i^j)_{j \geq i}\) be a reduced sequence of Banach manifolds and \((\omega_i)_{i \in \mathbb{N}}\) a sequence of coherent weak symplectic forms. Then \(\omega = \varprojlim \omega_i\) is a weak symplectic form on \(M = \varprojlim M_i\).

2. Let \(\omega\) be a 2-form on a projective limit \(M = \varprojlim M_i\) of a submersive sequence of manifolds \((M_i, \delta_i^j)_{j \geq i}\). Then \(\omega\) is a weak symplectic form if and only if each

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\(^1\) the projective sequence of typical fiber \((E_i, \lambda_i^j)_{j \geq i}\) is reductive

\(^2\)that is the projective sequence of typical fiber \((E_i, \lambda_i^j)_{j \geq i}\) is a reduced projective sequence of Banach spaces

\(^3\)cf. Definition\(^7\)

\(^4\)cf Definition\(^8\)
$T_x\delta_i : T_xM \to T_xM_i$ is symplectic submersion for each $i \in \mathbb{N}$.

Now, in the context of Theorem 2 Point (1), assume that each weak symplectic manifold $(M_i, \omega_i)$ satisfies the assumptions of Darboux-Bambusi Theorem (cf. Theorem 17) for each $i \in \mathbb{N}$, then it follows that the same is true for the projective limit $(M = \lim\limits_{\leftarrow} M_i, \omega = \lim\limits_{\leftarrow} \omega_i)$ (cf. Theorem 22).

Since the Darboux-Bambusi Theorem is then true for each $(M_i, \omega_i)$ it seems natural to look for the same result for $(M, \omega)$. A partial answer is given in Theorem 24.

The last section of this paper is devoted to a discussion on how the Moser’s method in the previous context can be applied. In particular Theorem 27 gives (very strong) sufficient conditions under which the Moser’s method can be applied and which is a kind of generalization of Kumar’s result ([16]).

Unfortunately such kind of results require so strong assumptions, that it seems there is no concrete applications outside elementary examples.

Finally, we give some examples for which the Darboux-Bambusi theorem is true and an example for which the Darboux-Bambusi Theorem is true on each manifold, but is not true on the projective limit of these manifolds. Note that this last section is analog to the same type of discussion in [25] in the context of direct limit of weak symplectic manifolds.

This work is self contained.

In section 2, after a survey on known results on symplectic forms on a Banach space (§2.1) we look for properties of coherent sequence of (linear) symplectic forms on a projective sequence of Banach spaces (§ 2.2). The precise context of Theorem 4 (resp. Theorem 2) can be found in § 2.3 (resp. §2.4). The proofs of all these results take place in §2.5.

Section 3 is devoted to show that under the assumption of Theorem 2, the assumption of Darboux-Bambusi assumptions which are satisfied for projective limit of weak symplectic manifold are also valid on the projective limit. The first subsection recall the Moser’s method and the Darboux-Bambusi Theorem. In the next subsection, under assumption of Theorem 4 (1), for such a projective limits of Banach bundles which satisfy a generalization of Darboux-Bambusi Theorem we show that the projective limit have the same properties. The last section is a discussion on the problem of existence of Darboux charts on a strong reduced projective sequence of Banach manifolds. Sufficient conditions are given in the first subsection. The announced discussion is developed in §4.2. Examples and contre-example about the existence of a projective limit of Darboux charts are given in §4.3. Finally we end this paper by a series of Appendices which sumrize all the definitions and properties on projective limits needed in this paper.

2. Projective limit of a coherent sequence of weak symplectic forms on a projective sequence of Banach bundle

2.1. Symplectic forms on Banach space. In this section we recall some well known results on linear symplectic form on a Banach space (cf. for instance [25]):

Definition 1. Let $E$ be a Banach space. A bilinear form $\omega$ is said to be weakly non degenerate if $\langle Y \in E, \omega(X, Y) = 0 \rangle \implies X = 0$.

Classically, to $\omega$ is associated the linear map $\omega^\flat : E \rightarrow E^*$ defined by $\omega^\flat(X)(Y) = \omega(X, Y), \forall Y \in E$.

Clearly, $\omega$ is weakly non degenerate if and only if $\omega^\flat$ is injective.

The 2-form $\omega$ is called strongly nondegenerate if $\omega^\flat$ is an isomorphism.
A fundamental result in finite dimensional linear symplectic space is the existence of a Darboux (linear) form for a symplectic 2-form:

If \( \omega \) is a symplectic form on a finite dimensional vector space \( E \), there exists a vector space \( L \) and an isomorphism \( A : E \to L \oplus L^* \) such that \( \omega = A^* \omega_L \) where
\[
\omega_L((u, \eta), (v, \xi)) = \langle \eta, v \rangle - \langle \xi, u \rangle
\]
(1)

This result is in direct relation with the notion of Lagrangian subspace which is a fundamental tool in the finite dimensional symplectic framework.

In the Banach framework, let \( \omega \) be a weak symplectic form on a Banach space.

A subspace \( F \) is isotropic if \( \omega(u, v) = 0 \) for all \( u, v \in F \). An isotropic subspace is always closed.

If \( F^{\perp_\omega} = \{ w \in E : \forall u \in F, \ \omega(u, v) = 0 \} \) is the orthogonal symplectic space of \( F \), then \( F \) is isotropic if and only if \( F \subset F^{\perp_\omega} \) and is maximal isotropic if \( F = F^{\perp_\omega} \). Unfortunately, in the Banach framework, a maximal isotropic subspace \( L \) can be not supplemented. Following Weinstein’s terminology ([27]), an isotropic space \( L \) is called a Lagrangian space if there exists an isotropic space \( L' \) such that \( E = L \oplus L' \). Since \( \omega \) is strong non degenerate, this implies that \( L \) and \( L' \) are maximal isotropic and then are Lagrangian spaces (see [27]).

Unfortunately, in general, for a given symplectic structure, Lagrangian subspaces need not exist (cf. [14]). Even for a strong symplectic structure on Banach space which is not Hilbertizable, the non existence of Lagrangian subspaces is an open problem to our knowledge. Following [27], a symplectic form \( \omega \) on a Banach space \( E \) is a Darboux (linear) form if there exists a Banach space \( L \) and an isomorphism \( A : E \to L \oplus L^* \) such that \( \omega = A^* \omega_L \) where \( \omega_L \) is defined in (1). Note that in this case \( E \) must be reflexive.

Let \( E \) be a Banach space provided with a norm \( \| \cdot \| \). We consider a symplectic form \( \omega \) on \( E \) and let \( \omega^\flat : E \to E^* \) be the associated bounded linear operator. Following [4] and [10], on \( E \), we consider the norm \( \| u \|_\omega = \| \omega^\flat(u) \|^\flat \) where \( \| \cdot \|^\flat \) is the canonical norm on \( E^* \) associated to \( \| \cdot \| \). Of course, we have \( \| u \|_\omega \leq \| \omega^\flat \|^\flat \| u \| \) (where \( \| \omega^\flat \|^\flat \) is the norm of the operator \( \omega^\flat \)) and so the inclusion of the normed space \( (E, \| \cdot \|_\omega) \) in \( (E, \| \cdot \|) \) is continuous. We denote by \( \hat{E} \) the Banach space which is the completion of \( (E, \| \cdot \|) \). Since \( \omega^\flat \) is an isometry from \( (E, \| \cdot \|_\omega) \) to \( E^* \), we can extend \( \omega^\flat \) to a bounded operator \( \tilde{\omega}^\flat \) from \( \hat{E} \) to \( E^* \). Assume that \( E \) is reflexive. Therefore \( \tilde{\omega}^\flat \) is an isometry between \( \hat{E} \) and \( E^* \) ([4] Lemma 2.7). Moreover, \( \omega^\flat \) can be seen as a bounded linear operator from \( E \) to \( E^* \) and is in fact an isomorphism ([4] Lemma 2.8). Note that since \( E^* \) is reflexive, this implies that \( \tilde{\omega}^\flat \) is also reflexive.

**Remark 2.** If \( \| \cdot \|^\star \) is an equivalent norm of \( \| \cdot \| \) on \( E \), then the corresponding \( \| \cdot \|^{\star \flat} \) and \( \| \cdot \|^{\flat \star} \) are also equivalent norm on \( E^* \) and so \( \| \cdot \|^\flat_{\omega} \) and \( \| \cdot \|_{\omega} \) are equivalent norms on \( E \) and so the completion \( \hat{E} \) depends only of Banach structure on \( E \) defined by equivalent the norms on \( E \).

### 2.2. Case of projective limit of Banach spaces.

Let \( \omega \) be a skew-symmetric bilinear form on a Banach space \( E \) and \( K \) a Banach subspace of \( E \). Recall that the \( \omega \)-orthogonal subspace \( K^{\perp_\omega} \) is defined by

\[
K^{\perp_\omega} = \{ x \in E : \forall y \in K, \ \omega(x, y) = 0 \}.
\]

When there is no ambiguity this set is simply denoted \( K^\perp \). Note that since \( \omega \) is skew-symmetric \( K^\perp = \{ x \in E : \forall y \in K, \ \omega(y, x) = 0 \} \) and so \( (K^\perp)^\perp = K \).

If \( K^0 = \{ \xi \in E^* : \forall u \in K, \ \xi(u) = 0 \} \) is the annihilator of \( K \), then \( K^\perp = (\omega^\flat)^{-1}(K^0) \).

Given two Banach subspaces \( K \) and \( K' \) of \( E \), the following relations are classical:

- If \( K \subset K' \) then \( K^\perp \subset K' \perp \) and, in particular, for any subspace \( K, E^\perp \subset K^\perp \).
- \( (K + K')^\perp = K^\perp \cap K'^\perp \).
- \( (K \cap K')^\perp = K^\perp + K'^\perp \).
Let $\omega$ (resp. $\omega'$) be a skew-symmetric bilinear form on a Banach space $E$ (resp. $E'$) and $\ell : E \to E'$ a continuous map. By analogy to the terminology for Finsler geometry (cf. [1]) we introduce

**Definition 3.** We say that $\ell$ is a weak isometry between $\omega$ and $\omega'$ if $\ell(E)$ is dense in $E'$ and we have:

$$\ker \ell \cap (\ker \ell)\perp = \{0\} \text{ and } \ell^* \omega' = \omega \text{ in restriction to } (\ker \ell)^\perp$$

(2)

Note that the condition "$\ker \ell \cap (\ker \ell)^\perp = \{0\}$" is equivalent to the condition "the restriction of $\omega$ to $(\ker \ell)^\perp$ is non degenerate".

**Proposition 4.** Let $\omega$ (resp. $\omega'$) be a weak skew-symmetric on a Banach space $E$ (resp. $E'$) and let $\ell : E \to E'$ be a continuous map. We set $K = (\ker \ell)$ and denote by $\overline{\omega}$ the restriction of $B$ to $K^\perp$. We have the following properties:

1. If $\ell$ is a weak isometry between $\omega$ and $\omega'$, then $\overline{\omega}$ and $\omega'_{|B}$ are non degenerate, and $\ker \ell^* \omega' = \ker \ell$.

2. If $\omega$ and $\overline{\omega}$ are non degenerate, $\ell$ is a weak isometry between $\omega$ and $\omega'$ if and only if $\ell^* \omega' = \omega$ on $(\ker \ell)^\perp$ and, in this case, the restriction of $\omega'$ to $\ell(E)$ is non degenerate.

3. Let $\omega''$ be a skew symmetric bilinear form on a Banach space $E''$ and $\ell' : E' \to E''$ a continuous linear map. if $\ell$ (resp. $\ell'$) is a weak isometry between $\omega$ and $\omega'$ (resp. $\omega'$ and $\omega''$) then $\ell' \circ \ell$ is a weak isometry between $\omega$ and $\omega''$.

Note that if $\ell$ is a weak symplectic isometry between $\omega$ and $\omega'$, the restriction $\overline{\ell}$ of $\ell$ to $(\ker \ell)^\perp$ is an isomorphism and

$$\overline{\ell}^* \omega' = \ell^* \omega'_{|\ker \ell} = \omega_{|\ker \ell}$$

(3)

**Proof.** (1) We have $\overline{\omega}(u,v) = 0$ for all $v \in K^\perp$ if and only if $u$ belongs $K \cap K^\perp$ which implies that $\overline{\omega}$ is non degenerate. Since $\overline{\ell}$ is an isomorphism, from (3), it follows that $\omega'_{|B}$ is non degenerate. Now, $v$ belongs to $\ker(\ell^* \omega')^\perp$ if and only if $\omega'(\ell(u), \ell(v))$, if and only if $\ell(u) = 0$.

(2) Assume $\omega$ and $\overline{\omega}$ are non degenerate. We must show that $E = K \oplus K^\perp$. But $(K \oplus K^\perp)^\perp = K^\perp \cap K = \{0\}$ and since $\omega$ is non degenerate, it follows that $\{0\}^\perp = E = K \oplus K^\perp$, which ends the proof of (2) according to (3).

(3) Under the assumptions of (3), we have $E = \ker \ell \oplus (\ker \ell)^\perp$ and $E' = \ker \ell' \oplus (\ker \ell')^\perp$. Now, since the inclusion of $E(\ell)$ in $E'$ continuous and we set $K' = \ker \ell' \cap \ell(E)$ then $(K')^\perp = (\ker \ell')^\perp \cap (\ker \ell)^\perp$ and so $\ell(E) = K' \oplus (K')^\perp$. Let $\overline{\ell}$ be the restriction of $\ell$ to $(\ker \ell)^\perp$; it is an isomorphism onto $(K')^\perp$. If $K = \overline{\ell}^{-1}(K')$ and $H = \overline{\ell}^{-1}(\ker \ell)^\perp$ then $(\ker \ell)^\perp = K \oplus H$. By construction, $\ker \ell' \circ \ell' = \ker \ell \oplus K$ and we have $(\ker \ell' \oplus K)^\perp = (\ker \ell)^\perp \cap K^\perp = H$. Indeed $H$ is contained in $(\ker \ell)^\perp$, and $H = \overline{\ell}^{-1}(\ker \ell)^\perp$ and $(\overline{\ell}^{-1})^* (\omega_{|\ker \ell})^\perp = \omega'_{|B}$, this implies that $H$ is the orthogonal of $K$ in $(\ker \ell)^\perp$. Now, the restriction of $\overline{\ell}^* \ell'$ of $\ell'$ to $(\ker \ell' \circ \ell)^\perp$ is an isomorphism. Consider for any $(u,v) \in H^\perp$ we have:

$$\omega'(u,v) = \overline{\omega}'(\overline{\ell}(u), \overline{\ell}(v)) = \omega''(\overline{\ell} \circ \ell(u), \overline{\ell} \circ \ell(v)).$$

So the proof is completed.

**Definition 5.** Let $(E_i, \ell^i_{i+1})_{i \geq 1}$ be a reductive \footnote{cf. Definition 3} projective sequence Banach spaces and $(\omega_i)_{i \in \mathbb{N}}$ be a sequence of (linear) weak symplectic forms $\omega_i$ on $E_i$. We say that $(\omega_i)_{i \in \mathbb{N}}$ is a coherent sequence of symplectic forms if each $\ell^i_{i+1}$ is a weak isometry between $\omega_{i+1}$ and $\omega_i$, for all $i \in \mathbb{N}$.
We then have the following property:

**Proposition 6.** Let \( (\omega_i)_{i \in \mathbb{N}} \) be a sequence of compatible symplectic forms on a reduced projective sequence \( (E_i, \ell_i^j)_{j \geq i} \). Then if \( u = \lim_{i \to \infty} u_i \) and \( v = \lim_{i \to \infty} v_i \) in \( E = \lim_{i \to \infty} E_i \),
\[
\omega(u, v) = \lim_{i \to \infty} \omega_i(u_i, v_i)
\]
defines a weak symplectic 2-form on \( E \).

Since the proof of this Proposition is very technical, the reader find it in Appendix [I].

**Remark 7.**

1. From the properties of the sequence \((\ell_i^j)_{j \geq i}\) and Proposition [II] (3), if \( (\omega_i)_{i \in \mathbb{N}} \) is a sequence of compatible weak symplectic 2-forms, then \( \ell_i^j \) is a weak isometry between \( \omega_j \) and \( \omega_i \), for all \( i \in \mathbb{N} \) and all \( j \geq i \).

2. Consider the assumptions of Proposition [II]. If \( (E_i, \ell_i^j)_{j \geq i} \) is a ILB sequence (cf. Appendix B), we have \( \ker \ell_i^j = \{0\} \) for all \( j \geq i \) and \( i \in \mathbb{N} \). Thus \( (\omega_i)_{i \in \mathbb{N}} \) is a sequence of compatible symplectic forms on this projective system if and only if, for all \( j \geq i \) and \( i \in \mathbb{N} \), \( \omega_j = (\lambda_i^j)^* \omega_i \), and \( \omega_0 \) is symplectic. But in general, if for some pair \((i, j)\), \( \ker \ell_i^j \neq \{0\} \), the condition \( \omega_j = (\ell_i^j)^* \omega_i \) implies that \( E_j^\perp \neq \{0\} \) and so \( \omega_j \) cannot be symplectic.

3. In Proposition [III] when \((E_i, \ell_i^j)_{j \geq i} \) is a surjective projective sequence, the symplectic form \( \omega \) on \( E \) has the property that the induced form on \( \ker \ell_i \) is sympelctic and so we have \( E = \ker \ell_i \oplus (\ker \ell_i)^\perp \) where \((\ker \ell_i)^\perp \) is the orthogonal of \( \ker \ell_i \) (relative to \( \omega \)).

As in finite dimension, we introduce:

**Definition 8.** Let \( E = \lim_{i \to \infty} E_i \) a projective limit of a surjective projective sequence \( (E_i, \ell_i^j)_{j \geq i} \).
Consider a (weak) symplectic form \( \omega \) on \( E \) such that \( E = \ker \ell_i \oplus (\ker \ell_i)^\perp \). We will say that \( \ell_i \) is a symplectic submersion.

**Remark 9.** In the context of Definition [IV], the restriction of \( \ell_i \) to \((\ker \ell_i)^\perp \) is an isomorphism onto \( \mathbb{R} \), and so we have a well symplectic form \( \omega_i \) on \( E_i \) such that \( \omega = \ell_i^* \omega_i \) in restriction to \((\ker \ell_i)^\perp \). Thus this definition is analog to the notion of isometric submersion between Finsler manifolds in finite dimension introduced in [IV].

We have the following type of converse of Proposition [III]:

**Proposition 10.** Let \((E_i, \ell_i^j)_{j \geq i} \) be a surjective projective sequence of Banach space and \( E = \lim_{i \to \infty} E_i \). If \( \omega \) is a symplectic form on \( E \) such that \( \ell_i : E \to E_i \) is a symplectic submersion for all \( i \in \mathbb{N} \), then \( \omega \) induces a symplectic form \( \omega_i \) on \( E_i \). Moreover, \((\omega_i)_{i \in \mathbb{N}} \) is a sequence of compatible symplectic forms and the projective limit associated to this sequence is precisely \( \omega \).

**Proof.** Since for \( j \geq i \), \( \ell_i = \ell_i^j \circ \ell_j \) this implies \( \ker \ell_i = \ker \ell_j \oplus (\ell_i^j)^{-1}(\ker \ell_i^j) \). Thus we have \( \ker \ell_j \subset \ker \ell_i \) and so \((\ker \ell_i)^\perp \subset (\ker \ell_j)^\perp \). As we have seen previously, there exists a (unique) symplectic form \( \omega_j \) on \( E_j \) such that \( \omega = \ell_j^* \omega_j \) on \((\ker \ell_j)^\perp \). Since for any \( j \in \mathbb{N} \), the restriction \( \ell_j^* \) to \((\ker \ell_j)^\perp \) is an isomorphism onto \( E_j \), we have:
\[
\omega_j(u_j, v_j) = \omega(\ell_j^*(u_j'), \ell_j^* (v_j')) \quad \text{for all } u_j, v_j' \in (\ker \ell_j)^\perp \text{ with } u_j = \ell_j^*(u_j') \text{ and } v_j = \ell_j^*(v_j').
\]

But since \((\ker \ell_i)^\perp \subset (\ker \ell_j)^\perp \), it follows that, for any \( u_i, v_i' \in (\ker \ell_i)^\perp \), we have
\[
\omega(\ell_i^*(u_i'), \ell_i^* (v_i')) = \omega(\ell_i^j \circ \ell_j^*(u_i'), \ell_i^j \circ \ell_j^*(v_i')) = (\ell_i^j)^* \omega(\ell_j^*(u_i'), \ell_j^* (v_i')).
\]

\[\text{that is each } \ell_i^j \text{ is surjective}\]
Thus we obtain
\[(ω_j) = (ℓ'_i)^*ω_j \text{ on } ℓ'_i \left(\ker ℓ_i\right)\].

The proof will be completed if we show that \(ℓ'_i \left(\ker ℓ_i\right) = \ker ℓ'_i\). But this results follows from \(ℓ'_i \left(\ker ℓ_i\right) = \ker ℓ'_i\).

\(\square\)

2.3. Case of projective sequence of Banach bundles.

**Definition 11.** Let \(\left(E_i, λ_i^j\right)\) be a projective sequence of Banach bundles over a projective sequence \(\left(M_i, δ_i^j\right)\) of manifolds and let \(ω_i\) be a sequence of weak symplectic forms \(ω_i\) on \(E_i\). If \(E_0\) is the typical fibre of \(E_i\), assume that the following properties are satisfied:

\((\text{RPSBS}):\) The sequence \(\left(E_i, λ_i^j\right)\) is a reduced projective sequence of Banach spaces.

We say that \(ω_i\) is a sequence of compatible symplectic forms if the sequence \(\left((ω_i)_{x_i}\right)\) is a sequence of compatible (linear) weak symplectic forms on the projective sequence \(\left(π_i^{-1}(x_i), (λ_i^j)_{x_i}\right)\) of Banach spaces.

Under the context of this Definition, we have

**Theorem 12.** Consider a projective sequence \(\left(E_i, λ_i^j\right)\) of Banach bundles over a projective sequence of Banach manifolds \(\left(M_i, δ_i^j\right)\) which satisfies the assumption (RPSBS).

1. Let \(ω_i\) be a sequence of compatible weak symplectic forms on \(\left(E_i, λ_i^j\right)\). Then \(ω = \lim ω_i\) is a well defined weak symplectic form on the Fréchet bundle \(E = \lim E_i\) over \(M = \lim M_i\).

2. Conversely, let \(ω\) be a weak symplectic form on a projective limit bundles \(E = \lim E_i, π = \lim π_i, M = \lim M_i\) of a submersive projective sequence of Banach bundles \(\left(E_i, π_i, M_i\right)\). Assume that for each \(x = \lim x_i\), the map \((λ_i)_x : π_i^{-1}(x) \to π_i^{-1}(x_i)\) is a submersion. Then \(ω\) induces a weak symplectic 2-form \(ω_i\) on \(E_i\) which gives rise to a family of compatible weak symplectic forms. Moreover, the 2-form on \(E\) defined by this sequence \(\left(ω_i\right)\) is precisely the given 2-form \(ω\).

We obtain directly the following Corollary:

**Corollary 13.** A 2-form \(ω\) on a Fréchet bundle, projective limit \(E = \lim E_i, π = \lim π_i, M = \lim M_i\) of a submersive sequence of Banach fibre bundles \(\left(E_i, λ_i^j\right)\) is a weak symplectic form if and only if there exists a sequence of compatible weak symplectic forms \(\left(ω_i\right)\) on \(E_i\) such that \(ω = \lim ω_i\).

Note that Theorem[1] in the introduction is Theorem[12] joined with Corollary[13]

2.4. Case of projective limit of weak symplectic Banach manifolds. By application of Theorem[12] when \(E_i\) is the tangent bundle \(TM_i\) of a Banach manifold \(M_i\), we obtain the following Theorem which is exactly Theorem[2] in the introduction:

**Theorem 14.**

\[\text{cf. Definition[19]}\]
Remark 15.

(1) Let \( \{M_i, \lambda_i^j\}_{j \geq i} \) be a reduced sequence of Banach manifolds and \( (\omega_i)_{i \in \mathbb{N}} \) a sequence of compatible weak symplectic forms. Then \( \omega = \lim_{i \to \infty} \omega_i \) is a weak symplectic form on \( M = \lim_{i \to \infty} M_i \).

(2) Let \( \omega \) be a 2-form on a projective limit \( M = \lim_{i \to \infty} M_i \) of a submersive sequence of manifolds \( \{M_i, \delta_i^j\}_{j \geq i} \). Then \( \omega \) is a weak symplectic form if and only if each \( T_z\delta_i : T_zM \to T_{\delta_i(z)}M_i \) is a symplectic submersion for each \( i \in \mathbb{N} \).

2.5. Proofs of results.

Proof. From Proposition \ref{prop:weak_symplectic_form} we know that \( \omega \) is well defined. Now \( \omega \) is a smooth 2-form since it is a projective limit of smooth 2 forms, which ends the proof of (1).

Now let \( \omega \) be a weak symplectic form on a projective limit bundle \( E = \lim_{i \to \infty} E_i, \pi = \lim_{i \to \infty} \pi_i, M = \lim_{i \to \infty} M_i \) which satisfies the assumptions of (2). Given some \( x = \lim_{i \to \infty} x_i \in M \), since \( (\lambda_i)_x : \pi_i^{-1}(x) \to \pi_i^{-1}(x_i) \) is a symplectic submersion of symplectic spaces, the restriction \( (\lambda_i)_x^j \) of \( \lambda_i \) to \( \ker(\lambda_i)_x^j \) is an isomorphism on \( \pi_i^{-1}(x_i) \) and so \( (\omega_i)_x^j = \left( (\lambda_i)_x^j \right)^* (\omega_j) \mid_{\ker(\lambda_i)_x^j} \) is a symplectic form on \( \pi_i^{-1}(x_i) \). It remains to show that \( x_i \mapsto \omega_{\pi_i^{-1}(x_i)} \) is smooth.

Fix some \( x = \lim_{i \to \infty} x_i \in M \). There exists \( \phi(U) \times E \) with the following commutative diagram

\[
\begin{array}{c}
\pi^{-1}(U) \\
\downarrow \pi^-1 \\
U \\
\downarrow \phi \\
\phi(U)
\end{array}
\begin{array}{c}
\phi(U) \times E \\
\downarrow \pi \times \pi \\
\phi(U) \times E_i \\
\downarrow \phi_i \\
\phi_i(U_i)
\end{array}
\]

(4)

Let \( \Omega \) be the symplectic form on \( \phi(U) \times E \) such that \( \omega = \pi^* \Omega \). According to Proposition \ref{prop:symplectic_submersion}, \( \ker \lambda_i \) is a sub-bundle of \( E \). Now, since \( \omega \) is a smooth symplectic form and the orthogonal \( \ker(\lambda_i)_x^j \) is a supplemented space of \( \ker(\lambda_i)_x \) for all \( z \in M \), it follows that \( \ker(\lambda_i)_x^j \) is a Banach sub-bunlde of \( E \) and so the Diagram (4) have the more precise version, after
shrinking $U$ if necessary:

\[
\pi^{-1}(U) \xrightarrow{\tau} \phi(U) \times \mathbb{K}_i \times \mathbb{H}_i
\]

where $\mathbb{K}_i$ is the Kernel of $\lambda_i$ and $\mathbb{H}_i$ is the orthogonal of $\mathbb{K}_i$ relative to $\Omega\phi(x)$ over $\phi(x)$. Since the restriction $\lambda_i'$ of $\lambda_i$ to $\mathbb{H}_i$ is an isomorphism onto $E_i$ and so $\delta_i \times \lambda_i'$ is an isomorphism from $\phi(U) \times \mathbb{H}_i$ onto $\phi(U) \times E_i$. Thus, $(\Omega_i) = (\delta_i \times \lambda_i')^\ast (\Omega)_{\phi(U) \times \mathbb{H}_i}$ is a symplectic form on $\phi(U) \times E_i$ and so $\omega_i = \tau_i^\ast (\Omega)$ is a smooth symplectic form.

Proof [Proof of Corollary 14] According to the assumption of this Corollary, after applying Theorem 12, the proof will be completed if we prove that the 2-form $\omega_i$ defined by the closed 2-form $\omega_i$ is also closed and if $\omega$ is a closed 2-form on the projective limit $M = \lim_{\leftarrow} M_i$ (each induced 2 form $\omega_i$ induced on $M_i$ is closed). Under the notations of the proof of Theorem 12, we have

1. $E_i = M_i$ and $E = \bar{M}$;
2. $\lambda_i' = T\delta_i$, $\ell_i' = \delta_i'$, $\lambda_i = T\delta_i$;
3. $\tau_i = T\phi_i$, $\tau = T\phi$.

We can apply the context of Lemma 57 and so if $M_i' = \ker \delta_i'$, then $M_i'$ is isomorphic to $\prod_{l=0}^{\infty} M_i'$ and so $M \equiv \prod_{l=0}^{\infty} M_i'$. According to Diagram 5 in our context, we have

\[
\pi^{-1}(U) \xrightarrow{\tau} \phi(U) \times \prod_{l=0}^{\infty} M_i' \times \prod_{l=0}^{i} M_i'
\]
and \( \phi(U) \) is an open set in \( \prod_{i=0}^{\infty} M_i' \) and \( \phi_i(U_i) \) is an open set of \( \prod_{i=0}^{\infty} M_i' \). Thus, \( \phi_i(U_i) \) is of type \( \prod_{i=0}^{\infty} U_i' \) where \( U_i' \) is an open set of \( M_i' \) and \( \phi(U) \) is of type \( \prod_{i=0}^{\infty} U_i' \) where \( U_i' \) is an open set of \( M_i' \) and with only a finite number of \( l \geq i \) for which \( U_l \neq M_l' \).

(1) Assume that \( \omega \) is a projective limit of the sequence \( (\omega_i)_{i\in\mathbb{N}} \). As in the proof of Theorem [12] let \( \Omega \) be the form on \( \phi_i(U_i) \) induced by \( \omega_i \) and we denote by \( \Omega \) the symplectic form on \( \phi(U) \) induced by \( \omega \) according to the context of Diagram [6]. If \( \eta_i \) be the natural inclusion of \( U_i' \) in \( \prod_{i=0}^{\infty} M_i' \), we set \( \Omega_i = \eta_i^* \Omega \) for \( l \geq i \). Note that \( \Omega_i \) does not depend on the choice of the integer \( i \geq l \). As \( \Omega_i \) is closed, it follows that \( \Omega_i \) is closed. Note that each subbundle \( U_i' \times E_i' \) is the tangent bundle of \( U_i' \). But, from the construction of \( \omega \) (and so \( \Omega \)), if \( X_1 \) and \( X_2 \) are vector fields on \( \phi(U) \) which are tangent to \( U_i' \) and \( U_i' \) respectively, we have \( \Omega(X_1, X_2) = 0 \) if \( l_1 \neq l_2 \) and \( \Omega(X_1, X_2) = \Omega_i(X_1, X_2) \) if \( l_2 = l_1 = l \). This implies that \( \Omega \) is closed.

(2) Assume that \( \omega \) is a symplectic form such that \( T_x \delta_i : T_x M \rightarrow T_x M_i \) is a symplectic submersion. Then from Theorem [6] (2), \( \omega \) induces a non degenerate 2-form \( \omega_i \) on \( M_i \).

Again let \( \Omega \) (resp. \( \Omega_i \)) be the 2-form on \( \phi(U) \) (resp. \( \phi_i(U_i) \)) according to the context of Diagram [6]. We must show that each \( \Omega_i \) is closed. Since \( \Omega \) is the projective limit of the sequence \( (\Omega_i)_{i\in\mathbb{N}} \), according to Theorem [12] (2). Thus, as previously, if \( X_1 \) and \( X_2 \) are vector fields on \( \phi(U) \) which are tangent to \( U_i' \) and \( U_i' \) respectively, we have \( \Omega_i(X_1, X_2) = 0 \) if \( l_1 \neq l_2 \) and \( \Omega_i(X_1, X_2) = \Omega_i(X_1, X_2) \) if \( l_2 = l_1 = l \). Thus \( \Omega_i = \eta_i^* \Omega \) if \( \eta_i \) is the natural inclusion of \( \phi_i(U_i) \) in \( \phi(U) \). It follows that each \( \omega_i \) is closed.

3. WEAK SYMPLECTIC FORMS ON A SUBMERSIVE PROJECTIVE SEQUENCE OF REFLEXIVE BANACH BUNDLES AND DARBOUX-BAMBUSI ASSUMPTION

3.1. Moser’s method and Darboux-Bambusi Theorem. In the Banach context, it is well known that a symplectic form can be strong or weak (non degenerate) (cf. Definition [1]). The Darboux Theorem was firstly proved for strong symplectic Banach manifolds by Weinstein ([27]). But Marsden ([19]) showed that the Darboux theorem fails for a weak symplectic Banach manifold. However, in [3], Bambusi found necessary and sufficient conditions for the validity of Darboux theorem for a weak symplectic Banach manifold (Darboux-Bambusi Theorem). The proof of all these versions of Darboux Theorem were established by Moser’s method.

We recall the following generalization of Moser’s Lemma (see [25]).

Let \( M \) be a manifold modeled on a reflexive Banach space \( \mathbb{M} \). Consider a weak symplectic form \( \omega \) on \( M \). Then \( \omega^3 : TM \rightarrow T^* M \) is an injective bundle morphism. According to section [2.1] we denote by \( T\mathbb{M} \) the Banach space which is the completion of \( T_x M \) provided with the norm \( \| \|_\omega \) associated to some norm \( \| \| \) on \( T_x M \). The Banach space \( T\mathbb{M} \) does not depend on this choice. Then \( \omega_x \) can be extended to a continuous bilinear map \( \hat{\omega}_x \) on \( T_x M \times T_x^* \mathbb{M} \) and \( \omega_x^* \) becomes an isomorphism from \( T_x M \) to \( (T\mathbb{M})^* \). We set \( T\mathbb{M} = \bigcup_{x \in M} T_x \mathbb{M} \) and \( (T\mathbb{M})^* = \bigcup_{x \in M} (T_x \mathbb{M})^* \).

Theorem 16 (Moser’s Lemma). Let \( \omega \) be a weak symplectic form on a Banach manifold \( M \) modeled on a reflexive Banach space \( \mathbb{M} \). Assume that we have the following properties:
(i) There exists a neighbourhood $U$ of $x_0 \in M$ such that $\hat{T}M|_U$ is a trivial Banach bundle whose typical fibre is the Banach space $(\hat{T}_xM, ||\omega_{x_0}|$);
(ii) $\omega$ can be extended to a smooth field of continuous bilinear forms on $TM|_U \times \hat{T}M|_U$.

Consider a family $\{\omega^t\}_{0 \leq t \leq 1}$ of closed 2-forms which smoothly depends on $t$ with the following properties:

- $\omega^0 = \omega$ and $\forall t \in [0, 1], \omega^t_{x_0} = \omega_{x_0}$;
- $\omega^t$ can be extended to a smooth field of continuous bilinear forms on $TM|_U \times \hat{T}M|_U$.

Then there exists a neighbourhood $V$ of $x_0$ such that each $\omega^t$ is a symplectic form on $V$ and there exists a family $\{F_t\}_{0 \leq t \leq 1}$ of diffeomorphisms $F_t$ from a neighbourhood $V_0 \subset V$ of $x_0$ to a neighbourhood $F_t(V_0) \subset V$ of $x_0$ such that $F_0 = \text{Id}$ and $F_t^* \omega^t = \omega$, for all $0 \leq t \leq 1$.

Proof [sketch for more details see [25]] Without loss of generality, we may assume that $U$ is an open neighbourhood of 0 in $M$ and $\hat{T}M|_U = U \times \hat{M}$. Therefore, $U \times \hat{M}$ is a trivial Banach bundle modeled on the Banach space $(\hat{M}, ||\omega||)$. Since $\omega$ can be extended to a non-degenerate skew symmetric bilinear form (again denoted $\omega$) on $U \times (\hat{M} \times \hat{M})$ then $\omega^b$ is a Banach bundle isomorphism from $U \times \hat{M}$ to $U \times \hat{M}^*$. We set $\hat{\omega}^t = \hat{\omega}^t|_U$. Since each $\hat{\omega}^t$ is closed for $0 \leq t \leq 1$, we have:

$$d\hat{\omega}^t = \frac{d}{dt}(d\omega^t) = 0$$

and so $\hat{\omega}^t$ is closed. After shrinking $U$ if necessary, from the Poincaré Lemma, there exists a 1-form $\alpha^t$ on $U$ such that $\hat{\omega}^t = d\alpha^t$ for all $0 \leq t \leq 1$. In fact $\alpha_t$ can be given by

$$\alpha^t_x = \int_0^1 s.\hat{\omega}^s_x(x)dx.$$

Since at $x = 0$, $(\hat{\omega}^s_x)^b$ is an isomorphism from $M$ to $\hat{M}^*$, there exists a neighbourhood $V$ of 0 such that $(\hat{\omega}^s_x)^b$ is an isomorphism from $M$ to $\hat{M}^*$ for all $x \in V$ and $0 \leq t \leq 1$. In particular, $\omega^t$ is a symplectic form on $V$. Moreover $x \mapsto (\hat{\omega}^s_x)^b$ is smooth and takes values in $\mathcal{L}(M, \hat{M}^*)$. We set $X^s_x := -((\hat{\omega}^s_x)^b)^{-1}(\alpha^s_x)$. It is a well defined time dependent vector field and let $F_t$ be the flow generated by $X^s$ defined on some neighbourhood $V_0 \subset V$ of 0. As for all $t \in [0, 1]$, $\hat{\omega}^s_{x_0} = 0$, then $X^s_{x_0} = 0$. Thus, for all $t \in [0, 1]$, $F_t(x_0) = x_0$. As classically, we have

$$\frac{d}{dt} F_t^* \omega^t = F_t^* (L_{X^t} \omega^t) + F_t^* \frac{d}{dt} \omega^t = F_t^* (-d\alpha^t + \omega^t) = 0.$$

Thus $F_t^* \omega^t = \omega$. $\square$

Now as a Corollary of Theorem [16] we obtain the Bambusi’s version of Darboux Theorem ([3], Theorem 2.1).

**Theorem 17** (Darboux-Bambusi Theorem). Let $\omega$ be a weak symplectic form on a Banach manifold $M$ modelled on a reflexive Banach space $\hat{M}$. Assume that the assumptions (i) and (ii) of Theorem [77] are satisfied. Then there exists a chart $(V, F)$ around $x_0$ such that $F^* \omega_0 = \omega$ where $\omega_0$ is the constant form on $F(V)$ defined by $(F^{-1})^* \omega_{x_0}$.

**Definition 18.** The chart $(V, F)$ in Theorem [77] will be called a Darboux chart around $x_0$. 
3.2. Projective sequence of weak symplectic bundle reflexive Banach bundle with Darboux-Bambusi assumptions.

Let \( E = \lim_{i \to \infty} E_i \) be a projective limit of a projective sequence of reflexive Banach spaces \( \left( E_i, \lambda_i^* \right)_{i \geq 1} \). We can provide each Banach space \( E_i \) with a norm \( \| \cdot \|_i \) such that \( \| \lambda_i^* x \|_i \leq 1 \) for \( i \in \mathbb{N} \).

We consider a sequence \( (\omega_i)_{i \in \mathbb{N}} \) of weak symplectic forms on \( E_i \) and let \( \omega_i^* : E_i \to E_i^* \) be the associated bounded linear operator. According to notations in Remark 2, we consider the norm \( \| u \|_{\omega_i} = \| \omega_i(u) \|_i^* \) where \( \| \cdot \|_i^* \) is the canonical norm on \( E_i^* \) associated to \( \| \cdot \|_i \). We have seen that the inclusion of the Banach space \((E_i, \| \cdot \|_i)\) in the normed space \((E_i, \| \cdot \|_{\omega_i})\) is continuous and we have denoted by \( \hat{E}_i \) the Banach space which is the completion of \((E_i, \| \cdot \|_{\omega_i})\). Recall that from Remark 2, the Banach space \( \hat{E}_i \) does not depend on the choice of the norm \( \| \cdot \|_i \) on \( E_i \). According to section 2.1 (before Remark indnorm), \( \omega_i^* \) can be extended to a symplectic submersion between \( \hat{E}_i \) and \( E_i^* \). Moreover, \( \omega_i^* \) is an isomorphism from \( E_i \) to \( E_i^* \).

**Lemma 19.**

1. The sequence \( \left( \hat{E}_i^* \right)_{i \in \mathbb{N}} \) is a projective sequence of Banach spaces and so \( \hat{E}^* = \lim_{i \to \infty} \hat{E}_i^* \) is well defined. Moreover, if \( \lambda_i^* \) is surjective and its kernel is split, then the bonding map \( \lambda_i^* : \hat{E}_i^* \to \hat{E}_i^* \) also satisfies this assumption.

2. The projective limit \( \omega^* = \lim_{i \to \infty} \omega_i^* \) is well defined and is an isomorphism from \( E \) to \( E^* \).

**Proof.** (1) It is sufficient to show that \( \lambda_i^* \) and \( \omega_i^* \) give rise to a map \( \lambda_i^* : \hat{E}_i^* \to \hat{E}_i^* \) and if \( \lambda_i^* \) is surjective and with a split kernel so is \( \lambda_i^* \). Indeed since \( \omega_i^* \) is an isomorphism from \( E_i \) to \( E_i^* \), the bonding map \( \lambda_i^* = \omega_i^* \circ (\omega_i^*)^{-1} \) satisfied the announced properties in (1).

(2) is obvious.

Now we consider a reduced projective sequence \( \left( E_i, \pi_i, M_i \right)_{i \in \mathbb{N}} \) of Banach vector bundles where the typical fibre \( E_i \) is reflexive. The projective limit \( \hat{E} = \lim E_i \) has a structure Fréchet bundle over \( M = \lim M_i \) with typical fibre \( \hat{E} = \lim E_i \) (cf. Proposition 10).

Consider a sequence \( (\omega_i)_{i,n \in \mathbb{N}} \) of compatible weak symplectic forms \( \omega_i \) on \( E_i \). According to the previous notations, since \( E_i \) is reflexive, we denote by \( \left( \hat{E}_i \right)_{x_i} \) the Banach space which is the completion of \( (E_i)_{x_i} \) provided with the norm \( \| \cdot \|_{(\omega_i)_{x_i}}. \) Then \( (\omega_i)_{x_i} \) can be extended to a continuous bilinear map \( (\hat{\omega}_i)_{x_i} \) on \( (E_i)_{x_i} \times (\hat{E}_i)_{x_i} \) and \( (\omega_i)_{x_i} \) becomes an isomorphism from \( (E_i)_{x_i} \) to \( (\hat{E}_i)_{x_i}^* \). We set

\[
\hat{E}_i = \bigcup_{x_i \in M_i} (E_i)_{x_i}, \quad \hat{E}_i^* = \bigcup_{x_i \in M_i} (\hat{E}_i)_{x_i}^*.
\]

According to the assumption of Theorem 17 we introduce the following terminology:

**Definition 20.** Let \( \left( E_i, \pi_i, M_i \right)_{i \in \mathbb{N}} \) be a reduced projective sequence of Banach bundles whose typical fibre \( E_i \) is reflexive. Consider a sequence \( (\omega_i)_{i,n \in \mathbb{N}} \) of compatible weak symplectic forms \( \omega_i \) on \( E_i \). We say that the sequence \( (\omega_i)_{i \in \mathbb{N}} \) satisfies the Bambusi-Darboux assumption around \( x^0 \in M \) if there exists a projective limit chart \( U = \lim U_i \) around \( x^0 \) such that:

(1): for each \( i \in \mathbb{N} \), \( (\hat{E}_i)_{U_i} \) is a trivial Banach bundle;
Theorem 22. Consider a sequence \((\omega_i)_{i \in \mathbb{N}}\) of compatible symplectic forms \(\omega_i\) on \(E_i\) which satisfies the Bambusi-Darboux assumption around \(x^0 \in M\). Then we have the following properties:

1. The projective limit \(\hat{E}_U^* = \lim\hat{E}_i^*|U_i\) is well defined and is a trivial Fréchet bundle with typical fibre \(\hat{F} = \lim\hat{E}_i\).

2. The sequence \(\left(\omega_i^\flat\right)\) of isomorphisms from \(E_{i|U_i}\) to \(\hat{E}_i^*|U_i\) induces an isomorphism from \(E_U^*\) to \(\hat{E}_U^*\).

Proof (1) From our assumptions, for each \(i\), we have a sequence of trivializations \(\hat{\tau}_i : (\hat{E}_i)|U_i \to U_i \times \hat{E}_i\). Thus we obtain a sequence \(\hat{\tau}_i^{-1} : U_i \times \hat{E}_i^* \to (\hat{E}_i^*)|U_i\) of isomorphisms of trivial bundles. Now, from the proof of Lemma [19] we have the bonding map \(\hat{\lambda}_j^\flat : \hat{E}_j^\flat \to \hat{E}_i^\flat\) and by restriction to \(U_j\) we have a bonding map \(\delta_i^j : U_j \to U_i\). So we get a bundle morphism \(\delta_i^j \times \hat{\lambda}_j^\flat\) from \(U_j \times \hat{E}_j^\flat\) to \(U_i \times \hat{E}_i^\flat\). Now the map

\[
\hat{\tau}_i\circ (\delta_i^j \times \hat{\lambda}_j^\flat) \circ \hat{\tau}_j
\]

is a bonding map for the projective sequence of trivial bundles \(\left((\hat{E}_i)|U_i\right)\). Therefore the projective limits \(\hat{\tau} = \lim\hat{\tau}_i\) and \(\hat{E}_U^* = \lim(\hat{E}_i^*)|U_i\) are well defined and \(\hat{\tau}\) is a Fréchet isomorphism bundle from \(U \times \hat{E}^\flat\) to \(\hat{E}_U^*\), which ends the proof of (1).

(2) At first, from Proposition [9] then \(\omega = \lim\omega_i\) is a 2-form on \(E\). From our assumption, since for each \(i \in \mathbb{N}\) we can extend \(\omega_i\) to a bilinear onto \((E_i)|U_i \times (\hat{E}_i)|U_i\), this implies that \(\omega_i^\flat\) is an isomorphism from \((E_i)|U_i\) to \((\hat{E}_i^*)|U_i\). Consider the sequence of bonding maps \(\hat{\lambda}_j^\flat\) for the projective sequence \(\left((\hat{E}_i^*)|U_i\right)\) previously defined. Then we have the following commutative diagram:

\[
\begin{array}{ccc}
U_j \times E_j & \xrightarrow{\tau_i^{-1}} & (E_j)|U_j \\
\downarrow \delta_i^j \times \tau_i^{-1} & & \downarrow \delta_i^j \\
U_i \times E_i & \xrightarrow{\tau_i^{-1}} & (E_i)|U_i \\
\end{array}
\]

\[
\begin{array}{ccc}
(E_j)|U_j & \xrightarrow{\omega_i^\flat} & (\hat{E}_j^*)|U_j \\
\downarrow \tau_i^{-1} & & \downarrow \tau_i^{-1} \\
(E_i)|U_i & \xrightarrow{\omega_i^\flat} & (\hat{E}_i^*)|U_i \\
\end{array}
\]

\[
\begin{array}{ccc}
U_j \times E_j & \xrightarrow{\tau_i} & (E)_j|U_j \\
\downarrow \delta_i^j & & \downarrow \delta_i^j \\
U_i \times E_i & \xrightarrow{\tau_i} & (E_i)|U_i \\
\end{array}
\]

\[
\begin{array}{ccc}
\hat{E}_j^* & \xrightarrow{\tau_i} & \hat{E}_i^* \\
\downarrow \tau_i & & \downarrow \tau_i \\
U_i \times E_i & \xrightarrow{\omega_i^\flat} & (\hat{E}_i^*)|U_i \\
\end{array}
\]

It follows that the projective limit \(\omega^\flat = \lim\omega_i^\flat\) is well defined and is an isomorphism from \(E_U^*\) to \(\hat{E}_U^*\). \(\square\)

4. Problem of existence of Darboux charts on a strong reduced projective sequence of Banach manifolds

4.1. Conditions of existence of Darboux charts. Let \((M_i, \delta_i^j)_{j \geq i}\) be a submersive or decreasing projective sequence of Banach manifolds where \(M_i\) is modeled on a Banach space \(\mathcal{M}_i\). We first apply the previous results for \(E_i = TM_i\).

Theorem 22.
(1) Consider a sequence \( (\omega_i)_{i \in \mathbb{N}} \) of compatible weak symplectic forms \( \omega_i \) on \( M_i \). Then, for each \( x \in M \), the projective limit \( \omega^\pi_x = \lim_{\pi} (\omega_i)_x \) is well defined and is an isomorphism from \( T_x M \) to \( (\widehat{T}_x M)^* \). Moreover \( \omega^\pi_x(u,v) = \omega^\pi_x(u)(v) \) defines a smooth weak symplectic form on \( M \).

(2) Let \( \omega \) be a symplectic form on a submersive projective limit manifold \( M = \lim_{\pi} M_i \). For all \( i \in \mathbb{N} \), assume that the canonical projection \( \delta_i : M \to M_i \) is a symplectic submersion. Then there exists a symplectic form \( \omega_i \) on \( M_i \) such that \( \delta_i^* \omega_i = \omega \) in restriction to \( (\ker \delta_i)^\perp \) and the sequence \( (\omega_i)_{i \in \mathbb{N}} \) is a sequence of compatible weak symplectic forms such that \( \omega \) is the projective limit of \( (\omega_i)_{i \in \mathbb{N}} \) on \( M \) is exactly \( \omega \).

Proof (1) Since \( \omega(u,v) = \omega^\pi(u)(v) \), by application of Proposition[21] to \( E_i = TM_i \), we obtain that \( \omega \) is non degenerate. The proof of that \( \omega \) is closed is formally the same as in the proof of Corollary[14](1).

(2) is a direct consequence of Corollary[14](2).

\( \square \)

As in the Banach context, we introduce the notion of Darboux chart:

**Definition 23.** Let \( \omega \) be a weak symplectic form on the direct limit \( M = \lim_{\pi} M_i \). We say that a chart \( (V, \psi) \) around \( x_0 \) is a Darboux chart if \( \psi^* \omega^0 = \omega \) where \( \omega^0 \) is the constant form on \( \psi(U) \) defined by \( (\psi^{-1})^* \omega_{x_0} \).

We have the following necessary and sufficient conditions of existence of Darboux charts on a submersive projective sequence of Banach manifolds:

**Theorem 24.** Let \( (M_i, \delta^i_{j \geq i}) \) be a submersive or decreasing projective sequence of Banach manifolds where \( M_i \) is modeled on a reflexive Banach space \( M_i \).

(1) Consider a sequence \( (\omega_i)_{i \in \mathbb{N}} \) of compatible symplectic forms \( \omega_i \) on \( M_i \) and let \( \omega \) be the symplectic form which is the projective limit of \( (\omega_i)_{i \in \mathbb{N}} \) on \( M = \lim_{\pi} M_i \). Assume that the following property is satisfied:

\( \textbf{(D)}: \) There exists a projective limit chart \( (U = \lim_{\pi} U_i, \phi = \lim_{\pi} \phi_i) \) around \( x_0 \) such that, for each \( x^0_i = \delta_i(x^0) \in M_i \), then \( (U_i, \phi_i) \) is a Darboux chart around \( x^0_i \) for \( \omega_i \).

Then \( (U, \phi) \) is a Darboux chart around \( x^0 \) for \( \omega \).

(2) Let \( \omega \) be a weak symplectic form on a submersive projective limit \( M = \lim_{\pi} M_i \) such that \( \delta_i : M \to M_i \) is a symplectic submersion. Assume that there exists a Darboux chart \( (V, \psi) \) around \( x^0 \) in \( M \).

If \( \omega_i \) is the symplectic form on \( M_i \) induced by \( \omega \), then there exists a projective limit chart \( (U = \lim_{\pi} U_i, \phi = \lim_{\pi} \phi_i) \) around \( x^0 \) such that the property \( \textbf{(D)} \) is satisfied.

Proof (1) Assume that the assumption \( \textbf{(D)} \) is true and that \( (M_i)_{i \in \mathbb{N}} \) is a reduced projective sequence of Banach manifolds. We fix some \( x^0 \in M \). We consider a projective limit chart \( (U = \lim_{\pi} U_i, \phi = \lim_{\pi} \phi_i) \) around \( x^0 \) such that, if \( x^0_i = \delta_i(x^0) \in U_i \), then \( (U_i, \phi_i) \)
is a Darboux chart around $x_i^0$ for $\omega_i$. Now we have the following commutative diagram:

\[
\begin{array}{c}
\pi^{-1}(U) \xrightarrow{T^\phi} \phi(U) \times M \\
\downarrow \quad \delta_i \times \delta_i \\
\pi_i^{-1}(U_i) \xrightarrow{T\phi_i} \phi_i(U_i) \times M_i
\end{array}
\]

According to this diagram and modulo the diffeomorphisms $\phi$ and $\phi_i$, we may assume that

- $U$ is an open neighbourhood of $x^0 \equiv 0 \in M_i$, and $U_i$ is a neighbourhood of $0 \in M_i$;
- $\omega$ is a smooth 2-form on $U$ and $\omega_i$ is a constant 2-form on $U_i$.

Now if $x = \lim x_i \in U$, $u = \lim u_i$ and $v = \lim v_i$, since $\omega_i$ is constant on $U_i$ it follows that $(\omega_i)_{x_i}(u_i, v_i)$ is independent of $x_i \in U_i$; so the value

$$
\omega_x(u, v) = \lim (\omega_i)_{x_i}(u_i, v_i)
$$

is independent of the point $x$, which ends the proof of (1).

(2) Let $\omega$ be a weak symplectic form on $M = \lim M_i$ such that, for all $i \in \mathbb{N}$, $\delta_i : M \to M_i$ is a symplectic submersion. Assume that we have a Darboux chart $(U = \lim U_i, \phi = \lim \phi_i)$ around $x^0$ for $\omega$. Fix some $i \in \mathbb{N}$. In the context of Diagram (7), we have $M_i \equiv \mathbb{K}_i \times \mathbb{H}_i$ where $\mathbb{K}_i$ is the kernel of $T_0 \delta_i$ and $\mathbb{H}_i$ is the orthogonal of $\mathbb{K}_i$ in $M_i \equiv T_0 M$ (cf. Diagram (4) with, for all $i \in \mathbb{N}$, $E_i = TM_i$). Thus again, modulo the diffeomorphisms $\phi$ and $\phi_i$, we may assume that

- $x^0 \equiv 0 \in U \subset \mathbb{K}_i \times \mathbb{H}_i$, $x_i^0 \equiv 0 \in U_i \subset M_i$;
- $\omega$ is a constant 2-form on $U$ and $\omega_i$ is a smooth 2-form on $U_i$.

Recall that the restriction of $\delta_i$ to $\mathbb{H}_i$ is an isomorphism onto $M_i$, thus we may also assume that $\mathbb{H}_i = M_i$. In this way, we have $\delta_i^* \omega_i = \omega$ in restriction to $\mathbb{H}_i = M_i$. Thus, with our identification, $\omega_i$ is nothing but the restriction of $\omega$ to $U_i \times M_i$, and so $\omega_i$ is a constant 2-form on $U_i$ whose value is fixed by the restriction of $\omega$ to $M_i$. 

4.2. **Problem of existence of Darboux chart in general.** In this subsection, we will explain why, even in the context of a submersive projective sequence of weak symplectic Banach manifolds which satisfies the assumption of Theorem [17], in general, there does not exist any Darboux chart for the induced symplectic form on the projective limit.

Let $(M_i, \delta_i^j)_{j \geq 1}$ be a projective sequence of Banach manifolds where $M_i$ is modeled on a reflexive Banach space $M_i$. Consider a sequence $(\omega_i)_{i \in \mathbb{N}}$ of compatible weak symplectic forms on $M_i$. Since $M_i$ is reflexive, we denote by $\hat{T}_{x_i} M_i$ the Banach space which is the completion of $T_{x_i} M_i$ provided with the norm $\| (\omega_i)_{x_i} \|$. Then $(\omega_i)_{x_i}$ can be extended to a continuous bilinear map $(\hat{\omega_i})_{x_i}$ on $\hat{T}_{x_i} M_i \times \hat{T}_{x_i} \hat{M}_i$ and $(\omega_i)_{x_i}^*$ becomes an isomorphism from $\hat{T}_{x_i} M_i$ to $(\hat{T}_{x_i} \hat{M}_i)^*$. We set

$$
\hat{T} M_i = \bigcup_{x_i \in M_i} \hat{T}_{x_i} M_i, \quad \hat{T} \hat{M}_i^* = \bigcup_{x_i \in M_i} \hat{T}_{x_i} \hat{M}_i^*.
$$

Then by application of Proposition [21] we have:
Proposition 25. Let \((M_i, \delta^i)\)\;_{i\geq 1} be a reduced projective sequence of Banach manifolds whose model is a reflexive Banach space \(M_0\). Consider a sequence \((\omega_i)\) of compatible weak symplectic forms \(\omega_i\) on \(M_i\). Assume that we have the following assumptions\footnote{These assumptions correspond to the Bambusi-Darboux assumptions in Definition 20.} at \(x^0 \in M = \varprojlim M_i:\)

\begin{enumerate}[(i)]
  \item there exists a limit chart \((U = \varprojlim_i U_i, \phi = \varprojlim_i \phi_i)\) around \(x^0\) such that \((\tilde{T}M_i)_{|U_i}\) is a trivial Banach bundle.
  \item \(\omega_i\) can be extended to a smooth field of continuous bilinear forms on \((TM_i)_{|U_i} \times (\tilde{T}M_i)_{|U_i}\) for all \(i \in \mathbb{N}\).
\end{enumerate}

Then \(\tilde{T}^*M_{|U}\) is a trivial bundle. If \(\omega\) is the symplectic form defined by the sequence \((\omega_i)_{i \in \mathbb{N}}\), then the morphism \(\omega^b : TM \to T^*M\) induces an isomorphism from \(TM_{|U}\) to \(\tilde{T}^*M_{|U}\).

Note that the context of Proposition 25 covers the particular framework of projective limit of strong symplectic Banach manifolds \((M_i, \omega_i)_{i \in \mathbb{N}}\).

We will expose which arguments are needed to prove a Darboux theorem in the context of reduced projective sequence of Banach manifolds under the assumptions of Proposition 25. In fact, we point out the problems that arise in establishing the existence of a Darboux chart by Moser’s method.

**Case 1.** Assume that \(M = \varprojlim M_i\) is a reduced projective limit. Fix some point \(a = \varprojlim a_n \in M\). In the context on Proposition 25 on the projective limit chart \((U, \phi)\) around \(a\), we can replace \(U\) by \(\phi(U)\), \(\omega\) by \(\phi^*\omega\) on the open subset \(\phi(U)\) of the Fréchet space \(M\). Thus, if \(\omega^0\) is the constant form on \(U\) defined by \(\omega_a\), we consider the 1-parameter family
\[
\omega^t = \omega^0 + \varepsilon^t, \quad \text{with } \varepsilon = \omega - \omega^0.
\]
Since \(\omega^t\) is closed and \(M\) is a Fréchet space, by [15] Lemma 33.20, there exists a neighbourhood \(V \subset U\) of \(a\) and a 1-form \(\alpha\) on \(V\) such that \(d\alpha = \varepsilon\) which is given by
\[
\alpha_x := \int_0^1 s \varepsilon_{sx}(x, \cdot) ds.
\]
Now, for all \(0 \leq t \leq 1\), \(\omega^t_{x_0}\) is an isomorphism from \(T_{x_0}M \equiv M\) onto \(\tilde{T}_{x_0}M \equiv \tilde{M}\). In the Banach context, using the fact that the set of invertible operators is open in the set of operators, after restricting \(V\), we may assume that \((\omega^t_{x_0})^*\) is a field of isomorphisms from \(M\) to \(\tilde{M}\). Unfortunately, this result is not true in the Fréchet setting. Therefore, the classical proof does not work in this way in general.

**Case 2.** Assume that \(M\) is a submersive projective limit. According to Theorem 22 assume that the canonical projection \(\delta_i : M \to M_i\) is a symplectic submersion, for all \(i \in \mathbb{M}\). Then \(\omega\) induces a symplectic form \(\omega_i\) on \(M_i\). Therefore, for each \(i\), let \(\alpha_i\) be the 1-form induced by \(\alpha\) on \(\phi_i(U_i \cap V)\). Then we have \(\omega_i = d\alpha_i\) and also
\[
(\alpha_i)_{x_i} = \int_0^1 s (\varepsilon^t_{x_{si}}(x_i, \cdot)) ds
\]
where \(\tilde{\omega}_i = \omega_i - \omega^0_i\) is associated to the 1-parameter family \(\omega_i^t = \omega^t_i + t\tilde{\omega}_i\). We are exactly in the context of the proof of Theorem 10 and so the local flow \(\text{Fl}_{x_i}^t\) of \(X_i^t = ((\omega_i^t)^*)^{-1}(\alpha_i)\) is a local diffeomorphism from a neighbourhood \(W_i\) of \(a_i\) in \(V_i\) and, in this way, we build a Darboux chart around \(a_i\) in \(M_i\). Therefore, after restricting each \(W_i\), if necessary, assume that:
**PLDC:** (projective limit Darboux chart) We have a projective sequence of such open sets \((W_i)_{i \in \mathbb{N}}\), then on \(W = \lim_{i \to \infty} W_i\), the family of local diffeomorphisms \(F^t = \lim_{i \to \infty} F^t_i\) is defined on \(W\).

Recall that \(\omega^i = \lim_{n \to \infty} \omega^i_n\) and \(\omega^i\) is an isomorphism. Thus according to the previous notations, we have a time dependent vector field

\[ X^t = ((\omega^1)^{-1}(\alpha)) \]

and again, we have \(L_{X^t} \omega^i = 0\). Of course, if the (PLDC) assumption on \((W_n)_{n \in \mathbb{N}}\) is true, then \(X^t = \lim_{i \to \infty} X^t_i\). So we obtain a Darboux chart as in the Banach context. Note that, in this case, we are in the context of Theorem 27.

**Remark 26.** In fact, under the assumption (PLDC), the flow \(Fl_t\) is the local flow (at time \(t \in [0, 1]\)) of \(X^t = \lim_{i \to \infty} X^t_i\) where \(X^t_i = ((\omega^i_n)^{-1}(\alpha_i))\) (with the previous notations). Unfortunately, according to Remark 41, outside particularity special cases, the "Darboux chart" assumption is not true in general, since, in general,

\[ \bigcap_{j \geq 0} \delta_j^i(W_j) \]

is not an open neighbourhood of \(a_{i_0}\).

Consider again the context of case 2.

Fix some norm \(\| \cdot \|_i\) on \(T_{a_i} M_i\) for all \(i \in \mathbb{N}\). Assume there exists \(K > 0\) such that \(\|((\omega^i_n)^{\delta})_i\|_i^{op} \leq K\), for all \(t \in [0, 1]\) and for all \(i \in \mathbb{N}\). Then, according to Theorem 56, there exists an open neighbourhood \(W\) of \(a \in M\) such that \((\omega^i_n)^{\delta}\) is uniformly bounded on \(W\) and so the same is true for \((\omega^i_n)^{-1}\). It follows that the time dependent vector field \(X^t\) defined in Remark 26 satisfies the assumption of Theorem 56 and so the assumption (PLDC) will be satisfied.

Conversely if \(\omega\) is a symplectic form on \(M\) such that \(\delta_i : M \to M_i\) is a symplectic submersion, then we can apply the previous arguments. Thus we have:

**Theorem 27.** Let \(M = \lim_{i \to \infty} M_i\) be a submersive projective limit of reflexive Banach manifolds \(M_i\) and \((\omega_i)_{i \in \mathbb{N}}\) a compatible sequence of symplectic forms (resp. \(\omega\) a symplectic form on \(M\) such that \(\delta_i : M \to M_i\) is a symplectic submersion).

If the assumptions of Proposition 22 are satisfied around some point \(a = \lim_{i \to \infty} a_i \in M\) and if

\[ \exists K > 0 : \forall t \in [0, 1], \forall i \in \mathbb{N}, \|((\omega^i_n)^{\delta})_i\|^{op}_i \leq K \]

then there exists a Darboux chart around \(a\).

**Remark 28.** A theorem of existence of a Darboux chart in the context of projective limit of Banach manifolds was firstly proved by Kumar (17), Theorem 5.1. The sufficient condition required in this Theorem for the existence of such Darboux Chart also implies the validity of "Darboux chart assumption". More precisely, under the previous notations, it is assumed that

\[ \exists K > 0 : \forall t \in [0, 1], \forall i \in \mathbb{N}, \|((\omega^i_n)^{-1}(\alpha_i))\|_i \leq K. \]

Note that the context of Kumar’s Theorem is the same as in Theorem 27 except that the previous last condition is stronger than the last condition of Theorem 27. The big problem of such results is that, without very particular case (cf 15), to our knowledge, there exists no general situation in which such a result can be applied.

4.3. Examples and contre-example about the existence of a projective limit of Darboux charts.
Example 29. According to [25] section 4, the set $L^k_\infty(S^1, M)$ of Sobolev loops of class $L^k_\infty$ has a Banach structure and manifold if where $(M, \omega)$ is a symplectic manifold, we can provide $L^k_\infty(S^1)$ with a weak symplectic form $\Omega_k$ and around any $\gamma \in L^k_\infty(S^1, M)$, we have a Darboux chart (cf. [26] Theorem 32). Moreover, $L^k_\infty(S^1, M)$ is a Hilbert space and $\Omega_k$ is a strong symplectic form. If we denote by $L^\infty(S^1, M)$ the set of smooth loops in $M$, we have $L^\infty(S^1, M) = \lim_{\to} L^k_\infty(S^1, M)$ and this space is a $L^\infty$-manifold. It is easy to see that the sequence of forms $(\Omega_k)_{k \in \mathbb{N}}$ are compatible and since the projective sequence $(L^k_\infty(S^1, M))_{k \in \mathbb{N}}$ is reduced, we get a weak symplectic form $\Omega = \lim_{\to} \Omega_k$ on $L^\infty(S^1, M)$. In fact, $\Omega$ can be defined directly in the same way as $\Omega_k$ on each $L^k_\infty(S^1, M)$.

When $M = \mathbb{R}^{2m}$, consider the canonical (linear) Darboux form $\omega$ on $\mathbb{R}^{2m}$. Then we have a global Darboux chart for $\Omega$ on $L^\infty(S^1, \mathbb{R}^{2m})$ (cf. [18]). Of course, since we also have a global Darboux chart on each $L^k_\infty(S^1, \mathbb{R}^{2m})$, we then get an example of projective limit of Darboux charts.

Example 30. Let $(M_i)_{i \in \mathbb{N}}$ be a sequence of Banach spaces. Consider the submersive projective sequence of Banach spaces $\big( \prod_{k=1}^\infty M_k \big)_{i \in \mathbb{N}^*}$ of Banach spaces where $\delta_i : \prod_{k=1}^\infty M_k \to \prod_{k=1}^i M_k$ is the canonical projection. Then the projective limit $\overline{M}$ is the product $\prod_{k=1}^\infty M_k$.

On $\overline{M}$ the projective limit topology is the product topology and it is also the topology of Fréchet manifold.

Now, assume that on each $M_k$ we have a weak symplectic form $\omega_k$ such that, for some $\bar{\varepsilon} = \lim_{\to} (x_1, \ldots, x_n) \in \overline{M}$, each symplectic form $\omega_k$ satisfies the assumptions (i) and (ii) of Theorem 16 at $x_k$ and for all $k \in \mathbb{N}^*$. Then from this Theorem, around the point $x_k \in M_k$, we have a Darboux chart $(V_k, F_k)$.

For any $\bar{x}_n := (x_1, \ldots, x_n) \in \overline{M}_n$ and $\bar{u}_n := (u_1, \ldots, u_n)$, $\bar{v}_n = (v_1, \ldots, v_n)$ in $T_{\bar{x}_n} \overline{M}_n$ we define the 2 form

$$\bar{\omega}_n(\bar{u}_n, \bar{v}_n) := \sum_{k=1}^n \omega_k(u_k, v_k).$$

Then $\bar{\omega}_n$ is also a weak symplectic form on $\overline{M}_n$ and it is easy to see that $(\overline{V}_n, \overline{F}_n)$ is a Darboux chart for $\bar{\omega}_n$ around $\bar{x}_n$. Now it is clear that the sequence $(\bar{\omega}_n)_{n \in \mathbb{N}}$ of weak symplectic forms are compatible and so give rise to a weak symplectic form $\bar{\omega}$ on $\overline{M}$. Then $(\overline{V} = \lim_{\to} \overline{V}_n, \lim_{\to} \overline{F}_n)$ is a Darboux chart around $\bar{x} := \lim_{\to} \bar{x}_n$ if $V$ is an open set if and only if $\overline{V}_n = \overline{M}_n$ for any $n \in \mathbb{N}$ outside a finite subset $J \subset \mathbb{N}$. Such a situation occurs for instance in the following contexts:

1. $\omega_k$ is a linear Darboux form on the Banach space $M_k$ for all $k \in \mathbb{N}$ eventually outside of finite set $J$ (cf. section 2.1).
2. $\omega_k$ is a weak linear symplectic form on the reflexive Banach space $M_k$ for all $k \in \mathbb{N}^*$ eventually outside of finite set $J$ (cf. [5] Proposition B.3 Point (3)).
3. $\mathbb{H}$ is a separable infinite-dimensional real Hilbert space and we consider:
   - $M_k = \mathbb{H}$ for each integer $k \in \mathbb{N}$;
   - $S_k : \mathbb{H} \to \mathbb{H}$ is a compact operator with dense range, but proper subset of $\mathbb{H}$, which is self adjoint and positive (such an operator is injective).
   - $\hat{\omega}$ a linear Darboux form on $\mathbb{H}$ and $\omega_k = S_k^* \hat{\omega}$ for at most a finite number of integers and otherwise $S_k = Id_\mathbb{H}$

From the example of [20], we can obtain the following example for which there is no Darboux chart on a submersive projective limit of Banach manifold contexts:

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9 such operators $S_k$ exist since the Hilbert space $\mathbb{H}$ is separable and infinite-dimensional.
Example 31. Let $\mathbb{H}$ be a separable infinite-dimensional real Hilbert space endowed with its inner product $\langle \cdot, \cdot \rangle$. If $g$ is a weak Riemannian metric on $\mathbb{H}$, we may use the trivialization $T\mathbb{H} = \mathbb{H} \times \mathbb{H}$ to define a weak symplectic form $\omega$ in the following way ([20]):

$$2\omega_{(x,e)}((u, v), (u', v')) = D_x g_x (c, u)(u' - D_x g_x (c, u')u + g_x (v', u) - g_x (v, u')).$$

Then the operator $\omega^h_{(x,e)} : T_{(x,e)}\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$ can be written as a matrix of type

$$\frac{1}{2} \begin{pmatrix} (T_{(x,e)} - g^h_x) & -g^h_x \\ g^h_x & 0 \end{pmatrix}.$$

Since $g^h_x$ is always injective by assumption, it follows that $\omega^h_{(x,e)}$ is always injective and is surjective if and only if $g^h_x$ is. It follows that if $\Sigma$ is the set of points $x \in \mathbb{H}$ where $g^h_x$ is not surjective, then the set of points $(x, e) \in T\mathbb{H}$ where $\omega(x,e)$ is not a strong symplectic form is precisely $\Sigma \times \mathbb{H}$.

As at the end of the above Example, let $S : \mathbb{H} \rightarrow \mathbb{H}$ be a compact operator with dense range, but proper subset of $\mathbb{H}$, which is self-adjoint and positive. Given a fixed $a \in \mathbb{H}$, then $A_x = \| x - a \|^2 Id_\mathbb{H} + S$ is a smooth field of bounded operators of $\mathbb{H}$ which is an isomorphism for all $x \neq a$ and $A_a(\mathbb{H}) = \mathbb{H}$ but $A_a(\mathbb{H})$ is dense in $\mathbb{H}$ (cf. [21]). Then $g_x(a, f) = \langle A_x(a), f \rangle$ is a weak Riemannian metric and the associated symplectic form $\omega^h(x,e)$ is not a strong symplectic form if and only if $(x,e)$ belongs to $\{ a \} \times \mathbb{H}$ and, in this case, the range of $\omega^h_{(x,e)}$ is dense in $T_{(x,e)}(\mathbb{H} \times \mathbb{H}) \equiv \mathbb{H} \times \mathbb{H}$.

For each $k \in \mathbb{N}^*$ and any $x \in \mathbb{H}$ we set

$$(A_k)_x = \| x - \frac{a}{k} \|^2 Id_\mathbb{H} + S.$$
Now, for each $n \in \mathbb{N}^*$, since $(0, 0)$ belongs to the open set $\mathcal{U}_n$, we have a Darboux chart $(\mathcal{V}_n, \mathcal{F}_n)$ around $(0, 0) \in T^n \mathbb{H}$ from the classical Darboux Theorem for strong symplectic Banach manifold (cf. [19] or [27] for instance). Since $\omega_n$ is a strong symplectic form on $\mathcal{V}_n$ we must have $\mathcal{V}_n \subset \mathcal{U}_n$. But from the definition of $\mathcal{U}_n$, it follows that

$$\delta_1^n(\mathcal{V}_n) \cap T^n \mathbb{H} \subset \{(x, u) \in \mathbb{H} \times \mathbb{H} : ||x|| < \frac{1}{n}\}.$$  

Therefore, according to Remark [33] the sequence $(\mathcal{V}_n, \mathcal{F}_n)$ is not a projective sequence of charts and so there is no Darboux chart for $\omega$ around $(0, 0) \in T^n \mathbb{H}$.

### Appendix A. Projective limits of topological spaces

**Definition 32.** A projective sequence of topological spaces is a sequence 

$$((X_i, \delta_i^j))_{(i,j) \in \mathbb{N}^2, j \geq i}$$

where

- **(PSTS 1):** For all $i \in \mathbb{N}$, $X_i$ is a topological space;
- **(PSTS 2):** For all $(i,j) \in \mathbb{N}^2$ such that $j \geq i$, $\delta_i^j : X_j \to X_i$ is a continuous map;
- **(PSTS 3):** For all $i \in \mathbb{N}$, $\delta_i^i = \text{Id}_{X_i}$;
- **(PSTS 4):** For all $(i,j,k) \in \mathbb{N}^3$ such that $k \geq j \geq i$, $\delta_i^j \circ \delta_j^k = \delta_i^k$.

**Notation 33.** For the sake of simplicity, the projective sequence $((X_i, \delta_i^j))_{(i,j) \in \mathbb{N}^2, j \geq i}$ will be denoted $(X_i, \delta_i^j)_{j \geq i}$.

An element $(x_i)_{i \in \mathbb{N}}$ of the product $\prod_{i \in \mathbb{N}} X_i$ is called a thread if, for all $j \geq i$, $\delta_i^j(x_j) = x_i$.

**Definition 34.** The set $X = \varprojlim_{i \in \mathbb{N}} X_i$ of all threads, endowed with the finest topology for which all the projections $\delta_i : X \to X_i$ are continuous, is called the projective limit of the sequence $(X_i, \delta_i^j)_{j \geq i}$.

A basis of the topology of $X$ is constituted by the subsets $(\delta_i)^{-1}(U_i)$ where $U_i$ is an open subset of $X_i$ (and so $\delta_i$ is open whenever $\delta_i$ is surjective).

**Definition 35.** Let $(X_i, \delta_i^j)_{j \geq i}$ and $(Y_i, \gamma_i^j)_{j \geq i}$ be two projective sequences whose respective projective limits are $X$ and $Y$.

A sequence $(f_i)_{i \in \mathbb{N}}$ of continuous mappings $f_i : X_i \to Y_i$, satisfying, for all $(i,j) \in \mathbb{N}^2$, $j \geq i$, the coherence condition

$$\gamma_i^j \circ f_j = f_i \circ \delta_i^j$$

is called a projective sequence of mappings.

The projective limit of this sequence is the mapping

$$f : X \ni (x_i)_{i \in \mathbb{N}} \mapsto (f_i(x_i))_{i \in \mathbb{N}} \ni Y.$$

The mapping $f$ is continuous if all the $f_i$ are continuous (cf. [?]).

### Appendix B. Projective limits of Banach spaces

Consider a projective sequence $(E_i, \delta_i^j)_{j \geq i}$ of Banach spaces.

**Remark 36.** Since we have a countable sequence of Banach spaces, according to the properties of bonding maps, the sequence $(\delta_i^j)_{(i,j) \in \mathbb{N}^2, j \geq i}$ is well defined by the sequence of bonding maps $(\delta_i^{i+1})_{i \in \mathbb{N}}$. 
Fix some norm $\| \cdot \|$ on $E_i$, for all $i \in \mathbb{N}$. If $x = \lim_{i} x_i$, then $p_n(x) = \max_{0 \leq i \leq n} \| x_i \|$ is a semi-norm on the projective limit $F = \lim_{\rightarrow} E_n$ which provides a structure of Fréchet space on this vector space (see [7]).

**Definition 37.** A projective sequence $(E_i, \delta^i_j)_{j \geq i}$ of Banach spaces is called reduced if the range of $\delta^{i+1}_i$ is dense for all $i \in \mathbb{N}$.

**Definition 38.** Two projective sequences $(E_i, \delta^i_j)_{j \geq i}$ and $(E'_i, \delta'^i_j)_{j \geq i}$ of Banach spaces are called equivalent if there exist isometries $A_i : E_i \to E'_i$ for all $i \in \mathbb{N}$ such that

$$\delta'^{i+1}_i = A_i^{-1} \circ \delta^{i+1}_i \circ A_{i+1}.$$ 

Of course, any projective sequence $(E_i, \delta^i_j)_{j \geq i}$ of Banach spaces is not reduced and, in general, such a sequence is not equivalent to a reduced one. However, by replacing each $E_i$ by the closure $E'_i$ in $E_i$ of $\delta^i_i(E_{i+1})$ and $\delta^{i+1}_i$ by the restriction $\delta'^{i+1}_i$ of $\delta^{i+1}_i$ to $E'_{i+1}$, we produce a reduced sequence of Banach spaces $(E'_i, \delta'^i_j)_{j \geq i}$ such that $\lim_{\rightarrow} E_i = \lim_{\leftarrow} E'_i$.

Conversely, any Fréchet space provided with a countable family of semi-norms is topologically isomorphic to the projective limit of a reduced projective sequence.

A particular important case of projective limit of a reduced projective sequence of Banach spaces corresponds to the case of a decreasing sequence:

$$E_0 \supset E_1 \supset \cdots \supset E_i \supset E_{i+1} \supset \cdots$$

fulfilling, for any $i \in \mathbb{N}$, the properties:

- (DecS 1): the inclusion $\delta^{i+1}_i : E_{i+1} \to E_i$ is continuous;
- (DecS 2): $E_{i+1}$ is dense in $E_i$.

Then the projective limit $\lim_{\rightarrow} E_i$ is the intersection $\bigcap_{i \in \mathbb{N}} E_i$; it is called an inverse limit of Banach spaces or ILB for short (cf. [24]). In fact, any Fréchet space is an ILB space (cf. Appendix A).

**Appendix C. Projective limits of differential maps**

The following proposition (cf. [10], Lemma 1.2) is essential

**Proposition 39.** Let $(E_i, \delta^i_j)_{j \geq i}$ be a projective sequence of Banach spaces whose projective limit is the Fréchet space $\bar{F} = \lim_{\leftarrow} E_i$ and $(f_i : E_i \to E_i)_{i \in \mathbb{N}}$ a projective sequence of differential maps whose projective limit is $f = \lim_{\leftarrow} f_i$. Then the following conditions hold:

1. $f$ is smooth in the convenient sense (cf. [15])
2. For all $x = (x_i)_{i \in \mathbb{N}}$, $df_x = \lim_{\leftarrow} (df_i)_{x_i}$.
3. $df = \lim_{\leftarrow} df_i$.

**Appendix D. Projective limits of Banach manifolds**

**Definition 40.** The projective sequence $(M_i, \delta^i_j)_{j \geq i}$ is called projective sequence of Banach manifolds if

- (PSBM 1): $M_i$ is a manifold modeled on the Banach space $M_i$;
- (PSBM 2): $(\bar{M}_i, \delta^i_j)$ is a projective sequence of Banach spaces;
- (PSBM 3): For all $x = (x_i) \in M = \lim_{\leftarrow} M_i$, there exists a projective sequence of local charts $(U_i, \varphi_i)_{i \in \mathbb{N}}$ such that $x_i \in \bar{U}_i$ where one has the relation $\varphi_i \circ \delta^i_j = \bar{\delta}^i_j \circ \varphi_j$. 


Definition 42. The sequence defined via bonding map $\delta$ on $PLB \phi$ is a homeomorphism (projective limit of homeomorphisms) and the charts changing $\delta_M$. More particular is the situation:

(PSBM 4): $U = \lim_{i \to \infty} U_i$ is a non empty open set in $M$.

Under the assumptions (PSBM 1) and (PSBM 2) in Definition 40, the assumptions (PSBM 3) and (PSBM 4) around $x \in M$ is called the projective limit chart property around $x \in M$ and $(U = \lim_{i \to \infty} U_i, \phi = \lim_{i \to \infty} \phi_i)$ is called a projective limit chart.

The projective limit $M = \lim_{i \to \infty} M_i$ has a structure of Fréchet manifold modeled on the Fréchet space $M = \lim_{i \to \infty} M_i$, and is called a PLB-manifold. The differentiable structure is defined via the charts $(U, \varphi)$ where $\varphi = \lim_{i \to \infty} \varphi_i : U \to (\varphi_i(U_i))_{i \in N}$.

\varphi is a homeomorphism (projective limit of homeomorphisms) and the charts changings $(\psi \circ \varphi_i^{-1})_{\varphi_i(U)} = \lim_{i \to \infty} \left( (\psi_i \circ (\varphi_i)^{-1})_{\varphi_i(U_i)} \right)$ between open sets of Fréchet spaces are smooth in the sense of convenient spaces.

**Remark 41.** If $M$ is the projective limit of the sequence $(M_i, \delta^i_j)_{j \geq i}$, then, as a set, $M$ can be identified with

$$\left\{ (x_i)_{i \in N} \in \prod_{i \in N} M_i : \forall j \geq i, x_i = \delta^i_j(x_j) \right\}.$$ 

Since each $M_i$ is a topological space, we can provide $\prod_{i \in N} M_i$ with the product topology and so, since each $\delta^i_j$ is continuous, it follows that $M$ is a closed subset in $\prod_{i \in N} M_i$ which can be provided with the induced topology generated by the open sets of type $\prod_{i \in N} V_i \bigcap M$ where $V_i$ is an open set of $M_i$ for a finite number of indices $i$ and otherwise $V_i = M_i$.

The sequence $(M_i, \delta^i_j)_{j \geq i}$ is called reduced projective sequence of Banach manifolds if the sequence $(M_i, \delta^i_j)_{j \geq i}$ is a reduced projective sequence of Banach spaces. Then $\delta^i_j(M_j)$ is dense in $M$, for all $j \geq i$. We will say that $(M_i, \delta^i_j)_{j \geq i}$ is a reduced projective sequence and $M = \lim_{i \to \infty} M_i$ is a reduced PLB-manifold. This situation occurs when the bonding map $\delta^i_j$ is a surjective submersion from $M_j$ onto $M_i$ for all $j \geq i$. In this case, we say that $(M_i, \delta^i_j)_{j \geq i}$ is a surjective projective sequence and $M = \lim_{i \to \infty} M_i$ is a surjective PLB-manifold. More particular is the situation:

**Definition 42.** The sequence $(M_i, \delta^i_j)_{j \geq i}$ is called submersive projective sequence of Banach manifolds if

- (SPSBM 1): $\forall (i,j) \in \mathbb{N}^2 : j \geq i, \delta^i_j : M_j \to M_i$ is a surjective submersion;
- (SPSBM 2): Around each $x \in M = \lim_{i \to \infty} M_i$, there exists a projective limit chart $(U = \lim_{i \to \infty} U_i, \varphi = \lim_{i \to \infty} \varphi_i)$;
- (SPSBM 3): For all $i \in \mathbb{N}$, there exists a decomposition $M_i = \ker \delta^{i+1}_i \oplus M_i'$ such that the following diagram is commutative:

$$
\begin{array}{c}
U_{i+1} \\
\delta^{i+1}_i \\
\phi_i \\
U_i \\
\end{array} \xymatrix{ \ar[r]^-{\varphi_i} & \ar[r]^-{\delta^{i+1}_i} & M_i \ar[r]^-{\delta^{i+1}_i} & M_i' \\
U_{i+1} \ar[r]^-{\varphi_i} & \ar[r]^-{\delta^{i+1}_i} & M_i \ar[r]^-{\delta^{i+1}_i} & M_i' }
$$

Such a chart is called a submersive projective limit chart around $x$.

The projective limit $M = \lim_{i \to \infty} M_i$ of a submersive projective sequence $(M_i, \delta^i_j)_{j \geq i}$ is called a submersive projective limit of Banach manifolds or for short a submersive PLB-manifold. In this case, we have the following results (cf. [4]).
Proposition 43. Let \((M_i, \delta_i^j)_{j\geq i}\) be a surjective (resp. submersive) projective sequence. Then, for each \(i \in \mathbb{N}\), the map \(\delta_i : M \rightarrow M_i\) is surjective (resp. is a submersion).

Under the assumptions of Proposition 43, in fact each \(\delta_i^j : M_j \rightarrow M_i\) is a surjective submersion for all \(j \geq i\) where \((i,j) \in \mathbb{N}^2\).

Another important situation of reduced PLB-manifold, is the case of PLB-manifold defined as follows:

Definition 44. A PLB-manifold \(M = \lim\leftarrow^\mathbb{N} M_i\) is called PLB-manifold if

1. \((ILBM\ 1)\): \(\forall i \in \mathbb{N}, M_{i+1} \subset M_i\);
2. \((ILBM\ 2)\): \(\forall i \in \mathbb{N}, \delta_i^{i+1} : M_{i+1} \rightarrow M_i\) is the canonical inclusion which is a weak immersion with dense range.

Note that this definition is stronger than the definition of PLB-manifold in the Omori’s sense (see [24]) since we impose the condition \((PSBM4)\). In this case, \(M = \bigcap_{i \in \mathbb{N}} M_i\).

Appendix E. Projective limits of Banach vector bundles

Let \((M_i, \delta_i^j)_{j\geq i}\) be a projective sequence of Banach manifolds where each manifold \(M_i\) is modeled on the Banach space \(M_i\).

For any integer \(i\), let \((E_i, \pi_i, M_i)\) be the Banach vector bundle whose type fibre is the Banach vector space \(E_i\), where \((E_i, \lambda_i^j)_{j\geq i}\) is a projective sequence of Banach spaces.

Definition 45. \(((E_i, \pi_i, M_i), (f_i^j, \delta_i^j))_{j\geq i},\) where \(f_i^j : E_j \rightarrow E_i\) is a morphism of vector bundles, is called a projective sequence of Banach vector bundles on the projective sequence of manifolds \((M_i, \delta_i^j)_{j\geq i}\) if for all \((i, x)\) there exists a projective sequence of trivializations \((U_i, \pi_i)\) of \((E_i, \pi_i, M_i)\), where \(\pi_i : (\pi_i)^{-1}(U_i) \rightarrow U_i \times \mathbb{E}_i\) are local diffeomorphisms, such that \(x_i \in U_i\) (open in \(M_i\)) and where \(U = \lim\leftarrow\mathbb{N} U_i\) is a non empty open set in \(M\) where, for all \((i,j) \in \mathbb{N}^2\) such that \(j \geq i\), we have the compatibility condition

\([PLBVB] : (\delta_i^j \times \lambda_i^j) \circ \tau_j = \tau_i \circ f_i^j\).

With the previous notations, \((U = \lim\leftarrow\mathbb{N} U_i, \tau = \lim\leftarrow\mathbb{N} \pi_i)\) is called a projective bundle chart limit. The triple of projective limit \((E = \lim\leftarrow\mathbb{N} E_i, \pi = \lim\leftarrow\mathbb{N} \pi_i, M = \lim\leftarrow\mathbb{N} M_i)\) is called a projective limit of Banach bundles or PLB-bundle for short.

The following proposition generalizes the result of [12] about the projective limit of tangent bundles to Banach manifolds.

Proposition 46. Let \(((E_i, \pi_i, M_i), (f_i^j, \delta_i^j))_{j\geq i}\) be a projective sequence of Banach vector bundles.

Then \((\lim\leftarrow\mathbb{N} E_i, \lim\leftarrow\mathbb{N} \pi_i, \lim\leftarrow\mathbb{N} M_i)\) is a Fréchet vector bundle.

Notation 47. From now on and for the sake of simplicity, the projective sequence of vector bundles \(((E_i, \pi_i, M_i), (f_i^j, \delta_i^j))_{j\geq i}\) will be denoted \((E_i, \pi_i, M_i)\).

Remark that GL \((E)\) cannot be endowed with a structure of Lie group. So it cannot play the role of structural group. We then consider, as in [11], the generalized Lie group \(H^0 (E) = \lim\leftarrow\mathbb{N} H_i^0 (E)\) which is the projective limit of the Banach-Lie groups

\[H_i^0 (E) = \left\{ (h_1, \ldots, h_i) \in \prod_{j=1}^i \text{GL} (E_j) : \lambda_k^j \circ h_j = h_k \circ \lambda_i^j, \text{ for } k \leq j \leq i \right\}.\]

We then obtain the differentiability of the transition functions \(T\).
As we have already seen, this result was firstly proved in [12].

Definition 49. A sequence \((E_i, \pi_i, M_i)\) is called a submersive projective sequence of Banach vector bundles if \((E_i, \pi_i, M_i)\) is a submersive projective sequence of Banach manifolds and if around each \(x \in M_i\), there exists a projective limit chart bundle \((U_i = \lim \pi_{i+1}^* U_{i+1}, \tau = \lim \tau_{i+1})\) such that for all \(i \in \mathbb{N}\), we have a decomposition \(E_{i+1} = \ker \tilde{\lambda}_{i+1} \oplus E_i\) such that the condition (PLBVB) is true.

The projective limit \((E, \pi, M)\) of a projective sequence of Banach vector bundles \((E_i, \pi_i, M_i)\) is called a submersive projective limit of Banach bundles or submersive PLB-bundle for short.

Now, we have the following result whose proof is similar to Proposition 43:

**Proposition 50.** Let \((E_i, \pi_i, M_i)\) be a submersive projective sequence of Banach bundles. Then, for each \(i \in \mathbb{N}\), the map \(\lambda_i : E \to E_i\) is a submersion.

**APPENDIX F. THE BANACH SPACE \(\mathcal{H}_b(F_1, F_2)\)**

Let \((F_1, \nu_1^B)\) (resp. \((F_2, \nu_2^B)\)) be a graded Fréchet space.

Recall that a linear map \(L : F_1 \to F_2\) is continuous if

\[
\forall n \in \mathbb{N}, \exists k_n \in \mathbb{N}, \exists C_n > 0 : \forall x \in F_1, \nu_2^B (L x) \leq C_n \nu_1^{B_{k_n}} (x).
\]

The space \(\mathcal{L}(F_1, F_2)\) of continuous linear maps between both these Fréchet spaces generally drops out of the Fréchet category. Indeed, \(\mathcal{L}(F_1, F_2)\) is a Hausdorff locally convex topological vector space whose topology is defined by the family of semi-norms \(\{p_{n, B}\}\):

\[
p_{n, B} (L) = \sup_{x \in B} \{ \nu_2^B (L x) \}
\]

where \(n \in \mathbb{N}\) and \(B\) is any bounded subset of \(F_1\). This topology is not metrizable since the family \(\{p_{n, B}\}\) is not countable.

So \(\mathcal{L}(F_1, F_2)\) will be replaced, under certain assumptions, by a projective limit of appropriate functional spaces as introduced in [11].

We denote by \(\mathcal{L}(\mathbb{B}_1^0, \mathbb{B}_2^0)\) the space of linear continuous maps (or equivalently bounded linear maps because \(\mathbb{B}_1^0\) and \(\mathbb{B}_2^0\) are normed spaces). We then have the following result ([11], Theorem 2.3.10).

**Theorem 51.** The space of all continuous linear maps between \(F_1\) and \(F_2\) which can be represented as projective limits

\[
\mathcal{H}(F_1, F_2) = \left\{ (L_n) \in \prod_{n \in \mathbb{N}} \mathcal{L}(\mathbb{B}_1^0, \mathbb{B}_2^0) : \lim L_n \text{ exists} \right\}
\]

is a Fréchet space.
For this sequence \((L_n)_{n \in \mathbb{N}}\) of linear maps, for any integer \(0 \leq n \leq m\), the following diagram is commutative

\[
\begin{array}{ccc}
\mathbb{B}_n^m & \xrightarrow{(\delta_1^n)^m} & \mathbb{B}_1^m \\
L_n & \downarrow & L_m \\
\mathbb{F}^m & \xrightarrow{(\delta_2^n)^m} & \mathbb{F}^n
\end{array}
\]

On \(\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)\), the topology can be defined by the sequence of seminorms \(p_n\) given by

\[
p_n(L) = \max_{0 \leq k \leq n} \sup \{ \nu_k^1(L.x), x \in \mathbb{F}_1, \nu_k^1(x) = 1 \}
\]

so that \((\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2), p_n)\) is a graded Fréchet space.

**Remark 52.** For \(l \in \{1, 2\}\) given a graduation \((\nu_k^l)\) on a Fréchet space \(\mathbb{F}_l\), let \(\mathbb{B}_k^l\) be the associated local Banach space and \(\delta_k^l : \mathbb{F}_l \rightarrow \mathbb{B}_k^l\) the canonical projection.

The quotient norm \(\tilde{\nu}_n^l\) associated to \(\nu_n^l\) is defined by

\[
\tilde{\nu}_n^l(\delta_n(z)) = \sup \{ \nu_n^l(y) : \delta_n(y) = \delta_n(z) \}.
\]

We denote by \((\tilde{\nu}_n^l)^{op}\) the corresponding operator norm on \(\mathcal{L}(\mathbb{B}_k^l, \mathbb{B}_m^l)\).

If \(L = \lim_{n \to \infty} L_n\) where \(L_n : \mathbb{B}_1^l \to \mathbb{B}_2^l\), then we have

\[
(\tilde{\nu}_n^l)^{op}(L_n) = \sup \{ \tilde{\nu}_n^l(L_n.x) : x \in \mathbb{B}_1^l, \tilde{\nu}_n^l(x) = 1 \} = \sup \{ \nu_n^l(L.x), x \in \mathbb{F}_1, \nu_n^l(x) = 1 \}.
\]

This implies that

\[
p_n(L) = \max_{0 \leq l \leq n} (\tilde{\nu}_n^l)^{op}(L_n).
\]

**Definition 53.** Let \((\mathbb{F}_1, \nu_1^l)\) and \((\mathbb{F}_2, \nu_2^l)\) be graded Fréchet spaces. A linear map \(L : \mathbb{F}_1 \to \mathbb{F}_2\) is called a uniformly bounded operator, if

\[
\exists C > O : \forall n \in \mathbb{N}, \nu_n(L(x)) \leq C \mu_n(x).
\]

We denote by \(\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)\) the set of uniformly bounded operators. Of course \(\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)\) is contained in \(\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)\) and \(L \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)\) belongs to \(\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)\) if and only if \(\sup_{n \in \mathbb{N}} p_n(L) < \infty\) and so

\[
\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2) = [\mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)]_b.
\]

When \(\mathbb{F} = \mathbb{F}_1 = \mathbb{F}_2\) and \(\nu_1^l = \nu_2^l\) for all \(n \in \mathbb{N}\), the set \(\mathcal{H}(\mathbb{F}, \mathbb{F})\) (resp. \(\mathcal{H}_b(\mathbb{F}, \mathbb{F})\)) is simply denoted \(\mathcal{H}(\mathbb{F})\) (resp. \(\mathcal{H}_b(\mathbb{F})\)).

We denote by \(\mathcal{I} \mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)\) (resp. \(\mathcal{S} \mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)\)) the set of injective (resp. surjective) operators of \(\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)\) with closed range.

**Proposition 54.** \([\mathbb{H}]\)

1. Each operator \(L \in \mathcal{H}(\mathbb{F}_1, \mathbb{F}_2)\) has a closed range if and only if, for each \(n \in \mathbb{N}\), the induced operator \(L_n : \mathbb{B}_1^l \to \mathbb{B}_2^l\) has a closed range.
2. \(\mathcal{I} \mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)\) is an open subset of \(\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)\).
3. \(\mathcal{S} \mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)\) is an open subset of \(\mathcal{H}_b(\mathbb{F}_1, \mathbb{F}_2)\).

We are in situation to end this section by the following result:

**Theorem 55.** \([\mathbb{H}]\)

1. The Banach space \(\mathcal{H}_b(\mathbb{F})\) has a Banach-Lie algebra structure and the set \(\mathcal{G} \mathcal{H}_b(\mathbb{F})\) of uniformly bounded isomorphisms of \(\mathbb{F}\) is open in \(\mathcal{H}_b(\mathbb{F})\).
2. \(\mathcal{G} \mathcal{H}_b(\mathbb{F})\) has a structure of Banach-Lie group whose Lie algebra is \(\mathcal{H}_b(\mathbb{F})\).
3. If \(\mathbb{F}\) is identified with the projective \(\lim \mathbb{B}^n\) we denote by \(\exp_n : \mathcal{L}(\mathbb{B}_n) \to \mathcal{G} \mathcal{L}(\mathbb{B}_n)\), then we have a well defined smooth map \(\exp := \lim \exp_n : \mathcal{H}_b(\mathbb{F}) \to \mathcal{G} \mathcal{H}_b(\mathbb{F})\) which is a diffeomorphism from an open set of \(0 \in \mathcal{H}_b(\mathbb{F})\) onto a a neighbourhood of \(\text{Id} \mathbb{F}\).
Appendix G. A theorem of existence of ODE

The following result is in fact a reformulation in our context of Theorem 1 in [?].

**Theorem 56.** Let $\mathbb{F}$ a Fréchet space realized as the limit of a surjective projective sequence of Banach spaces $(\mathbb{B}_n, \lambda_n^m)_{m \geq n}$ whose topology is defined by the sequence of seminorms $(\nu_n)_{n \in \mathbb{N}}$. Let $I$ be an open interval in $\mathbb{R}$ and $U$ be an open set of $I \times \mathbb{F}$. Then $U$ is a surjective projective limit of open sets $U_n \subset I \times \mathbb{B}_n$. Consider a smooth map $f = \lim f_n : U \rightarrow \mathbb{F}$, projective limit of maps $f_n : U_n \rightarrow \mathbb{B}_n$. Assume that for every point $(t, x) \in U$, and every $n \in \mathbb{N}$, there exists an integrable function $K_n > 0$ such that

$$\forall ((t, x), (t, x')) \in U^2, \quad \nu_n(\dot{f}(t, x) - \dot{f}(t, x')) \leq K_n(t) \nu_n(x - x').$$

(10) and consider the differential equation:

$$\dot{x} = \phi(t, x).$$

(11)

1. For any $(t_0, x_0) \in U$, there exists $\alpha > 0$ with $I_\alpha = [t_0 - \alpha, t_0 + \alpha] \subset I$, an open pseudo-ball $V = B(x_0, \alpha) \subset U$ and a map $\Phi : I_\alpha \times I_\alpha \times V \rightarrow \mathbb{F}$ such that $t \mapsto \Phi(t, \tau, x)$ is the unique solution of (11) with initial condition $\Phi(t, \tau, x) = x$ for all $x \in V$.

2. $V$ is the projective limit of the open balls $V_n$ of $\mathbb{B}_n$. For each $n \in \mathbb{N}$, the curve $t \mapsto \lambda_n \circ \Phi(t, \tau, \lambda_n(x))$ is the unique solution $\gamma : I_\alpha \rightarrow \mathbb{B}_n$ of the differential equation $\dot{x}_n = \phi_n(t, x_n)$ with initial condition $\gamma(\tau) = \lambda_n(x_0)$.

Appendix H. Proof of Proposition 6

The proof of this Proposition needs the following Lemma:

**Lemma 57.** Let $E$ be a projective limit of a reductive projective sequence $(E_i, \ell_i^j)_{j \geq i}$. Assume that, for all $(i, j) \in \mathbb{N}^2$ such that $j \geq i$, the kernel of $\ell_i^j$ is supplemented.

1. For each $i \in \mathbb{N}$ and each $j \geq i$ we have a decomposition

$$E_i = E_i^j \oplus E_i^{j+1} \oplus \cdots \oplus E_i^{j-1} \oplus \ker \ell_{j-1}^j$$

with the following properties for all $j \geq i$

(a) $\ker \ell_i^j = E_i^{j+1} \oplus \cdots \oplus E_i^{j-1} \oplus \ker \ell_{j-1}^j$;

(b) the restriction of $(\ell_i^j)^j_{i}$ of $\ell_i^j$ to $(E_i^1 \oplus \cdots \oplus E_i^j)$ is injective with dense range in $E_i^j$;

(c) $\ell_i^j(E_i^h)$ is dense in $E_i^j$ for all $i \leq h \leq l$, is dense in $\ker \ell_{j-1}^j$ for $h = l$ and $\ell_i^j(E_i^h) = \{0\}$ for $l < h \leq j - 1$.

2. Let $\ell_i : E \rightarrow E_i$ the canonical projection. Then $E = \ker \ell_i \oplus E_i$ and the restriction $\ell_i^j$ of $\ell$ to $E_i$ is a continuous map injective map into $E_i$ with dense range. Moreover, if $|| \ ||_i$ is a norm on $E_i$, then $\nu_i = || \ ||_i \circ \ell_i$ is a semi-norm on $E$ and the restriction of $\nu_i$ to $E_i$ is a norm and in this case, $\ell_i^j$ is an isometry. In particular, the completion $\overline{E_i}$ of $E_i$ is isomorphic to $E_i$.

3. We set $K_i = \ker \ell_{i-1}^i$ for $i \geq 1$ and $K_0 = E_0$. If $E_{i}' = \prod_{l=0}^{i-1} K_i$, then there exist bounding maps $\kappa_i^j : E_i' \rightarrow E_i'$ with dense range, so that $(E_i', \kappa_i^j)_{j \geq i}$ is a reduced projective sequence. If $E' = \lim E_i'$ there exists an injective continuous linear map $\lim \theta_i : E \rightarrow E'$ with dense range where $\theta_i$ is an injective linear map from $E_i$ into $E'_i$ with dense range. Moreover, if $(E_i', \ell_i^j)_{j \geq i}$ if a surjective projective sequence, then each $\theta_i$, $i \in \mathbb{N}$ and $\theta$ are isomorphisms.

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10 This means that we have: $\forall m \geq n$, $\lambda_m^m \circ f_m = f_m \circ (Id_m \times \lambda_m^m)$
Remark 58. Note that, if \((E_i, \ell_i^j)_{j \geq i}\) is a \(\mathcal{ILB}\) sequence\(^{11}\), the decomposition \([^12]\) is reduced to \(E'_i = E_i \cap E_j = (\ell^j_i)^{-1}(E_i)\). We have \(E'_i = E_0\) for all \(i \in \mathbb{N}\) and \(\theta_i\) is the inclusion of \(E_i\) in \(E_0\). Thus, in this case, the morphism \(\theta : E \to E'\) is simply the injection of \(E = \bigcap_{i \in \mathbb{N}} E_i\) into \(E_0\). This means that the only interesting context of Lemma \(^{[27]}\) is when \(\ell_i^j\) is not reduced to zero for some pairs \((j, i)\) and \(j \geq i\).

Proof. Fix some \(i \in \mathbb{N}\) and assume that, for all \(i \leq l \leq j\), we have a decomposition of type \([^12]\) with properties (b) and (c). It is clear that this assumption is true for \(l = i\) and \(l = i + 1\). At first, we have a decomposition

\[ E_{j+1} = \ker \ell_j^{i+1} \oplus F_{j+1}. \]

Therefore the restriction \((\ell'_j)^{i+1}\) of \(\ell_j^{i+1}\) to \(F_{j+1}\) is an injective continuous map from \(F_{j+1}\) into \(E_j\) and \((\ell'_j)^{i+1}\) and \(\ell_j^{i+1}\) have the same range, so \((\ell'_j)^{i+1}(F_{j+1})\) is dense in \(E_j\).

Therefore, according to \([^12]\), each vector space \(K_{j+1}^i = [(\ell'_j)^{i+1}]^{-1}(\ker \ell_{j+1}^i)\), \(K_{j+1}^i = [(\ell'_j)^{i+1}]^{-1}(E'_j)\) for all \(i \leq l < j\) are Banach subspaces of \(F_{j+1}\) and we have the following decomposition:

\[ F_{j+1} = E_{j+1} \oplus E_{j+1}^{i+1} \oplus E_{j+2}^{i+1} \oplus \cdots \oplus E_j^{i+1}. \]

It follows that, for \(j > l > i\), we have

\begin{align*}
(\ell'_j)^{i+1}(\ker \ell^i_j) &= (\ell_j^{i+1})^{-1}(E_{i+1} \oplus E_{i+2} \oplus \cdots \oplus E_{j-1} \oplus \ker \ell_{j-1}^i) \\
&= E_{i+1} \oplus E_{i+2} \oplus \cdots \oplus E_{j-1} \oplus \ker \ell_j^{i+1}
\end{align*}

and also:

\begin{align*}
\ell_i^{i+1}(E_{i+1}^{i+1} \oplus E_{i+1}^{i+1} \oplus \cdots \oplus E_{j}^{i+1}) &= \ell'_j(\ell_j^{i+1}(E_{i+1}^{i+1} \oplus E_{i+1}^{i+1} \oplus \cdots \oplus E_{j}^{i+1})) \\
&= \ell'_j(E_i \oplus E_{i+1}^{i+1} \oplus \cdots \oplus E_{j-1} \oplus \ker \ell_{j-1}^i) \\
&= \ell'_j(E_i)
\end{align*}

which is dense in \(E_j\).

For \(l = j\), we have

\[ \ell_j^{i+1}(E_{i+1}^{i+1} \oplus E_{i+1}^{i+1} \oplus \cdots \oplus E_{j+1}) = \ell_j^{i+1}(E_{j+1}) \]

which is dense in \(E_j\). Finally, according to the definition of \((\ell'_j)^{i+1}\) and the definition \(E_{j+1}^{i+1}\) it follows that \((\ell'_j)^{i+1}\) is an injective continuous map from \(E_{j+1}^{i+1}\) into \(E_{j+1}\) with dense range for all \(i \leq h < j + 1\). Thus (1) is proved.

For \(i\) fixed, on the one hand, the sequence \((\ker \ell_i, (\ell_i^j)_{j \geq i})\) is a projective system of Banach spaces according to properties (1) and (3) in the first part of Lemma \(^{[27]}\). Since \(\ell_i = \lim \ell_i^j\), then \(\ell_i^j = \lim \ker \ell_i^j\) which is a closed Fréchet subspace of \(E\).

On the other hand, we have \(E_j = \ker \ell_i^j \oplus E'_i\) and from (3), the sequence \((E'_i, (\ell_i^j)|_{E'_i})_{j \geq i}\) is a projective sequence of Banach spaces. Thus if \(F'_i = \lim E'_i\), we have \(E = \ker \ell_i \oplus F'_i\). It follows that the restriction \(\ell'_i\) of \(\ell_i\) to \(F'_i\) is an isomorphism onto \(\ell_i(F'_i)\) which is dense in \(E_i\). If \(\| \cdot \|\) is a norm on \(E_i\), then \(\nu_i = \| \cdot \| \circ \ell_i\) is semi-norm on \(E\) whose kernel is precisely \(\ker \ell_i\). Thus, the restriction of \(\nu_i\) to \(E'_i\) is a norm and so \(\ell'_i\) is an isometry which ends the proof of (2).

(3) We set \(E_i = \ker \ell_{i-1}^i \oplus E'_i\). We consider the follows maps:

\(^{11}\) cf. Definition \(^{[14]}\)
Thus we can apply lemma 57. Thus, for all $1 \leq i \leq j$ we can define a 2-form \( \omega_i \) such that \( \omega_i \) is also surjective and so \( \theta_j \) is isomorphic to \( \lim_{\leftarrow} \theta_{ij} \) from \( E \to \lim_{\leftarrow} E_i \). But \( E \) can be identified with (cf. Appendix A)

\[
\left\{ (x_i) \in \prod_{i \in \mathbb{N}} E_i', \ x_i = \ell_i'(x_i'), \ 0 \leq i \leq j, \ i, j \in \mathbb{N} \right\}
\]

In the same way, \( E' \) can be identified with

\[
\left\{ (x_i) \in \prod_{i \in \mathbb{N}} E_i', \ x_i = \kappa_i'(x_i'), \ 0 \leq i \leq j, \ i, j \in \mathbb{N} \right\}.
\]

This implies that \( \theta \) is a continuous injective map from \( \mathbb{E} \) to \( \mathbb{E}' \) with dense range.

Now assume that \( \ell_i' \) is surjective for all \( j \geq i \) and \( (i, j) \in \mathbb{N}^2 \). Then clearly this implies that \( \kappa_i' \) is also surjective and so \( \theta_j \) is an isomorphism for all \( j \geq i \). In this way, \( \theta \) is also an isomorphism.

**Proof** (Proof of Proposition 6) From our assumption, on the sequence \( (\omega_i)_{i \in \mathbb{N}} \) and Proposition 3 we have a decomposition \( E_i = \ker \ell_{i-1}' \oplus F_i \) with \( F_i = (\ker \ell_{i-1}')^\perp \) for all \( i \geq 1 \). Thus we can apply lemma 57. Thus, for all \( 1 \leq j \leq n \), we have a decomposition

\[
E_j = \ker \ell_{j-1}' \oplus F_j \text{ with } F_j = E_0' \oplus E_1' \oplus E_2' \oplus \cdots \oplus E_{j-1}'.
\]

At first, \( \omega_0 \) is a symplectic form \( \omega_{E_0} \) on \( E_0 = E_0' \). From the assumption on the sequence \( (\omega_i)_{i \in \mathbb{N}} \), by induction on \( i \), \( \omega_j \) induces a symplectic form \( \omega_{E_i} \) on \( E_i \). In this way, we obtain a symplectic form \( \omega_i \) on \( E_i' \) by

\[
\omega_i(u_i', v_i') = \sum_{l=0}^{i} \omega_{E_l}(u_l', v_l')
\]

if \( u_i' = (u_0', \ldots, u_i') \) and \( v_i' = (v_0', \ldots, v_i') \). According to the notations of the proof of Lemma 57 for all \( (i, j) \in \mathbb{N}^2 \) such that \( j \geq i \), from the definition of \( \kappa_i' \) and \( \theta_j \), it is easy to see that

\[
\omega_{(i)} : (u_i', v_i') \mapsto \left( (\kappa_i')^* \omega_i \right)_{|E_i'}. \forall \ 0 \leq i \leq j \text{ and } \omega_i = \theta_i^* \omega_i.'
\]

Now the sequence \( (E_i' \times E_i', \kappa_i' \times \kappa_i') \) is a reduced projective system. Thus according to 12, we can define a 2-form \( \omega \) on \( E' \) in the following way:

\[
\omega'(u', v') = \lim_{\leftarrow} \omega_i'((u_0', \ldots, u_i'), (v_0', \ldots, v_i')).
\]

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12 Take \( h = l \) in property (3)
It remains to show that $\omega'$ is symplectic. At first, note that by construction of $\omega'_l$ we have $\omega'_l(u, v) = 0$ for all $u \in K_j$ and $v \in \prod_{l \neq j, 0 \leq l \leq i} K_l$. Assume $\omega'(\overline{uv}) = 0$ for all $\overline{v} \in \mathbb{B}'$ this implies that for all $l \in \mathbb{N}$, we have $\omega'_l(u_l, v'_l) = 0$ for all $v_l \in K_l$ and so we must have have $u'_l = 0$ for all $l \in \mathbb{N}$. Now, since $\theta = \lim \theta_l$ is injective, and $\omega_i = \theta_i^* \omega'$ and the range of $\theta$ is dense this implies the results for $\omega$.

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