The Structure and Enumeration of Link Projections

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Abstract

We define a decomposition of link projections whose pieces we call atoroidal graphs. We describe a surgery operation on these graphs and show that all atoroidal graphs can be generated by performing surgery repeatedly on a family of well known link projections. This gives a method of enumerating atoroidal graphs and hence link projections by recomposing the pieces of the decomposition.

1 Introduction

The problem of enumeration of knots and links has always interested knot theorists. In this paper we introduce a method of enumerating link projections by first decomposing them into pieces called atoroidal graphs. We define surgery on these atoroidal graphs and show how they can be enumerated by performing surgery on a well known family of link projections. By recomposing these atoroidal graphs we can thus enumerate link projections. I have included an enumeration of atoroidal graphs to 12 crossings.

A link projection is given by a 4-valent planar graph $G$. To form a link we can replace each vertex of $G$ by a crossing. To enumerate links in this way we must first enumerate link projections. It was Kirkman’s success in enumerating link projections or polyhedra as he called them($[K1],[K2]$) that

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formed the basis of the knot tables of both Tait\([1]\) and Little\([L1,L2]\). In \([C]\), Conway introduced a notation which made it possible for him to enumerate knots to 11 crossings and links to 10 crossings in a single afternoon where before it had taken years. In his paper Conway defined a *basic polyhedron* to be a polyhedron with no bigon regions and showed that every link is obtained by replacing each vertex of a basic polyhedron by a rational tangle. These basic polyhedra are closely related to the atoroidal graphs defined in this paper and can be enumerated using the enumeration of atoroidal graphs described.

The decomposition of a link projection into atoroidal graphs is achieved by cutting the projection along certain non-trivial curves. We then define surgery on an atoroidal graph giving a new atoroidal graph with one more vertex. This gives a partial ordering on atoroidal graphs where \(G_1 \prec G_2\) if a surgery on \(G_1\) results in \(G_2\) and we show that a graph is initial if and only if it has no vertices of a given type. Using this we can list the initial objects and thus enumerate all atoroidal graphs by repeatedly performing surgery on these initial objects.

The motivation for the paper comes from orbifold theory and hyperbolic geometry but a background in these is not necessary here. For a reference see \([Th]\). For readers interested, these aspects are laid out in the section on orbifolds.

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## 2 Decomposition

### Link Projections

Given a link \(L\) a general position projection of \(L\) is a 4-valent graph \(G\) embedded in \(S^2\). As we are only considering such graphs we will use graph to mean a 4-valent graph embedded in \(S^2\).

Let \(G\) be a graph with vertex set \(V\).

**Definition:** 1 *An n-curve of \(G\) is a simple closed curve in \(S^2-V\) intersecting \(G\) n times.*

Let \(\alpha\) be an n-curve\((n = 0,2,4)\) of \(G\). Then \(\alpha\) splits \(S^2\) into two disks \(D_1,D_2\). We say a component \(D_i\) is trivial if \(D_i \cap G\) is either empty, a simple
arc, two disjoint simple arcs or two arcs crossing at a single vertex (figure 1). If $\alpha$ has a trivial component then $\alpha$ is trivial. Otherwise $\alpha$ is called non-trivial.

**Definition: 2** A graph $G$ is irreducible if all $n$-curves ($n = 0, 2$) are trivial.

**Definition: 3** A graph $G$ is atoroidal if all $n$-curves ($n = 0, 2, 4$) are trivial.

**Decomposition**

If $\alpha$ is a non-trivial $n$-curve ($n = 0, 2, 4$) of $G$ then we can decompose $G$ along $\alpha$ into graphs $G_1$ and $G_2$ as follows. First cut along $\alpha$, this splits the sphere into two disks $D_1, D_2$. To obtain the graph $G_i$ from $D_i$ we identify the boundary of $D_i$ to a single point. We say that $G$ decomposes into $G_1$ and $G_2$ along $\alpha$.

We now describe the decomposition of a link projection $G$ into atoroidal graphs. If all $n$-curves ($n = 0, 2, 4$) in $G$ are trivial the the decomposition is done. Otherwise decompose $G$ into $G_1$ and $G_2$ along a non-trivial $n$-curve ($n = 0, 2, 4$) where $n$ is chosen to be as small as possible. Now repeat the decomposition on the resultant graphs $G_1$ and $G_2$. It is obvious that this decomposition terminates.

The decomposition along non-trivial 0-curves is especially simple, corresponding to splitting a graph into its connected components and thus when dealing with connected graphs we only need concern ourselves with non-trivial 2-curves and 4-curves.
3 Structure of Atoroidal Graphs

Almost all Atoroidal Graphs are Hyperbolic

To investigate the type of atoroidal graphs possible we consider the cell division of $S^2$ given by $G$. We call the cells, faces of $G$ and a face $F$ is called an $n$-gon if it has $n$ vertices of $G$ on its boundary. Any $n$-gon $F$ of $G$ gives a $2n$-curve $\alpha_F$ by taking the boundary of a small neighborhood $N_F$ of $F$.

If $G$ has a 0-gon $F$ then $\alpha_F$ is a trivial 0-curve (i.e. it bounds a disk $D$ in $S^2 - G$) as $G$ is irreducible. Attach a disk to the boundary of $N_F \Rightarrow G$ is just a jordan curve or the unknot projection (figure 2).

If $G$ has a 1-gon (monogon) $F$ then $\alpha_F$ is a trivial 2-curve as $G$ is irreducible. Attach a neighborhood of an arc to $N_F \Rightarrow G$ is the graph having the form of the number 8 (figure 2).

If $G$ has a 2-gon (bigon) $F$ then $\alpha_F$ is a trivial 4-curve as $G$ is atoroidal. Attaching a neighborhood of a vertex to $N_F \Rightarrow G$ is the projection of the trefoil and attaching a neighborhood of two parallel arcs of $G$ to $N_F$ can be done in 2 ways to give two possible graphs but as can be seen in figure 2 only one is also irreducible. Thus $G$ is either the trefoil projection or the Hopf link projection.

Apart from these 4 exceptional graphs all other atoroidal graphs have faces that are at least triangular and are called hyperbolic graphs.

Surgery

Given any atoroidal graph $G$ which has a face $F$ with greater than 3 vertices then we can perform surgery on $G$ to give another atoroidal graph $G'$. We choose edges $e, e_1, e_2$ of $F$ with $e_1, e_2$ adjacent to $e$. $G'$ is obtained by pinching together $e_1, e_2$, i.e. take a simple arc $\alpha \subset F$ with endpoints in the interior of $e_1, e_2$ resp. and homotope $\alpha$ to a single point. The graph $G'$ obtained by performing surgery on $G$ has one more vertex than $G$ (figure 3).

Lemma: 1 $G'$ is atoroidal.

Proof: Let $\alpha'$ be an n-curve in $G'$ which decomposes $S^2$ into disks $D_1', D_2'$. We can get $G$ back by splitting open the new crossing $v$ and as this splitting can be done in a small neighborhood of $v$ and $\alpha'$ is outside such a neighborhood, we get an n-curve $\alpha$ in $G$ which splits $S^2$ into disks $D_1, D_2$. Note that $D_i'$ and
Figure 2: Exceptions

Figure 3: Surgery
$D_i$ are either the same or the former is obtained from the latter by pinching two edges together.

As $G$ is atoroidal then $\alpha$ is trivial for $n = 0, 2, 4$ and we can assume $D_1$ is trivial.

$n = 0, 2 \Rightarrow D'_1 = D_1$ as $D_1$ has at most one edge intersecting it so we haven’t enough edges to pinch $\Rightarrow D'_1$ trivial. $\Rightarrow \alpha'$ trivial.

$n = 4 \Rightarrow D_1$ either neighborhood of vertex or neighborhood of two non-intersecting arcs of $G$. If $D_1$ is neighborhood of vertex then the only edges that can be pinched are adjacent $\Rightarrow D'_1 = D_1 \Rightarrow \alpha'$ trivial. If $D_1$ is neighborhood of two parallel arcs $\Rightarrow D'_1 = D_1$ or $D'_1$ is obtained by pinching the parallel arcs of $D_1$ together $\Rightarrow D'_1$ is neighborhood of a vertex $\Rightarrow \alpha'$ trivial $\Rightarrow G'$ is atoroidal $\blacksquare$

### 4 Orbifolds

This section explains how the work arises out of considering certain orbifolds associated with a link projection. In this setting the decomposition and surgery we define are the torus decomposition and dehn surgery on these orbifolds. An orbifold is a generalization of a manifold in which the space is locally modeled on $\mathbb{R}^n$ modulo the action of a finite group. For example if a group $G$ acts properly discontinuously on a space $M$ then $M/G$ is an orbifold and is a manifold if the action is also free. For a reference on orbifolds see chapter 13 of [Th].

#### Associated Orbifolds

We associate two orbifolds $O_G$ and $O'_G$ to a graph $G$ as follows. We consider $G$ as a graph sitting on $S^2$ in $S^3$. Let $B$ be a ball in $S^3$ with boundary $S^2$ and $V$ be the vertex set of $G$. $O_G$ is a polyhedral orbifold with underlying space $X_{O_G} = B - V$, singular locus $\Sigma_{O_G} = S^2 - V$ and 1-dimensional singular locus $\Sigma^1_{O_G} = G - V$. The 1-dimensional singular locus is marked with $D_2$ indicating that any point on it is modeled by $D^3/D_2$ where $D_2$ acts by two reflections in planes meeting in right angles. $O'_G$ has underlying space $X_{O'_G} = S^3 - V$ and singular locus $\Sigma_{O'_G} = G - V$. Here the singular locus is 1-dimensional and is
marked with $Z_2$ to indicate any point on it is modeled on $D^3/Z_2$ where $Z_2$ acts by rotation of order two.

$O'_G$ is the double of $O_G$ in the sense of orbifolds.

**Decomposition and Surgery**

In [B] we show that the torus decomposition on the orbifolds $O_G$ and $O'_G$ is the decomposition we’ve described on $G$. By Andreev’s theorem (see [IT]) if $G$ is a hyperbolic graph then $O_G$ can be realized as an ideal hyperbolic polyhedron with all dihedral angles right angles. Taking the subgroup of orientation preserving elements of $\pi_1(O_G)$ shows us that $O'_G$ can also be realized as a hyperbolic orbifold.

We show ([B]) that for any graph $G$ the double cover of $O'_G$ is a link compliment denoted by $L_G$ with one component for each vertex of $G$ and is a hyperbolic link complement iff $G$ is a hyperbolic graph. Also if $G'$ is obtained by surgery on $G$ then link complement $L_{G'}$ is obtained from $L_G$ by removing a simple closed curve, i.e. by *dehn drilling*.

**5 Partial Ordering**

Surgery gives atoroidal graphs a partial ordering $< \prec$ by defining $G_1 \prec G_2$ iff $G_2$ is obtained by performing $r$ successive surgeries on $G_1, r = 0, 1, 2, \ldots$. Note that surgery cannot be performed on any of the four exceptional atoroidal graphs and they are never the resultant graph of surgery, therefore they are isolated objects (both initial and final). Thus $\prec$ restricts to a p.o. on hyperbolic graphs. To study $\prec$ we show that the initial objects are a well known family of graphs and we can generate all atoroidal graphs by performing surgery on these initial objects.

After surgery has been performed on a graph to give a graph $G$ with a new vertex $v$. The vertex $v$ is a vertex of a triangle $T$ which has adjacent faces $F_1, F_2$ meeting at $v$ each being greater than triangular (figure 4). A vertex with this local structure we call *simple*. To find a $G$ s.t. $G \prec G$ we might just look for a simple vertex $v$ and cut open at $v$ (there is a unique way to cut open a simple vertex) but this doesn’t necessarily give an atoroidal graph as the resultant may have non-trivial 4-curves (figure 5). What we will show is that if a graph has a simple vertex $v$ belonging to a triangle $T$ then the graph
can be cut open at some vertex of $T$ to give an atoroidal graph. This implies that an initial object cannot have the any simple vertices. Before proving the stated result we will use it to show what the initial objects are.

Since the exceptions are isolated, they never arise in a sequence of surgeries and all other initial objects are hyperbolic. Let $G$ be a hyperbolic initial object (not one of the exceptions), calculating the euler number of the cell division of $S^2$ into faces of $G$ we see that $G$ has a triangular face $T_1$. As $G$ is initial, $T_1$ has two adjacent triangular faces $T_2^l, T_2^r$ (figure 6). Again using the fact that $G$ is initial we have that $T_2^l, T_2^r$ both have a neighboring triangular face other than $T_1$ labeled $T_3^l, T_3^r$ respectively.

If $T_3^l = T_3^r$ then $G$ atoroidal $\Rightarrow$ $G$ is borromian ring projection. Also if $T_3^l, T_3^r$ have a common vertex then $G$ irreducible $\Rightarrow$ $G$ is again the Borromian ring projection which we call $T_3$ (figure 6).

If $T_3^l, T_3^r$ are disjoint then each has a neighboring triangular face $T_4^l, T_4^r$ other than the previous faces $T_2^l, T_2^r$. These are unique as the face adjacent
Figure 6: Initial Setup and the resulting Borromian Rings

Figure 7: Next Stage and the resulting graph $T_4$

to both $T_3^l$ and $T_3^r$ is at least 4 sided. If now $T_4^l, T_4^r$ have a common vertex then using $G$ atoroidal $\Rightarrow G$ is of the form given in figure 7 which we call $T_4$.

Continuing this we get the collection of graphs $\{T_n\}_{n \geq 3}$ (figure 8) which together with the exceptional atoroidal graphs are the initial objects of $\prec$ of which only $\{T_n\}_{n \geq 4}$ are non-terminal. Knowing the initial objects allows us enumerate all atoroidal graphs by performing surgery repeatedly.

**Lemma: 2** Let $G$ be an atoroidal graph with a simple vertex $v$ of triangle $T$ and let $G'$ be the graph obtained by cutting $G$ open at $v$. Let $e_1', e_2'$ be the two edges of face $F'$ in $G'$ pinched to get $G$ then

1. $G'$ is irreducible.
2. Any non-trivial 4-curve $\alpha'$ of $G'$ intersects the face $F'$ in a single arc $\beta'$ which separates $e_1', e_2'$.
Proof: As above let \( e'_1, e'_2 \) be the two edges of \( F' \) pinched together to give \( G \) with \( e' \) the edge adjacent to both. If \( \exists \) neighborhood \( N_{e'} \) of \( e' \) s.t. \( N_{e'} \cap \alpha' \) is empty \( \Rightarrow \) can pinch \( e'_1, e'_2 \) in \( N_{e'} \) with \( \alpha' \) giving an n-curve \( \alpha \) in \( G \). If \( n = 0, 2, 4 \) then \( \alpha \) is trivial and splits \( S^2 \) into two disks \( D_1, D_2 \) with \( D_1 \) trivial. Similarly \( \alpha' \) splits \( S^2 \) into \( D'_1, D'_2 \) with either \( D'_1 = D_1 \) or \( D'_1 \) obtained from \( D_1 \) by cutting open a crossing. In either case this gives \( D'_1 \) trivial \( \Rightarrow \alpha' \) trivial.

Therefore every non-trivial n-curve \( (n = 0, 2, 4) \) must intersect \( F' \) in an arc \( \beta' \) that has one endpoint on \( e' \) and the other on another edge \( e'_3 \) of \( F' \) with \( e'_3 \neq e', e'_1, e'_2 \).

If \( n = 0 \) then \( \alpha' \) doesn’t intersect \( G' \) \( \Rightarrow \alpha' \) trivial.

If \( n = 2 \) then \( \alpha' \) only intersects \( G' \) at the two endpoints of \( \beta' \). If we pinch \( e'_1, e'_2 \) together to get \( G \) we can do so by either leaving \( e'_1 \) fixed and pulling \( e'_2 \) through \( \alpha' \) or vise-versa. This gives 4-curves \( \alpha'^r, \alpha'^l \) resp. in \( G \) which are identical with \( \alpha' \) outside a neighborhood of the new vertex \( v \) and either go right or left around \( v \) as the curves approach \( v \) from inside \( T \) (figure 9). \( G \) atoroidal \( \Rightarrow \alpha'^r \) is trivial \( \Rightarrow \alpha'^r \) splits \( S^2 \) into \( D_1, D_2 \) s.t. \( D_1 \) trivial. The region containing \( v \) also contains another vertex of \( T \) so it can’t be trivial. Therefore \( D_1 \) is the region containing vertex \( v_2 \) of \( T \) and is neighborhood of \( v_2 \Rightarrow v \) not simple as one of faces is bigon. This contradiction implies that \( G' \) has no non-trivial 2-curves \( \Rightarrow G' \) is irreducible.

If \( n = 4 \) \( \Rightarrow \alpha' \cap F' \) consists of either 1 or 2 arcs. If it is 2 arcs \( \beta'_1, \beta'_2 \)
then traversing around $\alpha'$ we have 4 connected arcs $\beta'_1, \gamma'_1, \beta'_2, \gamma'_2$. We can join endpoints of $\gamma'_1$ by another arc $\delta'_1$ in $F''$ s.t. $\gamma'_1 \cup \delta'_1$ is a 2-curve in $G' \Rightarrow$ trivial. If endpoints of $\gamma'_1$ belong to different edges then both components of $S^2 - \gamma'_1 \cup \delta'_1$ contain vertices which contradicts it being trivial. Therefore both $\gamma'_1, \gamma'_2$ are contained in adjacent faces to $F''$ and $\alpha'$ is neighborhood of two parallel arcs of $G' \Rightarrow \alpha'$ trivial(figure [10]). Therefore any non-trivial 4-curve in $G'$ intersects $F'$ in a single arc $\beta'$ separating the two edges pinched in $G'$ to obtain $G \ □$

**Definition: 4** An $n$-curve $\alpha$ ($n \geq 4$) in $G$ is trivial iff either
- $\alpha$ is the boundary of a neighborhood of a vertex of $G$

or
- $\exists$ arc $\beta$ intersecting $G$ at most once s.t. $\beta \cap \alpha = \partial \beta$ and $\partial \beta$ splits $\alpha$ into $\alpha_1, \alpha_2$ each containing at least two points of $G$. $\beta$ is called a compression of $\alpha$. 

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If $G$ is atoroidal then the only trivial 6-curves can easily shown to be those curves that split $S^2$ into two disks one of which has one of four types given in figure 11. Disks of this kind bounding a 6-curve are called trivial.

**Lemma: 3** If $\alpha'$ is a non-trivial 4-curve in $G'$ then associated with it are two non trivial 6-curves $\alpha^r, \alpha^l$ in $G$.

**Proof:** $\alpha'$ intersects $F'$ in a single arc $\beta'$ separating edges $e'_1, e'_2$ and as before we can pinch $e'_1, e'_2$ together to get graph $G$. This can be done in two ways, either by fixing $e'_1$ and pushing $e'_2$ across $\beta'$ or vise-versa. We get two 6-curves $\alpha^r, \alpha^l$ in $G$ both identical to $\alpha'$ outside a neighborhood of the new vertex $v$ and either going right or left around the vertex inside the neighborhood of $v$ as before (figure 11).

$\alpha^r$ splits $S^2$ into disks $D_1, D_2$ with $D_1$ containing vertices $v, v_1$ of $T$ and $D_2$ containing vertex $v_2$. Therefore if $D_1$ is trivial then it must be the same type as the fourth disk described in figure 11. But then $\alpha'$ would bound a neighborhood of a vertex of $G$ which contradicts $\alpha'$ non-trivial. If $D_2$ is trivial then it is either the same type as the third or fourth disk in figure 11. If its the third type then as before $\alpha'$ would bound a neighborhood of a vertex of $G$ which contradicts $\alpha'$ non-trivial. If it is the fourth type then adjacent face $F_2$ of $T$ would be a triangle contradicting $v$ being simple. Therefore $\alpha^r$ is non-trivial and similarly $\alpha^l$. $\square$

**Theorem 1** If $G$ has a simple vertex $v$ of a triangle $T$ then either

- splitting open at $v$ gives an atoroidal graph $G'$
- or
- both other vertices $v_1, v_2$ of $T$ are simple and splitting at either gives an atoroidal graph.

**Proof:** If $G'$ is not atoroidal $\Rightarrow \exists \alpha'$ non-trivial 4-curve in $G'$ and $\alpha^r, \alpha^l$ non-trivial 6-curves in $G$. Triangle $T$ has adjacent faces $F_1, F_2, F_3$ with both
Figure 12: All vertices of $T$ are simple

$F_1, F_2$ non-triangular as $v$ is simple. If $F_3$ was triangular then $\alpha^r$ splits $F_3$ in two, one piece containing just one vertex say $v_1$ and the other containing two. Therefore $\alpha^r$ takes a clockwise path about $v_1$ from $T$ through $F_3$. If instead we take an anticlockwise path $\alpha'$ gives us another 4-curve in $\tilde{G}'$ called $\bar{\alpha}'$ (figure 12). Since $\bar{\alpha}'$ doesn’t intersect $e'$ then it is trivial. Therefore it splits $S^2$ into disks $\bar{D}_1', \bar{D}_2'$ with $\bar{D}_1'$ trivial. $\alpha'$ splits $S^2$ into $D_1', D_2'$ with $D_1'$ obtained from $\bar{D}_1'$ by crossing two adjacent ends so if $D_1'$ is neighborhood of two parallel arcs then $D_1'$ is neighborhood of vertex and if $\bar{D}_1'$ is neighborhood of vertex then $G'$ would contain a bigon. Therefore $F_3$ must be non-triangular and both $v_1$ and $v_2$ are simple.

If splitting at $v_1$ doesn’t give an atoroidal graph then there is a non-trivial 4-curve $\alpha'_1$ in $G'_1$ and non-trivial 6-curves $\alpha^r_1, \alpha^l_1$ in $G$. Considering the 6-curves $\alpha, \alpha_1$, where $\alpha = \alpha^l$ and $\alpha_1 = \alpha^r_1$ we will show that they can be isotoped to intersect in only two points. Then by showing that they cannot intersect in the given way (figure 13) the theorem is proven.

Firstly we will show that $\alpha, \alpha_1$ can be isotoped to only intersect twice. If they intersect any more then $S^2 - \alpha \cup \alpha_1$ contains at least four regions that are disks with boundary consisting of one arc of $\alpha$ and $\alpha_1$. As $\alpha \cup \alpha_1$ has 12 intersections with $G$ then each of these four regions cannot have boundaries being n-curves $n \geq 4$. Therefore one of these regions $D$ has $\gamma = \partial D$ either a trivial 0 or 2-curve and $\gamma = \beta \cup \beta_1$ where $\beta, \beta_1$ are arcs of $\alpha, \alpha_1$ resp.. If $\gamma$ is a 0-curve then either $D$ or $D_c$ is a trivial disk. If $D$ is trivial then can isotope to remove two intersections of $\alpha$ and $\alpha_1$ by pulling $\beta$ through $\beta_1$. If $D_c$ is trivial then any of the other 3 disks with boundary consisting of one arc of $\alpha$ and $\alpha_1$ are trivial and hence can reduce the number of intersections as in first case.
If $\gamma$ is a 2-curve then each of $\beta, \beta_1$ intersect $G$ as if say $\beta$ didn’t then it would be a compression for $\alpha_1$ which contradicts $\alpha_1$ being non-trivial. Therefore either $D$ or $D^c$ is trivial i.e. a neighborhood of an arc of $G$. If $D$ is trivial then can isotope by pulling $\beta$ through $\beta_1$ reducing number of intersections of $\alpha$ and $\alpha_1$. If $D^c$ is trivial then other 3 disks with boundary consisting of one arc of $\alpha$ and $\alpha_1$ cannot have boundaries being n-curves $n \geq 4$ as they can have a maximum of 10 intersections with $G$ between them. Therefore there is region $\bar{D}$ which either doesn’t intersect $G$ and thus we can isotope as before to reduce the number of intersections of $\alpha$ and $\alpha_1$ or is neighborhood of an arc of $G$ which can also be isotoped as before.

So we can assume $\alpha$ and $\alpha_1$ intersect twice and divide $S^2$ into 4 disks. We label these disks $D_i, i = 1, \ldots, 4$ where $D_1, D_2, D_3$ contain $v_1, v_2, v$ resp. and $\gamma^i = \partial D_i$. $\gamma^i$ is an $n_i$-curve where $\sum n_i = 24$ and $n_i \geq 4$. Also $\gamma^i = \beta^i \cup \beta_1^i$ where $\beta^i, \beta_1^i$ are arcs of $\alpha, \alpha_1$ resp. (figure 13). Note that the arcs $\beta^i, \beta_1^i$ have duplication with each of two arcs that $\alpha$ or $\alpha_1$ is divided repeated twice. This is for ease of labeling and can be thought of as the two sides of the same arc on $\alpha$ or $\alpha_1$.

**Case 1:** If $n_2 = 4$ then $D_2$ is a neighborhood of $v_2$ and both $\beta^2$ and $\beta_1^2$ intersect $G$ twice otherwise we get a compression of $\alpha$ or $\alpha_1$. Therefore $\gamma^1$ is a 6-curve and inside $D_1$ is a 4-curve $\tilde{\gamma}^1$ (figure 14). If it is boundary of neighborhood of two parallel arcs of $G$ then this implies either $F_3$ is a bigon or $\alpha_1$ is a trivial 6-curve. If it is boundary of neighborhood of a vertex then this would imply that $F_3$ was triangular. Therefore $n_2 \neq 4$.

**Case 2:** If $n_1 = 4$ then $\beta_1^1$ intersects $G$ only twice as otherwise $\beta^1$ is
a compression of $\alpha_1$. Therefore $\gamma^4$ is a 6-curve and $D_4$ contains a 4-curve $\tilde{\gamma}^4$ (figure 14). If it is boundary of neighborhood of two parallel arcs of $G$ then this implies that either $F_1$ is a bigon or $\alpha$ is trivial 6-curve. If it is the boundary of neighborhood of vertex then this implies that $F_1$ is triangular. Therefore $n_1 \neq 4$ and by symmetry $n_3 \neq 4$.

**Case 3:** If $n_4 = 4$ then $D_4$ must be neighborhood of parallel arcs of $G$ which implies $F_1$ is a bigon (figure 14). Therefore $n_4 \neq 4$.

**Case 4:** Therefore $n_i = 6$ and each arc $\beta^i, \beta^i_1$ intersects $G$ exactly 3 times. Therefore $\tilde{\gamma}^1, \tilde{\gamma}^4$ are both 4-curves. If $\tilde{\gamma}^4$ is boundary of neighborhood of two parallel arcs of $G$ then this implies either $F_1$ is a bigon or both $\alpha$ and $\alpha_1$ have compressions, contradicting them being non-trivial (figure 14). If $\tilde{\gamma}^4$ is boundary of a neighborhood of a vertex then $F_1$ would be triangular. Therefore we have shown that there cannot exist 6-curves intersecting as $\alpha$ and $\alpha_1$ do $\Rightarrow$ if $\alpha$ exists (i.e. $G'$ isn’t atoroidal) $\Rightarrow$ $\alpha_1$ can’t exist $\Rightarrow$ $G'_1$ is atoroidal. Similarly $G'_2$ is atoroidal also $\Box$

We have shown that the initial objects of $\prec$ are $\{T_n\}_{n \geq 3}$ along with the 4 exceptions. $T_n$ is the projection of the $(3,n)$ torus link with the link having three components if 3 divides $n$ and having one component otherwise. From this we see that $T_n$ has symmetries taking any directed edge of one of the non-triangular faces to any other. Therefore any surgery on $T_n$ gives the same graph which we call $T_n^+$. 

Figure 14: Cases
Lemma: 4 If $T_n \prec G$ and $T_n \neq G$ (n > 4) then $T_{n-1} \prec G$.

Proof: If $T_n \prec G$ and $T_n \neq G$ then $T^+_n \prec G$. As in figure 7 we can pinch together edges of $T^+_2$ to $T^+_4$ to get $T^+_n$. Only one vertex of $T_1$ is simple so we cut open at that vertex first. This reduces the pinched $T^+_2$ to a triangle which has only one simple vertex which we now cut open (figure 4). This resulting graph is $T^+_{n-1}$ so we have that $T_{n-1} \prec T^+_{n-1} \prec T^+_n \prec G$.

Therefore if $C_n$ is the set of proper descendants of $T_n$ (i.e. $T_n \not\in C_n$) then

$$C_4 \supseteq C_5 \supseteq C_6 \supseteq \cdots \supseteq C_n \cdots$$

6 Enumeration

We now have a way to enumerate atoroidal graphs up to any prescribed crossing number by performing surgery on the initial objects. Figure 10 is the enumeration of atoroidal graphs up to 12 crossings. To enumerate prime link projections we need only recombine the atoroidal graphs as follows. We choose two atoroidal graphs $G_1$ and $G_2$ with vertices $v_1$ and $v_2$ respectively. Now take the compliment of a neighborhood of each vertex and attach their boundaries, making sure to match up the strands of the graphs. In recombining we do not use the first 3 exceptions as either they have no vertices or the compliment of a neighborhood of a vertex is trivial. To enumerate the basic polyhedra of Conway the trefoil projection is also not used as the compliment of a vertex is a bigon.

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Figure 15: $T_{n-1}^+ \prec T_n^+$

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Figure 16: Atoroidal Graphs of 12 crossings or less