Boundary loop models and 2D quantum gravity

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Abstract. We study the O(n) loop model on a dynamically triangulated disc, with a new type of boundary condition, discovered recently by Jacobsen and Saleur. The partition function of the model is that of a gas of self-and mutually avoiding loops covering the disc. The Jacobsen–Saleur (JS) boundary condition prescribes that the loops that do not touch the boundary have fugacity \( n \in [-2, 2] \), while the loops touching the boundary at least once are given different fugacity \( y \). The class of JS boundary conditions, labeled by the real number \( y \), contains the Neumann \( (y = n) \) and Dirichlet \( (y = 1) \) boundary conditions as particular cases. Here we consider the dense phase of the loop gas, where we compute the two-point boundary correlators of the \( L \)-leg operators with mixed Neumann–JS boundary condition. The result coincides with the boundary two-point function in Liouville theory, derived by Fateev, Zamolodchikov and Zamolodchikov. The Liouville charge of the boundary operators match, by the KPZ correspondence, with the \( L \)-leg boundary exponents conjectured by JS.

Keywords: conformal field theory, correlation functions, loop models and polymers, solvable lattice models

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1. Introduction

The solvable statistical models that have a geometrical description in terms of self-and mutually avoiding clusters, like Ising, O(n) and Potts models, can be also formulated and solved on a dynamical lattice [1]–[7]. A statistical model defined on a dynamical lattice is said to be coupled to gravity, since the sum over lattices gives a discretization of the path integral over Riemann metrics on the world sheet. For each critical point described by a ‘matter’ CFT, the ‘coupling to gravity’ consists in adding a Liouville and ghost sectors and dressing the scaling operators by exponents of the Liouville field [8]–[10]. The description of the critical points based on Liouville theory allows us to interpret the wealth of exact results about statistical models on dynamical lattices obtained via matrix model or combinatorial techniques.

A statistical system on a random lattice exhibits qualitatively the same critical phases as that on a regular lattice, but with different critical exponents. At a critical point characterized by a conformal anomaly $c \leq 1$, the conformal weight $h$ of a matter field\(^2\) can be extracted from its ‘gravitational dimension’ $\Delta$, which determines the scaling

\(^2\) Only operators without spin ($h = \bar{h}$) survive after coupling to gravity.
properties of the correlation functions involving this field [9, 10]:
\[ h = \frac{\Delta(\Delta - \gamma_{\text{str}})}{1 - \gamma_{\text{str}}}, \quad c = 1 - 6\frac{\gamma_{\text{str}}^2}{1 - \gamma_{\text{str}}}. \] (1.1)

The exponent \( \gamma_{\text{str}} \) (gamma string) describes the critical fluctuations of the area of the random surface.

The solutions of the KPZ scaling relation (1.1) can be parameterized by a pair of numbers \( r \) and \( s \), not necessarily integers:
\[ h_{rs} = \bar{h}_{rs} = \frac{(r/b - sb)^2 - (1/b - b)^2}{4}, \quad \Delta_{rs} = \frac{r/b - sb - (1/b - b)}{2b}. \] (1.2)

The dependence on the matter central charge is through the positive parameter \( b < 1 \), defined as
\[ b^2 = \frac{1}{1 - \gamma_{\text{str}}} < 1. \] (1.3)

The conformal weights have the symmetry \( h_{rs} = h_{-r, -s} \), unlike the gravitational dimensions. To each conformal weight one can associate two gravitational dimensions, \( \Delta_{rs} \) and \( \Delta_{-r, -s} \), which are the two roots of the quadratic relation (1.1). They correspond to the two possible gravitational dressings of the matter conformal field by Liouville vertex operators.

The correspondence between the critical phenomena on flat and dynamical lattices is particularly useful in the presence of boundaries. The boundary critical exponents [11] on flat and dynamical lattices are again related by (1.1). Boundary 2D quantum gravity became a powerful tool for evaluating exact critical exponents [12], complementary to the Coulomb gas techniques [13] and the Schramm–Loewner evolution (SLE) [14]. For systems coupled to gravity it is possible, by cutting open the path integral along cluster boundaries, to solve analytically problems whose exact solution is inaccessible on a flat lattice. On the other hand, our understanding of the boundary phenomena in 2D gravity is much helped by the progress achieved in boundary Liouville theory over the last years [15]–[17].

In this paper we will focus on the possible boundary conditions and the spectrum of boundary exponents of the \( O(n) \) model coupled to 2D gravity [5]. Our main purpose is to check a very interesting conjecture made in [18] about the general boundary conditions for the \( O(n) \) model on a flat lattice.

The \( O(n) \) model has a continuum transition if the number of flavors is in the interval \(-2 \leq n \leq 2\), where it can be parameterized by an angle,
\[ n = 2\cos(\pi \theta). \] (1.4)

The boundary \( O(n) \) model was originally considered with Neumann boundary condition (the loops are reflected from the boundary). The boundary scaling dimensions of the \( L \)-leg operators, realized as sources of \( L \) open lines, were conjectured in [19] and then derived in [20]. Furthermore, the partition function of the \( O(n) \) model on the annulus with Neumann boundary conditions was evaluated in [21]. Another obvious boundary condition is the Dirichlet boundary condition, studied in [22, 23], for which there is an open line ending at each site of the boundary. The dimensions of the \( L \)-leg boundary operators

\[ \text{In these papers the loop gas was considered in the context of the SOS model, for which the Dirichlet and Neumann boundary conditions have the opposite meaning.} \]
with mixed Dirichlet and Neumann boundary conditions were computed in [24, 23] by coupling the model to 2D gravity and then using the KPZ scaling relation (1.1).

Recently, Jacobsen and Saleur put forward a proposal about the complete classification of the boundary conditions of the $O(n)$ loop gas model in the dense phase, described by, in general non-rational, CFT with $b^2 = 1 - \theta$. Their proposal is based on a previous work [25] and possibly overlaps, for rational $\theta$, with the results of [26]. The claim of [18] is that there is a continuum of boundary conditions characterized by a real variable $y$. The Jacobsen–Saleur boundary condition, which we will denote in the following by JS, is defined by counting the loops that touch at least once the boundary with different fugacity $y$, while the loops that do not touch the boundary are counted with fugacity $n$. The JS boundary conditions contain, as particular cases, Neumann ($y = n$) and Dirichlet ($y = 1$) boundary conditions for the $O(n)$ field.

In order to evaluate the $L$-leg exponents, the authors of [18] considered the loop gas on an annulus, with Neumann boundary condition on the inner rim, JS boundary condition on the outer rim and $L$ non-contractible loops separating the two boundaries. The loop model with these boundary conditions were called there the boundary loop model (BLM). According to [18], BLM with $L$ non-contractible lines has two sectors, the blobbed and the unblobbed one, characterized by two different critical exponents for $L \geq 1$. In the blobbed (unblobbed) sector the outmost non-contractible line touches at least once (does not touch) the outer rim. It is argued in [18] that the scaling exponents in the two sectors are characterized by the conformal weights

$$h_L^{\text{unblob}} = h_{-r,-r+L}$$
$$h_L^{\text{blob}} = h_{r,r+L}$$

(dense phase),

where the real parameter $r$ is related to the boundary fugacity $y$ by

$$y(r) = \frac{\sin[(r + 1)\pi\theta]}{\sin(r\pi\theta)}.$$

The boundary exponents for Neumann–Neumann [19] and Neumann–Dirichlet [23] boundary conditions appear as particular cases of (1.5) when $r = 1$ and $r = (1 - \theta)/2\theta$, correspondingly.

Inspired by [18], in this paper we analyze the JS boundary conditions for the $O(n)$ loop gas coupled to 2D gravity. Put in string theory terms, the problem we address is to describe the ensemble of D-branes in bosonic string theory whose target space is the $(n - 1)$-dimensional sphere. We restrict ourselves to the dense phase of the $O(n)$ model, where we derive loop equations in the form of recurrence relations between the boundary two-point functions of the $L$-leg and $(L - 1)$-leg operators. The loop equations are obtained by cutting open the world sheet along segments of loops that connect two points of the boundary. In the continuum limit, the solution of the loop equations is given, up to a normalization, by the two-point boundary correlator in boundary Liouville theory [15]. The scaling exponents characterizing the solution reproduce, via KPZ relation (1.1), the spectrum of boundary conformal weights conjectured by JS [18].

The paper is organized as follows. In section 2 we give a short review of the results we will need about the loop gas coupled to gravity. In particular, we derive the loop
equations for the disc partition function with Neumann boundary conditions. In section 3 we formulate the JS boundary conditions in terms of the boundary measure for the $O(n)$ spins and derive the loop equations for the two-point correlators with mixed Neumann–JS boundary conditions. In section 4 we take the continuum limit of the loop equations and evaluate the boundary $L$-leg exponents. Here we also show that the loop equations can be cast in the form of the functional equations, generated by the boundary ground ring of Liouville gravity. Summary of the results and some concluding remarks are presented in section 5.

2. The boundary $O(n)$ loop model coupled to 2D gravity

2.1. Disc partition function with Neumann boundary conditions

The $O(n)$ model [13], defined originally on a regular hexagonal lattice, can be considered on an arbitrary trivalent planar graph $T^*$, which is dual to a triangulation of the disc $T$. The local fluctuating field is a $n$-component spin $\vec{S}(r)$ associated with the vertices $r$ of $T^*$. The Boltzmann weight for given configuration is

$$\prod_{\langle rr' \rangle} (t + \vec{S}(r) \cdot \vec{S}(r')),$$

(2.1)

where the product is over the links $\langle rr' \rangle$ of $T^*$. Here we consider the maximally packed dense phase $t = 0$. The partition function for given triangulation is the trace over these weights, defined by $\operatorname{Tr}[1] = 1$, $\operatorname{Tr}[S_a(r)S_b(r)] = \delta_{ab}$ and $\operatorname{Tr}[S_a(r)] = \operatorname{Tr}[S_a(r)S_b(r)S_c(r)] = 0$. Expanding the trace as a sum of monomials, the trace for given triangulation $T$ can be written as a sum over all configurations of densely packed self-avoiding, mutually avoiding loops on $T^*$ (figure 1). Each loop is taken with a weight $n$.

The disc partition function of the $O(n)$ model coupled to 2D gravity is defined as the double average: with respect to the $O(n)$ field on given triangulation and with respect to a sufficiently large class of triangulations. We will consider all possible triangulations, including the degenerate ones, which are dual to trivalent planar graphs with several connected components. In such a triangulation a boundary edge can either belong to a triangle, or be identified with another edge of the boundary. The measure in the ensemble of triangulations of the disc is determined by the ‘cosmological constant’ $x$ coupled to the area and the ‘boundary cosmological constant’ $\zeta$ coupled to the boundary length. By definition the area of a triangulation is the number of its triangles and the boundary length is the number of edges along the boundary.
The basic observable in the $O(n)$ model coupled to 2D gravity is the disc partition function $\Phi(\zeta, x)$. Its derivative $W = -\partial_\zeta \Phi$ is given by the series

$$W(\zeta) = \sum_{k=0}^{\infty} \zeta^{-l} W_l(x), \quad (2.2)$$

where $W_l$ is the non-normalized expectation value of the Boltzmann factor (2.1) in the ensemble of triangulations $T_l$ with boundary of length $l$. Since there is no restriction for the $O(n)$ spins at the boundary, we have Neumann boundary condition. The loop expansion of $W_l$ is

$$W_l = \sum_{T_l} x^{-(\text{Area of } T_l)} \sum_{\text{loops on } T_l^*} n^{(\text{Number of loops})}, \quad (2.3)$$

Note that here the boundary has a marked point, hence there is no symmetry factor $1/l$ in the sum. The generating function (2.2) is the boundary one-point function of the identity operator.

### 2.2. Loop equation for the disc partition function

Here we remind the combinatorial derivation of the loop equations for the disc amplitude with Neumann boundary conditions (2.2). This derivation, first given in [6], is a useful exercise to do before passing to the more complicated case of mixed boundary conditions.

The triangulations filled by loops that enter in the sum in the rhs of (2.3) can be divided into two classes (figure 2). The first class comprises the degenerate triangulations for which the first boundary edge is not an edge of a triangle, but is connected directly to the $k + 1$th boundary edge. The contribution of such triangulations factorizes to $W_{k-1} W_{l-k-1}$. There are $l - 1$ such terms, with $k = 1, 2, \ldots, l - 1$. For the rest of the triangulations entering the sum in (2.3) the first edge belongs to a triangle, which must be visited by a loop, since the loops are densely packed. Now consider the ensemble of all triangles visited by this loop. These triangles form a closed strip that contains the loop. Let $q$ and $p$ be, respectively, the lengths of the internal and external boundaries of the strip. For given $q$ and $p$ the expectation value factorized to the contribution of the internal disc $W_q$, that of the external disc $W_{l-1-p}$ and the number of realizations of the strip, $(p+q)!/p!q!$. We have also a factor $n$ because of the loop contained in the strip.
Adding the contributions of the two classes of triangulations, we obtain the following bilinear equation for the $W_k$:

$$W_l = \sum_{k=1}^{l-1} W_{l-k} W_{k-1} + n \sum_{p,q \geq 0} (\frac{p+q}{p!q!}) (1/x)^{p+q+1} W_{l+p-1} W_{q},$$  \hspace{1cm} (2.4)$$

where by definition $W_0 = 1$. Equations of this type are usually called loop equations. In terms of the generating function $(2.2)$, the loop equation is

$$\left[ -\zeta W(\zeta) + W(\zeta)^2 + n W(x - \zeta) W(\zeta) \right]_< = 0,$$  \hspace{1cm} (2.5)$$

where $[.]_<$ denotes the negative part of the Laurent series in $\zeta$.

It is consistent to assume that the solution has a cut $[a, b]$ on the real axis, with $a < b < x/2$. Then one can write the projection $[.]_<$ in (2.5) as a contour integral

$$1 - \zeta W(\zeta) + W^2(\zeta) + n \oint_{\gamma} \frac{d\zeta'}{2\pi i} \frac{W(\zeta) - W(\zeta')}{z - z'} W(x - \zeta') = 0,$$  \hspace{1cm} (2.6)$$

where the contour of integration encircles the cut of $W(\zeta)$ and leaves outside the cut of $W(x - \zeta)$. Equation (2.6) yields a condition for the discontinuity across the real axis

$$\text{Disc} W(\zeta) \left[ -\zeta + W(\zeta + i0) + W(\zeta - i0) + n W(x - \zeta) \right] = 0, \quad (\zeta \in \mathbb{R}).$$  \hspace{1cm} (2.7)$$

The condition (2.7), after being symmetrized with respect to $\zeta \rightarrow x - \zeta$, implies that a certain bilinear combination of $W(\zeta)$ and $W(x - \zeta)$ has zero discontinuity on the real axis and therefore is analytic in the whole complex plane. This leads, taking into account the asymptotics $W(\zeta) \simeq 1/\zeta$ at infinity, to the functional identity [7]

$$W(\zeta)^2 + W(x - \zeta)^2 + n W(\zeta) W(x - \zeta) = \zeta W(\zeta) + (x - \zeta) W(x - \zeta) - 2.$$  \hspace{1cm} (2.8)$$

3. Boundary correlators with mixed Neumann/JS boundary conditions

3.1. The JS boundary conditions in terms of spins and loops

The JS boundary condition can be introduced by restricting the $O(n)$ spins on the boundary to take values in a submanifold of ‘dimension’ $y$. That is, the first $y$ components of the $O(n)$ are free, while the rest $n - y$ components are fixed. This is equivalent to replacing the Boltzmann weight (2.1) with

$$\prod_{\langle rr' \rangle \in \text{bulk}} \tilde{S}(r) \cdot \tilde{S}(r') \prod_{r \in \text{boundary}} \sum_{b=1}^{y} S_b(r) S_b(r).$$  \hspace{1cm} (3.1)$$

The Boltzmann weight (3.1) is invariant with respect to a subgroup $O(y) \subset O(n)$, which means that the JS boundary conditions are associated with the conjugacy classes of $O(n)$. This is a necessary condition to have conformal invariant boundary theory [27].

In the original formulation of the $O(n)$ model both $n$ and $y$ are integers\(^5\), but the result of evaluating the trace can be analytically continued for non-integer values of $y$ and $n$. The disc partition function is then formulated as a gas of fully packed loops on the

\(^5\) It is always possible to consider part of the components of the $O(n)$ vector as anticommuting variables, so that the restriction $0 \leq y \leq n$ can be avoided.
world sheet, having two different fugacities. The loops that do not touch the boundary have fugacity $n$, while the loops that are reflected from the boundary one or several times have different fugacity $y$.

There are two classes of local operators compatible with the boundary measure in (3.1), which we denote by $S_L^\parallel$ and $S_L^\perp$. They are defined as $O(y)$ invariant polynomials of the spin components:

$$S_L^\parallel = \sum_{1 \leq a_1 < \cdots < a_L \leq y} S_{a_1} \ldots S_{a_L}, \quad S_L^\perp = \sum_{y+1 \leq b_1 < \cdots < b_L \leq n} S_{b_1} \ldots S_{b_L}. \quad (3.2)$$

We will investigate the boundary correlation functions of the operators (3.2) with Neumann boundary condition on the left segment and JS boundary conditions on the right segment of the boundary. We denote these correlation functions by

$$D_L^\parallel(\zeta, \tilde{\zeta}) = \langle \zeta [S_L^\parallel]^\zeta [S_L^\parallel]^\zeta \rangle_{\text{disk}}, \quad D_L^\perp(\zeta, \tilde{\zeta}) = \langle \zeta [S_L^\perp]^\zeta [S_L^\perp]^\zeta \rangle_{\text{disk}}. \quad (3.3)$$

The symbol $\langle \rangle_{\text{disk}}$ means a double sum over the $O(n)$ spins and over the triangulations of the disc, characterized by the boundary cosmological constants $\zeta$ and $\tilde{\zeta}$ associated with the two segments of the boundary. In terms of the loop gas, the two-point functions (3.3) are the partition function of the loop gas with $L$ open lines connecting two marked points on the boundary, as shown in figure 3. In the loop expansion of $D_L^\parallel$, the configurations where the rightmost open line touches the JS boundary have the same weight as those in which it does not. In the case of $S_L^\perp$, the lines that touch the JS boundary have zero weight. In the terminology of [18], the correlators $D_L^\parallel$ and $D_L^\perp$ are the two-point functions of the $L$-leg operator, respectively, in the blobbed and unblobbed sectors.

3.2. Loop equations for the $L$-leg correlators with mixed N/JS boundary conditions

The correlation functions (3.3) are defined as the series expansions

$$D_L(\zeta, \tilde{\zeta}) = x^{-L} \sum_{l, \tilde{l}=0}^{\infty} D_{l, \tilde{l}}^L \zeta^{-l-1} \tilde{\zeta}^{-\tilde{l}-1}, \quad (3.4)$$

where $D_L$ can be either $D_L^\parallel$ or $D_L^\perp$. The coefficients $D_{l, \tilde{l}}^L$ are the two-point functions with fixed boundary lengths, $l$ and $\tilde{l}$. The functions $D_L$ and $D_{L-1}$ satisfy a simple recurrence
Figure 4. The recurrence equation for the boundary two-point correlators $(L \geq 2)$.

relation $[24,23]$. This relation in fact holds for Neumann boundary condition on the left segment and any boundary condition on the right segment of the boundary. It is obtained by taking into account all possible configurations of the strip containing the leftmost open line (figure 4).

Each such configuration is characterized by the lengths $p$ and $q$, respectively, of the left and the right boundaries of the strip. The length of the leftmost line is then $p + q$. The strip splits the triangulated disc into two pieces in such a way that the Boltzmann weights factorize, so that the sum over triangulations and loops can be done in each piece separately. The left piece contributes the disc amplitude $W_{i+p}$, while the right piece yields the boundary two-point function $D^{L-1}_{q,l}$, again with Neumann boundary condition on the left segment. Taking into account that the strip in the middle can be realized in $(p + q)!/p!q!$ ways as well as the factor $x^{-p-q}$ associated with the area $p + q$ of the strip we get

$$D^{L}_{i,l} = \sum_{p,q=0}^{\infty} W_{i+p} \frac{(p + q)!}{p!q!} x^{-p-q} D^{L-1}_{q,l}. \quad (3.5)$$

Written in terms of the generating functions (3.4), this equation takes the form

$$D_{L}(\zeta, \tilde{\zeta}) = [W(\zeta)D_{L-1}(x - \zeta, \tilde{\zeta})]_{<}, \quad (3.6)$$

where we used the same notations as in (2.5). We again express the projection $[]_{<}$ as a contour integral:

$$D_{L}(\zeta, \tilde{\zeta}) = -\oint \frac{d\zeta'}{2\pi i} \frac{W(\zeta) - W(\zeta')}{\zeta - \zeta'} D_{L-1}(x - \zeta', \tilde{\zeta}). \quad (3.7)$$

This gives for the discontinuity on the real axis

$$\text{Disc}_{\zeta} D_{L}(\zeta, \tilde{\zeta}) + \text{Disc}_{\zeta} W(\zeta) \cdot D_{L-1}(x - \zeta, \tilde{\zeta}) = 0. \quad (3.8)$$

Equation (3.5) was derived for $L \geq 2$, but we will extend the definition of the two-point functions to $L = 0$, so that it holds also for $L = 1$. The subtle point here is how to weight the degenerate triangulations where the left and right boundaries touch at one or several points. The two-point function $D^{0}_{0}$ is defined as a sum over all triangulations, including those where two boundaries touch, while in the loop expansion of the two-point function
$D_0^\perp$ the triangulations with touchings are excluded. Such partition functions behave as though local operators were inserted at the marked points \[11\]. We will think of $D_0^\parallel$ and $D_0^\perp$ as the boundary two-point function of the operators $S_0^\parallel$ and $S_0^\perp$, correspondingly.

The recurrence relations (3.5) allow us to determine all correlation functions (3.3), once the functions $A \equiv D_0^\perp$ and $B \equiv D_1^\parallel$ are known. Below we derive two independent nonlinear equations for $A$ and $B$ and find their unique solution in the continuum limit. These equations involve nontrivially only the cosmological constant $\zeta$ of the Neumann boundary, while the cosmological constant $\tilde{\zeta}$ of the JS boundary enters as a parameter. Therefore from now on the dependence on $\zeta$ will be implicit in our notation. The first equation is derived by splitting the sum of triangulations contributing to $A_{l,\tilde{l}}$, into four sets associated with the way the first edge of the JS boundary is connected to the rest of the triangulation (figure 5). First, there is the possibility that the JS boundary has zero length ($\tilde{l} = 0$). In this case $A_{l,\tilde{l}} = W_l$. If the JS boundary contains at least one edge, we count two possibilities. The first edge can be glued to another edge of the JS boundary, or it can be the edge of a triangle containing a segment of a loop. The last possibility is realized by two types of configurations: (a) a loop that touches the boundary only once and (b) a loop that touches the boundary at least twice.

In case (a) we get a product $A_{l+p,\tilde{l}}W_q$, as in the loop equation (2.4) for the disc amplitude. In case (b) we can apply the same argument as the one used in the derivation of (3.5). The strip that splits the disk into two here consists of the triangles visited by the most external arc of the loop. The left piece is bounded by the Neumann boundary, the left boundary of the strip of length $p$ and a piece of the JS boundary of length $k$. It yields a factor $A_{l+p,k}$. The right piece is the partition function of the disc with mixed Neumann/JS boundary conditions and an open line connecting the extremities of the two segments of the boundary. The open line can touch both boundaries an unrestricted number of times. By definition this is the two-point function $B_{q,l-k-2}$, where $q$ is the length of the right side of the strip.
Summing up the four terms and taking into account the combinatorial factors, we get

\[
A_{l,\tilde{l}} = W_l \delta_{l,0} + \sum_{k=0}^{l-2} A_{l-k-2} \tilde{W}_k + \sum_{p,q=0}^{\infty} x^{-p-q-1} (p+q)! \frac{1}{p!q!} A_{l+p,\tilde{l}-1} W_q \\
+ y \sum_{k=0}^{l-2} \sum_{p,q=0}^{\infty} x^{-p-q-2} (p+q)! \frac{1}{p!q!} A_{l+p,k} B_{q,\tilde{l}-k-2}.
\] (3.9)

The equation satisfied by \( B \) is very similar to (3.5). Here we distinguish two possibilities: the open line does not touch (touches at least once) the JS boundary (figure 6):

\[
B_{l,\tilde{l}} = \sum_{p,q=0}^{\infty} \frac{(p+q)!}{p!q!} x^{-p-q} W_{l+p} A_{q,\tilde{l}} + \sum_{p,q=0}^{\infty} \frac{(p+q)!}{p!q!} x^{-p-q-1} B_{l+p,\tilde{l}-k-1} A_{q,k}.
\] (3.10)

In terms of the generating functions (3.4) the loop equations (3.9) and (3.10) state

\[
\tilde{\zeta} A(\zeta) = W(\zeta) + A(\zeta) \tilde{W}(\tilde{\zeta}) + y [A(\zeta) (W(x - \zeta) + B(x - \zeta))]_<
\] (3.11)

\[
B(\zeta) = [W(\zeta) A(x - \zeta)]_< + [B(\zeta) A(x - \zeta)]_<.
\] (3.12)

Representing, as before, the projection [\( [\cdot]_< \)] as a contour integral and taking the discontinuity across the real axis, we obtain

\[
y \text{Disc}_\zeta A(\zeta) \cdot \left( W(x - \zeta) + B(x - \zeta) + \frac{1}{y} (\tilde{W}(\tilde{\zeta}) - \tilde{\zeta}) \right) + \text{Disc}_\zeta W(\zeta) = 0
\] (3.13)

\[
\text{Disc}_\zeta B(\zeta) = \text{Disc}_\zeta W(\zeta) \cdot A(x - \zeta) + \text{Disc}_\zeta B(\zeta) \cdot A(x - \zeta).
\] (3.14)

### 3.3. Final form of the loop equations for \( D_{11}^\parallel \) and \( D_0^\perp \)

The linear terms in equations (3.13) and (3.14) can be eliminated by shifting the observables \( A = D_0^\perp \) and \( B = D_{11}^\parallel \). We redefine \( D_0^\perp \) and \( D_{11}^\parallel \) as

\[
D_{11}^\parallel(\zeta) := B(\zeta) + W(\zeta) + \frac{1}{y} (\tilde{W}(\tilde{\zeta}) - \tilde{\zeta}),
\]

\[
D_0^\perp(\zeta) := A(\zeta) - 1.
\] (3.15)

The added terms only change the weight when one or both boundaries degenerate to a point and thus does not affect the critical behavior. For the shifted quantities (3.15),

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equations (3.13) and (3.14) simplify to

\[ \text{Disc}_\zeta D^\parallel_0(\zeta) \cdot D^\parallel_1(x - \zeta) + \frac{1}{y} \text{Disc}_\zeta W(\zeta) = 0, \]  

(3.16)

\[ \text{Disc}_\zeta D^\parallel_1(z) \cdot D^\perp_0(x - \zeta) + \text{Disc}_\zeta W(\zeta) = 0. \]  

(3.17)

The set of equations (3.8), (3.16) and (3.17) is overdetermined. Equation (3.17) is consistent with (3.8) for \( L = 1 \) under the condition

\[ D^\parallel_0 = -\frac{1}{D^\perp_0}. \]  

(3.18)

This condition has simple geometrical explanation. Indeed, it is easy to see that the correlation functions \( D^\parallel_0 \) and \( D^\perp_0 \) are, unlike their counterparts on a flat lattice, different. The difference comes from the fact that on a dynamical world sheet the Neumann boundary can come close to the JS boundary and touch it one or several times. Microscopically, the algebraic relation (3.18) can be understood as a geometric progression obtained by taking into account all possible touchings:

\[ D^\parallel_0 = 1 + A + A^2 + \cdots = \frac{1}{1 - A}. \]  

(3.19)

The two equations (3.16) and (3.17), together with the asymptotics at infinity that follow from the expansion (3.4), imply the following functional identity:

\[ D^\parallel_1(\zeta)D^\perp_0(x - \zeta) = -W(\zeta) - \frac{1}{y}W(x - \zeta) - \frac{1}{y}(\tilde{W}(\tilde{\zeta}) - \tilde{\zeta}). \]  

(3.20)

The loop equations (3.8), (3.16) and (3.17) obtained here allow us in principle to compute all boundary two-point correlators with mixed Neumann/JS boundary conditions. The three couplings, \( x, \zeta \) and \( \tilde{\zeta} \), associated respectively with the area of the triangulation, the length of the Neumann boundary and the length of the JS boundary, enter in the loop equations implicitly through the disc amplitude \( W(\zeta) \).

4. The continuum limit

In this section we will study the continuum limit of the solution, in which the three couplings are tuned close to their critical values, \( \xi^*, \zeta^* \) and \( \tilde{\zeta}^* \). The solution in the continuum limit depends on the three renormalized couplings

\[ \mu \sim x - x^*, \quad z \sim \zeta - \zeta^* \quad \text{and} \quad \tilde{z} \sim \tilde{\zeta} - \tilde{\zeta}^*. \]

4.1. Disc one-point function with Neumann boundary conditions

The power series (2.3) converges for \( x > x^* \), where

\[ x^* = 2\sqrt{2(2 + n)} \]

is the critical value of the cosmological constant [7]. We introduce a small cutoff parameter (elementary length) \( a \) and define the renormalized cosmological constant \( \mu \), boundary
cosmological constant $z$ and loop amplitude $w$ as follows$^6$:

$$
\mu := a^{-2} - \frac{8}{4 - n^2} \frac{x^2 - x^2}{x^2},
$$

$$
z := a^{-1/(1-\theta)} \left( \zeta - \frac{1}{2} x \right),
$$

$$
w(z) := a^{-1} \left( W(\zeta) - \frac{1}{2 - n} \zeta + \frac{n}{4 - n^2} x \right).
$$

(4.1)

Then equation (2.8) takes the form

$$
w(z)^2 + w(-z)^2 + nw(z)w(-z) = \mu \sin^2 \pi \theta + a^2 \frac{\theta}{1 - \theta} \frac{z^2}{2 - n}.
$$

(4.2)

In the continuum limit $a \to 0$, the second term on the rhs of (4.2) can be neglected provided $\frac{1}{2} < \theta < 1$, or $0 < n < 2$. Then the solution of (4.2) can be written in parametric form as

$$
z = M \cosh \tau,
$$

$$
w(z) = -M^{1-\theta} \cosh(1 - \theta) \tau,
$$

(4.3)

where $M = C \mu^{1/2(1-\theta)}$. The value of the constant $C$ is fixed by the normalization of the disc partition function $\Phi$. One possible choice is

$$
\partial_\mu \Phi = \frac{M^\theta}{\theta} \cosh \theta \tau, \quad \partial_z \Phi = -w(z), \quad M^{2 - 2\theta} = 2\mu.
$$

(4.4)

As a function of $z$, the loop amplitude $w(z)$ has a branch cut along the interval $[-\infty, -M]$. The solution (4.3) corresponds to Liouville gravity with matter central charge

$$
c = 1 - \frac{6}{1 - \theta} \theta^2.
$$

(4.5)

The susceptibility $u(\mu)$, which is, by definition, the partition function of the loop gas on a dynamically triangulated sphere with two punctures, scales as $u \sim \mu^{-\gamma_{str}}$, with

$$
\gamma_{str} = -\frac{\theta}{1 - \theta}.
$$

(4.6)

4.2. Boundary two-point functions of the $L$-leg operators

The continuum limit of the boundary two-point functions is obtained as a triple scaling limit in $x, \zeta$ and $\tilde{\zeta}$, in which the area of the triangulation as well as the lengths of the Neumann and JS boundaries diverge. The point $x = x^*, \zeta = \zeta^*, \tilde{\zeta} = \tilde{\zeta}^*$ is a singular point of equation (3.20), where the rhs vanishes:

$$
W(\zeta^*) + \frac{1}{y} W(\zeta^*) + \frac{1}{y} (\tilde{W}(\tilde{\zeta}^*) - \tilde{\zeta}^*) = 0.
$$

(4.7)

We define the renormalized cosmological constant $\tilde{z}$ and loop amplitude with JS boundary conditions as

$$
\tilde{z} := a^{-1/(1-\theta)} \left( \tilde{\zeta} - \tilde{\zeta}^* \right),
$$

$$
\tilde{w}(\tilde{z}) := a^{-1} \left( \tilde{W}(\tilde{\zeta}) - \tilde{W}(\tilde{\zeta}^*) \right).
$$

(4.8)

$^6$ Note that in the dense phase the boundary has anomalous dimension. This is a consequence of the fractal structure of the boundary in this phase.
In the continuum limit \( a \to 0 \), the linear terms in \( z \) and \( \tilde{z} \) can be neglected, since they are multiplied by \( a^{\frac{\theta}{1 - \theta}} \), and functional equation (3.20) takes the form

\[
D_1^\parallel(z)D_0^\perp(-z) = -w(z) - \frac{1}{y}w(-z) - \frac{1}{y}\tilde{w}(\tilde{z}).
\] (4.9)

Together with (3.18), this equation yields a linear relation between \( D_1^\parallel \) and \( D_0^\perp \):

\[
D_1^\parallel(z) = \left[ w(z) + \frac{1}{y}w(-z) + \frac{1}{y}\tilde{w}(\tilde{z}) \right] D_0^\perp(-z).
\] (4.10)

Equation (4.10) is the key result of this paper. It allows us to determine, up to a normalization, the boundary two-point functions with \( D_1^\parallel \) and \( D_0^\perp \). The two-point functions with \( L > 1 \) are then easily evaluated from the recurrence equations (3.8), which have, in the continuum limit, the form

\[
D_L^\parallel(z + i0) - D_L^\parallel(z - i0) = \left[ w(z + i0) - w(z - i0) \right] D_{L-1}(z) \quad (z < -M).
\] (4.11)

The solution, as a function of \( \mu \), \( z \) and \( \tilde{z} \), is expected to be of the form

\[
D_L^\parallel(\mu, z, \tilde{z}) = \mu^{1 - (1/2)\gamma_{str}} (\sqrt{\mu})^{2\Delta_B^L - 2} \hat{D}_L(z/M, \tilde{z}/M),
\] (4.12)

where the exponent \( \gamma_{str} \) is given by (4.6), \( \Delta_B^L \) is the boundary gravitational dimension of the \( L \)-leg operator, and \( M = (2\mu)^{1/(2 - 2\theta)} \).

Before giving the complete solution, we are going to solve a simpler problem: to determine the scaling behavior of the two-point function when \( \mu = \tilde{z} = 0 \). Restricted in this way, the two-point function (4.12) reduces to a power of the only non-zero coupling \( z \):

\[
D_L \sim z^{(1 - \theta)(2\Delta_B^L - \gamma_{str})}.
\] (4.13)

From here one can determine the conformal dimensions of the \( L \)-leg operators using the KPZ map (1.1).

### 4.3. Evaluation of the \( L \)-leg critical exponents for N–JS boundary conditions

At \( \mu = \tilde{z} = 0 \), the solution for the observables \( D_L \) and \( w \) must be given by powers of \( z \):

\[
D_\parallel \sim z^{\alpha_L}, \quad D_\perp \sim z^{\beta_L}, \quad w \sim z^{1 - \theta}.
\] (4.14)

Then (4.9) yields three identities relating \( \beta_0, \beta_1 \) and \( y \). The first one follows from comparing the powers of \( z \) on both sides:

\[
\beta_1 + \alpha_0 = 1 - \theta.
\] (4.15)

The other two identities arise when equating the imaginary part of both sides of (4.9), respectively, for \( z > 0 \) and \( z < 0 \):

\[
\sin(\pi \alpha_0) = -\frac{1}{y}\sin(\theta\pi), \quad \sin(\pi \beta_1) = -\sin(\theta\pi).
\] (4.16)

These equations determine \( y \) as a function of \( \alpha_0 \):

\[
y = \frac{\sin(\pi \beta_1)}{\sin(\pi \alpha_0)} = \frac{\sin(\pi(\alpha_0 + \theta))}{\sin(\pi \alpha_0)}.
\] (4.17)
We see that the expression (4.17) for \( y \) is of the form (1.6) with
\[
\alpha_0 = r \theta. \tag{4.18}
\]
To determine \( \alpha_0 \), we must invert the multi-value function \( y(\alpha_0) = y(\alpha_0 + 1) \). The relevant branch is given by the lowest positive value of \( \alpha_0 \), which must be in the interval
\[
0 \leq \alpha_0 < 1 \quad \text{or} \quad 0 < r < \frac{1}{\theta}. \tag{4.19}
\]
The fact that \( \alpha_0 \) is positive is a consequence of the representation (3.19) of \( D_0^\parallel \). The singularity of the lhs cannot be weaker than the singularity of each of the terms, hence \( \beta_0 \leq \alpha_0 \). Since \( \alpha_0 + \beta_0 = 0 \), this means that \( \alpha_0 \) is positive and \( \beta_0 \) is negative\(^7\).

Once we determined the critical exponent for \( L = 0 \), the other exponents follow from the recurrence relation (4.11):
\[
\alpha_L = L(1 - \theta) + r \theta \quad \beta_L = L(1 - \theta) - r \theta. \tag{4.20}
\]
Comparing these values with (4.13) we find for the gravitational dimensions \( \Delta_L^\parallel \) and \( \Delta_L^\perp \)
\[
\Delta_L^\parallel = \frac{\alpha_L - \theta}{2(1 - \theta)} = \frac{L(1 - \theta) + r \theta - \theta}{2(1 - \theta)} = \Delta_{r,r-L},
\]
\[
\Delta_L^\perp = \frac{\beta_L - \theta}{2(1 - \theta)} = \frac{L(1 - \theta) - r \theta - \theta}{2(1 - \theta)} = \Delta_{r,-r-L}. \tag{4.21}
\]
By the KPZ scaling relation (1.1), and using the symmetry \( h_{-r,-s} = h_{r,s} \), we find the conformal weights of the \( L \)-leg operators with mixed Neumann–JS boundary conditions:
\[
h_L^\parallel = h_{r,r+L}; \quad h_L^\perp = h_{r,r-L}, \tag{4.22}
\]
in accord with [18].

When \( r = 1 \), then \( y = n \) and the loops touching the boundary have the same fugacity as the loops in the bulk. In this case one obtains the well known \( L \)-leg exponents with Neumann–Neumann boundary condition on flat [19] and dynamical [22] lattices:
\[
h_{L,\text{leg}}^{N,D} = h_{0,1/2+L}. \notag
\]

### 4.4. Complete solution and relation to boundary Liouville theory

In the previous subsection we determined the prefactor in (4.12). Now we will evaluate the scaling functions \( \hat{D}_L \). First, using the expression (4.3) of the loop amplitude \( w(z) \) and the identity
\[
[w(\tau) + y^{-1} w(\tau + i \pi)]/M^{1-\theta} = -C \cosh[(1 - \theta)\tau + i \pi r \theta], \tag{4.23}
\]
with \( C = \sin \pi \theta / \sin(r + 1) \pi \theta \), we write (4.10) as
\[
\hat{D}_1^{\parallel}(\tau) = -\left( C \cosh[(1 - \theta)\tau + i \pi r \theta] + \frac{\tilde{w}(\tilde{z})}{yM^{1-\theta}} \right) \hat{D}_0^{\parallel}(\tau + i \pi). \tag{4.24}
\]
Next, we change the variable \( \tilde{z} \to \tilde{\tau} \) so that
\[
\frac{\tilde{w}(\tilde{z})}{M^{1-\theta}} = y C \cosh(1 - \theta) \tilde{\tau} = \frac{\sin \pi \theta}{\sin \pi r \theta} \cosh(1 - \theta) \tilde{\tau}. \tag{4.25}
\]
\(^7\) Let us stress that this argument is justified only for non-negative couplings, when all terms in the series are non-negative.
After that equation (4.11) takes the form
\[ \hat{D}_0^I(\tau, \bar{\tau}) = -C \left( \cosh \left( (1 - \theta)\tau + i\pi \theta \right) + \cosh(1 - \theta)\bar{\tau} \right) \hat{D}_0^I(\tau + i\pi, \bar{\tau}). \] (4.26)

Let us remark that equation (4.25) is just a change of variable and does not involve any assumption about the boundary one-point function with JS boundary conditions \( \hat{w}(\bar{z}) \), because the function \( \hat{z}(\bar{\tau}) \) is not yet determined. On the other hand, the only solution compatible with the world-sheet CFT is \( \hat{z} \sim M \cosh \bar{\tau} \). Indeed, there is only one non-zero boundary one-point function, that of the Liouville-dressed identity operator, which is given by (4.3).

For generic \( r \), one can recognize in (4.26) the functional equation derived by Fateev, Zamolodchikov and Zamolodchikov for the boundary two-point function in boundary Liouville theory [15]. This equation also appeared as the first member of an infinite series of functional identities characterizing the boundary ground ring in 2D quantum gravity [28, 29].

The boundary two-point function in Liouville gravity depends on the target-space momentum \( p \) and the two boundary parameters \( \tau \) and \( \bar{\tau} \). It is given, up to a normalization factor that depends only on \( p \), by [15]
\[ D(p; \tau, \bar{\tau}) = \mu^{p/2} \hat{D}(p; \tau, \bar{\tau}), \]
\[ \hat{D}(p; \tau, \bar{\tau}) = \exp \left( -\int_{-\infty}^{\infty} dt \left[ \frac{\sinh(\pi pt/b^2) \cos(\tau t) \cos(\bar{\tau} t)}{\sinh(\pi t) \sinh(\pi t/b^2)} - \frac{p}{\pi t} \right] \right). \] (4.27)

The scaling function \( \hat{D}(p; \tau, \bar{\tau}) \) satisfies the identity
\[ \hat{D}(p + b^2; \tau, \bar{\tau}) = \frac{1}{2} \left[ \cosh \left( b^2 \tau \mp i\pi p \right) + \cosh (b^2 \bar{\tau}) \right] \hat{D}(p; \tau \pm i\pi, \bar{\tau}), \] (4.28)
which is the same as (4.26), with \( b^2 = 1 - \theta \), \( p = -r\theta \) and \( C = 1/2 \). Therefore
\[ \hat{D}_0^I(\tau, \bar{\tau}) = \hat{D}(\theta r; \tau, \bar{\tau}), \quad \hat{D}_1^I(\tau, \bar{\tau}) = \hat{D}(\theta r + 1 - \theta; \tau, \bar{\tau}) \] (4.29)
is a solution of (4.26). The relation (3.18) then implies
\[ \hat{D}_0^I(\tau, \bar{\tau}) = \hat{D}(\theta r; \tau, \bar{\tau}). \] (4.30)

Is this the physical solution that corresponds to the series expansion (3.4)? Equation (4.28) does not determine uniquely the two-point function. In order to specify the unique solution, FZZ [15] used the duality symmetry, which supplies another equation of the same type, as well as the symmetry in \( \tau \leftrightarrow \bar{\tau} \). Neither of these symmetries is satisfied in the microscopic theory. If we assume that the physical solution enjoys these symmetries in the scaling limit, then (4.29) is the unique solution. In the particular cases \( r = 1 \) and \( r = (1 - \theta)/2\theta \), the function (4.30) reproduces correctly the expressions obtained previously for the disc partition function with mixed Neumann/Neumann and Neumann/Dirichlet boundary conditions.

Once the two-point functions with \( L = 0 \) are known, the rest can be determined from the recurrence equation (4.11), which can be cast into the form
\[ \hat{D}_L(\tau + i\pi, \bar{\tau}) - \hat{D}_L(\tau - i\pi, \bar{\tau}) = -2i \sin \pi \theta \sinh((1 - \theta)\tau) \hat{D}_{L-1}(\tau, \bar{\tau}). \] (4.31)

\(^8\) Our notations are related to the notations of [15] by \( 1/b + b - 2\beta = p/b, \tau = \pi s/b, b^2 = 1 - \theta \).

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This equation is compatible (up to normalization) with (4.28) upon the identification
\[ p_L^\parallel = -\theta r + (1 - \theta)L \quad \text{and} \quad p_L^\perp = \theta r + (1 - \theta)L, \] respectively, for $D_L^\parallel$ and $D_L^\perp$. When $L \geq 1$, the operators $S_L^\parallel$ and $S_L^\perp$ have different conformal weights. The two-point functions $D_L^\parallel$ and $D_L^\perp$ are related by Liouville reflection, and correspond to the ‘physical’ and ‘unphysical’ Liouville dress of the same boundary matter field with conformal weight $h_{r,r}$.

### 4.5. Dirichlet boundary conditions and twist operators

The Dirichlet boundary conditions for the O($n$) model is defined by fixing the O($n$) spin on the boundary to point to a given direction, say $\vec{S} = (1, 0, \ldots, 0)$. This is a particular case of the JS boundary condition, obtained by taking $y = 1$, or equivalently
\[ r = \frac{1 - \theta}{2\theta}. \]

This case deserves special attention, because here we can compare the general formula (4.30) with the exact results found in [22, 24, 23].

The Dirichlet boundary condition for the O($n$) model leads to the same loop gas expansion as the Neumann boundary condition for the SOS model, which was first studied in [22] and then given a world-sheet CFT interpretation in [24] and [23]. Namely, it is assumed that each point of the boundary is an endpoint of an open line, as sketched in figure 7. The boundary condition changing operator $\mathcal{T}$ separating Dirichlet and Neumann boundary conditions was called in [24] a twist operator, by analogy with the Gaussian field. The correlation function of two twist operators:
\[
\Omega(z, \bar{z}) = \langle [\mathcal{T}]_D^z [\mathcal{T}]_N^\bar{z} \rangle_{\text{disk}}
\] was first evaluated in [22], equations (4.34)–(4.37) of that paper. Afterwards this solution was identified [24] as a special case of the FZZ two-point function (4.27) with $p = (1 - \theta)/2$, which corresponds to $r$ given by (4.33).

It is easy to see that the sum over the configurations with open lines as the one in figure 7 can be interpreted, for this particular value of $r$, either as the loop expansion for $D_0^\perp$, or as the loop expansion for $D_1^\parallel$. Indeed, if we connect pairwise the endpoints of the open lines, as shown in figure 8(a), we obtain a configuration of the loop expansion for $D_0^\perp$. The Boltzmann weights also match under the condition that all loops that touch the boundary have fugacity $y = 1$. Alternatively, we can leave the first and the last open line endpoint free and connect the rest of the endpoints pairwise, as is shown in...
Figure 8. The two ways of closing the open lines.

Then the first and the last points are connected by an open line, and we obtain a configuration of the loop expansion for $D_{1}^{\parallel}$. Therefore, even at microscopic level,

$$\Omega = D_{0}^{\perp} = D_{1}^{\parallel} \quad (y = 1).$$

The two-point function of the boundary twist operator, $\Omega(\tau, \tilde{\tau})$, is obtained as the solution of a quadratic functional identity, equation (4.25) of [22], which is identical to (4.9) with $y = 1$. In this particular case the solution can be found without additional assumptions about symmetry, and the result [22] coincides with (4.27) for $p = (1 - \theta)/2$.

The conformal weights of the excited twist operators $T_{L}$, or the boundary $L$-leg operators with mixed Neumann/Dirichlet boundary conditions, were identified in [23] as

$$h_{T_{L}} = h_{0,L+1/2}. \quad (4.35)$$

One can check that $h_{T_{L}} = h_{l,L+1} = h_{\perp,L}$, with $r$ given by (4.33). The different Kac-table like identifications of these operators are possible because of the ambiguities of the representation (1.2).

Thus the results obtained here for general $r$ are in full agreement with those of [22, 24, 23]. There is, however, a difference in the interpretation of the results, which is due to the different form of the microscopic loop equations. Compared to equation (4.18) of [22], our equation (3.20) contains an extra term $\tilde{W}$, which takes into account the self-touchings of the JS boundary. This term was omitted in [22], hence the self-touchings of the Dirichlet boundary were not taken into account there. As a result, the two-point function (4.34), evaluated in [22], describes a sum over surfaces with self-touchings allowed for the Dirichlet boundary and forbidden for the Neumann boundary. This explains the puzzling observation, made in [22], that the fractal dimensions of the Dirichlet and Neumann boundaries are different for $\theta \neq 0$.

We find it here more natural to define the sum over surfaces so that both segments of the boundary have the same dimension, which is the case when the contact term in question is taken into account.

5. Conclusions

In this paper we evaluated the boundary two-point function for the $O(n)$ loop model on the dynamically triangulated disc with presumably the most general boundary conditions, constructed recently in [18]. We restricted ourselves to the dense phase of the loop gas, where both the bulk and the boundary are critical and the only parameters of the theory

\[9\]

In [23], the conformal weights for the excited twist operators were actually evaluated for the dilute phase, where they are given by $h_{L+1/2,0}$. 

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are the bulk and boundary cosmological constants. The scaling behavior of the two-point function confirms the $L$-leg exponents (4.22) conjectured in [18]. Our result for the two-point function implies the symmetry

$$S_L^\parallel \leftrightarrow S_L^\perp, \quad y \leftrightarrow n - y,$$

(5.1)

which looks quite natural given the definition (3.2) of these operators and resembles the duality symmetry that exchanges Dirichlet and Neumann boundary conditions. In the parameterization (1.6), exchanging $y$ and $n - y$ is equivalent to changing the sign of $r$, due to the identity

$$y(r) + y(-r) = n.$$ 

(5.2)

In the loop gas formulation, the symmetry (5.1) is spelled out as

$$\{\text{blobbed}, r\} \leftrightarrow \{\text{unblobbed}, -r\}$$

(5.3)

and is respected by the exponents (1.5).

The dilute phase is more intricate because for each $y$ there is a one-parameter family of boundary conditions and a fine tuning of the matter coupling constants should be done both in the bulk and on the boundary. Our preliminary results for the scaling dimensions [30] seem to be compatible with the unpublished results of Jacobsen and Saleur.

Finally, let us mention that the $O(n)$ model coupled to 2D gravity can be viewed as a solvable model of bosonic string theory with curved target space, representing the $(n - 1)$-dimensional sphere. The continuous spectrum of $D$-branes in this theory is presumably related to the fact that the target space curvature,

$$R = (n - 1)(n - 2),$$

is negative in the interval $1 < n < 2$. We believe that all our results can also be obtained on the basis of the dual $O(n)$ invariant matrix model [5].

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References

[1] Kazakov V K, Percolation on a fractal with the statistics of planar Feynman graphs: exact solution, 1989 Mod. Phys. Lett. A 4 1091
[2] Kazakov V A, 1986 Pis. Zh. Eksp. Teor. Fiz. 44 105
Kazakov V A, Exact solution of the Ising model on a random two-dimensional lattice, 1986 JETP Lett. 44 133 (translation)
[3] Boulatov D V and Kazakov V A, The Ising model on random planar lattice: the structure of phase transition and the exact critical exponents, 1987 Phys. Lett. B 186 379

doi:10.1088/1742-5468/2007/08/P08023
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[4] Duplantier B and Kostov I, 1988 Phys. Rev. Lett. 61 1433
   Duplantier B and Kostov I, 1990 Nucl. Phys. B 340 491
[5] Kostov I K, O(n) vector model on a planar random lattice: spectrum of anomalous dimensions, 1989 Mod. Phys. Lett. A 4 217
[6] Kostov I K, The ADE face models on a fluctuating planar lattice, 1989 Nucl. Phys. B 340 491
[7] Kostov I K, Strings with discrete target space, 1992 Nucl. Phys. B 376 539 [hep-th/9112059]
[8] Polyakov A, Quantum geometry of bosonic strings, 1981 Phys. Lett. B 103 207
[9] Knizhnik V, Polyakov A and Zamolodchikov A, 1988 Mod. Phys. Lett. A 3 819
   David F, 1988 Mod. Phys. Lett. A 3 1651
   Distler J and Kawai H, 1989 Nucl. Phys. B 321 509
[10] David F, Conformal field theories coupled to 2-D gravity in the conformal gauge, 1988 Mod. Phys. Lett. A 3 1651
   Distler J and Kawai H, Conformal field theory and 2-D quantum gravity or who’s afraid of Joseph Liouville?, 1989 Nucl. Phys. B 321 509
[11] Cardy J L, Conformal invariance and surface critical behavior, 1984 Nucl. Phys. B 240 514
[12] Duplantier B, Conformal fractal geometry and boundary quantum gravity, 2003 Preprint math-ph/0303034
[13] Nienhuis B, 1982 Phys. Rev. Lett. 49 1062
   Nienhuis B, 1984 J. Stat. Phys. 34 731
[14] See, for example, Lawler G, 2005 Conformally Invariant Processes in the Plane (Washington, DC: American Mathematical Society)
[15] Fateev V, Zamolodchikov A B and Zamolodchikov A B, Boundary Liouville field theory. I: boundary state and boundary two-point function, 2000 Preprint hep-th/0001012
[16] Ponsot B and Teschner J, Boundary Liouville field theory: boundary three point function, 2002 Nucl. Phys. B 622 309 [hep-th/0110244]
[17] Hosomichi K, Bulk-boundary propagator in Liouville theory on a disc, 2001 J. High Energy Phys. JHEP11(2001)044 [hep-th/0108093]
[18] Jacobsen J and Saleur H, Conformal boundary loop models, 2006 Preprint math-ph/0611078
[19] Duplantier B and Saleur H, 1987 Phys. Rev. Lett. 58 2325
[20] Saleur H and Bauer M, 1989 Nucl. Phys. B 320 591
[21] Cardy J L, The O(n) model on the annulus, 2006 Preprint math-ph/0604043
[22] Kazakov V and Kostov I, Loop gas model for open strings, 1992 Nucl. Phys. B 386 520
[23] Kostov I K, Ponsot B and Serban D, Boundary Liouville theory and 2D quantum gravity, 2004 Nucl. Phys. B 683 309 [hep-th/0307189]
[24] Kostov I K, Boundary correlators in 2D quantum gravity: Liouville versus discrete approach, 2003 Nucl. Phys. B 658 397 [hep-th/0212194]
[25] Nichols A, Rittenberg V and de Gier J, 2005 J. Stat. Mech. P03003 [cond-mat/0411152]
   Nichols A, 2006 J. Stat. Mech. P01003 [hep-th/0509069]
   Nichols A, 2006 J. Stat. Mech. L02004 [hep-th/0512273]
[26] Pearce P A, Rasmussen J and Zuber J B, Logarithmic minimal models, 2006 J. Stat. Mech. P11017 [hep-th/0607232]
[27] Alekseev A Y and Schomerus V, 1999 Phys. Rev. D 60 061901 [hep-th/9812193]
[28] Bershadsky M and Kutasov D, Scattering of open and closed strings in (1 + 1)-dimensions, 1992 Nucl. Phys. B 382 213 [hep-th/9204049]
[29] Kostov I K, Boundary ground ring in 2D string theory, 2004 Nucl. Phys. B 689 3 [hep-th/0312301]
[30] Kostov I and Zamolodchikov Al, 2007 work in progress