ON THE INTEGRAL OF THE ERROR TERM
IN THE DIRICHLET DIVISOR PROBLEM

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Abstract. Several results are obtained concerning the function \( \Delta_k(x) \), which represents the error term in the general Dirichlet divisor problem. These include the estimates for the integral of this function, as well as for the corresponding mean square integral. The mean square integral of \( \Delta_2(x) \) is investigated in detail.

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1. Introduction

Let as usual \( d_k(n) \) \((k \in \mathbb{N})\) denote the number of ways \( n \) may be written as a product of \( k \) fixed factors. Thus \( d_1(n) \equiv 1, d_2(n) = d(n) = \sum_{d|n} 1 \) is the number of divisors of \( n \), and in general

\[
\sum_{n=1}^{\infty} d_k(n)n^{-s} = \zeta^k(s) \quad (s = \sigma + it, \sigma = \Re s > 1),
\]

where \( \zeta(s) \) is the Riemann zeta-function. The general (Dirichlet) divisor problem (or the Piltz divisor problem, as it is also sometimes called) consists of the estimation of the quantity

\[
\Delta_k(x) := \sum_{n \leq x}^{'} d_k(n) - xP_{k-1}(\log x) - \zeta^k(0),
\]

which represents the error term in the asymptotic formula for the summatory function of \( d_k(n) \), where \( \sum_{n \leq x}^{'}, \) in general means that the last term in the sum is to be halved if \( x \in \mathbb{N} \), and \( \zeta(0) = -\frac{1}{2} \). The main term in the formula is

\[
xP_{k-1}(\log x) = \text{Res}_{s=1} \zeta^k(s)x^ss^{-1},
\]

where \( P_{k-1}(u) \) is a certain polynomial in \( u \) of degree \( k - 1 \) whose coefficients depend on \( k \), and \( \text{Res}_{z=z_0} F(z) \) denotes the residue of \( F(z) \) at the pole \( z = z_0 \). One has, for example,
$P_2(u) = u + 2\gamma - 1$, where $\gamma = 0.577\ldots$ is Euler’s constant. Sometimes the function $\Delta_k(x)$ is not defined by (1.2) but by

$$\Delta_k(x) := \sum_{n \leq x} d_k(n) - x P_{k-1}(\log x),$$

but we shall adhere to (1.2). In general, the coefficients of $P_{k-1}(u)$ may be found by the use of the Laurent expansion of $\zeta(s)$ near $s = 1$, namely

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \gamma_k (s-1)^k,$$

where $(\gamma \equiv \gamma_0)$ for $k \geq 0$

$$\gamma_k = \frac{(-1)^k}{k!} \lim_{N \to \infty} \left( \sum_{n \leq N} \frac{\log n}{n} - \frac{\log^{k+1} N}{k+1} \right)$$

are the so-called Stieltjes constants. The coefficients of $P_{k-1}(u)$ in (1.3) were evaluated explicitly in term of the $\gamma_k$’s by A.F. Lavrik [7].

A large literature exists on estimates for $\Delta_k(x)$, both pointwise and in the mean square sense (see e.g., [3, Chapters 3, 13] and [12, Chapter 12]). Of special interest is the function $\Delta_2(x) \equiv \Delta(x)$, which represents the error term in the classical Dirichlet divisor problem, and which admits an explicit formula, due to F.G. Voronoi (see e.g., [3, Chapter 3]), in term of the Bessel functions. The aim of this paper is to provide estimates for $\int_1^x \Delta_k(u) \, du$, both pointwise and in the mean square sense, for certain “small” values of $k$. This problem appears to be really of interest for $k > 4$. Namely from Voronoi’s formula one has ($c_1$ is a constant)

$$\int_1^x \Delta(u) \, du = \frac{1}{2\sqrt{2\pi} x^{3/4}} \sum_{n=1}^{\infty} d(n)n^{-5/4} \sin(4\pi \sqrt{nx} - \frac{\pi}{4})$$

$$+ \frac{15}{2^6 \sqrt{2\pi}^3 x^{1/2}} \sum_{n=1}^{\infty} d(n)n^{-7/4} \cos(4\pi \sqrt{nx} - \frac{\pi}{4}) + c_1 + O(x^{-1/4}).$$

The series in (1.4) are both absolutely convergent and (see e.g., [4]) and they are both also $\Omega_{\pm}(1)$. Moreover, from (1.4) one easily obtains

$$\int_1^X \left( \int_1^x \Delta(u) \, du \right)^2 \, dx \sim CX^{5/2} \quad (C > 0, \ X \to \infty).$$

This settles the case $k = 2$, and for $k = 3$ note that by the complex integration method (the Perron inversion formula) one has (see [12, Chapter 12]), for $X \leq x \leq 2X$,

$$\Delta_3(x) = \frac{1}{\pi \sqrt{3}} x^{1/3} \sum_{n \leq X^2} d_3(n)n^{-2/3} \cos\left(6\pi \sqrt{n} x^{1/3}\right) + O(x^\epsilon).$$

(1.5)
Here, as usual, $\varepsilon$ denotes arbitrarily small constants which are not necessarily the same ones at each occurrence. From (1.5) we obtain

$$
\int_{X}^{2X} \Delta_3(x) \, dx = \frac{1}{\pi \sqrt{3}} \sum_{n \leq X^2} d_3(n) n^{-2/3} \int_{X}^{2X} x^{1/3} \cos \left( 6\pi (nx)^{1/3} \right) \, dx + O_\varepsilon (X^{1+\varepsilon})
$$

$$
\ll \varepsilon X \sum_{n \leq X^2} d_3(n) n^{-2/3} n^{-1/3} + X^{1+\varepsilon} \ll \varepsilon X^{1+\varepsilon}
$$

(1.6)

by using the first derivative test ([3, Lemma 2.1]). Consequently (1.6) gives

$$
\int_{1}^{x} \Delta_3(u) \, du \ll \varepsilon x^{1+\varepsilon}, \quad \int_{1}^{X} \left( \int_{1}^{x} \Delta_3(u) \, du \right)^2 \, dx \ll \varepsilon X^{3+\varepsilon}.
$$

(1.7)

In general, one can obtain by complex integration methods the expression (1 $\ll N \ll x^C$)

$$
\Delta_k(x) = \frac{x^{k-1}}{\pi \sqrt{k}} \sum_{n \leq N} d_k(n) n^{-k+1} \cos \left( 2k\pi (xn)^{1/k} + \frac{(k-3)\pi}{4} \right)
$$

$$
+ O_{k,\varepsilon} \left( x^\varepsilon \left( 1 + x^{\frac{k-1}{k+1}} N^{-\frac{1}{k+1}} + (xN)^{\frac{2}{k+1} - \frac{1}{2}} \right) \right).
$$

(1.8)

However, already for $k = 4$ this formula does not lead to good results. Namely in evaluating $\int_{X}^{2X} \Delta_4(x) \, dx$ we shall encounter

$$
\frac{1}{2\pi} \sum_{n \leq N} d_4(n) n^{-5/8} \int_{X}^{2X} x^{3/8} \cos \left( 8\pi (xn)^{1/4} + \frac{\pi}{4} \right) \, dx
$$

$$
\ll \sum_{n \leq N} d_4(n) n^{-5/8} \cdot X^{9/8} n^{-1/4} \ll X^{9/8} N^{1/8} \log^3 N,
$$

which coupled with the contribution of the error terms in (1.8), will give a poor final result. For this reason in the next section we shall adopt another approach. We shall use power moment results for $\zeta(s)$ to derive results on the estimation of $\int_{1}^{x} \Delta_k(u) \, du$. In Section 3 mean square results will be discussed, and in Section 4 we shall discuss mean square results for $\Delta_k(x)$, with the accent on the most important case $k = 2$.

2. Estimates for the integral of the error term

We start from the classical Perron inversion formula which gives, for suitable $0 < c < 1$,

$$
\Delta_k(x) = \frac{1}{2\pi i} \int_{(c)} \zeta^k(s) \frac{x^s}{s} \, ds + (-\frac{1}{2})^k,
$$

(2.1)

where as usual

$$
\int_{(c)} F(s) \, ds = \lim_{T \to \infty} \int_{c-iT}^{c+iT} F(s) \, ds.
$$
Integration of (2.1) gives then, for $x > 1$,

$$
\int_1^x \Delta_k(u) \, du = \frac{1}{2\pi i} \int_\gamma \zeta^k(s) \frac{x^{s+1}}{s(s+1)} \, ds + O(x)
$$

$$
\ll x^{c+1} \int_{-\infty}^{\infty} \frac{|\zeta(c+it)|^k}{1+t^2} \, dt + x \ll x^{c+1}
$$

(2.2)

with $c = \eta_k + \varepsilon$, where $\eta_k$ is the infimum of $\eta$ for which one has

$$
\int_0^{2T} |\zeta(\eta + it)|^k \, dt \ll_{\varepsilon} T^{2+\varepsilon}.
$$

(2.3)

We shall obtain

$$
\eta_k \leq \frac{1}{2} - \frac{1}{k} \quad (4 \leq k \leq 8).
$$

(2.4)

The proof of (2.4) will be given now in the most interesting case $k = 8$. By using the functional equation ([3, Chapter 1])

$$
\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \asymp |t|^\frac{1}{2}-\sigma \quad (s = \sigma + it)
$$

we have, for $0 < \eta < 1$,

$$
\int_T^{2T} |\zeta(\eta + it)|^8 \, dt \ll T^{4-8\eta} \int_T^{2T} |\zeta(1-\eta + it)|^8 \, dt.
$$

(2.5)

Now we take $\eta = \frac{3}{8}$ and use the bound ([3, Chapter 8])

$$
\int_T^{2T} |\zeta(\frac{5}{8} + it)|^8 \, dt \ll_{\varepsilon} T^{1+\varepsilon}
$$

(2.6)

to obtain from (2.5)

$$
\int_T^{2T} |\zeta(\frac{3}{8} + it)|^8 \, dt \ll T \int_T^{2T} |\zeta(\frac{5}{8} + it)|^8 \, dt \ll_{\varepsilon} T^{2+\varepsilon},
$$

which gives $\eta_8 \leq \frac{3}{8}$, as asserted. One actually has $\eta_k = \frac{1}{2} - \frac{1}{k}$ for $2 \leq k \leq 8$, which is not difficult to see. Unfortunately the existing results on power moments of $\zeta(s)$ do not permit one to extend the validity of (2.4) to any $k$ satisfying $k > 8$. Nevertheless one can find an upper bound for $\eta_k$ for any given $k > 8$, but a general expression for this upper bound, considered as a function of $k$, would be rather complicated. For this reason we shall content ourselves with explicit bounds for “small” values of $k$, in particular for $k \leq 12$.

From the bounds (see [3, Chapter 8])

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^9 \, dt \ll_{\varepsilon} T^{13/8+\varepsilon}, \quad \int_0^T |\zeta(\frac{35}{51} + it)|^9 \, dt \ll_{\varepsilon} T^{1+\varepsilon}
$$
On the integral of the error term in the Dirichlet divisor problem

and convexity of mean values ([3, Lemma 8.3]) one obtains

\[
\int_0^T |\zeta(\sigma + it)|^9 \, dt \ll \varepsilon \, T^{239/64 + 279\varepsilon + \varepsilon} \quad (\frac{1}{2} \leq \sigma \leq \frac{35}{64}).
\]

This yields, for \( \frac{10}{54} \leq \eta \leq \frac{1}{2} \),

\[
\int_T^{2T} |\zeta(\eta + it)|^9 \, dt \ll T^{9(\frac{1}{2} - \eta)} \int_T^{2T} |\zeta(1 - \eta + it)|^9 \, dt
\]

\[
\ll \varepsilon \, T^{9(\frac{1}{2} - \eta) + \frac{279(\eta - 1) + 239}{64} + \varepsilon} \ll \varepsilon \, T^{2 + \varepsilon}
\]

for \( 129 \leq 306\eta \), giving

\[
\eta_9 \leq \frac{43}{102} = 0.421568627 \ldots .
\]

For the case \( k = 10 \) we use the bound of Zhang [13] (the bound of Ivić–Ouellet [6, p. 250] is slightly weaker, leading to \( \eta_{10} \leq \frac{73}{160} = 0.45625 \))

\[
\int_T^{2T} |\zeta(\sigma + it)|^{10} \, dt \ll \varepsilon \, T^{17 - 20\sigma + \varepsilon} \quad (\frac{9}{20} \leq \sigma \leq \frac{1}{2})
\]

to obtain that \( (17 - 20\sigma)/4 \leq 2 \) for \( \sigma \geq 9/20 \), giving

\[
\eta_{10} \leq \frac{9}{20} = 0.45.
\]

Similarly from the bounds ([3, Chapter 8])

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^{11} \, dt \ll \varepsilon \, T^{\frac{12}{8} + \varepsilon}, \quad \int_0^T |\zeta(\frac{7}{10} + it)|^{11} \, dt \ll \varepsilon \, T^{1 + \varepsilon}
\]

we obtain

\[
\eta_{11} \leq \frac{51}{106} = 0.481132075.
\]

The slightly better bound

\[
\int_0^T |\zeta(\frac{1232}{1771} + it)|^{11} \, dt \ll \varepsilon \, T^{1 + \varepsilon}, \quad \frac{1232}{1771} = 0.6956521 \ldots ,
\]

of [6] would give a further slight improvement of the bound for \( \eta_{11} \). Finally from

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^{12} \, dt \ll \varepsilon \, T^{2 + \varepsilon}
\]

it follows that \( \eta_{12} \leq \frac{1}{2} \).
Now let \( \theta_k \) denote the infimum of \( \theta > 0 \) for which
\[
\int_1^x \Delta_k(u) \, du \ll x^\theta,
\]
and as usual let \( \beta_k \) denote the infimum of \( b \) for which
\[
\int_1^x \Delta^2_k(u) \, du \ll x^{1+2b}.
\]
Then we have
\[
\theta_k \leq 1 + \eta_k \quad (k \geq 4),
\]
and from [3, Lemma 13.1] (with \( \eta_{2k} = \gamma_k \)) we have \( \eta_{2k} = \beta_k \), hence for even \( k \) we can bound \( \theta_k \) in terms of \( \beta_k \) and (2.9). Collecting the above results we obtain

**THEOREM 1.** We have the bounds
\[
\theta_3 \leq 1, \quad \theta_k \leq \frac{3}{2} - \frac{1}{k} \quad (4 \leq k \leq 8), \quad \theta_9 \leq \frac{145}{102}, \quad \theta_{10} \leq \frac{29}{20}, \quad \theta_{11} \leq \frac{157}{106}, \quad \theta_{12} \leq \frac{3}{2},
\]
and in general for \( k \geq 2 \) we have
\[
\theta_{2k} \leq 1 + \beta_k.
\]

It is known that \( \beta_k \geq (k - 1)/(2k) \) for \( k \geq 2 \), and in fact the Lindelöf hypothesis \( (\zeta(\frac{1}{2} + it) \ll \varepsilon \, |t|^\frac{1}{2}) \) is equivalent to \( \beta_k = (k - 1)/(2k) \) for every \( k \geq 2 \). We know at present that \( \beta_k = (k - 1)/(2k) \) holds for \( k = 2, 3, 4 \), while e.g. \( \beta_5 \leq \frac{3}{20} \) (see [13]) and \( \beta_6 \leq \frac{1}{2} \) (see [3]). It is not easy to surmise what is the true value of \( \theta_k \), and in particular to see how sharp is the inequality in (2.9).

3. The mean square of the integral of the error term

The approach to mean square estimates for \( \int_1^x \Delta_k(u) \, du \) is based on the use of Parseval’s formula for Mellin transforms (see E. C. Titchmarsh [11]). The Mellin transform of an integrable function \( f(x) \) is commonly defined as
\[
\mathcal{M}[f(x)] = F(s) = \int_0^{\infty} x^{s-1} f(x) \, dx \quad (s = \sigma + it).
\]

An important feature of Mellin transforms is the so-called inversion formula. It states that if \( F(s) = \mathcal{M}[f(x)], \ y^{\sigma-1} f(y) \in L^1(0, \infty) \) and \( f(y) \) is of bounded variation in a neighbourhood of \( y = x \), then
\[
\frac{f(x+0) + f(x-0)}{2} = \frac{1}{2\pi i} \int_{(\sigma)} F(s) x^{-s} \, ds.
\]
Conversely if (3.1) holds, then $F(s) = \mathcal{M}[f(x)]$. We recall that if $f(x)$ denotes measurable functions, then

$$L^p(a,b) := \left\{ f(x) \left| \int_a^b |f(x)|^p \, dx < +\infty \right. \right\}.$$ 

A form of Parseval’s formula for Mellin transforms is the relation

$$\int_0^\infty f(x)g(x)x^{2\sigma-1} \, dx = \frac{1}{2\pi i} \int_{(\sigma)} F(s)\overline{G(s)} \, ds, \quad (3.2)$$

which holds e.g., if

$$F(s) = \mathcal{M}[f(x)], \ G(s) = \mathcal{M}[g(x)], \ x^{\sigma-\frac{1}{2}}f(x) \in L^2(0, \infty), \ x^{\sigma-\frac{1}{2}}g(x) \in L^2(0, \infty).$$

The starting point for our mean square results is the bound

$$\int_X^{2X} \left( \int_1^x \Delta_k(u) \, du \right)^2 \, dx \ll \int_X^{2X} |f_k(x)|^2 \, dx + X^3, \quad (3.3)$$

which follows from (2.2) and (3.2) with

$$f_k(x) := \frac{1}{2\pi i} \int_{(c)} \frac{\zeta^k(s)x^{s+1}}{s(s+1)} \, ds \quad (c > \eta_k) \quad (3.4)$$

for $x \geq 1$, and $f_k(x) = 0$ for $x < 1$. From (3.1) and (3.4) we have

$$\frac{\zeta^k(s)}{s(s+1)} = \int_0^1 f_k \left( \frac{1}{x} \right) x \cdot x^{s-1} \, dx \quad (\sigma > \eta_k). \quad (3.5)$$

Consequently (3.2) yields

$$\int_0^1 \left| f_k \left( \frac{1}{x} \right) \right|^2 x^2 x^{2\sigma-1} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta(c+it)|^{2k}}{(c^2 + t^2)((c+1)^2 + t^2)} \, dt. \quad (3.6)$$

Since, by $L^2$–theory, the convergence of one integral in (3.6) implies the convergence of the other one, and

$$\int_0^1 \left| f_k \left( \frac{1}{x} \right) \right|^2 x^2 x^{2\sigma-1} \, dx = \int_1^\infty |f_k(x)|^2 x^{-3-2\sigma} \, dx, \quad (3.7)$$

it follows from (3.6) and (3.7) that

$$\int_X^{2X} |f_k(x)|^2 \, dx \ll X^{3+2c_k} \quad (X > 1), \quad (3.8)$$
provided that \( 0 < c_k < 1 \) is such a constant for which

\[
\int_{-\infty}^{\infty} |\zeta(c_k + it)|^{2k} \frac{dt}{1 + t^4} \ll 1.
\]

The last condition reduces to finding \( 0 < \sigma_k < 1 \) such that

\[
\int_{T}^{2T} |\zeta(\sigma_k + it)|^{2k} dt \ll_{\varepsilon} T^{4+\varepsilon},
\]

and then one can take \( c_k = \sigma_k + \varepsilon \) in (3.8). We trivially have \( c_3 = \varepsilon \) (see (1.7)), and also \( c_4 = \varepsilon \) (follows from \( \zeta(i t) \ll t^{1/2} \log t \)).

For \( k > 4 \) we have, by the functional equation for \( \zeta(s) \),

\[
\int_{T}^{2T} |\zeta(\sigma_k + it)|^{2k} dt \ll T^{k-2k\sigma_k} \int_{T}^{2T} |\zeta(1 - \sigma_k + it)|^{2k} dt \ll_{\varepsilon} T^{1+k-2\sigma_k+\varepsilon},
\]

provided that for a given \( k \) one can find \( 0 < \sigma_k < 1 \) for which one has

\[
\int_{0}^{T} |\zeta(1 - \sigma_k + it)|^{2k} dt \ll_{\varepsilon} T^{1+\varepsilon}.
\]

If we can take

\[
\sigma_k = \frac{k - 3}{2k},
\]

then from (3.11) we obtain

\[
\int_{T}^{2T} |\zeta(\sigma_k + it)|^{2k} dt \ll_{\varepsilon} T^{4+\varepsilon},
\]

which shows that \( c_k = (k - 3)/(2k) + \varepsilon \) is permissible. We can infer that (3.11) holds with (3.12) if \( k = 5 \) and \( k = 6 \), since we have the bound ([3, Chapter 8])

\[
\int_{0}^{T} |\zeta(\frac{3}{4} + it)|^{12} dt \ll_{\varepsilon} T^{1+\varepsilon}.
\]

It is very likely that \( c_k = \frac{k-3}{2k} + \varepsilon \) will hold for at least some \( k > 6 \), but this cannot be inferred from the existing results on power moments of \( \zeta(s) \). For \( k > 6 \) one will have to use a weaker bound than (3.11), and consequently we shall have a weaker bound than \( c_k = \frac{k-3}{2k} + \varepsilon \). A general formula for \( c_k \) is possible, but its form would be rather complicated. For this reason we shall content ourselves with the above bounds, which we formulate as

THEOREM 2. Let \( \rho_k \) be the infimum of \( \rho > 0 \) for which

\[
\int_{1}^{X} \left( \int_{1}^{x} \Delta_k(u) du \right)^2 dx \ll X^\rho.
\]
Then we have $\rho_3 \leq 3, \rho_4 \leq \frac{13}{7}, \rho_5 \leq \frac{17}{7}, \rho_6 \leq \frac{7}{2}$.

From the definition of $\rho_k$ it easily follows that

$$\rho_k \leq 3 + 2\beta_k, \quad (3.13)$$

where $\beta_k$ is as in Section 2. Note, however, that the bounds of Theorem 3 are much better than the bounds that one can derive from (3.13) and the sharpest known bounds for $\beta_k$.

4. The mean square formula for $\Delta_k(x)$

Let us define, for $k \geq 2$,

$$K_k(s) = \int_1^\infty \Delta_k^2(x)x^{-s} \, dx. \quad (4.1)$$

By an integration by parts and the use of (3.13) it follows that $K_k(s)$ is a regular function of $s$ for $\sigma = \Re s > 1 + 2\beta_k$. The analytic behaviour of $K_k(s)$ enables one to obtain information on the mean square of $\Delta_k(x)$ via the formula

$$\int_1^X \Delta_k^2(x) \, dx = \frac{1}{2\pi i} \int_{(1+2\beta_k+\varepsilon)} K_k(s) \frac{X^s}{s} \, ds \quad (X > 1). \quad (4.2)$$

Namely by using the classical integral ($c > 0$)

$$\frac{1}{2\pi i} \int_{(c)} \frac{y^s}{s} \, ds = \begin{cases} 1 & (y > 1), \\ \frac{1}{2} & (y = 1), \\ 0 & (0 < y < 1), \end{cases}$$

we have

$$\frac{1}{2\pi i} \int_{(1+2\beta_k+\varepsilon)} K_k(s) \frac{X^s}{s} \, ds = \int_1^\infty \left( \frac{1}{2\pi i} \int_{(1+2\beta_k+\varepsilon)} \left( \frac{X}{x} \right)^s \, ds \right) \Delta_k^2(x) \, dx$$

$$= \int_1^X \Delta_k^2(x) \, dx \quad (X > 1).$$

We can obtain analytic continuation of $K_k(s)$ to the left of the line $\sigma = 1 + 2\beta_k$ in two cases: $k = 2$ and $k = 3$, which follows from mean square results on $\Delta_k(x)$ (see [3, Chapter 13]). In the latter case we use the asymptotic formula

$$\int_1^x \Delta_3^2(y) \, dy = C x^{5/3} + R(x),$$

$$C = \frac{1}{10\pi^2} \sum_{n=1}^\infty d_3(n)n^{-4/3}, \quad R(x) \ll_x x^{14/3+\varepsilon}. \quad (4.3)$$
From (4.3) we obtain
\[K_3(s) = \int_1^\infty \Delta_3^2(x) x^{-s} \, dx = \int_1^\infty \left( \frac{5}{3} C x^{2/3} + R'(x) \right) x^{-s} \, dx \]
\[= \frac{5C}{3s - 5} + C_1 + s \int_1^\infty R(x) x^{-s-1} \, dx. \tag{4.4}\]

The formula (4.4) holds initially for \(\sigma > 5/3\), but the upper bound for \(R(x)\) in (4.3) shows that it provides analytic continuation of \(K_3(s)\) to the half-plane \(\sigma > 14/9\), where \(K_3(s)\) is regular except for a simple pole at \(s = 5/3\).

The case \(k = 2\) is even more interesting. We have
\[\int_1^x \Delta_2^2(y) \, dy = A x^{3/2} + F(x), \tag{4.5}\]
\[A = \frac{1}{6\pi^2} \sum_{n=1}^{\infty} d^2(n) n^{-3/2}, \quad F(x) \ll x \log x, \]
where the upper bound for \(F(x)\) is due to E. Preissmann [9]. Similarly to (4.4) we obtain from (4.5)
\[K_2(s) = \frac{3A}{2s - 3} + C_2 + s \int_1^\infty F(x) x^{-s-1} \, dx. \tag{4.6}\]

In view of the upper bound for \(F(x)\) in (4.5) it follows that (4.6) provides analytic continuation of \(K_2(s)\) to the half-plane \(\sigma > 1\), where \(K_2(s)\) is regular, except for a simple pole at \(s = 3/2\). In general, it seems very likely that \(K_k(s)\) possesses analytic continuation to the left of \(\sigma = 1 + 2\beta_k\), and that it has a simple pole at \(s = 1 + 2\beta_k\). However, the existing mean square results on \(\Delta_k(x)\) are not sharp enough to deduce this assertion. In the case of \(K_2(s)\) one can obtain analytic continuation of \(K_2(s)\) to the half-plane \(\sigma > 2/3\), as well as mean square results for \(\sigma > 1\). The results are contained in

THEOREM 3. The function \(K_2(s)\) possesses analytic continuation to the region \(\sigma > 2/3\), where it is a regular function of \(s\), except at \(s = 3/2\) where it has a simple pole, and at \(s = 1\) where it has a pole of order 3. In the region \(\sigma > 1\), without an \(\varepsilon\)-neighbourhood of \(s = 3/2\), it is of polynomial growth in \(|3m s|\). Moreover,
\[\int_0^T |K_2(s + it)|^2 \, dt \ll \begin{cases} T^{6-4\sigma} (\log T)^{12-8\sigma} & (1 < \sigma < 3/2), \\ 1 & (\sigma > 3/2). \end{cases} \tag{4.7}\]

**Proof.** We use the Laplace transform formula
\[\int_0^\infty \Delta_2^2(x) e^{-x/T} \, dx = \frac{B}{8} \left( \frac{T}{\pi} \right)^{3/2} + (A_1 \log^2 T + A_2 \log T + A_3)T + O_\varepsilon(T^{2/3+\varepsilon}), \tag{4.8}\]
On the integral of the error term in the Dirichlet divisor problem

where

\[ B = \sum_{n=1}^{\infty} d^2(n)n^{-3/2}, \quad A_1 = -\frac{1}{4\pi^2}, \]

which was proved in [5]. Since the integral defining \( \mathcal{K}_2(s) \) is absolutely convergent for \( \sigma > 3/2 \), it follows that for \( c > 3/2 \) and \( T > 0 \) one has

\[
\frac{1}{2\pi i} \int_{(c)} \Gamma(s)T^s \mathcal{K}_2(s) \, ds = \int_1^{\infty} \Delta^2(x) \left( \frac{1}{2\pi i} \int_{(c)} \left( \frac{x}{T} \right)^s \Gamma(s) \, ds \right) \, dx
\]

\[
= \int_1^{\infty} \Delta^2(x) e^{-x/T} \, dx = \int_0^{\infty} \Delta^2(x) e^{-x/T} \, dx + O(1). \tag{4.9}
\]

Hence (4.8) gives, for \( c > 3/2 \) and \( T \geq 1 \),

\[
\frac{1}{2\pi i} \int_{(c)} \Gamma(s)T^s \mathcal{K}_2(s) \, ds = \frac{B}{8} \left( \frac{T}{\pi} \right)^{3/2} + \left( A_1 \log^2 T + A_2 \log T + A_3 \right) T + O_\varepsilon(T^{2/3+\varepsilon}),
\]

or for \( 0 < x \leq 1, c > 3/2 \),

\[
f(x) := \frac{1}{2\pi i} \int_{(c)} \Gamma(s)x^{-s} \mathcal{K}_2(s) \, ds
\]

\[
= \frac{B}{8} \left( \frac{x}{\pi} \right)^{-3/2} + \left( A_1 \log^2 \left( \frac{1}{x} \right) + A_2 \log \left( \frac{1}{x} \right) + A_3 \right) \frac{1}{x} + O_\varepsilon \left( x^{-2/3-\varepsilon} \right). \tag{4.10}
\]

From (4.10) we deduce for \( \sigma > 3/2 \), by the Mellin inversion formula (3.1),

\[
\Gamma(s) \mathcal{K}_2(s) = \int_0^{\infty} f(x)x^{s-1} \, dx = \int_0^1 f(x)x^{s-1} \, dx + \int_1^{\infty} f(x)x^{s-1} \, dx = I_1(s) + I_2(s), \tag{4.11}
\]

say. Note that the definition of \( f(x) \) in (4.10) makes sense if \( x > 0 \), and (4.9) with \( T = 1/x \) yields

\[ f(x) \ll e^{-\frac{1}{2}x} \quad (x \geq 1), \tag{4.12} \]

hence (4.12) shows that \( I_2(s) \) is regular for \( \sigma > 0 \). To investigate \( I_1(s) \) we use (4.10) to deduce that

\[
I_1(s) = \frac{c_1}{s - \frac{3}{2}} + \frac{a_3}{(s-1)^3} + \frac{a_2}{(s-1)^2} + \frac{a_1}{s-1} + \int_0^1 h(x)x^{s-1} \, dx, \tag{4.13}
\]

where \( c_1, a_j \) are effectively computable constants, and

\[
h(x) \ll \varepsilon x^{-\frac{5}{3} - \varepsilon} \quad (0 < x \leq 1). \]

This means that the integral in (4.13) is regular for \( \sigma > 2/3 \), proving the first part of Theorem 3. From (4.6) we obtain

\[ \mathcal{K}_2(s) \ll \varepsilon \left| t \right| \quad (\sigma > 1, \left| s - \frac{3}{2} \right| \geq \varepsilon). \]
It is very likely that the error term in (4.8) can be sharpened to $O_\epsilon(T^{1/2+\epsilon})$. For this reason it also seems likely that $K_2(s)$ admits analytic continuation even to the half-plane $\sigma > \frac{1}{2}$, where it is of polynomial growth. However at present it does not seem possible to prove this assertion.

It remains to prove the mean square bounds of (4.7). From Parseval’s formula (3.2) one has, if $\sigma$ is sufficiently large,

$$\int_1^\infty \Delta^4(x)x^{1-2\sigma} \, dx = \frac{1}{2\pi} \int_{-\infty}^\infty |K_2(\sigma+it)|^2 \, dt. \quad (4.14)$$

Namely (4.14) follows from (4.1) and

$$\int_1^\infty f^2(x)x^{1-2\sigma} \, dx = \frac{1}{2\pi} \int_{-\infty}^\infty |F^*(\sigma+it)|^2 \, dt, \quad (4.15)$$

where

$$F^*(s) := \int_1^\infty f(x)x^{-s} \, dx. \quad (4.16)$$

One obtains (4.15) from Parseval’s formula (3.2) on replacing $f(x)$ and $g(x)$ by $\frac{1}{x}\tilde{f}(x)$, where $\tilde{f}(x) = f\left(\frac{1}{x}\right)$ if $0 < x \leq 1$ and $\tilde{f}(x) = 0$ otherwise.

Note that one has (see D.R. Heath-Brown [2])

$$\int_0^x \Delta^4(y) \, dy \sim Cx^2 \quad (C > 0), \quad (4.17)$$

hence the integral on the left-hand side of (4.14) is convergent for $\sigma > 3/2$. This implies that the integral on the right-hand side of (4.14) is also convergent for $\sigma > 3/2$, giving the second mean square bound in (4.7).

To obtain the first mean square bound in (4.7) write

$$K_2(s) = \int_1^\infty \Delta^2(x)x^{-s} \, dx = \int_1^X \Delta^2(x)x^{-s} \, dx + \int_X^{\infty} \Delta^2(x)x^{-s} \, dx,$$

where $X$ will be suitably chosen a little later. Using (4.5) we have

$$\int_X^{\infty} \Delta^2(x)x^{-s} \, dx = \int_X^{\infty} \left(\frac{3}{2}Ax^{1/2} + F'(x)\right)x^{-s} \, dx = \frac{3AX^{\frac{3}{2}-s}}{2s-3} + O(X^{1-\sigma}\log^4 X) + s \int_X^{\infty} F(x)x^{-s-1} \, dx, \quad (4.18)$$

which provides then the analytic continuation of $K_2(s)$ to $\sigma > 1$. To treat the mean square integral of $K_2(s)$ when $1 < \sigma < \frac{3}{2}$ we use the following method. Let us consider

$$I := \int_T^{2T} \left| \int_a^b g(x)x^{-s} \, dx \right|^2 \, dt \quad (s = \sigma + it, T \geq T_0 > 0, a \geq 1),$$
and set in (4.15) \( f(x) = g(x) \) if \( a \leq x \leq b \) and \( f(x) = 0 \) otherwise. Then \( F^*(s) \) in (4.16) becomes

\[
F^*(s) = \int_a^b g(x)x^{-s} \, dx.
\]

Consequently (4.15) (with \( f \equiv g \)) gives

\[
\frac{1}{2\pi} I \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |F^*(\sigma + it)|^2 \, dt = \int_a^b g^2(x)x^{1-2\sigma} \, dx.
\]

(4.19)

Returning to the mean square of \( K_2(s) \) we have from (4.18) and (4.19), when \( 1 < \sigma < \frac{3}{2} \),

\[
\int_T^{2T} |K_2(\sigma + it)|^2 \, dt \ll \int_T^{2T} \left| \int_1^X \Delta^2(x)x^{-s} \, dx \right|^2 \, dt + \int_T^{2T} \left| \int_X^{\infty} \Delta^2(x)x^{-s} \, dx \right|^2 \, dt
\]

\[
\ll \int_1^X \Delta^4(x)x^{1-2\sigma} \, dx + T^{-1}X^{3-2\sigma} + TX^{2-2\sigma}\log X + T^2 \int_X^{\infty} F^2(x)x^{-1-2\sigma} \, dx
\]

\[
\ll X^{3-2\sigma} + TX^{2-2\sigma}\log X + T^2X^{2-2\sigma}\log^4 X \ll T^{6-4\sigma}(\log T)^{12-8\sigma}
\]

with the choice \( X = T^2\log^4 T \). Here we used (4.17) and the bound of K.-M. Tsang [10] (with \( r = 2 \))

\[
\int_2^X |F(x) + (4\pi^2)^{-1}x\log^2 x - \kappa x\log x|^r \, dx \ll (cr)^{4r}X^{r+1},
\]

(4.20)

which is valid uniformly for \( X > 2, r \in \mathbb{N} \) and suitable constants \( \kappa \) and \( c \). This completes the proof of Theorem 3.

In concluding, it seems in order to discuss the shape of the mean square formula (4.5). In [8] and [10] some remarkable results on \( F(x) \) were proved by Lau and Tsang, which include the bound (4.20). From (4.20) Tsang deduces that, for almost all \( x \),

\[
F(x) = -\frac{1}{4\pi^2}x\log^2 x + \kappa x\log x + O(x),
\]

(4.21)

and conjectures that (4.21) holds for all \( x \geq 2 \). In view of Theorem 3 and (4.2) it seems that perhaps a formula sharper than (4.21) holds for all \( x \), namely

\[
F(x) = -\frac{1}{4\pi^2}x\log^2 x + \kappa x\log x + \lambda x + G(x), \quad G(x) = O(x^\alpha),
\]

(4.22)

for suitable constants \( \kappa, \lambda \) and \( 0 < \alpha < 1 \). Namely heuristically we use (4.2) with \( k = 2 \) and shift the line of integration to the left, passing over the poles at \( s = 3/2 \) (which yields the main term \( Ax^{3/2} \) in (4.5) and at \( s = 1 \) (which yields the main term in (4.22)). I conjecture that (4.22) holds with any \( \alpha \) satisfying \( \frac{3}{4} < \alpha < 1 \). The reason for the bound
\( \alpha > \frac{3}{4} \) is that (4.22) with \( \alpha < \frac{3}{4} \) is not possible, which will be shown now. We start from ([6, Lemma 2])

\[
\Delta(x) = H^{-1} \int_{x}^{x+H} \Delta(y) \, dy + O(H \log x) \quad (x^\varepsilon \leq H \leq x).
\] (4.23)

By the Cauchy-Schwarz inequality for integrals and (4.5), (4.23) implies

\[
\Delta^2(x) \ll H^{-1} \int_{x}^{x+H} \Delta^2(y) \, dy + H^2 \log^2 x
\]
\[
\ll x^{1/2} + H^{-1}(F(x+H) - F(x-H)) + H^2 \log^2 x.
\] (4.24)

Now suppose that (4.22) holds with some \( \alpha < \frac{3}{4} \). Since we have (see e.g., J.L. Hafner [1])

\[
\limsup_{x \to \infty} |\Delta(x)x^{-1/4}| = \infty,
\]

this means that, given any constant \( C > 0 \), there exist arbitrarily large values \( X_1 \) such that \( \Delta^2(X_1) > CX_1^{1/2} \). Therefore (4.24) yields

\[
X_1^{1/2} \ll H^{-1} \left( F(X_1 + H) - F(X_1 - H) \right) + H^2 \log^2 X_1
\]
\[
\ll H^{-1} X_1^\alpha + H^2 \log^2 X_1 \ll X_1^{2\alpha/3} \log^2 X_1
\] (4.25)

if \( H = X_1^{\alpha/3} \). But since \( \alpha < \frac{3}{4} \), (4.25) gives then a contradiction, proving the assertion.

On the other hand, the above proof shows that if (4.22) holds with \( \alpha = \frac{3}{4} + \varepsilon \), then (4.24) yields the conjectural bound \( \Delta(x) \ll \varepsilon x^{1/4+\varepsilon} \), which is very strong.

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