BERIKASHVILI’S FUNCTOR $D$ AND
THE DEFORMATION EQUATION

JOHANNES HUEBSCHMANN

Dedicated to Nodar Berikashvili on the occasion of his seventieth birthday

Abstract. Berikashvili’s functor $D$ defined in terms of twisting cochains is related to deformation theory, gauge theory, Chen’s formal power series connections, and the master equation in physics. The idea is advertised that some unification and understanding of the links between these topics is provided by the notion of twisting cochain and the idea of classifying twisting cochains.

Introduction

In [5], N. Berikashvili introduced the functor $D$ in terms of “twisting elements” or “twisting cochains” in a differential graded algebra. At that time twisting cochains already had a history in topology and differential homological algebra, see e. g. [41] (where the terminology “twisting morphism” is used) and the historical references to the work of E. H. Brown and to the Séminaire Cartan in that paper. With hindsight we see that twisting cochains make precise a certain piece of structure discovered by H. Cartan in [10] and [11] which is behind the notion of transgression. Berikashvili and his students subsequently studied the functor $D$ further, cf. e. g. [6, 7, 36, 44, 45]. In this note we show the connections of this functor with various other developments in the literature, in particular with some more recent ones related with deformation theory and the master equation. Our survey is far from complete at this point, though, and we apologize for all the relevant citations which have been omitted. The formal coincidences between the developments we will present here are startling and a general theory unifying and explaining them satisfactorily as incarnations of the same idea is yet to be found. Our “Leitmotiv” is the observation that some such unification and simplification is provided by the notion of twisting cochain and by the ideas underlying deformation theory. Indeed we will explain below how Berikashvili’s functor $D$ may be viewed as a version of the deformation functor.

I am indebted to J. Stasheff for a number of comments. It is a pleasure to dedicate this paper to Nodar Berikashvili for his 70’th birthday.
1. Berikashvili’s functor $D$

The ground ring will be written $R$; it is a commutative ring with one. Berikashvili’s functor $D$ may be described briefly in the following way: Let $A$ be a differential graded algebra. A homogeneous element $\tau$ of $A$ (necessarily of degree $-1$) is said to be a twisting element or twisting cochain provided it satisfies the equation

\[(1.1)\quad D\tau = \tau\tau,\]

where $D$ refers to the differential in $A$, and where $\tau\tau$ is the product in $A$ of $\tau$ with itself. Denote the set of twisting elements by $T(A)$. Next let $G$ be the group of invertible elements of $A_0$, and define an action of $G$ on $T(A)$, to be written $(x, y) \mapsto x \ast y$, $x \in G$, $y \in T(A)$, by means of the formula

\[(1.2)\quad x \ast y = xyx^{-1} + (Dx)x^{-1}.\]

This is well defined, that is, given $x \in G$ and $y \in T(A)$, $x \ast y$ satisfies the equation (1.1) as well. This is readily seen by a straightforward calculation relying on the formulas

\[(Dx)x^{-1} + xDx^{-1} = 0, \quad (Dx^{-1})x + x^{-1}Dx = 0\]

which, in turn, follow from $xx^{-1} = 1$. The set of orbits $(T(A))/G$ is written $D(A)$ and the assignment to $A$ of $D(A)$ is a functor from the category of differential graded algebras to the category of sets. For later reference, we will say that two twisting elements $x_1, x_2$ are equivalent provided they lie in the same $G$-orbit.

Berikashvili applied this functor $D$ in various situations; here is one example [7]: Consider a (Serre) fibration $F \to E \to B$. Let $RH_*F$ be a free (additive) resolution (over the ground ring) of the homology $H_*F$ of the fiber and consider the differential (trigraded) algebra

\[A = C^*(B, \text{End}(RH_*F))\]

with the ordinary cup product as algebra structure. Given $a \in A$, endow $Y = C_*B \otimes (RH_*F)$ with the twisted differential determined by $a$ and write $Y_a$ for the resulting chain complex. Berikashvili proved that the fibration determines an element $a \in A$ together with a chain equivalence $m$ from $Y_a$ to $C_*E$ satisfying the requisite compatibility properties in such a way that $Y_a$ is a model for the Serre fibration; he showed that, furthermore, given an arbitrary element $\overline{a}$ in the class $[a] \in D(A)$ of $a \in T(A)$, there is an isomorphism from $Y_{\overline{a}}$ to $Y_a$ whence there is again a chain equivalence $\overline{m}$ from $Y_{\overline{a}}$ to $C_*E$ such that $Y_{\overline{a}}$ is a model for the Serre fibration. The class of $a$ in $T(A)$ is called the predifferential of the fibration; it may be shown that, in a sense, the predifferential does not depend on the choice of resolution of the fibre. The standard filtration of the model $Y_a$ induces a spectral sequence which is the ordinary Serre spectral sequence of the fibration. There is also a cohomology version of the theory. These models have been used, for example, to study the section problem for fibrations. When the homology $H_*F$ of the fiber is free over the ground ring, the model $Y_a$ comes down to models of the kind introduced by Hirsch [27].

Prompted by Halperin-Stasheff [26], Saneblidze [45] created a multiplicative version of the functor $D$ over the rationals. To this end, he reworked the functor $D$ with a multiplicative resolution of the cohomology of the fibre. He established in particular
a multiplicative version of Berikashvili’s result, but now for the cochains of a fibre space. Still in [45], Saneblidze extended this approach so as to incorporate Hopf algebra structures, and he applied this to study the loop space.

Such a multiplicative resolution of the cohomology was already used by Schlessinger-Stasheff [46] for a similar purpose, that is, to study deformations of rational homotopy types of spaces and fibrations; we will explain this briefly further below.

Homological perturbation theory enabled the present author to construct small free resolutions by means of appropriate twisting cochains which, in turn, led to complete numerical calculations in group cohomology [29], [30], [31] which so far cannot be done by other methods.

2. The deformation equation

Henceforth we suppose that the ground ring \( R \) contains the rationals as a subring. The equation (1.1) occurs, perhaps up to a sign, in the literature as deformation equation or master equation. More customary is the corresponding Lie algebra deformation equation or Lie algebra master equation: As an equation on elements of a differential graded Lie algebra \( \mathcal{L} \), for an element \( \tau \) of \( \mathcal{L} \), the deformation equation has the form

\[
D\tau = \frac{1}{2}[\tau, \tau]
\]

(up to sign). In particular, given a cocommutative coaugmented differential graded coalgebra \( C \) and a differential graded Lie algebra \( \mathfrak{g} \), \( \mathcal{L} = \text{Hom}(C, \mathfrak{g}) \) inherits a differential graded Lie algebra structure in the standard way, and an element \( \tau \) thereof satisfying (2.1), that is to say, a morphism \( \tau: C \to \mathfrak{g} \) satisfying (2.1), is called a Lie algebra twisting cochain, cf. e. g. [41], [43]. When \( \mathcal{A} \) is the universal (enveloping) differential graded algebra of a general differential graded Lie algebra, for an element \( \tau \) of \( \mathcal{L} \), the equations (1.1) and (2.1) are manifestly equivalent. Inspection shows that, given a differential graded algebra \( \mathcal{A} \) and a solution \( \tau \) of the equation (1.1), the operator \( d_\tau \) on \( \mathcal{A} \) defined by

\[
d_\tau(a) = da - \tau a, \quad a \in \mathcal{A},
\]

yields a new differential graded algebra structure on \( \mathcal{A} \). Likewise, given a differential graded Lie algebra \( \mathcal{L} \) and a solution \( \tau \) of the equation (2.1), the operator \( d_\tau \) on \( \mathcal{L} \) defined by

\[
d_\tau(a) = da - [\tau, a], \quad a \in \mathcal{L},
\]

yields a new differential graded Lie algebra structure on \( \mathcal{L} \).
3. Gauge theory

In view of the well known spectacular results obtained by means of gauge theory since the 80’s there is no need to comment on the significance of gauge theory here. We only reproduce briefly some of the relevant notions and language in order to explain how gauge theory relates to Berikashvili’s functor.

From now on the ground ring $R$ is that of the reals or that of the complex numbers. Consider the space $\mathcal{A}(ξ)$ of connections on a principal bundle $ξ:P → M$ with structure group $G$ having Lie algebra $\mathfrak{g}$. Denote by $\text{ad}(ξ)$ the adjoint bundle; this is the bundle over $M$ with fiber $\mathfrak{g}$ which is associated to $ξ$ via the adjoint representation of $G$ on $\mathfrak{g}$. The Lie bracket on $\mathfrak{g}$ induces a graded Lie algebra structure on the $\text{ad}(ξ)$-valued de Rham forms $Ω^*(M, \text{ad}(ξ))$. Given a connection $A$ and an $\text{ad}(ξ)$-valued 1-form $η$, the sum $A + η$ is again a connection, and its curvature $F_{A+η}$ is given by the well known formula

$$F_{A+η} = F_A + d_Aη + \frac{1}{2}[η, η]$$

where $d_A$ denotes the operator of covariant derivative of the connection $A$. In particular, $A + η$ has the same curvature as $A$ if and only if $η$ satisfies the equation

$$d_Aη + \frac{1}{2}[η, η] = 0,$$

often referred to as the Maurer-Cartan equation; notice that this equation for $η$ is equivalent to one of the kind (2.1) for $τ = −η$. In particular, when $A$ is flat, $(Ω^*(M, \text{ad}(ξ)), d_A)$ is a differential graded Lie algebra, and $A + η$ is flat if and only if $η$ satisfies the equation (3.1). The operator $d_τ$ on $\mathcal{L} = (Ω^*(M, \text{ad}(ξ)))$ given in (2.3), where $τ = −η$, then coincides with the operator $d_{A+η}$ of covariant derivative for the flat connection $A + η$. Moreover, the definition of the operation (1.2) very much looks like the operation of the group of gauge transformations on the space of connections. Thus, for a general differential graded algebra $A$, the value $D(A)$ of Berikashvili’s functor $D$ formally looks like a space of gauge equivalence classes of flat connections.

4. Chen’s formal power series connections

A special case of twisting element or solution of the deformation or master equation was introduced by Chen [13]. Chen’s formalism involves formal power series.

We remind the reader that the ground ring is that of the reals or that of the complex numbers. In his paper [13], Chen considers the (bi)graded algebra $Ω^*(M, T[[V]])$ of smooth forms on a smooth manifold $M$ with values in the graded algebra of non-commutative formal power series $T[[V]]$ on a graded vector space $V$. In Chen’s terminology [13] (1.2), a formal power series connection is a formal power series in $Ω^*(M, T[[V]])$ of (total) degree $−1$ (the degree being appropriately interpreted). The curvature of a formal power series connection $ω$ is, then, by definition, the element $κ = dω + ω ∧ ω ∈ Ω^*(M, T[[V]])$ of degree $−2$ where $d$ refers to the ordinary de Rham differential and $∧$ to the cup pairing induced by the multiplication on $T[[V]]$; notice that

$$ω ∧ ω = \frac{1}{2}[ω, ω] ∈ Ω^*(M, T[[V]])$$
where \([ , ]\) denotes the bracket in \(\Omega^*(M, T[[V]])\) induced from the commutator pairing \(T[[V]] \otimes T[[V]] \to T[[V]]\) in the ordinary way. In Theorem 1.3.1 of [13], Chen proves the following: When \(V\) is taken to be the desuspension \(s^{-1}\tilde{H}_*(M)\) of the reduced (real) homology \(\tilde{H}_*(M)\) of \(M\), a splitting of the ordinary de Rham complex \(\Omega^*(M)\) into a direct sum of \(H^*(M)\), the coboundaries, and a residual summand \(W\) (written \(M\) in Chen’s paper), determines a formal power series connection \(\omega \in \Omega^*(M, T[[V]])\) together with a differential \(\partial\) on \(T[[V]]\) which turns the latter into a differential graded algebra in such a way that

\[
\kappa + \omega \partial = 0.
\]

This formal power series connection \(\omega\) has a certain uniqueness property. To reproduce it, suppose \(\omega\) written in the form \(\omega = \omega_1 + \omega_2 + \ldots\) where, for \(j \geq 1\), \(\omega_j\) is the component whose values lie in \(V^{\otimes j}\). Now \(\omega\) is uniquely determined by the requirement that the constituent \(\omega_1\) coincide essentially with the projection of \(\Omega^*(M)\) onto \(H^*(M)\) (determined by the splitting) and that, for \(j \geq 2\), the \(\omega_j\)’s be non-zero at most on the residual summand \(W\) of the (chosen) decomposition of \(\Omega^*(M)\). To explain the significance of the identity (4.1), we endow \(\Omega^*(M, T[[V]])\) with the differential \(D = d \pm \partial\) (appropriate signs being taken) where \(d\) refers to the de Rham differential and \(\partial\) to the operator on \(\Omega^*(M, T[[V]])\) induced by that on \(T[[V]]\) denoted by the same symbol. With this preparation out of the way, the identity (4.1) says that the formal power series connection \(\omega\) (or rather its negative, in view of our convention) is a solution of the deformation or master equation (1.1), where \(A = (\Omega^*(M, T[[V]]), D)\). Chen goes on to observe that, in fact, when \(M\) is simply connected, in view of classical algebraic topology results related with the cobar construction, \((T[[V]], \partial)\) is a kind of cobar construction and hence a model for the real chains of the loop space of \(M\).

Chen’s construction was reexamined, generalized, and simplified in the framework of homological perturbation theory by Gugenheim in [18]; Gugenheim’s starting point was the observation that Chen’s splitting of the ordinary de Rham complex quoted above in fact just amounts to a contraction of chain complexes in the sense of homological perturbation theory; cf. e. g. [35].

Suppose that \(V\) is of finite type over the reals (or complex numbers, according to the case considered). Appropriately dualized, Chen’s notion of formal power series connection then fits into the ordinary twisting cochains framework, cf. [20]. Indeed, by means of the tensor coalgebra \(T^c[V^*]\) on the dual \(V^*\) of \(V\), a formal power series connection may then be written as a morphism

\[
\omega: T^c[V^*] \to \Omega^*(M)
\]

of degree \(-1\) with values in the de Rham algebra \(\Omega^*(M)\). In particular, when \(V\) is taken to be the desuspension \(s^{-1}\tilde{H}_*(M)\) of the reduced homology \(\tilde{H}_*(M)\) so that \(V^*\) is the suspension \(s\tilde{H}^*(M)\) of the reduced cohomology, the tensor coalgebra \(T^c[V^*]\) carries an operator \(\delta\) compatible with the graded coalgebra structure (whose dual is the above operator \(\partial\) on \(T[[V]]\)) so that

\[
T^c_{\delta}[V^*] = (T^c[V^*], \delta)
\]
is a model for the de Rham algebra $\Omega^* M$, that is, a model for the real cochains on the loop space of $M$ when $M$ is simply connected; Chen's formal power series connection satisfying (4.1) is now precisely a twisting cochain

$$\omega: T^*_0[V^*] \to \Omega^*(M).$$

The operator $\delta$ and formal power series connection $\omega$ have been constructed in [20] by means of homological perturbation theory.

### 5. Deformation theory

In [24], Hain and Tondeur interpret Chen's notion of formal power series connection as the dual of the versal deformation of a trivial connection. There is indeed an intimate relationship between Berikashvili's functor, Chen's formal power series connections, and deformation theory, in particular the deformation theory of rational homotopy types developed by Schlessinger-Stasheff [46]. We now explain briefly some of the relevant deformation theory notions. See e.g. [16] for a recent more detailed account of deformation theory.

According to a point of view adopted by Deligne and others, any problem in deformation theory is controlled by a differential graded Lie algebra, unique up to homotopy equivalence of differential graded Lie algebras, in fact, up to $L_\infty$-equivalence.

Let $\mathcal{L}$ be a differential graded Lie algebra, the bracket being written $[\ , \ ]: \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$. Recall that the assignment

$$\mathcal{L}_0 \times \mathcal{L}_{-1} \to \mathcal{L}_{-1}, \quad (\zeta, \alpha) \mapsto [\zeta, \alpha] - d\zeta$$

yields an action of $\mathcal{L}_0$ on $\mathcal{L}_{-1}$, the latter being viewed as a vector space, by infinitesimal affine transformations. When $\mathcal{L}_0$ is nilpotent (or more generally, pro-nilpotent), the Campbell-Baker-Hausdorff formula turns $\mathcal{L}_0$ into a (pro-)nilpotent (sometimes Lie) group $\Gamma$ having Lie algebra $\mathcal{L}_0$, and the $\mathcal{L}_0$-action on $\mathcal{L}_{-1}$ integrates to an action of $\Gamma$ on $\mathcal{L}_{-1}$ by affine transformations.

Let $\mathfrak{g}$ be a differential graded Lie algebra, the bracket being written $[\ , \ ]: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. The usual definition of the functor $\text{Def}_\mathfrak{g}$ assigns to a local Artinian algebra $\mathcal{A}$ with maximal ideal $\mathfrak{m}$ the set

$$\text{Def}_\mathfrak{g}(\mathcal{A}) = \{ \gamma \in \mathfrak{g}_{-1} \otimes \mathfrak{m}; d\gamma + \frac{1}{2}[\gamma, \gamma] = 0 \}/\Gamma_\mathcal{A};$$

here $\mathcal{L} = \mathfrak{g} \otimes \mathfrak{m}$ is the induced differential graded $\mathcal{A}$-Lie algebra, so that the nilpotent Lie algebra $\mathcal{L}_0 = \mathfrak{g}_0 \otimes \mathfrak{m}$ acts on $\mathcal{L}_{-1} = \mathfrak{g}_{-1} \otimes \mathfrak{m}$ infinitesimally, and $\Gamma_\mathcal{A}$ is the corresponding nilpotent group which accordingly acts on $\mathcal{L}_{-1}$ by affine transformations. The points of $\text{Def}_\mathfrak{g}(\mathcal{A})$ are the isomorphism classes of the corresponding Deligne groupoid. Two elements $\gamma_1, \gamma_2 \in \mathfrak{g}_{-1} \otimes \mathfrak{m}$ are said to be equivalent provided they lie in the same $\Gamma_\mathcal{A}$-orbit.

There is a striking similarity between the functor $\text{Def}_\mathfrak{g}$ and Berikashvili's functor $\mathcal{D}$. This similarity is of course not a coincidence: Let $C = S^c[\xi]$ be the symmetric coalgebra with a single cogenerator $\xi$ of degree 0, that is, the symmetric coalgebra on a free $R$-module with a single basis element $\xi$ of degree 0. As an $R$-module,
C has free generators \( \xi_0 = 1, \xi_1 = \xi, \xi_2, \xi_3, \ldots \), with diagonal map \( \Delta \), counit \( \varepsilon \), and coaugmentation map \( \eta \) being given by

\[
\Delta \xi_n = \sum_{i+j=n} \xi_i \otimes \xi_j, \quad \varepsilon(\xi_0) = 1, \varepsilon(\xi_1) = \varepsilon(\xi_2) = \cdots = 0, \quad \eta(1) = \xi_0 = 1.
\]

Thus, when \( t: C \to R \) denotes the \( R \)-linear map which sends \( \xi \) to 1 and is zero on the other generators of \( C \), the ring \( \text{Hom}(C, R) \) coincides with the complete local ring \( R[[t]] \) of formal power series in the variable \( t \), and the ideal consisting of the formal power series without constant term is the unique maximal ideal of this ring. A twisting cochain \( \tau: C \to g \) is then precisely a formal deformation in the ordinary sense, cf. e. g. [16] for the notion of formal deformation. For example when \( g \) arises from the Hochschild complex of an algebra \( B \) in the usual way, the bracket being the Gerstenhaber bracket [15, 16], a twisting cochain \( \tau: C \to g \) is a formal deformation of \( B \). Furthermore, for a general differential graded Lie algebra \( g \), \( \text{Hom}(C, g) \) inherits a differential graded Lie algebra structure as we have already pointed out, and the coaugmentation map \( \eta \) induces surjective morphism \( \eta^* \) of differential graded Lie algebras from \( \text{Hom}(C, g) \) onto \( g \); thus, writing \( \mathcal{L}^C \) for the kernel of \( \eta^* \), we obtain an extension

\[
0 \to \mathcal{L}^C \to \text{Hom}(C, g) \xrightarrow{\eta^*} g \to 0
\]

of differential graded Lie algebras. With respect to the filtration induced from the coaugmentation filtration of \( C \), the differential graded Lie algebra \( \mathcal{L}^C \) is pronilpotent, i. e. an inverse limit of nilpotent Lie algebras. Thus the infinitesimal action of \( \mathcal{L}_0^C \) on \( \mathcal{L}_{-1}^C \) integrates to an affine action of the corresponding pronilpotent group \( \Gamma^C \) on \( \mathcal{L}_{-1}^C \). The twisting cochains constitute a subset of \( \mathcal{L}_{-1}^C \), the \( \Gamma^C \)-action carries twisting cochains to twisting cochains, and the value \( \text{Def}_g(A) \) of the algebra \( A = R[[t]] = \text{Hom}(C, R) \) under the functor \( \text{Def}_g \) is the orbit space of twisting cochains under the \( \Gamma^C \)-action; thus \( \text{Def}_g(A) \) classifies \( g \)-valued twisting cochains defined on \( C \). (Technically, one would have to replace the tensor product occurring in (5.2) by a completed tensor product when \( g_{-1} \) is not finitely generated.) On the other hand, we could take the differential graded algebra \( A = \text{Hom}(C, Ug) \), where \( Ug \) refers to the universal differential graded algebra of \( g \), and apply Berikashvili’s functor \( D \) to it; the resulting object \( D(A) \) again classifies twisting cochains defined on \( C \), but now with values in all of \( Ug \), and the two notions of equivalence are somewhat different since the action of the group \( \Gamma^C \) and that of the group of units \( G \) in \( A^0 \) are not in an obvious way related.

Berikashvili’s classification procedure, that is, that involving the group of units \( G \) in \( A^0 \), provides in fact a classification in terms of homotopy classes of twisting cochains with reference to the corresponding notion of homotopy of twisting cochains: More precisely, the equation (1.2) says that \( x \) provides a homotopy of twisting cochains between \( y \) and \( x \ast y \); see e. g. [42] for this notion of homotopy of twisting cochains.

The standard formal deformation theory classification procedure involves the group corresponding to the degree zero Lie algebra constituent and the action thereof on the Lie algebra constituent in degree \(-1\), though. Under appropriate circumstances, an interpretation of the latter classification in terms of a suitable notion of homotopy of twisting cochains (more intricate than the naive notion mentioned above) may be
found in [46]. This raises the question whether we can isolate the precise relationship between the two classification procedures. The whole discussion is of course valid for any cocomplete cocommutative differential graded coalgebra instead of the present $C$. Thus we see that Berikashvili’s functor may be viewed as a version of the deformation functor.

We now explain briefly some of the links between deformation theory and homological perturbation theory. Consider a smooth complex manifold $M$. The differential graded Lie algebra $L$ of $\partial\sigma$-forms on $M$ with values in the holomorphic tangent bundle $\tau_M$, the differential being the operator $\partial\sigma$, controls the deformations of the complex structure. This differential graded Lie algebra is called the Kodaira-Spencer algebra of $M$; its cohomology coincides with the sheaf cohomology $H^\ast(M, \tau_M)$ of $M$ with values in the holomorphic tangent bundle $\tau_M$ and plainly inherits a graded Lie algebra structure. A 1-cochain $\eta$ in $L$ determines a new complex structure, that is, a new $\partial\sigma$-operator, if and only if $\eta$ satisfies the deformation equation

$$\partial\eta + \frac{1}{2}[\eta, \eta] = 0$$

which, up to sign, is just the equation (2.1). Recall that a complex analytic family of complex manifolds over $M$ parametrized by a based open domain $(U, o)$ of $\mathbb{C}^n$ may be described as a map $\rho: U \to L_{-1}$ whose values lie in $\mathcal{F} = \{\eta; \partial\eta + \frac{1}{2}[\eta, \eta] = 0\}$ and which has the requisite properties so that twisting the complex analytic product $U \times M$ via $\rho$ yields a new complex analytic structure on $U \times M$ (the latter here being viewed as the product of two smooth manifolds only); in particular, the appropriate adjoint of $\rho$ is a smooth section of the smooth vector bundle on $U \times M$ arising from pulling back the smooth vector bundle underlying $L_{-1}$ via the projection $U \times M \to M$, and $\rho(o) = 0$. See e. g. [38, 39] for details. In coordinates $z_1, \ldots, z_n$ on $U$, $\rho$ may be written as a power series in these variables with coefficients from $L_{-1}$. This power series being viewed as a formal power series, the requirement that the values of $\rho$ lie in $\mathcal{F}$ is equivalent to the negative of $\rho$ being a twisting cochain $S_c[V] \to L$, where $V = T_oU \cong \mathbb{C}^n$ and where $S_c[V]$ refers to the symmetric coalgebra on $V$. Standard techniques reduce the study of such deformations to cohomology classes in $H^1(M, \tau_M)$, and such a class $y$ is unobstructed if and only if $[y, y]$ is zero where $[\ , \ ]$ refers to the induced graded Lie bracket on $H^\ast(M, \tau_M)$. In particular, according to Tian and Todorov [48,49], for a Calabi-Yau manifold $M$, the graded Lie bracket on $H^\ast(M, \tau_M)$ is zero, and this implies the fact, first observed by Bogomolov [8, 9], that the deformations of a Calabi-Yau manifold are parametrized by an open subset of $H^1(M, \tau_M)$.

The $\partial\sigma$-forms with values in the non-zero exterior powers of the holomorphic tangent bundle, endowed with the Frölicher-Nijenhuis bracket, constitute a differential Gerstenhaber algebra (see e. g. [33] for details). The Tian-Todorov Lemma says that, when $M$ is a Calabi-Yau manifold, this differential Gerstenhaber algebra has a differential Batalin-Vilkovisky algebra structure, the requisite generator being induced from a choice of holomorphic volume form. In [1], Barannikov and Kontsevich constructed a formal solution of the master equation for such a differential Batalin-Vilkovisky algebra. By means of this formal solution, they endowed the extended moduli space of complex structures with a formal Frobenius manifold structure. This development, in turn, was prompted by the mirror conjecture, cf. [50]. See
e. g. [40] for more details on Frobenius manifolds. A purely formal approach which explains the requisite differential Batalin-Vilkovisky algebra structures in terms of duality for Lie-Rinehart algebras may be found in [33] which also contains more references; some comments about the origin of Batalin-Vilkovisky algebras and their relevance in physics will be given in the next section. In [35], formal solutions of the master equation have been obtained much more generally by the methods of homological perturbation theory. For related constructions of twisting cochains see [22], [28] (2.11), [42] (2.2). The approach in [35] covers also the requisite twisting cochains needed for the deformation theory of rational homotopy types and rational fibrations and for the corresponding classification theory developed by Schlessinger and Stasheff in [46]. In Section 7 below we will briefly indicate how this relates to the functor $D$. For details and history on homological perturbation theory see e. g. [34].

6. The master equation in physics
In a series of seminal papers [2], [3], [4], Batalin and Vilkovisky studied the quantization of constrained systems and for that purpose introduced certain differential graded algebras which have later been named *Batalin-Vilkovisky algebras*. Batalin-Vilkovisky algebras have recently become important in string theory and elsewhere, cf. e. g. [1], [32], [35], [40], [47]. String theory leads to the mysterious mirror conjecture, see e. g. [50]. In this context, the quantum Batalin-Vilkovisky master equation has the form of the Maurer-Cartan equation for a flat connection while the classical version has the form of the integrability condition of deformation theory. See [47] for more details and references.

7. Concluding remarks
We have seen that Berikashvili’s functor may be viewed as a version of the deformation functor, and that (formal) deformations may be viewed as twisting cochains defined on coaugmented cocommutative differential graded coalgebras. Chen’s formal power series connections are also subsumed under the notion of twisting cochain, and the curvature of a Chen formal power series connection being zero amounts to the corresponding master equation being satisfied. Thus, when twisting cochains defined on not necessarily cocommutative coalgebras are viewed as non-commutative generalizations of deformations, we can perhaps give a meaning to the point of view of Hain and Tondeur that Chen’s construction of formal power series connection should be viewed as the versal deformation of something (or the dual thereof).

In the same rite, the deformation theory for rational homotopy types developed by Schlessinger and Stasheff [46] involves as an ingredient the (filtered multiplicative) Halperin-Stasheff model [26] for a differential graded commutative algebra and in particular the master equation. Saneblidze’s multiplicative version of the multiplicative functor $D$ [45] also relies on the Halperin-Stasheff model and the requisite master equation. According to Saneblidze’s Theorem 4.8 in [45], under the circumstances of that theorem, for appropriate spaces $X$, the object $D(X)$ introduced there (defined in terms of the corresponding Sullivan-de Rham complex) classifies fiber homotopy types over $X$ where the homotopy type of the fiber is fixed in advance, while Schlessinger-Stasheff obtain such a classification in Theorem 9.6 of [46] in terms of their moduli space arising from their differential graded Lie algebra. Thus the two
classification theories are plainly related, but can we make the relationship precise and explicit? Another intriguing observation, which may be found just before Theorem 5.5 in [45], is that there the requisite “Hopf predifferential” may be obtained as a Chen formal power series connection, in view of what is said in [23]. Thus, what is the precise relationship between the multiplicative functor $D$ and Chen’s formal power series connections?

Perhaps some of these questions related with Berikashvili’s functor will be answered in the future.

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