Nonexistence of proper $p$-biharmonic maps and Liouville type theorems I: case of $p \geq 2$

Yingbo Han, Yong Luo

Abstract

Let $u : (M, g) \rightarrow (N, h)$ be a map between Riemannian manifolds $(M, g)$ and $(N, h)$. The $p$-bienergy of $u$ is defined by $E_p(u) = \int_M |\tau(u)|^p d\nu_g$, where $\tau(u)$ is the tension field of $u$ and $p > 1$. Critical points of $E_p(\cdot)$ are called $p$-biharmonic maps. In this paper we will prove nonexistence result of proper $p$-biharmonic maps when $p \geq 2$. In particular when $M = \mathbb{R}^m$, we get Liouville type results under proper integral conditions, which extend the related results of Baird, Fardoun and Ouakkas [1].

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1 Introduction

In the past several decades harmonic map plays a central role in geometry and analysis. Let $u : (M, g) \rightarrow (N, h)$ be a map between Riemannian manifolds $(M, g)$ and $(N, h)$. The energy of $u$ is defined by

$$E(u) = \int_M \frac{|du|^2}{2} d\nu_g,$$

where $d\nu_g$ is the volume element on $(M, g)$. The Euler-Lagrange equation of $E(\cdot)$ is

$$\tau(u) = \sum_{i=1}^m \{ \hat{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i) \} = 0,$$

where $\hat{\nabla}$ is the Levi-Civita connection on the pullback bundle $u^{-1}TN$ and $\{e_i\}$ is a local orthonormal frame field on $M$.

In 1983, Eells and Lemaire [5] (see also [6]) proposed to consider the bienergy functional

$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2} d\nu_g,$$

where $\tau(u)$ is the tension field of $u$. Recall that $u$ is harmonic if $\tau(u) = 0$. The Euler-Lagrange equation of $E_2(\cdot)$ is [12]

$$\tau_2(u) := \Delta \tau(u) + \sum_{i=1}^m R^N(\tau(u), du(e_i), du(e_i)) = 0,$$

where $\Delta := Tr_g(\hat{\nabla})^2$ and $R^N$ is the Riemannian curvature tensor of $(N, h)$. To further generalize the notion of harmonic maps, Han and Feng [10] considered the $p$-bienergy ($p > 1$) functional as follows:

$$E_p(u) = \int_M |\tau(u)|^p d\nu_g.$$
Remark 1.1. In [10] Han and Feng defined a more general object called $F$-biharmonic maps. $p$-biharmonic maps are $F$-biharmonic maps with $F(t) = (2t)^{\frac{p}{2}}$.

We define the $p$-bitension field of $u$ by \((1.1)\)

$$
\tau_p(u) := p\{\triangle(|\tau(u)|^{p-2}\tau(u)) + \sum_{i=1}^{m} R^N(|\tau(u)|^{p-2}\tau(u), du(e_i)) du(e_i)\}.
$$

The Euler-Lagrange equation of $E_p(\cdot)$ is $\tau_p(u) = 0$ and a smooth map $u$ satisfying $\tau_p(u) = 0$ is called a $p$-biharmonic map.

### 2 Nonexistence result

It is obvious that harmonic maps are $p$-biharmonic maps when $p \geq 2$. We call $p$-biharmonic maps which are not harmonic proper $p$-biharmonic maps. It is natural to consider when $p$-biharmonic maps are harmonic maps. There are a lot of results in this direction when $p = 2$ (cf. [3] [18] [21] for recent surveys). Han and Feng [10] proved that $p$-biharmonic maps from a compact (oriented) manifold into a manifold with nonpositive curvature must be harmonic. In noncompact case nonexistence results of proper isometric $p$-biharmonic maps were proved in [2] [9] [10] [11] [14] [16]. In [11] Han and Zhang proved the following result.

**Theorem 2.1 (HZ).** Let $u : (M, g) \rightarrow (N, h)$ be a $p$-biharmonic map from a Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ with non-positive sectional curvature and $a \geq 0$ be a non-negative real constant.

(i) If 

$$
\int_M |\tau(u)|^{a+p} dv_g < \infty,
$$

and the energy is finite, that is 

$$
\int_M |du|^2 dv_g < \infty,
$$

then $u$ is harmonic.

(ii) If $Vol(M, g) = \infty$ and 

$$
\int_M |\tau(u)|^{a+p} dv_g < \infty,
$$

then $u$ is harmonic, where $p \geq 2$.

The first aim of this paper is to generalize the above theorem by releasing the integral conditions.

**Theorem 2.2.** Let $u : (M, g) \rightarrow (N, h)$ be a $p$-biharmonic map ($p \geq 2$) from a complete Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ of nonpositive sectional curvature and $1 \leq q \leq \infty$, $p - 1 < s$.

(i) If $|du|$ is bounded in $L^q(M)$ and 

$$
\int_M |\tau(u)|^s dv_g < \infty,
$$

then $u$ is harmonic.
(ii) If $\text{Vol}(M, g) = \infty$ and 
\[
\int_M |\tau(u)|^s \, dv_g < \infty,
\]
then $u$ is harmonic.

When the target manifold has strictly negative sectional curvature, we have

**Theorem 2.3.** Let $u : (M, g) \to (N, h)$ be a $p$-biharmonic map $(p \geq 2)$ from a complete Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ of strictly negative sectional curvature and 
\[
\int_M |\tau(u)|^s \, dv_g < \infty
\]
for some $p - 1 < s$. Assume that there is a point $q \in M$ such that $\text{rank}u(q) \geq 2$, then $u$ is a harmonic map.

**Remark 2.4.** Here and in the following the rank of $u$ at a point $q \in M$ is defined by the dimension of the linear space $\text{du}(T_qM)$, where $T_qM$ is the tangent bundle of $M$ at $q$.

**Remark 2.5.** When $p = 2$, Theorem 2.2 and Theorem 2.3 were proved in [15], which extended previous results of Luo [13], Maeta [17] and Nakauchi et al. [19].

Because from Schoen and Yau’s paper [24] we see that a harmonic map from a complete noncompact Riemannian manifold of nonnegative Ricci curvature to a Riemannian manifold of nonpositive sectional curvature with $\int_M |du|^q \, dv_g < \infty (q > 1)$ must be a constant map, as a corollary of Theorem 2.2 we have the following Liouville type result for $p$-biharmonic maps.

**Corollary 2.6.** Let $u : (M, g) \to (N, h)$ be a $p$-biharmonic map $(p \geq 2)$ from a complete Riemannian manifold $(M, g)$ with $\text{Ric}^M \geq 0$ into a Riemannian manifold $(N, h)$ of nonpositive sectional curvature such that 
\[
\int_M |\tau(u)|^s + |du|^q \, dv_g < \infty,
\]
where $s > p - 1$ and $q > 1$. Then $u$ is a constant map.

**Remark 2.7.** This Liouville type result was first proved when $p = 2$ by Baird et al. in [1]. Though they assumed $s = q = 2$, it is easy to see from their proofs that their Liouville type result holds whenever $s > 1$ and $q > 1$.

### 2.1 Proof of Theorem 2.2

First let’s prove a lemma.

**Lemma 2.8.** Assume that $u : (M, g) \to (N, h)$ is a $p$-biharmonic map $(p \geq 2)$ from a complete manifold $(M, g)$ to a nonpositively curved manifold $(N, h)$ and 
\[
\int_M |\tau(u)|^s \, dv_g < \infty
\]
for some $s > p - 1$. Then $|\tau(u)|$ is a constant and moreover $\nabla \tau(u) = 0$. 

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Proof. Let $\epsilon > 0$. A direct computation shows that

$$\Delta(|\tau(u)|^{2p-2} + \epsilon)^{\frac{1}{2}}$$

$$= (|\tau(u)|^{2p-2} + \epsilon)^{-\frac{1}{2}} (\frac{1}{2}(|\tau(u)|^{2p-2} + \epsilon) \Delta|\tau(u)|^{2p-2} - \frac{1}{4} |\nabla|\tau(u)|^{2p-2}|^2).$$

(2.1)

Moreover,

$$\Delta|\tau(u)|^{2p-2} = 2|\nabla(|\tau(u)|^{p-2}\tau(u))|^2 + 2h(|\tau(u)|^{p-2}\tau(u), \tilde{\Delta}(|\tau(u)|^{p-2}\tau(u)))$$

$$= 2|\nabla(|\tau(u)|^{p-2}\tau(u))|^2 - 2 \sum_{i=1}^{m} (R^N(|\tau(u)|^{p-2}\tau(u), du(e_i), du(e_i), |\tau(u)|^{p-2}\tau(u)))$$

$$\geq 2|\nabla(|\tau(u)|^{p-2}\tau(u))|^2.$$  

(2.2)

and

$$\frac{1}{4} |\nabla|\tau(u)|^{2p-2}|^2 = h^2(|\tau(u)|^{p-2}\tau(u), \nabla(|\tau(u)|^{p-2}\tau(u)))$$

$$\leq |\nabla(|\tau(u)|^{p-2}\tau(u))|^2 |\tau(u)|^{2p-2}.$$  

(2.3)

From (2.1)-(2.3) we see that

$$\Delta(|\tau(u)|^{2p-2} + \epsilon)^{\frac{1}{2}} \geq 0,$$

which by letting $\epsilon \to 0$ implies that

$$\Delta|\tau(u)|^{p-1} \geq 0.$$

Then if $\int_M |\tau(u)|^s dv < \infty$ for $s > p - 1$, by Yau’s [25] classical $L^p$ ($p > 1$) Liouville type theorem we have that there exists a constant $c$ such that $|\tau(u)| = c$.

If $c = 0$ then $\nabla \tau(u) = 0$. If $c \neq 0$, then from the proof we see that $\nabla(|\tau(u)|^{p-2}\tau(u)) = 0$, i.e. $\nabla \tau(u) = 0$. This completes the proof.

Now let us continue to prove Theorem 2.2. From the above lemma we see that $|\tau(u)| = c$ is a constant. Hence if $Vol(M) = \infty$, we must have $c = 0$, which proves (ii) of Theorem 2.2. To prove (i) of Theorem 2.2 we distinguish two cases. If $c = 0$, we are done. If $c \neq 0$, we see that $Vol(M) < \infty$ and we will get a contradiction in the following. Define a 1-form on $M$ by

$$\omega(X) := \langle du(X), \tau(u) \rangle, \ (X \in TM).$$

Then we have

$$\int_M |\omega| dv = \int_M (\sum_{i=1}^{m} |\omega(e_i)|^2)^{\frac{1}{2}} dv$$

$$\leq \int_M |\tau(u)||du| dv$$

$$\leq cVol(M)^{1-\frac{1}{q}} (\int_M |du|^q dv)^{\frac{1}{q}}$$

$$< \infty,$$

where if $q = \infty$ we denote $||du||_{L^{\infty}(M)} = (\int_M |du|^q dv)^{\frac{1}{q}}$. 

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In addition, we consider $-\delta \omega = \sum_{i=1}^{m} (\nabla_{e_{i}} \omega)(e_{i})$:

$$-\delta \omega = \sum_{i=1}^{m} \nabla_{e_{i}}(\omega(e_{i})) - \omega(e_{i}e_{i})$$

$$= \sum_{i=1}^{m} \{ \langle \tilde{\nabla}_{e_{i}} du(e_{i}), \tau(u) \rangle - \langle du(e_{i}), \tau(u) \rangle \}$$

$$= \sum_{i=1}^{m} \langle \tilde{\nabla}_{e_{i}} du(e_{i}) - du(\nabla_{e_{i}} e_{i}), \tau(u) \rangle$$

$$= |\tau(u)|^{2},$$

where in the second equality we used $\tilde{\nabla} \tau(u) = 0$. Now by Gaffney’s theorem ([7], see the appendix for precise statement) and the above equality we have that

$$0 = \int_{M} -\delta \omega = \int_{M} |\tau(u)|^{2} dv_{g} = c^{2} Vol(M),$$

which implies that $c = 0$, a contradiction. Therefore we must have $c = 0$, i.e. $u$ is a harmonic map. This completes the proof of Theorem 2.2.

2.2 Proof of Theorem 2.3

By Lemma 2.8, $|\tau(\phi)| = c$ is a constant. We only need to prove that $c = 0$. Assume that $c \neq 0$, we will get a contradiction. By the $p$-biharmonic equation and the Weitzenböck formula we have at $q \in M$:

$$0 = -\frac{1}{2} \Delta |\tau(u)|^{2p-2}$$

$$= -\langle \tilde{\Delta}(|\tau(u)|^{p-2} \tau(u)), \tau(u) \rangle - |\tilde{\nabla}(|\tau(u)|^{p-2} \tau(u))|^{2}$$

$$= \sum_{i=1}^{m} \langle R^{N}(|\tau(u)|^{p-2} \tau(u), du(e_{i}), du(e_{i}), |\tau(u)|^{p-2} \tau(u)) - |\tilde{\nabla}(|\tau(u)|^{p-2} \tau(u))|^{2}$$

$$= \sum_{i=1}^{m} \langle R^{N}(|\tau(u)|^{p-2} \tau(u), du(e_{i}), du(e_{i}), |\tau(u)|^{p-2} \tau(u)),$$

where in the first and fourth equalities we used Lemma 2.8 twice. Since the sectional curvature of $N$ is strictly negative, we must have that $du(e_{i})$ is parallel to $\tau(u)$ at $q \in M \forall i$, i.e. rank$u(q) \leq 1$, a contradiction. This completes the proof of Theorem 2.3.

3 Stress energy tensor and a growth formula for $p$-biharmonic maps

In the following we will derive Liouville type results for $p$-biharmonic maps ($p \geq 2$) from the $m$-dimensional Euclidean space $\mathbb{R}^{m}$. To do this we need to use a formula for the stress energy tensor of $p$-biharmonic maps, introduced in [10].
Let \( u : (M, g) \to (N, h) \) be a smooth map between two Riemannian manifolds. The stress \( p \)-bienergy tensor of \( u \) is defined by

\[
S_p(u) = [(1-p)|\tau(u)|^p + p\text{div} h(|\tau(u)|^{p-2}\tau(u), du)]g - 2p\text{sym} h(\nabla(|\tau(u)|^{p-2}\tau(u)), du),
\]
where \( \text{sym} T(X, Y) \) denotes symmetrization of a 2-tensor, that is \( \text{sym} T(X, Y) = \frac{1}{2}(T(X, Y) + T(Y, X)) \). We have

**Proposition 3.1** ([10], Theorem 4.3). For any smooth map \( u : (M, g) \to (N, h) \)

\[
(\text{div} S_p(u))(X) = -h(\tau_p(u), du(X)) - p(p-2)|\tau(u)|^{p-2}X(\frac{|\tau(u)|^2}{2}).
\]  

**Proof.** In Theorem 4.3 of [10], let \( F(t) = (2t)^\frac{p}{2} \).

In particular, if \( u \) is a smooth \( p \)-biharmonic map we have

\[
(\text{div} S_p(u))(X) = -p(p-2)|\tau(u)|^{p-2}X(\frac{|\tau(u)|^2}{2}).
\]

Let \( T \) be a symmetric covariant 2-tensor on a Riemannian manifold \( (M, g) \) and let \( X \) be a vector field on \( M \). Then

\[
\text{div} (T \lrcorner X) = (\text{div} T)(X) + \frac{1}{2} \langle \mathcal{L}_X g, T \rangle,
\]
where \( (T \lrcorner X)(Y) := T(X, Y) \), \( \mathcal{L} \) is the Lie derivative operator and

\[
\langle \mathcal{L}_X g, T \rangle = \langle \mathcal{L}_X g(e_i, e_j)T(e_i, e_j) \rangle,
\]
where \( \{e_i\} \) is an orthonormal basis. Integrating this formula over a compact domain \( U \) with smooth boundary, we obtain

\[
\int_{\partial U} T(X, n) d\sigma = \int_U (\text{div} T)(X) dv_g + \frac{1}{2} \int_U \langle \mathcal{L}_X g, T \rangle dv_g,
\]
where \( n \) is the outward pointing unit normal and \( dv \) is the volume element along \( \partial U \). From Proposition 3.1 taking \( T = S_p(u) \) in the above formula we have the following growth formula.

**Theorem 3.2.** Let \( u : V \subseteq \mathbb{R}^m \to (N, h) \) be a \( p \)-biharmonic map defined on a open subset \( V \) of Euclidean space \( \mathbb{R}^m \) with its canonical metric \( g \). Let \( B_r \) be a ball of radius of \( r \) contained in \( V \) and \( S_r = \partial B_r \). Then we have

\[
\int_{B_r} \text{div} |\tau(u)|^p dx = -m(\frac{m}{p}) \int_{S_r} |\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}) d\sigma - (1-\frac{1}{p})r \int_{S_r} |\tau(u)|^p d\sigma
\]

\[
- r \int_{S_r} \frac{\partial}{\partial r} h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r})) d\sigma + 2r \int_{S_r} h(\nabla(|\tau(u)|^{p-2}\tau(u)), \nabla \frac{\partial}{\partial r} du(\frac{\partial}{\partial r})) d\sigma
\]

\[
+ (p-2)r \int_{B_r} |\tau(u)|^{p-2} \frac{\partial}{\partial r} \frac{|\tau(u)|^2}{2} dx.
\]  

\[
(3.4)
\]
Proof. In (3.3) choose \( X = r \frac{\partial}{\partial r} \), \( T = S_p(u) \) and \( U = B_r \). Then we have

\[
    r \int_{S_r} S_p(u)(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})d\sigma = \int_{B_r} \text{div} S_p(u)(r \frac{\partial}{\partial r})dv_g + \int_{B_r} \langle g, S_p(u) \rangle dv_g, \tag{3.5}
\]

where we used \( \mathcal{L}_r g = 2g \). By definition of \( S_p(u) \) we see that

\[
    r \int_{S_r} S_p(u)(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})d\sigma = r \int_{S_r} (1 - p)|\tau(u)|^p + p \text{div} h(|\tau(u)|^{p-2}\tau(u), du) d\sigma \\
- 2rp \int_{S_r} h(\nabla \frac{\partial}{\partial r} |\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r})) d\sigma \\
- r \int_{S_r} (1 - p)|\tau(u)|^p + p \text{div} h(|\tau(u)|^{p-2}\tau(u), du) d\sigma \\
- 2rp \int_{S_r} \nabla \frac{\partial}{\partial r} h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r})) d\sigma \\
+ 2rp \int_{S_r} h(|\tau(u)|^{p-2}\tau(u), \nabla \frac{\partial}{\partial r} du(\frac{\partial}{\partial r})) d\sigma \tag{3.6}
\]

and by

\[(\text{div} S_p(u))(X) = -p(p - 2)|\tau(u)|^{p-2} X(\frac{\tau(u)^2}{2})\]

we have

\[
    \int_{B_r} \text{div} S_p(u)(r \frac{\partial}{\partial r})dv_g = -p(p - 2) \int_{B_r} |\tau(u)|^{p-2} r \frac{\partial}{\partial r}(\frac{\tau(u)^2}{2})dv_g. \tag{3.7}
\]

In addition

\[
    \int_{B_r} \langle g, S_p(u) \rangle dv_g = m(1 - p) \int_{B_r} |\tau(u)|^p dv_g + mp \int_{B_r} \text{div} h(|\tau(u)|^{p-2}\tau(u), du)dv_g \\
- 2p \int_{B_r} \sum_i h(\nabla_{e_i} |\tau(u)|^{p-2}\tau(u), du(e_i)) dv_g \\
= m(1 - p) \int_{B_r} |\tau(u)|^p dv_g + 2p \int_{B_r} |\tau(u)|^p dv_g \\
+ (m - 2)p \int_{B_r} \text{div} h(|\tau(u)|^{p-2}\tau(u), du)dv_g \\
= (2p + m(1 - p)) \int_{B_r} |\tau(u)|^p dv_g \\
+ (m - 2)p \int_{S_r} h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r})) d\sigma. \tag{3.8}
\]
From (3.5)-(3.8) we finish the proof of Theorem 3.2

When $p = 2$, this growth formula was stated (without a proof) in [1], where Baird et al. used this formula to prove several Liouville type theorems for biharmonic maps from $\mathbb{R}^m$. We will systematically extend their results in the next section to case of $p \geq 2$.

4 Liouville type theorem for $p$-biharmonic maps from $\mathbb{R}^m$

We suppose in this section that $(M, g)$ is the $m$-dimensional Euclidean space $\mathbb{R}^m$ with its canonical metric. For harmonic maps with $m \neq 2$, it is well known when $m = 1$([24]) and when $m \geq 3$([8],[23]) that if their energy is finite, then they must be constant. This result was extended to biharmonic maps by Baird et al.(1) when $m \neq 4$. We will further prove such Liouville type results under proper integral conditions for general $p$-biharmonic maps when $p \geq 2$. It is a surprise that when $p > 2$ we have Liouville type result in all dimensions (even if $m = 2p$, when the energy functional $E_p$ is scaling invariant).

We will deal with separately the case of $m = 1$ and $m \geq 2$. In the later case we will use the growth formula (3.4), and the hypotheses is stronger.

**Theorem 4.1.** Let $u : (\mathbb{R}, g) \to (N, h)$ be a $p$-biharmonic map $(p \geq 2)$ satisfying

$$\int_{\mathbb{R}} (|\tau(u)|^p + |\nu|^p)dx < \infty. \quad (4.1)$$

Then $u$ is constant.

**Proof.** Since $u$ is $p$-biharmonic, hence we have

$$\text{div} (S_p(u))(X) = -h(\tau_p(u), du(X)) - p(p - 2)|\tau(u)|^{p-2}X(\frac{|\tau(u)|^2}{2}) = -(p - 2)X|\tau(u)|^p.$$

Therefore

$$\text{div} (S_p(u)(X, \cdot)) = \frac{1}{2} \langle \mathcal{L}_X g, S_p(u) \rangle - (p - 2)X|\tau(u)|^p.$$  

Taking $X = \frac{\partial}{\partial x}$ and since $\mathcal{L}_{\frac{\partial}{\partial x}} g = 0$ we get

$$\frac{\partial}{\partial x}(S_p(u)(\frac{\partial}{\partial x}, \frac{\partial}{\partial x})) + (p - 2)|\tau(u)|^p = 0.$$  

Thus there exists a constant $C$ such that

$$S_p(u)(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) + (p - 2)|\tau(u)|^p = C. \quad (4.2)$$

By definition of $S_p(u)$ we see that

$$\begin{align*}
S_p(u)(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) &= [(1 - p)|\tau(u)|^p + p\text{div} h(|\tau(u)|^{p-2}\tau(u), du) - 2ph(\nabla_{\frac{\partial}{\partial x}}(|\tau(u)|^{p-2}\tau(u)), du(\frac{\partial}{\partial x}))] \\
&= (1 - p)|\tau(u)|^p + p\frac{\partial}{\partial x}h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial x})) \\
&- 2p\frac{\partial}{\partial x}h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial x})) + 2ph(|\tau(u)|^{p-2}\tau(u), \nabla_{\frac{\partial}{\partial x}} du(\frac{\partial}{\partial x})) \\
&= (1 + p)|\tau(u)|^p - p\frac{\partial}{\partial x}h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial x})),
\end{align*}$$

\[\square\]
Hence there exist sequences \( \{ \) \( \) such that \( \lim_{n \to \infty} R_n = \infty \) and \( \lim_{n \to \infty} R'_n = -\infty \) which satisfy

\[
\lim_{n \to \infty} h(|\tau(u)|^{p-2}\tau(u))(R_n), du\left( \frac{\partial}{\partial x} \right)(R_n) = 0,
\]

and

\[
\lim_{n \to \infty} h(|\tau(u)|^{p-2}\tau(u))(R'_n), du\left( \frac{\partial}{\partial x} \right)(R'_n) = 0.
\]

Therefore on integrating over \( (4.3) \) from \( R'_n \) to \( R_n \) we get

\[
\int_{R'_n}^{R_n} [1 + (2p - 2)]|\tau(u)|^p dx
\]

\[
= p[h(|\tau(u)|^{p-2}\tau(u))(R_n), du\left( \frac{\partial}{\partial x} \right)(R_n)]
\]

\[
- ph(|\tau(u)|^{p-2}\tau(u))(R'_n), du\left( \frac{\partial}{\partial x} \right)(R'_n)) + C(R_n - R'_n).
\]

Hence we have

\[
C = \frac{1}{R_n - R'_n} \left\{ \int_{R'_n}^{R_n} [1 + (2p - 2)]|\tau(u)|^p dx - p[h(|\tau(u)|^{p-2}\tau(u))(R_n), du\left( \frac{\partial}{\partial x} \right)(R_n)]
\]

\[
+ ph(|\tau(u)|^{p-2}\tau(u))(R'_n), du\left( \frac{\partial}{\partial x} \right)(R'_n)) \right\}.
\]

Letting \( n \to \infty \) in the above quality we get \( C = 0 \) and we obtain

\[
\int_{R'_n}^{R_n} [1 + (2p - 2)]|\tau(u)|^p dx
\]

\[
= p[h(|\tau(u)|^{p-2}\tau(u))(R_n), du\left( \frac{\partial}{\partial x} \right)(R_n)]
\]

\[
- ph(|\tau(u)|^{p-2}\tau(u))(R'_n), du\left( \frac{\partial}{\partial x} \right)(R'_n))].
\]

Letting \( n \to \infty \) again we get \( \int_{\mathbb{R}} |\tau(u)|^p dx = 0 \), implying that \( \tau(u) = 0 \), i.e. \( u \) is a harmonic map.

Recall that for harmonic maps we have the following Bochner formula \((4)\)

\[
\frac{1}{2} \Delta |du|^2 = |\nabla du|^2 + \langle \text{Ric}^M \nabla u, \nabla u \rangle - \sum_{i,j} \langle Rm^N(du(e_i), du(e_j)du(e_i), du(e_j)) \rangle.
\]
where \( \{e_i\} \) is a local orthonormal frame field on \( M \). Hence when \( M = \mathbb{R} \) we have \( \frac{1}{2} \Delta |du|^2 = |\nabla du|^2 \). Therefore
\[
|du|\Delta |du| = |\nabla du|^2 - |\nabla |du||^2 \geq 0,
\]
which implies that \( |du| \) is a subharmonic function on \( \mathbb{R} \). Then by Yau’s \( L^p \) Liouville type theorem \((23)\) for subharmonic functions we have \( |du| \) is a constant which is zero by \( \int_\mathbb{R} |du|^p dx < \infty \). Thus we have proved that \( u \) is a constant map. \( \square \)

When \( m \geq 2 \) we have

**Theorem 4.2.** Let \( u : (\mathbb{R}^m, g) \to (N, h) \) be a \( p \)-biharmonic map satisfying
\[
\int_{\mathbb{R}^m} (|\nabla du|^p + |du|^p) dx < \infty, \tag{4.7}
\]
where \( m \geq 2 \) and \( p > 2 \). Then \( u \) is a harmonic map.

Moreover \( u \) is a constant map if \( m \geq 3 \) and in addition we assume
\[
\int_{\mathbb{R}^m} |du|^q dx < \infty,
\]
where \( 2 \leq q \leq m \).

**Proof.** To prove this theorem we will need use the growth formula \((3.4)\). Notice that
\[
(p - 2) \int_{B_r} |\tau(u)|^{p-2} \frac{\partial}{\partial r} (|\tau(u)|^2) dx = \frac{p - 2}{p} \int_{B_r} \frac{\partial}{\partial r} |\tau(u)|^p dx \tag{4.8}
\]
\[
= \frac{p - 2}{p} \int_{S_r} |\tau(u)|^p d\sigma - \frac{(p - 2)(m - 1)}{p} \int_{B_r} |\tau(u)|^p dx.
\]

Equation \((4.8)\) is one of our main observations.

Then from the above equality and \((3.4)\) we have
\[
[2 - (1 - \frac{1}{p})m] \frac{1}{r} \int_{B_r} |\tau(u)|^p dx + \frac{(p - 2)(m - 1)}{p} \int_{B_r} |\tau(u)|^p dx = \frac{(m - 2)}{r} \int_{S_r} h(\vert\tau(u)\vert^{p-2}\tau(u), du(\frac{\partial}{\partial r})) d\sigma - \frac{(p - 2)}{p} \int_{S_r} |\tau(u)|^p d\sigma
\]
\[
- \int_{S_r} \frac{\partial}{\partial r} h(\vert\tau(u)\vert^{p-2}\tau(u), du(\frac{\partial}{\partial r})) d\sigma + 2 \int_{S_r} h(\vert\tau(u)\vert^{p-2}\tau(u), \nabla du(\frac{\partial}{\partial r})) d\sigma + \frac{(p - 2)}{p} \int_{S_r} |\tau(u)|^p d\sigma. \tag{4.9}
\]

Since
\[
|\tau(u)|^p \leq C(m, p) |\nabla du|^p,
\]
on applying the Young’s inequality we get
\[
\int_{\mathbb{R}^m} h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))) dx \leq C(p) \int_{\mathbb{R}^m} (|\tau(u)|^p dx + |du(\frac{\partial}{\partial r})|^p dx \leq C(m, p) \int_{\mathbb{R}^m} |\nabla du|^p + |du|^p dx < \infty. \tag{4.10}
\]
Hence by Lemma 3.5 in [1] there exists an increasing sequence of \((R_n) \to \infty\) and three positive constants \(C_1, C_2\) and \(C_3\) such that

\[
\int_{S_{R_n}} |h(\tau(u)|^{p-2}\tau(u), du(\partial_{\tau})| dx \leq \frac{C_1}{R_n},
\]

and

\[
C_2 \leq \ln \frac{R_n}{R_{n-1}} \leq C_3.
\]

Furthermore from (4.10) we have

\[
\lim_{n \to \infty} \int_{R_n}^{R_{n-1}} \int_{S_r} |h(\tau(u)|^{p-2}\tau(u), du(\partial_{\tau})| d\sigma dr = 0.
\]

Similarly,

\[
\int_{\mathbb{R}^m} |h(\tau(u)|^{p-2}\tau(u), \nabla \alpha du(\partial_{\tau})| dx \leq C(p) \int_{\mathbb{R}^m} |\nabla \alpha du(\partial_{\tau})|^{p} dx
\]

\[
\leq C(m, p) \int_{\mathbb{R}^m} |\tilde{\nabla} du|^p dx < \infty.
\]

Therefore

\[
\lim_{n \to \infty} \int_{R_n}^{R_{n-1}} \int_{S_r} |h(\tau(u)|^{p-2}\tau(u), \nabla \alpha du(\partial_{\tau})| d\sigma dr = 0.
\]

Again since

\[
\int_{\mathbb{R}^m} |\tau(u)|^{p} dx \leq C(m, p) \int_{\mathbb{R}^m} |\tilde{\nabla} du|^p dx < \infty,
\]

we have

\[
\lim_{n \to \infty} \int_{R_n}^{R_{n-1}} \int_{S_r} |\tau(u)|^{p} d\sigma dr = 0.
\]

Now integrating over (4.19) from \(R_{n-1}\) to \(R_n\) we get

\[
[2 - (1 - \frac{1}{p})m] \int_{R_{n-1}}^{R_n} \frac{1}{r} \int_{B_r} |\tau(u)|^p dxd\tau + \frac{(p-2)(m-1)}{p} \int_{R_{n-1}}^{R_n} \int_{B_r} \frac{|\tau(u)|^p}{|x|} dxd\tau
\]

\[
= \int_{R_{n-1}}^{R_n} \frac{-(m-2)}{r} \int_{S_r} h(|\tau(u)|^{p-2}\tau(u), du(\partial_{\tau})) d\sigma dr - (1 - \frac{1}{p}) \int_{R_{n-1}}^{R_n} \int_{S_r} |\tau(u)|^p d\sigma dr
\]

\[
- \int_{R_{n-1}}^{R_n} \int_{S_r} \frac{\partial}{\partial r} h(|\tau(u)|^{p-2}\tau(u), du(\partial_{\tau})) d\sigma dr + 2 \int_{R_{n-1}}^{R_n} \int_{S_r} h(|\tau(u)|^{p-2}\tau(u), \nabla \alpha du(\partial_{\tau})) d\sigma dr
\]

\[
+ \frac{(p-2)}{p} \int_{R_{n-1}}^{R_n} \int_{S_r} |\tau(u)|^p d\sigma dr
\]

\[
= \int_{R_{n-1}}^{R_n} \frac{1}{r} \int_{S_r} h(|\tau(u)|^{p-2}\tau(u), du(\partial_{\tau})) d\sigma - (1 - \frac{1}{p}) \int_{R_{n-1}}^{R_n} \int_{S_r} |\tau(u)|^p d\sigma dr
\]

\[
- \int_{S_{R_n}} h(|\tau(u)|^{p-2}\tau(u), du(\partial_{\tau})) d\sigma + \int_{S_{R_n}} h(|\tau(u)|^{p-2}\tau(u), du(\partial_{\tau})) d\sigma
\]

\[
+ 2 \int_{R_{n-1}}^{R_n} \int_{S_r} h(|\tau(u)|^{p-2}\tau(u), \nabla \alpha du(\partial_{\tau})) d\sigma dr + \frac{(p-2)}{p} \int_{R_{n-1}}^{R_n} \int_{S_r} |\tau(u)|^p d\sigma dr
\]

\[
= \int_{R_{n-1}}^{R_n} \frac{1}{r} \int_{S_r} h(|\tau(u)|^{p-2}\tau(u), du(\partial_{\tau})) d\sigma dr
\]

\[
+ \frac{(p-2)}{p} \int_{R_{n-1}}^{R_n} \int_{S_r} |\tau(u)|^p d\sigma dr,
\]

\[
(4.16)
\]
where in the second equality we used the following computations

\[
\int_{R_{n-1}}^{R_n} \int_{S_r} \frac{\partial}{\partial r} h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))d\sigma dr \\
= \int_{R_{n-1}}^{R_n} \int_{S_r} \frac{\partial}{\partial r} (h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))r^{m-1})d\omega dr \\
- (m-1) \int_{R_{n-1}}^{R_n} \frac{1}{r} \int_{S_r} h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))d\sigma dr \\
= \int_{R_{n-1}}^{R_n} \frac{\partial}{\partial r} \int_{S_r} h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))d\sigma dr \\
- (m-1) \int_{R_{n-1}}^{R_n} \frac{1}{r} \int_{S_r} h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))d\sigma dr \\
= \int_{S_{R_{n-1}}} h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))d\sigma - \int_{S_{R_{n-1}}} h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))d\sigma \\
- (m-1) \int_{R_{n-1}}^{R_n} \frac{1}{r} \int_{S_r} h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))d\sigma dr.
\]

Therefore we get

\[
\frac{(p-2)(m-1)}{p} (R_n - R_{n-1}) \int_{B_{R_{n-1}}} \frac{|\tau(u)|^p}{|x|} dx dr \\
\leq ||2 - (1 - \frac{1}{p})m|| \int_{R_{n-1}}^{R_n} \frac{1}{r} \int_{B_r} |\tau(u)|^p dx dr \\
+ C \int_{R_{n-1}}^{R_n} \int_{S_r} |h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))|d\sigma \\
+ C \int_{R_{n-1}}^{R_n} \int_{S_r} |\tau(u)|^p d\sigma dr + \int_{S_{R_{n-1}}} |h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))|d\sigma \\
+ \int_{S_{R_{n-1}}} |h(|\tau(u)|^{p-2}\tau(u), du(\frac{\partial}{\partial r}))|d\sigma + 2 \int_{R_{n-1}}^{R_n} \int_{S_r} |h(|\tau(u)|^{p-2}\tau(u), \nabla \frac{\partial}{\partial r}, du(\frac{\partial}{\partial r}))|d\sigma dr \\
+ \frac{(p-2)}{p} \int_{R_{n-1}}^{R_n} \int_{S_r} |\tau(u)|^p d\sigma dr.
\]

In addition,

\[
\int_{R_{n-1}}^{R_n} \frac{1}{r} \int_{B_r} |\tau(u)|^p dx dr \\
\leq \int_{B_{R_{n-1}}} |\tau(u)|^p dx \int_{R_{n-1}}^{R_n} \frac{1}{r} dr \\
= \ln \frac{R_n}{R_{n-1}} \int_{B_{R_{n}}} |\tau(u)|^p dx \leq C_3 \int_{B_{R_{n}}} |\tau(u)|^p dx.
\]

Then by (4.11), (4.13), (4.14), (4.15) and (4.19), letting $n \to \infty$, we see that the right hand of (4.18) is bounded by $C_3 \int_{\mathbb{R}^n} |\tau(u)|^p dx < \infty$, but the left hand side goes to $\infty$ since $\lim_{n \to \infty} (R_n - R_{n-1}) = \infty$. Therefore, (4.18) does not hold for large $n$. The proof is complete.
\( R_{n-1} = \infty \) by \( \ln \frac{R_n}{R_{n-1}} \geq C_2 > 0 \), if \( |\tau(u)| \) dose not vanish anywhere. That is we have proved that \( u \) is a harmonic map.

Furthermore if \( m \geq 3 \) and \( \int_{\mathbb{R}^m} |du|^q dx < \infty \) (\( 2 \leq q \leq m \)) we have \( u \) is a constant map by the well known result of \([8]\) and \([23]\) when \( q = 2 \), of \([22]\) when \( 2 \leq q < m \) and of \([20]\) when \( q = m \). \( \square \)

From Theorem 4.2 we can obtain the following Liouville type result.

**Theorem 4.3.** Let \( u : (\mathbb{R}^m, g) \to (N, h) \) be a \( p \)-biharmonic map satisfying

\[
\int_{\mathbb{R}^m} (|\nabla du|^p + |du|^p) dx < \infty, \tag{4.20}
\]

where \( m \geq 2 \) and \( p > 2 \). If \( \text{rank} u(x) \leq 1, \forall x \in \mathbb{R}^m \), \( u \) is a constant map.

In particular, if \( u : (\mathbb{R}^m, g) \to (N^1, h) \) is a \( p \)-biharmonic map satisfying

\[
\int_{\mathbb{R}^m} (|\tilde{\nabla} du|^p + |du|^p) dx < \infty, \tag{4.21}
\]

where \( m \geq 2 \) and \( p > 2 \). Then \( u \) is a constant map.

**Proof.** From Theorem 4.2 we see that \( u \) is a harmonic map. To prove that \( u \) is a constant map we follow the argument given at the last lines of the proof of Theorem 4.1.

Since \( u \) is a harmonic map, we have the following Bochner’s formula

\[
\frac{1}{2} \Delta |du|^2 = |\tilde{\nabla} du|^2 + \langle \text{Ric}^M \nabla u, \nabla u \rangle - \sum_{i,j} \langle R^{N_i} (du(e_i), du(e_j)) du(e_i), du(e_j) \rangle,
\]

where \( \{e_i\} \) is a local orthonormal frame field on \( M \). Hence when \( M = \mathbb{R}^m \) and \( \text{rank} u \leq 1 \) we have \( \frac{1}{2} \Delta |du|^2 = |\tilde{\nabla} du|^2 \). Therefore \( |du| \Delta |du| = |\tilde{\nabla} du|^2 - |\nabla |du||^2 \geq 0 \), which implies that \( |du| \) is a subharmonic function on \( \mathbb{R}^m \). Then by Yau’s \( L^p \) Liouville type theorem ([25]) for subharmonic functions we have \( |du| \) is a constant which is zero by \( \int_{\mathbb{R}^m} |du|^p dx < \infty \). Thus we have proved that \( u \) is a constant map. \( \square \)

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## 5 Appendix

**Theorem 5.1** (Gaffney’s theorem). Let \((M, g)\) be a complete Riemannian manifold. If a \( C^1 \) 1-form \( \omega \) satisfies that

\[
\int_M |\omega| dv_g + \int_M |\delta \omega| dv_g < \infty,
\]

or equivalently, a \( C^1 \) vector field \( X \) defined by \( \omega(Y) = \langle X, Y \rangle, (\forall Y \in TM) \) satisfies that

\[
\int_M |X| dv_g + \int_M |\text{div} X| dv_g < \infty,
\]

then

\[
\int_M \delta \omega dv_g = \int_M \text{div} X dv_g = 0.
\]
6 Ethical statements

The authors declare that they have no conflict of interest. This article does not contain any studies with human participants or animals performed by any of the authors. Informed consent was obtained from all individual participants included in the study.

References

[1] P. Baird, A. Fardoun and S. Ouakkas, Liouville-type theorems for biharmonic maps between Riemannian manifolds, *Adv. Calc. Var.* 3(2010), 49–68.

[2] X. Z. Cao and Y. Luo, On p-biharmonic submanifolds in nonpositively curved manifolds, *Kodai Math. J.* 39(2016), no. 3, 567–578.

[3] B. Y. Chen, Some open problems and conjectures on submanifolds of finite type: recent development, *Tamkang J. Math.* 45(2014), no.1, 87–108.

[4] J. Eells and L. Lemaire, A report on harmonic maps, *Bull. London Math. Soc.* 10(1978), no.1, 1–68.

[5] J. Eells and L. Lemaire, Selected topics in harmonic maps, *Amer. Math. Soc.*, CBMS, 50(1983).

[6] J. Eells and J.H. Sampson, Variational theory in fibre bundles, *Proc. U.S.-Japan Seminar in Differential Geometry, Kyoto*(1965), 22–33.

[7] M. P. Gaffney, A special Stokes’ theorem for complete riemannian manifolds, *Ann. Math.* 60(1954), 140–145.

[8] W. D. Garber, S. H. H. Ruijsenaas, E. Seller and D. Burns, On finite action solutions of the non-linear σ-model, *Annals of Physics* 119(1979), 305–325.

[9] Y. B. Han, Some results of p-biharmonic submanifolds in a Riemannian manifold of non-positive, *J. Geom.* 106(2015), 471–482.

[10] Y. B. Han and S. X. Feng, Some results of F-biharmonic maps, *Acta Math. Univ. Comenianae Vol. LXXXIII*, 1(2014), 47-66.

[11] Y. B. Han and W. Zhang, Some results of p-biharmonic maps into a non-positively curved manifold, *J. Korean Math. Soc.* 52(2015), 1097–1108.

[12] G. Y. Jiang, 2-harmonic maps and their first and second variational formulas, *Chinese Ann. Math. Ser. A* 7(1986), 389–402. Translated into English by H. Urakawa in *Note Mat.* 28 (2009), Suppl. 1, 209–232.

[13] Y. Luo, Liouville type theorems on complete manifolds and non-existence of bi-harmonic maps, *J. Geom. Anal.* 25(2015), 2436–2449.

[14] Y. Luo, The maximal principle for properly immersed submanifolds and its applications, *Geom. Dedicata* 181(2016), 103–112.
[15] Y. Luo, Remarks on the nonexistence of biharmonic maps, *Arch. Math. (Basel)* 107(2016), no. 2, 191–200.

[16] Y. Luo and S. Maeta, Biharmonic hypersurfaces in a sphere, *Proc. Amer. Math. Soc.* 145(2017), no. 7, 3109–3116.

[17] S. Maeta, Biharmonic maps from a complete Riemannian manifold into a non-positively curved manifold, *Ann. Glob. Anal. Geom.* 46(2014), 75–85.

[18] S. Montaldo and C. Oniciuc, A short survey on biharmonic maps between Riemannian manifolds, *Rev. Un. Mat. Argentina* 47(2006), no. 2, 1–22.

[19] N. Nakauchi, H. Urakawa and S. Gudmundsson, Biharmonic maps in a Riemannian manifold of non-positive curvature, *Geom. Dedicata* 164(2014), 263–272.

[20] N. Nakauchi and S. Takakuwa, A remark on $p$-harmonic maps, *Nonlinear Anal.* 25(1997), 169–185.

[21] Y. L. Ou, Some recent progress of biharmonic submanifolds. Recent advances in the geometry of submanifolds-dedicated to the memory of Franki Dillen (1963–2013), 127–139, *Contemp. Math.* 674, Amer. Math. Soc., Providence, RI, 2016.

[22] P. Price, A monotonicity formula for Yang-Mills fields, *Manuscripta Math.* 43(1983), 131–166.

[23] H. C. J. Sealey, Some conditions ensuring the vanishing of harmonic differential forms with applications to harmonic maps and Yang-Mills theory, *Math. Proc. Cambridge Philos. Soc.* 91(1982), no. 3, 441–452.

[24] R. Schoen and S. T. Yau, Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative Ricci curvature, *Comment. Math. Helv.* 51(1976), no. 3, 333–341.

[25] S. T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry, *Indiana. Uni. Math. J.* 25(1976), 659–670.

YINGBO HAN
School of Mathematics and Statistics, Xinyang Normal University, Xinyang, 464000, Henan, China.
yingbohan@163.com

YONG LUO
School of Mathematics and statistics, Wuhan University, Wuhan 430072, China.
yongluo@whu.edu.cn