Fusion $U_q(G_2^{(1)})$ vertex models and analytic Bethe ansätze

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We introduce fusion $U_q(G_2^{(1)})$ vertex models related to fundamental representations. The eigenvalues of their row to row transfer matrices are derived through analytic Bethe ansätze. By combining these results with our previous studies on functional relations among transfer matrices (the $T$-system), we conjecture explicit eigenvalues for a wide class of fusion models. These results can be neatly expressed in terms of a Yangian analogue of the Young tableaux.

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§1 Introduction

Recent years, the quantum group symmetry has been playing major roles in various branches of physics. Yang Baxter equation is the most fundamental relation in these studies[1, 2]:

\[ R_{W_2 W_3}(u)R_{W_1 W_3}(u+v)R_{W_1 W_2}(v) = R_{W_1 W_2}(v)R_{W_1 W_3}(u+v)R_{W_2 W_3}(u). \]  

(1)

Here \( R \) matrix, \( R_{W_1 W_2}(u) \) denotes the operator acting upon a triplex of spaces \( W_1 \otimes W_2 \otimes W_3 \) as \( R_{W_1 W_2}(u) \otimes 1 \). Viewed as a two dimensional(2D) vertex model, \( R_{W_1 W_2}(u) \) represents the vertex weights acting on the auxiliary (horizontal) space \( W_1 \) and the quantum (vertical) space \( W_2 \). Jimbo and Bazhanov[3,4] found a systematic way of constructing solution to eq.(1). Let \( V_\Lambda \) be a highest weight module of a Lie algebra \( g \) with the highest weight \( \Lambda \). Once a solution for \( W_1 = W_2 = W_3 = V_\Lambda \) is known, it is in principle possible to obtain \( R \) matrices acting on spaces \( W_1, W_2 \) and \( W_3 \) other than \( V_\Lambda \), and not necessary identical[5]. This procedure is quite general and called the fusion.

The fusion space, however, is not irreducible as the \( g \)- module in general[6,7,8]. For \( V_{m\bar{\Lambda}_u} \), we have the corresponding fusion space \( W_{m(a)} \), that is, the irreducible finite dimensional representation of quantum affine algebra \( U_q(\hat{g}) \)[3, 9]. It is expected that the irreducible finite dimensional representation of quantum affine algebra for generic \( q \) is smoothly connected to that of Yangian \( q \rightarrow 1 \) (for \( g = sl_2 \), see [10]). In this sense, we call \( W_{m(a)} \) Yangian space in the following. Then the decomposition of \( W_{m(a)} \) as the \( g \)-module reads,

\[ W_{m(a)} \big|_g = \oplus \{ n \} Z(m, \{ n \}, \Lambda') V_{\Lambda'}. \]  

(2)

where \( \Lambda' = m\bar{\Lambda}_a - \sum n_k \alpha_k \) and \( Z(m, \{ n \}, \Lambda') \) is a known numerical factor given in [8].

In this communication, we work on fusion \( U_q(G_2(1)) \) vertex models. The motivation comes from our previous study[11] on the functional identities among row to row transfer matrices(referred to as the \( T \)-system in the following). Let the quantum and the auxiliary space be \( W_{s(p)} \otimes N \) and \( W_{m(a)} \), respectively. Here \( a, p = 1, \cdots, \text{rank } g \) and \( m, s \) denote arbitrary integers. In the followings we fix \( p, s \) for these meanings. We define a transfer matrix \( T_{m(a)}(u) \) by

\[ T_{m(a)}(u) = \text{tr}_{W_{m(a)} W_{m(a)} W_{s(p)}(u-w_1) \cdots R_{W_{m(a)} W_{s(p)}(u-w_N)}.} \]  

(3)

The \( T \)-system states the relations among \( T_{m(a)}(u) \) for various \( a, m \), however, with identical \( W_{s(p)} \). (See eq.(13) for \( U_q(G_2(1)) \)). In ref[11], it is shown that the \( T \)-system Yang-Baxterizes the relations between Yangian characters \( Q_{m(a)}^{(a)} \). Moreover, it has been conjectured that any \( T_{m(a)}(u) \) and its eigenvalue \( r_{m(a)}(u) \) can be written as a polynomial of fundamental ones, \( a = 1, \cdots, \text{rank } g, m = 1 \), irrespective of \( W_{s(p)} \). Thus, by finding explicit forms for elementary ones, we can obtain those for arbitrary \( a, m \). This program has been recently completed for \( U_q(G_2(1)) \) model[12]. For \( U_q(G_2(1)) \) model, we have also two fundamental transfer matrices \( T_{1(1)}(u) \) and \( T_{1(2)}(u) \). Their explicit eigenvalues for \( s = 1 \) are conjectured in this report via analytic Bethe ansätze[13]. We find it convenient to introduce a Yangian
analogue of the Young tableaux in presenting these results. A conjecture on the explicit form for \( r_m^{(1)}(u) \), \( m \) arbitrary integer, is proposed in terms of tableaux generated by quite simple rules. Such tableaux are first introduced by Bazhanov and Reshetikhin [14] for \( sl_n \) models. Their tableaux coincide with the semi-standard tableaux. We believe that this coincidence is not accidental but universal. That is, the tableaux representing the eigenvalues of transfer matrices might be deserved as Yangian basis. Then the simple rules we find here imply that the Yangian theory is, though complicated in disguise, quite natural object.

§2 Fundamental Models

We fix notations. Let \( \alpha_1, \alpha_2 \) be simple roots. We introduce the bilinear form \( (\ast|\ast) \) with normalization \( (\alpha_1|\alpha_1) = 3(\alpha_2|\alpha_2) = 2 \) and \( (\alpha_1|\alpha_2) = -1 \). Fundamental weights \( \Lambda_1, \Lambda_2 \) are given by \( \Lambda_1 = 2\alpha_1 + 3\alpha_2, \Lambda_2 = \alpha_1 + 2\alpha_2 \). Let \( \epsilon_{-3} = \alpha_1 + 2\alpha_2, \epsilon_{-2} = \alpha_1 + \alpha_2, \epsilon_{-1} = \alpha_2, \epsilon_i = 0 \) and \( \epsilon_i = -\epsilon_{-i}, (i = 1, 2, 3) \). The corresponding weight vectors are denoted by \( v_i, i = 1, \cdots, 7 \). These 7 vectors form basis for the highest weight module \( V_{\Lambda_2} \) of \( G_2 \) with the highest weight vector \( \Lambda_2 \).

\[
[u] = \frac{(q^u-q^{-u})}{(q-q^{-1})} \quad \text{where} \quad q \neq \text{root of unity is the deformation parameter.}
\]

We first describe the most fundamental vertex model obtained by Ogievetsky (\( q = 1 \))[15] and Kuniba (\( q \) general)[16].

model 1. \( W_1 = W_2 = W_1^{(2)} = V_{\Lambda_2} \)

Let the above-mentioned 7 vectors be physical variables assigned to each bond. At each vertex, the conservation of weights is imposed: \( \epsilon_i + \epsilon_j = \epsilon_k + \epsilon_{\ell} \). This "ice rule" results 175 possible vertex configurations. Let \( P \) be a permutation operator: \( P(u \otimes v) = v \otimes u \). From Shur’s lemma, \( PR_{V_{\Lambda_2} \otimes V_{\Lambda_2}}(u) \) can be represented using the operators \( P_{2V_{\Lambda_2}}(PV_{\Lambda_1}, PV_{\Lambda_2}, PV_0) \) which project \( V_{\Lambda_2} \otimes V_{\Lambda_2} \) to \( V_{2\Lambda_2} \) \( (V_{\Lambda_1}, V_{\Lambda_2}, V_0) \), respectively,

\[
PR_{V_{\Lambda_2} \otimes V_{\Lambda_2}}(u) = \sum_{\Lambda=2\Lambda_2,\Lambda_1,\Lambda_2,0} \rho_{\Lambda}(u)P_{\Lambda} \quad (4)
\]

where

\[
[4][6]\rho_{\Lambda}(u) = \begin{cases} 
[1+u][4+u][6+u] & \Lambda = 2\Lambda_2 \\
[1-u][4+u][6+u] & \Lambda = \Lambda_1 \\
[1+u][4-u][6+u] & \Lambda = \Lambda_2 \\
[1-u][4+u][6-u] & \Lambda = 0.
\end{cases}
\]

Explicit expressions for projectors \( P_{\Lambda} \) can be found in ref[16]. Let us remark that RSOS counterpart is solved in [17].

From the above decomposition, one sees that \( PR_{V_{\Lambda_2} \otimes V_{\Lambda_2}}(u) \) possesses singular points at \( u = 1 \) etc. The Yangian space \( W_1^{(1)} \) is related to the singularity at \( u = 1 \). As the \( G_2 \) module, 15 dimensional space \( W_1^{(1)} \) decomposes as \( V_{\Lambda_1} \oplus V_0 \). The basis for the latter two modules in terms of \( v_i (i = -3, \cdots, 3) \) are obtained in ref.[16]. There, three vectors \( v_7^{(\Lambda_1)}, v_8^{(\Lambda_1)} \) and \( v_1^{(0)} \) having zero weight are constructed explicitly. In the module \( W_1^{(1)} \), these vectors are, after some modifications, constituents of a null space.
\( W^{(1)}_1 \) and \( W^{(2)}_1 \) deserve Yangian analogue of the spaces of fundamental representations\cite{18}. In the next section, we deal with the eigenvalue problems for transfer matrices of which the quantum and the auxiliary spaces are either of \( W^{(1)}_1 \) or \( W^{(2)}_1 \).

For convenience, we call,
model 2. \( W_1 = W^{(1)}_1, W_2 = W^{(2)}_1 \)
model 3. \( W_1 = W^{(2)}_1, W_2 = W^{(1)}_1 \)
model 4. \( W_1 = W_2 = W^{(1)}_1 \).

\section{Eigenvalues for transfer matrices}

Let us address the main problem in this report; what are the explicit forms for eigenvalues of row to row transfer matrices? Some years ago, Reshetikhin\cite{19} conjectured it for the model 1 through the analytic Bethe ansatz. To present his result, we prepare some notations. Let the vacuum state be such that all vertical edges assume \( \epsilon_{-3} \). By \( \phi_i(u) = \phi_i(u) \), we mean the eigenvalues, with respect to this vacuum state, of the transfer matrix of which the rightmost and the leftmost horizontal edges are assigned edge variable \( \epsilon_i \). Their explicit forms are given by,

\[
\phi_{-3}(u) = f(1 + u)f(4 + u)f(6 + u), \quad \phi_{-2}(u) = \phi_1(u) = f(u)f(4 + u)f(6 + u) \\
\phi_0(u) = f(u)f(3 + u)f(6 + u) \quad \phi_1(u) = \phi_2(u) = f(u)f(2 + u)f(6 + u) \\
\phi_3(u) = f(u)f(2 + u)f(5 + u) \\
f(u) = \prod_{j=1}^{N} [u - w_j], \quad (5)
\]

and \( w_j(j = 1, \ldots, N) \) are free parameters called inhomogeneity. Note that we adopt different notations from those in \cite{19}. Let us further introduce two functions,

\[
D^{(1)}(u) = \prod_{j=1}^{N_1} [u - iu_j^{(1)}] \quad D^{(2)}(u) = \prod_{j=1}^{N_2} [u - iu_j^{(2)}], \quad (6)
\]

where the parameters \( \{u_j^{(a)}\}(a = 1, 2 \quad j = 1, \ldots, N_1 \) or \( , N_2) \) are solutions to the Bethe ansatz equations,

\[
\frac{f(iu_j^{(a)} + \delta_{a,p} \frac{\epsilon_1}{t_p})}{f(iu_j^{(a)} - \delta_{a,p} \frac{\epsilon_1}{t_p})} = \prod_{b=1}^{2} \frac{D^{(b)}(iu_j^{(a)} + (\alpha_a|\alpha_b))}{D^{(b)}(iu_j^{(a)} - (\alpha_a|\alpha_b))} \quad (a = 1, 2). \quad (7)
\]

Here \( p, s \) specify the quantum space, now \( s = 1 \) and \( p = 2 \). The parameters \( t_1 \) and \( t_2 \) are related to the lengths of corresponding simple roots and \( t_1 = 1, t_2 = 3 \). The numbers of roots \( N_1, N_2 \) are chosen such that \( Ns\Lambda_1 - N_1\alpha_1 - N_2\alpha_2 \) is a non negative weight.
Finally we introduce the most significant object in this report, a Yangian analogue of the Young tableaux. We associate an expression to a box with a number in the following way:

\[
\begin{align*}
\boxed{3} & \leftrightarrow \phi_3(u) \frac{D^{(2)}(u - 1/2)}{D^{(2)}(u + 1/2)} \\
\boxed{-2} & \leftrightarrow \phi_{-2}(u) \frac{D^{(1)}(u - 1)D^{(2)}(u + 3/2)}{D^{(1)}(u + 2)D^{(2)}(u + 1/2)} \\
\boxed{-1} & \leftrightarrow \phi_{-1}(u) \frac{D^{(1)}(u + 5)D^{(2)}(u + 3/2)}{D^{(1)}(u + 2)D^{(2)}(u + 7/2)} \\
\boxed{0} & \leftrightarrow \phi_0(u) \frac{D^{(2)}(u + 9/2)D^{(2)}(u + 3/2)}{D^{(2)}(u + 5/2)D^{(2)}(u + 7/2)} \\
\boxed{1} & \leftrightarrow \phi_1(u) \frac{D^{(1)}(u + 1)D^{(2)}(u + 9/2)}{D^{(1)}(u + 4)D^{(2)}(u + 5/2)} \\
\boxed{2} & \leftrightarrow \phi_2(u) \frac{D^{(1)}(u + 7)D^{(2)}(u + 9/2)}{D^{(1)}(u + 4)D^{(2)}(u + 11/2)} \\
\boxed{3} & \leftrightarrow \phi_3(u) \frac{D^{(2)}(u + 13/2)}{D^{(2)}(u + 11/2)}.
\end{align*}
\]

(8)

The boxes in lhs are analogues to elements in \(V_{\Lambda_2}, \epsilon_i, (i = -3, \cdots, 3)\). We remark that these boxes carry spectral parameter dependencies through rhs.

These identifications stem from two reasons. First, we identify the box of \(i\) with the expression of which the vacuum expectation value is \(\phi_i(u)\). Second, we make an analogy between the actions of Chevalley generators \(E_{-\alpha_i}, i = 1, 2\) and the pole structure of expressions. For example, \(\epsilon_{-3}\) and \(\epsilon_{-2}\) are "connected" by the action of \(E_{-\alpha_2}\). On the other hand, we see that the corresponding expressions possess common poles at \(u = i\omega_{(j)}^{(2)} - 1/2, (j = 1, \cdots N_2)\) from eq.(8). The pole free condition for the sum of these two is given by BAE. Then we regard that these two expressions are "connected" by the pole free condition for \(\{u_{(a)}^{(j)}\}\).

In this way, we identify the "connection" by an action of \(E_{-\alpha_a}\) with that by the pole free condition for \(\{u_{(a)}^{(j)}\}\).

We further remark the "crossing symmetry" like property of above boxes. That is, we have a symmetry in box expressions,

\[
\boxed{i} = -\boxed{i} \quad \text{under} \quad u \rightarrow -6-u, \quad w_i \rightarrow -w_i.
\]

This might be a consequence from the crossing symmetry for vertex weights. At least, one can check that the configurations with the vacuum state in its quantum space satisfy this property.

Now the Reshetikhin’s result can be neatly described in terms of the box as,

\[
\tau_1^{(2)}(u) = \sum_{i=-3}^{3} \boxed{i}.
\]

(9)
Remark that we implicitly assume that algebra acts on the auxiliary space. Then the eigenvalues of two transfer matrices having identical auxiliary space but different quantum ones will have common combination of $D^{(u)}(u)$ functions, since pole structure should be same. Indeed, this is the case for $U_q(G_2)$ models[12].

In the following, we assume this pole free condition - $G_2$ action correspondence for general cases. The assumption is of great help in making conjecture for the eigenvalues of transfer matrices of model 2,3,4. We work out this program using weight space diagram for classical $G_2$ algebra together with BAE and the little information from explicit fusion procedure.

We first address the eigenvalue problem for the model 2.

Let us look at the 14 dimensional space $V_{\lambda_1}$ (Fig1). A left-down(right-down) arrow presents the action of $E_{-\alpha_1}$ ($E_{-\alpha_2}$). $W^{(1)}_1$ space has one extra vector having zero weight other than $v_7^{(\lambda_1)}$ and $v_8^{(\lambda_1)}$. And they are no longer identical to those in $V_{\lambda_1}$ as noted previously. Nevertheless, we can gain lots of insight from Fig(1) following the preceding hypothesis.

We succeed in making a conjecture for $\tau^{(1)}_1(u)$ using such arguments , together with the fact

$$\tau^{(2)}_1(u - 1/2)\tau^{(2)}_1(u + 1/2) = \tau^{(1)}_1(u) + \cdots.$$  \hspace{2cm} (10)

This is a natural consequence from the fusion procedure.

The result can be again neatly described in terms of boxes. For this purpose, we introduce a set of two boxes glued vertically. We assign the spectral parameter $u+1/2(u-1/2)$ to an upper (a lower) box . Each column corresponds to the product of two expressions with different spectral parameters. Then $\tau^{(1)}_1(u)$ is given by the sum of expressions corresponding to the following tableaux;

\[\begin{array}{cccccccccccc}
-3 & -3 & -3 & -3 & -3 & -3 & -2 & -2 & -2 \\
-2 & -1 & 0 & 1 & 2 & 3 & 1 & 2 & 3 \\
-1 & -1 & -1 & 0 & 1 & 2 \\
1 & 2 & 3 & 3 & 3 & 3
\end{array}\]  \hspace{2cm} (11)

(See Fig 2). Next we consider the model 3, 4. The model 3 possesses identical auxiliary space to the model 1. By the argument soon below eq.(9), the problem reduces to evaluate vacuum expectation values. They are explicitly calculable using a similar argument given for $C_r$ models [20]. We omit details of derivation and write only the modifications to $\phi_j(u)$ model 3

$$\phi_{-3}(u) = \phi_{-2}(u) = f(1 + u)f(5 + u)$$
$$\phi_{-1}(u) = \phi_0(u) = \phi_1(u) = f(1 + u)f(11 + u)$$
$$\phi_2(u) = \phi_3(u) = f(7 + u)f(11 + u).$$  \hspace{2cm} (12)
We verified that $\tau_1^{(2)}(u)$ is pole free by virtue of the BAE(7) with $p = 1$ and $s = 1$.
For the model 4, we can also evaluate the vacuum expectation values. Remarkably, the eigenvalue for model 4 transfer matrix obtained in this way agree with the expression from the same set of tableaux in eq(11), assuming $\phi_i(u)$ in eq.(12), in this turn. This is naturally expected from our arguments since algebra acts on the auxiliary space. And this coincidence supports validity of our hypothesis.
The expressions of $\tau$ for the model $1 \sim 4$ are checked by comparison with results of the brute force diagonalizations in $q \rightarrow 1$ limit for $N = 2$ and some cases $N = 3$.

§4 Solutions to the $T$-system
Before closing this report, let us consider the $T$-system problem. The $T$-system is a set of functional relations among transfer matrices having common quantum spaces, however, different auxiliary spaces. For the $G_2$ model in the present normalization of the spectral parameter, it reads[11]

$$
\begin{align*}
T_m^{(1)}(u - 3/2)T_m^{(1)}(u + 3/2) &= T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + g_m^{(1)}(u)T_{3m}^{(2)}(u) \\
T_{3m}^{(2)}(u - 1/2)T_{3m}^{(2)}(u + 1/2) &= T_{3m+1}^{(2)}(u)T_{3m-1}^{(2)}(u) + T_m^{(1)}(u - 1/3)T_m^{(1)}(u)T_m^{(1)}(u + 1/3) \\
T_{3m+1}^{(2)}(u - 1/2)T_{3m+1}^{(2)}(u + 1/2) &= T_{3m+2}^{(2)}(u)T_{3m}^{(2)}(u) + T_m^{(1)}(u - 1/2)T_m^{(1)}(u + 1/2)T_m^{(1)}(u) \\
T_{3m+2}^{(2)}(u - 1/2)T_{3m+2}^{(2)}(u + 1/2) &= T_{3m+3}^{(2)}(u)T_{3m+1}^{(2)}(u) + T_m^{(1)}(u)T_m^{(1)}(u - 1/2)T_m^{(1)}(u + 1/2)
\end{align*}
$$

(13)

where $g_m^{(1)}(u) = \prod_{j=1}^{m} g_1^{(1)}(u + \frac{3(m+1)}{2} - 3j)$ and

$$
g_1^{(1)}(u) = \begin{cases} f(-2 + u)f(3 + u)f(8 + u) & \text{for } p = 2, s = 1 \\
 f(-3/2 + u)f(15/2 + u) & \text{for } p = 1, s = 1. \end{cases}
$$

And the same relations hold replacing $T$ by $\tau$ since $T'$s constitute a commuting family.
We can, in principle, obtain the explicit expression for any $\tau_m^{(a)}(u)$ since now we have expressions for $a = 1, 2, m = 1$.
In the limit $u \rightarrow \infty$, we expect that $T_m^{(a)}(u)$ converges to $Q_m^{(a)}$ after appropriate renormalization. The latter gives the dimension of Yangian space $W_m^{(a)}$ by specialization. This means that the number of terms in $T_m^{(a)}(u)$ should agree with the dimension of $W_m^{(a)}$. The latter is known for small $m$. Thus this deserves as a check. By successive solving eq.(13), we obtain $T_1^{(1)}(u), T_1^{(2)}(u)(m = 1, 2)$, consisting of 34,133,92 terms respectively, which agree with $Q_m^{(a)}$[21].
In particular, the expression for $Q_m^{(1)}$ is conjectured in a beautiful form [21],

$$Q_m^{(1)} = \sum_{j=0}^{m} \chi(j, \Lambda_1)$$  \hspace{1cm} (14)

where $\chi(\Lambda)$ is a classical $G_2$ character for the highest weight module $V_{\Lambda}$.

Thus it might be tempting to find a "u-" version of this. Through studies on the $T$-system, we find a conjecture for the eigenvalue of transfer matrix $T_m^{(1)}(u)$. To express this, we prepare a set of tableaux $BW_m^{(1)}$.

Definition

Let $b(m, \{u\}, \{d\})$ be a table consisting of $2 \times m$ boxes and whose upper (lower) row arguments are $\{u_1, u_2, \cdots, u_m\}, \{\{d_1, d_2, \cdots, d_m\}\}$ where $-3 \leq u_i, d_i \leq 3$.

Then $b(m, \{u\}, \{d\})$ is a member of $BW_m^{(1)}$ if it satisfies

1. Each column is a member of tableaux in eq.(11).
2. Each adjacent columns should satisfy the condition,
   2.a $u_k \leq u_{k+1}, d_k \leq d_{k+1}$ (k = 1, \cdots, m - 1).
   2.b if $u_k = u_{k+1} = -3$ then either of $d_k$ or $d_{k+1}$ must be less than 0.
   2.c if $d_k = d_{k+1} = 3$ then either of $d_k$ or $d_{k+1}$ must be greater than 0.

From $b(m, \{u\}, \{d\}) \in BW_m^{(1)}$, we construct an expression $b(m, \{u\}, \{d\},u)$ in the following way,

1. For a box with figure $u_k, (d_k)$ we assign the spectral parameter \(u - \frac{3(m+1)}{2} + 3k - \frac{1}{2}\).
2. Take the all products of the corresponding expressions according to expressions in (boxes) and with the above mentioned spectral parameters.

Then we have,

Conjecture.

The eigenvalue $\tau_m^{(1)}(u)$ of $T_m^{(1)}(u)$ is given by

$$\tau_m^{(1)}(u) = \sum b(m, \{u\}, \{d\},u)$$  \hspace{1cm} (15)

where the summation should be taken over all $b(m, \{u\}, \{d\}) \in BW_m^{(1)}$.

We have the following supports to this conjecture.

1. We have verified that the number of elements in $BW_m^{(1)}$ agree with the Yangian dimension up to $m = 10$ (dimension = 73788).
2. The expression for $T_2^{(1)}(u)$ is analytically verified to obey the above rule for $p = 1, 2$ and $s = 1$. 
3 If \( \tau^{(1)}_m(u) \) defined by eq(15) is really the solution to the \( T \)-system(13), then this equality must hold good irrespective of forms for \( f(u), D^{(1)}(u) \) and \( D^{(2)}(u) \). We assume 3rd order polynomials in \( u \) for \( f(u), D^{(1)}(u), D^{(2)}(u) \), and choose their coefficients from random numbers. Then we have checked the agreement between the numerical solution for the \( T \)-system and the numerical result from the above rules up to \( m = 6 \).

§5 Conclusion
In this paper, we have reported studies on fusion \( U_q(G_2) \) vertex models. Under few assumptions, we have found analytic expressions for the transfer matrices’ eigenvalues of models related to \( W^{(1)}_1 \) or \( W^{(2)}_1 \). It has been shown how the result can be neatly expressed in terms of the tableaux. A conjecture for \( \tau^{(1)}_m(u) \) is given and several supports for this are presented. We have seen that the rules for tableaux are quite simple.
These facts encourage us to extend similar analyses on models based on other Lie algebras. Such program is now under progress and partial results are promising[22].
Contrary to the success for \( a = 1 \) case, we still have not yet determined general expressions for \( \tau^{(2)}_m(u) \) \( (m = 4, \cdots \infty) \). This is partly due to the lack of information even for \( Q^{(2)}_m \)[21].
We hope to report the further studies on this in near future.

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Figure Captions

Figure 1. 14 dimensional space for $V_{\Lambda_1}$. The numbers associated with boxes are assigned according to the indices of vectors $\in V_{\Lambda_1}$ given in ref(16).

Figure 2. 15 dimensional space for $W^{(1)}_1$. Boxes with $1 \leq i \leq 8$ denote $v_{i}^{\Lambda_1}$ in [16]. Those with $9 \leq i \leq 14$ represent $v_{i-8}^{\Lambda_1}$ by replacing $v_{\mu}$ by $v_{-\mu}$.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9405201v1