LIE PROPERTIES IN ASSOCIATIVE ALGEBRAS

Szilvia Homolya, Jenő Szigeti, Leon van Wyk, and Michal Ziembowski

Abstract. Let $K$ be a field, then we exhibit two matrices in the full $n \times n$ matrix algebra $M_n(K)$ which generate $M_n(K)$ as a Lie $K$-algebra with the commutator Lie product. We also study Lie centralizers of a not necessarily commutative unitary algebra and obtain results which we hope will eventually be a step in the direction of, firstly, proving that a Lie-nilpotent $K$-subspace (or a sub Lie $K$-algebra) of a finite-dimensional associative algebra over $K$ of index $k$ (say) generates a Lie-nilpotent associative subalgebra of much higher nilpotency index, and secondly, in the light of the sharp upper bound for the maximum ($K$-)dimension of a Lie-nilpotent $K$-subalgebra of $M_n(K)$ of index $k$ obtained in [13], finding an upper bound for the maximum dimension of a Lie-nilpotent (of index $k$) sub Lie $K$-algebra of $M_n(K)$. Finally, the constructive elementary proof of the Skolem-Noether theorem for the matrix algebra $M_n(K)$ in [14], in conjunction with the well-known characterization of Lie automorphisms of $M_n(K)$ (if the characteristic of $K$ is different from 2 and 3) in terms of, amongst others, automorphisms and anti-automorphisms of $M_n(K)$, leads us to a unifying approach to constructively describe automorphisms and anti-automorphisms of $M_n(K)$.

1. Introduction and Motivation

Throughout the paper an algebra $R$ means a not necessarily commutative unitary algebra over a field $K$ (in most of the results $K$ can be replaced by a commutative unitary ring satisfying certain mild extra conditions). The centralizer of an element $a \in R$ is denoted by $\text{Cen}(a) = \{ r \in R : ra = ar \}$, and the centre of $R$ by $Z(R) = \{ r \in R : rs = sr \text{ for all } s \in R \}$. Clearly, $Z(R) \subseteq \text{Cen}(a)$ are $K$-subalgebras of $R$.

We start with the following simple observation.

1.1. Proposition. If the elements $a_1, a_2, \ldots, a_t \in R$ generate $R$ as an associative algebra, then the intersection of their centralizers is trivial, i.e.,

$$\text{Cen}(a_1) \cap \text{Cen}(a_2) \cap \cdots \cap \text{Cen}(a_t) = Z(R).$$

2020 Mathematics Subject Classification. Primary 16S50, 16U70, 16W20, Secondary 16U80, 17B40.

Key words and phrases. Lie algebra, generator, centralizer, matrix algebra, automorphism, symplectic involution.
The full \( n \times n \) matrix algebra over \( K \) is denoted by \( M_n(K) \). The standard matrix unit in \( M_n(K) \) with 1 in the \((i,j)\) position and zeros in all other positions is denoted by \( E_{i,j} \), and \( I_n \) denotes the \( n \times n \) identity matrix.

The fact that \( M_n(K) \) can be generated as a \( K \)-algebra by the two matrices \( E_{n,1} \) and
\[
S := E_{1,2} + E_{2,3} + \cdots + E_{n-1,n},
\]
i.e.,
\[
(1.1) \quad M_n(K) = \langle E_{1,1}, S \rangle_K,
\]
played a prominent role in [14], in which a constructive elementary proof of the Skolem-Noether theorem (see, e.g., [8, 9] and [12]) for the matrix algebra \( M_n(K) \), \( K \) any field, was given. To be precise, given a \( K \)-automorphism \( \varphi \) of \( M_n(K) \), an invertible matrix \( A \in M_n(K) \) yielding the conjugation
\[
\varphi(X) = AXA^{-1}
\]
for all \( X \in M_n(K) \), was constructed from only the two \( \varphi \)-images, \( \varphi(E_{n,1}) \) and \( \varphi(S) \), of the matrices \( E_{n,1} \) and \( S \), respectively, and a nonzero vector \( \mathbf{a} \) in the kernel of the matrix \( I_n - (\varphi(S))^{n-1}\varphi(E_{n,1}) \in M_n(K) \), as follows:
\[
(1.2) \quad A = \left[ (\varphi(S))^{n-1}\varphi(E_{n,1})\mathbf{a} \mid (\varphi(S))^{n-2}\varphi(E_{n,1})\mathbf{a} \mid \cdots \mathbf{a} \mid (\varphi(E_{n,1})\mathbf{a} \mid \varphi(E_{n,1})\mathbf{a} \right].
\]

A Lie automorphism \( \psi \) of a \( K \)-algebra \( R \) is a one-to-one \( K \)-linear map from \( R \) onto itself which preserves the commutator Lie product (also called the Lie bracket in the literature), i.e.,
\[
\psi([x,y]) = [\psi(x), \psi(y)],
\]
equivalently,
\[
\psi(xy - yx) = \psi(x)\psi(y) - \psi(y)\psi(x),
\]
for all \( x, y \in R \). We note that \( M_n(K) \) with the commutator Lie product plays an exceptional role in the theory of finite dimensional Lie algebras. The fundamental Ado-Iwasawa theorem (see [5]) asserts that every finite-dimensional Lie \( K \)-algebra can be embedded into \( M_n(K) \) for some \( n \geq 1 \).

If \( K \) is any field of characteristic different from 2 and 3, then (see, e.g., [4, 6], [7] and [8]) every Lie automorphism \( \psi \) of \( M_n(K) \) can be presented as a sum
\[
(1.3) \quad \psi = \sigma + \tau,
\]
where \( \sigma \) is either an automorphism of \( M_n(K) \) (as a \( K \)-algebra) or the negative of an anti-automorphism of \( M_n(K) \), and \( \tau \) is an additive mapping from \( M_n(K) \) to \( K \) which maps commutators into zero. In the light of this significant result we apply (1.2) in Section 4, where we present a unifying approach to constructively describe automorphisms and anti-automorphisms of \( M_n(K) \).

First we show in Section 2, in a vein similar to [14], that the matrix \( E_{1,1} \) and the permutation matrix \( P = S + E_{n,1} \) generate \( M_n(K) \) as a Lie \( K \)-algebra.

In Section 3 we study Lie centralizers in a (not necessarily commutative) unitary algebra \( R \). We obtain results which we hope will eventually pave the way towards, firstly, proving that a Lie-nilpotent \( K \)-subspace (or a sub Lie \( K \)-algebra) of a finite-dimensional associative algebra over \( K \) of index \( k \) (say) generates a Lie-nilpotent
associative subalgebra (of much higher nilpotency index), and secondly, finding an upper bound (perhaps even a sharp upper bound) for the maximum dimension of a Lie-nilpotent (of index \( k \)) sub Lie \( K \)-algebra of \( M_n(K) \) (see Conjecture 3.8). In this context the sharp upper bound for the maximum dimension of a Lie-nilpotent \( K \)-subalgebra of \( M_n(K) \) of index \( k \geq 1 \) is important (see [13]).

2. Two matrices generating \( M_n(K) \) as a Lie algebra

We shall make use of the well known multiplication rule of standard matrix units:

\[
E_{i,j}E_{k,l} = \begin{cases} 
E_{i,l} & \text{if } j = k; \\
0 & \text{if } j \neq k.
\end{cases}
\]

The permutation matrix \( P \in M_n(K) \) is defined as follows:

\[
P = E_{1,2} + E_{2,3} + \cdots + E_{n-1,n} + E_{n,1}.
\]

We now show that \( M_n(K) \) can be generated as a Lie \( K \)-algebra by two matrices.

2.1. Theorem. The matrices \( P \) and \( E_{1,1} \) generate \( M_n(K) \) as a Lie algebra with the commutator Lie product.

**Proof.** Let \( \mathcal{G} = \langle P, E_{1,1} \rangle_{\text{Lie}} \) denote the Lie subalgebra of \( M_n(K) \) generated by the matrices \( P \) and \( E_{1,1} \). Clearly,

\[
E_{1,2} - E_{n,1} = E_{1,1}P - PE_{1,1} = [E_{1,1}, P] \in \mathcal{G}
\]

and

\[
E_{1,2} + E_{n,1} = E_{1,1}(E_{1,2} - E_{n,1}) - (E_{1,2} - E_{n,1})E_{1,1} = [E_{1,1}, E_{1,2} - E_{n,1}] \in \mathcal{G}
\]

ensure that

\[
E_{1,2} = \frac{1}{2} (E_{1,2} + E_{n,1}) \in \mathcal{G}
\]

and

\[
E_{n,1} = \frac{1}{2} (E_{1,2} + E_{n,1}) \in \mathcal{G}.
\]

Starting from \( E_{1,2} \in \mathcal{G} \), assume that \( E_{i,j} \in \mathcal{G} \) for some \( 2 \leq j \leq n - 1 \). Using

\[
S = E_{1,2} + E_{2,3} + \cdots + E_{n-1,n} = P - E_{n,1} \in \mathcal{G},
\]

we obtain that \( E_{i,j+1} = [E_{i,j}, S] \in \mathcal{G} \). Therefore, it follows that

\[
E_{i,1}, E_{1,2}, E_{1,3}, \ldots, E_{i,n} \in \mathcal{G}.
\]

Next, starting from \( E_{n,1} \in \mathcal{G} \), assume that \( E_{i,1} \in \mathcal{G} \) for some \( 3 \leq i \leq n \). Now

\[
E_{i-1,1} - E_{i,2} = SE_{i,1} - E_{i,1}S = [S, E_{i,1}] \in \mathcal{G}
\]

and

\[
E_{i,2} = E_{i,1}E_{1,2} - E_{1,2}E_{i,1} = [E_{i,1}, E_{1,2}] \in \mathcal{G}
\]

give that

\[
E_{i-1,1} = (E_{i-1,1} - E_{i,2}) + E_{i,2} \in \mathcal{G}.
\]

Consequently, we have that

\[
E_{n,1}, E_{n-1,1}, \ldots, E_{2,1}, E_{1,1} \in \mathcal{G}.
\]

Finally, if \( i \neq j \), then

\[
E_{i,j} = E_{i,1}E_{1,j} - E_{1,j}E_{i,1} = [E_{i,1}, E_{1,j}] \in \mathcal{G},
\]

Therefore, \( \mathcal{G} \) is the entire Lie algebra \( M_n(K) \).
and if \( i = j \), then
\[
E_{i,i} = E_{1,1} + (E_{i,1}E_{1,i} - E_{1,i}E_{i,1}) = E_{1,1} + [E_{i,1}, E_{1,i}] \in \mathcal{G}.
\]

Thus we have that \( E_{i,j} \in \mathcal{G} \) for all \( 1 \leq i, j \leq n \), whence we conclude that \( \mathcal{G} = M_n(K) \).

\[ \square \]

2.2. Remark. We note that in the above theorem \( K \) can be a commutative unitary ring such that \( \frac{1}{2} \in K \). An other observation is that the Lie generation of \( M_n(K) \) is much stronger than the associative generation. Indeed, \( E_{1,1} \) implies that \( S \) and \( E_{2,1} \) also generate \( M_n(K) \) as an associative \( K \)-algebra. Since \( S \) and \( E_{2,1} \) have zero traces, it follows that all matrices in \( \langle S, E_{2,1} \rangle_{\text{Lie}} \) have zero traces and \( \langle S, E_{2,1} \rangle_{\text{Lie}} \neq M_n(K) \).

3. The Lie centralizer

For a sequence \( x_1, x_2, \ldots, x_m \) of elements in a not necessarily commutative unitary algebra \( R \) over a field (or commutative ring) \( K \) with unity we use the notation
\[
[x_1, x_2, \ldots, x_m]_m = [\ldots [x_1, x_2], x_3], \ldots, x_m].
\]
The \( k \)-th Lie centralizer of a subset \( H \subseteq R \) is
\[
L_k(H) = \{ r \in R : [r, x_1, \ldots, x_k]_{k+1} = 0 \text{ for all } x_i \in H, 1 \leq i \leq k \},
\]
a \( K \)-subspace (submodule) of \( R \).

As a consequence of \([rs, x_1] = [r, sx_1] + [s, x_1r] \), we can see that the containment
\[
\{shr : s, r \in R \text{ and } h \in H\} \subseteq H
\]
implies that \( L_k(H) \) is a (unitary) \( K \)-subalgebra of \( R \). Clearly,
\[
\bigcap_{h \in H} \text{Cen}(h) = L_1(H) \subseteq L_2(H) \subseteq \cdots \subseteq L_k(H) \subseteq L_{k+1}(H) \subseteq \cdots
\]
follows from
\[
[r, x_1, \ldots, x_k, x_{k+1}]_{k+2} = [[r, x_1, \ldots, x_k]_{k+1}, x_{k+1}].
\]
The \( \omega \)-Lie centralizer of \( H \subseteq R \) is defined as
\[
L_\omega(H) = \bigcup_{k=1}^{\infty} L_k(H).
\]
A subset \( H \subseteq R \) is called Lie-nilpotent of index \( k \geq 1 \) if \( H \subseteq L_k(H) \). A natural further step is the following: \( H \) is called \( \omega \)-Lie-nilpotent (or almost Lie-nilpotent) if \( H \subseteq L_\omega(H) \).

3.1. Proposition. If \( r \in L_k(H) \) and \( 1 \leq j \leq k \), then
\[
[x_1, \ldots, x_j, r, x_{j+1}, \ldots, x_k]_{k+1} = 0
\]
for all \( x_i \in H, 1 \leq i \leq k \).
PROOF. It is a well known consequence of the Jacobian identity that in any Lie ring, $[x_1, \ldots, x_j, r]_{j+1}$ can be written as a sum of $2^{j-1}$ terms of the form
\[ \pm[r, x_{\pi(1)}, \ldots, x_{\pi(j)}]_{j+1}, \]
where $\pi$ is some permutation of \{1, 2, \ldots, j\}. We note that an easy induction on $j$ works. It follows that $[x_1, \ldots, x_j, r, x_{j+1}, \ldots, x_k]_{k+1}$ can be written as a sum of $2^{j-1}$ terms of the form
\[ \pm[r, x_{\pi(1)}, \ldots, x_{\pi(j)}, x_{j+1}, \ldots, x_k]_{k+1}, \]
whence $[x_1, \ldots, x_j, r, x_{j+1}, \ldots, x_k]_{k+1} = 0$ follows. \[ \square \]

3.2. Proposition. If $L_k(H) = L_{k+1}(H)$, then $L_{k+1}(H) = L_{k+2}(H)$.

PROOF. For the elements $x_1 \in H$ and $r_2 \in L_{k+2}(H)$ we have
\[ [[r, x_1], x_2, \ldots, x_{k+2}]_{k+3} = [r, x_1, \ldots, x_{k+2}]_{k+3} = 0 \]
for all $x_i \in H$, $2 \leq i \leq k+2$. Thus we obtain that $[r, x_1] \in L_{k+1}(H)$ for all $x_1 \in H$, whence $[r, x_1] \in L_k(H)$ and
\[ [r, x_1, \ldots, x_{k+1}]_{k+2} = [[r, x_1], x_2, \ldots, x_k, x_{k+1}]_{k+1} = 0 \]
follow for all $x_i \in H$, $1 \leq i \leq k+1$. In view of the above argument, $r \in L_{k+1}(H)$ and $L_{k+2}(H) = L_{k+1}(H)$ can be derived. \[ \square \]

3.3. Proposition. Let $R$ be a finite-dimensional algebra over a field $K$ with $\dim K(R) = d$. Then for any subset $H \subseteq R$ we have $L_\omega(H) = L_d(H)$.

PROOF. The finite-dimensionality of $R$ implies that
\[ \{0\} \subseteq L_1(H) \subseteq L_2(H) \subseteq \cdots \subseteq L_k(H) \subseteq L_{k+1}(H) \subseteq \cdots \]
cannot be a strictly ascending infinite chain of $K$-subspaces. In view of Proposition 3.2, the shape of the above chain is
\[ \{0\} \subseteq L_1(H) \subseteq L_2(H) \subseteq \cdots \subseteq L_t(H) = L_{t+1}(H) = L_{t+2}(H) = \cdots \]
for some $t \geq 1$ (notice that $1_R \in L_1(H)$). Now
\[ t \leq \dim K(L_t(H)) \leq \dim K(R) = d \]
and $L_\omega(H) = L_d(H)$ follows. \[ \square \]

3.4. Corollary. Let $R$ be a finite-dimensional algebra over a field $K$ with $\dim K(R) = d$. If $H \subseteq R$ is $\omega$-Lie-nilpotent (almost Lie-nilpotent), then $H$ is Lie-nilpotent of index $d$.

3.5. Theorem. For any subset $H \subseteq R$, we have $L_p(H)L_q(H) \subseteq L_{p+q-1}(H)$ for all $p, q \geq 1$, and $L_\omega(H)$ is a $K$-subalgebra of $R$. 

with 0 ≤ 6

\[ \sum_{1 \leq i_1 < i_2 < \cdots < i_t \leq k} [r, x_{i_1}, \ldots, x_{i_t}]_{t+1} \cdot [s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1}, \quad * (k) \]

where the sum is taken over all strictly increasing sequences

1 ≤ i_1 < i_2 < \cdots < i_t ≤ k and 1 ≤ j_1 < j_2 < \cdots < j_{k-t} ≤ k,

with 0 ≤ t ≤ k and

\[ \{j_1, j_2, \ldots, j_{k-t}\} = \{1, 2, \ldots, k\} \setminus \{i_1, i_2, \ldots, i_t\}. \]

In the above the empty and the full sequences are allowed with \([r, \emptyset]_{0+1} = r\) and \([s, \emptyset]_{0+1} = s\).

If k = 1, then

\[ [r, s, x_1]_2 = [r, s, x_1]_1 = [r, [s, x_1]_1] = [r, s, x_1]_1 + [r, x_1]_1 = [r, s, x_1]_1 + [r, x_1]_1 \cdot [s, \emptyset]_1 \]

is well known.

Assume that \(* (k)\) holds for some k ≥ 1. We use

\[ [a, b, x_{k+1}] = a [b, x_{k+1}] + [a, x_{k+1}] b \]

repeatedly in the following calculations:

\[ [r, s, x_1, \ldots, x_k, x_{k+1}]_{k+2} = [[r, s, x_1, \ldots, x_k]]_{k+1}, x_{k+1} \]

\[ = \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_t \leq k} [r, x_{i_1}, \ldots, x_{i_t}]_{t+1} \cdot [s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1}, x_{k+1} \right) \]

\[ = \sum_{1 \leq i_1 < i_2 < \cdots < i_t \leq k} [r, x_{i_1}, \ldots, x_{i_t}]_{t+1} \cdot [s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1}, x_{k+1} \]

\[ + \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_t \leq k} [r, x_{i_1}, \ldots, x_{i_t}]_{t+1} \cdot [s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1}, x_{k+1} \right) \]

\[ = \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_t \leq k} [r, x_{i_1}, \ldots, x_{i_t}]_{t+1} \cdot [s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1}, x_{k+1} \right) \]

\[ + \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_t \leq k} [r, x_{i_1}, \ldots, x_{i_t}]_{t+1} \cdot [s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1}, x_{k+1} \right) \]

\[ + \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_t \leq k} [r, x_{i_1}, \ldots, x_{i_t}]_{t+1} \cdot [s, x_{j_1}, \ldots, x_{j_{k-t}}]_{k-t+1}, x_{k+1} \right) \]
3.6. Remark. A property $\mathcal{P}$ which is defined for any finite sequence $x_1, \ldots, x_m$ of elements in $R$ is called hereditary if $\mathcal{P}$ holds for any subsequence $x_{i_1}, \ldots, x_{i_t}$ with $1 \leq i_1 < i_2 < \cdots < i_t \leq m$. Two typical examples are $\mathcal{D}$ and $\mathcal{L}$. For a sequence $x_1, \ldots, x_m \in R$ the meaning of $\mathcal{D}$ is that the elements $x_1, \ldots, x_m$ are distinct and the meaning of $\mathcal{L}$ is that the elements $x_1, \ldots, x_m$ are linearly independent over the base field $K$.

The $k$-th Lie centralizer of a subset $H \subseteq R$ with respect to the property $\mathcal{P}$ is $\mathcal{L}_k^\mathcal{P}(H) = \{ r \in R \mid [r, x_1, \ldots, x_k]_{k+1} = 0 \text{ for all } x_1, \ldots, x_k \in H \text{ having property } \mathcal{P} \}$.

Using the same calculations as in the above proof, the following interesting (and probably far reaching) generalization of Theorem 3.5 can be obtained: If $\mathcal{P}$ is a hereditary property, then for any subset $H \subseteq R$, we have $\mathcal{L}_k^\mathcal{P}(H) \subseteq \mathcal{L}_k^\mathcal{P}(H)$ (and the union $\mathcal{L}_\infty^\mathcal{P}(H) = \bigcup_{k=1}^\infty \mathcal{L}_k(H)$ is a $K$-subalgebra of $R$).

3.7. Remark. Unfortunately we were not able to prove the following:

Let $R$ be a finite dimensional algebra over a field $K$ with $\dim_K(R) = d$. If $V \subseteq R$ is a Lie-nilpotent $K$-subspace (or a sub Lie $K$-algebra) of index $k \geq 1$, then the associative $K$-subalgebra $\langle V \rangle_K$ of $R$ generated by $V$ is Lie-nilpotent of index $f(k, d)$. The main result in [13] states that if $K$ is any field and $R$ is any Lie-nilpotent $K$-subalgebra of $M_n(K)$ of index $k \geq 1$, then

$$\dim_K(R) \leq g(k + 1, n),$$
where \( g(k+1,n) \) is the maximum of

\[
\frac{1}{2} \left( n^2 - \sum_{i=1}^{k+1} n_i^2 \right) + 1,
\]

subject to the constraint \( \sum_{i=1}^{k+1} n_i = n \), with \( n_1, n_2, \ldots, n_{k+1} \) non-negative integers.

To be precise:

**Theorem.** (see [13]) If \( R \) is a Lie-nilpotent \( K \)-subalgebra of \( M_n(K) \) of index \( k \geq 1 \), with (according to the Division Algorithm)

\[
n = (k+1) \left\lfloor \frac{n}{k+1} \right\rfloor + r, \quad 0 \leq r < k+1,
\]

then

\[
\frac{1}{2} \left( n^2 - (k+1-r) \left\lfloor \frac{n}{k+1} \right\rfloor^2 - r \left( \left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right)^2 \right) + 1
\]

is a sharp upper bound for \( \dim_K(R) \).

Using the above Theorem and the statement formulated in Remark 3.7 would allow to get an upper bound for the maximum dimension of a Lie nilpotent (of index \( k \)) sub Lie \( K \)-algebra of the full matrix algebra \( M_n(K) \). We had hoped that the foregoing results would lead to a proof of the following conjecture, but unfortunately we fell short:

**3.8. Conjecture:** If \( L \subseteq M_n(K) \) is an \( \omega \)-Lie-nilpotent sub Lie \( K \)-algebra, then

\[
\dim_K(L) \leq 1 + \frac{1}{2}(n^2 - n)
\]

### 4. A unifying approach to constructively describe automorphisms and anti-automorphisms of matrix algebras

The importance of automorphisms and anti-automorphisms of a matrix ring \( M_n(K) \) over a field \( K \) is evident. We apply (1.2) in this section by presenting a unifying approach to constructively describe automorphisms and anti-automorphisms of \( M_n(K) \).

In particular, first consider the following setting: for an automorphism \( f \) of a field \( K \), i.e., for \( f \in \text{Aut}(K) \), and for any \( X \in M_n(K) \), let \( X_f \) denote the matrix obtained from \( X \) by applying \( f \) entrywise, i.e., \( X_f = [x_{i,j}]_f = [f(x_{i,j})] \), and let \( B \) be any invertible matrix in \( M_n(K) \). Then the function \( \beta : M_n(K) \to M_n(K) \), defined by

\[
\beta(X) = BX_fB^{-1}
\]

for all \( X \in M_n(K) \), is a ring automorphism of \( M_n(K) \), but it need not be a \( K \)-automorphism of \( M_n(K) \). In fact, it is easily verified that \( \beta \) is a \( K \)-automorphism of \( M_n(K) \) if and only if \( f \) is the identity automorphism of \( K \). Nevertheless, we obtain the following constructive description in the above vein (see also [11] Corollary 1.2):

**4.1. Proposition.** Let \( f \in \text{Aut}(K) \) (\( K \) any field), let \( B \) be any invertible matrix in \( M_n(K) \), and let \( \beta : M_n(K) \to M_n(K) \) be the function defined by

\[
\beta(X) = BX_fB^{-1}
\]
for all $X \in M_n(K)$. Then
\[ \beta(X) = \overline{BX_fB}^{-1} \]
for all $X \in M_n(K)$, where $\overline{B} \in M_n(K)$ is the invertible matrix
\[ \overline{B} = \left[ (\beta(S))^{n-1}\beta(E_{n,1})b \mid (\beta(S))^{n-2}\beta(E_{n,1})b \mid \cdots \mid \beta(S)\beta(E_{n,1})b \mid \beta(E_{n,1})b \right], \]
and $b$ is a nonzero vector in the kernel of $I_n - (\beta(S))^{n-1}\beta(E_{n,1}) \in M_n(K)$.

**Proof.** Since
\[ \beta(X_{f^{-1}}) = BX_{f^{-1}}B^{-1} \]
for all $X \in M_n(K)$, it follows that $\alpha : M_n(K) \to M_n(K)$, defined by
\[ \alpha(X) = \beta(X_{f^{-1}}) \]
for all $X \in M_n(K)$, is indeed a $K$-automorphism of $M_n(K)$. Hence, by (1.2), we can constructively find an invertible matrix $\overline{B}$ (say) in $M_n(K)$ such that
\[ \alpha(X) = \overline{BX_{f^{-1}}B^{-1}} \]
for all $X \in M_n(K)$, where $\overline{B} \in M_n(K)$ is the invertible matrix
\[ \overline{B} = \left[ (\alpha(S))^{n-1}\alpha(E_{n,1})b \mid (\alpha(S))^{n-2}\alpha(E_{n,1})b \mid \cdots \mid \alpha(S)\alpha(E_{n,1})b \mid \alpha(E_{n,1})b \right], \]
with $b$ a nonzero vector in the kernel of the matrix $I_n - (\alpha(S))^{n-1}\alpha(E_{n,1}) \in M_n(K)$. Since $\alpha(S) = \beta(S_{f^{-1}})$ and $\alpha(E_{n,1}) = \beta((E_{n,1})_{f^{-1}})$, and since every entry of $S$ and $E_{n,1}$ is 0 or 1, with $f \in \text{Aut}(K)$, we have that $\alpha(S) = \beta(S)$ and $\alpha(E_{n,1}) = \beta(E_{n,1})$. Therefore,
\[ \beta(X) = \beta((X_f)_{f^{-1}}) = \alpha(X_f) = \overline{BX_f}\overline{B}^{-1} \]
for all $X \in M_n(K)$, where
\[ \overline{B} = \left[ (\beta(S))^{n-1}\beta(E_{n,1})b \mid (\beta(S))^{n-2}\beta(E_{n,1})b \mid \cdots \mid \beta(S)\beta(E_{n,1})b \mid \beta(E_{n,1})b \right], \]
with $b$ a nonzero vector in the kernel of $I_n - (\beta(S))^{n-1}\beta(E_{n,1}) \in M_n(K)$.

For our purposes we state explicitly a result from [11], using our notation:

**4.2. Corollary.** (11 Corollary 1.2)] Let $K$ be an arbitrary field, and let $\phi : M_n(K) \to M_n(K)$ be a bijective additive function satisfying $\phi(XY) = \phi(X)\phi(Y)$ for all $X, Y \in M_n(K)$. Then there exists an automorphism $f$ of the field $K$ and an invertible matrix $A \in M_n(K)$ such that
\[ \phi(X) = AX_fA^{-1} \]
for all $X \in M_n(K)$.

Next, combining Proposition 4.1 and Corollary 4.2, and denoting the transpose of a matrix $X \in M_n(K)$ by $X^\top$, we also obtain the following constructive and explicit description of an invertible matrix yielding any anti-automorphism of $M_n(K)$. In this regard it is noteworthy that, just as $S$ and $E_{n,1}$ generate $M_n(K)$ as a $K$-algebra, so do their transposes $S^\top$ and $E_{1,n}$, respectively.
4.3. Theorem. If $\phi$ is a ring anti-automorphism of $M_n(K)$, then

$$\phi(X) = \overline{A}X^\top \overline{A}^{-1}$$

for all $X \in M_n(K)$, where $\overline{A} \in M_n(K)$ is the invertible matrix

$$\overline{A} = \left[ (\phi(S^\top))^n \phi(E_{1,n})a | (\phi(S^\top))^{n-2} \phi(E_{1,n})a | \cdots | \phi(S^\top) \phi(E_{1,n})a | \phi(E_{1,n})a \right],$$

with $a$ a nonzero vector in the kernel of $I_n - (\phi(S^\top))^{n-1} \phi(E_{1,n}) \in M_n(K)$.

**Proof.** Let $T$ denote the transposition map $X \mapsto X^\top$ on $M_n(K)$. Since $T$ is also a ring anti-automorphism of $M_n(K)$, the composition $\phi \circ T$ is a ring automorphism of $M_n(K)$, and so by Corollary 4.2, there is an automorphism $f$ of $K$ and an invertible matrix $A \in M_n(K)$ such that

$$\phi \circ T(X) = AXfA^{-1}$$

for all $X \in M_n(K)$. Hence, by Proposition 4.1,

$$\phi \circ T(X) = \overline{A}XfA^{-1}$$

for all $X \in M_n(K)$, where

$$\overline{A} = \left[ ((\phi \circ T)(S))^n \phi(E_{1,n})a | ((\phi \circ T)(S))^{n-2} \phi(E_{1,n})a | \cdots | ((\phi \circ T)(S)) \phi(E_{1,n})a | \phi(E_{1,n})a \right].$$

(Here $S^\top = E_{2,1} + E_{3,2} + \cdots + E_{n,n-1}$, and $a$ is a nonzero vector in the kernel of $I_n - (\phi(S^\top))^{n-1} \phi(E_{1,n})$.)

In particular,

$$\phi \circ T(X^\top) = AXf^\top A^{-1},$$

i.e.,

$$\phi(X) = AXf^\top A^{-1}$$

for all $X \in M_n(K)$. \qed

We illustrate the construction of $\overline{A}$ in Theorem 4.3 with the (canonical) symplectic involution as a special case of an anti-automorphism $\phi$.

**Example.** Consider the symplectic involution $\phi$ on $M_4(K)$ (see, e.g., [1] or [10]), i.e. $\phi$ is the anti-automorphism of $M_4(K)$ defined by

$$\phi \left( \begin{bmatrix} U & P \\ Q & V \end{bmatrix} \right) = \begin{bmatrix} V^\top & -P^\top \\ -Q & U^\top \end{bmatrix}$$

for all $U,V,P,Q \in M_4(K)$. In order to construct $\overline{A}$ above, we need certain powers of $\phi(S^\top)$. Since $S^\top = E_{2,1} + E_{3,2} + \cdots + E_{8,7}$, we have

$$\phi(S^\top) = E_{1,2} + E_{2,3} + E_{3,4} + E_{5,6} + E_{6,7} + E_{7,8} - E_{8,1}. \quad (4.1)$$
Instead of expressing the higher powers \((S^\top)^i\) and \((\phi(S^\top))^i\), \(i = 2, 3, \ldots, 7\), in the form of an expressions as in \(4.1\), which can obviously be done relatively easily, we have found the resulting expressions in terms of the \(E_{i,j}\)'s rather cumbersome to comprehend, and so, although explicit presentations of these matrices, as above, take considerably more space, we have opted for the latter, since the resulting presentations are much more illuminating to the reader. Moreover, it also makes it absolutely clear that this situation for \(M_8(K)\) can be generalized to \(M_n(K)\) for any even number \(n\).

Thus, we get the following:

\[
(S^\top)^2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}, \quad (\phi(S^\top))^2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
(S^\top)^3 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}, \quad (\phi(S^\top))^3 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
\[(S^\top)^4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad (\phi(S^\top))^4 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},\]

\[(S^\top)^5 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad (\phi(S^\top))^5 = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix},\]

\[(S^\top)^6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (\phi(S^\top))^6 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix},\]

and

\[(S^\top)^7 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad (\phi(S^\top))^7 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}.\]
Hence, \( \phi(S^T)^7 \phi(E_{1,8}) = (-E_{5,4})(-E_{4,5}) = E_{5,5} \), and so

\[
a := e_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

is a nonzero vector in the kernel of

\[
I_8 - \phi(S^T)^7 \phi(E_{1,8}) = E_{1,1} + E_{2,2} + E_{3,3} + E_{4,4} + E_{6,6} + E_{7,7} + E_{8,8}.
\]

(Here \( e_j \) denotes the \( 8 \times 1 \) column vector with 1 in position \( j \), and 0 elsewhere.) Therefore, since

\[
\phi(E_{1,8})a = -E_{4,5}e_5 = -e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]

the foregoing presentations of \( (\phi(S^T))^i, \ i = 2, 3, \ldots, 7 \), together with the construction of \( \overline{A} \) in Proposition ??, yields

\[
\overline{A} = \begin{bmatrix} (\phi(S^T))^7 \phi(E_{1,8})a & (\phi(S^T))^6 \phi(E_{1,8})a & \cdots & (\phi(S^T)) \phi(E_{1,n})a & \phi(E_{1,n})a \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & -1 & -1 & \cdots & 0 \\ 1 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_4 & 0 & \cdots & 0 & I_4 \end{bmatrix}
\]

the latter being the negative of the matrix \( y \) on the last line of the first page of [10]. (Of course, \( \overline{A} X^\top \overline{A}^{-1} = (\lambda \overline{A}) X^\top (\lambda \overline{A})^{-1} \) for every \( 0 \neq \lambda \in F \).) This concludes the example.

Next, consider the setting following (1.2), with the only difference that the function \( \beta : M_n(K) \to M_n(K) \) is defined by

\[
\beta(X) = BX^\top B^{-1}
\]
for all $X \in M_n(K)$ (instead of $\beta(X) = BX_f B^{-1}$). Then, as before, $\beta$ is a ring anti-automorphism of $M_n(K)$, but it need not be a $K$-anti-automorphism of $M_n(K)$. In this case we have the following result:

**4.4. Corollary.** Let $f \in \text{Aut}(K)$ ($K$ any field), let $B$ be any invertible matrix in $M_n(K)$, and let $\beta : M_n(K) \to M_n(K)$ be the function defined by

$$\beta(X) = BX_f^T B^{-1}$$

for all $X \in M_n(K)$. Then

$$\beta(X) = B\bar{X}_f^T \bar{B}^{-1}$$

for all $X \in M_n(K)$, where $\bar{B} \in M_n(K)$ is the invertible matrix

$$\bar{B} = \left[ (\beta(S^T))^{n-1} \beta(E_{1,n})b \mid (\beta(S^T))^{-n+1} \beta(E_{1,n})b \mid \cdots \mid (\beta(S^T))^{n-2} \beta(E_{1,n})b \mid \beta(E_{1,n})b \right]$$

and $b$ is a nonzero vector in the kernel of $I_n - (\beta(S^T))^{n-1} \beta(E_{1,n}) \in M_n(K)$.

**Proof.** Since

$$\beta(X_{f^{-1}}) = BX_f^T B^{-1}$$

for all $X \in M_n(K)$, it follows that $\alpha : M_n(K) \to M_n(K)$, defined by

$$\alpha(X) = \beta(X_{f^{-1}})$$

for all $X \in M_n(K)$, is a $K$-anti-automorphism of $M_n(K)$. Hence, by Theorem 4.3, we can constructively find an invertible matrix $\bar{B}$ (say) in $M_n(K)$ such that

$$\alpha(X) = \bar{B}X_f^T \bar{B}^{-1}$$

for all $X \in M_n(K)$, where $\bar{B} \in M_n(K)$ is the invertible matrix

$$\bar{B} = \left[ (\alpha(S^T))^{n-1} \alpha(E_{1,n})b \mid (\alpha(S^T))^{-n+1} \alpha(E_{1,n})b \mid \cdots \mid (\alpha(S^T))^{n-2} \alpha(E_{1,n})b \mid \alpha(E_{1,n})b \right]$$

and $b$ is a nonzero vector in the kernel of the matrix $I_n - (\alpha(S^T))^{n-1} \alpha(E_{1,n})$ in $M_n(K)$. We have $\alpha(S^T) = \beta(S_f^{f^{-1}})$ and $\alpha(E_{1,n}) = \beta((E_{1,n})_{f^{-1}})$, and so, since every entry of both $S^T$ and $E_{1,n}$ is 0 or 1, and since $f^{-1} \in \text{Aut}(K)$, we have that $\alpha(S^T) = \beta(S^T)$ and $\alpha(E_{1,n}) = \beta(E_{1,n})$. Therefore,

$$\beta(X) = \beta((X_f)_{f^{-1}}) = \alpha(X_f) = \bar{B}X_f^T \bar{B}^{-1}$$

for all $X \in M_n(K)$, where

$$\bar{B} = \left[ (\beta(S^T))^{n-1} \beta(E_{1,n})b \mid (\beta(S^T))^{-n+1} \beta(E_{1,n})b \mid \cdots \mid (\beta(S^T))^{n-2} \beta(E_{1,n})b \mid \beta(E_{1,n})b \right]$$

with $b$ a nonzero vector in the kernel of $I_n - (\beta(S^T))^{n-1} \beta(E_{1,n}) \in M_n(K)$. □

Consider again (1.3). By Proposition 4.1, Corollary 4.2 and Theorem 4.3 we have an exact description of $\sigma$ in (1.3) in the terms of the images of generators of $M_n(K)$. Regarding $\tau$, recall that it is an additive mapping from $M_n(K)$ to $K$ which maps commutators into zero. With $\text{tr}(X)$ denoting the trace of a matrix $X$ in $M_n(K)$, we have the following:

**4.5. Proposition.** Let $\tau$ be as in (1.3), and let $X = \sum_{i,j=1}^n k_{ij} E_{i,j} \in M_n(K)$. Then $\tau(X) = \tau(\text{tr}(X) \cdot E_{1,1})$. 
Proof. If $i \neq j$, then $[k_{ij} E_{i,j}, E_{j,j}] = k_{ij} E_{i,j}$, and so, since $\tau$ maps commutators to zero, we have $\tau(k_{ij} E_{i,j}) = 0$. Hence, $\tau(X) = \tau\left(\sum_{i=1}^{n} k_{ii} E_{i,i}\right)$. Note also that $k_{ii} E_{i,i} = [k_{ii} E_{i,1}, E_{1,i}] + k_{ii} E_{1,1}$ for every $i$, and so $\tau(k_{ii} E_{i,i}) = \tau(k_{ii} E_{1,1})$. Consequently,

$$\tau(X) = \tau\left(\sum_{i=1}^{n} k_{ii} E_{1,1}\right) = \tau(\text{tr}(X) \cdot E_{1,1}).$$

□

Unfortunately, we do not seem to be able to describe $\tau(\text{tr}(X) \cdot E_{1,1})$ any better. In general, if $\psi$ in (1.3) is not a Lie $K$-automorphism, then we may not have $\tau(\text{tr}(X) \cdot E_{1,1}) = \text{tr}(X) \tau(E_{1,1})$.

The following result by Dolinar et al. should be mentioned here:

Theorem. (see [2]) Let $K$ be a field, and let $\psi : M_n(K) \to M_n(K)$ be a bijective map which preserves the commutator Lie product. Then there is an invertible matrix $T \in M_n(K)$, a field automorphism $f$ of $K$, and a function $\tau : M_n(K) \to K$, where $\tau(X) = 0$ for all matrices of trace zero such that:

(i) for $n \geq 3$ and $K$ with a least $2^{n-1}$ elements, either

$$\psi(X) = TX^f T^{-1} + \tau(X)I$$

for all $X \in M_n(K)$, or

$$\psi(X) = -T(X^f)^T T^{-1} + \tau(X)I$$

for all $X \in M_n(K)$;

(ii) for $n = 2$ and $\text{char} K \neq 2$,

$$\psi(X) = TX^f T^{-1} + \tau(X)I$$

for all $X \in M_n(K)$.

Considering this theorem, we note that if we consider the functions

$$\sigma_1, \sigma_2 : M_n(K) \to M_n(K),$$

defined by

$$\sigma_1(X) = TX^f T^{-1} \quad \text{and} \quad \sigma_2(X) = -T(X^f)^T T^{-1}$$

for all $X \in M_n(K)$, then by the foregoing constructions and considerations, $\sigma_1$ is an automorphism of $M_n(K)$ and $\sigma_2$ is the negative of an anti-automorphism of $M_n(K)$ (in both cases as rings), and as before, we have exact descriptions of them in the terms of generators of $M_n(K)$. However, we know nothing more about $\tau$.

Funding

The second author was partially supported by the National Research, Development and Innovation Office of Hungary (NKFIH) K119934. The research of the fourth author was funded by the Polish National Science Centre Grant DEC-2017/25/B/ST1/00384.
References

[1] J. Dale Hill, Polynomial identities for matrices symmetric with respect to the symplectic involution, *J. Algebra* **349** (2012), 8–21.
[2] G. Dolinar, B. Kuzma and J. Marovt, Lie product preserving maps on $M_n(F)$, *Filomat* **31** (2017), 5335–5344.
[3] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*. Cambridge Studies in Advanced Mathematics, 101. Cambridge University Press, Cambridge, 2006.
[4] L. K. Hua, A theorem on matrices over a field and its applications, *J. Chinese Math. Soc. (N.S.)* **1** (1951), 110–163.
[5] N. Jacobson, Lie algebras. Interscience Tracts in Pure and Applied Mathematics, No. 10 *Interscience Publishers (a division of John Wiley & Sons)*, New York - London, 1962.
[6] W. S. Martindale, III, Lie isomorphisms of primitive rings, *Proc. Amer. Math. Soc.* **14** (1963), 909–916.
[7] W. S. Martindale, III, Lie isomorphisms of simple rings, *J. London Math. Soc.* **44** (1969), 213–221.
[8] W. S. Martindale, III, Lie isomorphisms of prime rings, *Trans. Amer. Math. Soc.* **142** (1969), 437–455.
[9] E. Noether, Nichtkommutative Algebra, *Math. Z.* **37** (1933), 514–541.
[10] L. H. Rowen and U. Schild, A scalar expression for matrices with symplectic involution, *Mathematics of Computation* **32** (1978), 607–613.
[11] P. Semrl, Maps on matrix spaces, *Linear Algebra Appl.* **413** (2006), 364–393.
[12] T. Skolem, Zur Theorie der assoziativen Zahlensysteme, *Skriver Oslo* **12** (1927), 50.
[13] J. Szigeti, J. van den Berg, L. van Wyk and M. Ziembowski, The maximum dimension of a Lie nilpotent subalgebra of $M_n(F)$ of index $m$. *Trans. Amer. Math. Soc.* **372** (2019), 4553–4583.
[14] J. Szigeti and L. van Wyk, A constructive elementary proof of the Skolem-Noether Theorem for matrix algebras, *Amer. Math. Monthly* **124** (2017), 966–968.