A FAMILY OF 3D STEADY GRADIENT SOLITONS THAT ARE FLYING WINGS

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Abstract. Hamilton conjectured that there exists a 3d steady gradient Ricci soliton that is a flying wing. We confirm his conjecture in this paper. We also found a family of $\mathbb{Z}_2 \times O(n-1)$-symmetric n-dimensional steady gradient Ricci solitons with positive curvature operator in all dimension $n \geq 3$.

1. Introduction

Ricci solitons are self-similar solutions of the Ricci flow equation, and they often arise as blow-up limits of singularities of Ricci flows. In particular, a steady gradient soliton is a smooth complete Riemannian manifold satisfying

\begin{equation}
\text{Ric} = \nabla^2 f
\end{equation}

for some smooth function $f$ on $M$, which is called a potential function of the soliton. It generates a Ricci flow for all times by $g(t) = \phi_t^*(g)$, where $\{\phi_t\}_{t \in (-\infty, \infty)}$ is the one-parameter group of diffeomorphisms generated by $-\nabla f$ with $\phi_0$ the identity.

In dimension 2, the only non-flat steady gradient soliton is Hamilton’s cigar on $\mathbb{R}^2$ [Ham88]. It is rotationally symmetric and has positive curvature. In dimension 3, a steady gradient soliton is always non-negatively curved by Hamilton’s Harnack inequality. If the sectional curvature vanishes somewhere, then by the strong maximum principle it is either flat or quotients of $\mathbb{R} \times \text{Cigar}$. The only known non-flat 3d steady gradient solitons before were the Bryant soliton, and quotients of $\mathbb{R} \times \text{Cigar}$. Bryant solitons were found by Bryant on $\mathbb{R}^n$ for $n \geq 3$ [Bry05]. They are rotationally symmetric and have positive curvature operator. It is an open problem whether there is any 3d steady gradient soliton other than the Bryant soliton and quotients of $\mathbb{R} \times \text{Cigar}$, see for example [DZ16, CCG07, CH18, CMM16].

Hamilton conjectured that there is a positively curved 3d steady gradient soliton that is a flying wing. We confirm Hamilton’s conjecture in this paper. Our first theorem finds a family of $\mathbb{Z}_2 \times O(n-1)$-symmetric n-dimensional steady gradient solitons with positive curvature operator and prescribed Ricci curvature at a critical point of the potential function in all dimension $n \geq 3$.

Theorem 1.1. Given any $\alpha \in (0,1)$, there exists an $n$-dimensional $\mathbb{Z}_2 \times O(n-1)$-symmetric steady gradient soliton $(M^n, g, f, p)$ with positive curvature operator, where $f$ is the potential function and $p$ is a critical point of $f$, such that $\lambda_1 = \alpha \lambda_2 = \cdots = \alpha \lambda_n$, where $\lambda_1, \ldots, \lambda_n$ are the $n$ eigenvalues of Ricci curvature at $p$. 
Theorem 1.1 also gives an affirmative answer to the open problem whether there exists non-rotationally symmetric steady Ricci solitons in dimensions \( n \geq 4 \) [Cao10], since the Bryant soliton is the only rotationally symmetric steady soliton in dimension \( n \geq 3 \). Moreover, the new family of 3d steady gradient solitons in Theorem 1.1 are all collapsed, since the Bryant soliton is the only non-collapsed and non-flat 3d steady gradient soliton [Bre13].

The term flying wing is used by Hamilton to describe certain translating solutions in mean curvature flow and steady solitons in Ricci flow. In mean curvature flow, a flying wing is a convex, locally uniformly convex translator that is contained in a slab region. For the mean curvature flow in dimension 3, there exist both \( \mathbb{Z}_2 \times O(2) \)-symmetric flying wings and non-flying wings, which are not rotationally symmetric [Wan11]. In Ricci flow, a flying wing is a steady gradient soliton with positive curvature operator, whose tangent cone at infinity is a sector with angle \( \alpha \in (0, \pi) \), i.e. a metric cone over the interval \([-\frac{\alpha}{2}, \frac{\alpha}{2}]\). The tangent cone at infinity of a 3d steady gradient soliton is either a ray or a sector.

Our second theorem says that a \( \mathbb{Z}_2 \times O(2)-\)symmetric 3d steady gradient soliton must be a flying wing if it is not a Bryant soliton. Combining with Theorem 1.1, we obtain an affirmative answer to Hamilton’s conjecture.

**Theorem 1.2.** Let \((M, g, f, p)\) be a \( \mathbb{Z}_2 \times O(2)-\)symmetric 3d steady gradient soliton. Suppose its tangent cone at infinity is a ray. Then it is isometric to the Bryant soliton.

Let \((M, g)\) be a \( \mathbb{Z}_2 \times O(2)-\)symmetric 3d flying wing, then the \( O(2)-\)action fixes a complete geodesic \( \Gamma \) which goes to infinity at both ends. We show that \((M, g)\) strongly dimension reduces to a cigar soliton along \( \Gamma \). That is to say for any sequence \( s_i > 0 \) and \( s_i \to \infty \), a subsequence of the rescaled ancient solution \((M, R(\Gamma(s_i))g(t), \Gamma(s_i))\) converges to the ancient solution of \( \mathbb{R} \times \text{Cigar} \). Moreover, our next theorem establishes a quantitative relation between the limit of scalar curvature along \( \Gamma \) and the angle of the tangent cone at infinity. In particular, the theorem says that the scalar curvature does not vanish at infinity in a \( \mathbb{Z}_2 \times O(2)-\)symmetric 3d flying wing.

**Theorem 1.3.** Given \( \alpha \in [0, \pi] \). Let \((M, g, f, p)\) be a \( \mathbb{Z}_2 \times O(2)-\)symmetric 3d steady gradient soliton. Suppose its tangent cone at infinity is a metric cone over the interval \([-\frac{\alpha}{2}, \frac{\alpha}{2}]\). Let \( \Gamma : (-\infty, \infty) \to M \) be the complete geodesic fixed by the \( O(2)-\)action, then

\[
\lim_{s \to \infty} R(\Gamma(s)) = R(p) \sin^2 \frac{\alpha}{2}.
\]

As an application of Theorem 1.2 and 1.3, the following corollary gives a sequence of \( \mathbb{Z}_2 \times O(2)-\)symmetric 3d flying wings, whose tangent cone at infinity are sectors with angles \( \alpha_i \to 0 \) as \( i \to \infty \).

**Corollary 1.4.** There exists a sequence of \( \mathbb{Z}_2 \times O(2)-\)symmetric 3d steady gradient solitons \( \{(M_i, g_i, p_i)\}_{i=1}^{\infty} \), such that the tangent cone at infinity of each \((M_i, g_i, p_i)\) is a sector with angle \( \alpha_i \in (0, \pi) \) and \( \lim_{i \to \infty} \alpha_i = 0 \).
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The structure of the paper is as follow: In Section 2 we prove Theorem 1.1. The proof uses a result by Deruelle of expanding gradient solitons. In Section 3 we study the dimension reduction of a $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady soliton at infinity. In Section 4 we prove Theorem 1.2 and 1.3. We first prove Theorem 1.3. The proof of Theorem 1.2 is by a bootstrap argument: Suppose it is not a Bryant soliton. Then under a bootstrap estimate we can show that the length of the killing field generated by the $O(2)$-action stays bounded at infinity. This would imply $\lim_{s \to \infty} R(\Gamma(s)) > 0$, which by Theorem 1.3 gives a contradiction.

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2. Existence of a family of steady gradient solitons

The main result in this section is Theorem 1.1, the existence of a family of steady gradient solitons in dimension $n \geq 3$. The outline of the proof is the following: We first construct a sequence of smooth families of expanding gradient solitons \( \{(M_i, \mu, g_i, 0, p_i, \mu), \mu \in [0, 1] \} \) with positive curvature operator, such that \((M_i, 0, g_i, 0, p_i, 0)\) converges to a Bryant soliton, and \((M_i, 1, g_i, 1, p_i, 1)\) converges to the product of $\mathbb{R}$ and an \((n-1)\)-dimensional Bryant soliton if $n \geq 4$, or a cigar soliton if $n = 3$. Moreover, we require that the asymptotic volume ratio of each expanding gradient solitons tends to zero uniformly as $i \to \infty$.

Let $\alpha_i(\mu)$ be the quotients of the smallest and largest eigenvalues of the Ricci curvature at $p_i, \mu$ in $(M_i, g_i, 0, p_i, \mu)$, then $\alpha_i(\mu)$ is a smooth function in $\mu$ for each fixed $i$. Then for any $\alpha \in (0, 1)$, there is some $\mu_i \in (0, 1)$ such that $\alpha_i(\mu_i) = \alpha$. Since the asymptotic volume ratio of $(M_i, g_i, 0, p_i, \mu_i)$ goes to zero, we can show that it subconverges to an $n$-dimensional steady gradient soliton $(M, g, p)$ with positive curvature operator. In particular, the quotients of the smallest and largest eigenvalues of the Ricci curvature at $p$ in $(M, g, p)$ is equal to $\alpha$.

To construct a smooth family of expanding gradient solitons $(M_i, \mu, g_i, \mu, p_i, \mu), \mu \in [0, 1]$, with positive curvature operator for each fixed $i$, we use Deruelle’s work on expanding gradient solitons [Der16]. He showed that for any \((n-1)\)-dimensional smooth simply connected Riemannian manifold $(X_1, g_{X_1})$ with $Rm > 1$, there exists a unique expanding gradient soliton $(M_1, g_1, p_1)$ with positive curvature operator that is asymptotic to the cone $(C(X_1), dr^2 + r^2 g_{X_1})$. Moreover, there is a one-parameter smooth family of expanding gradient solitons connecting $(M_1, g_1, p_1)$ to an expanding gradient soliton $(M_0, g_0, p_0)$, whose asymptotic cone is rotationally symmetric. By Chodosh’s work the soliton $(M_0, g_0, p_0)$ is rotationally symmetric, and hence is a Bryant expanding soliton [Cho14].

2.1. Preliminaries. In this subsection we fix some notions that will be frequently used. First, we recall some standard notions and facts from Alexandrov geometry: Let $(M, g)$ be a non-negatively curved Riemannian manifold, then for any triple of
points \( o, p, q \in M \), the comparison angle \( \tilde{\angle} poq \) is the corresponding angle formed by minimizing geodesics with lengths equal to \( d(o, p), d(o, q), d(p, q) \) in Euclidean space. Let \( op, oq \) be two minimizing geodesics in \( M \) between \( o, p \) and \( o, q \), and \( \angle poq \) be the angle between them at \( o \), then \( \angle poq \geq \tilde{\angle} poq \). Moreover, for any \( p' \in op \) and \( q' \in oq \), the monotonicity of angle comparison implies \( \tilde{\angle} p'oq' \geq \tilde{\angle} poq \).

Next, we define what we mean by a metric on \( \mathbb{R}^n \) to be \( \mathbb{Z}_2 \times O(n-1) \)-symmetric. First, we define an \( O(n-1) \)-action on the Euclidean space \( \mathbb{R}^n \) by extending the standard \( O(n-1) \)-action on \( \mathbb{R}^{n-1} = \{ x^n = 0 \} \subset \mathbb{R}^n \) in the way such that it fixes the \( x^n \)-axis. Then we define a \( \mathbb{Z}_2 \times O(n-1) \)-action on \( \mathbb{R}^n \) by further defining a \( \mathbb{Z}_2 \)-action to be generated by a reflection that fixes the hypersurface \( \{ x^n = 0 \} \). Then

**Definition 2.1.** We say that an \( n \)-dimensional Riemannian manifold \((M^n, g)\) is \( O(n-1) \)-symmetric (resp. \( \mathbb{Z}_2 \times O(n-1) \)-symmetric) if there exist an isometric \( O(n-1) \)-action (resp. a \( \mathbb{Z}_2 \times O(n-1) \)-action), and a diffeomorphism \( \Phi : M^n \to \mathbb{R}^n \) such that \( \Phi \) is equivariant with the two actions, where the action on \( \mathbb{R}^n \) is defined as above.

Suppose \((M^n, g)\) is an \( O(n-1) \)-symmetric Riemannian manifold, then there is a totally geodesic surface \( N \subset M \) diffeomorphic to the upper half-plane in \( \mathbb{R}^2 \), whose boundary is a geodesic \( \Gamma \) in \( M \), such that \( g = g_N + \varphi^2 g_{n-2} \) at all \( x \in M \setminus \Gamma \), where \( \varphi \) is a function on \( N \) with \( \varphi = 0 \) on \( \Gamma \) and \( \varphi > 0 \) on \( N \setminus \Gamma \).

Now suppose \((M^n, g)\) is \( \mathbb{Z}_2 \times O(n-1) \)-symmetric, and let \( \Gamma = \Phi^{-1}(\{ x^1 = \cdots = x^{n-1} = 0 \}) \), \( N = \Phi^{-1}(\{ x^1 = \cdots = x^{n-2} = 0, x^{n-1} > 0 \}) \) and \( \Sigma = \Phi^{-1}(x^n = 0) \). Then

1. \( \Gamma \) is a geodesic that goes to infinity at both ends, because it is the fixed point set of the \( O(n-1) \)-action.
2. \( \Sigma \) is a rotationally symmetric \((n-1)\)-dimensional totally geodesic submanifold, because it is the fixed point set of the \( \mathbb{Z}_2 \)-action.
3. \( N \) is a totally geodesic surface because it is a connected component of the fixed point set of an isometry in \( O(n-1) \). So there is a smooth positive function \( \varphi : N \to \mathbb{R} \) such that \( g = g_N + \varphi^2 g_{n-2} \) on \( M \setminus \Gamma \), where \( g_{n-2} \) is the standard round metric on \( S^{n-2} \).
4. \( \Phi^{-1}(0) \) is the unique point fixed by both actions, at which \( \Gamma \) intersects orthogonally with \( \Sigma \).

In this paper, we study \( n \)-dimensional expanding or steady gradient soliton \((M^n, g)\) with non-negative curvature operator, whose potential function \( f \) has a critical point \( p \). We denote it by a quadruple \((M^n, g, f, p)\) (and sometimes a triple \((M^n, g, p)\)). Note that \( R \) attains its maximum at \( p \) by the identity \( R + |\nabla f|^2 = \text{const.} \), and if the curvature operator is positive, then \( p \) is the unique critical point.

We assume \((M^n, g, f, p)\) is \( \mathbb{Z}_2 \times O(n-1) \)-symmetric, and fix the notions \( \Gamma, N, \varphi, \Sigma \) from above, and assume \( \Gamma : (-\infty, \infty) \to M \) has unit speed and \( \Gamma(0) = p \). If \( Rm > 0 \), it is easy to see that \( p \) is the unique point fixed by the \( \mathbb{Z}_2 \times O(n-1) \)-action. Moreover, by the soliton equation \( \nabla^2 f = \text{Ric} + cg \), \( c \geq 0 \) it follows that the potential function
f is invariant under the actions. Therefore, \( \Gamma \) and all unit speed geodesics in \( \Sigma \) are integral curves of \( \frac{\nabla f}{|\nabla f|} \).

Moreover, use \( i, j, k, l \) for indices on \( N \), and \( \alpha, \beta \) and \( g_{\alpha\beta} \) for indices and metric components on \( S^{n-2} \) with the standard round metric. Then by computation the nonzero components of the curvature tensor of \( (M \setminus \Gamma, g) \) are

\[
R^M_{ijkl} = R^N_{ijkl}, \quad R_{\alpha\beta} = -g_{\alpha\beta}(\nabla^2 \varphi)_{ij}. \tag{2.1}
\]

So by \( Rm \geq 0 \) we have \( \nabla^2 \varphi \leq 0 \) and \( \varphi \) is concave.

### 2.2. Proof of Theorem 1.1

To prove Theorem 1.1 we will take a limit of a sequence of expanding gradient solitons with \( R(p) = 1 \), where \( p \) is the critical point of the potential function. To do this, we need an injectivity radius lower bound and a uniform curvature bound. The curvature bounds follows directly from \( R_{\text{max}} = R(p) = 1 \), and the injectivity radius estimate follows from the next lemma.

**Lemma 2.2.** There exists \( C > 0 \) such that the following holds: Let \( (M^n, g, f, p) \) be a \( \mathbb{Z}_2 \times O(n-1) \)-symmetric \( n \)-dimensional expanding (or steady) gradient soliton with positive curvature operator. Suppose \( R(p) = 1 \). Then \( \text{vol}(B(p, 1)) \geq C^{-1} \) and \( \text{inj}(p) \geq C^{-1} \).

**Proof.** We shall use \( C \) to denote all positive constants, whose value may vary from in lines. Let \( \gamma : [0, \infty) \rightarrow \Sigma \) be a unit speed geodesic emanating from \( p \in \Sigma \) such that \( \gamma \) is contained in \( N \). Then by the curvature assumption and the Jacobi comparison we get \( \varphi(\gamma(1)) \geq c := \sin 1 > 0 \). Since \( \varphi(\gamma(s)) \) increases in \( s \) and \( d(p, \gamma(1)) \leq 1 \), we can find \( s_0 \geq 1 \) such that \( d(p, \gamma(s_0)) = 1 \) and \( \varphi(\gamma(s_0)) \geq c \).

Let \( q = \gamma(s_0) \). We claim \( d(q, \Gamma) \geq c \): First, suppose \( d(q, \Gamma) = d(q, x) \) for some \( x \in \Gamma \). Let \( \sigma \) be the unit speed minimizing geodesic from \( x \) and \( q \), then by the first variation formula we see that \( \sigma \) intersects with \( \Gamma \) orthogonally at \( x \). Consider all the preimages of \( \sigma \) under the Riemannian submersion \( M \rightarrow N \), which form a smooth submanifold with induced metric \( dr^2 + \varphi^2(\sigma(r))g_{S^{n-2}} \). Then by the vertical tangent condition at \( x \), we have

\[
\left. \frac{d}{dr} \right|_{r=0} \varphi(\sigma(r)) = 1,
\]

which by \( \varphi(q) \geq c \) and the concavity of \( \varphi \) implies \( d(q, \Gamma) \geq c \).

Choose some \( y \in \Gamma \) such that \( d(p, y) = 1 \). Let \( pq, yp, yq \) be minimizing geodesics between these points. By replacing \( pq \) with its image under a suitable \( \mathbb{Z}_2 \times O(n-1) \)-action, we may assume \( \angle ypq \leq \frac{\pi}{2} \). So by angle comparison we get

\[
\widetilde{\angle yqp} \leq \angle ypq \leq \frac{\pi}{2},
\]

and hence

\[
\widetilde{\angle yqp} \geq \frac{\pi}{4}
\]

since \( d(p, y) = d(p, q) \). So for some \( y' \in yq, p' \in pq \) such that

\[
d(y', q) = d(p', q) = c
\]

we have

\[
\widetilde{\angle y'qp} \geq \widetilde{\angle yqp} \geq \frac{\pi}{4},
\]

and hence \( d(y', p') \geq C^{-1} \), which by volume comparison implies \( \text{vol}_N(B_N(q, c)) \geq C^{-1} \). Moreover, since \( \varphi \) is concave and \( \varphi(q) = 1 \), we have \( \varphi \geq C^{-1} \) on \( B_N(q, \frac{c}{2}) \), integrating it on \( B_N(q, c) \) it implies

\[
\text{vol}_M(B_M(p, 1)) \geq C^{-1}.
\]

The assertion about the injectivity radius now follows from the volume lower bound and the curvature bound \( R \leq R(p) = 1 \). \( \Box \)
Recall that if \((M^n, g, f, p)\) is an expanding gradient soliton satisfying
\[
\text{(2.2)} \quad \text{Ric} + \lambda g = \nabla^2 f
\]
for some \(\lambda > 0\). Then it generates a Ricci flow \(g(t) := (2\lambda t)\phi^*_{t - \frac{1}{2\lambda}} g, \ t \in (0, \infty), \) where \(\{\phi_s\}_{s \in (-\infty, \infty)}\) is the one-parameter diffeomorphisms generated by the time-dependent vector field \(-\frac{1}{1 + 2\lambda s} \nabla f\) with \(\phi_0\) the identity. Moreover, \(g(t)\) is an expanding gradient soliton satisfying
\[
\text{(2.3)} \quad \text{Ric}(g(t)) + \frac{1}{2t} g(t) = \nabla^2 f_t,
\]
where \(f_t = \phi^*_{t - \frac{1}{2\lambda}} f\).

Let \((M^n_i, g_i, f_i, p_i)\) be a sequence of \(\mathbb{Z}_2 \times O(n - 1)\)-symmetric expanding gradient solitons with positive curvature operator, which satisfies \(R(p_i) = 1\) and the asymptotic volume ratio \(\text{AVR}(g_i) \to 0\) as \(i \to \infty\). Let \(C_i > 0\) be the constant such that \((M^n_i, g_i, f_i, p_i)\) satisfies the soliton equation
\[
\text{(2.4)} \quad \text{Ric}(g_i) + \frac{1}{2C_i} g_i = \nabla^2 f_i.
\]
Then the following lemma shows \(C_i \to \infty\) as \(i \to \infty\), and hence there is a subsequence of \((M^n_i, g_i, f_i, p_i)\) smoothly converging to a steady gradient soliton.

**Lemma 2.3.** Let \((M^n_i, g_i, f_i, p_i)\) be a sequence of \(\mathbb{Z}_2 \times O(n - 1)\)-symmetric expanding gradient solitons with positive curvature operator. Suppose \(R_{g_i}(p_i) = 1\) and \(\text{AVR}(g_i) \to 0\) as \(i \to \infty\). Then a subsequence of \((M_i, g_i, f_i, p_i)\) smoothly converges to an \(n\)-dimensional \(\mathbb{Z}_2 \times O(n - 1)\)-symmetric steady gradient soliton \((M, g, f, p)\).

**Proof.** Suppose \((M^n_i, g_i, f_i, p_i)\) satisfies
\[
\text{(2.5)} \quad \text{Ric}(g_i) + \frac{1}{2C_i} g_i = \nabla^2 f_i
\]
for some constant \(C_i > 0\). Let \((M_i, g_i(t), f_{i,t}, p_i)\), \(t \in (0, \infty), \) be the Ricci flow generated by \((M_i, g_i, f_i, p_i)\), where \(g_i(t) = \frac{t}{C_i} \phi^*_{t - C_i} g_i, \ f_{i,t} = \phi^*_{t - C_i} f_i, \) and \(\{\phi_{i,s}\}_{s \in (-C_i, \infty)}\) is the family of diffeomorphisms generated by \(-\frac{t}{C_i + C_i} \nabla f_i\) with \(\phi_0\) the identity. By a direct computation we can show
\[
\text{(2.6)} \quad \text{Ric}(g_i(t)) + \frac{1}{2t} g_i(t) = \nabla^2 f_{i,t},
\]
for all positive time \(t\). In particular, we have \(g_i(C_i) = g_i\) and \(R_{g_i(t)}(p_i) = C_i\).

We claim that \(C_i \to \infty\) as \(i \to \infty\): Suppose this is not true. Then by passing to a subsequence we may assume \(C_i \leq C\) for some constant \(C > 0\) and all \(i\). We shall use \(C\) to denote all positive constant that is independent of \(i\).

First, by Lemma 2.2 we have \(\text{inj}_{g_{i(t)}}(p_i) \geq C^{-1}\) and
\[
\text{(2.7)} \quad R_{\tilde{g}_{i(t)}}(x) \leq R_{\tilde{g}_{i(t)}}(p_i) \leq \frac{C}{t},
\]
for all $x \in M_i$ and $t \in (0, \infty)$. So by Hamilton’s compactness for Ricci flow we may assume after passing to a subsequence that $(M_i, \tilde{g}_i(t), p_i), t \in (0, \infty)$, converges to a smooth Ricci flow $(M_\infty, g_\infty(t), p_\infty)$ on $(0, \infty)$. Assume $f_{i,1}(p_i) = 0$, then by $|\nabla f_{i,1}|(p_i) = 0$ and $\text{Ric}_{\tilde{g}_i(1)} + \frac{1}{2}\tilde{g}_i(1) = \nabla^2 f_{i,1}$, we can apply Shi’s derivative estimates to get bounds for higher derivatives of curvatures, and thus bounds for higher derivatives of $f_{i,1}$. So we may assume $f_{i,1}$ converges to a smooth function $f_\infty$ satisfying $\text{Ric}_{g_\infty(1)} + \frac{1}{2}g_\infty(1) = \nabla^2 f_\infty$, which makes $(M_\infty, g_\infty(t), p_\infty)$ an expanding gradient soliton. Since $R_{\tilde{g}_i(t)} \leq \frac{C}{t}$, it follows that $R_{g_\infty(t)} \leq \frac{C}{t}$.

This curvature condition combined with Hamilton’s distance distortion estimate gives us a uniform double side control on $d_{\tilde{g}_i(t)}$ and $d_{g_\infty(t)}$, which implies the following pointed Gromov-Hausdorff convergences

$$\text{(2.8)} \quad (M_i, \tilde{g}_i(t), p_i) \xrightarrow{pGH} (C(X_i), o_i), \quad (M_\infty, g_\infty(t), p_\infty) \xrightarrow{pGH} (C(X), o),$$

where $X_i, X$ are some compact length spaces, and $o_i, o$ are the cone points of the metric cones $C(X_i), C(X)$. In particular, the first convergence is uniform for all $i$, which implies $(C(X_i), o_i) \xrightarrow{pGH} (C(X), o)$.

Let $\mathcal{H}_n(\cdot)$ denote the n-dimensional Hausdorff measure. Then since it is weakly continuous under the Gromov-Hausdorff convergence [BGP92], we have

$$\text{(2.9)} \quad \mathcal{H}_n(B(o, 1)) = \lim_{i \to \infty} \text{vol}(B(o_i, 1)) = \lim_{i \to \infty} \text{AVR}(C(X_i)) = \lim_{i \to \infty} \text{AVR}(g_i) = 0.$$

However, since $(M_\infty, g_\infty)$ is an expanding gradient soliton, it has positive asymptotic volume ratio [Ham93]. So by volume comparison we have

$$\text{(2.10)} \quad \mathcal{H}_n(B(o, 1)) \geq \lim_{i \to \infty} \mathcal{H}_n(B_i(p_\infty, 1)) \geq \text{AVR}(g_\infty(t)) > 0,$$

a contradiction. This proves the claim at beginning that $C_i \to \infty$ when $i \to \infty$.

Let $\tilde{g}_i(t) = \tilde{g}_i(t + C_i), t \in (-C_i, \infty)$, then $\tilde{g}_i(0) = g_i, R_{\tilde{g}_i(0)}(p_i) = 1$, and

$$\text{(2.11)} \quad R_{\tilde{g}_i(t)}(x) = R_{\tilde{g}_i(t+C_i)}(x) \leq \frac{C_i}{t+C_i} \leq 2,$$

for all $x \in M_i$ and $t \in (-\frac{C_i}{2}, \infty)$. By Lemma 2.2 there is a subsequence of $(M_i, \tilde{g}_i(t), p_i)$ which smoothly converges to a Ricci flow $(M, g(t), p), t \in (-\infty, \infty)$. Moreover, by the equation (2.3) and Shi’s derivative estimates we obtain uniform bounds for all higher derivatives of $f_i$. Since $C_i \to \infty$ as $i \to \infty$, we may assume by passing to a subsequence that $f_i$ smoothly converges to a function $f$ on $M$ which satisfies $\text{Ric}(g) = \nabla^2 f$. So $(M, g(0), f, p)$ is a steady gradient soliton. The $\mathbb{Z}_2 \times O(n-1)$-symmetry is an easy consequence of the smooth convergence.

Now we prove Theorem 1.1.
Proof of Theorem 1.1. We claim that there is a sequence of smooth families of $\mathbb{Z}_2 \times O(n-1)$-symmetric Riemannian manifolds $\{X_{i,\mu}, \mu \in [0, 1]\}_{i=0}^{\infty}$ diffeomorphic to $S^{n-1}$, satisfying the following:

1. $X_{i,0}$ is a rescaled round $(n-1)$-sphere;
2. $\text{diam}(X_{i,1}) \to \pi$ as $i \to \infty$;
3. $K(X_{i,\mu}) > 1$;
4. $\lim_{i \to \infty} \sup_{[0,1]} \text{vol}(X_{i,\mu}) = 0$.

We say $X_{i,\mu}$ is $\mathbb{Z}_2 \times O(n-1)$-symmetric if it is rotationally symmetric, and there is a $\mathbb{Z}_2$-isometry that maps the two centers of rotations to each other. We construct $\{X_{i,\mu}, \mu \in [0, 1]\}_{i=0}^{\infty}$ in dimension $n = 3$ below, and the case for $n > 3$ follows in a same and easier way.

First, we find a sequence of smooth $\mathbb{Z}_2 \times O(2)$-symmetric surfaces $\{X_{i,1}\}_{i=1}^{\infty}$ with $K(X_{i,1}) > 1$, $\text{diam}(X_{i,1}) \to \pi$ and $\text{vol}(X_{i,1}) \to 0$ as $i \to \infty$: For each large $i$, let $g_i$ be the metric of the surface of revolution $(i^{-1} \sin r \cos \theta, i^{-1} \sin r \sin \theta, r)$, $r \in [0, \pi]$ and $\theta \in [0, 2\pi]$. Then by the formula of Gaussian curvature we see that $K_{\min}(g_i) = (i^{-2} + 1)^{-2}$. Then we obtain a $\mathbb{Z}_2 \times O(2)$-symmetric $C^1$-metric $g_i'$ by cutting off $g_i$ at some small distances to the two points $r = 0$ and $\pi$, and gluing the middle part with suitable parts from two round spheres of radius $r_i$ such that $r_i \to 0$ as $i \to \infty$. Then by some standard ODE estimates, we can approximate $g_i'$ in $C^1$-sense by smooth $\mathbb{Z}_2 \times O(2)$-symmetric metrics with $K \geq (i^{-2} + 1)^{-2}$. Let $g_i''$ be such a metric very close to $g_i'$, and let $X_{i,1}$ be equipped with a suitable rescaling of $g_i''$, then the sequence $\{X_{i,1}\}_{i=0}^{\infty}$ satisfies all desired conditions.

Second, for each large $i$, let $h_i(t)$ be the Ricci flow with $h_i(0) = X_{i,1}$, and assume its curvature blows up at $T_i > 0$. Let $K_i(t)$ be the minimum of the Gauss curvature, and $V_i(t)$ be the volume of $h_i(t)$. Then we can find a smooth function $r_i : [0, T_i] \to \mathbb{R}_+$ such that $r_i(0) = 1$, $r_i(t) \leq \min\{\sqrt{\frac{K_i(0)}{V_i(0)}}, \sqrt{\frac{V_i(t)}{V_i(0)}}\}$ for all $t \in [0, T_i]$, and $r_i(t) = \sqrt{\frac{V_i(0)}{V_i(t)}}$ when $t$ is close to $T_i$ (note $\frac{V_i(0)}{V_i(t)} > \sqrt{\frac{K_i(0)}{K_i(t)}}$ when $i$ is sufficiently large since $\lim_{i \to \infty} V_i(0) = 0$ and $\limsup_{i \to \infty} K_i(0) \leq 1$). Then the rescaled Ricci flow $r_i^2(t)h_i(t)$ converges to a smooth round 2-sphere when $t \to T_i$. Moreover, by letting $X_{i,\mu} = r_i^2(T_i(1-\mu))h_i(T_i(1-\mu))$, $\mu \in [0, 1]$, we obtain a smooth family of $\mathbb{Z}_2 \times O(2)$-symmetric surfaces $\{X_{i,\mu}\}$ with $K(X_{i,\mu}) > 1$, $\text{vol}(X_{i,\mu}) \leq \text{vol}(X_{i,1})$, and $X_{i,0}$ is a round 2-sphere. So items (1)-(4) hold.

Therefore, for each fixed $i$, by applying Theorem 1.4 in [Der16] to the smooth family of metrics $X_{i,\mu}$, $\mu \in [0, 1]$, we obtain a smooth family of $n$-dimensional expanding gradient solitons $(M_{i,\mu}, g_{i,\mu}, p_{i,\mu})$, $\mu \in [0, 1]$, with positive curvature operator, and asymptotic to the cone $C(X_{i,\mu})$. Moreover, by Theorem 1.3 in [Der16], the Ricci flow generated by an expanding gradient soliton coming out of $C(X_{i,\mu})$ is unique. So any isometry of $C(X_{i,\mu})$ acts by isometry of the Ricci flow. In particular, it implies that $(M_{i,\mu}, g_{i,\mu}, p_{i,\mu})$ is $\mathbb{Z}_2 \times O(n-1)$-symmetric and $(M_{i,0}, g_{i,0}, p_{i,0})$ is rotationally symmetric.
ally symmetric, it follows that \((M, M)\) is a Bryant soliton, see e.g. [CLN06].

So for each \(d_n\), \(T\) and \(\lambda_q\) points of \(\{\alpha\}\), which is an \((n-1)\)-dimensional Bryant soliton if \(n > n = 3\), see e.g. [CLN06].

Lemma 2.3 and by passing to a subsequence, we may assume \((M,0,0,0,0)\) and \((M,0,1,0,0)\) smoothly converge to two steady gradient solitons \((M,0,0,0,0)\) and \((M,0,1,0,0)\) respectively. On the one hand, since \((M,0,0,0,0)\) is rotationally symmetric, it follows that \((M,0,0,0,0)\) is rotationally symmetric, and hence is a Bryant soliton, see e.g. [CLN06].

On the other hand, since \(\text{diam}(X,1) \to \pi\) when \(i \to \infty\), the tangent cone at infinity for each \(M,1\) converges to a half-plane, or equivalently a cone over the interval \([0, \pi]\).

So for each \(j \in \mathbb{N}\) and all sufficiently large \(i\), we can find points \(q_{i,j}, r_{i,j} \in M,1\) such that \(d(q_{i,j}, r_{i,j}, p_{i,j}) = d(q_{i,j}, r_{i,j}, p_{i,j}) = j\) and \(\tilde{\lambda}q_{i,j}p_{i,j}r_{i,j} \geq \pi - j^{-1}\). Passing to the limit we obtain points \(q_{j}, r_{j} \in M,1\) with \(d(q_{j}, p_{j,1}) = d(r_{j}, p_{j,1}) = j\) and \(\tilde{\lambda}q_{j}p_{j,1}r_{j} \geq \pi - j^{-1}\). Then letting \(j \to \infty\) and passing to a subsequence, the geodesics \(p_{j,1}q_{j}p_{j,1}r_{j}\) converge to two rays which together form a line passing through \(p_{i,j}\). Then by the strong maximum principle of Ricci flow, \((M,1,0,1)\) is the product of \(\mathbb{R}\) and an \((n-1)\)-dimensional rotationally symmetric steady gradient soliton with positive curvature operator, which is an \((n-1)\)-dimensional Bryant soliton if \(n > 3\), and a cigar soliton if \(n = 3\), see e.g. [CLN06].

For any \(Z \times O(n-1)\)-symmetric expanding or steady gradient soliton \((M,0,0)\) with non-negative curvature operator, we write \(\lambda_1(g), \lambda_2(g) = \cdots = \lambda_n(g)\) to be the \(n\) eigenvalues of the Ricci curvature at \(p\) in the directions of \(\Gamma'(0)\) and its orthogonal complement subspace \(T_p\Sigma = (\Gamma'(0))^\perp\). For any \(\alpha \in (0,1)\), since \(\frac{\lambda_1}{\lambda_2}(g,0,0) = 1\) and \(\frac{\lambda_1}{\lambda_2}(g,1,0) = 0\), we have \(\frac{\lambda_1}{\lambda_2}(g,0,0) > \alpha\) and \(\frac{\lambda_1}{\lambda_2}(g,1,0) < \alpha\) when \(i\) is sufficiently large. Since \(\frac{\lambda_1}{\lambda_2}(g,0,\mu)\) is a continuous function of \(\mu\) for each fixed \(i\), there is some \(\mu_i \in (0,1)\) such that \(\frac{\lambda_1}{\lambda_2}(g,0,\mu_i) = \alpha\). Applying Lemma 2.3 to the sequence \((M,0,0,0,0)\) and taking a limit, we obtain an \(n\)-dimensional \(Z \times O(n-1)\)-symmetric steady gradient soliton \((M,0,0)\) with \(\frac{\lambda_1}{\lambda_2}(g) = \alpha\). This proves Theorem 1.1.

3. Dimension reduction of 3d steady gradient solitons with symmetry

Hamilton proved a dimension reduction result for \((2n+1)\)-dimensional steady solitons with positive curvature operator and bounded curvature. It says that there exists a sequence of points \(\{p_i\}_{i=0}^\infty\) where \(p_i \to \infty\), such that the rescaled sequence \((M, R(p_i)g(t), p_i), t \in (-\infty,0],\) converges to a limit which splits as a product of \(\mathbb{R}\) and a \(2n\)-dimensional ancient solution with non-negative curvature operator and bounded curvature. See also [CDM20] for a discussion of dimension reductions of a 4d steady soliton with non-negative Ricci curvature outside of a compact subset.

**Definition 3.1.** Let \((M,0,0)\) be a 3d \(Z \times O(2)\)-symmetric steady gradient soliton. We say it **strongly dimension reduces** along \(\Gamma\) to a 2-dimensional ancient Ricci flow \((N,0)\), if for any sequence \(s_i \to \infty\), a subsequence of \((M, R(\Gamma(s_i))g(t), \Gamma(s_i)), t \in (-\infty,0],\) smoothly converges to \((N,0)\). We also say a 2d ancient Ricci flow
\( (N, h(t)) \) is a **dimension reduction** of \((M, g, p)\) along \(\Gamma\), if there exists \(s_i \to \infty\) such that \((M, R(\Gamma(s_i))g(t), \Gamma(s_i))\) smoothly converges to \((N, h(t), p_\infty)\).

Assume \((M, g, p)\) is not a Bryant soliton, the main result Theorem 3.3 in this section shows that all dimension reductions of \((M, g, p)\) are non-compact 2d ancient Ricci flows. In particular, assuming in addition \(\lim_{s \to \infty} R(\Gamma(s)) > 0\), then \((M, g, p)\) strongly dimension reduces along \(\Gamma\) to a cigar soliton. We will see by Theorem 1.3 that the assumption \(\lim_{s \to \infty} R(\Gamma(s)) > 0\) always holds true.

First we prove a lemma about the potential function and distance function for all steady gradient solitons \((M, g, f, p)\) with positive curvature operator in any dimension \(n \geq 3\), where \(p\) is a critical point of \(f\).

**Lemma 3.2.** Let \((M^n, g, f, p)\) be an \(n\)-dimensional steady gradient soliton with positive curvature operator. Suppose \(\gamma : (0, \infty) \to M\) is an integral curve of \(\nabla f\), and \(\lim_{s \to 0} \gamma(s) = p\). Then for any \(\epsilon > 0\), there exists \(s_0 > 0\) such that for any \(s_1 > s_2 > s_0\) we have

\[
(1 - \epsilon)(s_2 - s_1) \leq d(\gamma(s_1), \gamma(s_2)) \leq (s_2 - s_1).
\]

In particular, we have \((1 - \epsilon)s \leq d(p, \gamma(s)) \leq s\) for all \(s \geq s_0\). Moreover, let \(\sigma\) be a unit speed minimizing geodesic between \(p\) and \(\sigma(0) := \gamma(s)\). Then

\[
\angle(\sigma'(0), \nabla f) \leq \epsilon.
\]

**Proof.** Without loss of generality, we may also assume \(f(p) = 0\) and \(\lim_{s \to \infty} |\nabla f|(\gamma(s)) = 1\) after rescaling the metric. We use \(\epsilon = \epsilon(s)\) to denote all functions such that \(\lim_{s \to \infty} \epsilon(s) = 0\).

On the one hand, for any \(s_2 > s_1 \geq 0\), let \(\sigma : [0, D] \to M\) be a minimizing geodesic from \(\gamma(s_1)\) to \(\gamma(s_2)\), where \(D = d(\gamma(s_1), \gamma(s_2))\). Since \(\frac{d}{dr}(\nabla f, \sigma'(r)) = \nabla^2 f(\sigma'(r), \sigma'(r)) \geq 0\), we obtain

\[
f(\gamma(s_2)) - f(\gamma(s_1)) = \int_0^D \langle \nabla f, \sigma'(r) \rangle \, dr \leq D \langle \nabla f, \sigma'(D) \rangle,
\]

which by \(|\nabla f| \leq 1\) implies

\[
f(\gamma(s_2)) - f(\gamma(s_1)) \leq d(\gamma(s_1), \gamma(s_2)).
\]

On the other hand, since \(\lim_{s \to \infty} |\nabla f|(\gamma(s)) = 1\), there is \(s_0 > 0\) such that \(|\nabla f|(\gamma(s)) > 1 - \epsilon\) for all \(s \geq s_0\). Therefore, for all \(s_2 > s_1 \geq s_0\) we have

\[
f(\gamma(s_2)) - f(\gamma(s_1)) = \int_{s_1}^{s_2} \langle \nabla f, \gamma'(r) \rangle \, dr = \int_{s_1}^{s_2} |\nabla f|(\gamma(r)) \, dr \geq (1 - \epsilon)(s_2 - s_1),
\]

which together with (3.3) proves the first inequality in (3.1), where the second inequality is an easy consequence of \(|\gamma'(s)| = 1\). The inequality of \(d(p, \gamma(s))\) follows (3.1) and a triangle inequality.
Now let $\sigma : [0, d(p, \gamma(s))] \to M$ be a minimizing geodesic from $p$ to $\gamma(s)$. Then \((3.3)\) implies
\[
(3.6) \quad f(\gamma(s)) \leq d(p, \gamma(s)) \langle \nabla f, \sigma'(d(p, \gamma(s))) \rangle.
\]
Moreover, by \((3.5)\) and $\lim_{s \to \infty} f(\gamma(s)) = \infty$ we have
\[
(3.7) \quad d(\gamma(s_0), \gamma(s)) \leq s - s_0 \leq (1 + \epsilon)(f(\gamma(s)) - f(\gamma(s_0))) \leq (1 + \epsilon)f(\gamma(s))
\]
for all $s$ sufficiently large, which by triangle inequality and $\lim_{s \to \infty} d(p, \gamma(s)) = \infty$ implies
\[
(3.8) \quad d(p, \gamma(s)) \leq d(p, \gamma(s_0)) + d(\gamma(s_0), \gamma(s)) \leq (1 + \epsilon)d(\gamma(s_0), \gamma(s)) \leq (1 + \epsilon)f(\gamma(s)).
\]
This combining with \((3.6)\) and $|\nabla f| \leq 1$ yields
\[
(3.9) \quad \left\langle \frac{\nabla f}{|\nabla f|}, \sigma'(d(p, \gamma(s))) \right\rangle \geq \frac{f(\gamma(s))}{d(p, \gamma(s)) |\nabla f|} \geq 1 - \epsilon,
\]
which proves \((3.2)\).

The following lemma is a weaker version of our dimension reduction theorem \((3.5)\). It shows that all dilation sequence along $\Gamma$ converges to a limit after passing to a subsequence, where the limit may depend on the sequence.

**Lemma 3.3.** Let $(M, g, p)$ be a $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady soliton with positive curvature operator. Then there is $C > 0$ such that the following holds:

For any $s_i \to +\infty$, a subsequence of $(M, R(\Gamma(s_i))g(t), \Gamma(s_i))$, $t \in (-\infty, 0]$, smoothly converges to an ancient Ricci flow $(\mathbb{R} \times g_\infty(t), p_\infty)$, where $g_\infty(t)$ is a 2d ancient Ricci flow with positive curvature operator and $R \leq C$. Moreover, $R^{-1/2}(\Gamma(s_i))\Gamma'(s_i)$ smoothly converges to a unit vector in the $\mathbb{R}$-direction of $\mathbb{R} \times g_\infty(t)$, and $g_\infty(t)$ is rotationally symmetric around $p_\infty$.

**Proof.** Since the lemma clearly holds for $\mathbb{R} \times \text{Cigar}$, we may assume without loss of generality that $(M, g, p)$ has $\text{Rm} > 0$. Let $r(s) = \sup\{\rho > 0 : \text{vol}(B(\Gamma(s), \rho)) \geq \frac{\omega}{2} \rho^3\}$ where $\omega$ is equal to the volume of the ball of radius one in the Euclidean space. Since the asymptotic volume ratio of any 3d ancient Ricci flow with positive curvature is zero, see e.g. [CLN06] theorem 9.30, we have $r(s) < \infty$, and $\lim_{s \to \infty} \frac{r(s)}{s} = 0$ by a volume comparison argument. So $\text{vol}(B(\Gamma(s), r(s))) = \frac{\omega}{2} r^3(s)$.

For any $D > 0$ and any $x \in B(\Gamma(s), Dr(s))$, by a volume comparison argument we have $\text{vol}(B(x, r(s))) \geq C_1^{-1} r^3(s)$ for some $C_1(D) > 0$. Therefore, by applying Theorem 45.1(b) in [KL08], which gives upper curvature bound to a Ricci flow with non-negative curvature operator when there is a volume lower bound (or more general Proposition 3.2 in [Bam18]), there is $C_2(D) > 0$ such that $R \leq C_2 r^{-2}(s)$ in $B(\Gamma(s), Dr(s))$. In particular, there is some $C_0 > 0$ such that $R(\Gamma(s)) \leq C_0 r^{-2}(s)$, and $\text{inj}(\Gamma(s)) \geq C_0^{-1} r(s)$ by the volume bound together with the curvature bound in $B(\Gamma(s), r(s))$. 


Therefore, for any $s_i \to \infty$, by the Harnack inequality and Shi’s estimate we may assume by passing to a subsequence that $(r^{-2}(s_i)g(t), \Gamma(s_i))$, $t \in (-\infty, 0]$, converges to an ancient solution $h(\infty)$. Let $\Gamma_1(s) = \Gamma(r(s_i)s + s_i)$, $s \in (-\infty, \infty)$. Suppose $\Gamma_i$ converges to the geodesic $\Gamma_\infty$ in $h(\infty)$ as $i \to \infty$, modulo the diffeomorphisms. We claim that $\Gamma_\infty$ is a line: Since $\lim_{s \to \infty} \frac{r(s_i)}{r(s)} = 0$, we have $s_i - Dr(s_i) \to \infty$, by which we can apply Lemma 3.2 and deduce that for any $D > 0$ that $\Delta \Gamma_i(\infty)(D) \to \pi$ as $i \to \infty$. So $d(\Gamma_\infty(\infty), \Gamma_\infty(D)) = 2D$. Letting $D \to \infty$, this implies $\Gamma_\infty$ is a line.

Next we claim that there is some $C > 0$ such that $R^{-1/2}(\Gamma(s)) \leq Cr(s)$ for all large $s$. Then the assertion of the lemma follows immediately from replacing the scaling factor $r^{-2}(s_i)$ by $R(\Gamma(s_i))$ and taking the limit. Now suppose by contradiction the claim does not hold, then there is a sequence $s_i \to \infty$ such that $\lim_{i \to \infty} \frac{R^{-1/2}(\Gamma(s_i))}{r(s_i)} = 0$. Then by taking a subsequence we may assume $(r^{-2}(s_i)g, \Gamma(s_i))$ converges to $(\mathbb{R} \times g_\infty(t), p_\infty)$ where $g_\infty(t)$ is some 2d ancient solution. On the one hand, as a consequence of taking the limit, we have $\text{vol}(B(p_\infty, 1)) = \frac{2}{\omega}$ and $R(p_\infty) = 0$, which by the strong maximum principle implies that $g_\infty(t)$ is flat.

On the other hand, since $\Gamma_i$ converges to a line, we can find a sequence $D_i \to \infty$ such that $\Sigma_i := \exp_{\Gamma(s_i)}(\Gamma'(s_i)^{-1}) \cap B(\Gamma(s_i), D_i r(s_i))$ with the metric $g_{\Sigma_i}$ induced by $g$ is a smooth surface which is rotationally symmetric around $\Gamma(s_i)$, and $(r^{-2}(s_i)g_{\Sigma_i}, \Gamma(s_i))$ smoothly converges to $(g_\infty(0), p_\infty)$. So $g_\infty(0)$ is rotationally symmetric around $p_\infty$. Since $g_\infty(0)$ is flat, this implies $g_\infty(0)$ is isometric to $\mathbb{R}^2$, and hence $\text{vol}(B(p_\infty, 1)) = \omega > \frac{2}{\omega}$, a contradiction.

To rephrase the statement of Lemma 3.3 in another way, we introduce the definition of $\epsilon$-closeness between two Ricci flows.

**Definition 3.4.** For any $\epsilon > 0$, we say a pointed Ricci flow $(M_1, g_1(t), p_1)$, $t \in [-T, 0]$, is $\epsilon$-close to a pointed Ricci flow $(M_2, g_2(t), p_2)$, $t \in [-T, 0]$, if there is a diffeomorphism onto its image $\phi : B_{g_2(0)}(p_2, \epsilon^{-1}) \to M_1$, such that $\phi(p_2) = p_1$ and $\|\phi^* g_1(t) - g_2(t)\|_{C^{\infty}(\Upsilon)} < \epsilon$ for all $t \in [-\min\{T, \epsilon^{-1}\}, 0]$, where the norms and derivatives are taken with respect to $g_2(0)$.

Then Lemma 3.3 shows that $(R(\Gamma(s))g(t), \Gamma(s))$ is $\epsilon$-close to the product of $\mathbb{R}$ and a dimension reduction for all sufficiently large $s$. Moreover, a dimension reduction $(M_\infty, g_\infty(t), p_\infty)$ is a 2d ancient solution with positive curvature and rotationally symmetric around $p_\infty$.

Assume $(M, g, p)$ is not a Bryant soliton, the next theorem shows that $M_\infty$ is non-compact. In particular, assuming $\lim_{s \to \infty} R(\Gamma(s)) > 0$, then $(M, g, p)$ strongly dimension reduces along $\Gamma$ to a cigar soliton with $R(p_\infty, 0) = 1$.

**Theorem 3.5.** Let $(M, g, p)$ be a 3d $\mathbb{Z}_2 \times O(2)$-symmetric steady soliton with positive curvature operator, and assume $(M, g, p)$ is not a Bryant soliton. Then any dimension reduction of $(M, g, p)$ along $\Gamma$ is non-compact. In particular, if $\lim_{s \to \infty} R(\Gamma(s)) > 0$, then...
then \((M, g, p)\) strongly dimension reduces along \(\Gamma\) to a cigar soliton \((M_\infty, g_\infty(t), p_\infty)\), \(t \in (-\infty, 0]\), with \(R(p_\infty, 0) = 1\).

**Proof.** Let \(\epsilon > 0\) be sufficiently small. We denote by \(\epsilon_\#\) all constants that depend on \(\epsilon\) such that \(\epsilon_\# \to 0\) as \(\epsilon \to 0\). For each sufficiently large \(s\), by Lemma 3.3 there is a dimension reduction \((h_s(t), p_s)\) of \((M, g, p)\) along \(\Gamma\), such that \((R(\Gamma(s))g(t), \Gamma(s))\) is \(\epsilon\)-close to \((\mathbb{R} \times h_s(t), p_s)\). So \((h_s(t), p_s)\) is a 2d ancient Ricci flow rotationally symmetric around \(p_s\) and \(R(p_s, 0) = 1\). Note the choice of \(h_s(t)\) may not be unique for a fixed \(s\), but any two such solutions are \(\epsilon_\#\)-close to each other. Let

\[
F(s) = \text{diam}(h_s(0)) \in (0, \infty].
\]

First, if \(\limsup_{s \to \infty} F(s) < \frac{1}{100\epsilon_\#}\), then there is \(\kappa = \kappa(\epsilon) > 0\) such that all \(h_s(0)\) is \(\kappa\)-non-collapsed. This implies easily that \((M, g, p)\) is \(\kappa\)-non-collapsed, and hence by the classification result of non-collapsed 3d steady gradient solitons [Bre13] (or more generally the classification of 3d \(\kappa\)-solutions [Bre13, BKL19]), is a Bryant soliton, which is a contradiction. So \(\limsup_{s \to \infty} F(s) \geq \frac{1}{100\epsilon_\#} > 100\pi\).

Next, we claim that \(F(s) \geq D := \frac{1}{100\epsilon_\#}\) for all large \(s\): First, choose \(s_0\) such that \(F(s_0) \geq 3D\), and let

\[
s_1 = \sup\{s \geq s_0 \mid F(\mu) \geq 2D \text{ for all } \mu \in [s, s_0]\}.
\]

Then \(F(s_1) \in [2(1 - \epsilon_\#)D, 2(1 + \epsilon_\#)D]\) and \((h_{s_1}(t), p_{s_1})\) is a Rosenau solution by the classification of compact ancient 2d Ricci flows [DHS12]. Moreover, assume \(\epsilon\) is sufficiently small, then \(1 - \epsilon_\# \leq R(p_{s_1}, t) \leq 1\) for all \(t \leq 0\), see e.g. [CLN06, Chap 4.4], and

\[
\text{diam}(h_{s_1}(t))R^{1/2}(p_{s_1}, t) \geq (1 - \epsilon_\#)F(s_1) \geq 2(1 - \epsilon_\#)D
\]

for all \(t \leq 0\). Moreover, by a distance distortion estimate, see e.g. [KL08, Lem 27.8], we can find a \(t_1 \in [-\epsilon^{-1}, 0]\) such that

\[
\text{diam}(h_{s_1}(t_1))R^{1/2}(p_{s_1}, t_1) = 4D.
\]

Since \(g(t) = \phi_t^* (g)\), where \(\{\phi_t\}_{t \in (-\infty, 0]}\) is the flow of \(-\nabla f\) with \(\phi_0\) the identity. We see that \((g(t), \Gamma(s))\) is isometric to \((g, \phi_t(\Gamma(s)))\), and since \(\Gamma\) is the integral curve of \(-\frac{\nabla f}{\sqrt{g_{\nabla f}}}\), by a direct computation we obtain

\[
\phi_t(\Gamma(s)) = \Gamma \left( s - \int_0^t |\nabla f| (\phi_\mu(\Gamma(s))) \, d\mu \right).
\]

Let \(s_2 = s_1 - \int_0^{T_1} |\nabla f| (\phi_\mu(\Gamma(s_1))) \, d\mu\), where \(T_1 = t_1 R^{-1}(\Gamma(s_1)) < 0\). Then \(s_2 > s_1, \phi_{T_1}(\Gamma(s_1)) = \Gamma(s_2)\), and \((g(T_2), \Gamma(s_1))\) is isometric to \((g, \Gamma(s_2))\). The conditions \(3.12, 3.13\) imply \(F(s) \geq 2(1 - \epsilon_\#)D \geq D\) for all \(s \in [s_1, s_2]\), and \(F(s_2) \geq 4(1 - \epsilon_\#)D \geq 3D\). In particular, this implies \(s_2 - s_1 \geq R^{-1/2}(\Gamma(s_1)) \geq R^{-1/2}(p)\).

Therefore, by induction we find a sequence \(\{s_{2k}\}_{k=0}^\infty\), such that \(F \geq D\) in \([s_{2(k-1)}, s_{2k}]\), \(F(s_{2k}) \geq 3D\), and \(s_{2k} - s_{2(k-1)} \geq R^{-1/2}(p) > 0\). This implies \(F(s) \geq D = \frac{1}{100\epsilon_\#}\) for
all large $s$. Letting $\epsilon \to 0$, we obtain that any dimension reduction along $\Gamma$ is non-compact.

Now assume $\lim_{s \to \infty} R(\Gamma(s)) > 0$. It follows easily from the soliton equation $\text{Ric} = \nabla^2 f$ and Shi’s derivative estimates that all dimension reduction along $\Gamma$ is a steady soliton with positive curvature, which by the classification of 2d steady solitons must be a cigar soliton. This combining with Lemma 3.3 implies that $(M, g, p)$ strongly dimension reduces to a cigar soliton with scalar curvature equal to 1 at the center of rotation.

□

4. 3D STEADY GRADIENT SOLITONS WITH SYMMETRY ARE FLYING WINGS

In this section, we prove Theorem 1.2 and 1.3. Since a 3d steady gradient soliton is non-negatively curved and has zero asymptotic volume ratio, it follows easily that its tangent cone at infinity is either a ray or a sector, where a sector with angle $\alpha \in (0, \pi]$ is a metric cone over the interval $[-\frac{\alpha}{2}, -\frac{\alpha}{2}]$. Theorem 1.2 shows that a 3d steady gradient soliton with $\mathbb{Z}_2 \times O(2)$-symmetry, whose tangent cone at infinity is a ray, has to be a Bryant soliton. Therefore, the family of solitons we found in Theorem 1.1 are all flying wings, which by definition are steady gradient solitons with positive curvature operator whose tangent cone at infinity are sectors. The proof of Theorem 1.2 uses Theorem 1.3 which gives an equality between the angle of the tangent cone at infinity and the limit of the scalar curvature along $\Gamma$.

Throughout this section we assume $(M, g, p)$ is a $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady gradient soliton, and $\Gamma$ and $\Sigma$ are the fixed point sets of the $O(2)$ and $\mathbb{Z}_2$-action respectively.

The next lemma shows that the integral of scalar curvature in metric balls increases at least linearly in radius. We remark that this is also a consequence of [CMM16], which shows that the only 3d steady solitons satisfying $\lim \inf \frac{1}{s} \int_{B(p, s)} R \, d\text{vol}_M = 0$ are quotients of $\mathbb{R}^3$ and $\mathbb{R} \times \text{Cigar}$. The proof below is self-contained and more direct under the symmetric assumption.

**Lemma 4.1.** There exists $C > 0$ such that $\int_{B(p, s)} R \, d\text{vol}_M \geq C^{-1} s$ for sufficiently large $s$.

**Proof.** Fix some small $\epsilon > 0$ and let $s_0 > 0$ be large enough such that Lemma 3.2 holds for $\epsilon$. Consider the covering of $\Gamma([s_0, s])$ by $\{ \Gamma([\mu - R^{-1/2}(\Gamma(\mu)), \mu + R^{-1/2}(\Gamma(\mu))]) \}_{\mu \in [s_0, s]}$. Let $\{ \Gamma([\mu_i - R^{-1/2}(\Gamma(\mu_i)), \mu_i + R^{-1/2}(\Gamma(\mu_i))]) \}_{i=1}^m$ be a Vitali covering of it, which is disjoint from each other and $\Gamma([s_0, s])$ is covered by $\{ \Gamma([\mu_i - 5R^{-1/2}(\Gamma(\mu_i)), \mu_i + 5R^{-1/2}(\Gamma(\mu_i))]) \}_{i=1}^m$. So for any $\mu_i < \mu_j$,

$$\mu_j - \mu_i \geq R^{-1/2}(\Gamma(\mu_i)) + R^{-1/2}(\Gamma(\mu_j)) \geq R^{-1/2}(\Gamma(\mu_j)), \quad (4.1)$$

where
and
\[
(4.2) \quad s - s_0 \leq \sum_{i=1}^{m} 10R^{-1/2}(\Gamma(\mu_i)).
\]

Let \( c = \frac{1 - \epsilon}{4} \), we claim that \( B(\Gamma(\mu_i), cR^{-1/2}(\Gamma(\mu_i))) \) and \( B(\Gamma(\mu_j), cR^{-1/2}(\Gamma(\mu_j))) \) are disjoint: Suppose not, then \( d(\Gamma(\mu_i), \Gamma(\mu_j)) < 2cR^{-1/2}(\Gamma(\mu_j)) \), and by Lemma 3.2 we get
\[
(4.3) \quad \mu_j - \mu_i \leq (1 - \epsilon)^{-1}d(\Gamma(\mu_i), \Gamma(\mu_j)) \leq 2(1 - \epsilon)^{-1}cR^{-1/2}(\Gamma(\mu_j)) < R^{-1/2}(\Gamma(\mu_j)),
\]
which contradicts (4.1).

By Theorem 3.3 and Shi’s derivative estimates, there is some \( C_1 > 0 \) such that
\[
(4.4) \quad \int_{B(\Gamma(s), cR^{-1/2}(\Gamma(s)))} R \, dv_{vol_M} \geq C_1^{-1}R^{-1/2}(\Gamma(s)).
\]
Since \( \lim_{s \to \infty} \frac{R^{-1/2}(\Gamma(s))}{s} = 0 \), which can be seen from the proof of Lemma 3.3, we have \( B(\Gamma(\mu_i), cR^{-1/2}(\Gamma(\mu_i))) \subset B(p, 2s) \) for all \( i \). Therefore, by (4.2) and (4.4) we obtain
\[
(4.5) \quad \int_{B(p, 2s)} R \, dv_{vol_M} \geq \sum_{i=1}^{m} \int_{B(\Gamma(\mu_i), cR^{-1/2}(\Gamma(\mu_i)))} R \, dv_{vol_M} \geq C_2^{-1}s
\]
for some \( C_2 > 0 \).

For convenience, in the rest proofs we shall often use \( \epsilon(s) \) to denote all functions such that \( \lim_{s \to \infty} \epsilon(s) = 0 \), and use \( C \) for all constants that are independent of \( s \).

Recall that for a non-negatively curved Riemannian manifold, the tangent cone at infinity is the a metric space of the equivalent classes of geodesic rays, where the distance between two rays is the limit of their comparison angles at infinity. Moreover, the tangent cone at infinity is isometric to any Gromov-Hausdorff limit of a blow-down sequence of the manifold, see e.g. [KL08]. In the next lemma, we show that geodesics between \( p \) and points going to infinity along \( \Gamma \) and \( \Sigma \) converge to two rays, and their distance in the tangent cone at infinity is equal to \( \frac{\alpha}{2} \).

**Lemma 4.2.** Suppose the tangent cone at infinity of \((M, g, p)\) is a metric cone over the interval \([-\frac{\alpha}{2}, \frac{\alpha}{2}]\), for some \( \alpha \in [0, \pi] \). Then

1. For any sequence \( q_i \in \Sigma \) and \( q_i \to \infty \), the geodesics between \( p \) and \( q_i \) converge to the equivalent class \( 0 \in X \).

2. For any sequence \( s_i \to +\infty \), the geodesics between \( p \) and \( \Gamma(s_i) \) converge to the equivalent class \( \frac{\alpha}{2} \in X \).

3. For any \( q_i \in \Sigma \), \( q_i \to \infty \), and \( o_i = \Gamma(s_i) \), \( s_i \to \infty \), with \( C^{-1}d(p, o_i) \leq d(p, q_i) \leq C d(p, o_i) \), we have \( \lim_{i \to \infty} \tilde{d}(q_i, p) = \frac{\alpha}{2} \).

**Proof.** Since \( \Sigma \) is the fixed point set of the \( \mathbb{Z}_2 \)-action, assertion (1) follows immediately from the fact that the tangent cone at infinity is isometric to the Gromov-Hausdorff limit of \((M, \lambda_i g, p)\) for any sequence \( \lambda_i \to 0 \).
For assertion (2), it suffices to show \( \tilde{Z}_q p_q \vec{\gamma}_i \to \alpha \), where \( q_i = \Gamma(s_i), \vec{\gamma}_i = \Gamma(-s_i), \) \( s_i \to +\infty \). If \( \alpha = \pi \), then \( (M, g, p) \) is \( \mathbb{R} \times \text{Cigar} \) and the conclusion follows easily. So we may assume \( \alpha < \pi \) below. Let \( \gamma_i : [0, d(q_i, \vec{\gamma}_i)] \to M \) be a minimizing geodesic from \( q_i \) to \( \vec{\gamma}_i \), then we see that \( d(p, \gamma_i) \to \infty \) as \( i \to \infty \), because otherwise \( \gamma_i \) would converge to a line..

Let \( \sigma \) be a ray from \( p \) in the class \( \frac{\sigma}{2} \), then its image \( \overline{\sigma} \) under the \( \mathbb{Z}_2 \)-action is a ray in the class \( -\frac{\sigma}{2} \). We may assume by replacing \( \sigma \) with its image under some \( O(2) \)-action that \( \sigma \) (resp. \( \overline{\sigma} \)) intersects with \( \gamma_i \) at some point \( o_i = \gamma_i(D) \) (resp. \( \overline{o_i} = \gamma_i(d(q_i, \vec{\gamma}_i) - D) \)), for some \( D \in [0, \frac{d(q_i, \vec{\gamma}_i)}{2}] \). Then by triangle inequalities we get

\[
\tag{4.6} d(p, o_i) \geq d(p, q_i) - D, \quad d(p, \vec{\gamma}_i) \geq d(p, \vec{\gamma}_i) - D, \quad d(o_i, \vec{\gamma}_i) = d(q_i, \vec{\gamma}_i) - 2D,
\]

using which we can compute directly to show \( \tilde{Z}_q p_q \vec{\gamma}_i \geq \tilde{Z}_o p \vec{\gamma}_i \). Since \( d(p, o_i) \to \infty \) as \( i \to \infty \), it follows that \( \lim_{i \to \infty} \tilde{Z} o p \vec{\gamma}_i = \alpha \), which implies \( \liminf_{i \to \infty} \tilde{Z}_q p \vec{\gamma}_i \geq \alpha \), and hence \( \lim_{i \to \infty} \tilde{Z}_q p \vec{\gamma}_i = \alpha \).

Assertion (3) follows from (1) and (2) immediately.

From now on we fix a minimizing geodesic \( \gamma : [0, \infty) \to \Sigma \) starting from \( p \) such that \( \gamma((0, \infty)) \subset N \), and two functions \( h_1(s) = d(\gamma(s), \Gamma) \) and \( h_2(s) = \varphi(\gamma(s)) \) that can be thought of as “dimensions” of the soliton. For example, we have \( h_1(s) \approx s^{1/2}, h_2(s) \approx s^{1/2} \) in a Bryant soliton, and \( h_1(s) \approx s, \lim_{s \to \infty} h_2(s) < \infty \) in \( \mathbb{R} \times \text{Cigar} \). We establish inequalities between these two functions and \( R(\gamma(s)) \) in the following three lemmas, when \( s \) is sufficiently large.

**Lemma 4.3.** There exists \( C > 0 \) such that \( h_1^2(s) R(\gamma(s)) \leq C \) for all large \( s \).

**Proof.** Without loss of generality we may assume \( \alpha < \pi \), because otherwise \((M, g, p)\) is \( \mathbb{R} \times \text{Cigar} \), where the assertion follows from the exponential decay of the scalar curvature.

Let \( p_1 = \gamma(s) \) and \( p_2 = \Gamma(s \cos \frac{\alpha}{2}) \). On the one hand, since \( \alpha < \pi \), we have \( \Gamma(s \cos \frac{\alpha}{2}) \to \infty \) as \( s \to \infty \), which allows us to apply Lemma 3.22 and deduce \( \left| \frac{d(p_1 p_2)}{s} - 1 \right| + \left| \frac{d(p_2 p_2)}{s} \cos \frac{\alpha}{2} \right| \leq \epsilon(s) \). Moreover, since \( |\tilde{Z} p_1 p_2 - \tilde{\varphi} \frac{\alpha}{2}| \leq \epsilon(s) \) by Lemma 4.2, it follows that \( \left| \tilde{Z} p p_1 p_2 - \left( \frac{\alpha}{2} - \frac{\alpha}{2} \right) \right| \leq \epsilon(s) \). Choose \( p', p_2' \) in the minimizing geodesics between \( p, p_1 \) and \( p_1, p_2 \) such that \( d(p_1, p_2') = d(p_1, p') = h_1(s) \). Then by angle comparison \( \tilde{Z} p' p_1 p_2' \geq \tilde{Z} p p_1 p_2 \geq \frac{\alpha}{2} - \frac{\alpha}{2} - \epsilon(s) \), and hence \( \partial B_N(p_1, h_1(s)) \geq d(p', p_2') \geq C^{-1} h_1(s) \). So by volume comparison we get

\[
\tag{4.7} \text{vol}(B_N(p_1, h_1(s))) \geq C^{-1} h_1^2(s).
\]

On the other hand, let \( \tilde{M}_0 \to M_0 := M \setminus \Gamma \) be the universal covering, and \((\tilde{M}_0, \tilde{g}(t), \tilde{p}_1)\) be the pull-back Ricci flow of \((M_0, g(t), p_1), t \in (-\infty, 0), \) where \( g(t) \) is the steady soliton solution with \( g(0) = g \). Then \( \tilde{g}(0) = g_N + \varphi^2 d\theta^2, \theta \in (-\infty, \infty) \),
and by using (4.7) we get
\begin{equation}
\text{vol}(B_{\tilde{g}}(0)(\bar{p}_1, h_1(s))) \geq \frac{1}{2} h_1(s) \text{vol}(B_N(p_1, \frac{1}{2} h_1(s))) \geq C^{-1} h_1^3(s).
\end{equation}

So by applying Theorem 45.1(b) in [KL08], we obtain $R(p_1) = R(\bar{p}_1) \leq C h_1^{-2}(s)$. \hfill \square

**Lemma 4.4.** Suppose $(M, g, p)$ is not a Bryant soliton. Then $\frac{h_2(s)}{h_1(s)} \to 0$ as $s \to \infty$.

*Proof.* Suppose by contradiction that there is a sequence $s_i \to \infty$ such that $\frac{h_2(s_i)}{h_1(s_i)} \geq C^{-1} > 0$ for some $C > 0$ and all $i$. Let $\sigma_i$ be a minimizing geodesic from $\gamma(s_i, q_i)$. Then $\sigma_i$ intersects with $\Gamma$ orthogonally at $q_i$. Let $\Sigma_i = \phi^{-1}(\sigma_i)$, where $\phi : (M \setminus \Gamma, g) \to (N, g_N)$ is the Riemannian submersion. Then $(\Sigma_i, g_i)$ is a smooth rotationally symmetric surface with non-negative curvature, where $g_i$ is the metric induced by $g$. Then by Theorem 3.5 $(\Sigma_i, R(\Gamma(s_i))g_i)$ smoothly converges to the time-0-slice of a non-compact ancient Ricci flow $g_\infty(t)$.

Moreover, by Theorem 3.3 we know that any blow-down limit along $\Gamma$ is a product of $\mathbb{R}$ and a non-compact ancient Ricci flow, from which it follows that $\lim_{s \to \infty} h_1(s) R^{1/2}(\Gamma(s)) = \infty$. This combining with $\frac{h_2(s)}{h_1(s)} \geq C^{-1}$ and a volume comparison implies that the asymptotic volume ratio of $g_\infty(0)$ is positive, and hence $g_\infty(t)$ is flat, a contradiction. \hfill \square

**Lemma 4.5.** Suppose the tangent cone at infinity of $(M, g, p)$ is a ray. Then there is some $C > 0$ such that $h_1(s) h_2(s) \geq C^{-1} s$ for all large $s$.

*Proof.* The assertion clearly holds when $(M, g, p)$ is a Bryant soliton, so we may assume below that $(M, g, p)$ is not a Bryant soliton.

On the one hand, since $h_1(s) = d(\gamma(s), \Gamma)$, we have $d(q, \gamma(s)) = h_1(s)$ for some $q \in \Gamma$. Let $\overline{q}$ be the image of $q$ under the $\mathbb{Z}_2$-action, and $\sigma : [-\frac{1}{2} d(q, \overline{q}), \frac{1}{2} d(q, \overline{q})]$ be a minimizing geodesic from $q$ to $\overline{q}$. Then by the $\mathbb{Z}_2$-symmetry it follows that $\sigma$ intersects orthogonally with $\Sigma$ at $\sigma(0)$ and
\begin{equation}
\frac{d(q, \sigma(0)) = d(q, \Sigma) = \frac{1}{2} d(q, \overline{q}).
\end{equation}

Moreover, by replacing $\sigma$ with its image under some $O(2)$-action, we may assume $\sigma(0) \in \gamma$. So we have
\begin{equation}
\frac{1}{2} d(q, \overline{q}) = d(q, \gamma) \leq d(q, \gamma(s)) = h_1(s).
\end{equation}

Since the tangent cone at infinity is a ray, by Lemma 4.2 and $h_1(s) = d(\gamma(s), \Gamma) \leq d(\gamma(s), \Gamma(s))$, we see $h_1(s) \leq \epsilon(s)s$. So by Lemma 3.2 and using triangle inequality we obtain
\begin{equation}
d(p, \sigma(0)) \leq d(p, \gamma(s)) + d(\gamma(s), q) + d(q, \sigma(0)) \leq d(p, \gamma(s)) + 2h_1(s) \leq (1 + \epsilon(s))s.
\end{equation}
Suppose \( \sigma(0) = \gamma(s') \) for some \( s' > 0 \), then by Lemma 3.2, this implies \( s' \leq (1 + \epsilon(s))s \), which by the concavity of \( h_2 \) yields
\[
(4.12) \quad h_2(s) \geq (1 - \epsilon(s))h_2(s') \geq \frac{1}{2}h_2(s').
\]

On the other hand, let \( \Omega(s) \subset M \) be the domain bounded by \( \phi^{-1}(\sigma) \), where \( \phi : (M \setminus \Gamma, g) \to (N, g_N) \) is the Riemannian submersion, then
\[
(4.13) \quad d(\partial\Omega(s), p) \geq d(p, \sigma(0)) - d(q, \sigma(0)) \geq (1 - \epsilon(s))s - h_1(s) \geq (1 - \epsilon(s))s,
\]
which implies \( \Omega(s) \supset B(p, \frac{1}{2}s) \), So by Stokes’ theorem, \( R = \Delta f \), and Lemma 4.1 we obtain
\[
(4.14) \quad \text{Area}(\partial\Omega(s)) \geq \int_{\partial\Omega(s)} \langle \nabla f, \vec{n} \rangle = \int_{\Omega(s)} \Delta f \, d\text{vol}_M \geq \int_{B(p, \frac{1}{2}s)} R \, d\text{vol}_M \geq C^{-1}s.
\]
By the \( \mathbb{Z}_2 \)-symmetry we have \( \frac{d}{dr} |_{r=0} \varphi(\sigma(r)) = 0 \), which combining with the concavity of the warping function \( \varphi \) implies \( \varphi(\sigma(r)) \leq \varphi(\sigma(0)) = h_2(s') \) for all \( r \in [-\frac{1}{2}d(p, \overline{p}), \frac{1}{2}d(p, \overline{p})] \). So
\[
(4.15) \quad \text{Area}(\partial\Omega(s)) = \int_0^{2\pi} \int_{-\frac{1}{2}d(p, \overline{p})}^{\frac{1}{2}d(p, \overline{p})} \varphi(\sigma(r)) \, dr \, d\theta \leq 2\pi d(q, \overline{q})h_2(s') \leq Ch_1(s)h_2(s),
\]
where we used (4.10) and (4.12) in the last inequality. This together with (4.14) proves the lemma.

**Lemma 4.6.** Suppose the tangent cone at infinity is a ray, and \( \lim_{s \to \infty} h_2(s) < \infty \). Then \( \lim_{s \to \infty} R(\Gamma(s)) > 0 \).

**Proof.** Suppose \( s \) is sufficiently large, and \( \lim_{r \to \infty} \varphi(\gamma(r)) = \lim_{r \to \infty} h_2(r) = C \). Let \( p_1 = \Gamma(s) \), \( \overline{p_1} = \Gamma(-s) \), and \( \sigma : [-\frac{1}{2}d(p_1, \overline{p_1}), \frac{1}{2}d(p_1, \overline{p_1})] \to M \) be a minimizing geodesic from \( p_1 \) to \( \overline{p_1} \), which intersects \( \gamma \) at \( \sigma(0) \). Let \( pp_1, p\overline{p_1}, p\overline{p_1} = \sigma \) be minimizing geodesics between these points that are contained in \( N \). Then since \( \angle_{p_1p_1\overline{p_1}} \leq \epsilon(s) \), we have \( \angle_{pp_1\overline{p_1}} \geq \angle_{pp_1\overline{p_1}} \geq \frac{\pi}{2} - \epsilon(s) \).

For some \( s' >> s \), take \( q = \gamma(s') \), and let \( qp_1, q\overline{p_1} \) be minimizing geodesics between these point. By replacing \( \sigma = p_1\overline{p_1} \) and \( pp_1 \) with their image under suitable \( O(2) \)-actions, we may assume that \( \angle_{pp_1\overline{p_1}} + \angle_{qp_1p_1} \leq \pi \). Since by angle comparison \( \angle_{p_1p_1q} \geq \angle_{p_1\overline{p_1}p_1} \geq \frac{\pi}{2} - \epsilon(s) \), it follows that \( |\angle_{pp_1\overline{p_1}} - \frac{\pi}{2}| \leq \epsilon(s) \). Moreover, since \( \angle(\nabla f(p_1), pp_1) \leq \epsilon(s) \) by Lemma 3.2, it follows that the angle between the tangent vectors of \( \sigma \) and \( \Gamma \) at \( p_1 \) is \( \epsilon(s) \)-close to \( \frac{\pi}{2} \).

By the dimension reduction Theorem 3.3, it is easy to see \( R^{-1/2}(\Gamma(s)) < \frac{1}{2}d(p_1, \overline{p_1}) \) and
\[
(4.16) \quad \varphi(\sigma(R^{-1/2}(\Gamma(s)))) \geq C^{-1}R^{-1/2}(\Gamma(s)).
\]
By the symmetry it follows that \( \sigma \) intersects with \( \gamma \) orthogonally at \( \sigma(0) \), and \( \frac{d}{dr}|_{r=0} \varphi(\sigma(r)) = 0 \). So by the concavity of \( \varphi \) we get
\[
\varphi(\sigma(R^{-1/2}(\Gamma(s)))) \leq \varphi(\sigma(0)) \leq C,
\]
which together with (4.16) implies the lemma. \( \square \)

Now we prove Theorem 1.3, which says that the equation
\[
\text{(4.17)}
\]
holds if the tangent cone at infinity of \((M, g, p)\) is a sector of angle \( \alpha \in [0, \pi] \). The idea of the proof is the following: First, for each sufficiently large \( s \), we find a unit speed minimizing geodesic \( \sigma : [0, a] \to M \) with \( \sigma(0) = \Gamma(s) \) and \( \sigma(a) \in \gamma \), such that \( \angle(\nabla f, \sigma'(r)) \) is close to \( \frac{\pi}{2} \) at \( r = 0 \), and close to \( \frac{\pi - \alpha}{2} \) at \( r = a \). So \( \langle \nabla f, \sigma'(r) \rangle |_{\sigma} \) is close to \( \sin \frac{\alpha}{2} \). Then by the soliton equation \( \text{Ric} = \nabla^2 f \) we see that
\[
\text{(4.18)}
\]
which implies that the integral \( \int_{0}^{a} \text{Ric}(\sigma'(r), \sigma'(r)) \, dr \) is close to \( \sin \frac{\alpha}{2} \).

It remains to show the integral \( \int_{0}^{a} \text{Ric}(\sigma'(r), \sigma'(r)) \, dr \) is close to \( R^{1/2}(\Gamma(s)) \). To do this, we split it into two parts on \([0, b]\) and \([b, a]\) for some suitable \( b < a \). On the one hand, we show by the second variation formula that \( \int_{0}^{a} \text{Ric}(\sigma'(r), \sigma'(r)) \, dr \) is much smaller than \( R^{1/2}(\Gamma(s)) \), since the curvature near \( \sigma(b) \) and \( \sigma(a) \) is sufficiently smaller than \( R(\Gamma(s)) \). On the other hand, since \((M, g, p)\) strongly dimension reduces to a cigar soliton, we show that \( \int_{0}^{b} \text{Ric}(\sigma'(r), \sigma'(r)) \, dr \) is sufficiently close to \( R^{1/2}(\Gamma(s)) \).

**Proof of Theorem 1.3** First, if \( \alpha = \pi \), the soliton is \( \mathbb{R} \times \text{Cigar} \), which clearly satisfies the equation. So from now on we assume \( \alpha \in [0, \pi) \). We may also assume \( R(p) = 1 \).

We claim for each \( s \) sufficiently large, there is a minimizing geodesic \( \sigma : [0, d(p_1, p_2)] \to M \) from \( p_1 \) to \( p_2 \), such that \( p_1 = \Gamma(s) \), \( p_2 \in \gamma \), and
\[
\text{(4.20)}
\]
For \( \alpha = 0 \), we choose \( \sigma \) as in Lemma 4.6 then it satisfies all the conditions. So we may assume \( \alpha > 0 \) below.

Let \( p_2 = \gamma(\frac{\alpha}{\cos \frac{\alpha}{2}}) \). Then we have
\[
\left| d(p_1, p_2) - 1 \right| + \left| \frac{d(p_1, p_2)}{s} - \frac{1}{\cos \frac{\alpha}{2}} \right| \leq \varepsilon(s) \text{ by Lemma 3.2} \]
and
\[
\left| \tilde{\gamma}p_1 p_2 - \frac{\alpha}{2} \right| \leq \varepsilon(s) \text{ by Lemma 4.2} \]
So by cosine formula we obtain
\[
\text{(4.21)}
\]
Since \( \alpha > 0 \), this yields
\[
\text{(4.22)}
\]
Let $p_3 = \Gamma(\frac{s}{\cos^2(\frac{\alpha}{2})})$. Then by Lemma 3.2 and some triangle inequalities we get

\[(4.23) \quad \left| \frac{d(p_1, p_3)}{s} - \left( \frac{1}{\cos^2\left(\frac{\alpha}{2}\right)} - 1 \right) \right| + \left| \frac{d(p_2, p_3)}{s} - \tan\frac{\alpha}{2} \right| \leq \epsilon(s), \]

which together with (4.21) implies

\[(4.24) \quad \left| \angle p_3 p_1 p_2 \right| - \frac{\pi}{2} \leq \epsilon(s). \]

Let $pp_2, pp_1, p_1 p_2, p_1 p_3, p_2 p_3$ be minimizing geodesics between these points, and let $\sigma = p_1 p_2 : [0, d(p_1, p_2)] \to M$. In particular, by replacing geodesics $pp_1, p_1 p_3$ with their images under suitable $O(2)$-actions (note $p, p_1, p_3 \in \Gamma$ are fixed under $O(2)$-actions), we may assume $\angle p_2 p_1 p + \angle p_3 p_1 p_2 \leq \pi$. So by (4.24), (4.22) and angle comparisons we obtain

\[(4.25) \quad \left| \angle p_2 p_1 p - \frac{\pi}{2} \right| \leq \epsilon(s). \]

In the same way we can show

\[(4.26) \quad \left| \angle pp_2 p_1 - \left( \frac{\pi - \alpha}{2} \right) \right| \leq \epsilon(s). \]

By Lemma 3.2 we have $\angle (\nabla f(p_1), pp_1) < \epsilon(s)$, which combining with (4.23) yields $|\langle \nabla f(p_1), \sigma'(0) \rangle| \leq \epsilon(s)$. Moreover, by Lemma 4.3 and 4.4 we see that $\lim_{r \to \infty} R(\gamma(r)) = 0$, so $\lim_{r \to \infty} |\nabla f|^2(\gamma(r)) = 1$. Then $\angle (\nabla f(p_2), pp_2) < \epsilon(s)$ by Lemma 3.2 which combining with (4.26) yields $|\langle \nabla f, \sigma'(d(p_2, p_1)) \rangle| \leq \epsilon(s)$. So we find the geodesic $\sigma$ as claimed.

By the soliton equation $\nabla^2 f = \text{Ric}$ we have

\[(4.27) \quad \langle \nabla f, \sigma'(r) \rangle_{d(p_2, p_1)} = \int_{0}^{d(p_2, p_1)} \text{Ric}(\sigma'(r), \sigma'(r)) \, dr. \]

On the one hand, since $\sigma$ satisfies (4.20), it follows that

\[(4.28) \quad \left| \langle \nabla f, \sigma'(r) \rangle_{d(p_2, p_1)} - \sin\frac{\alpha}{2} \right| \leq \epsilon(s). \]

On the other hand, by Theorem 3.5, we can find $D(s) < d(p_2, p_1)$ such that $\lim_{s \to \infty} D(s) = \infty$ and $d(\sigma(D(s)), \Gamma) \geq \frac{D(s)}{2}$, and moreover

\[(4.29) \quad R^{-1/2}(\Gamma(s)) \int_{0}^{D(s)} \text{Ric}(\sigma'(r), \sigma'(r)) \, dr \leq C, \]

if $\lim_{s \to \infty} R(\Gamma(s)) = 0$ where $C > 0$ is a constant independent of $s$, and

\[(4.30) \quad \left| \left( R^{-1/2}(\Gamma(s)) \int_{0}^{D(s)} \text{Ric}(\sigma'(r), \sigma'(r)) \, dr \right) - 1 \right| \leq \epsilon(s) \]
if \( \lim_{s \to \infty} R(\Gamma(s)) > 0 \), where we used the fact that for a cigar soliton with the sectional curvature \( K \) at the center of rotation equal to \( \frac{1}{2} \), the integral of \( K \) along a geodesic emanating from \( p \) is \( \int_0^\infty K \, dr = \int_0^\infty \frac{1}{2} \text{sech}^2\left(\frac{1}{2} r\right) \, dr = 1 \).

Since \( d(\sigma(D(s)), \Gamma) \geq \frac{D(s)}{2} \), by the same argument as Lemma 4.3, we get \( R \leq C(D(s))^{-2} \) in the two balls \( B(\sigma(D(s)), D(s)) \) and \( B(\sigma(d(p_2, p_1)), D(s)) \). This implies by the second variation formula that

\[
(4.31) \quad \int_{D(s)}^{d(p_2, p_1)} \text{Ric}(\sigma'(r), \sigma'(r)) \, dr \leq \frac{C}{D(s)} \leq \epsilon(s) R^{1/2}(\Gamma(s)).
\]

If \( \lim_{s \to \infty} R(\Gamma(s)) = 0 \), this combining with (4.29), (4.28) and (4.27) implies \( \sin \frac{2}{r} = 0 \), \( \alpha = 0 \), and hence the theorem holds in this case. If \( \lim_{s \to \infty} R(\Gamma(s)) > 0 \), then (4.31) together with (4.30) implies

\[
(4.32) \quad \left| \int_0^{d(p_2, p_1)} \text{Ric}(\sigma'(r), \sigma'(r)) \, dr - R^{1/2}(\Gamma(s)) \right| \leq \epsilon(s) R^{1/2}(\Gamma(s)) \leq \epsilon(s).
\]

Combining (4.28) (4.32) (4.27) and letting \( s \to \infty \), the theorem follows immediately.

Now we prove Theorem 1.2 by a bootstrap argument: First we observe that by the identity satisfied by the killing field \( \frac{\partial}{\partial \theta} \) on \( M \setminus \Gamma \), we can establish a relation between some curvature and the warping function \( \varphi \) restricted on \( \gamma \subset \Sigma \):

\[
(4.33) \quad \text{Ric} \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right) = |\nabla f|(\gamma(s)) h_2(s) h_2'(s).
\]

Recall \( h_2(s) = \varphi(\gamma(s)) \).

Now suppose by contradiction that the soliton is not a Bryant soliton, then by combining the estimates from Lemma 4.3, 4.5 in the equation (4.33), we obtain that \( h_2(s) \) grows slower than \( s^{1/2} \). Next, replacing the weak estimate of \( h_2(s) \leq \epsilon(s) h_1(s) \) from Lemma 4.4 with our new upper bound \( h_2(s) \ll s^{1/2} \), and applying Lemma 4.3 again, we deduce that \( h_2(s) \) stays bounded when \( s \) goes to infinity. In particular, this implies \( \lim_{s \to \infty} R(\Gamma(s)) > 0 \), which by Theorem 1.3 implies that the tangent cone at infinity is a sector with positive angle, contradicting to our assumption.

**Proof of Theorem 1.2** Let \( \epsilon(s) \) be constants that converge to \( 0 \) as \( s \to \infty \), and let \( C \) denote all constants that are uniform for all large \( s \). Suppose by contradiction that \( M \) is not a Bryant soliton. We shall use the notations in Lemma 4.3, 4.5. Since \( g = g_N + \varphi^2 d\theta^2 \) on \( M \setminus \Gamma \), it follows that \( X := \frac{\partial}{\partial \theta} \) is a killing field. So by the identity of killing field we have

\[
(4.34) \quad \langle \nabla_X X, \nabla f \rangle + \langle \nabla \nabla f X, X \rangle = 0.
\]

Note that \( \langle X, \nabla f \rangle = 0 \) and \( \nabla^2 f = \text{Ric} \), this gives the identity

\[
(4.35) \quad \text{Ric} \left( \frac{X}{|X|}, \frac{X}{|X|} \right) = \frac{\nabla f(|X|)}{|X|}.
\]
Restrict the LHS of (4.35) on $\gamma(s)$ and abbreviate it by $\tilde{R}(s)$, then by the relations among $h_1(s), h_2(s)$ and $R(\gamma(s))$ from Lemma 1.3, 1.4 and 1.3 we obtain
\[
(4.36) \quad s \tilde{R}(s) \leq s R(\gamma(s)) \leq C h_1(s) h_2(s) R(\gamma(s)) \leq \epsilon(s) h_1(s)^2 R(\gamma(s)) \leq \epsilon(s),
\]
which by (4.35), $\lim_{s \to \infty} |\nabla f|(\gamma(s)) = C > 0$, $h_2(s) = |X|(\gamma(s))$, and $h_2'(s) \geq 0$ implies
\[
(4.37) \quad \frac{h_2'(s)}{h_2(s)} \leq \frac{\nabla f(h_2(s))}{|\nabla f| h_2(s)} = \frac{2 \tilde{R}(s)}{|\nabla f| h_2(s)} < \frac{\epsilon(s)}{C s} < \frac{2 \epsilon_0}{s},
\]
for all large $s$, where $\epsilon_0 > 0$ is chosen such that $2 \epsilon_0 < 1$. So $h_2(s) < C s^{-\epsilon_0}$ for all large $s$.

Next, by using $h_2(s) < C s^{2\epsilon_0}$ and applying Lemma 1.3 again we obtain $h_1(s) \geq C^{-1} s^{1-\epsilon_0}$, which combining with Lemma 1.3 again gives
\[
(4.38) \quad \tilde{R}(s) \leq R(\gamma(s)) \leq C s^{-2+2\epsilon_0}.
\]
Now substituting this into equation (4.35) we obtain
\[
(4.39) \quad \frac{h_2'(s)}{h_2(s)} < C s^{-2+2\epsilon_0},
\]
which implies $h_2(s) \leq C e^{-C s^{-1+2\epsilon_0}}$, and the warping function $\varphi$ satisfies $\lim_{s \to \infty} \varphi(\gamma(s)) < \infty$. This by Lemma 1.6 implies $\lim_{s \to \infty} R(\Gamma(s)) > 0$, which by Theorem 1.3 yields a contradiction. \hfill \Box

Proof of Corollary 1.4. First, by the proof of Theorem 1.1 there exist a $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady gradient soliton $(M_1, g_1, p_1)$ and a sequence of $\mathbb{Z}_2 \times O(2)$-symmetric expanding gradient solitons with positive curvature operator $(M_{1k}, g_{1k}, p_{1k})$ which smoothly converges to $(M_1, g_1, p_1)$, where $R_{g_{1k}}(p_{1k}) = R_{g_1}(p_1) = 1$ and $(M_1, g_1, p_1)$ is neither a Bryant soliton nor $\mathbb{R} \times \text{Cigar}$. So by Theorem 1.2 there exists $\alpha_1 \in (0, \pi)$ such that the tangent cone at infinity of $(M_1, g_1, p_1)$ is a sector with angle $\alpha_1$, and applying Theorem 1.3 we get $\lim_{s \to \infty} R_{g_1}(\Gamma(s)) = \sin^2 \frac{\alpha_1}{2}$.

Let $(M_0, g_0, p_0)$ be a Bryant soliton with $R_{g_0}(p_0) = 1$, since $\lim_{s \to \infty} R_{g_0} (\Gamma(s)) = 0$, we can find $s_1 > 0$ such that $R_{g_0}(\Gamma(s_1)) < \frac{1}{2} \sin^2 \frac{\alpha_1}{2}$. Choose a constant $\tilde{R} \in (R_{g_0}(\Gamma(s_1)), \frac{1}{2} \sin^2 \frac{\alpha_1}{2})$. Then as in the proof of Theorem 1.1, we can find a sequence of $\mathbb{Z}_2 \times O(2)$-symmetric expanding gradient solitons $(M_{2k}, g_{2k}, p_{2k})$ with positive curvature operator, which smoothly converges to a $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady gradient soliton $(M_2, g_2, p_2)$, with $R_{g_2}(p_2) = R_{g_2}(p_{2k}) = 1$ and $R_{g_2}(\Gamma(s_1)) = \tilde{R}$. Assume the tangent cone at infinity of $(M_2, g_2, p_2)$ is a sector with angle $\alpha_2 \in [0, \pi]$. Then $\alpha \in (0, \pi)$ by Theorem 1.2. Moreover, by Theorem 1.3 we have
\[
(4.40) \quad \sin^2 \frac{\alpha_2}{2} = \lim_{s \to \infty} R_{g_2}(\Gamma(s)) \leq \tilde{R} < \frac{1}{2} \sin^2 \frac{\alpha_1}{2}.
\]
Therefore, by induction we get a sequence of $\mathbb{Z}_2 \times O(2)$-symmetric steady gradient solitons $(M_i, g_i, p_i)$ whose tangent cone at infinity is a sector with angle $\alpha_i$ satisfying $\sin^2 \frac{\alpha_{i+1}}{2} < \frac{1}{2} \sin^2 \frac{\alpha_i}{2}$ for all $i$. So $\alpha_i \to 0$ as $i \to \infty$. \hfill \Box
References

[Bam18] Richard H. Bamler. Long-time behavior of 3-dimensional ricci flow A: Generalizations of perelman’s long-time estimates. Geometry and Topology, 22(2):775–844, 2018.

[BGP92] Yu Burago, M Gromov, and G Perelman. A.D. Alexandrov spaces with curvature bounded below. Russian Mathematical Surveys, 47(2):1–58, 1992.

[BK19] Richard H. Bamler and Bruce Kleiner. On the rotational symmetry of 3-dimensional $\kappa$-solutions. arxiv.org/abs/1904.05388, 2019.

[Bre13] Simon Brendle. Rotational symmetry of self-similar solutions to the Ricci flow. Inventiones Mathematicae, 194(3):731–764, 2013.

[Bre18] Simon Brendle. Ancient solutions to the Ricci flow in dimension 3. arxiv.org/abs/1811.02559, 2018.

[Bre05] RL Bryant. Ricci flow solitons in dimension three with SO (3)-symmetries. preprint, Duke Univ, pages 1–24, 2005.

[Cao10] Huai-Dong Cao. Recent Progress on Ricci Solitons. Advanced Lectures in Mathematics, 11:1–38, 2010.

[CCG07] Bennet Chow, SC Chu, and D Glickenstein. The Ricci flow: techniques and applications Volume 2– Part I: Geometric Aspects. Part I: Geometric Aspects . . . , 2, 2007.

[CDM20] Bennet Chow, Yuxing Deng, and Zilu Ma. On four-dimensional steady gradient Ricci solitons that dimension reduce. arXiv:2009.11456, 2020.

[CH18] Huai Dong Cao and Chenxu He. Infinitesimal rigidity of collapsed gradient steady Ricci solitons in dimension three. Communications in Analysis and Geometry, 26(3):505–529, 2018.

[Cho14] Otis Chodosh. Expanding Ricci solitons asymptotic to cones. Calculus of Variations and Partial Differential Equations, 51(1-2):1–15, 2014.

[CLN06] Bennett Chow, Peng Lu, and Lei Ni. Hamilton’s Ricci Flow. American Mathematical Society, 2006.

[CMM16] Giovanni Catino, Paolo Mastrolia, and Dario D. Monticelli. Classification of expanding and steady ricci solitons with integral curvature decay. Geometry and Topology, 20(5):2665–2685, 2016.

[Der16] Alix Deruelle. Smoothing out positively curved metric cones by Ricci expanders. Geometric and Functional Analysis, 26(1):188–249, 2016.

[DHS12] Panagiota Daskalopoulos, Richard Hamilton, and Natasa Sesum. Classification of ancient compact solutions to the ricci flow on surfaces. Journal of Differential Geometry, 91(2):171–214, 2012.

[DZ16] Yuxing Deng and Xiaohua Zhu. 3d steady Gradient Ricci Solitons with linear curvature decay. arXiv:1612.05713, 2016.

[Ham88] Richard Hamilton. The Ricci flow on surfaces. Contemporary Mathematics, 71:237–261, 1988.

[Ham93] Richard Hamilton. The formations of singularities in the Ricci Flow. Surveys in Differential Geometry, 2(1):7–136, 1993.

[KL08] Bruce Kleiner and John Lott. Notes on Perelman’s papers. Geom. Topol., 2, 2008.

[Wan11] Xu Jia Wang. Convex solutions to the mean curvature flow. Annals of Mathematics, 173(3):1185–1239, 2011.

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