Relaxation of twisted vortices in the Faddeev-Skyrme model

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Abstract

We study vortex knotting in the Faddeev-Skyrme model. Starting with a straight vortex line twisted around its axis we follow its evolution under dissipative energy minimization dynamics. With low twist per unit length the vortex forms a helical coil, but with higher twist numbers the vortex becomes knotted or a ring is formed around the vortex.

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Vortices are familiar objects in the physical world, from smoke rings to hurricanes and maelstroms. There is always an associated vector field, which in the above-mentioned examples is the velocity field of the underlying material, and which has a special value that determines the core of the vortex (e.g., the eye of a hurricane). But by its nature, velocity-vector fields can deform continuously to zero and therefore the corresponding vortex can disappear.

Recently there have been experimental observations of vortices in more exotic circumstances in which the relevant vector field is not associated to velocity but rather to spin or other such property. Then it is possible to have vortices that are both \textit{nonsingular} and \textit{conserved}, and the conservation follows from topological reasons. A typical example of such a case is a vortex in a material that is described by a unit-vector field $\mathbf{n} = (n_1, n_2, n_3)$, $|\mathbf{n}| = 1$. There are many different topological properties that could come to play, but here we are only interested in vector fields that are characterized by the Hopf charge. Then we have a nonsingular unit vector field for which one can \textit{locally} define a topological charge density whose integrated value is conserved. Thus, in this case, we assume that faraway the vector field is parallel, which is in contrast to topological charges determined from nontrivial asymptotic properties as is the case with monopoles. Physically such vortex
configurations can be obtained for example in the “continuous unlocked vortex” (CUV) structure of rotating superfluid $^3$He-A [1]: If the vortex is along the $z$-direction, then far away in the $(x,y)$-plane the $\hat{l}$-spin points to the $x$-direction, at the vortex core to the opposite direction, and in between the 3D unit vectors interpolate these values continuously. In general, topological vortex structures have been detected in liquid helium in the superfluid state [1 2 3], two dimensional electron gas systems like the quantum Hall ferromagnets [4], nematic liquid crystals [5], Bose-Einstein condensates [6] and superconductors [7]. They probably exist also in cosmic strings [8], the solar corona [9] and in SU(2) Yang-Mills model [10]. In many of these examples the unit vector field mentioned above is not manifest, but it can be seen when the standard physical variables are suitably parameterized [11].

Vortices in 3D are often constructed simply by stacking identical $(x,y)$-layers of 2D vortex structures. In the CUV model mentioned above, the asymptotic behavior and core location are invariant under a global rotation of the vector field around the $x$-direction, and the model itself is often invariant under such a gauge rotation as well. In that case, one can construct twisted vortices: the vortex core (e.g., the $z$-axis) is straight, the asymptotic direction of the vortex field (in the $(x,y)$-plane) is constant (e.g., along the $x$-axis), but the vector field on different $z$-layers is obtained from the initial 2D vortex by a $z$-dependent gauge rotation around the the asymptotic direction of the vector field. One hopes that such twisted vortices can also eventually be made in the laboratory, and that they can be seen to play a role in some of the physical processes mentioned above.

Once twisted vortices have been created one may ask about their stability, and, if they are unstable, how they tend to deform. Rubber-band experiments illustrate that the cost balance between twisting and stretching is sometimes such that it is advantageous to convert twisting into stretching. One may therefore expect that, when the vortex twist per unit length is high enough and the system is allowed to evolve into its nearest energy minimum, the resulting state may involve a bent or even knotted vortex core with less twisting along the core.

In this Letter we present the results of a study on twisted vortices in the Faddeev-Skyrme model. Previous numerical simulations of this model have concentrated on stable symmetric rings with the Hopf charge of 1 and 2 [12 13], on bent rings (un-knots) [14], and on linked un-knots and rings with multipole core [15 16 17 18]. In these studies the model has been shown to support various kinds of knotted vortex solitons with nonzero Hopf charge. Here we study the relaxation process and minimum-energy solutions of twisted vortices. The results will be helpful in verifying the existence of twisted vortices and hopefully also in constructing knotted structures in various media.

The Faddeev-Skyrme model is defined by Faddeev’s Lagrangian [19]

$$L = \frac{1}{2} \int \left[ (\partial_{\mu} n)^2 + g F_{\mu\nu}^2 \right] d^3x, \quad F_{\mu\nu} = \epsilon_{abc} n^a \partial_{\mu} n^b \partial_{\nu} n^c, \quad n^2 = 1. \quad (1)$$

If we assume that the unit vector field $n$ has a fixed value $n_\infty$ in all asymptotic spatial directions (which in the vortex case means that it closes eventually) we can compactify $\mathbb{R}^3$ (i.e., contract the spatial infinity to a point) and then the conserved topological charge is given by the Hopf charge $Q$. It characterizes the homotopy of the mapping $n : \mathbb{R}^3 \cup \{\infty\} \approx S^3 \to S^2$, which is given by $\pi_3(S^2) = \mathbb{Z}$, and therefore these maps are classified by an integer, which we call $Q$. 

2
The basic definition of Hopf charge is designed for finite-size objects, and it must be modified for vortices extending to infinity. We consider here only periodic vortices and adapt the definition straightforwardly to that case. The Hopf charge can be computed from its local differential-geometric definition

$$Q = \frac{1}{16\pi^2} \int \epsilon^{ijk} A_i F_{jk} d^3x,$$  \hspace{1cm} (2)

where $A_i$ is defined via $F_{ij} = \partial_i A_j - \partial_j A_i$ and which can be implemented numerically and computed for one $z$-period. An equivalent definition is based on the linking number of the preimages of two different points on the target space $S^2$ of the map $n$ [20] (see also [17]). This second definition is also directly applicable to our periodic case.

In this work the initial configurations were constructed from 2D vortex configurations, having the value $n_{\infty} = (0, 0, 1)$ far away from the $z$-axis and $n_{\text{core}} = (0, 0, -1)$ at the $z$-axis. These 2D layers were stacked with a $z$-dependent twist, in the cylindrical coordinates $(x = \rho \cos \theta, y = \rho \sin \theta)$ we have

$$n(x, y, z; n, m) = \begin{pmatrix} \sqrt{1 - f(\rho)^2} \cos(m\theta + 2\pi nz/L) \\ \sqrt{1 - f(\rho)^2} \sin(m\theta + 2\pi nz/L) \\ f(\rho) \end{pmatrix}.$$  \hspace{1cm} (3)

Here $L$ is the box length in the $z$-direction, and $f(\rho)$ is some profile function with $f(0) = -1, f(\infty) = +1$, for example $f(\rho) = 1 - 2a/(a - 1 + e^{b\rho^2})$. The values of $a$ and $b$ were selected to give the initial configuration as low energy as possible. In order to avoid singular symmetric situations we introduced a small bend to the vortex core-line.

The minimum energy configurations were obtained numerically using the steepest-descent method, which was improved by taking into account also the gradients of the previous step. The system was discretized on a rectangular lattice, with grid sizes from $240^2 \cdot 120$ to $480^2 \cdot 240$. The corresponding box size is 5.0 units in the shortest dimension (for details of our computations in cubic lattices, see [15]). At the $z$-boundaries we used periodic boundary conditions, so that we had an integer amount of twists within the box. At the $(x, y)$-boundaries the vector field was fixed to its vacuum value $n_{\infty}$. The existence of such a rigid boundary could in principle affect the evolution of the system, and therefore we used as large a box as possible. The sufficiency of the discretization density was followed by checking that the angle between two nearest-neighbor vectors never exceeded $30^\circ$; if such a situation was approaching the system was put into a denser grid. Computations were terminated whenever the system temperature was low enough.\footnote{The present computations were done on CSC - Scientific Computing’s IBM eServer Cluster 1600 (16 pSeries 690 nodes of 32 POWER4 CPUs at 1.1 GHz each) and IBM SP (32 WinterHawk II nodes of 4 POEWR3-II CPUs at 375 MHz each) parallel supercomputer clusters. Each round of iteration took about 15 milliseconds for the $240^2 \cdot 120$ grid on the (faster) pSeries 690 computer and 360 milliseconds for the $480^2 \cdot 240$ grid on the (slower) SP system. The total number of iterations was typically somewhat less than 200.000.}

We studied cases where the box contained 5 to 8 full twists along a single vortex, lower twist-per-unit-length initial states tend to form helical configurations without any interesting knotting. (For further discussions on the straight vortices of the Faddeev-Skyrme model, see [21, 22].) In Figures 1-4 we present snapshots of the deformation
process for the cases $Q = 5 \ldots 8$ that were studied in detail. In these Figures we have used the “isosurface view” and plotted a small isosurface around the $n_{\text{core}} = (0,0,-1)$ value, the coloring describes vector orientation on this surface, as described in [15].

In all cases the vortex string tends to lengthen and bend in space in order to decrease twist along the core, in the leftmost parts of Figures 1-4 this has already progressed to some extent. If there is enough twist per unit length some sections of the coil will eventually touch and pass through each other thereby changing the topology of the core line, this is illustrated, e.g., in Figure 2 (b) and (c). In Figure 3 this process takes place twice and the result is a knot in the vortex line. In Figure 4 the process itself is more intricate, but the result is also a knotted vortex core. The energies of the final states are as follows (subscript stands for the Hopf charge)

$$E_5 = 506.44, E_6 = 584.02, E_7 = 657.57, E_8 = 732.97.$$ In this range of $Q$ values the data can be fitted to the VK-bound $E_Q = 152.91|Q|^{3/4}$ (within 1%), but a linear fit to $E_Q = 130.72 + 75.312|Q|$ is even better (within 0.25%).

During minimization the whole vector field changes slowly and the behavior of the core line (which is the preimage of the south-pole of the target space of $n$) only tells part of the story. Note, e.g., that the deformation process is continuous and preserves the Hopf charge, but nevertheless the knottedness of the core line changes. Thus in addition to the core line we should also follow the twisting around it, indeed the proper mathematical description is obtained using “framed links”. In Figures 1-4 the twisting is described by the colors, but another point of view is obtained if we plot a ribbon whose center is at the core line [17], i.e., the preimage of a short line-segment passing through the south pole of the target space (actually the preimage of any line segment carries the same information).

In Figure 5 we have plotted the data used in Figure 2 using this “ribbon view” (for technical reasons the ribbon is represented by 5 nearby narrow tubes). We can see that in this particular case a one-component ribbon will cross itself and break into a twisted ring surrounding a less twisted ribbon. The details of the crossing process is given in Figure 6 (see also Figures 7-10 in [17]).

The essential feature of the ribbon process is that it converts between a twist in the ribbon and a crossing. The process is such that the Hopf charge is conserved, this is easily seen when we use the definition based on the crossing and twisting of the ribbon, or equivalently on linking numbers of two preimages (see [17], Sec. 6-7). For this purpose the curves have to be directed (for multicomponent curves the directions have to be consistent so that when the target points are moved and number of components change we do not have to reverse any directions). The curves of the different preimages cross on the projected figure, and at each crossing we assign a $+1$ or $-1$ depending whether the (directed) over-crossing curve has to be rotated counterclockwise or clockwise, respectively, to be parallel with the under-crossing curve. The linking number is one half of the sum of these crossing numbers. In Figure 7 we have used this “preimage view” and plotted three pairs of curves corresponding to various preimage pairs of the final state with $Q = 5$. In each case the linking number is found to be 5.

One interesting observation is that the vortex core of the final state in Figure 1 is not a simple curve, but seems to have a crossing. A similar situation is seen in the linked un-knots $1+2+2$ and $1+3+2$ in Figure 5 of [16]. One might hope that it is just due to the finite thickness of the plotted core-tube and it would disappear if we plotted a sufficiently narrow tube, but this is not necessarily the case. What really takes place is again clarified
in the ribbon view in Figure 8: In the presented final state the core-line seems to be in the middle of a ribbon going through the splitting process described in Figure 6. Indeed, such crossed pre-image curves are always present as intermediate states between preimages with a different number of components: if one connects these preimages in the target space by a curve then at least one point on this curve will interpolate between preimages with different number of components, which is only possible if its own preimage is a crossing curve. (Furthermore, the collection of points whose preimages are crossed curves will themselves form a continuous curve on the target space, this will be discussed in a future publication.)

In conclusion, we have studied twisted vortices in the Faddeev-Skyrme model and found how they deform under energy minimization. Twisting the vector field more than four times around the vortex core (with the chosen dimensions) makes the vortices unstable and the final configurations will be knotted or otherwise deformed. We have analyzed the deformation processes with different views of the data. The isosurface view, which emphasizes the vortex core position, has been augmented with the ribbon view, which adds further information on the details of the vector field during the deformation process. In the preimage view one can see how the Hopf charge, which is conserved under deformation, can be calculated from the linking of the different preimages (even when they have different number of components). Our numerical calculations show that in this model the deformation is a complicated but still well-defined process from a given initial twisted vortex to the final configuration with the same Hopf charge. One hopes that the corresponding structures can also be seen experimentally in some physical systems.

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References

[1] Ü. Parts, J. M. Karimäki, J. H. Koivuniemi, M. Krusius, V. M. H. Ruutu, E. V. Thuneberg, G. E. Volovik, Phys. Rev. Lett. 75 (1995) 3320.
[2] E. V. Thuneberg, Physica B 210 (1995) 287.
[3] V. M. H. Ruutu, V. B. Eltsov, A. J. Gill, T. W. B. Kibble, M. Krusius, Yu. G. Makhlin, B. Plaçais, G. E. Volovik, Wen Xu, Nature 382 (1996) 334.
[4] Steven M. Girvin, Physics Today, June (2000) 39.
[5] M. J. Bowick, L. Chandar, E. A. Schiff, A. M. Srivastava, Science 263 (1994) 943.
[6] E. Babaev, L. D. Faddeev, A. J. Niemi, Phys. Rev. B 65 (2002) 100512.
[7] E. Babaev, Phys. Rev. Lett. 88 (2002) 177002.

[8] M. B. Hindmarsh, T. W. B. Kibble, Rep. Prog. Phys. 58 (1995) 477.

[9] L. Faddeev, L. Freyhult, A. J. Niemi, P. Rajan, J. Phys. A 35 (2002) L133.

[10] L. Faddeev, A. J. Niemi, Phys. Rev. Lett. 82 (1999) 1624.

[11] L. Faddeev, A. J. Niemi, Phys. Rev. Lett. 85 (2000) 3416.

[12] L. Faddeev, A. Niemi, Nature 387, 1 May (1997) 58.

[13] J. Gladikowski, M. Hellmund, Phys. Rev. D 56 (1997) 5194.

[14] R. A. Battye, P. M. Sutcliffe, Phys. Rev. Lett. 81 (1998) 4798.

[15] J. Hietarinta, P. Salo, Phys. Lett. B 451 (1999) 60.

[16] J. Hietarinta, P. Salo, Phys. Rev. D 62 (2000) 081701.

[17] J. Hietarinta, J. Jäykkä, P. Salo, JHEP Conference Proceedings at [http://jhep.sissa.it/] PRHEP-unesp2002/017, 2003.

[18] For video animations, see [http://users.utu.fi/hietarin/knots]

[19] L. Faddeev, IAS Print (1975) 75-QS70; L. Faddeev, in: M. Pantaleo, F. De Finis (Eds.), Relativity, Quanta, and Cosmology, Vol.1, NY Springer, New York, 1979, pp. 247-266.

[20] R. Bott, L. Tu, Differential forms in algebraic topology, NY Springer, New York, 1982.

[21] M. Miettinen, A. J. Niemi, Y. Stroganov, Phys. Lett. B 474 (2000) 303.

[22] M. Lübcke, S. M. Nasir, A. J. Niemi, K. Torokoff, Phys. Lett. B 534 (2002) 195.

[23] A. F. Vakulenko and L. V. Kapitanskii, Sov. Phys. Dokl. 24 (1979) 433.
**Figure captions**

Figure 1: Different stages in the deformation process from a straight twisted vortex of charge $Q = 5$. In the first snapshot the vortex has already twisted considerably, in subsequent pictures different parts of the core line touch and deform, forming a ring attached to the vortex core.

Figure 2: As in Figure 1, but for $Q = 6$. Now the final state is a vortex of charge $Q = 2$ surrounded with a ring with $Q = 2$, together having $Q = 6$.

Figure 3: As in Figure 1, but for $Q = 7$. Now the final state is a knotted vortex, it has been reached by two crossing processes.

Figure 4: As in Figure 1, but for $Q = 8$. The deformation process is still more complicated, the result is the same knot as in Figure 3 but the vortex core carries more twist.

Figure 5: The ribbon view of the vector fields in Figures 2 (b) and (c), with an intermediate state. In the first picture ribbon edges at two different places have already touched and the purple edge has gone through the crossing-splitting deformation. In the middle picture the deformation has gone through half of the ribbon, the center part forming a cross. In the last picture the red edge is finally about to form a crossing thereby finishing the process in which a ribbon ring is formed around the vortex ribbon.

Figure 6: Details of the ribbon crossing process. We have chosen the ribbons to look identical at the boundaries of the figure and since the ribbons interact through edges of same color one of them has been partially twisted $180^\circ$. Whenever lines cross they rearrange and the process goes through the whole ribbon. As a result we have two non-crossing ribbons, one of which has developed a $360^\circ$ twist while the other one can be straightened.

Figure 7: Preimages of three arbitrary pairs of points for the final state of charge $Q = 5$ vortex. The Hopf charge can be computed from the linking numbers of two different preimages, when curves are consistently directed, and the result is independent of the number of components.

Figure 8: Final state of the $Q = 5$ case in the ribbon view. The vortex core (colored grey) is in the middle of a splitting ribbon.
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