ON BOUNDARY POINTS AT WHICH THE SQUEEZING FUNCTION TENDS TO ONE

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Abstract. J.E. Fornæss posed the question whether the boundary point of smoothly bounded pseudoconvex domain is strictly pseudoconvex, if the asymptotic limit of the squeezing function is 1. The purpose of this paper is to give an affirmative answer when the domain is in $\mathbb{C}^2$ with smooth boundary of finite type in the sense of D’Angelo [4].

1. Introduction

Let $B^n(p;r) := \{z \in \mathbb{C}^n : \|z - p\| < r\}$. For a domain $\Omega$ in $\mathbb{C}^n$ and $z_0 \in \Omega$, let $F_{\Omega}(z_0) := \{f: \Omega \to B^n(0;1) \mid f 1-1 \text{ holomorphic}, f(z_0) = 0\}$. Then the squeezing function $s_{\Omega}: \Omega \to \mathbb{R}$ is defined to be

$$s_{\Omega}(z) := \sup\{r : B^n(0;r) \subset f(\Omega), f \in F_{\Omega}(z)\}.$$  

Note that $0 < s_{\Omega}(z) \leq 1$ for any $z \in \Omega$.

This concept first appeared in [13, 14] in the context concerning the holomorphic homogeneous regular manifolds. But the name squeezing function comes from [5]; this concept is closely related to the concept of bounded geometry by Cheng and Yau [2], as one sees from [17]. It is obvious from its construction that the squeezing function is a biholomorphic invariant.

Recent studies have shown:

Theorem 1.1 ([12], See also [3, 5]). If a bounded domain $\Omega$ in $\mathbb{C}^n$ has a boundary point, say $p$, about which the boundary is strictly pseudoconvex, then $\lim_{\Omega \ni z \to p} s_{\Omega}(z) = 1$.

J. E. Fornæss asked recently whether its converse is true (cf. [7], Sections 1 and 4), i.e., he posed the following question.

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Question 1. If \( \Omega \) is a bounded domain with smooth boundary, and if 
\[
\lim_{\Omega \ni z \to p \in \partial \Omega} s_\Omega(z) = 1,
\]
then is the boundary of \( \Omega \) strictly pseudoconvex at \( p \)?

In a recent article, A. Zimmer has shown that the answer is affirmative if the bounded domain is also assumed to be convex [18].

The main purpose of this article is to present the following result.

**Theorem 1.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^2 \) with smooth pseudoconvex boundary. If \( p \) is a boundary point of \( \Omega \) of finite type, in the sense of D’Angelo [4], and if 
\[
\lim_{\Omega \ni z \to p} S_\Omega(z) = 1,
\]
then \( \partial \Omega \) is strictly pseudoconvex at \( p \).

We remark that our proof is only for complex dimension 2. It is mainly due to the limitation of current knowledge concerning the convergence of the scaling methods (cf. [8]). If the domain is convex and Kobayashi hyperbolic for instance, then there is no such restriction. Consequently in case the domain is bounded convex, as treated in [18], our proof-arguments also answer Question 1 affirmatively, which we shall put an explication of, in a remark at the end.

On the other hand, as shown recently in [7], the answer to Question 1 is negative if no other conditions on the bounded domain than the boundary being \( \mathcal{C}^2 \) are assumed. Also the high dimensions can imply some unexpected phenomenon [6]. Thus the hypothesis of the our theorem is in some sense reasonable.

2. Construction of the scaling sequence

Let \( \Omega \) and the boundary point \( p \in \partial \Omega \) be as in the hypothesis of the theorem.

Let \( v_p \) be the unit normal vector to \( \partial \Omega \) at \( p \) pointing outward. Then take a sequence \( \{p_j\} \subset \Omega \) such that \( p - p_j = t_j v_p \) for some \( t_j \) satisfying

1. \( 0 < t_{j+1} < t_j \) for every \( j = 1, 2, \ldots \) and
2. \( \lim_{j \to \infty} t_j = 0 \).

We also set \( \delta_j := 2(1 - s_\Omega(p_j)) \) for every \( j \). Since \( \lim_{j \to \infty} s_\Omega(p_j) = 1 \) by assumption, there exists, for each \( j \), an injective holomorphic map \( f_j: \Omega \to \mathbb{B}^2 \) such that \( f_j(p_j) = (0, 0) \) and \( \mathbb{B}^2(0; 1 - \delta_j) \subset f_j(\Omega) \) for all \( j \).

The next step is to use the dilation sequence \( \{\alpha_j : \Omega \to \mathbb{C}^2\} \) introduced in [10], Section 2.2. (We remark that this is a mild but necessary modification of Pinchuk’s stretching sequence [11, 10].) If the type of \( p \) is \( 2k \) for some integer \( k \), then there is a neighborhood \( U \) of \( p \) and \( \alpha_j \in Aut(\mathbb{C}^2) \) satisfying the following properties:
(1) The map $\alpha_j$ is the composition (in order) of a translation, a unitary map, a triangular map and a dilation map.

(2) $\alpha_j(p) = (0, 0)$ and $\alpha_j(p_j) = (-1, 0)$ for all $j$.

(3) The local defining function $\rho_j$ of $\alpha_j(\Omega \cap U)$ at $(0, 0)$ is represented by

$$\rho_j(w, z) = \text{Re} w + P_j(z, \bar{z}) + R_j(z, \bar{z}) + (\text{Im} w)Q_j(\text{Im} w, z, \bar{z}),$$

where:

- $P_j$ is a nonzero real-valued homogeneous subharmonic polynomial of degree $2k$ with no harmonic terms,
- $R_j$ and $Q_j$ are real-valued smooth functions satisfying the conditions on the vanishing: order $\nu(R_j(z, \bar{z})) > 2k$ and $\nu(Q_j(\text{Im} w, z, \bar{z})) \geq 1$.

Note that the convergences $P_j \to \hat{P}, R_j \to 0$ and $Q_j \to 0$ are uniform on compact subsets of $\mathbb{C}^2$, while $\hat{P}$ is a nonzero real-valued subharmonic polynomial of degree $2k$. This is proved in detail in Lemma 2.4 of [10]. Consequently, $\rho_j$ converges uniformly to $\hat{\rho} := \text{Re} w + \hat{P}(z, \bar{z})$ on compact subsets of $\mathbb{C}^2$.

Define $\sigma_j : \alpha_j(\Omega) \to f_j(\Omega)$ by $\sigma_j := f_j \circ \alpha_j^{-1}$. Note that $\sigma_j(-1, 0) = (0, 0)$ for every $j$. 

\begin{center}
\begin{tikzpicture}
  \node (p) at (0, 0) [circle, fill, inner sep=1pt, label={below left:$p$}] {};
  \node (pj) at (1.5, 0) [circle, fill, inner sep=1pt, label={below left:$p_j$}] {};
  \node (Omega) at (-2, 1) [circle, draw, label={above:$\Omega$}] {};
  \node (Rew) at (5, 0) [circle, draw, label={right:$\text{Re} w$}] {};
  \node (alphaOmega) at (-2, -1) [circle, draw, label={below:$\alpha_j(\Omega)$}] {};
  \node (alphaOmega1) at (-3, -1.5) [circle, draw, label={below:$\alpha_j(p) = (0, 0)$}] {};
  \node (alphaOmega2) at (-2.5, -1) [circle, draw, label={below:$\alpha_j(p_0) = (-1, 0)$}] {};
  \draw [->] (p) to [bend left] node [above] {$\alpha_j$} (pj);
  \draw [->] (Omega) to [bend left] node [above] {$\alpha_j$} (alphaOmega);
  \draw [->] (alphaOmega) to [bend left] node [below] {$\sigma_j := f_j \circ \alpha_j^{-1}$} (Rew);
  \draw [->] (alphaOmega1) to [bend left] node [below] {$\alpha_j$} (alphaOmega2);
  \draw [->] (pj) to [bend left] node [right] {$f_j$} (Rew);
  \draw [->] (pj) to [bend left] node [right] {$f_j$} (alphaOmega);
  \draw [->] (Omega) to [bend left] node [right] {$f_j$} (Rew);
  \draw [dashed] (0, 0) circle (1) node [right] {$B(0; 1)$};
  \draw [dashed] (0, 0) circle (1 - \delta_j) node [right] {$B(0; 1 - \delta_j)$};
  \draw [dashed] (fj(0), 0) circle (1) node [right] {$f_j(\Omega) \subset B(0, 1)$};
\end{tikzpicture}
\end{center}
3. Convergence of the “reverse” scaling sequence \( \{\sigma_j\} \)

Recall the concept of normal set-convergence introduced in [8], Section 9.2.2; it will give the necessary control for the convergence.

**Definition 3.1.** Let \( \Omega_j \) be domains in \( \mathbb{C}^n \) for each \( j = 1, 2, \ldots \). The sequence \( \Omega_j \) is said to converge normally to a domain \( \hat{\Omega} \), if the following two conditions hold:

1. For any compact set \( K \) contained in the interior of \( \bigcap_{j>m} \Omega_j \) for some positive integer \( m \), \( K \subset \hat{\Omega} \).
2. For any compact subset \( K' \) of \( \hat{\Omega} \), there exists a constant \( m > 0 \) such that \( K' \subset \bigcap_{j>m} \Omega_j \).

**Proposition 3.2.** If \( \Omega_j \) is a sequence of domains in \( \mathbb{C}^n \) that converges normally to the domain \( \hat{\Omega} \), then

1. If a sequence of holomorphic mappings \( f_j : \Omega_j \rightarrow \Omega'_j \) from \( \Omega_j \) to another domain \( \Omega' \) converges uniformly on compact subsets of \( \hat{\Omega} \), then its limit is a holomorphic mapping from \( \hat{\Omega} \) into the closure of the domain \( \Omega' \).
2. If a sequence of holomorphic mappings \( g_j : \Omega'_j \rightarrow \Omega_j \) converges uniformly on compact subsets of \( \Omega' \), if \( \hat{\Omega} \) is pseudoconvex, and if there are a point \( p \in \Omega' \) and a constant \( c > 0 \) so that the inequality \( |\det(g_j|_p)| > c \) holds for each \( j \), then \( \lim_{j \to \infty} g_j \) is a holomorphic mapping from the domain \( \Omega' \) into \( \hat{\Omega} \).

In our construction, the set-convergences \( f_j(\Omega) \rightarrow \mathbb{B}^2 \) and \( \alpha_j(\Omega) \rightarrow \hat{\Omega} \) are in accordance with the sense of normal set-convergence with

\[ \hat{\Omega} := \{(w, z) \in \mathbb{C}^2 | \hat{\rho} = \text{Re } w + \hat{P}(z, \bar{z}) < 0\}. \]

Notice that \( \hat{\Omega} \) is unbounded, so the convergence of the forward scaling sequence \( \{\sigma_j^{-1}\} \) is not immediately obvious. On the other hand, one easily observes that the inverse sequence (i.e., the reverse scaling sequence) \( \{\sigma_j\} \) converges, choosing a subsequence when necessary, by Proposition 3.2 and Montel’s theorem. So we take a convergent subsequence and call it \( \{\sigma_j\} \) again, and denote the limit map by \( \hat{\sigma} \). Since \( \hat{\sigma}(-1, 0) = (0, 0) \in \mathbb{B}^2 \), it holds that \( \hat{\sigma}(\hat{\Omega}) \subset \mathbb{B}^2 \).

Now we show:

**Proposition 3.3.** \( \hat{\sigma} : \hat{\Omega} \rightarrow \mathbb{B}^2 \) is a biholomorphic map.

**Proof.** The proof is almost the same as those for Propositions 2.8 and 2.10 of [10]. However, the surjectivity part of \( \hat{\sigma} \) requires a few, simple
but perhaps subtle adjustments. Therefore, we choose to include the detail here.

Suppose that \( \hat{\sigma} \) is not onto. Then there is a boundary point \( q \) of \( \hat{\sigma}(\hat{\Omega}) \) in \( \mathbb{B}^2 \). Notice that \( f_j^{-1}(q) \) converges to \( p \) since \( p \) is a peak point of \( \partial \Omega \). Denote by \( q_j := f_j^{-1}(q) \). Now we construct a new scaling sequence \( \sigma_j^q := f_j \circ (\alpha_j^q)^{-1} \) where \( \alpha_j^q \) is a stretching map (in the sense of Pinchuk) with respect to \( q_j \) as above. In the same way, there is a subsequential limit map \( \hat{\sigma}_q : \hat{\Omega}^q \to \mathbb{B}^2 \) of \( \{ \sigma_j^q \} \), where \( \hat{\Omega}^q \) is the limit domain of the sequence \( \alpha_j^q(\Omega) \) in the sense of normal set-convergence. We have already observed that \( \hat{\sigma}_q \) is 1-1. Taking a subsequence if necessary, we may assume that the uniform convergence holds for \( \sigma_j^{-1} \to \hat{\sigma}^{-1} \) and \( (\sigma_j^q)^{-1} \to \hat{\sigma}_q^{-1} \) on compact subsets of \( \hat{\sigma}(\hat{\Omega}) \) and \( \hat{\sigma}_q(\hat{\Omega}^q) \) respectively. (See Lemma 2.9 in [10].)

Denote by \( W := \hat{\sigma}(\hat{\Omega}) \cap \hat{\sigma}_q(\hat{\Omega}^q) \). Then the map \( \beta_j := (\sigma_j^q)^{-1} \circ \sigma_j : \sigma_j^{-1}(W) \to (\sigma_j^q)^{-1}(W) \) is well-defined. Actually, \( \beta_j \equiv \alpha_j^q \circ \alpha_j^{-1} \), and this, for each \( j \), is a polynomial automorphism of \( \mathbb{C}^2 \) with degree less than or equal to \( 2k \). On the other hand, \( \beta_j \) converges to \( \hat{\beta} := (\hat{\sigma}_q)^{-1} \circ \hat{\sigma} \) uniformly on compact subsets of \( \hat{\sigma}^{-1}(W) \). By calculation, \( \hat{\beta} \) turns out to be a polynomial automorphism of \( \mathbb{C}^2 \) of degree less than or equal to \( 2k \). Now Proposition 3.2 guarantees that the restriction \( \hat{\beta}|_{\hat{\Omega}} \) is a 1-1 holomorphic map from \( \hat{\Omega} \) into \( \hat{\Omega}^q \). In a similar way for the inverse sequence \( \{ \beta_j^{-1} \} \), the map \( (\hat{\beta})^{-1}|_{\hat{\Omega}^q} \) is a 1-1 holomorphic map from \( \hat{\Omega}^q \) into \( \hat{\Omega} \). Hence \( \hat{\beta} : \hat{\Omega} \to \hat{\Omega}^q \) is a biholomorphism.

On the other hand, we see that a sequence of points in \( W \) convergent to \( q \) gives rise to a sequence in \( \hat{\Omega} \) via \( \hat{\sigma} \) that approaches the boundary
∂Ω and also to a sequence in $\hat{Ω}^{q}$ via $\hat{σ}^{q}$ which, this time, converges to the interior point $(-1,0)$. This results in that the biholomorphism $\hat{β}: \hat{Ω} \to \hat{Ω}^{q}$ maps a sequence approaching the boundary to a sequence convergent to an interior point. So $\hat{β}$ fails to be proper, and the surjectivity of $\hat{σ}$ follows by this contradiction. □

4. Proof of the Theorem 1.2

We are ready to complete the proof of Theorem 1.2. Recall that $\hat{Ω} := \{(w,z) \in \mathbb{C}^2 \mid \text{Re } w + \hat{P}(z,\bar{z}) < 0\}$, where $\hat{P}$ is a nonzero real-valued subharmonic polynomial of degree $2k$. Note that the unit ball $\mathbb{B}^2$ is biholomorphic to the Siegel half space $\{(w,z) \in \mathbb{C}^2 \mid \text{Re } w + |z|^2 < 0\}$. So the theorem of Oeljeklaus in [15], which says that these two domains must be affinely biholomorphic in such a case, implies in particular that $\hat{P}(z,\bar{z}) = c|z|^2$ for some $c > 0$. Therefore, the origin must have been the boundary point of type 2 in the first place. Since the convergence $\rho_j \to \text{Re } w + c|z|^2$ is uniform on each jet-level, there exist positive constants $c$ and $j_0$ such that the smallest eigenvalue of the Levi form of each $\rho_j$ at $(0,0)$ is larger than $c$ whenever $j > j_0$. Consequently, the Levi form of $\rho$ at $q$ is strictly positive-definite. This completes the proof of Theorem 1.2. □

5. A remark for the convex case

Notice that the scaling sequence converges without difficulties in case the domain is bounded convex. See e.g., [11]. If the boundary point $p$ under consideration satisfies the condition $\lim_{Ω^{q} \to p} s_Ω(q) = 1$ and if $p$ were $C^\infty$ convex of infinite type, then the scaled limit turns out to be a convex domain that contains a one-dimensional disc with a positive radius in its boundary. (cf. [18], Theorem 3.7).
On the other hand, the preceding arguments imply that this domain has to be biholomorphic to the ball. But this is impossible, for instance by [9]. So the boundary point $p$ has to be of finite type. Notice that our arguments work in all dimensions in this case, as the scaling method for bounded convex domains converges regardless of dimension (cf., e.g., [11]). This reconfirms Zimmer’s affirmative answer to Question 1 for the smoothly bounded convex domains in $\mathbb{C}^n$ for all $n$.

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