Effects of anisotropic dynamics on cosmic strings

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Abstract

The dynamics of cosmic strings is considered in anisotropic backgrounds. In particular, the behaviour of infinitely long straight cosmic strings and of cosmic string loops is determined. Small perturbations of a straight cosmic string are calculated. The relevance of these results is discussed with respect to the possible observational imprints of an anisotropic phase on the behaviour of a cosmic string network.

1 Introduction

Topological defects are expected to arise naturally in the early universe, as the result of phase transitions followed by spontaneously broken symmetries. In the context of gauge theories, monopoles and domain walls may lead to disastrous consequences for cosmology, while cosmic strings may play a useful rôle. Cosmic strings are linear topological defects, analogous to flux tubes in type-II superconductors, or to vortex filaments in superfluid helium. Cosmic strings were shown [1] to be generically formed at the end of an inflationary era, within the framework of supersymmetric grand unified theories.

Observations indicate that the universe is nearly homogeneous and isotropic on large scales. This can naturally be achieved by a stage of de Sitter inflation at early times. In the presence of a positive cosmological constant the isotropic solution is an attractor solution, as was shown in the case of spatially homogeneous models which are described by the Bianchi models. All but Bianchi IX models approach the de Sitter solution at late times [2]. The study of observational imprints, such as the primordial curvature perturbation spectrum, has largely focused on the isotropic phase. However, recently the effects of an anisotropic stage on the anisotropies of the cosmic microwave

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background have been studied [3]–[6]. One of the motivations was the observation of anomalies in the WMAP data, such as the alignment of the moments in the lowest multipoles and the suppression of the quadrupole [3]. In Ref. [7] the WMAP first year data were fitted to a Bianchi VIIh model and thus accounting for the aforementioned anomalies by assuming a globally anisotropic background. The temperature anisotropies of the cosmic microwave background in spatially homogeneous and anisotropic universes were first studied in Ref. [8].

In the context of topological defects, the effects of an anisotropic stage of inflation on domain walls was studied in Ref. [9] and on cosmic string loops in Ref. [10]. In Ref. [11] the equations of motion for cosmic strings in a (3+1)-dimensional Friedmann-Lemaître-Robertson-Walker model with extra dimensions were presented which describe in general an anisotropic higher dimensional space-time. There the evolution of a cosmic string network was studied in a (D+1)-dimensional isotropic space-time as well as in a (3+1)-dimensional Friedmann-Lemaître-Robertson-Walker model with static extra dimensions. In what follows we investigate the fingerprint of anisotropic expansion on cosmic strings dynamics in four dimensions. Such a study may be relevant for models where the duration of inflation did not have enough time to reach the attractor solution, so that cosmic strings, formed at the end of inflation, live in an anisotropic background.

2 Strings in anisotropic backgrounds

The simplest spatially homogeneous, anisotropic model is of type Bianchi I which is described by the metric

\[ ds^2 = dt^2 - a_1^2 dx^2 - a_2^2 dy^2 - a_3^2 dz^2 , \]

where \( a_i(t) \) with \( i = 1, 2, 3 \) are the scale factors in the three different spatial directions. Inflationary solutions in this background coupled to a scalar field \( \varphi \) with a potential \( V(\varphi) \) were studied in detail in Ref. [5]. In particular, in the presence of a positive cosmological constant, corresponding to \( V = \text{const.}, \; \dot{\varphi} = 0 \), the solutions for the scale factors \( a_i \) are given by [5]

\[ a_i(t) = A \left[ \sinh \left( \frac{t}{t_*} \right) \right]^{\frac{2}{3}} \left[ \tanh \left( \frac{t}{2t_*} \right) \right]^{(2/3) \sin \beta_i} , \]

where \( A \equiv (\mathcal{K}^2 / 6V_0)^{1/6} \), where \( \mathcal{K}^2 \) is an integration constant related to the square of the shear, \( V_0 \equiv 8\pi V/(3M_P^2) \) with \( M_P \) the Planck mass, and \( t_* \equiv 1/3\sqrt{V_0} \). Moreover, \( \beta_i = \beta + (2\pi/3)i, \; i \in \{1, 2, 3\} \) with \( \beta \) a parameter, which we choose, following Ref. [5], to be between 0 and 2\( \pi/3 \).

In these backgrounds there is an initial singularity and the behaviour of the scale factors close to the singularity is described by a Kasner solution [5]

\[ a_i(t) = a_{i,*} \left( \frac{t}{2t_*} \right)^{p_i} , \]

where in the following \( t_* \) will be rescaled so to absorb the factor 2. The exponents \( p_i \) satisfy the Kasner relations

\[ \sum_i p_i = 1 , \quad \sum_i p_i^2 = 1 . \]

The Kasner exponents are given by \( p_i = (2/3) \sin \beta_i + (1/3) \) and are shown [5] in Fig. 1.
In the following we will use the metric in the form
\[ ds^2 = a_1^2(\tau)d\tau^2 - d x^2 \]
\[ - a_2^2(\tau)d y^2 - a_3^2(\tau)d z^2 , \]
where in the case of an isotropic space-time \( \tau \) corresponds to conformal time. Transforming these solutions to the time variable \( \tau \), the scale factors take the form
\[ a_i(\tau) = a_{i,*} \left( \frac{\tau}{\tau_*} \right)^{\alpha_i} , \]
where \( \alpha_i \equiv p_i/(1 - p_1) \).

It should be noted that \( p_1 < 1 \) (cf., Fig. 1) and therefore the expanding or contracting nature, respectively, of a direction only depends on the sign of \( p_i \) and does not change when changing from cosmic time to the conformal time-like variable \( \tau \). Moreover, it is interesting to note that the direction whose dynamics is determined by the Kasner exponent \( p_3 \) is always expanding. Whereas the direction associated with \( p_1 \) changes its behaviour from an expanding to a contracting universe, the direction determined by \( p_2 \) changes from a contracting to an expanding universe. The cross-over takes place for both directions at \( \beta = \pi/2 \), in which case these directions are actually static and \( p_3 \) reaches its maximum at \( p_3 = 1 \).

Our aim is to estimate the effects of an anisotropic stage on the dynamics of a cosmic string. Therefore, for simplicity, the Kasner solution is used in Sections 3 and 4 to describe the background dynamics. Moreover, in the numerical solutions in Section 3 we set \( \tau_* = 1 \) and \( a_{i,*} = 1 \). The discussion of string loops in Section 5 is done for general anisotropic space-times.

Neglecting the thickness of the string, the dynamics of the string is determined by the Nambu-Goto action (cf., e.g., Ref. [12])
\[ S = -\mu \int \sqrt{-\gamma}d^2\zeta , \]
where \( \mu \) is the string mass per unit length and \( \gamma \) is the determinant of the worldsheet metric. Varying the action, Eq. (2.7), w.r.t. the space-time coordinates \( x^\mu(\zeta^a) \) gives the string equation of motion. Usually the following gauge conditions are imposed

\[
\zeta^0 \equiv \tau, \quad \dot{x} \cdot x' = g_{\mu\nu} \dot{x}^\mu x'^\nu = 0, \quad (2.8)
\]

where a dot denotes \( \partial/\partial \tau \) and a prime \( \partial/\partial \zeta \) with \( \zeta \equiv \zeta^1 \). Then the string equation of motion is given by \cite{12}

\[
\frac{\partial}{\partial \tau} \left[ \dot{x}^\mu x'^2 \right] + \frac{\partial}{\partial \zeta} \left[ \frac{x'^\mu \dot{x}^2}{\sqrt{-\gamma}} \right] + \frac{1}{\sqrt{-\gamma}} \Gamma^\mu_{\nu\sigma} \left( x'^2 \dot{x}^\nu \dot{x}^\sigma + \dot{x}^2 x'^\nu x'^\sigma \right) = 0. \quad (2.9)
\]

The non-vanishing components of the world-sheet metric \( \gamma_{ab} = g_{\mu\nu} x^\mu_{,a} x^\nu_{,b} \) (latin indices run from 1 to 3) are given by

\[
\gamma_{\tau\tau} = \dot{x}^2 = a_1^2 - \sum_i a_i^2 \dot{x}_i^2, \\
\gamma_{\zeta\zeta} = x'^2 = -\sum_i a_i^2 x_i'^2. \quad (2.10)
\]

Defining

\[
\epsilon \equiv -\frac{x'^2}{\sqrt{-\gamma}}, \quad (2.11)
\]

Eq. (2.9) yields

\[
\dot{\epsilon} = -\epsilon \left[ \frac{\dot{a}_1}{a_1} + \sum_i \frac{a_i \dot{a}_i}{a_i^2} \left( \dot{x}_i^2 - \left( \frac{x_i'}{\epsilon} \right)^2 \right) \right], \quad (2.12)
\]

for \( \mu = 0 \) and

\[
\ddot{x}^i + \left[ \frac{2 \dot{a}_i}{a_i} + \frac{\dot{\epsilon}}{\epsilon} \right] \dot{x}^i = \frac{1}{\epsilon} \left( \frac{x_i'}{\epsilon} \right)', \quad (2.13)
\]

for \( \mu = i \). In the isotropic limit the usual equations of a cosmic string in a flat Friedmann-Lemaître-Robertson-Walker background are recovered (cf., e.g., Ref. \cite{12}). Moreover, these equations agree with those in Refs. \cite{10} and \cite{11}.

### 3 A moving straight string

One can show that here, as for an isotropic background, a static straight string \( x^i(\zeta) = c^i \zeta \) is a solution (cf., e.g., Ref. \cite{12}). Following Ref. \cite{12}, let us consider a moving straight string of the form

\[
x^i(\zeta, \tau) = b^i(\tau) + c^i \zeta, \quad (3.14)
\]

where \( c^i \) are constants. Defining \( v^i = \dot{b}^i \), Eq. (2.13) yields

\[
\ddot{v}^i + \left[ 2 \frac{\dot{a}_i}{a_i} - \frac{\dot{a}_1}{a_1} - \sum_j \frac{a_j \dot{a}_j}{a_j^2} \left( v^j 2 - \left( \frac{c^j}{\epsilon} \right)^2 \right) \right] v^i = 0. \quad (3.15)
\]
In an isotropic background with a scale factor $a(t)$ the variable $v^i$ corresponds to the proper velocity of the string $\vec{V} = a \frac{dx^i}{dt} = \vec{v}$. However, in an anisotropic background the physical velocity of the string is given by $V^i = a_i \frac{dx^i}{d\tau} = \frac{a}{a_1} v^i$. From the definition of $\epsilon$ (cf., Eq. (2.11)) it is found that

$$\epsilon = \left( \frac{\sum_i a_i^2 (c^i)^2}{a_1^2 - \sum_i a_i^2 (v^i)^2} \right)^{\frac{1}{2}},$$

so that $\epsilon = \epsilon(\tau)$. In the isotropic case Eq. (3.15) can be solved to find the behaviour of $v = |\vec{v}|$,

$$v(1 - v^2)^{-\frac{1}{2}} \propto a^{-2}. \quad (3.17)$$

In Figs. 2 and 3 solutions for anisotropic backgrounds are presented. The gauge condition $\dot{x} \cdot x' = \sum_i a_i^2 v^i c^i = 0$ is imposed by choosing as initial conditions $\vec{v} = (v_0, 0, v_0)$ for $\vec{c} = (0, 1, 0)$ and $\vec{v} = (0, v_0, v_0)$ for $\vec{c} = (1, 0, 0)$. These are two particular configurations chosen in order to illustrate the general behaviour of a moving straight string in an anisotropic background.

![Figure 2: Numerical solutions for the components of the string velocity (left), $|\vec{V}| = \sqrt{\sum_i \left( \frac{a_i}{a_1} \right)^2 (v^i)^2}$ (middle) and $\epsilon$ (right) for the Kasner solution for the choice $c = (1, 0, 0)$. The initial velocity vector is given by by $\vec{V} = (0, V_0, V_0)$ with $V_0 = 10^{-6}$.](image_url)

As can be appreciated in Figs. 2 and 3 the difference in the behaviour of the velocity components is most significant for small values of $\beta$. In particular, in the setting of Fig. 2 the string moves within one collapsing and one expanding direction. The velocity component in the direction which is collapsing for $\beta < \pi/2$ completely dominates the movement of the string. At $\beta = \pi/2$ this dimension becomes expanding. However, the corresponding value $p_2 < p_3$ (cf., Fig. 1), so that the scale factor is less growing in that direction. Moreover, $\epsilon$ is found to be approximately constant in $\tau$, independently of the value of the Kasner exponents. This can easily be understood by recalling that in this case the only non-vanishing component of $c^i$ is $c^1$ and therefore neglecting the contribution due the velocity components in Eq. (3.16) yields the approximately constant behaviour of $\epsilon$.

The solution in Fig. 3 picks out one direction which starts out as an expanding direction and becomes contracting at $\beta = \pi/2$ and one direction which always expands. The velocity of the string
Figure 3: Numerical solutions for the components of the string velocity (left), $|\vec{V}| = \sqrt{\sum_i \left( \frac{a_i}{a} \right)^2 (v_i)^2}$ (middle) and $\epsilon$ (right) for the Kasner solution for the choice $c = (0, 1.0, 0)$. The initial velocity vector is given by $\vec{V} = (V_0, 0, V_0)$ with $V_0 = 10^{-6}$.

is dominated by the direction whose scale factor is less growing and there is a cross over in the magnitudes of the velocity components at $\pi/6$, where the Kasner exponent $p_3$ becomes larger than the Kasner exponent $p_1$. So that the scale factor in the latter direction becomes smaller than in the direction associated with $p_3$. In this case $\epsilon$ is not a constant, but rather scales approximately as $\epsilon \sim a_2/a_1$ which is decreasing in $\tau$ for small values of $\beta$ and is slowly growing with time for $\beta > \pi/2$ (cf., Fig. 1), with a maximum growth $\propto \tau^{0.46}$ at $\beta = 2\pi/3$.

Therefore, these numerical solutions seem to indicate that the behaviour of a velocity component is inversely proportional to the scale factor in that direction. This is similar to the solution in the isotropic case (cf., Eq. (3.17)).

4 Perturbations of a straight string

Following Ref. [12] to describe small perturbations on a static straight string we assume

$$x^i = c^i \zeta + \delta x^i,$$

(4.18)

where $\delta x^i = \delta x^i(\tau, \zeta)$ describes a small perturbation. Then Eq. (2.13) yields after linearization, to first order in the perturbation,

$$\delta \ddot{x}^i + \left[ 2 \frac{\dot{a}_i}{a_i} - \frac{\dot{a}_1}{a_1} + M \right] \delta \dot{x}^i - N(\delta x^i)'' = 0,$$

(4.19)

where

$$M \equiv \frac{\sum_j a_j \dot{a}_j c_j^2}{\sum_m a_m^2 (c_m)^2}, \quad N = \frac{a_1^2}{\sum_m a_m^2 (c_m)^2}.$$

(4.20)
In an isotropic background this equation reduces to one discussed in [12] where the condition $\sum_i (c^i)^2 = 1$ is imposed. The gauge condition Eq. (2.8) leads at linear order to

$$\sum_i a_i^2 c_i \delta \dot{x}^i = 0.$$ (4.21)

In order to get a general understanding of the behaviour of the perturbations in an anisotropic background particular classes of solutions will be discussed. The simplest way to satisfy the gauge condition is to assume the following two different classes of configurations:

- **(a)** one component of $c^i$, call it $c^j$, is non zero and the two perturbation components $\delta x^i$, corresponding to $i \neq j$, are non vanishing
- **(b)** one perturbation amplitude $\delta x^i$, call it $\delta x^j$, is non vanishing and the two components of $c^i$, corresponding to $i \neq j$, are non zero.

In the first class are configurations of the form:

$$c^i = (1,0,0) \ ; \ \delta x^i = (0,\delta x^2,\delta x^3) ,$$
$$c^i = (0,1,0) \ ; \ \delta x^i = (\delta x^1,0,\delta x^3) ,$$
$$c^i = (0,0,1) \ ; \ \delta x^i = (\delta x^1,\delta x^2,0).$$ (4.22)

The last two cases are basically equivalent; the solutions of one case can be obtained by interchanging the indices 2 and 3. The first case is different from the other two due to the fact that the definition of the conformal time-like variable $\tau$ involves the scale factor $a_1$.

In the second class are solutions with

$$c^i = \frac{1}{\sqrt{2}}(1,1,0) \ ; \ \delta x^i = (0,0,\delta x^3) ,$$
$$c^i = \frac{1}{\sqrt{2}}(1,0,1) \ ; \ \delta x^i = (0,\delta x^2,0) ,$$
$$c^i = \frac{1}{\sqrt{2}}(0,1,1) \ ; \ \delta x^i = (\delta x^1,0,0).$$ (4.23)

In both classes the constant vector $c^i$ has been chosen such that $\sum_i (c^i)^2 = 1$, as was used in the isotropic case in [12].

In the following the perturbation equation, Eq. (4.19), will be solved in both cases assuming

$$\delta x^i = A^i(\tau) e^{ik\zeta}.$$ (4.24)

### 4.1 Case (a)

We begin with the case where the string space-time coordinates are given by

$$\bar{x} = \begin{pmatrix} \zeta \\ \delta x^2 \\ \delta x^3 \end{pmatrix},$$ (4.25)
so that the space-like world sheet coordinate $\zeta$ lies along the space-time $x$-direction. Thus the wavenumber $k$ lies along this direction as well. So $k$ could be thought of as $k_x$. The perturbation equation, Eq. (4.19), reads for $i = 2, 3$

$$\ddot{A}_i + 2\frac{\dot{a}_i}{a_i}\dot{A}_i + k^2 A^i = 0 ,$$

(4.26)

which can be solved in terms of Bessel functions [13] and thus the perturbation, assuming that it becomes a constant for small arguments (mimicking the isotropic case [12]) is given by

$$\delta x^i = (k\tau)^{-\mu_i} J_{\mu_i}(k\tau)e^{ik\zeta} ,$$

(4.27)

where $J_{\mu_i}(z)$ is a Bessel function of the first kind and $\mu_i \equiv (2\alpha_i - 1)/2$, where the $\alpha_i$ are the exponents in the scale factors (cf., Eq. (2.6)). The physical wavelength of the wave traveling in $x$-direction is given by $\lambda_{\text{phys}} = 2\pi a_1(\tau)/k$.

For $k\tau \ll 1$ the amplitudes $\delta x^i$ in the $y$- and $z$-directions approach constant values. Since these are the coordinate values, the physical amplitudes are determined by $\delta x^i_{\text{phys}} \equiv a_i(\tau)\delta x^i$, $(i = 2, 3)$. Therefore the ratio $\delta x^i_{\text{phys}}/\lambda_{\text{phys}} \sim a_i/a_1$. Thus contrary to the isotropic case the string is not conformally stretched, but rather becomes wigglier or straightens out, which means that the amplitude grows or diminishes with respect to the wavelength, respectively, depending on the ratio of the two scale factors.

In the opposite limit $k\tau \gg 1$ the physical amplitudes $\delta x^i_{\text{phys}}$ become constant. Therefore, since the physical wavelength grows as $a_1$, the string becomes straighter, which is the same behaviour as found in the isotropic case [12]. If, however, the $x$-direction is a collapsing direction the string becomes wigglier with time which is the case for $\beta > \pi/2$ (cf., Fig. 2).

Choosing a different configuration in this class the string will be described by

$$\vec{x} = \begin{pmatrix} \delta x^1 \\ \zeta \\ \delta x^3 \end{pmatrix} ,$$

(4.28)

So that the space-like world-sheet coordinate $\zeta$ lies along the space-time $y$-direction. Thus the wavenumber $k$ lies along this direction as well. So $k$ could be thought of as $k_y$. The perturbation equation, Eq. (4.19), reads for $i = 1, 3$

$$\ddot{A}^i + \left[ 2\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} \right] \dot{A}^i + \left( \frac{a_1}{a_2} \right)^2 k^2 A^i = 0 ,$$

(4.29)

which can be solved in terms of Bessel functions [13] and yields the perturbations

$$\delta x^i = (k\tau)^{\mu_i} J_{\mu_i}(\beta(k\tau)^{\gamma})e^{ik\zeta} ,$$

(4.30)
where
\[
\begin{align*}
\mu_1 &= \frac{1 - \alpha_1 - \alpha_2}{2}, \\
\mu_3 &= \frac{1 - 2\alpha_3 + \alpha_1 - \alpha_2}{2}, \\
\nu_1 &= \frac{1 - \alpha_1 - \alpha_2}{2(1 + \alpha_1 - \alpha_2)}, \\
\nu_3 &= \frac{1 - 2\alpha_3 + \alpha_1 - \alpha_2}{2(1 + \alpha_1 - \alpha_2)}, \\
\beta &= \frac{a_{1,\ast}}{a_{2,\ast}} \frac{(k\tau_\ast)^{\alpha_2 - \alpha_1}}{|1 + \alpha_1 - \alpha_2|}, \\
\gamma &= 1 + \alpha_1 - \alpha_2.
\end{align*}
\]

As can be seen from Fig. 1 the parameter \(\gamma = (1 - p_2)/(1 - p_1)\) is always positive in the case of the Kasner solutions considered here. Since \(\gamma > 0\) the comoving perturbation amplitudes \(\delta x_i\), for \(i = 1, 3\), approach a constant in the limit \(k\tau \ll 1\). However, the ratio of the physical amplitude over the physical wavelength becomes in this case \(\delta x_{\text{phys}}/\lambda_{\text{phys}} \sim a_i/a_2\). Therefore, it depends on the ratio of the scale factors if the string is straightened or becomes wigglier. In the limit \(k\tau \gg 1\) the physical amplitude becomes a constant. Therefore, the ratio of the physical perturbation amplitude over the physical wavelength goes as \(1/a_2\). So that in the case that this is an expanding direction the string straightens and in the opposite case, that is for a collapsing direction it becomes wigglier.

In Fig. 4 the total co-moving perturbation amplitude \(\delta x \equiv \sqrt{\delta x^1 + \delta x^3}\) is shown for \((p_1, p_2, p_3) = (2/3, -1/3, 2/3)\) which corresponds to \(\beta = \pi/6\). It is compared with the corresponding total perturbation amplitude in the isotropic backgrounds with \(p = p_1 = p_2 = -1/3\) and \(p = p_1 = p_2 = 2/3\), where \(p\) is the exponent of the scale factor using cosmic time.

Figure 4: The total co-moving perturbation amplitude \(\delta x \equiv \sqrt{\delta x^1 + (\delta x^3)^2}\) for the Kasner solution with \((p_1, p_2, p_3) = (2/3, -1/3, 2/3)\) (black, solid line), the isotropic solution \(p = p_1 = p_2 = -1/3\) (red, dot-dashed line) and \(p = p_1 = p_2 = 2/3\) (blue, dashed line).
4.2 Case (b)

It is assumed that the string is described by

$$\vec{x} = \begin{pmatrix} \frac{\zeta}{\sqrt{2}} \\ \frac{\zeta}{\sqrt{2}} \\ \delta x^3 \end{pmatrix}.$$  \hspace{1cm} (4.32)

So that the space coordinates $x$ and $y$ satisfy $x = y$, which implies that the string lies along this diagonal in the $(x,y)$-plane. Moreover, $k = \sqrt{k_x^2 + k_y^2}$. The perturbation equation reads

$$\ddot{A}_3 + \frac{1}{\tau} \left[ 2\alpha_3 - \alpha_1 + \frac{\alpha_1 + \alpha_2 \left( \frac{a_2}{a_1} \right)^2 \left( \frac{\tau}{\tau^*} \right)^{2(\alpha_2 - \alpha_1)}}{1 + \left( \frac{a_2}{a_1} \right)^2 \left( \frac{\tau}{\tau^*} \right)^{2(\alpha_2 - \alpha_1)}} \right] \dot{A}_3 + \frac{2k^2}{1 + \left( \frac{a_2}{a_1} \right)^2 \left( \frac{\tau}{\tau^*} \right)^{2(\alpha_2 - \alpha_1)}} A^3 = 0.$$  \hspace{1cm} (4.33)

In this case it seems more difficult to find an exact solution, so we will consider different regimes. Assuming $\tau^*$ is some initial time then $\tau/\tau^* \geq 1$. Now assuming that $\alpha_2 - \alpha_1 > 0$ then the third term in the coefficient of the $\dot{A}_3$ reduces to $\alpha_2$ and the time-dependent term dominates in the denominator of the coefficient of $A^3$. Thus the resulting equation corresponds to Eq. (4.29). In the case where $\alpha_2 - \alpha_1 < 0$ an equation similar (with an additional factor of $\sqrt{2}$ in the argument of the Bessel function) to Eq. (4.26) results.

Therefore the behaviour of the ratio between the physical amplitude and the physical wavelength is similar to the one discussed in case (a). The physical wavenumber in case (b) is given by

$$k^2_{\text{phys}} = \left( \frac{k_x}{a_1} \right)^2 + \left( \frac{k_y}{a_2} \right)^2,$$

implying that the physical wavelength reads

$$\lambda_{\text{phys}} \sim \frac{1}{k_{\text{phys}}} = \frac{a_2}{\left[ \left( \frac{a_2}{a_1} \right)^2 k_x^2 + k_y^2 \right]^{\frac{1}{2}}}.$$  \hspace{1cm} (4.34)

Thus, for $a_2/a_1 \gg 1$ the physical wavelength $\lambda_{\text{phys}} \sim a_1$, which implies that for $k\tau \ll 1$ the ratio of the physical perturbation over the physical wavelength, and thus the wiggliness of the string, evolves as $a_3/a_1$. In the opposite case $k\tau \gg 1$ this ratio evolves as $1/a_1$. Moreover, in the case for $a_2/a_1 \ll 1$ the physical wavelength evolves as $\lambda_{\text{phys}} \sim a_2$. The resulting behaviour of the ratio of the physical amplitude over physical wavelength is then described as in the previous discussion with $a_1$ replaced by $a_2$.

4.3 Discussion

We have found that in an anisotropic background it is no longer true that irregularities on a straight string are frozen-in on super-horizon scales as it is the case in isotropic backgrounds where the shape of a string remains unchanged. In the case of an anisotropic background, however, the wiggliness of the string can augment or diminish depending on the time evolution of the ratio of the two scale factors involved.
This might mean for example that strings from an anisotropic phase could have much more small scale structure at the time of re-entry into the horizon during the standard radiation or matter dominated phases. This would imply the generation of more gravitational radiation \[14\] which might be used to put limits on the global anisotropy of the early universe, which could imply constraints on the minimum duration of a inflationary era.

The behaviour on sub-horizon scales is basically the same as in the isotropic case, that is the wiggliness scales with \(1/a_i\). Here, due the nature of the Kasner solution, instead of decaying wiggliness might increase if the universe is contracting in that particular direction.

## 5 String loops

There are many different types of string loop solutions beginning with the simplest case which in an isotropic background is a circular loop lying in one of the coordinate planes. Most explicit solutions have been found in flat space-time, such as the two-parameter solutions \[15, 16\] and generalizations thereof (see for example Ref. \[12\]). However, loops have also been considered in curved backgrounds such as Friedmann-Lemaitre-Roberston-Walker \[17, 18\] or Kantowski-Sachs, Bianchi I and IX \[19\]. Loops in an axisymmetric background have been studied numerically in Ref. \[10\]. In this section the scale factors are assumed to be general and not necessarily the Kasner solutions used in the previous sections.

We will begin by considering periodic solutions in general. The gauge condition (Eq. (2.8)) and \(\epsilon\) (cf., Eq. (2.11)) take the form

\[
\sum_i a_i^2 \dot{x}^i \dot{x}^i = 0 ,
\]

\[
\sum_i \left( \frac{a_i}{a_1} \right)^2 \left[ \dot{x}^i \dot{x}^i + e^{-\epsilon}(x^i)'^2 \right] = 1 ,
\]

respectively. These two equations together with Eqs. (2.12) and (2.13) form the set of equations determining the dynamics of a cosmic string in an anisotropic background. The equations reduce in the case of a flat space-time to the known ones (cf., e.g., Ref. \[12\]). In flat space-time Eq. (2.13) reduces to a wave equation which is solved in general by (e.g., Ref. \[12\])

\[
\vec{x}(\zeta, \tau) = \frac{1}{2} \left[ \vec{b}_+(\zeta+) + \vec{b}_-(\zeta-) \right] ,
\]

where \(\zeta_\pm \equiv \zeta \pm \tau\) and \(b_\pm\) are arbitrary functions subject to the constraint \(\partial_+ \vec{b}_+^2 = \partial_- \vec{b}_-^2 = 1\) where \(\partial_\pm \equiv \partial/\partial \zeta_\pm\). Since we are interested in periodic solutions we use the form of Eq. (5.37) which yields to

\[
\sum_i \left( \frac{a_i}{a_1} \right)^2 (\partial_+ b_+^i)^2 = \sum_i \left( \frac{a_i}{a_1} \right)^2 (\partial_- b_-^i)^2 , (5.38)
\]

\[
\frac{1}{2} (1 + \epsilon^{-2}) \sum_i \left( \frac{a_i}{a_1} \right)^2 (\partial_- b_+^i)^2 + \frac{1}{2} (\epsilon^{-2} - 1) \sum_i \left( \frac{a_i}{a_1} \right)^2 \partial_+ b_+^i \partial_- b_-^i = 1 , (5.39)
\]

\[
\dot{\epsilon} = -\epsilon \left[ \frac{\dot{a}_1}{a_1} + \frac{1}{4} (1 - \epsilon^{-2}) \sum_i \frac{a_i \dot{a}_i}{a_1^2} [((\partial_- b_-^i)^2 + (\partial_+ b_+^i)^2] - \frac{1}{2} (1 + \epsilon^{-2}) \sum_i \frac{a_i \dot{a}_i}{a_1^2} \partial_- b_- \partial_+ b_+ \right] , (5.40)
\]

\[
(1 - \epsilon^{-2}) (\partial_-^2 b_-^i + \partial_+^2 b_+^i) = \left[ \frac{\dot{a}_i}{a_i} + \frac{\dot{\epsilon}}{\epsilon} - \frac{\epsilon'}{\epsilon^2} \right] \partial_- b_-^i - \left[ \frac{\dot{a}_i}{a_i} + \frac{\dot{\epsilon}}{\epsilon} + \frac{\epsilon'}{\epsilon^2} \right] \partial_+ b_+^i . (5.41)
\]
So far no approximations have been made and these equations are exact. Clearly the flat space-time solution can be recovered. In flat space-time loop solutions are described by \( \vec{b}_\pm(\zeta + L) = \vec{b}_\pm(\zeta) \) which leads to \( \vec{x}(\zeta + L/2, \tau + L/2) = \vec{x}(\zeta, \tau) \), where \( L \) is the invariant length of the loop \[12\]. Moreover, if the time-scale of oscillation is comparable to the loop length, in other words if the relation \( \vec{x}(\zeta + L/2, \tau + L/2) = \vec{x}(\zeta, \tau) \) is satisfied, one may deduce that the motion of the loop is relativistic. From Eqs. (5.38)-(5.41) it can be seen that the effect of the expansion of the universe, which is not a periodic function of time, is to inhibit the periodicity of the solutions. However, in the regime where \( L \) is smaller than the Hubble time, which is equivalent to considering string loops whose sizes are smaller than the horizon \[1\], the departure from periodicity of the solutions is small. Therefore considering sub-horizon solutions and time-scales where the evolution of the scale factors can be neglected, it is a good approximation to consider \( \epsilon \approx 1 \) a constant. In this case the last equation is identically satisfied and the functions \( b^i_\pm \) are arbitrary subject to the constraint given by Eqs. (5.38) and (5.39). Note that a similar approach has been considered in Ref. \[17\] for isotropic backgrounds.

In the anisotropic case the constraints imply

\[
\sum_i \left( \frac{a_i}{a_1} \right)^2 (\partial_+ b^i_\pm)^2 = \sum_i \left( \frac{a_i}{a_1} \right)^2 (\partial_- b^i_\pm)^2 = 1. \tag{5.42}
\]

Therefore, in the case of an anisotropic space-time periodic solutions are described by vector functions \( \partial_- \vec{b}_- \) and \( \partial_+ \vec{b}_+ \) which are closed functions on an ellipsoid, described in general by the equation \( x^2/A^2 + y^2/B^2 + z^2/C^2 = 1 \) where \( A, B \) and \( C \) are the equatorial radii and the polar radius, respectively. These radii are given by the inverse of the ratio of the scale factors in the different directions and the scale factor \( a_1 \). In the case of an isotropic background these solutions are on a unit sphere as it is the case in a flat space-time \[12\]. Considering the constraint equations, Eq. (5.42), formally the flat space-times solutions are recovered by assuming the relation

\[
b^i_\pm(\text{flat}) = \frac{a_i}{a_1} b^i_\pm(\text{aniso}), \tag{5.43}
\]

which holds on time-scales when the expansion of the universe can be neglected. This means that well inside the horizon the condition for intersection of two closed curves determined by \( \partial_+ \vec{b}_+ \) and \( -\partial_- \vec{b}_- \) is determined by the same condition as in flat space-time, that is \( \partial_+ \vec{b}_+^{(\text{flat})} = -\partial_- \vec{b}_-^{(\text{flat})} \). Therefore, it is expected that the number of cusps in this limit is the same as in the case of a flat background.

Calculating the square of the proper velocity of the string at such an intersection it is found that, using Eq. (5.42),

\[
\vec{V}^2 = \sum_i \left( \frac{a_i}{a_1} \right)^2 (\partial_+ b^i_\pm)^2 = 1, \tag{5.44}
\]

so that these are luminal points exactly as in an isotropic background.

Moreover, as in flat space-time the behaviour of the string at the cusp can be studied by expanding around \( \zeta_\pm = 0 \) \[20\] (cf., also Ref. \[12\], \[21\]). Thus choosing \( \zeta_\pm \) such that the cusp is at \( \zeta_\pm^{(c)} = 0 \) as well as \( \vec{x}(\zeta_\pm^{(c)}) = 0 \), truncating the expansion at third order, one gets

\[
\vec{b}_\pm(\zeta_\pm) = \vec{b}_\pm^{(c)} + \frac{1}{2} \vec{b}_\pm^{(2)} \zeta_\pm^2 + \frac{1}{6} \vec{b}_\pm^{(3)} \zeta_\pm^3, \tag{5.45}
\]

\[1\] In numerical simulations with periodic boundary conditions, all strings are closed; sub-horizon strings are thus often referred to as loops and super-horizon ones as infinite strings.
where $\vec{n}^{(c)} \equiv \partial_+ \vec{b}_+ = -\partial_- \vec{b}_-$ and the notation $\vec{b}_-^{(i)}$ indicates the $i$-th derivative with respect to the argument. Moreover, $\vec{n}^{(c)}$ and $\vec{b}_-^{(i)}$ are calculated at the cusp. Differentiating the constraint Eq. (5.42) and using that at the cusp $V_i = \frac{a_i}{a_1} \vec{n}^{(c)}_i$ results in

$$\sum_i V^i \frac{a_i}{a_1} x^{(2)i} = 0 , \quad (5.46)$$

$$\sum_i V^i \frac{a_i}{a_1} x^{(3)i} = \frac{1}{2} \sum_i \left( \frac{a_i}{a_1} \right)^2 \left[ \left( b_-^{(2)i} \right)^2 - \left( b_+^{(2)i} \right)^2 \right] . \quad (5.47)$$

Equation (5.46) implies that at the cusp two orthogonal directions can be defined using relation Eq. (5.43). This corresponds to locally defining an orthonormal frame. Therefore, expressing the approximate solution at the cusp in terms of the flat space solution at $\tau = 0$ yields to

$$x^i = \frac{1}{2} \frac{a_1}{a_i} x^{(\text{flat})(2)i}(0) \zeta^2 + \frac{1}{6} \frac{a_1}{a_i} x^{(\text{flat})(3)i}(0) \zeta^3 , \quad (5.48)$$

where $x^{(\text{flat})(2)i}$ (and $x^{(\text{flat})(3)i}$) denote the second (and third, respectively) derivative of the $i$-th component of the three-vector $x$ written in terms of left- and right-movers. Assuming that $x^{(\text{flat})(2i)}(0)$ and $x^{(\text{flat})(3i)}(0)$ are orthogonal directions, say $\dot{\xi}^{(2i)} = \dot{y}$ and $\ddot{\xi}^{(1i)} = \dot{x}$ then in the flat space-time or isotropic case the set of parametric equations is given by $x = (1/6)\zeta^3$ and $y = (1/2)\zeta^2$ which yields the standard behaviour $y \propto x^{2/3}$ [12][22]. However, in the anisotropic case the parametric equations include the ratio of the scale factors. Choosing the directions as before, one finds $x = (1/6)\zeta^3$ and $y = (a_1/2/a_2)\zeta^2$. Thus as can be seen in Fig. 5 the shape, namely the opening of the cusp is changed in the anisotropic case in comparison with the flat or isotropic case. Moreover, as it is expected the shape depends on the orientation of the cusp, that is in which direction it is assumed that the second and third derivatives lie at the cusp.

As a particular example the generalization of the elliptical loop solution in flat space-time [23] will be considered. Assuming that the string is already well inside the horizon, in the anisotropic background this solution takes the form in terms of the variables $\zeta$:

$$\vec{x} = \frac{1}{2} \left( \begin{array}{c}
\sin \zeta_+ + \sin \zeta_-
\frac{a_1}{a_2} \left( \cos \zeta_+ + \cos \zeta_- \right) \cos \frac{\Phi}{2}
\frac{a_1}{a_3} \left( \cos \zeta_- - \cos \zeta_+ \right) \sin \frac{\Phi}{2}
\end{array} \right) , \quad (5.49)$$

Figure 5: Shape of a cusp in the flat space-time (black, solid line), in an anisotropic space-time with $a_1 = 0.5a_2$ (red, dot-dashed line) and for $a_1 = 2a_2$ (blue, dashed line). $x^{(\text{flat})(3)}(0)$ lies in the direction $x$ and $x^{(\text{flat})(2)}(0)$ lies in the direction $y$.
where φ determines the shape of the ellipse. Whereas in an isotropic background φ = 0 and φ = π correspond to an oscillating circular loop that collapses to a point and a rotating double line, this is no longer the case in an anisotropic background.

We have thus shown that an anisotropic background may leave its fingerprints on the dynamics of cosmic string loops. In an isotropic geometry cusps are defined as points where the string moves momentarily at the speed of light. It was shown that this also holds in anisotropic backgrounds. The physical velocity of the string reaches the speed of light at a cusp. Thus cusps are important since they create strong gravitational pulses which propagate at the speed of light along the beam and they are extending from the cusp in the direction perpendicular to the string velocity. If the loops entering the horizon have a richer small-scale structure and less pronounced cusps, then one should expect a fan-like pattern of gravity wave beams, like the one emitted by kinks, rather than the single beam structure characteristic of radiation emitted by cusps. Moreover, the difference in the shape of the cusps should leave an imprint on the effect of gravitational pulses from cusps on particle trajectories.

6 Conclusions

Different configurations of cosmic strings have been considered in an anisotropic background. The motivation for considering such an anisotropic case is that the phase before the universe reached isotropy during inflation has left an imprint on a cosmic string network. This has been suggested in Ref. [10] where the equation for a string loop has been solved numerically in an axisymmetric background.

A static straight string is a solution of an anisotropic background. Perturbations of a straight string depend on the global anisotropy of the background. In particular, it was found that the wiggleness of a string on super-horizon scales depends on the ratio of two scale factors. Therefore, once the string re-enters the horizon during the standard radiation or matter domination it might be much more or much less wigglier than in the isotropic case changing the emitted gravitational radiation. Thus opening the possibility by using gravitational wave constraints to put limits on the global anisotropy in the very early universe.

Studying a static circular loop, it was found that the anisotropic evolution of the universe tends to impede periodic solutions. Writing the equations of motion and the constraint equations in terms of left- and right-moving solutions it was found that if the size of the string loop is smaller than the horizon or equivalently if the expansion of the universe can be neglected periodic solutions exist.

Moreover, it was found that the derivatives of the right- and left-moving vectors describe closed curves on an ellipsoid rather than the unit sphere as it is the case in an isotropic universe and flat space-time. Furthermore, the condition for the formation of cusps is the same in flat space-time, so that the number of cusps does not change. In an anisotropic space-time the formation of cusps is determined by the intersection of two-dimensional curves on an ellipsoid. However, with respect to the flat space-time the shape or opening of the cusp does change in an anisotropic space-time. Finally, the proper string velocity at the cusp reaches the speed of light just like in an isotropic background. Thus one might expect the gravity waves radiation from cusps are of the

\footnote{This is different to the case of cusps in models with higher dimensions where the numbers as well as the shape of cusps change \cite{24}. In fact for n extra dimensions the probability of cusp formation is zero since instead of intersections of a one dimensional object on a 2-sphere (in the isotropic case) one has to consider intersections on a (2 + n)-sphere and thus intersections are unlikely \cite{25}. This led \cite{24} to consider near cusp events which are rounded cusps.}
same magnitude as in an isotropic background whereas the importance of gravity waves from the small-scale structure of the strings as they enter the horizon may turn out to be more important.

If the size of the loop is of the size of the horizon or even larger in general the solutions are expected to be non periodic and so the number of loops should be very small. However, the distribution of shapes of loops formed during the anisotropic phase has to be left to a full numerical simulation. This also applies to the general shape of the strings, since one would not expect the strings to be described by simple random walks and the network characterized by just one characteristic length if a cosmic string network forms in an anisotropic background [12, 11].

Finally, the conical string geometry also has observational implications. However, the deficit angle in an axisymmetric Bianchi I model is the same as in an isotropic background. Thus one may conclude that for a sub-horizon string observational effects, such as the formation of double images of light sources located behind the string position, are similar to those of a string in a Minkowski background.

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