Empirical investigation of a quantum field theory of forward rates

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A new test of a wide class of interest rate models is proposed and applied to a recently developed quantum field theoretic model and the industry standard Heath-Jarrow-Morton model. This test is independent of the volatility function unlike other tests previously proposed in the literature. It is found that the HJM model is inconsistent with the data while the quantum field theoretic model is in significant agreement with data. We also show that a portion of the spread between long and short term interest rates is explicable in terms of this model.

Physicists have been working on several aspects of financial research over the last decade [1]. One of the most important and as yet unsolved problems in finance is the modelling of interest rates. The interest rates at any point in time form a usually continuous curve (current interest rates for different times in the future) called the forward rate curve (FRC). We denote these rates by \( f(t,x) \) where \( t \) is the current time and \( x \) is the time in the future for which the forward rate applies. For example, \( f(1,2) \) is the interest rate one year from now for an instantaneous deposit to be made 2 years into the future.

The earliest interest rate models (eg., Vasicek [2]) dealt only with the spot rate (current interest rate for the present time) and the forward rate curve was treated as a derived quantity. These were found to be inconsistent with the observed data. The Ho-Lee [3] and HJM [4] models were developed to deal with this problem by modelling the entire forward rate curve (FRC) rather than just the spot rate. The HJM model is, however, limited in the sense that the Brownian motions on which the HJM economy depends are independent of \( x \). One way of removing this restriction is by formulating the forward rate curve as a quantum mechanical string as in Baaquie [5] (we will refer to this model as the quantum field theoretic model).

Several empirical tests of the HJM model have been performed (eg., B"uhler, Uhrig-Homburg, Walter and Weber [6], Flesker [7], Sim and Thurston [8]) with mixed results. All of the tests assume a certain form for the volatility function \( \sigma \). In this paper, we propose a test which is independent of the volatility function. The test is applied to the HJM and quantum field theoretic model which was introduced in [5]. The HJM model can be formulated as a limit of the model in [5] which is briefly reviewed below.

The one factor quantum field theoretic model models the forward rates as

\[
\frac{\partial f(t,x)}{\partial t} = \alpha(t,x) + \sigma(t,x)A(t,x)
\]

where \( A(t,x) \) is a quantum field whose action is given by

\[
S[A] = \int_{t_0}^{\infty} dt \int_{t}^{t + T_{FR}} dx L[A]
\]

\[
L[A] = -\frac{1}{2} \left( A^2(t,x) + \frac{1}{\mu^2} \left( \frac{\partial A(t,x)}{\partial x} \right)^2 \right)
\]

where \( T_{FR} \) is the largest time to maturity for which the forward rates are defined (the domain is hence a semi-infinite parallelogram defined by \( t > t_0, t < x < t + T_{FR} \)). \( T_{FR} \) is introduced to ensure that the action is well defined but does not affect final results as the limit \( T_{FR} \to \infty \) must be taken. When \( \mu \to 0 \), this model reduces to the HJM model up to a rescaling (for details, please see [5]).

We assume that the function \( \sigma(t,x) \) depends only upon the variable \( \theta = x - t \). This is a theoretically reasonable assumption as it is the result of assuming that the theory is time translation invariant. Most of the functions used for \( \sigma(t,x) \) in the literature satisfy this condition.

The initial forward rate curve \( f(t_0,x) \) has to be specified. The field values of \( A(t,x) \) on the rest of the boundary points of the domain are arbitrary and are integration variables. The presence of the second term in the action given in (2) seems to be justified from the phenomenology of the forward rates [9] and is not ruled out by no arbitrage.

The moment generating functional for the quantum field theory is given by the Feynman path integral as

\[
Z[J] = \frac{1}{Z} \int DA e^{\int_{t_0}^{T_{FR}} dt \int_{t}^{t + T_{FR}} dx J(t,x)A(t,x)} e^{S[A]}
\]

On performing the calculation (details are provided in [5]), we obtain

\[
Z[J] = e^{\frac{1}{\mu^2} \int_{t_0}^{T_{FR}} d\theta d\theta' J(t,\theta) D(\theta,\theta',t,T_{FR}) J(t,\theta')}
\]
where \( \theta = x - t \), \( \theta' = x' - t \) and the propagator \( D(\theta, \theta'; t, T) \) is given by

\[
D(\theta, \theta'; t, T) = \frac{\mu T_{FR}}{\sinh^3(\mu T_{FR})} \left[ \sinh \mu (T_{FR} - \theta) \sinh \mu \theta' \right. \\
\left. \{1 + \sinh^2(\mu T_{FR}) \Theta(\theta - \theta')\} + \sinh \mu (T_{FR} - \theta') \sinh \mu \theta \right. \\
\left. \{1 + \sinh^2(\mu T_{FR}) \Theta(\theta' - \theta)\} + \cosh(\mu T_{FR}) \right. \\
\left. \{\sinh \mu \theta \sinh \mu \theta' + \sinh \mu (T_{FR} - \theta) \sinh \mu (T_{FR} - \theta')\} \right]
\]

(6)

The above results are for unconstrained boundary conditions. It is, however, well known that short term interest rates are heavily influenced by central banks. Hence, it is reasonable to treat the field at the boundary where \( t = x \) differently. If we assume that the field at that boundary (i.e. \( A(t, t) \)) is distributed normally with variance \( a \), we obtain the propagator

\[
D_1(\theta, \theta') = D(\theta, \theta') - \frac{D(0, \theta) D(0, \theta')}{D(0, 0) + a}
\]

(7)

(It can be readily seen that the mean of the field at the boundary does not affect any of the results due to the no arbitrage condition.)

To understand the significance of the propagator \( D(\theta, \theta'; t, T_{FR}) \) we note that the correlator of the field \( A(t, \theta) \), is given by

\[
E(A(t, \theta) A(t', \theta')) = \delta(t - t') D(\theta, \theta'; t, T_{FR})
\]

(8)

It can be readily shown that the no arbitrage condition is satisfied only when

\[
\alpha(t, x) = \sigma(t, x) \int_t^T dx' D(x, x'; t, T_{FR}) \sigma(t, x')
\]

(9)

In the limit \( \mu \to 0 \), \( D \to 1 \) and we obtain the well known result

\[
\alpha(t, x) = \sigma(t, x) \int_t^T dx' \sigma(t, x')
\]

(10)

for the one factor HJM model.

Following Bouchaud [3], we use the daily closing prices for eurodollar futures prices as a measure of the forward rates. The eurodollar futures prices are linearly interpolated to calculate the forward rates at 3 month intervals. The 3 month deposit rate that the eurodollar futures actually represents is taken to be a good approximation to the instantaneous forward rate. The data used for this paper are the same as that used in [4]. The data cover the 1990s and the length of the dataset is 846 trading days and forward rates 7 years into the future are available.

We parametrize the forward rates as \( f(t, \theta) \) rather than \( f(t, x) \) as this considerably simplifies the analysis considerably since the domain shape in the \( (t, \theta) \) variables is rectangular.

We concentrate mainly on the following quantities (again partially following [4])

\[
V(\theta) = \sqrt{< \delta f^2(t, \theta)>}
\]

(11)

\[
C(\theta) = \frac{< \delta f(t, \theta_{min}) \delta f(t, \theta) - \delta f(t, \theta_{min}) >}{< \delta f^2(t, \theta_{min}) >}
\]

(12)

\[
r(\theta) = \frac{V(\theta)}{C(\theta) + 1}
\]

(13)

with the differences being taken over one trading day \( (\epsilon) \), \( \delta f(t, \theta) = f(t + \epsilon, \theta) - f(t, \theta) \) and \( \theta_{min} \) being three months. We assume that there are 250 trading days in a year. In the following analysis, we use the discretization \( \delta(0) = \frac{1}{2} \).

Using the one factor HJM model, we can derive the following expressions for the above quantities which are accurate to zeroth order in \( \epsilon \)

\[
V_{HJM}(\theta) = \sigma(\theta) \sqrt{\epsilon}
\]

(14)

\[
C_{HJM}(\theta) = \frac{\sigma(\theta)}{\sigma(\theta_{min})} - 1
\]

(15)

\[
r_{HJM}(\theta) = \sigma(\theta_{min}) \sqrt{\epsilon}
\]

(16)

In deriving this equation, we have discretized the Brownian motion process \( W \) as \( W(t) = \sqrt{\frac{\epsilon}{2}} x \) where \( x \) is a random number with the standard normal distribution. We particularly note that the ratio \( r_{HJM}(\theta) \) is independent of \( \sigma(\theta) \) and
is in fact constant. The ratio as calculated from the data is shown in figure 1 and can be seen to be far from constant. Hence we see that the time translation invariant one factor HJM model is inconsistent with the real evolution of the FRC for any choice of function $\sigma(\theta)$.

Using the unconstrained quantum field theoretic model, we can again derive the expressions for the above quantities to zeroth order accuracy in $\epsilon$ to obtain

$$V_{QFT}(\theta) = \sigma(\theta) \sqrt{D(\theta, \theta; t, T_{FR})} \epsilon$$

$$C_{QFT}(\theta) = \frac{\sigma(\theta) D(\theta, \theta_{min}, t, T_{FR})}{\sigma(\theta_{min}) D(\theta_{min}, \theta_{min}, t, T_{FR})} - 1$$

FIG. 1. The observed and fitted $r(\theta)$ for the quantum field theoretic and HJM models. It can be seen that the quantum field theoretic model is in much better agreement with the data.

|                  | $\mu$ (year$^{-1}$) | $\sigma(\theta_{min})$ (year$^{-1}$) |
|------------------|----------------------|---------------------------------------|
| Least Squares    | 0.0822               | 0.0308                                |
| Bootstrap (90% C.I.) | (0.080, 0.085)         | (0.0299, 0.0317)                      |
| Serial Segments (90% C.I.) | (0.081, 0.099)       | (0.028, 0.033)                       |

TABLE I. Results obtained for the unconstrained quantum field theoretic model.
The ratio \( r(\theta) \) is thus given in this model by

\[
r_{QFT}(\theta) = \frac{\sigma(\theta_{min}) \sqrt{\epsilon D(\theta, \theta; t, T_{FR}) D(\theta_{min}, \theta_{min}; t, T_{FR})}}{D(\theta, \theta_{min}; t, T_{FR})}
\]

which is still independent of \( \sigma(\theta) \). However, we note that the ratio is no longer constant. Using this fact, we can fit the ratio to find \( \mu \) and \( \sigma(\theta_{min}) \). We took the limit \( T_{FR} \to \infty \) as required and used the Levenberg-Marquardt method \[11\] to obtain the non-linear least squares fit. The results are shown in table I. The confidence intervals were obtained through the bootstrap method \[12\]. An alternative confidence interval was obtained by dividing the data into series of 500 days starting from the first day, second day and so on and so forth. The function \( r(\theta) \) was calculated and the parameters fitted for the resulting 346 data sets. The confidence interval using these serial data sets are also shown in table I. The fitted ratio together with the observed values are shown in figure 1.

![Image 1](Sigma_from_C_and_V_for_the_QFT_model.png)

**FIG. 2.** \( \sigma(\theta) \) derived from \( V \) and from \( C \) for the quantum field theoretic model.

![Image 2](Sigma_from_C_and_V_for_HJM.png)

**FIG. 3.** \( \sigma(\theta) \) derived from \( V \) and from \( C \) for the one factor HJM model.

We can also use equations 17 and 18 to obtain two different estimates of the function \( \sigma(\theta) \). These two estimates are plotted in figure 2. Similar estimates of \( \sigma(\theta) \) for the one factor HJM model are plotted in figure 3. As can be readily seen, the HJM model is seen to be inconsistent with the data. On the other hand, the quantum field theoretic model is in good agreement with data. It is also interesting to note that the volatility function for the HJM model derived from the data is very far from the constant or exponential forms that are commonly used in the literature.

Performing the same procedure for the constrained quantum field theoretic model, we obtain the results in table I. The fitted ratio in this case is shown in figure 4. The two estimates of \( \sigma(\theta) \) are shown in figure 5. The agreement between the two functions is better than in the case of the unconstrained model as may be expected due to the additional parameter involved. However, it can be seen from the large confidence intervals that the model is probably overspecified since different values of the parameters give rise to very similar values for \( r(\theta) \).

Another quantity that is of great interest is the mean spread between the forward rates and the spot rate

\[
s(\theta) = \langle f(t, \theta) - f(t, \theta_{min}) \rangle
\]
The spread is the linear sum of two parts: the spread due to the market price of risk and the spread that results from the no arbitrage condition in the model. Since we assume that \( \sigma \) is only a function of \( \theta \), it follows that \( \alpha \) is also only a function of \( \theta \). To calculate the spread due to the model, we assume that the initial forward rate curve is flat or that the effect of the initial forward rate curve becomes negligible after a long time.

In that case, the mean spread due to the no arbitrage condition in the quantum field theoretic model is given by

\[
s_{QFT}(\theta) = (\theta - \theta_{min}) \lim_{t \to \infty} \alpha(t) - \int_{\theta_{min}}^{\theta} \alpha(t) dt
\]  

(21)

where

\[
\alpha_{QFT}(t) = \sigma(t) \int_{0}^{t} \sigma(\theta) D(t, \theta; t, T_{FR}) d\theta
\]  

(22)

| Method                | \( \mu \) (year\(^{-1}\)) | \( \sigma(\theta_{min}) \) (year\(^{-1}\)) | \( a \) (year\(^{-1}\)) |
|-----------------------|-----------------------------|------------------------------------------|------------------------|
| Least Squares         | 0.0174                      | 0.181                                    | 0.0024                 |
| Bootstrap (90% C.I.)  | (0.006, 0.020)              | (0.158, 0.548)                           | (0.0002, 0.0032)       |
| Serial Segments (90% C.I.) | (0.019, 0.055)             | (0.044, 0.203)                           | (0.002, 0.071)         |

TABLE II. Results obtained for the constrained quantum field theoretic model
FIG. 5. $\sigma(\theta)$ derived from $V$ and from $C$ for the constrained quantum field theoretic model

FIG. 6. The calculated spread due to no arbitrage and the observed mean spreads. The large difference is due to the spread due to risk aversion in the market.

Using one of the estimates of $\sigma(\theta)$ (using either gives very similar results), we can calculate the spread due to the no arbitrage condition by numerical integration. Due to the relative inaccuracy of the estimation of $\sigma(\theta)$ in the first place, a trapezoidal integration was considered sufficient. The result together with the observed spread is shown in figure 6. It is seen that the calculated spread is significantly smaller than the actual spread which is consistent with the existence of the spread due to risk aversion. However, we see that a significant portion of the spread might be derived from the way the forward rate curve evolves. A very similar result is obtained when the constrained quantum field theoretic model is used.

To summarize, we have proposed a new way to test the one factor, time translation invariant Heath-Jarrow-Morton and Baaquie’s one factor, time translation invariant quantum field theoretic model for the evolution of forward rates using historical eurodollar futures data. We have found that the one factor HJM model can be rejected while the quantum field theoretic model is consistent with the data. We also find that a quantum field theoretic model with constrained boundary conditions to reflect the special nature of the spot rate is also consistent with the data but the parameters of the model cannot be sufficiently accurately derived using this method. We also show that a significant portion of the spread can be explained by the quantum field theoretic model.

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