Asymptotic Dynamics and Asymptotic Symmetries of Three-Dimensional Extended AdS Supergravity

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Abstract

We investigate systematically the asymptotic dynamics and symmetries of all three-dimensional extended AdS supergravity models. First, starting from the Chern-Simons formulation, we show explicitly that the (super)anti-de Sitter boundary conditions imply that the asymptotic symmetry algebra is the extended superconformal algebra with quadratic nonlinearities in the currents. We then derive the super-Liouville action by solving the Chern-Simons theory and obtain a realization of the superconformal algebras in terms of super-Liouville fields. Finally, we discuss the possible periodic conditions that can be imposed on the generators of the algebra and generalize the spectral flow analysed previously in the context of the $N$-extended linear superconformal algebras with $N \leq 4$. The (2+1)-AdS/2-CFT correspondence sheds a new light on the properties of the nonlinear superconformal algebras. It also provides a general and natural interpretation of the spectral flow.
1 Introduction

The connection between diffeomorphism-invariant theories in $D + 1$ dimensions and conformal field theories in $D$ dimensions is a fascinating subject [1, 2, 3, 4]. One of the simplest contexts in which it has been studied is the case of $2 + 1$ dimensions, for which it has been proved more than ten years ago that pure Chern-Simons theory [5] on a manifold with boundary induces the Wess-Zumino-Witten model [6] on the boundary [7, 8, 9]. This context is also particularly rich since the conformal group on the boundary is infinite-dimensional.

As it has been realized, the detailed correspondence between $D + 1$ and $D$ dimensions strongly depends on the precise form of the boundary conditions. So, while one gets the WZW model with the natural boundary conditions of [8, 9], one gets different theories if one adopts other boundary conditions. This feature is relevant to $2 + 1$ gravity with a negative cosmological constant (“AdS gravity in $2 + 1$ dimensions”), which may be viewed as a Chern-Simons theory with gauge group $SO(2, 2)$ at least as far as the equations of motion are concerned [10, 11]. It has been shown in [12] that the anti-de Sitter asymptotics [13] lead to the Liouville theory at infinity (up to zero modes). In terms of the CS/WZW correspondence, one may understand this result as due to the fact that the AdS asymptotic conditions are stronger than the boundary conditions of [8, 9] and enforce the Hamiltonian reduction from $SL(2, R)$-WZW to Liouville theory [14, 15, 16, 17].

The purpose of this paper is to extend the analysis to the supersymmetric case. A first step in this direction was taken in [18, 19], where the case $N = 1$ was considered and shown to lead either to the Ramond or Neveu-Schwarz superalgebras at infinity, depending on the periodicity conditions on the gravitino field. The extended models were not treated in detail. This is done here.

After a brief review of AdS supergravity (section 2), we provide asymptotic conditions on all the fields. These conditions are geometrically motivated and chosen to enforce AdS asymptotics. We then show that they automatically imply that the asymptotic superconformal algebras are the
superconformal algebras with non linearities in the currents discussed in [20, 21, 22, 23, 24, 25] (section 3). Furthermore, the (classical) Virasoro central charge is equal to $6k = 3\ell/2G$ for all models. The analysis is done by identifying in the $(2+1)$-Chern-Simons formulation the generators of the asymptotic symmetries as appropriate surface terms at infinity and then computing their algebra. In sections 4 and 5, we give a dynamical explanation of the origin of these symmetries by relating explicitly the extended supergravity actions to the extended super-Liouville ones. A few fine points (e.g., implementation of the reduction constraints inside the action) are carefully analysed. Finally, in section 6, we discuss the “spectral flow” [26]. Different “muddings” for the generators of the superconformal algebras can be adopted, leading to apparently different algebras. However, some of these algebras are in fact related by the so-called spectral flow introduced in [26] for the linear superconformal algebras. We close our paper with a concluding section, followed by an appendix, in which we discuss the role of zero modes and holonomies in the Liouville model arising from $2+1$ gravity and their relation to the sign of the Liouville potential.

The interest of extended AdS supergravity models in the context of the AdS/CFT correspondence is at least threefold:

1. First, the treatment sheds a new light on the properties of the nonlinear algebras of [20, 21, 22, 23, 24, 25]. In particular, it provides a physical explanation for the relation between the superconformal algebras and superalgebras found algebraically in [21, 24, 25]. The superconformal algebra is the boundary symmetry of an AdS supergravity theory in the Chern-Simons formulation with the superalgebra as a gauge group.

2. Second, one automatically gets explicit realizations of these superconformal algebras in terms of Liouville fields.

3. Third, the spectral flow gets a general and natural interpretation in the Chern-Simons framework.
Some of the results reported here are implicitly contained in the literature (see e.g. [27, 28, 29]). However, no systematic and complete treatment is to our knowledge available. In view of the recent interest in the AdS/CFT correspondence, we feel that such a treatment can be useful.

2 Extended AdS Supergravities in Three Dimensions

2.1 Superalgebras

As shown in [10], supergravity in $2+1$ dimensions can be formulated as a pure Chern-Simons theory based on an appropriate supergroup. The dreibein, the gravitini and the gauge fields can be combined in a super Chern-Simons connection. We denote the superalgebra by $G = G^0 \oplus G^1$, where $G^0$ is the even part of the superalgebra and $G^1$ the odd part. Since we wish to describe AdS$_3$ space-time, the even subalgebra $G^0$ must contain $sl(2; R)$ and be of the form $G^0 \cong sl(2; R) \oplus \tilde{G}$. Furthermore, the fermionic generators must transform as $sl(2; R)$-spinors. The dimension of the “internal” algebra $\tilde{G}$ is denoted $D = \dim \tilde{G}$ and $\rho$ is the real, not necessarily irreducible representation of $\tilde{G}$ (of dimension $d = \dim \rho$) in which the fermionic generators transform.

These conditions are realized in only seven cases [31, 32, 33], which are the following

| $G$                   | $\tilde{G}$       | $\rho$ | $D$         |
|----------------------|-------------------|--------|-------------|
| Osp(N|2;R)              | so(N)             | N      | N(N-1)/2    |
| SU(1,1|N)$_{N\neq2}$     | su(N)+u(1)        | N+\bar{N}| N$^2$       |
| SU(1,1|2)/U(1)           | su(2)             | 2+\bar{2}| 3           |
| Osp(4$^\ast$|2M)       | su(2)+usp(2M)     | (2M,2) | M(2M+1)+3   |
| D$^1$(2, 1; $\alpha$)| su(2)+su(2)      | (2,2)  | 6           |
| G(3)                | $G_2$             | 7      | 14          |
| F(4)                | spin(7)           | $8_s$  | 21          |
In fact, this is also the list of superalgebras which can be associated with 2-dimensional superconformal algebras with quadratic non-linearities \[23, 24, 25\]. This is not an accident since, as we shall see, the two are related by the AdS/CFT correspondence.

### 2.2 Conventions

We follow \[24\] (see also \[34\]). We denote the basis of the representation \(\rho\) as \(\lambda^a = (\lambda^a)^\alpha\), \(\left[\lambda^a, \lambda^b\right] = f^{abc}\lambda^c\), and take the structure constants \(f^{abc}\) to be totally antisymmetric. The Killing metric for \(\tilde{G}\) is \(g^{ab} = -f^{acd}f^{bed} = -C_v\delta^{ab}\) where \(C_v\) is the eigenvalue of the second Casimir in the adjoint representation of \(\tilde{G}\).

We also denote by \(C_\rho\) the eigenvalue of the second Casimir in the representation \(\rho\):

\[
\lambda^a\lambda^b = -C_\rho I, \quad tr(\lambda^a\lambda^b) = -\frac{d}{D}C_\rho\delta^{ab} = \frac{d}{D}C_\rho g^{ab}.
\]

The representation \(\rho\) is orthogonal and admits a \(\tilde{G}\)-invariant symmetric metric \(\eta^{\alpha\beta} = \eta^{\beta\alpha}\), with inverse \(\eta_{\alpha\beta}\), which we will use to lower and raise the supersymmetry (Greek) indices. One has \(\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^{\gamma}\), \((\lambda^a)^{\alpha\beta} = (\lambda^a)^{\alpha}_{\gamma} \eta^{\gamma\beta} = - (\lambda^a)^{\beta\alpha}\), and also \((\lambda^a\lambda^b)^{\alpha\beta} = (\lambda^b\lambda^a)^{\beta\alpha}\).

The superalgebra generators and commutators are:

- **\(G_0\) generators**: The \(\text{sl}(2;\mathbb{R})\) generators \(t^a (a = 3, +, -)\) are equal to \(\frac{1}{2}\sigma^3, \sigma^+, \sigma^-\), and satisfy the commutation relations:

\[
\left[\frac{1}{2}\sigma^3, \sigma^\pm\right] = \pm\sigma^\pm \tag{2.1}
\]

\[
\left[\sigma^+, \sigma^-\right] = 2\left(\frac{1}{2}\sigma^3\right) \tag{2.2}
\]

The Killing metric for \(\text{sl}(2;\mathbb{R})\) is:

\[
\tilde{h}^{ab} = \begin{pmatrix}
2 & 0 & 0 \\
0 & 0 & 4 \\
0 & 4 & 0
\end{pmatrix}
\]
and in its 2-representation, $\text{Tr}(t^a t^b) = \frac{1}{4} h^{ab}$

The $\tilde{G}$ generators are denoted $T^a$ ($a = 1 \ldots D$), and satisfy:

$$[T^a, T^b] = f^{abc} T^c$$

These commute with all $\text{sl}(2;\mathbb{R})$ generators.

- $G_1$ generators: These are denoted by $R^\pm\alpha$ ($\alpha = 1 \ldots d$). Their commutators with the $\text{sl}(2;\mathbb{R})$ generators are:

$$\left[ \frac{1}{2} \sigma^3, R^\pm\alpha \right] = \pm \frac{1}{2} R^\pm\alpha$$

$$[\sigma^\pm, R^\pm\alpha] = 0$$

$$[\sigma^\pm, R^\mp\alpha] = R^\pm\alpha.$$ (2.4)

Their commutators with the $\tilde{G}$ generators are:

$$[T^a, R^\pm\alpha] = - (\lambda^a)^\alpha_\beta R^\pm\beta$$

The anticommutators of the fermion generators are:

$$\{ R^\pm\alpha, R^\pm\beta \} = \pm \eta^{\alpha\beta} \sigma^\pm$$

$$\{ R^\pm\alpha, R^\mp\beta \} = - \eta^{\alpha\beta} \left( \frac{1}{2} \sigma^3 \right) \pm \frac{d-1}{2C_0} (\lambda^a)^{\alpha\beta} T^a$$

The Jacobi identity for three fermion generators yields an identity involving the representation matrices, and is:

$$(\lambda^a)^{\beta\gamma} (\lambda^a)^\alpha_\delta + (\lambda^a)^{\alpha\gamma} (\lambda^a)^\beta_\delta = \frac{C_0}{d-1} \left( 2\eta^{\alpha\beta} \delta^\gamma_\delta - \eta^{\alpha\gamma} \delta^\beta_\delta - \eta^{\beta\gamma} \delta^\alpha_\delta \right)$$

It is fulfilled for all supergroups listed above.
The given superalgebras all admit a consistent, invariant, supersymmetric and nondegenerate bilinear form which we denote by $STr$ and which is defined by

$$
STr (t^a t^b) = \frac{1}{4} h^{ab} \quad ; \quad STr(R^{-\alpha} R^{+\beta}) = - STr(R^{+\alpha} R^{-\beta}) = \eta^{\alpha\beta} \quad ; \quad STr(T^a T^b) = \frac{2C_\rho}{d-1} \delta^{ab}
$$

(2.11)

2.3 Action

Using these conventions, we can parametrize a general element in $G$ by:

$$
\Gamma = \left( A^3, A^+, A^-, B^a T^a \right) + \left( \psi_{+a} R^{+\alpha} + \psi_{-a} R^{-\alpha} \right) \equiv A + \Psi
$$

(2.12)

where $A \equiv A^3 + A^+ + A^- \quad , \quad B \equiv B^a T^a \quad , \quad \Psi \equiv \psi_{+a} R^{+\alpha} + \psi_{-a} R^{-\alpha}$. Here $A^3, A^+, A^-, B^a$ are commuting parameters and $\psi_{+a}, \psi_{-a}$ anti-commuting Grassman parameters.

The action describing supergravity in $AdS_3$ space-time is a difference of two Chern-Simons actions $[10]$

$$
S[\Gamma, \tilde{\Gamma}] = S_{CS}[\Gamma] - S_{CS} [\tilde{\Gamma}] \quad (2.13)
$$

where $\Gamma, \tilde{\Gamma} \in G$ are superalgebra valued super-connections. We assume for simplicity that $\Gamma$ and $\tilde{\Gamma}$ take values in the same superalgebra, although this is not necessary. This choice leads to a non-chiral model in two dimensions to which the results of sections 4 and 5 apply. $S_{CS}$ is the super Chern-Simons action, defined by:

$$
S_{CS}[\Gamma] = \frac{k}{4\pi} \int_M STr \left( \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right) \quad (2.14)
$$

The integration is over $M$ - a 3-manifold, with topology $D \times \mathbb{R}$ and $k$ is related to the 3-dimensional Newton constant $G$ through $k = \ell^3/4G$ where $\ell$ is the anti-de Sitter radius. Furthermore, the product of two fermions in this formula (as well as in (3.10) below but in no other formula) differs by a factor $i$ from the standard Grassmann product fulfilling $(ab)^* = b^* a^*$ (e.g., (2.14) contains $i(k/4\pi) \psi_{+a} \wedge d\psi_{-b} STr(R^{+\alpha} R^{-\beta})$ in terms of the standard Grassmann product).
Writing the super-connection in component form and identifying the \( sl_2 \) components with the dreibeins and the connections:

\[
A^a_i = \omega^a_i + \frac{1}{\ell} e^a_i \tag{2.15}
\]

\[
\tilde{A}^a_i = \omega^a_i - \frac{1}{\ell} e^a_i \tag{2.16}
\]

we find the supergravity action for a general superalgebra:

\[
S \left[ \Gamma, \tilde{\Gamma} \right] = \frac{1}{8\pi G} \int_M d^3 x \left\{ \frac{1}{2} e R + \frac{e}{\ell^2} + \frac{i}{2} \varepsilon_ijk (\psi_i)_\mu \mathcal{D}^{\mu\nu}_j (\psi_k)_\nu + \frac{i}{2} \ell \varepsilon_ijk \left( \bar{\psi}_i \right)_\mu \tilde{\mathcal{D}}^{\mu\nu}_j \left( \bar{\psi}_k \right)_\nu + \frac{C_\rho}{d-1} \ell \varepsilon_ijk \left( B^a_i \partial_j B^a_k + \frac{1}{3} f^{abc} B^a_i B^b_j B^c_k \right) \right. \\
\left. - \frac{C_\rho}{d-1} \ell \varepsilon_ijk \left( \bar{B}^a_i \partial_j \bar{B}^a_k + \frac{1}{3} f^{abc} \bar{B}^a_i \bar{B}^b_j \bar{B}^c_k \right) - \frac{i}{2} \varepsilon_ijk \eta^{\alpha\beta} e^a_i \left( [\bar{\psi}_j]_\alpha t^a [\psi_k]_\beta - [\bar{\psi}_j]_\alpha t^a [\bar{\psi}_k]_\beta \right) \right\} \tag{2.17}
\]

where the square brackets stand for two-component \( sl_2 \)-spinors and where the operators \( \mathcal{D}^{\mu\nu}_j \) and \( \tilde{\mathcal{D}}^{\mu\nu}_j \) are respectively defined by

\[
\mathcal{D}^{\mu\nu}_j \equiv \begin{bmatrix}
2 \left( \eta^{\alpha\beta} \partial_j + (\lambda^a)^{\alpha\beta} D^a_j \right) \delta^\mu_+ \delta^\nu_- + \\
-\eta^{\alpha\beta} \left( \frac{1}{2} \omega^3_j \left[ \delta^\mu_- \delta^\nu_- + \delta^\mu_+ \delta^\nu_+ \right] + \omega^+ \delta^\mu_- \delta^\nu_- - \omega^- \delta^\mu_+ \delta^\nu_+ \right)
\end{bmatrix} \tag{2.18}
\]

\[
\tilde{\mathcal{D}}^{\mu\nu}_j \equiv \begin{bmatrix}
2 \left( \eta^{\alpha\beta} \partial_j + (\lambda^a)^{\alpha\beta} \tilde{B}^a_j \right) \delta^\mu_+ \delta^\nu_- + \\
-\eta^{\alpha\beta} \left( \frac{1}{2} \omega^3_j \left[ \delta^\mu_- \delta^\nu_- + \delta^\mu_+ \delta^\nu_+ \right] + \omega^+ \delta^\mu_- \delta^\nu_- - \omega^- \delta^\mu_+ \delta^\nu_+ \right)
\end{bmatrix} \tag{2.19}
\]

(From here on we work in units of \( \ell = 1 \), unless otherwise stated).

3  Asymptotic Symmetries - Extended superconformal algebras

3.1 Boundary conditions

The ground state of AdS supergravity is anti-de Sitter space with vanishing gravitini and gauge fields. AdS\(_3\) is the solution with the maximum number of isometries, namely \( 3 + 3 = 6 \) (3 per chiral
sector). It is also invariant under constant $\tilde{G} \oplus G$-transformations ($D + D$ transformations) as well as under $2d + 2d$ rigid supersymmetries. The corresponding Killing spinors are explicitly given in [18]. In short, AdS$_3$ is invariant under the superalgebra $G \oplus G$.

Besides AdS$_3$, there are other solutions, which differ from AdS$_3$ in their global properties [35]. The black-hole solution [36] is an important example and can be obtained from AdS$_3$ through appropriate identifications [37]. The boundary conditions to be imposed on the fields at infinity should allow for these physically interesting solutions.

It was shown in [13] that the appropriate conditions in the $sl(2; R)$-sector are, for $r \to \infty$,

$$A \sim \left\{ \frac{12\pi L(\theta,t)}{\sqrt{r}} \sigma^+ + r \sigma^- \right\} dx^+ + 0dx^- + \left\{ \frac{1}{\sqrt{r}} \right\} dr$$  \hspace{1cm} (3.1)

$$\tilde{A} \sim \left\{ \frac{12\pi \tilde{L}(\theta,t)}{\sqrt{r}} \sigma^- + r \sigma^+ \right\} dx^- + 0dx^+ + \left\{ -\frac{1}{\sqrt{r}} \right\} dr$$  \hspace{1cm} (3.2)

where we have switched to chiral coordinates: $x^\pm = t \pm \ell \theta$ and where $L$ and $\tilde{L}$ are arbitrary functions of $\theta$ and $t$. The factors of $2\pi$ are introduced for later convenience. (The asymptotic conditions were actually expressed in terms of the metric in [13] but are easily verified to be equivalent to those written here in terms of the connection).

In order to obtain the asymptotic boundary conditions for all components of the two superconnections, we follow the general scheme introduced in [38, 13]. Namely, we take a generic configuration having the above asymptotic behaviour with vanishing gravitini and gauge fields and act on it with the exact symmetries of AdS$_3$. In doing so, one generates new terms (since we start with configurations which are not AdS$_3$) which typically behave as

$$\Gamma \sim \left\{ \frac{12\pi L(\theta,t)}{\sqrt{r}} \sigma^+ + r \sigma^- \right\} dx^+$$

$$+ 0dx^-$$

$$+ \left\{ \frac{1}{\sqrt{r}} \right\} dr$$  \hspace{1cm} (3.3)

$$\tilde{\Gamma} \sim \left\{ \frac{12\pi \tilde{L}(\theta,t)}{\sqrt{r}} \sigma^- + r \sigma^+ \right\} dx^-$$

$$+ 0dx^+$$

$$+ \left\{ -\frac{1}{\sqrt{r}} \right\} dr$$
\begin{align}
+ 0dx^+ \\
+ \left\{ \frac{- \sigma^3}{r^2} \right\} dr
\end{align} (3.4)

where \( L(\theta, t), Q_{+\alpha}(\theta, t), B^a(\theta, t) \) and \( \bar{L}(\theta, t), \bar{Q}_{-\alpha}(\theta, t), \bar{B}^a(\theta, t) \) are functions of \( \theta, t \) \( (Q, \bar{Q} \text{ Grassmanian}) \), so we shall adopt (3.3) and (3.4) with arbitrary \( L, \bar{L}, Q, \bar{Q}, B \) and \( \bar{B} \) as boundary conditions. Here, we have scaled the terms proportional to \( B^a \) and \( \bar{B}^a \) with

\[ k_B = k \frac{2C^\rho}{d-1} \] (3.5)

for later convenience.

### 3.2 Asymptotic Symmetries

The boundary conditions at spatial infinity give us the asymptotic form of the superconnections. We work out in this subsection the gauge transformations that preserve the asymptotic form of the superconnections (“asymptotic symmetries”). In what follows, we focus only on the superconnection \( \Gamma \). The treatment for \( \bar{\Gamma} \) is analogous and eventually gives another copy of the superconformal algebra for the other chirality.

It is convenient to factor out the \( r \) dependence by performing a gauge transformation and to work with the equivalent connection \( \Delta_i \) defined by

\[ \Delta_i = b \partial_i b^{-1} + b \Gamma_i b^{-1} \] (3.6)

where \( b \) is a similarity transformation depending only on \( r \):

\[ b(r) = \exp \left[ \frac{\sigma^3}{2} \ln r \right]. \] (3.7)

The components of \( \Delta_i \) vanish asymptotically, except \( \Delta_u \) \( (u \equiv x^+) \), given by

\[ \frac{1}{2\pi} \Delta_u = \frac{1}{2\pi} b \Gamma_u b^{-1} = \left( \frac{L}{k} \sigma^+ + \frac{1}{2\pi} \sigma^- \right) + \left( \frac{B^a}{k_B} T^a \right) + \left( \frac{Q^{+\alpha}}{k} R^{+\alpha} \right). \] (3.8)
Two things should be mentioned about the asymptotic form of $\Delta u$. First, there is no $\sigma^3$ component and the term proportional to $\sigma^-$ is fixed to be one. Second, the fermion part of the superconnection contains no $R^{\alpha}$ components. These conditions turn out to be crucial for obtaining the superconformal algebra. Note that there is a similar “chirality condition” on the fermions in 4 (or higher) dimensions [38].

Acting on the superconnection with an infinitesimal gauge transformation:

$$\Lambda = \left( \chi^3 \sigma^3 + \chi^+ \sigma^+ + \chi^- \sigma^- \right) + (\omega^a T^a) + (\epsilon_{+\alpha} R^{+\alpha} + \epsilon_{-\alpha} R^{-\alpha})$$

we get:

$$\delta \Delta u = \partial_u \Lambda + [\Delta u, \Lambda]$$

$$= \left( \partial \chi^3 + 2\frac{2\pi L}{k} \chi^- - 2\chi^+ - i\eta^{\alpha\beta} \frac{2\pi Q^+}{k} \epsilon_{-\beta} \frac{1}{2} \sigma^3 \right)$$

$$+ \left( \partial \chi^+ - \frac{2\pi L}{k} \chi^3 + i\eta^{\alpha\beta} \frac{2\pi Q^+}{k} \epsilon_{+\beta} \right) \sigma^+$$

$$+ \left( \partial \chi^- - \chi^3 \right) \sigma^-$$

$$+ \left( \partial \omega_c + \frac{j^{abc}}{k_B} \omega^b \right)^c + i \frac{d-1}{2C_{\rho}} \left( \lambda^\alpha \right)^{\alpha\beta} \frac{2\pi Q^+}{k} \epsilon_{-\beta} \right)$$

$$+ \left( \partial \epsilon_{+\beta} + \epsilon_{-\beta} - \left( \lambda^a \right)^{\alpha\beta} \frac{2\pi B^a}{k_B} - \chi^3 \frac{2\pi Q^+}{2k} + \left( \lambda^a \right)^{\alpha\beta} \omega^a \frac{2\pi Q^+}{k} \right) R^{+\beta}$$

$$+ \left( \partial \epsilon_{-\beta} + \epsilon_{+\beta} - \left( \lambda^a \right)^{\alpha\beta} \frac{2\pi B^a}{k_B} - \chi^3 \frac{2\pi Q^+}{2k} \right) R^{-\beta}$$

In order to preserve the asymptotic form of the superconnection, the parameters of the gauge transformation $\Lambda$ are required to fulfill some relations. These relations are easily displayed by taking as independent parameters $\chi \equiv \chi^-$, $\epsilon_{\alpha} \equiv \epsilon_{-\alpha}$, and $\omega^a$. The relations in question are then just expressions for the other parameters,

$$\chi^3 = -\chi'$$

$$\chi^+ = -\frac{\chi''}{2} + \chi \frac{2\pi L}{k} - i\eta^{\alpha\beta} \frac{2\pi Q^+}{2k} \epsilon_{-\beta}$$

$$\epsilon_{+\alpha} = -\epsilon'_\alpha + \chi \frac{2\pi Q^+}{k_B} + \left( \lambda^a \right)^{\alpha\beta} \frac{2\pi B^a}{k_B}$$

$$\epsilon_{-\alpha} = -\epsilon'_\alpha + \chi \frac{2\pi Q^+}{k_B}$$
where the prime denotes derivative with respect to the argument. The transformations with parameters subject to (3.12), (3.13) and (3.14) are the asymptotic symmetries.

3.3 Superconformal Algebra

In order to exhibit the algebra of the asymptotic symmetry transformations, we shall work out the Poisson bracket algebra of their generators. To that end, we first need to identify these generators. This is done by following the Regge-Teitelboim method [39]. The gauge transformations of the Chern-Simons theory with parameters \( \Lambda^A \equiv (\chi^3, \chi^\pm, \omega^a, \epsilon_{\pm\alpha}) \) are generated in the equal-time Poisson bracket by the spatial integral \( G[\Lambda] = \int d^2x \Lambda^A G_A + S_\infty \), where (i) the \( G_A \) are the Chern-Simons constraints, equal to minus the factor of the temporal components of the superconnection in the action; and (ii) \( S_\infty \) is a boundary term at infinity chosen such that the variation \( \delta G[\Lambda] \) of the generator \( G[\Lambda] \) contains only undifferentiated field variations under the given boundary conditions [39] (“\( G[\Lambda] \) has well-defined functional derivatives”). If one follows this procedure, one gets (up to the bulk term that vanishes on-shell),

\[
G_L[\chi] = \int d\theta \chi(\theta)L(\theta) \tag{3.15}
\]
\[
G_B[\omega] = \int d\theta \omega^a(\theta)B^a(\theta) \tag{3.16}
\]
\[
G_Q[\epsilon] = \int d\theta \eta^{\alpha\beta}i\epsilon_\alpha(\theta)Q_{+\beta}(\theta). \tag{3.17}
\]

Thus, we see that \( L, Q_{+\alpha} \) and \( B^a \) are precisely the generators of the asymptotic symmetries. There is no factor in (3.15), (3.16) or (3.17) because we included the appropriate factors in the definitions of \( L, Q_{+\alpha} \) and \( B^a \).

Once the generators have been identified, one computes their Poisson bracket algebra by using the relationship \( \delta_A F = \{G[\Lambda], F\}_PB \), valid for any phase-space function, and working out \( \delta L, \delta Q_{+\alpha} \) and \( \delta B^a \) directly from the formulas of the previous subsections.

Under the asymptotic AdS subgroup, the transformation laws for the \( L, Q's \) and \( B's \) are given
by:

\[ \delta L = -\frac{k}{2\pi} \chi'' + \left[ (\chi L)' + \chi' L \right] - i\eta^{a\beta} \left[ \frac{1}{2} (Q_{+\alpha} \epsilon_{\beta})' + Q_{+\alpha} \epsilon_{\beta}' \right] \]

\[ - \frac{2\pi i}{k_B} (\lambda^a)_{a\beta} Q_{+\alpha} B^a \epsilon_{\beta} \]

\[ \delta Q_{+\alpha} = -\frac{k}{2\pi} \epsilon_{a\alpha} + \left[ (\chi Q_{+\alpha})' + \frac{1}{2} \chi' Q_{+\alpha} \right] + L_{\alpha} + \]

\[ + (\lambda^a)_{a\beta} \frac{k}{k_B} \left[ (\epsilon_{\beta} B^a)' + \epsilon_{\beta} B^a \right] + \]

\[ + (\lambda^a)_{a\beta} Q_{+\beta} - 2\pi \frac{k}{k_B} \chi (\lambda^a)_{a\beta} Q_{+\beta} B^a - 2\pi \frac{k}{2k_B^2} \left\{ (\lambda^a, \lambda^b) \right\}_{a\beta} \epsilon_{\beta} B^a B^b \]

\[ \delta B^a = \frac{k_B}{2\pi} \omega^{a\alpha} + f^{abc} B^b \omega^c + \frac{k_B}{k} \frac{d-1}{2C_{\rho}} (\lambda^a)_{a\beta} Q_{+\alpha} \epsilon_{\beta} \]

The formula \( \delta \Lambda F = \{G[\Lambda], F\}_{PB} \) gives then

\[ \{L(\theta), L(\theta')\}_{PB} = \frac{k}{4\pi} \delta'' (\theta - \theta') - (L(\theta) + L(\theta')) \delta' (\theta - \theta') \] (3.21)

\[ i \{Q_{+\alpha}(\theta), Q_{+\beta}(\theta')\}_{PB} = -\frac{k}{2\pi} \eta_{\alpha\beta} \delta'' (\theta - \theta') - (\lambda^a)_{a\beta} \frac{k}{k_B} \delta' (\theta - \theta') \left[ B^a (\theta) + B^a (\theta') \right] \]

\[ + \eta_{\alpha\beta} L(\theta) \delta (\theta - \theta') - 2\pi \frac{k}{2k_B^2} \left\{ (\lambda^a, \lambda^b) \right\}_{a\beta} B^a (\theta) B^b (\theta) \delta (\theta - \theta') \]

(3.22)

\[ \{B^a (\theta), B^b (\theta')\}_{PB} = -\frac{k_B}{2\pi} \delta^{ab} \delta (\theta - \theta') + f^{abc} \delta (\theta - \theta') B^c (\theta) \]

(3.23)

\[ \{L(\theta), Q_{+\alpha}(\theta')\}_{PB} = - \left[ Q_{+\alpha}(\theta) + \frac{1}{2} Q_{+\alpha}(\theta') \right] \delta' (\theta - \theta') \]

\[ - 2\pi (\lambda^a)_{a\beta} \frac{1}{k_B} Q_{+\beta}(\theta) B^a (\theta) \delta (\theta - \theta') \]

(3.24)

\[ \{B^a (\theta), Q_{+\alpha}(\theta')\}_{PB} = (\lambda^a)_{a\beta} Q_{+\beta}(\theta) \delta (\theta - \theta') \]

(3.25)

\[ \{L(\theta), B^a (\theta')\}_{PB} = 0 \]

(3.26)

As defined, the generator \( L \) does not act on the Kac-Moody currents \( B^a \). To get exactly the superconformal algebra in standard form, we must add to \( L \) the Sugawara energy-momentum operator for the \( B^a \), given classically by \( L_{SUG} = \frac{2\pi}{2k_B} B^a B^a \). Thus, redefining \( L \) by adding to it \( L_{SUG} \):

\[ \hat{L} \equiv L + L_{SUG} = L + \frac{2\pi}{2k_B} B^a B^a \] (3.27)
and rescaling $Q$: $\hat{Q}_{+\alpha} = \sqrt{2}Q_{+\alpha}$, the Poisson bracket algebra becomes

$$\{ \hat{L}(\theta), \hat{L}(\theta') \}_PB = \frac{k}{4\pi} \delta^{m}(\theta - \theta') - \left( \hat{L}(\theta) + \hat{L}(\theta') \right) \delta'(\theta - \theta') \quad (3.28)$$

$$i \left\{ \hat{Q}_{+\alpha}(\theta), \hat{Q}_{+\beta}(\theta') \right\}_PB = -\frac{k}{\pi} \eta_{\alpha\beta} \delta'(\theta - \theta') - 2(\lambda^\alpha)_{\alpha\beta} \frac{d-1}{2C_\rho} \delta'(\theta - \theta') \left[ B^a(\theta) + B^a(\theta') \right]$$

$$+ 2\eta_{\alpha\beta} \hat{L}(\theta) \delta(\theta - \theta')$$

$$- 2\pi k \left( \frac{d-1}{2kC_\rho} \right)^2 \left\{ \{ \lambda^a, \lambda^b \}_\alpha^\beta + \frac{2C_\rho}{d-1} \eta_{\alpha\beta} \delta_{ab} \right\} \times B^a(\theta) B^b(\theta) \delta(\theta - \theta') \quad (3.29)$$

$$\left\{ B^a(\theta), B^b(\theta') \right\}_PB = -\frac{k}{2\pi d-1} \varepsilon^{abc} \delta'(\theta - \theta') + f^{abc} \delta(\theta - \theta') B^c(\theta) \quad (3.30)$$

$$\{ \hat{L}(\theta), \hat{Q}_{+\alpha}(\theta') \}_PB = - \left[ \hat{Q}_{+\alpha}(\theta) + \frac{1}{2} \hat{Q}_{+\alpha}(\theta') \right] \delta'(\theta - \theta')$$

$$\left\{ B^a(\theta), \hat{Q}_{+\alpha}(\theta') \right\}_PB = (\lambda^\alpha)_{\alpha\beta} \hat{Q}_{+\beta}(\theta) \delta(\theta - \theta') \quad (3.31)$$

$$\{ \hat{L}(\theta), B^a(\theta') \}_PB = -B^a(\theta) \delta(\theta - \theta') \quad (3.32)$$

The Fourier mode form of this algebra is given, in quantum-mechanical notations, by \(^3\)

$$\{ \hat{L}_m, \hat{L}_n \} = (m - n) \hat{L}_{m+n} + \frac{k}{2m^3} \delta_{m+n,0} \quad (3.34)$$

$$\left\{ \hat{Q}_{+\alpha}_m, \hat{Q}_{+\beta}_n \right\} = 2\eta_{\alpha\beta} \hat{L}_{m+n} - 2i \frac{d-1}{2C_\rho}(m - n) (\lambda^a)_{\alpha\beta} (B^a)_{m+n}$$

$$+ 2k\eta_{\alpha\beta} m^2 \delta_{m+n,0} +$$

$$- k \left( \frac{d-1}{2kC_\rho} \right)^2 \left\{ \{ \lambda^a, \lambda^b \}_\alpha^\beta + \frac{2C_\rho}{d-1} \eta_{\alpha\beta} \delta_{ab} \right\} \left( B^a B^b \right)_{m+n}$$

$$\left[ B^a_m, B^b_n \right] = i f^{abc} B^c_{m+n} + \frac{2C_\rho k}{d-1} m \delta_{m+n,0} \quad (3.35)$$

$$\left[ \hat{L}_m, \hat{Q}_{+\alpha}_n \right] = (\frac{m}{2} - n) \hat{Q}_{+\alpha} \quad (3.36)$$

$$\left[ B^a_m, \hat{Q}_{+\alpha}_n \right] = i (\lambda^\alpha)_{\alpha\gamma} (\hat{Q}^{+\beta})_{m+n} = -i\eta_{\alpha\beta} (\lambda^\gamma)_{\beta\alpha} (\hat{Q}^{+\beta})_{m+n} \quad (3.37)$$

$$\left[ \hat{L}_m, B^a_n \right] = -n B^a_{m+n} \quad (3.38)$$

\(^2\)We set $A(\theta) = \frac{1}{2\pi} \sum_n A_n e^{i\alpha^a} \theta^n$ for any operator $A(\theta)$ and use the correspondence $\{ . \}_PB = -i \{ . \}$ (where $\{ . \}$ is the commutator) for any pair of operators, except when both are fermionic in which case one has $\{ . \}_PB = -i \{ . \}$ (anticommutator).
Here, by \( (B^a B^b)_{m+n} \), we mean \( B^a(\theta)B^b(\theta) = \left( \frac{1}{2\pi} \right)^2 \sum_n (B^a B^b)_n e^{in\theta} \). These are the non-linear superconformal algebras of [20, 21, 23, 24, 25], up to a constant shift of \( \hat{L}_0 \). The Virasoro central charge is equal to \( 6k \) for all models. It should be stressed, however, that we obtained the classical version of the algebra, where quantum effects of normal ordering are missing. Such effects modify the value of the coefficients in the algebra.

Thus, we have established that the boundary conditions of asymptotic AdS on \( \Gamma \) lead to an asymptotically superconformal symmetry algebra of the type discussed in [20, 21, 22, 23] and with classical Virasoro central charge equal to \( 6k \). Repeating this treatment for \( \bar{\Gamma} \) leads to another copy of the same superconformal algebra for the other chirality.

The above treatment gives a physical explanation for the relation between the superconformal algebras and superalgebras found algebraically in [23, 24, 25]. The superconformal algebra is the boundary symmetry of an AdS supergravity theory in the Chern-Simons formulation with the superalgebra as a gauge group.

4 Deriving the super-Liouville action from AdS supergravity

4.1 Super-WZW model

Because 3-dimensional supergravity possesses the superconformal algebra as asymptotic symmetry algebra, the dynamical theory describing its boundary degrees of freedom at infinity is expected to be superconformal. We show in this section that this is indeed the case and that the dynamical theory in question is (extended) super-Liouville theory. The procedure follows the approach of [12] for gravity and we shall thus describe here the main lines, emphasizing only the fine points (for related information concerning the relation between AdS\(_3\)-gravity and Liouville theory, see [10, 11]).

The boundary conditions (3.3) and (3.4) on the superconnections \( \Gamma \) and \( \bar{\Gamma} \) can be separated into two subsets:
1. First, the $v$-components $\Gamma_v$ ($v \equiv x^-$) and the $u$-components $\bar{\Gamma}_u$ of the superconnections along all the generators of the superalgebra are zero,

$$\Gamma_v = 0, \quad \bar{\Gamma}_u = 0 \quad (4.1)$$

2. Second, some of the $u$-components $\Gamma_u$ and the $v$-components $\bar{\Gamma}_v$ are constrained. More precisely, in terms of the redefined superconnections $\Delta_i$, one has

$$\Delta_u^- = 1, \quad \Delta_u^3 = 0, \quad \Delta_u, -\alpha = 0; \quad \bar{\Delta}_u^- = 1, \quad \bar{\Delta}_u^3 = 0, \quad \bar{\Delta}_v, +\alpha = 0. \quad (4.2)$$

It turns out that the analysis proceeds in much the same way if the currents $\Delta_u^-$ and $\bar{\Delta}_u^+$ are fixed to arbitrary non-vanishing values. So, we shall relax $\Delta_u^-$ and $\bar{\Delta}_u^+$ from their anti-de Sitter values and consider the more general conditions

$$\Delta_u^- = \mu, \quad \Delta_u^3 = 0, \quad \Delta_u, -\alpha = 0; \quad \bar{\Delta}_u^- = \nu, \quad \bar{\Delta}_u^3 = 0, \quad \bar{\Delta}_v, +\alpha = 0. \quad (4.3)$$

We shall first take care of the conditions (4.1).

It has been shown in [8, 9] that pure Chern-Simons theory on a manifold with a boundary is equivalent to a chiral WZW theory living on that boundary under conditions analogous to (4.1) (the analysis of [8, 9] is straightforwardly adapted to cover the boundary conditions (4.1), see [12]). In the present case, one gets two $G$-super-WZW theories on the cylinder $(t, \theta)$ at infinity, of opposite chiralities since the boundary conditions on $\Gamma$ and $\bar{\Gamma}$ are themselves of opposite parities. These two copies can be combined to yield the vector $G$-super-WZW theory coupled to zero modes whose explicit form depends on the topology of the manifold.

As a rule, we shall not write explicitly the zero mode terms in this section, so the action given below describes only part of the dynamics. This is sufficient for exhibiting the dynamical origin of the superconformal algebra at infinity. Note that the zero modes contain in particular holonomy terms, which are not zero even for the anti-de Sitter ground state since we are using the spinor representation of $sl_2$ [19]. A discussion on how to include the holonomies is given in the appendix.
Using the Gauss decomposition for a general supergroup element,

\[ g = E^+ E^0 E^- \]  

(4.4)

where

\[ E^+ = \exp(x \sigma^+ + \psi_+ R^{+\alpha}) = \exp(x \sigma^+) \exp(\psi_+ R^{+\alpha}) \]  

(4.5)

\[ E^0 = \exp(\varphi \sigma^3 + C^a T^a) = \exp(\varphi \sigma^3) \exp(C^a T^a) \]  

(4.6)

\[ E^- = \exp(y \sigma^- + \psi_- R^{-\alpha}) = \exp(y \sigma^-) \exp(\psi_- R^{-\alpha}) \]  

(4.7)

as well as the Polyakov-Wiegmann identity [42] for supermatrices, one gets the WZW action

\[ S^{SWW}(g) = k^2 \pi \int dx^+ dx^- \{ \partial_+ \varphi \partial_- \varphi + e^{-2\varphi} \left( \partial_+ y - i \psi_- \frac{\eta^{\alpha\beta}}{2} \partial_+ \psi_- \beta \right) \left( \partial_- x - i \psi_+ \frac{\eta^{\alpha\beta}}{2} \partial_- \psi_+ \beta \right) - ie^{-\varphi} \partial_- \psi_+ \alpha \partial_+ \psi_- \beta \} - \frac{2D}{d(d-1)} WZW[u] \]  

(4.8)

where \( u = \exp[C^a \lambda^a] \) and where \( WZW[u] \) is the standard WZW action for \( u \):

\[ WZW[u] = \frac{k}{4\pi} \int dx^+ dx^- Tr(\partial_- u u^{-1} \partial_+ u u^{-1}) + \Gamma[u] \]  

(4.9)

with \( \Gamma[u] \) the WZW term.

More precisely, one gets (4.8) in Hamiltonian form, i.e.,

\[ S^{WZW_H}(\varphi, x, y, u, \psi_+, \psi_-, \Pi_+, \Pi_-, \Pi^+ \alpha, \Pi^- \alpha) \]

\[ = \int dx^+ dx^- \{ \Pi_+ \partial_+ \varphi + \partial_- x \Pi_+ + \partial_+ y \Pi_+ + \partial_+ \psi_- \Pi^- \beta + \partial_- \psi_+ \Pi^+ \beta \}

\[ - \frac{1}{2} \left( \frac{\Pi_+^2}{k^2} + \frac{k^2}{\pi^2} \right) - \frac{2\pi}{k} \frac{\Pi_y \Pi_+ e^{2\varphi}}{\pi} - \frac{ik}{2\pi} e^{-\varphi} \Phi_- u^{\alpha\beta} \Phi_+ \beta \]

\[ - \frac{2D}{d(d-1)} WZW^H[u]. \]  

(4.10)

were we introduced the notations:

\[ \Phi_+ \alpha = e^{\varphi} \left( \frac{2\pi}{ik} \Pi_+ \psi_- \beta - \frac{\pi}{k} \Pi_y \psi_- \beta \right) (u^{-1})^3 \alpha \]  

(= \( \partial_- \psi_+ \alpha \) on-shell)  

(4.11)

\[ \Phi_- \alpha = e^{\varphi} \left( -\frac{2\pi}{ik} \Pi_- \beta + \frac{\pi}{k} \Pi_x \psi_+ \beta \right) u^\beta \alpha \]  

(= \( \partial_+ \psi_- \alpha \) on-shell)  

(4.12)
Indeed, the chiral actions are in first-order form, and one goes from the sum of the chiral actions to (4.10) by making a change of phase space variables [12, 43].

If one eliminates the conjugate momenta from (4.10) by using their own equations of motion, one gets (4.8). The relationship between the conjugate momenta and the temporal derivatives of the fields is explicitly given by (recall that \( x^\pm = t \pm \theta \), so that \( 2 \partial_x = \partial_t \pm \partial_\theta \) and \( dx^+ dx^- = 2 dt d\theta \))

\[
\Pi_\varphi = \frac{k}{2\pi} \dot{\varphi} \\
\Pi_x = \frac{k}{2\pi} e^{-2\varphi} \left( \partial_+ y - i \psi_+ \frac{\eta^{\alpha\beta}}{2} \partial_+ \psi_- \right) \\
\Pi_y = \frac{k}{2\pi} e^{-2\varphi} \left( \partial_- x - i \psi_- \frac{\eta^{\alpha\beta}}{2} \partial_- \psi_+ \right)
\]

and

\[
\Pi^{+\beta} = \frac{\partial^L L}{\partial \psi^{+\beta}_+} = \frac{ik}{2\pi} \frac{1}{2} e^{-2\varphi} \left( \partial_+ y - i \psi_+ \frac{\eta^{\gamma\delta}}{2} \partial_+ \psi_- \right) \psi_+ \eta^{\beta\alpha} - e^{-\varphi} u^{\beta\alpha} \partial_+ \psi_-
\]

\[
\Pi^{-\beta} = \frac{\partial^L L}{\partial \psi^{-\beta}_-} = \frac{ik}{2\pi} \frac{1}{2} e^{-2\varphi} \left( \partial_- x - i \psi_- \frac{\eta^{\gamma\delta}}{2} \partial_- \psi_+ \right) \psi_- \eta^{\beta\alpha} + e^{-\varphi} (u^{-1})^{\beta\alpha} \partial_- \psi_+
\]

### 4.2 Kac-Moody currents

As it is well known, the super WZW action has two sets ("left" and "right") of conserved currents, each set forming a super Kac-Moody algebra. With appropriate normalizations, the currents for the two chiralities can be taken to be \( \frac{k}{2\pi} \partial_+ gg^{-1} \) and \( -\frac{k}{2\pi} g^{-1} \partial_- g \) and we define the current components by

\[
\frac{k}{2\pi} \partial_+ gg^{-1} \equiv a^I G^I \equiv \left( J^3 \sigma^3 + J^+ \sigma^+ + J^- \sigma^- \right) + \left( \frac{d-1}{2C_\rho} B^a T^a \right) + (F^+ + F^- \sigma^+ + F_- \sigma^-) \quad (4.18)
\]
\[ -\frac{k}{2\pi}g^{-1}\partial_- g \equiv \tilde{a}^I G^I \equiv \left( \tilde{J}^\beta \sigma^\beta + \tilde{J}^+ \sigma^+ + \tilde{J}^- \sigma^- \right) + \left( \frac{d-1}{2C_\rho} \tilde{E}^a T^a \right) + \left( \tilde{F}_{+\alpha} R^{+\alpha} + \tilde{F}_{-\alpha} R^{-\alpha} \right) \]  

(4.19)

where \( a^I, \tilde{a}^I \) are generically the components of the currents and \( G^I \) the representation matrices of the superalgebra. The super K-M Poisson algebra (with \( \{q, p\} = 1 \)) between any two current components is then:

\[
\{ a^I(\theta), a^J(\theta') \} = g^{IJK} a^K(\theta) \delta(\theta - \theta') + \frac{k}{2\pi} h^{IJ} \delta'(\theta - \theta') \]

(4.20)

\[
\{ \tilde{a}^I(\theta), \tilde{a}^J(\theta') \} = g^{IJK} \tilde{a}^K(\theta) \delta(\theta - \theta') - \frac{k}{2\pi} h^{IJ} \delta'(\theta - \theta') \]

(4.21)

where

\[
\left[ G^I, G^J \right] = f^{IJK} G^K, \quad STr \left( G^I G^J \right) = g^{IJ}
\]

(4.22)

and the commutator of two superalgebra generators is understood as an anti-commutator for two odd generators. We have also defined \( h^{IJ} \) to be the inverse matrix to \( g^{IJ} \), \( h^{IJ} g^{JM} = \delta^M_\alpha \) and \( g^{IJK} = f^{LMN} h^{IL} h^{JM} g^{NK} \).

In terms of the fields, the currents for the two chiralities read explicitly:

\[
J^3 = \frac{k}{\pi} \partial_+ \varphi + 2\Pi_x x - \Pi^{+\alpha} \psi_{+\alpha}
\]

(4.23)

\[
J^- = \Pi_x
\]

(4.24)

\[
J^+ = -\Pi_y \left[ -e^{2\varphi} + i \frac{1}{2} \psi_+^{\alpha} \psi_+^{\alpha} \right] + e^{\varphi} \left( \psi_+^{\alpha} u^{\alpha\beta} \psi_-^{\beta} \right) + \frac{k}{2\pi} \left[ \partial_\theta x + \frac{i}{2} (\psi_+^{\alpha} u^{\alpha\beta} \partial_\theta \psi_-^{\beta}) - 2x \partial_+ \varphi - \frac{i}{2} \left( \partial_+ uu^{-1} \right)^a (\psi_+^{\alpha} \lambda^\alpha \beta \psi_-^{\beta}) \right]
\]

\[
- \Pi_x \left[ x^2 + \frac{1}{4!} \frac{d-1}{2C_\rho} (\psi_+^{\alpha} \lambda^\alpha \beta \psi_-^{\beta}) (\psi_+^{\beta} \lambda^\beta \delta \psi_-^{\delta}) \right] - \frac{1}{2} \Pi_x \psi_+^{\alpha} + i \Pi^{+\alpha} \eta_{\alpha\beta} \left[ i x \psi_-^{\beta} \psi_+^{\gamma} - \frac{1}{3!} \frac{d-1}{2C_\rho} (\lambda^\alpha \beta \gamma \psi_+^{\alpha} \psi_-^{\beta} \psi_+^{\gamma}) \right]
\]

(4.25)

\[
B^a = \frac{k}{2\pi} \frac{2C_\rho}{d-1} \left( \partial_+ uu^{-1} \right)^a - \Pi_{+\alpha} (\lambda^\alpha \beta \psi_+^{\beta})
\]

(4.26)

\[
F_{-\alpha} = i \eta_{\alpha\beta} \Pi^{+\beta} - \frac{1}{2} \Pi_x \psi_{+\alpha}
\]

(4.27)
\[ F_{\alpha} = -e^\varphi \left[ \frac{1}{2} \psi_\gamma \eta^{\gamma\beta} \Pi_y + i\Pi^{-\beta} \right] (u^{-1})_{\beta\alpha} + \frac{k}{2\pi} \left\{ \partial_\varphi \psi_\alpha - \partial_\gamma \varphi \psi_\alpha - \left( \partial_\varphi u u^{-1} \right)^a \left( \psi_+ (\lambda^a)_{\alpha}^{\beta} \right) \right\} + \Pi_x \left[ -x \psi_\alpha - \frac{i}{3!} \frac{d-1}{2C_\rho} \left( \psi_+ (\lambda^a)_{\alpha}^{\gamma\beta} \psi_+ (\lambda^a)_{\delta}^{\gamma} \right) \right] + \frac{1}{2} \Pi_x \psi_+ \eta^{\gamma\beta} + i\Pi^+ \left[ x - i \frac{d}{4} \psi_+ \psi_+ + \frac{i}{2} \frac{d-1}{2C_\rho} \left( (\lambda^a)_{\alpha}^{\gamma\beta} \psi_+ (\lambda^a)_{\delta}^{\gamma} \right) \right] \]}

and for the other chirality:

\[ \bar{J}^\beta = -\frac{k}{\pi} \partial_\gamma \varphi - 2\Pi_y y + \Pi^{-\alpha} \psi_\beta \] (4.29)

\[ \bar{J}^+ = -\Pi_y \] (4.30)

\[ \bar{J}^- = \Pi_x \left\{ -e^\varphi + \frac{i}{2} e^\varphi \left( \psi_+ u^{\alpha\beta} \psi_- \right) \right\} - \frac{i}{2} e^\varphi \left( \Pi_+ \psi_- \psi_+ \right) + \frac{k}{2\pi} \left\{ \partial_\gamma y - \frac{i}{2} \left( \psi_+ \eta^{\alpha\beta} \partial_\gamma \psi_- \right) + 2y \partial_\gamma \varphi - \frac{i}{2} \left( u^{-1} \partial_\gamma u \right)^a \left( \psi_- (\lambda^a)_{\alpha}^{\gamma\beta} \psi_- \right) \right\} + \Pi_y \left[ y^2 + \frac{1}{4!} \frac{d-1}{2C_\rho} \left( \psi_- (\lambda^a)_{\alpha}^{\gamma\beta} \psi_- \right) \right] \] (4.31)

\[ \bar{B}^a = - \left[ \frac{k}{2\pi} \frac{2C_\rho}{d-1} \left( u^{-1} \partial_\gamma u \right)^a + \left( \Pi_- (\lambda^a)_{\alpha}^{\alpha\beta} \psi_- \right) \right] \] (4.32)

\[ \bar{F}^- = -e^\varphi \left[ \frac{1}{2} \psi_+ \eta^{\gamma\beta} \Pi_x + i\Pi^+ \right] u_{\beta\alpha} \] (4.33)

\[ \bar{F}^- = -e^\varphi \left[ \frac{1}{2} \psi_+ \eta^{\gamma\beta} \Pi_x + i\Pi^+ \right] u_{\beta\alpha} + \frac{k}{2\pi} \left\{ \partial_\gamma \psi_- + \partial_\beta \varphi \psi_- + \left( u^{-1} \partial_\gamma u \right)^a \left( \psi_- (\lambda^a)_{\alpha}^{\gamma} \right) \right\} - \Pi_y \left[ -y \psi_- - \frac{i}{3!} \frac{d-1}{2C_\rho} \left( \psi_- (\lambda^a)_{\alpha}^{\gamma\beta} \psi_- \right) \right] \] (4.34)

4.3 Gauged theory

The variational principle that follows from 2+1 supergravity is in fact not (4.10), but rather, (4.10) supplemented by the constraints (4.13) since the only competing histories occurring in the variational
principle of supergravity are required to fulfill (4.3). In terms of the currents, the constraints are simply

\[ J^- = \frac{k}{2\pi \mu}, \quad \bar{J}^+ = \frac{k}{2\pi \nu} \quad (4.35) \]
\[ J^3 = 0, \quad \bar{J}^3 = 0 \quad (4.36) \]
\[ F_{-\alpha} = 0, \quad \bar{F}_{+\alpha} = 0 \quad (4.37) \]

(modulo holonomy terms discussed in the appendix). These are constraints on the canonical variables (with \( \partial_{\pm} \varphi \) expressed in terms of \( \Pi_{\varphi} \)), which we collectively denote by \( G_A \). Thus, the action for the theory is

\[
S[\varphi, x, y, u, \psi_{\pm \alpha}, \Pi_{\varphi}, \Pi_x, \Pi_y, \Pi_u, \Pi^{-\alpha}, \Pi^{+\alpha}, \xi^A] = S_{WZW}^H - \int dx^+ dx^- \xi^A G_A \quad (4.38)
\]

where \( \xi^A \) are Lagrange multipliers implementing the constraints.

It follows from the algebra of the currents that the constraints \( G_A = 0 \) are second class; one may split them into two subsets, \( G_A = (\phi_i, \chi_i) \), in such a way that the \( \phi_i \)'s are first class among themselves and generate therefore a gauge symmetry, while the \( \chi_i \)'s can be viewed as gauge conditions for the symmetry generated by the \( \phi_i \)'s (for information on constrained systems, see [44]). For instance, one may take for \( \phi_i \) the constraints \( J^- - \frac{k}{2\pi \mu} = 0, \quad \bar{J}^+ - \frac{k}{2\pi \nu} = 0 \) and the positive frequency part of the fermionic constraints [45]. The action (4.38) is thus the (gauge-fixed version of the) action for the gauged WZW model, in which one has gauged the subsupergroup generated by the first class constraints.

As it is well known, this model is equivalent to super-Liouville theory (see [14, 15, 16] for the bosonic case, and [45, 46, 47] for some of the fermionic cases). This is demonstrated in the next subsection for all extended models.
4.4 Super-Liouville action

The constraints $G_A = 0$ enable one to eliminate $\Pi_x, \Pi_y, \Pi^{-\alpha}, \Pi^{+\alpha}, x, y$ from the action. [As demonstrated by Lagrange himself, solving the constraints inside the action or taking them into account with the help of (Lagrange) multipliers are two equivalent procedures]. When doing so, one gets a reduced action $S_R^H$ which depends on the remaining variables, i.e., $\varphi, u, \psi_{+\alpha}, \psi_{-\alpha}, \Pi_{\varphi}, \Pi_u$ and which is precisely the super-Liouville action.

Indeed, the constraints imply

$$\Pi_x = \frac{k}{2\pi} \mu, \quad \Pi_y = -\frac{k}{2\pi} \nu,$$  \hspace{1cm} (4.39)

and

$$\Pi^{+\beta} = -\frac{i k \mu}{4\pi} \eta^{\beta\alpha} \psi_{+\alpha}, \quad \Pi^{-\beta} = \frac{i k \nu}{4\pi} \eta^{\beta\alpha} \psi_{-\alpha} \hspace{1cm} (4.40)$$

When substituting this inside (4.38), one gets, dropping a total derivative and integrating over the remaining momenta $\pi_\varphi$ and $\Pi_u$,

$$S_{SL} = \frac{k}{2\pi} \int dx^+ dx^- \left\{ \partial_+ \varphi \partial_- \varphi + \mu \nu \left( e^{2\varphi} - i e^{\varphi} \psi_{+\alpha} u^{\alpha\beta} \psi_{-\beta} \right) \right\}$$

$$+ \frac{i \mu}{2} \psi_{+\alpha} \eta^{\alpha\beta} \partial_- \psi_{+\beta} - \frac{i \nu}{2} \psi_{-\alpha} \eta^{\alpha\beta} \partial_+ \psi_{-\beta}$$

$$- \frac{2 D}{d (d - 1)} WZW(u) \hspace{1cm} (4.41)$$

which is just the super-Liouville action [48, 49].

The values of $\mu, \nu$ corresponding to the anti-de Sitter case are $\mu = 1$ and $\nu = 1$, leading to the action

$$S_{SL} = \frac{k}{2\pi} \int dx^+ dx^- \left\{ \partial_+ \varphi \partial_- \varphi + \left( e^{2\varphi} - i e^{\varphi} \psi_{+\alpha} u^{\alpha\beta} \psi_{-\beta} \right) \right\}$$

$$+ \frac{i}{2} \psi_{+\alpha} \eta^{\alpha\beta} \partial_- \psi_{+\beta} - \frac{i}{2} \psi_{-\alpha} \eta^{\alpha\beta} \partial_+ \psi_{-\beta}$$

$$- \frac{2 D}{d (d - 1)} WZW(u). \hspace{1cm} (4.42)$$

The sign in front of the exponential is not the one familiar from $2D$ gravity. This point is discussed more fully in the appendix.
5 Realization of superconformal generators in terms of Liouville fields

5.1 Construction of generators

The symmetries of the super-Liouville action can be understood in terms of the symmetries of the original, ungauged, WZW action.

The superWZW model is conformal, the generators $L, \tilde{L}$ being elements of the enveloping algebra of the current algebra defined by the Sugawara construction:

$$\frac{1}{2\pi} L = \frac{1}{k} \left( \left( \frac{J^3}{2} \right)^2 + J^+ J^- \right) + \frac{1}{k} \eta^{\alpha\beta} F_{+\alpha} F_{-\beta} + \frac{d - 1}{2C^'} \frac{B^a B^a}{2k}$$ (5.1)

$$- \frac{1}{2\pi} \tilde{L} = \frac{1}{k} \left( \left( \frac{\tilde{J}^3}{2} \right)^2 + \tilde{J}^+ \tilde{J}^- \right) + \frac{1}{k} \eta^{\alpha\beta} \tilde{F}_{+\alpha} \tilde{F}_{-\beta} + \frac{d - 1}{2C^'} \frac{\tilde{B}^a \tilde{B}^a}{2k}$$ (5.2)

The Poisson brackets of $L, \tilde{L}$ with the currents, measuring their dimension are:

$$\{ L(\theta), a^I(\theta') \} = a^I(\theta) \delta'(\theta - \theta')$$ (5.3)

$$\{ \tilde{L}(\theta), \tilde{a}^I(\theta') \} = \tilde{a}^I(\theta) \delta'(\theta - \theta')$$ (5.4)

The ungauged superWZW model is, however, not superconformal, in the sense that there are in general no superpartners to $L, \tilde{L}$ that would close with them according to the superconformal algebra. What is remarkable is that once we gauge the superWZW, i.e., impose the first class constraints

$$J^- = \frac{k}{2\pi} \mu, \quad F_{-\alpha} = 0, \quad \tilde{J}^+ = \frac{k}{2\pi} \nu, \quad \tilde{F}^\alpha_{+\alpha} = 0,$$ (5.5)

(where $>$ denotes the positive frequency part), we not only maintain conformal invariance but in fact gain superconformal invariance. Indeed, one can find polynomials in the Kac-Moody currents that (i) preserve the constraints; and (ii) close in the Poisson brackets according to the superconformal algebra modulo terms that vanish when (5.5) hold.
In the reduced theory obtained by strongly enforcing all the constraints (gauge constraints and gauge conditions $J^3 = 0$, $\tilde{J}^3 = 0$, $F_{+\alpha}^\leq = 0$, $\tilde{F}_{-\alpha}^\leq = 0$) and using Dirac brackets, the superconformal algebra is preserved since the generators are “first-class” (=gauge-invariant) so that their Dirac and Poisson brackets coincide.

We exemplify the procedure by constructing the superconformal generators $Q_\alpha, \tilde{Q}_\alpha$. In the actual construction, it is convenient to work in a “half-gauge-fixed” formulation in which $F_{+\alpha}^\leq = 0$, $\tilde{F}_{-\alpha}^\leq = 0$ are imposed, so that the constraints are

$$J^- = \frac{k}{2\pi} \nu, \quad F_{-\alpha} = 0.$$ (5.6)

$$\tilde{J}^+ = \frac{k}{2\pi} \mu, \quad \tilde{F}_{+\alpha} = 0.$$ (5.7)

One gets for $Q_\alpha$

$$\begin{align*}
\frac{k\nu}{2\pi} Q_\alpha &\equiv J^- \tilde{F}_{-\alpha} + \frac{J^3}{2} F_{-\alpha} + \frac{d-1}{2C_\rho} B^a \eta^{\beta\gamma} (\lambda^a)_{\alpha\beta} F_{-\gamma} + \frac{k}{2\pi} \partial_\theta F_{-\alpha}(\theta)
\end{align*}$$ (5.8)

and for $\tilde{Q}_\alpha$:

$$\begin{align*}
\frac{k\mu}{2\pi} \tilde{Q}_\alpha &\equiv \tilde{J}^+ \tilde{F}_{-\alpha} + \frac{\tilde{J}^3}{2} F_{+\alpha} + \frac{d-1}{2C_\rho} \tilde{B}^a \eta^{\beta\gamma} (\lambda^a)_{\alpha\beta} \tilde{F}_{+\gamma} - \frac{k}{2\pi} \partial_\theta \tilde{F}_{+\alpha}(\theta)
\end{align*}$$ (5.9)

The Poisson brackets of $Q_\alpha, \tilde{Q}_\alpha$ with the constraints are:

$$\{Q_\alpha(\theta), J^-(\theta') - \frac{k}{2\pi} \mu\} = 0, \quad \{\tilde{Q}_\alpha(\theta), \tilde{J}^+(\theta') - \frac{k}{2\pi} \nu\} = 0$$ (5.10)

and

$$\begin{align*}
\frac{k\nu}{2\pi} \{Q_\alpha(\theta), F_{-\beta}(\theta')\} &\equiv \frac{k}{2\pi} \eta_{\alpha\beta} \delta(\theta - \theta') \partial_\theta J^-(\theta) + \frac{1}{2} F_{-\beta}(\theta) F_{-\alpha}(\theta) \delta(\theta - \theta') + \\
&\quad - \frac{d-1}{2C_\rho} (\lambda^a)_{\alpha}' (\lambda^a)_{\beta}' F_{-\tau}(\theta) F_{-\epsilon}(\theta) \delta(\theta - \theta'),
\end{align*}$$ (5.11)

$$\begin{align*}
\frac{k\mu}{2\pi} \{\tilde{Q}_\alpha(\theta), \tilde{F}_{+\beta}(\theta')\} &\equiv \frac{k}{2\pi} \eta_{\alpha\beta} \delta(\theta - \theta') \partial_\theta \tilde{J}^+(\theta) - \frac{1}{2} \tilde{F}_{+\beta}(\theta) \tilde{F}_{+\alpha}(\theta) \delta(\theta - \theta') + \\
&\quad - \frac{d-1}{2C_\rho} (\lambda^a)_{\alpha}' (\lambda^a)_{\beta}' \tilde{F}_{+\tau}(\theta) \tilde{F}_{+\epsilon}(\theta) \delta(\theta - \theta'),
\end{align*}$$ (5.12)
expressions that vanish on the constraint surface. Note that the dimension of \( Q_\alpha, \tilde{Q}_\alpha \) are the difference between the dimension as measured by the Sugawara tensor and the the \( sl(2, R) \) spin.

The superconformal generators, as defined above, generate the superconformal algebra. Their Poisson brackets close on modified generators, all having (weakly) vanishing Poisson brackets with the constraints (the bracket of two first class functions is also first class). In particular the modified Virasoro generators \( \hat{L}, \tilde{L} \) which appear are

\[
\frac{k \mu}{2 \pi} \hat{L} \equiv -J^- (\theta)(L(\theta) - \partial_\theta J^3 / 2), \quad \frac{k \nu}{2 \pi} \tilde{L} \equiv -\tilde{J}^+ (\theta)(\tilde{L}(\theta) + \partial_\theta \tilde{J}^3 / 2) \tag{5.13}
\]

Their Poisson brackets with the constraints are:

\[
\{ \hat{L}(\theta), J^- (\theta') \} - \frac{k}{2 \pi} \mu \} = -J^- (\theta) \partial_\theta J^- (\theta) \delta(\theta - \theta') \tag{5.14}
\]

\[
\{ \hat{L}(\theta), \tilde{J}^+ (\theta') \} - \frac{k}{2 \pi} \nu \} = \tilde{J}^+ (\theta) \left( \partial_\theta \tilde{J}^+ (\theta) \right) \delta(\theta - \theta') \tag{5.15}
\]

\[
\{ \hat{L}(\theta), F^- (\alpha, \theta') \} = -\frac{1}{2} J^- (\theta) \left[ F^- (\alpha, \theta') \delta(\theta - \theta') - \partial_\theta F^- (\alpha, \theta) \delta(\theta - \theta') \right] \tag{5.16}
\]

\[
\{ \hat{L}(\theta), \tilde{F}^- (\alpha, \theta') \} = -\frac{1}{2} \tilde{J}^+ (\theta) \left[ \tilde{F}^- (\alpha, \theta') \delta(\theta - \theta') - \partial_\theta \tilde{F}^- (\alpha, \theta) \delta(\theta - \theta') \right] \tag{5.17}
\]

again vanishing on the manifold of the constraints. Similarly the modified Kac-Moody currents

\[
- \frac{2 \pi}{k \mu} J^- (\theta) B^\alpha (\theta) \text{ and } - \frac{2 \pi}{k \nu} \tilde{J}^+ (\theta) \tilde{B}^\alpha (\theta)
\]

appear.

The algebraic structure obtained is indeed the extended superconformal algebra provided we ignore terms which vanish with the constraints and we consider as equivalent, generators which reduce to the same expression on the constraint surface. This is of course the standard procedure for constrained systems and it lies at the heart of the Hamiltonian formulation of BRST theory \[44\].

When all the constraints (first and second class) are imposed, the generator \( Q_\alpha \) reduces to \( F_+ \alpha \) and \( \tilde{Q}_\alpha \) reduces to \( \tilde{F}_- \alpha \), while the generators \( \hat{L} \) and \( \tilde{L} \) reduce to \( -\mu J^- - \frac{\pi}{k \beta} B^\alpha B^\alpha \) and \( +\nu \tilde{J}^+ + \frac{\pi}{k \beta} \tilde{B}^\alpha \tilde{B}^\alpha \) respectively. Therefore, one can also view the superconformal generators as the “first-class extensions” \[44\] of the Kac-Moody currents that are not constrained by the reduction.
5.2 Explicit expressions

One can directly obtain the superconformal generators in terms of the super-Liouville fields by plugging the constraints into the above expressions (5.8), (5.13).

On the constraint surface, the currents reduce to

\[ J^3 = \frac{k}{\pi} \left[ \partial_+ \varphi + \mu x \right] \] (5.18)

\[ J^- = \frac{k}{2\pi} \mu \] (5.19)

\[ J^+ = \frac{k}{2\pi} \left[ -\nu e^{2\varphi} + i\nu e^{\varphi} \left( \psi_{+\alpha} u^\alpha_{\beta} \psi_{-\beta} \right) \right. \]
\[ \left. + \partial_\theta x + \frac{i}{2} \psi_{+\alpha} \eta^{\alpha\beta} \partial_\theta \psi_{+\beta} - 2\pi \partial_+ \varphi - \mu x^2 \right. \]
\[ \left. - \frac{i}{2} \left( \partial_+ uu^{-1} \right)^a \left( \psi_{+\alpha} (\lambda^a)^{\alpha\beta} \psi_{+\beta} \right) \right. \]
\[ \left. + \frac{\mu d - 1}{8} \left( \psi_{+\alpha} (\lambda^a)^{\alpha\beta} \psi_{+\beta} \right) \left( \psi_{+\gamma} (\lambda^a)^{\gamma\delta} \psi_{+\delta} \right) \right] \] (5.20)

\[ B^a = \frac{k}{2\pi} \left( \frac{2C_\rho}{d-1} \left( \partial_+ uu^{-1} \right)^a + \frac{i\mu}{2} \left( \psi_{+\alpha} (\lambda^a)^{\alpha\beta} \psi_{+\beta} \right) \right) \] (5.21)

\[ F_{-\alpha} = 0 \] (5.22)

\[ F_{+\alpha} = \frac{k}{2\pi} \left[ \nu e^{\varphi} \left( u \right)^{\alpha}_{\beta} \psi_{-\beta} + \partial_\theta \psi_{+\alpha} \right. \]
\[ \left. - \partial_+ \varphi \psi_{+\alpha} - \left( \partial_+ uu^{-1} \right)^a \left( \psi_{+\beta} (\lambda^a)^{\beta}_{\alpha} \right) \right. \]
\[ \left. + \frac{i\mu d - 1}{3} \left( \psi_{+\alpha} (\lambda^a)^{\gamma\beta} \psi_{+\beta} \right) \left( \psi_{+\beta} (\lambda^a)^{\delta}_{\alpha} \right) \right] \] (5.23)

\[ \tilde{J}^3 = -\frac{k}{\pi} \left[ \partial_+ \varphi - \nu y \right] \] (5.24)

\[ \tilde{J}^+ = \frac{k}{2\pi} \nu \] (5.25)

\[ \tilde{J}^- = \frac{k}{2\pi} \left[ -\nu e^{2\varphi} + i\nu e^{\varphi} \left( \psi_{+\alpha} u^\alpha_{\beta} \psi_{-\beta} \right) \right. \]
\[ \left. + \partial_\theta y + \frac{i}{2} \psi_{-\alpha} \eta^{\alpha\beta} \partial_\theta \psi_{-\beta} + 2\nu \partial_- \varphi - \nu y^2 \right. \]
\[ \left. - \frac{i}{2} \left( u^{-1} \partial_- \right)^a \left( \psi_{-\alpha} (\lambda^a)^{\alpha\beta} \psi_{-\beta} \right) \right. \]
\[ \left. + \frac{\nu d - 1}{8} \left( \psi_{-\alpha} (\lambda^a)^{\alpha\beta} \psi_{-\beta} \right) \left( \psi_{-\gamma} (\lambda^a)^{\gamma\delta} \psi_{-\delta} \right) \right] \] (5.26)

\[ \tilde{B}^a = -\frac{k}{2\pi} \left( \frac{2C_\rho}{d-1} \left( u^{-1} \partial_- u \right)^a + \frac{i\nu}{2} \left( \psi_{-\alpha} (\lambda^a)^{\alpha\beta} \psi_{-\beta} \right) \right) \] (5.27)
\[ \bar{F}_{\pm \alpha} = 0 \] (5.28)
\[ \bar{F}_{-\alpha} = \frac{k}{2\pi} j \mu \varphi \psi_+ u_\alpha^\beta + \partial_\theta \psi_- \alpha \\
+ \partial_- \varphi \psi_- \alpha + (u^{-1} \partial_- u)^a \left( \psi_- (\lambda^\alpha)^\beta \right) \\
+ \frac{i}{3} \nu \frac{d-1}{2 C_\rho} \left( \psi_- (\lambda^\alpha)^\beta \psi_- \beta \right) \left( \psi_- (\lambda^\alpha)^\beta \right) \] (5.29)

If one inserts these expressions into (5.8) and (5.13), one gets the following expressions for the superconformal generators:

\[ \hat{L} = \frac{k}{2\pi} \left[ \partial_+^2 \varphi - (\partial_+ \varphi)^2 - \frac{i\mu}{2} \psi_+ \eta^{\alpha\beta} \partial_+ \psi_+^\beta - \frac{C_\rho}{d-1} \left( \partial_+ uu^{-1} \right)^a \left( \partial_+ uu^{-1} \right)^a \right] \] (5.30)
\[ -B^a = -\frac{k}{2\pi} \left[ \frac{2C_\rho}{d-1} \left( \partial_+ uu^{-1} \right)^a + \frac{i\mu}{2} \left( \psi_+ (\lambda^\alpha)^\beta \psi_+^\beta \right) \right] \] (5.31)
\[ Q_\alpha = \frac{k}{2\pi} \left[ \partial_+ \psi_+ - \partial_+ \varphi \psi_+ - (\partial_+ uu^{-1})^a (\psi_+ (\lambda^\alpha)^\beta) \right] \\
+ \frac{i\mu}{3} \frac{d-1}{2 C_\rho} \left( \psi_+ (\lambda^\alpha)^\beta \psi_+^\beta \right) \left( \psi_+ (\lambda^\alpha)^\beta \right) \] (5.32)
\[ \tilde{L} = -\frac{k}{2\pi} \left[ \partial_-^2 \varphi - (\partial_- \varphi)^2 + \frac{i\nu}{2} \psi_- \eta^{\alpha\beta} \partial_- \psi_-^\beta - \frac{C_\rho}{d-1} \left( u^{-1} \partial_- u \right)^a \left( u^{-1} \partial_- u \right)^a \right] \] (5.33)
\[ -\tilde{B}^a = \frac{k}{2\pi} \left[ \frac{2C_\rho}{d-1} \left( u^{-1} \partial_- u \right)^a + \frac{i\nu}{2} \left( \psi_- (\lambda^\alpha)^\beta \psi_-^\beta \right) \right] \] (5.34)
\[ \tilde{Q}_\alpha = -\frac{k}{2\pi} \left[ \partial_- \psi_- - \partial_- \varphi \psi_- - (u^{-1} \partial_- u)^a (\psi_- (\lambda^\alpha)^\beta) \right] \\
- \frac{i\nu}{3} \frac{d-1}{2 C_\rho} \left( \psi_- (\lambda^\alpha)^\beta \psi_- \beta \right) \left( \psi_- (\lambda^\alpha)^\beta \right) \] (5.35)

where the time derivatives are expressed in terms of the canonical variables using the equations of motion. The Poisson brackets of the generators are those of the superconformal algebra. This can be directly verified from (5.30), (5.31) and (5.32) using the Poisson brackets for the super-Liouville field components and their spacetime derivatives that follow from the equations of motion and the basic canonical brackets, but the property actually also holds if one regards the super-Liouville field as free. This fact is familiar from chiral quantization [43].

The realization of the Poisson bracket superconformal algebra in terms of the super-Liouville fields can also be used in order to construct realizations for the quantum superconformal algebras.
The classical expressions (5.30), (5.31) and (5.32) become the quantum generators, some of the coefficients getting corrections of order $\hbar$. The expressions with the correct normalizations appear in a paper by Bina and Günaydın [34].

5.3 Symmetry transformations

One can work out the superconformal transformations of the fields by computing their Poisson brackets with the generators. For completeness, we list the infinitesimal symmetry transformations of all fields for the tilded chirality superconformal generators:

1. Transformations induced by the improved Virasoro generator, parametrized by $\epsilon = \epsilon(x^-)$:

$$
\delta \varphi = \partial_- \varphi \epsilon + \frac{1}{2} \partial_- \epsilon
$$

$$
\delta \psi_{+\alpha} = \partial_- \psi_{+\alpha} \epsilon
$$

$$
\delta \psi_{-\alpha} = \partial_- \psi_{-\alpha} + \frac{1}{2} \partial_- \psi_{-\alpha} \partial_- \epsilon
$$

$$
\delta u = \partial_- u \epsilon
$$

$$
\delta [u^{-1} \partial_- u]^a = \partial_- (\epsilon [u^{-1} \partial_- u]^a)
$$

2. Transformations induced by the superconformal generators, parametrized by $\zeta_\alpha = \zeta_\alpha(x^-)$:

$$
\delta \varphi = \frac{1}{2} (\zeta_\alpha \eta^{\alpha\beta} \psi_{-\beta})
$$

$$
\delta \psi_{+\beta} = -ie^\nu \zeta_\alpha [u^{-1}]^a_{\beta}
$$

$$
\delta \psi_{-\alpha} = -\frac{i}{\nu} (\partial_- \zeta_\alpha + \partial_- \varphi \zeta_\alpha - [u^{-1} \partial_- u]^a \zeta_\beta (\lambda^a)^{\beta}_{\alpha})

\frac{d-1}{2C_\rho} (\psi_{-\gamma} (\lambda^a)^\gamma_\beta \psi_{-\delta} (\lambda^a)^\delta_{\alpha}) - \frac{1}{2} (\zeta_\gamma \eta^{\gamma\beta} \psi_{-\beta} \psi_{-\alpha})

\delta u = -\frac{d-1}{2C_\rho} u T^a (\psi_{-\alpha} (\lambda^a)^{\alpha\beta} \zeta_\beta)

\delta [u^{-1} \partial_- u]^a = \frac{d-1}{2C_\rho} \left[ -\partial_- (\psi_{-\alpha} (\lambda^a)^{\alpha\beta} \zeta_\beta) + f^{abc} (\psi_{-\alpha} (\lambda^b)^{\alpha\beta} \zeta_\beta) [u^{-1} \partial_- u]^c \right]
$$
where we used the Jacobi identity (2.10) for the λ matrices.

3. Transformations induced by the Kac-Moody generators, parametrized by $\omega^a = \omega^a(x^-)$:

\[
\begin{align*}
\delta \varphi &= 0 \\
\delta \psi_{+\alpha} &= 0 \\
\delta \psi_{-\alpha} &= \omega^a \psi_{-\beta} \lambda^\alpha_{\beta} \\
\delta u &= u \omega^a T^a \\
\delta [u^{-1} \partial_- u]^a &= \partial_- \omega^a - f^{abc} \omega^b [u^{-1} \partial_- u]^c
\end{align*}
\] (5.47) (5.48) (5.49) (5.50) (5.51)

One can easily verify that the super-Liouville action is indeed invariant under these transformations.

A special situation occurs for the $SU(1,1|2)/U(1)$ (“small $N = 4$”) case. Writing the fermions as $2 \times 2$ matrices, the super-Liouville action has an alternative $SU(2)$ symmetry obtained by multiplication from the left of the fermionic matrix without involving the WZ action. This generates at the classical level an alternative $N = 4$ superconformal algebra. The quantum realization of this algebra was discussed in [48] and used recently in a physically interesting context in [50].

6 Classification of extended conformal algebras and spectral flow

Realizing the superconformal algebras as boundary theories of Chern-Simons actions allows a unified treatment of their classification following from different possible “moddings” of the generators. Some of these apparently different algebras are related by the so called “spectral flow” transformations [26]. This feature also gets a general and natural interpretation in the Chern-Simons framework.

We follow closely [26]. Both aspects mentioned above are related to the behaviour of the
fermionic generators under the extended Kac-Moody symmetry. This behaviour, in turn, must be such that the action (2.17) is single-valued. For convenience, we will denote in this section: \( \psi_\alpha \equiv \psi_{+\alpha}, \zeta_\alpha \equiv \psi_{-\alpha} \).

As discussed in the previous sections, the physical Hilbert space is realized in terms of representations of the boundary superconformal algebra. A given chiral representation is defined by \( h \), the eigenvalue of the \( L_0 \) Virasoro operator and a highest weight \( \lambda \) of the Kac-Moody algebra. Pairing the left and right representations, we construct the physical Hilbert space. The highest weight \( \tilde{\lambda} \) is measured by the holonomy \( P \exp(\int iB_\phi d\theta) \) around the boundary of the disk. For a given holonomy \( \exp(i2\pi\lambda) \), where \( \lambda \equiv \tilde{\lambda} \cdot \tilde{H} \), \( H_i \) being the generators in the Cartan subalgebra, regularity at the center of the disk ("origin") requires the coupling of the \( B_0 \) field to a source at the origin in the representation \( \lambda \) \( [9] \); the action of the source is given by:

\[
\int dt \, Tr[\lambda \omega^{-1}(\partial_0 + B_0)\omega(t)]
\]

Alternatively one can quantize the C-S theory on an annulus and require that the holonomies on the two boundaries are the same. The source at the center of the disk represents the zero-mode of the fields on the interior boundary.

We start by discussing the possible boundary conditions on the fields in (2.17). This will provide us with the classification of the superconformal algebras living on the boundary.

The topology of the \( AdS_3 \) manifold is \( D \times \mathbb{R} \). Parametrizing \( D \) by \( r, \theta \) the fields could have nontrivial periodicity conditions as a function of \( \theta \) provided the lagrangian density remains strictly periodic (single valued). In a real basis, the kinetic terms of the \( \psi_\alpha, \zeta_\alpha \) are invariant under a \( O(d) \) transformation, where \( d \) is the dimension of the real representation \( \rho \) of the compact gauge group \( \tilde{G} \) (extended symmetry) under which the gravitini transform. In this section, \( \tilde{G} \) denotes the internal group (and not its Lie algebra). Therefore one can have the nontrivial periodicity conditions:

\[
\psi(\theta + 2\pi) = g\psi(\theta) \quad \zeta(\theta + 2\pi) = \zeta(\theta)g^{-1}
\]
where \( g \) is an arbitrary element of \( O(d) \) and we made explicit just the \( \theta \) dependence of the fields. We can make, however, a change of variables in the functional integral \( \psi = \bar{g}\psi, \zeta = \zeta \bar{g}^{-1} \) which would leave the kinetic term form invariant. As a consequence \( g \) and \( \bar{g}^{-1}g \bar{g} \) in (6.2) are equivalent and the boundary conditions depend really only on conjugacy classes of \( O(d) \). Therefore the \( g \) in (6.2) can be taken to be a generic element in the Cartan torus of \( SO(d) \) or one fixed matrix \( g_0 \) in \( O(d) \) with determinant \(-1\).

We start analyzing the continuous, Cartan torus part. In the presence of the gauge couplings, \( O(d) \) is not anymore a symmetry. The continuous part is broken to \( \bar{G} \times F \), \( \bar{G} \) being the gauge group and \( F \) a residual “flavour” group. For the supergroups listed in Table 1 the flavour group exists just for \( SU(1,1|2)/U(1) \) when it is \( SU(2) \).

It is convenient to consider the fermionic fields in the Cartan basis of \( \bar{G} \), i.e. \( \psi_{\vec{\mu}}, \zeta_{\vec{\mu}} \), where \( \vec{\mu} \) is a weight of the gauge group \( \bar{G} \) in the representation \( \rho \). In this basis the transformations (6.2) become phase transformations. In order to keep the lagrangian strictly periodic also the gauge fields \( B \) should acquire nontrivial periodicity conditions: the Kac-Moody algebra must be “twisted” \[51\].

These are best studied if we use the Cartan basis for the gauge fields, i.e., we expand the connection \( B \) in terms of the raising and lowering operators \( E_{\vec{a}} \) and the operators in the Cartan subalgebra \( H_i \ i = 1, ..., r, \vec{\alpha} \) being roots and \( r \) the rank of the group \( \bar{G} \):

\[
B = B_{\vec{\alpha}}E_{\vec{\alpha}} + B_i H_i
\]

(6.3)

Then it is easy to see that the periodicity conditions which will leave the lagrangian single valued are:

\[
\psi_{\vec{\mu}}(\theta + 2\pi) = \exp(i\vec{\alpha}_{\vec{\mu}})\psi_{\vec{\mu}}(\theta) \quad \zeta_{\vec{\mu}}(\theta + 2\pi) = \exp(i\vec{\alpha}_{\vec{\mu}})\zeta_{\vec{\mu}}(\theta)
\]

(6.4)

and

\[
B_{\vec{\alpha}}(\theta + 2\pi) = \exp(i\vec{\alpha}_{\vec{\alpha}})B_{\vec{\alpha}}(\theta) \quad B_i(\theta + 2\pi) = B_i(\theta)
\]

(6.5)

where \( \vec{a} \) has \( r \) continuous components.
In addition, for the $SU(1,1|2)/U(1)$ case, when a flavour symmetry exists, there is an overall $U(1)$ phase transformation (the Cartan torus of the flavour group) affecting only the fermions:

$$\psi_{\mu}(\theta + 2\pi) = \exp(ib)\psi_{\mu}(\theta) \quad \zeta_{\mu}(\theta + 2\pi) = \exp(-ib)\zeta_{\mu}(\theta)$$

We discuss now the periodicity conditions related to the discrete element $g_0$. A necessary condition for the discrete operation $g_0$ to be compatible with the gauging is that each gauge field acts in a real representation. In particular, when the group $\tilde{G}$ is not simple, each component of $\rho$ corresponding to a factor of $\tilde{G}$ should be real.

The action of the discrete element $g_0$ on the fermions induces an action on the gauge fields through the conjugation of the representation matrices. When $d$ is odd, $g_0$ can be taken proportional to the unit matrix and therefore commutes with the representation matrices and the gauge fields are not affected. This situation is realized for $Osp(N|2;R)$ $N = odd$ and $G(3)$ in Table 1 which could have therefore, “discrete twisted” boundary conditions for the fermions. When $d$ is even the representation matrices are affected and the transformation is compatible with gauging if it corresponds to an outer automorphism of the group. This situation is realized only for the $Osp(N|2;R)$ $N = even$ case when the $SO(N)$ group has a $Z(2)$ automorphism: parity. Therefore among the superconformal algebras only $Osp(N|2;R)$ for all $N$, and $G(3)$ will have discrete twisted versions.

The number of continuous parameters characterizing the algebra can be easily read off from Table 1:

$[N/2]$ for $Osp(N|2;R)$, 4 for $F(4)$, 2 for $G(3)$, $N$ for $SU(1,1|N)$, 2 for $SU(1,1|2)/U(1)$, $M + 1$ for $Osp(4^*|2M)$ and 2 for $D(2,1;\alpha)$.

However, algebras having different values of the continuous parameters can be isomorphic. This feature, the “Spectral Flow”, can be easily understood in the C-S framework. Indeed, we can make a change of variable from the fields having nontrivial periodicity properties (6.4), (6.5), which have
an upper index \(a\), to fields which are strictly periodic and which we denote without an index:

\[
\psi^a_{\mu}(\theta) = \exp(i\vec{a}.\vec{\mu} \theta / 2\pi)\psi_{\mu}(\theta) \quad B^a_{\alpha}(\theta) = \exp(i\vec{a}.\vec{\alpha} \theta / 2\pi)B_{\alpha}(\theta)
\] (6.7)

We supplement (6.7) with a change of variable also for the fields \(B^a_i\) which were periodic:

\[
B^a_i(\theta) = B_i(\theta) + a_i/2\pi
\] (6.8)

The change of variables (6.7), (6.8) corresponds formally to a gauge transformation:

\[
u(\theta) = \exp(i\vec{a}.\vec{H} \theta / 2\pi)
\] (6.9)

Since the gauge group element \(u\) is not periodic, as a gauge transformation (6.9) is not legal. Nevertheless we can use it as a change of variable which will leave the bulk part, Eq.(2.17), of the C-S action form invariant when expressed in terms of the unindexed, strictly periodic variables.

The source term, Eq.(6.1), will not be invariant. The change can be easily calculated from the consistency requirement of the holonomy. Since through the change of variables an amount \(\exp(i\vec{a}.\vec{H}/2\pi)\) was removed from the holonomy, when written in terms of the unindexed variables the source term should have \(\lambda - a/2\pi\) instead of \(\lambda\).

Therefore all dependence on continuous parameters in the boundary conditions which can be related to the gauge group \(\tilde{G}\) is removable. It is obvious that a similar procedure for the continuous dependence related to the flavour symmetry or for the discrete boundary conditions would not apply.

The above discussion can be used in a straightforward fashion to obtain properties of the superconformal algebras. As discussed in Section 3 the C-S fields become directly the generators of the algebras. Therefore the possible periodicity conditions of the generators of the superconformal transformations \(Q\) are identical to the conditions (6.4) of the gravitino fields once a Cartan basis is used. The accompanying periodicity conditions of the Kac-Moody generators \(B\) follow from the periodicity conditions of the gauge fields (6.5). The enumeration given above of the possible boundary
conditions for the various C-S theories applies directly to the possible periodicities ("moddings") of the generators in the corresponding superconformal algebras.

Thus the change of variable removing dependences on continuous parameters gives the relation between algebras with different moddings, i.e. the spectral flow:

\[ Q_\mu^a(\theta) = \exp\left(\frac{i\vec{a}.\vec{\mu}}{2\pi}\right)Q_\mu(\theta) \tag{6.10} \]

\[ B_{\vec{\alpha}}^a(\theta) = \exp\left(\frac{i\vec{a}.\vec{\alpha}}{2\pi}\right)B_{\vec{\alpha}}(\theta) \quad B_i^a(\theta) = B_i(\theta) + \frac{a_i}{2\pi} \tag{6.11} \]

where we used the Cartan basis for the generators.

The highest weight \( \vec{\lambda}^a \) irrep of the algebra with modding "a", is mapped into an irrep with highest weight \( \vec{\lambda} \) of the Ramond (completely periodic) algebra. The weights are related by:

\[ \vec{\lambda}^a = \vec{\lambda} + \frac{\vec{a}}{2\pi} \tag{6.12} \]

We remark that since the spectral flow involves only the fields carrying Kac-Moody quantum numbers, it wouldn’t affect the part of the Virasoro generator coming from the \( sl(2) \) fields. The flow in the total Virasoro generator can be read off from the flow of the Sugawara part which was added to it, as discussed in Section 3. From the explicit Sugawara expression in terms of \( B \) and Eq. (6.11) we obtain the flow of the Virasoro generator \( L(\theta) \):

\[ \hat{L}^a(\theta) = L(\theta) + \frac{1}{\pi(k + h)} \vec{B}(\theta).\vec{a} + \frac{1}{4\pi^2(k + h)} \vec{a}.\vec{a} \tag{6.13} \]

Summarizing: besides the completely periodic (Ramond) modding for all the cases listed in Table 1 the superconformal algebras admit the following independent moddings:

a) discrete twisted modding for the \( Osp(N|2; R) \) series (including the \( N = 2, 3 \) and “large” \( N = 4 \) superconformal algebras studied before ) and \( G(3) \).

b) a modding depending on one continuous parameter for the \( SU(1,1|2)U(1) \) ( “small” \( N = 4 \) superconformal algebra.

All the other moddings for all the algebras are related to the above by spectral flow.
7 Conclusions

In this paper, we have analyzed in detail the AdS/CFT correspondence for extended supergravity in 2 + 1 dimensions. We have shown how the superconformal algebras with nonlinearities in the currents arise as asymptotic symmetry algebras. We have also shown that the dynamics of the asymptotic fields is controlled by the super-Liouville Lagrangian.

The AdS/CFT correspondence enlightens the properties of the non-linear superconformal algebras and provides a systematic discussion of the spectral flow.

We conclude by pointing out two questions that could be worth pursuing. First, it would be interesting to try to understand the mechanism of [52] from a pure three-dimensional point of view (see [53] in this context). Second, one might extend the analysis to the AdS\(_2\)/Super-conformal quantum mechanics along the lines of [54, 55, 56]. The extensions to the quasiconformal case (in the sense of [24]) and to Toda models (super W-algebras) [57] could also be of interest.

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Appendix

In this appendix, we indicate where zero modes and holonomies arise in the derivation of the super-Liouville action from the supergravity action. We show in particular that 2+1 supergravity only
yields the ordinary super-Liouville Lagrangian modulo zero mode and holonomy terms, one striking difference being that the holonomies in the right and left sectors are independent in supergravity. The analysis sheds as a by-product some interesting light on the sign of the exponential term in the Liouville action.

Since the subtleties arise already in the non-supersymmetric case, we shall drop the fermions in order to keep the formulas simple. Thus, our starting point is the action for ordinary gravity,

\[ S[A, \tilde{A}] = \frac{1}{8\pi G} \int_M d^3x \left\{ \frac{1}{2} eR + \frac{e}{\ell^2} \right\} \]  

where \( A \) and \( \tilde{A} \) are related to the dreibeins and the connection through (2.15) and (2.16). This action can be written as the difference between two \( sl(2,R) \)-Chern-Simons actions.

The first step follows the references [8, 9, 12] and consists in solving the Gauss constraints \( F^a_{ij} = 0, \tilde{F}^a_{ij} = 0 \) (\( i,j \) spatial indices) in the bulk, taking into account the first set of boundary conditions (4.1). We shall assume that the spatial sections have the annulus topology, as this is the case relevant to the black holes. As shown in [8], the solution of the Gauss constraints is then

\[ A_\theta = g^{-1}K(t)g + g^{-1}\partial_\theta g, \]  

\[ A_r = g^{-1}\partial_r g \]  

where \( g(t, r, \theta) \) is a \( sl(2,R) \) group element and where \( K(t) \) is a Lie-algebra element characterizing the holonomy. Under the transformation \( g(t, r, \theta) \rightarrow \alpha(t)g(t, r, \theta), K(t) \) is conjugated by \( \alpha(t) \).

Using this freedom, one can bring \( K(t) \) along \( \sigma^3 \) (the black hole holonomy is hyperbolic and we restrict therefore the analysis to this case),

\[ K(t) = K^3(t)\frac{\sigma^3}{2}. \]  

Similar formulas hold for the tilde sector.

If one inserts the solution of the Gauss constraints inside the Chern-Simons action supplemented
by the surface term appropriate to the boundary conditions \([1,1]\), one gets

\[
S = S^\infty + S^{Hor}
\]

(A.5)

where \(\infty\) and \(Hor\) refer respectively to the outer and inner boundaries. We are interested here in the piece \(S^\infty\) describing the dynamics at infinity, given by

\[
S^\infty = I + \tilde{I}
\]

(A.6)

with

\[
I[h(t, \theta), K(t)] = \frac{k}{4\pi} \int dt d\theta \left[ Tr \left( h^{-1} \partial_\theta hh^{-1} \partial_\theta h \right) - Tr E^2_\theta \right]
+ \frac{k}{2\pi} \int dt d\theta \left[ Tr \left( h^{-1} K(t) \partial_\theta h \right) \right]
+ \frac{k}{4\pi} \int dr dt d\theta Tr \left[ g^{-1} \partial_r g \left( g^{-1} \partial_t g g^{-1} \partial_\theta g - g^{-1} \partial_\theta g g^{-1} \partial_\theta g \right) \right].
\]

(A.7)

Here, \(h(t, \theta)\) is the value of \(g(t, r, \theta)\) on the outer boundary (we assume that the \(r\)-dependent similarity transformation of subsection 3.2 has been performed, so that the fields have well-defined asymptotics) and

\[
E_\theta = h^{-1} K(t) h + h^{-1} \partial_\theta h
\]

(A.8)

The holonomy \(K(t)\) appears also in the action \(S^{Hor}\) describing the dynamics at the inner boundary, but not \(h(t, \theta)\). Similarly, the action for the other chirality is

\[
\tilde{I}[\tilde{h}(t, \theta), \tilde{K}(t)] = \frac{k}{4\pi} \int dt d\theta \left[ Tr \left( -\tilde{h}^{-1} \partial_\theta \tilde{h} \tilde{h}^{-1} \partial_\theta \tilde{h} \right) - Tr \tilde{E}^2_\theta \right]
- \frac{k}{2\pi} \int dt d\theta \left[ Tr \left( \tilde{h}^{-1} \tilde{K}(t) \partial_\theta \tilde{h} \right) \right]
- \frac{k}{4\pi} \int dr dt d\theta Tr \left[ \tilde{g}^{-1} \partial_r \tilde{g} \left( \tilde{g}^{-1} \partial_t \tilde{g} \tilde{g}^{-1} \partial_\theta \tilde{g} - \tilde{g}^{-1} \partial_\theta \tilde{g} \tilde{g}^{-1} \partial_\theta \tilde{g} \right) \right]
\]

(A.9)

with

\[
\tilde{E}_\theta = \tilde{h}^{-1} \tilde{K}(t) \tilde{h} + \tilde{h}^{-1} \partial_\theta \tilde{h}
\]

(A.10)

and

\[
\tilde{K}(t) = \tilde{K}^3(t) \frac{\sigma^3}{2}
\]

(A.11)

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At this point, one can proceed in two different ways. One can either first recombine the two chiralities to get the vector WZW model (modulo holonomies and zero mode terms) and then, implement the Hamiltonian reduction enforced by the second set of constraints \(1.3\). This was the method followed in [12]. Alternatively, one can first implement the reduction constraints \(1.3\) and then discuss how to recombine the two (reduced) chirality sectors. This is the approach followed here.

Consider first the untilde sector. Using the Gauss decomposition\(^4\)

\[
h = \exp (\alpha \sigma^-) \exp (\beta \sigma^3) \exp (\gamma \sigma^+)\]  

one finds that the action takes the form

\[
I = \frac{k}{\pi} \int dt d\theta \left[ \partial_- \beta \partial_0 \beta + \partial_0 \alpha \partial_- \gamma \exp (2\beta) \right] + \frac{k}{\pi} \int dt d\theta \text{Tr} \left( h^{-1} K(t) \partial_- h \right) - \frac{k}{4} \int dt (K^3)^2 \]  

(A.13)

with \(\text{Tr} \left( h^{-1} K(t) \partial_- h \right)\) given explicitly by

\[
\text{Tr} \left( h^{-1} K(t) \partial_- h \right) = K^3(t) \left[ \partial_- \beta - \alpha \partial_- \gamma \exp (2\beta) \right] \]  

(A.14)

while the constraint \(1.3\) on the untilde current reads

\[
\partial_\theta \alpha \exp (2\beta) - \alpha K^3 \exp (2\beta) = \mu. \]  

(A.15)

\[
\partial_\theta \beta - \gamma \partial_\theta \alpha \exp (2\beta) + \frac{K^3}{2} + \alpha \gamma K^3 \exp (2\beta) = 0 \]  

(A.16)

The first of these constraints enables one to express \(\alpha\) in terms of \(K^3\) and \(\beta\); this can be verified by Fourier expansion of \(\alpha\) and \(\exp (-2\beta)\); note the crucial role played by the holonomy, which must be non-zero for the zero-mode part of the equation to make sense. The relationship between \(\alpha\) and \(\beta\) is non-local in \(\theta\). The second of these constraints can be rewritten as

\[
\partial_\theta \beta - \mu \gamma + \frac{K^3}{2} = 0 \]  

(A.17)

\(^4\)The global aspects of the Gauss decomposition are discussed in [58].
and enables one to express $\gamma$ in terms of $K^3$ and $\beta$ ($\mu \neq 0$).

Thus, one can completely get rid of $\alpha$ and $\beta$. If one does so, one gets a reduced action $I^R[\beta(t, \theta), K^3(t)]$ that involves only $\beta(t, \theta)$ and $K^3(t)$. Its expression is straightforward to work out but it is rather awkward and will not be given here. Suffice it to note that the $K^3$-independent part of $I^R$ is just the action for a free chiral boson

$$I^R = \frac{k}{\pi} \int dt d\theta \left[ \partial_+ \beta \partial_0 \beta \right] + (K^3 - \text{terms}) \quad (A.18)$$

and does not contain $\mu$ (in particular, it is independent of the sign of $\mu$). The (unwritten) piece involving $K^3$ in (A.18) is anyway incomplete since $K^3$ occurs also in the action $S^{Hor}$ describing the dynamics on the inner boundary. Similarly, the improved Virasoro generator is given by

$$\hat{L} = -\frac{k}{2\pi} \left[ (\partial_0 \beta)^2 + \partial_0 \partial_0 \beta \right] \quad (A.19)$$

(the minus sign is due to our conventions) and again, does not depend on $\mu$.

The same treatment can be worked out for the other chirality, yielding

$$\tilde{I}^R = -\frac{k}{\pi} \int dt d\theta \left[ \partial_+ \tilde{\beta} \partial_0 \tilde{\beta} \right] + (\tilde{K}^3 - \text{terms}) \quad (A.20)$$

and

$$\hat{\tilde{L}} = \frac{k}{2\pi} \left[ (\partial_0 \tilde{\beta})^2 - \partial_0 \partial_0 \tilde{\beta} \right]. \quad (A.21)$$

The first term is again the action for a chiral boson, of opposite chirality. Note that the Hamiltonians for the chiral bosons (dropping holonomy terms) are both positive definite.

One can now recombine the two chiralities. First, consider the solutions of the equations of motion. The chiral fields $\beta$ and $\tilde{\beta}$ depend on $x^+$ and $x^-$, respectively. Define

$$\exp 2\varphi = \frac{\partial_+ F \partial_- \tilde{F}}{[1 - F \tilde{F}]^2} \quad (A.22)$$

with $\partial_+ F = \exp(-2\beta)$ and $\partial_- \tilde{F} = \exp(2\tilde{\beta})$. It is easy to see that $\varphi$ is a solution of the Liouville equation

$$\partial_+ \partial_- \varphi - \exp 2\varphi = 0. \quad (A.23)$$
If one makes the corresponding formal change of phase space variables

$$\phi = -\beta + \tilde{\beta} - \log |1 - F\tilde{F}|, \quad F' = \exp (-2\beta), \quad \tilde{F}' = \exp 2\tilde{\beta}$$  \hspace{1cm} (A.24)

$$\Pi_{\phi} = \frac{k}{2\pi} \left( -\beta' - \tilde{\beta}' + \frac{\exp (-2\beta)\tilde{F} - F \exp 2\tilde{\beta}}{1 - FF'} \right)$$  \hspace{1cm} (A.25)

one gets from the sum of the chiral actions for \((\beta, \tilde{\beta})\) the Liouville action \((4.42)\) (with fermions and gauge fields dropped). Of course, the above transformation must be amended so as to be well-defined on the circle; this leads to additional terms in the super-Liouville action involving zero modes and holonomies, which are crucial (furthermore, we have not investigated the global features of the change of variables). Thus, the action coming from 2+1 gravity is not just the vector Liouville action, there are further zero modes and holonomy terms. In particular, the holonomies \(K^3\) and \(\tilde{K}^3\) are independent in 2+1 gravity. If the difference of the holonomies is an integer (i.e. the zero-mode conjugate to the difference is an angular momentum variable) the CFT on the boundary will still be local. We shall not discuss explicitly the extra terms here, because for practical purposes, it is more convenient not to recombine the chiralities and to regard the sum of the chiral actions (with the holonomy terms) as defining the Liouville model relevant to the asymptotic of 2+1 gravity\(^5\).

In the case of supergravity models with different number of supersymmetries in the right and left sectors, this is the only approach possible.

Note that alternatively, we could have defined

$$\exp 2\Phi = \frac{\partial_+ F \partial_- \tilde{F}}{[1 + FF']^2}$$  \hspace{1cm} (A.26)

The variable \(\Phi\) is a solution of the Liouville equation with standard sign,

$$\partial_+ \partial_- \Phi + \exp 2\Phi = 0.$$  \hspace{1cm} (A.27)

\(^5\)Thus, in particular, the super-Liouville action given in the text does not fully describe 2+1 supergravity. The zero modes and holonomies are different. Nevertheless, deriving the super-Liouville action from 2+1 supergravity (up to zero modes) is useful since it enables one to get straightforwardly the superconformal transformation laws.
The corresponding phase space change of variables, familiar from the analysis of the standard Liouville model [60, 61, 62]

\[ \Phi = -\beta + \tilde{\beta} - \log |1 + F\tilde{F}|, \quad F' = \exp (-2\beta), \quad \tilde{F}' = \exp 2\tilde{\beta} \quad (A.28) \]

\[ \Pi_\Phi = \frac{k}{2\pi} \left( -\beta' - \tilde{\beta}' - \frac{\exp (-2\beta)\tilde{F} - F \exp 2\tilde{\beta}}{1 + F\tilde{F}} \right) \quad (A.29) \]

brings the action to the standard Liouville action. Of course, the zero modes and holonomies enter differently this time. But, up to these terms, we see that one can change the sign in front of the Liouville exponential in the action by a (non-local, real) change of variables.

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