A ONE-DIMENSIONAL SYMMETRY RESULT FOR ENTIRE SOLUTIONS TO THE FISHER-KPP EQUATION

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Abstract. We consider the Fisher-KPP reaction-diffusion equation in the whole space. We prove that if a solution has, to main order and for all times (positive and negative), the same exponential decay as a planar traveling wave with speed larger than the minimal one at its leading edge, then it has to coincide with the aforementioned traveling wave.

1. Introduction

The Fisher-KPP equation

\[ u_t = \Delta u + f(u), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}, \quad N \geq 1, \quad (1.1) \]

with \( f(u) = u(1 - u) \) appears in the context of population dynamics to describe the spatial spread of an advantageous allele (see [2, 4]). For every wave speed \( c \geq 2 \) and \( \eta \in S^{N-1} \) it admits a unique, up to translations, planar traveling wave solution of the form

\[ u(x, t) = V(\xi) \quad \text{with} \quad \xi = \eta \cdot x - ct \]

such that

\[ V_\xi < 0, \quad \lim_{\xi \to -\infty} V(\xi) = 1, \quad \lim_{\xi \to +\infty} V(\xi) = 0. \]

We point out that \( V \) satisfies

\[ v_{\xi\xi} + cv_\xi + f(v) = 0, \quad \xi \in \mathbb{R}. \]

We note that such fronts move in the direction of \( \eta \) with constant speed \( c \), and their tail as \( \xi \to +\infty \) is frequently referred to as the \textit{leading edge} of the wave. More generally, there exists a \( c_s > 0 \) such that, for each \( c \geq c_s \), a completely analogous existence-uniqueness result holds for \( f \in C^1(\mathbb{R}) \) satisfying

\[ f(0) = f(1) = 0, \quad f > 0 \text{ in } (0, 1). \quad (1.2) \]

In fact, if one further assumes that

\[ f(s) < f'(0)s, \quad s \in (0, 1), \quad (1.3) \]
then \( c_* = 2\sqrt{f'(0)} \) holds and, possibly after a translation, \( V \) satisfies the following asymptotic behaviour at its leading edge:

\[
V(\xi) = \begin{cases} 
\xi e^{r\xi} (1 + o(1)) & \text{if } c = 2\sqrt{f'(0)}, \\
eo^{r\xi} (1 + o(1)) & \text{if } c > 2\sqrt{f'(0)},
\end{cases}
\quad \text{as } \xi \to +\infty,
\tag{1.4}
\]

where

\[
r = \frac{-c + \sqrt{c^2 - 4f'(0)^2}}{2} < 0.
\tag{1.5}
\]

For the above properties, we refer to the introduction of [3] and the many references therein.

The purpose of this note is to prove the following one-dimensional symmetry result.

**Theorem 1.1.** We assume that \( f \in C^2(\mathbb{R}) \) satisfies (1.2) and

\[
f'(s) \leq f'(0), \quad s \in (0, 1).
\tag{1.6}
\]

Let \( u \) be a solution to (1.1) with values in \((0, 1)\) such that for some \( \eta \in \mathbb{S}^{N-1} \) and \( c > 2\sqrt{f'(0)} \) it satisfies

\[
u(x, t) = e^{r(\eta \cdot x - ct)} (1 + o(1)) \quad \text{as } \eta \cdot x - ct \to +\infty,
\tag{1.7}
\]

uniformly in \( t \in \mathbb{R} \) and in the subspace of \( \mathbb{R}^N \) that is orthogonal to \( \eta \). Then, we have

\[u(x, t) \equiv V(\eta \cdot x - ct).\]

**Remark 1.1.** The assumptions on \( f \) in Theorem 1.1 imply that (1.3) holds, and thus \( c_* = 2\sqrt{f'(0)} \).

If \( f \) is of class \( C^1 \), satisfies (1.2) and \( f'(0) > 0, f'(1) < 0 \), it was shown in [1, Thm. 3.5] that if a solution of (1.1) satisfies

\[
V(\eta \cdot x - ct) \leq u \leq V(\eta \cdot x - ct - a), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R},
\tag{1.8}
\]

for some \( \eta \in \mathbb{S}^{N-1} \), \( c \geq c_* \) and \( a > 0 \), then

\[u \equiv V(\eta \cdot x - ct - b) \text{ for some } b \in [0, a].\]

If one further assumes that \( f \) is concave in \([0, 1]\) and \( c > 2\sqrt{f'(0)} \) (recall that this is \( c_* \) in the case of (1.3)), then the assertion (1.8) follows from the weaker assumptions that

\[0 < u < 1 \text{ and } u(x, t) \to 1 \text{ (resp. 0) unif. as } \eta \cdot x - ct \to -\infty \text{ (resp. } +\infty),\]

see [3] and [1, Rem. 3.6]. We refer to the former reference for the existence of an infinite-dimensional manifold of solutions to (1.1) which are not traveling waves (still for concave \( f \)).

Our method of proof is similar in spirit to the aforementioned references, in the sense that it relies on a sweeping argument and the strong maximum principle.
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However, there are substantial differences in the implementation of this general method. Most notably, we apply our sweeping argument to the linearized equation of (1.1) (after switching to traveling wave coordinates). A main observation is that, thanks to (1.6), \( e^{\eta \cdot x - ct} \) is a positive supersolution of the aforementioned equation if \( c > 2 \sqrt{f'(0)} \). In fact, we will sweep with this function. If \( c = 2 \sqrt{f'(0)} \) then \( (\eta \cdot x - ct)e^{\eta \cdot x - ct} \) (recall (1.4)) is still a supersolution but it is sign changing, which is the reason why our proof breaks down in that case. Related ideas in a different context can be found in our recent paper [8].

The rest of the paper is devoted to the proof of Theorem 1.1.

2. Proof of Theorem 1.1

Proof. Without loss of generality, we may assume that \( \eta = (1, 0, \cdots, 0) \).

Traveling wave coordinates. It is natural to study \( u \) in traveling wave coordinates. For this purpose, with a slight abuse of notation, we write

\[
u(x, t) = u(x_1, \cdots, x_N, t) = u(\xi, y, t) \quad \text{with} \quad \xi = x_1 - ct \quad \text{and} \quad y = (x_2, \cdots, x_N).
\]

In this frame of reference, \( u \) solves

\[u_t = u_{\xi \xi} + cu_{\xi} + \Delta_y u + f(u), \quad \xi \in \mathbb{R}, \quad y \in \mathbb{R}^{N-1}, \quad t \in \mathbb{R}, \tag{2.1}\]

and satisfies

\[u(\xi, y, t) = e^{r\xi} + w(\xi, y, t), \tag{2.2}\]

where \( w \) is such that

\[w(\xi, y, t) = o(e^{r\xi}) \quad \text{as} \quad \xi \to +\infty, \tag{2.3}\]

uniformly in \( y \in \mathbb{R}^{N-1} \) and \( t \in \mathbb{R} \). In these coordinates, which we will use throughout the rest of the proof, the assertion of the theorem reduces to

\[u_t \equiv 0 \quad \text{and} \quad \nabla_y u \equiv 0. \tag{2.4}\]

We will only show the first identity of the above relation since the other one can be established in a completely analogous fashion.

Gradient estimates. Our next objective is to see what the asymptotic behaviour (1.7) implies for \( u_t \). Since \( e^{r\xi} \) solves the linearized problem

\[v_{\xi \xi} + cv_{\xi} + f'(0)v = 0, \quad \xi \in \mathbb{R}, \tag{2.5}\]

we find that \( w \) solves

\[w_t - w_{\xi \xi} - cw_{\xi} - \Delta_y w = f(e^{r\xi} + w) - f'(0)e^{r\xi}. \tag{2.6}\]

The righthand side of the above equation can be written as

\[f(e^{r\xi} + w) - f(e^{r\xi}) - f(0) - f'(0)e^{r\xi} \equiv o(e^{r\xi}) \quad \text{as} \quad \xi \to +\infty, \tag{2.7}\]

uniformly in \( y \in \mathbb{R}^{N-1} \) and \( t \in \mathbb{R} \). Then, by applying standard interior parabolic \( W_{p,1} \) estimates (see for instance [6, Thm. 7.22]) in cylinders of the form \( C_{(\xi, y, t)} = \)
\(|\Xi - \xi| < 1, |Y - y| < 1, |T - t| < 1\), and using the parabolic Sobolev embedding (see [5, pgs. 80, 342]), we infer that
\[ |\nabla_{\xi,y} w| = o(e^{r\xi}) \text{ as } \xi \to +\infty, \] (2.8)
uniformly in \(y \in \mathbb{R}^{N-1}\) and \(t \in \mathbb{R}\).

Let \(z = w_{\xi}. \) (2.9)

Differentiation of (2.6) with respect to \(\xi\) yields
\[ z_t - z_{\xi\xi} - c z_{\xi} - \Delta_y z = f'(e^{r\xi} + w)(re^{r\xi} + z) - f'(0)re^{r\xi} \overset{(2.3),(2.8)}{=} o(e^{r\xi}), \]
as \(\xi \to +\infty\), uniformly in \(y \in \mathbb{R}^{N-1}\) and \(t \in \mathbb{R}\). By the same procedure as before, we get
\[ |\nabla_{\xi,y} z| = o(e^{r\xi}) \text{ as } \xi \to +\infty, \] (2.10)
uniformly in \(y \in \mathbb{R}^{N-1}\) and \(t \in \mathbb{R}\). For \(i = 2, \cdots, N\), let
\[ \omega = w_{x_i}. \] (2.11)

Differentiation of (2.6) now with respect to \(x_i\) yields
\[ \omega_t - \omega_{\xi\xi} - c \omega_{\xi} - \Delta_y \omega = f'(e^{r\xi} + w)\omega \overset{(2.8)}{=} o(e^{r\xi}), \]
as \(\xi \to +\infty\), uniformly in \(y \in \mathbb{R}^{N-1}\) and \(t \in \mathbb{R}\). By working in the usual way, we obtain
\[ |\nabla_{\xi,y} \omega| = o(e^{r\xi}) \text{ as } \xi \to +\infty, \] (2.12)
uniformly in \(y \in \mathbb{R}^{N-1}\) and \(t \in \mathbb{R}\).

Consequently, by combining (2.2), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12), we infer that
\[ u_t = o(e^{r\xi}) \text{ as } \xi \to +\infty, \] (2.13)
uniformly in \(y \in \mathbb{R}^{N-1}\) and \(t \in \mathbb{R}\).

**The sweeping argument.** We are now in position to apply a sweeping argument in order to show the desired relation (2.4). Let \(\psi = u_t. \) (2.14)

Differentiation of (2.1) with respect to \(t\) yields
\[ \psi_t = \psi_{\xi\xi} + c \psi_{\xi} + \Delta_y \psi + f'(u)\psi, \quad \xi \in \mathbb{R}, \ y \in \mathbb{R}^{N-1}, \ t \in \mathbb{R}, \] (2.14)
Moreover, in terms of \(\psi \) (2.13) becomes
\[ \psi = o(e^{r\xi}) \text{ as } \xi \to +\infty, \text{ uniformly in } y \in \mathbb{R}^{N-1} \text{ and } t \in \mathbb{R}. \] (2.15)

We observe that since \(e^{r\xi}\) is a solution of (2.5) (keep in mind (1.5)), and \(f\) satisfies (1.6), it is a supersolution of (2.14) (the fact that it is a supersolution of (2.1) is well known even under the weaker condition (1.3)). Indeed, we have
\[ (e^{r\xi})_t - (e^{r\xi})_{\xi\xi} - c(e^{r\xi})_{\xi} - \Delta_y e^{r\xi} - f'(u)e^{r\xi} = (f'(0) - f'(u)) e^{r\xi} \geq 0. \] (2.16)
Armed with the above information, we will show that $\psi \equiv 0$ by adapting Serrin’s sweeping principle (see [7, Thm. 2.7.1] for the elliptic case). Let us consider the set

$$\Lambda = \{ \lambda \geq 0 : \mu e^{r\xi} \geq \psi \text{ in } \mathbb{R}^{N+1} \text{ for every } \mu \geq \lambda \}.$$ 

Our goal is to show that $\Lambda = [0, \infty)$, which will yield $\psi \leq 0$. We can also apply the same argument, with $\psi$ replaced by $-\psi$, to obtain $\psi \geq 0$ and therefore conclude.

We first prove that $\Lambda \neq \emptyset$, and thus by continuity

$$\Lambda = [\tilde{\lambda}, \infty) \text{ for some } \tilde{\lambda} \geq 0. \quad (2.17)$$

To this end, we note that since $u$ is a bounded solution of (2.1) ($0 < u < 1$ as a matter of fact), standard interior estimates for linear parabolic equations [5, 6] and Sobolev embeddings imply that $u$ is bounded in $C^{2+\theta,1+\theta/2}(\mathbb{R}^N \times \mathbb{R})$ for any $\theta \in (0, 1)$. (2.18)

So, this implies that $\psi \in L^\infty(\mathbb{R}^N \times \mathbb{R})$. (2.19)

The above relation and (2.15) yield that there exists a $\tilde{\lambda} \gg 1$ such that

$$\tilde{\lambda} e^{r\xi} \geq \psi \text{ in } \mathbb{R}^{N+1}.$$ 

Hence, relation (2.17) holds for some $\tilde{\lambda} \in [0, \tilde{\lambda}]$. For future reference, we note that

$$\psi \leq \tilde{\lambda} e^{r\xi} \text{ in } \mathbb{R}^{N+1}. \quad (2.20)$$

In order to establish that $\tilde{\lambda} = 0$, as desired, we will argue by contradiction. So, let us suppose that $\tilde{\lambda} > 0$. To show that this is absurd, by the definition of the set $\Lambda$ and (2.17) it suffices to prove that there exists a small $\delta \in (0, \tilde{\lambda}/2)$ such that

$$\tilde{\lambda} - \delta \geq \psi(\xi, y, t), \quad (\xi, y, t) \in \mathbb{R}^{N+1}.$$ 

Suppose the above relation were false. Then, we could find $\lambda_n < \tilde{\lambda}$ with $\lambda_n \rightarrow \tilde{\lambda}$, $\xi_n \in \mathbb{R}$, $y_n \in \mathbb{R}^{N-1}$ and $t_n \in \mathbb{R}$ such that

$$\psi(\xi_n, y_n, t_n) \geq \lambda_n e^{r\xi_n}, \quad n \geq 1. \quad (2.21)$$

By virtue of (2.15), (2.19), and our assumption that $\tilde{\lambda} > 0$, we infer that the sequence $\{\xi_n\}$ is bounded. Hence, passing to a subsequence if necessary, we may assume that

$$\xi_n \rightarrow \xi_\infty \in \mathbb{R}. \quad (2.22)$$

Let us now consider the translated functions

$$U_n(\xi, y, t) = u(\xi, y + y_n, t + t_n) \text{ and } \Psi_n(\xi, y, t) = \psi(\xi, y + y_n, t + t_n), \quad n \geq 1.$$ 

Clearly, $0 < U_n < 1$ satisfies (2.18) uniformly with respect to $n$; while $\Psi_n$ solves

$$\Psi_t = \Psi_{\xi\xi} + c\Psi_\xi + \Delta_y \Psi + f'(U_n)\Psi; \quad \xi \in \mathbb{R}, \ y \in \mathbb{R}^{N-1}, \ t \in \mathbb{R}, \quad (2.23)$$
and
\[ \Psi_n \text{ is uniformly bounded with respect to } n \text{ (recall (2.19)).} \quad (2.24) \]

We also note that (2.15) becomes
\[ \Psi_n = o(e^{r\xi}) \text{ as } \xi \to +\infty, \text{ uniformly in } y \in \mathbb{R}^{N-1}, \ t \in \mathbb{R} \text{ and } n \geq 1. \]

Moreover, from (2.21) and (2.20) we obtain
\[ \Psi_n(\xi,0,0) \geq \lambda_n e^{r\xi_0} \text{ and } \Psi_n(\xi,y,t) \leq \tilde{\lambda} e^{r\xi}, \ \forall (\xi,y,t) \in \mathbb{R}^{N+1}, \ n \geq 1, \quad (2.25) \]

respectively.

By the aforementioned uniform Hölder estimates for \( U_n \) and a standard diagonal-compactness argument, passing to a further subsequence if necessary, we may assume that \( U_n \to U_\infty \) in \( C^{2,1}_{loc}(\mathbb{R}^{N+1}) \) for some \( 0 \leq U_\infty \leq 1 \) (actually, \( U_\infty \) solves (2.1) and satisfies (2.2)-(2.3) but we will not need this information). In turn, by (2.23), (2.24) and standard parabolic estimates, we deduce that \( \Psi_n \to \Psi_\infty \) in \( C^{2,1}_{loc}(\mathbb{R}^{N+1}) \) for some \( \Psi_\infty \in L^\infty(\mathbb{R}^{N+1}) \) that solves
\[ \Psi_t = \Psi_{\xi\xi} + c\Psi_{\xi} + \Delta_y \Psi + f'(U_\infty)\Psi, \ \xi \in \mathbb{R}, \ y \in \mathbb{R}^{N-1}, \ t \in \mathbb{R}, \quad (2.26) \]

and satisfies
\[ \Psi_\infty = o(e^{r\xi}) \text{ as } \xi \to +\infty, \text{ uniformly in } y \in \mathbb{R}^{N-1}, \ t \in \mathbb{R}. \quad (2.27) \]

Moreover, recalling (2.22), by letting \( n \to \infty \) in (2.25) we get
\[ \Psi_\infty(\xi_\infty,0,0) \geq \tilde{\lambda} e^{r\xi_0} \text{ and } \Psi_\infty(\xi,y,t) \leq \tilde{\lambda} e^{r\xi}, \ (\xi,y,t) \in \mathbb{R}^{N+1}. \]

On the other hand, since \( \tilde{\lambda} e^{r\xi} \) is a supersolution of (2.26) (by the same calculation as in (2.16) but with \( U_\infty \) in place of \( u \)), we infer by the strong maximum principle (see for instance [5, 6]) that \( \Psi_\infty \equiv \tilde{\lambda} e^{r\xi} \). However, the last relation contradicts (2.27), and thus the proof of the theorem is complete. \( \square \)

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