Homotopy invariance of higher $K$-theory for abelian categories

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Abstract

The main theorem in this paper is that the base change functor from an abelian category $A$ to its polynomial category in the sense of Schlichting $- \otimes_A \mathbb{Z}[t] : A \to A[t]$ induces an isomorphism on their $K$-theories if $A$ is noetherian and has enough projective objects. The main theorem implies the well-known fact that $A^1$-homotopy invariance of $K'$-theory for noetherian schemes.

1 Introduction

Contrary to the importance of $A^1$-homotopy invariance in the motivic homotopy theory [Voe98], [MV99] and [Voe00], the homotopy invariance of $K'$-theory for noetherian schemes still has been mysterious in the following sense. After [Sch11], every fundamental theorem except for homotopy invariance of $K'$-theory, the dévissage theorem in [Qui73] and the cell filtration theorem in [Wal85] are corollaries of Thomason-Schlichting localization theorem and in the view of non-commutative motive theory [CT09] or motive theory for $\infty$-categories [BGT10], non-connective $K$-theory is the universal localizing invariant. To relate motivic homotopy theory with motive theory for DG or $\infty$-categories, it is important to make clear the homotopy invariance of $K'$-theory in the view of motive theory for higher categories. Many authors have already defined affine line over certain categories as in [GM96], [Sch06] and [BGT10]. The main objective in the paper is to examine the homotopy invariance of $K$-theory for abelian categories by taking Schlichting polynomial categories. Let us recall the definition of polynomial categories.

Definition 1.1. (1) (Notation 2.1) For a category $C$, let us denote the category of endomorphisms in $C$ by $\text{End } C$. Namely, an object in $\text{End } C$ is a pair $(x, \phi)$ consisting of an object $x$ in $C$ and morphism $\phi : x \to x$ and a morphism between $(x, \phi) \to (y, \psi)$ is a morphism $f : x \to y$ in $C$ such that $\psi f = f \phi$.

(2) From now on, let $A$ be an abelian category. Let us denote the category of left exact functors from $A^{\text{op}}$ to the category of abelian groups $\text{Ab}$ by $\text{Lex } A$. The category $\text{Lex } A$ is a Grothendieck abelian category and the Yoneda embedding $y : A \to \text{Lex } A$ is exact and reflects exactness.

(3) (Notation 2.4) We say an object $x$ in $A$ is noetherian if every ascending filtration of subobjects of $x$ is stational. We say $A$ is noetherian if every object in $A$ is noetherian.

(4) (Definition 2.10) Let us assume that $A$ is a noetherian abelian category and let us denote the full subcategory of noetherian objects in $\text{End } \text{Lex } A$ by $A[t]$ and call it the noetherian polynomial category over $A$. We can prove that $A[t]$ is an abelian category. (See Lemma 2.5).
(5) For an object \( a \) in \( \mathcal{A} \), let us define an object \( a[t] (= (a[t], t)) \) in \( \text{End} \text{Lex} \mathcal{A} \) as follows. The underlying object \( a[t] \) is \( \bigoplus_{n=0}^{\infty} at^n \) where \( at^n \) is a copy of \( a \). The endomorphism \( t : a[t] \to a[t] \) is defined by the identity morphisms \( at^n \to at^{n+1} \) in each components. We can prove that if \( a \) is noetherian in \( \mathcal{A} \), then \( a[t] \) is noetherian in \( \mathcal{A}[t] \). (See Theorem 2.9). We call the association \( - \otimes_{\mathcal{A}} \mathbb{Z}[t] : \mathcal{A} \to \mathcal{A}[t], a \mapsto a[t] \) the base change functor which is an exact functor.

The main theorem is the following.

**Theorem 1.2** (Theorem [4.1]). Let \( \mathcal{A} \) be a noetherian abelian category which has enough projective objects. The functor \( - \otimes_{\mathcal{A}} \mathbb{Z}[t] : \mathcal{A} \to \mathcal{A}[t] \) induces the isomorphism on their \( K \)-theories \( K(\mathcal{A}) \cong K(\mathcal{A}[t]) \).

**Conventions.** In this note, basically we follow the notation of exact categories for [Kel90] and algebraic \( K \)-theory for [Qui73] and [Wal85]. For example, we call admissible monomorphisms (resp. admissible epimorphism and admissible short exact sequences) inflations (resp. deflations, conflations). We also call a category with cofibrations and weak equivalences a Waldhausen category. Let us denote the set of all natural numbers by \( \mathbb{N} \). We consider it to be totally ordered set with the usual order. For a Waldhausen category, we write the specific zero object by the same letter \( * \). Let us denote the 2-category of essentially small categories by \( \text{Cat} \). For categories \( \mathcal{X}, \mathcal{Y} \), we denote the (large) category of functors from \( \mathcal{X} \) to \( \mathcal{Y} \) by \( \text{Hom}(\mathcal{X}, \mathcal{Y}) \). For any (left) noetherian ring \( A \), we denote the category of finitely generated \( A \)-modules by \( \mathcal{M}_A \). Throughout the paper, we use the letter \( \mathcal{A} \) to denote an essentially small abelian category. For an object \( x \) in \( \mathcal{A} \) and a finite family \( \{x_i\}_{1 \leq i \leq m} \) of subobjects of \( x \), \( \sum_{i=1}^{m} x_i \) means the minimal sub object of \( x \) which contains all \( x_i \). For an additive category \( \mathcal{B} \), we write the category of chain complexes on \( \mathcal{B} \) by \( \text{Ch} \mathcal{B} \).

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## 2 Polynomial categories

In this section, let us recall the notation of polynomial abelian categories from [Sch00] or [Sch06].

### 2.1 End categories

**Notations 2.1.** For a category \( \mathcal{C} \), let us denote the category of endomorphisms in \( \mathcal{C} \) by \( \text{End} \mathcal{C} \). Namely, an object in \( \text{End} \mathcal{C} \) is a pair \( (x, \phi) \) consisting of an object \( x \) in \( \mathcal{C} \) and morphism \( \phi : x \to x \) and a morphism between \( (x, \phi) \to (y, \psi) \) is a morphism \( f : x \to y \) in \( \mathcal{C} \) such that \( \psi f = f \phi \). For any functor \( F : \mathcal{C} \to \mathcal{C}' \), we have the functor \( \text{End} F : \text{End} \mathcal{C} \to \text{End} \mathcal{C}' \), \( (x, \phi) \mapsto (Fx, F\phi) \). Moreover for any natural transformation \( \theta : F \to F' \) between functors \( F, F' : \mathcal{C} \to \mathcal{C}' \), we have a natural transformation \( \text{End} \theta : \text{End} F \to \text{End} F' \) defined by the formula \( \text{End} \theta(x, \phi) := \theta(x) \) for any object \( (x, \phi) \). This association gives a 2-functor

\[
\text{End} : \text{Cat} \to \text{Cat}.
\]
Remark 2.2. Let $C$ be a category and $F : T \to \text{End} C$, $i \mapsto (x_i, \phi_i)$ be a functor. Let us assume that there is a limit $\lim x_i$ (resp. colimit $\text{colim} x_i$) in $C$. Then we have $\lim F_i = (\lim x_i, \lim \phi_i)$ (resp. colimit $F_i = (\text{colim} x_i, \text{colim} \phi_i)$). In particular, if $C$ is additive (resp. abelian), then $\text{End} C$ is also additive (resp. abelian). Moreover if $C$ is an exact category (resp. a category with cofibration), then $\text{End} C$ naturally becomes an exact category (resp. a category with cofibration.) Here a sequence $(x, \phi) \to (y, \psi) \to (z, \xi)$ is a conflation if and only if $x \to y \to z$ is a conflation in $C$. (resp. a morphism $(x, \phi) \looparrowright (y, \psi)$ is a cofibration if and only if $u : x \to y$ is a cofibration in $C$. ) Moreover if $w$ is a class of morphisms in $C$ which satisfies the axioms of Waldhausen categories (and its dual), then the class of all morphisms in $\text{End} C$ which is in $w$ also satisfies the axioms of Waldhausen categories (and its dual).

Remark 2.3. In [GM96 III. 5.15], for a category $C$, the category $\text{End} C$ is called the polynomial category over $C$ and denoted by $C[T]$. For let $A$ be a commutative ring with unit and $\text{Mod}(A)$ the category of $A$-Modules, then we have the canonical category isomorphism

$$\text{Mod}(A[T]) \cong (\text{Mod}(A))[T], \ M \mapsto (M, T)$$

where $A[T]$ is the polynomial ring over $A$ and $T$ means the endomorphism $T : M \to M$, $x \mapsto Tx$. Moreover in general for any abelian category $A$, we have the equality

$$\text{hdim } A[T] = \text{hdim } A + 1$$

where $\text{hdim } A$ is the homological dimension of $A$ which is defined by

$$\text{hdim } A := \max \{n; \text{Ext}^n(x, y) \neq 0 \text{ for any objects } x, y\}.$$ 

But obviously for any left noetherian ring $A$, $(\mathcal{M}_A)[T]$ and $\mathcal{M}_{A[T]}$ are different categories. The main reasons is that $A[T]$ is not finitely generated as an $A$-module. In particular, the object $(A[T], T)$ is in $(\text{Mod}(A))[T]$ but not in $(\mathcal{M}_A)[T]$. In the subsection 2.3 we define the noetherian polynomial categories over noetherian abelian category which is introduced by Schlichting in [Sch06]. In the notion, we have the canonical category equivalence between $\mathcal{M}_{A[t]}$ and $(\mathcal{M}_A)[t]$. See [2.13]

2.2 Noetherian objects

In this subsection, we develop the theory of noetherian objects in an exact categories which is slightly different from the usual notation in the category theory.

Notations 2.4. Let $\mathcal{E}$ be an exact category and $x$ an object in $\mathcal{E}$. We say $x$ is a noetherian object if its every ascending filtration of admissible subobjects

$$x_0 \gg x_1 \gg x_2 \gg \cdots$$

is stational. We say $\mathcal{E}$ is a noetherian category if every object in $\mathcal{E}$ is noetherian.

We can easily prove the following lemmas.

Lemma 2.5. Let $\mathcal{E}$ be an exact category. Then we have the following.

1. Let $x \gg y \gg z$ be a conflation in $\mathcal{E}$. If $y$ is noetherian, then $x$ and $z$ are also.
(2) For noetherian objects \( x, y \) in \( \mathcal{E} \), \( x \oplus y \) is also noetherian.

(3) Moreover assume that \( \mathcal{E} \) is abelian, then the converse of (1) is true. Namely, in the notation (1), if \( x \) and \( z \) are noetherian, then \( y \) is also.

**Lemma 2.6.** For any exact faithful functor \( F : \mathcal{A} \to \mathcal{B} \) between abelian categories and an object \( x \) in \( \mathcal{A} \), if \( Fx \) is noetherian, then \( x \) is also noetherian.

### 2.3 Schlichting polynomial category

2.7. For an exact category \( \mathcal{E} \), let us denote the category of left exact functors from \( \mathcal{E}^{\text{op}} \) to the category of abelian groups \( \text{Ab} \) by \( \text{Lex} \mathcal{E} \). The category \( \text{Lex} \mathcal{E} \) is a Grothendieck abelian category and the Yoneda embedding \( y : \mathcal{E} \to \text{Lex} \mathcal{E} \) is exact and reflects exactness.

2.8. For an object \( a \) in an additive category \( \mathcal{A} \), let us define an object \( a[t] (= (a[t], t)) \) in \( \text{End} \text{Lex} \mathcal{A} \) as follows. The underlying object \( a[t] \) is \( \bigoplus_{n=0}^{\infty} at^n \) where \( at^n \) is a copy of \( a \). The endomorphism \( t : a[t] \to a[t] \) is defined by the identity morphisms \( at^n \to at^{n+1} \) in each components.

The following theorem is proved in [Sch00, 9.10 b]

**Theorem 2.9 (Abstract Hilbert basis theorem).** For any noetherian object \( a \) in an abelian category \( \mathcal{A} \), \( a[t] \) is also a noetherian object in \( \text{End} \text{Lex} \mathcal{A} \).

**Definition 2.10 (Schlichting polynomial category).** Let us assume that \( \mathcal{A} \) is a noetherian abelian category and let us denote the full subcategory of noetherian objects in \( \text{End} \text{Lex} \mathcal{A} \) by \( \mathcal{A}[t] \) and call it the noetherian polynomial category over \( \mathcal{A} \). By the virtue of 2.5 and 2.9, we learn that \( \mathcal{A}[t] \) is a noetherian abelian category.

**Remark 2.11.** We can easily prove that an object \( x \) in \( \text{End} \text{Lex} \mathcal{A} \) is in \( \mathcal{A}[t] \) if and only if there is a deflation \( a[t] \twoheadrightarrow x \) for some object \( a \) in \( \mathcal{A} \).

**Example 2.12.** For any noetherian objects \( a, b \) in \( \mathcal{A} \) and a morphism \( f : a[t] \to b[t] \) in \( \mathcal{A}[t] \), there is a positive integer \( m \) such that \( f(a) \) is in \( \bigoplus_{i=1}^{m} bt^i \). Since the morphism \( f \) is recovered by the restriction \( a \to a[t] \xrightarrow{f} b[t] \), \( f \) is determined by morphisms \( c_i : a \to b \ (0 \leq i \leq m) \) in \( \mathcal{A} \). We write \( f \) by \( \sum_{i=1}^{m} c_i t^i \).

**Example 2.13.** We have the category equivalence

\[ \mathcal{M}_{\mathcal{A}[t]} \cong (\mathcal{M}_{\mathcal{A}})[t], \ M \mapsto (M, t). \]

**Lemma 2.14.** For any projective object \( a \) in a noetherian abelian category \( \mathcal{A} \), \( a[t] \) is a projective object in \( \mathcal{A}[t] \).

**Proof.** Let us consider the diagram in \( \mathcal{A}[t] \) below.

```
\[
\begin{array}{c}
\text{a[t]} \\
\downarrow b \\
\downarrow f \\
\end{array}
\]
```

\begin{equation}
\begin{array}{c}
x \\
\downarrow g \\
y
\end{array}
\end{equation}

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Since there is an epimorphism $a \to f(a)$ from a noetherian object in $\text{Lex} \mathcal{A}$, the object $f(a)$ in $\text{Lex} \mathcal{A}$ is noetherian. Therefore $f(a)$ is in $\mathcal{A}$ by [Pop73, 5.8.8, 5.8.9]. Since $x$ is an object in $\mathcal{A}[t]$, there are an object $b$ in $\mathcal{A}$ and a epimorphism $s : b[t] \to x$ in $\text{End} \text{Lex} \mathcal{A}$. Then since $f(a)$ is a noetherian object in $\mathcal{A}$, there is a positive integer $m$ such that $\text{Im}(\bigoplus_{i=1}^{m} bt^i \to x \to y)$ contains $f(a)$. We put $u = s(\bigoplus_{i=1}^{m} bt^i)$ and $v = g(u)$. Then $u$ and $v$ are noetherian objects in $\text{Lex} \mathcal{A}$ and therefore they are in $\mathcal{A}$ by Ibid. We have the diagram in $\mathcal{A}$ below.

\[
\begin{array}{ccc}
a & \xrightarrow{h'} & u \\
\downarrow & & \downarrow g' \\
\downarrow f' & & \\
v & \xrightarrow{a} & y
\end{array}
\]

Therefore by projectivity of $a$, we have the dotted morphism $h' : a \to u$ which makes the diagram above commutative. The composition $a \xrightarrow{h'} u \to x$ induces the desired morphism $h : a[t] \to x$ which makes the first diagram above commutative. \(\square\)

## 3 Graded categories

In this section, we will introduce the notion of (noetherian) graded categories over categories and calculate the $K$-theory of noetherian graded categories over noetherian abelian categories.

### 3.1 Fundamental properties of graded categories

3.1. For a positive integer $n$, we define the category $<n>$ as follows. The class of objects of $<n>$ is just the set of all natural numbers $\mathbb{N}$. The class of morphisms of $<n>$ is generated by morphisms $\psi^i_m : m \to m + 1$ for any $m$ in $\mathbb{N}$ and $1 \leq i \leq n$ which subject to the equalities $\psi^i_{m+1} \psi^j_m = \psi^j_{m+1} \psi^i_m$ for each $m$ in $\mathbb{N}$ and $1 \leq i, j \leq n$.

**Definition 3.2 (Graded categories).** For a positive integer $n$ and a category $\mathcal{C}$, we put $\mathcal{C}_{gr}[n] := \text{HOM}(<n>, \mathcal{C})$ and call it the **category of (n-)graded category over $\mathcal{C}$**. For an object $x$, a morphism $f : x \to y$ in $\mathcal{C}_{gr}[n]$, we write $x(m), x(\psi^i_m)$ and $f(m)$ by $x_m, \psi^i_m$ and $f_m$ respectively.

**Remark 3.3.** We can calculate a (co)limit in $\mathcal{C}_{gr}[n]$ by term-wise (co)limit in $\mathcal{C}$. In particular, if $\mathcal{C}$ is additive (resp. abelian) then $\mathcal{C}_{gr}[n]$ is also additive (resp. abelian). Moreover if $\mathcal{C}$ is a category with cofibration (resp. an exact category), then $\mathcal{C}_{gr}[n]$ naturally becomes a category with cofibration (resp. an exact category). Here a sequence $x \to y \to z$ is a conflaction (resp. a morphism $x \to y$ is a cofibration) if it is term-wisely in $\mathcal{C}$. Moreover if $w$ is a class of morphisms in $\mathcal{C}$ which satisfies the axioms of Waldhausen categories (and its dual), then the class of all morphisms $lw$ in $\mathcal{C}_{gr}[n]$ those of morphisms $f$ such that $f_m$ is in $w$ for all natural number $m$ also satisfies the axioms of Waldhausen categories (and its dual).

3.4. For an exact category $\mathcal{E}$ and a positive integer $n$, we denote the full subcategory of all noetherian objects in $\mathcal{E}_{gr}[n]$ by $\mathcal{E}_{gr}^n[n]$. In particular if $\mathcal{E}$ is an abelian category then $\mathcal{E}_{gr}^n[n]$ is a noetherian abelian category by [2.5] In the case, we call $\mathcal{E}_{gr}[n]$ the **noetherian (n-)graded category over $\mathcal{E}$**.
**Remark 3.10.** For a noetherian commutative ring with unit $A$ and $\mathcal{E} = \mathcal{M}_A$, $\mathcal{E}_{gr}[n]$ is just the category of finitely generated graded $A[t_1, \ldots, t_n]$-modules.

**Proof.** Let $\mathcal{F}$ be a category of finitely generated graded $A[t_1, \ldots, t_n]$-modules. Any object $x$ in $\mathcal{F}$ is considered to be an object in $\mathcal{E}_{gr}[n]$ in the following way. Let us define the functor $x' : < n \to \mathcal{E}$ by $k \mapsto x_k$ and $(\psi^i : k \to k + 1) \mapsto (t_i : x_k \to x_{k+1})$. The association $x \mapsto x'$ induces a category equivalence $\mathcal{F} \sim \mathcal{E}_{gr}[n]$.

**Notations 3.6 (Degree shift).** Let $\mathcal{C}$ be a pointed category and $k$ an integer. We define the functor $(k) : C_{gr}[n] \to C_{gr}[n], x \mapsto x(k)$. For any object $x$ and any morphism $f : x \to y$ in $C_{gr}[n]$, we define the object $x(k)$ and the morphism $f(k) : x(k) \to y(k)$ as follows. We put

$$x(k)_m = \begin{cases} x_{m+k} & \text{if } m \geq -k, \\ 0 & \text{if } m < -k \end{cases}, \quad \psi^i_{m,x(k)} := \begin{cases} \psi^i_{m+k} & \text{if } m \geq -k, \\ 0 & \text{if } m < -k \end{cases} \text{ and } f(k)_m := \begin{cases} f_{m+k} & \text{if } m \geq -k, \\ 0 & \text{if } m < -k \end{cases}.$$

For any object $x$ in $C_{gr}[n]$ and a positive integer $k$, we have the canonical morphism $\psi^i_{x,k} = (\psi^i) : x(-k) \to x(-k + 1)$ defined by $\psi^i_{m-k} : x(-k)_m = x_{m-k} \to x(-k + 1)_m = x_{m-k+1}$ for each $m \in \mathbb{N}$.

**Notations 3.7.** For any natural numbers $m$ and $k$, an object $x$ in $C_{gr}[n]$ and a multi index $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$, we define the morphism $\psi^i_{x,k} = (\psi^i) : x(-\{\sum_{j=1}^n i_j + k\}) \to x(-k)$ by

$$\psi^i = (\psi^n)^{i_n}(\psi^{n-1})^{i_{n-1}} \cdots (\psi^1)^{i_1}.$$

**Notations 3.8 (Free graded object).** Let $\mathcal{C}$ be an additive category and $n$ a positive integer. We define the functor $\mathcal{F}[n] : \mathcal{C} \to C_{gr}[n]$ in the following way. For any object $x$ in $\mathcal{C}$, we define the object $\mathcal{F}[n](x) = x[\{\psi^i\}_{1 \leq i \leq m}]$ in $C_{gr}[n]$ as follows. We put

$$\mathcal{F}[n](x)_m := \bigoplus_{i=(i_1, \ldots, i_n) \in \mathbb{N}^n \atop \sum_{j=1}^n i_j = m} x_i,$$

where $x_i$ is a copy of $x$, $x_1 (\sum_{j=1}^n i_j = m)$ components of the morphisms $\psi^i_{m,\mathcal{F}[n](x)} : \mathcal{F}[n](x)_m \to \mathcal{F}[n](x)_{m+1}$ defined by id : $x_i \to x_i + e_k$ where $e_k$ is the $k$-th unit vector.

**3.9.** Let $\mathcal{C}$ be an additive category and $k$ a natural number. For any object $x$ in $C_{gr}[n]$, we have the canonical morphism $\mathcal{F}[n](x_k)(-k) \to x$ which is defined as follows. For each $m \geq k$ and $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$ such that $\sum_{j=1}^n i_j = m - k$, on the $x_i$ component of $\mathcal{F}[n](x_k)(-k)_m$, the morphism is defined by $\psi^i_{x_k} : x_i \to x_m$.

**Remark 3.10.** Let $\mathcal{C}$ be an additive category. Then the functor $\mathcal{F}[n] : \mathcal{C} \to C_{gr}[n]$ is the left adjoint functor of the functor $C_{gr}[n] \to \mathcal{C}$, $y \mapsto y_1$. Namely for any object $x$ in $\mathcal{C}$ and any object $y$ in $C_{gr}[n]$, we have the functorial isomorphism $\text{Hom}_\mathcal{C}(x, y_1) \sim \text{Hom}_{C_{gr}[n]}(\mathcal{F}[n](x), y)$, $f \mapsto (\mathcal{F}[n](x) \xrightarrow{\mathcal{F}[n](f)} \mathcal{F}[n](y_1) \xrightarrow{\text{Y}} y)$.

**Example 3.11.** For any objects $x$, $y$ in an additive category $\mathcal{C}$, a positive integer $k$, and family of morphisms $\{c_i\}_{i=1}^n \in \mathbb{N}^n$, $\sum_{j=1}^n i_j = k$ from $x$ to $y$, we define the morphism $\sum c_i \psi^i : \mathcal{F}[n](x)(-k) \to \mathcal{F}[n](y)$ by $c_i : x_i \to x_{i+1}$ on its $x_i$ component to $x_{i+1}$ component.

**Lemma 3.12.** Let $\mathcal{A}$ be a noetherian abelian category and $n$ a positive integer. We have the following assertions.
(1) For any object \( x \) in \( A \), \( \mathcal{F}[n](x) \) is a noetherian object in \( A_{gr}[n] \). In particular, we have the exact functor 
\[
\mathcal{F}_A[n] : A \to A'_{gr}[n].
\]

(2) For any object \( x \) in \( A'_{gr}[n] \), there is a natural number \( m \) such that the canonical morphism as in 3.9 
\[
\bigoplus_{k=0}^{m} \mathcal{F}[n](x_k)(-k) \to x
\]
is an epimorphism.

(3) If \( x \) is a projective object in \( A \), then \( \mathcal{F}[n](x) \) is also a projective object in \( A_{gr}[n] \).

Proof. (1) We define the functor 
\[
\Gamma : A_{gr}[n] \to \text{End}^n \text{Lex} A, \quad x \mapsto \left( \bigoplus x_i, \bigoplus \psi_1^1, \cdots, \bigoplus \psi_n^m \right)
\]
where \( \text{End}^n \) means the \( n \)-times iteration of the functor \( \text{End} \). Since \( \text{Lex} A \) is Grothendieck abelian, the functor \( \bigoplus \) is exact and therefore \( \Gamma \) is an exact functor. Moreover for a morphism \( f : x \to y \) in \( A_{gr}[n] \), the condition \( \Gamma(f) = 0 \) obviously implies the condition \( f = 0 \). Hence \( \Gamma \) is faithful. We can easily check that for any object \( x \) in \( A \), \( \Gamma(\mathcal{F}[n](x)) = x \) which is a noetherian object in \( \text{End}^n \text{Lex} A \) by 2.9. Therefore \( \mathcal{F}[n](x) \) is noetherian by 2.6.

(2) We put 
\[
z_1 = \text{Im}(\bigoplus_{k=0}^{l} \mathcal{F}[n](x_k)(-k) \to x).
\]
Let us consider the ascending chain of subobjects in \( x \)
\[
z_1 \hookrightarrow z_2 \hookrightarrow \cdots \hookrightarrow x.
\]
Since \( x \) is a noetherian object, there is a natural number \( m \) such that \( z_m = z_{m+1} = \cdots \). We claim that the canonical morphism 
\[
y := \bigoplus_{k=0}^{i} \mathcal{F}[n](x_k)(-k) \to x
\]
is an epimorphism. If \( k \geq m \), \( y_k \to x_k \) is obviously an epimorphism. If \( k > m \), then we have the equalities 
\[
\text{Im}(y_k \to x_k) = (z_m)_k = (z_k)_k = x_k.
\]
Therefore we get the desired result.

(3) Let us consider the left diagram in \( A_{gr}[n] \) below:

\[
\begin{array}{ccc}
\mathcal{F}[n](x) & \xrightarrow{f} & y \\
\downarrow h & & \downarrow g \\
0 & \xrightarrow{z} & 0
\end{array}
\]

where \( g \) is an epimorphism. Then we have the right diagram in \( A \) above. By projectivity of \( x \), we have the dotted morphism \( h_0 \) which makes the right diagram above commutative. Then by 3.10 we get \( h : \mathcal{F}[n](x) \to y \) which makes the left diagram above commutative. \( \square \)
Definition 3.13 (Canonical filtration). For any object $x$ in $A_{gr}'[n]$, we define the canonical filtration $F_\ast x$ as follows. $F_{-1} x = 0$ and for any $m \geq 0$,

$$
(F_m x)_k = \begin{cases} 
    x_k & \text{if } k \leq m \\
    \sum_{i=(i_1, \ldots, i_n) \in \mathbb{N}^n} \text{Im} \psi_m^i & \text{if } k > m.
\end{cases}
$$

Remark 3.14. Since every object $x$ in $A_{gr}'[n]$ is noetherian, there is the minimal integer $m$ such that $F_m x = F_{m+1} x = \cdots$. In the case, we can easily prove that $F_m x = x$. We call $m$ degree of $x$ and denote it by $\text{deg} x$.

3.2 Koszul homologies

In this subsection, we define the Koszul homologies of objects in $A_{gr}'[n]$ and as its application, we study the $K$-theory of $A_{gr}'[n]$.

Notations 3.15 (Koszul complex). Let $\mathcal{C}$ be an additive category and $n$ a positive integer. For any object $x$ in $\mathcal{C}_{gr}[n]$, we define the Koszul complex $\text{Kos}(x)$ associated with $x$ as follows. For each positive integer $(2)$

For each natural number $(3)$

For any natural number $(4)$

Definition 3.16 (Koszul homologies). Let $\mathcal{E}$ be an idempotent complete exact category and $n$ a positive integer. We put $B := \text{Lex} \mathcal{E}$. We define the family of functors $(T_i : \mathcal{E}_{gr}[n] \to B_{gr}[n])$ by $T_i(x) := H_i(\text{Kos}(x))$ for each $x$. $T_i(x)$ is said to be the $i$-th Koszul homology of $x$. Let us notice that for any conflations $x \to y \to z$ in $\mathcal{E}_{gr}[n]$, we have a long exact sequences

$$
\cdots \to T_{i+1}(z) \to T_i(x) \to T_i(y) \to T_i(z) \to T_{i-1}(x) \to \cdots.
$$

Definition 3.17 (Torsion free objects). An object $x$ in $A_{gr}'[n]$ is torsion free if $T_i(x) = 0$ for any $i > 0$. For each non-negative integer $m$, we denote the category of torsion free objects (of degree less than $m$) in $A_{gr}'[n]$ by $A'_{gr,tf}[n]$ (resp. $A'_{gr,tf,m}[n]$). Since $A'_{gr,tf}[n], A'_{gr,tf,m}[n]$ are closed under extensions in $A'_{gr}[n]$, they become exact categories in the natural way.

Proposition 3.18. For any objects $x$ in $A'_{gr}[n]$ and $y$ in $A$, we have the following assertions.

(1) For any natural number $k$, $F(n)(y)(-k)$ is torsion free.

(2) For each positive integer $s$, the assertion $T_0(x)_k = 0$ for any $k \leq s$ implies $x_k = 0$ for any $k \leq s$.

(3) We have the equality

$$
T_0(F_p x)_k = \begin{cases} 
    0 & \text{if } k > p \\
    T_0(x)_k & \text{if } k \leq p.
\end{cases}
$$

(4) For each natural number $p$, there is a canonical epimorphism

$$
\alpha^p : F[n](T_0(x)_p)(-p) \to F_p x / F_{p-1} x.
$$
(5) For each natural number \( p \), \( T_0(\alpha^p) \) is an isomorphism.

(6) If \( T_1(x) \) is trivial, then \( \alpha^p \) is an isomorphism.

**Proof.** (1) Since the degree shift functor is exact, we have the equality \( T_1(x(-k)) = T_1(x)(-k) \) for any natural numbers \( i, k \). Therefore we shall just check that \( \mathcal{F}[n](y) \) is torsion free. If \( \mathcal{A} \) is the category of finitely generated free \( \mathbb{Z} \)-modules \( \mathcal{F}_Z \) and \( y = \mathbb{Z} \), then \( \mathcal{F}[n](y) \) is just the \( m \)-th polynomial ring over \( \mathbb{Z} \), \( \mathbb{Z}[t_1, \ldots, t_m] \) and \( T_i(\mathcal{F}[n](y)) \) is the \( i \)-th homology group of the Koszul complex associated with the regular sequence \( t_1, \ldots, t_m \). In the case, it is well-known that \( T_i(\mathcal{F}[n](y)) = 0 \) for \( i > 0 \). For general \( \mathcal{A} \) and \( y \), there is the exact functor \( \mathcal{F}_Z \rightarrow \mathcal{A} \), \( \mathbb{Z} \rightarrow y \) which induces \( \text{Ch}(\mathcal{F}_Z^m[n]) \rightarrow \text{Ch}(\mathcal{A}_y^m[n]) \) and \( \text{Kos}(\mathcal{F}[n](\mathbb{Z})) \) goes to \( \text{Kos}(\mathcal{F}[n](y)) \) by the exact functor. Hence we learn that \( T_i(\mathcal{F}[n](y)) = 0 \) for \( i > 0 \).

(2) First notice that we have the equalities

\[
T_0(x)_k = \begin{cases} x_0 & \text{if } k = 0 \\ x_k / \text{Im}(\psi^1, \ldots, \psi^n) & \text{if } k > 0 \end{cases}.
\]

Therefore if \( T_0(x)_k = 0 \) for \( k \leq s \), then we have \( x_0 = 0 \) and \( x_k = \text{Im}(\psi^1, \ldots, \psi^n) \) for \( k \leq s \). Hence inductively we notice that \( x_k = 0 \) for \( k \leq s \).

The assertion (3) follows from direct calculation.

(4) We have the equality

\[
(F_p x / F_{p-1} x)_k \xrightarrow{\sim} \begin{cases} 0 & \text{if } k < p \\ x_p / \text{Im}(\psi^1, \ldots, \psi^n) = T_0(x)_p & \text{if } k = p \end{cases}.
\]

Therefore by [3,10] we have the canonical morphism

\[
\alpha^p : \mathcal{F}[n](T_0(x)_p)(-p) \rightarrow ((F_p x / F_{p-1} x)(p))(p) = F_p x / F_{p-1} x.
\]

One can easily check that the morphism is an epimorphism.

(5) By (1), we have the equalities

\[
F_p x / F_{p-1} x \xrightarrow{\sim} T_0(\mathcal{F}[n](T_0(x)_p)(-p))_k \xrightarrow{\sim} \begin{cases} 0 & \text{if } k \neq p \\ x_p / \text{Im}(\psi^1, \ldots, \psi^n) & \text{if } k = p \end{cases}
\]

and \( T_0(\alpha^p)_p = \text{id} \). Hence we get the assertion.

(6) Let \( K^p \) be the kernel of \( \alpha^p \), we have short exact sequences

\[
K^p \rightarrow \mathcal{F}[n](T_0(x)_p)(-p) \rightarrow F_p x / F_{p-1} x,
\]

\[
F_p x / F_{p-1} x \rightarrow F_p x \rightarrow F_p x / F_{p-1} x.
\]

We call the long exact sequences of Koszul homologies associated with short sequences above (I), (II) respectively. By (1) and the assertions (1) and (5), we have the isomorphism

\[
T_1(F_p x / F_{p-1} x) \xrightarrow{\sim} T_0(K^p).
\]

We claim that the following assertion.

**Claim.** \( T_1(F_p x / F_{p-1} x) = 0 \) and \( T_1(F_p x) = 0 \).
We prove the claim by descending induction of \( p \). For sufficiently large \( p \), we have \( T_1(F_p x) = T_1(x) \) and therefore it is trivial by the assumption. Then by (2) and (3), we have
\[
T_0(K^p) = T_1(F_p x/F_{p-1} x) = 0.
\]
Therefore by (2), we have \( K^p = 0 \). By (1) and (1), we have isomorphisms
\[
0 = T_2(F[n](T_0(F_p x/F_p^{-1} x)) \sim T_2(F_p x/F_{p-1} x).
\]
By (2), we get \( T_1(F_{p-1} x) = 0 \). Hence we prove the claim and by (2), we get the desired result.

**Theorem 3.19.** We have the canonical isomorphism
\[
\mathbb{Z}[\sigma] \otimes_{\mathbb{Z}} K(A) \xrightarrow{\sim} K(A'_{gr}[n])
\]
which makes the diagram below commutative for any natural number \( k \):
\[
\begin{array}{ccc}
K(A) & \xrightarrow{\sigma^k} & K(A')_{gr}[n] \\
\downarrow & & \downarrow \\
\mathbb{Z}[\sigma] \otimes_{\mathbb{Z}} K(A) & \xrightarrow{\sim} & K(A'_{gr}[n]).
\end{array}
\]

**Proof.** The inclusion functor \( A'_{gr,tf,m}[n] \to A'_{gr}[n] \) induce the isomorphism on their \( K \)-theories by (3.12), (3.18) and Corollary 3 of the resolution theorem in [Qui73]. For each \( m \), there are exact functors
\[
a : A'_{gr,tf,m}[n] \to A \times_{m+1}, \ x \mapsto (T_0(F_p x)_k)_{0 \leq k \leq m},
\]
\[
b : A \times_{m+1} \to A'_{gr,tf,m}[n], \ (x_k)_{0 \leq k \leq m} \mapsto \bigoplus_{k=0}^m F[n](x_k)(-k).
\]
Obviously \( ab \) induces the identity map on their \( K \)-theories. On the other hand, any \( x \) in \( A'_{gr,tf,m}[n] \) has an exact characteristic filtration \( F_n x \) with \( F_p x/F_{p-1} x \sim F[n](T_0(F_p x)_p)(-p) \) by (3.18) (6), so applying Corollary 2 of the additivity theorem in [Qui73], we learn that \( ba \) also induces the identity map on their \( K \)-theories. Therefore we have the isomorphism
\[
K(A'_{gr,tf,m}[n]) \xrightarrow{\sim} \bigoplus_{i=0}^m K(A) \sigma^i.
\]
Finally by taking the inductive limit, we get the desired isomorphism.

**4 Main theorem**

In this section, let us fix an essentially small noetherian abelian category \( A \) which has enough projective objects. There is an exact functor \(- \otimes_{A} \mathbb{Z}[t]\) from \( A \) to \( A[t] \), \( a \mapsto a[t] \). The purpose of this section is to study the induced map from \(- \otimes_{A} \mathbb{Z}[t]\) on \( K \)-theory. More precisely, we will prove the following theorem.

**Theorem 4.1.** The functor \(- \otimes_{A} \mathbb{Z}[t] : A \to A[t]\) induces the isomorphism on their \( K \)-theories
\[
K(A) \xrightarrow{\sim} K(A[t]).
\]
4.1 Nilpotent objects in $\mathcal{A}_{\text{gr}}'[2]$

In this subsection, we will define the category $\mathcal{A}_{\text{gr},\text{nil}}'[2]$ of nilpotent objects in $\mathcal{A}_{\text{gr}}'[2]$. We also study the relationship $\mathcal{A}_{\text{gr}}'[2]$ with $\mathcal{A}[t]$ and calculate the $K$-theory of $\mathcal{A}_{\text{gr},\text{nil}}'[2]$. For simplicity in this subsection we write $\psi = \psi^1$ and $\phi = \psi^2$ and for any object $x$ in $\mathcal{A}$, we write $F[2](x)$ by $x[\psi, \phi]$.

**Definition 4.2.** An object $x$ in $\mathcal{A}_{\text{gr}}'[2]$ is (ψ-) nilpotent if there is an integer $n$ such that

$$\psi^m_x = 0$$

for any non-negative integer $k$. We write the full subcategory of nilpotent objects in $\mathcal{A}_{\text{gr}}'[2]$ by $\mathcal{A}_{\text{gr},\text{nil}}'[2]$.

**Lemma 4.3.** The category $\mathcal{A}_{\text{gr},\text{nil}}'[2]$ is a Serre subcategory of $\mathcal{A}_{\text{gr}}'[2]$. In particular $\mathcal{A}_{\text{gr},\text{nil}}'[2]$ is an abelian category.

**Proof.** The assertion that $\mathcal{A}_{\text{gr},\text{nil}}'[2]$ is closed under sub and quotient objects and finite direct sum is easily proved. Now we intend to prove the following assertion. For a short exact sequence $0 \to y \to z$ in $\mathcal{A}_{\text{gr}}'[2]$, let $i, j$ be integers such that $\psi^i_x = 0$ and $\psi^j_x = 0$. Then we can easily prove that $\psi^{i+j} = 0$. Therefore $\mathcal{A}_{\text{gr},\text{nil}}'[2]$ is closed under extensions in $\mathcal{A}_{\text{gr}}'[2]$. □

**Proposition 4.4.** If $\mathcal{A}$ has enough projective objects, then there is a canonical isomorphism

$$\mathcal{A}_{\text{gr}}'[2]/\mathcal{A}_{\text{gr},\text{nil}}'[2] \cong \mathcal{A}[t].$$

**Proof.** We define the functor

$$\Theta : \mathcal{A}_{\text{gr}}'[2] \to \text{End Lex } \mathcal{A}, \quad x \mapsto (\text{Coker}(\bigoplus_{n=0}^{\infty} x_n \xrightarrow{id-\psi_x} \bigoplus_{n=0}^{\infty} x_n), \phi_x)$$

where $\psi_x = \bigoplus_{n=0}^{\infty} \psi_n$ and $\phi_x = \bigoplus_{n=0}^{\infty} \phi_n$. For any object $x$ in $\mathcal{A}_{\text{gr},\text{nil}}'[2]$, assume that $\psi^m_x = 0$ for any non-negative integer $k$. Then $\sum_{i=0}^{m-1} \psi^i_x$ is the inverse map of $id-\psi_x$. Therefore $\Theta(x) = 0$.

Next we prove that $id-\psi_x : \bigoplus_{n=0}^{m} x_n \to \bigoplus_{n=0}^{m} x_n$ is a monomorphism. Let $K$ be the kernel of $id-\psi_x$. Assume that $K$ is not the zero object. Then there is the maximum integer $m$ such that $K \cap \bigoplus_{n=0}^{\infty} x_n = 0$. Then we have equalities

$$0 = (id-\psi_x)(K \cap x_{m+1}) \cap x_{m+1} = K \cap x_{m+1}.$$

It contradicts the maximality of $m$. Therefore for any conflation $x \to y \to z$ in $\mathcal{A}_{\text{gr}}'[2]$, by applying the snake lemma to the commutative diagram below, we notice that $\Theta$ is an exact functor.

$$
\begin{array}{ccc}
\bigoplus x_i & \longrightarrow & \bigoplus y_i \\
\psi_x & \downarrow & \psi_y \\
\bigoplus x_i & \longrightarrow & \bigoplus z_i \\
\end{array}
$$

\[\begin{array}{ccc}
\bigoplus y_i & \longrightarrow & \bigoplus z_i \\
\psi_x & \downarrow & \psi_y \\
\bigoplus x_i & \longrightarrow & \bigoplus z_i \\
\end{array}\]
Moreover for any object \(x\) in \(\mathcal{A}\) and any positive integer \(k\), \(\Theta(x[\psi, \phi](-k)) = x[t]\). Therefore by [2.11] and [3.12] (2), we notice that \(\Theta\) induces the exact functor

\[
\Theta' : \mathcal{A}'_{\text{gr}}[2]/\mathcal{A}'_{\text{gr, nil}}[2] \to \mathcal{A}[t].
\]

Since \(\mathcal{A}\) has enough projective objects, for any object \(x\) in \(\mathcal{A}[t]\), there is a finite presentation

\[
a[t] \xrightarrow{\sum_{i=0}^m c_i^t} b[t] \to x \to 0
\]

where \(a\) and \(b\) are projective objects in \(\mathcal{A}\) and therefore \(a[t]\) and \(b[t]\) are also projective objects in \(\mathcal{A}[t]\) by [2.14]. (For the notation \(\sum_{i=0}^m c_i^t\), see [2.12]) Then we define the functor

\[
\Delta(x) := \text{Coker}(a[\psi, \phi](-m) \xrightarrow{\sum_{s=0}^m c_s^t \phi^s} b[\psi, \phi]).
\]

(For the notation \(\sum_{s=0}^m c_s^t \phi^s\), see [3.11].) Since \(a[\psi, \phi]\) and \(b[\psi, \phi]\) are projective in \(\mathcal{A}'_{\text{gr}}[2]\) by [3.12] (3), the association \(\Delta\) is well-defined and it gives the inverse functor of \(\Theta'\).

**Corollary 4.5.** We have a fibration sequence

\[
K(\mathcal{A}'_{\text{gr, nil}}[2]) \to K(\mathcal{A}'_{\text{gr}}[2]) \to K(\mathcal{A}[t]).
\]

**Proposition 4.6.** The inclusion functor \(\mathcal{A}'_{\text{gr}}[1] \hookrightarrow \mathcal{A}'_{\text{gr, nil}}[2]\) defined by \((x, \psi^1) \mapsto (x, \psi^1, \phi = 0)\) induces the isomorphism on their \(K\)-theories.

**Proof.** First notice that \(\mathcal{A}'_{\text{gr}}[1]\) is closed under admissible sub and quotient objects in \(\mathcal{A}'_{\text{gr, nil}}[2]\). Moreover for any \(x\) in \(\mathcal{A}'_{\text{gr, nil}}[2]\), let us consider the filtration \(\{\text{Im } \psi^k\}_{k \in \mathbb{N}}\) of \(x\). Then for each \(k\), \(\text{Im } \psi^k / \text{Im } \psi^{k+1}\) is isomorphic to an object in \(\mathcal{A}'_{\text{gr}}[1]\). Therefore we get the desired result by the dévissage theorem.

**Corollary 4.7.** We have the canonical isomorphism

\[
K(\mathcal{A}) \otimes_\mathbb{Z} \mathbb{Z}[\sigma] \cong K(\mathcal{A}'_{\text{gr, nil}}[2]).
\]

### 4.2 The proof of main theorem

In this subsection, we will finish the proof of the main theorem. The key lemma is the following.

**Lemma 4.8.** There is the commutative diagram below

\[
\begin{array}{ccc}
\mathbb{Z}[\sigma] \otimes_\mathbb{Z} K(\mathcal{A}) & \longrightarrow & K(\mathcal{A}'_{\text{gr, nil}}[2]) \\
(1-\sigma) \otimes_{\text{id}} & & \downarrow \\
\mathbb{Z}[\sigma] \otimes_\mathbb{Z} K(\mathcal{A}) & \longrightarrow & K(\mathcal{A}'_{\text{gr}}[2])
\end{array}
\]

**Proof.** An object \(a\) in \(\mathcal{A}\) goes to \((a[\psi], \psi, 0)\) by the functors \(\mathcal{A} \to \mathcal{A}'_{\text{gr, nil}}[2] \to \mathcal{A}'_{\text{gr}}[2]\) and goes to \(a[\psi, \phi]\) by the functor \(\mathcal{F}'[2] : \mathcal{A} \to \mathcal{A}'_{\text{gr}}[2]\). Moreover let us notice that the functor \(\mathcal{F}'[2](-k) : \mathcal{A} \to \mathcal{A}'_{\text{gr}}[2]\) induces \(\mathbb{Z}[\sigma] \otimes_\mathbb{Z} K(\mathcal{A}) \xrightarrow{\psi} \mathbb{Z}[\sigma] \otimes_\mathbb{Z} K(\mathcal{A}) \xrightarrow{\sim} K(\mathcal{A}'_{\text{gr}}[2])\) by [3.19]. On the other hand, for each object \(a\) in \(\mathcal{A}\), there is an exact sequence in \(\mathcal{A}'_{\text{gr}}[2]\)

\[
a[\psi, \phi](-1) \xrightarrow{\phi^*} a[\psi, \phi] \to (a[\psi], \psi, 0).
\]

By the additivity theorem, this implies that the diagram in the statement is commutative. \(\square\)
Proof of 4.1. The assertion follow from the commutative diagram of exact sequences below.

\[
\begin{array}{c}
\mathbb{Z}[\sigma] \otimes_{\mathbb{Z}} K(A) \xrightarrow{(1-\sigma) \otimes \text{id}} \mathbb{Z}[\sigma] \otimes_{\mathbb{Z}} K(A) \rightarrow K(A) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
K(A_{\text{gr,nil}}) \rightarrow K(A_{\text{gr}}) \rightarrow K(A'[t])
\end{array}
\]