Relating the Newman–Penrose constants to the Geroch–Hansen multipole moments

Thomas Bäckdahl
School of Mathematical Sciences, Queen Mary, University of London
E-mail: t.backdahl@qmul.ac.uk

Abstract. In these proceedings we investigate the relation between the Newman-Penrose constants and the Geroch-Hansen multipole moments for stationary spacetimes. This is a short description of the results of the paper [1] published in Classical and Quantum Gravity. This relation gives a simple characterization of which stationary spacetimes have vanishing Newman-Penrose constants.

1. Introduction
The Newman–Penrose (NP) constants were defined by Newman and Penrose in [2]. They are quantities of quadrupolar order defined on the null-infinities. Furthermore, they turn out to be conserved under time translations. Even though they have been studied for a long time, their meaning is still not fully understood. Lately it has been disputed whether the NP-constants are zero for stationary spacetimes or not. It has been shown that they are zero for all algebraically special stationary spacetimes including the Kerr solution [3]. The original paper [2] by Newman and Penrose, gives expressions of the NP-constants in terms of multipole moments. It is unclear however, how these moments were defined, if they are coordinate independent and if different moments can be specified independently. The Geroch–Hansen multipole moments have these properties, but were defined later [4, 5]. Therefore, this work is intended to clearly settle the matter by expressing the NP-constants in terms of the Geroch–Hansen multipole moments. These multipole moments also give a possibility of physical interpretation.

The Geroch–Hansen multipole moments can be freely specified under a simple convergence condition. That is, for any given choice of multipoles, satisfying the convergence condition, there is a unique stationary spacetime with these multipole moments. This was shown in [1] for the stationary axisymmetric case. Recently Herberthson [6] showed this for the general static case using results of Friedrich [7]. The result of Friedich states that for static spacetimes one can freely specify null data under a convergence condition. These null data are related to the multipole moments, but the relation is fairly complicated. The results by Friedrich have been extended to the stationary case by Aceña [8]. Hopefully the results by Herberthson can also be extended to the stationary case, but for now it is still an open problem.

The aim of this work is to express the NP-constants in terms of well studied quantities, namely the Geroch-Hansen multipole moments, and investigate if they vanish in general. These proceedings are not intended to give a complete argument, only to introduce the necessary notation and present the result. For a more complete exposition, see the full paper [1].
2. Notation

2.1. Asymptotic expansion

We will use Bondi–Sachs coordinates \((u, r, \zeta, \bar{\zeta})\), \(\zeta = e^{i\phi} \cot \theta\) and make expansions in terms of \(r^{-1}\). The Weyl-curvature is then expressed as

\[
\Psi_0 = \Psi_0^0 \frac{1}{r^5} + \Psi_0^1 \frac{1}{r^6} + O(r^{-7}), \\
\Psi_1 = \Psi_1^0 \frac{1}{r^5} + \Psi_1^1 \frac{1}{r^6} + O(r^{-7}), \\
\Psi_2 = \Psi_2^0 \frac{1}{r^5} + \Psi_2^1 \frac{1}{r^6} + O(r^{-7}).
\]

2.2. Newman–Penrose constants

The NP constants \(\{G_m\}\) are defined as

\[
G_m = \int_0^{2\pi} \int_0^\pi \Psi_0^1 \frac{1}{2Y_{2,m}} \sin \theta d\theta d\phi = \int_0^{2\pi} \int_0^\pi \left(\frac{10}{3} \Psi_0^1 - 5 \Psi_0^0 \Psi_0^0\right) \frac{1}{2Y_{2,m}} \sin \theta d\theta d\phi,
\]

where \(Y_{j,m}\) are spin-weighted spherical harmonics. The second expression is useful for the actual computation.

2.3. Asymptotic flatness

Now consider a conformal compactification \(V\) of the 3-manifold of trajectories of \(t^a\) (the time-like Killing vector) with metric \(h_{ab} = \Omega^2(-\lambda g_{ab} + t^a t^b)\). We want to choose \(\Omega\) such that we can add a point \(\Lambda\) (the infinity point) such that \(h_{ab}\) extends smoothly to \(\Lambda\). We also demand

\[
\Omega = 0, \quad D_a \Omega = 0, \quad D_a D_b \Omega = 2 h_{ab} \quad \text{at} \, \Lambda,
\]

where \(D_a\) is the covariant derivative on \(h_{ab}\).

2.4. Relativistic multipole moments

Define the complex potential

\[
P = \frac{1 - \lambda - i\omega}{(1 + \lambda + i\omega)\sqrt{\Omega}}.
\]

The Geroch-Hansen multipole moments are given by the limits of

\[
P_{a_1...a_n} = C \left[ D_{a_1} P_{a_2...a_n} - \frac{(n-1)(2n-3)}{2} R_{a_1a_2} P_{a_3...a_n} \right],
\]

as one approaches \(\Lambda\). Here \(C[\cdot]\) represents the totally symmetric and trace-free part.

3. Development of results

We expand \(\Psi_0^0, \Psi_1^0\) and \(\Psi_2^0\) in terms of spin-weighted spherical harmonics:

\[
\Psi_0^0 = \sum_{m=-2}^2 A_{m,2} Y_{2,m} = \sqrt{5} \frac{A_{-2} + 2\zeta A_{-1} + \sqrt{6} \zeta^2 A_0 + 2\zeta^3 A_1 + \zeta^4 A_2}{2\sqrt{\pi}(1 + \zeta^2)},
\]

\[
\Psi_1^0 = \sum_{m=-1}^1 B_{m,1} Y_{1,m} = -\sqrt{3} \frac{B_{-1} + \sqrt{2} \zeta B_0 + \zeta^2 B_1}{2\sqrt{\pi}(1 + \zeta)},
\]

\[
\Psi_2^0 = C.
\]
Here $C$ is real, $B_m$ and $A_m$ are complex.

The time-like Killing vector is computed to

$$t^a = \frac{\partial}{\partial u} + \mathcal{O}(r^{-5}) \frac{\partial}{\partial r} + \mathcal{O}(r^{-5}) \frac{\partial}{\partial \zeta} + \mathcal{O}(r^{-5}) \frac{\partial}{\partial \bar{\zeta}}.$$  \hspace{1cm} (5)

The following choice of conformal factor turns out to adequately simplify the computations

$$\Omega = (r^{-1} - C r^{-2} + \frac{11}{8} C^2 r^{-3})^2.$$ \hspace{1cm} (6)

We need good coordinates to verify the smoothness conditions. To this end we choose asymptotically Euclidean harmonic coordinates $(x, y, z)$, $\rho^2 = x^2 + y^2 + z^2$. The conformal metric and the conformal factor are then found to be

$$h_{ab} = dx^2 + dy^2 + dz^2 + \mathcal{O}(\rho^3), \quad \Omega = \rho^2 + \frac{1}{2} C^2 \rho^4 + \mathcal{O}(\rho^5).$$ \hspace{1cm} (7)

The limits in the definition of the multipole moments are found to be

$$\lim_{\rho \to 0} P = -C$$

$$\lim_{\rho \to 0} P_a = \frac{\sqrt{6}}{210 \pi} (B_{-1} - B_1) dx - \frac{i \sqrt{6}}{12 \sqrt{2}} (B_{-1} + B_1) dy + \frac{\sqrt{3}}{6 \sqrt{2}} B_0 dz$$

$$\lim_{\rho \to 0} P_{ab} = \frac{\sqrt{5}}{210 \sqrt{3}} (\sqrt{6} A_0 - 3 A_2 - 3 A_{-2}) dx^2 + \frac{i \sqrt{5}}{4 \sqrt{2}} (-A_2 + A_{-2}) dxdy + \frac{\sqrt{5}}{4 \sqrt{2}} (A_1 - A_{-1}) dydz - \frac{\sqrt{30}}{12 \sqrt{2}} A_0 dz^2$$ \hspace{1cm} (8)

By comparing the limits (8) with the definition of the moments $(M, C_a, Q_{ab})$, one can express $A_m$, $B_m$ and $C$ in terms of the multipole moments. Using this knowledge, one can compute the integrals (1).

4. Conclusion

The NP constants are

$$G_{-2} = -2 \sqrt{\frac{5}{\pi}} (3 C_y^2 - 3 C_x^2 + MQ_{xx} - MQ_{yy} + 2 i MQ_{xy} - 6 i C_x C_y),$$

$$G_{-1} = -4 \sqrt{\frac{5}{\pi}} (i MQ_{yz} - 3 C_x C_z - 3 i C_y C_z + MQ_{xz}),$$

$$G_0 = 2 \sqrt{30 \pi} (-C_x^2 - C_y^2 + 2 C_z^2 + MQ_{xx} + MQ_{yy}),$$

$$G_1 = -4 \sqrt{\frac{5}{\pi}} (i MQ_{yz} + 3 C_x C_z - 3 i C_y C_z - MQ_{xz}),$$

$$G_2 = -2 \sqrt{\frac{5}{\pi}} (3 C_y^2 - 3 C_x^2 + MQ_{xx} - MQ_{yy} - 2 i MQ_{xy} + 6 i C_x C_y).$$

As expected, this is the same form as in the original paper by Newman and Penrose [2], i.e., linear combinations of dipole squared and monopole times quadrupole. It is easy to see that the NP-constants are independent of the choice of conformal factor, hence invariant under translations. As the NP-constants are expansion coefficients for spin-weighted spherical harmonics, they will depend on the spin-frame through.

For the axisymmetric case we see that $G_{-2} = G_{-1} = G_1 = G_2 = 0$ and $G_0 = 2 \sqrt{30 \pi} (2 C_z^2 - MQ_{zz})$, where $Q_{zz} = -2 Q_{xx} = -2 Q_{yy}$ is the zz component of the quadrupole.

We can conclude that the NP-constants are in general not zero, but for some important solutions they are. For instance the Kerr solution has $C_z = i M a$, $Q_{zz} = -2 Q_{xx} = -2 Q_{yy} = -2 M a^2$, and all other components of $C_a$ and $Q_{ab}$ are zero. This yields the well known fact that all NP-constants are zero for the Kerr solution. In fact they are zero for all stationary, algebraically special solutions [3].
5. Acknowledgements
This work was supported by the Wenner-Gren foundations.

References
[1] Bäckdahl T 2009 Class. Quantum Grav. 26 175021
[2] Newman E T and Penrose R 1968 Proc. Roy. Soc. Lond. A 305 175
[3] Wu X and Shang Y 2007 Class. Quantum Grav. 24 679
[4] Geroch R 1970 J. Math. Phys. 11 2580
[5] Hansen R O 1974 J. Math. Phys. 15 46
[6] Herberthson M 2009 Class. Quantum Grav. 26 215009
[7] Friedrich H 2007 Ann. Henri Poincaré 8 817
[8] Aceña A E 2009 Ann. Henri Poincaré 10 275