Analysis for the reconstruction of attractors by reconstruction of phase spaces using delay times

F Mesa\(^1\), J R González Granada\(^1\), and G Correa Vélez\(^1\)

\(^1\) Departamento de Matemáticas, Universidad Tecnológica de Pereira, Pereira, Colombia

E-mail: femesa@utp.edu.co

Abstract. Being able to estimate the behavior of a system from observable data is one of the great difficulties that any system presents. This problem presents a challenge for researchers who perform scenario estimation and forecasting. In most problems it is proposed to perform data analysis, but in this article, we propose to perform synthesis in such a way that a diffeomorphic attractor is constructed that models the system. In the treatment of the analysis, we start from the inputs and assume some equations that describe the system, in the case of synthesis the most important thing is the data produced by the system, since these are real with some associated noise, so from those data and using Takens' theorem, we can build an attractor that models the system we model in a more real way.

1. Introduction

Very often there are phenomena in which the information necessary to express the behavior of the system cannot be visualized as a set of equations; this is the case in particular of the mathematical model that allows us to determine the population size of the predator and prey species that coexist, which is expressed through the use of the Lotka-Volterra equations [1,2]; many of these phenomena can have a large number (sometimes tens) of variables that interact in their process and modeling this becomes impractical. Therefore, the idea is to try to understand how this phenomenon works and consider only the set of variables that are basic in the process.

Based on that purpose, we proceed to model these phenomena in such a way that they are apparently simple; but the resulting system of equations that model it can be in general difficult to handle. In this way, if we establish a set of assumptions or hypotheses to the different equations, it is obtained that these systems will behave very differently since initially the system converges to an equilibrium point, which is a very particular state; but it would change its convergence towards the limit cycles, which are a group of states that are periodically repeating themselves or whose behavior is totally disconcerting and irregular. Thus, we have the presence of attractors that correspond to the groups where the solutions of the system converge [3].

The difficulty that results immediately when trying to describe the basic functioning of the phenomenon is due to the fact that quantitative information is not available on all the variables involved and information is only known about one or a few of them [4], therefore, the need arises to propose a methodology for the construction of attractors of deterministic systems in which there is not all the information necessary for the approach of the equations that model the system.
2. Phase space

An elegant and very powerful way in which a dynamic system is represented graphically, we find it in a phase space, which is obtained by identifying how many variables are essential in the construction of the original system and thus we build or construct a space (phase space) with as many dimensions as there are variables. The different coordinate axes of this space are used to represent each of the variables defined in the system, and thus with this vector structure we can determine all those states that allow the modeling of the system [5].

We can obtain a graphic description of the state of the system over time based on the points obtained from the values given by the variables evaluated at different times. The curve or trajectory plotted in this phase space obtained in this way represents an orbit corresponding to a state. With this representation of the system, we can then make a very good qualitative description of the temporal evolution of the model being studied. We will use the delay time method and thus reconstruct the phase space. This is achieved by constructing the vector shown in Equation (1).

\[ x(t_i), \ i = 1, ..., N. \]  

Obtained from a time series of a variable scalar. We can describe by means of Equation (2) [6] in a time \( t_i \) the corresponding phase space.

\[ X(t_i) = [x(t_i), x(t_i + \tau), x(t_i + 2\tau), ..., x(t_i + (m-1)\tau)]. \]  

In this Equation (2), each consecutive \( x \) corresponds to one of the observations, and \( i \) takes values from 1 to \( M, N - (m-1) \). \( \tau \) is the delay time, \( m \) is the embedding dimension and \( M = N - (m-1) \). \( \tau \) is the number of states (points) in the phase space. It is known that if the calculation of these parameters is correct, we can use the embedding theorem to reconstruct a dynamic whose formulation is equivalent to the dynamics of an attractor at the origin of the phase space, thus preserving some characteristics known as invariants of the system.

The embedding theorem tells us which are the criteria that the delay time \( \tau \) must satisfy for the reconstruction of the attractors; these are:

- That when comparing the sampling time of the series with the delay time, one is a multiple of the other.
- That the delay time is not too short or too small as \( x(t_i) \) and \( x(t_i + \tau) \) (the coordinates to be used in the data reconstruction vector) not be independent enough. This implies that it will not be possible to have an evolution in the state system.
- Do not choose the value of the delay time \( \tau \) too large, since any relationship between the coordinates \( x(t_i) \) and \( x(t_i + \tau) \) will be lost numerically and will be insignificant.

Based on the above, for the calculation of the delay time, we will propose in this article the application the use of the function that auto-correlates the sample. This is defined for a time series scalar \( x_n \) of \( N \) samples as Equation (3) and Equation (4).

\[ \rho(\tau) = \frac{\sum_{n=1}^{N}(x_{n+\tau} - \bar{x})(x_n - \bar{x})}{\sum_{n=1}^{N}(x_n - \bar{x})^2}, \]  

\[ \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x(t_i), \]  

where \( \bar{x} \) denotes the sample mean of the series \( x_n \). The criterion for the selection of \( \tau \) is to choose the lag time where \( \rho(\tau) \) reaches its first zero. In this way \( x_n \) and \( x_{n+\tau} \) will be linearly incorrect. In [7] it has been found that this criterion does not work in all situations, because long memory processes take a
long time to reach the value of zero for \( \rho(\tau) \), which makes their use impractical because Data samples are finite. Therefore, an alternative criterion is suggested which consists in choosing the lag time \( \tau \) where a value of \( 1/e \) is reached. This method is more robust to choose an adequate lag, however there is no evidence to show that \( 1/e \) is a universal factor of the autocorrelation function criterion for the choice of adequate lags [8].

When obtaining the lag time, it is sought to determine the entire global dimension where there are the necessary number of coordinates to deploy the observed orbits and eliminate the overlaps produced by the projection of the attractor in a space of smaller dimension. For this, the data set \( x(t) \) is analyzed and observed when these overlaps occur. It might think that the best decision is simply to assign a high dimension, however this may generate some mishaps such as the need for a greater amount of data to ensure a better trajectory modeling, in addition to that with each unit increased in the dimension Increase the amount of noise introduced [9].

Therefore, the smallest dimension that the attractor deploys must be determined. The lowest dimension in which the attractor is deployed without overlapping in the orbits is called the embedding dimension \( d \), where \( d \) is an integer. If you take two samples \( x_1(n) \) and \( x_2(n) \) from the same system, it is not guaranteed that when calculating the dimension, it will be the same for each of the samples. Each measurement with its lags provides a different non-linear combination of the original dynamic variables and can give a global non-linear mapping of the true space in the reconstructed space of dimension, where the smoothness and uniqueness of the trajectories are preserved [10].

For the estimation of the dimension of embedding, the use of the method of near false neighbors (FNN) described in [11] is proposed. The theoretical basis of the method is that for deterministic systems, the near points in the reconstructed phase space remain in this way in larger spaces; this is true if the immersion dimension is high enough to solve determinism [12]. Therefore, if the trajectory is projected over a very small space, there will be self-intersections and it is said that there is a state of false neighbor.

When the dimension of the reconstruction of the phase space increases, the number of paths that intersect the false neighbors decreases; however, if the dimension is large enough, both will disappear in their entirety. In [13] two criteria are mentioned to determine whether the neighbors are false or not. In our case, the fraction \( R(t_i) \) defined as was calculated Equation (5).

\[
R(t_i) = \frac{|x(t_i) - x^{NN}(t_i)|}{|x(t_i) - x^{NN}(t_i)|}, \tag{5}
\]

In practice the threshold of \( R_T \) used to define a false neighbor is a number around 15 [10]. This varies with the number of data for small sets, but to the extent that all regions of the attractor are sampled, the variation of false neighbors with the number of data is very small. If \( R(t_i) > R_T \) the states are considered false, neighbors. As a second criterion of falsehood of neighbors, the fraction can be considered. Equation (6).

\[
\frac{|x(t_i) - x^{NN}(t_i)|}{R_A} \geq A_T, \tag{6}
\]

where \( R_A \) is the attractor radius Equation (7).

\[
R_A^2 = \frac{1}{N} \sum_{i=1}^{N} [x(t_i) - \bar{x}]^2. \tag{7}
\]

3. Attractors

The main objective is to try to predict how the system will evolve, so the greatest interest is focused on seeing the trend of the different orbits that could occur. In this sense, a dynamic system is said to be dissipative if the volume of any set in the phase space decreases over time. That is, given a set of points \( S_0 \) in the phase space, with volume \( V(S_0) \), we will denote by \( S_t = \{ F(t,x) : x \in S_0 \} \) we will say that
the system is dissipative if for any moment of time $t > 0$, it is necessary that $V(S_t) < V(S_0)$, that is to say, the volume in the phase space is contracted under the action of the system of differential equations.

Therefore, in any dissipative system with a set of initial conditions, the trajectories eventually converge to a smaller and smaller volume set until they form a null volume set. In some way, if the system is dissipative, it can be ensured that finally the orbits will tend to approach a small set of states, which is what is formally known as a system attractor. An attractor $A$ is a set of null volume in the phase space that satisfies:

- $A$ is an invariant set: any trajectory that begins in $A$ remains permanently in $A$.
- $A$ attracts a set of initial conditions: there is a set, $U$, of non-zero volume, which contains $A$, such that if $x(0) \in U$, then Equation (8).

$$\lim_{t \to \infty} d(x(t), A) = 0.$$  \hspace{1cm} (8)

That is, $A$ attracts all paths that begin close enough to it. The largest $U$ that satisfies this property is called the attractor base. $A$ is minimal: there is no subset of $A$ that satisfies the above conditions.

For many natural and experimental phenomena, both the phase space and the mathematical description of the system are unknown; in these cases, the reconstruction of attractors is presented as a methodology that allows reconstructing the phase space and developing predictive methods based on incomplete information by means of a signal from the system (understanding as a signal the measurements of one of the variables of the phase space over time).

Suppose the study is a phenomenon for which, despite knowing that it is determined by a system of differential equations with $n$ variables, it is only possible to obtain discrete measures over time on the temporal evolution of one of its variables, which we can assume is the first $x_1$. That is, we can take a sample of values of the form. Equation (9).

$$s_0 = x_1(t), s_1 = x_1(t + \tau), s_2 = x_1(t + 2\tau), ..., s_k = x_1(k\tau),$$  \hspace{1cm} (9)

The theorem demonstrated by Takens is a powerful tool for the estimation of future values since it indicates that the orbit that the states follow in the $n$-dimensional phase space is equivalent (homeomorphic) to the orbit followed by the delay vectors. Therefore, the dynamics of the system and its behavior can be described by analyzing the attractor reconstruction of one of the system variables.

### 4. Description of the model

In order to perform the reconstruction of the phase space of a chaotic system, measurements were taken from a river source for 365 days, where a chaotic system can be understood as a system with periodic behavior. Figure 1 the measurements taken every day for a year are shown.

In Figure 1, the system exhibits aperiodic behavior, therefore, it is feasible to perform the reconstruction of the phase space for the estimation of future values of the system. For the choice of the lag factor or delay time $\tau$ the Equation (1) was used, for this the autocorrelation function was plotted and the value closest to $1/e$ was searched; as shown in Figure 2. In Figure 2, the first value closest to $1/e$ occurs at $\tau = 4$ therefore this value will be selected as the delay time for the reconstruction of the phase space; once the delay time is calculated, the embedding dimension is calculated from the Equation (3). For which a value $m = 14$ was obtained; once the embedding dimension is obtained, it is possible to perform the reconstruction of the time series, and in addition to the phase plane. Using the Takens theorem for the reconstruction of attractors, the reconstruction shown in Figure 3 was obtained.

It should be noted that once the attractor has been built, it is possible to make estimates of future values of the system, however, when it comes to making predictions of distant future values, errors may occur due to the system’s own uniqueness.
5. Conclusions
When it is possible to determine the attractor poles of the system, it is possible to think that for very large time values there is a very good prediction of the system; but this is not guaranteed, however, since the system has periodic orbits, it is possible to estimate the system's close values with a certain degree of veracity. The estimation of future values close to the system can be obtained after building the attractor; but due to the lack of periodicity of the system it is possible that errors will occur if you want to predict distant future values.

References
[1] Ferenti T 2017 Biomedical applications of time series analysis IEEE 30th Neumann Colloquium (NC) (Hungary: IEEE) p 000083
[2] Zhang Y, Tan X, Xi H, Zhao X 2008 Real-time risk management based on time series analysis 7th World Congress on Intelligent Control and Automation (China: IEEE) p 2518
[3] Deloya Vélez A H, González-Hernández H G, Reyes Guerra B 2014 Attractor reconstruction for plethysmographic biosignals International Conference on Electronics, Communications and Computers (CONIELECOMP) (México: IEEE) p 94
[4] Lyle J V, Charlton P H, Bonet E, Chaffey G, Christie M, Nandi M, Aston P J 2017 Beyond HRV: Analysis of ECG signals using attractor reconstruction Computing in Cardiology (CinC) (France: IEEE) p 1
[5] Kong L, Yang C, Wang Y, Gui W 2009 Generic phase space reconstruction method of multivariate time Chinese Control and Decision Conference (China: IEEE) p 3752
[6] Mesa F, Devia D M, Ospina R 2020 Problem of the P-Stockings applied to the location of facilities J. Phys. Conf. Ser. 1674(1) 012003:1
[7] Jiayu L, Guihong B, Hairui W, Xi W, Shilong C 2012 Principal component analysis and identification of power quality disturbance signal phase space reconstructed images Proceedings of the 31st Chinese Control Conference (China: IEEE) p 5229

[8] Xipan L, Zhaojiang Z, Jing H, Weijin D 2010 The study of time-variable law of surfaces deformation based on EMD and phase space reconstruction technique International Conference on E-Business and E-Government (China: IEEE) p 1978

[9] Yong L, Hongwei Z, Yourong L, Han X, Zhigang W, Sluming H 2010 Bearing fault diagnosis based on weighted phase space reconstruction International Conference on Digital Manufacturing & Automation (China: IEEE) p 315

[10] Xie Z, Wang K 2009 Selection of embedding parameters in phase space reconstruction Second International Conference on Intelligent Computation Technology and Automation (China: IEEE) p 637

[11] Jia H, Liang S, Lou S, Sheng X 2019 IEEE Sensors Journal 19(10) 3683

[12] Chen Y, Wei Z, Lu X, Qin B 2016 Behavior analysis of VanDerPol oscillator based on phase space reconstruction methods 9th International Symposium on Computational Intelligence and Design (ISCID) (China: IEEE) p 359

[13] Liu R, Hou Z, Shen J 2009 A method to determine the parameters of phase space reconstruction based on the neural network International Workshop on Chaos- Fractals Theories and Applications (China: IEEE) p 281