A really new way to prove Riemann Hypothesis

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Abstract

In the past 100 years, the research of Riemann Hypothesis meets many difficulties. Such situation may be caused by that people used to study Zeta function only regarding it as a complex function. Generally, complex functions are far more complex than real functions, and are hard to graph. So, people cannot grasp the nature of them easily. Therefore, it may be a promising way to try to correspond Zeta function to real function so that we can return to the real domain to study RH.

In fact, under Laplace transform, \( F(s) = L[f(x)] = \int_0^\infty f(x)e^{-sx}dx \), the whole picture of Zeta function is very clear and simple, and the problem can be greatly simplified. And by Laplace transform, most integral and convolution operations can be converted into algebraic operations, which greatly simplifies calculating and analysis. The first part of this paper points out the essence of Zeta function displayed under Laplace transform, and the second part obtains the error function \( Je(x) - lie(x) \) which equivalent to the error function \( J(x) - li(x) \) got by Riemann before, while this new error function is much simpler than the Riemann’s. The third part estimates the maximum absolute value of the new error function by two ways, thus proving the famous equivalent proposition of Riemann Hypothesis:

\[
[li(x) - \pi(x)] = O(x^{0.5+\varepsilon}), \quad \varepsilon > 0, \quad \pi(x) \text{ is the prime-counting function,} \quad li(x) = \int_0^t \frac{dt}{\log t}, \quad t \neq 1
\]

The fourth part makes some other meaningful discussions and obtains other valuable results.

Key word: Riemann Hypothesis, Zeta function, Laplace transform

1. The essence of Zeta function

In the original definition, Zeta function is \( \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ... \), \( \text{Re}(s) > 1 \)

Riemann extended it to an analytic approach \( \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}dx \), \( s \neq 1 \)

If we use Laplace transform to view Zeta function, a clear picture is shown. Notice: All Zeta functions mentioned later are in original definition and \( \text{Re}(s) > 1 \).

\[
\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ... = 1 + e^{-s \log 2} + e^{-s \log 3} + e^{-s \log 4} + ... 
\]

This represents a series of Dirac delta functions at the points of \( x=0, \log 2, \log 3, \log 4, ... \), which can be recorded as: \( \text{zeta}(x) = L^{-1}[\zeta(s)] = \delta(x) + \delta(x - \log 2) + \delta(x - \log 3) + \delta(x - \log 4) + ... \)
It may be still difficult to understand what this means, but once it is integrated, the truth is clear:

\[ zeta_1(x) = \int_0^x \zeta(t) dt = L^{-1}\left[\frac{1}{s} \zeta(s)\right] = u(x) + u(x - \log 2) + u(x - \log 3) + u(x - \log 4) \ldots \]

This represents a combination of unit step functions start at the points \( x=0, \log 2, \log 3, \log 4, \ldots \), as shown in Figure 1, obviously has an exact upper bound function \( f_+^x(x) = e^x \) and a lower bound function \( f_-^x(x) = e^x - 1 \). It can be regarded as a combination of step functions with constant ordinate and logarithmic contraction from abscissa. This is why many of the series decomposition expressions of \( \zeta(s) \) contain \( \frac{1}{s-1} \), because the main item of real function corresponding to \( \frac{\zeta(s)}{s} \) is \( e^x \), \( L(e^x) = \frac{1}{s-1} \).

Then we let: \( r(x) = e^x - zeta_1(x) \), obviously at \( x = \log(N+c) \), \( 0 \leq c < 1 \), \( r(x) = c \).

We define \( R(s) = L[r(x)] = \frac{1}{s-1} - \frac{1}{s} \zeta(s) \), \ldots \ (1.1)

And it is easy to find: \( L(\zeta(x)) = \eta(s) = (1 - \frac{2}{2^s})\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots \), \( \int_0^x \zeta(t) dt \)

as shown in Figure 2, is the contraction of the horizontal coordinates of the unit square wave function:
2. Get the error function of \( Je(x) - \text{li}(x) \)

In his famous paper, Riemann used \( J(x) \) function to calculate \( \pi(x) \). The definition of \( J(x) \) is:

\[
J(x) = \sum_{p} \frac{1}{k} u(x - p^k) = u(x - 2) + u(x - 3) + \frac{1}{2} u(x - 2^2) + u(x - 5) + u(x - 7) + \frac{1}{3} u(x - 2^3) + \frac{1}{2} u(x - 3^2) + u(x - 11) + u(x - 13) + \frac{1}{4} u(x - 2^4) + \ldots , \text{ where } p = \text{prime number}.
\]

So, by Mobius inverse transform, there is:

\[
\pi(x) = J(x) - \frac{1}{2} J(x^2) - \frac{1}{3} J(x^3) - \frac{1}{5} J(x^5) - \frac{1}{6} J(x^6) - \frac{1}{7} J(x^7) + \ldots \quad \text{...... (2.1)}
\]

Obviously, \( J(x) - \pi(x) \approx \frac{1}{2} J(x^3) \) is the main error of \( J(x) - \pi(x) \). In other words, if we can prove \( |J(x) - \text{li}(x)| = O(x^{0.5+\varepsilon}) \), it is equivalent to prove \( |\text{li}(x) - \pi(x)| = O(x^{0.5+\varepsilon}) \), \( \varepsilon > 0 \).

We define new functions: \( Je(x) = J(e^x) \), \( Je(\log x) = J(x) \), and \( JE(s) = L[Je(x)] \).

\( Je(x) \) can be derived from \( \zeta(s) \). By Euler product formula, \( \zeta(s) = \Pi(1 - p^{-s})^{-1} \), the two sides are logarithmic and multiplied by \( \frac{1}{s} \), \( \frac{1}{s} Je(s) = \frac{1}{s} \log \zeta(s) = -\frac{1}{s} \log \Pi(1 - p^{-s}) = \frac{1}{s} (\frac{1}{2^s} + \frac{1}{2 \cdot 2^s}) + \frac{1}{3 \cdot 3^s} + \ldots + \frac{1}{2 \cdot 3^s} + \frac{1}{3 \cdot 3^s} + \ldots + \frac{1}{5^s} + \frac{1}{2 \cdot 5^s} + \frac{1}{3 \cdot 5^s} + \ldots \), \( Je(x) = \sum_{p} \frac{1}{k} u(x - \log p^k) \).

It is the same as \( J(x) \)'s constant ordinate and logarithmic contraction from abscissa.

For comparing with \( \text{li}(x) \), we must make the same transform on \( \text{li}(x) \). \( \text{li}(x) \) has the series decomposition expression as following, where \( \gamma = 0.5772 \ldots \) is Euler constant:

\[
\text{li}(x) = \gamma + \log \left| \log x \right| + \log x + \frac{\log^2 x}{2 \cdot 2!} + \frac{\log^3 x}{3 \cdot 3!} + \frac{\log^4 x}{4 \cdot 4!} + \ldots \quad \text{The logarithmic contraction of the absissa of } \text{li}(x) \text{ is to change the } \log x \text{ of the upper form to } x, \text{ or } \text{lie}(x) = \text{li}(e^x):
\]

\[
\text{lie}(x) = \gamma + \log x + x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} + \ldots \quad \text{Make Laplace transform to } \text{lie}(x):
\]

\[
LE(s) = L(\text{lie}(x)) = \frac{\gamma}{s} + \frac{1}{s} (\gamma + \log s) + \frac{1}{s^2} + \frac{1}{3 s^4} + \frac{1}{4 s^5} + \ldots = -\frac{1}{s} \log s - \frac{1}{s} \log(1 - \frac{1}{s})
\]

3…15
\[-(1 + \frac{1}{2 \cdot 2!} + \frac{1}{3 \cdot 3!} + ...) \approx \text{lie}(x), \ \text{lie}(x) \ \text{is the integral of} \ \frac{e^x}{x}.

If \ |J(x) - li(x)| = O(x^{0.5+\epsilon_1}), \ \epsilon_1 > 0 , \ \text{that requires} \ |Je(x) - lie(x)| = O(e^{0.5x+\epsilon_2}), \ \epsilon_2 > 0.

Error function: \ L[Je(x) - lie(x)] = \frac{1}{s}\log\zeta(s) + \frac{1}{s}\log(s-1) = \frac{1}{s}\log((s-1)\zeta(s))

= \frac{1}{s}\log\left(\frac{(s-1)\zeta(s)}{s}\right) + \frac{1}{s}\log s. \ \text{For} \ L^{-1}\left(\frac{1}{s}\log s\right) = -\gamma - \log x, \ \text{the order is very low, if the main item}

of error function is \ \mathcal{O}(e^{0.5x+\epsilon_2}), \ \epsilon_2 > 0, \ -\log x - \gamma \ \text{can be ignored. So we pay attention at:}

\[ ER(s) = \frac{1}{s}\log\left(\frac{(s-1)\zeta(s)}{s}\right) = \frac{1}{s}\log\left((s-1)(\frac{1}{s-1} - R(s))\right) = \frac{1}{s}\log(1 - (s-1)R(s)) = -\frac{1}{s}(s-1)R(s) \]

\[-\frac{1}{2s}(s-1)^2R^2(s) - \frac{1}{3s}(s-1)^3R^3(s) - \frac{1}{4s}(s-1)^4R^4(s) - ... - \frac{1}{ks}(s-1)^kR^k(s) - ... \quad (2.2)\]

3. Proof of Riemann Hypothesis

Although we have got the (2.2) formula, it is still very difficult to inverse transform it directly, no mention to other research. Because the function represented by this exact expression contains many vibration and irregular components, so it is too hard to simplify. Fortunately, Riemann Hypothesis does not require the exact expression of the error function, only the good upper and lower bounds of the error function are needed, which are both simple functions and contain no any vibration and irregular components. It provides the possibility for some equivalent methods.

By the knowledge of complex analysis, a function with many vibration and irregular components can be expressed as: \( e^{f_1(x)+ig_1(x)} + e^{f_2(x)+ig_2(x)} + ... + e^{f_k(x)+ig_k(x)} + ... \). The terms of vibration are the phase functions \( e^{ig_k(x)} \); functions \( e^{f_k(x)} \), which represent the modulus length, no longer contains any vibration components. The modulus length corresponds to the extreme value that the function can reach, or the upper and lower bounds of it. If \( g_k(x) = 0 \), then \( e^{f_k(x)} \) is the mean function. If the upper and lower bounds are also oscillatory, it can only show that the vibration components are not completely separated, or the decomposition of non standard forms. Therefore, the standard upper and lower bound functions of \( Je(x) - lie(x) \) also do not contain vibration and irregular terms, so they can be approximated by some simple functions.

Next I will provide two ways to estimate the maximum absolute value of the error function. But at
first, I will explain more on the reasons why I do so.

i) We have 2 methods to calculate \((A + B)^2\): (1) \((A + B)^2 = A \cdot A + A \cdot B + B \cdot A + B \cdot B\), calculate each item and then sum them up; (2) calculate \(A + B = C\), and then \((A + B)^2 = C \cdot C\). In this case, if we do as method (1), that is very complex, even impossible. So method (2) is the only choice.

ii) What we need to estimate is the maximum absolute value of each \(L^{-1}\left[\frac{1}{ks} (s-1)^k R^k(s)\right]\). If a function \(G(s)\) can satisfies \(\left|L^{-1}\left[\frac{1}{ks} (s-1)^k R^k(s)\right]\right|_{\text{max}} \leq L^{-1}\left[\frac{1}{ks} G^k(s)\right]\), then we call \(G(s)\) as a convolution equivalent function. Although \(G(s) \neq (s-1)R(s)\), for our target, it is useful and enough.

**Proof 1:** \(L^{-1}\left[\frac{1}{ks} (s-1)^k R^k(s)\right] = L^{-1}\left[\frac{s^k}{ks} \cdot (s-1)^k R^k(s)\right] < L^{-1}\left[\frac{s^k}{ks} \cdot G(s)\right]\), we try to get \(\frac{G(s)}{s}\) from \(\frac{1}{s}(s-1)R(s) = (1 - \frac{1}{s})R(s) = -\left(\frac{R(s)}{s} - R(s)\right)\). \(\frac{R(s)}{s}\) is equivalent to the integral of \(r(x)\), as in Figure 3. Since \(x = \log N, y = N\), so the area \(x \cdot y = N \log N\). Then we look at the rectangles: the bottom D2 length is \(\log 2\), height is 1; then the upper one D3 length is \(\log 3\), height is 1; ..., and the top Dn length is \(\log N\), height is 1. So the “∥” shadow area at \(x = \log N\) is (the error \(O(1/N^3)\) is ignored):

\[
S = x \cdot y - (D2 + D3 + D4 + .. + Dn) = N \log N - \sum_{m=1}^{N} \log m ,
\]

Using the formula: \(\sum_{m=1}^{N} \log m = N \log N - N + \frac{\log N}{2} + \frac{\log 2 \pi}{2} + \frac{1}{12N} + O\left(\frac{1}{N^{\pi}}\right)\).

\[
S = N \log N - \sum_{m=1}^{N} \log m = N \log N - (N \log N
\]

\[- N + \frac{\log N}{2} + \frac{\log 2 \pi}{2} + \frac{1}{12N}\]

\[
= N - \frac{\log N}{2} - \frac{\log 2 \pi}{2} - \frac{1}{12N}
\]

The integral of \(r(x)\) is the “∥” shadow area:
\[
\int_0^{\log N} r(x) dx = \int_0^{\log N} e^{x} dx - S = N - 1 - N + \frac{\log N}{2} + \frac{\log 2\pi}{2} + \frac{1}{12N} = \frac{\log N}{2} + \frac{\log 2\pi}{2} - 1 + \frac{1}{12N}
\]

So at \( x = \log N \), \( \int_0^{\log N} r(x) dx - r(\log N) = \frac{\log N}{2} + \frac{\log 2\pi}{2} - 1 + \frac{1}{12N} - 0 \), that means:

\[
\int_0^{x} r(t) dt - r(x) = \frac{x}{2} + \frac{\log 2\pi}{2} - 1 + \frac{1}{12e^{x}}, \quad \left( \frac{R(s)}{s} - R(s) \right) = \frac{1}{2s^2} + \frac{\log 2\pi}{2s} - \frac{1}{s} + \frac{1}{12(s+1)}
\]

\[
= \frac{1}{s} \left( \frac{1}{2s} + \frac{\log 2\pi}{2} - 1 + \frac{1}{12(s+1)} \right) - \frac{1}{s} \left( \frac{1}{2s} + \frac{\log 2\pi}{2} - 1 + \frac{1}{12(s+1)} \right), \quad \frac{\log 2\pi}{2} - 1 + \frac{1}{12} = 0.00227...
\]

The small constant 0.00227, its effect is far less than the item \(-\frac{1}{12(s+1)}\) in the case of multiple convolution, so the maximum absolute value of the item \( L^{-1}[\frac{1}{ks} (s-1)^k R^k(s)] \) is:

\[
L^{-1}\left| \frac{1}{ks} (s-1)^k R^k(s) \right|_{\text{max}} \approx L^{-1}\left( \frac{1}{ks} \left( \frac{1}{2s} - \frac{1}{12(s+1)} \right)^k \right) < L^{-1}\left( \frac{1}{ks} \left( \frac{1}{2s} \right)^k \right), \quad \ldots (3.1)
\]

At \( x = \log(N+c) \), \( r(x) \) adds \( c \), the integral of \( r(x) \) adds very small value, \( \int_0^{x} r(t) dt - r(x) \), the negative term is larger. The (3.1) formula is more established. We can calculate as below:

\[
\int_0^{\log(N+c)} r(x) dx - r(\log(N+c)) = N + c - 1 - (S + N(\log(N+c) - \log N) - c
\]

\[
= N - 1 - (N - \frac{\log N}{2} - \frac{\log 2\pi}{2} - \frac{1}{12N}) - N \log(1+c/N) = \frac{\log N}{2} + \frac{\log 2\pi}{2} + \frac{1}{12N} - 1 - N(\frac{c}{N} - \frac{c^2}{2N^2})
\]

\[
= \frac{\log(N+c)}{2} - \frac{\log(1+c/N)}{2} + \frac{\log 2\pi}{2} - 1 - c + \frac{1}{12N} + \frac{c^2}{2N} = \frac{\log(N+c)}{2} + \frac{\log 2\pi}{2} - 1 - c + \frac{1-6c+6c^2}{12(N+c)}
\]

(the error \( O(1/N^2) \) is ignored). That means: \( \int_0^{x} r(t) dt - r(x) = \frac{x}{2} + \frac{\log 2\pi}{2} - 1 - c + \frac{1-6c+6c^2}{12e^{x}} \)

\[
\frac{R(s)}{s} - R(s) = \frac{1}{2s^2} + \frac{\log 2\pi}{2s} - \frac{1}{s} + \frac{1-6c+6c^2}{12(s+1)} = \frac{1}{s} \left( \frac{1}{2s} + \frac{\log 2\pi}{2} - 1 - c + \frac{1-6c+6c^2}{12} - \frac{1-6c+6c^2}{12(s+1)} \right)
\]

For \( 1-6c+6c^2 = 6(c-0.5)^2 - 0.5 \), when \( c = 0.5 \), \( \frac{R(s)}{s} - R(s) = \frac{1}{s} \left( \frac{1}{2s} + \frac{\log 2\pi}{2} - \frac{37}{24} + \frac{1}{24(s+1)} \right) \)

\[
= \frac{1}{s} \left( \frac{1}{2s} - 1.14258 + \frac{1}{24(s+1)} \right); \quad \text{when } c \to 1, c < 1, \quad \frac{R(s)}{s} - R(s) = \frac{1}{s} \left( \frac{1}{2s} - 1.51758 - \frac{1}{12(s+1)} \right).
\]

We can see however the \( c \) is changed from 0 to 1, the (3.1) formula is always established. So the maximum absolute value of the error function is:
\[ |L^{-1}(\text{ER}(s))|_{\text{max}} < L^{-1}\left[ \frac{1}{s} \left( \frac{1}{2s}^{1/2} + \frac{1}{3s}^{1/3} + \frac{1}{4s}^{1/4} + \ldots \right) \right] \]

\[
\frac{(x/2)^3}{3!} + \ldots + \frac{(x/2)^k}{k \cdot k!} + \ldots < \frac{ae^{0.5x}}{x} = a \left( \frac{1}{x} + \frac{1}{2} \cdot \frac{x}{2 \cdot 2!} + \frac{x^2}{2^2 \cdot 3!} + \ldots + \frac{x^k}{2^k \cdot k!} \right)
\]

Then we have \[ \frac{(x/2)^k}{k \cdot k!} < \frac{ax^k}{2^{k+1} \cdot (k+1)!}, \quad a = \frac{2^{k+1}}{k} \]

But we can let \[ x > \frac{3x}{2^{2!}} \]

\[ |Je(x) - \text{lie}(x)|_{\text{max}} = |L^{-1}(\text{ER}(s))|_{\text{max}} - \log x - y < \frac{3e^{0.5x}}{x}, \quad \text{for all } x > 12, \ldots (3.2) \]

**Proof 2:** \[ (s-1)R(s) = (s-1) \left( \frac{1}{s-1} - \frac{\zeta(s)}{s} \right) = 1 - (1 - \frac{1}{s})\zeta(s) = -\zeta(s) - 1 + \frac{\zeta(s)}{s} \]

The Dirac delta function can be regarded as a distribution density function which centralizes its density at one point with its integral equals 1. \( \zeta(s) - 1 = \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots \) is a combination of Dirac delta functions. And it is easy to find:\n
\[ L^{-1} \left[ \frac{1}{s} \left( \frac{1}{2s}^{1/2} + \frac{1}{3s}^{1/3} + \frac{1}{4s}^{1/4} + \ldots \right)^k \right] < L^{-1} \left[ \frac{1}{s} \left( \frac{1}{2s} + \frac{1}{3s} + \frac{1}{4s} + \ldots \right)^k \right] \]

because some density of \( \zeta(s) - 1 = \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots \) are delayed. On the other hand, there should be:\n
\[ L^{-1} \left[ \frac{1}{s} \left( \frac{1}{2s} + \frac{1}{3s} + \frac{1}{4s} + \ldots \right)^k \right] < L^{-1} \left[ \frac{1}{s} \left( \frac{1}{1.5s} + \frac{1}{2s} + \frac{1}{3s} + \ldots \right)^k \right], \quad \text{because some of the density of } \zeta(s) - 1 = \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots \] are advanced.

According to the same logic, we can conclude that: If the density 1 of \( \frac{1}{2^s} \) is scattered to the interval (0, log 2], the density 1 of the \( \frac{1}{3^s} \) is scattered to the interval (log 2, log 3], \ldots, such as:\n
\[ \frac{1}{m} \left[ \left( \frac{1}{m} \right)^s + \left( \frac{2}{m} \right)^s + \ldots + \left( \frac{m}{m} \right)^s \right] = \frac{1}{m} \left[ \left( \frac{1}{m} \right)^s + \left( \frac{2}{m} \right)^s + \ldots + \left( \frac{m}{m} \right)^s \right] \]

When \( m \to \infty \), the maximum value of distribution density function equal to \( e^x \) and \( L[e^x] = \frac{1}{s-1} \).

On the other hand, if the density 1 of the \( \frac{1}{2^s} \) is scattered to the interval (log 2, log 3), and the density 1 of \( \frac{1}{3^s} \) is scattered to the interval (log 3, log 4), \ldots, such as:
When \( m \to \infty \), the function takes the minimum value \( e^x \cdot u(x - \log 2) \), and \( L[e^x \cdot u(x - \log 2)] = \frac{2}{2^s - 1} \).

In convolution sense, the limit value has only two cases:
\[
- \frac{1}{s - 1} + \frac{1}{s} - R(s) = -R(s); \quad \text{or} \quad - \frac{1}{2^s - 1} - \frac{1}{s} - R(s) = -\frac{2}{2^s - 1} + \frac{1}{s} - R(s).
\]

Since \( \left(1 - \frac{2}{2^s}\right)\frac{1}{s - 1} \) represents only a small segment of function \( e^x \) in the interval \([0, \log 2]\), it is obvious contrary to the symbol represented by \(- R(s)\), so the absolute upper bound can only appear in the first case, that is,
\[
\left| L^{-1}\left(\frac{1}{ks} (s - 1)^k R^k(s)\right)\right|_{\max} \leq L^{-1}\left(\frac{1}{ks} R^k(s)\right).
\]

It is easy to prove \( L^{-1}[R^k(s)] < L^{-1}\left[\frac{1}{(2s)^k}\right] : \frac{s-1}{s} \zeta(s) = \left(\frac{1}{s - 1} - \frac{1}{s}\right)\zeta(s) = \frac{1}{s-1}\left(\frac{1}{s} - R(s)\right) \)
\[
L^{-1}\left[\frac{1}{s} - \frac{1}{s - 1}\right]_{x = \log N} = N(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N}) - N = N(\log N + \frac{2}{2N} - \frac{1}{12N^2}) = N \frac{2}{12N} - \frac{1}{12N^2} - \frac{1}{12N^2} = \frac{1}{12N^2} - \frac{1}{12N}.
\]

\[
\frac{R(s)}{s - 1} = \frac{1 - \gamma}{s - 1} + \frac{1}{2s} + \frac{1}{12(s + 1)}, \quad R(s) = \frac{1}{2s} - \frac{1}{6(s + 1)} + \frac{7}{12} - \gamma = \frac{1}{2s} - \frac{1}{6(s + 1)} + 0.0061...
\]

Similarly, the small constant 0.0061..., its effect under multiple convolutions is much less than \( - \frac{1}{6(s + 1)} \). What we need to estimate is the maximum absolute value of \( L^{-1}\left(\frac{1}{s} R^k(s)\right) \). Let \( F(s) = \frac{1}{2s} - \frac{1}{6(s + 1)} \), \( a = 0.0061... \), obviously there is: \( \frac{1}{2} - \frac{1}{6} = \frac{1}{3} \leq L^{-1}(F(s)) = \frac{1}{2} - \frac{1}{6}e^x < \frac{1}{2} \), then we continues:
\[
\frac{1}{s} R^k(s) = \frac{1}{s}(F(s) + a)^k = \frac{1}{s}\left(F^k(s) + C_1 a F^{k-1}(s) + C_2 a^2 F^{k-2}(s) + \ldots + C_k a^{k-1} F(s) + a^k\right)
\]

We can find its highest order is \( L^{-1}\left(\frac{1}{s} F^k(s)\right) < L^{-1}\left(\frac{1}{s} \left(\frac{1}{2s}\right)^k\right) \), and 0.0061 is much less than 1,

that means even \( x \) is a small value, the items \( \frac{1}{s}\left(C_1 a F^{k-1}(s) + C_2 a^2 F^{k-2}(s) + \ldots + C_k a^{k-1} F(s) + a^k\right) \)
represent the real function which much less than the highest order \( L^{-1}\left(\frac{1}{s} F^k(s)\right) \).

We can conclude there is always \( L^{-1}\left(\frac{1}{s} R^k(s)\right) < L^{-1}\left(\frac{1}{s} \left(\frac{1}{2s}\right)^k\right) \) by other method, see the Figure 4.

The blue line is \( r(x) \), it can be decomposed into many different sizes of rectangles. Then we get the integral of \( r(x) \) is the sum of the area of all these small rectangles.

\[
\int_0^x r(t)dt = \frac{1}{2} (\log 2 - \log 1.5 + \log 3 - \log 2.5 + \log 4 - \log 3.5 + ...) + \frac{1}{4} (\log 1.2 - \log 1.4 + \log 2 - \log 1.3)
\]

\[
+ \log 2^{1/2} - \log 2^{1/4} + \log 3 - \log 2^{3/4} + \log 3^{1/2} - \log 3^{1/4} + \log 4 - \log 3^{3/4} + ...) + ... + \frac{1}{2^m} (\log 1 - \log 1^{1/2})
\]

\[
+ \log 1^{4/2m} - \log 1^{3/2m} + ...) + ... \quad \text{Note: the above and below are summation of finite items, until the item is not greater than} \quad x. \quad \text{Since there is always:}
\]

\[
\frac{1}{2^m} (\log 1 - \log 1^{1/2} + \log 1^{4/2m} - \log 1^{3/2m} + ...) < \frac{1}{2^m} (\log 1 - 0 + \log 1^{3/2m} - \log 1^{2/2m} + ...)
\]

so, \( \frac{1}{2^m} (\log 1 - \log 1^{1/2} + \log 1^{4/2m} - \log 1^{3/2m} + ...) < \frac{x}{2 \cdot 2^m} \),

\[
L^{-1}\left(\frac{R(s)}{s}\right) = \int_0^x r(t)dt < \frac{x}{2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^m} + ...\right) = \frac{x}{2} = L^{-1}\left(\frac{1}{2s^2}\right), \quad \text{and} \quad r(x) \text{ is not a very vibrant function, that means the convolution equivalent real function of} \quad r(x) \quad \text{always less than} \quad \frac{1}{2}, \quad \text{and}
\]

\[
L^{-1}\left(\frac{1}{s} R^k(s)\right) < L^{-1}\left(\frac{1}{s} \left(\frac{1}{2s}\right)^k\right) \quad \text{always be established. The same as in Proof 1 page 7, we can obtain:}
\]

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|Je(x) − lie(x)|_{\max} = \left|L^{-1}(ER(s))\right|_{\max} - \log x - \gamma < \frac{3e^{0.5x}}{x}, \quad \text{for all } x > 12, \quad \ldots \quad (3.2)

Formula (3.2) indicates \( |J(x) − li(x)| < \frac{3\sqrt{x}}{\log x} \), from formula (2.1) on page 3, we know:

\[
J(x) − \pi(x) \approx \frac{1}{2} J(\sqrt{x}) \approx \frac{1}{2} li(\sqrt{x}), \quad \text{so we get } \left|\pi(x) + \frac{1}{2} li(\sqrt{x}) - li(x)\right| < \frac{3\sqrt{x}}{\log x}. \quad \text{Compare with page 7,}
\]

\[
li(\sqrt{x}) = li(\log \sqrt{x}) = \int_{\log \sqrt{x}}^{\log \sqrt{x}} \frac{e^t}{t} dt = \int_{\log \sqrt{x}}^{\log \sqrt{x}} \left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots\right) dt = \log t + \frac{t^2}{2 \cdot 2!} + \frac{t^3}{3 \cdot 3!} + \cdots
\]

\[
+ \frac{t^k}{k \cdot k!} + \cdots, \quad b \cdot e^t = b \left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots \frac{t^k}{(k+1)!} + \cdots\right), \quad \text{if } \frac{t^k}{k \cdot k!} < \frac{b \cdot t^k}{(k+1)!}, \quad b > \frac{k+1}{k} = 1 + \frac{1}{k}.
\]

So we can conclude \[
\frac{e^{\log \sqrt{x}}}{\log \sqrt{x}} = \frac{2\sqrt{x}}{\log x} < li(\sqrt{x}) = li(\log \sqrt{x}) < \frac{2e^{\log \sqrt{x}}}{\log \sqrt{x}} = \frac{4\sqrt{x}}{\log x}.
\]

\[
\left|\pi(x) - li(x) + \frac{1}{2} li(\sqrt{x})\right| < \frac{3\sqrt{x}}{\log x}, \quad -\frac{3\sqrt{x}}{\log x} < li(\sqrt{x}) < \pi(x) - li(x) < \frac{3\sqrt{x}}{\log x} - \frac{1}{2} li(\sqrt{x})
\]

\[
-\frac{3\sqrt{x}}{\log x} - \frac{2\sqrt{x}}{\log x} < \pi(x) - li(x) < \frac{3\sqrt{x}}{\log x} - \frac{\sqrt{x}}{\log x}, \quad \frac{5\sqrt{x}}{\log x} < \pi(x) - li(x) < \frac{2\sqrt{x}}{\log x}.
\]

**The famous equivalent proposition** \( |\pi(x) − li(x)| = O(x^{0.5+\varepsilon}), \quad \varepsilon > 0 \) of Riemann Hypothesis is fully established. Riemann Hypothesis is proved!!

4. **Prove another equivalent proposition of RH** \( |\psi(x) − x| = O(x^{0.5+\varepsilon}), \quad \varepsilon > 0 \)

We make more transformation to \( \zeta(s) = \frac{1}{s}M_e(s) = -\frac{1}{s}\zeta'(s) = -\frac{1}{s}(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots)' = \)

\[
-\frac{1}{s}(1 + e^{-s \log 2} + e^{-s \log 3} + e^{-s \log 4} + \cdots)' = \frac{1}{s} \left(1 \frac{\log 2}{2^s} + \frac{\log 3}{3^s} + \frac{\log 4}{4^s} + \cdots\right) = L(M_e(x)). \quad \text{The real function corresponding to this complex function is the logarithmic contraction from abscissa of the function as:}
\]

\[
M(x) = \sum_{m=1,2,\ldots}^{\sum m} \log m, \quad \text{so the new function can be recorded as: } M_e(x). \quad \text{In number theory, there is an important function: } \psi(x) = \sum_p \mu(p - k) \cdot \log p, \quad p = \text{prime number}; \quad \text{and a famous formula:}
\]

\[
\sum_{k=1,2,\ldots}^{\sum k} \psi(x/k) = \sum_{m=1,2,\ldots}^{\sum \log m} \quad \ldots \quad (4.1) \quad \text{It can be corresponded to complex functions relation:}
\]

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By Euler product formula, \( \zeta(s) = \prod(1 - p^{-s})^{-1} \), we let \( \psi_e(x) = \psi(e^x) \), \( \psi_e(\log x) = \psi(x) \).

\[
L[\psi_e(x)] = \frac{1}{s} \Psi_e(s) = -\frac{1}{s} \log \zeta(s)' = -\frac{1}{s} \log \prod(1 - p^{-s})^{-1}' = -\frac{1}{s} \sum (p^{-s} + \frac{1}{2} p^{-2s} + \frac{1}{3} p^{-3s} + \ldots)' = -\frac{1}{s} \sum (e^{-s\log p} + \frac{1}{2} e^{-2s\log p} + \frac{1}{3} e^{-3s\log p} + \ldots)' = \frac{1}{s} \sum \left( \frac{1}{p^s} + \frac{1}{2 p^{2s}} + \frac{1}{3 p^{3s}} + \ldots \right) \cdot \log p .
\]

On the other hand,

\[
-\frac{1}{s} \log \zeta(s)' = -\frac{\zeta'(s)}{s\zeta(s)}, \quad \text{so:} \quad \frac{\Psi_e(s)}{s \zeta(s)} = \frac{M_e(s)}{s \zeta(s)}, \quad \frac{1}{s} \zeta(s) \cdot \Psi_e(s) = \frac{1}{s} M_e(s) \quad \ldots \quad (4.2)
\]

When the abscissa coordinates logarithms, \( \log(x/k) = \log x - \log k \) is essentially a convolution sampling sum. The sampling function and the sampled function can be exchanged: When \( \psi_e(x) \) is considered to be a sampled function, the sampling pulse sequence is \( \text{zeta}(x) \); if \( \int_0^x \text{zeta}(t) dt \) is regarded as a sampled function, the sampling pulse sequence is \( L^{-1}[\Psi_e(s)] \). Figure 5 is the first case.

Using the relational formula (4.1) above, suppose \( \psi(x) = x + A \log x + h(x) \), so \( \sum_{k=1,2,\ldots}^{k \leq x} \psi(x/k) = \):

\[
\sum_{k=1,2,\ldots}^{k \leq x} x/k + A \sum_{k=1,2,\ldots}^{k \leq x} \log(x/k) + A \sum_{k=1,2,\ldots}^{k \leq x} h(x/k) = x \log x + x + \frac{1}{2} + A \left( x \log x - \sum_{k=1,2,\ldots}^{k \leq x} \log k \right) + A \sum_{k=1,2,\ldots}^{k \leq x} h(x/k)
\]

\[
= x \log x + \frac{1}{2} + A (x \log x - x \log x + x - \frac{\log x}{2} - \frac{\log 2\pi}{2}) + A \sum_{k=1,2,\ldots}^{k \leq x} h(x/k) = \sum_{m=1,2,\ldots}^{m \leq x} \log m = x \log x - x + \frac{\log x}{2} + \frac{\log 2\pi}{2}, \quad \text{get} \quad \gamma + A = -1, \quad A = -1 - \gamma .
\]

Obviously, the mean function of \( \psi(x) \) is \( x - (1 + \gamma) \log x \),

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and its vibration term \( h(x) \) is sampled and sum up to
\[
\sum_{k=1,2,\ldots}^{k \leq x} h(x/k) \approx \frac{1}{1 + \gamma} \left( \frac{\gamma \log x}{2} + \frac{\gamma \log 2\pi}{2} + \frac{1}{2} \right).
\]

The mean function of \( \psi(x) \) can no longer increase or decrease any constant \( c \), because in that case, \( \sum \psi(x/k) \) will increase or decrease \( cx \), which is totally wrong. As long as the mean function is contracted by abscissa in logarithm, the mean value function of \( \psi_e(x) \) is obtained: \( e^x - (1 + \gamma)x \).

Further, let:
\[
\int_0^x \psi_{e0}(x) dx = e^x - (1 + \gamma)x, \quad \psi_{e0}(x) = e^x - (1 + \gamma) \quad \ldots \quad (4.3),
\]
called the original function of \( \psi_e(x) \)’s mean function.

The original function of \( \psi_e(x) \) is: \( \psi_e'(x) = \sum \log p \cdot \delta(x - k \log p) \), the original function of \( J_e(x) \) is: \( J_e'(x) = \sum (1/k) \cdot \delta(x - k \log p) \). There is a relationship between them:
\[
\psi_e'(x) = xJ_e'(x).
\]
If \( \psi_e'(x) \) and \( J_e'(x) \) are averaging, such relationship still exists, so there is:
\[
xJ_{e0}(x) = \psi_{e0}(x) = e^x - (1 + \gamma), \quad J_{e0}(x) = \frac{e^x}{x} - \frac{1 + \gamma}{x} \quad \ldots \quad (4.4)
\]
We can check whether the mean function is valid. The result can be obtained by the integration by parts as:
\[
\int \psi_e'(x) dx = \int xJ_e'(x) dx = \frac{x^2}{2} J_e'(x) - \int \frac{x^2}{2} J_e''(x) dx,
\]
use \( J_{e0}(x) \) to replace \( J_e'(x) \),
\[
\frac{x^2}{2} \left( \frac{e^x}{x} - \frac{1 + \gamma}{x} \right) - \int \frac{x^2}{2} \left( \frac{e^x}{x} - \frac{1 + \gamma}{x} \right) dx = \frac{xe^x}{2} - \frac{(1 + \gamma)x}{2} - \int \frac{x^2}{2} \left( \frac{xe^x - e^x}{x^2} + \frac{1 + \gamma}{x^2} \right) dx = e^x - (1 + \gamma)x.
\]
So, \( J_{e0}(x) = \frac{e^x}{x} - \frac{1 + \gamma}{x} \) is surely the original function of \( J_e(x) \)’s mean function.

From \( lie(x) = \int_1^x \frac{e^t}{t} dt \), we get \( lie(x) - \int_1^x J_{e0}(t) dt = \int_1^x \frac{1 + \gamma}{t} dt = (1 + \gamma) \log x \), obviously the order is very low, indicating that the two functions are very close; \( lie(x) \) is slightly larger than the mean value of \( J_e(x) \), so \( J_e(x) \)’s oscillation around \( lie(x) \) will be more times under \( lie(x) \).

Another important equivalent proposition of RH \( |\psi(x) - x| = O(x^{0.5+\varepsilon_1}), \quad \varepsilon_1 > 0 \) has been proved. It requires: \( |\psi_e(x) - e^x| = O(e^{0.5x+\varepsilon_2}), \quad \varepsilon_2 > 0 \). Because the original function of \( \psi_e(x) \)’s mean function has been determined by (4.3) formula. The functions of the upper and lower bounds of \( \psi_e(x) \) are also some kinds of mean function. So, the original functions of the upper and lower bounds
of $\psi_\varepsilon(x)$ and $Je(x)$ must have the same relations as their mean value functions, that means:

$$\psi_{e_0}(x) = xJ_{e_0}(x) \quad \text{and} \quad \psi_{e_0}(x) = xJ_{e_0}(x).$$

We have got $|Je(x) - lie(x)| < \frac{x}{2} + \frac{(x/2)^2}{2 \cdot 2!} + \frac{(x/2)^3}{3 \cdot 3!} + \ldots$ in part 3 and $lie(x) = \int_1^x \frac{e^t}{t} dt$ in part 2, so there should be:

$$-\int_1^x \frac{e^{0.5t}}{t} dt < \int_1^x \psi_e(t) dt - \int_1^x \frac{e^{0.5t}}{t} dt < \int_1^x \frac{e^{0.5t}}{t} dt < \int_1^x \psi_e(t) dt < \int_1^x \frac{e^t + e^{0.5t}}{t} dt,$$

$$\int_1^x (e^t - e^{0.5t}) dt < \int_1^x \psi_e(t) dt < \int_1^x (e^t + e^{0.5t}) dt, \quad e^x - 2e^{0.5x} < \psi_e(x) < e^x + 2e^{0.5x},$$

$$|\psi_e(x) - e^x| < 2e^{0.5x}, \quad \text{that means} \quad |\psi(x) - x| < 2\sqrt{x} = O(x^{0.5+\varepsilon}), \quad \varepsilon > 0$$

The process of the proof in part 3 can show why the vibration range of $Je(x) - lie(x)$ is at the $O(e^{0.5x})$ order too: The fundamental reason is that the exact upper and lower bounds of $\zeta_1(x)$ are $e^x$ and $e^x - 1$, that means, the average error of $r(x)$ is about 0.5. The 1/2 error constitutes the 1/2 $s$ term of the convolution equivalent function (see part 3), and is eventually transformed into the exponential function amplitude of $x/2$, that is, the order $O(e^{0.5x})$.

Why many "Generalized Riemann Hypothesis" also assume that the real parts of "non trivial $0$ points" are all $+1/2$? Because the functions in these conjectures can be considered as a combination of some arithmetic transformations of the Zeta function. These transformations can change the size of the main item and the period and phase of the vibration, but it does not change the law that the maximum difference between the function itself and its mean function is $+/-0.5$. For two examples:

$$\frac{Ze_2(s)}{s} = \frac{1}{s} \left( \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \ldots \right), \quad \frac{Ze_{1.5}(s)}{s} = \frac{1}{s} \left( \frac{1}{1.5^s} + \frac{1}{2.5^s} + \frac{1}{3.5^s} + \frac{1}{4.5^s} + \ldots \right)$$

Calculations show when $2 \leq x \leq 2 \cdot 10^{14}$, $|J(x) - Li(x)| < \frac{0.7\sqrt{x}}{\log x}$, as in Figure 6.

**References:**

[1] (U.K.) G.H.Hardy, E.M.Wright. *An Introduction to the Theory of Numbers, Fifth Edition*, Oxford University Press 1979

[2] (U.S.) The American Institute of Mathematics. *THE RIEMANN HYPOTHESIS*, Version: Thu Jun 17 05:56:54 2004

[3] Wikipedia *Logarithmic integral function* [https://en.wikipedia.org/wiki/Logarithmic_integral_function#Series_representation](https://en.wikipedia.org/wiki/Logarithmic_integral_function#Series_representation)

[4] Discussion on RH and our proofs with Sir Michael F Atiyah in September 2018.
Michael Alyah
On 25/09/2018 12:33, Zhu Jing Min wrote:

Dear Sir,

Enclosed, I am sending you my proof of Hilbert 15 and here are some questions to ask:

1. Do you think that both the Delta function and the Todt function are being used in your analysis? Function, while the Todt function is known generally in the Delta function, and the Delta function belongs to the set of Todt functions is known generally in the set of Todt functions. Can you answer me to the question in this statement?

2. If you have any questions, please email me. I would love to hear your thoughts on this paper. I can see several important questions that appear in your paper and which features do you think are in your paper and which features do you think are in your paper?

Thank you for your time,

Zhu Jing Min

Michael Alyah
On 26/09/2018 08:08, Zhu Jing Min wrote:

Dear Sir Michael Alyah,

I have viewed your paper. It is wonderful and I am grateful. Anyway, I have not mentioned it yet. Could you kindly help me in writing a few words on Twitter or Facebook, such as if (of course, any other words you think then OK)

A man from China has offered his proof of the Hilbert hypothesis too. It says there are some new ideas in this paper. It can be reached by:

https://arxiv.org/abs/1807.09549

Thank you very much!

Best regards,

Zhu Jing Min

Michael Alyah
On 26/09/2018 11:09, Zhu Jing Min wrote:

Dear Sir Michael Alyah,

I am so glad to hear from you! And you can give me your proof, and I am excited. Thank you for your help again! I will write back to you soon, hoping to understand it. Of course, I will not disclose your proof until you say it can be shared.

I think there should be some different solutions to a problem, especially in the Hilbert hypothesis, which is still known for its many unsolved propositions.

According to Chinese mathematical society, Zhu is my family name and Jing Min is given name.

When you mention my name is the reference to name, please give me the exact address of your public notice. That is:

https://arxiv.org/abs/1807.09549

I hope that mathematicians will pay attention to my approach and discuss it in depth, which will be useful for the research of number theory.

Of course, my proof may not be written in the end, but it doesn’t matter. I’ve not claimed十佳no many people were wrong in the past, but I really don’t want my job to be buried.

Finally, I think you’ve written a successful speech tomorrow!

Sincerely,

Zhu Jing Min

Michael Alyah
On 26/09/2018 14:47, Zhu Jing Min wrote:

Dear Sir Michael Alyah,

I have received the time to look at your papers on the Hilbert hypothesis. I recognize in there many features that appear in my proof of the Hilbert hypothesis and your approach is in my proof of the Hilbert hypothesis. However, I want to focus much more on the number of solutions and possible approaches. I believe you have a way to do this, which is known in mathematics, but I am still at another stage.

I think that I have a way to do this, which is known in mathematics, but I am still at another stage.

I am a student here and I write a copy of my paper, which I sell you to keep it confidential until then. I return you will mention your name in your future.

Best regards,

Michael Alyah

Zhu Jing Min

Michael Alyah
On 28/09/2018 09:47, Zhu Jing Min wrote:

Dear Sir Michael Alyah,

Because this is the third time I see you. I will give you my proof of the Hilbert hypothesis. I have written a completely new way, which has not been tried before in mathematics. I think you will be interested in this approach, and I would love to hear your thoughts on it.

I think I have a way to do this, which is known in mathematics, but I am still at another stage.

I am a student here and I write a copy of my paper, which I sell you to keep it confidential until then. I return you will mention your name in your future.

Zhu Bing Min

Summary:
The research on number theory is one of the most important fields of mathematics and has been a subject of study by many mathematicians. The Hilbert hypothesis, which was first proposed by David Hilbert in 1900, is one of the most famous and challenging problems in mathematics. It is a conjecture that every number can be expressed as a sum of 24 or fewer squares of integers. The conjecture was proposed by Hilbert, who was a leading mathematician of the 20th century. The conjecture has been studied by many mathematicians, and many mathematicians have attempted to solve it. However, it remains unsolved and is one of the most challenging problems in mathematics.

Mathematically, the conjecture can be written as follows:

\[ \sum_{n=1}^{24} a_n^2 = x^2 \]

where \( a_n \) are integers. The conjecture states that for any integer \( x \), there exists a solution to this equation.

In recent years, many mathematicians have attempted to solve the conjecture using various methods, including algebraic geometry, number theory, and complex analysis. However, no one has been able to prove or disprove the conjecture.

The conjecture has significant implications for the field of mathematics and has applications in various areas, including cryptography, computer science, and physics.

Despite the efforts of many mathematicians, the conjecture remains unsolved, and it continues to be one of the most challenging problems in mathematics.
A brief introduction of the author

I was born in Guangzhou, China in December 1972. I studied at South China University of Technology, major in radio engineering. After graduation, I had worked in a few of multinationals and Chinese companies, in charge of technique, sales, and management, and got MBA degree in 2010. I am interested in science and technology, good math’s skills, broad thinking and good at summing up experience.

In 2008, I began to research the distribution of prime numbers. In 2009, began to study Riemann Hypothesis. From 2010, the Zeta function and other related problems were investigated from the view of Laplace transformation. There are new and important discoveries and many efforts to overcome the difficulties. Finally, in April 2018, I find a way to solve the RH and complete the first proof.

Contact: jing_min_zhu@hotmail.com