UNBORING IDEALS

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Abstract. Our main object of interest is the following notion: we say that a
topological space space $X$ is in $\text{FinBW}(I)$, where $I$ is an ideal on $\omega$, if for each
sequence $(x_n)_{n \in \omega}$ in $X$ one can find an $A \notin I$ such that $(x_n)_{n \in A}$ converges
in $X$.

We define an ideal $BI$ which is critical for $\text{FinBW}(I)$ in the following sense:
Under CH, for every ideal $I$, $BI \not\leq K$ (where $K$ denotes the Katětov preorder of
ideals) iff there is an uncountable separable space in $\text{FinBW}(I)$. We show that
$BI \not\leq K$ and $\omega_1$ with the order topology is in $\text{FinBW}(I)$, for all $\Pi^0_4$ ideals $I$.

We examine when $\text{FinBW}(I) \setminus \text{FinBW}(J)$ is nonempty: we prove under
$\text{MA}(\sigma\text{-centered})$ that for $\Pi^0_4$ ideals $I$ and $J$ this is equivalent to $J \not\leq K$ $I$.
Moreover, answering in negative a question of M. Hrušák and D. Meza-Alcántara, we show that the ideal $\text{Fin} \times \text{Fin}$ is not critical among Borel ideals
for extendability to a $\Pi^0_3$ ideal. Finally, we apply our results in studies of
Hindman spaces and in the context of analytic $P$-ideals.

1. Introduction

A nonempty collection $\mathcal{I}$ of subsets of a set $X$ is called an ideal on $X$ if it is
closed under subsets and finite unions of its elements. In this paper we also assume
throughout that $X \notin \mathcal{I}$ and $X = \bigcup \mathcal{I}$. All ideals considered in this paper are
defined on infinite countable sets. By $\text{Fin}$ we denote the ideal of all finite subsets
of $\omega = \{0, 1, \ldots\}$.

We treat the power set $\mathcal{P}(X)$ as the space $2^X$ of all functions $f : X \to 2$ (equipped
with the product topology, where each space $2 = \{0, 1\}$ carries the discrete topol-
ogy) by identifying subsets of $X$ with their characteristic functions. Thus, we can
talk about descriptive complexity of subsets of $\mathcal{P}(X)$ (in particular, of ideals on
$X$).

The classical Bolzano-Weierstrass theorem states that $[0, 1]$ is sequentially com-
 pact, that is, each sequence in it has a convergent subsequence. It is known that in
general there are compact topological spaces that are not sequentially compact. On
the other hand, $\omega_1$ with the order topology is a sequentially compact space which
is non-compact.

The following modification of the notion of sequentially compact space gives us
control on the size of the convergent subsequence.

Definition 1.1. Let $\mathcal{I}$ be an ideal on a countable set $M$. By $\text{FinBW}(I)$ we denote
the class of all Hausdorff spaces $X$ such that for every sequence $(x_n)_{n \in M} \subseteq X$ there

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space.
exists a set $A \notin I$ such that $(x_n)_{n \in A}$ converges in $X$. If $X$ is in this class we briefly write $X \in \text{FinBW}(I)$.

Observe that $\text{FinBW}(\text{Fin})$ is the class of all sequentially compact spaces. Note that the notation $\text{FinBW}(I)$ is close to the one introduced in [8], where $I \in \text{FinBW}$ means that $[0,1] \in \text{FinBW}(I)$ in the sense of Definition 1.1.

This subject has already been studied in the literature. There are at least two directions of those studies. The first one, related to the notions of Hindman space and Mrówka space, was initiated by M. Kojman. In [18] he considered spaces belonging to $\text{FinBW}(\mathcal{W})$, where $\mathcal{W}$ denotes the van der Waerden ideal, that is:

$$\mathcal{W} = \{A \subseteq \omega : (\exists k \in \omega) \text{A does not contain arithmetic progressions of length } k\}.$$

And in [17, Theorem 3] he showed that $\text{FinBW}({\mathcal{H}})$ is the class of all finite spaces, where $\mathcal{H}$ is the Hindman ideal. He also introduced there a new class of Hindman spaces, to which we’ll return in the last section of this paper. In [19, Theorem 3] M. Kojman and S. Shelah constructed under the continuum hypothesis (CH) an uncountable separable compact space from $\text{FinBW}(\mathcal{W})$, which is not a Hindman space. This result has been extended by A. L. Jones to the case of Martin’s axiom for $\sigma$-centered notions of forcing ($\text{MA}(\sigma\text{-centered})$) instead of CH. In both those results, the constructed space is a Mrówka space (for the definition of Mrówka space see subsection 2.4). Later J. Flášková in [11] examined $\text{FinBW}(I)$ for $\Sigma_2^\omega$ ideals $I$ (in her paper spaces from $\text{FinBW}(I)$ are called $I$-spaces). She showed under $\text{MA}(\sigma\text{-centered})$ that for every $\Sigma_2^\omega$ ideal there is an uncountable separable compact Mrówka space in $\text{FinBW}(I)$ which is not Hindman. As far as we know, there are no results concerning the necessity of the assumptions CH and $\text{MA}(\sigma\text{-centered})$ in the above studies.

The second direction concerns the following question: for which ideals $I$ do we have $[0,1] \in \text{FinBW}(I)$? These studies were initiated by R. Filipów, N. Mrożek, I. Recław and P. Szuca in [8] and then continued in [7], where it is proved, among other things, that if $I$ is an $\Sigma_2^\omega$ ideal then $[0,1] \in \text{FinBW}(I|A)$ for all $A \notin I$ [8, Proposition 3.4]. Finally, D. Meza-Alcántara in [28, Section 2.7] proved that the ideal conv on $[0,1] \cap \mathbb{Q}$ generated by all convergent sequences (this ideal is $\Sigma_4^\omega$) is critical for $\text{FinBW}$: $[0,1] \in \text{FinBW}(I)$ is equivalent to conv $\not\leq K I$, where $\leq_K$ denotes the Katétov preorder on ideals (i.e., $I \leq_K J$, if there is a $f : \bigcup J \to \bigcup I$ such that $f^{-1}[A] \in J$ for each $A \in I$).

This article is organized as follows. The most interesting results are presented in Sections 6 and 10. Section 2 contains some basic definitions needed throughout the paper, that were not mentioned in this Introduction. The longest and most technical Section 4 as well as Sections 3 and 5 are devoted to some preliminary discussions. Section 6 contains the main results of this part of the paper. In particular, we prove that $\omega_1$ with the order topology is in $\text{FinBW}(I)$, for all $\Pi_4^\omega$ ideals $I$, and find an ideal $\mathcal{B}I$ which serves as the boundary point for $\text{FinBW}(I)$ containing interesting spaces: Under CH, for every ideal $I$, $\mathcal{B}I \not\leq_K I$ if and only if there is an uncountable separable space in $\text{FinBW}(I)$. In Section 7 we pose some open problems and answer in negative a question of M. Hrušák and D. Meza-Alcántara about extendability to a $\Pi_4^\omega$ ideal. Sections 8 and 9 are devoted to preliminary results concerning the following question: when $\text{FinBW}(I) \setminus \text{FinBW}(J)$ is nonempty? In Section 10 we prove (under $\text{MA}(\sigma\text{-centered})$) that for $\Pi_4^\omega$ ideals $I$ and $J$ this is equivalent to
We say that every ideal is extendable to a maximal ideal. Extendable to a P-ideal" means that there is a P-ideal $J$ with $I \subseteq J$. In particular, we do not require that the sets $X_n$ are infinite or non-empty.

Let $A, B \in [\omega]^{\omega}$. We say that $A$ and $B$ are almost disjoint if $A \cap B$ is finite. A family $A \subseteq [\omega]^{\omega}$ is called an AD family if the members of $A$ are pairwise almost disjoint. If moreover $A$ is a maximal AD family with respect to inclusion, it is called a MAD family. Equivalently, $A$ is a MAD family if it is an AD family and for each infinite $B \subseteq \omega$ there is an $A \in A$ with $A \cap B$ infinite.

The restriction of an ideal $\mathcal{I}$ to $Y$ is given by $\mathcal{I}|Y = \{ A \cap Y : A \in \mathcal{I} \}$. Note that $\mathcal{I}|Y$ is an ideal on $Y$ if and only if $Y \notin \mathcal{I}$. We say that an ideal $\mathcal{I}$ on $X$ is generated by the family $\mathcal{F} \subseteq \mathcal{P}(X)$ if

$$\mathcal{I} = \{ A \subseteq X : (\exists n \in \omega) \ (\exists F_0, \ldots, F_n \in \mathcal{F}) \ A \subseteq F_0 \cup \ldots \cup F_n \}.$$  

**Definition 2.1.** If $\mathcal{I}$ is an ideal then $D_{\mathcal{I}}$ is the set of all functions $f : \bigcup \mathcal{I} \to \omega$ such that $f^{-1}[\{ n \}] \in \mathcal{I}$ for all $n \in \omega$. In particular, $D_{\text{Fin}}$ is the set of all finite-to-one functions from $\omega$ to $\omega$.

If $\mathcal{I}$ and $\mathcal{J}$ are ideals on $X$ and $Y$, respectively, then:

- the product of $\mathcal{I}$ and $\mathcal{J}$ is an ideal on $X \times Y$ given by:
  $$\mathcal{I} \otimes \mathcal{J} = \{ A \subseteq X \times Y : \{ x \in X : A(x) \notin \mathcal{J} \} \in \mathcal{I} \},$$
  where $A(x) = \{ y \in Y : (x, y) \in A \}$. Also with either $\mathcal{I}$ or $\mathcal{J}$ (but not both) replaced by $\emptyset$, the above formula defines ideals, denoted $\emptyset \otimes \mathcal{J}$ and $\mathcal{I} \otimes \emptyset$, respectively.
- the disjoint sum of $\mathcal{I}$ and $\mathcal{J}$ is an ideal on $(\{0\} \times X) \cup (\{1\} \times Y)$ given by:
  $$\mathcal{I} \oplus \mathcal{J} = \{ A \subseteq (\{0\} \times X) \cup (\{1\} \times Y) : A(0) \in \mathcal{I} \text{ and } A(1) \in \mathcal{J} \}.$$  

If $X$ is an infinite countable set then by $\text{Fin}(X)$ we will denote the ideal of all finite subsets of $X$. Hence, $\text{Fin} = \text{Fin}(\omega)$. For an ideal $\mathcal{I}$ and $A \subseteq \bigcup \mathcal{I}$ we say that $A$ is an $\mathcal{I}$-positive set whenever $A \notin \mathcal{I}$. An ideal $\mathcal{I}$ on $X$ is:

- tall if for each infinite $A \subseteq X$ there is a $B \subseteq A$ with $B \in \mathcal{I} \setminus \text{Fin}(A)$;
- a P-ideal if for each $(A_n) \subseteq \mathcal{I}$ there is an $A \in \mathcal{I}$ with $A_n \setminus A$ finite for all $n \in \omega$;
- maximal if for any $A \subseteq X$ either $A \in \mathcal{I}$ or $X \setminus A \in \mathcal{I}$ (equivalently, $\mathcal{I}$ is maximal with respect to inclusion). It is known that a maximal ideal cannot be Borel.

We say that $\mathcal{I}$ is extendable to $\mathcal{J}$ if $\mathcal{I} \subseteq \mathcal{J}$. Thus, for instance, the statement "$\mathcal{I}$ is extendable to a P-ideal" means that there is a P-ideal $\mathcal{J}$ with $\mathcal{I} \subseteq \mathcal{J}$. In particular, every ideal is extendable to a maximal ideal.

For $A \subseteq \mathcal{P}(X)$ we write

$$A^* = \{ X \setminus A : A \in A \}.$$
If $I$ is an ideal on $X$ then $I^*$ is a filter (i.e., a family closed under supersets and finite intersections of its elements). We call it the dual filter of $I$. Observe that the map $A \mapsto X \setminus A$ is continuous. Thus, $I$ and $I^*$ have the same descriptive complexity.

Dual filter of a maximal ideal is called an ultrafilter. Recall that an ultrafilter $U \subseteq \mathcal{P}(\omega)$ is a P-point if $U^*$ is a P-ideal. It is known that P-points exist under CH as well as under MA($\sigma$-centered).

2.2. Preorders on ideals. Let $I$ and $J$ be ideals on $X$ and $Y$, respectively (here we make an exception and allow $I = \mathcal{P}(X)$ or $J = \mathcal{P}(Y)$ – this will simplify some of our proofs). We say that:

- $J$ and $I$ are isomorphic and write $I \cong J$, if there is a bijection $f : Y \to X$ such that $(\forall A \subseteq X) f^{-1}[A] \in J \iff A \in I$;
- $J$ contains an isomorphic copy of $I$ and write $I \preceq J$, if there is a bijection $f : Y \to X$ such that $f^{-1}[A] \in J$ for each $A \in I$;
- $J$ is above $I$ in the Katětov-Blass preorder and write $I \leq_{KB} J$, if there is a finite-to-one $f : Y \to X$ (i.e., $f^{-1}[\{x\}] \in \text{Fin}(Y)$ for all $x \in X$) such that $f^{-1}[A] \in J$ for each $A \in I$;
- $J$ is above $I$ in the Katětov preorder and write $I \leq K J$, if there is an $f : Y \to X$ (not necessary finite-to-one) such that $f^{-1}[A] \in J$ for each $A \in I$;

Obviously, $I \cong J \implies I \preceq J \implies I \leq_{KB} J \implies I \leq K J$.

The preorders $\leq K$ and $\preceq$ were extensively studied e.g. in [2], [12], [15], [20], [21] and [28]. A property of ideals can often be expressed by finding a critical ideal (in some preorder on ideals) with respect to this property. This approach proved to be especially effective in the context of FinBW ideals.

Note that $I \leq_{KB} I/A$ (as witnessed by the identity function) for all ideals $I$ and all $A \notin I$.

**Definition 2.2** ([13] or [28, Subsection 2.1.2]). An ideal $I$ is:

- $\leq K$-uniform if $I/A \leq K I$ for all $A \notin I$;
- $\leq_{KB}$-uniform if $I/A \leq_{KB} I$ for all $A \notin I$;
- homogeneous if $I/A$ is isomorphic to $I$ whenever $A \notin I$.

Obviously, a homogeneous ideal is $\leq_{KB}$-uniform and a $\leq_{KB}$-uniform ideal is $\leq K$-uniform.

**Example 2.3.** The following ideals are homogeneous:

- $W$ (cf. [25, Example 2.6]);
- $\mathcal{H}$ (cf. [25, Example 2.6]);
- all maximal ideals (cf. [25, Example 1.4]);
- $\text{Fin}^n$ for all $n \geq 1$, where $\text{Fin}^1 = \text{Fin}$ and $\text{Fin}^{n+1} = \text{Fin} \otimes \text{Fin}^n$ (cf. [25, Remark after Proposition 2.9]).

**Example 2.4.** Define the density zero ideal

$$I_d = \left\{ A \subseteq \omega : \lim_{n \to \infty} \frac{|A \cap \{0,1,\ldots,n\}|}{n+1} = 0 \right\}.$$
It is a $\Pi^0_3$ P-ideal (cf. [4, Example 1.2.3(d)]). By [13, Proposition 2.1], it is also $\leq_K$-uniform (note that this fact is also stated in [28, Proposition 2.1.11], but with an incorrect proof). Actually, the proof of [13, Proposition 2.1] shows even more: $\mathcal{I}_d$ is $\leq_K$-uniform.

2.3. Related notions and their critical ideals. Consider the ideal:

$$\text{Fin}^2 = \text{Fin} \otimes \text{Fin} = \{ A \subseteq \omega^2 : \{ n \in \omega : A(n) \notin \text{Fin} \} \in \text{Fin} \}.$$  

This ideal is $\Sigma^0_4$. It is critical for the following property considered in the literature:

We say that an ideal $I$ on $X$ is a weak P-ideal, if for each partition $(A_n)$ of $X$ into sets belonging to $I$ there is an $S \not\in I$ with $S \cap A_n$ finite, for all $n \in \omega$ (note that all P-ideals are weak P-ideals).

**Proposition 2.5** (Essentially [26] and [28]). Let $I$ be an ideal.

(a) $[0,1] \in \text{FinBW}(I)$ if and only if $\text{conv} \nsubseteq I$;

(b) $I$ is a weak P-ideal if and only if $\text{Fin}^2 \nsubseteq I$.

**Proof.** Item (a) can be found in [28, Section 2.7]. For (b) see the proof of [26, Lemma 2]. □

In the whole paper we will use ”$\text{Fin}^2 \nsubseteq I$” and ”$I$ is a weak P-ideal” interchangeably.

**Proposition 2.6** ([2, Examples 4.1 and 4.4]). Let $I$ be any ideal and $J = \text{conv}$ or $J = \text{Fin}^2$. The following are equivalent:

- $J \leq_K I$;
- $J \leq_K B I$;
- $J \subseteq I$.

**Proposition 2.7** ([28, Theorem 2.8.7] and Proposition 2.6). Let $U$ be an ultrafilter.

The following are equivalent:

- $U$ is a P-point;
- $\text{conv} \nsubseteq_K U^*$;
- $\text{conv} \nsubseteq_K U^*$;
- $\text{Fin}^2 \nsubseteq_K U^*$;
- $\text{Fin}^2 \nsubseteq U^*$.

2.4. Mrówka spaces. For an infinite AD family $A$ of infinite subsets of $\omega$, define a topological space $\Phi(A)$ as follows:

- the underlying set of $\Phi(A)$ is $\omega \cup A \cup \{\infty\}$,
- the points of $\omega$ are isolated,
- each basic neighborhood of $A \in A$ is of the form $\{A\} \cup (A \setminus F)$, where $F \subseteq \omega$ is finite,
- each basic neighborhood of $\infty$ is of the form

$$\{\infty\} \cup (A \setminus G) \cup \left( \omega \setminus \left( F \cup \bigcup_{A \in G} A \right) \right),$$

where both $F \subseteq \omega$ and $G \subseteq A$ are finite.

Equivalently, $\Phi(A)$ is the one-point compactification of the space with $\omega \cup A$ as the underlying set and basic neighborhoods of $A \in A$ and $n \in \omega$ as above (this space
was introduced in [29]). Indeed, if $C$ is a compact subset of the latter space, then $C \cap A$ and $(C \cap \omega) \cup \bigcup_{A \in C \cap A} A$ have to be finite.

In this paper a space of the form $\Phi(A)$, for some AD family $A$, will be called Mrówka space.

Each Mrówka space is Hausdorff, separable and compact. Moreover, it is first countable at every point of $\Phi(A) \setminus \{\infty\}$. If $A$ is a MAD family, then $\Phi(A)$ is sequentially compact (see [9, Section 2] or [18, Theorem 6]). Observe also that $\Phi(A)$ is countable if and only if $A$ is countable. In particular, if $A$ is a MAD family, then $\Phi(A)$ is uncountable (as MAD families are uncountable). For more on Mrówka spaces see also [14] or [29].

3. Boring spaces

In this section we introduce a new class of topological spaces, which are in some sense the most boring examples of infinite sequentially compact spaces.

**Definition 3.1.** We say that a topological space $X$ is boring if $X$ is sequentially compact and there is a finite set $F \subseteq X$ such that each injective convergent sequence in $X$ converges to some point from $F$.

**Proposition 3.2.**

(i) For each cardinal $\kappa$, there is a Hausdorff boring space of cardinality $\kappa$.

(ii) Let $X$ be an infinite Hausdorff space. Then $X$ is a boring space if and only if $X$ is a disjoint union of finitely many one-point compactifications of discrete spaces.

(iii) Each boring space is compact.

(iv) A boring space is separable if and only if it is countable.

**Proof.** (i): If $\kappa$ is finite, it suffices to take any discrete space of cardinality $\kappa$. If $\kappa$ is an infinite cardinal number, fix a set $Y$ of cardinality $\kappa$ and let $X$ be the one-point compactification of the space $(Y, \mathcal{P}(Y))$. Then $X$ is as needed.

(ii): It is obvious that each disjoint union of finitely many one-point compactifications of discrete spaces is boring. Thus, we only need to prove the opposite implication.

Let $(X, \tau)$ be Hausdorff and boring. Let $\{x_0, \ldots, x_n\} \subseteq X$ be such that each injective convergent sequence in $X$ converges to some $x_i$, for $i \leq n$. Firstly, we will show that each $x \in X \setminus \{x_0, \ldots, x_n\}$ is isolated.

Fix any $x \in X \setminus \{x_0, \ldots, x_n\}$. Since $X$ is Hausdorff, there are pairwise disjoint open sets $V, V_0, \ldots, V_n$ such that $x \in V$ and $x_i \in V_i$, for $i \leq n$. Note that $V$ is finite, as otherwise it would contain an injective sequence which does not converge to any $x_i$. For each $y \in V \setminus \{x\}$ find open $U_y$ with $x \in U_y$ and $y \notin U_y$. Then $V \cap \bigcap_{y \in V \setminus \{x\}} U_y = \{x\}$ is open. Hence, $x$ is an isolated point.

Now we show that $X$ is a disjoint union of finitely many one-point compactifications of discrete spaces. Since $X$ is Hausdorff, there are pairwise disjoint open sets $U_0, \ldots, U_n$ such that $x_i \in U_i$, for $i \leq n$. Without loss of generality we may assume that $X = U_0 \cup \ldots \cup U_n$. Indeed, the set $X \setminus (U_0 \cup \ldots \cup U_n)$ is finite (otherwise it would contain an injective sequence with no convergent subsequence), so open (by the above paragraph) and we can add it to $U_0$. We claim that $U_i$, with the topology $\{V \cap U_i : V \in \tau\}$ is a one-point compactification of a discrete space, for each $i \leq n$.

Fix $i \leq n$. We already know that each $x \in U_i \setminus \{x_i\}$ is isolated. Thus, we only need to show that for each open neighborhood $V$ of $x_i$, the set $U_i \setminus V$ is finite. This
Definition 4.1. We define the ideal \( BI \) of 
\[ \mathcal{I} = A / \text{finite-to-one}. \]
Since \( f \) is almost all \( i \) intersects infinitely many \( \mathcal{I} \) is tall. Indeed, for each infinite \( A \subseteq \omega^3 \) either \( A(i) \) is finite for all \( i \in \omega \) or \( A \) intersects infinitely many \( \{i\} \times \omega^2 \). In the former case use tallness of \( \omega^2 \) to get an infinite \( B \subseteq A(i) \) with \( B \in \text{Fin}^2 \) and conclude that \( \{i\} \times B \) is an infinite subset of \( A \) belonging to \( BI \). In the latter case find any selector \( S \subseteq \omega^3 \) of the family \( (A \cap \{i\} \times \omega^2) \) and note that \( S \subseteq A \) and \( S \in BI \).

The next definition is fundamental for our considerations.

Definition 4.3. We say that \( \mathcal{I} \) is boring if \( BI \subseteq \mathcal{I} \). Otherwise we say that \( \mathcal{I} \) is unboring.

Throughout the entire paper we may use the next result without any reference.

Proposition 4.4. The following are equivalent for any ideal \( \mathcal{I} \):

(i) \( BI \not\leq_K \mathcal{I} \);
(ii) \( BI \not\leq_K \mathcal{I} \);
(iii) \( BI \not\leq \mathcal{I} \);
(iv) for any partition \( (X_{i,j}) \) of \( \bigcup \mathcal{I} \) into sets belonging to \( \mathcal{I} \) there is an \( A \notin \mathcal{I} \) such that \( A \cap X_{i,j} \) is finite for all \( (i,j) \in \omega^2 \) and \( A \cap \bigcup_j X_{i,j} \) is finite for almost all \( i \in \omega \);
(v) for each \( f : \bigcup \mathcal{I} \to \omega^2 \) there is an \( A \notin \mathcal{I} \) such that \( f[A] \in \text{Fin}^2 \) and \( f[A] \) is either constant or finite-to-one.
Proof. Without loss of generality, we can assume that $\mathcal{I}$ is an ideal on $\omega$.

(i) $\implies$ (ii): Obvious.

(ii) $\implies$ (iii): Obvious.

(iii) $\implies$ (iv): We will show that negation of (iv) implies $\mathcal{BI} \not\subseteq \mathcal{I}$. Suppose that there is a partition $(X_{i,j})$ of $\omega$ into sets belonging to $\mathcal{I}$ such that if $A \cap X_{i,j}$ is finite for all $(i, j) \in \omega^2$ and $A \cap \bigcup_j X_{i,j}$ is finite for almost all $i \in \omega$ then $A \in \mathcal{I}$.

Let $f : \omega \to \omega^3$ be any injective function satisfying $f[X_{i,j}] \subseteq \{(i, j)\} \times \omega$.

Observe that if $B \in \mathcal{BI}$ then $f^{-1}[B] \in \mathcal{I}$. Indeed, for each $B \in \mathcal{BI}$ we can find $k \in \omega$ such that $B(i) \in \text{Fin}^2$ for all $i < k$ and $B(i)$ is finite for all $i > k$. Thus, for every $i < k$ there is a $j_i \in \omega$ such that $B(i, j_i) \in \text{Fin}$ for all $j > j_i$. Hence, we get that $\mathcal{BI} \subseteq \mathcal{I}$.

Moreover, $B' = B \setminus \bigcup_{i < k} \bigcup_{j < j_i}(\{(i, j)\} \times \omega)$ intersects each $\{(i, j)\} \times \omega$ in a finite set (so also $f^{-1}[B'] \cap X_{i,j} \in \text{Fin}$ for all $(i, j) \in \omega^2$) and $B' \cap \{(i, j)\} \times \omega = B' \cap \bigcup_j X_{i,j}$ is finite for all $i \geq k$ (so $f^{-1}[B'] \cap \bigcup_j X_{i,j}$ is finite for almost all $i \in \omega$). Thus, $f^{-1}[B'] \in \mathcal{I}$, by our assumption and we can conclude that $f^{-1}[B] = f^{-1}[B'] \cup f^{-1}[B' \cap \bigcup_{i < k, j < j_i}(\{(i, j)\} \times \omega)] \in \mathcal{I}$.

As $\mathcal{BI}$ is tall and $f$ is injective such that $f^{-1}[B] \in \mathcal{I}$ for all $B \in \mathcal{BI}$, using [2, Lemma 3.3] we get that $\mathcal{BI} \subseteq \mathcal{I}$.

(iv) $\implies$ (v): Let $f : \omega \to \omega^2$ be such that $f^{-1}[\{(i, j)\}] \in \mathcal{I}$ for all $(i, j) \in \omega^2$, i.e., $f[A]$ being constant implies $A \in \mathcal{I}$ (if there is an $A \not\in \mathcal{I}$ with $f[A]$ constant, then we are done as $f[A] \in \text{Fin}(\omega^2) \subseteq \text{Fin}^2$). Consider the partition of $\omega$ given by $f^{-1}[\{(i, j)\}]$ for all $(i, j) \in \omega^2$. By (iv), there is an $A \not\in \mathcal{I}$ such that there is a $k \in \omega$ with $A \cap f^{-1}[\{(i, j)\}] \in \text{Fin}$ for all $i > k$ and $A \cap f^{-1}[\{(i, j)\}] \not\subseteq \text{Fin}$ for all $i \leq k$ and $j \in \omega$. Thus, $f[A]$ is finite-to-one (as $A \cap f^{-1}[\{(i, j)\}] \not\subseteq \text{Fin}$ for all $i, j$) and $f[A] \in \text{Fin}^2$ (as for each $i > k$ there are only finitely many $j$ such that $A \cap f^{-1}[\{(i, j)\}] \not= \emptyset$).

(v) $\implies$ (i): Assume to the contrary that $\mathcal{BI} \subseteq K \mathcal{I}$ and let $g : \omega \to \omega^3$ be the witnessing function. Define $f : \omega \to \omega^2$ by $g(n) \in \{f(n)\} \times \omega$. Then there is an $A \not\in \mathcal{I}$ such that $f[A] \in \text{Fin}^2$ and $f[A]$ is either constant or finite-to-one. However, $f[A]$ being constant implies that $A \not\subseteq g^{-1}[\{(i, j)\} \times \omega] \subseteq \mathcal{I}$ for some $(i, j) \in \omega$. Thus, $f[A]$ is finite-to-one. But then $A \cap g^{-1}[\{(i, j)\} \times \omega]$ is finite for all $(i, j) \in \omega$ and, as $f[A] \in \text{Fin}^2$, there is a $k \in \omega$ such that $A \cap g^{-1}[\{(i, j)\} \times \omega] = A \cap f^{-1}[\{(i, j)\} \times \omega] \not\subseteq \text{Fin}$ for all $i > k$. As $g$ witnesses $\mathcal{BI} \subseteq K \mathcal{I}$, we conclude that $A \in \mathcal{I}$, which contradicts the choice of $A$ and finishes the proof.

Remark 4.5. In the terminology of [2], equivalence of items (i) and (iii) from Proposition 4.4 means that $\mathcal{BI}$ has the property Kat.

Proposition 4.6. We have $\text{conv} \subseteq \mathcal{BI} \subseteq \text{Fin}^2$ and none of those inclusions can be reversed.

Proof. It is easy to check that $\mathcal{BI} \subseteq \text{Fin}^2$ is witnessed by any bijection $f : \omega^2 \to \omega^3$ satisfying $f([n] \times \omega) = \{n\} \times \omega^2$. 

By Proposition 2.6, in order to show \( \text{conv} \subseteq \mathcal{BI} \), it suffices to prove that \( \text{conv} \leq_{\mathcal{K}} \mathcal{BI} \), which in turn is equivalent to existence of a countable family \( \{X_n : n \in \omega\} \) such that for each \( A \notin \mathcal{BI} \) there is an \( \omega \) with \( |A \cap X_n| = |A \setminus X_n| = \omega \) (by [28, Theorem 2.4.3]). We claim that \( \{(i, j) \} \times \omega : i, j \in \omega \} \cup \{(i) \} \times \omega^2 : i \in \omega \) is the required family. Indeed, fix \( A \notin \mathcal{BI} \) and assume first that \( A \cap \{(i, j) \} \times \omega \) is infinite for some \( i, j \in \omega \). Then \( A \setminus \{(i, j) \} \times \omega \) has to be infinite as well since \( \{(i, j) \} \times \omega \in \mathcal{BI} \). Thus, \( \{(i, j) \} \times \omega \) is the required set. Assume now that \( A \cap \{(i, j) \} \times \omega \) is finite for all \( i, j \in \omega \). Then \( A \cap \{i\} \times \omega^2 \) is infinite for infinitely many \( i \in \omega \) (as \( A \notin \mathcal{BI} \)). Thus, we can find \( i \in \omega \) such that \( |A \cap \{i\} \times \omega^2| = |A \setminus \{i\} \times \omega^2| = \omega \).

Now we show that \( \mathcal{BI} \not\subseteq \text{conv} \), i.e., that \( \text{conv} \) is unborning. We will use item (v) of Proposition 4.4. Fix \( f : \mathbb{Q} \cap [0, 1] \rightarrow \omega^2 \). If \( f^{-1}[\{(i, j)\}] \notin \mathcal{I} \) for some \( (i, j) \in \omega^2 \) then put \( A = f^{-1}[\{(i, j)\}] \) and observe that \( f[A] \in \text{Fin}^2 \), \( A \notin \mathcal{I} \) and \( f[A] \) is constant. On the other hand, if \( f^{-1}[\{(i, j)\}] \in \mathcal{I} \) for each \( (i, j) \in \omega^2 \), then fix any bijection \( h : \omega \rightarrow \omega^2 \), enumerate all basic open sets in \([0, 1]\) as \( \{U_n : n \in \omega\} \) and pick \( q_n \in U_n \setminus \bigcup_{i<n} f^{-1}[\{h(i)\}] \) for all \( n \in \omega \). This can be done as \( U_n \cap \mathbb{Q} \notin \text{conv} \), for each \( n \in \omega \). Then \( B = \{q_n : n \in \omega\} \notin \text{conv} \) and \( f[B] \) is finite-to-one.

Observe that \( \text{Fin}^2 \not\subseteq \text{conv} \). Indeed, given any bijection \( g : B \rightarrow \omega^2 \) such that \( g^{-1}[\{n\}] \in \text{conv} \) for all \( n \in \omega \), one can inductively pick \( r_n \in (U_n \cap B) \setminus \bigcup_{i<n} g^{-1}[\{i\} \times \omega] \) (as \( U_n \cap B \notin \text{conv} \)). Then \( D = \{r_n : n \in \omega\} \subseteq B \), \( D \notin \text{conv} \) and \( g[D] \in 0 \otimes \text{Fin} \subseteq \text{Fin}^2 \).

By Proposition 2.6, \( \text{Fin}^2 \not\subseteq_{KB} \text{conv} \), so there is a \( C \in \text{Fin}^2 \) such that \( E = B \cap f^{-1}[C] \notin \text{conv} \). Then \( f[E] \in \text{Fin}^2 \) and \( f[E] \) is finite-to-one (as \( E \subseteq B \)). Hence, \( \text{conv} \) is unborning.

Finally, we show that \( \text{Fin}^2 \not\subseteq \mathcal{BI} \). Let \( f : \omega^3 \rightarrow \omega^2 \) be any bijection. Without loss of generality we can assume that \( f^{-1}[\{i\} \times \omega] \in \mathcal{BI} \) for all \( i \in \omega \) (otherwise we are done). Fix any \( g : \omega \rightarrow \omega \) such that \( g^{-1}[\{n\}] \) is infinite for all \( n \in \omega \). Pick inductively points \( x_n \in \omega^3 \) in such a way that

\[
\{x_n : n \in \omega\} \notin \mathcal{BI} \quad \text{(as \( X(n) \) is infinite for all \( n \in \omega \), but \( f[X] \in 0 \otimes \text{Fin} \subseteq \text{Fin}^2 \)).}
\]

\( \square \)

**Corollary 4.7.** Let \( \mathcal{I} \) be a maximal ideal. Then \( \mathcal{I}^* \) is a P-point if and only if \( \mathcal{I} \) is unborning.

**Proof.** Follows from Propositions 2.7 and 4.6. \( \square \)

### 4.2. Hereditary weak P-ideals.

The following definition will play a very important role in our considerations.

**Definition 4.8.** An ideal \( \mathcal{I} \) is a hereditary weak P-ideal if \( \text{Fin}^2 \not\subseteq \mathcal{I}[A] \) for each \( A \notin \mathcal{I} \).

An ideal \( \mathcal{I} \) is a P\(^+$\)-ideal if for every decreasing sequence \( (A_n) \) with \( A_n \notin \mathcal{I} \), for all \( n \in \omega \), there is an \( A \notin \mathcal{I} \) such that \( A \setminus A_n \) is finite for each \( n \in \omega \). It is known that each \( \Sigma_2^0 \)-ideal is a P\(^+$\)-ideal (cf. [7, Proposition 5.1]).

We say that an ideal \( \mathcal{I} \) on \( X \) is weakly selective if for each partition \( (A_n) \) of \( X \) such that \( A_0 \notin \mathcal{I}^* \) and \( A_{n+1} \in \mathcal{I} \), for all \( n \in \omega \), there is an \( S \notin \mathcal{I} \) with \( |S \cap A_n| \leq 1 \), for all \( n \in \omega \). It is easy to see that \( \mathcal{I} \) is weakly selective if and only if \( \mathcal{ED} \not\subseteq_{\mathcal{K}} \mathcal{I}[A] \).
for all $A \notin \mathcal{I}$ [28, Section 2.7], where
$$\mathcal{E}\mathcal{D} = \{ A \subseteq \omega^2 : (\exists k, m \in \omega) \ (\forall i > k) \ |A_{(i)}| < m \}.$$  

**Proposition 4.9.** Suppose that $\mathcal{I}$ satisfies at least one of the following conditions:

1. $\mathcal{I}$ is $\Pi^0_3$;
2. $\mathcal{I}$ is a $P^+$-ideal.
3. $\mathcal{I}$ is weakly selective.

Then $\mathcal{I}$ is a hereditary weak $P$-ideal.

**Proof.** (a): Observe that $\mathcal{I}|B$ is $\Pi^0_3$, for all $B \notin \mathcal{I}$ (as the identity function from $\mathcal{P}(B)$ to $\mathcal{P}(\bigcup \mathcal{I})$ is continuous). By [3, Theorems 7.5 and 9.1], a $\Pi^0_3$ ideal cannot contain an isomorphic copy of $\text{Fin}^2$. 

(b): Follows from [28, Theorem 2.4.2]. 

(c): Since $\mathcal{I}$ is weakly selective, $\mathcal{E}\mathcal{D} \not\subseteq \mathcal{I}|A$ for each $A \notin \mathcal{I}$. It follows that $\text{Fin}^2 \not\subseteq \mathcal{I}|A$ for each $A \notin \mathcal{I}$ (otherwise we would get $\mathcal{E}\mathcal{D} \subseteq \text{Fin}^2 \subseteq \mathcal{I}|A$).

**Proposition 4.10.** The following hold for any ideal $\mathcal{I}$:

1. $\mathcal{I}$ is a hereditary weak $P$-ideal $\implies$ $\mathcal{I}$ is unboring $\implies$ $\mathcal{I}$ is a weak $P$-ideal (equivalently, $\text{Fin}^2 \not\subseteq \mathcal{I} \implies \mathcal{B}\mathcal{I} \not\subseteq \mathcal{I} \implies (\exists \mathcal{A} \notin \mathcal{I}) \text{Fin}^2 \not\subseteq \mathcal{I}|A)$.
2. Both implications from item (a) cannot be reversed.
3. If $\mathcal{I}$ is extendable to an unboring ideal then $\mathcal{I}$ is unboring. In particular, if $\mathcal{I}$ is extendable to a hereditary weak $P$-ideal then $\mathcal{I}$ is unboring.

**Proof.** (a): If $\text{Fin}^2 \not\subseteq \mathcal{I}$ then $\mathcal{B}\mathcal{I} \not\subseteq \mathcal{I}$ by Proposition 4.6. This shows that $\mathcal{I}$ being unboring implies that $\mathcal{I}$ is a weak $P$-ideal.

To show the first implication, let $\mathcal{I}$ be a hereditary weak $P$-ideal. We need to show that $\mathcal{I}$ is unboring.

We will use item (v) of Proposition 4.4. Fix $f : \omega \to \omega^2$. If $f^{-1}[\{(i, j)\}] \notin \mathcal{I}$ for some $(i, j) \in \omega^2$ then put $A = f^{-1}[\{(i, j)\}]$ and observe that $f[A] \in \text{Fin}^2$, $A \notin \mathcal{I}$ and $f|A$ is constant. On the other hand, if $f^{-1}[\{(i, j)\}] \in \mathcal{I}$ for each $(i, j) \in \omega^2$, then $\{f^{-1}[\{(i, j)\}] : (i, j) \in \omega^2\}$ defines a partition of $\omega$ into sets from $\mathcal{I}$. As $\text{Fin}^2 \not\subseteq \mathcal{I}$, there is a $B \notin \mathcal{I}$ such that $f|B$ is finite-to-one. Moreover, since $\text{Fin}^2 \not\subseteq \mathcal{K} B \mathcal{I}|B$ (by Proposition 2.6), there is a $C \in \text{Fin}^2$ such that $A = B \cap f^{-1}[C] \notin \mathcal{I}|B$. Then $A \notin \mathcal{I}$, $f[A] \in \text{Fin}^2$ and $f|A$ is finite-to-one. 

(b): To show that $\text{Fin}^2 \not\subseteq \mathcal{I} \implies \mathcal{B}\mathcal{I} \not\subseteq \mathcal{I}$ cannot be reversed, it suffices to consider the ideal $\mathcal{B}\mathcal{I}$, which is boring, but $\text{Fin}^2 \not\subseteq \mathcal{B}\mathcal{I}$ (by Proposition 4.6).

Now we show that the other implication cannot be reversed. The ideal conv is a good example. By Proposition 4.6, conv is unboring. On the other hand, for each $n \in \omega$ pick a sequence $(x_{n, k})_k \subseteq \mathbb{Q} \cap (\frac{1}{n+2}, \frac{1}{n+1})$ converging to $\frac{1}{n+2}$. Then $\text{Fin}^2 \not\subseteq \text{conv}|A$, where $A = \{ x_{n, k} : n, k \in \omega \}$ (as witnessed by the function $f : A \to \omega^2$ given by $f(x_{n, k}) = (n, k)$).

(c): Let $\mathcal{J}$ be an unboring ideal containing $\mathcal{I}$. Then $\mathcal{I}$ cannot be boring as otherwise we would get $\mathcal{B}\mathcal{I} \subseteq \mathcal{I} \subseteq \mathcal{J}$, which contradicts the fact that $\mathcal{J}$ is unboring. The second part follows from item (a).

By the above result, if $\mathcal{I}$ is extendable to a hereditary weak $P$-ideal, then $\mathcal{I}$ is unboring. In the next subsection we show that this implication cannot be reversed even for Borel ideals. 

**Proposition 4.11.** $\mathcal{I}$ is a hereditary weak $P$-ideal if and only if $\mathcal{I}$ is hereditary unboring (that is, $\mathcal{B}\mathcal{I} \subseteq \mathcal{I}|B$ for no $B \notin \mathcal{I}$).
Proof. If $\mathcal{I}$ is a hereditary unboring ideal then the existence of $B \notin \mathcal{I}$ with $\Fin^2 \preceq \mathcal{I}B$ would imply $\mathcal{B}\mathcal{I} \preceq \mathcal{I}B$ (by Proposition 4.6). Thus, $\mathcal{I}$ is a hereditary weak $P$-ideal.

Conversely, assume that $\mathcal{I}$ is a hereditary weak $P$-ideal and fix $B \notin \mathcal{I}$. If $\mathcal{B}\mathcal{I} \preceq \mathcal{I}B$ and $f : B \to \omega^2$ is the witnessing bijection, then either $f^{-1}([i] \times \omega^2) \notin \mathcal{I}$ for all $i \in \omega$ or $f^{-1}([n] \times \omega^2) \notin \mathcal{I}$ for some $n \in \omega$. In the former case, $(f^{-1}([i] \times \omega^2))_{i \in \omega}$ is a partition of $B$ into sets belonging to $\mathcal{I}$ such that each $X \subseteq B$ with $X \cap f^{-1}([i] \times \omega^2) \in \Fin(B)$ for all $i \in \omega$ is in $\mathcal{I}$, i.e., $\Fin^2 \preceq \mathcal{I}B$. In the latter case, $(f^{-1}([n] \times \omega^2))_{i \in \omega}$ is a partition of $C = f^{-1}([n] \times \omega^2) \notin \mathcal{I}$ into sets belonging to $\mathcal{I}$ such that each $X \subseteq C$ with $X \cap f^{-1}([n] \times \omega^2) \in \Fin(C)$ for all $i \in \omega$ is in $\mathcal{I}$, i.e., $\Fin^2 \preceq \mathcal{I}C$. Thus, both cases contradict the assumption that $\mathcal{I}$ is a hereditary weak $P$-ideal. \hfill $\square$

Corollary 4.12. The following are equivalent for any $\leq_k$-uniform ideal $\mathcal{I}$:

(a) $\mathcal{I}$ is a weak $P$-ideal;
(b) $\mathcal{I}$ is unboring;
(c) $\mathcal{I}$ is a hereditary weak $P$-ideal;
(d) $\mathcal{I}$ is hereditary unboring (that is, $\mathcal{B}\mathcal{I} \preceq \mathcal{I}B$ for no $B \notin \mathcal{I}$).

Proof. (c)$\iff$(d): This is Proposition 4.11.
(c)$\implies$(b) and (b)$\implies$(a): This is item (a) of Proposition 4.10.
(a)$\implies$(c): Suppose to the contrary that $\Fin^2 \preceq \mathcal{I}A$ for some $A \notin \mathcal{I}$. Since $\mathcal{I}$ is $\leq_k$-uniform, we have $\Fin^2 \preceq \mathcal{I}A \leq_k \mathcal{I}$ which is equivalent to $\mathcal{I}$ not being a weak $P$-ideal (by Propositions 2.5 and 2.6). Thus, $\Fin^2 \nsubseteq \mathcal{I}A$ for all $A \notin \mathcal{I}$. \hfill $\square$

We end this subsection with an observation about extendability to hereditary weak $P$-ideals.

Proposition 4.13. If $\mathcal{I}$ is extendable to a hereditary weak $P$-ideal then there is a smallest (in the sense of inclusion) hereditary weak $P$-ideal $\widehat{\mathcal{I}}$ containing $\mathcal{I}$.

Proof. Define $\widehat{\mathcal{I}} = \bigcap\{\mathcal{J} : \mathcal{J}$ is a hereditary weak $P$-ideal containing $\mathcal{I}\}$. Then $\mathcal{I} \subseteq \widehat{\mathcal{I}}$ and $\widehat{\mathcal{I}}$ is an ideal as an intersection of ideals. We need to show that $\widehat{\mathcal{I}}$ is a hereditary weak $P$-ideal. Let $B \notin \widehat{\mathcal{I}}$. Then there is a hereditary weak $P$-ideal $\mathcal{J}$ containing $\mathcal{I}$ with $B \notin \mathcal{J}$. Thus, $\Fin^2 \preceq \mathcal{J}B$ would imply $\Fin^2 \preceq \mathcal{I}B \subseteq \mathcal{J}B$ which contradicts the fact that $\mathcal{J}$ is a hereditary weak $P$-ideal. \hfill $\square$

4.3. A technical notion. Following [27] we say that an ideal $\mathcal{I}$ on $X$ is $\omega$-diagonalizable by $\mathcal{I}^*$-universal sets, if one can find a family $\{Z_k : k \in \omega\}$ such that:

- for each $k \in \omega$ the set $Z_k \subseteq [X]^{<\omega} \setminus \{\emptyset\}$ is $\mathcal{I}^*$-universal, which means that for each $A \in \mathcal{I}$ there is a $Z \in Z_k$ with $Z \cap A = \emptyset$;
- for each $A \in \mathcal{I}$ there is a $k \in \omega$ such that $Z \nsubseteq A$ for every $Z \in Z_k$.

The following notion will be useful in considerations about replacing CH with MA($\sigma$-centered) in some of our results.

Definition 4.14. We say that an ideal $\mathcal{I}$ is strongly unboring if for each $f \in \mathcal{D}\mathcal{I}$ there is a $C \notin \mathcal{I}$ such that $f|C$ is finite-to-one and $\mathcal{I}C$ is $\omega$-diagonalizable by $(\mathcal{I}C)^*$-universal sets.

Lemma 4.15 (Essentially [26] and [27]). If $\mathcal{I}$ is a coanalytic ideal and $\Fin^2 \nsubseteq \mathcal{I}$ then $\mathcal{I}$ is $\omega$-diagonalizable by $\mathcal{I}^*$-universal sets.
Proof. Consider the game $G(I)$, defined by Laflamme (see [27]) as follows: Player I in his $n$'th move plays an element $C_n \in I$, and then Player II responds with any $F_n \in [\bigcup I]^{<\omega}$ such that $F_n \cap C_n = \emptyset$. Player I wins if $\bigcup_{n<\omega} F_n \in I$. Otherwise, Player II wins. By [24, Theorem 5.1], $G(I)$ is determined as $I$ is a coanalytic ideal. By [27, Theorem 2.16], Player I has a winning strategy in $G(I)$ if and only if $\text{Fin}^2 \subseteq I$. Thus, Player II has to have a winning strategy. Again by [27, Theorem 2.16], this is in turn equivalent to $I$ being $\omega$-diagonalizable by $\mathcal{I}^*$-universal sets. □

The above was first observed by M. Laczkovich and I. Reclaw in [26]. However, their observation concerned Borel ideals. In [24] it was shown, using an idea from [23], that this is also true for coanalytic ideals. See also [6] for some applications of this result.

Proposition 4.16. The following hold for any ideal $I$:

(a) If $I$ is extendable to a strongly unboring ideal then $I$ is strongly unboring.
(b) $I$ is extendable to a coanalytic hereditary weak P-ideal $\implies I$ is strongly unboring.$\implies I$ is unboring.
(c) There is a Borel strongly unboring (so also unboring, by the previous item) ideal not extendable to a hereditary weak P-ideal. In particular, the first implication from item (b) cannot be reversed even for Borel ideals.
(d) If $U$ is a $P$-point then $U^*$ is an unboring ideal which is not strongly unboring.

In particular, under $\text{MA}(\sigma$-centered) there are unboring ideals which are not strongly unboring.

Proof. (a): This is easy to verify using the definition of strongly unboring ideals.

(b): At first we will prove the first implication. Let $J$ be a coanalytic hereditary weak P-ideal containing $I$. We will show that $J$ is strongly unboring. By item (a), this will finish the proof.

Fix $f \in D_J$. Since $(f^{-1}[\{n\}])$ defines a partition of $\bigcup J$ into sets belonging to $J$ and $\text{Fin}^2 \not\subseteq J$, there is a $C \notin J$ such that $f|C$ is finite-to-one. Note that $J|C$ is a coanalytic ideal (as $J$ is coanalytic and the identity function from $\mathcal{P}(C)$ to $\mathcal{P}(\bigcup J)$ is continuous). Since $J$ is a hereditary weak P-ideal, $\text{Fin}^2 \not\subseteq J|C$. Thus, by Lemma 4.15, $J|C$ is $\omega$-diagonalizable by $(J|C)^*$-universal sets. Hence, $C$ is the required set.

Now we show the second implication. Suppose that $I$ is boring, i.e., $\mathcal{B} \subseteq I$. Without loss of generality we may assume that $\bigcup I = \omega^3$ and $\mathcal{B} \subseteq I$ (by considering an appropriate isomorphic copy of $I$). Fix any bijection $h : \omega \to \omega^2$ and define $f : \bigcup I \to \omega$ by $f([h(n)]) \times \omega = \{n\}$. Then $f \in D_{\mathcal{B}I} \subseteq D_I$. Let $B \notin I$ be such that $f|B$ is finite-to-one. Denote by $C = \{(i,j) \in \omega^2 : B \cap \{\{(i,j)\} \times \omega\} \neq \emptyset\}$. Then $C \notin \text{Fin}^2$ (as otherwise we would have $B \in \mathcal{B} \subseteq I$). As $\text{Fin}^2$ is homogeneous (see Example 2.3), $\text{Fin}^2|C \cong \text{Fin}^2$. Let $g : C \to \omega^2$ be the witnessing isomorphism and denote by $\pi_{1,2} : \omega^3 \to \omega^2$ the projection onto the first two coordinates (i.e., $\pi_{1,2}(x,y,z) = (x,y)$ for all $(x,y,z) \in \omega^3$). Then $g \circ \pi_{1,2}|B$ witnesses that $\text{Fin}^2 \leq_{KB} \mathcal{B}I|B \subseteq I|B$. By Proposition 2.6, this means that $\text{Fin}^2 \subseteq I|B$. Using the witnessing bijection it is easy to check that $I|B$ being $\omega$-diagonalizable by $(I|B)^*$-universal sets would imply that $\text{Fin}^2$ is $\omega$-diagonalizable by $(\text{Fin}^2)^*$-universal sets. However, this is impossible by [6, Theorem 6.2] (see also [3, Theorem 7.5] or [26, Theorem 5]). Hence, $I|B$ cannot be $\omega$-diagonalizable by $(I|B)^*$-universal sets. Therefore, each boring ideal is not strongly unboring.

(c): Consider the ideal $I$ on $\omega^4$ generated by:
• sets \( \{(i, j, k)\} \times \omega \), for all \((i, j, k) \in \omega^3\);
• sets \( B \subseteq \omega^4 \) such that \( B \cap \{(i, j, k)\} \times \omega \) is finite for all \((i, j, k) \in \omega^3\) and \( \{(i, j, k) \in \omega^3 : B \cap \{(i, j, k)\} \times \omega \neq \emptyset \} \in B(I) \).

Equivalently, \( I = (B(I) \otimes \emptyset) \cap \text{Fin}(\omega^3) \otimes \text{Fin} \). It is easy to see that \( I \) is Borel.

First we show that \( I \) is not extendable to a hereditary weak P-ideal. Suppose otherwise, i.e., that there is a hereditary weak P-ideal \( I' \supseteq I \). If \( \{(i, j)\} \times \omega^2 \notin I' \) for some \((i, j) \in \omega^2\), then \( I'_{i,j} = \{ A \subseteq \omega^4 : A \cap \{(i, j)\} \times \omega^2 \in I' \} \) is an ideal on \( \omega^4 \). It is easy to see that \( I' \) being a hereditary weak P-ideal implies that \( I'_{i,j} \) is a hereditary weak P-ideal as well. However, \( I'_{i,j} \supseteq \{ A \subseteq \omega^4 : A \cap \{(i, j)\} \times \omega^2 \in I \} \approx \text{Fin}^2 \)
and \( \text{Fin}^2 \) is not extendable to a hereditary weak P-ideal (by Propositions 4.6 and 4.10(c)). Therefore \( \{(i, j)\} \times \omega^2 \notin I' \) for all \((i, j) \in \omega^2\). However, this implies that \( I' \) is boring (as witnessed by any bijection \( f : \omega^4 \to \omega^3 \) with \( f\{(i, j)\} \times \omega^2 = \{(i, j)\} \times \omega \)). Recall that a hereditary weak P-ideal cannot be boring (by Proposition 4.10(a)). Thus, we get a contradiction which proves that \( I \) is not extendable to a hereditary weak P-ideal.

Now we show that \( I \) is strongly unboring. Fix \( f \in D_I \) and a bijection \( h : \omega \to \omega^3 \). Inductively pick points \( x_n \in \omega^3 \) such that:

• \( x_n \in \{ h(n) \} \times \omega \);
• if \( f\{h(n)\} \times \omega \setminus f\{\{x_i : i < n\}\} \) is nonempty then \( f(x_n) \) belongs to that set;
• if \( f\{h(n)\} \times \omega \subseteq f\{\{x_i : i < n\}\} \) then \( f(x_n) = \max f\{\{x_i : i < n\}\} \).

Then \( X = \{ x_n : n \in \omega \} \notin I \) since \( |X \cap \{(i, j, k)\} \times \omega| = 1 \) for all \((i, j, k) \in \omega^3\). Moreover, we have \( \text{Fin}^2 \not\subseteq I X \) (by Proposition 4.6, as \( B(I) \) and \( I \) are isomorphic).

Since \( I \) is Borel, so is \( I X \) (as the identity function from \( X \) to \( \omega^3 \) is continuous). Thus, by Lemma 4.15, \( I X \) is \( \omega \)-diagonalizable by \( (I X)^* \)-universal sets. To finish we only need to show that \( f X \) is finite-to-one.

Fix \( n \in \omega \). By the construction of \( X \), \( X \cap f^{-1}\{n\} \) can be infinite only if there were infinitely many \( m \in \omega \) with \( f\{h(m)\} \times \omega \subseteq \{0, 1, \ldots, n\} \), i.e., \( \{(h(m)) \times \omega \subseteq f^{-1}\{0, 1, \ldots, n\} \} \). However, \( f^{-1}\{0, 1, \ldots, n\} \in I \subseteq \text{Fin}(\omega^3) \otimes \text{Fin} \). Consequently, there are only finitely many \( m \in \omega \) covered by \( f^{-1}\{\{0, 1, \ldots, n\}\} \). Thus, \( f X \) is finite-to-one.

(d): Let \( \mathcal{U} \) be a P-point. Denote \( I = \mathcal{U}^* \). By Proposition 2.7, \( \text{Fin}^2 \not\subseteq I \). Since \( I \) is a maximal ideal, it is homogeneous (see Example 2.3). Thus, \( I \) is a hereditary weak P-ideal. This means that it is unboring (by Proposition 4.10(a)). On the other hand, we will show that for each \( C \notin I \) the ideal \( I \) is not \( \omega \)-diagonalizable by \( (I \mathcal{C})^* \)-universal sets (hence \( I \) cannot be strongly unboring). Fix any \( C \notin I \). By the proof of [26, Theorem 4], \( I \mathcal{C} \) being \( \omega \)-diagonalizable by \( (I \mathcal{C})^* \)-universal sets means that there is a \( \Sigma_2^\mathcal{C} \) set \( S \) with \( I \mathcal{C} \subseteq S \) and \( S \cap (I \mathcal{C})^* = \emptyset \). Since \( I \) is homogeneous, \( I \mathcal{C} \cong I \). In particular, \( I \mathcal{C} \) is a maximal ideal on \( C \), so \( I \mathcal{C} \cap (I \mathcal{C})^* = P(C) \) and \( S = I \mathcal{C} \). However, \( I \mathcal{C} \) is not Borel (as a maximal ideal). This contradiction finishes the proof.

5. Mrówka spaces

5.1. Results in ZFC. If \( A \subseteq [\omega^\omega] \) is an AD family then by \( \text{Fin}^2(A) \) we denote the ideal on \( \omega \) generated by sets belonging to \( A \) and by sets having finite intersection with each member of \( A \).
Proposition 5.1. The following are equivalent for any ideal $\mathcal{I}$ and any AD family $\mathcal{A}$:

(a) $\Phi(\mathcal{A})$ is in $\text{FinBW}(\mathcal{I})$;
(b) for every function $f \in \mathcal{D}_{\mathcal{I}}$ there is a $B \notin \mathcal{I}$ such that $f|B$ is finite-to-one and $f[B] \in \text{Fin}^2(\mathcal{A})$.

Proof. Without loss of generality we may assume that $\mathcal{I}$ is an ideal on $\omega$.
(a) $\implies$ (b): Let $X = \Phi(\mathcal{A})$ be in $\text{FinBW}(\mathcal{I})$. Assume to the contrary that (b) does not hold. Then there is an $f : \omega \to \omega$ satisfying $f^{-1}\{\{n\}\} \in \mathcal{I}$ for all $n \in \omega$ such that for each $B \notin \mathcal{I}$ with $f|B$ finite-to-one we have $f[B] \cap A \notin \text{Fin}$ for infinitely many $A \in \mathcal{A}$. We will show that the sequence $(f(n)) \subseteq X$ does not possess a convergent subsequence indexed by a set not belonging to $\mathcal{I}$.

Suppose that $B \notin \mathcal{I}$ and consider $(f(n))_{n \in B}$. Since $f^{-1}\{\{n\}\} \in \mathcal{I}$ for all $n \in \omega$, $(f(n))_{n \in B}$ cannot converge to any $n \in \omega$. Moreover, $(f(n))_{n \in B}$ cannot converge to any $A \in \mathcal{A}$, as there is an $A' \in \mathcal{A}$, $A \neq A'$ with $|f[B] \cap (A' \setminus A)| = |f[B] \cap A'| = \omega$. Finally, since $f[B] \cap A \notin \text{Fin}$ for some $A \in \mathcal{A}$, $(f(n))_{n \in B}$ cannot converge to $\omega$ (as $U = X \setminus \{\{A\} \cup A\}$ is an open neighborhood of $\omega$ such that infinitely many elements of the sequence $(f(n))_{n \in B}$ are outside $U$).

(b) $\implies$ (a): We need to show that $X = \Phi(\mathcal{A})$ is in $\text{FinBW}(\mathcal{I})$. Fix any $f : \omega \to X$. Without loss of generality we can assume that $f^{-1}\{\{n\}\} \in \mathcal{I}$ for all $x \in X$ (otherwise we are done as we would get a constant, so converging, sequence indexed by an $\mathcal{I}$-positive set). We have two possibilities: either $\text{Fin}^2 \nsubseteq \mathcal{I}f^{-1}[X \setminus \omega]$ or $\text{Fin}^2 \nsubseteq \mathcal{I}[f^{-1}[X \setminus \omega]$.

In the former case, we can find $B \notin \mathcal{I}$ such that $f[B] \subseteq X \setminus \omega$ and $f|B$ is finite-to-one. We claim that $(f(n))_{n \in B}$ converges to $\omega$. Indeed, let $U$ be a basic neighborhood of $\omega$. Then $(X \setminus \omega) \setminus U$ is finite. Since $f|B$ is finite-to-one and $f[B] \subseteq X \setminus \omega$, almost all elements of $(f(n))_{n \in B}$ have to belong to $U$.

In the latter case, there is a bijection $h : f^{-1}[X \setminus \omega] \to \omega^2$ witnessing $\text{Fin}^2 \subseteq \mathcal{I}[f^{-1}[X \setminus \omega]$. Define $g : f^{-1}[X \setminus \omega] \to \omega$ by $g(n) = \pi_1(h(n))$, for all $n \in f^{-1}[X \setminus \omega]$, where $\pi_1 : \omega^2 \to \omega$ is the projection onto the first coordinate. Let $g : \omega \to \omega$ be given by $g(n) = f(n)$, for all $n \in f^{-1}[\omega]$, and $g(n) = h(n)$ for all $n \in f^{-1}[\omega]$. Then $g^{-1}\{\{n\}\} \subseteq h^{-1}\{\{n\}\} \times \omega \cup f^{-1}\{\{n\}\} \in \mathcal{I}$, so there is a $B \notin \mathcal{I}$ such that $g[B] \in \mathcal{D}_{\text{FinBW}}$ and $g[B] \in \text{Fin}^2(\mathcal{A})$. Define $C = B \cap f^{-1}[\omega]$. Then $f[C] = g[C]$ is finite-to-one and $f[C] \in \text{Fin}^2(\mathcal{A})$.

We claim that $C \notin \mathcal{I}$. Indeed, denote $D = B \cap f^{-1}[\omega]$. Since $g[D] \in \mathcal{D}_{\text{FinBW}}$ and $D \subseteq f^{-1}[\omega]$, the function $h|D$ is finite-to-one. This implies $h[D] \in \Phi \cap \text{Fin} \subseteq \text{Fin}^2$ and hence $D \in \mathcal{I}[f^{-1}[\omega]$ (by the choice of $h$). As $C \cup D = B \notin \mathcal{I}$, we get that $C \notin \mathcal{I}$.

Since $C \notin \mathcal{I}$ and $f[C] \in \text{Fin}^2(\mathcal{A})$, there are two possibilities:

- either there are $C' \subseteq C$, $C' \notin \mathcal{I}$ and $A \in \mathcal{A}$ with $f[C'] \subseteq A$ (in this case $(f(n))_{n \in C'}$ converges to $A$);
- or there is a $C' \subseteq C$, $C' \notin \mathcal{I}$ with $f[C'] \cap A \in \text{Fin}$ for all $A \in \mathcal{A}$ (in this case $(f(n))_{n \in C'}$ converges to $\omega$).

This finishes the proof. \qed

Lemma 5.2. Let $\mathcal{I}$ be an ideal. If $\mathcal{A}$ is an AD family such that for every $f \in \mathcal{D}_{\mathcal{I}}$ there is a $B \notin \mathcal{I}$ such that $f|B$ is finite-to-one and $f[B] \subseteq A$ for some $A \in \mathcal{A}$, then $\mathcal{A}$ is a MAD family.
Proof. Suppose otherwise and let $C \subseteq \omega$ be infinite with $C \cap A \in \Fin$ for all $A \in \mathcal{A}$. Fix any bijection $f : \bigcup \mathcal{I} \to C$. Then $f \in \mathcal{D}_{\Fin(\bigcup \mathcal{I})} \subseteq \mathcal{D}_\omega$, but for any $B \subseteq \bigcup \mathcal{I}$ the condition $f[B] \subseteq A$, for some $A \in \mathcal{A}$, implies that $B \in \Fin(\bigcup \mathcal{I}) \subseteq \mathcal{I}$. Thus, the function $f$ would contradict the assumptions of this lemma. \hfill \box

5.2. Consequences of CH and MA.

**Theorem 5.3.** (CH) If $\mathcal{I}$ is un boring, then there is an uncountable Mrówka space in $\Fin B W(\mathcal{I})$.

**Proof.** By Proposition 5.1 and Lemma 5.2, it suffices to show that there is an AD family $\mathcal{A}$ such that for every $f \in \mathcal{D}_\omega$ there is a $B \notin \mathcal{I}$ such that $f[B]$ is finite-to-one and $f[B] \subseteq A$ for some $A \in \mathcal{A}$.

Fix a list $\{f_\alpha : \omega \leq \alpha < \omega_1\} = \mathcal{D}_\omega$. We will construct a family $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ by induction on $\alpha$. The enumeration may contain repetitions. At first let $\{A_\alpha : n \in \omega\}$ be any partition of $\omega$ into infinite sets.

Suppose that $\omega \leq \alpha < \omega_1$ and that $A_\beta$, for all $\beta < \alpha$, have been chosen. Using CH, enumerate $\alpha$ as $\langle \beta_i \rangle_{i \in \omega}$ and consider the partition $\langle B_i \rangle$ of $\omega$, where $B_0 = A_\beta_0$ and $B_i = A_\beta_i \setminus \bigcup_{j < i} A_\beta_j$, for all $0 < i < \omega$. Let $h : \omega \to \omega^2$ be any injection satisfying $h[B_i] \subseteq \{i\} \times \omega$. Define $f : \omega \to \omega^2$ by $f = h \circ f_\alpha$. Since $\mathcal{I}$ is un boring, there is a $C \notin \mathcal{I}$ such that $f[C] \in \Fin^2$ and $f[C]$ is either constant or finite-to-one (by Proposition 4.4(v)).

Observe that $f[C]$ cannot be constant, as $f_\alpha^{-1}[\{n\}] \in \mathcal{I}$ for all $n \in \omega$ and $h$ is an injection. Thus, $f[C]$ is finite-to-one. Since $f[C] \in \Fin^2$ and $C \notin \mathcal{I}$, either $C \cap f^{-1}[\{n\} \times \omega] \notin \mathcal{I}$ for some $n \in \omega$ or $C \cap f^{-1}[D] \notin \mathcal{I}$ for some $D \in \emptyset \otimes \Fin$.

In the former case put $B = C \cap f^{-1}[\{n\} \times \omega]$ and $A_\alpha = A_{\beta_n}$. Then $B \notin \mathcal{I}$, $f_\alpha[B]$ is finite-to-one and $f_\alpha[B] \subseteq B \subseteq A_\alpha$. In the latter case put $B = C \cap f^{-1}[D]$ and $A_\alpha = f_\alpha[B]$. Then $B \notin \mathcal{I}$, $f_\alpha[B]$ is finite-to-one and $f_\alpha[B] \subseteq A_\alpha$. Moreover, $A_\alpha \cap A_\beta_i$ is finite for each $i \in \omega$ as $D \in \emptyset \otimes \Fin$. \hfill \box

**Theorem 5.4.** (MA($\sigma$-centered)) If $\mathcal{I}$ is strongly un boring, then there is an uncountable separable Mrówka space in $\Fin B W(\mathcal{I})$.

**Proof.** By Proposition 5.1 and Lemma 5.2, it suffices to show that there is an AD family $\mathcal{A}$ such that for every $f \in \mathcal{D}_\omega$ there is a $B \notin \mathcal{I}$ such that $f[B]$ is finite-to-one and $f[B] \subseteq A$ for some $A \in \mathcal{A}$.

Let $\{f_\alpha : \omega \leq \alpha < \tau\}$ be an enumeration of $\mathcal{D}_\omega$. We will construct a family $\mathcal{A} = \{A_\alpha : \alpha < \tau\}$ by induction on $\alpha$. The enumeration may contain repetitions. At first let $\{A_\alpha : n \in \omega\}$ be any partition of $\omega$ into infinite sets.

Suppose that $\omega \leq \alpha < \tau$ and that $A_\beta$ for all $\beta < \alpha$ have been chosen. Consider the function $f_\alpha$. Since $\mathcal{I}$ is strongly un boring, we can find $C \notin \mathcal{I}$ such that $f_\alpha[C]$ is finite-to-one and $\mathcal{I}[C]$ is $\omega$-diagonalizable by $\langle \mathcal{I}[C] \rangle^\ast$-universal sets.

If there is a $\beta < \alpha$ such that $D = C \cap f^{-1}[A_\beta] \notin \mathcal{I}$ then put $A_\alpha = A_\beta$ and note that $f_\alpha[D]$ is finite-to-one and $f_\alpha[D] \subseteq A_\alpha$.

Otherwise, $C \cap f^{-1}[A_\beta] \in \mathcal{I}$ for all $\beta < \alpha$. Consider the following poset: conditions of $\mathcal{P}$ are pairs $p = (B_p, F_p) \in [C]^{<\omega} \times [\alpha]^{<\omega}$ and for $p, q \in \mathcal{P}$ let $q \vdash p$ if and only if $B_p \subseteq B_q, F_p \subseteq F_q$ and $B_q \setminus B_p \subseteq C \setminus f^{-1}_\alpha[\bigcup_{\beta \in F_p} A_\beta]$. Note that $\mathcal{P}$ is $\sigma$-centered as $\mathcal{P} = \bigcup_{B \in [C]^{<\omega}} \{p \in \mathcal{P} : B_p = B\}$ and given any $B \in [C]^{<\omega}$ and any $p$ and $q$ with $B_p = B_q = B$ we have $(B, F_p \cup F_q) \vdash p$ and $(B, F_p \cup F_q) \vdash q$.
Observe that for each $\beta < \alpha$ the open set $D_\beta = \{ p \in P : \beta \in F_p \}$ is dense in $P$.
Indeed, given any $p \in P$ put $q = (B_p \cup F_p \cup \{ \beta \})$ and note that $q \in D_\beta$ and $q \vdash p$.

Let $\{ Z_k : k \in \omega \}$ be the family witnessing $\omega$-diagonalizability of $\mathcal{I}|C$ by $(\mathcal{I}|C)^*$-universal sets. Recall that:

- for each $k \in \omega$ the family $Z_k \subseteq [C]^\omega \setminus \emptyset$ is such that for each $D \in \mathcal{I}|C$ there is a $Z \in Z_k$ with $Z \cap D = \emptyset$;
- for each $D \in \mathcal{I}|C$ there is a $k \in \omega$ such that $Z \not\subseteq D$ for every $Z \in Z_k$.

Define open sets $E_k = \{ p \in P : \exists Z \in Z_k \subseteq B_p \}$ for all $k \in \omega$. We need to show that each $E_k$ is dense. Let $k \in \omega$ and $p \in P$. Since $C \cap f^{-1}[A_\beta] \in \mathcal{I}$ for all $\beta < \alpha$, we have

$$f^{-1}_\alpha[\bigcup_{\beta \in F_p} A_\beta] \in \mathcal{I}|C.$$ 
Thus, there is a $Z \in Z_k$ with $Z \cap f^{-1}_\alpha[\bigcup_{\beta \in F_p} A_\beta] = \emptyset$.

Put $q = (B_p \cup Z, F_p)$. Clearly, $q \in E_k$ and $q \vdash p$. Thus, $E_k$ is dense.

Let $D = \{ E_n : n \in \omega \} \cup \{ D_\beta : \beta < \alpha \}$. Then $|D| < \mathfrak{c}$ and $D$ consists of open dense sets. By MA($\sigma$-centered), there is a $D$-generic filter $G \subseteq P$. Define $B = \bigcup \{ B_p : p \in G \}$.

Observe that $B \notin \mathcal{I}|C$. Indeed, $B$ contains some $Z \in Z_k$, for each $k \in \omega$ (since $G \cap E_k \neq \emptyset$). However, $B \in \mathcal{I}|C$ would imply that there is a $k \in \omega$ such that $Z \not\subseteq B$ for every $Z \in Z_k$.

Note also that $f_\alpha[B] \cap A_\beta$ is finite for all $\beta < \alpha$. Indeed, fix $\beta < \alpha$ and observe that there is a $p \in G$ with $F_p \ni \beta$ (since $G \cap D_\beta \neq \emptyset$). We claim that $f_\alpha[B] \cap A_\beta \subseteq B_p \in [C]^\omega$. To show this inclusion, fix any $q \in G$. Since $G$ is a filter, we can find $q' \in G$ with $q' \vdash q$ and $q' \vdash p$. Then $B_q \setminus B_p \subseteq B_{q'} \setminus B_p \subseteq C \setminus f^{-1}_\alpha[\bigcup_{\beta \in F_p} A_\beta] \subseteq C \setminus f^{-1}_\alpha[A_\beta]$.

Define $A_\alpha = f_\alpha[B]$ and note that $f_\alpha[B]$ is finite-to-one (since $B \subseteq C$). This finishes the induction step and it is clear that the family $\mathcal{A} = \{ A_\alpha : \alpha < \mathfrak{c} \}$ has the required properties.

6. Main Results

**Proposition 6.1.** If $\mathcal{I}$ is a hereditary weak $P$-ideal then $\omega_1$ with order topology is in $\text{FinBW}(\mathcal{I})$. In particular, there is a non-compact space in $\text{FinBW}(\mathcal{I})$.

**Proof.** Let $g : \omega \to \omega_1$. Without loss of generality we can assume that $g^{-1}(\{ \alpha \}) \in \mathcal{I}$ for all $\alpha < \omega_1$.

Find $x \in \omega_1$ such that $B = g^{-1}(\{0, x\}) \notin \mathcal{I}$ and $g^{-1}(\{0, x\}) \in \mathcal{I}$ for all $x' < x$.

Let $(\alpha_n)_n \subseteq x$ be any increasing sequence such that $\alpha_0 = 0$ and $\lim_n \alpha_n = x$. Fix any injection $h : B \to \omega^2$ such that $h[g^{-1}(\{\alpha_n, \alpha_{n+1}\})] \subseteq \{n\} \times \omega$, for all $n \in \omega$.

Observe that $h^{-1}(\{n\} \times \omega) \subseteq g^{-1}(\{0, \alpha_{n+1}\}) \in \mathcal{I}|B$ for all $n$, as $\alpha_{n+1} < x$. Since $\mathcal{I}$ is a hereditary weak $P$-ideal, $\text{Fin}^2 \subseteq \mathcal{I}|B$. Thus, there is a $C \notin \mathcal{I}$, $C \subseteq B$ such that $g[C] \cap [\alpha_n, \alpha_{n+1})$ is finite for all $n$. This means that $(g(n))_{n \in C}$ converges to $x$. \hfill \Box

**Proposition 6.2.** If $\mathcal{I}$ is unbounding then $\omega_2 + 1$ with order topology is in $\text{FinBW}(\mathcal{I})$.

**Proof.** Let $g : \omega \to \omega^2 + 1$. Without loss of generality we can assume that $g^{-1}(\{\alpha\}) \in \mathcal{I}$ for all $\alpha < \omega^2$.

Define $X_{0,0} = g^{-1}(\{0\}) \cup g^{-1}(\{\omega^2\})$ and $X_{i,j} = g^{-1}(\{i \cdot \omega + j\})$ for all $(i, j) \in \omega^2 \setminus \{(0, 0)\}$. Then each $X_{i,j}$ belongs to $\mathcal{I}$.

By Proposition 4.4(iv), there is an $A \notin \mathcal{I}$ such that $A \cap X_{i,j}$ is finite for all $i, j \in \omega$ and $A \cap \bigcup X_{i,j}$ is finite for almost all $i \in \omega$, i.e., $\{(i, j) : A \cap X_{i,j} \neq \emptyset\} \in \text{Fin}^2$. If $A_1 = A \cap \bigcup X_{i,j} \notin \mathcal{I}$ for some $i \in \omega$ then $i+1 \cdot \omega$ is a limit of $(g(n))_{n \in A_1}$. Otherwise, if $A_1 \in \mathcal{I}$ for all $i \in \omega$ then $\omega^2$ is a limit of $(g(n))_{n \in A'}$, where $A' = \bigcup \{ A_i : A_i \text{ is finite} \}$. \hfill \Box
Proposition 6.3. If $\mathcal{I}$ is boring then each space in $\text{FinBW}(\mathcal{I})$ is boring.

Proof. We will apply Proposition 4.4(v). Suppose that $\mathcal{I}$ is boring and $f : \omega \rightarrow \omega^2$ is the witnessing function. Observe that $f^{-1}([\{i, j\}]) \in \mathcal{I}$, for all $(i, j) \in \omega^2$ (otherwise for $A = f^{-1}([\{i, j\}]) \notin \mathcal{I}$ the function $f[A]$ would be constant and $f[A]$ would belong to $\text{Fin}^2$).

Fix any unboring $X$. Then one can find an infinite set $\{x_i : i \in \omega\} \subseteq X$ and a family of injective convergent sequences $\{(x_{i,j})_j : i \in \omega\}$ such that $\lim_j x_{i,j} = x_i$ for each $i \in \omega$. Without loss of generality we can assume that $x_{i,j} \neq x_{i',j'}$ whenever $(i, j) \neq (i', j')$ (since the intersection $\{x_{i,j} : j \in \omega\} \cap \{x_{i',j} : j \in \omega\}$ is finite, for all $i \neq i'$, as $(x_{i,j})$ and $(x_{i',j})$ have different limits).

Define a sequence $g : \omega \rightarrow X$ by $g(n) = x_{f(n)}$. We claim that for each $A \subseteq \omega$, if $(g(n))_{n \in A}$ is convergent, then $A \in \mathcal{I}$ (i.e., that $X$ is not in $\text{FinBW}(\mathcal{I})$).

Let $A \subseteq \omega$ be such that $(g(n))_{n \in A}$ is convergent. Since $f^{-1}([\{i, j\}]) \in \mathcal{I}$, for all $(i, j) \in \omega^2$, if $g[A]$ is finite, then $A \in \mathcal{I}$. Thus, as $(g(n))_{n \in A}$ is convergent, we can assume that $g[A]$ is finite-to-one. Then also $f[A]$ is finite-to-one. Moreover, if $(g(n))_{n \in A}$ converges, then $f[A] \in \text{Fin}^2$. Consequently, $A \in \mathcal{I}$.

□

Corollary 6.4. If $\mathcal{I}$ is boring then each space in $\text{FinBW}(\mathcal{I})$ is compact and there are no uncountable separable spaces in $\text{FinBW}(\mathcal{I})$.

Proof. If $\mathcal{I}$ is boring then each space in $\text{FinBW}(\mathcal{I})$ is boring by Proposition 6.3. Thus, by Proposition 3.2, each space in $\text{FinBW}(\mathcal{I})$ is compact and there are no uncountable separable spaces in $\text{FinBW}(\mathcal{I})$.

Recall that for any ideal $\mathcal{I}$, each finite space is in $\text{FinBW}(\mathcal{I})$ and each space from $\text{FinBW}(\mathcal{I})$ is sequentially compact.

Theorem 6.5. The following hold for any ideal $\mathcal{I}$:

(a) $\text{Fin}^2 \subseteq \mathcal{I}$ if and only if $\text{FinBW}(\mathcal{I})$ coincides with finite spaces.

(b) $\mathcal{I}$ is boring and $\text{Fin}^n \nsubseteq \mathcal{I}$ if and only if $\text{FinBW}(\mathcal{I})$ coincides with boring spaces.

(c) $\mathcal{I}$ is not tall if and only if $\text{FinBW}(\mathcal{I})$ coincides with sequentially compact spaces.

Proof. Without loss of generality we can assume that $\mathcal{I}$ is an ideal on $\omega$.

(a): Suppose first that $\text{Fin}^2 \subseteq \mathcal{I}$ and let $f : \omega \rightarrow \omega^2$ be a bijection witnessing it. Suppose that there is an infinite Hausdorff $X \in \text{FinBW}(\mathcal{I})$ and let $(x_n) \subseteq X$ be an injective sequence. Define $y_{f^{-1}(i,j)} = x_i$ for all $i, j \in \omega$ (note that the sequence $(y_n)$ is well-defined as $f$ is a bijection). Suppose that $(y_n)_{n \in A}$ converges to $z \in X$, for some $A \subseteq \omega$. Then $A \cap f^{-1}([\{i\} \times \omega])$ is finite whenever $x_i \neq z$ (because there is an open set $U \ni z$ with $x_i \notin U$, since $X$ is Hausdorff). Thus, either $z \notin \{x_i : i \in \omega\}$ and $A \cap f^{-1}([\{i\} \times \omega])$ is finite for all $i \in \omega$ or $z = x_n$ for some $n \in \omega$ and $A \cap f^{-1}([\{i\} \times \omega])$ is finite for all $i \in \omega \setminus \{n\}$. In both cases $A \notin \mathcal{I}$. Therefore, $X$ cannot be in $\text{FinBW}(\mathcal{I})$.

Suppose now that $\text{Fin}^2 \nsubseteq \mathcal{I}$. Consider the space $X = \{0\} \cup \{\frac{1}{n+1} : n \in \omega\}$ with the subspace topology inherited from $\mathbb{R}$. We claim that $X$ is an infinite space belonging to $\text{FinBW}(\mathcal{I})$. Let $f : \omega \rightarrow X$. Without loss of generality we may assume that $f^{-1}([x]) \in \mathcal{I}$ for all $x \in X$. The family $(f^{-1}([x]))_{x \in X}$ defines a partition of $\omega$ into sets belonging to $\mathcal{I}$. As $\text{Fin}^2 \nsubseteq \mathcal{I}$, we can find $A \notin \mathcal{I}$ such that $A \cap f^{-1}([x])$ is finite for all $x \in X$. Thus, $(f(n))_{n \in A}$ is a finite-to-one subsequence. In the space $X$ each such subsequence converges to $0 \in X$. Therefore, $X$ is in $\text{FinBW}(\mathcal{I})$. 


(b): If $\mathcal{I}$ is unboring then, by Proposition 6.2, there is an unboring Hausdorff space in $\text{FinBW}(\mathcal{I})$. If $\text{Fin}^2 \subseteq \mathcal{I}$ then $\text{FinBW}(\mathcal{I})$ coincides with finite spaces by item (a), so not every boring space is in $\text{FinBW}(\mathcal{I})$ (by Proposition 3.2(i)). Moreover, by Proposition 3.3, if $\text{Fin}^2 \not\subseteq \mathcal{I}$ then each boring space is in $\text{FinBW}(\mathcal{I})$. Finally, if $\mathcal{I}$ is boring then all Hausdorff spaces in $\text{FinBW}(\mathcal{I})$ are boring by Proposition 6.3.

(c): This is [9, Proposition 2.4]. □

Theorem 6.6. The following are equivalent for any ideal $\mathcal{I}$:

(a) $\mathcal{I}$ is unboring;
(b) there is a Hausdorff unboring space in $\text{FinBW}(\mathcal{I})$.

If additionally CH holds, then the above are equivalent to:

(c) there is an uncountable Mrówka space in $\text{FinBW}(\mathcal{I})$;
(d) there is an uncountable separable space in $\text{FinBW}(\mathcal{I})$.

Proof. (a) $\implies$ (b): This is Proposition 6.2.
(b) $\implies$ (a): This is Proposition 6.3.
(a) $\implies$ (c): Under CH, this is Theorem 5.3.
(c) $\implies$ (d): Every Mrówka space is separable.
(d) $\implies$ (a): This is Corollary 6.4. □

Since the implication (d) $\implies$ (a) in Theorem 6.6 is true in ZFC, we are able to state the following result.

Corollary 6.7. Let $\mathcal{I}$ be an ideal.

(a) If $\mathcal{I}$ is Borel, then the following are equivalent:
(i) $\mathcal{I}$ is unboring in some forcing extension;
(ii) $\mathcal{I}$ is unboring in every forcing extension;
(iii) in every forcing extension in which CH holds, there is an uncountable separable space in $\text{FinBW}(\mathcal{I})$;
(iv) there is a forcing extension in which there is an uncountable separable space in $\text{FinBW}(\mathcal{I})$.

(b) If either $\mathcal{I}$ being unboring holds in ZFC or $\mathcal{I}$ being boring holds in ZFC, then the following are equivalent:
(i) $\mathcal{I}$ is unboring;
(ii) under CH there is an uncountable separable space in $\text{FinBW}(\mathcal{I})$;
(iii) it is consistent that there is an uncountable separable space in $\text{FinBW}(\mathcal{I})$.

Proof. (a): Since $\mathcal{I}$ and $\mathcal{BI}$ are Borel, $\mathcal{I}$ being unboring is an absolute statement (by [12, Proposition 1.3]). Thus, (i) $\implies$ (ii) by Shoenfield’s absoluteness theorem, (ii) $\implies$ (iii) follows from Theorem 6.6, (iii) $\implies$ (iv) is trivial and (iv) $\implies$ (i) follows from Corollary 6.4.

(b): The implication (i) $\implies$ (ii) is true by Theorem 6.6, (ii) $\implies$ (iii) is trivial and if (iii) holds then it is consistent that $\mathcal{I}$ is unboring (by Corollary 6.4), so using our assumption we get item (i). □

Corollary 6.8. Suppose that $\mathcal{I}$ is extendable to a coanalytic weak P-ideal (in particular, to a $\Pi^0_4$ ideal). Then $\omega_1$ with the order topology is in $\text{FinBW}(\mathcal{I})$. In particular, there is a non-compact space in $\text{FinBW}(\mathcal{I})$. Moreover, if MA($\sigma$-centered) holds, then there is an uncountable separable Mrówka space in $\text{FinBW}(\mathcal{I})$.

Proof. Follows from Propositions 4.9, 4.16(b), 6.1 and Theorem 5.4. □
Corollary 6.9. Let $\mathcal{I}$ be a $\leq K$-uniform ideal. Then the following are equivalent:

(a) $\mathcal{I}$ is a weak P-ideal;
(b) there is a Hausdorff unboring space in $\text{FinBW}(\mathcal{I})$;
(c) $\omega_1$ with order topology is in $\text{FinBW}(\mathcal{I})$;
(d) there is a non-compact space in $\text{FinBW}(\mathcal{I})$.

If additionally CH holds, then the above are equivalent to:

(e) there is an uncountable Mrówka space in $\text{FinBW}(\mathcal{I})$;
(f) there is an uncountable separable space in $\text{FinBW}(\mathcal{I})$.

If $\mathcal{I}$ is coanalytic then in the above we can replace CH with $\text{MA}(\sigma$-centered).

Proof. As $\mathcal{I}$ is $\leq K$-uniform, $\mathcal{I}$ is a weak P-ideal if and only if $\mathcal{I}$ is unboring (by Corollary 4.12). This together with Theorem 6.6 shows the equivalence of (a), (b), (c) and (f) (under CH). Recall that each unboring $\leq K$-uniform ideal is a hereditary weak P-ideal (again by Corollary 4.12). Thus, Proposition 6.1 gives us the implication (a) $\implies$ (c). The implication (c) $\implies$ (d) is obvious and (d) $\implies$ (a) follows from Corollary 6.4.

If $\mathcal{I}$ is coanalytic, then (a) $\implies$ (e) under $\text{MA}(\sigma$-centered) follows from Corollaries 4.12 and 6.8. Finally, the implication (e) $\implies$ (f) is obvious (as every Mrówka space is separable) and (f) $\implies$ (a) is proved in Corollary 6.4. $\square$

Corollary 6.10. Suppose that $\mathcal{U}$ is an ultrafilter.

- If $\mathcal{U}$ is not a P-point then $\text{FinBW}(\mathcal{U}^*)$ coincides with finite spaces.
- If $\mathcal{U}$ is a P-point then the $[0, 1]$ interval and $\omega_1$ with the order topology are in $\text{FinBW}(\mathcal{U}^*)$. In particular, in $\text{FinBW}(\mathcal{U}^*)$ there is an uncountable separable space as well as a non-compact space.

Proof. By Proposition 2.7, the following are equivalent:

- $\mathcal{U}$ is a P-point;
- the $[0, 1]$ interval is in $\text{FinBW}(\mathcal{U}^*)$;
- $\mathcal{U}^*$ is a weak P-ideal.

Thus, the first statement is a consequence of Theorem 6.5(a). For the second statement, recall that each maximal ideal is $\leq K$-uniform (see Example 2.3). Thus, the thesis follows from Theorem 6.5 and Corollary 6.9. $\square$

7. Some (open) problems

We start this section with the following question posed by D. Meza-Alcántara in [28, Question 4.4.7]: Is it true that, if $\mathcal{I}$ is a Borel ideal then either $\text{Fin}^2 \not\leq K \mathcal{I}$ or there is a $\Pi^0_3$ ideal containing $\mathcal{I}$? This problem has been repeated by M. Hrušák in [12, Question 5.18] and recently in [15, Question 5.10]. In the next example we answer it in the negative.

Example 7.1. Consider the ideal $\mathcal{B}$. It is Borel (by Remark 4.2), $\text{Fin}^2 \not\leq K \mathcal{B}$ (by Propositions 2.6 and 4.6), but $\mathcal{B}$ cannot be extended to any $\Pi^0_3$ ideal. In fact, $\mathcal{B}$ even cannot be extended to any $\Pi^0_4$ ideal, as $\mathcal{B}$ is boring and each ideal extendable to a $\Pi^0_4$ ideal cannot be boring (by Propositions 4.9 and 4.10(c)).

The above example shows also that $\text{Fin}^2$ is not critical (in the sense of $\leq K$) for extendability to ideals of any Borel class (among Borel ideals): if there is a critical ideal for extendability to $\Pi^0_3$ ideals, then it must be $\leq K$-below $\mathcal{B}$ (by the above
example) and Fin$^2$ cannot be critical for extendability to $\Sigma^0_4$ ideals as Fin$^2$ is itself $\Sigma^0_4$.

However, there is still a chance that extendability to some Borel class is related to containing an isomorphic copy of Fin$^2$:

**Problem 7.2.** Is every Borel hereditary weak P-ideal extendable to a $\Pi^0_4$ ideal?

Note that the converse implication is false: $\mathcal{I} = \text{Fin} \oplus \text{Fin}^2$ is a Borel ideal extendable to a $\Sigma^0_4$ ideal $\{A \subseteq \{0\} \times \omega \cup \{(1) \times \omega^2 : A \cap \{0\} \times \omega \text{ is finite}\}$, but it is not a hereditary weak P-ideal (as $\mathcal{I}|\{(1) \times \omega^2\} \cong \text{Fin}^2$).

An equivalent formulation of the above question is the following: is it true that for each Borel ideal $\mathcal{I}$ the following are equivalent:

(a) $\mathcal{I}$ is extendable to a hereditary weak P-ideal;
(b) $\mathcal{I}$ is extendable to a $\Pi^0_4$ ideal?

Observe that the implication (b) $\Rightarrow$ (a) is true by Proposition 4.9.

Next example shows that in Question 7.2 we cannot omit the assumption that the ideal is Borel.

**Example 7.3.** If there are P-points (in particular, if $\text{MA}(\sigma\text{-centered})$ holds) then there are hereditary weak P-ideals which are not extendable to a $\Pi^0_4$ ideal. Indeed, let $U$ be a P-point and denote $\mathcal{I} = U^*$. By Proposition 2.7, Fin$^2 \not\subseteq \mathcal{I}$. Since $\mathcal{I}$ is a maximal ideal, it is homogeneous (see Example 2.3). Thus, $\mathcal{I}$ is a hereditary weak P-ideal. On the other hand, since $\mathcal{I}$ is maximal, the only ideal containing $\mathcal{I}$ is $\mathcal{I}$ itself. As no maximal ideal is Borel, $\mathcal{I}$ cannot be extended to a $\Pi^0_4$ ideal.

By Proposition 4.16(d), under $\text{MA}(\sigma\text{-centered})$ there are unboring ideals which are not strongly unboring. However, a positive answer to the following problem would mean that in Theorem 6.6 we can replace CH with $\text{MA}(\sigma\text{-centered})$ in the case of all Borel ideals (by Theorem 5.4).

**Problem 7.4.** Is every Borel unboring ideal strongly unboring?

We end this section with some comments about P-points under $\text{MA}(\sigma\text{-centered})$. By Corollary 6.10, for each P-point $U$ there is an uncountable separable space in $\text{FinBW}(U^*)$ (namely, the $[0, 1]$ interval). However, by Proposition 4.16(d), if $U$ is a P-point then $U^*$ is not strongly unboring. Thus, Theorem 5.4 does not give us an uncountable Mrówka space in $\text{FinBW}(U^*)$.

**Problem 7.5.** Assume $\text{MA}(\sigma\text{-centered})$. Is there a P-point $U$ such that there is no uncountable Mrówka space in $\text{FinBW}(U^*)$?

8. NEW PREORDER ON IDEALS

**Definition 8.1.** If $\mathcal{I}$ is an ideal and $(G_{i,j}) \subseteq \mathcal{I}$ is a partition of $\bigcup \mathcal{I}$ then $\hat{\mathcal{I}}(G_{i,j})$ is an ideal on $\omega^2$ consisting of all $A \subseteq \omega^2$ with:

$$\left( \forall X \subseteq \bigcup_{(i,j) \in A} G_{i,j} \right) \left( \left( \forall (i,j) \in A \right) X \cap G_{i,j} \text{ is finite} \implies X \in \mathcal{I} \right).$$

**Definition 8.2.** Let $\mathcal{I}$ and $\mathcal{J}$ be ideals. We write $\mathcal{J} \preceq \mathcal{I}$ if for each partition $(H_{i,j}) \subseteq \mathcal{J}$ with Fin$^2 \not\subseteq \hat{\mathcal{J}}(H_{i,j})$ there is a partition $(G_{i,j}) \subseteq \mathcal{I}$ such that $\hat{\mathcal{J}}(H_{i,j}) \cap \text{Fin}^2 \subseteq \hat{\mathcal{I}}(G_{i,j})$. 

One could think that in the above definition the condition $\text{Fin} \otimes \emptyset \subseteq \hat{J}(H_{i,j})$ is unnecessary or that the strange condition $\hat{J}(H_{i,j}) \cap \text{Fin}^2 \subseteq \hat{I}(G_{i,j})$ could be replaced by a simpler one: $\hat{J}(H_{i,j}) \subseteq \hat{I}(G_{i,j})$. As we will see in Proposition 10.1, the relation $\preceq$ is designed to have the following property: $\mathcal{J} \preceq \mathcal{I}$ implies $\text{FinBW}(\mathcal{I}) \subseteq \text{FinBW}(\mathcal{J})$, whenever $\mathcal{J}$ is a weak P-ideal. The mentioned above two modifications of $\preceq$ would also have the required property, but $\preceq$ is the weakest one among them and we wanted this new relation to be as accurate for this purpose as it can be.

**Proposition 8.3.** The relation $\preceq$ is a preorder on the set of ideals.

**Proof.** It is clear that $\preceq$ is reflexive, so we only show transitivity. Suppose that $\mathcal{I} \preceq \mathcal{I}'$ and $\mathcal{I}' \preceq \mathcal{I}''$. Fix any partition $(H_{i,j}) \subseteq \mathcal{I}$ with $\text{Fin} \otimes \emptyset \subseteq \hat{I}(H_{i,j})$. Then there is a partition $(H'_{i,j}) \subseteq \mathcal{I}'$ such that $\hat{I}(H_{i,j}) \cap \text{Fin}^2 \subseteq \hat{I}'(H'_{i,j})$. Since $\text{Fin} \otimes \emptyset \subseteq \hat{I}(H_{i,j}) \cap \text{Fin}^2 \subseteq \hat{I}'(H'_{i,j})$, there is a partition $(H''_{i,j}) \subseteq \mathcal{I}''$ such that $\hat{I}'(H'_{i,j}) \cap \text{Fin}^2 \subseteq \hat{I}''(H''_{i,j})$. Then we have $\hat{I}(H_{i,j}) \cap \text{Fin}^2 \subseteq \hat{I}'(H'_{i,j}) \cap \text{Fin}^2 \subseteq \hat{I}''(H''_{i,j})$. \hfill $\square$

**Proposition 8.4.** Let $\mathcal{I}$ and $\mathcal{J}$ be ideals.

(a) The following are equivalent:

(i) $\mathcal{I}$ is boring;

(ii) $\text{Fin}^2 \preceq \mathcal{I}$;

(iii) $\mathcal{J} \preceq \mathcal{I}$ for all ideals $\mathcal{J}$.

(b) If $\mathcal{J} \preceq \mathcal{I}$ then $\mathcal{J} \preceq_{ K} \mathcal{I}|A$ for some $A \notin \mathcal{I}$.

(c) Suppose that $\mathcal{I}$ is unborning. If $\mathcal{J} \preceq \mathcal{I}$ then $\mathcal{J} \preceq_{ K} \mathcal{I}|A$ for some $A \notin \mathcal{I}$.

**Proof.** Without loss of generality we may assume that $\mathcal{I}$ and $\mathcal{J}$ are ideals on $\omega$.

(a): The implication (iii) $\Rightarrow$ (ii) is obvious. If $\text{Fin}^2 \preceq \mathcal{I}$ then for the partition given by $H_{i,j} = \{(i, j)\} \in \text{Fin}^2$, there is a partition $(G_{i,j}) \subseteq \mathcal{I}$ such that $\text{Fin}^2 \subseteq \hat{I}(G_{i,j})$. Observe that this implies $\mathcal{I}$ being boring, by Proposition 4.4(iv). Indeed, $(G_{i,j}) \subseteq \mathcal{I}$ is a partition such that given any $X \subseteq \omega$, if $X \cap G_{i,j} \in \text{Fin}$ for all $(i, j) \in \omega^2$ and $X \cap \bigcup_{j \in \omega} G_{i,j} \in \text{Fin}$ for almost all $i \in \omega$, then $A = \{(i, j) \in \omega^2 : X \cap G_{i,j} \neq \emptyset\} \in \text{Fin}^2$, so also $A \in \hat{I}(G_{i,j})$ and, consequently, $X \in \mathcal{I}$ (as $X \cap G_{i,j} \in \text{Fin}$ for all $(i, j) \in \omega^2$).

On the other hand, if $\mathcal{I}$ is boring then, by Proposition 4.4(iv), there is a partition $(G_{i,j}) \subseteq \mathcal{I}$ such that for any $X \subseteq \omega$, if $X \cap G_{i,j} \in \text{Fin}$ for all $(i, j) \in \omega^2$ and $X \cap \bigcup_{j \in \omega} G_{i,j} \in \text{Fin}$ for almost all $i \in \omega$, then $X \in \mathcal{I}$. Fix any $A \in \text{Fin}^2$. Then for every $X \subseteq \omega$ such that $X \subseteq \bigcup_{(i, j) \in A} G_{i,j}$ and $X \cap G_{i,j} \in \text{Fin}$ for all $(i, j) \in \omega^2$ we get $X \in \mathcal{I}$. Thus, $A \in \hat{I}(G_{i,j})$. Thus, $\text{Fin}^2 \subseteq \hat{I}(G_{i,j})$ and $\mathcal{J} \preceq \mathcal{I}$ for every ideal $\mathcal{J}$.

(b): Let $f : \omega \rightarrow \omega$ witness $\mathcal{J} \preceq_{ K} \mathcal{I}$ and fix any partition $(H_{i,j}) \subseteq \mathcal{J}$ of $\omega$ with $\text{Fin} \otimes \emptyset \subseteq \hat{J}(H_{i,j})$. Define $G_{i,j} = f^{-1}[H_{i,j}]$ for all $(i, j) \in \omega^2$. Then $(G_{i,j}) \subseteq \mathcal{I}$ is a partition of $\omega$. Let $A \in \hat{J}(H_{i,j}) \cap \text{Fin}^2$. We need to prove that $A \in \hat{I}(G_{i,j})$. Let $X \subseteq \bigcup_{(i, j) \in A} G_{i,j}$ be such that $X \cap G_{i,j} \in \text{Fin}$ for all $(i, j) \in A$. Then $f[X] \subseteq \bigcup_{(i, j) \in A} H_{i,j}$ and $f[X] \cap H_{i,j} \in \text{Fin}$ for all $(i, j) \in A$. Since $A \in \hat{J}(H_{i,j})$, $f[X]$ is in $\mathcal{J}$. Therefore, $X \subseteq \hat{J}^{-1}[f[X]] \subseteq \mathcal{I}$ and hence $A \in \hat{I}(G_{i,j})$.

(c): Suppose that $\mathcal{I}$ is unborning and $\mathcal{J} \preceq_{ K} \mathcal{I}|A$ for all $A \notin \mathcal{I}$. We will show that $\mathcal{J} \not\preceq \mathcal{I}$. Define $H_{i,0} = \{i\}$ and $H_{i,j} = \emptyset$ for all $i \in \omega$ and $j \in \omega \setminus \{0\}$. Clearly, $\text{Fin} \otimes \emptyset \subseteq \hat{J}(H_{i,j})$. Fix any partition $(G_{i,j}) \subseteq \mathcal{I}$. Since $\mathcal{I}$ is unborning, by
There is a proof. Define $D = C \pi A$. We need to findLemma 8.6.

Since $I_i \in B A$ for all $i, j \in \omega^2$, we may assume that $I$ is a hereditary weak $P$-ideal on $\omega$ with $I \subseteq I$. Without loss of generality we may assume that $I$ and $J$ are ideals on $\omega$. Define $H_{i,0} = \{i\}$ and $H_{i,j} = \emptyset$ for all $i \in \omega$ and $j \in \omega \setminus \{0\}$.

Obviously, $\Fin I \subseteq \hat{J}(H_{i,j})$. Since $J \subseteq I$, there is a partition $(G_{i,j}) \subseteq I$ such that $\hat{J}(H_{i,j}) \cap \Fin^2 \subseteq \hat{\Fin}(G_{i,j})$.

Define $f : \omega \to \omega$ by

$$f(x) = i \iff x \in G_{i,j}$$

for all $x \in \omega$. We claim that $f$ witnesses $J \leq_K I$. Indeed, fix any $A \in J$. Notice that $B = f^{-1}[A] = \bigcup \{G_{i,j} : i \in A\}$. Suppose to the contrary that $B \notin I$. Since $I$ is a hereditary weak $P$-ideal, $\Fin B$ is unboring (by Proposition 4.11), so there is a $C \notin \Fin B$, $C \subseteq B$ such that $C \cap G_{i,j} \in \Fin$, for all $(i, j) \in \omega^2$, and $D = \{i, j) \in \omega^2 : \emptyset \subseteq G_{i,j} \neq \emptyset\} \subseteq \Fin^2$ (by Proposition 4.4(iv)). Observe that $D \notin \Fin(G_{i,j})$ (as $C \notin I$). However,

$$(i \in \omega : (i, j) \in D \text{ for some } j \in \omega) = \{i \in \omega : C \cap G_{i,j} \neq \emptyset \text{ for some } j \in \omega\} \\
\subseteq \{i \in \omega : B \cap G_{i,j} \neq \emptyset \text{ for some } j \in \omega\} = A \in J,$$

since $C \subseteq B$. Therefore, $D \in \hat{J}(H_{i,j})$, which contradicts the choice of $(G_{i,j})$ and finishes the proof.

A natural question is whether $\leq_K$ and $\prec$ coincide for all ideals. The remaining part of this section is devoted to that problem. It occurs that the answer is negative even for $\Sigma^0_0$ ideals. We need to prove three lemmas before providing a suitable example.

**Lemma 8.6.** Let $J$ be a hereditary weak $P$-ideal on $\omega$. Then the ideal $\Fin^2 \cap (J \otimes 0)$ is unboring.

**Proof.** Denote $I = \Fin^2 \cap (J \otimes 0)$ and let $\pi_1 : \omega^2 \to \omega$ be the projection onto the first coordinate, i.e., $\pi_1(i, j) = i$ for all $(i, j) \in \omega^2$. Fix any bijection $f : \omega^2 \to \omega^3$.

We need to find $A \in B\mathcal{L}$ such that $f^{-1}[A] \notin I$. There are three possible cases:

If $f^{-1}[\{(i, j)\} \times \omega] \notin I$ for some $(i, j) \in \omega^2$, then $\{(i, j)\} \times \omega \in B\mathcal{L}$ is the required set.

If $f^{-1}[\{(i, j)\} \times \omega] \notin I$ for all $(i, j) \in \omega^2$, but $A = \pi_1[f^{-1}[\{(i, j)\} \times \omega]] \notin J$ for some $i \in \omega$, then consider the family $\{\pi_1[f^{-1}[\{(i, j)\} \times \omega]] : j \in \omega\} \subseteq J(A)$ (note that it covers $A$). Since $J$ is a hereditary weak $P$-ideal, there is a $B \subseteq A$, $B \notin J$ such that $B \cap \pi_1[f^{-1}[\{(i, j)\} \times \omega]]$ is finite for all $j \in \omega$. For each $j \in \omega$ find a finite set $F_j \subseteq f^{-1}[\{(i, j)\} \times \omega]$ with $\pi_1[F_j] = B \cap \pi_1[f^{-1}[\{(i, j)\} \times \omega]]$. Define $C = f[\bigcup_j F_j]$. 


Then $f^{-1}[C] = \bigcup F_i \notin I$ (as $\pi_1[\bigcup F_i] = B \notin J$). On the other hand, $C \in BL$ as $C \subseteq \{i\} \times \omega^2$ and $C \cap \{(i,j)\} \times \omega = f[F_i]$ is finite for all $j \in \omega$.

If $\pi_1[f^{-1}[\{i\} \times \omega^2]] \in J$ for all $i \in \omega$, then the family $\{\pi_1[f^{-1}[\{i\} \times \omega^2]] : i \in \omega\} \subseteq J$ and, as $J$ is a weak P-ideal, we can find $B \notin J$ such that $B \cap \pi_1[f^{-1}[\{i\} \times \omega^2]]$ is finite for all $i \in \omega$. Similarly as in the previous paragraph, for each $i \in \omega$ find a finite set $F_i \subseteq f^{-1}[\{i\} \times \omega^2]$ with $\pi_1[F_i] = B \cap \pi_1[f^{-1}[\{i\} \times \omega^2]]$. Then for $C = f[\bigcup F_i]$ we have $f^{-1}[C] \notin I$, but $C \in BL$ as $C \cap \{(i) \times \omega^2\} = f[F_i]$ is finite for all $i \in \omega$.

\hspace{1cm} \Box

\textbf{Lemma 8.7.} Let $J$ be a tall P-ideal on $\omega$. Then $J \not\subseteq_K \text{Fin}^2 \cap (J \otimes \emptyset)$.

\textbf{Proof.} Denote $I = \text{Fin}^2 \cap (J \otimes \emptyset)$ and fix any $f : \omega^2 \rightarrow \omega$. If there are infinitely many $i \in \omega$ with $f[\{i\} \times \omega]$ finite, then denote $T = \{n \in \omega : (\exists i \in \omega) f(i,j) = n \text{ for infinitely many } j \in \omega\}$. If $T \in \text{Fin} \subseteq J$ then $f^{-1}[T] \notin I$ (as there are infinitely many $i \in \omega$ with $f[\{i\} \times \omega] \in \text{Fin}$) and we are done. On the other hand, if $T$ is infinite then, using the tallness of $J$, find an infinite $A \in J$ with $A \subseteq T$. Then $f^{-1}[A] \notin I$ as $f^{-1}[A] \subseteq \text{Fin}^2$.

Assume now that $f[\{i\} \times \omega]$ is infinite for almost all $i \in \omega$ and denote by $T$ the set of all those $i$. Since $J$ is tall, for each $i \in T$ we can find an infinite $A_i \in J$ with $A_i \subseteq f[\{i\} \times \omega]$. Using the fact that $J$ is a P-ideal, find $A \in J$ with $A \setminus A_i \text{ finite for all } i \in T$. Then $f^{-1}[A] \notin I$ as $f^{-1}[A] \cap \{(i) \times \omega\}$ is infinite for all $i \in T$.

\hspace{1cm} \Box

\textbf{Lemma 8.8.} If $J$ is a $\leq_{KB}$-uniform weak P-ideal on $\omega$, then $J \not\subseteq \text{Fin}^2 \cap (J \otimes \emptyset)$.

\textbf{Proof.} Denote $I = \text{Fin}^2 \cap (J \otimes \emptyset)$. Fix any partition $(H_{i,j}) \subseteq J$ with $\emptyset \otimes \text{Fin} \subseteq J(H_{i,j})$. Since $J$ is a weak P-ideal, there is a $C \notin J$ with $C \cap H_{i,j} \in \text{Fin}$ for all $i, j \in \omega$. Let $f : \omega \rightarrow C$ be a finite-to-one function witnessing $J[C] \not\subseteq_K J$ (which exists, as $J$ is $\leq_{KB}$-uniform). Define a partition $(G_{i,j}) \omega^2$ by $G_{i,j} = f^{-1}[C \cap H_{i,j}] \times \omega$. Since $f$ is finite-to-one and $C \cap H_{i,j} \in \text{Fin}$, the set $f^{-1}[C \cap H_{i,j}]$ is finite, so $G_{i,j} \in I$ for each $(i,j) \in \omega^2$. We claim that $J(H_{i,j}) \cap \text{Fin}^2 \subseteq \hat{I}(G_{i,j})$.

Let $A \in J(H_{i,j}) \cap \text{Fin}^2$. Then each subset of $\bigcup_{(i,j) \in A} H_{i,j}$ with finite intersection with each $H_{i,j}$ is in $J$. In particular, $C \cap \bigcup_{(i,j) \in A} H_{i,j} \in J[C]$ and consequently $f^{-1}[C \cap \bigcup_{(i,j) \in A} H_{i,j}] \in J$. We need to show that $A \in \hat{I}(G_{i,j})$. Fix any $X \subseteq \bigcup_{(i,j) \in A} G_{i,j}$ with $X \cap G_{i,j}$ finite for all $i, j \in \omega$. Then $X \in \emptyset \otimes \text{Fin} \subseteq \text{Fin}^2$ and

\[ X \subseteq \bigcup_{(i,j) \in A} G_{i,j} = f^{-1} \left[ C \cap \bigcup_{(i,j) \in A} H_{i,j} \right] \times \omega \in J \otimes \emptyset. \]

Hence, $X \in I$ and $A \in \hat{I}(G_{i,j})$.

\hspace{1cm} \Box

\textbf{Example 8.9.} Even for unboring $\Sigma^0_2$ ideals, $\leq_K$ and $\prec$ are not the same, i.e., there are unboring $\Sigma^0_2$ ideals $I$ and $J$ such that $J \not\leq_K I$ but $J \prec I$.

Indeed, let $J$ be a tall $\leq_{KB}$-uniform analytic P-ideal (for instance, $J = I_d$ see Example 2.4). Then $J$ is $\Pi^0_4$ (see [4, Lemma 1.2.2 and Theorem 1.2.5]), so it is a hereditary weak P-ideal (by Proposition 4.9). Moreover, $I = \text{Fin}^2 \cap (J \otimes \emptyset)$ is $\Sigma^0_4$ (as an intersection of two $\Sigma^0_4$ ideals). Thus, by Lemmas 8.6, 8.7 and 8.8, we obtain the thesis.
9. MRÓWKA SPACES REVISITED

In this section we turn our attention to two particular cases: pairs of ideals \((I, J)\) such that \(I\) is extendable to a hereditary weak P-ideal \(I'\) such that \(J \not\leq_K I'\) and pairs of ideals \((I, J)\) satisfying \(J \not\leq_K I|A\) for all \(A \not\in I\). Probably it is possible to state the results of this section in a more general way, however it would involve a lot of technicalities.

**Lemma 9.1.** If \(J\) is an ideal such that \(J \not\leq_K J'\) for some ideal \(J'\), then \(J\) is tall.

*Proof.* If \(J\) would not be tall then any bijection between \(\bigcup J'\) and the set \(B \not\in J\) with \(J|B = \text{Fin}(B)\) would witness \(J \leq_K J'\). \(\square\)

**Lemma 9.2.** Assume that there is an AD family \(A \subseteq J\) such that for every function \(f \in D_I\) there is a \(B \not\in I\) such that \(f|B\) is finite-to-one and \(f[B] \subseteq A\) for some \(A \in A\). Then \(\Phi(A) \in \text{FinBW}(I)\ \setminus \text{FinBW}(J)\).

*Proof.* The fact that \(\Phi(A) \in \text{FinBW}(I)\) follows from Proposition 5.1. We need to show that \(\Phi(A) \not\in \text{FinBW}(J)\). Without loss of generality we can assume that \(J\) is an ideal on \(\omega\). We claim that the sequence \((n)_{n \in \omega}\) does not have a convergent subsequence indexed by some \(B \not\in J\). Indeed, no subsequence of \((n)_{n \in \omega}\) converges to an element of \(\omega\) and if \((n)_{n \in B}\) converges to some \(A \in A\) then \(B \setminus A \in \text{Fin}\), so \(B \in J\) (as \(A \subseteq J\)). Finally, no \((n)_{n \in B}\) converges to \(\infty\) as \(\Phi(A)\) is Hausdorff and \(A\) is a MAD family, by Lemma 5.2 \((B \cap A)\) has to be infinite for some \(A \in A\), so the sequence \((n)_{n \in B}\) cannot converge to \(\infty\) as it has to have a subsequence \((n)_{n \in B \cap A}\) converging to \(A \in A\).

**Theorem 9.3.** Under CH, if \(I\) is unboring and \(J \not\leq_K I|A\) for all \(A \notin I\), then there is a Mrówka space in \(\text{FinBW}(I)\ \setminus \text{FinBW}(J)\). Moreover, if \(I\) is strongly unboring, then CH can be relaxed to \(\text{MA}(\sigma\text{-centered})\).

*Proof.* This proof is very similar to the proof of Theorem 5.3, so we will omit some details.

Enumerate \(D_I = \{f_\alpha : \omega \leq \alpha < \eta\}\). By induction on \(\alpha\), we will construct a family \(A = \{A_\alpha : \alpha < \eta\}\) as in Lemma 9.2. To start, find a partition \((A_\alpha)\) of \(\omega\) into sets belonging to \(J\) (which exists by Lemma 9.1).

Assume that \(\omega \leq \alpha < \eta\) and that \(A_\beta \not\in J\), for all \(\beta < \alpha\), have been chosen.

Suppose first that CH holds. If there is a \(\beta < \alpha\) with \(f_\alpha[C] \subseteq A_\beta\) for some \(C \notin I\) with \(f_\alpha[C]\) finite-to-one, then put \(A_\alpha = A_\beta\). Otherwise, enumerate \(\alpha\) as \((\beta_\iota)_{\iota \in \omega}\) and consider the partition \((B_\iota)\) of \(\omega\), where \(B_0 = A_{\beta_0}\) and \(B_\iota = A_{\beta_\iota} \setminus \bigcup_{\jmath < \iota} A_{\beta_\jmath}\) for all \(0 < \iota < \omega\). Since \(B I \subseteq I\), there is a \(C \notin I\) such that \(f_\alpha[C] \in D_{\text{Fin}(C)}\) and \(f_\alpha[C] \cap B_\iota \in \text{Fin}\) for all \(i \in \omega\). Since \(J \not\leq_K I|C\), there is a \(B \in J\) with \(f_\alpha^{-1}[B] \cap C \notin I|C\). Then \(f_\alpha[B]\) is finite-to-one. Put \(A_\alpha = f_\alpha[f_\alpha^{-1}[B] \cap C] \subseteq B \in J\) and observe that \(A_\alpha\) has finite intersection with each \(A_\beta\) for \(\beta < \alpha\) (as \(A_\alpha \subseteq f_\alpha[C]\)).

Suppose now that \(I\) is strongly unboring and \(\text{MA}(\sigma\text{-centered})\) holds. Then we can find \(C \notin I\) such that \(f_\alpha[C]\) is finite-to-one and \(I|C\) is \(\omega\)-diagonalizable by \((I|C)\)’-universal sets. If \(C \cap f^{-1}[A_\beta] \notin I\) for some \(\beta < \alpha\) then put \(A_\alpha = A_\beta\). Otherwise, perform the same construction as in the proof of Theorem 5.4 to get a set \(B \subseteq C\), \(B \not\in I|C\) such that \(f_\alpha[B] \in D_{\text{Fin}(B)}\) and \(f_\alpha[B] \cap A_\beta\) is finite for all \(\beta < \alpha\).
Since \( B \notin I \), \( J \not\subseteq K \mid B \). Thus, there is a \( D \in J \) such that \( f_{\alpha}^{-1}[D] \cap B \notin I \mid B \). Define \( A_{\alpha} = f_{\alpha}[f_{\alpha}^{-1}[D] \cap B] \subseteq D \in J \) and observe that \( A_{\alpha} \) has finite intersection with each \( A_{\beta} \) for \( \beta < \alpha \) (as \( A_{\alpha} \subseteq f_{\alpha}[B] \)).

**Theorem 9.4.** Under CH, if \( I \) is extendable to a hereditary weak P-ideal \( I' \) such that \( J \not\subseteq I' \), then there is a Mrówka space in \( \text{FinBW}(I) \setminus \text{FinBW}(J) \). Moreover, if \( I' \) is coanalytic, then \( CH \) can be relaxed to \( MA(\sigma\text{-centered}) \).

**Proof.** Note that \( \text{FinBW}(I') \subseteq \text{FinBW}(I) \) (as \( I \subseteq I' \)), so it suffices to find a Mrówka space in \( \text{FinBW}(I') \setminus \text{FinBW}(J) \).

Let \( \{f_{\alpha} : \omega \leq \alpha < \varsigma \} \) be an enumeration of \( D_{I'} \). By induction on \( \alpha \), we will construct a family \( A = \{A_{\alpha} : \alpha < \varsigma \} \) as in Lemma 9.2. To start, find a partition \( (A_{\alpha}) \) of \( \omega \) into sets belonging to \( J \) (which exists by Lemma 9.1).

Assume that \( \alpha < \varsigma \) and that \( A_{\beta} \in J \), for all \( \beta < \alpha \), have been chosen. Since \( J \not\subseteq I' \), there is a \( H \in J \) such that \( f_{\alpha}^{-1}[H] \notin I' \).

Suppose first that \( CH \) holds. If there is a \( \beta < \alpha \) with \( f_{\alpha}[C] \subseteq A_{\beta} \) for some \( C \not\in I' \mid f_{\alpha}^{-1}[H] \) with \( f_{\alpha}[C] \) finite-to-one, then put \( A_{\alpha} = A_{\beta} \). Otherwise, enumerate \( \alpha \) as \( (\beta_{i})_{i \in \omega} \). Consider the same partition \( (B_{i}) \) as in the proof of Theorem 9.3.

Since \( \exists \beta \in I' \mid f_{\alpha}^{-1}[H] \) (by Proposition 4.11), there is a \( C \not\in I' \mid f_{\alpha}^{-1}[H] \) such that \( f_{\alpha}[C] \in \text{FinBW}((C) \) and \( f_{\alpha}[C] \cap B_{i} \) is finite for all \( i \in \omega \) (see the proof of Theorem 5.3 for details). Moreover, since \( C \subseteq f_{\alpha}^{-1}[H] \), \( f_{\alpha}[C] \) is in \( J \). Put \( A_{\alpha} = f_{\alpha}[C] \) and observe that \( A_{\alpha} \cap A_{\beta} \) is finite for each \( i \in \omega \).

Suppose now that \( I' \) is coanalytic and \( MA(\sigma\text{-centered}) \) holds. Since \( I' \) is a hereditary weak P-ideal, \( \exists \beta \in I' \mid f_{\alpha}^{-1}[H] \) and we can find \( C \not\in I' \mid f_{\alpha}^{-1}[H] \) such that \( f_{\alpha}[C] \) is finite-to-one (and \( f_{\alpha}[C] \subseteq H \in J \)).

If there is a \( \beta < \alpha \) such that \( C \cap f_{\alpha}^{-1}[A_{\beta}] \notin I' \) then put \( A_{\alpha} = A_{\beta} \). If this is not the case, note that \( I' \) being a coanalytic hereditary weak P-ideal implies that \( I' \) is strongly un boring (by Proposition 4.16(b)). Hence, in the same way as in Theorem 5.4, we can produce \( B \subseteq C \) such that \( B \not\subseteq I' \mid C \) and \( f_{\alpha}[B] \cap A_{\beta} \) is finite for all \( \beta < \alpha \). Define \( A_{\alpha} = f_{\alpha}[B] \) and note that \( A_{\alpha} = f_{\alpha}[B] \subseteq f_{\alpha}[C] \subseteq H \in J \).

**10. Distinguishing ideals**

**Proposition 10.1.** If \( J \) is a weak P-ideal and \( J \not\subseteq I \) then \( \text{FinBW}(I) \subseteq \text{FinBW}(J) \).

**Proof.** We can assume that \( I \) and \( J \) are ideals on \( \omega \). Let \( X \in \text{FinBW}(I) \) and fix \( g : \omega \to X \). We need to find \( A \notin J \) such that \( (g(n))_{n \in A} \) converges. Without loss of generality we may assume that \( g^{-1}([x]) \in J \) for all \( x \in X \).

Suppose first that \( g[\omega] \) is a boring space. By Proposition 3.3, since \( J \) is a weak P-ideal, there is an \( A \notin J \) with \( (g(n))_{n \in A} \) convergent.

Assume now that \( g[\omega] \) is not boring and find \( \{x_{i} : i \in \omega \} \subseteq X \), \( x_{i} \neq x_{j} \) for all \( i \neq j \), and an infinite partition \( (B_{i})_{i \in \omega} \) of \( g[\omega] \) into infinite sets such that for each \( i \in \omega \) each injective sequence in \( B_{i} \) converges to \( x_{i} \). This can be done inductively. Indeed, let \( \{z_{i} : i \in \omega \} \) be an enumeration of \( g[\omega] \) and start the induction by finding any infinite \( B_{0}' \subseteq g[\omega] \) which is convergent to some \( x_{0} \in X \) (\( B_{0}' \) exists as \( X \) is sequentially compact) and putting \( B_{0} = B_{0}' \cup \{z_{0}\} \). Assume now that \( B_{j} \) and \( x_{j} \) for all \( j < i \) are already defined. Then for each \( j, j' < i, j \neq j' \) there is an open set \( V_{j, j'} \) such that \( x_{j} \in V_{j, j'} \) and \( x_{j'} \notin V_{j, j'} \). Define \( U_{j} = \bigcap_{j' < i, j' \neq j} V_{j, j'} \ni x_{j} \). Note that since \( g[\omega] \) is not boring, \( g[\omega] \setminus \bigcup_{j < i} B_{j} \) cannot be boring. Thus, either \( \left( g[\omega] \setminus \bigcup_{j < i} B_{j} \right) \setminus U_{i} \) is infinite and, using sequential compactness of \( X \), we
can find an infinite $B'_i \subseteq \left( g[\omega] \setminus \bigcup_{j<i} B_i \right) \setminus \bigcup_{j<i} U_i$ which is convergent to some $x_i \in X$, or $(g[\omega] \setminus \bigcup_{j<i} B_i) \setminus \bigcup_{j<i} U_i$ is finite, but there is a $j < i$ such that not every injective sequence in $U_j \cap (g[\omega] \setminus \bigcup_{j<i} B_i)$ converges to $x_j$, so we can find an infinite $B'_i \subseteq U_j \cap (g[\omega] \setminus \bigcup_{j<i} B_i)$ converging to some $x_i \notin \{x_j : j < i\}$. In both cases it suffices to put $B_i = B'_i \cup \{z_i\}$.

Define $H_{i,j} = g^{-1}([b_j]) \in \mathcal{J}$, where $B_i = \{b_j : j \in \omega\}$, for all $i, j \in \omega$.

If $\text{Fin} \otimes \emptyset \nsubseteq \hat{\mathcal{J}}(H_{i,j})$ then there is an $i \in \omega$ such that $\{i\} \times \omega \notin \hat{\mathcal{J}}(H_{i,j})$. Thus, we can find $A \notin \mathcal{J}$ with $A \subseteq \bigcup_{j \in \omega} H_{i,j}$ and $A \cap H_{i,j}$ finite for all $j \in \omega$. In this case $g[A]$ is finite-to-one (as $A \cap H_{i,j}$ is finite for all $j \in \omega$) and $(g(n))_{n \in A}$ converges to $x_i$ (as $g[A] \subseteq B_i$).

On the other hand, if $\text{Fin} \otimes \emptyset \subseteq \hat{\mathcal{J}}(H_{i,j})$, then by $\mathcal{J} \subseteq \mathcal{I}$ we can find a partition $(G_{i,j}) \subseteq \mathcal{I}$ such that $\hat{\mathcal{J}}(H_{i,j}) \cap \text{Fin}^2 \subseteq \hat{\mathcal{I}}(G_{i,j})$. Define $h : \omega \to X$ by

$$h(n) = b_j^n \iff n \in G_{i,j}$$

for all $n \in \omega$. Then $h^{-1}(\{x\}) \in \mathcal{I}$ for all $x \in X$.

Since $X \in \text{FinBW}(\mathcal{I})$, there is a $C \notin \mathcal{I}$ such that $(h(n))_{n \in C}$ converges in $X$. Since $X$ is Hausdorff, without loss of generality we may assume that $h[C]$ is finite-to-one (there may be at most one $y \in X$ with $C \cap h^{-1}(\{y\})$ infinite, so using the fact that $h^{-1}(\{y\}) \in \mathcal{I}$ it suffices to consider $C \setminus h^{-1}(\{y\}) \notin \mathcal{I}$ instead of $C$). Thus, $C \cap G_{i,j} \in \text{Fin}$ for all $i, j \in \omega$ (otherwise $h[C]$ would not be finite-to-one) and

$$D = \{(i,j) \in \omega^2 : C \cap G_{i,j} \neq \emptyset \} \subseteq \text{Fin}^2 \setminus \hat{\mathcal{I}}(G_{i,j}) \subseteq \text{Fin}^2 \setminus \hat{\mathcal{J}}(H_{i,j}).$$

Indeed, $D \in \text{Fin}^2$ as $|D \cap (\{i\} \times \omega)| = \omega$ for at most one $i \in \omega$ ($|D \cap (\{i\} \times \omega)| = \omega = |D \cap (\{i'\} \times \omega)|$) for two distinct $i,i' \in \omega$ would imply that $(h(n))_{n \in C}$ has two subsequences — one converging to $x_i$ and the other converging to $x_{i'}$. Moreover, $D \notin \hat{\mathcal{I}}(G_{i,j})$ as $C \subseteq \bigcup_{(i,j) \in D} G_{i,j}$ and $C \cap G_{i,j} \in \text{Fin}$ for all $i, j \in \omega$, but $C \notin \mathcal{I}$.

Finally, $D \notin \hat{\mathcal{J}}(H_{i,j})$ as otherwise $D$ would belong to $\hat{\mathcal{J}}(H_{i,j}) \cap \text{Fin}^2 \subseteq \hat{\mathcal{I}}(G_{i,j})$.

Since $D \notin \hat{\mathcal{J}}(H_{i,j})$, there is an $A \notin \mathcal{J}$ with $\{(i,j) \in \omega^2 : A \cap H_{i,j} \neq \emptyset \} \subseteq D$ and $A \cap H_{i,j} \in \text{Fin}$ for all $i, j \in \omega$. This implies that $g[A]$ is finite-to-one and $(g(n))_{n \in A}$ is a subsequence of $(h(n))_{n \in C}$, hence, converges to the same limit as $(h(n))_{n \in C}$.

**Corollary 10.2.** If $\mathcal{J} \subseteq K \mathcal{I}$ then $\text{FinBW} (\mathcal{I}) \subseteq \text{FinBW}(\mathcal{J})$.

**Proof.** Note that if $\text{Fin}^2 \nsubseteq \mathcal{J}$ then $\text{Fin}^2 \subseteq \mathcal{I}$ (by Proposition 2.6) and $\text{FinBW} (\mathcal{I}) = \text{FinBW}(\mathcal{J})$ by Theorem 6.5. On the other hand, if $\text{Fin}^2 \nsubseteq \mathcal{J}$ then we can apply Propositions 8.4(b) and 10.1.

Our next result shows that the question whether $\text{FinBW} (\mathcal{I})$ and $\text{FinBW}(\mathcal{J})$ can be distinguished is easy, whenever at least one of the ideal $\mathcal{I}$ and $\mathcal{J}$ is boring.

**Corollary 10.3.** Let $\mathcal{J}$ be boring.

(a) Suppose that $\text{Fin}^2 \subseteq \mathcal{J}$.

(i) If $\text{Fin}^2 \subseteq \mathcal{I}$ then $\text{FinBW}(\mathcal{I}) = \text{FinBW}(\mathcal{J})$.

(ii) If $\text{Fin}^2 \nsubseteq \mathcal{I}$ then $\text{FinBW}(\mathcal{J}) \nsubseteq \text{FinBW}(\mathcal{I})$.

(b) Suppose that $\text{Fin}^2 \nsubseteq \mathcal{J}$.

(i) If $\text{Fin}^2 \subseteq \mathcal{I}$ then $\text{FinBW}(\mathcal{I}) \nsubseteq \text{FinBW}(\mathcal{J})$.

(ii) If $\text{Fin}^2 \nsubseteq \mathcal{I}$ and $\mathcal{I}$ is boring then $\text{FinBW}(\mathcal{I}) = \text{FinBW}(\mathcal{J})$. 
(iii) If $I$ is unboring then $\text{FinBW}(J) \subsetneq \text{FinBW}(I)$.

Proof. Follows from Theorems 6.5 and 6.6. □

Theorem 10.4. (CH) The following are equivalent for any unboring ideal $J$ and any hereditary weak P-ideal $I$ (in particular, for any $\Pi_0^4$ ideal $I$):

(a) $\text{FinBW}(I) \setminus \text{FinBW}(J) \neq \emptyset$;
(b) there is a Mrówka space in $\text{FinBW}(I) \setminus \text{FinBW}(J)$;
(c) $J \not\leq_I I$;
(d) $J \not\lessdot I$.

Moreover, if $I$ is coanalytic, then it suffices to assume $\text{MA}(\sigma$-centered) instead of CH.

Proof. (c)$\iff$(d): This is Proposition 8.5.
(a)$\implies$(d): This is Proposition 10.1.
(c)$\implies$(b): This is Theorem 9.4.
(b)$\implies$(a): Obvious. □

Since the implication (a)$\implies$(c) in Theorem 10.4 is true in ZFC, we are able to state the following result.

Corollary 10.5. Let $I$ and $J$ be $\Pi_0^4$ ideals.

(a) The following are equivalent:
(i) $J \not\leq_K I$ in some forcing extension;
(ii) $J \not\leq_K I$ in every forcing extension;
(iii) $\text{FinBW}(I) \setminus \text{FinBW}(J) \neq \emptyset$ in every forcing extension in which $\text{MA}(\sigma$-centered) holds;
(iv) there is a forcing extension in which $\text{FinBW}(I) \setminus \text{FinBW}(J) \neq \emptyset$.

(b) If either $J \leq_K I$ holds in ZFC or $J \not\leq_K I$ holds in ZFC, then the following are equivalent:
(i) $J \not\leq_K I$;
(ii) under $\text{MA}(\sigma$-centered) there is an uncountable separable space in $\text{FinBW}(I)$;
(iii) it is consistent that there is an uncountable separable space in $\text{FinBW}(I)$.

Proof. The proof is entirely similar to the proof of Corollary 6.7 – it suffices to replace Theorem 6.6 with Theorem 10.4 and Corollary 6.4 with Propositions 8.5 and 10.1. □

Example 10.6. There are unboring $\Sigma_0^4$ ideals $I$ and $J$ such that $J \not\leq_K I$ but $\text{FinBW}(I) \subsetneq \text{FinBW}(J)$. Indeed, this follows from Example 8.9 and Proposition 10.1.

Theorem 10.4 characterizes $\text{FinBW}(I) \setminus \text{FinBW}(J) \neq \emptyset$ only in the case when $I$ is a hereditary weak P-ideal. Thus, a natural question is whether it is possible that $\text{FinBW}(I)$ and $\text{FinBW}(\bar{I})$ are not the same, where $\bar{I}$ is the smallest hereditary weak P-ideal containing $I$ (which exists by Proposition 4.13). The answer is yes (even for $\Sigma_0^4$ ideals). To show it we will need the following lemma.

Lemma 10.7. For each $B \notin \text{conv}$ there is a $C \subset B$, $C \notin \text{conv}$ with $\text{Fin}^2 \cong \text{conv}|C$.

Proof. As $B \notin \text{conv}$, there is an injective sequence $(x_n) \subseteq \overline{B}$ such that each $x_n$ is a limit of some injective sequence in $B$. Since $[0,1]$ is sequentially compact, we can assume that $(x_n)$ converges in $[0,1]$ to some $x \in [0,1]$. Let $(U_n)$ be a decreasing
sequence of open neighborhoods of \( x \) such that \( \lim_n \text{diam}(U_n) = 0 \) and \( x_n \in U_n \) for each \( n \in \omega \). For every \( n \) find \( B_n \subseteq B \) such that any injective sequence in \( B_n \) converges to \( x_n \). Define \( C_0 = U_0 \cap B_0 \) and \( C_{n+1} = U_{n+1} \cap B_{n+1} \setminus \bigcup_{i \leq n} B_i \) for all \( n \). Then \( \text{conv} \bigcup_n C_n \) is isomorphic to \( \text{Fin}^2 \).

\[ \text{Example 10.8.} \text{ Consider the } \Sigma_3^0 \text{ ideal } \text{conv}. \text{ It is extendable to the ideal } \text{null} = \bigl\{ A \subseteq \mathbb{Q} \cap [0,1]: \overline{A} \text{ is of Lebesgue measure } 0 \bigr\}. \text{ By Proposition 4.9, null is a hereditary weak P-ideal, as it is } \Pi_2^0 \text{ (see [5]). Thus, by Proposition 14.13, there is a smallest hereditary weak P-ideal } \overline{\text{conv}} \text{ containing conv.}

\text{We claim that } \text{FinBW(}\text{conv} ) \setminus \text{FinBW(}\overline{\text{conv}} ) \neq \emptyset \text{ under CH. In order to show it, we will prove that } \overline{\text{conv}} \not\subseteq K \text{ conv} |B \text{ for all } B \notin \text{conv}. \text{ As conv is unboring by Proposition 4.6, the thesis will follow from Theorem 9.3.}

\text{Fix } B \notin \text{conv}. \text{ By Lemma 10.7, there is a } C \notin \text{conv}, C \subseteq B \text{ with conv}|C \text{ isomorphic to } \text{Fin}^2. \text{ Thus, it suffices to show } \overline{\text{conv}} \not\subseteq K \text{ Fin}^2. \text{ It will follow that } \overline{\text{conv}} \not\subseteq K \text{ conv}|C \text{ and consequently } \overline{\text{conv}} \not\subseteq K \text{ conv}|B.

\text{Fix } f: \omega^2 \to \mathbb{Q} \cap [0,1]. \text{ Since } [0,1] \text{ is sequentially compact, for each } n \text{ we can find } x_n \in [0,1] \text{ and } A_n \in [\omega]^{\omega^2} \text{ such that either } f([n] \times A_n) = \{x_n\} \text{ or } f([n] \times A_n) \text{ is infinite and each injective sequence in } f([n] \times A_n) \text{ converges to } x_n. \text{ There are also } x \in [0,1] \text{ and } A \in [\omega]^{\omega^2} \text{ such that } (x_n)_{n \in A} \text{ converges to } x. \text{ Let } (U_n)_{n \in A} \text{ be a decreasing sequence of open neighborhoods of } x \text{ such that } \lim_n \text{diam}(U_n) = 0 \text{ and } x_n \in U_n \text{ for each } n \in A.

\text{Define } D = \bigcup_{n \in A} \{n\} \times (A_n \cap f^{-1}[U_n]). \text{ Then } \text{Fin}^2 \not\subseteq D \subseteq f^{-1}[f[D]]. \text{ However, } f[D] \in \overline{\text{conv}}. \text{ Indeed, either } f[D] \in \text{conv} \subseteq \overline{\text{conv}} \text{ or } \text{conv}|f[D] \text{ contains an isomorphic copy of } \text{Fin}^2. \text{ In the latter case, } f[D] \notin \overline{\text{conv}} \text{ would mean that } \text{Fin}^2 \subseteq \text{conv}|f[D] \subseteq \overline{\text{conv}}|f[D] \text{ and contradict the fact that } \overline{\text{conv}} \text{ is a hereditary weak P-ideal.}

\text{We end this section with an open problem.}

\text{Problem 10.9. Is it true that under CH for all Borel unboring ideals } I \text{ and } J \text{ the following are equivalent:}

\begin{align*}
(\text{i}) & \quad J \not\subseteq I; \\
(\text{ii}) & \quad \text{FinBW}(I) \setminus \text{FinBW}(J) \neq \emptyset.
\end{align*}

\text{Theorem 10.4 is a special case – it shows that this is true for hereditary weak P-ideals. By Proposition 10.1, the implication (ii) } \implies \text{ (i) is true in general (even in ZFC). However, we were not able to show under CH that } J \not\subseteq I \text{ implies existence of a Mrówka space in } \text{FinBW}(I) \setminus \text{FinBW}(J).}

11. \text{ Applications}

11.1. \text{ Simple density ideals.} \text{ Recall the definition of the density zero ideal: } I_d = \left\{ A \subseteq \omega : \lim_{n \to \infty} \frac{|A \cap \{0,1,\ldots,n\}|}{n+1} = 0 \right\}. \text{ In this subsection we want to apply our results to summarize some facts about } \text{FinBW}(I_d).

\text{We need to introduce two classes of ideals which have been considered in the literature.}

\begin{itemize}
\item If } g: \omega \to [0,\infty) \text{ is such that } \lim_n g(n) = \infty \text{ and the sequence } \left( \frac{n}{g(n)} \right) \text{ does not converge to } 0, \text{ then we define the simple density ideal } Z_g \text{ by:}
\end{itemize}

\[ A \in Z_g \iff \lim_n \frac{|A \cap \{0,1,\ldots,n\}|}{g(n)} = 0. \]
This class of ideals was first studied in [1] (see also [21] and [22]).

- If \( h : \omega \to [0, \infty) \) is such that \( \sum_{i=0}^{\infty} h(i) = \infty \) and \( \lim_{n} \frac{h(n)}{\sum_{i=0}^{n} h(i)} = 0 \), then we define an \( \text{Erdős-Ulam} \) ideal \( \mathcal{E}U_{h} \) by:

\[
A \in \mathcal{E}U_{h} \iff \lim_{n} \frac{\sum_{i \in A \cap \{0, \ldots, n\}} h(i)}{\sum_{i=0}^{n} h(i)} = 0
\]

(cf. [4, Example 1.2.3(d)]).

Both \( \text{Erdős-Ulam} \) ideals and simple density ideals are examples of analytic \( P \)-ideals (see [1] and [4, Example 1.2.3(d)]). Clearly, the ideal \( \mathcal{I}_{d} \) is both \( \text{Erdős-Ulam} \) and simple density. It is known that there are \( \varepsilon \) pairwise non-isomorphic simple density ideals which are not \( \text{Erdős-Ulam} \) as well as there are \( \varepsilon \) pairwise non-isomorphic \( \text{Erdős-Ulam} \) ideals which are not simple density (see [21, Proposition 5] and [22, Theorem 3]).

**Proposition 11.1.**

(a) \( \omega_{1} \) with order topology is in \( \text{FinBW}(\mathcal{I}_{d}) \). Moreover, under \( \text{MA}(\sigma\text{-centered}) \) there is an uncountable separable space in \( \text{FinBW}(\mathcal{I}_{d}) \).

(b) \( \text{FinBW}(\mathcal{E}U_{h}) = \text{FinBW}(\mathcal{I}_{d}) \) for all \( \text{Erdős-Ulam} \) ideals \( \mathcal{E}U_{h} \).

(c) \( \text{FinBW}(\mathcal{I}_{d}) \subseteq \text{FinBW}(\mathcal{Z}_{g}) \) for all simple density ideals \( \mathcal{Z}_{g} \).

(d) \( [0, 1] \notin \text{FinBW}(\mathcal{Z}_{g}) \setminus \text{FinBW}(\mathcal{I}_{d}) \) for each simple density ideal \( \mathcal{Z}_{g} \) which is not \( \text{Erdős-Ulam} \).

(e) (\( \text{MA}(\sigma\text{-centered}) \)) There is a family \( \mathcal{F} \) of size \( \varepsilon \) of simple density ideals such that \( \text{FinBW}(\mathcal{Z}_{g}) \setminus \text{FinBW}(\mathcal{Z}_{g'}) \neq \emptyset \) whenever \( \mathcal{Z}_{g}, \mathcal{Z}_{g'} \in \mathcal{F} \) are distinct.

**Proof.** (a): Follows from Corollary 6.8.

(b): By [4, Theorem 1.13.10], all \( \text{Erdős-Ulam} \) ideals are \( \leq_{K} \)-equivalent, i.e., \( \mathcal{E}U_{h} \leq_{K} \mathcal{E}U_{h'} \) whenever \( \mathcal{E}U_{h} \) and \( \mathcal{E}U_{h'} \) are \( \text{Erdős-Ulam} \) ideals (in fact, all \( \text{Erdős-Ulam} \) ideals are even \( \leq_{RB} \)-equivalent). Thus, it suffices to apply Corollary 10.2.

(c): Follows from Corollary 10.2, as \( \mathcal{Z}_{g} \leq_{KB} \mathcal{I}_{d} \) for each simple density ideal \( \mathcal{Z}_{g} \) (by [22, Theorem 4 and Remark 2]).

(d): By [8, Section 3] \( [0, 1] \notin \text{FinBW}(\mathcal{I}_{d}) \). On the other hand, each simple density ideal \( \mathcal{Z}_{g} \) which is not \( \text{Erdős-Ulam} \) is extendable to a \( \Sigma_{2}^{0} \) ideal (hence, \( [0, 1] \in \text{FinBW}(\mathcal{Z}_{g}) \) by [8, Theorem 4.2]). Indeed, each tall density ideal in the sense of Farah, which is not an \( \text{Erdős-Ulam} \) ideal (i.e., belonging to the class (24) from [4, Lemma 1.13.9]), is generated by some unbounded submeasure \( \phi \), hence, it can be extended to a \( \Sigma_{2}^{0} \) ideal \( \text{Fin}(\phi) \). Thus, it suffices to note that all simple density ideals are tall density ideals in the sense of Farah (by [1, Theorem 3.2]).

(e): By [4, Lemma 1.2.2 and Theorem 1.2.5], each analytic \( P \)-ideal is \( \Pi_{0}^{0} \). Thus, by Proposition 4.9, \( \text{Erdős-Ulam} \) ideals and simple density ideals are hereditary weak \( P \)-ideals. What is more, by [22, Theorem 3], there is a family of cardinality \( \varepsilon \) consisting of simple density ideals pairwise incomparable with respect to \( \leq_{K} \). Hence, the thesis follows from Theorem 10.4.

**11.2. Hindman spaces.**

**Definition 11.2.** For an infinite set \( M \subseteq \omega \) we write \( FS(M) = \{ \sum_{n \in F} n : F \in [M]^{<\omega}, F \neq \emptyset \} \). A topological space is \( \text{Hindman} \) if for each sequence \( (x_{n}) \subseteq X \) we can find \( x \in X \) and an infinite \( D \subseteq \omega \) such that for each open neighborhood \( U \) of \( x \) there is a \( m \in \omega \) with \( \{ x_{n} : n \in FS(D \setminus \{0, 1, \ldots, m\}) \} \subseteq U \).
Hindman spaces were introduced by M. Kojman in [17]. Since then this subject has been extensively studied among others in [9], [11], [16], [19] and recently in [10], where the following result is proved:

**Proposition 11.3** ([10, Proposition 1.1]). Every Mrówka space defined by a MAD family is not a Hindman space.

**Lemma 11.4.** Every boring space is Hindman.

**Proof.** By [17, Theorem 11], a space $X$ is Hindman whenever it satisfies condition $(\star)$, that is, whenever $X$ is Hausdorff and the closure in $X$ of each countable subset of $X$ is compact and first-countable. Using Proposition 3.2(ii), it is easy to see that each boring space satisfies condition $(\star)$.\hfill $\Box$

**Corollary 11.5.** (CH) The following are equivalent for every ideal $\mathcal{I}$:

- $\mathcal{I}$ is unboring;
- there is an uncountable Mrówka space in $\text{FinBW}(\mathcal{I})$ which is not a Hindman space;
- there is a space in $\text{FinBW}(\mathcal{I})$ which is not a Hindman space.

**Proof.** Follows from Theorem 6.6, Proposition 11.3 (as any uncountable Mrówka space given by Theorem 6.6 is defined by a MAD family) and Lemma 11.4.\hfill $\Box$

The following result generalizes the main theorem of [19] stating that under CH Hindman spaces do not coincide with the class $\text{FinBW}(\mathcal{W})$ (see also [16] where this is proved under MA($\sigma$-centered) instead of CH).

**Corollary 11.6.** (CH) There is no ideal $\mathcal{I}$ such that $\text{FinBW}(\mathcal{I})$ coincides with Hindman spaces.

**Proof.** If $\mathcal{I}$ is a boring ideal then $[0,1]$ is a Hindman space (cf. [17]) not belonging to $\text{FinBW}(\mathcal{I})$ (by Theorem 6.5). On the other hand, if $\mathcal{I}$ is unboring then it suffices to apply Corollary 11.5.\hfill $\Box$

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