MIRROR SYMMETRY AND QUANTUM COHOMOLOGY OF PROJECTIVE BUNDLES

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Abstract. In [E] we conjectured a relation between the quantum $D$-modules of a smooth variety $X$ and the projectivisation of a direct sum of line bundles over it. In this paper we prove the conjecture when $X$ is a semiample complete intersection in a toric variety. We use the conjecture to show that the relations of the small quantum cohomology ring of $X$ that come from differential operators lift to the projective bundle. The basic cohomology relation of the projective bundle deforms to a relation in the small quantum cohomology.

1. Introduction

Let $Y$ be a projective manifold. Denote by $Y_{k,\beta}$ the moduli stack of rational stable maps of class $\beta \in H_2(Y,\mathbb{Z})$ with $k$-markings \cite{FP} and $[Y_{k,\beta}]$ its virtual fundamental class \cite{BF,LT}. Throughout this paper we will be interested mainly in $k = 1$. Recall the following features

- $e : Y_{1,\beta} \to Y$ - the evaluation map.
- $\psi$ - the first chern class of the cotangent line bundle on $Y_{1,\beta}$.
- $ft : Y_{1,\beta} \to Y_{0,\beta}$ - the forgetful morphism.

Let $\hbar$ be a formal variable and

$$J_\beta(Y) := e_* \left( \frac{[Y_{1,\beta}]}{\hbar(\hbar - \psi)} \right) = \sum_{k=0}^{\infty} \frac{1}{\hbar^{2+k}} e_* (\psi^k \cap [Y_{1,\beta}]).$$

The sum is finite for dimension reasons. Let $p = \{p_1, p_2, ..., p_k\}$ be a nef basis of $H^2(Y,\mathbb{Q})$. For $t = (t_0, t_1, ..., t_k)$ let

$$tp := t_0 + \sum_{i=1}^{k} t_i p_i.$$ 

The $D$-module for the quantum cohomology of $Y$ is generated by \cite{G2}

$$J(Y) = \exp \left( \frac{tp}{\hbar} \right) \sum_{\beta \in H_2(Y,\mathbb{Z})} q^\beta J_\beta(Y).$$
where we use the convention $J_0 = 1$. The generator $J(Y)$ encodes all of the one marking Gromov-Witten invariants and gravitational descendants of $Y$.

For any ring $\mathcal{A}$, the formal completion of $\mathcal{A}$ along the semigroup $MY$ of the rational curves of $Y$ is defined to be

\begin{equation}
\mathcal{A}[[q^\beta]] := \{ \sum_{\beta \in MY} a_\beta q^\beta, \ a_\beta \in \mathcal{A}, \ \beta \ -\ \text{effective} \}.
\end{equation}

where $\beta \in H_2(Y, \mathbb{Z})$ is effective if it is a positive linear combination of rational curves. This new ring behaves like a power series since for each $\beta$, the set of $\alpha$ such that $\alpha$ and $\beta - \alpha$ are both effective is finite.

Alternatively, we may identify $q^\beta$ with $q^{d_1} \cdots q^{d_k}$ where $\{d_1, d_2, \ldots, d_k\}$ are the coordinates of $\beta$ relative to the dual basis of $\{p_1, \ldots, p_k\}$.

We regard $J(Y)$ as an element of $H^*(Y, \mathbb{Q})[[t]][[q^\beta]]$. Let $*$ denote the small quantum product of $Y$. The small quantum cohomology ring $(QH_*^s Y, *)$ is a deformation of the cohomology ring $(H^*(Y, \mathbb{Q}[q^\beta]), \cup)$. Its structural constants are three point Gromov-Witten invariants.

The generator $J(Y)$ arises naturally in the solution of the small quantum differential equation:

\begin{equation}
1 \leq i \leq k, \ h\partial / \partial t_i = p_i*.
\end{equation}

Furthermore $J(Y)$ may be used to generate relations in $QH_*^s Y$. Let $P(h, h\partial / \partial t_i, q_i)$ be a polynomial differential operator where $q_i$ and $h$ act via multiplication and $q_i = e^{t_i}$ are on the left of derivatives. If $P(h, h\partial / \partial t_i, q_i) J(Y) = 0$ then $P(0, p_i, q_i) = 0$ in $QH_*^s Y$.

If $Y$ is a toric variety, $J(Y)$ is related to an explicit hypergeometric series $I(Y)$ via a change of variables ([G1], [LLY3]). Furthermore, if $Y$ is Fano then the change of variables is trivial, i.e. $J(Y) = I(Y)$ thus completely determining the one point Gromov-Witten invariants and gravitational descendants of $Y$. In [E] we conjectured an extension of this result in the case of a projective bundle. We recall this conjecture below.

Let $X$ be a projective manifold. Following Grothendieck’s notation, let $\pi : \mathbb{P}(V) = \mathbb{P}(\oplus_{i=0}^n L_i) \to X$ be the projective bundle of hyperplanes of a vector bundle $V$. Assume that $L_0 = O_X$. The $H^*X$-module $H^*\mathbb{P}(V)$ is generated by $z := c_1(O_{\mathbb{P}(V)}(1))$ with the relation

\begin{equation}
\prod_{i=0}^n (z - c_1(L_i)) = 0.
\end{equation}
Let $s_i : X \to \mathbb{P}(V)$ be the section of $\pi$ determined by the $i$-th summand of $V$ and $X_i := s_i(X)$. Then $\mathcal{O}_{\mathbb{P}(V)}(1)|_{X_i} \simeq L_i$. Let $\{p_1, \ldots, p_k\}$ be a nef basis of $H^2(X, \mathbb{Q})$. In $[2]$ we showed that

**Lemma 1.0.1.** If the line bundles $L_i, i = 1, \ldots, n$ are nef then

(a) $\{p_1, \ldots, p_k, z\}$ is a nef basis of $H^2(\mathbb{P}(V), \mathbb{Q})$.
(b) The Mori cones of $X$ and $\mathbb{P}(V)$ are related via

$$M \mathbb{P}(V) = M X \oplus \mathbb{Z}_{\geq 0} \cdot [l]$$

where $[l]$ is the class of a line in the fiber of $\pi$.

Here $M X$ is embedded in $M \mathbb{P}(V)$ via the section $s_0$. If $C \in \mathbb{P}(V)$ is a rational curve, there exists a unique pair $(\nu \geq 0, \beta \in M X)$ such that $[C] = \nu[l] + \beta$. We will identify the homology class $[C]$ with $(\nu, \beta)$. The generator $J_{\mathbb{P}(V)}$ is an element of $H^*(\mathbb{P}(V), \mathbb{Q})[t, t_{k+1}][[q_1, q_2]]$.

For a line bundle $L$ and a curve $\alpha$ we denote $L(\alpha) := c_1(L) \cdot \alpha$.

Define the “twisting” factor

$$T_{\nu, \beta} := \prod_{i=0}^{n} \frac{\prod_{m=-\infty}^{0}(z - c_1(L_i) + mh)}{\prod_{m=-\infty}^{\nu - c_1(L_i)}(z - c_1(L_i) + mh)}.$$

Let $I_{\nu, \beta} := T_{\nu, \beta} \cdot \pi^* J_{\beta}$ where $\pi^*$ is the flat pull back and define a “twisted” hypergeometric series for the projective bundle $\mathbb{P}(V)$:

$$I(\mathbb{P}(V)) := \exp\left(\frac{tp + t_{k+1}z}{\hbar}\right) \cdot \sum_{\nu, \beta} q_1^\nu q_2^\beta I_{\nu, \beta}.$$

In $[2]$ we proposed the following

**Conjecture 1.0.1.** Let $L_i, i = 1, \ldots, n$ be nef line bundles such that $-K_X - c_1(V)$ is ample. Then $J(\mathbb{P}(V)) = I(\mathbb{P}(V))$.

In the next section we prove this conjecture when $X$ is a complete intersection in a toric variety.

In the last section we study the consequences of the proposed conjecture in the relation between $\mathbb{Q}H_X^* X$ and $\mathbb{Q}H^{*}_{\mathbb{P}(V)}$. Recall that $H^* \mathbb{P}(V)$ is an $H^* X$-module generated by $z := c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ with the relation

$$\prod_{i=0}^{n}(z - c_1(L_i)) = 0.$$  

We show that the relations of $\mathbb{Q}H_X^* X$ that come from the quantum differential equations lift to relations in $\mathbb{Q}H^{*}_{\mathbb{P}(V)}$. We also show that $[2]$ deforms into the relation

$$z \prod_{i=1}^{n}(z - c_1(L_i)) = q_1.$$
in $QH^*_s\mathbb{P}(V)$.

2. Toric case proof

Toric varieties and torus actions. Assume $Y$ is a toric variety determined by a fan $\Sigma \subset \mathbb{Z}^m$. Denote by $b_1, ..., b_r = m+k$ its one dimensional cones. Let $Z_Y \subset \mathbb{C}^r$ be the variety whose ideal is generated by the products of those variables which do not generate a cone in $\Sigma$. The toric variety $Y$ is the geometric quotient of $\mathbb{C}^r - Z(\Sigma)$ by a torus of dimension $k$.

Let $\tilde{L}_i$, $i = 0,1,...,n$ be toric line bundles and $\tilde{V} = \oplus_{i=0}^n \tilde{L}_i$. The projective bundle $\pi : \mathbb{P}(\tilde{V}) \to Y$ is also a toric variety and there is a canonical way to obtain its fan $\mathbb{O}$. Let $\mathbb{Z}^n$ be a new lattice with basis $\{f_1,...,f_n\}$. The edges $b_1, ..., b_r$ of $\Sigma$ are lifted to new edges $B_1, B_2, ..., B_r$ in $\mathbb{Z}^m \oplus \mathbb{Z}^n$ and subsequently $\Sigma$ is lifted in a new fan $\Sigma_1$ in the obvious way. Let $\Sigma_2 \subset 0 \oplus \mathbb{Z}^n$ be the fan of $\mathbb{P}(\mathbb{C}^n)$ with edges $F_0 = -\sum_{i=1}^n f_i, F_1 = f_1, ..., F_n = f_n$. The canonical fan associated to $\mathbb{P}(V)$ consists of the cones $\sigma_1 + \sigma_2$ where $\sigma_1, \sigma_2$ are cones in $\Sigma_1, \Sigma_2$. Let $N = r + n + 1$. The torus $\mathbb{T} = (\mathbb{C}^*)^N$ acts on both $Y$ and $\mathbb{P}(V)$ by scaling of coordinates in respectively $\mathbb{C}^r$ and $\mathbb{C}^N$. The one dimensional cones correspond to $\mathbb{T}$ invariant divisors. The edges $F_i$ in the canonical fan of the projective bundle, correspond to the divisors $z - c_1(L_i)$ where $z := c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ while $B_i$ correspond to the pullback of the base divisors associated with $L_i$.

Let $L'_a : a = 1,2,...,M$ be semiample line bundles (i.e. generated by sections). Let $X$ be the zero locus of a generic section $s$ of $E = \oplus_{a=1}^M L'_a$ and let $V$ be the restriction of $\tilde{V}$ to $X$. The total space of $\mathbb{P}(V)$ is the zero locus of the section $\pi^*(s)$ of the pull back bundle $\pi^*(\tilde{V})$. To assure that the conditions of the conjecture are met for the bundle $\mathbb{P}(V)$ over $X$ we assume that $\tilde{L}_i, i = 1,1,...,n$ are nef and $-K_Y - \sum_{a=1}^M c_1(L'_a) - \sum_{i=0}^n c_1(\tilde{L}_i)$ is ample.

Let $E_d$ be the bundle on $Y_{1,d}$ whose fiber over the moduli point $(C, x_1, f : C \to Y)$ is $\oplus_a H^0(f^*(L'_a))$. Denote by $s_E$ its canonical section induced by $s$, i.e.

$$s_E((C, x_1, f)) = f^*(s).$$

The stack theoretic zero section of $s_E$ is the disjoint union

$$(3) \quad Z(s_E) = \bigsqcup_{i, (\beta) = d} X_{1, \beta}.$$
The map \( i_s : H_2X \to H_2Y \) is not injective in general, hence the zero locus \( Z(s_E) \) may have more than one connected component. An example is the quadric surface in \( \mathbb{P}^3 \). The sum of the virtual fundamental classes \( [X,\beta] \) is the refined top Chern class of \( E_d \) with respect to \( s_E \).

There is a stack morphism \( \mathbb{P}(\tilde{V})_{1,(\nu,d)} \to Y_{1,d} \). Let \( \tilde{E}_{\nu,d} \) and \( \tilde{s}_E \) be the pull backs of \( E_d \) and \( s_E \). The zero section of \( \tilde{s}_E \) is the disjoint union

\[
z(\tilde{s}_E) = \bigsqcup_{i,(\beta)=d} \mathbb{P}(V)_{1,(\nu,\beta)}.
\]

It follows that

\[
\sum_{i,(\beta)=d} [\mathbb{P}(V)_{1,(\nu,\beta)}] = c_{\text{top}}(\tilde{E}_{\nu,d} \cap [\mathbb{P}(\tilde{V})_{1,(\nu,d)}]).
\]

Consider the following generating functions

\[
J(\mathbb{P}(\tilde{V}), E) = \exp\left(\frac{tp + t_{k+1}z}{\hbar}\right) \sum q_1 \nu q_2^d e_* \left( \frac{c_{\text{top}}(\tilde{E}_{\nu,d} \cap [\mathbb{P}(\tilde{V})_{1,(\nu,d)}])}{\hbar(\hbar - c)} \right)
\]

and

\[
\tilde{I}(\mathbb{P}(\tilde{V}), E) = \exp\left(\frac{tp + t_{k+1}z}{\hbar}\right) \sum q_1 \nu q_2^d \tau_{\nu,d} \pi_* e_* \left( \frac{c_{\text{top}}(E_d) \cap [Y_{1,d}]}{\hbar(\hbar - c)} \right).
\]

**Proposition 2.0.1.** If \(-K_Y - \sum_{a=1}^M c_1(L_a^\prime) - \sum_{i=0}^n c_1(L_i) \) is ample then

\[
J(\mathbb{P}(\tilde{V}), E) = \tilde{I}(\mathbb{P}(\tilde{V}), E).
\]

**Proof.** Let

\[
I_d(Y, E) = \prod_{a} \prod_{m=-\infty}^{L_a^\prime(d)} (L_a^\prime + m\hbar) \prod_{i} \prod_{m=-\infty}^{B_i(d)} (B_i + m\hbar).
\]

From [GL1, LLY2, LLY3] we know that \( J(\mathbb{P}(\tilde{V}), E) \) is related via a mirror transformation to

\[
I(\mathbb{P}(\tilde{V}), E) = \exp\left(\frac{tp + t_{k+1}z}{\hbar}\right) \cdot \sum q_1 \nu q_2^d \tau_{\nu,d} I_d(Y, E).
\]

Likewise

\[
J(Y, E) = \exp\left(\frac{tp}{\hbar}\right) \sum q_2^d e_* \left( \frac{c_{\text{top}}(E_d) \cap [Y_{1,d}]}{\hbar(\hbar - c)} \right)
\]

is related to

\[
I(Y, E) = \exp\left(\frac{tp}{\hbar}\right) \sum q_2^d I_d(Y, E).
\]
Since $-K_{F(V)} - \sum_a c_1(L'_a)$ and $-K_Y - \sum_a c_1(L'_a)$ are ample, the mirror transformations are particularly simple. Indeed, both series can be written as power series of $\hbar^{-1}$ as follows:

$$I(\mathbb{P}(\tilde{V}), E) = 1 + \frac{P_1(q_1, q_2)}{\hbar} + o(\hbar^{-1}), \quad I(Y, E) = 1 + \frac{P_2(q_2)}{\hbar} + o(\hbar^{-1}),$$

where $P_1(q_1, q_2), P_2(q_2)$ are both polynomials supported respectively in

$$\Lambda_1 := \{(\nu, d) \mid (-K_{F(V)} - \sum c_1(L'_a)) = 1; \quad z - c_1(\tilde{L}_j) \geq 0 \forall j = 1, 2, ..., n; \quad B_i \geq 0 \forall i = 1, 2, ..., r\}$$

and

$$\Lambda_2 := \{d \mid (-K_Y - \sum c_1(L'_a)) = 1; \quad B_i \geq 0 \forall i = 1, 2, ..., r\}.$$

Then

$$J(\mathbb{P}(\tilde{V}), E) = \exp \left( \frac{-P_1(q_1, q_2)}{\hbar} \right) I(\mathbb{P}(\tilde{V}), E)$$

and

$$J(Y, E) = \exp \left( \frac{-P_2(q_2)}{\hbar} \right) I(Y, E).$$

Let us examine the relation between $\Lambda_1$ and $\Lambda_2$. From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to \pi^*V^*(1) \to T_{F(V)} \to \pi^*T_X \to 0$$

we find that

$$-K_{F(V)} - \sum c_1(L'_a) = -K_Y - \sum_i (\tilde{L}_i) - \sum c_1(L'_a) + (n + 1)z.$$

Assume $(\nu, d) \in \Lambda_1$. Since $-K_Y - \sum_{a=1}^M c_1(L'_a) - \sum_{i=0}^n c_1(\tilde{L}_i)$ is ample then $\nu = 0$. Now $(z - c_1(\tilde{L}_j)) \cdot (0, d) \geq 0 \forall j = 1, 2, ..., n$ and $\tilde{L}_j$ are semipositive $\forall j = 1, 2, ..., n$. So $c_1(\tilde{L}_j) \cdot d = 0 \forall j = 1, 2, ..., n$. It follows that

$$(-K_Y - \sum c_1(L'_a)) \cdot d = (-K_{F(V)} - \sum c_1(L'_a)) \cdot d = 1,$$

so $d \in \Lambda_2$. Conversely, let $d \in \Lambda_2$. Then $c_1(\tilde{L}_j) \cdot d = 0 \forall j = 1, 2, ..., n$ since $-K_Y - \sum_{a=1}^M c_1(L'_a) - \sum_{i=0}^n c_1(\tilde{L}_i)$ is ample. It follows that

$$(-K_{F(V)} - \sum c_1(L'_a)) \cdot d = (-K_Y - \sum c_1(L'_a)) \cdot d = 1$$

and

$$(z - c_1(\tilde{L}_j)) \cdot d = 0, \forall j = 1, 2, ..., n$$

so $(0, d) \in \Lambda_1$.

We have thus shown that $c_1(\tilde{L}_j) \cdot d = 0, \forall d \in \Lambda_2, \forall j = 1, 2, ..., n$ and

$$\Lambda_1 = \{(0, d) \mid d \in \Lambda_2\}.$$
It follows that $T_{0,d} = 1$, $\forall d \in \Lambda_2$ hence $P_1(q_1, q_2) = P_2(q_2)$. Notice also that if we expand
\[
\exp \left( -\frac{P_2(q_2)}{\hbar} \right) = \sum_{\alpha} c_{\alpha} q_2^\alpha
\]
then
\[
c_1(\tilde{L}_j) \cdot \alpha = 0, \forall j = 1, 2, ..., n.
\]
Hence for each $(\nu, d) \in M_{\mathbb{P}}(\tilde{V})$ we have
\[
T_{\nu, d} = T_{\nu, d + \alpha}.
\]
Now the proposition follows easily. □

As we commented in the sentence after equation (3), in general the above proposition is not very relevant for our purpose. However, if we assume that the map
\[
i_* : H_2(X) \to H_2(Y)
\]
is injective, then one can easily show that
\[
i_*(J_{\nu, \beta}(\mathbb{P}(V))) = J_{\nu, i_*(\beta)}(\mathbb{P}(\tilde{V}), E)
\]
and
\[
i_*(I_{\nu, \beta}(\mathbb{P}(V))) = I_{\nu, i_*(\beta)}(\mathbb{P}(\tilde{V}), E).
\]
The proposition shows that the conjecture holds for complete intersection in toric varieties that satisfy condition (4).

3. Relations in the small quantum cohomology ring

In this section we use the proposed conjecture to study small quantum deformations of the cohomological relation
\[
H^*(\mathbb{P}(V)) = H^* X / \left( \prod_{i=0}^{n} (z - c_1(L_i)) \right) = 0.
\]

As explained in the introduction, some of the relations in the small quantum cohomology ring come from differential operators. Let
\[
c_1(L_i) = \sum_{j=1}^{k} a_{ij} p_j, \ i = 0, 1, ..., n.
\]

We obtain two kinds of relations in $QH^*_s \mathbb{P}(V)$.

First, the relations in $QH^*_s X$ that come from differential operators may be lifted to relations in $QH^*_s \mathbb{P}(V)$. Indeed, consider a polynomial differential operator
\[
\mathcal{P}(\hbar, \hbar \partial / \partial t_1, ..., \hbar \partial / \partial t_k, q_2) = \sum_{\alpha \in \Lambda} q_2^\alpha \mathcal{P}_\alpha
\]
where $\Lambda \subset MX$ is a finite set. Suppose that

$$0 = \mathcal{P}J(X) = \sum_{\alpha \in \Lambda} q_2^\alpha \sum_{\beta} \mathcal{P}_\alpha \left( \exp \left( \frac{pt}{h} \right) q_2^\beta \right) J_\beta(X)$$

$$= \sum_{\alpha \in \Lambda} q_2^\alpha \sum_{\beta} c_{\alpha,\beta} \exp \left( \frac{pt}{h} \right) q_2^\beta J_\beta(X) = \exp \left( \frac{pt}{h} \right) \sum_{\alpha \in \Lambda, \beta} q_2^{\alpha+\beta} c_{\alpha,\beta} J_\beta(X).$$

Let

$$\delta_\alpha = n \prod_{i=1}^L (z - c_1(L_i)),$$

$$\tilde{\mathcal{P}} = \sum_{\alpha \in \Lambda} q_2^\alpha \delta_\alpha \mathcal{P}_\alpha,$$

with the convention that the factors corresponding to $L_i$ are missing if $L_i(\alpha) = 0$. We compute

$$\tilde{\mathcal{P}}J(\mathbb{P}(V)) = \sum_{\alpha \in \Lambda} q_2^\alpha \delta_\alpha \sum_{\nu,\beta} \mathcal{P}_\alpha \left( q_2^\beta \exp \left( \frac{pt + zt_{k+1}}{h} \right) \right) q_1^\nu T_{\nu,\beta} J_\beta =$$

$$\sum_{\alpha \in \Lambda} q_2^\alpha \delta_\alpha \sum_{\nu,\beta} c_{\alpha,\beta} \exp \left( \frac{pt + zt_{k+1}}{h} \right) q_1^\nu q_2^\beta T_{\nu,\beta} J_\beta = 0.$$

A simple calculation shows that

$$\delta_\alpha \left( \exp \left( \frac{pt + zt_{k+1}}{h} \right) q_1^\nu q_2^\beta T_{\nu,\beta} \right) = \exp \left( \frac{pt + zt_{k+1}}{h} \right) q_1^\nu q_2^\beta T_{\nu,\alpha+\beta}. $$

It follows that

$$\tilde{\mathcal{P}}J(\mathbb{P}(V)) = \exp \left( \frac{pt + zt_{k+1}}{h} \right) \sum_{\nu} q_1^\nu \sum_{\alpha \in \Lambda, \beta} c_{\alpha,\beta} q_2^{\alpha+\beta} T_{\nu,\alpha+\beta} J_\beta(X) = 0.$$

Hence the relation $\mathcal{P}(0, p_1, ..., p_k, q_2) = 0$ in $QH^*_X$ lifts into the relation

$$\mathcal{P}(0, p_1, ..., p_k, q_2) \prod_{i=1}^n (z - c_1(L_i)) = 0$$

in $QH^*_\mathbb{P}(V)$, where

$$\left( \prod_{i=1}^n (z - c_1(L_i)) \right)^\alpha := \prod_{i=1}^n (z - c_1(L_i))^{L_i(\alpha)}, \forall \alpha \in MX.$$

Second, we derive a $q_1$-deformation of the relation $\prod_{i}(z - c_1(L_i)) = 0$. Consider the operator

$$\Delta(h \frac{\partial}{\partial t_1}, ..., h \frac{\partial}{\partial t_k}, h \frac{\partial}{\partial t_{k+1}}, q_1) := \prod_{i=0}^n (h \frac{\partial}{\partial t_{k+1}} - \sum_{j=1}^k a_{ij} h \frac{\partial}{\partial t_j}) - q_1.$$
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It is easy to show that it satisfies

\[ \Delta J(\mathbb{P}(V)) = 0. \]

It follows that \( \Delta(p_1, \ldots, p_k, z, q_1) = 0 \), in \( QH^*_s(\mathbb{P}(V)) \) i.e.

\[ \prod_{i=0}^{n}(z-c_1(L_i)) = q_1. \]

Much like \( z^{n+1} = q \) is the deformation in \( QH^*_s(\mathbb{P}^n) \) of \( z^{n+1} = 0 \), the above relation is the deformation of \( \prod_{i=0}^{n}(z-c_1(L_i)) = 0 \).

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