Measures of association between algebraic varieties, II: Self-correspondences

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Dedicated to Claire Voisin on the occasion of her sixtieth birthday

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1. Introduction

In our previous paper [LM23], we introduced and studied some invariants intended to measure how far from birationally isomorphic two given varieties $X$ and $Y$ of the same dimension might be. These were defined by studying the minimal birational complexity of correspondences between $X$ and $Y$. Following a suggestion of Jordan Ellenberg, the present note continues this line of thought by investigating self-correspondences of a given variety.

Let $X$ be a smooth complex projective variety of dimension $n$. By an auto-correspondence of $X$, we understand a smooth projective variety $Z$ of dimension $n$ sitting in a diagram

$$
\begin{array}{ccc}
X & \rightarrow & Z \\
\downarrow a & & \downarrow b \\
X & \rightarrow & X
\end{array}
$$

with $a$ and $b$ dominant and hence generically finite. We assume that $Z$ maps birationally to its image in $X \times X$ (so that general fibers of $a$ and $b$ are identified with subsets of $X$). The auto-correspondence degree of $X$ is defined to be

$$\text{autocorr}(X) = \min_{Z, \Delta} \{\deg(a) \cdot \deg(b)\},$$

the minimum being taken over all such $Z$ excluding those that map to the diagonal. Thus $\text{autocorr}(X) = 1$ if and only if $X$ admits non-trivial birational automorphisms. By considering the fiber square of a rational covering $X \rightarrow \mathbb{P}^n$, one sees that

$$\text{autocorr}(X) \leq (\text{irr}(X) - 1)^2,$$

where the degree of irrationality $\text{irr}(X)$ is defined to be the least degree of such a covering (see Section 2, below). Our intuition is that equality holding means that $X$ is “as far as possible” from having any interesting self-correspondences of low degree.

Our main results are as follows.

**Proposition A.** If $X$ is a very general curve of genus $g \geq 3$, then

$$\text{autocorr}(X) = (\text{gon}(X) - 1)^2,$$

and minimal correspondences arise from the fiber square of a gonal map.

**Theorem B.** Let $X \subseteq \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n + 2$. Then

$$\text{autocorr}(X) = (d - 2)^2 = (\text{irr}(X) - 1)^2,$$

and again minimal correspondences are birational to the fiber square of projection from a point.
In fact, we classify all self-correspondences in a slightly wider numerical range: see Theorem 4.

If \( X \) is a hyperelliptic curve of genus \( g \), then \( \text{autocorr}(X) = 1 \) since \( X \) has a non-trivial automorphism whose graph is a non-diagonal copy of \( X \) sitting in \( X \times X \). David Rhyd asked whether there are any unexpected hyperelliptic curves in this product. Our final result asserts that there are not.

**Theorem C.** Let \( X \) be a very general hyperelliptic curve of genus \( g \geq 2 \). The only hyperelliptic curves in \( X \times X \) are

- the fibers of the projection maps,
- the diagonal, and
- the graph of the hyperelliptic involution.

In particular, the image of any hyperelliptic curve in \( X \times X \) under the Abel–Jacobi map is geometrically degenerate in \( J(X) \times J(X) \); i.e., it generates a proper subtorus of that product.

By a hyperelliptic curve in \( X \times X \), we mean an irreducible curve \( Z \subseteq X \times X \) whose normalization is hyperelliptic.

As pointed out by the referee, Theorem C is closely related to work of Schoen from [Sch90]. In particular, [Sch90, Proposition 4.1(2)] implies Theorem C for genus 2 curves. Schoen also gives examples of hyperelliptic curves \( X \) such that \( X \times X \) contains finitely many hyperelliptic curves (see [Sch90, Lemma 1.5 and Proposition 2.2]).

A consequence of the proof of Theorem C (see Proposition 7) is that given a very general hyperelliptic curve \( X \) and any hyperelliptic curve \( C \) whose Jacobian dominates \( J(X) \), we have

\[
\text{Hom}(J(C), J(X)) \cong \mathbb{Z}.
\]

This rigidity statement complements recent results of Naranjo and Pirola concerning dominant morphisms from hyperelliptic Jacobians (see [NP18, Theorems 1.4 and 1.6]).

We work throughout over the complex numbers.

**Acknowledgments**

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It is an honor to dedicate this paper to Claire Voisin on the occasion of her sixtieth birthday. Her influence on both the field as a whole and the work of the two authors has been immense.

### 2. Preliminaries and proof of Proposition A

We start with some general remarks about the auto-correspondence degree. Given a smooth complex projective variety \( X \) of dimension \( n \), its auto-correspondence degree \( \text{autocorr}(X) \) is defined as in the introduction. Evidently, this is a birational invariant of \( X \).

Note that if \( X \) admits a rational covering \( X \to \mathbb{P}^n \) of degree \( \delta \), then

\[
\text{autocorr}(X) \leq (\delta - 1)^2.
\]

In fact, replacing \( X \) with a suitable birational model, we can suppose that \( X \to \mathbb{P}^n \) is an actual morphism. Then

\[
X \times_{\mathbb{P}^n} X \subseteq X \times X
\]

contains the diagonal \( \Delta_X \) as an irreducible component. The union of the remaining components \( Z' \subseteq X \times_{\mathbb{P}^n} X \) has degree \( \delta - 1 \) over each of the factors, and \((\ast)\) follows. In particular,

\[
\text{autocorr}(X) \leq (\text{irr}(X) - 1)^2,
\]

\((2.1)\)
where $\text{irr}(X)$ denotes the minimal degree of such a rational covering $X \to \mathbb{P}^n$. Our main results assert that in several circumstances equality holds in (2.1) and that the minimal correspondences arise as just described. We will say in this case that $Z$ is residual to the fiber square of a minimal covering of $\mathbb{P}^n$.

As in the earlier works [BCDP14, BDP+17, LM23], the action of a correspondence on cohomology plays a central role. In the situation of diagram (1.1), $Z$ gives rise to endomorphisms

$$Z_* = b_* \circ a^*, \quad Z^* = a_* \circ b^*$$

of the Hodge structure $H^u(X)$. We denote by

$$Z_*^{u,0} = \text{Tr}_b \circ a^*, \quad Z^*^{u,0} = \text{Tr}_a \circ b^*$$

the corresponding endomorphisms of the space $H^{u,0}(X)$ of holomorphic $n$-forms on $X$. In the cases of interest, these will act as a multiple of the identity, allowing us to use a variant of the arguments from the cited papers involving Cayley–Bacharach.

We now turn to the proof of Proposition A. We then suppose that $X$ is a very general curve of genus $g \geq 3$ and that $Z \to X \times X$ is a correspondence as in (1.1) that computes the auto-correspondence degree of $X$. The generality hypothesis on $X$ implies first of all that

$$\text{Pic}(X \times X) = a^* \text{Pic}(X) \oplus b^* \text{Pic}(X) \oplus Z \cdot \Delta,$$

and hence the image of $Z$ in $X \times X$ is defined by a section of

$$\text{(2.2)} \quad (B \boxplus A)(-m\Delta)$$

for some line bundles $A, B$ on $X$ and some $m \in \mathbb{Z}$. Note that then

$$\deg(a) = \deg(A) - m, \quad \deg(b) = \deg(B) - m.$$ 

Moreover, both maps

$$Z_*^{1,0}, Z^*^{1,0} : H^{1,0}(X) \to H^{1,0}(X)$$

are multiplication by $-m$.

We start by proving that

$$\deg(a), \deg(b) \geq \text{gon}(X) - 1,$$

which will imply that $\text{autocorr}(X) = (\text{gon}(X) - 1)^2$. We may suppose that $m \neq 0$, for if $m = 0$, then we are in the setting of [LM23, Example 1.7] and $\deg(a), \deg(b) \geq \text{gon}(X)$. Now fix a general point $y \in X$, and suppose that

$$b^{-1}(y) = x_1 + \ldots + x_\delta,$$

where $\delta = \deg(b)$. Then for any $\omega \in H^{1,0}(X)$, we have

$$\omega(x_1) + \ldots + \omega(x_\delta) = Z_*^{1,0}(\omega)(y) = -m \cdot \omega(y).$$

It follows that the $\delta + 1$ points $y, x_1, \ldots, x_\delta$ do not impose independent conditions on $H^{1,0}(X)$, and hence they move in at least a pencil. In other words, $\deg(b) + 1 \geq \text{gon}(X)$ and similarly $\deg(a) + 1 \geq \text{gon}(X)$, as required.

Assuming that $\deg(a) = \deg(b) = \text{gon}(X) - 1$, it remains to show that a minimal correspondence arises from the fiber square of a pencil. For this, we first of all rule out the possibility that $m < 0$. In fact, by intersecting $Z$ with the diagonal, one finds that $\deg(A) + \deg(B) \geq -m \cdot (2g - 2)$, and hence $\deg(a) + \deg(b) \geq -m \cdot (2g)$. But unless $g = 3$, this is impossible if $m < 0$ since $2 \cdot (\text{gon}(X) - 1) \leq g + 1$. When $g = 3$, one needs to rule out the existence of line bundles $A, B$ of degree 2 such that $r(A(y)) = r(B(y)) \geq 1$ for every $y \in X$, and this follows from the well-known description of pencils of degree 3 on a smooth plane quartic. (See also Remark 1.)

Returning to the setting of (2.2), now assume that $m > 0$. Then for every $x \in X$,

$$a^{-1}(x) \in |A(-m \cdot x)|, \quad b^{-1}(x) \in |B(-m \cdot x)|,$$
which implies that
\[(2.3) \quad r(A), r(B) \geq m.\]
We will use this to show that \(\deg(a)\) and \(\deg(b)\) are minimized when \(A\) and \(B\) move in pencils.

In fact, write \(d = \deg(A)\). If \(A\) is non-special, then \(r(A) = d - g\), so \(\deg(a) = d - m \geq g\) thanks to (2.3), and we get a map of smaller degree from a gonal pencil.\(^{(0)}\) Therefore assume that \(A\) is special, so that
\[A \in W^m_d(X).\]
We may suppose that \(X\) is Brill–Noether general, in which case
\[\rho(m,d,g) = g - (m + 1)(g - d + m) \geq 0.\]
It follows that
\[(m + 1)d \geq mg + m(m + 1)\]
and
\[d \geq \left(\frac{m}{m + 1}\right)g + m,\]
so that
\[\deg(a) = d - m \geq \left(\frac{m}{m + 1}\right)g.\]
This is minimized when \(m = 1\) and similarly for \(B\). Thus we can assume that \(r(A) = r(B) = 1\) and that the image of \(Z\) lies in the linear series
\[|(B \boxtimes A)(-\Delta)|\]
on \(X \times X\). But this series is empty unless \(A = B\), in which case it consists exactly of the residual to the diagonal in the fiber square of the pencil defined by \(A\). This completes the proof.

**Remark 1.** Suppose that \(X \subseteq \mathbb{P}^2\) is a smooth plane curve of degree \(d > 3\); then the correspondence defined as the closure of
\[\{(x,y) \in X \times X \mid y \neq x\text{ is in the embedded tangent line to } X \text{ at } x\}\]
dominates the first factor with degree \(d - 2\) but fails to arise from the fiber square of projection from a point on \(X\). The degree of the second projection is \(d(d - 1)\), which is much greater than \(d - 1 = \text{gon}(X)\).

**Remark 2.** As the referee remarks, the preceding result leads to two interesting questions to which we don’t know the answers. First, if \(X\) is a very general \(k\)-gonal curve, is it true that
\[\text{autocorr}(X) = (k - 1)^2?\]
We suspect that this should be the case. Second, does Proposition A hold if we replace “very general” with “general”?

### 3. Proof of Theorem B

In this section we prove the following refinements of Theorem B from the introduction.

**Theorem 3.** Let \(X \subseteq \mathbb{P}^{n+1}\) be a very general hypersurface of degree \(d \geq 2n + 2\), and consider a self-correspondence \(Z\) as in diagram (1.1):
\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{Z}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{b}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{X}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{X}
\end{array}
\end{array}
\]
\]
\(\)
\(^{(0)}\)The case \(g = 3\) requires a special argument here that we leave to the reader.
Assume that $Z$ does not map to the diagonal. Then

$$\deg(a) \geq d - 2, \quad \deg(b) \geq d - 2,$$

and hence $\text{autocorr}(X) = (d - 2)^2$.

**Theorem 4.** In the situation of Theorem 3, assume in addition that $\deg(a) \leq 2d - 2n - 3$.

(i) If $\deg(b) \leq d - 2$, then $\deg(a) = d - 2$ and $Z$ is birationally residual to the fiber square of projection from a point $x_0 \in X$.

(ii) If $\deg(b) = d - 1$, then either

(a) $Z$ is birational to the fiber product of two rational mappings $\phi_1, \phi_2: X \to \mathbb{P}^n$, or

(b) there exist an $n$-fold $Y$ and a dominant rational mapping $\phi: X \to Y$ of degree $d$ such that $Z$ is birationally residual to the diagonal in the fiber product $X \times_Y X$.

**Remark 5.** The various possibilities in Theorem 4 actually occur. For example, in (a) one considers projection from two different points in $X$, while (b) arises for a general projection $X \to \mathbb{P}^n$ from a point off $X$.

Turning to the proofs, the arguments follow the line of attack of [BCDP14, BDP+17, LM23], so we will be relatively brief. Fix $X$ and $Z$ as above, and write

$$\delta_a = \text{def} \deg(a) \quad \text{and} \quad \delta_b = \text{def} \deg(b).$$

The first point to observe is that we may—and do—assume that the endomorphism ring of the Hodge structure $H^n_{pr}(X, Z)$ is $Z$.

**Lemma 6.** If $X$ is a very general hypersurface in $\mathbb{P}^{n+1}$, then

$$\text{End}(H^n_{pr}(X, Z)) = Z \cdot \text{Id}.$$  

Equivalently, $H^n_{pr}(X \times X)$ is generated by the classes of the diagonal and the products $h_i^1 h_2^{n-i} (1 \leq i \leq n)$, where $h_j = \text{pr}_j^* c_1(\mathcal{O}(1)|_X)$.

**Proof.** The lemma follows from the computation of the algebraic monodromy group for the corresponding variations of Hodge structures (see [Bea86] or [PS08, Section 10.3]). In the cases $d = 1, 2$, the primitive cohomology has rank 0 and 1, respectively, so the statement holds trivially. For larger $d$, an element of $\text{GL}(H^n_{pr}(X, Z))$ is a morphism of Hodge structures if and only if it commutes with the orthogonal group ($n$ even) or the symplectic group ($n$ odd). The centralizer of both of these subgroups is $Z \cdot \text{Id}$. Alternative arguments were shown to us by Radu Lazia and Mark Green.

It follows from Lemma 6 that

$$Z^{n,0}_*: H^{n,0}(X) \to H^{n,0}(X)$$

is multiplication by some integer $c$. Note that then $Z^{n,0}_*: H^{n,0}(X) \to H^{n,0}(X)$ is multiplication by the same integer $c$. In fact, abusively writing $[Z]$ for the class of the image of $Z$ in $H^*(X \times X)$, one has

$$[Z] \in H^g^{n,n}(X \times X) = \langle \Delta, h_1^i h_2^{n-i} | 1 \leq i \leq n \rangle_{\mathbb{Q}}$$

and of these classes, only $\Delta$ gives rise to a non-zero map

$$H^{n,0}(X) \to H^{n,0}(X)$$

under the identification

$$H^{n,n}(X \times X) \cong \text{End}_{\mathbb{Q} - \text{HS}}(H^n(X)).$$

Moreover,

$$\Delta_* = \Delta^* = \text{Id}_{H^{n,0}(X)}.$$

As in the previous section, we will need to distinguish between the cases $c = 0$ and $c \neq 0$. 
We start by showing that
\[ \delta_a, \delta_b \geq d - 2, \]
by an argument parallel to that appearing in Section 2. First, observe that
\[ (3.1) \quad \delta_a, \delta_b \geq \begin{cases} d - n & \text{if } c = 0, \\ d - n - 1 & \text{if } c \neq 0. \end{cases} \]

Indeed, if \( c = 0 \), then \( Z \) is a traceless correspondence, so given general \( x, y \in X \), the sets \( a^{-1}(x) \) and \( b^{-1}(y) \) both satisfy the Cayley–Bacharach condition with respect to \( H^{n,0}(X) \). Similarly, when \( c \neq 0 \), the cycle \( Z - c \Delta \) is a traceless correspondence, and hence for general \( x, y \in X \), the sets \( a^{-1}(x) \cup \{x\} \) and \( b^{-1}(y) \cup \{x\} \) also both satisfy the Cayley–Bacharach condition. Inequality (3.1) then follows from [BCDP14, Theorem 2.4].

We next assume that \( \delta_b \leq d - 1 \), aiming for a contradiction when \( \delta_b \leq d - 3 \). Fix a general point \( y \in X \). The fiber of \( Z \) over \( y \) sits naturally as a subset of \( X \) and hence also \( \mathbb{P}^{n+1} \):
\[ Z_y = \{ x_{1}, \cdots, x_{b} \} = \text{def} \ b^{-1}(y) \subseteq X \subseteq \mathbb{P}^{n+1}. \]

Note that if \( y \) is general, then the points \( x_j \) are distinct. Since \( \delta_b + 1 \leq 2d - 2n + 1 \), it follows from [BCDP14, Theorem 2.5] and the vanishing of \((Z - c \cdot \Delta)_a\) that the finite set \( Z_y \) spans a line \( \ell_y \subseteq \mathbb{P}^{n+1} \). In a similar fashion, the generic fiber \( a^{-1}(x) \) spans a line \( \chi \ell \). Furthermore, if \( c \neq 0 \), the point \( y \) lies on \( \ell_y \) and \( x \) lies on \( \chi \ell \).

Write
\[ X \cdot \ell_y = \sum_{i=1}^{r} a_i z_i; \]
we denote by \( m(z) \) the multiplicity of \( z \) in \( X \cdot \ell_y \), and we note that the \( x_j \) appear among these points. Observe that \( m(x_j) \) does not depend on \( j \). Indeed, if \( m(x_j) \) were to vary, picking out the \( x_j \) with the highest multiplicity for each \( y \in X \) would define a non-trivial multisection of the generically finite map
\[ b: Z \rightarrow X, \]
thereby violating the irreducibility of \( Z \). Moreover, \( m(x_j) = 1 \) for every \( j \). Indeed, if \( c = 0 \), then \( \delta_b \geq d - n \) and thus \( 2\delta_b > d \). Since \( \sum m(x_j) \leq d \), we see that \( m(x_j) = 1 \). If \( c \neq 0 \), then \( \delta_b \geq d - n - 1 \) and \( 2\delta_b + 1 > d \). Since \( y \) is in the support of \( X \cdot \ell_y \), we get
\[ 1 + \sum m(x_j) \leq d, \]
which shows \( m(x_j) = 1 \) for all \( i \).

Let \( \psi: X \rightarrow \mathbb{G}(1, n+1) \) be the rational map that associates to a generic \( y \in X \) the line \( \ell_y \), and denote by \( \Gamma_\psi \subset X \times \mathbb{G}(1, n+1) \) the graph of \( \psi \). Consider the incidence correspondence
\[ I = \{(x, \ell) : x \in \ell_y \} \subseteq X \times \mathbb{G}(1, n+1). \]

We can assume that \( X \) does not contain any lines, and hence the projection \( I \rightarrow \mathbb{G}(1, n+1) \) is finite. Consider the cycle
\[ A = \text{def} \ \text{pr}_{X \times X} \left( \text{pr}^*_{\mathbb{G}(1, n+1) \times X} \Gamma_\psi \cdot \text{pr}^*_{X \times \mathbb{G}(1, n+1)} I \right) \]
on \( X \times X \). So the support of \( A \) is the set \( \{(x, y) : x \in \ell_y\} \). The image \( \tilde{Z} \) of \( Z \) in \( X \times X \) and possibly the diagonal \( \Delta \) are among the irreducible components of this cycle, and denoting by \( R \) the remaining components, we have
\[ A = \tilde{Z} + m\Delta + R. \]

By construction, \( R \) dominates the second factor, and we assert that it cannot dominate the first. Indeed, were \( R \) to dominate both factors, it would define a correspondence violating the degree bounds in (3.1).

Next, observe that \( A \) acts as the composition
\[ [A]^* = [I]^* \circ \psi_*: H^{n,0}(X) \rightarrow H^*(\mathbb{G}(1, n+1)) \rightarrow H^{n,0}(X), \]
and this composition is zero since $H^\bullet(G(1, n+1))$ is Hodge-Tate. Furthermore,
\[ [R]^* = 0 : H^{n,0}(X) \rightarrow H^{n,0}(X) \]
since $R$ does not dominate the first factor. Therefore $m = -c$, and in particular $c \leq 0$. If $c = 0$, we contend that $c = -1$. Indeed, given a general point $(x, y) \in Z$, the lines $x\ell$ and $\ell_y$ pass through $x$ and $y$ and thus
\[ x\ell = \ell_y. \]
Consequently, $x\ell \cdot X = \ell_y \cdot X$ and by the statements above, we see that $x$ and $y$ both appear with multiplicity 1 in this intersection. Accordingly, the diagonal must appear with multiplicity 1 in $A$, and thus $c = -1$.

To finish the proof, we need the following.

**Claim.** Every irreducible component of (the support of) $R$ is of the form $x_0 \times X$ for some $x_0 \in X$.

**Proof.** The proof proceeds exactly as the proof of [LM23, Theorem A]. In brief, if the projection of an irreducible component of $R$ to the first factor is $S$, one shows that sections of the canonical bundle of a desingularization of $S$ do not birationally separate many points. This contradicts computations of Ein [Ein88] and Voisin [Voi96] if $\dim S > 0$. \hfill $\square$

The claim implies that $R$ must be irreducible and reduced since lines meeting $X$ in any fixed zero-dimensional subscheme of $X$ of length 2 do not meet a general point of $X$. It follows that $\delta_b \geq d - 2$, and by symmetry that $\delta_a \geq d - 2$.

Theorem 4 also follows from this claim as follows.

**Proof of Theorem 4.** (i) If $\delta_b = d - 2$, we must have $c = -1$ and $R = x_0 \times X$ for some $x_0 \in X$. Then we have the equality
\[ Z = \text{closure}\{ (x, y) \mid x \neq x_0, y \neq x, x_0, \text{ and } x \in \overline{x_0 y} \}. \]
Indeed, every irreducible component of the right-hand side must dominate the second factor, and the degree of the projection of the right-hand side to the second factor is $d - 2$.

(ii) If $\delta_b = d - 1$ and $c = 0$, we will show that (a) is satisfied. There is a point $x_0 \in X$ such that $R = x_0 \times X$. Consider $(x, y) \in Z$ general, and let
\[ x\ell \cap X = \{ y_j \mid 0 \leq j \leq d - 1 \} \quad \text{and} \quad \ell_y \cap X = \{ x_j \mid 0 \leq j \leq \delta_a \}, \]
where $x = x_1$ and $y = y_1$, so that $(x_1, y), \ldots, (x_{d-1}, y)$ and $(x, y_1), \ldots, (x, y_\delta)$ are in $Z$. Since $(x, y) \in Z$ was chosen generically, $b^{-1}(y_2)$ consists of $d - 1$ points, one of which is $x$. Moreover, $b^{-1}(y_2)$ is contained in a line passing through $x_0$, and thus is contained in the line through $x_0$ and $x$. It follows that
\[ b^{-1}(y_2) = \{ (x_i, y_2) \mid 1 \leq i \leq d - 1 \}. \]
The same reasoning shows that
\[ \{ (x_i, y_j) \mid 1 \leq i \leq d - 1, 1 \leq j \leq \delta_a \} \subset Z. \]
Let $\varphi_1 : X \rightarrow \mathbb{P}^n$ be the projection from $x_0$, and consider the map
\[ \varphi_2 : X \rightarrow G(1, n+1) \]
\[ x \mapsto x\ell. \]
The maps $\varphi_1$ and $\varphi_2$ are generically finite of degree $d - 1$ and at least $\delta_a$, respectively. Considering degrees in the following diagram, we see that $\varphi_2$ had degree $\delta_a$ and that $(\varphi_1 \times \varphi_2)(Z) \subset \mathbb{P}^n \times \text{Im}(\varphi_2)$ maps
birationally to each factor:

\[
\begin{array}{c}
\text{X} \\
\downarrow_{\phi_1} \\
\downarrow_{\text{pr}_1} \\
\text{P}^n
\end{array}
\quad
\begin{array}{c}
Z \\
\downarrow \\
\text{(\(\phi_1 \times \phi_2\))}(Z) \\
\downarrow_{\text{pr}_2} \\
\text{Im}(\phi_2)
\end{array}
\quad
\begin{array}{c}
\text{X} \\
\downarrow_{\phi_2} \\
\downarrow_{\text{pr}_2} \\
\text{P}^n
\end{array}
\]

Hence, the subvariety

\[(\phi_1 \times \phi_2)(Z) \subset \text{P}^n \times \text{Im}(\phi_2)\]

is the graph of a birational isomorphism \(\psi: \text{P}^n \rightarrow \text{Im}(\phi_2)\). Accordingly, \(Z\) is the fiber product of \(\phi_1\) and \(\psi^{-1} \circ \phi_2\).

Finally, if \(\delta_b = d - 1\) and \(c \neq 0\), we show that (b) is satisfied. We must have \(c = -1\) and \(\text{deg}(a) = d - 1\).

Consider the rational map

\[\phi: X \dashrightarrow \text{G}(1, n + 1)\]

\[y \mapsto \ell_y.\]

Denoting by \(U\) an open on which \(\phi\) is defined, we contend that

\[Z = \{ (x, y) \in U^2: x \neq y, \phi(x) = \phi(y) \} \subset X \times X.\]

Given a generic \((x, y) \in Z\), the line \(\ell_y\) coincides with the line \(\ell_x\) as they both pass through \(x\) and \(y\). Write

\[\ell_x \cap X = \ell_y \cap X = \{ z_j: 1 \leq j \leq d \},\]

where \(z_1 = x\) and \(z_2 = y\). For any \(j > 1\), the point \((x, z_j)\) is on \(Z\), and \(b^{-1}(z_j)\) is contained in \(\ell_{z_j} = \ell_x = \ell_y\), so that

\[b^{-1}(z_j) = \ell_y \cap X \setminus \{ z_j \}\]

and

\[\{(z_i, z_j): i \neq j \} \subset Z.\]

It follows that \(\ell_x = \ell_y\) for a generic \(x \in X\) and that

\[Z = \{ (x, y) \in U^2: x \neq y, \phi(x) = \phi(y) \} \subset X \times X.\]

\[
\square
\]

4. Proof of Theorem C

Theorem C from the introduction follows easily from the following result.

**Proposition 7.** Let \(X\) be a very general hyperelliptic curve of genus \(g \geq 3\), and let \(Z \subseteq X \times X\) be a hyperelliptic curve. Then the image of (the normalization of) \(Z\) under the Abel–Jacobi map is geometrically degenerate; i.e., it generates a proper subtorus of \(J(X) \times J(X)\).

Note that we do not assume that \(Z\) is smooth; to say it is hyperelliptic means that its normalization is so.

We note that some genericity condition is necessary in Theorem C. For example, given a hyperelliptic curve \(X\), the graph of an automorphism \(X \rightarrow X\) which is neither the identity nor the hyperelliptic involution is a hyperelliptic curve sitting in \(X \times X\). The fact that such graphs map to geometrically degenerate curves in \(J(X) \times J(X)\), together with Proposition 7, suggests the following.
**Question.** Given an arbitrary hyperelliptic curve $X$ (resp. hyperelliptic curves $X$ and $Y$), does every hyperelliptic curve $Z \subseteq X \times X$ (resp. $Z \subseteq X \times Y$) map to a geometrically degenerate curve in $J(X) \times J(X)$ (resp. $J(X) \times J(Y)$)?

Let us first show how Theorem C follows from Proposition 7.

Consider a very general hyperelliptic curve $X$ and a hyperelliptic curve $Z \subseteq X \times X$ with normalization $Z'$. Abusing notation, we will call the image in $Z$ of Weierstrass points of $Z'$ Weierstrass points of $Z$. Such points map to Weierstrass points of $X$ under each projection. Consider a Weierstrass point $(x_0, y_0) \in Z$ and the embedding

$$ X \times X \to J(X) \times J(X) $$

$$(x, y) \mapsto ([x] - [x_0], [y] - [x_0]).$$

By Proposition 7, a translate of the image of $Z$ in $J(X) \times J(X)$ is contained in an abelian subvariety of $J(X) \times J(X)$. Since the image of $Z$ passes through

$$\tau = \text{def} \ (0, [y_0] - [x_0]) \in J(X)[2] \times J(X)[2],$$

it is in fact contained in $\tau + A$ for some proper abelian subvariety $A \subset J(X) \times J(X)$. Moreover, since $X$ is very general, the automorphism group of the Jacobian of $X$ is $\mathbb{Z}$, and thus there are integers $m, n \in \mathbb{Z}$, $m \geq 0$, such that $Z$ is contained in the image of

$$ J(X) \to J(X) \times J(X) $$

$$x \mapsto (mx, nx) + \tau. $$

Hence,

$$ Z \subset \{(x, x') \in X \times X : nx + x_0 = mx' + y_0 \in J(X)\} \subset X \times X. $$

Equivalently, $Z$ is contained in the fiber of the following map over $y_0 - x_0$:

$$ X \times X \to J(X) $$

$$(x, x') \mapsto mx - nx'. $$

Considering the differential of the map above and the fact that the Gauss map of $X$ embedded in its Jacobian has degree 2, it is easy to see that the only possibility is $n = \pm m$ and $x_0 = y_0$.

We have thus shown that $Z$ is contained either in the diagonal of $J(X)$ or in the anti-diagonal of $J(X)$. This completes the proof as the diagonal of $J(X)$ intersects $X \times X$ along the diagonal of $X$ and the anti-diagonal of $J(X)$ intersects $X \times X$ along the graph of the hyperelliptic involution of $X$.

Finally, we give the proof of Proposition 7.

**Proof of Proposition 7.** Consider $\mathcal{N}/S$, a locally complete family of hyperelliptic curves of genus $g$, and

$$\mathbb{Z} \subset J(\mathcal{N}/S) \times_S J(\mathcal{N}/S),$$

a family of hyperelliptic curves such that for very general $s \in S$, the curve $\mathcal{Z}_s$ generates $J(\mathcal{N}_s) \times J(\mathcal{N}_s)$. The idea is to arrive at a contradiction to the observation of Pirola [Pir89] that hyperelliptic curves on abelian varieties are rigid up to translation.

Specifically, specialize to loci $S_\lambda \subset S$ along which $J(\mathcal{N}_s)$ is isogenous to $A^1_g \times E$, where $E$ is a fixed elliptic curve and $A^1_g \to S_\lambda$ is a family of abelian $(g - 1)$-folds. For each $\lambda$, we have a map

$$ p_\lambda : \mathbb{Z} \to E \times E $$

which is the composition of the inclusion of $\mathbb{Z}$ in $J(\mathcal{N}_s)$ with the isogeny and the projection to the $E \times E$ factor.

**Claim.** The image of $\mathbb{Z}$ in $E \times E$ varies with $s \in S_\lambda$. 


But as we noted, this is impossible thanks to [Pir89], completing the proof.

The claim is established along the lines of [Voi18] and [Mar20]. Denoting by $G/S$ the relative Grassmanian of $(g-1)$-planes in $T_{J(X_s),0}$, [CP90] proves the density of the set

$$\{T_{A_s},0 \subseteq T_{J(X_s),0} \mid s \in S_{\lambda}\} \subseteq G.$$  

(In fact, [CP90] shows that the locus $\{T_E,0 \subset T_{J(X_s),0} \mid s \in S_{\lambda}\}$ is dense in the relative Grassmanian of lines in $T_{J(X_s),0}$. However, one can use the fact that Jacobians are isomorphic to their duals to get the stated assertion.) By a density argument, one can construct families of smooth curves $\overline{Z} \rightarrow G'$ and $\overline{Z}' \rightarrow G$ over a generically finite cover $G'$ of $G$ and a morphism

$$p: \overline{Z} \longrightarrow \overline{Z}'.$$

satisfying the following:

- Denoting by $\pi$ the map $G' \rightarrow S$, the curve $\overline{Z}_s$ is the normalization of $Z_{\pi(s)}$.
- For $s \in G'$ such that $\pi(s) \in S_{\lambda} \subset S$, the maps

$$p_{\lambda}: Z_s \longrightarrow p_{\lambda}(Z_s)$$

and

$$p: \overline{Z}_s \longrightarrow \overline{Z}_s'$$

agree birationally.

Now consider the composition

$$(4.1) \quad \text{Pic}^0(J(X_s) \times J(X_s)) \longrightarrow \text{Pic}^0(\overline{Z}_s) \overset{p_{\lambda}}{\longrightarrow} \text{Pic}^0(\overline{Z}_s'),$$

where the first map is the pullback by the composition

$$\overline{Z}_s \longrightarrow Z_s \longrightarrow J(X_s) \times J(X_s).$$

One easily checks that the composition (4.1) cannot be zero. Since $J(X_s)$ is simple for generic $s \in G'$, we deduce that the abelian variety $\text{Pic}^0(\overline{Z}_s)$ contains an abelian subvariety isogenous to $J(X_s)$ for all $s$ in an open set $U \subset G'$.

Finally, consider $\lambda$ such that $\pi^{-1}(S_{\lambda}) \cap U \neq \emptyset$. If $p_{\lambda}(Z_s)$ does not vary with $s \in S_{\lambda}$, the fixed abelian variety $\text{Pic}^0(\overline{Z}_s')$ contains an abelian subvariety isogenous to $J(X_s)$ for all $s \in S_{\lambda}$. This cannot be since the family $J(X_s)/S_{\lambda}$ is not isotrivial.

\[\square\]

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