Iwasawa $\mu$-invariants and congruence of Galois representations

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Abstract

In this paper, we compare the Selmer groups of congruent Galois representations over an admissible $p$-adic Lie extension. Namely, we show that the generalized Iwasawa $\mu$-invariants associated to the Pontryagin dual of the Selmer groups of two Galois representations are the same if the said Galois representations are congruent to each other for a high enough power. In the case when the admissible $p$-adic Lie extension is a multiple $\mathbb{Z}_p$-extension, we can even show that the $\pi$-primary submodule of the dual Selmer groups of the two Galois representations are pseudo-isomorphic to each other. We will also mention as a remark that our method here can be used to improve the results in a previous paper, and as a consequence, we are able to give a partial answer to a remark of Greenberg.

Keywords and Phrases: Selmer groups, $\mu$-invariant, admissible $p$-adic Lie extensions, pseudo-isomorphism.

Mathematics Subject Classification 2010: 11R23, 11R34, 11F80.

1 Introduction

The aim of this article is to study the variation of the Iwasawa invariants of the Selmer groups of two congruent Galois representations, or in general, a family of Galois representations with suitable congruence relations. Such studies were carried out over the cyclotomic $\mathbb{Z}_p$-extension in [EPW, GV, He, We] and over noncommutative $p$-adic Lie extensions in [Ch, SS]. One of the motivation behind these studies lies in the philosophy that the “Iwasawa main conjecture” should be preserved by congruences. We should mention that in the cyclotomic case, this philosophy is rather well understood (see [EPW, GV]), although the general noncommutative situation still seems much of a mystery (for instance, see [B, CS]).

An important observation made in the above cited works is that if the Iwasawa $\mu$-invariant of one of the Selmer groups vanishes, so does the other. It is then natural to consider the situation when the said Iwasawa $\mu$-invariants are nonzero and ask if one can relate the Iwasawa $\mu$-invariants. To the best of the author’s knowledge, such studies have only been considered in [B, BS]. As observed in these works and as we will see in this paper, to be able to even compare the Iwasawa $\mu$-invariants meaningfully, we require the congruence of the Galois representations to be high enough. We should mention that the preservation of the main conjecture by congruences seems to be a delicate problem when the Iwasawa
\(\mu\)-invariants are nonzero, and in this paper, we will only consider the more modest problem of how the Iwasawa \(\mu\)-invariants compare for Selmer groups of congruent Galois representations.

Our methods for comparing the \(\mu\)-invariants are inspired by those in [Lim2], where the author has adopted to compare the “\(\mu\)-component” of the algebraic functional equation. As in [Lim2], our approaches differ in the commutative and noncommutative cases. When the \(p\)-adic Lie extension is a multiple \(\mathbb{Z}_p\)-extension, we make use of the asymptotic formula of Cucuo and Monsky [CM, Mon] which allows us to deal with \(\mathbb{Z}_p[G]\)-modules for \(G \cong \mathbb{Z}_p^r\). Of course, these formulas are natural extension of the classical asymptotic formula of Iwasawa [Iw]. In fact, we can even show the stronger conclusion that the dual Selmer groups of the two congruent Galois representations have the same rank and the \(\pi\)-primary submodule of the dual Selmer groups are pseudo-isomorphic to each other when the power of the congruence is high enough. The key (and extra) input required to prove this assertion is Proposition [2.4] which gives a way to compare deeper structural properties of the \(\pi\)-primary submodules of two \(\mathcal{O}[G]\)-modules.

When the \(p\)-adic Lie extension is not commutative, we adopt the indirect approach as in [Lim2]. Namely, under the assumption of a certain torsion property of the Selmer groups in the sense of [CF^+], we show that the \(\mu\)-invariant of the Selmer group over the \(p\)-adic extension coincides with \(\mu\)-invariant of the said Selmer group over the cyclotomic \(\mathbb{Z}_p\)-extension. This conclusion in turn allows us to deduce the required equality of the \(\mu\)-invariants in the general noncommutative setting from the cyclotomic \(\mathbb{Z}_p\)-extension case. We should mention that it is perhaps more of a consequence of our approach that we can only establish an equality of the Iwasawa \(\mu\)-invariants in the noncommutative situation, and are not able to establish the pseudo-isomorphism of the \(\pi\)-primary submodule of the dual Selmer groups as in the commutative situation. We hope to come back to this topic in a future work.

For the remainder of the paper, we apply our results to study the variation of the Iwasawa \(\mu\)-invariants of the Selmer groups of certain specialization of a big Galois representation. We will also study the variation of certain torsion property of the dual Selmer groups.

Finally, at the end of the paper, we make a short note that we can apply Proposition [2.4] to improve the [Lim2 Theorem 4.2] which then allows us to answer the \(\pi\)-primary “component” of an assertion of Greenberg [Gr, P. 130, Equation (66)].

Acknowledgments. This work was written up when the author was a Postdoctoral fellow at the GANITA Lab at the University of Toronto. He would like to acknowledge the hospitality and conducive working conditions provided by the GANITA Lab and the University of Toronto while this work was in progress.

2 Algebraic Preliminaries

In this section, we recall some algebraic preliminaries that will be required in the later part of the paper. Fix a prime \(p\). Let \(\mathcal{O}\) be the ring of integers of some fixed finite extension \(K\) of \(\mathbb{Q}_p\). We fix a local parameter \(\pi\) for \(\mathcal{O}\) and denote the residue field of \(\mathcal{O}\) by \(k\). Let \(G\) be a compact pro-\(p\) \(p\)-adic Lie group
without \( p \)-torsion. It is well known that \( \mathcal{O}[G] \) is an Auslander regular ring (cf. \cite{VI} Theorems 3.26). Furthermore, the ring \( \mathcal{O}[G] \) has no zero divisors (cf. \cite{Neu}), and therefore, admits a skew field \( Q(G) \) which is flat over \( \mathcal{O}[G] \) (see \cite{GW} Chapters 6 and 10 or \cite{Lam} Chapter 4, §9 and §10). If \( M \) is a finitely generated \( \mathcal{O}[G] \)-module, we define the \( \mathcal{O}[G] \)-rank of \( M \) to be

\[
\text{rank}_{\mathcal{O}[G]} M = \dim_{Q(G)} Q(G) \otimes_{\mathcal{O}[G]} M.
\]

We will say that an \( \mathcal{O}[G] \)-module \( M \) is \textit{torsion} if \( \text{rank}_{\mathcal{O}[G]} M = 0 \).

Now suppose that \( N \) is a finitely generated \( k[G] \)-module, where one recalls that \( k \) is the residue field of \( \mathcal{O} \). We then define its \( k[G] \)-rank by

\[
\text{rank}_{k[G]} N = \frac{\text{rank}_{k[G_0]} N}{[G : G_0]},
\]

where \( G_0 \) is an open normal uniform pro-\( p \) subgroup of \( G \). It can be showed that this definition is independent of the choice of \( G_0 \) (see \cite{Ho} Proposition 1.6). Similarly, we will say that the module \( N \) is a \textit{torsion} \( k[G] \)-module if \( \text{rank}_{k[G]} N = 0 \).

For a given finitely generated \( \mathcal{O}[G] \)-module \( M \), we denote \( M(\pi) \) to be the \( \mathcal{O}[G] \)-submodule of \( M \) which consists of elements of \( M \) that are annihilated by some power of \( \pi \). Since the ring \( \mathcal{O}[G] \) is Noetherian, the module \( M(\pi) \) is finitely generated over \( \mathcal{O}[G] \). Therefore, one can find an integer \( r \geq 0 \) such that \( \pi^r \) annihilates \( M(\pi) \). Following \cite{Ho} Formula (33), we define

\[
\mu_{\mathcal{O}[G]}(M) = \sum_{i \geq 0} \text{rank}_{k[G]} (\pi^i M(\pi)/\pi^{i+1}).
\]

(For another alternative, but equivalent, definition, see \cite{VI} Definition 3.32.) By the above discussion and our definition of \( k[G] \)-rank, the sum on the right is a finite one. It is clear from the definition that \( \mu_{\mathcal{O}[G]}(M) = \mu_{\mathcal{O}[G]}(M(\pi)) \). Also, it is not difficult to see that this definition coincides with the classical notion of the \( \mu \)-invariant for \( \Gamma \)-modules when \( G = \Gamma \cong \mathbb{Z}_p \). When \( M \) is a finitely generated torsion \( \mathcal{O}[G] \)-module, it follows from \cite{VI} Theorem 3.40] that there is a \( \mathcal{O}[G] \)-homomorphism

\[
f : M(\pi) \longrightarrow \bigoplus_{i=1}^s \mathcal{O}[G]/\pi^{\alpha_i},
\]

whose kernel and cokernel are pseudo-null \( \mathcal{O}[G] \)-modules. It is straightforward to check that \( \mu_{\mathcal{O}[G]}(M) = \sum_{i=1}^s \alpha_i \). We will also set

\[
\theta_{\mathcal{O}[G]}(M) := \max_{1 \leq i \leq s} \{\alpha_i\}.
\]

We now record certain properties of these invariants which will be required in the subsequent of the paper.

**Lemma 2.1.** Let \( G \) be a compact pro-\( p \) \( p \)-adic Lie group with no \( p \)-torsion and let \( M \) be a finitely generated \( \mathcal{O}[G] \)-module. Then we have the following statements.
(a) For every finitely generated \( \mathcal{O}[G] \)-module \( M \), one has
\[
\mu_G(M) = \sum_{i \geq 0} (-1)^i \text{ord}_q(H_i(G, M(\pi)))
\]
where \( q \) is the cardinality of \( k \).

(b) Suppose that \( G \) has a closed normal subgroup \( H \) such that \( G/H \cong \mathbb{Z}_p \). If \( M \) is a \( \mathcal{O}[G] \)-module which is finitely generated over \( \mathcal{O}[H] \), then one has \( \mu_{\mathcal{O}[G]}(M) = 0 \).

(c) Suppose that we are given a short exact sequence of finitely generated \( \mathcal{O}[G] \) modules
\[
0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0.
\]

1. One has \( \mu_{\mathcal{O}[G]}(M) \leq \mu_{\mathcal{O}[G]}(M') + \mu_{\mathcal{O}[G]}(M'') \). Moreover, if \( M \), and hence also \( M' \) and \( M'' \), is \( \mathcal{O}[G] \)-torsion, the inequality is an equality.
2. If \( \mu_{\mathcal{O}[G]}(M'') = 0 \), then one has \( \mu_{\mathcal{O}[G]}(M') = \mu_{\mathcal{O}[G]}(M) \).
3. If \( M' \) is finitely generated over \( \mathcal{O}[H] \), then one has \( \mu_{\mathcal{O}[G]}(M) = \mu_{\mathcal{O}[G]}(M'') \) and \( \theta_{\mathcal{O}[G]}(M) = \theta_{\mathcal{O}[G]}(M'') \).

(d) Suppose that we are given an exact sequence of finitely generated \( \mathcal{O}[G] \)-modules
\[
A \rightarrow B \rightarrow C \rightarrow D
\]
such that \( A \) is finitely generated over \( \mathcal{O}[H] \) and \( \mu_G(D) = 0 \). Then one has the equality \( \mu_G(B) = \mu_G(C) \). Moreover, if \( D \) is also finitely generated over \( \mathcal{O}[H] \), we even have \( \theta_{\mathcal{O}[G]}(B) = \theta_{\mathcal{O}[G]}(C) \).

(e) Viewing \( M \) as a \( \mathbb{Z}_p[G] \)-module, we have \( [k : \mathbb{F}_p] \mu_{\mathcal{O}[G]}(M) = \mu_{\mathbb{Z}_p[G]}(M) \) and
\[
\left\lfloor \frac{\theta_{\mathcal{O}[G]}(M)}{e} \right\rfloor = \theta_{\mathbb{Z}_p[G]}(M),
\]
where \( \lfloor x \rfloor \) is the smallest integer not less than \( x \) and \( e \) is the ramification index of \( \mathcal{O}/\mathbb{Z}_p \).

(f) Assume further that \( G \cong \mathbb{Z}_p^r \) for \( r \geq 1 \). Suppose that \( M \) is a finitely generated \( \mathcal{O}[G] \)-module and \( M(\pi) \) is pseudo-isomorphic to
\[
\bigoplus_{i=1}^s \mathcal{O}[G]/\pi^{\alpha_i}.
\]
Then we have
\[
\mu_{\mathcal{O}[G]}(M/\pi^n) = n \rank_{\mathcal{O}[G]}(M) + \sum_{i=1}^s \min\{n, \alpha_i\} \quad \text{for } n \geq 1.
\]
In particular, we have \( \mu_{\mathcal{O}[G]}(M/\pi^n) \leq n \rank_{\mathcal{O}[G]}(M) + \mu_{\mathcal{O}[G]}(M) \) which is an equality if and only if \( n \geq \theta_G(M) \).
Proof. Statements (a), (b) and (c)(1) are proven in [Ho Corollary 1.7], [Ho Lemma 2.7] and [Ho Proposition 1.8] respectively. Statement (f) can be proven similarly as in [Lim Lemma 2.2(i)]. The remaining statements can be deduced from the previous statements without too much difficulties.

For the remainder of the section, we will specialize to the case $G \cong \mathbb{Z}_p^r$, where $r \geq 1$.

**Proposition 2.2.** Let $G \cong \mathbb{Z}_p^r$, where $r \geq 1$. Let $M$ and $N$ be two torsion $\mathcal{O}[G]$-modules such that $\mu_\mathcal{O}[G](M/\pi^0\mathcal{O}[G](M)) = \mu_\mathcal{O}[G](N/\pi^0\mathcal{O}[G](N))$. Then we have

$$\mu_\mathcal{O}[G](M) \leq \mu_\mathcal{O}[G](N).$$

Proof. By Lemma 2.1(f), we have

$$\mu_\mathcal{O}[G](M) = \mu_\mathcal{O}[G](M/\pi^0\mathcal{O}[G](M)) = \mu_\mathcal{O}[G](N/\pi^0\mathcal{O}[G](M)) \leq \mu_\mathcal{O}[G](N).$$

**Proposition 2.3.** Let $G \cong \mathbb{Z}_p^r$, where $r \geq 1$. Let $M$ and $N$ be two torsion $\mathcal{O}[G]$-modules such that $\theta_\mathcal{O}[G](M) = \theta_\mathcal{O}[G](N) \geq 1$ and $\mu_\mathcal{O}[G](M/\pi^i) = \mu_\mathcal{O}[G](N/\pi^i)$ for every $1 \leq i \leq \theta_\mathcal{O}[G](M)$.

Then $M(\pi)$ and $N(\pi)$ are pseudo-isomorphic to each other.

Proof. Write $\theta := \theta_\mathcal{O}[G](M) = \theta_\mathcal{O}[G](N)$. Suppose that $M(\pi)$ (resp., $N(\pi)$) is pseudo-isomorphic to

$$\bigoplus_{i=1}^s \mathcal{O}[G]/\pi^{\alpha_i} \quad \text{(resp., } \bigoplus_{i=1}^t \mathcal{O}[G]/\pi^{\beta_i}).$$

After rearranging, we may assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s = \theta$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_t = \theta$. To prove the proposition, we are reduced to showing that $s = t$ and $n \leq n' \leq s$. We first note the equality

$$s = \mu_\mathcal{O}[G](M/\pi) = \mu_\mathcal{O}[G](N/\pi) = t.$$ 

We now proceed to show $\alpha_n = \beta_n$ for every $1 \leq n \leq s$. We begin by showing that $\alpha_1 = \beta_1$ and we do this by the method of contradiction. Without loss of generality, we may assume that $\alpha_1 < \beta_1$. Then by Lemma 2.1(f), we have

$$\beta_1 = \mu_\mathcal{O}[G](N/\pi^{\beta_1}) = \mu_\mathcal{O}[G](M/\pi^{\beta_1}) = \alpha_1 + \sum_{i=2}^s \min\{\alpha_i, \beta_i\} \leq \alpha_1 + (s-1)\beta_1$$

which in turns implies $\beta_1 \leq \alpha_1$. This yields a contradiction. Hence we must have $\alpha_1 = \beta_1$. Now let $n > 1$ and suppose that $\alpha_i = \beta_i$ for $i < n$. We now proceed to show that $\alpha_n = \beta_n$ by the method of contradiction again. As before, without loss of generality, we may assume that $\alpha_n < \beta_n$. Then by an application of Lemma 2.1(f), we have

$$\beta_1 + \cdots + \beta_{n-1} + (s-n+1)\beta_n = \mu_\mathcal{O}[G](N/\pi^{\beta_n}) = \mu_\mathcal{O}[G](M/\pi^{\beta_n})$$

$$= \alpha_1 + \cdots + \alpha_n + \alpha_n + \sum_{i=n+1}^s \min\{\alpha_i, \beta_i\}$$

$$\leq \alpha_1 + \cdots + \alpha_n + \alpha_n + (s-n)\beta_n.$$
By our induction hypothesis, the above inequality simplifies to $\beta_n \leq \alpha_n$ which yields the required contradiction. Hence we must have $\alpha_n = \beta_n$. This completes the proof of the proposition.

The previous proposition may be difficult to apply due to the condition $\theta_{O[G]}(N) = \theta_{O[G]}(M)$ which is perhaps not easy to check. However, one can build on the proposition to obtain the following which is perhaps easier for application.

**Proposition 2.4.** Let $G \cong \mathbb{Z}_p^r$, where $r \geq 1$. Let $M$ and $N$ be two finitely generated $O[G]$-modules such that $M$ is a torsion $O[G]$-module and such that $\mu_{O[G]}(M/\pi^i) = \mu_{O[G]}(N/\pi^i)$ for every $1 \leq i \leq \theta_{O[G]}(M)$ + 1.

Then $N$ is torsion over $O[G]$ and we have the equality $\theta_{O[G]}(M) = \theta_{O[G]}(N)$. In particular, we have that $M(\pi)$ and $N(\pi)$ are pseudo-isomorphic to each other.

**Proof.** We first prove the proposition for the case when $\theta_{O[G]}(M) = 0$. Then we have $\mu_{O[G]}(N/\pi) = \mu_{O[G]}(M/\pi) = 0$. By Lemma 2.1(f), this in turns implies that $\text{rank}_{O[G]}(N) = 0$ and $\mu_{O[G]}(N) = 0$. Therefore, we have that $N$ is torsion over $O[G]$ and $\theta_{O[G]}(N) = 0$. Hence we have that $M(\pi)$ and $N(\pi)$ are both pseudo-null by [VI, Remark 3.33]. In particular, they are pseudo-isomorphic to each other.

Now suppose that $\theta_{O[G]}(M) \geq 1$. It then follows from the above discussion that we must also have $\theta_{O[G]}(N) \geq 1$. By Proposition 2.3, it suffices to show that $N$ is a torsion $O[G]$-module and that $\theta_{O[G]}(M) = \theta_{O[G]}(N)$. Suppose now that $N(\pi)$ is pseudo-isomorphic to the form

$$\bigoplus_{i=1}^t O[G]/\pi^{\beta_i}.$$ 

Write $r = \text{rank}_{O[G]}(N)$. By Lemma 2.1(f), we then have

$$\mu_{O[G]}(M) = \mu_{O[G]}(M/\pi^n) = \mu_{O[G]}(N/\pi^n) = nr + \sum_{i=1}^t \min\{n, \beta_i\}$$

for $n = \theta_{O[G]}(M), \theta_{O[G]}(M) + 1$. This in turn implies that

$$\theta_{O[G]}(M)r + \sum_{i=1}^t \min\{\theta_{O[G]}(M), \beta_i\} = (\theta_{O[G]}(M) + 1)r + \sum_{i=1}^t \min\{\theta_{O[G]}(M) + 1, \beta_i\}.$$ 

Since one always has $\theta_{O[G]}(M)r \leq (\theta_{O[G]}(M) + 1)r$ and $\min\{\theta_{O[G]}(M), \beta_i\} \leq \min\{\theta_{O[G]}(M) + 1, \beta_i\}$, in order for the above equality to hold, we must have $r = 0$ and $\min\{\theta_{O[G]}(M), \beta_i\} = \min\{\theta_{O[G]}(M) + 1, \beta_i\}$ for $1 \leq i \leq t$. The formal equality then shows that $N$ is a torsion $O[G]$-module, and the latter equalities show that $\beta_i \leq \theta_{O[G]}(M)$ for all $i$, or in other words, $\theta_{O[G]}(N) \leq \theta_{O[G]}(M)$. Therefore, we may repeat the above argument (noting that we have shown that $N$ is $O[G]$-torsion) replacing $\theta_{O[G]}(M)$ by $\theta_{O[G]}(N)$ and interchanging the roles of $M$ and $N$ to obtain the reverse inequality $\theta_{O[G]}(M) \leq \theta_{O[G]}(N)$. The remaining pseudo-isomorphic conclusion will now follow from an application of Proposition 2.3. 

□
We now quote the following result which is a special case of the asymptotic formulas of Iwasawa [Iw Theorem 4] (see also [NSW, Proposition 5.3.17]), and Cucuo and Monsky [CM Theorem 4.13] (see also [Mon. Theorem 3.12] for a finer statement). We say that a sequence of real numbers \((a_m)_{m \geq 1}\) is \(O(Q^m)\) for some nonnegative number \(Q\) if \(|a_m| \leq CQ^m\) for some constant \(C\) (independent of \(m\)) for all sufficiently large \(m\). We will write \(a_m = O(Q^m)\). If \((b_m)_{m \geq 1}\) is another sequence of real numbers, we sometimes write \(a_m = b_m + O(q^m)\) to mean \(a_m - b_m = O(q^m)\).

**Theorem 2.5** (Iwasawa, Cucuo-Monsky). Suppose that \(G \cong \mathbb{Z}_p^r\), where \(r \geq 1\). Denote \(G_m\) to be \(G^p^m\). Let \(M\) be a finitely generated \(O[G]\)-module such that \(M_{G_m}\) is finite for all \(m\). Then we have

\[
\text{ord}_p(M_{G_m}) = |k : F_p|\mu_G(M)p^{rm} + O(p^{(r-1)m}) \quad \text{for } m \gg 0.
\]

**Proof.** For large enough \(m\), we have

\[
\text{ord}_p(M_{G_m}) = \mu_{G_m}(M)p^{rm} + O(p^{(r-1)m}) = |k : F_p|\mu_G(M)p^{rm} + O(p^{(r-1)m}),
\]

where the first equality follows from [CM Theorem 4.13] by viewing \(M\) as a \(\mathbb{Z}_p[G]\)-module. We should mention that when \(r = 1\), this is also a consequence of the original asymptotic formula of Iwasawa (cf. [Iw, Theorem 4]).

### 3 Comparing Selmer groups over multiple \(\mathbb{Z}_p\)-extensions

As before, let \(p\) be a prime. We let \(F\) be a number field. If \(p = 2\), we assume further that \(F\) has no real primes. Denote \(O\) to be the ring of integers of some finite extension \(K\) of \(\mathbb{Q}_p\), and fix a local parameter \(\pi\) for \(O\). Suppose that we are given the following data:

(a) \(A\) is a cofinitely generated cofree \(O\)-module of \(O\)-corank \(d\) with a continuous, \(O\)-linear \(\text{Gal}(\bar{F}/F)\)-action which is unramified outside a finite set of primes of \(F\).

(b) For each prime \(v\) of \(F\) above \(p\), \(A_v\) is a \(\text{Gal}_(\bar{F}_v/F_v)\)-submodule of \(A\) which is cofree of \(O\)-corank \(d_v\).

(c) For each real prime \(v\) of \(F\), we write \(A_v^+ = A^{\text{Gal}_(\bar{F}_v/F_v)}\).

(d) The following equality

\[
\sum_{v|p}(d - d_v)[F_v : \mathbb{Q}_p] = dr_2(F) + \sum_{v \text{ real}}(d - d_v^+)
\]

holds. Here \(r_2(F)\) denotes the number of complex primes of \(F\).

We denote the above data as \((A, \{A_v\}_{v|p}, \{A_v^+\}_{v|R})\). We now consider the base change property of our data. Let \(L\) be a finite extension of \(F\). We can then obtain another data \((A, \{A_w\}_{w|p}, \{A_w^+\}_{w|R})\) over \(L\) as follows: we consider \(A\) as a \(\text{Gal}(\bar{F}/L)\)-module, and for each prime \(w\) of \(L\) above \(p\), we set \(A_w = A_v\).
where \( v \) is a prime of \( F \) below \( w \), and view it as a \( \text{Gal}(\bar{F}_w/L_w) \)-module. Then \( d_w = d_v \). For each real prime \( w \) of \( L \), one sets \( A^{\text{Gal}}(L_w/L_w) = A^{\text{Gal}}(\bar{F}_w/F_v) \) and writes \( d_w^+ = d_v^+ \), where \( v \) is a real prime of \( F \) below \( w \). In general, the \( d_w^- \)'s and \( d_w^+ \) need not satisfy equality \( \dagger \). We now record the following lemma which gives some sufficient conditions for equality \( \dagger \) to hold for the data \( (A, \{A_w\}_{w|p}, \{A_w^+\}_{w|R}) \) over \( L \) (see [Lim2, Lemma 4.1] for the proofs).

**Lemma 3.1.** Suppose that \( (A, \{A_v\}_{v|p}, \{A_v^+\}_{v|R}) \) is a data defined over \( F \). Suppose further that at least one of the following statements holds.

(i) \( [L : F] \) is odd.

(ii) \( F \) is totally imaginary.

Then we have the equality

\[
\sum_{w|p} (d - d_w)[L_w : \mathbb{Q}_p] = dr_2(L) + \sum_{w \text{ real}} (d - d_w^+).
\]

In this paper, we will always work with \( p \)-extension. Also, if \( p = 2 \), we always assume that \( F \) is totally imaginary. Therefore, Lemma 3.1 applies.

We now introduce the Selmer groups. We first remark that the Selmer group that we consider is \( \text{Greenberg} \) but all our main results also hold for another Selmer group of Greenberg and the Selmer complexes (see Section 6).

Let \( S \) be a finite set of primes of \( F \) which contains all the primes above \( p \), the ramified primes of \( A \) and all infinite primes. Denote \( F_S \) to be the maximal algebraic extension of \( F \) unramified outside \( S \) and write \( G(S(L)) = \text{Gal}(F_S/L) \) for every algebraic extension \( L \) of \( F \) which is contained in \( F_S \). Let \( L \) be a finite extension of \( F \) contained in \( F_S \) such that the data \( (A, \{A_w\}_{w|p}, \{A_w^+\}_{w|R}) \) satisfies \( \dagger \). For a prime \( w \) of \( L \) lying over \( S \), set

\[
H^1_{\text{str}}(L_w, A) = \begin{cases} \ker (H^1(L_w, A) \to H^1(L_w, A/A_w)) & \text{if } w \text{ divides } p, \\ \ker (H^1(L_w, A) \to H^1(L_w^{ur}, A)) & \text{if } w \text{ does not divide } p, \end{cases}
\]

where \( L_w^{ur} \) is the maximal unramified extension of \( L_w \). The (Greenberg strict) Selmer group attached to the data is then defined by

\[
S(A/L) := \text{Sel}^{\text{str}} A(L) := \ker \left(H^1(G_S(L), A) \to \bigoplus_{w \in S_L} H^1(L_w, A)\right),
\]

where we write \( H^1_S(L_w, A) = H^1(L_w, A)/H^1_{\text{str}}(L_w, A) \) and \( S_L \) denotes the set of primes of \( L \) above \( S \). It is straightforward to verify that \( S(A/L) = \lim_{\to n} S(A[\pi^n]/L) \), where \( S(A[\pi^n]/L) \) is the Selmer group defined similarly as above by replacing \( A \) by \( A[\pi^n] \) and \( A_w \) by \( A_w[\pi^n] \). Here the direct limit is taken over the maps \( S(A[\pi^n]/L) \to S(A[\pi^{n+1}]/L) \) which are induced by the natural injections \( A[\pi^n] \hookrightarrow A[\pi^{n+1}] \) and \( A_w[\pi^n] \hookrightarrow A_w[\pi^{n+1}] \). We will write \( X(A/L) \) for its Pontryagin dual.
We shall say that $F_\infty$ is an $S$-admissible $p$-adic Lie extension of $F$ if (i) $\text{Gal}(F_\infty/F)$ is compact $p$-adic Lie group, (ii) $F_\infty$ contains the cyclotomic $\mathbb{Z}_p$-extension $F_{\text{cyc}}$ of $F$ and (iii) $F_\infty$ is unramified outside $S$. Write $G = \text{Gal}(F_\infty/F)$, $H = \text{Gal}(F_\infty/F_{\text{cyc}})$ and $\Gamma = \text{Gal}(F_{\text{cyc}}/F)$. We define $S(A/F_\infty) = \lim_{L \to F_\infty} S(A/L)$, where the limit runs over all finite extensions $L$ of $F$ contained in $F_\infty$. We will write $X(A/F_\infty)$ for the Pontryagin dual of $S(A/F_\infty)$. It follows from [CS Lemma 2.2] that $X(A/F_\infty)$ is independent of the choice of $S$ as long as $S$ contains all the primes above $p$, the ramified primes of $A$, the primes that ramify in $F_\infty/F$ and all infinite primes.

Let $F_\infty$ be a $\mathbb{Z}_p'$-extension of $F$ which contains the cyclotomic $\mathbb{Z}_p'$-extension $F_{\text{cyc}}$ of $F$. We write $G = \text{Gal}(F_\infty/F)$. Let $F_m$ be the unique subextension of $F_\infty$ over $F$ with $\text{Gal}(F_m/F) \cong (\mathbb{Z}/p^m)^r$ and write $G_m = \text{Gal}(F_\infty/F_m)$. To state our result, we introduce another datum $(B, \{B_v\}_{v|p}, \{B_v^+\}_{v|R})$, which satisfies the conditions (a)–(d). To compare the Selmer groups, we need to expand the set $S$ to contain the ramified primes of $B$. We will also introduce the following important congruence condition on $A$ and $B$ which allows us to be able to compare the Selmer groups of $A$ and $B$.

$$(C_n) : \text{There is an isomorphism } A[\pi^n] \cong B[\pi^n] \text{ of } G_S(F)-\text{modules, which induces a } \text{Gal}(\bar{F}_v/F_v)-\text{isomorphism } A_v[\pi^n] \cong B_v[\pi^n] \text{ for each } v|p.\]$$

Clearly, $(C_n)$ implies $(C_i)$ for $i \leq n$. To simplify notation, we will write $\theta_G(A) = \theta_G(X(A/F_\infty))$ and $\theta_G(B) = \theta_G(X(B/F_\infty))$. We can now state the first theorem of this section.

**Theorem 3.2.** Let $F_\infty$ be an admissible $p$-adic Lie extension of $F$ with $G = \text{Gal}(F_\infty/F) \cong \mathbb{Z}_p'$. Assume that $X(A/F_\infty)$ and $X(B/F_\infty)$ are torsion over $\mathcal{O}[G]$. Suppose that condition $(C_{\theta_G(A)})$ holds. Then we have $\mu_{\mathcal{O}[G]} X(A/F_\infty) \leq \mu_{\mathcal{O}[G]} X(B/F_\infty)$.

Furthermore, suppose that $\theta_G(A) = \theta_G(B)$. Then $X(A/F_\infty)(\pi)$ and $X(B/F_\infty)(\pi)$ are pseudo-isomorphic to each other.

**Proof.** We first show that whenever $(C_i)$ holds, we then have the equality

$$\mu_{\mathcal{O}[G]}\left(X(A/F_\infty)/\pi^i\right) = \mu_{\mathcal{O}[G]}\left(X(B/F_\infty)/\pi^i\right).$$

As in the proof of [Lim2 Theorem 4.2], one can show that the order of the kernels and cokernels of the maps

$$S(D[\pi^i]/F_m) \xrightarrow{\sim} S(D/F_m)[\pi^i] \xrightarrow{\sim} \left(S(D/F_\infty)[\pi^i]\right)^G,$$

are $O(p^{r-1}m)$ for $D = A, B$ and $m >> 0$. Therefore, we have

$$\text{ord}_p\left(S(D[\pi^i]/F_m)\right) = \text{ord}_p\left(S(D/F_\infty)[\pi^i]^G\right) + O(p^{(r-1)m}).$$

for $D = A, B$. On the other hand, it follows from the validity of $(C_i)$ that

$$S(A[\pi^i]/F_m) = S(B[\pi^i]/F_m).$$

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Combining this observation with the above equality and Theorem 2.5, we obtain
\[
[k : \mathbb{F}_p][\mu_{\mathcal{O}[G]} \left( \frac{X(A/F_\infty)}{\pi} \right) p^m] = [k : \mathbb{F}_p][\mu_{\mathcal{O}[G]} \left( \frac{X(B/F_\infty)}{\pi} \right) p^m + O(p^{(r-1)m})]
\]
which in turn implies the required equality we set out to show.

The assertions of the theorem are now immediate from the above discussion and Propositions 2.2 and 2.3.

We record the second theorem of the section.

**Theorem 3.3.** Let \( F_\infty \) be an admissible \( p \)-adic Lie extension of \( F \) with \( G = \text{Gal}(F_\infty/F) \cong \mathbb{Z}_p^r \). Suppose that \( X(A/F_\infty) \) is torsion over \( \mathcal{O}[G] \) and that \( (\mathcal{C}_G(A) + 1) \) holds.

Then \( X(B/F_\infty) \) is torsion over \( \mathcal{O}[G] \), and \( X(A/F_\infty)(\pi) \) and \( X(B/F_\infty)(\pi) \) are pseudo-isomorphic to each other.

**Proof.** The proof is similar to that of the preceding theorem, where one makes use of Proposition 2.4 in place of Propositions 2.2 and 2.3.

**Remark 3.4.** If \( F_\infty \) is a general \( \mathbb{Z}_p^r \)-extension of \( F \) (that does not contain \( F_{\text{cyc}} \)) which has the property such that for each prime \( v \in S \), the decomposition group of \( \text{Gal}(F_\infty/F) \) at \( v \) has dimension \( \leq r - 1 \), then the argument of Theorems 3.2 and 3.3 carry over to give the same conclusion.

**Remark 3.5.** When \( F = \mathbb{Q} \) and \( F_\infty = F_{\text{cyc}} = \mathbb{Q}_{\text{cyc}} \), Theorem 3.2 was established in [BS] for Galois representations arising from elliptic curves with good ordinary reduction. Our results may therefore be seemed as a generalization of theirs. We also mention that our hypothesis is slightly weaker than the hypothesis in [BS], as it is based on \( \theta_{\mathcal{O}[G]} \)-invariant rather than the \( \mu_{\mathcal{O}[G]} \)-invariant as in [BS].

### 4 Comparing Selmer groups over noncommutative \( p \)-adic Lie extensions

In this section, we will compare the Iwasawa \( \mu \)-invariants of the Selmer groups of congruent Galois representations over a noncommutative \( p \)-adic Lie extension. As before, we shall say that \( F_\infty \) is an admissible \( p \)-adic Lie extension of \( F \) if (i) \( \text{Gal}(F_\infty/F) \) is compact pro-\( p \) \( p \)-adic Lie group without \( p \)-torsion, (ii) \( F_\infty \) contains the cyclotomic \( \mathbb{Z}_p \) extension \( F_{\text{cyc}} \) of \( F \) and (iii) \( F_\infty \) is unramified outside a set of finite primes. Write \( G = \text{Gal}(F_\infty/F) \), \( H = \text{Gal}(F_\infty/F_{\text{cyc}}) \) and \( \Gamma = \text{Gal}(F_{\text{cyc}}/F) \).

Let \( (A, \{ A_v \}_{v \mid p}, \{ A_v^+ \}_{v \mid \mathbb{R}}) \) and \( (B, \{ B_v \}_{v \mid p}, \{ B_v^+ \}_{v \mid \mathbb{R}}) \) denote the data defined as in Section 3. Let \( S \) denote a finite set of primes of \( F \) which contains all the primes above \( p \), the ramified primes of \( A \) and \( B \), the primes that are ramified in \( F_\infty/F \) and the infinite primes. As before, we denote the Pontryagin dual of the Selmer of \( A \) and \( B \) by \( X(A/F_\infty) \) and \( X(B/F_\infty) \) respectively.

To obtain a similar conclusion for the \( \mu \)-invariant over a general noncommutative \( p \)-adic Lie extension, we need to assume a stronger condition which was first introduced in [CF+]. For a finitely generated
$\mathcal{O}[G]$-module $M$, we say that $M$ belongs to $\mathfrak{N}_H(G)$ if $M/M(\pi)$ is finitely generated over $\mathcal{O}[H]$. Here $M(\pi)$ is the submodule of $M$ consisting of elements of $M$ annihilated by a power of $\pi$. Note that an $\mathcal{O}[G]$-module belonging to $\mathfrak{N}_H(G)$ is necessarily a torsion $\mathcal{O}[G]$-module. As before, we write $\theta_\Gamma(A) = \theta_{\mathcal{O}[\Gamma]}(X(A/F_{\text{cyc}}))$ and $\theta_\Gamma(B) = \theta_{\mathcal{O}[\Gamma]}(X(B/F_{\text{cyc}}))$

**Theorem 4.1.** Let $F_{\infty}$ be an admissible $p$-adic Lie extension. Assume that $X(A/F_{\infty})$ and $X(B/F_{\infty})$ belong to $\mathfrak{N}_H(G)$, and that $A(L_{\text{cyc}})$ and $B(L_{\text{cyc}})$ are finite for every finite extension $L$ of $F$ contained in $F_{\infty}$. Assume that condition $(C_{\theta_{\mathcal{O}[\Gamma]}(A)})$ is valid. Then we have $\mu_{\mathcal{O}[G]}(X(A/F_{\infty})) = \mu_{\mathcal{O}[G]}(X(B/F_{\infty}))$.

Furthermore, if either $\theta_\Gamma(A) = \theta_\Gamma(B)$ or condition $(C_{\theta_{\mathcal{O}[\Gamma]}(A)+1})$ is valid, then we have $\mu_{\mathcal{O}[G]}(X(A/F_{\infty})) = \mu_{\mathcal{O}[G]}(X(B/F_{\infty}))$.

**Proof.** The $\mathfrak{N}_H(G)$ hypothesis and the finiteness hypothesis allow us to apply [Lim2, Lemma 5.6] to conclude that $\mu_{\mathcal{O}[G],1}(X(A/F_{\infty})) = \mu_{\mathcal{O}[G],1}(X(A/F_{\text{cyc}}))$ and $\mu_{\mathcal{O}[G],1}(X(B/F_{\infty})) = \mu_{\mathcal{O}[G],1}(X(B/F_{\text{cyc}}))$. The conclusions of the theorem are now immediate consequences of Theorems 3.2 and 3.3.

**Remark 4.2.** We are not able to establish the stronger structure comparison of the $\pi$-primary component of the dual Selmer groups as in the commutative case. The main reason is that we are not able to establish the equalities of the $\mu$-invariants of the quotients $X(A/F_{\infty})/\pi^n$ and $X(B/F_{\infty})/\pi^n$.

We end the section mentioning a result which concerns with comparing the structural properties of the dual Selmer groups of two congruent Galois representations. We have seen in Proposition 3.3 that if $F_{\infty}$ is an abelian admissible extension, then one can deduce from the fact that $X(A/F_{\infty})$ is $\mathcal{O}[G]$-torsion to conclude that $X(B/F_{\infty})$ is also $\mathcal{O}[G]$-torsion. One may ask if one can establish an analogous result for the property of belonging to $\mathfrak{N}_H(G)$. This is the content of the next result.

**Proposition 4.3.** Let $F_{\infty}$ be an admissible $p$-adic Lie extension of $F$ with $G := \text{Gal}(F_{\infty}/F) \cong \mathbb{Z}_p^r$, where $r \geq 2$. Suppose that the following statements are satisfied.

(i) Suppose that $A(L_{\text{cyc}})$ and $B(L_{\text{cyc}})$ are finite for every finite extension $L$ of $F$ contained in $F_{\infty}$.

(ii) Assume that condition $(C_{\theta+1})$ is valid. Here $\theta = \max\{(\theta_{\mathcal{O}[\Gamma]}(A), \theta_{\mathcal{O}[G]}(A)\}$.

(iii) $H_i(H, X_f(B/F_{\infty}))$ is finite for all odd $i \leq r - 2$. Here $X_f(B/F_{\infty}) = X(B/F_{\infty})/X(B/F_{\infty})(\pi)$.

If $X(A/F_{\infty})$ belongs to $\mathfrak{N}_H(G)$, then $X(B/F_{\infty})$ belongs to $\mathfrak{N}_H(G)$.

**Proof.** The hypothesis of the proposition allows us to apply Theorem 3.2 to obtain $\mu_{\mathcal{O}[G]}(X(A/F_{\text{cyc}})) = \mu_{\mathcal{O}[G]}(X(B/F_{\text{cyc}}))$ and $\mu_{\mathcal{O}[G]}(X(A/F_{\infty})) = \mu_{\mathcal{O}[G]}(X(B/F_{\infty}))$. By virtue of the $\mathfrak{N}_H(G)$-hypothesis and assumption (i), we have $\mu_{\mathcal{O}[G]}(X(A/F_{\infty})) = \mu_{\mathcal{O}[G]}(X(A/F_{\text{cyc}}))$ (cf. [Lim1, Theorem 3.1] or [Lim2, Lemma 5.6]). Putting all the equalities together, we obtain $\mu_{\mathcal{O}[G]}(X(B/F_{\infty})) = \mu_{\mathcal{O}[G]}(X(B/F_{\text{cyc}}))$.

By assumption (iii), we may apply an analogous result to that of [Lim1, Theorem 3.1] to conclude that $X(B/F_{\infty})$ belongs to $\mathfrak{N}_H(G)$ which is as required.

As an immediate corollary, we have the following. We also mention that one may prove the corollary by appealing to [CS, Corollary 3.2].

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Corollary 4.4. Let $F_\infty$ be an admissible $p$-adic Lie extension of $F$ with $G := \text{Gal}(F_\infty/F) \cong \mathbb{Z}_p^2$. Suppose that $A(L_{\text{cyc}})$ and $B(L_{\text{cyc}})$ are finite for every finite extension $L$ of $F$ contained in $F_\infty$. Assume that condition $(C_{\theta^G})_{\infty}$ is valid. If $X(A/F_\infty)$ belongs to $\mathfrak{M}_H(G)$, then $X(B/F_\infty)$ belongs to $\mathfrak{M}_H(G)$.

5 Comparing specializations of big Galois representation

In this section, we will apply our previous results to compare the Selmer groups of specializations of a big Galois representation. As before, let $p$ be a prime. We let $F$ be a number field. If $p = 2$, we assume further that $F$ has no real primes. Denote $\mathcal{O}$ to be the ring of integers of some finite extension $K$ of $\mathbb{Q}_p$.

We write $R = \mathcal{O}[T]$ for the power series ring in one variable. Suppose that we are given the following data:

(a) $A$ is a cofinitely generated cofree $R$-module of $R$-corank $d$ with a continuous, $R$-linear $\text{Gal}(\bar{F}/F)$-action which is unramified outside a finite set of primes of $F$.

(b) For each prime $v$ of $F$ above $p$, $A_v$ is a $\text{Gal}(\bar{F}_v/F_v)$-submodule of $A$ which is cofree of $R$-corank $d_v$.

(c) For each real prime $v$ of $F$, we write $A_v^+ = A^{\text{Gal}(\bar{F}_v/F_v)}$ which we assume to be cofree of $R$-corank $d_v^+$.

(d) The following equality

$$\sum_{v | p} (d - d_v)[F_v : \mathbb{Q}_p] = dr_2(F) + \sum_{v \text{ real}} (d - d_v^+)$$

holds. Here $r_2(F)$ denotes the number of complex primes of $F$.

For any prime element $f$ of $\mathcal{O}[T]$ such that $\mathcal{O}[T]/f$ is a maximal order, then we can obtain a data $(A[f], \{A_v[f]\}_{v | p}, \{A_v^+[f]\}_{v \not| p})$ in the sense of Section 3. The next lemma has a easy proof which is left to reader.

Lemma 5.1. Let $f$ and $g$ be prime elements of $\mathcal{O}[T]$ with $\pi^n | f - g$ such that $\mathcal{O}[T]/f$ and $\mathcal{O}[T]/g$ are maximal orders. Then $A[f, \pi^n] = A[g, \pi^n]$.

The next two propositions compare the Iwasawa $\mu$-invariant of the specialization of the big Galois representations (compare with [B Corollary 4.37(1)], see also Section [B]). In the case when the admissible $p$-adic Lie extension is abelian, we even have the pseudo-isomorphism of the $\pi$-primary component of the dual Selmer groups.

Proposition 5.2. Let $F_\infty$ be an admissible $p$-adic Lie extension of $F$ with $G := \text{Gal}(F_\infty/F) \cong \mathbb{Z}_p^2$. Let $f$ be a prime element of $\mathcal{O}[T]$ such that $O' := \mathcal{O}[T]/f$ is a maximal order. Set $A = A[f]$ and suppose that $X(A/F_\infty)$ is torsion over $O'[G]$. Set

$$n := \left\lfloor \frac{\theta_{O'[G]}(X(A/F_\infty))}{e} + 1 \right\rfloor,$$

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where $e$ is the ramification index of $\mathcal{O}'/\mathcal{O}$. Then for every prime element $g$ of $\mathcal{O}[T]$ with $\pi^n|f - g$ such that $\mathcal{O}[T]/g$ is isomorphic to $\mathcal{O}'$, we have that $X(A[g]/F_{\infty})$ is torsion over $\mathcal{O}'[G]$, and that $X(A/F_{\infty})(\pi')$ and $X(A[g]/F_{\infty})(\pi')$ are pseudo-isomorphic.

**Proof.** Let $g$ be a prime element of $\mathcal{O}[T]$ which satisfies the hypothesis in the proposition. Let $\pi'$ be a prime element of $\mathcal{O}'$. We write $B = A[g]$. It follows from Lemma 5.1 that there is an isomorphism of $G_S(F)$-modules $A[\pi^{en}] \cong B[\pi^{en}]$ which induces an isomorphism of $\text{Gal}(\bar{F}_v/F_v)$-modules $A_v[\pi^{en}] \cong B_v[\pi^{en}]$ for each prime $v$ of $F$ above $p$. By our hypothesis of $n$, we have $en \geq \theta_{\mathcal{O}'}(G)(A) + 1$. Hence the conclusion of the proposition is now immediate from Theorem 3.3.

**Proposition 5.3.** Let $F_{\infty}$ be an admissible $p$-adic Lie extension of $F$. Let $f$ be a prime element of $\mathcal{O}[T]$ such that $\mathcal{O}' := \mathcal{O}[T]/f$ is a maximal order. Set $A = A[f]$ and suppose that $X(A/F_{\infty})$ belongs to $\mathcal{M}_H(G)$. Set

$$n := \left\lceil \frac{\theta_{\mathcal{O}'[G]}(X(A/F_{\infty})) + 1}{e} \right\rceil,$$

where $e$ is the ramification index of $\mathcal{O}'/\mathcal{O}$. Then for every prime element $g$ of $\mathcal{O}[T]$ with $\pi^n|f - g$ such that $\mathcal{O}[T]/g$ is isomorphic to $\mathcal{O}'$ and $X(A[g]/F_{\infty})$ belongs to $\mathcal{M}_H(G)$, we have that $X(A/F_{\infty})$ and $X(A[g]/F_{\infty})$ have the same $\mu_{\mathcal{O}'[G]}$-invariant.

When the admissible extension is a $\mathbb{Z}_p^2$-extension, we have the following proposition which refines [SS] Proposition 8.6 slightly.

**Proposition 5.4.** Write $B = A[g]$. Let $F_{\infty}$ be an admissible $p$-adic Lie extension of $F$ with $G := \text{Gal}(F_{\infty}/F) \cong \mathbb{Z}_p^2$. Let $f$ be a prime element of $\mathcal{O}[T]$ such that $\mathcal{O}' := \mathcal{O}[T]/f$ is a maximal order. Set $A = A[f]$ and suppose that $X(A/F_{\infty})$ belongs to $\mathcal{M}_H(G)$. Set

$$n := \left\lceil \frac{\theta + 1}{e} \right\rceil,$$

where $\theta = \max\{\theta_{\mathcal{O}'[T]}(A), \theta_{\mathcal{O}'[G]}(A)\}$ and $e$ is the ramification index of $\mathcal{O}'/\mathcal{O}$. Then for every prime element $g$ of $\mathcal{O}[T]$ with $\pi^n|f - g$ such that $\mathcal{O}[T]/g$ is isomorphic to $\mathcal{O}'$, we have that $X(A[g]/F_{\infty})$ belongs to $\mathcal{M}_H(G)$.

**Proof.** As noted in the previous proposition, it follows from Lemma 5.1 that there is an isomorphism of $G_S(F)$-modules $A[\pi^{en}] \cong B[\pi^{en}]$ which induces an isomorphism of $\text{Gal}(\bar{F}_v/F_v)$-modules $A_v[\pi^{en}] \cong B_v[\pi^{en}]$ for each prime $v$ of $F$ above $p$. Since $en \geq \theta + 1$, the conclusion of the proposition will follow from Corollary 4.4.

**Remark 5.5.** We should mention that although Proposition 5.4 allows us to deduce the $\mathcal{M}_H(G)$-property for certain specializations of the big Galois representation from the $\mathcal{M}_H(G)$-property for another specialization, we are not able to determine whether the dual Selmer group $X(A/F_{\infty})$ of the big Galois representation belongs to $\mathcal{M}_H(G)$ under the assumptions of the proposition.


6 Concluding Remarks

We end by making some remarks about the main results of this paper.

- By a standard comparison between the Selmer groups and the Greenberg Selmer groups (for instance, see [Gr, P. 121, P. 125], [Lim2, Section 7] or [SS, P. 591]), one obtain similar conclusion for the Greenberg Selmer groups of two congruent Galois representations.

- By another standard comparison between the Selmer groups and an appropriate Selmer complex (for instance, see [B, Proposition 3.24] or [Lim2, Section 7]), the results of the paper for Selmer groups can be carried over to the second cohomology group of the said Selmer complex or even the Selmer complex. This provide the link between our results in Section 5 and the results in [B] as mentioned the said section.

- Proposition 2.4 can be applied to improve the conclusion of [Lim2, Theorem 4.2]. The point is that the local-global Euler characteristic argument combined with the asymptotic formulas of Iwasawa and Cucuo-Monsky yield the equality

\[
\mu_{\mathcal{O}[G]}(X(A/\mathbb{F}_\infty)/\pi^i) = \mu_{\mathcal{O}[G]}(X(A^*/\mathbb{F}_\infty)/\pi^i)
\]

for all \(i \geq 1\) (see the proof of [Lim2, Theorem 4.2]). Here \(A^*\) is the dual of \(A\) in the sense of [Lim2]. Therefore, the hypothesis of the Proposition 2.4 is satisfied (since one may take \(i\) big enough), and hence one can conclude that \(X(A/\mathbb{F}_\infty)(\pi)\) and \(X(A^*/\mathbb{F}_\infty)(\pi)\) are pseudo-isomorphic.

In fact, when \(\mathbb{F}_\infty = \mathbb{F}_{\text{cyc}}\), Greenberg claimed that \(X(A/\mathbb{F}_{\text{cyc}})\) and \(X(A^*/\mathbb{F}_{\text{cyc}})\) should be pseudo-isomorphic up to an \(\iota\)-twist (cf. [Gr, P. 130, Equation (66)]) and gave some examples where this pseudo-isomorphism is known (see discussion after [Gr, P. 130, Equation (66)]). Therefore, our result above provides a positive answer to the \(\pi\)-primary part of this assertion of Greenberg.

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