FLAT FAMILIES OF POINT SCHEMES FOR CONNECTED GRADED ALGEBRAS

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Abstract. We study truncated point schemes of connected graded algebras as families over the parameter space of varying relations for the algebras, proving that the families are flat over the open dense locus where the point schemes achieve the expected (i.e. minimal) dimension.

When the truncated point scheme is zero-dimensional we obtain its number of points counted with multiplicity via a Chow ring computation. This latter application in particular confirms a conjecture of Brazfield to the effect that a generic two-generator, two-relator 4-dimensional Artin-Schelter regular algebra has seventeen truncated point modules of length six.

Introduction

The context for the present note is that of non-commutative projective algebraic geometry, in the sense of studying graded algebras and modules as (analogues of) homogeneous coordinate rings, as exemplified, for instance, by the seminal paper [1]. The follow-up work of [2, 3] introduced novel methods of handling the difficulties inherent in working with non-commutative rings by leveraging classical (as opposed to non-commutative) algebraic geometry to probe the nature of the “non-commutative projective schemes” embodied by the rings in question. We recall the relevant setup briefly.

To fix ideas and notation, consider an algebraically closed field $k$, an $r$-dimensional vector space $V$ whose dual is spanned by basis elements $x_i$, $1 \leq i \leq r$, and $s$ multilinear (of degree at least two) forms $f_j$ on $V$. The typical algebra we consider is then of the form

$$A = \frac{T(V^*)}{I} = \frac{k\langle x_1, \ldots, x_r \rangle}{(f_1, \ldots, f_s)}$$

(regarding the degree-one generators as elements of the dual $V^*$ is simply a matter of convention).

A point module of $A$ is a graded $A$-module that is cyclic and has Hilbert series $(1 - t)^{-1}$. If $A$ is commutative these correspond to the closed points of the projective scheme $\text{Proj} \, A$, justifying the nomenclature. One of the innovations of [2] was to introduce a scheme $\Gamma$ whose closed points parametrize the isomorphism classes of point modules over $A$; this is the so-called point scheme of $A$. The scheme $\Gamma$ is the inverse limit of the truncated point schemes $\{\Gamma_n\}_n$ defined as follows:

Regard the relations $f_j$ with degrees $d_j$ as elements of the respective tensor powers $(V^*)^{\otimes d_j}$. For every $n \geq 2$ we define

$$\Gamma_n \subset \mathbb{P}(V)^n \cong (\mathbb{P}^{r-1})^n$$

to be the zero scheme of the degree-$n$ component $I_n$ of the ideal $I$ generated by $f_j$s.

The closed points of $\Gamma_n$ parametrize the isomorphism classes of truncated point modules of length $n + 1$, defined as cyclic graded $A$-modules with Hilbert series $1 + t + t^2 + \cdots + t^n$. If the number $n$ is larger than or equal to the highest degree of the defining relations $f_1, \ldots, f_s$, then $\Gamma_n$ determines all truncated point schemes with indices larger than $n$ and hence the point scheme $\Gamma$.

In the present note we study the behavior of the truncated point schemes $\Gamma_n$ upon varying the relation space

$$(0-1) \quad \text{span}\{f_j\}$$

while keeping the degrees $d_j$ of the $f_j$ fixed. In other words, we regard $(0-1)$ as a point in the relevant product

$$(0-2) \quad G = \prod_{j=1}^{s} \mathbb{P}((V^*)^{\otimes d_j})$$

of projective spaces, and study $\Gamma_n$ as fibers of a family over the latter scheme.

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Theorem 1.3 below shows that under appropriate bounds on the degrees $d_j$, the locus $U \subset \mathbb{G}$ over which $\Gamma_n$ has the expected minimal dimension is open and dense in $\mathbb{G}$. Moreover, according to Theorem 1.6 the resulting family is flat over $U$. This implies that a suite of algebraic-geometric invariants one might compute for $\Gamma_n$ (e.g., the arithmetic genus) stay constant so long as the dimension of $\Gamma_n$ is that provided by the naive count.

When $\Gamma_n$ is zero-dimensional a simple computation in the Chow ring of $\mathbb{P}(V)^n$ returns the number of points of $\Gamma_n$ counted with multiplicity. We apply this procedure and our main results to algebras of the same “shape” (i.e. having the same number of generators and degrees of relations) as the four-dimensional Artin-Schelter regular algebras listed in [16, Proposition 1.4]. There are three types of such algebras, and in each case we compute the number of points (counted with multiplicity) of $\Gamma_n$ for the smallest number $n$ such that $\dim(\Gamma_n) = 0$. This includes via Proposition 1.7 a confirmation in Proposition 2.5 of [5, Conjecture IV.8.1]:

Conjecture 0.1. Let $A$ be a connected graded algebra with two degree-one generators. If the defining ideal of $A$ is generated by a generic cubic and a generic quartic relation, then $\Gamma_5$ consists of exactly seventeen distinct points.

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1. Main results

Fix positive integers $r$ and $s$. We consider the following family of algebras associated to an $s$-tuple:

Definition 1.1. For a tuple $$d = (d_1 \leq \cdots \leq d_s)$$ with $d_j \geq 2$ an algebra of type $(r, d)$ is a connected graded algebra with $r$ degree-one generators and $s$ relations of degrees $d_1, \ldots, d_s$.

We retain the notations in the introduction, and focus on $\Gamma_n$ for $n \geq d_s$, henceforth referred to as the stable range for $n$. Note that for stable $n$ the scheme $\Gamma_n$ is defined as the joint zero locus in $\mathbb{P}(V)^n$ of

$$\sum_{j=1}^{s} (n - d_j + 1)$$

multilinear equations whose respective degrees are indicated by the summands of (1-1): $n - d_j + 1$ equations of degree $d_j$.

Definition 1.2. Given $r$, $d$ and $n$ as above, the defect $\text{df}(r, d, n)$ attached to this data is the sum (1-1).

Recall that we are denoting by $\mathbb{G}$ the space (0-2) of relations for type-$(r, d)$ algebras. It is, in other words, the variety of tuples of homogeneous polynomials $f_j$ of prescribed degrees $d_j$ up to scaling. Now consider the closed subscheme $\mathbb{X}_n$ of $\mathbb{G} \times \mathbb{P}(V)^n$ defined by

$$\{ ((f_j), (a)_i) \in \mathbb{G} \times \mathbb{P}(V)^n \mid f_j(a_{i+1}, \ldots, a_{i+d_j}) = 0 \text{ for } 1 \leq j \leq s \text{ and } 0 \leq i \leq n - d_j \}.$$ 

The fiber $(\mathbb{X}_n)_R$ at $R \in \mathbb{G}$ along the projection $\pi: \mathbb{X}_n \to \mathbb{G}$ is the scheme $\Gamma_n$ attached to the type-$(r, d)$ algebra $TV^*/(R)$.

Our main results are as follows. First, we have the following observation to the effect that $\Gamma_n$ has “expected dimension” generically.

Theorem 1.3. Fix $r$, $d$ and $n$, and suppose the associated defect $\text{df}$ is $\leq n(r-1)$. Then,

(1) For each $R \in \mathbb{G}$, $\Gamma_n = (\mathbb{X}_n)_R$ is non-empty, and all components have dimension $\geq n(r-1) - \text{df}$;

(2) The locus $U$ of $R \in \mathbb{G}$ where all components of $\Gamma_n$ have dimension $n(r-1) - \text{df}$ is open and dense.

Proof. We prove the two claims separately.

(1) As observed above, $\Gamma_n$ is by definition the scheme-theoretic intersection of $\text{df}$ hypersurfaces in the $n(r-1)$-dimensional scheme $\mathbb{P}(V)^n$, so the lower bound $n(r-1) - \text{df}$ for the dimensions of the components is
a consequence, for instance, of [15, Proposition I.7.1]: that result is stated for subschemes of affine space, but our scheme \( \mathbb{P}(V)^n \) admits a cover by open patches isomorphic to \( A^n \).

The non-emptiness follows from the following computation in the Chow ring \( A^* = A^*(\mathbb{P}(V)^n) \). According to the Künneth theorem for Chow rings (e.g. [19, Propositions 1 and 2]) \( A^* \) is isomorphic to the \( n \)th tensor power of \( A^*(\mathbb{P}(V)) \), which is simply \( \mathbb{Z}[\varepsilon]/(\varepsilon^n) \) for the class \( \varepsilon \) of a hyperplane:

\[
A^* \cong \bigotimes_{i=1}^{n} \mathbb{Z}[\varepsilon_i]/(\varepsilon_i^r).
\]

Now consider the multilinearizations \( f_{i,j} \) of \( f_j \) on \( \mathbb{P}(V)^n \) with \( 0 \leq i \leq n - d_j \). \( \Gamma_n \) is the intersection of the respective zero loci \( V_{i,j} \) of \( f_{i,j} \), represented in the Chow ring by sums of the form

\[
(1-2) \quad \varepsilon_{i+1} + \varepsilon_{i+2} + \cdots + \varepsilon_{i+d_j}.
\]

The product of the elements (1-2) in the Chow ring will be shown to be non-zero in Lemma 1.4 below. In turn, this then implies that the intersection of the schemes \( V_{i,j} \) represented by (1-2) is non-empty.

To verify this last point, recall e.g. from [13, §8.1] that the product

\[
\prod_{j=1}^{s} \prod_{i=0}^{n-d_j} [V_{i,j}]
\]

can be obtained as the pushforward through

\[
\bigcap_{j=1}^{s} \bigcap_{i=0}^{n-d_j} V_{i,j} \to \mathbb{P}(V)^n
\]

of an element in the Chow ring of the left hand intersection (see especially [13, Example 8.1.9]). If this intersection were trivial then the element in question would vanish, hence the conclusion.

\[\text{(2) Since \( G \) is irreducible, it will be sufficient to prove that \( U \) is open and non-empty. We relegate non-emptiness to Lemma 1.5 below, and assuming it, we focus here on proving openness.}\]

We denote \( X = X_n \) for simplicity and write \( X_i \) for the irreducible components of \( X \). We can then apply [15, Exercise II.3.22] to each restriction and corestriction

\[
\pi_i : (X_{i})_{\text{red}} \to \pi(X_{i})_{\text{red}}.
\]

Part (d) of said exercise shows that the set of points of \( X_i \) lying in a fiber of \( \pi_i \) of dimension \( \geq n(r-1) - df + 1 \) is closed, and hence its image \( F_i \) through the projective morphism \( \pi_i \) is also closed. Since \( U \) is the complement of the closed finite union \( \bigcup F_i \), it is open. \( \blacksquare \)

**Lemma 1.4.** In the context of Theorem 1.3,

\[
\prod_{j=1}^{s} \prod_{i=0}^{n-d_j} (\varepsilon_{i+1} + \varepsilon_{i+2} + \cdots + \varepsilon_{i+d_j})
\]

is a nonzero element of the ring \( \bigotimes_{i=1}^{n} \mathbb{Z}[\varepsilon_i]/(\varepsilon_i^r) \).

**Proof.** We will prove the statement for all \( d = (d_1, \ldots, d_s) \) and \( n \) with the milder restriction \( 1 \leq d_j \leq n \) for all \( j \) and the same assumption \( df \leq n(r-1) \). We may assume \( 1 \leq d_1 \leq \cdots \leq d_s \leq n \) without loss of generality.

If we append \( d_{s+1} = n \) at the end of \( d \), then the defect is increased by one and the element in question is multiplied by \( \varepsilon_1 + \cdots + \varepsilon_n \). By applying this operation as many times as necessary, we can assume \( df = n(r-1) \).

Then the inequality \( r-1 \leq s \) follows from

\[
n(r-1) = \sum_{j=1}^{s} (n - d_j + 1) \leq \sum_{j=1}^{s} n = ns.
\]

The number of \( j \) with \( d_j = 1 \) is at most \( r - 1 \). Indeed, if \( 1 = d_1 = \cdots = d_{r-1} \), then

\[
n(r-1) = \sum_{j=1}^{s} (n - d_j + 1) = n(r-1) + \sum_{j=r}^{s} (n - d_j + 1)
\]

(1-2)
and \( n - d_j + 1 \geq 1 \). Hence \( s = r - 1 \) in this case.

Now we complete the proof by induction on \( n \). If \( n = 1 \), then \( d_1 = \cdots = d_s = 1 \) and hence \( s = r - 1 \). The element in question is \( \varepsilon_1^{r-1} \neq 0 \).

Let \( n \geq 2 \). For two elements \( P_1, P_2 \in \bigotimes_{i=1}^{n} \mathbb{Z}[\varepsilon_i]/(\varepsilon_i^r) = \mathbb{Z}[\varepsilon_1, \ldots, \varepsilon_n]/(\varepsilon_1^r, \ldots, \varepsilon_n^r) \), we write \( P_1 \leq P_2 \) if \( P_2 - P_1 \) is represented by a polynomial whose coefficients are all nonnegative. Then we have

\[
\prod_{j \leq r-1} \left( \prod_{i=0}^{n-d_j} (\varepsilon_{i+1} + \cdots + \varepsilon_{i+d_j}) \right) \geq \prod_{j \leq r-1} \left( \varepsilon_1 \prod_{i=1}^{n-d_j} (\varepsilon_{i+1} + \cdots + \varepsilon_{i+d_j}) \right), \quad \text{and}
\]

\[
\prod_{j \geq r} \left( \prod_{i=0}^{n-d_j} (\varepsilon_{i+1} + \cdots + \varepsilon_{i+d_j}) \right) \geq \prod_{j \geq r} \left( \varepsilon_1 \prod_{i=0}^{n-d_j} (\varepsilon_{i+2} + \cdots + \varepsilon_{i+d_j}) \right).
\]

Therefore

\[
(1-3) \quad \prod_{j=1}^{s} \left( \prod_{i=0}^{n-d_j} (\varepsilon_{i+1} + \cdots + \varepsilon_{i+d_j}) \right) \geq \varepsilon_1^{r-1} \prod_{j=1}^{s} \left( \prod_{i=0}^{(n-1)-d_j'} (\varepsilon_{i+2} + \cdots + \varepsilon_{i+d_j'}) \right)
\]

where \( d_j' = d_j \) for \( j \leq r - 1 \) and \( d_j' = d_j - 1 \) for \( j \geq r \). The right-hand side of (1-3) is of the form \( \varepsilon_1^{r-1} P \), where \( P \) is the element in question for the tuple \((d_1', \ldots, d_s')\) in variables \( \varepsilon_2, \ldots, \varepsilon_n \). Since the defect for this new tuple is

\[
\sum_{j=1}^{s} (n-1) - d_j' + 1 = \sum_{j=1}^{s} (n-d_j + 1) - (r-1) = (n-1)(r-1),
\]

the induction hypothesis implies that \( P \) is a nonzero element of \( \mathbb{Z}[\varepsilon_2, \ldots, \varepsilon_n]/(\varepsilon_2^r, \ldots, \varepsilon_n^r) \). Therefore both sides of (1-3) are \( \geq 0 \) and nonzero. This completes the proof. \[\blacksquare\]

**Lemma 1.5.** In the context of **Theorem 1.3** there are choices of relations \( f_j, 1 \leq j \leq s \) for which all components of \( \Gamma_n \) achieve the lower dimension bound of \( n(r-1) - df \).

**Proof.** Simply select the forms \( f_j \) to be of the form

\[
f_j = \prod_{i=1}^{d_j} \ell_{i,j}
\]

for linear forms \( \ell_{i,j} \) on \( \mathbb{P}(V) \), chosen so that the zero locus of any \( r \) is empty (i.e. the zero loci \( Z(\ell_{i,j}) \) are in general position in \( \mathbb{P}(V) \)).

The components of the joint zero locus of the multilinearizations of the \( f_j \) are obtained by imposing \( df \) linear constraints on the \( n \) components of points in \( \mathbb{P}(V)^n \), and the fact that such components have the requisite dimension \( n(r-1) - df \) follows from the generic choice of \( \ell_{i,j} \). \[\blacksquare\]

Additionally, the following result ensures that when \( \Gamma_n \) has the expected size, various invariants such as multi-degrees as subschemes of products of projective spaces, genus, etc. remain constant. The result is analogous to [9, Theorem 4.4], and its proof is similarly based on [12, Theorem 18.16].

**Theorem 1.6.** The restriction of the family \( \mathbb{X}_n \to \mathbb{G} \) to the open dense subscheme \( U \subseteq \mathbb{G} \) from **Theorem 1.3** is flat.

**Proof.** Once more denote \( \mathbb{X}_n \) by \( \mathbb{X} \); we indicate restriction of families by subscripts, as in \( \mathbb{X}_U \) for the restriction of \( \pi: \mathbb{X} \to \mathbb{G} \) to \( U \subseteq \mathbb{G} \). We will apply [12, Theorem 18.16 (b)] to the following setup.

Let \( x \in \mathbb{X}_U \). Then, the theorem in question applies to the local rings

\[
(R, P) = (\mathcal{O}_{U, \pi(x)}, m_{\pi(x)}) \to (\mathcal{O}_{\mathbb{X}, x}, m_x) = (A, Q).
\]

For this, we need

- \( R \) to be regular, which it is, being the local ring of a point on a product of projective spaces.
- \( A \) to be Cohen-Macaulay. This follows from the fact that \( \mathbb{X}_U \to U \) is a complete intersection in the family

\[
\prod_{j=1}^{s} \mathbb{P}((V^*)^{|d_j|}) \times U \to U.
\]
Theorem 1.6. Theorem 1.6

Section 1.

• The dimension of the fiber $A/PA$ equals the relative dimension $\dim(A) - \dim(R)$; this is simply a paraphrase of the fact that we are restricting to the locus $U$ where $\pi$ has fibers of the lowest possible expected dimension, i.e. $n(r - 1) - \text{df}(r, d, n)$.

This completes the proof. ■

Finally, we end with the following remark that will come in handy below, when we examine some examples.

**Proposition 1.7.** In the setting of Theorem 1.6, suppose furthermore that $n(r - 1) = \text{df}$. Then, the set $W \subseteq U$ over which $\Gamma_n$ is reduced is open and dense.

**Proof.** The irreducibility of $U$ means that it is sufficient to prove the set in question open and non-empty.

Under the present hypotheses, at each point in $U$ the scheme $\Gamma_n$ is finite, i.e. consists of several points, some, perhaps, with multiplicity. The flatness result in Theorem 1.6 ensures that the length $|\Gamma_n|$ is constant throughout $U$, counting multiplicity; we denote this common number by $\ell$.

By the functorial description of the Hilbert scheme of points (e.g. [14, p. 15], [4, Definition 2.1] or [17, Tag 0B94]), the flat family $\mathcal{X}_U \to U$ entails a map $\phi : U \to \text{Hilb}^\ell_{\mathbb{P}(V)^n}$.

The Hilbert scheme contains an open subscheme $\text{Hilb}^\circ$ consisting of $\ell$-tuples of distinct points in $\mathbb{P}(V)^n$ (see [4, Proposition 2.4] and the remarks following it). In conclusion, the openness and non-emptiness of $W$ will follow once we argue that $\phi(U)$ intersects $\text{Hilb}^\circ$, i.e. $\Gamma_n$ is reduced for at least one point of $U$.

To verify this last claim, note that $\Gamma_n$ will indeed be reduced for a generic choice of linear forms in the construction used in the proof of Lemma 1.5. ■

2. Examples and connections to prior work

The preceding material ties in with a number of results of similar flavor in the literature, as we now document.

We will be focusing on algebras with the same generator-relation pattern as the four-dimensional AS-regular ones classified in [16, Proposition 1.4]:

• four generators and six relations of degree 2;
• three generators, two degree-two relations and two degree-three relations;
• two generators and one relation in each degree 3 and 4.

Under the regularity assumptions of [16] the Betti numbers of these types of algebras are, respectively

• $1, 4, 6, 4, 1$;
• $1, 3, 4, 3, 1$;
• $1, 2, 2, 1$.

When applying the contents of Section 1, the relevant critical dimension $n(r - 1) - \text{df}(r, d, n)$ becomes zero for certain $n$, i.e. the $\Gamma_n$ in question will be non-empty finite (perhaps reduced) schemes.

2.1. 4 generators, 6 quadratic relations (type 14641). In this case, the results of Section 1 essentially recapture the main result of [11] to the effect that generically, such algebras have 20 point modules, counted with multiplicity.

In the absence of regularity conditions the scheme $\Gamma_2$ will be our stand-in for the scheme of point modules, and hence the $n$ to which Section 1 applies here is 2.

We thus have $r = 4$ and $s = 6$, and the vector space $V$ of the above discussion is dual to the span $V^*$ of linearly independent generators $x_1$ up to $x_4$. The scheme $\mathcal{G}$ is $\mathbb{P}(V^* \otimes V^*)^6$, all $d_j$ are equal to 2, and the defect is 6.

We then have

**Proposition 2.1.** Under the conventions of the present subsection, the scheme $\Gamma_2$ is non-empty, and the locus $U \subset \mathcal{G}$ where $\Gamma_2$ is zero-dimensional is open and dense.

For relation spaces $R \in U$, $\Gamma_2$ consists of twenty points, counted with multiplicity. These points are distinct for $R \in W$ as in Proposition 1.7.

**Proof.** Everything but the claim about the count of 20 is an immediate application of Theorems 1.3 and 1.6 and Proposition 1.7.

As for the count itself, it follows from the fact that examples with $|\Gamma_2| = 20$ exist, as first constructed in [11] (see also [18, 7, 10] and references therein) together with flatness; the latter ensures the constancy of the degree throughout the open parameter family $U$. 

Alternatively, one can avoid having to handle any examples at all by resorting to a Chow ring-based argument: 

\[ A^*(\mathbb{P}(V)^2) \] is in this case isomorphic to 

\[ \mathbb{Z}[\varepsilon_1]/(\varepsilon_1^4) \otimes \mathbb{Z}[\varepsilon_2]/(\varepsilon_2^4) \]

and each bilinearization of a relation cuts out a hypersurface \( V_i \), \( 1 \leq i \leq 6 \) of class \( \varepsilon_1 + \varepsilon_2 \). Since \( \varepsilon_1^4 = 0 \), this then implies that the product of the Chow classes \( [V_i] \) is 

\[ (\varepsilon_1 + \varepsilon_2)^6 = 20 \varepsilon_1^3 \varepsilon_2^3. \] (2-1)

On the other hand, [13, Example 8.2.1] implies that the product (2-1) is the Chow class of the scheme-theoretic intersection \( \bigcap V_i \). The assumption of the mentioned result is ensured by [13, Example 8.2.7] and the fact that each \( V_i \) is Cohen-Macaulay. Since \( \varepsilon_1^3 \varepsilon_2^3 \) is the Chow class of a point, this means that said intersection consists of twenty points with multiplicity. ■

**Remark 2.2.** It is the second proof of \( |\Gamma_2| = 20 \) given above that would presumably be more portable and flexible, as it is available even when \( \Gamma_n \) is not zero-dimensional. We will treat such a case in Proposition 2.6. ♦

**Remark 2.3.** The number \( n \) such that the critical dimension becomes zero is equal to \( \ell - 2 \), where \( \ell \) is the Gorenstein parameter of a four-dimensional AS-regular algebra of the same generator-relation pattern. Indeed, in the proof of [16, Proposition 1.4], it is observed that the AS-regular algebras considered there have Hilbert series \( 1/p(t) \), where \( p(t) \) has zero at \( t = 1 \) with multiplicity \( \geq 3 \). In our terminology, \( p(1) = 0 \) implies that \( s = 2r - 2 \) and \( p'(1) = 0 \) implies that the sum of \( d_j \)’s is \( (r - 1)\ell \). Thus the defect is 

\[ \sum_{j=1}^{s} (n - d_j + 1) = (2n + 2 - \ell)(r - 1) \]

and it is equal to \( n(r - 1) \) if and only if \( n = \ell - 2 \). ♦

### 2.2. 3 generators, quadratic and cubic relations (type 13431).

We now tackle the second bullet point listed at the beginning of the present section, corresponding to three-generator algebras with two quadratic and two cubic relations. We will then be studying \( \Gamma_3 \) (i.e. here \( n = 3 \)).

**Proposition 2.4.** Under the conventions of the present subsection, the scheme \( \Gamma_3 \) is non-empty, and the locus \( U \subset G \) where \( \Gamma_3 \) is zero-dimensional is open and dense.

For relation spaces \( R \in U \), \( \Gamma_3 \) consists of nineteen points, counted with multiplicity. These points are distinct for \( R \in W \).

**Proof.** The proof is entirely parallel to that of Proposition 2.1, only the count requiring modification.

This time, the relevant Chow ring is 

\[ A^*(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \cong \bigotimes_{i=1}^{3} \mathbb{Z}[\varepsilon_i]/(\varepsilon_i^3) \]

and the class of \( \Gamma_3 \) is the coefficient of \( \varepsilon_1^2 \varepsilon_2^2 \varepsilon_3^2 \) in 

\[ (\varepsilon_1 + \varepsilon_2)^2 (\varepsilon_2 + \varepsilon_3)^2 (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2. \]

This is easily seen to be 19 by direct computation. ■

### 2.3. 2 generators, cubic and quartic relations (type 12221).

This case is what in fact motivated the present note, and corresponds to the third bullet point the prefatory discussion to the present section.

This investigation is a follow-up to [8] (in turn inspired by [16]), and was prompted by our learning belatedly of the thesis [5], where some of the algebras of interest here are studied. Specifically, the following result resolves [5, Conjecture IV.8.1] (i.e. Conjecture 0.1) in the affirmative.

**Proposition 2.5.** Under the conventions of the present subsection, the scheme \( \Gamma_5 \) is non-empty, and the locus \( U \subset G \) where \( \Gamma_5 \) is zero-dimensional is open and dense.

For relation spaces \( R \in U \), \( \Gamma_5 \) consists of seventeen points, counted with multiplicity. The points are distinct for \( R \in W \).
Proof. Once more, the argument is precisely parallel to those of Propositions 2.1 and 2.4, except for inessential numerical differences in the last portion of the proof.

The Chow ring to consider here is
\[ A^*((\mathbb{P}^1)^5) \cong \bigotimes_{i=1}^{5} \mathbb{Z}[[\varepsilon_i]]/((\varepsilon_i^2)), \]
and the sought-after degree is the coefficient of \( \prod \varepsilon_i \) in
\[(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)(\varepsilon_2 + \varepsilon_3 + \varepsilon_4)(\varepsilon_3 + \varepsilon_4 + \varepsilon_5)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5);
\]this is indeed 17.

Alternatively, we can repeat example-based argument at the end of the proof of Proposition 2.1: the family \( X_U \to U \) is flat, and we know that its fiber has degree seventeen for at least one point in \( U \) via the examples in [5, Chapter V]. Flatness then ensures that the degree is seventeen throughout \( U \).

For the type of algebras of this subsection, we can apply our general result also to \( n = 4 \). In this case the expected dimension of \( \Gamma_4 \) is one.

Recall (e.g. [6, §2.1.1]) that the multidegree of a closed subscheme \( Y \) of a product \( \mathbb{P} = (\mathbb{P}^{r-1})^n \) of projective spaces consists of the tuple of cardinalities (with multiplicities)
\[(2-2) \left| Y \cap \bigcap_{i=1}^{\dim Y} H_i \right| \]
for generic choices of hypersurfaces in \( \mathbb{P} \) obtained as zeros of a single linear form on one of the factors \( \mathbb{P}^{r-1} \) of \( \mathbb{P} \).

When \( Y \) is a curve the codimension in (2-2) is one, and hence we can express the multidegree simply as a sequence of \( n \) non-negative integers
\[ |Y \cap H_i|, 1 \leq i \leq n \]
where
\[ H_i = (\mathbb{P}^{r-1})^{\times(i-1)} \times Z(\ell_i) \times (\mathbb{P}^{r-1})^{\times(n-i)} \]
for generic linear forms \( \ell_i \).

Proposition 2.6. Under the conventions of the present subsection, the scheme \( \Gamma_4 \) is non-empty, and the locus \( U \subset G \) where \( \Gamma_4 \) is one-dimensional is open and dense.

For relation spaces \( R \in U \), \( \Gamma_4 \) has multidegree \( (4, 3, 3, 4) \).

Proof. The proof is similar to that of Proposition 2.1, but now we consider the Chow ring
\[ A^*((\mathbb{P}^1)^4) \cong \bigotimes_{i=1}^{4} \mathbb{Z}[[\varepsilon_i]]/((\varepsilon_i^2)), \]
and compute the product
\[(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)(\varepsilon_2 + \varepsilon_3 + \varepsilon_4)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4).
\]The result is
\[4\varepsilon_1\varepsilon_2\varepsilon_3 + 3\varepsilon_1\varepsilon_2\varepsilon_4 + 3\varepsilon_1\varepsilon_3\varepsilon_4 + 4\varepsilon_2\varepsilon_3\varepsilon_4.
\]The same argument as the latter part of the proof of Proposition 2.1 implies that this is the Chow class of \( \Gamma_4 \).

Finally, the last statement follows from the fact that, as explained in [6, §2.1.1], the multidegree can be read off as the tuple of coefficients of the Chow class. ■

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