The soliton stars evolution.

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Abstract

The evolution of a soliton star filled with fermions is studied in the framework of general relativity. Such a system can be described by the surface tension $\sigma$, the bag constant $B$, and the fermion number density $\rho_0$. Usually one of these parameters prevails in the system and thus affects the spacetime inside the soliton. Whether it is described by Friedman or de Sitter metric depends on the prevailing parameter. The whole spacetime is divided by the surface of the soliton into the false vacuum region inside the soliton and the true vacuum region outside, the latter being described by the Schwarzschild line element. The aim of this paper is to study the equations of motion of the domain wall in two cases. In the first case the de Sitter metric describes the interior in the first case, and in the second case it is replaced by the Friedman metric. In both of them the Schwarzschild metric is outside the soliton. From the analysis of obtained equations one can draw conclusions concerning further evolution of a soliton star.
In this paper we study the evolution of a soliton stellar configuration. Let us consider the $SU_c$ colour QCD plus a phenomenological colourless scalar field $\Phi$ describing the colour confinement. The whole system is described by the scalar field $\Phi$, the fermion quark field $\psi_f$, the gluon field $A^a_{\mu}$ and the gravitational field $g_{\mu\nu}$. The generalized QCD Lagrangian function takes the form:

$$\mathcal{L} = -\frac{1}{4} (1 - \frac{\Phi^2}{\Phi^2_0}) F_{\mu\nu}^c F_{\mu\nu}^c + \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - U(\Phi)$$

and consists of two parts; the basic QCD Lagrangian supplemented by the Lagrangian of the dynamical scalar field. In the Lagrangian function (2) $R$ is the curvature scalar and

$$\kappa = 8\pi G \sim \frac{1}{M_{Pl}^2}$$

$G$ the gravitational constant and $U(\Phi)$ the potential. Due to nonlinearity of the potential $U(\Phi)$ the Lagrangian (2) leads to nontopological soliton solutions. The potential possesses the polynomial form of the fourth order (3)

$$U(\Phi) = \frac{a}{2!} \Phi^2 + \frac{b}{3!} \Phi^3 + \frac{c}{4!} \Phi^4 + B.$$  

The constants $a, b, c$ are parameters which are adopted in the restriction to enable fitting of the static properties of hadrons. The bag constant $B$ is a measure of the pressure of the physical vacuum on the perturbative one. The potential possesses two minima, one at $\Phi = 0$ which is a local minimum associated with the perturbative vacuum state and the second, the absolute minimum, at

$$\Phi_c = \frac{3b}{2c} [1 + (1 - \frac{8ac}{b^2})^{\frac{1}{2}}]$$

corresponding to the physical vacuum. The form of the potential is presented on Fig.1. The soliton solution exists between two limiting cases $B = 0$ which represents the degenerate vacuum state and $B = B_{max}$ when the local minimum and maximum overlap and an inflection point appears. This is the moment when the perturbative and physical vacuum states become metastable and also the limiting case in which the soliton solution still exists. The action

$$S = \int d^4x \sqrt{-g} \mathcal{L}$$
with $\mathcal{L}$ defined as (2) leads to the appropriate equations of motion. We are looking for spherically symmetric solution with the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^\nu dt^2 + e^\lambda dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(6)

Variation of the action (5) with respect to metric $g_{\mu\nu}$ gives the four-dimensional Einstain equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

(7)

with the energy-momentum tensor $T_{\mu\nu}$

$$T_{\mu\nu} = T_{\mu\nu}(\Phi) + T_{\mu\nu}(\psi)$$

(8)

where

$$T_{\mu\nu}(\Phi) = \partial_\mu \Phi \partial_\nu \Phi - g_{\mu\nu} \mathcal{L}(\Phi)$$

(9)

is the scalar field energy-momentum tensor and $T_{\mu\nu}(\psi)$ its fermion counterpart.

The soliton bag model in the thin-wall approximation describes the quark star model with a surface tension. The whole spacetime is divided by the surface of the soliton into the false vacuum region inside the soliton and the true vacuum region outside, the latter being described by the Schwarzschild line element. The aim of this paper is to study the dynamics of the domain wall in two cases. In both of them the Schwarzschild metric is outside the soliton.

Subscripts $\pm$ refer to values of the corresponding coordinates in the region outside and inside the soliton.

$$ds^2_\pm = -(1 - \frac{r_g}{r}) dt^2_\pm + \frac{1}{(1 - \frac{r_g}{r_+})} dr^2_\pm + r^2_\pm (d\theta^2 + \sin^2 \theta d\phi^2)$$

(10)

where $r_g = 2GM$ with $M$ being the total soliton mass.

The de Sitter metric describes the interior of the soliton in the case when the value of the cosmological constant $\Lambda$ dominates.

$$ds^2 = -(1 - \frac{1}{3} \Lambda r^2) dt^2 + \frac{1}{(1 - \frac{1}{3} \Lambda r^2)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(11)

In the second case when the fermion number density $\rho_0$ is the prevailing parameter, the interior metric is replaced by the Friedman solution (2). The line element has now the form

$$dl^2 = -\frac{dr^2}{(1 - \frac{r_0}{a})^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(12)
where $a$ is a curvature radius. After performing the following substitution $r = a \sin \chi$ where $\chi$ changes from 0 to $\pi$, the line element takes the form

$$dl^2 = a^2 \{d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)\}$$

(13)

For space with constant, positive curvature and for $a = \alpha r_g$ where $\alpha$ is the dimensionless parameter, the metric can be written in the following way

$$ds^2 = -dt^2 + \alpha^2 r_g^2 d\chi^2 + \alpha^2 r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

(14)

Because

$$r_g d\chi = \frac{dr}{1 - \frac{r^2}{r_g^2}}$$

finally one can get the metric

$$ds^2 = -dt^2 + \alpha^2 \left( \frac{dr^2}{1 - \frac{r^2}{r_g^2}} + r_g^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right).$$

(16)

The time evolution of the scale factor $a$ inside the soliton star resembles cosmological evolution of the universe but on a different scale. Inside the soliton star we have a nonvanishing cosmological constant $\Lambda = \kappa B$ and fermions which at rough approximation are treated as dust density $\rho_0$. The space-time in the soliton star will evolve according to the Einstein equation which gives, as in cosmology, the scale evolution equation

$$\dot{a}^2 + 1 = \frac{4}{3} \pi G \rho_0 a^2 + \frac{1}{3} \Lambda a^2$$

(17)

It is possible to interpret the time evolution of the metric inside the soliton as a movement of the fictitious particle with the vanishing energy under the influence of some effective potential

$$\frac{1}{2} \dot{a}^2 + U_{eff}(a) = E$$

(18)

For $E = 0$

$$U_{eff}(a) = \frac{1}{2} - \frac{8 \pi G}{6} (\rho_0 a_0^3) \frac{1}{a} - \frac{8 \pi G}{6} B a^2$$

(19)

where the dust mass $M_d = \rho_0 V$ with $V = 2\pi^2 a^3$. The "steady state" solution is obtained for

$$a_0 = \left( \frac{GM_d}{\pi \Lambda} \right)^{1/3} = \left( \frac{M_d}{8\pi^2 B} \right)^{1/3}$$

(20)
It is convenient to introduce the dimensionless potential

\[ U_{\text{eff}}(x) = \frac{1}{2} - A\left(\frac{1}{x} + \frac{1}{2}x^2\right) \]  

which depends only on one parameter \( A \) and \( a = a_0x \)

\[ A = \left(\frac{GM_\odot}{3\pi a_0}\right). \]

The critical point \( A_c = \frac{1}{3} \) represents the maximum of the potential \( U_{\text{eff}}(x) \) Fig. 2 presents the form of the potential function \( U_{\text{eff}}(x) \) for two values of parameters \( A; A = A_c \) and \( A = \frac{2}{3} \). Results obtained for different values of parameter \( B \) are presented in Table 1.

| \( B = (0.1 \text{GeV})^4 \) | \( B = (1 \text{GeV})^4 \) |
|--------------------------|--------------------------|
| \( a_0 = 1036 \text{ km} \) | \( a_0 = 0.48 \text{ km} \) |
| \( A = 0.001524 \) | \( A = 0.3284 \) |

The metric evolution is determined by the critical value of parameter \( A_c \) which produces the critical scale \( a_c \). If \( A > A_c \), the cosmological constant \( \Lambda \) can be neglected and the evolution is exactly the same as in the Friedman universe with gravitational collapse as a result. For the soliton star with \( B = 56 \text{MeV/fm}^3 \) we obtain \( \rho_c \approx 10^{11} \text{g/cm}^3 \) which is below the neutron star density. If \( A < A_c \), the cosmological constant \( \Lambda \) dominates and space-time expands. The empty soliton will expand exponentially according to the de Sitter solution with the Hubble constant

\[ H = \sqrt{\frac{8\pi G}{3}B}. \]

The bag constant \( B=56 \text{ MeV/fm}^3 \) determines the value of the parameter \( t_H \)

\[ t_H = \frac{1}{H} \sim 10^5 \text{s} \sim 1 \text{day}. \]

In order to derive the equations of motion for the surface of a soliton star we have to obtain the junction conditions between the inner and outer region of the soliton. Let \( \Sigma \) be a three-dimensional spacetime hypersurface swept out by the soliton surface. It is convenient to choose the Gaussian normal coordinates in the neighbourhood of \( \Sigma \). Coordinates of any point \( p \) belonging to the neighbourhood \( N \) of \( \Sigma \) are given by \( x_\mu = (x_i, \eta), x_i = (\tau, \theta, \varphi) \) where \( \varphi \)
and $\theta$ are angular variables, $\tau$ the proper time of an observer comoving with the domain wall, $\eta$ is the distance from $\Sigma$ to $p$ along the geodesics which are orthogonal to $\Sigma$ \[1\]. We also define $\xi^\mu$, as a unit vector field normal to $\eta = \text{const}$ hypersurface. The three dimensional metric intrinsic to the hypersurface is

$$h_{\mu\nu} = g_{\mu\nu} - \xi_\mu \xi_\nu$$  \hspace{1cm} (25)$$

where $g_{\mu\nu}$ is the four metric of the spacetime. According to Israel \[3\], in this case the 3-tensor $\gamma_{ij}$ defined as

$$\gamma_{ij} = K_{ij}^+ - K_{ij}^- \neq 0$$  \hspace{1cm} (26)$$

where $K_{ij}$ is the extrinsic curvature 3-tensor of $\eta = \text{const}$ hypersurface. This tensor can be calculated using the following relations

$$K_{ij} = -\Gamma^\alpha_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^\alpha}.$$  \hspace{1cm} (27)$$

We are interested in the situation when the energy-momentum tensor $T_{\mu\nu}$ of the four-dimensional spacetime has a $\delta$-function singularity on the hypersurface and can be expressed in the following form \[5\]

$$T_{\mu\nu} = S_{\mu\nu}(x_i)\delta(\eta)$$  \hspace{1cm} (28)$$

with $S_{\mu\nu}$ as the surface energy tensor. Taking this into account the extrinsic curvature $K_{ij}$ possesses a jump discontinuity across the hypersurface. Adopting the Gauss-Codazzi formalism \[5\] the $\gamma_{ij}$ tensor takes the form

$$\gamma_{ij} = -8\pi G (S_{ij} - \frac{1}{2} g_{ij} S)$$  \hspace{1cm} (29)$$

where $S_{ij}$ is defined as the integral of the energy-momentum tensor through the thickness of the wall.

$$S_{ij} = \lim_{\epsilon \to 0} \int_0^\epsilon T_{ij}dx^1$$  \hspace{1cm} (30)$$

We will limit our consideration to the following form of the surface energy tensor \[5\]

$$S_{\mu\nu} = \sigma(\tau)u_\mu u_\nu - \zeta(\tau)(h_{\mu\nu} + u_\mu u_\nu)$$  \hspace{1cm} (31)$$
where $\sigma$ is the surface energy density of the domain wall, $\zeta$ the surface tension and $u_\mu$ the four-velocity of the wall. For the domain wall $\sigma = \zeta$ and

$$S_{\mu\nu} = -\sigma h_{\mu\nu} \tag{32}$$

In order to describe the dynamics of the wall which separates the interior and the exterior of the soliton star we have to calculate the appropriate components of the extrinsic curvature; $K^+_{ij}$ - the extrinsic curvature for the Schwarzschild metric and $K^-_{ij}$ for the metric which describes the space inside the soliton. Taking into account the form of $S_{\mu\nu}$ (32)

$$\gamma^i_j = -4\pi G\sigma \delta^i_j \tag{33}$$

For the system described by the Lagrange function (2) it is possible to interpret the Einstein equation having the following form inside the soliton

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa B g_{\mu\nu} + \kappa T_{\mu\nu}(\psi) \tag{34}$$

or

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}(\psi) \tag{35}$$

with $\Lambda = \kappa B$ as the Einstein equation with the cosmological constant different from zero and with the fermion energy-momentum tensor. In the case when $T_{\mu\nu}(\psi) = 0$ (empty bag) the Einstein equation is reduced to the form with the solution expanding according to the de Sitter law. It is interesting to present the dynamical equation of the soliton surface. Owing to the spherical symmetry of $\Sigma$ the metric on the domain wall can be presented in the following way

$$ds^2 = -d\tau^2 + R(\tau)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{36}$$

where $\tau$ is a proper time along the world-line ($\theta, \phi = \text{const}$) of a dust particle. We adopt this metric as a matching condition of the interior and the exterior metric on the soliton surface. Now using the method presented in Israel’s paper [3] we obtain the dynamical equation for the soliton surface

$$(1 - \frac{1}{3} \Lambda R^2 + \dot{R}^2)^{\frac{1}{2}} - (1 - \frac{r_g}{R} + \dot{R}^2) = 4\pi \sigma R \tag{37}$$

In all these equations $\dot{R} = \frac{dR}{d\tau}$. This equation agrees with that obtained by A.Aurilia [6]. Once more the surface dynamics can be illustrated as fictitious
particle motion with the total energy equal to zero in the effective potential field.

\[ \frac{1}{2} \dot{R}^2 = -V_{\text{eff}}(R) \]  

(38)

where \( V_{\text{eff}}(R) \) can be expressed as

\[ V_{\text{eff}}(R) = \frac{1}{2} \left\{ 1 - R^2 \left( \frac{B}{3\sigma} + 2\pi G\sigma \right)^2 \right. \]

\[ - \frac{M}{R} \left( \frac{B}{6\pi G\sigma^2} + 1 \right) - \frac{M^2}{16\pi^2\sigma^2 R^4} \} \]  

(39)

substituting \( R = x r_g \), \( r_g \) being the gravitational radius, the effective potential \( V_{\text{eff}}(R) \) takes the following form

\[ V_{\text{eff}} = \frac{1}{2} \left\{ (1 - x^2 r_g^2) \left( \frac{B}{3\sigma} + 2\pi G\sigma \right)^2 \right. \]

\[ - \frac{1}{2x} \left( \frac{B}{6\pi G\sigma^2} + 1 \right) - \frac{1}{\left( 8\pi G r_g \sigma \right)^2 x^3} \} \]  

(40)

We can also present the mass \( M \) of the soliton

\[ M = \frac{4}{3} \pi R^3 B + 4\pi R^2 \sigma \left( 1 - \frac{8}{3} \pi GBR^2 + \dot{R}^2 \right)^{1/2} \]  

\[ - 8\pi^2 G\sigma^2 R^3 \]  

(41)

The subsequent terms in the expression for the mass are the volume energy and the surface energy with the relativistic correction and the surface-surface binding energy. Fig.3 presents the form of the potential \( V_{\text{eff}}(x) \) which depends only on one variable \( x \). The soliton described in this paper is not stable and such a configuration tends to obtain the equilibrium state by the emission of particles. Phenomenological description of this situation can be made by introducing the term identified as friction.

In the vicinity of point \( x_c \) which corresponds to the maximum of the potential \( V_{\text{eff}}(x) \), the function \( V_{\text{eff}}(x) \) can be approximated in the following way.

\[ V_{\text{eff},a}(x) = -\frac{1}{2} u_0 + \frac{1}{2} \omega^2 (x - x_0)^2 \]  

(42)

On Fig.3 the function \( V_{\text{eff},a}(x) \) is represented by the broken line. Substituting \( y = x - x_0 \) one can write

\[ \dot{y}^2 = \omega^2 (y + a^2) \]  

(43)
where $a^2 = \frac{m}{\omega^2}$. The general solutions of this equation have the following form

$$y_1 = \frac{1}{2} \{ e^{-t_0} e^{\omega t}(-a^2 + e^{2t_0} e^{-2\omega t}) \}$$

$$y_2 = \frac{1}{2} \{ e^{-t_0} e^{-\omega t}(-a^2 + e^{2t_0} e^{2\omega t}) \}$$

(44) (45)

where $y_1$ represents collapse and $y_2$ expansion of the soliton. For $t = 0$

$$y_0 = \frac{1}{2} \{ e^{-t_0}(-a^2 + e^{2t_0}) \} = \frac{1}{2} \{ -a^2 e^{-t_0} + e^{t_0} \}$$

(46)

so for this moment

$$x_p = x_0 + \frac{1}{2} \{ -a^2 e^{-t_0} + e^{t_0} \}$$

(47)

Substituting $e^{t_0} = az$

$$x_p = x_0 + \frac{1}{2} a (z^2 - 1)$$

(48)

The mass $M$ of the soliton is determined by the initial value $x_p$. The dimensionless parameter $m = \frac{M}{M_\odot}$

$$m = \frac{4}{3} \pi b x_p^3 + 4\pi s x_p^2$$

(49)

The equation of motion takes the following form

$$\ddot{x} + b \dot{x} = -\frac{\partial V_{\text{eff}, a}}{\partial x} = \omega^2 (x - x_c)$$

(50)

We can analyze the solution of the equation (50) in momentum space Fig.4. Solutions where $p > 0$, $p = \dot{x}$, describe the trajectories representing the expansion of the whole system whereas for solutions with $p < 0$ the trajectories lead to the collapses. Thus the values of the initial conditions are responsible either for the expansion or the collapse of the soliton. There is only one critical trajectory for which the whole system, after an infinite time, tends to the stable configuration $M = 0.92 M_\odot$ and $R = 16.33 \text{ km}$. Different types of trajectories are presented on Fig.4. The vertical straight line marks the position of the Schwarzschild horizon, two hyperbolas correspond to the trajectories if there is no friction in the equation of motion, broken line presents the trajectory which passes very close to the critical one. The straight line shows the position of the critical trajectory. The analysis of this figure leads
to the conclusion that the system reaches stable configuration before reaching the Schwarzschild horizon. Such a configuration corresponds to the situation when the soliton obtains the state of equilibrium between the emission of particles from the surface and the de Sitter expansion. Fig.5 shows the numerical solution of the equation of motion for the potential function $V_{\text{eff}}(x)$ and this solution confirms the analysis which was carried out on the basis of Fig.4.

Let us now concentrate on the situation when the cosmological constant $\Lambda$ is neglected inside the soliton. It has been already mentioned that this very solution corresponds with the case when the bag constant $B = 0$ and represents the degenerate vacuum state. The confinement is only due to the surface tension $\sigma$. This situation appears in the case of the Lee-Pang star [7]. The necessary and sufficient conditions for reality of this system is both the existence of the nontopological soliton which is guaranteed by the scalar field $\Phi$, the fermion field and the gravitational field. The soliton star in this case consists of a large interior and a much thinner surface shell of the width $\sim \mu(1 - 1)$, where $\mu$ is the mass of the scalar field $\Phi$. For the value of surface tension $\sigma = \frac{1}{n}(30 \text{GeV})^3$ the following parameters of the soliton star was obtained: mass $M \sim 10^{13} M_\odot$ and radius $R \sim 1$ light year.

The exterior geometry is still described by the Schwarzschild line element whereas the interior geometry is described by the Robertson-Walker metric. If fermions inside the soliton are treated in the rough approximation as dust with the density $\rho_0$, the energy momentum tensor $T_{\mu\nu}$ takes the form

$$T_{\mu\nu} = \rho_0 u_\mu u_\nu$$

where $u_\mu$ is the four-velocity vector of the particle. Once more the Israel’s equation for the motion of the soliton surface separating the two metrics was obtained

$$h^{-1}(1 + h\dot{R}^2)^{\frac{1}{2}} - (1 - \frac{2GM}{R} + \dot{R}^2)^{\frac{1}{2}} = 4\pi\sigma GR$$

where $h = \frac{\alpha^2}{1 - \frac{2GM}{R}}$, $\alpha$ is the scale factor. In order to study the dynamics of the soliton surface we identified the motion of the domain wall with the motion of a unit mass particle with vanishing total energy moving under the influence of the potential. The form of the potential function $W(\Phi)$ is presented on Fig.6. Closer analysis of this function reveals that once again if we introduce the friction the whole system behaves just like in the de Sitter case. One can notice the similarity with the results obtained in the previous
paragraph concerning the analysis of the function $V_{eff}(x)$. However, this time the critical trajectory reaches the Schwarzschild horizon $R = r_g$. Thus in this case the soliton collapses to the black hole.

1 Conclusions

The analysis of the soliton surface dynamics has been performed in this paper. It may be concluded that the star either collapses into a black hole or the whole system is characterized by the following parameters: the bag constant $B$, the surface tension $\sigma$, and the fermion number density $\rho_0$. If the cosmological constant $\Lambda$ is the prevailing parameter the space time inside the soliton can be described by the de Sitter solution. When $\rho_0$ prevails the Friedman solution is obtained. It turned out that in the de Sitter case there exists the stable configuration described by the macroscopic parameters: the mass $M = 0.92M_\odot$ and the radius $R = 16.33$ km. The radius of this configuration is bigger than the gravitational one and characteristic of the whole system resembles the ones that can be observed in neutron stars. In the case of Friedman solution the soliton which is characterized by the mass $M \sim 10^{13}M_\odot$ collapses exactly at the Schwarzschild horizon, so in the result we obtain the black hole.

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Figure captions

Fig.1. The form of the potential function $U(\Phi)$.
Fig.2. The effective potential $U_{eff}(x)$ for two values of parameter $A$: the continues line corresponds to the $A = A_c$ whereas dotted line to $A = \frac{2}{3}$.
Fig.3. The form of the effective potential $V_{eff}(x)$ - continues line, dotted line indicates the position of the function $V_{eff,a}(x)$.
Fig.4. The trajectories for different types of solutions presented in the phase space.
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Fig.6. The effective potential $W(\Phi)$. 
Figure 1: The form of the potential function $U(\Phi)$.

Figure 2: The effective potential $U_{\text{eff}}(x)$ for two values of parameter $A$: the continues line corresponds to the $A = A_c$ whereas dotted line to $A = \frac{2}{3}$. 
Figure 3: The form of the effective potential $V_{eff}(x)$ - continues line, dotted line indicates the position of the function $V_{eff,a}(x)$.

Figure 4: The trajectories for different types of solutions presented in the phase space.
Figure 5: Numerical solution of the equation of motion (50).

Figure 6: The effective potential $W(\Phi)$