Abstract. We study the $G_2$ analogue of the Goldberg conjecture on non-compact solvmanifolds. In contrast to the almost-Kähler case we prove that a 7-dimensional solvmanifold cannot admit any left-invariant calibrated $G_2$-structure $\varphi$ such that the induced metric $g_\varphi$ is Einstein, unless $g_\varphi$ is flat.

We give an example of 7-dimensional solvmanifold admitting a left-invariant calibrated $G_2$-structure $\varphi$ such that $g_\varphi$ is Ricci-soliton. Moreover, we show that a 7-dimensional (non-flat) Einstein solvmanifold $(S, g)$ cannot admit any left-invariant cocalibrated $G_2$-structure $\varphi$ such that the induced metric $g_\varphi = g$.

1. Introduction

A 7-dimensional smooth manifold $M^7$ is said to admit a $G_2$-structure if there is a reduction of the structure group of its frame bundle from $GL(7, \mathbb{R})$ to the exceptional Lie group $G_2$ which can actually be viewed naturally as a subgroup of $SO(7)$. Therefore a $G_2$-structure determines a Riemannian metric and an orientation. In fact, one can prove that the presence of a $G_2$-structure is equivalent to the existence of a certain type of a non-degenerate 3-form $\varphi$ on the manifold. By [11] a manifold $M^7$ with a $G_2$-structure comes equipped with a Riemannian metric $g$, a cross product $P$, a 3-form $\varphi$, and orientation, which satisfy the relation

$$\varphi(X, Y, Z) = g(P(X, Y), Z),$$

for every vector field $X, Y, Z$.

This is exactly analogue to the data of an almost Hermitian manifold, which comes with a Riemannian metric, an almost complex structure $J$, a 2-form $F$, and an orientation, which satisfy the relation $F(X, Y) = g(JX, Y)$.

Whenever this 3-form $\varphi$ is covariantly constant with respect to the Levi-Civita connection then the holonomy group is contained in $G_2$ and the 3-form $\varphi$ is closed and co-closed.

A $G_2$-structure is called calibrated if the 3-form $\varphi$ is closed and it can be viewed as the $G_2$ analogous of an almost-Kähler structure in almost Hermitian geometry. By the results in [6,8] no compact 7-dimensional manifold $M^7$ can support a calibrated $G_2$-structure $\varphi$ whose underlying metric $g_\varphi$ is Einstein unless $g_\varphi$ has holonomy contained in $G_2$. This could be considered to be a $G_2$ analogue of the Goldberg conjecture in almost-Kähler geometry. The result was generalized by R.L. Bryant to calibrated $G_2$-structures with too tightly pinched Ricci tensor and by R. Cleyton and S. Ivanov to calibrated $G_2$-structures with divergence-free Weyl tensor.
A non-compact complete Einstein (non-Kähler) almost-Kähler manifold with negative scalar curvature was constructed in [3] and in [14] it was shown that it is an almost-Kähler solvmanifold, that is, a simply connected solvable Lie group $S$ endowed with a left-invariant almost-Kähler structure [14]. In Section 3 we show that in dimension six this is the unique example of Einstein almost-Kähler (non-Kähler) solvmanifold and we classify the 6-dimensional solvmanifolds admitting a left-invariant (non-flat) Kähler-Einstein structure.

A natural problem is then to study the existence of calibrated $G_2$-structures inducing Einstein metrics on non-compact homogeneous Einstein manifolds. All the known examples of non-compact homogeneous Einstein manifolds belong to the class of solvmanifolds, that is, they are simply connected solvable Lie groups $S$ endowed with a left invariant metric (see for instance the survey [19]). A left-invariant metric on a Lie group $S$ will be always identified with the inner product $\langle \cdot, \cdot \rangle$ determined on the Lie algebra $\mathfrak{s}$ of $S$. According to a long standing conjecture attributed to D. Alekseevskii (see [4, 7.57]), these might exhaust the class of non-compact homogeneous Einstein manifolds.

On the other hand, Lauret in [20] showed that the Einstein solvmanifolds are standard, i.e. satisfy the following additional condition: if $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is the orthogonal decomposition of the Lie algebra $\mathfrak{s}$ of $S$ with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$, then $\mathfrak{a}$ is abelian.

A left-invariant Ricci-flat metric on a solvmanifold is necessarily flat [2], but solvmanifolds can admit incomplete metrics with holonomy contained in $G_2$ as shown in [12, 7].

In Section 4 by using the classification of 7-dimensional Einstein solvmanifolds and some obstructions to the existence of calibrated $G_2$-structures, we prove that a 7-dimensional solvmanifold cannot admit any left-invariant calibrated $G_2$-structure such that the induced metric $g_\varphi$ is Einstein, unless $g_\varphi$ is flat.

If $\varphi$ is co-closed, then the $G_2$-structure is called cocalibrated. In Section 5 we show that a 7-dimensional (non-flat) Einstein solvmanifold $(S, g)$ cannot admit any left-invariant cocalibrated $G_2$-structure $\varphi$ such that the induced metric $g_\varphi = g$.

## 2. Preliminaries on Einstein solvmanifolds

By [20] all the Einstein solvmanifolds are standard. Standard Einstein solvmanifolds constitute a distinguished class that has been deeply studied by J. Heber, who has obtained many remarkable structural and uniqueness results, by assuming only the standard condition (see [13]). In contrast to the compact case, a standard Einstein metric is unique up to isometry and scaling among left-invariant metrics [13, Theorem E]. The study of standard Einstein solvmanifolds can be reduced to the rank-one case, that is, to the ones with $\dim \mathfrak{a} = 1$ (see [13, Sections 4.5,4.6]) and everything is determined by the nilpotent Lie algebra $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$. Indeed, a nilpotent Lie algebra $\mathfrak{n}$ is the nilradical of a rank-one Einstein solvmanifold if and only if $\mathfrak{n}$ admits a nilsoliton metric (also called a minimal metric), meaning that its Ricci operator is a multiple of the identity modulo a derivation of $\mathfrak{n}$.

Any standard Einstein solvmanifold is isometric to a solvmanifold whose underlying metric Lie algebra resembles an Iwasawa subalgebra of a semisimple Lie algebra in the sense that $ad_A$ is symmetric and nonzero for any $A \in \mathfrak{a}, A \neq 0$. Moreover, if $H$ denotes the mean curvature vector of $S$ (i.e., the only element $H \in \mathfrak{a}$ such that $\text{tr}(ad_A) = \langle A, H \rangle$, for every $A \in \mathfrak{a}$), then the eigenvalues of $ad_H|\mathfrak{n}$ are all positive.
integers without a common divisor, say \( k_1 < \ldots < k_r \). If \( d_1, \ldots, d_r \) denote the corresponding multiplicities, then the tuple
\[
(k; d) = (k_1 < \ldots < k_r; d_1, \ldots, d_r)
\]
is called the eigenvalue type of \( S \). It turns out that \( \mathbb{R}H \oplus \mathfrak{n} \) is also an Einstein solvmanifold (with inner product the restriction of \( \langle \cdot, \cdot \rangle \) on it). It is thus enough to consider rank-one (i.e. \( \dim \mathfrak{a} = 1 \)) metric solvable Lie algebras since every higher rank Einstein solvmanifold will correspond to a unique rank-one Einstein solvmanifold and to a certain abelian subalgebra of derivations of \( \mathfrak{n} \) containing \( \text{ad}_H \). In every dimension, only finitely many eigenvalue types occur.

By [22, Lemma 11], [1] and [13, Proposition 6.12] it follows that if \((\mathfrak{s}, \langle \cdot, \cdot \rangle)\) is an Einstein (non-flat) solvable Lie algebra, such that \( \dim \mathfrak{a} = m \) and \([\mathfrak{s}, \mathfrak{s}]\) is abelian, then the eigenvalue type is \((1; k)\), with \( k = \dim [\mathfrak{s}, \mathfrak{s}] \geq m \).

In the case that \( \mathfrak{n} \) is non abelian, it is proved in [21] that any nilpotent Lie algebra of dimension \( \leq 5 \) admits an Einstein solvable extension. In [24] it is shown that the same is true for any of the 34 nilpotent Lie algebras of dimension 6, obtaining then a classification of all 7-dimensional rank-one Einstein solvmanifolds (see Table 2). A classification of 6 and 7-dimensional Einstein solvmanifolds of higher rank can be obtained by [25], where more in general there is a study of Ricci solitons up to dimension 7 on solvmanifolds. We recall that a Riemannian manifold \((M, g)\) is called Ricci soliton if the metric \( g \) is such that \( \text{Ric}(g) = \lambda g + L_X g \) for some \( \lambda \in \mathbb{R} \), and \( X \in \mathfrak{X}(M) \). Ricci solitons are called expanding, steady, or shrinking depending on whether \( \lambda < 0, \lambda = 0 \), or \( \lambda > 0 \). Any nontrivial homogeneous Ricci soliton must be non-compact, expanding and non-gradient (see for instance [21]). Up to now, all known examples are isometric to a left-invariant metric \( g \) on a simply connected Lie group \( G \) such that
\[
\text{Ric}(g) = \lambda I + D,
\]
for some \( \lambda \in \mathbb{R} \) and some derivation \( D \) of the Lie algebra \( \mathfrak{g} \) of \( G \). Conversely, any left-invariant metric \( g \) which satisfies (1) is automatically a Ricci soliton. If \( G \) is solvable, these metrics are also called solvsolitons.

3. Almost-Kähler structures

An almost Hermitian manifold \((M, J, g)\) is called an almost-Kähler manifold if the corresponding Kähler form \( F(\cdot, \cdot) = g(\cdot, J \cdot) \) is a closed 2-form. In this section we study the existence of Einstein almost-Kähler structures \((J, g, F)\) on 6-dimensional solvmanifolds.

Along all this work, the coefficient appearing in the rank-one Einstein extension of a Lie algebra will be denoted by \( a \) while the coefficients of the extension up to dimension 6 for almost-Kähler, and up to dimension 7 for \( G_2 \) manifolds, will be denoted by \( b_i \).

**Theorem 3.1.** A 6-dimensional solvmanifold \((S, g)\) admits a left-invariant Einstein (non-Kähler) almost-Kähler metric if and only if its Lie algebra \((\mathfrak{s}, g)\) is isometric to the rank-two Einstein solvable Lie algebra \( \mathfrak{g}_2 \) defined below. A 6-dimensional solvmanifold \((S, g)\) admits a left-invariant Kähler-Einstein structure if and only if the Lie algebra \((\mathfrak{s}, g)\) is isometric either to the rank-one Einstein solvable Lie algebra \( \mathfrak{k}_4 \) or to the rank-two Einstein solvable Lie algebra \( \mathfrak{f}_4 \) or to the
rank-three Einstein solvable Lie algebra \([1]\); both Lie algebras \([2]\) and \([3]\) are given below.

**Proof.** A 6-dimensional Einstein solvable Lie algebra \((\mathfrak{s}, g)\) is necessarily standard, so one has the orthogonal decomposition (with respect to \(g\))

\[
\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a},
\]

with \(\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]\) nilpotent and \(\mathfrak{a}\) abelian. We will consider separately the different cases according to the rank of \(\mathfrak{s}\), i.e., to the dimension of \(\mathfrak{a}\).

If \(\dim \mathfrak{a} = 1\) and \(\mathfrak{n}\) is abelian, then we know by [13, Proposition 6.12] that \(\mathfrak{s}\) has structure equations

\[
(\mathfrak{a} e_{16}, \mathfrak{a} e_{26}, \mathfrak{a} e_{36}, \mathfrak{a} e_{46}, \mathfrak{a} e_{56}, 0),
\]

where \(a\) is a non-zero real number. For this Lie algebra we get that any closed 2-form \(F\) is degenerate, i.e. satisfies \(F^3 = 0\) and so it does not admit symplectic forms.

If \(\dim \mathfrak{a} = 1\) but \(\mathfrak{n}\) is nilpotent (non-abelian), then \((\mathfrak{s}, g)\) is isometric to one of the solvable Lie algebras \(k_i\) \((i = 1, \ldots, 8)\) defined below in Table 1, endowed with the inner product \(g\) such that the basis \([e_1, \ldots, e_6]\) is orthonormal.

For \(k_1\), \(k_j\), \(5 \leq j \leq 8\), we get again that any closed 2-form \(F\) is degenerate.

The Lie algebras \(k_2\) and \(k_3\) admit symplectic forms. However, one can check that any almost complex structure \(J\) on \(k_i\) \((i = 2, 3)\) is such that \(g(\cdot, \cdot) \neq F(\cdot, J\cdot)\).

For \(k_4\) we get that a symplectic form is

\[
F = \mu_1 e_{12} + \mu_2 e_{16} + \mu_3 e_{26} + \mu_4 e_{36} + \mu_5 e_{46} + \mu_1 e_{56},
\]

where \(\mu_i\) are real numbers satisfying \(\mu_1 \neq 0\). The almost complex structures \(J\) such that \(g(\cdot, \cdot) = F(\cdot, J\cdot)\) are given, with respect to the basis \([e_1, \ldots, e_6]\), by

\[
Je_1 = \pm e_2, \quad Je_3 = \pm e_4, \quad Je_5 = \pm e_6,
\]

with \(e_i\) the dual of \(e^i\) via the inner product, and they are integrable. Therefore, \((J, g, F)\) are Kähler-Einstein structures on \(k_4\).

In order to determine all the 6-dimensional rank-two Einstein solvable Lie algebras, we need first to find the rank-one Einstein solvable extensions \(n_4 \oplus \mathbb{R}\langle e_5 \rangle\) of the 4-dimensional nilpotent Lie algebras \(n_4\).

Then we consider the standard solvable Lie algebra \(s_6 = n_4 \oplus a\), with \(a = \mathbb{R}\langle e_5, e_6 \rangle\) abelian and such that the basis \([e_1, \ldots, e_6]\) is orthonormal.

If \(n = [a, s]\) is abelian and dimension of \(a = 2\), we have to consider the structure equations

\[
\begin{align*}
d e^1 &= a e^{15} + b_1 e^{16}, \\
d e^2 &= a e^{25} + b_2 e^{26}, \\
d e^3 &= a e^{35} + b_3 e^{36}, \\
d e^4 &= a e^{45} + b_4 e^{46}, \\
d e^5 &= d e^6 = 0
\end{align*}
\]

and then to impose that the inner product for which \([e_1, \ldots, e_6]\) is orthonormal has to be Einstein and \(d^2 e^j = 0, \ j = 1, \ldots, 6\). Solving these conditions we find that
the structure equations are:

\[
\begin{align*}
de^1 &= ae^{15} + b_1 e^{16}, \\
de^2 &= ae^{25} + (-b_1 - b_3 - b_4) e^{26}, \\
de^3 &= ae^{35} + b_3 e^{36}, \\
de^4 &= ae^{45} + b_4 e^{46}, \\
de^5 &= de^6 = 0.
\end{align*}
\]

where \( a = \sqrt{2(b_1^2 + b_3^2 + b_4^2 + b_3 b_4 + b_4 b_1 + b_1 b_3)} \). This Lie algebra does not admit any symplectic form.

If \( \mathfrak{n} \) is nilpotent (non-abelian) and \( \dim \mathfrak{n} = 2 \), two cases should be considered for \( \mathfrak{n} = (0,0,e^{12},0) \) and \( (0,0,e^{12},e^{13}) \). We find that they have the following rank-one Einstein solvable extensions

\[
\left( \frac{1}{4} ae^{15}, \frac{1}{2} ae^{25}, \frac{1}{4} \sqrt{22 ae^{12} + ae^{35}}, \frac{3}{4} ae^{45}, 0 \right),
\]

if \( \mathfrak{n} = (0,0,e^{12},0) \); and

\[
\left( \frac{1}{4} ae^{15}, \frac{1}{2} ae^{25}, \frac{1}{4} \sqrt{5 ae^{12} + \frac{3}{4} ae^{35}}, \frac{1}{2} \sqrt{5 ae^{13} + ae^{45}}, 0 \right),
\]

if \( \mathfrak{n} = (0,0,e^{12},e^{13}) \). Now, to compute the rank-two Einstein extension of \( \mathfrak{n} = (0,0,e^{12},0) \) we should consider the Lie algebra

\[
\begin{align*}
de^1 &= \frac{1}{2} ae^{15} + b_1 e^{16} + b_2 e^{26} + b_3 e^{36} + b_4 e^{46}, \\
de^2 &= \frac{1}{2} ae^{25} + b_5 e^{16} + b_6 e^{26} + b_7 e^{36} + b_8 e^{46}, \\
de^3 &= \frac{1}{4} \sqrt{22 ae^{12} + ae^{35}} + b_9 e^{16} + b_{10} e^{26} + b_{11} e^{36} + b_{12} e^{46}, \\
de^4 &= \frac{3}{4} ae^{45} + b_{13} e^{16} + b_{14} e^{26} + b_{15} e^{36} + b_{16} e^{46}, \\
de^5 &= de^6 = 0.
\end{align*}
\]

Then we have to impose the Jacobi identity and that the inner product, such that the basis \( \{ e_1, \ldots, e_6 \} \) is orthonormal, has to be Einstein. We obtain the Einstein extension:

\[
\begin{align*}
de^1 &= \frac{1}{4} ae^{15} + b_1 e^{16} + b_2 e^{26}, \\
de^2 &= \frac{1}{2} ae^{25} + b_5 e^{16} + b_{10} e^{26}, \\
de^3 &= \frac{1}{4} \sqrt{22 ae^{12} + ae^{35}} + (b_1 + b_{10}) e^{36}, \\
de^4 &= \frac{3}{4} ae^{45} - 2(b_1 + b_{10}) e^{46}, \\
de^5 &= de^6 = 0,
\end{align*}
\]

where \( a = \frac{4 \sqrt{33}}{35} \sqrt{3b_1^2 + 5b_1 b_9 + b_2^2 + 3b_{10}^2} \), which admits the Kähler-Einstein structures given, in terms of the orthonormal basis \( \{ e_1, \ldots, e_6 \} \), by

\[
\begin{align*}
F &= \mu_1 (ae^{12} + 2 \sqrt{\frac{2}{11} ae^{35}} + 2 \sqrt{\frac{2}{11} (b_1 + b_{10}) e^{36}}) + \mu_2 (ae^{15} + 2b_1 e^{16} + 2b_2 e^{26}) \\
&\quad + \mu_3 (2b_2 e^{16} + ae^{25} + 2b_{10} e^{26}) + \mu_4 (3ae^{45} - 8(b_1 + b_{10} e^{46})) + \mu_5 e^{56}, \\
J e_1 &= e_2, \quad J e_2 = -e_1, \quad J e_3 = 2 \sqrt{\frac{2}{11}} e_5 + \sqrt{\frac{3}{11}} e_6, \quad J e_4 = \sqrt{\frac{3}{11}} e_5 - 2 \sqrt{\frac{2}{11}} e_6, \\
J e_5 &= -2 \sqrt{\frac{2}{11}} e_3 - \frac{3}{11} e_4, \quad J e_6 = -\frac{3}{11} e_3 + 2 \sqrt{\frac{2}{11}} e_4,
\end{align*}
\]
where \( \mu_i \) are real parameters satisfying \((b_1 + b_{10})\mu_2^2\mu_4 \neq 0\). The almost complex structure \( J \) is indeed complex i.e., the Nijenhuis tensor of \( J \) vanishes.

From the rank-one Einstein solvable extension of \( n = (0,0,e^{12},e^{13}) \) we get the 6-dimensional Einstein solvable Lie algebra of rank two:

\[
\begin{align*}
&\{ de^1 = \frac{3}{4}e^{15} + \frac{3}{2}ae^{16}, \\
&\quad de^2 = \frac{3}{2}e^{25} - ae^{26}, \\
&\quad de^3 = \frac{1}{2}\sqrt{3}ae^{12} + \frac{3}{4}ae^{35} - \frac{a}{2}e^{36}, \\
&\quad de^4 = \frac{1}{2}\sqrt{3}ae^{13} + ae^{45} + \frac{a}{2}e^{46}, \\
&\quad de^5 = de^6 = 0,
\end{align*}
\]

which admit the Einstein (non-Kähler) almost-Kähler structure given by

\[
F = \mu_1(-2\sqrt{3}e^{12} - 3e^{35} + e^{36}) + \mu_2(\sqrt{3}e^{13} + 2e^{45} + e^{46}) + \mu_3(e^{15} + 3e^{16}) + \mu_4(-e^{25} + 2e^{26}) + \mu_5e^{56},
\]

\[
Je_1 = e_3, \quad Je_2 = -\frac{1}{\sqrt{3}}e_5 + \frac{2}{\sqrt{6}}e_6, \quad Je_3 = \frac{1}{\sqrt{6}}e_5 + \frac{1}{\sqrt{6}}e_6,
\]

where \(\mu_2(4\mu_1^2 + \mu_2\mu_4) \neq 0\). The almost-Kähler structure is not integrable since

\[
N_J(e_1, e_2) = -\sqrt{3}ae_3, \quad N_J(e_1, e_5) = ae_1, \quad N_J(e_1, e_6) = -2ae_1.
\]

Now for the rank-three extensions we proceed as for the previous ones.

If \( \dim a = 3 \) and \( n \) is abelian, we have the Einstein solvable Lie algebra

\[
\begin{align*}
&\{ de^1 = \frac{ae^{14}}{\sqrt{6}} - \frac{\sqrt{2}}{2}ae^{15} + \frac{\sqrt{2}}{2}e^{16}, \\
&\quad de^2 = ae^{24} + \frac{\sqrt{2}}{2}ae^{25} + \frac{\sqrt{2}}{2}ae^{26}, \\
&\quad de^3 = ae^{34} - \sqrt{2}ae^{36}, \\
&\quad de^4 = de^5 = de^6 = 0,
\end{align*}
\]

which admits the almost-Kähler structure given by

\[
F = \mu_1(\sqrt{2}e^{14} - \sqrt{3}e^{15} + e^{16}) + \mu_2(\sqrt{2}e^{24} + \sqrt{3}e^{25} + e^{26}) + \mu_3(-e^{34} + \sqrt{2}e^{36}) + \mu_4e^{45} + \mu_5e^{46} + \mu_6e^{56},
\]

\[
Je_1 = \frac{1}{\sqrt{3}}e_4 - \frac{1}{\sqrt{2}}e_5 + \frac{1}{\sqrt{6}}e_6, \quad Je_2 = \frac{1}{\sqrt{3}}e_4 + \frac{1}{\sqrt{2}}e_5 + \frac{1}{\sqrt{6}}e_6, \quad Je_3 = -\frac{1}{\sqrt{3}}e_4 + \sqrt{\frac{2}{3}}e_6,
\]

where \(\mu_1\mu_2\mu_3 \neq 0\) and actually the almost complex structure is complex i.e., \( N_J = 0 \).

If \( \dim a = 3 \) and \( n \) is nilpotent (non-abelian) \( n \) is exactly \( h_3 \) (the 3-dimensional Heisenberg Lie algebra), having structure equations:

\[
(0,0,e^{12}).
\]

We find the following rank-one Einstein solvable extension

\[
(\frac{a}{4}e^{14}, \frac{a}{2}e^{24}, ae^{12} + ae^{34}, 0),
\]
proceeding in the same way as in the previous examples we find that \( h_3 \) does not admit a rank-three Einstein solvable extension unless it is flat.

\[
\begin{array}{|c|}
\hline
s_6 & 6\text{-dimensional Einstein solvable Lie algebras of rank one} \\
\hline
\ell_1 & (\frac{1}{2} \alpha e_{16}, \frac{1}{2} \alpha e_{26}, \frac{1}{2} \sqrt{\alpha} \alpha e_{12} + \frac{1}{2} \alpha e_{36}, \frac{1}{2} \sqrt{\alpha} \alpha e_{13} + \frac{1}{2} \alpha e_{46}, \frac{1}{2} \sqrt{\alpha} \alpha e_{14} + \frac{1}{2} \alpha e_{56}, 0) \\
\ell_2 & (\frac{1}{2} \alpha e_{16}, \frac{1}{2} \alpha e_{26}, \frac{1}{2} \sqrt{\alpha} \alpha e_{12} + \frac{1}{2} \alpha e_{36}, \frac{1}{2} \sqrt{\alpha} \alpha e_{13} + \frac{1}{2} \alpha e_{46}, \frac{1}{2} \sqrt{\alpha} \alpha e_{14} + \frac{1}{2} \alpha e_{56}, 0) \\
\ell_3 & (\frac{1}{2} \alpha e_{16}, \frac{1}{2} \alpha e_{26}, \frac{1}{2} \sqrt{\alpha} \alpha e_{12} + \frac{1}{2} \alpha e_{36}, \frac{1}{2} \sqrt{\alpha} \alpha e_{13} + \frac{1}{2} \alpha e_{46}, \frac{1}{2} \sqrt{\alpha} \alpha e_{14} + \frac{1}{2} \alpha e_{56}, 0) \\
\ell_4 & (\frac{1}{2} \alpha e_{16}, \frac{1}{2} \alpha e_{26}, \frac{1}{2} \sqrt{\alpha} \alpha e_{12} + \frac{1}{2} \alpha e_{36}, \frac{1}{2} \sqrt{\alpha} \alpha e_{13} + \frac{1}{2} \alpha e_{46}, \frac{1}{2} \sqrt{\alpha} \alpha e_{14} + \frac{1}{2} \alpha e_{56}, 0) \\
\ell_5 & (\frac{1}{2} \alpha e_{16}, \frac{1}{2} \alpha e_{26} + \frac{1}{2} \alpha e_{12} + \frac{1}{2} \alpha e_{36}, \frac{1}{2} \sqrt{\alpha} \alpha e_{13} + \frac{1}{2} \sqrt{\alpha} \alpha e_{46}, \frac{1}{2} \sqrt{\alpha} \alpha e_{14} + \frac{1}{2} \sqrt{\alpha} \alpha e_{56}, 0) \\
\ell_6 & (\frac{1}{2} \alpha e_{16}, \frac{1}{2} \alpha e_{26}, \frac{1}{2} \sqrt{\alpha} \alpha e_{12} + \frac{1}{2} \alpha e_{36}, \frac{1}{2} \sqrt{\alpha} \alpha e_{13} + \frac{1}{2} \alpha e_{46}, \frac{1}{2} \sqrt{\alpha} \alpha e_{14} + \frac{1}{2} \alpha e_{56}, 0) \\
\ell_7 & (\frac{1}{2} \alpha e_{16}, \frac{1}{2} \alpha e_{26}, \frac{1}{2} \sqrt{\alpha} \alpha e_{12} + \frac{1}{2} \alpha e_{36}, \frac{1}{2} \sqrt{\alpha} \alpha e_{13} + \frac{1}{2} \alpha e_{46}, \frac{1}{2} \sqrt{\alpha} \alpha e_{14} + \frac{1}{2} \sqrt{\alpha} \alpha e_{56}, 0) \\
\ell_8 & (\frac{1}{2} \alpha e_{16}, \frac{1}{2} \alpha e_{26}, \frac{1}{2} \sqrt{\alpha} \alpha e_{12} + \frac{1}{2} \alpha e_{36}, \frac{1}{2} \sqrt{\alpha} \alpha e_{13} + \frac{1}{2} \alpha e_{46}, \frac{1}{2} \sqrt{\alpha} \alpha e_{14} + \frac{1}{2} \sqrt{\alpha} \alpha e_{56}, 0) \\
\hline
\end{array}
\]

Table 1. Rank-one Einstein 6-dimensional solvable Lie algebras

4. **Calibrated** \( G_2 \)-**structures**

In this section we study the existence of calibrated \( G_2 \)-structures \( \varphi \) on 7-dimensional solvable Lie algebras whose underlying Riemannian metric \( g_\varphi \) is Einstein. We will use the classification of the 7-dimensional Einstein solvable Lie algebras and the following obstructions.

**Lemma 4.1.** [9] *If there is a non zero vector \( X \) in a 7-dimensional Lie algebra \( g \) such that \((i_X \varphi)^3 = 0 \) for all closed 3-form \( \varphi \in \Omega^3(g^*) \), then \( g \) does not admit any calibrated \( G_2 \)-structure.*

**Lemma 4.2.** Let \( g \) be a 7-dimensional Lie algebra and \( \varphi \) a \( G_2 \)-structure on \( g \). Then the bilinear form \( g_\varphi : g \times g \rightarrow \mathbb{R} \) defined by

\[
g_\varphi(X, Y)\text{vol} = \frac{1}{6}(i_X \varphi \wedge i_Y \varphi \wedge \varphi)
\]

has to be a Riemannian metric.

*Proof.* It follows by the fact that in general there is a 1–1 correspondence between \( G_2 \)-structures on a 7-manifold and 3-forms \( \varphi \) for which the 7-form-valued bilinear form \( B_\varphi \) defined by

\[
B_\varphi(X, Y) = (i_X \varphi \wedge i_Y \varphi \wedge \varphi)
\]

is positive definite (see [5], [15]).

**Lemma 4.3.** Let \( (s, g) \) be a 7-dimensional Einstein solvable Lie algebra endowed with a \( G_2 \)-structure \( \varphi \), then, for any \( A \in \mathfrak{a} = [s, s]^+ \) such that \( g_\varphi(A, A) = 1 \), the forms

\[
\alpha = i_A \varphi, \quad \beta = \varphi - \alpha \wedge A^*,
\]

define an \( SU(3) \)-structure on \((\mathbb{R}(A))^+\), where by \( A^* \in \mathfrak{s}^* \) we denote the dual of \( A \). So in particular \( \alpha \wedge \beta = 0 \) and \( \alpha^2 \neq 0 \).

*Proof.* It follows by Proposition 4.5 in [23].

In contrast with the almost-Kähler case, we can prove the following theorem.
Theorem 4.4. A 7-dimensional solvmanifold cannot admit any left-invariant calibrated $G_2$-structure $\varphi$ such that $g_{\varphi}$ is Einstein, unless $g_{\varphi}$ is flat. In particular, if the 7-dimensional Einstein (non-flat) solvmanifold $(S,g)$ has rank one, then $(S,g)$ has a calibrated $G_2$-structure if and only if the Lie algebra $\mathfrak{s}$ of $S$ is isometric to the Einstein solvable Lie algebras $\mathfrak{g}_1, \mathfrak{g}_4, \mathfrak{g}_9, \mathfrak{g}_{18}, \mathfrak{g}_{28}$ in Table 2.

Proof. A 7-dimensional Einstein solvable Lie algebra $(\mathfrak{s}, g)$ is necessarily standard, so one has the orthogonal decomposition (with respect to $g$)

$$\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a},$$

with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ nilpotent and $\mathfrak{a}$ abelian. We will consider separately the different cases according to the rank of $\mathfrak{s}$, i.e., to the dimension of $\mathfrak{a}$.

If $\dim \mathfrak{a} = 1$ and $\mathfrak{a}$ is abelian, then we know by [13, Proposition 6.12] that $\mathfrak{s}$ has structure equations

$$(ae^{17}, ae^{27}, ae^{37}, ae^{47}, ae^{57}, ae^{67}, 0),$$

where $a$ is a non-zero real number. Computing the generic closed 3-form on $\mathfrak{s}$ it is easy to check that $\mathfrak{s}$ cannot admit any calibrated $G_2$-structure.

If $\dim \mathfrak{a} = 1$ and $\mathfrak{a}$ is nilpotent (non-abelian), then $(\mathfrak{s}, g)$ is isometric to one of the solvable Lie algebras $\mathfrak{g}_i$, $i = 1, \ldots, 33$, in Table 2, endowed with the inner product such that the basis $\{e_1, \ldots, e_7\}$ is orthonormal. We may apply Lemma 4.1 with $X = e_6$ to all the Lie algebras $\mathfrak{g}_i$, $i = 1, \ldots, 33$, except to the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_4, \mathfrak{g}_9, \mathfrak{g}_{18}, \mathfrak{g}_{28}$, showing in this way that they do not admit any calibrated $G_2$-structure. For the remaining Lie algebras $\mathfrak{g}_1, \mathfrak{g}_4, \mathfrak{g}_9, \mathfrak{g}_{18}, \mathfrak{g}_{28}$ we first determine the generic closed 3-form $\varphi$ and then, by applying Lemma 4.3, we impose, that $\alpha \wedge \alpha \wedge \alpha \neq 0$ and $\alpha \wedge \beta = 0$, where

$$(5) \quad \alpha = i_{e_7} \varphi, \quad \beta = \varphi - e_7 \wedge \beta.$$ 

Moreover, we have that the closed 3-form $\varphi$ defines a $G_2$-structure if and only the matrix associated to the symmetric bilinear form $g_{\varphi}$, with respect to the orthonormal basis $\{e_1, \ldots, e_7\}$, is positive definite. Since the Einstein metric is unique up to scaling, a calibrated $G_2$-structure induces an Einstein metric if and only if the matrix associated to the symmetric bilinear form $g_{\varphi}$, with respect to the basis $\{e_1, \ldots, e_7\}$, is a multiple of the identity matrix. By a direct computation we have that then the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_4, \mathfrak{g}_9, \mathfrak{g}_{18}, \mathfrak{g}_{28}$ admit a calibrated $G_2$-structure (see Table 3) but they do not admit any calibrated $G_2$-structure inducing a Einstein (non-flat) metric.

Next, we show that result for the Lie algebra $\mathfrak{g}_{28}$. To this end, we see that any closed 3-form $\varphi$ on $\mathfrak{g}_{28}$ has the following expression:

$$\varphi = \rho_{1,2,7}e^{127} - \frac{1}{2} \rho_{5,6,7}e^{136} + \rho_{2,4,7}e^{137} + \frac{1}{2} \rho_{5,6,7}e^{145} - \rho_{2,3,7}e^{147} - \rho_{2,6,7}e^{157} + \rho_{2,5,7}e^{167} + \frac{1}{2} \rho_{5,6,7}e^{235} + \rho_{2,3,7}e^{237} + \frac{1}{2} \rho_{5,6,7}e^{246} + \rho_{2,4,7}e^{247} + \rho_{2,5,7}e^{257} + \rho_{2,6,7}e^{267} + \rho_{3,4,7}e^{347} + \rho_{3,5,7}e^{357} + \rho_{3,6,7}e^{367} + \rho_{3,6,7}e^{457} - \rho_{3,5,7}e^{467} + \rho_{5,6,7}e^{567},$$

where $\rho_{i,j,k}$ are arbitrary constants denoting the coefficients of $e^{ijk}$. 

In this case, one can check that the induced metric is given by the matrix $G$ with elements

\[
\begin{align*}
g(e_1, e_1) &= -\frac{1}{4}\rho_{1,2,7}\rho_{0,6,7}^2, \quad g(e_1, e_2) = 0, \quad g(e_1, e_3) = \frac{1}{2}\rho_{2,3,7}\rho_{5,6,7}^2, \\
g(e_1, e_4) &= \frac{1}{4}\rho_{2,4,7}\rho_{0,6,7}^2, \quad g(e_1, e_5) = \frac{1}{2}\rho_{2,5,6}\rho_{5,6,7}^2, \quad g(e_1, e_6) = \frac{1}{4}\rho_{2,6,7}\rho_{5,6,7}^2, \\
g(e_1, e_7) &= 0, \quad g(e_2, e_2) = -\frac{1}{4}\rho_{1,2,7}\rho_{5,6,7}^2, \quad g(e_2, e_3) = -\frac{1}{7}\rho_{2,4,7}\rho_{5,6,7}^2, \\
g(e_2, e_4) &= \frac{1}{4}\rho_{2,3,7}\rho_{0,6,7}^2, \quad g(e_2, e_5) = \frac{1}{4}\rho_{2,6,7}\rho_{5,6,7}^2, \quad g(e_2, e_6) = -\frac{1}{4}\rho_{2,5,7}\rho_{5,6,7}^2, \\
g(e_2, e_7) &= 0, \quad g(e_3, e_3) = -\frac{1}{4}\rho_{3,4,7}\rho_{5,6,7}^2, \quad g(e_3, e_4) = 0, \quad g(e_3, e_5) = \frac{1}{4}\rho_{3,6,7}\rho_{5,6,7}^2, \\
g(e_3, e_6) &= -\frac{1}{4}\rho_{3,5,7}\rho_{5,6,7}^2, \quad g(e_3, e_7) = 0, \quad g(e_4, e_4) = -\frac{1}{4}\rho_{3,4,7}\rho_{5,6,7}^2, \\
g(e_4, e_5) &= -\frac{1}{4}\rho_{3,5,7}\rho_{5,6,7}^2, \quad g(e_4, e_6) = -\frac{1}{4}\rho_{3,6,7}\rho_{5,6,7}^2, \quad g(e_4, e_7) = 0, \\
g(e_5, e_5) &= \frac{\rho_5^2}{4}, \quad g(e_5, e_6) = 0, \quad g(e_5, e_7) = 0, \quad g(e_6, e_6) = \frac{\rho_6^2}{4}, \quad g(e_6, e_7) = 0, \\
g(e_7, e_7) &= -\rho_{5,6,7}\rho_{3,7}^3 + \rho_{1,2,7}\rho_{5,6,7}^2 + \rho_{2,3,7}\rho_{5,6,7}^2 + \rho_{2,5,7}\rho_{3,4,7} + \rho_{2,6,7}\rho_{3,4,7} + \rho_{2,7,7}\rho_{3,5,7} - 2\rho_{2,6,7}\rho_{3,4,7} - 2\rho_{2,5,7}\rho_{3,4,7} + \rho_{2,6,7}\rho_{5,6,7}.
\end{align*}
\]

Now we have that the system $G = k \cdot I_G$ does not have solution, for any real number $k$, where $I_G$ is the identity matrix. This means that the Lie algebra $\mathfrak{g}_{28}$ does not admit any calibrated $G_2$-structure defining an Einstein metric. However, we can solve 48 from the 49 equations of the system $G = k \cdot I_G$, and we obtain the metric defined by the matrix

\[
G = 2 \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}
\]

Since this matrix is positive definite, the Lie algebra $\mathfrak{g}_{28}$ has a calibrated $G_2$ form

\[
\varphi = -2e^{127} - 2e^{347} - e^{136} + e^{145} + e^{235} + e^{246} + 2e^{567}
\]

which induces the metric defined by $G$.

In order to determine all the 7-dimensional rank-two Einstein solvable Lie algebras, we need first to find the rank-one Einstein solvable extensions $\mathfrak{s}_6 = \mathfrak{n}_5 \oplus \mathbb{R}\langle e_6 \rangle$ of any of the eight 5-dimensional nilpotent Lie algebras $\mathfrak{n}_5$ (see Table 1) and then consider the standard solvable Lie algebra $\mathfrak{s}_7 = \mathfrak{n}_5 \oplus \mathfrak{a}$, with $\mathfrak{a} = \mathbb{R}\langle e_6, e_7 \rangle$ abelian and such that the basis $\{e_1, \ldots, e_7\}$ is orthonormal. From $\mathfrak{t}_1$ we get the 7-dimensional Einstein Lie algebra of rank two with structure equations

\[
\begin{align*}
de^1 &= \frac{1}{4}\sqrt{6}ae^{16} + 4ae^{17}, \\
de^2 &= \frac{1}{2}\sqrt{6}ae^{26} - 7ae^{27}, \\
de^3 &= \frac{5}{3}\sqrt{18}ae^{12} + \frac{11}{16}\sqrt{6}ae^{36} - 3ae^{37}, \\
de^4 &= \frac{10}{3}\sqrt{6}ae^{13} + \frac{13}{6}\sqrt{6}ae^{46} + ae^{47}, \\
de^5 &= \frac{5}{3}\sqrt{18}ae^{14} + \frac{5}{2}\sqrt{6}ae^{56} + 5ae^{57}, \\
de^6 &= de^7 = 0.
\end{align*}
\]
By computing the generic closed 3-form $\varphi$ and by using Lemma 4.2 and Lemma 4.3 we get that the matrix associated to $g_\varphi$, with respect to the basis $\{e_1, \ldots, e_7\}$, cannot be a multiple of the identity matrix.

From $\mathfrak{f}_2$ we do not get any 7-dimensional Einstein Lie algebra of rank two. From $\mathfrak{f}_3$ we get the 7-dimensional Einstein Lie algebra of rank two with structure equations

$$d e^1 = \frac{1}{7} \sqrt{21} a e^{16} - ae^{17},$$
$$d e^2 = \frac{1}{7} \sqrt{21} a e^{26} + 2ae^{27},$$
$$d e^3 = \frac{2}{7} \sqrt{21} a e^{36} - 2ae^{47},$$
$$d e^4 = \frac{2}{21} \sqrt{30} \sqrt{21} a e^{12} + \frac{1}{3} \sqrt{21} ae^{46} + ae^{47},$$
$$d e^5 = \frac{2}{21} \sqrt{30} \sqrt{21} a e^{14} + \frac{2}{21} \sqrt{15} \sqrt{21} ae^{23} + \frac{19}{21} \sqrt{21} ae^{56},$$
$$d e^6 = d e^7 = 0.$$

By computing the generic closed 3-form $\varphi$ and by using Lemma 4.2 and Lemma 4.3 we get that the matrix associated to $g_\varphi$, with respect to the basis $\{e_1, \ldots, e_7\}$, cannot be a multiple of the identity matrix. From $\mathfrak{f}_4$ we get the 7-dimensional Einstein Lie algebras of rank two with structure equations

$$d e^1 = ae^{16} - b_7 e^{17} + b_2 e^{27} + b_3 e^{37} + b_4 e^{47},$$
$$d e^2 = ae^{26} + b_2 e^{17} + b_7 e^{27} + b_4 e^{37} - b_3 e^{47},$$
$$d e^3 = ae^{36} + b_3 e^{17} + b_4 e^{27} - b_19 e^{37} + b_{14} e^{47},$$
$$d e^4 = ae^{46} + b_4 e^{17} - b_3 e^{27} + b_{14} e^{37} + b_{19} e^{47},$$
$$d e^5 = 2ae^{12} + e^{34} + e^{56},$$
$$d e^6 = d e^7 = 0$$

where $a = \frac{1}{7} \sqrt{b_1^2 + b_2^2 + 2b_3^2 + 2b_4^2 + b_{14}^2 + b_{19}^2}$. We may then apply Lemma 4.1 with $X = e_5$.

From $\mathfrak{f}_5$ we get the 7-dimensional Einstein Lie algebras of rank two with structure equations

$$d e^1 = ae^{16} + b_{19} e^{17} + b_{20} e^{27},$$
$$d e^2 = ae^{26} + b_{20} e^{17} - b_{19} e^{27},$$
$$d e^3 = 4ae^{12} + 2ae^{36},$$
$$d e^4 = 2\sqrt{3} ae^{13} + 3ae^{46} + b_{19} e^{47} + b_{20} e^{57},$$
$$d e^5 = 2\sqrt{3} ae^{23} + 3ae^{56} + b_{20} e^{47} - b_{19} e^{57},$$
$$d e^6 = d e^7 = 0.$$

For these Lie algebras in order to study completely all possibilities we will study separately the two cases $b_{20} \neq 0$ and $b_{20} = 0$. By computing the generic closed 3-form $\varphi$ and using Lemma 4.3 we have that the system $g_\varphi(e_i, e_j) - k\delta^i_j = 0$ (where $k$ is a non zero positive real number) with variables the coefficients $c_{ijk}$ of $e^{ijk}$ in $\varphi$ has no solutions.
From $\mathfrak{t}_6$ we get the two families of 7-dimensional Einstein Lie algebras of rank two, namely $\mathfrak{t}_{6,1}$ and $\mathfrak{t}_{6,2}$ with respectively structure equations

$$
\begin{align*}
1) & \quad \left\{ \begin{array}{l}
de^1 = 2ae^{16} + 2(b_{19} + b_{25})e^{17}, \\
de^2 = 3ae^{26} - (b_{19} + 2b_{25})e^{27} + b_{12}e^{37}, \\
de^3 = 3ae^{36} + b_{12}e^{27} - (b_{19} + 2b_{25})e^{37}, \\
de^4 = 6ae^{12} + 5ae^{46} + b_{19}e^{47} + b_{12}e^{57}, \\
de^5 = 6ae^{13} + 5ae^{56} + b_{12}e^{47} + b_{25}e^{57}, \\
de^6 = de^7 = 0.
\end{array} \right. \\
\end{align*}
$$

and

$$
\begin{align*}
2) & \quad \left\{ \begin{array}{l}
de^1 = \sqrt{2}b_{25}e^{16} + 4b_{25}e^{17}, \\
de^2 = \frac{\sqrt{2}}{2}b_{25}e^{26} - 3b_{25}e^{27} - b_{12}e^{37}, \\
de^3 = \frac{\sqrt{2}}{2}b_{25}e^{36} + b_{12}e^{27} - 3b_{25}e^{37}, \\
de^4 = 3\sqrt{2}b_{25}e^{12} + \frac{\sqrt{2}}{2}b_{25}e^{46} + b_{25}e^{47} - b_{12}e^{57}, \\
de^5 = 3\sqrt{2}b_{25}e^{13} + \frac{\sqrt{2}}{2}b_{25}e^{56} + b_{12}e^{47} + b_{25}e^{57}, \\
de^6 = de^7 = 0.
\end{array} \right. \\
\end{align*}
$$

For 1) we compute first the generic closed 3-forms $\varphi$ and then, using Lemma 4.3 for $A = e_7$, we impose the condition $\alpha \wedge \beta = 0$. By this condition we get in particular that $\rho_{1,2,3} = 0$,

$$
\rho_{1,2,3} = 0,
$$

where by $\rho_{1,3,k}$ we denote the coefficient of $e^{ijk}$ in $\varphi$. One can immediately exclude the cases $\rho_{1,3,4} = 0$, since otherwise the element of the matrix associated to the metric $g_2$ has to be zero. Then we study separately the cases $\rho_{1,2,3} = 0$ and $b_{19} = -b_{25}$. In both cases we do not find any solution for the system $g_2(e_i, e_j) - k\delta_i^j = 0$.

For 2) we study separately the cases $b_{12}b_{25} \neq 0$, $b_{12} = 0$ and $b_{25} = 0$.

In the case $b_{12}b_{25} \neq 0$ we compute first the generic closed 3-forms $\varphi$ and then, using Lemma 4.3 for $A = e_7$, we impose the condition $\alpha \wedge \alpha \wedge \alpha \neq 0$, getting the condition $\rho_{1,2,5} \neq 0$. Thus, we take the system $S_{ij} = g_2(e_i, e_j) - k\delta_i^j = 0$ and get the values of $\rho_{2,3,4,6}, \rho_{3,4,5}, \rho_{3,5,6}$ and $\rho_{2,3,6}$ from $S_{5,5}, S_{3,5}, S_{4,4}$ and $S_{3,4}$. Now $S_{3,3} = -k$, and the system does not admit any solution.

In the case $b_{12} = 0$ we first compute the generic closed 3-forms $\varphi$ and then we use Lemma 4.3 for $A = e_7$, obtaining that $\rho_{2,3,6}, \rho_{2,4,6}, \rho_{3,5,6}$ and $\rho_{4,5,6}$ are all different from zero.

In the case $b_{25} = 0$ we first compute the generic closed 3-forms $\varphi$ and then we may apply Lemma 4.3 with $X = e_1, \ldots, e_5$.

From $\mathfrak{t}_7$ we get the four families of 7-dimensional Einstein Lie algebras of rank two, namely $\mathfrak{t}_{7,1}, \mathfrak{t}_{7,2}, \mathfrak{t}_{7,3}$ and $\mathfrak{t}_{7,4}$ with respectively structure equations

$$
\begin{align*}
i) & \quad \left\{ \begin{array}{l}
de^1 = ae^{16} + (-b_7 + b_{13})e^{17} + b_{6c}e^{27}, \\
de^2 = ae^{26} + b_{6c}e^{17} + b_7e^{27}, \\
de^3 = a(\sqrt{3}e^{12} + 2e^{36}) + b_{13}e^{37}, \\
de^4 = \frac{3}{2}ae^{46} - (2b_{13} + b_{25})e^{47} + b_{24}e^{57}, \\
de^5 = \frac{3}{2}ae^{56} + b_{24}e^{47} + b_{25}e^{57}, \\
de^6 = de^7 = 0.
\end{array} \right. \\
\end{align*}
$$
where \( a = \frac{2}{27} \sqrt{21b_4^2 - 21b_7f_{13} + 63b_{13}^2 + 21b_6^2 + 42b_{25}b_{13} + 21b_2^2 + 21b_{21}^2} \),

\[
\left\{
\begin{align*}
\text{de}^1 &= a e^{16} + (-b_7 + b_{13}) e^{17} + b_6 e^{27}, \\
\text{de}^2 &= a e^{26} + b_6 e^{17} + b_7 e^{27}, \\
\text{de}^3 &= a(\sqrt{7} e^{12} + 2 e^{36}) + b_{13} e^{37}, \\
\text{de}^4 &= \frac{4}{3} a e^{16} - b_{13} e^{47} - b_{24} e^{57}, \\
\text{de}^5 &= \frac{2}{3} a e^{56} + b_{24} e^{47} - b_{13} e^{57}, \\
\text{de}^6 &= \text{de}^7 = 0.
\end{align*}
\]

where \( a = \frac{2}{27} \sqrt{21b_4^2 - 21b_7b_{13} + 42b_{13}^2 + 21b_{13}^2} \),

\[
\left\{
\begin{align*}
\text{de}^1 &= a e^{16} + \frac{2}{3} b_{13} e^{17} - b_6 e^{27}, \\
\text{de}^2 &= a e^{26} + b_6 e^{17} + \frac{2}{3} b_{13} e^{27}, \\
\text{de}^3 &= a(\sqrt{7} e^{12} + 2 e^{36}) + b_{13} e^{37}, \\
\text{de}^4 &= \frac{4}{3} a e^{16} - (2b_{13} + b_{25}) e^{47} + b_{24} e^{57}, \\
\text{de}^5 &= \frac{2}{3} a e^{56} + b_{24} e^{47} + b_{25} e^{57}, \\
\text{de}^6 &= \text{de}^7 = 0.
\end{align*}
\]

where \( a = \frac{1}{27} \sqrt{231b_4^2 + 168b_{13}b_{25} + 84b_{25}^2 + 84b_{24}^2} \),

\[
\left\{
\begin{align*}
\text{de}^1 &= \frac{1}{4} \sqrt{3} b_{25} e^{16} - \frac{1}{2} b_{25} e^{17} - b_6 e^{27}, \\
\text{de}^2 &= \frac{1}{4} \sqrt{3} b_{25} e^{25} + b_6 e^{17} - \frac{1}{2} b_{25} e^{27}, \\
\text{de}^3 &= \frac{1}{4} \sqrt{3} b_{25}(\sqrt{7} e^{12} + 2 e^{36}) - b_{25} e^{37}, \\
\text{de}^4 &= \frac{1}{4} \sqrt{3} b_{25} e^{46} + b_{25} e^{47} - b_{24} e^{57}, \\
\text{de}^5 &= \frac{1}{4} \sqrt{3} b_{25} e^{56} + b_{24} e^{47} + b_{25} e^{57}, \\
\text{de}^6 &= \text{de}^7 = 0.
\end{align*}
\]

For all of them after computing the generic closed 3-forms we may apply Lemma 4.1 with \( X = e_3 \).

From \( t_8 \) we get the 7-dimensional Einstein Lie algebras of rank two with structure equations

\[
\left\{
\begin{align*}
\text{de}^1 &= a e^{16} + (-b_{13} + b_{19}) e^{17}, \\
\text{de}^2 &= a e^{26} + (2b_{13} - b_{19}) e^{27}, \\
\text{de}^3 &= a(\sqrt{26} e^{12} + 3 e^{36}) + b_{13} e^{37}, \\
\text{de}^4 &= a(\sqrt{26} e^{13} + 4 e^{46}) + b_{19} e^{47}, \\
\text{de}^5 &= 3 a e^{56} - (2b_{13} + b_{19}) e^{57}, \\
\text{de}^6 &= \text{de}^7 = 0.
\end{align*}
\]

where \( a = \frac{1}{39} \sqrt{390b_{13}^2 - 78b_{13}b_{19} + 156b_{19}^2} \).

We study separately the cases \( b_{13}b_{19} \neq 0, b_{13} = 0 \) and \( b_{19} = 0 \).

In the case \( b_{13}b_{19} \neq 0 \) using Lemma 4.3 (i.e. \( \alpha \wedge \alpha \wedge \alpha \neq 0 \), with \( A = e_7 \)) we may suppose \( \rho_{1,2,4,\rho_{1,3,5}} \neq 0 \) for the generic closed 3-form. Now, we consider the system \( S_{ij} = g(e_i, e_j) - \delta_{ij} = 0 \). We take \( \rho_{1,2,5}, \rho_{1,2,3} \) and \( a \) from \( S_{4,5} = 0 = S_{3,4} \) and \( S_{2,2} = 0 \). In the new system we can conclude that from equations \( S_{2,4} = 0 \) and \( S_{3,5} = 0 \) that there is no solution. Indeed from the two equations it follows that \( b_{13} = -\frac{1}{4} b_{19} \) and \( b_{13} = -\frac{1}{4} b_{19} \), which is a contradiction since \( b_{13}b_{19} \neq 0 \).

In the case \( b_{13} = 0 \) using Lemma 4.3 we may suppose \( \rho_{1,3,5,\rho_{3,4,7}} \neq 0 \) for the generic closed 3-form. Then we get \( \rho_{2,5,7}, k \) and \( \rho_{2,3,7} \) from \( S_{5,5} = 0 = S_{5,3} = S_{2,3} \). The
new system satisfies:

\[ S_{4,4} = \frac{7 \rho_{1,3,5} \left( 49 \rho_{1,2,5}^2 + 152 \rho_{1,4,7}^2 \right)}{76 \sqrt{78}} \]

In the case \( f_{19} = 0 \) using Lemma 4.4, we may suppose \( c_{1,3,5}c_{3,4,7} \neq 0 \) for the generic closed 3-form. Then we get \( c_{1,2,3}, c_{2,5,6}, c_{2,3,7} \) and \( c_{1,3,5} \) from \( S_{3,4} = 0 = S_{3,3} = S_{3,4} \) and \( S_{3,3} = 0 \). The new system satisfies:

\[ S_{4,4} = -\frac{959322 \rho_{1,2,5}^2 c_{3,4,7}^4 + 59711 k^2}{59711 k} \]

what implies again \( S_{4,4} \neq 0 \).

If \([s, s] = n\) is abelian and \( \dim n = 5 \), we have to consider the structure equations

\[
\begin{align*}
d e^1 &= a e^{16} + b_1 e^{17}, \\
d e^2 &= a e^{26} + b_2 e^{27}, \\
d e^3 &= a e^{36} + b_3 e^{37}, \\
d e^4 &= a e^{46} + b_4 e^{47}, \\
d e^5 &= a e^{56} + b_5 e^{57}, \\
d e^6 &= d e^7 = 0.
\end{align*}
\]

By imposing that \( s \) is a Einstein Lie algebra (the inner product is the one for which \( \{e_1, \ldots, e_7\} \) is orthonormal), we get the Lie algebras with structure equations

\[
\begin{align*}
d e^1 &= a e^{16} + (-b_2 - b_3 - b_4) e^{17}, \\
d e^2 &= a e^{26} + b_2 e^{27}, \\
d e^3 &= a e^{36} + b_3 e^{37}, \\
d e^4 &= a e^{46} + b_4 e^{47}, \\
d e^5 &= a e^{56}, \\
d e^6 &= d e^7 = 0,
\end{align*}
\]

where \( a = \sqrt{10 b_7^2 + 10 b_7 b_{13} + 10 b_7 b_{19} + 10 b_{13} b_{19} + 10 b_{15}} \). For these Lie algebras we first compute the generic closed 3-forms \( \varphi \) and then we may apply Lemma 4.1 with \( X = e_1, \ldots, e_5 \).

In order to determine all the 7-dimensional rank-three Einstein solvable Lie algebras, we need first to find the rank-one Einstein solvable extensions \( n_4 \oplus \mathbb{R}(e_5) \) of the two 4-dimensional nilpotent Lie algebras \( n_4 \) and then consider the standard solvable Lie algebra \( s_7 = n_4 \oplus a \) with \( a = \mathbb{R}(e_5, e_6, e_7) \) abelian and such that the basis \( \{e_1, \ldots, e_7\} \) is orthonormal. For any of the nilpotent Lie algebras \( n_4 \) we find the following rank-one Einstein solvable extensions

1) \( (\frac{1}{2} a e^{15}, \frac{1}{2} a e^{25}, \frac{1}{2} \sqrt{5} a e^{12} + \frac{1}{2} a e^{35}, \frac{1}{2} \sqrt{5} a e^{13} + a e^{45}, 0) \)

2) \( (\frac{1}{2} a e^{15}, \frac{1}{2} a e^{25}, \frac{1}{2} \sqrt{2} a e^{12} + a e^{35}, \frac{1}{2} a e^{45}, 0) \)

From 1) we do not get any 7-dimensional Einstein Lie algebra of rank three. From 2) we get the 7-dimensional Einstein Lie algebra of rank three with structure equations

\[
\begin{align*}
d e^1 &= \frac{1}{2} a e^{15} - (b_{10} + \frac{1}{2} b_{28}) e^{16} + b_2 e^{26} + (-b_{14} + b_{23}) e^{17} + b_6 e^{27}, \\
d e^2 &= \frac{1}{2} a e^{25} + b_6 e^{16} + b_{10} e^{26} + b_{13} e^{17} + b_{14} e^{27}, \\
d e^3 &= \frac{1}{2} \sqrt{2} a e^{12} + a e^{35} - \frac{1}{2} b_{28} e^{36} + b_{23} e^{37}, \\
d e^4 &= \frac{1}{2} a e^{45} + b_{28} e^{46} - 2 b_{23} e^{47}, \\
d e^5 &= d e^6 = d e^7 = 0.
\end{align*}
\]
satisfying the conditions $\frac{1}{2} d^2 e^i = 0, i = 1, \ldots, 4$, and one of the following:

(i) $a = \sqrt{\frac{32b_1^2 - 32b_1b_2 + 96b_2^2 + 32b_3^2}{33}},\ b_2 = b_9 = \pm \sqrt{\frac{11}{4b_1^4 - 4b_1b_2b_3 + b_2b_3 + 4b_3^2}}b_{13}b_{23},\ b_6 = b_{13}, b_{10} = \pm \frac{1}{11}\sqrt{\frac{11}{4b_1^4 - 4b_1b_2b_3 + b_2b_3 + 4b_3^2}}(-13b_{14}b_{23} + 6b_2^2 + 2b_4^2 + 2b_5^2),\ b_{28} = \pm \frac{2}{11}\sqrt{\frac{11}{4b_1^4 - 4b_1b_2b_3 + b_2b_3 + 4b_3^2}}(4b_{14}^2 - 4b_1b_2b_3 + b_2^2 + 4b_3^2)\ b_6 = b_{13} = b_{28} = 0, b_{14} = \frac{1}{2}b_{23}$. For the case (i) we consider the generic closed 3-form $\varphi$ and use all the time that $a \neq 0$ and the condition $\alpha \wedge \beta = 0$ (with $\alpha = i\varphi$). By imposing the vanishing of $g_\varphi(e_3, e_4)$ and the condition $g_\varphi(e_3, e_3) \neq 0$ we have always that either $g_\varphi(e_2, e_2) = 0$ or $g_\varphi(e_3, e_3) = 0$ so we cannot have calibrated $G_2$-structures associated to the Einstein metric.

For the case (ii) we start only to impose the conditions

$$a = \frac{2}{3}\sqrt{6}b_{23},\ b_6 = b_{13} = b_{28} = 0,\ b_{14} = \frac{1}{2}b_{23},$$

one needs for the Einstein condition still to impose that $b_2 = \pm \frac{1}{2}\sqrt{11b_{23}^2 - 4b_{10}^2}$. We consider the generic closed 3-form $\varphi$, using all the time that $b_{23} \neq 0$ (since $a \neq 0$) and we impose $\alpha \wedge \beta = 0$, where $\alpha = i\varphi$. We use that $g_\varphi(e_3, e_3) \neq 0$ and the equations

$$g_\varphi(e_3, e_4) = g_\varphi(e_2, e_4) = g_\varphi(e_1, e_3) = 0.$$

Studying separately the solutions of the above system we show that no calibrated $G_2$-structure can induce the Einstein metric.

If $n = [\mathfrak{s}, \mathfrak{s}]$ is abelian and $\text{dim } n = 4$, we get the Einstein solvable Lie algebras of rank three with the structure equations

$$\left\{\begin{array}{l}
\text{de}^1 = ae^{15} + b_1e^{16} + b_2e^{17}, \\
\text{de}^2 = ae^{25} + b_3e^{26} + b_4e^{27}, \\
\text{de}^3 = ae^{35} + b_5e^{36} + b_6e^{37}, \\
\text{de}^4 = ae^{45} + b_7e^{46} + b_8e^{47}, \\
\text{de}^5 = de^6 = de^7 = 0,
\end{array}\right.
$$

satisfying the conditions

$$b_7 = -b_1 - b_3 - b_5,\ b_8 = -b_2 - b_4 - b_6,$$

$$2b_1b_2 + 2b_3b_4 + 2b_5b_6 + b_2b_3 + b_2b_5 + b_1b_4 + b_4b_5 + b_1b_6 + b_3b_6 = 0,$$

$$2b_1^2 + 2b_3^2 + 2b_5^2 + 2b_1b_3 + 2b_1b_5 + 2b_3b_5 = 4a^2.$$

We impose for the generic 3-form $\varphi$ the conditions $d\varphi = 0$ and $\alpha \wedge \beta = 0$ (with $\alpha = i\varphi$), using all the time that $a \neq 0$. Then we consider the equations

$$g_\varphi(e_1, e_2) = g_\varphi(e_1, e_3) = g_\varphi(e_1, e_4) = g_\varphi(e_2, e_3) = 0.$$

By the conditions

$$g_\varphi(e_1, e_1) \neq 0,\ g_\varphi(e_2, e_2) \neq 0,\ g_\varphi(e_3, e_3) \neq 0,\ g_\varphi(e_4, e_4) \neq 0$$

we get respectively

$$\rho_{1, 2, 3, 4} \neq 0, \rho_{1, 2, 3, 5, 7} \neq 0, \rho_{1, 3, 5} \neq 0, \rho_{1, 4, 5} \neq 0, \rho_{1, 2, 3, 5} \neq 0, \rho_{1, 3, 5} \neq 0, \rho_{1, 4, 5} \neq 0.$$

$$\rho_{1, 2, 3, 5, 7} \neq 0, \rho_{1, 3, 5} \neq 0, \rho_{1, 4, 5} \neq 0, \rho_{1, 2, 3, 5} \neq 0, \rho_{1, 3, 5} \neq 0, \rho_{1, 4, 5} \neq 0.$$
In all the solutions of \( [3] \) we have always that either \( \rho_{1,2,5} = 0 \) or \( \rho_{1,3,5} = 0 \) or \( \rho_{1,4,5} = 0 \), which is not possible.

So, now we can conclude that there is not a counterexample for the \( G_2 \) analogue of the Goldberg conjecture in the class of solvmanifolds, but we can show the existence of a calibrated \( G_2 \)-structure whose underlying metric is a (non trivial) Ricci soliton.

**Example.** If we take the 6-dimensional nilpotent Lie algebra \( \mathfrak{n}_6 = (0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23}) \) and we compute the rank-one Ricci soliton solvable extension, we obtain the Lie algebra with structure equations:

\[
\mathfrak{s} = \left( -\frac{1}{2} e^{17}, -\frac{1}{2} e^{27}, e^{37}, e^{47}, -e^{24} - \frac{1}{2} e^{57}, e^{14} + e^{23} - \frac{1}{2} e^{67}, 0 \right)
\]

where the Ricci soliton metric \( g \) is the one making the basis \( \{e_1, \ldots, e_7\} \) orthonormal. The Lie algebra \( \mathfrak{s} \) admits the calibrated \( G_2 \)-form \( \varphi = -e^{136} + e^{145} + e^{235} + e^{246} + e^{567} - e^{127} - e^{347} \) such that \( g_\varphi = g \) and \( d * \varphi \neq 0 \). Therefore \( g_\varphi \) is a Ricci-soliton on \( \mathfrak{s} \) but \( \varphi \) is not parallel.

5. **Cocalibrated \( G_2 \)-structures**

In this section we study the existence of cocalibrated \( G_2 \)-structures \( \varphi \) on 7-dimensional Einstein solvable Lie algebras \( (\mathfrak{s}, g) \) whose underlying Riemannian metric \( g_\varphi = g \).

We will use the classification of the 7-dimensional Einstein solvable Lie algebras of the previous section, Lemma 4.2 and 4.3, together with the following obstructions.

**Lemma 5.1.** Let \( (\mathfrak{g}, g) \) be a 7-dimensional metric Lie algebra. If for every closed 4-form \( \Psi \in Z^4(\mathfrak{g}^*) \) there exists \( X \in \mathfrak{g} \) such that \((i_X(*\Psi))^3 = 0\), then \( \mathfrak{g} \) does not admit a cocalibrated \( G_2 \)-structure inducing the metric \( g \).

**Proof.** It is sufficient to prove that if a 3-form \( \varphi \) defines a \( G_2 \)-structure on \( \mathfrak{g} \) then for any \( X \in \mathfrak{g} \) we have

\[
i_X(*\varphi) \wedge \varphi \neq 0,
\]

where * is the Hodge star operator with respect to the metric \( g_\varphi \) associated to \( \varphi \). Since the 3-form \( \varphi \) defines a \( G_2 \)-structure on \( \mathfrak{g} \), then there exists a basis of \( \mathfrak{g} \) \( \{f_1, \cdots, f_7\} \) (which is orthonormal with respect to \( g_\varphi \)) such that

\[
\varphi = f^{124} + f^{235} + f^{346} + f^{457} + f^{156} + f^{267} + f^{137},
\]

where \( \{f^1, \cdots, f^7\} \) is the basis of \( \mathfrak{g}^* \) dual to \( \{f_1, \cdots, f_7\} \). Taking the Hodge star operator with respect to \( \{f_1, \cdots, f_7\} \) we have

\[
*\varphi = -f^{3567} + f^{1467} - f^{1257} + f^{1236} + f^{2347} - f^{1345} - f^{2456}.
\]

Thus, writing \( i_X \) the contraction by \( X \),

\[
i_{f_1}(*\varphi) = f^{467} - f^{257} + f^{236} - f^{345}
\]

and

\[
i_{f_1}(*\varphi) \wedge \varphi = 4 f^{234567}.
\]
Similarly, we have

\[
\begin{align*}
\iota_{f_2}(\ast\varphi) &= f^{157} - f^{136} + f^{447} - f^{456}, \quad \iota_{f_3}(\ast\varphi) \wedge \varphi = -4f^{134567}, \\
\iota_{f_3}(\ast\varphi) &= f^{357} + f^{126} - f^{247} + f^{145}, \quad \iota_{f_3}(\ast\varphi) \wedge \varphi = 4f^{124567}, \\
\iota_{f_4}(\ast\varphi) &= -f^{167} + f^{237} - f^{135} + f^{256}, \quad \iota_{f_4}(\ast\varphi) \wedge \varphi = -4f^{123567}, \\
\iota_{f_5}(\ast\varphi) &= f^{367} - f^{127} + f^{134} - f^{240}, \quad \iota_{f_5}(\ast\varphi) \wedge \varphi = 4f^{123467}, \\
\iota_{f_6}(\ast\varphi) &= -f^{357} + f^{147} - f^{123} + f^{245}, \quad \iota_{f_6}(\ast\varphi) \wedge \varphi = -4f^{123457}, \\
\iota_{f_7}(\ast\varphi) &= f^{156} - f^{146} + f^{125} - f^{234}, \quad \iota_{f_7}(\ast\varphi) \wedge \varphi = 4f^{123456}.
\end{align*}
\]

In general, for \(i = 1, \ldots, 7\), we see that

\[\iota_{f_i}(\ast\varphi) \wedge \varphi = (-1)^i 4^2f^{12\cdots(i-1)(i+1)\cdots7},\]

which is a non-zero 6-form for any \(i = 1, \ldots, 7\).

\[\Box\]

**Lemma 5.2.** Let \((\mathfrak{g}, g)\) be a 7-dimensional metric Lie algebra. If for any coclosed 3-form \(\varphi\) on \(\mathfrak{g}\), the differential form \(\tau_3 = \ast d\varphi|_{\Lambda^3_7\mathfrak{g}^*}\) satisfies the conditions

\[\varphi \wedge \tau_3 \neq 0 \text{ or } (\ast\varphi) \wedge \tau_3 \neq 0\]

then \(\mathfrak{g}\) does not admit a cocalibrated \(G_2\)-structure inducing the metric \(g\).

**Proof.** The expression of the differential and the codifferential of a \(G_2\) form \(\varphi\) are given in terms of the intrinsic torsion forms by

\[
d\varphi = \tau_0 \ast \varphi + 3\tau_1 \wedge \varphi + \ast\tau_3,
\]

\[
d \ast \varphi = 4\tau_1 \wedge \ast \varphi + \tau_2 \wedge \varphi.
\]

with \(\tau_0 \in \Lambda^0\mathfrak{g}^*, \tau_1 \in \Lambda^1\mathfrak{g}^*, \tau_2 \in \Lambda^2\mathfrak{g}^*, \) and \(\tau_3 \in \Lambda^3\mathfrak{g}^*.\) The cocalibrated condition \(d \ast \varphi = 0\) implies

\[d\varphi = \tau_0 \ast \varphi + \ast\tau_3.\]

Since

\[
\Lambda^2_7\mathfrak{g}^* = \{\rho \in \Lambda^3\mathfrak{g}^*|\rho \wedge \varphi = 0 = \rho \wedge \ast\varphi\},
\]

\[
\Lambda^3_7\mathfrak{g}^* = \{\gamma \in \Lambda^4\mathfrak{g}^*|\gamma \wedge \varphi = 0 = \gamma \wedge \ast\varphi\}
\]

it follows that

\[d\varphi \wedge \varphi = \tau_0|\varphi|^2 e^{1234567}.\]

Therefore

\[\tau_3 = - (d\varphi - \tau_0 \ast \varphi).
\]

Now as \(\tau_3 \in \Lambda^3\mathfrak{g}^*\) the conditions

\[\tau_3 \wedge \varphi = 0, \quad \tau_3 \wedge \ast\varphi = 0\]

must be fulfilled. \[\Box\]

We recall that a 5-dimensional manifold \(N\) has an \(SU(2)\)-structure if there exists a quadruplet \((\eta, \omega_1, \omega_2, \omega_3)\), where \(\eta\) is a 1-form and \(\omega_i\) are 2-forms on \(N\) satisfying \(\omega_i \wedge \omega_j = \delta_{ij} v, v \wedge \eta \neq 0\) for some nowhere vanishing 4-form \(v\), and

\[\iota_X\omega_3 = \iota_Y\omega_1 \implies \omega_2(X, Y) \geq 0.
\]

**Proposition 5.3.** Let \((\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}, g)\) be a 7-dimensional Einstein Lie algebra of rank two and let \(\{e_1, \ldots, e_7\}\) be an orthonormal basis of \((\mathfrak{g}, g)\) such that \(\mathfrak{a} = \mathbb{R}\langle e_6, e_7\rangle\), then a \(G_2\)-structure \(\varphi\) on \(\mathfrak{g}\) induces an \(SU(2)\)-structure on \(\mathfrak{n}\) such that the associated metric \(h\) is the restriction of \(g_{\varphi}\) to \(\mathfrak{n}\).
Proof. By Lemma 4.3 we know that the forms $F = e_6 \varphi$, $\psi_+ = \varphi - F \wedge e^7$ determine an $SU(3)$-structure on $\mathbb{R}(e_1, \ldots, e_6)$ and that the associated metric is the restriction of $g$ to $\mathbb{R}(e_1, \ldots, e_6)$. Now we can write $F = f^{12} + f^{34} + f^5 \wedge e^6$ and $\psi_+ + \psi_- = (f^1 + if^2) \wedge (f^3 + if^4) \wedge (f^5 + we^6)$ where $f_i \in \mathbb{R}(e_1, \ldots, e_5)$ and $\{f_1, \ldots, f_5, e_6\}$ is orthonormal. Then by [10, Proposition 1.4] the forms

$$\eta = f^5, \quad \omega_1 = f^{12} + f^{34}, \quad \omega_2 = f^{13} + f^{42}, \quad \omega_3 = f^{14} + f^{23}$$

define an $SU(2)$-structure on $n$. The basis $\{f_1, \ldots, f_5\}$ is orthonormal with respect to the metric $h$ induced by the $SU(2)$-structure. So, $h$ coincides with the restriction of $g_\varphi$ to $n$. □

Corollary 5.4. Let $(g = n \oplus a, g)$ be a 7-dimensional Einstein Lie algebra of rank two and let $\{e_1, \ldots, e_7\}$ be an orthonormal basis such that $a = \mathbb{R}(e_6, e_7)$. If for any coclosed 3-form $\varphi$ one of the following conditions

- $(\varphi_i^2 - \varphi_j^2) \wedge \eta \neq 0$ for some $i \neq j$;
- $\omega_i \wedge \eta \neq *_h \omega_i$ for some $i$, holds, where $(\varphi_1, \varphi_2, \varphi_3, \eta, h)$ is the $SU(2)$-structure as in Proposition 5.3, then $(\mathfrak{g}, \mathfrak{g})$ does not admit any cocalibrated $G_2$-structure $\varphi$ such that $g_\varphi = g$.

Proof. By Proposition 5.4 the $G_2$-structure induces an $SU(2)$-structure $(\varphi_1, \varphi_2, \varphi_3, \eta)$ on $n$. By definition of $SU(2)$-structure the forms $(\omega_1, \omega_2, \omega_3, \eta)$ have to satisfy the conditions $(\omega_i^2 - \omega_j^2) \wedge \eta \neq 0$ for all $i, j$ and $\omega_i \wedge \eta = *_h \omega_i$ for all $i = 1, 2, 3$. □

We know already that a 7-dimensional Enstein solvable Lie algebra cannot admit nearly-parallel $G_2$-structures since the scalar curvature has to be positive. For the cocalibrated $G_2$-structures we can prove the following

Theorem 5.5. A 7-dimensional (nonflat) Einstein solvmanifold $(S, g)$ cannot admit any left-invariant cocalibrated $G_2$-structure $\varphi$ such that $g_\varphi = g$.

Proof. For a 7-dimensional rank-one Einstein solvable Lie algebra $(\mathfrak{s}, g)$ we have the orthogonal decomposition (with respect to the Einstein metric $g$)

$$\mathfrak{s} = \mathfrak{n}_6 \oplus a,$$

with $\mathfrak{n}_6 = [\mathfrak{s}, \mathfrak{s}]$ a 6-dimensional nilpotent Lie algebra and $a = \mathbb{R}(e_7)$ abelian. If $\mathfrak{n}$ is abelian, then we know by [13, Proposition 6.12] that $\mathfrak{s}$ has structure equations

$$(ae^{17}, ae^{27}, ae^{37}, ae^{47}, ae^{57}, ae^{67}, 0),$$

where $a$ is a non-zero real number. Computing the generic co-closed 4-form on $\mathfrak{s}$ it easy to check that $\mathfrak{s}$ cannot admit any cocalibrated $G_2$-structure $g_\varphi$ such that $g_\varphi = g$.

If $\mathfrak{n}_6$ is nilpotent (non-abelian), then $(\mathfrak{s}, g)$ is isometric to one of the solvable Lie algebras $\mathfrak{g}_i, i = 1, \ldots, 33$, in Table 2, endowed with the Riemannian metric such that the basis $\{e_1, \ldots, e_7\}$ is orthonormal. We may apply Lemma 5.3 with $X = e_7$ to the Lie algebras $\mathfrak{g}_3$, $\mathfrak{g}_{13}$, $\mathfrak{g}_{23}$ and $\mathfrak{g}_j$, $25 \leq j \leq 33$, showing in this way that they do not admit any cocalibrated $G_2$-structure $\varphi$ such that $g_\varphi = g$. For the Lie algebras:

$$\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_4, \mathfrak{g}_5, \mathfrak{g}_6, \mathfrak{g}_20$$

we first determine the generic co-closed 3-form $\varphi$ and then, we compute the values of $\tau_0$, and of the 3-form $\tau_3$. We have that $\tau_3 \wedge \varphi \neq 0$ unless $\varphi = 0$ and so by applying
Lemma 5.2 we have that the Lie algebras do not admit a cocalibrated $G_2$-structure inducing an Einstein metric. For the Lie algebras

\[ g_7, g_8, g_9, g_{12}, g_{14}, g_{16}, g_{17}, g_{18}, g_{21}, g_{22}, g_{24} \]

we first determine the generic co-closed 3-form $\varphi$ and then, by applying Lemma 4.3, we impose, that $\alpha^\wedge \alpha^\wedge \alpha \neq 0$ and $\alpha^\wedge \beta = 0$, where $\alpha$ and $\beta$ are given in (5). Moreover, we have that the closed 3-form $\varphi$ defines a $G_2$-structure if and only if the matrix associated to the symmetric bilinear form $g_{\varphi}$, with respect to the orthonormal basis $\{e_1, \ldots, e_7\}$, is positive definite. Since the Einstein metric is unique up to scaling, a calibrated $G_2$-structure induces an Einstein metric if and only if the matrix associated to the symmetric bilinear form $g_{\varphi}$, with respect to the basis $\{e_1, \ldots, e_7\}$, is a multiple of the identity matrix. By a direct computation we have thus that the Lie algebras (5) cannot admit any cocalibrated $G_2$-structures inducing an Einstein metric.

For the 7-dimensional rank-two Einstein solvable Lie algebras, using the same notation as for the calibrated case we obtain the result for $\xi_1, \xi_3, \xi_5, \xi_{6,1}$ and $\xi_{6,2}$ by the first condition of Corollary 5.4. For the remaining Lie algebras, i.e., for $\xi_4, \xi_{7,1}, \xi_{7,2}, \xi_{7,3}, \xi_{7,4}, \xi_8$ and the extension of the abelian one, the result follows by using the second condition of Corollary 5.4.

In the rank-three case we have to study the extensions of the Lie algebras $n_4 = (0, 0, e^{12}, 0)$ and the four-dimensional abelian Lie algebra. For the first one we consider the structure quations (4), then we take a generic 3-form $\varphi$ such that all the coefficients of $e^{ijklm}$ in $d^* \varphi$ vanish except those of $e^{13467}$ and $e^{23467}$. Now if we compute the inner product $g_{\varphi}$ induced by $\varphi$ and we impose the conditions $g_{\varphi}(e_i, e_i) = g_{\varphi}(e_j, e_j)$ and $g_{\varphi}(e_i, e_j) = 0$ for all $i \neq j$ we obtain that $g_{\varphi}(e_6, e_6) = 0$. For the rank-three Einstein extension of the abelian Lie algebra we consider the structure equations (7) and we take a generic coclosed 3-form $\varphi$. By imposing $g_{\varphi}(e_i, e_i) = g_{\varphi}(e_j, e_j)$ and $g_{\varphi}(e_i, e_j) = 0$ for all $i \neq j$ we obtain $g_{\varphi}(e_6, e_6) = 0$.

\[ \square \]
Table 2. Rank-one Einstein 7-dimensional solvable Lie algebras
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Universidad del País Vasco, Facultad de Ciencia y Tecnología, Departamento de Matemáticas, Apartado 644, 48080 Bilbao, Spain.

marisa.fernandez@ehu.es
victormanuel.manero@ehu.es

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, Torino, Italy.
annamaria.fino@unito.it