Nonadiabatic Contribution to the Quasiparticle Self-Energy in
Systems with Strong Electron-Phonon Interaction

O.V. Danylenko\textsuperscript{1} and O.V. Dolgov\textsuperscript{2}

\textsuperscript{1} Institut für Theoretische Physik, Universität Tübingen,
Auf der Morgenstelle 14, D-72076 Tübingen, Germany

\textsuperscript{2} Max-Planck-Institut für Festkörperforschung,
Heisenbergstr. 1, D-70506 Stuttgart, Germany

Abstract

We investigate effects of a nonadiabatic electron-phonon(boson) interaction on the quasiparticle self-energy in the lowest order in the coupling constant. Existing approaches either overestimate, or underestimate these effects because of different approximations for momentum and frequency dependences of the vertex corrections. The connection between the nonadiabaticity and a possible instability of the interacting Fermi system is discussed as well.

I. INTRODUCTION

The standard description of electron-phonon interaction (EPI) in metals is based on the so-called Born-Oppenheimer\textsuperscript{1} (or adiabatic) theorem. Its main statement says that electrons are not sensitive to the motion of ions and are influenced only by their static electric field. One can to say that ”fast” electrons, due to electroneutrality, follow ”slow” ions. Narrow bands and a strong electron-phonon interaction make their velocities comparable in order of magnitude. An appropriate mathematical description is the standard Feynman-Dyson perturbation theory, where phonons are considered to be unaffected by the EPI. However,
there are some processes which in higher orders violate this approximation. Mathematically this can be taken into account by virtue of the so-called "vertex corrections". These effects were considered by Migdal who showed that their contributions are small, of the order of a small parameter \( \Omega_{ph}/\tilde{W} \), where \( \tilde{W} \) is a characteristic energy, defined by the bandwidth \( W \) or the Fermi energy \( E_F \).

It was noted by Migdal himself that at \( \vec{q} = 0 \) the vertex corrections are not small (so they are important for optical conductivity and Raman scattering). This case was considered in detail by Engelsberg and Schrieffer. Migdal’s theorem is also violated for a one-dimensional Fermi surface.

Traditionally nonadiabatic corrections were considered to be small because the Migdal parameter \( \Omega_{ph}/\tilde{W} \) was small. But recently some materials (for example, fullerenes and high-\( T_c \) superconductors) where \( \Omega_{ph} \sim \tilde{W} \) were discovered. So Migdal’s theorem about the smallness of the nonadiabatic corrections can be violated. This gives grounds for not only discussing these corrections, but even examining an antiadiabatic limit \( \Omega_{ph} \gg \tilde{W} \). But even with small \( \Omega_{ph}/\tilde{W} \) there are contradictory conclusions about the importance of such corrections in the normal and superconducting state (the authors of Refs. consider them to be negligible, while the authors of think the opposite is true). The most intriguing question is whether the violation of adiabaticity leads to an increase of the interelectron interaction (and, as a result, to an increase of \( T_c \)). There are contradictory results about the effect of the vertex corrections on \( T_c \). For example, Takada reported some increase of \( T_c \) for small parameters \( \Omega_{ph}/E_F \). Even more significant increase was claimed in. At the same time, in a decrease of \( T_c \) was obtained due to the vertex corrections.

One can distinguish between the two sorts of the nonadiabatic corrections to the electron self-energy: 1) the vertex ones, resulted from the higher order corrections of perturbation theory, as vertex function \( \Gamma \neq 1 \) even at an infinite electron (or hole) band \( -\infty < \varepsilon < +\infty \), and 2) the corrections due to the finite band width, \( -W < \varepsilon < W \).

Nonadiabatic corrections in the superconducting state enter the expression for \( T_c \) in two ways: indirectly through renormalization of \( Z \) and directly in the equation for the order
parameter $\Delta$. At the same time, analytical calculation of anomalous (crossing) diagrams is rather difficult. Therefore before proceeding to calculations in the superconducting state we want to clarify the role of such corrections in the normal state. The main goal of this paper is to compare different approaches to calculation of nonadiabatic corrections to the electron quasiparticle self-energy in the normal state (or more precisely to the mass renormalization factor $Z$). There are two methods to take nonadiabatic corrections into account. One of them, which can be called Migdal’s one, is based on the solution to the Bethe-Salpeter equation for the vertex function in the ladder approximation. In the lowest approximation, the first correction to the unity vertex is determined by the diagram in Fig.1. Then with the obtained vertex function the self-energy $\Sigma$ is calculated. This traditional method does not allow to take into account higher-order diagrams analytically because of complexity of the integration over momenta.

In 1989 the authors of the Ref. considered a non-ladder approximation, using the Ward identity. Later Y. Takada used the same idea in his method, which he called the gauge-invariant self-consistent (GISC) method. In this method, based on the Ward identity

$$i\omega_n\Gamma(i\omega_n, i\omega_n - i\omega_v, \vec{k}, \vec{k} - \vec{q}) - q\Gamma(i\omega_n, i\omega_n - i\omega_v, \vec{k}, \vec{k} - \vec{q}) = G^{-1}(i\omega_n, \vec{k}) - G^{-1}(i\omega_n - i\omega_v, \vec{k} - \vec{q}),$$

the vector term is neglected and the scalar vertex function is chosen to be a functional of the self-energy, which is supposed to be independent from momentum. Recently there appeared another method, aiming at improving the GISC approach as it takes into account the momentum dependence of the vertex function. So, when comparing these methods with the Migdal’s one, we also discuss the validity of these assumptions. In fact, for the self-energy we calculate all the diagrams up to, and including the second order. The parameter $\Omega_{ph}/W \equiv m_0$ is taken to be small.

For simplicity, we consider the model of the Fermi liquid with the usual electron-phonon Hamiltonian and Einstein phonon spectrum with the so-called Eliashberg spectral function $\alpha^2F_E(\Omega) = \frac{1}{2}\lambda\Omega_{ph}\delta(\Omega - \Omega_{ph})$, where $\Omega_{ph}$ is the phonon frequency independent from
momentum, and $\lambda$ is the electron-phonon interaction constant. It is assumed that the density of states is constant $N(0)$ for $-W < \varepsilon < W$ where $2W$ is the bandwidth. The chemical potential is zero, which corresponds to a half-filled band. It is assumed that $\lambda \sim 1$.

We are interested in the quasiparticle self-energy $\Sigma$ on the imaginary axis which is determined by the diagram in Fig.1. The phonon Green’s function is $D_0(i\omega_\nu) = -\Omega_{ph}^2/(\omega_\nu^2 + \Omega_{ph}^2)$, where $\omega_\nu = 2\nu\pi T$, and the electron Green’s function is $G(i\omega_n, \vec{p}) = (i\omega_n - \varepsilon_{\vec{p}} - \Sigma(i\omega_n, \vec{p}))^{-1}$, where $\omega_n = (2n+1)\pi T$, $\varepsilon_{\vec{p}}$ is the bare electron spectrum.

The vertex function itself (two-point one) is only of academic interest. We are, however, interested in the quasiparticle self-energy $\Sigma$ which affects many physical observables. Assuming that $\Sigma(i\omega_n, \vec{k})$ weakly depends on $\vec{k}$, we take $|\vec{k}| = p_F = \text{const}$, and then

$$
\Sigma(i\omega_n) = \int_0^\infty d\Omega \alpha^2 F_E(\Omega) T \sum_{\omega_\nu} \frac{2\Omega}{\omega_\nu^2 + \Omega^2} \frac{1}{N(0)} \times \sum_q G(i\omega_n - i\omega_\nu, \vec{k} - \vec{q}) \Gamma(i\omega_n, i\omega_n - i\omega_\nu, \vec{q}).
$$

The methods differ in the choice of the vertex function $\Gamma$, and this will be discussed later. One should note that $\Gamma(\vec{q})$ enters the integral for $\Sigma$, and all $\vec{q}$’s contribute to it. For simplicity we will obtain results for $T = 0$ and the variables will be changed as follows: $\omega_n \rightarrow \omega$, $\omega_\nu \rightarrow \nu$, remaining on the imaginary axis.

II. MIGDAL’S METHOD

In the method which can be called Migdal’s, the first correction to the unity vertex function $\Gamma^{(1)}$ is considered. This is shown in Fig1. It has been estimated in many papers, but now it is becoming especially significant because the cases were found where as the parameter $\lambda\Omega_{ph}/W$ may be not small.

The diagram in Fig.1 corresponds to the expression

$$
\Gamma^{(2)}(i\omega_n, i\omega_n - i\omega_\nu, \vec{q}) = T \sum_{\omega_\nu} \sum_q \int_0^\infty d\Omega \frac{\alpha^2 F_E(\Omega)2\Omega}{N(0)(\omega_\nu^2 + \Omega^2)} \times G^{(0)}(i\omega_n - i\omega_\nu, \vec{p} - \vec{q}) G^{(0)}(i\omega_n - i\omega_\nu - i\omega_\nu, \vec{p} - \vec{q} - \vec{q}).
$$
Assuming that $|\vec{p} - \vec{q}'| \sim p_F$ and expanding $\varepsilon(\vec{p} - \vec{q}' - \vec{q}) \approx \varepsilon(\vec{p} - \vec{q}') - q_F \cos \theta$ we get for $T = 0$

$$\Gamma^{(2)}(i\omega, i\omega - i\nu, \vec{q}) = \lambda \frac{1}{4} \int_{-1}^{1} d\eta \frac{1}{-im' + Q\eta} \times \left\{ \ln \frac{1 + im + \frac{1}{m_0}}{1 - im + \frac{1}{m_0}} + \ln \frac{1 - im}{1 + im} + \ln \frac{1 + im - im'}{1 - im + im'} + \ln \frac{1 - im + im' - Q\eta + \frac{1}{m_0}}{1 + im - im' + Q\eta + \frac{1}{m_0}} \right\},$$

(4)

where $m = \omega/\Omega_{ph}$, $m' = \nu/\Omega_{ph}$, $Q = qV_F/\Omega_{ph}$, $m_0 = \Omega_{ph}/W$.

This is a nonanalytical function at $\omega \to 0$, $\vec{q} \to 0$, i.e. the result depends on the order of taking the limits. Let us define the dynamical and static vertex functions

$$\Gamma_d = \Gamma(\vec{q} = 0, \nu \to 0, \omega);$$

$$\Gamma_s = \Gamma(\vec{q} \to 0, \nu = 0, \omega).$$

(5)

From (4) one can get

$$\Gamma^M_d = \Gamma_M(\vec{q} = 0, \nu \to 0, \omega) = \lambda \frac{1}{1 + (\frac{\omega}{\Omega_{ph}})^2} - \frac{1 + \frac{W}{\Omega_{ph}}}{(1 + \frac{W}{\Omega_{ph}})^2 + (\frac{\omega}{\Omega_{ph}})^2},$$

$$\Gamma^M_s = \Gamma_M(\vec{q} \to 0, \nu = 0, \omega) = -\lambda \frac{1 + \frac{W}{\Omega_{ph}}}{(1 + \frac{W}{\Omega_{ph}})^2 + (\frac{\omega}{\Omega_{ph}})^2}.$$

It is evident that the nonanalyticity mentioned above gives

$$\Gamma^M_d - \Gamma^M_s = \lambda \frac{1}{1 + (\frac{\omega}{\Omega_{ph}})^2}.$$

At $V_F|\vec{q}| \ll |\nu|$

$$\Gamma^{(2)}(i\omega, i\omega - i\nu, \vec{q})$$

$$= \lambda \frac{\Omega_{ph}}{\nu} \left( \arctan \frac{\omega}{\Omega_{ph}} - \arctan \frac{\omega - \nu}{\Omega_{ph}} - \arctan \frac{\omega}{W + \Omega_{ph}} + \arctan \frac{\omega - \nu}{W + \Omega_{ph}} \right) = -\frac{\Sigma^{(1)}(i\omega) - \Sigma^{(1)}(i\omega - i\nu)}{i\nu},$$

(6)

which satisfies the Ward identity (4).
At $V_F|q| \gg |\nu|$ one has $\Gamma^{(2)}(i\omega, i\omega - i\nu, \vec{q}) \sim \lambda \Omega_{ph}/W$, which complies with Refs.\textsuperscript{2,4}.

To calculate the self-energy we expand the last term of (4) in $Q$, leaving the first two terms as they are. Then

$$
\Gamma^{(2)}(i\omega, i\omega - i\nu, \vec{q}) = \lambda \frac{\Omega_{ph}}{qV_F} \arctan \frac{qV_F}{\nu} \\
\times \left( \frac{\omega}{\Omega_{ph}} - \arctan \frac{\omega - \nu}{\Omega_{ph}} - \arctan \frac{\omega}{W + \Omega_{ph}} + \arctan \frac{\omega - \nu}{W + \Omega_{ph}} \right) \\
- \lambda \frac{\Omega_{ph}^2}{qV_FW} \left( 1 + \frac{\Omega_{ph}}{W} \right) \left( \arctan \frac{qV_F}{\nu} \right) \\
\times \left[ 1 + 2 \frac{\Omega_{ph}}{W} + \left( \frac{\Omega_{ph}}{W} \right)^2 \left( 1 + \left( \frac{\omega}{\Omega_{ph}} \right)^2 - 2 \frac{\omega}{\Omega_{ph}} + \left( \frac{\omega}{\Omega_{ph}} \right)^2 \right) \right].
$$

(7)

In Fig. 2a we show $\Gamma^{(2)}(\nu)$ (numerical (4) and approximate analytical (7) results) for the case $\omega = 0$, as well as the result of the numerical integration of Eq.(4). One can see that there is a good agreement between these plots, so the expansion in small $Q$ is well justified.

The self-energy (2) consists then of three terms:

$$
\Sigma = \Sigma^{(1)} + \Sigma^{(2)}_v + \Sigma^{(2)}_r.
$$

(8)

The first order term is

$$
\Sigma^{(1)}(i\omega) = -i\lambda \Omega_{ph} \arctan \frac{\omega}{\Omega_{ph}} + i\lambda \Omega_{ph} \arctan \frac{\omega}{W + \Omega_{ph}}.
$$

(9)

The vertex correction $\Sigma^{(2)}_v$ is obtained when one substitutes $G = G^{(0)}$ and $\Gamma = \Gamma^{(2)}$ (4) into (3)

$$
\Sigma^{(2)}_v(i\omega) = \Sigma^{(2)a}_v(i\omega) + \Sigma^{(2)b}_v(i\omega),
$$

(10)

$$
\Sigma^{(2)a}_v(i\omega) = i\lambda^2 \frac{\Omega_{ph}}{p_FV_F} d^{(a)}(m, m_0, m'_0),
$$

$$
\Sigma^{(2)b}_v(i\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy \arctan \frac{1}{m_0(m - y)} \\
\times \left( \arctan m - \arctan(m - y) - \arctan \frac{m}{m_0} + \arctan \frac{m - y}{m_0} + 1 \right) \\
\times \left( 2 \arctan \frac{m}{ym'_0} - \frac{1}{2} \arctan \frac{4}{y^2m'_0^2} - 1 \right).
$$

(11)
\[
\Sigma^{(2)}_v(i\omega) = i\lambda^2 \frac{\Omega_{ph}}{p_FV_F} d_v^b(m, m_0, m'_0),
\]

where

\[
d_v^b(m, m_0, m'_0) = -\frac{1}{2\pi} m_0^2 (1 + m_0) \int^{+\infty}_{-\infty} \frac{1}{m^2 + 1} \arctan \frac{1}{m_0(m' - m)} \times \left\{ \frac{2}{m_0^2} - \frac{2m'}{m_0} \arctan \frac{2}{m' m_0} + \frac{m'^2}{2} \ln[1 + \left( \frac{2}{m' m_0} \right)^2] \right\}.
\]

(12)

For \( \omega/\Omega_{ph} \ll 1, \Omega_{ph}/W \ll 1 \) it gives

\[
\Sigma^{(2)}_v(i\omega) = \omega \lambda^2 \frac{\Omega_{ph}}{p_FV_F} \left[ \frac{\pi^2}{8} + \left( \frac{p_FV_F}{W} \right)^2 \right].
\]

(13)

The plot of \( \Sigma^{(2)}_v(i\omega) \) is shown in Fig.2b. The numerical calculations (the exact integration with Eq.(4)) give similar results for the small \( \omega \)’s. The existing discrepancy for larger frequencies is due to our approximation for \( \Gamma \).

Another term is the so-called rainbow diagram. As the inner part of the rainbow diagram corresponds to \( \Sigma^{(1)}(i\omega - i\nu) \), the second order rainbow diagram equals

\[
\Sigma^{(2)}_r(i\omega) = \int_0^{\infty} d\Omega \alpha^2 F_E(\Omega) T \sum_{\nu} \frac{2\Omega}{\omega^2 + \Omega^2} \frac{1}{N(0)} \times \sum_{\vec{q}} [G(i\omega_n - i\omega_\nu, \vec{k} - \vec{q})]^2 \Sigma^{(1)}(i\omega_n - i\omega_\nu),
\]

(14)

which results in

\[
\Sigma^{(2)}_r(i\omega) = i\lambda^2 \Omega_{ph} d_r(m, m_0),\text{ where}
\]

\[
d_r(m, m_0) = \frac{1}{m_0(m^4 + 2m^2 + 1 + \frac{1}{m_0^2} + 2(\frac{m}{m_0})^2 - \frac{2}{m_0^2})} \times (m \ln \frac{m^2 + 2}{m^2 + (2 + \frac{1}{m_0})^2} + (\arctan \frac{m}{2} - \arctan \frac{m}{2 + \frac{1}{m_0}})(m^2 - 1 + \frac{1}{m_0^2})
\]

\[ -2m \ln \frac{m_0 + 1}{m_0 + 2}. \]

(15)

At \( \omega/\Omega_{ph} \ll 1, \Omega_{ph}/W \ll 1 \), it equals

\[
\Sigma^{(2)}_r(i\omega) = \frac{1}{2} \lambda^2 \omega \frac{\Omega_{ph}}{W}.
\]

(16)
For $\omega/\Omega_{ph} \ll 1$, $\Omega_{ph}/W \ll 1$, $\Omega_{ph}/(p_FV_F) \ll 1$ one can get

$$Z_M \equiv 1 - \frac{\Sigma(i\omega)}{i\omega} = 1 + \lambda - \frac{\pi^2}{8} \lambda^2 \frac{\Omega_{ph}}{p_F V_F} - \frac{1}{2} \lambda^2 \frac{\Omega_{ph}}{W} - \lambda^2 \frac{p_F V_F}{W^2}.$$  

One can see that the last expression has three terms with different denominators. When, for example, $p_F V_F \sim \Omega_{ph}$, the main contribution comes from the term $-\frac{\pi^2}{8} \lambda^2 \Omega_{ph}/p_F V_F$ (even for an infinite bandwidth $W$). In Fig. 2b there are shown for comparison analytical and numerical plots for the self energy correction $\Sigma^{(2)}_v$. It is seen that the approximation used to calculate the correction is well justified.

If one assumes $W = p_F V_F$,

$$Z_M \approx 1 + 2\lambda^2 \frac{\Omega_{ph}}{W}.$$  

(17)

One can see that these corrections lower the renormalization function. Had this tendency persist in all higher orders, this would have resulted in an instability (see below).

### III. CAI, LEI, AND XIE’S APPROXIMATION

Following the paper, the authors of Ref. neglected the momentum dependence of the vertex function, considering $q \gg p_F \nu/W$, and interpolated it between zero-frequency and infinite-frequency limits:

$$\Gamma^{(2)}_{CLX}(i\omega_n, i\omega_{n'}) = (1 + 2 \int d\omega \frac{\omega \alpha^2 F(\omega)}{\omega^2 + (\omega_n - \omega_{n'})^2} \frac{\Lambda_0}{\omega_n^2 + \omega_{n'}^2 + 2(\Lambda_\infty/\Lambda_0)^{-1}} - 1,$$

where $\Lambda_0(\omega) = \frac{0.293\omega}{\omega + 0.667W}$, and $\Lambda_\infty(\omega)/\Lambda_0(\omega) = \frac{4\sqrt{2}}{3} \frac{0.667W + \omega}{0.586}$.

This results in an effective interaction

$$V_{e-ph}(\omega - \omega') = \frac{1}{N(0)} \frac{\lambda \Omega^2}{(\omega - \omega')^2 + \Omega^2} \times \frac{1}{1 + \frac{\lambda \Omega^2}{(\omega - \omega')^2 + \Omega^2} \times \frac{0.293\Omega}{\Omega + 0.667W}}.$$  

Combined with the contribution from the rainbow diagram, which is the same as in the Migdal method, this gives
\[ Z_{CLX} \approx 1 + \lambda - 1.38\lambda^2 \frac{\Omega}{W}. \]

The non-ladder approximation which was also discussed in \cite{Takada} was developed in detail by Takada, and this is considered below.

### IV. KOSTUR AND MITROVIĆ’S APPROXIMATION

There has been a series of papers, e.g. Ref. \cite{Kostur}, which considered the effect of the vertex corrections on \( T_c \) not only for an isotropic EPI, but also for spin (antiferromagnetic) fluctuations with a pronounced scattering at the wave vector \( \vec{Q}^* = (\pi, \pi) \). In the paper \cite{Kostur} the authors used the following equations for the self-energy \( \Sigma(\vec{k}, i\omega_n) = i\omega_n[1 - Z_n(\vec{k})] \):

\[
i\omega_n Z_n^{(2)}(\vec{k}) = -\frac{T^2}{4W} \sum_{n'n''} \lambda(n - n')\lambda(n - n'') \times \int_{-W}^{W} \frac{d\epsilon_k'}{\beta} \int_{-W}^{W} \frac{d\epsilon_{k''}}{\beta} \int_{-W}^{W} \frac{d\epsilon_{k'''}}{\beta} M(k, k'''; \epsilon_k', \epsilon_k'') \frac{i\omega_n i\omega_{n'} i\omega_{n''}}{Z_n'Z_n''Z_n'''} \times \frac{1}{[i\omega_{n'}^2 - (\epsilon_k'/Z_n')^2][i\omega_{n''}^2 - (\epsilon_{k''}/Z_n'')^2][i\omega_{n'''}^2 - (\epsilon_{k'''}/Z_n''')^2]}, \tag{18}
\]

where \( \lambda(m) = \int_0^\infty d\Omega \alpha^2 F(\Omega) \frac{20\Omega}{\nu_m + M}, \) and \( M(k, k'''; \epsilon_k', \epsilon_k'') \) is a geometrical factor being a complicated function of momenta which is considered in detail in Ref. \cite{Kostur}. For an isotropic EPI and a three-dimensional spherical Fermi-surface it is approximated by its value on the Fermi surface \( M(k, k'''; \epsilon_k', \epsilon_k'') = 1 \). It gives

\[ Z_n = 1 + \frac{\pi T}{|\omega_n|} \sum_{n'} \lambda(n - n') \Gamma_{KM}(n, n') s_n s_{n'} a_{n'}, \]

where \( \lambda(m) = \frac{\Omega_{ph}^2}{\nu_m + \Omega_{ph}}, \) \( a_n = 2/\pi \arctan(W/Z_n|\omega_n|), \) \( s_n = \text{sign} \omega_n, \) and correction to the vertex function equals

\[
\Gamma_{KM}^{(2)}(n, n') = -\frac{\pi^2}{2W} \sum_{n''} \lambda(n - n'') s_{n'+n''} s_{n'''} a_{n'+n''} a_{n'''} = \frac{\pi^2}{4W} \sum_{n''} \lambda(n - n'') s_{n'+n''} s_{n'''} a_{n'+n''} a_{n'''}.
\tag{19}
\]

One can find that

\[
\Gamma_{KM}^{(2)}(n, n') = 1 - \lambda \frac{\pi}{4 \frac{\Omega_{ph}}{W}} (\pi - 2|\arctan(\frac{\omega_{n'}}{\Omega_{ph}}) - \arctan(\frac{\omega_n}{\Omega_{ph}})|)
\]
and so does not depend from \( q \). However, for \( q \sim p_F \) which give the main contribution to the self-energy it presents the correct order of the correction to the vertex function (compare with Eq. (4)) and the self-energy. From Eq. (2) one has for \( \omega/\Omega_{ph} \ll 1, \Omega_{ph}/W \ll 1 \)

\[
Z_{KM} = 1 + \lambda + \lambda^2 \frac{\Omega_{ph}}{W} (-\frac{\pi^2}{8} - \frac{1}{2}) \approx 1 + \lambda - 1.73\lambda^2 \frac{\Omega_{ph}}{W}.
\]

This is similar to Eq. (17), but has a different numerical coefficient, due to neglecting the \( \vec{q} \)-dependence in the vertex function (19).

V. C. GRIMALDI, L. PIETRONERO, AND S. STRÄSSLER’S APPROXIMATION

In the paper the authors started with the same equation as in Migdal method, but made a series of approximate assumptions (for example, expanded in small \( q \)'s to take integrals) and gave the following estimate for the vertex function:

\[
\Gamma^{(2)}_{GPS}(\omega, Q, \Omega, W) = \frac{\lambda}{Q} \left[ \arctan m - \arctan \frac{m}{m_0} \right] \arctan \frac{Q}{m} \right]
\]

\[
-\left[ Q - m \arctan \frac{Q}{m} \left( \frac{1}{m_0} + 1 \right) \left( \frac{1}{m_0} + 1 \right)^2 + 2m^2 \right] \right].
\]

Here they set one of the external electronic frequencies to zero. In this case this result is similar to that in the Migdal’s approximation (4). However, to calculate \( \Sigma \) one needs dependences on both external frequencies.

There are some indications (see, e.g. 20) that for a small hole doping strong Coulomb correlations renormalize the EPI, giving rise to the strong forward (small-\( q \)) scattering peak, while the backward scattering is strongly suppressed. With this idea in mind, in Ref. 19 the electron-phonon coupling constant was assumed to have a cut-off in a momentum space \( |g_{\vec{p},\vec{k}}|^2 = g^2(2k_F/q_c)^2|2\theta(q_c - |\vec{p} - \vec{k}|)|. \) An approximation for \( q \to 0 \) was used and then \( q_c = Q_c \cdot 2p_F \) was set to \( q_c = 2p_F \) giving the following result for \( Z \):

\[
Z_{CGPS} \approx 1 + \lambda - \frac{\pi^2}{4} \frac{\Omega_{ph}}{W} \frac{1}{Q^2_c}.
\]

For \( Q_c = 1 \) it becomes
\[ Z_{CGPS} \approx 1 + \lambda - 0.8\lambda^2 \frac{\Omega_{ph}}{W}, \]

which gives the correct order of the correction to the self-energy, but underestimates it (compare with Eq. (17)). Probably this is due to the expansion in small \( q \)'s.

VI. TAKADA’S GISC METHOD

According to Takada’s gauge-invariant self-consistent (GISC) method, the vertex function can be chosen as a functional of the self-energy \( \Gamma_T = \Gamma_T[\Sigma_T] \). This choice is based on the Ward identity \( (\mathbb{I}) \) valid for all \( \omega_\nu \)'s and \( \vec{q} \)'s. In Ref. \( \mathbb{II} \) Takada proposes neglecting the vector term in the Ward identity and choosing

\[ \Gamma_T[\Sigma] = \Gamma_T(i\omega_n, i\omega_n - i\omega_\nu) = 1 + \left( -\frac{\Sigma_T(i\omega_n)}{2i\omega_n} - \frac{\Sigma_T(i\omega_n - i\omega_\nu)}{2(i\omega_n - i\omega_\nu)} \right), \]

which corresponds to the estimates of Refs. \( \mathbb{III}, \mathbb{IV} \) in this limit. If this method worked it would allow one to calculate also superconducting diagrams without complicated integrations over momenta.

The set of equations (1,2) is suggested to be solved by iterations. At the first step \( G = G^{(0)}, \Gamma = \Gamma^{(1)} = 1 \) give \( \Sigma_T = \Sigma_T^{(1)} = \Sigma^{(1)} \) from Eq.(2). At the second step \( G = G[\Sigma^{(1)}_T] \); and the correction \( \Gamma_T^{(2)} \) for \( \Omega_{ph}/W \ll 1 \) gives

\[ \Sigma_T^{(2)}(i\omega) = -i\lambda^2\Omega_{ph} \begin{cases} \left( 1 + \frac{\pi^2}{16} \right) \frac{\omega}{\Omega_{ph}}, & |\omega| \ll \Omega_{ph}, \\ \frac{\pi^2}{4} \Omega_{ph}, & |\omega| \gg \Omega_{ph}. \end{cases} \tag{21} \]

For small frequencies it results in

\[ Z_T = 1 + \lambda + \lambda^2(\frac{\pi^2}{16} + \frac{1}{4}) \approx 1 + \lambda + 0.9\lambda^2, \]

which has the incorrect sign and order of magnitude of the vertex correction (compare with (17)).

During this calculation, the correction to the vertex \( \Gamma_T^{(2)} \) is of the order of \( \lambda \) for any \( q \), which contradicts the Migdal’s theorem \( \mathbb{III} \) and can be valid only for \( q \sim 0 \) (see e.g. Ref. \( \mathbb{IV} \)).
So in Takada’s method the correction $\Sigma^{(2)}_T$ does not have the small parameter $\Omega_{ph}/W$. This can be explained by the fact that the region $V_F|\vec{q}| \ll |\nu|$ where $\Gamma \sim \lambda$ gives in fact a small contribution to $\Sigma$. If, on the other hand, $V_F|\vec{q}| \gg |\nu|$, the vertex $\Gamma$ is of the order of $\lambda \Omega_{ph}/W$, which gives $\Sigma^{(2)} \sim \lambda \Omega_{ph}^2/W$ (see (13)).

To explain this result, one can find the vector vertex function $\tilde{\Gamma}$ in the lowest approximation and show that it really cannot be neglected. The diagram in Fig. 1 corresponds to $\tilde{\Gamma}^{(2)}$ if $\Gamma^{(1)}$ is substituted by $\tilde{\Gamma}^{(2)}(i\omega_n, i\omega_n - i\nu, \vec{q}) - \tilde{\Gamma}^{(2)}(i\omega, i\omega - i\nu, -\vec{q})$. Using electron-hole symmetry, in our model $\tilde{\Gamma}(i\omega, i\omega - i\nu, \vec{k}, \vec{k} - \vec{q}) = \tilde{\Gamma}^{(2)}(i\omega, i\omega - i\nu, -\vec{q})$. Using electron-hole symmetry, in our model $\tilde{\Gamma}(i\omega, i\omega - i\nu, \vec{k}, \vec{k} - \vec{q}) = \tilde{\Gamma}^{(2)}(i\omega, i\omega - i\nu, -\vec{q})$.

Neglecting the $Q$-dependence in the last term of (22) one can get

$$q\tilde{\Gamma}^{(2)}(i\omega, i\omega - i\nu, \vec{q}) = \lambda \frac{1}{4} \int_{-1}^{1} d\eta \frac{\Omega Q\eta}{-im' + Q\eta} \times \{ \ln \frac{1 + im + \frac{1}{m_0}}{1 - im + \frac{1}{m_0}} + \ln \frac{1 - im}{1 + im} + \ln \frac{1 + im - im'}{1 - im + im'}$$

$$+ \ln \frac{1 - im + im' - Q\eta + \frac{1}{m_0}}{1 + im - im' + Q\eta + \frac{1}{m_0}} \}.$$  

(22)

We can find regions where scalar or vector terms in the Ward identity dominate.

If one uses at $\Omega_{ph}/W \ll 1$ the expression (7) for $\Gamma^{(2)}$, one can find that the terms $\nu \Gamma^{(2)}$ and $q\tilde{\Gamma}^{(2)}$ from (1) are of the same order if $\nu/qV_F = 0.43$. If, however, $\nu/qV_F < 0.43$, the vector term dominates and cannot be neglected.

VII. F. COSENZA, L. DE CESARE, AND M. FUSCO GIRARD’S IMPROVEMENT OF THE GISC METHOD

Taking into account a criticism of the GISC method, F. Cosenza, L. De Cesare, and M. Fusco Girard tried to improve the GISC method and did not neglect the vector term in the Ward identity. They employed the simplest choice of the solution to the Ward identity

$$\Gamma^{(2)}_{CDG}(\vec{k}, i\omega_n, \vec{k}', i\omega_{n'}) \approx \frac{(i\omega_{n'} - i\omega_n)[\Sigma(\vec{k}, i\omega_n) - \Sigma(\vec{k}', i\omega_{n'})]}{(i\omega_{n'} - i\omega_n)^2 - |\alpha(\vec{k}, \vec{k}')|^2|\vec{k} - \vec{k}'|^2},$$

(24)
where $\alpha(\vec{k}, \vec{k}') = (\vec{q}/2 + \vec{k})/m \approx V_F$. With $\Sigma(\vec{k}, i\omega_n)$ taken from Eq.(9) one gets for $|\vec{q}| = |\vec{k} - \vec{k}'| \sim p_F$

$$\Gamma^{(2)}_{CDG}(\vec{k}, i\omega_n, \vec{k}', i\omega_{n'}) \sim \lambda(\frac{\Omega_{ph}}{W})^2$$

which is different from the Migdal’s estimation $\Gamma^{(2)} \sim \lambda \frac{\Omega_{ph}}{W}$ (see (7)).

Substituting approximation (24) into the equation for the self-energy (2) one can get

$$\Sigma^{(2)}_{CDG}(i\omega \to 0) \approx 0.25i\lambda^2\omega^2(\frac{\Omega_{ph}}{W})^2(\ln \frac{W}{\Omega_{ph}})^2$$

which is different from (13) and gives

$$Z_{CDG} \approx 1 + \lambda - 0.25\lambda^2(\frac{\Omega_{ph}}{W})^2(\ln \frac{W}{\Omega_{ph}})^2.$$

This can be explained by the fact that the Ward identity is a one equation for two functions (the scalar and vector vertex functions) and thus allows for multiple solutions. So there is no particular preference in using (24) instead of any other solution.

**VIII. DISCUSSION**

So in the paper we considered consistently the contribution of nonadiabatic effects to the electron self-energy in the normal state at $T = 0$ in second orders of the EPI constant $\lambda$. Several methods of taking into account nonadiabatic corrections were compared which can be summarized in the Table ($p_F V_F = W$).

The results following from the approximations of V. N. Kostur and B. Mitrović, E. Cappelluti, C. Grimaldi, L. Pietronero, and S. Strässler give correct estimations of the order of magnitude and sign of the nonadiabatic corrections to the self-energy. The analytical results were compared with numerical calculations.

It is shown that the GISC method and its generalization give for the self-energy overestimated and underestimated results respectively. This is connected with the fact that the leading contribution to $\Sigma$ comes from the region $qv_F \gg \nu$ where $\Gamma \sim \lambda \Omega_{ph}/W$. 

13
This means that in the Ward identity one may not neglect the vector term as proposed by Takada. In the framework of the standard (Migdal’s) approach, in the lowest order, there is a calculable parameter $\nu/qv_F$ (the ratio of the frequency of an incoming phonon to its momentum) for which the contributions of the vector and scalar terms become of the same order. They are equal when $\nu/qv_F = 0.43$. So for the most metals, where the main contribution to the self energy comes from the region $\nu/qv_F \ll 1$, the GISC method may not be applied. One should also be careful when choosing a solution of the Ward identity, so as not to underestimate the scalar term, like in Ref.22. It seems that GISC method can work in systems with long range interaction, for example, in doped semiconductors.

The difference in the renormalization factors $Z$ in the Table are due with the different approximations used in the calculations of the momentum and frequency dependence of vertex function. We can illustrate this in Fig. 3 where we represent the results for $\Gamma^{(2)}$ in the Takada’s ($q$-independent) approximation, the Migdal approach for different $q$ (Eqs. (4,27) of the present paper), and the $q$-independent Kostur, Mitrović vertex function. We see that if the former overestimates the vertex corrections, the latter underestimates them.

For a stable system the condition $\text{Im} \Sigma(i\omega_n) < 0$ for $\omega_n > 0$ must be satisfied (it is equivalent to $Z(i\omega_n) > 1$). However, as one can see from the Table, the nonadiabatic corrections reduce the renormalization function $Z$, favoring the tendency towards instability. The critical $\lambda_{crit}$ at which it takes place (see Eqs. (10-16)) is given by

$$\lambda_{crit} = \frac{\text{arctan } m - \text{arctan } \frac{m}{1/m_0+1}}{m_0 d_v(m, m_0) + d_v(m, m_0)},$$

for $m \to 0$, where $d_v(m, m_0)$ and $d_v(m, m_0)$ were defined above. The dependence of $\lambda$ on the parameter $\Omega_{ph}/W$ is shown in Fig.4. Thus, for a stable system there are constraints on $\lambda$ for a fixed Migdal parameter. This can explain, for example, the existence of PbBi with $\lambda \sim 3$. At the same time, in some papers, e.g., it is stated that $\lambda \sim 1$ independently from the Migdal parameter. As one sees from Fig. 4, it may be the case only for $\Omega_{ph} \sim W$.

This possible violation of the analytical properties of the one-particle Green’s function could result in corresponding nonanalyticity of the two-particle Green’s function, and thus
in a charge response function. This can lead to a charge instability.

It follows from our analysis that different approximations to the vertex function give in the normal state quantitatively (and sometimes even qualitatively!) different estimates for the self-energy. This makes the conclusions of these theories doubtful, as the same approaches were used there to calculate $T_c$ in the superconducting state. This means that only direct Migdal-type calculations can give an answer to the question: Do nonadiabatic effects enhance critical temperature of the superconducting transition?\footnote{\textcopyright{}}

\textbf{IX. ACKNOWLEDGMENT}

We thank T. Dahm, O. Gunnarsson, M.L. Kulić, V.V. Losyakov, I.I. Mazin, E.G. Maksimov, and N. Schopohl for helpful discussions.
REFERENCES

1 M. Born, R. Oppenheimer, Ann. Phys. 84, 457 (1927)

2 A. B. Migdal, Sov. Phys.- JETP 7, 996 (1958)

3 S. Engelsberg and J. R. Schrieffer, Phys. Rev. 131, 993 (1963)

4 A. A. Abrikosov, L. P. Gorkov, I. E. Dzyaloshinskij, *Methods of Quantum Field Theory in Statistical Physics*, Prentice-Hall, Englewood, NY (1963)

5 P. B. Allen and B. Mitrović, in *Solid State Physics*, Ed. by H. Ehrenreich, F. Seitz and D. Turnbull, Academic N. Y. (1982), vol. 37, p. 1

6 J. R. Schrieffer, *Theory of Superconductivity*, W. A. Benjamin, INC. New York (1964)

7 D. J. Scalapino, in *Superconductivity*, Ed. by D. R. Parks, Dekker N. Y. (1969), vol. 1, p. 449.

8 D. A. Kirzhnits, in *The Problem of High-Temperature Superconductivity*, Ed. by V. L. Ginzburg and D. A. Kirzhnits (1977), Nauka, Moscow,- English transl., Consultant Bureau, N.Y. 1982

9 Y. Takada, J. Phys. Chem. Solids 54, 1779 (1993), Techn. Rep. of ISSP, Ser. A, No. 2732, September 1993, Y. Takada, J. Superconductivity 8, 628 (1995); Y. Takada, T. Higuchi, Phys. Rev. B 52, 12720 (1995)

10 O. V. Danylenko, O. V. Dolgov, V. V. Losyakov, Czechoslovak Journal of Physics 46, 925 (1996), Suppl. S2

11 O. V. Danylenko, O. V. Dolgov, V. V. Losyakov, Physics Letters A 230, 79 (1997)

12 O. V. Danylenko, O. V. Dolgov, M. L. Kulić and V. Oudovenko, Europhys. J. B 9, 201 (1999)

13 O. V. Dolgov, O. V. Danylenko, M. L. Kulić and V. Oudovenko, Int. Journ. of Modern
Physics B 12, 3083 (1998)

14 A. S. Alexandrov, V. V. Kabanov, Phys. Rev. B 54, 3655 (1996)

15 P. Benedetti, C. Grimaldi, L. Pietronero, and G. Varelogiannis, Europhys. Lett. 28, 351 (1994)

16 L. Pietronero, S. Strässler, and C. Grimaldi, Phys. Rev. B 52, 10516 (1995)

17 C. Grimaldi, L. Pietronero, and S. Strässler, Phys. Rev. B 52, 10530 (1995)

18 C. Grimaldi, L. Pietronero, and S. Strässler, Phys. Rev. Lett. 75, 1158 (1995)

19 C. Grimaldi, E. Cappelluti, and L. Pietronero, Europhys. Lett. 42, 667 (1998)

20 M.L. Kulić and R. Zeyher, Phys. Rev. B 49, 4395 (1994); (1994); R. Zeyher and M. L. Kulić, Phys. Rev. B 53, 2850 (1996); Phys. Rev. B 54, 8985 (1996)

21 P. Miller, J. K. Freericks, E. J. Nicol, Phys. Rev. B 58, 14498 (1998)

22 F. Cosenza, L. De Cesare, and M. Fusco Girard, Phys. Rev. B 59, 3349 (1999)

23 Formally, our conclusions are applicable to other low frequency boson-like excitations, e.g. spin-fluctuations.

24 In fact the authors used a cut-off in angles and not in momenta. Scattering by small momenta results instead in a nonexponential behavior of $T_c$ as a function of the coupling constant.

25 Y. Nambu, Phys. Rev. 117, 648 (1960)

26 M. Grabowski and L. J. Sham, Phys. Rev. B 29, 6132 (1984)

27 J. Cai, X. L. Lei, and L. M. Xie, Phys. Rev. B 39, 11618 (1989)

28 V. N. Kostur, B. Mitrović, Phys. Rev. B 50, 12774 (1994)

29 Kei-Ichi Kondo, Int. Journ. of Modern Physics A 7, 7239 (1992)
30 O. V. Dolgov, V. V. Losyakov, Physics Letters A 190, 189 (1994)

31 We should note that these values of $\lambda_{\text{crit}}$ are just the rough estimates, because our calculations were performed in the first order in $\lambda$.

32 O. V. Danylenko, O. V. Dolgov, to be published
| Method: | $Z^{(2)}$ |
|---|---|
| Migdal (numerical), with $\Gamma^{(2)}$ from Eq.(4) | $-2.7\lambda^2\Omega_{ph}/W$ |
| Migdal (analytical), with $\Gamma^{(2)}$ from Eq.(7) | $-2.7\lambda^2\Omega_{ph}/W$ |
| CLX, Ref.27 | $-1.38\lambda^2\Omega_{ph}/W$ |
| KM, Ref.28 | $-1.73\lambda^2\Omega_{ph}/W$ |
| CGPS, Ref.17 | $-0.8\lambda^2\Omega_{ph}/W$ |
| Takada’s GISC, Ref.9 | $+0.9\lambda^2$ |
| CDG, Ref.22 | $-0.25\lambda^2(\Omega_{ph}/W)^2(\ln W/\Omega_{ph})^2$ |
Figure Captions

Fig. 1. Equation for the vertex function $\Gamma^{(2)}$ in Migdal’s method. The outgoing lines are shown for clarity and are not included in the definition of $\Gamma$.

Fig. 2. a) Plot of $\Gamma^{(2)}(\omega = 0, i\nu)$ (Eq.(4), dashed line) and $\Gamma^{(2)}_{\text{appr}}(\omega = 0, i\nu)$ (Eq.(7), solid line) for $\lambda = 1, \Omega_{ph}/W = 0.1, \omega = 0, q/p_F = 0; 0, 2; 0, 4; 0, 6; 0, 8$. b) Corresponding frequency dependence of $\Sigma^{(2)}(i\omega)$ (dashed line) and $\Sigma^{(2)}_{\text{appr}}(i\omega)$ (solid line).

Fig. 3. $\Gamma^{(2)}(\omega = 0, \nu)$ in the Takada’s approximation (long dashed line); the Migdal approach for $q/p_F = 0, 0.5, 1$ (solid lines, from top to bottom); and the Kostur, Mitrović vertex function (short dashed line).

Fig. 4. $\lambda_{\text{crit}}$ at which $\Sigma(i\omega) = 0$ vs the parameter $\Omega_{ph}/W$. 
\[
\Gamma^{(2)}(i\omega_n, i\omega_n - i\omega', q) = i\omega_n, q \quad \Gamma^{(1)} \quad i\omega', q'
\]
$\Sigma^{(2)}(i\omega)$ vs $\omega/\Omega$
