OPEN ASPHERICAL MANIFOLDS NOT COVERED
BY THE EUCLIDEAN SPACE

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Abstract. We show that any open aspherical manifold of dimension \( n \geq 4 \) is tangentially homotopy equivalent to an \( n \)-manifold whose universal cover is not homeomorphic to \( \mathbb{R}^n \).

1. Introduction

Davis famously constructed, for every \( n \geq 4 \), a closed aspherical \( n \)-manifold whose universal cover is not homeomorphic to \( \mathbb{R}^n \) [Dav83]. We prove:

Theorem 1.1. If the universal cover of an open smooth manifold \( V \) is diffeomorphic to \( \mathbb{R}^n \) with \( n \geq 4 \), then the tangential homotopy type of \( V \) contains a continuum of open smooth \( n \)-manifolds whose universal covers are not homeomorphic to \( \mathbb{R}^n \).

Our proof also works in \( \text{pl} \) and \( \text{top} \) categories. Recall that a homotopy equivalence \( f: V \to W \) of \( n \)-manifolds is tangential if \( f^\#TW \) and \( TV \) are stably isomorphic, where \( TX \) denotes the tangent bundle if \( X \) is a smooth manifold and the tangent microbundle in the \( \text{pl} \) or \( \text{top} \) cases. By [Sie69, Theorem 2.3] this is equivalent to requiring that \( f \times \text{id}(\mathbb{R}^n) \) is homotopic to a \( \text{cat} \)-homeomorphism for some \( s \) (where \( \text{cat} \) equals \( \text{diff} \), \( \text{pl} \) or \( \text{top} \)).

In the simply-connected case Theorem 1.1 is due to Curtis-Kwun [CK65] for \( n \geq 5 \) and to Glaser [Gla67] for \( n = 4 \). The proof combines three ingredients:

1. A result of Curtis-Kwun [CK65] that for a boundary connected sum \( S \) of a countable family of compact \( n \)-manifolds, the homeomorphism type of \( \text{Int}(S) \) determines the isomorphism class of \( \pi_1(\partial S) \).

2. A recent result of Calcut-King-Siebenmann [CKS12] that any countable collection of \( \text{cat} \) properly embedded \( \mathbb{R}^{n-1} \)'s in \( \mathbb{R}^n \) is \( \text{cat} \) unknotted, which generalizes classical results of Cantrell and Stallings.

3. The existence of infinitely many smooth compact contractible \( n \)-manifolds whose boundary homology spheres have freely indecomposable fundamental groups. (Such examples can be found in [CK65, Gla67], and more examples are now known; see e.g. Casson-Harer [CH81] for aspherical homology 3-spheres that bound smooth contractible 4-manifolds, while Kervaire [Ker69] showed that the fundamental group of any homology 3-sphere appears as the fundamental group of the boundary of a smooth contractible \( n \)-manifold for any \( n \geq 5 \).)

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Proof. Since $V$ is open, it contains a CAT properly embedded ray whose CAT regular neighborhood is an embedded closed halfspace; see e.g. [CKS12 Section 3]. Hence $V$ is CAT isomorphic to the interior of a noncompact manifold $N$ whose boundary is an open disk. By the strong version of the Cantrell-Stalling hyperplane unknotting theorem proved in [CKS12 Corollary 9.3], the universal cover of $N$ can be compactified to $D^n$, the $n$-disk, in which the preimage of $\partial N$ becomes the union of a countable collection of round open disks with pairwise disjoint closures.

Let $\{C_i\}_{i \in \mathbb{N}}$ be an infinite sequence of compact contractible $n$-manifolds, such that $\pi_1(\partial C_i)$ are pairwise nonisomorphic and freely indecomposable. Given a subset $\alpha \subseteq \mathbb{N}$, let $C_\alpha$ be a boundary connected sum of $C_i$’s with indices in $\alpha$. (For our purposes the choices involved in defining boundary connected sums will always be irrelevant.) Fix a closed $(n-1)$-disk $\Delta \subset \partial C_\alpha$, and let $N_\alpha$ be a boundary connected sum of $N$ and $C_\alpha$ obtained by identifying $\Delta$ with a closed disk in $\partial N$. A deformation retraction $C_\alpha \to \Delta$ extends to a deformation retraction of $N_\alpha \to N$, so $V_\alpha := \text{Int}(N_\alpha)$ is tangentially homotopy equivalent to $V$.

If $Q_\alpha$ denotes a boundary connected sum of countably many copies of $C_\alpha$, then the interior of the universal cover of $N_\alpha$ is homeomorphic to the interior of a boundary connected sum of $D^n$ and $Q_\alpha$, which is homeomorphic to $\text{Int}(Q_\alpha)$. By [CK65 Theorem 4.1] if $Q_\alpha$, $Q_\beta$ have homeomorphic interiors, then $\partial Q_\alpha$, $\partial Q_\beta$ have isomorphic fundamental groups. Now $\pi_1(\partial Q_\alpha)$ is a free product in which each factor $\pi_1(\partial C_{i_k})$, $i_k \in \alpha$ appears countably many times. Each $\pi_1(\partial C_{i_k})$ is freely indecomposable, so $\alpha = \beta$ by Grushko’s theorem. Thus the universal covers of $\text{Int}(N_\alpha)$ lie in a continuum of homeomorphism types.

Remark 1.2. The proof of [CK65 Theorem 4.1] is quite technical, which may be due to the fact that the tools of Siebenmann’s thesis were not yet available at the time, so we summarize it in a modern language: Given $\alpha = \{i_1, \ldots, i_k, \ldots\}$ it is easy to construct a cofinal family $\{U_k\}_{k \geq 1}$ of neighborhoods of infinity in $Q_\alpha$ such that in the corresponding inverse sequence of fundamental groups, the group $\{\pi_1(U_k)\}$ is the free product of $\pi_1(C_{i_1}) * \cdots * \pi_1(C_{i_k})$, and the map $\pi_1(U_k) \leftarrow \pi_1(U_{k+1})$ is a retraction onto the first $k$ factors, and, in particular, is surjective. Hence the inverse sequence of groups is Mittag-Leffler, and therefore its pro-equivalence class depends only on the homeomorphism type of $Q_\alpha$. Now if $Q_\alpha$, $Q_\beta$ have homeomorphic interiors, then a simple diagram chase in the commutative diagram from the definition of pro-equivalence shows that each free factor of $\pi_1(Q_\alpha)$ occurs as a free factor of $\pi_1(Q_\beta)$, so $\alpha = \beta$.

Remark 1.3. Theorem 1.1 should hold for $n = 3$, but our proof fails. One could try substituting the boundary connected sum of $N$ and $Q_\alpha$ by the end sum of $V$ with a suitable Whitehead manifold, but the multiple hyperplane unknotting is no longer true, due to the existence of an exotic $[0,1] \times \mathbb{R}^2$ of ST89. This makes analyzing the fundamental group of infinity more delicate. For the same reason we do not attempt to prove Theorem 1.1 for $V$ whose universal cover is not $\mathbb{R}^n$.

Remark 1.4. By the Cartan-Hadamard theorem any (complete Riemannian) $n$-manifold of nonpositive curvature is covered by $\mathbb{R}^n$. So given an open $n$-manifold of nonpositive curvature with $n \geq 4$, Theorem 1.1 yields a continuum of $n$-manifolds in the same tangential homotopy type that admit no metric of nonpositive curvature. In Bel the author studied obstructions to nonpositive curvature on open manifolds that go beyond the Cartan-Hadamard theorem.
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