A Class of Fractional Degenerate Evolution Equations with Delay

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Abstract: We establish a class of degenerate fractional differential equations involving delay arguments in Banach spaces. The system endowed by a given background and the generalized Showalter–Sidorov conditions which are natural for degenerate type equations. We prove the results of local unique solvability by using, mainly, the method of contraction mappings. The obtained theory via its abstract results is applied to the research of initial-boundary value problems for both Scott–Blair and modified Sobolev systems of equations with delays.

Keywords: Gerasimov–Caputo fractional derivative; differential equation with delay; degenerate evolution equation; fixed point theorem

1. Introduction

During the last decades, fractional differential equations and their potential applications have gained a lot of importance, mainly because fractional calculus has become a powerful tool with more accurate and successful results when modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering [1]. It was found that various, especially interdisciplinary applications, can be elegantly modeled with the help of fractional derivatives which provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [2,3]. Advanced analysis and numerical simulations of several fractional-order systems have been shown to be very interesting, producing more useful results in applied sciences [4,5].

Delay differential equations are a type of equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. It arises in many biological and physical applications, and it often forces us to consider variable or state-dependent delays [6–8]. Integer or fractional-order degenerate differential equations, i.e., evolution equations not solved with respect to the highest order derivative, are often used to describe various processes in science and engineering: in [9,10] certain classes of the time-fractional order partial differential equations with polynomials differential with respect to the spatial variables elliptic self-adjoint operator, which contain some equations from hydrodynamics and the filtration theory, are studied. In [11] approximate controllability issues for such models are investigated; the unique solvability of similar equations with distributed order time derivatives are researched in [12].

In applications, fractional-order degenerate evolution equations with a delay are often successful. Such kinds of equations with a degenerate operator at the highest-order fractional derivative describe...
the dynamics of some fractional models of viscoelastic fluids (see the application in the last section of this work). There are very few papers dealing with essentially degenerate fractional-order equations with delay. Motivated by this fact, the purpose of this work is a step towards eliminating this gap.

We are concerned with the following fractional differential equations with delay

$$LD_t^\alpha x(t) = Mx(t) + \int_{-r}^0 K(s)x(t+s)ds + g(t), \quad t \in [0,T],$$

(1)

where $\mathcal{X}, \mathcal{Y}$ are Banach spaces, $L, M : \mathcal{X} \to \mathcal{Y}$ are linear operators, $L$ is continuous, $\ker L \neq \{0\}$ (for this reason such equations are called Sobolev type equations [13,14], or degenerate [15]), operator $M$ is closed and densely defined in $\mathcal{X}$, $D_t^\alpha$ is the Gerasimov-Caputo derivative of the order $\alpha \in (m-1,m]$, $m \in \mathbb{N}$. Equation (1) is endowed by a given background

$$Px(t) = h(t), \quad t \in [-r,0],$$

(2)

and by the generalized Showalter–Sidorov conditions

$$(Px)^{(k)}(0) = x_k, \quad k = 0,1, \ldots, m-1,$$

(3)

which are natural for degenerate evolution equations. Here, $P$ is a projector along the degeneration space of the homogeneous equation $LD_t^\alpha x(t) = Mx(t)$, it will be defined below. By the contraction mappings method, the local unique solvability of problems (1)–(3) is established.

Degenerate first-order evolution equations in Banach spaces were studied in [16,17] under various conditions on the operators $L, M$ and on the delay term. The unique solvability results for problems (1) and (2) with a strongly $(L,p)$-radial operator $M, g \equiv 0$ at $\alpha = 1$ were obtained in [18]. Here we use a similar approach, which is adapted to the case of a fractional derivative. The second section contains the preliminary results which are needed for supporting our results, in particular, the theorem on unique solvability of the Cauchy problem to the inhomogeneous linear Equation (1) with $K \equiv 0$. In the third section, we obtain the proof of the main result by means of the Banach fixed point theorem. The fourth and fifth sections demonstrate the applications of the obtained abstract results to the study of the unique solvability of initial-boundary value problems for time-fractional systems of partial differential equations with delay.

2. Solvability of Degenerate Inhomogeneous Equation

Let for $\delta > 0$, $t > 0$ $g_\delta(t) := \Gamma(\delta)^{-1}t^{\delta-1}, I_t^\delta h(t) := \int_0^t g_\delta(t-s)h(s)ds$, $m-1 < \alpha \leq m \in \mathbb{N}$, $D_t^m$ is the usual derivative of the order $m \in \mathbb{N}$, $I_0^0$ be the identical operator. The Gerasimov-Caputo derivative of a function $h$ is defined as

$$D_t^\alpha h(t) = D_t^m I_t^{m-\alpha} \left( h(t) - \sum_{k=0}^{m-1} h^{(k)}(0) \frac{t^k}{k!} \right).$$

Lemma 1. Ref. [19]. Let $Z$ be a Banach space, $l-1 < \beta \leq l \in \mathbb{N}$, $t > 0$. Then

$$\exists C_\beta > 0 \quad \forall h \in C^l([0,t]; Z) \quad \|D_t^\beta h\|_{C([0,t]; Z)} \leq C_\beta \|h\|_{C^l([0,t]; Z)}.$$

For Banach spaces, $\mathcal{X}$ and $\mathcal{Y}$ denote as $\mathcal{L}(\mathcal{X}; \mathcal{Y})$ the Banach space of all linear continuous operators, acting from $\mathcal{X}$ to $\mathcal{Y}$. Let $C^l(\mathcal{X}; \mathcal{Y})$ be the set of all linear closed operators, densely defined in $\mathcal{X}$, with the image in $\mathcal{Y}$.

In further consideration, we will assume that $L \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, ker $L \neq \{0\}$, $M \in C^l(\mathcal{X}; \mathcal{Y})$, $D_M$ is the domain of $M$ with the graph norm $\| \cdot \|_{D_M} := \| \cdot \|_\mathcal{X} + \|M \cdot \|_\mathcal{Y}$. Denote $\rho^L(M) := \{ \mu \in \mathbb{C} :$
An operator $M$ is called $(L, \sigma)$-bounded, if a ball $B_a(0) := \{ \mu \in \mathbb{C} : |\mu| < a \}$ with some $a > 0$ contains the set $\rho^L(M)$. If $M$ is $(L, \sigma)$-bounded, we have the projections

$$ P := \frac{1}{2\pi i} \int_{|\mu| = a} (\mu L - M)^{-1} L \, d\mu \in \mathcal{L}(\mathbf{X}), \quad Q := \frac{1}{2\pi i} \int_{|\mu| = a} L(\mu L - M)^{-1} \, d\mu \in \mathcal{L}(\mathbf{Y}) $$

(see [14] (pp. 89–90)). Let an operator $M$ be $(\sigma)$-bounded, i.e., if $D \in \mathcal{L}(\mathbf{X}; \mathbf{Y})$ for all $l = 0, 1, \ldots, p$, then $D^l \in \mathcal{L}(\mathbf{X}^l; \mathbf{Y})$, $k = 0, 1; (i)$ there exist operators $M_0^{-1} \in \mathcal{L}(\mathbf{Y}; \mathbf{X}^0)$, $L_1^{-1} \in \mathcal{L}(\mathbf{Y}^1; \mathbf{X})$.

Denote by $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $G := M_0^{-1} L_0$. For $p \in \mathbb{N}_0$ operator $M$ is called $(L, p)$-bounded, if it is $(L, \sigma)$-bounded, $G^p \neq 0$, $G^{p+1} = 0$.

Consider the degenerate inhomogeneous equation

$$ LD^p x(t) = Mx(t) + f(t), \quad t \in [0, T]. \tag{4} $$

A solution of this equation is a function $x \in C([0, T]; D_M)$, such that $D^l x \in C([0, T]; \mathbf{X})$ and equality (4) holds. A solution of the generalized Showalter–Sidorov problem

$$ (Px)^{(k)}(0) = x_k, \quad k = 0, 1, \ldots, m - 1, \tag{5} $$

to Equation (4) is a solution of the equation, such that conditions (5) are true.

Denote by $E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}$ the Mittag-Leffler function.

**Theorem 2.** Refs. [20,21]. Let $p \in \mathbb{N}_0$, an operator $M$ be $(L, p)$-bounded, $Qf \in C([0, T]; \mathbf{Y})$, for all $l = 0, 1, \ldots, p$, there exist $(GD^p)^1 M_0^{-1} (I - Q)f, D^p (GD^p)^1 M_0^{-1} (I - Q)f \in C([0, T]; \mathbf{X})$, $x_0, x_1, \ldots, x_{m-1} \in \mathbf{X}^1$. Then, problems (4) and (5) have a unique solution

$$ x_f(t) = \sum_{k=0}^{m-1} t^k E_{\alpha, 1}(L_1^{-1} M_1 t^k) P x_k + \int_0^t E_{\alpha, 1}(L_1^{-1} M_1 (t - s)^k) L_1^{-1} Qf(s) ds - \sum_{l=0}^{p} (GD^p)^1 M_0^{-1} (I - Q)f(t). \tag{6} $$

**Remark 1.** Due to Lemma 1 a function $f \in C^{m(p+1)}([0, T]; \mathbf{Y})$ satisfies the conditions of Theorem 2.

**3. Main Result**

Consider the problem

$$ Px(t) = h(t), \quad t \in [-r, 0], \quad (Px)^{(k)}(0) = x_k, \quad k = 0, 1, \ldots, m - 1, \tag{7} $$

for the degenerate fractional evolution equation with delay

$$ LD^p x(t) = Mx(t) + \int_{-r}^{0} K(s) x(t + s) ds + g(t), \quad t \in [0, T], \tag{8} $$

where $h(0) = x_0$, $K : [-r, 0] \to \mathcal{L}(\mathbf{X}; \mathbf{Y})$, $g : [0, T] \to \mathbf{Y}$.

A function $x \in C([0, T]; D_M) \cap C([-r, T]; \mathbf{X})$ is called a solution of problems (7) and (8), if $D^p x \in C([0, T]; \mathbf{X})$, it satisfies Equalities (7) and (8).
Theorem 3. Let \( p \in \mathbb{N}_0 \), an operator \( M \) be \((L, p)\)-bounded, \( h \in C([-r, 0]; \mathcal{X}^1) \), \( x_k \in \mathcal{X}^1 \), \( k = 0, 1, \ldots, m - 1 \), \( h(0) = x_0, T_0 > 0 \), \( g \in \mathcal{C}^{m(p+1)}([0, T_0]; \mathcal{Y}) \), \( \mathcal{K} \in \mathcal{C}^{m(p+1)}([-r, 0]; \mathcal{L}(\mathcal{X}; \mathcal{Y})) \), \( \mathcal{K}^{(n)}(-r) = \mathcal{K}^{(n)}(0) = 0 \) at \( n = 0, 1, \ldots, m(p+1) - 1 \). Then there exists \( T \in (0, T_0) \), such that problems (7) and (8) have a unique solution.

Proof. Fix \( T > 0 \) and consider on the segment \([0, T]\) Equation (4) with some \( f \in \mathcal{C}^{m(p+1)}([0, T]; \mathcal{Y}) \). Due to Theorem 2 and Remark 1 we have the solution \( x_f \) of problems (4) and (5) with the given \( x_{kT}, k = 0, 1, \ldots, m - 1 \). For brevity denote \( X_\beta(t) := E_{\alpha, \beta}(L^{-1}M_1 M_0^1)P \), \( \beta > 0 \), put at \( t \in [-r, 0) \) \( x_f(t) = h(t) + h_0(t) \) with some \( h_0 \in C([-r, 0]; \mathcal{X}^0) \) and define the operator

\[
[\Phi f](t) := \int_{-r}^{0} \mathcal{K}(s)x_f(t+s)ds + g(t) = \int_{-r}^{0} \mathcal{K}(s) \sum_{k=0}^{m-1} (t+s)^k X_{k+1}(t+s) x_k ds + \\
+ \int_{-r}^{0} \mathcal{K}(s) \int_{0}^{t+s} X_a(t+s-\tau)L^{-1}_1 Qf(\tau) d\tau ds - \int_{-r}^{0} \mathcal{K}(s) \sum_{l=0}^{p} (GD^l_1 G)^{l-1} M^{-1}_0 (I-Q)f(t+s) ds + \\
+ \int_{-r}^{0} \mathcal{K}(s) (h(s) + h_0(s))ds + g(t), \quad t \in [0, r),
\]

\[
[\Phi f](t) := \int_{-r}^{0} \mathcal{K}(s)x_f(t+s)ds + g(t) = \int_{-r}^{0} \mathcal{K}(s) \sum_{k=0}^{m-1} (t+s)^k X_{k+1}(t+s) x_k ds + \\
+ \int_{-r}^{0} \mathcal{K}(s) \int_{0}^{t+s} X_a(t+s-\tau)L^{-1}_1 Qf(\tau) d\tau ds - \int_{-r}^{0} \mathcal{K}(s) \sum_{l=0}^{p} (GD^l_1 G)^{l-1} M^{-1}_0 (I-Q)f(t+s) ds + g(t), \quad t \in [r, T],
\]

By induction, we can prove that at \( t \in [0, T], n = 0, 1, \ldots, m(p+1) \)

\[
[\Phi f]^{(n)}(t) = \frac{d^n}{dt^n} \int_{-r}^{t} \mathcal{K}(\tau-t)x_f(\tau) d\tau + g^{(n)}(t) = (-1)^n \int_{-r}^{0} \mathcal{K}^{(n)}(s)x_f(t+s)ds + g^{(n)}(t), \quad (9)
\]

since \( \mathcal{K}^{(n)}(-r) = \mathcal{K}^{(n)}(0) = 0 \) at \( n = 0, 1, \ldots, m(p+1) - 1 \). Therefore, for every \( f \) from the Banach space \( C^{m(p+1)}([0, T]; \mathcal{Y}) \) with the standard norm \( \| \cdot \|_{m(p+1)} \) we have \( \Phi f \in C^{m(p+1)}([0, T]; \mathcal{Y}) \).

Let \( t_r := \min \{ t, r \} \). For \( f_1, f_2 \in C^{m(p+1)}([0, T]; \mathcal{Y}), t \in [0, T], n = 0, 1, \ldots, m(p+1) \), due to (9)

\[
\frac{d^n}{dt^n} ([\Phi f_1](t) - [\Phi f_2](t)) = (-1)^n \int_{-r}^{0} \mathcal{K}^{(n)}(s) \sum_{l=0}^{p} (GD^l_1 G)^{l-1} M^{-1}_0 (I-Q)(f_1(t+s) - f_2(t+s)) ds,
\]

therefore, using Theorem 1 and Lemma 1, we obtain

\[
\| \Phi f_1 - \Phi f_2 \|_{m(p+1)} \leq C_1 \int_{-r}^{0} (t+s) \sum_{n=0}^{m(p+1)} \| \mathcal{K}^{(n)}(s) \|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} ds \| f_1 - f_2 \|_0 + \\
+ C_2 \int_{-r}^{0} \sum_{n=0}^{m(p+1)} \| \mathcal{K}^{(n)}(s) \|_{\mathcal{L}(\mathcal{X}; \mathcal{Y})} ds \| f_1 - f_2 \|_{mp} \leq CF(T) \| f_1 - f_2 \|_{m(p+1)},
\]
where, for the monotonously non-decreasing non-negative function

\[ F(t) := \frac{1}{m(p+1)} \sum_{n=0}^{m(p+1)} \|K^{(n)}(s)\|_{L(X,Y)}ds \]

we have \( F(t) \to 0 \) as \( t \to 0+ \). So, the inequality \( \|\Phi f_1 - \Phi f_2\|_{m(p+1)} \leq q\|f_1 - f_2\|_{m(p+1)} \) with some \( q \in (0,1) \) is valid for sufficiently small \( T > 0 \) and there exists a unique fixed point \( f_0 \) of the operator \( \Phi \) in \( C^{m(p+1)}([0,T];Y) \). Therefore,

\[ LD^\alpha t x_{f_0}(t) - Mx_{f_0}(t) = f_0(t) = [\Phi f_0](t) = \int_{-r}^{0} K(s)x_{f_0}(t + s)ds + g(t), \]

and the function \( x_{f_0} \), which is defined as in the beginning of this proof, is a solution of problems (7) and (8).

Note that the choice of function \( h_0 \) does not affect the proof, hence, we can choose \( h_0(t) \equiv (I - P)x_{f_0}(0) \), then the obtained \( x_0 \) is continuous on \([r,T]\).

Let there exist two solutions \( x_1, x_2 \) of the problem, denoted as \( f_i(t) = \int_{-r}^{0} K(s)x_i(t + s)ds, i = 1,2 \).

As before, we have \( f_i \in C^{m(p+1)}([0,T];Y) \) and \( LD^\alpha t x_i - Mx_i = f_i \), hence, by the construction \( \Phi f_i = f_i \), \( i = 1,2 \). Thus, \( \Phi \) has two fixed points, it is a contradiction. Consequently, \( f_1 \equiv f_2 \), for \( y := x_1 - x_2 \) we have \( LD^\alpha t y - My = 0, y^{(k)}(0) = 0, k = 0, 1, \ldots, m - 1 \), therefore, \( y \equiv 0 \) due to Theorem 2. So, the solution of problems (7) and (8) is unique. \( \square \)

4. A Scott–Blair Type System

Consider the problem

\[ \frac{\partial^k\nu}{\partial t^k}(x,0) = z_k(x), \quad x \in \Omega, \quad k = 0,1,\ldots,m-1, \]

(10)

\[ \nu(x,t) = h(x,t), \quad x \in \Omega, \quad t \in [-r,0], \]

(11)

\[ \nu(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \]

(12)

\[ (1 - \chi\Lambda)D^\alpha t \nu(x,t) = -(\partial \cdot \nabla)\nu(x,t) - (\nu \cdot \nabla)\nu(x,t) - r(x,t) + \]

\[ + \int_{-r}^{0} (K_1(s)\nu(t + s) + K_2(s)r(t + s))ds, \quad (x,t) \in \Omega \times [0,T], \]

(13)

\[ \nabla \cdot \nu(x,t) = 0, \quad (x,t) \in \Omega \times [0,T], \]

(14)

where \( \Omega \subset \mathbb{R}^n \) is a bounded region with a smooth boundary \( \partial\Omega, \chi \in \mathbb{R}, \partial \) is a given function. Function of the fluid velocity \( \nu = (v_1,v_2,\ldots,v_n) \) and of the pressure gradient \( r = (r_1,r_2,\ldots,r_n) = \nabla p \) are unknown.

This system without delay can be obtained, if the dynamics of a Scott–Blair medium [22] are described by using a fractional derivative of the same order as in the rheological relation for this medium, with subsequent linearization.

Let \( L_2 = (L_2(\Omega))^n, H_1 := (W_1^2(\Omega))^n, H_2 := (W_2^2(\Omega))^n \). The closure of \( \{\nu \in (C_0^\infty(\Omega))^n : \nabla \cdot \nu = 0\} \) in the space \( L_2 \) will be denoted by \( H_\nu \), and in the space \( H_1 \) it will be \( H_\nu^1 \). We have the decomposition \( L_2 = H_\nu \oplus H_\pi, \) where \( H_\pi \) is the orthogonal complement for \( H_\nu \). Denote by \( \Pi : L_2 \to H_\pi \) the corresponding to this decomposition orthoprojector, \( \Sigma = I - \Pi, H_\nu^2 = H_\nu^1 \cap H_\pi^2 \).

The operator \( \Lambda := \Sigma\Lambda \) with the domain \( H_\nu^2 \) in the space \( H_\nu \) has a real, negative, discrete spectrum with a finite multiplicity, condensing at \( -\infty \) [23].
At $\tilde{\omega} \in \mathbb{H}^1$ by the formula $Dw = -(\tilde{\omega} \cdot \nabla)w - (\omega \cdot \nabla)\tilde{\omega}$ operator $D \in \mathcal{L}(\mathbb{H}_r^2; \mathbb{L}_2)$ is defined. Put
\begin{equation}
\mathcal{X} = \mathbb{H}_r^2 \times \mathbb{H}_r, \quad \mathcal{Y} = \mathbb{L}_2 = \mathbb{H}_r \times \mathbb{H}_r,
\end{equation}
\begin{equation}
L = \begin{pmatrix}
I - \chi A & 0 \\
-\chi \Pi \Delta & 0
\end{pmatrix} \in \mathcal{L}(\mathcal{X}; \mathcal{Y}), \quad M = \begin{pmatrix}
\Sigma D & 0 \\
\Pi D & -I
\end{pmatrix} \in \mathcal{L}(\mathcal{X}; \mathcal{Y}).
\end{equation}

By the choice of the space $\mathcal{X}$ we take into account Equation (14) and condition (12). The function $r(\cdot, t)$ is a gradient, since it belongs to the space $\mathbb{H}_r$ at $t \geq 0$.

**Lemma 2.** Ref. [24]. Let $\chi \neq 0, \chi^{-1} \notin \sigma(A)$, the spaces $\mathcal{X}$ and $\mathcal{Y}$ and the operators $L$ and $M$ be defined by (15) and (16) respectively. Then the operator $M$ is $(L, 0)$-bounded and the projectors have the form
\begin{equation}
P = \begin{pmatrix}
I & 0 \\
\chi \Pi \Delta (I - \chi A)^{-1} & \Pi D
\end{pmatrix}, \quad Q = \begin{pmatrix}
I & 0 \\
-\chi \Pi \Delta (I - \chi A)^{-1} & 0
\end{pmatrix}.
\end{equation}

The form of the projectors $P$ and $Q$ implies that $\mathcal{X}^0 = \{0\} \times \mathbb{H}_r$, $\mathcal{X}^1 = \{(w_1, w_2) \in \mathbb{H}_r^2 \times \mathbb{H}_r : w_2 = (\chi \Pi \Delta (I - \chi A)^{-1} \Sigma D + \Pi D)w_1\}$, $\mathcal{Y}^0 = \{0\} \times \mathbb{H}_r$, $\mathcal{Y}^1 = \{(w_1, w_2) \in \mathbb{H}_r \times \mathbb{H}_r : w_2 = -\chi \Pi \Delta (I - \chi A)^{-1} w_1\}$.

**Theorem 4.** Let $h \in C([-r, 0]; \mathbb{H}_r^2), z_k \in \mathbb{H}_r^2, k = 0, 1, \ldots, m - 1$, $h(\cdot, 0) = z_0(\cdot), K_i \in C^m([-r, 0]; \mathbb{R}), K_i^{(n)}(-r) = K_i^{(n)}(0) = 0$ at $n = 0, 1, \ldots, m - 1, i = 1, 2$. Then there exists $T > 0$, such that problems (10)--(14) have a unique solution.

**Proof.** Due to Lemma 2 and Theorem 3 at $p = 0, g \equiv 0$ we obtain the required statement. \qed

5. A Modified Sobolev System

Consider another problem
\begin{equation}
\frac{\partial^k v}{\partial x^k}(x, 0) = z_k(x), \quad x \in \Omega, \quad k = 0, 1, \ldots, m - 1,
\end{equation}
\begin{equation}
v(x, t) = h(x, t), \quad x \in \Omega, \quad t \in [-r, 0],
\end{equation}
\begin{equation}
v_n(x, t) := \sum_{i=1}^{3} v_i(x, t) n_i(x) = 0, \quad (x, t) \in \partial \Omega \times [0, T],
\end{equation}
\begin{equation}
D^r_t v(x, t) = [v(x, t), \overline{\omega}] - r(x, t) + \int_{-r}^{0} (K_1(s)v(t + s) + K_2(s)r(t + s))ds, \quad (x, t) \in \Omega \times [0, T],
\end{equation}
\begin{equation}
\nabla \cdot v(x, t) = 0, \quad (x, t) \in \Omega \times [0, T],
\end{equation}
where $\Omega \subset \mathbb{R}^3$, is a bounded region with a smooth boundary $\partial \Omega, \overline{\omega} \in \mathbb{R}^3$.

Such a system without delay and at $\alpha = 1$ describes the dynamics of small internal movements of a stratified fluid in an equilibrium state [25].

Following the approach of S.L. Sobolev [25], we use the generalized statement of the problem (17)--(21), replacing incompressibility Equation (21) and boundary condition (19) with the equation
\begin{equation}
\Pi v(\cdot, t) = 0, \quad t \in [0, T],
\end{equation}
\end{document}
where $\Pi$ is the same orthoprojector as in the previous section. Indeed, the set $\{\nabla \varphi : \varphi \in C^\infty(\Omega)\}$ is dense in the subspace $H_\Pi$, and the integral identity

$$\int_\Omega (\nabla v, \nabla \varphi) = \int_\Omega v \varphi ds - \int_\Omega (\nabla \cdot v) \varphi dx$$

is true for all $\varphi \in C^\infty(\Omega), v \in H^1$, hence, for every $v \in H^1$ the satisfaction of conditions (19), (21) is equivalent to the inclusion $v \in H^1$. Rejecting the restriction $H^1$ we obtain condition (22).

Define by $Bw = [w, \bar{w}]$ at a fixed $\sigma \in R^3$ the linear operator $B \in L(L_2, L_2)$. Put $X = Y = L_2 = H_\sigma \times H_\Pi$,

$$L = \begin{pmatrix} I & 0 \\ \Omega & \Pi \end{pmatrix} \in L(X, Y),
M = \begin{pmatrix} \Sigma B & O \\ \Pi B & -I \end{pmatrix} \in L(X, Y).$$

Then it can be shown directly (see [26]), that the operator $M$ is $(L, 0)$-bounded and the projectors have the form

$$P = \begin{pmatrix} I & 0 \\ \Pi B & 0 \end{pmatrix},
Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $X^0 = \{0\} \times H_\Pi$, $X^1 = \{(w_1, w_2) \in H^1_\sigma \times H^1_\Pi : w_2 = \Pi Bw_1\}$, $Y^0 = \{0\} \times H_\Pi$, $Y^1 = H_\sigma \times \{0\}$. As in the previous section Theorem 3 at $p = 0$, $s \equiv 0$ implies the next result.

**Theorem 5.** Let $h \in C([r, 0]; H^1_\sigma)$, $z_k \in H^1_\sigma$, $k = 0, 1, \ldots, m - 1$, $h(\cdot, 0) = z_0(\cdot), K_i \in C^m([-r, 0]; R)$, $K_i^{(n)}(-r) = K_i^{(n)}(0) = 0$ at $n = m, 1, \ldots, m - 1$, $i = 1, 2$. Then there exists $T > 0$, such that problems (17), (18), (20), (22) have a unique solution.

6. Conclusions

We studied the local unique solvability of the problem with the generalized Showalter–Sidorov conditions, which is associated by a given background for degenerate fractional evolution equations in Banach spaces with delay, including the Gerasimov–Caputo derivative and a relatively bounded pair of linear operators. The complexity of the studied problem is the simultaneous presence of a fractional derivative, a degenerate operator at it, and a delay argument in the equation. The obtained result shows that by the methods of the theory of resolving families of operators for degenerate evolution equations, this complex problem can be solved. Abstract results can be used for investigating problems for partial differential equations, demonstrated on a problem for Scott–Blair and modified Sobolev systems of equations with delays.

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