AN ALGEBRAIC APPROACH TO THE ALGEBRAIC WEINSTEIN CONJECTURE

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Abstract. How does one measure the failure of Hochschild homology to commute with colimits? Here I relate this question to a major open problem about dynamics in contact manifolds — the assertion that Reeb orbits exist and are detected by symplectic homology. More precisely, I show that for polarizably Weinstein fillable contact manifolds, said property is equivalent to the failure of Hochschild homology to commute with certain colimits of representation categories of tree quivers. So as to be intelligible to algebraists, I try to include or black-box as much of the geometric background as possible.

Existence of closed geodesics on a compact Riemannian manifold $M$ is guaranteed for topological reasons [14, 25]. Let us recall the argument. Morse theory tells us that the homology of the free loop space $L M = \text{Maps}(S^1 \to M)$ can be computed by a complex generated by geodesics. The trivial loops contribute a subcomplex computing the homology of the original manifold, so there must be nontrivial loops unless $H_\bullet(M) \to H_\bullet(L M)$ is an isomorphism. It is obviously not an isomorphism unless $M$ is simply connected, and in this case we study the based loop space $\Omega M := \text{Maps}((S^1, 0) \to (M, m))$ and the fibration $\Omega M \to LM \to M$. As this is split by constant loops $M \to LM$, we find $\pi_k(\Omega M) = \pi_k(M) \oplus \pi_{k+1}(M)$, so by Hurewicz the first nontrivial homotopy group $\pi_{k+1}(M)$ contributes nontrivially to $H_k(LM)$, while $H_k(M)$ vanishes. In fact, in the simply connected case, one can obtain rather more refined information [42].

We would like to think of the map $H_\bullet(M) \to H_\bullet(L M)$ as arising from the following local-to-global construction. On $M$, we can consider the constant cosheaf of spaces, which we denote $\Omega$. By definition, this assigns a point to any contractible open set, and sends covers to colimits. By the $(\infty)$ van Kampen theorem, $\Omega(U)$ is the path groupoid of $U$, explaining the notation. Composition of loops gives $H_\bullet(\Omega(M))$ the structure of a ring, and its Hochschild homology gives the homology of the free loop space [17, 20]: $HH_\bullet(H_\bullet(\Omega M)) = H_\bullet(L M)$. The inclusion of constant loops is:

$$H_\bullet(M) = \text{colim}_U HH_\bullet(H_\bullet(\Omega(U))) \to HH_\bullet(\text{colim}_U H_\bullet(\Omega(U))) = HH_\bullet(L M)$$

The following is the first example of the problem we are interested in:

Question 1. Can the failure of (1) to be an isomorphism be seen in terms of some general machinery measuring the failure of Hochschild homology to commute with homotopy colimit?

We turn to contact geometry. The contact-geometric formulation of the geodesic flow is the following. On a cotangent bundle $T^* M$, there is the tautological 1-form $\lambda$, which at a given covector $\xi$ is the function on tangent vectors given by $\xi$. Fixing a metric on $M$, we may restrict $\lambda$ to the cosphere bundle $S^* M$; here it is contact, meaning that $\lambda \wedge (d\lambda^{n-1})$ is nowhere vanishing. The Reeb vector field $R$ on $S^* M$ is characterized by lying in the kernel of $d\lambda$ and normalized by $\lambda(R) = 1$. Its flow is naturally identified with the geodesic flow.

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1We abuse terminology to respect intuition and write ‘homology’ when we mean that we think of the complex as an object in the derived category. Similarly we write ‘$=$’ to mean e.g. identified by a canonical quasi-isomorphism, etc. We always work with $\infty$-categories, etc., and (pre)sheaves or cosheaves should be understood accordingly.
More generally, the same formulas define the Reeb flow for any contact form on any (odd dimensional) manifold \( V \). The Weinstein conjecture asserts the existence of a closed trajectory \([47]\).

It has long been a central problem in contact geometry \([44][22][23][21]\), and known in general only in dimension 3, by an argument whose ingredients have no known analogue in higher dimension \([40]\). The result is also known for flexible contact structures in general \([4]\); we will be interested here in what is in some sense the opposite setting \([31]\), of Weinstein fillable contact manifolds.

It is natural to try and generalize the Morse theoretic approach to geodesics to the study of Reeb orbits. That is, one wants a complex generated by orbits, so that nonvanishing of the homology groups implies the existence of orbits. The reason to impose a differential is to provide invariance under deformations: these homologies depend only on \( \ker \lambda \) rather than \( \lambda \) itself, i.e. on the contact structure rather than the contact form\(^2\).

Just as the cosphere bundle bounds the codisk bundle, we may ask that some general contact \((V, \lambda)\) is the boundary of some \( W \) to which \( \lambda \) extends, and over which \((d\lambda)^n\) is everywhere nonvanishing. We also ask that the ‘Liouville’ vector field \( Z \) characterized by \( d\lambda(Z, \cdot) = \lambda \) points out at the boundary; in the case of the cotangent bundle this vector field is radial in the fiber directions. Such \( W \) are called Liouville domains, and determine a symplectic cohomology \( \text{SH}^\bullet(W) \), which may be taken to be generated by the Reeb orbits of \((V, \lambda)\) and the critical points of a Morse function on \( W \) \([46]\). (See \([33]\) for a leisurely introduction to Liouville manifolds and symplectic cohomology.) The differential is such that these Morse critical points form a subcomplex on which

\[
\text{SH}^\bullet(W) \rightarrow \text{SH}^\bullet(W) \rightarrow \widehat{\text{SH}}^\bullet(W) \xrightarrow{[1]} \]

In particular, if \( \text{H}^\bullet(W) \rightarrow \text{SH}^\bullet(W) \) fails to be an isomorphism, then the Weinstein conjecture holds for \( V \), as \( \widehat{\text{SH}}^\bullet(W) \), which is generated by the Reeb orbits of \( V \), must be nonzero.

Viterbo’s algebraic Weinstein conjecture is the assertion that \( \widehat{\text{SH}}^\bullet(W) \) always detects an orbit. It is not typically easy to compute \( \text{SH}^\bullet(W) \). But when \( W = T^* M \), one knows:

**Theorem 2.** \([45][43]\) *There is an isomorphism \( \text{SH}^{\bullet+n}(T^* M) \cong H_{\ ·}(\mathcal{L}M) \).*\(^3\)

This isomorphism has seen many further developments; see e.g. \([10][8]\).

The composition \( \text{H}^{\bullet+n}(T^* M) \rightarrow \text{SH}^{\bullet+n}(T^* M) \cong H_{\ ·}(\mathcal{L}M) \) is identified with the inclusion of constant loops \( H_{\ ·}(M) \rightarrow H_{\ ·}(\mathcal{L}M) \) under Poincaré duality \([2]\) Lem. 3.6. So, the symplectic homology detects geodesics in essentially the same way as the Morse homology of the loop space did. However already in this case it does more: it shows that a contact level of \( T^* M \), not necessarily the unit cosphere bundle for any Riemannian metric, will also have Reeb orbits.

A class of Liouville domains including but rather more general than codisk bundles are the Weinstein domains\(^4\). By definition, these are those for which the Liouville vector field is gradient-like for a Morse-Bott function. In this case the critical points of \( Z \) have index \( \leq \dim W/2 \), and union

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\(^2\)Taubes’s celebrated work in 3 dimensions \([40]\) also can be understood in terms of such a homology theory \([41]\), though the theory he constructs (embedded contact homology) turns out to in fact be completely independent of \( \lambda \), and in fact isomorphic to a topological invariant (Seiberg-Witten homology) known to be sufficiently nontrivial to imply existence of orbits. Unfortunately, in higher dimensions, there are no known analogues of the embedded contact homology or the Seiberg-Witten homology.

\(^3\)There are many differing convention for the grading of symplectic cohomology, and also for which item to call symplectic cohomology and which symplectic homology. We follow \([1][19]\).

\(^4\)When \( M \) is not spin, it is necessary to twist one side or the other by a local system \([27]\).

\(^5\)Weinstein manifolds and the Weinstein conjecture have the same eponym, but a priori no other relation.
of descending level sets is a singular isotropic subset termed the core or skeleton. Stein domains from complex analysis are Weinstein when viewed as symplectic manifolds, and conversely any Weinstein domain is deformation equivalent to a Stein domain \[9\].

Weinstein domains for which the indices of critical points are \(< \dim W/2\) are said to be sub-critical, and for these it is known that \(\text{SH}^*\left(W\right) = 0\); in particular, the Weinstein conjecture holds for their contact boundaries \[43\]. Simple examples: the ball; the cotangent bundle of a noncompact manifold. Beyond these, the Weinstein conjecture is not known for contact boundaries of Weinstein domains in any reasonable generality, and it would be a major advance to establish it.

One available tool for computing symplectic homology is the open-closed morphism from the Hochschild homology of the wrapped Fukaya category \[6\]\[34\][15][32][35][6][1][19]:

\[
\text{HH}_n\left(\text{Fuk}(W)\right) \rightarrow \text{SH}^{*+n}(W)
\]

By either \[1\][16] or \[6\][13][12] plus a ‘generation’ result \[8\][18], this morphism is by now known to be an isomorphism for Weinstein domains.

For cotangent bundles, there is an object (the cotangent fiber) of \(F \in \text{Fuk}(T^*M)\), which generates the category and for which \(\text{Hom}(F, F) = \Omega_{-\cdot}(\Omega M)\). Thus the open-closed map induces

\[
\text{HH}_n\left(\Omega_{-\cdot}(\Omega M)\right) = \text{HH}_n\left(\text{Hom}(F, F)\right) \rightarrow \text{SH}^{*+n}(T^*M) = \Omega_{-\cdot}(LM)
\]

This is the same as the corresponding such morphism mentioned above \[7\].

How can the open closed map help us? At first, it does not look promising. We have not said what the Fukaya category is, but its definition involves the same sort of geometrical structures as are involved in symplectic homology. On top of this we have now added the nontrivial step of taking Hochschild homology. However, just as \(\Omega M\) has better local-to-global behavior than \(LM\), we also have (as anticipated by \[26\]):

**Theorem 3.** \[19][18][17][37][30\] The Fukaya category of a Weinstein manifold is the global sections of a constructible cosheaf of categories over the skeleton. Moreover, this cosheaf is isomorphic to the cosheaf of microlocal sheaves \[8\].

We have not said what microlocal sheaves are and it will not be relevant; but for a definition see \[30\], which is built on the technology of \[24\]. What is relevant is that microlocal sheaves are in principle combinatorial-topological in nature, but in practice the stalks of the above cosheaf may be complicated categories at complicated singularities of the skeleton. When the skeleton is smooth, the cosheaf is locally constant with stalk the category of chain complexes.

For \(W = T^*M\), the cosheaf is simply the path groupoid \(\Omega\) (twisted by a local system if \(M\) is not spin). More generally, Nadler found an explicit collection of so-called ‘arboreal’ singularities \[29][28\], with the property that:

**Theorem 4.** \[29\] When the skeleton is arboreal, the cosheaf of microlocal sheaves has stalks given by representation categories of tree quivers. The cogeneration morphisms are explicit.

When \(\dim W = 2\), the skeleton of \(W\) will be a (ribbon) graph, and ‘arboreal’ essentially amounts to asking that the graph is trivalent. The cosheaf \(\mathcal{A}\) will assign the category of chain

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6 \[34][19\] contain the relevant definitions. We will soon cite some results which compute the Fukaya category in all relevant cases, so the gist of the article will not be lost to the reader with no idea what the Fukaya category is.

7 I am not certain whether this follows from \[2\], but in any case it certainly does from \[19][18][17\].

8 Strictly speaking, what is presently in the literature requires that the Weinstein manifold is ‘stably polarizable’. It is known to experts how to remove this hypothesis; on the other hand we will impose it later for different reasons.
complexes at smooth points, and the category of exact triangles (aka $\text{Perf}(\bullet \to \bullet)$) at the trivalent vertices, with the obvious cogeneration morphisms. This case was studied in [11]. For $\dim W = 4$, the skeleton is two dimensional; the topology of the typical new kind of singularity is depicted in Figure 1. More geometric pictures can be found in [39, 5].

By an arboreal space, we will mean a pair $(X, \mathcal{A})$ of a space and a constructible cosheaf, which are locally given by Nadler’s explicit models. In particular, the stalks are representation categories of tree quivers. An explicitly combinatorial exposition of this notion can be found in [38].

Some further restrictions on the local models lead to the notion of positive arboreal space [5]. These provide skeleta for a large class of Weinstein manifolds:

**Theorem 5.** [5] There is an equivalence of categories between:
- Stably polarized Weinstein manifolds & their homotopies
- Positive arboreal spaces & their concordances

Here, a stable polarization is a choice of Lagrangian sub-bundle of $TW \oplus \mathbb{C}^n$ for some $n$.

**Remark.** It is not clearly understood what weaker hypothesis corresponds to taking all arboreal spaces; in dimension 4, none is needed [39], but this is apparently not the case in general [5]. It is expected that this can be repaired by adding some further explicit list of presently unknown singularities. There are also other tricks for reducing problems to the stably polarized case, like [30, Sec. 10]. In any case, I expect that any technique which works for stably polarized Weinstein manifolds should work for Weinstein manifolds in general.

In some sense we have already arrived at a reduction to algebra: one could try and develop tools for computing the colimit of categories giving $\mathcal{A}(X)$ or its Hochschild homology $HH_\bullet(\mathcal{A}(X))$. Indeed, as far as anyone knows, $SH^\bullet(W)$ is always either infinite dimensional, or zero. If one could show this zero or infinite property for $HH_\bullet(\mathcal{A}(X))$, the algebraic Weinstein conjecture would follow. Or if one could show that the natural circle action on any nonvanishing $HH_\bullet(\mathcal{A}(X))$ is nontrivial, the result would again follow.

Here we want to point out that in fact it is possible to directly generalize Equation (1) and make direct contact with Equation (2). Consider a cosheaf of categories $\mathcal{A}$ over a space $X$. The Hochschild homologies form a precosheaf $HH_\bullet(\mathcal{A})$ which is not generally a cosheaf, since Hochschild homology does not commute with colimit. We may cosheafify it and obtain a cosheaf $\mathcal{H}H_\bullet(\mathcal{A})$. There is a natural map

$$\Gamma(X, \mathcal{H}H_\bullet(\mathcal{A})) \to HH_\bullet(\mathcal{A}(X))$$

Because $\mathcal{A}$ is constructible, the LHS is the colimit of the Hochschild homologies of the stalks of $\mathcal{A}$, and the RHS is the Hochschild homology of the colimit of the stalks.
Moreover, $\mathcal{HH}_\bullet(A)$ can be computed explicitly. Indeed, for a tree quiver $T$, it is the case that $\mathcal{HH}_\bullet(\text{Perf}(T))$ is concentrated in degree zero, and is a free module whose rank is the number of vertices of $T$. This gives the stalks of $\mathcal{HH}_\bullet(A)$, and in fact one can show:

**Theorem 6.** [38] When $(X,A)$ arises from the skeleton of a stably polarizable Weinstein manifold, $\mathcal{HH}_{\bullet-n}(A)$ is the cosheaf of compactly supported cohomologies. As $X$ is compact, we have $\Gamma(X, \mathcal{HH}_{\bullet-n}(A)) \cong \mathcal{H}^\bullet(X)$.

In fact, it is possible to show that the resulting map

$\mathcal{H}^\bullet(X) \cong \Gamma(X, \mathcal{HH}_{\bullet-n}(A)) \to \mathcal{HH}_{\bullet-n}(A(X)) \to \mathcal{SH}^\bullet(W)$

agrees with the original $\mathcal{H}^\bullet(X) = \mathcal{H}^\bullet(W) \to \mathcal{SH}^\bullet(W)$. This follows from [19 Eq. 1.7] given that the local arboreal models are Weinstein sectors, as was shown in [36]. In some more detail: in [36] it is shown that the nondegenerate arboreal sectors are stopped; hence the top row of [19 Eq. 1.7] consists of isomorphisms. Degenerate arboreal singularities are obtained by stop removal; the desired commutativity descends using the stop removal localization sequence and the fact that Hochschild homology sends localizations to exact triangles.

Putting all this together we have:

**Theorem 7.** The algebraic Weinstein conjecture for contact manifolds with stably polarizable Weinstein filling is equivalent to the assertion that $\Gamma(X, \mathcal{HH}_\bullet(A)) \to \mathcal{HH}_\bullet(A(X))$ is never an isomorphism for positive arboreal spaces $(X,A)$.

We are left with the following:

**Question 8.** What measures the failure of Hochschild homology to commute with colimits?

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**References**

[1] Mohammed Abouzaid. A geometric criterion for generating the Fukaya category. *Publications Mathématiques de l’IHÉS*, 112:191–240, 2010.

[2] Mohammed Abouzaid. A cotangent fibre generates the Fukaya category. *Advances in Mathematics*, 228(2):894–939, 2011.

[3] Mohammed Abouzaid. Symplectic cohomology and Viterbo’s theorem. *arXiv:1312.3354*, 2013.

[4] Peter Albers and Helmut Hofer. On the Weinstein conjecture in higher dimensions. *arXiv:0705.3953*.

[5] Daniel Alvarez-Gavela, Yakov Eliashberg, and David Nadler. Positive arborealization of polarized Weinstein manifolds. *arXiv:2011.08962*.

[6] Frédéric Bourgeois, Tobias Ekholm, and Yakov Eliashberg. Effect of Legendrian surgery. *Geometry & Topology*, 16(1):301–389, 2012.

[7] Dan Burghelea and Zbigniew Fiedorowicz. Cyclic homology and algebraic k-theory of spaces—ii. *Topology*, 25(3):303–317, 1986.

[8] Baptiste Chantraine, Georgios Dimitroglou Rizell, Paolo Ghiggini, and Roman Golovko. Geometric generation of the wrapped Fukaya category of Weinstein manifolds and sectors. *arXiv:1712.09126*.

[9] Kai Cieliebak and Yakov Eliashberg. *From Stein to Weinstein and back: symplectic geometry of affine complex manifolds*. American Mathematical Soc., 2012.

[10] Kai Cieliebak and Janko Latschev. The role of string topology in symplectic field theory. *New perspectives and challenges in symplectic field theory*, 49:113–146, 2009.

[11] Tobias Dyckerhoff and Mikhail Kapranov. Triangulated surfaces in triangulated categories. *arXiv:1306.2545*. 
[12] Tobias Ekholm. Holomorphic curves for Legendrian surgery. arXiv:1906.07228.
[13] Tobias Ekholm and Yanki Lekili. Duality between Lagrangian and Legendrian invariants. arXiv:1701.01284.
[14] Abram Fet and Lazar Lyusternik. Variational problems on closed manifolds. Dokl. Akad. Nauk. SSSR, 81:17–18, 1951.
[15] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Lagrangian intersection Floer theory: anomaly and obstruction. American Mathematical Soc., 2010.
[16] Tobias Ekholm and Yanki Lekili. Duality between Lagrangian and Legendrian invariants. arXiv:1701.01284.
[17] Abram Fet and Lazar Lyusternik. Variational problems on closed manifolds. Dokl. Akad. Nauk. SSSR, 81:17–18, 1951.
[18] Abram Fet and Lazar Lyusternik. Variational problems on closed manifolds. Dokl. Akad. Nauk. SSSR, 81:17–18, 1951.