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Component mode synthesis methods using partial interface modes: Application to tuned and mistuned structures with cyclic symmetry

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We present new component mode synthesis methods using partial interface modes which are the structure normal modes resulting from the static condensation of the structure to the interface between the substructures and which are possibly clamped at a part of this interface. These methods are the generalization of the classical component mode synthesis methods and those using the interface modes. These methods allow to reduce the number of the interface coordinates and at the same time to keep some of the physical interface displacements. These methods are applied to a structure with cyclic symmetry in both tuned and mistuned cases.

1. Introduction

Component mode synthesis (CMS) or dynamic substructuring methods consist in performing the dynamic analysis of structures by decomposing the structure into substructures and by projecting the equation of motion of each substructure on a projection basis to obtain the reduced systems of the substructures before performing the substructure coupling to obtain the reduced system of the whole structure. In the classical CMS methods, the substructure projection basis is composed, on one hand, by the normal modes of the substructure with various boundary conditions at the interface, such as fixed interface [18,19,36,39,40], free interface [3,14,18,20,32,36,37,49,50,61], hybrid interface [20,28,49,73,74], or loaded interface [6], and on the other hand, by Ritz vectors derived from the static deformation shapes commonly called the static modes, such as the constraint modes, the attachment modes, the residual attachment modes etc. CMS methods have been described in several text books [5,31,41,52,54], many insights, variants and improvements have been proposed [1,8,23,26,27,29,38,42–46,48,56,58–60,64,66,69–71], and CMS methods have been widely used for a large range of applications [4,7,9,10,15–17,22,24,30,35,47,51,53,55,62,63,65,67,76–80]. A history, review and classification of CMS methods can be found in [25].

In the classical CMS methods, the generalized coordinates associated with the static modes are in most of the cases the displacements at the interface between the substructures, leading to reduced systems with large size due to the important number of degrees of freedom (DOF) at the interface. In order to reduce the number of interface DOF, the CMS methods using interface modes has first been developed for the fixed interface CMS method [2,12,13,21] and then extended to the free and hybrid interface CMS methods [72]. In these methods, the static modes are replaced by the interface modes, also called the junction modes or the eigen modes of the Poincaré–Steklov operator, which are the first few normal modes of the whole structure after performing the Guyan static condensation [33] to the interface between the substructures. The interface displacements associated with the static modes in the classical CMS methods are then replaced by a few generalized coordinates associated with the interface modes. Alternative approaches for reducing the interface DOF were also proposed in [2,11,21,34].

Although the CMS methods using interface modes produce reduced systems with very small size, one drawback is that all the interface DOF are removed from the reduced system. The presence of a part of the interface DOF in the reduced system is however sometimes desirable and even essential, either because these DOF are not numerous and they can provide quick and useful information, or because one needs to deal directly with them while solving the reduced system, for example to impose prescribed motions or to take into account local non-linearities such as contact, friction or free-play. The aim of this paper, which is a continuation of the work in [72], is to develop new CMS methods using partial interface modes which fix this drawback. These methods allow at the same time an important reduction of the number of the
interface DOF like in the CMS methods using interface modes, and the conservation of some interface DOF in the reduced system like in the classical CMS methods. To reach this aim, the approach in this work is that, instead of computing the interface modes, the latter are approximated by applying a second level CMS method on Guyan’s reduced system resulting from the static condensation of the whole structure to the interface between the substructures. The DOF of Guyan’s reduced system are partitioned into two sets containing respectively the interface DOF to be eliminated and those to be kept in the final reduced system, the former being considered as the interior DOF and the latter as the interface DOF in the second level CMS method. The choice of the kept interface DOF depends on the need of the user to keep them in the reduced system. The partial interface modes are defined as a first few normal modes of Guyan’s reduced system in which some of the kept interface DOF can be clamped, depending upon which CMS method, i.e. with fixed, free or hybrid interface, is applied to Guyan’s reduced system. The partial interface modes are completed with the static modes of Guyan’s reduced system, whose associated generalized coordinates are precisely the kept interface DOF, and together they replace the interface modes or the substructure static modes in the projection basis. The classical methods and the methods using interface modes are particular cases of the new methods using partial interface modes, the former are obtained when all the interface DOF are kept, and the latter when all the interface DOF are eliminated.

This paper is organized as follows: In Section 2, the classical CMS methods and the methods using interface modes are reminded. In Section 3, the new CMS methods using partial interface modes are presented. Section 4 deals with the case of structures with cyclic symmetry. In Section 5, the new CMS methods are applied to compute the eigen frequencies and modes and the frequency response of a tuned and mistuned bladed disk, with several selections of partial interface modes and kept interface DOF. They are compared with the whole structure computations and also with the classical methods and the methods using interface modes.

2. Classical methods and methods using interface modes

2.1. Substructure description, reduced system and coupling

We consider a structure $S$ which is decomposed into $n_s$ substructures $S_j$ ($j = 1, \ldots, n_s$) which do not overlap. We denote by $L^S$ the part of $S$ which consists in the frontier between the substructures and by $L$ the frontier of $S_j$ with the adjacent substructures. $L^S$ and $L$ will be called the interface of $S$ and $S_j$. The interface $L^S$ is partitioned into $L^S_{e}$, the interface DOF to be eliminated and $L^S_{c}$, the interface DOF to be kept, in the final reduced coupled system, and $L^S_{x}$, the interface DOF to be eliminated. The number of DOF in $L^S_{e}$ is very small compared to the number of DOF in $L^S_{c}$. The keep interface $L^S_{c}$ is then partitioned into the fixed keep interface $L^S_{ck}$ and the free keep interface $L^S_{ck}$.

For each substructure $S_j$, the interface $L$ is also partitioned into the fixed interface $L_e$ and the free interface $L$, thus $L$ can be fixed ($L_c = L$ and $L_e = L$), free ($L_c = \emptyset$ and $L_e = L$) or hybrid ($L_c \neq \emptyset$, $L_c \neq \emptyset$, $L_e = L_c \cup L$), in the latter case $S_j$ is supposed to be constrained when $L_e$ is fixed. The choice of $L_c$ and $L$ can be different from one substructure to another, and it is completely independent of the choice of $L^S_{ck}$ and $L^S_{c}$.

The equilibrium equation of the isolated substructure $S_j$ is written as:

$$Kx + Cx + Mx = f_e - P_a f_e.$$  \hfill (1)

$K, C$ and $M$ are the stiffness, damping and mass matrices of $S_j$, $f_e$ are the external forces applied on $S_j$ and $f_e$ are the interface reactions exerted by $S_j$ on $L_c$. The left superscript $^t$ denotes the transpose of a vector or a matrix.

The CMS methods consist in expressing the displacements of the substructure as a linear combination of the Ritz vectors in a projection basis $\mathbf{Q}$, i.e. $x = \mathbf{Q} \mathbf{q}$, where $\mathbf{q}$ is the vector of the generalized coordinates. By projecting the equilibrium equation (1) on the projection basis $\mathbf{Q}$, we obtain a reduced system:

\begin{align*}
\text{(a) Fixed interface method (CB)} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + \\
\text{Fixed interface normal modes $\Phi$} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + x = 1
\end{align*}

\begin{align*}
\text{(b) Free interface method} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + \\
\text{Free interface normal modes $\Phi$} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + f = 1
\end{align*}

\begin{align*}
\text{(c) Hybrid interface method} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + \\
\text{Hybrid interface normal modes $\Phi$} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + f = 1
\end{align*}

\begin{align*}
\text{Homogenized free interface normal modes $\Phi^h = \Phi - \Psi^a a^t \Phi$} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + \\
\text{Normalized attachment modes $\Psi^a = \Psi_c \rightarrow \Psi^a$ (CB)} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + f = 1
\end{align*}

\begin{align*}
\text{Constraint modes $\Psi_c$} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + \\
\text{Attachment modes $\Psi_a$} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + f = 1
\end{align*}

\begin{align*}
\text{Homogenized hybrid interface normal modes $\Phi^h = \Phi - \Psi^a a^t \Phi$} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + \\
\text{Constraint modes $\Psi_C$ (CB)} & \quad \begin{array}{c}
S_1 \quad L_1 \\
S_2
\end{array} + f = 1
\end{align*}

\begin{align*}
\text{Fig. 1. Classical CMS methods.}
\end{align*}
The coupling of the substructures is fairly simple, it is performed by using the continuity of the interface displacements and the equilibrium of the interface reactions. It corresponds the primal coupling formulation described in [25] which consists in assembling the substructure reduced matrices to form the reduced matrices of the whole structure, similarly to the assembly of the elementary matrices in the finite-element procedure. In the assembled reduced matrices, the substructure reduced matrices are coupled through the interface generalized coordinates which are at least common to two substructures, such as the interface displacements or the generalized coordinates associated with the interface modes or the partial interface modes.

2.2. Classical CMS methods

In the classical hybrid interface methods using attachment modes [28,72,74], the displacements of $S_j$ are expressed as (Fig. 1c):

$$
X = (\Phi - \Psi_j^a - \Phi)X_0 + (\Psi_j^c - \Psi_j^a - \Phi)X_L + \Psi_j X_a
= \Phi X_0 + \Psi_j X_L + \Psi_j X_a,
$$

(3)

where the left superscripts $s() = P_s(\cdot)$ and $c() = P_c(\cdot)$ denote the restrictions to the free interface $L_a$ and the fixed interface $L_f$. $\Phi$ are the hybrid interface normal modes of $S_j$, i.e. with $L_a$ fixed and $L_f$ free; $\Psi_j$ are the constraint modes obtained by imposing unit displacements on $L_a$ and with $L_f$ fixed; $\Psi_j^c = \Psi_j^c - \Psi_j^a - \Phi$ are the homogenized constraint modes; and $\Phi = \Phi - \Psi_j^a - \Phi$ are the homogenized normal modes. We have: $\Psi_j^a = 0$, $\Phi = 0$, $\Psi_j^c = 1$, $\Phi = 0$ and $\Phi = 0$.

The Craig and Bampton (CB) classical fixed interface method [19] (Fig. 1a) is a particular case of Eq. (3) in which $\Psi_j$ and $\Psi_j^c$ do not exist, $\Phi$ are the fixed interface normal modes, and $\Psi_j^c = \Psi_j^c - \Phi$, where $\Psi_j^c$ are the constraint modes of the CB method.

The classical free interface method using the attachment modes [36,74] (Fig. 1b) is also a particular case of Eq. (3) in which $\Psi_j$ and $\Psi_j^c$ do not exist, $\Phi$ are the free interface normal modes. For unconstrained substructures, the rigid body modes should be included in $\Phi$, and a special treatment should be used when computing $\Psi_j^c$, which consists in balancing the applied unit forces with the inertia forces and in orthogonalizing the static solutions to the rigid body modes [72].

It has been showed in [72] that in the hybrid interface methods, the Ritz vectors $[\Psi_j^c, \Psi_j^a]$ associated with the interface displacements in Eq. (3) are exactly the constraint modes $\Psi_j^c$ of the CB method. In the free interface method, $\Psi_j^c$ are equal to $\Psi_j^c$ for constrained substructures, while for unconstrained substructures a variant can be obtained by replacing $\Psi_j^c$ by $\Psi_j^c$. By using this variant of the free interface method, Eq. (3) can be written for the three types of interface as:

$$
X = \Phi X_0 + \Psi_j X_L.
$$

(4)

The reduced system Eq. (2) is obtained by projecting Eq. (1) on the projection basis $Q = [\Phi, \Psi_j^c]$, the unknowns are the modal coordinates $X_0$ and the interface displacements $X_L$, and the substructure coupling is performed through $X_L$.

Some variants [49,61,73,74] use the residual attachment modes $\Psi_j = \Psi_j^c + \Phi(\Phi^T K \Phi)^{-1} \Phi$ instead of the attachment modes. Although the attachment modes and the residual attachment modes should theoretically give the same results since the two sets $[\Phi, \Psi_j^a]$ and $[\Phi, \Psi_j]$ span the same subspace, the redundant contribution of $\Phi$ in $\Psi_j^a$ being simply removed in $\Psi_j$, it is easier to compute $\Psi_j$ than $\Psi_j^a$, and the inversion of $\Phi^T K \Phi$ for obtaining $\Psi_j$ is subjected to numerical problems since the terms of $\Phi^T K \Phi$ are very small when the number of the normal modes retained in $\Phi$ is important. Moreover, contrarily to the attachment modes, the residual attachment modes do not satisfy the relationships $\Psi_j^c = \Phi$ for the free interface method and $[\Psi_j^c, \Psi_j^a] = \Phi$ for the hybrid interface method.

2.3. CMS methods using interface modes

At the whole structure level, the displacements of $S$ are expressed from Eq. (4) as:

$$
x^s = \Phi^x X_0^s + \Psi_j X_L^s.
$$

(5)

where the homogenized normal modes $\Phi^x$ and the constraint modes $\Psi_j^c$ of $S$ are the extensions of the substructure homogenized normal modes $\Phi$ and constraint modes $\Psi_j^c$ to $S$ by completing with zeros on the other substructures. $\Psi_j^c$ are also the global constraint modes of $S$ obtained by imposing unit displacements on $L^s$. 

![Fig. 2. CMS methods using interface modes and partial interface modes. (G) is Guyan's reduced system Eq. (6).](image-url)
In order to reduce the number of interface DOF, the constraint modes $\Psi_p^i$ in Eq. (5) are replaced by the interface modes which are obtained by performing Guyan’s static condensation of the whole structure $S$ on the interface $L^i$, i.e. by expressing the displacements of $S$ as $\mathbf{x}^i = \Psi^i \mathbf{x}_i^1$. By projecting the free equation of motion of $S$ on $\Psi^i$, we obtain Guyan’s reduced system:

$$K^i \mathbf{x}_i^1 = M^i \mathbf{x}_i^1 \Omega^i,$$

where $K^i = M^i \Psi^i \mathbf{K} \Psi^i$ and $M^i = \Psi^i \mathbf{M} \Psi^i$. $\mathbf{K}$ and $\mathbf{M}$ being the stiffness and mass matrices of $S$.

The interface modes are the eigen vectors $\mathbf{x}_i^1$ of Guyan’s reduced system Eq. (6). The structure interface modes $\Phi_i$ are the approximated normal modes of $S$ obtained by the expansion of $\mathbf{x}_i^1$ to $S$ using $\Psi^i$, and the substructure interface modes $\Phi_0$ of $S$ are the restriction of $\Phi_i$ to $S_0$ (Fig. 2a):

$$\Phi_i = \Psi^i \mathbf{x}_i^1 \quad \text{and} \quad \Phi_0 = P^i \Phi_i = \Psi^i \mathbf{x}_i^1,$$

where the left superscript ‘$i$‘ ($P^i$) denotes the restriction to the interface $L^i$.

In practice, the interface modes are computed by projecting the stiffness and mass matrices of $S_0$ on $\Psi^i$ and then by assembling the resulting reduced matrices to obtain the reduced system Eq. (6), like in the CB method but without the normal modes $\Phi_i$. Since the whole set of all the structure interface modes $\Phi_i$ and the structure constraint modes $\Psi^i$ span the same subspace, a truncation of the subspace spanned by $\Psi^i$ is obtained by keeping among the solutions of Eq. (6) only the few interface modes $\mathbf{x}_i^1, \Phi_i$ and $\Phi_0$ corresponding to the lowest frequencies in $\Omega^i$.

The CMS method using interface modes \{12,21,72\} consists in replacing the constraint modes $\Psi^i$ by the interface modes $\Phi_i$ in Eq. (5), which is the same as replacing the constraint modes $\Psi^i$ by the interface modes $\Phi_0$ in Eq. (4). The physical displacements of $S_0$ are expressed as:

$$\mathbf{x} = \Phi_0 \mu + \Phi_0 \mu_0.$$

The generalized coordinates $\mu_0$ are not associated with any particular substructure, they are in contrary common to all the substructures. The reduced system of $S_0$ is obtained by projecting Eq. (1) on the projection basis $Q = [\Phi_0, \Phi_i]$ and the substructure coupling is performed through the interface coordinates $\mu_0$, Eq. (8) provides reduced coupled systems with a much smaller size than Eqs. (3) and (4), which however do not contain any interface displacements of $\mathbf{x}_i$ and $\mathbf{x}_i^1$.

3. CMS methods using partial interface modes

In the classical CMS methods or the methods using interface modes, the interface displacements $\mathbf{x}_i^1$ are either kept or eliminated in their totality in the final reduced coupled system. If we want to keep the displacements of the interface $L_i^1$ and eliminate those of $L_i^0$, the idea is to replace the interface modes by another set of vectors whose some of the associated generalized coordinates are precisely the displacements $\mathbf{x}_i^1$ of $L_i^1$. To this aim, we consider Guyan’s reduced system Eq. (6) as the free equation of motion of a structure whose DOF are the displacements $\mathbf{x}_i^1$ of $L_i^1$. We can then apply again a second level CMS method to Eq. (6) by considering $L_i^1$ as the interior DOF and $L_i^0$ as the interface DOF. The partial interface modes are the normal modes of Eq. (6) which are possibly clamped at a part of the kept interface $L_i^1$, depending upon which CMS method, i.e. with fixed, free or hybrid interface, is applied to Eq. (6). These partial interface modes are completed with the static modes of Eq. (6) associated with $L_i^2$ both sets are then expanded to the whole structure and then restricted to the substructures in order to form the projection basis of the substructure displacements. This approach is similar to the Ritz reduction of junction coordinates described in [21], where the Ritz vectors were however not specified in the general case and they were obtained from analytical functions in an application.

For instance, if the hybrid interface method given in Eq. (3) is applied to Guyan’s reduced system Eq. (6), the kept interface $L_i^1$ is partitioned into the fixed kept interface $L_i^1$ and the free kept interface $L_i^1$. The hybrid partial interface modes $\mathbf{x}_i^2$ are the normal modes of Guyan’s reduced system Eq. (6) with hybrid interface, i.e. with $L_i^0$ fixed and $L_i^1$ free, while the constraint modes $\mathbf{x}_i^1$ of Eq. (6) are obtained by imposing unit displacements on $L_i^0$ and with $L_i^1$ free, and the attachment modes $\mathbf{x}_i^0$ of Eq. (6) are obtained by applying opposite of unit forces on $L_i^0$ and with $L_i^1$ fixed (Fig. 2d). Similarly to Eq. (3), the interface modes $\mathbf{x}_i^1$ of Eq. (6) are then expressed as:

$$\mathbf{x}_i^1 = \mathbf{x}_i^0 \mathbf{B} + \mathbf{x}_i^0 \mathbf{C}_{\alpha k} \mathbf{x}_i^1 + \mathbf{x}_i^0 \mathbf{C}_{\alpha k} \mathbf{x}_i^1,$$

where the superscripts $\alpha$, $\alpha k$, and $\alpha k$ denote the restrictions to $L_i^0$, $L_i^1$, and $L_i^1$, and with $\mathbf{x}_i^0$ are the hybridized generalized constraint modes of Eq. (6). Since $\mathbf{x}_i^1 = 0$, $\mathbf{x}_i^0 = 1$, and $\mathbf{x}_i^1 = 0$, $\mathbf{x}_i^0 = 0$ and $\mathbf{x}_i^1 = 1$, it is clear that the generalized coordinates associated with $\mathbf{x}_i^0$ and $\mathbf{x}_i^1$ in Eq. (9) are the restrictions $\mathbf{x}_i^0$ and $\mathbf{x}_i^1$ of $\mathbf{x}_i^1$ to $L_i^0$ and $L_i^1$, respectively.

In order to build the reduced systems of the substructures, we only need the homogenized partial interface modes $\Phi_0$, the homogenized constraint modes $\Phi_0$, and the attachment modes $\Phi_0$, which are obtained in the same way as in Eq. (7), by the expansion using $\Phi_0$, $\Phi_0$, and $\Phi_0$ to $S$ to obtain $\Phi_0$, $\Phi_0$, and $\Phi_0$, and by the restriction of the latter to $S_0$:

$$\Phi_0 = \Psi_0 \mathbf{x}_i^0 \mathbf{B} + \Psi_0 \mathbf{C}_{\alpha k} \mathbf{x}_i^1 + \Psi_0 \mathbf{C}_{\alpha k} \mathbf{x}_i^1,$$

$$\Phi_0 = \Psi_0 \mathbf{x}_i^0 \mathbf{B} + \Psi_0 \mathbf{C}_{\alpha k} \mathbf{x}_i^1 + \Psi_0 \mathbf{C}_{\alpha k} \mathbf{x}_i^1,$$

Since the expansion using $\Phi_0$ of $L_i^1$ to the whole structure keeps the displacements at $L_i^1$ unchanged, we have: $\mathbf{x}_i^1 = 0$, $\mathbf{x}_i^0 = 1$, $\mathbf{x}_i^1 = 0$, $\mathbf{x}_i^0 = 0$ and $\mathbf{x}_i^1 = 1$. From Eqs. (7) and (9), the interface modes of $S_0$ and $S_0$ become:

$$\Phi_0 = \Psi_0 \mathbf{x}_i^0 \mathbf{B} + \Psi_0 \mathbf{C}_{\alpha k} \mathbf{x}_i^1 + \Psi_0 \mathbf{C}_{\alpha k} \mathbf{x}_i^1,$$

$$\Phi_0 = \Psi_0 \mathbf{x}_i^0 \mathbf{B} + \Psi_0 \mathbf{C}_{\alpha k} \mathbf{x}_i^1 + \Psi_0 \mathbf{C}_{\alpha k} \mathbf{x}_i^1,$$

In practice, the partial interface modes $\mathbf{x}_i^0$, the constraint modes $\mathbf{x}_i^1$, and the attachment modes $\mathbf{x}_i^0$ of Guyan’s reduced system Eq. (6) are computed, as well as $\mathbf{x}_i^0$, $\mathbf{x}_i^0$, and $\mathbf{x}_i^0$, with a special treatment for $\mathbf{x}_i^0$, if the whole structure is unconstrained. Only $\Phi_0$, $\Phi_0$, and $\Phi_0$ are then deduced by using Eq. (11).

The CMS method using hybrid partial interface modes consists in replacing the constraint modes $\Psi^i$ by the interface modes $\Phi_i$ given by Eq. (12) in Eq. (5), which is the same as substituting the expression Eq. (13) of the interface modes $\Phi_0$ in Eq. (8). The displacements of the substructure $S_0$ are expressed as:

$$\mathbf{x} = \Phi_0 \mu + \Phi_0 \mu_0 + \Phi_0 \mathbf{x}_i^1 \mathbf{C}_{\alpha k} + \Phi_0 \mathbf{x}_i^1 \mathbf{C}_{\alpha k}.$$  (14)
Table 1: Formulations of CMS methods.

| Number | Method | Type of interface | Formulation |
|--------|--------|-------------------|-------------|
| 1      | CB     | Fixed interface   | \( x = \Phi_{\mu} + \Psi_{\mu} x_{\mu} \) |
| 2      | FA     | Free interface    | \( x = \Phi_{\mu} + \Psi_{\mu} x_{\mu} \) with \( \Psi_{\mu} = \Psi_\nu \) and \( \Phi = \Phi - \Psi_{\nu} \) |
| 3      | HA     | Hybrid interface  | \( x = \Phi_{\mu} + \Psi_{\mu} x_{\mu} \) with \( \Psi_{\mu} = \Psi_{\nu} \) and \( \Phi = \Phi - \Psi_{\nu} \) |
| 4      | FA     | Free interface    | \( x = \Phi_{\mu} + \Psi_{\mu} x_{\mu} \) (identical to FA if constrained substructure) |
| 5      | HA     | Hybrid interface  | \( x = \Phi_{\mu} + \Psi_{\mu} x_{\mu} \) (identical to HA) |

If the fixed interface method \((l_{1b} = l_{3b} = 0)\) or the free interface method \((l_{1b} = l_{3b} = 0)\) is applied to Guyan’s reduced system Eq. (6), we obtain the CMS method using fixed partial interface modes (Fig. 2b) and the CMS method using free partial interface modes (Fig. 2c). Eq. (14) becomes respectively:

\[
\begin{align*}
\text{(15)} & : \quad x = \Phi_{\mu} + \Phi_{\nu} \mu_{\nu} + \Psi_{\mu} x_{\mu} \\
\text{(16)} & : \quad x = \Phi_{\mu} + \Phi_{\nu} \mu_{\nu} + \Psi_{\mu} x_{\mu} 
\end{align*}
\]

Eq. (16) is an improved variant of the CMS method using interface modes given by Eq. (8), since the free partial interface modes \( \Phi_{\nu} \) are exactly the interface modes \( \Phi_{\mu} \), while the attachment modes \( \Psi_{\mu} \) represent a static correction to the truncation of the interface modes. In some cases however, the free partial interface modes \( \Phi_{\nu} \) are not the same as the interface modes \( \Phi_{\mu} \), for example when the cyclic symmetry properties are used in combination with the CMS methods.

The formulations of all CMS methods are summarized in Table 1.

3.1. Remarks

(1) All the DOF of the kept interface \( l_{1b} \), i.e. \( x_{1b} \), are involved in the expressions Eqs. (14)–(16) of the displacements of the substructure \( S_{b} \), and not only the DOF of \( l_{3b} \) who belong to \( S_{b} \), even when \( S_{b} \) does not contain any DOF of \( l_{3b} \). Since the constraint modes \( \Psi_{kk}, \Psi_{Xk} \) and the attachment modes \( \Psi_{\mu}, \Psi_{\nu} \) result from the deformations of the whole structure \( S \) under the solicitations applied or imposed on \( l_{3b} \), a substructure \( S_{b} \) can be deformed even if the solicitations are not applied or imposed on its interface, except when all of its interface DOF are fixed like in the constraint modes \( \Psi_{\mu} \) of the CB method.

(2) The CMS methods using partial interface modes are the generalization of the classical methods and the methods using interface modes. Indeed, if all the interface displacements are eliminated in the reduced system, i.e. \( l_{3b} = 0 \) and \( l_{1b} = L_{1} \), we obtain the methods using interface modes given by Eq. (8). On the other hand, if all the interface displacements are kept in the reduced system, i.e. \( l_{3b} = L_{3} \) and \( l_{1b} = 0 \), we obtain the classical methods given by Eq. (4).

3 If all the interface modes or all the partial interface modes of Guyan reduced system Eq. (6) are kept in \( X_{b} \) and \( X_{\nu} \), i.e. if no truncation is made on these sets, the methods using interface modes and partial interface modes should provide the same results as the classical methods given by Eq. (4). Otherwise, the methods using interface modes and partial interface modes are less accurate than the classical methods. With the same number of interface modes and partial interface modes, the latter are expected to provide better results than the former, since the static modes of Guyan’s reduced system represent a static correction to the truncation of the interface modes.

4 Unlike in the classical Guyan static condensation where the choice of the master and the slave DOF is made without a clear criterion, the choice of the kept interface DOF \( L_{1b} \) and by consequent of the eliminated interface DOF \( L_{3b} \), depends only on the need of the user to keep them in the reduced system. The methods using partial interface modes should therefore be considered as an improvement of the methods using interface modes which, beside the reduction of the size of the reduced system, offers the possibility to keep some interface DOF in the latter. Also unlike in Guyan’s condensation where an important number of the master DOF should be kept to obtain good results, the presence of only a few interface DOF in the reduced system is sufficient to improve significantly the accuracy and the convergence of the results over the methods using interface modes, with only a small additional computation cost, as it will be showed in the numerical example. Therefore, even in the case the interface DOF are not needed in the reduced system, it is better to keep them and use the partial interface modes rather than the interface modes.

4. Case of structures with cyclic symmetry

A structure with cyclic symmetry is composed of \( N \) identical sectors \( S_{0}, S_{1}, \ldots, S_{N-1} \), it is obtained by \( N-1 \) repeated rotations of the reference sector \( S_{0} \) through the angle \( \alpha = \frac{2\pi}{N} \) rd to form
We impose the condition that (but not necessarily), the interface the same condition. The kept interface the same DOF on
ment modes

The normalized attachment modes

with

are expressed in the traveling wave coordinates as:

The reduced systems of the substructures are obtained by projecting the equilibrium equation of the substructures on the projection basis

The coupling of the substructure reduced systems through the generalized coordinates

and the keep interface displacements

provides the reduced system of

where

contains the generalized coordinates

of all the substructures composing

The reduced system is solved by applying the cyclic symmetry conditions

on the left and right frontiers

and

The solutions of Eqs. (22) and (23) provide the generalized coordinates

from which the traveling wave coordinates

of the substructures and

are deduced by using Eq. (21). The real physical displacements of each sector

are obtained as the real part of a summation on

Let us notice that the cyclic symmetry boundary conditions Eq. (17) are applied in two stages, at first on the eliminated left and right frontiers

and

in Eq. (19) when computing the partial interface modes and the static modes of Guyan’s reduced system Eq. (18), and secondly on the kept left and right frontiers

in Eq. (22) when solving the reduced system Eq. (22) of

In two particular cases, the cyclic symmetry conditions are imposed only once on the whole frontiers

and

in the classical methods where

since there is no interface modes or partial interface modes [35]; and (ii) when solving Guyan’s reduced system Eq. (18) in the methods using interface modes where

Moreover, the free partial interface modes are computed by imposing the cyclic symmetry conditions on the eliminated frontiers

and

in Guyan’s reduced system, so they are different to the interface modes for which these conditions are imposed on

and

These two sets will be the same if all the DOF of

and

are eliminated, i.e. if

5. Numerical application

5.1. Test case description and computation cases

The structure

consists of a cyclically symmetric bladed disk (Fig. 3a) which is composed of 15 repetitive sectors with data given in Table 2. The structure is clamped on the inner circle, so the DOF on the latter will not be taken into account. The number of DOF of the whole structure

is 10350 (3 DOF per node, only the flexural motion is considered).

At first, we consider the tuned case where only the reference sector

is modeled. For the CMS methods, 

is decomposed into two substructures, the reference disk sector

and the reference blade

(Fig. 3b). The interface

of

is also the interface

of

is composed of the right frontier

, the left frontier

and the disk-blade interface

. The numbers of DOF of

are 750, 660, 99, 129, 60, 60, and 9.

In a second stage, we introduce a mistuning in the structure, which consists in the random coefficients

and

for
The results of the CMS methods are compared to the reference results obtained by using the cyclic symmetry (CS) without CMS for the tuned case, and by performing the computations on the whole structure (WS) for the mistuned cases.

5.2. Component mode synthesis methods

The CMS methods used in the numerical application are organized into three groups, in each group the same type of interface (fixed, free or hybrid) is used for both substructure modes and partial interface modes: the fixed interface group (CB); the free interface using attachment modes group (FA); and the hybrid interface using attachment modes group (HA). Other combinations such as the CB method with free or hybrid partial interface modes are of course possible, but they are not used here since the number of the considered CMS methods is already too important.

In each group, we consider three methods (Table 1):

- the classical method (CB, FA and HA, or C in short) and the variants using \( \Psi_c \) of the free and hybrid interface classical methods (FA' and HA', or C in short);
- the method using interface modes (CBI, FAI and HAI);
- the method using partial interface modes (CBP, FAP and HAP, number 9, 13 and 17 in Table 1), with two selections of the kept interface DOF (Fig. 3b). In the first selection P1, nodes 105 and 167 on the disk–blade interface \( L_{b0} \) are kept in the reduced system. In the second selection P2, nodes 105, 167 and also nodes 21, 261 on the left and right frontiers of \( S_0 \) are kept. The numbers of the kept interface DOF are respectively 6 and 12 in the tuned case. The same selections are made for the other sectors with the corresponding nodes in the mistuned case, the numbers of the kept interface DOF are respectively 90 and 135 for P1 and P2.

In the hybrid interface group (HA), the following choice of the fixed interface and the free interface is made in the tuned case (Fig. 3b): concerning the substructure modes, \( L_1 \) and node 105 are fixed (\( L_1 \) and the other nodes of \( L_{b0} \) are free) for \( D_0 \), while 105 is free (the other nodes of \( L_{b0} \) are fixed) for \( B_0 \); concerning the partial interface modes, node 105 is fixed while nodes 21, 167 and 261 are free. The same choice is made for the other disk sectors \( D_0 \), blades \( B_0 \) and sectors \( S_0 \), with the corresponding nodes in the mistuned case. The above choice is made so that not only the substructures have the hybrid interface, but the substructure coupling is also performed between the fixed interface and the free interface.

5.3. Mode selections

We are interested to the eigen frequencies of the structure up to \( f_{\text{max}} = 3000 \text{ Hz} \). For the substructure normal modes, we use Rubin’s criterion [61] which consists in selecting all the free interface normal modes whose frequencies are smaller than a cut-out frequency defined by \( f_{\text{cs}} = 1.5f_{\text{max}} = 4500 \text{ Hz} \). In the tuned case, five modes are selected for the disk sector \( D_0 \) and four modes for the blade \( B_0 \), including three rigid body modes for the latter (Fig. 4). The same numbers of modes are used for the fixed and hybrid interface modes, and also in the mistuned case for each disk sector \( D_i \) and each blade \( B_i \).

The stiffness and mass matrices of the mistuned structure are obtained by multiplying the stiffness matrix of the tuned structure (Table 3). The stiffness and mass matrices of the mistuned structure are obtained by using the cyclic symmetry properties.

![Fig. 3. Structure, reference sector and decomposition into substructures.](image)

Table 2
Bladed disk data.

| Parameter                          | Value       |
|------------------------------------|-------------|
| Inner radius of the disk           | 12.7 x 10^-3 m |
| Outer radius of the disk           | 101.6 x 10^-3 m |
| Outer radius of the bladed disk    | 152.4 x 10^-3 m |
| Blade length                       | 50.8 x 10^-3 m |
| Blade width                        | 8.5 x 10^-3 m |
| Thickness                          | 1.982 x 10^-3 m |
| Young’s modulus                    | 2 x 10^11 N/m² |
| Poisson’s ratio                    | 0.3         |
| Density                            | 7860 kg/m³  |

Table 3
Coefficients for mistuning.

| Sector | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|--------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| \( c_{ii} \) | 0.88 | 0.80 | 1.19 | 0.61 | 1.12 | 1.25 | 1.27 | 1.13 | 0.89 | 0.93 | 0.74 | 1.03 | 0.81 | 1.25 | 1.09 |
| \( c_{ii} \) | 0.91 | 1.20 | 0.79 | 0.94 | 1.36 | 1.15 | 0.74 | 1.28 | 1.10 | 0.68 | 0.74 | 1.08 | 1.10 | 0.88 | 1.05 |
A similar criterion is used with a different cut-out frequency $f_{ci} = c_{if_{max}}$ for the interface modes and the partial interface modes \cite{72} (Fig. 5). Three selections (a, b and c) are used, corresponding respectively to $c_i = 1.5, 2.5$ and $3.5$ ($f_{ci} = 4500, 7500$ and $10,500$ Hz). In order to select the same number of interface modes and partial modes for all the methods, the maximum number of modes given by the criterion is used for each selection. They are respectively 4, 5 and 6 in the tuned case for each phase number, and 44, 62, 81 in the mistuned case. A suffix (for instance CBlb, CBP1b) indicates which selection is used.

Figs. 4 and 5 show that the frequencies of the substructure and partial interface modes in the CB and HA cases (fixed and hybrid interface) are much higher than those in the FA case (free interface). If we apply Rubin’s criterion to the CB and HA cases with the same cut-out frequency, the number of the selected modes will be very small. As Rubin’s criterion was initially used in \cite{61} to select the free interface substructure modes, we apply the criterion first to the FA case, and then use the same number of the selected modes for the CB and HA cases.

The size of the system in the reference cases CS and WS and the size of the coupled reduced system in all the CMS methods are given in Table 4.

\subsection*{5.4. Undamped frequencies and modes}

The 40 frequencies of the structure up to 3000 Hz obtained in the reference CS and WS case are plotted in Fig. 6 for both tuned and mistuned cases. The tuned frequencies are double for the phase numbers $n \neq 0$. The mistuned frequencies are no longer double, however they are very close to those of the tuned case. The difference between the frequencies of the tuned and mistuned cases are comprised between ±8%. The results of the CMS methods are not plotted since they are not distinguishable from the reference results.

Fig. 7 shows the relative errors on each frequency obtained with CMS methods and with Selection (a) of interface modes or partial interface modes. The classical methods give the smallest errors which are practically zeros, while the methods using interface modes give largest errors. As usually observed with CMS and other projection methods, the results are excellent on the low frequencies and deteriorate when the frequencies go up, due to the truncation of the projection basis. The truncation effect is emphasized after the 30th modes where the errors increase rapidly, in particular for the methods using interface modes.

Fig. 8 shows the evolution of the mean relative errors on the frequencies and modes obtained with the CMS methods using interface modes and partial interface modes in function of the number of the selected modes, the first three values corresponding

\begin{table}[h]
\centering
\caption{Size of the assembled reduced system and mean adimensioned CPU time (reduction and solution) of the frequency response.}
\begin{tabular}{|l|l|l|l|l|l|l|l|l|l|l|l|}
\hline
\textbf{Method} & \textbf{Reference CS, WS Classical C, C} & \textbf{Interface mode} & \textbf{Partial interface mode} \\
\hline
\hline
\textbf{Size of the assembled reduced system} & & & & & & & & & & & \\
Tuned & 750 & 138 & 13 & 14 & 15 & 19 & 20 & 21 & 25 & 26 & 27 \\
Mistuned & 10,350 & 1170 & 179 & 197 & 216 & 289 & 287 & 306 & 314 & 332 & 351 \\
\hline
\textbf{Mean CPU time (adimensioned by the smallest value, 3.26 s)} & & & & & & & & & & & \\
Tuned & 328.5 & 46.3 & 1 & 1.1 & 1.3 & 1.2 & 1.3 & 1.4 & 3.0 & 3.3 & 3.6 \\
MIS1 & 765.7 & 166.9 & 11.7 & 14.6 & 17.3 & 30.5 & 35.4 & 40.0 & 42.2 & 47.4 & 53.3 \\
MIS2 & 161.6 & 6.7 & 9.9 & 12.5 & 24.8 & 31.2 & 35.3 & 35.8 & 44.7 & 48.8 & \\
\hline
\end{tabular}
\end{table}
to the mode selections (a–c) derived from Rubin’s criterion. The errors on the mode $x_i$, normalized so that $k_{x_i} = 1$, are obtained by $e_i = k_{x_i} / C_0(t_{x_i(d)} - t_{x_i(r)})$. The three mode selections (a–c) provide very good results, for both interface modes and partial interface modes. Using more modes than in Selection (c) only improves slightly the results. With the same number of modes, the methods using interface modes are less
accurate than those using partial interface modes, as expected. The partial interface modes improve not only the accuracy of the results, but also their convergence when the number of modes increases. Indeed, the interface modes, although provide very good results, do not converge very well in the mistuned cases, except with the free interface method. Selection P2 provides better results than Selection P1 but not that much, despite that the number of the kept interface DOF is multiplied by 2 in the tuned case and by 1.5 in the mistuned case.

Fig. 9 shows the mean relative errors on the frequencies and modes obtained with all the CMS methods. For all the CMS methods, the results in the tuned case are much better than those in the mistuned cases, probably because the projection bases in the tuned case are more appropriate than in the mistuned case, since Rubin’s criterion is used to select the substructure modes in the tuned case, and the same numbers of modes are then kept for the mistuned cases. The MIS1 mistuned case gives much better results than the MIS2 mistuned case, this means that the tuned projection bases do not represent the mistuned structure as accurately as the mistuned projection bases. The mean errors in the tuned, MIS1 and MIS2 cases are respectively smaller than 0.08%, 0.2% and 0.6% on the frequencies, and 1%, 6% and 8% on the modes. The classical methods provide the best results, with less than 0.02% mean error on the frequencies and less than 0.3% mean error on the modes. Variants FA and HA give the same results than the FA and HA methods. This confirms that the Ritz vectors associated with the interface DOF in the free and hybrid interface classical methods are either identical to or can be replaced by the constraint modes of the CB methods, and justifies the replacement of the latter by the interface modes or the partial interface modes as in the CB method. The free interface methods are more accurate than the fixed interface and the hybrid interface methods, and the latter give similar results.

5.5. Frequency response to harmonic forces

Three harmonic forces, $f_1 = 10 \cos \omega t - 10 \sin \omega t$, $f_2 = 20 \cos \omega t - 20 \sin \omega t$ and $f_3 = 30 \cos \omega t - 30 \sin \omega t$, are applied to the DOF $u_i$ of nodes 21, 105 and 261 of sector $S_0$ (Fig. 3b). A proportional damping matrix $C = \alpha K + bM$ is introduced, with $\alpha = 5 \times 10^{-5}$ and $b = 1$. The frequency response is computed for both tuned and mistuned cases with the excitation frequency $\omega$ varying from 2 to 1000 Hz with a step of 2 Hz.

Fig. 10 shows the frequency response of the amplitudes $U_i$ of the displacements $u_i = U_i \cos(\omega t + \phi)$ of node 105 obtained in the reference cases (CS and WS). Three main resonance peaks are observed before 200 Hz, with a slight shift of the peaks towards the smaller frequencies in the mistuned case. The peak amplitudes are also a little different between the tuned and mistuned cases.

Fig. 11 shows the absolute errors $||U_{CMS} - U_{ref}||$ obtained with the CMS methods and with Selection (a) of interface modes or partial interface modes. All the CMS methods give excellent amplitudes in the tuned and the MIS1 mistuned cases, the absolute errors are smaller than $10^{-4}$, i.e. 4 order smaller than the amplitudes. The results are less accurate in the MIS2 mistuned case, with absolute errors smaller than $10^{-2}$.

Fig. 12 shows the relative errors on the amplitudes $U_i$, which are defined as the ratio of the euclidian norms $||U_{CMS} - U_{ref}||/||U_{ref}||$, where $U_i$ is the vector containing the amplitudes $U_i$ for all the excitation frequencies. All the CMS methods provide excellent results with relative errors smaller than 0.03%, 0.05% and 6% for the tuned, MIS1 and MIS2 cases respectively. Like in the frequency and mode computations, the classical methods and their variants are the most accurate. With the same mode selection (a–c), the methods using partial interface modes are more accurate than those using interface modes, and the accuracy increases with the number of interface modes and partial interface modes. Selection P2 of the kept interface DOF is only slightly better than Selection P1, and the free interface methods are better than the fixed and hybrid interface methods, but only in the tuned and MIS1 mistuned cases.

The mean CPU times, adimensioned by their smallest values 3.26 s, are presented in Table 4. For the CMS methods, the CPU time includes, on the one hand, the model reduction, i.e. the
computation of the substructure, interface and partial interface modes, the projection and the coupling of the substructures, and, on the other hand, the solution of the reduced system, including the restitution of the physical displacements.

Thanks to the use of the cyclic symmetry properties, the CPU times of the tuned case compared to the mistuned cases is reduced in average by a factor of 2.3, 3.5, 10.6 and 17.1 respectively with the reference methods, the classical CMS methods, the methods using interface modes and the methods using partial interface modes. The CMS methods using interface modes and partial interface modes are in average 40 and 20 times faster than the classical methods in the tuned case, 11 and 4 times faster in the MIS1 mistuned case, and 17 and 4 times faster in the MIS2 mistuned case. Due to the additional interface DOF in the reduced system, the methods using partial interface modes require more CPU times than the methods using interface modes. In the tuned case, the former methods with Selection P1 are only slightly slower than the latter methods, but they provide much better results. In the mistuned cases, the former methods with Selection P1 are about 3 times slower than the latter methods. Selection P2 requires about 1.5 times more CPU time than Selection P1, but the results are not much better.

The use of the tuned modes in the MIS2 mistuned case in place of the mistuned modes in the MIS1 mistuned case deteriorates significantly the results and does not reduce the CPU times as much as we may expect, since the saving on the CPU times only comes from the model reduction, or more precisely on the computation of the substructure modes and the interface modes or partial interface modes, and not from the solution of the reduced system.

6. Conclusion

Several CMS methods using partial interface modes which allows to reduce the number of interface generalized coordinates and at the same time to keep a few physical displacements in the reduced coupled system have been developed. They are applied on a bladed disk with cyclic symmetry and they are compared with the classical CMS methods and the CMS methods using interface modes. The results obtained with all the CMS methods are in very good agreement with the reference results.

According to the results obtained in this example, the following recommendations can be made:

- if the number of the interface DOF is not an issue, the classical CMS methods give the best results. The free interface methods are recommended since they are the most accurate, although
the fixed interface methods are also easy to implement and they work well with a large range of applications. The number of the interface DOF is too important and has to be reduced, the new CMS methods using partial interface modes are recommended. And even in the case the physical displacements are not needed in the reduced coupled system, it is better to keep some of them and use the partial interface modes rather than the interface modes, since the former are much better concerning the accuracy and the convergence of the results. A small number of the kept interface DOF and the mode selections derived from Rubin's criterion are sufficient to obtain very good results, and the increase of the number of the kept interface DOF or the selected modes do not improve very much the results.

for the mistuned case, the use of the tuned substructure, interface and partial interface modes in the CMS methods is not recommended, since it deteriorates significantly the results. The use of the mistuned modes provides much better results with only a small additional computational cost.

The development of CMS methods based on the proper orthogonal decomposition [57] is in progress. The future works will consist in using CMS methods for further investigation on mistuned bladed disk systems with non-linearity and aeroelastic coupling.

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