Logarithmic price of buffer downscaling on line metrics

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We consider the reordering buffer problem on a line consisting of \( n \) equidistant points. We show that, for any constant \( \delta \), an (offline) algorithm that has a buffer \((1 - \delta) \cdot k\) performs worse by a factor of \( \Omega(\log n) \) than an offline algorithm with buffer \( k \). In particular, this demonstrates that the \( O(\log n) \)-competitive online algorithm Moving-Partition by Gamzu and Segev (ACM Trans. on Algorithms, 6(1), 2009) is essentially optimal against any offline algorithm with a slightly larger buffer.

1 Introduction

In the reordering buffer problem, requests arrive in an online fashion at the points of a metric space and have to be served by an algorithm. An algorithm has a single server kept at a point of the metric space and is equipped with a finite buffer, whose capacity is denoted by \( k \). The buffer is used to give the algorithm a possibility of serving requests in a different order. Namely, at any time, there can be at most \( k \) unprocessed requests, and once their number is exactly \( k \), the algorithm has to move its server to the position of a pending request of its choice. The request is then considered served and removed from the buffer. The goal is to minimize the total distance traveled by the server.

The problem was coined by Räcke et al. [RSW02] and the currently best algorithm for general metric spaces is a randomized \( O(\log n \cdot \log k) \)-competitive strategy [ERW10, ER17], where \( n \) is the number of points in a metric space. Most of the research focused however on specific metric spaces, such as uniform metrics [RSW02, AR13, EW05], stars [ACER11, AIMR15, AR10] or lines of \( n \) equidistant points [GS09, KP10].

1.1 Two-stage approach

One of the straightforward ways to attack the problem is by a two-stage approach: (i) compare an online algorithm to an optimal offline algorithm that has a smaller buffer \( h < k \) and (ii) bound the ratio between optimal offline algorithms equipped with buffers \( h \) and \( k \). Such an approach was successfully executed by Englert and Westermann [EW05] for uniform metrics. They constructed

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an online algorithm Map (with a buffer $k$) that is $4$-competitive against $Orr(k/4)$, where $Orr(s)$ denotes the optimal offline algorithm with buffer $s$. Subsequently, they showed that on any instance the costs of $Orr(k/4)$ and $Orr(k)$ can differ by a factor of at most $O(log k)$. This implies that Map is $O(4 \cdot log k) = O(log k)$-competitive.

As shown by Aboud [Abo08], for the uniform metric space the relation between $Orr(k/4)$ and $Orr(k)$ cannot be asymptotically improved, which excludes the option of beating the ratio $O(log k)$ by the described two-stage process. Note that this does not rule out the possibility of decreasing the ratio by another approach. Indeed, subsequent works gave improved results, essentially resolving the uniform metric case for deterministic algorithms: Adamaszek et al. [ACER11] gave the ratio by another approach. Indeed, subsequent works gave improved results, essentially resolving the uniform metric case for deterministic algorithms: Adamaszek et al. [ACER11] gave an $O(\sqrt{\log k})$-competitive deterministic strategy and showed that the competitive ratio of every deterministic algorithm is $\Omega(\sqrt{\log k} / \log \log k)$.

1.2 Two-stage approach for line metrics (our result)

Arguably, the most uncharted territory is the line metric, or more specifically, a line graph consisting of $n$ equidistant sites. There, the lower bound on the competitive ratio is only $2.154$ [GS09]. On the other hand, the best known strategy is an $O(log n)$-competitive algorithm MovingPartition by Gamzu and Segev [GS09]. Note that achieving an upper bound of $O(k)$ is possible for any metric space [EW05], and hence the competitive ratio for line metrics is $O(\min(k, log n))$. In this paper, we show that it is not possible to improve this bound by the two-stage approach described above, by proving the following result.

Theorem 1. Fix a line consisting of $n$ equidistant sites, any $k$, and any constant $\delta \in (0, 1)$. There exists an input sequence on which the ratio between the costs of $Orr(k)$ and $Orr((1 - \delta) \cdot k)$ is at least $\Omega(\min(k, log n))$.

Our result has additional consequences for the two known online algorithms for the reordering buffer problem: MovingPartition [GS09] and Pay [ER17].

A straightforward modification of the analysis in [GS09] shows that MovingPartition is in fact $O((h/k) \cdot log n)$-competitive against an optimal offline algorithm that has a larger buffer $h > k$. Our result implies that, assuming $log n = O(k)$, for $h = k \cdot (1 + \epsilon)$ and a fixed $\epsilon > 0$, this ratio is asymptotically optimal.

Similarly, algorithm Pay [ER17] achieves a competitive ratio of $O((h/k) \cdot (log D + log k))$ on trees with hop-diameter $D$, and hence it is $O((h/k) \cdot (log n + log k))$-competitive when applied to a line metric. Therefore, Pay is asymptotically optimal for line metrics when $n = O(k)$ and $h = k \cdot (1 + \epsilon)$ for a fixed $\epsilon > 0$.

2 Main construction

The main technical contribution of this paper is to show Theorem 1 for the case when the line consists of exactly $n = 2^k + 1$ sites. For such a setting, the key lemma given below yields the cost separation of $\Omega(k) = \Omega(log n)$. In the next section, we show how request grouping extends this result from the case of $n = 2^k + 1$ sites to arbitrary values of $k$ and $n$.

Lemma 2. Fix two integers $0 < \ell' < \ell$ and a constant $\epsilon$ such that $\ell' = (1 - \epsilon) \cdot \ell$. Assume that the line consists of $2^{\ell'} + 1$ equidistant sites. There exists an input sequence on which the ratio between the costs of $Orr(\ell)$ and $Orr(\ell')$ is at least $\ell \cdot \epsilon^2 \cdot (1 + \epsilon)^{-1} \cdot log_2^{-1}(1 + 1/\epsilon)$. 

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We number the sites from 0 to $2^\ell$. For simplicity of the description, we associate times with request arrivals: all requests arrive only during a step between two consecutive integer times. For a single step, we will explicitly specify the positions of the requests arriving in this step but not their time ordering, which is irrelevant. (If needed, one may assume that they arrive in the increasing site numbering order.) The input sequence consists of phases, one of which we describe below. Then, any subsequent phase is “mirrored” with respect to the previous one, i.e., the description is identical except for reversing the site numbering.

The phase consists of $2^\ell$ steps and it begins at time 0. Each request is either regular or auxiliary. Each regular request has a rank $i$ from 0 to $\ell - 1$ and their positions are defined by the recursive construction of blocks described below.

For any integers $q \in \{1, \ldots, \ell\}$ and $s \in \{0, \ldots, 2^{\ell - q} - 1\}$, a $(q,s)$-block is a square portion of space-time, consisting of all times in $(2^q \cdot s, 2^q \cdot (s+1)]$ and all sites in $(2^q \cdot s, 2^q \cdot (s+1)]$, where $q$ is called the rank of the block. For instance, the $(\ell,0)$-block is the whole phase. Moreover, $(q,s)$-block contains two sub-blocks: the $(q-1,2s)$-block and the $(q-1,2s+1)$-block, see Figure 1. In each $(q,s)$-block a unique request of rank $q-1$ arrives at site $2^q \cdot (s+1)$ in the step following time $2^q \cdot s$; note that this request is not contained in any of its two sub-blocks. Graphically, it is near the top left corner of the block, and it is depicted as a disk in Figure 1.

In turn, auxiliary requests are grouped in sets of $\ell + 1$ called anchors (squares in Figure 1). For all $j$ from 0 to $2^\ell - 1$, the $j$-th anchor arrives at site $j$ in the step following time $j$.

Throughout the paper, any relative directions in the text pertain to this figure, e.g., “up” and
“right” mean “with growing site numbers” and “forward in time”, respectively.

2.1 Upper bound on the cost of the optimal algorithm with a larger buffer

Let the basic trajectory (depicted as a thick line in Figure 1) be the movement of a server that, for all \( j \) from 0 to \( 2^\ell - 1 \), remains at site \( j \) throughout the step between the integer times \( j \) and \( j + 1 \), and moves directly from \( j \) to \( j + 1 \) at the integer time \( j + 1 \). Note that in the subsequent phase the last site becomes site 0, and thus the basic trajectory remains continuous over multiple phases as well.

**Lemma 3.** The cost of \( \text{Orr}(\ell) \) in a single phase is at most \( 2^\ell \).

**Proof.** We prove that an algorithm with buffer size \( \ell \) that follows the basic trajectory, denoted \( \text{BT}(\ell) \), yields a feasible solution. As its cost is clearly \( 2^\ell \), the lemma follows. First, all anchors are served immediately by \( \text{BT}(\ell) \), without storing them in the buffer, as they arrive at a current position of the server. Second, we will show that for each of \( \ell \) possible ranks, \( \text{BT}(\ell) \) has at most one regular request of the given rank in its buffer, and hence its buffer capacity is not exceeded. To this end, observe that the \( j \)-th request of rank \( i \) arrives at site \( 2^{r+1} \cdot j \) in a step following time \( 2^{r+1} \cdot (j-1) \). It is served at time \( 2^{r+1} \cdot j \) by \( \text{BT}(\ell) \), i.e., right before the next request of this rank arrives.

In fact, the cost of \( \text{BT}(\ell) \) is optimal as the anchors placed at all sites force any algorithm with buffer \( \ell \) to traverse the whole line at least once per phase.

2.2 Lower bound on the cost of the optimal algorithm with a smaller buffer

In this part, we construct a lower bound that holds for any algorithm that has buffer \( \ell \) or smaller. Later, we apply this bound to \( \text{Orr}(\ell’) \); recall that \( \ell’ = (1 – \varepsilon) \cdot \ell \). Observe that any such algorithm has to process each anchor as soon as it arrives, since the anchor has \( \ell + 1 \) requests. Therefore, its trajectory within the step between times \( j \) and \( j + 1 \) has to pass through site \( j \). Informally speaking, its trajectory has to touch or intersect the basic trajectory in all steps of the phase.

Regular requests inside a \((q,s)\)-block are called new for this block, whereas regular requests that arrived at any site in \((2^q \cdot s, 2^q \cdot (s + 1)]\) before time \( 2^q \cdot s \) are called old for this block. Note that all the old requests for the \((q,s)\)-block are at a single site, namely at \( 2^q \cdot (s + 1) \). Recall that \((q,s)\)-block contains only one request not contained in its sub-blocks (with rank \( q - 1 \) and position at site \( 2^q \cdot (s + 1) \)).

For \( p, r \geq 0, q \geq 1 \), let \( T(p, q, r) \) be the minimal cost of any algorithm for serving a block of rank \( q \), assuming that at the beginning of the block, its buffer has space for \( p \) (new) requests and \( r \) old requests for this block are already stored in the buffer. If the algorithm trajectory leaves the block, \( T(p, q, r) \) accounts only for these parts of its trajectory that are contained completely inside the block. We show the following lower bound on \( T(p, q, r) \).

**Lemma 4.** \( T(p, q, r) \geq 2^q - 1 \). Furthermore, for \( q \geq 2 \),

\[
T(p, q, r) \geq \min \{ 2^q + 2 \cdot T(p + r, q - 1, 0), T(p - 1, q - 1, 0) + T(p - 1, q - 1, r + 1) \} \quad \text{if } p \geq 1,
\]

\[
T(p, q, r) \geq 2^q + 2 \cdot T(p + r, q - 1, 0) \quad \text{if } p = 0.
\]

**Proof.** The first inequality holds, because a block of rank \( q \) contains anchors at all its sites, and the distance between the two most distant of those is \( 2^q - 1 \). For the remaining two inequalities (for \( q \geq 2 \)), we consider two possible behaviors of the algorithm.
1. The algorithm moves the server to site $2^l \cdot (s + 1)$ before time $2^l \cdot s + 2^{l-1}$. That is, it moves the server to the upper boundary of the block (or beyond) within the first half-time of the block. The length of the part of the trajectory contained in the current block but not in its sub-blocks is then at least $2 \cdot 2^{l-1}$ (to the upper boundary and back). We may assume that such movement serves all $r$ old requests and also the unique new request of rank $q - 1$. For the purpose of the lower bound, we may assume that this happens instantly after the block begins. Consequently, for both sub-blocks there are no buffered old requests and the buffer part occupied by them was freed. Therefore, in this case

$$T(p, q, r) \geq 2^l + 2 \cdot T(p + r, q - 1, 0).$$

2. The strategy does not make such movement before time $2^l \cdot s + 2^{l-1}$. Note that this is only possible for $p \geq 1$ as the unique request of rank $q - 1$ (the top left corner of the block) will be stored in the buffer, decreasing empty buffer space to $p - 1$. This request becomes old for the later sub-block, but for the earlier sub-block there are no old requests. Therefore, in this case we get

$$T(p, q, r) \geq T(p - 1, q - 1, 0) + T(p - 1, q - 1, r + 1).$$

Combining these two cases yields the lemma. □

Note that the cost of $\text{Orr}(\ell)$ in a single phase is at least $T(\ell, \ell, 0)$. In this case, it is always possible to take the second case of the recurrence relation above, which corresponds to the basic trajectory behavior. $T(\ell, \ell, 0)$ then expands into a sum of $2^{l-1}$ terms of the form $T(1, 1, r)$. In our reasoning we only need $T(1, 1, r) \geq 1$ but a more careful argument could double this amount, resulting in an (otherwise trivial) lower bound of $2^l$ for the cost of $\text{Orr}(\ell)$.

Lemma 5. Fix any $\eta \in (0, 1)$. Let $a = (1 + \eta) \cdot \log_2(1 + 1/\eta)$, $b_i = 2(2^i - 1) \cdot \eta$ for all $i \geq 0$, and

$$\tau(p, q, r) = \frac{2^l}{a} \cdot (q - (1 + \eta)p - b_r).$$

Then, $T(p, q, r) \geq \tau(p, q, r)$ for any $p \geq 0$, $q \geq 1$ and $r \geq 0$.

Proof. The lemma follows by induction on $q$. Its basis corresponds to the case $q = 1$, where

$$T(p, 1, r) \geq 1 \geq 2/a = \tau(0, 1, 0) \geq \tau(p, 1, r).$$

The second inequality follows as $a > 2$ for any $\eta \in (0, 1)$, whereas the last one follows from the fact that $\tau$ decreases in both $p$ and $r$.

To show the inductive step, by Lemma 4, it suffices to prove the following two inequalities (for $q \geq 2$ and $r \geq 0$):

1. $2^l + 2\tau(p + r, q - 1, 0) \geq \tau(p, q, r)$ for $p \geq 0$, (1)
2. $\tau(p - 1, q - 1, 0) + \tau(p - 1, q - 1, r + 1) \geq \tau(p, q, r)$ for $p \geq 1$. (2)

For (1), we first determine a bound on $f(r) = (1 + \eta) \cdot r - b_r + 1$. By taking its derivative in $r$ (i.e., $1 + \eta - \eta \cdot 2^{r+1} / \log_2 e$), we observe that $f$ is maximized at $r_0 = \log_2(1 + 1/\eta) + \log_2 \log_2 e - 1$, and therefore

$$f(r) \leq f(r_0) = (1 + \eta) \cdot (\log_2(1 + 1/\eta) + \log_2 \log_2 e - \log_2 e - 1) + 2\eta + 1 \leq (1 + \eta) \log_2(1 + 1/\eta) = a.$$
Consequently, applying $b_0 = 0$ as well, we obtain
\[
2^q + 2\tau(p + r, q - 1, 0) = \frac{2^q}{a}(a + q - 1 - (1 + \eta)(p + r) - b_0) \geq \frac{2^q}{a}(q - (1 + \eta)p - b_r) = \tau(p, q, r) .
\]

For (2), from the definition of $b_i$ it holds that $(b_0 + b_{r+1})/2 = b_{r+1}/2 = (2^{r+1} - 1) \cdot \eta = b_r + \eta$, which implies
\[
\tau(p - 1, q - 1, 0) + \tau(p - 1, q - 1, r + 1) = \frac{2^q}{a}\left(q - 1 - (1 + \eta)(p - 1) - \frac{b_0 + b_{r+1}}{2}\right) = \frac{2^q}{a}\left(q - (1 + \eta)p + \eta - \frac{b_0 + b_{r+1}}{2}\right) = \frac{2^q}{a}(q - (1 + \eta)p - b_r) = \tau(p, q, r) .
\]

We now show how to prove the key lemma using Lemma 3 and applying Lemma 5 to $\text{Orr}(\ell')$.

**Proof of Lemma 2.** The whole phase is the block of rank $\ell$, with no buffered old requests and hence, by the definition of $T$, $T(\ell', \ell, 0) = T((1 - \epsilon) \cdot \ell, \ell, 0)$ is a lower bound for the cost of $\text{Orr}(\ell')$ in a single phase. Using Lemma 5 with $\eta = \epsilon$, we obtain
\[
T((1 - \epsilon) \cdot \ell, \ell, 0) \geq 2^q \cdot \frac{(\ell - (1 + \epsilon)) \cdot (1 - \epsilon) \cdot \ell}{(1 + \epsilon) \log_2(1 + 1/\epsilon)} = 2^q \cdot \ell \cdot \frac{\epsilon^2}{(1 + \epsilon) \log_2(1 + 1/\epsilon)} .
\]

On the other hand, the cost of $\text{Orr}(\ell)$ is at most $2^q$ by Lemma 3, and hence the lemma follows. \hspace{1cm} \Box

### 3 Cost separation for arbitrary buffer sizes

First, we show that Lemma 2 may be “scaled up” appropriately, and then we extend it to show Theorem 1.

**Lemma 6.** Fix positive integers $\ell$ and $\beta$, and a constant $\epsilon \in (0, 1)$. Assume that the line consists of at least $2^\ell + 1$ equidistant sites. There exists an input sequence on which the ratio between the costs of two optimal algorithms, one with a buffer of at least $\beta \cdot \ell$ and one with a buffer of at most $\beta \cdot (1 - \epsilon) \cdot \ell$, is at least $\Omega(\ell)$.

**Proof.** First, we fix the input sequence whose existence is asserted by Lemma 2. In this input, we replace each request by a packet consisting of $\beta$ requests at the same site. In any optimal solution, without loss of generality, all requests belonging to such a packet are processed together. Therefore, what matters is how many packets can be kept in the buffers of both algorithms, i.e., at least $\ell$ in case of the first algorithm and at most $(1 - \epsilon) \cdot \ell$ in case of the second one. Hence, for the assumed buffer capacities the cost separation is at least $\ell \cdot \epsilon^2 \cdot (1 + \epsilon)^{-1} \cdot \log_2^{-1}(1 + 1/\epsilon)$, which is $\Omega(\ell)$ for a fixed $\epsilon$.

Second, the result of Lemma 2 is still valid when the total number of line sites is not equal but greater than $2^\ell + 1$. The requests arrive then only at the first $2^\ell + 1$ sites and the remaining ones cannot help nor hinder the performance of any algorithm. \hspace{1cm} \Box

**Proof of Theorem 1.** Let $m = \lfloor \log_2(n - 1) \rfloor$, i.e., $m$ is the largest integer such that $2^m + 1 \leq n$. We consider two cases, and in each of them we lower-bound the cost ratio between $\text{Orr}(k)$ and $\text{Orr}((1 - \delta) \cdot k)$.

1. If $k < m$, let $\ell = k$, $\beta = 1$ and $\epsilon = \delta$. Lemma 6 now implies that the cost ratio is $\Omega(\ell) = \Omega(k)$.
2. If $k \geq m$, we may assume $k \geq 4/\delta$, as otherwise $\Omega(\min\{k, \log n\}) = \Omega(1)$ and the theorem follows trivially. Let $\ell = \lceil m \cdot \delta/4 \rceil$, $\beta = \lceil k/\ell \rceil$, and $\varepsilon = \delta/2$, so that $\beta \cdot \ell \leq k$. On the other hand, $\ell < k \cdot \delta/4 + 1 \leq k \cdot \delta/2$, hence $\beta \cdot \ell > (k/\ell - 1) \cdot \ell = k - \ell > k \cdot (1 - \delta/2)$, which finally yields $(1 - \varepsilon) \cdot \beta \cdot \ell > (1 - \delta/2)^2 \cdot k > (1 - \delta) \cdot k$. Therefore $\mathsf{Opt}(k)$ and $\mathsf{Opt}(1 - \delta) \cdot k$ fulfill the conditions of Lemma 6, which now implies that the cost ratio is $\Omega(\ell) = \Omega(m)$.

In both cases we obtain that the cost ratio is $\Omega(\min\{k, m\}) = \Omega(\min\{k, \log n\})$. □

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