APPROXIMATE CONTROLLABILITY OF THE SEMILINEAR REACTION-DIFFUSION EQUATION GOVERNED BY A MULTIPlicative CONTROL

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Abstract. In this paper we are concerned with the approximate controllability of a multidimensional semilinear reaction-diffusion equation governed by a multiplicative control, which is locally distributed in the reaction term. For a given initial state we provide sufficient conditions on the desirable state to be approximately reached within an arbitrarily small time interval. Our approaches are based on linear semigroup theory and some results on uniform approximation with smooth functions.

1. Introduction. Our goal in this paper is to study the global approximate controllability properties of the following semilinear Dirichlet boundary value problem

\[
\begin{cases}
y_t(t) = \Delta y(t) + v(x,t)1_O y(t) + f(t,y(t)), & \text{in } Q = \Omega \times (0,T_0) \\
y(0) = y_0 \in L^2(\Omega), & \text{on } \Sigma = \partial \Omega \times (0,T_0)
\end{cases}
\]

where \( T_0 > 0, \Omega \) is a bounded open domain of \( \mathbb{R}^d, d \geq 1 \) with smooth boundary \( \partial \Omega \) and \( O \) is an open subset of \( \Omega \) with a characteristic function denoted by \( 1_O \). The nonlinear term \( f : \mathcal{H} = [0,T_0] \times L^2(\Omega) \rightarrow L^2(\Omega) \) is Lipschitz continuous in both variables, i.e. there is a constant \( L > 0 \) such that

\[
\| f(t_1, y_1) - f(t_2, y_2) \| \leq L (|t_1 - t_2| + \| y_1 - y_2 \|), \quad \forall (t_i, y_i) \in \mathcal{H}, i = 1, 2,
\]

where \( \| \cdot \| \) refers to the conventional norm of \( L^2(\Omega) \). Here, for each time \( t \geq 0 \), the state \( y(t) \) is given by the function \( y(t) = y(\cdot, t) \in L^2(\Omega) \) and \( v(\cdot) \) is the control function which can be chosen in appropriate spaces. In terms of applications, equation (1) provides practical description of various real problems such as chemical reactions, nuclear chain reactions, and biomedical models... (see [1, 7, 23, 24, 30, 41, 42, 43, 44] and the references therein). Research in the controllability of distributed systems by additive (linear) controls have been the subject of several works (see for instance [17, 19, 20, 21, 25, 26, 32, 46]).

It is well understood that systems with additive control do not behave as those with multiplicative controls even in the absence of the nonlinearity \( f \) (i.e. bilinear

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systems). Indeed, it is immediate that the truncated version of (1) with \( f = 0 \) cannot be steered anywhere from the zero initial state. Furthermore, according to the maximum principle, the steering from a nonnegative initial state to a positive target state is impossible (see [27]). A similar negative result has been given for abstract infinite bilinear systems in [6]. According to these negative results and the nonlinear dependence of the state with respect to multiplicative controls, it appears that the duality approach and related techniques as the Carleman’s estimates, used to study the controllability of linear systems, do not systematically apply to the multiplicative control case. Given the impossibility of using the duality method for systems like (1), it seems natural to look for studying the possibility of reaching suitable classes of target states (without any linear structure). The question of controllability of PDEs equations by multiplicative controls has attracted many researchers in the context of various type of equations, such as rod equation [6, 31], Beam equation [9], Schrödinger equation [8, 31, 35], wave equation [6, 10, 28, 29, 30, 36] and heat equation [12, 13, 22, 30, 33, 37, 38]. In [12], the approximate controllability properties have been derived for the one-dimensional version of (1) for \( f = 0 \) and initial and target states with finitely many changes of sign. The exact controllability of the bilinear part of equation (1) with inhomogeneous Dirichlet conditions has been considered in [33, 37]. However, the assumptions of [33, 37] are not compatible when dealing with homogeneous Dirichlet conditions. In [27], Khapalov studied the global approximate controllability of the semilinear convection-diffusion-reaction equation by bilinear controls while dealing with nonnegative initial and target states. In [13], Cannarsa, Floridia and Khapalov have studied the global approximate controllability properties of the one dimensional version of (1) with a time-independent nonlinear term when the initial and target states admit no more than finitely many changes of sign. In [38], the question of multiplicative controllability of the bilinear variant of the system (1) has been discussed when the initial and target states \( y_0, y^d \) are such that \( y_0(x)y^d(x) \geq 0 \) for almost every \( x \in \Omega \). We observe that, in the aforementioned papers, the control acts all over the evolution domain. The case where the control action is localized in a subregion \( O \) of the geometrical domain \( \Omega \) has been considered by Fernández and Khapalov in [22] for \( f = 0 \). Thus they showed that when dealing with initial and target states which are nonnegative all over \( \Omega \), the approximate steering becomes impossible for the whole state unless the control support \( O \) is allowed to move with time (i.e. \( O = O(t) \)). Otherwise, one can only expect to have the so called ‘regional controllability’, that is \( y(T) \) can be steered to \( y^d \) only on \( O \). Here, we will study the approximate controllability for the multidimensional semilinear equation (1) for initial and target states that can change their sign. Furthermore, the steering control is constructed using an explicit approximation procedure, relying on Bernstein polynomials, combined with some density and approximation arguments.

The paper is organized as follows: In the next section, we present the main results. In the third section, we provide some preliminary results that will be needed along the paper. The fourth section is devoted to the proofs of the main results.

2. Main results. We are interested in studying the approximate controllability of system (1). More precisely, we will first provide a locally supported control that can steer the system (1) from its initial state \( y_0 \) to a final state \( y(T) \) which is sufficiently close to the desirable state \( y^d \) at a suitable time \( 0 < T < T_0 \) that depends on the choice of \((y_0, y^d)\) and the precision of steering \( \epsilon > 0 \). Everywhere
below we will consider only non-zero initial states $y_0 \in L^2(\Omega)$, for which we consider the set $\Lambda = \{ x \in \Omega/ y_0(x) \neq 0 \}$.

Our main results are as follows.

**Theorem 2.1.** Let $f$ be Lipschitz in both variables, let $y_0 \in L^2(\Omega) \setminus \{0\}$ be fixed and let $y^d \in L^2(\Omega)$ be a desired state such that: (i) $\{ x \in \Omega/ y_0(x) \neq y^d(x) \} \subset O$, a.e. and $a := \ln(\frac{y^d}{y_0}) 1_{\Lambda \cap O} \in L^\infty(O)$, and (ii) for a.e. $x \in O$, $y_0(x)y^d(x) \geq 0$ and $y_0(x) = 0 \iff y^d(x) = 0$.

Then for any $\epsilon > 0$, there are a time $0 < T = T(y_0, y^d, \epsilon) < T_0$ and a static control $v \in L^\infty(\Omega)$ such that for the respective solution to (1), we have the following estimate:

$$||y(T) - y^d|| < \epsilon.$$  \hfill (2)

As a consequence of Theorem 2.1, we have the following result.

**Corollary 1.** Let $O = \Omega$ and let $f$ be Lipschitz in both variables. If $y_0 \in L^\infty(\Omega)$ and $y^d \in L^2(\Omega)$ are such that $h = \frac{y^d}{y_0} 1_{\Lambda} \in L^2(\Omega)$ and if assumption (ii) of Theorem 2.1 holds, then for any $\epsilon > 0$, there are a time $0 < T = T(y_0, y^d, \epsilon) < T_0$ and a static control $v \in L^\infty(\Omega)$ such that for the respective solution to (1), we have the estimate (2).

**Remark 1.** In [27], approximate controllability results have been established for initial and target states which are not allowed to vanish in $\Omega$. Moreover, the one dimension version of equation (1) has been studied in [12, 13] with a nonlinearity which is independent of time and also the points of “change of sign” of $y_0$ and $y^d$ are supposed finite. Note that, in one-dimensional case, if the nonlinearity $f$ is time-independent and if $y_0$ and $y^d$ have opposite signs in a sub-interval of $\Omega$, then our results are not applicable while those of [12, 13] are applicable, provided the number of change of sign is finite and respect some order related to the maximum principle.

Next, we provide an example of initial and target states that can change their sign through a negligible set with non-isolated points.

**Example.** Let us consider the system (1) with $d = 2$, $f(t, y)(x) = \tilde{f}(y(x))$, a.e. $x \in \Omega := (0, 1)^2$, where $\tilde{f}$ is Lipschitz from $\mathbb{R}$ to $\mathbb{R}$. Let $O \subset \Omega$, $y_0 = (x_1 - x_2) 1_{O}$, a.e. $x = (x_1, x_2) \in \Omega$, let $y^d = k(x)y_0 1_{O}$ with $k \in L^\infty(\Omega)$ and $k(x) > 0$, a.e. $x \in O$. We can observe that $y_0$ and $y^d$ have the same sign a.e. in $O$. More precisely, $y_0$ and $y^d$ vanish on $\Gamma$ and are positive in $\Gamma^+ := \{(x_1, x_2) \in O/ x_1 > x_2\}$ and negative in $\Gamma^- := \{(x_1, x_2) \in O/ x_1 < x_2\}$. According to Theorem 2.1, the initial state $y_0$ can be approximately steered to $y^d$ at a small time $T_1$ which depends on $y_0$ and $y^d$.

**Outline and main ideas for the proofs.**

The proofs of the main results in Section 4 consist in establishing the estimate (2) in several steps by distinguishing various cases on smoothness of the initial state and the considered static control. The main idea for the proof of Theorem 2.1 consists in looking for a time $T = T(y_0, y^d, \epsilon)$ depending on $(y_0, y^d)$ and the precision of steering $\epsilon > 0$, and a static control $v(x, t) = v_T(x) \in L^\infty(\Omega)$ depending on $T > 0$ such that $e^{Tv_T}y_0 = y^d$ a.e. in $\Omega$, and showing that the respective solution to (1) is such that $y(T) - y^d \to 0$, as $T \to 0^+$. This goal will be achieved by selecting a static control $v(x, t) = v_T(x)$ that enables us to write

$$y(T) - y^d = \int_0^T e^{T-t}a(x)(Ay(s) + f(s, y(s)))ds,$$  \hfill (3)
and showing that the right-hand side of this relation tends to 0 as $T \to 0^+$.

The proof of Theorem 2.1 amounts to estimate the right-hand side in the above formula in order to prove that it can be made arbitrarily small as long as the static control $v_T(x)$ and the steering time $T$ are well-chosen. At that point, smoothness assumptions are required on the expected control and the respective solution. Then, based on the variation of constants formula and linear semigroup theory, we can conclude by density and approximation arguments.

Note that the idea of exploiting the relation (3) to establish the approximate steering was first introduced by Khapalov in [27] for initial and target state that have the same signs, and was exploited later in [12, 13] for $d = 1$ to study the case of initial and target states that change their sign in a finitely number of points. However, our methods differ from those of [12, 13, 27] in the way to show that the right-hand side of (3) goes to 0 as $T \to 0^+$, when dealing with a locally supported control.

3. Preliminaries.

3.1. Preliminary results on linear semigroups and evolution equations. Let us remind the reader that a one parameter family $S(t)$, $t \geq 0$, of bounded linear operators from a Banach space $X$ into $X$ is a semigroup on $X$ if (i) $S(0) = I$, (the identity operator on $X$) and (ii) $S(t + s) = S(t)S(s)$ for every $t, s \geq 0$. A semigroup $S(t)$ of bounded linear operators on $X$ is a $C_0$–semigroup if in addition $\lim_{t \to 0^+} S(t)x = x$ for every $x \in X$. This property guarantees the continuity of the semigroup on $R^+$. Moreover, one can show (see [40], p. 4) that for every $C_0$–semigroup $S(t)$, there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|S(t)\| \leq Me^{\omega t}, \forall t \geq 0.$$  \hspace{1cm} (4)

If $\omega = 0$ and $M = 1$, $S(t)$ is called a $C_0$–semigroup of contractions.

The linear operator $A$ defined by $Ax = \lim_{t \to 0^+} S(t)x - x$ for $x \in X$ such that $\lim_{t \to 0^+} S(t)x - x$ exists in $X$, is the infinitesimal generator of the $C_0$–semigroup $S(t)$. The linear space $D(A) := \{x \in X : \lim_{t \to 0^+} S(t)x - x \in X\}$ is the domain of $A$.

- An infinitesimal generator of a $C_0$–semigroup of contractions is dissipative, i.e., for every $y \in D(A)$ there is $y^* \in J(y)$ such that $\Re\langle Ay, y^* \rangle \leq 0$, where $J$ is the duality map from $X$ to $X^*$, which, to each $y \in X$, corresponds the set $J(y)$ of all $\phi \in X^*$ such that $\langle y, \phi \rangle = \|y\|^2 = \|\phi\|^2$, and where the dual $X^*$ of $X$ is the set of all bounded linear functionals on $X$ and $(y, \phi)$ is the duality pairing between $y \in X$ and $\phi \in X^*$. Conversely, if $A$ is a densely defined closed linear operator such that both $A$ and its adjoint operator $A^*$ are dissipative, then $A$ is the infinitesimal generator of a $C_0$–semigroup of contractions on $X$ (see [40], pp. 14-15).

- The resolvent set $\rho(A)$ of an unbounded linear operator $A$ in a Banach space $X$ is the set of all complex numbers $\lambda$ for which: $(\lambda I - A)^{-1}$ is a bounded linear operator in $X$. The family $R(\lambda; A) := (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ is called the resolvent of $A$. The operator $R(\lambda; A)$ commutes with $A$ and $S(t)$, and for all $y \in X$, we have $\lambda R(\lambda; A)y \rightarrow y$, as $\lambda \rightarrow +\infty$. We also have that $AR(\lambda; A) \in \mathcal{L}(X)$ and for all $y \in D(A)$, $\lambda AR(\lambda; A)y \rightarrow Ay$, as $\lambda \rightarrow +\infty$ (see [40], pp. 9-10).

- We have the following properties regarding $C_0$–semigroups ([40], pp. 4-5)

1. For every $x \in X$, $t \geq 0$: $\lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} S(s)xds = S(t)x$. 

(2) For every \( x \in X, t \geq 0 \): \( \int_0^t S(s)x ds \in \mathcal{D}(A) \) and \( A(\int_0^t S(s)x ds) = T(t)x - x \).

(3) For every \( x \in \mathcal{D}(A) \) and \( 0 \leq s \leq t \): \( S(t)x - S(s)x = \int_s^t S(\tau)Ax d\tau = \int_s^t AS(\tau)x d\tau \).

- From the above properties, one can deduce that if \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup \( S(t) \), then \( \mathcal{D}(A) \) (the domain of \( A \)) is dense in \( X \) and \( A \) is a closed linear operator. Moreover, according to Hille-Yosida’s Theorem (see for instance [40], p. 20), a linear operator \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup \( S(t) \) satisfying (4) if and only if (i) \( A \) is closed and \( \mathcal{D}(A) \) is dense in \( X \), and (ii) the resolvent set \( \rho(A) \) of \( A \) contains the ray \((\omega, +\infty)\) and \( \|R(\lambda; A)^n\| \leq \frac{M}{(\lambda-\omega)^n} \) for \( \lambda > \omega, n = 1, 2, ... \). In particular, a closed operator \( A \) with densely domain \( \mathcal{D}(A) \) in \( X \) is the infinitesimal generator of a \( C_0 \)-semigroup of contractions on \( X \) if and only if the resolvent set \( \rho(A) \) of \( A \) contains \( \mathbb{R}^+ \) and for all \( \lambda > 0 \): \( \|R(\lambda; A)\| \leq \frac{1}{\lambda} \) (see [40], p. 8).

- For \( x \in \mathcal{D}(A) \): \( Ax = \frac{d}{dt} S(t)x \big|_{t=0} \) and that \( y(t) := S(t)y_0 \) is differentiable and lies in \( \mathcal{D}(A) \) for all \( t > 0 \), and is the unique solution of the Cauchy problem: \( \dot{y}(t) = Ay(t), t > 0, y(0) = y_0 \). Moreover, for every \( y_0 \in X \); \( y(t) = S(t)y_0 \) is called mild solution of this Cauchy problem.

We now consider the nonhomogeneous initial value problem

\[
\begin{align*}
y_t(t) &= Ay(t) + f(t, y(t)), \quad t \in [0, T] \\
y(0) &= y_0
\end{align*}
\]

where \( T > 0, A \) is the infinitesimal generator of a \( C_0 \)-semigroup \( S(t) \) on \( X \) and \( f : [0, T] \times X \to X \) is a possibly non linear function.

Let us recall the notion of weak solution from [5].

**Definition 3.1.** A function \( y \in C([0, T]; X) \) is a weak solution of (5) if for every \( \varphi \in \mathcal{D}(A^*) \) (the domain of the adjoint operator \( A^* \) of \( A \)), the function \( t \mapsto \langle y(t), \varphi \rangle \) is absolutely continuous on \([0, T]\) and

\[
\frac{d}{dt} \langle y(t), \varphi \rangle = \langle y(t), A^* \varphi \rangle + \langle f(t, y(t)), \varphi \rangle, \quad \text{for a.e. } t \in [0, T],
\]

where \( \langle \cdot, \cdot \rangle \) is the duality pairing between the Banach space \( X \) and its dual \( X^* \).

A function \( y \in C([0, T]; X) \) is a weak solution of (5) on \([0, T]\) if and only if \( f(., y(., \cdot)) \in L^1(0, T; X) \) and \( y \) satisfies the variation of constants formula (see [5]):

\[
y(t) = S(t)y_0 + \int_0^t S(t-s)f(s, y(s))ds, \quad \forall t \in [0, T].
\]

Functions \( y \) satisfying the above formula are called “mild solutions” of the system (5). Moreover, the function \( y \) is a classical solution of (5) if \( y(t) \in \mathcal{D}(A) \), for \( t \in (0, T) \), \( y \) is continuous on \([0, T]\), \( y \) is continuously differentiable on \((0, T)\) and satisfies (5) (see [40], p. 126 & pp. 183-184). The mild solution \( y \) is a strong solution of (5) if it is differentiable almost everywhere on \([0, T]\), \( y_t \in L^1(0, T; X) \) and satisfies (5) a.e. on \([0, T]\) (see [40], p. 109).

The next result discusses the well-posedness for the problem (5) in the case of Lipschitz continuous functions \( f \).
Theorem 3.2. ([40], p. 184). Let \( f : [0, T] \times X \to X \) be continuous in \( t \) on \([0, T]\) and uniformly Lipschitz continuous on \( X \). Then for every \( y_0 \in X \) the system (5) has a unique mild solution \( y \in C([0, T] : X) \). Furthermore, the mapping \( y_0 \mapsto y \) is Lipschitz continuous from \( X \) into \( C([0, T] : X) \).

A sufficient condition for the mild solution of (5) to be a classical solution is given next.

Theorem 3.3. ([40], p. 187). Let \( f : [0, T] \times X \to X \) be continuously differentiable on \([0, T] \times X\). Then the mild solution of (5) with \( y_0 \in \mathcal{D}(A) \) is a classical solution of (5).

If \( f \) is only Lipschitz continuous, then the mild solution of (5) is not in general a classical one. However, in the context of a reflexive space \( X \), this may suffice to assure that the mild solution \( y \) with initial state \( y_0 \in \mathcal{D}(A) \) is a strong solution. We have:

Theorem 3.4. ([40], p. 189). Assume that \( X \) is a reflexive Banach space and that \( f : [0, T] \times X \to X \) is Lipschitz continuous in both variables. Then the mild solution \( y \) of the initial value problem (5) with \( y_0 \in \mathcal{D}(A) \) is a strong solution of (5).

3.2. Technical lemmas. Let us give the following lemma which concerns the uniform approximation of continuous functions using Bernstein polynomials [14, 15, 34].

Lemma 3.5. Let \( u : [0, 1] \to X \) be a continuous function from \([0, 1]\) to a Banach space \((X, \| \cdot \|_X)\), and let \( B_n(u) \) be the \( n \)th Bernstein polynomial for \( u \):

\[
B_n(u)(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} u\left(\frac{k}{n}\right), \quad n \geq 1.
\]

Then the sequence \( B_n(u) \) tends uniformly to \( u \), i.e., \( \sup_{t \in [0, 1]} \| B_n(u)(t) - u(t) \|_X \to 0 \), as \( n \to +\infty \).

Remark 2. For all \( n \geq 1 \), we have ([14], pp. 112-113)

\[
B_n(u)(t) = n \sum_{k=0}^{n-1} \binom{n-1}{k} t^k (1-t)^{n-1-k} (u\left(\frac{k}{n}\right) - u\left(\frac{k+1}{n}\right)), \quad (6)
\]

where \( B_n(u)(t) \) is the derivative of \( B_n(u)(t) \) with respect to \( t \).

Based on the mollification technique (see [11], pp. 69-71), we can prove the following smoothness lemma.

Lemma 3.6. Let \( \Omega \) be an open bounded set of \( \mathbb{R}^n \), \( n \geq 1 \). For all \( h \in L^\infty(\Omega) \) such that \( h \geq 0 \), a.e. in \( \Omega \), there exists \( (h_r) \subset C^\infty(\mathbb{R}^n) \) such that

(i) \( (h_r|_{\Omega}) \) is uniformly bounded with respect to \( r \), (where \( h_r|_{\Omega} \) designs the restriction of \( h_r \) to \( \Omega \)),

(ii) for all \( r > 0 \), \( h_r \geq 0 \), a.e in \( \Omega \),

and

(iii) \( h_r|_{\Omega} \to h \) in \( L^2(\Omega) \), as \( r \to 0^+ \).

4. The proof of the main results. In the sequel, we consider the system (1) on the Hilbert state space \( H := L^2(\Omega) \) equipped with its natural norm denoted by \( \| \cdot \| \), and let us introduce the unbounded operator \( A = \Delta \) with domain \( \mathcal{D}(A) = H^2(\Omega) \cap H^4(\Omega) \), endowed with the following graph norm: \( \| y \|_{\mathcal{D}(A)} = (\| y \|^2 + \| Ay \|^2)^{\frac{1}{2}} \), \( y \in \mathcal{D}(A) \). The operator \( A \) generates a contraction semigroup \( S(t) \) in \( H \). Then \( A \) is
dissipative, i.e., $\Re \langle Az, z \rangle \leq 0$, $\forall z \in \mathcal{D}(A)$, and we have $\sup_{t \geq 0} \|S(t)\| \leq 1$ and $\sup_{\lambda > 0} \|\lambda R(\chi; A)\| \leq 1$.

4.1. **Proof of Theorem 2.1.** Let us first observe that in the case where $a(x) = 0$, a.e. $x \in \Omega$, one can just use the null control, since $S(T)y_0 + \int_0^T S(T-s)f(s, y(s))ds \to y_0$, as $T \to 0^+$. Indeed, observing that $S(T)y_0 \to y_0$, as $T \to 0^+$, it suffices to show that: $\int_0^T S(T-s)f(s, y(s))ds \to 0$, as $T \to 0^+$.

Let $T > 0$. For all $t \in [0, T]$ and $y \in L^2(\Omega)$, we have

$$\|f(t, y)\| \leq \|f(t, y) - f(0, 0)\| + \|f(0, 0)\|$$

$$\leq L(t + \|y\|) + \|f(0, 0)\|,$$

where $L$ is a Lipschitz constant of $f$. It follows that

$$\|f(t, y)\| \leq L(T + \|y\|) + \|f(0, 0)\|, \quad \forall t \in [0, T]. \quad (7)$$

The mild solution $y$ satisfies the following variation of constants formula

$$y(t) = S(t)y_0 + \int_0^t S(t-s)f(s, y(s))ds, \quad \forall t \in [0, T]. \quad (8)$$

Thus, using (7) we have

$$\int_0^T \|f(s, y(s))ds\| \leq T(L + \|f(0, 0)\|) + L\int_0^t \|y(s)\|ds. \quad (9)$$

Then, it comes from (8)

$$\|y(t)\| \leq \|y_0\| + T(L + \|f(0, 0)\|) + L\int_0^t \|y(s)\|ds,$$

which gives via Gronwall’s inequality

$$\|y(t)\| \leq \left(\|y_0\| + T(L + \|f(0, 0)\|)\right)e^{TL}.$$ 

This together with (9) and the fact that $S(t)$ is a semigroup of contractions, gives

$$\left\|\int_0^T S(T-s)f(s, y(s))ds\right\| \leq T(L + \|f(0, 0)\|) + LT\left(\|y_0\| + T(L + \|f(0, 0)\|)\right)e^{TL},$$

which gives the claimed result. Thus we shall in the following assume $a(\cdot) \neq 0$. Moreover, for a time of steering $T > 0$ (which is to be determined later) we consider the control $v(x, t) = v_T(x) := \frac{a(x)}{T}$. Then the system (1) admits a unique mild solution $y(t)$ in $[0, T_0]$ in the state space $L^2(\Omega)$ (see [40], Theorem 1.2, p. 184), which is given by the following variation of constants formula

$$y(t) = S(t)y_0 + \int_0^t S(t-s)\left(\frac{a(x)}{T}y(s) + f(s, y(s))\right)ds, \quad \forall t \in [0, T_0]. \quad (10)$$

Furthermore since $a \in L^\infty(\Omega)$ and $f$ is Lipschitz, it follows from Gronwall’s inequality [16, 45] that the mapping $y_0 \mapsto y(t)$ is Lipschitz in $H$.

Now, from the assumptions (i)-(ii) of Theorem 2.1, we can derive that: $e^{\alpha}y_0 = y^d$. Indeed, for a.e. $x \in \Omega$ the formulae is obvious for $x \in O \cap \Lambda$. Moreover, the case $x \notin O$ follows from the fact that $\{x \in \Omega/ y_0(x) \neq y^d(x)\} \subset O$ a.e. Now if $x \notin \Lambda$, then $y_0(x) = 0$ and so by assumption (ii) of Theorem 2.1, we have $y^d(x) = 0$. Thus
If we, formally, use the following formula

\[ y(T) - y^d = \int_0^T e^{(T-s)a(s)} (Ay(s) + f(s, y(s))) ds, \]
	hen it suffices to show that the term in the right-hand side of the last relation tends to zero in \( L^2(\Omega) \) as \( T \to 0^+ \). To prove this, we need to show that the mild solution \( y(t) \) of (1) can be approximated by a classical one, and then we conclude by an argument of density. We will distinguish several cases.

4.1.1. The case \( a \in W^{2, \infty}(\Omega) \) and \( y_0 \in \mathcal{D}(A) \). This first case consists in three steps.

**Step 1.** In order to approximate the mild solution \( y(t) \) with a classical one, we will approximate the continuous function \( t \mapsto f(t, y(t)) \) with a \( C^1 \) function.

Without loss of generality, we assume in the sequel that \( T_0 = 1 \). Also, for any \( T \in (0, T_0) \), the letter \( C \) will be used to denote a generic positive constant (which is independent of \( T \)).

Since \( S(t) \) is a semigroup of contractions, we deduce from (10) that

\[ \|y(t)\| \leq \|y_0\| + \frac{\|a\|_{L^{\infty}(\Omega)}}{T} \int_0^T \|y(s)\| ds + \int_0^t \|f(s, y(s))\| ds. \]

Then, using (7), it comes:

\[ \|y(t)\| \leq \|y_0\| + T \left( L + \|f(0, 0)\| \right) + \left( \frac{\|a\|_{L^{\infty}(\Omega)}}{T} + L \right) \int_0^t \|y(s)\| ds, \]

which, by Gronwall’s inequality, leads to

\[ \|y(t)\| \leq \left( \|y_0\| + T \left( L + \|f(0, 0)\| \right) \right) e^{\left( \frac{\|a\|_{L^{\infty}(\Omega)}}{T} + TL \right)}. \]

Hence there exists a positive constant \( C = C(\|a\|_{L^{\infty}(\Omega)}) \) (which is independent of \( T \in (0, 1) \)) such that

\[ \|y(t)\| \leq C(1 + \|y_0\|), \forall t \in [0, T]. \quad (11) \]

Let us consider the continuous function \( F : t \mapsto f(t, y(t)) \). Then, using (7) and (11) and the fact that \( T < 1 \), we get

\[ \|F(t)\| \leq C(1 + \|y_0\|), \forall t \in [0, T], \quad (12) \]

where \( C = C(\|a\|_{L^{\infty}(\Omega)}) > 0 \) is independent of \( T \).

Let us show that \( F \) is Lipschitz in \([0, T]\). For all \( h, t \in [0, T] \) such that \( t+h \in [0, T] \), we have

\[
y(t+h) - y(t) = S(t+h)y_0 - S(t)y_0 + \int_0^h S(t+h-s)\left( \frac{a}{T}y(s) + F(s) \right) ds \\
+ \int_0^{t+h} S(t-s) \left( \frac{a}{T} (y(s+h) - y(s)) + (F(s+h) - F(s)) \right) ds.
\]

Let us estimate the first and the last terms of the right side of (13). For the first term, we have (since \( y_0 \in \mathcal{D}(A) \)):

\[ \|S(t+h)y_0 - S(t)y_0\| = \| \int_t^{t+h} S(s)Ay_0 ds \| \leq h \| Ay_0 \|. \]
Moreover, from the definition of $F$, we have:

$$\|F(s + h) - F(s)\| \leq L(h + \|y(s + h) - y(s)\|),$$

where $L$ is a Lipschitz constant of $f$.

Then using the two last estimates and inequalities (11)-(12) and the fact that $S(t)$ is a contraction semigroup, we derive from (13)

$$\|y(t + h) - y(t)\| \leq h A y_0 + h C (\|a\|_{L^\infty(\Omega)}(1 + 1 + \|y_0\|)$$

$$\quad + \int_0^t (L h + (\|a\|_{L^\infty(\Omega)} + L \|y(s + h) - y(s)\|) ds.$$

Then using $0 < T < 1 < \frac{1}{4}$, we deduce that:

$$\|y(t + h) - y(t)\| \leq h A y_0 + C (1 + \|y_0\|) + L \|a\|_{L^\infty(\Omega)} + L \int_0^t \|y(s + h) - y(s)\| ds,$$

where $C = C(\|a\|_{L^\infty(\Omega)})$ is independent of $T$, which by Gronwall's inequality gives the following estimate

$$\|y(t + h) - y(t)\| \leq C (1 + \|y_0\| D(A)) h, \quad \forall t \in [0, T],$$

where $C = C(\|a\|_{L^\infty(\Omega)})$ is independent of $T$.

Then using the last estimate and the fact that $f$ is Lipschitz, this gives

$$\|F(t) - F(s)\| \leq L (|t - s| + \|y(t) - y(s)\|)

\leq L (1 + C(1 + \|y_0\| D(A))) |t - s|, \quad \forall t, s \in [0, T].$$

This leads (for $0 < T < 1$) to

$$\|F(t) - F(s)\| \leq \frac{M_1}{T} |t - s|, \quad \forall t, s \in [0, T],$$

where $M_1 = M_1(\|a\|_{L^\infty(\Omega)}, \|y_0\| D(A)).$

Then given $\epsilon > 0$, we have for $\eta := \frac{\epsilon e^{-\|a\|_{L^\infty(\Omega)}}}{4 M_1}$

$$\forall t, s \in [0, T], \quad |t - s| < \eta \Rightarrow \|F(t) - F(s)\| < \frac{\epsilon e^{-\|a\|_{L^\infty(\Omega)}}}{4}.$$ (15)

Using Lemma 3.5, we can uniformly approach $F(t)$ on $[0, 1]$ with the following sequence of polynomials:

$$F_n(t) = \sum_{k=0}^n \binom{n}{k} t^k (1 - t)^{n-k} F\left(\frac{k}{n}\right), \quad n \geq 1.$$

From the expression of $F_n(t)$, we have by virtue of (12)

$$\sup_{t \in [0, T]} \|F_n(t)\| \leq C(1 + \|y_0\|),$$

where $C = C(\|a\|_{L^\infty(\Omega)}) > 0$ is independent of $T \in (0, 1)$.

By separating the terms of $F_n(t)$ for which $|\frac{k}{n} - t| < \eta$ and those for which $|\frac{k}{n} - t| \geq \eta$, we derive from (12) & (15) that:

$$\|F_n(t) - F(t)\| \leq \frac{\epsilon e^{-\|a\|_{L^\infty(\Omega)}}}{4} + \frac{C(1 + \|y_0\|)}{2n \eta^2}, \quad \forall t \in [0, T],$$

where $C = C(\|a\|_{L^\infty(\Omega)}) > 0$ is independent of $T \in (0, 1)$. 


Thus (recall that $\eta := \frac{\|e^{-\|a\|L^\infty(\Omega)}\|}{4M_t}$)
\[
\|F_n(t) - F(t)\| \leq \frac{\epsilon e^{-\|a\|L^\infty(\Omega)}}{4} + \frac{M_2}{nT^2\epsilon}, \quad \forall t \in [0, T],
\]
where $M_2 = M_2(\|a\|L^\infty(\Omega), \|y_0\|_{D(A)})$ is independent of $T$ and $n$. Hence we have
\[
\|F_n(t) - F(t)\| \leq \frac{\epsilon e^{-\|a\|L^\infty(\Omega)}}{2}, \quad \forall t \in [0, T],
\]
whenever
\[
nT^2 > \frac{4M_2\epsilon\|a\|L^\infty(\Omega)}{\epsilon^3}. \tag{18}
\]
Let $y_n(t)$ be the solution of the system:
\[
\frac{d}{dt}y_n(t) = Ay_n(t) + \frac{a(x)}{T} y_n(t) + F_n(t), \quad t \in (0, 1), \quad y_n(0) = y_0. \tag{19}
\]
Then we have
\[
y_n(t) = S(t)y_0 + \int_0^t S(t-s)\left(\frac{a(x)}{T} y_n(s) + F_n(s)\right)ds, \quad \forall t \in [0, 1]. \tag{20}
\]
Thus, using (16) and the Gronwall inequality, we get
\[
\|y_n(t)\| \leq C(1 + \|y_0\|), \quad \forall t \in [0, T],
\]
for some constant $C = C(\|a\|L^\infty(\Omega))$ which is independent of $T$ and $n$.
Moreover, it follows from (10) and (20) that:
\[
\|y(t) - y_n(t)\| \leq \int_0^t \frac{\|a\|L^\infty(\Omega)}{T} \|y_n(s) - y(s)\|ds + \int_0^T \|F_n(s) - F(s)\|ds, \quad \forall t \in [0, T].
\]
Then under (18) we have,
\[
\|y_n(t) - y(t)\| \leq \int_0^t \frac{\|a\|L^\infty(\Omega)}{T} \|y_n(s) - y(s)\|ds + \frac{T\epsilon e^{-\|a\|L^\infty(\Omega)}}{2},
\]
which gives via Gronwall inequality
\[
\|y(t) - y_n(t)\| < \frac{T\epsilon}{2}, \quad \forall t \in [0, T] \subset [0, 1], \tag{21}
\]
and hence
\[
\sup_{t \in [0, T]} \|y(t) - y_n(t)\| < \frac{\epsilon}{2}.
\]
**Step 2.** Here, we will establish an upper bound for the solution $y_n(t)$ of (20) with respect to the graph norm.
Since $y_0 \in D(A)$ and $F_n \in C^1([0, 1]; L^2(\Omega))$, we have that $y_n(t)$ is a classical solution (see [40], Theorem 1.5, p. 187). Then for all $t > 0$, we have $y_n(t) \in D(A)$ and
\[
\frac{d}{dt} \langle y_n(t), \phi \rangle = \langle y_n(t), \frac{a(x)}{T} \phi \rangle + \langle Ay_n(t) + F_n(t), \phi \rangle, \quad \forall t \in (0, 1), \forall \phi \in H. \tag{22}
\]
In other words, $y_n(t)$ is a weak solution of (19). We know that $Ay_n \in L^2(0, 1; L^2(\Omega))$ (see for instance [18], pp. 360-361). Hence $y_n(t)$ (see [3, 4]) satisfies the following variation of constants formula:
\[
y_n(t) = e^{t\frac{a(x)}{T}}y_0 + \int_0^t e^{(t-s)\frac{a(x)}{T}}(Ay_n(s) + F_n(s))ds, \quad \forall t \in [0, 1]. \tag{23}
\]
In particular, we have
\[ y_n(T) - y^d = \int_0^T e^{\frac{t-s}{T}a(x)}(Ay_n(s) + F_n(s))ds. \] (24)

Applying the bounded operator \( A_\lambda = \lambda R(\lambda; A)A \) to (20), we get
\[
A_\lambda y_n(t) = S(t)A_\lambda y_0 + \frac{1}{T} \int_0^t A_\lambda S(t-s)(a(x)y_n(s) + F_n(s))ds
\]
\[= S(t)A_\lambda y_0 + \frac{1}{T} \int_0^t A_\lambda S(t-s)(a(x)y_n(s))ds + \int_0^t \lambda R(\lambda; A)S'(t-s)F_n(s)ds
\]
\[= S(t)A_\lambda y_0 + \frac{1}{T} \int_0^t A_\lambda S(t-s)(a(x)y_n(s))ds - \int_0^t \frac{d}{ds}(\lambda R(\lambda; A)S(t-s)F_n(s))ds + \int_0^t \lambda R(\lambda; A)S(t-s)F'_n(s)ds
\]
\[= S(t)A_\lambda y_0 + \frac{1}{T} \int_0^t A_\lambda S(t-s)(a(x)y_n(s))ds + \lambda R(\lambda; A)(S(t)F_n(0) - F_n(t)) + \int_0^t \lambda R(\lambda; A)S(t-s)F'_n(s)ds.
\]

Since \( y_n(t) \in D(A) \) and \( a \in W^{2,\infty}(\Omega) \), we also have \( ay_n(t) \in D(A) \) for all \( t \in [0,1] \).

Hence, using the properties of the semigroup and the resolvent associated to \( A \),
we deduce from the above expression that
\[
\|A_\lambda y_n(t)\| \leq \|Ay_0\| + \frac{1}{T} \int_0^t \|A(a(x)y_n(s))\|ds + \|F_n(t)\| + \|F_n(0)\| + \int_0^t \|F'_n(s)\|ds.
\]
Then, letting \( \lambda \to +\infty \), we get
\[
\|A y_n(t)\| \leq \|Ay_0\| + \frac{1}{T} \int_0^t \|A(a(x)y_n(s))\|ds + \|F_n(t)\| + \|F_n(0)\| + \int_0^t \|F'_n(s)\|ds,
\] (25)
where the constant \( C = C(\|a\|_{W^{2,\infty}(\Omega)}) \) is independent of \( T \) and \( n \).

Let us now study the terms of right hand of inequality (25). We have by (6) that
\[
F'_n(t) = n \sum_{k=0}^{n-1} \left( \begin{array}{c} n-1 \\ k \end{array} \right) t^k (1-t)^{n-1-k} (F \left( \frac{k+1}{n} \right) - F \left( \frac{k}{n} \right)),
\]
which by (14) gives
\[
\sup_{0 \leq t \leq T} \|F'_n(t)\| \leq \frac{M_1}{T}, \quad (M_1 = M_1(\|a\|_{L^\infty(\Omega)}, \|y_0\|_{D(A)})).
\] (26)
Moreover, for every \( y \) in \( H^2(\Omega) \) we have the following second order Leibniz rule
\[
\Delta(ay) = y \Delta a + 2 \nabla a \cdot \nabla y + 2 \Delta a \cdot y, \text{ a.e. in } \Omega.
\] (27)
Now for every \( y \in H^1_0(\Omega) \cap H^2(\Omega) \), it comes from the Green formula that \( \langle \Delta y, y \rangle = -\|\nabla y\|^2 \), from which we deduce via Cauchy-Schwarz’s and Poincaré’s inequalities that
\[
\|\nabla y\| \leq C\|\Delta y\|.
\] (28)
for some constant \( C > 0 \) which depends only on \( \Omega \).

Taking into account (28) and the fact that \( a \in W^{2,\infty}(\Omega) \), we derive from (27)
\[
\int_0^t \|A(a(x)y_n(s))\|ds \leq \|\Delta a\|_{L^\infty(\Omega)} \int_0^t \|y_n(s)\|ds + C \int_0^t \|Ay_n(s)\|ds, \forall t \in [0,T],
\] (29)
where \( C = C(\|a\|_{W^{2,\infty}(\Omega)}) \) is independent of \( T \) and \( n \).
Then reporting (16), (26) and (29) in (25), we deduce, via Gronwall’s inequality
\[ \|y_n(t)\|_{D(A)} \leq M_3, \quad \forall t \in [0, T], \] (30)
where \( M_3 = M_3(\|a\|_{W^2,\infty}(\Omega), \|y_0\|_{D(A)}) \) is independent of \( T \) and \( n \).

**Step 3.** We now show that, for \( n \) large enough, one can choose \( T \) small enough so that \( y_n(T) \) (and so is \( y(T) \)) approaches \( y^d \) with any a priori fixed precision.

Using the estimates (16) and (30), we get from the relation (24)
\[ \|y_n(T) - y^d\| \leq M_4 T \] (31)
for some constant \( M_4 = M_4(\|a\|_{W^2,\infty}(\Omega), \|y_0\|_{D(A)}) \) which is independent of \( T \) and \( n \). We deduce that
\[ \|y_n(T) - y^d\| < \frac{\epsilon}{2}, \]
whenever
\[ 0 < T < \frac{\epsilon}{2M_4}. \] (32)

Finally, we can observe that \( F_n \) depends implicitly via \( y(t) \) on \( T \), but \( n \) is independent of \( T \). Then taking \( n \) and \( T \), respectively, such that
\[ 2\left(\frac{M_2\|a\|_{L^\infty(\Omega)}}{n\epsilon^4}\right)^\frac{1}{2} < \frac{\epsilon}{2M_4} \]
and
\[ 2\left(\frac{M_2\|a\|_{L^\infty(\Omega)}}{n\epsilon^3}\right)^\frac{1}{2} < T < \inf(1, \frac{\epsilon}{2M_4}), \]
so that (18) and (32) hold. Hence, we have
\[ \|y(T) - y^d\| \leq \|y_n(T) - y(T)\| + \|y_n(T) - y^d\| < \epsilon. \]

4.1.2. The case \( a \in W^{2,\infty}(\Omega) \) and \( y_0 \in L^2(\Omega) \). For all \( \lambda > 0 \), we set \( \tilde{y}_\lambda := \lambda R(\lambda; A)y_0 \in D(A) \), and let \( \tilde{y}_\lambda \) be the mild solution to (1) corresponding to the initial state \( \tilde{y}_\lambda \) with the \( \lambda \)-independent control \( v(x, t) = \frac{1}{\lambda} \ln(e^{T_0}) \mathbf{1}_{\Lambda \cap \Omega} \). We have
\[ \|y(T) - y^d\| \leq \|y(T) - \tilde{y}_\lambda(T)\| + \|\tilde{y}_\lambda(T) - e^a(x)\tilde{y}_\lambda - y^d\|. \] (33)

It follows from the variation of constants formula that
\[ \tilde{y}_\lambda(t) - y(t) = S(t)\tilde{y}_\lambda - S(t)y_0 + \frac{1}{T} \int_0^T S(t-s)\left(a(x)(\tilde{y}_\lambda(s) - y(s)) + f(s, \tilde{y}_\lambda(s)) - f(s, y(s))\right)ds. \]

Then, using the contraction property of the semigroup \( S(t) \), it comes
\[ \|\tilde{y}_\lambda(t) - y(t)\| \leq \|\tilde{y}_\lambda - y_0\| + \frac{\|a\|_{L^\infty(\Omega)}}{T} \int_0^T \|\tilde{y}_\lambda(s) - y(s)\|ds + L \int_0^T \|\tilde{y}_\lambda(s) - y(s)\|ds, \quad \forall t \in [0, T]. \]

Thus, the Gronwall’s inequality gives
\[ \|\tilde{y}_\lambda(T) - y(T)\| \leq C\|\tilde{y}_\lambda - y_0\|, \quad (C = C(\|a\|_{L^\infty(\Omega)})). \]

Moreover, we have
\[ \|e^{a(x)}\tilde{y}_\lambda - y^d\| \leq e \|a\|_{L^\infty(\Omega)} \|\tilde{y}_\lambda - y_0\|, \]
we deduce that there is a \( \lambda > 0 \), which is independent of \( T \) in \( (0, 1) \), such that
\[ \|\tilde{y}_\lambda(T) - y(T)\| + \|e^{a(x)}\tilde{y}_\lambda - y^d\| < \frac{\epsilon}{2}. \] (34)
For such a $\lambda$, we have:

$$y_\lambda(T) - e^{a(x)}\tilde{y}_0 = \int_0^T e^{\frac{T-s}{a(x)}}(Ay_\lambda(s) + f(s, y_\lambda(s)))ds.$$

According to the case discussed in the previous subsection, there exists $0 < T < 1$ for which

$$\|y_\lambda(T) - e^{a(x)}\tilde{y}_0\| < \frac{\epsilon}{2}.$$  (35)

From (33)-(35), we conclude that

$$\|y(T) - y^d\| < \epsilon.$$  

4.1.3. The general case: $a \in L^\infty(\Omega)$ and $y_0 \in L^2(\Omega)$. By Lemma 3.6, there exists $(h_r) \subset C^\infty(\mathbb{R}^d)$ such that $h_r|_{\Omega} \to h = e^a$ in $L^2(\Omega)$, as $r \to 0^+$ and $h_r > 0$ for a.e. in $\Omega$. Moreover, since $e^a \in L^\infty(\Omega)$, the sequence $(h_r)$ can be chosen such that $(h_r|_{\Omega})$ is bounded in $\Omega$ uniformly w.r.t $r > 0$.

Let us define the function: $a_r = \ln(h_r) \in W^{2,\infty}(\Omega)$. Since $a_r \in L^\infty(\Omega)$, there is a unique mild solution $y(t)$ corresponding to the control $v(x, t) = \frac{a_r(x)}{T}$ and initial state $y(0) = y_0$, and $y$ depends continuously on $y_0$. Let $(y_{0s}) \in L^\infty(\Omega)$ be such that $y_{0s} \to y_0$ in $L^2(\Omega)$, as $s \to 0^+$. Then we have

$$\|y(T) - y^d\| \leq \|y(T) - e^{a_r}y_0\| + \|e^{a_r}y_0 - e^{a_{0s}}y_{0s}\| + \|e^{a_{0s}}y_{0s} - e^{a_{0s}}y_0\| + \|e^{a_{0s}}y_{0s} - e^{a_r}y_0\|.$$

Moreover,

$$\|e^{a_{0s}}y_{0s} - e^{a_{0s}}y_{0s}\| + \|e^{a_{0s}}y_{0s} - e^{a_{0s}}y_0\| \leq \left(\sup_{r > 0} \|e^{a_r}\|_{L^\infty(\Omega)} + \epsilon\right)\|y_{0s} - y_0\|.$$

Since $(a_r)$ is uniformly bounded w.r.t $r$, there exists $s > 0$ be such that

$$\|e^{a_r}y_0 - e^{a_{0s}}y_{0s}\| + \|e^{a_{0s}}y_{0s} - e^{a_s}y_0\| < \frac{\epsilon}{3},$$

and for such value of $s$, we consider a $r > 0$ such that

$$\|e^{a_r} - e^{a_s}\|_{L^\infty(\Omega)} < \frac{\epsilon}{3}.$$

Finally, for this value of $r$, it comes from the case of the previous subsection that there exists $T > 0$ such that

$$\|y(T) - e^{a_r}y_0\| < \frac{\epsilon}{3}.$$

Hence we have $\|y(T) - y^d\| < \epsilon$.

4.2. Proof of corollary 1. We know from Lemma 3.6 that there exists $(h_r) \subset C^\infty(\mathbb{R}^d)$ such that for all $r > 0$, we have $h_r > 0$, a.e. in $\Omega$ and $h_r|_{\Omega} \to h$ in $L^2(\Omega)$, as $r \to 0^+$. Let $\epsilon > 0$ be fixed, and let $r > 0$ be such that

$$\|h_r - h\|_{L^\infty(\Omega)} < \frac{\epsilon}{2}.$$

Using the control $v(x, t) = \frac{a_r(x)}{T_1}$ with $0 < T_1 = T_1(\epsilon, y^d, y_0) < T$ is small enough and $a_r := \ln(h_r) \in C^\infty(\Omega)$, we get from the proof of Theorem 2.1

$$\|y(T_1) - e^{a_r}y_0\| < \frac{\epsilon}{2}.$$
Hence, observing that $h y_0 = y^d$, we deduce that
\[
\|y(T_1) - y^d\| \leq \|y(T_1) - h y_0\| + \|h_\tau y_0 - h y_0\| \\
\leq \|y(T_1) - e^{a \tau} y_0\| + \|h_\tau - h\| \|y_0\|_{L_\infty(\Omega)} \\
< \epsilon.
\]
This completes the proof.

**Remark 3.** • Based on the previous controllability results and inspired by the idea of [39], we can show that similar controllability results are still valid for a system of reaction-diffusion equations of the form (1), where the bilinear control is acting on all the equations.
• If the nonlinearity $f(t, y)$ is replaced by a function $f = f(t, y, \nabla y)$, which is globally Lipschitz continuous w.r.t all its arguments, then the same approach enables us to derive the approximate steering for every initial state $y_0 \in D(A) = H^1_0 \cap H^2(\Omega)$ as only the weaker norm $\|\cdot\|_{D(A)}$ is involved in the controllability proofs. Note that the Lipschitz continuity assumption of $f$ guarantees the conditions of well-posedness of [Pazy, p. 244-247].
• This work leaves open the case of blowing-up nonlinearities. For instance, for additive control, Fernández-Cara and Zuazua in [20] have showed that for weakly blowing up nonlinearities, the semilinear equation is small time globally null-controllable.

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