Dynamics of Spatial Heterogeneity in Landfill - A Stochastic Analysis

Amit Kumar Chattopadhyay
Nonlinearity and Complexity Research Group, Aston University, Birmingham B4 7ET, England
Prasanta Kumar Dey
Aston Business School, Aston University, Birmingham B4 7ET, England
Sadhan Kumar Ghosh
Mechanical Engineering Department, Centre for Quality Management System, Jadavpur University, Kolkata 700 032, India

Abstract
A landfill represents a complex and dynamically evolving structure that can be stochastically perturbed by exogenous factors. Both thermodynamic (equilibrium) and time varying (non-steady state) properties of a landfill are affected by spatially heterogeneous and nonlinear subprocesses that combine with constraining initial and boundary conditions arising from the associated surroundings. While multiple approaches have been made to model landfill statistics by incorporating spatially dependent parameters on the one hand (data based approach) and continuum dynamical mass-balance equations on the other (equation based modelling), practically no attempt has been made to amalgamate these two approaches while also incorporating inherent stochastically induced fluctuations affecting the process overall. In this article, we will implement a minimalist scheme of modelling the time evolution of a realistic three dimensional landfill through a reaction-diffusion based approach, focusing on the coupled interactions of four key variables - solid mass density, hydrolysed mass density, acetogenic mass density and methanogenic mass density, that themselves are stochastically affected by fluctuations in ambient surroundings. In a remarkable departure from previous predictions, our results indicate that close to the linearly stable limit, the large time steady state properties, arising out of a series of complex coupled interactions between the stochastically driven variables, are scarcely affected by the biochemical growth-decay statistics. Our results clearly show that an equilibrium landfill structure is primarily determined by the solid and hydrolysed mass densities only rendering the other variables as statistically “irrelevant” in this (large time) asymptotic limit. The other major implication of incorporation of stochasticity in the landfill evolution dynamics is in the hugely reduced production times of the plants that are now approximately 20-30 years instead of the previous (incorrect) deterministic model predictions of 50 years and above. The predictions from this stochastic model are in conformity with available experimental observations.

Keywords: Stochastic, Landfill, Probability density function, Ensemble, Root-mean-square, Hydrolysed mass, Acetogenic mass, Methanogenic mass

1. Introduction
Municipal waste management (MSW) has traditionally been a supply chain based facility primarily focused on ascertaining the most cost effective way of disposing household waste, including bio-waste. The concept of a modern
landfill though stems from the idea of not only cost optimising bulk bio-waste disposal, but also to recycle the bio-
disposables to convert chemical energy to industrially usable electric energy. From a supply chain perspective, this
constitutes a feedback architecture where the disposable waste produces usable energy that is then fed back to the
system itself for self-sustenance of the energy production process while simultaneously trafficking the extra energy
generated for industrial usage [1]. Such operational management of power production from disposable bio-waste
fundamentally relies on the engineering novelty that could ensure maximum energy production at minimum bio-filler
consumption while also maximising the profit generated by appropriate disbursement of the energy through the as-
associated supply chain network [2]. The success of such an “alternative energy” based industry then is inherently
determined by the accuracy at which the following two factors can be probabilistically evaluated - the start time of the
production process and the end time line up to which bulk production can be ensured from a plant.

Collection rates of the output landfill (methane) gas and associated collection efficiency are pivotal in quantifying
the quality of a production plant and in future planning deliverables based on such production. Results at real methane
production sites (methanogenic phase) have shown that the production rate and volume could drastically change
depending on the nature and quality of clay covers, geosynthetic clay liners and geomembrane composite covers with
the CH₄ [3] emission rates varying from 2.2 to 10,000 mg/m²/d. Aside of the core landfill engineering, alternative
(methane) production methods in the form of microbial oxidation have been proposed as a cost efficient measure [4].
Numerical models, focusing on the methane production rate with respect to the height dependence of landfill sites
have supported such observations [5] with additional information such as 99% of the methane gas flow at the bottom
being oxidized across the 0.8 m soil compost column with bulk oxidation of methane methane occurring within the
top 0.2 m. In a recent benchmark work, landfill gas generation data from residual municipal solid waste (RMSW)
have been utilized to estimate the anaerobic gas generation rate constants. Without having been explicitly mentioned
in this article [6], the numbers obtained (0.0347-0.0803 y⁻¹) seem clearly to indicate the importance of incorporation
of stochasticity in the landfill gas related mathematical models. Order of magnitude estimates made in the context
of the United Kingdom landfill data also agree with such numbers [7]. The methane production time lines of real
plants as indicated by numbers in this article estimate time periods between 12.5-33 years [6] that are obtained by
inverting these rate constants ¹. Such accurate estimations of methane production time lines from landfill sites have
evaded estimations from available deterministic theoretical models [8, 9, 10, 11, 12, 13] in which these numbers come
grossly overestimated by up to 150% further confirming the need for better theoretical models.

This is where stochastic mathematical modelling of the degradation rates of landfilled waste and consequent emer-
gence of the hydro-carbon gases (e.g. methane) from the facility assumes utmost importance. It is only through appro-
priate choice of the mathematical techniques involved that one could expect to reciprocate the real life experimental
facility that converts bio-waste to leachate and biogas. In the mathematical parlance, one needs to create the correct
algorithm based on the right choice of “boundary conditions”. While traditional models [8, 9, 10, 11, 12, 13, 14] have
been successfully able to predict the correct deterministic core of the processes defining a landfill facility, almost none
of these reliable works have made any explicit allusion to stochasticity mimicking the spatiotemporal fluctuation in
the values of the parameters governing the processes. A seminal work in this context by Zacharof & Butler [13] appre-
ciated the need for such an effect and did make a structured attempt to incorporate the effect of stochasticity in a four
phased biosolid→hydrolysed leachate→acetogenic compounds→bigas (methane) dynamical model. The inclusion of
such stochastic uncertainty infused the much needed consideration of phasal heterogeneity in the mathematical model,
but the absence of any explicit stochastic (noise) term in the dynamical model essentially implied that the results pre-
dicted from this model could at best be true only at the “mean field” probabilistic model. This picture thereby neglects
the presence of any long ranged hydrodynamic mode(s) that could connect a stochastically fluctuating quantity with
another in a different phase (e.g. uncertainty in the production volume of the hydrolysed leachate leads to a followup
uncertainty in the level of mass density of the produced biogas).

The premise of this article is to bridge this information gap between realistic stochastic fluctuations of variables as
seen in actual landfill sites and assumed deterministic approximation of the same in theoretical modelling as have been
done. The target is to ensure that not only qualitative facts concerning the landfill dynamics are correctly accounted
for but also accurate quantitative estimates of decay times of gas production facilities be estimated from the theoretical
model. This is the objective of this article and would be studied using well established reaction-diffusion formalism
as detailed below.

¹Production time periods are roughly equal to the inverse of the gas generation rate constants.
Starting from the skeleton laid out in the minimalist Zacharof-Butler model [13], we have introduced two key factors that were missing in this model. Firstly, explicit noise (stochastic) terms accompany each of our four dynamical equations representing each phase. Secondly, in order to incorporate the natural tendency of any physical system to neutralise the presence of any heterogeneity, we have incorporated diffusion terms in each of the phases that lead to a more generalised multiphase approach where the phases can mix with each other through stochasticity driven diffusion.

The article is organized as follows. After the derivation and description of the core stochastic model in the following section (Section 2), the temporal dynamics will be analyzed in details (Section 3) where the focus will be on autocorrelation functions, the squared terms of which will clone the stochastic temporal dynamics. This then will be followed by a conclusion section (Section 4) where a summary of the main results will be drawn.

2. The Reaction-Diffusion Model

As indicated already, this is a non-reductionist scheme geared toward optimised management of resources related to a waste management site, primarily focusing on the linear kinetics of the individual variables - solid mass density \(n_1^{(s)}\), hydrolysed mass density \(n_2^{(h)}\), acetogenic mass density \(n_3^{(a)}\) and methanogenic mass density \(n_4^{(m)}\) - together with their mutual coupled statistics that are often stochastically perturbed by the surrounding environment as well as through spatiotemporal parametric fluctuations. This will be a non-conventional approach that will address the role of complexity in the (heirarchical) biochemical pathways’ network. Unlike its predecessors [8, 9, 10, 11, 12] that have generally focused on data based phenomenological models, our continuum model will be structured around the well established reaction-diffusion scheme that has so often been successfully employed in addressing problems in biology [15, 16] and in fluid mechanics [17]. An important work in this lineage was the one by Zacharof & Butler [13] where the basic growth-decay kinetics of the respective variables, based on a mass-balance assumption, was correctly modelled that led to a framework incorporating heterogeneous spread of the different landfill variables within the ambit of a continuum model. The model, however, seriously lacked in two aspects. Firstly, no mechanism was correctly modelled that led to a framework incorporating heterogeneous spread of the different landfill variables that have generally focused on data based phenomenological models, our continuum model will be structured around the well established reaction-diffusion scheme that has so often been successfully employed in addressing problems in biology [15, 16] and in fluid mechanics [17]. An important work in this lineage was the one by Zacharof & Butler [13] where the basic growth-decay kinetics of the respective variables, based on a mass-balance assumption, was correctly modelled that led to a framework incorporating heterogeneous spread of the different landfill variables within the ambit of a continuum model. The model, however, seriously lacked in two aspects. Firstly, no mechanism was correctly modelled that led to a framework incorporating heterogeneous spread of the different landfill variables that have generally focused on data based phenomenological models, our continuum model will be structured around the well established reaction-diffusion scheme that has so often been successfully employed in addressing problems in biology [15, 16] and in fluid mechanics [17]. An important work in this lineage was the one by Zacharof & Butler [13] where the basic growth-decay kinetics of the respective variables, based on a mass-balance assumption, was correctly modelled that led to a framework incorporating heterogeneous spread of the different landfill variables within the ambit of a continuum model. The model, however, seriously lacked in two aspects. Firstly, no mechanism was correctly modelled that led to a framework incorporating heterogeneous spread of the different landfill variables that have generally focused on data based phenomenological models, our continuum model will be structured around the well established reaction-diffusion scheme that has so often been successfully employed in addressing problems in biology [15, 16] and in fluid mechanics [17]. An important work in this lineage was the one by Zacharof & Butler [13] where the basic growth-decay kinetics of the respective variables, based on a mass-balance assumption, was correctly modelled that led to a framework incorporating heterogeneous spread of the different landfill variables.

In line with the scheme enunciated by Eastman & Ferguson [12] and later adopted by Zacharof & Butler [13], we introduce a generalised hydrolysing growth/decay rate as the solution of first order reaction-diffusion kinetics in [12] as follows

\[
R(t) = A t^\alpha \exp(-kt),
\]

where \(R(t)\) = reaction rate at time \(t\) (kilogram/year) while \(A\) and \(k\) are respetively the amplitude and decay rates expressed in non-dimensional units. The constant \(\alpha\) subjectively characterises the specific landfill concerned (\(\alpha = 1\) represents the Zacharof-Butler [13] phase). Starting from the Zacharof-Butler model [13], we model our reaction-diffusion set of Langevin equations as follows:

\[
\frac{\partial n_1^{(s)}}{\partial t} = v_1 \frac{\partial^2 n_1^{(s)}}{\partial x^2} - k_0 n_1^{(s)} + \eta_1(x,t)
\]

(2a)

\[
\frac{\partial n_2^{(h)}}{\partial t} = v_2 \frac{\partial^2 n_2^{(h)}}{\partial x^2} + k_0 n_1^{(s)} - A_d t^\alpha \exp(-k_d t) + \eta_2(x,t)
\]

(2b)

\[
\frac{\partial n_3^{(a)}}{\partial t} = v_3 \frac{\partial^2 n_3^{(a)}}{\partial x^2} + k_0' n_4^{(m)} + A_d t^\alpha \exp(-k_d t) - A_m t^\alpha \exp(-k_m t) + \eta_3(x,t)
\]

(2c)

\[
\frac{\partial n_4^{(m)}}{\partial t} = v_4 \frac{\partial^2 n_4^{(m)}}{\partial x^2} - k_0' n_4^{(m)} + A_m t^\alpha \exp(-k_m t) + n_{H_2S}(t) + \eta_4(x,t).
\]

(2d)
In the above description, $k_h$ represents the degradation rate of the solid phase while $k_h'$ represents the same for the biogas (methane) phase. The $A$'s are the biochemical growth (decay) rate amplitudes and the $k$'s represent the corresponding decay time scales. As mentioned earlier, the diffusion terms $\nabla^2 n_i$ reflect the force of homogeneity working against the evident heterogeneous phase mixture that is present in a landfill. Although $H_i S$ is an unavoidable end product, since it comes with low relative percentage contribution ($\approx 0.05\%$), we would drop this term for subsequent calculations at this level of modelling.

Here the respective diffusion terms $(\frac{\partial^2 n_i}{\partial x^2})$ ($i = 1, 2, 3$ and $4$; the superscripts have been neglected to simplify notation, a convention that we will follow throughout this article henceforth) represent the heterogeneous relaxation of each variable ($n_1(x,t) \neq n_2(x,t) \neq n_3(x,t) \neq n_4(x,t)$) defined through diffusion constants $\nu_i$'s while the $\eta_i$'s characterise uncorrelated Gaussian white noises (an assumption) in three spatial dimensions corresponding to each variable $n_i$, as follows:

$$< \eta_i(x,t)\eta_j(x',t') > = 2D_i \delta^3(x-x') \delta(t-t') \delta_{ij},$$

(3)

The above continuum coupled model represented by equations (2a,2b,2c,2d) abide the ensemble averaged mass-balance relation: $\frac{\partial}{\partial t} < n_1^{(*)} + n_2^{(*)} + n_3^{(*)} + n_4^{(*)} > = 0$, the curly bracket “<>" representing the ensemble average over all noise realisations.

It must be noted that at this minimalist level, we are neglecting effects from noise cross-correlations (e.g. $< \eta_1 \eta_2 > = 0$), an assumption which implies that the origin of stochasticity in solid mass density will not incite an identical stochastic response in the hydrolysed, acetogenic or methanogenic mass densities, a reasonable assumption at this stage.

In what follows, we will calculate the ensemble averaged root-mean-square values of the respective autocorrelation functions like $\sqrt{< n_1(x,t)n_1(x,t+\tau) >}$ in the large time ($t \rightarrow \infty$) steady state equilibrium limit. These autocorrelation functions represent the experimentally measured stochastic equivalents of their deterministic counterparts (as in [13], only for $\nu_i = 0$ though).

3. Temporal Dynamics: Results

As detailed in the previous section 2), we will start with equations (2a,2b,2c,2d) and then analyse these equations in the $k - \omega$ Fourier space.

Starting with the the $n_1$-equation, we get

$$\frac{\partial n_1}{\partial t} = \nu_1 \frac{\partial^2 n_1}{\partial x^2} - k_h n_1 + \eta_1(x,t)$$

$$-i\omega \bar{n}_1(k,\omega) = -\nu k^2 \bar{n}_1 - k_h \bar{n}_1 + \bar{\eta}_1$$

$$\bar{n}_1 = \frac{\bar{\eta}_1(k,\omega)}{-i\omega + \nu k^2 + k_h},$$

(4)

followed by

$$C_1(\tau) = < n_1(x,t) \ast n_1(x,t+\tau) > = \int d^3k \int d\omega \ e^{-i\omega \tau} < \bar{n}_1(k,\omega) \bar{n}_1(-k,-\omega) > .$$

(5)

In the experimentally viable large time equilibrium limit ($t \rightarrow \infty$), this leads to the following solution for the $n_1$-autocorrelation function:

$$< n_1(x,t) \ast n_1(x,t+\tau) > = 4\pi^2 D_1 \int_0^\infty dk \frac{k^2 e^{-i(\nu k^2 + k_h)\tau}}{\nu_1 k^2 + k_h}$$

$$\approx \frac{4\pi^2 D_1}{\nu_1} \left[ \frac{\sqrt{\pi}}{2} \frac{\exp(-k_h \tau) + \frac{\pi k_h e^{-i(\nu_1 k_h)\tau}}{\nu_1 k_h}}{\sqrt{\nu_1 k_h}} \right],$$

for ($\nu_1 \leq 1$).

(6a)

$$< n_1(x,t) \ast n_1(x,t+\tau) > \\ \approx \frac{4\pi^2 D_1}{\nu_1} \left[ \frac{\sqrt{\pi}}{2} \frac{\exp(-k_h \tau) + \frac{\pi k_h e^{-i(\nu_1 k_h)\tau}}{\nu_1 k_h}}{\sqrt{\nu_1 k_h}} \right],$$

for ($\nu_1 \leq 1$).

(6b)
The above form given in equation (6b) is based on a complex integration within aforementioned limits of the wave vector k. For a more accurate expression valid for all k-limits, we will use the form given in equation (6a). This is the formula used in the plots shown later.

Starting from equation (2b), we get

\[-i \omega \tilde{r}_2(k, \omega) = -v_{2k}^2 \tilde{r}_2 + k_0 \tilde{r}_2 - A_a h(k, \omega) + \tilde{f}_2(k, \omega),\]  

(7)

where \( h(k, \omega) = \frac{1}{\sqrt{2\pi}} \Gamma(1 + \alpha) e^{-\frac{(1 + \alpha) \pi}{2}} \left(1 + \frac{x_0^2}{k_0^2}\right)^{-\frac{1}{2}} \left[-k_0^\alpha + (1 + \frac{1}{2}) \pi k_0^\alpha \right].\)

Without much loss of generality we may use the value \( \alpha = 1 \) as in [13] to get \( h(k, \omega) = i \sqrt{2\pi} \delta(\omega + ik_0) \), where \( \delta \) alludes to the celebrated Dirac-Delta function as is widely known in the literature [16, 17].

The above prescription leads to

\[\tilde{r}_2(k, \omega) = \frac{k_0 \tilde{f}_1(k, \omega)}{-i\omega + v_{2k}^2(-i\omega + v_{1k}^2 + k_0)} - \frac{i \sqrt{2\pi} A_a \delta(\omega + ik_1)}{(\omega + ik_0)(-i\omega + v_{2k}^2)} + \frac{\tilde{f}_2(k, \omega)}{-i\omega + v_{2k}^2}\]

(8)

that in turn gives

\[< \tilde{n}_2^*(k, \omega) \tilde{n}_2(-k, -\omega) > = \frac{k_0 \tilde{f}_1(k, \omega) \tilde{f}_1(-k, -\omega)}{(\omega + v_{2k}^2)(\omega + (v_{1k}^2 + k_0)^2)} - \frac{2\pi A_a^2 \delta(\omega + ik_1)}{(\omega^2 + k_0^2)(\omega^2 + v_{2k}^2)} + \frac{\tilde{f}_2(k, \omega) \tilde{f}_2(-k, -\omega)}{\omega^2 + v_{2k}^2}\]

(9)

As previously, for \( v_1 > v_2 \), the above equations (9) may be complex integrated around all 5 poles in any of the halves of the respective Argand diagram to obtain

\[< n_2(x, t) \* n_2(x, t + \tau) > = \int d^3 k \int d \omega \ e^{-i\omega \tau} < \tilde{n}_2^*(k, \omega) \tilde{n}_2(-k, -\omega) >\]

\[= \int_0^\infty dk \frac{2\pi D_k}{v_2(v_1k^2 + k_0)} \left[ \frac{1}{v_2(k_1^2 + k_0)} \left( e^{-(v_1k^2 + k_0)\tau} \right) \right.\]

\[+ \left. e^{-v_2^2\tau} + \frac{1}{(v_1 - v_2)k^2k_0} (e^{-v_1^2\tau} - e^{-v_2^2\tau}) \right] + \frac{4\pi D_e}{v_2} e^{-v_2^2\tau}\]

(10)

In line with derivations for the autocorrelation functions corresponding to variables \( n_1 \) and \( n_2 \), it should be noted that for the other two variables, our model defines the variable \( n_3 \) as the independent one while \( n_3 \) depends on \( n_4 \). Starting from equations (2c) and (2d) and following similar algebra as before, we can now evaluate the corresponding autocorrelation functions as follows:

\[< n_4(x, t) \* n_4(x, t + \tau) > = \int d^3 k \int d \omega \ e^{-i\omega \tau} < \tilde{n}_4^*(k, \omega) \tilde{n}_4(-k, -\omega) >\]

\[= 4\pi^2 D_4 \int_0^\infty dk \frac{k^2 e^{-(v_4k^2 + k_0)\tau}}{v_4k^2 + k_0} \]

\[\approx \frac{4\pi^2 D_4}{v_4} \left[ \frac{\sqrt{\pi}}{2 \sqrt{v_2}} \exp(-k_0^2 \tau) + \frac{\pi k_0^2 e^{-(v_4k_0^2 + k_0)\tau}}{2 \sqrt{v_4k_0^2}} \right] \text{ for } (v_4 < 1).\]

(11)

In the limit \( k_0' \to 0 \), the above correlation function takes the limiting value \( < n_4(x, t) \* n_4(x, t + \tau) > \mid_{k_0' \to 0} = \frac{2D_e v_4^{1/2}}{v_4^{1/2} k_0^{1/2}}.\)

Using the information from equation (11) above, the autocorrelation function defining the acetogenic decay dynamics in the limit \( k_0' \to 0 \) can be obtained as follows

\[< n_3(x, t) \* n_3(x, t + \tau) > = \int d^3 k \int d \omega \ e^{-i\omega \tau} < \tilde{n}_3^*(k, \omega) \tilde{n}_3(-k, -\omega) >\]

\[= 4\pi^2 |k_0|^2 D_4 \int_0^\infty dk \frac{k^2 e^{-(v_4k^2 + k_0)\tau}}{5[v_4 + v_3]k^2 + k_0^2} \left[ (v_4 - v_3)k^2 + k_0^2 \right] + \frac{D_3}{v_3} (k_{\text{max}} - k_{\text{min}}).\]

(12)
where $k_{\text{max}}$ and $k_{\text{min}}$ are respectively defined as the inverse of the smallest and largest length scales in the problem. In the context of the landfill model, $k_{\text{min}}$ is the inverse of the landfill diameter while $k_{\text{max}}$ is the inverse of the landfill height. So the difference is a small finite number.

In the above and all future formulations, we will consider same noise strengths, that is $D_1 = D_2 = D_3 = D_4 = D_0$ without any loss of generality. Using $\nu_1 = 1.0$, $\nu_2 = 0.8$, $k_h = 1.0$, $D_1 = 0.1$, $D_2 = 0.05$, $k_h' \to 0$ as the parameter values, the solutions of eqn(6b) and eqn(10) when plotted gives the following time decay profiles. It is to be noted that an inherent part of this conclusion relies on the fact that $\nu_1 \neq \nu_2$; in other words, on a heterogeneous spread through diffusional relaxation.

The appended figures compare the root-mean-squared profiles of all four variables after averaging over all stochastic realisations. More specifically, we plot $n_{i \text{rms}}(\tau) = \sqrt{\langle n_i(x,t) \ast n_i(x,t+\tau) \rangle}$ for $i=1,2,3,4$ against the time difference $\tau$. As can be seen, this are Gaussian stationary processes [19] implying that in the large time equilibrium limit ($t \to \infty$), the respective autocorrelation functions depend only on the time difference between two specific points of

![Figure 1](image-url)

**Figure 1:** The solid line represents the decay profile of the root mean square solid mass density autocorrelation function (represented by eqn(6b)) with time.

Fig 1 and Fig 2 show the time decay of solid and hydrolysed waste that are used to generate the acetogenic and methanogenic phases shown in Fig 3 and Fig 4. As is to be expected, the solid waste decay rate is steeper than the hydrolysed phase indicating that follow-up (methane) production necessitates a slow build-up leading to the target deliverables.
Figure 2: The solid line represents the decay profile of the root mean square hydrolysed density autocorrelation function (represented by eqn(10)) with time.
Figure 3: The solid line represents the decay profile of the root mean square solid acetogenic density autocorrelation function (represented by eqn(12)) with time.
Figure 4: The solid line represents the decay profile of the root mean square solid methanogenic density autocorrelation function (represented by eqn(11)) with time.
The comparison of all four plots shown in Fig 5 encapsulates the summary of this theoretical model. While the decay rates of the individual phases vary, the final decay lines for all phases converge to the golden number of 20 years for all estimated phases. This number is subject to the parameter values used (indicated above). We have tested for other realistic parameter values availed from other publications [6] to confirm that the decay time line always conforms to the time window of 15-30 years which is in line with this reference. This result is a huge improvement on benchmarked deterministic models [13, 14].

![Graphs showing time variations of mass densities of all four variables.](image)

**Figure 5:** Comparison of time variations of the mass densities of all four variables.

4. Conclusions

To summarise, our stochastic linearly stable model is primarily affected by the time dependent hydrolysable decay rate \( R(t) \) at finite time scales \( t < 10 \) years. For larger times, the steady state statistics remains unchanged with respect to changes in \( R(t) \). As indicated above, the steady methanogenic gas production rate is primarily determined by the production rate of hydrolysed mass starting from solid mass with the acetogenic density contributing the least in the process. Unlike the previous models [13, 14], our model satisfies the mass-balance relation at an ensemble averaged level and not for all deterministic realizations. This ensures dynamical equilibrium for all finite times: \( \frac{\delta}{\delta t}(n_1 + n_2 + n_3 + n_4) \geq 0 \). Another important conclusion that we draw is also exemplified by the fact that the solid mass decay rate is much faster than the other three decay processes. In arriving at the plots, although specified fixed values of the noise strengths were used, but the linear stability ensured that the qualitative deductions obtained from
Inverting the decay constants estimated in this analysis, we reassuringly arrive at the production time line limitations collaboration with Aston University, UK and Jadavpur University, India.

References

The implication of this theoretical analysis goes beyond the estimation of accurate gas production decay times and favorable comparisons with experimental data. The major benefit of such a study will be in the development of a robust business model in which quantitative dependence of such decay rates with varying system parameters are all part of the analytical description now. As is not so very difficult to conceive, landfill site structures and engineering depend on the ambience and country specific facilities that may imply wide variation in parameter values. The results presented here incorporate all such provisions, including fluctuating parameter values. A pragmatic underpinning with regard to landfill engineering will be precise quantitative control of parameters and clear ideas about the right parametric regime that will ensure gas production at a specified rate. As these rates may vary between different sites, as also on the country concerned, such numerical control would ensure easier and more direct improvement of existing landfill engineering frameworks. The work will also help municipalities and city councils to make right decision for landfill gas mining and implementing a sustainable landfill gas extraction as well energy recovery project.

A reassuring quantitative confirmation of the analysis presented here as well as its superiority over other similar attempts comes from a comparison with real landfill descriptions as presented independently by Gioannis, et al [6]. Inverting the decay constants estimated in this analysis, we reassuringly arrive at the production time line limitations as being between 12.5-33 years that is perfectly in harmony with numbers presented in this article (e.g. Fig 5). This quantitative ramification also stipulates the need for extending the present linearly stable model in to the more realistic nonlinear regime together with all stochastic fluctuations. This is the next plane along with a multivariate analysis of the outputs from the stochastic model in a Fokker-Planck structure.

5. Acknowledgment

The authors acknowledge support of British Council funding under UKIERI thematic partnership 2012-14 in collaboration with Aston University, UK and Jadavpur University, India.

References

[1] Scheutz, C., Bogner, J., De Visscher, A., Gebert, J., Hilger, H., Huber-Humer, M., Kjeldsen, P. and Spokas K., 2009. Microbial Methane Oxidation Processes and Technologies for Mitigation of Landfill gas Emissions. Waste Management 30(5), 409-455.

[2] Bariaz, M. A., Chanton, J. P. and Green, R. B., 2012. Controls on Landfill Gas Collection Efficiency: Instantaneous and Lifetime Performance. Journal of the Air and Waste Management Association 59(12), 1399-1404.

[3] Spokas, K., Bogner, J., Chanton, J. P., Moret, M., Aran, C., Graff, C., Moreau-Le Golvan, Y. and Hebe, L., 2006. Methane mass balance at three landfill sites: What is the efficiency of capture by gas collection systems? Waste Management 26(5), 519-525.

[4] Humer, M. P. and Lechner, F. P., 2008. Alternative approach to the elimination of greenhouse gases from old landfills. Waste Management and Research 17(6), 443-452.

[5] Perera, L. A. K., Acharai, G. and Hettiaratchi, J. P. A., 2002. Determination of Source Strength of Landfill Gas: A Numerical Modeling Approach. Journal of Environmental Engineering 128, 461-471.

[6] De Gioannis, G., Muntoni, A., Cappai, G. and Milia, S., 2008. Landfill gas generation after mechanical biological treatment of municipal solid waste. Estimation of gas generation rate constants. Waste Management 29(3), 1026-1034.

[7] Allen, M. R., Brathwaite, A. and Hills, C. C., 1997. Trace organic compounds in landfill gas at seven U.K. waste disposal sites. Environmental Science and Technology 31(4), 1054-1061.

[8] Cvetkovic, V., Dagan, G., 1994. Transport of kinetically sorbing solute by steady random velocity in heterogeneous porous formations. Journal of Fluid Mechanics 48(1), 193-201.

[9] El-Fadel, M., Findikakis, A. N., Leckie, J. O., 1997. Modelling leachate generation and transport in solid waste landfills. Environmental Technology 18, 669-686.
[10] El-Fadel, M., Findikakis, A. N., Leckie, J. O., 1997. Gas simulation models for solid waste landfills. Critical Reviews on Environmental Science and Technology 27(3), 237-283.
[11] E-Fadel, M., Khoury, R., 2000. Modelling settlement in MSW landfills: a critical review. Critical Reviews in Environmental Science and Technology 30(3), 327-361.
[12] Eastman, J. A., Fergusson, J. F., 1981. Solubilization of particulate organic carbon during the acid phase of anaerobic digestion. Journal of the Water Pollution Control Federation 53(3), 352-366.
[13] Zacharof, A. I., Butler, A. P., 2004. Stochastic modelling of landfill processes incorporating waste heterogeneity and data uncertainty. Waste Management 24, 241-250.
[14] Young, A., 1989. Mathematical modelling of landfill gas extraction. Journal of Environmental Engineering 115(6), 1073-1087.
[15] Murray, J. D., 2011. Mathematical Biology I & II: Springer.
[16] Chattopadhyay, A. K., Burroughs, N. J., 2007. Close contact fluctuations: The seeding of signalling domains in the immunological synapse. Europhysics Letters 77, 48003.
[17] Chattopadhyay, A. K., Bhattacharjee, J. K., 2000. Wall-bounded turbulent shear flow: Analytic result for a universal amplitude. Physical Review E 63, 016306.
[18] Chattopadhyay, A. K., Basu, A., Bhattacharjee, J. K., 2000. Coupled non-equilibrium growth equations: Self-consistent mode coupling using vertex renormalization. Physical Review E 61(2), 2086-2088.
[19] Bush, D. J., Chattopadhyay, A. K., 2014. Contact time periods in immunological synapse. Accepted in Physical Review E.