The Quantum $G_2$ Link Invariant

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Abstract

We derive an inductive, combinatorial definition of a polynomial-valued regular isotopy invariant of links and tangled graphs. We show that the invariant equals the Reshetikhin-Turaev invariant corresponding to the exceptional simple Lie algebra $G_2$. It is therefore related to $G_2$ in the same way that the HOMFLY polynomial is related to $A_n$ and the Kauffman polynomial is related to $B_n$, $C_n$, and $D_n$. We give parallel constructions for the other rank 2 Lie algebras and present some combinatorial conjectures motivated by the new inductive definitions.

This paper is divided into two parts. In the first part we derive from first principles some variants of the link invariant known as the Jones polynomial. In the second part we show that these invariants are the same known invariants constructed using rank 2 Lie algebras, and we discuss some of their properties.

1 Invariants of links and graphs

The simplest known definition of the Jones polynomial is the Kauffman bracket \cite{Kauffman}, which in this paper will be denoted by $\langle \cdot \rangle_{A_1}$ and will be called the $A_1$ bracket. The $A_1$ bracket is given by the following recursive rules:

\[
\begin{align*}
\langle \bigcirc \rangle_{A_1} &= -(q^{1/2} + q^{-1/2}) \langle \bigcirc \rangle_{A_1} \\
\langle \bigotimes \rangle_{A_1} &= -q^{1/4} \langle \bigotimes \rangle_{A_1} - q^{-1/4} \langle \bigotimes \rangle_{A_1}
\end{align*}
\]

The goal of this part of the paper is to derive definitions of the following three variants of the Jones polynomial:

**Theorem 1.1.** There is an invariant for regular isotopy of projections of links and tangled trivalent graphs called $\langle \cdot \rangle_{G_2}$ which is given by the following recursive rules:

\[
\begin{align*}
\langle \bigcirc \rangle_{G_2} &= a \langle \bigcirc \rangle_{G_2} \\
\langle \bigotimes \rangle_{G_2} &= 0 \\
\langle \bigotimes \rangle_{G_2} &= b \langle \bigotimes \rangle_{G_2} \\
\langle \bigotimes \rangle_{G_2} &= c \langle \bigotimes \rangle_{G_2} \\
\langle \bigotimes \rangle_{G_2} &= d_1 (\langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2}) + d_2 (\langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2}) \\
\langle \bigotimes \rangle_{G_2} &= e_1 (\langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2}) + e_2 (\langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2}) \\
\langle \bigotimes \rangle_{G_2} &= f_1 (\langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2}) + f_2 (\langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2}) \\
\langle \bigotimes \rangle_{G_2} &= g_1 (\langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2}) + g_2 (\langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2} + \langle \bigotimes \rangle_{G_2}) \\
\end{align*}
\]

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where the coefficients are defined as:

\[
\begin{align*}
a &= q^5 + q^4 + q + 1 + q^{-1} + q^{-4} + q^{-5} \\
b &= -q^3 - q^2 - q - q^{-1} - q^{-2} - q^{-3} \\
c &= q^2 + 1 + q^{-2} \\
d_1 &= -q - q^{-1} \\
d_2 &= q + 1 + q^{-1} \\
e_1 &= 1 \\
e_2 &= -1 \\
f_1 &= \frac{1}{1 + q^{-1}} \\
g_1 &= 1 + q \\
f_2 &= \frac{q}{1 + q^{-1}} \\
g_2 &= \frac{q^{-1}}{1 + q}
\end{align*}
\]

$q$ is an indeterminate, and $\langle \emptyset \rangle_{A_2} = 1$.

**Theorem 1.2.** There is an invariant for regular isotopy of projections of links and tangled trivalent graphs called $\langle \cdot \rangle_{A_2}$ which is given by the following recursive rules:

\[
\begin{align*}
| \begin{array}{c}
(\bigcirc) \\
1
\end{array} \rangle_{A_2} &= (q+1+q^{-1}) | \begin{array}{c}
1
\end{array} \rangle_{A_2} \\
| \begin{array}{c}
(\bigotimes) \\
1
\end{array} \rangle_{A_2} &= (q^{1/2}-q^{-1/2}) | \begin{array}{c}
1
\end{array} \rangle_{A_2} \\
| \begin{array}{c}
(\bigstar) \\
1
\end{array} \rangle_{A_2} &= | \begin{array}{c}
1
\end{array} \rangle_{A_2} + | \begin{array}{c}
1
\end{array} \rangle_{A_2} \\
| \begin{array}{c}
(\bigtriangledown) \\
1
\end{array} \rangle_{A_2} &= -q^{1/6} | \begin{array}{c}
q^{1/3}
\end{array} \rangle_{A_2} + q^{-1/3} | \begin{array}{c}
q^{1/3}
\end{array} \rangle_{A_2} \\
| \begin{array}{c}
(\bigtriangledown) \\
1
\end{array} \rangle_{A_2} &= -q^{-1/6} | \begin{array}{c}
q^{1/3}
\end{array} \rangle_{A_2} + q^{1/3} | \begin{array}{c}
q^{1/3}
\end{array} \rangle_{A_2}
\end{align*}
\]

where $q$ is an indeterminate and $\langle \emptyset \rangle_{A_2} = 1$.

**Theorem 1.3.** There is an invariant for regular isotopy of projections of links and tangled trivalent graphs
called \( \langle \cdot \rangle_{c_2} \) which is given by the following recursive rules:

\[
\begin{align*}
| \bigcirc \rangle_{c_2} &= -(q^2+q+q^{-1}+q^{-2})| \langle c_2 \\
| \bigotimes \rangle_{c_2} &= (q^3+q+q^{-1}+q^{-3})| \langle c_2 \\
| \Rightarrow \rangle_{c_2} &= 0 \\
| \Rightarrow\Rightarrow \rangle_{c_2} &= -(q+q^{-1})| \Rightarrow \rangle_{c_2} \\
| \Rightarrow \rangle_{c_2} &= 0 \\
| \bigotimes \rangle_{c_2} - | \bigotimes \rangle_{c_2} &= | \bigotimes \rangle_{c_2} - | \bigotimes \rangle_{c_2} \\
| \bigotimes \rangle_{c_2} &= 0 \\
| \bigotimes \rangle_{c_2} - | \bigotimes \rangle_{c_2} &= | \bigotimes \rangle_{c_2} - | \bigotimes \rangle_{c_2} \\
\bigotimes \rangle_{c_2} &= | \bigotimes \rangle_{c_2} - | \bigotimes \rangle_{c_2}
\end{align*}
\]

where \( q \) is an indeterminate and \( \langle \emptyset \rangle_{c_2} = 1 \). Part of the invariant can also be defined by the recursive rules:

\[
\begin{align*}
| \bigcirc \rangle_{c_2} &= -(q^2+q+q^{-1}+q^{-2})| \langle c_2 \\
| \bigotimes \rangle_{c_2} &= (q^3+q+q^{-1}+q^{-3})| \langle c_2 \\
| \Rightarrow \rangle_{c_2} &= 0 \\
| \Rightarrow\Rightarrow \rangle_{c_2} &= -(q+q^{-1})| \Rightarrow \rangle_{c_2} \\
| \Rightarrow \rangle_{c_2} &= 0 \\
| \bigotimes \rangle_{c_2} &= 0 \\
| \bigotimes \rangle_{c_2} - | \bigotimes \rangle_{c_2} &= | \bigotimes \rangle_{c_2} - | \bigotimes \rangle_{c_2} \\
| \bigotimes \rangle_{c_2} &= 0 \\
| \bigotimes \rangle_{c_2} - | \bigotimes \rangle_{c_2} &= | \bigotimes \rangle_{c_2} - | \bigotimes \rangle_{c_2} \\
\bigotimes \rangle_{c_2} &= | \bigotimes \rangle_{c_2} - | \bigotimes \rangle_{c_2}
\end{align*}
\]

which are related to the first set of rules by:

\[
| \bigotimes \rangle_{c_2} = | \bigotimes \rangle_{c_2} - | \bigotimes \rangle_{c_2} \langle c_2
\]

These invariants are not new. In fact, all three invariants are special cases of the Reshetikhin-Turaev invariants \[11\]. (It should be emphasized that these invariants are part of a wide literature of closely related ideas. Two other influential papers are \[2\] and \[16\].) According to Reshetikhin and Turaev, for each simple Lie algebra \( g \), there exists an invariant \( RT_g \) of appropriately colored tangled ribbon graphs. Each edge is colored by an irreducible representation of \( g \) and each vertex should be colored by a tensor of a certain kind. The invariants we define all satisfy \( \langle G \rangle_g = RT_g(G) \) if the graph is colored in a certain simple way. (By our notation, the Kauffman bracket also satisfies this equation.) This connection will be explained in detail in the second part of the paper.

In addition, the \( A_2 \) bracket is essentially a specialization of the HOMFLY polynomial \[3\], because the HOMFLY polynomial describes \( RT_{a_2} \); and the \( C_2 \) bracket is essentially a specialization of the Kauffman polynomial, because the Kauffman polynomial \[8\]
describes \( RT_{b_2}, RT_{c_2}, \) and \( RT_{d_2} \). By the relation \( B_2 = C_2 \), the \( C_2 \) bracket encompasses a second specialization of the Kauffman polynomial. (The Kauffman polynomial should not be confused with the
Kauffman bracket.) Thus, the most interesting case is the $G_2$ bracket. The Reshetikhin-Turaev definition in the $G_2$ case is also cast in an explicit form in \cite{10}.

The author recently learned that many of the results presented here, in particular the definition of the $A_2$ and $G_2$ brackets, were obtained independently by Francois Jaeger \cite{5,7,6}.

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2.1 Topological preliminaries

By a planar graph we mean a finite combinatorial graph which is embedded in the sphere $S^2$. We allow planar graphs to have edges which go in a circle and have no vertices, as well as a multiple edges between vertices and vertices connected to themselves. A face of a planar graph is a connected component of the complement of the graph. When interpreting the illustrations of planar graphs in this paper, the reader should identify the border of the page to a point.

Let $D$ be a disk in $S^2$. We define a planar graph with boundary to be the intersection of $D$ with a planar graph which is transverse to the boundary of $D$. Thus a planar graph with boundary is a graph which is embedded in a disk such that the edges of the graph meet the boundary of the disk transversely at special vertices of degree 1, called endpoints. The boundary of a planar graph in $D$ is defined to be the boundary of $D$ together with the endpoints and an inward-pointing normal. If $b$ is a boundary, the opposite boundary $-b$ is defined to be the same object with the distinguished normal vector reversed. In this paper we will consider graphs decorated in various ways and it is understood that the endpoints of a graph with boundary should be decorated correspondingly. For example, if the edges of a graph are colored, the endpoints should be assigned the same colors. If the edges are oriented, each endpoint should be assign a distinguished normal vector. The idea is that if two possibly decorated planar graphs have opposite boundaries, their union should be a planar graph without boundary decorated in the same way.

We define a graph projection to be a planar graph with special tetravalent vertices, called crossings, such that one pair of opposite incoming edges is labeled as passing “over” the other pair, like so:

We define a graph projection with boundary similarly.

We define a regular isotopy of a graph projection to be a sequence of operations consisting of ambient isotopy of the 2-sphere and the following combinatorial moves:

1. $\xymatrix{\ar@{-}[r] & \ar@{.}[r] & }$ \quad 2. $\xymatrix{\ar@{-}[r] & \ar@{.}[r] & }$ \quad 3. $\xymatrix{\ar@{-}[r] & \ar@{-}[r] & }$ \quad 4. $\xymatrix{\ar@{-}[r] & \ar@{-}[r] & }$

There are versions of the last two moves for every possible kind of vertex in the graph, although the moves are only illustrated for a trivalent vertex. We define a tangled ribbon graph (introduced in \cite{11}) to be an equivalence class of graphs projections under regular isotopy. Regular isotopy moves for graph projections with boundary are the same, except that an ambient isotopy must leave the boundary of the graph fixed.
We can also consider regular isotopy of graphs decorated in various ways; there is a version of each regular isotopy move for every possible decoration of the arcs involved in the move.

Two important classes of graph projections are those with no vertices, which are called link projections, and those with only trivalent vertices, which we will call freeway projections. (A link projection with boundary is allowed to have univalent vertices at the boundary.) Here is an example of a freeway projection:

A ribbon link (also called a framed link) is an equivalence class of link projections, a freeway is an equivalence class of freeway projections, a ribbon link with boundary (also called a framed tangle) is an equivalence class of ribbon links with boundary, and a freeway with boundary is an equivalence class of freeway projections with boundary. Here is an example of a freeway projection with boundary:

A tangled ribbon graph has a natural 3-dimensional interpretation as (the equivalence class under ambient isotopy of) a graph embedded in $S^3$ together with a (continuous) distinguished normal vector at each point. Alternatively we can think of a tangled ribbon graph as a graph embedded in $S^3$ together with a ribbon (a long, thin rectangle) assigned to each edge and a disk assigned to each vertex with the incident ribbons attached at the boundary in such a way that an oriented surface results. A graph projection is then the projection of the graph in the usual sense, where the normal vector everywhere points upward out of the plane of the projection, or the surface consisting of the ribbons and disks lies flat against the plane of the projection. A tangled ribbon graph with boundary can be thought of as a graph, with its ribbons or normal vectors, embedded in a ball in $S^3$ and with its endpoints lying at the boundary of the ball.

We define a freeway invariant to be a function defined on freeway projections which depends only on the underlying freeway.

2.2 Linearly recurrent invariants

The topological invariants presented in this paper are best understood as linearly recurrent invariants.

Let $I(G)$ be some $k$-valued invariant of some tangled graphs $G$ which are possibly decorated in some way. There is a natural way to extend the invariant $I$ to an invariant $\tilde{I}(B)$ for graph with boundary $B$ which takes values in a certain vector space over $k$ which depends only on $\partial B$. If $b$ is a boundary, then we let $F(b)$ be the vector space of all $k$-valued functions on the set of all links with boundary $b$. Then if $\partial B = b$, we can choose $\tilde{I}(B) \in F(b)$ to simply be the function which takes $B'$ to $I(B \cup B')$. We call the resulting invariant $\tilde{I}$ the vector invariant associated to the scalar invariant $I$.

So far, the notion of $\tilde{I}$ is not very interesting because the vector space $F(b)$ may be very large. However, it may happen that only a small portion of $F(b)$ is needed to understand the original invariant $I$. We define the spanning space $\text{Span}(b)$ to be the span of $\tilde{I}(B)$ for all $B$ such that $\partial B = b$. We say that $I$ is a linearly recurrent invariant, or a recurrent invariant for short, if $\text{Span}(b)$ is finite-dimensional for all $b$. The motivation for the name is that if the dimension of the space $\text{Span}(b)$ is small, then it is often possible to define the original invariant $I$ recursively by describing the invariant $\tilde{I}(b)$ and the vector space it lives in for only a few choices of $B$, a strategy which is similar to the definition of a linearly recurrent sequence of numbers.

Another more suggestive notation for scalar and vector invariants is bra-ket notation, by analogy with quantum mechanics. In this notation the value of the scalar invariant for some link or tangled graph $G$ is $\langle G \rangle$, while the corresponding vector invariant for some link or tangled graph with boundary $B$ is denoted $|B\rangle$.

As a warm-up for the rest of the paper, we review the derivation and definition of the Kauffman bracket, a linearly recurrent invariant of regular isotopy of links. We denote the Kauffman bracket by $\langle \cdot \rangle_A$, and by
the bra-ket convention we denote the corresponding vector invariant by \(| \cdot \rangle_{A_1}\). We assume that the 4-point spanning space is 2-dimensional and that the two crossingless tangles with 4 endpoints are a basis, and we assume that the 0-point spanning space is 1-dimensional. In this case there are necessarily relations:

\[
\begin{align*}
| \circ \rangle_{A_1} &= a | \rangle_{A_1} \\
| \times \rangle_{A_1} &= b | \rangle_{A_1} + c | \leadsto \rangle_{A_1}
\end{align*}
\]

If we assume that the bracket is invariant under the first of the regular isotopy moves, we can compute:

\[
| \bowtie \bowtie \rangle_{A_1} = bc | \leadsto \rangle_{A_1} + (b^2 + c^2) | \rangle_{A_1} + bc | \bowtie \bowtie \rangle_{A_1}
\]

to obtain the consistency equations:

\[
\begin{align*}
bc &= 1 \\
abc + b^2 + c^2 &= 0
\end{align*}
\]

We must also check the second regular isotopy move. However, by a ubiquitous trick in the theory of linearly recurrent invariants, if the ket of a crossing is a linear combination of kets of crossingless tangles (or in other contexts crossingless graphs), the second regular isotopy move is a corollary of the first regular isotopy move (more generally, a corollary of other moves involving vertices). We parameterize the solutions by setting \(b = -q^{1/4}\), which is slightly peculiar but is natural with hindsight. We obtain the recursive rules described in the introduction. The invariant is completely determined except for its value at the empty link, which lies at the base of the recursive rules. The choice of this value is equivalent to the choice of a global normalization factor. The most natural normalization in the context of this paper is to choose \(\langle \emptyset \rangle_{A_1} = 1\).

### 2.3 Constructing a freeway invariant

We assume that there is a scalar freeway invariant \(\langle \cdot \rangle_{G_2}\) whose corresponding vector invariant is \(| \cdot \rangle_{G_2}\), and which satisfies the following condition: For \(n < 6\), the bracket of the acyclic, crossingless freeways with \(n\) endpoints are linearly independent vectors, and that the bracket of a freeway which consists of a face with \(n\) sides is a linear combination of these acyclic freeways. These freeways are precisely the ones listed in the statement of Theorem 1.1. Thus, we assume the existence of coefficients \(a, b, c, d_1, d_2, e_1, \) and \(e_2\) as given in the statement of Theorem 1.1 (but we do not yet assume the formulas given for those coefficients).

We call this assumption the 1,0,1,1,4,10 Ansatz because there are 1,0,1,1,4, and 10 acyclic freeways with 0,1,2,3,4, and 5 endpoints, respectively. Note that by symmetry there can only be two distinct coefficients for the relation for a square and pentagon, for otherwise the bracket of a square or a pentagon would satisfy more than one linear relation, which would violate the linear independence assumption.

**Theorem 2.1.** Every freeway invariant which satisfies the 1,0,1,1,4,10 Ansatz is equal to the invariant defined in Theorem 1.1 for some value of \(q\).

We will proceed by proving Theorem 2.1. Along the way all of the conditions for the existence of the invariant will be met, thereby proving Theorem 1.1.

The following lemma shows that the coefficients \(a, \ldots, e_2\) completely determine \(\langle \cdot \rangle_{G_2}\) for crossingless freeways, provided that the coefficients are chosen so that the bracket exists, up to a choice of global normalization. As in the case of the Kauffman bracket, we choose for the sake of convention \(\langle \emptyset \rangle_{G_2} = 1\).

**Lemma 2.2.** Let \(Z\) be a non-empty crossingless freeway. The graph \(Z\) has at least one simply-connected face with five or fewer sides which does not share an edge with itself.

**Proof.** If we allot each face one third of each of its vertices and one half of each of its edges, then the Euler characteristic of a simply-connected face with \(n\) sides is \(1 - n/6\) and the Euler characteristic of a multiply-connected face is negative or zero. If no face shares an edge with itself, then since the Euler characteristic of the sphere is 2, we must have at least one face with positive Euler characteristic.
Suppose instead that at least one face does share an edge with itself. Then we may draw a circle which is inside this face except at one point:

If we draw a circle for each such face, then at least one of the circles must be innermost, i.e. it must bound a region containing no such faces. The total Euler characteristic of the faces in this region is at least $5/6$, so once again there must be a face in the region with the desired property.

A variation of the proof of this lemma yields a self-justification for the $1,0,1,1,4,10$ Ansatz: If a crossingless freeway has $n$ endpoints, then either the freeway is acyclic or the total Euler characteristic of its faces is at least $1 - n/6$. If $n < 6$, then any freeway with $n$ endpoints is a linear combination of acyclic freeways. The Ansatz implies that the acyclic freeways are bases for the first six spanning spaces of the bracket.

We now derive the values for the coefficients $a$, $b$, $c$, $d_i$, and $e_i$ which are consistent for the values of the bracket on all crossingless freeways. Let $P$ be a minimal crossingless freeway projection for which the recursive definition contradicts itself. Suppose that the projection $P$ has two faces $A$ and $B$ with five or fewer sides which are not adjacent. Then reducing the face $A$ must yield the same result as reducing the face $B$, because after we reduce $A$ we are still free to reduce $B$, and vice versa. Thus, the only possibility of a contradiction arises when $P$ has two adjacent faces with at most five sides.

Suppose, for example, that $P$ has a triangle next to a pentagon; it suffices to consider the case when $P$ is a triangle next to a pentagon and nothing more, like so:

We can in this case reduce either the pentagon or the triangle:

$$|\phantom{\text{G}}\rangle_{G_2} = c|\text{G}\rangle_{G_3} = cd_1(|\text{G}\rangle_{G_2} + |\text{G}\rangle_{G_2}) + cd_2(|\text{G}\rangle_{G_2} + |\text{G}\rangle_{G_2})$$

$$= e_1(|\text{G}\rangle_{G_2} + |\text{G}\rangle_{G_2} + |\text{G}\rangle_{G_2} + |\text{G}\rangle_{G_2} + |\text{G}\rangle_{G_2}) + e_2(|\text{G}\rangle_{G_2} + |\text{G}\rangle_{G_2} + |\text{G}\rangle_{G_2} + |\text{G}\rangle_{G_2} + |\text{G}\rangle_{G_2})$$

By the linear independence assumptions, we obtain the scalar equations:

$$cd_1 = e_1d_1 + e_1c + e_1b + e_2$$
$$cd_2 = e_1d_2 + e_2b$$

We can perform this procedure for all cases of a face with $n$ sides adjacent to a face with $m$ sides except $n = m = 5$. We obtain the equations:
\[ b^2 = bd_1 + d_2 + ad_2 \]
\[ c^2 = bcl + cd_1 + d_2 \]
\[ bc = 2e_1b + 2e_2 + ae_2 + e_1c \]
\[ cd_1 = d_1e_1 + ce_1 + be_1 + e_2 \]
\[ cd_2 = d_2e_1 + be_2 \]

\[ d_1e_1 + d_2 = e_1^2 \]

\[ d_1e_1 + d_1^2 = e_1^2 + ce_1 + d_1e_1 \]

\[ d_1e_1 = e_1^2 + d_1e_1 + e_2 \]

\[ d_1e_2 + d_2c = e_1e_2 + be_2 + 2d_2e_1 \]

\[ d_1e_2 = e_1e_2 + d_2e_1 \]

\[ d_1e_2 + d_2d_2 = e_1e_2 + e_2c \]

Since we have not made any linear independence assumptions for freeways with six endpoints, we do not obtain any equations which the coefficients must satisfy a priori in the case of two neighboring pentagons. Nevertheless, we must check the equations anyway, since the relevant freeways might be linearly independent. Here are the equations which result:

\[ e_2d_1 = e_1d_2 + e_1e_2 \]

\[ e_2 = e_1^2 \]

The only remaining possibility is that there are two faces with five or fewer sides which have two or more edges as their common border. This case is inconsequential and the reason is left as an exercise to the reader.

Undeterred by the sprawl of these equations and by the fact that there are more constraints than unknowns, we can proceed by reducing them to a sequence of relatively simple relations. First, we note that one solution can be varied in an uninteresting way to obtain another: We can multiply \( b, c, d_1, \) and \( e_1 \) by a constant and \( d_2 \) and \( e_2 \) by the square of that constant. To eliminate this degree of freedom, we normalize by declaring that \( e_1 = 1 \). (The case \( e_1 = 0 \) produces only trivial solutions.) Then we can eliminate variables by using some of the equations to simplify the algebra: We find that \( e_2 = -1 \) by equation (5), then express \( d_2 \) in terms of \( d_1 \) by equation (5), then \( c \) in terms of \( d_1 \) by equation (5), then we find \( b \) by equation (5), and finally \( a \) by equation (5). In the process we see that the other equations are satisfied automatically, and we are left with a second free parameter. We make the useful reparameterization \( d_1 = -q - q^{-1} \), with \( q \) an indeterminate, to obtain the solutions given in the statement of Theorem 1.1.

To extend the invariant to links and to freeways with crossings, we further assume that a crossing is a linear combination of the four acyclic, crossingless freeways with four endpoints. To find the four coefficients \( f_1, f_2, g_1, \) and \( g_2 \) for this linear dependence, we must investigate the behavior of the bracket under the regular isotopy moves. We first reduce the “before” and “after” pictures of the first regular move:

\[
\begin{align*}
&| \langle \phantom{G_2} \rangle G_2 = f_1 g_1 | \langle \phantom{G_2} \rangle G_2 + f_1^2 | \langle \phantom{G_2} \rangle G_2 + f_1 g_2 | \langle \phantom{G_2} \rangle G_2 + f_1 f_2 | \langle \phantom{G_2} \rangle G_2 \\
&+ g_1^2 | \langle \phantom{G_2} \rangle G_2 + g_1 f_1 | \langle \phantom{G_2} \rangle G_2 + g_1 g_2 | \langle \phantom{G_2} \rangle G_2 + g_1 f_2 | \langle \phantom{G_2} \rangle G_2 \\
&+ f_2 g_1 | \langle \phantom{G_2} \rangle G_2 + f_2 f_1 | \langle \phantom{G_2} \rangle G_2 + f_2 g_2 | \langle \phantom{G_2} \rangle G_2 + f_2 f_2 | \langle \phantom{G_2} \rangle G_2 \\
&+ g_2 g_1 | \langle \phantom{G_2} \rangle G_2 + g_2 f_1 | \langle \phantom{G_2} \rangle G_2 + g_2 g_2 | \langle \phantom{G_2} \rangle G_2 + g_2 f_2 | \langle \phantom{G_2} \rangle G_2 | \circ \langle \phantom{G_2} \rangle G_2
\end{align*}
\]

This yields the equations:

\[
\begin{align*}
f_1 g_1 d_1 + f_1^2 c + g_1^2 c + f_1 g_1 b + g_1 g_2 + f_1 f_2 &= 0 \\
f_1 g_1 d_1 + f_1 g_2 + f_2 g_1 &= 0 \\
f_1 g_1 d_2 + f_1 f_2 b + f_2^2 + g_1 g_2 b + g_2^2 + f_2 g_2 a &= 0 \\
f_1 g_1 d_2 + f_2 g_2 &= 1
\end{align*}
\]
the third and fourth moves result in the equations:

\begin{align*}
  f_1^2 d_1 + f_1 g_1 c + g_1 f_1 e_1 g_1^2 d_1 & = 0 \\
  g_1 f_1 e_1 + g_1 f_2 & = f_1 \\
  g_1 f_1 e_1 + g_2 f_1 & = g_1 \\
  g_1 f_1 e_1 + g_1^2 d_1 + g_2 g_1 & = 0 \\
  f_1^2 d_1 + f_1 f_2 + g_1 f_1 e_1 & = 0 \\
  g_1 f_1 e_1 + g_2 f_2 & = 0 \\
  f_1^2 d_2 + f_1 g_2 b + g_1 f_1 e_2 + g_1 g_2 c + g_2^2 & = 0 \\
  g_1 f_1 e_2 + g_1^2 d_2 + f_2 g_1 b + f_2 f_1 c + f_2^2 & = 0 \\
  f_1^2 d_2 + g_1 f_1 e_2 & = f_2 \\
  g_1 f_1 e_2 + g_1^2 d_2 & = g_2
\end{align*}

(7)

Comparing equations (6) and (7), we obtain:

\begin{align*}
  f_1 g_1 & = f_2 g_2 \\
  1 + d_2 & = 1 + q + 2 + q^{-1}
\end{align*}

Multiplying equations (8) and (9) and substituting the formula (2.3), we obtain:

\begin{align*}
  f_1^2 + g_1^2 = \frac{q + q^{-1}}{q + 2 + q^{-1}}
\end{align*}

Since we now know the sum and the product of $f_1^2$ and $g_1^2$, we obtain a quadratic equation. The result is the solutions given in the statement of Theorem 1.1.

It remains to check the second regular isotopy move. As in the case of two adjacent pentagons, we do not obtain any equations which the coefficients must satisfy a priori, but we must check the move anyway. In this case the ubiquitous trick which was left as an exercise to the reader in the case of the Kauffman bracket appears again. If we have invariance under the other moves, and if a crossing is a linear combination of crossingless freeways, then the second regular isotopy move is automatically satisfied.

In solving the quadratic equation we must break a Galois symmetry which maps the parameter $q$ to $q^{-1}$.

A rationale for making the choice presented here, as well as for choosing the parameter $q$ in the first place, will be presented in the second part of the paper.

## 2.4 Two other polynomial invariants

We can repeat the approach of Section 2.3 two more times by changing the notion of a freeway and picking a new Ansatz. To avoid confusion, let us define a $G_2$ freeway to be what we previously called a freeway.

The solutions will be polynomial invariants of a single variable. However, for the purpose of a uniform treatment of all of the invariants, it is desirable in general to make this variable a certain root of a variable $q$ which will be reused for each of the invariants. Therefore the final answers will be elements of the field $\mathbb{C}(q, q^{1/2}, q^{1/3}, q^{1/4}, \ldots)$. For every rational number $a$ we distinguish a power $q^a$ of the indeterminate $q$ in such a way that for every integer $n$, $q^{an} = (q^a)^n$.

We define an $A_2$ freeway to be an oriented planar graph with crossings and junctions that look like this:

We assume the existence of a regular isotopy invariant $\langle \cdot \rangle_{A_2}$ which satisfies the 1,0,1,1,2 Ansatz: The acyclic, crossingless $A_2$ freeways with 0,1,2,3, or 4 endpoints are assumed to be bases for their respective spanning spaces. This Ansatz, although simpler, comes with some technicalities. In the 2-endpoint case, one endpoint must be oriented inward and the other outward. In the 4-endpoint case, two endpoints must be oriented
inward and two outward, and we get two different bases which differ in the ordering of the endpoints in the projection:

\[ \begin{array}{c}
\text{or }
\end{array} \]

The two possible boundaries are equivalent, but not canonically, so there is no canonical formula for a change of basis. The endpoints of a 3-endpoint freeway must all be oriented inward or all outward, but in fact the 3-endpoint case of the Ansatz is not needed in the derivation of the invariant. The 1-endpoint case is even less relevant to the derivation because there are no \( A_2 \) freeways whatsoever with only 1 endpoint.

By a derivation similar to that of Section 2.3, the only possibilities for \( \langle \cdot \rangle_{A_2} \) are the ones described by Theorem 1.2. As before, the rules given in the theorem suffice as a recursive definition because any trivalent planar graph must have a face with at most 5 sides, except the situation is simpler because all faces of an \( A_2 \) freeway must have an even number of sides. The invariant \( \langle \cdot \rangle_{A_2} \) is identically equal to \( RT_{A_2} \) if all edges are colored with the 3-dimensional representation \( V_{1,0} \). The edges are oriented because the representation is not self-dual. The invariant is a special case of the HOMFLY polynomial with a normalization that makes it a regular isotopy invariant rather than an isotopy invariant:

\[
q^{1/6} \langle \rangle_{A_2} - q^{-1/6} \langle \rangle_{A_2} = (q^{1/2} - q^{-1/2}) \langle \rangle_{A_2}
\]

We also define a \( C_2 \) freeway to be a planar graph with two kinds of edges, illustrated as double and single edges, with junctions and crossings that look like this:

\[
\begin{array}{c}
\Rightarrow \quad \times \quad \times \quad \times \quad \times
\end{array}
\]

The Ansatz for \( \langle \cdot \rangle_{C_2} \) has a somewhat different form. We first assume the existence of a relation which we call a switching relation:

\[
\langle \rangle_{C_2} - \langle \rangle_{C_2} = k \langle \rangle_{C_2} - \langle \rangle_{C_2}
\]

Associated to the switching relation is a switching move:

\[
\Rightarrow \quad \Rightarrow
\]

We define a switching class of \( C_2 \) freeways to be an equivalence class under switching moves. We say that a double edge of a \( C_2 \) freeway is external if at least one of its vertices is an endpoint of the freeway, otherwise the edge is internal. Then the Ansatz says that when \( s + 2d < 8 \), a collection of acyclic, extended \( C_2 \) freeways with \( s \) single endpoints and \( d \) double endpoints and no internal double edges are a basis if the collection has one representative from each switching class. The caveat about ordering of endpoints applies here just as in the \( A_2 \) case.

By a third iteration of Section 2.3, the only possibilities for \( \langle \cdot \rangle_{C_2} \) are those of Theorem 1.3. To show that these rules suffice as an inductive definition, we define the angles of a (trivalent) junction to be \( 3\pi/4, 3\pi/4, \) and \( \pi/2 \), like so:

\[
\begin{array}{c}
135^\circ \quad 135^\circ
\end{array}
\]

We define the curvature of a face to be \( 2\pi \) minus the sum of the exterior angles. By an Euler characteristic argument, every crossingless \( C_2 \) freeway without boundary must have a face with positive curvature, and such a face can be reduced by the switching move to one for which one of the other moves applies.
Another approach to the $C_2$ bracket is to define an extended $C_2$ freeway to be one which can also have tetravalent vertices with no incident double edges. A tetravalent freeway is one with no double edges whatsoever. We assume the existence of a relation:

$$\langle \times \rangle_{C_2} = \langle \times \rangle_{C_2} - \langle \times \rangle_{C_2}$$

where the constants $\alpha$ and $\beta$ are arbitrary except that $\alpha$ must be non-zero to avoid degeneracy. If we interpret the four angles at a tetravalent vertex to be 90 degrees, we can derive recurrence rules for tetravalent freeways by reducing faces with positive curvature, namely faces with at most 3 sides. A face with $n$ sides now has $2^n$ endpoints, and for $n = 0, 1, 2, 3$ the number of acyclic tetravalent graphs with $2n$ endpoints is 1, 1, 3, and 14. But by the Ansatz for the $C_2$ bracket, these acyclic graphs must be a basis provided that $\alpha \neq 0$. Thus we obtain an example of a linearly recurrent theory for tetravalent graphs; double edges and trivalent vertices are avoided. (Note that a tetravalent vertex should not be confused with a crossing.) Moreover, some of the choices of the constants $\alpha$ and $\beta$ result in a simpler recurrent calculus than others. A particularly fortuitous choice is $\alpha = 1/2$ and $\beta = 1/2$, whose first advantage is a simpler relationship between the trivalent and tetravalent theories:

$$\langle \times \times \rangle_{C_2} = (q^{-1/2} - q^{1/2})(\langle \times \rangle_{C_2} - \langle \times \times \rangle_{C_2})$$

Its second advantage is that in the recurrence relations, the coefficients of 1 of the acyclic graphs with 4 endpoints and 10 of the acyclic graphs with 6 endpoints vanish. The resulting recurrence relations are those given in the second half of the statement of Theorem 1.3.

**Question 2.3.** Currently the tetravalent $C_2$ bracket, for various values of $\alpha$ and $\beta$, merely represents examples of linearly recurrent invariants that satisfy the 1,1, 3,14 Ansatz. What are all the invariants of tangled tetravalent graphs that satisfy this Ansatz?

The $C_2$ bracket constitutes two special cases of the Kauffman polynomial, one for plain edges and one for double edges. The two relations it satisfies, both in “Dubrovnik” form, are:

(Figure 27)

2.5 A few calculations

In this section we present the results of a few calculations with $\langle \cdot \rangle_{G_2}$. We begin with the auxiliary recursive rules:

(Figure 28)

Note that these rules can be inferred from [10, p. 61], and that they allow one to compute the bracket for any knot with 7 or fewer crossings. The simplest link which seems to defeat these rules is the Borromean rings, with 6 crossings and 3 components. Unfortunately, the Borromean rings have also defeated the author’s attempts at calculations by hand. However, some other possibly interesting values of the bracket and relations of the knot follow:

(Figure 29)

Note that all of the links and graphs are evaluated with the projection framing.

3 Invariants of Lie algebra representations

In this part of the paper we place the invariants $\langle \cdot \rangle_\mathfrak{g}$ in the proper context of the invariants $RT_\mathfrak{g}$ defined by Reshetikhin and Turaev in [11] and study their properties from the point of view of representation theory of Lie algebras.
3.1 Invariants of colored ribbon graphs

We begin with a summary of Reshetikhin and Turaev’s definition and some basic properties.

We review some of the theory of finite-dimensional left modules, or representations, of a Hopf algebra $H$. If we have two representations $V$ and $W$, the tensor product $V \otimes W$ is also a representation by the coalgebra structure of $H$. There is a trivial representation $I$ such that $I \otimes V \cong V \otimes I \cong V$ canonically. However, there is no canonical isomorphism between $V \otimes W$ and $W \otimes V$ as representation of $H$, and indeed the two modules need not be isomorphic at all. In particular, the switching map, which takes $x \otimes y$ to $y \otimes x$, is in general not a module homomorphism. On the other hand, we do have $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$, and it is still true that if $A$ and $B$ are morphisms between representations, $A \otimes B$ is also. Also, for each representation $V$ there is a representation $V^*$ and a canonical identification of $\text{Hom}(U \otimes V^*, W)$ and $\text{Hom}(U, W \otimes V)$. There is also a representation $V^\circ$ with a canonical identification of $\text{Hom}(V^\circ \otimes U, W)$ with $\text{Hom}(U, V \otimes W)$. Although there are canonical isomorphisms $V^{\circ *} \cong V^{* \circ} \cong V$, $V^{**}$ need not be isomorphic to $V$. Finally, we define $\text{Inv}(V) = \text{Hom}(I, V)$ by analogy with the case when $H$ is a group algebra.

There is a visual notation for morphisms in the category of $H$-modules and their compositions. If $f$ is a morphism from $V_1 \otimes \ldots \otimes V_n$ to $W_1 \otimes \ldots W_n$, we can represent $f$ by a diagram with one vertex:

(Figure 30)

Similarly for compositions of morphisms. For example, if $f : V_1 \to V_3 \otimes V_4$, $g : V_4 \otimes V_2 \to V_5$, and $h : V_3 \otimes V_5 \to V_6 \otimes V_7$ are morphisms, then we can express $h \circ (I \otimes g) \circ (f \otimes I)$ by the diagram:

(Figure 31)

Note that the facts stated in the previous paragraph allow us to erase internal edges labeled with the representation $I$; by a slight abuse of notation we may, for example, draw a morphism $f : I \to V \otimes W$ as:

(Figure 32)

In general these diagrams must be planar graphs in which none of the edges pass through a slope of zero. However, it is sometimes possible to augment the structure of a Hopf algebra to obtain a ribbon Hopf algebra, in which the graphs can have edges that bend arbitrarily and can have edges that cross each other, provided one arm of each crossing is labeled “over” and the other “under”. First, there is a distinguished element $R$ of $H \otimes H$ which we can use to construct, for each $V$ and $W$, an isomorphism $\tilde{R} : V \otimes W \to W \otimes V$ which is denoted by a right-handed crossing:

(Figure 33)

The inverse $\tilde{R}^{-1}$, which need not equal $\tilde{R}$, is denoted by a left-handed crossing:

(Figure 34)

Second, there is a distinguished element $K$ of $H$ which we can use to construct an isomorphism $\tilde{K} : V^{**} \to V$. It follows that there exist canonical morphisms which can be denoted as follows:

(Figure 34)

In light of the existence of these morphisms and the identification of $V^*$ and $V^\circ$, we can orient the edges of the diagrams and declare that an edge labeled with $V^*$ is equivalent to an oppositely oriented edge labeled with $V$. In a ribbon Hopf algebra the elements $R$ and $K$ must satisfy certain axioms which can be translated to various identities for the corresponding morphisms. Here are some of these identities:

(Figure 34)

These identities allow us to interpret the finite-dimensional representation theory of a given ribbon Hopf algebra $H$ as an invariant of colored ribbon graphs, where each edge is colored by a representation of $H$ and each vertex is colored by an element of $\text{Inv}(V_1 \otimes V_2 \otimes \ldots V_n)$, where $V_1, \ldots, V_n$ are the colors of the incoming edges going clockwise around the vertex. (The vertex must also be marked to identify which edge...
corresponds to the first factor in the tensor product; however there is an identification between the sets of colors allowed for two different markings. Although there is no single preferred isomorphism between $V \otimes W$ and $W \otimes V$, there is such an isomorphism between $\text{Inv}(V \otimes W)$ and $\text{Inv}(W \otimes V)$. It is constructed from the $K$ morphisms. We may interpret the value of a ribbon graph with no endpoints as a morphism from $I$ to $I$, i.e. as a scalar invariant. But the invariant is also a linearly recurrent invariant; the spanning space of endpoints colored with $V_1, \ldots, V_n$ is again $\text{Inv}(V_1 \otimes V_2 \otimes \ldots V_n)$. This invariant of colored ribbon graphs is the Reshetikhin-Turaev invariant corresponding to the ribbon Hopf algebra $H$.

We remark that for a fixed graph, the invariant is multilinear under direct sums, and more generally under extensions, of the colors of the edges, so it suffices to consider irreducible representations as colors. We also consider the consequences of an irreducible representation $V$ being self-dual. If $V$ is symmetrically self-dual, i.e. there is an element $B$ of $\text{Inv}(V \otimes V)$ which is invariant under the switching map, then we can allow an unoriented edge colored by $V$ by interpreting it as two oriented edges using $B$:

(Figure 36)

On the other hand, if $V$ is anti-symmetrically self-dual, i.e. the invariant tensor $B$ negates under the switching map, then this construction has a sign ambiguity. But suppose that all representations of $H$ are self-dual, some of them anti-symmetrically self-dual. Then we may resolve the ambiguity by observing that the Reshetikhin-Turaev theory works equally well for Hopf superalgebras. If we reinterpret anti-symmetrically self-dual representations as being negatively graded, they become supersymmetric. It is possible to replace the Hopf algebra $H$ with a Hopf superalgebra $H'$ for which these are the natural gradings.

The other essential ingredient in the Reshetikhin-Turaev theory is a collection of examples of ribbon Hopf algebras. For each simple Lie algebra $g$ there is a Hopf algebra $U_q(g)$, the quantized universal enveloping algebra of $g$, over the field $\mathbb{C}(q, q^{1/2}, q^{1/3}, \ldots)$ (see [3]). The representation ring of this Hopf algebra is the same as that of $g$ or $U(g)$. Consequently we will refer to the irreducible representations $U_q(g)$ as $V_\lambda$, where $\lambda$ is the highest weight of the corresponding representation of $g$. However, the calculus of morphisms between the representations is only the same as that for $g$ at the specialization $q = 1$. We denote the corresponding ribbon graph invariant by $RT_g$.

The most important first case of the ribbon graph invariants is their value on the untwisted unknot colored with some $V_\lambda$. The value of $RT_g$ for this graph is known as the quantum dimension of $V_\lambda$ and equals the trace of the element $K$. This element is a member of the subalgebra of $U_q(g)$ generated by the Cartan subalgebra of $g$, and this subalgebra is the same as the corresponding subalgebra of a certain completion $U(g)$, the ordinary universal enveloping algebra. If $\rho$ is half of the sum of the positive roots of the root system of $g$ and $H_\rho$ is the corresponding element of the Cartan subalgebra, then $K = q^{H_\rho}$. The upshot is that the trace of $K$ on $V_\lambda$ is $\sum_\alpha m(\alpha)q^{\rho(\alpha)}$, where $\alpha$ runs over the weights of $V_\lambda$ and $m(\alpha)$ is the multiplicity of $\alpha$. Another formula for the quantum dimension is the quantum Weyl dimension formula:

$$\dim_q V_\lambda = \frac{\prod_{\alpha > 0} [\lambda + \rho, \alpha)]}{\prod_{\alpha > 0} [\rho, \alpha]}$$

where $[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$ is a “quantum integer” (see [1] ch. 6, p. 151) for a proof that this is the trace of $K$).

For example, when $g = G_2$, we obtain:

$$\dim_q V_{1,0} = q^5 + q^4 + q + 1 + q^{-1} + q^{-4} + q^{-5} = \frac{[7][12][2]}{[6][4]}$$

as the quantum dimension of the smallest non-trivial irreducible representation. Note, however, that when we perform a supersymmetric conversion of a ribbon Hopf algebra $H$ as described above, the quantum dimension of a negatively graded representation negates.

The next most important computation is the evaluation of a right-handed twist:

(Figure 37)

where $c(V_\lambda)$ is the Casimir number for $V_\lambda$. A derivation of this equation is given in [4].
3.2 Comparison with $\langle \cdot \rangle_\theta$

We restrict our attention to the exceptional simple Lie algebra $G_2$. All representations of $G_2$ are symmetrically self-dual. The vector space $\text{Inv}(V_{1,0}^{\otimes n})$ is 1-dimensional; let us choose some element $T$ in it. We interpret a $G_2$ freeway as a colored ribbon graph by coloring all edges with the 7-dimensional representation $V_{1,0}$ and all vertices with $T$. In this way we obtain a value of $RT_{G_2}$ for $G_2$ freeways.

We can verify by a computation with weight diagrams that $\dim \text{Inv}(V_{1,0}^{\otimes n})$ is 1,0,1,1,4, and 10 as $n$ ranges from 0 to 5 (see [4]). A lengthier calculation shows that the values of $RT_{G_2}$ on acyclic graphs with $n$ endpoints are indeed a basis for $n < 5$ and $q = 1$ (and therefore also for $q$ an indeterminate). The author and Rena Zieve performed this calculation explicitly. The tensor $T$ was chosen as a trilinear form $T(a,b,c)$ on $V_{1,0}$. It is well-known that for the compact real Lie group $G_2$, $V_{1,0}$ can be interpreted as the imaginary octonions, or Cayley numbers, which are equipped with a positive-definite inner product and a cross product, a bilinear function “$\times$” with the property that if $a$ and $b$ are orthogonal unit vectors, $a \times b$ is a unit vector which is orthogonal to both $a$ and $b$. We define $T(a,b,c) = (a \times b) \cdot c$.

The conclusion is that the invariant $RT_{G_2}$ satisfies the 1,0,1,1,4,10 Ansatz. It remains only to show that the parameter $q$ in $RT_{G_2}$ agrees with the parameter $q$ in $\langle \cdot \rangle_{G_2}$. To do this we first check that the quantum dimension of $V_{1,0}$ equals the constant $a$ in the recurrence relation of $\langle \cdot \rangle_{G_2}$, and we compute the factor gained by a right-hand twist:

(Figure 38)

We verify that $c(V_{1,0}) = 6$. Thus, the two parameterizations are the same, up to a constant factor in the choice of $T$. (The choice of $T$ above for the explicit $q = 1$ computation agrees with the right choice for $\langle \cdot \rangle_{G_2}$.) Notice in particular that the Casimir element is always positive, which assures us that the ambiguity between $q$ and $q^{-1}$ has been resolved in agreement with convention.

By similar but simpler reasoning we can show that $\langle \cdot \rangle_{A_2} = RT_{A_2}$, where all edges of an $A_2$ freeway are colored with the 3-dimensional representation $V_{1,0}$, whose dual is $V_{0,1}$. The orientations of the edges are needed because $V_{1,0}$ is not self-dual. The colors for the vertices can be recognized as the determinant, or the usual 3-dimensional cross product.

We also have $\langle \cdot \rangle_{C_2} = RT_{C_2}$, where the single edges are colored with the 4-dimensional representation $V_{1,0}$ and the double edges are colored with the 5-dimensional representation $V_{0,1}$. The representation $V_{1,0}$ is the canonical representation of $\text{sp}(4) \cong C_2$. Note that it is anti-symmetrically self-dual. To convert from $\langle \cdot \rangle_{C_2}$, where edges are not oriented, we must use the Hopf superalgebra $U_q(\text{sp}(4))'$ instead of $U_q(\text{sp}(4))$. This superalgebra can also be thought of as $U_q(\text{sp}(0,4))$, where $\text{osp}(0,4)$ is an orthosymplectic Lie superalgebra. (Similarly, the Kauffman bracket $\langle \cdot \rangle_{A_1}$ more properly describes invariants from the representation theory of $\text{sp}(0,2)$ rather than the theory of $\text{sl}(2) = \text{sp}(2)$.) Finally, the representation $V_{0,1}$ is the canonical representation of $\text{so}(5) \cong B_2 \cong C_2$.

3.3 Questions open to the author

The coefficients $a$, $b$, $c$, $d_1$, $d_2$, $e_1$, and $e_2$ in the definition of the $G_2$ bracket, as well as their analogues for the $A_2$ and $C_2$ brackets, are all Laurent polynomials in $q$ with either nonpositive or nonnegative integer coefficients. This is a coincidence which cannot be explained by the freedom in choosing the parameter $q$. The coefficients could be complicated rational or even algebraic functions a priori. Moreover, using the notion of a quantum integer given above, we can write:
a = $\begin{bmatrix} 12 & 7 & 2 \\ 6 & 4 \end{bmatrix}$

b = $\begin{bmatrix} 8 & 3 \\ 4 \end{bmatrix}$

c = $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

d = $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

d = $\begin{bmatrix} 3 \end{bmatrix}$

e = $1$

e = $-1$

The coefficients for the $A_2$ and $C_2$ brackets have similar expressions. In general, saying that a Laurent polynomial in $q$ invariant under $q \mapsto q^{-1}$ with rational coefficients is a ratio of products of quantum integers is equivalent to saying that the zeroes of the polynomial are all roots of unity.

**Question 3.1.** Why do the coefficients in the definition of $\langle \cdot \rangle_{A_2}$, $\langle \cdot \rangle_{C_2}$, and $\langle \cdot \rangle_{G_2}$ have such a nice form?

The known properties of quantum dimension provide the answer to this question in the case of the coefficient $a$ as well as the value of the unknot for the other brackets. However, the author knows no analogous reasoning for the other coefficients.

Let us say that a crossingless $G_2$ freeway (also an $A_2$ freeway) has non-positive curvature if all interior faces have at least six sides. From the definition of $\langle \cdot \rangle_{G_2}$, it is clear that the set of freeways of non-positive curvature spans the spanning space of tangles with $n$ endpoints. It is not so clear when $n > 5$ that the same freeways span the corresponding spanning space for $RT_{G_2}$, which can be identified with the vector space $\text{Inv}(V_{1,0}^{\otimes n})$. Nevertheless, it follows from the fundamental theorem of invariant theory for $G_2$:

**Theorem 3.2.** (Schwarz) The set of all multilinear invariant forms on the imaginary octonions is generated by addition and multiplication of an invariant bilinear form $B(a,b)$, an invariant trilinear form $T(a,b,c)$ (both unique up to a scalar multiple), together with the Hodge dual of $T(a,b,c)$.

Let us call the third form $Q$. Then the theorem says that graphs of this form:

*(Figure 39)*

span $\text{Inv}(V_{1,0}^{\otimes n})$. Strictly speaking, the theorem only applies when $q = 1$, but by genericity, it must also hold for $q$ an indeterminate. Also, although the graphs mentioned by the theorem have crossings and also a new tetralinear form, both of these are some linear combination of the four acyclic $G_2$ freeways with four endpoints. Thus, the set of all freeways span, and therefore the ones with non-positive curvature do too.

**Conjecture 3.3.** The set of $G_2$ freeways of non-positive curvature with $n$ endpoints is a basis for $\text{Inv}(V_{1,0}^{\otimes n})$ via the function $\langle \cdot \rangle_{G_2}$.

It suffices to show that the number of such freeways equals $\dim \text{Inv}(V_{1,0}^{\otimes n})$. The latter number is also equal to the coefficient of the $x^2y^3$ term in the polynomial:

$$(1 + x + y + xy + x^{-1} + y^{-1} + (xy)^{-1})^n(x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} + x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^{-2}y^{-1} + x^3y - x^3y^2)$$

by the character formulas in [4].

The author has verified the conjecture up to $n = 9$ by this method; the number of acyclic, crossingless freeways with $n$ endpoints for $n$ from 0 to 9 is 1, 0, 1, 1, 4, 10, 35, 120, 455, and 1728, respectively.

The situation is similar for $A_2$ and $C_2$. It follows from the classical fundamental theorem of invariance theory for $\text{sl}(3)$ and $\text{sp}(4)$ that freeways with non-positive curvature span. However, the endpoints of an $A_2$ freeway must be labeled as either “incoming” or “outgoing” depending on the orientation of the edges containing the points, and the endpoints of a $C_2$ freeway are labeled as “single” or “double”. Recall that in the $C_2$ case we have extra equivalences which are conveyed by switching moves.
Conjecture 3.4. Consider a fixed set of $k$ incoming and $n$ outgoing points on a circle. The set of $A_2$ freeways of nonpositive curvature having these points as endpoints is a basis for $\text{Inv}(V_{1,0}^\otimes \otimes V_0^\otimes V_1^\otimes)$. It suffices to show that the number of such freeways equals the coefficient of the $xy^2$ term in the polynomial:

$$(xy + x^{-1} + y^{-1})^k (x^{-1}y - x + y)^n (xy^2 - x^{-2}y - x^{-1}y^{-2} - xy^{-1} - x^2y)$$

Conjecture 3.5. Consider a fixed set of $k$ points labeled “single” and $n$ points labeled “double” on a circle. A collection of $C_2$ freeways of nonpositive curvature, with one member in each switching class, constitutes a basis for $\text{Inv}(V_{1,0}^\otimes \otimes V_0^\otimes V_1^\otimes)$. It suffices to show that the number of switching classes equals the coefficient of the $xy^2$ term in the polynomial:

$$(x + y + x^{-1} + y^{-1})^k (1 + xy + x^{-1}y + xy^{-1} + x^{-1}y^{-1})^n (xy^2 - x^{-2}y - x^{-1}y^{-2} - xy^{-1} - x^2y)$$

Note that these conjectures have an analogue for the Kauffman bracket as well. In this case we may conjecture that crossingless tangles with $2n$ endpoints form a basis for $\text{Inv}(V_1^\otimes)^{\otimes 2n}$. By the fundamental theorem of invariant theory they must form a spanning set; it is well-known that the both the number of tangles and the dimension of the invariant space equal the Catalan number $\frac{1}{n+1} \binom{2n}{n}$.

Richard Stanley and John Stembridge have shown the author a proof of conjecture 3.5 in the case $n = 0$ by a variant of the Berele insertion algorithm [14], which plays the same role in the combinatorics of $\text{sp}(2d)$ that the Robinson-Schensted algorithm does in the combinatorics of $\text{sl}(d)$. Recall that when $n = 0$ we can consider tetravalent freeways instead of switching classes of trivalent freeways.

Proof. (Sketch) Let $F$ be a tetravalent freeway with nonpositive curvature. Let $v_1, v_2, \ldots, v_{2n}$ be the $2n$ endpoints of $F$. We interpret $F$ as a collection of crossing line segments whose endpoints are the $v_i$'s, so that the $v_i$'s are paired by arcs of the freeway. We let $m(i) = j$ if $v_i$ is connected to $v_j$. Then the function $m$ is a perfect matching, a fixed-point-free involution of the numbers from 1 to $2n$. Let $M(F)$ be the matching $m$, constructed using $F$.

It is easy to show by induction on $n$ that the matching $M(F)$ satisfies what we will call the 6-point condition: For every three numbers $n_1, n_2, \text{ and } n_3$, we cannot simultaneously have $M(F)(n_1) > M(F)(n_2) > M(F)(n_3) > n_1 > n_2 > n_3$. Moreover, $m$ is a bijection between freeways with non-positive curvature and matchings that satisfy the 6-point condition.

For each $i$ from 0 to $2n$, we construct a Young tableau $T_i$, an infinite array whose entries are positive integers or infinity. All but finitely-many entries of a Young tableau must be infinity, and the rows and columns must be non-decreasing. Here is an example of a Young tableau with the infinities omitted:

$$\begin{array}{cccc}
1 & 2 & 4 & 7 \\
3 & 8 & & \\
6 & 9 & & \\
\end{array}$$

The Young tableaux $T_0$ and $T_{2n}$ are both the empty tableau, the one with no finite entries. We proceed to construct $T_i$ from $T_{i-1}$ by considering $M(F)(i)$. If $M(F)(i) > i$, we insert $M(F)(i)$ into $T_{i-1}$ by Schensted insertion: We replace the highest entry $e$ in the first column which is greater than $M(F)(i)$ with $M(F)(i)$. If $e$ is finite, we then insert $e$ by Schensted insertion into the tableau consisting of the second column onward; if $e$ is infinity, we stop. If $M(F)(i) < i$, we remove $i$ by Berele deletion: We replace $i$ in the tableau by infinity, and then we repeatedly switch the misplaced infinite entry by the entry below or the entry to the right, whichever is smaller, until the array becomes a Young tableau again. Let $F_i$ be the support of $T_i$, the set of place where $T_i$ is non-zero. The sequence of $F_i$'s is called an up-down tableau, and the algorithm described here is a bijection between matchings of $2n$ elements and up-down tableaux of length $2n$. See [14] for a proof of this fact.

Matchings which happen to satisfy the 6-point condition correspond to tableaux with at most two rows. To see this, we observe that if some $T_i$ has two rows and the entry in the first column of the second row is $j$, then $j$ is the smallest number such that $M(F)(j) < i$ and for some $k > j$, $M(F)(k) < M(F)(j)$. 

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Therefore if we consider the first $T_i$ with three rows, we see that only a violation of the 6-point condition could have resulted in transforming $T_{i-1}$ to $T_i$. Conversely, if a matching does violate the 6-point condition, the corresponding up-down tableau must at some point have three rows by the same principle.

If an element of an up-down tableau has at most two rows, then that element can be described by a pair of integers $(a, b)$ with $a \geq b \geq 0$, where $a$ is the number of elements in the first row and the $b$ is the number in the second row. Thus, we obtain, for each matching, a sequence of pairs of integers $(a_i, b_i)$ with $a \geq b \geq 0$ such that $a_0 = b_0 = a_{2n} = b_{2n} = 0$ and such that either $a_i$ or $b_i$ differs by 1 from $a_{i-1}$ or $b_{i-1}$ and the other is equal. This sequence is a lattice path in the Weyl chamber of the Lie algebra $C_2$, and by the character theory of Lie algebras, the number of such paths is the dimension of $\text{Inv}(V_{1,0}^{\otimes 2n})$.

We conclude with the question about the present work that most bothers the author:

**Question 3.6.** If $\mathfrak{g}$ is a simple Lie algebra of rank greater than 2, does the invariant $RT_\mathfrak{g}$ have an inductive definition based on linear recurrence, planar graphs, and Euler characteristic?

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