EXPOENTIAL STABILIZATION OF A LINEAR KORTEweg-de VRIES EQUATION WITH INPUT SATURATION

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Abstract. This article deals with the issue of the exponential stability of a linear Korteweg-de Vries equation with input saturation. It is proved that the system is well-posed and the origin is exponentially stable for the closed loop system, by using the classical argument used in this kind of problems.

1. Introduction. During the last decades, the linear Korteweg-de Vries (KdV) equation

\[
\begin{cases}
  u_t(t, x) + u_x(t, x) + u_{xxx}(t, x) + f = 0, & (t, x) \in \mathbb{R}_+ \times [0, L]; \\
  u(t, 0) = u(t, L) = u_x(t, L) = 0, & t \in \mathbb{R}_+ \\
  u(0, x) = u_0(x),
\end{cases}
\]

where \( u \) stands for the state, \( f \) for the control and \( L > 0 \), have received a considerable attention. Indeed, in the literature, we find several works which study this equation (see e.g [21], [16], [12], [1], [3] and the references within). In general, the control \( f \) is taken to achieve some specific objectives. Consequently, it must always obey to predefined restrictive constraints. In particular, (1) is studied in [9, 2] with boundary control and in [12, 14] with the control is distributed. In [12], the authors show that the feedback control \( f(t, x) = a(x)u(t, x) \), where \( a \) is a positive function whose support is an nonempty open subset of \([0, L]\), makes the origin exponentially stable. When \( a = 0 \), it is also proved that the linear KdV equation is exponentially stable provided that

\[
L \notin \left\{ 2\pi \sqrt{K^2 + kl + l^2}/k, l \in \mathbb{N}^* \right\}.
\]

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One of most known constraints which act on the feedback law is saturation (see [18, 10, 7, 20, 11]). In [10], the following KdV equation
\[
\begin{align*}
\begin{cases}
  u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + asat(u(t,x)) = 0, & (t,x) \in \mathbb{R}_+ \times [0,L]; \\
  u(t,0) = u(t,L) = u_x(t,L) = 0, & t \in \mathbb{R}; \\
  u(0,x) = u_0(x),
\end{cases}
\end{align*}
\] (2)
has been studied. The saturation function sat(⋅) is given by
\[
sat(s) = \begin{cases} 
  -u_0, & \text{if } s \leq -u_0, \\
  s, & \text{if } -u_0 \leq s \leq u_0, \\
  u_0, & \text{if } s \geq u_0.
\end{cases}
\] (3)
where \(a\) and \(u_0\) are positive constants.

The well-posedness of the closed loop system (2) is proved by applying the nonlinear semigroup theory. It is also stated that the origin is asymptotically stable using a sector condition and lyapunov theory for infinite dimensional systems. In [8], the authors proposed to study a wave equation by using an observability hypothesis in the presence of saturating control law. Unlike [8] where the strong stability of a wave equation is investigated, we study the exponential stability of the linear Korteweg-de Vries equation (1) with a saturating feedback law.

In our paper we define the feedback control by,
\[
f(t, x) = sat(a(x)u(t, x)).
\] (4)
where \(a = a(x) \in L^\infty([0, L])\) satisfying
\[
\begin{cases}
  a_1 \geq a = a(x) \geq a_0 > 0 \quad \text{on } \omega \subseteq [0,L], \\
  \omega \quad \text{is a nonempty open subset of } [0,L],
\end{cases}
\]
and the function sat(⋅) is defined by
\[
sat(s) = \begin{cases} 
  s, & \text{if } \|s\|_{L^2(0,L)} \leq 1, \\
  \frac{1}{\|s\|_{L^2(0,L)}}, & \text{if } \|s\|_{L^2(0,L)} \geq 1.
\end{cases}
\] (5)
In particular, we analyze the well-posedness and exponential stability for the linear KdV equation. The well-posedness of the closed loop system is proved by using the nonlinear semigroup theory. By using an argument by contradiction, we show that the origin of the KdV equation (1) in closed loop system with the saturated control (4) is exponentially stable, which is the mean contribution of this work.

The article is organized as follows. In section 2, we study the well-posedness of (1). Section 3 is devoted to the exponential stabilization of (1). Finally, we give some conclusions in section 4.

**Notation.** \(u_t\) (resp. \(u_x\)) stands for the partial derivative of the function \(u\) with respect to \(t\) (resp. \(x\)). \(R_+\) (resp. \(i\)) denotes the real (resp. imaginary) part of a complex number. \(B^*\) is the adjoint operator of linear operator \(B\). Given \(L > 0\) and \(J=[0, L]\), \(\|\cdot\|_{L^2(J)}\) (resp \(<\cdot, \cdot>_L\) \(L^2(J)\)), denotes the norm (resp the product scalar) in \(L^2(J)\). \(H^1(J)\) denotes the set of all functions \(u \in L^2(J)\) such that \(u_x \in L^2(J)\). The space \(H^1(J)\) is equipped with the norm \(\|u\|_{H^1(J)}^2 = \|u_x\|_{L^2(J)}^2 + \|u\|_{L^2(J)}^2\). \(H^2(J)\) is the set of all functions \(u \in L^2(J)\) such that \(u_x, u_{xx} \in L^2(J)\). The space \(H^2(J)\) is equipped with the norm \(\|u\|_{H^2(J)}^2 = \|u_x\|_{L^2(J)}^2 + \|u_{xx}\|_{L^2(J)}^2\). \(H^3(J)\) is the set of all functions \(u \in L^2(J)\) such that \(u_x, u_{xx}, u_{xxx} \in L^2(J)\). The space \(H^3(J)\) is equipped with the norm \(\|u\|_{H^3(J)}^2 = \|u_x\|_{L^2(J)}^2 + \|u_{xx}\|_{L^2(J)}^2 + \|u_{xxx}\|_{L^2(J)}^2\). \(H^1_0(J)\) is the closure in \(L^2(J)\) of the set of smooth functions that are vanishing at \(x = 0\).
0 and \( x = L \). The space \( H_0^1(J) \) is equipped with the norm \( \|u\|^2_{H_0^1(J)} = \|u_x\|^2_{L^2(J)} \). \( H^{-1}(J) \) is the dual space of \( H_0^1(J) \).

2. Well-posedness. In this section we study the well-posedness of the solution of the following system:

\[
\begin{align*}
 u_t(t,x) + u_x(t,x) + u_{xx}(t,x) + \text{sat}(a(x)u(t,x)) = 0, & \quad (t,x) \in \mathbb{R}_+ \times [0,L]; \\
u(t,0) = u(t,L) = u_x(t,L) = 0, & \quad t \in \mathbb{R}; \\
u(0,x) = u_0(x),
\end{align*}
\]

with \( a = a(x) \in L^\infty([0,L]) \) satisfying

\[
\begin{align*}
a_1 \geq a \geq a(x) \geq a_0 > 0 \text{ for all } x \in \omega \subseteq [0,L], \\
\omega \text{ is a nonempty open subset of } [0,L],
\end{align*}
\]

**Remark 1.** The hypothesis \( a = a(x) \in L^\infty([0,L]) \) satisfying (7), is the standard assumption used in this type of problems (see [11, 12]). In [10], the authors assume that \( a \) is a positive constant.

Before studying the well-posedness, we recall the following definitions.

Consider the abstract system in a Hilbert space \( H \)

\[
\begin{align*}
 \dot{u}(t) &= Au(t) + f(t), \\
u(0) &= u_0 \in H,
\end{align*}
\]

where \( A \) is an infinitesimal generator of linear \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) defined on its domain \( D(A) \subseteq H \), and \( f \in L^1_{loc}([0,+,\infty[, H) \).

**Definition 2.1.** (Weak solution) A continuous function \( u : [0,+,\infty[ \to H \) is called a weak solution of (8) if

1. \( u(0) = u_0 \),

2. For all \( x^* \in D(A^*) \), the function \( t \mapsto <u(t), x^* > \) is absolutely continuous on all intervals \( [0,T], T > 0 \), and

\[
\frac{d}{dt} <u(t), x^*> = <u(t), A^* x^* > + <f(t), x^*> \text{ a.e on } [0,+,\infty[.
\]

**Definition 2.2.** (Classical solution) A function \( u(\cdot) \) is said to be a classical solution of (8) if \( u(\cdot) \in C^1([0,T], H) \), \( u(t) \in D(A) \) \( \forall t \geq 0 \) and \( u(\cdot) \) satisfies (8). \( u(\cdot) \) is said a classical solution of (8) on \( [0,+,\infty] \), if \( u(\cdot) \) is a classical solution of (8) on \( [0,T], \forall T > 0 \).

**Remark 2.** The function given by

\[
u(t) = T(t)u_0 + \int_0^t T(t-s)f(u(s))ds,
\]

is called a mild solution. By [4, Theorem 3.1.7, page 106], for every \( u_0 \in H \) and every \( f \in L^p([0,+,\infty[, H) \), a mild solution is also a weak solution.

**Remark 3.** From [15, Corollary 2.11, page 109], in a Hilbert space, if \( u_0 \in D(A) \) and \( f \) is lipschitz continuous, the weak solution (9) is also a classical solution.

For more details about weak (resp classical) solution see [6, Theorem 2.18, page 71](resp [15]).

In the following lemma we recall that the saturation function is Lipschitzian in \( L^2([0,L]) \).
Lemma 2.3. [20, Theorem 5.1] For all \((s_1, s_2) \in L^2(J)\), we have
\[ \|\text{sat}(s_1) - \text{sat}(s_2)\|_{L^2(J)} \leq 3\|s_1 - s_2\|_{L^2(J)}. \]

Let us state the main result of this section.

Theorem 2.4. (well-posedness). Assume that \(a = a(x) \in L^\infty(J)\) satisfying (7). Then for any \(u_0 \in L^2(J)\), there exists a unique weak solution \(u : [0, +\infty[ \to L^2(J)\), to (6).

Proof. Let us consider the following operators given by
\[
D(A) = \{ u \in H^3(J)/u(0) = u(L) = u_x(L) = 0 \}, \\
Au = -u_x - u_{xxx} - \text{sat}(a(x))u \quad \forall u \in D(A), \\
D(T_1) = D(A) \subset L^2(J), \\
T_1u = -u_x - u_{xxx},
\]
and
\[
D(T_2) = L^2(J), \\
T_2u = -\text{sat}(a(x))(u).
\]
Thus
\[
(T_1 + T_2)u = Au \quad \forall u \in D(A).
\]
First we will show that the operator \(T_1\) and its adjoint operator defined by
\[
T_1^*u = u_x + u_{xxx},
\]
(see [11]), are both dissipative. Indeed, let \(u \in D(T_1)\), we get
\[
\langle T_1u, u \rangle_{L^2(J)} = \langle -u_x - u_{xxx}, u \rangle_{L^2(J)} = - \int_0^L u_x \overline{u}dx - \int_0^L u_{xxx} \overline{u}dx
\]
Integrating by parts \(\int_0^L u_x \overline{u}dx\) and \(\int_0^L u_{xxx} \overline{u}dx\), we get
\[
-R_e \left\{ \int_0^L u_x \overline{u}dx \right\} = 0, \quad (14)
\]
and
\[
-R_e \left\{ \int_0^L u_{xxx} \overline{u}dx \right\} = -\frac{1}{2}|u_x(0)|^2 \leq 0. \quad (15)
\]
We deduce from (14) and (15) that
\[
R_e \langle T_1u, u \rangle_{L^2(J)} = -\frac{1}{2}|u_x(0)|^2 \leq 0.
\]
Let \(u \in D(T_1^*)\), we get
\[
R_e \langle T_1^*u, u \rangle_{L^2(J)} = R_e \langle u_x + u_{xxx}, u \rangle_{L^2(J)} = R_e \left\{ \int_0^L u_x \overline{u}dx \right\} + R_e \left\{ \int_0^L u_{xxx} \overline{u}dx \right\} = -\frac{1}{2}|u_x(L)|^2 \leq 0.
\]
It is well known that, \( T_1 \) is a closed operator (see \([16, Proposition\ 3.1]\)) and \( D(A) \) is dense in \( L^2(J) \), therefore from \([15, Corollary\ 4.4,\ page\ 15]\), the operator \( T_1 \) generates a strongly continuous linear semigroup of contractions which we denote by \( (T(t))_{t \geq 0} \). \([15, Theorem\ 4.3,\ page\ 14]\) implies that \( T_1 \) is a m-dissipative operator.

Moreover we will show that the operator \( T_2 \) is dissipative and globally lipschitzian. Indeed, for all \( u, v \in L^2(J), \) we have

\[
< sat(u) - sat(v), u - v >_{L^2(J)} = \int_0^L (sat(u) - sat(v))(u - v)dx 
\geq 0,
\]

this is clear as \([20]\)

\[
(sat(x) - sat(y))(x - y) \geq 0 \quad \forall \in \mathbb{R}. \quad (16)
\]

For our operator

\[
< sat(au) - sat(av), u - v >_{L^2(J)} = \int_0^L (sat(au) - sat(av))(u - v)dx 
= \int_{a \neq 0 \cap [0, L]} (sat(au) - sat(av))(au - av)\frac{1}{a} dx.
\]

By assumption \((7), \frac{1}{a} \geq 0 \) and by \((16), (sat(au) - sat(av))(au - av) \geq 0. \) Then

\[
< sat(au) - sat(av), u - v >_{L^2(J)} \geq 0. \quad (17)
\]

Therefore we deduce from \((17)\) that the operator \( T_2 = -sat(a(x)u) \) is dissipative in \( L^2(J). \)

Next by using lemma 2.3 and assumptions on \( a(x) \), we will show that \( T_2 \) is globally lipschitzian. Indeed for all \( u, v \in L^2(J), \) we have

\[
\|sat(a(x)u) - sat(a(x)v)\|_{L^2(J)} \leq 3\|a(x)(u - v)\|_{L^2(J)} 
\leq 3\|a(x)\|_{L^\infty([0, L])}\|u - v\|_{L^2(J)} 
\leq 3a_1\|u - v\|_{L^2(J)}. 
\]

Therefore, we deduce from \([19, Lemma\ 2.1,\ page\ 165]\) that \( A \) is a m-dissipative operator. \([13, Theorem\ 4.20,\ page\ 103]\), implies that the operator \( A \) generates a strongly continuous semigroup of contractions denoted by \( (T_{sat}(t))_{t \geq 0} \). Thus \((6)\) has a unique mild solution given by

\[
u(t) = T_{sat}(t)u_0, \quad (18)\]

and by \([6, Theorem\ 2.18,\ page\ 71]\) this mild solution is also a weak solution. \(\square\)

**Remark 4.** For every \( u_0 \in D(A) \), \((18)\) is a classical solution (see \([15, Corollary\ 2.11,\ page\ 109]\)).

**Remark 5.** For all \( u, v \in L^2(J), \) we have

\[
\|T_{sat}(t)u - T_{sat}(t)v\|_{L^2(J)} \leq \|u - v\|_{L^2(J)} \quad \forall t \geq 0. \quad (19)
\]

We deduce from \((19)\) that following map

\[
t \mapsto \|u(t, \cdot)\|_{L^2(J)},
\]

is non increasing, ie

\[
\|u(t, \cdot)\|_{L^2(J)} \leq \|u_0\|_{L^2(J)} \quad \forall u_0 \in D(A), \forall t \geq 0. \quad (20)
\]
3. **Exponential stability.** Before studying the exponential stability, we recall the following definition.

**Definition 3.1.** System (6) is said to be semi-globally exponentially stable in $L^2(J)$, if for any $r > 0$ there exists two constants $\mu_r := \mu_r(r) > 0$ and $K_r := K_r(r) > 0$ such that for any $u_0 \in L^2(J)$ with $\|u_0\|_{L^2(J)} \leq r$, the weak solution $u = u(t,x)$ to (6) satisfies

$$\|u(t,\cdot)\|_{L^2(J)} \leq K_r \|u_0\|_{L^2(J)} e^{-\mu_r t} \quad \forall t \geq 0.$$ 

The following lemmas play an important role to prove the exponential stability of the system (6).

**Lemma 3.2.** Suppose that $a = a(x)$ satisfies (7), $u_0 \in D(A)$ and $u$ is the classical solution of (6), then we have

$$\|u(t,.)\|_{L^2(0,T,H^1(J))}^2 \leq \frac{4T + L}{3} \|u_0\|_{L^2(J)}^2$$

with $T > 0$ and $L > 0$.

**Proof.** We multiply the equation (6) by $u$ and we integrate on $[0,L]$, we get

$$\int_0^L uu_t dx + \int_0^L uu_x dx + \int_0^L uu_{xxx} dx + \int_0^L sat(au).udx = 0$$

After some integrations by parts, we get

$$\int_0^L u(t,x)u_t(t,x)dx = \frac{1}{2} \frac{d}{dt}\|u(t,.)\|_{L^2([0,L])}^2; \quad \int_0^L uu_x dx = 0$$

and

$$\int_0^L uu_{xxx} dx = \frac{1}{2}\|u_x(t,0)\|^2.$$ 

Therefore, equation (22) becomes

$$\frac{1}{2} \frac{d}{dt}\|u(t,.)\|_{L^2(J)}^2 + \frac{1}{2}\|u_x(t,0)\|^2 + \int_0^L sat(au).udx = 0.$$ 

(23)

Then we multiply (6) by $xu$ and we integrate on $J \times [0,T]$, we obtain

$$\int_0^T \int_0^L xu u_t dx + \int_0^T \int_0^L xuu_x dx + \int_0^T \int_0^L xu_{xxx} dx + \int_0^T \int_0^L sat(au).xudx = 0$$

After some integrations by parts and by using the boundary conditions, we get

$$\int_0^T \int_0^L xu u_t dx dt = \frac{1}{2} \int_0^L xu^2(T,x) dx - \frac{1}{2} \int_0^L xu^2(0,x) dx;$$

(25)

$$\int_0^T \int_0^L xu u_x dx dt = -\frac{1}{2} \int_0^T \int_0^L u^2 dx dt$$

(26)

and

$$\int_0^T \int_0^L xu u_{xxx} dx dt = \frac{3}{2} \int_0^T \int_0^L u_x^2 dx dt$$

(27)
We multiply the last three equations and \( \int_0^T \int_0^L sat(au).xdx \) by \( \frac{2}{3} \) and we add it up, we obtain
\[
\begin{align*}
\int_0^T \int_0^L u_t^2 dxdt + \frac{1}{3} \int_0^T L x u(T,x)dx + \frac{2}{3} \int_0^T \int_0^L sat(au).xdxdt &= \frac{1}{3} \int_0^T \int_0^L u_t^2 dxdt + \frac{1}{3} \int_0^L x u^2(0,x). \\
(28)
\end{align*}
\]
Since \( a > 0 \) and \( sat(au).u \geq 0 \), we get
\[
\int_0^T \int_0^L u_t^2 dxdt = \frac{1}{3} \int_0^T \int_0^L u_t^2 dxdt + \frac{1}{3} \int_0^L x u^2(0,x)dx \\
- \frac{1}{3} \int_0^L x u^2(T,x)dx - \frac{2}{3} \int_0^T \int_0^L x sat(au).udxdt = 0
\]
(29)
Therefore we deduce from (23) and (29) that
\[
\|u(t,\cdot)\|_{L^2(0,T;H^1(J))} \leq \frac{4T + L}{3} \|u_0\|_{L^2(J)}^2
\]

**Lemma 3.3.** Suppose that \( a = a(x) \) satisfies (7), \( u_0 \in D(A) \) and \( u \) is the classical solution of (6), then we have
\[
\int_0^L |u_0|^2 dx \leq \frac{1}{T} \int_0^T \int_0^L |u(t,x)|^2 dxdt + \int_0^T |u_x(t,0)|^2 dt \\
+ 2 \int_0^T \int_0^L sat(au).udxdt
\]
(30)
with \( T > 0 \) and \( L > 0 \).

**Proof.** We multiply equation (6) by \((T-t)u\) and we integrate on \( J \times [0,T], \) we obtain
\[
\begin{align*}
\int_0^T \int_0^L (T-t)uu_t dxdt + \int_0^T \int_0^L (T-t)uu_x dxdt + \int_0^T \int_0^L (T-t)uu_{xx} dxdt \\
+ \int_0^T \int_0^L (T-t)u.sat(au)dxdt &= 0.
\end{align*}
\]
(31)
After some integrations by parts, we get
\[
\begin{align*}
\int_0^T \int_0^L (T-t)uu_t dxdt &= -\frac{T}{2} \int_0^L \left| u(0,x) \right|^2 dx + \frac{1}{2} \int_0^T \int_0^L \left| u(t,x) \right|^2 dtdx; \\
\int_0^T \int_0^L (T-t)uu_x dxdt &= 0
\end{align*}
\]
and
\[
\int_0^T \int_0^L (T-t)uu_{xx} dxdt = \frac{1}{2} \int_0^T (T-t)|u_x(t,0)|^2 dt,
\]
Then equation (31) becomes
\[
-T\frac{1}{2} \int_0^T \int_0^L |u(0,x)|^2 dx + \frac{1}{2} \int_0^T \int_0^L |u(t,x)|^2 dx \, dt + \frac{1}{2} \int_0^T (T-t)|u_x(t,0)|^2 dt \\
+ \int_0^T \int_0^L (T-t)\text{sat}(au) \, dx \, dt = 0.
\]
We multiply the last equation by 2, we get
\[
T \int_0^L |u_0|^2 dx = \int_0^T \int_0^L |u(t,x)|^2 dx \, dt + \int_0^T (T-t)|u_x(t,0)|^2 dt \\
+ 2T \int_0^L \int_0^L (T-t)\text{sat}(au) \, dx \, dt.
\]
Consequently,
\[
\int_0^L |u_0|^2 dx \leq \frac{2}{T} \int_0^T \int_0^L |u(t,x)|^2 dx \, dt + \int_0^T |u_x(t,0)|^2 dt \\
+ 2 \int_0^T \int_0^L \text{sat}(au) \, dx \, dt.
\]
Which completes the proof. \(\Box\)

**Lemma 3.4.** Suppose that \(a = a(x)\) satisfies (7) and \(u_0 \in D(A)\) with \(\|u_0\|_{L^2(J)} \leq r\). Then for any \(T > 0\), there exists \(c_1 = c_1(T) > 0\) such that
\[
\int_0^T \int_0^L |u(t,x)|^2 dx \leq c_1 \left\{ \int_0^T |u_x(t,0)|^2 dx + 2 \int_0^T T \int_0^L \text{sat}(au) \, dx \, dt \right\}.
\] (33)
Where \(u\) is the classical solution of (6).

**Proof.** We suppose that (33) fails to be true, then there exists a sequence of solution \((u_n)_{n \in \mathbb{N}}\) of (6) with
\[
\|u_n(0,\cdot)\|_{L^2(J)} \leq r,
\] (34)
and such that
\[
\lim_{n \to +\infty} \frac{\|u_n\|_{L^2(0,T,L^2(J))}^2}{\int_0^T \left| \frac{\partial}{\partial x} u_n(t,0) \right|^2 dt + 2 \int_0^T T \int_0^L \text{sat}(au_n) \, u_n \, dx \, dt} = +\infty.
\] (35)
We deduce from (20) and (34) that
\[
\|u_n(t,\cdot)\|_{L^2(J)} \leq r.
\] (36)
Let \(\lambda_n = \|u_n\|_{L^2(0,T,L^2(J))}^2\), \(v_n = \frac{u_n(t,x)}{\lambda_n}\) and \(v_n(0,x) = \frac{u_n(0,x)}{\lambda_n}\). By using (34), we deduce that \((\lambda_n)_{n \in \mathbb{N}}\) is bounded. Hence there exists a subsequence of \((\lambda_n)_{n \in \mathbb{N}}\) also denoted by \((\lambda_n)_{n \in \mathbb{N}}\) such that
\[
\lambda_n \to \lambda \geq 0 \text{ as } n \to +\infty.
\] (37)
\(u_n(t,x) = \lambda_n \circ v_n(t,x)\) is a solution of system (6), then we see that for all \(n \in \mathbb{N}\), the function \(v_n\) is a solution of the following system
\[
\begin{cases}
(v_n)_t(t,x) + (v_n)_x(t,x) + (v_n)_{xxx}(t,x) + \frac{\text{sat}(au_n(t,x))}{\lambda_n} = 0, & (t,x) \in \mathbb{R}_+ \times J, \\
v_n(t,0) = v_n(t,L) = (v_n)_x(t,L) = 0, & t \in \mathbb{R}_+, \\
v_n(0,x) = (v_n)_0(x),
\end{cases}
\] (38)
with

\[ \|v_n\|_{L^2(0,T,L^2(J))}^2 = 1 \]  \hfill (39)

We deduce from (35) and (39) that

\[ \lim_{n \to +\infty} \int_0^T \left| \frac{\partial}{\partial x} v_n(t,0) \right|^2 dt + 2 \int_0^T \int_0^L \frac{sat(a \lambda_n v_n)}{\lambda_n} v_n dx dt = 0 \]  \hfill (40)

We deduce from lemma 3.3, (39) and (40) that

\[ \|v_n(0,\cdot)\|_{L^2(J)}^2 = \frac{1}{T} \int_0^T \int_0^L |v_n(t,x)|^2 dx dt + \int_0^T |(v_n)_x(t,0)|^2 dt 
+ 2 \int_0^T \int_0^L sat(a v_n) v_n dx dt 
\leq \frac{1}{T} \|v_n\|_{L^2(0,T,L^2(J))}^2 + \int_0^T |(v_n)_x(t,0)|^2 dt 
+ 2 \int_0^T \int_0^L a(x)|v_n(t,x)|^2 dx dt 
\leq \frac{1}{T} + \int_0^T |(v_n)_x(t,0)|^2 dt + 2a_1 \int_0^T \int_0^L |v_n(t,x)|^2 dx dt 
= \frac{1}{T} + K(T) + 2a_1 = M \]  \hfill (41)

where \( M \) is positive constant. Consequently \( v_n(0,\cdot) \) is bounded in \( L^2(J) \). Then by using (20) and (41) we get

\[ \|v_n(t,\cdot)\|_{L^2(J)} \leq \sqrt{M} = M_1 \quad \forall t \in [0,T] \]  \hfill (42)

We deduce from lemma 3.2 that

\[ \|v_n\|_{L^2(0,T,L^2(J))}^2 = \left\| \frac{u_n(t,\cdot)}{\lambda_n} \right\|_{L^2(0,T,H^1(J))}^2 
\leq \frac{4T + L}{3} \left\| \frac{u_n(0,\cdot)}{\lambda_n} \right\|_{L^2(J)}^2 
\leq \frac{4T + L}{3} \|v_n(0,\cdot)\|_{L^2(J)}^2 
\leq \frac{4T + L}{3} M \]  \hfill (43)

Moreover \( \left\{ \frac{sat(a \lambda_n v_n)}{\lambda_n} \right\}_n \) is bounded in \( L^2(0,T,L^2(J)) \). Indeed from lemma 2.3, we deduce that

\[ \left\| \frac{sat(a \lambda_n v_n)}{\lambda_n} \right\| \leq 3 \left\| \frac{a \lambda_n v_n}{\lambda_n} \right\|_{L^2(0,T,L^2(J))} = 3 \|av_n\|_{L^2(0,T,L^2(J))} \leq 3a_1 \sqrt{L} \|v_n\|_{L^2(0,T,H^1(J))} \]  \hfill (44)
Therefore \( \left\{ \frac{sat(a\lambda_n v_n)}{\lambda_n} \right\}_{n \in \mathbb{N}} \) is subset of \( L^2(0, T, H^{-2}(J)) \), since \( L^2(J) \subset L^1(J) \subset H^{-1}(J) \subset H^{-2}(J) \). It follows by (38), (39), (43) and (44) that

\[
(v_n)_t = -(v_n)_x - (v_n)_{xxx} - \frac{sat(a\lambda_n v_n)}{\lambda_n} \quad (45)
\]

is bounded in \( L^2(0, T, H^{-2}(J)) \). Since the injection of \( H^1_0(J) \hookrightarrow L^2(J) \) is compact and \( (v_n)_n \) is bounded in \( L^2(0, T, H^1_0(J)) \), then \( (v_n)_n \) is relatively compact in \( L^2(0, T, L^2(J)) \). Then \( (v_n)_n \) has a convergent subsequence, denoted by \( (v_n)_n \) also, such that

\[
v_n \xrightarrow{n \to +\infty} v \quad \text{in} \quad L^2(0, T, L^2(J)) \quad (46)
\]

and

\[
\|v_n\|_{L^2(J)} \leq M_1. \quad (47)
\]

Using the weak lower semicontinuity, we have

- If \( \|a\lambda_n v_n\|_{L^2(J)} \leq 1 \), then \( sat(a\lambda_n v_n) = a\lambda_n v_n \), we get

\[
0 = \liminf_{n \to +\infty} \left[ \int_0^T \left| \frac{\partial}{\partial x} v_n(t, 0) \right|^2 dt + 2 \int_0^T \int_0^L sat(a\lambda_n v_n) \lambda_n \, v_n \lambda_n \, v_n \, dx \, dt \right]
= \liminf_{n \to +\infty} \left[ \int_0^T \left| \frac{\partial}{\partial x} v_n(t, 0) \right|^2 dt + 2 \int_0^T \int_0^L a\lambda_n v_n \lambda_n \, v_n \, dx \, dt \right]
= \liminf_{n \to +\infty} \left[ \int_0^T \left| \frac{\partial}{\partial x} v_n(t, 0) \right|^2 dt + 2 \int_0^T \int_0^L a|v_n|^2 \, dx \, dt \right]
\geq \int_0^T \left| \frac{\partial}{\partial x} v(t, 0) \right|^2 dt + 2 \int_0^T \int_0^L a \, |v|^2 \, dx \, dt
\]

we deduce from this last inequality that \( \frac{\partial}{\partial x} v(t, 0) = 0 \) and \( a(x)v = 0 \), in particular \( v = 0 \) on \( \omega \times [0, T] \).

- If \( \|a\lambda_n v_n\|_{L^2(J)} \geq 1 \),

\[
\frac{sat(a\lambda_n v_n)}{\lambda_n} = \frac{a\lambda_n v_n}{\|a\lambda_n v_n\|_{L^2(J)}} = \frac{a\lambda_n v_n}{\lambda_n \|a v_n\|_{L^2(J)}} = \frac{av_n}{\|av_n\|_{L^2(J)}}
\]

Moreover

\[
\|a(x)\|_{L^\infty(J)} \leq a_1 \quad \forall x \in J,
\]

then

\[
\frac{1}{\|a(x)\|_{L^\infty([0,L])}} \geq \frac{1}{a_1}. \quad (49)
\]

According to (37), in this case, we have two cases to study.
1. if \( \lambda_n \to 0 \), then \( \left( \frac{1}{\lambda_n} \right) \to +\infty \). By using (47) and (49), we get

\[
0 = \liminf_{n \to +\infty} \left[ \int_0^T \left| \frac{\partial}{\partial x} v_n(t, 0) \right|^2 dt + 2 \int_0^T \int_0^L \frac{a(\lambda_n v_n)}{\lambda_n} v_n dx dt \right] 
= \liminf_{n \to +\infty} \left[ \int_0^T \left| \frac{\partial}{\partial x} v_n(t, 0) \right|^2 dt + 2 \int_0^T \int_0^L \frac{a v_n}{\lambda_n \|a v_n\|_{L^2(J)}} v_n dx dt \right] 
\geq \liminf_{n \to +\infty} \left[ \int_0^T \left| \frac{\partial}{\partial x} v_n(t, 0) \right|^2 dt + 2 \left( \frac{1}{\lambda_n} \right) \int_0^T \int_0^L \frac{a|v_n|^2}{\|a(x)\|_{L^\infty} \|v_n\|_{L^2(J)}} dx dt \right] 
\geq \liminf_{n \to +\infty} \left[ \int_0^T \left| \frac{\partial}{\partial x} v_n(t, 0) \right|^2 dt + 2 \left( \frac{1}{\lambda_n} \right) \left( \frac{2}{M_1} \right) \left( \frac{1}{a_1} \right) \int_0^T \int_0^L a |v_n|^2 dx dt \right] 
= \int_0^T \left| \frac{\partial}{\partial x} v(t, 0) \right|^2 dt + \left( \frac{2}{M_1} \right) \left( \frac{1}{\lambda} \right) \left( \frac{1}{a_1} \right) \int_0^T \int_0^L a |v|^2 dx dt 
= + \infty 
\] 

(50)

senseless, thus \( \lambda_n \) does not converges to 0.

2. if \( \lambda_n \to \lambda > 0 \), then \( \left( \frac{1}{\lambda_n} \right) \to \left( \frac{1}{\lambda} \right) \). By using (47) and (49), we get

\[
0 = \liminf_{n \to +\infty} \left[ \int_0^T \left| \frac{\partial}{\partial x} v_n(t, 0) \right|^2 dt + 2 \int_0^T \int_0^L \frac{a(\lambda_n v_n)}{\lambda_n} v_n dx dt \right] 
= \liminf_{n \to +\infty} \left[ \int_0^T \left| \frac{\partial}{\partial x} v_n(t, 0) \right|^2 dt + 2 \int_0^T \int_0^L \frac{a v_n}{\lambda_n \|a v_n\|_{L^2(J)}} v_n dx dt \right] 
\geq \liminf_{n \to +\infty} \left[ \int_0^T \left| \frac{\partial}{\partial x} v_n(t, 0) \right|^2 dt + 2 \left( \frac{1}{\lambda_n} \right) \int_0^T \int_0^L \frac{a|v_n|^2}{\|a(x)\|_{L^\infty} \|v_n\|_{L^2(J)}} dx dt \right] 
\geq \liminf_{n \to +\infty} \left[ \int_0^T \left| \frac{\partial}{\partial x} v_n(t, 0) \right|^2 dt + 2 \left( \frac{1}{\lambda_n} \right) \left( \frac{2}{M_1} \right) \left( \frac{1}{a_1} \right) \int_0^T \int_0^L a |v_n|^2 dx dt \right] 
\geq \int_0^T \left| \frac{\partial}{\partial x} v(t, 0) \right|^2 dt + \left( \frac{2}{M_1} \right) \left( \frac{1}{\lambda} \right) \left( \frac{1}{a_1} \right) \int_0^T \int_0^L a |v|^2 dx dt 
\] 

(51)

In all cases, \( \frac{\partial}{\partial x} v(t, 0) = 0 \) and \( a(x)v = 0 \), in particular \( v = 0 \) on \( \omega \times [0, T] \). The limit satisfies

\[
v_t + v_x + v_{xxx} = 0
\]

By Holmgreen’s theorem ([5]), we deduce that \( v = 0 \) on \( [0, L] \times [0, T] \), this contradicts the fact that

\[
\|v\|_{L^2(0, T; L^2(J))} = 1
\]

Therefore (33) is fair. \( \square \)
Lemma 3.5. Suppose that \( a = a(x) \) satisfies (7) and \( u_0 \in D(A) \) with \( \|u_0\|_{L^2(J)} \leq r \). Then for any \( T > 0 \), there exists \( c = c(T) > 0 \) such that

\[
\|u_0\|^2_{L^2([0,L])} \leq c \left\{ \int_0^T |u_x(t,0)|^2 dt + \int_0^T \int_0^L sat(u) \cdot udxdt \right\}
\]  

(52)

Proof. By using lemmas 3.3 and 3.4, we have that

\[
\int_0^L |u_0|^2 dx \leq \frac{1}{T} \int_0^T \int_0^L |u(t,x)|^2 dx dt + \int_0^T |u_x(t,0)|^2 dt + 2 \int_0^T \int_0^L sat(u) \cdot udxdt
\]

\[
\leq \frac{1}{T} \left( c_1 \int_0^T |u_x(t,0)|^2 dt + 2c_1 \int_0^T \int_0^L sat(u) \cdot udxdt \right) + \int_0^T |u_x(t,0)|^2 dt + 2 \int_0^T \int_0^L sat(u) \cdot udxdt
\]

\[
= \left( 1 + \frac{c_1}{T} \right) \int_0^T |u_x(t,0)|^2 dt + 2 \left( 1 + \frac{c_1}{T} \right) \int_0^T \int_0^L sat(u) \cdot udxdt
\]

\[
= c(T) \left\{ \int_0^T |u_x(t,0)|^2 dt + 2 \int_0^T \int_0^L sat(u) \cdot udxdt \right\}
\]

with \( c(T) = 1 + \frac{c_1}{T} \).

Thus it concludes the proof of Lemma 3.5. \( \Box \)

Remark 6. Our proof of Lemmas 3.2, 3.3 and 3.4 follows the same strategy of the proof of [12, Theorem 2.2]. Unlike [12] where the authors assume that \( f(t, x) = a(x)u(t, x) \), in this paper the control operator is given by \( f(t, x) = sat(a(x)u(t, x)) \).

Lemma 3.6. (semiglobal exponential stability). For any \( r > 0 \) there exist two constants \( \mu_r = \mu_r(r) > 0 \) and \( K_r = K_r(r) > 0 \) such that for any \( u_0 \in D(A) \) satisfying \( \|u_0\|_{L^2(J)} \leq r \), the solution \( u = u(t, x) \) to (6) satisfies

\[
\|u(t, \cdot)\|_{L^2(J)} \leq K_r \|u_0\|_{L^2(J)} e^{-\mu_r t} \quad \forall t \geq 0.
\]

Proof. By using lemma 3.5, we have for any \( u_0 \in D(A) \), with \( \|u_0\|_{L^2(J)} \leq r \), (52) holds. Then we deduce from (52) and (23) that there exists a constant \( \lambda \in [0, 1] \) such that

\[
\|u(kT, \cdot)\|^2_{L^2(J)} \leq \lambda k \|u_0\|^2_{L^2(J)} \quad \forall k \geq 0.
\]

Since

\[
\|u(t, \cdot)\|_{L^2(J)} \leq \|u(kT, \cdot)\|_{L^2(J)} \quad \text{for } kT \leq t \leq (k + 1)T,
\]

then we get

\[
\|u(t, \cdot)\|^2_{L^2(J)} \leq \frac{1}{\lambda} \|u_0\|^2_{L^2(J)} e^{\frac{\log(\lambda)}{T}},
\]

from which the result stated in Lemma 3.6 follows. \( \Box \)

Now we are able to state and prove the main result of this section.

Theorem 3.7. Suppose that \( a = a(x) \) satisfies (7). Then for any \( L > 0 \), there exists \( \mu > 0 \) and for all \( u_0 \in L^2(J) \) there is \( c_2(u_0) > 0 \), such that

\[
\|u(t, \cdot)\|^2_{L^2(J)} \leq c_2(u_0) \|u_0\|^2_{L^2(J)} e^{-\mu t}
\]  

(53)
for all \( t \geq 0 \) and for any weak solution of (6).

**Proof.** Let \( u_0 \in D(A) \). By lemma 3.6 and (20) if
\[
\|u_0\|_{L^2(J)} \leq \frac{1}{a_1},
\]
then, for all \( t \geq 0 \),
\[
\|u(t,\cdot)\|_{L^2(J)} \leq \|u_0\|_{L^2(J)} \leq \frac{1}{a_1}.
\] (54)

Thus, \( sat(au) = au \), then from [12, Theorem 2.2], the corresponding solution \( u \) to (6) satisfies
\[
\|u(t,\cdot)\|_{L^2(J)} \leq \|u_0\|_{L^2(J)} e^{-\mu t}.
\] (55)

In addition, for a given \( r > 0 \), there exists a positive constant \( \mu_r \) such that if \( \|u_0\|_{L^2(J)} \leq r \), then any classical solution \( u \) to (6) satisfies
\[
\|u(t,\cdot)\|_{L^2(J)} \leq \|u_0\|_{L^2(J)} e^{-\mu_r t}.
\] (56)

Therefore, setting \( T_r = \mu_r^{-1} \ln(a_1 r) \), we have
\[
\|u_0\|_{L^2(J)} \leq r \Rightarrow \|u(T_r,\cdot)\|_{L^2(J)} e^{-\mu(T_r) r} = \frac{1}{a_1}.
\]

Thus using (55), we obtain
\[
\|u(t,\cdot)\|_{L^2(J)} \leq \|u(T_r,\cdot)\|_{L^2(J)} e^{-\mu(T_r) t} \quad \forall t \geq T_r
\]
\[
\leq \|u_0\|_{L^2(J)} e^{-\mu t} e^{\mu T_r} \forall t \geq 0.
\] (57)

Therefore (6) is exponentially stable in \( D(A) \). By using the fact that \( D(A) = L^2(J) \), we conclude that (6) is exponentially stable in \( L^2(J) \).

**Remark 7.** Our proof of Theorem 3.7 follows the same strategy of the proof of [17, Theorem 3.7].

4. **Conclusion.** In this paper it is proved that the KdV equation (1) in closed-loop system with the saturated control (4) is well-posed and is exponentially stable. Thanks to the theory of the nonlinear semigroup, the well-posedness of the closed-loop system has been demonstrated. The exponential stability is proved by using the classical argument used in this type of problems.

A possible future research line could be the exponential stability of the linear KdV equation in closed-loop system with a saturating control in \( L^p \) with \( p \geq 1 \).

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