LOCAL FINITENESS OF THE CURVE GRAPH VIA
SUBSURFACE PROJECTIONS AND A UNIFORM BOUND OF
TIGHT GEODESICS

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Abstract. We prove that if \( \xi(S) \geq 1 \), given \( l > 0 \) and \( k > 1 \), there exist a computable \( N_S(l, k) \) such that if \( C \subseteq C(S) \) and \( |C| > N_S(l, k) \), there exists \( C' \subseteq C \) such that \( |C'| \geq k \) and \( Z \subseteq S \) so that \( d_Z(x, y) > l \) for all \( x, y \in C' \).

As a corollary, for \( \xi(S) > 1 \), we compute a uniform bound of the slice of tight geodesics between any pair of curves in the sense of Bowditch [Bow08].

1. Introduction

Let \( S_{g,n} \) be a surface with \( g \) genus and \( n \) boundary components and \( \xi(S_{g,n}) = 3g + n - 3 \) be the complexity of \( S_{g,n} \). In this paper, we always assume that all curves are simple, closed, essential, and non-peripheral. Harvey introduced the curve complex [Har81]. If \( \xi(S) > 1 \), the vertices are isotopy classes of curves in \( S \), the simplices are collections of curves that can be mutually realized as disjoint curves in \( S \). If \( \xi(S) = 1 \), any two distinct curves intersect at least once. The vertices of the curve complex are isotopy classes of curves on \( S \), the simplices are collections of curves that mutually intersect the minimal possible number. We observe the dimension of \( C(S) \) is bounded by \( \xi(S) - 1 \). Indeed, our main object in this paper will be the 1-skelton of \( C(S) \), the curve graph which is a locally infinite and connected graph. In this paper, we mean \( a \in C(S) \) as \( a \) is an element of 0-skelton of \( C(S) \). Let \( x, y \in C(S) \), the distance between \( x \) and \( y \), \( d_S(x, y) \) is the minimal number of edges in any edge path in \( C(S) \) between \( x \) and \( y \) and we call the edges which realize the minimal number geodesics. With this metric, the curve graph is hyperbolic in the sense of Gromov, which was first proved by Masur and Minsky [MM99]. By different approaches, Bowditch [Bow06] and Hamenstädt [Ham07] also proved this result. Recently, the hyperbolicity has been improved so that it is uniform for all surfaces. This result was independently proved by Aougab [Aou], by Bowditch [Bow], by Clay, Rafi and Schleimer [MCS] and by Hensel, Przytycki and Webb [HPW].

One of the main results of this paper is to overcome the fact that the curve graph is locally infinite. We show that the curve graph is indeed “locally finite” under the subsurface projection defined by Masur and Minsky [MM00] in the following sense.

Theorem 1.1. Suppose \( \xi(S) \geq 1 \), given \( l > 0 \) and \( k > 1 \), there exist a computable \( N_S(l, k) \) such that if \( C \subseteq C(S) \) and \( |C| > N_S(l, k) \), then there exists \( C' \subseteq C \) such that \( |C'| \geq k \) and \( Z \subseteq S \) so that \( d_Z(x, y) > l \) for all \( x, y \in C' \).

We observe that if the curve graph was locally finite, the statement would be simply true when \( Z = S \).
As a direct corollary of this, we show there exists a uniform bound of the number of tight geodesics between any pair of curves. We recall the results on tight geodesics prior to this paper and some applications of the results. Tight geodesics were introduced by Masur and Minsky, and they proved that there exists at least one and only finitely many tight geodesics between any pair of curves [MM00]. Bowditch defined a slice of the union of tight geodesics between a pair of curves and showed that there exists a uniform but non-computable bound for the slice by using 3-dimensional hyperbolic geometry [Bow08], we also refer [Bow07] which involves a geometric limit argument. Schackleton showed that there exists a computable bound for the slice which depends on the intersection number of a given pair of curves [Sha12], the idea of his proof extends to show that there exists an algorithm to compute the distance between a given pair of curves. For this result, we also refer the unpublished thesis of Leasure [Lea02]. Independently, Webb [Weba] and the author [Wat] showed that there exists a computable bound for the slice which depends on the distance between a given pair of curves, and it is one of the key points in [Weba], where he showed that there exists a uniform and computable bound for the slice by a combinatorial argument. These studies of tight geodesics have many applications, including Thurston’s ending lamination conjecture [BCM12] [Min10], the asymptotic dimension of the curve graph [BF08] and the stable lengths of pseudo-Anosov elements [Bow08] [Sha12] [Weba].

We review the definition of a slice [Bow08] and state our result. Let \( N_i(x) \) denote the \( i \)-ball around \( x \in C(S) \). Suppose \( a, b \in C(S) \), let \( \mathcal{L}_T(a, b) \) be the set of all tight geodesics between \( a \) and \( b \), and \( G(a, b) = \cup \mathcal{L}_T(a, b) \subseteq C(S) \). Also, suppose \( A, B \subseteq C(S) \), let \( \mathcal{L}_T(A, B) = \cup_{a \in A, b \in B} \mathcal{L}_T(a, b) \) and \( G(A, B) = \cup_{a \in A, b \in B} G(a, b) \subseteq C(S) \). Suppose \( a, b \in C(S) \) and \( r > 0 \), let \( G(a, b; r) = G(N_r(a), N_r(b)) \).

The following result is due to Bowditch [Bow08] without computable bounds. Here, we state the recent result by Webb.

Suppose \( a, b \in C(S) \), we let \( g_{a,b} \) be a geodesic between \( a \) and \( b \).

**Theorem 1.2 (Weba).** Suppose \( \xi(S) > 1 \).

1. For any \( a, b \in C(S) \) and any \( c \in C(S) \) such that \( c \) lies on some \( g_{a,b} \),
   \[ |G(a, b) \cap N_\delta(c)| \leq K^{\xi(S)} \] where \( K \) is a uniform constant.

2. For any \( r \geq 0 \), \( a, b \in C(S) \) such that \( d_S(a, b) \geq 2r + 2j + 1 \) (where \( j = 10\delta + 1 \)), then for any \( c \in C(S) \) such that \( c \) lies on some \( g_{a,b} \) and \( c \notin N_{r+j}(a) \cup N_{r+j}(b) \), \[ |G(a, b; r) \cap N\delta(c)| \leq K^{\xi(S)} \] where \( K \) is a uniform constant.

We observe that \( N_\delta(c) \) intersects all (tight) geodesics between \( a \) and \( b \), \( N_\delta(c) \) intersects all (tight) geodesics between \( N_r(a) \) and \( N_r(b) \) by the hyperbolicity of the curve graph, so we call \( G(a, b) \cap N_\delta(c) \) and \( G(a, b; r) \cap N\delta(c) \) slices.

We also show that there exist a uniform and computable bounds for the slices, our bound will be weaker, yet the proof will be simpler as it is a direct corollary of Theorem 1.1. Also, the hypothesis of the second statement will be weaker, i.e. \( j \) will be \( 3\delta + 2 \) instead of \( 10\delta + 1 \). We will prove

**Theorem 1.3.** Suppose \( \xi(S) > 1 \).
For any $a, b \in C(S)$ and any $c \in C(S)$ such that $c$ lies on some $g_{a,b}$,

$$|G(a, b) \cap N_4(c)| \leq K_1^{(K_2^{(S)})}$$

where $K_1$ and $K_2$ are uniform constants.

For any $r \geq 0$, for any $a, b \in C(S)$ such that $d_S(a, b) \geq 2r + 2j + 1$ (where $j = 3\delta + 2$), then for any $c \in C(S)$ such that $c$ lies on some $g_{a,b}$ and $c \notin N_{r+j}(a) \cup N_{r+j}(b)$,

$$|G(a, b; r) \cap N_{2\delta}(c)| \leq K_1^{(K_2^{(S)})}$$

where $K_1$ and $K_2$ are uniform constants.

Theorem 1.1 and Theorem 1.3 will be respectively written as Theorem A and Theorem B in the rest of this paper.

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## 2. Background

In this section, we recall the definitions of tight geodesics and subsurface projections. Let $A, B \subset C(S)$, we define $d_S(A, B) = \max_{a \in A, b \in B} d_S(a, b)$. A multicurve $V \subset C(S)$ is the set of curves that form a simplex in $C(S)$. Suppose $V$ and $W$ are multicurves, we say $V$ and $W$ fill $S$ if there is no curve in $S - (V \cup W)$. If $\xi(S) > 1$, we observe that $d_S(V, W) \geq 3$ if and only if $V$ and $W$ fill $S$. A unique subsurface that $V$ and $W$ fill can be constructed by first taking the regular neighborhood of $V \cup W$, and filling in a disk for every component in the compliment of the regular neighborhood which is a disk. We denote the subsurface as $S(V, W)$.

**Definition 2.1.** A multigeodesic is a sequence of multicurves $V_0, V_1, \ldots, V_k$ such that $d_S(x, y) = |j - i|$ for all $i \neq j$ and all $x \in V_i$ and $y \in V_j$.

**Definition 2.2.** A tight geodesic is a multigeodesic $V_0, V_1, \ldots, V_k$ such that $V_i = \partial S(V_{i-1}, V_{i+1})$ for all $0 < i < k$.

If $\xi(S) \leq 1$, every geodesic is defined to be a tight geodesic.

**Theorem 2.3 (MM00).** There exists a tight geodesic between any two vertices in $C(S)$.

Let $x, y \in C(S)$ such that $d_S(x, y) = k$, then a tight geodesic between $x$ and $y$ is a tight geodesic $V_0, V_1, \ldots, V_k$ with $V_0 = \{x\}$ and $V_k = \{y\}$ and by choosing the starting point, denoting $X_0^T = \{x\}$, $X_i^T$ is the set of multicurves that are on a tight geodesic between $x$ and $y$ and distance $i$ apart from $x$.

We define the subsurface projection. First, we define the close relatives of the curve complex, the arc complex, $A(S)$ and the arc and curve complex, $AC(S)$. In this paper, we always assume that all arcs are simple, essential, and non-peripheral. We always assume isotopy of arcs is relative to the boundaries setwise unless we say relative to the boundaries pointwise.
Suppose $\xi(S) \geq 0$, the vertices of arc complex (arc and curve complex) are isotopy classes of arcs (arcs and curves) and the simplices are collections of arcs (arcs and curves) that can be mutually realized to be disjoint in $S$.

In this paper, $a \in A(S)$ and $a \in AC(S)$ means that $a$ is an element of 0-skelton in each complex.

**Definition 2.4.** Let $x, y \in AC(S)$, the intersection number, $i(x, y)$ is the minimal number of $|x \cap y|$ under the isotopy classes of $x$ and $y$. We say $x$ and $y$ are in minimal position if they realize the intersection number.

Let $A \subset S$, then $R(A)$ is a regular neighborhood of $A$ in $S$. Let $\mathcal{P}(C(S))$ and $\mathcal{P}(AC(S))$ be the set of finite subsets in each complex. Now, we define the subsurface projection.

Let $Z$ be a subsurface of $S$ such that $\xi(Z) > 0$. Let $x \in AC(S)$, assume $x$ and $\partial Z$ are in minimal position then we define the map, $i_Z : AC(S) \to \mathcal{P}(AC(Z))$ by taking isotopy classes of $\{x \cap Z\}$. Also define the map, $p_Z : AC(Z) \to \mathcal{P}(C(Z))$ as following.

- If $x \in C(Z), p_Z(x) = x$.
- If $x \in A(Z)$ and $\partial x$ lies on two distinct boundaries of $Z$, $z, z' \subset \partial Z$, then $p_Z(x) = \partial R(x \cup z \cup z')$.
- If $x \in A(Z)$ and $\partial x$ lies on one boundary of $Z$, $z \in \partial Z$, then $p_Z(x) = \partial R(x \cup z)$. We observe that $p_Z(x)$ has at most two components.

If $C \subset AC(S)$, we define $p_Z(C) = \cup_{c \in C} p_Z(c)$.

The subsurface projection on $Z$ is the map, $\pi_Z = p_Z \circ i_Z : AC(S) \to \mathcal{P}(C(Z))$. We observe that $\pi_S = p_S$.

If $C \subset AC(S)$, we define $\pi_Z(C) = \cup_{c \in C} \pi_Z(c)$. If $A, B \subset AC(S)$, we define $d_Z(A, B) = \max_{a \in \pi_Z(A), b \in \pi_Z(B)} d_Z(a, b)$.

Suppose $Z$ is an essential annulus in $S$ and if the core curve of $Z$ is $a \subset C(S)$ then we let $S^Z$ be the annular cover of $S$ determined by $\pi_1(a)$ such that it is compactified with $\partial \mathbb{H}^2$. We define the vertices of $C(Z)$ to be the set of isotopy classes of arcs whose endpoints lie on two boundaries of $S^Z$, here the isotopy is relative to $\partial S^Z$ pointwise. We define the metric, $x, y \in C(Z)$ are distance one apart if they are disjoint in the interior of $S^Z$. By fixing an orientation of $S$ and an ordering the components of $\partial S^Z$, algebraic intersection number of $x$ and $y$, $x \cdot y$ is well defined and we observe that $d_Z(x, y) = |x \cdot y| + 1$ by induction.

Suppose $b \in AC(S)$ such that $i(a, b) > 0$, then there exists $\beta$ which is a lift of $b$ in $S^Z$ which connects two boundaries of $S^Z$. We define the subsurface projection on $Z$, which is the map, $\pi_Z : AC(S) \to \mathcal{P}(C(Z))$. If $b \in AC(S)$ such that $i(a, b) > 0$, then $\pi_Z(b)$ is the set of all arcs obtained by the lift of $b$ which connects two boundaries of $S^Z$. If $i(a, b) = 0$, $\pi_Z(b) = \emptyset$.

As in the previous case, if $C \subset AC(S)$, we define $\pi_Z(C) = \cup_{c \in C} \pi_Z(c)$. If $A, B \subset AC(S)$, we define $d_Z(A, B) = \max_{a \in \pi_Z(A), b \in \pi_Z(B)} d_Z(a, b)$.

**Lemma 2.5 (MM00).** Suppose $Z$ is an essential annulus in $S$ and the core curve of $Z$ is $x \subset C(S)$. Let $T_x$ be the dehn twist of $x$, if $y \subset C(S)$ is such that $\pi_Z(y) \neq \emptyset$ then $d_Z(y, T_x^*(y)) = |n| + 2$ for $n \neq 0$.

If $y$ intersects $x$ exactly twice with opposite orientation, a half twist to $y$ is well defined to obtain a curve $H_x(y)$, which is taking $x \cup y$ and resolving the intersections
in a way consistent with the orientation (see [Luo10] for a generalization). Then
\[ H^2_x(y) = T_x(y), \text{ and } d_Z(y, H^n_x(y)) = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ for } n \neq 0. \]

Suppose \( Z \subseteq S \), we say \( C \subseteq AC(S) \) projects nontrivially on \( Z \) if \( \pi_Z(C) \neq \emptyset \).

Lastly, we observe the Bounded Geodesic Image Theorem. It was first proved by Masur and Minsky [MM00]. A more direct approach by Webb shows that the bound depends only on the hyperbolicity constant, which implies that the bound is uniform for all surfaces. Here, we state Webb’s version of the theorem.

**Theorem 2.6** (Webb). (Bounded Geodesic Image Theorem) Let \( \delta \) be a hyperbolicity constant of the curve graph of \( S \). There exists \( M(\delta) \) such that if \( \{x_i\}_{0}^{n} \) is a geodesic and \( x_i \) projects nontrivially on \( Z \subset S \) for all \( 0 \leq i \leq n \), then \( d_Z(x_0, x_n) \leq M \).

In the rest of the paper, we mean \( M \) as \( M \) in the statement of the Bounded Geodesic Image Theorem.

### 3. Outline

**Definition 3.1.** Suppose \( C \subseteq C(S) \). Let \( l > 0 \), \( k > 1 \) and \( Z \subseteq S \), we say \( C \) satisfies the property \( P(l, k, Z) \) if there do not exist more than \( k - 1 \) curves in \( C \) whose projections on \( Z \) are mutually more than \( l \) apart in \( C(Z) \). We say \( C \) satisfies the property \( P(l, k) \) if \( C \) satisfies the property \( P(l, k, Z) \) for all \( Z \subseteq S \).

First, we prove the following theorem.

**Theorem A.** Suppose \( \xi(S) \geq 1 \). Given \( l > 0 \) and \( k > 1 \), if \( A \subseteq C(S) \) satisfies \( P(l, k) \), then there exists a computable \( N_S(l, k) \) such that \( |A| \leq N_S(l, k) \).

Let \( L = l + 2M \) and \( N_{S_0,4}(l, k) = (2Lk)^{l+1} \). We will show that \( N_S(l, k) = (2N_{S_0,4}(L, k))^{((l+1)^{\xi(S)})} \).

With Theorem [A] we have Theorem [B]. For the readers’ convenience, we state the theorem with explicit bounds in this section. The bounds will follow from the computation given in Section 5.

**Theorem B.** Suppose \( \xi(S) > 1 \).

1. For any \( a, b \in C(S) \) and any \( c \in C(S) \) such that \( c \) lies on some \( g_{a,b} \),
   \[ |G(a, b) \cap N_S(c)| \leq (2N_{S_0,4}(4M, 3))^{((2M+1)^{\xi(S)})}. \]

2. For any \( r \geq 0 \), for any \( a, b \in C(S) \) such that \( d_S(a, b) > 2r + 2j + 1 \) (where \( j = 3\delta + 2 \)), then for any \( c \in C(S) \) such that \( c \) lies on some \( g_{a,b} \) and \( c \notin N_{r+j}(a) \cup N_{r+j}(b) \),
   \[ |G(a, b; r) \cap N_{2\delta}(c)| \leq (2N_{S_0,4}(6M, 3))^{((4M+1)^{\xi(S)})}. \]
4. Theorem A implies Theorem B

We observe the following important property for tight geodesics.

Lemma 4.1. Suppose $\xi(S) > 1$. Let $x, y \in C(S)$ and consider tight geodesics between $x$ and $y$ with $x = X_0^T$, let $V \in X_j^T$ for some $0 < j < d_S(x, y)$. If $\pi_Z(V) \neq \emptyset$ for some $Z \subseteq S$, then $d_Z(x, V) \leq M$ or $d_Z(V, y) \leq M$.

Proof. Let $\{V_i\}$ be a tight geodesic between $x$ and $y$ which contains $V$. By Definition 2.2, $\pi_Z(V_{j+1}) \neq \emptyset$ or $\pi_Z(V_{j+1}) \neq \emptyset$. Without loss of generality, if $\pi_Z(V_k) = \emptyset$ for some $k \leq j - 1$, with the fact that two curves which are distance more than two apart fill $S$, $\pi_Z(V_n) \neq \emptyset$ for all $n \geq j + 1$, so $d_Z(V, y) \leq M$ by the Bounded Geodesic Image Theorem. □

Definition 4.2. Suppose $x, y \in C(S)$, we let $g^i_{x, y}$ be a tight geodesic between $x$ and $y$. Here, a tight geodesic is a geodesic such that the vertex which is $i$ apart from $x$ is contained in the multicurve $V_i$ in the sense of Definition 2.2 where $V_0 = \{x\}$ for all $i$.

By Lemma 4.1, we have

Corollary 4.3. Suppose $\xi(S) > 1$.

1. For any $a, b \in C(S)$ and any $c \in C(S)$ such that $c$ lies on some $g_{a, b}$, $G(a, b) \cap N_{\delta}(c)$ satisfies $P(2M, 3)$.

2. For any $r \geq 0$, $a, b \in C(S)$ such that $d_S(a, b) > 2r + 2j + 1$ (where $j = 3\delta + 2$), then for any $c \in C(S)$ such that $c$ lies on some $g_{a, b}$ and $c \notin N_{r+j}(a) \cup N_{r+j}(b)$, $G(a, b; r) \cap N_{2\delta}(c)$ satisfies $P(4M, 3)$.

Proof. It suffices to show that for any $x$ in the slices, if $x$ projects nontrivially on some $Z \subseteq S$, $d_Z(x, a) \leq P$ or $d_Z(x, b) \leq P$ where $P = M$ for the first statement and $P = 2M$ for the second statement.

Since $M > 4\delta$ [Webb], we may exclude the case when $Z = S$.

The first statement directly follows from Lemma 4.1.

For the second statement, we also recall the fact that two curves which are distance more than two apart fill $S$.

Let $x_a \in N_r(a)$, $x_b \in N_r(b)$ such that there exist $g^i_{x_a, x_b}$ which contains $x$ and Lemma 4.1 applies to. Also, we may assume that $g^i_{x_a, x_b} \cap (N_r(a) \cup N_r(b)) = \{x_a, x_b\}.$

Let $g^i_{x_a, x_b}$ be the subsegment of $g^i_{x_a, x_b}$ between $x_a$ and $x$, and $g^i_{x, x_b}$ be the subsegment of $g^i_{x_a, x_b}$ between $x$ and $x_b$. If every vertex of $g^i_{x, x_b} \cup g_{c, x}$ projects nontrivially on $Z$, we are done since $d_Z(x, b) \leq d_Z(x, x_a) + d_Z(x_a, b) \leq 2M$. Therefore, we consider the case when there exists $q \in g^i_{x_a, x_b} \cup g_{c, x}$ such that $\pi_Z(q) = \emptyset$. In this case, we claim that $d_Z(q, x) \leq 2M$.

Case 1: Suppose $q \in g^i_{x_a, x_b} \setminus \{x \cup x_b\}$. By Lemma 4.1, we have $d_Z(x_a, x) \leq M$. Therefore, it suffices to show that every vertex of $g_{c, x}$ projects nontrivially on $Z$, which is to say $q \notin N_{r+2}(a)$. We claim $q \in N_{d_S(c, b)} + 3\delta(b)$, then by the hypothesis of $c$, we are done since $N_{d_S(c, b) + 3\delta}(b) = \emptyset$. To show the claim, we consider the 4-gon whose edges are $g^i_{x, x_a}$, $g_{x_a, b}$, $g_{b, c}$ and $g_{c, x}$. Additional geodesic, $g_{c, x}$ decomposes the 4-gon into two triangles, and by using the hyperbolicity, we have
q ∈ ND_{s(a,c)+δ}(a).

Case 2: Suppose q ∈ g_{x,b}. As in the case 1, we have g_{x,b} ⊆ ND_{s(a,c)+δ}(a).

By the hypothesis of c, we are done since ND_{s(a,c)+δ}(a) ∩ N_{r+2}(b) = ∅.

We observe that Theorem B follows from Corollary 4.3 and Theorem A.

5. The proof of Theorem A

We prove Theorem A by double induction. We first show for S_{1,1} and S_{0,4}, the curve graphs of them are agreeable graphs where vertices are identified with \( \mathbb{Q} \cup \{ \frac{1}{\pm} = \infty \} \). There is an isometry between C(S_{1,1}) and C(S_{0,4}). For a detailed treatment, see [FM12].

Suppose x ∈ C(S), we let C_{i}(x) = \{ y ∈ C(S) | d_{S}(x, y) = i \}.

We observe the following lemma which is the heart of the proof of Theorem A.

Lemma 5.1. Suppose \( \xi(S) \geq 1 \). Let x ∈ C(S) and B ⊆ C_{i}(x) for i > 1.

Let Z ⊆ S such that

\begin{itemize}
  \item if \( \xi(S) = 1 \), Z = R(x).
  \item if \( \xi(S) > 1 \), Z ⊆ S - x.
\end{itemize}

Then, if B satisfies \( P(l, k, Z) \), there exist B' ⊆ C_{1}(S) which satisfies \( P(l + 2M, k, Z) \) and with B ⊆ ∪_{y ∈ B'}C_{1}(y).

Proof. The proof will be the combination of tightness, Lemma 4.1 (for \( \xi(S) > 1 \)) and the Bounded Geodesic Image Theorem.

If \( \xi(S) = 1 \): Let b ∈ B, then every vertex of g_{x,b} - \{ x \} projects nontrivially on R(x). Therefore, if \{ b' \} = g_{x,b} ∩ C_{1}(x), we have d_{R(x)}(b', b) ≤ M by the Bounded Geodesic Image Theorem. Furthermore, suppose c ∈ B then if \{ c' \} = g_{x,c} ∩ C_{1}(x), we have

\[
  d_{R(x)}(b', c') \leq d_{R(x)}(b', b) + d_{R(x)}(b, c) + d_{R(x)}(c, c') \leq M + l + M = l + 2M.
\]

Therefore, B' = ∪_{w ∈ B}g_{x,w} ∩ C_{1}(x) satisfies \( P(l + 2M, k, R(x)) \) and by the definition of B', B ⊆ ∪_{y ∈ B'}C_{1}(y).

If \( \xi(S) > 1 \): Let b ∈ B, then by Lemma 4.1 we may take a \( g_{x,b} \) so that if \{ b' \} = g_{x,b} ∩ C_{1}(x) projects nontrivially on Z, every vertex of \( g_{x,b} - \{ x \} \) projects nontrivially on Z. As in the previous case, B' = ∪_{w ∈ B}g_{x,w} ∩ C_{1}(x) satisfies \( P(l + 2M, k, Z) \) and B ⊆ ∪_{y ∈ B'}C_{1}(y).

First, we prove Theorem A for \( \xi(S) = 1 \).

Theorem 5.2. Suppose S = S_{1,1}. Given l > 0 and k > 1, if A ⊆ C(S) satisfies \( P(l, k) \), then |A| ≤ (Lk)^{l+1}, where L = l + 2M.

Proof. Since A satisfies \( P(l, k, S) \), there would not be more than \( k-1 \) curves in A which are mutually more than l apart in C(S), so it suffices to understand a bound for A ∩ N_{l}(x) where x ∈ C(S).
Let \( L = l + 2M \). We claim that \(|A \cap C_i(x)| \leq (Lk)^i\) for all \( 1 \leq i \leq l \) by induction on \( i \). With the claim, we observe that

\[
|A| \leq (1 + \sum_{i=1}^{l} (Lk)^i) \cdot (k-1) = \left( \sum_{i=0}^{l} (Lk)^i \right) \cdot (k-1) = \left( \frac{(Lk)^{i+1} - 1}{Lk - 1} \right) \cdot (k-1) \leq (Lk)^{i+1}.
\]

**Base case:** If \( \frac{p}{q}, \frac{p}{q} \in C(S_{1,1}), i(\frac{p}{q}, \frac{p}{q}) = |sq - tp| \). We may assume \( x = \frac{1}{q} \) then \( C_1(x) = \mathbb{Z} \), let \( T_x \) be the dehn twist along \( x \). Suppose \( y \in C_1(x) \) then \( d_{R(x)}(T_x^y(y), y) = |i| + 2 \) by Lemma 2.5 so we have

\[
|A \cap C_1(x)| \leq (l - 1)(k-1) \leq Lk.
\]

**Inductive step:** Let \( B = A \cap C_i(x) \), then \( B \) satisfies \( \mathcal{P}(l, k, R(x)) \). By Lemma 5.1 there exists \( B' \) which satisfies \( \mathcal{P}(l + 2M, k, R(x)) \) and with \( B \subseteq \bigcup_{y \in B'} C_{i-1}(y) \). By the base case, we have

\[
|B'| \leq (l + 2M - 1)(k-1) \leq Lk.
\]

With our inductive hypothesis, we have

\[
|B| \leq (Lk) \cdot (Lk)^{i-1} \leq (Lk)^i.
\]

\( \square \)

For \( S_{0,4} \), analogous proof works, the only difference is that we use the half twist along \( x \), \( H_x \) instead of \( T_x \) and use Lemma 2.5. With the same setting as in Theorem 5.2 we have \(|A \cap C_i(x)| \leq (2(L - 1)k)^i \leq (2Lk)^i\) for all \( 1 \leq i \leq l \). Therefore,

\[
N_{S_{0,4}}(l, k) = (2Lk)^{l+1}.
\]

Now, we complete the proof of Theorem A.

**Proof of Theorem A.** Let \( N_S(l, k) = \max_{\xi \in S_{g,n} \leq \xi(S)} N_{S_{g,n}}(l, k) \).

Since \( A \) satisfies \( \mathcal{P}(l, k, S) \), it suffices to understand a bound for \( A \cap N_i(x) \) where \( x \in C(S) \).

Let \( L = l + 2M \). We claim that \(|A \cap C_i(x)| \leq (2N_S(L, k))^i\) for all \( 1 \leq i \leq l \) by induction on \( i \). With the claim, we observe that

\[
|A| \leq \left( 1 + \sum_{i=1}^{l} (2N_S(L, k))^i \right) \cdot (k-1) = \left( \sum_{i=0}^{l} (2N_S(L, k))^i \right) \cdot (k-1) = \left( \frac{(2N_S(L, k))^{i+1} - 1}{2N_S(L, k) - 1} \right) \cdot (k-1).
\]

Since \( k - 1 \leq 2N_S(L, k) - 1 \), we have

\[
|A| \leq (2N_S(L, k))^{l+1}.
\]
Base case: Suppose $S - x = \{S_1, S_2\}$, then we may assume $\xi(S_1), \xi(S_2) \geq 1$ since $C(S_{0,3}) = \emptyset$. If $S - x$ has one component, we let $S - x = \{S_1\}$, and treat $S_2 = \emptyset$.

Therefore,

$$|A \cap C_1(x)| \leq N_{S_1}(l, k) + N_{S_2}(l, k) \leq 2N_{S'}(l, k) \leq 2N_{S'}(L, k).$$

Inductive step: Let $B = A \cap C_1(x)$, since $B$ satisfies $\mathcal{P}(l + 2M, k, Z)$ for all $Z \subseteq S - x$, by Lemma 5.1 there exists $B' \subseteq C_1(x)$ which satisfies $\mathcal{P}(l + 2M, k, Z)$ for all $Z \subseteq S - x$ and with $B \subseteq \cup_{y \in B'} C_{i-1}(y)$. Therefore,

$$|B'| \leq 2N_{S'}(l + 2M, k) = 2N_{S'}(L, k)$$

and we have

$$|B| \leq 2N_{S'}(L, k) \cdot (2N_{S'}(L, k))^{i-1} = (2N_{S'}(L, k))^i.$$

Therefore, we have

$$|A| \leq (2N_{S'}(L, k))^{i+1}.$$

Now, we compute $(2N_{S'}(L, k))^{i+1}$. We observe that $N_{S_{0,4}}(l, k) \geq N_{S_{1,4}}(l, k)$.

Therefore, we have

$$N_S(l, k) = 2^{\sum_{i=0}^{\xi(S)} (l+1)^i} \cdot N_{S_{0,4}}(L, k) \left( (l+1)^{\xi(S)} \right)$$

$$\leq 2 \left( (l+1)^{\xi(S)} \right) \cdot N_{S_{0,4}}(L, k) \left( (l+1)^{\xi(S)} \right)$$

$$\leq \left( 2N_{S_{0,4}}(L, 3) \right)^{\xi(S)} \right).$$

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