Temperley-Lieb Algebra, Yang-Baxterization and universal Gate

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Received: date / Accepted: date

Abstract A method of constructing $n^2 \times n^2$ matrix realization of Temperley-Lieb algebras is presented. The single loop of these realizations are $d = \sqrt{n}$. In particular, a $9 \times 9$-matrix realization with single loop $d = \sqrt{3}$ is discussed. A unitary Yang-Baxter $\hat{R}(\theta, q_1, q_2)$ matrix is obtained via the Yang-Baxterization process. The entanglement properties and geometric properties (i.e., Berry Phase) of this Yang-Baxter system are explored.

Keywords Temperley-Lieb Algebra · Entanglement · Yang-Baxter system

PACS 03.65.Vf · 02.10.Kn · 03.67.Lx

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This work was supported by NSF of China (Grant No. 10875026).

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1 Introduction

Quantum Entanglement (QE)\[1, 2, 3, 4\], the most surprising non-classical property of a quantum system, plays a key role in quantum information and quantum computation processing. Similarly, topological entanglement (TE)\[5\] is described in terms of link diagrams and via the Artin braid group. There are natural relationships between QE and TE\[6, 7\]. Kauffman and his co-workers have explored the role of the unitary solutions to the Yang-Baxter Equation (YBE)\[8, 9, 10\] in quantum computation. According to their theories, the unitary Yang-Baxter $\hat{R}$ matrices are both universal for quantum computation and are also solutions to the condition for topological braiding. This motivates a novel way to study YBE (as well as braid relations)\[11, 12, 13, 14, 15, 16, 17\]. A set of size $4 \times 4$ universal quantum gates are constructed in terms of unitary $\hat{R}$ matrices, for example, the CNOT gate\[6\], DCNOT gate (i.e., Double CNOT gate)\[18\]. By means of universal $\hat{R}$ matrix, entanglement swapping and Yang-Baxter Hamiltonian are investigated in Ref.\[13\]. In Ref.\[15\], Chen et al. point out that all pure two-qudit entangled states can be achieved via a universal Yang-Baxter $\hat{R}$ matrix assisted by local unitary transformation. Later on, the geometric properties of this Yang-Baxter system is studied in Ref.\[19\].

Temperley-Lieb algebras (TLA) grew out of a study of solvable lattice models in two-dimensional Statistical Mechanics\[20\] and is related to link and knot invariants\[21\], a recent study\[22\] shows that TLA is found to present a suitable mathematical framework for describing quantum teleportation, entanglement swapping, universal quantum computation and quantum computation flow. Additionally, the systems of qutrits or more generally qudits are more powerful than the systems of qubits habitually used in quantum computer\[23, 24, 25, 26, 27, 28, 29\]. Due to the importance of TLA in quantum information processing, we find matrix realizations of TLA in high dimension in this paper. Consequently, by means of Yang-Baxterization approach, a family of universal $n^2 \times n^2$ $\hat{R}$ matrices associated with TLA can be constructed.

This paper is organized as follows: In Sec 2 we recall the method of constructing matrix realizations of TLA which is given by P.P.Kulish. Then we present a method of constructing $n^2 \times n^2$ matrix realizations of TLA with $n^3$ nonzero matrix elements. In Sec 3 a unitary $n^2 \times n^2$ Yang-Baxter $\hat{R}$ matrix is constructed via Yang-Baxterization\[30\] acting on the $n^2 \times n^2$ matrix realizations of TLA. In Sec 4 when $n=3$, we investigate the entanglement properties of $\hat{R}(\theta, q_1, q_2)$-matrix. We show that arbitrary degree of entanglement for two-qutrit entangled states can be generated via the unitary $\hat{R}(\theta, q_1, q_2)$-matrix acting on the standard basis. Then we can construct a Hamiltonian from the unitary $\hat{R}(\theta, q_1, q_2)$-matrix. Furthermore, the Berry phase of the system is investigated, and the results show that the Berry phase of this system can be interpreted under the framework of SU(2) algebra. This result is consistent with that given in Ref.\[19\].

2 An extended method of constructing realizations of TLA

In this paper, the matrix realization of TLA U-matrix and YBE solution $\hat{R}$-matrix are $n^2 \times n^2$ matrices acting on the tensor product space $\mathcal{V} \times \mathcal{V}$, where $\mathcal{V}$ is a $n$-dimensional vector space. As $U$ and $\hat{R}$ act on the tensor product $\mathcal{V}_i \times \mathcal{V}_{i+1}$, we denote them by $U_i$ and $\hat{R}_i$, respectively.
We first briefly review the theory of TLA [20]. For each natural number $m$, the TLA $TL_m(d)$ is generated by \{I, U_1, U_2 \cdots U_{m-1}\} with the TLA relations:

$$
\begin{align*}
U_i^2 &= dU_i & 1 \leq i \leq m - 1 \\
U_iU_{i \pm 1}U_i &= U_i & 1 \leq i \leq m \\
U_iU_j &= U_jU_i & |i - j| \geq 2
\end{align*}
$$

(1)

where the notation $U_i \equiv U_{i,i+1}$ is used. The $U_i$ represents $1_1 \otimes 1_2 \otimes \cdots \otimes 1_{i-1} \otimes U \otimes 1_{i+2} \otimes \cdots \otimes 1_m$, and $1_j$ represents the unit matrix in the $j$th space $V_j$. In topology, the parameter $d$ corresponds to a single loop “O”. In addition, the TLA is easily understood in terms of knot diagrams in Ref.[6].

In Ref.[31], P.P.Kulish et al. showed a method of constructing matrix realizations of TLA. Let us review it briefly. For a given invertible $n \times n$ matrix $A$, a $n^2 \times n^2$ matrix solution can be constructed in terms of $A$ and $A^{-1}$ with $U_{cd} = A_{ab}^cA_{de}^{-d}$. Hereafter, $U_{cd}$ denotes $U_{ab,cd}$ and $A_{ab}^c$ denotes $A_{a,b}$ with $a,b,c,d = 0,1,2, \cdots , n - 1$. One can verify that $U$ is a matrix realization of TLA. Let $Tr(M)$ denote the trace of matrix $M$, and $M^T$ denote the transpose of matrix $M$. In terms of $A$ and $A^{-1}$, the single loop $d$ can be determined by $d = Tr(A^TA^{-1})$. By means of this method, many realizations of TLA can be constructed. For example, we set

$$
A = \begin{pmatrix}
q^{1/2} & 0 & 0 & 0 \\
0 & q^{-1/2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix}
q^{-1/2} & 0 & 0 & 0 \\
0 & q^{1/2} & 0 & 0 \\
0 & 0 & q^{1/2} & 0 \\
0 & 0 & 0 & q^{1/2}
\end{pmatrix}.
$$

(2)

Then a $4 \times 4$ matrix realization of TLA can be constructed as follows

$$
U = \begin{pmatrix}
1 & 0 & 0 & q \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
q^{-1} & 0 & 0 & 1
\end{pmatrix}.
$$

However, not all solutions can be constructed with this method (for example, the solution associated with eight vertex model can’t be constructed). In the following, we introduce an extended method of constructing matrix realizations of TLA.

In order to construct a matrix realization of TLA, we introduce two $n \times n$ invertible matrices $A$ and $B$. We assume that matrix $U$ can be constructed as $U_{cd} = A_{ab}^cB_{de}^a$. Substituting this relation into TLA relations (1), the limited conditions for $A$ and $B$ can be derived. The relation $U_i^2 = dU_i$ yields $d = Tr(B^TA)$. Then $U$ is a realization of TLA relations (1) if and only if $A$ and $B$ respect the following conditions

$$
(BA)^T(AB) = (AB)(BA)^T = I_{n \times n}.
$$

(3)

Where $I_{n \times n}$ represents the unit matrix in $n$ dimension. In particular, if we take $B = A^{-1}$, we note that the condition (3) is obviously satisfied. Thus we re-obtain P.P.Kulish’s method of constructing matrix realizations of TLA.

In order to find matrices $A$ and $B$ satisfying these conditions, a special matrix structure is adopted, that is, each row and each column has one matrix element, and the matrix element locations are on the main diagonal symmetry. In addition, $A$ and $B$ satisfy the relation $B_{de}^a = (A_{ab}^c)^{-1}$ for the non-vanishing entries. In addition, we can verify that relations $A^TB = B^TA = AB = BA^T = BA = I_{n \times n}$ hold. In this case, Eq.(3) hold. Then we obtain a matrix realization of TLA. One can verify that $Tr(I_{n \times n}) = n$. 



In fact, we can select \( n \) matrices which satisfy these conditions, and all their matrix elements occupy different locations. Let \( i \) denote the \( i \)th matrix. Namely, the non-vanishing matrix elements of \( A^{(i)} \) are \( (A^{(i)})_{i-1}^{0}, (A^{(i)})_{i-2}^{1}, (A^{(i)})_{i-3}^{2}, \ldots, (A^{(i)})_{0}^{n-1}, (A^{(i)})_{n-1}^{(i)}=\ldots, (A^{(i)})_{n-1}^{n-1}, \ldots, (A^{(i)})_{n}^{0} \). For example, if \( n=4 \) and \( i=2 \), the non-vanishing matrix elements of \( A^{(2)} \) are \( (A^{(2)})_{1}^{0}, (A^{(2)})_{b}^{1}, (A^{(2)})_{a}^{2} \) and \( (A^{(2)})_{a}^{3} \). There are \( n \) invertible matrices \( B^{(i)} \) which is determined by \( [B^{(i)}]_{i,j} = [(A^{(i)})_{i,j}]^{-1} \) for non-vanishing matrix elements. In terms of matrices \( A^{(i)} \) and \( B^{(i)} \), a matrix representation of TLA can be constructed as \( U^{(i)} = [A^{(i)}]_{i,j} [B^{(i)}]_{i,j} \). By means of these \( n \) matrix realizations of TLA \( U^{(i)} \), we can construct a \( n^2 \times n^2 \) matrix realization of TLA with \( n^2 \) nonzero matrix elements. Taking the summation of these \( n \) matrices, we can obtain a combined matrix

\[
U = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U^{(i)}. \tag{4}
\]

In fact, the realizations of TLA matrix of this form can be represented in terms of Dirac’s “bra” and “ket”. This notation will appear in Sec.2. If we substitute Eq. (4) into Eqs. (1), the first equation in Eqs. (1) is satisfied automatically, and one can check that \( d = \sqrt{n} \). Then the other two relations are satisfied by the following limiting conditions,

\[
\sum_{j=1}^{n} (B^{(i)} A^{(j)})^T (A^{(k)} B^{(j)}) = 0_{n \times n} \tag{5}
\]

\[
\sum_{j=1}^{n} (A^{(j)} B^{(i)})(B^{(j)} A^{(k)})^T = 0_{n \times n}.
\]

Where \( i \neq k \) and \( i, k = 1, 2, \ldots, n \), and \( 0_{n} \) denotes \( n \times n \) matrix with all matrix elements are zero. This limiting condition together with the special matrix structure are used to determine \( U \) matrix. Two examples are shown to illustrate the application of this method in detail.

2.1 Example I: The case \( n = 2 \)

The simplest example which illustrates the method is the case \( n=2 \). According to the above analysis, when \( n = 2 \), we choose two sets of \( 2 \times 2 \) invertible matrices as follows,

\[
A^{(1)} = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} a_1^{-1} & 0 \\ 0 & b_1^{-1} \end{pmatrix}
\]

\[
A^{(2)} = \begin{pmatrix} 0 & a_2 \\ b_2 & 0 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 0 & a_2^{-1} \\ b_2^{-1} & 0 \end{pmatrix}.
\]

Where \( a_i \) and \( b_i \) are parameters which will be determined by the conditions in Eq. (5). Then two \( U \) matrices can be obtained as follows (we choose \( \{|00\rangle, |01\rangle, |10\rangle, |11\rangle \} \) as standard basis),

\[
U^{(1)} = \begin{pmatrix} 1 & 0 & 0 & a_1 b_1^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1^{-1} b_1 & 0 & 0 & 1 \end{pmatrix}, \quad U^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & a_2 b_2^{-1} & 0 \\ 0 & a_2^{-1} b_2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]
The trace of these two solutions is 2 (i.e., $d_1 = d_2 = 2$).

In order to obtain a solution associated with eight-vertex model, we consider the combinatorial structure of $U^{(1)}$ and $U^{(2)}$. The combinatorial form reads

$$ U = \frac{1}{\sqrt{2}} (U^{(1)} + U^{(2)}). $$

If we substitute this relation into Eqs.(5). Then we can derive a strong limiting condition $a_2 b_2^{-1} = \epsilon i (\epsilon = \pm)$. Let $M^*$ denote complex conjugation of matrix $M$. We can introduce a new parameter $q$ with $q = a_1 b_1^{-1}$, which is complex and has norm 1 (i.e. $q^* = q^{-1}$). Then an eight-vertex matrix representation with $d = \sqrt{2}$ is obtained as follows,

$$ U = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & q \\
0 & 1 & \epsilon i & 0 \\
0 & -\epsilon i & 1 & 0 \\
q^{-1} & 0 & 0 & 1
\end{pmatrix}. \quad (8) $$

Let

$$ |\psi_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle + q^{-1}|11\rangle), $$

$$ |\psi_2\rangle = \frac{1}{\sqrt{2}} (|01\rangle - \epsilon i|10\rangle). $$

Then, in terms of “bra” and “ket”, the $U$ matrix takes the following form

$$ U = \sqrt{2} (|\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2|). $$

This realization of TLA is associated with eight vertex model[9]. And this solution has been applied to many fields, such as topological quantum computation[32] and two dimensional representation of YBE[16].

2.2 Example II: The case $n = 3$

Let $A^{(i)}$ and $B^{(i)}$ (i=1,2,3) are three sets of 3×3 matrices with standard basis (i.e., $|00\rangle, |01\rangle, |11\rangle$). We set

$$ A^{(1)} = \begin{pmatrix}
0 & 0 & a_1 \\
0 & b_1 & 0 \\
c_1 & 0 & 0
\end{pmatrix}, \quad B^{(1)} = \begin{pmatrix}
0 & 0 & a_1^{-1} \\
0 & b_1^{-1} & 0 \\
c_1^{-1} & 0 & 0
\end{pmatrix}, $$

$$ A^{(2)} = \begin{pmatrix}
0 & a_2 & 0 \\
b_2 & 0 & 0 \\
0 & 0 & c_2
\end{pmatrix}, \quad B^{(2)} = \begin{pmatrix}
0 & a_2^{-1} & 0 \\
b_2^{-1} & 0 & 0 \\
0 & 0 & c_2^{-1}
\end{pmatrix}, $$

$$ A^{(3)} = \begin{pmatrix}
a_3 & 0 & 0 \\
0 & b_3 & 0 \\
0 & c_3 & 0
\end{pmatrix}, \quad B^{(3)} = \begin{pmatrix}
a_3^{-1} & 0 & 0 \\
0 & b_3^{-1} & 0 \\
0 & c_3^{-1} & 0
\end{pmatrix}. \quad (9) $$

Where $a_i$, $b_i$, and $c_i$ are also undetermined parameters. Thus we note that the relation Eq.(3) is clearly satisfied. If we choose $\{00, 01, 02, 10, 11, 12, 20, 21, 22\}$ as
standard basis, then we can obtain three sets of $3^2 \times 3^2$ matrices $U^{(1)}$, $U^{(2)}$ and $U^{(3)}$. In this case, their single loop $d_i = 3(i=1, 2, 3)$. Then the combined form of $U$ matrix $U = (U^{(1)} + U^{(2)} + U^{(3)})/\sqrt{3}$. Substituting this combined form into Eqs.\(\text{(5)}\), the undetermined parameters follows from the limited conditions,

\[
\begin{align*}
a_1b_1^{-1} &= \frac{q_1}{q_2} \
a_2b_2^{-1} &= \omega \
a_3b_3^{-1} &= \omega q_1 \
a_1c_1^{-1} &= 1, \
a_2c_2^{-1} &= \omega q_2, \
a_3c_3^{-1} &= q_1.
\end{align*}
\] (10)

Where $q_i = e^{i\varphi_i}$ and $\omega$ satisfies the relation $\omega^2 + \omega + 1 = 0$ (i.e., $\omega = e^{i\frac{2\pi}{3}}$). On the standard basis $U$ has the matrix form

\[
U = \frac{1}{\sqrt{3}} \left( \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & \omega q_1 & 0 & q_1 \\
0 & 1 & 0 & \omega & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & q_2 & q_2 \\
0 & \frac{1}{q_2} & 0 & 1 & 0 & 0 & 0 & q_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{q_2} \\
\frac{1}{q_2} & 0 & 0 & \frac{1}{q_2} & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \frac{1}{q_2} & 0 & 0 \\
0 & 0 & 0 & 0 & \omega & 0 & 1 & 0
\end{array} \right).
\] (11)

The single loop of this solution is $d = \sqrt{3}$. In fact, we can introduce three sets maximally entangled states as

\[
|\psi_1\rangle = \frac{1}{\sqrt{3}}(|02\rangle + q_1q_2^{-1}|11\rangle + |20\rangle)
\]

\[
|\psi_2\rangle = \frac{1}{\sqrt{3}}(|01\rangle + \omega^{-1}|10\rangle + \omega^{-1}q_2^{-1}|22\rangle)
\]

\[
|\psi_3\rangle = \frac{1}{\sqrt{3}}(|00\rangle + \omega^{-1}q_1^{-1}|12\rangle + q_1^{-1}|21\rangle).
\]

In terms of these maximally entangled states, the $U$ matrix \(\text{(11)}\) can be written in an elegant form

\[
U = \sqrt{3}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|).
\]

2.3 Remarks

We close this section with some remarks. Juan Ospina made a Mathematica implementation of this method, and the results in this article were re-obtained\[33\]. When $n = 2$, the solution \(\text{(8)}\) has been discussed in many works. As we all know, when $n = 3$, the solution \(\text{(11)}\) is not discussed. We note that the solutions \(\text{(8)}\) and \(\text{(11)}\) are Hermitian matrices (i.e., $U^\dagger = U$)(This fact will be used in the process of Yang-Baxterization approach).
In order to discuss the non-maximally entangled states, the author in Ref. [13], the unitary $\tilde{R}$ matrix has been introduced in Ref. [13]. To make the paper self-contained, we briefly review it in the following. In this work, we utilize the so-called relativistic Yang-Baxter Equation (YBE) [16]. The relativistic YBE reads,

$$\tilde{R}_i(u)\tilde{R}_{i+1}\left(\frac{u+v}{1+\beta^2uv}\right) = \tilde{R}_{i+1}(v)\tilde{R}_i\left(\frac{u+v}{1+\beta^2uv}\right) \tilde{R}_{i+1}(u)$$

(12)

where $\tilde{R}_i$ represents $1_1 \otimes 1_2 \otimes 1_3 \otimes \cdots \otimes 1_{i-1} \otimes \tilde{R} \otimes 1_{i+2} \otimes \cdots \otimes 1_m$. The variables $u$ and $v$ are called as the spectral parameters. The $\beta$ is a constant with $\beta^{-1} = ic$ ($c$ is light velocity).

Let the unitary $\tilde{R}(u)$ matrix take the form

$$\tilde{R}(u) = F(u)[I_n \times n + G(u)U].$$

(13)

Where the functions $F(u)$ and $G(u)$ are to be determined. Substituting Eq. (13) into Eq. (12), we obtain the relation

$$G(u) + G(v) + G\left(\frac{u+v}{1+\beta^2uv}\right) [G(u)G(v) - 1] + \sqrt{n}G(u)G(v) = 0$$

(14)

Following Hu et al. [16], we set

$$G(u) = \frac{a\beta u}{b\beta^2 u^2 + c\beta u + d}.$$  

If we substitute it to the relation Eq. (14). Then we obtain equations for undetermined parameters $a, b, c$ and $d$.

$$\begin{cases} 
  a^2 + \sqrt{n}ac + c^2 + 3bd + d^2 = 0 \\
  \sqrt{n}a + 2c = 0 \\
  b = d 
\end{cases}$$

After some algebra, a solution of $G(u)$ is obtained as follows

$$G(u) = \frac{4i\beta u}{\sqrt{4 - n(\beta^2 u^2 - 2\sqrt{n}/(4-n)i\beta u + 1)}}.$$  

(15)

We note that $n \neq 4$. The case $d = \sqrt{n}$ has been discussed in Ref. [16]. In addition, the unitary relation $\tilde{R}^\dagger(u)\tilde{R}(u) = \tilde{R}(u)\tilde{R}^\dagger(u) = I_{n \times n}$ leads to the relation $F^*(u)F(u) = 1$ and $G(u) + G^*(u) + \sqrt{n}G(u)G^*(u) = 0$, where $*$ denotes complex conjugation. Consider these relations, one can introduce a new variable $\theta$ with $G(u) = (e^{-2i\theta} - 1)/\sqrt{n}$, which equivalent to the relation

$$\frac{\beta^2 u^2 + 2\sqrt{n}/(4-n)i\beta u + 1}{\beta^2 u^2 - 2\sqrt{n}/(4-n)i\beta u + 1} = e^{-2i\theta}.$$
We set \( F(u) = e^{i\theta} \) with \( \theta \) is real. In terms of the new variable, we rewrite the Yang-Baxter matrix in a new form
\[
\tilde{R}(\theta, q_1, q_2) = e^{i\theta} I_{n \times n} - \frac{2i\sin \theta}{\sqrt{n}} U.
\] (16)

The case of \( n=2 \) has been discussed in Ref. (16). If \( n=3 \), on the standard basis the unitary solution of \( \tilde{R} \) matrix is
\[
\tilde{R} = \frac{1}{3}
\begin{pmatrix}
    f & 0 & 0 & 0 & \omega g q_1 & 0 & g q_1 & 0 \\
    0 & f & 0 & \omega g & 0 & 0 & 0 & \omega g q_2 \\
    0 & 0 & f & 0 & g q_1 & 0 & g q_1 & 0 \\
    \frac{\omega g}{q_1} & 0 & 0 & 0 & f & 0 & g q_1 & 0 \\
    0 & 0 & g & 0 & g q_1 & 0 & f & 0 \\
    \frac{\omega g}{q_2} & 0 & 0 & 0 & \omega g & 0 & f & 0 \\
    0 & \frac{\omega g}{q_1} & 0 & 0 & 0 & \omega g & 0 & f \\
    0 & \frac{\omega g}{q_2} & 0 & 0 & 0 & \omega g & 0 & f \\
\end{pmatrix}.
\] (17)

Where \( f \equiv f(\theta) = (e^{-i\theta} + 2e^{i\theta})/\sqrt{3} \) and \( g \equiv g(\theta) = (e^{-i\theta} - e^{i\theta})/\sqrt{3} \).

4 Entanglement and Hamiltonian

By Brylinski's theorem, a \( 4 \times 4 \) Yang-Baxter \( \tilde{R} \) matrix is universal for quantum computation, if and only if this Yang-Baxter \( \tilde{R} \) matrix can generate entangled states from separable states. The proof of universality for \( n^2 \times n^2 \) Yang-Baxter matrix is presented in Ref. (15). Via a unitary universal Yang-Baxter \( \tilde{R} \) matrix acting on the standard basis, one can obtain a set of entangled states. For example, if one lets \( \tilde{R}(\theta) \) act on the separable state \(|lm\rangle \) (i.e., \(|l\rangle \otimes |m\rangle \)), this yields the following family of states
\[
|\psi\rangle_{lm} = \sum_{n=1}^{n-1} \tilde{R}^{ij}_{lm} |l,m\rangle (l,m = 0, 1, \ldots, n-1).
\]

These unitary matrices may be universal for quantum computation, hence they can entangle states. The case \( n=2 \) has been discussed in Ref. (13).

Hereafter we focus on the case \( n=3 \). For example, if \( l=0 \) and \( m=0 \), then \(|\psi\rangle_{00} = (f(00) + \omega^{-1} g q_1^{-1} |12\rangle + g q_1^{-1} |21\rangle)/3 \). By means of negativity, we study these entangled states. It should be noted that the negativity criterion is necessary and sufficient only for \( 2 \otimes 2 \) and \( 2 \otimes 3 \) quantum systems. However, negativity is well-defined for calculation, and it has been widely applied to evaluation of entanglement. (35,36,37).

The negativity criterion for two qutrits is given by
\[
N(\rho) \equiv \frac{1}{2} \|\rho^{TA}\|_1 - 1,
\] (18)

where \( \|\rho^{TA}\|_1 \) denotes the trace norm of \( \rho^{TA} \), \( \rho^{TA} \) denotes the partial transpose of the bipartite state \( \rho \). The \( N(\rho) \) corresponds to the absolute value of the sum of negative eigenvalues of \( \rho^{TA} \), and negativity vanishes for unentangled states. Then negativity of the state \(|\psi\rangle_{00}\) yields
\[
N(\theta) = \frac{4}{9} (\sin^2 \theta + |\sin \theta| \sqrt{1 + 8\cos^2 \theta}).
\] (19)
If \( |g| = |f| \) (i.e. \( x = e^{i\pi/2} \)), then the state \( |\psi\rangle_00 \) becomes the maximally entangled state for two qutrits
\[
|\psi\rangle_00 = \frac{1}{\sqrt{3}} (e^{i\pi/6} |00\rangle - i \omega^{-1} q_1^{-1} |12\rangle - i q_1^{-1} 1 |21\rangle).
\]

In general, the unitary Yang-Baxter matrix \( \tilde{R}(\theta) \) acts on the basis \{\( |00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle \}, we obtain the same range of negativity as Eq (19). It is easy to check that the negativity ranges from 0 to 1 when the parameter \( \theta \) runs from 0 to \( \pi \). But for \( \theta \in [0, \pi] \), the negativity is not a monotonically increasing function of \( \theta \). And when \( \theta = \pi/3 \), \( \tilde{R}(\theta) \) generate nine complete and orthogonal maximally entangled states for two qutrits.

In fact, we can introduce a unitary transformation \( Y = Y_1 \otimes Y_2 \). \( Y_1 \) and \( Y_2 \) take the form
\[
Y_1 = \begin{pmatrix}
e^{i\pi/6} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-i\pi/6}
\end{pmatrix}, \quad Y_2 = \begin{pmatrix}
e^{-i\pi/6} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-i\pi/6}
\end{pmatrix}.
\]

By means of this local transformation, the universal \( \tilde{R} \) matrix (17) is locally equivalent to \( R \) matrix in Ref. [19].

A Hamiltonian of the Yang-Baxter system can be constructed from the \( \tilde{R}(\theta, \varphi_1, \varphi_2) \)-matrix. As shown in Ref. [16], the Hamiltonian is obtained through the Schrödinger evolution of the entangled states. Let the parameters \( \varphi_1 \) be time-dependent as \( \varphi_1 = \omega_1 t \). The Hamiltonian is
\[
\hat{H} = i\hbar \frac{\partial \tilde{R}(\theta, \varphi_1, \varphi_2)}{\partial t} \tilde{R}^\dagger(\theta, \varphi_1, \varphi_2).
\]

This Hamiltonian is equivalent to the Hamiltonian in Ref. [19], so one can obtain the same results as in Ref. [19]. The Berry phase of this system can also be explained in the framework of SU(2) algebra. The Berry phase can be explained as solid angle which is expanded in the parameter space. We will not discuss this in detail in this paper. But we should note that the meaning of the parameter \( \theta \) is different. The \( \theta \) in Ref. (19) arises from trigonometrical parameterization, and the \( \theta \) in this work arises from the relativistic rational parameter.

5 Summary

In this paper, we present a method of constructing \( n^2 \times n^2 \) matrix realization of TLA. This matrix realization of TLA has \( n^3 \) nonzero matrix elements. Applying Yang-Baxterization approach to the matrix realization of TLA, one can obtain a \( n^2 \times n^2 \) Yang-Baxter \( \check{R} \) matrix. When a Yang-Baxter \( \check{R} \) matrix acts on the standard basis, one can obtain a family of entangled states. Yang-Baxter \( \check{R} \) matrix is universal for quantum computation.

We believe that this family of Yang-Baxter \( \check{R} \) matrices associated with U matrices will be applied in quantum information, quantum computation and so on. We will investigate these applications in subsequent papers.

Acknowledgements The authors gratefully acknowledge Juan Ospina for helpful comments on this paper. Special thanks to the first referee for his advice and criticism on our manuscript.
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