RESEARCH ARTICLE

Concavity of solutions to semilinear equations in dimension two

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Abstract
We consider the Dirichlet problem for a class of semilinear equations on two-dimensional convex domains. We give a sufficient condition for the solution to be concave. Our condition uses comparison with ellipses, and is motivated by an idea of Kosmodem’yanskii. We also prove a result on propagation of concavity of solutions from the boundary, which holds in all dimensions.

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1 | INTRODUCTION

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded convex domain with smooth boundary and let \( u \in C^\infty(\Omega) \) solve the semilinear equation

\[
-\Delta u = f(u), \quad \text{in } \Omega, \\
\quad u = 0, \quad \text{on } \partial \Omega
\]

for a smooth positive function \( f : \mathbb{R} \to \mathbb{R} \).

In the special case \( f = 1 \), the corresponding Dirichlet problem,

\[
-\Delta u = 1, \quad \text{in } \Omega, \\
\quad u = 0, \quad \text{on } \partial \Omega
\]

is known as the 

torsion problem.
A natural question is whether the solution $u$ should inherit concavity properties from the domain $\Omega$. In general, $u$ is not concave, even in the case of the torsion problem. Indeed, the solution $u$ to (1.2) on an equilateral triangle is not concave [13]. One can obtain a non-concave solution with smooth boundary by taking $\Omega = \{ u \geq \varepsilon \}$ for a small $\varepsilon > 0$.

Recently Steinerberger [20] asked:

**Question 1.1.** Under what conditions on $f$ and $\Omega$ is the function $u$ concave on $\Omega$?

There appear to be very few results on this. For the torsion problem (1.2), Kosmodem’yanskii [17] showed that $u$ is concave if $\Omega$ satisfies a third-order contact parabola condition, which we discuss below in Section 4.

For the torsion problem, the answer is clearly yes when $\Omega$ is the interior of an ellipse, say, given by $a^2 x^2 + b^2 y^2 = 1$. Indeed, the solution is the concave quadratic function

$$u = c(1 - a^2 x^2 - b^2 y^2), \quad \text{for } c = \left(2a^2 + 2b^2\right)^{-1}.$$

Solutions will also be concave if $\Omega$ is a small perturbation of an ellipse (see [10], for example). On the other hand, as discussed above, $u$ is not concave for some domains where $\partial \Omega$ is close to a triangle. It is then natural to ask whether concavity for $u$ holds when $\Omega$ is “not too far” from being an ellipse.

Our main result gives a quantitative sufficient condition on $\Omega$ for concavity for a large class of $f$ which includes the case $f = 1$. Our condition uses comparison with ellipses.

**Theorem 1.1.** Let $u$ be a solution of (1.1). Assume:

1. For each point $p \in \partial \Omega$, there is an ellipse $E$, enclosing a finite area $A$, containing $\Omega$ and tangent to $\partial \Omega$ at $p$ such that, at $p$,

$$K_E \left( \frac{\pi^{2/3}}{A^{2/3} K_E^{4/3}} + 1 + \left( \frac{\partial_s K_E}{9K^4} \right)^2 \right) \geq K \left( 1 + \left( \frac{\partial_s K}{9K^4} \right)^2 \right),$$

where $K$ and $K_E$ are the curvatures of $\partial \Omega$ and $E$ respectively, and $\partial_s$ is the derivative with respect to arc length.
2. $f(0) = 1$, and on $[0, \max_{\Omega} u]$ we have $f > 0$, $f'' \leq 0$, and $f' \leq 0$.

Then $u$ is concave on $\overline{\Omega}$.

**Remark 1.1.**

(i) Condition (1) above is clearly satisfied when $\partial \Omega$ is itself an ellipse. Indeed, given any $p \in \partial \Omega$ we simply take $E = \partial \Omega$ in (1) in which case we replace $K_E, \partial_s K_E$ with $K, \partial_s K$ in (1.3), leaving a strict inequality that is obviously satisfied since $\pi^{2/3} / (A^{2/3} K_E^{4/3}) > 0$. In particular, condition (1) will continue to hold for all domains whose $C^3$ distance to a given ellipse is comparable to the term $\pi^{2/3} / (A^{2/3} K_E^{4/3})$.

(ii) Our proof is motivated by the argument of Kosmodem’yanskii [17] who used a comparison with parabolas that have a third-order contact with the boundary of $\Omega$. In Section 4 below, we describe a key difference between these two conditions: unlike (1) above, Kosmodem’yanskii’s condition cannot hold in any $C^3$ neighborhood of any domain. In this way,
one can produce domains that satisfy our conditions but not those of [17], see Remark 4.1 below.

(iii) Examples of functions \( f \) satisfying (2) and for which there exist smooth solutions to (1.1) are \( f(u) = 1 - cu^k \) for \( k = 1, 2, 3, \ldots \) and \( c \) a sufficiently small positive constant, depending only on \( \Omega \) and \( k \).

(iv) When \( f = 1 \) the conclusion can be strengthened to \( u \) being strongly concave, namely \( D^2 u < 0 \) on \( \overline{\Omega} \) (see Remark 2.1 (ii) below). In particular, this implies that the conditions in Theorem 1.1 are not optimal in general.

(v) It is not clear whether the analogue of Theorem 1.1 holds for dimensions larger than 2. Using our approach, new difficulties arise from the third-order derivatives of \( u \).

By our methods, we also obtain a short proof of the following “propagating concavity from the boundary” result, which holds in all dimensions.

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded convex domain with smooth boundary and let \( u \in C^\infty(\overline{\Omega}) \) solve (1.1). Assume \( f > 0 \) and \( f'' \leq 0 \). If \( D^2 u \leq 0 \) on \( \partial \Omega \) then \( u \) is concave in \( \Omega \).

This extends a recent result of Steinerberger [20], who proved the same result under additional assumptions on \( f' \). Earlier work of Keady–McNabb [13] established it for the torsion problem.

We conclude the introduction with some historical remarks. Instead of asking whether \( u \) is concave, one could ask whether weaker concavity properties hold. Lions [18] conjectured in 1981 that for a general \( f \) the superlevel sets of \( u \) should be convex (i.e. \( u \) is quasiconcave), and this has led to many further investigations. By a classical result of Gidas–Ni–Nirenberg [7], \( u \) is quasiconcave in the special case when \( \Omega \) is a disc, since then \( u \) is radially symmetric. Makar–Limanov showed that solutions of the two-dimension torsion problem (1.2) are quasiconcave since \( \sqrt{u} \) is concave [19]. In a well-known paper, Brascamp and Lieb [4] showed that if \( f(u) = \lambda u \) for \( \lambda > 0 \) then \( \log u \) is concave. There have been many further extensions and generalizations. We refer the reader to, for example, [1–3, 5, 6, 8, 11, 12, 14–16] and the references therein. On the other hand, a 2016 counterexample of Hamel–Nadirashvili–Sire [9] shows that quasiconcavity is false for general \( f \).

The outline of this paper is as follows. In Sections 2 and 3 we prove Theorems 1.1 and 1.2 respectively. In Section 4 we compare our condition (1) in Theorem 1.1 to the parabola condition of Kosmodem’yanskii [17].

## 2 | PROOF OF THEOREM 1.1

Assume for a contradiction that \( u \) is not concave on \( \overline{\Omega} \). Then there exists a vector \( e = (e_x, e_y) \) and a point \( p \in \overline{\Omega} \) such that the quantity

\[
H = u_{ee}
\]

achieves a strictly positive maximum at \( p \). Here and henceforth, we are using subscripts to denote partial derivatives.

We first show that we may assume that \( p \) lies in the boundary \( \partial \Omega \). Consider the set

\[
M = \{ x \in \Omega \mid H(x) = \sup_{\Omega} H \}.
\]
This is a closed set in $\Omega$. But it is also open, since if $x \in M$ then in a small ball $B$ with $x \in B \subset \Omega$ we have

$$\Delta H = \Delta u_{ee} = -f'(u)u_{ee} - f''(u)u_e^2 \geq 0,$$

since $f' \leq 0$ and $f'' \leq 0$. The strong maximum principle implies that $H$ is constant on $B$ and hence $B \subset \Omega$. Hence $M$ is closed and open in $\Omega$, so $M = \Omega$ or $M$ is empty. Either way, $H$ achieves its maximum at a boundary point $p \in \partial \Omega$.

We may rotate and translate the coordinates so that $p$ is the origin and $\partial \Omega$ is locally given by $y = \rho(x)$ for $\rho$ a convex function with $\rho(0) = 0$ and $\rho'(0) = 0$. Assumption (1) of the theorem implies that $\rho''(0) > 0$.

**Definition 2.1.** We say that such a coordinate system is a *preferred coordinate system* at $p \in \partial \Omega$.

We make the following claim.

**Claim.** At $p$, we have

$$0 < u_y \leq \frac{9(\rho'')^3}{9(\rho'')^4 + (\rho''')^2}.$$

(2.1)

where the second inequality is an equality if and only if $u$ is quadratic and $\partial \Omega$ is an ellipse.

**Proof of Claim.** The first inequality is an immediate consequence of the maximum principle and the Hopf Lemma. Indeed since $\Delta u = -f(u) < 0$, we have $u > 0$ in $\Omega$ and $\frac{\partial u}{\partial \nu} < 0$ on $\partial \Omega$, where $\nu$ is the outward pointing normal.

We will prove the second inequality using conditions (1) and (2) of Theorem 1.1. We recall a basic fact that any ellipse in the plane is given by an equation

$$a^2(x-h)^2 + b(x-h)(y-k) + c^2(y-k)^2 = 1,$$

(2.2)

for constants $a, b, c, h,$ and $k$ with $D := 4a^2c^2 - b^2 > 0$. Moreover, if $A$ is the area of the ellipse then

$$D = \frac{4\pi^2}{A^2}.$$

(2.3)

Let $E$ be the ellipse as in condition (1) of the theorem. We may and do assume without loss of generality that $a$ and $c$ are positive.

By the assumptions on $E$, the origin is a point on the ellipse and the tangent line to $E$ at the origin coincides with the $x$-axis. Hence, if we express $E$ near the origin as a graph $y = g(x)$ then $g(0) = 0$ and $g'(0) = 0$. Since the ellipse contains $\Omega$, which is in the upper half plane, we have $g''(0) > 0$.

Evaluating (2.2), and its implicit derivative with respect to $x$, at the origin, we can solve for $h$, $k$ to obtain

$$h = -\frac{b}{a\sqrt{D}}, \quad k = \frac{2a}{\sqrt{D}}.$$
Differentiating (2.2) implicitly another two times and evaluating at the origin, we obtain

\[ g''(0) = \frac{2a^3}{\sqrt{D}}, \quad g'''(0) = \frac{6a^4b}{D}. \]  

(2.4)

Define the function \( w = w(x, y) \) by

\[
w = \frac{1}{2(a^2 + c^2)} \left( 1 - a^2 \left(x + \frac{b}{a\sqrt{D}} \right)^2 \right.
\]
\[ - b \left(x + \frac{b}{a\sqrt{D}} \right) \left(y - \frac{2a}{\sqrt{D}}\right) - c^2 \left(y - \frac{2a}{\sqrt{D}}\right)^2 \left. \right). \]

Then \( \Delta w = -1 \) and the set \( \{w > 0\} \) coincides with the interior of the ellipse \( E \). By assumption the ellipse contains the set \( \Omega \). Hence

\[ \Delta(w - u) = f(u) - 1 \leq 0, \quad \text{on } \Omega, \]

and \( w - u \geq 0 \) on \( \partial \Omega \), where we used the condition that \( f' \leq 0 \). Moreover \( w - u = 0 \) at the origin. The strong maximum principle then implies that either \( w - u = 0 \) on \( \Omega \) in which case \( u = w \) is a quadratic and \( \partial \Omega \) is the ellipse \( E \), or else \( w - u > 0 \) on \( \Omega \) and \( w - u = 0 \) at \( p \in \partial \Omega \). Hence

\[ u_y(0,0) \leq w_y(0,0) = \frac{\sqrt{D}}{2a(a^2 + c^2)}. \]  

(2.5)

By the Hopf Lemma, the first inequality of (2.5) is strict unless \( u \) is quadratic. We now wish to write the right-hand sides of (2.1) and (2.5) in terms of the intrinsic quantities \( K, K_E, \partial_s K, \partial_s K_E \) and \( A \) featured in (1.3). Note that, evaluating at \( p \), using (2.4),

\[ K_E = g''(0) = \frac{2a^3}{\sqrt{D}}, \quad \partial_s K_E = g'''(0) = \frac{6a^4b}{D}. \]  

(2.6)

Recalling the formula \( D = 4a^2c^2 - b^2 \), we have

\[
\frac{\sqrt{D}}{2a(a^2 + c^2)} = \frac{\sqrt{D}}{2a^3 + (2a)^{-1}(D + b^2)}
\]
\[ = K^{-1}_E \left( \frac{\pi^{2/3}}{K^{4/3}_E A^{2/3}} + 1 + \frac{(\partial_s K_E)^2}{9K^4_E} \right)^{-1}, \]  

(2.7)

after substituting for \( D, a \) and \( b \) from (2.3) and (2.6), and simplifying. On the other hand, at \( p \) we have

\[ K = \rho''(0), \quad \partial_s K = \rho'''(0), \]
and hence
\[
\frac{9(\rho''')^3}{9(\rho'')^4 + (\rho''')^2} = K^{-1} \left( 1 + \left( \frac{\partial_x K}{9K^4} \right)^2 \right)^{-1}. \tag{2.8}
\]

Then (2.1) follows from combining (2.5), (2.7), (2.8) and (1.3), completing the proof of the claim.

Differentiating the equation \( u(x, \rho(x)) = 0 \) and evaluating at \( x = 0 \), we obtain, recalling that \( \rho'(0) = 0 \),
\[
\begin{align*}
    u_x &= 0 \\
    u_{xx} + u_y \rho'' &= 0 \\
    u_{xxx} + 3u_{xy} \rho'' + u_y \rho''' &= 0.
\end{align*} \tag{2.9}
\]

We recall that \( e \) is the unit vector used in the definition of \( H \). If the vector \( e \) is proportional to the vector \((1, 0)\) then we have \( u_{xx} > 0 \) at \( p \). But since \( u_y > 0 \) and \( \rho''(0) > 0 \) this contradicts the second equation of (2.9).

If \( e \) is proportional to the vector \((0, 1)\) then \( u_{yy} > 0 \) at \( p \). But then at \( p \) we have
\[
u_{xx} = -f(u) - u_{yy} < -1
\]
from the equation \( \Delta u = -f(u) \) and the fact that \( f(0) = 1 \). But by (2.1) and the second equation of (2.9) we have
\[
u_{xx} = -u_y \rho'' \geq \frac{9(\rho''')^4}{9(\rho'')^4 + (\rho''')^2} \geq -1,
\]
a contradiction.

Hence we may assume, after rescaling, that \( e = (1, \tau) \) for a real nonzero number \( \tau \). Write
\[
H = u_{ee} = u_{xx} + \tau^2 u_{yy} + 2\tau u_{xy}. \tag{2.10}
\]

Note that at \( p \) we have \( u_x = 0 \) and hence, differentiating the equation \( u_{xx} + u_{yy} = -f(u) \) we get
\[
\begin{align*}
    u_{xxx} + u_{yyy} &= 0, \\
    u_{xyy} + u_{yyy} &= -f'(u) u_y.
\end{align*} \tag{2.11}
\]

Then, recalling that \( H \) achieves a maximum at \( p \),
\[
0 = H_x = u_{exx}
= u_{xxx} + \tau^2 u_{xyy} + 2\tau u_{xyx}
= (1 - \tau^2) u_{xxx} - 2\tau u_{yyy} - 2\tau f'(u) u_y.
\]
Hence
\[ u_{yyy} = \frac{1 - \tau^2}{2\tau} u_{xxx} - f'(u)u_y. \]

Next, using this and (2.11),
\[ 0 \geq H_y = u_{xxy} + \tau^2 u_{yyy} + 2\tau u_{xyy} \]
\[ = (\tau^2 - 1)u_{yyy} - 2\tau u_{xxx} - f'(u)u_y \]
\[ = - \frac{(1 - \tau^2)^2 + 4\tau^2}{2\tau} u_{xxx} - (\tau^2 - 1)f'(u)u_y - f'(u)u_y \]
\[ = - \frac{(1 + \tau^2)^2}{2\tau} u_{xxx} - \tau^2 f'(u)u_y. \]

Hence
\[ \tau u_{xxx} \geq - \frac{2\tau^4}{(1 + \tau^2)^2} f'(u)u_y. \]

Compute using (2.10) and (2.9),
\[ - \frac{2\tau^4}{(1 + \tau^2)^2} f'(u)u_y \leq \tau u_{xxx} = -3\tau u_{xy}\rho'' - \tau u_y\rho''' \]
\[ = \frac{3\rho''}{2} (-H + u_{xx} + \tau^2 u_{yy}) - \tau u_y\rho'''. \]

Using \( u_{xx} + u_{yy} = -1 \) at \( p \) (using \( f(0) = 1 \)) and \( u_{xx} = -u_y\rho'' \) we obtain, recalling that \( H \) is positive at \( p \),
\[ 0 < 3\rho''H \leq 3\rho''(u_{xx} + \tau^2(-1 - u_{xx})) - 2\tau u_y\rho'' + \frac{4\tau^2}{(1 + \tau^2)^2} f'(u)u_y \]
\[ = 3\rho''(-u_y\rho'' + \tau^2(-1 + u_y\rho'')) - 2\tau u_y\rho''' + \frac{4\tau^2}{(1 + \tau^2)^2} f'(u)u_y \]
\[ \leq - u_y \left\{ \left( \frac{3\rho''}{u_y} - 3(\rho'')^2 \right) \tau^2 + 2\rho''\tau + 3(\rho'')^2 \right\}, \tag{2.12} \]

where for the last line we used \( f' \leq 0 \) and \( u_y > 0 \).

The inequality (2.12) implies that the quadratic function
\[ q(\tau) = \left( \frac{3\rho''}{u_y} - 3(\rho'')^2 \right) \tau^2 + 2\rho'''\tau + 3(\rho'')^2, \]
is negative for some \( \tau \) and hence has two roots. Here, we are using the fact that from (2.1) we have
\[ \frac{3\rho''}{u_y} - 3(\rho'')^2 = \frac{3\rho''}{u_y} (1 - \rho'' u_y) > 0. \]
where we have assumed without loss of generality that \( u \) is not quadratic, in which case the theorem would be trivial.

Since \( q \) has two roots, we have

\[
4(\rho'''^2 - 4 \left( \frac{3\rho''}{u_y} - 3(\rho'')^2 \right) 3(\rho'')^2 > 0,
\]

namely

\[
(\rho''')^2 + 9(\rho'')^4 > \frac{9(\rho'')^3}{u_y},
\]

which contradicts the claim. This completes the proof of the theorem.

**Remark 2.1.**

(i) Kosmodem’yanskii [17] uses some similar arguments to those above. He compares using a parabola instead of an ellipse, and applies the maximum principle to the determinant of the Hessian of \( u \) rather than \( u_{ee} \) for some fixed \( e \).

(ii) In the case of the torsion problem (1.2), one can strengthen the conclusion of Theorem 1.1 to \( u \) being strongly concave, namely that \( D^2 u < 0 \) on \( \overline{\Omega} \). Indeed, at the beginning of the proof, one can assume for a contradiction that \( H = u_{ee} \) has a nonnegative (rather than strictly positive) maximum. Since \( \Delta H = 0 \), this must occur at a boundary point. The rest of the proof goes through in the same way.

### 3 PROOF OF THEOREM 1.2

We may assume without loss of generality that \( \overline{\Omega} \) lies in the half space \( \{x_1 > 0\} \). Fix a unit vector \( e \). We wish to show that \( u_{ee} \leq 0 \).

Define, for a constant \( A > 0 \),

\[
Q_A = u_{ee} - Au - \sigma Ax_1,
\]

where

\[
\sigma = \frac{f(0)}{\sup_{x \in \Omega} |x_1 f'(u(x))| + 1} > 0.
\]

By our assumptions \( Q_A < 0 \) on \( \partial \Omega \) for any \( A > 0 \). Choose \( A \) sufficiently large, depending on \( u \), so that \( Q_A < 0 \) on \( \Omega \). We can do this because \( x_1 \geq c > 0 \) on \( \overline{\Omega} \), for a uniform \( c > 0 \).

Let us assume for a contradiction that if we let \( A \) decrease toward zero, there is some \( A = A_0 > 0 \) at which \( Q_A \) first attains 0 at some point in the interior. Indeed if not then \( Q_A < 0 \) for all \( A > 0 \), and letting \( A \to 0 \) proves that \( u_{ee} \leq 0 \), as required.
Fix this $A = A_0$. By assumption, there is some interior point $p \in \Omega$ at which $0 = Q_A(p) \geq Q_A(x)$ for all $x \in \Omega$. We compute at this $p$, recalling the definition of $\sigma$,

$$0 \geq \Delta Q_A = \Delta u_{ee} - A \Delta u = -f'(u)u_{ee} - f''(u)u^2_e + Af(u) \geq -Af'(u)u - \sigma Af'(u)x_1 + Af(u) > A(-f'(u)u + f(u)) - Af(0) \geq 0,$$

a contradiction. We used the concavity of $f$ twice. First to observe that $-f''(u)u^2_e \geq 0$ and second to see that

$$\frac{f(u) - f(0)}{u} \geq f'(u),$$

which is the same as

$$-uf'(u) + f(u) \geq f(0).$$

This completes the proof.

4 | A COMPARISON TO THE CONTACT PARABOLA CONDITION

Our proof of Theorem 1.1 is motivated by the proof of the main theorem in [17]. Kosmodem’yanskii showed that solutions to the torsion problem (1.2) are concave provided the following contact parabola condition (CPC) holds: for each point $p \in \partial \Omega$, the domain $\Omega$ is contained in the third order contact parabola at $p$.

Working in preferred coordinates $x, y$ at $p$ (as in Definition 2.1), the third order contact parabola at $p$ is the curve with equation $y = (Ax + By)^2$ for constants $A, B$ with $A > 0$ satisfying $y^{(k)}(0) = \rho^{(k)}(0)$ for $0 \leq k \leq 3$.

A key difference with condition (1) in Theorem 1.1, abbreviated hereby (CEC) (contact ellipse condition), is that (CEC) requires only first-order contact between the ellipses with the boundary. Indeed, in preferred coordinates around any $p \in \partial \Omega$, the ellipse in (CEC) will have the general equation (2.2) where the implicit function $y(x)$ must satisfy $y^{(k)}(0) = \rho^{(k)}(0)$ for $0 \leq k \leq 1$ while the second and third derivatives are required only to be comparable, but not necessarily equal, through (1.3).

To highlight the difference between these conditions, we will show that given any compact convex domain satisfying (CPC), an arbitrarily small $C^{3+\gamma}$ perturbation of $\Omega$, for $\gamma \in (0, 1)$ may result in a domain which violates (CPC), while this is not the case for (CEC) (see Remark 1.1 (i)).

Let $\Omega$ be a smooth compact convex domain satisfying (CPC). We will construct a sequence of compact domains $\Omega_c$ such that for all $c > 0$ sufficiently small, such that

(i) $\Omega_c$ is compact and convex and contains $\Omega$
(ii) $\Omega_c$ does not satisfy (CPC)
(iii) $\partial \Omega_c \rightarrow \partial \Omega$ uniformly in $C^{3+\gamma}$ as $c \rightarrow 0$, for any $\gamma \in (0, 1)$. 
Let $p$ be any point on $\partial \Omega$, and $y = (Ax + By)^2$ the third-order contact parabola at $p$ where $x, y$ are the preferred coordinates at $p$. Writing this parabola near $x = 0$ as $y = P(x)$ we compute,

$$P(x) = A^2 x^2 + 2A^3 Bx^3 + O(x^4).$$

If we write the boundary $\partial \Omega$ near the origin as $y = \rho(x)$, the assumption (CPC) implies that for $|x| \leq c_0$, for a small constant $c_0 > 0$,

$$\rho(x) = A^2 x^2 + 2A^3 Bx^3 + E(x),$$

where $E$ satisfies

$$|D^k E(x)| \leq C x^{4-k}, \quad k = 0, 1, \ldots, 4,$$

for a uniform constant $C$.

Define new smooth locally defined functions $\rho_c(x)$, for $0 < c \leq c_0$, as follows. Let $\xi : \mathbb{R} \to [0, 1]$ be a smooth function equal to 1 on $[-1/2, 1/2]$ and equal to zero outside $[-1, 1]$. Then define for $|x| \leq c_0$,

$$\rho_c(x) = \rho(x) - a \xi(x/c)x^4,$$

for a constant $a > 0$ chosen sufficiently large so that

$$\rho_c(x) = \rho(x) - ax^4 < P(x), \quad \text{for } |x| \leq c/2. \quad (4.1)$$

Note that $\rho_c \leq \rho$ on $|x| \leq c_0$ and $\rho_c = \rho$ outside $[-c, c]$. Hence, $\rho_c$ defines a new compact domain $\Omega_c$ which contains $\Omega$.

On $[-c, c]$ we have, for a uniform $C$,

$$\rho_c''(x) = \rho''(x) - a \left( \frac{\xi''(x/c)}{c^2} x^4 - \frac{8\xi''(x/c)}{c} x^3 \right) \geq 2A^2 - Cc^2 \geq A^2 > 0, \quad (4.2)$$

shrinking $c_0$ if necessary. Hence $\Omega_c$ is convex, and so (i) is satisfied.

The assertion (ii) follows from (4.1), which implies that $\Omega_c$ is not contained in the parabola $y = (Ax + By)^2$.

Finally, for (iii), we can estimate

$$|\rho_c^{(k)}| \leq C, \quad \text{for } k = 0, 1, \ldots, 4,$$

by a similar argument to that of (4.2) above. Since $\rho_c \to \rho$ in $C^0([-c_0, c_0])$ as $c \to 0$, we have $\rho_c \to \rho$ in $C^{3+\gamma}([-c_0, c_0])$ and this establishes (iii).

Remark 4.1. We give here an explicit example of a domain which satisfies the assumption (CEC) of Theorem 1.1 but not (CPC). Let $\Omega$ be the unit disc, which satisfies (CPC) by [17, Theorem 2]. Hence by the above argument and part (i) of Remark 1.1, there is a small perturbation $\Omega_c$ of $\Omega$ which satisfies (CEC) but not (CPC).
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