On free elementary $\mathbb{Z}_p C_p$-lattices

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Abstract. We show that all elementary lattices that are free $\mathbb{Z}_p C_p$-modules admit an orthogonal decomposition into a sum of a free unimodular and a $p$-modular $\mathbb{Z}_p C_p$-lattice.

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1. Introduction. Let $R := \mathbb{Z}_p C_p$ denote the group ring of the cyclic group of order $p$ over the localisation of $\mathbb{Z}$ at the prime $p$. The present paper considers free $R$-lattices $L \cong R^a$. The main observation in this situation is Theorem 2.2: Given two free $R$-modules $M$ and $L$ with $pM \subseteq L \subseteq M$, there is an $R$-basis $(g_1, \ldots, g_a)$ of $M$ and $0 \leq t \leq a$ such that $(g_1, \ldots, g_t, pg_{t+1}, \ldots, pg_a)$ is an $R$-basis of $L$. So these lattices do admit a compatible basis. Applying this observation to Hermitian $R$-lattices shows that free elementary Hermitian $R$-lattices admit an invariant splitting (see Theorem 4.1) as the orthogonal sum of a free unimodular lattice and a free $p$-modular lattice.

The results of this note have been used in the thesis [1] to study extremal lattices admitting an automorphism of order $p$ in the case that $p$ divides the level of the lattice.

2. Existence of compatible bases. For a prime $p$, we denote by

$$\mathbb{Z}_p := \left\{ \frac{a}{b} \in \mathbb{Q} \mid p \text{ does not divide } b \right\}$$

the localisation of $\mathbb{Z}$ at the prime $p$. The following arguments also apply accordingly to the completion of this discrete valuation ring. Let $R := \mathbb{Z}_p C_p$ denote the group ring of the cyclic group $C_p = \langle \sigma \rangle$ of order $p$. Then $e_1 := \frac{1}{p}(1 + \sigma + \cdots + \sigma^{p-1}) \in \mathbb{Q} C_p$ and $e_\zeta := 1 - e_1$ are the primitive idempotents in the group algebra $\mathbb{Q} C_p$ with $\mathbb{Q} C_p = \mathbb{Q} C_p e_1 \oplus \mathbb{Q} C_p e_\zeta \cong \mathbb{Q} \oplus \mathbb{Q} [\zeta_p]$, where $\zeta_p$ is
a primitive $p$-th root of unity. The ring $T := \mathbb{Z}_p[\zeta_p]$ is a discrete valuation ring in the $p$-th cyclotomic field $\mathbb{Q}[\zeta_p]$ with prime element $\pi := (1 - \zeta_p)$ and hence

$$Re_1 \oplus Re_\zeta \cong \mathbb{Z}_p \oplus \mathbb{Z}_p[\zeta_p] =: S \oplus T$$

is the unique maximal $\mathbb{Z}_p$-order in $\mathbb{Q}C_p$.

**Remark 2.1.** With the notation above, $T/(\pi) \cong \mathbb{Z}_p/(p) \cong \mathbb{F}_p$ and via this natural ring epimorphism,

$$R = \{(x, y) \in \mathbb{Z}_p \oplus \mathbb{Z}_p[\zeta_p] \mid x + p\mathbb{Z}_p = y + \pi\mathbb{Z}_p[\zeta_p]\}.$$ 

$R$ is generated as $\mathbb{Z}_p$-algebra by $1 = (1, 1)$ and $1 - \sigma = (0, \pi)$. Moreover $Re_1 \cap R = pRe_1 = pS$ and $Re_\zeta \cap R = \pi Re_\zeta = \pi T$ and the radical $J(R) := pS \oplus \pi T$ of $R$ is the unique maximal ideal of the local ring $R$.

By [6], the indecomposable $R$-lattices are the free $R$-module $R$, the trivial $R$-lattice $\mathbb{Z}_p = Re_1 = S$, and the lattice $\mathbb{Z}_p[\zeta_p] = Re_\zeta = T$ in the rational irreducible faithful representation of $C_p$. The theorem by Krull-Remak-Schmidt-Azumaya [2, Chapter 1, Section 11] ensures that any finitely generated $R$-lattice $L$ is a direct sum of indecomposable $R$-lattices

$$L \cong R^a \oplus T^b \oplus S^c.$$ 

In this note, we focus on the case of free $R$-lattices. Though $R$ is not a principal ideal domain, for certain sublattices of free $R$-lattices, there do exist compatible bases:

**Theorem 2.2.** Let $M \cong R^a$ be a free $R$-lattice of rank $a$. Assume that $L$ is a free $R$-lattice with $pM \subseteq L \subseteq M$. Then there is an $R$-basis $(g_1, \ldots, g_a)$ of $M = Rg_1 \oplus \cdots \oplus Rg_a$ and $0 \leq t \leq a$ such that

$$L = Rg_1 \oplus \cdots \oplus Rg_t \oplus pRg_{t+1} \oplus \cdots \oplus pRg_a.$$ 

**Proof.** Let $\tilde{S} := Me_1$ and $\tilde{T} := Me_\zeta$. Now $M \cong R^a$ is a free $R$-lattice, so, as in Remark 2.1, $M$ is a sublattice of $\tilde{S} \oplus \tilde{T}$ of index $p^a$, $\tilde{S} \cap M = p\tilde{S}$, and $\tilde{T} \cap M = \pi \tilde{T}$. The Jacobson radical is $J(M) = J(R)M = p\tilde{S} \oplus \pi \tilde{T}$ and of index $p^a$ in $M$. We proceed by induction on $a$.

If $a = 1$, then $M = R$, $\tilde{S} = S$, $\tilde{T} = T$. As $M/pM \cong \mathbb{F}_p C_p \cong \mathbb{F}_p[x]/(x - 1)^p$ is a chain ring, the $R$-sublattices of $M$ that contain $pM$ form a chain:

$$M \supset p\tilde{S} \oplus \pi \tilde{T} \supset p\tilde{S} \oplus \pi^2 \tilde{T} \supset \cdots \supset p\tilde{S} \oplus \pi^{p-2} \tilde{T} \supset p\tilde{S} \oplus p\tilde{T} \supset pM.$$ 

The only free $R$-lattices among these are $M$ and $pM$.

Now assume that $a > 1$. If $L \not\subseteq J(M)$, then we may choose $g_1 \in L \setminus J(M)$. As $g_1 \not\in J(M)$, the $R$-submodule $Rg_1$ of $M$ is a free submodule of both modules $L$ and $M$, so $M = Rg_1 \oplus M'$, $L = Rg_1 \oplus L'$ where $M'$ and $L'$ are free $R$-lattices of rank $a - 1$ satisfying the assumption of the theorem and the theorem follows by induction. So we may assume that

$$L \subseteq J(M) = p\tilde{S} \oplus \pi \tilde{T}.$$

The element $e_1 \in \mathbb{Q}C_p$ is a central idempotent in $\text{End}_R(J(M))$ projecting onto $p\tilde{S} = J(M)e_1$. The assumption that $pM \subseteq L \subseteq J(M)$ implies that

$$p\tilde{S} = pMe_1 \subseteq Le_1 \subseteq J(M)e_1 = p\tilde{S}.$$
So $L_1 = pM_1 = p\tilde{S}$.

To show that $L = pM$, we first show that $Le_\zeta = pMe_\zeta$.

As $pM \subseteq L$, we clearly have that $pMe_\zeta \subseteq Le_\zeta$.

To see the opposite inclusion, put $K := L \cap Le_\zeta$ to be the kernel of the projection $e_1 : L \to L_1$. As $L$ is free, we get, as in Remark 2.1, that $K = \pi Le_\zeta$. Let $k$ be maximal such that $K \subseteq \pi^kT$. Then $k \geq 2$ because $Le_\zeta \subseteq \pi T$ (see equation (1)).

Assume that $k \leq p - 1$. There is $\ell \in L$ such that $y = \ell e_\zeta \not\in \pi^kT$. As $pM_1 = L_1$, there is $m \in M$ such that $pme_1 = \ell e_1$. Now $pM \subseteq L$, so $pm \in L$ and $\ell - pm \in K = Ke_\zeta$.

We compute $\ell - pm = (\ell - pm)e_\zeta = y - pme_\zeta$.

As $pMe_\zeta = p\tilde{T} = \pi^{p-1}\tilde{T}$ and $y \not\in \pi^k\tilde{T}$, the assumption that $k \leq p - 1$ shows that $\ell - pm \not\in \pi^k\tilde{T}$, which contradicts the definition of $k$.

Therefore $k \geq p$ and $Le_\zeta \subseteq pMe_\zeta$.

Now $pM$ and $L$ both have index $p^n$ in $pM_1 \oplus pMe_\zeta = L_1 \oplus Le_\zeta$ (again by Remark 2.1 as $L$ and $M$ are free). So the assumption $pM \subseteq L$ implies that $pM = L$. \qed

Remark 2.3. Let $M \cong T^b \oplus S^c$ and let $L$ be a sublattice of $M$ again isomorphic to $T^b \oplus S^c$. Then $M = Me_\zeta \oplus M_1$ and $L = Le_\zeta \oplus L_1$. By the main theorem for modules over principal ideal domains, there is a $T$-basis $(x_1, \ldots, x_b)$ of $Me_\zeta$ and a $\mathbb{Z}_p$-basis $(y_1, \ldots, y_c)$ of $M_1$, as well as $0 \leq n_1 \leq \cdots \leq n_b$, $0 \leq m_1 \leq \cdots \leq m_c$, such that $L = \bigoplus_{i=1}^b \pi^{n_i}T x_i \oplus \bigoplus_{i=1}^c p^{m_i}\mathbb{Z}_p y_i$.

Example 2.4. For general modules $M$, however, Theorem 2.2 has no appropriate analogue. To see this, consider $M \cong R \oplus S$ and choose a pseudo-basis $(x, y)$ of $M$ such that $x$ generates a free direct summand and $y$, its complement isomorphic to $S$. Let $L$ be the $R$-sublattice generated by $pxe_1$ and $x(1 - \sigma) + y$. As $x(1 - \sigma) + y$ generates a free $R$-sublattice of $M$ and $R(pxe_1) \cong S$, we have $L \cong S \oplus R$. For $p > 2$, we compute that $pM \subseteq L \subseteq M$. Then the fact that $|M/L| = p^2$ implies that these two modules do not admit a compatible pseudo-basis.

3. Lattices in rational quadratic spaces. From now on, we consider $\mathbb{Z}_p$-lattices $L$ in a non-degenerate rational quadratic space $(V, B)$. The dual lattice of $L$ is

$$L^\# := \{x \in V \mid B(x, \ell) \in \mathbb{Z}_p \text{ for all } \ell \in L\}.$$  

The lattice $L$ is called integral if $L \subseteq L^\#$, and elementary if $pL^\# \subseteq L \subseteq L^\#$.

Following O’Meara [5, Section 82 G], we call a lattice $L$ unimodular if $L = L^\#$, and $p^2$-modular if $p^2L^\# = L$.

We now assume that $\sigma$ is an automorphism of $L$ of order $p$, so $\sigma$ is an orthogonal mapping of $(V, B)$ with $L\sigma = L$. Then also the dual lattice $L^\#$ is a $\sigma$-invariant lattice in $V$. As the dual basis of a lattice basis of $L$ is a lattice basis of $L^\#$, the symmetric bilinear form $B$ yields an identification between $L^\#$ and the lattice $\text{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p)$ of $\mathbb{Z}_p$-valued linear forms on $L$. The $\sigma$-invariance of $B$ shows that this is an isomorphism of $\mathbb{Z}_p[\sigma]$-modules.
Remark 3.1. As a $\mathbb{Z}_p[\sigma]$-module, we have $L^\# \cong \text{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p)$.

As all indecomposable $\mathbb{Z}_p[\sigma]$-lattices are isomorphic to their homomorphism lattices, we obtain

**Proposition 3.2 ([4, Lemma 5.6]).** If $L \cong R^a \oplus T^b \oplus S^c$ as $\mathbb{Z}_p[\sigma]$-lattice, then also $L^\# \cong R^a \oplus T^b \oplus S^c$.

The group ring $R$ comes with a natural involution, the unique $\mathbb{Z}_p$-linear map $- : R \to R$ with $\overline{\sigma^i} = \sigma^{-i}$ for all $0 \leq i \leq p - 1$. This involution is the restriction of the involution on the maximal order $S \oplus T$ that is trivial on $S$ and the complex conjugation on $T$.

**Remark 3.3.** The $\mathbb{Z}_p$-lattice $R$ is unimodular with respect to the symmetric bilinear form

$$R \times R \to \mathbb{Z}_p, (x, y) \mapsto \frac{1}{p} \text{Tr}_{reg}(x \overline{y})$$

where $\text{Tr}_{reg} : \mathbb{Q}C_p \to \mathbb{Q}$ denotes the regular trace of the $p$-dimensional $\mathbb{Q}$-algebra $\mathbb{Q}C_p$. We thus obtain a bijection between the set of $\sigma$-invariant $\mathbb{Z}_p$-valued symmetric bilinear forms on the $R$-lattice $L$ and the $R$-valued Hermitian forms on $L$: If $h : L \times L \to R$ is such a Hermitian form, then $B = \frac{1}{p} \text{Tr}_{reg} \circ h$ is a symmetric bilinear $\sigma$-invariant form on $L$. As $R = R^\#$, these forms yield the same notion of duality. In particular, the dual lattice $L^\#$ of a free lattice $L = \oplus_{i=1}^a R_{g_i}$ is again free $L^\# = \oplus_{i=1}^a R_{g_i}^*$ with the Hermitian dual basis $(g_1^*, \ldots, g_a^*)$ as a lattice basis, giving a constructive argument for Proposition 3.2 for free lattices.

4. Free elementary lattices. In this section, we assume that $L$ is an elementary lattice and $\sigma$ an automorphism of $L$ of prime order $p$. Recall that $R$ is the commutative ring $R := \mathbb{Z}_p[\sigma]$, so $L$ is an $R$-module.

**Theorem 4.1.** Let $p$ be a prime and let $L$ be an elementary lattice with an automorphism $\sigma$ such that $L \cong R^a$ is a free $R$-module. Then also $L^\# \cong R^a$ and there is an $R$-basis $(g_1, \ldots, g_a)$ of $L^\#$ and $0 \leq t \leq a$ such that $(g_1, \ldots, g_t, pg_{t+1}, \ldots, pg_a)$ is an $R$-basis of $L$. In particular, $L$ is the orthogonal sum of the unimodular free $R$-lattice $L_0 := Rg_1 \oplus \cdots \oplus Rg_t$ and a $p$-modular free $R$-lattice $L_1 := L_0^\perp$.

**Proof.** Under the assumption, both lattices $L$ and $M := L^\#$ are free $R$-modules satisfying $pM \subseteq L \subseteq M$. So, by Theorem 2.2, there is a basis $(g_1, \ldots, g_a)$ of $M$ such that $(g_1, \ldots, g_t, pg_{t+1}, \ldots, pg_a)$ is a basis of $L$. Clearly $L$ is an integral lattice and $L_0 := Rg_1 \oplus \cdots \oplus Rg_t$ is a unimodular sublattice of $L$. By [3, Satz 1.6], unimodular free sublattices split as orthogonal summands, so $L = L_0 \perp L_1$ with $L_1^\# = \frac{1}{p} L_1$, i.e. $L_1$ is $p$-modular.

\[ \square \]

Note that the assumption that the lattice is elementary is necessary, as the following example shows.
Example 4.2. Let $L = R g_1 \oplus R g_2$ be a free lattice of rank 2 with $R$-valued Hermitian form defined by the Gram matrix

$$
\begin{pmatrix}
(p, 0) & (0, \pi) \\
(0, \pi) & (p, 0)
\end{pmatrix}.
$$

Here we identify $R$ as a subring of $S \oplus T$, so $(p, 0) = pe_1 = 1 + \sigma + \cdots + \sigma^{p-1}$ and $(0, \pi) = (0, (1 - \zeta_p)) = 1 - \sigma \in R$. Then $L$ is orthogonally indecomposable because $Le_{\zeta}$ is an orthogonally indecomposable $T$-lattice, but $L$ is not modular. Note that the base change matrix between $(g_1, g_2)$ and the dual basis, an $R$-basis of $L^\#$, is the inverse of the Gram matrix above, so

$$
\begin{pmatrix}
(p^{-1}, 0) & (0, -\overline{\pi}^{-1}) \\
(0, -\pi^{-1}) & (p^{-1}, 0)
\end{pmatrix}.
$$

As $(1, 0) = e_1 \not\in R$, this shows that $pL^\# \not\subseteq L$, so $L$ is not an elementary lattice.

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