On a connection used in deformation quantization

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Abstract

Using natural lifting operations, we give a coordinate-free proof of the fact that the connection used by Bordemann, Neumaier and Waldmann [2] to construct the Fedosov standard ordered star product on the cotangent bundle of a Riemannian manifold is obtained by symplectification of the complete lift of the corresponding Levi-Civita connection, in the sense of Yano and Patterson [13]. In terms of local coordinates, this has already been shown by Plebański, Przanowski and Turrubiates [10].

1 Introduction

In Fedosov deformation quantization [3] of models living on the cotangent bundle $T^*Q$ over a Riemannian manifold $Q$, one uses a lift of the Levi-Civita connection to $T^*Q$ which is torsionless, symplectic and homogeneous to construct a so-called Fedosov derivation. The latter is crucial for the definition of the Fedosov star product defining the deformation quantization structure.

In more detail, let $Q$ be a manifold endowed with a Riemannian metric $g$ and let $\hat{\nabla}$ denote the corresponding Levi-Civita connection. Denote by $T^*Q$ the cotangent bundle of $Q$, by $\pi : T^*Q \to Q$ the natural projection, by $\theta$ the tautological 1-form on $T^*Q$, and by $\omega := d\theta$ the canonical symplectic form. Recall that a torsion-free linear connection $\nabla$ on $T^*Q$ is called

1. a lift of $\hat{\nabla}$ if $\pi' \circ (\nabla_X Y) = (\nabla_{\hat{X}} \hat{Y}) \circ \pi$ for all vector fields $X, Y$ on $T^*Q$ and $\hat{X}, \hat{Y}$ on $Q$ satisfying $\pi' \circ X = \hat{X} \circ \pi$ and $\pi' \circ Y = \hat{Y} \circ \pi$,

2. symplectic if $\nabla \omega = 0$,

3. homogeneous if $[\lambda, \nabla_X Y] = \nabla_{[\lambda, X]} Y - \nabla_X [\lambda Y] = 0$ for all vector fields $X, Y$ on $T^*Q$, where $\lambda$ denotes the Liouville vector field on $T^*Q$. 
It turns out that torsionless, symplectic and homogeneous lifts are not unique, see e.g. [1]. As shown in [2], one option to make the lift unique is to impose the additional condition that

\[ \omega(X_1, R(Y, X_2)X_3 + R(Y, X_3)X_2) + \text{(cyclic permutations of } X_1, X_2, X_3) = 0, \]

where \( R \) denotes the curvature tensor of \( \nabla \), viewed as a 2-form on \( T^*Q \) with values in the 1, 1-tensor fields on \( T^*Q \). In [2], the authors used this connection to construct the Fedosov standard ordered star product on \( T^*Q \). Let us refer to this connection as the BNW lift of \( \hat{\nabla} \). Recently, the BNW lift was used in the context of homological reduction, see [9].

Starting from the classical papers of Yano and Patterson [13, 14], the problem of lifting geometric objects living on a manifold \( Q \) to its cotangent bundle \( T^*Q \), or more generally to some tensor bundle over \( Q \), has been addressed frequently, see [4, 7, 6, 8] and further references therein. Very generally speaking, this problem is related to the discussion of natural geometric operations as treated in [5]. For our purposes, the notion of complete lift of a connection from \( Q \) to \( T^*Q \) as invented in [13] will be crucial. Let us denote this lift by \( \nabla^c \). It is defined as the Levi-Civita connection of a certain pseudo-Riemannian metric \( g^\nabla \) on \( T^*Q \), called the Riemann extension of \( \nabla \), see below for the definition.

Using the above techniques, we prove that the BNW lift \( \nabla \) coincides with the symplectification of the complete lift \( \nabla^c \). After having completed this paper, we learnt that this result has already been obtained earlier by Plebański, Przanowski and Turrubiates [10] using local coordinates.

## 2 Lifting operations for tensor fields

Let us recall the following natural lifting operations turning objects on \( Q \) into objects on \( T^*Q \) [13, 14]. Let \( \langle \cdot, \cdot \rangle \) denote the natural pairing between vectors and covectors, and between 1-forms and vector fields on \( Q \). Every vector field \( X \) on \( Q \) defines a function \( \tilde{X} \) on \( T^*Q \) by

\[ \tilde{X}(p) := \langle p, X_{\pi(p)} \rangle, \quad p \in T^*Q. \]

We will refer to this function as the tautological function defined by \( X \). The lift of a function \( f \) on \( Q \) to \( T^*Q \) is given by the pull-back under \( \pi \),

\[ \nu f := \pi^* f. \]

The lift of a 1-form \( \alpha \) on \( Q \) to \( T^*Q \) is the vertical vector field \( \nu \alpha \) on \( T^*Q \) induced by the complete flow

\[ T^*Q \times \mathbb{R} \to T^*Q, \quad (p, t) \mapsto p + t\alpha(\pi(p)). \]

The lift of 1-forms and the operation sending vector fields \( X \) on \( Q \) to their tautological functions \( \tilde{X} \) on \( T^*Q \) combine to a lifting operation turning 1, 1-tensor fields \( T \) on \( Q \) into...
vector fields \( vT \) on \( T^*Q \). By definition, for 1, 1-tensor fields of the form \( T = X \otimes \alpha \) with a vector field \( X \) and a 1-form \( \alpha \),
\[
v(X \otimes \alpha) = \tilde{X}(v\alpha).
\]

There are two further lifting operations turning vector fields \( X \) on \( Q \) into vector fields on \( T^*Q \), the complete lift \( cX \) induced by the natural symplectic structure of \( T^*Q \) and the horizontal lift \( hX \) defined by the Levi-Civita connection for any tensor bundle over \( Q \). The complete lift \( cX \) is the Hamiltonian vector field generated by the tautological function \( \tilde{X} \), i.e., the unique vector field on \( T^*Q \) satisfying
\[
\omega(\cdot, cX) = d\tilde{X}.
\]

The horizontal lift \( hX \) is uniquely determined by the conditions
\[
\pi' \circ (hX) = X \circ \pi, \quad \tilde{K} \circ (hX) = 0,
\]
where \( \tilde{K} : T(T^*Q) \to T^*Q \) is the connection mapping of \( \nabla \), see eg. Section 1.5 in [12].

Let us derive the relation between \( cX \) and \( hX \). Computations are simplified by the following observation.

**Lemma 1.** If two vector fields \( V \) and \( W \) on \( T^*Q \) coincide on all tautological functions defined by vector fields on \( Q \), then \( V = W \).

A proof using local coordinates was given in [13].

**Proof.** It suffices to show that for all tangent vectors \( V_p \) of \( T^*Q \) based at some \( p \) outside the zero section, the following holds. If \( V_p\tilde{X} = 0 \) for all vector fields \( X \) on \( Q \), then \( V_p = 0 \). To prove this, let such \( p \) and \( V_p \) be given. Put \( q = \pi(p) \) and choose a 1-form \( \alpha \) on \( Q \) such that \( \alpha(q) = p \). Then, \( \alpha'\pi'V_p \) is based at \( p \) and we may take the difference \( V_p - \alpha'\pi'V_p \).

Since \( \pi'(V_p - \alpha'\pi'V_p) = 0 \), there exists \( \xi \in T_qQ \) such that \( V_p - \alpha'\pi'V_p \) is represented by the curve \( t \mapsto p + t\xi \). It suffices to show that \( \pi'V_p = 0 \) and \( \xi = 0 \). For that purpose, we note that
\[
(\alpha'\pi'V_p)\tilde{X} = (\pi'V_p)(\alpha(X)), \tag{2}
\]
\[
(V_p - \alpha'\pi'V_p)\tilde{X} = \xi(X_q) \tag{3}
\]
for every vector field \( X \) on \( Q \). Let \( X_q \in T_qQ \) be given. Using a chart and a bump function centered at \( q \), we can extend \( X_q \) to a vector field \( X \) on \( Q \) in such a way that \( \alpha(X) \) is constant in some neighbourhood of \( q \). Then, (2) implies \( (\alpha'\pi'V_p)\tilde{X} = 0 \), so that (3) yields \( \xi(X_q) = 0 \). Since this holds for all \( X_q \in T_qQ \), we conclude that \( \xi = 0 \). Now, (2) and (3) imply that
\[
(\pi'V_p)(\alpha(X)) = 0 \tag{4}
\]
for all vector fields \( X \) on \( Q \). Let a smooth function \( f \) on \( Q \) be given. Using once again a chart and a bump function centered at \( q \), we can construct a vector field \( X \) on \( Q \) such that \( \alpha(X) = f \) in some neighbourhood of \( q \). Then, (4) yields that \( (\pi'V_p)f = 0 \). Since this holds true for all smooth functions on \( Q \), we obtain \( \pi'V_p = 0 \). This yields the assertion. \( \square \)
The following formulae will be needed throughout the paper.

**Lemma 2.** Let $X$, $Y$ be vector fields, $\alpha$ a 1-form, and $T$ a 1,1-tensor field on $Q$.

1. $\tilde{X} \circ \alpha = \langle \alpha, X \rangle$.
2. $(v\alpha)\tilde{X} = v\langle \alpha, X \rangle$.
3. $(vT)\tilde{X} = (T(X))^\sim$.
4. $\pi' \circ (cX) = X \circ \pi$.
5. $(cX)\tilde{Y} = \omega(cX, cY) = [X, Y]^\sim$.

**Proof.** Points 1–3 are immediate.

4. We evaluate both sides at $p \in T^*Q$ and apply them to a smooth function $f$ on $Q$. For the left hand side, this yields

$$\pi'((cX)_p)f = (cX)_p(vf).$$

Let $H_{\tilde{X}}$ and $H_{vf}$ denote the Hamiltonian vector fields of the functions $\tilde{X}$ and $vf$, respectively. By [11, Prop. 8.3.11], we have $H_{vf} = -v(df)$. Using this and point 2 we find

$$(cX)(vf) = H_{\tilde{X}}(vf) = -H_{vf}\tilde{X} = v(df)\tilde{X} = v(Xf) = \pi^*(Xf).$$

Hence, $\pi'((cX)_p)f = X_{\pi(p)}f$ for all $p$ and $f$. This yields the assertion.

5. By definition of the complete lift,

$$(cX)\tilde{Y} = \langle d\tilde{Y}, cX \rangle = \omega(cX, cY).$$

(5)

From this, we read off that

$$(cX)\tilde{Y} = -(cY)\tilde{X}.$$ (6)

Then, we rewrite

$$\omega(cX, cY) = d\theta(cX, cY) = (cX)\langle \theta, cY \rangle - (cY)\langle \theta, cX \rangle - \langle \theta, [cX, cY] \rangle.$$  

By point 4, we have $\langle \theta, cX \rangle = \tilde{X}$. According to Prop. 3.1.5 in [11], point 4 also implies $\pi' \circ [cX, cY] = [X, Y] \circ \pi$ and hence $\langle \theta, [cX, cY] \rangle = [X, Y]^\sim$. In view of (5), this yields $\omega(cX, cY) = 2(cX)\tilde{Y} - [X, Y]^\sim$. The assertion now follows from (5). □

**Proposition.** For every vector field $X$ on $Q$, one has

$$hX = cX + v(\nabla X).$$
Proof. We have to show that the right hand side satisfies the conditions (1), i.e.,
\[
\pi' \circ (cX + v(\hat{\nabla}X)) = X \circ \pi, \quad \hat{K} (cX + v(\hat{\nabla}X)) = 0.
\] (7)
The first condition follows from point 4 of Lemma 2. To prove the second condition, we use that for every vector field \(X\) on \(Q\), every 1-form \(\alpha\) on \(Q\) and every \(q \in Q\) one has [12, Prop. 1.5.6]
\[
\hat{K}(\alpha'X_q) = (\hat{\nabla}_X\alpha)_q
\] (8)
and that for every \(p \in T^*Q\), \(\hat{K}\) acts on the linear subspace \(T_p(T^*_\pi(p)Q) \subset T_p(T^*Q)\) as the
classical identification of that subspace with the fibre \(T^*_\pi(p)Q\). In view of the definition of
the vertical lift of 1-forms, the latter implies that
\[
\hat{K}((v\alpha)_p) = \alpha(\pi(p))
\] (9)
for all 1-forms \(\alpha\) on \(Q\) and all \(p \in T^*Q\). To evaluate (8), let \(X\), \(\alpha\) and \(q\) be given and
denote \(p = \alpha(q)\). We claim that
\[
\alpha'X_q = (cX)_p + (v(\hat{\nabla}X))_p + (v(\hat{\nabla}_X\alpha))_p.
\] (10)
By Lemma [1] it suffices to evaluate both sides on \(\tilde{Y}\) for an arbitrary vector field \(Y\) on \(Q\).
For the left hand side, point [1] of Lemma [2] yields
\[
(\alpha'X_q)\tilde{Y} = X_q(\tilde{Y} \circ \alpha) = X_q(\alpha(Y)).
\]
For the right hand side, using in addition points [2, 3] and [5] of that lemma and the fact that
\(\hat{\nabla}\) is torsion-free, we find
\[
\left\{(cX)_p + (v(\hat{\nabla}X))_p + (v(\hat{\nabla}_X\alpha))_p\right\}\tilde{Y} = [X,Y]^\sim(p) + (\hat{\nabla}_YX)^\sim(p) + (v(\hat{\nabla}_X\alpha,Y))(p)
= (\hat{\nabla}_XY)^\sim(p) + (v(\hat{\nabla}_X\alpha,Y))(p)
= (v(\alpha, \hat{\nabla}_XY))(p) + (v(\hat{\nabla}_X\alpha,Y))(p)
= (v(X(\alpha,Y)))(p)
= X_q(\alpha,Y).
\]
This proves (10). Next, (9) yields
\[
\hat{K}\left((v(\hat{\nabla}_X\alpha))_p\right) = (\hat{\nabla}_X\alpha)_q.
\] (11)
Now, applying \(\hat{K}\) to both sides of eq. (10) and using (8) and (11), we obtain that the
second condition in (7) holds true. \(\square\)
3 \ BNW lift and complete lift of the Levi-Civita connection

In the sequel, it will be convenient to view 1-forms as mappings $TQ \to \mathbb{R}$ and 1,1-tensor fields as mappings $TQ \to TQ$. According to [2], the BNW lift of $\hat{\nabla}$ is given by

$$\nabla v \alpha (v \beta) := 0, \quad \nabla v(hX) := 0, \quad \nabla hX (v \alpha) := v \left( \hat{\nabla}_X \alpha \right),$$

(12)

$$\nabla hX (hY) := h \left( \hat{\nabla}_X Y \right) + v \left( \frac{1}{2} \hat{\nabla}R(X, Y) + \frac{1}{6} \hat{\nabla}R(X, \cdot) Y + \frac{1}{6} \hat{\nabla}R(Y, \cdot) X \right)$$

(13)

for all vector fields $X, Y$ on $Q$ and 1-forms $\alpha, \beta$ on $Q$, where $\hat{\nabla}$ denotes the curvature tensor of $\hat{\nabla}$. The complete lift of $\hat{\nabla}$ will be denoted by $\hat{\nabla}$. According to [13], this is the Levi-Civita connection of the pseudo-Riemannian metric $\hat{g}$ on $T^*Q$ given by

$$\hat{g}(v \alpha, v \beta) = 0, \quad \hat{g}(v \alpha, cX) = v(\alpha(X)), \quad \hat{g}(cX, cY) = -v(\hat{\nabla}_X Y + \hat{\nabla}_Y X)$$

for all vector fields $X, Y$ on $Q$ and 1-forms $\alpha, \beta$ on $Q$. This metric is referred to as the Riemann extension of $\hat{\nabla}$ in [13]. Explicitly,

$$\hat{\nabla} v \alpha (v \beta) = 0, \quad \hat{\nabla} v \alpha (cX) = -v(\alpha \circ \hat{\nabla}X), \quad \hat{\nabla} cX (v \alpha) = v(\hat{\nabla}_X \alpha)$$

$$\hat{\nabla} cX (cY) = c(\hat{\nabla}_X Y) + v \left( \hat{\nabla}_X \circ \hat{\nabla}Y + \hat{\nabla}Y \circ \hat{\nabla}X - \hat{\nabla}R(X, \cdot) Y - \hat{\nabla}R(Y, \cdot) X \right)$$

for all vector fields $X, Y$ on $Q$ and 1-forms $\alpha, \beta$ on $Q$.

We will show that the BNW lift $\nabla$ arises from the complete lift $\hat{\nabla}$ by symplectification in the sense of [1]. We proceed by first rewriting $\hat{\nabla}$ in terms of horizontal lifts and then applying the symplectification procedure. For that purpose, we need knowledge on how $\hat{\nabla}$ acts on the vertical lifts of 1,1-tensor fields on $Q$.

Lemma 3. Let $X$ be a vector field on $Q$, $\alpha$ a 1-form on $Q$ and let $T, S$ be 1,1-tensor fields on $Q$.

1. $\hat{\nabla} v \alpha (v T) = v(\alpha \circ T)$,
2. $\hat{\nabla} v \alpha (cX) = 0$,
3. $\hat{\nabla} cX (v T) = v(\hat{\nabla}_X T) - v(\hat{\nabla}X \circ T)$,
4. $\hat{\nabla} v \alpha (cS) = -v(T \circ \hat{\nabla}X)$,
5. $\hat{\nabla} v \alpha (v S) = v(T \circ S)$.
\textbf{Proof.} It suffices to prove all formulae for $T$ being of the form $T = Y \otimes \beta$ with a vector field $Y$ on $Q$ and a 1-form $\beta$ on $Q$. In the computations, we use the properties of connection and the formulae of Lemma \ref{le:connectionproperties}.

1. $\hat{\nabla}_v(Y \otimes \beta) = \hat{\nabla}_v(\hat{Y} \otimes \beta) = (v(V)\hat{Y}) \otimes \beta = (v(\alpha, Y)) \otimes \beta = v(\alpha \circ Y \otimes \beta)$.
2. $\hat{\nabla}_v(Y \otimes \beta)(\nu) = \hat{\nabla}_{\hat{Y}}(\nu) = \hat{Y} \hat{\nabla}_v(\nu) = 0$.

3. We find
   \[
   \hat{c}_X(v(Y \otimes \beta)) = [X, Y]^{\circ} v\beta + \hat{Y}v(\hat{\nabla}_X \beta) = v\left([X, Y] \otimes \beta + Y \otimes \hat{\nabla}_X \beta\right). 
   \]

The first summand can be replaced by $v(\hat{\nabla}_X Y \otimes \beta - \hat{\nabla}_Y X \otimes \beta)$. Thus
   \[
   \hat{c}_X(v(Y \otimes \beta)) = v\left(\hat{\nabla}_X(Y \otimes \beta)\right) - v\left(\hat{\nabla}_X \circ (Y \otimes \beta)\right). 
   \]

4. $\hat{c}_v(Y \otimes \beta)(cX) = \hat{Y} \hat{\nabla}_v(cX) = -\hat{Y}v(\beta \circ \hat{\nabla}_X) = -v((Y \otimes \beta) \circ \hat{\nabla}_X)$.
5. $\hat{c}_v(Y \otimes \beta)(cS) = \hat{Y} \hat{\nabla}_v(cS) = \hat{Y}v(\beta \circ S) = v((Y \otimes \beta) \circ S).$ \hfill \qed

Now, we are prepared for rewriting $c$ in terms of horizontal lifts.

\textbf{Lemma 4.} In terms of the horizontal lift operation, the complete lift of $\hat{\nabla}$ is given by

\[
\begin{align*}
\hat{c}_v(v\beta) & = 0, & \hat{c}_v(hX) & = 0, & \hat{c}_hX(\nu) & = v(\hat{\nabla}_X \nu) \\
\hat{c}_hX(hY) & = h(\hat{\nabla}_X Y) - v\left(\hat{R}(Y, \cdot)X\right) & 
\end{align*}
\]

for all vector fields $X$, $Y$ on $Q$ and 1-forms $\alpha$, $\beta$ on $Q$.

The last formula may be written more symmetrically in the form

\[
\hat{c}_hX(hY) = h(\hat{\nabla}_X Y) - \frac{1}{2}v\left(\hat{R}(X, Y) + \hat{R}(X, \cdot)Y + \hat{R}(Y, \cdot)X\right). \tag{14} 
\]

\textbf{Proof.} The first formula holds by definition of $\hat{\nabla}$ and the second and the third formula follow immediately from the proposition and Lemma \ref{le:connectionproperties}. To prove the last formula, we compute

\[
\hat{c}_hX(hY) = h(\hat{\nabla}_X Y) + v\left(\hat{\nabla}Y \circ \hat{\nabla}_X - \hat{\nabla}_X(\hat{\nabla}Y) - \hat{R}(X, \cdot)Y - \hat{R}(Y, \cdot)X\right), 
\]

where the argument of $v(\cdot)$ is a 1,1-tensor field on $Q$. Evaluation of this term on $Z$ for some vector field $Z$ on $Q$ yields the tautological function of the vector field on $Q$ given by

\[
\hat{\nabla}_Z X Y - \hat{\nabla}_Z \hat{\nabla}_X Y + \hat{\nabla}_X \hat{\nabla}_Z Y - \hat{\nabla}_X \hat{c}_X Y - \hat{R}(X, Z) Y - \hat{R}(Y, Z) X. 
\]

Since $\hat{\nabla}$ is torsion-free, the first 4 terms combine to $\hat{R}(X, Z) Y$. This yields the last formula. \hfill \qed
Next, we symplectify $\nabla$ according to [1]. For that purpose, we define a 1,2-tensor field $N$ on $T^*Q$ by
\[
\omega(N(V,W),U) = (\nabla_V\omega)(W,U)
\]
for all vector fields $U$, $V$, $W$ on $T^*Q$. It is easy to check that
\[
\nabla^s V := \nabla V + \frac{1}{3}N(V,W) + \frac{1}{3}N(W,V)
\]
defines a connection on $T^*Q$ and that this connection is symplectic. To determine $\nabla^s$, we have to compute $N$. For that purpose, we have to evaluate $\omega$ on vertical lifts of 1-forms and 1,1-tensor fields on $Q$, and on horizontal lifts of vector field on $Q$.

**Lemma 5.** Let $X, Y$ be vector fields on $Q$, let $\alpha, \beta$ be 1-forms on $Q$, and let $T, S$ be 1,1-tensor fields on $Q$.

1. $\omega(v\alpha, v\beta) = \omega(v\alpha, vT) = \omega(vT, vS) = 0$.
2. $\omega(v\alpha, hX) = \langle \alpha, X \rangle$.
3. $\omega(vT, hX) = (T(X))\tilde{}$.
4. $\omega(hX, hY) = 0$.

**Proof.**

1. This follows from the fact that the fibres of $T^*Q$ are isotropic.

2. We have
\[
\omega(v\alpha, hX) = (v\alpha)\langle \theta, hX \rangle - (hX)\langle \theta, v\alpha \rangle - \langle \theta, [v\alpha, hX] \rangle.
\]
The second term vanishes, because $v\alpha$ is vertical. Formula (11) implies $\langle \theta, hX \rangle = \tilde{}X$, so that point 2 of Lemma 2 yields $\langle \alpha, X \rangle$ for the first term. For the last term, we evaluate $[v\alpha, hX]\tilde{Y}$ for an arbitrary vector field $Y$ on $Q$. Decomposing $hX$ according to the proposition and using the formulae of Lemma 2, we find
\[
[v\alpha, hX]\tilde{Y} = v\langle (\alpha, [X, Y]) + \langle \alpha, \tilde{\nabla}_Y X \rangle - X(\alpha, Y) \rangle.
\]
Since $\tilde{\nabla}$ is torsion-free, the terms on the right hand side combine to $-v(\tilde{\nabla}_X \alpha, Y)$. Thus,
\[
[v\alpha, hX] = -v(\tilde{\nabla}_X \alpha).
\]

Since this is vertical, the last term in (16) vanishes, and the assertion follows.

3. It suffices to check this for $T = Y \otimes \alpha$ for any vectors field $Y$ on $Q$ and any 1-form $\alpha$ on $Q$. By point 2
\[
\omega(v(Y \otimes \alpha), hX) = \tilde{Y} \omega(v\alpha, hX) = \tilde{Y} v\langle \alpha, X \rangle.
\]
Using the formulae of Lemma 2 this can be rewritten as $(v(Y \otimes \alpha)(X))\tilde{}$.

4. Using the proposition and the fact that the fibres of $T^*Q$ are isotropic, we can rewrite
\[
\omega(hX, hY) = \omega(cX, cY) + \omega(v(\tilde{\nabla}X), hY) + \omega(hX, v(\tilde{\nabla}Y))
\]
By point 3 and the fact that $\tilde{\nabla}$ is torsion-free, the last two terms yield
\[
(\tilde{\nabla}_Y X - \tilde{\nabla}_X Y)\tilde{} = [Y, X]\tilde{}.
\]
By point 5 of Lemma 2 the first term evaluates to $\omega(cX, cY) = [X, Y]\tilde{}$. 

\[\square\]
Remark. Point 4 states that the distribution on $T^*Q$ consisting of the horizontal subspaces is isotropic (in fact, Lagrangian). It thus provides a Lagrangian complement to the Lagrangian distribution of the fibre tangent spaces. This comes as no surprise, as the Riemannian metric on $Q$ has a natural lift to $T^*Q$ and the latter combines with the symplectic form to a Kähler structure on $T^*Q$.

Now, we can determine $N$.

Lemma 6. Let $X, Y$ be vector fields on $Q$ and let $\alpha, \beta$ be 1-forms on $Q$.

1. $N(\alpha, \alpha) = N(\alpha, hX) = N(hX, \alpha) = 0$.
2. $N(hX, hY) = 2v(\hat{R}(Y, \cdot)X)$.

Proof. For every combination of arguments, we have to compute $\omega(N(\cdot, \cdot), v\gamma)$ for any 1-form $\gamma$ on $Q$ and $\omega(N(\cdot, \cdot), hZ)$ for every vector field $Z$ on $Q$.

By definition of $N$ and the derivation property of connection,

$$\omega(N(\alpha, \alpha), \alpha) = \omega(\alpha, \alpha) = 0,$$

$$\omega(N(\alpha, hX), \alpha) = \omega(\alpha, hX) = 0,$$

$$\omega(N(hX, \alpha), \alpha) = 0.$$

According to Lemmas 4 and 5, each of the terms on the right hand side vanishes, no matter what $\#_1$ and $\#_2$ are. Thus, $N(\alpha, \alpha) = 0$ and $N(\alpha, hX) = 0$. Analogous calculations yield $\omega(N(hX, \alpha), v\gamma) = 0$ and $\omega(N(hX, h\alpha), h\alpha) = 0$, due to the derivation property of connection. Here, we have also used that $(hX)(vf) = v(Xf)$ for all smooth functions on $Q$, which follows at once from the first of the defining relations for $hX$ given in (1). Thus, $N(hX, \alpha) = 0$. Finally, we find

$$\omega(N(hX, \alpha), \alpha) = 0,$$

$$\omega(N(hX, hY), \alpha) = 0,$$

$$\omega(N(hX, hY), h\alpha) = 0,$$

$$\omega(N(hX, hY), hZ) = 0,$$

$$\omega(N(hX, hY), hZ) = 0.$$

Since

$$\omega(v(\hat{R}(Y, \cdot)X), v\gamma) = 0,$$

$$\omega(v(\hat{R}(Y, \cdot)X), h\alpha) = 0,$$

$$\omega(v(\hat{R}(Y, \cdot)X), hZ) = 0,$$

this yields the formula asserted for $N(hX, hY)$. 

By plugging the formulae of Lemmas 4 and 5, together with (14), into (15) and comparing the resulting formulae for $\nabla^s$ with (12) and (13), we finally obtain

Theorem. The BNW lift of $\hat{\nabla}$ is obtained from the complete lift by symplectification in the sense of [1].
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