A general formula for the correlation function in redshift space is derived in linear theory. The formula simultaneously includes wide-angle effects and cosmological distortions. The formula is applicable to any pair with arbitrary angle \( \theta \) between lines of sight, and arbitrary redshifts \( z_1, z_2 \), which are not necessarily small. The effects of the spatial curvature both on geometry and on the fluctuation spectrum are properly taken into account, and thus our formula holds in a Friedman-Lemaître universe with arbitrary cosmological parameters \( \Omega_0 \) and \( \lambda_0 \). We illustrate the pattern of the resulting correlation function with several models, and also show that the validity region of the conventional distant observer approximation is \( \theta \leq 10^\circ \).

Subject headings: cosmology: theory — galaxies: distances and redshifts — quasars: general — large-scale structure of universe — methods: statistical

1. INTRODUCTION

Ever since the pioneering work of Totsuji & Kihara (1969) and Peebles (1974), the two-point correlation function of galaxies has been one of the most fundamental tools for analyzing the large-scale structure of the universe. Recently, prominent advances in galaxy redshift surveys have been taking place (for a review, see Strauss 1999), and upcoming large-scale galaxy and QSO surveys, notably the Two-Degree Field Survey (2dF; Colless 1998; Folkes 1999), and the Sloan Digital Sky Survey (SDSS; Gunn & Weinberg 1995; Margon 1998), will provide three-dimensional, large-scale redshift maps for large-scale galaxy and QSO surveys, notably the Two-Degree Field Survey (2dF; Colless 1998; Folkes 1999), and the Sloan Digital Sky Survey (SDSS; Gunn & Weinberg 1995; Margon 1998), will provide three-dimensional, large-scale redshift maps of galaxies and QSOs. The two-point correlation function will play a central role in the analysis of such large-scale redshift maps.

In redshift surveys, the distances to objects are measured by recession velocities, and thus the distribution of objects in redshift space is not identical to that in real space. The clustering pattern is distorted by peculiar-velocity fields (Kaiser 1987) and, for high-redshift objects, by the cosmological warp of real space on a light cone (Alcock & Paczyński 1979). Such effects are called redshift distortions. These effects on the linear power spectrum and on the linear two-point correlation function have been investigated by many authors (for a review, see Hamilton 1998).

Most of the work that has been done addresses redshift distortions in the nearby universe and assumes \( z \ll 1 \). Kaiser (1987), in his seminal paper, derived the linear redshift distortions of the power spectrum for the nearby universe, employing the distant-observer approximation, which assumes that the scales of fluctuations of interest are much smaller than the distances to the objects. Generally, redshift distortions caused by peculiar velocities are along the line of sight. The radial nature of this distortion introduces a statistical inhomogeneity into the redshift space. In the distant-observer approximation, such inhomogeneity is neglected and the statistical homogeneity is recovered, while anisotropy is introduced instead. The Fourier spectrum has maximal advantage when statistical homogeneity does exist; for this reason, Kaiser’s formula has a very simple form. Hamilton (1992) transformed Kaiser’s formula for the power spectrum to the formula of the two-point correlation function, using the Legendre expansion (see also Lilje & Efstathiou 1989; McGill 1990).

Despite the simple form of the formula, the distant-observer approximation is not desirable because we cannot make use of all the information of the survey within this approximation. The wide-angle effect, which is the contribution of an angle \( \theta \) between lines of sight of two objects to the correlation function, affects the analysis of the redshift maps when we intend to use the entire data of the survey. Therefore, the analysis of wide-angle effects on the correlation function in linear theory (Hamilton & Culhane 1996; Zaroubi & Hoffmann 1996; Szalay, Matsubara, & Landy 1998; Bharadwaj 1999), as well as spherical harmonic analysis (Fisher, Scharf, & Lahav 1994; Heavens & Taylor 1995), are of great importance.

All the above studies are for the nearby universe, and assume \( z \ll 1 \). This condition is not appropriate for modern galaxy redshift surveys (\( z \sim 0.2 \)) or QSO redshift surveys (\( z \sim 2 \)). The evolution of clustering and the nonlinearity in the redshift-distance relation introduce a cosmological redshift distortion on the correlation function. Ballinger, Peacock, & Heavens (1996) and Matsubara & Suto (1996) explored this effect in power spectra and correlation functions, respectively, both employing the distant-observer approximation (see also Nakamura, Matsubara & Suto 1998; de Laix & Starkman 1998; Popowski et al. 1998; Suto et al. 1999; Nishioka & Yamamoto 1999; Nair 1999). The more the depth of the redshift surveys increases, the more important the cosmological redshift distortion becomes.

So far, the previous formulas for the two-point correlation function in redshift space have been restricted to either nearby universe or distant-observer approximations. The purpose of the present paper is to derive a unified formula for the linear redshift distortions of the correlation function, fully and simultaneously taking into account wide-angle effects and cosmo-
logical distortions. In this way, we do not have to think about which formula we should use, depending on the value of \( z \) and \( \theta \). In addition, the effects of spatial curvature are included in our formulation, and our formula applies to a Friedman-Lemaître universe with arbitrary cosmological parameters \( \Omega_0 \) and \( \Lambda_0 \). Our formula turns out to correctly reproduce the known results if we take appropriate limits of the formula.

In § 2, the redshift-space distortions of density fluctuations in radial directions are derived on a light cone, which include the evolutionary effects. In § 3, the formula for the correlation function in redshift space is derived separately for open, flat, and closed models. Demonstrations of redshift distortions in several cases and the validity region of the conventional distant observer approximation are given in § 4. Conclusions are summarized in § 5.

2. LINEAR REDSHIFT-SPACE DISTORTIONS ON A LIGHT CONE

2.1. Radial Distortion of the Linear Density Field on a Light Cone

Throughout the present paper, we assume the Friedman-Lemaître-Robertson-Walker (FLRW) metric as the background spacetime of the universe. According to the sign of the spatial curvature \( K \), the metric is given by

\[
ds^2 = -c^2 dt^2 + \begin{cases} \frac{a^2(t)}{-K} \left[ d\chi^2 + \sinh^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2) \right] & K < 0 , \\
\frac{a^2(t)}{2} \left[ d\chi^2 + \chi^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] & K = 0 , \\
\frac{a^2(t)}{K} \left[ d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2) \right] & K > 0 ,
\end{cases}
\]  

where \( \theta \) and \( \phi \) are usual angular coordinates, and \( \chi \) is a radial comoving coordinate. In the following discussion, we choose a unit system \( c = 1 \) and \( H_0 = 1 \), where \( H_0 = 100 \) \( h \) km s\(^{-1}\) Mpc\(^{-1}\) is the Hubble constant, so that comoving distances are measured in units of \( cH_0^{-1} = 2997.9 \) h\(^{-1}\) Mpc. In this unit system, the spatial curvature \( K \) is then given by

\[
K = \Omega_0 + \Lambda_0 - 1 ,
\]  

where the scale factor at present is normalized as \( a_0 = 1 \). From the FLRW metric, we define the three-dimensional metric \( \gamma_{ij} \) as

\[
ds^2 = -dt^2 + a^2 \gamma_{ij} dx^i dx^j .
\]  

That is, the tensor \( \gamma_{ij} \) is the metric for a 3-space of uniform spatial curvature \( K \). In the following discussion, vectors with Latin indices represent the 3-vector in the 3-space specified by the metric \( \gamma_{ij} \).

The comoving distance \( x \) at redshift \( z \) is given by

\[
x(z) = \int_0^z \frac{dz'}{H(z')} ,
\]  

where \( H(z) \) is the Hubble parameter at (comoving) redshift \( z \):

\[
H(z) = \sqrt{(1 + z)^3 \Omega_0 + (1 + z)^2(1 - \Omega_0 - \Lambda_0) + \Lambda_0} .
\]  

The radial comoving coordinate \( \chi \) and the comoving distance \( x \) are related by

\[
\chi = \begin{cases} |K|^{1/2} x(z) & K \neq 0 , \\
x(z) & K = 0 .
\end{cases}
\]  

Provided that a cosmological model is fixed, the comoving distance \( \chi \) and the redshift \( z \) are related through equation (2.6) on the observable past-light cone, and we interchangeably use these variables. Therefore, we use the redshift \( z \) as an alternative spatial coordinate. On a light-cone surface, it is also considered as an alternative to the time variable, \( t \).

In this paper, a light ray is assumed to be transmitted on an unperturbed metric, for simplicity. This point is discussed later.

Then, in redshift space, the angular coordinates \( (\theta, \phi) \) are common to those in real space. Only the radial distances are distorted in redshift space. Let us consider an object located at comoving coordinates \( (t, x^i) \), having a 4-velocity \( u^\mu \), normalized as \( u^\mu u_\mu = -1 \), as usual. In terms of the three-dimensional peculiar velocity, \( v^i \), this 4-velocity is given by

\[
u^\mu = \frac{(1, v^i)}{\sqrt{1 - a^2 v^i v_i}} .
\]  

The wave 4-vector, \( k^\mu \), satisfying the null geodesic equations, \( k^\mu k_\mu = 0 \) and \( k^\mu k_\nu = 0 \), is given by

\[
k^\mu \propto (a^{-1}, -a^{-3} n^i) , \quad k_\mu \propto (-a^{-1}, -n_i) ,
\]  

where \( n^i \) is a three-dimensional normal vector, representing the line of sight, and \( n_i = \gamma_{ij} n^j \). The frequencies of the light at the source and the observer are given by \( v_0 = (k_\mu u^\mu)_0 \), and \( v_0 = (k_\mu u^\mu)|_{\text{ob}} \), respectively. Assuming that the peculiar velocities are nonrelativistic, the redshift the observer actually observes, \( z_{\text{obs}} \), is given by

\[
1 + z_{\text{obs}} = \frac{v_0}{v_0} = (1 + z)(1 + W - W_0) ,
\]
where $W = an_i v^i$ and $W_0 = n_i v_0^i$ are the line-of-sight components of the peculiar velocities of the source and the observer, respectively. The peculiar velocity $W$ is evaluated on a light cone, and $W_0$ can be estimated from the value of the dipole anisotropy of the cosmic microwave background (CMB) radiation.

For convenience, we define the redshift-space physical comoving distance as $s(z) = x(z_{obs})$, where $z_{obs}$ is given by equation (2.9), and the function $x(z)$ is formally the same as equation (2.4), by definition. Explicitly, $s(z)$ is defined by

$$s(z) = \int_0^{z + (1 + z)(W - W_0)} \frac{dz'}{H(z')}.$$  \hspace{1cm} (2.10)

In other words, the redshift-space physical comoving distance $s$ defined by equation (2.10) is the apparent physical comoving distance of an object in redshift space that is originally at redshift $z$ in real space and is shifted by its own peculiar velocity. In the limit $z_{obs} \to 0$, equation (2.10) reduces to the usual relation for the nearby universe (Kaiser 1987), $s = z_{obs}/H_0 = x + (W - W_0)/H_0$. From the redshift-space physical comoving distance $s$, we also define the redshift-space analog of the comoving coordinate, $x_s$, as $x_s(z) = x(z_{obs})$, or equivalently,

$$x_s = \begin{cases} |K|^{1/2}s(z), & K \neq 0, \\ s(z), & K = 0. \end{cases} \hspace{1cm} (2.11)$$

The difference between the number density of observed objects in real space, $n^0(\chi, \theta, \phi, t)$, and that in redshift space, $n^{00}(\chi_s, \theta, \phi)$, are related by the number conservation

$$n^{00}(\chi_s, \theta, \phi) \sin^2 \chi \sinh^2 \chi_s d\chi_s d\Omega = n^0(\chi, \theta, \phi) \sin^2 \chi d\chi d\Omega$$  \hspace{1cm} (2.12)

for $K < 0$,

$$n^{00}(\chi_s, \theta, \phi) \chi_s^2 d\chi_s d\Omega = n^0(\chi, \theta, \phi) \chi^2 d\chi d\Omega$$  \hspace{1cm} (2.13)

for $K = 0$,

$$n^{00}(\chi_s, \theta, \phi) \sin^2 \chi \sinh^2 \chi_s d\chi_s d\Omega = n^0(\chi, \theta, \phi) \sin^2 \chi d\chi d\Omega$$  \hspace{1cm} (2.14)

for $K > 0$,

where $d\Omega = \sin \theta d\theta d\phi$, and $n^{00}$ is evaluated on a light cone. Therefore, $\delta^{00}(\chi_s, \theta, \phi)$ is formally the same as equation (2.4), by definition. Explicitly, $\delta^{00}(\chi_s, \theta, \phi)$ is related to $\delta(\chi, \theta, \phi)$ by

$$\delta^{00}(\chi_s, \theta, \phi) = \frac{n^{00}(\chi_s, \theta, \phi)}{\rho(\chi_s, \theta, \phi)} - 1 = \delta(\chi, \theta, \phi) - \frac{\delta}{\delta \chi} \left( \frac{U}{aH} \right) U - \left[ \frac{A(\chi)}{aH} - \tilde{q}_{dec} \right] U_0,$$  \hspace{1cm} (2.15)

where $U = |K|^{1/2}W$ and $U_0 = |K|^{1/2}W_0$ for a nonflat universe, and $U = W$ and $U_0 = W_0$ for a flat universe. With this expansion, the density contrast, $\delta^{00}(\chi, \theta, \phi)$, up to the first order is given by

$$\delta^{00}(\chi, \theta, \phi) = \frac{n^{00}(\chi_s, \theta, \phi)}{\rho(\chi_s, \theta, \phi)} - 1 = \delta(\chi, \theta, \phi) - \frac{\delta}{\delta \chi} \left( \frac{U}{aH} \right) U - \left[ \frac{A(\chi)}{aH} - \tilde{q}_{dec} \right] U_0,$$  \hspace{1cm} (2.16)

where

$$A(\chi) = \begin{cases} \frac{\cosh \chi}{\sin \chi} \left( 2 + \frac{\delta}{\delta \ln \Phi} \right) & K < 0, \\ \frac{1}{\chi} \left( 2 + \frac{\delta}{\delta \ln \Phi} \right) & K = 0, \\ \frac{\cos \chi}{\sin \chi} \left( 2 + \frac{\delta}{\delta \ln \Phi} \right) & K > 0, \end{cases} \hspace{1cm} (2.17)$$

and $\tilde{q}_{dec}(z) = -d(a^{-1}H^{-1})/d\chi, \Omega = 8\pi G \rho/(3H^2)$, and $\Lambda = \Lambda/(3H^2)$ are the normalized time-dependent deceleration parameter, the time-dependent density parameter, and the dimensionless cosmological term, respectively. The parameter $\tilde{q}_{dec}$ is equal to...
the usual time-dependent deceleration parameter $\Omega/2 - \lambda$ for a flat universe, and $|K|^{-1/2}$ times the usual deceleration parameter for a nonflat universe. The light-cone effect of density contrast in redshift space is represented by equation (2.16). It is obvious that this equation is a generalization of corresponding formula derived by Kaiser (1987) for $z \to 0$.

In the above equations, the light ray is assumed to be transmitted on an unperturbed metric. In reality, the frequency of the light is altered by the Sachs-Wolfe effect (Sachs & Wolfe 1967), and the path is bent by the gravitational lensing effect. The Sachs-Wolfe effect is the contribution of the potential fluctuations to the estimate of the redshift. The potential fluctuations are negligible except on scales comparable to the Hubble distance, and in reality, observational determination of the fluctuations on Hubble scales is not easy. Therefore, the Sachs-Wolfe effect on the correlation function of the density field is not supposed to be important in practice.

Gravitational lensing changes the position on the sky of observable objects. Weak lensing (Kaiser 1992, 1998; Bernardeau, Waerbeke, & Mellier 1997) is relevant in our linear analysis. The estimate of the number density turns out to be affected by the local convergence of the gravitational lensing. The local convergence is given by the integration of the density fluctuations along the line of sight (Bernardeau et al. 1997), and it is efficient for $z \gtrsim 1$. Therefore, it is possible that gravitational lensing affects the correlation function for $z \gtrsim 1$, in which case we should add a correction term to equation (2.16). A quantitative estimate of this effect is beyond the scope of this paper, and will be investigated in a future paper.

2.2. Redshift-Space Distortion Operator

The equations of motion relate the density contrast $\delta$ and the velocity field $U$. Since we intend to include the curvature effect, the Newtonian equations of motion are not appropriate. Instead, we employ the perturbed Einstein equation,

$$\delta G^a_\nu = 8\pi G \delta T^a_\nu,$$

(2.18)
on a background FLRW metric. In order to avoid the inclusion of spurious gauge modes in the solution, Bardeen (1980) introduced the gauge-invariant formalism for the above perturbed equation (see also Kodama & Sasaki 1984; Abbott & Schaefer 1986; Hwang & Vishniac 1990). In Appendix A, we review the gauge-invariant formalism for the pressureless fluid, introduced the gauge-invariant formalism for the above perturbed equation (see also Kodama & Sasaki 1984; Abbott & Schaefer 1986; Hwang & Vishniac 1990). In Appendix A, we review the gauge-invariant formalism for the pressureless fluid, and derive complete equations to determine the evolution of the density contrast and the velocity field. These are given by equations (A22)–(A24) as

$$\dot{\delta} + (\Delta + 3K)\delta = 0,$$

(2.19)

$$\dot{\psi} + 2H\psi + \Phi = 0,$$

(2.20)

$$\Delta + 3K)\Phi = \frac{3}{2}H^2\Omega \delta,$$

(2.21)

where an overdot denotes differentiation with respect to the proper time, $d/dt$, and $\Delta = \nabla^i \nabla_i$ is the Laplacian on the 3-metric, $\gamma_{ij}$ of the FLRW metric, which is explicitly given in Appendix B, equation (B1). As shown in Appendix A, the transverse part of the velocity field decays with time as $a^{-2}$ and can be neglected; thus, the velocity is characterized by the velocity potential $\psi$, so that the velocity is given by $v^i = \nabla^i \psi$. The variables $\delta$, $\psi$, and $\Phi$ in the above equations are actually gauge-invariant linear combinations, defined in Appendix A, and are guaranteed not to have a spurious gauge mode in the solution. These variables correspond to the density contrast, velocity potential, and gravitational potential inside the particle horizon. In general, $\delta$ and $\psi$ correspond to the density contrast and velocity in the velocity-orthogonal isotropic gauge, $V = B, H_T = 0$, in the notation of Appendix A. It is obvious that equations (2.20)–(2.21) correspond to the continuity, Euler, and Poisson equations in Newtonian linear theory. The only difference is the appearance of the curvature term in the Laplacian. The curvature term would not be important in practice, because the correlation function on curvature scales is too small to be practically detectable. However, we retain the curvature term for theoretical consistency in the following discussion.

Eliminating the variables $\psi$ and $\Phi$ from the equations, we obtain the evolution equation for the Bardeen gauge-invariant density contrast $\delta$, given by

$$\dot{\delta} + 2H\delta - \frac{3}{2}H^2\Omega \delta = 0.$$

(2.22)

This equation is equivalent to that in Newtonian theory for the density contrast, and the solution of this equation is well known (Peebles 1980). The time dependence of the growing solution is given by

$$D(t) \propto \Omega \int_0^1 \frac{dx}{(\Omega/x + \lambda x^2 + 1 - \Omega - \lambda)^{3/2}}.$$

(2.23)

In the following discussion, we normalize this growing factor as $D(t_0) = 1$, where $t_0$ is the present time. Thus, the growing solution of equation (2.22) is given by

$$\delta(t, \theta, \phi, t) = D(t)\delta(\chi, \theta, \phi),$$

(2.24)

where $\delta(\chi, \theta, \phi) = D(t_0)\delta_0(\chi, \theta, \phi)$ is the mass-density contrast at the present time.

Equation (2.20) gives the solution of the velocity potential as

$$\psi = -HDf(\Delta + 3K)^{-1}\delta_0,$$

(2.25)

where $f = [(a/D)da]/da = D/(HD)$, and $H = \dot{a}/a$ is the Hubble parameter at time $t$. The inverse operator $(\Delta + 3K)^{-1}$ is evaluated by spectral decomposition in the next section. It turns out that the function $f$ depends only on $\Omega$ and $\lambda$, and is approximately given by (Lahav et al. 1991)

$$f = \Omega^{0.6} + \frac{\lambda}{70} \left(1 + \frac{\Omega}{2}\right).$$

(2.26)
From equation (2.25),
\[ W = an^2 \nabla \cdot \psi = aHDF \frac{\partial}{\partial x} (\Delta + 3K)^{-1} \delta_0 . \]  
(2.27)

Since \( \frac{\partial}{\partial x} = |K|^{1/2} \frac{\partial}{\partial \chi} \) for a nonflat universe and \( \frac{\partial}{\partial x} = \frac{\partial}{\partial \chi} \) for a flat universe, equation (2.27) is equivalent to
\[ U(x, \theta, \phi) = aHDF \left\{ \frac{|K|}{\Delta + 3K} \left( \frac{\partial}{\partial \chi} \right) \right\}_{\Delta - 1} \delta_0 (x, \theta, \phi) \]
(2.28)
where \( \frac{\partial}{\partial \chi} = \frac{\partial}{\partial \chi} \).

Equations (2.16) and (2.28) indicate the linear operator that transforms the density contrast at present in real space to that in redshift space on a light cone. That is, we can define the redshift distortion operator as
\[ \hat{R} = 1 + \beta \left\{ \frac{|K|}{\Delta + 3K} \left( \frac{\partial}{\partial \chi} \right) \right\}_{\Delta - 1} \delta_0 (x, \theta, \phi) \]
where
\[ \alpha(\chi) = \begin{cases} \frac{\cosh \chi}{\sinh \chi} \left[ 2 + \frac{\partial \ln (D/\Phi)}{\partial \ln \chi} \right] & K < 0, \\ \frac{1}{\chi} \left[ 2 + \frac{\partial \ln (D/\Phi)}{\partial \ln \chi} \right] & K = 0, \\ \frac{\cos \chi}{\sin \chi} \left[ 2 + \frac{\partial \ln (D/\Phi)}{\partial \ln \chi} \right] & K > 0. \end{cases} \]
(2.30)

The time-dependent redshift-distortion parameter is defined by \( \beta(z) = f(z)/b(z) \), where \( b(z) \) is the time-dependent bias parameter. In the following discussion, we present the result for the case in which there is neither stochasticity nor scale-dependence in the biasing. It is straightforward to include the scale dependence and stochasticity, although the expression becomes more tedious. In the linear regime, however, the stochasticity is shown to vanish asymptotically on large scales, and biasing is scale-independent beyond the scale of galaxy formation, except in some special cases (Matsubara 1999).

For surveys of the nearby universe, \( z \ll 1 \), the functions \( D \) and \( f \) do not vary as rapidly as the selection function, \( \Phi \), and the factor \( DF \) in the above equations can be omitted, as is done in the literature. However, if the selection function varies as slowly as the factor \( DF \), the latter factor cannot be neglected. With the redshift distortion operator, we can reexpress the density contrast in redshift space as
\[ \delta(x, \theta, \phi) = bD\hat{R}_0 \delta_0 (x, \theta, \phi) + \left[ \frac{1 + z}{H(z)} A(\chi) - q_{dec}(z) \right] U_0 . \]
(2.31)

This expression depends on the peculiar motion of the observer, \( U_0 \), which somewhat complicates the analysis. Since we know the value of \( U_0 \) by measuring the dipole anisotropy of the CMB radiation (Kogut et al. 1993; Lineweaver et al. 1996), we can subtract the term that depends on \( U_0 \) from the above expression. In the following discussion, we derive the correlation function in redshift space from the observer in the CMB frame and drop the \( U_0 \) term. In a proper comparison of our result below and observations, one should be sure that the correlation function is corrected by such a transformation to the CMB frame (Hamilton 1998).

The correlation function in redshift space of two points \( x_1 \) and \( x_2 \) is thus given by
\[ \bar{\xi}^{(r)}(x_1, x_2) = b_1 b_2 D_1 D_2 \hat{R}_1 \hat{R}_2 \bar{\xi}(\chi) . \]
(2.32)

where \( b_1 = b(z_1), b_2 = b(z_2), D_1 = D(z_1), D_2 = D(z_2), z_1 \) and \( z_2 \) are the redshifts of the two points, and \( \bar{\xi}(\chi) \) is the mass correlation function in real space at the present time. The redshift-distortion operators \( \hat{R}_1 \) and \( \hat{R}_2 \) operate the density contrast at points \( x_1 \) and \( x_2 \), respectively, and \( \chi \) is a comoving separation of the two points. If we do not omit the local velocity term, \( U_0 \), the square of the second term in equation (2.31) is added to equation (2.32). In the next section, we obtain an explicit expression for equation (2.32).

3. THE CORRELATION FUNCTION IN REDSHIFT SPACE IN FRIEDMAN-LEMAÎTRE UNIVERSES

In this section, we derive an explicit expression for the correlation function in redshift space. We separately consider an open universe, a flat universe, and a closed universe in the following subsections.

3.1. An Open Universe

In an open universe (\( K < 0 \)), the correlation function in real space is given by equation (B51):
\[ \bar{\xi}(\chi) = \int \frac{v^2 dv}{2\pi^2} X_0(v, \chi) S(v) , \]
(3.1)

where
\[ X_0 = \frac{\sin v\chi}{v \sinh \chi} , \]
(3.2)
and $S(v)$ is the power spectrum of the gauge-invariant density contrast (see Appendix B for details). In this subsection, we omit the superscript $(-)$ of $X_l$, which distinguish the difference of functional forms according to the sign of curvature in Appendix B. Equation (2.32) implies that we only need to calculate $\tilde{R}_1, \tilde{R}_2 X_0(v, \chi)$ to obtain the formula for the correlation function in redshift space. Since the Laplacian is the invariant operator under transformation of the coordinate system, the inverse operation to is simply given by where $q_{i3} = (\ast)_{i3} K^{-1}$, $(\ast)_{i3} K^{-1} X_0 = \chi X_0$, because $X_0$ is the fundamental eigenfunction of the Laplacian (Appendix B, eq. [B2]).

The calculationally nontrivial part of the redshift-space distortion operator is finding the spatial derivatives along the lines of sight, $\partial_x$, which is not an invariant operation, unlike the Laplacian. We illustrate in the rest of this subsection how the calculation can be accomplished. The comoving separation $s$ between $x_1$ and $x_2$ is related to the comoving distances $x_1$ and $x_2$ of these two points from the observer, and the angle, $\theta$, between the lines of sight of these points with respect to the observer (Fig. 1), through the standard relation

$$\cosh \chi = \cosh \chi_1 \cosh \chi_2 - \sinh \chi_1 \sinh \chi_2 \cos \theta .$$  

(3.3)

The derivatives of this equation yield

$$\frac{\partial \chi}{\partial \chi_1} = \frac{1}{\sinh \chi} (\sinh \chi_1 \cosh \chi_2 - \cosh \chi_1 \sinh \chi_2 \cos \theta) = \cos \gamma_1 ,$$  

(3.4)

$$\frac{\partial \chi}{\partial \chi_2} = \frac{1}{\sinh \chi} (\cosh \chi_1 \sinh \chi_2 - \sinh \chi_1 \cosh \chi_2 \cos \theta) = \cos \gamma_2 ,$$  

(3.5)

where $\gamma_1$ is an angle between the geodesics of $\chi_1$ and $\chi$, and $\gamma_2$ is an angle between the geodesics of $\chi_2$ and $\chi$ (Fig. 1). The following equations are useful for our purpose:

$$\frac{\partial}{\partial \chi_1} (\sinh \chi \cos \gamma_1) = \cosh \chi ,$$  

(3.6)

$$\frac{\partial}{\partial \chi_2} (\sinh \chi \cos \gamma_1) = - \cosh \chi \cos \theta ,$$  

(3.7)

where

$$\cos \hat{\theta} = \frac{\sin \gamma_1 \sin \gamma_2}{\cosh \chi} - \cos \gamma_1 \cos \gamma_2 = \frac{\cosh \chi_1 \cosh \chi_2 \cos \theta - \sin \chi_1 \sin \chi_2}{\cosh \chi_1 \cosh \chi_2 - \sin \chi_1 \sin \chi_2 \cos \theta} .$$  

(3.8)

One can prove that $|\cos \hat{\theta}| \leq 1$ with this definition, and $\theta \rightarrow \hat{\theta}$ for scales much less than the curvature scale, $\chi \ll |K|^{-1}$. We also use the derivatives of the radial part of the harmonic function, $X_l(v, \chi)$, given by equation (B13), and the recursion relation, given by equation (B16). The explicit form of the radial part of the harmonic function, $X_l$, is presented in Appendix B, equations (B19)–(B23).

Note that the power spectrum near the horizon scale depends on the gauge choice. The power spectrum $S(v)$ is defined by Bardeen’s gauge-invariant density contrast, i.e., the density contrast in a gauge with velocity-orthogonal slicing, $V = R$. 

\[\text{FIG. 1.—Geometry of the relative position of the observer and the two points. The space is not necessarily Euclidean.}\]
After straightforward but somewhat tedious algebra, using equations (3.3)–(3.8), (B13), and (B16), one obtains the spatial derivatives along the lines of sight as

\[
\frac{\partial X_0}{\partial \chi} = \cos \gamma_1 X_1 ,
\]
\[
\frac{\partial^2 X_0}{\partial \chi^2} = X_0 - \frac{1}{3} (v^2 + 4)X_0 + \left( \cos^2 \gamma_1 - \frac{1}{3} \right)X_2 ,
\]
\[
\frac{\partial^2 X_0}{\partial \chi \partial X_2} = -\cos \theta X_0 + \frac{1}{3} \cos \theta (v^2 + 4)X_0 + \left( \cos \gamma_1 \cos \gamma_2 + \frac{1}{3} \cos \theta \right)X_2 ,
\]
\[
\frac{\partial^3 X_0}{\partial \chi^2 \partial X_2} = \cos \gamma_1 X_1 + \frac{1}{5} (2 \cos \gamma_1 \cos \theta - \cos \gamma_2)(v^2 + 4)X_1 + \frac{1}{5} (2 \cos \gamma_1 \cos \theta + 5 \cos^2 \gamma_1 \cos \gamma_2 - \cos \gamma_2)X_3 ,
\]
\[
\frac{\partial^4 X_0}{\partial \chi^2 \partial X_2^2} = X_0 - \frac{2}{15} (4 + 3 \cos^2 \theta)(v^2 + 4)X_0 + \frac{1}{15} (1 + 2 \cos^2 \theta)(v^2 + 4)^2X_0
\]
\[
- \frac{1}{21} [4 - 6 \cos^2 \theta - 27(\cos^2 \gamma_1 + \cos^2 \gamma_2) - 60 \cos \gamma_1 \cos \gamma_2 \cos \theta]X_2
\]
\[
+ \frac{1}{21} [2 + 4 \cos^2 \theta - 3(\cos^2 \gamma_1 + \cos^2 \gamma_2) + 12 \cos \gamma_1 \cos \gamma_2 \cos \theta](v^2 + 4)X_2
\]
\[
+ \frac{1}{35} [1 + 2 \cos^2 \theta - 5(\cos^2 \gamma_1 + \cos^2 \gamma_2) + 20 \cos \gamma_1 \cos \gamma_2 \cos \theta + 35 \cos^2 \gamma_1 \cos^2 \gamma_2]X_4 .
\]

Thus, \( \bar{R}_1 \bar{R}_2 X_0(v, \chi) \) is expanded as

\[
\bar{R}_1 \bar{R}_2 X_0(v, \chi) = \sum_{n,l} c_l^{(n)}(\chi_1, \chi_2, \theta) \left[ \frac{(-1)^n X_0(v, \chi)}{\sinh^{2n} \chi(v^2 + 4)^n} \right]
\]

where \((n, l) = (0, 0), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (2, 3), \) and \((2, 4)\), and the coefficients \( c_l^{(n)} \) are given by

\[
c_l^{(0)} = 1 + \frac{1}{2} \beta_1 + \beta_2 + \frac{1}{2} \beta_1 \beta_2 (1 + 2 \cos^2 \theta) ,
\]
\[
c_l^{(1)} = \left[ \beta_1 + \beta_2 + \frac{1}{2} \beta_1 \beta_2 (4 + 3 \cos \theta) \right] \sinh^2 \chi - \frac{1}{2} \beta_1 \beta_2 \tilde{a}_1 \tilde{a}_2 \cos \theta ,
\]
\[
c_l^{(2)} = \beta_1 \tilde{a}_1 \cos \gamma_1 + \beta_2 \tilde{a}_2 \cos \gamma_2 + \frac{1}{2} \beta_1 \beta_2 [\tilde{a}_1 (\cos \gamma_1 - 2 \cos \gamma_2 \cos \theta) + \tilde{a}_2 (\cos \gamma_2 - 2 \cos \gamma_1 \cos \theta)] ,
\]
\[
c_l^{(3)} = \beta_1 (\cos^2 \gamma_1 - \frac{1}{3}) + \beta_2 (\cos^2 \gamma_2 - \frac{1}{3})
\]
\[- \frac{1}{2} \beta_1 \beta_2 \tilde{a}_2 \sinh \chi - \cos \gamma_1 + \cos \gamma_2 \cos \theta ,
\]
\[
c_l^{(4)(0)} = \frac{1}{2} \tilde{a}_1 \tilde{a}_2 (\cos \gamma_1 \cos^2 \gamma_2 - \cos \gamma_1 + 2 \cos \gamma_2 \cos \theta) + \tilde{a}_2 (5 \cos \gamma_2 \cos^2 \gamma_1 - \cos \gamma_2 + 2 \cos \gamma_1 \cos \theta) ,
\]
\[
c_l^{(5)(0)} = -\frac{1}{2} \tilde{a}_1 \tilde{a}_2 (\cos \gamma_1 \cos^2 \gamma_2 - \cos \gamma_1 + 2 \cos \gamma_2 \cos \theta) + \tilde{a}_2 (5 \cos \gamma_2 \cos^2 \gamma_1 - \cos \gamma_2 + 2 \cos \gamma_1 \cos \theta) ,
\]

where

\[
\tilde{a}_1(\chi_1, \chi_2) = \alpha(\chi_1) \sinh \chi = \sinh \chi, 
\]
\[
\tilde{a}_2(\chi_1, \chi_2) = \alpha(\chi_2) \sinh \chi = \sinh \chi,
\]

and \( \beta_1 = \beta(\alpha_1), \beta_2 = \beta(\alpha_2), \alpha_1 = f(\alpha_1), \alpha_2 = f(\alpha_2), \Phi_1 = \Phi(\alpha_1), \) and \( \Phi_2 = \Phi(\alpha_2) \). Since these coefficients do not depend on \( v \), the correlation function in redshift space of equation (2.32) finally reduces to

\[
\xi(\chi_1, \chi_2) = b_1 b_2 D_1 D_2 \sum_{n,l} c_l^{(n)}(\chi_1, \chi_2, \theta) \Xi_l^{(n)}(\chi) ,
\]

where

\[
\Xi_l^{(n)}(\chi) = \frac{(-1)^n}{\sinh^{2n} \chi} \int \frac{\nu^2 \sinh \nu \sinh^2 \nu}{2 \pi n^2 (\nu^2 + 4^n)} S(v) .
\]
to be “scale invariant,” which corresponds to constant fluctuations in the gravitational potential per logarithmic interval in wavenumber (Lyth & Stewart 1990; Ratra & Peebles 1994; White & Bunn 1995),

\[ S(v) \propto \frac{(v^2 + 4)^2}{v(v^2 + 1)} T^2 \left( \frac{|K|^{1/2}}{\Gamma} \right), \] (3.28)

where

\[ T(p) = \frac{\ln(1 + 2.34p)}{2.34p} \left[ 1 + 3.89p + (16.1p)^2 + (5.46p)^3 + (6.71p)^4 \right]^{-1/4}, \] (3.29)

and the shape parameter is set to be \( \Gamma/(h^{-1} \text{ Mpc}) = 0.2 \), which corresponds to \( \Gamma = 6 \times 10^2 (cH_0^{-1})^{-1} \) in our unit system. On scales smaller than the horizon scale, \( v \gg 1 \), the power spectrum given in equation (3.28) reduces to the usual CDM-type power spectrum with a Harrison-Zeldovich primordial spectrum.

The set of equations (3.15)–(3.27) is our final formula for the correlation function in redshift space, which takes into account the light cone and spatial-curvature effect without the distant-observer approximation.

### 3.2. A Flat Universe

In a flat universe \( (K = 0) \), the correlation function in real space is given by equation (B51):

\[ \xi(\chi) = \frac{1}{2\pi^2} X_0(v, \chi) S(v), \] (3.30)

where

\[ X_0 = \sin \frac{v\chi}{v\chi}. \] (3.31)

We can repeat the calculation of the previous subsection for a flat universe \( (K = 0) \). The corresponding formula can also be used if we only consider a nearby universe, where the separation \( \chi \) and the distances \( \chi_1 \) and \( \chi_2 \) are much smaller than the curvature scale, \( |K|^{-1/2} \), which roughly corresponds to the horizon scale.

The calculation for a flat universe is similar to that of the previous subsection; alternatively, we can obtain the formula in a flat limit from the formula for an open universe, taking the limit \( \chi_1, \chi_2, \chi \to 0, v \to \infty \), with \( v\chi \) fixed. In any case, equation (3.3) reduces to the formula in Euclidean geometry,

\[ \chi^2 = \chi_1^2 + \chi_2^2 - 2\chi_1 \chi_2 \cos \theta. \] (3.32)

The meaning of \( \gamma_1 \) and \( \gamma_2 \) is the same, and they are explicitly given by

\[ \cos \gamma_1 = \frac{\chi_1 - \chi_2 \cos \theta}{\chi}, \quad \cos \gamma_2 = \frac{\chi_2 - \chi_1 \cos \theta}{\chi}. \] (3.33)

In a flat universe, the variable \( \bar{\theta} \), defined by equation (3.8) for the case of an open universe, reduces to \( \theta \), the angle between the lines of sight of \( \chi_1 \) and \( \chi_2 \) with respect to the observer.
Thus, the correlation function in redshift space is given by equation (3.26), where the coefficients $c^{(n)}$ in a flat universe are

$$c_0^{(n)} = 1 + \frac{1}{2}(\beta_1 + \beta_2) + \frac{1}{15}\beta_1 \beta_2 (1 + 2\cos^2 \theta),$$

(3.34)

$$c_1^{(n)} = \frac{1}{3}\beta_1 \beta_2 \tilde{a}_1 \tilde{a}_2 \cos \theta,$$

(3.35)

$$c_1^{(1)} = \beta_1 \tilde{a}_1 \cos \gamma_1 + \beta_2 \tilde{a}_2 \cos \gamma_2 + \frac{1}{2}\beta_1 \beta_2 \tilde{a}_1 \cos [\gamma_1 - 2\cos \gamma_2 \cos \theta] \tilde{a}_2 \cos \gamma_2 - 2\cos \gamma_1 \cos \theta],$$

(3.36)

$$c_2^{(1)} = \beta_1 \cos^2 \gamma_1 - \frac{1}{3} \beta_1 \beta_2 \cos^2 \gamma_2 - \frac{1}{3} \beta_1 \beta_2 \times \left[ \frac{3}{3} + \frac{3}{3} \cos^2 \theta - (\cos^2 \gamma_1 + \cos^2 \gamma_2) + 4 \cos \gamma_1 \cos \gamma_2 \cos \theta \right],$$

(3.37)

$$c_0^{(n)} = 0,$$

(3.38)

$$c_1^{(n)} = 0,$$

(3.39)

$$c_2^{(1)} = \beta_1 \beta_2 \tilde{a}_1 \tilde{a}_2 \cos \gamma_1 \cos \gamma_2 + \frac{1}{3} \cos \theta,$$

(3.40)

$$c_3^{(1)} = \frac{1}{3}\beta_1 \beta_2 \tilde{a}_1 (5 \cos \gamma_1 \cos^2 \gamma_2 - \cos \gamma_1 + 2 \cos \gamma_2 \cos \theta) + \tilde{a}_2 (5 \cos \gamma_2 \cos^2 \gamma_1 - \cos \gamma_2 + 2 \cos \gamma_1 \cos \theta),$$

(3.41)

$$c_4^{(1)} = \frac{1}{3}\beta_1 \beta_2 \left[ \frac{1}{3} + \frac{1}{3} \cos^2 \theta - (\cos^2 \gamma_1 + \cos^2 \gamma_2) + 4 \cos \gamma_1 \cos \gamma_2 \cos \theta + 7 \cos^2 \gamma_1 \cos^2 \gamma_2 \right],$$

(3.42)

where

$$\tilde{a}_i(x, \gamma) = u(x) \frac{\partial}{\partial \ln x} - \frac{1}{\cos \gamma} \frac{\partial}{\partial \ln x} \right],$$

(3.43)

$$\tilde{a}_i(x, x, \gamma) = u(x) \frac{\partial}{\partial \ln x} - \frac{1}{\cos \gamma} \frac{\partial}{\partial \ln x} \right].$$

(3.44)

The corresponding equation (3.27) is

$$\Xi^{(n)}(x) = \frac{(-1)^s}{x_{\Gamma n}} \int \frac{d^2 y}{2\pi^2} \frac{X_0(x, y)}{n^{2n}} S(v) = \frac{(-1)^{s+l}}{x_{\Gamma n}^2} \int \frac{k^2 dk_j k_j P(k)}{2\pi^2 \Gamma^{2n}}.$$

(3.45)

In the middle panel of Figure 2, some examples of the function $\Xi^{(n)}(x)$ for a flat model are plotted for a scale-invariant primordial power spectrum with a CDM-type transfer function,

$$S(v) \propto vT^2 \left( \frac{v \Gamma}{\Gamma} \right),$$

(3.46)

where we again set the shape parameter $\Gamma/(h^{-1} \text{ Mpc}) = 0.2$.

Alternatively, the coefficients $c^{(n)}$ can also be represented by using the variables $\gamma = \gamma_2 + \theta$ and $\theta_1 = \theta_2/2$. These variables are introduced in Szalay, Matsubara, & Landy (1998) in a calculation of wide-angle effects of the nearby universe ($z < 1$), except that their original definition of $\gamma$ is $\pi - \gamma$. The meaning of $\gamma$ is the angle between $x_2 - x_1$ and a symmetry axis that halves the angle $\theta$. With these new variables, equations (3.34)–(3.42) are transformed as

$$c_0^{(1)} = 1 + \frac{1}{2}(\beta_1 + \beta_2) + \frac{1}{2}\beta_1 \beta_2 - \frac{8}{15}\beta_1 \beta_2 \cos^2 \theta \sin^2 \theta,$$

(3.47)

$$c_1^{(n)} = -\frac{1}{2}\beta_1 \beta_2 \tilde{a}_1 \tilde{a}_2 \cos \theta,$$

(3.48)

$$c_1^{(1)} = \left[ (\beta_1 \tilde{a}_1 + \beta_2 \tilde{a}_2) + \frac{1}{3}\beta_1 \beta_2 (\tilde{a}_1 + \tilde{a}_2)(3 - 4 \cos^2 \theta_0) \right] \sin \theta \sin \gamma$$

$$+ \left[ (\beta_1 \tilde{a}_1 + \beta_2 \tilde{a}_2) + \frac{1}{3}\beta_1 \beta_2 (\tilde{a}_1 - \tilde{a}_2)(3 - 4 \sin^2 \theta_0) \right] \cos \theta \cos \gamma,$$

(3.49)

$$c_2^{(1)} = \left[ \frac{3}{3}(\beta_1 + \beta_2) + \frac{1}{3}\beta_1 \beta_2 \cos \theta \sin \theta \sin \gamma \right.$$

$$- 2\beta_1 \beta_2 \cos \theta \sin \theta \sin \gamma,$$

(3.50)

$$c_0^{(2)} = 0,$$

(3.51)

$$c_1^{(2)} = 0,$$

(3.52)

$$c_2^{(2)} = \frac{1}{2}\beta_1 \beta_2 \tilde{a}_1 \tilde{a}_2 \sin^2 \theta_0 - 2P_2(\cos \gamma),$$

(3.53)

$$c_3^{(2)} = -\frac{1}{2}\beta_1 \beta_2 (\tilde{a}_1 + \tilde{a}_2) \sin \theta_0 \left[ \cos \theta_0 P_1(\sin \gamma) - 2P_3(\sin \gamma) \right]$$

$$- (\tilde{a}_1 + \tilde{a}_2) \cos \theta_0 \sin^2 \theta_0 P_1(\cos \gamma) - 2P_3(\cos \gamma),$$

(3.54)

$$c_4^{(2)} = \frac{1}{3}\beta_1 \beta_2 \left( \frac{3}{3} \frac{P_2(\cos \gamma)}{\sin \gamma \sin^2 \theta_0} - \frac{1}{3} \sin^2 \theta_0 P_2(\cos \gamma) - \frac{1}{3} \cos \gamma \sin^2 \theta_0 \right),$$

(3.55)

where $P_0(x)$ is the Legendre function.

3.3. A Closed Universe

In a closed universe ($K > 0$), the correlation function in real space is given by equation (B52):

$$\xi(x) = \sum_{0}^{\infty} \frac{v^2}{2\pi^2} X_0(x, \gamma) S(v),$$

(3.56)
where

$$X_0 = \frac{\sin v\chi}{v \sin \chi}. \quad (3.57)$$

Repeating a similar calculation as for an open universe, or formally putting \( \chi \to i\chi, \chi_1 \to i\chi_1, \chi_2 \to i\chi_2, \) and \( v \to -iv \) into the calculation for an open universe, we obtain the formula for a closed universe. In this subsection, we just summarize the difference of the formula from the case of an open universe.

In the right panel of Figure 2, some examples of the function are plotted for a closed model. As in the open case, the distance between two points is much smaller than the distances of these points from the observer. The meaning of \( \gamma_1 \) and \( \gamma_2 \) are unchanged, and they are given by

$$\cos \gamma_1 = \cos \chi_1 \cos \chi_2 + \sin \chi_1 \sin \chi_2 \cos \theta. \quad (3.58)$$

$$\cos \gamma_2 = \frac{1}{\sin \chi} \left( \sin \chi_1 \sin \chi_2 - \sin \chi_1 \cos \chi_2 \cos \theta \right). \quad (3.59)$$

$$\cos \gamma_2 = \frac{1}{\sin \chi} \left( \cos \chi_1 \sin \chi_2 - \sin \chi_1 \cos \chi_2 \cos \theta \right). \quad (3.60)$$

The variable \( \theta \) in a closed universe is defined by

$$\cos \theta = \frac{\sin \gamma_1 \sin \gamma_2}{\cos \chi} - \cos \gamma_1 \cos \gamma_2 = \cos \chi_1 \cos \chi_2 + \sin \chi_1 \sin \chi_2 \cos \theta. \quad (3.61)$$

Then, the form of the coefficients \( c^{(n)} \) of equation (3.15)–(3.23) is almost the same, and we do not repeat the formula here. The only difference is that \( \sinh \) should be replaced by \( \sin \), and the definition of \( \Xi \) is changed as

$$\Xi_1(\chi, \chi) = \alpha(\chi_1) \sin \chi = \sin \chi \frac{\cos \chi_1}{\sin \chi_1} \left[ 2 + \frac{\partial \ln (D_1 f_1 \Phi_1)}{\partial \ln \sin \chi_1} \right], \quad (3.62)$$

$$\Xi_2(\chi, \chi) = \alpha(\chi_2) \sin \chi = \sin \chi \frac{\cos \chi_2}{\sin \chi_2} \left[ 2 + \frac{\partial \ln (D_2 f_2 \Phi_2)}{\partial \ln \sin \chi_2} \right]. \quad (3.63)$$

The formula for a closed universe is also given by the equation (3.26), where the definition of \( \Xi^{(n)} \) is replaced by

$$\Xi^{(n)}(\chi) = \frac{1}{\sin 2^n \chi} \sum_{n=3}^{\infty} \frac{v^2}{2\pi^2} \frac{X_4(v, \chi)}{(v^2 - 4)^n} S(v) \cdot \quad (3.64)$$

In the right panel of Figure 2, some examples of the function \( \Xi^{(n)}(\chi) \) are plotted for a closed model. As in the open case, the primordial power spectrum on large scales is assumed to be scale invariant, which corresponds to constant fluctuations in the gravitational potential per logarithmic interval in wavenumber (White & Scott 1996),

$$S(v) \propto \frac{(v^2 - 4)^2}{v(v^2 - 1)} T^2 \left( |k|^1/2 \frac{v}{c} \right), \quad (3.65)$$

where we again set the shape parameter as \( \Gamma/(h^{-1} \text{ Mpc}) = 0.2 \).

3.4. Recovery of the Known Formulas of the Redshift Distortions of the Correlation Function

It is an easy exercise to derive the previously known formulas of the redshift distortions of the correlation function from our general formula. We illustrate how our formula reduces to the known formulas by taking appropriate limits. Here we consider two approximations, \( \chi \ll \chi_1, \chi_2 \) and/or \( z \ll 1 \). The first approximation corresponds to the distant-observer approximation. In this approximation, the distance between two points is much smaller than the distances of these points from the observer. The second approximation corresponds to only considering the nearby universe. The redshift-distortion formulas known so far are restricted in these cases.

First, consider a limit \( \chi \ll \chi_1, \chi_2 \), with \( \gamma_1 \) and \( \gamma_2 \) fixed. We also set \( \beta_1 = \beta_2 = \beta \), and the distance between the two points \( \chi \) is much smaller than the curvature scale. Then, irrespective of the spatial curvature, \( \theta, \theta \to 0 \), and \( \gamma_2 \to \pi - \gamma_1 \equiv \gamma \). The coefficients given in equations (3.15)–(3.23) and (3.34)–(3.42) reduce to

$$c^{(0)}_0 = 1 + \frac{3}{2} \beta + \frac{1}{2} \beta^2, \quad (3.66)$$

$$c^{(0)}_2 = \left( \frac{3}{2} \beta + \frac{3}{2} \beta^2 \right) P_2(\cos \gamma), \quad (3.67)$$

$$c^{(2)}_4 = \frac{8}{5} \beta^2 P_4(\cos \gamma), \quad (3.68)$$

and all the other coefficients are zero. These coefficients are equivalent to the result that Hamilton (1992) derived in the \( z \to 0 \) limit with the distant-observer approximation, which is a direct Fourier transform of Kaiser's original form in Fourier space (Kaiser 1987). For finite \( z \), the equivalent result is obtained by Matsubara & Suto (1996) for the correlation function and by Ballinger, Peacock & Heavens (1996) for the power spectrum (see also Nakamura, Matsubara, & Suto 1998; de Laix &
Starkman 1998; Nair 1999). All these previous studies are based on the distant-observer approximation, and our general formula correctly preserves the limit of these cases.

The wide-angle effects on the redshift distortion of the correlation function without the distant-observer approximation have already been derived for the nearby universe, \( z < 0 \), by Szalay, Matsubara, & Landy (1998); Bharadwaj (1999) also derived an equivalent result with another parameterization. It is easy to see that our general results have the correct limit of Szalay et al. (1998). In the limit \( z \to 0 \), the geometry reduces a flat case, and in fact, equations (3.47)–(3.55) with \( \beta_1 = \beta_2 = \beta \) in that limit are completely equivalent to the result of Szalay et al. (1998) after correcting their typographical errors,\(^3\) noting the minor difference of definition, in that \( \gamma \) and \( \Xi_1^{(m)} \) here correspond to \( \pi - \gamma \) and \( (-1)^{m+1} \pi - i \xi_{s}^{2n} \) in Szalay et al. (1998), respectively.

### 4. Redshift Distortions and the Spatial Curvature of the Universe

It is well known that the redshift distortions of the correlation function for the nearby universe are good probes of the density parameter modulo the bias factor at the present time, \( \beta_0 = \Omega_0^{1/6}/b_0 \). The parameter \( \beta_0 \) is the redshift-distortion parameter of the nearby universe. Redshift distortions of the nearby universe depend almost not at all on spatial curvature, because the redshift distortions of the nearby universe are purely caused by the peculiar-velocity field, which is almost independent of the spatial curvature. However, redshift distortions at high redshifts, say in a quasar catalog, definitely depend on the spatial curvature of the universe, and are called the cosmological redshift distortion. Matsubara & Suto (1996) explicitly show the dependence of the cosmological distortion of the correlation function on cosmological parameters, employing the distant-observer approximation. To more properly probe the spatial curvature with cosmological distortions, it is not desirable to rely on the distant-observer approximation.

In this section, we numerically calculate the formula obtained in the previous section for several models, concentrating on geometrical effects. We plot the correlation function in directly observable velocity space. In the following discussion, the shape parameter for the CDM-type transfer function is fixed to \( \Gamma = 0.6 \times 10^2 \) in our unit system, which corresponds to \( \Gamma/(h^{-1} \text{Mpc}) = 0.2 \), in spite of the fact that CDM models predict \( \Gamma/(h^{-1} \text{Mpc}) = \Omega_0 h \). We fix the spectrum simply because we are interested in the pure distortion effects on the difference among the models, while the difference of the shape of the underlying power spectrum interests us less here. The primordial spectrum is assumed to be scale invariant, which corresponds to constant fluctuations in the gravitational potential per logarithmic interval in wavenumber (eqs. [3.28], [3.46], and [3.65]). For simplicity, we assume no bias (\( b = 1 \)) and \( \alpha = 0 \). The latter assumption corresponds to the case in which \( D(f\phi) \propto (\sinh \chi)^{-2}, \chi^{-2}, \) and \( (\sin \chi)^{-2} \), for open, flat, and closed models, respectively. Although this form of selection function is not physically motivated, it is not so unrealistic for merely illustrative purpose. In actual application, the selection function is individually determined for each redshift survey.

Figures 3–6 show the contour plots of the correlation function. Each figure consists of 12 panels. In each figure, from top to bottom, the cosmological models are standard (STD), flat, and open, i.e., \( (\Omega_0, \lambda_0) = (1, 0), (0.2, 0.8), \) and \( (0.2, 0) \), respectively. From left to right, the redshifts of the first points are \( z = 0.1, 0.3, 1.0, \) and 3.0.

In Figures 3 and 4, the correlation function with purely geometrical distortions is plotted for illustration, assuming that there are no peculiar velocities at all, by setting \( \beta = 0 \) in our formula. The first point, \( x_1 \), of equation (3.26) is at the center on the \( y \)-axis in each figure. The contour plot shows the value of the correlation function depending on the position of the second point, \( x_2 \). In Figure 3, the observer is located at the origin \((0,0)\), and the global distortions are shown. In Figure 4, the scale of \( x \) around the first object is fixed, and the observer is located at \((0, -z_4)\), outside the plots.

For lower redshifts, \( z = 0.1 \) and 0.3, the geometrical distortion is not significant. For higher redshifts, \( z = 1.0 \) and 3.0, due to the nonlinear relations of redshift and comoving distance, the contours are elongated in the direction of the line of sight in all three models. The extent of the elongation for the STD and open models is similar, but the elongation for flat model is smaller than in the other models. This is because the acceleration due to the cosmological constant squashes the \( z \)-space along the line of sight (Alcock & Paczyński 1979).

In Figures 5 and 6, the correlation function in redshift space is plotted, taking into account the velocity distortions. As in Figures 3 and 4, the observer is located at \((0,0)\) and \((0, -z_4)\), respectively.

For lower redshifts, the redshift distortions are mainly from peculiar-velocity fields, which depend only on the density parameter, \( \Omega_0 \). Thus, the STD model can be distinguished from other models by the correlation function at lower redshifts, but the flat and open models are similar. For higher redshifts, the flat and open models become different because of the squashing by the cosmological constant.

To illustrate the differences among profiles of the correlation function in redshift space for different models and different redshifts, we define the parallel correlation function, \( \xi_\parallel(z_\parallel; z) \), and the perpendicular correlation function, \( \xi_\perp(z_\perp; z) \), at a given \( z \). In terms of the correlation function in redshift space \( \xi^{(m)}(z_1, z_2, \theta) \), they are defined by

\[
\xi_\parallel(z_\parallel; z) = \xi^{(0)} \left( z_\parallel - \frac{z_\parallel}{2}, z + \frac{z_\parallel}{2}, 0 \right),
\]

\[
\xi_\perp(z_\perp; z) = \xi^{(0)} \left( z, z, \arccos \left[ 1 - \frac{z_\perp^2}{2z^2} \right] \right).
\]

\(^3\) The factor 4/15 in the last term of their equation (15) should be replaced by 8/15, and the left-hand side of their equation (20) should be replaced by 1/3 \( \alpha_1 x_2 \beta^2 \cos 2\theta \).
Fig. 3.—Contour plots of the correlation function in real space with purely geometrical distortions, without velocity distortions. The observer is sitting at the origin, and the first point at redshift $z_1$ is sitting at the center on the $y$-axis. The contour map shows the value of the correlation function depending on the position of the second point at redshift $z_2$ and angle $\theta$. Solid lines indicate positive correlation, and dotted lines indicate the negative correlation. The normalization is arbitrary, and the contour spacings are $\Delta \log_{10} |\xi| = 0.5$. Top: STD model ($\Omega_0 = 1, \lambda_0 = 0$); middle: flat model ($\Omega_0 = 0.2, \lambda_0 = 0.8$); bottom: open model ($\Omega_0 = 0.2, \lambda_0 = 0$). From left to right, the redshifts of the first point $z_1$ are 0.1, 0.3, 1, and 3.

Fig. 4.—Same as Fig. 3, but the scale of $z$ around the first point is fixed. The observer is located at $(0, -z_1)$, which is outside the plots.
Fig. 5.—Contour plots of the correlation function in redshift space. The panels are the same as in Fig. 3, but velocity distortions are included.

Fig. 6.—Contour plots of the correlation function in redshift space with fixed scale of $z$. The panels are the same as in Fig. 4, but velocity distortions are included.
Fig. 7.—Geometrical meaning of the definition of the parallel redshift interval $z_{\parallel}$ and the perpendicular redshift interval $z_{\perp}$.

Fig. 8.—Parallel correlation function, $\xi_{\parallel}$ (solid lines), and perpendicular correlation function, $\xi_{\perp}$ (dashed lines), in redshift space. The redshifts of the first point are 0.1, 0.3, 1, and 3 (left to right). The cosmological models are STD (top), flat (middle), and open (bottom).
The geometrical meaning of the definition of the parallel redshift interval, $z_{||}$, and the perpendicular redshift interval, $z_{\perp}$, are illustrated in Figure 7. As can be seen from Figures 5 or 6, these sections of the correlation function are supposed to be maximally distorted in opposite direction, and that is why we introduce them as illustrations. These functions are the generalization of the similar functions introduced by Matsubara & Suto (1996) for the case of the distant-observer approximation.

In Figure 8, those parallel and perpendicular correlation functions are plotted for STD, flat, and open models. The correlation function in real space is normalized as $\sigma_8 = 1$. Note that while the perpendicular correlation functions are not significantly different among three models, the parallel correlation functions are quite different. For lower redshifts, profiles of the parallel correlation function for the flat and open models are similar, as usual. For higher redshifts, the relative amplitude of $\xi_{||}$ compared to $\xi_{\perp}$ in the open model is higher than that in the flat model. In addition, the zero-crossing point of $\xi_{||}$ in the open model is larger than in the flat model. Those tendencies are both explained by cosmological-constant squashing, because the squashing shifts the profile toward small scales, or left. Even if the determination of the zero-crossing point of $\xi_{||}$ is observationally difficult, the former effect on the relative amplitude is a promising way to discriminate the spatial curvature by the observation of redshift distortions.

The distant-observer approximation has been widely used in analyses of galaxy redshift surveys. With our general formula, we can figure out when this approximation is valid and when it is not. The distant-observer approximation of the two-point correlation function has been derived by Hamilton (1992) for a nearby universe, $z \ll 1$. Matsubara & Suto (1996) generalize his formula to arbitrary redshifts, in which the distant-observer approximation is still adopted. In Figure 9, we plot the ratio of the value of those two previous formulas and that of our formula. We choose the geometry of the two points as follows: we fix the separation $z_{12} \equiv (z_1^2 + z_2^2 - 2z_1 z_2 \cos \theta)^{1/2}$ of the two points in velocity space. The angle $\gamma_2$ between the symmetric line that halves the lines of sight and the line between the two points is also fixed. The meaning of the angle $\gamma_2$ is the inclination of $z_{12}$ relative to the line of sight in velocity space. Explicitly, $\gamma_2$ is given by $\tan \gamma_2 = (z_1 + z_2) / |z_1 - z_2|^{-1} \tan (\theta/2)$. The angle $\theta$ between the lines of sight is varied in the figure. We plot the cases $z_{12} = 0.003$, $0.006$, $0.012$, and $0.024$, and $\gamma_2 = 0^\circ$, $45^\circ$, and $80^\circ$. There is some irregular behavior that corresponds to the zero-crossings of the correlation function.

The Hamilton formula (Fig. 9, thin lines) is valid when the separation $z_{12}$ is not so large. The reason Hamilton's formula deviates even in small angles is that when the angle is small enough, the redshift becomes large, and the evolutionary and geometrical effects are not negligible. In fact, the formula of Matsubara & Suto (Fig. 9, thick lines) is perfectly identical to our general formula when the angle is small.
The validity region of the distant-observer approximation depends on the inclination angle $\gamma$. However, one can conserva-
vitely estimate the validity region as $\theta \lesssim 10^\circ$. One can use $\theta \sim 20^\circ$ for some cases, while the blind application of the approximation for $\theta \sim 30^\circ$ can cause over 100% error.

5. CONCLUSIONS

In this paper we have derived for the first time a unified formula of the correlation function in redshift space in linear theory, which simultaneously includes wide-angle effects and the cosmological redshift distortions. The effects of the spatial curvature both on geometry and on the fluctuation spectrum are properly taken into account, and our formula applies to an arbitrary Friedman-Lemaître universe.

The distant-observer approximation of Matsubara & Suto (1996), which is a generalization of the work of Hamilton (1992), can be used when the angle $\theta$ between the lines of sight is less than $10^\circ$. Beyond that range, our formula provides a unique form for the correlation function in redshift space with geometrical distortions.

The correlation function in redshift space is uniquely determined if the cosmological parameters $\Omega_0$ and $\lambda_0$, the power spectrum $P(k)$, and the bias evolution $b(z)$ are specified. The Hubble constant does not affect the correlation function, provided that we do not adopt a specific model to the power spectrum and/or to the bias evolution that may depend on the Hubble constant. Our formula predicts the correlation function for a fixed model of these variables, and one can test any model by directly comparing the correlation function of the data and that of the theoretical prediction in redshift space.

Now we comment on some caveats to our formula. First, when we try to apply our formula to scales $\lesssim 20 \ h^{-1} \ Mpc$, it apparently lacks the finger-of-God effect, or the nonlinear smearing of the correlation along the line of sight. While it is still difficult to analytically include this effect in our formula, one can phenomenologically evaluate the effect by numerically smearing the formula along the line of sight. Second, our formula does not include the Sachs-Wolfe and gravitational lensing effects. The Sachs-Wolfe effect could affect our formula only on scales comparable to the Hubble distance, where the correlation function is too small to be practically detectable, as discussed in § 2.1. However, it is possible that the gravitational lensing effect affects the observable correlation function for $z \gtrsim 1$, in which case one should add correction terms to our formula. Even so, our formula for $z \lesssim 1$ would not be affected by those terms and still corresponds to the observable quantity. Those correction terms for gravitational lensing will be given in a future paper. Third, our formula assumes that the selection function is uncorrelated with the density fluctuations. In addition, the precise form of the selection function is practically difficult to determine. If the luminosities and/or the surface brightnesses are correlated with density fluctuations, they could mimic large-scale structure.

It might be the case that one could blindly seek models of the cosmological quantities $\Omega_0$, $\lambda_0$, $P(k)$, and $b(z)$. Each quantity has different effects on the correlation function in redshift space. Roughly speaking, the cosmological parameters $\Omega_0$ and $\lambda_0$ mostly affect the distortions of the contour of the correlation function, the power spectrum $P(k)$ mostly affects the profile of the correlation function, and the bias evolution mostly affects the $z$-dependence of the amplitude. Since all the pairs of objects in the redshift survey can be used, we expect that these quantities can be determined with small errors by performing proper-likelihood analyses, such as the Karhunen-Loève mode decomposition (Vogeley & Szalay 1996; Matsubara, Szalay, & Landy 1999). The application of the formula to actual data is a straightforward task. We believe that the formula presented in this paper will be one of the most fundamental theoretical tools in understanding the data of deep redshift surveys.

I wish to thank Alex Szalay, Naoshi Sugiyama, and Yasushi Suto for stimulating discussions. This work was supported by Japan Society for the Promotion of Science (JSPS) postdoctoral fellowships for research abroad.

APPENDIX A

GAUGE-IN Variant EVOLUTION EQUATIONS FOR THE DENSITY CONTRAST AND THE VELOCITY FIELD

In this Appendix, we review the derivation of the equations of motion for the density contrast and the velocity field in a general relativistic context. We assume that the matter is a pressureless, perfect fluid, and the background metric is given by a homogeneous, isotropic FLRW metric. For the perturbed Einstein equation,

$$\delta G^\alpha_\nu = 8\pi G \delta T^\alpha_\nu , \quad (A1)$$

Bardeen (1980) introduced the gauge-invariant formalism. Here we briefly review the formalism in the case of pressureless fluid, and derive relevant equations for our purpose, i.e., equations to relate the density contrast and the velocity field. In this Appendix, we use the conformal time $\tau$ defined by $d\tau = dt/a$, so that the metric is given by

$$ds^2 = a^2(\tau)(-d\tau^2 + \gamma_{ij}(x^i dx^j)(x^j)), \quad (A2)$$

in the coordinates $(\tau, x^i)$. A prime denotes differentiation with respect to the conformal time.

The perturbations are classified into scalar, vector, and tensor types. The scalar perturbations are expanded by the complete set of scalar harmonics $Q(x^i)$ satisfying the Helmholtz equation

$$\left(\nabla^2 + q^2\right) Q = 0,$$  

(A3)
where $\Delta = \nabla^i \nabla_i$ is the Laplacian and $\nabla_i$ is the three-dimensional covariant derivative with respect to the metric $\gamma_{ij}$. The explicit form of each mode of this solution is given in Appendix B. In the following discussion, the indices distinguishing different modes are omitted, and each mode function is represented by $Q$.

For each mode, the scalar perturbations in the metric of equation (A2) are defined by

\[
\begin{align*}
    h_{00} &= -2a^2 A(\tau)Q, \\
    h_{0i} &= -a^2 B(\tau)Q_i, \\
    h_{ij} &= 2a^2 [H_\tau(\tau)Q_i\gamma_{ij} + H_T(\tau)Q_{ij}],
\end{align*}
\]

where

\[
\begin{align*}
    Q_i &= -\frac{1}{q} \nabla_i Q, \\
    Q_{ij} &= \frac{1}{q^2} \nabla_i \nabla_j Q + \frac{1}{3} \gamma_{ij} Q.
\end{align*}
\]

The 3-velocity $w^i$ associated with the 4-velocity $u^\mu$ is represented by

\[
w^i = \frac{u^i}{u^0} = V(\tau)Q^i,
\]

and to first order the normalization $u^\mu u_\mu = -1$ gives

\[
u^0 = \frac{1 - A(\tau)Q}{a}.
\]

The pressureless energy-momentum tensor is given by $T^\mu_\nu = \rho u^\mu u_\nu$; thus, the scalar perturbations in the pressureless energy-momentum tensor are

\[
\begin{align*}
    \delta T^0_0 &= -\dot{\rho}\Delta(\tau)Q, \\
    \delta T^0_j &= \dot{\rho}[V(\tau) - B(\tau)]Q_j, \\
    \delta T^i_j &= 0.
\end{align*}
\]

In terms of the gauge-dependent variables $A$, $B$, $H_L$, $H_T$, $V$, and $\Delta$, Bardeen (1980) defined the gauge-invariant combinations

\[
\begin{align*}
    \Phi_A &= A + \frac{1}{q} B' + \frac{1}{q} \frac{a'}{a} B - \frac{1}{q^2} \left( H_T' + \frac{a'}{a} H_T \right), \\
    \Phi_H &= H_L + \frac{1}{3} H_T + \frac{1}{q} \frac{a'}{a} B - \frac{1}{q^2} \frac{a'}{a} H_T, \\
    \delta &= \Delta + \frac{3}{q} \frac{a'}{a} (V - B), \\
    V_s &= V - \frac{1}{q} H_T.
\end{align*}
\]

The perturbation equations (A1) for these scalar gauge-invariant variables are given by Bardeen (1980):

\[
\begin{align*}
    &\Phi_A + \Phi_H = 0, \\
    &\Phi_A + \Phi_H = 0, \\
    &V_s + \frac{a'}{a} V_s = q \Phi_A, \\
    &\delta' + (q^2 - 3K) \frac{1}{q} V_s = 0.
\end{align*}
\]

We define new variables, $\psi = -a^{-1}q^{-1}V_s$ and $\Phi = a^{-2}\Phi_A$. After superposing the modes of the above equations, we obtain

\[
\begin{align*}
    &\delta + (\Delta + 3K)\psi = 0, \\
    &\dot{\psi} + 2H\psi + \Phi = 0.
\end{align*}
\]
where an overdot denotes differentiation with respect to the proper time, $d/dt$, and $\Delta$ is the Laplacian on the FLRW metric, equation (B1). This is the complete set of equations to treat the scalar perturbations.

The velocity $w^i$ corresponds to $dx^i/dt = adx^i/d\tau = av_i$, and equation (A9) suggests $w^i = a\nabla^i \psi$, so that $v^i = \nabla^i \psi$, i.e., $\psi$ is the velocity potential and represents the longitudinal part of the velocity $v^i$. The transverse part of $v^i$ belongs to the vector perturbations.

The vector perturbations are expanded by the divergenceless vector harmonics $Q_{ij}^{(1)}$, satisfying

\begin{align}
(\Delta + q^2)Q_{ij}^{(1)} &= 0, \\
\nabla^i Q_{ij}^{(1)} &= 0.
\end{align}

For each mode, the vector perturbations are given by

\begin{align}
Q_{00} &= 0, \\
Q_{0i} &= -a^2B^{(1)}(\tau)Q_{i}^{(1)}), \\
Q_{ij} &= 2a^2H^{(1)}(\tau)Q_{ij}^{(1)},
\end{align}

where

\begin{equation}
Q_{ij}^{(1)} = -\frac{1}{2q} [\nabla_i Q_j^{(1)} + \nabla_j Q_i^{(1)}].
\end{equation}

The vector perturbations in the 3-momentum are

\begin{equation}
w^i = \frac{u^i}{u^0} = V^{(1)}(\tau)Q^i,
\end{equation}

and in the energy momentum tensor they are

\begin{align}
\delta T^{0}{}_{0} &= 0, \\
\delta T^{0}{}_{j} &= \bar{\rho}[V^{(1)}(\tau) - B^{(1)}(\tau)]Q_j, \\
\delta T^{i}{}_{j} &= 0.
\end{align}

For vector perturbations, Bardeen (1980) defined the gauge-invariant combinations

\begin{align}
\Psi &= B^{(1)} - \frac{1}{q} H^{(1)} \nabla, \\
V_c &= V^{(1)} - B^{(1)}.
\end{align}

The perturbation equations (A1) for these vector gauge-invariant variables are given by Bardeen (1980):

\begin{equation}
(q^2 - 2K)\Psi = 16\pi G\bar{\rho}V_c,
\end{equation}

\begin{equation}
V_c^2 + \frac{a'}{a} V_c = 0.
\end{equation}

We define a new variable, $v_T = a^{-1}V_c$, which is the transverse part of the velocity $v^i$. After superposing the modes of the equation (A38), we obtain

\begin{align}
\dot{v}_T + 2Hv_T &= 0, \\
\nabla_i v_T &= 0.
\end{align}

Since the density contrast and the velocity field are irrelevant to the tensor perturbations, we do not repeat here equations of motion for tensor perturbations. Complete formulas to determine the density contrast and the velocity field are given by equations (A22)–(A24) and (A39), where the velocity field is decomposed into the longitudinal part and the transverse part, as $v^i = \nabla^i \psi + v_T$. Equation (A39) is immediately integrated to give

\begin{equation}
v_T \propto a^{-2},
\end{equation}

i.e., the transverse part decays with time. This result is the consequence of the fact that the matter is pressureless. In summary, neglecting the decaying transverse mode, the complete formulas to determine the density contrast and the velocity field are given by equations (A22) and (2.21), where the velocity consists of the purely longitudinal part.
APPENDIX B

ORTHONORMAL LAPLACIAN SET IN THREE-DIMENSIONAL SPACES WITH CONSTANT CURVATURES

In this Appendix, we construct the orthonormal Laplacian set in a space with or without spatial curvature and derive the expression of the correlation function in real space in terms of the power spectrum, when the spatial curvature is not negligible.

B.1. HARMONIC FUNCTIONS

The Laplacian of a function $Q$ in the metric $\gamma_{ij}$ from equations (2.3) and (2.1) is given by

$$\Delta Q = \begin{cases} -K \sinh^2 \chi \left( \frac{\partial}{\partial \chi} \left( \sinh^2 \chi \frac{\partial Q}{\partial \chi} \right) + \frac{1}{\sin \theta \theta \frac{\partial}{\partial \theta} \left( \sin \theta \theta \frac{\partial Q}{\partial \theta} \right) + \frac{1}{\sin^2 \theta \theta \frac{\partial^2 Q}{\partial \phi^2}} \right) & K < 0 , \\ \frac{1}{\chi^2} \left( \frac{\partial}{\partial \chi} \left( \chi^2 \frac{\partial Q}{\partial \chi} \right) + \frac{1}{\sin \theta \theta \frac{\partial}{\partial \theta} \left( \sin \theta \theta \frac{\partial Q}{\partial \theta} \right) + \frac{1}{\sin^2 \theta \theta \frac{\partial^2 Q}{\partial \phi^2}} \right) & K = 0 , \\ \frac{K}{\sin^2 \chi \chi} \left( \frac{\partial}{\partial \chi} \left( \sin^2 \chi \frac{\partial Q}{\partial \chi} \right) + \frac{1}{\sin \theta \theta \frac{\partial}{\partial \theta} \left( \sin \theta \theta \frac{\partial Q}{\partial \theta} \right) + \frac{1}{\sin^2 \theta \theta \frac{\partial^2 Q}{\partial \phi^2}} \right) & K > 0 . \end{cases} \tag{B1}$$

To evaluate the inverse Laplacian, it is useful to obtain a complete set of scalar harmonic functions satisfying the Helmholtz equation (Harrison 1967; Wilson 1983)

$$(\Delta + q^2)Q = 0 . \tag{B2}$$

If we separate variables, the angular part of the solution is just a spherical harmonic, $Y_l^m(\theta, \phi)$. The radial part, $X_l(x)$, associated with $Y_l^m$ satisfies the radial equation

$$\frac{1}{\sinh^2 \chi \chi} \frac{\partial}{\partial \chi} \left( \sinh^2 \chi \frac{\partial X_l}{\partial \chi} \right) + v^2 + 1 - \frac{l(l+1)}{\sinh^2 \chi} X_l = 0 , \quad \text{for } K < 0 ; \tag{B3}$$

$$\sinh^2 \chi \frac{\partial}{\partial \chi} \left( \sinh^2 \chi \frac{\partial X_l}{\partial \chi} \right) + v^2 - \frac{l(l+1)}{\chi^2} X_l = 0 , \quad \text{for } K = 0 ; \tag{B4}$$

$$\sin^2 \chi \frac{\partial}{\partial \chi} \left( \sin^2 \chi \frac{\partial X_l}{\partial \chi} \right) + v^2 - 1 - \frac{l(l+1)}{\sin^2 \chi} X_l = 0 , \quad \text{for } K > 0 , \tag{B5}$$

where we introduce a new variable $v$ as

$$q^2 = -K(v^2 + 1) , \quad \text{for } K < 0 ; \tag{B6}$$

$$q^2 = v^2 , \quad \text{for } K = 0 ; \tag{B7}$$

$$q^2 = K(v^2 - 1) , \quad \text{for } K > 0 . \tag{B8}$$

The solutions of the radial equations (B3)–(B5) that are regular at the origin are given by conical functions, Bessel functions, and toroidal functions for negative, zero, and positive curvature, respectively (e.g., Harrison 1967; Abbott & Schaefer 1986):

$$X_l^{-}(v, \chi) = (-1)^l M_l^{-}(v) \sqrt{\frac{\pi}{2 \sinh \chi}} \mathcal{P}^{\frac{1}{2} - \frac{1}{2} - l} \cosh \chi$$

$$= \sinh^l \chi \frac{d^l}{d(\cosh \chi)^2} \left( \frac{\sin v \chi}{\sin \chi} \right) , \quad \text{for } v^2 \geq 0 , \ K < 0 ; \tag{B9}$$

$$X_l^{(0)}(v, \chi) = (-1)^l Y_l^0(v \chi) = \chi^l \frac{1}{v} \left( \frac{\partial}{\partial \chi} \frac{\sin v \chi}{\chi} \right) , \quad \text{for } v^2 \geq 0 , \ K = 0 ; \tag{B10}$$

$$X_l^{(+)}(v, \chi) = (-1)^l M_l^{(+)}(v) \sqrt{\frac{\pi}{2 \sin \chi}} P^{-\frac{1}{2} + \frac{1}{2} + l} \cos \chi$$

$$= (-1)^l \sin^l \chi \frac{d^l}{d(\cos \chi)^2} \left( \frac{\sin v \chi}{\sin \chi} \right) , \quad \text{for } v = 2, 3, 4, \ldots , \ K > 0 , \tag{B11}$$

where $\mathcal{P}^m$ is the associated Legendre function, and $P^m_{\nu}(x) = e^{i\pi/2} \mathcal{P}^m(x + i0)$ is the associated Legendre function on the cut (Magnus, Oberhettinger, & Soni 1966). The spectrum for the space of positive curvature is discrete, because of the condition that the function must be periodic, or single-valued. We introduce the notations $M_l^{(\pm)}(v) = (v^2 \pm 1^2)(v^2 \pm 2^2) \cdots (v^2 \pm l^2)$ and
We can see that $X_i^{(+)}(v, \chi) = i^i X_i^{-}(iv, i\chi)$, by analytic continuation with a natural choice of branches. For $\chi \to 0$, all these functions behave like $\chi'$ with appropriate constants, and thus $X_i(0, \chi) = \delta_{0i}$ (e.g., Harrison 1967). For the variable $v$ as positive integers in a closed universe, $X_i^{(+)\ast}$ can also be represented by Gegenbauer (or ultraspherical) polynomials, $C_{l+1}^{(+)\ast}(\cos \chi)$, as

$$X_i^{(+)\ast}(v, \chi) = \left(\frac{-2i!}{v}\right)^l C_{l+1}^{(+)\ast}(\cos \chi). \tag{B12}$$

We choose normalizations such that the $K \to 0$ limit of either equation (B9) or (B11) reduces to equation (B10). In these normalizations, the derivatives and recursion relations for $X_i$ are particularly simple. In fact, the derivatives of the radial functions are simply given by

$$\frac{\partial}{\partial \chi} \left[ \frac{X_i^{-}(v, \chi)}{\sinh^l \chi} \right] = \frac{X_i^{(-1)}(v, \chi)}{\sinh^l \chi}, \quad \text{for } K < 0; \tag{B13}$$

$$\frac{\partial}{\partial \chi} \left[ \frac{X_i^{(0)}(v, \chi)}{\chi^l} \right] = \frac{X_i^{(0)}(v, \chi)}{\chi^l}, \quad \text{for } K = 0; \tag{B14}$$

$$\frac{\partial}{\partial \chi} \left[ \frac{X_i^{(+)}(v, \chi)}{\sin^l \chi} \right] = \frac{X_i^{(+1)}(v, \chi)}{\sin^l \chi}, \quad \text{for } K > 0, \tag{B15}$$

and the recursion relations, which are derived through the fact that the terms $X_i$ are the solutions of the radial equations (B3)–(B5), are

$$(v^2 + l^2)X_i^{(-1)}(v, \chi) + (2l + 1) (\cosh \chi \sin \chi) X_i^{(-)}(v, \chi) + X_i^{(+)\ast}(v, \chi) = 0, \quad \text{for } K < 0; \tag{B16}$$

$$v^2 X_i^{(0)}(v, \chi) + \frac{2l + 1}{\chi} X_i^{(0)}(v, \chi) + X_i^{(+)\ast}(v, \chi) = 0, \quad \text{for } K = 0; \tag{B17}$$

$$(v^2 - l^2)X_i^{(+1)}(v, \chi) + (2l + 1) (\cosh \chi \sin \chi) X_i^{(+)}(v, \chi) + X_i^{(+)\ast}(v, \chi) = 0, \quad \text{for } K > 0. \tag{B18}$$

In this paper, we need only $l = 0, 1, 2, 3$, and $4$ for the evaluation of redshift distortion. They are given by

$$X_0^{(-)} = \frac{\sin v\chi}{v \sin \chi}, \tag{B19}$$

$$X_1^{(-)} = \frac{1}{v \sinh^2 \chi} (-\cosh \chi \sin v\chi + v \sin \chi \cos v\chi), \tag{B20}$$

$$X_2^{(-)} = \frac{1}{v \sinh^3 \chi} \{[3 - (v^2 - 2) \sinh^2 \chi] \sin v\chi - 3v \sin \chi \cosh \chi \cos v\chi\}, \tag{B21}$$

$$X_3^{(-)} = \frac{1}{v \sinh^4 \chi} \{\cosh \chi[15 + 6(v^2 - 1) \sinh^2 \chi] \sin v\chi + v \sin \chi[15 - (v^2 - 11) \sinh^2 \chi] \cos v\chi\}, \tag{B22}$$

$$X_4^{(-)} = \frac{1}{v \sinh^5 \chi} \{[105 - 15(3v^2 - 8) \sinh^2 \chi + (v^4 - 35v^2 + 24) \sinh^4 \chi] \sin v\chi - v \sin \chi[105 - 10(v^2 - 5) \sinh^2 \chi] \cos v\chi\}, \tag{B23}$$

for $K < 0$;

$$X_0^{(0)} = \frac{\sin v\chi}{v \chi}, \tag{B24}$$

$$X_1^{(0)} = \frac{1}{v \chi^2} (\sin v\chi + v \chi \cos v\chi), \tag{B25}$$

$$X_2^{(0)} = \frac{1}{v \chi^3} [(3 - v^2 \chi^2) \sin v\chi - 3v \chi \cos v\chi], \tag{B26}$$

$$X_3^{(0)} = \frac{1}{v \chi^4} [(15 + 6v^2 \chi^2) \sin v\chi + v \chi[15 - v^2 \chi^2] \cos v\chi], \tag{B27}$$

$$X_4^{(0)} = \frac{1}{v \chi^5} [(105 - 45v^2 \chi^2 + v^4 \chi^4) \sin v\chi - v \chi[105 - 10v^2 \chi^2] \cos v\chi], \tag{B28}$$
for $K = 0$; and

\[
X_0^{(+)} = \frac{\sin \chi}{v \sin \chi},
\]

\[
X_1^{(+)} = \frac{1}{\sin^2 \chi} (\cos \chi \sin v \chi + v \sin \chi \cos v \chi),
\]

\[
X_2^{(+)} = \frac{1}{\sin^3 \chi} \left[ (3 - (v^2 + 2) \sin^2 \chi) \sin v \chi - 3v \sin \chi \cos \chi \cos v \chi \right],
\]

\[
X_3^{(+)} = \frac{1}{\sin^4 \chi} \left[ \cosh \chi \left[ -15 + 6(v^2 + 1) \sin^2 \chi \right] \sin v \chi + v \sin \chi \left[ 15 - (v^2 + 11) \sin^2 \chi \right] \cos v \chi \right],
\]

\[
X_4^{(+)} = \frac{1}{\sin^5 \chi} \left[ 105 - 15(3v^2 + 8) \sin^2 \chi + (v^4 + 35v^2 + 24) \sin^4 \chi \right] \sin v \chi
\]

\[- v \sin \chi \left[ 105 - 10(v^2 + 5) \sin^2 \chi \right] \cos v \chi ,
\]

for $K > 0$.

Although our normalizations in equations (B9)–(B11) have simple relations for derivatives and recursion relations, it is also convenient to introduce further normalizations, as

\[
\hat{X}_1^{(+)}(v, \chi) = \frac{X_1^{(\pm)}(v, \chi)}{\sqrt{M^{(\pm)}}(v)}, \quad \hat{X}_3^{(0)}(v, \chi) = \frac{X_1^{(0)}(v, \chi)}{v} = (-1)j_l(v \chi).
\]

From the orthogonality and completeness of conical functions, Bessel functions, and Gegenbauer polynomials (Magnus et al. 1966), we can find

\[
4\pi \int \sinh^2 \chi d\chi \hat{X}_1^{(+)}(v, \chi) \hat{X}_1^{(-)}(v', \chi) = \frac{2\pi^2}{v^2}, \quad \text{for } K < 0 ;
\]

\[
4\pi \int \chi^2 d\chi \hat{X}_1^{(0)}(v, \chi) \hat{X}_1^{(0)}(v', \chi) = \frac{2\pi^2}{v^2}, \quad \text{for } K = 0 ;
\]

\[
4\pi \int \sin^2 \chi d\chi \hat{X}_1^{(+)}(v, \chi) \hat{X}_1^{(+)}(v', \chi) = \frac{2\pi^2}{v^2}, \quad \text{for } K > 0 ;
\]

and

\[
\int \frac{v^2 dv}{2\pi^2} \hat{X}_1^{(-)}(v, \chi) \hat{X}_1^{(-)}(v, \chi') = \frac{\delta(\chi - \chi')}{4\pi \sinh^2 \chi}, \quad \text{for } K < 0 ;
\]

\[
\int \frac{v^2 dv}{2\pi^2} \hat{X}_1^{(0)}(v, \chi) \hat{X}_1^{(0)}(v, \chi') = \frac{\delta(\chi - \chi')}{4\pi \chi^2}, \quad \text{for } K = 0 ;
\]

\[
\sum_{v=2}^{\infty} \int \frac{v^2 dv}{2\pi^2} \hat{X}_1^{(+)}(v, \chi) \hat{X}_1^{(+)}(v, \chi') = \frac{\delta(\chi - \chi')}{4\pi \sin^2 \chi}, \quad \text{for } K > 0 .
\]

B2. THE CORRELATION FUNCTION IN REAL SPACE AT THE PRESENT TIME

The density contrast $\delta(\chi, \theta, \phi)$ is expanded in terms of the eigenfunctions as

\[
\delta(\chi, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int \frac{v^2 dv}{2\pi^2} \hat{X}_1^{(-)}(v, \chi) Y_l^m(\theta, \phi), \quad \text{for } K < 0 ;
\]

\[
\delta(\chi, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int \frac{v^2 dv}{2\pi^2} \hat{X}_1^{(0)}(v, \chi) Y_l^m(\theta, \phi), \quad \text{for } K = 0 ;
\]

\[
\delta(\chi, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{v=3}^{\infty} \frac{v^2}{2\pi^2} \hat{X}_1^{(+)}(v, \chi) Y_l^m(\theta, \phi), \quad \text{for } K > 0 .
\]
For a positive-curvature model, the term $v = 2$ is omitted, because $v = 2$ corresponds to a mode that is a pure gauge term (Lifshitz & Khalatnikov 1963; Bardeen 1980), as indicated by equation (A18). The inverse of these expansions are

$$
\delta_{lm}(v) = 4\pi \int \sinh^2 \chi d\chi \int \sin \theta d\theta d\phi \delta(\chi, \theta, \phi) \hat{X}_l^{-}\!(v, \chi) Y_{lm}^*(\theta, \phi) , \quad \text{for } K < 0 ;
$$

(B44)

$$
\delta_{lm}(v) = 4\pi \int \chi^2 d\chi \int \sin \theta d\theta d\phi \delta(\chi, \theta, \phi) \hat{X}_l^{(0)}(v, \chi) Y_{lm}^*(\theta, \phi) , \quad \text{for } K = 0 ;
$$

(B45)

$$
\delta_{lm}(v) = 4\pi \int \sin^2 \chi d\chi \int \sin \theta d\theta d\phi \delta(\chi, \theta, \phi) \hat{X}_l^{+}\!(v, \chi) Y_{lm}^*(\theta, \phi) \quad \text{for } K > 0 .
$$

(B46)

Since the correlation function $\xi(\chi)$ depends only on the separation of two points, and does not depend on the particular choice of the coordinate system, we can write it as

$$
\xi(\chi) = \langle \delta(0, \theta, \phi) \delta(\chi, \theta, \phi) \rangle
= \sum_{l,m} \sum_{l',m'} \int \frac{v^2 dv}{2\pi^2} \int \frac{v'^2 dv'}{2\pi^2} \hat{X}_l^{-}(v,0) \hat{X}_l^{-}(v', \chi) Y_{lm*}(\theta, \phi) Y_{l'm'}^{*}(\theta, \phi) \langle \delta_{lm}(v) \delta_{l'm'}(v') \rangle , \quad \text{for } K \leq 0 ;
$$

(B47)

$$
\xi(\chi) = \langle \delta(0, \theta, \phi) \delta(\chi, \theta, \phi) \rangle
= \sum_{l,m} \sum_{l',m'} \sum_{v,v'} \frac{v^2}{2\pi^2} \frac{v'^2}{2\pi^2} \hat{X}_l^{+}(v,0) \hat{X}_l^{+}(v', \chi) Y_{lm*}(\theta, \phi) Y_{l'm'}^{*}(\theta, \phi) \langle \delta_{lm}(v) \delta_{l'm'}(v') \rangle , \quad \text{for } K > 0 .
$$

(B48)

Statistical homogeneity and isotropy of the universe suggest that the correlation of modes has the form

$$
\langle \delta_{lm}(v) \delta_{l'm'}(v') \rangle = (2\pi)^3 \delta_{ll'} \delta_{mm'} \frac{\delta(v - v')}{v^2} S(v) , \quad \text{for } K \leq 0 ;
$$

(B49)

$$
\langle \delta_{lm}(v) \delta_{l'm'}(v') \rangle = (2\pi)^3 \delta_{ll'} \delta_{mm'} \frac{\delta(v - v')}{v^2} S(v) , \quad \text{for } K > 0 ,
$$

(B50)

where $S(v)$ is a power spectrum of the number density field at present. From this equation and $X(0,0) = \delta_{10}$, equations (B47) and (B48) reduce to

$$
\xi(\chi) = \int \frac{v^2 dv}{2\pi^2} X_0^{-}(v, \chi) S(v) , \quad \text{for } K \leq 0 ;
$$

(B51)

$$
\xi(\chi) = \int \frac{v^2 dv}{2\pi^2} X_0^{+}(v, \chi) S(v) , \quad \text{for } K > 0 .
$$

(B52)

Equation (B51) is an equivalent formula previously derived by Wilson (1983). These formulas can even be transformed to forms more like the usual formula:

$$
\xi(\chi) = \int \frac{k^2 dk}{2\pi^2} P(k) \frac{\sqrt{-K} \sin (k\chi)}{k \sinh (\sqrt{-K} x)} , \quad \text{for } K < 0 ;
$$

(B53)

$$
\xi(\chi) = \int \frac{k^2 dk}{2\pi^2} P(k) \frac{\sin (k\chi)}{kx} , \quad \text{for } K = 0 ;
$$

(B54)

$$
\xi(\chi) = \sum_{i=1}^{\infty} \frac{k^2 \sqrt{K}}{2\pi^2} P(k_i) \frac{\sqrt{K} \sin (k_i \chi)}{k_i \sin (\sqrt{K} x)} , \quad \text{for } K > 0 ,
$$

(B55)

where the discrete wavenumber, $k_i = (i + 2)K^{1/2}$, is for the space of positive curvature. The relations between the usual power spectrum, $P(k)$, and the power spectrum $S(v)$, are given by

$$
P(k) = \frac{S(|K|^{-1/2})}{|K|^{3/2}} , \quad \text{for } K \neq 0 ;
$$

(B56)

$$
P(k) = S(k) , \quad \text{for } K = 0 .
$$

(B57)

When the scale of interest is much smaller than the curvature scale, $x \ll |K|^{-1/2}$, both equation (B53) for negative curvature and equation (B55) for positive curvature reduce to the familiar expression of equation (B54).

Finally, we comment that comparing the expression $\xi(\chi) = \langle \delta(\chi_1, \theta_1, \phi_1) \delta(\chi_2, \theta_2, \phi_2) \rangle$ and equations (B51) or (B52) proves the addition theorem for $\hat{X}_l = \hat{X}_l^{\pm,0}$:

$$
\hat{X}_0(v, \chi) = \sum_T (2l + 1) \hat{X}_l(v , \chi_1) \hat{X}_l(v , \chi_2) P(d \cos \theta) .
$$

(B58)

For the flat case, $K = 0$, the above equation is nothing but the well-known addition theorem of the Bessel function; therefore, the addition theorem for $K \neq 0$ is the generalizations of it to the constant-curvature space.
REFERENCES

Abbott, L. F., & Schaefer, R. K. 1986, ApJ, 308, 546
Alcock, C., & Paczynski, B. 1979, Nature, 281, 358
Ballinger, W. E., Peacock, J. A., & Heavens, A. F. 1996, MNRAS, 282, 877
Bardeen, J. M. 1980, Phys. Rev. D, 22, 1882
Bardeen, J. M., Bond, J. R., Kaiser, N., & Szalay, A. S. 1986, ApJ, 304, 15
Bernardeau, F., van Waerbeke, L., & Mellier, Y. 1997, A&A, 322, 1
Bharadwaj, S. 1999, ApJ, 516, 507
Colless, M. 1998, Philos. Trans. R. Soc. London, submitted (preprint astro-ph/9804079)
de Laix, A. A., & Starkman, G. 1998, ApJ, 501, 427
Fisher, K. B., Scharf, C., & Lahav O. 1994, MNRAS, 266, 219
Folkes, S. R., et al. 1999, MNRAS, 308, 459
Gunn, J. E., & Weinberg, D. H. 1995, in Wide-Field Spectroscopy and the Distant Universe, ed. S. J. Maddox & A. Aragon-Salamanca (Singapore: World Scientific)
Hamilton, A. J. S. 1992, ApJ, 385, L5
———. 1998, in The Evolving Universe: Selected Topics on Large-Scale Structure and on the Properties of Galaxies, ed. D. Hamilton (Dordrecht: Kluwer), 185
Hamilton, A. J. S., & Culhane, M. 1996, MNRAS, 278, 73
Harrison, E. R. 1967, Rev. Mod. Phys., 39, 862
Heavens, A. F., & Taylor, A. N. 1995, MNRAS, 275, 483
Hwang, J.-C., & Vishniac, E. T. 1990, ApJ, 353, 1
Kaiser, N. 1987, MNRAS, 227, 1
———. 1992, ApJ, 388, 272
———. 1998, ApJ, 498, 26
Kodama, H., & Sasaki, M. 1984, Prog. Theor. Phys. Suppl., 78, 1
Kogut, A., et al. 1993, ApJ, 419, 1
Lahav, O., Lilje, P. B., Primack, J. R., & Rees, M. J. 1991, MNRAS, 251, 128
Lifshitz, E., & Khalatnikov, I. 1963, Adv. Phys., 12, 185
Lilje, P. B., & Efstathiou, G. 1989, MNRAS, 236, 851
Lineweaver, C. H., Tenorio, L., Smoot, G. F., Kegstra, P., Banday, A. J., & Lubin, P. 1996, ApJ, 470, 38
Lyth, D. H., & Stewart, E. D. 1990, Phys. Lett. B, 252, 336
Magnus, W., Oberhettinger, F., & Soni, R. P. 1966, Formulas and Theorems for the Special Functions of Mathematical Physics (New York: Springer)
Margon, B. 1998, Philos. Trans. R. Soc. London A, submitted (astro-ph/9805314)
Matsubara, T. 1999, ApJ, 525, 543
Matsubara, T., & Suto, Y. 1996, ApJ, 470, L1
Matsubara, T., Szalay, A. S., & Landy, S. D. 1999, ApJ, submitted (preprint astro-ph/9911151)
McGill, C. 1990, MNRAS, 242, 428
Nair V. 1999, preprint (astro-ph/9904312)
Nakamura, T. T., Matsubara, T., & Suto, Y. 1998, ApJ, 494, 13
Nishioka, H., & Yamamoto, K. 1999, ApJ, 520, 426
Peebles, P. J. E. 1974, A&A, 32, 197
———. 1980, The Large-Scale Structure of the Universe (Princeton: Princeton Univ. Press)
Popowski, P. A., Weinberg, D. H., Ryden, B. S., & Osmer, P. S. 1998, ApJ, 498, 11
Ratra, B., & Peebles, P. J. E. 1994, ApJ, 432, L5
Sachs, R. W., & Wolfe, A. M. 1967, ApJ, 147, 73
Strauss, M. A. 1999, in Formation of Structure in the Universe, ed. A. Dekel & J. P. Ostriker (Cambridge: Cambridge Univ. Press), 172
Suto, Y., Magira, H., Jing, Y. P., Matsubara, T., & Yamamoto, K. 1999, Prog. Theor. Phys. Suppl., 133, 183
Szalay, A. S., Matsubara, T., & Landy, S. D. 1998, ApJ, 498, L1
Totsuji, H., & Kihara, T. 1969, PASJ, 21, 221
Vogeley, M. S., & Szalay, A. S. 1996, ApJ, 465, 34
White, M., & Scott, D. 1996, ApJ, 459, 415
Wilson, M. L. 1983, ApJ, 273, 2
Zaroubi, S., & Hoffman, Y. 1996, ApJ, 462, 25