The Laplacian and $\bar{\partial}$ operators on critical planar graphs

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Abstract

On a periodic planar graph whose edge weights satisfy a certain simple geometric condition, the discrete Laplacian and $\bar{\partial}$ operators have the property that their determinants and inverses only depend on the local geometry of the graph.

We obtain explicit expressions for the logarithms of the (normalized) determinants, as well as the inverses of these operators.

We relate the logarithm of the determinants to the volume plus mean curvature of an associated hyperbolic ideal polyhedron.

In the associated dimer and spanning tree models, for which the determinants of $\bar{\partial}$ and the Laplacian respectively play the role of the partition function, this allows us to compute the entropy and correlations in terms of the local geometry.

In addition, we define a continuous family of special discrete analytic functions, which, via convolutions gives a general process for constructing discrete analytic functions and discrete harmonic functions on critical planar graphs.

1 Introduction

Let $G$ be a graph and $c$ a nonnegative weight function on the edges of $G$ defining the edge conductances. We define the Laplacian operator $\Delta : \mathbb{C}^{|G|} \to \mathbb{C}^{|G|}$ by

$$\Delta f(u) = \sum_{w \sim u} c(uw)(f(u) - f(w))$$

where the sum is over vertices $w$ adjacent to $u$. This is one of the most basic and useful operators on $G$.

On the superposition of $G$ and its planar dual $G^*$, we can define another operator, the $\bar{\partial}$ operator, which defines “discrete analytic functions” (those which satisfy $\bar{\partial} f = 0$).

In fact we will define $\bar{\partial}$ on more general bipartite planar graphs, see section 3 below.

In the form we use here the definition comes from a statistical mechanics model, the dimer model: the determinant of the $\bar{\partial}$ operator is the partition function of the dimer model (see section 10).

We show that under a certain geometric condition, called isoradiality, on the embedding of a planar graph, the local structure of these two operators $\Delta$ and $\bar{\partial}$ takes on a surprising simplicity. In particular one can write down explicitly the determinants and inverses of these operators, which only depend on local quantities: the determinants

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only depend on the edge weights (not the combinatorics) and the value of the inverse on two vertices \( v_1, v_2 \) only depends on a path between the vertices.

In particular for the determinants we have the following results. For the definitions see below.

**Theorem 1.1** Let \( \mathcal{G}_T \) be an isoradial embedding of a periodic planar graph and \( c \) the associated weight function (giving the conductances on the edges). Then the logarithm of the normalized determinant of the Laplacian is

\[
\log \det_1 \Delta = \frac{1}{N} \sum_{\text{edges } e} \frac{2}{\pi} \left( L(\theta) + L\left(\frac{\pi}{2} - \theta\right) \right) + \frac{2\theta}{\pi} \log \tan \theta,
\]

where \( N \) is the number of vertices in a fundamental domain, the sum is over the set of edges in a fundamental domain, \( c(e) = \tan(\theta(e)) \) and \( L \) is the Lobachevsky function,

\[
L(x) = -\int_0^x \log 2 \sin t \, dt.
\]  

(1)

The determinant of the Laplacian is useful since it computes the partition function for several other statistical mechanical models, among them the spanning tree model (see e.g. [1]) and the discrete Gaussian free field. For the \( \bar{\partial} \) operator we have

**Theorem 1.2** Let \( \mathcal{G}_D \) be an isoradial embedding of a periodic bipartite planar graph and \( \nu \) the associated weight function on edges. Then the normalized determinant of the discrete \( \bar{\partial} \) operator satisfies

\[
\log \det_1 \bar{\partial} = \frac{1}{N} \sum_{\text{edges } e} \frac{1}{\pi} L(\theta) + \frac{\theta}{\pi} \log 2 \sin \theta,
\]

where \( \nu(e) = 2 \sin(\theta(e)) \), \( N \) is the number of vertices in a fundamental domain, and the sum is over the edges in a fundamental domain.

For results on and explicit expressions for the inverses of these operators, see Theorems 4.2 and 7.1 below.

Similar statements can be made for nonperiodic graphs, on condition that the right-hand sides makes sense; for this one needs a unique ergodicity assumption on the translation action on the graph. We will restrict ourselves to periodic cases here.

To a graph \( \mathcal{G}_D \) with critical weight function \( \nu \) as in Theorem 1.2 is naturally associated a 3-dimensional ideal hyperbolic polyhedron (with infinitely many vertices): it is the polyhedron \( P \) in the upper-half space model of \( \mathbb{H}^3 \) whose vertices are those of \( \mathcal{G}_D^\ast \), the dual graph of \( \mathcal{G}_D \). We give a geometric realization of a discrete analytic function as a deformation of the associated embedding of the graph, alternatively as a deformation of \( P \). Furthermore there is a relation between the above determinants and the volume of the polyhedra, which identifies the entropy of the corresponding dimer model to the volume, and the mean energy to the mean curvature.

These results were inspired by the work of and discussions with Christian Mercat [10] who used isoradial embeddings (“critical” embeddings in his terminology) in the study of discrete analytic functions and the Ising model. Isoradial embeddings apparently first appeared in the work of Duffin [1]. Our definition of “elementary” harmonic functions, see section 8, also appeared independently in [11], where a more detailed study of these functions is undertaken.

The outline of the paper is as follows.
• Section 2. We define “isoradial” embeddings and critical weight functions for planar graphs.

• Section 3. We define the $\bar{\partial}$ operator, discrete analytic functions, and show in what sense they give perturbations of the embedding. We also prove the existence, unicity and asymptotics of the inverse $\bar{\partial}^{-1}$, as well as giving an explicit local formula.

• Section 4. We define the $\bar{\partial}$ operator, discrete analytic functions, and show in what sense they give perturbations of the embedding. We also prove the existence, unicity and asymptotics of the inverse $\bar{\partial}^{-1}$, as well as giving an explicit local formula.

• Section 5. We define the normalized determinant $\det_{1} \bar{\partial}$ and prove Theorem 1.2.

• Section 6. We point out the relation between the Laplacian and our $\bar{\partial}$ operator in the case of graphs $G_{T}$.

• Section 7. We compute the Green’s function on $G_{T}$ and its asymptotics.

• Section 8. We prove that the set of isoradial embeddings of a graph is convex (and usually infinite dimensional), and log $\det_{1} \Delta$ and log $\det_{1} \bar{\partial}$ are concave functions on it.

• Section 9. We discuss a general construction of discrete analytic and discrete harmonic functions on critical graphs.

• Section 10. We discuss connections between the $\bar{\partial}$ operator and the dimer model. Here we relate the volume and mean curvature of the polyhedron $P$ with the entropy and mean energy of the dimer model.

2 Critical weight functions

2.1 Polyhedral embeddings

Let $G$ be a planar graph, and $G^{*}$ its planar dual. For a vertex $v \in G$ we denote $v^{*}$ its dual face of $G^{*}$. Similarly for an edge $e$ we denote by $e^{*}$ its dual edge.

A polyhedral embedding of $G$ is an embedding of $G$ in the plane such that every face is a cyclic polygon, that is, is inscribable in a circle. A regular polyhedral embedding is a polyhedral embedding in which the circumcenters of the faces are contained in the closures of the faces.

A polyhedral embedding of a graph $G$ defines a map from $G^{*}$, the dual of $G$, to the plane, by sending a dual vertex to the circumcenter of the corresponding face of $G$, and a dual edge to the segment joining its endpoints. If the embedding of $G$ is regular, this map will be an embedding of $G^{*}$ unless some circumcenters are on the boundary of their faces, in which case some dual vertices may have the same image in the plane. In other words some edges of $G^{*}$ may have length 0.

The term polyhedral refers to the fact that a polyhedral embedding gives rise to a three-dimensional ideal hyperbolic polyhedron, in the following way. The upper half-space model of hyperbolic space $\mathbb{H}^{3}$ is identified with $\{(x, y, z) \in \mathbb{R}^{3}|z > 0\}$. A polyhedral embedding of $G$ on the $xy$-plane is the vertical projection of the edges of an (infinite) ideal polyhedron $P(G)$ whose vertices are the vertices of $G$. If the polyhedral embedding is regular then $P(G)$ is convex; the dihedral angle at an edge is the angle of intersection of the circumspheres of the two adjacent faces. This polyhedron will play a role in section 10.
2.2 Isoradial embeddings

An isoradial embedding is a regular polyhedral embedding in which all circumcircles have the same radius.

If the embedding of $G$ is isoradial then the embedding of $G^*$ will also be isoradial, with the same radius. We will always take the common radius to be 1. To each edge $e$ of $G$ we associate a rhombus $R(e)$ whose vertices are the vertices of $e$ and the vertices of its dual edge. The rhombus is therefore of edge length 1 and the angle at a vertex of $G$ is in $[0, \pi]$. See Figure 1.

A chain of rhombi is an bi-infinite sequence of rhombi $\{\ldots, x_i, x_{i+1}, \ldots\}$ with $x_{i+1}$ adjacent to $x_i$ along the edge opposite the edge along which $x_{i-1}$ and $x_i$ are adjacent. The rhombi in a chain all contain an edge parallel to a given direction. When $G$ is bipartite, the vertices of $G$ in a rhombus chain are alternately black and white, so that a chain has a “black” side and a “white” side.

Note that a rhombus chain cannot cross itself (since it is monotone) and two rhombus chains cannot cross each other more than once, since at the crossing point the rhombus which both chains have in common has orientation determined by the relative orientations of the two chains.

Which planar graphs have isoradial embeddings? Here is an explicit conjecture. A zig-zag path in a planar graph is an edge path which alternately turns right and left, that is, upon arriving at a vertex it takes the first edge to the left, then upon arriving at the next vertex takes the first edge to the right, and so on. If the graph has an isoradial embedding a zig-zag path corresponds to a chain of rhombi. Consequently two necessary conditions for the existence of an isoradial embedding are that a zig-zag path cannot cross itself, and two zig-zag paths cannot cross each other more than once. We conjecture that these two conditions are sufficient as well.

Conjecture 1 A planar graph has an isoradial embedding if and only if no zig-zag path crosses itself, and no two zig-zag paths cross each other more than once.
2.3 Critical weights

Suppose we have an isoradial embedding of $G$. For each edge $e$ of $G$ define $\nu(e) = 2\sin \theta$ and $c(e) = \tan \theta$, where $2\theta$ is the angle of the rhombus $R(e)$ at the vertices it has in common with $e$. We refer to $\theta$ as the half-angle of $R$. The function $\nu$ is the weight function for the $\bar{\partial}$ matrix and $c$ is the weight function (conductance) for the Laplacian matrix. A weight function $\nu$ or $c$ which arises in this way is called critical. Note that $\nu(e)$ is the length of $e^*$, the dual edge of $e$.

Note that a critical weight function determines the embedding of $G$ and $G^*$. It is useful to allow the rhombus half-angles to be in $[0, \pi/2]$, that is, to allow degenerate rhombi with half-angles 0 or $\pi/2$. In this case the embeddings of $G$ or $G^*$ can be degenerate in the sense that two or more vertices may map to the same point.

If the half-angles of the rhombi in a chain are in $[0, \pi/2]$ but bounded away from 0 and $\pi/2$ we can deform the angles of all the rhombi in the chain, simply by changing slightly the direction of the common parallel edge. The new rhombus angles define a new critical weight function. Moreover one can show that every critical weight function can be obtained from any other by deforming in this way along chains: see section 8.

2.4 Periodic graphs

For the rest of the paper we suppose that $G$ is an infinite periodic graph, that is, the embedding (and therefore the weight function) is invariant by translates by a two-dimensional lattice $\Lambda$ in $\mathbb{R}^2$, and the quotient $G/\Lambda$ is finite. We denote this quotient $G_1$.

We use periodicity in a fundamental way in defining and proving the formulas for the determinants, Theorems 1.1 and 1.2. For the inverses of the operators, however, periodicity is not necessary. As long as the graph has a reasonable regularity, for example if there are only a finite number of rhombus angles, then the results on the inverses extend (with essentially no modifications to the proofs).

3 The $\bar{\partial}$ operator

3.1 Definition

Let $G_D$ be a bipartite planar graph, that is, the vertices of $G_D$ can be divided into two subsets $B \cup W$ and vertices in $B$ (the black vertices) are only adjacent to vertices in $W$ (the white vertices) and vice versa. Suppose that $G_D$ has an isoradial embedding. Let $\nu$ be a critical weight function on $G_D$.

We define a symmetric matrix $\bar{\partial}$ indexed by the vertices of $G_D$ as follows. If $v_1$ and $v_2$ are not adjacent $\bar{\partial}(v_1,v_2) = 0$. If $w$ and $b$ are adjacent vertices, $w$ being white and $b$ black, then $\bar{\partial}(w,b) = \bar{\partial}(b,w)$ is the complex number of length $\nu(wb)$ and direction pointing from $w$ to $b$. If $w$ and $b$ have the same image in the plane then the $|\bar{\partial}(w,b)| = 2$, and the direction of $\bar{\partial}(w,b)$ is that which is perpendicular to the corresponding dual edge (which has nonzero length by definition), and has sign determined by the local orientation. Another useful way to say this is as follows. If rhombus $R(wb)$ has vertices $w, x, b, y$ in counterclockwise (ccw) order then $\bar{\partial}(w,b)$ is $i$ times the complex vector $x - y$. 
The matrix \( \bar{\partial} \) defines the \( \bar{\partial} \) operator. The \( \bar{\partial} \) matrix is also called a **Kasteleyn matrix** for the underlying dimer model, see section [10]. Since \( \bar{\partial} \) maps \( \mathbb{C}^B \) to \( \mathbb{C}^W \) and \( \mathbb{C}^W \) to \( \mathbb{C}^B \), it really consists of two operators \( \bar{\partial}_{BW} : \mathbb{C}^B \to \mathbb{C}^W \) and its transpose \( \bar{\partial}_{WB} : \mathbb{C}^W \to \mathbb{C}^B \).

As an example, note that when \( G_D = \mathbb{Z}^2 \) with edge lengths \( r = \sqrt{2} \) the \( \bar{\partial} \) operator has a more recognizable form,

\[
\bar{\partial} f(v) := \sqrt{2} \left( f(v + r) - f(v - r) + if(v + ri) - if(v - ri) \right).
\]

### 3.2 Discrete analytic functions and perturbations

A function \( f : B \to \mathbb{C} \) satisfying \( \bar{\partial} f \equiv 0 \), that is \( \sum_{b \in B} \bar{\partial}(w, b)f(b) = 0 \) for all \( w \), is called a **discrete analytic function**. This generalizes the definition given in [10]. Similarly a function \( f : W \to \mathbb{C} \) satisfying \( \bar{\partial} f \equiv 0 \) is also called a discrete analytic function. So a critical bipartite planar graph supports two independent families of discrete analytic functions.

The simplest example of a discrete analytic function is a constant function (as the reader may check).

A discrete analytic function \( F : B \to \mathbb{C} \) defines a perturbation of the embedding of \( G_D^* \), as follows (this idea generalizes an example in [10]). The perturbation is a complex homothety (rotation, scaling and translation) on each black face but deforms the white faces in a more general way. See Figure 2.

![Figure 2](image-url)
The perturbation is defined as follows. Fix a small \( \epsilon > 0 \). To each face \( b^* \) of \( G^*_D \) we apply the map \( z \mapsto z(1 + \epsilon F(b)) + t_b \), where \( t_b \) is a certain translation. The \( t_b \) are chosen so that if two black faces meet at a vertex, their images still meet at the (image of the) same vertex. This is of course only possible if for each white face the polygonal path formed by the translates of the images of its edges is still a closed polygon: let \( w \) be a white vertex of \( G^*_D \), and \( b_1, \ldots, b_m \) its neighboring black vertices. Let \( e_1^*, \ldots, e_m^* \) be the edges running ccw around the face \( w^* \) of \( G^*_D \), so that \( e_j^* \) is the dual edge to \( wb_j \). Because the edges and dual edges are perpendicular, as a complex vector we have

\[
e_j^* = i \partial(w, b_j).
\]

Then \( \partial F = 0 \) implies that

\[
\partial(w, b_1)F(b_1) + \cdots + \partial(w, b_m)F(b_m) = 0,
\]

or

\[
e_1^*F(b_1) + \cdots + e_m^*F(b_m) = 0
\]

and since \( e_1^* + \cdots + e_m^* = 0 \), we have

\[
e_1^*(1 + \epsilon F(b_1)) + \cdots + e_m^*(1 + \epsilon F(b_m)) = 0.
\]

In conclusion we can find \( t_b \) so that each white face is again a closed polygon.

The inverse \( \bar{\partial}^{-1} \) which we define in the next section is a discrete analytic function of \( b \) for each fixed \( w \), except for a single singularity: it satisfies

\[
\sum_{b \in B} \partial(w', b)\bar{\partial}^{-1}(w, b) = \delta_{w',w} = \begin{cases} 0 & \text{if } w \neq w' \\ 1 & \text{if } w = w'. \end{cases}
\]

Therefore for a fixed \( w \) it defines a perturbation of the embedding except that the polygon \( w^* \) is not closed: at the face \( w^* \) the right-hand side of (2) is 1 not 0. So we can therefore define only a map from the branched cover of \( G^*_D \), branched around \( w \), to \( \mathbb{C} \); it has a non-trivial holonomy around \( w \). See Figure 3 for \( \bar{\partial}^{-1} \) in the case \( G_D = \mathbb{Z}^2 \).

4 The \( \bar{\partial}^{-1} \) operator

For critical bipartite graphs \( G_D \) we define \( \bar{\partial}^{-1} \) to be the unique function satisfying

1. \( \bar{\partial}\bar{\partial}^{-1} = Id \)
2. \( \bar{\partial}^{-1}(w, b) \to 0 \) when \( |b - w| \to \infty \).

We first show unicity, then existence in the following section.

4.1 Unicity

**Theorem 4.1** There is at most one \( \bar{\partial}^{-1} \) satisfying the above two properties.

**Proof:** We first show that for all \( w, b \) the argument of \( \bar{\partial}^{-1}(w, b) \) is well defined up to additive multiples of \( \pi \). To do this we conjugate \( \bar{\partial} \) by a diagonal unitary matrix to make it real. This conjugation simply involves multiplying the edge weights on edges incident to each vertex by a constant (a constant depending on the vertex).

Fix a vertex \( v_0 \) of \( G_D \). Let \( v \) be any other vertex of \( G_D \) and \( v_0, v_1, \ldots, v_2k = v \) be a path of rhombus edges from \( v_0 \) to \( v \). Each edge \( v_jv_{j+1} \) has exactly one vertex of
Figure 3: The function $\bar{\partial}^{-1}(w, \cdot)$ on the graph $\mathbb{Z}^2$ defines a perturbation of the standard embedding (here we took the $\epsilon$ of (3) to be 3.5).
$\mathcal{G}_D$ (the other is a vertex of $\mathcal{G}_D^*$. Direct the edge away from this vertex if it is white, and towards this vertex if it is black. Let $e^{i\alpha_j}$ be the corresponding vector (which may point contrary to the direction of the path from $v_0$ to $v$). We multiply all the edges incident to $v$ with weight $s_v = \pm e^{-\frac{i}{2}\sum \alpha_j}$ (either choice of sign will do). Note that these multipliers are well-defined up to $\pm 1$: the product of the multipliers on a path surrounding a rhombus is $\pm 1$. Let $S$ be the diagonal matrix of these weights (with an arbitrary choice of signs). Then we claim that each entry of $S^*\partial S$ is real: if $w, b$ are adjacent in $\mathcal{G}_D$, and $e^{i\theta}, e^{i\phi}$ are the edge vectors of the rhombus edge $R(wb)$, so that $\partial(w, b) = i(-e^{i\theta} + e^{i\phi})$, then

$$S^*\partial S(w, b) = s^*_ws_s\partial(w, b) = \pm s^*_w(s_w e^{-i(\theta+\phi)/2})(i(-e^{i\theta} + e^{i\phi})) \in \mathbb{R}.$$  

We now claim that $S^*\partial S$ has at most one inverse tending to zero at infinity. If $F_1, F_2$ are two such inverses, then $F_1 - F_2$ satisfies $S^*\partial S(F_1 - F_2) \equiv 0$. Moreover by reality of $S^*\partial S$ we can suppose $F_1 - F_2$ is real (otherwise take the real or imaginary parts separately).

Let $w_0$ be a fixed white vertex of $\mathcal{G}_D$. As in section 3.2 we associate to each black vertex $b$ a polygon similar to the face $b^*$ of $\mathcal{G}_D^*$. That is we associate a translate of $[S(F_1 - F_2)S^*(w_0, b)] \cdot b^*$. These translates are chosen so that if polygons $b_1^*, b_2^*$ meet at a vertex then their images meet at the the image of the same vertex. For each white face $w^*$ of $\mathcal{G}_D^*$, the images of its edges all have the same direction: if $b$ is a neighbor of $w$ then the dual edge $(wb)^* = e^{i\theta} - e^{i\phi}$ is mapped to the vector

$$(e^{i\theta} - e^{i\phi})S(F_1 - F_2)S^*(b, w_0) = (e^{i\theta} - e^{i\phi})(s_w e^{-i(\theta+\phi)/2})s^*_w,$$

which is $s^*_ws_w$ times a real number. Hence up to $\pm 1$ the direction is independent of the choice of neighbor $b$ of $w$. The image of these edges is therefore a segment. Thus we have a map from $\mathcal{G}_D^*$ to the plane, such that each black face is mapped to a similar copy of itself (positively oriented) and each white face is mapped to a segment. Each such segment has a neighborhood in $\mathbb{R}^2$ of its interior covered by the images of its neighboring black faces. If $(F_1 - F_2)(b, w_0)$ is non-zero at some $b$, the image of this map is open, since the image extends across the boundary of each black polygon. But this is impossible if $F_1 - F_2$ tends to zero at $\infty$.

4.2 Existence

For a critical weight function $\nu$ on a bipartite graph $\mathcal{G}_D$ we have the following explicit description of $\partial^{-1}$.

Let $v_0$ be a white vertex. Define for every vertex $v$ a rational function $f_v(z)$ as follows. Let $w_0 = v_0, v_1, v_2, \ldots, v_k = v$ be a path in the rhombus tiling from $w_0$ to $v$. Each edge $v_jv_{j+1}$ has exactly one vertex of $\mathcal{G}_D^*$ (the other is a vertex of $\mathcal{G}_D^*$). Direct the edge away from this vertex if it is white, and towards this vertex if it is black. Let $e^{i\alpha_j}$ be the corresponding vector (which may point contrary to the direction of the path). We define $f_v$ inductively along the path, starting from

$$f_{v_0} = 1.$$  

If the edge $v_jv_{j+1}$ leads away from a white vertex or towards a black vertex, then

$$f_{v_{j+1}} = \frac{f_{v_j}}{(z - e^{i\alpha_j})}.  \tag{5}$$
there is an ambiguity in the choice of angle, which is only defined up to a multiple of analytic functions.

Theorem 4.2 We have

\[ f_{v_{j+1}} = f_{v_j} \cdot (z - e^{i\alpha_j}). \]  

To see that \( f_v(z) \) is well-defined, it suffices to show that the multipliers for a path around a rhombus come out to 1. For a rhombus \( R(wb) \) with vertices \( w, x, b, y \) such that the vector \( wx \) is \( e^{i\theta} \) and the vector \( wy \) is \( e^{i\phi} \), we have

\[ f_w(z) = f_y(z)(z - e^{i\theta}) = f_b(z)(z - e^{i\theta})(z - e^{i\phi}) = f_x(z)(z - e^{i\theta}). \]

This shows that \( f \) is well-defined.

Any function \( f_v(z) \) satisfying (5), (6) (but not necessarily (4)) is a discrete analytic function for \( G_D \), see Theorem 2.1 below. We call such functions special discrete analytic functions.

For a black vertex \( b \) the coupling function \( \bar{\partial}^{-1}(w_0, b) \) is defined roughly as the sum over the poles of \( f_b(z) \) of the residue of \( f_b \) times the angle of \( z \) at the pole. However there is an ambiguity in the choice of angle, which is only defined up to a multiple of \( 2\pi \).

To make this definition precise, we have to assign angles (in \( \mathbb{R} \), not in \( [0, 2\pi) \)) to the poles of \( f_b(z) \). We work on the branched cover of the plane, branched over \( w_0 \), so that for each black vertex \( b \) in this cover we can assign a real angle \( \theta_0 \) to the complex vector \( b - w_0 \), which increases by \( 2\pi \) when \( b \) winds once around \( w_0 \). Each pole of \( f_b \) corresponds to a rhombus chain separating \( b \) from \( w_0 \), with \( b \) being on the black side of the chain. Because the chain is monotone we can assign an angle to the common parallel (recall that this points from the white side to the black side of the chain) which is in \( (\theta_0 - \pi, \theta_0 + \pi) \). This assigns a real angle to each pole of \( f_b \).

**Theorem 4.2** We have

\[ \bar{\partial}^{-1}(w_0, b) = -\frac{1}{2\pi i} \sum_{\text{poles } e^{i\theta}} \theta \cdot \text{Res}_{z=e^{i\theta}}(f_b), \]  

where the angles \( \theta \in \mathbb{R} \) are chosen as above for some lift of \( b \). This can be written

\[ \frac{1}{4\pi^2} \int_C f_b(z) \log zdz, \]  

where \( C \) is a closed contour surrounding ccw the part of the circle \( \{e^{i\theta} \mid \theta \in [\theta_0 - \pi + \epsilon, \theta_0 + \pi - \epsilon]\} \) which contains all the poles of \( f_b \), and with the origin in its exterior.

**Proof:** Let \( F(b) \) denote the right-hand side of (5). Note that if we choose a different lift of \( b \), the angles all change by a constant multiple of \( 2\pi \); however the sum of all residues of \( f \) is zero since \( f_b(z) \) has a double zero at \( \infty \). Therefore the right-hand side is independent of the lift of \( b \).

We need to show that \( \sum_{b \in B} \bar{\partial}(w, b) F(b) = \delta_{w_0}(w) \), and \( F(b) \) tends to zero when \( b \to \infty \) (see section 4.1).

Take a white vertex \( w, w \neq w_0 \), and let \( b_1, \ldots, b_k \) be the neighbors of \( w \). Let \( e^{i\theta_j}, e^{i\phi_j} \) be the edges of \( R(w, b_j) \), so that \( \partial(w, b_j) = i(-e^{i\theta_j} + e^{i\phi_j}) \). For each pole \( e^{i\theta} \) of some \( f_{b_j} \), the value of the angle \( \theta \) is the same for each of the \( b_j \) (they are part of the
same chain for which $e^{i\theta}$ is the common parallel). Let $C_\theta$ be a small loop around $e^{i\theta}$ in $\mathbb{C}$. We have

$$
\sum_{b \in B} \partial(b, w) F(b) = \sum_{j=1}^{k} i(-e^{i\theta_j} + e^{i\phi_j}) F(b_j),
$$

$$
= -\frac{1}{2\pi i} \sum_{j=1}^{k} i(-e^{i\theta_j} + e^{i\phi_j}) \sum_{\text{poles } e^{i\theta}} \theta \cdot \text{Res}_{z=e^{i\theta}} (f_{b_j})
$$

$$
= -\frac{1}{2\pi} \sum_{\text{poles } e^{i\theta}} \theta \cdot \frac{1}{2\pi i} \int_{C_\theta} f_{b_j}(z) dz
$$

$$
= -\frac{1}{2\pi} \sum_{\text{poles } e^{i\theta}} \theta \cdot \frac{1}{2\pi i} \int_{C_\theta} f_{w}(z) \left( \frac{1}{z - e^{i\theta_j}} - \frac{1}{z - e^{i\phi_j}} \right) dz
$$

$$
= -\frac{1}{2\pi} \sum_{\text{poles } e^{i\theta}} \theta \cdot \frac{1}{2\pi i} \int_{C_\theta} 0 dz = 0.
$$

However when $w = w_0$ we have

$$
F(b_j) = -\frac{1}{2\pi i} \left( \frac{\theta_j - \phi_j}{e^{i\theta_j} - e^{i\phi_j}} \right),
$$

so that

$$
\sum_{j=1}^{k} i(-e^{i\theta_j} + e^{i\phi_j}) F(b_j) = \frac{1}{2\pi} \sum_{j=1}^{k} \theta_j - \phi_j = 1,
$$

since the angles increase by $2\pi$ around $w_0$.

Finally to show that $F(b)$ tends to zero see Theorem 4.3 below. \qed

### 4.3 Asymptotics of $\partial^{-1}$

Let $w = v_0, v_1, \ldots, v_{2k} = b$ be a path of rhombus edges from $w$ to $b$ which crosses each rhombus chain at most once. Such a path exists since two chains cross each other at most once. Every vertex $v_{2j}$ is a vertex of $\mathcal{G}_D$, and the edges coming to and leaving $v_{2j}$ contribute a pole and a zero to $f_b$ (we are assuming $f_w(z) = 1$). So $f_b(z)$ has the form

$$
f_b(z) = \frac{1}{(z - e^{i\theta_1})(z - e^{i\theta_2}) \prod_{j=1}^{k-1} (z - e^{i\beta_j})}.
$$

By periodicity of $\mathcal{G}_D$, the angles $\beta_j$ are all in $[\theta_0 - \pi + \epsilon, \theta_0 + \pi - \epsilon]$ for some fixed $\epsilon > 0$ independent of $w, b$.

**Theorem 4.3** We have

$$
\partial^{-1}(w, b) = \frac{1}{2\pi} \left( \frac{1}{b - w} + \frac{\gamma}{(b - w)^3} \right) + \frac{1}{2\pi} \left( \frac{\xi_2}{(b - w)^3} + \frac{\bar{\xi}_2}{\gamma(b - w)^3} \right) + O \left( \frac{1}{|b - w|^3} \right),
$$

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where \( \gamma = f_b(0) = e^{-i(\theta_1 + \theta_2)} \prod e^{i(-\beta_j + \alpha_j)} \) and \( \xi_2 = e^{2i\theta_1} + e^{2i\theta_2} + \sum e^{2i\beta_j} - e^{2i\alpha_j} \) in the above notation.

**Proof:** By Theorem 4.2, \( \bar{\partial}^{-1}(w, b) \) is given by an integral

\[
\frac{1}{(2\pi)^2} \int_C f_b(z) \log z \, dz
\]

on a simple closed curve \( C \) which surrounds in a ccw sense the set \( \{ e^{i\theta} : \theta \in [\theta_0 - \pi + \epsilon, \theta_0 + \pi - \epsilon] \} \) and has the origin in its exterior. In fact since \( f \) tends to zero at \( \infty \) we can homotope \( C \) to the curve from \( \infty \) to the origin and back to \( \infty \) along the two sides of the ray \( R \) from the origin in direction \( \theta_0 + \pi \). On the two sides of this ray, \( \log z \) differs by \( 2\pi i \): it is \( 2\pi i \) less on the ccw side, and the integral therefore becomes

\[
\frac{1}{2\pi} \int_R f_b(z) \, dz,
\]

the integral being from \( \infty \) to 0 along \( R \).

We show that when \( |b - w| \) is large the only contributions to this integral come from a neighborhood of the origin and from a neighborhood of \( \infty \). Suppose without loss of generality that \( \theta_0 = 0 \), so that \( R \) is the negative real axis. Define

\[
N = |b - w| = |e^{i\theta_1} + e^{i\theta_2} + \sum_{j=1}^{k-1} e^{i\beta_j} - e^{i\alpha_j}|.
\]

For small \( t \) we have

\[
\frac{t - e^{i\alpha}}{t - e^{i\beta}} = \frac{e^{i\alpha}}{e^{i\beta}} \cdot \exp \left( (e^{-i\beta} - e^{-i\alpha})t + (e^{-2i\beta} - e^{-2i\alpha})t^2/2 + O(t^3) \right),
\]

giving

\[
f_b(t) = \gamma \exp \left( (\bar{b} - \bar{w})t + \xi_2 t^2/2 + O(t^3) \right). \tag{8}
\]

Since \( \beta \in [-\pi + \epsilon, \pi - \epsilon] \), we have \( \cos \beta > 1 - \delta \) for some fixed \( \delta = \delta(\epsilon) \). For all \( t < 0 \) we can use the bound

\[
\left| \frac{t - e^{i\alpha}}{t - e^{i\beta}} \right| = \sqrt{\frac{t^2 + 1 - 2t \cos \alpha}{t^2 + 1 - 2t \cos \beta}} 
\leq \sqrt{1 + \frac{2t(\cos \beta - \cos \alpha)}{t^2 + 1 - 2t(1 - \delta)}} 
\leq \exp \left( \frac{t(\cos \beta - \cos \alpha)}{t^2 + 1 - 2t(1 - \delta)} \right),
\]

which gives

\[
|f_b(t)| \leq \frac{e^{t^2 + 1 - 2t(1 - \delta)} \sum \cos \beta_j - \cos \alpha_j}{|(t - e^{i\theta_1})(t - e^{i\theta_2})|} \leq e^{t(N-2)/(1+t^2-2t(1-\delta))}. \tag{9}
\]

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For large $t$ we have
\[ \frac{t - e^{i\alpha}}{t - e^{i\beta}} = \exp \left( \frac{(e^{i\beta} - e^{i\alpha})}{t} + \frac{(e^{2i\beta} - e^{2i\alpha})}{2t^2} + O(\frac{1}{t^3}) \right), \]
which gives
\[ f_b(t) = e^{(b-w)/t + \xi_2/(2t^2) + O(t^{-3})}. \]  

(10)

When $-1/\sqrt{N} \leq t \leq 0$, we can use the approximation (8), when $-N^{1/2} \leq t \leq 1/\sqrt{N}$ we can use (9) and when $t \leq -N^{1/2}$ we can use (10).

The small-$|t|$ integral gives
\[ \gamma \int_{-1/\sqrt{N}}^{0} e^{t(b-w)/t + \xi_2/2t^2 + O(N/t^3)} dt = \gamma \left( \frac{1}{b-w} + \frac{\xi_2}{(b-w)^3} \right) + O\left( \frac{1}{N^3} \right). \]

The intermediate integral is negligible, and the large-$t$ integral gives
\[ \frac{1}{2\pi} \int_{-N^{1/2}}^{-\infty} \frac{e^{(b-w)/t + \xi_2/2t^2 + O(N/t^3)}}{t^2} dt = \frac{1}{2\pi} \left( \frac{1}{b-w} + \frac{\xi_2}{(b-w)^3} \right) + O\left( \frac{1}{N^3} \right). \]

\[ \square \]

5 Determinants

For a finite graph $G$ we define the normalized determinant $\det_1 M$ of an operator $M: \mathbb{C}^G \to \mathbb{C}^G$ to be
\[ \det_1 M \overset{\text{def}}{=} |\det M|^{1/|G|}, \]
where $|G|$ is the number of vertices of $G$.

For an operator $M$ on a finite graph, $\det M$ is a function of the matrix entries $M(i,j)$ which is linear in each entry separately. In particular for an edge $e = ij$, as a function of the matrix entry $M(i,j)$ we have $\det M = \alpha + \beta M(i,j)$, where $\beta$ is $(-1)^{i+j}$ times the determinant of the minor obtained by removing row $i$ and column $j$. That is,
\[ \frac{\partial (\det M)}{\partial M(i,j)} = M^{-1}(j,i) \cdot \det M, \]
or
\[ \frac{\partial (\log \det M)}{\partial M(i,j)} = M^{-1}(j,i). \]  

(11)

Suppose that $G$ is periodic under translates by a lattice $\Lambda$. Let $G_n = G/n\Lambda$, the finite graph which is the quotient of $G$ by $n\Lambda$. Now if we sum (11) for all $\Lambda$-translates of edge $ij$ in $G_n$, and then divide both sides by $|G_n|$, it yields
\[ \frac{\partial (\log \det_1 M)}{\partial M(i,j)} = \frac{1}{|G_1|} M^{-1}(j,i), \]  

(12)

where $M(i,j)$ is now the common weight of all translates of edge $ij$, that is, the left-hand side is the change in $\log \det_1 M$ when the weight of all translates of $ij$ changes. Note that this equation is independent of $n$. 
We can now use (12) to define the normalized determinant of $\bar{\partial}$ on an infinite periodic graph $G_D$ in terms of $\bar{\partial}^{-1}$. Rather, it defines the change in the normalized determinant. To complete the definition we need to define the normalized determinant of the $\bar{\partial}$ operator at some initial condition. If the half-angles of a graph degenerate by tending to 0 or $\pi/2$, the edge weights tend to either 0 or 2. The edges with weight 0 contribute nothing, the edge of weight 2 each are defined to contribute $\sqrt{2}$ to the normalized determinant. Thus we define the limit of $\log \det_1 \bar{\partial}^{-1}$ to be $\frac{n}{2} \log 2$, where $n$ is the number of edges per fundamental domain with weight 2. This makes sense since under this degeneration, removing the zero weight edges, the graph consists of copies of $\mathbb{Z}$, each consisting of weight 2 edges.

**Proof of Theorem 1.2:** Suppose $G_D$ is periodic with lattice $\Lambda$. Take a rhombus chain and all its $\Lambda$-translates in $G_D$. Let $\phi_0$ be the smallest half-angle of a rhombus on these chains, and $\phi_1$ the largest half-angle: they exist by periodicity. If $\phi_0 \leq \pi/2 - \phi_1$, deform these chains by changing the common parallels to decrease $\phi_0$ to zero: at that point the corresponding edge weight is zero and we can remove the edge completely from $G_D$, simplifying the graph in the process. If on the other hand $\pi/2 - \phi_1 < \phi_0$, deform the chain by increasing $\phi_1$ to $\pi/2$. At this point the rhombus $R_1$ of angle $\phi_1$ has flattened out to become a segment. We now repeat with another chain, but in future deformations we always keep $R_1$ as a segment: the angle $\phi_1$ remains $\pi/2$. We need to show that this is always possible. If we deform along a chain or union of chains whose common parallels do not contain an edge of $R_1$, the deformation will not affect $R_1$. If an edge of $R_1$ is in the common parallel of a chain being deformed, we deform simultaneously and in the same direction both chains which pass through $R_1$. So a deformation involves changing the common angle of parallels of chains, all of which have a parallel in the same direction. We can deform in this way because for any set of chains having parallels in the same direction, there is a deformation which tilts all the common parallels and is a translation on each complementary component of these chains. We continue deforming until all angles are zero or $\pi/2$. We have proved that any rhombus tiling of the plane can be deformed through rhombus tilings to flatten out all the rhombi: the rhombus vertices all map to $\mathbb{Z}$, the rhombus edges map to edges of $\mathbb{Z}$, and the edges of $G_D$ either have length 2 (and weight zero) or length 0 (and weight 2).

We can compute the change in $\log \det_1 \bar{\partial}$ as in (12) under such a deformation since it only depends on the angles of the rhombi in the chains.

To complete the calculation it suffices to compute the contribution for a deformation along a single chain. In fact we can treat each rhombus of the chain separately. Suppose we deform a chain whose common parallel is $e^{2i\theta}$. Suppose the rhombus $R(wb)$ has edges $e^{i\alpha}$ and $e^{i(\alpha+2\theta)}$ leading away from $w$. Then the term in

$$|G_1| \frac{d \log \det_1 \bar{\partial}}{d\theta}$$

which involves this rhombus is

$$|G_1| \frac{d(\log \det_1 \bar{\partial})}{d(\bar{\partial}(w, b))} \cdot \frac{d(\bar{\partial}(w, b))}{d\theta} = \bar{\partial}^{-1}(w, b) \frac{d(\bar{\partial}(w, b))}{d\theta} = -\frac{1}{2\pi i} \left( \frac{2\theta}{e^{i(\alpha+2\theta)} - e^{i\alpha}} \right) \frac{d}{d\theta} (i(-e^{i(\alpha+2\theta)} + e^{i\alpha}))$$

$$= \frac{\theta e^{i\theta}}{\pi \sin \theta}. $$
Suppose in the above deformations this rhombus has half-angle going to zero. Integrating the real part of this for $\theta$ from 0 to $\phi$ yields
\[ \frac{1}{\pi} \int_0^\phi \theta \cot \theta d\theta = \frac{\phi}{\pi} \log \sin \phi - \frac{1}{\pi} \int_0^\phi \log \sin \theta d\theta \]
(integrating by parts), which is
\[ = \frac{\phi}{\pi} \log 2 \sin \phi + \frac{1}{\pi} L(\phi). \]

On the other hand if the rhombus half-angle goes to $\pi/2$, we should integrate from $\pi$ down to $\phi$. This yields
\[ -\frac{1}{2} \log 2 + \frac{\phi}{\pi} \log 2 \sin \phi + \frac{1}{\pi} L(\phi), \]

since when $\phi = \pi/2$ we have
\[ \frac{\phi}{\pi} \log 2 \sin \phi + \frac{1}{\pi} L(\phi) = \frac{1}{2} \log 2. \]

This completes the proof. \(\square\)

Note that for $\mathbb{Z}^2$ in its standard embedding this gives the classical result of Kasteleyn [6, 3], except for an extra factor $\frac{1}{2} \log 2$ per edge since our edge weights are $\nu(e) = \sqrt{2}$, not 1.

A more classical way of defining the normalized determinant is via an exhaustion of $G_D$ by finite graphs. On an infinite graph $G_\infty$ which is a limit of a growing sequence of finite graphs $G_\infty = \lim_{n \to \infty} G_n$, and supposing that an operator $M_\infty$ on $G_\infty$ is a limit in an appropriate sense of $M_n$ on $G_n$, one would wish to define $\det_1 M_\infty \overset{def}{=} \lim_{n \to \infty} \det_1 M_n$, assuming this limit exists. Unfortunately it is not so easy to prove that the limit exists in the case of the operator $\bar{\partial}$, even when the graph is $\mathbb{Z}^2$ with periodic weights: see [3]. That is why we chose the previous definition instead.

6 Relating the Laplacian to $\bar{\partial}$

Let $G_T$ be a planar graph (not necessarily bipartite) with an isoradial embedding in the plane. To an edge $e$ with rhombus angle $2\theta \in [0, \pi]$ we associate a weight $c(e) = \tan \theta$. This weight $c(e)$ is the conductance in the underlying electrical network. This is the situation considered in [10].

Given $G_T$ we define another graph $G_D$ to be the superposition of $G_T$ and its dual, that is, $G_D$ has a vertex for each vertex, edge and face of $G_T$, and an edge for each half-edge of $G_T$ and each half dual-edge as well. See Figure 3. The graph $G_D$ is a bipartite planar graph, with black vertices of two types: vertices of $G_T$ and vertices of $G_T^*$, and white vertices of $G_D$ correspond to the edges of $G_T$.

An isoradial embedding of $G_T$ gives rise to an isoradial embedding of $G_D$: each rhombus of $G_T$ with half-angle $\theta$ is divided into four congruent rhombi in $G_D$, two of half-angle $\theta$ and two of half-angle $\frac{\pi}{2} - \theta$: see Figure 4.

An edge of $G_T$ of rhombus half-angle $\theta$ has weight $c(e) = \tan \theta$. This defines weight $\tan \theta$ on the two “halves” of each edge in $G_T \cup G_T^*$ (which are edges of $G_D$), and the dual edges are naturally weighted 1 (as in [3]).
The $\bar{\partial}$ operator on $G_D$ and the Laplacian on $G_T$ are related as follows. Let $\bar{\partial}$ be obtained from $\partial$ by multiplying edge weights of $G_D$ around each white vertex (coming from an edge of $G_T$ of half-angle $\theta$) by $1/(2\sqrt{\sin \theta \cos \theta})$, that is, $\bar{\partial}(w, b) = S\partial(w, b)S^*$ where $S$ is the diagonal matrix defined by $S(w, w) = 1/(2\sqrt{\sin \theta(w) \cos \theta(w)})$ and $S(b, b) = 1$.

Then an edge of $G_D$ coming from a “primal” edge of $G_T$ has weight $\sqrt{\tan \theta}$ for $\bar{\partial}$, and an edge coming from a dual edge has weight $1/\sqrt{\tan \theta}$ for $\bar{\partial}$. (These edges have weight $2\sin \theta, 2\cos \theta$ respectively for $\partial$.)

**Lemma 6.1** Restricted to vertices of $G_T$ we have $\bar{\partial}^* \bar{\partial} = \Delta_{G_T}$. Restricted to faces of $G_T$, we have $\bar{\partial}^* \bar{\partial} = \Delta_{G_T^*}$, the Laplacian on $G_T^*$.

**Proof:** Let $b \in G_T$ have neighbors $b_1, \ldots, b_k$ and let $\theta_i$ be the half-angle of edge $bb_i$. Then in $G_D$, $b$ has neighbors $w_1, \ldots, w_k$ with these same half-angles, and $\bar{\partial}(b, w_j) = \alpha_j \sqrt{\tan \theta_j}$ where $|\alpha_j| = 1$.

We have

$$\bar{\partial}^* \bar{\partial}(b, b) = \sum_{j=1}^k |\alpha_j \sqrt{\tan \theta_j}|^2 = \sum_{j=1}^k \tan \theta_j = \Delta(b, b).$$

If $b, b'$ are adjacent in $G_T$ we have

$$\bar{\partial}^* \bar{\partial}(b, b') = (\alpha \sqrt{\tan \theta})(-\bar{\alpha} \sqrt{\tan \theta}) = -\tan \theta = \Delta(b, b').$$

If $b, b'$ are at distance 2 in $G_D$ but correspond to a vertex and a face of $G_T$, we have two contributions to $\bar{\partial}^* \bar{\partial}(b, b')$ which cancel: in $G_T$ let $e_1, e_2$ be the two edges from $b$ bounding the face $b'$, with $e_1$ on the right. Then

$$\bar{\partial}^* \bar{\partial}(b, b') = \bar{\partial}(b, w_1)\bar{\partial}^*(w_1, b') + \bar{\partial}(b, w_2)\bar{\partial}^*(w_2, b')$$

$$= (\alpha_1 \sqrt{\tan \theta_1})(\frac{-i\alpha_1}{\sqrt{\tan \theta_1}}) + (\alpha_2 \sqrt{\tan \theta_2})(\frac{i\alpha_2}{\sqrt{\tan \theta_2}})$$

$$= 0.$$
Finally, if $b'$ is at distance at least 2 in $G_T$, $\bar{D}^* \bar{D}(b, b') = 0$.

The proof for $G_T^*$ is identical. \qed

## 7 Green’s function

As we did for $\bar{\partial}^{-1}$ we can give an explicit expression for the Green’s function on a critical graph.

The Green’s function $G(v, w)$ is defined to be the function satisfying

1. $\Delta G(v, w) = \delta_v(w)$ (the Laplacian is taken with respect to the second variable),
2. $G(v, w) = O(\log(|v - w|))$, and
3. $G(v, v) = 0$.

We define for each vertex $v$ of $G_T$ and $G_T^*$ a rational function $g_v(z)$ of $z$ as follows.

We define $g_v$ inductively starting from $g_{v0} = \frac{1}{z}$ and if $v, v'$ are adjacent vertices of a rhombus then

$$g_{v'} = g_v \cdot \frac{z + e^{i\theta}}{z - e^{i\theta}},$$

(13)

where $e^{i\theta}$ is the complex vector from $v$ to $v'$. Now $g_v$ is well-defined since the product of the multipliers around a rhombus is 1.

**Theorem 7.1**

$$G(v_0, v_1) = \frac{1}{4\pi i} \sum_{\text{poles } e^{i\phi} \neq 0} \theta \cdot \text{Res}_{z=e^{i\phi}}(g_{v_1}(z)) = \frac{1}{8\pi^2 i} \int_C g_{v_1}(z) \log zdz,$$

where the angles $\theta$ and curve $C$ are chosen as in Theorem 4.2. For $v_0 \in G_T$ and $v_1 \in G_T^*$ this formula defines $i$ times the harmonic conjugate of $G$.

**Proof:** The proof is similar to the proof of Theorem 4.2. Clearly $G(v, v) = 0$ by definition. To show that $\Delta G(v, w) = \delta_v(w)$, we show that for each edge the discrete Cauchy-Riemann equations are satisfied (10), that is, we show that for an edge $vv'$ with half-angle $\theta$ and with dual edge $ww'$, we have

$$(G(v_0, v') - G(v_0, w))i \tan \theta = G(v_0, w') - G(v_0, w),$$

assuming $ww'$ is obtained by turning cclw from $vv'$. It suffices that for each pole $e^{i\phi}$ we have

$$(g_{v'}(z) - g_v(z))i \tan \theta = g_{w'}(z) - g_w(z).$$

(14)

By (13), we have

$$g_{w'}(z) = g_v(z)\frac{z + e^{i\alpha}}{z - e^{i\alpha}}, \quad g_w(z) = g_v(z)\frac{z + e^{i\beta}}{z - e^{i\beta}}, \quad g_{v'}(z) = g_v(z)\frac{(z + e^{i\alpha})(z + e^{i\beta})}{(z - e^{i\alpha})(z - e^{i\beta})},$$

so that (14) is

$$g_v(z)\left(\frac{z + e^{i\alpha}}{z - e^{i\alpha}} - \frac{z + e^{i\beta}}{z - e^{i\beta}} - 1\right) i \tan \frac{\alpha - \beta}{2} = g_v(z)\left(\frac{z + e^{i\alpha}}{z - e^{i\alpha}} - \frac{z + e^{i\beta}}{z - e^{i\beta}}\right).$$
This identity holds, therefore \( G(v_0, w) \) is harmonic for \( w \neq v_0 \). Moreover for a dual vertex \( v^* \) adjacent to \( v_0 \) we have
\[
g_{v^*}(z) = \frac{z + e^{i\alpha}}{z - e^{i\alpha}},
\]
whose residue at \( e^{i\alpha} \) is 2. Thus the value of \( G(v_0, v) \) is \(-\alpha i/2\pi\). In particular for one cclw turn around the vertex \( v_0 \), the dual function increases by \(-i\), which implies that the Laplacian at \( v_0 \) is 1. Finally to show that \( G(v, w) = O(\log(|v - w|)) \), see Theorem 7.3 below.

From Lemma 6.1 there is a relation between \( G \) and \( \bar{D}^{-1} \):

**Corollary 7.2** When edge \( b_1b'_1 \) of \( G_T \) corresponds to vertex \( w_1 \) of \( G_D \), we have
\[
\frac{1}{D^*(w_1, b_1)} \bar{D}^{-1}(w_1, b_2) = G(b_1, b_2) - G(b'_1, b_2),
\]

**Proof:** By Lemma 6.1 taking the Laplacian of \( \bar{D}^{-1}(w_1, v) \) with respect to \( v \) gives
\[
\Delta_{G_T} \bar{D}^{-1}(w_1, v) = \bar{D}^* \bar{D} \bar{D}^{-1}(w_1, v) = \bar{D}^* \delta_{w_1}(v) = \bar{D}^*(w_1, b_1) \left( \delta_{b_1}(v) - \delta_{b'_1}(v) + \frac{i}{\tan \theta} (\delta_{f_1}(v) - i \delta_{f_2}(v)) \right),
\]
where \( f_1, f_2 \) are the two faces adjacent to \( e_1 \). In particular \( \bar{D}^{-1}(w_1, b) \) is harmonic as a function of \( b \in G_T \) except at \( b_1 \) and \( b'_1 \), where its Laplacian is \( \pm \bar{D}^*(w_1, b_1) \) respectively.

Since \( \bar{D}^{-1}(w_1, b_2) \rightarrow 0 \) as \( b_2 \rightarrow \infty \) this proves (15). \( \square \)

### 7.1 Asymptotics of the Green’s function

**Theorem 7.3**
\[
G(v_1, v_2) = -\frac{1}{2\pi} \log |v_2 - v_1| - \frac{\gamma_{Euler}}{2\pi} + O\left(\frac{1}{|v_2 - v_1|}\right).
\]

Note that our Laplacian is positive semidefinite, which results in the minus sign in this formula.

**Proof:** Let \( v_0 = w_0, w_1, \ldots, w_k = v_1 \) be a path of rhombus edges from \( v_0 \) to \( v_1 \). Let \( e^{i\theta_j} \) be the complex vector from \( w_j \) to \( w_{j+1} \). As in the proof of Theorem 4.3 we assume without loss of generality that \( \theta_j \in [-\pi + \epsilon, \pi - \epsilon] \) for some fixed \( \epsilon \) independent of \( v_1, v_2 \).

Let \( N = |v_1 - v_0| \). We take \( r \ll 1/N \) and \( R \gg N \). We integrate
\[
-\frac{1}{8\pi^2 i} \int_C \frac{1}{z} \prod_{j=0}^{k-1} \frac{z + e^{i\theta_j}}{z - e^{i\theta_j}} \log zdz,
\]
where \( C \) is a curve which runs cclw around the ball of radius \( R \) around the origin, from angle \(-\pi\) to \( \pi \), then along the negative \( x\)-axis from \(-R\) to \(-r\), then clockwise around the ball of radius \( r \) from angle \( \pi \) to \(-\pi\), and then back along the \( x\)-axis from \(-r\) to \(-R\).
The integral around the ball of radius \( r \) is
\[
- \frac{1}{8\pi^2} \int_{-\pi}^{\pi} (-1)^k (1 + O(Nr)) (\log r + i\theta) i d\theta = \frac{(-1)^k \log r}{4\pi} (1 + O(Nr)).
\]

The difference between the value of \( \log z \) above and below the \( x \)-axis is \( 2\pi i \), so that we can combine the two parts of the integral along the \( x \)-axis into the integral of
\[
- \frac{1}{4\pi} \int_{-R}^{r} \frac{1}{z} \prod_{j=0}^{k-1} \frac{z + e^{i\theta_j}}{z - e^{i\theta_j}} dz.
\]

This integral we split into a part from \(-R\) to \(-\sqrt{N}\), from \(-\sqrt{N}\) to \(-1/\sqrt{N}\) and from \(-1/\sqrt{N}\) to \(-r\).

For small \( t < 0 \) we have
\[
\prod_{j=0}^{k-1} \frac{t + e^{i\theta_j}}{t - e^{i\theta_j}} = (-1)^k e^{2 \sum e^{-i\theta_j} t + O(t^2)} = (-1)^k e^{2(\theta_1 - \theta_0) t + O(t^2)}
\]
so that the integral near the origin is
\[
- \frac{(-1)^k}{4\pi} \int_{-1/\sqrt{N}}^{-r} \frac{e^{\alpha t + O(t^2)}}{t} dt.
\]

where \( \alpha = 2(v_1 - v_0) \),
\[
= - \frac{(-1)^k}{4\pi} (1 + O(1/N)) \int_{-\alpha/\sqrt{N}}^{-\alpha} \frac{e^s}{s} ds
= - \frac{(-1)^k}{4\pi} (1 + O(1/N)) \left( \int_{-\alpha/\sqrt{N}}^{-1} \frac{e^s}{s} ds + \int_{-1}^{-\alpha} \frac{e^s - 1}{s} ds + \int_{-1}^{-\alpha} \frac{ds}{s} \right)
= - \frac{(-1)^k}{4\pi} \left( \log(r\bar{\alpha}) + \gamma_{\text{Euler}} \right) + O(1/N).
\]
The Euler \( \gamma \) constant \( \gamma_{\text{Euler}} \) comes from evaluating the first two of these integrals in the limit \( \alpha \rightarrow 0 \) and \( \alpha/\sqrt{N} \rightarrow \infty \).

In the intermediate range the integral is negligible (see the proof of Theorem 4.3). Near \( t = -R \) we have
\[
\prod_{j=0}^{k-1} \frac{t + e^{i\theta_j}}{t - e^{i\theta_j}} = e^{2 \sum e^{i\theta_j} / t + O(t^{-2})} = e^{2(v_1 - v_0) / t + O(t^{-2})}
\]
so that the integral near \(-R\) is
\[
- \frac{1}{4\pi} \int_{-\sqrt{N}}^{-R} \frac{e^{\alpha t + O(t^{-2})}}{t} dt = - \frac{1}{4\pi} \int_{-R/\alpha}^{-\sqrt{N}/\alpha} \frac{e^{1/s}}{s} ds,
= - \frac{1}{4\pi} \left( \int_{-R/\alpha}^{-1} \frac{e^{1/s} - 1}{s} ds + \int_{-1}^{-R/\alpha} \frac{ds}{s} + \int_{-1}^{-\sqrt{N}/\alpha} \frac{e^{1/s}}{s} ds \right)
= - \frac{1}{4\pi} \left( - \log(R/\alpha) + \gamma_{\text{Euler}} \right) + O(\frac{1}{N}),
\]
These are precisely the same integrals as in the “near \( r \)” case, under the change of variable \( s \rightarrow 1/s \).

Finally the integral around the ball of radius \( R \) gives

\[-\frac{1}{4\pi}(\log \alpha + (1)^k \log \bar{\alpha}) - (1 + (1)^k)\gamma_{\text{Euler}} \frac{k}{4\pi} + O\left(\frac{1}{N}\right)\]

which when \( k \) is even is

\[-\frac{1}{2\pi} \log |v_1 - v_0| - \gamma_{\text{Euler}} \frac{1}{2\pi} + O\left(\frac{1}{N}\right) = -\frac{1}{2\pi} \text{Re}(v_1 - v_0) - \gamma_{\text{Euler}} \frac{1}{2\pi} + O\left(\frac{1}{N}\right)\]

and when \( k \) is odd it is

\[-\frac{1}{4\pi} \log \frac{v_1 - v_0}{v_1 - \bar{v}_0} + O\left(\frac{1}{N}\right) = -\frac{i}{2\pi} \text{Im} \log(v_1 - v_0) + O\left(\frac{1}{N}\right).\]

\[\square\]

### 7.2 Determinant of Laplacian

Using Lemma 6.1 we can compute the determinant of \( \Delta \) in terms of the determinant of \( \bar{\partial} \). However it is as simple to compute it directly.

We define the normalized determinant \( \det_1 \Delta \) to be the function of the conductances satisfying, for \( w \) adjacent to \( v \),

\[
\frac{\partial(\log \det_1 \Delta)}{\partial(\Delta(v, w))} = \frac{1}{|G_1|} (G(v, v) - G(v, w) - G(w, v) + G(w, w)) = -\frac{2G(v, w)}{|G_1|},
\]

and for a graph with zero conductances, \( \det_1 \Delta := 1 \).

Let \( G_T \) be a periodic planar graph, periodic under \( \Lambda \) with fundamental domain \( G_T/\Lambda = G_1 \).

**Proof of Theorem 1.1** The method of proof is the same as that of Theorem 1.2. Take a chain of rhombi and all its translates under \( \Lambda \), and change their common parallel, decreasing the smallest angle to zero. If an edge has angle which goes to zero, its conductance goes to 0 as well and so it can be removed from \( G_T \). Similarly if any rhombus angle increases to \( \pi/2 \), the corresponding conductance becomes infinite and one can remove the corresponding edge, gluing the two vertices together. In this way we can simplify the graph until no edges remain.

However there is a problem which is that the determinant blows up when a conductance goes to \( \infty \). So instead we compute the change in the difference

\[E := \log \det_1 \Delta - \frac{1}{|G_1|} \sum_{e \in G_1} \frac{2\theta}{\pi} \log \tan \theta\]

as we reduce the graph in this manner. (The quantity \( E \) is the entropy of the underlying spanning tree model.)

To compute the change under the perturbations we can treat each rhombus independently. For a rhombus \( R vw \) of angle \( 2\theta \), with edges from \( v \) given by \( e^{i\alpha} \) and \( e^{i(\alpha + 2\theta)} \), a short calculation yields

\[G(v, w) = -\frac{\theta}{\pi \tan \theta},\]
so that the part of $\partial E / \partial \theta$ involving the rhombus $R$ is $1 / |G_1|$ times

$$-2G(v,w) \frac{\partial(\Delta(v,w))}{\partial \theta} - \frac{\partial}{\partial \theta} \left( \frac{2\theta}{\pi} \log \tan \theta \right) = \frac{2\theta}{\pi \tan \theta} \cdot \frac{\partial \tan \theta}{\partial \theta} - \left( \frac{2\log \tan \theta}{\pi} + \frac{2\theta}{\pi \tan \theta \cos^2 \theta} \right) = -\frac{2\log \tan \theta}{\pi}. $$

The integral from 0 to $\phi$

$$-\frac{2}{\pi} \int_0^\phi \log \tan \phi d\phi = \frac{2}{\pi} (L(\phi) + L(\pi/2 - \phi)), $$

where we used $L(\pi/2) = 0$. Similarly the integral from $\pi/2$ down to $\phi$

$$\frac{2}{\pi} (L(\phi) + L(\pi/2 - \phi)), $$

using $L(0) = L(\pi/2) = 0$. □

8 Convexity

**Proposition 8.1** When parametrized by the rhombus angles the set of isoradial embeddings of a graph $\mathcal{G}$ is convex.

**Proof:** Each rhombus has angle in $[0, \pi]$. The set of allowed angles is the subset of $[0, \pi]^{|E|}$ defined by the linear constraints that the sum of the angles around each vertex be $2\pi$. It is therefore convex. □

**Theorem 8.2** A periodic graph which has an isoradial embedding has a unique critical weight function $\nu$ maximizing $\det \tilde{\partial}$, and a unique critical weight function $c$ maximizing $\det \Delta$.

**Proof:** Given that the set of critical weight functions is convex, and $Z := \det \tilde{\partial}$ or $\det \Delta$ is continuous as a function of the rhombus angles, we know that $Z$ attains its maximum. To show unicity, it suffices to show that $\log Z$ is a strictly concave function of the rhombus angles. Suppose there are $n$ rhombi in a fundamental domain. Then $\log Z$ is the sum over the rhombi in the fundamental domain of a function depending only on the rhombus angle. Moreover, the functions

$$f_D(\theta) = \frac{1}{\pi} L(\theta) + \frac{\theta}{\pi} \log 2 \sin \theta$$

and

$$f_T(\theta) = \frac{2}{\pi} \left( L(\theta) + L\left(\frac{\pi}{2} - \theta\right) \right) + \frac{2\theta}{\pi} \log \tan \theta$$

are strictly concave as a function of $\theta$. So we can view $\log Z$ as a function on $[0, \pi]^n$ (the set of rhombus angles), subject to the linear restrictions that the sum of the angles around each vertex be $2\pi$. Thus $\log Z$ is the restriction to a linear subspace of a strictly concave function on $[0, \pi]^n$. As such it is strictly concave and so has a unique maximum. □

What are the geometric interpretations of these maxima?
9 Discretization

Recall that a special discrete analytic function is a function \( f_{b}(z) \) satisfying (5) and (6). Note that it is defined by its value at any vertex. Other discrete analytic functions can be built up from these by convolutions:

**Theorem 9.1** Let \( f_{b}(z) \) be a special discrete analytic function and \( \mu \) be any measure in the plane. Then the function \( F(b) = \int f_{b}(z)d\mu(z) \) is a discrete analytic function.

**Proof:** The proof was already given, in the proof of Theorem 4.2; we repeat it here. Let \( w \) be a white vertex with neighbors \( b_1, \ldots, b_k \), so that \( R(wb_j) \) has edges \( e^{i\theta_j} \) and \( e^{i\phi_j} \).

We have

\[
\sum_{b \in B} \partial(w, b)F(b) = \sum_{j=1}^{k} i(-e^{i\theta_j} + e^{i\phi_j})F(b_j)
\]

\[
= \sum_{j=1}^{k} i(-e^{i\theta_j} + e^{i\phi_j}) \int f_{b_j}(z)d\mu(z)
\]

\[
= \int f_{w}(z) \sum_{j=1}^{k} \frac{(-e^{i\theta_j} + e^{i\phi_j})}{(z - e^{i\theta_j})(z - e^{i\phi_j})} d\mu(z)
\]

\[
= \int f_{w}(z) \left( \sum_{j=1}^{k} \frac{1}{(z - e^{i\phi_j})} - \frac{1}{(z - e^{i\theta_j})} \right) d\mu(z)
\]

\[
= \int 0 d\mu(z) = 0.
\]

\[
\square
\]

There is corresponding version for discrete harmonic functions, which also appear in a different form in [11]:

**Corollary 9.2** On \( G_T \), let \( h_{v_0}(z) \) be given and if \( h_{v}(z) \) is defined and \( v' \) is adjacent to \( v \), define

\[
h_{v'}(z) = h_{v}(z) \frac{(z + e^{i\theta})(z + e^{i\phi})}{(z - e^{i\theta})(z - e^{i\phi})},
\]

where \( \theta \) and \( \phi \) are the angles of the edges of rhombus \( R(vv') \). This defines \( h_{v}(z) \) for all \( v \in G_T \), and \( H(v) = \int h_{v}(z)d\mu(z) \) is discrete harmonic.

10 The dimer model

10.1 Definitions

The dimer model on planar graphs was initiated by Fisher, Kasteleyn and Temperley [6, 13], who “solved” the model on \( \mathbb{Z}^2 \) by finding a closed-form expression for the partition function.

A dimer covering, or perfect matching, of a graph \( G \) is a set of edges \( M \) of \( G \) which covers every vertex exactly once, that is every vertex is an endpoint of exactly one
element of $M$. Let $\nu : E \to [0, \infty)$ be a weight function on the edges of $G$. To an edge $e$ we associate an energy $-\log \nu(e)$; this defines a natural probability measure $\mu$, the Boltzman measure, on the set of all dimer coverings $M$, where the probability of a configuration is proportional to the product of its dimer weights (the exponent of minus the sum of the energies), $\mu(C) = \prod_{e \in C} \nu(e)$. The dimer model is the study of this measure.

The dimer partition function is by definition

$$Z(G, \nu) = \sum_{M \in M} \nu(M).$$

The partition function per site is by definition $Z = Z(G, \nu)^{1/|G|}$, where $|G|$ is the number of vertices of $G$.

For infinite graphs one can define the partition function per site by taking limits of the partition functions per site on finite pieces: if $\{G_n\}$ is an exhaustion of $G$ by finite subgraphs, one can define

$$Z = \lim_{n \to \infty} Z(G_n, \nu)^{1/|G_n|},$$

when this limit exists. Usually this limit will depend on the sequence $\{G_n\}$: see e.g. [3].

One can also define a probability measure $\mu$ on $M(G)$ which is a weak limit $\mu = \lim_{n \to \infty} \mu_n$ where $\mu_n$ is the Boltzmann measure on $M(G_n)$, assuming this limit exists. This limit is even more dependent on the sequence than the partition function per site.

One general situation where these limits are known to exist is when $G_D$ is of the form $G_T \cup G^*_T$ as described in section 6. For a periodic $G_T$ the Green’s functions are known to converge to the corresponding Green’s function on the plane as defined in section 7, see [3]. From corollary 15 we can compute the $\bar{\partial}^{-1}$ operator from the Green’s function. In fact in this situation one can exhibit a measure-preserving isomorphism from the dimer model on $G_D$ to the corresponding spanning tree model on $G_T$ [12, 8]. In particular the partitions functions are equal.

The utility of the $\bar{\partial}$ operator in the dimer model is the subject of the next section.

10.2 Dimers on finite graphs

A subgraph $G_1$ of $G$ is said to be simply connected if it consists of the set of edges and vertices contained on or inside a simple polygonal path in $G$.

**Theorem 10.1** Let $G_1$ be a simply connected subgraph of a graph $G$ (where $G$ has a critical weight function), and $\bar{\partial}_1$ the submatrix of $\bar{\partial}$ corresponding to $G_1$. The dimer partition function on $G_1$ is $Z_1 = \sqrt{\det \bar{\partial}_1}$. The probability of edges $\{w_1 b_1, \ldots, w_k b_k\}$ occurring in a configuration chosen with respect to the Boltzmann measure $\mu$ is

$$\left( \prod_{i=1}^k \bar{\partial}_1(w_i, b_i) \right) \det_{1 \leq i, j \leq k} ((\bar{\partial}_1)^{-1}(w_i, b_j)).$$

The function $\bar{\partial}^{-1}$ is called the coupling function or inverse Kasteleyn matrix.

**Proof:** For the first statement it suffices to show that $\bar{\partial}$ is Kasteleyn-flat [8]. That is, we must show that for each face of $G_1$ (except the outer face) with vertices $u_1, v_1, \ldots, u_m, v_m$ in cyclic order, we have

$$\arg(\bar{\partial}(u_1, v_1) \cdots \bar{\partial}(u_m, v_m)) = \arg((-1)^{m-1} \bar{\partial}(v_1, u_2) \cdots \bar{\partial}(v_{m-1}, u_m)\bar{\partial}(v_m, u_1)).$$
(This identity implies that two dimer configurations which only differ around a single face have the same argument in the expansion of the determinant. By [13], any two configurations can be obtained from one another by such displacements.) To prove this identity, note that it is true if the points are regularly spaced along a 2m-gon, and note that it remains true if you move one point at a time.

For the second statement, see [7]. □

10.3 Volume and mean curvature of $P(G)$

**Proposition 10.2 (Milnor [14])** The volume of a hyperbolic simplex in the upper half space model with vertices at $\infty$, (0,0,1), (1,0,0) and $(\cos \theta, \sin \theta, 0)$ is $L((\pi - \theta)/2)$, where $L$ is the Lobachevsky function [14].

Let $G$ be a bipartite planar graph with an isoradial embedding, and $P(G^*)$ the hyperbolic polyhedron associated to its dual (vertices of $P$ are vertices of $G^*$). For each rhombus $R(wb)$ of $G$ let $w,x,b,y$ be its four vertices; we can decompose $P$ into simplices, so that the part of $P$ over $R$ consists of two simplices with vertices $w, x, y, \infty$ and $b, x, y, \infty$. Here vertices $x, y$ are ideal whereas $w$ and $b$ are in the interior of hyperbolic three-space (at Euclidean distance 1 from the $xy$-plane). Each of these simplices has volume $L(\theta)$, where $\theta$ is the half-angle of the rhombus. Therefore in the expression for the determinant of $\partial$, the sum of the terms involving $L$ is $\frac{1}{2\pi}$ times the volume of $P$.

The mean curvature of a compact convex polyhedron in $\mathbb{H}^3$ is by definition the sum over the edges of $(\pi - \theta_{dih})\ell$, where $\theta_{dih}$ is the dihedral angle and $\ell$ is the hyperbolic edge length. For an ideal polyhedron, the edge lengths are infinite, so one first assigns a small horosphere to each vertex and then computes the edge lengths in the exterior of these horospheres. This definition of course depends on the radii of the horospheres. In the case of a $P$ arising from an isoradial embedding of $G$ we can choose for a fixed small $\epsilon$ all the horospheres to be Euclidean balls of diameter $\epsilon$; in this case the hyperbolic length of an edge between points $v_1$ and $v_2$ is exactly

$$2\log \frac{|v_2 - v_1|}{\epsilon},$$

where $|v_2 - v_1|$ is the Euclidean distance from $v_1$ to $v_2$.

For a rhombus of half-angle $\theta$, the dihedral angle of $P$ at the dual edge is $\pi - 2\theta$. Therefore the mean curvature per vertex of $P$ is

$$\frac{4}{|G_1|} \sum_{e \in G_1} \theta \log \frac{2\sin \theta}{\epsilon} = -4\pi \log \epsilon + \frac{4}{|G_1|} \sum_{e \in G_1} \theta \log 2 \sin \theta$$

(16)

this last equality holds since at each vertex the sum of the half-angles is $\pi$. Finally it makes sense to define the normalized mean curvature to be the above formula (16) without the $-4\pi \log \epsilon$ term.

10.4 Geometry and Statistical mechanics

If $G_D$ arises as $G_T \cup G_T^*$ for a periodic isoradial graph $G_T$, we can identify $\text{det}_1 \tilde{\partial}$ with the partition function of the associated dimer model, and $\partial^{-1}$ as the coupling function, as discussed at the end of section 10.1.

Moreover we have
Theorem 10.3  The volume per site of $P(G_D^*)$ is $2\pi$ times the entropy per site of the dimer model. The normalized mean curvature per site of $P(G_D^*)$ is $4\pi$ times the mean energy per site of a dimer configuration.

Proof:  In the dimer model the probability of occurrence of an edge $e = wb$ is $\nu(e)|\hat{\partial}^{-1}(w,b)| = \frac{\theta}{\pi}$. Recall that the energy contribution for this edge is $\log 2 \sin \theta$. The average energy per vertex $\bar{E}$ is therefore $1/|G_1|$ times the sum over the edges in a fundamental domain of $\frac{d}{\theta} \log 2 \sin \theta$. By the argument of Proposition 9.1 of [3], the fact that the edge correlations tend to zero implies that the entropy per vertex of the dimer model is precisely $(\log \det \hat{\partial}) - \bar{E}$, or

$$\frac{1}{|G_1|} \sum_e \frac{1}{\pi} L(\theta),$$

that is, $1/2\pi$ times the volume per vertex of $P$.

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