A BRIEF DESCRIPTION OF OPERATORS ASSOCIATED TO THE QUANTUM HARMONIC OSCILLATOR ON SCHATTEN-VON NEUMANN CLASSES

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Abstract. In this note we study pseudo-multipliers associated to the harmonic oscillator (also called Hermite multipliers) belonging to Schatten classes on $L^2(\mathbb{R}^n)$. We also investigate the spectral trace of these operators. MSC 2010. Primary 81Q10 ; Secondary 47B10, 81Q05.

1. Introduction

1.1. Outline of the paper. Pseudo-multipliers and multipliers associated to the harmonic oscillator arise from the study of Hermite expansions for complex functions on $\mathbb{R}^n$ (see Thangavelu [21, 22, 23, 24, 25, 26], Epperson [10] and Bagchi and Thangavelu [1]). In this note, we are interested in the membership of pseudo-multipliers associated to the harmonic oscillator (also called Hermite pseudo-multipliers) in the Schatten classes, $S_r(L^2)$ on $L^2(\mathbb{R}^n)$. With this paper we finish the classification of pseudo-multipliers in classes of $r$-nuclear operators on $L^p$-spaces (see Barraza and Cardona [2, 3]), which on $L^2(\mathbb{R}^n)$ coincide with the Schatten-von Neumann classes of order $r$. Our main result is Theorem 1.1 where we establish some criteria in order that pseudo-multipliers belong to the classes $S_r(L^2)$, $0 < r \leq 2$. In order to present our main result we recall some notions. Let us consider the sequence of Hermite function on $\mathbb{R}^n$, $\phi_\nu = \prod_{j=1}^n \phi_{\nu_j}$, $\phi_{\nu_j}(x_j) = (2^{\nu_j} \nu_j! \sqrt{\pi})^{-\frac{1}{2}} H_{\nu_j}(x_j) e^{-\frac{1}{2} x_j^2}$ (1.1) where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n$, and $H_{\nu_j}(x_j)$ denotes the Hermite polynomial of order $\nu_j$. It is well known that the Hermite functions provide a complete and orthonormal system in $L^2(\mathbb{R}^n)$. If we consider the operator $L = -\Delta + |x|^2$ acting on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, where $\Delta$ is the standard Laplace operator on $\mathbb{R}^n$, then we have the relation $L \phi_\nu = \lambda_\nu \phi_\nu$, $\nu \in \mathbb{N}_0^n$. The operator $L$ is symmetric and positive in $L^2(\mathbb{R}^n)$ and admits a self-adjoint extension $H$ whose domain is given by $\text{Dom}(H) = \left\{ \sum_{\nu \in \mathbb{N}_0^n} \langle f, \phi_\nu \rangle_{L^2} \phi_\nu : \sum_{\nu \in \mathbb{N}_0^n} |\lambda_\nu \langle f, \phi_\nu \rangle_{L^2}|^2 < \infty \right\}$ (1.2).

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So, for \(f \in \text{Dom}(H)\), we have
\[
(Hf)(x) = \sum_{\nu \in \mathbb{N}_0} \lambda_\nu \widehat{f}(\phi_\nu)\phi_\nu(x), \quad \widehat{f}(\phi_\nu) = \langle f, \phi_\nu \rangle_{L^2}. \tag{1.3}
\]

The operator \(H\) is precisely the quantum harmonic oscillator on \(\mathbb{R}^n\) (see [12]).

The sequence \(\{\widehat{f}(\phi_\nu)\}\) determines the Fourier-Hermite transform of \(f\), with corresponding inversion formula
\[
f(x) = \sum_{\nu \in \mathbb{N}_0} \widehat{f}(\phi_\nu)\phi_\nu(x). \tag{1.4}
\]

On the other hand, pseudo-multipliers are defined by the quantization process that associates to a function \(m\) on \(\mathbb{R}^n \times \mathbb{N}_0^\infty\) a linear operator \(T_m\) of the form:
\[
T_m f(x) = \sum_{\nu \in \mathbb{N}_0^n} m(x, \nu) \widehat{f}(\phi_\nu)\phi_\nu(x), \quad f \in \text{Dom}(T_m). \tag{1.5}
\]

The function \(m\) on \(\mathbb{R}^n \times \mathbb{N}_0^\infty\) is called the symbol of the pseudo-multiplier \(T_m\).

If in (1.5), \(m(x, \nu) = m(\nu)\) for all \(x\), the operator \(T_m\) is called a multiplier.

Multipliers and pseudo-multipliers have been studied, for example, in the works [1, 18, 19, 20, 21, 22] (and references therein) principally by its mapping properties on \(L^p\) spaces. In order that the operator \(T_m : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)\) belongs to the Schatten class \(S_r(L^2)\), in this paper we provide some conditions on the symbol \(m\).

1.2. Pseudo-multipliers in Schatten classes. By following A. Grothendieck [11], we can recall that a linear operator \(T : E \to F\) (\(E\) and \(F\) Banach spaces) is \(r\)-nuclear, if there exist sequences \((e'_n)_{n \in \mathbb{N}_0}\) in \(E'\) (the dual space of \(E\)) and \((y_n)_{n \in \mathbb{N}_0}\) in \(F\) such that
\[
Tf = \sum_{n \in \mathbb{N}_0} e'_n(f)y_n, \quad \text{and} \quad \sum_{n \in \mathbb{N}_0} \|e'_n\|_{E'}\|y_n\|_F < \infty. \tag{1.6}
\]

The class of \(r\)-nuclear operators is usually endowed with the quasi-norm
\[
n_r(T) := \inf \left\{ \left\{ \sum_n \|e'_n\|_{E'}\|y_n\|_F \right\}^{\frac{1}{r}} : T = \sum_n e'_n \otimes y_n \right\} \tag{1.7}
\]

In addition, when \(E = F\) is a Hilbert space and \(r = 1\) (resp. \(r = 2\)) the definition above agrees with the concept of trace class operators (resp. Hilbert-Schmidt).

For the case of Hilbert spaces \(H\), the set of \(r\)-nuclear operators agrees with the Schatten-von Neumann class of order \(r\) (see Pietsch [13, 14]). We recall that a linear operator \(T\) on a Hilbert space \(H\) belong to the Schatten class of order \(r\), \(S_r(H)\) if
\[
s_r(T) := \sum_{n \in \mathbb{N}_0} \lambda_n(T)^r < \infty, \tag{1.8}
\]

where \(\{\lambda_n(T)\}\) denotes the sequence of singular values of \(T\), which are the eigenvalues of the operator \(\sqrt{T^*T}\). It was proved in [2] that a multiplier \(T_m\) with symbol satisfying conditions of the form
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\[ \kappa(m, p_1, p_2) := \sum_{s=0}^{n} \sum_{\nu \in I_s} \alpha_{r, p_1, p_2}(s, \nu) |m(\nu)|^r < \infty, \]  

(1.9)

where \( \{I_s\}_{s=0}^{n} \) is a suitable partition of \( \mathbb{N}_0^n \), and \( \alpha_{r, p_1, p_2}(s, \nu) \) is a suitable kernel, can be extended to a \( r \)-nuclear operator from \( L^{p_1}(\mathbb{R}^n) \) into \( L^{p_2}(\mathbb{R}^n) \). Although it is easy to see that similar necessary conditions apply for pseudo-multipliers, the \( r \)-nuclearity for these operators in \( L^p \)-spaces was characterized in [3] by the following condition,

- a pseudo-multiplier \( T_m \) can be extended to a \( r \)-nuclear operator from \( L^{p_1} \) into \( L^{p_2} \) if and only if there exist functions \( h_k \) and \( g_k \) satisfying

\[ m(x, \nu) = \phi_\nu(x)^{-1} \sum_{k=1}^{\infty} h_k(x) \hat{g}(\phi_\nu), \, \text{a.e.} \, w. \, x, \, \text{with} \sum_{k=0}^{\infty} \|g_k\|_{L^{p_2}'} \|h_k\|_{L^{p_1}} < \infty. \]  

(1.10)

If we consider \( p_1 = p_2 = 2 \), and a multiplier \( T_m \), the conditions above can be replaced by the following more simple one,

\[ \kappa(m, 2, r) := \sum_{\nu \in \mathbb{N}_0^n} |m(\nu)|^r < \infty, \]  

(1.11)

because the set of singular values of a multiplier \( T_m \) consists of the elements in the sequence \( \{|m(\nu)|\}_{\nu \in \mathbb{N}_0^n} \). The condition (1.10) characterizes the membership of pseudo-multipliers in Schatten classes in terms of the existence of certain measurable functions. However, in this paper we provide explicit conditions on \( m \) in order to guarantee that \( T_m \in S_r(L^2) \), because explicit conditions allow us to known information about the distribution of the spectrum of these operators. Our main result is the following theorem.

**Theorem 1.1.** Let \( T_m \) be a pseudo-multiplier with symbol \( m \) defined on \( \mathbb{R}^n \times \mathbb{N}_0^n \). Then we have,

- \( T_m \) is a Hilbert-Schmidt operator on \( L^2(\mathbb{R}^n) \), i.e., \( T_m \in S_2(L^2) \), if and only if

\[ \sum_{\nu \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} |m(x, \nu)|^2 \phi_\nu(x)^2 dx < \infty. \]  

(1.12)

- If \( T_m \) is a positive and self-adjoint operator, then \( T_m \) is trace class, i.e., \( T_m \in S_1(L^2) \), if and only if

\[ \sum_{\nu \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} m(x, \nu) \phi_\nu(x)^2 dx < \infty. \]  

(1.13)

- \( T_m \in S_r(L^2) \), \( 0 < r \leq 1 \), if

\[ \sum_{\nu \in \mathbb{N}_0^n} \left( \int_{\mathbb{R}^n} |m(x, \nu)|^2 \phi_\nu(x)^2 dx \right)^{\frac{r}{2}} < \infty. \]  

(1.14)
• If $1 < r < 2$ and there exists $\sigma > n\left(\frac{1}{r} - \frac{1}{2}\right)$ such that

$$\sum_{\nu \in \mathbb{N}_0^n} |\nu|^{2\sigma} \int_{\mathbb{R}^n} |m(x, \nu)|^2\phi_\nu(x)^2 \, dx < \infty,$$

(1.15)

then $T_m \in S_r(L^2)$.

1.3. Related works. Now, we include some references on the subject. Sufficient conditions for the $r$-nuclearity of spectral multipliers associated to the harmonic oscillator, but, in modulation spaces and Wiener amalgam spaces have been considered by J. Delgado, M. Ruzhansky and B. Wang in [7, 8]. The Properties of these multipliers in $L^p$-spaces have been investigated in the references S. Bagchi, S. Thangavelu [1], J. Epperson [10], K. Stempak and J.L. Torrea [18, 19, 20], S. Thangavelu [21, 22] and references therein. Hermite expansions for distributions can be found in B. Simon [17]. The $r$-nuclearity and Grothendieck-Lidskii formulae for multipliers and other types of integral operators can be found in [6, 8].

On Hilbert spaces the class of $r$-nuclear operators agrees with the Schatten-von Neumann class $S_r(H)$; in this context operators with integral kernel on Lebesgue spaces and, in particular, operators with kernel acting of a special way with an-harmonic oscillators of the form $E_a = -\Delta_x + |x|^a$, $a > 0$, has been considered on Schatten classes on $L^2(\mathbb{R}^n)$ in J. Delgado and M. Ruzhansky [9]. The proof of our results will be presented in the next section.

2. Pseudo-multipliers in Schatten-von Neumann classes

In this section we prove our main result for pseudo-multipliers $T_m$. Our criteria will be formulated in terms of the symbols $m$. First, let us observe that every multiplier $T_m$ is an operator with kernel $K_m(x, y)$. In fact, straightforward computation show that

$$T_m f(x) = \int_{\mathbb{R}^n} K_m(x, y) f(y) \, dy, \quad K_m(x, y) := \sum_{\nu \in \mathbb{N}_0^n} m(x, \nu)\phi_\nu(x)\phi_\nu(y)$$

(2.1)

for every $f \in \mathcal{D}(\mathbb{R}^n)$. We will use the following result (see J. Delgado [4, 5]).

**Theorem 2.1.** Let us consider $1 \leq p_1, p_2 < \infty$, $0 < r \leq 1$ and let $p'_1$ be such that $\frac{1}{p_1} + \frac{1}{p'_1} = 1$. An operator $T : L^{p_1}(\mu_1) \to L^{p_2}(\mu_2)$ is $r$-nuclear if and only if there exist sequences $(g_n)_n$ in $L^{p_2}(\mu_2)$, and $(h_n)_n$ in $L^{q_1}(\mu_1)$, such that

$$\sum_n \|g_n\|_{L^{p_2}}^r \|h_n\|_{L^{q_1}} < \infty,$n and $T f(x) = \int (\sum_n g_n(x) h_n(y)) f(y) \, d\mu_1(y)$, a.e.w. $x$,

(2.2)

for every $f \in L^{p_1}(\mu_1)$. In this case, if $p_1 = p_2$ (see Section 3 of [4]) the nuclear trace of $T$ is given by

$$\text{Tr}(T) := \int \sum_n g_n(x) h_n(x) \, d\mu_1(x).$$

(2.3)

Now, we prove our main theorem.
Proof of Theorem 1.1. Let us consider a pseudo-multiplier \( T_m \). By definition, \( T_m \) is a Hilbert-Schmidt operator if and only if there exists an orthonormal basis \( \{ e_\nu \} \) of \( L^2(\mathbb{R}^n) \) such that

\[
\sum_\nu \| T_m e_\nu \|_{L^2}^2 < \infty. \tag{2.4}
\]

In particular, if we choose the system of Hermite functions \( \{ \phi_\nu \} \), which provides an orthonormal basis of \( L^2(\mathbb{R}^n) \), from the relation \( T_m(\phi_\nu) = m(x, \nu)\phi_\nu \), we conclude that \( T_m \) is of Hilbert-Schmidt type, if and only if

\[
\sum_\nu \| m(\cdot, \nu)\phi_\nu \|_{L^2}^2 = \sum_{\nu \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} |m(x, \nu)|^2 \phi_\nu(x)^2 dx < \infty. \tag{2.5}
\]

So, we have proved the first statement. Now, if we assume that \( T_m \) is positive and self-adjoint, then \( T_m \) is of class trace if and only if there exists an orthonormal basis \( \{ e_\nu \} \) of \( L^2(\mathbb{R}^n) \) such that

\[
\sum_\nu \langle T_m e_\nu, e_\nu \rangle_{L^2} < \infty. \tag{2.6}
\]

As in the first assertion, if we choose the basis formed by the Hermite functions, \( T_m \) is of class trace if and only if

\[
\sum_\nu \langle T_m e_\nu, e_\nu \rangle_{L^2} = \sum_{\nu \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} m(x, \nu)\phi_\nu(x)^2 dx < \infty,
\]

which proves the second assertion. Now, we will verify that (1.1) implies that \( T_m \in S_r(L^2) \) for \( 0 < r \leq 1 \). For this, we will use Delgado’s Theorem (Theorem 2.1) to the representation (2.1) of \( K_m \)

\[
K_m(x, y) := \sum_{\nu \in \mathbb{N}_0^n} m(x, \nu)\phi_\nu(x)\phi_\nu(y). \tag{2.8}
\]

So, \( T_m \in S_r(L^2) \) if

\[
\sum_{\nu} \| m(\cdot, \nu) \|_{L^2}^r \| \phi_\nu \|_{L^2}^r < \infty,
\]

where we have used that the \( L^2 \)-norm of every Hermite function \( \phi_\nu \) is normalised.

In order to finish the proof, we only need to prove that (1.15) assures that \( T_m \in S_r(L^2) \) for \( 1 < r < 2 \). This can be proved by using the following multiplication property on Schatten classes:

\[
S_p(H)S_q(H) \subset S_r(H), \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{2.10}
\]

So, we will factorize \( T_m \) as

\[
T_m = T_m H^\sigma H^{-\sigma}, \quad \sigma > 0, \tag{2.11}
\]

where \( H \) is the harmonic oscillator. Let us note that the symbol of \( A = T_m H^\sigma \) is given by \( a(x, \nu) = m(x, \nu)(2|\nu| + n)^\sigma \). So, from the second assertion, \( A \in S_2(L^2) \)
if and only if
\[ \sum_{\nu \in \mathbb{N}_0} |\nu|^{2\sigma} \int_{\mathbb{R}^n} |m(x,\nu)|^2 \phi_\nu(x)^2 dx \lesssim \sum_{\nu \in \mathbb{N}_0} (2|\nu| + n)^{2\sigma} \int_{\mathbb{R}^n} |m(x,\nu)|^2 \phi_\nu(x)^2 dx < \infty. \]

In order to prove that $T_m \in S_r(L^2)$, in view of the multiplication property
\[ S_2(L^2) S_{\frac{2r}{2-r}}(L^2) \subset S_r(L^2), \tag{2.12} \]
we only need to prove that $H^{-\sigma} \in S_p(L^2)$ with $p = \frac{2r}{2-r}$. The symbol of $H^{-\sigma}$ is given by $a'(\nu) = (2|\nu| + n)^{-\sigma}$. By using the hypothesis $\sigma > n(\frac{1}{r} - \frac{1}{2})$ we have that
\[ \sum_{\nu} |a'(\nu)|^p = \sum_{\nu} (2|\nu| + n)^{-\sigma p} < \infty \]
because $\sigma p = \sigma(\frac{1}{r} - \frac{1}{2})^{-1} > n$. So, we finish the proof. \hfill \Box

2.1. **Trace class pseudo-multipliers of the harmonic oscillator.** In order to determine a relation with the eigenvalues of $T_m$ we recall the following result (see [15]).

**Theorem 2.2.** Let $T : L^p(\mu) \to L^p(\mu)$ be a $r$-nuclear operator as in (1.6). If $\frac{1}{r} = 1 + \frac{1}{p} - \frac{1}{2}$, then,
\[ \text{Tr}(T) := \sum_{n \in \mathbb{N}_0^*} f_n^p = \sum_n \lambda_n(T) \tag{2.13} \]
where $\lambda_n(T), n \in \mathbb{N}$ is the sequence of eigenvalues of $T$ with multiplicities taken into account.

As an immediate consequence of the preceding theorem, if $T_m : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a trace class (1-nuclear) then,
\[ \text{Tr}(T_m) = \int_{\mathbb{R}^n} \sum_{\nu \in \mathbb{N}_0^*} m(x,\nu) \phi_\nu(x)^2 dx = \sum_n \lambda_n(T), \tag{2.14} \]
where $\lambda_n(T), n \in \mathbb{N}$ is the sequence of eigenvalues of $T_m$ with multiplicities taken into account.

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