ABSTRACT

A model of a long optical communication line consisting of alternating segments with anomalous and normal dispersion, whose lengths are picked up randomly from a certain interval, is considered. As the first stage of the analysis, we calculate small changes of parameters of a quasi-Gaussian pulse passing a double-segment cell by means of the variational approximation (VA) and approximate the evolution of the pulse passing many cells by smoothed ODEs with random coefficients, which are solved numerically. Next, we perform systematic direct simulations of the model. Results are presented as dependences of the pulse's mean width, and standard deviation of the width from its mean value, on
the propagation distance. Averaging over 200 different realizations of the random length set reveals slow long-scale dynamics of the pulse, frequently in the form of long-period oscillations of its width. It is thus found that the soliton is most stable in the case of the zero path-average dispersion (PAD), less stable in the case of anomalous PAD, and least stable in the case of normal PAD. The soliton’s stability also strongly depends on its energy, the soliton with small energy being much more robust than its large-energy counterpart.

1 Introduction

The use of dispersion management (DM) for improvement of pulse transmission in long optical-fiber links has attracted a great deal of attention, see, e.g., Refs. [1]-[12] and references therein. Almost all the works on this topic published thus far were dealing with conditions for stationary propagation of a pulse in a link consisting of periodically alternating fixed-length pieces of the fiber with anomalous and normal dispersion. However, really exiting terrestrial optical communication webs are patchwork systems, which include links with different values of the dispersion and, what is especially important, very different lengths. If the distribution of the lengths may be assumed random, upgrading the patchwork systems by means of DM makes it necessary to consider transmission of pulses in long communication lines subject to random DM. Besides the obvious significance for the applications, this issue if also of considerable interest by itself, in the context of soliton propagation in strongly inhomogeneous media (see, e.g., the book [13]). The objective of this work is to develop both semi-analytical variational and fully numerical approaches to the description of pulses’ dynamics in a random-DM system. The varia-
tional approximation (VA), which was first applied to the DM models in [14], has then become a commonly accepted tool for theoretical consideration of DM solitons, see, e.g., Refs. [3, 4, 8].

Most promising results, viz., solitons that are most robust against random variation of the fiber-segment lengths, will be found in this work for the case of zero path-average dispersion (PAD) $\beta_0$. Note that the case of zero or slightly normal ($\beta_0 > 0$) PAD has recently attracted special attention in the context of regular (periodic) DM, after it had been demonstrated that stable propagation of pulses is possible in this case too [4, 5, 8].

There are two different natural models of random DM. In the two-lengths (2L) model, the lengths of the alternating anomalous- and normal-dispersion pieces are selected randomly from a certain interval independently from each other. In the other, one-length (1L) model, equal lengths of the anomalous- and normal-dispersion pieces inside each DM cell are selected randomly from the same interval. In the 2L model, $\beta_0$ is the global average value, while in the 1L model $\beta_0$ is the mean dispersion in each cell. Final results are quite similar for both models, therefore in this work they will be presented for the 1L version.

Evolution of the soliton is a random setting is quantified by a dependence of its width on the propagation distance. In the random-DM links, the soliton’s width demonstrates rapid erratic oscillations. However, averaging over a large number of the random-DM realizations, and taking average deviations from the mean values, we obtain smooth results which can be understood. Roughly, the main conclusions are that robustness of a randomly dispersion-managed soliton crucially depends on the value of PAD and on the soliton’s energy. First, the soliton’s degradation (decay) is fast at $\beta_0 > 0$, slower at $\beta_0 < 0$ (recall this corresponds to the normal and anomalous dispersion, respectively), and much slower at $\beta_0 = 0$. Second, degradation is much slower for solitons with small energy than
for those with large energy. Both these findings can be understood if one notices that, in the exactly solvable linear model (corresponding to the small-energy limit) with random DM, a chirped Gaussian pulse is chaotically vibrating with *no systematic degradation* (spreading out), provided that PAD is exactly zero (see, e.g., Ref. [4]).

A preliminary version of the present work was published in [9]. Recently, another random-DM model was also considered by Abdullaev and Baizakov [8] (see also [15]). However, in that work emphasis was made on a different model, in which the local values of the dispersion, rather than the fiber-segment lengths, were subject to a random distribution (the case of the randomly distributed lengths was only briefly touched within the framework of VA in Ref. [8]). Unlike the results reported in the present work, sharp differences between robust and degrading soliton propagation regimes were not found in the model considered in Ref. [8].

The rest of the paper is organized as follows. In section 2, the random-DM model is formulated. In section 3, we recapitulate basic results of VA for the periodic-DM model, and then develop VA for the random-DM one; the approximation is based on the Gaussian *ansatz* for the pulse. In section 4, results of systematic numerical simulations of the VA-generated equations with randomly varying coefficients are displayed. Section 5 presents results of *direct* numerical simulations of the underlying nonlinear Schrödinger (NLS) equation with random DM, which are, generally, found to be in accord with predictions of VA. Section 6 concludes the paper.
2 The random-dispersion-management model

We start with a model which is a straightforward generalization of the well-known DM schemes based on the NLS equation [1]-[5],

\[ iu_z - \frac{1}{2} \beta(z) u_{rr} + |u|^2 u = 0, \]  

which is written in the standard notation [17]. The dispersion coefficient is taken in the form \( \beta(z) = \beta_0 + \beta_1(z) \), where \( \beta_0 \) is its average value (PAD), and \( \beta_1(z) \) is a variable part with the zero average, which is taken in the following form inside the \( n \)-th DM cell:

\[ \beta_1(z) = \begin{cases} 
\beta_-, & z_n < z < z_n + L_n^{(-)}, \\
\beta_+, & z_n + L_n^{(-)} < z < z_n + L_n^{(-)} + L_n^{(+)} \equiv z_{n+1}.
\end{cases} \]  

Here, \( \beta_- + \beta_0 < 0 \) and \( \beta_+ + \beta_0 > 0 \) are, respectively, the dispersion coefficient in the anomalous- and normal-dispersion fibers and \( L_n^{(\pm)} \) are lengths of the corresponding pieces.

In the case of periodic DM, the lengths \( L_n^{(\pm)} \) are the same in all the cells, while in the case of random DM, they vary stochastically from a cell to a cell. The condition that the average value of \( \beta_1(z) \) is zero implies that \( \beta_- L^{(-)} + \beta_+ L^{(+)} = 0 \), the overbar standing for the averaging. We assume that random values of both \( L^{(-)} \) and \( L^{(+)} \) are distributed uniformly within a certain interval, so that their mean values are equal. Then, the above condition requires that \( |\beta_-| = \beta_+ \), which will be assumed to hold.

The model (1) conserves the total energy \( E \), which we define as

\[ E \equiv \sqrt{2/\pi} \int_{-\infty}^{+\infty} |u(\tau)|^2 d\tau. \]  

Following many earlier works [1]-[10], the model does not include losses and gain, presuming that they are mutually compensated at a smaller scale. The analysis presented below can be extended to include the losses and gain, but this is postponed to another
work, as it is necessary first of all to understand properties of the conservative model.
Other effects which are not included into the present work but may be relevant in certain cases, as it is known that they may help to stabilize the transmission of pulses in the long periodic-DM link, are the third-order dispersion [10] and filtering [11].

We are concerned with the case when DM is not too weak, hence the shape of the pulse is well-known to be close to a Gaussian [1]-[10]. Accordingly, VA may be based on the Gaussian ansatz [3]-[8]. We will here follow a version of the Gaussian-based VA developed (for periodic DM) in detail in [5]. Using the scaling invariance of Eq. (1), the following normalizations are adopted in Ref. [5],

\[ L^(-) + L^+ \equiv 1, \ |\beta_-| L^(-) = \beta_+ L^+ \equiv 1. \]  \hspace{1cm} (4)

In the present work, we apply these normalizations to the mean values of the random lengths. Since our model assumes \( L^(-) = L^+ \), Eq. (4) yields \( L^+ = 1/2 \), and \( |\beta_\pm| = 2 \).

To comply with the former normalization, we choose the interval from which the random lengths \( L^\pm \) are picked up as

\[ 0.1 < L < 0.9. \]  \hspace{1cm} (5)

The minimum length 0.1 is introduced here because, in reality, the length can be neither very large (say, larger than 200 km) nor very small (shorter than 20 km).

3 The variational approximation

3.1 The Gaussian ansatz

The strong-DM regime implies that, locally, the dispersion is much stronger than the nonlinearity, and that \( |\beta_0| \ll |\beta_\mp| \). In the zero-order approximation, completely neglecting
the nonlinearity, one has an exact Gaussian solution to the linearized equation (1) 

\[ u_0 = \frac{\tau_0 \sqrt{P_0}}{\sqrt{\tau_0^2 + 2i\Delta(z)}} \exp \left[ -\frac{\tau^2}{\tau_0^2 + 2i\Delta(z)} + i\phi \right]. \] (6)

Here, \( P_0 \) and \( \tau_0 \) are, respectively, the peak power and minimum width of the pulse, \( -\Delta(z) \equiv -\Delta_0 + \int_{z_n}^{z} \beta(z')dz' \) is the accumulated dispersion defined inside the \( n \)-th DM cell, and \( \Delta_0 \) and \( \phi \) are real constants.

The parameter \( \tau_0^{-2} \) in the expression (6) is proportional, with regard to the normalizations (4), to the well-known DM strength, which (in the case of more general normalizations) is defined as

\[ S = \frac{|\beta_-| L^(-) + \beta_+ L^(+)}{\tau_{\text{FWHM}}^2} \] ,

where \( \tau_{\text{FWHM}} \) is the full width at half maximum (FWHM) of the pulse at the midpoint of the anomalous-dispersion segment, where the pulse is narrowest. Applying the notation adopted here, we replace \( L^{(\pm)} \) in the definition of \( S \) by the above mean values \( 1/2 \), and also insert \( |\beta_\pm| = 2 \), obtaining

\[ S = 1.443/\tau_0^2. \] (7)

The strength \( S \) is the most important characteristic of the DM schemes. It determines their basic properties, which virtually do not depend on other parameters (such as, e.g., \( L_+/L_- \)), provided that \( S \) is fixed. In particular, detailed numerical simulations reveal that stable DM pulses do not exist at \( S > S_{\text{max}} \simeq 10 \), the propagation at zero or slightly normal PAD is possible is \( S > S_{\text{cr}} \approx 4 \) \([4, 5, 8]\), and strongest suppression of the interaction between the pulses is attained at \( S \sim 1.6 \) \([12]\).

The exact solution (3) for the linear model will be used below as an ansatz on which VA for the nonlinear model is based. In most other versions of VA \([3]\), the Gaussian ansatz is also used, but in a different form,

\[ u_0 = a(z) \exp \left[ -\frac{\tau_0^2}{W^2(z)} + ib(z)\tau^2 + i\phi \right]. \] (8)
The complex amplitude $a(z)$ and real width $W(z)$ and chirp $b(z)$ introduced in this expression are related to parameters of the ansatz (3) as follows:

$$a^2(z) = \tau_0^2 P \left[ \frac{1}{\tau_0^2 + 2i\Delta(z)} \right]^{-1}, \quad W(z) = \tau_0^{-1} \sqrt{\tau_0^4 + 4\Delta^2(z)}, \quad b(z) = 2\Delta \left[ \tau_0^4 + 4\Delta^2(z) \right]^{-1}. \quad (9)$$

Using the VA technique, one can derive equations for the nonlinearity-induced evolution of the parameters $P$, $\tau_0$, and $\Delta_0$, that were constant within the framework of the exact Gaussian solution in the absence of the nonlinearity [5]. First, in accord with the energy conservation, we obtain $P_0 \tau_0 \equiv E = \text{const}$ (this coincides with the conserved energy defined by Eq. (3)), and then

$$\frac{d\tau_0}{dz} = \frac{\sqrt{2}E\tau_0 \Delta(z)}{W^3(z)} , \quad \frac{d\Delta_0}{dz} = -\beta_0 + \frac{E [4\Delta^2(z) - \tau_0^4]}{2\sqrt{2}W^3(z)}. \quad (10)$$

Because the average dispersion is small in the DM regime, it is also treated (to derive the second equation in (10)) as a weak perturbation.

The changes of the parameters $\tau_0$ and $\Delta_0$ per one DM cell can be calculated as

$$\delta\tau_0 = \oint \frac{d\tau_0}{dz} dz, \quad \delta\Delta_0 = \oint \frac{d\Delta_0}{dz} dz, \quad (11)$$

where $\oint$ stands for the integration over a full cell, from $z = z_n$ to $z = z_n + L_n^{(-)} + L_n^{(+)}$. In the spirit of the perturbation theory, the changes (11) are assumed small, hence, calculating the integrals in Eq. (11), it is sufficient to take into regard only the rapid variation of $\Delta(z)$, while $\tau_0$ and $\Delta_0$ are treated as constants.

### 3.2 Revisiting the case of the periodic dispersion management

To develop VA for the random-DM system, it is first necessary to recapitulate basic results for the usual periodic case. In that case, to obtain conditions providing for the stationary
transmission of the Gaussian pulse in the long DM line, one equates to zero $\delta \tau_0$ and $\delta \Delta_0$, evaluated as per Eq. (11), with $\tau_0$ and $\Delta_0$ kept constant inside the integrals. This yields

$$\Delta_0 = -\frac{1}{2} \frac{\beta_0}{E} = \frac{\sqrt{2}}{4 \tau_0^3} \left[ \ln \left( \sqrt{1 + \tau_0^{-4} + \tau_0^{-2}} \right) - 2 \left( \tau_0^4 + 1 \right)^{-1/2} \right].$$

(12)

In particular, Eqs. (12) predict that the DM soliton propagates steadily at anomalous average dispersion, $\beta_0 < 0$, provided that $\tau_0^2 > (\tau_0^2)_{cr} \approx 0.301$, at $\beta_0 = 0$ if $\tau_0^2 = (\tau_0^2)_{cr}$, and at normal average dispersion, $\beta_0 > 0$, if $\tau_0^2 < (\tau_0^2)_{cr}$. $(\tau_0^2)_{min} \approx 0.148$

In the case $\beta_0 > 0$, Eq. (12) predicts that the solution exists in a limited interval of the normal-PAD values, viz.,

$$0 \leq \beta_0/E \leq (\beta_0/E)_{max} \approx 0.0127.$$

(13)

Moreover, inside this interval Eq. (12) yields two different values of the minimum width $\tau_0$ for a given $\beta_0/E$ (while in the anomalous-PAD region, $\tau_0$ is a uniquely defined function of $\beta_0/E$) [3]. On the basis of general stability criteria [16], one can immediately conclude that the solution (i.e., DM soliton) corresponding to the larger value of $\tau_0$ is stable, while the one corresponding to the smaller $\tau_0$ is unstable. The border between the stable and unstable solitons corresponds to $\beta_0/E = (\beta_0/E)_{max}$, and it is at $\tau_0^2 = (\tau_0^2)_{min} \approx 0.148$.

Translating $\tau_0^2$ into $S$ according to Eq. (7) (in particular, $(\tau_0^2)_{min}$ gives rise to $S_{max} \approx 9.75$), we eventually conclude that VA based on Eqs. (3) and (10) predicts the following:

- stable DM solitons at anomalous path-average dispersion if $S < S_{cr} \approx 4.79$;
- stable DM solitons at zero path-average dispersion if $S = S_{cr} \approx 4.79$;
- stable DM solitons at normal path-average dispersion if $4.79 < S < S_{max} \approx 9.75$;
• no stable DM soliton if $S > S_{\text{max}} \approx 9.75$.

Below, we will use, instead of $P_0$, the power normalized to that of the fundamental sech soliton having the same width as a given pulse [8], which is $P \equiv 4 \cdot 1.12 P_0$ (the factor 1.12 is the ratio of the FWHM for the sech-shaped and Gaussian pulses). To further illustrate the properties of the solitons in the periodic-DM model, in Fig. 1a we show the normalized power $P$ vs. the map strength $S$ for different values of $\beta_0$, as predicted by Eqs. (12) and (7) (the dependences are shown only in the region $S < 9.75$, where the solitons are expected to be stable). For comparison, Fig. 1b shows the same dependence obtained from direct simulations of the full equation (1). The stars mark in Fig. 1 particular solutions whose response to random variations of the fiber segment lengths will be displayed below. The curves in Fig. 1b corresponding to normal PAD ($\beta_0 > 0$) terminate at points where the corresponding DM soliton becomes unstable.

The comparison of Figs. 1a and 1b shows that VA based on Eqs. (13) and (14) yields quite acceptable results (for periodic DM) just in the range of small energies/powers, for which this approximation was devised. In particular, the VA-predicted $S_{\text{cr}} \approx 4.79$ is different from but nevertheless close to the critical DM strength $S_{\text{cr}} \approx 4$ which the direct simulations give for the small-power case. With the increase of power, the numerically found $S_{\text{cr}}$ grows, as is evident in Fig. 1b. It is also noteworthy that the value $S_{\text{max}} \approx 9.75$, predicted by VA as the stability limit for the DM solitons, is indeed close to what is given by the direct simulations for small powers, see Fig. 1b.

At larger powers, there is a considerable discrepancy between VA and the direct numerical results. However, simulations displayed below show that, in the random-DM system, the soliton is strongly unstable at large energies anyway, so we are really interested only
in the small-energy range, for which the above VA is adequate.

4 The randomly dispersion-managed system

In the general case, when the pulse transmission is not steady (including the random-DM case), its evolution from a cell to a cell can be described in terms of a map, \( \tau_0 \rightarrow \tau_0 + \delta \tau_0, \Delta_0 \rightarrow \Delta_0 + \delta \Delta_0 \). Because the changes are small, many iterations of the map, corresponding to the propagation distance comprising many DM cells, may be approximated by smoothed differential equations, \( d\tau_0/dz = \delta \tau_0/\left( L_n^{-} + L_n^{+}\right) \) and \( d\Delta_0/dz = \delta \Delta_0/\left( L_n^{-} + L_n^{+}\right) \). A straightforward calculation, using Eqs. (11) and (10) and taking into regard the normalization \( |\beta_-| = |\beta_+| = 2 \) adopted above, leads to a final form of the smoothed equations:

\[
\frac{d\tau_0}{dz} = \frac{\sqrt{2}E\tau_0^3}{8\left[L^{-} + L^{+}\right]} \left\{ \frac{1}{\sqrt{\tau_0^4 + 4\Delta_0^2}} + \frac{1}{\sqrt{\tau_0^4 + 4\left[\Delta_0 + 2L^{-} - 2L^{+}\right]^2}} \right\}
\]

\[
\frac{d\Delta_0}{dz} = -\beta_0 + \frac{\sqrt{2}E\tau_0^3}{8\left[L^{-} + L^{+}\right]} \left\{ \frac{2\Delta_0}{\sqrt{\tau_0^4 + 4\Delta_0^2}} + \frac{2\left[\Delta_0 + 2L^{-} - 2L^{+}\right]}{\sqrt{\tau_0^4 + 4\left[\Delta_0 + 2L^{-} - 2L^{+}\right]^2}} \right\}
\]

\[
-\frac{4\left[\Delta_0 + 2L^{-}\right]}{\sqrt{\tau_0^4 + 4\left[\Delta_0 + 2L^{-}\right]^2}} - \frac{1}{2} \ln \left( 2\Delta_0 + \sqrt{\tau_0^4 + 4\Delta_0^2} \right)
\]

\[
-\frac{1}{2} \ln \left( 2\left[\Delta_0 + 2L^{-} - 2L^{+}\right] + \sqrt{\tau_0^4 + 4\left[\Delta_0 + 2L^{-} - 2L^{+}\right]^2} \right)
\]

\[
+ \ln \left( 2\left[\Delta_0 + 2L^{-}\right] + \sqrt{\tau_0^4 + 4\left[\Delta_0 + 2L^{-}\right]^2} \right) \right\}.
\]

To check these equations, one can get back to the case of periodic DM, with \( L^{-} = L^{+} = 1/2 \), as per Eqs. (4). In this case, a fixed point of Eqs. (14) and (15) \( (d\tau_0/dz = d\Delta_0/dz = 0) \) has exactly the same values of \( \Delta_0 \) and \( \tau_0 \) as given by Eqs. (12). In the next section, we will display results of numerical integration of Eqs. (14) and (15) for the
5 Numerical simulations of the variational equations

Equations (14) and (15) with random coefficients were numerically integrated, with initial conditions corresponding to the chirpless pulse, whose parameters were taken as per the fixed point (12) of the allied periodic-DM model. The most essential single characteristic of the pulse propagation at given values of $\beta_0$ and $E$ is the rms cell-average pulse’s width $\overline{W}$, which we define as

$$\overline{W} \equiv L^{-1} \int W(z) dz \quad (16)$$

where the relation (1) and normalizations (4) have been used. It is noteworthy that, in the case of periodic DM, the steady propagation regime corresponding to the fixed point (12) with $\Delta_0 = -1/2$ gives rise to the minimum rms width at $\tau_0^2 = 1/\sqrt{3}$ (i.e., in the anomalous-PAD region, as $1/\sqrt{3} > (\tau_0^2)_{cr} \approx 0.30$).

Simulations of Eqs. (14) and (15) reveal that there are two different dynamical regimes. In the case when the soliton’s energy is sufficiently low, i.e., one is indeed close to the quasi-linear approximation for which the above derivation is relevant, and PAD is anomalous or zero, i.e., $\beta_0 \leq 0$ (especially, if $\beta_0 = 0$), the pulse performs random vibrations but remains, as a matter of fact, fairly stable over long propagation distances. In the case when the energy is higher, as well as when PAD is normal, $\beta_0 > 0$, the pulse demonstrates fast degradation and spreading out.

To present the results in a more systematic form, in Fig. 2 we display the results for a typical case of the effectively stable propagation with small normalized power of
the soliton, $P = 0.16$ (in this case, $E = 0.5$), and anomalous PAD, $\beta_0 = +0.2$ (which is 10% of the local dispersion). The simulations of Eqs. (14) and (15) were performed 200 times, with the same initial conditions but different realizations of the random length set. Using numerical data for the 200 runs, we computed the evolution of $\langle \tau_0 \rangle$ and the averaged dependence $\langle W(z) \rangle$ ($\langle \ldots \rangle$ stands for the averaging over 200 runs), along with the corresponding standard deviations. Both dependences are displayed in Fig. 2.

It is clearly seen in Fig. 2 and in a number of similar plots not displayed here that, on top of the random vibrations, which are directly induced by random DM but are eliminated by averaging over 200 realizations, the soliton demonstrates slow (long-scale) dynamics. Systematic degradation of the oscillations and of the soliton itself takes place too, the degradation being slower for lower powers. For the particular case shown in Fig. 2, the pulses remain certainly usable over the propagation distance $\approx 100$ DM cells. Further simulations of Eqs. (14) and (15) for propagation distances essentially larger than 1000 DM cells (not shown here; see an example in Ref. [9]) show that, in fact, the sluggish spreading out of the soliton suddenly ends up with its blowup (complete decay into radiation). It is interesting to note that VA has predicted the same scenario for the evolution of the soliton in the early work [14] for a model with regular sinusoidal modulation of the local dispersion: a long span of chaotic but nevertheless quasi-stable vibrations is suddenly changed by rapid irreversible decay.

Fig. 3 shows a drastic difference in the soliton’s evolution which takes place if the power is increased to $P = 0.44$, (the corresponding energy is $E = 2.5$), without changing parameters of the random-DM fiber link. In this case, which is typical for high powers, rapid decay of the soliton without long-scale vibrations is observed. Note that, since $\langle \tau_0 \rangle$ grows quite slowly, and the spectral width of the pulse is $\sim \tau_0^{-1}$, most of the pulse’s
broadening in the temporal domain observed in Fig 3 is due to chirping of the pulse, rather than directly to a change in its spectral width.

Taking $\beta_0 = 0$, instead of anomalous PAD, \textit{radically improves} the situation, as is seen in Fig. 4. For the high-power case with $P = 0.44$ ($E = 3.6$), which gave rise to rapid degradation of the soliton in the presence of the anomalous PAD, the pulse now survives over much longer distances, see Fig 4a. For lower values of the power, we observer still more robust propagation with zero PAD. Fig 4b shows the case $P = 0.12$ ($E = 0.1$), where the pulse shows slow dynamics induced by the random length variation in a strongly dispersion-managed link ($S \approx 4.8$) with very little degradation over 1000 DM cells.

The PAD’s value $\beta_0 = 0$ turns out to be a point of a \textit{sharp optimum}: taking any tangible small normal value of PAD, $\beta_0 > 0$, we always observed rapid decay (without long-scale oscillations) of the soliton at virtually all the values of the energy, see for instance Fig 5, which displays the case with $P = 0.27$ ($E = 2.5$) and $\beta_0 = 0.02$ (1% of the local dispersion). In this case the soliton character of the pulse is lost quickly. The broadening of the pulse in the spectral domain is enhanced by its spectral broadening, manifested in the decrease of $\tau_0$. The broadening rate, however, is slower than in Fig 3a due to the lower magnitude of PAD.

6 Direct simulations

It is necessary to compare the predictions of VA with direct simulations of Eq. [1]. The pattern of the simulations was the same as in the previous section, i.e., simulations were performed for 200 different realizations of the random length set, in order to evaluate averaged evolution of the pulse’s parameters. First, the shape of the DM soliton in the
allied periodic-DM link was numerically determined, to be used as the initial configuration. The randomness was the same as above, i.e., the lengths were uniformly distributed between 0.2 and 1.8 of the average length. The general trends predicted by the VA are confirmed by the direct simulations: lower energy and anomalous or, especially, zero PAD enhance the stability, see details below.

In direct simulations, the width must be defined with special care. The usual FWHM definition may be misleading in the case of random DM, as the soliton can sometimes split (see below); besides that, this definition ignores the presence of a radiation component. Therefore, we adopted an integral definition, with which the width is a size of the temporal region on both sides of the soliton’s center that contains 76% of the net energy. For a pulse with the Gaussian shape the width defined this way coincides with FWHM. We will assume that optimum dispersion compensation can be applied at the receiver, i.e., any linear chirp is removed from the pulse by a dispersion compensating element installed before the receiver. Note that in the framework of the above VA, \( \tau_0 \) represented the pulse’s width at the chirp-free points.

Starting with anomalous PAD, in Fig 6 we show results for the same case for which results obtained by means of VA were displayed in Fig 2, i.e., the magnitude of the PAD is 10% of the local dispersion and the power is low, \( P = 0.18 \). This time, we show (by bold curves) not only the averaged evolution, but also (by thin curves) the set of the particular evolutions corresponding to different realizations of the random length set. Comparing Figs. 6 and 2, we conclude that the averaged results are qualitatively similar. In particular, internal vibrations are present in the first 100 cells in both cases, and the systematic temporal-domain broadening of the pulse takes place in both cases, although the broadening rate is overestimated by VA (hence, the full numerical results,
predicting slower degradation of the soliton, seem considerably better for the applications than the less accurate results generated by VA). The latter discrepancy is, most plausibly, accounted for by radiative losses that are ignored by the VA.

Since we allow for dispersion compensation at the receiving end, this can change the link’s PAD. The effective PAD, corrected with regard to the optimal receiving-end dispersion compensator, may therefore be regarded as a function of the propagation distance. The inset to Fig. 6 shows that the thus redefined effective PAD stabilizes at the value 0.2, which turns out to be the same as for the unperturbed DM soliton. This means that no post-transmission dispersion compensation is needed in order to receive a chirp-free pulse; the dispersion of the link is balanced by its nonlinearity. For higher powers, however, this is no longer the case, see an example for $P = 0.47$ in Fig. 7. The pulse broadens and decays fast and the optimum end dispersion drops rather than stabilizing, i.e., the pulse acquires a chirp during the propagation and extra post-transmission dispersion compensation will be necessary. Compared to the variational results, displayed for the same case in Fig 3, we conclude that the direct simulations of Eq. (1) reveal a rapid decay of the pulse shape. Moreover, detailed examination of the pulse shapes generated by the direct simulations shows they often develop a multi-peak structure, so that it can sometimes be hard to identify the pulse proper. This is why Fig. 7 shows the evolution for the first 50 DM cells only.

It is necessary to mention that essentially longer direct simulations of Eq. (1) (not shown here) demonstrate that the gradual decay of the soliton is, at a final stage, suddenly interrupted by splitting of the residual pulse into two smaller ones. Recall that, although VA by definition cannot predict the eventual sudden splitting of the pulse, it does predict, as it was mentioned above, something similar, viz., sudden fast decay of the soliton after
a long stage of chaotic vibrations. It is noteworthy that gradual evolution of the soliton in the model with sinusoidal periodic modulation of the dispersion also ends up with sudden splitting, which approximately corresponds to the sudden decay predicted by VA in the same model [18, 19].

To further illustrate the internal dynamics of the pulse, we picked a case where PAD is anomalous and equal 1% of the local dispersion, and the power is \( P = 0.1 \). Fig. 8 shows the average trajectory on the dynamical plane, where the coordinates are the width and chirp of the soliton at the midpoint of the anomalous-fiber section. The chirp is in this case defined as the second derivative of the phase of the pulse taken at its center. The trajectory demonstrates a quasi-circular motion around a center which is drifting to the right, towards a broader pulse. The trajectory clearly demonstrates the nonlinear character of the system, cf. the inset to Fig. 8 which shows the trajectory if the nonlinearity are dropped. Note that a stationary periodic-DM soliton would be represented by a single point in this plane, and a perturbed (nonstationary) soliton in the periodic-DM system would trace a circle around a fixed center shifted to a broader pulse, as compared to the stationary DM soliton [20]. The random length variation is a permanently acting perturbation which generates the persistent drift of the center in Fig. 8.

Numerical simulations at zero PAD also agree well with the results generated by VA: zero PAD provides the slowest pulse degradation in the random DM link. Fig. 9 shows the evolution of the pulse width when \( \beta_0 = 0 \) and the power is \( P = 0.15 \). Almost no broadening (< 5%) can be observed, on the average, after having passed 200 DM cells. The broadening does increase with the power, but remains, as predicted by VA, much less than in the case of nonzero PAD. For instance, taking \( P = 0.45 \) results only in a
doubling of the width on the propagation distance equal 200 DM cells. Comparison to
the variational results for the same case displayed in Fig. 4 again shows that full numerical
results are qualitatively similar but somewhat better, predicting slower degradation of the
pulse. One of the reasons for this may be that the radiation remains trapped within the
pulse.

Lastly, for the case of normal PAD the direct simulations confirm the prediction of VA
according to which the solitons are least stable in this case. This is partly explained by the
fact that there is, effectively, a minimum (threshold) power necessary for the existence
of stable solitons at normal PAD even with periodic DM, see Fig 1b, and the larger
power always stimulates the degradation of the pulse. Detailed examination of the direct
numerical results for the normal-PAD case also shows that the pulse’s spreading out is
boosted by its spectral broadening during the initial stage of propagation. As a result,
even at very weak normal PAD, $\beta_0 = 0.02$ (1% of the magnitude of the local dispersion),
the pulse stays intact for no more than 10 DM cells.

**Conclusion**

In this work, we have put forward a model of a long optical communication line subject to
random dispersion management (DM). The line consists of alternating fiber pieces with
anomalous and normal dispersion, whose lengths are picked up randomly from a certain
interval, while the absolute values of the dispersion coefficients in both pieces are always
equal. By means of the variational approximation, we calculated small changes of param-
eters of the propagating quasi-Gaussian pulse per one DM cell, and then approximated
the evolution of the pulse passing many cells by smoothed ordinary differential equations
with randomly varying coefficients. The equations were solved numerically, and results were presented as the dependences of the pulse’s mean width, and average deviation of the width from the mean value, on the propagation distance. Averaging the results over 200 different realizations of the random length set removes rapid oscillations of the width and reveals slow long-scale dynamics of the pulse, frequently in the form of long-period oscillations. The second part of the work is based on direct numerical simulation of the same model. Comparing the results, we have concluded that essential features of the soliton dynamics are the same in the variational approximation and in direct numerical simulations: the propagating soliton is most stable in the case of zero path-average dispersion (PAD), less stable in the case of anomalous PAD, and least stable in the case of normal PAD. The soliton’s stability also strongly depends on its energy, so that the soliton with small energy is much more robust than its large-energy counterpart.

7 Acknowledgments

We appreciate valuable discussions with M. Tur (Tel Aviv University), M. Böhm and F. Mitschke (University of Rostock), and F. Lederer (University of Jena).
References

[1] M. Suzuki, I.Morita I, N. Edagawa , S. Yamamoto, H. Taga, and S. Akiba, “Reduction of Gordon-Haus timing jitter by periodic dispersion compensation in soliton transmission”, Electron. Lett. 31, 2027-2029 (1995); M. Nakazawa and H. Kubota, “Optical soliton communication in a positively and negatively dispersion-allocated optical-fiber transmission-line”, Electron. Lett. 31, 216-217 (1995); F.M. Knox, N.J. Doran, K.J. Blow and I. Bennion, “Enhanced power solitons in optical fibres with periodic dispersion management”, Electron. Lett. 32, 54-55 (1996); I. Gabitov and S.K. Turitsyn, “Averaged pulse dynamics in a cascaded transmission system with passive dispersion compensation”, Opt. Lett. 21, 327-329 (1996); T. Okamawari, Y. Ueda, A. Maruta, Y. Kodama, and A. Hasegawa, “Interaction between guiding centre solitons in a periodically dispersion compensated optical transmission line”, Electron. Lett. 33, 1063-1065 (1997); G.M. Carter, J.M. Jacob, C.R. Menyuk, E.A. Golovchenko and A.N. Pilipetskii, “Timing-jitter reduction for a dispersion-managed soliton system: Experimental evidence”, Opt. Lett. 22, 513-515 (1997); M.K. Chin and X.Y. Tang, “Quasi-stable soliton transmission in dispersion managed fiber links with lumped amplifiers”, IEEE Photon. Technol. Lett. 9, 538-540 (1997).

[2] New Trends in Optical Soliton Transmission Systems. A. Hasegawa, editor (Kluwer Academic Publishers: Dordrecht/Boston/London, 1998).

[3] A. Berntson, D. Anderson, M. Lisak, M.L. Quiroga-Teixeiro and M. Karlsson, “Self-phase modulation in dispersion compensated optical fibre transmission systems”, Opt. Comm. 130, 153-162 (1996); I. Gabitov, E.G. Shapiro and S.K. Turitsyn, “Optical pulse dynamics in fiber links with dispersion compensation”, Opt. Comm. 134,
317-329 (1997); B.A. Malomed, “Pulse propagation in a nonlinear optical fiber with periodically modulated dispersion: Variational approach”, Opt. Comm. 136, 313-319 (1997); T.-S. Yang and W.L. Kath, “Analysis of enhanced-power solitons in dispersion-managed optical fibers”, Opt. Lett. 22, 985-987 (1997); M. Matsumoto, “Theory of stretched-pulse transmission dispersion-managed fibers”, Opt. Lett. 22, 1238-1240 (1997); S.K. Turitsyn, I. Gabitov, E.W. Laedke, V.K. Mezentsev, S.L. Musher, E.G. Shapiro, T. Schäfer, and K.H. Spatschek, “Variational approach to optical pulse propagation in dispersion compensated transmission systems”, Opt. Comm. 151, 117-135 (1998).

[4] J.H.B. Nijhof, N.J. Doran, W. Forysiak and F.M. Knox, “Stable soliton-like propagation in dispersion managed systems with net anomalous, zero and normal dispersion”, Electron. Lett. 33, 1726-1727 (1997); S.K. Turitsyn and E.G. Shapiro, “Dispersion-managed solitons in optical amplifier transmission systems with zero average dispersion”, Opt. Lett. 23, 682-684 (1998); J.N. Kutz and S.G. Evangelides, “Dispersion-managed breathers with average normal dispersion”, Opt. Lett. 23, 685-687 (1998).

[5] T.I. Lakoba, J. Yang, D.K. Kaup, and B.A. Malomed, “Conditions for stationary pulse propagation in the strong dispersion management regime”, Opt. Comm. 149, 366 (1998).

[6] F.Kh. Abdullaev and B.B. Baizakov, Disintegration of a soliton in a dispersion-managed optical communication line with random parameters, Opt. Lett. 25, 93-95 (2000).
[7] B.A. Malomed, F. Matera, and M. Settembre, *Reduction of the jitter for return-to-zero signal*, Opt. Comm. **143**, 193-198 (1997).

[8] A. Berntson, N.J. Doran, W. Forysiak, and J.H.B. Nijhof, “Power dependence of dispersion-managed solitons for anomalous, zero, and normal path-average dispersion”, Opt. Lett. **23**, 900 (1998).

[9] B.A. Malomed and A. Berntson, “Propagation of a pulse in a fibre link with random dispersion management”, in *Nonlinear Guided Waves and Their Applications*, OSA Technical Digest, p. 289-291. (Optical Society of America: Washington, 1999).

[10] D. Frantzeskakis, K. Hizanidis, B.A. Malomed, and H.E. Nistazakis, *Stabilizing soliton transmission by the third-order dispersion in long links with dispersion management*, Pure Appl. Opt. **7**, L57-L62 (1998); T.I. Lakoba and G.P. Agrawal, “Effects of third-order dispersion on dispersion-managed solitons”, J. Opt. Soc. Am. B **16**, 1331-1343 (1999).

[11] B.A. Malomed, “Jitter suppression by guiding filters in combination with dispersion management”, Opt. Lett. **23**, 1250 (1998); A. Berntson and B.A. Malomed, “Dispersion-management with filtering”, Opt. Lett. **24**, 507 (1999); L.F. Mollenauer, P.V. Mamyshev, and J.P. Gordon, “Effect of guiding filters on the behavior of dispersion managed solitons”, Opt. Lett. **24**, 220-222 (1999).

[12] S. Kumar, M. Wald, and F. Lederer, “Soliton interaction in strongly dispersion-managed optical fibers”, Opt. Lett. **23**, 1019 (1998); M. Wald, B. Malomed, and F. Lederer, *Interaction of moderately dispersion managed solitons*, Opt. Comm. **72**, 31 (1999).
[13] V.V. Konotop and L. Vázquez. *Nonlinear Random Waves* (World Scientific: Singapore, 1994).

[14] B.A. Malomed, D.F. Parker and N.F. Smyth, *Resonant shape oscillations and decay of a soliton in periodically inhomogeneous nonlinear optical fiber*, Phys. Rev. E **48**, 1418-1425 (1993).

[15] F.Kh. Abdullaev, J. Bronski, and G.C. Papanicolaou, “*Soliton perturbations and the random Kepler problem*”, Physica D **135**, 369 (1999).

[16] G. Iooss and D.D. Joseph. *Elementary Stability and Bifurcation Theory* (Springer: New York, 1990).

[17] G.P. Agrawal. *Nonlinear Fiber Optics* (Academic Press: San Diego, 1995).

[18] R. Grimshaw, J. He and B.A. Malomed, *Decay of a soliton in a periodically modulated nonlinear waveguide*, Physica Scripta **53**, 385-393 (1996).

[19] F.Kh. Abdullaev and J.C. Caputo, *Validation of the variational approach for chirped pulses in fibers with periodic dispersion*, Phys. Rev. E **58**, 6637-6648 (1998).

[20] D. Anderson, A. Berntson, M. Lisak..., in *Nonlinear Guided Waves and Their Applications*, OSA Technical Digest, p. .... (Optical Society of America: Washington, 1999). [ANDERS: PLEASE COMPLETE THE REFERENCE - I DO NOT HAVE DIJON PROCEEDINGS AT HAND]
FIGURE CAPTIONS

Fig. 1. The normalized power vs. the map strength for stationary solitons in the periodic DM system: (a) the analytical result, Eq. (12), produced by the variational approximation; (b) direct numerical solution of the NLS equation (1). Each line corresponds to a constant value of PAD: from left to right, $\beta_0 = -0.2, -0.02, 0$ and $0.02$. The stars mark particular cases for which response to random length variations are further displayed in Figs. 2 through 9.

Fig. 2. The cell-average pulse width (top) and minimum-width parameter $\tau_0$ (bottom) vs. the propagation distance, generated by numerical integration of the variational equations (14) and (15) for the power $P = 0.16$ and $\beta_0 = -0.2$ (anomalous PAD). The mean values (solid curve) and standard deviations from them (dashed curves) are produced by averaging over 200 different realizations of the random length set.

Fig. 3. The same as in Fig. 2 but for higher power, $P = 0.44$.

Fig. 4. Evolution of the cell-average pulse width for zero PAD propagating over 1000 DM cells, in the cases of high power $P = 0.47$ (top) and low power $P = 0.1$ (bottom). The mean values (solid curve) and standard deviations from them (dashed curves) are produced by averaging over 200 different realizations of the random length set.

Fig. 5. The same as Fig. 2 but with normal PAD, $\beta_0 = 0.02$, and $P = 0.27$.

Fig. 6. The pulse width vs. the propagation distance for anomalous PAD, $\beta_0 = -0.2$, and low power, $P = 0.18$, generated by direct simulations of Eq. (1). Shown are both the mean values (solid curve) and standard deviations from them (dashed curves) as produced by averaging over 200 different realizations of the random length set, and grey curves corresponding to particular realizations of the random set. The inset shows the effective
PAD corrected with regard to the optimal dispersion compensation at the receiving edge (average of 200 simulations).

Fig. 7. The same as in Fig 6. but for high power, \( P = 0.47 \).

Fig. 8. Results of direct simulations of the NLS equation (1) with anomalous PAD, \( \beta_0 = -0.02 \). The plot shows the pulse dynamics in the dynamical plane whose coordinates are the chirp at the center of the pulse and its width, evaluated at the midpoint of the each anomalous-fiber section. The inset shows the same but in the absence of the nonlinear term in Eq. (1).

Fig. 9. The same as in Fig. 6 for the case of zero PAD and \( P = 0.15 \).
Map strength, $S$

Normalized power

$\beta_0 = -0.2$

$\beta_0 = -0.02$

$\beta_0 = 0$

$\beta_0 = 0.02$
\[ \beta_0 = -0.02 \]
\[ \beta_0 = -0.2 \]
\[ \beta_0 = 0.02 \]
\[ \beta_0 = 0 \]
\[ \langle W(z) \rangle / \langle W(0) \rangle \]
Distance of propagation, unit cells

$\langle W(z) \rangle / \langle W(0) \rangle$

$P=0.47$

$P=0.1$
\[ \langle W(z) \rangle / \langle W(0) \rangle \]

Distance of propagation, unit cells
Distance of propagation, unit cells

Pulse width

Optimal $\beta_0$

0.19
0.2
0.21

0
50
100
150
200

0
20
40
60
80
100
120
140
160
180
200

0.8
0.9
1
1.1
1.2
1.3
1.4
1.5
1.6
Distance of propagation, unit cells

Pulse width