QUANTIZATION OF (VOLUME-PRESERVING) ACTIONS ON $\mathbb{R}^d$

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Abstract. We associate a space of (formal) representations on $C^\infty(\mathbb{R}^d)[[\hbar]]$ (which we call quantizations) with an action of a group on $\mathbb{R}^d$ by smooth diffeomorphisms. If the action is further volume preserving, these quantizations can be realized as unitary representations on $L^2(\mathbb{R}^d)$ by bounded $\hbar$-dependent Fourier integral operators, the formal case corresponding to the asymptotics in the limit $\hbar \to 0$. We construct DGA’s controlling these quantizations and prove existence and rigidity results for them.

Contents

1. Introduction 1
Acknowledgments 3
2. Preliminaries 3
2.1. Fourier integral operators 4
2.2. Pseudodifferential operators 5
2.3. A class of bounded FIOs 5
2.4. Asymptotic expansions and formal operators 7
3. Quantization of $G$-actions 8
3.1. Unitary $G$-systems 9
3.2. Formal $G$-systems 11
3.3. Unitary $G$-systems independent of $\xi$ 12
4. DGAs of $G$-amplitudes 14
4.1. MC elements in $\mathcal{A}_\varphi$ and $G$-systems 14
4.2. MC elements in $\mathcal{P}_{\varphi}$ and formal $G$-systems 16
5. Existence and rigidity Theorem 17
5.1. Trivial action 19
References 20

1. INTRODUCTION

The question of quantizing a smooth action $\varphi$ of a Lie group $G$ on a manifold $M$ has received different (although related) answers depending on the particular structures at hand on the manifold, the type of Lie groups acting, the type of actions, and the quantization theory used.
When the manifold is symplectic, and the action is Hamiltonian and admits a momentum map, both geometric quantization theory (see [11, 14] for instance) and deformation quantization theory (see [4, 7, 6, 23, 20] for instance) have their own notion of quantization. On the other hand, Rieffel in [19], using ideas of deformation quantization, introduced a notion of action quantization that supposes no symplectic structure on the manifold to begin with if the group acting is \(\mathbb{R}^d\). This program has been extended to various other groups and cases (see [7, 8, 22]).

In this paper, we propose a quantization scheme for a general action \(\varphi\) of a group \(G\) (not necessarily a Lie group) on \(\mathbb{R}^d\) by smooth diffeomorphisms. More precisely, we associate to such an action a space \(\text{Rep}_\varphi(G)\) of representations (which we call quantizations) by certain formal operators on the space \(C^\infty(\mathbb{R}^d)[[\hbar]]\) of formal power series in \(\hbar\) with coefficients in the smooth functions on \(\mathbb{R}^d\). In particular, the trivial quantization, obtained by the pullback of functions, \(T_g\psi(x) = \psi(\varphi^{-1}_g(x))\) with \(g \in G\), is always in \(\text{Rep}_\varphi(G)\), and the other representations in \(\text{Rep}_\varphi(G)\) can be seen as “deformations” of this trivial quantization.

The main result of this paper (Theorem 26) gives cohomological obstructions to the existence of such “deformations” as well as information with regards to their rigidity (i.e. when all the quantizations in \(\text{Rep}_\varphi(G)\) are equivalent to the trivial one). The main ingredient to prove these existence and rigidity results is a Differential Graded Algebra (DGA) whose Maurer-Cartan elements are in one-to-one correspondence with the quantizations of the action.

When the action \(\varphi\) is further volume preserving and bounded (which means that \(|\varphi'_g(x)| = 1\) for all \(g \in G\) and \(x \in \mathbb{R}^d\) with the additional condition that \(\varphi_g\) and all of its derivatives are bounded for all \(g \in G\)), \(\text{Rep}_\varphi(G)\) can be realized as a space of unitary representations on \(L^2(\mathbb{R}^d)\) by certain bounded Fourier Integral Operators, or FIOs for short (see [9, 10, 12, 15] for general references), which depend on a parameter \(\hbar\). In this non-formal setting, there is also a DGA controlling quantization.

Actually, the formal quantizations associated with an action by smooth diffeomorphisms are constructed by taking the asymptotic expansion in the limit \(\hbar \to 0\) of the FIOs used in the volume preserving case and forgetting that these expansions come from honest bounded operators. What results is a set of formal operators of infinite order, which may not be “resummable” if the action we start with is not bounded.

We also explain how geometric quantization (Example 9) and deformation quantization (Section 3.2) are related to our quantization scheme for actions.

This paper is organized as follows:

In Section 2, we introduce the class of \(\hbar\)-dependent FIOs we use to quantize volume-preserving actions. These operators are of the form

\[
\text{Op}(a, \varphi)\psi(x) = \int \psi(\pi) a(x, \xi) e^{i \frac{\hbar}{2} \xi \cdot \varphi^{-1}(x) - \pi} d\xi d\pi / (2\pi \hbar)^d,
\]
where \( \varphi \) is a (bounded) diffeomorphism of \( \mathbb{R}^d \). We give results on the continuity of these operators as well as their asymptotics in the limit \( h \to 0 \), which we interpret as formal operators of infinite order.

In Section 3, we introduce the space \( \text{Rep}_\varphi(G) \) of quantizations associated with an action together with their corresponding \( G \)-systems. The starting point is the observation that the trivial quantization can be rewritten in terms of the FIOs of the previous section as follows:

\[
T_g \psi = \text{Op}(1, \varphi_g)\psi.
\]

When the action is bounded, a \( G \)-system is a system \( \{a_g(x, \xi)\}_{g \in G} \) of amplitudes such that the operators

\[
T^a_g \psi = \text{Op}(a_g, \varphi_g)\psi
\]

form a representation of \( G \) on \( L^2(\mathbb{R}^d) \) by bounded operators. We explain that the asymptotic expansion of these quantizations yields a notion of formal \( G \)-systems and formal quantizations that can be used when the action is no longer bounded. When the action is further volume preserving, we can require the \( G \)-system to be such that the corresponding representations are unitary. There are a number of examples of this in the literature, but, mostly, when the amplitudes of the \( G \)-system do not depend on \( \xi \).

Because of this, we conclude this section by a study of these special \( G \)-systems, yielding to Theorem 14, which is an analog of our main result for formal \( G \)-systems (Theorem 26) in this special case.

In Section 4, we construct two DGAs controlling, respectively, \( G \)-systems and their formal versions. We show that Maurer-Cartan elements are in one-to-one correspondence with \( G \)-systems (both in the formal and non-formal case) and that gauge equivalent Maurer-Cartan elements give equivalent quantizations.

In Section 5, we state and prove the main theorem of this paper (Theorem 26), which gives cohomological conditions with regards to the existence and rigidity of formal \( G \)-systems. We spell out this theorem in the case the action we start with is trivial, obtaining results (Theorem 27) very close to those of Pinzcon [18] on deformations of representations.

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2. Preliminaries

In this section, we review a class of \( h \)-dependent Fourier integral operators that we will use in Section 3 for action quantization purposes. We discuss the continuity of these operators as well as the closeness of their composition. We also give the asymptotics of these operators in the limit \( h \to 0 \), which we will use later on to define a notion of "formal
quantization" of actions. Along the way, we review some facts about pseudo-differential operators.

Throughout this paper, we will consider $\mathbb{R}^d$ with its canonical coordinates $x = (x_1, \ldots, x_d)$, and we will identify its cotangent bundle $T^*\mathbb{R}^d$ with $\mathbb{R}^{2d} = \mathbb{R}^d \times (\mathbb{R}^d)^*$, where $(\mathbb{R}^d)^*$ is the dual to $\mathbb{R}^d$ with dual coordinates $\xi = (\xi_1, \ldots, \xi_d)$. The paring between $\mathbb{R}^d$ and $(\mathbb{R}^d)^*$, will be denoted by $\langle \cdot, \cdot \rangle$ so that $\langle x, \xi \rangle = \sum_{i=1}^{d} x_i \xi_i$. Also, $\frac{d\xi d\bar{x}}{(2\pi\hbar)^d}$ will stand for the Lebesgue measure on $T^*\mathbb{R}^d$. We will also make use of the multi-index notation: For $\alpha \in \mathbb{N}^d$, we define

$$|\alpha| = \alpha_1 + \cdots + \alpha_d, \quad y^\alpha = y^{\alpha_1} \cdots y^{\alpha_d} \text{ for } y \in \mathbb{R}^d,$$

$$\partial^\alpha_x = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_d}}, \quad \partial^\alpha_\xi = \frac{\partial^{|\alpha|}}{\partial \xi^{\alpha_1} \cdots \partial \xi^{\alpha_d}},$$

$$D^\alpha_x = \frac{1}{i|\alpha|} \partial^\alpha_x, \quad D^\alpha_\xi = \frac{1}{i|\alpha|} \partial^\alpha_\xi.$$ 

2.1. Fourier integral operators. A Fourier Integral Operator (or FIO) on $\mathbb{R}^d$ is an integral operator, denoted by $\text{Op}(a, S)$, of the form

$$\text{Op}(a, S)\psi(x) = \int \psi(\overline{x}) \bar{a}(x, \xi) e^{iS(x, \xi)} \frac{d\xi d\overline{x}}{(2\pi\hbar)^d}$$

from the space $C^\infty_0(\mathbb{R}^d)$ of compactly supported smooth functions on $\mathbb{R}^d$ to the space $\mathcal{D}'(\mathbb{R}^d)$ of distribution on $\mathbb{R}^d$, where

- $\hbar$ is a fixed real number in the interval $[0,1]$ (later on, we will be interested in taking the limit $\hbar \to 0$ and in considering $\hbar$ as a formal parameter in the resulting asymptotic expansion),
- $a$ is a smooth function on $\mathbb{R}^d \times (\mathbb{R}^d)^*$ called the amplitude or the (total) symbol of the Fourier integral operator,
- $S$ is a smooth function on $(\mathbb{R}^d)^* \times \mathbb{R}^d \times \mathbb{R}^d$ called the phase of the operator. (More generally, one can define the phase on $\Lambda \times \mathbb{R}^d \times \mathbb{R}^d$, where $\Lambda$ is a more general space of parameters than $(\mathbb{R}^d)^*$; see [11] for a presentation of the full theory.)

A general problem is to find suitable conditions on both the amplitudes and the phases so as to obtain a class of FIOs that enjoys the following nice properties:

- the operator composition is closed when restricted to this class of FIOs (which is in general not the case)
- the operators can be extended to continuous operators on the space $L^2(\mathbb{R}^d)$ of square integrable functions on $\mathbb{R}^d$

We now present two classes of FIOs that have these good properties.
2.2. Pseudodifferential operators. A pseudodifferential operator is a Fourier integral operator with phase $S(\xi, \vec{x}, x) = \langle \xi, x - \vec{x} \rangle$. In other words, it is an integral operator of the form

$$\text{(2.2)} \quad (\text{Op}(a)\psi)(x) = \int \psi(\vec{\tau})a(x, \vec{\xi})e^{i\vec{\xi} \cdot (x - \vec{\tau})} \frac{d\vec{\xi} d\vec{\tau}}{(2\pi i h)^d}.$$ 

Following [15, p. 12], we define $S_n(1)$ to be the set of bounded symbols (or amplitudes) on $\mathbb{R}^n$, that is, the set of families of smooth functions on $\mathbb{R}^n$ parametrized by some $h \in (0, h_0]$ that are uniformly bounded together with all their derivatives.

Unless necessary, we will not write explicitly the dependence on $h$ (i.e. we will write $a(z)$ instead of $a(z; h)$ for symbols in $S_n(1)$, where $z \in \mathbb{R}^n$).

We will make use of the following result, which is a weaker version of [15, Thm. 2.8.1, p. 43]:

**Theorem 1.** If $a \in S_{2d}(1)$, then $\text{Op}(a)$ is a continuous operator on $L^2(\mathbb{R}^d)$.

The class of pseudodifferential operators with bounded symbols is closed under composition: Namely, we have that

$$\text{Op}(a) \circ \text{Op}(b) = \text{Op}(a \ast b),$$

where

$$\text{(2.3)} \quad (a \ast b)(x, \xi) = \int a(x, \vec{\xi})b(x, \xi)e^{i\vec{\xi} \cdot (x - \vec{\xi})} \frac{d\vec{\xi} d\vec{\xi}}{(2\pi i h)^d},$$

is the **Standard product** between (bounded) symbols (see [10] for instance).

2.3. A class of bounded FIOs. Let $\text{Diff}(\mathbb{R}^d)$ be the group of diffeomorphisms of $\mathbb{R}^d$. We will now consider Fourier integral operators $\text{Op}(a, S)$ for which the phase is of the form

$$S(\xi, \vec{x}, x) = \langle \xi, \varphi^{-1}(x) - \vec{x} \rangle,$$

where $\varphi$ is a diffeomorphism on $\mathbb{R}^d$ of a special type. More precisely, we focus on the cases when $\varphi$ lies in the following subgroups of the diffeomorphisms on $\mathbb{R}^d$:

**Definition 2.** We define

1. the subgroup of **bounded diffeomorphisms** $\text{Diff}_b(\mathbb{R}^d)$ to be the diffeomorphisms of $\mathbb{R}^d$ that have all of their derivatives bounded, i.e. $\sup_{x \in \mathbb{R}^d} |\partial^\beta \varphi(x)| < \infty$ for all multi-indices $\beta \in \mathbb{N}^d \setminus (0, \ldots, 0)$;

2. the subgroup of **volume preserving diffeomorphisms** $\text{Diff}_{b,v}(\mathbb{R}^d)$ to be the subgroup of $\text{Diff}_b(\mathbb{R}^d)$ such that $|\varphi'(x)| = 1$.

**Proposition 3.** Given $a \in S_{2d}(1)$ and $\varphi \in \text{Diff}_b(\mathbb{R}^d)$, the corresponding FIO

$$\text{(2.4)} \quad \text{Op}(a, \varphi)\psi(x) = \int \psi(\vec{\tau})a(x, \vec{\xi})e^{i\vec{\xi} \cdot \varphi^{-1}(x) - \vec{\tau}} \frac{d\vec{\xi} d\vec{\tau}}{(2\pi i h)^d},$$
is a continuous linear operator on $L^2(\mathbb{R}^d)$. Moreover, if $\varphi \in \text{Diff}_{b,v}(\mathbb{R}^d)$ and the amplitude satisfies the additional condition

\begin{equation}
(2.5) \quad \frac{1}{(2\pi \hbar)^{d/2}} \int a^*(\varphi(x), \xi) a(\varphi(x), \tilde{\xi}) e^{i\langle x, \xi - \tilde{\xi} \rangle} dx = \delta(\xi - \tilde{\xi}),
\end{equation}

where $\delta$ is the delta function, then $\text{Op}(a, \varphi)$ is unitary.

**Proof.** First consider the action of $\text{Diff}(\mathbb{R}^d)$ on $C^\infty(T^*\mathbb{R}^d)$ defined by

\begin{equation}
(2.6) \quad (\varphi a)(x, \xi) = a(\varphi^{-1}x, \xi), \quad \varphi \in \text{Diff}(\mathbb{R}^d)
\end{equation}

and the action by pullback of $\text{Diff}(\mathbb{R}^d)$ on the functions on $\mathbb{R}^d$:

\[ t_\varphi(\psi)(x) := \psi(\varphi^{-1}(x)). \]

These two actions have the following properties:

- the action \((2.6)\) restricted to the subgroup $\text{Diff}_b(\mathbb{R}^d)$ preserves the space of bounded symbols $S_{2d}(1)$;
- if $\varphi \in \text{Diff}_b(\mathbb{R}^d)$, $t_\varphi$ is a continuous operator on $L^2(\mathbb{R}^d)$ and, if $\varphi \in \text{Diff}_{b,v}(\mathbb{R}^d)$, it is also unitary.

The proof of the Proposition will follow now from the following identity

\[ \text{Op}(a, \varphi) = t_\varphi \circ \text{Op}(\varphi^{-1}a), \]

together with Theorem 1 and from the fact that, for $\varphi \in \text{Diff}_b(\mathbb{R}^d)$ (resp. $\varphi \in \text{Diff}_{b,v}(\mathbb{R}^d)$), $t_\varphi$ is continuous (resp. unitary) while $\varphi^{-1}a$ remains bounded when $a$ is bounded. A direct computation shows that \((2.5)\) implies unitarity. □

We now show that the FIOs of the form \((2.4)\) are closed under operator composition by defining a product-like operation for their symbols. (Note that we will not obtain an algebra of symbols here, since this new symbol product depends on the particular underlying diffeomorphisms.)

**Proposition 4.** Let $\varphi_1, \varphi_2 \in \text{Diff}_b(\mathbb{R}^d)$ and $a, b \in S_{2d}(1)$, then

\begin{equation}
(2.7) \quad \text{Op}(a, \varphi_1) \circ \text{Op}(b, \varphi_2) = \text{Op}(a_{\varphi_1} \ast_{\varphi_2} b, \varphi_2 \circ \varphi_1)
\end{equation}

where $((a_{\varphi_1} \ast_{\varphi_2} b)(x, \xi))$ is given by the integral

\begin{equation}
(2.8) \quad \int a(x, \xi)b(\tilde{x}, \tilde{\xi}) e^{\frac{i}{\hbar} \left(\langle \xi, \varphi_1^{-1}(\tilde{x}) - \varphi_1^{-1} \circ \varphi_2^{-1}(x) \rangle + \langle \tilde{\xi}, \varphi_1^{-1}(x) - \tilde{x} \rangle \right)} \frac{d\tilde{x} d\tilde{\xi}}{(2\pi \hbar)^d}.
\end{equation}

**Proof.** We first compute the composition

\[ \left( \text{Op}(a, \varphi_1) \text{Op}(b, \varphi_2) \psi \right)(x) \]

directly, and we obtain

\[ \int \psi(\tilde{x})a(x, \xi)b(\tilde{x}, \tilde{\xi}) e^{\frac{i}{\hbar} \left(\langle \xi, \varphi_1^{-1}(\tilde{x}) - \tilde{x} \rangle + \langle \tilde{\xi}, \varphi_2^{-1}(\tilde{x}) - \tilde{\xi} \rangle \right)} \frac{d\tilde{x} d\tilde{\xi}}{(2\pi \hbar)^d}. \]
The phase of the oscillatory exponential in the line above can be rewritten as follows
\[
\langle \xi, (\varphi_2 \circ \varphi_1)^{-1}(x) - \tilde{x} \rangle + \left( \langle \xi, \varphi_2^{-1}(x) - \varphi_1^{-1} \circ \varphi_2^{-1}(x) \rangle + \langle \xi, \varphi_1^{-1}(x) - x \rangle \right)
\]
so that, defining the product \((a_{\varphi_1} \star_{\varphi_2} b)\) as in (2.8), we obtain (2.7). \(\square\)

**Lemma 5.** Let \(\varphi_1, \varphi_2, \varphi_3 \in \text{Diff}_b(\mathbb{R}^d)\) and \(a, b, c \in S_{2d}(1)\). Then
\[
(2.9) \quad a_{\varphi_1} \star_{\varphi_2} \varphi_3 (b \star_{\varphi_2} \varphi_3 c) = (a_{\varphi_1} \star_{\varphi_2} b) \varphi_1 \star_{\varphi_3} c.
\]

**Proof.** This comes immediately from the fact that
\[
\text{Op}(a, \varphi_1) \circ \left( \text{Op}(b, \varphi_2) \circ \text{Op}(c, \varphi_3) \right) = \left( \text{Op}(a, \varphi_1) \circ \text{Op}(b, \varphi_2) \right) \circ \text{Op}(c, \varphi_3)
\]
together with (2.7). \(\square\)

### 2.4. Asymptotic expansions and formal operators.

We now work out the asymptotic expansion of the bounded operators (2.3) in the limit \(h \to 0\). First, we fix the dependence in \(h\) for the amplitude as follows
\[
(2.10) \quad a(x, \xi) = a^0(x, \xi) + a^1(x, \xi)h + a^2(x, \xi)h^2 + \cdots,
\]
where the \(a^n \in S_{2d}(1)\) do not depend on \(h\) for all \(n\). Namely, the Borel summation lemma (see [15], Prop. 2.3.2, p. 14) for instance) guarantees then that there exists an amplitude in \(S_{2d}(1)\) depending on \(h\) whose asymptotic expansion in \(h\) yields back (2.10).

Now, changing the variable \(\tilde{\xi} = \xi/h\) and letting \(h \to 0\) (which allows us to perform a Taylor’s series of the amplitude at \((x, 0))\), we obtain that
\[
\text{Op}(a(x, \xi), \varphi)\psi(x) = \int \psi(x, \xi) a(x, h\xi) e^{i(\xi \cdot \varphi^{-1}(x) - \xi \cdot \varphi)(x)} \frac{d\xi d\varphi}{(2\pi)^d},
\]

\[
= \sum_{n \geq 0} h^n \text{Op}_1(P^n, \varphi)\psi(x),
\]

where \(\text{Op}_1\) is the same integral operator as \(\text{Op}\) except with the parameter \(h\) in the phase set to 1, and where
\[
P^n(x, \xi) = \sum_{k=0}^n f_\alpha(x)\xi^\alpha,
\]
are polynomial in \(\xi\) of order \(n\) with coefficients in \(S_{2d}(1)\) (actually, \(f_\alpha(x) = \frac{1}{\alpha!} \partial^\alpha_x a_{n-|\alpha|}(x, 0)\)).

Since, for a polynomial \(P^n(x, \xi)\) in \(\xi\) as above, the corresponding operator
\[
\text{Op}_1(P^n, \varphi)\psi(x) = \sum_{|\alpha| \leq n} f_\alpha(x) (D_x^\alpha \psi)(\varphi^{-1}(x)) = (P^n(x, D) \psi)(\varphi^{-1}(x))
\]
is a differential operator of order \(n\) (composed with a pullback), we obtain for \(\text{Op}(a, \varphi)\) an asymptotic expansion in terms of infinite order differential operators of the form:
\[
(2.11) \quad \text{Op}(a, \varphi)\psi(x) = P^0(x)\psi(\varphi^{-1}(x)) + \sum_{n \geq 1} h^n (P^n(x, D) \psi)(\varphi^{-1}(x)).
\]
Remark 6. This derivation for the asymptotic (2.11) is a shortcut for the usual stationary phase expansion. One recovers (2.11) by using the usual stationary phase expansion (see [10]) for quadratic phase using the following change of variable \( \bar{y} = \varphi^{-1}(x) - \bar{x} \).

In the following definition, we retain only the formal aspects of the asymptotics, forgetting that the operators (2.4) are actually bounded operators (i.e. the amplitudes are in \( S_{2d}(1) \) and the action is in \( \text{Diff}_b(\mathbb{R}^d) \)). This will allows us later on to consider quantizations of actions that are not necessarily volume-preserving nor bounded.

**Definition 7.** We define the algebra \( \mathcal{D} \) of formal operators of the form

\[
\text{Op}_1(P, \varphi)\psi(x) = P^0(x)\psi(\varphi^{-1}(x)) + \sum_{n \geq 1} \hbar^n \left( P^n(x, D) \psi \right)(\varphi^{-1}(x)), \quad \varphi \in \text{Diff}(\mathbb{R}^d)
\]

which acts on the formal space of functions \( C^\infty(\mathbb{R}^d)[[\hbar]] \), and where

\[
P^n(x, D) = \sum_{|\alpha| \leq n} f_\alpha(x) D^\alpha,
\]

is a differential operator of order \( n \) with coefficients \( f_\alpha \in C^\infty(\mathbb{R}^d) \). The corresponding space of symbols \( \mathcal{P} \) is the space of formal functions of the form

\[
P(x, \xi) = P^0(x) + \sum_{n \geq 1} \hbar^n \sum_{|\alpha| \leq n} f_\alpha(x) \xi^\alpha,
\]

with \( P^0(x), f_\alpha \in C^\infty(\mathbb{R}^d) \).

Note that, as before, we obtain a composition of formal symbols of thanks to (2.13)

\[
\text{Op}_1(P, \varphi_1) \circ \text{Op}_1(K, \varphi_2) = \text{Op}_1(P, \varphi_1 \circ \varphi_2 K, \varphi_1 \circ \varphi_2).
\]

Again, this does not define an algebra structure on \( \mathcal{P} \) since the composition depends on the underlying bounded diffeomorphisms \( \varphi_1 \) and \( \varphi_2 \).

### 3. Quantization of \( G \)-actions

In this section, we quantize a given action \( \varphi \) of a group \( G \) on \( \mathbb{R}^d \) (using the Fourier integral operators of the previous section as well as their asymptotics). By this, we mean to associate with \( \varphi \) a set \( \text{Rep}_q(G) \) of infinite dimensional representations of \( G \) on an appropriate space of “functions” on \( \mathbb{R}^d \). We call a quantization of the action a representation in \( \text{Rep}_q(G) \).

The actual implementation of \( \text{Rep}_q(G) \) (i.e. the choice of the functional space on which we represent the group as well as the properties of the operators forming the quantization) depends on the type of actions at hand. We distinguish between three cases, all of which contain what we call the trivial quantization, i.e. the representation obtained by pullback of functions:

\[
(T_g \psi)(x) = \psi(\varphi_g^{-1} x), \quad g \in G.
\]
Quantizations in $\text{Rep}_\varphi(G)$ can be regarded, in a sense, as “deformations” of the trivial quantization.

Here are the three cases we are interested in:

- An action of a group $G$ on $\mathbb{R}^d$ by smooth diffeomorphisms (i.e. $\varphi_g \in \text{Diff}(\mathbb{R}^d)$ for all $g \in G$), which we call here simply an action.
- An action of a group on $G$ on $\mathbb{R}^d$ by bounded smooth diffeomorphisms (i.e. $\varphi_g \in \text{Diff}_b(\mathbb{R}^d)$ for all $g \in G$), which we call a bounded action.
- An action of a group $G$ on $\mathbb{R}^d$ by bounded and volume-preserving smooth diffeomorphisms (i.e. $\varphi_g \in \text{Diff}_{b,v}(\mathbb{R}^d)$ for all $g \in G$), which we call a volume-preserving action.

3.1. **Unitary G-systems.** If the action is a volume-preserving and bounded (i.e $\varphi_g \in \text{Diff}_{b,v}(\mathbb{R}^d)$ for all $g \in G$), $\text{Rep}_\varphi(G)$ is a set of representations by bounded unitary operators on the Hilbert space $L^2(\mathbb{R}^d)$ of the square integrable functions on $\mathbb{R}^d$. These operators are of the form (2.4), with amplitudes in $S_{2d}(1)$ and satisfying the unitarity condition (2.5). Observe that the trivial quantization in this case is formed by bounded unitary operators on $L^2(\mathbb{R}^d)$. We define:

**Definition 8.** A unitary $G$-system of amplitudes (associated with a volume-preserving and bounded action $\varphi$ of a group $G$ on $\mathbb{R}^d$) is a map

$$a : G \rightarrow S_{2d}(1)$$

such that the collection of operators

$$T^a_g := \text{Op}(a_g, \varphi_g)$$

(defined in (2.4)) forms a unitary representation of $G$ by bounded operators on $L^2(\mathbb{R}^d)$.

**Example 9.** Unitary $G$-systems from geometric quantization. Suppose we have a volume-preserving action $\varphi$ of a Lie group $G$ on $\mathbb{R}^{2n}$ (endowed with its canonical symplectic form $\omega = \sum_i dp_i \wedge dx_i$), which is hamiltonian and which admits a momentum map

$$J : g \rightarrow C^\infty(\mathbb{R}^{2n}).$$

The condition $|\varphi'(x)| = 1$ is always satisfied, since $\varphi_g : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a symplectomorphism for all $g \in G$; so, here, the volume-preserving condition on $\varphi$ is only really a condition on the boundedness of $\varphi_g$ as well as on its derivatives.

Geometric quantization prescribes then a way (as explained in [21] ch. 8. sec. 4) for instance) to associate a unitary flow on $L^2(\mathbb{R}^{2n})$ with the hamiltonian flow $\varphi_t$ integrating the hamiltonian vector field $X_f$ of a function $f \in C^\infty(\mathbb{R}^{2n})$; namely,

$$U_t(f)\Psi(x) = \exp \left( i \int_0^t L_f(\varphi^{-1}_s(x))ds \right) \Psi(\varphi^{-1}_t(x)),$$

where $L_f = \theta(X_f) - f$, $\theta$ is the canonical Liouville 1-form on $\mathbb{R}^{2n}$ and $x = (p, q) \in \mathbb{R}^{2n}$. If $f$ is a complete function (i.e. $X_f$ is a complete vector field), then $U_t(f)$ forms a 1-parameter group.
Now, if the Lie group $G$ is nilpotent for instance and the Hamiltonian vector fields $X_{J(v)}$ are complete for all $v \in \mathfrak{g}$, we obtain a unitary representation of $G$ on $L^2(\mathbb{R}^{2n})$ by taking
\[
\rho_g := U_1(J(v)), \quad g = \exp(v),
\]
where $\exp$ is the exponential map from $\mathfrak{g}$ to $G$, which is a diffeomorphism for nilpotent groups.

Observe that, if we set
\[
a_g(x) = \exp \left( i \int_0^1 \mathcal{L}_{J(v)}(\varphi^{-1}_s(x)) \, ds \right), \quad g = \exp(v),
\]
the representation $\rho_g$ can be regarded as a quantization $T^a_g \in \text{Rep}_\varphi(G)$ associated with the $G$-system $a_g$, which is independent of $\xi$.

**Example 10. Unitary $G$-systems from galilean covariance.** Consider the space-time $\mathbb{R}^4 = \mathbb{R}_t \times \mathbb{R}^3$. The additive group $\mathbb{R}^3$ of translations acts on $\mathbb{R}^4$ by galilean boost
\[
\varphi_v(t, x) = (t, x + vt).
\]
In (non-relativistic) quantum mechanics, dynamics is described by square integrable functions $\Psi : \mathbb{R}^4 \to \mathbb{C}$ satisfying the Schrödinger equation $i \partial_t \Psi = H \Psi$, where $H$ is the Hamiltonian operator. It turns out that this equation is not covariant with respect to the trivial quantization of the galilean boost. To obtain covariance, one needs to use the following unitary $G$-system
\[
a_g(t, x, \xi) = e^{-i(\frac{1}{2}mv^2 t - mvx)},
\]
which yields the quantization
\[
T^a_g \Psi(t, x) = e^{-i(\frac{1}{2}mv^2 t - mvx)} \psi(t, x - vt).
\]

There seems to be many examples in the literature of unitary $G$-systems that are independent of $\xi$ as in the previous examples (see also for instance the representation in [20, p. 544, p. 557]). Because of this, we devote Section 3.3 to the study of these special $G$-systems.

Let us give an example of unitary $G$-system that also depends on $\xi$.

**Example 11.** Consider the multiplicative group $\mathbb{R}^+$ of the strictly positive real numbers and its trivial action on $\mathbb{R}$, i.e. $\varphi_g(x) = x$. Then
\[
a_g(\xi) = e^{i\xi \ln(g)}
\]
is a unitary $G$-system. Namely, one verifies that the corresponding operator is then given by
\[
T_g \psi(x) = \psi(x + \ln(g)),
\]
which is a unitary representation on $L^2(\mathbb{R})$.

If the action is only bounded, (i.e. $\varphi_g \in \text{Diff}_b(\mathbb{R}^d)$ for all $g \in G$), $\text{Rep}_\varphi(G)$ is again a set of infinite dimensional representations on $L^2(\mathbb{R}^d)$, except that now the operators (2.1) forming the representations are no longer unitary. Condition (2.5) on the amplitudes is then dropped, but we still require that the amplitudes are in $S_{2d}(1)$. In this case, the operators $T_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ forming the trivial quantization are only bounded but no longer unitary, since the action is not volume-preserving. We define:
Definition 12. A (non-unitary) $G$-system of amplitudes (associated with a bounded action $\varphi$ of a group $G$ on $\mathbb{R}^d$) is a map
\begin{equation}
(3.3) \quad a : G \rightarrow S_{2d}(1)
\end{equation}
such that the collection of operators
\[ T_a^g := \text{Op}(a_g, \varphi_g) \]
(defined in (2.4)) forms a representation of $G$ by bounded operators on $L^2(\mathbb{R}^d)$.

3.2. Formal $G$-systems. The operators in (2.4) that we used to define quantizations of actions in the two previous cases depend on a parameter $\hbar$ and, thus, have an asymptotic expansion in terms of formal operators as discussed in Section 2.4.

We can now forget that these expansions comes from well-defined bounded operators on $L^2(\mathbb{R}^d)$ and use the formal operators (2.12) to define formal quantizations when the action is neither bounded nor volume-preserving. In this case, $\text{Rep}_\varphi(G)$ is a set of formal representations by formal operators of the form (2.12) on the space $C^\infty(\mathbb{R}^d)[[\hbar]]$ of formal power series in the formal parameter $\hbar$ with value in the smooth function on $\mathbb{R}^d$. More precisely, we define:

Definition 13. A formal $G$-system of amplitudes (associated with an action of a group $G$ on $\mathbb{R}^d$) is a map
\[ a : G \rightarrow \mathcal{P} \]
such that the formal operators
\[ T_a^g = \text{Op}_1(a_g, \varphi_g) \in \mathcal{D} \]
form a representation of $G$ on $C^\infty(\mathbb{R}^d)[[\hbar]]$, where $\text{Op}_1(a, \varphi_g)$ is defined as in (2.12).

Formal $G$-systems seem to be related to both deformation quantization ($G$-equivariant star-products) and deformation theory of Lie morphisms, when the action we start with is a smooth action of a Lie group on $\mathbb{R}^d$. Let us comment here briefly on these points.

In deformation quantization ([6]), one quantizes an action (by Poisson diffeomorphisms) of a Lie group $G$ on a Poisson manifold $M$ by constructing $G$-equivariant star-products $\ast$ on $M$. For us, $M = \mathbb{R}^d$. (This notion is somewhat different whether one considers formal deformations, as in [6], or strict ones, as in [20].) The idea is to find star-products $\ast$ (quantizing a Poisson structure on $\mathbb{R}^d$ that is invariant with respect to the group action), which has the following property ($G$-equivariance):
\[ T_g\psi_1 \ast T_g\psi_2 = T_g(\psi_1 \ast \psi_2), \]
where $\psi_1, \psi_2 \in C^\infty(\mathbb{R}^d)[[\hbar]]$ and $T$ is the aforementioned trivial quantization of the action.

Despite compatibility between the action and the Poisson structure, the star-products quantizing the Poisson structure are generally not $G$-equivariant ($G$-equivariant star-products may even not exist at all; see [3]). Thus, in some cases, one also needs to "deform" the trivial quantization to obtain $G$-equivariance (for the deformed action), as
in [3, 4] in the formal case (for the corresponding infinitesimal action), or as in [20] for strict quantization of the Heisenberg manifolds.

The latter case is specially interesting for us, since the deformation of the action $\varphi$ of the Heisenberg group $G$ on the Heisenberg manifolds is of the form (1.1) for a certain $G$-system independent of $\xi$ (see [20, p. 557]).

It would be interesting to see if, for a given (strict) star-product on a Poisson manifold on which a Lie group $G$ acts by Poisson diffeomorphisms, one can always find a deformation of the trivial quantization in our space of quantization $\text{Rep}_{\varphi}(G)$ that is $G$-equivariant.

There is also a way in which quantizations of actions by $G$-systems as defined above may be related to the general theory of Lie morphism deformations as in [16], and, more specifically, to the work of Ovsienko and collaborators ([2, 17]) on embeddings of the Lie algebra of vector fields into various Lie algebras (and, in particular, the Lie algebra of pseudodifferential operators).

Namely, the infinitesimal version of the trivial quantization of an action yields an embedding from a Lie subalgebra of the vector fields on the manifold into its Lie algebra of pseudodifferential operators. Then the infinitesimal representations associated with quantizations in $\text{Rep}_{\varphi}(G)$ (i.e., the space of Lie algebra representations corresponding to the unitary/formal representations in $\text{Rep}_{\varphi}(G)$) should, in a sense, be related to deformations of this embedding.

It would also be interesting to compare the various obstructions (and actual deformations) obtained in this infinitesimal context with the obstructions we obtain in Section 5.

3.3. Unitary $G$-systems independent of $\xi$. Let $\varphi$ be a volume-preserving action of $G$ on $\mathbb{R}^d$. We are looking for $G$-systems associated with this action for which the amplitudes do not depend on $\xi$. Example 3 from geometric quantization and Example 10 from the galilean covariance of the Schrödinger equation are of this type. In the context of strict deformation quantization the representations in [20, p. 544, p. 557] are also of this type.

Let us study these $G$-systems independently. We start by defining a useful complex:

Denote by $\mathcal{B}$ the space of smooth functions on $\mathbb{R}^d$ with all of their derivatives bounded. One verifies that $\mathcal{B}$ is a left $G$-module with respect to the action

$$(g \cdot S)(x) = S(\varphi_g^{-1}(x)), \quad S \in \mathcal{B};$$

Observe that, in contrast with $S_d(1)$, we do not require that a function in $\mathcal{B}$ be bounded (only its derivatives). We further turn $\mathcal{B}$ into a $G$-bimodule by considering the right action of $G$ on $\mathcal{B}$. Now consider the group cohomology with values in the bimodule $\mathcal{B}$. The corresponding space $C^k_{\varphi}(G, \mathcal{B})$ of (normalized) $k$-cochains is given by the smooth maps

$$S : G^k \rightarrow \mathcal{B}, \quad k \geq 0$$

such that $S_{g_1, \ldots, g_k} = 0$ if one of the $g_i$'s is the group unit. The differential

$$\delta : C^k_{\varphi}(G, \mathcal{B}) \rightarrow C^{k+1}_{\varphi}(G, \mathcal{B})$$
is given by the usual formula

\[(\delta c)_{g_1,\ldots,g_{k+1}} = g_1 \cdot c_{g_2,\ldots,g_{k+1}} - c_{g_1g_2,\ldots,g_k} + \cdots \pm c_{g_1,\ldots,g_{k+1}} \oplus c_{g_1,\ldots,g_k},\]

where the right action by \(g_{k+1}\) on the last term is the trivial action.

**Theorem 14.** A \(G\)-system is independent of \(\xi\) iff it is of the form

\[(3.4) \quad a_g(x) = e^{iS_g(x)},\]

where \(S_g\) is a 1-cocycle in \(C^\bullet_\varphi(G,B)\). The corresponding operators are given by

\[(3.5) \quad T_g^a \psi(x) = e^{iS_g(x)}\psi(x).\]

Moreover, cocycles in the same cohomology class induce equivalent representations. In other words, \(H^1_\varphi(G,B)\) controls the deformations by unitary multiplication operators of the trivial quantization: If this first cohomology group vanishes, all deformations of the form \((3.5)\) are equivalent to the trivial quantization.

**Proof.** Suppose \(a_g\) is of the form \((2.1)\). Since \(S_g \in B\), we have that \(a_g \in S_{2d}(1)\), and Proposition \(2.3\) guarantees that \(T_g^a\) is a continuous operator on \(L^2(\mathbb{R}^d)\). The unitarity follows from the fact that \(a_g^*(x)a_g(x) = 1\) for all \(x \in \mathbb{R}^d\). Conversely, the operators corresponding to a \(G\)-system \(a_g(x)\) that is independent of \(\xi\) are of the form

\[T_g^a \psi(x) = a_g(x)\psi(\varphi_g^{-1}(x)).\]

The unitarity condition for these operators is equivalent to the condition

\[\int (1 - a_g^*(\varphi_g(x))a_g(\varphi_g(x)))\psi_1^*(x)\psi_2(x) = 0,\]

for all \(\psi_1, \psi_2 \in L^2(\mathbb{R}^d)\), which in turns is equivalent to \(a_g^*(x)a_g(x) = 1\). The only functions satisfying this last condition are of the form \(e^{iS_g(x)}\). Now observe that, for an amplitude of this form, \(a_g \in S_{2d}(1)\) if and only if \(S_g \in B\).

Let us check now that \(a_g(x) = e^{iS_g(x)}\) with \(S \in C^1(G,B)\) is a \(G\)-system if and only if \(\delta S = 0\). For this, we observe that

\[T_{g_1g_2}^a \psi(x) = e^{iS_{g_1g_2}(x)}\psi(\varphi_{g_1g_2}^{-1}(x))\]

is equal to

\[T_{g_1}^a T_{g_2}^a \psi(x) = e^{i(S_{g_1}(x)+S_{g_2}(\varphi_{g_1}^{-1}(x))))\psi(\varphi_{g_1g_2}^{-1}(x))\]

if and only if

\[S_{g_2}(\varphi_{g_1}^{-1}(x)) - S_{g_1g_2}(x) + S_{g_1}(x) = 0,\]

that is if and only if \(\delta S = 0\). At last, let us notice that the normalization condition for cochains \(a \in C^1(G,B)\) is equivalent to \(T_e^a = \text{id}\).

Let us show now that if \(S - \breve{S} = \delta K\), where \(S\) and \(\breve{S}\) are 1-cocycle and \(K\) is a 0-cochain, then the induced representations \(T^a\) and \(\breve{T}^a\) are equivalent. Consider the
Quantization of (Volume-Preserving) Actions on $\mathbb{R}^d$

bounded operator $\hat{K}\psi(x) = e^{iK(x)}\psi(x)$. Then
\[
(T^{\phi}_{\tilde{g}} \circ \hat{K})\psi(x) = e^{i(S_{\tilde{g}}(x) + K(\varphi^{-1}_g(x)))}\psi(\varphi^{-1}_g(x)),
\]
\[
(\tilde{K} \circ T^{\phi}_{\tilde{g}})\psi(x) = e^{i(S_{\tilde{g}}(x) + K(x)))}\psi(\varphi^{-1}_g(x)).
\]
Therefore, the relation $\tilde{S}_{\tilde{g}}(x) - S_{\tilde{g}}(x) = K(\varphi^{-1}_g(x)) - K(x) = (\delta K)_g(x)$ implies that $T^{\phi}_{\tilde{g}} \circ \hat{K} = \hat{K} \circ T^{\phi}_{\tilde{g}}$. □

In the next section, we will work out a similar cohomological equation (a Maurer-Cartan equation) for general $G$-systems (i.e. with a dependence on $\xi$).

Example 15. Let $h \in \mathcal{B}$ be invariant under the action of $G$ (i.e. $h(\varphi^{-1}_g(x)) = h(x)$ for all $x$ and $g$). For any smooth function $c : G \to \mathbb{R}$ that satisfies
\[(3.6)\]
\[c_1 = 0 \quad \text{and} \quad c_{g_{1}g_{2}} = c_{g_{1}} + c_{g_{2}},\]
we verify that $S_{\tilde{g}}(x) = h(x)c_{\tilde{g}}$ is a cocycle. As a consequence, the family of amplitudes
\[a_{g}(x) = e^{i\sum_{k} h_{k}(x)c_{g}^{k}}, \quad g \in G,
\]
is a $G$-system, where $h_{1}, \ldots, h_{n}$ are invariant functions in $\mathcal{B}$ and $c_{1}, \ldots, c_{n}$ are smooth functions from $G$ to $\mathbb{R}$ satisfying (3.6).

4. DGAs of $G$-Amplitudes

In this section, we construct two DGAs, $A_{\varphi}$ and $P_{\varphi}$, associated with, respectively, a bounded action and a smooth action $\varphi$ of a group $G$ on $\mathbb{R}^d$. We show that the Maurer-Cartan elements in $A_{\varphi}$ correspond to $G$-systems while Maurer-Cartan elements in $P_{\varphi}$ correspond to formal $G$-systems. One can regards $P_{\varphi}$ as the “asymptotic” version of $A_{\varphi}$. We also show that, in both cases, gauge equivalent Maurer-Cartan elements yields equivalent quantizations.

4.1. MC elements in $A_{\varphi}$ and $G$-systems. We define here a Differential Graded Algebra (or DGA for short) associated with a bounded action $\varphi$ of a Lie group $G$ on $\mathbb{R}^d$ whose Maurer-Cartan elements correspond to (nonunitary) $G$-systems of amplitudes. Roughly, the elements of degree $k$ in this DGA are amplitudes depending on $k$ group variables, and the graded product corresponds to the composition of the Fourier integral operators that one can naturally associate with these amplitudes using the action as a phase.

More precisely, for any $k \geq 0$, we define the space of $k$-cochains by
\[(4.1)\]
\[A_{\varphi}^k = \{a : G \times \cdots \times G \to S_{2d}(1)\}, \quad A_{\varphi}^0 = S_{2d}(1),\]
such that $a_{e, \ldots, e} = 1$ with $e$ being the group unit. The differential $d : A_{\varphi}^k \to A_{\varphi}^{k+1}$ is defined by
\[(4.2)\]
\[(da)(g_1, \ldots, g_k) = \sum_{i=1}^{k} (-1)^ia(g_1, \ldots, g_ig_{i+1}, \ldots, g_{k+1}),\]
which we extend by $\mathbb{C}$-linearity to $A^\cdot_\varphi = \oplus_{k \geq 0} A^k_\varphi$. This turns $(A^\cdot_\varphi, d)$ it into a complex, which we call the complex of G-amplitudes.

Let us now define a graded associative product on $A^\cdot_\varphi$. To an element $a \in A^k_\varphi$, we can assign the following collection of Fourier integral operators

$$T^a_1, \ldots, g_k := \text{Op}(a_{g_1, \ldots, g_k}, \varphi_{g_1, \ldots, g_k}), \quad (g_1, \ldots, g_k) \in G^k.$$ 

The composition of these operators for $a \in A^k_\varphi$ and $b \in A^l_\varphi$ yields

$$(4.3) \quad T^a_{g_1, \ldots, g_k} \circ T^b_{g_{k+1}, \ldots, g_{k+l}} = \text{Op}(a_{g_1, \ldots, g_k}, \varphi_{g_1, \ldots, g_k}) \circ \text{Op}(b_{g_{k+1}, \ldots, g_{k+l}}, \varphi_{g_{k+1}, \ldots, g_{k+l}}) = \text{Op}(a_{g_1, \ldots, g_k} \star b_{g_{k+1}, \ldots, g_{k+l}}, \varphi_{g_1, \ldots, g_{k+l}}),$$

where $a_{g_1, \ldots, g_k} \star b_{g_{k+1}, \ldots, g_{k+l}}$ is a shorthand for the product of amplitudes defined in (2.8):

$$\varphi_{g_1, \ldots, g_k} \star \varphi_{g_{k+1}, \ldots, g_{k+l}} = b(g_{k+1}, \ldots, g_{k+l}).$$

This leads us to define a graded associative product on the complex of G-amplitudes

$$\star : A^k_\varphi \times A^l_\varphi \longrightarrow A^{k+l}_\varphi$$

in the following way: Given $a \in A^k_\varphi$ and $b \in A^l_\varphi$, we define

$$(4.4) \quad (a \star b)(g_1, \ldots, g_{k+l}) = a_{g_1, \ldots, g_k} \star b_{g_{k+1}, \ldots, g_{k+l}},$$

which turns $A^\cdot_\varphi$ into a graded algebra with the nice property that

$$T^a \circ T^b = T^{a \star b}.$$ 

**Lemma 16.** $(A^\cdot_\varphi, d, \star)$ is a DGA.

**Proof.** The fact that $d$ squares to zero is clear from its formula (it is the usual group cohomology differential without the boundary terms). The associativity of the product $\star$ comes from the associativity of the operator composition in (4.3)

Let us check that $d$ is a derivation for $\star$:

$$\delta(a \star b)_{g_1, \ldots, g_{k+l+1}} = \sum_{i=1}^{k+l} (-1)^i (a \star b)_{g_1, \ldots, g_i, g_{i+1}, \ldots, g_{k+l+1}},$$

$$= \left( \sum_{i=1}^{k} (-1)^i a_{g_1, \ldots, g_i, g_{i+1}, \ldots, g_k} \varphi_{g_{i+1}, \ldots, g_{k+l}} b_{g_{k+1}, \ldots, g_{k+l+1}} \right) + (-1)^k a_{g_1, \ldots, g_k} \varphi_{g_1, \ldots, g_{k+1}} \varphi_{g_{k+1}, \ldots, g_{k+l+1}} \left( \sum_{i=1}^{l} (-1)^i b_{g_{k+i}, \ldots, g_{k+i+1}, \ldots, g_{k+l+1}} \right),$$

$$= ((\delta a) \star b)_{g_1, \ldots, g_{k+l+1}} + (-1)^k (a \star (\delta b))_{g_1, \ldots, g_{k+l+1}}.$$

Let us now remind the following definition:
Definition 17. Let \((\mathcal{A}, \*, d)\) be a DGA. The solutions of the Maurer-Cartan equation
\[ da + a \* a = 0 \]
are called Maurer-Cartan elements. The set of all Maurer-Cartan elements of \( \mathcal{A} \) will be denoted by \( \text{MC}(\mathcal{A}) \).

Two Maurer-Cartan elements \( a, b \in \text{MC}(\mathcal{A}) \) are called gauge equivalent if there exists an invertible \( u \in \mathcal{A}^0 \) such that \( au = ua = du \).

Remark 18. The set \( \text{MC}(\mathcal{A}) \) is a subset of \( \mathcal{A}^1 \).

Proposition 19. Let \( \varphi \) be a bounded action of a group \( G \) on \( \mathbb{R}^d \). There is a one-to-one correspondence between the set \( \text{MC}(\mathcal{A}_\varphi^\bullet) \) of Maurer-Cartan elements in the complex of \( G \)-amplitudes and the set of (nonunitary) \( G \)-systems of amplitudes associated with the action \( \varphi \). Moreover, gauge equivalent Maurer-Cartan elements induce equivalent representations.

Proof. Let \( a \in \mathcal{A}^1 \). Then the associated collection of operators \( T^a_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) is a representation of \( G \) if and only if
\[
0 = T^a_{g_1 g_2} - T^a_{g_2 g_1},
0 = \text{Op}(a_{g_1 g_2}, \varphi_{g_1 g_2}) - T^a_{g_1 g_2},
0 = \text{Op}((da)_{g_1 g_2} - (a \* a)_{g_1 g_2}, \varphi_{g_1 g_2}),
\]
that is iff \( da + a \* a = 0 \). The unitality condition \( T^a_e = 1 \) is taken care of by the requirement on the cochains that \( a_e = 1 \).

Let us check now that two gauge equivalent Maurer-Cartan elements induce equivalent representations. First off, we note that, in \( \mathcal{A}_\varphi^\bullet \), all elements of degree zero are cocycles. This means that \( a, b \in \text{MC}(\mathcal{A}_\varphi^\bullet) \) are gauge equivalent if there is an invertible \( u \in \mathcal{A}_\varphi^0 \) such that \( au = ua \). Since \( u \) is of degree zero, neither \( u \) nor \( T^u \) depend on group variables. The commutation \( au = ua \) on the level of amplitudes implies that
\[
T^a_g \circ T^u = T^u \circ T^b, \quad g \in G,
\]
on the level of operators. That is, \( T^u \) intertwines the two representations; since \( T^u \) is invertible, because \( u \) is invertible, the representations \( T^a \) and \( T^b \) are equivalent. □

Remark 20. The Maurer-Cartan equation applied to an ansatz of the form \( 3.4 \) yields back the cocycle condition of Proposition 14. Namely,
\[
(de^iS)_{g_1 g_2}(x) = -e^{iS_{g_1 g_2}(x)} \quad \text{and} \quad (e^{iS \* e^{iS}})_{g_1 g_2}(x) = e^{i(S_{g_1}(x) + S_{g_2}(\varphi^{-1}(x)))},
\]
which implies that \( e^{iS} \in \text{MC}(\mathcal{A}^\bullet) \) if and only if \( \delta S = 0 \).

4.2. MC elements in \( \mathcal{P}_\varphi \) and formal \( G \)-systems. We define now a formal version of the amplitude complex by replacing the bounded symbols \( S_{2d}(1) \) by their formal version \( \mathcal{P} \).

The complex of formal \( G \)-amplitudes \( \mathcal{P}_\varphi^\bullet \) is defined in the following way. The space of \( k \)-cochain is given by
\[
\mathcal{P}_\varphi^k = \{ a : G \times \cdots \times G \to \mathcal{P} \}.\]
The differential \( d : P_k^\phi \rightarrow P_{k+1}^\phi \) is obtained from (4.2) by linear extension. Similarly, we obtained a graded associative product \( \star : P_k^\phi \times P_l^\phi \rightarrow P_{k+l}^\phi \) from (4.4) by linear extension, turning \( P_\phi^\bullet \) into a DGA (the proof of this is similar to that of Lemma 16). Mimicking the proof of Proposition 19, we obtain:

**Proposition 21.** Let \( \phi \) be an action of \( G \) on \( \mathbb{R}^d \). Then Maurer-Cartan elements in \( P_\phi^\bullet \) are in one-to-one correspondence with formal \( G \)-systems associated with \( \phi \). Moreover, gauge equivalent Maurer-Cartan elements yields equivalent representations.

**Proposition 22.** Let \( a = P^0(x) + \hbar P^1 + \cdots \in P_1^\phi \) be a Maurer-Cartan element in \( P_\phi^\bullet \). It defines a new differential on \( P_\phi^\bullet \) as follows:

\[
(4.5) \quad d_{P^0} a = da + [P^0, a] = da + P^0 \star a - (-1)^{|a|} a \star P^0.
\]

Moreover, \( P^1 \) is a cocycle with respect to this new differential, and we get the following recursive equations for the higher order terms

\[
(4.6) \quad d_{P^0} P^n = - \sum_{i+j=n \atop i, j \geq 1} P^i \star P^j.
\]

**Proof.** The Maurer-Cartan equation at order zero in \( \hbar \) reads

\[
dP^0 + P^0 \star P^0 = 0,
\]

which means that \( P^0 \) is itself a Maurer-Cartan element. Now it is a general fact that a differential \( d \) twisted by a Maurer-Cartan element as in (4.5) is again a differential.

The Maurer-Cartan equation at order 1 in \( \hbar \) reads

\[
dP^1 + P^0 \star P^1 + P^1 \star P^0 = 0,
\]

which is exactly \( d_{P^0} P^1 = 0 \) because \( P^1 \) is of degree 1 (it has only one group variable). At last, we obtain (4.6) by looking at the MC equation at order \( n \geq 2 \). □

### 5. Existence and Rigidity Theorem

In this section, we give cohomological conditions for the existence of formal \( G \)-systems, that is, Maurer-Cartan elements in \( P_\phi^\bullet \). The discussion that follows is based on appendix A of [1]. The main fact is that \( P_\phi^\bullet \) is a complete DGA in the sense of [11]; complete DGA have neat cohomological conditions governing the existence and obstruction of Maurer-Cartan elements.

**Definition 23.** We define \( \text{Pol}_d(n) \) for \( n \geq 0 \) to be the space of polynomial in \( \xi \) of the form

\[
P(x, \xi) = \sum_{|\alpha| \leq d} f_\alpha(x) \xi^\alpha,
\]

where \( f_\alpha \in C^\infty(\mathbb{R}^d) \).
First of all, $\mathcal{P}^\bullet$ has a natural filtration
\[
\cdots \subset F^{k+1} \mathcal{P}^\bullet \subset F^k \mathcal{P}^\bullet \subset \cdots \subset F^1 \mathcal{P}^\bullet \subset F^0 \mathcal{P}^\bullet = \mathcal{P}^\bullet,
\]
for which each of the $(F^k \mathcal{P}^\bullet, d)$ is a subcomplex and such that
\[
* : F^k \mathcal{P}^\bullet \times F^l \mathcal{P}^\bullet \to F^{k+l} \mathcal{P}^\bullet
\]
This filtration is given by
\[
F^k \mathcal{P}^\bullet = \{ \sum_{n \geq k} h^n P^n : P^n \in \text{Pol}_d(n) \}, \quad k \geq 1.
\]
We have then a tower
\[
\mathcal{P}^\bullet / F^1 \mathcal{P}^\bullet \leftarrow \mathcal{P}^\bullet / F^1 \mathcal{P}^\bullet \leftarrow \cdots,
\]
whose inverse limit is exactly $\mathcal{P}^\bullet$. This makes $\mathcal{P}^\bullet$ a complete DGA in the sense of the Appendix A of [1].

**Definition 24.** Define the graded vector space $\text{Pol}^k_d(n)$ to be
\[
\text{Pol}^k_d(n) := \{ P : G^k \to \text{Pol}_d(n) \}, \quad n, k \geq 0.
\]
Observe that, as graded vector space, we have that
\[
F^n \mathcal{P}^\bullet / F^{n+1} \mathcal{P}^\bullet \simeq \text{Pol}^n_d(n)
\]
and the following decomposition of the complex of formal $G$-amplitudes:
\[
\mathcal{P}^\bullet = \text{Pol}_d^0(0) \oplus h \text{Pol}_d^1(1) \oplus h^2 \text{Pol}_d^1(2) \oplus \cdots
\]
Let $P^0 \in \text{Pol}_d^0(0)$ be a Maurer-Cartan element. Then the twisted differential $d_{P^0}$ defined by formula (4.5), respects this decomposition and $(\text{Pol}_d^k(n), d_{P^0})$ is a complex for each $n \geq 0$. These complexes will be the main ingredients in our existence and rigidity results for formal $G$-systems.

From Proposition 22 we get that if
\[
P^0 + h P^1 + h^2 P^2 + \cdots
\]
is a Maurer-Cartan element, then $P^0$ is a Maurer-Cartan element in $\mathcal{P}^\bullet$ and $P^1$ is a 1-cocyle in $(\text{Pol}_d^1(n), d_{P^0})$. Now if we start with a Maurer-Cartan element $P^0$ and and a 1-cocyle $P^1$, in general $P^0 + h P^1$ is not a Maurer-Cartan element in $\mathcal{P}^\bullet$, and we may wonder whether it is possible to find higher terms to get a Maurer-Cartan element.

Another question is whether the representation obtained from (5.2) is equivalent to the one obtained by the first term only, i.e. when a Maurer-Cartan element is gauge equivalent to its first term.

**Definition 25.** A Maurer-Cartan element $P^0$ in $\mathcal{P}^\bullet$ is called rigid if all Maurer-Cartan elements having as first term $P^0$ are gauge equivalent to this first term.
In terms of the induced representations, \( P^0 \) being rigid means that all the representations obtained from Maurer-Cartan elements of the form (5.2) are equivalent as representations to the representation
\[
T_g^P \psi(x) = P^0(x) \psi(\varphi_g^{-1}(x)).
\]

The following theorem gives cohomological conditions answering the questions mentioned above.

**Theorem 26.** Let \( P^0 \in \text{Pol}_d^0(0) \) be a Maurer-Cartan element and \( P^1 \in \text{Pol}_d^1(1) \) a one-cocycle (i.e., \( d_{P^0} P^1 = 0 \)). If
\[
H^2(\text{Pol}_d^0(n), d_{P^0}) = 0, \quad n \geq 2,
\]
then there exists a Maurer-Cartan element \( \omega \) in \( \mathcal{P}_\varphi^\bullet \) such that
\[
\omega = P^0 + \hbar P^1 + \mathcal{O}(\hbar^2).
\]
Moreover if
\[
H^1(\text{Pol}_d^0(n), d_{P^0}) = 0, \quad n \geq 1,
\]
the Maurer-Cartan element \( P^0 \) is rigid.

**Proof.** The proof relies on Proposition A.3 and A.6 of the Appendix A of \([1]\). Since \( \gamma = P^0 + \hbar P^1 \) is a Maurer-Cartan element modulo \( \mathcal{F}^2 \mathcal{P}_\varphi^\bullet \), Proposition A.3 tells us that there exist a Maurer-Cartan element \( \omega = P^0 + \hbar P^1 + \mathcal{O}(\hbar^2) \) provided
\[
H^2(\mathcal{F}^n \mathcal{P}_\varphi^\bullet / \mathcal{F}^{n+1} \mathcal{P}_\varphi^\bullet, d_\gamma) = 0, \quad n \geq 2,
\]
where \( d_\gamma \) is the operator \( d_\gamma a = da + [\gamma, a] \), which becomes a differential on the quotient \( \mathcal{F}^n \mathcal{P}_\varphi^\bullet / \mathcal{F}^{n+1} \mathcal{P}_\varphi^\bullet \). The first part of the theorem follows from (5.1) and the fact that \( d_\gamma \) becomes \( d_{P^0} \) when passing to the quotient (because \( P^1 \) has one power of \( \hbar \), which will make this term disappear in the quotient). The rigidity part of the theorem is a direct application of Proposition A.6 with the same observations as above. \( \square \)

5.1. **Trivial action.** Consider the case when group action \( G \) is trivial \( \varphi_g = \text{id} \) as well as the first term of the deformation, that is, we are looking at \( G \)-systems of the form
\[
a_g = 1 + \hbar P^1_g + \hbar^2 P^2_g + \cdots
\]
The corresponding operators implementing the representation are then deformations of the trivial representations of \( G \) in \( C^\infty(\mathbb{R}^d)[[\hbar]] \); they are of the form
\[
T_g^n \psi(x) = \text{id} + \sum_{n \geq 1} \hbar^n P^n_g(x, D),
\]
where \( P^n_g(x, D) \) is a differential operator of order \( n \) with nonconstant bounded coefficients. The following theorem gives a simplification of the existence and rigidity result for general deformations. This result is very close to that of Pinzcon \([18]\) on obstructions and rigidity of deformations of representations.
Theorem 27. Consider the cohomology $H^\bullet(G, C^\infty(\mathbb{R}^d))$ of $G$ with coefficients in the smooth functions on $\mathbb{R}^d$, which we consider as a trivial $G$-bimodule. If $H^2(G, C^\infty(\mathbb{R}^d)) = 0$ then there exists representation of $G$ into $C^\infty[[\hbar]]$ of the form (5.3). Moreover, if $H^1(G, C^\infty(\mathbb{R}^d)) = 0$, all these representations are equivalent.

Proof. As a graded vector space $\text{Pol}^\bullet\mathbb{R}^d(n)$ can be identified with the following direct sum with $n$-terms
$$C^\bullet(G, C^\infty(\mathbb{R}^d)) \oplus \cdots \oplus C^\bullet(G, C^\infty(\mathbb{R}^d)),$$
since, for a cochain $P = \sum_{|\alpha| \leq n} f^\alpha(x)\xi^\alpha$, we have that $f^\alpha \in C^\bullet(G, C^\infty(\mathbb{R}^d))$ for all multi-indices $\alpha$. Using Theorem 26, we only need to show that, in the case the action is trivial, $d_1$ respects this splitting. Let us compute the differential of $P \in \text{Pol}^k\mathbb{R}^d(n)$:
$$d_1 P_{g_1,\ldots,g_{k+1}} = (dP)_{g_1,\ldots,g_{k+1}} + 1_{g_1} \ast P_{g_2,\ldots,g_{k+1}} - (-1)^k P_{g_1,\ldots,g_k} \ast 1_{g_{k+1}},$$
$$= P_{g_2,\ldots,g_{k+1}} + \sum_{i=1}^{k} (-1)^i P_{g_1,\ldots,g_i,g_{i+1},\ldots,g_{k+1}} + (-1)^{k+1} P_{g_1,\ldots,g_{k}},$$
since the product $\ast$ is now the standard product (associated with the standard quantization) because the action is trivial. Since only the $f^\alpha$’s depend on the group variables, we obtain that
$$d_1 P = \sum_{|\alpha| \leq n} (\tilde{\delta} f^\alpha)\xi^\alpha,$$
where $\tilde{\delta}$ is the differential of the group cohomology of $G$ in $S_d(1)$ considered as a trivial bimodule. \qed

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