SKIN EFFECT PROBLEM WITH DIFFUSION BOUNDARY CONDITIONS IN MAXWELL PLASMA

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The analytical method of solving the boundary problems for a system of equations describing the behaviour of electrons and an electric field in the Maxwell plasma half-space is developed. Here the diffusion reflection of electrons from the surface as a boundary condition is supposed. The exact solution for boundary value problem of the Vlasov – Maxwell equations through the eigenfunctions of the discrete spectrum and the continuous spectrum is obtained. The method of decomposition by eigenfunctions of discrete and continuous spectrum is used. The exact expression for the surface impedance also is obtained.
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1. Introduction. Statement of problem. The skin effect is the plasma’s response to an external constant-amplitude variable electromagnetic field that is tangential to the surface [1]. This problem represents essential interest [1–5], with the focal point being on the computation of surface impedance.

In [4] showed that electrons can transport the plasma current away from the skin layer due their thermal motion over distances of order $v_T/\omega$. In [5] this numerical study provides through specific numerical comparisons a useful perspective on the limitations and capabilities of the various Krook collisional models. This information should be useful to future studies that try to find a compromise in handling the yet unresolved problem of how to unify the kinetic response and collisions in wave phenomena.

The paper further develops the analytical method of solving boundary problem for a system of equations describing behaviour of electrons and electric field in Maxwell plasma. The method is based on the idea of the
decomposition of the solution the problem by singular eigenfunctions of the corresponding characteristic equations.

We find the analytical solution to the boundary problem of the skin effect theory for electron plasma that fills the half-space. The analytical solution has a form of the sum of an integral and two (or one) exponentially decreasing particular solutions of the initial system. Depending on index the problem one of the particular solutions disappears.

We consider a collision plasma with behaviour is described by modelling kinetic equation with integral of collisions in the form BGK–model (BGK= Bhatnagar, Gross, Krook).

Let’s Maxwell plasma fills the half-space \( x > 0 \). Here \( x \) is the orthogonal coordinate to the plasma boundary. Let’s the external electric field has only \( y \) component. Then the self-consistent electric field inside in plasma also has only \( y \) component \( E_y(x, t) = E(x)e^{-i\omega t} \). We consider the kinetic equation for the electron distribution function:

\[
\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + eE(x)e^{-i\omega t} \frac{\partial f}{\partial p_y} = \nu(f_0 - f(t, x, v)). \quad (1.1)
\]

In equation (1.1) \( \nu \) is the frequency of electron collisions with ions, \( e \) is the charge of electron, \( f_0(v) \) is the equilibrium Maxwell distribution function, \( p \) is the momentum of electron,

\[
f_0(v) = n \left( \frac{\beta}{\pi} \right)^{3/2} \exp(-\beta^2 v^2), \quad \beta = \frac{m}{2k_B T}.
\]

Here \( m \) is the mass of electron, \( k_B \) is the Boltzmann constant, \( T \) is the temperature of plasma, \( v \) is the velocity of the electron, \( n \) is the concentration of electrons, \( c \) is the speed of light.

The electric field \( E(x) \) satisfies to the equation:

\[
E''(x) + \frac{\omega^2}{c^2} E(x) = -\frac{4\pi ie^{i\omega t} \omega e}{c^2} \int v_y f(t, x, v) \, d^3v. \quad (1.2)
\]

We assume that intensity of an electric field is such that linear approximation is valid. Then distribution function can be presented in the form:

\[
f = f_0 (1 + Cy \exp(-i\omega t)h(x, \mu)),
\]
where \( C = \sqrt{\beta} v \) is the dimensionless velocity of electron, \( \mu = C_x \). Let \( l = v_T \tau \) is the mean free path of electrons, \( v_T = 1/\sqrt{\beta}, v_T \) is the thermal electron velocity, \( \tau = 1/\nu \). We introduce the dimensionless parameters and the electric field:

\[
t_1 = \nu t, \quad x_1 = \frac{x}{l}, \quad e(x_1) = \frac{\sqrt{2} e}{\nu \sqrt{m k_B T}} E(x_1).
\]

Later we substitute \( x_1 \) for \( x \). The substitution produces the following form of the kinetic equation (1.1) and the equation on a field with the displacement current (1.2):

\[
\mu \frac{\partial h}{\partial x} + z_0 h(x, \mu) = e(x), \quad z_0 = 1 - i \omega \tau, \quad (1.3)
\]

\[
e''(x) + Q^2 e(x) = -i \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu'^2) h(x, \mu') \, d\mu', \quad Q = \frac{\omega l}{c}, \quad (1.4)
\]

where \( \delta = \frac{c^2}{2\pi \omega \sigma_0} \), \( \delta \) is the classical depth of the skin layer, \( \sigma_0 = \frac{e^2 n}{mv}, \sigma_0 \) is the electric conductance, \( \alpha = \frac{2l^2}{\delta^2} \), \( \alpha \) is the anomaly parameter.

We formulate the boundary conditions for the distribution function of the electron in case of the diffusion electron reflection from the surface:

\[
h(0, \mu) = 0, \quad 0 < \mu < +\infty. \quad (1.5)
\]

We use the condition that function \( h(x, \mu) \) vanishes far from the surface:

\[
h(+\infty, \mu) = 0, \quad -\infty < \mu < +\infty, \quad (1.6)
\]

and conditions for electric field on the interface and far from it:

\[
e'(0) = e_s', \quad e(+\infty) = 0, \quad (1.7)
\]

where \( e_s' \) is the given value of the gradient of the electric field on the plasma interface.

So, the skin effect problem is formulated completely. We seek solution of system of the equations (1.3) and (1.4) in this problem, that satisfy to the boundary conditions (1.5) – (1.7).
2. Eigenfunctions and eigenvalues. The separation of variables \([2]–[4]\)

\[ h_{\eta}(x, \mu) = \exp(-z_0 \frac{x}{\eta}) \Phi(\eta, \mu), \quad (2.1) \]
\[ e_{\eta}(x) = \exp(-z_0 \frac{x}{\eta}) E(\eta), \quad (2.2) \]

where \(\eta\) is the spectral parameter (generally, complex-valued), reduces system of the equation (1.3) and (1.4) to the characteristic system:

\[ (\eta - \mu) \Phi(\eta, \mu) = \frac{\eta}{z_0} E(\eta), \quad (2.3) \]
\[ \left[ z_0^2 + Q^2 \eta^2 \right] E(\eta) = -\frac{i \alpha \eta^2}{\sqrt{\pi}} n(\eta), \quad (2.4) \]

where

\[ n(\eta) = \int_{-\infty}^{\infty} e^{-\mu^2} \Phi(\eta, \mu) d\mu. \]

From equations (2.3) and (2.4) for \(\eta \in (-\infty, \infty)\), we find the eigenfunctions of the continuous spectrum \([2], [6]\):

\[ \Phi(\eta, \mu) = \frac{a}{\sqrt{\pi}} \eta^3 e^{-\eta^2} P \frac{1}{\eta - \mu} + \lambda(\mu) \delta(\eta - \mu), \quad (2.6) \]
\[ E(\eta) = \frac{az_0}{\sqrt{\pi}} \eta^2 e^{-\eta^2}, \quad a = -\frac{i \alpha}{z_0^3}, \quad (2.7) \]

In the equation (2.6) the symbol \(P x^{-1}\) denotes the distribution, i.e. the principal value of the integral of \(x^{-1}\), \(\delta(x)\) is the Dirac delta function.

Since the distribution and electric field functions decline as we go further from the boundary, we regard the continuous as positive real half-axis: \(0 < \eta < +\infty\). The eigen solution of the continuous spectrum \(h_{\eta}(x, \mu), e_{\eta}(x)\) are decreasing from value \(x\), since \(\text{Re } z_0 > 0\).

In the equations (2.6) and (2.7) there is the normalization condition:

\[ \int_{-\infty}^{\infty} e^{-\mu^2} \Phi(\eta, \mu) d\mu = \left[ 1 + \left( \frac{\omega l c}{e} \right)^2 \eta^2 \right] e^{-\eta^2}, \]

\(\lambda(z)\) is the dispersion function,

\[ \lambda(z) = 1 + \left( \frac{Q}{z_0} \right)^2 z^2 + \frac{az_0^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} d\mu \int_{-\infty}^{\infty} \frac{e^{-\mu^2} \mu - z}{\mu - z}, \quad (2.8) \]
Consider the dispersion function in the following form:

\[ \lambda(z) = 1 + b z^2 - a p(z), \]

Here \[ b = \frac{Q^2}{z_0^2} = \frac{(\omega l)^2}{c^2 \left(1 - i \frac{\omega}{\nu}\right)^2}, \]

\[ p(\mu) = -\mu^3 \exp(-\mu^2), \quad q(\mu) = \sqrt{\pi} \mu^3 \exp(-\mu^2), \]

\[ b = \frac{\omega_1^2}{\nu_1^2 \left(1 - i \frac{\omega_1}{\nu_1}\right)^2} \left(\frac{\nu_T}{c}\right)^2 = \frac{\omega_1^2}{\nu_1^2 \left(\nu_1 - i \omega_1\right)^2} \left(\frac{\nu_T}{c}\right)^2, \]

\[ a = -i \frac{\omega_1}{\nu_1^3 \left(1 - i \frac{\omega_1}{\nu_1}\right)^3} = \frac{-i \omega_1}{\nu_1^3 \left(\nu_1 - i \omega_1\right)^3}. \]

The fact that \( \lambda(\mu) \) has a double pole at \( z = \infty \) follows from the asymptotic series in neighborhood of infinity:

\[ \lambda(z) = (b - a)z^2 + \left(1 - \frac{a}{2}\right) - \frac{3a}{4} \cdot \frac{1}{z^2} - \frac{15a}{8} \cdot \frac{1}{z^4} \cdots. \quad (2.13) \]

3. **Mode of plasma in Maxwell plasma.** Find out the structure of discrete spectrum, this spectrum consists of the zeros of the dispersion function. Each zero corresponds to the own decision of the discrete spectrum, named also mode of plasma.

Take two line \( \Gamma_{\varepsilon}^{\pm} \), parallel real axis and defending from it on distance \( \varepsilon, \varepsilon > 0 \). The value \( \varepsilon \) shall choose so small that all zeroes to dispersion function lay outside of narrow band, concluded between direct \( \Gamma_{\varepsilon}^{+} \) and \( \Gamma_{\varepsilon}^{-} \).

According to principle of the argument difference between number of the zeroes and number pole to dispersion function is an incrementation of its logarithm:

\[ N - P = \frac{1}{2\pi i} \left[ \int_{\Gamma_{\varepsilon}^{+}} + \int_{\Gamma_{\varepsilon}^{-}} \right] d \ln \lambda(z). \quad (3.1) \]

In (3.1) each zero and pole are considered so much once as their multiplicity, the lines \( \Gamma_{\varepsilon}^{+} \) and \( \Gamma_{\varepsilon}^{-} \) are passed accordingly in positive and negative directions. Then the dispersion function in infinitely removed point has a pole of the second order, i.e. \( P = 2 \).
When $\varepsilon \to 0$ from (3.1) we get:

$$N - 2 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\ln \frac{\lambda^+(\mu)}{\lambda^-(\mu)}. \quad (3.2)$$

Integral from (3.2) represented as:

$$\int_{-\infty}^{\infty} d\ln \frac{\lambda^+(\mu)}{\lambda^-(\mu)} = \int_{0}^{\infty} d\ln \frac{\lambda^+(\mu)}{\lambda^-(\mu)} + \int_{-\infty}^{0} d\ln \frac{\lambda^+(\mu)}{\lambda^-(\mu)}.$$

In the second integral we shall do change the variable: $\tau \to -\tau$, we have:

$$\lambda^+(-\tau) = \lambda^-(\tau), \quad \lambda^-(\tau) = \lambda^+(\tau).$$

Consequently, second integral is first. Really,

$$\int_{0}^{\infty} d\ln \frac{\lambda^+(\mu)}{\lambda^-(\mu)} = - \int_{0}^{\infty} d\ln \frac{\lambda^+(-\mu)}{\lambda^-(-\mu)} = - \int_{0}^{\infty} d\ln \frac{\lambda^-(\mu)}{\lambda^+(\mu)} = \int_{0}^{\infty} d\ln \frac{\lambda^+(\mu)}{\lambda^-(\mu)}.$$

Thereby,

$$N - 2 = \frac{1}{\pi i} \int_{0}^{\infty} d\ln \frac{\lambda^+(\mu)}{\lambda^-(\mu)}. \quad (3.3)$$

On complex plane we consider line $\Gamma : z = G(t), \ 0 \leq t \leq +\infty$, where $G(t) = \frac{\lambda^+(t)}{\lambda^-(t)}$. Not difficult check that

$$G(0) = 1, \quad \lim_{t \to +\infty} G(t) = 1.$$

These equality mean that curve $\Gamma$ is closed: it comes out of points $z = 1$ and ends in this point. According to (3.3):

$$N - 2 = \frac{1}{\pi i} \left[ \ln |G(\tau)| + i \arg G(\tau) \right]_{0}^{+\infty} = \frac{1}{\pi} \left[ \arg G(\tau) \right]_{0}^{+\infty}.$$

Thence we get:

$$N - 2 = \frac{1}{\pi} \left[ \arg G(t) \right]_{0}^{+\infty} = 2 \varkappa(G), \quad (3.4)$$

where $\varkappa = \varkappa(G)$ is the index of the problem, $G(t)$ is the number of revolution the curve $\Gamma$ comparatively begin coordinates, made in positive direction.

From (3.4) we see

$$N = 2 + \frac{1}{\pi} \left[ \arg G(+\infty) - \arg G(0) \right] = 2 + \frac{1}{\pi} \arg G(+\infty), \quad (3.5)$$
since \( \arg G(0) = 0 \).

\( \arg G(t) \) is the regular branch of the argument, fixed in zero condition: \( \arg G(0) = 0 \), and determined in cut complex plane with cut along positive part of real axis. This branch complies with the main by importance of the argument.

We present the dispersion function in form (the displacement current is absent):

\[
\lambda(z) = -ic \left( \frac{i}{c} + \frac{z^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-\mu^2)}{\mu - z} d\mu \right) = -ic \omega(z),
\]

where

\[
\omega(z) = \delta + \frac{z^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-\mu^2)}{\mu - z} d\mu, \quad \delta = \delta_1 + \frac{i}{c}.
\] (3.6)

According to (3.6) border values of the function \( \omega(z) \) overhand and from below on real axis are: \( \omega^\pm(\mu) = \omega(\mu) \pm i\sqrt{\pi} \mu^3 \exp(-\mu^2) \).

Then, the \( G(\mu) \) is equal:

\[
G(\mu) = \frac{\delta_1 - p(\mu) + i[\delta_2 + q(\mu)]}{\delta_1 - p(\mu) + i[\delta_2 - q(\mu)]},
\] (3.7)

Where

\[
\delta = \delta_1 + i\delta_2 = -\frac{4c^2}{\pi v_T^2 \omega_p^2} \cdot \frac{(\omega + i\nu)^3}{\omega},
\]

whence

\[
\delta_1 = -\frac{\omega^3 - 3\omega^2 \nu^2}{v_T^2 \omega_p^2}, \quad \delta_2 = -\frac{3\omega^2 \nu - \nu^3}{v_T^2 \omega_p^2}, \quad v_c = \sqrt{\frac{\pi v_T}{4c}}.
\]

Select in equality (3.7) for function \( G(\mu) \) real and imaginary part: \( G(\mu) = G_1(\mu) + iG_2(\mu) \). We have

\[
G_1(\mu) = \frac{[\delta_1 - p(\mu)]^2 + [\delta_2 - q^2(\mu)]}{[\delta_1 - p(\mu)]^2 + [\delta_2 - q(\mu)]^2},
\]

\[
G_2(\mu) = \frac{2q(\mu) [\delta_1 - p(\mu)]}{[\delta_1 - p(\mu)]^2 + [\delta_2 - q(\mu)]^2}.
\]

On complex \( \delta \)-planes shall enter curve \( \Lambda \):

\[
\Lambda: \quad \delta_1 = p(\mu), \quad \delta_2 = \pm q(\mu), \quad 0 \leq \mu \leq +\infty.
\]
Entrails the curve $\Lambda$ shall mark through $\Delta^+$, and $\Delta^-$ is the external of curve $\Lambda$. The domains $\Delta^+$ and $\Delta^-$ are unlimited (fig. 5.1).

Possible prove that:

1) if $\delta \in \Delta^+$, that $\arg G(+\infty) = 2\pi$ (the curve $\Lambda$ does one turn in positive direction comparatively begin coordinates),

2) if $\delta \in \Delta^-$, that $\arg G(+\infty) = 0$ (the curve $\Lambda$ does not cover begin coordinates).

![Fig.1. The domain $\Delta^\pm$ on $\delta$-plane, $\delta = (\delta_1, \delta_2)$.](image)

According to (3.5) we get

1) if $\delta \in \Delta^+$, then $N = 4$ – dispersion function has four zeroes, and

2) if $\delta \in \Delta^-$, then $N = 2$ dispersion function has two zeroes.

Go from $\delta$-planes to planes $(\omega_1, \nu_1)$, where $\omega_1 = \gamma/v_c$, $\nu_1 = \varepsilon/v_c$, and $\gamma = \omega/\omega_p$, $\varepsilon = \nu/\omega_p$. Find images the unlimited domain $\Delta^\pm$ and curve $\Lambda$ under such image.

Present the parametric equations crooked $\Lambda$ in form

$$L_1: \quad -\omega_1^2 + 3\nu_1^2 = p(\mu), \quad -3\omega_1\nu_1 + \frac{\nu_1^3}{\omega_1} = \pm q(\mu). \quad (3.8)$$

From the first equation from (3.8) find:

$$\omega_1 = \pm \sqrt{3\nu_1^2 - p(\mu)}. \quad (3.9)$$

Substitute (3.9) in the second equation from (3.8). We Get:

$$\nu_1^3 - 3\nu_1(3\nu_1^2 - p(\mu)) = \pm q(\mu)\sqrt{3\nu_1^2 - p(\mu)}.$$
Fig. 2. Unlimited domain $D_i^+$ on $(\omega_1, \nu_1)$-plane. Border of the domain $D_i^+$ is the curve $L_1 = \partial D_i^+$.

Involve this equation in square:

$$64\nu_1^6 - 48\nu_1^4p(\mu) + 3\nu_1^2[3p^2(\mu) - q^2(\mu)] + q^2(\mu)p(\mu) = 0. \quad (3.10)$$

Under each fixed $\mu \in [0, +\infty]$ the equation (3.10) has a single zero, which we mark through $Y(\mu)$. Ensemble of all such zeroes forms is function $\nu_1 = Y(\mu), \ 0 \leq \mu \leq +\infty$.

Hence we find the first parametric equation of curve $L_1$:

$$\varepsilon = v_cY(\mu), \quad 0 \leq \mu \leq +\infty. \quad (3.11)$$

Substituting (3.11) in the second from equations (3.8), find:

$$\omega_1 = \sqrt{3Y^2(\mu) - p(\mu)}, \quad 0 \leq \mu \leq +\infty. \quad (3.12)$$

According to (3.11) and (3.12) we get parametric equations curve $L_1$ on planes $(\omega_1, \nu_1)$:

$$L_1 : \omega_1 = \sqrt{3Y^2(\mu) - p(\mu)}, \nu_1 = Y(\mu), \ 0 \leq \mu \leq +\infty.$$

Go from plane $(\omega_1, \nu_1)$ to planes parameter $(\gamma, \varepsilon)$. Present the parametric equations crooked $\Lambda$ in form:

$$-\gamma^2 + 3\varepsilon^2 = v_c^2p(\mu), \quad -3\gamma\varepsilon + \frac{\varepsilon^3}{\gamma} = \pm v_cq(\mu). \quad (3.13)$$
where \[ \gamma = \frac{\omega}{\omega_p}, \quad \varepsilon = \frac{\varepsilon}{\omega_p}. \]

From the first equation from (3.13) find:
\[ \gamma = \pm \sqrt{3\varepsilon^2 - v_c^2p(\mu)}. \tag{3.14} \]

Substitute (3.14) in the second equation from (3.13). We Get:
\[ \varepsilon^3 - 3\varepsilon(3\varepsilon^2 - v_c^2p(\mu)) = \pm v_c^2q(\mu)\sqrt{3\varepsilon^2 - v_c^2p(\mu)}. \]

Involve this equation in square:
\[ 64 \left( \frac{\varepsilon}{v_c} \right)^6 - 48 \left( \frac{\varepsilon}{v_c} \right)^4 p(\mu) + 3 \left( \frac{\varepsilon}{v_c} \right)^2 \left[ 3p^2(\mu) - q^2(\mu) \right] + q^2(\mu)p(\mu) = 0. \tag{3.15} \]

Under each fixed \( \mu \in [0, +\infty] \) the equation (3.15) has a single zero, which we mark through \( Y(\mu) \). Ensemble of all such zeroes forms is function
\[ \frac{\varepsilon}{v_c} = Y(\mu), \quad 0 \leq \mu \leq +\infty. \]

Consequently, we have found one parametric equation of curve \( L \), being image of curve \( \Lambda \):
\[ \varepsilon = v_cY(\mu), \quad 0 \leq \mu \leq +\infty. \tag{3.16} \]

Substituting (3.16) in the second from equations (3.13), we find:
\[ \gamma = v_c\sqrt{3Y^2(\mu) - p(\mu)}, \quad 0 \leq \mu \leq +\infty. \tag{3.17} \]

According (3.16) and (3.17) on \((\gamma, \varepsilon)\)-plane we have the parametric equations of curve \( L(v_c) \):
\[ L(v_c) : \quad \gamma = v_c\sqrt{3Y^2(\mu) - p(\mu)}, \quad \varepsilon = v_cY(\mu), \quad 0 \leq \mu \leq +\infty. \]

We find the general solution to system (1.3), (1.4) in the form of an expansion in the eigenfunctions of the discrete and continuous spectrum; this solution automatically satisfies the boundary conditions at infinity:
\[
\begin{align*}
  h(x, \mu) &= \frac{a}{\sqrt{\pi}} \sum_{k=0}^{1} \frac{A_k\eta_k^3}{\eta_k - \mu} \exp \left( -\eta_k^2 - \frac{z_0x}{\eta_k} \right) + \\
  &+ \int_{0}^{\infty} \exp \left( -\frac{z_0x}{\eta} \right) A(\eta)\Phi(\eta, \mu) \, d\eta, \quad (2.10)
\end{align*}
\]
Fig. 3. The unlimited domain $D^\pm$ on the $(\gamma, \varepsilon)$-plane. For curves 1, 2, 3 $v_c = 0.0006$ ($T = 1000^\circ K$), 0.001 ($T = 3000^\circ K$), 0.013 ($T = 5000^\circ K$) respectively.

\[ e(x) = \frac{az_0}{\sqrt{\pi}} \sum_{k=0}^{1} A_k \eta_k^2 \exp \left( -\eta_k^2 - \frac{z_0 x}{\eta_k} \right) + \frac{az_0}{\sqrt{\pi}} \int_{0}^{\infty} \exp \left( -\eta^2 - \frac{z_0 x}{\eta} \right) \eta^2 A(\eta) \, d\eta. \] (2.11)

Here $A_k (k = 0, 1)$ are the unknown coefficients corresponding to the discrete spectrum, $A_1 = 0$, if $\delta \in D^-$; and $A(\eta)$ is an unknown function called the coefficient of the continuous spectrum $\text{Re} \left( \frac{z}{\eta_k} \right) > 0 \ (k = 0, 1)$, $\text{Re} \ z_0 = 1$.

4. The index of the problem equals zero. We consider the case when the problem index is equal to zero. Substituting expressions (2.10) and (2.11) into (1.5), we obtain the singular integral equation with the Cauchy kernel:

\[ \frac{a}{\sqrt{\pi}} \int_{0}^{\infty} \frac{\eta^3 \exp(-\eta^2) A(\eta)}{\eta - \mu} \, d\eta + \lambda(\mu) A(\mu) + a \varphi(\mu) = 0, \quad 0 < \mu < \infty, \] (3.1)

where

\[ \varphi(\mu) = \frac{A_0 \eta_0^3 \exp(-\eta_0^2)}{\sqrt{\pi}} \left( \eta_0 - \mu \right). \]
We define the auxiliary function in the complex plane

\[ N(z) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\eta^2 \exp(-\eta^2) A(\eta)}{\eta - z} d\eta. \]

Then the equation (3.1) is possible to transform to the Riemann—Hilbert boundary value problem:

\[ \lambda^+(\mu) \left[ N^+(\mu) + \varphi(\mu) \right] = \lambda^-(\mu) \left[ N^-(\mu) + \varphi(\mu) \right], \quad 0 < \mu < \infty. \quad (3.2) \]

We consider the Riemann—Hilbert boundary value problem on the half-axis:

\[ \frac{X^+(\mu)}{X^-(\mu)} = G(\mu), \quad 0 < \mu < +\infty. \]

where

\[ G(\mu) = \frac{\lambda^+(\mu)}{\lambda^-(\mu)}, \]

\( G(\mu) \) is the coefficient of the boundary value problem.

The index in this problem is \( \psi(G) = 0 \), then the solution of problem (3.2) has the form [4]:

\[ X(z) = \exp V(z), \quad V(z) = \frac{1}{2\pi i} \int_0^\infty \ln G(\tau) \frac{d\tau}{\tau - z}. \quad (3.3) \]

We use (3.3) to reduce problem (3.2) for determining an analytic function \( N(z) \) from its zero jump problem on the cut:

\[ X^+(\mu) \left[ N^+(\mu) + \varphi(\mu) \right] = X^-(\mu) \left[ N^-(\mu) + \varphi(\mu) \right], \quad 0 < \mu < \infty. \quad (3.4) \]

A general solution of (3.4) is given by the formula

\[ N(z) = \frac{A_0}{\sqrt{\pi}} \frac{\eta_0^3 \exp(-\eta_0^2)}{z - \eta_0} + \frac{C_0}{(z - \eta_0)X(\eta_0)}. \quad (3.5) \]

where \( C_0 \) is an arbitrary constant.

Let’s eliminate poles at the decision (3.5), we have:

\[ C_0 = -\frac{A_0}{\sqrt{\pi}} \exp(-\eta_0^2) \eta_0^3 X(\eta_0). \quad (3.6) \]
Employing the Sokhotsky formula for difference from the boundary values of function $N(z)$, we find the coefficient of the continuous spectrum:

$$2\sqrt{\pi} i \eta^3 \exp(-\eta^2) A(\eta) = \frac{C_0}{\eta - \eta_0} \left[ \frac{1}{X^+(\eta)} - \frac{1}{X^-(\eta)} \right].$$

From the definition of the auxiliary function $N(z)$ we have:

$$N(0) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \eta^2 \exp(-\eta^2) A(\eta) d\eta. \quad (3.7)$$

Considering equality (3.7), we write (2.11) in form:

$$-\frac{C_0}{\eta_0 X(\eta_0)} + N(0) = \frac{1}{az_0}.$$

From general solution (3.5) the constant $C_0$ is found:

$$C_0 = -\frac{\eta_0}{az_0} X(0). \quad (3.8)$$

From comparison (3.6) and (3.8) we get the unknown coefficients corresponding to the discrete spectrum:

$$A_0 = \frac{\sqrt{\pi} X(0)}{az_0 X(\eta_0) \eta_0^2 \exp(-\eta_0^2)}.$$

The derivative of expression (2.11) leads us to equality

$$e^{\prime}(x) = \frac{C_0}{\eta_0^2 X(\eta_0)} \exp(-\frac{z_0 x}{\eta_0}) - \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \exp(-\eta^2 - \frac{z_0 x}{\eta}) \eta A(\eta) d\eta.$$

From here we find the electric field derivative on plasma boundary $x = 0$:

$$e^{\prime}(0) = \frac{C_0}{\eta_0 X(\eta_0)} - \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \exp(-\eta^2) \eta A(\eta) d\eta. \quad (3.9)$$

For calculation of integral from the expression (3.9) we shall use a derivative of auxiliary function

$$N^{\prime}(\mu) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{\eta^3 \exp(-\eta^2) A(\eta)}{(\eta - \mu)^2} d\eta. \quad (3.10)$$
Using the expression (3.10) we find the derivative in zero:

\[ N'(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \eta \exp(-\eta^2) A(\eta) \, d\eta. \]  

(3.11)

Using expression for coefficient of the continuous spectrum by (3.11) we find:

\[ N'(z) = -\frac{C_0}{(z - \eta_0)^2} \left( \frac{1}{X(z)} - \frac{1}{X(\eta_0)} \right) - \frac{C_0}{z - \eta_0} \frac{X'(z)}{X^2(z)}, \]

whence

\[ N'(0) = -\frac{C_0}{\eta_0} \left( \frac{1}{X(0)} - \frac{1}{X(\eta_0)} \right) - \frac{C_0}{\eta_0} \frac{X'(0)}{X(0)} \]  

(3.12)

Substituting (3.12) into (3.9) we obtain the equation:

\[ e'(0) = \frac{a z_0^2 C_0}{\eta_0 X(0)} \left[ \frac{1}{\eta_0} - \frac{X'(0)}{X(0)} \right] = z_0 \left[ \frac{X'(0)}{X(0)} - \frac{1}{\eta_0} \right]. \]  

(3.13)

In [2] the following formula for the impedance is given:

\[ Z = \frac{4\pi i \omega l}{c^2 z_0} \cdot \frac{e(0)}{e'(0)}. \]  

(3.14)

Substituting (3.13) into (3.14), we obtain the exact expression for the impedance:

\[ Z = \frac{4\pi i \omega l}{c^2 z_0} \left[ \frac{X'(0)}{X(0)} - \frac{1}{\eta_0} \right]^{-1}. \]  

(3.15)

The equation (3.15) is the exact expression for calculation of the surface impedance. This equation expresses value of the impedance in terms of the function \( X(z) \) and zeros of the dispersion function of the problem.

5. The index of the problem equals one. Now we consider case when the problem index is equal one, i.e. \( \kappa(G) = 1 \).

Substituting expressions (2.10) and (2.11) into (1.5), we obtain the singular integral equation with the Cauchy kernel:

\[ a\varphi(\mu) + \frac{a}{\sqrt{\pi}} \int_0^\infty \frac{\eta^3 \exp(-\eta^2) A(\eta)}{\eta - \mu} \, d\eta + \lambda(\mu) A(\mu) = 0, \quad 0 < \mu < \infty, \quad (4.1) \]

where

\[ \varphi(\mu) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^1 \frac{A_k \eta_k^3 \exp(-\eta_k^2)}{\eta_k - \mu}. \]
We define the auxiliary function in the complex plane:

$$N(z) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{\eta^3 \exp(-\eta^2) A(\eta)}{\eta - z} d\eta,$$

for which boundary values from above and from below on the valid axis formulas of Sokhotskii are carried out:

$$N^+(\mu) - N^-(\mu) = 2\sqrt{\pi} i \mu^3 \exp(-\mu^2) A(\mu).$$

Using the boundary values $N^\pm(\mu)$ and $\lambda^\pm(\mu)$, we pass from the singular equation (4.1) to the Riemann — Hilbert boundary value problem:

$$\lambda^+(\mu)[N^+(\mu) + \varphi(\mu)] = \lambda^-(\mu)[N^-(\mu) + \varphi(\mu)], \quad 0 < \mu < \infty. \quad (4.2)$$

Let’s solve a corresponding Riemann — Hilbert boundary value problem:

$$\frac{X^+(\mu)}{X^-(\mu)} = G(\mu), \quad 0 < \mu < +\infty, \quad (4.3)$$

where

$$G(\mu) = \frac{\lambda^+(\mu)}{\lambda^-(\mu)}, \quad 0 < \mu < \infty.$$

Since problem index is equal one $\kappa(G) = 1$ as the solution of (4.3) we take function

$$X(z) = \frac{1}{z} \exp V(z), \quad V(z) = \frac{1}{2\pi i} \int_{0}^{\infty} \frac{\ln G(\tau) - 2\pi i}{\tau - z} d\tau. \quad (4.4)$$

Using (4.4) we transform the boundary value problem (4.2) to determining an analytic function from its zero jump problem on the cut:

$$X^+(\mu)[N^+(\mu) + \varphi(\mu)] = X^-(\mu)[N^-(\mu) + \varphi(\mu)], \quad 0 < \mu < \infty. \quad (4.5)$$

The general solution of (4.5) is following

$$N(z) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{1} \frac{A_k \eta_k^3 \exp(-\eta_k^2)}{z - \eta_k} + \sum_{k=0}^{1} \frac{C_k}{(z - \eta_k)X(z)}, \quad (4.6)$$

where $C_0$, $C_1$ is the arbitrary values.

Since $N(\infty) = 0$, from here we receive that $C_0 + C_1 = 0$. 

Let’s eliminate poles at the decision (4.6), we get:

\[ C_0 = -\frac{A_0}{\sqrt{\pi}} \eta_0^3 \exp(-\eta_0^2)X(\eta_0), \quad C_1 = -\frac{A_1}{\sqrt{\pi}} \eta_1^3 \exp(-\eta_1^2)X(\eta_1). \] (4.7)

Substituting the general solution (4.6) in the Sokhotskii formula for (4.6) we find unknown function called the coefficient of the continuous spectrum:

\[ \eta^3 \exp(-\eta^2)A(\eta) = \frac{1}{2\sqrt{\pi}i} \left[ \frac{1}{X^+(\eta)} - \frac{1}{X^-(\eta)} \right] \sum_{k=0}^{\infty} \frac{C_k}{\eta - \eta_k}. \] (4.8)

Using (4.7) we reduce (2.10) to the form:

\[-\frac{C_0}{\eta_0 X(\eta_0)} - \frac{C_1}{\eta_1 X(\eta_1)} + \frac{1}{\sqrt{\pi}} \int_0^\infty \eta^2 \exp(-\eta^2)A(\eta) d\eta = \frac{1}{az_0}, \] (4.9)

From equation (4.9) we will find last integral using definition of function \( N(z) \):

\[ N(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \eta^2 \exp(-\eta^2)A(\eta) d\eta. \]

Using the common decision (4.6), we found \( N(0) \):

\[ N(0) = C_0 \left[ \frac{1}{\eta_0 X(\eta_0)} - \frac{1}{\eta_1 X(\eta_1)} - \frac{1}{\eta_0 X(0)} + \frac{1}{\eta_1 X(0)} \right]. \]

We substitute this expression into the equation (4.9):

\[ C_0 = \frac{X(0)}{az_0 \left( \frac{1}{\eta_1} - \frac{1}{\eta_0} \right)}. \] (4.10)

Differentiating expansion for electric field (2.11) and using (4.7), where \( x = 0 \) we found:

\[ e'(0) = \frac{C_0 az_0^2}{\eta_0^2 X(\eta_0)} - \frac{C_0 az_0^2}{\eta_1^2 X(\eta_1)} - \frac{az_0^2}{\sqrt{\pi}} \int_0^\infty \eta \exp(-\eta^2)A(\eta) d\eta. \] (4.11)

Integral from (4.11) is the auxiliary function derivative \( N'(0) \):

\[ N'(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \eta \exp(-\eta^2)A(\eta) d\eta. \]
The common solution (4.6) we present in form:

\[
N(z) = \frac{1}{z - \eta_0} \left[ \frac{C_0}{X(z)} + \frac{1}{\sqrt{\pi}} A_0 \eta_0^3 \exp(-\eta_0^2) \right] + \\
\frac{1}{z - \eta_1} \left[ \frac{C_1}{X(z)} + \frac{1}{\sqrt{\pi}} A_1 \eta_1^3 \exp(-\eta_1^2) \right].
\]

Derivative of this expression is equal:

\[
N'(z) = -\frac{1}{(z - \eta_0)^2} \left[ \frac{1}{X(z)} - \frac{1}{X(\eta_0)} \right] - \frac{C_0}{z - \eta_0} \frac{X'(z)}{X(z)} + \\
\frac{C_0}{(z - \eta_1)^2} \left[ \frac{1}{X(z)} - \frac{1}{X(\eta_1)} \right] + \frac{C_0}{z - \eta_1} \frac{X'(z)}{X(z)}.
\]

Hence, we obtain:

\[
N'(0) = -\frac{1}{\eta_0^2} \left[ \frac{1}{X(0)} - \frac{1}{X(\eta_0)} \right] + \frac{C_0}{\eta_0} \frac{X'(0)}{X(0)} + \\
\frac{C_0}{\eta_1^2} \left[ \frac{1}{X(0)} - \frac{1}{X(\eta_1)} \right] - \frac{C_0}{\eta_1} \frac{X'(0)}{X(0)}.
\]

Using this equality for the electric field derivative we have:

\[
e'(0) = \frac{C_0 a z_0^2}{X(0)} \left[ \left( \frac{1}{\eta_0} - \frac{1}{\eta_1} \right) \left( \frac{1}{\eta_0} + \frac{1}{\eta_1} \right) + \frac{X'(0)}{X(0)} \left( \frac{1}{\eta_0} - \frac{1}{\eta_1} \right) \right].
\]

Therefore according to (4.10) we receive that

\[
e'(0) = z_0 \left[ \frac{X'(0)}{X(0)} - \frac{1}{\eta_0} - \frac{1}{\eta_1} \right].
\]

Hence, expression for the impedance is equal:

\[
Z = \frac{4\pi i \omega l}{c^2 z_0} \cdot \left[ \frac{X'(0)}{X(0)} - \frac{1}{\eta_0} - \frac{1}{\eta_1} \right]^{-1}.
\]

This equation expresses value of the impedance in terms of factor function \(X(z)\) and zeros of dispersive function of the problem.

CONCLUSIONS
A closed form solution of a system of two equations Boltzmann — Vlasov and Maxwell arising in the skin effect problem for a Maxwillian plasmas is presented. The kinetic Boltzmann — Vlasov equation with a \( \tau \)-model collision operator is considered. Case’s \cite{7} method and Riemann — Hilbert boundary value problem \cite{8} with the coefficient \( G(\mu) = \frac{\lambda^{+}(\mu)}{\lambda^{-}(\mu)} \), where \( \lambda(z) \) is the dispersion function of the problem. The discrete fashion is obtained. Consider the domain \( D^{+} \), that if the frequency lies in this domain, there are four discrete solution, and if the frequency is outside this domain, there are two discrete solutions. The exact solution of the initial boundary value problem with diffusion scattering of electrons from plasma boundary is constructed. The exact formula for calculation of the surface impedance is obtained.

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