ENUMERATION OF TREE-LIKE MAPS WITH ARBITRARY NUMBER OF VERTICES

AARON CHUN SHING CHAN

ABSTRACT. This paper provides the generating series for the embedding of tree-like graphs of arbitrary number of vertices, according to their genus. It applies and extends the techniques of Chan [3], where it was used to give an alternate proof of the Goulden and Slofstra formula. Furthermore, this greatly generalizes the famous Harer-Zagier formula [2], which computes the Euler characteristic of the moduli space of curves, and is equivalent to the computation of one vertex maps.

1. Introduction

Let $n$ and $k$ be integers such that $0 \leq n, k$. We use $[n]$ to denote the set $\{1, \ldots, n\}$, $[n]^k$ to denote the Cartesian product of $[n]$ with itself $k$ times, and $[n;k]$ to denote the set of all $k$-subsets of $[n]$. If $S$ is a set of even cardinality, then a pairing $\mu$ of $S$ is a partition of $S$ into disjoint subsets of size 2. Next, a partial pairing $T$ of a set $S$ is a pairing on a subset $S' \subseteq S$ of even cardinality. If $|S'| = 2k$, then $T$ is called a $k$-partial pairing of $S$. The set $S'$ is called the support of the partial pairing $T$. Finally, the set of all $k$-partial pairings of $[n]$ is denoted as $T_{n,k}$, which has cardinality $|T_{n,k}| = \binom{n}{2k}(2k-1)!$, where $(2k-1)! = \prod_{i=1}^{k} (2j-1)$ is the double factorial, with the convention that $-1!! = 1$.

Let $p$ and $n$ be positive integers. We use $[p]^n$ to denote the set $\{1^2, 2^2, \ldots, p^2\}$, whose elements $i^2$, $i = 1, \ldots, p$, are regarded as a labelled version of the integer $i$, labelled by the “$i$” in the superscript position. Then, suppose $p = (p_1, \ldots, p_n)$ is a vector of length $n$ of positive integers, we let $[p_1, \ldots, p_n]$ to be the set $[p_1]^1 \cup \cdots \cup [p_n]^n$. For example, $[3, 5, 2]$ is the set $\{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2\}$. Furthermore, if $p_1 + \cdots + p_n$ is even, then the set of all pairings of $[p_1, \ldots, p_n]$ is denoted as $P_{p_1, \ldots, p_n}$. Now, if $\mu$ is a pairing of $[p_1, \ldots, p_n]$, then a pair $\{x^2, y^2\}$ in $\mu$ is a mixed pair if $i \neq k$, and a non-mixed pair otherwise. To describe the number of mixed and non-mixed pairs in a pairing $\mu$, we introduce the parameters $q$ and $s$. Let $q = (q_1, \ldots, q_n)$ be a vector of length $n$, and $s = (s_{1,2}, s_{1,3}, \ldots, s_{n-1,n})$ be an $n \times n$ strictly upper triangular matrix, where for ease of notation we let $s_{i,k} = s_{k,i}$ for $i > k$ and $s_i = \sum_{i \neq k} s_{i,k}$. If $p_i = 2q_i + s_i$ is positive for $1 \leq i \leq n$, we define $P_n(q,s) \subseteq P_{p_1, \ldots, p_n}$ to be the subset of the pairing such that for $\mu \in P_n(q,s)$, $\mu$ has $q_i$ non-mixed pairs of the form $\{x^2, y^2\}$ and $s_{i,k}$ mixed pairs of the form $\{x^2, y^2\}$. When convenient, we will sometimes treat $s$ as a vector of length $\frac{n(n-1)}{2}$. Furthermore, the support graph of $s$ is the graph $G$ with the vertex set $[n]$, such that $\{i, k\}$ is an edge of $G$ if and only if $s_{i,k} > 0$.

Let $\gamma_{p_1, \ldots, p_n}$ be the canonical cycle permutation of $P_{p_1, \ldots, p_n}$ given by $\gamma_{p_1, \ldots, p_n} = (1^2, \ldots, p_1^2) \cdots (1^2, \ldots, p_n^2)$.

For $L \geq 1$, we define $A_{n,L}^{(q,s)} \subseteq P_n(q,s)$ to be the subset of pairings such that for $\mu \in A_{n,L}^{(q,s)}$, $\mu \gamma_{p_1, \ldots, p_n}$ has exactly $L$ cycles, and let $a_{n,L}^{(q,s)} = |A_{n,L}^{(q,s)}|$. Our result can be stated as follows.

**Theorem 1.** Let $n \geq 1$, $q = (q_1, \ldots, q_n)$ and $s = (s_{1,2}, s_{1,3}, \ldots, s_{n-1,n})$ be vectors of non-negative integers, and suppose that the support graph $G$ of $s$ is a tree with edges $e_1, \ldots, e_{n-1}$. Then, the generating series $A_{n,L}^{(q,s)}(x) = \sum_{L \geq 1} a_{n,L}^{(q,s)} x^L$ satisfies

$$A_{n,L}^{(q,s)}(K) = \sum_{t=0}^{n} \prod_{i=1}^{n} \frac{(2q_i + s_i)!}{2^t_i t_i! (s_i + q_i - t_i)!} \cdot v_{n,K,q-t+1}^{(s)}$$

where $v_{n,K,q-t+1}^{(s)}$ is the number of ways to distribute $n$ distinct objects into $K$ boxes with $q-t+1$.
for all $K \geq 1$, where

$$v_{n,K,R}^{(s)} = \sum_{A_{n-1}=0}^{\min(s_{n-1},K)-1} \cdots \sum_{A_1=0}^{\min(s_1,K)-1} \prod_{j=1}^{n-1} \frac{(K - A_{e_j} - 1)!}{(K + s_{e_j} - A_{e_j} - 1)!} \times \prod_{i=1}^{n} \frac{(K + \sum_{k<i} (s_{i,k} - A_{i,k} - 1))! (R_i - 1 + \sum_{k<i} s_{i,k})!}{(K - R_i - \sum_{k<i} A_{i,k})! (R_i + \sum_{k<i} (s_{i,k} - 1))!}$$

Furthermore, for fixed $n$, $q$, and $s$, this expression can be written as a polynomial in $K$.

In this expression, the sum $\sum_{k<i}$ is over all indices $k$ that are adjacent to $i$ in the support graph of $s$. Furthermore, for each edge $e_j = \{i,k\}$, the summation variable $A_{e_j}$ is equivalently written as $A_{i,k}$ and $A_{k,i}$ in parts of the expression. Now, the fact that this expression can be written as a polynomial in $K$ for fixed parameters $n$, $q$, and $s$ means that we can substitute $K = x$ into our expression for $A_n^{(q,s)}(K)$ to obtain $A_n^{(q,s)}(x)$.

In the language of enumerating maps, this generating series counts the number of combinatorial maps with $n$ vertices and $L$ faces, such that there are $q_i$ loop edges incident to vertex $i$, and $e_{i,j}$ edges between vertices $i$ and $j$. Furthermore, the combinatorial maps counted in this series are connected if and only if the support graph of $s$ is connected. A survey on the relationship between maps and the products of permutations can be found in [11].

This theorem generalizes a number of theorems already existing in the literature. In particular, the $n = 1$ case of our theorem is the Harer-Zagier formula for computing the Euler characteristic of the moduli space of curves, which can be written as follows.

**Theorem 2. (Harer-Zagier [6])** Let $q$ be a positive integer, and $A_L^{(q)}$ be the subset of pairings of $\mathcal{P}_2q$ such that for $\mu \in A_L^{(q)}$, $\mu_{\gamma_{2q}^{-1}}$ has exactly $L$ cycles. If we let $a_L^{(q)} = |A_L^{(q)}|$, then the generating series for $a_L^{(q)}$ is given by

$$A^{(q)}(x) = (2q - 1)!! \sum_{k \geq 1} 2^{k-1} \binom{q}{k} \binom{x}{k} \left( \frac{q}{k-1} \right)$$

There are numerous proofs of this formula in the literature, both algebraic and combinatorial. A selection of the proofs can be found in the papers by Goulden and Nica [4], Itzykson and Zuber [7], Jackson [8], Kerov [9], Kontsevich [10], Lass [12], Penner [13], and Zagier [14]. The original proof of Harer-Zagier uses matrix integration, and there are numerous other algebraic proofs for this same result. Some subsequent proofs used purely combinatorial approaches, such as the use of Eulerian tours by Lass, and the use of trees by Goulden and Nica. To reduce [Theorem 1] to the Harer-Zagier formula, we can simply take $q_1 = q$, $s$ to be empty, and then reversing the sum with $t = q - k - 1$.

The $n = 2$ case of our theorem was proved by Goulden and Slofstra [5] using a combinatorial technique that we will extend in this paper.

**Theorem 3. (Goulden-Slofstra [5])** Let $q_1$ and $q_2$ be non-negative integers, and $s$ be a positive integer. Let $A_L^{(q_1,q_2;s)}$ be the subset of pairings of $\mathcal{P}^{(q_1,q_2;s)}$ such that for $\mu \in A_L^{(q_1,q_2;s)}$, $\mu_{\gamma_{2q_1+s,2q_2+s}^{-1}}$ has exactly $L$ cycles. If we let $a_L^{(q_1,q_2;s)} = |A_L^{(q_1,q_2;s)}|$, then the generating series for $a_L^{(q_1,q_2;s)}$ is given by

$$A^{(q_1,q_2;s)}(x) = p_1!p_2! \sum_{k=1}^{d+1} \sum_{i=0}^{\left\lfloor \frac{p_1}{i} \right\rfloor} \sum_{j=0}^{\left\lfloor \frac{p_2}{j} \right\rfloor} \frac{1}{(d - i - j)! (d - i - j)!} \binom{x}{k} \binom{d - i - j}{k - 1} \Delta_k^{(q_1,q_2;s)}$$

where $p_1 = 2q_1 + s$, $p_2 = 2q_2 + s$, $d = q_1 + q_2 + s$, and

$$\Delta_k^{(q_1,q_2;s)} = \begin{pmatrix} k-1 & k-1 & k-1 \\ q_1-i & q_2-j & q_1+s-i \end{pmatrix} \begin{pmatrix} k-1 \\ q_1+i \\ q_2+s-j \end{pmatrix}$$

In this expression, $p_1$ and $p_2$ are the degrees of vertices 1 and 2, respectively, and $d$ is the total number of pairs in the pairing.
Unlike the $n = 1$ case, the most direct way to show that Theorem 1 can be reduced to Theorem 3 is to delve into the combinatorial proof itself. By noting the differences in the definitions of vertical arrays between Goulden and Slofstra and our subsequent definitions, we can relate their cardinalities using inclusion-exclusion. Further algebraic manipulations then shows that the two formulas are equivalent. As the proof is rather lengthy, readers interested in the proof can consult [2].

2. Paired Functions and Paired Arrays

For proving our main theorem, we will use a combinatorial object called paired functions, which are related to the paired surjections introduced in Goulden and Slofstra [5]. The difference between the two objects is that we reject the non-empty condition here, which makes our object equivalent to the $K$-colouring cycles used in some of the algebraic techniques in [11]. This brings together the algebraic and combinatorial techniques, as they effectively count the same set of objects.

Definition 4. Let $n, K \geq 1$, $q = (q_1, \ldots, q_n) \geq 0$, $s = (s_{1,2}, s_{1,3}, \ldots, s_{n-1,n}) \geq 0$, and $p_i = 2q_i + \sum_{k \neq i} s_{k,i}$ for $1 \leq i \leq n$. An ordered pair $(\mu, \pi)$ is a paired function if $\mu \in \mathcal{P}_{n}^{(q,s)}$ and $\pi: [p_1, \ldots, p_n] \rightarrow [K]$ is a function satisfying

$$\pi(\mu(v)) = \pi(\gamma_{p_1,p_2,\ldots,p_n}(v)) \quad \text{for all } v \in [p_1, \ldots, p_n]$$

We denote the set of paired functions satisfying the parameters $n, K, q$, and $s$ as $\mathcal{F}_{n,K}^{(q,s)}$, and we let $f_{n,K}^{(q,s)} = \left| \mathcal{F}_{n,K}^{(q,s)} \right|$.

By substituting in $u = \gamma_{p_1,p_2,\ldots,p_n}(v)$, we have $\pi(u) = \pi(\mu_{\gamma_{p_1,p_2,\ldots,p_n}}^{-1}(u))$ for all $u \in [p_1, \ldots, p_n]$. This implies that the cycles of $\mu_{\gamma_{p_1,p_2,\ldots,p_n}}^{-1}$ are preserved by $\pi$. In other words, each of the cycles of $\mu_{\gamma_{p_1,p_2,\ldots,p_n}}^{-1}$ is coloured with one of $K$ colours. Hence, for any given pairing $\mu \in \mathcal{A}_{n,L}^{(q,s)}$, there are $K^L$ functions $\pi: [p_1, \ldots, p_n] \rightarrow [K]$ such that $(\mu, \pi)$ is a paired function. Furthermore, by applying the definition to all pairs $\{x^k, y^k\}$ of $\mu$, we have that $(\mu, \pi)$ is a paired function if and only if

$$(\pi(\mu(y^k)), \pi(\gamma_{p_1,p_2,\ldots,p_n}(x^k))) = (\pi(\gamma_{p_1,p_2,\ldots,p_n}(y^k)), \pi(\mu(x^k)))$$

$$(\pi(x^k), \pi((x + 1)^k)) = (\pi((y + 1)^k), \pi(y^k))$$

(1)

holds for all pairs $\{x^k, y^k\}$ of $\mu$, where addition is done modulo $p_i$ and $p_k$ on the left and right hand side, respectively.

Recall that $a_{n,L}^{(q,s)}$ is the number of pairings $\mu \in \mathcal{P}_{n}^{(q,s)}$ such that $\mu_{\gamma_{p_1,p_2,\ldots,p_n}}^{-1}$ has exactly $L$ cycles. Hence, for each pairing $\mu$, there are $K^L$ functions $\pi$ such that $(\mu, \pi)$ is a paired function. This gives us

(2) $$A_{n,K}^{(q,s)}(K) = \sum_{L \geq 1} a_{n,L}^{(q,s)} K^L = f_{n,K}^{(q,s)}$$

for $K \geq 1$. Therefore, if we can find an expression for $f_{n,K}^{(q,s)}$ that is a polynomial in $K$, we can substitute $K = x$ into that expression to obtain $A_{n,K}^{(q,s)}(x)$.

To represent paired functions, we use a graphical representation introduced in Goulden and Slofstra, called the labelled array. This is an $n \times K$ array of cells arranged in a grid. Each element $x^k$ of $\mu$ is represented as a vertex, where the vertex labelled $x^k$ is placed into cell $(i, j)$ if $\pi(x^k) = j$. The vertices are arranged horizontally within a cell, in increasing order of the labels. Furthermore, for each pair $\{x^k, y^k\}$ in $\mu$, an edge is drawn between their corresponding vertices.
Definition 5. Let \( n, K \geq 1, \mathbf{q} = (q_1, \ldots, q_n) \geq \mathbf{0}, \mathbf{s} = (s_{1,2}, \ldots, s_{n-1,n}) \geq \mathbf{0}, \) and \( \mathbf{R} = (R_1, \ldots, R_n) \in [K]^n \). We define \( \mathcal{P}_n^{(\mathbf{q}, \mathbf{s})} \) to be the set of \textit{paired arrays}, which are arrays of cells and vertices subject to the following conditions.

- A paired array is an array of cells, arranged in \( n \) rows and \( K \) columns.
- Each cell \((i, j)\) contains an ordered list of vertices, arranged left to right, so that row \( i \) contains \( p_i := 2q_i + s_i = 2q_i + \sum_{k<i} s_{k,i} + \sum_{k>i} s_{i,k} \) vertices in total.
- Each vertex \( u \) is paired with exactly one other vertex \( v \), which is called the \textit{partner} of \( u \). Exactly \( 2q_i \) vertices of row \( i \) are paired with other vertices of row \( i \), and for \( i < k \), exactly \( s_{i,k} \) vertices of row \( i \) are paired with vertices of row \( k \). Graphically, the pairings are denoted as edges between vertices.
- Each row \( i \) has exactly \( R_i \) marked cells, which are denoted by marking the cell with a box in its upper right corner.
• A vertex $v$ is critical if it is the rightmost vertex of a cell, and the cell it belongs to is not marked. A pair $\{u, v\}$ that contains a critical vertex is a critical pair.
• A pair of vertices $\{u, v\}$ is a mixed pair if $u$ and $v$ belong to different rows. The vertices $u$ and $v$ are called mixed vertices.
• An object of a paired array refers to either a vertex, or the box used to indicate that a cell is marked. If a cell both contains vertices and a box, the box is to be taken as the rightmost object of the cell.

Generally, we use $\alpha \in \mathcal{PA}_{n,K,R}^{(q,s)}$ to denote a paired array. Before introducing the conditions used in Goulden and Slofstra, we will first introduce a number of useful notations and conventions.

**Convention 6.** For notational convenience, we introduce the following:

- We use calligraphic letters to denote columns or sets of columns. For generic columns or sets of columns, we use the letters $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$.
- For each calligraphic letter, we use the corresponding upper case letter to denote the number of columns in the set. For example, $X = |\mathcal{X}|$.
- For each calligraphic letter, we use the corresponding lower case letter, subscripted by the row number, to denote the total number of vertices in those columns for a given row. For example, $x_i$ is the total number of vertices in row $i$ of the columns of $\mathcal{X}$.
- We generally use $i,j,k,\ell$ as index variables, with $i$ and $k$ for rows, and $j$ and $\ell$ for columns. Furthermore, we use cell $(i, j)$ to denote the cell in row $i$, column $j$ of the array.
- We use $K$ to denote the set of all columns, and $n$ to denote the total number of columns.
- We use $R_i$ to denote the set of columns that are marked in row $i$, and $R_i$ to denote the number of columns that are marked in row $i$.
- We use $F_i$ to denote the set of columns that have at least one vertex in row $i$, and $F_i$ to denote the number of columns that are marked in row $i$.
- We use $w_{i,j}$ to denote the number of vertices in cell $(i, j)$, and $w$ to denote a matrix of $w_{i,j}$ describing the number of vertices in each cell of row $i$.
- We let $s_{i,k} = s_{k,i}$ for $i > k$, and $s_i = \sum_{k \neq i} s_{i,k}$ be the total number of mixed vertices of row $i$. This means that row $i$ contains $p_i = 2q_i + s_i$ vertices.

With these conventions, we are ready to define the two conditions that allow us to create a bijection between labelled arrays and paired arrays.

**Definition 7.** Let $\alpha \in \mathcal{PA}_{n,K,R}^{(q,s)}$ be a paired array.

- $\alpha$ is said to satisfy the balance condition if for each cell $(i, j)$, the number of mixed vertices in cell $(i, j)$ is equal to the number of mixed pairs $\{u, v\}$ such that $u$ is in row $i$ and $v$ is in column $j$ (but not row $i$).
- For each row $i$, the forest condition function $\psi_i : F_i \setminus R_i \mapsto K$ is defined as follows: For each column $j \in F_i \setminus R_i$, if the rightmost vertex $v$ is paired with a vertex $u$ in column $\ell$, then $\psi_i(j) = \ell$. $\alpha$ is said to satisfy the forest condition if for each row $i$, the functional digraph of $\psi_i$ on the vertex set $F_i \cup \psi_i(F_i) \cup R_i$ is a forest with root vertices $R_i$. That is, for each column $j \in F_i \setminus R_i$, there exists some positive integer $t$ such that $\psi_i^t(j) \in R_i$. Note that we always include $R_i$ in the vertex set of the functional digraph of $\psi_i$, regardless of whether they are in the domain or range of $\psi_i$.

A paired array is proper if it satisfies the balance and forest conditions. A paired array is called a canonical array if it is proper and $R = 1$. We denote the set of canonical arrays as $\mathcal{CA}_{n,K}^{(q,s)}$, and we let $c_{n,K}^{(q,s)} = |\mathcal{CA}_{n,K}^{(q,s)}|$. A paired array is called a vertical array if for every pair $\{u, v\}$, $u$ and $v$ are in different rows, and is proper if it satisfies the balance and forest conditions. We denote the set of vertical arrays as $\mathcal{VA}_{n,K,R}^{(s)} = \mathcal{PA}_{n,K,R}^{(0,s)}$ and the set of proper vertical arrays as $\mathcal{PV}_{n,K,R}^{(s)}$. We also let $v_{n,K,R}^{(s)} = |\mathcal{PV}_{n,K,R}^{(s)}|$. For notational convenience, we extend our definition of $v_{n,K,R}^{(s)}$ to all $R \geq 1$ by letting $v_{n,K,R}^{(s)} = 0$ if $R_i > K$ for some $1 \leq i \leq n$. Again, unlike in Goulden and Slofstra, we do not have the non-empty condition in our definition of the proper paired array.

Note that we will generally not work directly with paired arrays that do not satisfy the forest condition. However, as vertical arrays not satisfying the forest condition are vital for extending paired arrays, we
have separated the forest condition from the definition of vertical arrays itself. Of the two conditions in

[Definition 7](#) the forest condition is more fundamental, and all the arrays we define in this paper will satisfy
some form of this condition. The balance condition is in general difficult to handle, but can be radically
simplified if the support graph of \(s\) forms a tree. For convenience, arrays that have such property are called
tree-shaped. With tree-shaped arrays, we can reduce the balance condition to a condition that only depends
on the number of mixed vertices in a cell, essentially allowing us to ignore it.

**Lemma 8.** Let \(\alpha \in PA_{n,K}^{(q,s)}\) be a tree-shaped paired array, and suppose that \(s_{i,k,j}\) is the number of vertices in
cell \((i,j)\) that are paired with a vertex in row \(k\) for all \(1 \leq i, k \leq n\) and \(1 \leq j \leq K\). Then, \(\alpha\) satisfies the
balance condition if and only if \(s_{i,k,j} = s_{k,i,j}\) for all \(i \neq k\).

**Proof.** First, note that \(s_{i,k}\) is the number of mixed pairs \(\{u,v\}\) with \(u\) in row \(i\) and \(v\) in row \(k\), so \(s_{i,k} = \sum_j s_{i,k,j}\). Also, let \(x_{i,j}\) be the number of mixed vertices in cell \((i,j)\), and observe that \(x_{i,j} = \sum_{k \neq i} s_{i,k,j}\).

Suppose \(s_{i,k,j} = s_{k,i,j}\) for all \(1 \leq i, k \leq n\) and \(1 \leq j \leq K\). Then, by summing over all \(k \neq i\), we have
\[
\sum_i x_{i,j} = \sum_{k \neq i} \sum_j s_{i,k,j} = \sum_{k \neq i} s_{k,i,j}.
\]

As \(s_{k,i,j}\) is the number of mixed vertices in cell \((k,j)\) that are paired with a vertex in row \(i\), the latter sum counts the number of mixed pairs \(\{u,v\}\) such that \(u\) is in row \(i\) and \(v\) is in row \(k\). Therefore, \(\alpha\) satisfies the balance condition.

Conversely, suppose \(\alpha\) satisfies the balance condition. By the same reasoning, we have \(x_{i,j} = \sum_{k \neq i} s_{i,k,j} = \sum_{k \neq i} s_{k,i,j}\). We will show by induction that \(s_{i,k,j} = s_{k,i,j}\) for all \(i \neq k\).

Let \(G\) be the support graph of \(s\) and suppose \(G\) is a tree. Without loss of generality, let the vertex \(n\) be a leaf of \(G\), and assume that it is adjacent to the vertex \(n-1\). As \(n\) is not joined to other vertices in \(G\), we have \(s_{n,k,j} = s_{n,n,j} = 0\) for all \(1 \leq k \leq n - 2\) and \(1 \leq j \leq K\). Substituting this into \(\sum_{k \neq n} s_{n,k,j} = \sum_{k \neq i} s_{n,k,j}\), we obtain \(s_{n,n-1,j} = s_{n-1,n,j}\). This gives \(s_{n,k,j} = s_{k,n,j}\) for \(1 \leq k \leq n - 1\) and \(1 \leq j \leq K\).

Now, let \(s'_{i,k} = \sum_i s_{i,k,j}\) and \(x'_{i,j} = \sum_{k \neq i} s_{i,k,j}\) for \(1 \leq i, k \leq n - 1\), \(i \neq k\), and \(1 \leq j \leq K\). That is, we have effectively removed the last row of \(\alpha\). Then,
\[
\sum_{k \neq i} s_{i,k,j} = \sum_{k \neq i} s_{i,k,j} - s_{i,n,j} = \sum_{k \neq i} s_{k,i,j} - s_{n,i,j} = \sum_{k \neq i} s_{k,i,j}
\]
by using the fact that \(s_{i,n,j} = s_{n,i,j}\), and substituting in the identity for \(x_{i,j}\). Furthermore, as \(s'_{i,k} = s_{i,k}\) for \(1 \leq i, k \leq n - 1\), the support graph given by \(s'\) is \(G\setminus\{n\}\). As \(n\) is a leaf of \(G\), \(G\setminus\{n\}\) is also a tree. By the inductive hypothesis, \(s_{i,k,j} = s_{k,i,j}\) for all \(1 \leq i, k \leq n - 1\) and \(1 \leq j \leq K\), where \(i \neq k\).

Therefore, \(\alpha\) satisfies the balance condition if and only if \(s_{i,k,j} = s_{k,i,j}\) for all \(i \neq k\), as desired.

Now that we have defined the necessary framework for paired arrays, we will state the relationship between
canonical arrays and labelled arrays.

**Theorem 9.** For \(n,K \geq 1\), \(q \geq 0\), and \(s \geq 0\), we have \(f_{n,K}^{(q,s)} = c_{n,K}^{(q,s)}\).

The proof is essentially the same as that in Goulden and Slofstra, but without the non-empty condition.
To obtain the canonical array from labelled array, we simply mark the cell that contains \(1^{(*)}\) in each row \(i\),
then remove the labels. To reconstruct the labelled array and prove that it is a bijection, we use the same
label recovery procedure introduced in their paper. As an example of this bijection, we have transformed the
labelled array depicted in Figure 1 into the canonical array depicted in Figure 2.

Now that we know that canonical arrays are in bijection with labelled arrays with the same parameters,
the problem of enumerating maps on surfaces reduces to that of enumerating canonical arrays. To solve the
latter problem, we will extend the procedures in Goulden and Slofstra to remove all non-mixed pairs. Then,
we will decompose the resulting paired arrays via induction, removing one row at a time.

**Theorem 10.** Let \(n,K \geq 1\), \(q \geq 0\), and \(s \geq 0\). We have
\[
c_{n,K}^{(q,s)} = \sum_{t=0}^{n} \prod_{i=1}^{n} \frac{(2q_i + s_i)!}{2^t t! (s_i + q_i - t)!} c_{n,K; q-t+1}^{(s)}
\]
Furthermore, if $v^{(s)}_{n,K,R}$ can be written as a polynomial expression in $K$ for all $R_i$, where $1 \leq R_i \leq q_i + 1$, then $c^{(q,s)}_{n,K}$ can be written as a polynomial expression in $K$.

Proof. Despite not having the non-empty condition in our definition of the paired functions and paired array, the proof of this theorem is essentially the same as that of Goulden and Slofstra. The polynomiality of $c^{(q,s)}_{n,K}$ follows from the fact that the summation bounds are independent of $K$, so $c^{(q,s)}_{n,K}$ as expressed above is a polynomial combination of $v^{(s)}_{n,K,q-t+1}$, with coefficients that are also independent of $K$. □

For example, by decomposing the canonical array in Figure 2, we can obtain the vertical array in Figure 3. Then, by combining the theorems we have so far, we can write the generating series in terms of the number of vertical arrays.

Corollary 11. Let $n, K \geq 1$, $q \geq 0$, and $s \geq 0$. We have

$$A^{(q,s)}_n(K) = \sum_{t=0}^q \prod_{i=1}^n \frac{(2q_i + s_i)!}{2^{q_i} t_i! (s_i + q_i - t_i)!} \cdot v^{(s)}_{n,K,q-t+1}$$

Proof. By combining (2) Theorem 9 and Theorem 10 the result immediately follows. □

3. Definitions and Terminology of Arrowed Arrays

In this section, we will extend two-row paired arrays by the addition of arrows, which represent hypothetical critical vertices. This will allow us recursively decompose vertical arrays into arrowed arrays and smaller vertical arrays. Some of the definitions and theorems are taken directly from [3], while others are direct extensions. For the sake of length, we will omit the proofs of those theorems.

Definition 12. Let $K \geq 1$, $s \geq 0$, and $1 \leq R_1, R_2 \leq K$. An arrowed array is a pair $(\alpha, \phi)$, where $\alpha \in VA^{(s)}_{2,K,R_1,R_2}$ is a two-row vertical array, and $\phi : K \setminus R_1 \to K$ is a partial function from $H \subseteq K \setminus R_1$ to $K$, with $R_1$ being the set of marked columns in row 1 of $\alpha$. Graphically, $\phi$ is denoted by arrows drawn above row 1, where an arrow from $j$ to $j'$ is drawn if $j \in H$ and $\phi(j) = j'$. For convenience, the two ends of the arrow belonging to columns $j$ and $j'$ are called the arrow-tail and arrow-head respectively, and column $j$ is said to point to column $j'$. Furthermore, both the arrow-tail and arrow-head belong to row 1 of their respective columns.

With the generalization of paired arrays to arrowed arrays, there are corresponding generalizations of the terms and conventions used to describe paired arrays. These generalizations will be compatible with the conventions for paired arrays if the partial function $\phi$ is empty.
Figure 3. Proper vertical array from the decomposition of Figure 2

- An object of \((\alpha, \phi)\) refers to either a vertex, a box, or an arrow-tail. If a cell both contains vertices and a box, or vertices and an arrow-tail, either the box or the arrow-tail is to be taken as the rightmost object of the cell.

- A vertex \(v\) of an arrowed array is **critical** if it is the rightmost vertex of a cell, and the cell it belongs to is neither marked nor contains an arrow-tail.

- \((\alpha, \phi)\) is said to satisfy the **non-empty condition** if for each column \(j\), there exists at least one cell that contains an object.

- \((\alpha, \phi)\) is said to satisfy the **balance condition** if for each column \(j\), the number of vertices in cell \((1, j)\) is equal to the number of vertices in cell \((2, j)\).

- Let \(F_i\) be the set of columns in row \(i\) that contain at least one vertex. The **forest condition function** \(\psi_1: (H \cup F_1) \setminus R_1 \mapsto K\) for row 1 is defined as follows: For each column \(j \in H\), let \(\psi_1(j) = \phi(j)\); for \(j \in F_1 \setminus (H \cup R_1)\), if the rightmost vertex \(v\) is paired with a vertex \(u\) in column \(j'\), let \(\psi_1(j) = j'\). The forest condition function \(\psi_2\) for row 2 is defined to be the same as the one for paired arrays in [Definition 7]. \((\alpha, \phi)\) is said to satisfy the **forest condition** if the functional digraph of \(\psi_1\) on the vertex set \(H \cup F_1 \cup \psi_1(H \cup F_1) \cup R_1\) is a forest with root vertices \(R_1\), and the functional digraph of \(\psi_2\) on the vertex set \(F_2 \cup \psi_2(F_2) \cup R_2\) is a forest with root vertices \(R_2\). That is, for each column \(j \in (H \cup F_1) \setminus R_1\), there exists some positive integer \(t\) such that \(\psi_1^t(j) \in R_1\), and for each column \(j \in F_2 \setminus R_2\), there exists some positive integer \(t\) such that \(\psi_2^t(j) \in R_2\).

- Additionally, \((\alpha, \phi)\) is said to satisfy the **full condition** if every cell contains at least one object.

The set of arrowed arrays that satisfies the forest condition is denoted \(\mathcal{A}\mathcal{R}_{K; R_1, R_2}^{(s)}\).

Notice in particular that a cell cannot contain both an arrow-tail and be marked at the same time. Unless otherwise stated, we will continue to use the conventions for paired arrays defined in [Convention 6] for arrowed arrays. However, we will be using the definition of critical vertex defined here instead of the one in [Definition 5]. As with paired arrays, we will always include the columns \(R_i\) in the vertex set for the functional digraph of \(\psi_i\), regardless of whether they are in the range of \(\psi_i\). Note that permuting the columns of an arrowed array does not change whether the array satisfies the balance or forest conditions, as all this action does is to relabel the vertices of the functional digraph. Furthermore, to reduce cluttering, we will draw the boxes for row 2 at the lower right corner instead of the upper right. An example of an arrowed array that satisfies the forest condition can be found in [Figure 4].

While the parameters used for defining the set of arrowed arrays is natural with respect to paired arrays, it does not easily lend itself to a formula. To make it manageable for summation, we need to partition the set of arrowed arrays by adding further constraints, which will take for form of three different substructures.
Definition 13. Let $K \geq 1$, $s \geq 0$, and $1 \leq R_1, R_2 \leq K$. A substructure $\Theta$ of $\mathcal{AR}^{(s)}_{K; R_1, R_2}$ is a set of constraints that defines a subset of $\mathcal{AR}^{(s)}_{K; R_1, R_2}$. For convenience, an arrowed array $(\alpha, \phi)$ is said to satisfy $\Theta$ if $(\alpha, \phi)$ satisfies the constraints given by $\Theta$. In particular, here are the three substructures that we will use in this paper.

- Let $w$ be a non-negative matrix of size $2 \times K$, $H_1$ and $H_2$ be $R_1$ and $R_2$ subsets of $K$, and $\phi$ be a partial function from $H \subseteq K \setminus H_1$ to $K$. The substructure $\Gamma = (w, H_1, H_2, \phi)$ is defined to be the set of substructures $\mathcal{AR}^{(s)}_{K; R_1, R_2}$, such that for each pair $(\alpha', \phi') \in \mathcal{AR}^{(s)}_{K; R_1, R_2}$, $\alpha'$ contains $w_{i,j}$ vertices in cell $(i,j)$, the marked cells in row 1 and 2 of $\alpha'$ are $H_1$ and $H_2$, respectively, and $\phi' = \phi$.

- Let $w$ be a non-negative vector of size $K$, $H_1$ be an $R_1$ subset of $K$, and $\phi$ be a partial function from $H \subseteq K \setminus H_1$ to $K$. The substructure $\Delta = (w, H_1, \phi)$ is defined to be the subset of $\mathcal{AR}^{(s)}_{K; R_1, R_2}$, such that for each pair $(\alpha', \phi') \in \mathcal{AR}^{(s)}_{K; R_1, R_2}$, $(\alpha', \phi')$ satisfies the balance condition, $\alpha'$ contains $w_j$ vertices in both cells $(1,j)$ and $(2,j)$, the marked cells in row 1 of $\alpha'$ is $H_1$, and $\phi' = \phi$. Furthermore, for $A \geq 0$, we define $\Delta_A$ to be the substructure that describes the subset of arrowed arrays that satisfies $\Delta$, and have exactly $A$ columns of type $A$.

- Let $\mathcal{P}$ be a subset of $K$ with $|\mathcal{P}| \geq R_1 \geq 1$, $x$ be a non-negative vector of size $K$, and $\phi: K \setminus \mathcal{P} \to K$ be a partial function from $H \subseteq K \setminus \mathcal{P}$ to $H \cup \mathcal{P}$. Suppose that $x_j = 0$ for all $j \notin H \cup \mathcal{P}$ and $s$ be such that $\sum_j x_j = s - |\mathcal{P}| + R_1$. The substructure $\lambda = (x, \mathcal{P}, \phi)$ is defined to be the subset of $\mathcal{AR}^{(s)}_{K; R_1, R_2}$, such that for each pair $(\alpha', \phi') \in \mathcal{AR}^{(s)}_{K; R_1, R_2}$, $(\alpha', \phi')$ satisfies the balance condition, the set of marked cells in row 1 of $\alpha'$ is a subset of $\mathcal{P}$, and $\phi' = \phi$. Furthermore, for each column $j \in H \cup \mathcal{P}$, both cells $(1,j)$ and $(2,j)$ contains $x_j + 1$ vertices if $j \in \mathcal{P}$ and is unmarked, and $x_j$ vertices otherwise.

For convenience, we say a substructure $\Theta$ is a refinement of another substructure $\Theta'$ if the set of arrowed arrays satisfying $\Theta$ is a subset of the arrowed arrays satisfying $\Theta'$. We denote it as $\Theta \hookrightarrow \Theta'$. Furthermore, if $\Theta_1, \ldots, \Theta_t$ is a set of substructures that are refinements of a substructure $\Theta'$, we say that $\Theta_1, \ldots, \Theta_t$ partitions $\Theta'$ if the sets of arrowed arrays satisfying the $\Theta_i$’s are mutually disjoint, and their union is the set of arrowed arrays that satisfy $\Theta$. Finally, we will use arrowed array terminologies such as critical vertex, arrow-head, and points to with substructures when they are applicable.

Note that the latter substructures in Definition 13 can be partitioned using the substructure directly above. Furthermore, the substructures $\Delta_A$ are refinements of substructures $\Delta$. Also, for substructure $\lambda$, the vertices are restricted to the columns $H \cup \mathcal{P}$, and $x$ represents the number of non-critical vertices in row 1.

Lemma 14. Let $\Gamma = (w, H_1, H_2, \phi)$ be a substructure of $\mathcal{AR}^{(s)}_{K; R_1, R_2}$, and suppose that $\phi$ contains a column $X$ that points to a column $Y$, with cell $(1,Y)$ marked. Let $\Gamma' = (w, H_1 \cup \{X\}, H_2, \phi')$ be a substructure of
\[ AR_{K:R_1+1,R_2}^{(s)} \text{, such that} \]

\[
\phi' (j) = \begin{cases} 
\text{undefined} & j = X \\
\phi(j) & j \in H \setminus X
\end{cases},
\]

that is, instead of pointing to \( Y \), we mark cell \((1,X)\) of \( \Gamma' \). Then, the number of arrowed arrays satisfying \( \Gamma \) and the number of arrowed arrays satisfying \( \Gamma' \) are equal. Furthermore, \( \Gamma \) satisfies the balance, non-empty, and full conditions if and only if \( \Gamma' \) satisfies them, respectively.

The proof of this lemma can be found in [3], and by changing the proof slightly, we can show that for substructure \( \Delta \), the number of arrowed arrays satisfying \( \Delta = (w, R_1, \phi) \) and \( \Delta' = (w, R_1 \cup \{X\}, \phi') \) are equal.

**Lemma 15.** Let \( \Gamma = (w, R_1, R_2, \phi) \) be a substructure of \( AR_{K:R_1+1,R_2}^{(s)} \), and suppose that \( \phi \) contains a column \( X \) that points to a column \( Y \), and the column \( Y \) points to another column \( Z \). Let \( \Gamma' = (w, R_1, R_2, \phi') \) be a substructure of \( AR_{K:R_1+1,R_2}^{(s)} \) such that

\[
\phi' (j) = \begin{cases} 
Z & j = X \\
\phi(j) & j \in H \setminus X
\end{cases},
\]

that is, instead of pointing to \( Y \), \( X \) now points to \( Z \) in \( \phi' \). Then, the number of arrowed arrays satisfying \( \Gamma \) and the number of arrowed arrays satisfying \( \Gamma' \) are equal. Furthermore, \( \Gamma \) satisfies the balance, non-empty, and full conditions if and only if \( \Gamma' \) satisfies them, respectively.

Similarly, the proof of this lemma can be adapted to show that number of arrowed arrays satisfying \( \Delta = (w, R_1, \phi) \) and \( \Delta' = (w, R_1, \phi') \) are equal, and the same for the number of arrowed arrays satisfying \( \Lambda = (x, P, \phi) \) and \( \Lambda' = (y, P, \phi') \).

Collectively, these are the *arrow simplification lemmas*, and pictures describing the applications of these lemmas can be found in Figure 5 and Figure 6. Furthermore, applying these lemmas to the array in Figure 4 gives us Figure 7. Note that these lemmas can be applied repeatedly to simplify a substructure, until either all arrow-heads are in cells that are unmarked and have no arrow-tails, or an arrow-head is in the same cell as its own arrow-tail. We are only interested in the former, as the latter implies that there is a cycle in the functional digraph of \( \phi \), which violates the forest condition. This gives rise to the following definition.

**Definition 16.** A substructure \( \Theta \) is irreducible if the functional digraph of \( \phi \) is acyclic, and \( \Theta \) cannot be further simplified with the application of arrow simplification lemmas. Any cell of an irreducible substructure containing an arrow-head must be unmarked in row 1, and cannot contain an arrow-tail. Furthermore, it follows from definition that if an irreducible substructure satisfies the full condition, then any cell containing an arrow-head must also contain a critical vertex in row 1.

Note that for substructure \( \Lambda \), only the second arrow simplification lemma applies. Furthermore, for substructure \( \Lambda = (x, P, \phi) \), the arrow-heads must be in cells of \( H \cup P \), but they cannot be in \( H \) for \( \Lambda \) to be irreducible. Hence, if \( \Lambda \) is irreducible, then \( \phi \) must be a function from \( H \) to \( P \).

**Definition 17.** If \( \Gamma = (w, R_1, R_2, \phi) \) is an irreducible substructure, then we can categorize the columns of \( \Gamma \) as follows: Let \( A, B, C, D \) be a partition of the columns of \( K \setminus H \), where

- Columns in \( A \) have both row 1 and row 2 unmarked
- Columns in \( B \) have row 1 marked and row 2 unmarked
- Columns in \( C \) have row 1 unmarked and row 2 marked
- Columns in \( D \) have both row 1 and row 2 marked

Furthermore, if \( X \) is a column or a set of columns, let \( \overline{X} \) and \( \overline{X} \) be the sets of columns that have arrows pointing to \( X \), and that have row 2 unmarked and marked, respectively. In particular, \( \overline{A} \) and \( \overline{A} \) denotes the sets of columns pointing to \( A \), and \( \overline{C} \) and \( \overline{C} \) denotes the sets of columns pointing to \( C \), with row 2 unmarked and marked, respectively. These sets of columns implicitly defined by \( \Gamma \) are referred to as *column types*, and a diagram with all the column types can be found in Figure 8.
By applying the arrow simplification procedure to the left figure, we arrive at the right figure. R1 and R2 can be arbitrary in whether they are marked, but they must be the same between the two figures.

**Figure 5. Arrow Simplification 1**

By applying the arrow simplification procedure to the top figure, we arrive at the bottom figure. R1, R2, R3, and R4 can be arbitrary in whether they are marked, but they must be the same between the two figures. The same holds for the optional arrow with Z as its tail.

**Figure 6. Arrow Simplification 2**
Figure 7. Simplification of the arrowed array in Figure 4 into an irreducible array

Figure 8. Column types and variables for the number of vertices

These eight column types form a partition of $\mathcal{K}$ on irreducible substructures, and knowing the number of columns and the number of vertices for each column type of $\Gamma$ is sufficient to count the number of arrowed arrays satisfying it.

4. Enumeration of Substructures

Now, we have everything we need to provide formulas for the number of arrowed arrays satisfying the substructures defined in Definition 13. The first formula is proved in [3], and enumerates arrays satisfying substructure $\Gamma$.

**Theorem 18.** Given an irreducible substructure $\Gamma = (w, R_1, R_2, \phi)$ that satisfies the full condition with $s \geq A + 2$, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{A}R_{K,R_1,R_2}^{(s)}$ that satisfy $\Gamma$ is given by the formula

$$T(\Gamma) = (s - 1)! \left[ \frac{(b_2 + d_2)(\bar{a}_1 + c_1 + \bar{c}_1 + d_1)}{s - A} + \frac{b_1(c_2 + \bar{c}_2 + \bar{c}_1) - \bar{c}_1(b_2 + d_2)}{(s - A)(s - A - 1)} \right]$$

In the case where $s = A + 1$, the formula reduces to

$$T(\Gamma) = (s - 1)! (b_2 + d_2)(\bar{a}_1 + c_1 + \bar{c}_1 + d_1)$$

By the convention set out in Convention 6 we let a lower case variable $x_i$ represent the total number of points in row $i$ of the columns of type $X$, and $A$ represent the number of columns of type $A$. For convenience,
we will drop the subscripts of the formula from here on, as we will only deal with arrays that satisfy the balance condition.

Next, we provide a formula for substructure $\Delta$. This substructure allows us to mark the cells of row 2 arbitrarily, while keeping the positions of the marked cells in row 1 and the vertices fixed.

**Theorem 19.** Let $R_1, R_2 \geq 1$, and let $\Delta = (w, R_1, \phi)$ be an irreducible substructure that satisfies the balance condition. Furthermore, suppose $w_j > 0$ for $1 \leq j \leq K$. Then, the number of arrowed arrays $(\alpha, \phi) \in AR_{K; R_1, R_2}$ with substructure $\Delta$ is given by the formula

$$T(\Delta) = s! \sum_{A=0}^{s-1} \frac{r}{s-A} \binom{M}{M-A} \binom{K-M-1}{R_2-M+A-1}$$

where $r$ is the total number of vertices in row 1 of the columns of $R_1$, and $M$ is the number of columns that contain a critical vertex in row 1.

**Proof.** To prove this theorem, we sum $T(\Gamma)$ over all substructures $\Gamma = (w, R_1, R_2, \phi)$ that are refinements of $\Delta$. Since $\Delta$ satisfies the balance conditions and $w_j > 0$ for $1 \leq j \leq K$, all substructures $\Gamma$ satisfy the full condition, so we can use the formula of $T(\Gamma)$ given by Theorem 18. Note that $T(\Gamma)$ only depends on the number of columns of type $A$, even though it depends on the number of vertices of other column types. Therefore, we first sum over all $\Gamma$ with $A$ columns of type $A$ to obtain $T(\Delta_A)$, then we sum $A$ from 0 to $s-1$ to obtain $T(\Delta)$. As $\Delta$ satisfies the balance condition, so must all $\Gamma$ that are refinements of $\Delta$. This implies that we can drop the subscripts from $T(\Gamma)$.

Let $M$ be the set of columns that contains a critical vertex in row 1, and $H$ be the set of columns that contains an arrow-tail. Then, $R_1, M, H$ and $R_1$ partitions $K$. As $R_1 \geq 1$, we have $M < K$. In the case where $M = 0$, we have $M = H = \emptyset$ and $R_1 = K$. Therefore, by simplifying and substituting in the formula for $T(\Gamma)$, we have

$$T(\Delta) = \sum_{1 \leq i \leq \Delta} d(s-1)!$$

Note that a vertex $v$ in cell $(1, X)$ contributes to $d$ if $X$ is marked in row 2. As there are $\binom{K-1}{R_2-1}$ ways to mark the columns of $K$ in row 2 with $X$ marked, and $s$ vertices in row 1, we have

$$T(\Delta) = s! \binom{K-1}{R_2-1}$$

This result agrees with substituting $M = A = 0$ into the formula for $T(\Delta)$.

In the case where $1 \leq M \leq K-1$, we have $R_1 = B \cup D$. This gives us $r = b + d$, and allows us to rewrite $T(\Gamma)$ as

$$T(\Gamma) = (s-1)! (T_1(\Gamma) + T_2(\Gamma) + T_3(\Gamma) + T_4(\Gamma))$$

where

$$T_1(\Gamma) = \frac{rc}{s-A}$$

$$T_2(\Gamma) = \frac{r(\bar{a} + \bar{c} + d)}{s-A}$$

$$T_3(\Gamma) = \frac{b(c + \bar{c})}{(s-A)(s-A-1)}$$

$$T_4(\Gamma) = -\frac{r\bar{c}}{(s-A)(s-A-1)}$$

for $0 \leq A \leq s-2$, with $T_3(\Gamma) + T_4(\Gamma) = 0$ for $s = A-1$. As the substructures $\Gamma = (w, R_1, R_2, \phi)$ with $A$ columns of type $A$ partitions $\Delta_A$, we can let $T_i(\Delta_A) = \sum_{\Gamma \rightarrow \Delta_A} T_i(\Gamma)$ for $i = 1, 2, 3, 4$, which gives us

$$T(\Delta) = (s-1)! \left( \sum_{A=0}^{s-1} (T_1(\Delta_A) + T_2(\Delta_A)) + \sum_{A=0}^{s-2} (T_3(\Delta_A) + T_4(\Delta_A)) \right)$$

To evaluate each of the $T_i(\Delta_A)$, we look at the number of substructures $\Gamma$ such that a vertex or a pair of vertices contributes to the numerator of $T_i(\Delta_A)$. Note that we can ignore $r$ since it is the number of vertices in $R_1$, which is a constant with respect to $\Delta$. Of the three sets of columns, only the columns of $M$ can become columns of type $A$. Therefore, if a substructure $\Gamma$ is a refinement of $\Delta_A$, it must have exactly $M - A$ marked cells in row 2 of $\mathcal{M}$. It must also have exactly $R_2 - M + A$ marked cells in row 2 of $R_1 \cup \mathcal{H}$. This
means in total, there are \((\frac{M}{M-A})(\frac{K-M}{R_2-M+A})\) substructures of the form \(\Gamma = (w, R_1, R_2, \phi)\) that are refinements of \(\Delta_A\).

Now, a vertex \(v\) in row 1 of a column \(X\) contributes to \(c\) if \(X \in M\) and \(X\) is marked in row 2. As there are \((\frac{M-1}{M-A-1})\) ways to mark the columns of \(M\) in row 2 with \(X\) marked, and \((\frac{K-M}{R_2-M+A})\) ways to mark the columns of \(K\backslash M\), \(v\) contributes \((\frac{M-1}{M-A-1})(\frac{K-M}{R_2-M+A})\) times to \(c\). Let \(m\) be the total number of vertices in \(M\), we have

\[
T_1(\Delta_A) = T_1(\Gamma) \cdot \frac{m}{c} \left( \frac{M-1}{M-A-1} \right) \left( \frac{K-M}{R_2-M+A} \right)
\]

Next, a vertex \(v\) in row 1 of a column \(X\) contributes to \(\bar{a} + \bar{c} + d\) if \(X \in K\backslash M\) and \(X\) is marked in row 2. As there are \((\frac{K-M-1}{R_2-M+A-1})\) ways to mark the columns of \(K\backslash M\) in row 2 with \(X\) marked, and \((\frac{M}{M-A})\) ways to mark the columns of \(M\), \(v\) contributes \((\frac{M}{M-A})(\frac{K-M-1}{R_2-M+A-1})\) times to \(\bar{a} + \bar{c} + d\). Given that there are \(s - m\) vertices in \(K\backslash M\), we have

\[
T_2(\Delta_A) = \frac{r(s-m)}{s-A} \left( \frac{M}{M-A} \right) \left( \frac{K-M-1}{R_2-M+A-1} \right)
\]

Similarly, let \([v, u]\) be a pair of vertices with \(v\) in row 1 of a column \(X\) and \(u\) in row 2 of a column \(Y\). Then, \([v, u]\) contributes to \(b(\bar{c} + \bar{c} + \bar{d})\) if the following conditions hold. First, we have \(X \in R_1, \ Y \in K\backslash R_1,\) and \(X\) unmarked in row 2. Furthermore, let \(Z\) be the column \(Y\) if \(Y \in M\), and \(Z\) be the column that \(\gamma\) points to if \(\gamma \in H\). Then, \(Z\) must be a column of \(M\) and must also be marked. Now, as there are \((\frac{M-1}{M-A-1})\) ways to mark the columns of \(M\) with \(Z\) marked, and \((\frac{K-M-1}{R_2-M+A-1})\) ways to mark the columns of \(K\backslash M\) in row 2 with \(X\) unmarked, \([v, u]\) contributes \((\frac{M-1}{M-A-1})(\frac{K-M-1}{R_2-M+A-1})\) times to \(b(\bar{c} + \bar{c} + \bar{d})\). Given that there are \(r(s-r)\) such pairs of \([v, u]\), we have

\[
T_3(\Delta_A) = \frac{r(s-r)(s-A)(s-A-1)}{(s-A)(s-A) \left( \frac{M}{M-A-1} \right) \left( \frac{K-M-1}{R_2-M+A} \right)}
\]

Finally, a vertex \(v\) in row 1 of a column \(X\) contributes to \(\bar{d}\) if \(X \in H\), \(X\) is unmarked in row 2, and the column \(Z\) that \(X\) points to is marked in row 2. As there are \((\frac{M-1}{R_2-M+A-1})\) ways to mark the columns of \(K\backslash M\) in row 2 with \(X\) unmarked, and \((\frac{M-1}{M-A-1})\) ways to mark the columns of \(M\) with \(Z\) marked, \(v\) contributes \((\frac{M-1}{M-A-1})(\frac{K-M-1}{R_2-M+A-1})\) times to \(\bar{d}\). Given that there are \(s - m - r\) vertices in \(H\), we have

\[
T_4(\Delta_A) = -\frac{r(s-m-r)(s-A)(s-A-1)}{(s-A)(s-A) \left( \frac{M}{M-A-1} \right) \left( \frac{K-M-1}{R_2-M+A} \right)}
\]

Now, let \(T_{3+4}(\Delta_A) = T_3(\Delta_A) + T_4(\Delta_A)\), and observe that

\[
T_{3+4}(\Delta_A) = \frac{rm}{(s-A)(s-A-1)} \left( \frac{M-1}{M-A-1} \right) \left( \frac{K-M-1}{R_2-M+A} \right)
\]

and

\[
T_1(\Delta_A) + T_2(\Delta_A) = T_r(\Delta_A) + T_{m1}(\Delta_A) + T_{m2}(\Delta_A)
\]

where

\[
T_r(\Delta_A) = \frac{rs}{s-A} \left( \frac{M}{M-A} \right) \left( \frac{K-M-1}{R_2-M+A-1} \right)
\]

\[
T_{m1}(\Delta_A) = \frac{rm}{s-A} \left( \frac{M-1}{M-A-1} \right) \left( \frac{K-M-1}{R_2-M+A} \right)
\]

\[
T_{m2}(\Delta_A) = -\frac{rm}{s-A} \left( \frac{M-1}{M-A-1} \right) \left( \frac{K-M-1}{R_2-M+A-1} \right)
\]
By substituting these formulas into $T(\Delta)$, we have
\[
T(\Delta) = (s - 1)! \left( \sum_{A=0}^{s-1} (T_r(\Delta_A) + T_{m1}(\Delta_A) + T_{m2}(\Delta_A)) + \sum_{A=0}^{s-2} T_{3+4}(\Delta_A) \right)
\]
Next, we will show that $\sum_{A=0}^{s-1} (T_{m1}(\Delta_A) + T_{m2}(\Delta_A)) + \sum_{A=0}^{s-2} T_{3+4}(\Delta_A) = 0$. Note that
\[
T_{m1}(\Delta_A) + T_{3+4}(\Delta_A) = \frac{rm}{s - A - 1} \left( \frac{M - 1}{M - A - 1} \left( K - M - 1 \right) \left( R_2 - M + A \right) \right)
\]
for $0 \leq A \leq s - 2$. Therefore, by shifting the index of $\sum_{A=0}^{s-1} T_{m2}(\Delta_A)$ by one and noting that $T_{m2}(\Delta_0) = 0$, we have
\[
\sum_{A=0}^{s-1} (T_{m1}(\Delta_A) + T_{m2}(\Delta_A)) + \sum_{A=0}^{s-2} T_{3+4}(\Delta_A)
\]
This proves our formula for $T(\Delta)$.

Recall from Definition 7 that proper vertical arrays do not require a vertex in each cell, so the formula in Theorem 19 does not apply to all arrow arrays that will result in our subsequent decomposition. In our previous paper [3], which covers the case $n = 2$, we only needed the formula for two-row vertical arrays, equivalently arrowed arrays without arrows. Hence, we bypassed this issue by removing the columns with no vertices, then summed over all possible ways to add the empty columns. However, that approach does not work here, as arrowed arrays may have arrows in columns that are otherwise empty. Therefore, we need to extend Theorem 19 to cover a wider range of arrowed arrays.

Definition 20. An irreducible substructure $\Delta = (w, R_1, \phi)$ is admissible if each cell that contains an arrowhead also contains at least one vertex. This means that if an arrowed array $(\alpha, \phi)$ satisfies an admissible substructure $\Delta$, then $\psi_i(j)$ must have at least one vertex. In particular, the only way to violate the forest condition of row $i$ is for there to be a cycle in the functional digraph of $\psi_i$.

Note that the definition of admissible for substructure $\Delta$ is compatible with the definition of irreducible for substructure $\Lambda$. That is, if $\Lambda = (x, \mathcal{P}, \phi)$ is an irreducible substructure and $\Delta = (w, R_1, \phi)$ is a refinement of $\Lambda$, then $\Delta$ can be reduced to an admissible substructure $\Delta'$ by the application of Lemma 14. Therefore, we will provide a formula for admissible substructure $\Delta$ as follows.

Theorem 21. Let $R_1, R_2 \geq 1$, and let $\Delta = (w, R_1, \phi)$ be an admissible substructure. Then, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{A}(s)_{K, R_1, R_2}$ with substructure $\Delta$ is given by the same formula as in Theorem 19. That is,
\[
T(\Delta) = s! \sum_{A=0}^{s-1} \frac{r}{s - A} \left( \frac{M}{M - A} \right) \left( \frac{K - M - 1}{R_2 - M + A - 1} \right)
\]
Proof. As permuting the columns of an arrowed array does not change whether it satisfies the forest condition, we can without loss of generality assume that the first $k$ of the $K$ columns of $\Delta$ are the ones that contain at least one vertex. In particular, it means that $\phi_i(j) \in [k]$. Now, let $\Delta^R$ be the subset of arrowed arrays...
that satisfies $\Delta$, and have exactly $R$ marked cells in the first $k$ columns of row 2. Furthermore, let $\Delta_{R,k} = (w', R_1 \cap [k], \phi')$ be the restriction of $\Delta_R$ to the first $k$ columns. In other words, $\Delta_{R,k} = (w', R_1 \cap [k], \phi')$ is a substructure of $\mathcal{AR}_{R,R_1\cap[k],R}^{(s)}$, where $w_j' = w_j$ and $\phi_j'(j) = \phi_j(j)$ for $1 \leq j \leq k$. Note that $\phi_j(j) \in [k]$ implies that $\phi'_j(j) \in [k]$, so this is well defined. We will show that there is a $(K-k)$ to 1 correspondence between arrowed arrays satisfying $\Delta$ and arrowed arrays satisfying $\Delta_{R,k}$.

Let $(\alpha, \phi)$ be an arrowed array satisfying $\Delta$ and consider the cell $(i,j)$, where $k+1 \leq j \leq K$. As $\Delta$ is admissible, there cannot be another column $j'$ such that $\psi_j(j') = j$. So, by deleting this column, we have either deleted an isolated root vertex, deleted a leaf, or done nothing to the functional digraph of $\psi_i$. Hence, we can remove the column $j$ from the array without violating the forest condition. Therefore, we can simply cut off the rightmost $K-k$ columns of $(\alpha, \phi)$ to obtain an arrowed array $(\alpha', \phi')$ that satisfies $\Delta_{R,k}$.

Conversely, given an arrowed array $(\alpha', \phi')$ satisfying $\Delta_{R,k}$, we can add $K-k$ columns with no vertices to obtain an arrowed array $(\alpha, \phi)$ satisfying $\Delta$. Note that the positions of arrows and marked cells in row 1 is completely fixed by $\Delta_R$. However, only the first $k$ columns of $(\alpha, \phi)$ are predetermined in row 2, as given by $(\alpha', \phi')$. For the remaining $K-k$ columns, we can mark $R_2-R$ cells arbitrarily and satisfy the forest condition, as adding columns with no vertices does not change $\psi_2$. Therefore, for each arrowed array $(\alpha', \phi')$ satisfying $\Delta_{R,k}$, there are exactly $(K-k)_{R_2-R}$ arrowed arrays satisfying $\Delta_R$.

By construction, each of the $\Delta_{R,k}$ has $w_j > 0$ for $1 \leq j \leq k$, so we can use Theorem 19 to obtain $T(\Delta_{R,k})$. Furthermore, $\Delta^1, \ldots, \Delta^{\min(k,R_2)}$ partitions $\Delta$, and for $R = 0$ or $R > k$, we have $T(\Delta_{R,k}) = 0$. Therefore, we can change the bounds to $0 \leq k \leq R_2$, and use the Chu-Vandermonde identity (pg. 67 of [1]) to obtain

$$T(\Delta) = \sum_{R=0}^{\min(k,R_2)} T(\Delta_{R,k}) \binom{K-k}{R_2-R}$$

which is the formula for $T(\Delta)$ as given by Theorem 19.

Next, we will rewrite this formula using hypergeometric transformations, as that will simplify our work later.

**Theorem 22.** Let $R_1, R_2 \geq 1$, and let $\Delta = (w, R_1, \phi)$ be an admissible substructure that satisfies the balance condition. Then, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{AR}_{R,R_1,R_2}^{(s)}$ with substructure $\Delta$ is given by the formula

$$T(\Delta) = r \sum_{A=0}^{\min(s,K)-1} \frac{M!(K-A-1)!(s-A-1)!}{(M-A)!(K-R_2-A)!(R_2-1)!}$$

where $r$ is the total number of vertices in row 1 of the columns of $R_1$, and $M$ is the number of columns that contain a critical vertex in row 1.

**Proof.** First, we rewrite $T(\Delta)$ using factorials to obtain

$$T(\Delta) = r \sum_{A=0}^{s-1} \frac{s!M!(s-A-1)!(K-M-1)!}{(s-A)!(M-A)!(R_2-M+A-1)!(K-R_2-A)!}$$

If $M \geq s$, then $r = 0$, as each column of $M$ requires a critical vertex, and there are only $s$ vertices in row 1. In this case, the theorem is true as both the original formula and the new formula imply that $T(\Delta) = 0$. Otherwise, we have $M \leq s-1$ and $(M-A)!$ in the denominator, which allows us to lower the upper bound of the summation to $M$. We can then write it using the standard notation for hypergeometric series to
obtain

\[
T(\Delta) = r \cdot \frac{3F_2\left(-M, -s, -K + R_2; 1\right)}{(R_2 - M - 1)! (K - R_2)!} \frac{(s - 1)! (K - M - 1)!}{(R_2 - M - 1)! (K - R_2)!}
\]

\[
= r \cdot \frac{3F_2\left(-M + 1, -K + R_2; 1\right)}{1 - K, -s + 1; 1} \frac{(K - M)^{(M)} (s - 1)! (K - M - 1)!}{(R_2 - M)^{(M)} (R_2 - M - 1)! (K - R_2)!}
\]

\[
= r \cdot \frac{\min(s, K) - 1}{M! (K - A - 1)! (s - A - 1)! (M - A)! (K - R_2 - A)! (R_2 - 1)!}
\]

where we use the $3F_2$ identity

\[
3F_2\left(-N, b, c; 1\right) = \frac{(d - e)^{(N)}}{d^{(N)}} 3F_2\left(-N, e - b, c; 1\right)
\]

for non-negative integer $N$, and $a, b, c, d \in \mathbb{C}$. This identity can be found on pg. 142 of [1].

Now, as $(M - A)!$ is again part of the new denominator, we can raise the summation index without changing the value of the sum. Note that we know $M \leq s - 1$, and we can deduce that $M \leq K - 1$ as $R_1 \geq 1$. This allows us to raise the upper bound to $\min(s, K) - 1$, while keeping the numerator well defined.

The benefit of this new formula is that we are no longer required to keep $M \leq \min(s, K) - 1$. While taking $M \geq \min(s, K)$ for $\Delta$ makes no sense combinatorially, the value for $T(\Delta)$ is well defined and finite. This frees up $M$ for manipulation and summation if we can multiply $T(\Delta)$ with an expression that is zero if $M \geq s$ or $M \geq K$. When we do the induction on the number of vertical arrays, this fact will become extremely useful.

With the formula for admissible substructures $\Delta$, we can now provide a formula for the number of arrowed arrays satisfying substructure $\Lambda$.

**Theorem 23.** Given a substructure $\Lambda = (\mathcal{X}, \mathcal{P}, \phi)$ such that the functional digraph of $\phi$ on $\mathcal{H} \cup \mathcal{P}$ is a rooted forest with root vertices $\mathcal{P}$, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{AR}^{(s)}_{K; R_1; R_2}$ satisfying substructure $\Lambda$ is given by the formula

\[
T(\Lambda) = \sum_{A=0}^{\min(s, K) - 1} \frac{(s - P + R_1) (K - A - 1)! (s - A - 1)! (P - 1)!}{(P - R_1 - A)! (K - R_2 - A)! (R_1 - 1)! (R_2 - 1)!}
\]

where $P$ is the number of columns of $\mathcal{P}$.

**Proof.** First, we suppose that $\Lambda$ is irreducible. We prove this by substituting into the formula for $T(\Delta)$ given by Theorem 22. Let $\mathcal{R}_1$ be an $R_1$-subset of $\mathcal{P}$, and consider the substructure $\Delta = (\mathcal{X}, \mathcal{R}_1, \phi)$, where $x'_i = x_i + 1$ if $x \in \mathcal{P} \setminus \mathcal{R}_1$, and $x'_i = x_i$, otherwise. Now, note that $\Delta$ may not be irreducible, as there can be arrows pointing to the columns of $\mathcal{R}_1$. Therefore, we have to reduce $\Delta$ using the arrow simplification lemma defined in Lemma 14. This gives us an irreducible substructure $\Delta' = (\mathcal{X}', \mathcal{R}_1 \cup \mathcal{H}_1, \phi')$, where $\mathcal{H}_1 \subseteq \mathcal{H}$ is the set of columns that points to $\mathcal{R}_1$, and $\phi'$ is $\phi$ restricted to the columns of $\mathcal{H} \setminus \mathcal{H}_1$.

Now, $\Delta'$ satisfies the balance condition by construction. Furthermore, any cell of $\Delta'$ that contains an arrow-head must be in $\mathcal{P} \setminus \mathcal{R}_1$, as otherwise $\Delta'$ will not be irreducible. Since the columns of $\mathcal{P} \setminus \mathcal{R}_1$ must each contain at least one vertex, $\Delta'$ is an admissible substructure, so we can use the formula for $T(\Delta)$ given by Theorem 22. As $\Delta'$ satisfies the balance condition, we can take $r$ to be the number of vertices in row 1 of $\mathcal{R}_1$. Observe that the $P - R_1$ vertices added to row 1 of $\mathcal{P} \setminus \mathcal{R}_1$ are all critical vertices, regardless of the choice of $\mathcal{R}_1$. Hence, they never contribute to $T(\Delta)$. This means that we only need to consider the non-critical vertices of row 1, which are given by $\mathcal{X}$. Now, a non-critical vertex $u$ in row 1 of a column $\mathcal{X}$ contributes to $r$ of the formula for $T(\Delta)$ if $\mathcal{X} \in \mathcal{R}_1$, or $\mathcal{X} \in \mathcal{H}$ and $\mathcal{X}$ points to a column in $\mathcal{R}_1$. In either case, there are $\binom{P - 1}{R_1 - 1}$ different subsets $\mathcal{R}_1$ such that $\mathcal{X}$ is marked in $\Delta'$, out of the $\binom{P}{R_1}$ possible $R_1$-subsets of $\mathcal{P}$. Given that all non-critical vertices of row 1 are in $\mathcal{P} \cup \mathcal{H}$, and that there are $s - P + R_1$ non-critical vertices in row 17.
1, we have

\[ T(\Lambda) = T(\Delta') \cdot \frac{s - P + R_1}{r} \left( \frac{P - 1}{R_1 - 1} \right) \]

\[ = \min(s,K) - 1 \sum_{A=0}^{(s-P+R_1)} \frac{(s-P+R_1)(K-A-1)!}{(P-A-1)!} \frac{(K-A-1)!}{(R_1-1)!} \frac{(P-1)!}{(R_2-1)!} \]

where we substitute in \( M = P - R_1 \) as the number of critical vertices in row 1.

Finally, if \( \Lambda \) is not irreducible, we can repeatedly apply Lemma 15 to obtain an irreducible substructure \( \Lambda' = (y, P, \phi') \). As \( s, K, R_1, R_2, \) and \( P \) all remain the same, we have \( T(\Lambda) = T(\Lambda') \), so the result follows. \( \Box \)

5. Enumeration of Vertical Arrays

At this point, we are ready to decompose proper vertical arrays. Recall that a paired array \( \alpha \in PA^{(q,s)}_{n,K;R} \) is tree-shaped if the support graph of \( s \) is a tree. With tree-shaped vertical arrays, we can delete a row that is a leaf in the support graph while keeping the support graph a tree. This allows us to recursively decompose tree-shaped vertical arrays into smaller tree-shaped vertical arrays and arrowed arrays. Then, by using Theorem 23 we can provide a formula for \( v^{(s)}_{n,K;R} \) when the support graph of \( s \) is a tree.

We start off with a number of preliminary definitions and facts.

**Fact 24.** Let \( \alpha \in PA^{(q,s)}_{n,K;R} \) be a proper paired array. Suppose cell \((i, j)\) of \( \alpha \) is an unmarked cell containing at least one vertex, then \( \alpha' \) that is formed by marking cell \((i, j)\) of \( \alpha \) is also a proper paired array.

Note that the converse of Fact 24 is not true. For example, if \( \alpha' \) has only one marked cell in row \( i \), then unmarking that cell violates the forest condition for that row. This fact allows us to mark cells containing critical vertices, making those vertices non-critical and removing them from the forest condition. This leads to our next definition.

**Definition 25.** If \( n, K \geq 1 \), then a partially-paired array \( \alpha \) is an \( n \times K \) array of cells, where each cell contains zero or more vertices, and is either marked or unmarked. Furthermore, each vertex of the array may be paired with another vertex. However, only the rightmost vertices of unmarked cells are required to be paired with another vertex, and we call the vertices not paired with any other vertices unpaired vertices. Terms for paired arrays such as critical vertices and parameters like \( q_i \) and \( R_i \) carry over from Definition 5 and Convention 6.

By definition, all paired arrays are partially-paired arrays. Also, as unpaired vertices are neither mixed nor critical, they do not affect the balance or forest conditions. However, we do consider unpaired vertices as objects in a partially-paired array.

Now, our main reason for using partially-paired arrays is so that we can unpair vertices of a paired array. That is, if \( \{u, v\} \) is a pair of non-critical vertices in a partially-paired array \( \alpha \), we can unpair them to create a new partially-paired array \( \alpha' \) that is otherwise identical to \( \alpha \), but with \( u \) and \( v \) unpaired. Then, we can remove \( u \) and \( v \) separately without impacting the balance and forest conditions. We will adapt a technique from Goulden and Slofstra for labelling the objects in a row of a partially-paired array with a set of positive integers. This allows us to insert or remove a subset of the unpaired vertices while keeping track of their positions.

**Procedure 26.** Let \( \alpha \) be a partially-paired array with \( p_i \) vertices and \( R_i \) marked cells in row \( i \), where \( 1 \leq i \leq n \). We describe the following three procedures:

1. Let \( S \) be a set of positive integers of size \( p_i + R_i \). To label row \( i \) of \( \alpha \) with \( S \) is to assign from left to right elements of \( S \) to the objects of row \( i \), from smallest to largest. As described in Definition 5, in a cell that contains both vertices and a box, the box is to be taken as the rightmost object of the cell.

2. Let \( V \) be a subset of the unpaired vertices in row \( i \). To extract \( V \) from \( \alpha \) is to create a partially-paired array \( \alpha' \) and a set of positive integers \( W \), where \( \alpha' \) is \( \alpha \) with \( V \) deleted, and \( W \) is a \( |V| \)-subset of \( [p_i + R_i - 1] \). This is done by labelling row \( i \) of \( \alpha \) with \( [p_i + R_i] \), then deleting \( V \) from \( \alpha \). We let \( W \) be the labels of the vertices deleted. Note that \( W \) cannot contain \( p_i + R_i \) as the deleted vertices cannot be the rightmost objects of their cells.
(3) Let $\mathcal{W}$ be a $y$-subset of $[p_i + R_i + y - 1]$, where $y \geq 0$. To insert $\mathcal{W}$ into row $i$ of $\alpha$ is to add $y$ unpaired vertices to row $i$ of $\alpha$ to create a partially-paired array $\alpha'$.

This is done by labelling row $i$ of $\alpha$ with $[p_i + R_i + y] \setminus \mathcal{W}$. Then, for each $w \in \mathcal{W}$, we find the smallest $w' \notin \mathcal{W}$ such that $w' > w$, and place a vertex to the left of and in the same cell as the object labelled $w'$. As the new vertex is not the rightmost object of a cell, it is non-critical. Furthermore, if there is more than one vertex to be inserted to the left of an object, they should be inserted in increasing order from left to right.

In the end, row $i$ of $\alpha'$ contains $p_i + R_i + y$ objects, labelled from left to right by $1$ to $p_i + R_i + y$ in increasing order. Finally, we let $\mathcal{V}$ denote the set of vertices inserted, to mirror the extraction procedure.

Notice that in both the extraction and insertion procedures, the vertices involved are unpaired. Furthermore, the use of the same variables $\mathcal{V}$ and $\mathcal{W}$ between procedure 2 and 3 is deliberate, as we shall now show that the extraction and insertion procedures are inverses of each other.

**Proposition 27.** Let $\alpha$ be a partially-paired array with $p_i$ vertices and $R_i$ marked cells in row $i$, and $\mathcal{V}$ be a subset of the unpaired vertices in row $i$, where $1 \leq i \leq n$. Let $\beta$ be the partially-paired array and $\mathcal{W}$ be the subset of $[p_i + R_i - 1]$ created by extracting $\mathcal{V}$ from $\alpha$. Suppose $\alpha'$ is the partially-paired array formed by reinserting $\mathcal{W}$ into row $i$ of $\beta$, and $\mathcal{V}'$ is the set of vertices inserted, then $\alpha = \alpha'$ and $\mathcal{V} = \mathcal{V}'$. Conversely, let $\beta$ be a partially-paired array with $p_i$ vertices and $R_i$ marked cells in row $i$, where $1 \leq i \leq n$, and suppose $\mathcal{W}$ is a $y$-subset of $[p_i + R_i + y - 1]$, with $y \geq 0$. Let $\alpha$ be the partially-paired array formed by inserting $\mathcal{W}$ into row $i$ of $\beta$, and $\mathcal{V}$ be the set of inserted vertices. Suppose $\beta'$ and $\mathcal{V}'$ is the pair of objects created from extracting $\mathcal{V}$ from $\alpha$, then $\beta = \beta'$ and $\mathcal{W} = \mathcal{W}'$. In both cases, $\alpha$ is proper if and only if $\beta$ is proper.

**Proof.** Note that when we extract $\mathcal{V}$ from $\alpha$, we obtain the partially-paired array $\beta$ and the set $\mathcal{W}$ that is a $|\mathcal{V}|$-subset of $[p_i + R_i - 1]$. As $\beta$ has $p_i + R_i - |\mathcal{V}|$ objects in row $i$, we can insert $\mathcal{W}$ into $\beta$ to obtain the partially-paired array $\alpha'$ and the set $\mathcal{V}'$ of inserted vertices. Furthermore, the objects remaining in $\beta$ are labelled with the same labels $[p_i + R_i - 1] \setminus \mathcal{W}$ during the extraction and insertion procedures. Finally, each vertex $v \in \mathcal{V}$ is in the same cell and to the left of some other object in $\alpha$, and is reinserted into that same cell in $\alpha'$ in increasing order of labels. Therefore, $\alpha = \alpha'$ and $\mathcal{V} = \mathcal{V}'$.

Conversely, when we insert $\mathcal{W}$ into row $i$ of $\beta$, the vertices inserted by $\mathcal{W}$ are non-critical vertices, and are the objects in $\alpha$ that are labelled from left to right with $[p_i + R_i + |\mathcal{W}|]$. As the set of inserted vertices retains the same labels when in the extraction procedure, we have $\beta = \beta'$ and $\mathcal{W} = \mathcal{W}'$.

Finally, in both the extraction and insertion procedures, the vertices involved are non-critical and unpaired. Therefore, they do not impact the balance or the forest conditions. Hence, $\alpha$ is proper if and only if $\beta$ is proper. □

Next, we define the compatibility condition that allows us to combine arrowed arrays and vertical arrays together.

**Definition 28.** Let $\alpha \in \mathcal{PV}_n^{|s|}$ be an $n$-row proper vertical array with $\mathcal{R}_s$ as its set of marked cells in row $i$, and $\psi_i$ as its forest condition function for row $i$. A substructure $\Lambda = (\mathcal{X}, \mathcal{P}, \phi)$ as defined in [Definition 13] is $\Lambda$-compatible with row $i$ of $\alpha$ if $\mathcal{P} = \mathcal{R}_i$, and $\phi = \psi_i$. Furthermore, let $R_1' \leq R_2' \leq K$, and suppose that $\mathcal{W}$ is a $x$-subset of $[s_i + R_i + x - 1]$ for some $x \geq 0$. We define $\Lambda_{\alpha,i,\mathcal{W}}$ to be the substructure of $\mathcal{A}_{|s_i|}^{(x+R_i-R_i')}$ with parameters $\Lambda_{\alpha,i,\mathcal{W}} = (\mathcal{X}, \mathcal{R}_i, \psi_i)$, where $\mathcal{X} = (x_1, \ldots, x_K)$ and $x_j$ is the number of vertices inserted into cell $(i,j)$ of $\alpha$ if $\mathcal{W}$ is inserted into row $i$ of $\alpha$ by the insertion procedure defined in [Procedure 26].

By definition, $\Lambda_{\alpha,i,\mathcal{W}}$ is $\Lambda$-compatible with row $i$ of $\alpha$. Also, by summing over the number of vertices inserted into cell $(i,j)$, we have $|\mathcal{W}| = \sum_j x_j$.

With substructure compatibility defined, we can now decompose tree-shaped vertical arrays. Let $\alpha \in \mathcal{PV}_{n+1}^{|s|}$ be an $(n+1)$-row proper vertical array, and without loss of generality assume that row $n+1$ is a leaf vertex adjacent to row $n$ in the support graph of $s$. To extract row $n+1$ from $\alpha$, we mark the cells in row $n$ containing the critical vertices matched with vertices in row $n+1$. Then, we remove all pairs between rows $n$ and $n+1$, and subsequently delete row $n+1$. To keep track of the removed vertices in row $n$, we use a $(s_n+1 - P + \mathcal{R}_n)$-subset to represent the positions of the non-critical vertices, and an arrowed array to represent the critical vertices and pairings of the vertices removed.
Theorem 29. Let $n, K \geq 1$, $s = (s_{1, 2}, s_{1, 3}, \ldots, s_{n, n+1}) \geq 0$, and $R = (R_1, \ldots, R_{n+1}) \in [K]^{n+1}$. Suppose the support graph of $s$ is a tree with the vertex $n+1$ as a leaf adjacent to the vertex $n$. Then, there exists a decomposition

$$
\zeta : PVA^{(s)}_{n+1,K,R} \rightarrow \bigcup_{P=R_n} \bigcup_{\beta \in PVA^{(s')}_{n,K,R'}} \bigcup_{W \in [s_n + R_n - 1; R_{n+1} - P + R_n]} (\beta, W, \Lambda_{\beta,n,W})
$$

of proper vertical arrays into a triple of smaller vertical arrays, $(s_{n+1} - P + R_n)$-subsets, and arrowed arrays. Here, $\Lambda_{\beta,n,W}$ are substructures of $AR^{(s_{n+1})}_{n,K}$, $s'$ is $s$ restricted to an $n \times n$ matrix by removing the last row and column, $s_i = \sum_{k \neq i} s_{i,k}$ for $1 \leq i \leq n+1$, and $R'$ is a vector of length $n$ given by

$$
R'_k = \begin{cases} R_k & k < n \\ P & k = n \end{cases}
$$

Furthermore, this decomposition is a bijection.

Note that we can apply this theorem to any pair of rows $i$ and $k$, such that $i$ is a leaf in the support graph. Also, $s_n$ includes the vertex pairs between rows $n$ and $n+1$, and the marked cells in row $n$ of $\beta$ are given by $R'_n$, which is a set of size $P$ that contains $R_n$ as a subset.

Proof. We will prove the bijection by providing the decomposition and show that it is invertible. Conceptually, we take the mixed pairs between row $n$ and row $n+1$ of $\alpha$, and put them into an arrowed array $(\sigma, \phi)$. Then, we add marked cells and arrows to $(\sigma, \phi)$ in such a way that rows $n$ and $n+1$ of $\alpha$ have the same forest condition functions as rows 1 and 2 of $(\sigma, \phi)$, respectively. To record the position of the non-critical vertices in row $n$, we extract and record these vertices as a $s_{n+1} - P + R_n$-subset of $[s_n + R_n - 1]$. Finally, we mark the cells of $\alpha$ containing the critical vertices of row $n$ that are paired with vertices of row $n+1$, so as to preserve the forest condition for row $n$.

Let $V$ be the set of non-critical vertices that are paired with vertices of row $n+1$, and $U$ be the set of critical vertices that are paired with vertices of row $n+1$. Note that the vertices of $U$ and $V$ must be in row $n$ by our assumption, and that $|U \cup V| = s_{n+1}$. Therefore, if we let $P = R_n + |U|$, we have $R_n \leq P \leq K$. Furthermore, since $|U| \leq s_{n+1}$, we have $P \leq s_{n+1} + R_n$, which combines to give $R_n \leq P \leq \min(s_{n+1} + R_n, K)$.

To construct the proper vertical array $\beta \in PVA^{(s')}_{n,K,R'}$, and the subset $W \in [s_n + R_n - 1; s_{n+1} - P + R_n]$, we first mark the cells containing the vertices of $U$. Next, we unpair all vertex pairs with one vertex in row $n+1$, delete row $n+1$, and call the resulting array $\alpha'$. This leaves all other mixed pairs unchanged, $s'$ describes the number of mixed pairs of $\alpha'$. Then, as the support graph of $s'$ is the support graph of $s$ with the vertex $n+1$ removed, the support graph of $s'$ is also a tree. Also, note that deleting row $n+1$ removes the variables $s_{n+1,k,j}$ and $s_{k,n+1,j}$ from $\alpha$, but leaves the remaining $s_{i,k,j}$ the same for all $1 \leq i, k \leq n$, $i \neq k$. Therefore, the conditions of Lemma 8 remain satisfied in $\alpha'$, so $\alpha'$ satisfies the balance condition. In addition, since we marked the cells containing $U$, the forest condition remains satisfied when we unpair the vertices of $U \cup V$ and delete row $n+1$. This means that $\alpha'$ is a proper partially-paired array. Next, we remove the vertices of $U$ from $\alpha'$ to obtain the partially-paired array $\alpha''$, and we extract $V$ from $\alpha''$ as described in Procedure 26 to obtain the subset $W$ and the vertical array $\beta$. Note that $\alpha''$ has $R_n + |U|$ marked cells, $s_n - |U|$ total vertices, and $|V| = s_n - P + R_n$ unpaired vertices in row $n$. Therefore, $W$ is a $s_n - P + R_n$-subset of $s_n + R_n - 1$. Furthermore, by Proposition 27, $\beta$ is also a proper paired array. By construction, $\beta$ satisfies the parameters $R'$ and $s'$, and contains no non-mixed pairs, so $\beta \in PVA^{(s')}_{n,K,R'}$ as desired.

To preserve information on the pairs we removed, we construct an arrowed array $(\sigma, \phi) \in \Lambda_{\beta,n,W}$ such that $\psi_n = \psi_1'$ and $\psi_{n+1} = \psi_2'$, where $\psi_n$ and $\psi_{n+1}$ are the forest condition functions for rows $n$ and $n+1$ of $\alpha$, while $\psi_1'$ and $\psi_2'$ are the forest condition functions for rows 1 and 2 of $(\sigma, \phi)$, respectively. For each vertex $v \in U \cup V$ that is in cell $(n, j)$, we place a corresponding vertex $x_{i,j}$ into cell $(1, j)$ of $\sigma$. Similarly, for each vertex $u$ in cell $(n+1, j)$, we place a corresponding vertex $x_u$ in cell $(2, j)$ of $\sigma$. If we need to place more than one vertex into the same cell, we place them in the same order in $\sigma$ as they are in $\alpha$. Then, for each pair $(v, u)$ between row $n$ and $n+1$, we pair their corresponding vertices $x_u$ and $x_v$ in $\sigma$. Next, we mark cell $(1, j)$ of
σ if cell (n, j) of α is marked, and we mark cell (2, j) of σ if cell (n + 1, j) of α is marked. Finally, suppose (n, j) of α contains a critical vertex u \notin U. Then, it must be paired with some vertex v in some cell (k, j'), where 1 ≤ k < n − 1. In this case, we let φ(j) = j'. This completes the construction of (σ, φ).

By construction, (σ, φ) is in ARK,Rn,Rn+1 for all marked cells and vertex pairs between rows n and n + 1 to (σ, φ). This also implies that cell (1, j) of σ satisfies s_{n+1,j} = s_{n,j} for all j, so (σ, φ) satisfies the balance condition. Furthermore, by replacing the critical pairs of row n of α with arrows, we ensure that rows 1 and 2 of (σ, φ) have the same set of marked cells and forest condition functions as rows n and n + 1 of α. Therefore, (σ, φ) satisfies the forest condition, as α is a proper vertical array.

Finally, we need to show that (σ, φ) ∈ Λ_{β,n,W} = (x, R′_n, θ_n), where θ_n is the forest condition function for row n of β. By construction, θ_n(j) and φ(j) are only defined for cells (n, j) with critical vertices u \notin U. Furthermore, we have θ_n(j) = φ(j) in those cases, so θ_n = φ. Then, note that the set of marked cells in row 1 of σ is R_n, which is a subset of R′_n. Now, if we reinsert W into row n of β, we recover α″ and the set Ω of extracted vertices by Proposition 27. These are all the non-critical vertices in row 1 of (σ, φ), in the same cells as in V. Furthermore, if cell (n, j) is marked in β, then it must either be marked in α, in which case cell (1, j) is marked in σ, or contain a vertex u \in U, in which case it is unmarked and contains the vertex x_u. In both cases, (σ, φ) satisfies x and R′_n. Therefore, we have (σ, φ) ∈ Λ_{β,n,W}, as desired.

Conversely, let β ∈ PVA^\{s′\}_n,K,R, where W \in [s_n + R_n - 1; s_{n+1} - P + R_n], and (σ, φ) ∈ ARK,Rn,Rn+1 that satisfies Λ_{β,n,W}. We first construct partially-paired array β′ by inserting W into row n of β as described in Procedure 26. This gives us a set Ω of unpaired vertices in β′, labelled with the elements of W. By Proposition 27, β′ is a proper partially-paired array. Furthermore, by the definition of Λ_{β,n,W}, the vertices of Ω are the same columns as the non-critical vertices in row 1 of σ. Therefore, for each vertex v \in Ω, we can let x_v be the non-critical vertex in row 1 of σ that corresponds to v. Next, consider each cell (1, j) of σ that contains a critical vertex. Since Λ_{β,n,W} is Λ-compatible with β, cell (n, j) is marked in β, and by extension β′. This means that we can add an unpaired vertex u to cell (n, j), which we place to the right of all other vertices in that cell. Similarly, to the vertices of Ω, we let the corresponding vertex in cell (1, j) of σ be x_u. After adding these vertices, we let the resulting partially-paired array be β″, and let the set of vertices added to obtain β″ be U. By Proposition 27, β″ is a proper partially-paired array. Since row n of β″ has P marked cells, while row 1 of σ has R_n marked cells, we have |U| = P - R_n. Also, since W is a (s_{n+1} - P + R_n)-subset, we have |U| + |Ω| = s_{n+1} as desired.

Next, we extend β″ by adding row n + 1. For each cell (2, j) of σ that is marked, we mark cell (n + 1, j) of β″. Similarly, for each vertex x_v in cell (2, j) of σ, we add a corresponding vertex v in row (n + 1, j) of β″. Then, for each pair (x_v, v) in σ, we pair their corresponding vertices u \in U \cup Ω and v in row n + 1. Finally, we unmark the cells containing the vertices of U to recover α. By construction, α satisfies the parameters R and s, and contains no non-mixed pairs, so α ∈ PVA^\{s\}_n,K,R as desired.

As with the other direction, we copied all marked cells and vertex pairs of (σ, φ) into rows n and n + 1 of α. As σ satisfies the balance condition, we have s_{n+1,j} = s_{n+1,n,j} and s_{n+1,k,j} = s_{k,n+1,j} = 0 for k < n. By Lemma 8, the statement that β satisfies the balance condition means that s_{i,k,j} = s_{i,k+1,j} for all 1 ≤ i < k ≤ n and 1 ≤ j ≤ K. Hence, α satisfies the balance condition. By the compatibility condition, φ is the same as θ_n, the forest condition function for row n of β. Furthermore, by replacing marked cells with critical pairs of (σ, φ), we have ensured that rows n and n + 1 of α have the same set of marked cells and forest condition functions as rows 1 and 2 of (σ, φ). As the forest condition of the other rows are unchanged, α is a proper vertical array.

Finally, we have to show that the two operations presented are inverses of each other. By Proposition 27, the extraction and insertion procedures are inverses. Furthermore, if we extract U and reinsert it, the vertices inserted acquire the same labels as before the extraction. Therefore, we can correspond the non-critical vertices in row 1 of (σ, φ) with the vertices of Ω. Then, the columns which contain the critical vertices U are exactly the columns of (σ, φ) that contain critical vertices in row 1. This allows us to recover the columns of U, so that we can add critical vertices and unmark cells. Similarly, the vertices in row 2 of (σ, φ) correspond to the vertices of row n + 1 of α. As we have a correspondence between the vertices of U \cup Ω and vertices of row n + 1 with the vertices in row 1 and 2 of (σ, φ), respectively, we can recover the pairing of the removed vertices via the pairing of vertices in (σ, φ). Therefore, ζ as described, is a bijection. □

21
Figure 9. A tree-shaped, 3-row vertical array

Figure 10. Partially-paired array $\alpha'$ and $\alpha''$ corresponding to the decomposition of row 3 of Figure 9

Note that in the proof of Theorem 29, $\alpha'$ and $\beta''$ correspond to each other, so does $\alpha''$ and $\beta'$. Also, the decomposition works with any row that is a leaf vertex of the support graph. With this decomposition, we can iteratively pick a row where the support graph of $s$ is a leaf, and remove that row. This leaves arrowed arrays with support graph $s'$, which is a tree with $n$ rows, so we can repeat the process.

As an example, we will decompose the tree-shaped vertical array in Figure 9. By following the decomposition described in Theorem 29, we can decompose row 3 and arrive at the partially-paired array $\alpha'$ and $\alpha''$, as depicted in Figure 10. For clarity, we have marked the vertices of $U$ and $V$ in $\alpha'$, and labelled the objects in row 2 of $\alpha''$. After the decomposition, we obtain the minimal array $\beta$ and the arrowed array $(\sigma, \phi)$, depicted in Figure 11 as well as the subset $W = \{1, 2, 5\} \in [8; 3]$ and the value $P = 3$.

Now that we have a decomposition of tree-shaped vertical arrays, we can provide an explicit formula for $v^{(s)}_{n,K,R}$ via induction. We start with the following corollary.

**Corollary 30.** Let $n, K \geq 1$, $s = (s_1, 2, s_1, 3, \ldots, s_n, n + 1) \geq 0$, and $R = (R_1, \ldots, R_{n+1}) \in [K]^{n+1}$. Suppose the support graph of $s$ is a tree with the vertex $n+1$ as a leaf adjacent to the vertex $n$. Then,

$$v^{(s)}_{n+1,K,R} = \sum_{P=R_n}^{\min(s_{n+1},R_n,K) \min(s_{n+1}+1,K)-1} \sum_{A_{n+1}=0}^{s_n + R_n - 1} \binom{s_n + R_n - 1}{s_{n+1} - P + R_n} v^{(s')}_{n,K,R'} \times$$

$$\frac{(s_{n+1} - P + R_n)(K - A_{n+1} - 1)}{(P - R_n - A_{n+1})!(K - R_{n+1} - A_{n+1})!(R_n - 1)!(R_{n+1} - 1)!}$$
where $\mathbf{s}'$ is $\mathbf{s}$ restricted to an $n \times n$ matrix by removing the last row and column, $s_i = \sum_{k \neq i} s_{i,k}$ for $1 \leq i \leq n + 1$, and $\mathbf{R}'$ is a vector of length $n$ given by

$$R'_k = \begin{cases} R_k & k < n \\ P & k = n \end{cases}$$

Proof. Let $P$ be such that $R_n \leq P \leq \min (s_{n+1} + R_n, K)$, $\beta \in \mathcal{PVA}_{n,K;\mathbf{R}'}$ be an $n$-row vertical array with parameters as defined in [Theorem 29] $W$ be a $(s_{n+1} - P + R_n)$-subset of $[s_n + R_n - 1]$, and $\Lambda_{\beta,n,W}$ be a substructure of $AR^{(s_{n+1})}_{K;R_n,R_n+1}$. As $\beta$ is a proper vertical array, the forest condition function $\theta_n$ for row $n$ is a forest with root vertices $P$. Furthermore, the constraint $R_n \leq P \leq \min (s_{n+1} + R_n, K)$ matches with the definition of substructure $\Lambda$. Then, for a given $P$, there are $(s_{n+1} - P + R_n)$ distinct $(s_{n+1} - P + R_n)$-subsets of $[s_n + R_n - 1]$. Finally, for a given $R'_n = P$, there are $\psi_{n,K;\mathbf{R}'}$ proper vertical arrays. Combining these gives the formula of our corollary as desired.

As we have assumed that the support graph $G$ of $\mathbf{s}$ is a tree, we can repeatedly select a row that corresponds to a leaf vertex in $G$, and iterate the decomposition in [Theorem 29] Then, by taking the cardinality of both sides, we obtain the following theorem.

**Theorem 31.** Let $n, K \geq 1$, $\mathbf{s} \geq 0$, and $\mathbf{R} \geq 1$. Suppose the support graph $G$ of $\mathbf{s}$ is a tree. Then,

$$v^{(s)}_{n,K;\mathbf{R}} = \sum_{A_{n+1} = 0}^{\min(s_{n+1},K)-1} \cdots \sum_{A_{n+1} = 0}^{\min(s_{n+1},K)-1} \prod_{j=1}^{n-1} \left( \frac{K - A_{e_j} - 1)!}{(K + s_{e_j} - A_{e_j} - 1)!} \right) \times \prod_{i=1}^{n} \left( \frac{K + \sum_{k\sim i} (s_{i,k} - A_{i,k} - 1))! (R_i - 1 + \sum_{k\sim i} s_{i,k})!}{(K - R_i - \sum_{k\sim i} A_{i,k})! (R_i + \sum_{k\sim i} (s_{i,k} - 1))!} \right)$$

where $e_1, \ldots, e_{n-1}$ are the edges of $G$. Furthermore, for each edge $e_j = \{i,k\}$ in $G$, the summation variable $A_{e_j}$ can be equivalently written as $A_{i,k}$ and $A_{k,i}$. Finally, the sum $\sum_{k\sim i}$ is over all indices $k$ that are adjacent to $i$ in the support graph of $\mathbf{s}$.
Hence, we can safely increase the upper bound of the denominator, the summation term is zero if \( s_i = n \) do not appear.

\[
v_{n,K,R}^{(s)} = \frac{\min(s_{1,1}, K-1) \min(s_{n+1}, K-1)}{\sum_{A_{1,1}=0}^{n-1} \sum_{A_{2,1}=0}^{n-1} \prod_{j=1}^{n-1} \frac{(K-A_{e_j}-1)!}{(K+s_e-A_{e_j}-1)!}} \times \frac{(K+s_i+1-A_{i,1}-\delta_i)!}{(R_i+s_i+1-1)!} \times \frac{(K+s_n-s_{n+1}-A_n+A_{n+1}-\delta_n+1)!}{(P+s_n-s_{n+1}-1)!} \times \frac{(P+1)!}{(K-P-A_n+A_{n+1})!} \times \frac{(P+s_n-s_{n+1}-\delta_n+1)!}{(P+1)!}.
\]

As we shall later see, we can remove the upper bounds of \( K-1 \), but upper bounds of \( s_{e_j}-1 \) are necessary and cannot be removed.

**Proof.** Note that if \( R_i > K \) for some \( i \), the term \((K-R_i - \sum_{k=i}^{\infty} A_{i,k})! \) in the denominator causes the entire sum to be zero. Otherwise, we prove this theorem via induction on the number of rows.

**Base case:**

Suppose \( n = 1 \), then we have no vertices, so there are \((K_{R_1})\) vertical arrays in \( PV(A_{1,K,R_1}) \). This matches with our formula for \( v_{1,K,R_1}^{(s)} \), as the variables \( s_{e_j} \) and summations \( A_{e_j} \) do not appear.

**Inductive step:**

Let \( s = (s_{1,2}, s_{1,3}, \ldots, s_{n,n+1}) \geq 0, R = (R_1, \ldots, R_{n+1}) \geq 1, \) and assume that the functional digraph of \( s \) has vertex \( n+1 \) as a leaf, and is adjacent to vertex \( n \). Then, for convenience of notation, let \( s_i = \sum_{k=i}^{\infty} s_{i,k}, A_i = \sum_{k=i}^{\infty} A_{i,k} \), and \( \delta_i = \sum_{k=i}^{\infty} 1 \) for \( 1 \leq i \leq n+1 \). Furthermore, let the \( e_1, \ldots, e_n \) be the edges of the support graph of \( s \), with \( e_n \) being the edge between vertex \( n \) and \( n+1 \). This means that \( s_{n+1} = s_{n,n+1}, A_{n+1} = A_{n,n+1}, \) and for \( 1 \leq i \leq n-1, A_i \) does not contain the variable \( A_{n,n+1} \).

By applying \( s' = s - s_{n+1} \) to our inductive hypothesis, we have

\[
v_{n,K,R'}^{(s')} = \sum_{A_{e_1}, \ldots, A_{e_{n-1}}} C(A_{e_1}, \ldots, A_{e_{n-1}}) \times \frac{(K+s_n-s_{n+1}-A_n+A_{n+1}-\delta_n+1)!}{(P+1)!} \times \frac{(P+s_n-s_{n+1}-1)!}{(K-P-A_n+A_{n+1})!} \times \frac{(P+s_n-s_{n+1}-\delta_n+1)!}{(P+1)!}.
\]

Note that \( A_n \) and \( \delta_n \) are substituted with \( A_n - A_{n+1} \) and \( \delta_n - 1 \), respectively, as the support graph of \( s' \) does not contain the edge \( e_n = \{n,n+1\} \). To simplify the expression for further manipulation, we let \( C(A_{e_1}, \ldots, A_{e_{n-1}}) \) to be the first two products inside the sum. That is, we rewrite the above expression as

\[
v_{n,K,R'}^{(s')} = \sum_{A_{e_1}, \ldots, A_{e_{n-1}}} C(A_{e_1}, \ldots, A_{e_{n-1}}) \times \frac{(K+s_n-s_{n+1}-A_n+A_{n+1}-\delta_n+1)!}{(P+1)!} \times \frac{(P+s_n-s_{n+1}-1)!}{(K-P-A_n+A_{n+1})!} \times \frac{(P+s_n-s_{n+1}-\delta_n+1)!}{(P+1)!}.
\]

Then, we can substitute this expression into Corollary 30, which gives

\[
v_{n+1,K,R}^{(s)} = \sum_{A_{e_1}, \ldots, A_{e_{n-1}}} \sum_{A_{n+1}=0}^{P} \sum_{P=0}^{\min(s_{n+1}, K-R_n)} C(A_{e_1}, \ldots, A_{e_{n-1}}) \times \frac{(K+s_n-s_{n+1}-A_n+A_{n+1}-\delta_n+1)!}{(P+1)!} \times \frac{(P+s_n-s_{n+1}-1)!}{(K-P-R_n-A_n+A_{n+1})!} \times \frac{(P+s_n-s_{n+1}-\delta_n+1)!}{(P+1)!} \times \frac{(s_{n+1}-P-1)!}{(K-P-A_n+1)!} \times \frac{(P+1)!}{(K-P-R_n-A_n+1)!} \times \frac{(R_n+1)-1)!}{(R_n-1)!}.
\]

after shifting the summation index \( P \) down by \( R_n \). As the terms \((s_{n+1}-P-1)! \) and \((K-P-R_n-A_n+A_{n+1})! \) are in the denominator, the summation term is zero if \( P > \min(s_{n+1}, K-R_n) \), noting that \( A_n \geq A_{n+1} \). Hence, we can safely increase the upper bound of the \( P \) summation to infinity. Furthermore, if \( P < A_{n+1} \),
then the summation term is also zero, as we have \((P - A_{n+1})!\) in the denominator. This allows us to substitute \(P = Q + A_{n+1}\) and sum over \(Q \geq 0\) instead. By doing these substitutions, we obtain

\[
v^{(s)}_{n+1,K;R} = \sum_{\mathbf{A}_{e_1},\ldots,\mathbf{A}_{e_{n-1}}} \sum_{A_{n+1} = 0}^{\min(s_{n+1},K)-1} \sum_{Q \geq 0} C(A_{e_1},\ldots,A_{e_{n-1}}) \times \]

\[
\frac{(K + s_n - s_{n+1} - A_n + A_{n+1} - \delta_n + 1)! (s_n + R_n - 1)!}{(K - Q - R_n - A_n)! (Q + R_n + A_{n+1} + s_n - s_{n+1} - \delta_n + 1)!} \times \]

\[
\frac{(s_{n+1} - A_{n+1} - Q - 1)!Q! (K - R_{n+1} - A_{n+1})! (R_n - 1)! (R_{n+1} - 1)!}{(R_n - 1)! (K - R_n - A_n)! (R_n + s_n - \delta_n)! (K - R_{n+1} - A_{n+1})! (R_{n+1} - 1)!} \]

using the Chu-Vandermonde identity (pg. 67 of [1]). By noting that \(\delta_{n+1} = 1\) and simplifying the formula we obtained for \(v^{(s)}_{n+1,K;R}\), we obtain

\[
v^{(s)}_{n+1,K;R} = \sum_{\mathbf{A}_{e_1},\ldots,\mathbf{A}_{e_{n-1}}} \sum_{A_{n+1} = 0}^{\min(s_{n+1},K)-1} C(A_{e_1},\ldots,A_{e_{n-1}}) \times \]

\[
\frac{(K + s_i - A_i - \delta_i)! (R_i + s_i - 1)!}{(K - R_i - A_i)! (R_i + s_i - \delta_i)!} \]

which proves our induction as desired. 

To remove the upper bounds of \(K - 1\) in Theorem 31, we will for each edge \(e\) of the support graph of \(s\), assign a vertex \(v\) that is incident to \(e\). This will allow us to regroup the factorial terms in \(v^{(s)}_{n,K;R}\), which will allow us to rewrite the expression with rising factorials.

**Corollary 32.** Let \(n, K \geq 1\), \(s \geq 0\), and \(R \geq 1\). Suppose that the support graph \(G\) of \(s\) is a tree with edges \(e_1,\ldots,e_{n-1}\), such that \(e_j\) is incident with vertex \(j\) in \(G\) for \(1 \leq j \leq n - 1\). Then,

\[
v^{(s)}_{n,K;R} = \prod_{i=1}^{n} \frac{(R_i - 1 + \sum_{k=1}^{s_i} s_{i,k})!}{(R_i + 1 + \sum_{k=1}^{s_i} (s_{i,k} - 1))!} \times \]

\[
\sum_{A_{e_1} = 0}^{s_{e_1}-1} \cdots \sum_{A_{e_{n-1}} = 0}^{s_{e_{n-1}}-1} \left( K - R_n - \sum_{k=1}^{\sum_{k=1}^{A_{e_1}}} A_{n,k} + 1 \right) \times \]

\[
\prod_{j=1}^{n-1} \left( K - R_j - \sum_{k=1}^{A_{j,k}} A_{j,k} + 1 \right) \times \]

\[
(K + s_{e_j} - A_{e_j})^{-A_{j,k} - A_{e_j} - 1} \times \]

\[
\left( K + s_{e_j} - A_{e_j} \right)^{A_{j,k} + A_{e_j} + 1} \]

where for each edge \(e_j = \{j, \ell\} \in G\), the summation variable \(A_{e_j}\) can be equivalently written as \(A_{j,\ell}\) and \(A_{\ell,j}\). As in Theorem 31, the sum \(\sum_{k=1}^{s_{e_j}}\) is over all indices \(k\) that are adjacent to \(j\) in the support graph of \(s\). Furthermore, \(v^{(s)}_{n,K;R}\) as expressed in this corollary is a polynomial in \(K\).

Note that given an arbitrary tree with vertices \(1,\ldots,n\), we can achieve the incidence condition in Corollary 32 by repeatedly taking a leaf vertex \(k \neq n\), label the edge incident to vertex \(k\) as \(e_k\), then delete both vertex \(k\) and edge \(e_k\).
Proof. First, we rearrange the expression for \( v_{n,K:R}^{(s)} \) in [Theorem 31] to obtain

\[
v_{n,K:R}^{(s)} = \prod_{i=1}^{n} \frac{(R_i - 1 + \sum_{k=1}^{n} s_{i,k})!}{(R_i - 1)! (R_i + \sum_{k=1}^{n} (s_{i,k} - 1))!} \times \sum_{A_{n,i}=0} \cdots \sum_{A_{n,-i}=0} \frac{(K + \sum_{k=n}^{n} (s_{n,k} - A_{n,k} - 1))!}{(K - R_n - \sum_{k=n}^{n} A_{n,k})!} \times \frac{(K - A_{n,i} - 1)!}{(K - R_n - \sum_{k=n}^{n} A_{n,k})!} \times \left( \frac{K + \sum_{k=n}^{n} (s_{n,k} - A_{n,k} - 1))!}{(K - R_n - \sum_{k=n}^{n} A_{n,k})!} \times \frac{(K - A_{n,i} - 1)!}{(K - R_n - \sum_{k=n}^{n} A_{n,k})!} \times \left( \frac{K + \sum_{k=n}^{n} (s_{n,k} - A_{n,k} - 1))!}{(K - R_n - \sum_{k=n}^{n} A_{n,k})!} \times \frac{(K - A_{n,i} - 1)!}{(K - R_n - \sum_{k=n}^{n} A_{n,k})!} \right)^{n-1} \right]^{j=1}
\]

As the sums \( \sum_{k=n}^{n} \) are over the support graph of \( s \), and \( e_j \) is incident to vertex \( j \), we have \( \sum_{k=n}^{n} (s_{n,k} - 1) \geq 0 \), \( A_{e_j} = \sum_{k=n}^{n} A_{e_j,k} \), and \( s_{e_j} - A_{e_j} - 1 \leq \sum_{k=n}^{n} (s_{e_j,k} - A_{e_j,k} - 1) \). Hence, we can rewrite the ratios of factorials in rows 2 and 3 into rising factorials. This gives us the expression in [4] but still retaining the upper bounds

of \( K - 1 \). It remains to show that the summation term is equal to zero if \( A_{e_j} \geq K \) for some edge \( e_j \).

Now, suppose \( 0 \leq A_{e_j} \leq s_{e_j} - 1 \) holds for all edges \( e_j \), but there exists some edge \( e_j \) such that \( A_{e_j} \geq K \). Let \( G' = (V, E') \) be the subgraph of \( G \) such that \( \{i, j\} \in E' \) if and only if \( A_{e_j} \geq K \). As \( G' \) is a forest, there must be a vertex \( \ell \) that is incident to some edge \( \{j, \ell\} \in G' \), but \( e_\ell \notin G' \). If \( \ell = n \), then

\[
\left( K - R_n - \sum_{k=n}^{n} A_{n,k} + 1 \right) \left( \frac{\sum_{k=n}^{n} (s_{n,k} - 1) + R_n}{\sum_{k=n}^{n} (s_{n,k} - 1)} \right) = \frac{(K + \sum_{k=n}^{n} (s_{n,k} - A_{n,k} - 1))!}{(K - R_n - \sum_{k=n}^{n} A_{n,k})!}
\]

as \( A_{n,k} \leq s_{n,k} - 1 \) and \( K - \sum_{k=n}^{n} A_{n,k} \leq K - A_{n,k} \leq 0 \). Otherwise, we have \( \ell \neq n \), which yields

\[
\left( K - R_\ell - \sum_{k=\ell}^{n} A_{\ell,k} + 1 \right) \left( \frac{R_\ell + \sum_{k=\ell}^{n} A_{\ell,k} - A_{e_\ell} - 1}{\sum_{k=\ell}^{n} (s_{n,k} - 1)} \right) = \frac{(K - A_{\ell,j} - 1)!}{(K - R_\ell - \sum_{k=\ell}^{n} A_{\ell,k})!}
\]

as \( e_\ell \notin G' \) implies that \( A_{e_\ell} \leq K - 1 \), and \( \{ j, \ell \} \in G' \) implies that \( K - \sum_{k=\ell}^{n} A_{\ell,k} \leq K - A_{\ell,j} - 1 \). In both cases, at least one of the rising factorials is zero within the summation term, so the entire term is zero if \( A_{e_j} \geq K \).

Finally, since the number of terms in the rising factorials is independent of \( K \), and the number of summation terms is determined by the \( s_{e_j} \)'s, the expression for \( v_{n,K:R}^{(s)} \) as written in this corollary is a polynomial in \( K \), as desired.

For example, suppose \( n = 3 \) and \( s_{1,2} = 0 \). Then, our formula for \( v_{n,K:R}^{(s)} \) in [3] can be written as

\[
v_{n,K:R}^{(s)} = \frac{(R_1 + s_{1,2} + s_{2,3} - 1)!}{(R_1 - 1)! (R_2 - 1)! (R_2 - 1)! (R_3 - 1)! (R_1 + s_{1,2} + s_{2,3} - 2)!} \times \sum_{A_{1,3}=0}^{s_{1,3}-1} \sum_{A_{2,3}=0}^{s_{2,3}-1} (K - R_3 - A_{1,3} - A_{2,3} + 1) (R_1 + s_{1,3} + s_{2,3} - 3)! \times \left[ (K - R_1 - A_{1,3} + 1) (R_1 - 1) (K - R_2 - A_{2,3} + 1)(R_2 - 1) \right]
\]

which is a polynomial in \( K \).

With this corollary, we have obtained an expression for \( v_{n,K:R}^{(s)} \) that is a polynomial in \( K \) for all \( R \geq 1 \), if the support graph of \( s \) is a tree. We can substitute this into [Theorem 10] to obtain a polynomial expression for \( f_{n,K}^{(q,s)} \) by [Theorem 9]. Then, using [2], we can substitute \( K = x \) into the expression for \( f_{n,K}^{(q,s)} \) to obtain \( A_n^{(q,s)}(x) \), proving [Theorem 1] as desired.
6. ACKNOWLEDGEMENTS

Many thanks for the help of I.P. Goulden for supporting me in my doctoral studies, during which this research is conducted, as well as the editing and verifying of the results in this paper.

REFERENCES

[1] G.E. Andrews, R. Askey, and R. Roy. Special Functions. Cambridge University Press, 1999.
[2] A.C.S. Chan. Combinatorial Methods for Enumerating Maps in Surfaces of Arbitrary Genus. PhD thesis, University of Waterloo, 2016.
[3] A.C.S. Chan. Methods of enumerating two vertex maps of arbitrary genus. forthcoming, 2017.
[4] I.P. Goulden and A. Nica. A direct bijection for the Harer-Zagier formula. Journal of Combinatorial Theory, Series A, 111(2):224–238, August 2005.
[5] I.P. Goulden and W. Slofstra. Annular embeddings of permutations for arbitrary genus. Journal of Combinatorial Theory, Series A, 117(3):272–288, April 2010.
[6] J. Harer and D. Zagier. The Euler characteristic of the moduli space of curves. Inventiones Mathematicae, 85:457–486, 1986.
[7] C. Itzykson and J.-B. Zuber. Matrix integration and combinatorics of modular groups. Communications in Mathematical Physics, 134(3):197–207, 1990.
[8] D.M. Jackson. On an integral representation for the genus series for 2-cell embeddings. Transactions of the American Mathematical Society, 344(2):755–772, August 1994.
[9] S. Kerov. Rook placements on ferrer boards and matrix integrals. Journal of Mathematical Sciences, 96(5):3531–3536, October 1999.
[10] M. Kontsevich. Intersection theory on the moduli space of curves and matrix airy functions. Communications in Mathematical Physics, 147:1–23, 1992.
[11] S.K. Lando and A.K. Zvonkin. Graphs on Surfaces and Their Applications, volume 141 of Encyclopaedia of Mathematical Sciences. Springer, 2004.
[12] B. Lass. Démonstration combinatoire de la formule de Harer-Zagier. Comptes Rendus de l’Académie des Sciences, Series I, 333:155–160, 2001.
[13] R.C. Penner. Perturbative series and the moduli space of Riemann surfaces. Journal of Differential Geometry, 27:35–53, 1988.
[14] D. Zagier. On the distribution of the number of cycles of elements in symmetric groups. Nieuw archief voor wiskunde, 13:489–495, 1995.