Quantum resonances in a single plaquette of Josephson junctions: excitations of Rabi oscillations

M. V. Fistul

Max-Planck Institut für Physik Komplexer Systeme, D-01187, Dresden, Germany

(March 22, 2022)

We present a theoretical study of a quantum regime of the resistive (whirling) state of dc driven anisotropic single plaquette containing three small Josephson junctions. The current-voltage characteristics of such a system display resonant steps that are due to the resonant interaction between the time dependent Josephson current and the excited electromagnetic oscillations (EOs). The voltage positions of the resonances are determined by the quantum interband transitions of EOs. We show that in the quantum regime as the system is driven on the resonance, coherent Rabi oscillations between the quantum levels of EOs occur. At variance with the classical regime the magnitude and the width of resonances are determined by the frequency of Rabi oscillations that in turn, depends in a peculiar manner on an externally applied magnetic field and the parameters of the system.

Various quantum effects predicted and observed in macroscopic systems have attracted a great attention as it allows to understand the foundation of quantum mechanics and the applicability of quantum mechanics to dissipative systems. Moreover, the interest to this field has been boosted by the possibility to use macroscopic quantum coherent effects for quantum computation.

In this field of study Josephson coupled systems consisting of a few interacting Josephson junctions, are of special interest. These systems contain a large number of particles and still, their behaviour is determined by the macroscopic variables, namely Josephson phases \( \phi_i(t) \). Moreover, the dynamics of Josephson phases can be controlled by an externally applied magnetic field \( H_{\text{ext}} \) and dc bias \( \gamma \), and at low temperatures the number of quasi-particles is extremely small and therefore, the dissipation caused by the quasi-particle current is also small.

Indeed, the peculiar macroscopic quantum effects such as tunneling and resonant tunneling of Josephson phase, discrete energy levels have been observed in single Josephson junctions, SQUID systems etc. In the presence of an externally applied microwave radiation the enhancement of both tunneling and resonant tunneling of Josephson phase, have been also observed. These effects can be considered as an evidence of the quantum coherent dynamics, i.e. the presence of coherent Rabi oscillations in macroscopic systems. The majority of macroscopic quantum effects have been studied as the Josephson junctions were biased in the superconducting state, i.e. zero dc voltage state, and the quantum mechanical behaviour of the resistive state of Josephson coupled systems has not been analyzed.

It is also well known that in the classical regime the resistive state of Josephson coupled systems displays intrinsic Josephson current oscillations. The oscillating Josephson current can excite the electromagnetic oscillations (EOs) in the superconducting loops, and in turn, these EOs resonantly interact with the time dependent Josephson current. In the case of a weak damping such an interaction leads to a pronounced resonant step in the current-voltage characteristics (I-V curves). The voltage position of the resonance \( V \) is determined by the characteristic frequency of EOs \( \omega_0 \) as

\[
V = \frac{h\omega_0}{2e}.
\]

The magnitude of the resonance depends on an externally applied magnetic field \( H_{\text{ext}} \) and the width of the resonance is determined by the damping parameter that, in the classical regime, is due to the presence of a quasi-particle (dissipative) current.

In this paper we present a theoretical (semiclassical) analysis of the quantum coherent effects in the resistive (whirling) state of a dc driven single anisotropic plaquette containing three small Josephson junctions. This system consists of two vertical junctions parallel to the bias current \( \gamma \) and a horizontal junction in the transverse direction, as presented in Fig. 1.

The dynamics of the system crucially depends on two parameters: the anisotropy \( \eta = \frac{L_H}{L_V} \), where \( L_H \) and \( L_V \) are respectively the critical currents of horizontal and vertical junctions, and the discreteness parameter (normalized inductance of the cell), \( \beta \). Moreover, the quantum effects are enhanced in the limit of a small Josephson energy \( E_{J,t} = \frac{h\Delta}{2e} \leq h\omega_p \), where \( \omega_p \) is the plasma frequency. In this limit it is also naturally to apply an external charge \( Q \) to the horizontal junction (see, Fig. 1). This charge controls the frequency of transitions between different quantum levels of EOs. Such a system presents a simplest case allowing to couple the Josephson current oscillations with a nonlinear oscillator (horizontal junction), and therefore, to remove the quantum-classical correspondence of a harmonic oscillator and to observe the quantum effects in the resistive state. Note here, that the coherent quantum-mechanical behaviour of a single plaquette of Josephson junctions biased in the superconducting state, have been studied in details in Refs.
The frequency \( \omega = 2eV/h \) is determined by the dc voltage \( V \) across the junction. As a result we find that the supercurrent flowing through the vertical junctions \( I_s \) is expressed in the form:

\[
I_s = I_{cV} < \sin(\omega t) \cos(\pi f + \xi(t)) > ,
\]

where \(< ... >\) means the time-average procedure.

Next, to simplify the analysis, we consider a small plaquette of Josephson junctions as the discreteness parameter \( \beta_L \ll 1 \). In this case the relationship \( \xi(t) = \varphi_h/2 \) is valid, and the system is characterized by one degree of freedom \( \xi \). Introducing the canonical momentum \( \hat{p}_\xi = \partial \hat{L}/\partial \dot{\xi} \) and the corresponding operator of momentum \( \hat{\rho}_\xi = -i\hbar \partial /\partial \xi \), we arrive at the time-dependent Hamiltonian

\[
\dot{\hat{H}}(t) = \hat{H}_0 - 2E_f \cos(\pi f + \xi) \cos(\omega t) ,
\]

\[
\hat{H}_0 = \frac{\omega^2}{E_f(4 + 8\eta)}(\hat{p}_\xi - 4\eta \alpha v_g)^2 - E_f \eta \cos 2\xi .
\]

Here, \( \hat{H}_0 \) is the Hamiltonian of the autonomous nonlinear oscillator, where the first term presents the total charging energy of the system and the second term is the Josephson energy of the horizontal junction. The last term in \( \hat{H}(t) \) presents an intrinsic magnetic field dependent coupling between the time dependent Josephson current and EOs.

We are interested in the resonant interaction between the ac Josephson current and EOs, and thus, two relevant energy levels \( E_m \) and \( E_n \) of the Hamiltonian \( \hat{H}_0 \), namely \( \omega_{nm}(v_g) = \omega_{nm} - \omega \), are important for our problem. These energy levels may be controlled by an externally applied gate voltage \( v_g \). Because a nonlinear oscillator has no coinciding frequency differences \( \omega_{nm} \), we may truncate our system to the two-level system. With this crucial assumption the Hamiltonian \( \hat{H}(t) \) is written in a simple form:

\[
\hat{H}(t) = \frac{\omega_{nm}}{2} \hat{\sigma}_z + E_f(a_{nn} - a_{mm}) \cos(\omega t) \hat{\sigma}_z
\]

\[-2E_f(a_{nm}) \cos(\omega t) \hat{\sigma}_x ,
\]

where the matrix elements \( a_{nm} \) are

\[
a_{nm} = \int_0^{2\pi} d\xi \psi_{n,m}^*(\xi; v_g)\psi_{m,n}(\xi; v_g) \cos(\pi f + \xi) .
\]

Here, \( \psi_{n,m}(x; v_g) \) is the gate-voltage dependent wave functions of the autonomous nonlinear oscillator, and \( \hat{\sigma}_x, \hat{\sigma}_z \) are the Pauli matrices.

Next, we use the standard density matrix approach. In the case of a weak damping the corresponding time-dependent equation for the density matrix \( \hat{\rho}(t) \) is taken in the form

\[
\hat{h}\dot{\hat{\rho}}(t) = -i[\hat{H}(t), \hat{\rho}(t)] + [\hat{H}_R, (\hat{\rho}(t) - \rho_s)] ,
\]

where \( \rho_s \) is the steady state density matrix.
where $\rho_{\beta}$ is the equilibrium density matrix, and the dissipative operator $\hat{H}_R$ characterizes the various relaxation processes. In a simplest case this operator is described by two damping parameters $\nu_{1,2}$. By making use of (9) the supercurrent $I_s$ is expressed through the quantum-mechanical average of the operators $\hat{a}_{x,z}$ as

$$I_s = I_{cV} \leq |a_{nm}| \sin(\omega t) \text{Tr} \{ \hat{\rho}(t) \hat{a}_x \} + \frac{a_{nm} - a_{nm}}{2} \sin(\omega t) \text{Tr} \{ \hat{\rho}(t) \hat{a}_x \}.$$  

Eq. (9) is a particular case of the well-known Bloch equations and by using the rotation wave approximation we finally obtain $I_s$ as

$$I_s = \frac{2eE_j^2}{\hbar^2} |a_{nm}|^2 \frac{\nu_2}{(\omega - \omega_{nm})^2 + \nu_2^2 + 2(\frac{E_j^2}{\hbar^2})^2(\frac{\nu_1}{\nu_2})|a_{nm}|^2}.$$  

(10)

Thus, Eq. (10) shows that the current-voltage characteristics of the plaquette with three small Josephson junctions can display a number of resonances. The physical origin of these resonances is the resonant absorption of the ac Josephson oscillations by the horizontal Josephson junction being in the superconducting state. The voltage positions of the resonances $V \approx \hbar\omega_{nm}/2e$ are mapped to the various transitions occurring in the spectrum of EOs (the Josephson phase of the horizontal junction). The width of the resonances is determined by the relaxation processes as $\nu \gg E_j|a_{nm}|/\hbar$, or the frequency $\omega_R \approx \frac{E_j}{\hbar a_{nm}}$ of coherent Rabi oscillations in the opposite limit $(\nu \ll E_j|a_{nm}|/\hbar)$. The maximum value of the magnitude of the resonance depends on the damping parameters $\nu_{1,2}$, and may reach the value $E_{nm}^{max} \approx e\nu_2$. Note here that we assumed the low temperature regime ($T \leq \hbar\omega_{nm}$) and did not take into account processes involving multi-photon interactions between the ac Josephson current and EOs. These multi-photon interactions lead to additional subharmonic resonances ($\omega \approx \omega_{nm}/k$) with smaller magnitude.

The spectrum $E_n(v_g)$ and the corresponding wave functions are found as periodic solutions of the Schrödinger equation:

$$\hat{H}_0 \psi_n(\xi; v_g) = E_n(v_g) \psi_n(\xi; v_g).$$  

(11)

It is well known that the spectrum $E_n(v_g)$ of Eq. (11) contains an infinite number of bands and is controlled by the gate-voltage $v_g$. Although the solutions of (11) can be analyzed by making use of the Mathieu functions for arbitrary ratio $E_j/\hbar\omega_p$, here we consider the regime of small Josephson energy, $E_j \ll \hbar\omega_p$, where all results are simplified. In this limit, and at low temperatures as the transitions between the ground state and the excited levels are important, we obtain

$$\omega_{n0} \approx \frac{\omega_p^2}{E_j(4 + 8\eta)}|n(n - 8\eta\alpha v_g)|$$  

(12)

The typical dependence of $E_n(v_g)$ and a number of transitions are shown in Fig. 2a.

| FIG. 2. a) The dependence of $\epsilon_n(v_g) = \frac{E_j(4 + 8\eta)}{k^2\omega_p^2}$ in the limit of zero Josephson energy. The induced charge $Q = 4\epsilon_0\alpha v_g$. The arrows show possible transitions in the spectrum of EOs. b) The quantum resonances in the current-voltage characteristics. The resonances correspond to the transitions presented in Fig. 2a. The voltage is normalized to the plasma frequency $\hbar\omega_p/2e$. The parameters are $\eta = 1$, $\nu_1 = \nu_2 = 0.1\omega_p$, $\frac{E_j}{\hbar\omega_p} = 0.3$, $Q = 0.1$. |

By making use of a perturbation theory the relevant matrix elements $a_{nm}$ are obtained in this limit. Thus, e.g. $a_{\pm 1 0} \approx 1$, $a_{\pm 3 0} \approx \left(\frac{E_j}{\hbar\omega_p}\right)^2$. The transition $0 \rightarrow 2$ is not appearing in the $I-V$ curve because the matrix element $a_{2 0}$ is rather small. It is due to a specific symmetry of the potential energy ($\propto \cos 2\xi$) in the Hamiltonian $\hat{H}_0$. The calculated resonant current-voltage characteristics is presented in Fig. 2b. As the Josephson energy $E_j$ is small, the voltage positions of the resonances are strongly affected by the gate voltage $v_g$ but the widths and the magnitudes of the resonances weakly depend on the externally applied magnetic field $H_{ext}$. In the opposite case as the Josephson energy is large, $E_j \gg \hbar\omega_p$, the situation is reversed: the voltage positions of the resonances are weakly altered by $v_g$ but the width of the resonances strongly depends on $H_{ext}$. In the case of intermediate values of $E_j \approx \hbar\omega_p$ the strong dependence of the resonant current-voltage characteristics on both parameters $v_g$ and $H_{ext}$ is found.

In conclusion we have shown that a particular system
of a single plaquette containing three small Josephson junctions display resonances in the \( I-V \) curve. These resonances are due to the resonant absorption of intrinsic ac Josephson oscillations and are the fingerprints of various transitions between the discrete energy levels of the macroscopic Josephson phase. The coherent quantum-mechanical dynamics of these transitions may be controlled by variation of the bias current \( \gamma \), the gate voltage \( V_g \) and an externally applied magnetic field \( H_{\text{ext}} \). Finally, we note that similar quantum resonances may be found also in more complex Josephson (or mixed) coupled systems, e.g. the inductively coupled dc and RF SQUIDs, dc SQUID and quantum dots, etc. The measurements of these resonances and their dependence on the parameters of the system should allow to study in detail the coherent quantum-mechanical dynamics of macroscopic variables.

I thank S.-G. Chung, S. Flach, P. Hakonen, and A. V. Ustinov for useful discussions.

---

1. A. O. Caldeira and A. T. Leggett, Ann. Phys. 149, 374 (1983).
2. M. Tinkham, *Introduction to Superconductivity*, McGraw-Hill, New York, 1996.
3. Devoret M. H., J. M. Martinis and J. Clarke, Phys. Rev. Lett., 55 1908 (1985).
4. T. P. Orlando, J. E. Mooij, L. Tian, C. H. van der Wal, L. S. Levitov, S. Lloyd, J. J. Mazo, Phys. Rev. B, 60, 15398 (1999).
5. C, H. van der Wal, A. C. J. ter Haar, F. K. Wilhelm, R. N. Schouten, C. J. P. M. Harmans, T. P. Orlando, S. Lloyd and J. E. Mooij, Science, 290, 773 (2000).
6. Friedman J. R., Patel V., Chen W., Tolpygo S. K., and Lukens J. E., *Nature*, 406, (2000) 43-46; Rouse R., Han S. Y., and Lukens J. E., *Phys. Rev. Lett.* 75, (1995) 1614 ;
7. T. Dittrich, P. Hänggi, G. Ingold, B. Kramer, G. Schön, and W. Zwerger *Quantum Transport and Dissipation*, Wiley-VCH, 1998.
8. H. Dekker, Phys. Rep., 80, 1 (1981).
9. A. O. Caldeira and A. J. Leggett, Physica 121A, 587 (1983).
10. 42. A. Wallraff, Y. Koval, M. Levitchev, M. V. Fistul, and A. V. Ustinov, J. Low Temp. Phys. 118, 543 (2000)
11. A similar spectroscopy of discrete levels of macroscopic Josephson phase has been carried out experimentally for a SQUID loop coupled to a single small Josephson junction, see R. Lindell, J. Penttilä, M. Paalanen, and P. Hakonen, Bulletin of the APS, 46, No. 1 (2001).
12. G. -L. Ingold and Yu. V. Nazarov, in *Single Charge Tunneling Coulomb Blockade Phenomena in Nanostructures*, edited by H. Grabert and M. H. Devoret (Plenum Press, New York, 1992, Chap. 2.
13. K. Blum, *Density matrix theory and applications*, Plenum, NY, (1981).
14. R. Loudon, *The quantum theory of light*, Oxford, Clarendon