The ultraviolet and infrared perturbative finiteness of massless QED$_3$

O.M. Del Cima,* D.H.T. Franco,† and O. Piguet‡

Universidade Federal de Viçosa (UFV),
Departamento de Física - Campus Universitário,
Avenida Peter Henry Rolfs s/n - 36570-000 - Viçosa - MG - Brazil.
(Dated: May 3, 2018)

In memory of my beloved mother Victoria Monteiro Del Cima (1920-2013)

The massless QED$_3$ is ultraviolet and infrared perturbatively finite, parity and infrared anomaly free to all orders in perturbation theory.

PACS numbers: 11.10.Gh 11.15.-q 11.15.Bt 11.15.Ex

The perturbative finiteness is one of the most peculiar properties of topological field theories in three space-time dimensions [1]. Thanks to a by-product of superrenormalizability and the presence of topological terms, Yang-Mills-Chern-Simons and BF-Yang-Mills theories are also finite at all orders in perturbation theory – in the sense of vanishing β-function [2]. In spite of not being a topological field theory, the massless QED$_3$ is also perturbatively finite, exhibiting quite interesting and subtle properties as superrenormalizability, parity invariance and the presence of infrared divergences. The issue of “how superrenormalizable interactions cure their infrared divergences” has been analyzed in [3], and a possible parity breaking at the quantum level, in the literature called parity anomaly, has been discarded [4–6].

The algebraic proof we are presenting in this letter on the ultraviolet and infrared finiteness, and absence of parity and infrared anomaly, in the massless QED$_3$, is based on general theorems of perturbative quantum field theory [7–10], where the Lowenstein-Zimmermann subtraction scheme is adopted. Here we summarize the main results skipping the intermediate steps of the Lowenstein-Zimmermann subtraction scheme in the framework of BPHZL renormalization method [10]. Such subtraction scheme has to be introduced, thanks to the presence of massless (gauge and fermion) fields, in order to subtract infrared divergences that should arise in the process of the ultraviolet subtractions.

The discussion of the extension of the theory in the tree-approximation to all orders in perturbation theory is organized according to two independent parts: in the first step, we study the stability of the classical action. For the quantum theory the stability corresponds to the fact that the radiative corrections can be reabsorbed by a redefinition of the initial parameters of the theory. Next, one computes the possible anomalies through an analysis of the Wess-Zumino consistency condition, then one checks if the possible breakings induced by radiative corrections can be fine-tuned by a suitable choice of non-invariant local counterterms.

The gauge invariant action for the massless QED$_3$, with the gauge invariant Lowenstein-Zimmermann mass terms added, is given by:

\[ \Sigma_{\text{inv}}^{(s-1)} = \int d^3x \left\{ -\frac{1}{4} F_{\mu\nu}^2 + i\bar{\psi} \gamma^\mu \partial_\mu \psi + \frac{\mu}{2} (s-1) \epsilon_{\mu
u\rho} A_\mu \partial_\nu A_\rho - m(s-1)\bar{\psi}\psi \right\} , \]

(1)

where \( \bar{\psi} \equiv (\bar{\psi} + icA)\psi \) and \( \epsilon \) is a dimensionful coupling constant with mass dimension \( \frac{1}{2} \). The Lowenstein-Zimmermann parameter \( s \) lies in the interval \( 0 \leq s \leq 1 \) and plays the role of an additional subtraction variable (as the external momentum) in the BPHZL renormalization program, such that the massless QED$_3$ is recovered for \( s = 1 \).

In the BPHZL scheme a subtracted (finite) integrand, \( R(p,k,s) \), is written in terms of the unsubtracted (divergent) one, \( I(p,k,s) \), as

\[ R(p,k,s) = (1 - t_{p,s-1})I(p,k,s) \]

\[ = (1 - t_{0,s-1}^0 + t_{1,s-1}^1)I(p,k,s) , \]

where \( t_{d,s} \) is the Taylor series about \( x = y = 0 \) to order \( d \) if \( d \geq 0 \). Thus, for our purposes, by assuming \( s = 1 \), a subtracted integrand, \( R(p,k,s) \), reads

\[ R(p,k,1) = I(p,k,1) - \underbrace{I(0,k,1)}_{\text{parity-even}} - \underbrace{p^\rho \frac{\partial}{\partial p^\rho}}_{\text{parity-odd terms}} I(0,k,0) . \]

In order to quantize the system (1) one has to add a gauge-fixing action, \( \Sigma_{\text{gf}} \), and an action term, \( \Sigma_{\text{ext}} \), coupling the non-linear BRS transformations to external sources:

\[ \Sigma_{\text{gf}} = \int d^3x \left\{ b\partial^\mu A_\mu + \frac{\xi}{2} b^2 + \tau c \right\} , \]

(2)

\[ \Sigma_{\text{ext}} = \int d^3x \left\{ \bar{\Omega} s - s\bar{\psi}\right\} . \]
No Lowenstein-Zimmermann mass has to be introduced to the Faddeev-Popov ghosts since they are free fields, therefore, they decouple.

The BRS transformations are given by:
\[ s\psi = ic\psi, \quad s\overline{\psi} = -ic\overline{\psi}, \]
\[ sA_\mu = -\frac{1}{e}\partial_\mu c, \quad sc = 0, \]
\[ s\overline{c} = \frac{1}{e}b, \quad sb = 0, \]  
(4)

where \( c \) is the ghost, \( \overline{c} \) is the antighost and \( b \) is the Langevin multiplier field.

The following nilpotency identities hold:
\[ \delta S = 0, \]
(5)

The UV and IR dimensions – these dimensions are those which are involved in the Lowenstein-Zimmermann subtraction scheme \([10]\) – \( d \) and \( r \), respectively, as well as the ghost numbers, \( \Phi \Pi \), and the Grassmann parity, \( GP \), of all fields are collected in Table I.

The BRS invariance of the action is expressed in a functional way by the Slavnov-Taylor identity
\[ S(\Sigma^{(s-1)}) = 0, \]  
(6)

where the Slavnov-Taylor operator \( S \) is defined, acting on an arbitrary functional \( F \), by
\[ S(F) = \int d^3x \left\{ -\frac{1}{e} \partial_\mu c \frac{\delta F}{\delta A^\mu} + \frac{1}{e} b \frac{\delta F}{\delta \overline{c}} + \delta F \frac{\delta F}{\delta \Omega} \right\}. \]  
(7)

The corresponding linearized Slavnov-Taylor operator reads
\[ S_F = \int d^3x \left\{ -\frac{1}{e} \partial_\mu c \frac{\delta \delta F}{\delta A^\mu} + \frac{1}{e} b \frac{\delta \delta F}{\delta \overline{c}} + \delta F \frac{\delta \delta F}{\delta \Omega} \right\}. \]  
(8)

The following nilpotency identities hold:
\[ S_F S(F) = 0, \quad \forall F, \]
\[ S_F S_F = 0 \quad \text{if} \quad S(F) = 0. \]  
(9) \tag{10}

In particular, \( (S_\Sigma)^2 = 0 \), since the action \( \Sigma^{(s-1)} \) obeys the Slavnov-Taylor identity (6). The operation of \( S_\Sigma \) upon the fields and the external sources is given by
\[ S_\Sigma \phi = s\phi, \quad \phi = \psi, \overline{\psi}, A_\mu, c, \overline{c}, b, \]
\[ S_\Sigma \Omega = -\frac{\delta \Sigma}{\delta \psi}, \quad S_\Sigma \overline{\Omega} = \frac{\delta \Sigma}{\delta \psi}. \]  
(11)

The classical action \( \Sigma^{(s-1)} \) is moreover characterized by the gauge condition, the ghost equation and the antighost equation, given by:
\[ \frac{\delta \Sigma^{(s-1)}}{\delta \delta b} = \partial^\mu A_\mu + \xi b, \]  
(12)
\[ \frac{\delta \Sigma^{(s-1)}}{\delta \delta c} = \Box c, \]  
(13)
\[ -i \frac{\delta \Sigma^{(s-1)}}{\delta \delta c} = i \Box c + \overline{\Omega} \psi + \overline{\psi} \Omega. \]  
(14)

The action is invariant also with respect to the rigid symmetry
\[ W_{\text{rigid}} \Sigma^{(s-1)} = 0, \]  
(15)

where the Ward operator, \( W_{\text{rigid}} \), is defined by
\[ W_{\text{rigid}} = \int d^3x \left\{ \psi \frac{\delta}{\delta \psi} - \overline{\psi} \frac{\delta}{\delta \overline{\psi}} + \Omega \frac{\delta}{\delta \Omega} - \overline{\Omega} \frac{\delta}{\delta \overline{\Omega}} \right\}. \]  
(16)

The classical action for the massless QED \( s = 1 \) is also invariant under parity, \( P \), its action upon the fields and external sources is fixed as below:
\[ x_\mu \xrightarrow{P} x_\mu = (x_0, -x_1, x_2), \]
\[ \psi \xrightarrow{P} \psi = -\gamma_1 \psi, \quad \overline{\psi} \xrightarrow{P} \overline{\psi} = -i \psi \gamma_1, \]
\[ A_\mu \xrightarrow{P} A_\mu = (A_0, -A_1, A_2), \]
\[ \phi \xrightarrow{P} \phi = \phi, \quad \phi = c, c, b, \]
\[ \Omega \xrightarrow{P} \Omega = -\gamma_1 \Omega, \quad \overline{\Omega} \xrightarrow{P} \overline{\Omega} = i \overline{\gamma} \psi \gamma_1. \]  
(17)

In order to verify if the action in the tree-approximation is stable under radiative corrections, we perturb it by an arbitrary integrated local functional (counterterm) \( \Sigma^{c(s-1)} \), such that
\[ \Sigma^{c(s-1)} = \Sigma^{(s-1)} + \varepsilon \Sigma^{c(s-1)}, \]  
(18)

where \( \varepsilon \) is an infinitesimal parameter. The functional \( \Sigma^{(s)} \equiv \Sigma^{c(s)}_{s=1} \) has the same quantum numbers as the action in the tree-approximation at \( s = 1 \).

The deformed action \( \Sigma^{c(s-1)} \) must still obey all the constraints listed above, Eqs.(12-15). Then \( \Sigma^{c(s-1)} \) is subjected to the following set of constraints:
\[ S_\Sigma \Sigma^{c(s-1)} = 0, \]  
(19)
\[ \frac{\delta \Sigma^{c(s-1)}}{\delta \delta b} = \frac{\delta \Sigma^{c(s-1)}}{\delta \delta c} = \frac{\delta \Sigma^{c(s-1)}}{\delta \delta c} = 0, \]  
(20)
\[ W_{\text{rigid}} \Sigma^{c(s-1)} = 0. \]  
(21)

We find that the most general invariant counterterm \( \Sigma^{c(s-1)} \), i.e., the most general field polynomial with UV and IR dimensions bounded by \( d \leq 3 \) and \( r \geq \frac{d}{2} \), with ghost number zero and fulfilling the conditions displayed in Eqs.(19-21), is given by:
\[ \Sigma^{c(s-1)} = \int d^3x \left\{ a_1 F^{\mu\nu} F_{\mu\nu} + a_2 \overline{\psi} \overline{\psi} \right\}. \]  
(22)
However, there are other restrictions due to the super-renormalizability of the theory and its parity invariance – the massless QED$_3$ recovered for $s = 1$. From the superrenormalizability, the coupling constant-dependent power-counting formula [2] is given by:

$$
\left( \frac{\delta(\gamma)}{\rho(\gamma)} \right) = 3 - \sum_{\Phi} \left( \frac{d_{\Phi}}{r_{\Phi}} \right) N_{\Phi} - \frac{1}{2} N_e ,
$$

(23)

for the UV $(\delta(\gamma))$ and IR $(\rho(\gamma))$ degrees of divergence of a 1-particle irreducible Feynman graph, $\gamma$. Here $N_{\Phi}$ is the number of external lines of $\gamma$ corresponding to the field $\Phi$, $d_{\Phi}$ and $r_{\Phi}$ are the UV and IR dimensions of $\Phi$, respectively, as given in Table I, and $N_e$ is the power of the coupling constant $e$ in the integral corresponding to the diagram $\gamma$. Since the counterterms are generated by loop graphs, they are of order two in $e$ at least. Hence, the effective UV and IR dimensions of the counterterm $\Sigma^{(s-1)}$ are bounded by $d \leq 2$ and $r \geq \frac{3}{2}$, by this reason, $a_1 = a_2 = 0$. Moreover, since the counterterm $\Sigma^c \equiv \Sigma^{|s=1}$ is also parity invariant, it yields that $a_3 = a_4 = 0$. It can be concluded that there is no possibility for any local deformation, implying the absence of any counterterm:

$$
\Sigma^c = \Sigma^{|s=1} = 0 .
$$

(24)

This result means that the usual ambiguities due to the renormalization procedure do not appear in the present model. Because the classical stability does not imply in general the possibility of extending the theory to the quantum level, our purpose now is to show the absence of anomalies. This result, combined with the previous one (24), concerning the absence of counterterms, completes the proof of the perturbative finiteness and absence of a parity anomaly in massless QED$_3$.

At the quantum level the vertex functional, $\Gamma^{(s-1)}$, which coincides with the classical action (5) at order 0 in $\hbar$,

$$
\Gamma^{(s-1)} = \Sigma^{(s-1)} + O(\hbar) ,
$$

(25)

has to satisfy the same constraints as the classical action does, namely, Eqs.(12-15).

According to the Quantum Action Principle [7, 9] the Slavnov-Taylor identity (6) may get a quantum breaking

$$
\mathcal{S}(\Gamma^{(s-1)}) = \Delta \cdot \Gamma^{(s-1)}|_{s=1} = \Delta + O(\hbar \Delta) ,
$$

(26)

where $\Delta \equiv \Delta|_{s=1}$ is an integrated local functional, taken at $s = 1$, with ghost number 1 and UV and IR dimensions bounded by $d \leq \frac{3}{2}$ and $r \geq 3$.

The nilpotency identity (9) together with

$$
\mathcal{S}_\Gamma = \mathcal{S}_\Sigma + O(\hbar)
$$

(27)

implies the following consistency condition for the breaking $\Delta$:

$$
\mathcal{S}_\Sigma \Delta = 0 ,
$$

(28)

beyond that, $\Delta$ satisfies:

$$
\frac{\delta \Delta}{\delta b} = \frac{\delta \Delta}{\delta c} = \int d^3 x \frac{\delta}{\delta c} \Delta = W_{\text{rigid}} \Delta = 0 .
$$

(29)

The Wess-Zumino consistency condition (28) constitutes a cohomology problem in the sector of ghost number one. Its solution can always be written as a sum of a trivial cocycle $\mathcal{S}_\Sigma \hat{\Delta}^{(0)}$, where $\hat{\Delta}^{(0)}$ has ghost number 0, and of nontrivial elements belonging to the cohomology of $\mathcal{S}_\Sigma$ (8) in the sector of ghost number one:

$$
\Delta^{(1)} = \hat{\Delta}^{(1)} + \mathcal{S}_\Sigma \Delta^{(0)} .
$$

(30)

It should be stressed that it still remains a possible parity violation at the quantum level induced by a parity-odd noninvariant counterterm. Due to the fact that the Lowenstein-Zimmermann subtraction scheme breaks parity during the intermediary steps, the Slavnov-Taylor identity breaking, $\Delta^{(1)}$, is not necessarily parity invariant. In any case, $\Delta^{(1)}$ must obey the conditions imposed by Eqs.(28-29). The trivial cocycle $\mathcal{S}_\Sigma \hat{\Delta}^{(0)}$ can be absorbed into the vertex functional $\Gamma^{(s-1)}$ as a noninvariant integrated local counterterm, $-\hat{\Delta}^{(0)}$. On the other hand, a nonzero $\hat{\Delta}^{(1)}$ would represent an anomaly. If there was any parity-odd $\Delta^{(1)}$, a parity anomaly would be present induced by noninvariant counterterm, $-\hat{\Delta}^{(0)}$.

By analysing the Slavnov-Taylor operator $\mathcal{S}_\Sigma$ (8) and the Eq.(26), one sees that the breaking $\Delta^{(1)}$ has UV and IR dimensions bounded by $d \leq \frac{7}{2}$ and $r \geq 3$. But being an effect of the radiative corrections, the insertion $\Delta^{(1)}$ possesses a factor $e^2$ at least, and thus its effective dimensions are in fact bounded by $d \leq \frac{5}{2}$ and $r \geq 2$.

From the antighost equation

$$
\int d^3 x \frac{\delta}{\delta c} \Delta^{(1)} = 0 ,
$$

(31)

it can be concluded that $\Delta^{(1)}$ is given by

$$
\Delta^{(1)} = \int d^3 x (\partial^\mu c) K_\mu ,
$$

(32)

where $K_\mu$ has UV and IR dimensions bounded by $d \leq \frac{3}{2}$ and $r \geq 1$, the ghost $c$ is dimensionless. Now, $\Delta^{(1)}$ can be split into two pieces which are even and odd under parity by writing $K_\mu$ as

$$
K_\mu = r_v V_\mu + r_\rho P_\mu ,
$$

(33)

in such a way that $V_\mu$ is a vector and $P_\mu$ a pseudo-vector. Bearing in mind that $K_\mu$ has its UV and IR dimensions bounded by $d \leq \frac{3}{2}$ and $r \geq 1$, we conclude that there are no $V_\mu$ satisfying these dimensional constraints, therefore, $\{V_\mu\} = \emptyset$, which means the absence of parity-even Slavnov-Taylor breaking. However, still remains the odd sector represented by $P_\mu$, and by a dimensional analysis a candidate for $P_\mu$ is found. The only candidate which survives all the constraints above is

$$
P_\mu = F_\mu = \frac{1}{2} \epsilon_{\mu \nu \rho} F^{\nu \rho} .
$$

(34)
| $A_\mu$ | $\psi$ | $c$ | $\bar{c}$ | $b$ | $\Omega$ | $s-1$ | $s$ |
|-------|-------|-----|-------|-----|--------|-------|-----|
| $d$   | 1/2   | 1   | 0     | 1   | 3/2    | 2     | 1   |
| $r$   | 1/2   | 1   | 0     | 1   | 3/2    | 2     | 1   |
| $\Phi\Pi$ | 0     | 0   | 1     | -1  | 0     | -1    | 0   |
| $GP$  | 0     | 0   | 1     | 1   | 0     | 1     | 0   |

TABLE I: UV and IR dimensions, $d$ and $r$, ghost numbers, $\Phi\Pi$, and Grassmann parity, $GP$. It turns out that there is only one parity-odd candidate, $\Delta^{(1)}_{\text{odd}}$, which could be a parity anomaly, surviving all the constraints above:

$$\Delta^{(1)} = \Delta^{(1)}_{\text{odd}} = \frac{r_p}{2} \int d^3 x (\partial^\mu c) \epsilon_{\mu\nu\rho} F^{\nu\rho} ,$$

(35)

where integrating by parts it shows that

$$\Delta^{(1)} = \Delta^{(1)}_{\text{odd}} \equiv 0 .$$

(36)

Hence, there is no radiative corrections to the insertion describing the breaking of the Slavnov-Taylor identity, $\{ \Delta^{(1)} \} = 0$, which means that there is no possible breaking to the Slavnov-Taylor identity, and neither parity is violated nor infrared anomaly stems by non-invariant counter-terms that could be induced due to the Lowenstein-Zimmermann subtraction scheme – which breaks parity.

We finally conclude that the massless QED$_3$ is infrared and ultraviolet finite (vanishing coupling constant $\beta_c$-function and anomaly dimensions of the fields), infrared and parity anomaly free at all orders in perturbation theory. The latter being a by-product of super-renormalizability and absence of parity-odd noninvariant counterterms.

O.M.D.C. dedicates this work to his daughter, Vittoria, to his son, Enzo, and to his mother, Victoria (in memoriam 01/05/1920–26/08/2013).

[1] E. Guadagnini, M. Martellini and M. Mintchev, Phys.Lett. B227 (1989) 111; Nucl.Phys. B330 (1990) 575; A. Blasi and R. Collina, Nucl.Phys. B345 (1990) 472; F. Delduc, C. Lucchesi, O. Piguet and S.P. Sorrela, Nucl.Phys. B346 (1990) 313; C. Lucchesi and O. Piguet, Nucl.Phys. B381 (1992) 281; N. Maggiore and S.P. Sorrela, Nucl.Phys. B377 (1992) 236; C. Lucchesi, O. Piguet and S.P. Sorrela, Nucl.Phys. B395 (1993) 325; O.M. Del Cima, J.M. Grimstrup and M. Schweda, Phys.Lett. B463 (1999) 48.

[2] O.M. Del Cima, D.H.T. Franco, J.A. Helayel-Neto and O. Piguet, JHEP 9802 (1998) 002; Lett.Math.Phys. 47 (1999) 265; JHEP 9804 (1998) 010.

[3] R. Jackiw and S. Templeton, Phys.Rev. D23 (1981) 2291.

[4] S. Rao and R. Yahalom, Phys.Lett. B172 (1986) 227.

[5] R. Delbourgo and A.B. Waites, Phys.Lett. B300 (1993) 241 and Austral.J.Phys. 47 (1994) 465.

[6] B.M. Pimentel and J.L. Tomazelli, Prog.Theor.Phys. 95 (1996) 1217.

[7] J.H. Lowenstein, Phys.Rev. D4 (1971) 2281 and Comm.Math.Phys. 24 (1971) 1; Y.M.P. Lam, Phys.Rev. D6 (1972) 2145 and Phys.Rev. D7 (1973) 2943; T.E. Clark and J.H. Lowenstein, Nucl.Phys. B113 (1976) 109.

[8] C. Becchi, A. Rouet and R. Stora, Comm.Math.Phys. 42 (1975) 127 and Ann.Phys.(N.Y.) 98 (1976) 287; O. Piguet and A. Rouet, Phys.Rep. 76 (1981) 1.

[9] O. Piguet and S.P. Sorella, *Algebraic Renormalization*, Lecture Notes in Physics, m28, Springer-Verlag (Berlin-Heidelberg), 1995; see also references therein.

[10] W. Zimmermann, Comm.Math.Phys. 15 (1969) 208 and *Lectures on Elementary Particles and Quantum Field Theory*, 1970 Brandeis lectures, eds. S. Deser, M. Grisaru and H. Pendleton, MIT Press (Cambridge-USA), 1971; J.H. Lowenstein and W. Zimmermann, Nucl.Phys. B86 (1975) 77; J.H. Lowenstein, Comm.Math.Phys. 47 (1976) 53 and *Renormalization Theory*, eds. G. Velo and A.S. Wightman, D. Reidel (Dordrecht-Holland), 1976; P. Breitenlohner and D. Maison, Comm.Math.Phys. 52 (1977) 55.