Space and time complexity of integration of FLINSPACE computable real functions in Ko–Friedman model

Sergey V. Yakhontov

Abstract

In the present paper it is shown that real function \( g(x) = \int_0^x f(t)dt \) is a linear-space computable real function on interval \([0,1]\) if \( f \) is a linear-space computable \( C^2[0,1] \) real function on interval \([0,1]\), and this result does not depend on any open question in the computational complexity theory. The time complexity of computable real functions and integration of computable real functions is considered in the context of Ko–Friedman model which is based on the notion of Cauchy functions computable by Turing machines.

Keywords: Computable real functions, Cauchy function representation, polynomial-time computable real functions, linear-space computable real functions, \( C^2[0,1] \) real functions, integration of computable real functions.

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1 Introduction

In the present paper, we consider computable real numbers and functions that are represented by Cauchy functions computable by Turing machines [1].

Main results regarding computable real numbers and functions can be found in [1–4]; main results regarding computational complexity of computations on Turing machines can be found in [5].

As is usual, the set of real functions whose 2-nd derivative exists and is continuous on interval \([0,1]\) is denoted by \( C^2[0,1] \), and the set of \( C^k[0,1] \) real functions for all \( k \geq 1 \) is denoted by \( C^\infty[0,1] \).

It is known [1] that real function \( g(x) = \int_0^x f(t)dt \) is polynomial-time computable real function on interval \([0,1]\) iff \( FP = \#P \) wherein \( f \) is a polynomial-time computable real function on interval \([0,1]\).
It means integration of polynomial-time computable real functions is as hard as string functions from complexity class \(#P\).

So this result from \([1]\) is relativized to question whether \(\text{FP} = \#P\) or not which is one of the open questions in the computational complexity theory.

In the present paper it is shown that real function \(g(x) = \int_0^x f(t)dt\) is a linear-space computable real function on interval \([0, 1]\) if \(f\) is a linear-space computable \(C^2[0, 1]\) real function on interval \([0, 1]\), and this result does not depend on any open question in the computational complexity theory.

### 1.1 CF computable real numbers and functions

Cauchy functions in the model defined in \([1]\) are functions binary converging to real numbers. A function \(\phi : \mathbb{N} \to \mathbb{D}\) (here \(\mathbb{D}\) is the set of dyadic rational numbers) is said to binary converge to real number \(x\) if

\[
|\phi(n) - x| \leq 2^{-n}
\]

for all \(n \in \mathbb{N}\); \(CF_x\) denotes the set of all functions binary converging to \(x\).

Real number \(x\) is said to be a \(CF\) computable real number if \(CF_x\) contains a computable function \(\phi\).

Real function \(f\) on interval \([a, b]\) is said to be a \(CF\) computable function on interval \([a, b]\) if there exists a function-oracle Turing machine \(M\) such that for all \(x \in [a, b]\) and for all \(\phi \in CF_x\) function \(\psi\) computed by \(M\) with oracle \(\phi\) is in \(CF_{f(x)}\).

The input of functions \(\phi\) and \(\psi\) is \(0^n\) (0 repeated \(n\) times) when a number or a function is evaluated to precision \(2^{-n}\).

**Definition 1.** \([1]\) Function \(f : [a, b] \to \mathbb{R}\) is said to be computable in time \(t(n)\) real function on interval \([a, b]\) if for all computable real numbers \(x \in [a, b]\) function \(\psi \in CF_{f(x)}\) (\(\psi\) is from the definition of \(CF\) computable real function) is computable in time \(t(n)\).

**Definition 2.** \([1]\) Function \(f : [a, b] \to \mathbb{R}\) is said to be computable in space \(s(n)\) real function on interval \([a, b]\) if for all computable real numbers \(x \in [a, b]\) function \(\psi \in CF_{f(x)}\) (\(\psi\) is from the definition of \(CF\) computable real function) is computable in space \(s(n)\).

\(\text{FP}\) denotes the class of string functions computable in polynomial time on Turing machines, \(\text{FLINSPACE}\) denotes the class of string functions computable in linear space on Turing machines, and \(\text{FEXPTIME}\) denotes the class of string functions computable in exponential time on Turing machines.

According to these notations, polynomial-time computable real functions are said to be \(\text{FP}\) computable real functions, linear-space computable real functions are said to be \(\text{FLINSPACE}\) computable real functions, and exponential-time computable real functions are said to be \(\text{FEXPTIME}\) computable real functions.

The set of \(\text{FP}\) computable real functions on interval \([a, b]\) is denoted by \(\text{FP}_{C[a,b]}\); the set of \(\text{FLINSPACE}\) computable real functions on interval \([a, b]\) is denoted by \(\text{FLINSPACE}_{C[a,b]}\); and the set of \(\text{FEXPTIME}\) computable real functions on interval \([a, b]\) is denoted by \(\text{FEXPTIME}_{C[a,b]}\).

The set of the \(C^2[0, 1]\) real functions from class \(\text{FP}_{C[a,b]}\) is denoted by \(\text{FP}_{C^2[a,b]}\), and the set of the \(C^\infty[0, 1]\) real functions from class \(\text{FP}_{C[a,b]}\) is denoted by \(\text{FP}_{C^\infty[a,b]}\).

### 1.2 Integration of \(\text{FP}\) computable real functions

The main results from \([1]\) regarding integration of \(\text{FP}\) computable real functions are the following.

**Theorem 1.** \([1\ 5.33]\) The following are equivalent:

a) Let \(f\) be in \(\text{FP}_{C[0,1]}\). Then, the function \(g(x) = \int_0^x f(t)dt\) is polynomial-time computable.

b) Let \(f\) be in \(\text{FP}_{C^\infty[0,1]}\). Then, the function \(g(x) = \int_0^x f(t)dt\) is polynomial-time computable.

c) \(\text{FP} = \#P\).
It means if $\text{FP} \neq \#\text{P}$ then the integral of a polynomial-time computable real function $f$ is not necessarily polynomial-time computable even if $f$ is known to be infinitely differentiable. But if $f$ is polynomial-time computable and is analytic on $[0, 1]$ then the integral of $f$ must be computable in polynomial time.

Some additional results regarding the time complexity of integration of computable real functions can be found in [6]. For example, the computation of the volume of a one-dimensional convex set $K$ is $\#\text{P}$-complete if $K$ is represented by a polynomial-time computable function defining its boundary.

## 2 Upper bound of the time complexity of integration

Let $f$ be a linear-space computable $C^2[0, 1]$ real function on interval $[0, 1]$. Let’s consider the composite trapezoidal rule [7] for function $f$ on interval $[a, x]$ for $a < x \leq b$:

$$g(x) = \int_a^x f(t)dt = \frac{h}{2} \left( f(t_0) + \sum_{i=1}^{k-1} 2 \cdot f(t_i) + f(t_k) \right) - \frac{(b-a)^3}{12 \cdot k^2} f''(\xi)$$

wherein $k$ is a natural number, $h = \frac{x-a}{k}$, $t_0 = a$, $t_i = a + i \cdot h$, $t_k = b$, and $a < \xi < x$. This equation on interval $[0, x]$ is as follows:

$$g(x) = \int_0^x f(t)dt = \frac{h}{2} \left( f(t_0) + \sum_{i=1}^{k-1} 2 \cdot f(t_i) + f(t_k) \right) - \frac{1}{12 \cdot k^2} f''(\xi)$$

wherein $h = \frac{x}{k}$, $t_0 = 0$, $t_i = i \cdot h$, $t_k = x$, and $0 < \xi < x$.

To compute approximations $g^*(x)$ of function $g$ on interval $[0, 1]$, let’s compute approximations $f^*(x_i)$ of function $f$ at points $x_i$, $i \in \{0, \ldots, k\}$, to precision $2^{-m}$ wherein $m$ is a natural number. In that case, the following holds:

$$|\Delta(g; x)| = |g^*(x) - g(x)| = \frac{h}{2} \left( f^*(t_0) + \sum_{i=1}^{k-1} 2 \cdot f^*(t_i) + f^*(t_k) \right) - \frac{1}{12 \cdot k^2} f''(\xi) \leq \frac{h}{2} \left( |f^*(t_0) - f(t_0)| + \sum_{i=1}^{k-1} 2 \cdot |f^*(t_i) - f(t_i)| \right) + \frac{1}{12 \cdot k^2} f''(\xi).$$
So, we have:

$$|\Delta(g; x)| = |g^*(x) - g(x)| \leq \frac{h}{2} \left( \frac{1}{2^m} + \frac{2(k-1)}{2^m} \right) + \frac{1}{12 \cdot k^2} C_1 \leq \frac{1}{k^2} \frac{k}{2^m} + \frac{1}{12 \cdot k^2} C_1 = \frac{1}{2^m} + \frac{1}{k^2} C_2;$$

here the fact that $f''$ is bounded above if $f \in C^2[0, 1]$ is taken into account. If we take $m = 2n$ and $k = C2^2^n$ then

$$|\Delta(g; x)| = \frac{1}{2^m} + \frac{1}{(C_2)^2 2^2n} C_2 < 2^{-n}.$$  

It means it is sufficient to compute in a loop for $i \in [0,k]$ approximations $g^*(x)$ of function $g$ on interval $[0,1]$ using formula (1) wherein $k = C2^2^n$ and $f^*(x_i)$ are the approximations of function $f$ at points $x_i$, $i \in [0..k]$, to precision $2^{-m}$ wherein $m = 2n$.  

So, the following theorems holds.

**Theorem 2.** If $f$ is a linear-space computable $C^2[0, 1]$ real function on interval $[0, 1]$ then real function $g(x) = \int_x^0 f(t)dt$ is a linear-space computable real function on interval $[0, 1]$.

**Theorem 3.** If $f$ is a polynomial-time and linear-space computable $C^2[0, 1]$ real function on interval $[0, 1]$ then real function $g(x) = \int_0^x f(t)dt$ is an exponential-time and linear-space computable real function on interval $[0, 1]$.

### 3 Conclusion

In the present paper, it is shown that real function $g(x) = \int_0^x f(t)dt$ is a linear-space computable real function on interval $[0, 1]$ whenever $f$ is a linear-space computable $C^2[0, 1]$ real function on interval $[0, 1]$. This result differs from the result from [1] regarding the time complexity of integration of polynomial-time computable real functions in the sense that integration of a polynomial-time computable real function may be not polynomial-time computable (if $f$ is not analytic and $\text{FP} \neq \text{NP}$), but integration of linear-space computable $C^2$ real functions is always linear-space computable.

Regarding further investigations, it is interesting to derive results regarding the space complexity of other operators, for example, the space complexity of differentiation of computable functions.

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