Electrified Dp-Branes Intersections

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Abstract

In this review, we introduce the electrified Dp-branes intersections in the low energy effective theory. We focus on D1-D3, D1-D5 and D0-D2 branes. We give the solutions configurations in the low energy effective theory in the absence and the presence of electric field by exciting one, two and three scalars in D3 system. The solutions from D3 point of view in the last two cases are given as a spike which is interpreted as an attached bundle of a superposition of coordinates of another brane given as a collective coordinate a long which the brane extends away from the D3-brane. The lowest energy in both cases is higher than the energy found in the case of D1⊥D3 branes. We also find space-time dependent solutions in D1-D3 system that are a natural generalization of those found without the electric field. Then, we show that, in the flat background, the D1-D3 and D1-D5 branes obey Neumann boundary conditions by discussing the fluctuations of fuzzy funnel solutions in both systems. Also, we give the broken duality in D1-D3 system and the unbroken duality in D1-D5 system. In D0-D2 system, we consider generalized Maxwell theory by introducing a generalized connection put at the origin of the spherical D2-brane to describe anyons instead of Chern-Simons term. The D2-brane got a higher energy and the static potential for two opposite charged exotic particles is found to have a screening nature on a fuzzy two-sphere instead of confinement which is a special property of the system on the plane.

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1 Introduction

D-branes have brought about significant advances in string theory. They are known as extended objects on which string endpoints can live. Various brane configurations have attracted much attention over recent years and several papers have been devoted to the study of the relationship between noncommutative geometry [1] and string theory [2] and the relationship between D-branes with different dimensions as well [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. The appearance of noncommutative geometry in string theory can be understood from a different point of view. For example, in type IIA theory, a D2-brane can be constructed from multiple D0-branes by imposing a noncommutative relation on their coordinates in matrix theory or under the strong magnetic field the world volume coordinates of a D2-brane become noncommutative by considering the quantum Hall system [11] and the magnetic field charge is interpreted as the number of D0-branes. On other hand, in type IIB theory, much of the progress has come about by directly studying the low energy dynamics of the D-branes world volume which is known to be governed by the Born-Infeld (BI) action [4, 8, 13, 14] which has many fascinating features. Among these there is the possibility for D-branes to morph into other D-branes of different dimensions by exciting some of the scalar fields [5, 6]. The subject of intersecting branes in string theory is very rich and has been studied for a long time [15, 16]. The dynamics of their solutions (bion spike [5, 16, 14, 17] and fuzzy funnel [18, 14]) were studied by considering linearized fluctuations around the static solutions [19, 20].

Among the goals of this review is to show that, in the presence of a world volume electric field, space-time dependent solutions can be generalised nicely and for certain condition we observe brane collapses with a speed less than that of light. We have also obtained a generalisation of the BPS solution. Another goal is to give the lowest energy in D3-brane theory by exciting two and three scalars [7] and also to determine the boundary conditions of D1⊥D3 and D1⊥D5 branes in solving the equations of motion of the fuzzy funnel’s fluctuations and discussing the associated potentials [20, 21]. We remark that the variation of the potential $V$ in terms of electric field $E$ and the spatial coordinate $\sigma$ in non-zero modes for both overall and relative transverse fluctuations in D1⊥D3 system and in zero mode of overall transverse fluctuations in D1⊥D5 system shows a singularity at some stage of $\sigma$. This is more clearly seen at the presence of electric field which leads to separate the system into two regions; small and large $\sigma$ depending on electric field. This implies that these intersecting branes obey Neumann boundary conditions and the end of open string can move freely on the brane. Consequently, the idea that the end of a string ending on a Dp-brane can be seen as an electrically charged particle is supported by this result. The obtained result in D1⊥D3 system is agree with its dual discussed in [17] considering Born-Infeld action dealing with the fluctuation of the bion spike in D3⊥D1 branes case. The dual case of D1⊥D5 branes is not yet discussed.

Another interesting subject we discuss, in this work, concerns the duality of D1⊥D5 branes in the presence of electric field [7]. By considering the energy of our system from D1 and D5 branes descriptions we find that the energies obtained from the two theories match and the presence of an electric field doesn’t spoil this duality. By contrary, in the case of D1⊥D3 branes showed in [22], the duality is broken because of the presence of electric field.

Then, another important point we deal with in this review concerns the spherical D2-brane [10]. We consider exotic particles described by generalized Maxwell theory in
which we introduce a generalized connection on a fuzzy two-sphere which has a dual
description in terms of an abelian gauge field on a spherical D2-brane and is interpreted
as a bound state of a spherical D2-brane and D0-branes. The exotic particles are known
as excitations and quasi-particles or anyons; i.e. fermions (bosons) carrying odd (even)
number of elementary magnetic flux quanta \[23\]. They are living in two-dimensional space
as composite particles having arbitrary spin, and they are characterized by fractional
statistics which are interpolating between bosonic statistics and fermionic one \[23, 24\].
Among the main results in this section is the change of the potential’s nature; there is no
confinement any more, and the disappearance of the confinement in the two-sphere case
for the exotic system is very interesting result. We also find that the energy of the gauge
field dominates when the radius of the fuzzy two-sphere goes to infinity, then the energy
of flat D2-brane which is a dual of the fuzzy two-sphere becomes high which is different
from the case of the quantum Hall effect (QHE) where the energy of the flat D2-brane
goes to zero. An important remark is that our system could be identified with the QHE in
higher dimensions only if the radius \(r\) and the number of charges \(N\) go to zero. Also, what
makes the model very different and very special is that the potential loses the confining
nature in the fuzzy two-sphere case.

This review is organized as follows. We begin in section 2 with a brief review on
D1 \(\perp\) D3 and D1 \(\perp\) D5 branes in dyonic case by using the abelian and non-abelian BI actions.
In section 3, we investigate the two and three excited scalars in D3-brane theory. In
section 4, we give the generalization space-time dependent solutions in the presence of
a world volume electric field. In section 5, we study the electrified fluctuations of fuzzy
funnel solutions corresponding to D3 and D5 branes. In the first subsection, we review the
zero and non-zero modes of the overall transverse electrified fluctuations of fuzzy funnel
solutions given in \[20\]. Then we discuss the zero and non-zero modes of the relative
transverse fluctuations in the second subsection. In the third subsection, we treat the
electrified fluctuations of the fuzzy funnel in D1 \(\perp\) D5 system. We study the solutions of
the linearized equations of motion of the overall transverse fluctuations in zero mode and
we discuss its associated potential at the extremities of \(\sigma\). In section 6, we review in brief
the broken duality of D1 \(\perp\) D3 in the dyonic case and the unbroken duality of intersecting
D1-D5 branes in the presence of an electric field. The section 7, the fuzzy two-sphere is
realized as one of D2-brane descriptions with special properties because of the generalized
Maxwell theory. The conclusion is presented in section 8.

2 Intersecting Dp-Branes

In this section, we review in brief the intersection of D1 branes with D3 and D5 branes.
We focus our study on the presence of electric field and its influence on the potentials and
the fluctuations of the fuzzy funnels.

2.1 Electrified D1 \(\perp\) D3 System

We start by giving in brief the known solutions of intersecting D1-D3 branes. From
the point of view of D3 brane description the configuration is described by a monopole on
its world volume. We use the abelian BI action and one excited transverse scalar in dyonic
case to give the bion solution [8, 26]. The system is described by the following action

\[ S = -T_3 \int d^4 \sigma \sqrt{-\det(\eta_{ab} + \lambda^2 \partial_a \phi^i \partial_b \phi^j + \lambda F_{ab})} \]

\[ = -T_3 \int d^4 \sigma \left[ 1 + \lambda^2 \left( | \nabla \phi |^2 + \vec{B}^2 + \vec{E}^2 \right) \right] \]

\[ + \lambda^4 \left( (\vec{B} \cdot \nabla \phi)^2 + (\vec{E} \cdot \vec{B})^2 + | \vec{E} \wedge \nabla \phi |^2 \right) \right)^{1/2} \]

(1)

in which \( F_{ab} \) \((a, b = 0, \ldots, 3)\) is the field strength and the electric field is denoted as \( F_{0a} = E_a \). \( \sigma^a \) denote the world volume coordinates while \( \phi^i \) \((i = 4, \ldots, 9)\) are the scalars describing transverse fluctuations of the brane and \( \lambda = 2\pi \ell_s^2 \) with \( \ell_s \) is the string length. In our case we excite just one scalar so \( \phi^i = \phi^9 \equiv \phi \). Following the same process used in the reference [8] by considering static gauge, we look for the lowest energy of the system. Accordingly to (1) the energy of dyonic system is given as

\[ \Xi = -T_3 \int d^3 \sigma \left[ 1 + \lambda^2 \left( | \nabla \phi |^2 + \vec{B}^2 + \vec{E}^2 \right) \right] \]

\[ + \lambda^4 \left( (\vec{B} \cdot \nabla \phi)^2 + (\vec{E} \cdot \vec{B})^2 + | \vec{E} \wedge \nabla \phi |^2 \right) \right]^{1/2}. \]

(2)

Then if we require \( \nabla \phi + \vec{B} + \vec{E} = 0 \), \( \Xi \) reduces to

\[ \Xi_0 = -T_3 \int d^3 \sigma \left[ 1 + \lambda^2 \left( | \nabla \phi |^2 + \vec{B}^2 + \vec{E} \cdot \vec{E} \right) \right] \}

\[ + \lambda^4 \left( (\vec{E} \cdot \vec{B})^2 + | \vec{E} \wedge \nabla \phi |^2 \right) \right]^{1/2}. \]

(3)

as minimum energy. By using the Bianchi identity \( \nabla \vec{B} = 0 \) and the fact that the gauge field is static, the bion solution is then

\[ \phi = \frac{N_m + N_e}{2r}, \]

(4)

with \( N_m \) is magnetic charge and \( N_e \) electric charge.

Now we consider the dual description of the D1⊥D3 branes from D1 branes point of view. To get D3-branes from D-strings, we use the non-abelian BI action. The natural definition of this action suggested in [35] is based on replacing the field strength in the BI action by a non-abelian field strength and adding the symmetrized trace \( STr(\ldots) \) in front of the \( \sqrt{\det} \) action. The precise prescription proposed in [35] was that inside the trace one takes a symmetrized average over orderings of the field strength. We refer the reader to [35] for more details on this action.

The non-abelian BI action describing D-string opening up into a D3-brane is given by

\[ S = -T_1 \int d^2 \sigma STr \left[ -\det(\eta_{ab} + \lambda^2 \partial_a \phi^i Q^{-1} \partial_b \phi^j \det Q^{ij}) \right]^{1/2}. \]

(5)

where \( Q_{ij} = \delta_{ij} + i\lambda[\phi_i, \phi_j] \). Expanding this action to leading order in \( \lambda \) yields the usual non-abelian scalar action

\[ S \approx -T_1 \int d^2 \sigma \left[ N + \lambda^2 Tr(\partial_a \phi^i + \frac{1}{2}[\phi_i, \phi_j][\phi_j, \phi_i]) + \ldots \right]^{1/2}. \]
We deal with the leading order in $N$ when we expand the symmetrized trace and we consider large $N$ limit. The solutions of the equation of motion of the scalar fields $\phi_i$, $i = 1, 2, 3$ represent the D-string expanding into a D3-brane analogous to the bion solution of the D3-brane theory \[5, 6\]. The solutions are
\[ \phi_i = \pm \frac{\alpha_i}{2\sigma}, \quad [\alpha_i, \alpha_j] = 2i\epsilon^{ijk}\alpha_k, \]
with the corresponding geometry is a long funnel where the cross-section at fixed $\sigma$ has the topology of a fuzzy two-sphere.

The dyonic case is presented by considering $(N,N_f)$-strings. We introduce a background U(1) electric field on the $N$ D-strings, corresponding to $N_f$ fundamental strings dissolved on the world sheet \[14\]. The theory is described by the action
\[ S = -T_1 \int d^2 \sigma STr \left[ -\det(\eta_{ab} + \lambda^2 \partial_a \phi^i \partial_b \phi^j + \lambda F_{ab}) \det Q_{ij} \right]^\frac{1}{2}. \]  \tag{6}

The action can be rewritten as
\[ S = -T_1 \int d^2 \sigma STr \left[ -\det \left( \eta_{ab} + \lambda F_{ab} \right) \right]^\frac{1}{2}. \]  \tag{7}

By computing the determinant, the action becomes
\[ S = -T_1 \int d^2 \sigma STr \left[ (1 - \lambda^2 E^2 + \alpha_i \alpha_i \check{R}^2)(1 + 4\lambda^2 \alpha_j \alpha_j \check{R}^4) \right]^\frac{1}{2}, \]  \tag{8}

in which we replaced the field strength $F_{\tau \sigma}$ by $EI_N$ ($I_N$ is $N \times N$-matrix) and the following ansatz were inserted
\[ \phi_i = \check{R}\alpha_i. \]  \tag{9}

Hence, we get the funnel solution for dyonic string by solving the equation of variation of $\check{R}$ as follows
\[ \phi_i = \frac{\alpha_i}{2\sigma \sqrt{1 - \lambda^2 E^2}}. \]  \tag{10}

### 2.2 Electrified D1 ⊥ D5 Branes

The fuzzy funnel configuration in which the D-strings expand into orthogonal D5-branes shares many common features with the D3-brane funnel. The action describing the static configurations involving five nontrivial scalars is
\[ S = -T_1 \int d^2 \sigma STr \left[ 1 + \lambda^2 (\partial_\sigma \Phi_i)^2 + 2\lambda^2 \Phi_{ij} \Phi_{ji} + 2\lambda^4 (\Phi_{ij} \Phi_{ji})^2 - 4\lambda^4 \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{li} \right. \\
+ 2\lambda^4 (\partial_\sigma \Phi_{ij})^2 \Phi_{jk} \Phi_{kl} - 4\lambda^4 \partial_\sigma \Phi_i \Phi_{ij} \partial_\sigma \Phi_k + \frac{\lambda^6}{4} (\epsilon_{ijklm} \partial_\sigma \Phi_i \Phi_{jklm} \Phi_{lm})^2 \left. \right]^\frac{1}{2}, \]  \tag{11}

where $\Phi_{ij} \equiv \frac{1}{2} [\Phi_i, \Phi_j]$ and the funnel solution is given by suggesting the following ansatz
\[ \Phi_i(\sigma) = \mp \check{R}(\sigma)G_i, \]  \tag{12}

$i = 1, \ldots, 5$, where $\check{R}(\sigma)$ is the (positive) radial profile and $G_i$ are the matrices constructed in \[27, 28\]. We note that $G_i$ are given by the totally symmetric $n$-fold tensor product of
4×4 gamma matrices, and that the dimension of the matrices is related to the integer \( n \) by \( N = \frac{(n+1)(n+2)(n+3)}{6} \). The Funnel solution (12) has the following physical radius

\[
R(\sigma) = \frac{\lambda}{N} \sqrt{\text{Tr}(\Phi_i)^2} = \sqrt{c} \lambda \hat{R}(\sigma),
\]

(13)

with \( c \) is the Casimir associated with the \( G_i \) matrices, given by \( c = n(n+4) \) and the resulting action for the radial profile \( R(\sigma) \) is

\[
S = -NT_1 \int d^2 \sigma \sqrt{1 + (R')^2(1 + 4 \frac{R^4}{c \lambda^2})}.
\]

(14)

We note that this result only captures the leading large \( N \) contribution at each order in the expansion of the square root.

To extend the discussion to dyonic strings we consider \( (N,N_f) \)-strings. Thus, the electric field is on and the system dyonic is described by the action

\[
S = - T_1 \int d^2 \sigma S T r \left[ - \text{det} (\eta_{ab} + \lambda^2 \partial_a \Phi^i Q^{-1}_{ij} \partial_b \Phi^j + \lambda F_{ab}) \text{det} Q^{ij} \right]^{\frac{1}{2}}.
\]

(15)

The action can be rewritten as

\[
S = - T_1 \int d^2 \sigma S T r \left[ - \text{det} \left( \begin{array}{cc} \eta_{ab} + \lambda F_{ab} & \lambda \partial_a \Phi^j \\ - \lambda \partial_b \Phi^i & Q^{ij} \end{array} \right) \right]^{\frac{1}{2}},
\]

(15')

with \( Q_{ij} = \delta_{ij} + i \lambda [\Phi_i, \Phi_j] \) and \( i,j = 1, \ldots, 5, a,b = \tau, \sigma \). We insert the ansatz (12) and \( F_{\tau \sigma} = EI_N \) (\( I_N \) is \( N \times N \)-matrix) in the action (15). Then we compute the determinant and we obtain

\[
S = - NT_1 \int d^2 \sigma \sqrt{1 - \lambda^2 E^2 + (R')^2(1 + 4 \frac{R^4}{c \lambda^2})}.
\]

(15'')

The funnel solution is

\[
\Phi_i(\sigma) = \mp \frac{R(\sigma)}{\lambda \sqrt{c}} G_i.
\]

(16)

From (15'') We can derive the lowest energy

\[
E = NT_1 \int d \sigma \sqrt{ \left( \sqrt{1 - \lambda^2 E^2} \pm R' \sqrt{\frac{8R^4}{c \lambda^2} + \frac{16R^8}{c^2 \lambda^4}} \right)^2 + \left( R' \mp \sqrt{1 - \lambda^2 E^2} \sqrt{\frac{8R^4}{c \lambda^2} + \frac{16R^8}{c^2 \lambda^4}} \right)^2}
\]

\[
\geq NT_1 \int d \sigma \left( \sqrt{1 - \lambda^2 E^2} \pm R' \sqrt{\frac{8R^4}{c \lambda^2} + \frac{16R^8}{c^2 \lambda^4}} \right).
\]

This is obtained when

\[
R' = \mp \sqrt{1 - \lambda^2 E^2} \sqrt{\frac{8R^4}{c \lambda^2} + \frac{16R^8}{c^2 \lambda^4}}.
\]

This equation can be explicitly solved in terms of elliptic functions. For small \( R \), the \( R^4 \) term under the square root dominates, and we find the funnel solution. Then the physical radius of the fuzzy funnel solution (16) is found to be

\[
R \approx \frac{\lambda \sqrt{c}}{2 \sqrt{2} \sqrt{1 - \lambda^2 E^2} \sigma}.
\]

(17)
In the dual point of view of the D5-brane world volume theory, the action describing the system is the Born-Infeld action

\[ S = -T_5 \int d^6\sigma S Tr \sqrt{-\det(G_{ab} + \lambda^2 \partial_a \phi \partial_b \phi + \lambda F_{ab})}, \]

with \( a, b = 0, 1, ..., 5 \) and \( \phi \) the excited transverse scalar. To get a spike solution with electric field switched on from D5-brane theory we follow the analogous method of bion spike in D3-brane theory as discussed in [8, 12, 28]. We use spherical polar coordinates and the metric is

\[ ds^2 = G_{ab} d\sigma^a d\sigma^b = -dt^2 + dr^2 + r^2 g_{ij} d\alpha^i d\alpha^j, \]

with \( r \) the radius and \( \alpha^i, i = 1, ..., 4 \), Euler angles. \( g_{ij} \) is the diagonal metric on a four-sphere with unit radius

\[ g_{ij} = \begin{pmatrix} 1 & \sin^2(\alpha^1) & \sin^2(\alpha^1)\sin^2(\alpha^2) & \sin^2(\alpha^1)\sin^2(\alpha^2)\sin^2(\alpha^3) \\ \sin^2(\alpha^1) & \sin^2(\alpha^2) & \sin^2(\alpha^1)\sin^2(\alpha^2) & \sin^2(\alpha^2)\sin^2(\alpha^3) \\ \sin^2(\alpha^1)\sin^2(\alpha^2) & \sin^2(\alpha^1)\sin^2(\alpha^2) & \sin^2(\alpha^1)\sin^2(\alpha^2)\sin^2(\alpha^3) & \sin^2(\alpha^1)\sin^2(\alpha^2)\sin^2(\alpha^3) \\ \sin^2(\alpha^1)\sin^2(\alpha^2)\sin^2(\alpha^3) & \sin^2(\alpha^1)\sin^2(\alpha^2)\sin^2(\alpha^3) & \sin^2(\alpha^1)\sin^2(\alpha^2)\sin^2(\alpha^3) & \sin^2(\alpha^1)\sin^2(\alpha^2)\sin^2(\alpha^3) \end{pmatrix}. \]

In this theory, we add the electric field \( E \) as a static radial field in the \( U(1) \) sector. The scalar \( \phi \) is only a function of the radius by considering bion spike solutions with a "nearly spherically symmetric" ansatz. In the same time to compare with the radial profile obtained in D1-brane theory we identify the physical transverse distance as \( \sigma = \lambda \phi \), and the radius \( r = R \) which fixes the coefficients. Thus the scalar is found to be

\[ \phi(r) = \pm \int \frac{dr}{\lambda(1 - \alpha^2)(\frac{r^2}{N^2} + 1)^2 - 1}, \]

where \( \alpha = \frac{g_s N_f}{\sqrt{N^2 + g_s^2 N_f^2}} \).

### 3 Exciting Two and Three Scalars

By following the same mechanism used for D1\( \perp \)D3 branes, this section is devoted to finding the funnel solutions when two and three excited scalars are involved. Thus, we restrict ourselves to using BPS arguments to find some solutions which have the interpretation of a D2-brane ending on D3-brane and D3-brane ending on other D3-brane. In these studies we consider the absence and the presence of an electric field. We found that the investigation of excited D3-brane in each case leads to the fact that by exciting 2 and 3 of its transverse directions in the absence or the presence of electric field, the brane develops a spike which is interpreted as an attached bundle of a superposition of coordinates of another brane given as a collective coordinate a long which the brane extends away from the D3-brane. Then, by considering the lowest energy states of our system we remark that the lowest energy in the intersecting branes case is obtained by the D1-D3 branes intersection and the energy is higher if we excite more scalar fields and even more in the presence of an electric field.

Thus, we use the abelian Born-Infeld action for the world volume gauge field and transverse displacement scalars to explore some aspects of D3-brane structure and dynamics. We deal with magnetic and dyonic cases [7].
3.1 Absence of Electric Field

We consider the case where D3-brane has more than one scalar describing transverse fluctuations. We denote the world volume coordinates by \( \sigma^a, a = 0, 1, 2, 3 \), and the transverse directions by the scalars \( \phi^i, i = 4, \ldots, 9 \). In D3-brane theory construction, the low energy dynamics of a single D3-brane is described by the BI action by using static gauge

\[
S_{BI} = \int L = -T_3 \int d^4\sigma \sqrt{-\det(\eta_{ab} + \lambda^2 \partial_\sigma \phi^i \partial_\sigma \phi^j + \lambda F_{ab})}
\]  

(18)

with \( F_{ab} \) is the field strength of the \( U(1) \) gauge field on the brane. By exciting two scalar fields and setting to zero the other scalars, the energy is evaluated for the fluctuations through two directions and for static configurations as follows

\[
\zeta = -L = T_3 \int d^3\sigma \left[ 1 + \lambda^2 \left( |\nabla \phi_4|^2 + |\nabla \phi_5|^2 + \vec{B}^2 \right) + \lambda^4 \left( (\vec{B} \cdot \nabla \phi_4)^2 + (\vec{B} \cdot \nabla \phi_5)^2 \right) 
\right.

\[
+ \lambda^4 \left| \nabla \phi_4 \wedge \nabla \phi_5 \right|^2 \right]^{\frac{1}{2}}.
\]

(19)

If we introduce a complex scalar field \( C = \phi_4 + i\phi_5 \), we can rewrite the energy as

\[
\zeta = T_3 \int d^3\sigma \left[ \lambda^2 (\nabla C + \vec{B})(\nabla C + \vec{B})^* + (1 - \lambda^2 (\nabla C \cdot \vec{B}))(1 - \lambda^2 (\nabla C \cdot \vec{B}))^* 
\right.

\[
+ \frac{1}{2} \lambda^4 \left| \nabla C \wedge \nabla C^* \right|^2 \right]^{\frac{1}{2}}.
\]

(20)

In this case, we observe that to get minimum energy we can set the first term to zero and this will lead to

\[
\nabla C = -\vec{B} = \nabla \phi_4 + i\nabla \phi_5.
\]

(21)

We know that \( \vec{B} \) is real, then \( \nabla \phi_5 = 0 \) and thus in the "bi-excited scalar" system the lowest energy is identified to \( D3 \perp D1 \) system. This suggests that to study the minimum energy configuration of the D3-brane system it is only worthwhile to excite just one scalar.

Now, requiring \( \nabla \phi_5 \neq 0 \) we get different energy bound. The energy (20) can be rewritten as follows

\[
\zeta = T_3 \int d^3\sigma \left[ \lambda^2 \left| \nabla \phi_4 + \nabla \phi_5 + \vec{B} \right|^2 + (1 \pm \lambda^2 \vec{B} \cdot (\nabla \phi_4 + \nabla \phi_5))^2 
\right.

\[
\pm 2\lambda^2 \nabla \phi_4 \cdot \nabla \phi_5 + \lambda^4 \left| \nabla \phi_4 \wedge \nabla \phi_5 \right|^2 \right]^{\frac{1}{2}}.
\]

(22)

The new bound is now found with the following constraint

\[
\nabla \phi_4 + \nabla \phi_5 = \pm \vec{B},
\]

(23)

and \( \pm 2\lambda^2 \nabla \phi_4 \cdot \nabla \phi_5 \geq 0 \) should also be satisfied. Then the energy is

\[
\tilde{\zeta} = T_3 \int d^3\sigma \left[ (1 \pm \lambda^2 \vec{B} \cdot (\nabla \phi_4 + \nabla \phi_5))^2 + \lambda^4 \left| \nabla \phi_4 \wedge \nabla \phi_5 \right|^2 \pm 2\lambda^2 \nabla \phi_4 \cdot \nabla \phi_5 \right]^{\frac{1}{2}}.
\]

(24)

We choose without loss of generality \( \nabla \phi_4 \perp \nabla \phi_5 \).
Thus we get $\nabla^2 \phi_4 + \nabla^2 \phi_5 = 0$ (by using the Bianchi identity), and the solution is

$$\phi_4 + \phi_5 = \pm \frac{N_m}{2r}. \quad (25)$$

This solution could be generalized by considering three excited scalars. The energy is then found to be

$$\zeta_3 = T_3 \int d^3 \sigma \left[ 1 + \lambda^2 \left( \sum_{i=4}^6 |\nabla \phi_i|^2 + \vec{B}^2 \right) + \lambda^4 \left( \sum_{i=4}^6 |\vec{B} \cdot \nabla \phi_i|^2 + \frac{1}{2} \sum_{i,j=4}^6 |\nabla \phi_i \wedge \nabla \phi_j|^2 \right) \right]^{\frac{1}{2}},$$

$$\geq T_3 \int d^3 \sigma \left[ (1 \pm \lambda^2 \vec{B} \cdot \sum_{i=4}^6 \nabla \phi_i)^2 + \frac{1}{2} \sum_{i,j=4}^6 |\nabla \phi_i \wedge \nabla \phi_j|^2 \pm \frac{\lambda^2}{2} \sum_{i,j=4}^6 \nabla \phi_i \cdot \nabla \phi_j \right]^{\frac{1}{2}}. \quad (26)$$

We should also consider $\pm \lambda^2 \sum_{i,j=4}^6 \nabla \phi_i \cdot \nabla \phi_j \geq 0$ to get the lowest energy configuration. This should be found by canceling some of the terms in the second line of the expression given in (26). The simplest way is to require the orthogonality of $\nabla \phi_i$ and $\nabla \phi_j$. Then the lowest energy in the static gauge is

$$\tilde{\zeta}_3 = T_3 \int d^3 \sigma \left[ (1 \pm \lambda^2 \vec{B} \cdot \sum_{i=4}^6 \nabla \phi_i)^2 + \frac{1}{2} \sum_{i,j=4}^6 |\nabla \phi_i \wedge \nabla \phi_j|^2 \right]^{\frac{1}{2}}, \quad (27)$$

with the constraints

$$\sum_{i=4}^6 \nabla \phi_i = \pm \vec{B}, \quad (28)$$

and the solution is similar to the "bi-excited system", we find

$$\sum_{i=4}^6 \phi_i = \pm \frac{N_m}{2r}. \quad (29)$$

The solution obtained for each excited scalar field has one collective coordinate in the D3-brane world volume theory. This is the direction along which the brane extends away from the D3-brane. Thus, this collective coordinate represents a "ridge" solution in the D3-brane theory.

Now, we will look at another case in which the electric field is present, and to see how the energy of the system could be minimized and what kind of solutions we could obtain.

### 3.2 Addition of Electric Field

First we start by exciting two transverse directions ($\phi_4$ and $\phi_5$) with the electric field $\vec{E}$ switched on. We consider as previous that $\nabla \phi_4 \perp \nabla \phi_5$. Then the energy of our system
is

\[ E_3 = T_3 \int d^3 \sigma \left[ 1 + \lambda^2 \left( | \nabla \phi_4 |^2 + | \nabla \phi_5 |^2 + \vec{B}^2 + \vec{E}^2 \right) \right. \]

\[ \left. + \lambda^4 \left[ (\vec{B} \cdot \nabla \phi_4)^2 + (\vec{B} \cdot \nabla \phi_5)^2 + (\vec{E} \cdot \vec{B})^2 + | \nabla \phi_4 \wedge \nabla \phi_5 |^2 + | \vec{E} \wedge \nabla \phi_4 |^2 + | \vec{E} \wedge \nabla \phi_5 |^2 \right] \right. \]

\[ \left. + \lambda^6 | \vec{E} \cdot (\nabla \phi_4 \wedge \nabla \phi_5) |^2 \right]^{\frac{1}{2}} \]

\[ = T_3 \int d^3 \sigma \left( \lambda^2 | \nabla \phi_4 + \nabla \phi_5 \pm (\vec{B} + \vec{E}) |^2 + [1 \mp \lambda^2 (\nabla \phi_4 + \nabla \phi_5) \cdot (\vec{B} + \vec{E})]^2 \right. \]

\[ \left. - 2 \lambda^2 \vec{B} \cdot \vec{E} + \lambda^4 \vec{E} \cdot (\vec{E} \cdot \vec{B})^2 \right) \]

\[ + \lambda^4 \left[ | \nabla \phi_4 \wedge \nabla \phi_5 |^2 + | \vec{E} \wedge \nabla \phi_5 |^2 - (\vec{E} \cdot \nabla \phi_4)^2 - (\vec{E} \cdot \nabla \phi_5)^2 \right. \]

\[ \left. - 2 \left( \nabla \phi_4 \cdot (\vec{E} + \vec{B}) \right) \nabla \phi_5 \left( \vec{E} + \vec{B} \right) + \left( \nabla \phi_4 \cdot \vec{B} \right) \left( \nabla \phi_4 \cdot \vec{E} \right) + \left( \nabla \phi_5 \cdot \vec{B} \right) \left( \nabla \phi_5 \cdot \vec{E} \right) \right] \]

\[ \left. + \lambda^6 | \vec{E} \cdot (\nabla \phi_4 \wedge \nabla \phi_5) |^2 \right) \sqrt{2}. \] (30)

By taking into consideration the previous analysis in the absence of an electric field, the energy \( E_3 \) becomes

\[ E_3 \geq T_3 \int d^3 \sigma \left( \left[ 1 + \lambda^2 (\nabla \phi_4 + \nabla \phi_5) \cdot (\vec{B} + \vec{E}) \right] - 2 \lambda^2 \vec{B} \cdot \vec{E} + \lambda^4 (\vec{E} \cdot \vec{B})^2 \right. \]

\[ \left. + \lambda^4 \left[ \vec{E}^2 | \nabla \phi_4 |^2 + \vec{E}^2 | \nabla \phi_5 |^2 \right] + \lambda^6 | \vec{E} \cdot (\nabla \phi_4 \wedge \nabla \phi_5) |^2 \right) \frac{1}{2}. \] (31)

This expression is consistent with the fact that

\[ \nabla \phi_4 + \nabla \phi_5 \pm (\vec{B} + \vec{E}) = 0 \] (32)

and

\[ - 2 \lambda^2 \vec{B} \cdot \vec{E} + \lambda^4 (\vec{E} \cdot \vec{B})^2 \geq 0. \] (33)

We also used the following expression

\[ | \vec{E} \wedge \nabla \phi_5 |^2 = \vec{E}^2 | \nabla \phi_5 |^2 - (\vec{E} \cdot \nabla \phi_5)^2. \] (34)

Then to get the lowest energy in the presence of an electric field we require \(-2 \lambda^2 \vec{B} \cdot \vec{E} + \lambda^4 (\vec{E} \cdot \vec{B})^2 = 0\); i.e. \( \vec{E} \cdot \vec{B} = \frac{2}{\lambda^2} \) or \( \vec{E} \perp \vec{B} \). With these simplifications, the energy becomes

\[ E_3 = T_3 \int d^3 \sigma \left( \left[ 1 + \lambda^2 (\nabla \phi_4 + \nabla \phi_5) \cdot (\vec{B} + \vec{E}) \right]^2 + \lambda^4 \vec{E}^2 \left( | \nabla \phi_4 |^2 + | \nabla \phi_5 |^2 \right) \right. \]

\[ \left. + \lambda^6 | \vec{E} \cdot (\nabla \phi_4 \wedge \nabla \phi_5) |^2 \right) \frac{1}{2}. \] (35)

In the following we consider \( \vec{E} \cdot \vec{B} = \frac{2}{\lambda^2} \). We remark that the energy of D2 \( \perp \) D3 brane is increased by the presence of the electric field and by switching it off we obtain the lowest energy configuration obtained previously.
By solving (32) in the static gauge, using the Bianchi identity, we obtain the solution
\[ \phi_4 + \phi_5 = \mp \frac{N_m + N_e}{2r}, \] (36)
with \( r^2 = \sum_{a=1}^{3} (\sigma^a)^2 \), \( N_m \) and \( N_e \) the magnetic and the electric charges respectively.

Now, by exciting three scalars, the energy is generalized to the following
\[
\xi_3 = T_3 \int d^3\sigma \left[ 1 + \lambda^2 \left( \sum_{i=4}^{6} \left| \nabla \phi_i \right|^2 + \vec{B}^2 + \vec{E}^2 \right) 
+ \lambda^4 \left( (\vec{E} \cdot \vec{B})^2 + \sum_{i=4}^{6} \left| \vec{B} \cdot \nabla \phi_i \right|^2 + \sum_{i=4}^{6} \left| \vec{E} \wedge \nabla \phi_i \right|^2 + \frac{1}{2} \sum_{i,j=4}^{6} \left| \nabla \phi_i \wedge \nabla \phi_j \right|^2 \right) 
+ \lambda^6 \left( \sum_{i,j=4}^{6} \left| \nabla \phi_i \wedge \nabla \phi_j \right|^2 \right)^{1/2} \right] \] (37)

We also consider as before \( \nabla \Phi_i \perp \nabla \Phi_j \) and \( \vec{E} \cdot \vec{B} = \frac{2}{\lambda^2} \) with \( i, j = 4, 5, 6 \). Then the energy will be reduced to the lowest energy for three excited directions
\[
\tilde{\xi}_3 = T_3 \int d^3\sigma \left[ 1 \pm \lambda^2 \sum_{i=4}^{6} \left| \nabla \phi_i \right|^2 + \vec{B}^2 + \vec{E}^2 \right] + \lambda^4 \sum_{i,j=4}^{6} \left| \vec{E} \wedge \nabla \phi_i \right|^2 + \frac{1}{2} \sum_{i,j=4}^{6} \left| \nabla \phi_i \wedge \nabla \phi_j \right|^2 \] (38)

with \( i < j \) in the summations and where we assume the following condition
\[
\sum_{i=4}^{6} \nabla \phi_i \pm (\vec{B} + \vec{E}) = 0. \] (39)

The last equation is easily solved to give the solution
\[
\sum_{i=4}^{6} \phi_i = \mp \frac{N_m + N_e}{2r}. \] (40)

Again if we require that \( \vec{E} \) is parallel to one of \( \nabla \phi_i \) the energy will be minimized as the last term in the expression (38) of the energy vanishes. Then, accordingly to (38) and (40) with
\[
\left| \vec{B} \right| = \left| \frac{N_m}{2r^2} \right|, \quad \left| \vec{E} \right| = \left| \frac{N_e}{2r^2} \right| \]
the energy becomes
\[
\tilde{\xi}_3 = T_3 \int d^3\sigma \left[ 1 + \lambda^2 \left( \frac{N_m + N_e}{4r^4} \right)^2 + \lambda^4 \frac{N_m^2 + N_e^2}{8r^6} \right]^{1/2} \] (41)

This is the lowest energy for both cases where two or three transverse scalars are excited since the solution is a superposition of the scalars.
4 Dyonic and Non-Static D1\perp D3 Funnels

In the D1 worldvolume point of view, the intersection of D1 and D3-brane is described as a funnel of increasing radius as we approach the D3 brane, where the D-strings expand into a fuzzy two-sphere. In the D3-picture the worldvolume solution includes a BPS magnetic monopole and the Higgs field is interpreted as a transverse spike. To enlarge the discussion of this subject, we lift the static condition and we keep the electric field on.

4.1 Space-Time dependent Solution

We start by D1-picture. The action is given by the non-abelian BI action of \((N - N_f)\) strings. By considering again the ansatz (9) the action (6) became

\[
S = -T_1 \int d^2 \sigma S T r \sqrt{1 + \lambda^2 c \dot{R}^2 - \lambda^2 c \ddot{R}^2 - \lambda^2 E^2 \sqrt{1 + 4 \lambda^2 c \dot{R}^4}},
\]

with the notations \(\dot{R} = \partial_\tau \hat{R}\) and \(\dot{\hat{R}} = \partial_r \hat{R}\). By varying this with respect to \(\hat{R}\) we recover the full equations of motion given by

\[
2\lambda^2 c \ddot{R} R' R''(1 - \lambda^2 \dot{c}^2 \ddot{R}) - \ddot{R}(1 + \lambda^2 c \dot{R}^2) = 8\hat{R}^3(\frac{1 + \lambda^2 c \dot{R}^2 - \lambda^2 c \ddot{R}^2 - \lambda^2 E^2}{1 + 4 \lambda^2 c \dot{R}^4}).
\]

To get these equations in terms of dimensionless variables we consider the following rescalings

\[
r = \sqrt{2\lambda c \dot{R}} \quad \tilde{\tau} = \sqrt{\frac{2}{\lambda c}} \tau \quad \tilde{\sigma} = \sqrt{\frac{2}{\lambda c}} \sigma.
\]

Then the above equations of motion can be written in a Lorentz-invariant form

\[
\partial_\mu \partial^\nu r + (\partial_\mu \partial^\nu r)(\partial_\nu r) - (\partial_\mu \partial^\nu r)(\partial_\nu r) = 2r^3 \frac{1 + (\partial_\mu r)(\partial^\nu r) - \lambda^2 E^2}{1 + r^4},
\]

where \(\mu\) and \(\nu\) can take the values \(\tilde{\tau}\) and \(\tilde{\sigma}\). The re-scaled action and energy density of the configuration are

\[
\tilde{S} = -T_1 \int d^2 \sigma S T \tilde{r} \sqrt{1 + r^2 - \dot{\tilde{r}}^2 - \lambda^2 E^2 \sqrt{1 + r^4}},
\]

\[
T_{\tau \tau} = E = (1 + r^2 - \lambda^2 E^2) \sqrt{\frac{1 + r^4}{1 + r^2 - \dot{r}^2 - \lambda^2 E^2}},
\]

and the pressure is given by

\[
T_{\sigma \sigma} = (1 - \dot{r}^2 - \lambda^2 E^2) \sqrt{\frac{1 + r^4}{1 + r^2 - \dot{r}^2 - \lambda^2 E^2}},
\]

with dots and primes implying differentiation with respect to the re-scaled time and space respectively.

Now if we consider \(r_0\) is the initial radius of the collapsing configuration where \(\dot{r} = 0\), the energy is \(E = (1 + r_0^2 - \lambda^2 E^2) \sqrt{\frac{1 + r_0^4}{1 + r_0^2 - \lambda^2 E^2}}\). The conserved energy density leads then at large \(N\) to the following equation

\[
\dot{r}^2 = (1 + r^2 - \lambda^2 E^2) - \frac{1 + r^4}{1 + r_0^2 \left(1 + r_0^2 - \lambda^2 E^2\right)^2} \left(1 + r_0^2 - \lambda^2 E^2\right).
\]
In purely time dependence, we get

\[ r^2 = \frac{(1 - \lambda^2 E^2) r_0^4 - r_1^4}{1 + r_0^4}. \]  

(49)

Then we can write

\[ \int_0^t dt = \sqrt{\frac{1 + r_0^4}{1 - \lambda^2 E^2}} \int_{r_0}^r \frac{dr}{\sqrt{r_0^4 - r^4}}. \]  

(50)

Now if we consider the initial radius at \( t = 0 \) is \( r_1 \) at \( \dot{r} = u \), the equation (48) becomes

\[ r^2 = \frac{r_1^4(1 - \lambda^2 E^2) - r_1^4(1 - u^2 - \lambda^2 E^2) + u^2}{1 + r_1^4}, \]  

(51)

and we can solve the following equality

\[ \int_0^t dt = \sqrt{\frac{1 + r_1^4}{1 - u^2 - \lambda^2 E^2}} \int \frac{dr}{r_1 \sqrt{r_1^4 - r^4}}. \]  

(52)

we get

\[ it \sqrt{\frac{1 - u^2 - \lambda^2 E^2}{1 + r_1^4}} = \int_{r_1}^{r_2} \frac{dr}{\sqrt{r^4 - 1}} + \int_{r_2}^{r_0} \frac{dr}{\sqrt{r^4 - 1}}. \]  

(53)

Then this leads to

\[ i\tilde{t} = it \sqrt{2} \sqrt{\frac{1 - u^2 - \lambda^2 E^2}{1 + r_1^4}} r_2 = cn^{-1} \left( \frac{r_1}{r_2}, \frac{1}{\sqrt{2}} \right) + cn^{-1} \left( \frac{r_2}{r_1}, \frac{1}{\sqrt{2}} \right). \]  

(54)

Let \( T = cn^{-1} \left( \frac{r_1}{r_2}, \frac{1}{\sqrt{2}} \right) \) the the solution is

\[ r(t) = r_2 cn \left( \tilde{t} + T, \frac{1}{\sqrt{2}} \right). \]  

(55)

If \( u = 0 \) we find

\[ it \sqrt{2} \sqrt{\frac{1 - \lambda^2 E^2}{1 + r_1^4}} r_1 = cn^{-1} \left( \frac{r_1}{r_2}, \frac{1}{\sqrt{2}} \right). \]  

(56)

Then if \( E = 0 \) we get

\[ \dot{r}^2 = \frac{r_1^4 - r_1^4(1 - u^2)}{1 + r_1^4}, \]  

(57)

and

\[ it \sqrt{2} \sqrt{\frac{1 - \lambda^2 E^2}{1 + r_1^4}} r_2 = cn^{-1} \left( \frac{r_1}{r_2}, \frac{1}{\sqrt{2}} \right) + cn^{-1} \left( \frac{r_2}{r_1}, \frac{1}{\sqrt{2}} \right). \]  

(58)

The last case is \( E = 0 \) and \( u = 0 \) and this was discussed in the reference [3]. For the first and the second cases, we remark that if \( r_1 \rightarrow \infty \) we get \( \dot{r}^2 = 1 - \lambda^2 E^2 \) but for the third and the forth cases we have \( \dot{r}^2 = 1 \), so for at the presence of electric field the brane collapses lesser that the speed of light. For the both cases \( E = 0 \) and \( E \neq 0 \) a collapsing speed does not depend on the initial speed.
In the purely spatial independence we follow the same way as time dependence. Let’s consider at 
\( t = 0, \ r' = v \) and \( \dot{r} = 0 \) and the initial radius is \( r_1 \) and the purely spatial 
solution for \( T_{\sigma\sigma} \) is gotten by solving the following, first we have 
\[
    r'^2 = \frac{r^4(1 + v^2 - \lambda^2 E^2) - r_1^4(-\lambda^2 E^2) + v^2}{1 + r_1^4},
\]
and
\[
    \int_0^\sigma d\sigma = \sqrt{\frac{1 + r_1^4}{1 - v^2 - \lambda^2 E^2}} \int_{r_1}^r \frac{dr}{\sqrt{r^4 - r_1^4}},
\]
where \( r'^4 = r_1^4 - \frac{v^2(1 + r_1^4)}{1 + v^2 - \lambda^2 E^2} \) and \( \int = \int_{r_1}^{r'} \int_{r_1}^{r'} \) so we find
\[
    \frac{\sigma \sqrt{1 + v^2 - \lambda^2 E^2}}{\sqrt{1 + r_1^4}} = \int_{r_1}^r \frac{dr}{\sqrt{r^4 - r_1^4}} - \int_{r_1}^{r'} \frac{dr}{\sqrt{r'^4 - r_1^4}},
\]
then we get
\[
    \frac{\sigma \sqrt{2r'_1 \sqrt{1 + (v^2 - \lambda^2 E^2)}}}{\sqrt{1 + r_1^4}} = cn^{-1}(\frac{r'}{r_1}, \frac{1}{\sqrt{2}}) - cn^{-1}(\frac{r'_1}{r_1}, \frac{1}{\sqrt{2}}),
\]
and
\[
    = cn^{-1}\left(\frac{r'^2}{r_1^2} + \sqrt{\frac{1 - (\frac{r'}{r_1})^4(1 - (\frac{r'_1}{r_1})^4)}{1 + (\frac{r'}{r_1})^4(1 - r_1^2)}}, \frac{1}{\sqrt{2}}\right).
\]
For \( v^2 = r_1^4(1 - \lambda^2 E^2) \) we get
\[
    r'^2 = r^4(1 - \lambda^2 E^2)
\]
which leads to the following
\[
    \mp r = \frac{1}{\sigma(1 - \lambda^2 E^2) \mp \frac{1}{r}}.
\]
Thus for particular initial conditions of \( r' \) and \( r \), we do not find the infinite periodic brane- 
anti-brane array, but rather a solution with a decaying \( r \). In the presence of an electric field 
it is attenuated by the factor of \( 1/\sqrt{1 - \lambda^2 E^2} \). The equation \( r_0^2 = r^4(1 - \lambda^2 E^2) \) provides 
a generalization of the BPS equation in presence of an electric field. Note that for \( E = 0 \) we 
get the usual BPS equation \( r_0^2 = r^4 \). The BPS equation can be obtained by studying 
the condition to have an unbroken supersymmetry, which amounts to the condition for 
the vanishing of the variation of gaugino on the world volume of the intersection \[30\].

For equations (51) and (59), consider the Wick rotation \( \tau \rightarrow i\sigma \). This provides interesting 
results in the cases where \( E, u \) and \( v \) are all zero. Here we find (51) \( \rightarrow \) (59) for 
\( \tau \rightarrow i\sigma \) and \( u \rightarrow iv \). Thus in the presence of the electric field, the space time dependent 
solutions generalize nicely.

4.2 Automorphisms

Another interesting feature of these solutions is the \( r \leftrightarrow 1/r \) duality, which arises as the 
consequence of the invariance of the complex curve under an \( r \leftrightarrow 1/r \) automorphism. More 
precisely, in the absence of any worldvolume electric field and with \( u = 0 \), we find (51)
reduces to the equation of a complex curve \( s^2 = \frac{r_0^4(1 - \lambda^2 E^2) - r^4(1 - u^2 - \lambda^2 E^2) + u^2}{1 + r_0^4} \) where we have defined \( \dot{r} = s \) and \( r \) and \( s \) are interpreted as complex variables. For this curve the relevant automorphisms have been studied in [30] and a connection to the \( \tau \rightarrow i\sigma \) duality has been made. Here we find a similar duality. Towards this end, we study an automorphism of the curve (51) for \( E \neq 0 \) and \( v \neq 0 \). After defining \( s = \dot{r} \) we find

\[
s^2 = \frac{r_0^4(1 - \lambda^2 E^2) - r^4(1 - u^2 - \lambda^2 E^2) + u^2}{1 + r_0^4}.
\]

The automorphism acts as \( r \rightarrow R = 1/r \), \( r_0 \rightarrow R_0 = 1/r_0 \), \( s \rightarrow \tilde{s} = is/r^2 \), \( u \rightarrow u' = -iuR^2 \). This automorphism acts at fixed \( E \). For \( u = 0 \), \( r \rightarrow R = r_0^2/r \), \( s \rightarrow \tilde{s} = isr_0^2/r^2 \), is still a good automorphism of (64).

5 Dyonic Fluctuations

In this section, we give an examination of the propagation of the fluctuations on the fuzzy funnel. The setup is similar to both \( D1 \perp D5 \) and \( D1 \perp D3 \) systems. We notice that there are two basic types of funnel’s fluctuations, the overall transverse ones in the directions perpendicular to both the Dp-brane (\( p=3,5 \)) and the string (i.e., \( X^{p+1,..,8} \)), and the relative transverse ones which are transverse to the string, but parallel to the Dp-brane world volume (i.e., along \( X^{1,..,p} \)).

Thus, we treat the dynamics of the funnel solutions. We solve the linearized equations of motion for small and time-dependent fluctuations of the transverse scalars around the exact background in dyonic case.

5.1 Overall Transverse Fluctuations in \( D1 \perp D3 \) System

5.1.1 Zero Mode

We deal with the fluctuations of the funnel (10) discussed in the previous section. By plugging into the full \((N,N_f)\)-string action (6,7) the ”overall transverse” \( \delta \phi^m(\sigma,t) = f^m(\sigma,t)I_N, m = 4, ..., 8 \) which is the simplest type of fluctuation with \( I_N \) the identity matrix, together with the funnel solution, we get

\[
S = -T_1 \int d^2\sigma Str[(1 + \lambda E)(1 + \frac{\lambda^2 \alpha^i\alpha^i}{4(1 - \lambda^2 E^2)^2 \sigma^4})^{\frac{1}{2}}
(1 - (1 - \lambda E)\lambda^2(\partial\delta\phi^m)^2 + \lambda^2(\partial_\sigma \delta\phi^m)^2)] \approx -NT_1 \int d^2\sigma H[(1 + \lambda E) - (1 - \lambda^2 E^2)\frac{\lambda^2}{2}(f^m)^2 + \frac{(1 + \lambda E)\lambda^2}{2H}(\partial_\sigma f^m)^2 + ...]
\]

where

\[
H = 1 + \frac{\lambda^2 C}{4(1 - \lambda^2 E^2)^2 \sigma^4}
\]

and \( C = Tr\alpha^i\alpha^i \). For the irreducible \( N \times N \) representation we have \( C = N^2 - 1 \). In the last line we have only kept the terms quadratic in the fluctuations as this is sufficient to determine the linearized equations of motion

\[
(1 - \lambda E)(1 + \lambda^2 \frac{N^2 - 1}{4(1 - \lambda^2 E^2)^2 \sigma^4})\partial^2_\sigma - \partial^2_\sigma) f^m = 0.
\]
In the overall case, all the points of the fuzzy funnel move or fluctuate in the same direction of the dyonic string by an equal distance $\delta x^m$. Thus, the fluctuations $f^m$ could be rewritten as follows

$$f^m(\sigma,t) = \Phi(\sigma)e^{-i\omega t}\delta x^m,$$  \hspace{1cm} (67)

where $\Phi$ is a function of the spatial coordinate. With this ansatz the equation of motion (66) becomes

$$((1 - \lambda E)(1 + \lambda^2 \frac{N^2 - 1}{4(1 - \lambda^2 E^2)^2 \sigma^4})w^2 + \partial^2_\sigma)\Phi(\sigma) = 0.$$  \hspace{1cm} (68)

Then, the problem is reduced to finding the solution of a single scalar equation.

We consider the physical phenomenon which is defined by the fact that the electric field $E$ is in the interval $[0, \frac{1}{\lambda}]$ (contrary to what was treated in [31], such that $E$ was tending to $\infty$).

The equation (68) is an analog one-dimensional Schrödinger equation. Let’s rewrite it as

$$\left(\frac{1}{w^2(1 - \lambda E)}\partial^2_\sigma + 1 + \frac{\lambda^2 N^2}{4(1 - \lambda^2 E^2)^2 \sigma^4}\right)\Phi(\sigma) = 0,$$  \hspace{1cm} (69)

for large $N$. If we suggest

$$\tilde{\sigma} = w\sqrt{1 - \lambda E}\sigma,$$  \hspace{1cm} (70)

the equation (69) becomes

$$\left(\partial^2_{\tilde{\sigma}} + 1 + \frac{\kappa^2}{\tilde{\sigma}^4}\right)\Phi(\tilde{\sigma}) = 0,$$  \hspace{1cm} (71)

with the potential is

$$V(\tilde{\sigma}) = \frac{\kappa^2}{\tilde{\sigma}^4},$$  \hspace{1cm} (72)

and

$$\kappa = \frac{\lambda N w^2}{2(1 + \lambda E)}.$$  \hspace{1cm} (73)

The equation (71) is a Schrödinger equation for an attractive singular potential $\propto \tilde{\sigma}^{-4}$ and depends on the single coupling parameter $\kappa$ with constant positive Schrödinger energy. The solution is then known by making the following coordinate change

$$\chi(\tilde{\sigma}) = \int_{\sqrt{\kappa}}^{\tilde{\sigma}} dy \sqrt{1 + \frac{\kappa^2}{y^2}},$$  \hspace{1cm} (74)

and

$$\Phi = (1 + \frac{\kappa^2}{\tilde{\sigma}^4})^{-\frac{1}{4}}\tilde{\Phi}.$$  \hspace{1cm} (75)

Thus, the equation (71) becomes

$$\left(-\partial^2_{\chi} + V(\chi)\right)\tilde{\Phi} = 0,$$  \hspace{1cm} (76)

with

$$V(\chi) = \frac{5\kappa^2}{(\tilde{\sigma}^2 + \frac{\kappa^2}{\tilde{\sigma}^4})^3}.$$  \hspace{1cm} (77)

Accordingly to the variation of this potential (Fig.1), the system looks like separated into two regions depending on $\sigma$. In small $\sigma$ region $V$ is close to 0 with a constant value
for all $E$. In large $\sigma$ region, specially when $\sigma$ reaches 0.7, $V$ increases too fast as we jump to a new region and gets a maximum value when $E \approx 0.5$.

Then, the fluctuation is found to be

$$\Phi = (1 + \frac{\kappa^2}{\sigma^4})^{-\frac{1}{2}} e^{\pm i\chi(\sigma)}.$$  \hspace{1cm} (78)

This fluctuation has the following limits; at large $\sigma$, $\Phi \sim e^{\pm i\chi(\sigma)}$ and if $\sigma$ is small $\Phi = \frac{\sigma}{\sqrt{\kappa}} e^{\pm i\chi(\sigma)}$. These are the asymptotic wave function in the regions $\chi \to \pm \infty$, while around $\chi \sim 0$; i.e. $\sigma \sim \sqrt{\kappa}$, $\Phi \sim 2^{-\frac{1}{2}}$. Also we find that \( \Phi \) has different expressions in small and large $\sigma$ regions.

5.1.2 Non-Zero Modes

The fluctuations discussed above could be called the zero mode $\ell = 0$ and for non-zero modes $\ell \geq 0$, the fluctuations are $\delta \phi^m(\sigma, t) = \sum_{\ell=0}^{N-1} \psi_{\ell, i_1 \ldots i_\ell} \alpha^{i_1} \ldots \alpha^{i_\ell}$ with $\psi_{\ell, i_1 \ldots i_\ell}$ are completely symmetric and traceless in the lower indices.

The action describing this system is

$$S \approx -NT \int d^2 \sigma \left[(1 + \lambda E)H - (1 - \lambda^2 E^2)H \frac{\lambda^2}{2}(\partial_\sigma \delta \phi^m)^2 \right.$$  

$$+ \frac{(1+\lambda E)\lambda^2}{HI} (\partial_\sigma \delta \phi^m)^2 - (1 - \lambda^2 E^2) \frac{\lambda^2}{2} [\phi^i, \delta \phi^m]^2$$  

$$- \frac{\lambda^4}{12} (\partial_\sigma \phi^i, \partial_\sigma \delta \phi^m)^2 + ... \right].$$

(79)

Now the linearized equations of motion are

$$\left[(1 + \lambda E)H \partial^2_\sigma - \partial^2_\sigma\right] \delta \phi^m + (1 - \lambda^2 E^2) [\phi^i, [\phi^i, \delta \phi^m]] - \frac{\lambda^2}{6} [\partial_\sigma \phi^i, [\partial_\sigma \phi^i, \partial^2_\sigma \delta \phi^m]] = 0,$$  \hspace{1cm} (80)

with $H = 1 + \frac{\lambda^2 N^2 - 1}{4(1 - \lambda^2 E^2) \sigma^4}$. Since the background solution is $\phi^i \propto \alpha^i$ and we have $[\alpha^i, \alpha^j] = 2i \epsilon_{ijk} \alpha^k$, we get

$$[\alpha^i, [\alpha^i, \delta \phi^m]] = \sum_{\ell < N} \psi_{\ell, i_1 \ldots i_\ell} [\alpha^i, [\alpha^{i_1} \ldots \alpha^{i_\ell}]]$$  

$$= \sum_{\ell < N} 4\ell (\ell + 1) \psi_{\ell, i_1 \ldots i_\ell} \alpha^{i_1} \ldots \alpha^{i_\ell}.$$  \hspace{1cm} (81)

To obtain a specific spherical harmonic on 2-sphere, we have

$$[\phi^i, [\phi^i, \delta \phi^m]] = \frac{\ell(\ell + 1)}{1 - \lambda^2 E^2 \sigma^2} \delta \phi^m,$$  

$$[\partial_\sigma \phi^i, [\partial_\sigma \phi^i, \partial^2_\sigma \delta \phi^m]] = \frac{\ell(\ell + 1)}{1 - \lambda^2 E^2 \sigma^2} \partial^2_\sigma \delta \phi^m.$$  \hspace{1cm} (82)

Then for each mode the equations of motion are

$$\left[((1 + \lambda E)(1 + \lambda^2 \frac{N^2 - 1}{4(1 - \lambda^2 E^2) \sigma^4}) - \frac{\lambda^2 \ell(\ell + 1)}{6(1 - \lambda^2 E^2 \sigma^4)} \partial^2_\sigma - \partial^2_\sigma + \frac{\ell(\ell + 1)}{\sigma^2}\right] \delta \phi^m = 0.$$  \hspace{1cm} (83)

The solution of the equation of motion can be found by taking the following proposal. Let’s consider $\phi^m_\ell = f_\ell^m(\sigma)e^{-i\omega t} \delta x^m$ in direction $m$ with $f_\ell^m(\sigma)$ is some function of $\sigma$ for each mode $\ell$.  

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The last equation can be rewritten as
\[
\left[ -\partial^2_\sigma + V(\sigma) \right] f^m_\ell(\sigma) = w^2(1 + \lambda E)f^m_\ell(\sigma), 
\]  
(84)
with
\[
V(\sigma) = -w^2((1 + \lambda E)\frac{\lambda^2 N^2}{4(1 - \lambda^2 E^2)^2\sigma^4} - \frac{\lambda^2 \ell(\ell + 1)}{6(1 - \lambda^2 E^2)^2\sigma^4}) + \frac{\ell(\ell + 1)}{\sigma^2}.
\]

In small \( \sigma \) region, this potential is reduced to
\[
V(\sigma) = -w^2\frac{\lambda^2}{(1 - \lambda^2 E^2)^2\sigma^4}\left(\frac{(1 + \lambda E)N^2}{4} - \frac{\ell(\ell + 1)}{6}\right).
\]
This potential (Fig.2(a)) is close to 0 for almost of \( \sigma \) and \( E \) until that \( E \approx 0.87 \) we remark that \( V \) changes at \( \sigma \approx 0.04 \) and then goes up too fast to be close to 0 again for the other values of \( \sigma \).

In small \( \sigma \) limit, we reduce the equation (84) to the following form
\[
\left[ w^2((1 + \lambda E)(1 + \lambda^2\frac{N^2 - 1}{4(1 - \lambda^2 E^2)^2\sigma^4}) - \frac{\lambda^2 \ell(\ell + 1)}{6(1 - \lambda^2 E^2)^2\sigma^4}) + \partial^2_\sigma \right] f^m_\ell(\sigma) = 0. 
\]  
(85)
and again as
\[
\left[ 1 + \frac{1}{(1 - \lambda^2 E^2)^2\sigma^4}\left(\frac{\lambda^2 N^2 - 1}{4} - \frac{\lambda^2 \ell(\ell + 1)}{6(1 + \lambda E)}\right) + \frac{1}{w^2(1 + \lambda E)\partial^2_\sigma} \right] f^m_\ell(\sigma) = 0. 
\]  
(86)
We define new coordinate \( \tilde{\sigma} = w\sqrt{1 + \lambda E}\sigma \) and the latter equation becomes
\[
\left[ \partial^2_{\tilde{\sigma}} + 1 + \frac{\kappa^2}{\tilde{\sigma}^4} \right] f^m_\ell(\sigma) = 0, 
\]  
(87)
where
\[
\kappa^2 = \frac{w^2(1 + \lambda E)}{(1 - \lambda^2 E^2)^2\sigma^4}\left(\lambda^2\frac{N^2 - 1}{4} - \frac{\lambda^2 \ell(\ell + 1)}{6(1 + \lambda E)}\right)^{\frac{1}{2}},
\]
such that
\[
N > \sqrt[3]{\frac{2\ell(\ell + 1)}{3(1 + \lambda E)}} + 1.
\]
By following the same setup of zero mode, we get the solution by using the steps (74-78) with new \( \kappa \). Since we considered small \( \sigma \) we get
\[
V(\chi) = \frac{5\tilde{\sigma}^6}{\kappa^4}, 
\]  
(88)
and the fluctuation is found to be
\[
f^m_\ell = \frac{\tilde{\sigma}}{\sqrt{\kappa}} e^{\pm i\chi(\tilde{\sigma})}. 
\]  
(89)
in small \( \sigma \) region.

Now, let’s check the case of large \( \sigma \). In this case, the equation of motion (84) of the fluctuation can be rewritten in the following form
\[
\left[ -\partial^2_\sigma + V(\sigma) \right] f^m_\ell(\sigma) = w^2(1 + \lambda E)f^m_\ell(\sigma), 
\]  
(90)
with
\[ V(\sigma) = \frac{\ell(\ell + 1)}{\sigma^2}. \]

We remark that, in large \( \sigma \) limit (Fig. 2(b)), the potential \( V \) is independent of \( E \) and going down as \( \sigma \) is going up. The figures 2(a) and 2(b) show that the system in non-zero modes is separated to two totally different regions and the main remark is that the potential gets a singularity at some level of \( \sigma \) which is considered the intersection of small and large \( \sigma \) regions. In our calculations we took small \( \sigma \) from zero until the half of the unit of \( \lambda = 1 \) and the large \( \sigma \) region from 0.5 until 1 with \( w = 1, l = 1 \) and \( N = 10 \).

The \( f^m_\ell \) is now a Sturm-Liouville eigenvalue problem. The fluctuation is found to be
\[ f^m_\ell(\sigma) = \alpha \sqrt{\sigma} \text{BesselJ}\left( \frac{1}{2} \sqrt{1 + 4(\ell + 1)}, w\sigma\sqrt{1 + \lambda E} \right) + \beta \sqrt{\sigma} \text{BesselY}\left( \frac{1}{2} \sqrt{1 + 4(\ell + 1)}, w\sigma\sqrt{1 + \lambda E} \right), \]
with \( \alpha, \beta \) are constants. Again, it’s clear that the fluctuation solution in this case is totally different from the one gotten in small \( \sigma \) limit (89) supporting the idea that the system is divided to two regions. In the following, we continue the study of D1⊥D3 branes by dealing with the relative transverse fluctuations.

### 5.2 Relative Transverse Fluctuations in D1⊥D3 System

#### 5.2.1 Zero Mode

In this subsection, we consider the "relative transverse" fluctuations \( \delta \phi^i(\sigma, t) = f^i(\sigma, t)I_N, i = 1, 2, 3 \), and the action describing the system has the expression
\[ S = -T_1 \int d^2\sigma \text{STr}\left[ -\det\left( \begin{array}{cc} \eta_{ab} + \lambda F_{ab} & \lambda \partial_a(\phi^j + \delta \phi^j) \\ -\lambda \partial_b(\phi^j + \delta \phi^j) & Q^{ij} \end{array} \right) \right]^{\frac{1}{2}}, \]
with
\[ Q^{ij} = Q^{ij} + \lambda\left( [\phi_i, \delta \phi_j] + [\delta \phi_i, \phi_j] + [\delta \phi_i, \delta \phi_j] \right). \]

As done above, we keep only the terms quadratic in the fluctuations and the action becomes
\[ S \approx -NT_1 \int d^2\sigma \left[ (1 - \lambda^2E^2)H - (1 - \lambda E) \frac{\lambda^2}{2} (\dot{f}^i)^2 + \frac{(1 + \lambda E)\lambda^2}{2H}(\partial_\sigma f^i)^2 + \ldots \right], \]
with \( H = (1 + \lambda^2 \frac{N^2 - 1}{4(1 - \lambda^2E^2)^2\sigma^4}). \)

Then we define the relative transverse fluctuation as \( f^i = \Phi^i(\sigma)e^{-iwt}\delta x^i \) in the direction of \( x^i \), with \( \Phi \) is a function of \( \sigma \), and the equations of motion of the fluctuations are found to be
\[ \left( -\partial_\sigma^2 - \frac{w^2\lambda^2(1 - \lambda E)(N^2 - 1)}{4(1 + \lambda E)(1 - \lambda^2E^2)^2\sigma^4}\right)\Phi^i = w^2 \frac{1 - \lambda E}{1 + \lambda E} \Phi^i, \]
where the potential is
\[ V(\sigma) = -\frac{w^2\lambda^2(1 - \lambda E)(N^2 - 1)}{4(1 + \lambda E)(1 - \lambda^2E^2)^2\sigma^4}. \]

We remark that the presence of \( E \) is quickly increasing the potential from \(-\infty\) to zero. Then, when \( E \) is close to the inverse of \( \lambda \) the potential is close to zero for all \( \sigma \);
\[ E \sim 0, \quad V(\sigma) \sim -\lambda^2 \frac{N^2-1}{4\sigma} w^2 \]
\[ E \sim \frac{1}{\lambda}, \quad V(\sigma) \sim -\frac{1-\lambda E}{2} \lambda^2 \frac{N^2-1}{4(1-\lambda^2 E^2)^2 \sigma^2} w^2. \]

This case is seen as a zero mode of what is following so we will focus on its general case known as non-zero modes.

### 5.2.2 Non-Zero Modes

Let’s give the equation of motion of relative transverse fluctuations of non-zero \( \ell \) modes with \((N, N_j)\)-strings intersecting D3-branes. The fluctuation is given by \( \delta \phi^i(\sigma, t) = \sum_{\ell=1}^{N-1} \psi^i_{\ell} \alpha^i_{\ell} \cdots \alpha^i_1 \) with \( \psi^i_{\ell} \) are completely symmetric and traceless in the lower indices.

The action describing this system is
\[
S \approx -NT \int d^2 \sigma \left[ (1 - \lambda^2 E^2) \partial_t^2 \delta \phi^i - (1 - \lambda E) \frac{H^2}{2} (\partial_\sigma \delta \phi^i)^2 - (1 - \lambda E) \frac{\lambda^2}{2} [\phi^i, [\phi^i, \delta \phi^i]] \right],
\]

The equation of motion for relative transverse fluctuations in non-zero modes is
\[
\left[ 1 - \frac{\lambda E}{1 + \lambda E} \frac{H}{\partial^2} - \partial^2 \right] \delta \phi^i + (1 - \lambda E) [\phi^i, [\phi^i, \delta \phi^i]] = \frac{\lambda^2}{6} [\partial_\sigma \phi^i, [\partial_\sigma \phi^i, \partial^2 \phi^i]] = 0. \quad (95)
\]

By the same way followed in overall case the equation of motion for each mode \( \ell \) is found to be
\[
\left[ -\partial^2 + \left( 1 - \frac{\lambda E}{1 + \lambda E} \frac{H}{(1 + \lambda E)^2 \sigma^4} \right) + \frac{\lambda^2 \ell (\ell + 1)}{6(1 - \lambda^2 E^2)^2 \sigma^4} \right] \partial^2 + \frac{\ell (\ell + 1)}{1 + \lambda E \sigma^2} \delta \phi^i_\ell = 0. \quad (96)
\]

We write \( \delta \phi^i_\ell = f^i_\ell e^{-i\omega t} \delta x^i \) in the direction of \( x^i \), then the equation (97) becomes
\[
\left[ -\partial^2 - \left( 1 - \frac{\lambda E}{1 + \lambda E} \frac{N^2 - 1}{4(1 - \lambda^2 E^2)^2 \sigma^4} \right) - \frac{\lambda^2 \ell (\ell + 1)}{6(1 - \lambda^2 E^2)^2 \sigma^4} \right] f^i_\ell = 0. \quad (97)
\]

To solve this equation we start, for simplicity, by considering small \( \sigma \). The equation (98) is reduced to
\[
\left[ -\partial^2 - \frac{\lambda^2 \omega^2}{1 - \lambda^2 E^2} \left( 1 - \frac{\lambda E}{1 + \lambda E} \frac{N^2 - 1}{4} - \frac{\ell (\ell + 1)}{6} \right) \right] f^i_\ell = \frac{1 - \lambda E}{1 + \lambda E} \omega^2 f^i_\ell, \quad (99)
\]

with the potential
\[
V = \frac{-\lambda^2 \omega^2}{1 - \lambda^2 E^2} \left( 1 - \frac{\lambda E}{1 + \lambda E} \frac{N^2 - 1}{4} - \frac{\ell (\ell + 1)}{6} \right).
\]

The potential \( V \) is quite zero for all \( E \) and only at \( \sigma \approx 0.02 \) that we see \( V \) varies in terms of \( E \) and goes up too fast to be close to zero as a constant function (Fig.4(a)).
The equation of motion (99) can be rewritten as follows

\[
\left[ -\frac{1 + \lambda E}{1 - \lambda E} \partial_\sigma^2 - \left( 1 + \lambda^2 \frac{N^2 - 1}{4(1 - \lambda^2 E^2)^2} \right) - \frac{1 + \lambda E}{1 - \lambda E} \frac{\lambda^2 \ell (\ell + 1)}{6(1 - \lambda^2 E^2)^2} \right] f_i^\ell = 0. \tag{100}
\]

We change the coordinate to \( \tilde{\sigma} = \sqrt{1 + \frac{\lambda E}{1 + \lambda E}} \) \( w \sigma \) and the equation (100) is rewritten as

\[
\left[ \partial_{\tilde{\sigma}}^2 + 1 + \frac{\kappa^2}{\tilde{\sigma}^4} \right] f_i^\ell (\tilde{\sigma}) = 0, \tag{101}
\]

with

\[
\kappa^2 = w^4 \lambda^2 \frac{3(1 - \lambda E)^2 (N^2 - 1) - 2(1 - \lambda^2 E^2) \ell (\ell + 1)}{12(1 + \lambda E^2)(1 - \lambda^2 E^2)^2}.
\]

Then we follow the suggestions of WKB by making a coordinate change;

\[
\beta(\tilde{\sigma}) = \int \frac{d\tilde{\sigma}}{\sqrt{\kappa}} \sqrt{1 + \frac{\kappa^2}{y^4}}., \tag{102}
\]

and

\[
f_i^\ell (\tilde{\sigma}) = (1 + \frac{\kappa^2}{\tilde{\sigma}^4})^{-\frac{1}{4}} \tilde{f}_i^\ell (\tilde{\sigma}). \tag{103}
\]

Thus, the equation (101) becomes

\[
\left( - \partial_{\beta}^2 + \frac{5\kappa^2}{(\tilde{\sigma}^2 + \frac{\kappa^2}{\tilde{\sigma}^4})^3} \right) \tilde{f}_i = 0, \tag{104}
\]

with

\[
V(\beta) = \frac{5\kappa^2}{(\tilde{\sigma}^2 + \frac{\kappa^2}{\tilde{\sigma}^4})^3}. \tag{105}
\]

Then

\[
f_i^\ell = (1 + \frac{\kappa^2}{\tilde{\sigma}^4})^{-\frac{1}{4}} e^{i\beta(\tilde{\sigma})}. \tag{106}
\]

Since we are dealing with small \( \sigma \) case the obtained fluctuation becomes

\[
f_i^\ell = \frac{\tilde{\sigma}}{\sqrt{\kappa}} e^{i\beta(\tilde{\sigma})}.
\]

This is the asymptotic wave function in the regions \( \beta \to -\infty \), while around \( \beta \sim 0 \); i.e. \( \tilde{\sigma} \sim \sqrt{\kappa} \), \( f_i^\ell \sim 2^{-\frac{1}{4}} \). The variation of this fluctuation in terms of small \( \sigma \) and the electric field is well shown in Fig.3(a) by considering the real part of the function. The variation of \( f_i^\ell \) in terms of \( \sigma \) has positive values and goes up as \( \sigma \) goes up for all \( E \) in general. The influence of \( E \) on \( f_i^\ell \) appears at \( \sigma \approx 0.2 \).

Now, if \( \sigma \) is too large the equation of motion (98) becomes

\[
\left[ - \partial_\sigma^2 + \frac{\ell (\ell + 1)}{1 + \lambda E \sigma^2} \right] f_i^\ell = \frac{1 - \lambda E}{1 + \lambda E} w^2 f_i^\ell. \tag{107}
\]

The fluctuation solution of this equation is

\[
f_i^\ell (\sigma) = \alpha \sqrt{\sigma} BesselJ \left( \frac{1}{4} \sqrt{1 + \frac{4(\ell + 1)}{1 + \lambda E}}, w \sigma \sqrt{\frac{1 - \lambda E}{1 + \lambda E}} \right) \\
+ \beta \sqrt{\sigma} BesselY \left( \frac{1}{4} \sqrt{1 + \frac{4(\ell + 1)}{1 + \lambda E}}, w \sigma \sqrt{\frac{1 - \lambda E}{1 + \lambda E}} \right), \tag{108}
\]

with \( \alpha, \beta \) are constants. The variation of this fluctuation in terms of large \( \sigma \) and \( E \) is given by Fig.3(b). The values of \( f_i^\ell \) are negative and they are going down as \( E \) going up.
By dealing with the fluctuations (106) (Fig.3(a)) and (108) (Fig.3(b)) in small and large \( \sigma \) regions respectively, we remark that it is clear that we get different fluctuations from small to large \( \sigma \) with a singularity at some stage of \( \sigma \) and consequently the system is separated into two regions depending on the electric field.

The potential associated to (108) is

\[
V(\sigma) = \ell(\ell + 1) \frac{(1 + \lambda E)\sigma^2}{\lambda^2}.
\]

Accordingly to Fig.4(b) describing the variation of \( V \), we remark that, \( V \) goes down as \( \sigma \) goes up and more down as \( E \) goes up; i.e. the potential becomes more small as the electric field appears between zero and \( E \sim \frac{1}{\lambda} \). The potentials represented by the two figures Fig.4(a) and Fig.4(b) don’t have an intersecting point at some stage of \( \sigma \). This leads the system to get a singularity which supports the idea that the system is separated into two regions in non-zero modes of relative transverse fluctuations.

Consequently, the D1⊥D3 system has Neumann boundary conditions and this is more clear at the presence of electric field. This is proved through this section by discussing different modes and different directions of the fluctuations of the funnel solution and their associated potentials. In the figures representing these variations we set \( w, \ell \) equals to the unit of \( \lambda = 1 \) in all the treated equations and \( N = 10 \).

### 5.3 Overall Transverse Fluctuations in D1⊥D5 System

We extend this study to discuss the electrified fluctuations in D1⊥D5 system. we give the equations of motion of the fluctuations and their solutions. Then, we discuss the variation of the potential and the fluctuations in terms of electric field and the spatial coordinate.
We start by considering overall transverse fluctuations in zero mode. This type of fluctuations is given as \( \delta \phi^m(\sigma, t) = f^m(\sigma, t)I_N, \) \( m = 6, 7, 8. \) Plugging this fluctuation into the full \((N, N_f)-\)string action \((15'), \) together with the funnel \((16)\) the action is found to be

\[
S = -NT_1 \int d^2 \sigma \left( (1 + \lambda E)A - \frac{1}{2}(1 - \lambda^2 E^2)\lambda^2 A(f^m)^2 + \frac{1}{2}(1 + \lambda E)\lambda^2 (\partial_\sigma f^m)^2 + ... \right). \tag{109}
\]

where

\[
A = (1 + \frac{4R(\sigma)^4}{c\lambda^2})^2, \tag{110}
\]

with the quadratic terms in \( f^m \) were the only terms retained in the action. The linearized equation of motion of the fluctuation is then

\[
\left[ (1 - \lambda E)\left( 1 + \frac{4R(\sigma)^4}{c\lambda^2} \right)^2 \partial_t^2 - \partial_\sigma^2 \right] f^m = 0. \tag{111}
\]

We consider small \( R \) which is given by \((17)\) in the second section. We insert its expression in the last equation and the equation of motion becomes

\[
\left[ (1 - \lambda E)\left( 1 + \frac{n^2 \lambda^2}{16(1 - \lambda^2 E^2)^2 \sigma^4} \right)^2 \partial_t^2 - \partial_\sigma^2 \right] f^m = 0 \tag{112}
\]

for large \( n. \)

Let’s consider the fluctuation in the following form

\[
f^m = \phi(\sigma)e^{-iwt}\delta x^m, \tag{113}
\]

with \( \delta x^m, \ m = 6, 7, 8, \) the direction of the fluctuation. The equation \((112)\) becomes

\[
\left[ - \partial_\sigma^2 - w^2(1 - \lambda E)(\frac{n^2 \lambda^2}{8(1 - \lambda^2 E^2)^2 \sigma^4} + \frac{n^4 \lambda^4}{16^2(1 - \lambda^2 E^2)^4 \sigma^8}) \right] \phi = w^2(1 - \lambda E)\phi. \tag{114}
\]

The potential of this system is

\[
V = -w^2(1 - \lambda E)(\frac{n^2 \lambda^2}{8(1 - \lambda^2 E^2)^2 \sigma^4} + \frac{n^4 \lambda^4}{16^2(1 - \lambda^2 E^2)^4 \sigma^8}), \tag{115}
\]

depending on the electric field \( E \) with \( E \in [0, \frac{1}{\lambda}]. \)

5.3.1 Small \( \sigma \) Limit

The equation \((114)\) is complicated and to simplify the calculations we start by considering the small \( \sigma \) and \( \frac{1}{\sigma^3} \) dominates in \((114)\) and \((115). \) We then discuss the equation

\[
\left[ - \partial_\sigma^2 - w^2(1 - \lambda E)(\frac{n^4 \lambda^4}{16^2(1 - \lambda^2 E^2)^4 \sigma^8}) \right] \phi = w^2(1 - \lambda E)\phi, \tag{116}
\]

and the potential is reduced to

\[
V = -w^2(1 - \lambda E)\frac{n^4 \lambda^4}{16^2(1 - \lambda^2 E^2)^4 \sigma^8}. \tag{117}
\]
As shown in Fig.5(a), the potential $V$ tends to $-\infty$ until some values of $E$ when $E$ and $\sigma$ are close to zero, and once $E$ is close to the inverse of $\lambda$ the potential is zero for all small $\sigma$. We consider in this case $\sigma \in [0, 0.5]$ in the unit of $\lambda$ with $\lambda = 1$, $w = 1$, $n = 10$ and $E \in [0, 1]$.

To solve the differential equation (116), we consider the total differential on the fluctuation. Let’s denote $\partial_{\sigma} \phi \equiv \phi'$. Since $\phi$ depends only on $\sigma$ we find $\frac{d\phi}{d\sigma} = \partial_{\sigma} \phi$. We rewrite the equation (116) in this form

$$
\frac{1}{\phi} \frac{d\phi'}{d\sigma} = -w^2(1 - \lambda E)\left[-\frac{n^4\lambda^4}{16^2(1 - \lambda^2 E^2)^4\sigma^8} + 1\right].
$$

(118)

An integral formula can be written as follows

$$
\int_0^{\sigma} \frac{d\phi'}{\phi} = -\int_0^\sigma w^2(1 - \lambda E)\left[-\frac{n^4\lambda^4}{16^2(1 - \lambda^2 E^2)^4\sigma^8} + 1\right]d\sigma,
$$

(119)

which gives

$$
\frac{\phi'}{\phi} = -w^2(1 - \lambda E)\left[-\frac{n^4\lambda^4}{16^2(1 - \lambda^2 E^2)^4\sigma^8} + \sigma\right] + \alpha.
$$

(120)

We integrate again the following

$$
\int_0^\sigma \frac{d\phi}{\phi} = -\int_0^\sigma (w^2(1 - \lambda E)\left[-\frac{n^4\lambda^4}{16^2(1 - \lambda^2 E^2)^4\sigma^8} + \sigma\right] + \alpha)d\sigma.
$$

(121)

We get

$$
\ln \phi = -w^2(1 - \lambda E)\left[-\frac{n^4\lambda^4}{16^2 \times 7(1 - \lambda^2 E^2)^4\sigma^7} + \sigma\right] + \alpha \sigma + \beta,
$$

(122)

and the fluctuation in small $\sigma$ region is found to be

$$
\phi(\sigma) = \beta e^{-w^2(1 - \lambda E)\left[-\frac{n^4\lambda^4}{16^2 \times 42(1 - \lambda^2 E^2)^4\sigma^6} + \frac{1}{2}\sigma^2\right] + \alpha \sigma}.
$$

(123)

with $\beta$ and $\alpha$ are constants.

We plot the progress of the obtained fluctuation in (Fig.6(a)). First we consider the constants $\beta = 1 = \alpha$, then the small spatial coordinate in the interval $[0, 0.5]$ with the unit of $\lambda = 1$, $w = 1$ and $n = 4$. As above the electric field is in $[0, 1]$. We see that at the absence of the electric field there is no fluctuations at all and this phenomenon continues for the small values of $E$. When $E \approx 0.5$ the fluctuation appears from $\sigma = 0.15$ and goes down as $\sigma$ and $E$ go up.

5.3.2 Large $\sigma$ Limit

In the large $\sigma$ case, the equation (114) becomes

$$
\left[-\partial_{\sigma}^2 - w^2(1 - \lambda E)\frac{n^2 \lambda^2}{8(1 - \lambda^2 E^2)^2\sigma^4}\right]\phi = w^2(1 - \lambda E)\phi
$$

(124)

and the potential is

$$
V = -w^2(1 - \lambda E)\frac{n^2 \lambda^2}{8(1 - \lambda^2 E^2)^2\sigma^4}.
$$

(125)
By plotting the progress of this potential (Fig.5(b)) we consider the large spatial coordinate in the interval \([0.5, 1]\) and \(E \in [0, 1]\) in the unit of \(\lambda = 1, \ w = 1\) with \(n = 10\). The obtained figure shows that \(V\) has in general higher values than the ones obtained in small \(\sigma\) case (Fig.5(a) describing (117)). Specially, for the first values of \(\sigma\), \(V\) goes up from negative values to be close to zero for almost values of \(E\) until \(E\) is close to \(1/\lambda\), approximately from \(E = 0.8\) where \(V \approx -0.02\), we remark that \(V\) has small variation in \([0.8, 1]\) region of \(\sigma\). By contrary, in figure 5(a), when \(\sigma = 0.5\) which is the last value of \(\sigma\) in that case we find \(V\) is already zero for all \(E\). Consequently, these two potentials (117) and (125) show a big gap to go from one system to other that they describe, meaning that our system is separated into two regions; small and large \(\sigma\) depending on \(E\).

Now, we should solve the equation of motion of the relative transverse fluctuations (124), in the case of large \(\sigma\). We start by defining a new coordinate

\[ \tilde{\sigma}^2 = w^2(1 - \lambda E)\sigma^2 \]

and (124) becomes

\[
\left(1 + \frac{n^2 \lambda^2 w^4(1 - \lambda E)^2}{8(1 - \lambda^2 E^2)^2 \tilde{\sigma}^4} + \partial_{\tilde{\sigma}}^2\right)\phi(\tilde{\sigma}) = 0,
\]

with the potential is

\[ V(\tilde{\sigma}) = \frac{\kappa^2}{\tilde{\sigma}^4}, \]

and

\[ \kappa^2 = \frac{n^2 \lambda^2 w^4(1 - \lambda E)^2}{8(1 - \lambda^2 E^2)^2}. \]

The equation (126) is a Schrödinger equation for an attractive singular potential \(\propto \tilde{\sigma}^{-4}\) and depends on the single coupling parameter \(\kappa\) with constant positive Schrödinger energy. The solution is then known by making the following coordinate change

\[ \chi(\tilde{\sigma}) = \int \frac{d\tilde{\sigma}}{\sqrt{\kappa}} \sqrt{1 + \frac{\kappa^2}{y^2}}, \]

and

\[ \Phi = (1 + \frac{\kappa^2}{\tilde{\sigma}^4})^{-\frac{1}{2}} e^{\pm i\chi(\tilde{\sigma})}. \]

Thus, the equation (126) becomes

\[ \left( - \partial_{\chi}^2 + V(\chi) \right)\Phi = 0, \]

with

\[ V(\chi) = \frac{5\kappa^2}{(\tilde{\sigma}^2 + \frac{\kappa^2}{\tilde{\sigma}^2})^3}. \]

Then, the fluctuation is found to be

\[ \Phi = (1 + \frac{\kappa^2}{\tilde{\sigma}^4})^{-\frac{1}{2}} e^{\pm i\chi(\tilde{\sigma})}. \]

This fluctuation has the following limit; since we are in large \(\sigma\) region \(\Phi \sim e^{\pm i\chi(\tilde{\sigma})}\). This is the asymptotic wave function in the regions \(\chi \to +\infty\), while around \(\chi \sim 0\); i.e. \(\tilde{\sigma} \sim \sqrt{\kappa}\),
Φ \sim 2^{-\frac{1}{4}}. Owing to the plotting of the progress of this fluctuation given by Fig.6(b), by considering the real part of the function, we remark that Φ goes down fast as E goes up for all σ. When σ = 0.5 the fluctuation gets different values for all E compared to the values gotten in small σ region by (123) (Fig.6(a)). These tow figures show that the fluctuations of fuzzy funnel of D1⊥D5 branes have a singularity at some stage of σ separating the system into two regions; small and large σ.

6 Duality in Electrified D1⊥D3 and D1⊥D5 Systems

6.1 Duality in D1⊥D3 Branes

Now, we see if D1 and D3 descriptions [26] in D1⊥D3 system discussed in section 2 match or not. As we showed in [22], the two descriptions D1 and D3 don’t have a complete agreement in the presence of a world volume electric field since the energies don’t match as we will see in the following, even if their profiles match very well.

The energy is easily derived from the action (8) for the static solution (10). The minimum energy condition is

\[ \frac{d\phi_i}{d\sigma} = \pm \frac{i}{2} \epsilon_{ijk} [\phi_j, \phi_k], \]

which can be identified as the Nahm equations [37]. We insert the ansatz (9) and this implies

\[ \hat{R}' = \pm 2\sqrt{1 - \lambda^2 E^2 \hat{R}^2}. \] (133)

Using this condition and evaluating the Hamiltonian, \( \int d\sigma (DE - L) \), for the dyonic funnel solutions (\( L \) is the Lagrangian), the energy is expressed as

\[ E_1 = T_1 \int d\sigma Str \left[ \frac{\lambda^2 E^2}{\sqrt{1 - \lambda^2 E^2}} + \sqrt{1 - \lambda^2 E^2} [1 + 4\lambda^2 \alpha \alpha \hat{R}^4] \right]. \]
We can manipulate this result by introducing the physical radius \( R = \lambda \sqrt{C} |\hat{R}| \) and using \( T_1 = 4\pi^2 \ell_s T_3 \). It’s also useful to consider the electric displacement \( D \) Consequently, the energy from the D1 brane theory is found to be

\[
E_1 = T_1 \int d\sigma \sqrt{N_m^2 + g_s N_e^2 + T_3 (1 - \frac{1}{N_m^2})^{\frac{1}{2}}} \int dR 4\pi R^2, \tag{134}
\]

with \( g_s \) is the string coupling with \( T_1 = (\lambda g_s)^{-1} \). The first term comes from collecting the contributions independent of \( \hat{R} \). The second term gotten from the terms containing \( \hat{R} \) and is used to put these in the form \( \hat{R}^2 |\hat{R}'| \). Then we have repeatedly applied (133) in producing the second term.

If we consider large \( N_m \) the energy is reduced to the following

\[
E_1 = T_1 N_m \int d\sigma, \tag{135}
\]

which can be rewritten in terms of physical radius \( R \) as

\[
E_1 = T_3 N_m \int 4\pi R^2 dR, \tag{136}
\]

with \( T_3 = \frac{T_1}{4\pi \ell_s^2} \). In D3-brane description the energy (3) becomes

\[
E_3 = T_3 \int d^{3}\sigma \sqrt{1 + \lambda^4 \frac{N_m^2 [(N_m + N_e)^2 + N_e^2]}{16r^8} + 2\lambda^2 \frac{N_m (N_m + N_e)}{4r^4} + 2\lambda^2 \frac{N_e^2}{4r^4}}, \tag{137}
\]

such that the magnetic and the electric fields are given by

\[
\vec{B} = \frac{N_m}{2r^2} \vec{r}, \quad \vec{E} = \frac{N_e}{2r^2} \vec{r}. \tag{138}
\]

In the large \( N_m \) limit and fixed \( N_e \), the energy (137) of the spherically symmetric BPS configuration is reduced to

\[
E_3 = T_3 \frac{N_m \sqrt{(N_m + N_e)^2 + N_e^2}}{N_m + N_e} \int 4\pi r^2 dr. \tag{139}
\]

Again we consider large \( N_m \) limit and fixed \( N_e \) and we get

\[
\frac{\sqrt{(N_m + N_e)^2 + N_e^2}}{N_m + N_e} \rightarrow 1.
\]

Consequently, for fixed \( N_e \) and large \( N_m \) limit we have agreement from both sides (D1 and D3 descriptions) and the energy is

\[
E_3 = T_3 N_m \int 4\pi r^2 dr, \tag{140}
\]
in which we identify the physical radius $R$ from D1 description and $r$ from D3 description.

Now, if we take large $N_m$ limit keeping $N_e/N_m = K$ fixed at any arbitrary $K > 0$ the result will be different. Thus, from D1 description the energy becomes

$$E_1 = T_3 N_m \sqrt{1 + g_s K^2} \int 4\pi R^2 dR,$$

(141)

and from D3 description the energy is

$$E_3 = T_3 \frac{N_m \sqrt{(1 + K)^2 + K^2}}{1 + K} \int 4\pi R^2 dR.$$

(142)

Then we have disagreement. Consequently, the presence of electric field spoils the duality between D1 and D3 descriptions of intersecting D1-D3 branes.

### 6.2 Duality in D1⊥D5 Branes

Although D1⊥D5 branes system \cite{3} \cite{12} is not supersymmetric, the fuzzy funnel configuration in which the D-strings expand into orthogonal D5-branes shares many common features with the D3-brane funnel. Thus, we are interested in establishing whether a similar result holds also in the case of D1⊥D5 branes meaning the presence of a world volume electric field leads to broken duality or not.

From the D1 description the system is described by the action (15”). We consider static configurations involving five (rather than three) nontrivial scalars, $\phi_i$ with $i = 1, ..., 5$ with the proposed ansatz (16).

The electric field is fixed by the quantization condition on the displacement field, $D = \frac{N f}{N}$, where

$$D = \frac{1}{N} \frac{\delta S}{\delta E} = \frac{\lambda^2 T_1 E}{\sqrt{1 - \lambda^2 E^2}^2}$$

(143)

after using the equations of motion, the energy ($\tilde{E}_1 = \int d\sigma (DE - L)$) of the system is evaluated to be

$$\tilde{E}_1 = \sqrt{N^2 + g_s^2 N_f^2 T_1} \int d\sigma + \frac{6N}{c} T_5 \int \Omega_4 R^4 dR + NT_1 \int dR + \Delta E,$$

(144)

with $T_5 = \frac{T_1}{(2\pi \ell_s)^4}$ and the first and the second terms correspond to the energies of $N$ semi-infinite strings stretching from $\sigma = 0$ to infinity and of $6N/c$ D5-branes respectively. The contribution of the last terms to the energy indicates that the configuration is not supersymmetric. The last contribution is a finite binding energy $\Delta E = 1.0102 N c^{1/4} T_1 \ell_s$.

The energy of the D1-D5 system from D5 description (subsection 2.2) is evaluated to be

$$\tilde{E}_5 = \sqrt{NT_1 \left(1 - \left(\frac{g_s N_f}{\sqrt{N^2 + g_s^2 N_f^2}}\right)^2\right)} \int d\sigma + \frac{6N}{c} T_5 \int \Omega_4 R^4 dR + NT_1 \int dR + \Delta E,$$

(145)

with $\Delta E$ is the same one found above from D1 description. In the absence of electric field, it’s clear that by identifying the profiles of D1 and D5 descriptions in the limit of $N$ we could get complete agreement for the geometry and the energy determined by
the two dual approaches. Now, in the presence of an electric field it seems there is also agreement. We compare the energy from the D1 description (144) and the energy from the D5 description (145). If we consider the large \( N \) limit and fixed \( N_f \) the first term of \( \tilde{E}_1 \) becomes

\[
\sqrt{N^2 + g_s^2 N_f^2} T_1 \int d\sigma \rightarrow N T_1 \int d\sigma,
\]

(146)

and the first term of \( \tilde{E}_5 \) goes to the following value

\[
\frac{N T_1}{\sqrt{1 - \left( \frac{g_s N_f}{\sqrt{N^2 + g_s^2 N_f^2}} \right)^2}} \int d\sigma \rightarrow N T_1 \int d\sigma,
\]

(147)

which proves the agreement at large \( N \).

Now, let’s fix the value \( \frac{g_s N_f}{N} \) to be one value \( M \) which can’t be neglected at large limit of \( N \). Thus, if \( N \) is large the last two limits (146) and (147) become

\[
\sqrt{N^2 + g_s^2 N_f^2} T_1 \int d\sigma \rightarrow N T_1 \sqrt{1 + M^2} \int d\sigma
\]

(148)

and

\[
\frac{N T_1}{\sqrt{1 - \left( \frac{g_s N_f}{\sqrt{N^2 + g_s^2 N_f^2}} \right)^2}} \int d\sigma \rightarrow \frac{N T_1}{\sqrt{1 - \frac{M^2}{1+M^2}}} \int d\sigma.
\]

(149)

The right hand term of (149) is equal to the right hand term of (148)

\[
\frac{N T_1}{\sqrt{1+M^2}} = N T_1 \sqrt{1+M^2}.
\]

Then, this implies agreement of the two duals at the level of energy of the two descriptions. Consequently, the duality in D1⊥D5 branes is unbroken by switching on the electric field.

7 D2-Brane and Generalized Maxwell Theory

One of the field-theories describing anyons is the model where the matter is interacting with the Chern-Simons (CS) gauge field [25]. In reference [32], Stern has introduced another approach to treat anyons that does not require the CS term, but introduces a generalized connection to which the conserved U(1) current is coupled in a gauge invariant way [33]. In this model the gauge field is dynamical and the potential has the confining nature which makes the model different [34].

In this section, we treat the same system but on the two-sphere. Among the main results in this work is the change of the potential’s nature; there is no confinement any more, and the disappearance of the confinement in the two-sphere case for the exotic system is very interesting result. It was shown in [38] that compact Maxwell theory in (2+1)-dimensions confines permanently electric test charges and the usual two-dimensional Coulomb potential is \( V(R) \sim \ln R \). Since the electrostatic potential has the form \( V(R) \sim R \) and holds for all values of the gauge coupling, the compact (2+1)-dimensional Maxwell theory does not exhibit any phase transition, i.e., the confinement is permanent. In the present paper, things are changed by treating the exotic system in
higher dimensions and $V(R) \sim \frac{1}{R}$ with $R$ is the distance between two opposite charged exotic particles. Another important result we get is at the level of energy; D2-brane gets a higher energy if the radius $r$ of the two-sphere goes to infinity and it is higher if the number of charges is large which makes the system very special.

7.1 Generalized Connection and Anyons

The simplest way to realize fractional statistics characterizing anyons in (2+1) dimensional space-time is usually by adding a Chern-Simons term to the action. Recently, a novel way was introduced in [32] to describe anyons without a Chern-Simons term. Thus, a generalized connection was considered in (2+1)-dimensions denoted $A_\theta^\mu, \mu = 0, 1, 2$. The gauge theory is defined by the following Lagrangian

$$L_\theta = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\theta^\mu$$  \hspace{1cm} (150)

with $A_\theta^\mu \equiv A_\mu + \frac{\theta}{2} \epsilon_{\mu
u\rho} F^{\nu\rho}$ and $\theta$ is real parameter in Minkowski space. The Lagrangian $L_\theta$ describes Maxwell theory that couples to the current via the generalized connection rather than the usual one. This coupling is gauge invariant as long as $J^\mu$ is a conserved external current. In this theory, the gauge fields are dynamical and the canonical momenta are $\pi^\mu = F^{\mu 0} + \theta \epsilon^{0\mu \nu} J_\nu$ which results in the usual primary constraint $\pi^0 = 0$ and $\pi^i = F^{i0} + \theta \epsilon^{0ij} J_j$ ($i, j = 1, 2$). Thus the magnetic field is $B = \epsilon_{ij} \partial^i A^j$ and the electric field is $E^i = \pi^i - \theta \epsilon^{ij} J_j$.

Now, accordingly to (150), the equations of motion for $A_\mu$ is

$$\partial^\nu \partial_\nu A_\mu = J_\mu + \theta \epsilon_{\mu\nu\rho} \partial^\nu J^\rho.$$ \hspace{1cm} (151)

Then, we consider the simplest case of a static pointlike particle located at the origin which is described by $J^0 = e\delta^{(2)}(x)$. By solving (151) for the gauge field one finds

$$A_0 = \frac{\ln r}{2\pi}, \quad A_1 = \frac{\theta x_2}{2\pi r^2}, \quad A_2 = \frac{\theta x_1}{2\pi r^2},$$

with $r^2 = x_1^2 + x_2^2$. This background describes one unit of an electrically charged particle and an infinitely thin magnetic flux with total flux $\theta$ both located at the origin and the shift in the statistics of the particle is fixed by the Aharonov-Bohm effect to be

$$\Delta \phi = \theta.$$  \hspace{1cm} (152)

with $\phi = \int_a^b A_i dx_i$ the phase acquired by the pointlike particle traveling along some path $P$. $\Delta \phi$ is the phase difference between any two paths at the same endpoints. We note that in the case of Chern-Simons theory, the phase is two times $\theta$ and this is due to the fact that the charged particle is winding around a magnetic flux while in the present theory we also have the contribution of a flux tube winding around the charged particle. Another reason is that with the $A_\theta^\mu$ construction a long range electric field is also generated which couples to the current and gives exactly the same phase.

For a static charged particle located at the origin and $J^i = 0$, the static electromagnetic fields are

$$B(x) = e\theta \delta^{(2)}(x)$$

$$E_i(x) = -\frac{\theta \delta_i^2}{2\pi r^2}$$ \hspace{1cm} (153)
and the total magnetic flux attached to $N$ charged particles is

$$\Phi = \int_V d^2 x B(x) = e \theta N.$$  \hspace{1cm} (154)

We note that the both $L_{CS}$ (the lagrangian in Chern-Simons theory) and $L_\theta$ lead to fractional statistics by the same mechanism of attaching a magnetic flux to the charged particles but the physics they describe is quite different. We remark, for example, that in this theory the interaction potential is an object of considerable interest [34]. The potential has confining nature; it grows to infinity when the natural separation of the physical degrees of freedom grow, but in the Maxwell-Chern-Simons theory, the Chern-Simons term turns the electric and magnetic fields massive leading to a screening potential between static charges.

### 7.2 Anyons and Fuzzy Two-Sphere

Now, let us consider exotic particle moving on a two-sphere instead of a plane in the background of a monopole put at the origin. First, the two-sphere is $S^2 \sim CP^1 = SU(2)/U(1)$ and the representations of $SU(2)$ are given by the standard angular momentum theory.

The coordinates of fuzzy two-sphere are given by the $SU(2)$ algebra

$$[X_i, X_j] = i \alpha \epsilon_{ijk} X_k, \quad X_i = \alpha L_i,$$  \hspace{1cm} (155)

$L_i$ is the total angular momentum with the representation to be the spin $\ell$ and $\alpha$ is a dimensionful constant. We note that around the north pole of $S^2$ labeled by $L_3 = \ell$, the fuzzy two-sphere algebra becomes a noncommutative plane if $\ell \rightarrow \infty$,

$$[X_i, X_j] = i \alpha^2 \ell \epsilon_{ij} I,$$  \hspace{1cm} (156)

with $I$ is the identity.

#### 7.2.1 Connection

To construct the connection which goes to the generalized connection given above when the radius of fuzzy two-sphere goes to infinity we use the first Hopf map as known in the literature which is a map from $S^3$ to $S^2$ and naturally introduces a $U(1)$ bundle on $S^2$. Then, the two-sphere can be parameterized by two complex coordinates $u_\alpha$ such that $u_\alpha^* u_\alpha = 1$ with $u_\alpha^*$ is the complex conjugate of $u_\alpha$. A spatial coordinate $x_i$ on $S^2$ with radius $r$ is written in terms of $u_\alpha$’s as

$$x_i = r u^\dagger \sigma_i u$$  \hspace{1cm} (157)

with $\sigma_i$ are Pauli matrices. The vector potential on $S^2$ is

$$A_i dx_i = -i \gamma u_\alpha^* du_\alpha,$$  \hspace{1cm} (158)

with $\gamma$ is integer due to the Dirac quantization rule.

Thus, the Hopf spinor satisfying (157) is given by

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{\sqrt{2r(r + x_3)}} \left( \begin{array}{c} r + x_3 \\ x_1 + ix_2 \end{array} \right) e^{i \chi},$$  \hspace{1cm} (159)
$e^{ix}$ is a $U(1)$ phase. The connection is defined as

$$A_i dx_i = -\frac{\hbar}{e} u_{\alpha}^* d u_{\alpha} = \frac{\hbar}{2 e r (r + x_3)} \epsilon_{ij3} x_j dx_i. \quad (160)$$

By considering the motion of an exotic particle (charged particle-magnetic flux composite) on two-sphere, the monopole charge is $\frac{\theta}{4\pi} = \frac{\theta}{4\pi}$ which is identified with the connection in two dimensional space for $r \to \infty$ discussed in section 2. By generalizing the spinor to $(2S + 1)$-components spinor $u^{(S)}$, the monopole charge becomes $\theta = \frac{\hbar S}{e}$ and

$$x_i = \frac{1}{S} r u^{(S)}_i \sigma_i^{(S)},$$

where $\sigma_i^{(S)}$ is the spin $S$ representation of $SU(2)$.

Now, for simplicity we consider a static particle at $x'$. The magnetic and electric fields are given in (135) and the charge-magnetic dipole is defined by the current

$$J_0 = e \delta^{(3)} (x - x') \quad J_i = \frac{\mu m}{e} \epsilon_{im} \partial_m J_0, \quad (161)$$

$\mu_m$ is the dipole’s moment.

### 7.2.2 Generalized Maxwell Theory

The Hamiltonian of this system is written as follows

$$H = \frac{1}{2m r^2} M_i M_i + \int d^3 x \left( \frac{1}{2} F_{00} F^{00} + \frac{1}{4} F_{ij} F^{ij} - \frac{\theta}{2} \epsilon^{0ij} J^0 F^{ij} \right) \quad (162)$$

such that for a static point like particle $J_i = 0$ and the primary constraint is

$$\pi^0 = 0$$

which leads to the secondary constraint

$$\partial_i \pi^i - J^0 = 0$$

with $\pi^\mu$ is the canonical momentum of gauge field $A^\mu$. $M_i$ is the orbital angular momentum of the charged particle

$$M_i = \epsilon_{ijk} x_j \left( -ih \partial_k + e A_k^0 \right)$$

$$= \epsilon_{ijk} x_j \left( -ih \partial_k + e A_k + \frac{e}{2} \epsilon_{kmm} F^{mm} + \frac{e}{2} \epsilon_{kn0} F^{n0} \right), \quad (163)$$

where $i, j, k, n, m = 1, 2, 3$ and $A_k^0$ is the generalized connection. The strength field $F^{\mu\nu}$ is

$$F^{nm} = -\frac{\theta}{4\pi} \epsilon_{nmll} x_l \quad F^{n0} = \frac{\theta}{4\pi r (r + x_3)} \epsilon_{nll3} (\dot{x}_l - \dot{x}_3 \frac{2r + x_3}{r (r + x_3)}), \quad (164)$$

we note that $\epsilon_{kn0} F^{n0} = 0$ since $\epsilon_{nl3} \epsilon_{kn0} = 0$ because $l \neq 0$. Then

$$M_i = \epsilon_{ijk} x_j \left( -ih \partial_k + e A_k - \frac{e \theta^2}{4\pi} \frac{x_k}{r^3} \right), \quad (165)$$

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Thus the Hamiltonian (162) of this system is reduced to

$$H = \frac{1}{2mr^2}M_iM_i + \frac{1}{2}\int d^3x(E_i^2 + B^2),$$

(166)

with $\epsilon_{ij}^0F_{ij} = -\frac{e\theta}{4\pi}\epsilon_{ijk}\epsilon^{ijk}J_0^0x_k = -\frac{e\theta}{2\pi}\delta_{0k}J_0^0x_k = 0$ in (13) since $k = 1, 2, 3$. Accordingly to (153,154), we calculate the second term of $H$ in three-dimensional space and the Hamiltonian is

$$H = H_0 + e^2\theta^2N + \frac{e^2r}{3\pi},$$

(167)

with

$$H_0 = \frac{1}{2mr^2}M_iM_i.$$  

(168)

The conclusion we get from this subsection is that the Hamiltonian is different from the one describing the QHE and they are identified ($H \sim H_0$) only if $N, r \to 0$.

7.2.3 Realization of Fuzzy Two-Sphere

First, we remark that the orbital angular momentum of the particle $M_i$ given by (165) satisfy the following deformed commutation relations

$$[M_i, M_j] = i\hbar\epsilon_{ijk}(M_k + \frac{e\theta}{2\pi r}x_k).$$

(169)

This means that the total angular momentum generalizing the $SU(2)$ algebra should be defined as

$$L_i = M_i - \frac{e\theta}{2\pi r}x_i$$

(170)

and we get

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$$
$$[L_i, M_j] = i\hbar\epsilon_{ijk}M_k$$
$$[L_i, x_j] = i\hbar\epsilon_{ijk}x_k.$$  

(171)

Consequently, by simple calculation we find that

$$[L_i, H] = 0,$$

then $SU(2)$ symmetry is generated by $L_i$. We also see that

$$M_iM_i = L_iL_i - \left(\frac{e\theta}{2\pi}\right)^2$$
$$= \hbar^2(l(l + 1) - 4S^2),$$

(172)

with $\ell$ is the eigenvalue of $L_3$. We set, in what follows, $\ell = n + 2S$ ($n = 0, 1, 2,...$) with $n$ plays the role of the level index which could be identified with the Landau level index in the Chern-Simons theory and $n = 0$ corresponds to the lowest level.

The noncommutative Geometry known as fuzzy two-sphere is described by the coordinates $X_i$ defined as

$$X_i = \frac{2\pi r}{e\theta}L_i.$$  

(173)
and they are related to the commutative coordinates \( x_i \) by
\[
X_i = \frac{2\pi r}{e\theta} M_i - x_i. \tag{174}
\]
They satisfy the following commutation relations
\[
[X_i, X_j] = i\hbar \epsilon_{ijk} \frac{2\pi r}{e\theta} X_k, \tag{175}
\]
and the fuzzy two-sphere is realized for the motion of exotic particle on a two-sphere. Its radius is given by the quadratic Casimir of \( SU(2) \)
\[
r^2 = \hbar^2 \left( \frac{2\pi r}{e\theta} \right)^2 \ell(\ell + 1). \tag{176}
\]

The exotic particles obey the cyclotron motion as well-known in the planar system, in which the magnetic field causes them to follow a circular path or cyclotron orbit. According to (172,173), we get the radius of the cyclotron motion \( r_n^c \) in the \( n \)-th level
\[
r_n^c = \frac{2\pi r}{e\theta} \hbar \sqrt{2S(2n + 1) + n(n + 1)}, \tag{177}
\]
which is given by \((r_n^c)^2 = \left( \frac{2\pi r}{e\theta} \right)^2 M_i M_i \). Owing to (174), this radius is related to the coordinates \( X_i \) and \( x_i \) and it is given by \((r_n^c)^2 = (X_i + x_i)(X_i + x_i) \).

For the lowest level we get
\[
r_0^c = \frac{r}{\sqrt{2S}}, \tag{178}
\]
which is identified with the one obtained in the lowest Landau Level discussed in [11]. Also we remark that \( r_0^c \) is much smaller than \( r \) in the lowest level \( n = 0 \) and in a strong magnetic field limit, and \( x_i \) are identified with \( X_i \).

### 7.2.4 Energy

Owing to (172), the energy eigenvalue of \( H \) (167) is
\[
E_n = \frac{\hbar^2}{2mr^2} (2S(2n + 1) + n(n + 1)) + e^2 \theta^2 N + \frac{e^2 r}{3\pi}. \tag{179}
\]
Then we notice that this model could be identified with the one treated in references [11] only in the following case: If both of the radius of fuzzy two-sphere and the number of charges \( N \) are vanishingly small; i.e. \( r, N \rightarrow 0 \). Also, we note that \( n \) in (179) indicates the level index which could be identified with the Landau level index in the Chern-Simons theory only for small \( r \) and \( N \). The variance energy between the lowest level \( n = 0 \) and the first level is
\[
\Delta E = \frac{\hbar^2 (2S + 1)}{mr^2}.
\]
As a remark, the lowest level in our system may be identified with the Lowest Landau level phenomena if the number of charges is small and \( r \rightarrow 0 \) with \( \frac{S}{mr} \gg 1 \) and the energy induced by the dynamical gauge field is ignored. Otherwise, if the above case is
not satisfied; i.e. \( r \gg 1 \) or \( N \gg 1 \), the model is now totally different. Thus the variance energy of the system is

\[ \Delta E = 0, \]

and the energy is dominated by one of the two last terms of (177) or both. Then the energy is

\[ E_n = e^2 \theta^2 N + \frac{e^2 r}{3\pi}, \quad \forall n. \quad (180) \]

We notice here that the variance energy of the system will depend only on the variance of the number of particles or the radius.

Consequently, dealing with the case of exotic particles system in which we introduce a generalized connection put at the origin of two-sphere we get a noncommutative geometry. The energy obtained in this model is very special and very different from the one obtained in QHE case since the gauge field in this system is dynamical. We notice that the energy of the gauge field dominates when the radius of the fuzzy two-sphere goes to infinity; i.e. the flat D2-brane which is a dual of the fuzzy two-sphere has higher energy. We remark that this result is definitely different from the one could be obtained in the case of QHE. In this latter case if \( r \to \infty \) the fuzzy two-sphere goes to flat D2-brane having an energy which goes to zero in this limit.

7.2.5 Potential

We complete this section by making another interesting remark. As known, the potential has a confining nature in two-dimensional space when the generalized connection is introduced instead of adding CS-term; i.e., the potential grows to infinity when the natural separation of the physical degrees of freedom grows.

After giving the energy we may now proceed to discuss the interaction energy between pointlike sources in the model under consideration. This can be done by computing the expectation value of the energy operator \( H \) in a physical state \( |\Omega\rangle \) by following the mechanism used in [39]. We consider the stringy gauge-invariant \( |\bar{\Psi}(y)\Psi(y')\rangle \) state,

\[ |\Omega\rangle \equiv |\bar{\Psi}(y)\Psi(y')\rangle = |\bar{\psi}(y)e^{-i\int \frac{dz^i A_i (z)}{y}} \psi(y')|0\rangle, \quad (181) \]

where \(|0\rangle\) is the physical vacuum state and the integral is over the linear spacelike path starting at \( y \) and ending at \( y' \), on a fixed time slice. Note that the strings between exotic particles have been introduced to have a gauge-invariant state \(|\Omega\rangle\), in other terms, this means that the elementary particles (bosons or fermions) are now dressed by a cloud of gauge fields.

From the foregoing Hamiltonian discussion, we first note that

\[ \pi_i |\bar{\Psi}(y)\Psi(y')\rangle = \bar{\Psi}(y)\Psi(y')\pi_i|0\rangle + e \int y'_{y} dz_i \delta^3(x - z)|\bar{\Psi}(y)\Psi(y')\rangle. \quad (182) \]

Owing to (166,180) and the fact that we consider a static pointlike particle; so \( \pi_i = F_{0i} = E_i \), we get the expectation value of the Hamiltonian as

\[ \langle \Omega | H | \Omega \rangle = \langle 0 | H | 0 \rangle + e^2 \int d^3 x \left( \int y_{y} dz_i \delta^3(x - z) \right)^2, \quad (183) \]
with \(x\) and \(z\) are three-dimensional vectors. Remembering that the integrals over \(z\) are zero except on the contour of integrations.

The last term of (183) is nothing but the Coulomb interaction plus an infinite self-energy term. In order to carry out this calculation we write the path as \(z = y + \alpha(y - y')\) where \(\alpha\) is the parameter describing the contour. By using the spherical coordinates the integral under square becomes

\[
\int_y^{y'} dz_i \delta^3(x - z) = \frac{y - y'}{||y - y'||^2} \int_0^1 d\alpha \frac{1}{\alpha} \delta(||y - x||, \alpha||y' - y||) \sum_{\ell,m} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi). \tag{184}
\]

Using the usual properties for the spherical harmonics and after subtracting the self-energy term, we obtain the potential as

\[
V = -\frac{e^2}{4\pi} \frac{1}{||y' - y||}. \tag{185}
\]

This result lets us to draw attention to the fact that with fuzzy two-sphere the generalized Maxwell theory doesn’t have confining nature any more which was a special property for anyons described by generalized Maxwell theory in two-dimensional space. Thus the problem of confinement could be solved by considering the two-sphere in stead of two-dimensional flat space.

### 8 Discussion and Conclusion

We considered the abelian and non-abelian BI dynamics of the dyonic string, from section 2 until section 6, such that the electric field \(E\) has a limited value. The limit of \(E\) attains a maximum value

\[
E_{\text{max}} = T_1 = \frac{1}{\lambda} \tag{186}
\]

(for simplicity we dropped \(2\pi\) in all the calculations). This limiting value arises because if \(E > \frac{1}{\lambda}\) the action ceases to make physical sense\[35\]. The system becomes unstable. Since the string effectively carries electric charges of equal sign at each of its endpoints, as \(E\) increases the charges start to repel each other and stretch the string. For \(E\) larger than the critical value (186), the string tension \(T_1\) can no longer hold the strings together.

The investigation of excited D3-branes through the two cases of absence or presence of an electric field lead to the fact that by exciting 2 and 3 of its transverse directions the brane develops a spike which is interpreted as an attached bundle of a superposition of coordinates of another brane given as a collective coordinate a long which the brane extends away from the D3-brane. In our study of D3-branes, by exciting 2 and 3 transverse directions we found that a magnetic monopole produces a singularity in the D3-branes transverse displacement which can be interpreted as a superposition of coordinates describing Dp-branes \((p = 2, 3)\) attached to the D3-brane and the same for the dyonic case. We also obtained another important result that the lowest energy in the intersection branes case is obtained at the level of D1\(\perp\)D3 branes and the energy is higher if we excite more scalar fields and even more so in the presence of an electric field.

Further in the presence of a world volume electric field, space time dependent solutions can be generalised nicely and for \(r_0 = 0\) we observe brane collapses with a speed
less than that of light. We have also obtained a generalisation of the BPS solution. An automorphism of our solutions relevant for the $r \rightarrow 1/r$ duality has been discussed.

We also showed that certain excitations of $D1 \perp D3$ and $D1 \perp D5$ systems can be shown to obey Neumann boundary conditions. In this context, by considering $E \in [1, \frac{1}{\lambda}]$ in $D1 \perp D3$ and $D1 \perp D5$ branes, we treated the fluctuations of the fuzzy funnel solutions and discussed the associated potentials $V$ in terms of the electric field $E$ and the spatial coordinate $\sigma$. We considered the unit of $\lambda$ in all the figures representing the variations. We limited $\sigma$ to be in the interval $[0,0.5]$ for small $\sigma$ and $[0.5,1]$ for large $\sigma$.

Concerning $D1 \perp D3$ system, we gave the variation of $V$ in zero mode of overall transverse fluctuations in Fig.1; this figure shows that the system is looking like separated into two regions depending on the electric field. The potential is stable for $E$ varies from 0 until 0.7 and then $V$ goes up quickly as $E$ close to $\frac{1}{\lambda}$. Then we dealt with the general case of the overall transverse fluctuations which is the non-zero modes. In this case, the idea that the system is divided into two regions appears more clear. We gave the figure representing the potential in Fig.2; we see that at $\sigma = 0.5$ the potential gets a singularity. We continued to treat the other kind of the fluctuations, it’s the relative transverse fluctuations. These fluctuations are represented by Fig.3 and the associated potentials by Fig.4; we obtain the same remark that both fluctuations and potential have a singularity at $\sigma = 0.5$. This supported the idea that the system is separated into two regions.

We extended our study to the case of higher dimensions. We treated the electrified fluctuations of $D1 \perp D5$ branes and we studied only the zero mode of overall transverse fluctuations. We notice that when the electric field is going up and down the potential of the system is changing and the appearance of the singularity is more clear (Fig.5) and we have the same remarks for the fluctuations of fuzzy funnel solutions as well (Fig.6) which cause the division of the system into two regions depending on small and large $\sigma$ and also on $E$.

Consequently, the end point of the dyonic strings moves on the brane which means we have Neumann boundary conditions in $D1 \perp D3$ and $D1 \perp D5$ branes. The physical interpretation is that a string attached to the D3 and D5 branes manifests itself as an electric charge, and the waves on the string cause the end point of the string to freely oscillate. Thus, we realize Polchinski’s open string Neumann boundary conditions dynamically by considering non-abelian BI action in $D1 \perp D3$ and $D1 \perp D5$ systems.

The duality between dyonic D1 and D3 descriptions is no longer valid. We have further argued that the D-string description breaks down. This is perhaps not surprising. Indeed, in this limit [13] have argued that the effective tension of the string goes to zero. Thus, excited strings modes will not be very heavy compared to massless string modes and one might question the validity of the Dirac-Born-Infeld action which retains only the massless modes. Our other interest was the fate of the duality of $D1 \perp D5$ branes. We found that the duality between the D1 and D5 descriptions is unbroken in the presence of an electric field. Then, the duality in D1-D5 case is still valid.

In type IIA theory, we have used the generalized Maxwell theory on a two-sphere instead of a flat two-dimensional space. This leads to results totally different from those obtained in the case of QHE in higher dimensions [11]. By considering the exotic particles described by generalized Maxwell theory, the energy produced by the gauge field contributes to the energy of the system (18) depending on the number of charges and the radius of the sphere. We remark that the energy of gauge field dominates when the radius of two-sphere goes to infinity; i.e. the energy of flat D2-brane is generated by the gauge
field leading to increasing energy. We also notice that the energy becomes higher if the number of charges is large. Another important remark is that with a fuzzy two-sphere the static potential for two opposite charged exotic particles loses its confining nature without adding the CS-term; i.e. by plunging the generalized Maxwell theory in a higher dimensional theory, the potential has a screening nature.

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