A quantitative study of orbit counting and discrete spectrum for anti-de Sitter 3-manifolds

By Kazuki KANNAKA
RIKEN iTHEMS, 2-1 Hirosawa, Wako, Saitama 351-0198, Japan
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Abstract: Let $\Gamma$ be a discontinuous group for the 3-dimensional anti-de Sitter space $\text{AdS}^3 := \text{SO}_0(2,2)/\text{SO}_0(2,1)$. In this article, we discuss a growth rate of the counting of $\Gamma$-orbits at infinity and the discrete spectrum of the hyperbolic Laplacian of the complete anti-de Sitter manifold $\Gamma\backslash\text{AdS}^3$.

Key words: Anti-de Sitter manifold; anti-de Sitter space; discontinuous group; counting problem; hyperbolic Laplacian.

1. Introduction. The 3-dimensional anti-de Sitter space $\text{AdS}^3 := \text{SO}_0(2,2)/\text{SO}_0(2,1)$ is a Lorentzian manifold with constant sectional curvature $-1$ of which the identity component of the isometry group is the Lie group $\text{SO}_0(2,2)$. Discontinuous groups for $\text{AdS}^3$ and their deformation theory have been developed by renowned mathematicians, William Goldman, Toshiyuki Kobayashi, and Fanny Kassel, among others.

In this article, we discuss a growth rate of the counting of orbits of a discontinuous group $\Gamma$ for $\text{AdS}^3$ at infinity and the discrete spectrum of the hyperbolic Laplacian of the complete anti-de Sitter manifold $\Gamma\backslash\text{AdS}^3$. Detailed proofs of the results will appear elsewhere.

2. Relationship between the sharpness of the $\Gamma$-action and a growth rate of the counting at infinity. In old days, the terminology “discontinuous groups” was used to denote the same meaning of discrete subgroups. Indeed, the action of a discrete group of isometries is automatically properly discontinuous in the Riemannian setting. In his study of the action of discrete groups beyond the Riemannian setting, Kobayashi [13] advocated to make a difference of two terminologies: discontinuous groups for the property of actions, and discrete subgroups for the property of groups. Following this principle, we call a discrete subgroup $\Gamma$ of a Lie group $G$ a discontinuous group for a homogeneous manifold $G/H$ if the natural $\Gamma$-action on $G/H$ from the left is properly discontinuous and free [13, Def. 1.3]. Then any $\Gamma$-orbit meets a compact subset of $G/H$ in at most finitely many points, and thus we may consider the number of the intersection points. Kassel-Kobayashi [6] introduced a compact subset $B(R)$ called a pseudoball of radius $R > 0$ in any semisimple symmetric space $G/H$, in particular, in $\text{AdS}^3$, of which the volume is of exponential growth as $R \to \infty$. Moreover, they studied a growth rate of the counting

$$N_\Gamma(x,R) := \#(\Gamma x \cap B(R))$$

of the $\Gamma$-orbit through $x \in G/H$ as $R \to \infty$.

When the metric tensor is indefinite as in the anti-de Sitter space $\text{AdS}^3$, an isotropy subgroup of the isometry group is not necessarily compact and an orbit of a discrete subgroup $\Gamma$ of isometries may have accumulation points. In particular, $\Gamma$ may not act on $G/H$ properly discontinuously. Generalizing a pioneering work of Kobayashi [10] on the properness criterion by means of the Cartan projection for homogeneous manifolds of reductive type, Kobayashi [11] and Benoist [1] established a criterion for a general discrete subgroup $\Gamma$ of a reductive Lie group $G$ to act properly discontinuously on $G/H$. As a slightly stronger condition than this criterion, Kassel-Kobayashi [6] introduced the notion of $(c,C)$-sharpness ($c > 0$, $C \geq 0$) of a discontinuous group which quantifies proper discontinuity. Loosely speaking, the parameter $c > 0$ indicates that the “degree of proper discontinuity” of the $\Gamma$-action is weaker if $c$ approaches to 0. Then they gave an upper estimate of the counting for
(c,C)-sharp discontinuous groups for any semi-simple symmetric space G/H, in particular, for AdS3 by means of the two constants c and C, and proved that the counting N_τ(x,R) is of exponential growth uniformly with respect to x ∈ G/H as R → ∞:

**Fact 1** (Kassel-Kobayashi [6, Lem. 4.6 (4)]). There exists A > 0 such that for any c > 0, C ≥ 0, and torsion-free (c,C)-sharp discontinuous group Γ for AdS3, one has

\[ \forall x \in \text{AdS}^3, \forall R > 0, N_\tau(x,R) \leq A \exp \left( \frac{4(R+C)}{c} \right). \]

On the other hand, there has been no existing literature about the counting for a non-sharp discontinuous group (the case c = 0) to the best knowledge of the author. We find non-sharp discontinuous groups Γ with various behaviors of the counting of Γ-orbits:

**Theorem 2.** There exists a non-sharp discontinuous group Γ for AdS3 such that

\[ \forall x \in \text{AdS}^3, \forall R > 0, N_\tau(x,R) \leq 4^R. \]

In particular, N_τ(x,R) is of exponential growth uniformly with respect to x ∈ AdS^3 as R → ∞.

**Theorem 3.** For any monotone increasing function f: R → R_{>0} and any x ∈ AdS^3, there exists a discontinuous group Γ = Γ_{f,x} for AdS^3 satisfying

\[ \lim_{R \to \infty} \frac{N_\tau(x,R)}{f(R)} = \infty. \]

For example, applying Theorem 3 to f(R) = exp(e^R), we can construct a discontinuous group Γ satisfying

\[ \lim_{R \to \infty} \frac{\#(\Gamma x \cap B(R))}{\text{vol}(B(R))} = \infty. \]

It should be noted that Eskin-McMullen [2] also considered the counting of a Γ-orbit Γx for a general semisimple symmetric space G/H. They dealt with the case where Γ is a lattice of G and x is a special point in G/H, and thus their setting is completely different from [6] and also from ours.

### 3. Construction of non-sharp discontinuous groups.

In this section, we describe how to construct non-sharp discontinuous groups for AdS^3 used in the proofs of Theorems 2 and 3. We note that the product group SL(2,R) × SL(2,R) acts isometrically on AdS^3 = SO_0(2,2)/SO_0(2,1) via the double covering SL(2,R) × SL(2,R) → SO_0(2,2).

Generalizing a non-sharp example of Guéritaud-Kassel [3, Sect. 10.1], we construct a family of infinitely generated subgroups of SL(2,R) × SL(2,R). Our subgroup has four sequences \((a_-, a_+, r, R)\) as parameters. We find a properness criterion and a sharpness criterion for the actions of our subgroups on AdS^3 using the asymptotic behaviors of these sequences.

For a quadruple of real-valued sequences \((a_-, a_+, r, R)\), we define \(\alpha_k, \beta_k \in SL(2,R)\) by

\[ \alpha_k = \frac{1}{r(k)} \begin{pmatrix} a_+ + (a_-)(r(k)+R(k)^2) & -a_- \end{pmatrix}, \]

\[ \beta_k = \frac{1}{r(k)} \begin{pmatrix} a_+ + (a_-)(-r(k)+R(k)^2) & -a_- \end{pmatrix}, \]

and denote by \(\Gamma_{\nu}(a_-, a_+, r, R)\) for sufficiently large \(\nu \in \mathbb{N}\) the subgroup generated by \(\alpha_k, \beta_k \in SL(2,R) \times SL(2,R)\) for all \(k = \nu, \nu+1, \ldots\).

Let \(A_{\nu}^-\) and \(B_{\nu}^-\) for \(\epsilon \in \{-, +\}\) be respectively the half-disks in the upper half plane \(H^2 = \{ z \in \mathbb{C} | \text{Im } z > 0 \}\) defined by

\[ A_{\nu}^- := \{ z \in H^2 | |z - a_{\nu}(k)| \leq r(k) \}, \]

\[ B_{\nu}^- := \{ z \in H^2 | |z - a_{\nu}(k)| \leq R(k) \}, \]

see Fig. 1. Then we note

\[ \alpha_k(A_{\nu}^-) \subset H^2 \setminus A_{\nu}^+, \beta_k(B_{\nu}^-) \subset H^2 \setminus B_{\nu}^+, \]

where SL(2,R) acts on \(H^2\) as linear fractional transformations. One can see by an elementary argument of general topology called the ping-pong argument that the subgroup \(\Gamma_{\nu}(a_-, a_+, r, R)\) is discrete and free if the half-disks \(A_{\nu}^+, A_{\nu+1}^-, \ldots\) (resp. \(B_{\nu}^+, B_{\nu+1}^-, \ldots\)) are disjoint.

Let \(p(x)\) be a real-valued monotone increasing \(C^2\)-function defined for sufficiently large \(x \in \mathbb{R}\) such that \(\lim_{x \to \infty} p(x) = \infty\) and that the second derivative \(p''(x)\) is nowhere vanishing. In this article, for simplicity, we assume that the pair of sequences \((a_+(k), a_-(k))\) can be expressed as

![Fig. 1. A^-_k and B^-_k in H^2.](image-url)
for sufficiently large $k \in \mathbb{N}$. Moreover, we suppose
\begin{equation}
R(k) > r(k),
\end{equation}
\begin{equation}
\lim_{k \to \infty} \min \{p'(k-1), p'(k+1)\} = 0.
\end{equation}
Then $B_k^+ \supset A_k^+$ holds and an easy calculation shows that the half-disks $B_k^+, B_{k+1}^+ , \ldots$ are disjoint for sufficiently large $\nu \in \mathbb{N}$, see Fig. 2.

The following are a properness criterion and a sharpness criterion for the action on $\text{AdS}^3$ of the discrete subgroup $\Gamma_\nu(a_-, a_+, r, R)$:

**Proposition 4.** Let $(a_-, a_+, r, R)$ be a quadruple of sequences satisfying (3.1)–(3.3) as above. The action on $\text{AdS}^3$ of the discrete subgroup
\begin{equation}
\Gamma_\nu(a_-, a_+, r, R)
\end{equation}
for sufficiently large $\nu \in \mathbb{N}$ is:

1. properly discontinuous if and only if
\[ \lim_{k \to \infty} \frac{R(k)}{r(k)} = \infty; \]
2. sharp if and only if
\[ \liminf_{k \to \infty} \log \left( \frac{R(k)}{r(k)} \right) \left( \log \frac{a_-(k) a_+(k)}{r(k)} \right)^{-1} \neq 0. \]

**Example 5.** For the triples $(p(x), r(k), R(k))$ in Table I, we form the subgroups $\Gamma_\nu \equiv \Gamma_\nu(a_-, a_+, r, R)$ with (3.1)–(3.3). Then Proposition 4 shows that $\Gamma_\nu$ are all discontinuous groups for $\text{AdS}^3$ for sufficiently large $\nu \in \mathbb{N}$ but not always sharp as summarized in Table I.

4. Discrete spectrum of non-sharp anti-de Sitter manifolds. Next we consider discrete spectrum of the Laplacian of the noncompact anti-de Sitter manifold $\Gamma \setminus \text{AdS}^3$ for a non-sharp discontinuous group $\Gamma$.

Let us recall some basic notions. A pseudo-Riemannian manifold is a $C^\infty$-manifold equipped with a smooth non-degenerate symmetric bilinear tensor of signature $(p, q)$. It is called Riemannian if $q = 0$ and Lorentzian if $q = 1$. As in the Riemannian case, $\Box = \text{div} \circ \text{grad}$ defines a second order differential operator (the Laplacian) on a pseudo-Riemannian manifold. In contrast to the Riemannian setting, the Laplacian on a Lorentzian manifold is not an elliptic differential operator but a hyperbolic differential operator, and its eigenfunction is not analytic in general.

We write $L^2(M)$ for the Hilbert space of square integrable functions with respect to the volume form induced by the pseudo-Riemannian structure of $M$, and denote by $L^2(M)$ for $\lambda \in \mathbb{C}$ the space of square integrable eigenfunctions
\[ \{ f \in L^2(M) \mid \Box f = \lambda f \text{ in the weak sense} \}. \]
Then the set of $L^2$-eigenvalues
\[ \text{Spec}_d(\Box_M) := \{ \lambda \in \mathbb{C} \mid L^2(M) \neq 0 \} \]
is called the discrete spectrum of the Laplacian of $M$.

We recall the theory of Kassel-Kobayashi [6] on the discrete spectrum of “intrinsic” differential operators on locally semisimple symmetric spaces by limiting ourselves to the case $\text{AdS}^3$. Let $\Gamma$ be a discontinuous group for $\text{AdS}^3$. Then the quotient space $\Gamma \setminus \text{AdS}^3$ is a $C^\infty$-manifold and the quotient map $\text{AdS}^3 \to \Gamma \setminus \text{AdS}^3$ is a covering map of $C^\infty$-class. The quotient manifold $\Gamma \setminus \text{AdS}^3$ admits a Lorentzian structure with constant sectional curvature $-1$ via this covering map. Kassel-Kobayashi [6] and Kobayashi [14] initiated the study of spectral analysis on locally symmetric spaces, in particular, that of the discrete spectrum $\text{Spec}_d(\Box)$ of the hyperbolic Laplacian $\Box$ on the anti-de Sitter manifold $\Gamma \setminus \text{AdS}^3$.

They introduced “the $\Gamma$-averages of non-periodic eigenfunctions” as a generalization of Poincaré series to construct $L^2$-eigenvalues. If an eigenfunction $\varphi$ of the Laplacian on $\text{AdS}^3$ is integrable, then the generalized Poincaré series
\[ \varphi^\Gamma(\Gamma x) := \sum_{\gamma \in \Gamma} \varphi(\gamma^{-1} x) \]
defines an integrable function on the anti-de Sitter manifold $\Gamma \setminus \text{AdS}^3$, and is an eigenfunction of the Laplacian with same eigenvalue. It is known that the Laplacian on $\text{AdS}^3$ has the following $L^2$-eigenvalues:

$$\lambda_m := 4m(m - 1) \quad (m \in \mathbb{Z} \text{ and } m \geq 2).$$

As an application of an upper estimate of the counting as in Fact 1, they proved $L^2$-convergence and non-vanishing of the generalized Poincaré series of eigenfunctions for sufficiently large eigenvalue $\lambda_m$, and obtained the following theorem:

**Fact 6** [6]. For any sharp discontinuous group $\Gamma$ for $\text{AdS}^3$, there exists a constant $m_0(\Gamma) > 0$ such that

$$\text{Spec}_d(\square_{\Gamma \setminus \text{AdS}^3}) \supset \{\lambda_m \mid m \in \mathbb{Z}, \ m > m_0(\Gamma)\}.$$  

A natural question would be whether the Laplacian on an anti-de Sitter manifold $\Gamma \setminus \text{AdS}^3$ still has an $L^2$-eigenvalue if the discontinuous group $\Gamma$ is non-sharp. As an application of an upper estimate of the counting as in Theorem 2, we see that there exist countably many $L^2$-eigenvalues for some non-sharp $\Gamma$ by applying the machinery developed in [6]:

**Theorem 7.** There exist a non-sharp discontinuous group $\Gamma$ for $\text{AdS}^3$ and a constant $m_0'(<0)$ such that

$$\text{Spec}_d(\square_{\Gamma \setminus \text{AdS}^3}) \supset \{\lambda_m \mid m \in \mathbb{Z}, \ m > m_0'(\Gamma)\}.$$  

5. Multiplicity of the discrete spectrum. In the final section we discuss the multiplicity of the $L^2$-eigenvalue $\lambda_m$ of the Laplacian of an anti-de Sitter manifold $\Gamma \setminus \text{AdS}^3$ constructed by the generalized Poincaré series. Here, for a pseudo-Riemannian manifold $M$,

$$\mathcal{N}_M(\lambda) := \dim_C L^2_\lambda(M) \in \mathbb{N} \cup \{\infty\}$$

is called the multiplicity of an $L^2$-eigenvalue $\lambda$. The Laplacian on a Riemannian manifold is an elliptic differential operator and the multiplicity of an $L^2$-eigenvalue is always finite if $M$ is compact. However, in the Lorentzian setting, the multiplicity may be finite or may not even if $M$ is compact (e.g., [8,14]).

If a discontinuous group $\Gamma$ for $\text{AdS}^3$ is standard [6, Def. 1.4] and torsion-free, $\mathcal{N}_{\Gamma \setminus \text{AdS}^3}(\lambda_m) = \infty$ for sufficiently large $m \in \mathbb{N}$, which is derived from the results in Kassel-Kobayashi [7,8]. On the other hand, there exists a non-standard discontinuous group $\Gamma$, for example a finitely generated discontinuous group $\Gamma$ which is Zariski-dense in the Lie group $\text{SO}(2,2)$ [9,12]. However, it is not known whether the multiplicities of the Laplacian are finite in this case. We see that the multiplicities of the Laplacian on the anti-de Sitter manifold $\Gamma \setminus \text{AdS}^3$ for such $\Gamma$ are unbounded as follows:

**Theorem 8.** For any finitely generated discontinuous group $\Gamma$ for $\text{AdS}^3$, there exists a constant $c_\Gamma > 0$ such that

$$(5.1) \quad \mathcal{N}_{\Gamma \setminus \text{AdS}^3}(\lambda_m) \geq \log_3 m - c_\Gamma.$$

In particular,

$$\lim_{m \to \infty} \mathcal{N}_{\Gamma \setminus \text{AdS}^3}(\lambda_m) = \infty.$$  

To prove this theorem, we use $\text{SO}(2) \times \text{SO}(2)$-finite $L^2$-eigenfunctions of the Laplacian on $\text{AdS}^3$ with eigenvalue $\lambda_m$ vanishing at the origin. We note that such eigenfunctions decay more rapidly at infinity than at the origin with respect to geodesic parameters. We choose an $L^2$-eigenfunction with eigenvalue $\lambda_m$ for each $j = 0, 1, \ldots, k - 1$ which decays at the origin as rapidly as $R^j$ when a “pseudo-distance” $R$ from the origin tends to zero, and show the linear independence of their generalized Poincaré series when $m > 3^{k+c_\Gamma}$, which proves (5.1).

Finally we discuss a lower bound of the multiplicities of $L^2$-eigenvalues under a small deformation of a discrete subgroup. The general study of local rigidity and stability of discontinuous groups for non-Riemannian homogeneous manifolds was initiated by Kobayashi [12] and Kobayashi-Nasrin [15], and has been further developed by Kassel [5] and others in specific settings. In our $\text{AdS}^3$ setting, any cocompact discontinuous group is not locally rigid and its proper discontinuity is stable under any small deformation [9,12]. Moreover, Kassel-Kobayashi [6] constructed infinitely many stable $L^2$-eigenvalues of the Laplacian of any compact anti-de Sitter manifold $\Gamma \setminus \text{AdS}^3$ under any small deformation of $\Gamma$. More specifically, for sufficiently large $m \in \mathbb{N}$, one has

$$\lambda_m \in \bigcap_{\Gamma'} \text{Spec}_d(\square_{\Gamma' \setminus \text{AdS}^3}),$$

where $\Gamma'$ runs over a sufficiently small neighborhood of $\Gamma$ in the compact-open topology [6, Cor. 9.10], see [6, Def. 1.6] for the definition of stable eigenvalues in a much more general setting. We introduce
a function \( \tilde{N}_{\Gamma \backslash \text{AdS}^3} : \mathbb{C} \to \mathbb{N} \cup \{\infty\} \) satisfying the following for the multiplicities of stable eigenvalues:

- \( \tilde{N}_{\Gamma \backslash \text{AdS}^3}(\lambda) \neq 0 \) if and only if \( \lambda \) is a stable \( L^2 \)-eigenvalue of \( \square_{\Gamma \backslash \text{AdS}^3} \);
- \( \tilde{N}_{\Gamma \backslash \text{AdS}^3}(\lambda) \geq \tilde{N}_{\Gamma \backslash \text{AdS}^3}(\lambda') \) for any \( \Gamma' \) sufficiently close to \( \Gamma \).

**Theorem 9.** For any cocompact discontinuous group \( \Gamma \) for \( \text{AdS}^3 \),

\[
\lim_{m \to \infty} \tilde{N}_{\Gamma \backslash \text{AdS}^3}(\lambda_m) = \infty.
\]

The constant \( c_\Gamma \) also plays a crucial role in the proof of Theorem 9. Here recall (5.1). The geometric constant \( c_\Gamma \) is defined by using

- a growth rate of the counting \( N_\Gamma(x,R) \) as \( R \to \infty \);
- the “injective radius” of the anti-de Sitter manifold \( \Gamma \backslash \text{AdS}^3 \).

We control these two quantities simultaneously using Lipschitz constants associated to \( \Gamma \) introduced in Kassel [4] and Kassel-Kobayashi [6], and further investigated by Guéritaud-Kassel [3], and show that \( c_\Gamma \) depends “continuously” on a small deformation of \( \Gamma \). We prove that the larger \( m \in \mathbb{N} \) is, the more linearly independent \( L^2 \)-eigenfunctions of the Laplacian of the compact anti-de Sitter manifold \( \Gamma \backslash \text{AdS}^3 \) can be constructed and that their construction is stable under any small deformation of \( \Gamma \).

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