The effects of focused transducer geometry and sample size on the measurement of ultrasonic transmission properties

T J Atkins\textsuperscript{1,2}, V F Humphrey\textsuperscript{2}, F A Duck\textsuperscript{1} and M A Tooley\textsuperscript{1}

\textsuperscript{1}Department of Medical Physics and Bioengineering, Royal United Hospital, Combe Park, Bath BA1 3NG, UK

\textsuperscript{2}Institute of Sound and Vibration Research, University of Southampton, Southampton SO17 1BJ, UK

Email: timothy.atkins@nhs.net

Abstract. The response of two coaxially aligned weakly focused ultrasonic transducers, typical of those employed for measuring the attenuation of small samples using the immersion method, has been investigated. The effects of the sample size on transmission measurements have been analyzed by integrating the sound pressure distribution functions of the radiator and receiver over different limits to determine the size of the region that contributes to the system response. The results enable the errors introduced into measurements of attenuation to be estimated as a function of sample size. A theoretical expression has been used to examine how the transducer separation affects the receiver output. The calculations are compared with an experimental study of the axial response of three unpaired transducers in water. The separation of each transducer pair giving the maximum response was determined, and compared with the field characteristics of the individual transducers. The optimum transducer separation, for accurate estimation of sample properties, was found to fall between the sum of the focal distances and the sum of the geometric focal lengths as this reduced diffraction errors.

1. Introduction

Spherically focused ultrasonic transducers are used in many applications including materials characterization and medical imaging. In a biomedical context, coaxially aligned weakly focused transducers are often used to measure the acoustic properties of tissue \textit{in-vitro} and of tissue-like media \cite{1}. An advantage of using focused transducers in this context is that it helps to reduce the required sample size and increase sensitivity; this paper quantifies the effect of sample size on diffraction error.

A number of workers have investigated the output of a pair of coaxial planar circular transducers as a function of separation distance \cite{2, 3}. The resulting diffraction loss curves are useful for selecting separation distances in immersion measurements in order to minimise diffraction loss errors \cite{1}.

The present work explores the equivalent geometry and sample size issues related to measurement of sample properties, using coaxially aligned spherical transducers with weak focusing, by modelling and measuring the combined diffraction effects for combinations of three specific transducers.

2. Theoretical expression for receiver response

Consider an unapodized spherically concave source with radius \(a_1\) and radius of curvature \(D_1\) producing sound at angular frequency \(\omega\) in cylindrical coordinate system \((z, r, \phi)\). The radiation is
generated by uniform motion normal to the transducer surface. A second transducer is located coaxially with the source at a separation \( Z = z_1 + z_2 \) (see figure 1).

Figure 1. Cross section view of two coaxially aligned cylindrically symmetric concave transducers, showing coordinates and symbols used in the analysis.

Following Gavrilov et al [4], the sound pressure averaged over the active area of the receiver, \( p_{\text{rec}}(Z) \), can be calculated from the convolution of the sound pressure distribution functions of the two transducers over an infinite plane between the transducers. This receiver response can be estimated by

\[
p_{\text{rec}}(Z, R) = \frac{2}{P_0 a_z^2} \int_{r_0}^R p_1(r, z) p_2(r, z) r dr
\]

where \( p_1(r, z) \) is the complex sound pressure produced by the radiator at radial coordinate \( r \) from the \( z \) axis in a plane at a distance \( z_1 \) from the source. Likewise \( p_2(r, z) \) is the complex sound pressure produced by the second transducer operating in transmit mode at a distance \( z_2 \) from the source when \( p_2 \) is the average pressure amplitude across the surface of the second transducer.

For such a system the measured free field receiver response will be proportional to the calculated value \( p_{\text{rec}}(Z) \) in the limit \( R \to \infty \). This solution is invariant over the choice of integration plane. For a given transducer separation \( Z, z_1 \), and hence \( z_2 \), may be varied without affecting the result.

Restricting the integration to a finite radial distance \( R \) introduces an error into the calculation of \( p_{\text{rec}}(Z) \), and hence the measured receiver response. This enables the effect of using finite size samples on the received sound pressure to be investigated by analysing the radius of the restricted integral region required to accurately estimate the receiver response.

With \( P_0 \) as the average pressure amplitude at the surface of the transducer, Gavrilov et al [4] give a solution for the sound pressure of a focusing source of radius \( a \) and focal length \( D \) in the focal plane as

\[
p(r) = -i P_0 \frac{a}{r} \exp(ikD) J_0 \left( \frac{kar}{D} \right)
\]

(2)

Lucas and Muir [5] provide a more general solution at a distance \( z \) from a source as

\[
p(r, z) = -ik \frac{P_0}{z} \exp(ikz) \exp \left( \frac{ikr^2}{2z} \right) \left[ \frac{1}{2} \int_0^1 \exp \left( \frac{ikx^2}{2} \left( 1 - \frac{1}{D} \right) \right) J_0 \left( \frac{krx}{z} \right) dx \right].
\]

(3)

A significant feature of equation (3) is the presence of the phase term \( \exp(ikr^2/2z) \). This indicates that the phase is not constant across the field. The phase term is not important when considering point measurements in a field; however, in this case the subsequent convolution of two of these functions means that the phase needs to be carefully considered. It is omitted from equation (2) as this is an approximation based on the geometrical optics work of Debye [6].

Introducing a cylindrical sample into the measurement system during attenuation measurements can be considered to split the integration for the receiver response, \( p_{\text{rec}}(Z) \), into two parts. The first part consists of an integration from the axis up to the sample radius \( s \), for which there will be an attenuation introduced, and an integration beyond the sample radius \( s \). The introduction of the sample may change the phase of the ultrasonic wave through the sample area.

Let the average receiver pressure without the sample be \( p_{\text{rec}}(Z) \) and the sample have a transmission coefficient \( T \) (allowing for attenuation and transmission at the sample interfaces). The error in estimating \( T \) by the measured transmission coefficient \( T' \) can be shown to be

\[
T' = (1 - T) \left[ 1 + p_{\text{rec}}(Z, s)/p_{\text{rec}}(Z) \right].
\]

(4)

The magnitude of the error in estimating \( T \) by \( T' \) is therefore minimised by ensuring \( p_{\text{rec}}(Z, s)/p_{\text{rec}}(Z) \) is close to unity. In this paper the ratio \( |p_{\text{rec}}(Z, R)|/|p_{\text{rec}}(Z)| \) will be used to illustrate how close \( p_{\text{rec}}(Z, R) \) is to the infinite value \( p_{\text{rec}}(Z) \). To assist in the analysis \( R_{\text{min}} \) is defined as the minimum integration limit
required to ensure that the ratio of $|p_{rec}(Z, R)|/|p_{rec}(Z)|$ is greater than a specified value for all $R > R_{lim}$.

Similar analysis and integration limits are used for the phase variation of $p_{rec}(Z, R)/p_{rec}(Z)$.

The maximum error introduced by using a finite sized sample can be calculated from the ratio $|p_{rec}(Z, R)|/|p_{rec}(Z)|$ and the sample transmission coefficient; alternatively the minimum sample size required to ensure that these errors are below a specified level can be calculated. This analysis does not include the effects of sample impedance on the diffracted wave or sample inhomogeneity.

Three weakly-focused transducers of approximate centre frequency 3.5 MHz were selected for this investigation; Table 1 shows the properties of the transducers estimated from acoustic measurements of the transducers using an NPL Beam Calibrator [7]. Each transducer was modelled using equation (2) or by numerically integrating equation (3). The models for $p(r, z)$ were then used in equation (1) to calculate the receiver response $p_{rec}(Z)$ and the estimate for a restricted integration plane $p_{rec}(Z, R)$.

Table 1. Measured transducer characteristics used in theoretical model.

| Transducer | Radius of curvature $(D)$ / cm | Transducer radius $(a)$ / cm | Acoustic focal length / cm | -6dB Beamwidth at acoustic focus / cm |
|------------|-------------------------------|-----------------------------|---------------------------|-------------------------------------|
| A          | 7.6                           | 0.61                        | 5.0                       | 0.27                                |
| B          | 11.6                          | 0.62                        | 6.0                       | 0.39                                |
| C          | 13.5                          | 0.91                        | 9.5                       | 0.35                                |

3. Results

3.1. Debye approximation at sum of geometric foci

Consider a pair of transducers separated by the sum of their radii of curvature. With a substitution of $X = ka_1r/D_1$ and $\beta X = ka_2r/D_2$ at the plane of their coincident geometric foci, equation (2) becomes

$$p_{rec}(D_1 + D_2, R) = -2P_0 \frac{a_1}{a_2} \exp[ik(D_1 + D_2)] \int_{X=0}^{X=X_{lim}} \frac{J_1(X)J_1(\beta X)}{X} dX.$$  \hspace{1cm} (5)

The integral $I = \int_{X=0}^{X=X_{lim}} \frac{J_1(X)J_1(\beta X)}{X} dX$ has a value of $\beta/2$ for $0 < \beta \leq 1$ and $1/2\beta$ for $\beta \geq 1$.

The integrand and the integral $I$ are plotted in figures 2 and 3 for $\beta = 1$ and $\beta = 0.669$ respectively. For $\beta = 1$ (matched transducers) the integral is a monotonically increasing function which reaches a value of 0.5 when integrated over an infinite range. For $\beta \neq 1$ (non-identical transducers) the received signal is less than that for matched transducers; however the received signal approaches the value for infinite $R$ faster than for identical transducers.

![Figure 2](image2.png)  \hspace{1cm} ![Figure 3](image3.png)

**Figure 2.** Plot of the functions $J_1^2(X)/X$ and $\int J_1^2(X)/X dX$ used in the calculation of the receiver response for coaxially aligned transducers ($X$ is dimensionless).

**Figure 3.** Plot of the functions $J_1(X)J_1(\beta X)/X$ and $\int J_1(X)J_1(\beta X)/X dX$ used in the calculation of the receiver response for coaxially aligned transducers with $\beta = 0.669$ ($X$ is dimensionless).
3.2. Location of diffraction peak
To reduce the effect of diffraction error on attenuation measurements the transducers should be located at a position of diffraction maxima or minima [1]. This ensures that the diffraction changes caused by the introduction of a medium of different sound speed are minimized when a sample is placed between the transducers. For coaxially aligned strongly focused transducers this has conventionally been accepted as requiring a transducer separation of the sum of the geometric foci [1, 8]. For weakly focused transducers, however, the diffraction peak does not occur when the transducers are separated by the sum of their geometric foci since the geometric and acoustic foci do not coincide. Figure 4 shows the modelled variation in the integrated response varies with separation for transducers A and C. The sum of the geometric foci is shown as a vertical line on the figure. The transducer separation has to be reduced from the sum of the geometric foci to reach the position of the diffraction maximum. This agrees with measurements of the receiver output obtained using the coaxial transducers A and C, shown in figure 5. The sum of geometric foci is indicated at 21 cm and the sum of acoustic foci at 14.5 cm. This shows that the position of the diffraction peak occurs at a position between the sum of acoustic and the sum of the geometric focal lengths; this was seen for all three transducer pairings.

3.3. Full expression for identical and non-identical transducers
One of the properties of equation (1) is that for integration to infinity the average pressure $p_{\text{rec}}(Z)$ (and hence receiver output) is not affected by the integration plane that is chosen. Figure 6 shows how the magnitude of $p_{\text{rec}}(Z, R)$ varies with $R$ for transducers A and C for different distances of the integration plane from the receiver. When the integration limit is greater than approximately 2.0 cm, the magnitude of $p_{\text{rec}}(Z, R)$ is the same for all integration planes.

Figure 4. Example of the modelled variation in the received response $p_{\text{rec}}$, normalized to the average pressure amplitude across the surface of the transmitting transducer $P_0$, as a function of the separation between the transducers A and C. The sum of geometric foci is shown at 21 cm.

Figure 5. Example of the measured variation in the received response $p_{\text{rec}}$, as a function of separation between the transducers A and C. The sum of geometric foci is shown at 21 cm and the sum of acoustic foci shown at 14.5 cm.

Figure 6. Plot showing the variation in received pressure $p_{\text{rec}}(R)$ with integration limit $R$ for different distances of integration plane from the receiving transducer for transducers A and C.
Figure 7 demonstrates the effect of the integration radius on the solution at the diffraction peak when equation (3) is used in the model. For different values of the $|p_{rec}(Z, R)|/|p_{rec}(Z)|$ ratio there are different transducer combinations which give the smallest value of $R_{lim}$. For each transducer separation, the choice of integration plane affects the value of $R_{lim}$, and the displayed standard deviations quantify the spread obtained if the integration plane is varied over a 4 cm range. Figure 8 shows similar data for the phase difference between the restricted and infinite integration.

Figure 7. Plot showing the minimum integration limit $R_{lim}$ required for selected ratios of $|p_{rec}(Z, R)|/|p_{rec}(Z)|$ for matched transducer pairs (AA, BB, CC) and transducers of different geometry (AB, AC, BC) at the position of diffraction loss peak, calculated using equation (3). Example error bars (1 SD) are shown for transducer pairings AC and BB.

Figure 8. Plot showing the minimum integration limit $R_{lim}$ required for selected phase differences between $p_{rec}(Z, R)$ and $p_{rec}(Z)$ for matched transducer pairs (AA, BB, CC) and transducers of different geometry (AB, AC, BC) at the position of diffraction loss peak, calculated using equation (3). Example error bars (1 SD) are shown for transducer pairings AC and BB.

4. Discussion

This work highlights the fact that the Debye solution quoted by Gavrilov et al [4] is an approximation obtained in the geometrics optics limit. It agrees with fuller formulations in terms of the focal plane amplitudes but not phases. The results presented here show it is important to include the transverse phase term $\exp(ikr^2/2z)$ when modelling opposed pairs of weakly focused transducers, as the calculation of the receiver response involves an integration of the product of two complex sound pressure distribution functions over the radial coordinate $r$. Analysis of the two solutions used in this paper indicates that the phase term reduces the size of the integration limit $R_{lim}$ required to ensure an accurate estimation of $p_{rec}$ by a restricted integration. The phase term does introduce a phase difference between the restricted integral $p_{rec}(Z, R)$ and the infinite integral $p_{rec}(Z)$. However, the phase differences are small and should not significantly affect results (figure 8).

The difference between the integration results for a limited radius and infinite radius gives an indication of the error that is likely to be obtained by using a coaxial circular sample of a given radius in immersion measurements of acoustic attenuation. The results obtained here can be used to estimate the minimum size of sample required to measure a given attenuation to a given precision.

The criterion quoted in the literature [1, 8] to minimize errors in measurement due to diffraction losses is to place opposed transducers at the diffraction peak. For the transducers used in this paper both the theoretical model and measurements indicate that the separation between the transducers should be reduced from the position of the sum of the geometric focal lengths (figures 4 and 5). This is due to the transducers being weakly focused as opposed to strongly focused transducers where the geometric and acoustic foci would coincide. Placing weakly focused transducers at the sum of their geometric foci would lead to increased diffraction errors.

When using coaxially aligned spherically focused transducers to measure ultrasonic attenuation properties the measurement system needs to be designed to minimize uncertainties in the measurement of the attenuation of the sample. The analysis presented here for the average receiver pressure
$p_{rec}(Z, R)$ shows that if $R$ is small then $p_{rec}(Z, R)$ is not an accurate estimate of $p_{rec}(Z)$. To provide an accurate estimate of $p_{rec}(Z)$ (small error) the integration must be performed over a large radial distance ($R > R_{lim}$).

When measuring ultrasonic attenuation, it has been shown [9] that the optimum attenuation for measurement is approximately 1 neper (8.68 dB). The errors introduced by sample size effects calculated here may have large effects on such measurements of attenuation. Therefore relatively large samples are required to ensure sample size effects do not contribute a significant error. For the transducers used in this paper sample sizes of approximately 0.8cm radius are required to ensure the ratio of $|p_{rec}(Z, R)|/|p_{rec}(Z)|$ is greater than 0.98 (figure 7). This is equivalent to a sample diameter of 4 to 6 times the focal beam diameter. The actual effect this will have is dependent on the transmission properties of the sample itself (see equation (4)). The phase difference between $p_{rec}(Z, R)$ and $p_{rec}(Z)$ will be negligible with samples of this size.

When the difference between the estimated and actual value of the average receiver pressure is small (ratio $|p_{rec}(Z, R)|/|p_{rec}(Z)|$ is close to one) there does not seem to be an obvious choice of integration plane to minimize the required sample size (figures 6 and 7). The choice of integration plane is analogous to the physical positioning of the sample between the two transducers, indicating that for relatively large samples, the axial positioning of the sample is not critical.

5. Conclusions
Two models have been used to calculate the average pressure on the receiver of a pair of coaxial focused transducers. Results for three specific transducers operating at 3.5 MHz have been utilized to analyze the effects of sample size and geometry on the measurement of ultrasonic properties.

A potential source of error in attenuation measurements due to finite sample size has been discussed. This source may introduce significant errors into the measurement of attenuation and should be considered when designing attenuation measurement experiments. For the transducers used in this paper sample sizes of approximately 0.8cm radius, equivalent to a sample diameter of 4 to 6 times the focal beam diameter, are required to ensure an ratio $|p_{rec}(Z, R)|/|p_{rec}(Z)|$ is greater than 0.98. This will limit the errors due to sample size effects introduced into attenuation measurements.

Both the theoretical model and measurements of specific transducers indicate that for weakly focused transducers the optimum separation between the transducers should be reduced from the sum of the geometric focal lengths in order to minimise diffraction loss corrections.

Acknowledgements
This work was funded under the UK Department of Health NEAT programme project J008.

References
[1] Bamber J C 2004 Attenuation and absorption Physical Principles of Medical Ultrasonics ed C R Hill, J C Bamber and G R ter Harr (Chichester: J Wiley) pp93-166
[2] Rhyne T L 1977 J. Acoust. Soc. Am. 61 218-324
[3] Harrison J A, Cook-Martin G N and Challis R E 1984 J. Acoust. Soc. Am. 76 1008-1022
[4] Gavrilov L R, Dmitriew V N and Solontsova L V 1988 J. Acoust. Soc. Am. 83 1167-1179
[5] Lucas B G and Muir T G 1982 J. Acoust. Soc. Am. 72 1289-1296
[6] Debye P 1909 Ann. Phys. 335 775-776
[7] Preston R C 1988 IEEE Trans. Ultrason. Ferroelect. and Freq. Contr. 35 122-139
[8] Penttinen A and Luukkala M 1977 J Phys. D: Appl. Phys. 10 665-669
[9] Kalashnikov A N and Challis R E 2005 IEEE Trans. Ultrason. Ferroelect. And Freq. Contr. 52 1754-1768