THE CONVERGENCE OF CIRCLE PACKINGS ON TORI

YANWEN LUO AND XU XU

ABSTRACT. The notion of circle packings has been explored intensively since Thurston [17] proposed it as an approximation to conformal maps. One of the fundamental questions about circle packings is whether it converges to the Riemann mapping if the radius of the circles tends to zero. In this paper, we prove the convergence of circle packings to the maps induced by the uniformization theorem in a different sense proposed by Gu-Luo-Wu [3] for surfaces of genus one. This is the first result of the convergence of circle packings on surfaces with nontrivial topology.

1. INTRODUCTION

In 1985, Thurston [18] proposed “the Finite Riemann Mapping Theorem”, claiming that the maps induced from circle packings converge to the Riemann mapping. This conjecture initiated a flurry of research about the convergence of maps constructed from circle packings to the corresponding conformal maps. Rodin-Sullivan [10] firstly confirmed the uniform convergence of the piecewise linear maps on a simply connected domain in the plane induced from the standard hexagonal circle packings to the Riemann mapping. A series of results including He [6], Doyle-He-Rodin [2] and He-Schramm [7] proved the convergence of the first and second derivatives of the induced maps from the circle packings extended by Möbius transformations to their smooth counterparts. Eventually, He-Schramm [8] showed the $C^\infty$ convergence of hexagonal circle packings to the Riemann mapping. It also leads to the study of discrete analogs of concepts and theorems in complex analysis and Riemann surfaces. See [11] for a comprehensive survey on this topic.

In contrast to most of the results above concerning about planar domains in the plane, little is known about the convergence of circle packings to conformal diffeomorphisms provided by the classical uniformization theorem for Riemannian surfaces with non-trivial topology. In this case, it is difficult to define a canonical map from the circle packings on general surfaces to surfaces with constant curvature. Meanwhile, computing such a map

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on general surfaces has a wide range of applications in surface parameterizations [19]. Therefore, we need to justify the convergence of the discrete conformal maps induced from circle packings to the smooth conformal maps on general surfaces.

The goal of this paper is to demonstrate the convergence of circle packings to uniformization maps in an alternative framework proposed in Gu-Luo-Wu [3] and Wu-Zhu [13] when the surface is a smooth torus. This scheme of convergence is proposed for a new theory of discrete conformal geometry called vertex scaling. The fundamental results about this new theory include a discrete uniformization theorem for polyhedral surfaces [4, 5, 15], an infinite rigidity theorem similar to the hexagonal circle packings in the plane [14], and the convergence of discrete conformal maps to their smooth counterparts [9]. Effective algorithms are also available to compute conformal maps [12].

We will apply this convergence scheme and prove that the uniformization factor given by the uniformization theorem can be approximated by the discrete uniformization factor produced by the deformation of radii of circle packings. The main idea comes from the work by Wu-Zhu [13].

1.1. Set Up. Assume that $M$ is a closed orientable smooth surface of genus one, and $T$ is a triangulation of $M$. Denote the set of vertices, edges, and faces of $T$ as $V = V(T), E = E(T), F = F(T)$ respectively. Suppose $M$ is equipped with a Riemannian metric $g$, $T$ is a geodesic triangulation if any edge in $T$ is an embedded geodesic segment in $(M, g)$. Therefore, each face in $F$ of a geodesic triangulation is a geodesic triangle in $(M, g)$. It induces an edge length function $l \in \mathbb{R}^E_{>0}$. We denote the geodesic triangulation with an edge length function on $(M, g)$ as $(T, l)$. The area of $(M, g)$ is denoted as $\text{Area}(M, g)$.

Conversely, let $l$ be a positive function on $E$ satisfying triangle inequalities for each face of $T$. Then we can determine a Euclidean triangular mesh $(T, l)_E$ by gluing Euclidean triangles with prescribed edge lengths $l$ by isometries. Note that a Euclidean triangular mesh produces a piecewise flat metric on $M$ with all the singular points in $V(T)$. The area of $(T, l)_E$, which is the sum of areas of Euclidean triangles in the mesh, is denoted as $\text{Area}((T, u \ast l)_E)$.

Given $(T, l)_E$, let $\theta^i_{jk}$ denote the inner angle at $i$ in the Euclidean triangle $\triangle ijk \in F$, and discrete curvature $K_i$ at a vertex $i \in V$ be the angle defect

$$K_i = 2\pi - \sum_{jk: \triangle ijk \in F} \theta^i_{jk}.$$ 

Note that $(T, l)_E$ is flat if and only if $K_i = 0$ for any $i \in V$. 

Thurston [17] introduced circle packings on surfaces with non-positive Euler characteristic. We summarize relevant concepts as follows.

**Definition 1.1.** The edge length function of a Euclidean triangular mesh \((T, l)_E\) is a circle packing metric if for any \(ij \in E(T)\)

\[ l_{ij}^2 = e^{2\rho_i} + e^{2\rho_j} + 2e^{\rho_i + \rho_j} \cos \Theta_{ij}, \]

for a function \(\Theta : E \to [0, \pi/2]\) and a function \(\rho : V \to \mathbb{R}\). The function \(\Theta\) is called a conformal weight, and the function \(\rho\) is called a logarithmic radius. A circle packing metric is uniform if \(\rho\) is a constant function. Two circle packing metrics \((T, l)_E\) and \((T, l'_E)\) are discrete conformal if their conformal weights are identical.

Note that \(r_i = e^{\rho_i}\) is the radius of the circle centered at vertex \(i\). If we draw a triangle \(\triangle ijk\) with lengths given by \(l_{ij}, l_{ik}, l_{jk}\) and three circles of radius \(r_i, r_j, r_k\) at vertices \(i, j, k\) respectively in the plane, then \(\Theta_{ij}\) will be the intersection angle of the two circles centered at \(i\) and \(j\).

Any two discrete conformal circle packing metrics \(l\) and \(l'\) are differed only by their logarithmic radii \(u_i = \rho'_i - \rho_i\). Such a vector \(u \in \mathbb{R}^V\) is called a discrete conformal factor, and we denote this relation as \(l' = u \ast l\). If we start with a circle packing metric \((T, l)_E\) with a conformal weight \(\Theta\) and logarithmic radius \(\rho\), the discrete conformal factor \(u\) acts on the initial logarithmic radius by \(u \rightarrow \rho + u\), which is equivalent to \(l' = u \ast l\). We can produce all the other circle packing metrics discrete conformal to the initial metric \(l\) through this action.

Let \(\theta'_{jk}(u)\) and \(K_i(u)\) denote the inner angle and the discrete curvature at the vertex \(i\) in \((T, u \ast l)_E\) respectively. Set \(K(u) = [K_i(u)]_{i \in V} \in \mathbb{R}^V\). Then \(u \in \mathbb{R}^V\) is called a discrete uniformization factor if \(K(u) = 0\).

1.2. **Main results.** By the uniformization theorem, for any \((M, g)\) there exists a unique function \(\bar{u}\) on \(M\) such that \((M, e^{2\bar{u}}g)\) is a flat torus with unit area. We will prove that the discrete uniformization factor \(u \in \mathbb{R}^V\) induced by a sufficiently regular and dense geodesic triangulation \((T, l)\) on \((M, g)\) approximates the smooth uniformization factor \(\bar{u} \in C^\infty(M)\). To characterize the regularity of a geodesic triangulation, we use the following definition.

**Definition 1.2.** A Euclidean triangular mesh \((T, l)_E\) is called \(\epsilon\)-regular if

(a) The discrete metric \(l\) is a circle packing metric with some conformal weight \(\Theta\) so that \(\cos \Theta_{ij} \geq \epsilon\) for all \(ij \in E\);
(b) Any inner angle $\theta_{jk}^i$ in $(T, l)_E$ satisfies $\theta_{jk}^i \geq \epsilon$.

For a geodesic triangulation $(T, l)$ on a Riemannian surface, it is $\epsilon$-regular if the corresponding Euclidean mesh $(T, l)_E$ is $\epsilon$-regular.

**Remark 1.3.** Note that an $\epsilon$-regular geodesic triangulation $(T, l)$ has a uniform upper bound on the degrees of vertices by $\lceil 2\pi/\epsilon \rceil$.

Let $|x| = \max_{i \in A} |x_i|$ denote the maximal norm of a vector $x \in \mathbb{R}^A$ when $A$ is finite. Define the size of a geodesic triangulation $(T, l)$ to be $|l|$. The main result of this paper is

**Theorem 1.4.** Suppose $(M, g)$ is a closed orientable Riemannian surface of genus one, and $\bar{u} = \bar{u}_{M,g} \in C^\infty(M)$ is the unique function on $M$ such that $e^{2\bar{u}} g$ is the metric of constant curvature with unit area. Assume $T$ is a geodesic triangulation in $(M, g)$ with edge length $l \in \mathbb{R}_{E(T)}$. Then for any $\epsilon > 0$, there exists a constant $\delta = \delta(M, g, \epsilon) > 0$ such that if $(T, l)_E$ is $\epsilon$-regular, uniform, and $|l| < \delta$, then

(a) there exists a unique discrete conformal factor $u \in \mathbb{R}^V$, such that $(T, u \ast l)_E$ is globally flat and $\text{Area}((T, u \ast l)_E) = 1$, and

(b) $|u - \bar{u}|_{V(T)} \leq C|l|$ for some constant $C = C(M, g, \epsilon)$.

Theorem 1.4 implies that the discrete conformal factor $u$ defined on the set of vertices approaches the smooth conformal factor $\bar{u}$ restricted to the set of vertices as the size of geodesic triangulations tends to zero.

**Remark 1.5.** The uniqueness of the discrete uniformization factor was proved by Thurston [17] in terms of the rigidity of circle packing metrics with respect to the discrete curvature. The existence of arbitrarily dense, $\epsilon$-regular, and uniform geodesic triangulations on any $(M, g)$ can be deduced from the work by Colin de Verdiere [1]. It provides a family of geodesic triangulations on any Riemannian torus whose angles of triangles are arbitrarily closed to $\pi/3$ and sizes tend to zero.

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Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

2. Preliminaries and Preparations

We summarize the main tools in Wu-Zhu [13] to prove Theorem 1.4 in this section. It consists of four subsections, containing basic facts about calculus on graphs, properties of geodesic triangles and geodesic triangulations, and the curvature deformation with respect to discrete conformal factors. In addition, the last subsection provides a proof of a cubic estimation for controlling the deviation between smooth conformal deformation of metrics and discrete conformal deformation of metrics.

2.1. Discrete calculus of graphs. Let $G = (V, E)$ be a graph with the set of vertices $V$ and the set of edges $E$. Define three vector spaces $\mathbb{R}^V$, $\mathbb{R}^E$ and $\mathbb{R}_A^E$, where

(a) the vector $x \in \mathbb{R}^E$ is symmetric, i.e., $x_{ij} = x_{ji}$;

(b) the vector $x \in \mathbb{R}_A^E$ is anti-symmetric, i.e., $x_{ij} = -x_{ji}$.

A vector $x \in \mathbb{R}_A^E$ is called a flow on $G$. A positive vector in $\mathbb{R}^E$ is called an edge weight $\eta$ on $G$. Given an edge weight $\eta$ and a vector $x \in \mathbb{R}^V$, the gradient $\nabla x = \nabla_\eta x$ is a flow defined by

$$(\nabla x)_{ij} = \eta_{ij}(x_j - x_i).$$

The divergence $div(x)$ of a flow $x \in \mathbb{R}_A^E$ is a vector in $\mathbb{R}^V$ defined by

$$div(x)_i = \sum_{j \sim i} x_{ij}.$$ Given an edge weight $\eta$, the Laplacian $\Delta = \Delta_\eta : \mathbb{R}^V \rightarrow \mathbb{R}^V$ is defined as $\Delta x = \Delta_\eta x = div(\nabla_\eta x)$. Specifically,

$$(\Delta x)_i = \sum_{j \sim i} (\nabla x)_{ij} = \sum_{j \sim i} \eta_{ij}(x_j - x_i).$$

The Laplacian is a linear operator on $\mathbb{R}^V$. It could be represented as a $|V| \times |V|$ symmetric matrix. It is well-known that if $G$ is connected, $ker(\Delta)$ is an 1-dimensional space defined by

$$1 = \{x \in \mathbb{R}^V : x = (a, ..., a) \text{ for some } a \in \mathbb{R}\}.$$ Note that $\Delta_\eta$ is an invertible operator on $1^\perp$ for any edge weight $\eta \in \mathbb{R}_{>0}^E$. Let $\Delta^{-1}_\eta$ denote this inverse of $\Delta_\eta$, and we have
Lemma 2.1 (Lemma 2.2 in [13]). The map \((\eta, y) \mapsto \Delta^{-1} y\) is smooth from \(\mathbb{R}^E_{>0} \times 1^\perp\) to \(1^\perp\).

Let \((G, l)\) be a metric graph with the length defined by a positive vector \(l \in \mathbb{R}^E_{>0}\). A notion of the \(C\)-isoperimetric condition on it was introduced in [13] as a discrete version of the isoperimetric inequality on Riemannian surface. Set the boundary for any \(U \subset V\) as
\[
\partial U = \{ij \in E : i \in U, \text{ and } j \notin U\},
\]
and define the \(l\)-perimeter of \(U\) and the \(l\)-area of \(U\) respectively as
\[
|\partial U|_l \doteq \sum_{ij \in \partial U} l_{ij} \quad \text{and} \quad |U|_l \doteq \sum_{ij \in E(U)} l_{ij}^2.
\]
A metric graph \((G, l)\) satisfies the \(C\)-isoperimetric condition for some constant \(C > 0\), if for any \(U \subset V\),
\[
\min\{|U|_l, |V|_l - |U|_l\} \leq C |\partial U|^2.
\]
This notion of \(C\)-isoperimetric metric graphs is an essential tool to prove the main theorem due to the following discrete elliptic estimation.

Lemma 2.2 (Lemma 2.3 in [13]). Assume \((G, l)\) is \(C_1\)-isoperimetric, and \(x \in \mathbb{R}^E_{>0}, \eta \in \mathbb{R}^E_{>0}, C_2 > 0, C_3 > 0\) are such that
\begin{enumerate}[(i)]
\item \(|x_{ij}| \leq C_2 l_{ij}^2\) for any \(ij \in E\);
\item \(\eta_{ij} \geq C_3\) for any \(ij \in E\).
\end{enumerate}
Then
\[
|\Delta^{-1} \circ \text{div}(x)| \leq \frac{4C_2 \sqrt{C_1 + 1}}{C_3} |l| \cdot |V|_l^{1/2}.
\]

2.2. Lemmas about geodesic triangles and geodesic triangulations. We will constantly replace small geodesic triangles on Riemannian surfaces by Euclidean triangles in this paper. Hence, it is necessary to estimate the difference of angles, areas, and curvatures between them. The following two lemmas provide a bound on this difference.

Given a geodesic triangle \(\triangle ABC\) on a Riemannian surface, we will denote its inner angles as \(A, B, C\), and its lengths of edges opposite to \(A, B, C\) as \(a, b, c\). The area of \(\triangle ABC\) will be denoted as \(|\triangle ABC|\).
Lemma 2.3 (Lemma 3.5 in [13]). Given a Euclidean triangle \( \triangle ABC \) with edge lengths \( a, b, \) and \( c, \) if all the angles in \( \triangle ABC \) are at least \( \epsilon > 0, \) \( \delta < \epsilon^2/48, \) and
\[
|a' - a| \leq \delta a, \quad |b' - b| \leq \delta b, \quad |c' - c| \leq \delta c,
\]
then

1. \( a', b', c' \) form a Euclidean triangle with opposite inner angles \( A', B', C' \) respectively;

2. \[
|A' - A| \leq \frac{24}{\epsilon} \delta;
\]

3. \[
\left| |\triangle A'B'C'| - |\triangle ABC| \right| \leq \frac{576}{\epsilon^2} \delta \cdot |\triangle ABC|,
\]
where \( |\triangle ABC| \) is the area of \( \triangle ABC. \)

The next lemma on geodesic triangles provides a weak equivalence between edge lengths and areas in a Euclidean triangle with lower bounds on angles.

Lemma 2.4 (Lemma 4.1 [13]). Suppose \( \Delta_E ABC \) is a Euclidean triangle with the edge lengths \( a, b, c. \) If all the inner angles in \( \triangle_E ABC \) are at least \( \epsilon > 0, \) then
\[
\frac{\epsilon}{8} a^2 \leq |\Delta_E ABC| \leq \frac{1}{\epsilon} a^2.
\]

Starting from a geodesic triangulation \((T, l)\) on a Riemannian surface \((M, g)\), we will deform the Riemannian metric \( g \) in the smooth conformal class to the metric \( e^{2u}g. \) The following lemma ensures that when the size of the geodesic triangulation \((T, l)\) is sufficiently small, there exists a geodesic triangulation \((T', \tilde{l})\) isotopic to \((T, l)\) relative to \( V \) with respect to the metric \( e^{2u}g \) such that the angle deviation is small. Moreover, the 1-skeleton of \((T', \tilde{l})\), when regarded as a metric graph, is \( C\)-isoperimetric.

Lemma 2.5 (Lemma 4.4 in [13]). Suppose \((M, g)\) is a closed Riemannian surface, and \( T \) is a geodesic triangulation of \((M, g)\), and \( l \in \mathbb{R}^E(T) \) denotes the geodesic lengths of the edges.

(a) Given a conformal factor \( u \in C^\infty(M) \), there exists a constant \( \delta = \delta(M, g, u, \epsilon) > 0 \) such that if \( |l| < \delta \) then there exists a geodesic triangulation \( T' \) with edge length
in $(M, e^{2u} g)$ where $V(T') = V(T)$, and $T'$ is isotopic to $T$ relative to $V(T)$. Furthermore,

$$|\bar{\theta}_{jk}^i - \theta_{jk}^i| \leq \frac{\epsilon}{2},$$

where $\theta_{jk}^i$ is an angle in $(T, l)_E$, and $\bar{\theta}_{jk}^i$ is the corresponding angle in $(T', \bar{l})_E$.

(b) If $(T, l)_E$ is $\epsilon$-regular, there exists a constant $\delta = \delta(M, g, \epsilon)$ such that if $|l| < \delta$, $(T, l)$, as a metric graph on the 1-skeleton of $T$, is $C$-isoperimetric for some constant $C = C(M, g, \epsilon)$.

2.3. Differential of the discrete curvature. The explicit formula for the derivative of angles in Euclidean triangles with respect to discrete conformal factors is given in [20, 16].

**Lemma 2.6** ([20], Lemma 2.5 in [16]). Let $\theta_{jk}^i(u)$ be the inner angle at $i$ in triangle $\triangle ijk$ in $(T, u \ast l)_E$, and $K(u) \in \mathbb{R}^V$ is the discrete curvature. Then

$$\frac{\partial \theta_{jk}^i}{\partial u_j} = \frac{\partial \theta_{ik}^j}{\partial u_i} - r_i^2 r_j^2 \sin^2 \Theta_{ij} + r_j^2 r_k^2 (\cos \Theta_{jk} + \cos \Theta_{ij} \cos \Theta_{ik}) + r_i^2 r_j^2 r_k^2 (\cos \Theta_{ik} + \cos \Theta_{ij} \cos \Theta_{jk})$$

$$\times \frac{1}{l_{ik} l_{ij}^3 \sin \theta_{jk}^i}.$$

Define an edge weight $\eta \in \mathbb{R}^E$ as

$$\eta_{ij}(u) = \frac{\partial \theta_{jk}^i}{\partial u_j} + \frac{\partial \theta_{jm}^i}{\partial u_j},$$

where $\triangle ijk$ and $\triangle ijm$ are triangles in $T$. Then we have

$$\frac{\partial K}{\partial u} = -\Delta_{\eta(u)}.$$

This connection between the differential of the discrete curvatures and the Laplacian operator will be crucial to construct a flow to locate the discrete uniformization factor.

2.4. Cubic estimation. The following lemma illustrates the deviation between distances induced by discrete conformal factors and smooth conformal factors. It is an analogue to Proposition 5.2 in [13] in the setting of circle packing metrics.
Lemma 2.7. Suppose \((M, g)\) is a closed Riemannian surface, and \(u \in C^\infty(M)\) is a conformal factor. Assume that \(x, y \in M\) such that
\[
d_g(x, y) = l = \sqrt{e^{2\rho} + e^{2\rho} + 2e^{2\rho}I} = e^\rho \sqrt{2 + 2I},
\]
where \(I \geq 0\). Then there exists \(C = C(M, g, u, I) > 0\) such that
\[
|d_{e^{2u}g}(x, y) - \sqrt{e^{2(\rho+u(x))} + e^{2(\rho+u(y))} + 2e^{2\rho+u(x)+u(y)}I}| \leq C d_g(x, y)^3.
\]

Proof. By replacing \(g\) with \(e^{2u}g\) and consider \(g\) as a conformal change of \(e^{2u}g\) by the conformal factor \(-2u\), it suffices to prove that
\[
d_{e^{2u}g}(x, y) - \sqrt{e^{2(\rho+u(x))} + e^{2(\rho+u(y))} + 2e^{2\rho+u(x)+u(y)}I} \leq Cl^3
\]
for some constant \(C = C(M, g, u) > 0\).

Let \(\gamma : [0, l] \to (M, g)\) be a shortest geodesic connecting \(x, y\). Let \(h(t) = u(\gamma(t))\) and \(s_{e^{2u}g}(\gamma)\) be the length of the curve \(\gamma\) in the new metric \(e^{2u}g\). Then
\[
d_{e^{2u}g}(x, y) - \sqrt{e^{2(\rho+u(x))} + e^{2(\rho+u(y))} + 2e^{2\rho+u(x)+u(y)}I} \\
\leq s_{e^{2u}g}(\gamma) - \sqrt{e^{2(\rho+u(0))} + e^{2(\rho+u(l))} + 2e^{2\rho+u(0)+u(l)}I}
\]
(3)
\[
= \left[ \int_0^l e^{h(t)} dt - l \cdot e^{h(l/2)} \right] + \left[ l \cdot e^{h(l/2)} - \sqrt{e^{2(\rho+u(0))} + e^{2(\rho+u(l))} + 2e^{2\rho+u(0)+u(l)}I} \right].
\]
The first term in Equation (3) is estimated by the mid-point rule approximation of definite integral of a function \(f(t) \in C^2[0, l]\) given by
\[
\left| \int_0^l f(x) dx - l \cdot f(\frac{l}{2}) \right| \leq \frac{l^3}{24} \max_{0 \leq t \leq l} |f''(t)|.
\]
Note that \(h(t)\) and its first two derivatives are determined by \(u, \gamma,\) and their derivatives up to the second order. Since \(M\) is compact, and all the functions involved are smooth functions, then there exists a constant \(C_1 = C_1(M, g, u) > 0\) such that the maximum of \(h\) and its first and second derivatives are bounded by \(C_1\). Therefore,
\[
\left| \int_0^l e^{h(t)} dt - l \cdot e^{h(l/2)} \right| \leq C_1 l^3.
\]
The second term in Equation (3) is
\[
\begin{align*}
    l \cdot e^{h(l/2)} & - \sqrt{e^{2(\rho+h(0))} + e^{2(\rho+h(l))} + 2e^{2\rho+h(0)+h(l)} I} \\
    &= e^\rho e^{h(l/2)} \frac{\sqrt{2 + 2I}}{e^h(0) + e^{2h(l)} + 2e^{h(0)+h(l)} I} \\
    &= e^\rho \frac{(2e^{2h(l/2)} - e^{2h(0)} - e^{2h(l)}) + 2I(e^{2h(l/2)} - e^{h(0)+h(l)})}{e^{h(l/2)} \sqrt{2 + 2I} + e^{2h(0)} + e^{2h(l)} + 2e^{h(0)+h(l)} I}.
\end{align*}
\]

It can be bounded with help of the following inequality for a function in \( f \in C^2[0, l] \) given by
\[
\left| f\left(\frac{l}{2}\right) - \frac{1}{2} [f(0) + f(l)] \right| \leq \frac{l^2}{4} \max_{0 \leq t \leq l} |f''(t)|.
\]

There exist \( C_2 = C_2(M, g, u) > 0 \), and \( C_3 = C_3(M, g, u) > 0 \) such that
\[
\left| 2e^{2h(l/2)} - e^{2h(0)} - e^{2h(l)} \right| \leq C_2 l^2
\]
and
\[
\left| e^{2h(l/2)} - e^{h(0)+h(l)} \right| \leq e^\xi \cdot \left| 2h(l/2) - h(0) - h(l) \right| \leq C_3 l^2,
\]
where \( \xi \) is between \( 2h(l/2) \) and \( h(0) + h(1) \).

Set \( \mu \) be the minimum of \( u \) in \( M \), then
\[
l \cdot e^{h(l/2)} - \sqrt{e^{2(\rho+h(0))} + e^{2(\rho+h(l))} + 2e^{2\rho+h(0)+h(l)} I} \leq \frac{e^\rho l^2 (C_2 + C_3)}{2e^\mu \sqrt{2 + 2I}} \leq C_4 l^3,
\]
for some \( C_4 = C_4(M, g, u, I) > 0 \).

Hence,
\[
\left| d_{e^{2u}g}(x, y) - \sqrt{e^{2(\rho+u(x))} + e^{2(\rho+u(y))} + 2e^{2\rho+u(x)+u(y)} I} \right| \leq C d_g(x, y)^3,
\]
for some \( C = C(M, g, u, I) > 0 \).

\(\square\)

**Remark 2.8.** Lemma 2.7 shows that if \( x \) and \( y \) are two vertices in a geodesic triangulation determining one edge, the length of this edge in the new conformal metric \( e^{2u}g \) is approximated by the length of this edge in the discrete conformal metric, which is obtained from the discrete conformal change by \( u \).

The same estimate holds if the condition on \( \rho \) is relaxed to the assumption that there exists a constant \( C = C(M, g, u) \) such that \( |\rho_i - \rho_j| \leq C I \) for all \( i, j \in V \) and the condition on \( I \) is relaxed to \( I > -1 \). We omit the proof here.
3. PROOF OF THE MAIN THEOREM

In this section, we will prove the main theorem following the framework in [13].

Recall that the main theorem is

**Theorem 1.4.** Suppose \((M, g)\) is a closed orientable Riemannian surface of genus one, and \(\bar{u} = \bar{u}_{M, g} \in C^\infty(M)\) is the unique function on \(M\) such that \(e^{2\bar{u}}g\) is the metric of constant curvature with unit area. Assume \(T\) is a geodesic triangulation in \((M, g)\) with edge length \(l \in \mathbb{R}^{E(T)}\). Then for any \(\epsilon > 0\), there exists a constant \(\delta = \delta(M, g, \epsilon) > 0\) such that if \((T, l)_E\) is \(\epsilon\)-regular, uniform, and \(|l| < \delta\), then

(a) there exists a unique discrete conformal factor \(u \in \mathbb{R}^V\), such that \((T, u \ast l)_E\) is globally flat and \(\text{Area}((T, u \ast l)_E) = 1\), and

(b) \(\left|u - \bar{u}\right|_{V(T)} \leq C|l|\) for some constant \(C = C(M, g, \epsilon) > 0\).

In the rest of this paper, the notion \(a = O(b)\) represents that if \(\delta = \delta(M, g, \epsilon)\) is sufficiently small, then \(|a| \leq C \cdot b\) for some constant \(C = C(M, g, \epsilon)\). Moreover, we will replace \(\bar{u}_{V(T)}\) by \(\bar{u}\) for simplicity when there is no confusion.

The proof consists of the following three steps.

1. Prove that \((T, \bar{u} \ast l)_E\) is close to a flat Euclidean triangular mesh \((T, \bar{l})_E\), which means that

\[K(\bar{u}) = div(x) = O(|l|^2)\]

for some flow \(x \in \mathbb{R}^E_A\) with \(x_{ij} = O(|l|^2_{ij})\).

2. Construct a smooth deformation of the discrete conformal factor \(u(t) : [0, 1] \to \mathbb{R}^V\) starting at \(u(0) = \bar{u}\) such that

\[K(u(t)) = (1 - t)K(\bar{u}).\]

Note that at the end of this deformation, the curvature \(K(u(1)) \equiv 0\). As a result, we locate a discrete uniformization factor. Furthermore, this deformation is bounded by the size of the regular geodesic triangulations, in the sense that

\[|u(1) - \bar{u}| = O(|l|).\]

3. Deform \(u(1)\) by a small constant vector to \(\tilde{u}\) so that the resulting flat torus has unit area, then

\[|u(1) - \tilde{u}| = O(|l|).\]
Proof. We assume that \((T, l)\) is \(\epsilon\)-regular, uniform, and \(|l| \leq \delta\) where \(\delta = \delta(M, g, \epsilon)\) is a sufficiently small constant to be determined. Then \(l\) is a circle packing metric given by a constant logarithmic radii \(\rho\) and a conformal weight \(\Theta\).

If \(\delta\) is sufficiently small, then by Lemma \(2.5\), there exists a geodesic triangulation \(T'\) of \((M, e^{2\bar{u}}g)\) homotopic to \(T\) relative to \(V(T) = V(T')\). Let \(\bar{l} \in \mathbb{R}^E(T) \simeq \mathbb{R}^E(T')\) denote the geodesic lengths of edges of \(T'\) in \((M, e^{2\bar{u}}g)\). Since \(T'\) and \(T\) are combinatorial equivalent and isotopic, we will denote \((T', \bar{l})\) as \((T, \bar{l})\). Clearly, \((T, \bar{l})_E = (T, \bar{l})\) is isometric to \((M, e^{2\bar{u}}g)\) and flat since \((M, e^{2\bar{u}}g)\) is a flat torus.

3.1. Step 1. By Lemma \(2.7\),
\[
\bar{l}_{ij} - (\bar{u} * l)_{ij} = O(|l|^3).
\]
Then by Lemma \(2.3\) the angle deviation satisfies
\[
\alpha_{jk}^i := \bar{\theta}_{jk}^i - \theta_{jk}(\bar{u}) = O(|l|^2),
\]
where \(\bar{\theta}_{jk}^i\) denotes the inner angle in \((T, \bar{l})_E\), and \(\theta_{jk}(\bar{u})\) denotes the inner angle in \((T, \bar{u} * l)_E\). Define a flow \(x \in \mathbb{R}^E_A\) as
\[
x_{ij} := \frac{\alpha_{jk}^i - \alpha_{ik}^j - \alpha_{ik}^j}{3} = O(l_{ij}^2),
\]
where \(\triangle_{ijk}\) and \(\triangle_{ijk'}\) are adjacent triangles. Since \(\alpha_{jk}^i + \alpha_{ik}^j + \alpha_{ij}^k = 0\), then
\[
div(x)_i = \sum_{j:j \sim i} x_{ij} = \frac{\sum_{jk: \triangle_{ijk} \in F(T)} \left( \frac{\alpha_{jk}^i - \alpha_{ik}^j}{3} + \frac{\alpha_{jk}^i - \alpha_{ij}^k}{3} \right)}{j: \triangle_{ijk} \in F(T)} = \sum_{jk: \triangle_{ijk} \in F(T)} \alpha_{ij}^i = K_i(\bar{u}).
\]
Since degrees of vertices of \( \epsilon \)-regular triangulations are uniformly bounded, we have

\[ K_i(\bar{u}) = \sum_{j: j \sim i} x_{ij} = O(|l|^2). \]

It implies that the curvature of the metric \((T, \bar{u} \ast l)\) is closed to zero when \( \delta \) is sufficiently small. By Lemma 2.5, angles in \((T, \bar{l})_E\) have a lower bound by \( \epsilon/2 \), although \((T, \bar{l})_E\) might not be a circle packing metric. By equation (5), \((T, \bar{u} \ast l)_E\) is \( \epsilon/3 \)-regular for sufficiently small \( \delta > 0 \).

3.2. **Step 2.** Define

\[ \tilde{\Omega} = \{ u \in 1^1 : (T, u \ast l)_E \text{ is } \epsilon/5 \text{-regular}\}, \]

and

\[ \Omega = \{ u \in \tilde{\Omega} : |u - \bar{u}| \leq 1, (T, u \ast l)_E \text{ is } \epsilon/4 \text{-regular}\}. \]

Then \( \bar{u} \) is in the interior of \( \Omega \). Consider the system of differential equations on \( int(\tilde{\Omega}) \),

\[
\begin{cases}
    u'(t) = \Delta_{\eta(u)}^{-1} K(\bar{u}) = \Delta_{\eta(u)}^{-1} \circ \text{div}(x), \\
    u(0) = \bar{u},
\end{cases}
\]

where the edge weight \( \eta(u) \) is given by Theorem 2.6. By Lemma 2.1, the right-hand side of (6) is a smooth function of \( u \), so the equation (6) has a unique solution \( u(t) \) with the maximum existing open interval in \( \Omega \) given by \([0, T_0]\) for \( T_0 > 0 \). Moreover, by Theorem 2.6

\[ K(u(t)) = (1 - t)K(\bar{u}). \]

Therefore, we only need to show that \( T_0 \geq 1 \) for sufficiently small \( \delta > 0 \). Then \( u(1) \) is a discrete uniformization factor.

By Lemma 2.5 \((T, l)\) is \( C \)-isoperimetric for some constant \( C = C(M, g, \epsilon) \). For any \( u \in \Omega \), \((T, u \ast l)\) is \((e^{4|u|+1}C)\)-isoperimetric since \(|u| \leq |\bar{u}| + 1\). To apply Lemma 2.2, we need to bound related quantities \( \eta \) and \(|V| \) as follows.

Define \( r_i(u) = e^{\rho + u_i} \) when \( u \in \Omega \). There exists a constant \( R = R(M, g) \) such that for any \( i, j \in V \),

\[ r_i/r_j \leq e^{u_i - u_j} \leq e^{|\bar{u}_i - \bar{u}_j| + 2} \leq R. \]
Then
\[
\frac{\partial \theta_{jk}^i}{\partial u_j} = r_i^2 r_j^2 \sin^2 \Theta_{ij} + r_i^2 r_j r_k (\cos \Theta_{jk} + \cos \Theta_{ij} \cos \Theta_{ik}) + r_i r_j^2 r_k (\cos \Theta_{ik} + \cos \Theta_{ij} \cos \Theta_{jk}) \geq \frac{\epsilon r_i r_j r_k (r_i + r_j)}{l_{ik}^2} \geq \frac{\epsilon}{8 R^3}.
\]

Hence,
\[
\eta_{ij}(u) = \frac{\partial \theta_{jk}^i}{\partial u_j} + \frac{\partial \theta_{im}^i}{\partial u_j} \geq \frac{\epsilon}{4 R^3}.
\]

In addition, by Lemma 2.4 and the fact that angles of \((T, \bar{l})_E\) has a lower bounded by \(\epsilon/2\),
\[
|V_t| = \sum_{ij \in E} l_{ij}^2 = O(\sum_{ij \in E} \bar{l}_{ij}^2) = O(\sum_{ijk \in F} (\bar{l}_{ij}^2 + \bar{l}_{ik}^2 + \bar{l}_{jk}^2)) = O(\sum_{ijk \in F} |(\triangle ij k, \bar{l})_E|) = O(|(T, \bar{l})_E|) = O(1).
\]

Here \(|(T, \bar{l})_E|\) denotes the area of the piecewise flat metric on \(M\). Combining these two bounds with the fact that \(x_{ij} = O(|l|^2)\) in Step 1, we can apply Lemma 2.2 to conclude that
\[
|u'(t)| = O(|l|),
\]
for any \(t \in [0, T_0)\). If \(T_0 < 1\),
\[
|u(T_0) - \bar{u}| = O(|l|).
\]

Therefore, \(u(T_0) \in \text{int}(\Omega)\) if \(\delta\) is sufficiently small, which contradicts with the maximal property of \(T_0\). Thus, \(T_0 \geq 1\) and \((T, u(1))_E\) is globally flat and
\[
|u(1) - \bar{u}| = O(|l|).
\]

3.3. **Step 3.** Let \(\tilde{u}\) be the discrete uniformization factor with \(\text{Area}(T, \tilde{u} \ast l)_E = 1\) and
\[
\tilde{u} - u(1) \in 1 = \{ x \in \mathbb{R}^V : x = (a, ..., a) \} \text{ for some } a \in \mathbb{R} \}.
\]

To estimate their difference, notice first that
\[
e^{-|u(1) - \bar{u}|} (\bar{u} \ast l)_{ij} \leq (u(1) \ast l)_{ij} \leq e^{|u(1) - \bar{u}|} (\bar{u} \ast l)_{ij}.
\]
By equation (4),
\[
|(u(1) \ast l)_{ij} - \bar{l}_{ij}| = |(u(1) \ast l)_{ij} - (\bar{u} \ast l)_{ij} + |(\bar{u} \ast l)_{ij} - \bar{l}_{ij}|
\leq (e^{2|u(1) - \bar{u}|} - 1)(\bar{u} \ast l)_{ij} + O(l_{ij}^3) = O(|l| \cdot \bar{l}_{ij}).
\]
Applying Lemma 2.3, we deduce that
\[ |((\Delta ijk, u(1) * l)_E | - |((\Delta ijk, \bar{u})_E | = O(|l|) |(\Delta ijk, \bar{u})_E |. \]
Since \( \text{Area}((T, \bar{l})_E) = \text{Area}((T, \tilde{u} * l)_E) = 1 \), we take the sum over all triangles in \( F \), then
\[ |\text{Area}(T, u(1) * l)_E - \text{Area}(T, \tilde{u} * l)_E| = O(|l|). \]
This implies that
\[ |	ilde{u} - u(1)| = O(|l|). \]
Therefore, we have
\[ |	ilde{u} - \tilde{u}| \leq |\tilde{u} - u(1)| + |\tilde{u} - u(1)| = O(|l|). \]
Then \( \tilde{u} \) is the desired conformal factor, and the proof of Theorem 1.4 is completed.

\[ \square \]

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ, 08854

Email address: yl1594@math.rutgers.edu

SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, WUHAN 430072, P.R. CHINA

Email address: xuxu2@whu.edu.cn