Critical Binder cumulant and universality: Fortuin-Kasteleyn clusters and order-parameter fluctuations

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We investigate the dependence of the critical Binder cumulant of the magnetization and the largest Fortuin-Kasteleyn cluster on the boundary conditions and aspect ratio of the underlying square Ising lattices. By means of the Swendsen-Wang algorithm, we generate numerical data for large system sizes and we perform a detailed finite-size scaling analysis for several values of the aspect ratio \( r \), for both periodic and free boundary conditions. We estimate the universal probability density functions of the largest Fortuin-Kasteleyn cluster and we compare it to those of the magnetization at criticality. It is shown that these probability density functions follow similar scaling laws, and it is found that the values of the critical Binder cumulant of the largest Fortuin-Kasteleyn cluster are upper bounds to the values of the respective order-parameter’s cumulant, with a splitting behavior for large values of the aspect ratio. We also investigate the dependence of the amplitudes of the magnetization and the largest Fortuin-Kasteleyn cluster on the aspect ratio and boundary conditions. We find that the associated exponents, describing the aspect ratio dependencies, are different for the magnetization and the largest Fortuin-Kasteleyn cluster, but in each case are independent of boundary conditions.

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I. INTRODUCTION

According to the universality hypothesis \[1\textsuperscript{–}3\], all critical systems with the same dimensionality, the same symmetry of the order parameter, and the same range of interactions are expected to share the same set of critical exponents. For the two-dimensional (2d) Ising model (square and some other lattices) all critical exponents are known exactly \[4,5\]. These exponents are expected to be obeyed by the Ising model on all 2d lattices and also by all other models, which according to the hypothesis are expected to belong in the same universality class. Furthermore, there is strong evidence that, in addition to critical exponents, certain critical-point ratios are universal \[8,11\] and of particular interest is the value of the critical Binder cumulant of the order parameter, discussed also in the present work.

The fourth-order cumulant of some thermodynamic parameter \( Q \) of a finite lattice system, known as the Binder cumulant, is defined as \[8\]

\[
U_Q(T, L) = 1 - \frac{\langle Q^4 \rangle_L}{3\langle Q^2 \rangle_L^2},
\]

with \( L \) the linear lattice size. The critical value of the Binder cumulant of the order parameter of an Ising system is then

\[
U_M^* = \lim_{L \to \infty} U_M(T = T_c, L),
\]

with \( M \) the magnetization

\[
M = \langle 1/N \rangle \sum_{i=1}^{N} \sigma_i,
\]

\( \sigma_i \) the spin variable, and \( N \) the number of lattice sites. This parameter is a measure of the deviation of the universal probability density function from a Gaussian function. It is well known that the characteristic behavior of \( U_M \) \[8,11\] near criticality provides a traditional route to obtain transition temperatures (from the intersection of the cumulants of systems with different sizes) and may also be used to extract the critical exponent \( \nu \) of the correlation length \[8–10\]. Its critical value, \( U_M^* \), was originally believed to fully characterize a given universality class. As discussed by various authors, the same value seems to be shared by several 2d models, such as the XY models with an easy axis, the nearest-neighbor spin-1 Ising model, and the isotropic nearest-neighbor Ising-like models, including also the nearest-neighbor “border \( \phi^4 \) model” with softened spins \[12–18\]. It appears also to be independent of the lattice details, such as the lattice structure \[19,20\]. However, this “universality” applies only in a limited sense. The value of \( U_M^* \) does depend on the boundary conditions \[8,11\], the shape of the system \[13,17,21,22\], as well as on the symmetry of the interactions \[19\].

An accurate estimation of critical point ratios for the ferromagnetic Ising model on the square and triangular lattices has been provided via the transfer-matrix technique \[13\]. In this paper, Kamieniarz and Blöte estimated \( U_M^* \) as a function of the aspect ratio \( r \) (see discussion below for the definition of \( r \) ), reporting, in particular for the square Ising model with periodic boundary conditions and \( r = 1 \), the value \( U_M^* = 0.61069 \ldots \). The influence of the anisotropic interactions on the critical Binder cumulant was studied, analytically, by Dohm and Chen \[24,25\] and, numerically, by Selke and
Sluchêr \cite{21,22}, indicating that $U_M^*$ depends continuously on the anisotropy, in the case of periodic boundary conditions and $r = 1$. Furthermore, Kastening \cite{20} obtained a renormalization-group quantitative description of the anisotropy dependence of $U_M^*$.

In the present paper, we investigate certain aspects of the critical Binder cumulant and, in particular, its dependence on the boundary conditions and the relevant aspect ratio of the lattice. We concentrate our interest on numerical observations illustrating parallel behavior to that of the critical Binder cumulant of the largest Fortuin-Kasteleyn cluster (LFKC). The rest of the paper is organized as follows: In the next section, we define the model and outline the numerical details. Then, in Sec. III we present and discuss our numerical findings. Finally, we summarize our conclusions in Sec. IV.

II. MODEL AND SIMULATION DETAILS

Let the size of the LFKC be denoted by $S_{\text{LFKC}}$ and let us define, in analogy to the magnetization, $l_\infty = S_{\text{LFKC}}/N$. Then, the relevant critical Binder cumulant may be denoted as $U_{l_\infty}^*$. To be concrete, we consider the square Ising model in zero field, with the standard Hamiltonian

$$\mathcal{H} = -J \sum_{(ij)} \sigma_i \sigma_j,$$

where the spin variables $\sigma_i$ take the Ising values $\pm 1$ and $(ij)$ denotes summation over all nearest-neighbor pairs of sites. For the needs of our study, we construct ferromagnetic nearest-neighbor square Ising systems with $L$ rows and $L_1 = rL$ columns, corresponding to $N = L \times L_1 = L \times rL \equiv (L^*)^2$ sites. Furthermore, we consider several values of the aspect ratio $r = \{1, 4, 9, 16, 25, 36, 50, 64, 100\}$, and investigate both periodic (PBC) and free boundary conditions (FBC). As our numerical vehicle, we implement the Swendsen-Wang algorithm \cite{27,30}, and we identify clusters by the Hoshen-Kopelman procedure \cite{29,31}.

The comprehensive Monte Carlo study of De Meo et al. \cite{32} presented a review of the connections of Fortuin and Kasteleyn’s work \cite{33} to the Swendsen-Wang algorithm and a review of the relevant literature. In this study, the authors investigated the scaling properties of the cluster size distribution and provided a numerical verification of the theoretical results given by Hu \cite{34}. In particular, they showed that the relevant bond-correlated percolation model has the Ising critical temperature and critical exponents. Thus, it is generally assumed, that the LFKC corresponds to the magnetization, but the distribution functions, and accordingly, the Binder cumulants, are quite different.

We concentrate on the dependency of the critical Binder cumulants, $U_M^*$ and $U_{l_\infty}^*$, on the boundary conditions and the aspect ratio and compare our results with previous work when available \cite{13,19,35}. We also illustrate and compare the corresponding probability density functions (pdfs) observing their evolution as a function of the aspect ratio. For PBC and $r = 4$, a pronounced double-peak structure is observed in the pdf of LFKC, and we give for this a geometrical explanation, involving the probability that the LFKC percolates along both (short and long) directions of the lattice simultaneously. Finally, we discuss the critical-exponent equivalence \cite{32,34} and the scaling properties of $Q = |M|$ and $Q = l_\infty$, for both PBC and FBC. In particular, we estimate the amplitudes $A_Q$ of the power law $(Q) = A_Q L^{-\beta/\nu}$ for all values of the aspect ratio considered. It is shown that these amplitudes follow a power law with $r$, and the corresponding exponents are determined. This analysis is related to the interesting superscaling concepts reported by Watanabe et al. \cite{36} in their study of percolation on rectangular domains, as will be further discussed below.

The square Ising systems under study were simulated only at the exact critical temperature $k_BT_c/J = 2.2691853 \cdots$. In our numerical approach, we define a Swendsen-Wang Monte Carlo step to consist of $10 - 20$ (depending on $L^*$) Swendsen-Wang moves, in which all Fortuin-Kasteleyn clusters attempt to flip with probability $1/2$ \cite{27,28}. A number of Swendsen-Wang Monte Carlo steps, denoted as $n_{\text{eq}}$, is used for equilibration and a large number of such steps, denoted as $n_{\text{rec}}$, is used for the recording of the data related to the Fortuin-Kasteleyn cluster decomposition. Typical values of the parameters $n_{\text{eq}}$ and $n_{\text{rec}}$, used in our simulations, are $n_{\text{eq}} = 1600$

| BC  | $r$ | $U_M^*$ \cite{13} | $U_M^*$ | $U_{l_\infty}^*$ | $U_{l_\infty}^*$ |
|-----|----|-----------------|--------|-----------------|------------------|
| PBC | 1  | 0.61069 \cdot | 0.6167(2) |  |  |  |
| PBC | 4  | 0.48723 \cdot | 0.4995(5) |  |  |  |
| PBC | 9  | 0.27054 \cdot | 0.3880(8) |  |  |  |
| PBC | 16 | 0.1539(6)     | 0.4317(10)|  |  |  |
| PBC | 25 | 0.0984(10)    | 0.4760(6) |  |  |  |
| PBC | 36 | 0.0685(10)    | 0.5044(10)|  |  |  |
| PBC | 50 | 0.04920 \cdot | 0.5258(15)|  |  |  |
| PBC | 64 | 0.0375(12)   | 0.5385(8) |  |  |  |
| PBC | 100| 0.02454 \cdot| 0.5585(10)|  |  |  |
| FBC | 1  | 0.396(2) \cite{19} | 0.4370(10)|  |  |  |
| FBC | 4  | 0.2365(15)    | 0.3898(10)|  |  |  |
| FBC | 9  | 0.1188(12)    | 0.4394(80)|  |  |  |
| FBC | 16 | 0.0680(5)     | 0.4800(40)|  |  |  |
| FBC | 25 | 0.0452(14)    | 0.5160(20)|  |  |  |
| FBC | 36 | 0.0330(6)     | 0.5365(5) |  |  |  |
| FBC | 50 | 0.0214(10)    | 0.5533(12)|  |  |  |
| FBC | 64 | 0.0178(15)    | 0.5631(15)|  |  |  |
| FBC | 100| 0.0123(10)   | 0.5790(10)|  |  |  |

$^a$Best estimates from the literature.
and \( n_{\text{rec}} = 64000 \) for \( L^* = 20 \), whereas \( n_{\text{eq}} = 7200 \) and \( n_{\text{rec}} = 115200 \) for \( L^* = 120 \). In each case, we used 10 independent runs, restarted from new random spin configurations. The statistical errors, of the corresponding data were set equal to 3 standard deviations of the 10 independent runs. In order to achieve good accuracy in the estimation of the above mentioned amplitudes, via an extrapolation finite-size scaling scheme, the above described simulations were carried out for all values of aspect ratio \( r \). In all cases, approximately the range \( L^* = 20 - 120 \) was covered by \( 8 - 10 \) different widths \( L \), and for \( r = 1 \) and \( r = 4 \), we also simulated systems with linear sizes \( L = 160 \) and \( 200 \).

**III. NUMERICAL RESULTS AND DISCUSSION**

All our estimates for the critical Binder cumulants are given in Table I together with the existing ones from the corresponding literature. In Fig. 1 we illustrate the finite-size behavior of the critical cumulants for a selected set of the cases (as indicated on the panel), and the rather smooth linear extrapolation, which provides us with the limiting values of the critical Binder cumulants. The values listed in Table I were obtained by applying the expected leading correction term \( aL^{-1.75} \) [13]. Note that, in almost all cases, these values and the ones obtained by a linear extrapolation agree within error bars. From the Table one can observe a very good agreement with previous estimates regarding the magnetization’s critical Binder cumulants and the estimates of the present study. The critical Binder cumulants of the LFKC \( U_{M}(\infty) \) are found, in all cases, to be upper bounds to the values of the critical Binder cumulants of the order parameter. The smallest difference, between the two cumulants corresponds to the case of PBC with \( r = 1 \) and this difference is enhanced as the order-parameter cumulant deviates from the value 2/3, approaching the limiting (Gaussian) value 0, as \( r \to \infty \).

This strong splitting behavior is presented in Fig. 2 which gives a full illustration of the dependence of critical Binder cumulants for magnetization and the LFKC on the aspect ratio for both cases of boundary conditions considered. Several interesting conclusions can be drawn from this figure. First, as should be expected, and shown by Kamieniarz and Blöte [13] for PBC, the limiting magnetization cumulants \( U_{M}^{*} \) agree with the Gaussian value 0, describing linear systems, as \( r \to \infty \) for both PBC and FBC. For large \( r \), \( U_{M}^{*}(r) \) becomes linear in \( r^{-1} \), and as pointed out by Kamieniarz and Blöte [13], the product \( A_{U}(r) = rU_{M}(r) \) approach exponentially fast the universal amplitude \( A_{U} = \lim_{r \to \infty} [U_{M}(r)r] \). The estimates for this universal amplitude of the transfer-matrix technique in Ref. [13], agree to \( \sim 5 \) significant figures with the value \( A_{U} = 2.46044(2) \), obtained from conformal invariance [3]. The statistical Monte Carlo errors permit here a, moderately accurate, estimate of the order of \( A_{U} = 2.466(7) \), as can be seen by a linear fit of the \( r = 16 - 100 \) data of Table I. For FBC, \( U_{M}^{*}(r) \) becomes also linear in \( r^{-1} \), and the corresponding fit, in the range \( r = 16 - 100 \), provides the estimate \( A_{U} = 1.055(26) \).

The cumulant \( U_{M}^{*} \) of the LFKC shows a non-monotonic behavior approaching finally, as \( r \to \infty \), a non-trivial value different to 2/3 (describing the ordered phase). To estimate the limiting behavior, we assume the law \( U_{M}^{*}(r) = U_{M}^{*}(\infty) - Br^{-x} \). Fitting the \( r = 16 - 100 \), or the \( r = 25 - 100 \) data of Table I we find quite stable estimates. For the \( r = 25 - 100 \) range, this estimation scheme gives \( U_{M}^{*}(r) = 0.626(4) - 0.96(4) r^{-0.57(2)} \) for PBC, and \( U_{M}^{*}(r) = 0.643(10) - 0.63(7) r^{-0.50(6)} \) for FBC. Although even for \( r = 100 \), the values of the cumulants deviate significantly from their limiting values, the exponents \( x \) agree with the value 0.5 within error bars and appear to be independent of the boundary conditions. Thus, the exponents controlling the limiting r
behavior are different for $U_M(r)$ and $U_{\infty}^r (r)$, but are independent of the boundary conditions. Note that, the above limiting values are indicated by the dashed (PBC) and dotted (FBC) lines in the panel of Fig. 2 together with the full line corresponding to the value 2/3. In the limit $r \to \infty$, cumulative universality between PBC and FBC is reflected in the exponents, and the role of the boundary conditions appears to diminish in that limit. As noted above, for moderate values of the aspect ratio, both $U_M$ and $U_{\infty}^r$ have a strong dependence on the boundary conditions. This non-monotonic behavior, and the smooth final approach in the limit $r \to \infty$, are reflections of the evolution features of the corresponding pdfs, which are further illustrated bellow.

Figure 3 illustrates the scaling of the order-parameter pdfs, while Fig. 4 illustrates the scaling of the relevant functions of the LFKC for the cases with PBC and FBC, for which their Binder’s cumulant finite-size behavior is illustrated in Fig. 1. The scaled distributions have been constructed by using as scaling variables the $x = Q/\sqrt{\langle Q^2 \rangle}$, with $Q = |M|$ or $Q = l_{\infty}$. In the scaling limit (system size going to infinity) these functions are expected to be universal and characterize the given universality class [8, 32]. The scaled density functions are then obtained from $p(x)dx = p_Q(Q)dQ$, i.e., $p(x) = p_Q(Q) \cdot \sqrt{\langle Q^2 \rangle}$, and also a smoothing process of the fluctuations has been applied. The pronounced double-peak structure for PBC and $r = 4$, the left shoulders for PBC and $r = 1$ observed in Fig. 4, and also the non-monotonic behavior of the critical cumulant of the LFKC of Fig. 2, are interesting findings reflecting geometrical features of the present bond correlated percolation model. A brief qualitative description of these features is attempted below, observing the variation of certain percolation probabilities with the aspect ratio ($r$).

Throughout the years, different percolation probabilities have appeared in the literature [43, 44], using different terms such as spanning probability [41], crossing probability [43] or existence probability [36, 42]. It is well known that above the percolation threshold there exists an infinite cluster with probability one [41], while exactly at the percolation threshold the “crossing probabilities” need not be one and their study is an important topic with many and famous contributions, such as Cardy’s exact result [43] on the square lattice with FBC. These critical probabilities depend on the boundary conditions and the aspect ratio [36, 43]. For the square systems considered here, with $L$ rows and $L_1 = rL$ columns, we define $p_{\text{short}}$ to be the probability that the LFKC percolates only in the short direction, visiting every row of the lattice (that is having at least one point in every row). Respectively, the corresponding probability that the LFKC percolates only in the long direction, visiting all columns will be denoted by $p_{\text{long}}$, and the probability of simultaneously percolating in both the short and long directions by $p_{\text{both}}$. We may note here that for the present bond-correlated LFKC percolation, at the critical point, the sum $p_{\text{span}} = p_{\text{both}} + p_{\text{long}} + p_{\text{short}}$ (spanning probability in some direction) will also depend on the boundary conditions and the aspect ratio and need not be one. The behavior of $p_{\text{both}}$ and $p_{\text{short}}$ as a function of $r$ is illustrated in Fig. 4 for both PBC and FBC. Clearly, a strong variation with respect to the aspect ratio $r$ is observed. To construct Fig. 5 we have used systems with approximately 3600 lattice sites. For instance for $r = 2$, lattices of $42 \times 84$ with 3528 lattice points were used, while for $r = 36$, lattices of $10 \times 360$ with 3600 lattice points. Additionally, for $r = 1$, we carried out a brief finite-size scaling analysis using data in the range $L = 30 – 100$. The resulting limiting values are indicated with the dashed lines in Fig. 5 and demonstrate that the finite-size behavior in the main panel, using systems with 3600 lattice sites, is already a genuine representation of the behavior in the thermodynamic limit. As can be seen, for moderate values of $r$ and PBC, the LFKC percolates with significant probabilities in both directions of the square lattice, giving rise to the double-peak struc-
the similarity in the shape for larger values (with PBC and FBC for moderate aspect-ratio values and versatility in the shape of the pdfs of the LFKC for the cases directions.

this shoulder-like behavior to a restricted pdf describing only the LFKC that percolate simultaneously in both directions. This existence of a non-vanishing contribution of LFKC percolating only in the short direction of the lattice. This peak structure, while the absence of such a structure in the case of FBC, is due to the very early and large separation of the corresponding probabilities. The presence of the left shoulders for PBC and \( r = 1 \), in the pdf of the LFKC shown in Fig. 4 may also be explained by the existence of a non-vanishing contribution of LFKC percolating only in the short direction of the lattice. This is illustrated in the inset of Fig. 4 where we compare this shoulder-like behavior to a restricted pdf describing only the LFKC that percolates simultaneously in both directions.

Subsequently, Fig. 5 clarifies in the main panel the diversity in the shape of the pdfs of the LFKC for the cases with PBC and FBC for moderate aspect-ratio values and the similarity in the shape for larger values \( (r = 36) \). The inset of the same figure points out the similarity in the shape of the magnetization’s pdfs for \( r = 36 \).

FIG. 5: (color online) Percolation probabilities \( p_{\text{both}} \) and \( p_{\text{short}} \) with respect to \( \ln (r) \). Dashed lines indicate the \( r = 1 \) values in the thermodynamic limit (see discussion in the text), and the expected large-\( r \) behavior. The crossover behavior for the PBC is associated with the appearance of the double-peak structure in Fig. 4. In the inset we compare the \( r = 1 \) shoulder-like behavior in Fig. 4 with a restricted pdf describing only the LFKC that percolate simultaneously in both directions.

ture. Then, as \( r \) grows, the probability for percolation along the width of the rectangular lattices increases substantially, whereas, it declines along the length direction, leading to the evaporation of the right peak. The crossing of the probabilities for PBC, in the main panel of Fig. 4 gives a clear explanation of the observed double-peak structure, while the absence of such a structure in the case of FBC, is due to the very early and large separation of the corresponding probabilities. The presence of the left shoulders for PBC and \( r = 1 \), in the pdf of the LFKC shown in Fig. 4 may also be explained by the existence of a non-vanishing contribution of LFKC percolating only in the short direction of the lattice. This is illustrated in the inset of Fig. 4 where we compare this shoulder-like behavior to a restricted pdf describing only the LFKC that percolates simultaneously in both directions.

Subsequently, Fig. 6 clarifies in the main panel the diversity in the shape of the pdfs of the LFKC for the cases with PBC and FBC for moderate aspect-ratio values and the similarity in the shape for larger values \( (r = 36) \). The inset of the same figure points out the similarity in the shape of the magnetization’s pdfs for \( r = 36 \).

Differences in the shapes of the universal pdfs of the magnetization and the LFKC should be expected from the theoretical arguments of Hu [34] and the Monte Carlo study of De Meo et al. [32]. As shown numerically in Ref. [32], below \( T_c \) the magnetization susceptibility differs from the corresponding percolation susceptibility. Thus, the above illustrations reveal these differences at \( T_c \), but also show a variability in the behavior for moderate values of the aspect ratio. The illustrations in Fig. 7 give a sketch of the evolution of both pdfs, as we increase the aspect ratio from \( r = 1 \) to \( r = 16 \) in the case of PBC. From the first panel (a) of Fig. 6 for the case \( r = 1 \), we observe a small but noticeable difference in the left-tails, which is however enough to produce the small difference of the cumulant values in Table I. The double-peak structure of the pdf of the LFKC for \( r = 4 \), in panel (b) of Fig. 6 is also associated to the pronounced left tail in the pdfs of the magnetization. For larger values of the aspect ratio [see panels (c) and (d) in Fig. 6] the pdfs of the LFKC tend to to the shape illustrated in the main panel of Fig. 6 for \( r = 36 \), and the corresponding pdfs of the magnetization to the one illustrated in the inset of Fig. 6 for \( r = 36 \). Now, according to Refs. [32, 34], the behavior of \( \langle Q \rangle \), with \( Q = |M| \) or \( Q = l_{\infty} \), should be expected to be described by the same power law of the form \( \langle Q \rangle = AL^{-\beta/\nu} \), with the critical exponent having the 2d Ising value \( \beta/\nu = 0.125 \). Thus, as we vary the aspect ratio, the corresponding leading amplitudes, \( A_{|M|} \) and \( A_{l_{\infty}} \), decrease and describe the asymptotic shifts to smaller mean values. These shifts are considerable and are responsible for the development of a Gaussian-like shape in the small \( M \) behavior of the pdfs of the magnetization, as also illustrated in the inset of Fig. 6 for \( r = 36 \).

From the shape of the pdfs of the LFKC in Fig. 4 and also from the comparative plot of Fig. 6 we observe that larger fluctuations (widths) of the LFKC enhances, in all cases, the left tail of the magnetization’s pdf, since smaller Fortuin-Kasteleyn clusters favor the mixing of positive and negative spin clusters. However, for large values of the aspect ratio, the pronounced \( M = 0 \) behavior of the universal pdfs is mainly due to the shifts of the density functions of the LFKC, discussed above. Comparing their shapes, Figs. 6 and 4 we can appreciate this evolution for the case of FBC with \( r = 1 \) and also.

FIG. 6: (color online) Main panel: Scaled pdfs of the LFKC, \( x = l_{\infty}/\sqrt{(\langle M \rangle)} \), for PBC and FBC. Illustration of striking differences for \( r = 4 \) and approach to the same universal pdf as \( r \) increases \( (r = 36) \). Inset: Illustration of the similarity of the magnetization’s pdfs, \( x = |M|/\sqrt{(M^2)} \), for large \( r \) \( (r = 36) \).
FIG. 7: (color online) Scaled pdfs of the magnetization, $x = |M|/\sqrt{\langle M^2 \rangle}$, and the LFKC, $x = l_\infty/\sqrt{\langle Q^2 \rangle}$, for PBC as we vary the aspect ratio in the window $r = 1 - 16$. This figure elucidates the similarity among the two functions for $r = 1$, but also highlights their striking differences with increasing $r$.

for the case with PBC and $r = 4$. The large fluctuations of the LFKC and their shifts to smaller mean values induce similar behavior on the statistically significant part of the Fortuin-Kasteleyn clusters, which contribute and enhance the small $M$ behavior of the order-parameter pdf.

Let us now consider the scaling properties of $|M|$ and $l_\infty$, and the critical-exponent equivalence [32, 34]. In order to observe quantitatively the scaling properties of the above distributions, we attempted to apply to our data a simple power law, $\langle Q \rangle = A_Q L^{-\beta/\nu}$, and furthermore other expansions, including algebraic powers, but also logarithmic terms, $\langle Q \rangle = A_Q L^{-\beta/\nu}(1 + B \ln (L)/L + C/L + \cdots)$. For PBC and almost all values of $r$, the simple power law produced stable estimates with effective exponents converging to the 2d Ising value $\beta/\nu = 0.125$. Furthermore, by fixing the exponent to the above expected value, and varying the width range $L = (l_{\min} - l_{\max})$, we obtained smoothly behaving effective values for the corresponding amplitudes $A_Q$. Employing a linear extrapolation in $1/L_{\min}$ to the above effective values, we found very accurate estimates for the amplitudes in the thermodynamic limit. The same procedure was also followed using the simplest correction form $\langle Q \rangle = A_Q L^{-\beta/\nu}(1 + B/L)$, giving final estimates which are the same, within error bars, to those obtained by the simple power law. For the case of FBC, the fitting attempts to the simple power law produce, in general, overestimated values of the exponent, and by fixing $\beta/\nu = 0.125$ the resulting effective values of amplitudes $A_Q$, deviate significantly from their asymptotic values. However, the fitting attempts to the form $\langle Q \rangle = A_Q L^{-\beta/\nu}(1 + B/L)$, gave a smooth behavior of effective amplitudes with small deviations of their asymptotic values, allowing an accurate estimation of the amplitudes.

In particular, for the case of PBC with $r = 1$, we found by applying a simple power law, $\beta/\nu = 0.1248(4)$ from the magnetization data and $\beta/\nu = 0.1249(2)$ from the $l_\infty$ data, with corresponding amplitudes $A_{|M|} = 1.008(2)$ and $A_{l_\infty} = 1.007(2)$. However, this was an exceptionally good case, while for FBC with $r = 1$, the simple power law, when applied in the range $L = 20 - 96$, produces the results $\langle |M| \rangle = 0.581(5) L^{-0.146(2)}$ and $\langle l_\infty \rangle = 0.556(4) L^{-0.145(2)}$. Moving to larger values of...
Aromatic magnetization follow closely, for large enough general behavior is illustrated in Fig. 8. The amplitudes are accurate estimation of the amplitudes for all present model. In order to verify these concepts, in

perscaling exponent and thus for the relevant amplitudes we associate a su-

difference grows with increasing r in general different, and in the case of FBC the simplest extrapolation schemes verifies that for all aspect ratios l∞, whereas for FBC the simplest correction term B/L, is very effective since already the effective estimates of the range L = 20 − 96 are very close to their asymptotic values.

The systematic application of the above described extrapolation schemes verifies that for all aspect ratios r the amplitude A|M| is higher from A|∞, and in fact their difference grows with increasing r. This is an interesting topic related to the theoretical arguments of Hu and to the superscaling concepts reported by Watanabe et al. in their study of percolation on rectangular domains. Adapting this superscaling proposal of Ref. to our study at criticality, we assume that the above amplitudes follow for large r a power law. This is equivalent to the proposal \( \langle Q \rangle = ar^{-z}L^{-\beta/\nu}(1 + \cdots) \) and thus for the relevant amplitudes we associate a superscaling exponent z. Watanabe et al. have pointed out the interest of a study of their superscaling concept to correlated percolation models, such as the present model. In order to verify these concepts, in the realm of the present study, we have carried out an accurate estimation of the amplitudes for all r considered here. Our estimates are given in Table II and the general behavior is illustrated in Fig. 8. The amplitudes of the magnetization follow closely, for large enough r, say r ≥ 9, a simple power law of the form A|M|(r) = ar−z, with small corrections. Thus, for PBC we obtain A|M|(r) = 1.361(9)r−0.50(2), whereas for FBC A|M|(r) = 0.762(9)r−0.50(2). For the amplitudes A|∞ of the LFKC, the deviations from the simple power law are larger. Now, a correction term of the form B/r stabilizes the behavior of the effective estimates. Thus, in the case of PBC we find A|∞(r) = 3.01(10)r−0.816(14)(1 − 1.71(20)/r), while A|∞(r) = 1.42(5)r−0.816(14)(1 − 1.52(22)/r) for the case of FBC. The above asymptotic behaviors have been illustrated in Fig. 8 by drawing the corresponding lines through the estimates in the range r ≥ 3. As expected, see also Ref. 36, the power laws apply only for systems with large r.

IV. CONCLUSIONS

To summarize our conclusions, the amplitudes of \( \langle |M| \rangle \) and \( \langle |M| \rangle \) are, in general, different and depend on the boundary conditions. Their dependence on the aspect ratio r can be meaningfully described by the superscaling concepts of Ref. 36, and by estimating the corresponding exponents. These superscaling exponents (z) are certainly different for \( \langle |M| \rangle \) and \( \langle |M| \rangle \), but are independent of the boundary conditions. This universality with respect to the boundary conditions appears to be also valid, as we have showed, for the approach of the cumulants to their limiting values for large r. Our illustrations of the distribution functions allow for a better understanding of the different behaviors of the Binder cumulants and provide an interpretation showing the dominance of the fluctuations of the LFKC and the importance of their shifts for the corresponding order-parameter’s universal distribution functions. Larger fluctuations of the LFKC and their shifts to smaller values induce similar behaviors on the statistically significant part of the Fortuin-Kasteleyn clusters, enhancing the small order-parameter behavior, which is mainly responsible for large deviations of the critical cumulant from the value 2/3 of the ordered phase. A straightforward future challenge emerging from the current work would be the test of the above findings for different lattice geometries and higher dimensions.

The main issue of this work was to explain, by looking

TABLE II: Amplitudes of the power law \( \langle Q \rangle = A_{Q}L^{-\beta/\nu} \) obtained by the schemes detailed in the text using the data of magnetization and the LFKC.

| r   | PBC            | FBC            |
|-----|----------------|----------------|
|     | A|M|           | A|∞|           | A|M|           | A|∞|           |
| 1   | 1.008(2)       | 1.007(2)       | 0.5239(4)       | 0.5031(4)     |
| 4   | 0.7204(1)      | 0.7073(1)      | 0.3550(10)      | 0.3250(50)    |
| 9   | 0.4674(2)      | 0.4251(3)      | 0.2470(60)      | 0.1970(30)    |
| 16  | 0.3436(4)      | 0.2826(4)      | 0.1896(2)       | 0.1335(2)     |
| 25  | 0.2727(2)      | 0.2033(2)      | 0.1530(4)       | 0.0965(1)     |
| 36  | 0.2259(2)      | 0.1542(2)      | 0.1278(2)       | 0.0732(1)     |
| 50  | 0.1912(12)     | 0.1198(3)      | 0.1088(3)       | 0.0567(4)     |
| 64  | 0.1689(9)      | 0.0988(9)      | 0.0962(4)       | 0.0466(2)     |
| 100 | 0.0764(2)      | 0.0697(6)      | 0.0764(8)       | 0.0325(2)     |
at the geometrical sensitivity of the LFKC upon varying the boundary conditions and the aspect ratio, the interesting behavior of critical Binder cumulants of the order parameter. As shown, these features are reflected in the distribution functions of the LFKC and it should be underlined at this point that one aspect of the fundamental achievement in the theory of equilibrium critical phenomena, i.e., the confirmation of universality and the calculation of critical exponents, has been obtained via the pdfs of the main thermodynamic variables of the system at criticality. The use of the universal character of the order-parameter pdf in describing critical properties has been shown to be quite valuable [8, 37–40, 47], has been extended to the study of pure and disordered magnetic systems and is of current interest (for a recent review and update on the topic see Ref. [47]).

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