Extension of the Shirafuji model for Massive Particles with Spin

Sergey Fedoruk\textsuperscript{+a}, Andrzej Frydryszak\textsuperscript{†b}, Jerzy Lukierski \textsuperscript{†c},
Cèsar Miquel-Espanya\textsuperscript{† ‡d}

\textsuperscript{+} Ukrainian Engineering-Pedagogical Academy, Kharkov, Ukraine
\textsuperscript{†} Institute of Theoretical Physics, Wroclaw University, 50-204 Wroclaw, Poland
\textsuperscript{‡} Departamento de Física Teórica and IFIC (Centro Mixto CSIC-UVEG), 46100-Burjassot (Valencia), Spain

Abstract

We extend the Shirafuji model for massless particles with primary spacetime coordinates and composite four-momenta to a model for massive particles with spin and electric charge. The primary variables in the model are the spacetime four-vector, four scalars describing spin and charge degrees of freedom as well as a pair of Weyl spinors. The geometric description proposed in this paper provides an intermediate step between the free purely twistorial model in two-twistor space in which both spacetime and four-momenta vectors are composite, and the standard particle model, where both spacetime and four-momenta vectors are elementary. We quantize the model and find explicitly the first-quantized wavefunctions describing relativistic particles with mass, spin and electric charge. The spacetime coordinates in the model are not commutative; this leads to a wavefunction that depends only on one covariant projection of the spacetime four-vector (covariantized time coordinate) defining plane wave solutions.

1 Introduction

There are known three equivalent ways of describing massless relativistic particles:

\textsuperscript{a}E-mail: fed@postmaster.co.uk
\textsuperscript{b}E-mail: amfry@ift.uni.wroc.pl
\textsuperscript{c}E-mail: lukier@ift.uni.wroc.pl
\textsuperscript{d}E-mail: Cesar.Miquel@ific.uv.es
i) **Purely twistorial description** - with primary twistor variables and composite both spacetime and four-momenta. A free point particle moving in twistor space \( Z^A = (\omega^\alpha, \pi_\beta^A) \in \mathbb{C}^4 \) \((A = 1, \ldots, 4; \alpha, \beta = 1, 2)\) (see e.g. [11]) is described by the action

\[
S_1 = \frac{i}{2} \int d\tau \left[ (\bar{Z}^A Z_A - h.c.) + \lambda (\bar{Z}^A Z_A) \right],
\]

where \( \bar{Z}_A \) denotes the complex conjugation of \( Z^A \), the conformal \( SU(2, 2) \) scalar product is \( \bar{Z}^A Z_A \equiv \bar{Z}_A g^{AB} Z_B \) and the conformal metric \( g^{AB} \) in twistor space is usually chosen to be \( g^{AB} = \begin{pmatrix} o & -iI_2 \\ iI_2 & 0 \end{pmatrix} \).

ii) **Mixed twistorial-spacetime description** - with primary spacetime coordinates and composite four-momenta. The relativistic phase space \( (x^{\alpha\beta}, P_{\alpha\beta}) = (\frac{1}{2}(\sigma_\mu)^{\alpha\beta} x^\mu, \frac{1}{2}(\sigma_\mu)^{\alpha\beta} P_\mu) \) is determined by the basic relations of the Penrose theory\(^1\)

\[
P_{\alpha\beta} = \pi_\alpha \pi_\beta, \quad \omega^\alpha = i x^{\alpha\beta} \pi_\beta, \quad (1.2a) \]

\[
\omega^\alpha = i x^{\alpha\beta} \pi_\beta, \quad (1.2b)
\]

where the constraint \( Z^A \bar{Z}_A = 0 \) is required if we wish \( x^{\alpha\beta} \) to be Hermitian \((i.e. \ x^\mu = (\sigma^\mu)^{\alpha\beta} x^{\alpha\beta} \text{is real})\). In the mixed twistor-spacetime approach we use only the relation (1.2b), and we obtain from (1.1) (modulo divergence term) the Shirafuji model \([2]\) \((\check{a} \equiv \frac{d\alpha}{d\tau})\)

\[
S_1' = \int d\tau \pi_\alpha \pi_\beta \check{x}^{\alpha\beta},
\]

which was extensively used by the Kharkov group (see e.g. [3]).

iii) **Standard geometric description** - with primary relativistic phase space variables (spacetime coordinates and four-momenta). Inserting in (1.3) the relation (1.2a) we obtain the known action for the massless relativistic particle moving in Minkowski space

\[
S'' = \int d\tau \left( P_{\alpha\beta} \check{x}^{\alpha\beta} - e P^2 \right),
\]

\(^1\)We use the following notation. The metric is mostly minus \( \eta_{\mu\nu} = \text{diag}(+---)\). The Weyl two-spinor indices are risen and lowered in the following way \( \varphi^\alpha = c^\alpha_\beta \varphi_\beta \), \( \varphi_\alpha = \varphi^\beta \epsilon_\beta^\alpha \), \( \varphi^\alpha = \epsilon^{\alpha\beta} \varphi_\beta \), \( \varphi_\alpha = \varphi^\beta \epsilon_\beta^\alpha \) where \( c^\alpha_\beta \epsilon_\beta^\gamma = -\delta^\gamma_\alpha \), \( c^\alpha_\beta \epsilon_\beta^\gamma = -\delta^\gamma_\beta \). The algebra for the \( \sigma \)-matrices \( \sigma^{\mu}_{\alpha\beta} = \overline{(\sigma_{\mu})^\alpha_\beta} \) and \( \sigma_{\mu}^{\alpha\beta} = c^{\beta}_\gamma c^{\gamma}_\alpha c^{\alpha\beta} \) is \( \sigma_{\mu\alpha} \sigma_{\nu\beta} + \sigma_{\mu\beta} \sigma_{\nu\alpha} = 2 \eta_{\mu\nu} \delta_{\alpha\beta} \).

\( \sigma_{\mu\alpha} \sigma_{\nu\beta} = 2 c^{\alpha\beta} \epsilon_{\alpha\beta}^\mu \) Also we define \( A_\mu = \sigma_{\mu\alpha} A^{\alpha\beta} \), and therefore \( A_{\alpha\beta} = \frac{1}{2} A_\mu \sigma_{\alpha\beta}^\mu \). \( A^{\alpha\beta} = \frac{1}{2} A_\mu \sigma_{\alpha\beta}^\mu \) for any vector \( A_\mu \).
where the term \( eP^2 \) encodes the constraint \( P^2 = 2P_{\alpha\beta}P^{\alpha\beta} = 0 \) following algebraically from (1.2a). If we eliminate \( P_\mu \), we obtain the standard Lagrangian for a free massive relativistic particle

\[
S''_1 = \frac{1}{2} \int d\tau \frac{1}{e} \dot{x}_\mu \dot{x}^\mu.
\]

(1.5)

The three equivalent models (1.1), (1.3) and (1.4) describe three possible different geometric set-ups in the theory of massless relativistic particles: a purely spinorial (twistorial) framework, a hybrid formulation using simultaneously spinorial and spacetime elementary coordinates, and the description in the standard relativistic phase space (\( x_\mu, P_\mu \)).

The extension of these three geometric levels to the two-twistor sector has been presented recently \([4, 5, 6]\) in terms of the corresponding Liouville one-forms. If we introduce two twistors \((i = 1, 2; A = 1, \ldots, 4)^2\)

\[
Z_{Ai} = (\omega^{\alpha}_i, \bar{\pi}^{\dot{\alpha}}_i),
\]

(1.6)

the free Liouville one-form extending (modulo constraints) the action (1.1) to the two-twistor case is the following

\[
\Theta_2 = \frac{i}{2} \left( \omega^{\alpha}_i d\pi_{\alpha i} + \bar{\pi}^{\dot{\alpha}}_i d\omega^{\dot{\alpha}i} - \text{h.c.} \right).
\]

(1.7)

After using the two-twistor generalization of (1.2a), (1.2b)

\[
P_{\alpha\beta} = \pi^{i}_{\alpha} \bar{\pi}^{i}_{\beta}, \quad \omega^{\alpha}_i = iz^{\alpha\beta} \bar{\pi}^{\dot{\alpha}}_i,
\]

(1.8a)

\[
(1.8b)
\]

where

\[
z^{\alpha\beta} = x^{\alpha\beta} + iy^{\alpha\beta},
\]

(1.9)

one obtains

\[
\Theta'_2 = \pi^{i}_{\alpha} \bar{\pi}^{i}_{\beta} dx^{\alpha\beta} + iy^{\alpha\beta} \left( \pi^{i}_{\alpha} d\bar{\pi}^{i}_{\beta} - \bar{\pi}^{i}_{\beta} d\pi^{i}_{\alpha} \right).
\]

(1.10)

We introduce the new variables

\[
s^{ij} = -2y^{\alpha\beta} \pi^{i}_{\alpha} \bar{\pi}^{j}_{\beta} = (s^j_i),
\]

(1.11)

\[\text{The indices } i = 1, 2 \text{ describe an internal } SU(2) \text{ symmetry. The complex conjugation implies the change from covariant (lower) indices to contravariant (upper) indices.}\]
and define \( f \) and \( \tilde{f} \) satisfying

\[
\begin{align*}
\bar{\pi}^{\dot{\alpha}} \bar{\pi}_i^{\dot{\beta}} &= -\epsilon^{\dot{ij}} f, \\
\pi_\alpha^i \pi_\alpha^j &= -\epsilon_{ij} \tilde{f}, \\
\bar{\pi}_{\dot{\alpha}}^{i} \bar{\pi}_{\dot{\beta}}^i &= \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{f}, \\
\pi_{\alpha}^i \pi_{\alpha}^j &= \epsilon_{\alpha \beta} \bar{f}.
\end{align*}
\] (1.12a-d)

Inverting (1.11)

\[
y^{\dot{\alpha} \dot{\beta}} = -\frac{1}{2|f|^2} s_i^j \pi^{\dot{\alpha}} \pi^{\dot{\beta}},
\] (1.13)

and using (1.8a) one obtains from (1.10)

\[
\Theta'_2 = P_\mu dx^\mu + \frac{i}{2} s_i^j \left[ \frac{1}{f} \pi^\alpha_k d\pi_\alpha^j \epsilon^{ki} + \frac{1}{f} \bar{\pi}^{\dot{\alpha}} d\bar{\pi}^{\dot{\alpha}} \epsilon_{kj} \right].
\] (1.14)

The formula (1.14) determines the two-twistor generalization of the Shirafuji action (1.3). We see that the primary, or equivalently elementary, variables are now the following ones

\[
N = 1 \Rightarrow N = 2
\]

\[
x^{\dot{\alpha} \dot{\beta}}, \pi_\alpha, \bar{\pi}_{\dot{\alpha}} \Rightarrow x^{\dot{\alpha} \dot{\beta}}, \pi_{\alpha}^i, \pi_{\dot{\alpha}}^{\dot{i}}, s_i^j.
\] (1.15)

The particle model described by the Liouville one-form (1.14) provides a framework to describe the mass, spin and electric charge but does not specify their values. We shall introduce further their numerical values by postulating suitable physical constraints. One concludes that the quantum-mechanical solution of the model (1.14) may describe infinite-dimensional higher spin and electric charge multiplets, linked with the field-theoretic formulation of higher spin theories (see e.g. \[7, 8, 9\]).

The plan of our paper is the following:

In Sect. 2 we define our model by its kinematic part following from (1.14) and by adding four physical constraints. We shall describe the model in the corresponding phase space with the enlarged spacetime sector \((Q_L; L = 1, \ldots, 16)\) and the enlarged momenta \((P_L; L = 1, \ldots, 16)^3\)

\[
\begin{align*}
Q_L &= (x^{\dot{\alpha} \dot{\beta}}, \pi_{\alpha}^i, \bar{\pi}_{\dot{\alpha}}^{\dot{i}}, s_i^j), \\
P_L &= (P_{\dot{\alpha} \dot{\beta}}, P_{\alpha}^i, P_{\dot{\alpha}}^{\dot{i}}, P_{(s)}^i j).
\end{align*}
\] (1.16a-b)

We also present the complete classical analysis of the constraints.

\[3\]The subindex \( A \) enumerates the real degrees of freedom.
In Sect. 3 we show how to eliminate all the second class constraints by a non-linear change of the variables \((x^\alpha, P^\alpha_\beta, P^\alpha_\dot{\beta}, \bar{P}^\dot{\alpha}_i, \bar{P}^\dot{\alpha}_{\dot{i}})\), i.e., by choosing a set of suitable coordinates in the phase space (1.16a), (1.16b), and by introducing Dirac brackets.

In Sect. 4 we introduce the first quantization of the model and provide the solution of the first class constraints. In such a way we obtain the wave equations for the massive particles with spin and electric charge. It appears that the wavefunction is determined in the four-momentum space, because in twistor formalism the composite four-momenta are commuting. The composite spacetime coordinates \(x^\mu\), defined by (1.8b) and (1.9), are non-commutative due to the following Poisson bracket [10]

\[
\{x^\mu, x^\nu\} = \frac{1}{m^4} \epsilon_{\mu\nu\rho\tau} P^\rho W^\tau,
\]

where \(W^\tau\) is the Pauli-Lubański four-vector. It follows, however, from (1.17) that

\[
\{P^\mu x_\mu, x^\nu\} = 0,
\]

what implies that our quantum-mechanical wavefunction for the non-vanishing spin case can depend on the projection \(\tilde{\tau} = P^\mu x_\mu m\) of the spacetime four-vector defining covariantized scalar time coordinate.

In Sect. 5 we present a brief outlook.

2 The classical model - analysis of constraints in phase space

2.1 Action, conservation laws and physical constraints

We describe the dynamics of a massive spinning particle by its trajectory in the generalized coordinate space

\[
Q_L(\tau) = (x^\mu(\tau), \pi_{\alpha k}(\tau), \bar{\pi}_k^\dot{\alpha}(\tau), s_{k j}(\tau)),
\]

where \(x^\mu\) is the spacetime vector of position, \(\pi_{\alpha k}, \bar{\pi}_k^\dot{\alpha} = (\pi_{\alpha k})\) \((k, j = 1, 2)\) are two pairs of commuting Weyl spinors and the four quantities \(s_{k j}\), satisfying the reality condition \(s_{k j} = (s_j k)\), are Lorentz scalars. The action derived from (1.14) has the following form \((a = 1, \ldots, 4)\)

\[
S = \int d\tau \mathcal{L} = \int d\tau \left[ P^\mu \dot{x}^\mu + \frac{i}{2} s_{k j} \left( \frac{1}{f} \pi_{\alpha k} \dot{\bar{\pi}}_{\dot{\alpha} j} + \frac{1}{f} \bar{\pi}_k^\dot{\alpha} \dot{\pi}_{\alpha j} \right) + \lambda^a T_a \right],
\]

5
where the $T_\alpha$ are algebraic constraints on the coordinates (2.1) to be specified later, and the quantities

$$P^\alpha = \pi^{\alpha k} \bar{n} A_k; \quad f = \frac{1}{2} \bar{\pi} A_k \pi^k, \quad \bar{f} = \frac{1}{2} \bar{\pi} A_k \pi^k,$$

are bilinear functions of the spinors $\pi_{\alpha k}$ and $\bar{\pi}^k$. From the Lagrangian (2.2) we obtain

$$P_\mu = \frac{\partial L}{\partial \dot{x}_\mu} = \sigma_{\alpha \beta} \bar{\pi} A_k \pi^k \pi^\alpha \bar{\pi} A_k; \quad (2.5a)
$$

$$P^{\alpha j} = \frac{\partial L}{\partial \dot{\pi}_j} = \frac{i}{2f} \pi^{\alpha k} \bar{s}_{k j} \bar{\pi} A_k; \quad (2.5b)
$$

$$P_{(s) k j} = \frac{\partial L}{\partial \dot{\bar{s}}_{k j}} = 0. \quad (2.5c)
$$

The formulae (2.5a)-(2.5c), defining the momenta, give us the following sixteen primary constraints

$$P_\mu - \sigma_{\alpha \beta} \pi^{\alpha k} \pi_\beta = 0, \quad (2.6a)
$$

$$P^{\alpha j} - \frac{i}{2f} \pi^{\alpha k} \bar{s}_{k j} \bar{\pi} A_k \pi^\alpha \bar{\pi} A_k \approx 0, \quad (2.6b)
$$

$$P_{(s) k j} \approx 0. \quad (2.6c)
$$

In order to determine the variables which describe the spin degrees of freedom we calculate the Noether charges $M_{\mu \nu}$ corresponding to the Lorentz symmetries for the action (2.2). We obtain

$$M_{\mu \nu} = \epsilon_{\mu \nu \lambda \rho} P_\nu M_{\lambda \rho} = \frac{i}{2} P^{\alpha k} \pi^\alpha \bar{\pi} A_k \pi^\beta \bar{\pi} A_k + \frac{i}{2} \bar{\pi} A_k \pi^k \bar{\pi} A_k \bar{P}, \quad (2.7)
$$

where $P_\mu$, $P^{\alpha k}$, $\bar{P}$ are taken from (2.5a)-(2.5c) and the definitions for $\sigma_{\mu \nu}$ and $\bar{\sigma}^\mu$ are

$$(\sigma_{\mu \nu})^\alpha \beta \equiv \frac{1}{2} (\sigma_{\alpha \gamma} \sigma^\gamma \beta - \sigma^\alpha \gamma \sigma_{\mu \beta} \gamma), \quad (\bar{\sigma}^\mu)_{\hat{\alpha}} \hat{\beta} \equiv \frac{1}{2} (\sigma_{\mu \gamma} \sigma^\gamma \beta - \sigma_{\mu \gamma} \sigma^\gamma \beta). \quad (2.8)
$$

The Pauli-Lubański four-vector is given by the formula

$$W^\mu = \epsilon_{\mu \nu \lambda \rho} P_\nu M_{\lambda \rho} = \frac{i}{2} P^{\alpha k} \pi^\alpha \bar{\pi} A_k \pi^\beta \bar{\pi} A_k + \frac{i}{2} \bar{\pi} A_k \pi^k \bar{\pi} A_k \bar{P} P_\nu. \quad (2.9)\]
where the relations $\epsilon^{\mu\nu\lambda\rho} \sigma_{\lambda\rho} = -2i\sigma^{\mu\nu}$, $\epsilon^{\mu\nu\lambda\rho} \bar{\sigma}_{\lambda\rho} = 2i\bar{\sigma}^{\mu\nu}$ have been used. Inserting the expressions (2.5a)-(2.5c) in (2.9) we obtain

$$W^\mu = -\frac{1}{2} P^\mu_{\ k} l (s_l^k + \epsilon^{kj} s_j^m \epsilon_{ml}) ,$$

where

$$P^\mu_{\ k} l \equiv \pi^\alpha_{\ k} \sigma^\mu_{\ \alpha\beta} \bar{\pi}^{\beta l}.$$ (2.11)

In fact, only the traceless part of $s_l^k$ is present in (2.10). So, if we insert in (2.10) the decomposition $s_l^k = s_0 \delta_l^k + s_r (\tau_r) l^k$ we obtain

$$W^\mu = -P^\mu_{\ k} l (\tau_r) l^k s_r .$$ (2.12)

Now, using

$$P^\mu_{\ k} l P_{\mu l} = 2 \epsilon_{ki} e^{lj} f \bar{f} ,$$

we obtain

$$W^\mu W_\mu = -4f \bar{f} s^2 .$$ (2.14)

But from (2.3)-(2.4) we can derive

$$4f \bar{f} = P_\mu P^\mu = P^2 ,$$

and one obtains

$$W^\mu W_\mu = -P^2 s^2 .$$ (2.16)

We determine now the form of the four algebraic constraints $T_a$ ($a = 1, 2, 3, 4$) present in the action (2.2).

The variables $\lambda^a(\tau)$ are Lagrange multipliers for the four physical constraints $T_a$. We take the physical constraints in the following form

\begin{align*}
T_1 : & \quad T \equiv 4f \bar{f} - m^2 \approx 0 , \quad (2.17a) \\
T_2 : & \quad S \equiv s^2 - s(s + 1) \approx 0 , \quad (2.17b) \\
T_3 : & \quad S_3 \equiv s_3 - m_3 \approx 0 , \quad (2.17c) \\
T_4 : & \quad Q \equiv s_0 - q \approx 0 . \quad (2.17d)
\end{align*}

The real quantities $s = (s_r) = (s_1, s_2, s_3)$ and $s_0$ which are present in (2.17b)-(2.17d) are defined in terms of the Lagrangian variables $s_l^k$ as follows

$$s_0 = \frac{1}{2} s_l^k , \quad s_r = \frac{1}{2} s_j^l (\tau_r) j^k , \quad r = 1, 2, 3 ,$$

(2.18)

\footnote{The justification of the form of the constraints $T_a$ can be obtained by considering the symmetries of the action (2.2). It appears that the choice (2.17a)-(2.17d) and the interpretation of $s_i$ (see (2.18)) as covariant spin projection is related with the formulae for the corresponding Noether charges.}
where \((\tau^r)_j^k\) are the Pauli matrices.

The constraint (2.17a) defines the mass \(m\) of the particle because using it together with (2.15) we obtain that

\[ P_\mu P^\mu = m^2. \]  

The constraints (2.17b) and (2.17c) are introduced in the action (2.2) in order to obtain a definite spin \(s\) and the covariant spin projection \(s_3\) whereas the constraint (2.17d) defines the \(U(1)\) charge \(q\) of the particle.

In the subsection 2.3 we shall see, from the preservation of the constraints in time, that secondary constraints do not appear in our model. Thus, the full set of constraints is given by the physical constraints (2.17a)-(2.17d) and by the primary ones (2.6a)-(2.6c).

### 2.2 Analysis of the primary constraints

If we transform the twelve constraints (2.6a), (2.6b) to equivalent Lorentz-invariant expressions by contracting them with the spinors \(\pi_\alpha^k\) and \(\bar{\pi}_{\dot{\alpha}}^k\) (the matrices \(\pi_\alpha^k, \bar{\pi}_{\dot{\alpha}}^k\) are invertible due to (1.12a)-(1.12d)) the discussion of the constraints is simplified and their splitting into first and second class is clearer. After such contractions, the eight expressions (2.6b) take the form

\[ \pi_\alpha^k P_{\alpha j} - i \frac{1}{2} s_k^j \approx 0, \quad \bar{P}_{j}^{\dot{\alpha} k} \bar{\pi}_{\dot{\alpha}}^j \approx 0. \]  

By considering the sum and the difference of the expressions above, we obtain the following set of eight constraints

\[ D_k^j \equiv D_k^j + s_k^j \approx 0, \quad B_k^j \equiv B_k^j \approx 0, \]  

where the quantities

\[ D_k^j \equiv i(\pi_\alpha^k P_{\alpha j} - \bar{P}_{k}^{\dot{\alpha} j} \bar{\pi}_{\dot{\alpha}}), \quad B_k^j \equiv i(\pi_\alpha^k P_{\alpha j} + \bar{P}_{k}^{\dot{\alpha} j} \bar{\pi}_{\dot{\alpha}}), \]  

contain only spinorial phase space variables.

The four constraints (2.6a), after contraction with spinors, take the form

\[ C_k^l \equiv \mathcal{P}_k^l + m^2 \delta_k^l \approx 0, \]  

where we take into account (2.15) and (2.3a) and introduce the following notation

\[ \mathcal{P}_k^l \equiv 4 \pi_\alpha^k P^{\alpha \beta} \bar{\pi}_{\dot{\beta}}^l. \]
Note that the spinorial bilinears introduce an orthogonal basis $P_{\mu k}^l$ defined by (2.11) or equivalently

$$P_{\mu}^l = (\tau^{(r)})^k_l P_{\mu k}^l = (\tau^{(r)})^k_l \pi^\alpha_k (\sigma_\mu)^{\alpha\beta} \bar{\pi}^\beta_l,$$  \hspace{1cm} (2.25)

with $P_{\mu l} \equiv P_{\mu}$. Then, the relation (2.23) can be written also as follows

$$P_{k}^l = 2 P^\mu {_{k}^l} P^\mu = -m^2 \delta^l_k.$$  \hspace{1cm} (2.26)

The algebra of the constraints (2.21), (2.23) and (2.6c) becomes more transparent if we introduce the following $SU(2)$ scalar and vector quantities:

$$D_r = \frac{1}{2} P^j_l (\tau_r)^j_k, \quad D_0 = \frac{1}{2} P^k_l; \quad B_r = \frac{1}{2} B^j_l (\tau_r)^j_k, \quad B_0 = \frac{1}{2} B^k_l;$$  \hspace{1cm} (2.27a)

$$P_r = \frac{1}{2} P^j_l (\tau_r)^j_k, \quad P_0 = \frac{1}{2} P^k_l;$$  \hspace{1cm} (2.27c)

$$C_r = \frac{1}{2} C^j_l (\tau_r)^j_k, \quad C_0 = \frac{1}{2} C^k_l;$$  \hspace{1cm} (2.27d)

where $(\tau_r)^j_k$, $r = 1, 2, 3$ are the isospin Pauli matrices. In terms of the variables (2.27a)-(2.27d) and (2.18) the constraints (2.21), (2.23) and (2.6c) take the form

$$R_r \equiv P_{(s)r} \approx 0, \quad R_0 \equiv P_{(s)0} \approx 0,$$  \hspace{1cm} (2.28a)

$$D_r \equiv D_r + s_r \approx 0, \quad D_0 \equiv D_0 + s_0 \approx 0,$$  \hspace{1cm} (2.28b)

$$B_r \equiv B_r \approx 0, \quad B_0 \equiv B_0 \approx 0,$$  \hspace{1cm} (2.28c)

$$C_r \equiv C_r \approx 0, \quad C_0 \equiv C_0 + m^2 \approx 0.$$  \hspace{1cm} (2.28d)

Thus, our full set of the constraints is described now by the four physical constraints (2.17a)-(2.17d) and by the sixteen primary constraints (2.28a)-(2.28d).
We present now the canonical Poisson brackets of the coordinates (2.1) and their momenta (2.5a)-(2.5c)

\[
\{x^\mu, P_\nu\} = \delta^\mu_\nu, \quad \{s^k_j, P_{(s)}^n_l\} = \delta^k_l \delta^j_n, \quad (2.29a)
\]

\[
\{\pi^\mu, P_\nu\} = \delta^\mu_\nu, \quad \{\pi^k_{\alpha}, \vec{P}^\alpha_j\} = \delta^k_j \delta^\alpha_\delta, \quad (2.29b)
\]

\[
\{s_0, P_{(s)0}\} = \frac{1}{2}, \quad \{s_r, P_{(s)q}\} = \frac{1}{2} \delta_{rq}. \quad (2.29c)
\]

These allow us to compute the Poisson brackets between the quantities (2.27b)-(2.27c). The the non-vanishing ones are

\[
\{\mathcal{D}_r, \mathcal{D}_p\} = -\epsilon_{rqp} \mathcal{D}_q, \quad \{\mathcal{D}_r, \mathcal{B}_p\} = -\epsilon_{rqp} \mathcal{B}_q, \quad \{\mathcal{B}_r, \mathcal{B}_p\} = -\epsilon_{rqp} \mathcal{D}_q, \quad \{\mathcal{P}_r, \mathcal{D}_p\} = -\epsilon_{rqp} \mathcal{P}_q, \quad \{\mathcal{P}_r, \mathcal{B}_p\} = i\delta_{rp} \mathcal{P}_0, \quad \{\mathcal{P}_0, \mathcal{B}_r\} = i\mathcal{P}_r, \quad \{\mathcal{P}_r, \mathcal{B}_0\} = i\mathcal{P}_r, \quad \{\mathcal{P}_0, \mathcal{B}_0\} = i\mathcal{P}_0. \quad (2.30a)
\]

From (2.30a) we see that the three quantities \(\mathcal{D}_r\) are the generators of \(SO(3)\) and the three quantities \(\mathcal{B}_r\) extend the \(SO(3)\) algebra to the Lorentz symmetry \(SO(3,1) \simeq sl(2; \mathbb{C})\). Because the generators \(\mathcal{D}_r, \mathcal{B}_r\) are scalars we shall call them internal symmetry generators.

From (2.30b), (2.30c) we see that the quantities \(\mathcal{P}_0, \mathcal{P}_r\) which describe the covariant projections of the four-momentum on the composite four-vectors (2.11) extend the internal Lorentz generators (\(\mathcal{D}_r, \mathcal{B}_r\)) to an internal Poincaré algebra.

Finally, we can write the complete list of non-vanishing Poisson brackets between all twenty constraints in our model (four physical constraints (2.17a)-(2.17d) and sixteen primary ones (2.28a)-(2.28d)):

\[
\{D_r, D_p\} = -\epsilon_{rqp} D_q + \epsilon_{rqp} s_q, \quad (2.31a)
\]

\[
\{D_r, B_p\} = -\epsilon_{rqp} B_q, \quad (2.31b)
\]

\[
\{B_r, B_p\} = -\epsilon_{rqp} D_q + \epsilon_{rqp} s_q, \quad (2.31c)
\]

\[
\{C_r, D_p\} = -\epsilon_{rqp} C_q, \quad (2.31d)
\]

\[
\{C_r, B_p\} = i\delta_{rp} C_0 - i\delta_{rp} m^2, \quad (2.31e)
\]

\[
\{C_r, B_0\} = iC_r, \quad (2.31f)
\]

\[
\{C_0, B_r\} = iC_r, \quad (2.31g)
\]

\[
\{C_0, B_0\} = iC_0 - im^2, \quad (2.31h)
\]
\[ \{T, B_0\} = 2iT + 2im^2, \quad (2.31i) \]
\[ \{D_r, R_p\} = \frac{1}{2} \delta_{rp}, \quad (2.31j) \]
\[ \{D_0, R_0\} = \frac{1}{2}, \quad (2.31k) \]
\[ \{S, R_p\} = s_p, \quad (2.31l) \]
\[ \{S_3, R_3\} = \frac{1}{2}, \quad (2.31m) \]
\[ \{Q, R_0\} = \frac{1}{2}. \quad (2.31n) \]

### 2.3 Time evolution of constraints and their split into first and second class

The action (2.2) is invariant under an arbitrary rescaling on the world line \( \tau \to \tau' = \tau'(\tau) \) and the canonical Hamiltonian vanishes

\[ H = \mathcal{P} \dot{Q}_L - L = 0. \quad (2.32) \]

The total Hamiltonian is given, therefore, by a linear combination of all the constraints

\[ H^C = \lambda_r^{(D)} D_r + \lambda_0^{(D)} D_0 + \lambda_r^{(B)} B_r + \lambda_0^{(B)} B_0 + \lambda_r^{(C)} C_r + \lambda_0^{(C)} C_0 + \]
\[ + \lambda_r^{(R)} R_r + \lambda_0^{(R)} R_0 + \lambda^{(T)} T + \lambda^{(S)} S + \lambda^{(S_3)} S_3 + \lambda^{(Q)} Q. \quad (2.33) \]

Imposing the preservation of all the constraints in time (see Appendix A) we find that four out of twenty Lagrange multipliers are not determined. The Hamiltonian (2.33) takes the following final form

\[ H = \lambda_0^{(C)} F + \lambda^{(S)} S + \lambda^{(S_3)} S_3 + \lambda^{(Q)} Q, \quad (2.34) \]

where

\[ F = C_0 + \frac{1}{2} T \simeq 0, \quad (2.35a) \]
\[ S = S - 2s_r D_r \simeq 0, \quad (2.35b) \]
\[ S_3 = S_3 - D_3 - 2\epsilon_{3rq} s_q R_r \simeq 0, \quad (2.35c) \]
\[ Q = Q - D_0 \simeq 0 \]  
(2.35d)

describe the four first class constraints. The other sixteen constraints can be presented as eight pairs of canonically conjugated second class constraints

\[ D_r \equiv D_r + s_r \approx 0 \quad \Leftrightarrow \quad R_r \equiv P_{(s)r} \approx 0, \quad (2.36a) \]
\[ D_0 \equiv D_0 + s_0 \approx 0 \quad \Leftrightarrow \quad R_0 \equiv P_{(s)0} \approx 0, \quad (2.36b) \]
\[ B_r \equiv B_r \approx 0 \quad \Leftrightarrow \quad C_r \equiv P_r \approx 0, \quad (2.36c) \]
\[ B_0 \equiv B_0 \approx 0 \quad \Leftrightarrow \quad T \approx 0. \quad (2.36d) \]

The subset of constraints \( D_r, B_r \) does not close under the PB operation (see Eqs. (2.31a)-(2.31c)). This can be avoided if we introduce the following linear combination of constraints

\[ D'_r \equiv D_r - \epsilon_{rps} s_p R_q = D_r + s_r - \epsilon_{rps} s_p P_{(s)q} \approx 0, \quad (2.37a) \]
\[ B'_r \equiv B_r + \frac{i}{2m^2} \epsilon_{rps} s_p C_q = B_r + \frac{i}{2m^2} \epsilon_{rps} s_p P_q \approx 0. \quad (2.37b) \]

These have the following Poisson brackets

\[ \{ D'_r, D'_p \} = -\epsilon_{rps} D'_q + \frac{1}{2}(s_r R_p - s_p R_r), \quad (2.38a) \]
\[ \{ D'_r, B'_p \} = -\epsilon_{rps} B'_q + \frac{i}{4m^2}(\delta_{rp} s_q C_q - s_p C_r), \quad (2.38b) \]
\[ \{ B'_r, B'_p \} = -\epsilon_{rps} D'_q - (s_r R_p - s_p R_r) + \frac{1}{m^2} \epsilon_{rps} s_q C_0, \quad (2.38c) \]

which vanish on the surface of the constraints.

### 3 Solving the second class constraints

From the relations (2.31a)-(2.31k) we see that the four pairs of constraints described by Eqs. (2.36a)-(2.36b) satisfy canonical PB, i.e. they have the so-called resolution form\(^5\). Therefore, if we exclude the variables \( s_r, s_0 \) and

\(^5\)A pair of constraints \( A \approx 0, B \approx 0 \) have the resolution form in the phase space \((x_i, p_i)\) \( i = 1, \ldots, N \) if they have the form given by the following formulae:

\[ A = x_1 - f(x_r, p_r) \approx 0 \quad B = p_1 \approx 0 \quad (r = 2, 3, \ldots, N) \quad (3.1) \]

This form of the constraints was considered by Dirac [11]. In such a case the Dirac brackets are identical with the canonical PB.
$P_{(s)r}, P_{(s)0}$ by means of the constraints (2.36c), (2.36d) the Dirac brackets for the remaining variables will coincide with the canonical ones (see Eqs. (2.29a)-(2.29c)). Consequently, in all expressions we should insert

$$s_r = -D_r, \quad s_0 = -D_0, \quad P_{(s)r} = 0, \quad P_{(s)0} = 0. \quad (3.2)$$

In order to treat the three pairs of second class constraints (2.36c) it is convenient to introduce in the phase space $(Q_L, P_L)$ the new variables canonically conjugated to the constraints $P_r$. In such a way the constraints (2.36c) will also have the resolution form, and the introduction of corresponding Dirac brackets will not change the PB relations for the other variables. For this purpose we pass from the twenty-four initial canonical variables (we recall that $s_r, s_0$ and $P_{(s)r}, P_{(s)0}$ are not present due to (3.2))

$$(x^\mu; P_\mu), \quad (\pi_{\alpha k}; P^{\alpha k}), \quad (\bar{\pi}_\alpha^k; \bar{P}_k^\alpha),$$

to the new twenty-four canonical variables

$$(\bar{x}_0, \bar{x}_r; P_0, P_r), \quad (\pi_{\alpha k}'; P^{\alpha k}), \quad (\bar{\pi}_\alpha^k; \bar{P}_k^\alpha),$$

where the variables $P_0, P_r$ are given by the expressions (2.24), (2.27c) describing covariant momentum projections, i.e.

$$P_0 = \pi_{\alpha k} \sigma^{\alpha \beta} \bar{\pi}_\beta^k P_\mu, \quad P_r = (\tau_r)_j^k \pi_{\alpha k} \sigma^{\alpha \beta} \bar{\pi}_\beta^j P_\mu. \quad (3.3)$$

Besides, we take

$$\pi_{\alpha k}' = \pi_{\alpha k}, \quad \bar{\pi}_\alpha^k = \bar{\pi}_\alpha^k. \quad (3.4)$$

and hence we will omit the prime in the transformed spinors. One can check that the new coordinates and spinorial momenta, which satisfy canonical commutation relations

$$\{\bar{x}_0, P_0\} = 1, \quad \{\bar{x}_r, P_q\} = -\delta_{rq}, \quad (3.5a)$$

$$\{\pi_{\alpha k}', P^{\beta j}\} = \delta^{\beta \delta}_j \delta^k_\alpha, \quad \{\bar{\pi}_\alpha^k, \bar{P}_j^\beta\} = \delta_{\delta \delta}^{\beta \delta}_j, \quad (3.5b)$$

are given by the formulae

$$\bar{x}_0 = \frac{1}{2|f|^2} \pi^{\alpha k} x_{\alpha \beta} \bar{\pi}_\beta^k, \quad \bar{x}_r = -\frac{1}{2|f|^2} (\tau_r)_k^j \pi^{\alpha k} x_{\alpha \beta} \bar{\pi}_\beta^j. \quad (3.6)$$

---

6The generating function of this canonical transformation has the form

$$F(P^\mu, \pi_{\alpha k}, \bar{\pi}_\alpha^k; \bar{x}_0, \bar{x}_r, P^{\alpha k}, \bar{P}_k^\alpha) =$$

$$= -[\pi_{\alpha k} \sigma^{\alpha \beta} \bar{\pi}_\beta^k P_\mu] \bar{x}_0 + [(\tau_r)_j^k \pi_{\alpha k} \sigma^{\alpha \beta} \bar{\pi}_\beta^j P_\mu] \bar{x}_r + \pi_{\alpha k} P^{\alpha k} + \bar{\pi}_\alpha^k \bar{P}_k^\alpha.$$  

In this generating function there are encoded the expressions (3.8), (3.13) by $P_0 = -\frac{\partial F}{\partial \bar{x}_0}, P_r = \frac{\partial F}{\partial \bar{x}_r}, \pi_{\alpha k}' = \frac{\partial F}{\partial \bar{P}_k^\alpha}, \bar{\pi}_\alpha^k = \frac{\partial F}{\partial P^{\alpha k}}$. From $x_\mu = -\frac{\partial F}{\partial \pi_\mu}, P^{\alpha k} = \frac{\partial F}{\partial \pi_{\alpha k}}, \bar{P}_k^\alpha = \frac{\partial F}{\partial \bar{\pi}_\alpha^k}$ we obtain the expressions (3.8), (3.17).
\[ P^{\alpha k} = P^{\alpha k} - \frac{2}{f} \pi^{\gamma k} x_{\gamma \beta} P^\beta \alpha, \quad \bar{P}^\alpha_k = \bar{P}^\alpha_k + \frac{2}{f} P^{\alpha \beta} x_{\beta k} \bar{\pi}_k^\gamma. \] (3.7)

We see from (3.6) that the new covariant spacetime coordinates \((\bar{x}_0, \bar{x}_r)\) are described by four covariant projections on the composite four-vectors (2.11). Using Eq. (3.7) we obtain the following useful relations

\[ \pi_{\alpha k} P^{\alpha k} = \pi_{\alpha k} P^{\alpha k} - \bar{x}_0 P_0 + \bar{x}_r P_r, \] (3.8a)

\[ \bar{P}^\alpha_k \bar{\pi}^k_\alpha = \bar{P}^\alpha_k \bar{\pi}^k_\alpha - \bar{x}_0 \bar{P}_0 + \bar{x}_r \bar{P}_r, \] (3.8b)

\[ (\tau_r)^k \pi_{\alpha k} P^{\alpha l} = (\tau_r)^k \pi_{\alpha k} P^{\alpha l} - \bar{x}_0 \bar{P}_r + x_r P_0 - i \epsilon_{rpq} \bar{x}_p \bar{P}_q, \] (3.8c)

\[ (\tau_r)^k \bar{P}^\alpha_k \bar{\pi}^l_\alpha = (\tau_r)^k \bar{P}^\alpha_k \bar{\pi}^l_\alpha - \bar{x}_0 \bar{P}_r + \bar{x}_r \bar{P}_0 + i \epsilon_{rpq} \bar{x}_p \bar{P}_q. \] (3.8d)

In terms of the new variables the expressions (2.22) contain an additional term depending on \(\bar{x}_0, \bar{x}_r, P_0, P_r\), namely

\[ B_0 \rightarrow B_0 - i \bar{x}_0 P_0 + i \bar{x}_r P_r, \quad B_r \rightarrow B_r - i \bar{x}_0 P_r + i \bar{x}_r P_0, \] (3.9a)

\[ D_0 \rightarrow D_0, \quad D_r \rightarrow D_r + \epsilon_{rpq} \bar{x}_p \bar{P}_q, \] (3.9b)

where in the r.h.s. of these relations the terms \(B_0, B_r, D_0, D_r\) are obtained from (2.22) by the replacements \(P^{\alpha k} \rightarrow P^{\alpha k}, \bar{P}^\alpha_k \rightarrow \bar{P}^\alpha_k\).

Thus, in the new variables \((\bar{x}_0, \bar{x}_r, P_0, P_r, \pi_{\alpha i}, \bar{P}^\alpha_i, \bar{\pi}^k_\alpha, \bar{P}^\alpha_k)\) the three \(B_r\) constraints in (2.36c) take the form

\[ B_r = B_r - i \bar{x}_0 P_r + i \bar{x}_r P_0 \approx 0. \] (3.10)

The pairs of constraints (2.36c) satisfy the canonical PB, rescaled by \(P_0\) (see (2.30b)), what is a trivial extension of the resolution form. If we introduce Dirac brackets consistent with the constraints (2.36c) one can exclude the variables \(\bar{x}_r\) and \(P_r\) by setting

\[ \bar{x}_r = - \frac{i}{P_0} B_r, \quad P_r = 0, \] (3.11)

again without any modification of the PB for the remaining variables \((\bar{x}_0, P_0, \pi_{\alpha i}, P_{\alpha i}, \bar{\pi}^k_\alpha, \bar{P}^\alpha_k)\).

The only two remaining second class constraints have the form

\[ T = 4 f \bar{f} - m^2 = 0, \] (3.12a)

\[ B_0 = B_0 - i \bar{x}_0 P_0 = 0. \] (3.12b)
Subsequently, we introduce the Dirac brackets (DB) as follows

\[ \{ y, y' \}_D = \{ y, y' \} + \{ y, B_0 \} \frac{i}{2(T + m^2)} \{ T, y' \} - \{ y, T \} \frac{i}{2(T + m^2)} \{ B_0, y' \}, \]

(3.13)

where \( y, y' \in Y_R = (\tilde{x}_0, \mathcal{P}_0, \pi_{\alpha k}, \bar{\pi}_k, \mathcal{P}_k^\alpha); \ R = 1, 2, \ldots, 18. \) It is easy to check that \( \{ y, T \} \neq 0 \) only if \( y \in (\mathcal{P}^{\alpha k}, \mathcal{P}^\alpha_k), i.e., \) only those canonical PB that include the spinorial momenta are modified. We obtain from (3.13) the following explicit Dirac brackets for the phase variables in \( Y_R \) (we present only the non-vanishing ones):

\[ \{ x_0, \mathcal{P}_0 \}_D = 1, \]

(3.14a)

\[ \{ x_0, \mathcal{P}^\beta_j \}_D = \frac{2f}{m^2} x_0 \pi^\beta_j, \quad \{ x_0, \bar{\mathcal{P}}^\beta_j \}_D = -\frac{2\bar{f}}{m^2} x_0 \bar{\pi}^\beta_j, \]

(3.14b)

\[ \{ \mathcal{P}_0, \mathcal{P}^\beta_j \}_D = -\frac{2f}{m^2} \mathcal{P}_0 \pi^\beta_j, \quad \{ \mathcal{P}_0, \bar{\mathcal{P}}^\beta_j \}_D = \frac{2\bar{f}}{m^2} \mathcal{P}_0 \bar{\pi}^\beta_j, \]

(3.14c)

\[ \{ \pi_{\alpha k}, \mathcal{P}^\beta_j \}_D = \delta^\alpha_\beta \delta^j_k - \frac{f}{m^2} \pi_{\alpha k} \pi^\beta_j, \quad \{ \pi_{\alpha k}, \bar{\mathcal{P}}^\beta_j \}_D = \frac{\bar{f}}{m^2} \pi_{\alpha k} \bar{\pi}^\beta_j, \]

(3.14d)

\[ \{ \bar{\pi}_k^\alpha, \mathcal{P}^\beta_j \}_D = \delta^\alpha_\beta \delta^j_k + \frac{\bar{f}}{m^2} \bar{\pi}_k^\alpha \bar{\pi}^\beta_j, \quad \{ \bar{\pi}_k^\alpha, \bar{\mathcal{P}}^\beta_j \}_D = -\frac{f}{m^2} \bar{\pi}_k^\alpha \bar{\pi}^\beta_j, \]

(3.14e)

\[ \{ \mathcal{P}^{\alpha k}, \mathcal{P}^\beta_j \}_D = -\frac{f}{m^2} (\bar{\pi}^{\alpha k} \mathcal{P}^\beta_j - \pi^\beta_j \mathcal{P}^{\alpha k}), \]

(3.14f)

\[ \{ \bar{\mathcal{P}}_k^\alpha, \bar{\mathcal{P}}_j^\beta \}_D = \frac{\bar{f}}{m^2} (\bar{\pi}_k^\alpha \bar{\pi}_j^\beta - \bar{\pi}_j^\beta \bar{\pi}_k^\alpha), \]

(3.14g)

\[ \{ \mathcal{P}^{\alpha k}, \bar{\mathcal{P}}_j^\beta \}_D = -\frac{f}{m^2} \pi^{\alpha k} \bar{\pi}_j^\beta - \frac{\bar{f}}{m^2} \pi^\beta_j \mathcal{P}^{\alpha k}. \]

(3.14h)

The brackets (3.14a)-(3.14h) are consistent with the second class constraints (3.12a)-(3.12b) i.e. for all the variables \( Y_R \) (\( R = 1 \ldots 18 \)) we have

\[ \{ T, Y_R \}_D = \{ D, Y_R \}_D = 0 \]

(3.15)

We observe that the relation (3.12a) reduces one spinorial degree of freedom, i.e. we are left with seven unconstrained spinorial coordinates.
4 First quantization and solution of the first class constraints

4.1 First class constraints

After taking into account all the sixteen second class constraints (2.36a)-(2.36d) there remain the following eighteen phase space variables

\[ \tilde{x}_0, P_0; \pi_{\alpha k}, P^{\alpha k}; \tilde{\pi}_k^{\dot{\alpha}}, P^\alpha_k, \] (4.1)

which are constrained by two algebraic relations (3.12a), (3.12b) and satisfy the Dirac brackets (3.14a)-(3.14h) (the remaining ones are canonical). After performing the quantization of the canonical Dirac brackets \( \{y, y'\}_D \rightarrow \frac{i}{\hbar}[\hat{y}, \hat{y}'] \) (we put \( \hbar = 1 \)) one obtains the corresponding commutation relations, where we should keep the order of the quantized momenta as it is written in the formulae (3.14f)-(3.14h), i.e. we use the ‘qp-ordering’.

The sixteen independent degrees of freedom described by the variables (4.1) are additionally restricted by the four first class constraints (2.35a)-(2.35d). These, after the use of some identities following from the second class constraints, can be written in the following form

\[ P_0 + m^2 \approx 0, \] (4.2a)

\[ \mathcal{D}_r \mathcal{D}_r - s(s+1) \approx 0, \] (4.2b)

\[ \mathcal{D}_3 + m_3 \approx 0, \] (4.2c)

\[ \mathcal{D}_0 + q \approx 0, \] (4.2d)

where the numerical values of \( m, s, m_3 \) and \( q \) describe mass, spin, spin projection and internal Abelian (electric) charge.

4.2 Covariant solution of the constraints

In order to write the first class constraints (4.2a)-(4.2d) as the wave equations we take the Schrödinger realization of the quantized variables (4.1) on the commuting generalized coordinate space \( (\tilde{x}_0, \pi_{\alpha j}, \tilde{\pi}_j^{\dot{\alpha}}) \). The corresponding generalized momenta \((P_0, P^{\beta j}, \mathcal{P}^{\dot{\beta} j})\) have the following differential realizations:

\[ P_0 = -i \frac{\partial}{\partial \tilde{x}_0}, \] (4.3a)
\[ P^\beta_j = -i \frac{\partial}{\partial \pi_\beta} + i \frac{f}{m^2} \pi^\beta_j \left( \pi_{\alpha k} \frac{\partial}{\partial \pi_{\alpha k}} + \bar{\pi}^k_\alpha \frac{\partial}{\partial \bar{\pi}^k_\alpha} - 2 \bar{x}_0 \frac{\partial}{\partial \bar{x}_0} \right), \quad (4.3b) \]

\[ \bar{P} \dot{\pi}_j = -i \frac{\partial}{\partial \bar{\pi}_j} - i \frac{\bar{f}}{m^2} \pi^\beta_j \left( \pi_{\alpha k} \frac{\partial}{\partial \pi_{\alpha k}} + \bar{\pi}^k_\alpha \frac{\partial}{\partial \bar{\pi}^k_\alpha} - 2 \bar{x}_0 \frac{\partial}{\partial \bar{x}_0} \right). \quad (4.3c) \]

The consistency of our quantization procedure can be obtained a posteriori by checking that the realizations (4.3a)-(4.3c) satisfy the commutation relations obtained by the quantization of Dirac brackets (3.14a)-(3.14h). One can also show that the relations (3.12a)-(3.12b) are satisfied. Subsequently, we obtain the following simple differential realizations of the operators \( D_r, D_0 \) defining the three first class constraints (4.2b)-(4.2d):

\[ D_0 = \frac{1}{2} \left( \pi_{\alpha k} \frac{\partial}{\partial \pi_{\alpha k}} - \bar{\pi}^k_\alpha \frac{\partial}{\partial \bar{\pi}^k_\alpha} \right), \quad (4.4a) \]

\[ D_r = \frac{1}{2} \tau_r j^k \left( \pi_{\alpha k} \frac{\partial}{\partial \pi_{\alpha j}} - \bar{\pi}^j_\alpha \frac{\partial}{\partial \bar{\pi}^j_\alpha} \right), \quad r = 1, 2, 3 \quad (4.4b) \]

The wavefunction has the following coordinate dependence

\[ \Psi = \Psi(\bar{x}_0, \pi_{\alpha k}, \bar{\pi}^k_\alpha). \quad (4.5) \]

Substituting (4.3a), (4.4a) and (4.4b) in (4.2a)-(4.2d) we obtain four generalized wave equations. We shall solve them consecutively:

i) **Mass shell constraint (4.2a).**

The general solution of the constraint (4.2a)

\[ i \frac{\partial}{\partial \bar{x}_0} \Psi(\bar{x}_0, \pi_{\alpha k}, \bar{\pi}^k_\alpha) = m^2 \Psi(\bar{x}_0, \pi_{\alpha k}, \bar{\pi}^k_\alpha), \quad (4.6) \]

is the following

\[ \Psi(\bar{x}_0, \pi_{\alpha k}, \bar{\pi}^k_\alpha) = e^{-im^2 \bar{x}_0} \Phi(\pi_{\alpha k}, \bar{\pi}^k_\alpha). \quad (4.7) \]

Using the expression (3.6) for \( \bar{x}_0 \) and (2.6a) for \( P_\mu \) and the constraint (2.17a) we obtain the following formula for the covariantized time coordinate \( \bar{\tau} = m \bar{x}_0 \)

\[ m^2 \bar{x}_0 \equiv m \bar{\tau} = \frac{m^2}{2|f|^2} \pi_{\alpha k} x_{\alpha \beta} \bar{\pi}^\beta_k = 2 \pi_{\alpha k} x_{\alpha \beta} \bar{\pi}^\beta_k = P^\mu x_\mu. \quad (4.8) \]

Therefore, the exponent in the wavefunction (4.7) has the standard form of a plane wave

\[ \Psi(\bar{x}_0, \pi_{\alpha k}, \bar{\pi}^k_\alpha) = e^{-ix_\mu P^\mu} \Phi(\pi_{\alpha k}, \bar{\pi}^k_\alpha), \quad (4.9) \]
where the four-momentum $P_\mu$ is composite, i.e.

$$P^\mu = \pi^{\alpha k}_\alpha \sigma^\mu_{\alpha \beta} \bar{\pi}^{\beta \dot{\alpha}}_k.$$  \hfill (4.10)

**ii) Normalized spinors and electric charge.**

The eight (real) variables $\pi^{\alpha k}_\alpha, \bar{\pi}^{k \dot{\alpha}}_\dot{\alpha}$ define two variables $f, \bar{f}$ as follows

$$\pi^{\alpha k}_\alpha \pi^{\alpha k}_\alpha = 2 \bar{f}, \quad \bar{\pi}^{k \dot{\alpha}}_\dot{\alpha} \bar{\pi}^{k \dot{\alpha}}_\dot{\alpha} = 2 f$$  \hfill (4.11)

and the remaining six degrees of freedom can be described by the normalized spinors

$$u^{\alpha i}_\alpha = \left( \frac{\bar{f}}{f} \right)^{-1/4} \pi^{\alpha i}_\alpha, \quad \bar{u}^{\dot{\alpha} i}_\dot{\alpha} = \left( \frac{\bar{f}}{f} \right)^{1/4} \bar{\pi}^{\dot{\alpha} i}_\dot{\alpha}. \quad (4.12)$$

Due to the constraint (3.12a) the modulus of $f$ is given by the mass parameter $|f| = \frac{m}{2}$ and the variable $y \in S^1$

$$y \equiv \frac{\bar{f}}{f}, \quad (4.13)$$

defines the phase of $f$ which will be eliminated by the constraint (4.2d).

From the definition (4.12) the variables $u^{\alpha i}_\alpha, \bar{u}^{\dot{\alpha} i}_\dot{\alpha}$ satisfy the relations

$$u^{\alpha k}_\alpha u^{\alpha k}_\alpha = 2m, \quad \bar{u}_{\dot{\alpha} k} \bar{u}^{\dot{\alpha} k} = 2m. \quad (4.14)$$

Three out of six degrees of freedom can be expressed via the formula (1.8a) by the components of the four-momentum vector. One can also add that the variables $m^{-1/2}u^{\alpha i}_\alpha$ form the $SL(2,C)$ matrix of and play the rôle of spinorial Lorentz harmonics (see e.g. [12, 13]).

Let us observe that using the expression (2.3) for $P^{\alpha \dot{\alpha}} = \pi^{k \dot{\alpha}}_\dot{\alpha} \bar{\pi}^{\alpha k}$ we obtain that the spinors $\pi^{\alpha k}_\alpha, \bar{\pi}^{k \dot{\alpha}}_\dot{\alpha}$ satisfy Dirac-type equations with complex mass $2f$

$$P^{\alpha \dot{\alpha}} \pi^{\dot{\alpha} i}_\dot{\alpha} = f \pi^{\dot{\alpha} i}_\dot{\alpha}, \quad P^{\dot{\alpha} \alpha} \pi^{\alpha i}_\alpha = \bar{f} \bar{\pi}^{\dot{\alpha} i}_\dot{\alpha}. \quad (4.15)$$

Substituting (4.12) we obtain from (4.15) the standard Dirac equations with real mass $m$ in two-component (Weyl) form

$$P^{\alpha \dot{\alpha}} \bar{u}^{\dot{\alpha} i}_\dot{\alpha} = \frac{m}{2} u^{\alpha i}_\alpha, \quad P^{\dot{\alpha} \alpha} u^{\alpha i}_\alpha = \frac{m}{2} \bar{u}^{\dot{\alpha} i}_\dot{\alpha}. \quad (4.16)$$

where $m$ is real because of (4.14).

Using the variables $y, u^{\alpha i}_\alpha, \bar{u}^{\dot{\alpha} i}_\dot{\alpha}$ (we recall that $|f| = \frac{m}{2}$) the differential operators (4.4a), (4.4b) take the form

$$\mathcal{D}_0 = 2y \frac{\partial}{\partial y}, \quad (4.17a)$$
The first class constraint \((4.2d)\) looks as follows
\[
(2y \frac{\partial}{\partial y} + q) \Phi_m(y, u_{\alpha k}, \bar{u}_\alpha^k) = 0,
\]
and has the following solution
\[
\Phi_m(y, u_{\alpha k}, \bar{u}_\alpha^k) = y^{-q/2} \tilde{\Phi}_m(u_{\alpha k}, \bar{u}_\alpha^k),
\]
where the function \(\tilde{\Phi}_m(u_{\alpha k}, \bar{u}_\alpha^k)\) depends on the normalized spinors \(u_{\alpha i}\) and \(\bar{u}_i^\alpha\) only.

**iii) Spin description.**

Let us find now the solution of the constraints (4.2b), (4.2c) for the function \(\Phi(u_{\alpha k}, \bar{u}_\alpha^k)\) using the polynomial expansion in spinor variables
\[
\tilde{\Phi}(u_{\alpha k}, \bar{u}_\alpha^k) = \sum_{k,n=0}^{\infty} \frac{1}{k! n!} u_{\alpha_1 \bar{i}_1} \ldots u_{\alpha_k \bar{i}_k} \bar{u}_{\beta_1} \ldots \bar{u}_{\beta_n} \phi^{\alpha_1 \ldots \alpha_k \bar{\beta}_1 \ldots \bar{\beta}_n}_{i_1 \ldots i_k j_1 \ldots j_n}(P_\mu).
\]

The coefficient fields \(\phi^{\alpha_1 \ldots \alpha_k \bar{\beta}_1 \ldots \bar{\beta}_n}_{i_1 \ldots i_k j_1 \ldots j_n}(P_\mu)\) depend on \(P_\mu = \sigma_{\mu \bar{\alpha}} u_{\alpha k} \bar{u}_k^\bar{\alpha} = \sigma_{\mu \bar{\alpha}} P_{\alpha k} \bar{u}_k^\bar{\alpha}\) and one can show that \(D_r P_\mu = D_0 P_\mu = 0\). The wavefunctions \(\phi\) in (4.20) depending on the four-momentum are symmetric in all indices of the same type
\[
\phi^{\alpha_1 \ldots \alpha_k \bar{\beta}_1 \ldots \bar{\beta}_n}_{i_1 \ldots i_k j_1 \ldots j_n}(P_\mu) = \phi^{(\alpha_1 \ldots \alpha_k \bar{\beta}_1 \ldots \bar{\beta}_n)}_{(i_1 \ldots i_k j_1 \ldots j_n)}(P_\mu),
\]
and they do satisfy the following traceless condition
\[
\phi^{\alpha_1 \ldots \alpha_k \bar{\beta}_1 \ldots \bar{\beta}_n}_{i_1 \ldots i_k j_1 \ldots j_n}(P_\mu) = 0.
\]
Using the following identities

$$\frac{1}{2} u_{\gamma k} \frac{\partial}{\partial u_{\gamma l}} u_{\delta l} \frac{\partial}{\partial u_{\delta k}} u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}_{\beta_1}^j \ldots \bar{u}_{\beta_n}^j \phi^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_n i_1 \ldots i_k} \mu =$$

$$= \frac{k(k + 1)}{2} u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}_{\beta_1}^j \ldots \bar{u}_{\beta_n}^j \phi^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_n i_1 \ldots i_k} (P_\mu), \quad (4.24a)$$

$$\frac{1}{2} u_{\gamma l} \frac{\partial}{\partial u_{\gamma k}} u_{\delta k} \frac{\partial}{\partial u_{\delta l}} u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}_{\beta_1}^j \ldots \bar{u}_{\beta_n}^j \phi^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_n i_1 \ldots i_k} (P_\mu) =$$

$$= \frac{n(n + 1)}{2} u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}_{\beta_1}^j \ldots \bar{u}_{\beta_n}^j \phi^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_n i_1 \ldots i_k} (P_\mu), \quad (4.24b)$$

$$\bar{u}_{\gamma l} \frac{\partial}{\partial u_{\gamma k}} u_{\delta k} \frac{\partial}{\partial u_{\delta l}} u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}_{\beta_1}^j \ldots \bar{u}_{\beta_n}^j \phi^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_n i_1 \ldots i_k} (P_\mu) \sim$$

$$\sim \phi^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_n i_1 \ldots i_k} (P_\mu) = 0, \quad (4.24c)$$

$$-\frac{1}{4} \left( u_{\gamma k} \frac{\partial}{\partial u_{\gamma l}} - u_{\delta l} \frac{\partial}{\partial u_{\delta k}} \right)^2 u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}_{\beta_1}^j \ldots \bar{u}_{\beta_n}^j \phi^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_n i_1 \ldots i_k} (P_\mu) =$$

$$= -\frac{(k - n)^2}{4} u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}_{\beta_1}^j \ldots \bar{u}_{\beta_n}^j \phi^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_n i_1 \ldots i_k} (P_\mu), \quad (4.24d)$$

we obtain the action of the operator $\mathcal{D}_r \mathcal{D}_r$ on the polynomials in the expansion [4.20]

$$\mathcal{D}_r \mathcal{D}_r u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}_{\beta_1}^j \ldots \bar{u}_{\beta_n}^j \phi^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_n i_1 \ldots i_k} (P_\mu) =$$

$$= \frac{k + n}{2} (\frac{k + n}{2} + 1) u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}_{\beta_1}^j \ldots \bar{u}_{\beta_n}^j \phi^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_n i_1 \ldots i_k} (P_\mu). \quad (4.25)$$

Thus, the solution of Eq. [4.2b] is

$$\tilde{\phi}(u_{\alpha k}, \bar{u}_{\alpha k}^k) = \sum_{k,n; k + n = 2s} \frac{1}{k! n!} u_{\alpha_1 i_1} \ldots u_{\alpha_k i_k} \bar{u}_{\beta_1}^j \ldots \bar{u}_{\beta_n}^j \phi^{\alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_n i_1 \ldots i_k} (P_\mu), \quad (4.26)$$

where in this expansion only spinorial polynomials of order $2s$ ($k = k_1 + k_2$, $n = n_1 + n_2$) are present,

$$k + n = 2s. \quad (4.27)$$

where $k_i (i = 1, 2)$ denotes the number of spinors $u_{\alpha}^i$, and $n_i (i = 1, 2)$ the number of spinors $\overline{\eta}^i$. 

20
We can also derive the consequences of the relations (4.16) between the spinors $u_{\alpha k}$, $\bar{u}^k_{\dot{\alpha}}$. Inserting these relations in (4.26) we find that the component fields $\phi^{\alpha_1...\beta_1...i_1...i_k}_{j_1...j_n}(P_\mu)$ in expansion (4.26) satisfy the generalized Dirac equations

$$P_{\beta\dot{\beta}}\phi^{\alpha_1...\alpha_k}_{\beta_1...\beta_n}(P_\mu) + \frac{m}{2} \phi^{\alpha_1...\alpha_k}_{\dot{\beta}_1...\dot{\beta}_n}(P_\mu) = 0,$$

(4.28a)

$$P^{\dot{\alpha}\alpha}\phi^{\alpha_1...\alpha_k}_{\dot{\beta}_1...\dot{\beta}_n}(P_\mu) + \frac{m}{2} \phi^{\alpha_2...\alpha_k}_{\dot{\alpha}_1...\dot{\beta}_n}(P_\mu) = 0,$$

(4.28b)

as well as the transversality condition

$$P_{\alpha\dot{\beta}}\phi^{\alpha_1...\alpha_k}_{\beta_1...\beta_n}(P_\mu) = 0.$$  

(4.29)

In order to describe covariant projection of the spin, given by the eigenvalue equation (4.2c), we observe that

$$(D_3 - s_3) (u_{\alpha_1 i} \cdots u_{\alpha_k i} \bar{u}^{\dot{j}_1}_{\dot{\beta}_1} \cdots \bar{u}^{\dot{j}_n}_{\dot{\beta}_n}) = 0,$$

(4.30)

if

$$s_3 = k_1 - k_2 - (n_1 - n_2).$$

(4.31)

5 Examples: $s = \frac{1}{2}$ and $s = 1$

i) Spin $s = 1/2$. In this case the field (4.26) is the following

$$\tilde{\Phi}(u_{\alpha k}, \bar{u}^k_{\dot{\alpha}}) = u_{\alpha i} \phi^{\alpha i} + \bar{u}^{\dot{i}}_{\dot{\alpha}} \phi^{\dot{i}}_{\dot{\alpha}} = -u_{\alpha i} \phi^{\alpha i} + \bar{u}^{\dot{i}}_{\dot{\alpha}} \phi^{\dot{i}}_{\dot{\alpha}}.$$  

(5.1)

Inserting in (5.1) $u^i_{\alpha} = \frac{2}{m} P_{\alpha \dot{\alpha}} \bar{u}^{\dot{i}}_{\dot{\alpha}}$, $\bar{u}^{\dot{i}}_{\dot{\alpha}} = \frac{2}{m} P^{\dot{\alpha} \alpha} u_{\alpha i}$ (see (4.10)) we obtain

$$\tilde{\Phi}(u_{\alpha k}, \bar{u}^k_{\dot{\alpha}}) = -\frac{2}{m} P_{\alpha \dot{\alpha}} \bar{u}^{\dot{i}}_{\dot{\alpha}} \phi^{\alpha i} + \frac{2}{m} P^{\dot{\alpha} \alpha} u_{\alpha i} \phi^{\dot{i}}_{\dot{\alpha}} = -\frac{2}{m} \bar{u}^{\dot{i}}_{\dot{\alpha}} P^{\dot{\alpha} \alpha} \phi^{\alpha i} + \frac{2}{m} P^{\dot{\alpha} \alpha} u_{\alpha i} \phi^{\dot{i}}_{\dot{\alpha}}.$$  

(5.2)

From (5.1) and (5.2) we obtain

$$u_{\alpha i} (\phi^{\alpha i} - \frac{2}{m} P^{\dot{\alpha} \alpha} \phi^{\dot{i}}_{\dot{\alpha}}) + \bar{u}^{\dot{i}}_{\dot{\alpha}} (\phi^{\alpha i} + \frac{2}{m} P^{\dot{\alpha} \alpha} \phi^{\dot{i}}_{\dot{\alpha}}) = 0,$$

7In this section all fields $\phi_{\alpha i}$, $\phi^{\dot{i}}_{\dot{\alpha}}$ etc. depend on the composite four-momenta $P_\mu$ (see (4.10)), what we shall not indicate explicitly.
or equivalently
\[ P_{\alpha\hat{\alpha}} \phi_{\hat{\alpha}i} + \frac{m}{2} \phi_{\alpha i} = 0, \quad P^{\alpha\hat{\alpha}} \phi_{\hat{\alpha}i} + \frac{m}{2} \phi_{\alpha i} = 0. \] (5.3)

Denoting
\[ \phi_{\alpha}^1 = \phi_{\alpha2} \equiv \phi_{\alpha}, \quad \phi_{\hat{\alpha}}^2 = -\phi_{\alpha1} \equiv \chi_{\alpha}, \] (5.4a)
\[ \phi_{\hat{\alpha}1} = -\phi_{\alpha}^2 \equiv \phi_{\hat{\alpha}}, \quad \phi_{\hat{\alpha}2} = \phi_{\alpha}^1 \equiv \chi_{\alpha}, \] (5.4b)

\[(\epsilon^{12} = +1 \text{ in our notation}) \text{ the equations (5.3) are (we remind that in our paper } P_{\alpha\hat{\alpha}} = \frac{1}{2} P_\mu \sigma^\mu_{\alpha\hat{\alpha}}) \]
\[ P_\mu \sigma^\mu_{\alpha\hat{\alpha}} \chi_{\hat{\alpha}} + m \phi_{\alpha} = 0, \quad P_\mu \sigma^{\mu\hat{\alpha}\alpha} \phi_{\alpha} + m \chi_{\hat{\alpha}} = 0, \] (5.5a)
\[ P_\mu \sigma^\mu_{\alpha\hat{\alpha}} \phi_{\hat{\alpha}} - m \chi_{\alpha} = 0, \quad P_\mu \sigma^{\mu\hat{\alpha}\alpha} \chi_{\alpha} - m \phi_{\hat{\alpha}} = 0. \] (5.5b)

If we define
\[ \phi_{i\hat{\alpha}} = \bar{\phi}_{i\hat{\alpha}} = (\bar{\phi}_{\alpha})_{\hat{\alpha}}, \] (5.6)
i.e.,
\[ \phi_{\hat{\alpha}} = \bar{\phi}_{\hat{\alpha}} = (\bar{\phi}_{\alpha}), \quad \chi_{\hat{\alpha}} = \chi_{\hat{\alpha}} = (\chi_{\alpha}), \] (5.7)
once can pass to four-component Dirac spinors \((\bar{\psi} = \psi^\dagger \gamma_0)\)
\[ \psi_1 = \psi \equiv \left( \begin{array}{c} \phi_\alpha \\ \chi_{\hat{\alpha}} \end{array} \right), \quad \psi_2 = \psi^C = C \bar{\psi} \equiv \left( \begin{array}{c} \chi_\alpha \\ -\phi_{\hat{\alpha}} \end{array} \right), \] (5.8)
and use the Dirac matrices \(\gamma_\mu\) in Weyl representation
\[ \gamma_\mu = \left( \begin{array}{cc} 0 & \sigma^\mu_{\alpha\beta} \\ \sigma^{\mu\beta}_{\alpha\hat{\beta}} & 0 \end{array} \right), \quad \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \] (5.9)

where
\[ C \gamma_\mu^T = -\gamma_\mu C \Rightarrow C = \gamma_2 \gamma_0 = \left( \begin{array}{cc} \epsilon_{\alpha\beta} & 0 \\ 0 & -\epsilon_{\hat{\alpha}\hat{\beta}} \end{array} \right), \] (5.10)
and then, the equations (5.5a), (5.5b) take the form
\[ (P_\mu \gamma^\mu + m) \psi_1 = 0, \quad (P_\mu \gamma^\mu + m) \psi_2 = 0, \] (5.11)
where the field \(\psi_1\) describes free relativistic spin \(\frac{1}{2}\) particles, and \(\psi_2\) its charge-conjugated counterpart.
One can note that the two Dirac fields \((5.8)\) form a \(SU(2)\)-pseudo-Majorana spinor (these spinors are in \(D = 1 + 3\) dimension [14])

\[
\psi^i \equiv \left( \frac{\phi^i_\alpha}{\bar{\phi}^{\dot{\alpha}i}} \right),
\]

which satisfies the reality condition

\[
\psi^{iT} C \gamma_5 = \epsilon^{ij} \bar{\psi}_j,
\]

where

\[
\gamma_5 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.
\]

The presence of \(\gamma_5\) in (5.13) reflects the 'pseudo' reality.

\[\text{ii) Spin } s = 1.\]

In this case the field \((4.26)\) is

\[
\tilde{\Phi}(u^{\alpha}_k, \bar{u}^{\dot{\alpha}}_a) = \frac{1}{2} u^{\alpha}_\beta u^{\beta}_j \phi^{\alpha\beta ij} + u^{\alpha}_\beta \bar{u}^{\dot{\alpha}}_\bar{\beta} \phi^{\dot{\alpha}\dot{\beta} i j} + \frac{1}{2} u^{\alpha}_a \bar{u}^{\dot{\alpha}}_a \phi^{\dot{\alpha}\dot{\beta} i j}.
\]

Inserting in this expression \(u^{\alpha}_a = \frac{2}{m} P^{\alpha} u^{\dot{\alpha}}_a, \bar{u}^{\dot{\alpha}}_i = -\frac{2}{m} P^{\alpha} u^{\dot{\alpha}}_i\), we obtain

\[
\tilde{\Phi}(u^{\alpha}_k, \bar{u}^{\dot{\alpha}}_a) = \frac{1}{2} \frac{2}{m} \left( u^{\alpha}_\beta u^{\beta}_j P^{\dot{\alpha}\dot{\beta}} \phi^{\dot{\alpha}\dot{\beta} i j} + u^{\alpha}_\beta \bar{u}^{\dot{\alpha}}_\bar{\beta} \left(-P^{\dot{\alpha}\dot{\beta}} \phi^{\dot{\alpha}\dot{\beta} i}_j + P^{\dot{\alpha}\dot{\beta}} \phi^{\dot{\alpha}\dot{\beta} i}_j \right) - \bar{u}^{\dot{\alpha}}_a \bar{u}^{\dot{\alpha}}_a P^{\dot{\alpha}\dot{\beta}} \phi^{\dot{\alpha}\dot{\beta} i j} \right).
\]

Comparing (5.15) and (5.16) we obtain the following equations

\[
P^{\dot{\alpha}\dot{\beta}} \phi^{\dot{\alpha}\dot{\beta} i j} + \frac{m}{2} \varphi^{\dot{\alpha} ij} = 0, \quad P^{\dot{\alpha}\dot{\beta}} \phi^{\dot{\alpha}\dot{\beta} i j} + \frac{m}{2} \varphi^{\dot{\alpha} ij} = 0,
\]

\[
\frac{1}{2} \left( P^{\dot{\alpha}\dot{\beta}} \phi^{\dot{\alpha}\dot{\beta} i j} + P^{\dot{\alpha}\dot{\beta}} \phi^{\dot{\alpha}\dot{\beta} i j} \right) + \frac{m}{2} \varphi^{\dot{\alpha} ij} = 0.
\]

The antisymmetric parts of equations (5.17) provide the transversality condition for fields \(\phi^{\alpha\beta i j}\)

\[
P^{\dot{\alpha}\dot{\beta}} \phi^{\dot{\alpha}\dot{\beta} i j} = 0.
\]

Using \(P^{\alpha\beta} P^{\dot{\alpha}\dot{\beta}} = \frac{1}{4} m^2 \delta^{\alpha}_{\dot{\alpha}}\) we obtain further

\[
P^{\alpha\beta} \phi^{\alpha\beta i j} + \frac{m}{2} \phi^{\alpha ij} = 0, \quad P^{\dot{\alpha}\dot{\beta}} \phi^{\alpha\beta i j} + \frac{m}{2} \phi^{\alpha ij} = 0.
\]

The equations (5.17a)-(5.19) are Bargman-Wigner equations written in two-spinor notation. One can pass to four-component Dirac spinor notation
if one constructs from the fields $\phi_{\alpha\beta}^{ij}$, $\dot{\phi}_{\alpha}^{\dot{\beta}ij}$, $\phi_{\alpha}^{\dot{\beta}ij}$ and $\dot{\phi}_{\alpha}^{\dot{\beta}ij}$ the following Bargman-Wigner fields

$$\psi_{ab}^{ij} = \left( \begin{array}{c} \phi_{ab}^{ij} \\ \phi_{b}^{\dot{\alpha}ij} \\ \phi_{\alpha}^{\dot{\beta}ij} \\ \phi_{\alpha}^{\dot{\beta}ij} \end{array} \right),$$

(5.20)

with double Dirac indices $a,b = 1,2,3,4$. Since $\phi_{\alpha\beta}^{ij} = \phi_{\beta\alpha}^{ij}$, $\phi_{\alpha}^{\dot{\beta}ij} = \phi_{\beta}^{\dot{\alpha}ij}$ the fields (5.20) are symmetric, $\psi_{ab}^{ij} = \psi_{ba}^{ij}$. Due to the equations (5.17a)-(5.19) the fields (5.20) satisfy the Bargmann-Wigner-Dirac equation for massive spin 1 fields

$$P_{\mu}^{\gamma\mu}b_{\nu}^{ij} + m\psi_{ac}^{ij} = 0.$$  

We obtain Proca fields if we define the fields

$$A_{\mu}^{ij} = \sigma_{\mu}^{\alpha\beta}\phi_{\alpha}^{\dot{\beta}ij} , \quad F_{\mu\nu}^{ij} = m(\sigma_{\mu\nu}^{\alpha\beta}\phi_{\alpha}^{\dot{\beta}ij} + \sigma_{\mu\nu}^{\dot{\alpha}\dot{\beta}}\phi_{\alpha}^{\dot{\beta}ij}).$$ 

(5.21)

Inserting (5.21) into the equations (5.17a)-(5.19) we obtain the Proca equations

$$P_{\mu}^{\gamma\mu}A_{\mu}^{ij} = 0 ,$$ 

(5.22a)

$$P_{\mu}A_{\nu}^{ij} - P_{\nu}A_{\mu}^{ij} = F_{\mu\nu}^{ij} ,$$ 

(5.22b)

$$P_{\mu}^{\gamma\nu}F_{\mu\nu}^{ij} - m^2 A_{\nu}^{ij} = 0 ,$$ 

(5.22c)

as well as the identity

$$P_{[\mu}F_{\nu\lambda]}^{ij} = 0 .$$ 

(5.23)

We obtained three complex fields (internal $SU(2)$-triplet) with spin $s = 1$. On the function (5.15) we should impose the reality condition $\tilde{\Phi} = \bar{\Phi}$ which gives

$$\tilde{\phi}_{\alpha}^{\dot{\beta}ij} = \phi_{\alpha}^{\dot{\beta}ij} = (\phi_{\alpha}^{\alpha\betaij}) , \quad \tilde{\phi}_{\alpha}^{\dot{\beta}i} = (\phi_{\alpha}^{\alpha\dot{\beta}i}).$$ 

(5.24)

The relations (5.22b) can be written down as the following $SU(2)$-Majorana reality conditions

$$\psi^{ij}_{\alpha\dot{\beta}}(C\gamma_{5})_{\alpha}(C\gamma_{5})_{\dot{\beta}} = \epsilon_{k}^{i}e^{i\gamma_{5}}\psi^{klab}_{klab}$$ 

(5.25)

and the fields (5.21) satisfy the reality conditions

$$\bar{A}_{\mu}^{ij} = A_{\mu}^{ij} , \quad (\bar{F}_{\mu\nu}^{ij}) = F_{\mu\nu}^{ij} .$$ 

(5.26)

The relation (5.26) defines three real vector fields and the corresponding three real field strengths.
6 Conclusions

In the present paper we have described a classical and first-quantized model of massive relativistic particles with spin based on a hybrid geometry of phase space, with primary spacetime coordinates $x_\mu$ and composite four-momenta $P_\mu$ expressed in terms of fundamental spinorial variables. These spinorial coordinates describe half of the two-twistor spinorial degrees of freedom. One can say that the employed geometric framework is half way between the purely twistorial and the standard spacetime approaches. We would like to point out that a model for massive particles with spin in an enlarged spacetime derived from two-twistor geometry, with primary both spacetime coordinates and four-momenta $P_\mu$, has been recently described in [4]-[6]. It should be added that the two-twistor degrees of freedom were studied recently by one of the authors and applied to the spacetime description of massive spinning particles [15]-[17]. The difference with our approach here consists in the choice of the primary geometric variables which in [15]-[17] contains, besides two-twistor degrees of freedom, a primary internal $SU(2)$ spinor, called the index spinor [18]. In the present paper all the degrees of freedom describing massive particles with spin and internal charge are derived entirely from the two-twistor geometry.

One of the features of the description of spin in the twistor framework as well as in our model is the use of an orthogonal reference frame in four-momentum space, with three basic four-vectors $P^{(r)}_\mu$ orthogonal to the fourth four-vector $P_\mu$ (see (2.15)). In such a way the relativistic spin in any frame is described by the $SU(2)$ algebra of Lorentz-invariant spin projection operators. Consequently, we describe the state with definite values of spin square $s^2$ and invariant spin projection $s_3$ by a Lorentz-covariant wave function.

In order to quantize the classical system we have introduced a complete set of commuting observables, which determine the generalized coordinates of the wavefunction. In our case the set of commuting generalized coordinates does not contain all the spacetime coordinates, because in our geometric framework they do not commute (see (1.17)). As a result, only the Lorentz-invariant projection $\tilde{x}_0 = x_\mu P^\mu$ can be included into the quantum-mechanical commuting coordinates. In such a way we are allowed to use plane waves $e^{ix_\mu P^\mu}$ as describing the spacetime dependence of the wavefunction. We conclude, therefore, that although in our framework the spacetime coordinates of spinning massive particles are non-commutative, we are able to obtain the standard plane wave solutions.
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A Appendix: Time evolution of the constraints

The equations describing the time evolution of all the constraints are

$$\dot{D}_r = \{H, D_r\} = \epsilon_{pq} \lambda^{(D)}_p (D_q - s_q) + \epsilon_{pq} \lambda^{(B)}_p B_q + \epsilon_{pq} \lambda^{(C)}_p C_q - \frac{1}{2} \lambda^{(R)}_r \approx$$

$$\approx -\epsilon_{pq} \lambda^{(D)}_p s_q - \frac{1}{2} \lambda^{(R)}_r = 0 , \quad (A.1a)$$

$$\dot{D}_0 = \{H, D_0\} = -\frac{1}{2} \lambda^{(R)}_0 = 0 , \quad (A.1b)$$

$$\dot{B}_r = \{H, B_r\} = \epsilon_{pq} \lambda^{(B)}_p (D_q - s_q) + \epsilon_{pq} \lambda^{(D)}_p B_q - i \lambda^{(C)}_r (m^2 - C_0) + \lambda^{(C)}_0 C_r \approx$$

$$\approx -\epsilon_{pq} \lambda^{(B)}_p s_q - im^2 \lambda^{(C)}_r = 0 , \quad (A.1c)$$

$$\dot{B}_0 = \{H, B_0\} = i \lambda^{(C)}_r C_r - i \lambda^{(C)}_0 (m^2 - C_0) + 2i \lambda^{(T)} (m^2 + T) \approx$$

$$\approx -im^2 \lambda^{(C)}_0 + 2im^2 \lambda^{(T)} = 0 , \quad (A.1d)$$

$$\dot{C}_r = \{H, C_r\} = \epsilon_{pq} \lambda^{(D)}_p C_q + i \lambda^{(B)}_r (m^2 - C_0) - i \lambda^{(B)}_0 C_r \approx im^2 \lambda^{(B)}_r = 0 , \quad (A.1e)$$

$$\dot{C}_0 = \{H, C_0\} = -i \lambda^{(B)}_r C_r + i \lambda^{(B)}_0 (m^2 - C_0) \approx im^2 \lambda^{(B)}_0 = 0 , \quad (A.1f)$$

$$\dot{R}_r = \{H, R_r\} = \frac{1}{2} \lambda^{(D)}_r + \lambda^{(S)} s_r + \frac{1}{2} \delta_{r3} \lambda^{(S)} = 0 , \quad (A.1g)$$

$$\dot{R}_0 = \{H, R_0\} = \frac{1}{2} \lambda^{(D)}_0 + \frac{1}{2} \lambda^{(Q)} = 0 , \quad (A.1h)$$

$$\dot{T} = \{H, T\} = -2i \lambda^{(B)}_0 (m^2 + T) \approx -2im^2 \lambda^{(B)}_0 = 0 , \quad (A.1i)$$

$$\dot{S} = \{H, S\} = -\lambda^{(R)}_r s_r = 0 , \quad (A.1j)$$

$$\dot{S}_3 = \{H, S_3\} = -\frac{1}{2} \lambda^{(R)}_3 = 0 , \quad (A.1k)$$

$$\dot{Q} = \{H, Q\} = -\frac{1}{2} \lambda^{(R)}_0 = 0 . \quad (A.1l)$$
One obtains from (A.1a)-(A.1l) the relations which imply the preservation of the constraints in time:

\begin{align}
\lambda_0^{(R)} &= \lambda_r^{(B)} = \lambda_0^{(B)} = \lambda_r^{(C)} = 0, \quad (A.2a) \\
\lambda_3^{(R)} &= 0, \quad (A.2b)
\end{align}

\begin{align}
\lambda_0^{(C)} &= 2\lambda^{(T)}, \quad (A.3a) \\
\lambda_0^{(Q)} &= -\lambda_0^{(D)}, \quad (A.3b) \\
\lambda_r^{(R)} &= -2\epsilon_{rst}\lambda_3^{(D)}s_t, \quad (A.3c) \\
\lambda_r^{(D)} &= -2\lambda^{(S)}s_r - \delta_{3r}\lambda^{(S_3)}. \quad (A.3d)
\end{align}

From (A.3c)-(A.3d) one obtains

\[ \lambda_r^{(R)} = 2\epsilon_{r3t}\lambda^{(S_3)}s_t, \quad (A.4) \]

in consistency with the relation (A.2b).

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