MAPS OF DEGREE 1 AND LUSTERNIK–SCHNIRELMANN CATEGORY

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Abstract. Given a map \( f : M \to N \) of degree 1 of closed manifolds, is it true that the Lusternik–Schnirelmann category of the range of the map is not more than the category of the domain? We discuss this and some related questions.

1. Introduction

Below \( \text{cat} \) denotes the (normalized) Lusternik–Schnirelmann category, \( [2] \).

1.1. Question Let \( M, N, \dim M = \dim N = n \), be two closed connected orientable manifolds, and let \( f : M \to N \) be a map of degree \( \pm 1 \). Is it true that \( \text{cat} M \geq \text{cat} N \)?

There are several reasons to conjecture that the above-mentioned question has the affirmative answer for all maps \( f \) of degree 1. Indeed, informally, the domain of \( f \) is more “massive” than the range of \( f \). For example, it is well known that the induced maps \( f_* : H_* (M) \to H_* (N) \) and \( f_* : \pi_1 (M) \to \pi_1 (N) \) are surjective.

In [18] I proved some results that confirm the conjecture under some suitable hypotheses, and now people speak about the Rudyak conjecture, cf. [2, Open Problem 2.48], [3], (although I prefer to state questions rather than conjectures). To date we do not know any counterexample.

Here we demonstrate more situations when the conjecture holds, and discuss some variations of the conjecture.

2. Preliminaries

2.1. Definition Let \( M, N, \dim M = \dim N = n \), be two closed connected oriented smooth manifolds, and let \( [M] \in H_n (M) = \mathbb{Z} \) and \( [N] \in H_n (N) = \mathbb{Z} \) be the corresponding fundamental classes. Given a map \( f : M \to N \), we define the degree of a map \( f : M \to N \) as the number \( \deg f \in \mathbb{Z} \) such that \( f_* [M] = (\deg f) [N] \).

Let \( E \) be a commutative ring spectrum (cohomology theory).

2.2. Lemma If \( M \) is \( E \)-orientable and \( f : M \to N \) is a map of degree 1 then \( f_* : E_* (M) \to E_* (N) \) is an epimorphism and \( f^* : E^* (N) \to E^* (M) \) is a monomorphism.

Proof. See [17] Theorem V.2.13, cf. also Dyer [6]. \( \square \)

2.3. Remark An analog of Lemma 2.2 holds for cohomology with local coefficients. Let \( A \) be a local coefficient system of abelian groups on \( N \), and let \( f^* (A) \) be the induced coefficient system. Then \( f_* : H_* (M; f^* A) \to H_* (N; A) \) is a split epimorphism and \( f^* : H^* (N; A) \to H^* (M; f^* A) \) is a split monomorphism. The proofs are based on Poincaré duality with local coefficients and the equality \( f_* (f^* x \cup y) = x \cup f_* y \) for \( x \in H^* (N; A), y \in H_* (M; f^* A) \). See e.g. [1].
2.4. Definition Given a CW space $X$, the \textit{Lusternik–Schnirelmann category} $\text{cat } X$ of $X$ is the least integer $m$ such that there exists an open covering $\{A_0, A_1, \ldots, A_m\}$ with each $A_i$ contractible in $X$ (not necessary in itself). If no such covering exist, we put $\text{cat } X = \infty$. Note the inequality $\text{cat } X \leq \dim X$ for $X$ connected.

In future we abbreviate Lusternik–Schnirelmann to LS. A good source for LS theory is [2]. Given a closed smooth manifold $M$ and a smooth function $f : M \to \mathbb{R}$, the number of critical points of $f$ can’t be less than cat $M$, [13] [14]. This result turned out to be the starting point of LS theory. Currently, the LS theory is a broad area of intensive topological research.

Let $X$ be a path connected space. Take a point $x_0 \in X$, set

$$PX = P(X, x_0) = \{\omega \in X^{[0,1]} \mid \omega(0) = x_0\}$$

and consider the fibration $p : PX \to X, p(\omega) = \omega(1)$. Let $p_k = p_k^X : P_k(X) \to X$ be the $k$-fold fiberwise join $PX \ast_X \cdots \ast_X XP \to X$. According to the Ganea–Svarc theorem, [2] [20], the inequality $\text{cat } X < k$ holds if and only if the fibration $p_k : P_k(X) \to X$ has a section. In other words, the number $\text{cat } X$ is the least $k$ such that the fibration $p_{k+1} : P_{k+1}(X) \to X$ has a section.

3. Approximations

Recall (see the Introduction) that we discuss whether the existence of a map $f : M \to N$ of degree 1 implies the inequality $\text{cat } M \geq \text{cat } N$. Here we present two results appearing when we approximate the LS category by the cup-length and Toomer invariant.

Recall the following cup-length estimate of LS category. Let $E$ be a commutative ring spectrum (cohomology theory). The cup-length of $X$ with respect to $E$ is the number

$$\text{cl}_E(X) := \sup\{m \mid u_1 \sim \cdots \sim u_m \neq 0 \text{ where } u_i \in \tilde{E}^*(X)\}.$$ 

The well-known cup-length theorem [2] asserts that $\text{cl}_E(X) \leq \text{cat } X$.

We give another estimate of LS category.

3.1. Definition Define the (cohomological) Toomer invariant

$$e_E^*(X) = \sup\{k \mid \ker(p_k^* : E^*(X)) \to E^*(P_k(X)) \neq 0\}.$$ 

Note the decreasing sequence $\cdots \supset \ker(p_k^*) \supset \ker(p_{k+1}^*) \supset \cdots$. Moreover,

$$e_E^*(X) = \inf\{k \mid \ker(p_k^* : E^*(X)) \to E^*(P_k(X)) = 0\} - 1.$$ 

In the definition of the Toomer invariant, $E$ does not need to be a ring spectrum, it can be an arbitrary spectrum.

3.2. Proposition We have $\text{cl}_E(X) \leq e_E^*(X) \leq \text{cat } X$.

\textbf{Proof.} First, $p_m = p_m^X$ has a section for $m > \text{cat } X$, and so $\ker p_m^* = 0$ for all $m > \text{cat } X$. Hence, $e_E^*(X) \leq \text{cat } X$. Now, we put $\text{cl}_E(X) = l$, $e_E^*(X) = k$ and prove that $l \leq k$. Take $u_1, \ldots, u_l \in \tilde{E}^*(X)$ such that $u_1 \sim \cdots \sim u_l \neq 0$. Then, since $p_{k+1}^*$ is monic, we conclude that $p_{k+1}^*(u_1 \sim \cdots \sim u_l) \neq 0$. In other words,

$$(p_{k+1}^* u_1) \sim \cdots \sim (p_{k+1}^* u_l) \neq 0$$

in $P_{k+1}X$. Hence, $\text{cat } P_{k+1}(X) \geq l$ because of the cup-length theorem. It remains to note that $\text{cat } P_{k+1}(X) \leq k$ for all $k$, see [18] Prop. 1.5(ii)] or [2].
3.3. Proposition Let $M$ be a closed connected $E$-orientable manifold, and let $f : M \to N$ be a map of degree $\pm 1$. Then $cl_E(M) \geq cl_E(N) \geq e^*_E(N)$.

Proof. Put $cl_E(N) = l$ and take $u_1, \ldots, u_l \in \tilde{E}^*(N)$ such that $u_1 \cup \cdots \cup u_l \neq 0$. Then $f^*(u_1 \cup \cdots \cup u_l) \neq 0$ by \ref{2.2}. So, $f^*(u_1) \cup \cdots \cup f^*(u_l) \neq 0$ in $M$, and thus $cl_E(M) \geq l$.

Now, let $e^*_E(M) = k$. Then $(p_1^M)^*$ is a monomorphism for $i > k$. Consider the commutative diagram
\[
\begin{array}{ccc}
P_i M & \xrightarrow{P_i f} & P_i N \\
p_i^M & & \downarrow p_i^N \\
M & \xrightarrow{f} & N.
\end{array}
\]
and note that, by \ref{2.2}, $f^*((p_1^M)^*)$ is monic for $i > k$. Hence, because of commutativity of the diagram, $(p_i^N)^*$ is monic for $i > k$. Thus, $e^*_E(N) \leq k$.

\[\square\]

3.4. Corollary Let $M$ be a closed connected $E$-orientable manifold, and let $f : M \to N$ be a map of degree $\pm 1$.

(i) Suppose that $e^*_E(N) = \text{cat} \, N$. Then $\text{cat} \, M \geq \text{cat} \, N$.

(ii) Suppose that $cl_E(N) = \text{cat} \, N$. Then $\text{cat} \, M \geq \text{cat} \, N$.

Proof. (i) We have $\text{cat} \, M \geq e^*_E(M) \geq e^*_E(N) = \text{cat} \, N$, the second inequality following from Proposition \ref{3.3}.

(ii) Because of (i), it suffices to prove that $e^*_E(N) = \text{cat} \, N$. But this holds since $cl_E(N) \leq e^*_E(N) \leq \text{cat} \, N$.

\[\square\]

4. LOW-DIMENSIONAL MANIFOLDS

We prove that for $n \leq 4$, $\text{cat} \, M^n \geq \text{cat} \, N^n$ provided that there exists a map $f : M \to N$ of degree $1$. The inequality holds trivially for $n = 1$.

The case $n = 2$ is also simple. Denote by $g(X)$ the genus of a closed connected orientable surface $X$. Then $g(M) \geq g(N)$ because of surjectivity of $f_* : H_2(M) \to H_2(N)$, see \ref{2.2} Furthermore, $\text{cat} \, X = 1$ if $g(X) = 0$ ($X = S^2$) and $\text{cat} \, X = 2$ for $g > 1$.

The case $n = 3$ is considered in \cite{16} Corollary 1.3].

The case $n = 4$. First, if $\text{cat} \, N = 4$ then $\text{cat} \, N = \text{cat} \, M$ by \cite{18} Corollary 3.6(ii)].

Next we consider $\text{cat} \, N = 3$ and prove that $\text{cat} \, M \geq 3$. By way of contradiction, assume that $\text{cat} \, M \leq 2$, and hence the group $\pi_1(M)$ is free by \cite{3}. Now, $\pi_1(N)$ is free since $\text{deg} \, f = 1$, see \cite{5}. But then $N$ has a CW decomposition whose 3-skeleton is a wedge of spheres, \cite{3}, and hence $\text{cat} \, N \leq 2$, a contradiction. Finally, the case $\text{cat} \, M = 1 < 2 = \text{cat} \, N$ is impossible for trivial reasons ($M$ should be a homotopy sphere, and therefore $N$ should be a homotopy sphere).

5. SOME EXEMPLIFICATIONS

The following result is a weak version of \cite{18} Theorem 3.6(i)].

5.1. Theorem Let $M, N$ be two smooth closed connected stably parallelizable manifolds, and assume that there exists a map $f : M \to N$ of degree $\pm 1$. If $N$ is $(q - 1)$-connected and $\dim \, N \leq 2q \text{cat} \, N - 4$ then $\text{cat} \, M \geq \text{cat} \, N$.

\[\square\]
5.2. Remark In [18, Theorem 3.6(i)] we use [18, Corollary 3.3(i)] where, in turn, we require \( \dim N \geq 4 \). However, the case \( \dim N \leq 4 \) is covered by Section 4.

5.3. Theorem Let \( T^k \) denote the \( k \)-dimensional torus. Let \( M, N \) be two smooth closed connected stably parallelizable manifolds, and assume that there exists a map \( f : M \to N \) of degree \( \pm 1 \). Then there exists \( k \) such that \( \text{cat}(M \times T^k) \geq \text{cat}(N \times T^k) \).

Proof. Put \( \dim M = n \) and note that \( \text{cat}(T^k \times M) \geq \text{cat}T^k = k \). Now, if \( k \geq n + 4 \) then
\[
2 \text{cat} M - 4 \geq 2k - 4 \geq k + n,
\]
and we are done by Theorem 5.1.

Another example. Consider the exceptional Lie group \( G_2 \). Recall that \( \dim G_2 = 14 \). Note that \( G_2 \) is parallelizable being a Lie group.

5.4. Proposition Let \( M \) be a stably parallelizable 14-dimensional closed manifold that admits a map \( f : M \to G_2 \) of degree \( \pm 1 \). Then \( \dim M \geq \text{cat}G_2 \).

Proof. The group \( G_2 \) is 2-connected and \( \dim G_2 = 4 \), [11]. Now the result follows from Theorem 5.1 with \( N = G_2 \) and \( q = 3 \) because \( 14 = \dim G_2 \leq 2q \text{cat} G_2 - 4 = 20 \).

Let \( SO_n \) denote the special orthogonal group, i.e., the group of the orthogonal \( n \times n \)-matrices of determinant 1. Recall that \( \dim SO_n = n(n - 1)/2 \).

5.5. Theorem Let \( M \) be a closed connected smooth manifold, and let \( f : M \to SO_n \) be a map of degree \( \pm 1 \). Then \( \dim M \geq \text{cat}(SO_n) \) for \( n \leq 9 \).

Proof. We apply Corollary 3.3 for the case \( E = HZ/2 \), i.e., to arbitrary closed connected manifolds. Below cl denotes \( \text{cl}_{HZ/2} \). Because of Corollary 3.3 it suffices to prove that \( \text{cat} SO_n = \text{cl}(SO_n) \) for \( n \leq 9 \). Recall that \( H^*(SO_n; \mathbb{Z}/2) \) is the polynomial algebra on generators \( b_i \) of odd degree \( i < n \), truncated by the relations \( b_i^{p_i} = 0 \) where \( p_i \) is the smallest power of 2 such that \( b_i^{p_i} \) has degree \( \geq n \), [7]. In other words,
\[
H^*(SO_n; \mathbb{Z}/2) = \mathbb{Z}/2[b_1, \ldots , b_k, \ldots ]/(b_1^{p_1}, \ldots , b_k^{p_k}, \ldots )\]
Note that \( p_k = 1 \) for \( 2k - 1 > n \), and so \( H^*(SO_n) \) is really a truncated polynomial ring (not a formal power series ring). Hence,
\[
\text{cl}(SO_n) = (p_1 - 1) + \ldots + (p_k - 1) + \ldots
\]
and the sum on the right is finite because \( p_k = 1 \) for all but finitely many \( k \)’s.

For sake of simplicity, we use the notation \( S_n \) for \( H^*(SO_n; \mathbb{Z}/2) \). We have:
\[
\begin{align*}
\dim SO_3 &= 3, \text{cl}(SO_3) = 3 \text{ because } S_3 = \mathbb{Z}/2[b_1]/(b_1^4). \\
\dim SO_4 &= 6, \text{cl}(SO_4) = 4 \text{ because } S_4 = \mathbb{Z}/2[b_1, b_3]/(b_1^4, b_3^2). \\
\dim SO_5 &= 10, \text{cl}(SO_5) = 8 \text{ because } S_5 = \mathbb{Z}/2[b_1, b_3]/(b_1^8, b_3^2). \\
\dim SO_6 &= 15, \text{cl}(SO_6) = 9 \text{ because } S_6 = \mathbb{Z}/2[b_1, b_3, b_5]/(b_1^8, b_3^2, b_5^2). \\
\dim SO_7 &= 21, \text{cl}(SO_7) = 11 \text{ because } S_7 = \mathbb{Z}/2[b_1, b_3, b_5]/(b_1^8, b_3^2, b_5^2). \\
\dim SO_8 &= 28, \text{cl}(SO_8) = 12 \text{ because } S_8 = \mathbb{Z}/2[b_1, b_3, b_5, b_7]/(b_1^8, b_3^2, b_5^2, b_7^2). \\
\dim SO_9 &= 36, \text{cl}(SO_9) = 20 \text{ because } S_9 = \mathbb{Z}/2[b_1, b_3, b_5, b_7]/(b_1^8, b_3^4, b_5^2, b_7^2).
\end{align*}
\]

The values of \( \text{cat} SO_n, n \leq 9 \) are calculated in [12], see also [10]. Compare these values with the above-noted values of \( \text{cl}(SO_n) \) and see that \( \text{cat} SO_n = \text{cl}(SO_n) \) for \( n \leq 9 \).

5.6. Remarks 1. The anonymous referee noticed that, probably, the method of Theorem 5.5 can also be applied to other Lie groups \( (U_n, SU_n, \text{etc.}) \). This is indeed true, but we do not develop these things here.
2. In above-mentioned Theorem 5.1 we can relax the assumption on $M$ by requiring that the normal bundle of $M$ be stably fiber homotopy trivial, i.e., that $M$ is $S$-orientable where $S$ is the sphere spectrum. However, we can’t provide the same weakening for $N$ because our proof in [18] uses surgeries on $N$.

6. Theme and Variations: Other Numerical Invariants Similar to LS Category

Let $\text{Crit}_X$ denote the minimum number of critical points of a smooth function $f : X \to \mathbb{R}$ on the closed smooth manifold $X$, and let $\text{ballcat}(X)$ be the minimum $m \in \mathbb{N}$ such that there is a covering of $X$ by $m + 1$ smooth open balls. It is known that $\text{cat}_X \leq \text{ballcat}(X) \leq \text{Crit}_M - 1$, see [2]. Note that there are examples with $\text{cat}_M < \text{ballcat}(M)$. Indeed, there are examples of manifolds $M$ such that $\text{cat}_M = \text{cat}(M \setminus \text{pt})$, [9], while $\text{cat}(M \setminus \text{pt}) = \text{cat} M - 1$ whenever $\text{cat} M = \text{ballcat}(M)$. On the other hand, there are no known examples with $\text{ballcat} M + 1 < \text{Crit}_M$.

Now, we can pose an open question whether $\text{ballcat}_M \geq \text{ballcat}_N$ and $\text{Crit}_M \geq \text{Crit}_N$ provided there exists a map $f : M \to N$ of degree 1.

We can also consider the number $\text{Crit}^*(X)$, that is, the minimum number of nongenerate critical points of a smooth function $f : X \to \mathbb{R}$ on the closed smooth manifold $X$. There is a big difference between $\text{Crit} X$ and $\text{Crit}^* X$. For example, if $S_g$ is a surface of genus $g \geq 1$ then $\text{Crit} S_g = 3$ while $\text{Crit}^* S_g = 2g$. So, we can ask if $\text{Crit}^* M \geq \text{Crit}^* N$ provided there exists a map $f : M \to N$ of degree 1. One of the lower bounds of $\text{Crit}^*(X)$ is the sum of Betti numbers $SB(X)$, the inequality $SB(X) \leq \text{Crit}^*(X)$ being a direct corollary of Morse theory, [15], and we can regard $SB(X)$ as an approximation of $\text{Crit}^* X$. Now, if $f : M \to N$ is a map of degree 1 then the inequality $SB(M) \geq SB(N)$ follows from [2].

If $M, N$ are closed simply-connected manifolds of dimension $\geq 6$ and there exists a map $M \to N$ of degree 1 then $\text{Crit}^*(M) \geq \text{Crit}^*(N)$. Indeed, for every Morse function $h : X \to \mathbb{R}$ on a closed connected smooth manifold $X$ we have the Morse inequalities

$$m_\lambda \geq r_\lambda + t_\lambda + t_{\lambda-1}$$

where $m_\lambda$ is the number of critical points of index $\lambda$ of $h$ and $r_\lambda, t_\lambda$ are the rank and the torsion rank of $H_\lambda(X)$, respectively. Now, if $X$ is a closed simply connected manifold with dim $M \geq 6$ then $X$ possesses a Morse function for which the above-mentioned Morse inequalities turn out to be equalities. This is a well-known Smale Theorem [19]. Now, the inequalities $r_\lambda(M) \geq r_\lambda(N)$ and $t_\lambda(M) \geq t_\lambda(N)$ follow from [2], and we are done.

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