STRONGLY LECH-INDEPENDENT IDEALS AND LECH’S CONJECTURE

CHENG MENG

Abstract. We introduce the notion of strongly Lech-independent ideals as a generalization of Lech-independent ideals defined by Lech and Hanes, and use this notion to derive inequalities on multiplicities of ideals. In particular we prove that if \((R, m) \to (S, n)\) is a flat local extension of local rings with \(\dim R = \dim S\), the completion of \(S\) is the completion of a standard graded ring over a field \(k\) with respect to the homogeneous maximal ideal, and the completion of \(mS\) is the completion of a homogeneous ideal, then \(e(R) \leq e(S)\).

1. Introduction

Around 1960, Lech made the following remarkable conjecture on the Hilbert-Samuel multiplicities in \([8]\):

**Conjecture 1.1.** Let \((R, m) \to (S, n)\) be a flat local extension of local rings. Then \(e(R) \leq e(S)\).

As the Hilbert-Samuel multiplicity measures the singularity of a ring, this conjecture roughly means that the singularity of \(R\) is no worse than that of \(S\) if \((R, m) \to (S, n)\) is a flat local extension. This conjecture has now stood for more than sixty years and remains open in most cases. It has been proved in the following cases:

1. \(\dim R \leq 2\) \([8]\);
2. \(S/mS\) is a complete intersection \([8]\);
3. \(R\) is a strict complete intersection \([9]\);
4. \(\dim R = 3\) and \(R\) has equal characteristic \([11]\);
5. \(R\) is a standard graded ring over a perfect field (localized at the homogeneous maximal ideal) \([12]\).

For other results see \([2]\), \([3]\), \([4]\) and \([10]\). In this paper the key concept is a new notion called strongly Lech-independence, which is a natural generalization of Lech-independence introduced in \([9]\) and explored in \([2]\). By definition, an ideal \(I \subset S\) is strongly Lech-independent if for any \(i\), \(I^i/I^{i+1}\) is free over \(S/I\), and a sequence of elements is strongly Lech-independent if it forms a minimal generating set of a strongly Lech-independent ideal. Under strongly Lech-independence assumption, we can calculate the colength of powers of an ideal using the data on the monomials of a minimal generating set of the ideal, thus we can derive inequalities on multiplicities. The main result on multiplicities of ideals is the following particular case of Lech’s conjecture:

\[ e(R) \leq e(S) \]
Theorem (See Theorem 4.7). Let \((R, m) \rightarrow (S, n)\) be a flat local extension of local rings with \(\dim R = \dim S = d\). Suppose the completion of \(S\) is the completion of a standard graded ring over a field \(k\) with respect to the homogeneous maximal ideal, and the completion of \(mS\) is the completion of a homogeneous ideal generated by homogeneous elements of degree \(t_1 \leq t_2 \leq \ldots \leq t_r\). Then \(e(S) \geq e(R) t_1 t_2 \ldots t_{r-d}\).

This theorem will lead to the inequality \(e(S) \geq e(R)\) because we always have \(r \geq d\) and \(t_1 \geq 1\).

We can also derive an inequality of the other direction, that is, we can find an upper bound of \(e(S)\) using strongly Lech-independence condition. For \(f \in S\) where \((S, n)\) is a Noetherian local ring, let \(\text{ord}(f) = t\) if \(f \in n^n \setminus n^{t+1}\) and \(\text{ord}(f) = \infty\) if \(f = 0\). Let \(\bar{v}(x) = \lim_{n \to \infty} \text{ord}(x^n)/n\), then \(\bar{v}\) is a well-defined function from \(S\) to \(\mathbb{R}\) called the asymptotic Samuel function. It is either a nonnegative rational number or \(\infty\). Then we have the following upper bound of \(e(S)\):

**Theorem (See Theorem 4.11).** Let \(I\) be an \(n\)-primary ideal in \(S\) which is strongly Lech-independent. Let \(d = \dim S\). Assume \(I\) is minimally generated by \((x_1, \ldots, x_r)\), \(\bar{v}(x_i) = s_i\) and \(s_1 \leq s_2 \leq \ldots \leq s_r\). Then \(e(S) \leq e(I)/s_1 \ldots s_{d-1} s_d\) and \(s_d < \infty\).

The paper is organized in the following way. In Section 2 we start with the definition of a standard set, along with some basic definitions and properties on the set of monomials in a polynomial ring. In Section 3 we define strongly Lech-independence and expansion property and prove some equivalent conditions. There are also some examples showing the relation between strongly Lech-independence and other notions. In Section 4 we use strongly Lech-independence to analyze the colength of powers of ideals and derive inequalities on multiplicities.

## 2. STANDARD SETS IN A POLYNOMIAL RING

Let \(r\) be a positive integer, \(k\) be a field. Let \(P = k[T_1, \ldots, T_r]\) be a polynomial ring in \(r\) variables where \(T_i\)'s are indeterminates.

**Definition 2.1.** An ideal \(I\) of \(P\) is called a monomial ideal, if \(I\) is generated by monomials. A set of monomials \(\Gamma\) is called a standard set of monomials, or a standard set for short, if \(\Gamma\) is a subset of monomials in \(P\) such that if \(u\) is in \(\Gamma\), then every monomial dividing \(u\) is in \(\Gamma\).

Let \(\text{Mon}(\cdot)\) be the set of all the monomials in a polynomial ring or a monomial ideal. For a standard set \(\Gamma\), let \(\Gamma_i\) be the monomials of degree \(i\) in \(\Gamma\). A standard set is closed under taking factors, hence its complement is closed under taking multiples, which means that the complement is just the set of all monomials in a monomial ideal. Hence we have:

**Proposition 2.2.** \(\Gamma\) is a standard set if and only if for some monomial ideal \(I_{\Gamma}\), \(\text{Mon}(P) \setminus \Gamma = \text{Mon}(I_{\Gamma})\). This builds a bijection between the set of standard sets and the set of monomial ideals in \(P\).

The next proposition shows that some data of the graded ring \(P/I_{\Gamma}\) can be computed explicitly using \(\Gamma\) where \(\Gamma\) is a standard set. The definition of multigraded Hilbert series can be seen in [1].

**Proposition 2.3.** Let \(\Gamma\) be a standard set in a polynomial ring \(P\). Let \(z = (z_1, \ldots, z_r)\). For a monomial \(u = T_1^{a_1} T_2^{a_2} \ldots T_r^{a_r} \in P\), let \(u(z) = z_1^{a_1} z_2^{a_2} \ldots z_r^{a_r}\). The multigraded Hilbert series of \(P/I_{\Gamma}\) is \(HS_{P/I_{\Gamma}}(z) = \sum_{u \in \Gamma} u(z)\). This is a...
power series in variables \(z_1, ..., z_r\). The Hilbert series of \(P/I_\Gamma\) is \(HS_{P/I_\Gamma}(z) = HS_{P/I_\Gamma}(z, z, ..., z)\). The dimension \(d\) of \(P/I_\Gamma\) is the order of \(HS_{P/I_\Gamma}(z)\) at the pole \(z = 1\); the multiplicity of \(P/I_\Gamma\) is \(\lim_{z \to 1} HS_{P/I_\Gamma}(z)(1 - z)^d\).

Sometimes we only care about the standard set \(\Gamma\), not the monomial ideal \(I_\Gamma\). So we make the following convention.

**Definition 3.1.** We say that a sequence of elements of Lech-independence in [9] and generalize it to strongly Lech-independence:

- Lech-independent, if it forms a minimal generating set of an ideal which is Lech-independent; strongly Lech-independent, if it forms a minimal generating set of an ideal which is Lech-independent.

**Proposition 3.2.** The following are equivalent for \(I\).

1. \(I\) is Lech-independent.
2. \(\sum a_i x_i = 0\) be a relation between the minimal generators \(x_i\) of \(I\). Then \(a_i \in I\) for all \(i\).
3. \(\phi\) be a presentation matrix for a minimal presentation of the ideal \(I\) viewed as an \(S\)-module, then \(\phi\) has entries in \(I\).

We have the following equivalent conditions for strongly Lech-independence.

**Proposition 3.3.** The following are equivalent for \(I\).

1. \(I\) is strongly Lech-independent.
(2) \( gr_1(S) \) is free over \( S/I \).
(3) \( gr_1(S) \) is flat over \( S/I \).

**Proof.** It suffices to prove (3) \( \Rightarrow \) (1). If \( gr_1(S) \) is flat over \( S/I \), then for any \( i \), \( I^i/I^{i+1} \) is flat over \( S/I \) because it is a direct summand of \( gr_1(S) \). But it is finitely generated over the local ring \( S/I \), so it is free. So \( I \) is strongly Lech-independent by definition.

We introduce one kind of expansion property for elements in the ring \( S \). For a sequence \( x_1, ..., x_r \) of \( r \) elements in \( S \) and \( u = T_1^{a_1}T_2^{a_2}...T_r^{a_r} \), let \( u(x) = x_1^{a_1}x_2^{a_2}...x_r^{a_r} \in S \). For a monomial ideal \( J \subset P \), let \( J(x) = (u(x), u \in Mon(J)) \). It is an ideal in \( S \).

**Definition 3.4.** We say a map \( \sigma : S/I \to S \) is a **lifting which preserves 0**, or a lifting for short, if \( \sigma(0) = 0 \) and the composition of \( \sigma \) with the natural quotient map \( \pi : S \to S/I \) is the identity map.

Roughly speaking, \( \sigma \) picks a representative for each coset in \( S/I \). We always choose 0 as a representative for simplicity.

**Definition 3.5.** Let \( i < j \) be two positive integers, \( x_1, ..., x_r \) be a sequence of \( r \) elements in \( S \), \( I \) be the ideal \( (x_1, ..., x_r) \), \( \Gamma \) a subset of \( Mon(P) \). Assume \( x_1, ..., x_r \) is a minimal generating set of \( I \). We say \( x_1, ..., x_r \) is \( \Gamma \)-expandable from degree \( i \) to \( j \), if for any lifting \( \sigma : S/I \to S \), every element \( f \in I^i \) has a unique representation

\[
  f = \sum_{u \in \Gamma_i, i \leq k \leq j-1} f_u u(x) \mod I^i,
\]

such that for any \( u, f_u \in \sigma(S/I) \). If \( S \) is complete, we say that \( x_1, ..., x_r \) is \( \Gamma \)-expandable from degree \( i \) to \( \infty \), if for any lifting \( \sigma : S/I \to S \) and every element \( f \in I^i \) there is a unique representation

\[
  f = \sum_{u \in \Gamma_i, i \leq k} f_u u(x)
\]

such that for any \( u, f_u \in \sigma(S/I) \). We say that \( x_1, ..., x_r \) is \( \Gamma \)-expandable if it is expandable from degree 0 to \( \infty \). The two expressions \( f = \sum_{u \in \Gamma_i, i \leq k \leq j-1} f_u u(x) \) modulo \( I^i \) and \( f = \sum_{u \in \Gamma_i, i \leq k} f_u u(x) \) are called the expansion of \( f \) with respect to \( \Gamma \) and the lifting \( \sigma \), or simply the expansion of \( f \) if \( \Gamma \) and \( \sigma \) are clear. We say an ideal is \( \Gamma \)-expandable from degree \( i \) to \( j \) or \( \infty \) if one minimal generating sequence of the ideal is \( \Gamma \)-expandable from degree \( i \) to \( j \) or \( \infty \).

By definition the expansion property depends on the choice of the minimal generators and the order. When we say “an ideal \( I \) is \( \Gamma \)-expandable” without pointing out a minimal generating sequence of \( I \) which is \( \Gamma \)-expandable, we implicitly choose such a sequence and in this case the notation \( u(x), u \in \Gamma \) will make sense. Also when we say \( x_1, ..., x_r \) is \( \Gamma \)-expandable for \( \Gamma \subset Mon(P) \), we always assume that the length of the sequence \( r \) is equal to the number of variables in \( P \).

For the consistency of the notation, we denote \( I^\infty = 0 \). Note that we always assume \( S \) is complete when we talk about “\( \Gamma \)-expandable from degree \( i \) to \( \infty \)”.

**Remark 3.6.** Suppose \( x_1, ..., x_r \) is \( \Gamma \)-expandable from degree \( i \) to \( j \), \( I = (x_1, ..., x_r) \), and take \( f, g \in I^i \) such that \( f - g \in P \). Then let \( f = \sum_{u \in \Gamma_i, i \leq k \leq j-1} f_u u(x) \mod I^i \) be the unique expansion, we have \( g = \sum_{u \in \Gamma_i, i \leq k \leq j-1} f_u u(x) \mod I^i \), so the unique expansion of \( f \) and \( g \) are the same, that is, it only depends on the coset \( f + P \).
Strongly Lech-independence can be described using the expansion property. We start with two lemmas:

**Lemma 3.7.** Let \( i_1, i_2 \) be positive integers, and \( i_3 \) is either a positive integer or \( \infty \) such that \( i_1 < i_2 < i_3 \). Consider 3 conditions on a sequence \( x_1, \ldots, x_r \).

1. \( x_1, \ldots, x_r \) is \( \Gamma \)-expandable from degree \( i_1 \) to \( i_2 \)
2. \( x_1, \ldots, x_r \) is \( \Gamma \)-expandable from degree \( i_1 \) to \( i_3 \)
3. \( x_1, \ldots, x_r \) is \( \Gamma \)-expandable from degree \( i_2 \) to \( i_3 \)

Then two of them imply the third one.

**Proof.** Let \( I = (x_1, \ldots, x_r) \). Obviously \( u \in \Gamma_k \) implies \( u(x) \in I^k \).

Assume (1) and (2) are true, then for any \( f \in I^{i_2} \subset I^{i_1} \), by (2) we have

\[
 f = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_3 - 1} f_u u(x) \text{ modulo } I^{i_3}.
\]

Let

\[
 f' = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f_u u(x),
\]

then \( f' = f = 0 \) modulo \( I^{i_2} \). By (1) the unique expansion of \( f' \) modulo \( I^{i_2} \) exists and it must be 0. So \( f_u = 0 \) for all \( u \in \Gamma_k, i_1 \leq k \leq i_2 - 1 \) and hence we have

\[
 f = \sum_{u \in \Gamma_k, i_2 \leq k \leq i_3 - 1} f_u u(x).
\]

This shows the existence. The uniqueness just follows from (2) because an expansion from degree \( i_2 \) to \( i_3 \) can be viewed as an expansion from degree \( i_1 \) to \( i_3 \) by adding 0’s.

Assume (1) and (3) are true. Let \( f \in I^{i_1} \), then by (1)

\[
 f = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f_u u(x) + g,
\]

where \( g \in I^{i_2} \). By (3),

\[
 g = \sum_{u \in \Gamma_k, i_2 \leq k \leq i_3 - 1} g_u u(x) + h,
\]

where \( h \in I^{i_3} \). Thus

\[
 f = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f_u u(x) + \sum_{u \in \Gamma_k, i_2 \leq k \leq i_3 - 1} g_u u(x) + h
\]

is a representation of \( f \). This shows the existence. For uniqueness, let

\[
 f' = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f'_u u(x) + \sum_{u \in \Gamma_k, i_2 \leq k \leq i_3 - 1} g'_u u(x)
\]

be another representation of \( f \) modulo \( I^{i_3} \). Then

\[
 f = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f_u u(x) + \sum_{u \in \Gamma_k, i_2 \leq k \leq i_3 - 1} g_u u(x)
\]

\[
 = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f'_u u(x) + \sum_{u \in \Gamma_k, i_2 \leq k \leq i_3 - 1} g'_u u(x) \text{ modulo } I^{i_3}.
\]

So

\[
 f = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f_u u(x) = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f'_u u(x) \text{ modulo } I^{i_2}.
\]
Hence by (1), \( f_u = f_u' \) for any \( u \in \Gamma_k, i_1 \leq k \leq i_2 - 1 \). Cancelling these terms, we get
\[
\sum_{u \in \Gamma_k, i_2 \leq k \leq i_1 - 1} g_u u(x) = \sum_{u \in \Gamma_k, i_2 \leq k \leq i_1 - 1} g'_u u(x) \mod I^{i_3}.
\]
By (3) \( g_u = g'_u \), which proves the uniqueness.

Assume (2) and (3) are true. Then for any \( f \in I^{i_1} \), by (2)
\[
f = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f_u u(x) \mod I^{i_3}.
\]
Then
\[
f = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f'_u u(x) \mod I^{i_2},
\]
so the representation exists. Suppose there is another expression
\[
f = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f'_u u(x) + g, g \in I^{i_2}.
\]
Then by (3)
\[
g = \sum_{u \in \Gamma_k, i_2 \leq k \leq i_3 - 1} g_u u(x) \mod I^{i_3}.
\]
So
\[
f = \sum_{u \in \Gamma_k, i_1 \leq k \leq i_2 - 1} f'_u u(x) + \sum_{u \in \Gamma_k, i_2 \leq k \leq i_3 - 1} g_u u(x) \mod I^{i_3}.
\]
Hence \( f'_u = f_u \) for any \( u \in \Gamma_k, i_1 \leq k \leq i_2 - 1 \) by the uniqueness of (2), so the uniqueness of (1) is proved. \( \square \)

**Lemma 3.8.** Assume \( S \) is complete. Let \( i \) be an integer. Let \( i'_1 < i'_2 < \ldots \) be a sequence of integers going to infinity and assume that \( i < i'_1 \). Suppose \( x_1, \ldots, x_r \) is \( \Gamma \)-expandable from degree \( i \) to \( i'_j \) for any \( j \). Then \( x_1, \ldots, x_r \) is \( \Gamma \)-expandable from degree \( i \) to \( \infty \).

**Proof.** Let \( I = (x_1, \ldots, x_r) \) and take \( f \in I^i \). Let
\[
f = \sum_{u \in \Gamma_k, i \leq k \leq i'_j - 1} f_{j,u} u(x) + g_j, g_j \in I^{i'}.
\]
Suppose \( j < j' \). Then
\[
\sum_{u \in \Gamma_k, i \leq k \leq i'_j - 1} f_{j,u} u(x) + g_j = \sum_{u \in \Gamma_k, i \leq k \leq i'_j - 1} f_{j',u} u(x) + g_{j'},
\]
so
\[
\sum_{u \in \Gamma_k, i \leq k \leq i'_j - 1} f_{j,u} u(x) = \sum_{u \in \Gamma_k, i \leq k \leq i'_j - 1} f_{j',u} u(x) \mod I^{i'_j}.
\]
By the uniqueness of the representation, \( f_{j,u} = f_{j',u} \) for any \( j, j', u \). So for any \( u \), \( f_{j,u} \) is independent of the choice of \( j \) so we can denote it by \( f_u \). The expression \( \sum_{u \in \Gamma_k, i \leq k < \infty} f_u u(x) \) makes sense because the ring is complete. We have \( f - \sum_{u \in \Gamma_k, i \leq k < \infty} f_u u(x) \in I^{i'_j} \) for any \( j \), so it is 0. Therefore,
\[
f = \sum_{u \in \Gamma_k, i \leq k < \infty} f_u u(x)
\]
is a representation of \( f \). The uniqueness can be proved modulo \( I^{i'_j} \) for any \( j \). \( \square \)
The previous two lemmas lead to the following proposition which characterizes strongly Lech-independence.

**Proposition 3.9.** The following are equivalent.

(1) $I$ is strongly Lech-independent.

(2) For every minimal generating sequence $x_1, \ldots, x_r$ of $I$ there is a standard subset $\Gamma$ of $\text{Mon}(P)$ such that $\Gamma / I^{i+1}$ is free over $S/I$ with basis $u(x)$, with $u \in \Gamma_i$.

(3) For every minimal generating sequence $x_1, \ldots, x_r$ of $I$ there is a standard subset $\Gamma$ of $\text{Mon}(P)$ such that for any $i$, $x_1, \ldots, x_r$ is $\Gamma$-expandable from degree $i$ to $i+1$.

(4) For every minimal generating sequence $x_1, \ldots, x_r$ of $I$ there is a standard subset $\Gamma$ of $\text{Mon}(P)$ such that for any $i$, $x_1, \ldots, x_r$ is $\Gamma$-expandable from degree $i$ to $j$.

(5) For every minimal generating sequence $x_1, \ldots, x_r$ of $I$ there is a standard subset $\Gamma$ of $\text{Mon}(P)$ such that for any $i$, $x_1, \ldots, x_r$ is $\Gamma$-expandable from degree $i$ to $\infty$.

**Proof.** (1) implies (2): Let $I = (x_1, \ldots, x_r)$. Since $I / I^{i+1}$ is free, the preimage of a $k$-basis of $I / I^{i+1} \otimes_S S/I$ forms an $S/I$-basis of $I / I^{i+1}$. Consider the special fibre ring $F_I(S) = gr_I(S) \otimes_S S/I$, then it is standard graded over the field $S/I$. We may write $F_I(S) = k[T_1, \ldots, T_r] / J$ for some homogeneous ideal $J$ such that the image of $x_i$ is $T_i + J$ for $1 \leq i \leq r$. Let $\Gamma = \text{Mon}(k[T_1, \ldots, T_r]) \setminus \text{Mon}(\text{in}(J))$, where the initial is taken with respect to any term order which is a refinement of the partial order given by the total degree. Then by the basic propositions of the initial ideal in $\mathfrak{I}$, the monomials in $\Gamma_i$ is a $k$-basis of $I / I^{i+1} \otimes_S S/I$. So taking the preimage, we know that $u(x), u \in \Gamma_i$ is an $S/I$-basis of $I / I^{i+1}$.

(2) implies (1): trivial.

(2) implies (3): Suppose (2) is true. Let $f \in I^i$. Since $I^i / I^{i+1}$ is generated by $u(x), u \in \Gamma_i$, $f + I^{i+1} = \sum_{u \in \Gamma_i} f_u u(x) + I^{i+1}$. So $f = \sum_{u \in \Gamma_i} f_u u(x) + g$, $g \in I^{i+1}$. If there is another representation $\sum_{u \in \Gamma_i} f'_u u(x) + g', g' \in I^{i+1}$, then in $I^i / I^{i+1}$ we have that $\sum_{u \in \Gamma_i} f'_u u(x) = \sum_{u \in \Gamma_i} f'_u u(x)$. But $u(x), u \in \Gamma_i$ is an $S/I$-basis, so $f_u = f'_u$ modulo $I$. But $f_u, f'_u \in \sigma(S/I)$. So $f_u = \sigma(f_u + I) = \sigma(f'_u + I) = f'_u$. This proves (3).

(3) implies (2): Suppose (3) is true. By the existence and the uniqueness of the representation of every element in $I$ modulo $I^{i+1}$, we know that $I^i / I^{i+1}$ is free over $S/I$ with basis $u(x)$, with $u \in \Gamma_i$.

(3) implies (4): use Lemma 3.7 and induct on $j - i$.

(4) implies (3): trivial.

(4) implies (5): use Lemma 3.8.

(5) implies (4): use Lemma 3.7 for $i_3 = \infty$. $\square$

**Remark 3.10.** Let $I$ be a strongly Lech-independent ideal. By Proposition 3.9 $I$ is $\Gamma$-expandable for some $\Gamma$. So it makes sense to talk about the expansion with respect to such $\Gamma$ and a lifting $\sigma$.

Such $\Gamma$ here for which $I$ is expandable is not unique, but $|\Gamma| = \text{rank}_{S/I} I^i / I^{i+1}$ is independent of the choice of $\Gamma$, which means that $\dim(\Gamma)$ and $e(\Gamma)$ are independent of the choice of $\Gamma$. More precisely, we have:

**Proposition 3.11.** Let $I$ be a strongly Lech-independent ideal of a local ring $(S, n)$. Then $\dim(\Gamma)$ and $e(\Gamma)$ are independent of the choice of $\Gamma$ whenever $I$ is $\Gamma$-expandable from degree $i$ to $j$ for any $i < j$. If moreover $S/I$ is Artinian, then $\dim(\Gamma) = \dim S$ and $e(I) = l(S/I)e(\Gamma)$. In particular, if $I$ is the maximal ideal $n$, then $e(\Gamma) = e(S)$.\[\]
Proposition 3.13. Let I be an ideal in S such that I is Γ-expandable for some Γ. Then T₁, ..., T_r ∈ Γ.

Proof. Let x₁, ..., x_r be a sequence of minimal generators of I which is Γ-expandable, then they also form a set of minimal generators of I/I². Suppose T_i ∉ Γ. Since Γ is a standard set, it only contains monomials not involving T_i, so expanding x_i uniquely we get y + z where y ∈ σ(S/I) and z ∈ (x₁, ..., x_{i-1}, x_{i+1}, ..., x_r). Since y = x_i - z ∈ I, y = 0. So x_i ∈ (x₁, ..., x_{i-1}, x_{i+1}, ..., x_r) which is a contradiction because x_i is a minimal generator.

The following proposition and corollary on Lech-independence are taken from [9] by Lech.

Proposition 3.13. Let x₁, x₂, ..., x_r be Lech-independent in S and I = (x₁, x₂, ..., x_r). Suppose x₁ = yy'. Then:
(1) y, x₂, ..., x_r is Lech-independent.
(2) I : y = (y', x₂, ..., x_r).
(3) There is an exact sequence 0 → S/(y', x₂, ..., x_r) → S/I → S/(y, x₂, ..., x_r) → 0.
(4) If I is n-primary, then l(S/I) = l(S/(y, x₂, ..., x_r)) + l(S/(y', x₂, ..., x_r)).

Corollary 3.14. Let x₁, x₂, ..., x_r be elements of S and a₁, ..., a_r be positive integers. Suppose x₁^{a₁}, x₂^{a₂}, ..., x_r^{a_r} is Lech-independent. Then so is x₁, ..., x_r.

There is an analogue of Corollary 3.14 for the expansion property.

Definition 3.15. Let Γ be a standard set. Let a = (a₁, ..., a_r) be a set of positive integers. Let Γ' be the following set of monomials {u(x₁^{a₁}, ..., x_r^{a_r})x₁^{b₁}x₂^{b₂}...x_r^{b_r}|u ∈ Γ, 0 ≤ bᵢ < aᵢ}. Then Γ' is a standard set. We denote Γ' = aΓ.
Remark 3.16. This multiplication on the set of standard sets can be derived
from an action on the monomial ideals. Actually, let $\phi_a$ be an endomorphism
of $P$ which sends $T_i$ to $T_{i+1}$, then $\phi_a$ maps a monomial to a monomial,
but it extends to monomial ideals. Now the multiplication satisfies $I_1I_2 =
\phi_a(I_1)P$. Since the set of actions $\phi_a \in N^r$ is a commutative and
associative monoid, the action of $N^r$ on the set of standard sets is commutative
and associative.

Using the notation above, we have the following proposition:

Proposition 3.17. Let $x_1, x_2, \ldots, x_r$ be elements of $S$ and $a_1, \ldots, a_r$ be positive
integers. Suppose $x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}$ is $\Gamma$-expandable and Lech-independent. Let
$\mathbf{a} = (a_1, \ldots, a_r)$, then $x_1, \ldots, x_r$ is $\mathbf{a}$-$\Gamma$-expandable.

Proof. Let $I = (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$ and $J = (x_1, x_2, \ldots, x_r)$. For any lifting
$\sigma : S/J \to S$, we associate a lifting $\sigma' : S/I \to S$: by Lemma 3.18 below every element $f \in S$
has a unique expression $f = \sum_{u \in \text{Mon}(P)} f_u(x)$ modulo $I$ such that $f_u \in \sigma(S/J)$
for any $u$. Let $\sigma'(f) = \sum_{u \in \text{Mon}(P)} f_u(x)$. The image of $\sigma'$ only depends on the coset $f + I$ and it is a lifting $\sigma' : S/I \to S$. Now $x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}$ is $\Gamma$-expandable, so every element $f \in S$ can be expand uniquely as

$$
\sum_{v \in \Gamma} g_v(x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}) = \sum_{v \in \Gamma, u \in \text{Mon}(P) \setminus \text{Mon}((T_1^{a_1}, \ldots, T_r^{a_r}))} g_{uv}(x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})
$$

where $g_v \in \sigma'(S/I)$, $g_{uv} \in \sigma(S/J)$. As $u$ ranges over $\text{Mon}(P) \setminus \text{Mon}((T_1^{a_1}, \ldots, T_r^{a_r}))$ and $v$ ranges over $\Gamma$, $\sigma'(u(x)v(x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}))$ ranges over $u(x), u \in 2\Gamma$, so we are done. \qed

Lemma 3.18. Let $x_1, x_2, \ldots, x_r$ be elements of $S$ and $a_1, \ldots, a_r$ be positive
integers. Suppose $x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}$ is Lech-independent. Let $P = k[T_1, T_2, \ldots, T_r]$, $J = (T_1^{a_1}, \ldots, T_r^{a_r})$, $l = l(P/J) = a_1a_2\ldots a_r$, $I = (x_1, x_2, \ldots, x_r)$. Then the following holds:
(1) Every prime filtration of $P/J$ given by $J = J_l \subset J_{l-1} \subset \ldots \subset J_0 = P$ such that $J_l/J_{l+1} \cong k$ for any $l$.
(2) There exists one prime filtration $\mathcal{F}$ of $P/J$ given by $J_i$ such that every $J_i$ is monomial and $J_i(x)/J_i+1(x) \cong S/I'$.
(3) Suppose $\mathcal{F}_0$ is a prime filtration of $P/J$ given by monomial ideals $J_i$, then there is a one-to-one correspondence between $J_i, 0 \leq i \leq l-1$ and $\text{Mon}(P) \setminus \text{Mon}(J)$ which maps $J_i$ to the monomial generator of $J_i/J_{i+1}$. Denote this map by $M_{\times,\mathcal{F}_0} : \{0, 1, 2, \ldots, l-1\} \to \text{Mon}(P)$.
(4) For any lifting $\sigma : S/I' \to S$ and $f \in S$ there is a unique expansion modulo $I$, that is, an equation of the form

$$
f = \sum_{u \in \text{Mon}(P) \setminus \text{Mon}(J)} f_u(x) \text{ modulo } I
$$

such that $f_u \in \sigma(S/I)$.
(5) For any prime filtration $\mathcal{G}$ of $J$ given by monomial ideals $J_i$, $J_i(x)/J_{i+1}(x) \cong S/I'$.

Proof. (1) The prime filtration always exists for ideals in a Noetherian ring. Since $J$
is $(T_1, \ldots, T_r)$-primary and $(T_1, \ldots, T_r)$ is maximal, every factor is $P/(T_1, \ldots, T_r) \cong k$.
The length is $l$ by the definition of length.
(2) Applying Proposition 3.13 inductively we know the following proposition: Let $x_1, x_2, \ldots, x_r, x_{r+1}^r$ be Lech-independent, then there exists a filtration of the quotient ring $S/(x_1, x_2, \ldots, x_r, x_{r+1})$ given by ideals $((x_1, x_2, \ldots, x_r, x_{r+1}))$, $0 \leq i \leq r$ and $((x_1, x_2, \ldots, x_{r+1}))$ given by ideals $((x_1, x_2, \ldots, x_{r+1}, x_{r+1}^r)) \cong S/((x_1, x_2, \ldots, x_r, x_{r+1}))$. So if $x_1, x_2, \ldots, x_r$ is Lech-independent, we can first get a filtration of the quotient $S/((x_1, x_2, \ldots, x_r, x_{r+1}))$ by changing the power of $x_r$; then we refine this filtration by changing the power of $x_{r-1}$; and refine it by changing the power of $x_{r-2}, \ldots, x_1$. Finally we get a filtration of $S/((x_1, x_2, \ldots, x_r, x_{r+1}))$ such that all the factors are isomorphic, so every factor is isomorphic to the first factor which is $S/(x_1, x_2, \ldots, x_r)$. Let $<P$ be the pure lexicographic order on $P$ with $1 < T_1 < T_2 < \ldots < T_r$, then this filtration is just of the form $J_i(x)$ where $J_i$ is a monomial generated by $Mon(P)$ except for the largest $i$ monomials not in $J$. In particular $J_i$ is a prime filtration of $P/J$.

(3) The quotient $J_i/J_i+1$ can be generated by monomials and is isomorphic to $k$ as a $P$-module, so there is only one monomial generator and is unique. For every monomial $u \in Mon(P)\setminus Mon(J)$, there is a largest $i$ such that $u \in J_i, u \notin J_i+1$. So $u \neq 0$ in $J_i/J_i+1$, and since $J_i/J_i+1 \cong k$, $u$ is the generator of $J_i/J_i+1$.

(4) Take $F_0 = F$ in (3), then $M_F$ is well-defined. For any $f \in S$ we pick $f_u$ inductively. Suppose $f_u$ is already defined for $u = M_F(0), M_F(1), \ldots, M_F(l-1)$ for $0 \leq i \leq l$ such that

$$f - \sum_{0 \leq j \leq l-1} f_{M_F(j)} M_F(j)(x) \in J_i(x).$$

This is trivial for $i = 0$ because in this case $f \in J_0(x) = S$. Now

$$f - \sum_{0 \leq j \leq l-1} f_{M_F(j)} M_F(j)(x) \in g \cdot M_F(i)(x) + J_i+1(x)$$

for some $g \in S$. Find the image of $g \cdot M_F(i)(x)$ in $J_i(x)/J_i+1(x) \cong S/I' \cdot M_F(i)(x)$; thus

$$(g-\sigma(g+I'))M_F(i)(x) = f - \sum_{0 \leq j \leq l-1} f_{M_F(j)} M_F(j)(x) - \sigma(g+I')M_F(i)(x) \in J_i+1(x).$$

So we find $f_u$ for $u = M_F(0), M_F(1), \ldots, M_F(l)$ by choosing $f_{M_F(i)} = \sigma(g)$. So by induction we find $f_u$ for $u = M_F(0), M_F(1), \ldots, M_F(l-1)$ such that

$$f - \sum_{0 \leq j \leq l-1} f_u u(x) \in J_i(x) = J(x) = I.$$

We claim that an expression of this kind is unique; otherwise

$$\sum_{0 \leq j \leq l-1} f_{M_F(j)} M_F(j)(x) = \sum_{0 \leq j \leq l-1} g_{M_F(j)} M_F(j)(x) \bmod J$$

and $f_{M_F(j)}, g_{M_F(j)}$ are not all equal. Find smallest $i$ such that $f_u \neq g_u$ for $u = M_F(i)$. By cancelling the first $i$ terms we may assume $f_u = 0$ for $u = M_F(j), j < i$. Then take the image in $J_i(x)/J_i+1(x) \cong S/I' M_F(i)(x)$ we get $f_{M_F(i)} M_F(i)(x) = g_{M_F(i)} M_F(i)(x)$. So $f_{M_F(i)} = g_{M_F(i)} \bmod I'$. But $f_{M_F(i)}, g_{M_F(i)}$ are both liftings by $\sigma$ of the same coset, so they are equal, which leads to a contradiction. Thus the expansion for every element modulo $I$ is unique.

(5) A generating set of $J_i(x)$ can be given by a generating set of $J_j(x)/J_j+1(x), i \leq j \leq l-1$ and a generating set of $I = J(x) = J_l(x)$. We know each $J_i(x)/J_i+1(x)$ is a quotient of $S/I'$ generated by $M_\sigma(i)(x)$. If this quotient is not faithful, then there
is a relation $aM_G(i)(x) = 0$ in $J_i(x)/J_{i+1}(x)$ where $a \neq 0$ in $S/I'$. Lift $a \neq 0$ to $b = \sigma(a)$, then $bM_G(i)(x) \in J_{i+1}(x)$, so there exist $g_a \in \sigma(S/I')$, $u = M_G(j), i+1 \leq j \leq l - 1$ such that
\[ bM_G(i)(x) + \sum_{i \leq j \leq l-1} gM_G(j)G(j)(x) \in I. \]

But there is another expansion which is $0 \in I$ and $b \neq 0$ because $a \neq 0$, so we get two distinct expansions of $0$ modulo $I$, which leads to a contradiction by (4). \hfill \Box

There are two typical examples of strongly Lech-independent ideals.

**Example 3.19.** Suppose $I$ is generated by a regular sequence, or $I$ is the maximal ideal $m$, then $J$ is strongly Lech-independent.

Strongly Lech-independence implies Lech-independence, but not conversely by the following example.

**Example 3.20.** Let $S_0$ be an Artinian local ring which is not a field and let $n_0$ be the maximal ideal of $S_0$. Let $S = S_0[[x]]/n_0x^2$ and $I = (x)$. Then $I$ is Lech-independent, but not strongly Lech-independent.

**Proof.** We have $gr_I(S) = S_0[x]/n_0x^2$, $S/I = S_0$, $I/I^2 = S_0x$ is free over $S_0$, but $I^2/I^3 = (S_0/n_0)x^2$ is not free over $S_0$. \hfill \Box

There are also some other strongly Lech-independent ideals given by the following proposition:

**Proposition 3.21.** Suppose $(R, m) \to (S, n)$ is a flat local map, and $J$ is a strongly Lech-independent ideal in $R$. Pick any $\Gamma$ such that $J$ is $\Gamma$-expandable from degree $i$ to $j$ for any $i < j$. Such $\Gamma$ exists by Proposition 3.9. Then $I = JS$ is strongly Lech-independent in $S$, and $I$ is $\Gamma$-expandable from degree $i$ to $j$ for any $i < j$. In particular if $J = m$, then $I = mS$ is strongly Lech-independent. Moreover for any $\Gamma$ such that $mS$ is $\Gamma$-expandable from degree $i$ to $\infty$ for any $i$, we have $e(\Gamma) = e(R)$.

**Proof.** If $(R, m) \to (S, n)$ is flat local map, then there is an isomorphism $I^i/I^{i+1} \cong J^i/J^{i+1} \otimes_{R/J} S/I$. Note that freeness and a basis of a module is preserved under any base change. Let $x_1, x_2, \ldots, x_r$ be a minimal generating set of $J$, and $y_i$ be the image of $x_i$, then $y_1, y_2, \ldots, y_r$ is a minimal generating set of $I$ because the map is local. So if $J$ is $\Gamma$-expandable from degree $i$ to $j$ for any $i < j$, or equivalently $J$ is $\Gamma$-expandable from degree $i$ to $i + 1$ for any $i$, then $u(x_i), u \in \Gamma_i$ is a basis of $J^i/J^{i+1}$ over $R/J$. This means $u(y_i), u \in \Gamma_i$ is a basis of $I^i/I^{i+1}$ over $S/I$. Hence $I$ is $\Gamma$-expandable from degree $i$ to $i + 1$ for any $i$, so $I$ is $\Gamma$-expandable from degree $i$ to $j$ for any $i < j$. If $J = m$, we can pick a $\Gamma'$ such that $J$ is $\Gamma'$-expandable from degree $i$ to $j$ for any $i < j$, then $I$ is also $\Gamma'$-expandable from degree $i$ to $j$ for any $i < j$. Then $e(\Gamma) = e(\Gamma') = e(R)$ by Proposition 3.11. \hfill \Box

**Example 3.22.** Let $S$ be a Noetherian local ring, $x_1, \ldots, x_r$ be strongly Lech-independent elements in $S$. Let $S' = S[T]/(T^k - x_1)$. Then the natural inclusion $S \to S'$ is flat local, hence $x_1, \ldots, x_r$ is still strongly Lech-independent in $S'$. We will show later that $T, x_2, \ldots, x_r$ may not be strongly Lech-independent in Example 3.27.

We provide an important source of strongly Lech-independent ideals, that is, find a flat local map and extend the maximal ideal of the source ring to the target. However, these do not provide all the strongly Lech-independent ideals.
Proposition 3.25. Let $k$ be a field, $S = k[[t,x,y]]/(t^2, x^2 - ty^2)$, $I = (x,y)$. Then $I$ is strongly Lech-independent in $S$. Let $R$ be the subring generated over $k$ by $x$ and $y$. Then $R = k[[x,y]]/(x^4)$ and $S$ is not flat over $R$.

Proof. We have $gr_1(S) = k[[t,x,y]]/(t^2, x^2 - ty^2)$. It is a standard graded ring with deg $t = 0$, deg $x = deg y = 1$. Let $S_0 = gr_1(S)_{0} = k[t]/t^2$, then $gr_1(S)_{1} = S_0x + S_0y$ is free over $S_0$. For $i \geq 2$,

$$gr_1(S)_{i} = \sum_{0 \leq j \leq i} S_0 x^j y^{i-j} / \sum_{2 \leq i \leq 1} S_0 (x^j y^{i-j} - tx^{i-2} y^{i-j+2}).$$

The set $\{x^j y^{i-j} - tx^{i-2} y^{i-j+2}\}$ is part of a minimal basis of the free module $\sum_{0 \leq j \leq i} S_0 x^j y^{i-j}$, so the quotient is still a free $S_0$-module, which implies that $I$ is strongly Lech-independent. Let $\phi : k[[x,y]] \to S$. Then $R = k[[x,y]]/\ker \phi$ and $\ker \phi = (t^2, x^2 - ty^2) \cap k[[x,y]]$. Let $< be the pure lexicographic order such that $1 > t > x > y$. Then for a power series $f \in k[[t,x,y]]$, $f \in k[[x,y]]$ if and only if the largest term of $f$ is in $k[[x,y]]$. We apply the Buchberger’s algorithm to compute the ideal of largest terms. The Gröbner basis of the ideal $(t^2, x^2 - ty^2)$ is $t^2, x^2 - ty^2, x^4$, so $(t^2, x^2 - ty^2) \cap k[[x,y]] = (x^4)$ which implies that $R = k[[x,y]]/(x^4)$. So $S$ has a minimal generating set $1, t$ as an $R$-module and a nontrivial relation $x^2 - ty^2 = 0$, so $S$ is not free over $R$. Since $S$ is module-finite over $R$ and $R$ is local, $S$ is not flat over $R$.

There is another example where the residue fields of the local rings are different.

Example 3.24. Let $S = \mathbb{C}[[x_1, x_2]]/(x_1^n + \sqrt{-1}x_2^n)$ and $R = \mathbb{R}[[x_1, x_2]]/(x_1^n + x_2^n)$. Then $R$ is a subring of $S$ with maximal ideal $m = (x_1, x_2)$, $mS = (x_1, x_2)S$ is strongly independent, and $S$ is not flat over $R$.

Proof. We have $\mathbb{C}[[x_1, x_2]]$ is a UFD. It is easy to see $x_1^n + \sqrt{-1}x_2^n = l_1 l_2$ where $l_1, l_2$ are two linear forms which are not real and not conjugate to each other, so $l_1, l_2, l_1^*, l_2^*$ are pairwise relatively prime where $^*$ denotes the conjugate of a complex polynomial. So if $f$ is a real polynomial and can be divided by $(x_1^n + \sqrt{-1}x_2^n)$, it can also be divided by $l_1 l_2$, and this implies $f$ is a multiple of $l_1 l_2 x_1^n + x_2^n$. Hence $R$ is a subring of $S$. The fact that $(x_1, x_2)S$ is strongly Lech-independent and $R \to S$ is not flat can be proved in the same way as in the last example.

Corollary 3.14 allows us to replace Lech-independent elements with their roots and the new sequence is still Lech-independent. However, its converse does not hold, so in general we cannot replace elements with their powers while preserving the independence property. The following proposition shows that “stays Lech-independent after raising to any power” is equivalent to being a regular sequence.

Proposition 3.25. Let $I$ be an ideal of a complete local ring $S$ which contains a field $k$, and $x_1, ..., x_r$ be a set of minimal generators of $I$. Then the following are equivalent.

1. For any positive integer $a_1, ..., a_r$, $x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r}$ is Lech-independent.
2. For any positive integer $a_1, ..., a_r$, $x_1^{a_1}, x_2^{a_2}, ..., x_r^{a_r}$ is strongly Lech-independent.
3. $x_1, ..., x_r$ is Mon(P)-expandable.
4. $x_1, ..., x_r$ forms a regular sequence.

Proof. (2) implies (1) is trivial.
is free of rank over $S/I_J$. In particular if $I_k \in \text{Mon}(P)$ is a monomial ideal and $K_j/K_{j+1} \cong k$ for every $j$. By Lemma 3.18, $K_j(x)/K_{j+1}(x) \cong S/I$. So $J_i(x)/J_{i+1}(x)$ has a filtration such that each factor of the form $K_j(x)/K_{j+1}(x)$ which is free over $S/I$, thus $J_i(x)/J_{i+1}(x)$ is free over $S/I$. The number of factors is just the length of $J_i/J_{i+1}$, so $J_i(x)/J_{i+1}(x)$ is free of rank $l(J_i/J_{i+1})$ over $S/I$. The set \{u(x), u \in J_i \setminus J_{i+1}\} is a generator of $J_i(x)/J_{i+1}(x)$ and its cardinality is equal to the rank of $J_i(x)/J_{i+1}(x)$, so it is a free basis. In particular if $i < u = \min \{a_i\}$, then $J_i = J_i^1, I_i = I_i(x) = P, J_{i+1} = J_{i+1}^1, I_{i+1} = I_{i+1}(x) = P^1$ and $l(J_i/J_{i+1}) = \text{dim}_k P_i$. So $I^1/I'$ is free with rank equal to $\text{dim}_k P_i$. As we let $a \to \infty$, we know that this is true for all $i$; thus $I$ is strongly Lech-independent which is $\text{Mon}(P)$-expansible.

(3) implies (4): Pick a $k$-linear lifting $\sigma$. Take an element $\bar{f} \in S/(x_1, \ldots, x_j)$ for some $j$ and let $f$ be a preimage of $\bar{f}$ in $S$. Suppose $x_{j+1} f \in (x_1, \ldots, x_j)$, we want to prove $f \in (x_1, \ldots, x_j)$. Expand $f = \sum f_u u(x)$, then $x_{j+1} f = \sum f_u \cdot (uT_{j+1})(x)$. This expansion satisfies $f_u \in \sigma(S/(x_1, \ldots, x_r))$, so it must be the unique expansion. We claim that for any $g \in (x_1, \ldots, x_j)$ with an expansion $\sum g_u u(x), g_u \neq 0$ only if $u \in (T_1, \ldots, T_j)$. Let $g = \sum_{1 \leq i \leq j} g_i x_i$. The expansion of $g_i$ exists, and is of the form $\sum_{u \in \text{Mon}(P)} g_{i,u} u(x)$; then

$$g = \sum_{1 \leq i \leq j, u \in \text{Mon}(P)} g_{i,u}(uT_i)(x) = \sum_{1 \leq i \leq j, u/T_i \in \text{Mon}(P)} g_{i,u/T_i} u(x).$$

But fixing $u$,

$$\sum_{1 \leq i \leq j, u/T_i \in \text{Mon}(P)} g_{i,u/T_i} \in \sigma(S/(x_1, \ldots, x_r))$$

because $\sigma$ is $k$-linear, hence additive. So $\sum_{1 \leq i \leq j, u/T_i \in \text{Mon}(P)} g_{i,u/T_i} u(x)$ is an expansion of $g$, so it must be the unique expansion, and in this expansion the coefficient of $u(x)$ is $\sum_{1 \leq i \leq j, u/T_i \in \text{Mon}(P)} g_{i,u/T_i}$; it is nonzero only if $u \in (T_1, \ldots, T_j)$. Apply the claim to $x_{j+1} f$, we see that $f_u \neq 0$ implies $uT_{j+1} \in (T_1, \ldots, T_j)$, so $u \in (T_1, \ldots, T_j)$ and in this case $u(x) \in (x_1, \ldots, x_j)$, so $f \in (x_1, \ldots, x_j)$. Since this is true for any $j$, we get (4).

(4) implies (2): if $x_1, \ldots, x_r$ forms a regular sequence, then $\text{gr}_{(x_1, \ldots, x_r)} S \cong S/(x_1, \ldots, x_r)[T_1, \ldots, T_e]$, so we get (2). \hfill \Box

Remark 3.26. The proof of Kunz’s theorem in \cite{Kunz} uses the equivalence of (1) and (4) in the previous proposition. To be precise, suppose $R$ is a local ring of positive characteristic $p$ such that the Frobenius action on $R$ is flat. Let $x_1, \ldots, x_r$ be a minimal generating set of the maximal ideal of $R$, then it is strongly Lech-independent, so after a flat base change $F^e$ it is still strongly Lech-independent. But after a flat base change the minimal generating set becomes $x_1^{p^e}, x_2^{p^e}, \ldots, x_r^{p^e}$. Let $e$ go to $\infty$ and notice that Lech-independence property passes to factors by Proposition 3.13, hence any power of $x_1, \ldots, x_r$ is Lech-independent. So $x_1, \ldots, x_r$ forms a regular sequence, hence the ring is regular. The above proof can also be seen in standard textbooks or lecture notes, for instance, \cite{Lipman}. 

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Let $x_1, x_2, \ldots, x_r$ be elements of $S$ and $a_1, \ldots, a_r$ be positive integers. Let $I = (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r})$, $I' = (x_1, x_2, \ldots, x_r)$. In the above paragraph we know $I$ is strongly Lech-independent implies that $I'$ is $\Gamma$-expandable for some $\Gamma$. Also $I$ is Lech-independent implies that $I'$ is Lech-independent. So it is natural to ask whether $I$ is strongly Lech-independent implies that $I'$ is strongly Lech-independent, and by Proposition 3.17 it suffices to prove the following: $I'$ is $\Gamma$-expandable implies $I'$ is $\Gamma$-expandable from degree $i$ to $j$ for any $i < j$. However, both implications are wrong. This is the reason to introduce the complicated notion $\Gamma$-expandable from degree $i$ to $j$ to describe strongly Lech-independence.

Example 3.27. Let $S = k[[x, y, t]]/(t^2, ty^2 - x^8)$ and $I = (x, y)$, $I' = (x^4, y)$. Let $P = k[T_1, T_2]$, $\Gamma = Mon(P) \setminus Mon((T_1^8))$. Then $I'$ is strongly Lech-independent; $I'$ is $\Gamma$-expandable, but it is not strongly Lech-independent. In particular for some $i$ $I'$ is not $\Gamma$-expandable from degree $i$ to $\infty$.

Proof. Set $S' = k[[X, y, t]]/(t^2, ty^2 - X^2)$ and $J = (X, y)$. Then by Example 3.23 $J$ is strongly Lech-independent. There is a map $S' \to S : X \to x^4, y \to y, t \to t$ and it is flat local. So $I' = JS$ is strongly Lech-independent. To prove $I'$ is $\Gamma$-expandable, we may apply Proposition 3.17, but we may also apply the theory of initial ideal and Gröbner basis. Choose a local monomial order $<$ on $S$ such that the initial ideal of $K = (t^2, ty^2 - x^8)$ is $(t^2, x^8)$. The initial of an element is the smallest term in that element and the initial ideal is the ideal generated by smallest terms of elements in an ideal. For example, we can choose $<$ to be the pure lexicographic order on $x, y, t$ such that $x < y < t < 1$. Then the initial terms of the two generators are $t^2$ and $x^8$, and they are relatively prime, so they form a Gröbner basis of $K$. So every element $f$ in $S = k[[x, y, t]]/I$ can be expressed uniquely as a (possibly infinite) sum

$$f = \sum_{i=0,1,0 \leq j \leq 7, k} f_{i,j,k} t^i x^j y^k = \sum_{0 \leq j \leq 7, k} (f_{0,j,k} + t f_{1,j,k}) x^j y^k.$$ 

Also $S/I = k[t]/t^2$ so we may choose the lifting $\sigma : S/I \to S$ which maps $a + bt + I$ to $a + bt$ for any $a, b \in k$. Then we know $I$ is $\Gamma$-expandable by the unique expression of $f$. However, $I^2/I^3$ is minimally generated by $x^2, xy, y^2$ with a nontrivial relation $ty^2 = 0$, so it is not free over $S/I$, so $I$ is not strongly Lech-independent. \qed

Example 3.28. Let $S = k[[x, y, t]]/(t^2, ty - x^2)$, $I = (x, y)$, $P = k[T_1, T_2]$, and $\Gamma = Mon(P) \setminus Mon((T_1^2))$. Under the pure lexicographic order such that $x < y < t < 1$, the initial ideal of $(t^2, ty - x^2)$ is $(t^2, x^2)$. So by the same token above $I$ is $\Gamma$-expandable. But $I$ is not Lech-independent because $I/I^2$ is minimally generated by $x, y$ with $ty = 0$, so it is not free over $S/I$. So being $\Gamma$-expandable does not imply Lech-independence.

There is a special implication; being strongly Lech-independent implies being Ratliff-Rush.

Definition 3.29. Let $S$ be a local ring, $I$ an ideal of $S$. Then $\bar{I} = \bigcup_i I^{i+1} : I^i$ is called the Ratliff-Rush closure of $I$. We say that $I$ is Ratliff-Rush if its Ratliff-Rush closure is itself.

Now the following proposition is trivial.

Proposition 3.30. $I$ is Ratliff-Rush if and only if $\text{Ann}_{S/I}(I^i/I^{i+1}) = 0$ for any $i$. In particular, strongly Lech-independence implies being Ratliff-Rush.
Remark 3.31. The converse of Proposition 3.30 does not hold. For example, consider $S = k[[t_1, t_2, x, y]]/(t_1^2, t_2, t_1x^2 - t_2y^2)$ and $I = (x, y)$. Then $I$ is Lech-independent. $S/I, f/I^2$ is free over $S/I$. For $i \geq 2$,

$$I^i/I^{i+1} = \sum_{0 \leq j \leq i} (S/I)x^iy^{i-j} / \sum_{2 \leq j \leq i} k \cdot (t_1x^j - t_2x^{j-2}y^2).$$

The set $x^iy^{i-j}$ is a minimal generating set, but not a basis, so $I^i/I^{i+1}$ is not free, so $I$ is not strongly Lech-independent. We claim that $I$ is Ratliff-Rush. It suffices to prove $\text{Ann}_{S/I}(x^i + I^{i+1}) = 0$. Suppose this is not true, then there exist $a, b, c \in k$ not all $0$ such that $(a + bt_1 + ct_2)x^i = 0$ in $I^i/I^{i+1}$. Equivalently, there exist $a_j \in k$ such that $(a + bt_1 + ct_2)x^i + \sum a_j(t_1x^j - t_2x^{j-2}y^2) = 0$ in $k[[t_1, t_2, x, y]]/(t_1^2, t_2, t_1t_2)$. But the elements $\{t_1x^j - t_2x^{j-2}y^2, x^i, t_1x^i, t_2x^i\}$ are $k$-linearly independent in $k[[t_1, t_2, x, y]]/(t_1^2, t_2, t_1t_2)$, thus $a = b = c = 0$, which is a contradiction.

4. **Strongly Lech-independence and inequalities on multiplicities of ideals**

Throughout this section, we keep the same assumptions as the last section, that is, $S$ is a Noetherian local ring with maximal ideal $m$, $I$ is an ideal of $S$, and $P = k[T_1, \ldots, T_r]$ is a polynomial ring in $r$ variables. Moreover, we assume that $(S, m)$ is a complete local ring with a coefficient field $k$ unless otherwise stated. We begin with a lemma which is a reformulation of the expansion property.

**Lemma 4.1.** Let $\Gamma$ be a standard set, $I$ be an $n$-primary ideal in $S$ which is $\Gamma$-expandable. Take $f_1, f_2, \ldots, f_t \in S$ such that their images in $S/I$ form a $k$-basis of $S/I$, and define a $k$-linear map $\sigma : S/I \to S$ which maps $f_i + I$ to $f_i$. Then $\sigma$ is a lifting, and expanding $f$ as a linear combination of $f_i \cdot u(x)$ gives a $k$-linear isomorphism

$$S \cong \prod_{1 \leq i \leq t, u \in \Gamma} k \cdot f_i u(x).$$

**Proof.** Since $\sigma$ is $k$-linear, $\sigma(0) = 0$. Every element in $S/I$ is $\sum_{1 \leq i \leq t} a_i f_i + I$ for some $a_1, \ldots, a_t \in k$. Let $\pi : S \to S/I$ be the projection, then $\pi \sigma(\sum_{1 \leq i \leq t} a_i f_i + I) = \pi(\sum_{1 \leq i \leq t} a_i f_i) = \sum_{1 \leq i \leq t} a_i f_i + I$. So $\sigma$ is a lifting. For every $f \in S$, $f = \sum_{u \in \Gamma} f_u u(x)$. We write $f_u = \sum_{1 \leq i \leq t} c_{i,u} f_i$ modulo $I$ for $c_{i,u} \in k$. But $\sigma$ is $k$-linear, so $\sum_{1 \leq i \leq t} c_{i,u} f_i \in \sigma(S/I)$, so $f_u = \sum_{1 \leq i \leq t} c_{i,u} f_i$ in $S$. So $f = \sum_{u \in \Gamma} f_u u(x) = \sum_{1 \leq i \leq t, u \in \Gamma} c_{i,u} f_i u(x)$. This defines the map, and it is well-defined by the uniqueness of the expansion. The map is surjective since the preimage of an expansion is just the value of the sum, and it exists when $S$ is complete. It is injective because if two elements give the same expansion then they are both equal to the sum, hence they must be equal. It suffices to prove linearity. Suppose $f = \sum_{u \in \Gamma} f_u u(x), g = \sum_{u \in \Gamma} g_u u(x), c \in k$. Then $f + cg = \sum_{u \in \Gamma} (f_u + cg_u) u(x)$. By the assumption on the expansion $f_u = \sigma(f_u + I), g_u = \sigma(g_u + I), so f_u + cg_u = \sigma(f_u + cg_u + I) \in \sigma(S/I)$. Hence $f + cg = \sum_{u \in \Gamma} (f_u + cg_u) u(x)$ is the unique expansion of $f + cg$. This proves the lemma.

**Corollary 4.2.** With the same assumptions as in Lemma 4.1, let $t$ be a positive integer. Set

$$A_{l,t} = \{ f_i u(x) | 1 \leq i \leq l, u \in \Gamma, f_i u(x) \notin n' \}.$$
and \( A_{2,t} = \{ f_i u(x) | 1 \leq i \leq l, u \in \Gamma, \operatorname{ord}(f_i) + \sum_{1 \leq j \leq r} \operatorname{ord}(x_j) \deg_{T_j}(u) < t \} \).

Then we have:

1. \( S/n^t \) can be spanned over \( k \) by \( A_{1,t} \).
2. \( A_{1,t} \subset A_{2,t} \). So \( S/n^t \) can be spanned by \( A_{2,t} \).
3. If the set \( A_{2,t} \) is linearly independent modulo \( n^t \), then it is a \( k \)-basis of \( S/n^t \).

So \( \dim_k S/n^t = |A_{2,t}| \).

Proof. Every element in \( S/n^t \) is of the form \( f + n^t \), and we can represent \( f \) as \( f = \sum_{1 \leq i \leq l, u \in \Gamma} c_i u f_i u(x) \) by the unique expansion property. Since \( I \neq S \), \( I^t \subset n^t \). So \( u \in \Gamma_j, j \geq t \) implies \( u(x) \in I^t \subset n^t \). Thus \( f = \sum_{1 \leq i \leq l, u \in \Gamma, j < t} c_i u f_i u(x) \) in \( S/n^t \) and this is a finite linear combination. This means that \( f + n^t \) is in the span of all the \( f_i u(x) \), so it’s in the span of \( f_i u(x) \) such that \( f_i u(x) \notin n^t \) because \( f_i u(x) \in n^t \) means that \( f_i u(x) = 0 \) in \( S/n^t \). This proves (1). For the second claim, note that if \( f_i u(x) \notin A_{2,t} \), then \( \deg(f_i) + \sum_{1 \leq j \leq r} \operatorname{ord}(x_j) \deg_{T_j}(u) \geq t \), so \( \deg(f_i u(x)) \geq t \), \( f_i u(x) \in n^t \), and \( f_i u(x) \notin A_{1,t} \). This proves (2). (3) is obvious by (2). \( \square \)

Recall that the Hilbert series of \( S \) is \( H_{S_S}(z) = \sum_{i \geq 0} \dim_k(n^i/n^{i+1})z^i \). Define a partial order \( \leq \) on \( \mathbb{R}[[z]] \) to be degreewise comparison, that is, \( \sum_{i \geq 0} a_i z^i \leq \sum_{i \geq 0} b_i z^i \) if \( a_i \leq b_i \) for all \( i \). We have an embedding \( \mathbb{R}[[z]](a-z) \hookrightarrow \mathbb{R}[[z]] \) for any \( a \neq 0 \). That means if \( z = 0 \) is not a pole of a rational series \( a(z) \) then we can view \( a(z) \) as an element in \( \mathbb{R}[[z]] \), while at the same time \( a(z) \) is defined over \( \mathbb{C} \) except for finitely many poles of \( a(z) \), so we can take limits in \( \mathbb{C} \).

**Lemma 4.3.** Let \( d \) be a positive integer, \( a(z) = \sum_{i \geq 0} a_i z^i \) be a rational series satisfying the following properties:

(P1) \( a(z) \) only has poles at roots of unity;
(P2) \( z = 1 \) is a pole of \( a(z) \) with order \( d \);
(P3) \( d \) The orders of poles of \( a(z) \) except for 1 are less than \( d \).

Then we have

\[
(4.1) \quad \lim_{z \to 1} \sum_{i \geq 0} a_i z^i (1 - z)^d = \lim_{k \to \infty} \frac{(d - 1)!}{(d + k - 1)!} \frac{\partial^k a(0)}{\partial z^k}.
\]

Proof. We can express \( a(z) \) using partial-fraction decomposition. To be precise, let \( U \) be the set of poles of \( a(z) \), then there exist finitely many real numbers \( e_{i,\xi}, 1 \leq i \leq d - 1, \xi \in U, \) a real number \( e_0 \neq 0 \), and a polynomial \( b(z) \) such that

\[
(4.2) \quad a(z) = \sum_{1 \leq i \leq d - 1, \xi \in U} e_{i,\xi}(\xi - z)^{i - d} + e_0(1 - z)^{-d} + b(z).
\]

Let \( L \) be the map \( a(z) \to \lim_{k \to \infty} \frac{(d - 1)!}{(d + k - 1)!} \frac{\partial^k a(0)}{\partial z^k} \). Then it is \( \mathbb{Q} \)-linear when it is well-defined. We apply \( L \) to each term in the right side of (4.2). If \( 1 \leq i \leq d - 1 \),

\[
L((\xi - z)^{i - d}) = \lim_{k \to \infty} \frac{(d - 1)!}{(d - i - 1)!}(d - i + k - 1)! (\xi - 0)^{i - d - k} = 0
\]

as \((\xi - 0)^{i - d - k}\) is bounded and \(\frac{(d - 1)!}{(d - i - 1)!} \frac{(d - i + k - 1)!}{(d + k - 1)!}\) goes to 0,

\[
L((1 - z)^{-d}) = \lim_{k \to \infty} \frac{(d - 1)!}{(d - k)!} (d + k - 1)! (1 - 0)^{i - d - k} = 1,
\]

\[
L(b(z)) = \lim_{k \to \infty} \frac{1}{(d - 1)!} \frac{\partial^k b(0)}{\partial z^k}.
\]

Therefore we have

\[
\sum_{i \geq 0} a_i z^i (1 - z)^d = \lim_{k \to \infty} \frac{1}{(d - 1)!} \frac{\partial^k a(0)}{\partial z^k}.
\]
and \( L(b(z)) = 0 \) as \( b(z) \) is a polynomial. This means the right side of (4.1) is \( L(a(z)) = e_0 \). The left side is also \( e_0 \), so they are equal. \( \square \)

**Lemma 4.4.** Let \( \sum_{i \geq 0} a_i z^i, \sum_{i \geq 0} b_i z^i \) be two rational series satisfying (P1), (P2), and (P3). Assume \( \sum_{i \geq 0} a_i z^i / (1 - z) \leq \sum_{i \geq 0} b_i z^i / (1 - z) \), then

\[
\lim_{z \to 1} \sum_{i \geq 0} a_i z^i (1 - z)^d \leq \lim_{z \to 1} \sum_{i \geq 0} b_i z^i (1 - z)^d.
\]

**Proof.** Let \( \sum_{i \geq 0} a_i' z^i = \sum_{i \geq 0} a_i z^i / (1 - z), \sum_{i \geq 0} b_i' z^i = \sum_{i \geq 0} b_i z^i / (1 - z) \). It suffices to prove that

\[
\lim_{z \to 1} \sum_{i \geq 0} a_i' z^i (1 - z)^{d+1} \leq \lim_{z \to 1} \sum_{i \geq 0} b_i' z^i (1 - z)^{d+1}.
\]

Now \( \sum_{i \geq 0} a_i' z^i \) is a rational series satisfying (P1), (P2), (P3), so by Lemma 4.3 the limit on the left side of (4.3) is equal to \( \lim_{z \to 1} \frac{d!}{(d + k)!} \frac{\partial^k a'(0)}{\partial z^k} \), and similar for the right side. By assumption \( \sum_{i \geq 0} a_i' z^i \leq \sum_{i \geq 0} b_i' z^i \) and the partial order on the power series is preserved by taking derivatives, multiplying a positive constant, and evaluate at 0. So

\[
\frac{d!}{(d + k)!} \frac{\partial^k a'(0)}{\partial z^k} \leq \frac{d!}{(d + k)!} \frac{\partial^k b'(0)}{\partial z^k},
\]

and take the limit when \( k \to \infty \). \( \square \)

**Theorem 4.5.** Let \( I \) be an \( n \)-primary ideal in \( S, x_1, \ldots, x_r \) be a minimal generating sequence of \( I \) such that the order of \( x_i \) is \( t_i \) and \( t_1 \leq t_2 \leq \ldots \leq t_r \). Denote \( d = \dim S \). Assume \( x_1, \ldots, x_r \) is \( \Gamma \)-expandable for some \( \Gamma \). We choose \( f_i, 1 \leq i \leq l \) such that their images form a homogeneous \( k \)-basis of \( gr_n(S/I) \).

1. Let \( c(z) = \sum_{t \geq 0} c_t z^t \), where \( c_t \) is the number of \( f_i u(x) \) such that \( 1 \leq i \leq l, u \in \Gamma, \text{ord}(f_i) + \sum_{1 \leq j \leq r} \text{ord}(x_j) \text{deg}_{T_j}(u) = t \). Then

\[
\frac{d!}{(d + k)!} \frac{\partial^k a'(0)}{\partial z^k} \leq \frac{d!}{(d + k)!} \frac{\partial^k b'(0)}{\partial z^k},
\]

and \( c(z) \) satisfies (P1), (P2), (P3).

2. We have

\[
HS_S(z)/(1 - z) \leq c(z)/(1 - z).
\]

If moreover for any \( t \), the set

\[
A_{2,t} = \{ f_i u(x) | 1 \leq i \leq l, u \in \Gamma, \text{ord}(f_i) + \sum_{1 \leq j \leq r} \text{ord}(x_j) \text{deg}_{T_j}(u) < t \}
\]

is \( k \)-linearly independent modulo \( n^t \), then

\[
HS_S(z)/(1 - z) = c(z)/(1 - z).
\]

3. We have:

\[
l(S/I)e(\Gamma)/t_r t_{r-1} \ldots t_{r-d+1} \leq \lim_{z \to 1} c(z)(1 - z)^d \leq l(S/I)e(\Gamma)/t_1 t_2 \ldots t_d.
\]
There is an upper bound of the multiplicity of the maximal ideal:
\[ e(n) \leq e(\Gamma)l(S/I)/t_1...t_{d-1}t_d. \]

If moreover the set
\[ A_{2,t} = \{ fu(x) | 1 \leq i \leq l, u \in \Gamma, \text{ord}(f_i) + \sum_{1 \leq j \leq r} \text{ord}(x_j) \deg_{T_j}(u) < t \} \]
is \( k \)-linearly independent modulo \( n^t \) for any \( t \), then there is also a lower bound:
\[ e(n) \geq e(\Gamma)l(S/I)/t_rt_{r-1}...t_{r-d+1}. \]

**Proof.**

(1) By definition,
\[
c(z) = \sum_{1 \leq i \leq l, u \in \Gamma} z^{\text{ord}(f_i)+\sum_{1 \leq j \leq r} \text{ord}(x_j) \deg_{T_j}(u)} = \sum_{1 \leq i \leq l} z^{\text{ord}(f_i)} \sum_{u \in \Gamma} z^{\sum_{1 \leq j \leq r} \text{ord}(x_j) \deg_{T_j}(u)} = HS_{S/I}(z) u(z^{t_1}, z^{t_2}, ..., z^{t_r}) = HS_{S/I}(z)HS_T(z^{t_1}, z^{t_2}, ..., z^{t_r}).
\]

Let \( (u, S_i)_{i \in \Lambda} \) be a Stanley decomposition of \( \Gamma \). Then by Proposition 2.6
\[
HS_T(z) = \sum_{i \in \Lambda} \frac{u_i(z^{t_1}, z^{t_2}, ..., z^{t_r})}{\Pi_{T_j \in S_i}(1-1/z)}.
\]

The right side of (4.4) has two factors. The first factor \( HS_{S/I}(z) \) is a polynomial with \( HS_{S/I}(1) = l(S/I) > 0 \), so it is regular at \( z = 1 \). The other factor is a finite sum, and we compute the order of each term in the sum. Note that
\[
\frac{u_i(z^{t_1}, z^{t_2}, ..., z^{t_r})}{\Pi_{T_j \in S_i}(1-1/z)} = \frac{u_i(z^{t_1}, z^{t_2}, ..., z^{t_r})}{(\Pi_{T_j \in S_i}(1+z+...+z^{t_j-1})(1-z)|S_i|}
\]
so the order at \( z = 1 \) of the \( i \)-th term is just \( |S_i| \), and the other poles are given by \( t_j \)-th roots of unity; every \( t_j \)-th root of unity is a single pole of \( 1/(1+z+...+z^{t_j-1}) \), so the order of the \( i \)-th term at every pole is at most \( |S_i| \). So the order of the sum at \( z = 1 \) is at most \( \max |S_i| = d \), but after multiplying \( (1-z)^d \) and evaluating at 1 each term is positive, so they do not cancel, so the order at 1 is equal to \( d \). The orders of the sum at the other poles are at most \( d \). This means that \( c(z) \) satisfies (P1), (P2d), (P3d+1).

(2) If the images of \( f_i \)'s form a homogeneous \( k \)-basis of \( gr_u(S/I) \) then \( f_i \)'s form a \( k \)-basis of \( S/I \). The \((t-1)\)-th coefficient of \( HS_{S}(z)/(1-z) \) is the sum of the coefficients of \( 1, z, ..., z^{t-1} \) in \( HS_{S}(z) \), which is \( l(S/n^t) \). The \((t-1)\)-th coefficient of \( c(z)/(1-z) \) is the sum of the coefficients of \( 1, z, ..., z^{t-1} \) in \( c(z) \), so it is the number of \( f_i u(x) \) such that \( \text{ord}(f_i) + \sum_{1 \leq j \leq r} \text{ord}(x_j) \deg_{T_j}(u) < t \), which is \( |A_{2,t}| \). It is no less than the length of \( S/n^t \) by Corollary 4.2, and the equality holds if the additional assumption of (2) holds. So \( HS_{S}(z)/(1-z) \leq c(z)/(1-z) \), and the equality holds if for any \( t \), \( A_{2,t} \) is \( k \)-linearly independent modulo \( n^t \).
By (1),
\[
\lim_{z \to 1} c(z)(1 - z)^d = \lim_{z \to 1} HS_{S/I}(z) \sum_{i \in A} \frac{u_i(z^{t_1}, z^{t_2}, \ldots, z^{t_r})}{\prod_{j \in S_I(z)}(1 - z^{t_j})}(1 - z)^d
\]
\[
= l(S/I) \lim_{z \to 1} \sum_{i \in A} \frac{u_i(1, 1, \ldots, 1)}{\prod_{j \in S_I(z)}(1 - z^{t_j})}(1 - z)^d
\]
\[
= l(S/I) \sum_{i \in A, |S| = d} \frac{1}{\prod_{j \in S_I(z)}}
\]
Also \(e(\Gamma) = \sum_{i \in A, |S| = d} 1\). By the choice of \(t_1, \ldots, t_r\),
\[
t_1t_2\ldots t_d \leq \prod_{j \in S_I(z)} t_j \leq t_{r-d+1}\ldots t_{r-1} t_r
\]
whenever \(|S| = d\). So
\[
1/t_1 t_2 \ldots t_d \geq 1/\prod_{j \in S_I(z)} t_j \geq 1/t_{r-d+1}\ldots t_{r-1} t_r.
\]
Take the sum over \(i\) where \(|S| = d\) and multiply by \(l(S/I)\), we get the conclusion.

(4) By Lemma 3.11 \(\dim S = \dim \Gamma = d\), so \(e(n) = \lim_{z \to 1} HS_{S/I}(z)(1 - z)^d\). In (2) we get \(HS_{S/I}(z)/(1 - z) = e(z)/(1 - z)\). The series \(c(z)/(1 - z)\) satisfies (P1), (P2d+1) and (P3d+1) by (1); \(HS_{S/I}(z)/(1 - z)\) has a single pole at \(z = 1\) of order \(d + 1\) so it also satisfies (P1), (P2d+1) and (P3d+1). So we can apply Lemma 4.4 to get
\[
\lim_{z \to 1} HS_{S/I}(z)(1 - z)^d \leq \lim_{z \to 1} c(z)(1 - z)^d \leq l(S/I)e(\Gamma)/t_1t_2\ldots t_d.
\]
So the first inequality is true. If the additional assumption holds, then \(HS_{S/I}(z)/(1 - z) = c(z)/(1 - z)\) by (2), so
\[
\lim_{z \to 1} HS_{S/I}(z)(1 - z)^d = \lim_{z \to 1} c(z)(1 - z)^d \geq e(\Gamma)l(S/I)/t_t t_{r-1}\ldots t_{r-d+1}.
\]

The condition in Theorem 4.5(4) is quite strong and is false in general. However, it can be satisfied in the standard graded case. The following lemma builds a relation between the standard graded case, the local case and the complete local case.

**Proposition 4.6.** Let \((S_g, n_g)\) be a standard graded ring over a field \(k\), let \((S, n)\) be its completion with respect to \(n_g\). Let \((S_L, n_L)\) be a local ring such that there is a flat map \((S_g, n_g) \to (S_L, n_L)\), \(n_L = n_g S_L\) and the completion of \((S_L, n_L)\) with respect to \(n_L\) is equal to \(S\). Let \(I_g\) be a homogeneous ideal in \(S_g\), and let \(I_L = I_g S_L, I = I_g S\). Choose a set of homogeneous minimal generators \(y_1, \ldots, y_e\) of \(n_g\). Then:
1. There is a homogeneous ideal \(K_g\) such that \(S_g = k[y_1, \ldots, y_e]/K_g k[[y_1, \ldots, y_e]]\), and in this case \(S = k[[y_1, \ldots, y_e]]/K_g k[[y_1, \ldots, y_e]]\).
2. We have embeddings of rings \(S_g \xrightarrow{i_1} S_L \xrightarrow{i_2} S\). More generally, for any homogeneous \(S_g\)-ideal \(J\) we have injections \(S_g/J \hookrightarrow S_L/J S_L \hookrightarrow S/J S\).
3. Either \(I_g, I_L, I\) are all Artinian or none of them is Artinian.
4. Assume that \(I_g\) is Artinian, then for any \(t\), \(I^t_g/I^{t+1}_g \cong I^t_L/I^{t+1}_L \cong I^t/I^{t+1}\) where these isomorphisms are induced by \(i_1\) and \(i_2\).
(5) Assume that \( J_g \subset S_g \) is homogeneous and Artinian, \( J_L = J_gS_L \) and \( J = J_LS \), then either \( J_L, J \) are both strongly Lech-independent or none of them is strongly Lech-independent. If they are strongly Lech-independent and one of them is \( \Gamma \)-expandable from degree \( i \) to \( j \) for any \( i < j \), then both of them are \( \Gamma \)-expandable from degree \( i \) to \( j \) for any \( i < j \).

(6) The notion \( \text{ord}(f) \) is well-defined for nonzero elements \( f \) in \( S_g, S_L, S \) and the different orders are compatible via \( i_1 \) and \( i_2 \).

(7) If \( I_g, I_L, I \) are all Artinian then \( e(I_g) = e(I_L) = e(I) \). In particular \( e(n_g, S_g) = e(S_L) = e(S) \).

Proof. (1) This is trivial.

(2) It suffices to prove that for any ideal homogeneous ideal \( J \) of \( S_g, S_g/J_g \hookrightarrow S/J_gS \) and for any ideal \( J_L \) of \( S_L, S_L/J_L \hookrightarrow S/J_LS \). The first map is injective because it is the completion map and \( S_g/J_g \) is standard graded, hence \( n_g \)-separated. The second map is injective because it is a faithfully flat ring map.

(3) The dimension of a standard graded ring over a field \( k \) is equal to the dimension of its localization at the homogeneous maximal ideal, and the dimension of any local ring is equal to the dimension of its completion. This implies that \( \dim S_g/J_g = \dim(S_g/J_g)_{n_g} = \dim S/I = \dim S_L/I_L \), so either they are all 0 or they are all nonzero.

(4) Note that \( I_g^t/I_g^{t+1} \) is \( n_g \)-primary, so it is isomorphic to its completion which is \( I^t/I^{t+1} \). Similarly \( I_L^t/I_L^{t+1} \cong I^t/I^{t+1} \).

(5) This can be proved by (4) and the definition of strongly Lech-independence and expansion property.

(6) It suffices to check that \( n^t \cap S_L = n_L^t \) and \( n^t \cap S_g = n_g^t \). This is proved in (2) by taking \( I = n_g \).

(7) We have \( e(I_g) = e(I_L) = e(I) \) by (4), so the first part is true. The second part of (7) can be proved by taking \( I_g = n_g \) in the first part. \( \square \)

**Theorem 4.7.** Let \((S, n)\) be a local ring which is not necessarily complete. Assume the completion of \( S \) is the completion of a standard graded ring over a field \( k \) with respect to the homogeneous maximal ideal. Let \( I \) be an \( n \)-primary Lech-independent \( S \)-ideal whose completion is the completion of a homogeneous ideal \( I_g \) with homogeneous minimal generators \( x_1, \ldots, x_r \) such that every \( x_1 \) is homogeneous in \( S_g \) of degree \( t_1 \) and \( t_1 \leq t_2 \leq \ldots \leq t_r \). Assume moreover that there is a standard set \( \Gamma \) such that \( x_1, \ldots, x_r \) is \( \Gamma \)-expandable. Then \( e(S) \geq e(\Gamma)_{t_1 \ldots t_r - d} \). In particular, if the completion of \( S \) is the completion of a standard graded ring over a field \( k \) with respect to the homogeneous maximal ideal, there is a flat local map \( (R, \mathfrak{m}) \to (S, n) \) such that for some homogeneous ideal \( I_g \), \( I = \mathfrak{m}S \) and \( I_g \) are the same after taking completion, and \( I_g \) is minimally generated by elements of degree \( t_1 \leq t_2 \leq \ldots \leq t_r \), then \( e(S) \geq e(R)_{t_1 \ldots t_r - d} \).

Proof. By Proposition 4.6, \( e(S), \Gamma, t_1, \ldots, t_r \) remains the same after we replace \( S \) by its completion, so we may always complete \( S \) to assume that \( S \) is the completion of \( S_g \) with respect to \( n_g \). Moreover in \( S \) we have \( \text{ord}(x_i) = t_i \). Since \( I \) is homogeneous, we may choose a \( k \)-basis \( f_1 + I \) of \( S/I \) such that each \( f_i \) is homogeneous in \( S_g \); here we view \( S_g \) as a subring of \( S \). Also the homogeneous minimal generators \( x_1, \ldots, x_r \) are in \( S_g \). Let \( \sum c_{i,u} f_i u(x) \) be a sum satisfying \( c_{i,u} \in k, u \in \Gamma \), where \( c_i \)'s are not all 0, and \( \text{ord}(f_i) + \sum_{1 \leq j \leq r} \text{ord}(u) \text{deg}_{f_j}(u) < t \) for any \( c_{i,u} \neq 0 \). Then the sum is nonzero by unique expansion property. Also, each term is in \( S_g \) and we
can view the sum as an element in \( S_q \). Since each term has nonzero components only in degree smaller than \( t \), the sum has nonzero components in degree smaller than \( t \), and in particular, it does not lie in \( n^t \), so it does not lie in \( n^t \) because \( n^t \cap S_q = n^t \). So \( \{ f_i u(x), \text{ord}(f_i) + \sum_{1 \leq j \leq r} \text{ord}(x_j) \text{deg}_{S_q}(u) < t \} \) is \( k \)-linearly independent modulo \( n^t \). Since this is true for any \( t \), Theorem 4.5(4) implies that \( e(S) = e(n) \geq e(\Gamma) l(S/I) / t_1 t_2 \ldots t_r - d + 1 \). For the second part of the theorem, assume there is a flat local map \(( R, m ) \to ( S, n )\) such that \( I = m S \). Then \( m \) and \( I = m S \) are strongly Lech-independent. In particular, it is Lech-independent, so by Hanes’ result in \([2]\), \( l(S/I) \geq t_1 t_2 \ldots t_r \). So \( e(n) \geq e(\Gamma) t_1 t_2 \ldots t_r - t_1 t_2 \ldots t_r - d + 1 = e(\Gamma) t_1 t_2 \ldots t_r - d \). Also \( m \) is \( \Gamma' \)-expandable for some \( \Gamma' \), and in this case \( I \) is also \( \Gamma' \)-expandable. This implies \( e(R) = e(\Gamma') = e(\Gamma) \) by Proposition 3.11. So by the first part of the theorem, \( e(S) = e(n) \geq e(R) t_1 t_2 \ldots t_r - d \). \( \square \)

**Remark 4.8.** Theorem 4.7 is a generalization of some of Hane’s results, for example, Corollary 3.2 of \([2]\). We make no assumptions on the minimal reduction of \( m \) or \( m S \). For example, consider \( R = k[[x, y]]/xy^2 \to S = k[[x, y]]/xy^2 \). Then neither \( x \) or \( y^2 \) can be a minimal reduction of \( m \). The minimal reduction consists of one element which is a linear combination of \( x \) and \( y^2 \) which is not homogeneous in \( S \). So we cannot use Hane’s result, but we can apply Theorem 4.7 to prove \( e(R) \leq e(S) \).

We can strengthen the first inequality in Theorem 4.5 (4) using the asymptotic Samuel function.

**Definition 4.9.** The asymptotic Samuel function is \( \tilde{v} : S \to \mathbb{R} \cup \{ \infty \} \) such that \( \tilde{v}(x) = \lim_{n \to \infty} \text{ord}(x^n)/n \).

**Proposition 4.10.** Let \( S \) be a local ring.
1. \( \tilde{v} \) is well-defined, that is, the limit exists for any \( x \in S \).
2. \( \tilde{v} \) has values in \( \mathbb{Q} \cup \{ \infty \} \).
3. \( \tilde{v}(x) \geq \text{ord}(x) \).

**Proof.** For (1) (2) see Chapter 6 and 10 of \([6]\). (3) is true as \( \text{ord}(x^n) \geq n \cdot \text{ord}(x) \). \( \square \)

**Theorem 4.11.** Let \( I \) be an \( n \)-primary ideal in \( S \). Assume \( I \) is minimally generated by \((x_1, \ldots, x_r)\) and the sequence \( x_1, \ldots, x_r \) is \( \Gamma \)-expandable with \( \dim(\Gamma) = d > 0 \). Denote \( \tilde{v}(x_i) = s_i \) and assume that \( s_1 \leq s_2 \leq \ldots \leq s_r \). Then \( e(S) \leq e(\Gamma) l(S/I)/s_1 \ldots s_d - 1 8_d \) and \( s_d < \infty \). If moreover \( I \) is strongly Lech-independent, then \( e(S) \leq e(I)/s_1 \ldots s_d - 1 8_d \).

**Proof.** Choose any positive rational number \( q_i < s_i \) such that \( q_1 \leq q_2 \leq \ldots \leq q_r \). Choose a positive integer \( C \) such that \( Cq_i \) is an integer for any \( i \). Take \( f_1, f_2, \ldots, f_t \) such that their images form a \( k \)-basis of \( S/I \). By definition of \( s_i = \tilde{v}(x_i) \), there exists a constant \( D_i \in \mathbb{Z} \) such that \( \text{ord}(x_1^n) \geq nq_1 + D_i \) for each \( i \). So if \( u = T_1^{q_1} T_2^{q_2} \ldots T_r^{q_r}, \) \( \text{ord}(u(x)) \geq q_1 a_1 + q_2 a_2 + \ldots + q_r a_r + D, \) where \( D = D_1 + D_2 + \ldots + D_r. \) Let \( Z = z^{1/C} \) and view \( \mathbb{R}[[z]] \) as a subring of \( \mathbb{R}[[Z]] \).

Let \( b(z) = \sum_{t \geq 0} b_t z^t \) where \( b_t \) is the number of \( f_t u(x) \) satisfying the condition \( \sum_{1 \leq j \leq r} q_j \text{deg}_{S_q}(u) = t \) then similar to the proof of Theorem 4.5 (1) we can prove
\[
\tilde{b}(z) = l(S/I) HS_{q_1, q_2, \ldots, q_r}.
\]
The exponents of terms in \( b(t) \) is in \( 1/CZ \), so we can view
\[
b(z) = b(Z^C) = l(S/I) HS_{1}(Z^{Cq_1}, Z^{Cq_2}, \ldots, Z^{Cq_r})
\]
as an element of $\mathbb{R}[[Z]]$.

Assume $t \in 1/CZ$. Set

$$A_{3,t} = \{ f_i u(x) | 1 \leq i \leq l, u \in \Gamma, \sum_{1 \leq j \leq r} q_j \deg_{T_j}(u) < t \}. $$

Note that $A_{3,t} \subset A_{3,t+1/C}$ and

$$A_{3,t+1/C} \setminus A_{3,t} = \{ f_i u(x) | 1 \leq i \leq l, u \in \Gamma, \sum_{1 \leq j \leq r} q_j \deg_{T_j}(u) = t \}$$

because $C \sum_{1 \leq j \leq r} q_j \deg_{T_j}(u)$ is always an integer. This implies $|A_{3,t+1/C}| - |A_{3,t}| = b_t$. Consider the series

$$b'(z) = \sum_{t \geq 0, t \in 1/CZ} |A_{3,t}| z^t = \sum_{t \geq 0, t \in 1/CZ} |A_{3,t}| Z^t. $$

Then $b'(z) = b(z)(1 - z^{1/C})$ or equivalently, $b'(Z^C) = b(Z^C)(1 - Z)$. The Hilbert series of $S$ is

$$HS_S(z) = \sum_i \dim_k(n^i/n^{i+1}) z^i = HS_S(Z^C) = \sum_i \dim_k(n^i/n^{i+1}) Z^i. $$

Let $a(Z) = \sum_i a_i Z^i = HS_S(Z^C)(1 - Z)$. Then $a_i = \dim_k(S/n^{i/C})$ where $\lfloor . \rfloor$ is the floor function.

Suppose $t \in \mathbb{Z}$. Since $\text{ord}(f_i u(x)) \geq \sum_{1 \leq j \leq r} q_j \deg_{T_j}(u) + D$, $\sum_{1 \leq j \leq r} q_j \deg_{T_j}(u) \geq t$ implies $f_i u(x) \in n^{t+D}$, so $S/n^{t+D}$ can be spanned by $A_{3,t}$. This means that $\dim_k(S/n^{t+D}) \leq |A_{3,t}|$. So if $t$ is an integer

$$a_{Ct+CD-C} = \dim_k(S/n^{t+D}) \leq |A_{3,t}|. $$

As $|A_{3,t}|$ is increasing in terms of $t$ and $a_i$ only depends on $i/C$,

$$a_{Ct+CD-C} = a_{C[t] + CD-C} \leq |A_{3,t}| \leq |A_{3,t+C}| $$

for any $t \in 1/CZ$, or equivalently, $a_{t+CD-C} \leq |A_{3,t+C}|$ for any $t \in \mathbb{Z}$. This means that

$$\sum_{t \geq 0, t \in \mathbb{Z}} a_{t+CD-C} Z^t \leq \sum_{t \geq 0, t \in \mathbb{Z}} |A_{3,t+C}| Z^t.$$

So

$$Z^{C-CD}HS_S(Z^C)/(1 - Z) - P(Z) \leq b(Z^C)/(1 - Z)$$

where $P(z)$ is the term of $Z^{C-CD}HS_S(Z^C)/(1 - Z)$ with negative exponents; in particular $P(z)$ is a Laurent polynomial in $z$. On the left side of (4.5), $HS_S(z)$ has a single pole at $z = 1$ of order $d$; so $HS_S(Z^C)$ has a pole at $z = \xi$ of order $d$ for every $C$-th root of unity where we view $Z$ as the variable. This implies that $Z^{C-CD}HS_S(Z^C)/(1 - Z) - P(Z)$ has a pole at $Z = 1$ of order $d + 1$ and a pole at $Z = \xi$ of order $d$ for every $C$-th root of unity $\xi \neq 1$. This means that $Z^{C-CD}HS_S(Z^C)/(1 - Z) - P(Z)$ satisfies (P1), (P2$^{d+1}$), and (P3$^{d+1}$). On the right side of (4.5), we have

$$b(Z^C)/(1 - Z) = l(S/I)HS_T(Z^{Cq_1}, Z^{Cq_2}, ..., Z^{Cq_r})/(1 - Z)$$

and by the same proof in Theorem 4.5 (3) we know $b(Z^C)/(1 - Z)$ also satisfies (P1), (P2$^{d+1}$), and (P3$^{d+1}$). Now apply Lemma 4.4, we get

$$\lim_{Z \to 1} (Z^{C-CD}HS_S(Z^C))/(1 - Z) - P(Z))(1 - Z)^{d+1} \leq \lim_{Z \to 1} b(Z^C)(1 - Z)^d.$$
The left side of (4.6) is equal to
\[
\lim_{z \to 1} (z^{1-D} HS_S(z)/(1 - z^{1/C} - P(z^{1/C}))(1 - z^{1/C})^{d+1} = \lim_{z \to 1} HS_S(z)(1 - z^{1/C})^d = 1/C^d \cdot \lim_{z \to 1} HS_S(z)/(1 - z)^d = 1/C^d e(n).
\]
The right side of (4.6) is equal to
\[
l(S/I) \sum_{i, |S|_i = d} \frac{1}{\prod_{T \in S_i} Cq_j} = l(S/I)/C^d \cdot \sum_{i, |S|_i = d} \frac{1}{\prod_{T_j \in S_i} q_j}
\]
which is no greater than $1/C^d \cdot e(\Gamma)l(S/I)/q_d$ by a similar proof in Theorem 4.5 (3) and (4). So multiplying (4.6) by $C^d$ we get $e(n) \leq e(\Gamma)l(S/I)/q_d$. Let $q_d$ goes to $s_d$, we get $e(n) \leq e(\Gamma)l(S/I)/s_d$. But $e(n) > 0$, so $s_d < \infty$. If $I$ is strongly Lech-independent, then $e(I) = e(\Gamma)l(S/I)$ and $e(n) \leq e(I)/s_d$. By definition $e(S) = e(n)$, so we are done. 

By proposition 4.10 (3) $s_i = \tilde{v}(x_i) \geq t_i = ord(x_i)$, so Theorem 4.11 is stronger than Theorem 4.5 (4).

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