Correspondence between modified gravity and 5D Ricci-flat cosmologies

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Abstract

We study the correspondence between two theoretical frameworks for describing dark energy, \( f(R) \) gravity and higher-dimensional Space-Time-Matter (STM) or induced-matter theory. We show that the Hubble expansion parameter in \( f(R) \) gravity can be associated with a combination of metric functions in STM theory, and consider a specific example whose properties are consistent with late-time acceleration.

1 Introduction

The question of dark energy and the accelerating universe has been the focus of a large amount of activities in recent years. This expansion has directly been measured from the light-curves of several hundred type Ia supernovae [1, 2] and independently from observations of the cosmic microwave background (CMB) by the WMAP satellite [3] and other CMB experiments [4, 5]. All these results strongly suggest that the universe is permeated smoothly by dark energy which has a negative pressure and violates the strong energy condition. Dark energy and the accelerating universe have been discussed extensively from various point of views over the past few years [6, 7, 8]. In principle, a natural candidate for dark energy could be a small cosmological constant. One approach in this direction is to employ what is known as modified gravity where an arbitrary function of the Ricci scalar is added to the Einstein-Hilbert action. It has been shown that such a modification may account for the late time acceleration and the initial inflationary period in the evolution of the universe [9]. One of the first proposals in this regard was suggested in [10] where a term of the form \( R^{-1} \) was added to the usual Einstein-Hilbert action. However, several models of \( f(R) \) gravity, proposed in [9, 10], have been shown to suffer from unwanted instabilities [11] or violate solar system constraints [12]. These instabilities arise because of an extra propagating scalar degree of freedom, and hence any viable \( f(R) \) model must be constructed in such a way as to render the dynamics of this scalar field carefully controllable. This is in principle possible, and indeed \( f(R) \) theories can be made to satisfy solar system and laboratory tests by invoking the so-called Chameleon mechanism [13]. Recently, the
authors in [14] have tried to resolve the singularity problem arising in the strong gravity regime of the otherwise viable $f(R)$ theories by imposing appropriate fine-tuning on their models.

In $f(R)$ gravity, Einstein equations possess extra terms induced from geometry which, when moved to the right hand side, may be interpreted as a matter source represented by the energy-momentum tensor $T_{\text{Curv}}$, see equation (3). In a similar fashion, the Space-Time-Matter (STM) theory, discussed below, results in Einstein equations in 4D with some extra geometrical terms which may be interpreted as induced matter. It therefore seems plausible to make a correspondence between the geometrical terms in STM and $T_{\text{Curv}}$ resulting in $f(R)$ gravity. We shall explore this idea to show that different choices of the parameter $\mu(t)$ in STM may correspond to different choices of $f(R)$ in curvature quintessence models in modified gravity.

The correspondence discussed above is based on the idea of extra dimensions. The idea that our world may have more than four dimensions is due to Kaluza [15], who unified Einstein’s theory of General Relativity with Maxwell’s theory of Electromagnetism in a 5D manifold. Since then, higher dimensional or Kaluza-Klein theories of gravity have been studied extensively [16] from different angles. Notable amongst them is the STM theory mentioned above, proposed by Wesson and his collaborators, which is designed to explain the origin of matter in terms of the geometry of the bulk space in which our 4D world is embedded, for reviews see [17]. More precisely, in STM theory, our world is a hypersurface embedded in a five-dimensional Ricci-flat ($R_{AB} = 0$) manifold where all the matter in our world can be thought of as being manifestations of the geometrical properties of the higher dimensional space. The fact that such an embedding can be done is supported by Campbell’s theorem [18] which states that any analytical solution of the Einstein field equations in $N$ dimensions can be locally embedded in a Ricci-flat manifold in $(N + 1)$ dimensions. Since the matter is induced from the extra dimension, this theory is also called the induced matter theory. Applications of the idea of induced matter or induced geometry can also be found in other situations [19]. The STM theory allows for the metric components to be dependent on the extra dimension and does not require the extra dimension to be compact. The sort of cosmologies stemming from STM theory is studied in [20, 21, 22].

In this paper we consider the correspondence between $f(R)$ gravity and STM theory. In section 2 we present a short review of 4D dark energy models in the framework of $f(R)$ gravity. In section 3 the field equations are solved in STM theory by fixing a suitable metric and the resulting geometric terms are interpreted as dark energy. The cosmological evolution in STM are considered in section 4. Section 5 deals with an example for a special form of $f(R)$. Conclusions are drawn in the last section.

## 2 4D dark energy models in modified gravity

To begin with, let us start by a brief review of $f(R)$ gravity theory. We start by writing the four dimensional action as

$$S = \int d^4x \sqrt{-g} f(R) + S_m,$$

where $f(R)$ is a function of the Ricci scalar $R$ and $S_m$ is the action for the matter fields. We use units such that $8\pi G_N = c = \hbar = 1$. The field equations are

$$f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} = f'(R)^{\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}) + \tilde{T}^m_{\mu\nu},$$

which can be recast into the more expressive form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}^{\text{Curv}} + T^m_{\mu\nu},$$

where an stress-energy tensor has been defined for the curvature contribution

$$T^{\text{Curv}}_{\mu\nu} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\mu\nu} \left[ f(R) - R f'(R) \right] + f'(R)^{\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}) \right\},$$
and
\[ T_{\mu\nu}^m = \frac{1}{f'(R)} \tilde{T}_{\mu\nu}^m, \] (5)
is the stress-energy tensor of the matter. To derive the field equations we consider the Robertson-Walker metric for the evolution of the cosmos
\[ ds^2 = dt^2 - a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \] (6)
where \( k \) is the curvature of the space, namely, \( k = 0, 1, -1 \) for the flat, closed and open universes respectively. Substituting the above metric in equation (3) we obtain the 4D, spatially flat Friedmann equations as follows
\[ H^2 = \frac{1}{3} (\rho_m + \rho_{\text{Curv}}), \] (7)
and
\[ \dot{H} = -\frac{1}{2} \left[ (\rho_m + p_m) + \rho_{\text{Curv}} + p_{\text{Curv}} \right], \] (8)
where a dot represents derivation with respect to time. Such a universe is dominated by a barotropic perfect fluid with the equation of state (EOS) given by \( p_m = w_m \rho_m \) (\( w_m = 0 \) for pressureless cold dark matter and \( w_m = 1/3 \) for radiation) and a spatially homogenous curvature quintessence.

The energy density and pressure of the curvature quintessence are
\[ p_{\text{Curv}} = \frac{1}{f'(R)} \left\{ 2 \left( \frac{\dot{a}}{a} \right) \dot{R} f''(R) + \ddot{R} f''(R) + \dot{R}^2 f'''(R) - \frac{1}{2} \left[ f(R) - R f'(R) \right] \right\}, \] (9)
and
\[ \rho_{\text{Curv}} = \frac{1}{f'(R)} \left\{ \frac{1}{2} \left[ f(R) - R f'(R) \right] - 3 \left( \frac{\dot{a}}{a} \right) \dot{R} f''(R) \right\}, \] (10)
respectively. The equation of state of the curvature quintessence is
\[ w_{\text{Curv}} = \frac{p_{\text{Curv}}}{\rho_{\text{Curv}}}. \] (11)
Recently, cosmological observations have indicate that our universe is undergoing an accelerated expanding phase. This could be due to the vacuum energy or dark energy which dominates our universe against other forms of matter such as dark matter and Baryonic matter. We thus concentrate on the vacuum sector \( i.e. \rho_m = p_m = 0 \), from which the evolution equation of curvature quintessence becomes
\[ \dot{\rho}_{\text{Curv}} + 3H (\rho_{\text{Curv}} + p_{\text{Curv}}) = 0, \] (12)
yielding
\[ \rho_{\text{Curv}}(z) = \rho_{\text{Curv}}^0 \exp \left[ 3 \int_0^z (1 + w_{\text{Curv}}) d\ln(1 + z) \right] \equiv \rho_{\text{Curv}}^0 E(z), \] (13)
where, \( 1 + z = \frac{\omega_0}{a} \) is the redshift and the subscript 0 denotes the current value. In terms of the redshift, the first Friedmann equation can be written as
\[ H(z)^2 = H_0^2 \Omega_{\text{Curv}}^0 E(z), \] (14)
where \( \Omega_{\text{Curv}}^0 \) and \( H_0 \) are the current values of the dimensionless density parameter and Hubble parameter, respectively. Equation (14) is the Friedmann equation in terms of the redshift, \( z \), which is suitable for cosmological observations. In fact, equations (14) and (26), obtained in section 4, are the cosmological connections between \( f(R) \) gravity and STM theory.
3 Dark energy in 5D models

In the context of STM theory, a class of exact 5D cosmological solutions has been investigated and discussed in [23]. This solution was further pursued in [20] where it was shown to describe a cosmological model with a big bounce as opposed to the ubiquitous big bang. The 5D metric of this solution reads

\[ dS^2 = B^2 dt^2 - A^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) - dy^2, \]  

where \( d\Omega^2 \equiv (d\theta^2 + \sin^2 \theta d\phi^2) \)

and

\[ A^2 = (\mu^2 + k) y^2 + 2\nu y + \frac{\nu^2 + K}{\mu^2 + k}, \]

\[ B = \frac{1}{\mu} \frac{\partial A}{\partial t} = \frac{\dot{A}}{\mu}. \]  

Here \( \mu = \mu(t) \) and \( \nu = \nu(t) \) are two arbitrary functions of \( t \), \( k \) is the 3D curvature index \((k = \pm 1, 0)\), and \( K \) is a constant. This solution satisfies the 5D vacuum equation \( R_{AB} = 0 \). The Kretschmann curvature scalar

\[ I_3 = R_{ABCD}R^{ABCD} = \frac{72K^2}{A^8}, \]

shows that \( K \) determines the curvature of the 5D manifold. Such a solution was considered in [23] with a different notation.

Using the 4D part of the 5D metric (15) to calculate the 4D Einstein tensor, we obtain

\[ (4)G^0_0 = \frac{3(\mu^2 + k)}{A^2}, \]

\[ (4)G^1_1 = (4)G^2_2 = (4)G^3_3 = \frac{2\mu\dot{\mu}}{\dot{A}A} + \frac{\mu^2 + k}{A^2}. \]

As was mentioned earlier, since the recent observations show that the universe is currently going through an accelerated expanding phase, we assume that the induced matter contains only dark energy with \( \rho_{DE} \), i.e. \( \rho_m = 0 \). We then have

\[ \frac{3(\mu^2 + k)}{A^2} = \rho_{DE}, \]

\[ \frac{2\mu\dot{\mu}}{\dot{A}A} + \frac{\mu^2 + k}{A^2} = -p_{DE}. \]

From equations (19) and (20), one obtains the EOS of dark energy

\[ w_{DE} = \frac{p_{DE}}{\rho_{DE}} = -\frac{2\mu\dot{\mu}/\dot{A}A + (\mu^2 + k)/A^2}{3(\mu^2 + k)/A^2}. \]

The Hubble and deceleration parameters are given in [20, 22] and can be written as

\[ H = \frac{\dot{A}}{AB} = \frac{\mu}{A}, \]

and

\[ q(t, \theta) = -A^2 \frac{d^2 A}{dt^2} \left( \frac{dA}{dt} \right)^2 = -\frac{A\dot{\mu}}{\mu\dot{A}}. \]

from which we see that \( \dot{\mu}/\mu > 0 \) represents an accelerating universe while \( \dot{\mu}/\mu < 0 \) represents a decelerating one. The function \( \mu(t) \) therefore plays a crucial role in defining the properties of the universe at late times.
4 Cosmological evolution of STM theory

In this section we will concentrate on the predictions of the cosmological evolution in the spatially flat case \( k = 0 \). To avoid having to specify the form of the function \( \nu(t) \), we change the parameter \( t \) to \( z \) and use \( A_0 / A = 1 + z \) and define \( \mu_0^2 / \mu^2 = F(z) \), noting that \( F(0) \equiv 1 \). We then find that equations (21)-(23) reduce to

\[
w_{DE}(z) = -\frac{1 + (1 + z) d \ln F(z) / dz}{3},
\]

(24)
and

\[
q_{DE}(z) = \frac{1 + 3 \Omega_{DE} w_{DE}}{2} = -\frac{(1 + z) d \ln F(z)}{dz}.
\]

(25)

There is an arbitrary function \( \mu(t) \) in the present 5D model. Different choices of \( \mu(t) \) may correspond to different choices of \( f(R) \) in curvature quintessence models in modified gravity. Various choices of \( \mu(t) \) correspond to the choices of \( F(z) \). This enables us to look for the desired properties of the universe via equations (24) and (25). Using these definitions, the Friedmann equation becomes

\[
H^2 = H_0^2 (1 + z)^2 F(z)^{-1}.
\]

(26)

This would allow us to use the supernovae observational data to constrain the parameters contained in the model or the function \( F(z) \). By comparing equation (26) with equation (14), we find that there exists a correspondence between the functions \( f(R) \) and \( F(z) \). We thus take \( F(z) \) as

\[
F(z) = (1 + z)^2 \left[ \Omega_{0 \text{curv}} E(z) \right]^{-1}.
\]

(27)

According to (13), it is easy to see that the function \( E(z) \) is determined by the particular choice of \( f(R) \) which, in turn, determines the function \( F(z) \) through equation (27). The evolution of the density components and EOS of dark energy may now be derived. In the next section we will consider the correspondence between modified gravity and the Ricci-flat cosmology for a generic \( f(R) \).

5 Example

To continue, we must determine the functional form of \( f(R) \). Thus, for example, we choose \( f(R) \) as a generic power law of the scalar curvature and assume for the scale factor a power law solution in 4D, investigated in the first reference in [9]. Therefore

\[
f(R) = f_0 R^n, \quad a(t) = a_0 \left( \frac{t}{t_0} \right)^{\alpha}.
\]

(28)

The interesting cases are for the values of \( \alpha \) satisfying \( \alpha > 1 \) which would lead to an accelerated expansion of our universe. Let us now concentrate on the case \( \rho_m = 0 \). Inserting equation (28) into the dynamical system (7) and (8), for a spatially flat space-time we obtain an algebraic system for parameters \( n \) and \( \alpha \)

\[
\begin{align*}
\alpha \left[ \alpha(n - 2) + 2n^2 - 3n + 1 \right] &= 0, \\
\alpha \left[ n^2 - n + 1 + \alpha(n - 2) \right] &= n(n - 1)(2n - 1),
\end{align*}
\]

(29)

from which the allowed solutions are

\[
\alpha = 0 \rightarrow n = 0, \frac{1}{2}, 1,
\]

\[
\alpha = \frac{2n^2 - 3n + 1}{2 - n}, \quad \forall n, \quad n \neq 2.
\]

(30)
The solutions with $\alpha = 0$ are not interesting since they provide static cosmologies with a non-evolving scale factor. Note that this result matches the standard General Relativity result, $n = 1$, in the absence of matter. On the other hand, the cases with generic $\alpha$ and $n$ furnish an entire family of significant cosmological models. Using equations (9) and (10) we can also deduce the equation of state for the family of solutions $\alpha = \frac{2n^2 - 3n + 1}{2 - n}$ as

$$w_{\text{Curv}}(n) = -\left(\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3}\right),$$

where $w_{\text{Curv}} \rightarrow -1$ as $n \rightarrow \infty$. This shows that an infinite $n$ is compatible with recovering an infinite cosmological constant. Thus, using equation (31), the functions $E(z)$ and $F(z)$ are given by

$$E(z) = (1 + z)^3 \left[\frac{-2n + 4}{6n^2 - 9n + 3}\right],$$

$$F(z) = (1 + z)^2 \left[\Omega_{0\text{Curv}}(1 + z)^3 \left[\frac{-2n + 4}{6n^2 - 9n + 3}\right]\right]^{-1}.$$  

Now, using the above equations, we see that equations (24) and (25) can be written as

$$w_{\text{DE}}(n) = -\left(\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3}\right),$$

and

$$q_{\text{DE}}(n) = -\frac{A\mu}{\mu A} = -\frac{-2n^2 + 2n + 1}{2n^2 - 3n + 1}.$$  

Therefore, within the context of the present investigation, the accelerating, dark energy dominated universe can be obtained by using the correspondence between $F(z)$ and $f(R)$ in modified gravity theories. We observe that in STM theory, 5D dark energy cosmological models correspond to 4D curvature quintessence models. This result is consistent with the correspondence between exact solutions in Kaluza-Klein gravity and scalar tensor theory [24]. We also note that, as is well known, with a suitable conformal transformation, $f(R)$ gravity reduces to scalar tensor theory.

From equations (32) and (33), we can rewrite equation (26) as

$$h(z, n) = \Omega_{0\text{Curv}}(1 + z)^3 \left[\frac{-2n + 4}{6n^2 - 9n + 3}\right],$$

where $h(z, n) \equiv \frac{H(z)^2}{H_0^2}$ and the contribution of ordinary matter has been neglected. Figure 1 shows the behavior of $h(n)$ as a function of $n$ for $z \sim 1.5$ and $\Omega_{0\text{Curv}} \simeq 0.70$. As can be seen, for $n \rightarrow \pm\infty$ and $z \rightarrow 0$ we have $h(z, n) \rightarrow 0$, that is, the universe finally approaches the curvature dominant state, thus undergoing an accelerated expanding phase. Figure 2 shows the behavior of $h(z)$ as a function of $z$ for $n = 2, 10, -10$ and $\Omega_{0\text{Curv}} \simeq 0.70$. We see that for small $z$, $h(z) \rightarrow 0$. Thus, we have obtained late-time accelerating solutions only by using the correspondence between $f(R)$ gravity and STM theory. Here, we have interpreted the properties of 5D Ricci-flat cosmologies by dark energy models in modified gravity.

6 Conclusions

In this paper we have investigated the present accelerated expanding phase of the universe using a general class of 5D Ricci-flat cosmological models. Such an exact solution contains two arbitrary
Figure 1: Behavior of $h(n)$ as a function of $n$ for $z \sim 1.5$ and $\Omega_{\text{Curv}} \simeq 0.70$. An accelerating universe occurs for $n \lesssim -2$ and $n \gtrsim 2$.

Figure 2: Behavior of $h(z)$ as a function of $z$ for $n = 2$ (solid line), $n = 10$ (dashed line), $n = -10$ (dot-dashed line) and $\Omega_{\text{Curv}} \simeq 0.70$. It can be seen that for the above values of $n$, as $z \to 0$, the curves converge to the point $h(z) \to 0.7$. 
functions, $\mu(t)$ and $\nu(t)$ which are analogous to different forms of $f(R)$ in curvature quintessence models. Also, once the forms of the arbitrary functions are specified, the characteristic parameters determining the evolution of our universe are specified. We have noted that the correspondence between the functions $F(z)$ and $f(R)$ plays a crucial role and defines the form of the function $F(z)$. Finally, by taking a specific form for $f(R)$ we obtained solutions that are consistent with late-time acceleration of the universe.

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