Stabilization of Controller-Driven Nonuniformly Sampled Systems via Singular Value Assignment

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Abstract: The stabilization problem of nonuniformly sampled systems is considered, provided that the controller/scheduler can select the sampling period. We proposed a method to construct the stabilizing state feedback controller using singular value assignment. The closed-loop system is guaranteed to be stable for arbitrary selection of bounded sampling periods. This method also provides the upper bound for the maximum sampling period.

Keywords: Sampled-data control, Nonuniform sampling, Networked control systems, Stability, Singular value assignment

1. INTRODUCTION

Study of nonuniformly sampled systems may originate from network control systems (Suh (2008), Hespanha et al. (2007), Baillieul and Antsaklis (2007)), event based control (Lunze and Lehmann (2010)), or they can be treated as a subset of discrete switching or hybrid systems (Naghshtabrizi et al. (2006)).

The stability results and control techniques are well-established when the sampling time is uniform, i.e. constant for all times (Kuo (1995), Chen and Francis (1995)). But since nonuniformly sampled systems are time-varying, finding stabilizing controllers for these systems is proved to be difficult and many efforts have been made in recent years Lin and Antsaklis (2009), Kao and Fujioka (2013), Fujioka and Nakai (2010), Seuret (2012), Zhang et al. (2007), Baillieul and Antsaklis (2007), event based control (Lunze and Lehmann (2010)), or they can be treated as a subset of discrete switching or hybrid systems (Naghshtabrizi et al. (2006)).

While most of the literature focus on stabilizing a nonuniformly sampled system when the sampling intervals are unknown using a time-invariant controller, we consider a different setting. We assume that the controller/scheduler also selects the next sampling interval for the system online. This setting is also discussed in Haimovich and Osella (2013) and stabilizing controllers are found when the number of inputs is at least 1 less than the unstable eigenvalues of the open loop system.

In our previous work Sevim and Goren-Sumer (2016), we proposed a sampling-period-varying state feedback controller that guarantees stability for bounded and arbitrarily varying sampling intervals using digital redesign technique. The key idea of that study was to use eigenvector matrix of the desired closed-loop system matrix to construct an induced norm that guarantees stability.

The same idea of constructing an induced matrix norm using similarity transformation is also used in this paper to establish an alternative method to design a stabilizing state feedback controller using singular value assignment.

The proposed method is based on the main theorem given in Martin and Wang (2009), which gives the necessary and sufficient conditions for the existence of a state feedback gain to assign the desired singular values of the closed-loop system. The authors also provide an algorithm to calculate that state feedback gain.

To the best of our knowledge, this is the first work that uses singular value assignment to stabilize non-uniformly sampled systems.

1.1 Notation and Terminology

For $x \in \mathbb{C}^n$, $|x|$ denotes any vector norm and for $A \in \mathbb{C}^{n \times n}$, $||A||$ denotes a norm induced by some vector norm $||\cdot||$, i.e. $||A|| := \max_{x \in \mathbb{C}^n} |Ax|/|x|$. $\sigma(A)$ denotes the maximum singular value of $A$.

2. PROBLEM FORMULATION

Consider the continuous time system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, rank $B = m \leq n$, and $(A, B)$ is stabilizable.

Define the sequence of time instances $\{t_k\}_{k \in \mathbb{N}}$ where

$$0 = t_0 < t_1 < \cdots < t_k < \cdots$$

with $\lim_{k \to \infty} t_k = \infty$. We assume that $\{t_k\}_{k \in \mathbb{N}}$ are known a priori and sampling intervals $h_k := t_{k+1} - t_k$ are bounded, i.e. $h_k \in [h_{\min}, h_{\max}]$, $\forall k \in \mathbb{N}$ for some $h_{\max} > h_{\min} > 0$.

The objective is to design a sampled state feedback controller $K(t_k) \in \mathbb{R}^{m \times n}$ such that the system

$$\dot{x}(t) = Ax(t) + BK(t_k)x(t_k), \quad \forall t \in [t_k, t_{k+1})$$

is stable for any selection of $\{t_k\}_{k \in \mathbb{N}}$ as long as $h_k \in [h_{\min}, h_{\max}]$.
Consider the discretized model of (1)

\[ x_{k+1} = F_k x_k + G_k u_k \]  

where \( u_k := u(t_k) \),

\[ F_k := e^{Ah_k} \quad \text{and} \quad G_k := \left( \int_0^{h_k} e^{At}\,d\tau \right) B. \]

The following result is standard:

**Lemma 1.** Define \( K_k := K(t_k) \) where \( K(t_k) \) is given in (2). Then the solutions of (2) and

\[ x_{k+1} = (F_k + G_k K_k) x_k \]  

are same at the sampling instances, i.e. \( x_k = x(t_k), \forall k \in \mathbb{N} \), assuming \( x(t_0) = x_0 \).

**Proof.** We prove with induction. \( x(t_0) = x_0 \) by assumption. Assume that \( x(t_k) = x_k \), so

\[ x(t_{k+1}) = e^{At_{k+1}} x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} B u(\eta) \,d\eta \]

\[ = e^{Ah_k} x(t_k) + \left( \int_0^{h_k} e^{A\tau} \,d\tau \right) B u(t_k) \]

\[ = F_k x_k + G_k K_k x_k \]

\[ = x_{k+1} \]

So the problem becomes finding \( K_k \) and \( h_{\max} \) such that system (4) is guaranteed to be stable for \( h_k \in [h_{\min}, h_{\max}] \). We are going to use singular value assignment to guarantee the stability of (4).

3. BACKGROUND

It is well-known that the stability conditions for nonuniformly sampled systems are not trivial (Liberzon and Morse (1999)). There are many sufficient stability results exist, such as Lin and Antsaklis (2009), Kao and Fujoka (2013), Seuret (2012), Liberzon et al. (1999).

We use the following definition and sufficient condition for stability.

**Definition 1.** The system

\[ x_{k+1} = F_k x_k \]  

is globally asymptotically stable if, for any \( x_0 \in \mathbb{R}^n \)

\[ \lim_{k \to \infty} |x_k| = 0 \]

for some vector norm \( |\cdot| \).

**Theorem 1.** If there exists a matrix norm \( \|\cdot\| \) induced over some vector norm \( |\cdot| \) such that \( \|F_k\| < 1, \forall k \in \mathbb{N} \), then the system (5) is globally asymptotically stable.

**Proof.** Let \( V(x_k) := |x_k| \). Then,

\[ V(x_{k+1}) = |x_{k+1}| \leq \|F_k\| |x_k| < |x_k| = V(x_k). \]

Hence \( V(\cdot) \) is a Lyapunov function for the system (5).

**Lemma 2.** Let \( \|\cdot\| \) be a matrix norm induced over some vector norm \( |\cdot| \). Then for any \( A \in \mathbb{C}^{n \times n} \)

\[ \|A\|_T := \|T^{-1} A T\| \]  

is an induced norm over the vector norm \( |\cdot|_T := \|T^{-1} \cdot\| \) (7)

where \( x \in \mathbb{C}^n \) and \( T \in \mathbb{C}^{n \times n} \) is an invertible matrix.

**Proof.** It is easy to see that \( |\cdot|_T \) is a vector norm. Then,

\[ \max_{x \in \mathbb{C}^n} \frac{|Ax|_T}{|x|_T} = \max_{y \in \mathbb{C}^n} \frac{|ATy|_T}{|Ty|_T} \]

\[ = \max_{y \in \mathbb{C}^n} \frac{|T^{-1}ATy|}{|T^{-1}Ty|} \]

\[ = \|A\|_T \]

**Corollary 1.** If there exists an invertible matrix \( T \in \mathbb{C}^{n \times n} \) such that \( \sigma(T^{-1}F_k T) < 1, \forall k \in \mathbb{N} \), then the system (5) is globally asymptotically stable, where \( \sigma(\cdot) \) is the maximum singular value.

**Proof.** It is well-known that the maximum singular value of a matrix is the induced matrix norm over Euclidean vector norm. Then the result follows from Lemma 2.

The necessary and sufficient conditions for assigning singular values using state feedback is given in Martin and Wang (2009). The following theorem is given without proof for completeness.

**Theorem 2.** For any given \( n \times n \) matrix \( A \) and \( n \times m \) full rank matrix \( B \) [over reals] with \( 1 \leq m \leq n \), let the singular values of

\[ (I - B(B^T B)^{-1}B^T) A \]

be

\[ 0 = a_1 = \cdots = a_m \leq a_{m+1} \leq \cdots \leq a_n. \]

For any given set of values

\[ 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \]

there exists a real \( F \) such that the singular values of \( A + BF \) are \( \{s_1, \ldots, s_n\} \) if and only if

\[ a_j \leq s_j \leq a_{j+m} \]

for all \( j = 1, \ldots, n \) (with the convention that \( a_j = \infty \) if \( j > n \)).

4. MAIN RESULTS

The following Corollary is immediate from Theorem 2 and Corollary 1.

**Corollary 2.** If there exists an invertible matrix \( T \in \mathbb{R}^{n \times n} \) such that

\[ \sigma \left( \left\{ I - \hat{G}(h)[\hat{G}^T(h)\hat{G}(h)]^{-1}\hat{G}^T(h) \right\} \hat{F}(h) \right) < 1 \]

for all \( h \in (0, h_{\max}) \) for some \( h_{\max} > 0 \), where

\[ \hat{F}(h) := T^{-1} e^{Ah} T \quad \text{and} \quad \hat{G}(h) := T^{-1} \left( \int_0^h e^{A\tau} \,d\tau \right) B, \]

then there exists \( \hat{K}(h) \in \mathbb{R}^{m \times n} \) such that

\[ \sigma \left( \hat{F}(h) + \hat{G}(h)\hat{K}(h) \right) < 1 \]

for all \( h \in (0, h_{\max}) \). Also the state feedback gain

\[ K_k := \hat{K}(h_k) T^{-1} \]

guarantees the stability of (4) for any given \( \{t_k\}_{k \in \mathbb{N}} \) such that \( h_k \in (0, h_{\max}) \).

**Theorem 3.** Let \( (A, B) \) is given in (1) and there exists \( K \in \mathbb{R}^{m \times n} \) such that \( A + BK \) is stable and has real distinct eigenvalues. Let \( T \) be the matrix such that

\[ D := T^{-1}(A + BK)T \]

is diagonal. Then, there exists an interval \( (0, h_{\max}) \) such that (11) is satisfied with this \( T \). Moreover, \( h_{\max} \) is determined by the eigenvalues of \( D \).
Proof. Let \( F(h) := e^{(A+BK)h} \). As a result of Theorem 9 in Sevim and Goren-Sumer (2016) we can state that there exists a \( K(h) \) such that
\[
\| F(h) + G(h)K(h) - F_d(h) \| < \delta
\]
for \( h \in (0, h_{\text{max}}) \) and arbitrary \( \delta > 0 \). Then,$$
\begin{align*}
\| T^{-1} [F(h) + G(h)K(h) - F_d(h)]T \| &= \| T^{-1} [F(h) + G(h)K(h) - F_d(h)] \| \| T \| \\
&\leq \| T^{-1} \| \| F(h) + G(h)K(h) - F_d(h) \| \| T \| \\
&< \delta \| T^{-1} \| \| T \| := \epsilon
\end{align*}
$$
This implies
\[
\| F(h) + G(h)K(h) \| < \| e^{Dh} \| + \epsilon
\]
By selecting
\[
\delta = 1 - \| e^{Dh} \| \| T^{-1} \| \| T \|
\]
we can see that there exists a \( K(h) \) that assigns the maximum singular value of \( F(h) + G(h)K(h) \) less than 1. From Theorem 2, we can conclude that
\[
\sigma \left( (I - \hat{G}(h)[\hat{G}^T(h)\hat{G}(h)]^{-1}\hat{G}^T(h))\hat{F}(h) \right) < 1
\]
Algorithim 1. The following design algorithm is proposed based on Theorem 3.

1. Find a \( K \in R^{n \times n} \) such that \( A+BK \) is Hurwitz and has real distinct eigenvalues.
2. Find \( T \) such that \( D := T^{-1}(A+BK)T \) is diagonal.
3. Calculate
\[
\hat{F}(h) := T^{-1}e^{Ah}T \quad \text{and} \quad \hat{G}(h) := T^{-1}\left( \int_{0}^{h} e^{A\tau} d\tau \right) B.
\]
4. Calculate the singular values of \( \hat{P}(h) := (I - \hat{G}(h)[\hat{G}^T(h)\hat{G}(h)]^{-1}\hat{G}^T(h))\hat{F}(h) \).

Due to Theorem 3, there exists \( h_{\text{max}} \) such that \( \sigma(\hat{P}(h)) < 1 \) for all \( h \in (0, h_{\text{max}}) \).
5. Determine the desired singular values according to Theorem 2.
6. Calculate \( \hat{K}(h) \) such that \( \hat{F}(h) + \hat{G}(h)\hat{K}(h) \) has desired singular values using the algorithm given in Martin and Wang (2009).
7. The controller \( \hat{K}_k := \hat{K}(h_k)T^{-1} \) is a stabilizing controller for the system (4).

5. NUMERICAL EXAMPLE

Example 1. We consider the example given in Haimovich and Osella (2013) with an additional unstable eigenvalue. This example is also given in our previous work Sevim and Goren-Sumer (2016).
\[
A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 2 \\ 1 \end{bmatrix}
\]
We selected the following state feedback gain to stabilize the system:
\[
K = \begin{bmatrix} 1128 & -1064 & -105 \\ 289 & 289 & 34 \end{bmatrix}
\]
The nonzero singular values of \( \hat{P}(h) \) are calculated as in Figure 1. From the figure, we can conclude that \( h_{\text{max}} \approx 0.62 \) and the singular values of \( \hat{F}(h) + \hat{G}(h)\hat{K}(h) \) can be arbitrarily assigned anywhere between the solid curves.

In Figure 2, the state response of the closed loop system \( \hat{x}(t) = Ax(t) + BK(t_k)x(t_k), \forall t \in [t_k, t_{k+1}) \) under random sampling is given.

6. CONCLUSIONS

We considered the stabilization problem of nonuniformly sampled systems with the assumption that sampling periods are known to the controller. We presented a theorem for the existence of a state feedback controller that assigns the singular values of the closed loop system matrix under similarity transformation, to desired values. Also, we proposed an algorithm to calculate the sampling-period-varying state feedback controller that guarantees stability for bounded and arbitrarily varying sampling intervals. We illustrated the accuracy of the method in numerical examples.
REFERENCES

Baillieul, J. and Antsaklis, P.J. (2007). Control and Communication Challenges in Networked Real-Time Systems. *IEEE Proc.*, 95(1), 9–28.

Chen, T. and Francis, B.A. (1995). *Optimal Sampled-Data Control Systems*. Springer London.

Fujioka, H. and Nakai, T. (2010). Stabilising systems with aperiodic sample-and-hold devices: state feedback case. *IET Control Theory & Applications*, 4(2), 265. doi: 10.1049/iet-cta.2009.0012.

Haimovich, H. and Osella, E.N. (2013). On controller-driven varying-sampling-rate stabilization via Lie-algebraic solvability. *Nonlinear Analysis: Hybrid Systems*, 7(1), 28–38. doi:10.1016/j.nahs.2012.04.001.

Hespanha, J., Naghshtabrizi, P., and Xu, Y.Y. (2007). A Survey of Recent Results in Networked Control Systems. *Proceedings of the IEEE*, 95(1). doi: 10.1109/JPROC.2006.887288.

Kao, C.Y. and Fujioka, H. (2013). On stability of systems with aperiodic sampling devices. *IEEE Transactions on Automatic Control*, 58(8), 2085–2090. doi: 10.1109/TAC.2013.2246491.

Kuo, B.C. (1995). *Digital Control Systems*. Oxford University Press, 2nd edition.

Liberzon, D. and Morse, A.S. (1999). Basic problems in stability and design of switched systems. *IEEE Control Systems*, 19(5), 59–70. doi:10.1109/37.793443.

Liberzon, D., Hespanha, J.P., and Morse, A.S. (1999). Stability of switched systems: a Lie-algebraic condition. *Systems and Control Letters*, 37, 117–122.

Lin, H. and Antsaklis, P.J. (2009). Stability and stabilization of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 54(2), 308–322. doi:10.1109/TAC.2008.2012009.

Lunze, J. and Lehmann, D. (2010). A state-feedback approach to event-based control. *Automatica*, 46(1), 211–215. doi:10.1016/j.automatica.2009.10.035.

Martin, C.F. and Wang, X.a. (2009). Singular Value Assignment. *SIAM Journal on Control and Optimization*, 48(4), 2388–2406. doi:10.1137/070704915.

Naghshtabrizi, P., Hespanha, J.P., and Teel, A.R. (2006). On the robust stability and stabilization of sampled-data systems: A hybrid system approach. *Decision and Control, 2006 45th IEEE Conference on*, 4873–4878. doi:10.1109/CDC.2006.377315.

Seuret, A. (2012). A novel stability analysis of linear systems under asynchronous samplings. *Automatica*, 48(1), 177–182. doi:10.1016/j.automatica.2011.09.033.

Sevim, U. and Goren-Sumer, L. (2016). Stabilization of Controller-Driven Nonuniformly Sampled Systems via Digital Redesign. *IFAC-PapersOnLine*, 49(9), 142–145. doi:10.1016/j.ifacol.2016.07.515.

Suh, Y.S. (2008). Stability and stabilization of nonuniform sampling systems. *Automatica*, 44(12), 3222–3226. doi: 10.1016/j.automatica.2008.10.002.

Zhang, W., Branicky, M.S., and Phillips, S.M. (2001). Stability of networked control systems. *Control Systems, IEEE*, 21(February), 84–99. doi:10.1109/37.898794.