Intersection complex via residue

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Graphical abstract

A normal crossing divisor gives rise to a stratification of a smooth scheme, and a logarithmic connection of a vector bundle along the divisor induces residue maps along each stratum.

Public summary

- We provide an intrinsic definition of intersection subcomplex via these residues.
- We present an explicit geometric description of it.

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Abstract: We provide an intrinsic algebraic definition of the intersection complex for a variety.

Keywords: algebraic geometry; intersection complex; weight filtration

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1 Introduction

Intersection homology theory is a generalization of singular homology for singular algebraic varieties.

In Ref. [1], Sheng and Zhang established a positive characteristic analog of an intersection cohomology theory for polarsed variations of Hodge structures and proposed an algebraic definition of the intersection complex, but with the help of coordinate systems. Here, we provide an intrinsic definition of the intersection complex via residues and provide a geometric description of it.

The remainder of this paper is organized as follows. Section 2 establishes notations and presents key definitions. Section 3 provides the main theorem and its proof. Finally, in Section 4, an explicit computation following the spirit of proof in surface case is made, and a counterexample is discussed.

2 Intersection complex

Let $(X, D)$ be a smooth scheme over a regular locally Noetherian scheme $S$ with a reduced smooth normal crossing divisor $D = \sum D_i$, where $I$ is a finite index set, and $\varepsilon$ be a locally free coherent sheaf with an integrable logarithmic $\lambda$-connection $\nabla$ along $D$.

We introduce some natural morphisms of log-differential sheaves before providing our definitions.

Suppose $X$ is of relative dimension $n$ over $S$. Owing to smoothness of $X$ and the definition of simple normal crossing divisors, for any $x \in X$, there exists a neighborhood $U$ of $x$ such that we can find a coordinate system $(t_1, \ldots, t_s, \varepsilon)$ such that $D \cap U$ is defined by the equation $t_1 = \cdots = t_s = 0$. As an immediate result, $\Omega^r_{\varepsilon, D}$ admits an $\Omega_D$ basis

$$\{\omega_i = \text{dlog}t_i, \ldots, \omega_s = \text{dlog}t_s; \omega_{s+1} = \text{d}t_{s+1}, \ldots, \omega_n = \text{d}t_n\}.$$

Moreover, it induces a free system of generators for $\Omega^r_{\varepsilon, D}(\text{log}D)$.

For $1 \leq i < j \leq r$ and $a \geq 1$, we define

$$\beta^j_\varepsilon : \Omega^r_\varepsilon(\text{log}D) \to \Omega^r_{\varepsilon, D}(\text{log}(D - D_i)_{|D_i})$$

where $\phi' \in \text{span} \omega_i$, with $i \notin I$.

One can consider $\beta^j_\varepsilon$ as taking the residual part of a log differential form along $D_i$, and $\gamma^j_\varepsilon$ is the restriction of the $D_i$ regular log differential forms to $D$. Obviously, $\beta^j_\varepsilon$ and $\gamma^j_\varepsilon$ are surjective and independent of the coordinate system, respectively. For simplicity, we omit the upper symbol $a$.

Clearly, for any log connection $\nabla$, the composite map $(\beta \otimes \text{Id}) \circ \nabla$ factors through $\gamma_i$.

$$\varepsilon \to \Omega_D \otimes \varepsilon \to \Omega_D \otimes \varepsilon.$$

We call the second map the residue map of $\nabla$ along $D_i$, and denote it as $\text{Res}_i(\nabla)$.

We can generalize morphisms above to the multi-indices case as follows. For a subset $I = \{j_1, \ldots, j_r\} \subseteq \{1, 2, \ldots, r\}$ with $j_1 < j_2 < \cdots < j_r$, set $D_I = \cap_{i \in I} D_i$, and define the residue $\text{Res}_I$ of the connection $\nabla$ along $D_I$ as follows:

$$\text{Res}_I(\nabla) \circ \text{Res}_I(\nabla) \circ \cdots \circ \text{Res}_I(\nabla).$$

We define $\beta_I$ and $\gamma_I$ in a similar manner.

The following diagram naturally commutes.

$$\begin{array}{c}
\varepsilon \otimes \Omega^r_{\varepsilon, D}(\text{log}(D - D_I)) \xrightarrow{\nabla(\text{inclusion})} \varepsilon \otimes \Omega^r_{\varepsilon, D}(\text{log}(D)), \\
\downarrow l \otimes \gamma_I \\
\varepsilon \otimes \Omega^r_{\varepsilon, D}(\text{log}(D - D_I))_{|D_I} \xrightarrow{\gamma_I \otimes \beta_I} \varepsilon \otimes \Omega^r_{\varepsilon, D}(\text{log}(D - D_I))_{|D_I}.
\end{array}$$

where $l : \varepsilon \to \varepsilon_{|D_I}$ is the canonical restriction map.

Now we can define the intersection complex. Set
\[ X_{n-s} = \bigcup_{s \leq |I|} X_{n-s} \cap D_f, \quad 1 \leq s \leq |I|, \]
then the following descending chain gives rise to a stratification of \( X \):
\[ X := X \supset X_{n-1} \supset \cdots \supset X_{n-s} \supset X_{n-I} = \emptyset. \]

And let \( j_i : U_i := X - X_{n-1} \to U_{n-1} = X - X_{n-I} \) be the natural inclusion for \( n - |I| \leq j_i \leq -\infty \).

**Definition 2.1.** Notations as above. We inductively define res-intersection complex \( IC \) as follows:
- \( IC^r(\mathcal{E}, \nabla)(U_i) \) is defined. A section \( \beta \in j_i \)
- \( IC^r(\mathcal{E}, \nabla)(U_i) \) belongs to \( IC^r(\mathcal{E}, \nabla)(U_{n-s}) \) when the following two conditions are satisfied:
  1. \( \beta \) has log pole along \( D_{n-s} \), and
  2. \( \text{Res}_{X_{n-s}} \beta \in \text{Im}(\text{Res}_{X_{n-s}} \nabla : \Omega^{a+s}_{X_{n-s}} \to \Omega^{a+s}_{X_{n-s}}) \)

Then we provide a geometric description of res-intersection complex in the sequel of this section. For any subset \( j \) of \( I \), let \( D_j = \cap_{i \in I} D_i, \quad D_j = D_j - \cup_{i \in j} (D_i \cap D_j) \) and let \( D_j = X - D_j \). Then theoretically, we have the equation \( X = \bigcup_{|j|} D_j \). Each \( D_j \) is a locally closed subspace of \( X \), and thus we can endow \( D_j \) with reduced subsheaf structure.

**Proposition 2.1.** If \( \text{Res}_i(\nabla) \) are bundle morphisms for all \( i \in \{1, 2, \cdots, r\} \), then the res-intersection complex is a complex of locally free sheaves if it is restricted to each stratum \( D_j \), where \( I \) is an index subset of \( \{1, 2, \cdots, r\} \) and \( D_j \) is endowed with a reduced subsheaf structure.

We employ the following lemma to prove Proposition 2.1 [3].

**Lemma 2.1.** Let \( X \) be a reduced Noetherian scheme, and let \( \mathcal{F} \) be a coherent sheaf on \( X \). Consider the function
\[ \phi_i(x) = \dim_{k(x)} \mathcal{F} \otimes_{O_{\mathcal{O}_x}} k(x), \]
where \( k(x) = O_x/m \) is the residue field at point \( x \). If \( \phi \) is constant, then \( \mathcal{F} \) is locally free.

**Proof of Proposition 2.1.** Consider the reduced scheme \( D_j \) and its associated coherent sheaf \( IC^r(\mathcal{E}, \nabla)|_{D_j} \). Because of the assumption the divisors are reduced, Lemma 2.1, the proposition is proven if we can show that the dimension of the fibre of sheaf, which is \( \phi|_{D_j} \), is constant over \( D_j \).

For each \( x \in D_j \), \( IC^r(\mathcal{E}, \nabla)|_{D_j} \) is an \( \Omega_{X_{n-s}} \) module spanned by basis \( [\text{Res}_i(\nabla)|_{D_j}] \otimes \otimes_{t} k(x) \), where \( \text{Res}_i(\nabla) \) is the fibre of \( \mathcal{E} \) and \( J \) is a subset of \( I \).

**3 Main theorem**
In the following, we show that the res-intersection subcomplex above coincides with the intersection subcomplex defined in Ref. [1].

Let \( X, D, \mathcal{E} \) be as in the previous section. Given a coordinate system
\[ \{t_1, t_2, \cdots, t_s, t_{s+1}, \cdots, t_r\} \]
of \( U \), locally we can write
\[ \nabla = \sum_{i \in I} \nabla_{d\log t_i} + \sum_{i \in I} \nabla_{d\log t_i}, \]
due to that the set \( \{d\log t_i | i \in I, 1 \leq k \leq n\} \) forms a basis of the log sheaf. For subset \( I = \{j_1, \cdots, j_s\} \subseteq I \) with \( j_1 < j_2 < \cdots < j_s \), let \( \nabla = \nabla_{j_1} \nabla_{j_2} \cdots \nabla_{j_s} \). We can generalize diagram (1) as follows:

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In Ref. [1], the intersection complex is defined as follows:  

**Definition 3.1.** $IC (X, e)$ is an $\Omega$ graded submodule of $\Omega (\log D) \otimes e$ generated by the abelian sub sheaf

\[ \sum_{i \in M} \nabla e_i \otimes \omega_i, \]

where $U$ is an open subset of $X$, and $M = \{1, 2, \ldots, r\}$.  

Our main theorem is as follows:  

**Theorem 3.2.** If $Res (\nabla) : \epsilon|_{U} \to \epsilon|_{U}$ are bundle morphisms for all $i \in I$, then $IC (X, e) = IC (X, e)$.  

This proof makes essential use of the weight filtration of the log complex.  

**Definition 3.3.** Weight filtration $W$ of the logarithmic complex is defined as follows:

\[ (l_1 \otimes \beta_1) (\nabla, \omega) = (Res (\nabla)) (\epsilon|_{U}) \otimes \Omega_{e, \beta_1} \]

where $l_1 : e \to \epsilon|_{U}$ is the canonical restriction map. And if $Res_i$ is a bundle morphism then

\[ (l_i \otimes \beta_i) (\nabla, \omega) = (Res (\nabla)) (\epsilon|_{U}) \otimes \Omega_{e, \beta_i} \]

Proof. It is a basic fact of weight filtration, for a rigorous proof of this lemma the reader is referred to [6] and [7].  

Firstly, one have to verify that the upper arrow is well defined. That is, one have to show that $\nabla, \omega \in (e \otimes \Omega_{e, \beta_1}^\oplus)$ is contained in $W^n (\Omega (\log D) \otimes e)$. It is straightforward because the source the map $e \otimes \Omega_{e, \beta_1}^\oplus$ is a weighted zero and the map $\nabla, \omega$ is of weight $[l]$.  

Note that we have $\beta_i (l_i \otimes \beta_1 (\nabla, \omega) = (Res (\nabla)) (\epsilon|_{U}) \otimes \Omega_{e, \beta_i}^\oplus$, hence the vertical arrow on the right is well-defined. The commutativity of the diagram follows from restricting the diagram 1 on sub bundle $e \otimes \Omega_{e, \beta_1}^\oplus \subset e \otimes \Omega_{e, \beta_1}^\oplus (\log D - \sum D_i|_{\Delta_0})$. It remains to show the equation 6. It is easy to see the sheaf on right side is contained in left side. By the commutativity of the diagram 1, one has the left side of the equation 6 is contained in

\[ (Res_i (\nabla)) (\epsilon|_{U}) \otimes \Omega_{e_i, \beta_i} (\log (D - \sum D_i|_{\Delta_0})). \]

Therefore, the equation 6 follows from the following claim.  

Claim: If $Res_i$ is a bundle morphism, then we have the equation

\[ (Res_i (\nabla)) (\epsilon|_{U}) \otimes \Omega_{e_i, \beta_i} (\log (D - \sum D_i|_{\Delta_0})) \cap (\epsilon|_{U}) \otimes \Omega_{e, \beta_i}^\oplus \]

\[ (Res_i (\nabla)) (\epsilon|_{U}) \otimes \Omega_{e_i, \beta_i} \]

For the "$\supseteq" direction, it is obvious. For the other direction, the sheaf $Res_i (\nabla) (\epsilon|_{U})$ is locally free due to the assumption that $Res_i$ is a bundle morphism, thus it has no torsion along $D_i \cap D_j$, where $j \in J = I - I$. This completes the proof of the claim.  

The second provides a local description of the weight filtration along divisor in terms of the coordinates.  

We now return to the proof of the main theorem.  

**Proof of Theorem 3.1.** Without a loss of generality, we as-
Let $X = \text{Spec}(k[t_1, t_2])$ be a surface and let $D = D_1 + D_2$, defined by the equation $t_1 t_2 = 0$. The divisor gives rise to a stratification of the surface as $X = D_1 t_1 D_2 D_2$. With the help of the coordinates $t_i$, we can write $\nabla = \nabla_1 d \log t_1 + \nabla_2 d \log t_2$.

1. $IC^0(X, \mathcal{E}) = IC^0(X, \mathcal{E})$, because both are equal to $E$.
2. In the one-degree term, note that the sections of the sheaf $IC^1(X, \mathcal{E})$ are of the form:

$$s = \nabla_1(e_1) d \log t_1 + \nabla_2(e_2) d \log t_2 + f_1 d t_1 + f_2 d t_2,$$

Let $s \in IC^1(X, \mathcal{E}) (X)$. To verify $s \in IC^1(X, \mathcal{E})$, we aim to find $e_1$ and $e_2$. By definition, $s$ satisfies $\beta(s) \in \text{Im}(\text{Res}_1)$: Consider a commutative diagram:

$$\begin{array}{ccc}
E & \xrightarrow{\gamma} & E \otimes \Omega^1_X (\log(D)) \\
\downarrow \quad \gamma & & \downarrow \Box_{\beta_1} \\
E_{|S_0} & \xrightarrow{\text{Res}_1} & E_{|S_0}
\end{array}$$

The first vertical arrow is surjective, so we can find a section $e_1 \in E$ such that $s - \nabla_1(e_1) d \log t_1$ is in the kernel of the second vertical, which is $E \otimes \Omega^1_X (\log(D_2))$, replacing 1 with 2, and we obtain a section $e_1$ of $E$, by the exact sequence (4),

$$s_1 = s - (\nabla_1(e_1) d \log t_1 + \nabla_2(e_2) d \log t_2)$$

is of weight zero. In other words, it is regular, which allows us to write:

$$s = \nabla_1(e_1) d \log t_1 + \nabla_2(e_2) d \log t_2 + f_1 d t_1 + f_2 d t_2,$$

where $e_1, e_2, f_1, f_2 \in \mathfrak{S}(U)$.

3. Using the same pattern, a section $\omega$ in $IC^2(X, \mathcal{E})(X)$ is of the form:

$$t = (\nabla_{12}(e_{12}) d \log t_1 \wedge d \log t_1 + \nabla_1(e_1) d \log t_1 \wedge d t_1 + \nabla_2(e_2) d \log t_2 \wedge d t_2 + \nabla_{22}(e_{22}) d t_2 \wedge d t_2)$$

By definition. For any section $s \in IC^2(X, \mathcal{E})(X)$, we aim to get section $e_{12}, e_1, e_2, f$. By (5), we have the following commutative diagram:

$$\begin{array}{ccc}
E & \xrightarrow{\gamma_{12}} & E \otimes \Omega^2_X (\log(D)) \\
\downarrow \quad \gamma_{12} & & \downarrow \Box_{\beta_{12}} \\
E_{|S_{01}} & \xrightarrow{\text{Res}_{12}} & E_{|S_{01}}
\end{array}$$

where both vertical arrows are surjective, we obtain $e_{12}, e_1, e_2$ such that $s_2 := s - (\nabla(e_{12}))_0 \omega_0 \in \text{ker} E \otimes \Box_{\beta_{12}}$, which is of weight one by short exact sequence (4).

Then consider diagram (5) and set $a = 2, I = \{1, 2\}$. Using the same argument as above, we obtain $\tilde{e}_i$ and $\tilde{e}_i \in E \otimes \Omega^2_X$, such that $s_2 - \nabla(e_i) d \log t_i \wedge d t_i - \nabla(e_i) d \log t_i \wedge d t_i$ in $\text{ker}(\beta_{12} \otimes \Box_{\beta_1})$, which is exactly $\Omega^2_X \otimes E$ by (4). Therefore, $e_{12}, e_1, e_2, f$ are the section we want. This completes our proof.

In the sequel of the section we will present an example, which is provided by Ref. [1], to show that the main theorem will be wrong if the residue morphisms are not assumed to be bundle morphisms. Let $k$ be a perfect field of character $p$, $(X, D)$ is as above. Define logarithmic connection over:

$$\nabla : O_X \rightarrow \Omega^1_{X/k}(\log D)$$

$$f \rightarrow df + (f t_2) \cdot \frac{dt_1}{t_1^p}.$$

One can verify that $\nabla$ is integral, and the residue morphisms are as follows:

$$\text{Res}_1(\nabla) = t_2^p; \text{Res}_2(\nabla) = 0.$$ (8)

One can see that $\text{Res}_2(\nabla)$ has torsion at $t_2 = 0$, hence it is not a bundle morphism. By definition, we have $IC^2 = (t_2^p d \log t_1 \wedge d t_1 O_X + (f_2 d t_2) \wedge \mathcal{O}_X$. Consider the section $e_{12} d \log t_1 \wedge d t_2$, it is a section in $IC^2$, but not in $IC^2$.

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Conflict of interest
The author declares that he has no conflict of interest.

Biographies
Xiaojin Lin is currently a graduate student under the tutelage of Prof. Mao Sheng at the University of Science and Technology of China. His research interests focus on Hodge theory and vector bundle.

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