Brownian local minima and other random dense countable sets

Boris Tsirelson

Abstract

We compare two examples of random dense countable sets, Brownian local minima and unordered uniform infinite sample. They appear to be identically distributed. A framework for such notions is proposed. In addition, random elements of other singular spaces (especially, reals modulo rationals) are considered.

Introduction

For almost every Brownian path \( \omega = (b_t)_{t \in [0,1]} \) on \([0,1]\), the set

(0.1) \[ M_\omega = \{ s \in (0,1) : \exists \varepsilon > 0 \ \forall t \in (s - \varepsilon, s) \cup (s, s + \varepsilon) \ b_s < b_t \} \]

of local minimizers on \((0,1)\) is a dense countable subset of \((0,1)\). Should we say that \((M_\omega)_\omega\) is a random countable dense set? Can we give an example of an event of the form \(\{ \omega : M_\omega \in A \}\) possessing a probability different from 0 and 1? No, we cannot (see also Corollary 5.1). All dense countable subsets of \((0,1)\) are a set DCS(0, 1) (of sets), just a set, not a Polish space, not even a standard Borel space. What should we mean by a DCS(0, 1)-valued random variable and its distribution? Apart from such conceptual questions we have specific examples and questions; here is one. A 'uniform infinite sample', that is, an infinite sample from the uniform distribution on \((0,1)\) may be described by the product \(((0,1)^\infty, \text{mes}^\infty)\) of an infinite sequence of copies of the probability space \(((0,1), \text{mes})\), where 'mes' stands for the Lebesgue measure on \((0,1)\). For almost every point \(u = (u_1, u_2, \ldots)\) of this product space, the set

(0.2) \[ S_u = \{u_1, u_2, \ldots\} = \{ s \in (0,1) : \exists n \ u_n = s \} \]

is a dense countable subset of \((0,1)\). It appears that \((M_\omega)_\omega\) and \((S_u)_u\) are identically distributed in the following sense (see Th. 4.7).
Theorem. There exists a joining $J$ between the Brownian motion on $[0, 1]$ and the uniform infinite sample such that $M_\omega = S_u$ for $J$-almost all pairs $(\omega, u)$.

The theorem follows from a more general theory presented below. If you consider the theory too general, try to find a better proof of this theorem or maybe its two-point corollary; namely, construct (at least) two independent uniform random variables $U_1, U_2$ coupled with the Brownian motion $(B_t)_t$ in such a way that almost surely $U_1, U_2$ are (some of the) local minimizers of $(B_t)_t$.

1 Definitions

The set $DCS(0, 1)$ is a singular space in the sense of Kechris [2, §2]: a ‘bad’ quotient space of a ‘good’ space by a ‘good’ equivalence relation. (A simpler example of a singular space is $\mathbb{R}/\mathbb{Q}$, reals modulo rationals.) Namely,

$$DCS(0, 1) = (0, 1)^{\infty}/E.$$

Here $(0, 1)^{\infty}$ is the set of all sequences $u = (u_1, u_2, \ldots)$ of pairwise different points of $(0, 1)$, and $E$ is the following equivalence relation on $(0, 1)^{\infty}$:

$$E = \{(u, v) : S_u = S_v\},$$

$S_u$ being defined by (1.2). (In fact, equivalence classes are orbits of a natural action of the infinite permutation group, see [4, Sect. 2e].) Note that $(0, 1)^\infty$ is a standard Borel space and $E$ is a Borel subset of $(0, 1)^{\infty} \times (0, 1)^{\infty}$. It is possible to equip the quotient space with its natural $\sigma$-field (of sets whose inverse images are measurable) and define random variables and distributions accordingly. Is it a good idea? I do not know. (See also Sect. 5.) I prefer another concept of a random element in a singular space, sketched in [4, Sect. 2e] and formalized below.

Throughout Sections 1–4, either by assumption or by construction, all probability spaces are standard. Recall that a standard probability space (known also as a Lebesgue-Rokhlin space) is a probability space isomorphic (mod 0) to an interval with the Lebesgue measure, a finite or countable collection of atoms, or a combination of both.

1.3 Definition. Let $B$ be a standard Borel space, $E \subseteq B \times B$ an equivalence relation on $B$, and $\Omega$ (or rather $(\Omega, \mathcal{F}, P)$) a probability space. A map $X :
Ω → B/E is called *measurable*, if there exists a measurable map \( Y : \Omega \to B \)
such that the following diagram is commutative:

\[
\begin{array}{ccc}
\Omega & \xrightarrow{Y} & B \\
\downarrow{X} & & \downarrow{\pi} \\
B/E & &
\end{array}
\]

Note that \( B \) and \( E \) have to be given. We do not touch on the question,
what happens if (in some sense) \( B/E = B_1/E_1 \). Note also that Def. 1.3
is in the spirit of the ‘diffeology’ (see [1], especially Sect. 1.14 ‘Quotient of
manifolds’ and 1.15 ‘The irrational torus’).

Equivalence classes of measurable maps \( \Omega \to B \) are elements of the set
\( L_0(\Omega \to B) \) of \( B \)-valued random variables on \( \Omega \). Similarly, we define
\( L_0(\Omega \to B/E) \) as the set of all equivalence classes of measurable maps \( \Omega \to B/E \)
(the equivalence being the equality almost everywhere, as usual). Being equipped
with the natural \( \sigma \)-field, the set \( L_0(\Omega \to B) \) is a standard Borel space. The
set \( L_0(\Omega \to B/E) \) may be treated as a singular space,

\[
L_0(\Omega \to B/E) = L_0(\Omega \to B)/L_0(\Omega \to E)
\]

where \( L_0(\Omega \to E) \) is the following equivalence relation on \( L_0(\Omega \to B) \):
\( (f,g) \in L_0(\Omega \to E) \iff (f(\omega),g(\omega)) \in E \) for almost all \( \omega \).

Def. 1.3 is compatible with the usual definition in the following sense. Let
\( A, B \) be two standard Borel spaces, \( f : B \to A \) a Borel function, and \( E = \{(x_1, x_2) : f(x_1) = f(x_2)\} \). Then \( B/E = A \) (after the evident identification).

It is easy to check that

\[
L_0(\Omega \to B/E) = L_0(\Omega \to A)
\]

(after the evident identification); here \( L_0(\Omega \to A) \) is defined as usual, while
\( L_0(\Omega \to B/E) \) is defined by 1.3.

Waiving the \( \sigma \)-field on \( B/E \) we lose the usual definition of a distribution
on \( B/E \). Instead we may define the notion ‘identically distributed’ as follows.

1.4 Definition. Let \( B \) be a standard Borel space, \( E \subset B \times B \) an equiva-
lence relation on \( B \), and \( \Omega_1, \Omega_2 \) probability spaces. Random variables
\( f \in L_0(\Omega_1 \to B/E), g \in L_0(\Omega_2 \to B/E) \) are *identically distributed*, if
there exist a probability space \( \Omega \) and measure preserving maps \( T_1 : \Omega \to \Omega_1, \)
\( T_2 : \Omega \to \Omega_2 \) such that \( f(T_1(\omega)) = g(T_2(\omega)) \) for almost all \( \omega \in \Omega \).
That is, the following diagram must be commutative (mod 0):

\[ \begin{array}{ccc}
\Omega & \xrightarrow{T_1} & \Omega_1 \\
& \searrow & \nwarrow \\
& & \Omega_2 \\
& \swarrow & \nearrow \\
& B/E & \\
\end{array} \]

The joint distribution of \( T_1(\omega), T_2(\omega) \) is a joining, that is, a measure \( J \) on \( \Omega_1 \times \Omega_2 \) with given marginals \( P_1, P_2 \). Here is a definition equivalent to Def. 1.4: \( f, g \) are identically distributed, if there exists a joining \( J \) between \( \Omega_1 \) and \( \Omega_2 \) such that \( f(\omega_1) = g(\omega_2) \) for \( J \)-almost all pairs \( (\omega_1, \omega_2) \).

Def. 1.4 is compatible with the usual definition, similarly to Def. 1.3. Namely, let \( B/E = A \) be a standard Borel space. Then \( f \in L_0(\Omega_1 \to B/E), g \in L_0(\Omega_2 \to B/E) \) are identically distributed according to Def. 1.4 if and only if \( f \in L_0(\Omega_1 \to A), g \in L_0(\Omega_2 \to A) \) are identically distributed in the usual sense.

**1.5 Definition.** Let \( B \) be a standard Borel space and \( E \subset B \times B \) an equivalence relation on \( B \). A distribution on \( B/E \) is an equivalence class of \( B/E \)-valued random variables on \( ((0,1), \mes) \); here equivalence of two random variables means that they are identically distributed.

Def. 1.5 is compatible with the usual definition (similarly to 1.3, 1.4).

We return to DCS\((0,1)\) treated as \( B/E \) according to (1.1), (1.2). The first example of a DCS\((0,1)\)-valued random variable is the unordered uniform infinite sample. We define it as the \( B/E \)-valued random variable corresponding to the ordered uniform infinite sample. The latter is the \( B \)-valued random variable \( (U_1, U_2, \ldots) \); here \( B = (0,1)_{\infty}^\mathbb{N} \) and \( U_1, U_2, \ldots \) are i.i.d. random variables uniform on \((0,1)\). The unordered uniform infinite sample depends on the choice of \( U_1, U_2, \ldots \) and the underlying probability space, but its distribution is uniquely determined.

## 2 Main lemma

**2.1 Lemma.** Let \( X_1, X_2, \ldots \) be real-valued random variables (on some probability space) such that for every \( n = 1, 2, \ldots \) the conditional distribution of \( X_n \) given \( X_1, \ldots, X_{n-1} \) has a density \( (x, \omega) \mapsto f_n(x, \omega) \). If

\[
\sum_{n=1}^{\infty} f_n(x, \omega) = \begin{cases} 
\infty & \text{for } 0 < x < 1, \\
0 & \text{otherwise}
\end{cases}
\]

then...
for almost all \( x \) and \( \omega \), then the DCS\((0,1)\)-valued random variable

\[
\omega \mapsto \{X_1(\omega), X_2(\omega), \ldots \} = \{ x \in \mathbb{R} : \exists n \ X_n(\omega) = x \}
\]

is distributed like an unordered uniform infinite sample.

The proof is given below after some discussion. We see that the distribution of an unordered (not just uniform) infinite sample does not depend on the underlying one-dimensional distribution on \((0,1)\) provided that the latter distribution has a strictly positive density on \((0,1)\). The same holds for independent (not just identically distributed) \(X_n\), provided that each \(X_n\) has a density \(f_n\) and \(f_1 + f_2 + \cdots = \infty\) almost everywhere on \((0,1)\). Especially, the case \(f_1 = f_3 = \ldots, f_2 = f_4 = \ldots\) leads to the following fact.

**2.2 Corollary.** If \(\Omega_1 \ni \omega_1 \mapsto A(\omega_1) \in \text{DCS}(a,b)\) is an unordered uniform infinite sample on \((a,b)\) and \(\Omega_2 \ni \omega_2 \mapsto B(\omega_2) \in \text{DCS}(b,c)\) is an unordered uniform infinite sample on \((b,c)\) then

\[
\Omega_1 \times \Omega_2 \ni (\omega_1, \omega_2) \mapsto A(\omega_1) \cup B(\omega_2) \in \text{DCS}(a,c)
\]

is distributed like an unordered uniform infinite sample on \((a,c)\).

**Proof of Lemma 2.1.** We introduce a Poisson random subset of the strip \((0,1) \times (0,\infty)\) on some probability space \(\Omega, \Omega \ni \omega \mapsto A(\omega) \subset (0,1) \times (0,\infty)\), whose intensity measure is the (two-dimensional) Lebesgue measure on the strip. Almost surely, \(A(\omega)\) is a countable, locally finite set. We define functions \(f_n : (0,1)^n \to [0,\infty)\) by

\[
f_n(x, \omega) = g_n(X_1(\omega), \ldots, X_{n-1}(\omega), x)
\]

(some ambiguity in \(g_n\) is harmless) and construct random variables \(Y_1, Y_2, \cdots : \Omega \to (0,1)\) and \(T_1, T_2, \cdots : \Omega \to (0,\infty)\) step by step, as follows.

The first step:

\[
T_1(\omega) = \min \{ t > 0 : \exists y \in (0,1) \ (y, tg_1(y)) \in A(\omega) \} = \min_{(y,h) \in A(\omega)} \frac{h}{g_1(y)};
\]

this random variable is distributed Exp\((1)\), since \(\int_0^1 g_1(y) \, dy = 1\). The corresponding point \(y\) (evidently unique a.s.) gives us \(Y_1(\omega)\),

\[
(Y_1(\omega), T_1(\omega)g_1(Y_1(\omega))) \in A(\omega).
\]
The random variable $Y_1$ is distributed like $X_1$ (since $g_1$ is its density) and independent of $T_1$.

Probabilistic statements about the second step (below) are conditioned on $T_1$ and $Y_1$. The conditioning does not perturb the Poisson set $A$ above the graph of the function $T_1g_1(\cdot)$.

The second step:

$$ T_2(\omega) = \min\{t > 0 : \exists y \in (0,1) \ (y, T_1(\omega)g_1(y) + tg_2(Y_1(\omega), y)) \in A(\omega)\} $$

is distributed Exp(1) (since $\int_0^1 g_2(Y_1(\omega), y) \, dy = 1$ a.s.), and we define $Y_2$ as the unique $y$,

$$ (Y_2(\omega), T_1(\omega)g_1(Y_2(\omega)) + T_2(\omega)g_2(Y_1(\omega), Y_2(\omega))) \in A(\omega). $$

Random variables $T_2, Y_2$ are independent; $T_2$ is distributed Exp(1), while $Y_2$ has the density $g_2(Y_1, \cdot)$. These relations are conditional; unconditionally, the pair $(Y_1, Y_2)$ is distributed like $(X_1, X_2)$ and independent of the pair $(T_1, T_2) \sim \text{Exp}(1) \otimes \text{Exp}(1)$. Conditioning on $T_1, Y_1, T_2, Y_2$ does not perturb the Poisson set $A$ above the graph of the function $T_1g_1(\cdot) + T_2g_2(Y_1, \cdot)$.

Continuing the process we get random variables $T_n, Y_n \ (n = 1, 2, \ldots)$ on $\Omega$ such that the sequence $(Y_1, Y_2, \ldots)$ is distributed like $(X_1, X_2, \ldots)$ and independent of the i.i.d. sequence $(T_1, T_2, \ldots)$ of Exp(1) random variables. Conditioning on all $Y_n$ and $T_n$ does not perturb the Poisson set $A$ above the graph of the function $\sum_n T_ng_n(Y_1, \ldots, Y_{n-1}, \cdot)$ (a void claim if the sum is infinite everywhere).
Now we use the condition \( \sum_n f_n(x, \omega) = \infty \). It gives us
\[
\sum_n g_n(Y_1(\omega), \ldots, Y_{n-1}(\omega), x) = \infty
\]
for almost all \( \omega \in \Omega, x \in (0, 1) \). It follows that
\[
\sum_n T_n(\omega)g_n(Y_1(\omega), \ldots, Y_{n-1}(\omega), x) = \infty
\]
for almost all \( \omega \in \Omega, x \in (0, 1) \) (since the relation holds conditionally, given \( (Y_1, Y_2, \ldots) \)). We see that almost no points of the strip remain above the graph of this sum, and therefore, no one point of \( A(\omega) \) does (a.s.). All points of \( A(\omega) \) are used in our construction. Therefore the DCS(0, 1)-valued random variable \( \omega \mapsto \{Y_1(\omega), Y_2(\omega), \ldots\} \) is just the projection of the Poisson set \( A(\omega) \), therefore, an unordered uniform infinite sample. On the other hand, \( \{Y, Y_2, \ldots\} \) is distributed like \( \{X, X_2, \ldots\} \).

### 3 A sufficient condition

The condition \( \sum f_n(x, \omega) = \infty \) of Lemma 2.1 may be checked pointwise. For every \( x \) we have a series of random variables \( f_n(x, \cdot) \geq 0 \), and check its divergence a.s. If this holds for all (or almost all) \( x \in (0, 1) \), Lemma 2.1 is applicable.

Let \( Y_1, Y_2, \ldots : \Omega \to [0, \infty) \) be a sequence of random variables (generally, interdependent). We seek a sufficient condition for the property

\[
(3.1) \quad \sum Y_n = \infty \quad \text{a.s.}
\]

If \( \sum Y_n < \infty \) a.s. then \( Y_n \to 0 \) a.s., which implies \( \mathbb{P}(Y_n < \varepsilon) \to 1 \) for any \( \varepsilon > 0 \) (since indicators \( 1_{[\varepsilon, \infty)}(Y_n) \) converge to 0 a.s.). Given an event \( A \subset \Omega, \mathbb{P}(A) > 0 \), we may apply the remark above to the probability space \( A \) (with the conditional measure). The case \( A = \{\sum Y_n < \infty\} \) leads to

\[
\liminf_n \mathbb{P}(Y_n < \varepsilon) \geq \mathbb{P}(\sum Y_n < \infty),
\]

thus,

\[
\mathbb{P}(\sum Y_n < \infty) \leq \liminf_{\varepsilon \to 0^+} \mathbb{P}(Y_n < \varepsilon) \leq \lim_{\varepsilon \to 0^+} \sup_n \mathbb{P}(Y_n < \varepsilon).
\]

We see that the condition

\[
\lim_{\varepsilon \to 0^+} \sup_n \mathbb{P}(Y_n < \varepsilon) = 0
\]

is sufficient.
is sufficient for (3.1). Unfortunately, this sufficient condition is too strong for our purpose. We assume a weaker condition

\[(3.2) \lim_{\varepsilon \to 0^+} \sup_n \mathbb{P}(0 < Y_n < \varepsilon) = 0.\]

Surely, (3.2) does not imply (3.1), since \(Y_n\) may vanish. We seek an additional condition on the events \(\{Y_n = 0\}\).

Once again, if \(\sum Y_n < \infty\) a.s. then \(\mathbb{P}(Y_n < \varepsilon) \to 1\) for any \(\varepsilon\); combined with (3.2) it gives \(\mathbb{P}(Y_n = 0) \to 1\). As before, we condition on the event \(\{\sum Y_n < \infty\}\) (which does not invalidate (3.2); of course we assume here that the event is of positive probability). The straightforward conclusion

\[\lim inf_n \mathbb{P}(Y_n = 0) \geq \mathbb{P}(\sum Y_n < \infty)\]

is of little interest; instead, we introduce the condition

\[(3.3) \mathbb{P}(A \cap \{Y_n > 0\}) \to 0 \quad \text{implies} \quad \mathbb{P}(A) = 0\]

for all measurable sets \(A \subset \Omega\).

**3.4 Lemma.** Every sequence \((Y_n)\) satisfying both (3.2) and (3.3) satisfies (3.1).

*Proof.* Otherwise \(\mathbb{P}(Y_n = 0 \mid A) \to 1\) where \(A = \{\sum Y_n < \infty\}, \mathbb{P}(A) > 0\). Thus, \(\mathbb{P}(Y_n > 0 \mid A) \to 0\) in contradiction to (3.3). \(\square\)

Now we need a condition sufficient for (3.3). Let \(T : \Omega \to \Omega\) be a (strongly) mixing measure preserving transformation and \(B \subset \Omega\) a measurable set, \(\mathbb{P}(B) > 0\). Then

\[\mathbb{P}(A \cap T^{-n}(B)) \to \mathbb{P}(A)\mathbb{P}(B)\]

for every measurable set \(A \subset \Omega\), which ensures (3.3) if the events \(\{Y_n > 0\}\) are of the form \(T^{-n}(B)\). However, we need a more general case,

\[\{Y_n > 0\} = T^{-n}(A_n)\]

where \(\{A_1, A_2, \ldots\}\) is a precompact set (of events). The precompactness means that every subsequence \((A_{n_k})_k\) contains a subsequence \((A_{n_{k_i}})_i\) such that \(\mathbb{P}(A_{n_{k_i}} \setminus A_{n_j}) \to 0\) as \(i, j \to \infty\). Or equivalently, all indicator functions \(1_{A_n}\) belong to a single compact subset of \(L_2(\Omega, \mathcal{F}, \mathbb{P})\).
3.5 Lemma. Let $T : \Omega \to \Omega$ be a mixing measure preserving transformation and $A_n \subset \Omega$ measurable sets such that

$$\limsup_n \mathbb{P}(A_n) > 0$$

and $\{A_1, A_2, \ldots\}$ is a precompact set. Then $\mathbb{P}(A \cap T^{-n}(A_n)) \to 0$ implies $\mathbb{P}(A) = 0$ for all measurable sets $A \subset \Omega$.

Proof. The isometric operator $U : L_2 \to L_2$ defined by $(Uf)(\omega) = f(T\omega)$ satisfies

$$U^n \to \mathbb{E} \quad \text{weakly as } n \to \infty$$

where $\mathbb{E}$ is the expectation treated as the projection onto the one-dimensional space of constants. It follows that for every $g \in L_2$ the convergence

$$\langle U^n f, g \rangle - \langle \mathbb{E} f, g \rangle \to 0$$

is uniform in $f$ as long as $f$ runs over a compact set. We take $f = f_n = 1_{A_n}$, $g = 1_A$ and get

$$\mathbb{P}(T^{-n}(A_n) \cap A) - \mathbb{P}(A_n)\mathbb{P}(A) = \langle U^n f_n, g \rangle - \langle \mathbb{E} f_n, g \rangle \to 0.$$ 

If $A$ satisfies $\mathbb{P}(A \cap T^{-n}(A_n)) \to 0$ then $\mathbb{P}(A_n)\mathbb{P}(A) \to 0$ which implies $\mathbb{P}(A) = 0$. \qed

The following proposition combines the ideas of 3.4, 3.5 and introduces one more idea (the transition from $Z_n$ to $Y_n$) needed for the next section.

3.6 Proposition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $T : \Omega \to \Omega$ a mixing measure preserving transformation, $Y_1, Y_2, \ldots : \Omega \to [0, \infty)$ and $Z_1, Z_2, \ldots : \Omega \to [0, \infty)$ random variables, and $A_1, A_2, \ldots \subset \Omega$ a precompact sequence of measurable sets such that $\limsup_n \mathbb{P}(A_n) > 0$. Assume that

(a) $\{Y_n \neq 0\} = \{Z_n \neq 0\} = T^{-n}(A_n)$ for each $n$,

(b) $Y_n = \mathbb{E}(Z_n | Y_n)$ for each $n$,

(c) $\lim_{\varepsilon \to 0+} \sup_n \mathbb{P}(0 < Z_n < \varepsilon) = 0$.

Then $\sum Y_n = \infty$ a.s.

Proof. First, we claim that

$$\mathbb{P}(0 < Y_n < \varepsilon) \leq 2\mathbb{P}(0 < Z_n < 2\varepsilon) \quad (3.7)$$

for all $n$ and $\varepsilon$. Proof: conditioning on the event $\{Y_n \neq 0\} = \{Z_n \neq 0\}$ reduces (3.7) to a simpler claim: $\mathbb{P}(Y_n < \varepsilon) \leq 2\mathbb{P}(Z_n < 2\varepsilon)$ for any random variables $Y, Z : \Omega \to [0, \infty)$ such that $Y = \mathbb{E}(Z | Y)$. We note that

$$\mathbb{P}(Z \geq 2\varepsilon | Y) \leq \frac{1}{2\varepsilon} \mathbb{E}(Z | Y) = \frac{1}{2\varepsilon} Y,$$
thus,
\[
P(Y < \varepsilon) = P\left(\frac{1}{2\varepsilon}Y < \frac{1}{2}\right) \leq P\left(\frac{1}{2}P(Z \geq 2\varepsilon | Y) < \frac{1}{2}\right) = \\
= P\left(P\left(Z < 2\varepsilon | Y \right) > \frac{1}{2}\right) \leq \left(\frac{1}{2}\right)^{-1}E\left(P\left(Z < 2\varepsilon | Y \right)\right) = 2P(Z < 2\varepsilon),
\]
which proves the claim.

Combining (3.7) with (c) we see that the sequence \((Y_n)_n\) satisfies (3.2). On the other hand, Lemma 3.5 combined with (a) gives (3.3). Lemma 3.4 completes the proof. □

3.8 Remark. Condition 3.6(a) may be relaxed: \(\{Y_k \neq 0\} = \{Z_k \neq 0\} = T^{-n_k}(A_k)\) for some \(n_1 < n_2 < \ldots\)

4 Main theorem

We consider the usual one-dimensional Brownian motion \((t, \omega) \mapsto B_t(\omega)\) for \(t \in [0, 1]; \omega\) runs over a probability space \((\Omega, \mathcal{F}, P)\). The set \(M_\omega\) of all local minimizers of the path \(t \mapsto B_t(\omega)\) on \((0, 1)\) is well-known to be a dense countable set,

\[M_\omega \in \text{DCS}(0, 1)\quad \text{for almost all } \omega.\]

4.1 Lemma. The map \(\omega \mapsto M_\omega\) from \(\Omega\) to \(\text{DCS}(0, 1)\) is measurable (as defined by 1.3 using (1.1)).

Proof. We need a measurable enumeration of \(M_\omega\), that is, a sequence of random variables \(X_1, X_2, \ldots : \Omega \rightarrow (0, 1)\) such that

\[(4.2)\quad M_\omega = \{X_1(\omega), X_2(\omega), \ldots\},\]

\[X_k(\omega) \neq X_l(\omega) \quad \text{for } k \neq l\]

for almost all \(\omega\). We enumerate all dyadic intervals by the numbers 2, 3, 4, \ldots,

\[
\begin{array}{cccccccccc}
 n & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
 I_n & (0, 1) & (0, \frac{1}{2}) & \left(\frac{1}{2}, 1\right) & (0, \frac{1}{4}) & \left(\frac{1}{4}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{3}{4}\right) & \left(\frac{3}{4}, 1\right) & (0, \frac{1}{8}) & \ldots \\
\end{array}
\]

For each \(n > 1\) we consider the left half \(I'_n = I_{2n-1}\) and the right half \(I''_n = I_{2n}\) of \(I_n\), the corresponding Brownian minimizers \(X'_n, X''_n\),

\[X'_n \in I'_n, \quad B_{X'_n} = \inf_{t \in I'_n} B_t, \quad X''_n \in I''_n, \quad B_{X''_n} = \inf_{t \in I''_n} B_t,\]

and define \(X_n\) as the minimizer that corresponds to the greater minimum,

\[X_n = \begin{cases} 
X'_n & \text{if } B_{X'_n} > B_{X''_n}, \\
X''_n & \text{if } B_{X'_n} < B_{X''_n}.
\end{cases}\]
In addition we define $X_1$ as the Brownian minimizer on the whole $(0,1)$.

For every $k = 0, 1, 2, \ldots$ the $2^k$ numbers $X_1, \ldots, X_{2^k}$ are nothing but the Brownian minimizers on the $2^k$ dyadic intervals $I_n$ for $2^k < n \leq 2^{k+1}$, that is, the intervals $((i-1)/2^k, i/2^k)$ for $i = 1, \ldots, 2^k$ (randomly rearranged, of course). Therefore (4.2) is satisfied.

4.3 Lemma. The random variables $X_1, X_2, \ldots$ introduced in the proof of Lemma 4.1 are such that for every $n = 1, 2, \ldots$ the conditional distribution of $X_n$ given $X_1, \ldots, X_{n-1}$ has a density $(x, \omega) \mapsto f_n(x, \omega)$.

Proof. We define by $\mathcal{E}_n$ the sub-$\sigma$-field of $\mathcal{F}$ generated by $X_1, \ldots, X_{n-1}$, and by $C_n$ the event $\{X_n \in I_n^\prime\}$. Note that $C_n \in \mathcal{E}_{n-1}$, since $C_n = \{X_1 \in I_1^\prime\} \cup \cdots \cup \{X_{n-1} \in I_{n-1}^\prime\}$. Note also that $C_n = \{X_n \in I_{2n-1}\}$ and $\Omega \setminus C_n = \{X_n \in I_{2n}\}$. We define by $\mathcal{G}_n$ the sub-$\sigma$-field of $\mathcal{F}$ generated by all $B_s$ for $s \in [0,1] \setminus I_n$, and by $\mathcal{F}_n$ the sub-$\sigma$-field of $\mathcal{F}$ that contains $C_n$, coincides with $\mathcal{G}_{2n-1}$ on $C_n$ and with $\mathcal{G}_{2n}$ on $\Omega \setminus C_n$. In other words, $\mathcal{F}_n$ consists of sets of the form $(A \cap \{X_n \in I_{2n-1}\}) \cup (B \cap \{X_n \in I_{2n}\})$ for $A \in \mathcal{G}_{2n-1}, B \in \mathcal{G}_{2n}$.

We claim that $\mathcal{E}_{n-1} \subset \mathcal{F}_n$. Proof: both $\sigma$-fields contain $C_n$; on $C_n$ the inclusion holds since here $X_1, \ldots, X_{n-1} \in [0,1] \setminus I_n'$; on $\Omega \setminus C_n$ the inclusion holds since here $X_1, \ldots, X_{n-1} \in [0,1] \setminus I_n''$.

The conditional distribution of $X_n$ given $\mathcal{F}_n$ is easy to describe. On $C_n$ it is the conditional distribution of the Brownian minimizer on $I_n'$ under three conditions. Two conditions are boundary values of the Brownian path on the two endpoints of $I_n'$. The third condition is a lower bound on (the minimum of) the Brownian path on $I_n'$; it must exceed the minimum on $I_n''$. A similar description holds on $\Omega \setminus C_n$. Clearly, the conditional distribution of $X_n$ given $\mathcal{F}_n$ has a density $(x, \omega) \mapsto g_n(x, \omega)$ (see also (4.4) below).

Taking into account that $\mathcal{E}_{n-1} \subset \mathcal{F}_n$ we conclude that the conditional distribution of $X_n$ given $\mathcal{E}_{n-1}$ has a density $(x, \omega) \mapsto f_n(x, \omega)$,

$$f_n(x, \cdot) = \mathbb{E}\left(g_n(x, \cdot) \mid \mathcal{E}_{n-1}\right).$$

Here is an explicit formula for the conditional density $g_n$ introduced above: for $\omega \in C_n$ and $x \in I_n' = (u, v)$,

$$g_n(x, \omega) = \frac{1}{v-u} \varphi\left(\frac{1}{\sqrt{v-u}}\left(B_u(\omega) - \min_{I_n''} B(\cdot, \omega)\right), \frac{1}{\sqrt{v-u}}\left(B_v(\omega) - \min_{I_n''} B(\cdot, \omega)\right), \frac{x-u}{v-u}\right),$$

where the function $\varphi$ is defined by

$$\varphi(a, b, t) = \text{const}(a, b) \int_0^{\min(a, b)} (a-y)(b-y) \exp\left(-\frac{(a-y)^2}{2t} - \frac{(b-y)^2}{2(1-t)}\right) dy.$$
for \( t \in (0, 1) \) and \( a, b > 0 \); the normalizing constant, \( \text{const}(a, b) \), ensures that 
\[
\int_0^1 \varphi(a, b, t) \, dt = 1. 
\]
(For \( \omega \in \Omega \setminus C_n \) the formula is similar.) The formula follows easily from the description of the conditional distribution given in the proof of Lemma 4.3, the Brownian scaling, and the well-known joint distribution of the minimizer \( T \) and the minimum \( Y = B_T \) of a Brownian path on \([0, 1]\) conditioned by \( B_0 = a, B_1 = b \). Namely, the conditional density of \( (T, Y) \) is
\[
(4.5) \quad (t, y) \mapsto \sqrt{\frac{2}{\pi}} \frac{(a - y)(b - y)}{(t - t^2)^{3/2}} \exp \left( \frac{(a - b)^2}{2t} - \frac{(a - y)^2}{2t} - \frac{(b - y)^2}{2(1 - t)} \right)
\]
for \( 0 < t < 1, -\infty < y < \min(a, b) \).

We need the (unconditional) distribution of the random variable \( g_n(x, \cdot) \) in order to check 3.6(c); the distribution should not concentrate near the origin. However, the infimum of \( \varphi(a, b, t) \) over all \( t \in (0, 1) \) vanishes (unless \( a = b \)). We restrict ourselves to a subinterval, say, the inner half \([1/4, 3/4]\) of \([0, 1]\); clearly, \( \varphi(a, b, t) \geq \psi(a, b) > 0 \) for \( 1/4 \leq t \leq 3/4 \), therefore
\[
(4.6) \quad \mathbb{P} \left( 0 < g_n(x, \cdot) < \varepsilon \right) \leq \xi(\varepsilon), \quad \xi(\varepsilon) \to 0 \text{ as } \varepsilon \to 0
\]
for some \( \xi(\cdot) \) (not depending on \( n \) and \( x \)), provided that \( x \) belongs to the inner half of \( I'_n \) or \( I''_n \). (In fact we get much more, namely, \( \mathbb{P} \left( 0 < \text{Length}(I_n) \cdot g_n(x, \cdot) < \varepsilon \right) \leq \xi(\varepsilon) \).)

4.7 Theorem. The DCS(0,1)-valued random variable \( \omega \mapsto M_\omega \) is distributed like an unordered uniform infinite sample.

Proof. We will prove that the random variables \( X_n \) introduced in the proof of Lemma 4.1 satisfy the conditions of Lemma 2.1. First, we note that almost every \( x \in (0, 1) \) belongs to the inner half of \( I_n \) for infinitely many \( n \). Let \( x \) be such a number; we will prove that \( \sum f_n(x, \cdot) = \infty \) a.s.

We take \( n_1 < n_2 < \ldots \) such that \( x \) belongs to the inner half of \( I'_{n_k} \) or \( I''_{n_k} \) for each \( k \), and define random variables \( Y_k, Z_k \) by
\[
Y_k = f_{n_k}(x, \cdot), \quad Z_k = g_{n_k}(x, \cdot),
\]
where \( f_n, g_n \) are the conditional densities introduced in the proof of Lemma 4.3 (They are continuous in \( x \).) The relation \( f_n(x, \cdot) = \mathbb{E} \left( g_n(x, \cdot) \mid \mathcal{F}_{n-1} \right) \), noted there, shows that \( Y_k = \mathbb{E} \left( Z_k \mid X_1, \ldots, X_{n_k-1} \right) \) which gives us 3.6(b). Condition 3.6(c) follows from (4.6). Taking into account Remark 3.8 it remains to prove that \( \{Y_k \neq 0\} = \{Z_k \neq 0\} = T^{-m_k}(A_k) \) for some \( m_1 < m_2, \ldots \), some precompact sequence \( (A_k)_k \) such that \( \limsup_k \mathbb{P}(A_k) > 0 \), and some mixing \( T : \Omega \to \Omega \).
We define $T$ on the probability space of two-sided Brownian paths as the Brownian scaling centered at $x$,

$$B_{x+2s}(T\omega) = \sqrt{2} \left( B_{x+s}(\omega) - B_{x/2}(\omega) \right) \text{ for } s \in \mathbb{R};$$

it is well-known to be mixing. Recalling the events $C_n$ introduced in the proof of Lemma 4.3 we see that $\{Y_k \neq 0\} = \{Z_k \neq 0\} = C_{n_k}$ for all $k$ such that $x \in I'_{n_k}$. (Other $k$ satisfy $x \in I''_{n_k}$ and $\{Y_k \neq 0\} = \{Z_k \neq 0\} = \Omega \setminus C_{n_k}$; they are left to the reader.) We have

$$C_{n_k} = \left\{ \min_{I'_{n_k}} B > \min_{I''_{n_k}} B \right\},$$

thus, $C_{n_k} = T^{-m_k}(A_k)$ where $m_k$ are such that the length of $I_{n_k}$ is $2^{-m_k}$, and $A_k$ are defined by

$$A_k = \left\{ \min_{(a_k-1/2,a_k]} B > \min_{[a_k,a_k+1/2]} B \right\},$$

$a_k \in (x, x + 1/2)$ being such that

$$I_{n_k} = [x - 2^{-m_k}(x - a_k + \frac{1}{2}), x + 2^{-m_k}(a_k + \frac{1}{2} - x)].$$

Clearly, $\mathbb{P}(A_k) = 1/2$ for all $k$. Precompactness of the sequence $(A_k)_k$ is ensured by continuity of the map $a \mapsto \{\min_{[a-1/2,a]} B > \min_{[a,a+1/2]} B\}$ from $\mathbb{R}$ to the space of events. \qed

5 The alternative way

In this section I abandon (temporarily!) my principle (formulated before Def. 1.3) and try nonstandard probability spaces. Given a standard Borel space $B$ and an equivalence relation $E \subset B \times B$, the quotient set $B/E$ is equipped with the $\sigma$-field $\mathcal{F}_{B/E}$ of all sets $A \subset B/E$ whose inverse images in $B$ (w.r.t. the canonical projection $B \to B/E$) are measurable. Thus, $B/E$ is a Borel space (nonstandard, in general).

In order to avoid ambiguity, concepts of Sect. 1 will be called ‘strong’, while concepts of this section — ‘weak’. For example, a map $\Omega \to B/E$ is strongly measurable, if it is measurable according to 1.3, and weakly measurable, if it is a measurable map from $(\Omega, \mathcal{F}, P)$ to $(B/E, \mathcal{F}_{B/E})$ according to the usual definition. (Still, $(\Omega, \mathcal{F}, P)$ is a standard probability space.) Another example: weak distributions on $B/E$ are just probability measures on $(B/E, \mathcal{F}_{B/E})$. Strong distributions are much less customary objects (recall 1.5).
A strongly measurable map \( \Omega \to B/E \) evidently is weakly measurable. The converse is wrong in general (since \( E \) need not be measurable). Maybe it holds under some reasonable condition on \( E \); I do not know.

If strongly measurable \( f : \Omega_1 \to B/E, g : \Omega_2 \to B/E \) are strongly identically distributed, then evidently they are weakly identically distributed. We get a map from strong distributions on \( B/E \) to weak distributions on \( B/E \). Is it injective? Is it surjective? I do not know.

Theorem 4.7 considers two strong DCS(0, 1)-valued random variables and states that they are strongly identically distributed. Therefore they are weakly identically distributed, which allows us to transfer the Hewitt-Savage zero-one law from the infinite sample to the Brownian minimizers, as follows.

5.1 Corollary. Let \( U_1, U_2, \ldots \) be i.i.d. random variables uniform on \((0, 1)\); random variables \( X_1, X_2, \ldots \) be all the Brownian local minimizers on \((0, 1)\) (enumerated as in Sect. 4 or otherwise); and \( A \subset (0, 1)^\infty \) a Borel set invariant under permutations. Then

\[
\mathbb{P}\left( (X_1, X_2, \ldots) \in A \right) = \mathbb{P}\left( (U_1, U_2, \ldots) \in A \right) \in \{0, 1\}.
\]

5.2 Question. Let two strong DCS(0, 1)-valued random variables be weakly identically distributed. Does it follow that they are strongly identically distributed? (See also 5.11.)

5.3 Proposition. If two strong \( \mathbb{R}/\mathbb{Q} \)-valued random variables are weakly identically distributed then they are strongly identically distributed.

The proof is given after Proposition 5.10. Of course, by \( \mathbb{R}/\mathbb{Q} \) I mean reals modulo rationals, that is, \( \mathbb{R}/E \) where \( E = \{(x, y) \in \mathbb{R}^2 : x - y \in \mathbb{Q}\} \).

5.4 Corollary. Let probability measures \( \mu, \nu \) on \( \mathbb{R} \) be absolutely continuous (w.r.t. the Lebesgue measure). Then there exists a probability measure \( J \) on \( \mathbb{R}^2 \), whose marginals are \( \mu, \nu \), such that

\[
x - y \in \mathbb{Q} \quad \text{for } J\text{-almost all } (x, y).
\]

You may try to construct such \( J \) explicitly, say, when \( \mu \) is uniform and \( \nu \) is exponential.

Given a standard Borel space \( B \), we introduce the algebra \( \mathcal{A} \) of subsets of \( B \times B \) generated by all product sets \( U \times V \) where \( U, V \subset B \) are Borel sets. That is, elements of \( \mathcal{A} \) are of the form \( U_1 \times V_1 \cup \ldots \cup U_n \times V_n \). The following lemma is a slight modification of the well-known ‘marriage lemma’.

By a positive measure I mean a \([0, \infty)\)-valued Borel measure (the measure of the whole space is finite, and may vanish).

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5.5 Lemma. Let $\mu, \nu$ be positive measures on $B$, and $W \in \mathcal{A}$. Then

$$\sup_{m: m_1 \leq \mu, m_2 \leq \nu} m(W) = \inf_{U,V: W \subset U \times B \cup B \times V} (\mu(U) + \nu(V)),$$

here $m$ runs over positive measures on $W$; $U,V$ run over Borel subsets of $B$; and $m_1, m_2$ stand for the marginals of $m$ (that is, $m_1(U) = m(U \times B)$ and $m_2(V) = m(B \times V)$).

Proof. Clearly, $\sup m(W) \leq \inf (\mu(U) + \nu(V))$ (since $m(W) \leq m(U \times B) + m(B \times V)$); we have to prove that $\sup m(W) \geq \inf (\mu(U) + \nu(V))$. First, we reduce the general case to the elementary case of a finite set $B$. To this end we take a finite partition $B = B_1 \cup \ldots \cup B_n$ such that $W$ is the union of $B_k \times B_l$ (over some pairs $(k,l)$) and consider linear combinations of product measures $(\mu \cdot 1_{B_k}) \times (\nu \cdot 1_{B_l})$.

For a finite $B$ we apply the usual duality argument in the finite-dimensional space $\mathbb{R}^B$:

$$\sup m(W) = \inf_{f,g} \left( \int f \, d\mu + \int g \, d\nu \right)$$

where the infimum is taken over all pairs of functions $f,g : B \to [0, \infty)$ such that $f(x) + g(y) \geq 1$ for all $(x,y) \in W$. It remains to prove that

$$\inf \left( \int f \, d\mu + \int g \, d\nu \right) \geq \inf (\mu(U) + \nu(V)).$$

Introducing $U_\theta = \{ x \in B : f(x) \geq \theta \}$, $V_\theta = \{ y \in B : g(y) \geq 1 - \theta \}$ for $\theta \in (0,1)$, we get $W \subset U_\theta \times B \cup B \times V_\theta$ for each $\theta$ and

$$\int_0^1 \mu(U_\theta) \, d\theta = \int f \, d\mu, \quad \int_0^1 \nu(V_\theta) \, d\theta = \int g \, d\nu,$$

therefore $\int f \, d\mu + \int g \, d\nu \geq \inf_\theta (\mu(U_\theta) + \nu(V_\theta))$. 

Here is a slight modification of a well-known result of Strassen [3, Sect. 6] about measures with given marginals, concentrated on a given closed subset of a product space. A set of class $\mathcal{A}_\delta$ is, by definition, a set of the form $W_1 \cap W_2 \cap \ldots$ where $W_1, W_2, \ldots \in \mathcal{A}$ (and $\mathcal{A}$ is introduced before 5.5).

5.6 Lemma. Let $\mu, \nu$ be positive measures on $B$, and $W \in \mathcal{A}_\delta$. Then

$$\sup_{m: m_1 \leq \mu, m_2 \leq \nu} m(W) = \inf_{U,V: W \subset U \times B \cup B \times V} (\mu(U) + \nu(V)),$$

here $m$ runs over positive measures on $W$ and $U,V$ run over Borel subsets of $B$ (and $m_1, m_2$ stand for the marginals of $m$, as before).
Proof. Once again, ‘≤’ is evident; we have to prove ‘≥’. We take \( W_1, W_2, \ldots \in \mathcal{A} \) such that \( W_n \Downarrow W \) (that is, \( W_1 \supset W_2 \supset \ldots \) and \( W = W_1 \cap W_2 \cap \ldots \)). Lemma 5.5 applied to each \( W_n \) separately gives us measures \( m_n \) on \( W_n \) satisfying the restriction on marginals \((m_n)_1 \leq \mu, (m_n)_2 \leq \nu\) and such that

\[
m_n(W_n) + \frac{1}{n} \geq \inf_{U,V: W_n \subset U \times B \cup B \times V} (\mu(U) + \nu(V)) \geq \inf_{U,V: W \subset U \times B \cup B \times V} (\mu(U) + \nu(V)).
\]

The space of joinings, equipped with an appropriate topology, is a compact metrizable space, and functions \( J \mapsto J(W) \) are continuous as long as \( W \in \mathcal{A} \); see the digression ‘The compact space of joinings’ in [4, Sect. 4b]. This fact (and its proof) holds also for the space of all positive measures \( m \) on \( B \times B \) satisfying \( m_1 \leq \mu, m_2 \leq \nu \) (rather than \( m_1 = \mu, m_2 = \nu \)). Taking a convergent subsequence \( m_{n_k} \to m \) we get

\[
m(W_n) \geq \inf_{U,V: W \subset U \times B \cup B \times V} (\mu(U) + \nu(V))
\]

for all \( n \); however, \( m(W_n) \Downarrow m(W) \). \( \square \)

5.7 Lemma. The following two conditions are equivalent for every \( W \subset B \times B \):

(a) \( \inf \{ \mu(U) + \nu(V) : U \times B \cup B \times V \supset W \} = 0 \);

(b) \( W \subset U \times B \cup B \times V \) for some \( U, V \) such that \( \mu(U) = 0, \nu(V) = 0 \).

Proof. (b) \implies (a): trivial.

(a) \implies (b): We take \( U_n, V_n \) such that \( W \subset U_n \times B \cup B \times V_n \) for each \( n \), and \( \sum (\mu(U_n) + \nu(V_n)) < \infty \). Then \( \mu(\limsup U_n) = 0 \) and \( \nu(\limsup V_n) = 0 \); here \( \limsup U_n \) is the set of all \( x \in B \) such that \( x \in U_n \) for infinitely many \( n \). It remains to note that \( W \subset (\limsup U_n) \times B \cup B \times (\limsup V_n) \). \( \square \)

We turn to measures with given marginals, concentrated on a given equivalence relation \( E \subset B \times B \). By \( \mathcal{F}_E \) we denote the \( \sigma \)-field of all Borel sets \( A \subset B \) that are \( E \)-saturated, that is, \( (x, y) \in E \& x \in A \implies y \in A \). If a measure \( m \) on \( B \times B \) is concentrated on \( E \) (that is, \( (B \times B) \setminus E \subset A \), \( m(A) = 0 \) for some Borel set \( A \subset B \times B \) then the marginal measures \( m_1, m_2 \) are equal on \( \mathcal{F}_E \) (that is, \( m_1(A) = m_2(A) \) for all \( A \in \mathcal{F}_E \)), since the symmetric difference between \( A \times B \) and \( B \times A \) is contained in \( (B \times B) \setminus E \). By a nonzero positive measure I mean that the measure of the whole space does not vanish.

5.8 Lemma. The following two conditions on \( E \) are equivalent:

(a) for every pair \( (\mu, \nu) \) of probability measures on \( B \) equal on \( \mathcal{F}_E \) there exists a probability measure \( m \) concentrated on \( E \) such that \( m_1 = \mu, m_2 = \nu \).
(b) for every pair \((\mu, \nu)\) of nonzero positive measures on \(B\) equal on \(\mathcal{F}_E\) there exists a nonzero positive measure \(m\) concentrated on \(E\) such that \(m_1 \leq \mu, m_2 \leq \nu\).

**Proof.** \((a) \implies (b): \)** We note that \(\mu(B) = \nu(B)\), apply \((a)\) to \((1/\mu(B))\mu\) and \((1/\nu(B))\nu\) and use \(\mu(B)m\).

\((b) \implies (a): \)** We consider the set \(M\) of all positive measures \(m\) concentrated on \(E\) such that \(m_1 \leq \mu, m_2 \leq \nu\). The set \(M\) contains a maximal element \(m\), since \(M\) contains the limit of every increasing sequence of elements of \(M\). We have to prove that \(m(B \times B) = 1\). Assume the contrary: \(m(B \times B) < 1\). The nonzero positive measures \(\mu - m_1, \nu - m_2\) are equal on \(\mathcal{F}_E\) Item \((b)\) \(\implies \)(a) gives us a nonzero positive measure \(\Delta m\) concentrated on \(E\) such that \((\Delta m)_1 \leq \mu - m_1, (\Delta m)_2 \leq \nu - m_2\). Thus, \(m + \Delta m\) belongs to \(M\), in contradiction to the maximality of \(m\). \(\Box\)

**5.9 Remark.** Let \(\mu, \nu\) be probability measures on \(B\) equal on \(\mathcal{F}_E\). Then the following condition is sufficient for the existence of a probability measure \(m\) concentrated on \(E\) such that \(m_1 = \mu, m_2 = \nu\):

(a) for every nonzero positive measures \(\mu_0, \nu_0\) equal on \(\mathcal{F}_E\) and satisfying \(\mu_0 \leq \mu, \nu_0 \leq \nu\) there exists a nonzero positive measure \(m'\) concentrated on \(E\) such that \(m'_1 \leq \mu_0, m'_2 \leq \nu_0\).

The proof is basically the same as the proof of \((\text{b)} \implies \text{(a)}\) in Lemma 5.8.

The saturation of a set \(A\) (w.r.t. a given equivalence relation \(E\)) is, by definition, \(\{y \in B : \exists x \in A \ (x, y) \in E\}\). (It need not be a Borel set even if \(A\) and \(E\) are Borel sets.) A set of class \(\delta\) is, by definition, a set of the form \(W_1 \cup W_2 \cup \ldots\) where \(W_1, W_2, \ldots \in \mathcal{A}_\delta\) (and \(\mathcal{A}_\delta\) is introduced before 5.6).

**5.10 Proposition.** Let \(B\) be a standard Borel space, \(E \subset B \times B\) an equivalence relation of class \(\delta\) such that for every Borel set its saturation is also a Borel set, and \(\mu, \nu\) probability measures on \(B\) equal on \(\mathcal{F}_E\). Then there exists a probability measure \(m\) concentrated on \(E\) such that \(m_1 = \mu, m_2 = \nu\).

**Proof.** Assume that the sufficient Condition 5.9(a) is violated for some nonzero \(\mu_0 \leq \mu, \nu_0 \leq \nu\) equal on \(\mathcal{F}_E\). We take \(W_n \in \mathcal{A}_\delta\) such that \(E = W_1 \cup W_2 \cup \ldots\) and note that each \(W_n\) violates the condition, that is, \(m = 0\) is the only positive \(m\) concentrated on \(W_n\) such that \(m_1 \leq \mu_0, m_2 \leq \nu_0\). We apply Lemma 5.6 to \(\mu_0, \nu_0, W_n\); the supremum vanishes, therefore the infimum vanishes. Lemma 5.7 gives us \(U_n, V_n\) such that \(\mu_0(U_n) = 0, \nu_0(V_n) = 0\) and \(W_n \subset U_n \times B \cup B \times V_n\). Taking \(U = U_1 \cup U_2 \cup \ldots\) and \(V = V_1 \cup V_2 \cup \ldots\) we get

\[
\mu_0(U) = 0, \quad \nu_0(V) = 0, \quad E \subset U \times B \cup B \times V.
\]
The latter means that a point of $B \setminus U$ is never equivalent to a point of $B \setminus V$, that is, the saturation $A$ of the set $B \setminus U$ is a subset of $V$. We have $A \in \mathcal{F}_E$ (the saturation of a Borel set is Borel, as assumed), therefore $\mu_0(A) = \nu_0(A) \leq \nu_0(V) = 0$ and $\mu_0(B \setminus U) \leq \mu_0(A) = 0$, in contradiction to the fact that $\mu_0(B \setminus U) = \mu_0(B) > 0$.

Proposition 5.3 is basically a special case of Proposition 5.10. The equivalence relation $E = \{(x, y) \in \mathbb{R}^2 : x - y \in \mathbb{Q}\}$ belongs to the class $F_\sigma$ (which means, the union of a sequence of closed sets), therefore, to the class $\mathcal{A}_{\delta\sigma}$ (since every closed set belongs to $\mathcal{A}_\delta$). The saturation $A + \mathbb{Q}$ of any Borel set $A$ is Borel (since $A + q$ is, for each $q \in \mathbb{Q}$). We have two strong $\mathbb{R}/\mathbb{Q}$-valued random variables that are weakly identically distributed. They arise from two $\mathbb{R}$-valued random variables whose distributions $\mu, \nu$ are equal on $\mathcal{F}_E$. Proposition 5.10 gives us $m$ concentrated on $E$ whose marginals are $\mu, \nu$. This $m$ is a joining between $(\mathbb{R}, \mu)$ and $(\mathbb{R}, \nu)$. It remains to lift the joining to the probability spaces, the domains of our random variables, which is easy to do by means of the conditional measures on these spaces.

In contrast, the equivalence relation (1.2) is of the class $F_{\sigma\delta}$, therefore, $\mathcal{A}_{\delta\sigma\delta}$.

5.11 Question. Find a generalization of Proposition 5.10 to equivalence relations of the class $\mathcal{A}_{\delta\sigma\delta}$, applicable to (1.2). Is it possible? (See also 5.2.)

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Boris Tsirelson

School of Mathematics
Tel Aviv University
Tel Aviv 69978, Israel

mailto:tsirel@post.tau.ac.il
http://www.tau.ac.il/~tsirel/