Coherent State Functional Integrals in Quantum Cosmology

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Coherent state functional integrals for the minisuperspace models of quantum cosmology are studied. By the well-established canonical theories, the transition amplitudes in the path-integral representations of Wheeler-DeWitt quantum cosmology and loop quantum cosmology can be formulated through group averaging. The effective action and Hamiltonian with higher-order quantum corrections are thus obtained in both models within the scheme of Gaussian coherent states. It turns out that for a non-symmetric Hamiltonian constraint operator, the Moyal (star)-product emerges naturally in the effective Hamiltonian. This reveals the intrinsic relation among coherent state functional integral, effective theory and star-product. Moreover, both the resulted effective theories imply a possible quantum cosmological effect in large scale limit under certain condition.

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I. INTRODUCTION

One of the most fundamental issues in modern physics is quantum gravity. Among various approaches to quantum gravity, the viewpoint of nonperturbative quantization has been received increased attention. The first significant attempt of this kind was presented in 1960s [1]. Based on the ADM Hamiltonian formalism of general relativity (GR) and the Dirac’s generalized Hamiltonian quantization theory, which is in principle applicable to constrained system, Wheeler and DeWitt proposed a wave functional description of gravitational field. In this quantum geometrodynamical approach, the whole universe is described by a wave function which is defined on superspace - the space of all 3-metrics and matter field configurations. The dynamics is encoded in the second-order differential Wheeler-DeWitt (WDW) equation. Despite the elegant form of WDW theory, it encountered a number of fundamental problems, such as the physical meaning of quantum constraint equation, ordering of operators and problem of time [2]. To overcome these obstacles, especially to impose boundary conditions on WDW equation to find solutions, there were some attempts appealing to the path integral formalism. Hartle and Hawking suggested an Euclidean path integral representation [3], which gave the “no-boundary” proposal for the wave function. However, this approach can not be evaluated exactly in general cases as the integral was usually badly divergent. To alleviate the difficulty, Halliwell derived a Lorenzian path integral formalism for minisuperspace models [4], which revealed the relationship between the choice of measure in the path integral and the operator ordering in the WDW equation.

In the last two decades, an alternative background independent approach developing rapidly is loop quantum gravity (LQG) [5–8]. The starting point of LQG is the Hamiltonian connection dynamics of GR rather than the ADM formalism. In this framework, GR looks like a gauge field theory with $SU(2)$ as its internal gauge group. By taking the holonomy of $su(2)$-connection $A_a^i$ and flux of densitized triad $E_b^i$ as basic variables, the quantum kinematical framework of LQG has been rigorously constructed, and the Hamiltonian constraint operator can also be well defined to represent quantum dynamics. Moreover, a few physically significant results, especially the resolution of big bang singularity, have been obtained in the minisuperspace models of loop quantum cosmology (LQC) [9, 10]. The quantum bounce replacement of big bang and its properties are being studied from different prescriptions of LQC [11–16]. Effective equations were also derived in isotropic models [19–21], which predict evolution of universe with quantum corrections and shed new lights on the singularity resolution. We refer to [17, 18] for overviews on the recent progress in LQC.

Besides the canonical formalism, the so-called spin foam models were proposed as the path integral formalism of LQG [5]. However, whether the two approaches are equivalent to each other is a longstanding open question. Thanks to the development of LQC, we have much simple models to address this question. As symmetry-reduced models, there are only finite number of degrees of freedom in LQC. Following the conventional method in quantum mechanics, one can find the path integral formalism of LQC starting with the canonical formalism. This approach is being implemented by a series of papers [22, 23] with the scheme of simplified LQC [24]. In these papers, the transition amplitude

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were casted into path integrals by using either *de-parameterized* Hamiltonian or the *group averaging* method. Here one employed the complete basis of eigen-states of the volume operator to formulate a path integral with somehow *decrete-steps*, which inherited certain properties of spin foams [25]. Moreover, the first-order effective action for the path integral was derived by this approach [22, 23], which implied the origin of singularity resolution of LQC in the path integral representation. In canonical LQC, the effective Hamiltonian constraint with higher-order quantum corrections could even be obtained by the semiclassical analysis using coherent states of Gaussian type, which implied a possible effect of quantum gravity on large scale cosmology [11, 21]. Moreover, a systematic approach to the effective theories of quantum cosmology in the canonical framework is being developed [26, 27]. It is thus interesting to see whether the effective Hamiltonian in the canonical theories can be confirmed by some path integral representation. Since the higher-order corrections of the Hamiltonian come from the quantum fluctuations, a natural attempt to achieve them is to employ coherent state path integral [28].

In both WDW quantum cosmology and LQC, by coupling with a massless scalar field, the Hamiltonian constraint equations can be reformulated as Klein-Gordon-like equations, where the corresponding gravitational Hamiltonian operators, as multiplications of several self-adjoint operators, are non-symmetric in the kinematical Hilbert space [10, 24]. While this treatment is essential in order to obtain the physical states satisfying the constraint equation, it also provides elegant physical models to examine the so-called *Moyal *-product in quantum mechanics. At the very beginning, Moyal proposed the *-product in order to clarify the role of statistical concepts in quantum mechanical system [29]. Then this idea were generalized to many situations including quantum spacetime itself. It provided the core concept of *non-commutative geometry*, one of promising and interesting new tools in the study of quantum field theory and quantum gravity [30]. In canonical quantum theories, the *-product can also be understood by coherent state approach [31]. In fact, on a coherent state, the expectation value of the multiplication of two non-commutative variables, where the conjugate momentum satisfies \[ \{q_a, p_b\} = i\hbar \delta_{ab}, \] and \( q \) is then determined by a scale factor \( a \) satisfying \( q_a = a^3 p_a \). It is convenient to introduce an elementary cell \( V \) and restrict all integrations to this cell since the spatial slice is non-compact. The volume of \( V \) with respect to \( q_{ab} \) is denoted as \( V_o \) and the physical volume is \( V = a^3 V_o \). Then the geometrical pair \( (a, p_a) \) can be used as canonical variables, where the conjugate momentum satisfies \( p_a \propto a \dot{a} \).

In order to study the WDW cosmology and LQC on the same footing, we employ the new canonical variables \( (A^i_a, E^o_i) \) in both theories. Due to the homogeneity and isotropy, we can fix a set of orthonormal cotriad and triad \( \{\omega^i_a, \omega_i^a\} \) compatible with \( q_{ab} \) and adapted to \( V \). Then the cotriad \( \omega^i_a \) which are orthonormal with respect to physical metric \( q_{ab} \) can be written as \( \omega^i_a = \chi^a \omega^a_i \), where \( \chi = 1 \) if \( \omega^i_a \) has the same orientation as the fiducial \( \omega^a_i \) and \( \chi = -1 \) if the orientation is opposite. The basic canonical variables take the simple form [32]

\[
A^i_a = eV_o^{-1/3}\omega^i_a, \quad E^o_i = p_i V_o^{-2/3}q^o_{a}. \tag{2.2}
\]

The dynamical variables are thus reduced to \( (c, p) \) with the Poisson bracket: \( \{c, p\} = 8\pi G \gamma / 3 \), where \( \gamma \) is the Barbero-Immirzi parameter. Following the \( \mu \)-schema of "improved dynamics" [10], the regulator \( \mu \) used in holonomies is given by \( \mu = \sqrt{\Delta / |p|} \), where \( \Delta = 4\sqrt{3} \pi G \gamma / \ell_p^2 \) is a minimum nonzero eigenvalue of the area operator [33]. In order to do the semiclassical analysis, it is convenient to introduce new dimensionless conjugate variables [11, 24]:

\[
b := \frac{\mu c}{2} = \frac{4 \cdot 3^3 \pi G (\gamma) \ell_p p_a}{3V_o a^2}, \quad v := \frac{\text{sgn}(p)|p|^2}{2\gamma \ell_p^2 \sqrt{\Delta}} = \frac{\chi a^3 V_o}{4 \cdot 3^3 (\pi \gamma)^{2} \ell_p^2}, \tag{2.3}
\]

with the Poisson bracket \( \{v, b\} = \frac{1}{\hbar} \), where the Planck length \( \ell_p \) is given by \( \ell_p^2 = G \hbar \). From the matter part of action (2.1), we can get the momentum of \( \phi \) as \( p_\phi = a^3 V_o \dot{\phi} \) and the Poisson bracket: \( \{\phi, p_\phi\} = 1 \). The kinematical
Hilbert space of the quantum theory is supposed to be a tensor product of the gravitational and matter parts. In WDW quantum cosmology, one employed the standard Schrödinger representation for both matter and gravity to construct Hilbert space $\mathcal{H}^{\text{WDW}}_\text{kin}$. The (generalized) orthonormal basis in $\mathcal{H}^{\text{WDW}}_\text{kin}$ is given by $|v\rangle \otimes |\phi\rangle$ (or denoted as $|v, \phi\rangle$) with the inner product: $\langle v', \phi'| v, \phi \rangle = \delta(v', v)\delta(\phi', \phi)$. The fundamental operators act on a quantum state $\Psi(v, \phi) \in \mathcal{H}^{\text{WDW}}_\text{kin}$ as in standard quantum mechanics. To obtain the physical states, one has to solve the quantum Hamiltonian constraint equation [24]:

$$\hat{C}^{\text{WDW}} \cdot \Psi(v, \phi) = \left( -\frac{\hat{p}_v^2}{\hbar^2} + \hat{\Theta}^{\text{WDW}} \right) \Psi(v, \phi) - 12\pi G \cdot v \partial_v (v \partial_v) \Psi(v, \phi) = 0. \quad (2.4)$$

However, in LQC, while the Schrödinger representation was still used for the matter, gravity was quantized by the polymer-like representation [32]. Thus quantum states in the gravitational Hilbert space of LQC are functions $\Psi(v, \phi)$ with support on a countable number of points and with finite norm $\|\Psi\|^2 := \sum_n |\Psi(n)|^2$ [34]. Hence the inner product is defined by a Kronecker delta $\langle v' \| v \rangle = \delta(v', v)$. The basic operators act on a quantum state $\Psi(v, \phi)$ in the kinematical Hilbert space $\mathcal{H}^{\text{LQC}}_\text{kin}$ as:

$$\hat{v} \Psi(v, \phi) = v \Psi(v, \phi), \quad \hat{\sigma} \Psi(v, \phi) = \Psi(v + 1, \phi). \quad (2.5)$$

The quantum Hamiltonian constraint equation becomes

$$\hat{C}^{\text{LQC}} \cdot \Psi(v, \phi) = \left( -\frac{\hat{p}_v^2}{\hbar^2} + \hat{\Theta}^{\text{LQC}} \right) \Psi(v, \phi) = 0, \quad (2.6)$$

where $\hat{\Theta}^{\text{LQC}}$ is a positive second-order difference operator defined by a simplified scheme as [24]:

$$\hat{\Theta}^{\text{LQC}} \cdot \Psi(v, \phi) = -\frac{3\pi G}{4}v [(v + 2)\Psi(v + 4, \phi) - 2v\Psi(v, \phi) + (v - 2)\Psi(v - 4, \phi)]. \quad (2.7)$$

Solutions to the constraint equations and their physical inner products can be obtained through the group averaging procedure. It is demonstrated in [22] that in a timeless framework the most important entity is the amplitude

$$A(v_f, \phi_f; v_i, \phi_i) := \int_{-\infty}^{\infty} d\alpha \langle v_f, \phi_f \| e^{i\alpha \hat{C}} \| v_i, \phi_i \rangle, \quad (2.8)$$

which contains all the dynamical information.

We are going to concern about coherent state functional integrals. The (generalized) coherent state of the matter part is labeled by a complex variable $z_o := \frac{1}{\sqrt{2\sigma}}(\phi_o + \frac{i}{\hbar}\sigma \phi_o)$ and defined by

$$|\Psi_{z_o}\rangle := \int_{-\infty}^{\infty} d\phi \ e^{-\frac{(\phi - \phi_o)^2}{2\sigma^2}} e^{\frac{i}{\hbar}\sigma \phi_o} \langle \phi - \phi_o \| \phi \rangle,$$

which is the eigenstate of the annihilation operator $\hat{\sigma} \equiv \frac{1}{\sqrt{2\sigma}}(\hat{\phi} + \frac{i}{\hbar}\hat{\sigma} \phi^2)$, where $\sigma$ describes the width of the wave-packet or quantum fluctuation. It satisfies the key properties of a coherent state, namely, saturation of Heisenberg’s uncertainty relation, resolution of identity and peakness property. Similarly, we can introduce $\eta_o := \frac{1}{\sqrt{2d}}(v_o + ib_o \delta^2)$ labeling the coherent state of the gravitational part of WDW theory, which is defined by

$$|\Psi_{\eta_o}\rangle := \int_{-\infty}^{\infty} dv \ e^{-\frac{(v - v_o)^2}{2d^2}} e^{-ib_o(v - v_o)} |v\rangle. \quad (2.10)$$

It also has the analogous properties of the coherent state of matter part, especially the resolution of the identity:

$$\int_{-\infty}^{\infty} dv_o \int_{-\infty}^{\infty} db_o \langle \Psi_{\eta_o} \| \Psi_{\eta_o} \rangle = 1. \quad (2.11)$$

The whole coherent state of WDW theory reads $|\Psi_{z_o}\rangle \Psi_{\eta_o} \rangle = |\Psi_{z_o}\rangle \otimes |\Psi_{\eta_o}\rangle$. On the other hand, due to the polymer-like structure, the coherent state of LQC is different from that of WDW. Here one can define $\zeta_o := \frac{1}{\sqrt{2d}}(v_o + ib_o d^2)$ to label the generalized coherent state [21, 34]:

$$|\Psi_{\zeta_o}\rangle := \sum_{v \in \mathbb{R}} e^{-\frac{(v - v_o)^2}{2d^2}} e^{-ib_o(v - v_o)} |v\rangle. \quad (2.12)$$
where $d$ is the characteristic width of the wave packet and $1 \ll d \ll v_o$ because of the semiclassical feature. For practical use, one defines the projection of this state on some lattice of variable $v$, saying the shadow state [34]:

$$\Psi_{\text{shad}} := \sum_{k=-\infty}^{\infty} e^{-\frac{(k-v_o)^2}{2d^2}} e^{ib_0(k-v_o)} |k\rangle, \quad k \in \mathbb{Z},$$

(2.13)

where we chose the regular lattice $\{v = k, k \in \mathbb{Z}\}$. This shadow state also has the analogous properties of a coherent state. Note that our final result will not depend on the particular choice of regular lattice, since the "Hamiltonian operator" in Eq. (2.7) is a difference operator with step of size "4". The resolution of identity now reads

$$\int_{-\infty}^{\infty} dv_o \int_{-\pi}^{\pi} \frac{db_0}{2\pi} \frac{|\Psi_{\text{shad}}\rangle \langle\Psi_{\text{shad}}|}{\langle\Psi_{\text{shad}}|\Psi_{\text{shad}}\rangle} = \sum_{k=-\infty}^{\infty} |k\rangle \langle k| \equiv \mathbb{I},$$

(2.14)

where the identity $\mathbb{I}$ is in the subspace in which the states have support only on the regular lattice. It should be noticed that the states with support on semilattice only in a single semi-axis of the real line was studied in Ref. [14], which are superselected by an alternative Hamiltonian constraint operator suitable to deal with some delicate issues in LQC. With this Hamiltonian operator of special symmetrized ordering, the singularity decouples in the kinematical Hilbert space and hence can be removed. It would be interesting to study also the effective equations of this Hamiltonian operator by coherent states supported on semilattice. However, we remark that, for practical calculations in our coherent state functional integral, the dynamical difference equation in [14] still needs to be suitably "simplified".

III. COHERENT STATES FUNCTIONAL INTEGRALS

In the path integral of the conventional non-relativistic quantum mechanics, one needs to compute the matrix element of the evolution operator $e^{-i\Delta t\hat{H}}$ within the time interval $\Delta t$. However, the situation of cosmology of GR is very different since both WDW cosmology and LQC are totally constrained systems, and the operator $\hat{C}$ is not a true Hamiltonian. Instead, we start from the physical inner product, i.e., the transition amplitude, of coherent states with normalization:

$$A([\Psi_f, [\Psi_i]) = \frac{\langle \Psi_{f_1} | \int_{-\infty}^{\infty} d\alpha e^{i\alpha C} | \Psi_{f_2}\rangle \langle \Psi_{i_1} | \int_{-\infty}^{\infty} d\alpha e^{i\alpha C} | \Psi_{i_2}\rangle}{\|\Psi_{f_1}\|\|\Psi_{f_2}\|\|\Psi_{i_1}\|\|\Psi_{i_2}\|}. $$

(3.1)

To calculate the transition amplitude, we split a fictitious time interval $\Delta \tau = 1$ into $N$ pieces $\epsilon = \frac{1}{N}$ and thus get $e^{i\alpha C} = e^{iN\epsilon=1 \epsilon_n C} = \prod_{k=1}^{N} e^{i\epsilon_n C}$. Inserting $N$ times of coherent states resolution of identity of $|\Psi_{z_n}\rangle$ and Eq. (2.11) (or Eq. (2.14)), Eq. (3.1) can be casted into

$$A([\Psi_f, [\Psi_i]) = \int_{-\infty}^{\infty} d\alpha A^{\text{matt}}_{\alpha,N}\langle \Psi_{f_1} | \int_{-\infty}^{\infty} d\epsilon_1 e^{i\epsilon_1 C} | \Psi_{f_2}\rangle A^{\text{grav}}_{\alpha,N}\langle \Psi_{i_1} | \int_{-\infty}^{\infty} d\epsilon_1 e^{i\epsilon_1 C} | \Psi_{i_2}\rangle, $$

(3.2)

where

$$A^{\text{matt}}_{\alpha,N} = \int_{-\infty}^{\infty} d\phi_{N-1} \ldots d\phi_1 \int_{-\infty}^{\infty} dp_{\phi_{N-1}} \ldots \frac{dp_{\phi_1}}{2\pi \hbar} \cdots \prod_{n=1}^{N} \frac{\langle \Psi_{z_n} | e^{i\epsilon_n C} | \Psi_{z_{n-1}}\rangle}{\|\Psi_{z_n}\|\|\Psi_{z_{n-1}}\|}, $$

(3.3a)

$$A^{\text{grav}}_{\alpha,N} = \int_{-\infty}^{\infty} dv_{N-1} \ldots dv_1 \int_{-\pi}^{\pi} d\phi_{N-1} \ldots \frac{d\phi_1}{2\pi} \cdots \prod_{n=1}^{N} \frac{\langle \Psi_{\eta_n} | e^{-i\epsilon_n \Theta} | \Psi_{\eta_{n-1}}\rangle}{\|\Psi_{\eta_n}\|\|\Psi_{\eta_{n-1}}\|}, $$

(3.3b)

with $z_N \equiv z_f, z_0 \equiv z_i, \eta_N \equiv \eta_f,$ and $\eta_0 \equiv \eta_i$. Notice that the characteristic widths $\sigma$ and $\delta$ at different steps are not necessarily the same. So we have to denote $\sigma_n$ and $\delta_n$ in the semiclassical states $|\Psi_{z_n}\rangle$ and $|\Psi_{\eta_n}\rangle$ respectively at the "n-step". Now the main task is to calculate the matrix element of the exponential operators on coherent states. The

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1 Hereafter, without confusion of notations we omit the superscript "shad" of the shadow state for convenience.
exponential operator $e^{i\alpha C}$ can be expanded as $1 + i\alpha C + \mathcal{O}(\alpha^2)$. For the purpose of a concise writing, we introduce some intermediate-step notations,

$$p_{\phi_n} = \frac{\sigma_n^2 p_{\phi_n} + \sigma_{n-1}^2 p_{\phi_{n-1}}}{\sigma_n^2 + \sigma_{n-1}^2}, \quad \sigma_n^2 \equiv \frac{2\sigma_n^2 \sigma_{n-1}^2}{\sigma_n^2 + \sigma_{n-1}^2}.$$ 

Through a detailed calculation shown in the appendix, we get for the matter part

$$\prod_{n=1}^N \frac{\langle \Psi_{z_n} | e^{i\alpha C} | \Psi_{z_{n-1}} \rangle}{\| \Psi_{z_n} \| \| \Psi_{z_{n-1}} \|} = \left( \prod_{n=1}^N \frac{\langle \Psi_{z_n} | \Psi_{z_{n-1}} \rangle}{\| \Psi_{z_n} \| \| \Psi_{z_{n-1}} \|} \right) \exp \left[ \frac{i\alpha}{\hbar^2} \sum_{n=1}^N \left( p_{\phi_{n-1}}^2 + \frac{\hbar^2}{\sigma_{n-1}^2} \right) \right],$$

(3.4)

where the inner product of two adjacent states is

$$\langle \Psi_{z_n} | \Psi_{z_{n-1}} \rangle = \sqrt{\pi} \sqrt{\frac{\sigma_n^2}{\sigma_{n-1}^2}} \exp \left[ - \left( \frac{\sigma_n^2}{\sigma_{n-1}^2} \right) \right],$$

(3.5)

and hence the product of series $\langle \Psi_{z_n} | \Psi_{z_{n-1}} \rangle$ can be expressed as

$$\prod_{n=1}^N \frac{\langle \Psi_{z_n} | \Psi_{z_{n-1}} \rangle}{\| \Psi_{z_n} \| \| \Psi_{z_{n-1}} \|} = \exp \left[ \left\{ \sum_{n=1}^N - \left( \frac{2\sigma_n^2 + \sigma_{n-1}^2}{\sigma_{n-1}^2} \right) \right\} \right] \left( \prod_{n=1}^N \frac{2\sigma_n^2 \sigma_{n-1}^2}{\sigma_n^2 + \sigma_{n-1}^2} \right) \exp \left[ \left\{ \sum_{n=1}^N \frac{\rho_{\phi_n}^2}{2\sigma_n^2 + \sigma_{n-1}^2} \right\} \right] \exp \left[ \left\{ \sum_{n=1}^N \frac{\rho_{\phi_{n-1}}^2}{2\sigma_{n-1}^2 + \sigma_{n-1}^2} \right\} \right].$$

(3.6)

Here we introduced a virtual width $\sigma_{N+1}$ by hand, satisfying $\sigma_{N+1} - \sigma_N = \sigma_N - \sigma_{N-1}$, in order to get the tidy sum in the exponential position. In the limit of $N \to \infty$, $\sigma_{N+1}$ will approach $\sigma_{N} \equiv \sigma_f$ and hence does not effect the quantum dynamics.

For the gravitational part, a careful calculation in WDW quantum cosmology shown in the appendix yields

$$\prod_{n=1}^N \frac{\langle \Psi_{\eta_n} | e^{i\alpha \Theta_{WDW}} | \Psi_{\eta_{n-1}} \rangle}{\| \Psi_{\eta_n} \| \| \Psi_{\eta_{n-1}} \|} = \left( \prod_{n=1}^N \frac{\langle \Psi_{\eta_n} | \Psi_{\eta_{n-1}} \rangle}{\| \Psi_{\eta_n} \| \| \Psi_{\eta_{n-1}} \|} \right) \exp \left[ -i\alpha \cdot 12\pi G \sum_{n=1}^N \left( \left( \frac{1}{\delta^2_{n-1} + b_{n-1}^2} \right) \left( \frac{3}{2\delta^2_{n-1}} \right) - \frac{2\pi^2 \delta^2_{n-1}}{2\delta^2_{n-1}} \right) \right],$$

(3.7)

where $\tau_n \equiv \frac{\delta_{n-1} + \delta_{n-1}^2}{\delta_{n-1} + \delta_{n-1}^2}, \quad b_n \equiv \frac{\delta_{n-1}^2 + \delta_{n-1}^4}{\delta_{n-1} + \delta_{n-1}^2}, \quad \beta_n \equiv \frac{2\delta_{n-1}^2}{\delta_{n-1} + \delta_{n-1}^2}$, and $\prod_{n=1}^N \langle \Psi_{\eta_n} | \Psi_{\eta_{n-1}} \rangle$ takes the form similar to Eq. (3.6). Now we take the limit $N \to \infty$ and substitute $\int_0^1 dt$ for $\sum_{n=1}^N \epsilon$ to get the functional integral formalism of the amplitude:

$$A([\Psi_f] | [\Psi_i]) = e^{\frac{i}{2} \int (|\dot{z}|^2 - |z|^2 - |\eta|^2 - |\bar{\eta}|^2) \int d\tau \int [\mathcal{D}\phi(\tau) | [\mathcal{D}p_\phi(\tau)] | [\mathcal{D}v(\tau)] | [\mathcal{D}b(\tau)] \right] e^{i(S_{\text{matt}}^\alpha + S_{\text{grav}}^\alpha)},$$

(3.8)

where

$$S_{\text{matt}}^\alpha = \int_0^1 dt \left( \frac{\partial^2}{4\sigma^2} + i \frac{d}{dt} \left( \frac{\sigma^2 p_\phi^2}{4\hbar^2} \right) + \frac{p_\phi^2}{\hbar^2} + \frac{\alpha}{\hbar^2} \left( \frac{p_\phi^2}{\hbar^2} + \frac{\hbar^2}{2\sigma^2} \right) \right),$$

(3.9)

$$S_{\text{grav}}^\alpha = \int_0^1 dt \left( \frac{\partial^2}{4\sigma^2} + i \frac{d}{dt} \left( \frac{v^2}{4\sigma^2} \right) + \frac{\partial^2}{4\sigma^2} - \frac{b^2}{2\sigma^2} - \frac{\alpha}{\hbar^2} \left( \frac{v^2}{2} + \frac{b^2}{2\sigma^2} \right) \right).$$

(3.10)

Here the "dots" over $\phi$ and $v$ stand for the time derivative with respect to the fictitious time $\tau$. The functional measures are defined on continuous paths by taking the limit of $N \to \infty$:

$$\int [\mathcal{D}\phi(\tau)] | [\mathcal{D}p_\phi(\tau)] := \lim_{N \to \infty} \left( \prod_{n=1}^N \frac{2\sigma_n \sigma_{n-1}}{\sigma_n^2 + \sigma_{n-1}^2} \right) \int \prod_{n=1}^N \frac{d\phi_n dp_\phi_n}{2\pi \hbar},$$

(3.11a)

$$\int [\mathcal{D}v(\tau)] | [\mathcal{D}b(\tau)] := \lim_{N \to \infty} \left( \prod_{n=1}^N \frac{2\delta_n \delta_{n-1}}{\delta_n^2 + \delta_{n-1}^2} \right) \int \prod_{n=1}^N \frac{dv_n db_n}{2\pi}.$$ 

(3.11b)
Ignoring the total derivatives with respect to $\tau$ in Eqs. (3.9) and (3.10), we can read out the total effective Hamiltonian constraint in WDW quantum cosmology as:

$$\mathcal{H}_{\text{eff}} = -\frac{p^2_D}{\hbar^2} - \frac{1}{2\sigma^2} + 12\pi G \left[ \left( v^2 + \frac{\delta^2}{2} \right) (b^2 + \frac{1}{2\delta^2}) - ivb \right].$$  (3.12)

Note that $\frac{\delta^2}{2}$ and $\frac{1}{2\delta^2}$ are the square of fluctuations of $\hat{v}$ and $\hat{b}$ respectively. They can be seen as quantum corrections to the leading term: $v^2b^2 - ivb$.

On the other hand, careful calculations in LQC shown also in the appendix give

$$\prod_{n=1}^{N} \frac{\langle \Psi_n | e^{i\alpha \Theta_{\text{LQC}}} | \Psi_{n-1} \rangle}{\| \Psi_n \| \| \Psi_{n-1} \|} = \left( \prod_{n=1}^{N} \frac{\langle \Psi_n | \Psi_{n-1} \rangle}{\| \Psi_n \| \| \Psi_{n-1} \|} \right) \cdot \exp \left[ -i\alpha \frac{3\pi G}{\hbar} \sum_{n=1}^{N} \left( \left( \frac{\tau_n^2 + \frac{d_n^2}{2}}{2} \right) \left( \sin^2(2\overline{\tau_n}) (1 - \frac{8}{d_n^2 + d_{n-1}^2}) + \frac{4}{d_n^2 + d_{n-1}^2} \right) + i \sin(4\overline{\tau_n}) \frac{2d_n^2}{d_n^2 + d_{n-1}^2} (1 - \frac{8}{d_n^2 + d_{n-1}^2}) \right) \right].$$  (3.13)

where $\tau_n = \frac{d_n^2}{d_n^2 + d_{n-1}^2} v_n + d_n^2/v_{n-1}, \overline{\tau}_n = \frac{d_n^2}{d_n^2 + d_{n-1}^2} b_n - b_{n-1}, \overline{\tau}_n = \frac{d_n^2}{d_n^2 + d_{n-1}^2},$ and $\prod_{n=1}^{N} \frac{\langle \Psi_n | \Psi_{n-1} \rangle}{\| \Psi_n \| \| \Psi_{n-1} \|}$ also takes the form similar to Eq. (3.6). It should be noted that, in the above calculation, the inner product of two shadow states is

$$\langle \Psi_n | \Psi_{n-1} \rangle = \sum_{k'} e^{-\frac{(k'-v_n)^2}{2\tau_n^2} + ib_n(k'-v_n)} \langle k' | \sum_k e^{-\frac{(k-v_n)^2}{2\overline{\tau}_n^2} + ib_{n-1}(k-v_n)} | k \rangle = \exp \left[ -\frac{(v_n - v_{n-1})^2}{2(d_n^2 + d_{n-1}^2)} + i\overline{\tau}_n(v_n - v_{n-1}) \right] \sum_k \exp \left[ -\frac{(k - \tau_n)^2}{\overline{\tau}_n} - i(b_n - b_{n-1})(k - \overline{\tau}_n) \right].$$

Using the so-called Poisson re-sum formula:

$$\sum_{k=-\infty}^{\infty} g(k + x) = \sum_{k=-\infty}^{\infty} e^{i2\pi kx} \int_{-\infty}^{\infty} dy \ g(y) e^{-i2\pi ky}, \ k \in \mathbb{Z},$$  (3.14)

we have the summation about $v$ as

$$\sum_k \exp \left( -(k - \tau_n)^2/\overline{\tau}_n - i(b_n - b_{n-1})(k - \overline{\tau}_n) \right)$$

$$= \sum_k e^{-i2\pi k\tau_n} \int dy \ e^{-y^2/\overline{\tau}_n^2 - i(b_n - b_{n-1})y - i2\pi ky}$$

$$= \sqrt{\pi} \sqrt{\overline{\tau}_n^2} \sum_k e^{-i2\pi k\tau_n} \exp \left( -(b_n - b_{n-1} + 2\pi k)^2/4 \right)$$

and hence the inner product as

$$\langle \Psi_n | \Psi_{n-1} \rangle \approx \sqrt{\pi} \sqrt{\overline{\tau}_n^2} \exp \left( -\frac{(v_n - v_{n-1})^2}{2(d_n^2 + d_{n-1}^2)} - \frac{(b_n - b_{n-1})^2}{4} + i\overline{\tau}_n(v_n - v_{n-1}) \right).$$  (3.15)

where we kept only the $k = 0$ term, because in the continuous limit $N \rightarrow \infty$, one has $b_n - b_{n-1} \rightarrow 0$ and the terms corresponding to non-zero integer $k$ are of the same or higher orders of $O \left( e^{-\pi^2 \overline{\tau}_n^2} \right)$ and hence negligible under the semiclassical condition $d \gg 1$. The matrix element $\langle \Psi_n | e^{i\alpha \Theta_{\text{LQC}}} | \Psi_{n-1} \rangle$ can be obtained by similar method. Thus we obtain:

$$A([\Psi_f | [\Psi_i]) = e^{\frac{\hat{\theta}}{\hbar} \left( |x|^2 + |y|^2 + |z|^2 + |\zeta|^2 \right)} \int d\alpha \int [D\phi(\tau)] [Dp_\phi(\tau)] [Dv(\tau)] [Db(\tau)] e^{i \int_{\text{mat}} + \text{grav}}.$$

(3.16)
The effective Hamiltonian constraint in LQC can be read out as:

\[
\mathcal{H}_{\text{eff}} = -\frac{p_v^2}{\hbar^2} - \frac{1}{2\sigma^2} + 3\pi G \left[ \left( v^2 + \frac{d^2}{2} \right) \left( \sin^2(2b) \left( 1 - \frac{4}{d^2} \right) + \frac{2}{d^2} \right) + iv \sin(4b) \left( 1 - \frac{4}{d^2} \right) \right],
\]

wherein the terms \( \frac{d^2}{2} \) and \( \frac{2}{d^2} \) are the square of fluctuations of \( \dot{v} \) and \( \sin(2b) \) respectively. They are also quantum corrections to the leading term.

At the first sight, both \( \mathcal{H}_{\text{eff}} \) and \( \mathcal{H}_t \) look problematic due to the **imaginary part**. One might even suspect the validity of the coherent state path integral in the models. However, a careful observation reveals that the real and imaginary parts of the leading terms can be synthesized into a *Moyal-*product [31] in both models, i.e.,

\[
\begin{align*}
\varepsilon b^2 - ivb &= \varepsilon e^{-\frac{1}{2} \left( \delta \bar{\delta} - \delta \bar{\delta}^* \right)} (bvb) = v * (bvb), \\
v^2 \sin^2(2b) + iv \sin(4b) &= \varepsilon e^{\frac{1}{2} \left( \delta \bar{\delta} - \delta \bar{\delta}^* \right)} (\sin(2b)v \sin(2b)) = v * (\sin(2b)v \sin(2b)).
\end{align*}
\]

Therefore the effective Hamiltonian constraint in WDW theory takes the form:

\[
\mathcal{H}_{\text{eff}} = -\frac{p_v^2}{\hbar^2} - \frac{1}{2\sigma^2} + 12\pi G \left( v * (bvb) + \frac{\hbar^2 \delta^2}{2} + \frac{v^2}{2\delta^2} + \frac{1}{4} \right),
\]

while that in LQC becomes

\[
\mathcal{H}_{\text{eff}} = -\frac{p_v^2}{\hbar^2} - \frac{1}{2\sigma^2} + 3\pi G \left( v * (\sin(2b)v \sin(2b)) \left( 1 - \frac{4}{d^2} \right) + \frac{\sin^2(2b)d^2}{2} \left( 1 - \frac{4}{d^2} \right) + \frac{2v^2}{d^2} + 1 \right).
\]

To understand how the *Moyal-*product emerges in the gravitational part of the Hamiltonian, recall that both \( \hat{\Theta}_{\text{WDW}} \propto \hat{\varepsilon}(bvb) \) and \( \hat{\Theta}_{\text{LQC}} \propto \hat{\varepsilon}(\sin(2b)\hat{\varepsilon}(\sin(2b)) \) are non-symmetric operators which can be regarded as a product of two self-adjoint operators. Thus, the coherent state functional integrals suggest the *Moyal-*product to express the effective Hamiltonian for the quantum system with a non-*symmetric* Hamiltonian operator. Now we explore the motivation of a non-symmetric gravitational Hamiltonian operator through the LQC prescription. It should be noted that the initial Hamiltonian constraint operator in LQC is actually self-adjoint in the kinematical Hilbert space [10, 24]. To resolve the constraint equation and find physical states, one feasible method is to rebuild the constraint equation as a Klein-Gordon-like equation and treat the scalar \( \phi \) as an *internal time*. As a result, the constrained quantum system was recast into an unconstrained system of non-relativistic particle whose dynamics is governed by a Klein-Gordon-like equation with an *emergent time* variable [10, 24]. The price to get this Klein-Gordon-like equation is that the new gravitational Hamiltonian operator \( \hat{\Theta}_{\text{LQC}} \) becomes a multiplication of two self-adjoint operators, and hence it is no longer symmetric. But this does not indicate that one could not employ \( \hat{\Theta} \) in the intermediate step to find physical states. On the other hand, this non-symmetric \( \hat{\Theta}_{\text{LQC}} \) just provides a suitable arena to examine the *Moyal-*product from the path integral perspective. The appearance of *Moyal-*product in our path integral formalism indicates a possible duality between the path integral formulation on a non-commutative Moyal plane (see e.g. Ref.[35]) and the canonical quantization on an usual phase space with a "non-symmetric" Hamiltonian operator.

We can also take another practical way to symmetrize \( \hat{\Theta}_{\text{LQC}} \) at the beginning. For example, one can define a symmetric version of \( \hat{\Theta}_{\text{LQC}} \) by

\[
\hat{\Theta}'_{\text{LQC}} := \frac{1}{2} (\hat{\Theta}_{\text{LQC}} + \hat{\Theta}^*_{\text{LQC}}) \propto [\hat{\varepsilon}(\sin(2b)\hat{\varepsilon}(\sin(2b)) + (\sin(2b)\hat{\varepsilon}(\sin(2b))\hat{\varepsilon}),
\]

and then carry out the same procedure of above coherent state functional integral. In the calculation of matrix element \( \langle \Psi_{\alpha} | \hat{\Theta}'_{\text{LQC}} | \Psi_{\alpha-1} \rangle \), we could think that the operators \( \hat{\varepsilon} \) and \( \sin(2b)\hat{\varepsilon}(\sin(2b)) \) in \( \hat{\Theta}_{\text{LQC}} \) act on bra \( \langle \Psi_{\alpha} \rangle \) and ket \( | \Psi_{\alpha-1} \rangle \) respectively, while \( \sin(2b)\hat{\varepsilon}(\sin(2b)) \) and \( \hat{\varepsilon} \) in \( \hat{\Theta}^*_{\text{LQC}} \) act on bra \( \langle \Psi_{\alpha} \rangle \) and ket \( | \Psi_{\alpha-1} \rangle \) respectively. Then it is not difficult to see that the imaginary parts generated by \( \hat{\Theta}_{\text{LQC}} \) and \( \hat{\Theta}^*_{\text{LQC}} \) cancel each other. Hence for the symmetric Hamiltonian operator corresponding to \( \hat{\Theta}_{\text{LQC}} \), we can get a real effective Hamiltonian constraint without imaginary part for LQC. Similarly, we can also get a real effective Hamiltonian constraint in WDW quantum cosmology by employing a symmetric version of \( \hat{\Theta}_{\text{WDW}} \).
IV. ON THE EFFECTIVE DYNAMICS

Using the effective Hamiltonian constraints $\mathcal{H}_{\text{eff}}$ and $\mathcal{H}_{\text{eff}}^*$ which contain Moyal $*$-product, one may derive the corresponding dynamical equations. For LQC, we can define the evolution equations by:

$$f(v, b) := \frac{1}{i\hbar} (f * \mathcal{H}_{\text{eff}} - \mathcal{H}_{\text{eff}} * f),$$

for any dynamical quantity $f(v, b)$. Especially, the evolution of basic variables can be obtained as:

$$\dot{v} = \frac{12\pi G}{\!\hbar}\left[v * (v \sin (2b) \cos (2b)(1 - 4\varepsilon^2) + \frac{\sin (2b) \cos (2b)(1 - 4\varepsilon^2)}{2\varepsilon^2}\right]$$

$$+ \left(\frac{v^2}{2} - \left(v^2 + \frac{1}{2} \varepsilon^2\right) \sin^2 (2b) - \frac{\sin^2 (2b)(1 - 4\varepsilon^2)}{8\varepsilon^4}\right) \partial_b \varepsilon^2,$$

and

$$\dot{b} = -\frac{3\pi G}{\!\hbar}\left[2(v(1 - 4\varepsilon^2) \sin (2b)) * \sin (2b) + 4v\varepsilon^2\right]$$

$$- \frac{1}{\varepsilon^2}\left(\frac{\sin^2 (2b)}{2}(1 - 4\varepsilon^2) + (v^2 + \frac{1}{2} \varepsilon^2)4 \sin^2 (2b)\varepsilon^4 - 2v^2\varepsilon^4\right) \partial_b \varepsilon^2,$$

where $\varepsilon \equiv 1/d$ denote the quantum fluctuation of $b$, $\partial_b \varepsilon^2 \equiv \partial(\varepsilon^2)/\partial b$ and $\partial_v \equiv \partial/\partial v$. Similarly, we can use the effective Hamiltonian (3.19) in WDW quantum cosmology to get the evolution of basic variables as:

$$\dot{v} = -\frac{12\pi G}{\!\hbar}\left[2v \ast (vb) + \frac{b}{\varepsilon^2} + \left(\frac{v^2}{2} - \frac{b^2}{2\varepsilon^4}\right) \partial_b \varepsilon^2\right],$$

$$\dot{b} = \frac{12\pi G}{\!\hbar}\left[2(vb) \ast b + v \varepsilon^2 - \frac{1}{\varepsilon^4}\left(\frac{b^2}{2} - \frac{v^2 \varepsilon^4}{2}\right) \partial_v \varepsilon^2\right],$$

where $\varepsilon \equiv 1/\delta$ denotes the quantum fluctuation of $b$. However, there seem no way to understand Eqs. (4.2a)-(4.3b) directly as effective classical equations because of the $*$-product therein. To get physically predictable effective equations, we have to appeal to other possibilities.

Since the Moyal $*$-product originates from the non-commutativity of operators, one can symmetrize the operator $\hat{\Theta}_{\text{LQC}}$ as Eq. (3.21) and repeat the procedure of coherent state functional integrals in last section. Then it is not difficult to get the effective Hamiltonian constraint for LQC as:

$$\mathcal{H} := -\frac{\rho_c^2}{\hbar^2} - \frac{1}{2\sigma^2} + 3\pi G \left(\frac{v^2}{2} + \frac{d^2}{2}\right) \left(\sin^2 (2b)(1 - 4\varepsilon^2) + \frac{2}{d^2}\right),$$

which takes the same form as Eq.(3.20) but without the $*$-product. Note that this effective Hamiltonian constraint is different from that obtained in Ref.[21] where a different Hamiltonian constraint operator was employed. Using the conventional Poisson bracket, we can get the evolution of $v$ as

$$\dot{v} = -\frac{12\pi G}{\hbar}\left[(v^2 + \frac{1}{2} \varepsilon^2) \sin (2b) \cos (2b)(1 - 4\varepsilon^2) + \left(\frac{v^2}{2} - \left(v^2 + \frac{1}{2} \varepsilon^2\right) \sin^2 (2b) - \frac{\sin^2 (2b)(1 - 4\varepsilon^2)}{8\varepsilon^4}\right) \partial_b \varepsilon^2\right].$$

Then a modified Friedmann equation can be derived as

$$H_{\text{LQC}}^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_c}{3}$$

$$\cdot \left[1 + \frac{1}{2\varepsilon^2} \sin (2b) \cos (2b)(1 - 4\varepsilon^2) + \frac{1}{2} - \left(1 + \frac{1}{2\varepsilon^2} \sin (2b) - \frac{\sin^2 (2b)(1 - 4\varepsilon^2)}{8\varepsilon^2 \varepsilon^2}\right) \partial_b \varepsilon^2\right]^2$$

(4.6)

where $\rho_c = \rho_c \sqrt{\frac{\hbar}{2\pi G^2 k_B}}$ is a constant. To annihilate $\sin (2b)$ and $\cos (2b)$ in Eq. (4.6), we use the constraint equation (4.4) to get

$$\sin^2 (2b) = \frac{1}{1 - 4\varepsilon^2} \left(\frac{\rho F}{\rho_c E} - 2\varepsilon^2\right),$$

(4.7)
where \( \rho = \frac{p^2}{2\hbar^2} \) is the density of matter, \( F = 1 + \frac{\hbar^2}{2\pi^2 \rho c} \), and \( E = 1 + \frac{1}{2\pi^2 \rho c} \). However, Eq. (4.6) looks problematic since it depends on the volume \( v \) of the chosen fiducial cell. This originates from the fact that we have to use the coherent states peaked on the phase points \((v, b)\) in the path integral. In the final picture we have to remove the infrared regulator by letting the cell occupy full spatial manifold. In this limit, the irrelevant correction terms proportional to \( 1/(v_{\#})^2 \) could be neglected, while the relevant terms proportional to \( \varv^2 \) would be kept, since \( \varv \) was understood as the fluctuation of \( \sin b \) which does not depend on the fiducial cell. We finally get

\[
H_{LQC}^2 = \frac{8\pi G \rho_c}{3} \left[ \pm \sqrt{\left( \frac{\rho}{\rho_c} F - 2\varv^2 \right) \left( 1 - 4\varv^2 - \left( \frac{\rho}{\rho_c} F - 2\varv^2 \right) \right)} + \frac{1}{2} - \frac{1}{1 - 4\varv^2} \left( \frac{\rho}{\rho_c} F - 2\varv^2 \right) \partial_b \varv^2 \right]^2, \tag{4.8}
\]

where the positive and negative signs correspond to the expanding and contracting universe respectively. Since \( \frac{\hbar^2}{2\pi^2 \rho_c} \) is the square of fluctuation of \( \hat{\phi} \), one has \( \frac{\hbar^2}{2\pi^2 \rho_c^2} \ll 1 \). If one ignored the higher order quantum corrections, the modified Friedmann equation (4.8) would be simplified to the well-known form: \( H_{LQC}^2 = \frac{8\pi G \rho_c}{3} \left( 1 - \frac{\varv}{\rho_c} \right) \). However, Eq. (4.8) implies significant departure from classical GR, which is manifested in the bounce or re-collapse points determined by \( H_{LQC} = 0 \). For a contracting universe, the quantum bounce happens when

\[
\frac{1}{1 - 4\varv^2} \left( \frac{\rho}{\rho_c} F - 2\varv^2 \right) = \frac{1}{2} + \frac{1}{2} \sqrt{\frac{(1 - 4\varv^2)^2}{(1 - 4\varv^2)^2 + (\partial_b \varv^2)^2}}. \tag{4.9}
\]

Because \( \varv \ll 1 \) and \( \partial_b \varv^2 \ll 1 \), the right hand side of Eq. (4.9) could be very close to but no bigger than 1. Hence the so-called quantum bounce of LQC will occur when \( \rho \) increases to \( \rho_{\text{boun}} \approx \rho_c \). On the other hand, for an expanding universe, the positive sign should be chosen in Eq. (4.8). As a result, the Hubble parameter would always keep non-zero unless \( \partial_b \varv^2 \) approaches 0 asymptotically. Assuming this is the case, in the asymptotic regime we would get

\[
H_{LQC}^2 = \frac{8\pi G \rho_c}{3} \left( \frac{\rho}{\rho_c} F - 2\varv^2 \right) \left( 1 - 4\varv^2 - \left( \frac{\rho}{\rho_c} F - 2\varv^2 \right) \right), \tag{4.10}
\]

Therefore, under above assumption, a re-collapse would occur if \( \rho \) decrease to \( \rho_{\text{coll}} \approx 2\varv^2 \rho_c \), which coincides with the result in the canonical theory \([11, 21]\). As pointed out in Ref. [21], in this case the inferred re-collapse is almost in all probability as viewed from the parameter space characterizing the quantum fluctuation \( \varv \).

For WDW quantum cosmology, one can also symmetrize the operator \( \Theta_{\text{WDW}} \) and repeat the procedure of coherent state functional integrals. Then it is not difficult to get the effective Hamiltonian constraint:

\[
\mathcal{H} := -\frac{p^2_{\phi}}{\hbar^2} - \frac{1}{2\sigma^2} + 12\pi G \left( \frac{v^2 + \delta^2}{2} \right) \left( b^2 + \frac{1}{2\sigma^2} \right), \tag{4.11}
\]

which takes the same form as Eq. (3.19) but without the \( * \)-product. Using this Hamiltonian constraint, we can get the modified Friedmann equation for WDW cosmology as:

\[
H_{WDW}^2 = \frac{8\pi G \rho_c}{3} \left[ \pm \sqrt{\left( \frac{\rho}{\rho_c} F - 2\varv^2 + \frac{\partial_b \varv^2}{2} \right)} \right]^2, \tag{4.12}
\]

where the positive and negative signs correspond to the expanding and contracting universe respectively. It is obvious from Eq. (4.12) that there would be no bounce for a contracting universe. For an expanding universe, the Hubble parameter \( H_{WDW}^2 \) might vanish only if \( \partial_b \varv^2 \) approaches 0 asymptotically. In this case, Eq. (4.12) implies that a re-collapse would also happen if the density of matter could decrease to \( \rho \approx \rho_{\text{coll}} = 2\varv^2 \rho_c \). Hence, once higher-order quantum corrections are included, the inferred re-collapse is a common effect in both WDW cosmology and LQC under the condition that quantum fluctuations approach constant asymptotically. Intuitively, as the universe expands unboundedly, the matter density would become so tiny that its effect could be comparable to that of quantum fluctuations of the spacetime geometry. Then the Hamiltonian constraint may force the universe to contract back.

V. CONCLUDING REMARKS

The minisuperspace models of quantum cosmology provide the good avenue for testing the ideas and constructions of quantum gravity theories. A few physically significant results have been obtained in both WDW cosmology
and LQC. In LQC, the big bang singularity is resolved by the quantum bounce, and the effective Hamiltonian constraint with higher-order quantum corrections could even be obtained by the semiclassical analysis, which implied a possible effect of quantum gravity on large scale cosmology. It is desirable to study such kind of predictions from different perspectives and in different frameworks. Since the higher-order corrections of the Hamiltonian come from the quantum fluctuations, a natural attempt to achieve them is to employ coherent state path-integral. On the other hand, the so-called Moyal $\ast$-product in quantum mechanics is generalized to many situations including quantum spacetime itself. It is also possible and desirable to derive the $\ast$-product by coherent state functional integral approach within quantum cosmological models. These issues have been addressed in previous sections.

We summarize our main results with a few remarks. First, by the well-established canonical theories, the coherent state functional integrals for both WDW cosmology and LQC have been formulated by group averaging. As far as we know, this is the first attempt to apply coherent state functional integral to the models of quantum cosmology. Second, the main calculation results of our coherent state functional integrals are the effective Hamiltonian constraints (3.19) and (3.20) for WDW cosmology and LQC respectively. These show that the Moyal (star)-product can emerge naturally in the path integral approach via the effective Hamiltonian with higher-order quantum corrections. That is the main reason why we start with the non-symmetric gravitational Hamiltonian constraint operators $\hat{\Theta}_{\text{WDW}}$ and $\hat{\Theta}_{\text{LQC}}$ for the path integrals. Whether the resulted Hamiltonian constraints with $\ast$-product could make some physical prediction would be an interesting open issue. Tentatively, the appearance of Moyal $\ast$-product in our coherent state path integral indicates a possible duality between the path integral formulation on a non-commutative Moyal plane and the canonical quantization on an usual phase space with a “non-symmetric” Hamiltonian operator. Third, for symmetric Hamiltonian constraint operators, the effective theories and modified Friedmann equations have been obtained by the coherent state path integrals in both WDW cosmology and LQC. For LQC, the effective equation (4.8) can reduce to the first-order modified Friedmann equation when higher order quantum corrections are neglected. Hence the quantum bounce resolution of big bang singularity can also be obtained by the path integral representation. On the other hand, if higher order corrections are included, under the condition that quantum fluctuations approach constant asymptotically, there is great possibility for the re-collapse of an expanding universe due to the quantum gravity effect, which coincides with the result obtained in canonical LQC. Moreover, the effective equations imply that the inferred effect of re-collapse is common in both WDW cosmology and LQC under above condition. Finally, it should be noted that, as we used the coherent states of Gaussian type, the effective equations and hence the inferred effect of re-collapse are only valid with the assumption that these coherent states can faithfully represent the semiclassical behaviors of WDW cosmology and LQC. Whether there is a similar result for other semiclassical states is still an interesting open issue.

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Appendix A: Calculation of the functional integrals

We will use the following Gaussian integrals:

\[
\begin{align*}
\int_{-\infty}^{+\infty} dx \ e^{-a^2 x^2} &= \frac{\sqrt{\pi}}{a}, \\
\int_{-\infty}^{+\infty} dx \ x^2 e^{-a^2 x^2} &= \frac{\sqrt{\pi}}{a} \frac{1}{2a^2}, \\
\int_{-\infty}^{+\infty} dx \ x^3 e^{-a^2 x^2} &= \frac{\sqrt{\pi}}{a} \frac{3}{4a^4}, \\
\int_{-\infty}^{+\infty} dx \ \cos(bx) e^{-a^2 x^2} &= \frac{\sqrt{\pi}}{a} \frac{e^{-\frac{b^2}{4a^2}}}{\sqrt{a}}, \\
\int_{-\infty}^{+\infty} dx \ x^2 \cos(bx) e^{-a^2 x^2} &= \frac{\sqrt{\pi}}{a} \left( \frac{1}{2a^2} - \frac{b^2}{4a^4} \right) e^{-\frac{b^2}{4a^2}}, \\
\int_{-\infty}^{+\infty} dx \ x \sin(bx) e^{-a^2 x^2} &= \frac{\sqrt{\pi}}{a} \left( \frac{b}{2a^2} \right) e^{-\frac{b^2}{4a^2}}, \\
\int_{-\infty}^{+\infty} dx \ x^3 \sin(bx) e^{-a^2 x^2} &= \frac{\sqrt{\pi}}{a} \left( \frac{3b}{4a^4} - \frac{b^3}{8a^6} \right) e^{-\frac{b^2}{4a^2}}, \\
\int_{-\infty}^{+\infty} dx \ x^4 \cos(bx) e^{-a^2 x^2} &= \frac{\sqrt{\pi}}{a} \left( \frac{3}{4a^4} - \frac{3b^2}{4a^6} + \frac{b^4}{16a^8} \right) e^{-\frac{b^2}{4a^2}},
\end{align*}
\]

with constant parameters \( a > 0, \ b \in \mathbb{R} \). For the simplicity of notation, we denote the momentum conjugate to \( \phi \) by \( p_\phi \) in this appendix.

The main task is to calculate the matrix elements of exponentiated operators: \( \langle \Psi_{zn} | e^{i\alpha_\theta} | \Psi_{z_n-1} \rangle \), \( \langle \Psi_{zn} | e^{-i\alpha_\theta}\Theta_{WDW} | \Psi_{z_n-1} \rangle \) and \( \langle \Psi_{zn} | e^{-i\alpha_\theta}\Theta_{LQC} | \Psi_{z_n-1} \rangle \). For matter part, we get

\[
\langle \Psi_{zn} | \Psi_{z_n-1} \rangle = \exp \left( -\frac{(\phi_n - \phi_{n-1})^2}{2(\sigma_n^2 + \sigma_{n-1}^2)} + \frac{i}{\hbar} \frac{\sigma_n^2 p_n + \sigma_{n-1}^2 p_{n-1}}{\sigma_n^2 + \sigma_{n-1}^2} (\phi_n - \phi_{n-1}) \right) \cdot \int d\phi \exp \left[ -\frac{\sigma_n^2 + \sigma_{n-1}^2}{2\sigma_n^2 \sigma_{n-1}^2} \left( \phi - \frac{\sigma_{n-1}^2 \phi_n + \sigma_n^2 \phi_{n-1}}{\sigma_n^2 + \sigma_{n-1}^2} \right)^2 - \frac{i}{\hbar} (p_n - p_{n-1}) \left( \phi - \frac{\sigma_{n-1}^2 \phi_n + \sigma_n^2 \phi_{n-1}}{\sigma_n^2 + \sigma_{n-1}^2} \right) \right].
\]

Introducing some intermediate-step notations:

\[
\overline{\phi}_n = \frac{\sigma_{n-1}^2 \phi_n + \sigma_n^2 \phi_{n-1}}{\sigma_n^2 + \sigma_{n-1}^2}, \quad \overline{p}_n = \frac{\sigma_n^2 p_n + \sigma_{n-1}^2 p_{n-1}}{\sigma_n^2 + \sigma_{n-1}^2}, \quad \overline{\sigma}_n = \frac{2\sigma_n^2 \sigma_{n-1}^2}{\sigma_n^2 + \sigma_{n-1}^2},
\]

we can get a concise writing:

\[
\langle \Psi_{zn} | \Psi_{z_n-1} \rangle = \exp \left( -\frac{(\phi_n - \phi_{n-1})^2}{2(\sigma_n^2 + \sigma_{n-1}^2)} + \frac{i}{\hbar} \overline{p}_n(\phi_n - \phi_{n-1}) \right) \cdot \int d\phi \exp \left( -\frac{(\phi - \overline{\phi}_n)^2}{\sigma_n^2} \right) \exp \left( -\frac{i}{\hbar} (p_n - p_{n-1})(\phi - \overline{\phi}_n) \right),
\]

and the integration wherein is easy to be done by changing the integrated variable \( \phi \) to \( \tilde{\phi} = \phi - \overline{\phi}_n \):

\[
\int d\tilde{\phi} \cos \left( \frac{i}{\hbar} (p_n - p_{n-1})\tilde{\phi} \right) \exp \left( \frac{\tilde{\phi}^2}{\sigma_n^2} \right) = \sqrt{\pi} \sqrt{\sigma_n^2} \exp \left( -\frac{(p_n - p_{n-1})^2}{4\hbar^2} \sigma_n^2 \right).
\]

Finally we get the inner product:

\[
\langle \Psi_{zn} | \Psi_{z_n-1} \rangle = \sqrt{\pi} \sqrt{\sigma_n^2} \exp \left[ \frac{(\phi_n - \phi_{n-1})^2}{2(\sigma_n^2 + \sigma_{n-1}^2)} - \frac{(p_n - p_{n-1})^2}{4\hbar^2} \sigma_n^2 + \frac{i}{\hbar} \overline{p}_n(\phi_n - \phi_{n-1}) \right]. \quad (A2)
\]
To get Eq. (3.6), we need to deal with the product
\[
\prod_{n=1}^{N} \frac{\langle \Psi_{z_n} | \Psi_{z_{n-1}} \rangle}{\| \Psi_{z_n} \| \| \Psi_{z_{n-1}} \|} = \left( \prod_{n=1}^{N} \sqrt{\frac{2\sigma_{n}\sigma_{n-1}}{\sigma_{n}^2 + \sigma_{n-1}^2}} \right) \exp \left[ \sum_{n=1}^{N} i \hbar \pi_{n}(\phi_{n} - \phi_{n-1}) \right] \exp \left[ \sum_{n=1}^{N} \left( - \frac{\phi_{n}^2 - 2\phi_{n}\phi_{n-1}}{2(\sigma_{n}^2 + \sigma_{n-1}^2)} - \frac{\phi_{n-1}^2}{2(\sigma_{n}^2 + \sigma_{n-1}^2)} - \frac{\phi_{n}^2 - 2\phi_{n}\phi_{n-1}}{4\hbar^2} - \frac{\phi_{n-1}^2}{4\hbar^2} \right) \right] \quad (A3)
\]
where the summation in the last part of exponential in Eq. (A3) can be re-organized as
\[
\sum_{n=1}^{N} \left( - \frac{\phi_{n}^2 - 2\phi_{n}\phi_{n-1}}{2(\sigma_{n}^2 + \sigma_{n-1}^2)} - \frac{\phi_{n-1}^2}{2(\sigma_{n}^2 + \sigma_{n-1}^2)} \right) = \frac{\phi_{N}^2}{2(\sigma_{N}^2 + \sigma_{N-1}^2)} - \frac{\phi_{0}^2}{2(\sigma_{1}^2 + \sigma_{0}^2)} - \sum_{n=1}^{N} \frac{2(\sigma_{n+1}^2 + \sigma_{n}^2)\phi_{n}(\phi_{n} - \phi_{n-1}) - (\sigma_{n+1} + \sigma_{n})(\sigma_{n+1} - \sigma_{n})\phi_{n}^2}{2(\sigma_{n+1}^2 + \sigma_{n}^2)(\sigma_{n}^2 + \sigma_{n+1}^2)} \quad (A4)
\]
and similarly
\[
\sum_{n=1}^{N} \left( - \frac{p_{n}^2 - 2p_{n}p_{n-1}}{4\hbar^2} - \frac{p_{n-1}^2}{4\hbar^2} \right) = \frac{p_{N}^2}{2\hbar^2(\sigma_{N}^2 + \sigma_{N-1}^2)} - \frac{p_{0}^2}{2\hbar^2(\sigma_{1}^2 + \sigma_{0}^2)} - \sum_{n=1}^{N} \frac{4(\sigma_{n+1}^2\sigma_{n}^2 + \sigma_{n}^2\sigma_{n-1}^2)\frac{2}{\hbar^2} \cdot \\pi \cdot (\phi_{n} - \phi_{n-1}) + 2\sigma_{n}^2(\sigma_{n+1}^2 + \sigma_{n}^2)(\sigma_{n+1} - \sigma_{n})^2}{4\hbar^2(\sigma_{n+1}^2 + \sigma_{n}^2)(\sigma_{n}^2 + \sigma_{n+1}^2)} \quad (A5)
\]
Collecting the above results of Eq. (A3), Eq. (A4) and Eq. (A5), we can finally get Eq. (3.6). The matrix element \( \langle \Psi_{z_{N}} | i\sigma_{N}^2 \rangle \Psi_{z_{n-1}} \) is proportional to
\[
\langle \Psi_{z_{N}} | \Psi_{z_{n-1}} \rangle = \exp \left( - \frac{(\phi_{n} - \phi_{n-1})^2}{2(\sigma_{n}^2 + \sigma_{n-1}^2)} + i \frac{\hbar}{\sigma_{n}} \pi_{n}(\phi_{n} - \phi_{n-1}) \right) \]
\[
\cdot \int \frac{dp_{n-1}}{\sigma_{n-1}} \frac{d\phi}{\sigma_{n}} \frac{\hbar^2}{\sigma_{n}} (\phi - \phi_{n-1})^2 + \frac{2i\hbar p_{n-1}}{\sigma_{n-1}} (\phi - \phi_{n-1}) \exp \left( - \frac{(\phi - \bar{\phi}_{n})^2}{\sigma_{n}^2} - i \frac{\hbar}{\sigma_{n}} (p_{n} - p_{n-1})(\phi - \bar{\phi}_{n}) \right) \]
To do the above integral, we have to rewrite the integrand as a function of \( \phi - \bar{\phi}_{n} \). Except for the exponential function, the left terms are polynomials of \( \phi \). The integral of zeroth power term \( p_{n-1}^2 + \frac{\hbar^2}{\sigma_{n}} \) gives
\[
\sqrt{\pi} \sqrt{\frac{2\hbar^2}{\sigma_{n}}} \exp \left( - \frac{(p_{n} - p_{n-1})^2}{4\hbar^2} \right) \cdot \left( p_{n-1}^2 + \frac{\hbar^2}{\sigma_{n}} \right) ,
\]
and hence its contribution to matrix element \( \langle \Psi_{z_{N}} | \Psi_{z_{n-1}} \rangle \) is:
\[
\langle \Psi_{z_{N}} | \Psi_{z_{n-1}} \rangle \left( p_{n-1}^2 + \frac{\hbar^2}{\sigma_{n}} \right) .
\]
To deal with the linear and square terms of \( \phi \), we first rewrite
\[
\phi - \phi_{n-1} = \phi - \bar{\phi}_{n} + \bar{\phi}_{n} - \phi_{n-1} = \phi - \bar{\phi}_{n} + \frac{\sigma_{n-1}^2}{\sigma_{n}^2 + \sigma_{n-1}^2} (\phi_{n} - \phi_{n-1}) \equiv \phi - \bar{\phi}_{n} + \phi'_{n},
\]
\[
(\phi - \phi_{n-1})^2 = (\phi - \bar{\phi}_{n} + \bar{\phi}_{n} - \phi_{n-1})^2 = (\phi - \bar{\phi}_{n})^2 + 2\phi_{n}(\phi - \bar{\phi}_{n}) + \phi'_{n}^2 ,
\]
Similar to the matter part, it is easy to get the inner product of two coherent states as
\[
\langle \Psi_n | c_{\alpha \alpha} \hat{\sigma}^2 | \Psi_{n-1} \rangle = \sqrt{\pi} \frac{\sigma_n^2}{\sigma_n^2} \exp \left( -\frac{(\phi_n - \phi_{n-1})^2}{\sigma_n^2} - \frac{i}{\hbar} \langle \sigma_n^2 \rangle - \frac{1}{\hbar^2} (p_n - p_{n-1})^2 \frac{\sigma_n^2}{\sigma_n^2} \right),
\]

where some notations are defined as before:
\[
\phi_n = \frac{\sqrt{\sigma_n^2}}{\hbar} p_n, \quad \psi_n = \frac{\sqrt{\sigma_n^2}}{\hbar} \phi_n.
\]

Combining all the above results, the matrix element \( \langle \Psi_{zn} | e^{ic_{\alpha \alpha} \hat{\sigma}^2} | \Psi_{zn-1} \rangle \) is
\[
\langle \Psi_{zn} | e^{ic_{\alpha \alpha} \hat{\sigma}^2} | \Psi_{zn-1} \rangle = (zn-1, zn, zn) \exp \left[ \frac{ic_{\alpha \alpha}}{\hbar^2} \left( p_{n-1}^2 + \frac{\hbar^2}{\sigma_n^2} \frac{2\sigma_n^2}{\sigma_n^2} \right) - \frac{\sigma_n^2}{\sigma_n^2} + \frac{2ih\sigma_n^2}{\sigma_n^2} \phi_n \right] + O(\epsilon^2)
\]
\[
= \langle \Psi_{zn} | \Psi_{zn-1} \rangle \exp \left[ \frac{ic_{\alpha \alpha}}{\hbar^2} \left( p_{n-1}^2 + \frac{\hbar^2}{\sigma_n^2} \frac{2\sigma_n^2}{\sigma_n^2} + P_{n,n-1} \right) \right],
\]
up to \( O(\epsilon^2) \). Here \( P_{n,n-1} \) denotes a polynomial of \( p_n - p_{n-1} \) and \( \phi_n - \phi_{n-1} \) without zeroth order terms. To get a functional integral formalism, one has to take the number of steps \( N \to \infty \) and so \( \phi_n - \phi_{n-1} \to 0, \ h\epsilon \to 0 \). Furthermore, there is already an infinitesimal \( \epsilon \) multiplied to \( P_{n,n-1} \). This means that \( P_{n,n-1} \) does not become a \textit{time derivative} in the limit of \( N \to \infty \). As a result, \( P_{n,n-1} \) does not contribute to \textit{effective action} \( S_{\alpha}^{\text{matt}} \).

For gravitational part of WDW theory, we get
\[
\langle \Psi_{\eta_n} | e^{-ic_{\alpha \alpha} \hat{\Theta}_{\text{WDW}}} | \Psi_{\eta_{n-1}} \rangle = \langle \Psi_{\eta_n} | \Psi_{\eta_{n-1}} \rangle + i\alpha (12\pi G) \int dv \Psi^*_\eta_n(v) v \partial v \left[ v \partial_v \Psi_{\eta_{n-1}}(v) \right] + O(\epsilon^2).
\]

Similar to the matter part, it is easy to get the inner product of two coherent states as
\[
\langle \Psi_{\eta_n} | \Psi_{\eta_{n-1}} \rangle = \sqrt{\pi} \frac{\sigma_n^2}{\sigma_n^2} \exp \left( -\frac{(v_n - v_{n-1})^2}{2(\sigma_n^2 + \sigma_{n-1}^2)} - \frac{4(\sigma_n^2 + \sigma_{n-1}^2)}{\delta_n^2} \right),
\]
where some notations are defined as before:
\[
\sigma_n = \frac{\delta_n^2 b_n + \delta_{n-1}^2 b_{n-1}}{\delta_n^2 + \delta_{n-1}^2}, \quad \sigma_{n-1} = \frac{\delta_n^2 \delta_{n-1}^2}{\delta_n^2 + \delta_{n-1}^2}.
\]

The second term of Eq. (A7) is proportional to
\[
\int dv \Psi^*_\eta_n(v) v \partial_v \left[ v \partial_v \Psi_{\eta_{n-1}}(v) \right] = e^{(v_n-v_{n-1})^2} \frac{\sigma_n(v_n-v_{n-1})}{\sigma_n^2} \int dv e^{-\frac{(v-v_n)^2}{\sigma_n^2}} \frac{2ib_n v^2 (v - v_{n-1})}{\delta_n^2 - \frac{v^2 (v - v_{n-1})^2}{\delta_n^2} - \frac{1}{ib_n v_{n-1} v_{n-1}}},
\]
\[
\int dv e^{-\frac{(v-v_n)^2}{\sigma_n^2}} \frac{2ib_n v^2 (v - v_{n-1})}{\delta_n^2 - \frac{v^2 (v - v_{n-1})^2}{\delta_n^2} - \frac{1}{ib_n v_{n-1} v_{n-1}}}.
\]
where $\nu_n = \frac{\delta_{2n-1}v_n + \delta_{2n}v_{n-1}}{\delta_{2n-1} + \delta_{2n-1}}$. To do the integral, we first have to rewrite the integrand as a function of $v - \nu_n$ as follows:

$$
- \left( \frac{1}{\delta_{2n-1}^2} + b_{n-1}^2 \right) v^2 + \frac{v^2(v - v_{n-1})^2}{\delta_{2n-1}^2} + \frac{2i\nu_{n-1}v^2(v - v_{n-1})}{\delta_{2n-1}^2} - \frac{v(v - v_{n-1})}{\delta_{2n-1}^2} - ib_{n-1}v
$$

$$
= - \left[ \left( \frac{1}{\delta_{2n-1}^2} + b_{n-1}^2 \right) \nu_n^2 + ib_{n-1}\nu_n \right] + \frac{(v - \nu_n)^4}{\delta_{2n-1}^4} + \frac{2(\nu_n + \nu_{n-1})^2 + 2i\nu_{n-1}}{\delta_{2n-1}^2} (v - \nu_n)^3
$$

$$
- \left[ \left( \frac{1}{\delta_{2n-1}^2} + b_{n-1}^2 \right) \nu_n^2 + (\nu_{n-1})^2 + 4\nu_n\nu_{n-1} \right] - \frac{2i\nu_{n-1}(2\nu_n + \nu_{n-1})}{\delta_{2n-1}^2} + \frac{1}{\delta_{2n-1}^2} (v - \nu_n)^2
$$

$$
- \frac{2\nu_n}{\delta_{2n-1}^2} \left( \frac{1}{\delta_{2n-1}^2} + b_{n-1}^2 \right) \nu_n + \nu_{n-1} - \frac{2i\nu_{n-1}\nu_n}{\delta_{2n-1}^2} + \frac{4i\nu_{n-1}\nu_n}{\delta_{2n-1}^2} + \frac{1}{\delta_{2n-1}^2} (v - \nu_n)
$$

and finally the matrix element:

$$
\langle \Psi_{\eta n} | e^{-i\alpha \hat{A}_{\text{WQD}}} | \Psi_{\eta n-1} \rangle = \langle \Psi_{\eta n} | \Psi_{\eta n-1} \rangle \exp \left[ -i\alpha \cdot 12\pi G \left( \frac{1}{\delta_{2n-1}^2} + b_{n-1}^2 \right) \nu_n + \nu_{n-1} - \frac{3(\nu_n^2)}{4\delta_{2n-1}^4} \right]
$$

where $\nu_n = \frac{\delta_{2n-1}(v_n - v_{n-1})}{\delta_{2n-1}}$. Doing the integral term by term, we can obtain the result of Eq. (A9):

$$
\int dv \Psi_{\eta n}^*(v) v \partial_\eta \left[ v \partial_\eta \Psi_{\eta n-1}(v) \right] = -\langle \Psi_{\eta n} | \Psi_{\eta n-1} \rangle \left[ \left( \frac{1}{\delta_{2n-1}^2} + b_{n-1}^2 \right) \nu_n + \nu_{n-1} - \frac{3(\nu_n^2)}{4\delta_{2n-1}^4} \right]
$$

and finally the matrix element:

$$
\langle \Psi_{\eta n} | e^{-i\alpha \hat{A}_{\text{WQD}}} | \Psi_{\eta n-1} \rangle = \langle \Psi_{\eta n} | \Psi_{\eta n-1} \rangle \exp \left[ -i\alpha \cdot 12\pi G \left( \frac{1}{\delta_{2n-1}^2} + b_{n-1}^2 \right) \nu_n + \nu_{n-1} - \frac{3(\nu_n^2)}{4\delta_{2n-1}^4} \right]
$$

where $D_{n,n-1}^{\text{WQD}}$ is a polynomial of $v_n - v_{n-1}$, $b_n - b_{n-1}$ and $\delta_n - \delta_{n-1}$ without the zeroth order term. As before, this quantity does not contribute to the effective action under the continuous limit $\epsilon \equiv 1/N \rightarrow 0$.

For gravitational part of LQC, we have to calculate the inner product $\langle \Psi_{\eta n} | \Psi_{\eta n-1} \rangle$ and matrix element $\langle \Psi_{\eta n} | e^{-i\alpha \hat{A}_{\text{LQC}}} | \Psi_{\eta n-1} \rangle$. The inner product of two shadow states has been obtained in Eq. (3.15). The order of $O(\epsilon)$ of the matrix element $\langle \Psi_{\eta n} | e^{-i\alpha \hat{A}_{\text{LQC}}} | \Psi_{\eta n-1} \rangle$ is

$$
\langle \Psi_{\eta n} | e^{-i\alpha \hat{A}_{\text{LQC}}} | \Psi_{\eta n-1} \rangle = i\alpha \frac{3\pi G}{4} \sum_k \left[ k(k + 2)\Psi_{\eta n}^*(k)\Psi_{\eta n-1}(k) - 2k^2\Psi_{\eta n}^*(k)\Psi_{\eta n-1}(k) + k(k - 2)\Psi_{\eta n}^*(k)\Psi_{\eta n-1}(k) \right]
$$

Now we need to deal with the three terms $D_{n,n-1}^+, D_{n,n-1}^0, D_{n,n-1}^-$ separately. First, we get

$$
D_{n,n-1}^+ \equiv \sum_k (k^2 + 2k) e^{-\frac{i\nu_n(v_n - v_{n-1})^2}{2\nu_n^2}} e^{-ib_n_k(v_n - v_{n-1})} e^{ib_n^+(k + 4 - v_n - 1)}
$$

$$
= \exp \left[ \frac{4(v_n - v_{n-1})}{d_{2n-1}^2 + d_{2n-1}^2} - \frac{8}{d_{2n-1}^2 + d_{2n-1}^2} + \frac{i4b_n}{2(d_{2n-1}^2 + d_{2n-1}^2)} \right]
$$

$$
\cdot \sum_k (k^2 + 2k) \exp \left[ (-k - \nu_n^+)^2/d_{2n-1}^2 - i(b_n - b_{n-1})(k - \nu_n^+) \right],
$$

where $\nu_n^+ = \frac{\delta_{2n-1}v_n + \delta_{2n}(v_{n-1} - 1)}{d_{2n-1}^2 + d_{2n-1}^2}$. To do the summation in the above equation, we first have to rewrite $k^2 + 2k$ as a
function of $k - \Phi_n^+$:

$$
k^2 + 2k = (k - \Phi_n^+)^2 + (2\Phi_n^+ + 2)(k - \Phi_n^+) + \Phi_n^+ + 2\Phi_n^+
= (k - \Phi_n^+)^2 + 2 \left( \Phi_n - \frac{3d_n^2 - d_n^2 - d_{n-1}^2}{d_n^2 + d_{n-1}^2} \right) (k - \Phi_n^+) + \Phi_n^+ - 2\Phi_n + \frac{3d_n^2 - d_n^2 - d_{n-1}^1}{d_n^2 + d_{n-1}^2} + \frac{8d_n^2 (d_n^2 - d_{n-1}^2)}{(d_n^2 + d_{n-1}^2)}
$$

where $\bar{v}_n = \frac{d_n^2 v_n + d_{n-1}^2 v_{n-1}}{d_n^2 + d_{n-1}^2}$. The sum about the square term is

$$
\sum_k (k - \Phi_n^+)^2 \exp \left( -(k - \Phi_n^+)^2 / \bar{v}_n - i(b_n - b_{n-1})(k - \Phi_n^+) \right)
= \sqrt{\pi} \sqrt{\bar{v}_n} \exp \left( -(b_n - b_{n-1})^2 \bar{v}_n / 4 \right) \left( \frac{d_n^2}{2} - (b_n - b_{n-1})^2 (\bar{v}_n)^2 / 4 \right).
$$

The sum about the linear term is proportional to

$$
\sum_k (k - \Phi_n^+) \exp \left( -(k - \Phi_n^+)^2 / \bar{v}_n - i(b_n - b_{n-1})(k - \Phi_n^+) \right)
= -i \sqrt{\pi} \sqrt{\bar{v}_n} \exp \left( -(b_n - b_{n-1})^2 \bar{v}_n / 4 \right) (b_n - b_{n-1}) \bar{v}_n / 2,
$$

and the contributions of the zeroth order terms are proportional to

$$
\sum_k \exp \left( -(k - \Phi_n^+)^2 / \bar{v}_n - i(b_n - b_{n-1})(k - \Phi_n^+) \right) = \sqrt{\pi} \sqrt{\bar{v}_n} \exp \left( -(b_n - b_{n-1})^2 \bar{v}_n / 4 \right).
$$

Collecting the above results, we get

$$
D_{n,n-1}^+ = \langle \Psi_n | \Phi_{n-1} \rangle e^{-\frac{2\Phi_n^2}{\bar{v}_n} + i \Phi_n} \left( \frac{\bar{v}_n^2}{2} - 2\Phi_n^2 \frac{2d_n^2}{d_n^2 + d_{n-1}^2} + \frac{d_n^2}{2} + P_{n,n-1}^+ \right),
$$

(A15)

where $P_{n,n-1}^+$ denotes a polynomial of $v_n - v_{n-1}, b_n - b_{n-1}$ and $d_n - d_{n-1}$ without the zeroth order term. Here we have expanded the factor $\exp \left( \frac{-4(b_n - b_{n-1})}{d_n^2 + d_{n-1}^2} \right)$ in Eq. (A14) as $1 - \frac{4(b_n - b_{n-1})}{d_n^2 + d_{n-1}^2} + \cdots$. Except for the leading term 1, all the other terms can be conflated with $P_{n,n-1}^+$. Analogous to the matter part, under the continuous limit $N \longrightarrow \infty$, this $P_{n,n-1}^+$ does not contribute to the effective action of gravity. With the experience of computing $D_{n,n-1}^+$, it is easy to calculate $D_{n,n-1}^0$ and $D_{n,n-1}^-$ as follows:

$$
D_{n,n-1}^0 = 2 \langle \Psi_n | \Phi_{n-1} \rangle \left( \frac{\bar{v}_n^2}{2} + P_{n,n-1}^0 \right),
$$

(A16)

$$
D_{n,n-1}^- = \langle \Psi_n | \Phi_{n-1} \rangle e^{-\frac{2\Phi_n^2}{\bar{v}_n} + i \Phi_n} \left( \frac{\bar{v}_n^2}{2} + 2\Phi_n^2 \frac{2d_n^2}{d_n^2 + d_{n-1}^2} + \frac{d_n^2}{2} + P_{n,n-1}^- \right).
$$

(A17)

Taking the expansion $e^{-\frac{2\Phi_n^2}{\bar{v}_n}} = 1 - \frac{8}{d_n^2 + d_{n-1}^2} + \mathcal{O} \left( \frac{1}{d_n^2} \right)$ and neglecting the higher order terms than $\left( \frac{1}{d_n^2} \right)$, we can get the combination

$$
D_{n,n-1}^+ - D_{n,n-1}^0 + D_{n,n-1}^-
= -4 \langle \Psi_n | \Phi_{n-1} \rangle \left( \frac{\bar{v}_n^2}{2} \right) \left( \sin^2 (2\Phi_n) \left( 1 - \frac{8}{d_n^2 + d_{n-1}^2} + \frac{4}{d_n^2 + d_{n-1}^2} \right) + i \sin (4\Phi_n) \pi_n \left( 1 - \frac{8}{d_n^2 + d_{n-1}^2} \right) + P_{n,n-1}^{\text{grav}} \right),
$$

and hence the matrix element $\langle \Phi_{n-1} | e^{-ic\alpha_n \tilde{\Theta}_{n-1}} | \Phi_{n-1} \rangle$ is

$$
\langle \Phi_{n-1} | e^{-ic\alpha_n \tilde{\Theta}_{n-1}} \rangle \exp \left[ -i \alpha_n \cdot 3\pi G \left( \left( \frac{\bar{v}_n^2}{2} \right) \left( \sin^2 (2\Phi_n) \left( 1 - \frac{8}{d_n^2 + d_{n-1}^2} + \frac{4}{d_n^2 + d_{n-1}^2} \right) + i \sin (4\Phi_n) \pi_n \left( 1 - \frac{8}{d_n^2 + d_{n-1}^2} \right) + P_{n,n-1}^{\text{grav}} \right) \right),
$$

(A18)
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