SURREAL NUMBERS, EXPONENTIATION AND DERIVATIONS

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Abstract. We give a presentation of Conway’s surreal numbers focusing on the connections with transseries and Hardy fields and trying to simplify when possible the existing treatments.

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Date: 15 August 2020.

2010 Mathematics Subject Classification. 03C64, 03E10,16W60,26A12,41A58.

Key words and phrases. Surreal numbers, transseries.

Partially supported by the Italian research project PRIN 2017, “Mathematical logic: models, sets, computability”, Prot. 2017NWTM8RPRIN.
1. Introduction

Conway’s field \(\mathbb{No}\) of surreal numbers [13] includes both the field of real numbers \(\mathbb{R}\) and the class \(\mathbb{On}\) of all ordinal numbers. The surreals originally emerged as a subclass of the larger class of games, comprising for instance the game of “Go” and similar combinatorial games. In this paper we shall however be interested in the more recent connections with transseries and Hardy fields.

The field \(\mathbb{T}\) of transseries [44, 45] (see [27, 30] for a different variant) is an extension of the field of Puiseux series that plays an important role in Ecalle’s positive solution of the problem of Dulac [16, 17]: the finiteness of limit cycles in polynomial planar vector fields. Unlike the Puiseux series, the transseries are closed under formal integration and admit an exponential and a logarithmic function. It is possible to consider \(\mathbb{T}\) as a universal domain for the existence of solutions of an important class of formal differential equations of non-oscillatory nature. This is made precise in [4], where the first-order theory of \(\mathbb{T}\) is shown to be recursively axiomatisable and model complete.

A first connection between \(\mathbb{T}\) and \(\mathbb{No}\) comes from the fact that \(\mathbb{No}\) admits a representation in terms of generalised series [13] and has an exponential function \(\exp: \mathbb{No} \to \mathbb{No}\) extending the real exponential function [21]. This makes it possible to interpret surreal numbers as asymptotic expansions, as in [8]. We can then see \(\mathbb{T}\) as a substructure of \(\mathbb{No}\), with the ordinal \(\omega\) playing the role of the formal variable, and introduce a strongly additive surjective derivation \(\partial: \mathbb{No} \to \mathbb{No}\) compatible with the exponential function and extending the derivation of \(\mathbb{T}\) [9]. It turns out that \(\mathbb{T}\) is an elementary substructure of \(\mathbb{No}\) both as a differential field [6], and as an exponential field [40] (based on [34, 43]). Very recently Kaplan has announced an axiomatisation and a model completeness result in the language with both \(\partial\) and \(\exp\), thus showing that \(\mathbb{T}\) is an elementary substructure of \(\mathbb{No}\) in this language [24]. Another connection with transseries comes from the work of Costin, Ehrlich, and Friedman [14], where it is shown that Ecalle-Borel transseriable functions extend naturally to \(\mathbb{No}\).

A remarkable feature of \(\mathbb{No}\), not shared by the field of transseries, is its universality. Every divisible ordered abelian group is isomorphic to an initial subgroup of \(\mathbb{No}\), and every real-closed field is isomorphic to an initial subfield of \(\mathbb{No}\) [18]. In a similar spirit it can be shown that every Hardy field can be embedded in \(\mathbb{No}\) as a differential field [6].

In this paper we shall give a detailed presentation of the rich structure on \(\mathbb{No}\), including the exponential and differential structure, and describe some of the connections with transseries and Hardy fields. This paper can be read as a supplement to [32].

2. Simplicity and order

Let \(\mathbb{On}\) be the class of von Neumann ordinals. We recall that an ordinal coincides with the set of all smaller ordinals. Conway [13] defined the surreals as a subclass of the class of “games”. He then showed that it is possible to represent each surreal as a transfinite binary sequence, called its sign-expansion. Following Gonshor [21] we define the surreal numbers directly as sign-expansions. More precisely, a surreal number is a function \(x: \alpha \to \{-+,+\}\) from some ordinal \(\alpha\) to \(\{-+,+\}\). If \(x\) is as above, we call \(\alpha\) the birthday of \(x\) and we write \(\alpha = \text{birthday}(x)\). We also say that \(x\) is born on day \(\alpha\).
The class \( \mathbf{No} \) of all surreal numbers has a natural structure of a complete binary tree whose nodes are the sign-expansions. The ancestors of a surreal \( y : \alpha \rightarrow \{-1,1\} \) are the restrictions of \( y \) to a smaller ordinal \( \beta < \alpha \). If \( x \) is an ancestor of \( y \), we say that \( x \) is simpler than \( y \), or that \( y \) is a descendant of \( x \). If \( x \) is simpler than \( y \), then clearly \( \text{birthday}(x) < \text{birthday}(y) \). The empty sequence, born on day 0, is the root of the tree and coincides with the simplest surreal number. Each surreal \( x : \alpha \rightarrow \{-,+,\} \) has two successors: a left-successor \( "x-" \) obtained by appending the sign \( "-" \) at the end of \( x \), and a right-successor \( "x+" \) obtained by appending the sign \( "+" \) at the end of \( x \). These are the immediate descendants of \( x \). Each descendant of \( x \) is either an immediate descendant, or a descendant of an immediate descendant.

We can now introduce a total order \( < \) on \( \mathbf{No} \) which essentially consists in projecting the tree on a horizontal axis parallel to a line of “siblings” in the tree. More precisely, a surreal \( x \) is bigger than its left-successor and its descendants, and smaller than its right-successor and its descendants. Thus \( x- < x < x+ \) and the same inequalities hold if we append arbitrary sequences after \( x- \) and \( x+ \), so for instance \(+ -- + < ++ + -\).

It should be remarked that \( \mathbf{No} \), like \( \mathbf{On} \), is not a set but a proper class. As an ordered class, \( \mathbf{No} \) has a remarkable universal property: every totally ordered set can be embedded in \((\mathbf{No}, <)\).

3. Left and right options

A subclass \( C \subseteq \mathbf{No} \) is convex if whenever \( x < y < z \) are surreal numbers and \( x, z \) are in \( C \), also \( y \) belongs to \( C \). Every non-empty convex subclass of \( \mathbf{No} \) has a simplest element, given by the element with smallest birthday. If \( L \) and \( R \) are sets of surreals, we write \( L < R \) if each element of \( L \) is smaller than each element of \( R \). In this case, the class of all elements \( x \in \mathbf{No} \) satisfying \( L < x < R \) is non-empty and convex. We write

\[ x = L \mid R \]
to express the fact that $x$ is the simplest surreal such that $L < x < R$. This representation is not unique, but it can be made unique adding the condition that $L \cup R$ is the set of all of all surreals simpler than $x$. This is called the canonical representation of $x$. In this case, the elements of $L$ are called the left-options of $x$, while the elements of $R$ are its right-options.

We can now name a few surreals (see Figure 1 on page 3). The root of the tree is the simplest surreal, and it is written as $0 = \emptyset | \emptyset$. Then we have its right-successor $1 = \{0\} | \emptyset$ and its left-successor $-1 = \emptyset | \{0\}$. The simplest surreal between $0$ and $1$ is $1/2 = \{0\} | \{1\}$, which coincides with the left-successor of $1$. With these definitions we have $-1 < 0 < 1/2 < 1$. These labels are consistent with the ring operations that we shall define below.

4. Sum and product

We shall define the sum $x + y$ and the product $xy$ of two surreal numbers $x$ and $y$ by induction on simplicity (using the fact that simplicity is a well founded relation) so as to obtain an ordered field. Assume that we have already defined $a + y$, $x + b$, $a + b$ for all $a$ simpler than $x$ and $b$ simpler than $y$. The axioms of ordered rings dictate that the operation $+$ should be strictly increasing in both arguments. This motivates the definition

$$x + y = \{x^L + y, x + y^L\} | \{x^R + y, x + y^R\}$$

where $x^L$ ranges over the left-options of $x$ and $x^R$ ranges over its right-options.

We define $-x$ exchanging recursively the left and right options:

$$-x = \{-x^R\} | \{-x^L\}.$$ 

In terms of sign-expansions, $-x$ is obtained from $x$ exchanging all plus signs with minus signs. It can be verified that $(\mathbb{No}, <, +)$ is an ordered abelian group and $-x$ is the opposite of $x$, that is $x + (-x) = 0$. As usual we write $x - y$ for $x + (-y)$.

To define the product $xy$ we assume that we have already defined $ay$, $xb$, $ab$ for all $a$ simpler than $x$ and all $b$ simpler than $y$. We now impose the distributivity law $(x - a)(y - b) = xy - xb - ay + ab$ and observe that this equality, together with the axioms of ordered rings, determines the sign of the difference $xy - (xb + ay - ab)$ given the relative order of $x, y, a, b$. We define $xy$ as the simplest surreal such that these signs are respected. More formally, we put

$$xy = \{x^L y + xy^L - x^L y^R, x^R y + xy^R - x^R y^L\} | \{x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\},$$

where as above $x^L$ ranges over the left-options of $x$, $x^R$ ranges over its right-options, and similarly for $y^L$ and $y^R$.

Conway [13] showed that with these operations $(\mathbb{No}, <, +, \cdot)$ is an ordered field. Moreover he showed that $\mathbb{No}$ is real closed: every polynomial over $\mathbb{No}$ which changes sign has a root in $\mathbb{No}$.

5. Embedding the reals

Since $\mathbb{No}$ is an ordered field, it contains a unique subfield $\mathbb{Q} \subset \mathbb{No}$ isomorphic to the rational numbers. The subgroup of the dyadic rationals $m/2^n \in \mathbb{Q}$, with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, correspond exactly to the surreal numbers $s : k \to \{0, 1\}$ whose birthday is a finite ordinal $k \in \mathbb{N}$.
Now, given a real number $r \in \mathbb{R}$, let $L \subseteq \mathbb{Q}$ be the set of rationals $< r$ and let $R \subseteq \mathbb{Q}$ be the set of rationals $> r$. If we identify $r$ with the surreal $L \mid R$ we obtain inclusions of ordered fields

$$\mathbb{Q} \subset \mathbb{R} \subset \text{No}.$$  

Under this identification, a surreal number $x : \alpha \to \{+, -\}$ belongs to $\mathbb{R}$ if and only its birthday is $\leq \omega$ and its sign-expansion is not eventually constant (see [13] or [21, p. 33]). By the classical work of Tarski, the theory of real closed ordered field is model complete, so the field $\mathbb{R}$ is an elementary substructure of $\text{No}$.

6. Embedding the ordinals

The ordinal numbers admit a natural embedding in the surreals. The ordinal $\alpha$ is mapped to the sign-expansion $x : \alpha \to \{-, +\}$ consisting entirely of plus signs. The image of this embedding is given by the surreals which admit a representation of the form $L \mid \emptyset$. When there is no risk of confusion we identify the ordinals with their image in $\text{No}$ and write $\text{On} \subset \text{No}$. Note however that we cannot make this identification when speaking about the birthday of a surreal number.

The natural numbers $\mathbb{N} \subset \mathbb{Q} \subset \text{No}$ coincide with the finite ordinals under the above embedding. The simplest surreal bigger than all natural numbers is the ordinal $\omega = \{0, 1, 2, \ldots \} \mid \emptyset$; its successor is the ordinal $\omega + 1 = \{0, 1, 2, \ldots , \omega \} \mid \emptyset$.

The surreal sum and product, when restricted to $\text{On} \subset \text{No}$, coincide with the Hessenberg sum and product of ordinal numbers [13, p. 28]. Unlike the usual sum and product of ordinals, the Hessenberg operations are commutative.

Since $\text{No}$ is a real closed ordered field which includes both $\mathbb{R}$ and $\text{No}$, it contains some strange numbers like $\omega - 1 = \{0, 1, 2, \ldots \} \mid \{\omega\}$ or $1/\omega = \{0\} \mid \{2^{-n}\}_{n \in \mathbb{N}}$, or $\sqrt{\omega}$, and we shall later see that it also contains $\log(\omega)$ and $\exp(\omega)$.

7. Asymptotic notations

Given $f, g$ in an ordered abelian group, we write $f \leq g$ if $|f| \leq n|g|$ for some $n \in \mathbb{N}$; if $f \leq g$ holds, we say that $f$ is dominated by $g$. If both $f \leq g$ and $g \leq f$ hold, we say that $f$ and $g$ belong to the same Archimedean class, and we write $f \asymp g$. We say that $f$ is strictly dominated by $g$, written $f \prec g$, if we have both $f \leq g$ and $f \neq g$. We define $f \sim g$ as $f - g \prec f$ and we say in this case that $f$ is asymptotic to $g$. Notice that $\sim$ is a symmetric relation. Indeed assume $f - g \prec f$ and let us prove that $f - g \prec g$. This is clear if $f \leq g$. On the other hand if $g \prec f$, then $f - g \asymp f$, contradicting the assumption.

We shall write $O(f)$ for the set of all $g$ such that $g \leq f$ and $o(f)$ for the set of all $g$ such that $g \prec f$.

Now let $K$ be an ordered field. We can use the above notations for elements of $K$ referring to the underlying structure of additive ordered group. Given a multiplicative subgroup $\mathfrak{M} \subseteq K_{>0}$, we say that $\mathfrak{M}$ is a group of monomials of $K$ if for every $f \in K \setminus \{0\}$ there is one and only one $n \in \mathfrak{M}$ with $f \asymp n$.

Every ordered field $K$ admits a Krull valuation $v : K^* \to v(K^*)$ whose value ring is the subring of finite elements $O(1)$. This is called the natural valuation, or Archimedean valuation. The restriction of the natural valuation to a group of monomials $\mathfrak{M}$ is an isomorphism from $\mathfrak{M}$ to $v(K^*)$. In other words, a group of monomials is a section of the natural valuation.
Example 7.1. Let $\mathbb{R}(x)$ be the field of rational functions ordered by $x > \mathbb{R}$ and let $f, g \in \mathbb{R}(x)$. We have $f < g$ if $f/g$ tends to 0; $f \sim g$ if $f(x)/g(x)$ tends to 1; and $f \asymp g$ if $f/g$ tends to a non-zero limit in $\mathbb{R}$. The multiplicative group $x^{\mathbb{Z}}$ is a group of monomials of $\mathbb{R}(x)$.

If $K$ is a real closed field, its value group with respect to the Archimedean valuation is a $\mathbb{Q}$-vector space. From the existence of basis in vector spaces it follows that every real closed field admits a group of monomials, but this property may fail if we relax the assumption that the field is real closed.

8. Generalised power series

Let $\langle \mathfrak{M}, <, \cdot, 1 \rangle$ be an abelian ordered group, written in multiplicative notation. We write $\mathbb{R}(\langle \mathfrak{M} \rangle)$ to denote Hahn’s field of generalised power series with monomials in $\mathfrak{M}$ and we recall that $\mathbb{R}(\langle \mathfrak{M} \rangle)$ is a maximal ordered field with a group of monomials isomorphic to $\mathfrak{M}$ (Kaplansky [25]).

Definition 8.1 (Hahn [22]). An element of $\mathbb{R}(\langle \mathfrak{M} \rangle)$ is a function $f : \mathfrak{M} \to \mathbb{R}$ whose support $\{ n \in \mathfrak{M} \mid f(n) \neq 0 \}$ is a reverse well ordered subset of $\mathfrak{M}$. To denote such a map, we use the notation $f = \sum_{n \in \mathfrak{M}} n f(n)$, or the notation

$$f = \sum_{i < \alpha} n_i r_i$$

where $\alpha \in \text{On}$, $(n_i)_{i < \alpha}$ is a decreasing enumeration of the support of $f$, and $0 \neq r_i = f(n_i) \in \mathbb{R}$ for all $i < \alpha$ (if $\alpha = 0$, the sum is empty, and $f = 0$).

The pointwise addition of functions makes $\mathbb{R}(\langle \mathfrak{M} \rangle)$ into a group. The multiplication $fg$ is defined by the usual convolution formula: the coefficient of $n$ in the product $fg$ is the sum $\sum f(a) g(b)$ taken over all pairs $(a, b)$ with $ab = m$. Since the supports of $f$ and $g$ are reverse well ordered, there are only finitely many pairs $(a, b)$ such that the real number $f(a) g(b)$ is non-zero, so $fg$ is well defined. With these operations $\mathbb{R}(\langle \mathfrak{M} \rangle)$ is obviously a ring, and we shall see below that it is a field. We order $\mathbb{R}(\langle \mathfrak{M} \rangle)$ in the obvious way: if $f = \sum_{i < \alpha} n_i r_i \neq 0$, then $f > 0 \iff r_0 > 0$.

Example 8.2. The field of Laurent series in descending powers of $x$ coincides with the Hahn field $\mathbb{R}(\langle x^{\mathbb{Z}} \rangle)$ ordered by $x > \mathbb{R}$.

We introduce a notion of infinite sum in $\mathbb{R}(\langle \mathfrak{M} \rangle)$ as follows.

Definition 8.3. A family $(f_i : i \in I)$ in $\mathbb{R}(\langle \mathfrak{M} \rangle)$ is summable if each $n \in \mathfrak{M}$ belongs to the support of finitely many $f_i$ and there is no strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ in $\mathfrak{M}$ such that each $n_k$ belongs to the support of some $f_i$. The sum

$$f = \sum_{i \in I} f_i$$

is then defined adding the coefficients of the corresponding monomials.

Given a multi-index $i = (i_1, \ldots, i_n) \in \mathbb{N}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}(\langle \mathfrak{M} \rangle)^n$, let $x^i = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$. We write $x < 1$ if $x_i < 1$ for all $i = 1, \ldots, n$.

Lemma 8.4 (Neumann’s lemma [33, 1]). For every $\varepsilon < 1$ in $\mathbb{R}(\langle \mathfrak{M} \rangle)^n$ and every $\{r_i\}_{i \in \mathbb{N}^n} \subseteq \mathbb{R}$ the family $(r_i \varepsilon^i)_{i \in \mathbb{N}^n}$ is summable.
If \( f \neq 0 \in \mathbb{R}(\mathbb{N}) \), we can find the multiplicative inverse of \( f \) as follows. Factoring out the leading monomial we write \( f = nr(1 + \varepsilon) \) with \( n \in \mathbb{N}, r \in \mathbb{R}^* \) and \( \varepsilon < 1 \). Then \( f^{-1} = n^{-1}r^{-1}(1 + \varepsilon)^{-1} \) where \( (1 + \varepsilon)^{-1} = \sum_{n \in \mathbb{N}} (-1)^n \varepsilon^n \) (the sum exists by Neumann’s lemma).

9. The omega-map

For \( x \in \mathbb{N}_0 \), we say that \( x \) is a surreal monomial if \( x \) is the simplest positive surreal in its Archimedean class. The surreals monomials form a group under the multiplication of \( \mathbb{N}_0 \), so they are indeed a group of monomials of \( \mathbb{N}_0 \) according to the previous definitions. Since \( \mathbb{N}_0 \) is a proper class, its group of monomials is also a proper class. What is more interesting is that the class of monomials is order isomorphic to \( \mathbb{N}_0 \) and it can be parametrised as follows.

**Theorem 9.1** (Conway [13]). There is an increasing map

\[
x \in \mathbb{N}_0 \mapsto \omega^x \in \mathbb{N}_0
\]

whose image \( \omega^{\mathbb{N}_0} \) is the class of surreal monomials and such that

1. \( \omega^0 = 1 \);
2. \( \omega^1 = \omega \) (the ordinal \( \omega \) seen as a surreal);
3. \( \omega^{x+y} = \omega^x \omega^y \).

This implies in particular that \( \omega^{(\varepsilon n)} = (\omega^x)^n \) for each \( n \in \mathbb{Z} \), so we can write \( \omega^{x^n} \) without parenthesis.

The definition of \( \omega^x \) is the following.

**Definition 9.2.** For \( x \in \mathbb{N}_0 \), let \( \omega^x = \{0, k\omega^{x'}\} \mid \{2^{-k}\omega^{x''}\} \) where \( x' \) ranges over the left-options of \( x \), \( x'' \) ranges over its right-options, and \( k \) ranges in \( \mathbb{N} \).

It follows from the definition that \( \omega^x \) is the simplest representative \( > 0 \) of its Archimedean class and that if \( x \) is simpler than \( y \), then \( \omega^x \) is simpler than \( \omega^y \).

10. Conway normal form

Let \( \mathcal{M} = \omega^{\mathbb{N}_0} \) be the class of surreal monomials and let \( \mathbb{R}(\omega^{\mathbb{N}_0})_{\mathsf{On}} = \mathbb{R}(\mathcal{M})_{\mathsf{On}} \) be the field of generalised power series with monomials in \( \mathcal{M} \); the subscript “\( \mathsf{On} \)” is meant to emphasise that, although the group \( \omega^{\mathbb{N}_0} \) is a proper class, in the definition of \( \mathbb{R}(\omega^{\mathbb{N}_0})_{\mathsf{On}} \) we only consider series \( \sum_{i<\alpha} m_i r_i \) whose support is a set (indexed by an ordinal). In this section we shall see that \( \mathbb{N}_0 = \mathbb{R}(\omega^{\mathbb{N}_0})_{\mathsf{On}} \) via a canonical identification.

**Remark 10.1.** Let \( \kappa \) be a regular uncountable cardinal and consider the subfield \( \mathbb{N}_0(\kappa) \subseteq \mathbb{N}_0 \) whose elements are the surreal numbers born before day \( \kappa \). It can be shown that \( \mathbb{N}_0(\kappa) \) is isomorphic to a field of the form \( \mathbb{R}(\mathcal{M})_{\kappa} \), where \( \mathbb{R}(\mathcal{M})_{\kappa} \subseteq \mathbb{R}(\mathcal{M}) \) consists of the series with support of cardinality \( < \kappa \) [40]. So \( \mathbb{N}_0 \) should not be thought as a field of the form \( \mathbb{R}(\mathcal{M}) \), but rather as a field of the form \( \mathbb{R}(\mathcal{M})_{\kappa} \) with \( \kappa \) being an inaccessible cardinal living in some larger set theoretic universe.

In §12 we shall see that \( \mathbb{N}_0 \) admits an exponential map. This should be contrasted with a result of Kuhlmann, Kuhlmann, and Shelah [26] where it is shown that a field of the form \( \mathbb{R}(\mathcal{M}) \) (where \( \mathcal{M} \) is a set of monomials) never admits an exponential map.
Theorem 10.2 (Conway [13]). We can make the identification \( \mathbb{N}_0 = \mathbb{R}((\omega \mathbb{N}_0))_{\mathbb{O}_n} \) via a canonical isomorphism of ordered fields.

Proof. By induction on the ordinal \( \alpha \), we will associate to a generalised power series \( \sum_{i<\alpha} m_i r_i \in \mathbb{R}((\omega \mathbb{N}_0))_{\mathbb{O}_n} \) a surreal number \( f \in \mathbb{N}_0 \) and we call \( \sum_{i<\alpha} m_i r_i \) the normal form of \( f \). In this case we write

\[
f = \sum_{i<\alpha} m_i r_i,
\]

identifying the surreal \( f \) with its normal form. The strategy is to first define a field embedding from \( \mathbb{R}((\omega \mathbb{N}_0))_{\mathbb{O}_n} \) to \( \mathbb{N}_0 \), and then show that it is surjective. Clearly 0 goes to 0 under the embedding. If \( \alpha = \beta + 1 \), we put

\[
\sum_{i<\alpha} m_i r_i = \sum_{i<\beta} m_i r_i + m_\beta r_\beta
\]

where “+” is the addition in \( \mathbb{N}_0 \); if \( \alpha \) is a limit ordinal, then the image of \( \sum_{i<\alpha} m_i r_i \) in \( \mathbb{N}_0 \) is the simplest \( z \in \mathbb{N}_0 \) such that \( z - \sum_{i<\beta} m_i r_i \simeq m_\beta \) for all \( \beta < \alpha \) (we can consider \( \sum_{i<\beta} m_i r_i \) as an element of \( \mathbb{N}_0 \) by the inductive hypothesis). The existence of \( z \) follows from the fact that every convex class of surreal numbers has a simplest element.

We have thus defined a field embedding from \( \mathbb{R}((\omega \mathbb{N}_0))_{\mathbb{O}_n} \) to \( \mathbb{N}_0 \). To prove that it is surjective we define the inverse map, that is, we compute the normal form of a surreal number. So let \( f \in \mathbb{N}_0 \). If \( f = 0 \), then \( f \) is already in normal form (represented by the empty sum). If \( f \neq 0 \) there is a unique monomial \( m \) and a unique real number \( r \in \mathbb{R}^\ast \) such that \( f = rm + g \) with \( g < f \). Then \( rm \) is the first term of the normal form of \( f \) and to find the other terms we iterate the process. More precisely, suppose we have defined the \( i \)-th term \( r_i m_i \) of \( f \) for each \( i < \alpha \), so that we can write

\[
f = \sum_{i<\alpha} r_i m_i + g_{\alpha}
\]

with \( g_{\alpha} < m_i \) for all \( i < \alpha \). If \( g_{\alpha} = 0 \) we have finished. In the opposite case, the \( \alpha \)-th term of \( f \) is first term of \( g_{\alpha} \). It can be shown that

\[
\alpha \leq \text{birthday}(\sum_{i<\alpha} r_i m_i) \leq \text{birthday}(f),
\]

so the process must stop in a number of steps \( \leq \text{birthday}(f) \).

Definition 10.3. Let \( f = \sum_{i<\alpha} m_i r_i \in \mathbb{N}_0 \) be written normal form, as in the proof of Theorem 10.2. Since the omega-map parametrises the surreal monomials, we can also write

\[
f = \sum_{i<\alpha} \omega^{x_i} r_i
\]

where \((x_i)_{i<\alpha}\) is a decreasing sequence in \( \mathbb{N}_0 \). This is called the Conway normal form of \( f \).

Thanks to the identification \( \mathbb{N}_0 = \mathbb{R}((\mathbb{N}_0))_{\mathbb{O}_n} \), we can define the support of a surreal number and the sum of a summable family of surreal numbers as in the context of generalised series. Note that the support of 0 is the empty set.

Proposition 10.4 ([13]). Conway’s omega-map extends the homonymous map on the ordinal numbers. If \( f \in \mathbb{O}_n \subset \mathbb{N}_0 \) is an ordinal, its Conway normal form
\[ \sum_{i < \alpha} \omega^{r_i} x_i \] coincides with the Cantor normal form. So in this case \( x_i \in \mathbb{O}_n \), \( r_i \in \mathbb{N} \), and \( \alpha < \omega \).

The class \( \mathcal{O}(1) \) of the finite elements of \( \mathbb{N}_o \) is an \( \mathbb{R} \)-vector subspace of \( \mathbb{N}_o \), and as such it has many complementary spaces. We can however select one specific complement as follows.

**Definition 10.5.** Let \( \mathfrak{M} = \omega^{\mathbb{N}_o} \) be the class of all surreal monomials and notice that \( \mathfrak{M}^> = \omega^{\mathbb{N}_o^> \times \mathbb{N}} \) is the class of all infinite monomials. A surreal \( x = \sum_{i < \alpha} m_i r_i \in \mathbb{R}(\mathfrak{M})_{\mathbb{O}_n} \) is purely infinite if all monomials \( m_i \) in its support are \( > 1 \) (hence infinite). Let \( \mathbb{N}_o^\dagger \) be the (non-unitary) ring of purely infinite surreals. Every \( x \in \mathbb{N}_o \) can be written in a unique way in the form

\[ x = x^+ + x^0 + x^\perp \]

where \( x^+ \in \mathbb{N}_o^\dagger \), \( x^0 \in \mathbb{R} \) and \( x^\perp \prec 1 \). This yields a direct sum decomposition

\[ \mathbb{N}_o = \mathbb{N}_o^\dagger + \mathbb{R} + \mathcal{O}(1) \]

of \( \mathbb{R} \)-vector spaces, where \( \mathcal{O}(1) \) is the set of elements \( \prec 1 \). Note that \( \mathbb{N}_o^\dagger \) is a complement of the ring of finite elements \( \mathcal{O}(1) = \mathbb{R} + \mathcal{O}(1) = \{ x \in \mathbb{N}_o \mid x \leq 1 \} \).

### 11. Restricted analytic functions

In this section we show that every real analytic function \( f : \mathbb{R} \to \mathbb{R} \) has a natural extension to a function \( f : O(1) \to \mathbb{N}_o \) where \( O(1) \) is the class of finite surreal numbers. More generally, a real analytic functions defined on an open subset \( U \subseteq \mathbb{R}^n \), has an extension to a function \( f : U + \mathcal{O}(1) \to \mathbb{N}_o \) as in the following definition.

**Definition 11.1.** Let \( U \subseteq \mathbb{R}^n \) be an open set and let \( f : U \to \mathbb{R} \) be a real analytic function. Now let

\[ \tilde{U} = U + \mathcal{O}(1) \]

be the infinitesimal neighbourhood of \( U \) in \( \mathbb{N}_o^n \). There is a natural extension of \( f \) to a function

\[ \tilde{f} : \tilde{U} \to \mathbb{N}_o \]

defined as follows. For \( r \in U \), let \( \sum_{i \in \mathbb{N}^n} \frac{D^i f(r)}{i!} X^i \) be the Taylor series of \( f \) around \( r \), where \( i = (i_1, \ldots, i_n) \) is a multi-index. Now for \( \varepsilon \in \mathcal{O}(1) \subseteq \mathbb{N}_o^n \), define \( \tilde{f}(r + \varepsilon) = \sum_{i \in \mathbb{N}^n} \frac{D^i f(r)}{i!} \varepsilon^i \), where the summability is ensured by Lemma 8.4 (Neumann’s lemma). Since \( f \) is analytic on \( U \), the function \( \tilde{f} : \tilde{U} \to \mathbb{N}_o \) extends \( f \).

**Remark 11.2.** We have \( \tilde{f} \circ g = \tilde{f} \circ \tilde{g} \) whenever the image of \( g \) is contained in the domain of \( f \).

**Definition 11.3.** Let \( \mathbb{R} \) be the field of real numbers, let \( \mathbb{R}_{an} \) be the expansion of \( \mathbb{R} \) with all analytic functions restricted to some box \([-1, 1]^n \), and let \( \mathbb{R}_{an}(\exp) \) be the expansion of \( \mathbb{R}_{an} \) with the real exponential function. Now let \( T_{an} \) be the theory of \( \mathbb{R}_{an} \), let \( T_{\exp} \) be the theory of \( (\mathbb{R}, \exp) \) and \( T_{an}(\exp) \) be the theory of \( \mathbb{R}_{an}(\exp) \).

We recall that \( T_{\exp}, T_{an} \) and \( T_{an}(\exp) \) are model complete by \([47, 15, 43]\) respectively.
exp to be an isomorphism of ordered groups from \((\mathbb{R}, +, <)\) to \((\mathbb{R}^{>0}, +, <)\). The expansion of \(\mathbb{R}^{>0}\) with all these restricted functions is a model of \(T\).

The proof is based on the axiomatisation of \(T\) given in [43].

12. Exponentiation

The next goal is to expand \(\mathbb{R}^{>0}\) to a model of \(\mathbb{R}\) through the introduction of an exponential function. We already know how to extend the real exponential function \(\exp : \mathbb{R} \to \mathbb{R}\) to a function \(\exp : O(1) \to \mathbb{R}\) using Definition 11.1, so the problem is how to define \(\exp(x)\) when \(x \in \mathbb{R}^{>0}\) is infinite. Since we want the surreal \(\exp\) to be an isomorphism of ordered groups from \((\mathbb{R}, +, <)\) to \((\mathbb{R}^{>0}, +, <)\), the image \(\exp(\mathbb{R}^{>0})\) of the class of purely infinite elements (Definition 10.5) must be a complement of \(O(1)^{>0}\) in the multiplicative group \(\mathbb{R}^{>0}\). We already have a natural choice for such a complement, namely the class \(\mathcal{M} = \omega^{\mathbb{R}^{>0}}\) of surreal monomial. It is then natural to require that

\[
\exp(\mathbb{R}^{>0}) = \mathcal{M},
\]

so in particular \(\exp(\mathcal{M}^{>1}) \subset \mathcal{M}\), i.e. the class of infinite monomials is closed under \(\exp\). To achieve this goal, for \(x > 0\) we define \(\exp(x) = \omega^{\exp(x)}\) for a suitable increasing bijection \(g : \mathbb{R}^{>0} \to \mathbb{R}\). To ensure that \(\exp\) grows faster than any polynomial, the function \(g\) is chosen in such a way that for \(m \in \mathcal{M}^{>1}\), we have \(\exp(m) > m^n\) for all \(n \in \mathbb{N}\). This translates into the condition \(\omega^{g(x)} > x\), so we define \(g\) as the simplest increasing map with this property. The following definition formalises the idea.

**Definition 12.1** (Gonshor [21, p. 169]). The index of \(x \in \mathbb{R}^{>0}\) is the unique \(c = \text{index}(x)\) such that \(x \simeq \omega^c\). In particular \(\text{index}(\omega^c) = c\). Define \(g : \mathbb{R}^{>0} \to \mathbb{R}\) by the recursive equation

\[
g(x) = \{\text{index}(x), g(x')\} \cup \{g(x'')\}
\]

where \(x'\) ranges over the left options of \(x\) and \(x''\) ranges over its right options.

**Remark 12.2.** It can be proved that \(g\) is an increasing bijection \(g : \mathbb{R}^{>0} \to \mathbb{R}\) (with inverse given by Definition 13.1). Since \(g(x) > \text{index}(x)\), we have \(\omega^{g(x)} > x\).

The function \(g\) can be difficult to compute, but Gonshor [21] showed that \(g(n) = n\) for every \(n \in \mathbb{N}\). More generally, if \(\alpha\) is an ordinal, then \(g(\alpha) = \alpha\) unless there is an epsilon number \(\varepsilon\) such that \(\varepsilon \leq \alpha < \varepsilon + \omega\), in which case \(g(\alpha) = \alpha + 1\) [21, Thm. 10.14].

**Definition 12.3.** Given \(x \in \mathbb{R}^{>0}\), we write \(x = x^1 + x^0 + x^1\) as in Definition 10.5 and define \(\exp(x) = \exp(x^1) \exp(x^0) \exp(x^1)\) where the factors on the right-hand side are defined as follows:

1. The restriction of \(\exp\) to \(\mathbb{R}\) coincides with the real exponential function.
2. \(\exp(x) = \sum_{n \in \mathbb{N}} x^n\) for \(\varepsilon < 1\).
3. \(\exp(\omega^c) = \omega^{\exp(g(x))}\) for \(x > 0\). This defines the restriction of \(\exp\) to the class \(\mathcal{M}^{>1}\) of all infinite monomials.
(4) The extension to $\text{No}^+$ is given by the formula
\[
\exp(\sum_{i<\alpha} \omega^{x_i} r_i) = \omega^{\sum_{i<\alpha} \omega^{x_i} r_i}
\]
where $x = \sum_{i<\alpha} \omega^{x_i} r_i \in \text{No}^+$ (i.e. $x_i > 0$ for all $i$).

The above definition is equivalent to the one of Gonshor [21, Thms. 10.2, 10.3, 10.13]. Note that (1) and (2) agree with Definition 11.1.

**Theorem 12.4.** We have:

1. $\exp$ is a group isomorphism from $(\text{No}, +, \prec)$ to $(\text{No}^+ > 0, \cdot, \prec)$;
2. $\exp$ extends the real exponential function;
3. $\exp(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n / n!$ for $\varepsilon \prec 1$;
4. $\exp(\text{No}^+)$ is the class $\mathfrak{M}$ of surreal monomials;
5. $\exp(x) > x^n$ for all $x \in \text{No}^+ > N$ and all $n \in \mathbb{N}$.

**Proof.** All points follow easily from the definitions except possibly (5). By Definition 12.1 we have $g(z) > \text{index}(z)$ for all $z \in \text{No}^+ > 0$. We thus obtain $\omega^{g(z)} > \omega^{\text{index}(z)} \prec z$. It follows that for $z > 0$ we have $\exp(\omega^z) = \omega^{\omega^{g(z)}} > \omega^{2^n}$ for all $n \in \mathbb{N}$, so (5) holds whenever $x = \omega^z$ is an infinite monomial and the general case easily follows.

13. **Logarithm**

The surreal logarithm $\log : \text{No}^+ > 0 \to \text{No}$ is the inverse of $\exp : \text{No} \to \text{No}^+ > 0$. To be able to give a direct definition we need an auxiliary function.

**Definition 13.1.** Define $h : \text{No} \to \text{No}^+ > 0$ by
\[
h(x) = \{0, h(x') \mid \{h(x''), \omega^{x}/2^n\}\}
\]
where $x'$ ranges over the left-options of $x$, $x''$ ranges over its right-options, while $n$ ranges over the natural numbers.

**Remark 13.2.** Gonshor [21] showed that $h$ is the inverse of the function $g : \text{No}^+ > 0 \to \text{No}$ in Definition 12.1.

We are now ready to define $\log : \text{No}^+ > 0 \to \text{No}$.

**Definition 13.3.** Given $x \in \text{No}^+ > 0$, we write $x = rm(1 + \varepsilon)$, with $r \in \mathbb{R}^+ > 0$, $m \in \mathfrak{M}$ and $\varepsilon \prec 1$, and we define $\log(x) = \log(r) + \log(m) + \log(1 + \varepsilon)$, where the right-hand side is defined as follows.

1. The restriction of $\log$ to $\mathbb{R}^+ > 0$ agrees with the natural logarithm on $\mathbb{R}$;
2. $\log(1 + \varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \varepsilon^n / n$ for $\varepsilon \prec 1$;
3. $\log(\omega^{\omega^x}) = \omega^{h(x)}$;
4. $\log(\sum_{i<\alpha} \omega^{x_i} r_i) = \sum_{i<\alpha} \omega^{h(x_i)} r_i$.

The following proposition is an easy consequence of Remarks 11.2 and 13.2.

**Proposition 13.4.** $\log : \text{No}^+ > 0 \to \text{No}$ is the inverse of $\exp : \text{No} \to \text{No}^+ > 0$.

In particular, since $\exp(\text{No}^+) = \mathfrak{M}$, we have $\log(\mathfrak{M}) = \text{No}^+$. 

14. Ressayre’s axioms

To prove that \( \mathbb{N}_o \) is an elementary extension of \( \mathbb{R}_\exp \) we need the following result of Ressayre [34], which can also be derived from the axiomatisation of \( T_{an}(\exp) \) given by van den Dries, Macintyre, and Marker [45].

**Theorem 14.1** ([34, 45]). A real closed ordered field \( K \) endowed with an isomorphism of ordered groups \( E : (K, +, <) \to (K^{>0}, \cdot, <) \) is a model of \( T_{\exp} \) if and only if the following axioms hold:

(i) \( E(x) > x^n \) for all \( x \in K^{>N} \) and all \( n \in \mathbb{N} \);

(ii) the restriction of \( E \) to \([0, 1]\) makes \( K \) into a model of the theory of \( (\mathbb{R}, \exp \upharpoonright [0, 1]) \).

**Remark 14.2.** If \( (K, E \upharpoonright [0, 1]) \) is elementary equivalent to \( (\mathbb{R}, \exp \upharpoonright [0, 1]) \), then these two structures have isomorphic ultrapowers \( K^* \) and \( \mathbb{R}^* \). Since the ultrapower construction yields elementary extensions in any language, we can use the isomorphism to expand \( K^* \) to a model of \( T_{an}(\exp) \). It follows that any structure \( (K, E) \) satisfying the hypothesis of Theorem 14.1 (Ressayre’s axioms) has an elementary extension which can be expanded to a model of \( T_{an}(\exp) \).

**Corollary 14.3** (van den Dries and Ehrlich [40]). The field \( \mathbb{N}_o \) of surreal numbers, with Gonshor’s \( \exp \) and the natural interpretation of the restricted analytic function, is a model of \( T_{an}(\exp) \).

Since \( T_{an}(\exp) \) is model complete, it follows that \( \mathbb{N}_o \) is an elementary extension of \( \mathbb{R} \) as an exponential field with all restricted analytic functions.

**Remark 14.4.** In the presence of the other hypothesis, condition (i) in Theorem 14.1 is equivalent to: \( E(x) \geq x + 1 \) for all \( x \in K \). The latter is a first order axiom, so the first-order theory \( T_{\exp} \) is finitely axiomatisable over the theory of restricted exp. As a consequence, these two theories are either both decidable or both undecidable. Macintyre and Wilkie [31] prove the decidability of \( T_{\exp} \) assuming “Schanuel’s conjecture”, but the unconditional decidability remains an open problem.

15. Exponential normal form

The presence of \( \exp \) and the omega-map generates a conflict of notation, as \( \omega^x \) could be interpreted either as the omega-map applied to \( x \) or as \( \exp(x \log(\omega)) \). To avoid ambiguities, when there is a risk of confusion we write \( \omega^x \) (with a little dot) for the omega-map.

**Definition 15.1.** Given \( a, b \in \mathbb{N}_o \) with \( a > 0 \), we define

\[
a^b = \exp(b \log(a))
\]

and we write \( \omega^b \) or \( \Omega(b) \) for the image of \( b \) under the omega-map.

In general \( \Omega(b) = \omega^b \neq \omega = \exp(b \log(\omega)) \). Indeed \( \omega^b \) is always a monomial, while \( \omega^b \) can be any positive surreal.

From \( \mathbb{N}_o = \mathbb{R}(\mathfrak{M})_{\mathfrak{R}} \) (Theorem 10.2) and \( \mathfrak{M} = \exp(\mathbb{N}_o^+) = \omega^{\mathbb{N}_o} \) (Theorem 12.4), we obtain the following result.

**Proposition 15.2.** Every surreal number \( f \) can be written in a unique way in the form

\[
f = \sum_{i < \alpha} e^{\gamma_i} r_i
\]
where \( \alpha \) is an ordinal, \((\gamma_i)_{i<\alpha}\) is a decreasing sequence in \(\mathbb{N}^\downarrow\) and \(r_i \in \mathbb{R}^\ast\). This is called the Ressayre form of \(f\), or the exponential normal form.

To convert the Conway normal form to the Ressayre form we need to show how to pass from the representation \(\mathfrak{M} = \omega \mathbb{N}^\downarrow\) to the representation \(\mathfrak{M} = \exp(\mathbb{N}^\downarrow)\). It is convenient to introduce the following definition.

**Definition 15.3.** Let \(g : \mathbb{N}^\ast > 0 \rightarrow \mathbb{N}^\ast\) and \(h : \mathbb{N}^\ast \rightarrow \mathbb{N}^\ast > 0\) be as in Definitions 12.1 and 13.1 and recall that \(h\) is the inverse of \(g\). Define \(G : \mathbb{N}^\downarrow \rightarrow \mathbb{N}^\downarrow\) and \(H : \mathbb{N} \rightarrow \mathbb{N}^\uparrow\) as follows.

1. \(G(\sum_{i<\alpha} \omega^y r_i) = \sum_{i<\alpha} \omega^{g(y_i)} r_i;\)
2. \(H(\sum_{i<\alpha} \omega^x r_i) = \sum_{i<\alpha} \omega^{h(x_i)} r_i.\)

Clearly \(H\) and \(G\) are strongly \(\mathbb{R}\)-linear (i.e. they are \(\mathbb{R}\)-linear and distributive over infinite sums) and \(H\) is the inverse of \(G\).

The following proposition is a rephrasing of Definitions 12.3(4) and 13.3(4).

**Proposition 15.4.** We have:

1. \(e^\gamma = \omega^{G(\gamma)}\) for all \(\gamma \in \mathbb{N}^\downarrow;\)
2. \(\omega^x = e^{H(x)}\) for all \(x \in \mathbb{N}^\ast.\)

**Corollary 15.5.** For \(r \in \mathbb{R}\) and \(x \in \mathbb{N}^\ast\), \(\Omega(x)^r = \Omega(xr)\), so a real power of a monomial is a monomial. In particular, \(\omega^r = \omega^r.\)

**Proof.** By the \(\mathbb{R}\)-linearity of \(H\), \(\Omega(x)^r = (e^{H(x)})^r = e^{H(x)r} = e^{H(xr)} = \Omega(xr)\)

16. Normal forms as asymptotic expansions

Given a function \(f(x)\) defined as a composition of algebraic operations, log and exp, we may consider \(f(\omega)\) as a surreal number, and the normal form of \(f(\omega)\) corresponds to an asymptotic development of \(f(x)\) for \(x \rightarrow +\infty\).

**Example 16.1.** To find the normal form of \((\omega + 1)^\omega \in \mathbb{N}^\ast\) we write

\[
(\omega + 1)^\omega = \exp(\omega(\log(1 + \omega)))
\]

\[
= \exp(\omega \left( \log(\omega) + \log(1 + \omega^{-1}) \right))
\]

\[
= \exp \left( \omega \log(\omega) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \omega^{-n+1} \right)
\]

\[
= \exp \left( \omega \log(\omega) + 1 - \frac{1}{2} \omega^{-1} + \ldots \right)
\]

\[
= \omega^e \exp(-2^{-1} \omega^{-1} + \ldots)
\]

\[
= e^\omega (1 - 2^{-1} \omega^{-1} + \ldots)
\]

\[
= e^\omega - e^2 \omega^{-1} + \ldots
\]

Replacing \(\omega\) with a real variable \(x\) this corresponds to the asymptotic development

\[
(x + 1)^x = e^x - e^2 \omega^{-1} + \ldots
\]

for \(x \rightarrow +\infty.\)

The idea of normal forms as asymptotic expansions in exploited in [8] to study the possible limits at \(+\infty\) of the quotients of two “Skolem functions”, improving some results of [42] on a problem of Skolem [39].
In \( \mathbb{No} \) we have a notion of infinite sum due to the identification \( \mathbb{No} = \mathbb{R}(\mathbb{M})_{\mathbb{On}} \), but we also have an exp and a log function, so we can introduce infinite products via the formula

\[
\prod_{i \in I} f_i = \exp \left( \sum_{i \in I} \log(f_i) \right)
\]

where \( (f_i)_{i \in I} \) in \( \mathbb{No}^{>0} \) is such that \( (\log(f_i))_{i \in I} \) is summable. We say in this case that the product of the family \( (f_i)_{i \in I} \) exists. It follows at once from the definition that if \( (x_i)_{i \in I} \) is summable, then the product of the family \( (e^{x_i})_{i \in I} \) exists and

\[
\prod_{i \in I} e^{x_i} = e^{\sum_{i \in I} x_i}.
\]

As observed in [6], a similar statement holds for the omega-map: if \( (x_i)_{i \in I} \) is summable, then the product of the family \( (\omega^{x_i})_{i \in I} \) exists and

\[
\prod_{i \in I} \omega^{x_i} = \omega^{\sum_{i \in I} x_i}.
\]

For the proof it suffices to observe that \( \omega^{x_i} = e^{H(x_i)} \) where \( H : \mathbb{No} \to \mathbb{No}^I \) is the strongly \( \mathbb{R} \)-linear isomorphism in Definition 15.3. In particular we have:

**Proposition 17.1.** Given a family of monomials \( (m_i)_{i \in I} \), if the product \( \prod_{i \in I} m_i \) exists, it is a monomial.

The notation of infinite products will be employed in the definition of a surreal derivation. In general the infinite distributivity law fails, namely \( \prod_{i \in I} \sum_{j \in J} x_{i,j} \neq \sum_{f : I \to J} \prod_{i \in I} x_{i,f(i)} \). Indeed it is easy to find examples in which the left-hand side is well defined, but the right-hand side is not because of summability problems.

18. Levels and Log-atomic Numbers

**Definition 18.1.** Given \( x, y \in \mathbb{No}^{>\mathbb{N}} \), we say that \( x \) and \( y \) have the same level if \( \log_n(x) \asymp \log_n(y) \) for some \( n \in \mathbb{N} \). If \( x \) and \( y \) have different levels and \( x < y \), we say that \( x \) has lower level than \( y \), or that \( y \) has higher level than \( x \).

By definition, \( x \) has lower level than \( y \) if and only if \( \log_n(x) \prec \log_n(y) \) for every \( n \in \mathbb{N} \), or equivalently \( \exp_n(k \log_n(x)) < y \) for every \( n, k \in \mathbb{N} \).

To be in the same level is a coarser equivalence relation than to be in the same Archimedean class: given \( x \in \mathbb{No}^{>\mathbb{N}} \), the elements \( x \) and \( x^n \) are in the same level for each \( n \in \mathbb{N} \), while \( \exp(x) \) is in a higher level. The class of levels has the following density property.

**Proposition 18.2.** Given two subsets \( A \) and \( B \) of \( \mathbb{No}^{>\mathbb{N}} \) such that the level of every element of \( A \) is smaller than the level of every element of \( B \), there is \( z \in \mathbb{No} \) which has higher level than all elements of \( A \) and lower level than all elements of \( B \).

**Proof.** It suffices to take \( z = A' \mid B' \) where \( A' \supseteq A \) and \( B' \supseteq B \) are defined by \( A' = \{ \exp_n(k \log_n(a)) \mid n, k \in \mathbb{N}, a \in A \} \) and \( B' = \{ \exp_n(2^{-k} \log_n(b)) \mid n, k \in \mathbb{N}, b \in B \} \).
It follows from the above density properties that there are intermediate levels between \( \omega \) and \( \exp(\omega) \). Our next goal is to parametrize a class of representative of the levels by \( \mathbb{N} \) itself.

**Definition 18.3.** By induction on simplicity, we define \( \lambda_x \in \mathbb{N}^{>\mathbb{N}} \) as the simplest element such that:

1. \( \lambda_x \) is the simplest element of its level,
2. \( \lambda_{x'} < \lambda_x < \lambda_{x''} \) for all left-options \( x' \) of \( x \) and all right-options \( x'' \) of \( x \).

The above is equivalent to [9, Definition 5.12].

**Proposition 18.4.** The map \( x \mapsto \lambda_x \) is increasing and its image \( \lambda_{\mathbb{N}} = \{ \lambda_x \mid x \in \mathbb{N} \} \) contains one and only one representative for each level.

**Remark 18.5.** We have:

1. The elements \( \lambda_{-\alpha} \) with \( \alpha \in \mathbb{O} \) are coinitial in \( \mathbb{N}^{>\mathbb{N}} \).
2. The elements \( \lambda_{\alpha} \) with \( \alpha \in \mathbb{O} \) are cofinal in \( \mathbb{N} \).

**Definition 18.6.** For \( n \in \mathbb{N} \) we define:

1. \( \log_n(x) = x, \log_{n+1}(x) = \log(\log_n(x)) \);
2. \( \exp_0(x) = x, \exp_{n+1}(x) = \exp(\exp_n(x)) \).

We extend the definition to the case when \( n \in \mathbb{Z} \) putting \( \log_n = \exp_{-n} \).

**Proposition 18.7** ([9]). We have \( \lambda_{-n} = \log_n(\omega) \) for all \( n \in \mathbb{Z} \). In particular, \( \lambda_{-1} = \log(\omega), \lambda_0 = \omega, \lambda_1 = \exp(\omega) \).

**Definition 18.8.** Given \( x \in \mathbb{N}^{>\mathbb{N}} \), we say that \( x \) is log-atomic if, for each \( n \in \mathbb{N} \), \( \log_n(x) \) is a monomial.

**Theorem 18.9** ([9]). For \( z \in \mathbb{N}^{>\mathbb{N}} \) the following are equivalent:

1. \( z \) is the simplest element of its level;
2. \( z \) is log-atomic;
3. \( z \in \lambda_{\mathbb{N}} \).

Since \( x \mapsto \lambda_x \) is increasing, it follows that \( \lambda_{1/2} \) represents an intermediate level between \( \omega = \lambda_0 \) and \( \exp(\omega) = \lambda_1 \), while \( \lambda_{-\omega} \) is lower than \( \lambda_{-n} = \log_n(\omega) \) for all \( n \in \mathbb{N} \). Proposition 18.7 can be extended as follows.

**Theorem 18.10** ([6]). We have \( \lambda_{x-1} = \log(\lambda_x) \) for all \( x \in \mathbb{N} \).

When \( \alpha \) is an ordinal it is suggestive to think of \( \lambda_{-\alpha} \) as the \( \alpha \)-times iterated log applied to \( \omega \) and write \( \lambda_{-\alpha} = \log_{\alpha}(\omega) \) as in [6].

**Remark 18.11** ([6]). We have:

1. For each ordinal \( \alpha \), \( \lambda_{-\alpha} = \omega^{\omega^{-\alpha}} \);
2. For each limit ordinal \( \alpha, \lambda_{-\alpha} \) is the simplest surreal \( x > \mathbb{N} \) such that \( x < \lambda_{-\beta} \) for all ordinals \( \beta < \alpha \).

The search for a satisfactory definition of \( \log_{\alpha}(x) \) for a general \( x \in \mathbb{N}^{>\mathbb{N}} \) is connected to the problem of representing \( \mathbb{N} \) as a field of “hyperseries”. For some recent work on this project see [5, 46, 7].
19. Simplicity and the Omega-Map

We have encountered two important representations of surreal numbers: sign-expansions and Conway’s normal forms. In general it is not easy to convert one to the other, but the following theorem (and related results in [19]) shows that normal forms are well behaved with respect to the simplicity relation.

Definition 19.1. Given \( f, g \in \text{No} \), we say that \( g \) is a truncation of \( f \), if \( f = \sum_{i<\alpha} m_i r_i \) and \( g = \sum_{i<\beta} m_i r_i \) for some \( \beta < \alpha \), where both expressions are in normal form.

Theorem 19.2 (Conway [13], Gonshor [21, §6]). We have:

1. If \( x \) is simpler than \( y \), then \( \omega^x \) is simpler than \( \omega^y \).
2. If \( m \) is a monomial and \( r \in \mathbb{R}^* \), then \( \pm m \) is simpler or equal to \( mr \), where \( \pm 1 \) is the sign of \( r \).
3. \( \text{birthday}(x) \leq \text{birthday}(\omega^x) \).
4. If \( g \) is a proper truncation of \( f \), then \( g \) is simpler than \( f \).
5. If \( m_ir_i \) is a term of \( f = \sum_{i<\alpha} m_ir_i \), then \( \text{birthday}(m_ir_i) \leq \text{birthday}(f) \); if \( i+1 < \alpha \) the inequality is strict because in this case \( m_ir_i \) is a term of a proper truncation of \( f \).

Corollary 19.3. If \( f = \sum_{i<\alpha} \omega^{x_i} r_i \) is in normal form, then for every \( i < \alpha \) we have \( \text{birthday}(x_i) \leq \text{birthday}(f) \) and the inequality is strict if \( i+1 < \alpha \).

If \( f \) is as in the corollary, the only possibility to have \( \text{birthday}(x_i) = \text{birthday}(f) \), is that \( \alpha \) is a successor ordinal and \( \omega^{x_i} \) is the smallest monomial in the support of \( f \), that is, \( i+1 = \alpha \). Moreover in this case \( r_{i+1} = \pm 1 \), so we can write \( f = \sum_{j<i} \omega^{x_j} r_{i} \pm \omega^{x_i} \). A special case of this situation is when \( f = \omega^f \). The surreal numbers with this property are called epsilon-numbers, and they include the ordinal numbers with the homonymous property.

20. Simplicity and Exp

In general it is not true that if \( x \) is simpler than \( y \), then \( \exp(x) \) is simpler than \( \exp(y) \); it suffices to take \( x = \omega \) and \( y = \log(\omega) \). It can however be proved that if \( x \) is a truncation of \( y \), then \( \exp(x) \) is simpler then \( \exp(y) \). This suggests the following definition, which can be useful in inductive proofs based on the exponential normal form.

Definition 20.1. Define \( \npre \) as the smallest preorder on \( \text{No}^* \) such that

1. if \( g \) is a non-zero truncation of \( f \), then \( g \npre f \);
2. if \( g \) and \( f \) are purely infinite and \( g \npre f \), then \( \pm e^g \npre e^f r \), where \( r \in \mathbb{R}^* \) and \( \pm \) is the sign of \( r \). More generally, \( x \pm e^g \npre x + e^f r \) where \( x \in \text{No} \) is such that all the monomials in the support of \( x \) are greater than both \( e^g \) and \( e^f \).

If \( f \npre g \), we say that \( f \) is a nested truncation of \( g \). We write \( f \npre g \) if we have both \( f \npre g \) and \( f \not\npre g \).

Theorem 20.2 ([9, Thm. 4.26]). If \( y \npre x \), then \( y \) is simpler than \( x \).

It follows that \( \npre \) is a well founded relation, so we have an associated notion of rank defined by transfinite induction.

Definition 20.3. The nested truncation rank \( \text{NR}(x) \in \text{On} \) of \( x \in \text{No}^* \) is recursively defined by \( \text{NR}(x) = \sup\{\text{NR}(y) + 1 \mid y \npre x \} \).
We have now an analogue of Theorem 19.2 and Corollary 19.3.

**Proposition 20.4** ([9]). We have:

1. If \( g \) is a proper non-zero truncation of \( f \), then \( \text{NR}(g) < \text{NR}(f) \).
2. If \( f = \sum_{i<\alpha} e^{r_i} \) is in normal form, then for every \( i < \alpha \) we have \( \text{NR}(e^{r_i}) \leq \text{NR}(f) \) and the inequality is strict if \( i + 1 < \alpha \).
3. \( \text{NR}(\pm e^{r}) = \text{NR}(\gamma) \) for all \( \gamma \in \text{No}^1 \).
4. If \( r \neq \pm 1 \), then \( \text{NR}(e^{r}) = \text{NR}(e^{\gamma}) + 1 \) for all \( \gamma \in \text{No}^1 \).
5. For \( x \in \text{No}^* \), we have \( \text{NR}(x) = 0 \) if and only if \( x \) has the form \( \pm \lambda^{\pm 1} \) where \( \lambda \) is log-atomic.

**Corollary 20.5.** If \( f = \sum_{i<\alpha} e^{r_i} \) is in exponential normal form, then \( \text{NR}(\gamma_i) \leq \text{NR}(f) \) for all \( i < \alpha \), and \( \text{NR}(\gamma_i) < \text{NR}(f) \) if \( i + 1 < \alpha \). If \( f \) has the same nested truncation rank of one of the exponents \( \gamma_i \) in its exponential normal form, then \( f = \sum_{i<\alpha} e^{r_i} \) with \( \gamma_i \) being the smallest exponent.

21. **Branches in the tree of iterated exponents**

Given a surreal number \( f = \sum_{i<\alpha} e^{r_i} \) written in exponential normal form we call each \( \gamma_i \in \text{No}^1 \) an **exponent** of \( f \). Taking the exponents of the exponents and iterating the process we obtain a tree-like structure, where the children of \( f \) are its exponents. We can then consider the infinite branches through this tree, as in the definition below.

**Definition 21.1.** A **branch** is a sequence \( (B_n)_{n \in \mathbb{N}} \) of surreal numbers such that \( B_{n+1} \) is an exponent of \( B_n \) for each \( n \in \mathbb{N} \).

Notice that there are no branches \( B \) starting with a real number \( r = B_0 \in \mathbb{R} \), because a non-zero real number \( r = e^{0}r \) has only zero as a possible exponent, and zero has no exponents at all. On the other hand any \( f \in \text{No} \setminus \mathbb{R} \) has at least one branch, because it has at least one non-zero exponent and such an exponent is necessarily in \( \text{No}^1 \setminus \{0\} \subset \text{No} \setminus \mathbb{R} \). A log-atomic number \( x \in \text{No} \) has exactly one branch, given by the iterated logarithms \( \log_n(x) \) of \( x \), which are themselves log-atomic elements.

**Proposition 21.2.** Let \( B \) be a branch such that \( B_{n+1} \) is the greatest exponent of \( B_n \) for every sufficiently large \( n \in \mathbb{N} \). Then there is \( m \in \mathbb{N} \) such that \( B_m \) is log-atomic.

**Proof.** Since there are no infinite decreasing sequences of ordinals, by Corollary 20.5, for all large enough \( n \in \mathbb{N} \) we have \( \text{NR}(B_{n+1}) = \text{NR}(B_n) \). This equation implies that \( B_n = \sum_{j<i} e^{r_j} \pm e^{B_{n+1}} \) with \( B_{n+1} \) being the smallest exponent. On the other hand for \( n \) sufficiently large \( B_{n+1} \) is also the greatest exponent, so \( B_n = \pm e^{B_{n+1}} \). Moreover, for \( n > 0 \), \( B_n \) is purely infinite (being an exponent), so if \( B_n = \pm e^{B_{n+1}} \), then \( B_{n+1} \) is positive and we can write \( B_{n+1} = e^{B_{n+2}} \). It follows that, for all large enough \( n \in \mathbb{N} \), \( B_n = e^{B_{n+1}} \), and therefore \( B_n \) is log-atomic.

22. **Transseries**

**Definition 22.1.** Given a class \( \Delta \subseteq \lambda_{\text{No}} \) of log-atomic numbers, let \( \mathbb{R}(\langle \Delta \rangle) \) be the class of all \( f \in \text{No} \) such that every branch \( B \) starting at \( B_0 = f \) reaches an element of \( \Delta \), i.e. there is \( n \in \mathbb{N} \) with \( B_n \in \Delta \).
We define:

Definition 23.1. A term is a monomial multiplied by a non-zero real number. We shall write a term in the form \( c \gamma r \), or \( r e^{\gamma} \), where \( \gamma \in \mathbb{R}^\ast \) and \( r \in \mathbb{R}^\ast \).

(2) We say that \( e^{\gamma} r \) is a term of \( f \in \mathbb{R}^\ast \) by \( \mathbb{R}^\ast \) if it is one of the terms \( e^{\gamma} r_i \) in its normal form \( f = \sum_{i<\alpha} e^{\gamma} r_i \).

(3) A term \( e^{\gamma} r \) is non-constant if it does not belong to \( \mathbb{R} \), namely \( \gamma \neq 0 \).

Proposition 22.2 ([9]). \( \mathbb{R}^{\langle \Delta \rangle} \) is the smallest subfield of \( \mathbb{N} \) containing \( \Delta \) and closed under \( \exp \) and infinite sums of summable families. If \( \log(\Delta) \subset \Delta \), then \( \mathbb{R}^{\langle \Delta \rangle} \) is also closed under \( \log \). The containment \( \mathbb{R}^{\langle \Delta \rangle} \subset \mathbb{N} \) is always proper, because in \( \mathbb{R} \) there are branches that do not reach any log-atomic number.

Taking \( \Delta = \lambda_{\mathbb{N}} = \{ \log_n(\omega) \mid n \in \mathbb{N} \} \), we obtain the field of omega-series \( \mathbb{R}^{\langle \lambda_{\mathbb{N}} \rangle} \), which will also be denoted \( \mathbb{R}^{\langle \omega \rangle} \) as in [9]. Thus \( \mathbb{R}^{\langle \omega \rangle} \) is the smallest subfield of \( \mathbb{N} \) containing \( \mathbb{R} \cup \{ \omega \} \) and closed under \( \sum, \exp, \log \). The field of omega-series is a proper class, like \( \mathbb{N} \) itself, but it has interesting subfields which are sets, among which an isomorphic copy of the field of transseries.

Example 22.4. The monomial \( \prod_{n \in \mathbb{N}} \lambda_{\mathbb{N}} = \prod_{n \in \mathbb{N}} \log_n(\omega) = \exp(\sum_{n \in \mathbb{N}} -\log_n(\omega)) \) is in \( T^EL \), but not in \( T \).

As exponential fields we have \( \mathbb{R}^{\exp} \subset T \subset T^{EL} \subset \mathbb{N} \) (where “\( \prec \)” means “elementary substructure”), however \( T \neq T^{EL} \) when considered as differential fields with their natural derivation. Specifically, \( T \) is closed under integration (i.e. the derivative is surjective), while \( T^{EL} \) is not because there is no element in \( T^{EL} \) whose derivative is \( \prod_{n \in \mathbb{N}} \lambda_{\mathbb{N}} \). Later we shall introduce a surjective strongly additive derivation \( \partial : \mathbb{N} \to \mathbb{N} \) with \( \partial \omega = 1 \) which extends the natural derivation on these transserial fields.

23. Paths

In the previous section we have defined the notion of branch, using the iterated exponents in the exponential normal form. For the purpose of introducing a derivation, it is convenient to define a similar notion which however takes into account not only the exponents, but also the coefficients appearing in the normal form. This gives rise to the notion of path, defined in this section.

Definition 23.1. We define:

1. A term is a monomial multiplied by a non-zero real number. We shall write a term in the form \( c \gamma r \), or \( re^{\gamma} \), where \( \gamma \in \mathbb{N}^{\ast} \) and \( r \in \mathbb{R}^{\ast} \).

2. We say that \( e^{\gamma} r \) is a term of \( f \in \mathbb{R}^{\ast} \) if it is one of the terms \( e^{\gamma} r_i \) in its normal form \( f = \sum_{i<\alpha} e^{\gamma} r_i \).

3. A term \( e^{\gamma} r \) is non-constant if it does not belong to \( \mathbb{R} \), namely \( \gamma \neq 0 \).
Definition 23.2. A path $P$ is a sequence $(P_n \mid n \in \mathbb{N})$ of non-constant terms such that, for each $n \in \mathbb{N}$, if $P_n = r_n e^{\gamma_n}$, then $P_{n+1}$ is one of the non-constant terms of $\gamma_n \in \mathbb{No}^+$, so we can write

$$P_n = r_n \exp(x_{n+1} + P_{n+1} + \delta_{n+1})$$

where $\gamma_n = x_{n+1} + P_{n+1} + \delta_{n+1} \in \mathbb{No}^+$, $\delta_{n+1} < P_{n+1}$, and all the monomials of $x_{n+1}$ are $\succ P_{n+1}$.

The following rather surprising property about paths follows from the existence of the nested truncation rank.

Proposition 23.3. Let $P$ be a path. In the above notations, for every sufficiently large $n \in \mathbb{N}$ we have $r_n = \pm 1$ and $\delta_{n+1} = 0$, so

$$P_n = \pm \exp(x_n + P_{n+1}).$$

Proof. For every $n \in \mathbb{N}$, $\text{NR}(P_{n+1}) \leq \text{NR}(P_n)$, hence for every sufficiently large $n \in \mathbb{N}$ we have $\text{NR}(P_{n+1}) = \text{NR}(P_n)$. For these values of $n \in \mathbb{N}$ we must have $P_n = \pm \exp(x_n + P_{n+1})$. □

Proposition 23.3 can be rephrased by saying that $\mathbb{No}$ satisfies axiom T4 in [38] and therefore $\mathbb{No}$ is a field of (generalized) transseries in the sense of that paper.

Remark 23.4. The field of transseries $\mathbb{T}$ satisfies an even stronger property: every branch reaches a log-atomic number. This stronger axiom is called ELT4 in [28] and it is also satisfied by the field of omega-series $\mathbb{R}(\omega)$, which is in fact the largest subfield of $\mathbb{No}$ satisfying the stronger property. The surreal field $\mathbb{No}$ contains nested transseries such as $y = \sqrt{\omega} + e^{\sqrt{\log(\omega)} + e^{\sqrt{\log(\omega)} + \ldots}}$ which violate ELT4, while still respecting T4. In this example, the element $y = y(\omega)$ is a solution of the functional equation $y(\omega) = \sqrt{\omega} + e^{y(\log(\omega))}$. The perturbed equation $y(\omega) = \sqrt{\omega} + e^{y(\log(\omega))} + \log \omega$ cannot have solutions in $\mathbb{No}$ since it would violate T4 (the example is taken from [5]).

Definition 23.5. Let $P$ be a path.

1. We say that $P$ is a path of $f \in \mathbb{No}$, if $P_0$ is a non-constant term of $f$. Let

$$\mathcal{P}(f) = \{ P \mid P \text{ is a path of } f \}.$$  

2. Given $f \in \mathbb{No} \setminus \mathbb{R}$, the dominant path of $f$ is the unique path $P \in \mathcal{P}(f)$ such that $P_0 = r_0 e^{\gamma_0}$ is the leading term of $f - f^0$ (where $f^0$ is the constant term of $f$, see Definition 10.5) and for each $n$, if $P_n = r_n e^{\gamma_n}$, then $P_{n+1}$ is the leading term of $\gamma_n$. For the dominant path we can thus write

$$P_n = r_n \exp(P_{n+1} + \delta_{n+1})$$

with $\delta_{n+1} < P_{n+1}$.

Remark 23.6. We have:

1. Given $f \in \mathbb{No}$, we have $\mathcal{P}(f) = \emptyset$ if and only if $f \in \mathbb{R}$;
2. Any log-atomic element $\lambda_x$ has exact one path $P$, given by $P_n = \log_n \lambda_x$.

Proposition 23.7 ([9, Lemma 6.23]). Let $P$ be a dominant path. Then there is $n \in \mathbb{N}$ such that $P_n$ is log-atomic.
Proof. By Proposition 23.3 if \( n \in \mathbb{N} \) is sufficiently large, \( P_n = \pm \exp(x_n + P_{n+1}) \).
On the other hand, since \( P \) is dominant, we also have \( P_n = r_n \exp(P_{n+1} + \delta_{n+1}) \).
Matching the two equations, we obtain \( P_n = \pm \exp(P_{n+1}) \) for every sufficiently large \( n \).
This implies that \( P_{n+1} \) is purely infinite, and therefore its exponent \( P_{n+2} \)
is positive. It follows that for \( n \) sufficiently large, \( P_n = \exp(P_{n+1}) \), so \( P_n \) is log-atomic.
\( \square \)

24. Surreal derivations

A derivation on a field \( K \) is a linear map \( \partial : K \to K \) satisfying Leibniz’s rule
\( \partial(xy) = x\partial y + y\partial x \).

**Definition 24.1.** A surreal derivation is a derivation \( \partial : \text{No} \to \text{No} \) satisfying:

1. (SD1) if \( x > \mathbb{N} \), then \( \partial x > 0 \);
2. (SD2) \( \ker(\partial) = \mathbb{R} \);
3. (SD3) \( \partial e^f = e^f \partial f \);
4. (SD4) if \( (f_i)_{i \in I} \) is summable, then so is \( (\partial f_i)_{i \in I} \) and \( \partial(\sum_{i \in I} f_i) = \sum_{i \in I} \partial f_i \).

The motivation for (SD1) comes from the theory of Hardy fields discussed in Section 25.

To define a surreal derivation, a crucial problem is to decide its restriction to the log-atomic numbers. If we stipulate that \( \partial \) is a derivation on \( \text{No} \), the log-atomic numbers. If we stipulate that \( \partial \omega \) is an ordinal, the condition on \( \beta \) becomes \( \beta < x + \omega \). As special cases of (SD5) we have

1. \( \partial \omega = 1 \);
2. \( \partial \log_n(\omega) = \frac{1}{\prod_{m \in \omega} \log_{m}(\omega)} \);
3. \( \partial \lambda_{-\alpha} = \frac{1}{\prod_{\beta < \alpha} \lambda_{-\beta}} \) for every ordinal \( \alpha \)

(\( \) where \( \beta \) ranges over the ordinals < \( \alpha \)).

**Remark 24.3.** The present form of (SD5) appears in [6] and is a notational simplification of the equivalent formula in [9, Def. 6.7]. To prove that (SD5) implies (1)–(3) we reason as follows. When \( \alpha \) is an ordinal, the fraction in (SD5) simplifies as \( \partial \lambda_{-\alpha} = \frac{1}{\prod_{\beta < \alpha + \omega} \lambda_{-\beta}} = \frac{1}{\prod_{\beta < \alpha} \lambda_{-\beta}} \). Recalling that \( \lambda_{-n} = \log_n(\omega) \) for \( n \in \mathbb{N} \), we obtain, \( \partial \log_n(\omega) = \frac{1}{\prod_{m \in \omega} \log_{m}(\omega)} \), so in particular \( \partial \omega = 1 \).

To justify (SD5) in the general case we argue as follows. It can be shown that (SD1)–(SD4) imply that \( \log \partial x - \log \partial y \leq x - y \leq \max\{x, y\} \) for all \( x, y \in \text{No}^{>\mathbb{N}} \) such that \( x - y > \mathbb{N} \) [9, Prop. 6.5]. Accordingly, we define a prederivation as a map
$D : \lambda_{\mathbb{N}_0} \to \mathbb{N}_0^{>0}$ satisfying $\partial e^x = e^x D(x)$ and $\log D(x) - \log D(y) < \max\{x, y\}$ for $x > y$ in $\lambda_{\mathbb{N}_0}$ (recall that the elements of $\lambda_{\mathbb{N}_0}$ and their differences are positive infinite). Given two prederivations, we say that one is simpler than the other if it takes a simpler value on any element of minimal simplicity where they differ. It turns out that the simplest prederivation is given by (SD5) [9, Thm. 9.6].

To prove Theorem 24.2 we need to show how to extend a prederivation $D : \lambda_{\mathbb{N}_0} \to \mathbb{N}_0$ to a surreal derivation. To this aim we observe that every surreal derivation must satisfy
\[
\partial\left(\sum_{i<\alpha} e^{\gamma_i} r_i\right) = \sum_{i<\alpha} e^{\gamma_i} r_i \partial \gamma_i,
\]
so the derivative of $f = \sum_{i<\alpha} e^{\gamma_i} r_i$ is a sum of contributions (possibly with cancellations) given by the terms $e^{\gamma_i} r_i$ multiplied by the derivative of the corresponding exponents $\gamma_i$. An iteration of the process involves the terms appearing in the exponents of $f$, then the exponents of the exponents, and so on. This process never ends, unless we start with a real number. Eventually, we are led to write $\partial f$ as a sum of contributions $\partial D(P) \in \mathbb{N}_0$, one for each path $P \in \mathcal{P}(f)$, as in the following definition (adapted to the surreals from a similar construction in [38]).

**Definition 24.4.** Let $D : \lambda_{\mathbb{N}_0} \to \mathbb{N}_0$ be a prederivation and let $P = (P_n \mid n \in \mathbb{N})$ be a path. We define its path-derivative $\partial D(P) \in \mathbb{N}_0$ as follows. If one of the terms $P_n$ of $P$ is a log-atomic number, we put
\[
\partial D(P) = \left( \prod_{m<n} P_m \right) D(P_n)
\]
and we observe that the above definition does not depend on the choice of $n$ (because if $P_n$ is log-atomic $D(P_n) = P_n D(P_{n+1})$). On the other hand, if $P$ does not reach a log-atomic number, we define its path-derivative $\partial D(P)$ to be zero. Under the simplifying assumption that $D$ takes values in $\mathcal{M} \mathbb{R}^*$, it can be proved that the family $(\partial D(P) \mid P \in \mathcal{P}(f))$ is summable [9, Prop. 6.20, Thm. 6.32], so we can define
\[
\partial f = \sum_{P \in \mathcal{P}(f)} \partial D(P)
\]
and we call $\partial$ the derivation induced by the prederivation $D$.

**Proposition 24.5** ([9, Thm. 6.32]). Given a prederivation $D : \lambda_{\mathbb{N}_0} \to \mathcal{M} \mathbb{R}^*$, the map $\partial : \mathbb{N}_0 \to \mathbb{N}_0$ defined as above is a surreal derivation.

**Sketch of Proof.** We omit the verification that $\partial$ is a strongly additive derivation compatible with $\exp$ and we only make a comment on the proof that $\ker \partial = \mathbb{R}$. Clearly $\mathbb{R} \subseteq \ker \partial$ because for $f \in \mathbb{R}$ we have $\mathcal{P}(f) = \emptyset$. On the other hand if $f \notin \mathbb{R}$ and $P \in \mathcal{P}(f)$ is the dominant path, then $P$ reaches a log-atomic number by Proposition 23.7, so $\partial D(P) \neq 0$. Moreover, in the formula $\partial f = \sum_{P \in \mathcal{P}(f)} \partial D(P)$, the contribution of the dominant path dominates the contributions of the other paths in $\mathcal{P}(f)$, so it cannot be cancelled.

To complete the proof of Theorem 24.2 one needs to establish the following proposition, for the proof of which we refer to the original paper.

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1The assumption can probably be relaxed, but we do not insist on this point since we are mainly interested in the simplest prederivation, whose values are in $\mathcal{M}$. 

---
Proposition 24.6 ([9, Prop. 7.6]). The derivation $\partial : \mathbb{N} \to \mathbb{N}$ induced by the simplest pre-derivation is surjective, hence it is as prescribed in Theorem 24.2.

In the rest of the paper, we let $\partial : \mathbb{N} \to \mathbb{N}$ be the surreal derivation induced by the simplest pre-derivation. If $\partial f = g$, we say that $f$ is an integral of $g$. Since $\ker \partial = \mathbb{R}$, the integral is defined up to an additive real constant.

Remark 24.7. If we modify (SD5) putting $\partial \lambda - x = \prod_{n \in \mathbb{N}} \lambda - x - n$, we obtain a different surreal derivation which however is not surjective as there would be no integral of 1 (see [9, Def. 6.6]). Despite the fact that the formula $\partial \lambda - x = \prod_{n \in \mathbb{N}} \lambda - x - n$ looks intuitively simpler than (SD5), the pre-derivation given by (SD5) is in fact simpler according to our definitions.

25. Hardy fields

In this section we discuss the connections between surreal numbers and Hardy fields.

Definition 25.1 (See [12]). A Hardy field is a set of germs at $+\infty$ of real valued differentiable functions on positive half-lines of $\mathbb{R}$ that form a field under the usual addition and multiplication of germs and it is closed under differentiation of germs.

There is a large literature on Hardy fields, see for instance the pioneering work of Hardy [23] and Rosenlicht [35, 36, 37]. If the germ of $f$ belongs to a Hardy field, then $f$ is eventually always positive, or negative, or zero (we cannot have $\sin(x)$), so each Hardy field is an ordered field: its positive elements are the germs of the eventually positive functions.

Example 25.2. The field $\mathbb{R}(x)$ of rational functions is a Hardy field (with the order induced by $x > \mathbb{R}$). The set of germs at $+\infty$ of functions definable in an o-minimal expansion of the reals is a Hardy field.

The notion of H-fields is an algebraic counterpart of the notion of Hardy field.

Definition 25.3 ([2, 3]). An H-field is an ordered field $K$ with a derivation $\partial : K \to K$ satisfying

1. If $f > c$ for every $c \in \ker(\partial)$, then $\partial f > 0$;
2. If $|f|$ is bounded by some constant $c \in \ker \partial$, then $f$ is equal to a constant plus an element smaller in absolute value than any positive constant).

We say that an H-field has small derivation if whenever $|f|$ is smaller than any constant, so is $|\partial f|$.

Note that if an H-field $K \supseteq \mathbb{R}$ has a derivation $\partial$ with $\ker \partial = \mathbb{R}$ and there is an element $f \in K$ with $\partial f = 1$, then the derivation is small. If $\ker \partial = \mathbb{R}$, then (1) takes the form $f > \mathbb{N} \implies \partial f > 0$, which coincides with property (SD1) in the definition of surreal derivation (Definition 24.1).

Example 25.4. Any Hardy field is an H-field. Other examples include: the field of Laurent series, the field of Puiseux series, the field $\mathbb{T}$ of transseries, the field $\mathbb{N}$ of surreal numbers (with the derivation of Proposition 24.6).

We have already mentioned the fact that every linear order (whose domain is a set) embeds in $\mathbb{N}$. Similarly, it can be proved that many ordered algebraic structures embed nicely in $\mathbb{N}$ [18, 20]. In a similar spirit, we have the following result.
Theorem 25.5 ([6]). Every Hardy field extending \( \mathbb{R} \) (and more generally every \( H \)-field with a small derivation and constant field \( \mathbb{R} \)) admits an embedding into \((\mathbb{N}_0, \partial)\) as a differential field.

Aschenbrenner, van den Dries, and van der Hoeven [4] proved that every system of algebraic differential equations (in several variables) over the field \( T \) of transseries which has a solution in a larger \( H \)-field, has already a solution in \( T \). Moreover there is an algorithm which decides when this happens. Indeed the authors of [4] showed that the complete (first–order) theory of \( T \) as a differential field is recursively axiomatisable, hence decidable. Combining this with Theorem 25.5, we deduce that \( \mathbb{N}_0 \) is closed under a large class of differential equations compatible with the theory of \( H \)-fields. For instance, if \( P(y) \) is a polynomial over \( \mathbb{N}_0 \), there is some \( f \in \mathbb{N}_0 \) such that \( \partial f = P(f) \) (it suffices to observe that the corresponding result holds in \( T \) and the coefficients of \( P(y) \) lie in some small \( H \)-field contained in \( \mathbb{N}_0 \), where “small” means that the domain is a set).

26. Composition

Given two formal power series \( f(x) \) and \( g(x) \) such that \( g(x) \) has no constant term, we can define the composition \( f(g(x)) \) simply by substituting all occurrences of \( x \) in \( f(x) \) with \( g(x) \) and expanding the resulting expression. In a similar way, we can define the composition \( f \circ g \) of two transseries \( f, g \in T \) provided \( g > N \). More generally, given two omega-series \( f, g \in \mathbb{R}(\langle \omega \rangle) \) with \( g > N \) we can define \( f \circ g \) replacing each occurrence of \( \omega \) in \( f \) with \( g \) and putting the resulting expression in normal form, where the hypothesis \( g > N \) is needed to ensure the summability of the development. We can even allow the second argument of the composition to be a surreal number, as in the following result.

Theorem 26.1 ([10]). There is a unique map \( \circ : \mathbb{R}(\langle \omega \rangle) \times \mathbb{N}_0^{>N} \to \mathbb{N}_0 \) satisfying

(1) \( r \circ x = r \) if \( r \in \mathbb{R} \);
(2) \( \omega \circ x = x \);
(3) \( \left( \sum_{i \in I} f_i \right) \circ x = \sum_{i \in I} (f_i \circ x) \);
(4) \( \exp(f) \circ x = \exp(f \circ x) \).

Note that (3) also implies \( \log(h) \circ x = \log(h \circ x) \), so in particular \( \log_n(\omega) \circ x = \log_n(x) \). Since every branch of an element of \( \mathbb{R}(\langle \omega \rangle) \) reaches a log-atomic number of the form \( \log_n(\omega) \), points (1)–(4) determine the value of the composition \( f \circ x \) for \( f \in \mathbb{R}(\langle \omega \rangle) \) and \( x \in \mathbb{N}_0^{>N} \), thus proving the uniqueness part of the theorem. The existence part is more complicated, as one needs to verify inductively that if \( f_i \circ x \) has been defined for every \( i \in I \) and the family \( (f_i)_{i \in I} \) is summable, then also \( (f_i \circ x)_{i \in I} \) is summable.

The associativity property \( (f \circ g) \circ x = f \circ (g \circ x) \) holds whenever it makes sense, i.e. when both \( f \) and \( g \) are omega-series (while \( x \in \mathbb{N}_0^{>N} \)). It follows that we can interpret an omega-series \( f \) (and in particular a transseries), as a surreal function \( \mathbb{N}_0^{>N} \to \mathbb{N}_0 \) sending \( x \) to \( f \circ x \). We also write \( f(x) \) instead of \( f \circ x \).

Since \( \mathbb{N}_0 \) is an ordered field, it makes sense to ask whether the function \( x \mapsto f \circ x \) is differentiable. Point (1) of the following proposition shows that the answer is positive and the derivative of this function coincides with the the function \( x \mapsto (\partial f) \circ x \).

Proposition 26.2 ([10]). For $f, g \in \mathbb{R}((\omega))$ with $g > N$, $x, y \in \mathbb{N}^{>N}$ and $\varepsilon \in \mathbb{N}$, we have:

1. $\partial f \circ x = \lim_{\varepsilon \to 0} \frac{f_0(x+\varepsilon) - f_0 x}{\varepsilon}$;
2. $\partial (f \circ g) = (\partial f \circ g) \partial g$;
3. $f \circ (b+\varepsilon) = \sum_{n \in \mathbb{N}} \frac{1}{n!} ((\partial (n) f \circ b) \varepsilon^n)$, provided $\varepsilon$ is sufficiently small;
4. if $\partial f = 0$, then $f \circ x = f$;
5. if $\partial f > 0$ and $x < y$, then $f \circ x < f \circ y$.

A natural question is whether we can extend the composition $f, x \mapsto f \circ x$ allowing both $f$ and $x$ to be surreal numbers (with $x > N$). We shall return to this problem in the next section.

27. Conclusions

The following table compares $\mathbb{N}$ with other rings and fields of formal series which can be embedded in $\mathbb{N}$ (with the ordinal $\omega$ playing the role of a formal variable).

| Power series | Laurent | Puiseux | Transseries | Surreals |
|--------------|---------|---------|-------------|----------|
| Ordered ring | ✓       | ✓       | ✓           | ✓        |
| Ordered field| ✓       | ✓       | ✓           | ✓        |
| Derivation   | ✓       | ✓       | ✓           | ✓        |
| Integrals    | ✓       | ✓       | ✓           | ✓        |
| Infinite sums| ✓       | ✓       | restricted  | restricted |
| exp and log  | ✓       | ✓       | ✓           | ✓        |
| Composition  | ✓       | ✓       | ✓           | ✓        |

In the table, “Integrals” means that the derivation is surjective and “Restricted” means that we do not consider all the possible sums of summable families, as illustrated by the following examples:

- $\sum_{n \in \mathbb{N}} \omega^1/n$ is a transseries (in $\omega$), but not a Puiseux series;
- $\sum_{n \in \mathbb{N}} \log_n (\omega)$ is a surreal, but not a transseries.

It is an open problem whether there is an associative composition $\circ : \mathbb{N} \times \mathbb{N}^{>N} \to \mathbb{N}$ satisfying the properties in Theorem 26.1. This is connected to the problem of representing $\mathbb{N}$ as a field of “hyperseries” [5]. Given a composition and a surreal derivation, we may require that they satisfy the compatibility relations expressed by Proposition 26.2. In this case the derivation is actually definable in terms of the composition using the formula $\partial f = \lim_{\varepsilon \to 0} \frac{f_0(\omega+\varepsilon) - f_0 \omega}{\varepsilon}$.

In [10, Thm. 8.4] it is shown that the surreal derivation considered in this paper (the derivation defined in Proposition 24.6) is not compatible with a composition. There could however be other derivations, obtained by modifying (SD5) in Theorem 24.2, which are compatible with a composition. In [41] it is shown that, up to isomorphism, the derivation of Proposition 24.6 is the unique one satisfying some natural properties. Those properties however do not include the fact that the derivation distributes over infinite sums. The problem whether there is a composition and a compatible “better derivation” is therefore still unresolved.
References

[1] Norman L. Alling. *Foundations of Analysis over Surreal Number Fields*, volume 141 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1987. ISBN 0-444-70226-1.

[2] Matthias Aschenbrenner and Lou van den Dries. *H*-fields and their Liouville extensions. *Math. Zeitschrift*, 242(3):543–588, 2002. doi:10.1007/s002090000358.

[3] Matthias Aschenbrenner and Lou van den Dries. Liouville closed *H*-fields. *J. Pure Appl. Algebra*, 197(1-3):83–139, may 2005. doi:10.1016/j.jpaa.2004.08.009.

[4] Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven. *Asymptotic Differential Algebra and Model Theory of Transseries*. Annals of Mathematical Studies. Princeton University Press, Princeton, dec 2017. ISBN 9781400885411. doi:10.1515/9781400885411.

[5] Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven. On numbers, germs, and transseries. In *Proc. Int. Congr. Math. (ICM 2018)*, pages 1–23. WORLD SCIENTIFIC, may 2019. doi:10.1142/9789813272880_0042.

[6] Matthias Aschenbrenner, Lou Van Den Dries, and Joris Van Der Hoeven. The surreal numbers as a universal *H*-field. *J. Eur. Math. Soc.*, 21(4):1179–1199, 2019. doi:10.4171/JEMS/858.

[7] Vincent Bagayoko and Joris Van Der Hoeven. Surreal substructures. *hal-02151377*, 2019.

[8] Alessandro Berarducci and Marcello Mamino. Asymptotic analysis of Skolem’s exponential functions. *arXiv*, (November):1–23, 2019. URL http://arxiv.org/abs/1911.07576.

[9] Alessandro Berarducci and Vincenzo Mantova. Surreal numbers, derivations and transseries. *J. Eur. Math. Soc.*, 20(2):339–390, jan 2018. doi:10.4171/JEMS/769.

[10] Alessandro Berarducci and Vincenzo Mantova. Transseries as germs of surreal functions. *Trans. Am. Math. Soc.*, 371(5):3549–3592, dec 2019. ISSN 0002-9947. doi:10.1090/tran/7428.

[11] Alessandro Berarducci, Salma Kuhlmann, Vincenzo Mantova, and Mickaël Matutinski. Exponential fields and Conway’s omega-map. *Proc. Amer. Math. Soc, to Appear*, pages 1–15, 2018. URL http://arxiv.org/abs/1810.03029.

[12] Nicolas Bourbaki. * Fonctions d’une variable réelle: Théorie élémentaire*. Eléments de mathématique. Diffusion C.C.L.S., Paris, 1976. ISBN 9783540340386.

[13] John H. Conway. *On number and games*, volume 6 of *London Mathematical Society Monographs*. Academic Press, London, 1976. ISBN 0-12-186350-6.

[14] Ovidiu Costin, Philip Ehrlich, and Harvey M. Friedman. Integration on the Surreals: a Conjecture of Conway, Kruskal and Norton. may 2015. URL http://arxiv.org/abs/1505.02478.

[15] J Denef and Lou van den Dries. *p*-adic and real subanalytic sets. *Ann. Math.*, 128(1):79–138, 1988. URL http://www.jstor.org/stable/1971463.

[16] H. Dulac. Sur les cycles limites. *Bull. la Société Mathématique Fr.*, 51:45–188, 1923. doi:10.24033/bsmf.1031.

[17] Jean Ecalle. *Introduction aux fonctions analyzables et preuve constructive de la conjecture de Dulac*. Actualités Mathématiques. Hermann, Paris, 1992. ISBN 2-7056-6199-9.
[18] Philip Ehrlich. Number Systems with Simplicity Hierarchies: A Generalization of Conway’s Theory of Surreal Numbers. J. Symb. Log., 66(3):1231–1258, 2001. URL http://journals.cambridge.org/abstract_S0022481200010604.

[19] Philip Ehrlich. Conway names, the simplicity hierarchy and the surreal number tree. J. Log. Anal., 1(January):1–26, 2011. doi:10.4115/jla.2011.3.1.

[20] Philip Ehrlich and Elliot Kaplan. Surreal ordered exponential fields. pages 1–25, 2020. URL http://arxiv.org/abs/2002.07739.

[21] Harry Gonshor. An introduction to the theory of surreal numbers. London Mathematical Society Lecture Notes Series. Cambridge University Press, Cambridge, 1986. doi:10.1017/CBO9780511629143.

[22] Hans Hahn. Über die nichtarchimedischen Größensysteme. Sitz. der K. Akad der Wiss., Math. Nat. KL., 116(IIa):601–655, 1907.

[23] G. H. Hardy. Orders of infinity, The ‘infinitärrechn’ of Paul du Bois-Reymond. Cambridge University Press, 1910.

[24] Elliot Kaplan. Model completeness for the differential field of transseries with exponentiation. ArXiv:2004.1495v2, pages 1–27, May 2020. URL http://arxiv.org/abs/2004.14957.

[25] Irvin Kaplansky. Maximal fields with valuations. Duke Math. J., 9:303–321, 1942.

[26] Franz-Viktor Kuhlmann, Salma Kuhlmann, and Saharon Shelah. Exponentiation in power series fields. Proc. Am. Math. Soc., 125(11):3177–3183, 1997. doi:10.1090/S0002-9939-97-03964-6.

[27] Salma Kuhlmann. Ordered Exponential Fields, volume 12 of Fields Institute Monographs. American Mathematical Society, Providence, Rhode Island, 2000. ISBN 0-8218-0943-1.

[28] Salma Kuhlmann and Mickaël Matusinski. The Exponential-Logarithmic Equivalence Classes of Surreal Numbers. Order, 32(1):53–68, mar 2015. doi:10.1007/s11083-013-9315-3.

[29] Salma Kuhlmann and Saharon Shelah. $\kappa$-bounded exponential-logarithmic power series fields. Ann. Pure Appl. Log., 136(3):284–296, nov 2005. ISSN 01680072. doi:10.1016/j.apal.2005.04.001.

[30] Salma Kuhlmann and Marcus Tressl. Comparison of exponential-logarithmic and logarithmic-exponential series. Math. Log. Q., 58(6):434–448, nov 2012. ISSN 09425616. doi:10.1002/malq.201100113.

[31] Angus Macintyre and Alex J. Wilkie. On the decidability of the real exponential field. In Odefreddi, editor, Kreiseliana About Around Georg Kreisel, pages 441–467. A K Peters, 1996. ISBN 156881061X.

[32] Vincenzo Mantova and Mickaël Matusinski. Surreal numbers with derivation, Hardy fields and transseries: a survey. In Ordered Algebr. Struct. Relat. Top., volume 697 of Contemporary Mathematics, pages 265–290. Amer. Math. Soc., Providence, RI, 2017. doi:10.1090/conm/697/14057.

[33] Bernhard Hermann Neumann. On ordered division rings. Trans. Amer. Math. Soc., 66(1):202–252, 1949.

[34] Jean-Pierre Ressayre. Integer parts of real closed exponential fields (extended abstract). In Peter Clote and J. Krajíček, editors, Arith. Proof Theory, Comput. Complex. (Prague, 1991), volume 23 of Oxford Logic Guides, pages 278–288. Oxford University Press, New York, 1993. ISBN 978-0-19-853690-1.
[35] Maxwell Rosenlicht. The rank of a Hardy field. *Trans. Am. Math. Soc.*, 280 (2):659–671, 1983. doi:10.2307/1999639.

[36] Maxwell Rosenlicht. Hardy fields. *J. Math. Anal. Appl.*, 93(2):297–311, may 1983. doi:10.1016/0022-247X(83)90175-0.

[37] Maxwell Rosenlicht. Growth properties of functions in Hardy fields. *Trans. Am. Math. Soc.*, 299(1):261–261, jan 1987. doi:10.2307/2000493.

[38] Michael Ch. Schmeling. *Corps de transséries*. Phd, Université de Paris 7, 2001. URL http://cat.inist.fr/?aModele=afficheN&cpsidt=14197291.

[39] Thoralf Skolem. An ordered set of arithmetic functions representing the least $\epsilon$-number. *Nor. Vid. Selsk. Forh., Trondheim*, 29:54–59, 1956.

[40] Lou van den Dries and Philip Ehrlich. Fields of surreal numbers and exponentiation. *Fundam. Math.*, 167(2):173–188, 2001. doi:10.4064/fm167-2-3.

[41] Lou van den Dries and Philip Ehrlich. Homogeneous universal $H$-fields. *Proc. Am. Math. Soc.*, 147(5):2231–2234, 2019. doi:10.1090/proc/14424.

[42] Lou van den Dries and Hilbert Levitz. On Skolem’s exponential functions below $2^x$. *Trans. Am. Math. Soc.*, 286(1):339–349, 1984.

[43] Lou van den Dries, Angus Macintyre, and David Marker. The elementary theory of restricted analytic fields with exponentiation. *Ann. Math.*, 140(1):183–205, 1994. URL http://www.jstor.org/stable/2118545.

[44] Lou van den Dries, Angus Macintyre, and David Marker. Logarithmic-Exponential Power Series. *J. London Math. Soc.*, 56(3):417–434, dec 1997. doi:10.1112/S0024610797005437.

[45] Lou van den Dries, Angus Macintyre, and David Marker. Logarithmic-exponential series. *Ann. Pure Appl. Log.*, 111(1-2):61–113, jul 2001. ISSN 01680072. doi:10.1016/S0168-0072(01)00035-5.

[46] Lou van den Dries, Joris van der Hoeven, and Elliot Kaplan. Logarithmic hyperseries. *Trans. Am. Math. Soc.*, 372(7):5199–5241, jul 2019. doi:10.1090/tran/7876.

[47] Alex J. Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *J. Amer. Math. Soc.*, 9(4):1051–1094, 1996.