ELECTRICAL RESISTANCE OF THE LOW DIMENSIONAL CRITICAL BRANCHING RANDOM WALK

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Abstract. We show that the electrical resistance between the origin and generation \( n \) of the incipient infinite oriented branching random walk in dimensions \( d < 6 \) is \( O(n^{1-\alpha}) \) for some universal constant \( \alpha > 0 \). This answers a question of Barlow, Járai, Kumagai and Slade [2].

1. Introduction

We study the electrical resistance of the trace of oriented critical branching random walk (BRW) in low dimensions. This trace is obtained by drawing a critical Galton-Watson tree \( T \) conditioned to survive forever and randomly mapping it into \( \mathbb{Z}^d \times \mathbb{Z}^+ \) in the following manner: we initialize by mapping the root of \( T \) to \((0,0)\) and recursively, if \( V \in T \) was mapped to \((x,n)\) and \( U \in T \) is a child of \( V \), then we map \( U \) to \((y,n+1)\) where \( y \) is chosen according to a symmetric random walk distribution (we assume that this distribution has an exponential moment). Denote by \( \Phi : T \to \mathbb{Z}^d \times \mathbb{Z}^+ \) this random mapping. The trace we consider in this paper is the graph induced by the set of edges \( \{\Phi(V), \Phi(U)\} \) for every edge \( \{U,V\} \) of \( T \).

It follows from the work of Barlow, Járai, Kumagai and Slade [2] (who studied the much more difficult model of critical oriented percolation (OP)) that when \( d > 6 \), the electrical resistance between the root and generation \( n \) in the BRW is linear in probability. This enabled them to calculate various exponents describing the behavior of the simple random walk on the trace. In particular, they show that the mean hitting time at graph distance \( n \) is \( \Theta(n^3) \), that the spectral dimension equals \( 4/3 \) and more, see [2].

They asked [2, Section 1.4 (iii)] whether the resistance of the critical BRW is still linear in \( n \) in dimensions \( 4 < d \leq 6 \), that is, in any dimension above the critical dimension 4 of OP [6,7,8,9]. Here we answer their question by showing that the resistance is \( O(n^{1-\alpha}) \) when \( d \leq 5 \).

Theorem 1. Let \( R(n) \) denote the expected effective resistance between the origin and generation \( n \) of a branching random walk in dimension \( d < 6 \) with progeny distribution that has mean 1, positive variance and finite third moment, conditioned to survive forever. There exists a universal constant \( \alpha > 0 \) such that

\[
R(n) = O(n^{1-\alpha}).
\]
Unlike our firm understanding of anomalous diffusion in high dimensions \cite{2,11,13}, random fractals in low dimensions are not (stochastically) finitely ramified. That is, we do not see pivotal edges at every scale. This makes their analysis more challenging, even in the case of the critical BRW which is one of the simplest models of statistical physics. Our argument heavily relies on the built-in independence and self-similarity of the model to obtain recursive inequalities for the resistance. We first show that intersections within the trace occur at every scale (see Figure \ref{fig:intersection} and Theorem \ref{thm:intersection}); these intersections exist only when $d < 6$. Secondly, we show that the branches leading to each intersection are themselves distributed as BRW, allowing us to bound the electrical circuit using the parallel law and to form recursive estimates (Theorem \ref{thm:recursive}). There are additional technical difficulties to overcome. For instance, when intersections do not occur, the resistance is stochastically larger than it is unconditionally and one needs to get adequate bounds on it. Calculating the precise polynomial exponent which determines the growth of $R(n)$ when $d < 6$ remains a challenging open problem.

As mentioned before, it is believed that OP in $d = 5$ behaves similarly to BRW hence we expect an analogue of Theorem \ref{thm:main} to hold. Presumably, the general setup (illustrated in Figure \ref{fig:intersection}) and proving existence of intersections (Theorem \ref{thm:intersection}) can be done for OP (based on results of \cite{6,7}). However, due to the lack of distributional self-similarity in OP it seems difficult to obtain recursive bounds (that is, an analogue of Theorem \ref{thm:recursive}). Furthermore, we do not know whether the exponent determining the growth of the resistance in OP in $d = 5$ should be the same as the one for BRW (assuming they both exist).

It is easy to see (and stated in \cite{2}) that the volume up to generation $n$ of the BRW trace is of order $\Theta(n^2)$ in probability. Hence, Theorem \ref{thm:main} together with the commute time identity \cite{14} shows that the mean exit time of the simple random walk on the BRW trace from the ball of radius $n$ in graph distance is at most $O(n^{3-\alpha})$, i.e., much faster than the $\Theta(n^3)$ in dimensions $d > 6$, see \cite{2}. In fact, if one calculated the exponent determining the growth of the resistance, then many other random walk exponents (such as the spectral dimension, walk dimension etc.) could be determined, see \cite{14}. In particular, if the resistance exponent exists, it follows from our results that the spectral dimension is strictly larger than $4/3$.

**Remark 1.** We emphasize that the exponent $\alpha > 0$ of Theorem \ref{thm:main} is universal in the sense that it does not depend on the progeny or random walk distributions.

**Remark 2.** By projecting the trace to $\mathbb{Z}^d$ we get a similar result for the usual (non-oriented) branching random walk: the effective resistance between the origin and the particles of generation $n$ is $O(n^{1-\alpha})$ when $d \leq 5$. This is because the projection only decreases the effective resistance. By a similar argument, projecting $\mathbb{Z}^5$ into $\mathbb{Z}^d$ with $d < 5$, we learn that it suffices to prove Theorem \ref{thm:main} for $d = 5$. 

1.1. Incipient infinite branching process. Let \( \{p(k)\}_{k \geq 0} \) be a progeny distribution of a Galton-Watson branching process. Our assumptions on \( \{p(k)\} \) are the following.

(i) Criticality: \( \sum_k k p(k) = 1 \).
(ii) Finite variance: \( \sum_k k(k - 1)p(k) = \sigma^2 \in (0, \infty) \).
(iii) Bounded third moment: \( \sum_k k^3 p(k) \leq C_3 < \infty \).

It is classical that under condition (i) (and that \( p(1) < 1 \)) the branching process dies out with probability 1. To construct the incipient infinite branching process (IIBP), we simply condition on survival up to level \( n \), and take the weak limit of the measures obtained as \( n \to \infty \). However, it will be convenient for us to use an equivalent construction of the IIBP (see [11, 16]).

Consider an infinite path \((V_0, V_1, \ldots)\) and attach to each vertex \( V_i \) a critical branching process with progeny distribution \( \tilde{p} \) in the first generation and \( p \) afterwards, where \( \tilde{p} \) is the size biased law of \( p \) minus 1, that is,

\[ \tilde{p}(k) = \frac{k + 1}{p(k+1)}. \]

1.2. Incipient infinite branching random walk. Let \( p^1(x, y) \) denote the 1-step transition probability of a random walk on \( \mathbb{Z}^d \). We assume the following:

(i) Exponential moment: \( \sum_{x \in \mathbb{Z}^d} e^{b|x|} p^1(o, x) < \infty \) for some \( b > 0 \).
(ii) Non-degeneracy: \( \{x \in \mathbb{Z}^d : p^1(o, x) > 0\} \) generates \( \mathbb{Z}^d \) as a group.
(iii) Symmetry: \( p^1(x, y) = p^1(y, x) \).

We remark that we did not try to obtain the optimal condition on \( p^1 \). In fact, conditions (ii) and (iii) are not essential for our proof, and (i) can plausibly be replaced with a weaker condition, however, we opted to make the calculations smoother. Likewise, we have not tried to optimize the moment condition on \( p(k) \).

Given a rooted tree \( T \) we define a random mapping \( \Phi : T \to \mathbb{Z}^d \times \mathbb{Z}_+ \) which we will call henceforth a “random walk” mapping. Firstly, \( \Phi \) maps the root of \( T \) to \((o, 0)\) and recursively, given a vertex \( V \) of \( T \) at height \( h \) and its mapping \( \Phi(V) = (x, h) \) we map each upward neighbor \( U \) of \( V \), independently, by drawing a random neighbor \( y \) of \( x \), according to \( p^1(x, \cdot) \), and putting \( \Phi(U) = (y, h+1) \). The incipient infinite branching random walk (IIBRW) is obtained by taking \( T \) to be the IIBP.

For any tree \( T \) we consider \( \Phi(T) \) as a graph on the vertex set \( \mathbb{Z}^d \times \mathbb{Z}_+ \) and we add the edge \( \{\Phi(U), \Phi(W)\} \) for any tree edge \( \{U, W\} \) (there may be parallel edges). The trace of the IIBRW is simply \( \Phi(T) \) where \( T \) is the IIBP.

1.3. Electrical resistance. We provide a brief background on the electric effective resistance of a network, for further information see [15]. Let \( G = (V, E) \) be a finite connected graph with two marked vertices \( a \) and \( z \) (we assume here that all edge weights are 1). The effective resistance between
a and z, denoted \( R_{\text{eff}}(a \leftrightarrow z) \), is the minimum energy \( \mathcal{E}(\theta) \) over all unit flows from \( a \) to \( z \), where \( \mathcal{E}(\theta) = \sum_{e \in E} \theta(e)^2 \). The connection between this quantity and the simple random walk \( \{S_t\}_{t \geq 0} \) on \( G \) is evident via the identity (see [15]),

\[
R_{\text{eff}}(a \leftrightarrow z) = \frac{1}{\deg(a)P_a(\tau_z < \tau_+ + a)},
\]

where \( \deg(a) \) is the vertex degree of \( a \), \( \tau_z \) is the first visit time to \( z \), \( \tau_+ \) is the first positive visit time to \( a \) and \( P_a \) is the simple random walk probability measure conditioned on \( X_0 = a \). Another useful connection is the commute time identity asserting that

\[
E_a\tau_z + E_z\tau_a = 2|E|R_{\text{eff}}(a \leftrightarrow z),
\]

(1.1)

We will frequently use the easy fact that the resistance satisfies the triangle inequality, that is, for any three vertices \( x, y, z \) we have

\[
R_{\text{eff}}(x \leftrightarrow z) \leq R_{\text{eff}}(x \leftrightarrow y) + R_{\text{eff}}(y \leftrightarrow z).
\]

(1.2)

Lastly, we will use the parallel law for effective resistance stating that if \( G_1 = (V, E_1) \) and \( G_2 = (V, E_2) \) are two connected graphs on the same vertex set and \( a, z \in V \), then the effective resistance \( R_{1 \cup 2} \) between \( a \) and \( z \) in \((V, E_1 \cup E_2)\) (where we allow multiple edges in this union) satisfies

\[
\frac{1}{R_{1 \cup 2}} \geq \frac{1}{R_1} + \frac{1}{R_2},
\]

(1.3)

where \( R_1, R_2 \) are the effective resistances between \( a \) and \( z \) in \( G_1, G_2 \), respectively.

1.4. **Finite approximations.** We use the following finite approximations to the IIBRW in order to establish recursions.

**Definition.** Suppose \( n \geq 1 \) and \( m \geq 2n \). Let \( T_{n,m} \) denote the following random tree:

(i) A path of length \( n \) (the backbone): \((V_0, \ldots, V_n)\) with a marked root \( \rho = V_0 \).

(ii) For each \( 0 \leq i \leq n - 1 \) attach to \( V_i \) a critical branching process with progeny distribution \( \tilde{p} \) in the first generation and \( p \) afterwards, conditioned to die out before generation \( m - i \) (i.e. none of the vertices of the attached trees reach distance \( m \) from \( \rho \)).

The following is an important quantity in the proof. For \( x \in \mathbb{Z}^d \) define

\[
\gamma(n, x) = \sup_{m \geq 2n} \mathbf{E}_{T_{n,m}}\left[R_{\text{eff}}((0,0) \leftrightarrow \Phi(V_n)) \mid \Phi(V_n) = (x, n)\right],
\]

where we consider the resistance in the graph \( \Phi(T_{n,m}) \).

It will be convenient to introduce the following norm on \( \mathbb{R}^d \) adapted to the “typical size” of the random walk displacements. We do this in order to conveniently obtain a universal estimate on \( \alpha \) of Theorem [1] but the reader
may just assume that $p^1$ is the transition matrix of the nearest-neighbor simple random walk and that the norm below is the Euclidean norm.

Let $Q_{ij} = \sum_{x \in \mathbb{Z}^d} x_i x_j p^1(o, x)$ be the covariance matrix of the step distribution, and let $Q^{-1}$ denote the inverse of $Q$. We define

$$\|x\| := \sqrt{\frac{1}{d} \sum_{i,j=1}^d x_i Q^{-1}_{ij} x_j}.$$  \hfill (1.4)

The main effort in this paper is the following theorem.

**Theorem 2.** Assume $d \leq 5$. There exists a universal constant $\alpha \in (0, 1/2)$ and also $A = A(\sigma^2, C_3, p^1) < \infty$ such that for all $n \geq 1$

$$\gamma(n, x) \leq An^{1-\alpha} \left( \frac{\|x\|^2}{n} \bigvee 1 \right)^\alpha.$$

**Remark.** Note that we cannot expect $\gamma(n, x)$ to be $O(n^{1-\alpha})$ for all $x$. Indeed, when $\|x\| = \Theta(n)$, then conditioned on $\Phi(V_n) = (x, n)$ the projection of the path $\Phi(V_0), \ldots, \Phi(V_n)$ onto $\mathbb{Z}^d$ has positive speed and since this conditioning does not affect the mapping of the trees hanging on $V_i$, there will be little intersections and we expect the resistance then to be linear in $n$. Theorem 2 will be proved by induction, hence it has to contain an estimate valid for all $x \in \mathbb{Z}^d$.

**Proof of Theorem 1 assuming Theorem 2.** Recall that in the construction of the IIBRW we attach to the backbone unconditional critical trees, whereas in the definition of $T_{n,m}$ we attach critical trees conditioned not to reach a certain level. However, when $n$ is fixed and $m \to \infty$ the distribution of these critical trees tends to the distribution of an unconditional critical tree. Hence,

$$R(n) \leq \lim_{m \to \infty} \mathbb{E}_{T_{n,m}}[R_{\text{eff}}((o, 0) \leftrightarrow \Phi(V_n))],$$

where we bounded the resistance to generation $n$ by the resistance to a single vertex $\Phi(V_n)$. Therefore,

$$R(n) \leq \sup_{m \geq 2n} \mathbb{E}_{T_{n,m}}[R_{\text{eff}}((o, 0) \leftrightarrow \Phi(V_n))] = \sum_{x \in \mathbb{Z}^d} p^n(o, x) \gamma(n, x).$$

By Theorem 2 we have

$$R(n) \leq An^{1-\alpha} \sum_{x: \|x\| \leq \sqrt{n}} p^n(o, x) + An^{1-\alpha} \sum_{x: \|x\| \geq \sqrt{n}} p^n(o, x) \left( \frac{\|x\|^2}{n} \right)^\alpha.$$  

The first sum is bounded by $An^{1-\alpha}$. For the second sum we bound by

$$\sum_x p^n(o, x) \left( \frac{\|x\|^2}{n} \right) = 1,$$

(see Section 1.3), concluding the proof. \qed
1.5. Some random walk estimates. We provide here some standard random walk estimates that will be useful throughout the proof. We denote by \( \{S(n)\}_{n \geq 0} \) a random walk with step distribution \( p^1 \) and \( S(0) = o \). Let \( (S^1(n), \ldots, S^d(n)) \) denote the coordinates of \( S(n) \) in a coordinate system that diagonalizes \( Q^{-1} \) (lower indices will be used for the Euclidean coordinates). Due to independent and mean zero increments and the definition of the norm, we have

\[
E\left[ \|S(n)\|^2 \right] = n E\left[ \|S(1)\|^2 \right] = \frac{n}{d} \sum_{i,j=1}^d E\left[ S(1)_i Q^{-1}_{ij} S(1)_j \right] = n.
\]

Applying Chebyshev’s inequality, we get

\[
\sum_{x \in \mathbb{Z}^d : \|x\| \leq \sqrt{2}} p^1(o,x) = 1 - P(\|S(1)\|^2 > 2) \geq 1/2.
\] (1.5)

The central limit theorem [3, Theorem 2.9.6] implies that for any \( 0 < L < \infty \) and any \( v \in \mathbb{R}^d \), we have

\[
\lim_{n \to \infty} P(\|S(n) - \sqrt{n}v\| \leq L\sqrt{n}) = C(v, d, L) \in (0,1),
\] (1.6)

with the constant \( C(v, d, L) \) independent of \( p^1 \).

The following proposition summarizes some estimates we will need on the random walk \( S \) conditioned on the event \( \{S(n) = x\} \).

**Proposition 3.** There exists \( k_1 = k_1(p^1), \ C > 0 \) and \( \delta_1 = \delta_1(d) > 0 \) such that the following hold.

(i) Whenever \( k_1 \leq k \leq n, \|x\| \leq 4n/\sqrt{k} \), we have

\[
E\left[ \|S(k)\|^2 \bigg| S(n) = x \right] \leq Ck.
\] (1.7)

(ii) Whenever \( k_1 \leq k \leq \delta_1 n, \|x\| \leq 4n/\sqrt{k} \), we have

\[
E\left[ \|S(k)\|^2 \bigg| S(n) = x, \|S(k)\| > \sqrt{k} \right] \leq Ck.
\] (1.8)

(iii) Whenever \( k_1 \leq k \leq \delta_1 n, k_1 \leq k' \leq n-k \) and \( \|x\| \leq \min\{4n/\sqrt{k}, 4n/\sqrt{k'}\} \), we have

\[
E\left[ \|S(k + k') - S(k)\|^2 \bigg| S(n) = x, \|S(k)\| > \sqrt{k} \right] \leq CK'.
\] (1.9)

For the proof of this proposition, we will use the exponential moment assumption from Section 1.2. Let \( b_1 > 0 \) be such that when \( \|\beta\| < b_1 \) we have

\[
Z_\beta := \sum_{y \in \mathbb{Z}^d} e^{\beta \cdot y} p^1(o,y) < \infty,
\]

where \( \cdot \) in the exponent denotes inner product with respect to the quadratic form \( \sum_{i,j} x_i Q^{-1}_{ij} y_j \). Define the exponentially tilted step distribution

\[
p^1_\beta(o,y) = \frac{1}{Z_\beta} e^{\beta \cdot y} p^1(o,y)
\]
Let $X(1), \ldots, X(n)$ be i.i.d. distributed according to $p^1$, so that $S(n) = X(1) + \cdots + X(n)$, and let $X_\beta(1), \ldots, X_\beta(n)$ be i.i.d. distributed according to $p^{1_\beta}$. Let $S_\beta(n) = X_\beta(1) + \cdots + X_\beta(n)$ and let

$$m_\beta := E[X_\beta(1)] = \frac{1}{Z_\beta} \sum_{y \in \mathbb{Z}^d} ye^{y_\beta} p^1(o, y) = \nabla \log Z_\beta.$$ 

Since the Jacobian of $\beta \mapsto m_\beta$ at $\beta = 0$ is non-singular, for $v \in \mathbb{R}^d$ sufficiently close to 0 there exists a unique $\beta$ such that $m_\beta = v$. We write $Q_\beta$ for the covariance matrix of $X_\beta(1)$, $D_\beta = \det(Q_\beta)^{1/2d}$, and $\| \cdot \|_\beta$ for the norm arising from $Q_\beta^{-1}$. Note that $D_\beta$ and $\| \cdot \|_\beta$ depend continuously on $\beta$ in a neighbourhood of 0. In particular, for $\beta$ in a neighbourhood of 0 we have

$$\|v\|_\beta \leq 2\|v\|. \quad (1.10)$$

Since $E[\|X(1)\|^2] = 1$, for $\beta$ sufficiently close to 0, we have

$$\Sigma^2_\beta := E\left[\|X_\beta(1) - m_\beta\|^2\right] \leq 2. \quad (1.11)$$

We will need the following local limit theorem that is uniform in small $\beta$.

**Lemma 1.1.** There exists $C = C(d)$ and $0 < b_2 = b_2(p^1) < b_1$ such that the following hold.

(i) There exists $n_1 = n_1(p^1)$ such that for all $y \in \mathbb{Z}^d$ we have

$$P(S_\beta(n) = y) \leq \frac{2C}{D_\beta^d n^{d/2}}, \quad (1.12)$$

when $n \geq n_1$, $\|\beta\| \leq b_2$.

(ii) For any $0 < \epsilon < 1$ and $0 < L < \infty$ there exists $n_2 = n_2(p^1, \epsilon, L)$ such that for all $y \in \mathbb{Z}^d$ such that $\|y - nm_\beta\| \leq L \sqrt{n}$ we have

$$P(S_\beta(n) = y) \leq \frac{C(1 + \epsilon)}{D_\beta^d n^{d/2}} e^{-d\|y - nm_\beta\|_\beta^2/(2n)},$$

$$P(S_\beta(n) = y) \geq \frac{C(1 - \epsilon)}{D_\beta^d n^{d/2}} e^{-d\|y - nm_\beta\|_\beta^2/(2n)}, \quad (1.13)$$

when $n \geq n_2$, $\|\beta\| \leq b_2$.

We assumed above that the walk has period 1. Trivial modifications can be made to handle the case of period 2, and we will not make this explicit in our arguments.

**Proof of Lemma 1.1.** The lemma can be proved by appealing to a local central limit theorem for lattice distributions [3, Theorem 2.5.2]. Note that the standard proof in [3] can be followed, and this gives uniformity in $\beta$. □

Specializing to $\beta = 0$, we denote

$$D := \det(Q)^{1/2d}.$$
Observe that with the norm introduced in (1.4), we have
\[ \sum_{x : ||x|| \leq L} 1 \geq cD^dL^d, \tag{1.14} \]
for some constant \( c > 0 \) and all \( L \geq 1 \). When \( d \geq 3 \), the Green function
\[ G(x) := \sum_{n=0}^{\infty} p^n(a, x) \]
satisfies (see [10, Theorem 4.3.5]):
\[ G(x) \leq \frac{C(d)}{D^d} ||x||^{2-d}, \quad \text{when } ||x|| \geq L_1 = L_1(p^1). \tag{1.15} \]

It follows from Lemma 1.1 that there exist \( 0 < b_3 = b_3(p^1) < b_1, k_2 = k_2(p^1) \) and \( c = c(d) < 1 \) such that for \( k \geq k_2 \) and \( ||\beta|| \leq b_3 \) we have
\[ \mathbb{P}(||S_\beta(k) - km_\beta|| \leq \sqrt{k}) \leq c. \tag{1.16} \]

We now choose \( 0 < b_0 = b_0(p^1) \leq \min\{b_2, b_3\} \) so that Lemma 1.1 (1.10), (1.11) and (1.16) all hold when \( ||\beta|| \leq b_0 \). We also choose now \( r_0 = r_0(p^1) > 0 \) such that \( ||v|| \leq r_0 \) implies \( ||\beta|| \leq b_0 \) for the unique \( \beta \) such that \( m_\beta = v \). The constants \( b_0 \) and \( r_0 \) will now be fixed for the remainder of the paper.

We are ready to prove Proposition 3.

**Proof of Proposition 3** Choose \( k_1 = k_1(p^1) \) in such a way that \( 4/\sqrt{k_1} < r_0 \) and \( k_1 \geq k_2(p^1) \) for the constant \( k_2 \) of (1.16). We also require that \( k_1 \geq n_1 \) and \( k_1 \geq n_2(\epsilon = 1/2, L = 1) \) for the constants \( n_1, n_2 \) from Lemma 1.1. Fix \( x, n \) and \( k \), and let \( \beta \) be such that \( m_\beta = x/n \). Note that the choice of \( k_1 \) and the conditions on \( x \) and \( n \) imply that \( ||\beta|| \leq b_0 \).

It is easy to check that conditional on \( S(n) = x \), the joint distribution of \( X(1), \ldots, X(n) \) is the same as the joint distribution of \( X_\beta(1), \ldots, X_\beta(n) \) conditioned on \( S_\beta(n) = x \). Consequently, the joint distribution of \( X(1) - x/n, \ldots, X(n) - x/n \), given \( S(n) = x = 0 \) is the same as the joint distribution of \( X_\beta(1) - m_\beta, \ldots, X_\beta(n) - m_\beta \), given \( S_\beta(n) - nm_\beta = 0 \). Therefore, since \( \mathbb{E}[S(k) \mid S(n) = x] = (k/n)x \), we have
\[ \mathbb{E}\left[||S(k)||^2 \mid S(n) = x\right] = \mathbb{E}\left[||S(k) - \frac{k}{n}x + \frac{k}{n}x||^2 \mid S(n) = x\right] \]
\[ = \mathbb{E}\left[||S(k) - \frac{k}{n}x||^2 \mid S(n) = x\right] + \frac{k^2}{n^2}||x||^2. \tag{1.17} \]

The second term on the right hand side is at most \( 16k \), by our assumption on \( ||x|| \). The first term on the right hand side of (1.17) equals
\[ \mathbb{E}\left[||S_\beta(k) - km_\beta||^2 \mid S_\beta(n) = nm_\beta\right]. \]

Conditional on \( S_\beta(n) = nm_\beta \), the variables \( X_\beta(1), \ldots, X_\beta(n) \) are exchangeable, and it is easy to use \( S_\beta(n) - nm_\beta = 0 \) (expanding the variance) that \( X_\beta(k_1) \) and \( X_\beta(k_2) \) are negatively correlated for all \( 1 \leq k_1 < k_2 \leq n \) and \( j = 1, \ldots, d \). It follows that
\[ \mathbb{E}\left[||S_\beta(k) - km_\beta||^2 \mid S_\beta(n) = nm_\beta\right] \leq k\mathbb{E}\left[||X_\beta(1) - m_\beta||^2 \mid S_\beta(n) = nm_\beta\right]. \]
It remains to estimate the conditional expectation on the right hand side. Using Lemma 1.1, this is at most
\[ \sum_{y \in \mathbb{Z}^d} \left\| y - m_\beta \right\|^2 \frac{p'_\beta(o, y)}{p'_\beta(o, x)} \leq \sum_{y \in \mathbb{Z}^d} \left\| y - m_\beta \right\|^2 \frac{p'_\beta(o, y)}{p'_\beta(o, x)} \frac{4(n - 1)^{d/2}}{n^{d/2}} \]
\[ \leq C E \left[ \left\| X_\beta(1) - m_\beta \right\|^2 \right] \leq C \Sigma^2_\beta. \]

By (1.11), we obtain the first statement (1.7) of the proposition.

In order to prove (1.8) it is sufficient, due to the just proven part (i), to show that \( \delta_1 \) can be chosen such that \( \mathbb{P} \left( \left\| S(k) \right\| > \sqrt{k} \mid S(n) = x \right) \geq c' > 0 \). For this, let \( c \) be the constant in (1.16) and recall the constant \( C_1 \) in (1.10). Observe that if \( \left\| y \right\| \leq \sqrt{k} \), we have
\[ \left\| (x - y) - (n - k)m_\beta \right\| = \left\| y - km_\beta \right\| \leq \left\| y \right\| + k \frac{\left\| x \right\|}{n} \leq \sqrt{k} + k \frac{4n}{\sqrt{kn}} = 5 \sqrt{k}. \]
Hence due to (1.10), \( \left\| y - km_\beta \right\|^2 \leq 100k \). We now use Lemma 1.1(ii) with \( \epsilon > 0 \) satisfying
\[ c(1 + \epsilon)/(1 - \epsilon) < (1 + c)/2. \]
We write
\[ \mathbb{P} \left( \left\| S(k) \right\| \leq \sqrt{k} \mid S(n) = x \right) \]
\[ = \sum_{y : \left\| y \right\| \leq \sqrt{k}} \frac{p'_\beta(o, y)}{p'_\beta(o, x)} \leq \sum_{y : \left\| y \right\| \leq \sqrt{k}} \frac{1 + \epsilon}{1 - \epsilon} e^{-50d \delta_1/(1 - \delta_1) \left(1 + \delta_1\right)^{d/2}} \leq \frac{1 + \epsilon}{1 - \epsilon} e^{-50d \delta_1/(1 - \delta_1) \left(1 + \delta_1\right)^{d/2}}. \]
We choose \( \delta_1 = \delta_1(d) \) so that the right hand side is < 1, and this proves part (ii) of the proposition.

The last statement (1.9) now follows easily. Due to exchangability, and part (i), we have
\[ \mathbb{E} \left[ \left\| S(k + k') - S(k) \right\|^2 \mid S(n) = x \right] = \mathbb{E} \left[ \left\| S(k') \right\|^2 \mid S(n) = x \right] \leq C k'. \]
Hence the statement follows from \( \mathbb{P} \left( \left\| S(k) \right\| > \sqrt{k} \mid S(n) = x \right) \geq c' > 0 \) proved in part (ii).

2. Setting up the induction scheme

We begin by introducing some useful notation. Given an instance of \( T_{n,m} \), consider some small \( \delta > 0 \), where we assume that \( \delta n \) is an integer. We write
\[ X_i = V_{i \delta n} \quad i = 0, 1, \ldots, \lfloor \delta^{-1} \rfloor, \]
and write \( x_i \in \mathbb{Z}^d, i = 0, 1, \ldots, \lfloor \delta^{-1} \rfloor \) for the random spatial location of \( X_i \), that is, \( x_i \) is the unique vertex satisfying \( \Phi(X_i) = (x_i, i\delta n) \). Write \( T_{n,m}(\ell) \) for the subtree of \( T_{n,m} \) emanating from \( V_\ell \) off the backbone (including the vertex \( V_\ell \)).

We let \( N = \lfloor (K\delta)^{-1} \rfloor - 1 \), and subdivide the backbone into \( N \) stretches of length \( K\delta n \), and a remaining part of length at least \( K\delta n \) and less than \( 2K\delta n \).

We begin with some definitions that are depicted in Figure [1].

**Definition 2.1.** For \( \ell \) satisfying \( i\delta n \leq \ell < (i+1)\delta n \) we say that a backbone vertex \( V_\ell \) has the unique descendant property (UDP) if in \( T_{n,m}(\ell) \) it has a unique descendant at level \( (i+1)\delta n \) that reaches level \( (i+2)\delta n \). For any other vertex \( V \) of \( T_{n,m} \) at level \( i\delta n \) we say that \( V \) has UDP if it has a unique descendant at level \( (i+1)\delta n \) that reaches level \( (i+2)\delta n \).

**Definition 2.2.** Given an integer \( K \geq 1 \), a number \( \delta > 0 \) such that \( K\delta \leq (1/2) \) and an instance of \( T_{n,m} \) we say that a sequence \((i, i+1, \ldots, i+K)\) of length \( K+1 \) is \( K \)-tree-good if the following holds:

1. There exists a unique \( i\delta n \leq \ell_1 < (i+1)\delta n \) such that \( T_{n,m}(\ell_1) \) reaches height \( (i+2)\delta n \). Moreover, this unique \( \ell_1 \) satisfies \( (i+1/4)\delta n \leq \ell_1 \leq (i+3/4)\delta n \).
2. \( V_{\ell_1} \) has UDP and we call the unique descendant \( Y_{i+1} \). For all \( i' \) satisfying \( i + 2 \leq i' \leq i + K \) we inductively define the vertices \( Y_{i'} \) of \( T_{n,m}(\ell_1) \) as follows. We require that \( Y_{i'-1} \) has UDP and call the unique descendant \( Y_{i'} \).
3. There exists a unique \( (i+K-1)\delta n \leq \ell_2 < (i+K)\delta n \) such that \( T_{n,m}(\ell_2) \) reaches height \( (i+K+1)\delta n \). Moreover, this unique \( \ell_2 \) satisfies \( (i+K-3/4)\delta n \leq \ell_2 \leq (i+K-1/4)\delta n \).
4. \( V_{\ell_2} \) has UDP, and we call the unique descendant \( X_{i+K} \). The vertex \( X_{i+K} \) has UDP, and we call the unique descendant \( Y_{i+K+1} \). Similarly, \( Y_{i+K} \) has UDP, and we call the unique descendant \( Y_{i+K+1} \).

Given a \( K \)-tree-good sequence \((i, \ldots, i+K)\) we denote by \( V_{\ell_1}^+ \) (respectively \( V_{\ell_2}^+ \)) the child of \( V_{\ell_1} \) (respectively \( V_{\ell_2} \)) leading to \( Y_{i+1} \) (respectively \( X_{i+K} \)).

We further define the spatial locations \( y_{i'} \) by \( \Phi(Y_{i'}) = (y_{i'}, i'\delta n) \) for \( i + 1 \leq i' \leq i + K \), and similarly define \( x_{i+K}^+, x_{i+K+1}^+, v_{\ell_1}, v_{\ell_2}, v_{\ell_2}^+ \).

We will write \( U \prec W \) to denote that \( W \) is a descendant of \( U \), and write \( h(U), h(W) \) for their respective heights in the tree (in particular, \( h(W) > h(U) \)).

**Definition 2.3.** Let \( U \prec W \) be two tree vertices and let \( u, w \in \mathbb{Z}^d \) be defined by \( \Phi(U) = (u, h(U)) \) and \( \Phi(W) = (w, h(W)) \). We say that \( U \) and \( W \) are typically-spaced if \( \|w - u\| \leq \sqrt{h(W) - h(U)} \). Denote this event by \( TS(U, W) \).

**Definition 2.4.** We say that a \( K \)-tree-good sequence \((i, \ldots, i+K)\) is \( K \)-spatially-good if the following holds.
Figure 1. (a) Illustration of $K$-tree-good. (b) Illustration of $K$-spatially-good. All spatial distances between consecutive vertices are at most $\sqrt{t}$ time difference and the spatial distance between $x_{i+K}'$ and $y_{i+K}$ is at most $\sqrt{\delta n}$.

(5) $\mathcal{T}S(X_i, V_{\ell_1})$
$\mathcal{T}S(V_{\ell_1+1}, X_{i+1})$
$\mathcal{T}S(X_{i+K-1}, V_{\ell_2})$
$\mathcal{T}S(V_{\ell_2+1}, X_{i+K})$
(6) $\mathcal{T}S(V_{\ell_1}', Y_{i+1})$
$\mathcal{T}S(V_{\ell_2}', X_{i+K}')$
$\|x_{i+K}' - y_{i+K}\| \leq \sqrt{\delta n}$

Definition 2.5. When a sequence $(i, \ldots, i+K)$ is both $K$-tree-good and $K$-spatially-good we say that it is $K$-good. Let $\mathcal{A}(i)$ be the event that $(i, \ldots, i+K)$ is $K$-good.

Next, let $(i, \ldots, i+K)$ be a $K$-good sequence and let $U_1, U_2$ be two vertices at the same height such that $U_1 \succ X_{i+K}'$ and $U_2 \succ Y_{i+K}$. Given these, we write $Z_1$ for the highest common ancestor of $U_1$ and $X_{i+K}'$ and $Z_2$ for the highest common ancestor of $U_2$ and $Y_{i+K}$ (see Figure 2). Further, we
Figure 2. The labelling of vertices in the two (potentially) intersecting trees emanating from $X'_{i+K}$ and $Y_{i+K}$.

denote by $Z_1^+$ (respectively $Z_2^+$) the child of $Z_1$ (respectively $Z_2$) leading to $U_1$ (respectively $U_2$).

**Definition 2.6.** We say that $U_1, U_2$ intersect-well if the following conditions hold:

1. $U_1 \succ X'_{i+K}, U_2 \succ Y_{i+K}$,
2. $(i + K + (5/6))\delta n \leq h(U_1) = h(U_2) \leq (i + K + 1)\delta n$;
3. $(i + K + (1/2))\delta n \leq h(Z_1), h(Z_2) \leq (i + K + (4/6))\delta n$;
4. $T S(X'_{i+K}, Z_1), T S(Z_1^+, U_1), T S(Y_{i+K}, Z_2), T S(Z_2^+, U_2)$;
5. $\Phi(U_1) = \Phi(U_2)$.

And define the random set $\mathcal{I}$ by

$$\mathcal{I} = \{(U_1, U_2) : U_1 \text{ and } U_2 \text{ intersect-well}\}, \quad (2.1)$$

Lastly, we define the event $B(i, c_0)$ where $c_0 > 0$ is a constant

$$B(i, c_0) = \mathcal{A}(i) \cap \{|\mathcal{I}| \geq \frac{c_0\sigma^4}{D^d}(\delta n)^{(6-d)/2}\}.$$

Our first theorem is that $K$-good runs $(i, i + 1, \ldots, i + K)$ occur with positive density and in each, the probability of seeing many intersections occurs with positive probability.

**Theorem 4** (Intersections exist). Assume that $d = 5$. There exist constants $c_0, c_1 > 0$ and for any $K \geq 2$ there exists $c_2 = c_2(K) > 0$, and $n_3 = n_3(a^2, C_3, p^1, K)$ such that for any $0 < \delta < (K + 4)^{-1}$, whenever $\delta n \geq n_3$ and $x$ satisfies $\|x\| \leq \sqrt{2n/\delta}$, we have

$$P(\mathcal{A}(i) \mid \Phi(V_n) = (x, n)) \geq c_2.$$
and
\[ \mathbf{P}(B(i, c_0) \mid A(i), \Phi(V_n) = (x, n)) \geq c_1 \]
for \( i = 0, K, 2K, \ldots, (N - 1)K \).

To proceed let us define
\[ \gamma(n) = \sup_{x : \|x\| \leq \sqrt{n}} \gamma(n, x). \]

When all the good events occur, it is immediate by definition and the triangle inequality (1.2) that the resistance between \( X_i \) and \( X_{i + K} \) is bounded above by \( K\gamma(\delta n) \). The following theorem shows that the intersections create a “short-cut” in the electric circuit, allowing us to bound the resistance between the two ends of the run \((i, \ldots, i + K)\) using the parallel law of electric resistance (1.3) essentially by \( \frac{3}{4}K\gamma(\delta n) \). This multiplicative constant improvement allows the induction argument to work.

**Theorem 5 (Analysis of good blocks).** There exists \( K_0 < \infty \) and \( n_4 = n_4(\sigma^2, C_3, p^1) \) such that if \( K \geq K_0 \) and \( \delta n \geq n_4 \), we have
\[ \mathbf{E}\left[ \text{Res}_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i + K})) \mid A(i), B(i, c_0), \Phi(V_n) = (x, n) \right] \leq \frac{3K}{4} \max_{1 \leq k \leq \delta n} \gamma(k) \]
for \( i = 0, K, 2K, \ldots, \lfloor (K\delta)^{-1} \rfloor - 2 \).

To complete the induction step we also need a bound on the resistance conditioned on \( A(i)^c \cup B(i, c_0)^c \). This is rather lengthy, since for each reason that either \( A(i) \) or \( B(i, c_0) \) fail, we provide a different bound on the resistance which we eventually collect together at the proof of the induction step.

### 2.1. Organization

The proof of Theorem 4 is done in Section 3 and the proof of Theorem 5 is presented in Section 6. The analysis of the resistance when the good events fail to occur is presented in Sections 4 and 5.

### 3. Existence of intersections

In this section we prove Theorem 4. In Section 3.2 we show that \( K \)-good runs \((i, \ldots, i + K)\) occur with positive probability, proving the first statement of Theorem 4. In Section 3.3 we show that given a \( K \)-good run, there are “enough” intersections with positive probability, proving the second statement of Theorem 4.

#### 3.1. Preliminaries

Recall that \( \tilde{p}(k) = (k + 1)p(k + 1) \) for \( k \geq 0 \). We denote by \( \{N_n\}_{n \geq 0} \) a branching process with \( N_0 = 1 \) and progeny distribution \( p(k) \), and by \( \{	ilde{N}_n\}_{n \geq 0} \) a branching process with \( \tilde{N}_0 = 1 \) and progeny \( \tilde{p}(k) \) in the first generation and progeny \( p(k) \) afterwards. Note that for all \( n \geq 1 \) we have \( \mathbf{E}N_n = 1 \) and
\[ \mathbf{E}\tilde{N}_n = \sum_{k \geq 0} k\tilde{p}(k) = \sum_{k \geq 1} k(k - 1)p(k) = \text{Var}(N_1) = \sigma^2. \]
We denote by $f(s)$ and $\tilde{f}(s) = f'(s)$ the generating functions of $p$ and $\tilde{p}$, respectively. Then the generating functions of $N_n$ and $\tilde{N}_n$ are $f_n(s)$ and $\tilde{g}_n(s) := \tilde{f}(f_{n-1}(s))$, respectively, where $f_n(s)$ is the $n$-fold composition of $f$ with itself.

We denote by $\theta(n) = f_n(0) = P(N_n > 0)$ the survival probability of the branching process up to time $n$. It is well known \cite{12, 1} that

$$\frac{\theta(n) \sigma^2 n}{2} \to 1 \text{ as } n \to \infty. \quad (3.1)$$

Furthermore, there exists $n_5 = n_5(C_3) < \infty$ such that

$$\frac{1}{\sigma^2 n} \leq \theta(n) \leq \frac{3}{\sigma^2 n}, \quad n \geq n_5. \quad (3.2)$$

Moreover, there exists $n'_5 = n'_5(C_3)$ such that we have

$$\theta(n) - \theta(m) \geq \frac{1}{2\sigma^2 n}, \quad \text{whenever } m \geq 2n, \, n \geq n'_5. \quad (3.3)$$

**Lemma 3.1.** For any $C > 0$ there exists $c' = c'(C) > 0$ and $n_6 = n_6(C, \sigma^2, C_3) < \infty$ such that for all $n \geq n_6$ we have

$$f'_n \left(1 - \frac{C}{\sigma^2 n}\right) \geq c'$$

and

$$g'_n \left(1 - \frac{C}{\sigma^2 n}\right) \geq c' \sigma^2. \quad (3.4)$$

**Proof.** We have that

$$f'_n(s) \geq (f'(s))^n \geq (1 - (1 - s)f''(1))^n. \quad (3.5)$$

Indeed, the first inequality follows by appealing to the chain rule and using the fact that $f_n(s) \geq s$ for $s \in [0,1]$ and that $f$ is convex. The second inequality follows from the mean-value theorem together with the fact that $f''$ is increasing (the coefficients of the Taylor series of $f$ are non-negative by definition). Substituting $s = 1 - C/\sigma^2 n$ gives the first statement (recall that $f''(1) = \sigma^2$). For the second statement, observe that

$$g'_n(s) = \tilde{f}'(f_{n-1}(s)) f'_{n-1}(s) \geq \tilde{f}'(s) (f'(s))^{n-1} = f''(s) (f'(s))^{n-1} \geq (f''(1) - (1 - s)f'''(1)) (1 - (1 - s)f''(1))^{n-1}. \quad (3.6)$$

Substituting $s = 1 - C/\sigma^2 n$ and using that $f'''(1) \leq C_3$ yields the result. \qed

**Lemma 3.2.** There exist $n_7 = n_7(C_3) < \infty$ such that

$$\frac{1}{2n} \leq P(\tilde{N}_n > 0) \leq \frac{3}{n}, \quad \text{whenever } n \geq n_7, \quad (3.5)$$

and

$$\frac{1}{4n} \leq P(\tilde{N}_n > 0 \mid \tilde{N}_m = 0) \leq \frac{6}{n}, \quad \text{whenever } n \geq n_7, \, m \geq 2n. \quad (3.6)$$
Proof. For the upper bound in (3.5), if the process survives \( n \) generations, one of the particles at generation 1 needs to survive \( n - 1 \) generations, so by (3.2) we bound this probability by
\[
\sum_{k=1}^{\infty} \tilde{p}(k) \frac{3}{\sigma^2 n} \leq \frac{3}{n}.
\]
For the lower bound in (3.5), we write
\[
P(\tilde{N}_n > 0) = 1 - \tilde{f}(f_{n-1}(0)) = 1 - f'(1 - \theta(n-1))
\geq 1 - f' \left( 1 - \frac{1}{\sigma^2(n-1)} \right)
\geq \frac{1}{\sigma^2(n-1)} f'' \left( 1 - \frac{1}{\sigma^2(n-1)} \right),
\]
where the last inequality is due to the mean-value theorem. As before, \( f''(s) \geq (f''(1) - (1-s)f'''(1)) \) and \( f''(1) = \sigma^2 \) and \( f'''(1) \leq C_3 \) gives the lower bound.

The proof of (3.6) is quite similar. The upper bound follows easily, since by (3.5) we have for large enough \( n \) the inequality
\[
P(\tilde{N}_n > 0 | \tilde{N}_m = 0) \leq 2P(\tilde{N}_n > 0).
\]
For the lower bound, using (3.3) we write:
\[
P(\tilde{N}_n > 0 | \tilde{N}_m = 0) = \tilde{f}(f_{m-1}(0)) = f'(1 - \theta(m-1))
\geq (\theta(n-1) - \theta(m-1)) f''(1 - \theta(n-1))
\geq \frac{1}{2\sigma^2 n} f'' \left( 1 - \frac{c}{\sigma^2 n} \right).
\]
This is now bounded from below as in (3.7).

3.2. \( K \)-good runs occurs. The proof is broken down into a series of lemmas showing that each of the conditions involved in a run \((i, \ldots, i+K)\) being \( K \)-tree-good and \( K \)-spatially-good (that is, the conditions in Definitions 2.2 and 2.4) holds with probability bounded away from 0. For \( a = 1, \ldots, 6 \) let \( D_{(a)} \) denote the event that condition \( a \) in Definitions 2.2 and 2.4 is satisfied.

We start by analyzing the conditions in Definition 2.2(1)–(4). Recall that these only involve the branching process, hence here the conditioning on \( \{ \Phi(V_n) = (x,n) \} \) present in Theorem 4 is irrelevant. Therefore we omit it in the lemmas below.

Lemma 3.3. There exists \( c > 0 \) such that we have
\[
P(D_{(1)}) \geq c
\]
whenever \( \delta n \geq n_7 \).
Proof. Assume without loss of generality that \( i = 0 \) (the proof will be the same for any \( 0 \leq i < \delta^{-1} \)). For any \( \ell \) satisfying \( 0 \leq \ell < \delta n \), let \( D_{(2)}(\ell) \) be the event that the random tree attached to \( V_\ell \), that is \( T_{n,m}(\ell) \), reaches level \( 2\delta n \). So \( D_{(1)} \) is the event that exactly one of the events \( \{ D_{(1)}(\ell) \} \) occurs, and that the index \( \ell_1 \) of that event lies between \( (1/4)\delta n \) and \( (3/4)\delta n \). The events \( \{ D_{(1)}(\ell) \} \) are independent, and due to (3.6) each has probability between \( \frac{1}{6\delta n} \) and \( \frac{1}{6n} \). Hence \( P(D_{(1)}) \geq c > 0 \).

**Lemma 3.4.** There exists \( c = c(K) > 0 \) such that we have

\[
P(D_{(2)} | D_{(1)}, \ell_1) \geq c > 0
\]

whenever \( \delta n \geq \text{max}\{n_5, n_5', 4n_6(3, \sigma^2, C_3), n_7\} \).

Proof. Again we assume that \( i = 0 \) (the reader will notice that we only use the fact that \( m - i\delta n \geq n \)). The probability that \( V_\ell \) has UDP given \( D_{(1)} \) equals

\[
\sum_{k=1}^{\infty} P(\bar{N}_{\delta n-\ell_1} = k) k \left( \theta(\delta n) - \theta(m - \delta n) \right) (1 - \theta(\delta n))^{k-1}
\]

\[
\frac{P(\bar{N}_{2\delta n-\ell_1} > 0, \bar{N}_{m-\ell_1} = 0)}{P(\bar{N}_{2\delta n-\ell_1} > 0)}
\]

since \( T_{n,m}(\ell_1) \) is now conditioned to survive \( 2\delta n - \ell_1 \) generations, but that the \( m - \ell_1 \)-th generation died out. Hence

\[
P(V_\ell \text{ has UDP} | D_{(1)}, \ell_1) \geq \frac{\theta(\delta n) - \theta(m - \delta n)}{\frac{P(\bar{N}_{2\delta n-\ell_1} > 0)}{P(\bar{N}_{2\delta n-\ell_1} > 0)}} P(\bar{N}_{\delta n} > 0, \bar{N}_{m-\delta n} = 0)
\]

Due (3.3) and Lemmas 3.2 and 3.3 the right hand side of (3.9) is at least a universal constant \( c' > 0 \).

Now, conditioned on \( V_\ell \) having UDP, the descendant tree emanating from \( Y_1 \) is a critical tree conditioned to survive \( \delta n \) generations but not \( m - \delta n \) generations. So the conditional probability that \( Y_1 \) has UDP equals

\[
\sum_{k=1}^{\infty} P(\bar{N}_{\delta n} = k) k \left( \theta(\delta n) - \theta(m - \delta n) \right) (1 - \theta(\delta n))^{k-1}
\]

\[
P(\bar{N}_{\delta n} > 0, \bar{N}_{m-\delta n} = 0)
\]

and similarly this is bounded below by \( c' \). Iterating this argument over \( Y_2, Y_3, \ldots, Y_{K-1} \) gives a probability of at least \( c(K) = c^K \), as required. \qed

**Lemma 3.5.** There exists \( c > 0 \) such that

\[
P(D_{(3)} | D_{(1)}, \ell_1, D_{(2)}) = P(D_{(3)}) \geq c
\]

whenever \( \delta n \geq n_7 \).

Proof. The proof is the same as the proof of Lemma 3.3 \qed

**Lemma 3.6.** There exists \( c > 0 \) such that

\[
P(D_{(4)} | D_{(1)}, \ell_1, D_{(2)}, D_{(3)}, \ell_2) \geq c
\]

whenever \( \delta n \geq \text{max}\{n_5, n_5', 4n_6(3, \sigma^2, C_3), n_7\} \).

Proof. This is proved almost identically to Lemma 3.3 \qed
We next show that the conditions in Definition 3.4 (5)–(6) also hold with probability bounded away from 0.

**Lemma 3.7.** There exists $c = c(K) > 0$ and $n_8 = n_8(p^1, K)$ such that whenever $0 < \delta < 1/(K + 4)$, $\delta n \geq n_8$ and $\|x\| \leq \sqrt{2n/\delta}$, we have

$$P(D_5, D_6 | D_1, \ell_1, D_2, D_3, \ell_2, D_4, \Phi(V_n) = (x, n)) \geq c.$$  \hspace{1cm} (3.10)

**Proof.** Let us condition on the entire branching process tree $T_{n,m}$ in which $D_1 \cdots D_4$ hold. It will be convenient to consider the event $D_5' \subset D_5$ where we replace the requirements in Definition 2.4 (5)–(6) by

(i) $\|x_i - v_{\ell_1}\| \leq (1/2)\sqrt{\ell_1 - i\delta n}$,

(ii) $\|v_{\ell_{i+1}} - x_{i+1}\| \leq (1/2)\sqrt{i\delta n - \ell_1 - 1}$,

(iii) For each $i + 1 \leq j \leq i + K - 2$ we have $\|x_i - x_{i+1}\| \leq (1/2)\sqrt{\delta n}$,

(iv) $\|x_{i+K-1} - v_{\ell_2}\| \leq (1/2)\sqrt{\ell_2 - (i + K - 1)\delta n}$,

(v) $\|x_{i+K} - v_{\ell_{i+1}}\| \leq (1/2)\sqrt{(i + K)\delta n - \ell_2 - 1}$,

(vi) $\|v_{\ell_1} - v_{\ell_{i+1}}\|, \|v_{\ell_2} - v_{\ell_{i+1}}\| \leq \sqrt{2}$

We will show that

$$P(D_5', D_6|T_{n,m}) \geq c,$$  \hspace{1cm} (3.11)

and that

$$P(\Phi(V_n) = (x, n)|D_5', D_6, T_{n,m}) \geq cP(\Phi(V_n) = (x, n)|T_{n,m}),$$  \hspace{1cm} (3.12)

which will conclude our proof. To prove (3.11) we first note that the events of $D_5'$ are all independent and each occurs with probability bounded below by a constant, by (1.6) and (1.3). Conditioned on $x_i, x_{\ell_1}, x_{\ell_1 + 1}, x_{i+1}, \ldots, x_{i+K}$ that satisfy $D_5'$, the event $D_6$ has probability at least $c = c(K) > 0$, indeed, because of the factors $1/2$ in the definition of $D_5'$, repeated application of the central limit theorem yields that the displacement requirements in $D_6$ can be satisfied.

To prove (3.12) we condition on the value of $z = x_{i+K} - x_i$. Choose $n_8$ large enough so that the conditions $\|x\| \leq \sqrt{2n/\delta}$ and $\delta n \geq n_8$ imply $\|x/n\| \leq r_0$ (where $r_0$ is the constant chosen in Section 1.3). Let $\beta$ be such that $m_\beta = x/n$.

Observe that

$$\|z - K\delta n m_\beta\| \leq (K + 4)\sqrt{\delta n} + K\delta n\|x\|/n \leq 5\sqrt{K}\sqrt{K}\delta n.$$  

We now also require $n_8 \geq n_2(p^1, \epsilon = 1/2, L = 5\sqrt{K})$, where $n_2$ is the constant in Lemma 1.4. Then Lemma 1.1(ii) implies

$$P(\Phi(V_n) = (x, n)|D_5', D_6, T_{n,m}) = P(S_\beta(n - K\delta n) = x - z) \geq \frac{c(K)}{D_\beta^d(n - K\delta n)^{d/2}} \geq \frac{c'(K)}{D_\beta^d n^{d/2}}.$$
On the other hand,
\[ P\left( \Phi(V_n) = (x, n) \mid T_{n,m} \right) \leq \frac{2C}{D^{\frac{d}{2}n^{d/2}}} . \]

The last two inequalities imply (3.12).

\[ \square \]

3.3. Abundant intersections. We proceed with proving the second part of Theorem 4. To ease the presentation of this calculation let \( T_1 \) and \( T_2 \) be independent random trees distributed as \( T_{0n,2n} \) and rooted at \( \rho_1, \rho_2 \), respectively. Let \( \Phi_1 \) and \( \Phi_2 \) be independent random walk mappings of \( T_1 \) and \( T_2 \), respectively, into \( \mathbb{Z}^d \times \mathbb{Z}_+ \) such that \( \Phi_1(\rho_1) = (o, 0) \) and \( \Phi_2(\rho_2) = (x, 0) \). Then on the event \( A(i) \cap \{ y_1 + K - x = x \prime + K \} \), the random variable \( |I| \) introduced in (2.1) has the same distribution as the random variable (also denoted \( |I| \) here):

\[ |I| = \sum_{U_1 \in T_1, U_2 \in T_2} 1_{(U_1, U_2) \text{ intersect-well}} . \]

Here we have tacitly adapted the definition of “intersect-well” to the present setting, by replacing \( X' + K \) by \( \rho_1 \) and \( Y + K \) by \( \rho_2 \). Our goal in this section is to show that when \( d = 5 \) we have \( |I| \geq c \sigma^4 D^{-5}(\delta n)^{1/2} \) with positive probability.

**Theorem 6.** Assume \( d = 5 \) and that \( \|x\| \leq \sqrt{\delta n} \). There exist constants \( C < \infty, c > 0 \) and \( n_9 = n_9(\sigma^2, C_3, p^1) < \infty \) such that for \( \delta n \geq n_9 \) we have

\[ E|I| \geq \frac{c \sigma^4}{D^{5/2}} \sqrt{\delta n} , \]

and

\[ E|I|^2 \leq \frac{C \sigma^8}{D^{10}} \delta n . \quad (3.13) \]

Recall that for a tree vertex \( V \) we write \( h(V) \) for its distance from the root. Also recall the vertices \( Z_1, Z_1^+, Z_2, Z_2^+ \) introduced before Definition 2.6 and the constant \( n_2 \) of Lemma 1.1(ii).

**Lemma 3.8.** Given instances of \( T_1 \) and \( T_2 \), let \( U_1 \in T_1 \) and \( U_2 \in T_2 \) be vertices both at height \( (5/6)\delta n \leq h(U_1) = h(U_2) \leq \delta n \), and such that \( (1/2)\delta n \leq h(Z_1), h(Z_2) \leq (4/6)\delta n \). There exists \( c = c(d) > 0 \) such that whenever \( \delta n \geq 6n_2(p^1, \epsilon = 1/2, L = 1) \) and \( \|x\| \leq \sqrt{\delta n} \) we have

\[ P\left( (U_1, U_2) \text{ intersect-well} \mid T_1, T_2 \right) \geq \frac{c}{D^{d}(\delta n)^{d/2}} . \]

**Proof.** Denote the spatial locations of \( Z_1, Z_1^+, Z_2, Z_2^+ \) by \( z_1, z_1^+, z_2, z_2^+ \), and denote the common spatial location of \( U_1 \) and \( U_2 \) be \( u \). Let us choose the
spatial locations so that the inequalities
\[
\|z_1\| \leq \sqrt{(1/2)\delta n} \quad \|z_2 - x\| \leq \sqrt{(1/2)\delta n} \\
\|z_1^+ - z_1\| \leq \sqrt{2} \quad \|z_2^+ - z_2\| \leq \sqrt{2} \\
\|z_1^+ - u\| \leq \sqrt{(1/6)\delta n} \quad \|z_2^+ - u\| \leq \sqrt{(1/6)\delta n}
\]
are satisfied — this guarantees that the required events $T_S(\cdot, \cdot)$ all occur. Fix the displacements $z_1^+ - z_1$ and $z_2^+ - z_2$. Since $\sqrt{1/2} + \sqrt{1/6} > 1/2$, there are $\geq cD^{3d}(\delta n)^{3d/2}$ choices for $(z_1, u, z_2)$ satisfying the requirements above. Due to Lemma 1.1, each choice has probability at least $c \left( D - d(\delta n) - d/2 \right)$ occurring. Combined with (1.5) to handle the displacements $z_1^+ - z_1$ and $z_2^+ - z_2$, this proves the statement of the lemma. □

**Lemma 3.9.** We have
\[
E|I| \geq \frac{c\sigma^4}{D^d} (\delta n)^{(6-d)/2}.
\] (3.14)
whenever $\delta n \geq \max\{n_2(p^1, \epsilon = 1/2, L = 1), n_5, 6n_6\}$.

**Proof.** By Lemma 3.8 we have
\[
E|I| \geq \frac{c}{D^d(\delta n)^{d/2}} \sum_{h=5\delta n/6}^{\delta n} \sum_{k_1} \sum_{k_2} E\mathcal{L}_{h,k_1} E\mathcal{L}_{h,k_2} (3.15)
\]
where $\mathcal{L}(h, k)$ counts the number of $U_1 \in T_1$ at level $h$ such that $Z_1$ is at level $k$. Note that since $Z_1$ is a backbone vertex, we have that
\[
E\mathcal{L}_{h,k_1} = E \left[ \tilde{N}_{h-k_1} | \tilde{N}_{2\delta n-k_1} = 0 \right].
\]
We have that
\[
E \left[ \tilde{N}_{h-k_1} | \tilde{N}_{2\delta n-k_1} = 0 \right] = (1 - \theta(2\delta n - k_1))^{-1} g_{h-k_1} (1 - \theta(2\delta n - h)) \geq c\sigma^2,
\]
by Lemma 3.2 and Lemma 3.1. Summing this estimate in (3.15) concludes the proof. □

The remainder of this section is devoted to the proof of the second moment estimate in Theorem 6. Given numbers $h_u, h_w, k_1$ satisfying
\[
\delta n/2 \leq k_1 \leq h_u, h_w \leq \delta n,
\]
we write $\mathcal{L}(h_u, h_w, k_1)$ for the variable counting the number of pairs of tree vertices $U, W$ such that their highest common ancestor in the tree is at level $k_1$.

**Lemma 3.10.** We have
\[
E_{T_{\delta n,2\delta n}} \mathcal{L}(h_u, h_w, k_1) \leq (C_3 + 2\sigma^4\delta n) 1_{\{h_u > k_1, h_w > k_1\}} + (1 + 2\sigma^2 \delta n) 1_{\{h_u = k_1 \text{ or } h_w = k_1\}}.
\]
Proof. Let $\mathcal{T}_{\delta n, \infty}$ be a random tree obtained similarly to $\mathcal{T}_{\delta n, 2\delta n}$ dropping the requirement that the critical trees hanging on the backbone are conditioned not to reach level $2\delta n$. By the FKG inequality \cite{FKG, JLR} we have

$$E_{\mathcal{T}_{\delta n, 2\delta n}} \mathcal{L}(h_u, h_w, k_1) \leq E_{\mathcal{T}_{\delta n, \infty}} \mathcal{L}(h_u, h_w, k_1),$$

indeed, the measure $\mathcal{T}_{\delta n, 2\delta n}$ is obtained from $\mathcal{T}_{\delta n, \infty}$ by conditioning on a monotone decreasing event in a product measure (all the independent progeny random variables) and the random variable $\mathcal{L}$ is monotone increasing. From here we will always calculate with respect to $\mathcal{T}_{\delta n, \infty}$ and we drop the corresponding subscript.

For two vertices $U, W$ at heights $h_u, h_w$ we write $S$ for their highest common ancestor at height $k_1$. There is a slight difference in the calculation depending on whether $S$ is in the backbone of $\mathcal{T}_{\delta n, \infty}$ or not. Write $\mathcal{L}^1(h_u, h_w, k_1)$ for the number of such $U, W$ such that $S$ is not on the backbone and $\mathcal{L}^2(h_u, h_w, k_1)$ when $S$ is on the backbone. We first estimate $E\mathcal{L}^1$. When $h_u > k_1$ and $h_w > k_1$, the expected number of pairs $U, W$ emanating from a fixed $S$ at height $k_1$ is at most

$$\sum_{k=2}^{\infty} p(k) k (k-1) \mathbb{E}N_{h_u - k_1 - 1} \mathbb{E}N_{h_w - k_1 - 1} = \sigma^2.$$ 

When either $h_u = k_1$ or $h_w = k_1$ (that is, either $U$ or $W$ equal $S$) the expected number of such pairs is at most 1. By summing over the backbone vertex from which $S$ emanates we have that

$$E\mathcal{L}^1(h_u, h_w, k_1) \leq \sigma^2 \delta n \mathbb{1}_{\{h_u > k_1, h_w > k_1\}} + \sigma^2 \delta n \mathbb{1}_{\{h_u = k_1 \text{ or } h_w = k_1\}}.$$ 

To estimate $E\mathcal{L}^2$ we assume now that $S$ is the unique vertex on the backbone at height $k_1$, and when $h_u > k_1$ and $h_w > k_1$ the expected number of $U, W$ in $\mathcal{T}_{\delta n, \infty}(k_1)$ is

$$\sum_{k=2}^{\infty} \hat{p}(k) k (k-1) \mathbb{E}N_{h_u - k_1 - 1} \mathbb{E}N_{h_w - k_1 - 1} \leq C_3.$$ 

The expected number of $U, W$ such that $U \in \mathcal{T}_{\delta n, \infty}(k_1)$ but $W$ emanates from some other backbone vertex at height $k_1$ is at most $\sigma^2 \delta n$. Similarly, the expected number of $U, W$ in which $h_u = k_1$ (and so $U = S$) is at most $\sigma^2 \delta n$. Putting these together gives

$$E\mathcal{L}^1(h_u, h_w, k_1) \leq (C_3 + \sigma^2 \delta n) \mathbb{1}_{\{h_u > k_1, h_w > k_1\}} + (1 + \sigma^2 \delta n) \mathbb{1}_{\{h_u = k_1 \text{ or } h_w = k_1\}}.$$ 

Recall the constant $n_1(p^1)$ of Lemma \ref{lem1} and the constant $L_1(p^1)$ of \cite{JLR}. \qed
Lemma 3.11. Suppose \( d \geq 3 \). There are constants \( C = C(d) > 0 \) and \( C_2 = C_2(p^1) \) such that

\[
\sum_{h: k_1 \vee k_2 \leq h \leq \delta n} p^{2h-k_1-k_2}(z_1, z_2) \leq \frac{C}{D^d} f(k_1, k_2, z_1, z_2),
\]

where

\[
f(k_1, k_2, z_1, z_2) := \begin{cases} 
|k_1 - k_2|^{(2-d)/2} & \text{if } \|z_1 - z_2\| \leq |k_1 - k_2|^{1/2} \\
C_2 & \text{if } \|z_1 - z_2\| \leq |k_1 - k_2|^{1/2} < n_1; \\
\|z_1 - z_2\|^{2-d} & \text{if } \|z_1 - z_2\| > |k_1 - k_2|^{1/2} \\
C_2 & \text{if } |k_1 - k_2|^{1/2} < \|z_1 - z_2\| < L_1.
\end{cases}
\]

Proof. Suppose first we are in the case \( \|z_1 - z_2\| \leq |k_1 - k_2|^{1/2} \). Then for all \( h \geq k_1 \vee k_2 \) we have \( 2h - k_1 - k_2 \geq |k_1 - k_2| \geq \|z_1 - z_2\|^2 \). Hence due to Lemma 1.1 in the case when \( |k_1 - k_2| \) is large enough, we have

\[
\sum_{h: k_1 \vee k_2 \leq h \leq \delta n} p^{2h-k_1-k_2}(z_1, z_2) \leq \sum_{k=|k_1-k_2|}^{\infty} p^k(z_1, z_2) \\
\leq \frac{C}{D^d} \sum_{k=|k_1-k_2|}^{\infty} k^{-d/2} \\
\leq \frac{C}{D^d} |k_1 - k_2|^{(2-d)/2}.
\]

When \( |k_1 - k_2| \) is not large, the bound follows trivially.

Suppose now we are in the other case \( \|z_1 - z_2\| > |k_1 - k_2|^{1/2} \). Then due to (1.15), in the case when \( \|z_1 - z_2\| \) is large enough, we have

\[
\sum_{h: k_1 \vee k_2 \leq h \leq \delta n} p^{2h-k_1-k_2}(z_1, z_2) \leq \sum_{k=0}^{\infty} p^k(z_1, z_2) \leq \frac{C}{D^d} \|z_1 - z_2\|^{2-d}.
\]

The bound is trivial in the case when \( \|z_1 - z_2\| \) is not large.

Proof of Theorem 3. The lower bound on the first moment is Lemma 3.9 (we require that \( n_9 \geq \max\{6n_2(p^1, \epsilon = 1/2, L = 1), n_5, 6n_6\} \)). We are left to prove the upper bound on the second moment. First we drop the requirements of “typically spaced” from the definition of \( \mathcal{I} \). This gives that

\[
E |\mathcal{I}^2| \leq \sum_{k_1, k_2 = \delta n/2}^{\delta n} \sum_{h_1, h_2 = k_1 \vee k_2}^{\delta n} E_{\mathcal{I}}(h_1, h_2, k_1) E_{\mathcal{I}}(h_1, h_2, k_2) \\
\times p(h_1, h_2, k_1, k_2),
\]

(3.16)
where \( p(h_u, h_w, k_1, k_2) \) is the probability that \( \Phi_1(U_1) = \Phi_2(U_2) \) and \( \Phi_1(W_1) = \Phi_2(W_2) \) where \( U_1, U_2, W_1, W_2 \) are any tree vertices satisfying that the highest common ancestor of \( U_1 \) and \( W_1 \) is at height \( k_1 \) and the highest common ancestor of \( U_2 \) and \( W_2 \) is at height \( k_2 \) and \( h(U_1) = h(U_2) = h_u \) and \( h(W_1) = h(W_2) = h_w \). Note that this probability only depends on the corresponding heights and not on the vertices. We have that

\[
p(h_u, h_w, k_1, k_2) = \sum_{z_1, z_2 \in \mathbb{Z}^d} \sum_{u, w \in \mathbb{Z}^d} p^{k_1}(o, z_1)p^{h_u - k_1}(z_1, u)p^{h_w - k_1}(z_1, w) \\
\times p^{k_2}(x, z_2)p^{h_u - k_2}(z_2, u)p^{h_w - k_2}(z_2, w). \tag{3.17}
\]

We can perform the summations over \( u, w \) yielding the expression

\[
\sum_{z_1, z_2 \in \mathbb{Z}^d} p^{k_1}(o, z_1)p^{2h_u - k_1 - k_2}(z_1, z_2)p^{2h_w - k_1 - k_2}(z_1, z_2)p^{k_2}(x, z_2). \tag{3.18}
\]

Using Lemmas 3.10 and 3.11 we sum (3.16) over \( h_u, h_w > k_1 \lor k_2 \) and we get a bound of

\[
Y_1 = \frac{C(\sigma^4 \delta n)^2}{D^{2d}} \sum_{k_1, k_2 = \delta n/2}^{\delta n} \sum_{z_1, z_2} f(k_1, k_2, z_1, z_2)^2 p^{k_1}(o, z_1)p^{k_2}(x, z_2).
\]

Similarly, we sum over \( h_u > k_1 \lor k_2 \) and \( h_w = k_1 \lor k_2 \) and when the roles of \( h_u \) and \( h_w \) exchanged, getting a bound of

\[
Y_2 = \frac{C(\sigma^3 \delta n)^2}{D^d} \sum_{k_1, k_2 = \delta n/2}^{\delta n} \sum_{z_1, z_2} f(k_1, k_2, z_1, z_2) p^{k_1 - k_2}(z_1, z_2)p^{k_1}(o, z_1)p^{k_2}(x, z_2).
\]

And finally our third bound is when \( h_u = h_w = k_1 \lor k_2 \) giving

\[
Y_3 = C(\sigma^2 \delta n)^2 \sum_{k_1, k_2 = \delta n/2}^{\delta n} \sum_{z_1, z_2} p^{k_1}(o, z_1)p^{k_1 - k_2}(z_1, z_2)p^{k_1 - k_2}(z_1, z_2)p^{k_2}(x, z_2),
\]

so that \( \mathbb{E}[Z]^2 \leq Y_1 + Y_2 + Y_3 \). We start with bounding \( Y_1 \). We split the summation over \( z_1, z_2 \) into two parts:

(I) \( \|z_2 - z_1\| \leq |k_1 - k_2|^{1/2} \);

(II) \( \|z_2 - z_1\| > |k_1 - k_2|^{1/2} \).

For the bounds we are going to require \( n_0 \geq 2n_1 \). We first bound case (I), and initially restrict to \( |k_1 - k_2| \geq n_1 \) where \( n_1 \) is from Lemma 3.11. Using Lemma 3.14 and \( k_1 \geq \delta n/2 \geq n_0/2 \geq n_1 \) in the first step, the sum over \( z_1, z_2 \)
in \( Y_1 \) is at most
\[
\sum_{z_2 \in \mathbb{Z}^d} \sum_{z_1 : \|z_1 - z_2\| \leq |k_1 - k_2|^{1/2}} p^{k_1}(0, z_1) p^{k_2}(x, z_2) |k_1 - k_2|^{2-d} \\
\leq \frac{C}{D^d(\delta n)^{d/2}} \sum_{z_2 \in \mathbb{Z}^d} \sum_{z_1 : \|z_1 - z_2\| \leq |k_1 - k_2|^{1/2}} p^{k_2}(x, z_2) |k_1 - k_2|^{2-d} \\
\leq \frac{C}{(\delta n)^{d/2}} \sum_{z_2 \in \mathbb{Z}^d} p^{k_2}(x, z_2) |k_1 - k_2|^{2-d} = \frac{C|k_1 - k_2|^{2-d}}{(\delta n)^{d/2}}. \tag{3.19}
\]

Now we sum this over \( k_1, k_2 \) and get a bound of \( C(\delta n)^{4-d} \). Similarly, when summing over \( k_1, k_2 \) satisfying \( |k_1 - k_2| \leq n_1 \) we get a bound of \( C(\delta n)^{1-d/2} \) which is negligible since \( d < 6 \). Putting all these together gives a contribution to \( Y_1 \) from case 1 that is of order \( D^{-2d} \sigma^8(\delta n)^{6-d} \).

In case (II) we initially restrict to \( \|z_1 - z_2\| \geq L_1 \). We have
\[
\sum_{z_2 \in \mathbb{Z}^d} \sum_{z_1 : \|z_1 - z_2\| > |k_1 - k_2|^{1/2}} p^{k_1}(0, z_1) p^{k_2}(x, z_2) \|z_1 - z_2\|^{4-2d} \\
\leq \frac{C}{D^d(\delta n)^{d/2}} \sum_{z_2 \in \mathbb{Z}^d} \sum_{z_1 : \|z_1 - z_2\| > |k_1 - k_2|^{1/2}} p^{k_2}(x, z_2) \|z_1 - z_2\|^{4-2d} \\
\leq \frac{C}{(\delta n)^{d/2}} \sum_{z_2 \in \mathbb{Z}^d} p^{k_2}(x, z_2) \|z_1 - z_2\|^{(4-d)/2} = \frac{C|k_1 - k_2|^{2-d}}{(\delta n)^{d/2}}.
\]

The case \( \|z_1 - z_2\| \leq L_1 \) is dealt with similarly, and all together we get that \( n_9 \) can be chosen in such a way that
\[ Y_1 \leq CD^{-2d} \sigma^8(\delta n)^{6-d}. \]

Very similar calculations yield that
\[ Y_2 \leq CD^{-2d} \sigma^6(\delta n)^{3-d/2}, \quad Y_3 \leq CD^{-2d} \sigma^4(\delta n)^{3-d/2}, \]
concluding the proof.

**Proof of Theorem 4** The first part of the theorem is just a combination of Lemmas 3.3, 3.4, 3.5, 3.6 and 3.7 where we take
\[ n_3 = \max\{n_5, n_5', 6n_6(3, \sigma^2, C_3), n_7, n_8(p^1, K), n_9(\sigma^2, C_3, p^1)\}. \]

For the second part of the theorem we now choose \( c_0 = c/2 \), where \( c \) is the constant in the lower bound on the first moment in Theorem 6. Then the second statement of Theorem 4 follows immediately from Theorem 5 together with the inequality
\[ \mathbb{P}\left(V \geq \frac{1}{2}\mathbb{E}V\right) \geq \frac{(\mathbb{E}V)^2}{4\mathbb{E}V^2}, \]
valid for any non-negative random variable \( V \). \( \square \)
4. Analysis of tree bad blocks

In this section we bound the resistance between $\Phi(X_i)$ and $\Phi(X_{i+K})$ conditioned on one of the good events in Definition 2.2 not occurring. We will give a bound in terms of the following quantity, which later we will bound inductively. For any $k \leq n$ define

$$\bar{\gamma}(k; (x, n)) = \sum_{y \in \mathbb{Z}^d} \frac{p^{k}(o, y)p^{n-k}(y, x)}{p^n(o, x)} \gamma(k, y).$$ (4.1)

For $a = 1, \ldots, 6$ we define $\mathcal{E}_{(a)}$ to be the event that conditions (1) to $(a-1)$ in Definitions 2.2 and 2.3 are satisfied, but condition $(a)$ is not. Then we may write the disjoint union

$$\mathcal{A}(i)^c \cup \mathcal{B}(i, c_0)^c = \bigcup_{a=1}^{6} \mathcal{E}_{(a)} \cup (\mathcal{A}(i) \cap \mathcal{B}(i, c_0))^c.$$ 

Recall the constants $n_5, n_7$ introduced in (3.2) Lemma 3.2.

**Lemma 4.1.** There exist $C_4 > 0$ and $\delta_2 > 0$ such that

$$\mathbb{E} \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{E}_{(1)}, \Phi(V_n) = (x, n) \right]$$

$$\leq (1 + C_4\delta)\bar{\gamma}(\delta n; (x, n)) + (K - 1)\bar{\gamma}(\delta n; (x, n))$$

whenever $0 < \delta < \delta_2$, $\delta n \geq \max\{n_5(C_3), n_7(C_3)\}$.

**Proof.** By the triangle inequality of effective resistance (1.2) we have

$$\mathbb{E} \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{E}_{(1)}, \Phi(V_n) = (x, n) \right]$$

$$\leq \sum_{i'=i}^{i+K-1} \mathbb{E} \left[ R_{\text{eff}}(\Phi(X_{i'}) \leftrightarrow \Phi(X_{i'+1})) \mid \mathcal{E}_{(1)}, \Phi(V_n) = (x, n) \right].$$ (4.2)

The terms $i' = i + 1, \ldots, i + K - 1$ are not affected by the conditioning on $\mathcal{E}_{(1)}$, and hence we get the $(K - 1)\bar{\gamma}(\delta n; (x, n))$ term. So it remains to prove that

$$\mathbb{E} \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1})) \mathbf{1}_{\mathcal{E}_{(1)}} \mid \Phi(V_n) = (x, n) \right]$$

$$\leq (1 + O(\delta))\bar{\gamma}(\delta n; (x, n))\mathbb{P}(\mathcal{E}_{(1)} \mid \Phi(V_n) = (x, n)).$$

If $\mathcal{E}_{(1)}$ occurs, then precisely one of the following three disjoint events must happen:

(i) There are no levels in $[i\delta n, (i + 1)\delta n)$ that reach height $(i + 2)\delta n$,
(ii) There are more than one such levels,
(iii) There is a unique such level $\ell_1$ but $\ell_1 \notin [(i + 1/4)\delta n, (i + 3/4)\delta n]$.

We handle each of these separately. If (i) occurs, then the trees emanating from each level are conditioned not to reach level $(i + 2)\delta n$. Hence,

$$\mathbb{E} \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1})) \mathbf{1}_{(i)} \mid \Phi(V_n) = (x, n) \right]$$

$$\leq \bar{\gamma}(\delta n; (x, n))\mathbb{P}(\mathcal{E}_{(1)})\mathbb{P}((i) \mid \Phi(V_n) = (x, n)).$$
since in the definition of $\gamma(\delta n; (x,n))$ we take a supremum over $m \geq 2\delta n$.

In handling the event (ii), the following notation will be convenient. We write $R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1}))$ for the effective resistance evaluated in a given graph $G$. If (ii) occurs, then let $j_1, \ldots, j_k$ be the set of levels in $[i\delta n, (i+1)\delta n]$ such that $k \geq 2$ and $T_{n,m}(j_s)$ reaches level $(i+2)\delta n$ but not level $m$ for all $s = 1, \ldots, k$ and denote by $F(j_1, \ldots, j_k)$ this event. Let $T_{n,\infty}$ be defined as $T_{n,m}$ only without the conditioning on the side branches. We have 

$$\mathbb{E}\left[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1})) \mid (ii), \Phi(V_n) = (x,n)\right] = \sum_{k \geq 2} \mathbb{E}\left[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1})) \mid F(j_1, \ldots, j_k), \Phi(V_n) = (x,n)\right] \times P(F(j_1, \ldots, j_k) \mid (ii)), $$

since the events in question require that all side branches emanating from $V_{i\delta n}$ to $V_{(i+K)\delta n}$ do not reach level $m$. During the rest of the proof of (ii) we work where $T_{n,\infty}$ is the background measure.

Write $F'(j_1, \ldots, j_k)$ for the same event as $F(j_1, \ldots, j_k)$ except that the trees $T_{n,\infty}(j_s)$ are now only required to reach level $(i+2)\delta n$ (and may perhaps reach level $m$ as well). Since $F \subseteq F'$ we have

$$\mathbb{E}\left[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1}))1_{F(j_1, \ldots, j_k)} \mid \Phi(V_n) = (x,n)\right] \leq \mathbb{E}\left[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1}))1_{F'(j_1, \ldots, j_k)} \mid \Phi(V_n) = (x,n)\right].$$

(4.3)

Since $F'$ is an increasing event and $R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1}))$ is a decreasing random variable, the FKG inequality \cite{FKG, Gr99} implies that the right hand side of (4.3) is at most

$$\mathbb{E}\left[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1})) \mid \Phi(V_n) = (x,n)\right] \mathbb{P}(F'(j_1, \ldots, j_k)) \leq \mathbb{P}(F'(j_1, \ldots, j_k)) \gamma(\delta n; (x,n)),$$

where in the last step we are using that $T_{n,\infty}$ is the weak limit as $m \to \infty$ of $T_{n,m}$.

We need to bound the ratio between the probability of $F$ and $F'$. Write $N$ for the total number of progeny at level $(i+2)\delta n$ of $T_{n,\infty}(j_1), \ldots, T_{n,\infty}(j_k)$. Then,

$$\mathbb{P}(F) \geq \mathbb{P}(F')\mathbb{E}\left[(1 - \theta(m - (i+2)\delta n))^N \mid F'\right] \geq \mathbb{P}(F')\mathbb{E}\left[(1 - 6(\sigma^2 n)^{-1})^N \mid F'\right],$$

where the last inequality is by $m - (i+2)\delta n \geq (1 - 2\delta)n \geq n/2 \geq n_5$ and our estimate on $\theta$ \cite{Gr99}. Note that $N = N^{(1)} + \ldots + N^{(k)}$ where $N^{(1)}, \ldots, N^{(k)}$ are independent and $N^{(s)}$ has the distribution of $\tilde{N}_{(i+2)\delta n - j_s}$, $s = 1, \ldots, k$. 


Hence,

\[ \mathbb{E}\left[ (1 - 6(\sigma^2 n)^{-1})^N \mid \mathcal{F} \right] \geq \prod_{s=1}^{k} \mathbb{E}\left[ (1 - 6(\sigma^2 n)^{-1})^{\mathcal{N}(s)} \mid \mathcal{N}(s) > 0 \right] \]

\[ \geq \prod_{s=1}^{k} \mathbb{E}\left[ (1 - \mathcal{N}(s)6(\sigma^2 n)^{-1}) \mid \mathcal{N}(s) > 0 \right] \]

\[ \geq (1 - O(\delta))^k, \]

since \( \mathbb{E}[\mathcal{N}(s)|\mathcal{N}(s) > 0] = \mathbb{E}[\mathcal{N}_2(\delta n - j_s) > 0]^{-1} = O(\sigma^2 \delta n) \) when \( \delta n \geq n_\gamma \), by (3.5). Hence,

\[ \mathbb{P}(\mathcal{F}'(j_1, \ldots, j_k)) \leq (1 + O(\delta))^k \mathbb{P}(\mathcal{F}(j_1, \ldots, j_k)). \]

Therefore,

\[
\mathbb{E}\left[ \text{R}_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1})) \mid \mathcal{F}(j_1, \ldots, j_k), \Phi(V_n) = (x,n) \right]
\leq (1 + O(\delta))^k \bar{\gamma}(\delta n; (x,n)).
\]

In the tree \( T_{n,\infty} \), and hence in the tree \( T_{n,m} \), the number of vertices \( V_k \) on the backbone that reach \((i + 2)\delta n\) is stochastically bounded above by a Binomial random variable with parameters \( \delta n \) and \( p = \frac{c}{n} \), by (3.5). Hence, the probability that there are precisely \( k \) such vertices is at most \( e^{-ck} \) for some \( c > 0 \). We get that as long as \( \delta > 0 \) is small enough (as a function of \( c \)) we have

\[
\mathbb{E}\left[ \text{R}_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1})) 1_{(\text{ii})} \mid \Phi(V_n) = (x,n) \right]
\leq (1 + O(\delta))^k \bar{\gamma}(\delta n; (x,n)) \mathbb{P}(1_{(\text{ii})} \mid \Phi(V_n) = (x,n)),
\]

concluding the analysis of (ii).

If (iii) occurs, then there is a unique \( \ell_1 \) which reaches level \((i + 2)\delta n\) but not \( m \) and all other levels do not reach level \((i + 2)\delta n\). A similar analysis as in (ii) with \( k = 1 \) using the FKG inequality gives that

\[
\mathbb{E}\left[ \text{R}_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+1})) 1_{(\text{iii})} \mid \Phi(V_n) = (x,n) \right]
\leq (1 + O(\delta))^k \bar{\gamma}(\delta n; (x,n)) \mathbb{P}(1_{(\text{iii})} \mid \Phi(V_n) = (x,n)).
\]

\[ \square \]

**Lemma 4.2.**

\[
\mathbb{E}\left[ \text{R}_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{E}(2), \ell_1, \Phi(V_n) = (x,n) \right]
\leq \bar{\gamma}(\ell_1 - i \delta n; (x,n)) + 1 + \bar{\gamma}((i + 1)\delta n - \ell_1 - 1)
+ (K - 1)\bar{\gamma}(\delta n; (x,n)).
\]
Proof. As in the previous lemma, we use the triangle inequality as in \((4.2)\) with \(E(1)\) now replaced by \(E(2)\). Again, the terms containing \(R_{\text{eff}}(\Phi(X_{i'}) \leftrightarrow \Phi(X_{i'+1}))\) for \(i' = i + 1, \ldots, i + K - 1\) are unaffected by the conditioning, and hence contribute the term \((K - 1)\bar{\gamma}(\delta n; (x, n))\). The rest of the lemma is much easier than the previous one, since on the event that Definition 2.2(1) is satisfied, the backbone \(V_{\delta n}, \ldots, V_{\ell_1}\) together with its side branches (not counting the side branch of \(V_{\ell_1}\)) is distributed as \(T_{\ell_1 - i\delta n, 2\delta n}\), and the backbone \(V_{\ell_1 + 1}, \ldots V_{(i + 1)\delta n}\) together with its side branches (again, not counting the side branch of \(V_{(i + 1)\delta n}\)) is distributed as \(T_{(i + 1)\delta n - \ell_1 - 1, 2\delta n - \ell_1 - 1}\). Hence we get

\[
\mathbb{E}\left[R_{\text{eff}}(\Phi(X_{i}) \leftrightarrow \Phi(X_{i+1})) \mid E(2), \ell_1, \Phi(V_n) = (x, n)\right] \\
\leq \bar{\gamma}(\ell_1 - i\delta n; (x, n)) + 1 + \bar{\gamma}((i + 1)\delta n - \ell_1 - 1),
\]
as required. \(\square\)

Lemma 4.3. We have

\[
\mathbb{E}\left[R_{\text{eff}}(\Phi(X_{i}) \leftrightarrow \Phi(X_{i+K})) \mid E(3), \ell_1, \Phi(V_n) = (x, n)\right] \\
\leq \bar{\gamma}(\ell_1 - i\delta n; (x, n)) + 1 + \bar{\gamma}((i + 1)\delta n - \ell_1 - 1) \\
+ (K - 2)\bar{\gamma}(\delta n; (x, n)) + (1 + C_4\delta)\bar{\gamma}(\delta n; (x, n))
\]
whenever \(0 < \delta < \delta_2, \delta n \geq \max\{n_5(C_3), n_7(C_3)\}\).

Proof. We again start with the triangle inequality as in \((4.2)\), with \(E(1)\) now replaced by \(E(3)\). An argument almost identical to that of Lemma 4.1 yields that the term involving \(R_{\text{eff}}(\Phi(X_{i+K-1}) \leftrightarrow \Phi(X_{i+K}))\) is bounded by \((1 + C_4\delta)\bar{\gamma}(\delta n; (x, n))\). The rest of the terms are bounded as in Lemma 4.2. \(\square\)

Lemma 4.4.

\[
\mathbb{E}\left[R_{\text{eff}}(\Phi(X_{i}) \leftrightarrow \Phi(X_{i+K})) \mid E(4), \ell_1, \ell_2, \Phi(V_n) = (x, n)\right] \\
\leq \bar{\gamma}(\ell_1 - i\delta n; (x, n)) + 1 + \bar{\gamma}((i + 1)\delta n - \ell_1 - 1) \\
+ (K - 2)\bar{\gamma}(\delta n; (x, n)) + \bar{\gamma}(\ell_2 - ((i + K - 1)\delta n; (x, n)) + 1 \\
+ \bar{\gamma}((i + K)\delta n - \ell_2 - 1; (x, n)).
\]

Proof. Similarly to the proof of Lemma 4.2 we bound the resistance using subgraphs that conditioned on Definition 2.2(1),(2),(3) holding (and conditioned on the values of \(\ell_1, \ell_2\) are independent of whether (4) holds or not. \(\square\)

5. Analysis of spatially bad blocks

In this section we analyze what happens when condition (5) or (6) in Definition 2.3 fails, that is, some spatial displacement is “not typical”, and
also what happens when $B(i, c_0)$ fails. Let us introduce some notation. We write $G_{\text{tree}}$ for the event
\[ G_{\text{tree}} = \{(1)-(4), \ell_1, \ell_2, \Phi(V_n) = (x, n)\}. \]
We define a set of times $i\delta n = T_0 < T_1 < \cdots < T_{K+4} = (i + K)\delta n$, time differences $t_1, t_2, \ldots, t_{K+4}$ and spatial locations $z_0, \ldots, z_{K+4} \in \mathbb{Z}^d$ by
\[
\begin{align*}
z_0 &= x_i & T_0 &= i\delta n \\
z_1 &= v_{\ell_1} & t_1 &= \ell_1 - i\delta n & T_1 &= \ell_1 \\
z_2 &= v_{\ell_1+1} & t_2 &= 1 & T_2 &= \ell_1 + 1 \\
z_3 &= x_{i+1} & t_3 &= (i + 1)\delta n - \ell_1 - 1 & T_3 &= (i + 1)\delta n \\
z_4 &= x_{i+2} & t_4 &= \delta n & T_4 &= (i + 2)\delta n \\
z_5 &= x_{i+3} & t_5 &= \delta n & T_5 &= (i + 3)\delta n \\
\vdots & & \vdots & & \vdots \\
z_{K+1} &= x_{i+K-1} & t_{K+1} &= \delta n & T_{K+1} &= (i + K - 1)\delta n \\
z_{K+2} &= v_{\ell_2} & t_{K+2} &= \ell_2 - (i + K - 1)\delta n & T_{K+2} &= \ell_2 \\
z_{K+3} &= v_{\ell_2+1} & t_{K+3} &= 1 & T_{K+3} &= \ell_2 + 1 \\
z_{K+4} &= x_{i+K} & t_{K+4} &= K\delta n - \ell_2 - 1 & T_{K+4} &= (i + K)\delta n
\end{align*}
\]
Observe that conditional on $G_{\text{tree}}$, the times $T_s$ and time differences $t_s$ are non-random but the spatial locations $z_s$ are random. Furthermore, we define for any $s = 1, \ldots, K+4$
\[
q_s(z) = \sum_{\substack{\|y_r\| \leq \sqrt{T_r} \\|y_{s-1}\| = \sqrt{T_{s-1}}} \atop r = 1, \ldots, s-1} \prod_{r=1}^{s-1} p_{\ell-r}(0, y_r).
\]
Finally, for any $s = 1, \ldots, K+4$ we define the event $E_s^{(5)}$ by
\[
E_s^{(5)} = \left\{ \left\{ \|z_r - z_{r-1}\| \leq \sqrt{T_r} \right\} \cap \left\{ \|z_s - z_{s-1}\| > \sqrt{T_s} \right\} \right\}.
\]
Note that
\[
P(E_s^{(5)} \mid G_{\text{tree}}) = \sum_{z,y: \|y\| > \sqrt{T_s}} q_s(z) \frac{p_{\ell-s}(z, z+y)p^{n-(T_s-T_0)}(z+y, x)}{p^n(o, x)}.
\]
\[ \mathcal{R}_{s,s'} \leq \sum_z q_s(z) \sum_{y: \|y\| > \sqrt{s}} \frac{p^{ts}(o,y)p^{n-T_s+yT_0}(y,x-z)}{p^n(o,x)} \gamma(t_s, y), \]

when \( s' = s, s' \neq 2, K + 3, \)

\[ \mathcal{R}_{s',s} \leq \sum_{z: y_s, y_s' \neq 0,\|y\| > \sqrt{T_s}} q_s(z) p^{ts}(o,y_s)p^{t's}(o,y_s)p^{n-T_s}(y,x-z-y_s) \gamma(t_s', y), \]

when \( s' > s, s' \neq 2, K + 3. \)

Proof. The case \( s' = 2, K + 3 \) is trivial, so assume \( s' \neq 2, K + 3. \) The case \( s' < s \) is easy. Condition on \( E^{s}(5) \) and on the spatial locations \( z_0, z_1, \ldots, z_s \) such that \( E^{s}(5) \) holds. Since \( \|z_s' - z_{s'-1}\| \leq \sqrt{T_s} \) we may bound the resistance between the corresponding points by \( \gamma(t_s'). \)

In order to handle the case \( s' = s \), we condition on \( z_0, z_1, \ldots, z_s \) such that \( E^{s}(5) \) holds. With this conditioning the required resistance is bounded above by \( \gamma(t_s, z_s - z_{s-1}) \). So the required expectation is bounded above by

\[ \sum_{z_0, z_1, \ldots, z_s: \|z_r - z_{r-1}\| \leq \sqrt{T_r}, \quad r = 1, \ldots, s-1} \sum_{\|z_s - z_{s-1}\| > \sqrt{T_s}} \frac{p^{T_0}(o,z_0)}{p^n(o,x)} \left[ \prod_{r=1}^{s} p^{l_r}(z_{r-1}, z_r) \right] p^{n-T_s}(z_s, x) \gamma(t_s, z_s - z_{s-1}). \]

By changing variables \( y_1 = z_1 - z_0, y_2 = z_2 - z_1, \ldots, y_{s-1} = z_s - z_{s-2} \) and \( y = z_s - z_{s-1} \) and \( z = y_1 + \cdots + y_{s-1} \) this equals

\[ \sum_{z_0, y_1, \ldots, y_{s-1}, y} \frac{p^{T_0}(o,z_0)}{p^n(o,x)} \left[ \prod_{r=1}^{s-1} p^{l_r}(o,y_r) \right] p^{l_s}(o,y)p^{n-T_s}(z_0 + z + y, x) \gamma(t_s, y), \]

where \( \sum' \) indicates the restriction \( \|y_1\| \leq \sqrt{T_1}, \ldots, \|y_{s-1}\| \leq \sqrt{T_s}, \|y\| > \sqrt{T_s}. \) By summing over \( z_0 \) this simplifies to

\[ \sum_z q_s(z) \sum_{y: \|y\| > \sqrt{T_s}} \frac{p^{l_s}(o,y)p^{n-T_s+yT_0}(y,x-z)}{p^n(o,x)} \gamma(t_s, y) \]

as required.

The case \( s' > s \) is done similarly. The required expectation is bounded above by

\[ \sum_{z_0, z_1, \ldots, z_s: \|z_r - z_{r-1}\| \leq \sqrt{T_r}, \quad r = 1, \ldots, s-1} \sum_{\|z_s - z_{s-1}\| > \sqrt{T_s}} \frac{p^{T_0}(o,z_0)}{p^n(o,x)} \left[ \prod_{r=1}^{s} p^{l_r}(z_{r-1}, z_r) \right] p^{T_{s-1}-T_s}(z_s, z_{s'-1}) \]

\[ \times p^{l_{s'}}(z_{s'-1} + y)p^{n-T_{s'}}(z_{s'-1} + y, x) \gamma(t_{s'}, y), \]
Summing over $z_0, z_{s'-1}$ and recalling (5.1) simplifies this to
\[
\sum_{z \neq y_s, y_{s'}} q_s(z) \frac{p_s o, y_s p_{s'} o, y_p^n o, y p_{n-t_{s'-T_s}}(y, x - z - y_s)}{p^n o, x} \gamma(t_{s'}, y)
\]

Next, to handle part (6) of Definition 2.4 recall that we defined
\[
\mathcal{E}_{(6)} = \bigcap_{s=1}^{K+4} \{ ||z_s - z_{s-1}|| \leq \sqrt{T_s} \} \bigcap \{ \text{one of the conditions in (6) fails} \}.
\]
We also define $\mathcal{E}_{(7)} = \mathcal{A}(i) \cap \mathcal{B}(i, c_0)^c$.

**Lemma 5.2.**
\[
\mathbb{E} \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{E}_{(6)} \cup \mathcal{E}_{(7)}, \mathcal{G}_{\text{tree}} \right]
\leq (K - 2)\gamma(\delta n) + \gamma(\ell_1 - i\delta n) + 1 + \gamma((i + 1)\delta n - \ell_1 - 1)
+ \gamma(\ell_2 - (i + K - 1)\delta n) + 1 + \gamma((i + K)\delta n - \ell_2 - 1).
\]

**Proof.** Condition on $\mathcal{G}_{\text{tree}}$ and $\mathcal{E}_{(6)} \cup \mathcal{E}_{(7)}$. We have that $||z_s - z_{s-1}|| \leq \sqrt{T_s}$ for all $s = 1, \ldots, K + 4$. Hence, under this conditioning, we may bound the resistance between $(z_{s-1}, T_{s-1})$ and $(z_s, T_s)$ by $\gamma(t_s)$, concluding the proof.

We close this section with a bound on the resistance on the “final stretch” between $X_{i_{\text{last}}}$ and $V_n$, where $i_{\text{last}} = K(\lfloor(K\delta)^{-1}\rfloor - 1)$. Observe that $K\delta n \leq n - i_{\text{last}}\delta n < 2K\delta n$, and write
\[n - i_{\text{last}}\delta n = K'\delta n + \tilde{n},\]
where $K \leq K' \leq 2K - 2$ and $\delta n \leq \tilde{n} < 2\delta n$.

**Lemma 5.3.**
\[
\mathbb{E} \left[ R_{\text{eff}}(\Phi(X_{i_{\text{last}}}) \leftrightarrow \Phi(V_n)) \mid \Phi(V_n) = (x, n) \right]
\leq K'\gamma(\delta n; (x, n)) + \tilde{\gamma}(\tilde{n}; (x, n)).
\]

**Proof.** This follows from the triangle inequality for resistance.

6. **Analysis of good blocks**

In this section we will estimate expectations of resistances given the event
\[
\mathcal{G}_{\text{good}} = \{ \Phi(V_n) = (x, n), \mathcal{A}(i), \mathcal{B}(i, c_0), \ell_1, \ell_2 \}.
\]

**Lemma 6.1.** **Conditional on $\mathcal{G}_{\text{good}}$, we have**
\begin{enumerate}
  \item \( \mathbb{E} \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(V_{\ell_1})) \mid \mathcal{G}_{\text{good}} \right] \leq \gamma(\ell_1 - i\delta n). \)
  \item \( \mathbb{E} \left[ R_{\text{eff}}(\Phi(V_{\ell_1}) \leftrightarrow \Phi(X_{i+1})) \mid \mathcal{G}_{\text{good}} \right] \leq \gamma((i + 1)\delta n - \ell_1 - 1) + 1. \)
  \item For all $i + 1 \leq j \leq i + K - 2$ we have
    \[ \mathbb{E} \left[ R_{\text{eff}}(\Phi(X_j) \leftrightarrow \Phi(X_{j+1})) \mid \mathcal{G}_{\text{good}} \right] \leq \gamma(\delta n). \)
\end{enumerate}
(iv) $\mathbb{E}[R_{\text{eff}}(\Phi(X_{i+K-1}) \leftrightarrow \Phi(V_{\ell_2})) \mid G_{\text{good}}] \leq \gamma(\ell_2 - (i + K - 1)\delta n)$.

(v) $\mathbb{E}[R_{\text{eff}}(\Phi(V_{\ell_2}) \leftrightarrow \Phi(X_{i+K})) \mid G_{\text{good}}] \leq \gamma((i + K)\delta n - \ell_2 - 1) + 1$.

(vi) $\mathbb{E}[R_{\text{eff}}(\Phi(V_{\ell_2}) \leftrightarrow \Phi(X'_{i+K})) \mid G_{\text{good}}] \leq \gamma((i + K)\delta n - \ell_2 - 1) + 1$.

(vii) For all $i + 1 \leq j \leq i + K - 1$ we have $\mathbb{E}[R_{\text{eff}}(\Phi(Y_j) \leftrightarrow \Phi(Y_{j+1})) \mid G_{\text{good}}] \leq \gamma(\delta n)$. 

Proof. The proof of (i), (iii), (iv) and (viii) is immediate by Definition 2.4. The other estimates follow almost as quickly by Definition 2.4 and triangle inequality for resistance. □

Recall the constant $n_1$ from Lemma 1.1(i) and the constant $n_9$ from Theorem 6.

Lemma 6.2. Assume $d = 5$. There exists $C_5 < \infty$ such that we have

$$\mathbb{E}[R_{\text{eff}}(\Phi(Y_j) \leftrightarrow \Phi(Y_{j+1})) \mid G_{\text{good}}] \leq C_5 \max_{1 \leq \ell \leq \delta n} \gamma(\ell)$$

whenever $\delta n \geq \max\{n_1(p^1), n_9(\sigma^2, C_3, p^1)\}$.

For convenience we will prove Lemma 6.2 under the assumption that there exists an $M$ such that the progeny distribution is bounded by $M$ with probability 1. Then by taking $M \to \infty$ and keeping $n$ fixed we obtain Lemma 6.2 in our usual generality. This is possible, since $C_5$ does not depend on $M$, and the restriction on $\delta n$ only depends on $\sigma^2, C_3$, so it is sufficient to approximate $\{p(k)\}$ by some $\{p_M(k)\}$ in such a way that

$$1 = \sum_{k=0}^{M} p_M(k) = \sum_{k=0}^{M} kp_M(k),$$

$$\sigma^2 = \lim_{M \to \infty} \sum_{k \geq 1} k(k - 1)p_M(k),$$

$$C_3 \geq \sup_{M \geq 1} \sum_{k \geq 1} k^3p_M(k).$$

Therefore in the rest of this section we assume the bound $M$.

Given any $n$ and $m$ such that $m \geq 2n$ we regard the random tree $T_{n,m}$ as a subtree of an infinite $M$-ary tree $T_M$ with root $\rho$ as follows: the root of $T_{n,m}$ is mapped to $\rho$ and if $W$ is a vertex of $T_{n,m}$ with $k$ children we map the $k$ edges randomly amongst the $\binom{M}{k}$ possible choices in $T_M$. Denote by $V_n \in T_M$ the random vertex where the last backbone vertex of $T_{n,m}$ was mapped to. The triple $(T_{n,m}, \rho, V_n)$ is a doubly rooted tree. Define

$$q(k) := \binom{M}{k}^{-1} p(k).$$
Lemma 6.3. For a fixed triple \((t, \rho, V)\) where \(t \subset T_M\) is a tree and \(V \in T_M\) at height \(n\) such that \(t\) does not reach level \(m\) and \(V\) has no children in \(t\), we have
\[
\mathbf{P}(\mathbf{T}_{n,m}, \rho, V_n) = (t, \rho, V) \propto \prod_{W \in t \setminus \{V_0, \ldots, V_n\}} \left( \begin{array}{c} \delta_n - 1 \\ \deg_t^+(V_i) - 1 \\ \deg_t^+(W) \end{array} \right),
\]
where \(\deg_t^+(W)\) is the number of children of \(W\) in \(t\).

Proof. Let \(\rho = V_0, V_1, \ldots, V_n = V\) be the unique path in \(t\) from \(\rho\) to \(V\). The probability that \(T_{n,m} = t\) with this backbone equals
\[
\frac{1}{Z} \prod_{i=0}^{n-1} \tilde{\rho}(\deg_t^+(V_i) - 1) \cdot \prod_{W \in t \setminus \{V_0, \ldots, V_n\}} p(\deg_t^+(W)) ,
\]
where \(Z = \prod_{i=0}^{n-1} \theta(m - i)\). Hence, the probability that \((T_{n,m}, \rho, V_n) = (t, \rho, V)\) as subtrees of \(T_M\) equals
\[
\frac{1}{Z} \prod_{i=0}^{n-1} \left( \begin{array}{c} M - 1 \\ \deg_t^+(V_i) - 1 \\ \deg_t^+(W) \end{array} \right)^{-1} \tilde{\rho}(\deg_t^+(V_i) - 1) \prod_{W \in t \setminus \{V_0, \ldots, V_n\}} \left( \begin{array}{c} M \\ \deg_t^+(W) \end{array} \right)^{-1} p(\deg_t^+(W)) .
\]
Manipulating with \(\tilde{\rho}(k - 1) = kp(k)\) finishes the proof. \(\square\)

For the statement of the next lemma we fix
\[
0 \leq k_1 \leq \delta n - 1 \quad \quad k_1 + 1 \leq h_u \leq \delta n
\]
Given \(V \in T_M\) at level \(\delta n\) and \(U \in T_M\) at level \(h_u\) let \(Z \in T_M\) be the highest common ancestor of \(V\) and \(U\) and let \(Z^+\) be the unique child of \(Z\) leading towards \(U\). Given a tree \(t \subset T_M\) such that \(V, U \in t\) and \(V\) does not have any children in \(t\), we have a unique decomposition of \(t\) into edge disjoint trees \((t^A, \rho, Z), (t^B, Z^+, U), t^C\) and \(t^D\), see figure 3. The doubly rooted tree \((t^A, \rho, Z)\) contains all the descendants of \(\rho\) that are not descendants of \(Z\). The doubly rooted tree \((t^B, Z^+, U)\) contains all the descendants of \(Z^+\) that are not descendants of \(U\). The tree \(t^C\) contains all the descendants of \(U\) and finally the tree \(t^D\) contains all other edges, namely, all the descendants of \(Z\) that are not descendants of \(Z^+\) (in particular, the edge \(Z, Z^+\) is in \(t^D\)).

For \(W \in T_M\) let \(\Theta_W\) denote the tree isomorphism that takes \(W\) to \(\rho\) and the descendants subtree of \(W\) onto \(T_M\).

Lemma 6.4. Let \(V, U \in T_M\) be at heights \(\delta n\) and \(h_u\), respectively and \((T, \rho, V)\) be distributed as \((T_{\delta n, 2\delta n}, \rho, V_{\delta n})\). Conditionally on the event \(\{V = V, U \in T\}\) we have that
\[
(T^A, \rho, Z) \overset{d}{=} (T_{k_1, 2\delta n}, \rho, V_{k_1}) \mid V_{k_1} = Z,
\]
and

\[ \Theta_{Z^+}( (T^B, Z^+, U) ) \overset{d}{=} (T_{h_u-k_1-1,2\delta n-k_1-1}, \rho, V_{h_u-k_1-1}) \mid V_{h_u-k_1-1} = \Theta_{Z^+}(U). \]

**Proof.** For any \( t \subset T_M \) that contains \( V \) and \( U \) (and \( V \) has no children in \( t \)) by lemma 6.3 we have

\[ P((T, \rho, V) = (t, \rho, V)) \propto \prod_{W \in t, W \neq V} q(\deg_t^+(W)). \]

We factorize the right hand side so it equals

\[ \prod_{W \in t^A, W \neq Z} q(\deg_t^+(W)) \cdot \prod_{W \in t^B, W \neq U} q(\deg_t^+(W)) \cdot \prod_{W \in t^C} q(\deg_t^+(W)) \cdot \prod_{W \in t^D} q(\deg_t^+(W)). \]

By summing over all the possible values of \( t^B, t^C \) and \( t^D \) we get that

\[ P((T^A, \rho, Z) = (t^A, \rho, Z)) \propto \prod_{W \in t^A, W \neq Z} q(\deg_t^+(W)), \]
which gives the claim for $\mathcal{T}^A$ by Lemma 6.3. The same argument works similarly for $\mathcal{T}^B$ noting that under the shift $\Theta_{Z^+}$ the degrees do not change. □

Proof of Lemma 6.2. All our expectations in the following proof are conditioned on the event $A(i), \Phi(V_n) = (x, n)$.

Let $(T_1, \rho, V_1)$ and $(T_2, \rho, V_2)$ be two independent copies of $(T_{\delta n}, 2\delta n, \rho, V_{\delta n})$ randomly embedded into $T_M$ as before. Conditionally on $A(i)$ let $\Phi_1$ and $\Phi_2$ be independent random walk mappings of $T_M$ such that $\Phi_1(\rho) = \Phi'(X_{i+K})$ and $\Phi_2(\rho) = \Phi(Y_{i+K})$, so that $\|\Phi_1(\rho) - \Phi_2(\rho)\| \leq \sqrt{\delta n}$. In this way, the required quantity $R_{\text{eff}}(\Phi(X_{i+K}) \leftrightarrow \Phi(Y_{i+K}))$ is distributed as $R_{\text{eff}}(\Phi_1(\rho) \leftrightarrow \Phi_2(\rho))$ where in the latter, the resistance is computed in the graph $\Phi_1(T_1) \cup \Phi_2(T_2)$.

For notational convenience, and without loss of generality, we assume that $\Phi_1(\rho_1) = (o, 0)$, $\Phi_2(\rho_2) = (x, 0)$, with $|x| \leq \sqrt{\delta n}$. Recall the notation $Z_1, Z_1^+, Z_2, Z_2^+$ introduced after Definition 2.5 Definition 2.6 adapted to the current setting reads as follows:

Definition 6.1. We say that the vertices $U_1, U_2 \in T_M$ intersect-well if:
1. $U_1 \in T_1$ and $U_2 \in T_2$;
2. $(5/6)\delta n \leq h(U_1) = h(U_2) \leq \delta n$;
3. $(1/2)\delta n \leq h(Z_1) = h(Z_2) \leq (4/6)\delta n$;
4. $\mathcal{T}S(\rho_1, Z_1), \mathcal{T}S(Z_1^+, U_1), \mathcal{T}S(\rho_2, Z_2), \mathcal{T}S(Z_2^+, U_2)$;
5. $\Phi_1(U_1) = \Phi_2(U_2)$.

Define $\tilde{\mathcal{I}}$ by

$$\tilde{\mathcal{I}} = \{(U_1, U_2) : U_1, U_2 \text{intersect-well}\}.$$ (6.2)

Then it is clear that $|\tilde{\mathcal{I}}|$ has the same distribution as $|\mathcal{I}|$ introduced earlier. Recall that $\mathcal{B} = \mathcal{B}(i, c_0)$ is the event $\{|\mathcal{I}| \geq c_0 \frac{a_{\text{eff}}}{D^{1/2}} (\delta n)^{6-\frac{d}{2}}\}$. Conditional on $\mathcal{T}_1, \mathcal{T}_2, \Phi_1, \Phi_2$, and the event $\mathcal{B}$, draw a pair $(U_1, U_2)$ from the set $\tilde{\mathcal{I}}$, uniformly at random. This is possible, since on the event $\mathcal{B}$ we have $|\tilde{\mathcal{I}}| > 0$. Denote

$$\mathcal{R} = R_{\text{eff}}(\Phi_1(\rho_1) \leftrightarrow \Phi_2(\rho_2)).$$

Writing for short

$$1_V = 1_{V_1=v_1, V_2=v_2},$$

$$1_U = 1_{U_1=u_1, U_2=u_2},$$

we have

$$\mathbb{E}[\mathcal{R}1_\mathcal{B}] = \sum_{V_1, V_2 \in T_M} \sum_{U_1, U_2 \in T_M} \mathbb{E}[\mathcal{R}1_\mathcal{B}1_V1_U].$$ (6.3)

Recall that $V_1$ and $U_1$ determine the vertices $Z_1, Z_1^+$, and $V_2$ and $U_2$ determine $Z_2, Z_2^+$. The first sum in (6.3) is over all pairs $V_1, V_2 \in T_M$ at height
δn. The second sum is over all pairs $U_1, U_2 \in T_M$ such that

$$\delta n / 2 \leq k_1 := h(Z_1) \leq (4/6) \delta n,$$

$$\delta n / 2 \leq k_2 := h(Z_2) \leq (4/6) \delta n.$$

Given $z_1, z_1^+, z_2, z_2^+, u \in \mathbb{Z}^d$, write $1_\Phi$ for short for the indicator function of the intersection of the following six events:

$$\begin{align*}
\Phi_1(Z_1) &= (z_1, k_1) \\
\Phi_2(Z_1) &= (z_1, k_2) \\
\Phi_1(Z_1^+) &= (z_1^+, k_1 + 1) \\
\Phi_2(Z_1^+) &= (z_1^+, k_2 + 1) \\
\Phi_1(U_1) &= (u, h_u) \\
\Phi_2(U_2) &= (u, h_u)
\end{align*}$$

This allows us to rewrite (6.3) in the form:

$$E[R_{1T}] = \sum_{V_1, V_2} \sum_{U_1, U_2} \sum_{u, z_1, z_1^+, z_2, z_2^+} E[R_{1B} 1_{U_1} 1_{1_\Phi}].$$

Here the prime on the summation over $z_1, z_1^+, z_2, z_2^+, u$ indicates that these vertices are restricted to choices that are compatible with the occurrence of $TS(\rho_1, Z_1)$, $TS(Z_1^+, U_1)$, $TS(\rho_2, Z_2)$, $TS(Z_2^+, U_2)$, that is, $|z_1| \leq \sqrt{k_1}$, $|u-z_1^+| \leq \sqrt{h_u-k_1-1}$, $|z_2-x| \leq \sqrt{k_2}$, $|u-z_2^+| \leq \sqrt{h_u-k_2-1}$.

In the presence of the indicators on the right hand side of (6.4) we can also insert the indicator $1_T = 1_{U_1 \in T_1, U_2 \in T_2}$ as this event already occurs. Hence the expectation on the right hand side of (6.4) equals

$$E[R_{1B} 1_{U_1} 1_{1_\Phi}].$$

Observe that we have

$$E[1_{U} | B, (T_1, V_1), (T_2, V_2), \Phi_1, \Phi_2] = \frac{1}{|I|} \leq \frac{D^d}{c_0 \sigma^4 (\delta n)^{(6-d)/2}},$$

and that $R$ and the other indicators in (6.5) are measurable with respect to the conditioning in (6.6). Hence

$$E[R_{1B} 1_{V} 1_{U} 1_{1_\Phi}] \leq \frac{D^d}{c_0 \sigma^4 (\delta n)^{(6-d)/2}} E[R_{1B} 1_{V} 1_{U} 1_{1_\Phi}],$$

$$\leq \frac{D^d}{c_0 \sigma^4 (\delta n)^{(6-d)/2}} E[R_{1V} 1_{U} 1_{1_\Phi}].$$

In order to bound $R$ from above, we define

$$R_1 = R_{\text{eff}} (\Phi_1(\rho_1) \leftrightarrow \Phi_1(U_1))$$

and

$$R_2 = R_{\text{eff}} (\Phi_2(\rho_2) \leftrightarrow \Phi_2(U_2)),$$

and by the triangle inequality for effective resistance (1.2) we have

$$R \leq R_1 + R_2.$$
on the event $1_T1_\Phi$. Inserting this into (6.5) yields
\[
E[R1_B] \leq \frac{D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}} \sum_{V_1,V_2, U_1,U_2} \sum'_{u,z_1,z_1^++z_2,z_2^+} \left( E[R1V1T1_\Phi] + E[R2V1T1_\Phi] \right).
\]

We only analyze the term containing $R_1$, since the arguments for handling $R_2$ are identical. We bound $R_1$ from above by,
\[
R_1 \leq R_{\text{eff}}(\Phi_1(\rho_1) \leftrightarrow \Phi_1(Z_1)) + 1 + R_{\text{eff}}(\Phi_1(Z_1^+) \leftrightarrow \Phi_1(U_1)).
\]

Due to Lemma 6.4, conditioned on the events in the indicators $1_V1_T$, the distribution of $T_1^A$ is the same as that of $T_{k_1,2\delta n}$, and the distribution of $T_1^B$ is the same as the distribution of $T_{h_u-k_1-1,2\delta n-k_1-1}$. Due to the presence of the indicator $1_\Phi$, that fixes the spatial locations of $\Phi_1(Z_1), \Phi_1(Z_1^+), \Phi_1(U_1)$ (respectively) to be $z_1, z_1^+, u$ (respectively), we have
\[
E[R_{\text{eff}}(\Phi_1(\rho_1) \leftrightarrow \Phi_1(Z_1))1_V1_T1_\Phi] = \gamma(k_1,z_1)E[1_V1_T1_\Phi]
\]
\[
\leq \gamma(k_1)E[1_V1_T1_\Phi],
\]
and
\[
E[R_{\text{eff}}(\Phi_1(Z_1^+) \leftrightarrow \Phi_1(U_1))1_V1_T1_\Phi] = \gamma(h_u-k_1-1,u-z_1^+)E[1_V1_T1_\Phi]
\]
\[
\leq \gamma(h_u-k_1-1)E[1_V1_T1_\Phi].
\]

Together with analogous bounds for $R_2$, this yields
\[
E[R1_B] \leq \frac{D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}} \sum_{V_1,V_2, U_1,U_2} \sum'_{u,z_1,z_1^++z_2,z_2^+} E[1_V1_T1_\Phi]
\]
\[
\times (\gamma(k_1) + \gamma(h_u-k_1-1) + \gamma(k_2) + \gamma(h_u-k_2-1) + 2)
\]
\[
\leq \frac{(4\max_0<k<\delta n)\gamma(k) + 2)D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}} \sum_{V_1,V_2, U_1,U_2} \sum'_{u,z_1,z_1^++z_2,z_2^+} E[1_V1_T1_\Phi].
\]

We have
\[
E[1_V1_T1_\Phi] = E[1_V1_T]|p^{k_1}(o,z_1)p^{h_u-k_1-1}(z_1^+, u)
\]
\[
\times p^{k_2}(x,z_2)p^{h_u-k_2-1}(z_2^+, u).
\]

Removing the restrictions involved in the primed summation in (6.8) we can perform the convolutions of the transition probabilities and get that
\[
E[R1_B] \leq \frac{4\max_0<k<\delta n)\gamma(k) + 2)D^d}{c_0\sigma^4(\delta n)^{(6-d)/2}} E[1_V1_T]|p^{2h_u}(o,x). 
\]
By the local central limit theorem, and due to \( \|x\| \leq \sqrt{\delta n}, \ h_u \geq (5/6)\delta n \geq n_1/2 \), we have
\[
\mathbf{p}^{2h_u}(o, x) \leq \frac{C}{D^d(\delta n)^{d/2}}. 
\]

(6.10)

Now, fix \( V_1, V_2 \) and \( h_u \) and sum \( 1_T \) on \( U_1, U_2 \). This number is bounded by the product of the number of vertices of \( T_1 \) and \( T_2 \) at height \( h_u \), respectively. Note that this random variable is independent of \( 1_V \), and is a product of two independent variables that have the same distribution, namely, the number of vertices of \( T_{\delta n, 2\delta n} \) at level \( h_u \). The latter is stochastically smaller than the number of vertices of \( T_{\delta n, \infty} \) at level \( h_u \), which has expectation \( \sum_{k_1 < h_u} \mathbb{E}N_{h_u - k_1} \leq \sigma^2\delta n \). Finally, note that
\[
\sum_{V_1, V_2} \mathbb{E}[1_V] = 1. 
\]

(6.11)

Putting together (6.9), (6.10), and (6.11) we get:
\[
\mathbb{E}[R_{1B}] \leq \frac{(4 \max_{1 \leq k \leq \delta n} \gamma(k) + 2)D^d}{c_0\sigma^4(\delta n)^{6 - d}/2} \sum_{h_u =(6/4)\delta n}^{\delta n} \frac{(\sigma^2 \delta n)^2}{D^d(\delta n)^{d/2}} \leq C \max_{1 \leq k \leq \delta n} \gamma(k). 
\]
An appeal to the second part of Theorem 4 concludes the proof. □

**Proof of Theorem 5.** We choose \( n_5 = \max\{n_1(p^1), n_9(\sigma^2, C_3, p^1)\} \). Note the elementary inequality \( \frac{1}{\gamma_{i+1} + \gamma_2} \leq \frac{1}{4} \). We apply this inequality to the resistances of the two graphs "in parallel" between \( V_{\ell_1} \) and \( V_{\ell_2} \): one via the backbone and one via the vertices \( Y_{1+1}, \ldots, Y_{i+K}, X_{i+K} \). The parallel law (1.3) and Lemmas 6.1 and 6.2 gives
\[
\mathbb{E}[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}) | G_{\text{good}})] 
\leq \gamma(\ell_1 - i\delta n) + 1 + \gamma((i + K)\delta n - \ell_2 - 1)
\]
\[
+ \frac{1}{4} [1 + \gamma((i + 1)\delta n - \ell_1 - 1) + (K - 2)\gamma(\delta n) + \gamma(\ell_2 - (i + K - 1)\delta n)] + \frac{1}{4} [1 + \gamma((i + 1)\delta n - \ell_1 - 1) + (K - 1)\gamma(\delta n) \]
\[
+ 1 + \gamma((i + K)\delta n - \ell_2 - 1) + C_5 \max_{1 \leq k \leq \delta n} \gamma(k)].
\]
Choosing \( K \) large with respect to \( C_5 \) concludes the proof of the theorem. □

7. **Proof of Theorem 2**

Let \( K_0 \) be the constant in Theorem 5. We fix \( K = K_0 \) for the remainder of the proof. Let
\[
n_0 = \max\{n_3(\sigma^2, C_3, p^1, K), n_4(\sigma^2, C_3, p^1), 4k_1(p^1)\},
\]
where \( n_3 \) and \( n_4 \) are the constants from Theorems 4 and 5, and \( k_1 \) is the constant from Proposition 3. Let \( \delta_0 > 0, \alpha \in (0, 1/2) \) and \( A > 0 \) be
constants. These will be chosen below in the order: \( \delta_0, \alpha, A \), and among others we will require that

\[
\delta_0 \leq (K + 4)^{-1}, \quad \delta_0 \leq \delta_1, \quad (7.1)
\]

where \( \delta_1 \) is the constant from Proposition 3(ii). Once \( \delta_0 \) and \( \alpha \) will be chosen, we choose \( A \) to satisfy:

\[
A \geq \frac{n_0}{\delta_0}, \quad A^{-1/(2\alpha)} \leq 1/\sqrt{n_0}. \quad (7.2)
\]

We prove the theorem by induction. Since \( A \geq n_0/\delta_0 \), the theorem holds for all \( n < n_0/\delta_0 \), so we may assume \( n \geq n_0/\delta_0 \). Our induction hypothesis is that for all \( n' < n \) and all \( x \in \mathbb{Z}^d \) we have

\[
\gamma(n', x) \leq A(n')^{1-\alpha} \left( \frac{\|x\|^2}{n'} \lor 1 \right)^{\alpha},
\]

and given the hypothesis we prove it for \( n \). Since \( \gamma(n, x) \leq n \) it suffices to prove when \( \|x\| \leq nA^{-1/(2\alpha)} \). Note that this implies \( \|x\| \leq n/\sqrt{n_0} \). Now, given such \( x \) fix

\[
\delta = \min \left\{ \eta \geq \min\{\delta_0, n/\|x\|^2\} : \eta n \text{ is an integer} \right\}. \quad (7.3)
\]

Note that

\[
\delta n \geq \min \left\{ \delta_0 n, \frac{n^2}{\|x\|^2} \right\} \geq n_0, \quad (7.4)
\]

and

\[
\|x\| \leq \sqrt{2n/\delta}, \quad (7.5)
\]

so Theorem 3 can be applied to \((x, n)\).

Consider the sequences

\((0, \ldots, K), (K, \ldots, 2K), \ldots, ((N-1)K, \ldots, NK)\),

where \( N = \lfloor (\delta K)^{-1} \rfloor - 1 \) is the number of sequences. Fix any integer \( m \geq 2n \) and define \( \gamma_m(n, x) \) to be

\[
\gamma_m(n, x) = \mathbb{E}_{\mathcal{T}_{n, m}} \left[ R_{\text{eff}}((o, 0) \leftrightarrow \Phi(V_n)) \mid \Phi(V_n) = (x, n) \right],
\]

where we consider the resistance in the graph \( \Phi(\mathcal{T}_{n, m}) \), so that \( \gamma(n, x) = \sup_{m \geq 2n} \gamma_m(n, x) \). We bound \( \gamma_m(n, x) \) by estimating

\[
\mathbb{E} \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \mid \Phi(V_n) = (x, n) \right],
\]

for each \( i = 0, K, 2K, \ldots, (N-1)K \) and then adding these up using the triangle inequality for resistance (1.2), also adding the estimate for the final stretch from \( NK\delta n \) to \( n \).
Fix such an \( i \). We split the above expectation according to whether \( A(i) \cap B(i, c_0) \) occurred. By Theorem 5 we have that
\[
\mathbb{E}[R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K}))1_{A(i) \cap B(i, c_0)} | \Phi(V_n) = (x, n)] \\
\leq 3K \max_{1 \leq k \leq \delta n} \frac{\gamma(k)}{4} \mathbb{P}(A(i) \cap B(i, c_0) | \Phi(V_n) = (x, n)) \\
\leq \frac{3AK(\delta n)^{1-\alpha}}{4} \mathbb{P}(A(i) \cap B(i, c_0) | \Phi(V_n) = (x, n)),
\]
where the last inequality is due to our induction hypothesis.

We now proceed to estimate the expectation on the event that either \( A(i) \) or \( B(i, c_0) \) fail. Recall that we may write
\[
A(i)^c \cup B(i, c_0)^c = \bigcup_{a=1}^{6} \mathcal{E}(a) \cup (A(i) \cap B(i, c_0)),
\]
where \( \mathcal{E}(a) \) for \( a = 1, \ldots, 6 \) were defined in Section 4. For these estimate we will need the following lemmas.

**Lemma 7.1.** There exists \( C_6 > 0 \) such that, assuming the induction hypothesis, for all \( \delta n/4 \leq k \leq 2\delta n \) we have
\[
\bar{\gamma}(k; (x, n)) \leq (1 + C_6 \alpha)Ak^{1-\alpha},
\]
where \( \bar{\gamma} \) is defined at 111.

**Proof.** By the induction hypothesis
\[
\bar{\gamma}(k; (x, n)) \leq Ak^{1-\alpha}G_1(\alpha),
\]
where
\[
G_1(\alpha) = \sum_{y \in \mathbb{Z}^d} \mathbb{P}^k(y, o) \mathbb{P}^{n-k}(y, x) \left( \frac{\|y\|^2}{k} \right)^{1/2} \alpha.
\]
We have that \( G_1(0) = 1 \), and that
\[
G_1'(\alpha) = \sum_{y \in \mathbb{Z}^d, \|y\| > \sqrt{k}} \mathbb{P}^k(y, o) \mathbb{P}^{n-k}(y, x) \left( \frac{\|y\|^2}{k} \right) \alpha \log(\|y\|^2/k).
\]
We bound \( (\|y\|^2/k)^\alpha \log(\|y\|^2/k) \leq C\|y\|^2/k \) since \( \alpha \leq 1/2 \) and get that
\[
G_1'(\alpha) \leq Ck^{-1} \mathbb{E}[\|S(k)\|^2 | S(n) = x].
\]
Since \( k \geq \delta n/4 \geq n_0/4 \geq k_1 \) and \( \|x\| \leq \sqrt{2n/\delta} = \sqrt{2n} / \sqrt{\delta n} \leq 4n / \sqrt{k} \), we can apply Proposition 3(i) to the expectation on the right hand side of (7.7). This gives that \( G_1'(\alpha) \leq C \), and the lemma follows. \( \square \)

For the next lemma, recall the notation of Section 5.

**Lemma 7.2.** There exists \( C_6 > 0 \) such that, assuming the induction hypothesis, for all \( s' \geq s \) we have
\[
\mathbb{E}[R_{\text{eff}}((z_{s'-1}, T_{s'-1}) \leftrightarrow (z_{s'}, T_{s'}))1_{\mathcal{E}_s^{\|z\|}} | \mathcal{G}_{\text{tree}}] \leq (1 + C \alpha)A_{s'}^{1-\alpha} \mathbb{P}(\mathcal{E}_s^{\|z\|} | \mathcal{G}_{\text{tree}}).
\]
As in the previous lemma, we have

\[ G(3) = \sum_{z} q_s(z) \sum_{y:|y|>\sqrt{t_s}} \frac{p^{t_s}(o,y)p^{n-T_0}(y,x-z)}{p^{n}(o,x)} \left( \frac{||y||^2}{t_s} \vee 1 \right)^{\alpha}. \]

By (5.2) we have that

\[ G(2) = \frac{1}{t_3} \sum_{z} q_s(z) \sum_{y:|y|>\sqrt{t_s}} \frac{p^{n-T_{s-1}+T_0}(o,z-x)}{p^{n}(o,x)} \]

\[ \times \sum_{y:|y|>\sqrt{t_s}} ||y||^2 \frac{p^{t_s}(o,y)p^{n-T_0}(y,x-z)}{p^{n-T_{s-1}+T_0}(o,x-z)}. \]

Fix \( z \) and observe that

\[ ||x-z|| \leq ||x|| + ||z|| \leq \sqrt{2n/\delta} + (K+4)\sqrt{\delta n} = n \left( \frac{\sqrt{2}}{\sqrt{\delta n}} + \frac{\delta(K+4)}{\sqrt{\delta n}} \right) \]

where in the last step we used (7.1). This implies that \( ||x-z|| \leq 3n/\sqrt{\delta n} \leq 3n/\sqrt{t_s} \). We also have \( t_s \geq \delta n/4 \geq n_0/4 \geq k_1 \), where \( k_1 \) is the constant chosen in Proposition 3 and \( t_s \leq \delta n \leq \delta_1 n \), due to (7.1). Hence we can apply Proposition (ii) to the sum over \( y \) in (7.8), and get that

\[ G'(2) = C \sum_{z:y:|y|>\sqrt{t_s}} q_s(z) \frac{p^{t_s}(o,y)p^{n-T_0}(y,x-z)}{p^{n}(o,x)} \]

\[ = C \mathcal{P}(\mathcal{E}_1^{(3)}|G_{tree}). \]

This gives the statement of the lemma in the case \( s' = s \).

The case \( s' > s \) is similar. We appeal to the last statement of Lemma 5.1 and obtain that the required quantity is at most \( At_s^{1-\alpha} G_3(\alpha) \) where

\[ G_3(\alpha) = \sum_{z,y,ys:|y|>\sqrt{t_s}} q_s(z) \frac{p^{t_s}(o,y)p^{t_s'}(y,x-z)}{p^{t_s'}(o,ys)} \left( \frac{||y||^2}{t_{s'}} \vee 1 \right)^{\alpha}. \]

By (5.2) we see (performing the sum over \( y \)) that \( G_3(0) = \mathcal{P}(\mathcal{E}_1^{(3)}|G_{tree}) \). The derivative \( G_3'(\alpha) \) can be analyzed similarly to \( G_2' \), this time using Proposition 3(iii). This yields \( G_3'(\alpha) \leq C \mathcal{P}(\mathcal{E}_1^{(3)}|G_{tree}) \), and proves the statement of the lemma in the case \( s' > s \). \( \square \)
We now proceed with bounding the resistance given $A(i)^c \cup B(i, c_i)^c$. Lemmas 5.1 and 7.2 and the induction hypothesis give that

$$
E \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{E}(1), \Phi(V_n) = (x, n) \right] \\
\leq A(K + C_4\delta)(1 + C_6\alpha)(\delta n)^{1-\alpha}.
$$

Lemmas 4.2 and 7.1 give

$$
E \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{E}(2), \ell_1, \Phi(V_n) = (x, n) \right] \\
\leq A(1 + C_6\alpha) \left[ (\ell_1 - i\delta n)^{1-\alpha} + ((i + 1)\delta n - \ell_1)^{1-\alpha} + (K - 1)(\delta n)^{1-\alpha} \right] + 1.
$$

Since $\frac{4}{\delta n} - i \in [1/4, 3/4],

$$(\ell_1 - i\delta n)^{1-\alpha} + ((i + 1)\delta n - \ell_1)^{1-\alpha} \leq (1 + C_7\alpha)(\delta n)^{1-\alpha},$$

where $C_7 = (1/2)\log 4 + \sqrt{(3/4)\log(4/3)}$. Hence

$$
E \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{E}(2), \ell_1, \Phi(V_n) = (x, n) \right] \\
\leq A(1 + (C_6 + C_7 + C_6C_7\alpha)K)(\delta n)^{1-\alpha} + 1.
$$

Lemmas 4.3 and 7.1 give

$$
E \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{E}(3), \ell_1, \Phi(V_n) = (x, n) \right] \\
\leq A(1 + C_6\alpha) \left[ (\ell_1 - i\delta n)^{1-\alpha} + ((i + 1)\delta n - \ell_1)^{1-\alpha} + (K - 1 + C_4\delta)(\delta n)^{1-\alpha} \right] + 1 \\
\leq A(K + C_4\delta + C_7\alpha)(1 + C_6\alpha)(\delta n)^{1-\alpha} + 1.
$$

Lemmas 4.4 and 7.1 give

$$
E \left[ R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \mid \mathcal{E}(4), \ell_1, \ell_2, \Phi(V_n) = (x, n) \right] \\
\leq A(1 + C_6\alpha) \left[ (\ell_1 - i\delta n)^{1-\alpha} + ((i + 1)\delta n - \ell_1)^{1-\alpha} + (K - 2)(\delta n)^{1-\alpha} + (\ell_2 - (i + K - 1)\delta n)^{1-\alpha} + ((i + K)\delta n - \ell_2)^{1-\alpha} \right] + 2 \\
\leq A(1 + C_6\alpha)(K + 2C_7\alpha)(\delta n)^{1-\alpha} + 2.
$$

Lemmas 5.1, 7.2 and the induction hypothesis give that for any $s = 1, \ldots, K + 4$ and $s' = 1, \ldots, K + 4$ we have that

$$
E \left[ R_{\text{eff}}((z_{s-1}, T_{s-1}) \leftrightarrow (z_{s'}, T_{s'})) \mid \mathcal{E}(5), \mathcal{G}_{\text{tree}} \right] \\
\leq \begin{cases} 
1 & \text{if } s' = 2, K + 3, \\
A(t_{s'})^{1-\alpha} & \text{if } s' < s, s' \neq 2, K + 3, \\
A(1 + C_6\alpha)(t_{s'})^{1-\alpha} & \text{if } s' \geq s, s' \neq 2, K + 3.
\end{cases}
$$
By the triangle inequality for resistance we get that for all $s = 1, \ldots, K + 4$

$$\mathbb{E}
\left[
R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \bigg| \mathcal{E}^s_{(5)}, \mathcal{G}_{\text{tree}}
\right] \leq AK(1 + C_6\alpha)(1 + 2C_7\alpha)(\delta n)^{1-\alpha} + 2.$$  

Hence

$$\mathbb{E}
\left[
R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \bigg| \mathcal{E}^s_{(5)}, \ell_1, \ell_2, \Phi(V_n) = (x, n)
\right] \leq AK(1 + C_6\alpha)(1 + 2C_7\alpha)(\delta n)^{1-\alpha} + 2.$$  

By Lemma 5.2 and the induction hypothesis (recall $\mathcal{E}^s_{(7)} = A(i) \cap B(i, c_0)^c$):

$$\mathbb{E}
\left[
R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \bigg| \mathcal{E}^s_{(6)} \cup \mathcal{E}^s_{(7)}, \mathcal{G}_{\text{tree}}
\right] \leq A(K + 2C_7\alpha)(\delta n)^{1-\alpha} + 2.$$  

Putting these together gives that there exists $C_8 = C_8(K) > 0$ such that

$$\mathbb{E}
\left[
R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \bigg| \mathcal{E}^s_{(6)}, \mathcal{G}_{\text{tree}}
\right] \leq A(K + C_8(\delta + \alpha))(\delta n)^{1-\alpha} + 2.$$  

By Theorem 4 there exists a constant $c = c(K) \in (0, 1)$ such that the last quantity is at most

$$A(\delta n)^{1-\alpha}\left[ c\frac{3K}{4} + (1 - c)(K + C_8(\delta + \alpha)) \right].$$  

We now choose $\delta_0$ and $\alpha$ (depending only on $K = K_0$). In addition to the already required (7.1), let $\delta_0$ satisfy:

$$2K\delta_0(1 + C_6) + 2\delta_0(1 + C_6) < \frac{c}{16}.$$  

Let $\alpha > 0$ satisfy:

$$\alpha \leq \delta_0, \quad \left(1 - \frac{c}{16}\right)^{\delta_0^-\alpha} \leq 1.$$  

The first condition on $\delta_0$ in (7.3) gives that

$$\mathbb{E}
\left[
R_{\text{eff}}(\Phi(X_i) \leftrightarrow \Phi(X_{i+K})) \bigg| \Phi(V_n) = (x, n)
\right] \leq AK(1 - c/8)(\delta n)^{1-\alpha} \quad \text{and} \quad \leq An^{1-\alpha}\delta^{-\alpha}K\delta(1 - c/8).$$
for \( i = 0, K, \ldots, (N - 1)K \). For the final stretch, Lemmas 5.3 and 7.1 and the induction hypothesis gives:

\[
E\left[ R_{\text{eff}}(\Phi(X_{i\text{last}}) \leftrightarrow \Phi(V_n)) \mid \Phi(V_n) = (x, n) \right] \\
\leq K' A(\delta n) 1^{1-\alpha} (1 + C_6\alpha) + A(\tilde{n}) 1^{1-\alpha} (1 + C_6\alpha) .
\]

(7.12)

Using \( K' \leq 2K \) and \( \tilde{n} \leq 2\delta n \) and the second requirement on \( \delta_0 \) in (7.9), the right hand side of (7.12) is at most

\[
A 1^{1-\alpha} (2K(1 + C_6) + (2)(1 + C_6)) \\
\leq A 1^{1-\alpha} (c/16) .
\]

(7.13)

We sum (7.11) over all sequences using the triangle inequality and add (7.12). This gives

\[
\gamma(n, x) = \sup_{m \geq n} \gamma_m(n, x) \\
\leq A 1^{1-\alpha} (1 - c/16) ,
\]

(7.13)

where we used \( NK\delta \leq 1 \).

To conclude, note that if in the definition (7.3) of \( \delta \) we have \( \delta_0 \leq n/\|x\|^2 \), then \( \delta^{-\alpha} \leq \delta_0^{-\alpha} \), and due to the choice of \( \alpha \), (7.13) yields \( \gamma(n, x) \leq A n^{1-\alpha} \). If we have \( n/\|x\|^2 \leq \delta_0 \), then \( \delta^{-\alpha} \leq (\|x\|^2/n)^\alpha \), and (7.13) yields \( \gamma(n, x) \leq A n^{1-\alpha} (\|x\|^2/n)^\alpha \). This concludes the induction and our proof. □

Acknowledgements

We thank Ori Gurel-Gurevich, Gady Kozma and Gordon Slade for useful conversations. This research was supported by NSF and NSERC grants.

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