NEW RESULTS FROM MIXED-ACTION LATTICE GAUGE THEORY

1 Introduction

These days, lattice gauge theory is able to provide quantitative results for an increasing number of problems in field theory. It has taken some two decades of trial, error and insight to reach this position. Nonetheless, it is still vital to test the formalism for any possible problems lurking just around the corner from the results presented as QCD phenomenology: in fact, there is less excuse than ever for ignorance.

Many such tests probe the area of universality: the choice of a lattice-regularised theory is far from unique, but the same physics should result in every case when one goes to the continuum limit and hence the details of behaviour at the cut-off scale are expected to become irrelevant. In this case, we investigate the dependence on the representation of the gauge group used in the standard Wilson lattice action:

\[ S = \beta \sum_{\text{plaq}} \text{Tr}_{\text{rep}}(U_{\text{plaq}}). \] (1)

Here, \( U_{\text{plaq}} \) is an element of the gauge group corresponding to the smallest closed path on the lattice, the plaquette; \( \beta \) is the inverse coupling for the representation chosen. The representation itself is hidden within the trace \( \text{Tr}_{\text{rep}} \). If we choose our \( SU(2) \) group elements to be two-by-two unitary complex matrices, which we can write as four real numbers with one constraint, the fundamental (spin-1/2) representation of the gauge group just corresponds to the trace of the matrix up to a factor 1/2. This is the most natural choice, and the one that is usually made in lattice gauge theory.

The adjoint or spin-1 representation, however, contains the trace of the matrix squared. This means that the action is insensitive to a change of sign in the matrices. Pictured in terms of the \( SU(2) \) group manifold, a three-sphere, the fundamental representation treats all points on the manifold as distinct, while the adjoint representation identifies the opposite ends of diameters of the sphere and corresponds to the group \( SO(3) \) (as do all whole-integer representations of \( SU(2) \)).

The continuum limit here is where the lattice spacing \( a \) goes to zero in physical units. Because of asymptotic freedom, this occurs as the inverse coupling \( \beta \) goes to infinity, which is the perturbative limit for the unrenormalised degrees of freedom (though not for the physical ones). Universality implies that the topology of the gauge manifold should be irrelevant in this limit.

In the early eighties, Bhanot and Creutz realised that one could gain even more information by combining together terms in the action with a trace in both representations,

\[ S = \sum_{\text{plaq}} \left( \beta_F \text{Tr}_F(U_{\text{plaq}}) + \beta_A \text{Tr}_A(U_{\text{plaq}}) \right), \] (2)

where \( F \) and \( A \) stand for the fundamental and adjoint representations. They then investigated the phase diagram in the mixed-coupling plane \((\beta_F, \beta_A)\).

The resulting diagram was dominated by two first-order bulk transitions, which joined to form another transition that ended abruptly (see top left of figure 1). As these are bulk (volume) transitions, there is no physical scale associated with them and they remain at a fixed coupling \( \beta \): they cannot be taken to the continuum limit and are therefore artifacts of some sort.

The transition intersecting the \( \beta_A \) axis was explained in terms of monopoles by Caneschi, Halliday and Schwimmer: the monopoles were present because of non-contractable loops in the gauge manifold, namely semi-circumferences whose end points were in this case identified due to the \( SO(3) \) invariance. It was further shown that the high-\( \beta_A \) phase, beyond the monopole transition, was the same as the phase in the standard \( \beta_A = 0 \) theory which has no bulk transitions.

In fact, one can think of the separate area at the top left of the phase diagram as being the region where two
‘copies’ of the underlying gauge system with its continuum limit exist, near the identity $I$ and its negative $-I$ in the gauge manifold (these are of course identical in the pure-adjoint case), related by a $Z(2)$ symmetry (i.e. the regions are really the same gauge system reflected) which can be broken by increasing $\beta_F$. In the rest of the diagram, there is only one ‘copy’, near $I$. As one goes towards the continuum limit of small lattice spacing and large $\beta$, only the gauge fields lying near $I$ and (in the one case) the image near $-I$ are important. Consequently, universality is not in danger; the lines due to the extra $Z(2)$ degree of freedom are merely an annoyance.

2 Finite temperature effects

This complicated but comprehensible picture was spoilt recently when Gavai, Grady and Mathur investigated the finite temperature transition known to exist in fundamental ($\beta_A = 0$) SU(2). This is different in nature to the bulk transitions described above: it depends on a physical scale, here the critical temperature $T_c$. In the usual formalism on a lattice of physical size $(N_s a)^3 \times N_t a = L^3 \times L_t$, where $N_s$ and $N_t$ are integers, $T_c = 1/L_t$. Increasing $N_t$, one therefore needs to decrease $a$ and hence increase $\beta$ to recover $T_c$, which leads one again to the continuum limit. In other words, as one increases the number of links $N_t$ in the time dimension of the lattice, the transition should move to larger $\beta$. This has been confirmed and quantitatively analysed for the usual fundamental case.

Universality requires that this transition extend into the mixed coupling plane: in fact the most naïve picture in the perturbative limit would be an ellipse extending from the $\beta_F$ to the $\beta_A$ axis. In the continuum limit it should also clearly separate itself from the bulk effects, which have no such limit.

However, refs. show that, on the contrary, the transition, which is second order for $\beta_A = 0$, becomes first order and furthermore seems to merge with the line with the endpoint seen by Bhanot and Creutz (which appeared to retain a finite temperature nature), in an apparently clear violation of universality. This is the problem we try to elucidate here.

3 Simulations

We have actually simulated using the SO(3)-invariant action proposed by Halliday and Schwimmer. Instead of a pure adjoint part of the action they added an auxiliary $Z(2)$-valued field $\sigma$, defined on the plaquettes:

$$S = \sum_{\text{plaq}} (\beta_F \text{Tr}_F(U_{\text{plaq}}) + \beta_V \text{Tr}_V(U_{\text{plaq}}) \times \sigma_{\text{plaq}}),$$

where the path integral measure is extended to include a sum over all values of $\sigma = \pm 1$. They called this the Villain form of the action.

The $\beta_V$ term does not correspond to an irreducible representation of SU(2) and indeed when decomposed includes all whole-integer representations (in the language of spin) of the group. However, the SO(3) nature is presumably the dominant effect at work and indeed the phase diagram found for the $(\beta_F, \beta_V)$ plane is very similar to that of Bhanot and Creutz, although it should be noted that the scale on the SO(3)-invariant axis is somewhat different.

The main advantage of the Villain action in this work is technical: its form (Tr is linear in the SU(2) matrix algebra used while TrA is not) means an efficient heatbath-plus-overrelaxation updating scheme can be used for the Monte Carlo analysis in both the gauge and Z(2) degrees of freedom rather than the less efficient Metropolis scheme required in the adjoint case.

4 Results

We have found the phase transition in $\beta_F$ for a range of different $\beta_V$. We have used both $N_t = 2$ and $N_t = 4$; on the latter, our results so far only consist of one spatial volume with $N_s = 8$, so extracting the order of the phase transition is less precise than on the $N_t = 2$ lattices, where we have $N_s = 6$ (for exploratory work) $N_s = 8$ and $N_s = 16$. Our simulations typically consist of 80,000 sweeps, each consisting of one heatbath plus four over-relaxation steps, so that we expect autocorrelations to be low (although they are taken into account in the bootstrap error analysis).

The results, obtained from the peak in the susceptibility of the Polyakov loop, are shown in table. As for the order of the phase transition, in the $N_t = 2$ case it appears to be second order up to $\beta_V = 2.5$ and in the $N_t = 4$ case up to $\beta_V = 2.2$, and first order thereafter. In these intermediate cases, this is not easy to determine definitely and we quote the best estimate. We shall require more simulations right on the phase transition for this to be clearer. In the other cases we can be more definite, as discussed below.

The results are also shown in figure. Only the squares and crosses represent actual results; the lower lines are drawn by hand as a guide. The upper lines are

| $\beta_F$ | $N_t = 2$ | $N_t = 4$ |
|----------|----------|----------|
| $\beta_F$ | $N_t = 2$ | $N_t = 4$ |
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also drawn by hand after the results of ref. [3]. Presumably the lines join somewhere (see below).

The basic picture is the same as that of ref. [4], and the trends seem clear despite the provisional nature of the results presented. At small SO(3) coupling $\beta_V$, the transition is second order and the analysis is standard [1]. At large $\beta_V$ the transition becomes first order, with a clear two-state signal, and increasingly strong in the sense that it takes many Monte Carlo sweeps before changing from one phase into another. In fact on the $8^3 \times 4$ lattice at $\beta_V = 3.0$ we are only able to quote an upper and a lower limit (although these are firm) for the phase transition as between these values of $\beta_F$ both phases are stable over several tens of thousands of sweeps. Nonetheless, perhaps the central observation of this paper is this: we do definitely observe an $N_t$ dependence in this region, confirming that the transition is not bulk even though it is first order.

Note, however, that even with $N_t = 2$ we could establish the position of the phase transition accurately only for $N_s = 6$, which leaves the possibility of finite spatial size effects, the more so when one takes into account the convergence noted in the next paragraph.

One major difference from the previous results is that it is now much clearer that the phase transition lines are roughly straight over a wide range—itself a departure from naive expectations of universality—and that they converge towards the end point of the bulk lines. However, we have not verified the monopole and $Z(2)$-symmetry-breaking bulk transitions in detail and the point of convergence is unknown. Indeed, as those transitions are also strongly first order, it is very difficult to pin the transitions down for lattices larger than those used in ref. [3] and finite size effects in this region are impossible to rule out.

5 Implications

It now seems clear that the (presumably physical) finite size transition somehow becomes entangled with artifacts, and that this requires some sorting out. A truly analytic understanding is some way off; we make the following remarks merely in an effort to make the first steps.

Under the assumption that universality of the gauge theory can be salvaged, the second order finite temperature transition must actually be present, but somehow hidden by the artifacts. Therefore, we suggest that the first order effects are distortions of the mixed-coupling plane due to long-range effects of the bulk transition lines. If these effects were not present, the finite temperature lines at increasing $N_t$ would simply move further out to large $\beta$. As they seem to be funnelled in towards the meeting point of the two bulk lines, we suggest that this is covering up a whole range of physics of different lattice spacings $a$: for some (hypothetical) small $a$, the scale of $T_c$ takes one left of the point, and for some large $a$, this scale emerges again and starts moving to larger $\beta$. Near the point, the distortions caused by this actually crease the plane somehow, showing the first order effects we see which hide the physical transition.

This is more natural than it may appear at first sight. First of all, the two bulk lines presumably have the effect of a fold—that is, a range of physics is covered by a single value of $\beta_V$ or $\beta_F$. Their conjunction might then be expected to have this effect in both directions, acting as the funnel observed. Secondly, long range remnants of the bulk effect certainly seem to be present anyway. There will be more discussion of this in the eventual paper.

References

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