OPTIMAL RATE LIST DECODING OVER BOUNDED ALPHABETS USING ALGEBRAIC-GEOMETRIC CODES

VENKATESAN GURUSWAMI AND CHAOPING XING

Abstract. We give new constructions of two classes of algebraic code families which are efficiently list decodable with small output list size from a fraction $1 - R - \varepsilon$ of adversarial errors where $R$ is the rate of the code, for any desired positive constant $\varepsilon$. The alphabet size depends only on $\varepsilon$ and is nearly-optimal.

The first class of codes are obtained by folding algebraic-geometric codes using automorphisms of the underlying function field. The list decoding algorithm is based on a linear-algebraic approach, which pins down the candidate messages to a subspace with a nice “periodic” structure. The list is pruned by pre-coding into a special form of subspace-avoiding sets. Instantiating this construction with the explicit Garcia-Stichtenoth tower of function fields yields codes list-decodable up to a $1 - R - \varepsilon$ error fraction with list size bounded by $O(1/\varepsilon)$, matching the existential bound for random codes up to constant factors. Further, the alphabet size of the codes is a constant depending only on $\varepsilon$ — it can be made $\exp(\tilde{O}(1/\varepsilon^2))$ which is not much worse than the lower bound of $\exp(\Omega(1/\varepsilon))$.

The parameters we achieve are thus quite close to the existential bounds in all three aspects — error-correction radius, alphabet size, and list-size — simultaneously. Our code construction is Monte Carlo and has the claimed list decoding property with high probability. Once the code is (efficiently) sampled, the encoding/decoding algorithms are deterministic with a running time $O_c(N^c)$ for an absolute constant $c$, where $N$ is the code’s block length.

The second class of codes are obtained by restricting evaluation points of an algebraic-geometric code to rational points from a subfield. Once again, the linear-algebraic approach to list decoding to pin down candidate messages to a “periodic” subspace. We develop an alternate approach based on subspace designs is used to pre-code messages and prune the subspace of candidate solutions. Together with the subsequent explicit constructions of subspace designs, this yields the first deterministic construction of an algebraic code family of rate $R$ with efficient list decoding from $1 - R - \varepsilon$ fraction of errors over an alphabet of constant size $\exp(\tilde{O}(1/\varepsilon^2))$. The list size is bounded by a very slowly growing function of the block length $N$; in particular, it is at most $O(\log^{(r)}(N))$ (the $r$’th iterated logarithm) for any fixed integer $r$. The explicit construction avoids the shortcoming of the Monte Carlo sampling at the expense of a slightly worse list size.

Extended abstracts announcing these results were presented at the 2012 and 2013 ACM Symposia on Theory of Computing (STOC) [16, 17]. This is a merged and revised version of these conference papers, that accounts for the explicit subspace designs that were constructed in [8] subsequent to [17], and makes some simplifications and improvements in the construction of h.s.e sets in Section 6 compared to [16].

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1. Introduction

An error-correcting code $C$ of block length $N$ over a finite alphabet $\Sigma$ maps a set $\mathcal{M}$ of messages into codewords in $\Sigma^N$. The rate of the code $C$, denoted $R$, equals $\frac{1}{N}\log_{|\Sigma|} |\mathcal{M}|$. In this work, we will be interested in codes for adversarial noise, where the channel can arbitrarily corrupt any subset of up to $\tau N$ symbols of the codeword. The goal will be to correct such errors and recover the original message/codeword efficiently. It is easy to see that information-theoretically, we need to receive at least $RN$ symbols correctly in order to recover the message (since $|\mathcal{M}| = |\Sigma|^{RN}$), so we must have $\tau \leq 1 - R$.

Perhaps surprisingly, in a model called list decoding, recovery up to this information-theoretic limit becomes possible. Let us say that a code $C \subseteq \Sigma^N$ is $(\tau, \ell)$-list decodable if for every received word $y \in \Sigma^N$, there are at most $\ell$ codewords $c \in C$ such that $y$ and $c$ differ in at most $\tau N$ positions. Such a code allows, in principle, the correction of a fraction $\tau$ of errors, outputting at most $\ell$ candidate codewords one of which is the originally transmitted codeword.

The probabilistic method shows that a random code of rate $R$ over an alphabet of size $\exp(O(1/\varepsilon))$ is with high probability $(1 - R - \varepsilon, O(1/\varepsilon))$-list decodable [2]. However, it is not known how to construct or even randomly sample such a code for which the associated algorithmic task of list decoding (i.e., given $y \in \Sigma^N$, find the list of codewords within fractional radius $1 - R - \varepsilon$) can be performed efficiently. This work takes a big step in that direction, giving a randomized construction of such efficiently list-decodable codes over a slightly worse alphabet size of $\exp(\tilde{O}(1/\varepsilon^2))$. We note that the alphabet size needs to be at least $\exp(\Omega(1/\varepsilon))$ in order to list decode from a fraction $1 - R - \varepsilon$ of errors, so this is close to optimal. For the list-size needed as a function of $\varepsilon$ for decoding a $1 - R - \varepsilon$ fraction of errors, the best lower bound is only $\Omega(\log(1/\varepsilon))$ [9], but as mentioned above, even random coding arguments only achieve a list-size of $O(1/\varepsilon)$, which our construction matches up to constant factors. We also give a fully deterministic construction with a list-size that is very slowly growing as a function of the block length.

We now review some of the key results on algebraic list decoding leading up to this work. A more technical comparison with related work appears in Section 1.1. The first construction of codes that achieved the optimal trade-off between rate and list-decoding radius, i.e., enabled list decoding up to a fraction $1 - R - \varepsilon$ of worst-case errors with rate $R$, was due to Guruswami and Rudra [11]. They showed that a variant of Reed-Solomon (RS) codes called folded RS codes admit such a list decoder. For a decoding radius of $1 - R - \varepsilon$, the code was based on bundling together disjoint windows of $m = \Theta(1/\varepsilon^2)$ consecutive symbols of the RS codeword into a single symbol over a larger alphabet. As a result, the alphabet size of the construction was $N^{\Omega(1/\varepsilon^2)}$. Ideas based on code concatenation and expander codes can be used to bring down the alphabet size to $\exp(\tilde{O}(1/\varepsilon^4))$, but this compromises some nice features such as list recovery and soft decoding of the folded RS code. Also, the decoding time complexity as well as proven bound on worst-case output list size for these constructions were $N^{\Omega(1/\varepsilon)}$.

Our main final result statement is the following, offering two constructions, one randomized and one deterministic, of variants of algebraic-geometric (AG) codes that are list-decodable with optimal rate. More detailed statements can be found in the technical sections of the paper.

**Theorem 1.1 (Main).** For any $R \in (0, 1)$ and positive constant $\varepsilon \in (0, 1)$, there is

(i) a Monte Carlo construction of a family of codes of rate at least $R$ over an alphabet size $\exp(O(\log(1/\varepsilon)/\varepsilon^2))$ that are encodable and $(1 - R - \varepsilon, O(1/(R\varepsilon))$-list decodable
in $O_\varepsilon(N^c)$ time, where $N$ is the block length of the code and $c$ is an absolute positive constant.

(ii) a deterministic construction of a family of codes of rate at least $R$ over an alphabet size $\exp(O(\log^2(1/\varepsilon)/\varepsilon^2))$ that are encodable and $(1 - R - \varepsilon, L(N))$-list decodable in $O_\varepsilon(N^c)$ time, for a list size that satisfies $L(N) = o(\log^{(r)} N)$ (the $r$'th iterated logarithm) for any fixed integer $r$.

The first part of Theorem 1.1 is achieved through folded algebraic-geometric codes and hierarchical subspace-evasive sets. To fold algebraic-geometric codes, we first find suitable automorphisms of the ground function field. Consequently, the list of possible candidate messages has an exponential size, but is well structured. To prune down the list size, we only encode messages in a subspace-evasive set which has small intersection with the original list. To make use of subspace-evasive sets efficiently, we have to: (i) give an explicit pseudorandom construction of these sets; and (ii) encode the messages to subspace-evasive sets efficiently. We refer to Section 2 for details.

The second part of Theorem 1.1 is obtained through usual algebraic-geometric codes with evaluation points over subfields and subspace design. As in the first part, the list of possible candidate messages has an exponential size, but is well structured. The approach based on hierarchical subspace-evasive sets in the first part leads to excellent list size; however, we only know randomized constructions of hierarchical subspace-evasive sets. To obtain a deterministic list decoding, we prune down the list of possible solutions through subspace designs (see Section 2 for details).

We note that our Monte Carlo construction gives codes that are quite close to the existential bounds in three aspects simultaneously — the trade-off between error fraction $1 - R - \varepsilon$ and rate $R$, the list-size as a function of $\varepsilon$, and the alphabet size of the code family (again as a function of $\varepsilon$). Even though these codes are not fully explicit, they are “functionally explicit” in the sense that once the code is (efficiently) sampled, with high probability the polynomial time encoding and decoding algorithms deliver the claimed error-correction guarantees for all allowed error pattern. The explicit construction avoids this shortcoming at the expense of a slightly worse list size. Our algorithms can also be extended to the “list recovery” setting in a manner similar to [11, 7]; we omit discussion of this aspect and the straightforward details.

1.1. Prior and related work. Let us recap a bit more formally the construction of folded RS codes from [11]. One begins with the Reed-Solomon encoding of a polynomial $f \in \mathbb{F}_q[X]$ of degree $< k$ consisting of the evaluation of $f$ on a subset of field elements ordered as $1, \gamma, \ldots, \gamma^{N-1}$ for some primitive element $\gamma \in \mathbb{F}_q$ and $N < q$. For an integer “folding” parameter $m \geq 1$ that divides $N$, the folded RS codeword is defined over alphabet $\mathbb{F}_q^m$ and consists of $n/m$ blocks, with the $j$'th block consisting of the $m$-tuple $(f(\gamma^{(j-1)m}), f(\gamma^{(j-1)m+1}), \ldots, f(\gamma^{jm-1}))$. The algorithm in [11] for list decoding these codes was based on the algebraic identity $\overline{f(\gamma X)} = \overline{f(X)^q}$ in the residue field $\mathbb{F}_q[X]/(X^{q-1} - \gamma)$ where $\overline{f}$ denotes the residue $f$ mod $(X^{q-1} - \gamma)$. This identity is used to solve for $f$ from an equation of the form $Q(X, f(X), f(\gamma X), \ldots, f(\gamma^{s-1}X)) = 0$ for some low-degree nonzero multivariate polynomial $Q$. The high degree $q > n$ of this identity, coupled with $s \approx 1/\varepsilon$, led to the large bounds on list-size and decoding complexity in [11].

One possible approach to reduce $q$ (as a function of the code length) in this construction would be to work with algebraic-geometric codes based on function fields $K$.
over $\mathbb{F}_q$ with more rational points. However, an automorphism $\sigma$ of $K$ that can play the role of the automorphism $f(X) \mapsto f(\gamma X)$ of $\mathbb{F}_q(X)$ is only known (or even possible) for very special function fields. This approach was used in [6] to construct list-decodable codes based on cyclotomic function fields using as $\sigma$ certain Frobenius automorphisms. These codes improved the alphabet size to polylogarithmic in $N$, but the bound on list-size and decoding complexity remained $N^{\Omega(1/\varepsilon)}$. Recently, a linear-algebraic approach to list decoding folded RS codes was discovered in [27, 7]. Here, in the interpolation stage, which is common to all list decoding algorithms for algebraic codes [26, 13, 21, 11], one finds a linear multivariate polynomial $Q(X, Y_1, \ldots, Y_s)$ whose total degree in the $Y_i$’s is 1. The simple but key observation driving the linear-algebraic approach is that the equation $Q(X, f(X), \ldots, f(\gamma^{s-1}X)) = 0$ now becomes a linear system in the coefficients of $f$. Further, it is shown that the solution space has dimension less than $s$, which again gives a list-size upper bound of $q^{s-1}$. Finally, since the list of candidate messages fall in an affine space, it was noted in [7] that one can bring down the list size by carefully “pre-coding” the message polynomials so that their $k$ coefficients belong to a “subspace-evasive set” (which has small intersection with every $s$-dimensional subspace of $\mathbb{F}_q^k$). This idea was used in [14] to give a randomized construction of $(1 - R - \varepsilon, O(1/\varepsilon))$-list decodable codes of rate $R$. However, the alphabet size and runtime of the decoding algorithm both remained $N^{\Omega(1/\varepsilon)}$. Similar results were also shown in [14] for derivative codes, where the encoding of a polynomial $f$ consists of the evaluations of $f$ and its first $m-1$ derivatives at distinct field elements.

Concurrently with our work reported in [16], Dvir and Lovett gave an elegant construction of explicit subspace evasive sets based on certain algebraic varieties [1]. This yields an explicit version of the codes from [7], albeit with a worse list size bound of $(1/\varepsilon)^{O(1/\varepsilon)}$. This work and [1] are incomparable in terms of results. The big advantage of [1] is the deterministic construction of the code. The benefits in our work are: (i) both constructions give codes over an alphabet size that is a constant independent of $N$, whereas in [1] the $N^{\Omega(1/\varepsilon^2)}$ alphabet size of folded RS codes is inherited; (ii) our first Monte Carlo construction ensures list-decodability with a list-size of $O(1/\varepsilon)$ that is much better and in fact matches the full random construction up to constant factors\(^2\) and (iii) our second construction gives a deterministic algorithm as well with almost constant list size (and constant alphabet size). Another important feature is that both our work and [1] achieve a decoding complexity of $O(\varepsilon(N^c))$ with exponent independent of $\varepsilon$.

Our paper presents two classes of codes: folded algebraic-geometric codes and usual algebraic-geometric codes with evaluation points over subfields. For both the classes of codes, we can apply hierarchical subspace-evasive sets as well as subspace design to prune down the list size by taking certain subcodes. This is because of the “periodic” structure of the subspace in which the candidate messages are pinned down by the linear-algebraic list decoder is similar in both cases. Thus, we can obtain both randomized and deterministic algorithms from each of the two classes of codes. In total, we have four combinations of constructions. To illustrate both algebraic approaches, we decide to focus on two combinations, i.e., (i) folded algebraic-geometric codes with hierarchical subspace-evasive sets; and (ii) usual algebraic-geometric codes with evaluation points over subfields with subspace designs. These are listed in Figure 1. We note that the other two combinations are also possible, as the pruning of the subspace of solutions is “black-box” with respect to its periodic structure.

\(^2\)As mentioned above, the bound in [1] is $(1/\varepsilon)^{O(1/\varepsilon)}$ and it seems very difficult to get a sub-exponential dependence on $1/\varepsilon$ with the algebraic approach relying on Bezout’s theorem to construct subspace-evasive sets.
In the table presented in Figure 1, we list previous results and those in this paper. The major improvement of this work is to bring down the alphabet size to constant, while at the same time ensuring small list size and low decoding complexity where the exponent of the polynomial run time does not depend on $\varepsilon$. Our folded algebraic-geometric subcodes achieve a list size matching the fully random constructions up to constant factors, together with alphabet size not much worse than the lower bound $\exp(\Omega(1/\varepsilon^2))$. On the last line, our algebraic-geometric subcodes give a deterministic list decoding with almost constant list size and optimal decoding radius.

1.2. Open questions. The challenge of decoding up to radius approaching the optimal bound $(1 - R - \varepsilon)$ with rate $R$ along with good list and alphabet size is, for the most part, solved by our work. There are still some goals that have not been met. One is to get a fully deterministic construction with constant list-size and alphabet size (as a function of $\varepsilon$), and construction/decoding complexity $O(1/\varepsilon)$. In this regard, in a recent personal communication, Kopparty, Ron-Zewi, and Saraf [19] report a construction that achieves constant list and alphabet size with complexity $N^{\varepsilon}(\log \log N)^{O(1/\varepsilon)}$ (the list size, while a constant, grows doubly exponentially in $1/\varepsilon$). Another challenge is to construct a $(1 - R - \varepsilon, L)$-list decodable code of rate $R$ (for list size $L$ bounded by a polynomial in the block length), over an alphabet of size $\exp(\tilde{O}(1/\varepsilon))$, which is the asymptotically optimal size. Note that all constructions over a constant-sized alphabet known so far have alphabet size at least $\exp(\Omega(1/\varepsilon^2))$. Finally, the various algebraic and expander-based techniques that have led to progress on list decoding only work over large alphabets. The challenge of efficient optimal rate list decoding over say the binary alphabet, even for the simpler model of erasures, remains wide open. The best known constructions are obtained via concatenation, and are list-decodable up to the so-called Blokh-Zyablov bound [12].

1.3. Organization. The paper is organized as follows. In Section 2 we describe the detailed techniques of our paper including algebraic approaches and pseudorandomness. Following the section on techniques, in Section 3 we introduce periodic and ultra-periodic subspaces, give definitions and basic properties. In Section 4 we recall some basis results on function fields and algebraic-geometric codes. To illustrate our ideas in an algebraically simpler (and perhaps more practical) setting, in Section 5 we give a construction based
on a tower of Hermitian field extensions \[22\]. This is capable of giving a similar result to our best ones based on the Garcia-Stichtenoth tower, albeit with alphabet size and list-size upper bound polylogarithmic in the code length. In Section \[6\] we first introduce hierarchical subspace-evasive sets, then show that random sets are hierarchical subspace-evasive with high probability. We also present a pseudorandom construction of hierarchical subspace-evasive sets, which also allow for efficient encoding and efficient computation of intersection with periodic subspaces.

Folded algebraic-geometric codes from the Garcia-Stichtenoth tower are studied in Section \[7\]. The list size, decoding radius and decoding algorithm via local expansion are also discussed in this section. Section \[8\] is devoted to the discussion of pruning down the list size for folded codes from both the Hermitian and the Garcia-Stichtenoth towers using hierarchical subspace-evasive sets. The second class of our codes, namely usual algebraic-geometric codes with evaluation points over subfields is presented in Section \[9\]. In this section, we first discuss list decoding for the simpler Reed-Solomon case, and then generalize it to list decoding of arbitrary algebraic geometric codes and finally instantiate the approach with the codes from the Garcia-Stichtenoth tower. In Section \[10\] we introduce subspace designs and cascaded subspace designs, and discuss parameters of random and explicit constructions of those. In the last section, the explicit construction of subcodes of RS and AG subcodes based on subspace designs is presented.

2. Our techniques

We describe some of the main new ingredients that go into our work. We need both new algebraic insights and constructions, as well as ideas in pseudorandomness relating to (variants of) subspace-evasive sets. We describe these in turn below.

2.1. Algebraic ideas. It is shown in \[13\] that one can list decode the usual algebraic-geometric codes up to the Johnson bound. Thus, to list decode the algebraic-geometric codes beyond the Johnson bound, we have to consider some variants of usual algebraic-geometric codes. In this work, we present two variants of algebraic-geometric codes–folded algebraic geometric codes and usual algebraic geometric codes with evaluation points over subfields. We describe these in turn.

2.1.1. Folding AG codes. The first approach is to use suitable automorphisms of function fields to fold the code. This approach was used for Reed-Solomon codes in \[11\] and for cyclotomic function field in \[6\], though this was done using the original approach in \[11\] where the messages to be list decoded were pinned down to the roots of a higher degree polynomial over a large residue field. As mentioned earlier, effecting this “non-linear” approach in \[11\] \[6\] with automorphisms of more general function fields seems intricate at best. In this work we employ the linear-algebraic list decoding method of \[13\]. However, the correct generalization of the linear-algebraic list decoding approach to the function field case is also not obvious. One of the main algebraic insights in this work is noting that the right way to generalize the linear-algebraic approach to codes based on algebraic function fields is to rely on the local power series expansion of functions from the message space at a suitable rational point. (The case for Reed-Solomon codes being the expansion around 0, which is a finite polynomial form.)

Working with a suitable automorphism which has a “diagonal” action on the local expansion lets us extend the linear-algebraic decoding method to AG codes. Implementing this for specific AG codes requires an explicit specification of a basis for an associated message (Riemann-Roch) space, and the efficient computation of the local expansion of
the basis elements at a special rational point on the curve. We show how to do this for two towers of function fields: the Hermitian tower [22] and the asymptotically optimal Garcia-Stichtenoth tower [4, 5]. The former tower is quite simple to handle — it has an easily written down explicit basis, and we show how to compute the local expansion of functions around the point with all zero coordinates. However, the Hermitian tower does not have bounded ratio of the genus to number of rational points, and so does not give constant alphabet codes (we can get codes over an alphabet size that is polylogarithmic in the block length though). Explicit basis for Riemann-Roch spaces of the Garcia-Stichtenoth tower were constructed in [23]. Regarding local expansions, one major difference is that we work with local expansion of functions at the point at infinity, which is fully “ramified” in the tower. For both these towers, we find and work with a nice automorphism that acts diagonally on the local expansion, and use it for folding the codes and decoding them by solving a linear system.

2.1.2. Restricting evaluation points to a subfield. The second approach is to work with “normal” algebraic-geometric codes, based on evaluating functions from a Riemann-Roch space at some rational places, except we use a constant field extension of the function field for the function space, but restrict to evaluating at rational places over the original base field. Let use give a brief idea why restricting evaluation points to a subfield enables correcting more errors. The idea behind list decoding results for folded RS (or derivative) codes in [11, 14] is that the encoding of a message polynomial \( f \in \mathbb{F}_q[X] \) includes the values of \( f \) and closely related polynomials at the evaluation points. Given a string not too far from the encoding of \( f \), one can use this property together with the “interpolation method” to find an algebraic condition that \( f \) (and its closely related polynomials) must satisfy, eg. \( A_0(X) + A_1(X)f(X) + A_2(X)f'(X) + \cdots + A_s(X)f^{(s-1)}(X) = 0 \) in the case of derivative codes [14] (here \( f^{(i)} \) denotes the \( i^{th} \) formal derivative of \( f \), and the \( A_0, A_1, \ldots, A_s \) are low-degree polynomials found by the decoder). The solutions \( f(X) \) to this equation form an affine space, which can be efficiently found (and later pruned for list size reduction when we pre-code messages into a subspace-evasive set).

For Reed-Solomon codes as in Definition [12] the encoding only includes the values of \( f \) at \( \alpha_1, \alpha_2, \ldots, \alpha_n \). But since \( \alpha_i \in \mathbb{F}_q \), we have \( f(\alpha_i)^q = f^\sigma(\alpha_i) \) where \( f^\sigma \) is the polynomial obtained by the action of the Frobenius automorphism that maps \( y \mapsto y^q \) on \( f \) (formally, \( f^\sigma(X) = \sum_{j=0}^{k-1} f_j^q X^j \) if \( f(X) = \sum_{j=0}^{k-1} f_j X^j \)). Thus the decoder can “manufacture” the values of \( f^\sigma \) (and similarly \( f^{\sigma^2}, f^{\sigma^3}, \text{etc.} \)) at the \( \alpha_i \). Applying the above approach then enables finding a relation \( A_0(X) + A_1(X)f(X) + A_2(X)f^\sigma(X) + \cdots + A_s(X)f^{\sigma^{s-1}}(X) = 0 \), which is again an \( \mathbb{F}_q \)-linear condition on \( f \) that can be used to solve for \( f \). We remark here that this approach can also be applied effectively to linearized polynomials, and can be used to construct variants of Gabidulin codes that are list-decodable up to the optimal \( 1 - R \) fraction of errors (where \( R \) is the rate) in the rank metric [15].

To extend this idea to algebraic-geometric codes, we work with constant extensions \( \mathbb{F}_{q^m} \cdot F \) of algebraic function fields \( F/\mathbb{F}_q \). The messages belong to a Riemann-Roch space over \( \mathbb{F}_{q^m} \), but they are encoded via their evaluations at \( \mathbb{F}_q \)-rational points. For decoding, we recover the message function \( f \) in terms of the coefficients of its local expansion at some rational point \( P \). (The Reed-Solomon setting is a special case when \( F = \mathbb{F}_q(X) \), and \( P = 0 \), i.e., the zero of \( X \).) To get the best trade-offs, we use AG codes based on a tower of function fields due to Garcia and Stichtenoth [4, 5] which achieve the optimal trade-off between the number of \( \mathbb{F}_q \)-rational points and the genus. For this case, we recover messages in terms of their local expansion around the point at infinity \( P_\infty \) which is also
used to define the Riemann-Roch space of messages. So we treat this setting separately (Section 9.3), after describing the framework for general AG codes first (Section 9.2).

2.2. **Pseudorandomness.** The above algebraic ideas enable us to pin down the messages into a structured subspace of dimension linear in the message length. The specific structure of the subspace is a certain “periodicity” — there is a subspace $W \subset \mathbb{F}_q^m$ such that once $f_0, f_1, \ldots, f_{i-1}$ (the first $i$ coefficients of the message polynomial) are fixed, $f_i$ belongs to a coset of $W$. We now describe our ideas to prune this list, by restricting (or “pre-coding”) the message polynomials to belong to carefully constructed pseudorandom subsets that have small intersection with any periodic subspace.

2.2.1. **Hierarchical subspace-evasive sets.** The first approach to follows along the lines of [14] and only encode messages in a **subspace-evasive set** which has small intersection with low-dimensional subspaces. Implementing this in our case, however, leads to several problems. First, since the subspace we like to avoid intersecting much has large dimension, the list size bound will be linear in the code length and not a constant like in our final result. More severely, we cannot go over the elements of this subspace to prune the list as that would take exponential time. To solve the latter problem, we observe that the subspace has a special “periodic” structure, and exploit this to show the existence of large “hierarchically subspace evasive” (h.s.e) subsets which have small intersection with the projection of the subspace on certain prefixes. Isolating the periodic property of the subspaces, and formulating the right notion of evasiveness w.r.t to such subspaces, is an important aspect of this work.

We also give a pseudorandom construction of good h.s.e sets using limited wise independent sample spaces, in a manner enabling the efficient iterative computation of the final list of intersecting elements. Further our construction allows for efficient indexing into the h.s.e set which leads to an efficient encoding algorithm for our code). As a further ingredient, we note that the number of possible subspaces that arise in the decoding is much smaller than the total number of possibilities. Using this together with an added trick in the h.s.e set construction, we are able to reduce the list size to a constant.

2.2.2. **Subspace designs.** The approach based on h.s.e sets leads to excellent list size; however, we only know randomized constructions of h.s.e sets with the required properties. Our second approach to prune the subspace of possible solutions is based on **subspace designs** and leads to deterministic subcode constructions. Recall that the coefficients $f_0, f_1, \ldots, f_{k-1}$ of the message polynomial (which belong to the extension field $\mathbb{F}_{q^m}$) are pinned down by the linear-algebraic list decoder to a periodic subspace with the property that there is an $\mathbb{F}_q$-subspace $W \subset \mathbb{F}_{q^m}$ such that once $f_0, f_1, \ldots, f_{i-1}$ are fixed, $f_i$ belongs to a coset of $W$. Our idea then is to restrict $f_i$ to belong to a subspace $H_i$ where $H_1, H_2, \ldots, H_k$ are a collection of subspaces in $\mathbb{F}_{q^m}$ such that for any $s$-dimensional subspace $W \subset \mathbb{F}_{q^m}$, only a small number of them have non-trivial intersection with $W$. More precisely, we require that $\sum_{i=1}^k \dim(W \cap H_i)$ is small. We call such a collection as a **subspace design** in $\mathbb{F}_{q^m}$. We feel that the concept of subspace designs is interesting in its own right, and view the introduction of this notion in Section 10 as a key contribution in this work. Indeed, subsequent work by Forbes and Guruswami [3] highlighted the central role played by subspace designs in “linear-algebraic pseudorandomness” and in particular how they lead to rank condensers and dimension expanders.

A simple probabilistic argument shows the existence of $q^{\Omega(\varepsilon m)}$ random subspaces of dimension $(1 - \varepsilon)m$ have small total intersection with every $s$-dimensional $W$. This construction can also be derandomized, though the construction complexity of the resulting
codes becomes quasi-polynomial with this approach for the parameter choices needed in the construction.

Fortunately, in a follow-on to [17], Guruswami and Kopparty gave explicit constructions of subspace designs with parameters nearly matching the random constructions [8]. One can pre-code with this subspace design to get explicit list-decodable sub-codes of Reed-Solomon codes whose evaluation points are in a subfield (Section 11.1). However, this construction inherits the large field size of Reed-Solomon codes.

For explicit subcodes of algebraic-geometric codes using subspace designs we need additional ideas. The dimension $k$ in the case of AG codes is much larger than the alphabet size $q^m$ (in fact that is the whole point of generalizing to AG codes). So we cannot have a subspace design in $\mathbb{F}_q^m$ with $k$ subspaces. We therefore use several “layers” of subspace designs in a cascaded fashion (Section 10.2) — the first one in $\mathbb{F}_q^m$, the next one in $\mathbb{F}_q^{m_1}$ for $m_1 \gg q^{\sqrt{m}}$, the third one in $\mathbb{F}_q^{m_2}$ for $m_2 \gg q^{\sqrt{m_1}}$ and so on. Since the $m_i$’s increase exponentially, we only need about $\log^* k$ levels of subspace designs. Each level incurs about a factor $1/\epsilon$ increase in the dimension of the “periodic subspace” ($W$ when we begin) at the corresponding scale. With a careful technical argument and choice of parameters, we are able to obtain the bounds of Theorem 1.1(ii).

3. Periodic subspaces

In this section we formalize a certain “periodic” property of affine subspaces that will arise in our list decoding application.

We begin with some notation. For a vector $y = (y_1, y_2, \ldots, y_m) \in \mathbb{F}_q^m$ and positive integers $t_1 \leq t_2 \leq m$, we denote by $\text{proj}_{[t_1, t_2]}(y) \in \mathbb{F}_q^{t_2-t_1+1}$ its projection onto coordinates $t_1$ through $t_2$, i.e., $\text{proj}_{[t_1, t_2]}(y) = (y_{t_1}, y_{t_1+1}, \ldots, y_{t_2})$. When $t_1 = 1$, we use $\text{proj}_1(y)$ to denote $\text{proj}_{[1, t]}(y)$. These notions are extended to subsets of strings in the obvious way: $\text{proj}_{[t_1, t_2]}(S) = \{\text{proj}_{[t_1, t_2]}(x) \mid x \in S\}$.

**Definition 1** (Periodic subspaces). For positive integers $r, b, \Delta$ and $\kappa := b\Delta$, an affine subspace $H \subset \mathbb{F}_q^\kappa$ is said to be $(r, \Delta, b)$-periodic if there exists a subspace $W \subseteq \mathbb{F}_q^\Delta$ of dimension at most $r$ such that for every $j = 1, 2, \ldots, b$, and every “prefix” $a \in \mathbb{F}_q^{(j-1)\Delta}$, the projected affine subspace of $\mathbb{F}_q^\Delta$ defined as

$$\{\text{proj}_{[j-1, j\Delta]}(x) \mid x \in H \text{ and } \text{proj}_{(j-1)\Delta}(x) = a\}$$

is contained in an affine subspace of $\mathbb{F}_q^\Delta$ given by $W + v_a$ for some vector $v_a \in \mathbb{F}_q^\Delta$ dependent on $a$.

The motivation of the above definition will be clear when we present our linear-algebraic list decoders, which will pin down the messages that must be output within an $(s-1, m, k)$-periodic (affine) subspace of $\mathbb{F}_q^{mk}$ (where $q^m$ will be the alphabet size of the code, $k$ its dimension, and $s$ a parameter of the algorithm that governs how close the decoding performance approaches the Singleton bound).

The following properties of periodic affine spaces follow from the definition.

**Claim 3.1.** Let $H$ be an $(r, \Delta, b)$-periodic affine subspace. Then for each $j = 1, 2, \ldots, b$,

1. *the projection of $H$ to the first $j$ blocks of $\Delta$ coordinates, $\text{proj}_j(\Delta)(H) = \{\text{proj}_j(\Delta)(x) \mid x \in H\}$, has dimension at most $jr$. (In particular $H$ has dimension at most $br$.)*
(2) for each \( a \in \mathbb{F}_q^{(j-1)\Delta} \), there are at most \( q^r \) extensions \( y \in \text{proj}_{j\Delta}(H) \) such that \( \text{proj}_{(j-1)\Delta}(y) = a \).

For an affine space \( H \), its underlying subspace is the subspace \( S \) such that \( H \) is a coset of \( S \).

**Definition 2** (Representing periodic affine subspaces). The canonical representation of an \((r, \Delta, b)\)-periodic subspace \( H \) consists of a matrix \( B \in \mathbb{F}_q^{\Delta \times \Delta} \) such that \( \ker(B) \) has dimension at most \( r \), and vectors \( a_i \in \mathbb{F}_q \) and matrices \( A_{i,j} \in \mathbb{F}_q^{\Delta \times \Delta} \) for \( 1 \leq i \leq b \) and \( 1 \leq j < i \), such that \( x \in H \) if and only if for every \( i = 1, 2, \ldots, b \) the following holds:

\[
a_i + \left( \sum_{j=1}^{i-1} A_{i,j} \cdot \text{proj}_{(j-1)\Delta+1,j\Delta}(x) \right) + B \cdot \text{proj}_{(i-1)\Delta+1,i\Delta}(x) = 0.
\]

**Ultra-periodic subspaces.** For our result on pre-coding algebraic-geometric codes with subspace designs, we will exploit an even stronger property that holds for the subspaces output by the linear-algebraic list decoder. We formalize this notion below.

**Definition 3.** An affine subspace \( H \) of \( \mathbb{F}_q^n \) is said to be \((r, \Delta)\)-ultra periodic if for every integer \( \ell, 1 \leq \ell \leq \frac{\Delta}{r} \), setting \( b_{\ell} = \left\lceil \frac{\Delta}{\ell} \right\rceil \), we have \( \text{proj}_{b_{\ell},\ell\Delta}(H) \) is \((\ell r, \ell \Delta, b_{\ell})\)-periodic.

The definition captures the fact that the subspace is periodic not only for blocks of size \( \Delta \), but also for block sizes that are multiples of \( \Delta \). Thus the subspace looks periodic in all “scales” simultaneously.

### 4. Preliminaries on function fields and algebraic-geometric codes

For convenience of the reader, we start with some background on global function fields over finite fields. The reader may refer to [25, 20] for detailed background on function fields and algebraic-geometric codes.

For a prime power \( q \), let \( \mathbb{F}_q \) be the finite field of \( q \) elements. An algebraic function field over \( \mathbb{F}_q \) in one variable is a field extension \( F \supset \mathbb{F}_q \) such that \( F \) is a finite algebraic extension of \( \mathbb{F}_q(x) \) for some \( x \in F \) that is transcendental over \( \mathbb{F}_q \). The field \( \mathbb{F}_q \) is called the full constant field of \( F \) if the algebraic closure of \( \mathbb{F}_q \) in \( F \) is \( \mathbb{F}_q \) itself. Such a function field is also called a global function field. From now on, we always denote by \( F/\mathbb{F}_q \) a function field \( F \) with the full constant field \( \mathbb{F}_q \).

Let \( \mathbb{P}_F \) denote the set of places of \( F \). The divisor group, denoted by \( \text{Div}(F) \), is the free abelian group generated by all places in \( \mathbb{P}_F \). An element \( G = \sum_{P \in \mathbb{P}_F} n_P P \) of \( \text{Div}(F) \) is called a divisor of \( F \), where \( n_P = 0 \) for almost all \( P \in \mathbb{P}_F \). We denote \( n_P \) by \( \nu_P(G) \). The support, denoted by \( \text{Supp}(G) \), of \( G \) is the set \( \{ P \in \mathbb{P}_F : n_P \neq 0 \} \). For a nonzero function \( z \in F \), the principal divisor of \( z \) is defined to be \( \text{div}(z) = \sum_{P \in \mathbb{P}_F} \nu_P(z)P \), where \( \nu_P \) denotes the normalized discrete valuation at \( P \). The zero and pole divisors of \( z \) are defined to be \( \text{div}(z)_0 = \sum_{\nu_P(z) > 0} \nu_P(z)P \) and \( \text{div}(z)_{\infty} = -\sum_{\nu_P(z) < 0} \nu_P(z)P \), respectively.

**Riemann-Roch space.** For a divisor \( G \) of \( F \), we define the Riemann-Roch space associated with \( G \) by

\[
\mathcal{L}(G) := \{ f \in F^* : \text{div}(f) + G \geq 0 \} \cup \{0\},
\]

where \( F^* \) denotes the set of nonzero elements of \( F \). Then \( \mathcal{L}(G) \) is a finite dimensional space over \( \mathbb{F}_q \) and its dimension \( \ell(G) \) is determined by the Riemann-Roch theorem which gives

\[
\ell(G) = \deg(G) + 1 - g + \ell(W - G),
\]
where \( g \) is the genus of \( F \) and \( W \) is a canonical divisor of degree \( 2g - 2 \). Therefore, we always have that \( \ell(G) \geq \deg(G) + 1 - g \) and the equality holds if \( \deg(G) \geq 2g - 1 \).

Consider finite extension \( \mathbb{F}_q^m \) over \( \mathbb{F}_q \) and the constant extension \( F_m := \mathbb{F}_q^m \cdot F \) over \( F \). A divisor \( G \) of \( F \) can be viewed as a divisor of \( F_m \). Thus, we can consider the Riemann-Roch space in \( F_m \) given by

\[
\mathcal{L}_m(G) := \{ f \in F_m^*: \text{div}(f) + G \geq 0 \} \cup \{ 0 \}.
\]

Then it is clear that \( \mathcal{L}_m(G) \) contains \( \mathcal{L}(G) \) and \( \mathcal{L}_m(G) \) is a finite dimensional vector space over \( \mathbb{F}_q^m \). Furthermore, \( \mathcal{L}_m(G) \) is the tensor product of \( \mathcal{L}(G) \) with \( \mathbb{F}_q^m \) (see [24, Proposition 5.8 of Chapter II]). This implies that

\[
\dim_{\mathbb{F}_q^m}(\mathcal{L}_m(G)) = \dim_{\mathbb{F}_q}(\mathcal{L}(G))
\]

and an \( \mathbb{F}_q \)-basis of \( \mathcal{L}(G) \) is also an \( \mathbb{F}_q^m \)-basis of \( \mathcal{L}_m(G) \).

**Automorphism.** For a function \( f \) and a place \( P \in \mathbb{P}_F \) with \( \nu_P(f) \geq 0 \), we denote by \( f(P) \) the residue class of \( f \) in the residue class field \( F_p \) at \( P \). For an automorphism \( \phi \in \text{Aut}(F/\mathbb{F}_q) \) and a place \( P \), we denote by \( \phi(P) \) the place \( \{ \phi(x) : x \in P \} \). For a function \( f \in F \), we denote by \( f^\phi \) the action of \( \phi \) on \( f \). If \( \nu_P(f) \geq 0 \) and \( \nu_{\phi(P)}(f) \geq 0 \), then one has that \( \nu_P(f^\phi) \geq 0 \) and \( f^\phi = f^{\phi^{-1}}(P) \). Furthermore, for a divisor \( G = \sum_{P \in \mathbb{P}_F} m_P P \) we denote by \( G^\phi \) the divisor \( \sum_{P \in \mathbb{P}_F} m_P P^\phi \). Therefore, we have

\[
\phi(\mathcal{L}(G)) := \{ f^\phi : f \in \mathcal{L}(G) \} = \mathcal{L}(G^\phi).
\]

Assume that \( E/\mathbb{F}_q \) is a subfield of \( F \) and \( \phi \) is an automorphism of \( \text{Aut}(F/E) \). Then for a divisor \( G \) of \( F \) that is invariant under \( \phi \), then we have \( \phi(\mathcal{L}(G)) = \mathcal{L}(G) \). Next we consider the constant extension \( F_m = \mathbb{F}_q^m \cdot F \). Let \( \sigma \) be the Frobenius automorphism \( \mathbb{F}_q^m \), i.e., \( \sigma(\alpha) = \alpha^q \) for any \( \alpha \in \mathbb{F}_q^m \). Then \( \sigma \) can be extended to an automorphism of \( \text{Aut}(F_m/F) \).

**Local expansion (or power series).** Let \( F/\mathbb{F}_q \) be a function field and let \( P \) be a rational place. An element \( t \) of \( F \) is called a local parameter at \( P \) if \( \nu_P(t) = 1 \) (such a local parameter always exists). For a nonzero function \( f \in F \) with \( \nu_P(f) \geq v \), we have \( \nu_P \left( \frac{f}{t} \right) \geq 0 \). Put \( a_v = \left( \frac{f}{t} \right) (P) \), i.e., \( a_v \) is the value of the function \( f/t^v \) at \( P \). Note that the function \( f/t^v - a_v \) satisfies \( \nu_P \left( \frac{f}{t} - a_v \right) \geq 1 \), hence we know that \( \nu_P \left( \frac{f - a_v t^v}{t^{v+1}} \right) \geq 0 \). Put \( a_{v+1} = \left( \frac{f - a_v t^v}{t^{v+1}} \right) (P) \). Then \( \nu_P(f - a_v t^v - a_{v+1} t^{v+1}) \geq v + 2 \).

Assume that we have obtained a sequence \( \{ a_r \}_{r=v}^m (m > v) \) of elements of \( \mathbb{F}_q \) such that \( \nu_P(f - \sum_{r=v}^k a_r t^r) \geq k + 1 \) for all \( v \leq k \leq m \). Put \( a_{m+1} = \left( \frac{f - \sum_{r=v}^m a_r t^r}{t^{m+1}} \right) (P) \). Then \( \nu_P(f - \sum_{r=v}^{m+1} a_r t^r) \geq m + 2 \). In this way we continue our construction of \( a_r \). Then we obtain an infinite sequence \( \{ a_r \}_{r=v}^\infty \) of elements of \( \mathbb{F}_q \) such that \( \nu_P(f - \sum_{r=v}^m a_r t^r) \geq m + 1 \) for all \( m \geq v \). We summarize the above construction in the formal expansion

\[
f = \sum_{r=v}^\infty a_r t^r,
\]

which is called the **local expansion** of \( f \) at \( P \).

It is clear that the local expansions of a function depends on the choice of the local parameters \( t \). Note that if a power series \( \sum_{i=v}^\infty a_i t^i \) satisfies \( \nu_P(f - \sum_{i=v}^m a_i t^i) \geq m + 1 \) for all \( m \geq v \), then it is a local expansion of \( f \). The above procedure shows that finding a local expansion at a rational place is very efficient as long as the computation of evaluations of functions at this place is easy.
If $f$ belongs to a Riemann-Roch space $\mathcal{L}(G)$ with $\deg(G) = d$. Denote $\nu_P(G)$ by $v$, then the first $d+1$ coefficients $a_v, a_{v+1}, \ldots, a_{v+d}$ in (1) determines the function $f$. To see this, assume that $g$ is a function of $\mathcal{L}(G)$ with the same first $d+1$ coefficients in its local expansion. Then we have $f - g \in \mathcal{L}(G - (d+1)P)$ which is the zero vector space. This implies that $f = g$.

**Algebraic-geometric codes.** Let $P = \{P_1, P_2, \ldots, P_N\}$ be a set of $N$ distinct rational places of a function field $F/\mathbb{F}_q$ of genus $g$. Let $G$ be a divisor of $F$ with $\text{Supp}(G) \cap P = \emptyset$. Then the algebraic-geometric codes defined by

$$C(P, G) := \{(f(P_1), f(P_2), \ldots, f(P_N)) : f \in \mathcal{L}(G)\}$$

is an $\mathbb{F}_q$-linear code of length $N$. Furthermore, the dimensions of $C(P, G)$ is equal to $\ell(G)$ if $N > \deg(G)$.

The codes considered in this paper are variations of the above algebraic-geometric codes, namely, folded algebraic-geometric codes and algebraic-geometric codes with evaluation points in a subfield.

5. Folded codes from the Hermitian tower

In this section, we will describe a family of folded codes based on the Hermitian function field (or rather a tower of such fields).

5.1. Background on Hermitian tower. In what follows, let $r$ be a prime power and let $q = r^2$. We denote by $\mathbb{F}_q$ the finite field with $q$ elements. The Hermitian function tower that we are going to use for our code construction was discussed in [22]. The reader may refer to [22] for the detailed background on the Hermitian function tower, and Stichtenoth’s book [25] for general background on algebraic function fields and their use in constructing algebraic-geometric codes. The Hermitian tower is defined by the following recursive equations

$$x_{i+1}^r + x_{i+1} = x_i^{r+1}, \quad i = 1, 2, \ldots, e - 1.$$ 

Put $F_e = \mathbb{F}_q(x_1, x_2, \ldots, x_e)$ for $e \geq 2$. We will assume that $r \geq 2e$.

**Rational places.** The function field $F_e$ has $r^{e+1} + 1$ rational places. One of these is the “point at infinity” which is the unique pole $P_{\infty}$ of $x_1$ (and is fully ramified). The other $r^{e+1} - 1$ come from the rational places lying over the unique zero $P_\alpha$ of $x_1 - \alpha$ for each $\alpha \in \mathbb{F}_q$. Note that for every $\alpha \in \mathbb{F}_q$, $P_\alpha$ splits completely in $F_e$, i.e., there are $r^{e-1}$ rational places lying over $P_\alpha$. Intuitively, one can think of the rational places of $F_e$ (besides $P_{\infty}$) as being given by $e$-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_e) \in \mathbb{F}_q^e$ that satisfy $\alpha_{i+1} + \alpha_{i+1} = \alpha_i^{r+1}$ for $i = 1, 2, \ldots, e - 1$. For each value of $\alpha \in \mathbb{F}_q$, there are precisely $r$ solutions to $\beta \in \mathbb{F}_q$ satisfying $\beta^r + \beta = \alpha^{r+1}$, so the number of such $e$-tuples is $r^{e+1}$ ($q = r^2$ choices for $\alpha_1$, and then $r$ choices for each successive $\alpha_i$, $2 \leq i \leq e$).

**Riemann-Roch spaces.** For a place $P$ of $F_e$, we denote by $\nu_P$ the discrete valuation of $P$: for a function $h \in F_e$, if $h$ has a zero at $P$, then $\nu_P(h)$ gives the number (multiplicity) of zeroes, if $h$ has a pole at $P$, then $-\nu_P(h)$ gives the pole order of $h$ at $P$, and $\nu_P(h) = 0$ if $h$ has neither a zero or a pole at $P$.

For an integer $l$, we consider the Riemann-Roch space defined by

$$\mathcal{L}(lP_{\infty}) := \{h \in F_e \setminus \{0\} : \nu_{P_{\infty}}(h) \geq -l\} \cup \{0\}.$$
Then the dimension $\ell(lP_\infty)$ is at least $l - g_e + 1$ and furthermore, $\ell(lP_\infty) = l - g_e + 1$ if $l \geq 2g_e - 1$. A basis over $\mathbb{F}_q$ of $\mathcal{L}(lP_\infty)$ can be explicitly constructed as follows

$$\left\{ x_1^{j_1} \cdots x_e^{j_e} : (j_1, \ldots, j_e) \in \mathbb{Z}_{\geq 0}^e, \sum_{i=1}^e j_i r^{e-i}(r+1)^{i-1} \leq l \right\}. \tag{2}$$

We stress that evaluating elements of $\mathcal{L}(lP_\infty)$ at the rational places of $F_e$ (other than $P_\infty$) is easy: we simply have to evaluate a linear combination of the monomials allowed in (2) at the tuples $(\alpha_1, \alpha_2, \ldots, \alpha_e) \in \mathbb{F}_q^e$ mentioned above. In other words, it is just evaluating an $e$-variate polynomial at a specific subset of $r^{e+1}$ points of $\mathbb{F}_q^e$, and can be accomplished in polynomial time.

**Genus.** The genus $g_e$ of the function field $F_e$ is given by

$$g_e = \frac{1}{2} \left( \sum_{i=1}^{e-1} r^e \left( 1 + \frac{1}{r} \right)^{i-1} - (r+1)^{e-1} + 1 \right) \leq \frac{r^e}{2} \sum_{i=1}^e \left( \frac{e}{i} \right) \frac{1}{r^{i-1}} \leq \frac{er^e}{2} \sum_{i=1}^e \left( \frac{e}{r} \right)^{i-1} \leq er^e$$

where the last step used $r \geq 2e$.

**A useful automorphism.** Let $\gamma$ be a primitive element of $\mathbb{F}_q$ and consider the automorphism $\sigma \in \text{Aut}(F_e/\mathbb{F}_q)$ defined by

$$\sigma : x_i \mapsto \gamma^{(r+1)i-1} x_i \text{ for } i = 1, 2, \ldots, e.$$ 

The order of $\sigma$ is $q - 1$ and furthermore, we have the following facts:

(i) Let $P_0$ be the unique common zero of $x_1, x_2, \ldots, x_e$ (this corresponds to the $e$-tuple $(0, 0, \ldots, 0)$), and $P_\infty$ the unique pole of $x_1$. The automorphism $\sigma$ keeps $P_0$ and $P_\infty$ unchanged, i.e., $P_0^\sigma = P_0$ and $P_\infty^\sigma = P_\infty$.

(ii) Let $\mathcal{P}$ be the set of all the rational places which are neither $P_\infty$ nor zeros of $x_1$. Then $|\mathcal{P}| = (q - 1)r^{e-1}$. Moreover, $\sigma$ divides $\mathcal{P}$ into $r^{e-1}$ orbits and each orbit has $q - 1$ places. For an integer $m$ with $1 \leq m \leq q - 1$, we can label $Nm$ distinct elements $P_1, P_1^\sigma, \ldots, P_1^{m-1}, \ldots, P_N, P_N^\sigma, \ldots, P_N^{m-1}$ in $\mathcal{P}$, as long as $N \leq r^{e-1} \left\lfloor \frac{q-1}{m} \right\rfloor$.

**Definition 4** (Folded codes from the Hermitian tower). Assume that $m, l, N$ are positive integers satisfying $1 \leq m \leq q - 1$ and $l/m \leq N \leq r^{e-1} \left\lfloor \frac{q-1}{m} \right\rfloor$. The folded code from $F_e$ with parameters $N, l, q, e, m$, denoted by $\widetilde{\mathcal{F}} H(N, l, q, e, m)$, encodes a message function $f \in \mathcal{L}(lP_\infty)$ as

$$f \mapsto \left( \begin{array}{c} f(P_1) \\ f(P_1^\sigma) \\ \vdots \\ f(P_1^{m-1}) \\ f(P_2) \\ f(P_2^\sigma) \\ \vdots \\ f(P_2^{m-1}) \\ \vdots \\ f(P_N) \\ f(P_N^\sigma) \\ \vdots \\ f(P_N^{m-1}) \end{array} \right) \in \left( \mathbb{F}_q^m \right)^N. \tag{4}$$

**Lemma 5.1.** The above code $\widetilde{\mathcal{F}} H(N, l, q, e, m)$ is an $\mathbb{F}_q$-linear code over alphabet size $q^m$, rate at least $\ell - g_e + 1 \frac{1}{Nm}$, and minimum distance at least $N - \frac{1}{m}$.

**Proof.** It is clear that the map (4) is an $\mathbb{F}_q$-linear map. The dimension over $\mathbb{F}_q$ of the message space $\mathcal{L}(lP_\infty)$ is at least $l - g_e + 1$ by the Riemann-Roch theorem, which gives the claimed lower bound on rate. For the distance property, observe that if the $i$-th column is zero, then $f$ has $m$ zeros. This implies that the encoding of a nonzero function $f$ can have at most $l/m$ zero columns since $f \in \mathcal{L}(lP_\infty)$. \qed
5.2. **Redefining the code in terms of local expansion at** $P_0$. For our decoding, we will actually recover the message $f \in \mathcal{L}(lP_\infty)$ in terms of the coefficients of its power series expansion around $P_0$

$$f = f_0 + f_1 x + f_2 x^2 + \cdots$$

where $x := x_1$ is the local parameter at $P_0$ (which means that $x_1$ has exactly one zero at $P_0$, i.e., $\nu_{P_0}(x_1) = 1$). In fact, realizing that one must work in this power series representation is one of the key insights in this work.

Let us first show that one can efficiently move back-and-forth between the representation of $f \in \mathcal{L}(lP_\infty)$ in terms of a basis for $\mathcal{L}(lP_\infty)$ and its power series representation $(f_0, f_1, \ldots)$ around $P_0$. Since the mapping $f \mapsto (f_0, f_1, \ldots)$ is $\mathbb{F}_q$-linear, it suffices to compute the local expansion at $P_0$ of a basis for $\mathcal{L}(lP_\infty)$.

**Lemma 5.2.** For any $n$, one can compute the first $n$ terms of the local expansion of the basis elements $(2)$ at $P_0$ using $\text{poly}(n)$ operations over $\mathbb{F}_q$.

**Proof.** By the structure of the basis functions in $(2)$, it is sufficient to find an algorithm of efficiently finding local expansions of $x_i$ at $P_0$ for every $i = 1, 2, \ldots, e$. We can inductively find the local expansions of $x_i$ at $P_0$ as follows.

For $i = 1$, $x_1$ is the local parameter $x$ of $P_0$, so $x$ is the local expansion of $x_1$ at $P_0$.

Now assume that we know the local expansion of $x_i = \sum_{j=1}^\infty c_{i,j} x^j$ at $P_0$ for some $c_{i,j} \in \mathbb{F}_q$. Then we have

$$\sum_{j=1}^\infty c_{i+1,j} x^j + \sum_{j=1}^\infty c_{i+1,j} x^j = x_{i+1} + x_{i+1} = x_{i+1} = \left( \sum_{j=1}^\infty c_{i,j} x^j \right) \left( \sum_{j=1}^\infty c_{i,j} x^j \right).$$

By comparing the coefficients of $x^j$ in the above identity, we can easily solve $c_{i+1,j}$'s from $c_{i,j}$'s. More specifically, the coefficient of $x^j$ at the left of the identity is

$$\begin{cases} c_{i+1,j} & \text{if } r \not| j \\ c_{i,j} + c_{i+1,j/r} & \text{if } r | j. \end{cases}$$

Thus, all $c_{i+1,j}$'s can be easily solved recursively. \qed

To keep the list output by the algorithm at a controllable size, we will combine the code with certain special subspace evasive sets. For this purpose, we will actually need to index the messages of the code by the first $k$ coefficients $(f_0, f_1, \ldots, f_{k-1})$ of the local expansion of the function $f$ at $P_0$. This requires that for every $(f_0, f_1, \ldots, f_{k-1})$ there is a $f \in \mathcal{L}(lP_\infty)$ whose power series expansion has the $f_i$ as the first $k$ coefficients. This is easy to ensure by taking $l = k + 2g_e - 1$ as we argue below. Note that to ensure that $\mathcal{L}(lP_\infty)$ has dimension $k$, it suffices to pick $l = k + g_e - 1$ by the Riemann-Roch theorem. We pick $l$ to be $g_e$ more than this bound. Since the genus will be much smaller than the code length, we can afford this small loss in parameters.

Let us define the local expansion map $\text{ev}_{P_0} : \mathcal{L}((k+2g_e-1)P_\infty) \to \mathbb{F}_q^k$ that maps $f$ to $(f_0, f_1, \ldots, f_{k-1})$ where $f = f_0 + f_1 x + f_2 x^2 + \cdots$ is the local expansion of $f$ at $P_0$.

**Claim 5.3.** $\text{ev}_{P_0}$ is an $\mathbb{F}_q$-linear surjective map. Further, we can compute $\text{ev}_{P_0}$ using $\text{poly}(k, g_e)$ operations over $\mathbb{F}_q$ given a representation of the input $f \in \mathcal{L}((k+2g_e-1)P_\infty)$ in terms of the basis $(2)$.

**Proof.** The $\mathbb{F}_q$-linearity of $\text{ev}_{P_0}$ is clear. The kernel of $\text{ev}_{P_0}$ is $\mathcal{L}((k+2g_e-1)P_\infty - kP_0)$ which has dimension exactly $g_e$ by the Riemann-Roch theorem. By the rank-nullity theorem,
the image must have dimension $k$, and so the map is surjective. The claimed complexity of computation follows immediately from Lemma 5.2.

For each $(f_0, f_1, \ldots, f_{k-1}) \in \mathbb{F}_q^k$, we can therefore pick a pre-image in $L((k + 2g_e - 1)P_\infty)$. For convenience, we will denote an injective map making such a unique choice by $\kappa_{P_0} : \mathbb{F}_q^k \rightarrow L((k + 2g_e - 1)P_\infty)$. By picking the pre-images of a basis of $\mathbb{F}_q^k$ and extending it by linearity, we can assume $\kappa_{P_0}$ to be $\mathbb{F}_q$-linear, and thus specify it by a $(k + g_e) \times k$ matrix. We record this fact for easy reference below.

Claim 5.4. The map $\kappa_{P_0} : \mathbb{F}_q^k \rightarrow L((k + 2g_e - 1)P_\infty)$ is $\mathbb{F}_q$-linear and injective. We can compute a representation of this linear transformation using $\text{poly}(k, g_e)$ operations over $\mathbb{F}_q$, and the map itself can be evaluated using $\text{poly}(k, g_e)$ operations over $\mathbb{F}_q$.

We will now redefine a version of the folded Hermitian code that maps $\mathbb{F}_q^k$ to $(\mathbb{F}_q^m)^N$ by composing the folded encoding $\mathcal{F}_H$ from the original Definition 4 with $\kappa_{P_0}$.

Definition 5 (Folded Hermitian code using local expansion). The folded Hermitian code $\mathcal{F}_H(N, k, q, e, m)$ maps $f = (f_0, f_1, \ldots, f_{k-1}) \in \mathbb{F}_q^k$ to $\mathcal{F}_H(N, k + 2g_e - 1, q, e, m)(\kappa_{P_0}(f)) \in (\mathbb{F}_q^m)^N$.

The rate of the above code equals $k/(Nm)$ and its distance is at least $N - (k + 2g_e - 1)/m$.

5.3. List decoding folded codes from the Hermitian tower. We now present a list decoding algorithm for the above codes. The algorithm follows the linear-algebraic list decoding algorithm for folded Reed-Solomon codes. Suppose a codeword $\mathcal{F}_H$ encoding $f \in \text{Im}(\kappa_{P_0}) \subseteq L((k + 2g_e - 1)P_\infty)$ is transmitted and received as

$$y = \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{N,1} & y_{N,2} & \cdots & y_{N,m} \end{pmatrix}$$

where some columns are erroneous. Let $s \geq 1$ be an integer parameter associated with the decoder.

Lemma 5.5. Given a received word as in (5), using $\text{poly}(N)$ operations over $\mathbb{F}_q$, we can find a nonzero linear polynomial in $F_\sigma[y_1, y_2, \ldots, y_s]$ of the form

$$Q(Y_1, Y_2, \ldots, Y_s) = A_0 + A_1Y_1 + A_2Y_2 + \cdots + A_sY_s$$

satisfying

$$Q(y_{i,j}, y_{i,j+1}, \ldots, y_{i,j+s-1}) = A_0(P_1^\sigma) + A_1(P_1^\sigma)y_{i,j+1} + \cdots + A_s(P_1^\sigma)y_{i,j+s} = 0$$

for $i = 1, 2, \ldots, N$ and $j = 0, 1, \ldots, m-s$. The coefficients $A_i$ of $Q$ satisfy $A_i \in L(DP_\infty)$ for $i = 1, 2, \ldots, s$ and $A_0 \in L((D + k + 2g_e - 1)P_\infty)$ for a “degree” parameter $D$ chosen as

$$D = \left\lfloor \frac{N(m - s + 1) - k + (s - 1)g_e - 1 + k}{s + 1} \right\rfloor.$$

Proof. If we fix a basis of $L(DP_\infty)$ (of the form $2$) and extend it to a basis of $L((D + k + 2g_e - 1)P_\infty)$, then the number of freedoms of $A_0$ is at least $D + k + g_e$ and the number of freedoms of $A_i$ is at least $D - g_e + 1$ for $i \geq 1$. Thus, the total number of freedoms in the polynomial $Q$ equals

$$s(D - g_e + 1) + D + k + g_e = (s + 1)(D + 1) - (s - 1)g_e - 1 + k > N(m - s + 1)$$

$$\square$$
for the above choice $\mathbf{y}$ of $D$. The interpolation requirements on $Q \in \mathbb{F}_q[Y_1, \ldots, Y_s]$ are the following:

(10) $Q(y_{i,j}, y_{i,j+1}, \ldots, y_{i,j+s-1}) = A_0(P_{i}^{\sigma^j}) + A_1(P_{i}^{\sigma^j})y_{i,j+1} + \cdots + A_s(P_{i}^{\sigma^j})y_{i,j+s} = 0$

for $i = 1, 2, \ldots, N$ and $j = 0, 1, \ldots, m - s$. The interpolation requirements on $Q$ give a total of $N(m - s + 1)$ homogeneous linear equations that the coefficients of the $A_i$'s w.r.t. the chosen basis of $\mathcal{L}(D + k + 2g_e - 1)P_\infty$ must satisfy. Since the number of such coefficients (degrees of freedom in $Q$) exceeds $N(m - s + 1)$, we can conclude that such a linear polynomial $Q$ as required by the lemma must exist, and can be found by solving a homogeneous linear system over $\mathbb{F}_q$ with about $N(m - s + 1)$ variables and constraints. □

Similar to earlier interpolation based list decoding algorithms, the following lemma gives an algebraic condition that the message functions $f \in \mathcal{L}(k + 2g_e - 1)P_\infty$ we are interested in list decoding must satisfy. The proof is a standard argument comparing the pole order to the number of zeroes.

**Lemma 5.6.** If $f$ is a function in $\mathcal{L}(k + 2g_e - 1)P_\infty$ whose encoding (4) agrees with the received word $y$ in at least $t$ columns with $t > \frac{D + k + 2g_e - 1}{m-s+1}$, then

(11) $Q(f, f^{\sigma^{-1}}, \ldots, f^{\sigma^{-(s-1)}}) = A_0 + A_1 f + A_2 f^{\sigma^{-1}} + \cdots + A_s f^{\sigma^{-(s-1)}} = 0.$

**Proof.** The proof proceeds by comparing the number of zeroes of the function $Q(f, f^{\sigma^{-1}}, \ldots, f^{\sigma^{-(s-1)}}) = A_0 + A_1 f + A_2 f^{\sigma^{-1}} + \cdots + A_s f^{\sigma^{-(s-1)}}$ with $D + k + 2g_e - 1$. Note that $Q(f, f^{\sigma^{-1}}, \ldots, f^{\sigma^{-(s-1)}})$ is a function in $\mathcal{L}(D + k + 2g_e - 1)P_\infty$. If column $i$ of the encoding (4) of $f$ agrees with $y$, then for all $j = 0, 1, \ldots, m-s$, we have

\[
0 = A_0(P_{i}^{\sigma^j}) + A_1(P_{i}^{\sigma^j})y_{i,j+1} + A_2(P_{i}^{\sigma^j})y_{i,j+2} + \cdots + A_s(P_{i}^{\sigma^j})y_{i,j+s} = A_0(P_{i}^{\sigma^j}) + A_1(P_{i}^{\sigma^j})f(P_{i}^{\sigma^j}) + A_2(P_{i}^{\sigma^j})f(P_{i}^{\sigma^{j+1}}) + \cdots + A_s(P_{i}^{\sigma^j})f(P_{i}^{\sigma^{j+s-1}}) = A_0(P_{i}^{\sigma^j}) + A_1(P_{i}^{\sigma^j})f(P_{i}^{\sigma^j}) + A_2(P_{i}^{\sigma^j})f^{\sigma^{-1}}(P_{i}^{\sigma^j}) + \cdots + A_s(P_{i}^{\sigma^j})f^{\sigma^{-(s-1)}}(P_{i}^{\sigma^j}) = (A_0 + A_1 f + A_2 f^{\sigma^{-1}} + \cdots + A_s f^{\sigma^{-(s-1)}})(P_{i}^{\sigma^j}).
\]

Note that here we use the fact that $f^\sigma(P) = f(P)^\sigma = f(P)$, or equivalently $f(P^\sigma) = f^\sigma(P)$. In other words, $Q(f, f^{\sigma^{-1}}, \ldots, f^{\sigma^{-(s-1)}})$ has $(m-s+1)$ distinct zeroes from this agreeing column. Thus, there are a total of at least $t(m-s+1)$ zeroes for all the agreeing columns. Hence, $Q(f, f^{\sigma^{-1}}, \ldots, f^{\sigma^{-(s-1)}})$ must be the zero function when $t(m-s+1) > D + k + 2g_e - 1$. □

**Solving the functional equation for $f$.** Our goal next is to recover the list of solutions $f$ to the functional equation (11). Recall that our message functions lie in $\text{Im}(\kappa P_0)$, so we can recover $f$ by recovering the top $k$ coefficients $(f_0, f_1, \ldots, f_{k-1})$ of its local expansion $f = \sum_{j=0}^{\infty} f_j x^j$ at $P_0$. We now prove that $(f_0, f_1, \ldots, f_{k-1})$ for $f$ satisfying Equation (11) belong to a “periodic” subspace (in the sense of Definition 4) of not too large dimension.

**Lemma 5.7.** The set of solutions $(f_0, f_1, \ldots, f_{k-1}) \in \mathbb{F}_q^k$ such that $f = f_0 + f_1 x + f_2 x^2 + \cdots \in \mathcal{L}(k + 2g_e - 1)P_\infty$ obeys equation

(12) $A_0 + A_1 f + A_2 f^{\sigma^{-1}} + \cdots + A_s f^{\sigma^{-(s-1)}} = 0,$

when the $A_i$’s obey the pole order restrictions of Lemma 5.5 and at least one $A_i$ is nonzero, is an $(s-1, q-1, \lceil \frac{k}{q-1} \rceil)$-periodic subspace (note that we do not require $(q-1)|k$ here and one could extend the subspace to length $(q-1)\lceil \frac{k}{q-1} \rceil$ by padding with 0’s). Further, there are at most $q^{Nm+s+1}$ possible choices of this subspace over varying choices of the $A_i$’s.
Proof. Let \( u = \min \{ \nu_{P_i}(A_i) : i = 1, 2, \ldots, s \} \). Then it is clear that \( u \geq 0 \) and \( \nu_{P_0}(A_0) \geq u \). Each \( A_i \) has a local expansion at \( P_0 \):

\[ A_i = x^u \sum_{j=0}^{\infty} a_{i,j} x^j \]

for \( i = 0, 1, \ldots, s - 1 \), which can be efficiently computed from the basis representation of the \( A_i \)'s. From the definition of \( u \), one knows that the polynomial

\[ B_0(X) := a_{1,0} + a_{2,0}X + \cdots + a_{s,0}X^{s-1} \]

is nonzero. Assume that at \( P_0 \), the function \( f \) has a local expansion \( \sum_{j=0}^{\infty} f_j x^j \). Then \( f^{\sigma^{-1}} \) has a local expansion at \( P_0 \) as follows

\[ f^{\sigma^{-1}} = \sum_{j=0}^{\infty} \xi^{ij} f_j x^j, \]

where \( \xi = 1/\gamma \). The coefficient of \( x^{d+u} \) in the local expansion of \( Q(f, f^{\sigma^{-1}}, \ldots, f^{\sigma^{-(s-1)}}) \) is

\[ 0 = B_0(\xi^d)f_d + \sum_{i=0}^{d-1} b_if_i + a_{0,d}, \]

where \( b_i \in \mathbb{F}_q \) is a linear combination of \( a_{i,j} \) which does not involve \( f_j \). Hence, \( f_d \) is uniquely determined by \( f_0, \ldots, f_{d-1} \) as long as \( B_0(\xi^d) \neq 0 \). Let \( S := \{ 0 \leq d \leq q - 2 : B_0(\xi^d) = 0 \} \). Then it is clear that \( |S| \leq s - 1 \) since the order of \( \xi \) is \( q - 1 \) and \( B_0(X) \) has degree at most \( s - 1 \). Thus, \( B_0(\xi^j) \neq 0 \) if and only if \( j \mod (q - 1) \notin S \); and in this case \( f_j \) is a fixed affine linear combination of \( f_i \) for \( 0 \leq i < j \).

Let \( W \) be the set of solutions \((f_0, f_1, \ldots, f_{k-1})\). The fact that \( W \) is \((s, q - 1, \frac{k}{q - 1})\)-periodic follows from (13). Note that the coefficients \( b_{d-j} \) for \( j \geq 1 \) in that equation are given by \( B_j(\xi^{d-j}) \) where \( B_j(X) := a_{1,j} + a_{2,j}X + \cdots + a_{s,j}X^{s-1} \). Therefore, once the values of \( f_i \), \( 0 \leq i < (j-1)(q-1) \) are fixed, the possible choices for the next block of \( (q - 1) \) coordinates, \( f_{j(q-1)}, \ldots, f_{j(q-1)-1}, \) lie in an affine shift of a fixed subspace of dimension at most \( (q - 1) \). Further, this shift is an easily computed affine linear combination of the \( f_i \)'s in the previous blocks. This implies the efficient computability of the claimed representation of \( W \).

Finally, by the choice of \( D \) in (8), the total number of possible \((A_0, A_1, \ldots, A_s)\) and hence the number of possible functional equations (12), is at most \( q^{N(m-s+1)+s+1} \leq q^{Nm+s+1} \). Therefore, the number of possible candidate subspaces \( W \) is also at most \( q^{Nm+s+1} \). \( \square \)

Combining Lemmas 5.6 and 5.7, we conclude, after some simple calculations, that one can find a representation of the \((s, q - 1)\)-periodic subspace containing all candidate messages \((f_0, f_1, \ldots, f_{k-1})\) in polynomial time, when the fraction of errors \( \tau = 1 - t/N \) satisfies

\[ \tau \leq \frac{s}{s+1} - \frac{s}{s+1} \frac{k}{N(m-s+1)} - \frac{3m}{m-s+1} \frac{g_6}{mN}. \]

Pruning the subspace. Applying Lemma 9.8 directly we would get a list size bound of \( \approx q^{sk/q} \) which would be super-polynomial in the code length unless \( k = O(q) \). Thus this idea does not directly allow us to get good list decodable codes while keeping the base field size small or achieve a list size that grows polynomially in \( s \). Instead what we show
is that by only encoding \((f_0, f_1, \ldots, f_{k-1}) \in \mathbb{F}_q^k\) that are restricted to belong to a special subspace-evasive set, we can (i) bring down the list size, and (ii) find this list efficiently in polynomial time (and further the exponent of the polynomial is independent of \(\varepsilon\), the gap to capacity). To this end, we develop the necessary machinery concerning subspace evasive sets next. Later, in Section 8.1, we combine these subspace evasive sets with our folded Hermitian codes to get good list-decodable codes.

6. Subspace evasive sets with additional structure

Let us first recall the notion of “ordinary” subspace-evasive sets from [7].

**Definition 6.** A subset \(S \subseteq \mathbb{F}_q^k\) is said to be \((d, \ell)\)-subspace-evasive if for all \(d\)-dimensional affine subspaces \(W \subseteq \mathbb{F}_q^k\), we have \(|S \cap W| \leq \ell\).

We next define the notion of evasiveness w.r.t a collection of subspaces instead of all subspaces of a particular dimension.

**Definition 7.** Let \(\mathcal{F}\) be a family of (affine) subspaces of \(\mathbb{F}_q^k\), each of dimension at most \(d\). A subset \(S \subseteq \mathbb{F}_q^k\) is said to be \((\mathcal{F}, d, \ell)\)-evasive if for all \(W \in \mathcal{F}\), we have \(|S \cap W| \leq \ell\).

6.1. Hierarchical subspace-evasive sets. The key to pruning the list to a small size is the notion of a hierarchical subspace-evasive set, which is defined as a subset of \(\mathbb{F}_q^k\) with the property that some of its prefixes are subspace-evasive with respect to \((s, \Delta, b)\)-periodic subspaces. We will show how the special subspace-evasive sets help towards pruning the list in our list decoding context in Section 6.4.

**Definition 8.** Let \(\mathcal{F}\) be a family of \((s, \Delta, b)\)-periodic subspaces of \(\mathbb{F}_q^k\) with \(k = b\Delta\). A subset \(S \subseteq \mathbb{F}_q^k\) is said to be \((\mathcal{F}, s, \Delta, b, L)\)-h.s.e (for hierarchically subspace evasive for block size \(\Delta\)) if for every affine subspace \(W \in \mathcal{F}\), the following bound holds for \(j = 1, 2, \ldots, b\):

\[
|\text{proj}_{j\Delta}(S) \cap \text{proj}_{j\Delta}(W)| \leq L.
\]

6.2. Random sets are hierarchically subspace evasive. Our goal is to give a randomized construction of large h.s.e sets that works with high probability, with the further properties that one can index into elements of this set efficiently (necessary for efficient encoding), and one can check membership in the set efficiently (which is important for efficient decoding).

An easy probabilistic argument, see [7], shows that a random subset of \(\mathbb{F}_q^k\) of size about \(q^{(1-\zeta)k}\) is \((d, O(d/\zeta))\)-subspace evasive with high probability. As a warmup, let us work out the similar proof for the case when we have only to avoid a not too large family \(\mathcal{F}\) of all possible \(d\)-dimensional affine subspaces. The advantage is that the guarantee on the intersection size is now \(O(1/\zeta)\) and independent of the dimension \(d\) of the subspaces one is trying to evade.

**Lemma 6.1.** Let \(\zeta \in (0, 1)\) and \(k\) be a large enough positive integer. Let \(\mathcal{F}\) be a family of affine subspaces of \(\mathbb{F}_q^k\), each of dimension at most \(d \leq \zeta k/2\), with \(|\mathcal{F}| \leq q^{ck}\) for some positive constant \(c\).

Let \(\mathcal{W}\) be a random subset of \(\mathbb{F}_q^k\) chosen by including each \(x \in \mathbb{F}_q^k\) in \(\mathcal{W}\) with probability \(q^{-ck}\). Then with probability at least \(1 - q^{-ck}\), \(\mathcal{W}\) satisfies both the following conditions: (i) \(|\mathcal{W}| \geq q^{(1-2\zeta)k}\), and (ii) \(\mathcal{W}\) is \((\mathcal{F}, d, 4c/\zeta)\)-evasive.
Proof. The first part follows by noting that the expected size of $W$ equals $q^{(1−ζ)k}$ and a standard Chernoff bound calculation. For the second part, fix an affine subspace $S \subseteq F$ of dimension at most $d$, and a subset $T \subseteq S$ of size $t$, for some parameter $t$ to be specified shortly. The probability that $W \supseteq T$ equals $q^{−ζkt}$. By a union bound over the at most $q^k$ choices for the affine subspace $S \in F$, and the at most $q^d$ choices of $t$-element subsets $T$ of $S$, we get that the probability that $W$ is not $(F, d, t)$-evasive is at most $q^{k+d} \cdot q^{-ζkt} \leq q^{ck} q^{-ckt/2}$ since $d \leq ζk/2$. Choosing $t = \lceil 4c/ζ \rceil$, this quantity is bounded from above by $q^{-ck}$.

6.3. Pseudorandom construction of large h.s.e subsets. We next turn to the pseudorandom construction of large h.s.e subsets. Suppose, for some fixed subset $F$ of $(s, Δ, b)$-periodic subspaces of $\mathbb{F}_q^k$ with $k = bΔ$, we are interested in an $(F, s, Δ, b, L)$-h.s.e subset of $\mathbb{F}_q^k$ of size $\approx q^{(1−ζ)k}$ for a constant $ζ$, $1/Δ < ζ < 1/3$. For simplicity, let us assume that the block size $Δ$ divides $k$, though arbitrary $k$ can be easily handled. (We will also ignore floors and ceilings in the description to avoid notational clutter; those are easy to accommodate and do not affect any of the claims.) The parameters $b, Δ, k$ and field size $q$ will be considered fixed for the rest of the discussion in this section.

Denote $Δ' = (1−ζ)Δ$, $b' = (1−ζ)b$, and $k' = b'Δ = (1−ζ)k$.

The random part of the construction will consist of mutually independent, random univariate polynomials $P_1, P_2, \ldots, P_b$ and $Q$, where $P_j \in \mathbb{F}_{q^{Δ'}}[T]$ for $1 \leq j \leq b'$ and $Q \in \mathbb{F}_{q^{Δ'}}[T]$ are random polynomials of degree $λ$\footnote{We will assume that representations of the necessary extension fields $\mathbb{F}_q^{Δ'}$ are all available. For this purpose, we only need irreducible polynomials over $\mathbb{F}_q$ of appropriate degrees, which can be constructed by picking random polynomials and checking them for irreducibility. Our final construction is anyway randomized, so the randomized nature of this step does not affect the results.} The degree parameter will be chosen to be $λ = Θ(k)$\footnote{The degree of $Q$ can in fact be just $O(1/ζ)$, but for uniformity we fix the degree of all polynomials to be the same.}

The key fact we will use about random polynomials is the following, which follows by virtue of the $λ$-wise independence of the values of a random degree $λ$ polynomial.

Fact 6.2. Let $P \in \mathbb{K}[T]$ be a polynomial of degree $λ$ whose coefficients are picked uniformly and independently at random from the field $\mathbb{K}$. For a fixed subset $T \subseteq \mathbb{K}$ with $|T| \leq λ$, the values $\{P(α)\}_{α \in T}$ are independent random values in $\mathbb{K}$.

We remark that this property of low-degree polynomials was also the basis of the pseudorandom construction of subspace evasive sets in [14]. However, since we require the h.s.e property, and need to exploit the periodicity of the subspaces we are trying to evade (which can have large dimension), the construction here is more complicated, and needs to use several polynomials $P_j$’s evaluated in a nested fashion, and one further polynomial $Q$ to further bring down the list size to a constant (this final use of $Q$ is similar in spirit to the construction in [14]). We remark that the construction presented here is a bit simpler and cleaner than the one in the conference version [16], and comes with efficient encoding automatically by construction. In contrast, the construction in [16] required some additional work in order to allow for efficient encoding.

In what follows we assume that, for $j = 1, 2, \ldots, b'$, some fixed bases of the fields $\mathbb{F}_{q^{Δ'}}$ have been chosen, giving us some canonical $\mathbb{F}_q$-linear injective maps

$$ρ_j : \mathbb{F}_q^{Δ'} \rightarrow \mathbb{F}_{q^{Δ'}}.$$
Also, for \( j = 1, 2, \ldots, b' \), let 
\[
\xi_j : \mathbb{F}_{q^{\Delta'}} \rightarrow \mathbb{F}_{q}^{\Delta}
\]
be some arbitrary \( \mathbb{F}_{q} \)-linear surjective map (thus \( \xi_j \) just outputs the first \( \zeta \Delta \) coordinates of the representation of elements of \( \mathbb{F}_{q^{\Delta'}} \) as vectors in \( \mathbb{F}_{q}^{\Delta} \) w.r.t some fixed basis). Finally, let \( \rho : \mathbb{F}_{q'}^{k'} \rightarrow \mathbb{F}_{q'}^{k'} \) be some fixed \( \mathbb{F}_{q} \)-linear injective map, and \( \xi : \mathbb{F}_{q'}^{k'} \rightarrow \mathbb{F}_{q}^{ck} \) be an arbitrary \( \mathbb{F}_{q} \)-linear surjective map.

We are now ready to describe our construction of h.s.e set based on the random polynomials \( P_1, P_2, \ldots, P_{b'}, Q \).

**Definition 9** (h.s.e set construction). Given the polynomials \( P_j \in \mathbb{F}_{q^{\Delta'}}[T] \) for \( i = 1, 2, \ldots, b' \) and \( Q \in \mathbb{F}_{q'}[T] \), define the subset \( \Gamma(P_1, P_2, \ldots, P_{b'}; Q) \) by

\[
\{(y_1, z_1, y_2, z_2, \ldots, y_{b'}, z_{b'}; w) \in \mathbb{F}_{q}^k \mid \text{for } j = 1, 2, \ldots, b' : y_j \in \mathbb{F}_{q}^{\Delta'}, \text{ and } w = \xi(Q(\rho(y_1, y_2, \ldots, y_{b'}))) \in \mathbb{F}_{q}^{ck} \text{; and } \}
\]

By construction, once suitable representations of the extension fields are available by pre-processing and the choice of \( P_1, \ldots, P_{b'}; Q \) is made, we can efficiently compute a bijective encoding map \( \text{HSE} : \mathbb{F}_{q}^{1-\zeta jk} \rightarrow \Gamma(P_1, P_2, \ldots, P_{b'}; Q) \). Indeed, we can view the input \( y \in \mathbb{F}_{q}^{b'} \) as \( (y_1, y_2, \ldots, y_{b'}) \) with \( y_j \in \mathbb{F}_{q}^{\Delta'} \) and then compute the \( z_j \)'s and \( w \) efficiently using \( \text{poly}(k) \) operations over \( \mathbb{F}_{q} \) (recall that the degree of the polynomials is \( \lambda = \Theta(k) \)).

We now move on to the main claim about the h.s.e property of our construction.

**Theorem 6.3.** Let \( \zeta \in (0, 1/3) \) and \( s \) be a positive integer satisfying \( s < \zeta \Delta / 10 \). Let \( \mathcal{F} \) be a subset of at most \( q^{ck} \) \((s, \Delta, b)\)-periodic subspaces of \( \mathbb{F}_{q}^k \) for \( k = b\Delta \) and some positive constant \( c \). Suppose that the parameters satisfy the condition \( q^{\Delta} \geq (2q^{2ck})^{10/9} \). Then with probability \( 1 - q(-\Omega(k)) \) over the choice of random polynomials \( \{P_i\}_{1 \leq i \leq b} \) and \( Q \) each of degree \( \lambda = \lfloor ck \rfloor \), the set \( \Gamma(P_1, P_2, \ldots, P_{b'}; Q) \) from Definition 9 is \((\mathcal{F}, s, \Delta, b, L)\)-h.s.e and \((\mathcal{F}, sb, \ell)\)-evasive

for \( L = \lfloor ck \rfloor \) and \( \ell = \lfloor 4c/\zeta \rfloor \).

**Proof.** Note that the first \( k' = (1-\zeta)k \) symbols of vectors in \( \Gamma(P_1, \ldots, P_{b'}; Q) \) only depend on the \( P_j \)'s. We will first prove that with high probability over the choice of the \( P_j \)'s the following holds (call such a choice of \( P_j \)'s as good):

For every \( W \in \mathcal{F} \), \( |\text{proj}_k(W) \cap \text{proj}_{k'}(\Gamma)| < L \), where we denote \( \Gamma \) as shorthand for \( \Gamma(P_1, \ldots, P_{b'}; Q) \).

Then, conditioned on a good choice of \( P_j \)'s, we will prove that with high probability over the choice of the random polynomial \( Q \), \( |W \cap \Gamma| < \ell \). Together, these steps will imply that the set \( \Gamma(P_1, P_2, \ldots, P_{b'}; Q) \) is \((\mathcal{F}, sb, \ell)\)-evasive. (Note that every subspace in \( \mathcal{F} \) has dimension at most \( sb \) by Claim 3.1.) We will return to the \((\mathcal{F}, s, \Delta, b, L)\)-h.s.e property at the end of the proof.

Let us first establish the second step. Fix a good choice of \( P_1, \ldots, P_{b'} \), and suppose we pick \( Q \) randomly. Fix a subspace \( W \in \mathcal{F} \). Since \( |\text{proj}_k(W) \cap \text{proj}_{k'}(\Gamma)| < L \) (recall that \( \text{proj}_{k'}(\Gamma) \) only depends on the \( P_j \)'s and thus is already determined), the number of elements of \( W \) that could possibly belong to \( \Gamma \) (after the choice of \( Q \)) is at most \( L \cdot q^{(s-b')k} = L q^{sb} \).
there are at most \(q^{s(b-b')}\) extensions that can fall in \(W\) since \(W\) is \((s,\Delta,b)\)-periodic. Further, the probability over the choice of \(Q\) that any such fixed extension belongs to \(\Gamma\) is at most \(q^{-ck}\), and any \(\ell\) of these events are independent. (Note that for a fixed prefix, there can be at most extension that falls in \(\Gamma\), so for \(\ell\) different strings to fall in \(\Gamma\), their prefixes must be distinct and are mapped to independent locations by the random polynomial \(\Gamma\).) Therefore, the probability over the choice of \(Q\) that \(|W \cap \Gamma| \geq \ell\) is at most \((Lq^{cb})^\ell q^{-ck\ell}\). By a union bound over all \(W \in \mathcal{F}\), we conclude that \(|W \cap \Gamma| < \ell\) for every \(W \in \mathcal{F}\) simultaneously, except with probability at most

\[q^{ck}\ell L^\ell q^{c(s-\Delta)b}\ell \leq q^{ck}(ck)^\ell q^{-c\Delta b\ell/2} = q^{ck}q^{-c\ell b\ell/2}\]

where in the first inequality we used \(s \leq \Delta/2\) and in the next one \(ck \leq q^{ck}\) both of which hold comfortably. For \(\ell \geq 4c/\zeta\), the above probability upper bound is at most \(q^{-ck}\).

We now turn to the first step, on the \(P_j\)’s being good with high probability. Fix some \(W \in \mathcal{F}\); we will prove by induction on \(j\) that

\[|\text{proj}_{j\Delta}(W) \cap \text{proj}_{j\Delta}(\Gamma)| < L\]

w.h.p over the choice of \(P_1, P_2, \ldots, P_j\), for \(1 \leq j \leq b'\) (note that \(\text{proj}_{j\Delta}(\Gamma)\) only depends on \(P_1, \ldots, P_j\), so this event is well defined). For the base case \(j = 1\), \(|\text{proj}_{j\Delta}(W)| \leq q^s\) as \(W\) is \((s,\Delta,b)\)-periodic, and the probability that some \(L\) of these \(q^s\) elements belong to \(\text{proj}_{j\Delta}(\Gamma)\) is at most \(q^{gL}\) times the probability that \(L\) distinct elements in \(\mathbb{F}_q^\Delta\) are mapped to specific values in \(\mathbb{F}_q^\Delta\) by \(\xi_1 \circ P_1\), which is at most \((q^{-c\Delta})^L\). So the overall probability that \(|\text{proj}_{j\Delta}(W) \cap \text{proj}_{j\Delta}(\Gamma)| \geq L\) is at most \(q^{(s-\zeta\Delta)\ell}\).

Now let \(j \geq 2\) and assume \(|\text{proj}_{(j-1)\Delta}(W) \cap \text{proj}_{(j-1)\Delta}(\Gamma)| < L\). By the \((s,\Delta,b)\)-periodicity of \(W\), for each of the (less than \(L\)) prefixes in \(\text{proj}_{(j-1)\Delta}(W) \cap \text{proj}_{(j-1)\Delta}(\Gamma)\), there are at most \(q^s\) extensions that fall in \(\text{proj}_{j\Delta}(W)\). Similarly to the argument used for second step above, the probability that some \(L\) of these belong to \(\text{proj}_{j\Delta}(\Gamma)\) is at most \((Lq^s)^L \cdot q^{-c\Delta L}\). Thus, the probability that \(|\text{proj}_{j\Delta}(W) \cap \text{proj}_{j\Delta}(\Gamma)| \geq L\) is at most \((L \cdot q^{(s-\zeta\Delta)})^L\).

Combining these arguments, we conclude that the probability over the choice of the \(P_j\)’s that \(|\text{proj}_{j\Delta}(W) \cap \text{proj}_{j\Delta}(\Gamma)| \geq L\) is at most

\[b'(L \cdot q^{(s-\zeta\Delta)})^L \leq (2ckq^{-0.9\zeta\Delta})^L \leq q^{-2L}\]

where the last step used the assumption that \(q^{\zeta\Delta} \geq (2q^{-2ck})^{10/9}\).

Finally, since there are at most \(q^{ck}\) subspaces \(W \in \mathcal{F}\), by a union bound we have that for all \(W \in \mathcal{F}\) simultaneously, \(|\text{proj}_{k\Delta}(W) \cap \text{proj}_{k\Delta}(\Gamma)| < L\) with probability at least \(1 - q^{ck}q^{-L} = 1 - q^{-ck}\) over the choice of \(P_1, \ldots, P_{b'}\).

To finish the proof, we need to verify the \((\mathcal{F}, s, \Delta, b, L)\)-h.s.e property. That is, we need to prove that w.h.p, \(|\text{proj}_{j\Delta}(W) \cap \text{proj}_{j\Delta}(\Gamma)| \leq L\) for every \(W \in \mathcal{F}\) and \(j = 1, 2, \ldots, b\). By (15), this holds for \(j = 1, 2, \ldots, b'\). By construction, the last \(\zeta k\) symbols of any vector in \(\Gamma\) is a function of the first \((1 - \zeta)k = b'\Delta\) symbols, so \(|\text{proj}_{j\Delta}(W) \cap \text{proj}_{j\Delta}(\Gamma)| \leq L\) also holds for \(b' < j \leq b\). \(\square\)

6.4. Efficient computation of intersection with h.s.e. subsets. The key aspect which makes h.s.e subsets useful in our context to prune the affine space of candidate messages, and indeed motivated the exact specifics of the definition and aspects of its construction, is the following claim which shows that intersection of a \((s,\Delta,b)\)-periodic subspace with our h.s.e set can found efficiently.
Lemma 6.4. There is an algorithm running in time $\poly(k, q^{c/}\Delta)$ that provides the following guarantee. Given as input the polynomials $P_1, \ldots, P_r$ and $Q$ underlying the construction of an $(F, s, \Delta, b, L)$-h.s.e and $(F, sb, \ell)$-evasive set $\Gamma = \Gamma(P_1, \ldots, P_r; Q)$ and an $(s, \Delta, b)$-periodic subspace $W \subseteq F^k$ belonging to $F$, the algorithm computes the at most $\ell$ elements of $W \cap \Gamma$.

Proof. The proof essentially follows from the observations made in the proof of Theorem 6.3. First note that claim that $|W \cap \Gamma(P_1, \ldots, P_r; Q) \leq \ell$ just follows from the $(F, sb, \ell)$-easiveness of $\Gamma$. To compute $W \cap \Gamma$, the algorithm iteratively computes the intersections $\text{proj}_{j\Delta}(W) \cap \text{proj}_{j\Delta}(\Gamma)$ for $1 \leq j \leq \ell$. As $\Gamma$ is $(F, s, \Delta, b, L)$-h.s.e, this intersection has size at most $L$. To compute $\text{proj}_{j\Delta}(W) \cap \text{proj}_{j\Delta}(\Gamma)$, the algorithm runs over the at most $q^s$ possible extensions of each element of $\text{proj}_{(j-1)\Delta}(W) \cap \text{proj}_{(j-1)\Delta}(\Gamma)$ that can belong to $\text{proj}_{j\Delta}(W)$ (due to the $(s, \Delta, b)$-periodicity of $W$), and checks which ones also belong to $\text{proj}_{j\Delta}(\Gamma)$. The complexity amounts to $q^{O(s)}$ evaluations of degree $O(k)$ polynomials, and thus takes $q^{O(\Delta/\ell)} \poly(k)$ time. To compute $W \cap \Gamma$ from $\text{proj}_{j\Delta}(W) \cap \text{proj}_{j\Delta}(\Gamma)$, we recall the earlier observation that the construction of $\Gamma$ implies that there is a unique extension of an element in $\text{proj}_{j\Delta}(\Gamma)$ that belongs to $\Gamma$. □

We conclude this section by recording in convenient form all necessary properties of our h.s.e set construction, which follow from Theorem 6.3 and Lemma 6.4.

Theorem 6.5. Let $\zeta \in (0, 1)$, and $b, \Delta, s$, and $k = b\Delta$ be positive integers satisfying $s < \zeta \Delta/10$. Let $F$ be a family of at most $q^k$ $(s, \Delta, b)$-periodic subspaces of $F^k$. Then the $\poly(k, \log q)$ time randomized construction from Definition 6 of the injective map $\text{HSE} : F^{(1-\zeta)^2k} \rightarrow F^k$ satisfies the following properties:

1. Given $x \in F^{(1-\zeta)^2k}$, $\text{HSE}(x)$ can be computed using $\poly(k)$ operations over $F_q$.
2. For every $W \in F$, the set $\{x \in F^{(1-\zeta)^2k} | \text{HSE}(x) \in W\}$ has size at most $O(c/\zeta)$, and can be computed in $\poly(k, q^{c/\Delta})$ time.

7. Folded codes from the Garcia-Stichtenoth tower

Compared with the Hermitian tower of function fields, the Garcia-Stichtenoth tower of function fields yields folded codes with better parameters due to the fact that the Garcia-Stichtenoth tower is an optimal one in the sense that the ratio of number of rational places against genus achieves the maximal possible value. The construction of folded codes from the Garcia-Stichtenoth tower is almost identical to the one from the Hermitian tower except for one major difference: the redefined code from the Garcia-Stichtenoth tower is constructed in terms of the local expansion at point $P_\infty$, while in the Hermitian case local expansion at $P_0$ is considered. For convenience of the reader, we give a parallel description of folded codes from the Garcia-Stichtenoth tower, while only sketching the identical parts.

7.1. Background on Garcia-Stichtenoth tower. Again let $r$ be a prime power and let $q = r^2$. We denote by $F_q$ the finite field with $q$ elements. The Garcia-Stichtenoth towers that we are going to use for our code construction were discussed in [4, 5]. The reader may refer to [3, 7] for the detailed background on the Garcia-Stichtenoth function tower. There are two optimal Garcia-Stichtenoth towers that are equivalent. For simplicity, we introduce the tower defined by the following recursive equations [5]

$$x_{i+1}^r + x_{i+1} = \frac{x_i^r}{x_i^{r-1} + 1}, \quad i = 1, 2, \ldots, e - 1.$$ (16)
Put \( K_e = \mathbb{F}_q(x_1, x_2, \ldots, x_e) \) for \( e \geq 2 \).

**Rational places.** The function field \( K_e \) has at least \( r^{e-1}(r^2 - r) + 1 \) rational places. One of these is the “point at infinity” which is the unique pole \( P_\infty \) of \( x_1 \) (and is fully ramified). The other \( r^{e-1}(r^2 - r) \) come from the rational places lying over the unique zero of \( x_1 - \alpha \) for each \( \alpha \in \mathbb{F}_q \) with \( \alpha^r + \alpha \neq 0 \). Note that for every \( \alpha \in \mathbb{F}_q \) with \( \alpha^r + \alpha \neq 0 \), the unique zero of \( x_1 - \alpha \) splits completely in \( K_e \), i.e., there are \( r^{e-1} \) rational places lying over the zero of \( x_1 - \alpha \). Let \( \mathbb{P} \) be the set of all the rational places lying over the zero of \( x_1 - \alpha \) for all \( \alpha \in \mathbb{F}_q \) with \( \alpha^r + \alpha \neq 0 \). Then, intuitively, one can think of the \( r^{e-1}(r^2 - r) \) rational places in \( \mathbb{P} \) as being given by \( e \)-tuples \( (\alpha_1, \alpha_2, \ldots, \alpha_e) \in \mathbb{F}_q^e \) that satisfy \( \alpha_{i+1}^r + \alpha_{i+1} = \frac{\alpha_i^r + 1}{\alpha_i^{r-1} + 1} \) for \( i = 1, 2, \ldots, e - 1 \) and \( \alpha_1^{r+1} + 1 \neq 0 \). For each value of \( \alpha \in \mathbb{F}_q \), there are precisely \( r \) solutions to \( \beta \in \mathbb{F}_q \) satisfying \( \beta^r + \beta = \frac{\alpha^r}{\alpha^{r+1}} \), so the number of such \( e \)-tuples is \( r^{e-1}(r^2 - r) \) \( (r^2 - r) \) choices for \( \alpha_1 \), and then \( r \) choices for each successive \( \alpha_i, 2 \leq i \leq e \).

**Riemann-Roch spaces.** As shown in [23], every function of \( K_e \) with a pole only at \( P_\infty \) has an expression of the form

\[
\begin{align*}
\alpha_1 \left( \sum_{i=0}^{r-1} c_i \pi_1^{x_1 \cdots x_e} \prod_{i=0}^{r-1} \pi_i \right),
\end{align*}
\]

where \( a \geq 0, c_i \in \mathbb{F}_q \), and for \( 1 \leq j < e \), \( h_j = x_j^{r-1} + 1 \) and \( \pi_j = h_1 h_2 \ldots h_j \). Moreover, Shum et al. [24] present an algorithm running in time polynomial in \( l \) that outputs a basis of over \( \mathbb{F}_q \) of \( \mathcal{L}(LP_\infty) \) explicitly in the above form.

We stress that evaluating elements of \( \mathcal{L}(LP_\infty) \) at the rational places of \( \mathbb{P} \) is easy: we simply have to evaluate a linear combination of the monomials allowed in \( \mathcal{L}(LP_\infty) \) at the tuples \( (\alpha_1, \alpha_2, \ldots, \alpha_e) \in \mathbb{F}_q^e \) mentioned above. In other words, it is just evaluating an \( e \)-variate polynomial at a specific subset of \( r^{e-1}(r^2 - r) \) points of \( \mathbb{F}_q^e \) and can be accomplished in polynomial time.

**Genus.** The genus \( g_e \) of the function field \( K_e \) is given by

\[
\begin{align*}
g_e = \begin{cases} 
(\frac{r^e}{2} - 1)^2 & \text{if } e \text{ is even} \\
(\frac{(r^e-1)^2}{2} - 1)(\frac{(r^{e+1})^2}{2} - 1) & \text{if } e \text{ is odd}.
\end{cases}
\end{align*}
\]

Thus the genus \( g_e \) is at most \( r^e \). (Compare this with the \( er^e \) bound for the Hermitian tower; this smaller genus is what allows to pick \( e \) as large as we want in the Garcia-Stichtenoth tower, while keeping the field size \( q \) fixed.)

**A useful automorphism.** Let \( \gamma \) be a primitive element of \( \mathbb{F}_r \) and consider the automorphism \( \sigma \in \text{Aut}(K_e/\mathbb{F}_q) \) defined by

\[
\sigma : x_i \mapsto \gamma^{(r+1)x_{i-1}^r} x_i \quad \text{for } i = 1, 2, \ldots, e.
\]

Then the order of \( \sigma \) is \( r - 1 \) and furthermore, we have the following facts:

1. \( \sigma \) keeps \( P_\infty \) unchanged, i.e., \( P_\infty^\sigma = P_\infty \);
2. Let \( \mathbb{P} \) be the set of all the rational places lying over \( x_1 - \alpha \) for all \( \alpha \in \mathbb{F}_q \) with \( \alpha^r + \alpha \neq 0 \). Then \( |\mathbb{P}| = (r - 1)r^e \). Moreover, \( \sigma \) divides \( \mathbb{P} \) into \( r^e \) orbits and each orbit has \( r - 1 \) places. For an integer \( m \) with \( 1 \leq m \leq r - 1 \), we can label \( Nm \) distinct elements

\[
P_1, P_1^\sigma, \ldots, P_1^{r^e-1}, \ldots, P_N, P_N^\sigma, \ldots, P_N^{r^e-1}
\]

in \( \mathbb{P} \), as long as \( N \leq r^e \lfloor \frac{r-1}{m} \rfloor \).
The folded codes from the Garcia-Stichtenoth tower are defined similarly to the Hermitian case.

**Definition 10 (Folded codes from the Garcia-Stichtenoth tower).** Assume that \( m, k, N \) are positive integers satisfying \( 1 \leq m \leq r - 1 \) and \( l/m < N \leq r^{c} \left\lfloor \frac{c}{m} \right\rfloor \). The folded code from \( K_{e} \) with parameters \( N, l, q, e, m \), denoted by \( \overline{FGS}(N, l, q, e, m) \), encodes a message function \( f \in \mathcal{L}(lP_{\infty}) \) as

\[
\begin{align*}
(18) \quad f \mapsto \left( \begin{array}{c}
\begin{bmatrix}
 f(P_{1}^{1}) \\
 f(P_{1}^{m}) \\
 \vdots \\
 f(P_{1}^{m^{n-1}})
\end{bmatrix},
\begin{bmatrix}
 f(P_{2}^{1}) \\
 f(P_{2}^{m}) \\
 \vdots \\
 f(P_{2}^{m^{n-1}})
\end{bmatrix},
\ldots,
\begin{bmatrix}
 f(P_{N}^{1}) \\
 f(P_{N}^{m}) \\
 \vdots \\
 f(P_{N}^{m^{n-1}})
\end{bmatrix}
\end{array} \right) \in \left( \mathbb{F}_q^m \right)^N.
\end{align*}
\]

Then we have a similar result on parameters of \( \overline{FGS}(N, l, q, e, m) \).

**Lemma 7.1.** The above code \( \overline{FGS}(N, l, q, e, m) \) is an \( \mathbb{F}_q \)-linear code over alphabet size \( q^m \), rate at least \( \frac{l - q + 1}{Nm} \), and minimum distance at least \( N - \frac{l}{m} \).

### 7.2. Redefining the code in terms of local expansion at \( P_{\infty} \)

In the Hermitian case, we use coefficients of its power series expansion around \( P_{0} \). However, for the Garcia-Stichtenoth tower we do not have such a nice point \( P_{0} \). Fortunately, we can use point \( P_{\infty} \) to achieve our mission.

Again for our decoding, we will actually recover the message \( f \in \mathcal{L}(lP_{\infty}) \) in terms of the coefficients of its power series expansion around \( P_{\infty} \)

\[
f = T^{-l}(f_{0} + f_{1}T + f_{2}T^{2} + \cdots)
\]

where \( T := \frac{1}{x_{\infty}} \) is the local parameter at \( P_{\infty} \) (which means that \( x_{e} \) has exactly one pole at \( P_{\infty} \), i.e., \( \nu_{P_{\infty}}(x_{e}) = -1 \)).

In this case we can also show that one can efficiently move back-and-forth between the representation of \( f \in \mathcal{L}(lP_{\infty}) \) in terms of a basis for \( \mathcal{L}(lP_{\infty}) \) and its power series representation \( (f_{0}, f_{1}, \ldots) \) around \( P_{\infty} \). Since the mapping \( f \mapsto (f_{0}, f_{1}, \ldots) \) is \( \mathbb{F}_q \)-linear, it suffices to compute the local expansion at \( P_{\infty} \) of a basis for \( \mathcal{L}(lP_{\infty}) \).

**Lemma 7.2.** For any \( n \), one can compute the first \( n \) terms of the local expansion of the basis elements \( (17) \) at \( P_{\infty} \) using \( \text{poly}(n) \) operations over \( \mathbb{F}_q \).

**Proof.** First let \( h \) be a nonzero function in \( \mathbb{F}_q(x_{1}, x_{2}, \ldots, x_{e}) \) with \( \nu_{P_{\infty}}(h) = v \in \mathbb{Z} \). Assume that the local expansion \( h = T^{v} \sum_{j=0}^{\infty} a_{j}T^{j} \) is known. To find the local expansion \( \frac{1}{h} = T^{-v} \sum_{j=0}^{\infty} c_{j}T^{j} \). Consider the identity

\[
1 = \left( \sum_{j=0}^{\infty} c_{j}T^{j} \right) \left( \sum_{j=0}^{\infty} a_{j}T^{j} \right).
\]

Then by comparing the coefficients of \( T^{i} \) in the above identity, one has \( c_{0} = a_{0}^{-1} \) and \( c_{i} = -a_{0}^{-1}(c_{i-1}a_{1} + \cdots + c_{0}a_{i}) \) can be easily computed recursively for all \( i \geq 1 \).

Thus, by the structure of the basis functions in \( (17) \), it is sufficient to find an algorithm of efficiently finding local expansions of \( x_{i} \) at \( P_{\infty} \) for every \( i = 1, 2, \ldots, e \). We can inductively find the local expansions of \( x_{i} \) at \( P_{\infty} \) as follows. We note that \( \nu_{P_{\infty}}(x_{i}) = -r^{e-i} \) for \( i = 1, 2, \ldots, e \).

For \( i = e \), \( x_{e} \) has the local expansion \( \frac{1}{T} \) at \( P_{\infty} \).
Now assume that we know the local expansion of \(x_i\). Then we can easily compute the local expansion of \(x_i^r + x_i\) and hence the local expansion of \(1/(x_i^r + x_i)\). Let us assume that 

\[
1/(x_i^r + x_i) \text{ has local expansion } 1/(x_i^r + x_i) = T^e + 1 + \sum_{j=0}^{\infty} \alpha_j T^j \text{ at } P_\infty
\]

for some \(\alpha_j \in \mathbb{F}_q\). Assume that \(1/x_{i-1}\) has the local expansion \(1/x_{i-1} = T^{e-i-1} + \sum_{j=0}^{\infty} \beta_j T^j\). To find \(\beta_j\), we consider the identity

\[
T^{e-i-1} \sum_{j=0}^{\infty} \beta_j T^j + T^{e-i-2} \sum_{j=0}^{\infty} \beta_j T^j = \frac{1}{x_{i-1}} + \left(\frac{1}{x_{i-1}}\right)^r = \frac{1}{x_i^r + x_i} = T^{e-i-1} \sum_{j=0}^{\infty} \alpha_j T^j.
\]

By comparing the coefficients of \(T^j + r^{e-i-1}\) in the above identity, we have that \(\beta_0 = \alpha_0\) and \(\beta_j\) can be easily computed recursively by the following formula for all \(i \geq 1\).

\[
\beta_j = \begin{cases} 
\alpha_j & \text{if } r \nmid j \\
\alpha_j - \beta_{j/r} & \text{if } r | j.
\end{cases}
\]

Therefore, the local expansion of \(x_{i-1}\) at \(P_\infty\) can be easily computed. \(\square\)

As in the Hermitian case, we will actually need to index the messages of the code by the first \(k\) coefficients \((f_0, f_1, \ldots, f_{k-1})\) of the local expansion of the function \(f\) at \(P_\infty\).

Let us define the local expansion map \(\text{ev}_{P_\infty} : \mathcal{L}((k + 2g_e - 1)P_\infty) \rightarrow \mathbb{F}_q^k\) that maps \(f\) to \((f_0, f_1, \ldots, f_{k-1})\) where \(f = T^{-(k+2g_e-1)}(f_0 + f_1 T + f_2 T^2 + \cdots)\) is the local expansion of \(f\) at \(P_\infty\).

**Claim 7.3.** \(\text{ev}_{P_\infty}\) is an \(\mathbb{F}_q\)-linear surjective map. Further, we can compute \(\text{ev}_{P_\infty}\) using \(\text{poly}(k, g_e)\) operations over \(\mathbb{F}_q\) given a representation of the input \(f \in \mathcal{L}((k + 2g_e - 1)P_\infty)\) in terms of the basis \(\{1, T, T^2, \ldots, T^{k-1}\}\).

The proof of this claim is similar to Claim 5.3. Note that the kernel of \(\text{ev}_{P_\infty}\) is \(\mathcal{L}((2g_e - 1)P_\infty)\) which has dimension exactly \(g_e\) by the Riemann-Roch theorem.

For each \((f_0, f_1, \ldots, f_{k-1}) \in \mathbb{F}_q^k\), we can therefore pick a pre-image in \(\mathcal{L}((k + 2g_e - 1)P_\infty)\). For convenience, we will denote an injective map making such a unique choice by \(\kappa_{P_\infty} : \mathbb{F}_q^k \rightarrow \mathcal{L}((k + 2g_e - 1)P_\infty)\). By picking the pre-images of a basis of \(\mathbb{F}_q^k\) and extending it by linearity, we can assume \(\kappa_{P_\infty}\) to be \(\mathbb{F}_q\)-linear, and thus specify it by a \((k + g_e) \times k\) matrix. We record this fact for easy reference below.

**Claim 7.4.** The map \(\kappa_{P_\infty} : \mathbb{F}_q^k \rightarrow \mathcal{L}((k + 2g_e - 1)P_\infty)\) is \(\mathbb{F}_q\)-linear and injective. We can compute a representation of this linear transformation using \(\text{poly}(k, g_e)\) operations over \(\mathbb{F}_q\), and the map itself can be evaluated using \(\text{poly}(k, g_e)\) operations over \(\mathbb{F}_q\).

Now we redefine a version of the folded Garcia-Stichtenoth code that maps \(\mathbb{F}_q^k\) to \((\mathbb{F}_q^m)^N\) by composing the folded encoding \(\text{fold}^{\text{e}}\) from the original Definition 10 with \(\kappa_{P_\infty}\).

**Definition 11** (Folded Garcia-Stichtenoth code using local expansion). The folded Garcia-Stichtenoth code (FGS code for short) \(\text{FGS}(N, k, q, e, m)\) maps \(\mathbb{F}_q^k\) to \(\text{FGS}(N, k + 2g_e - 1, q, e, m) (\kappa_{P_\infty}(f))\) \((\mathbb{F}_q^m)^N\).

The rate of the above code equals \(k/(Nm)\) and its distance is at least \(N - (k + 2g_e - 1)/m\).

### 7.3. List decoding FGS codes

The list decoding part for the codes from the Garcia-Stichtenoth tower is almost identical to the Hermitian tower. We only sketch this part briefly.
If \( f \) is a function in \( \mathcal{L}((k + 2g_e - 1)P_\infty) \) whose encoding \([18]\) agrees with the received word \( y \) in at least \( t \) columns with \( t > \frac{D + k + 2g_e - 1}{m - s + 1} \) and

\[
D = \left\lfloor \frac{N(m - s + 1) - k + (s - 1)g_e + 1}{s + 1} \right\rfloor,
\]

then there exist \( A_i \in \mathcal{L}(DP_\infty) \) for \( i = 1, 2, \ldots, s \) and \( A_0 \in \mathcal{L}((D + k + 2g_e - 1)P_\infty) \) such that they are not all zero and

\[
Q(f, f^{\sigma^{-1}}, \ldots, f^{\sigma^{-(s-1)}}) = A_0 + A_1 f + A_2 f^{\sigma^{-1}} + \cdots + A_s f^{\sigma^{-(s-1)}} = 0. \tag{19}
\]

**Solving the functional equation for \( f \).** As in the Hermitian case, our goal next is to recover the list of solutions \( f \) to the functional equation \([19]\). Recall that our message functions lie in \( \text{Im}(\kappa_{P_\infty}) \), so we can recover \( f \) by recovering the top \( k \) coefficients \((f_0, f_1, \ldots, f_{k-1})\) of its local expansion.

\[
f = T^{-(k+2g_e-1)} \sum_{j=0}^{\infty} f_j T^j \tag{20}
\]

at \( P_\infty \). We now prove that \((f_0, f_1, \ldots, f_{k-1})\) for \( f \) satisfying Equation \((19)\) belong to a “periodic” subspace (in the sense of Definition \([1] \)) of not too large dimension.

**Lemma 7.5.** The set of solutions \((f_0, f_1, \ldots, f_{k-1}) \in \mathbb{F}_q^k\) such that

\[
f = T^{-(k+2g_e-1)} \sum_{j=0}^{\infty} f_j T^j \in \mathcal{L}((k + 2g_e - 1)P_\infty)
\]

obeys equation

\[
A_0 + A_1 f + A_2 f^{\sigma^{-1}} + \cdots + A_s f^{\sigma^{-(s-1)}} = 0 \tag{21}
\]

when at least one \( A_i \) is nonzero is an \((s - 1, r - 1, \lceil \frac{k}{r - 1} \rceil )\)-periodic subspace.

Further, there are at most \( q^{Nm+s+1} \) possible choices of this subspace over varying choices of the \( A_i \)'s.

**Proof.** Let \( u = \min \{ \nu_{P_\infty}(A_i) : i = 1, 2, \ldots, s \} \). Then it is clear that \( u \leq 0 \) and \( \nu_{P_\infty}(A_0) \geq u - (k + 2g_e - 1) \). Each \( A_i \) has a local expansion at \( P_\infty \):

\[
A_i = T^u \sum_{j=0}^{\infty} a_{i,j} T^j
\]

for \( i = 1, \ldots, s - 1 \) and \( A_0 \) has a local expansion at \( P_\infty \):

\[
A_0 = T^{u-(k+2g_e-1)} \sum_{j=0}^{\infty} a_{0,j} T^j
\]

From the definition of \( u \), one knows that the polynomial

\[
B_0(X) := a_{1,0} + a_{2,0} X + \cdots + a_{s,0} X^{s-1}
\]

is nonzero.

Assume that at \( P_\infty \), the function \( f \) has a local expansion \((20)\). Then \( f^{\sigma^{-i}} \) has a local expansion at \( P_\infty \) as follows

\[
f^{\sigma^{-i}} = \xi^{-(k+2g_e-1)i} T^{-(k+2g_e-1)} \sum_{j=0}^{\infty} \xi^{ji} f_j T^j,
\]
where $\xi = 1/\gamma$.

The coefficient of $T^{d+u-(k+2g_e-1)}$ in the local expansion of $Q(f, f^{\sigma-1}, \ldots, f^{\sigma-(s-1)})$ is

\begin{equation}
0 = B(\xi^{d-(k+2g_e-1)})f_d + \sum_{i=0}^{d-1} b_if_i + a_{0,d},
\end{equation}

where $b_i \in \mathbb{F}_q$ is a linear combination of $a_{i,j}$ which does not involve $f_j$. Hence, $f_d$ is uniquely determined by $f_0, \ldots, f_{d-1}$ as long as $B(\xi^{d-(k+g_e-1)}) \neq 0$.

Let $S := \{0 \leq d \leq r-2 : B(\xi^{d-(k+g_e-1)}) = 0\}$. Then it is clear that $|S| \leq s-1$ since the order of $\xi$ is $r-1$ and $B_0(X)$ has degree at most $s-1$. Thus, $B(\xi^{d-(k+g_e-1)}) \neq 0$ if and only if $j \mod (r-1) \notin S$; and in this case $f_j$ is a fixed affine linear combination of $f_i$ for $0 \leq i < j$. Note that $B_0(X)$ has at most $(s-1)\left\lceil \frac{k}{r-1} \right\rceil$ roots among $\{\xi^i : i = 0, 1, \ldots, k-1\}$. It follows that the set of solutions $(f_0, f_1, \ldots, f_{k-1})$ is an affine space $W \subset \mathbb{F}_q^k$, and the dimension of $W$ is at most $(s-1)\left\lceil \frac{k}{r-1} \right\rceil$.

The fact that $W$ is $(s,r-1,\left\lceil \frac{k}{r-1} \right\rceil)$-periodic subspace follows from (22). Note that the coefficients $b_{d-j}$ for $j \geq 1$ in that equation are given by $B_j(\xi^{d-j-(k+2g_e-1)})$, where $B_j(X) := a_{1,j}X + a_{2,j}X + \cdots + a_{s,j}X^{s-1}$. Therefore, once the values of $f_i$, $0 \leq i < (j-1)(r-1)$ are fixed, the possible choices for the next block of $(r-1)$ coordinates, $f_{(j-1)(r-1)+1}, \ldots, f_{j(r-1)-1}$, lie in an affine shift of a fixed subspace of dimension at most $(s-1)$. Further, the affine shift is an affine linear combination of the $f_i$’s in the previous blocks.

Finally, by the choice of $D$, the total number of possible $(A_0, A_1, \ldots, A_s)$ and hence the number of polynomial solutions \cite{22}, is at most $q^{N(m-s+1)+s+1} \leq q^{Nm+s+1}$. Therefore, the number of possible candidate subspaces $W$ is also at most $q^{Nm+s+1}$.

Similar to the bound \cite{14} for the Hermitian case, we conclude, after some simple calculations and using the upper bound on genus $g_e \leq e^2$, that one can find a representation of the $(s,r-1)$-periodic subspace containing all candidate messages $(f_0, f_1, \ldots, f_{k-1})$ in polynomial time, when the fraction of errors $\tau = 1 - t/N$ satisfies

\begin{equation}
\tau \leq \frac{s}{s+1} \left(1 - \frac{k}{N(m-s+1)}\right) - \frac{3m}{m-s+1} \frac{r^e}{mN}.
\end{equation}

8. Pruning list for folded AG codes using h.s.e sets

8.1. Combining folded Hermitian codes and h.s.e sets. Instead of encoding arbitrary $f \in \mathbb{F}_q^k$ by the folded Hermitian code (Definition \text{5}), we can restrict the messages $f$ to belong to the range of our h.s.e set, so that the affine space of solutions guaranteed by Lemma \text{5.7} can be efficiently pruned to a small list. The formal claim is below.

\textbf{Theorem 8.1}. Let $e \geq 2$ be an integer, $r \geq 2e$ a large enough prime power, $q = r^2$, and $\xi \in (1/q, 1)$. Let $k \leq q^{e^{1/2}}$ be a positive integer. Let $s, m$ be positive integers satisfying $1 \leq s \leq m \leq q-1$ and $s < \xi q/12$. Finally let $N$ be an integer satisfied $k+2er^e \leq Nm \leq (q-1)r^e$.

Consider the code $C_1$ with encoding $E_1 : \mathbb{F}_q^{(1-\xi)^{2k}} \to (\mathbb{F}_q^m)^N$ defined as

$$E_1(x) = FH(N, k, q, e, m)(\text{HSE}(x)),$$
for a random map \( HSE : \mathbb{F}_q^{(1-\zeta)2k} \to \mathbb{F}_q^k \) as constructed in Definition 4 for a period size \( \Delta = q-1 \) and \( b = \lceil \frac{k}{q-1} \rceil \) (where technically we pad the input \( \mathbb{F}_q^k \) with 0’s to make its length \( b\Delta \) to feed into HSE).

Then, the code \( C_1 \) code has rate \( R = (1-\zeta)^2k/(Nm) \), can be encoded in \( \text{poly}(Nm\zeta^4) \) time, and with high probability over the choice of \( HSE \), it is \((\tau, \ell)\)-list decodable in time \( \text{poly}(Nm\zeta^4) \) for \( \ell \leq O(1/(R\zeta)) \) and

\[
\tau = \frac{s}{s+1} \left( 1 - \frac{k}{N(m-s+1)} \right) - \frac{3m}{m-s+1} \frac{\epsilon r}{mN}.
\]

Proof. The claim about the rate is clear, and the encoding time follows from the time to compute HSE recorded in Theorem 6.5.

By (4), the genus \( g_e \leq \epsilon r \), and so the condition on \( N, m \) meets the requirement for the construction of the folded Hermitian tower based code in Definition 11 and the claimed value of the error fraction \( \tau \) satisfies (14). By Lemma 5.7 we know that the candidate messages found by the decoder lie in one of at most \( q^{2Nm} \) possible \((s, q-1, \lceil \frac{k}{q-1} \rceil)\)-periodic subspaces.

One can check that the conditions of Theorem 6.5 are met for our choice of \( \zeta, s, q, k, \Delta \). Appealing to Theorem 5.5 with the choice \( c = 2Nm/k = O(1/R) \), we conclude that, with high probability over the choice of \( HSE \), there is a decoding algorithm running in time \( \text{poly}(Nm\zeta^4) \) to list decode \( C_1 \) from a fraction \( \tau \) of errors, outputting at most \( O(1/(R\zeta)) \) messages in the worst-case.

Let \( \epsilon > 0 \) be a small positive constant, and a family of codes of length \( N \) (assumed large enough) and rate \( R \in (0, 1) \) is sought. Pick \( n \) to be a growing parameter.

By picking \( s = \Theta(1/\epsilon) \), \( m = \Theta(1/\epsilon^2) \), \( r = \lceil \log n \rceil \), \( e = \lceil \log n \log \log n \rceil \), \( \zeta = (\log n \log \log n)^{-1} \), \( N = \lceil \frac{(r^2-1)\epsilon}{m} \rceil \), and \( k \) proportional to \( Nm \) in Theorem 8.1, we can conclude the following.

**Corollary 8.2.** For any \( R \in (0, 1) \) and positive constant \( \epsilon \in (0, 1) \), there is a Monte Carlo construction of a family of codes of rate at least \( R \) over an alphabet size \( (\log N)^{O(1/\epsilon^2)} \) that are encodable and \((1-R-\epsilon, O(R^{-1} \log N \log \log N))\)-list decodable in \( \text{poly}(N, 1/\epsilon) \) time, where \( N \) is the block length of the code.

Our promised main result (Theorem 11) achieves better parameters than the above — an alphabet size of \( \exp(O(1/\epsilon^2)) \) and list-size of \( O(1/(R\epsilon)) \). This is based on the Garcia-Stichtenoth tower and is described next.

### 8.2. Combining folded Garcia-Stichtenoth codes and h.se sets.

Similarly to Section 8.1, we now show how to pre-code the messages of the FGS code with a h.se subset. The approach is similar, though we need one additional idea to ensure that we can pick parameters so that the base field \( \mathbb{F}_q \) can be constant-sized and obtain a final list-size bound that is a constant independent of the code length. This idea is to work with a larger “period size” \( \Delta \) for the periodic subspaces, based on the following observation.

**Observation 8.3.** Let \( W \) be an \((s, \Delta, b)\)-periodic subspace of \( \mathbb{F}_q^k \) for \( k = b\Delta \). Then \( W \) is also \((su, \Delta u, b)\)-periodic for every integer \( u \), \( 1 \leq u \leq b \).

---

6Technically, it will belong to proj\(_b\)(\( W \)) of such a periodic subspace \( W \), but we may pretend that there are \((q-1)[k/(q-1)] - k \) extra dummy coordinates which we decode. Or we can just assume for convenience that \( k \) is divisible by \( q-1 \).
As in the Hermitian case, instead of encoding arbitrary \( f \in \mathbb{F}_q^k \) by the folded Garcia-Stichtenoth code (Definition 18), we will restrict the messages \( f \) to belong to the range of our h.s.e set. This will ensure that the affine space of solutions guaranteed by Lemma 7.5 can be efficiently pruned to a small list.

**Theorem 8.4.** Let \( r \) be a prime power, \( q = r^2 \), and \( e \geq 2 \) be an integer, and \( \zeta \in (0, 1) \). Let \( k \leq q^r \Delta /2 \) be a positive integer. Let \( \Delta \leq k \) be a multiple of \( (r-1) \), say \( \Delta = u(r-1) \) for a positive integer \( u \).

Let \( s, m \) be positive integers satisfying \( 1 \leq s \leq m \leq r-1 \) and \( s < \zeta r/12 \). Finally let \( N \) be an integer satisfying \( k + 2r^e \leq Nm \leq (r-1)r^e \).

Consider the code \( C_2 \) with encoding \( E_2 : \mathbb{F}_q^{(1-\zeta)^2k} \rightarrow (\mathbb{F}_q^m)^N \) defined as

\[
E_2(x) = \text{FGS}(N, k, q, e, m)(\text{HSE}(x)) ,
\]

for a random map \( \text{HSE} : \mathbb{F}_q^{(1-\zeta)^2k} \rightarrow \mathbb{F}_q^k \) as constructed in Definition 8 for a period size \( \Delta \) and \( b = \lceil \frac{k}{\Delta} \rceil \) (where we pad the input \( \mathbb{F}_q^k \) with 0’s to make its length \( b\Delta \) to feed into \( \text{HSE} \)).

The code \( C_2 \) has rate \( R = (1-\zeta)^2k/(Nm) \), can be encoded in \( \text{poly}(Nmq^{r}\Delta) \) time, and w.h.p over the choice of \( \text{HSE} \), it is \((\tau, \ell)-\text{list decodable in time} \text{ poly}(Nmq^{r}\Delta)\) for \( \ell \leq O(1/(R\zeta)) \) and

\[
\tau = \frac{s}{s+1} \left( 1 - \frac{k}{N(m-s+1)} \right) - \frac{3m}{m-s+1} \frac{r^e}{mN} .
\]

**Proof.** The proof is very similar to that of Theorem 8.1. The claim about the rate is clear, and the encoding time follows from the time to compute \( \text{HSE} \) recorded in Theorem 6.5.

The genus \( g_\ell \) is now upper bounded by \( r^e \), and so the condition on \( N, m \) meets the requirement for the construction of the folded Hermitian tower based code in Definition 4 and the claimed value of the error fraction \( \tau \) satisfies (14). By Lemma 5.7, we know that the candidate messages found by the decoder lie in one of at most \( q^{2Nm} \) possible \((s, r-1, \lceil \frac{k}{r-1} \rceil)\)-periodic subspaces[^7]. Now by Observation 8.3, each of these subspaces is also \((su, \Delta, \lceil \frac{k}{\Delta} \rceil)\)-periodic. One can check that the conditions of Theorem 6.5 are met for our choice of \( \zeta, s, q, k, \Delta \) and taking \( su \) to play the role of \( s \) (since \( s < \zeta r/12 \), we have \( su < \zeta \Delta/10 \)).

Appealing to Theorem 6.5 with the choice \( c = 2Nm/k = O(1/R) \), we conclude that there is a decoding algorithm running in time \( \text{poly}(Nmq^r\Delta) \) to list decode \( C_2 \) from a fraction \( \tau \) of errors, outputting at most \( O(1/(R\zeta)) \) messages in the worst-case. \( \square \)

Finally, all that is left to be done is to pick parameters to show how the above can lead to optimal rate list-decodable codes over a constant-sized alphabet which further achieve very good list-size.

Let \( \varepsilon > 0 \) be a small positive constant, and a family of codes of length \( N \) (assumed large enough) and rate \( R \in (0, 1) \) is sought. Pick \( n \) to be a growing parameter.

Let us pick \( s = \Theta(1/\varepsilon), m = \Theta(1/\varepsilon^2), \zeta = \varepsilon/12, r = \Theta(1/\varepsilon), q = r^2, \) and \( e = \lceil \frac{\log n}{\log r} \rceil \), \( N = \left( \frac{r-1}{m^r} \right), \) and \( k = RNm(1 + \varepsilon) \). This ensures that (i) there are at least \( n = Nm \) rational places and so we get a code of length at least \( n/m = N \), (ii) the rate of the code \( C_2 \) is at least \( R \), and (iii) the error fraction (24) is at least \( 1 - R - \varepsilon \).

[^7]: Technically, it will belong to \( \text{proj}_s(W) \) of such a periodic subspace \( W \), but we may pretend that there are \( (r-1)(k/r-1) \) extra dummy coordinates which we decode. Or we can just assume for convenience that \( r - 1 \) divides \( k \).
The remaining part is to pick a multiple $\Delta$ of $(r - 1)$ so that the $k \leq q^{\zeta \Delta/2}$ condition is met. This can be achieved by choosing $u = \lceil \log n / \log (1/\varepsilon) \rceil$ and $\Delta = (r - 1)u$. With these choices, we can conclude the following, which is one of the main end goals of this paper.

**Theorem 8.5** (Main; Corollary to Theorem 8.4 with above choice of parameters). For any $R \in (0, 1)$ and positive constant $\varepsilon \in (0, 1)$, there is a Monte Carlo construction of a family of codes of rate at least $R$ over an alphabet size $\exp(O(\log(1/\varepsilon)/\varepsilon^2))$ that are encodable and $(1 - R - \varepsilon, O(1/(R\varepsilon))-\text{list decodable})$ in poly$(N)$ time, where $N$ is the block length of the code.

It may be instructive to recap why the Hermitian tower could not give a result like the above one. In the Hermitian case, the ratio $g_e/n$ of the genus to the number of rational places was about $e/r = e/\sqrt{q}$, and thus we needed $q > e^2$. Since the period $\Delta$ was about $q$, the running time of the decoder was bigger than $q^{O(\sqrt{q})}$, whereas the length of the code was at most $q^{O(\sqrt{q})}$. This dictated the choice of $q \approx \log^2 n$, and then to keep the running time polynomial, we had to take $\zeta \approx (\log n \log \log n)^{-1}$.

## 9. List decoding AG codes with subfield evaluation points

In this section, we will present a linear-algebraic list decoding algorithm for algebraic-geometric (AG) codes based on evaluations of functions at rational points over a subfield. The algorithm will manage to correct a large fraction of errors, and pin down the possible messages to a well-structured affine subspace of dimension much smaller than that of the code. For simplicity, we begin with the case of Reed-Solomon codes in Section 9.1. We then extend it to a general framework for decoding AG codes based on constant field extensions in Section 9.2. Finally, in Section 9.1, we instantiate the general framework (with a slight twist) to codes based on the Garcia-Stichtenoth tower.

### 9.1. Decoding Reed-Solomon codes

Our list decoding algorithm will apply to Reed-Solomon codes with evaluation points in a subfield, defined below.

**Definition 12.** [Reed-Solomon code with evaluations in a subfield] Let $\mathbb{F}_q$ be a finite field with $q$ elements, and $m$ a positive integer. Let $n, k$ be positive integers satisfying $1 \leq k < n \leq q$. The Reed-Solomon code $\text{RS}^{(q,m)}[n,k]$ is a code over alphabet $\mathbb{F}_q^m$ that encodes a polynomial $f \in \mathbb{F}_q^m[X]$ of degree at most $k - 1$ as

$$f(X) \mapsto (f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n))$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are an arbitrary sequence of $n$ distinct elements of $\mathbb{F}_q$.

Note that while the message polynomial has coefficients from $\mathbb{F}_q^m$, the encoding only contains its evaluations at points in the subfield $\mathbb{F}_q$. The above code has rate $k/n$, and minimum distance $(n - k + 1)$.

We now present a list decoding algorithm for the above Reed-Solomon codes. Suppose the codeword $(f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n))$ is received as $(y_1, y_2, \ldots, y_n) \in \mathbb{F}_{q^m}^n$ with at most $e = \tau n$ errors (i.e., $y_i \neq f(\alpha_i)$ for at most $e$ values of $i \in \{1, 2, \ldots, n\}$). The goal is to recover the list of all polynomials of degree less than $k$ whose encoding is within Hamming distance $e$ from $y$. As is common in algebraic list decoders, the algorithm will have two steps: (i) interpolation to find an algebraic equation the message polynomials must satisfy, and (ii) solving the equation for the candidate message polynomials.
Interpolation step. Let \( 1 \leq s \leq m \) be an integer parameter of the algorithm. Choose the “degree parameter” \( D \) to be
\[
D = \left\lfloor \frac{n - k + 1}{s + 1} \right\rfloor.
\]

**Definition 13** (Space of interpolation polynomials). Let \( \mathcal{P} \) be the space of polynomials \( Q \in \mathbb{F}_{q^m}[X, Y_1, Y_2, \ldots, Y_s] \) of the form
\[
Q(X, Y_1, Y_2, \ldots, Y_s) = A_0(X) + A_1(X)Y_1 + A_2(X)Y_2 + \cdots + A_s(X)Y_s,
\]
with each \( A_i \in \mathbb{F}_{q^m}[X] \) and \( \deg(A_0) \leq D + k - 1 \) and \( \deg(A_i) \leq D \) for \( i = 1, 2, \ldots, s \).

The lemma below follows because for our choice of \( D \), the number of degrees of freedom for polynomials in \( \mathcal{P} \) exceeds the number \( n \) of interpolation conditions (27). We include the easy proof for completeness.

**Lemma 9.1.** There exists a nonzero polynomial \( Q \in \mathcal{P} \) such that
\[
Q(\alpha_i, y_i, y_i^q, y_i^{q^2}, \ldots, y_i^{q^{s-1}}) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.
\]
Further such a \( Q \) can be found using \( O(n^3) \) operations over \( \mathbb{F}_{q^m} \).

**Proof.** Note that \( \mathcal{P} \) is an \( \mathbb{F}_{q^m} \)-vector space of dimension
\[
(D + k) + s(D + 1) = (D + 1)(s + 1) + k - 1 > n,
\]
where the last inequality follows from our choice (25). The interpolation conditions required in the lemma impose \( n \) homogeneous linear conditions on \( Q \). Since this is smaller than the dimension of \( \mathcal{P} \), there must exist a nonzero \( Q \in \mathcal{P} \) that meets the interpolation conditions
\[
Q(\alpha_i, y_i, y_i^q, y_i^{q^2}, \ldots, y_i^{q^{s-1}}) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.
\]
Finding such a \( Q \) amounts to solving a homogeneous linear system over \( \mathbb{F}_{q^m} \) with \( n \) constraints and at most \( \dim(\mathcal{P}) \leq n + s + 2 \) unknowns, which can be done in \( O(n^3) \) time. \( \square \)

Lemma 9.3 below shows that any polynomial \( Q \) given by Lemma 9.1 yields an algebraic condition that the message functions \( f \) we are interested in list decoding must satisfy.

**Definition 14** (Frobenius action on polynomials). For a polynomial \( f \in \mathbb{F}_{q^m}[X] \) with \( f(X) = f_0 + f_1X + \cdots + f_{k-1}X^{k-1} \), define the polynomial \( f^\sigma \in \mathbb{F}_{q^m}[X] \) as \( f^\sigma(X) = f_0^\sigma + f_1^\sigma X + \cdots + f_{k-1}^\sigma X^{k-1} \).

For \( i \geq 2 \), we define \( f^{\sigma^i} \) recursively as \( (f^{\sigma^{i-1}})^\sigma \).

The following simple fact is key to our analysis.

**Fact 9.2.** If \( \alpha \in \mathbb{F}_q \), then \( f(\alpha)^q = (f^\sigma)(\alpha) \) for all \( j = 1, 2, \ldots \).

**Lemma 9.3.** Suppose \( Q \in \mathcal{P} \) satisfies the interpolation conditions (27). Suppose \( f \in \mathbb{F}_{q^m}[X] \) of degree less than \( k \) satisfies \( f(\alpha_i) \neq y_i \) for at most \( e \) values of \( i \in \{1, 2, \ldots, n\} \) with \( e \leq \frac{s}{s+1}(n-k) \). Then \( Q(X, f(X), f^\sigma(X), f^{\sigma^2}(X), \ldots, f^{\sigma^{s-1}}(X)) = 0 \).

**Proof.** Define the polynomial \( \Phi \in \mathbb{F}_{q^m}[X] \) by \( \Phi(X) := Q(X, f(X), f^\sigma(X), f^{\sigma^2}(X), \ldots, f^{\sigma^{s-1}}(X)) \). By the construction of \( Q \) and the fact that \( \deg(f) \leq k-1 \), we have \( \deg(\Phi) \leq D + k - 1 \leq \frac{n-k+1}{s+1} + k - 1 = \frac{n}{s+1} + \frac{s}{s+1}(k-1) \).
Suppose \( y_i = f(\alpha_i) \). By Fact \( \text{[9.2]} \) we have \( y_i^q = f(\alpha_i)^q = (f^q)(\alpha_i) \), and similarly \( y_i^{q^2} = (f^{q^2})(\alpha_i) \) for \( j = 2, 3, \ldots \). Thus for each \( i \) such that \( f(\alpha_i) = y_i \), we have

\[
\Phi(\alpha_i) = Q(\alpha_i, f(\alpha_i), f^q(\alpha_i), \ldots, f^{q^{s-1}}(\alpha_i)) = Q(\alpha_i, y_i, y_i^q, \ldots, y_i^{q^{s-1}}) = 0. 
\]

Thus \( \Phi \) has at least \( n - e \geq \frac{s^+}{s+1} + \frac{s^+}{s+1} \) zeroes. Since this exceeds the upper bound on the degree of \( \Phi \), \( \Phi \) must be the zero polynomial. \( \square \)

**Finding candidate solutions.** The previous two lemmas imply that the polynomials \( f \) whose encodings differ from \( (y_1, \ldots, y_n) \) in at most \( \frac{s}{s+1}(n - k) \) positions can be found amongst the solutions of the functional equation \( A_0 + A_1 f + A_2 f^q + \cdots + A_s f^{q^{s-1}} = 0 \). We now prove that these solutions form a well-structured affine space over \( \mathbb{F}_q \).

**Lemma 9.4.** For integers \( 1 \leq s \leq m \), the set of solutions \( f = \sum_{i=0}^{k-1} f_i X^i \in \mathbb{F}_q[X] \) to the equation

\[
A_0(X) + A_1(X) f(X) + A_2(X) f^q(X) + \cdots + A_s(X) f^{q^{s-1}}(X) = 0
\]

when at least one of \( \{A_0, A_1, \ldots, A_s\} \) is nonzero is an affine subspace over \( \mathbb{F}_q \) of dimension at most \( (s - 1)k \). Further, fixing an \( \mathbb{F}_q \)-basis of \( \mathbb{F}_q^m \) and viewing each \( f_i \) as an element of \( \mathbb{F}_q^m \), the solutions are an \( (s - 1, m, k) \)-periodic subspace of \( \mathbb{F}_q^{mnk} \). A canonical representation of this periodic subspace (in the sense of Definition 2) can be computed in \( \text{poly}(k, m, \log q) \) time.

**Proof.** If \( f, g \) are two solutions to (28), then so is \( \alpha f + \beta g \) for any \( \alpha, \beta \in \mathbb{F}_q \) with \( \alpha + \beta = 1 \). So the solutions to (28) form an affine \( \mathbb{F}_q \)-subspace. We now proceed to analyze the structure of the subspace.

First, by factoring out a common powers of \( X \) that divide all of \( A_0(X), A_1(X), \ldots, A_s(X) \), we can assume that at least one \( A_{i^*}(X) \) for some \( i^* \in \{0, 1, \ldots, s\} \) is not divisible by \( X \), and has nonzero constant term. Further, if \( A_1(X), \ldots, A_s(X) \) are all divisible by \( X \), then so is \( A_0(X) \), so we can take \( i^* > 0 \).

Let us denote \( A_i(X) = a_{i,0} + a_{i,1}X + a_{i,2}X^2 + \cdots \) for \( i = 0, 1, 2, \ldots, s \). For \( l = 0, 1, 2, \ldots, k - 1 \), define the linearized polynomial

\[
B_l(X) = a_{l,0}X + a_{l,1}X^q + a_{l,2}X^{q^2} + \cdots + a_{l,s}X^{q^{s-1}}.
\]

We know that \( a_{l,s,0} \neq 0 \), and therefore \( B_0 \neq 0 \). This implies that the solutions \( \beta \in \mathbb{F}_q^m \) to \( B_0(\beta) = 0 \) is a subspace, say \( W \), of \( \mathbb{F}_q^m \) of dimension at most \( s - 1 \).

Fix an \( i \in \{0, 1, \ldots, k - 1\} \). Expanding the equation (28) and equating the coefficient of \( X^i \) to be 0, we get

\[
a_{0,i} + B_i(f_0) + B_{i-1}(f_1) + \cdots + B_1(f_{i-1}) + B_0(f_i) = 0.
\]

This implies \( f_i \in W + \theta_i \) for some \( \theta_i \in \mathbb{F}_q^m \) that is determined by \( f_0, f_1, \ldots, f_{i-1} \). Therefore, for each choice of \( f_0, f_1, \ldots, f_{i-1}, f_i \) must belong to a fixed coset of the subspace \( W \) of dimension at most \( s - 1 \). Thus, the solutions belong to an \( (s - 1, m, k) \)-periodic subspace. Also, it is clear from (30) that a canonical representation of the periodic subspace can be computed in \( \text{poly}(k, m, \log q) \) time. \( \square \)

Combining Lemmas 9.3 and 9.4, we see that one can find an affine space of dimension \( (s - 1)k \) that contains the coefficients of all polynomials whose encodings differ from the input \( (y_1, \ldots, y_n) \) in at most a fraction \( \frac{s^+}{s+1}(1 - R) \) of the positions. Note the dimension of the message space of the Reed-Solomon code \( RS(q,m)[n,k] \) over \( \mathbb{F}_q \) is \( km \). The above lemma pins down the candidate polynomials to a space of dimension \( (s - 1)k \). For \( s \ll m \),
this is a lot smaller. In particular, it implies one can list decode in time sub-linear in
the code size (the proof follows by taking \( s = \lceil 1/\varepsilon \rceil \) and \( m > \frac{\varepsilon}{2} \)).

**Corollary 9.5.** For every \( R \in (0,1) \), and \( \varepsilon, \gamma > 0 \), there is a positive integer \( m \) such that
for all large enough prime powers \( q \), the Reed-Solomon code \( C = RS(q,m)[q,Rq] \) can be list
decoded from a fraction \( (1 - R - \varepsilon) \) of errors in \( |C|^{\gamma} \) time, outputting a list of size at most
\( |C|^{\gamma} \).

Since the dimension of the subspace guaranteed by Lemma 9.4 grows linearly in \( k \),
we still cannot afford to list this subspace as the decoder’s output for polynomial time
decoding. Instead, we will use a “pre-code” that only allows polynomials with coefficients
in a carefully chosen subset that is guaranteed to have small intersection with the space
of solutions to any equation of the form (28). Further, this intersection can be found
quickly without going over all solutions to (28). In Section 10, we will see the approach
to accomplish this based on subspace designs.

### 9.2. Decoding algebraic-geometric codes

In this section we generalize the Reed-
Solomon algorithm to algebraic-geometric codes. The description in this section will be
for a general abstract AG code. So we will focus on the algebraic ideas, and not mention
complexity estimates. The next subsection will focus on a specific AG code based on
Garcia-Stichtenoth function fields, which will require a small change to the setup, and
where we will also mention computational aspects. We assume familiarity with the basic
terminology and notation; the reader is referred to Stichtenoth’s book for basic background
setup of algebraic function fields and codes based on function fields, and use standard
notation; the reader is referred to Stichtenoth’s book for basic background.

Let \( F/F_q \) be a function field of genus \( g \). Let \( P_\infty, P_1, P_2, \ldots, P_N \) be \( N + 1 \) distinct
\( F_q \)-rational places. Let \( \sigma \in \text{Gal}(F_q^m/F_q) \) be the Frobenius automorphism, i.e, \( \alpha^\sigma = \alpha^q \) for
all \( \alpha \in F_q^m \). Then we can extend \( \sigma \) to an automorphism in \( \text{Gal}(F_m/F) \), where \( F_m \) is the
constant extension \( F_q^m \cdot F \). Note that \( P^\sigma = P \) for any place of \( F \).

For a place \( P \) of \( F \), we denote by \( \nu_P \) the discrete valuation of \( P \). For an integer \( l \), we
consider the Riemann-Roch space over \( F_q \) defined by

\[
\mathcal{L}(lP_\infty) := \{ h \in F \setminus \{ 0 \} : \nu_{P_\infty}(h) \geq -l \} \cup \{ 0 \}.
\]

Then the dimension \( \ell(lP_\infty) \) is at least \( l - g + 1 \) and equality holds if \( l \geq 2g - 1 \). Furthermore,
we define the Riemann-Roch space over \( F_q^m \) by

\[
\mathcal{L}_m(lP_\infty) := \{ h \in F_m \setminus \{ 0 \} : \nu_{P_\infty}(h) \geq -l \} \cup \{ 0 \}.
\]

Consider the Goppa geometric code defined by

\[
C(m;l) := \{ (f(P_1), f(P_2), \ldots, f(P_N)) : f \in \mathcal{L}_m(lP_\infty) \}.
\]

The following result is a fundamental fact about algebraic-geometric codes.

**Lemma 9.6.** The above code \( C(m;l) \) is an \( F_q^m \)-linear code over \( F_q^m \), rate at least \( \frac{1-g+1}{m} \),
and minimum distance at least \( N - 1 \).

We now present a list decoding algorithm for the above codes. The algorithm follows
the linear-algebraic list decoding algorithm for RS codes. Suppose a codeword encoding
\( f \in \mathcal{L}_m((k + 2g - 1)P_\infty) \) is transmitted and received as \( y = (y_1, y_2, \ldots, y_N) \).

Given such a received word, we will interpolate a nonzero linear polynomial over \( F_m \)

\[
Q(Y_1, Y_2, \ldots, Y_s) = A_0 + A_1Y_1 + A_2Y_2 + \cdots + A_sY_s
\]

(31)
where \( A_i \in \mathcal{L}_m(DP_\infty) \) for \( i = 1, 2, \ldots, s \) and \( A_0 \in \mathcal{L}_m((D + k + 2g - 1)P_\infty) \) with the degree parameter \( D \) chosen to be

\[
D = \left\lfloor \frac{N - k + (s - 1)g + 1}{s + 1} \right\rfloor.
\]

If we fix a basis of \( \mathcal{L}_m(DP_\infty) \) and extend it to a basis of \( \mathcal{L}_m((D + k + 2g - 1)P_\infty) \), then the number of freedoms of \( A_0 \) is at least \( D + k + g \) and the number of freedoms of \( A_i \) is at least \( D - g + 1 \) for \( i \geq 1 \). Thus, the total number of freedoms in the polynomial \( Q \) equals

\[
s(D - g + 1) + D + k + g = (s + 1)(D + 1) - (s - 1)g - 1 + k > N.
\]

for the above choice (32) of \( D \). The interpolation requirements on \( Q \in F_m[Y_1, \ldots, Y_s] \) are the following:

\[
Q(y_i, y_i^\sigma, \ldots, y_i^{\sigma^{s-1}}) = A_0(P_i) + A_1(P_i)y_i + A_2(P_i)y_i^\sigma + \cdots + A_s(P_i)y_i^{\sigma^{s-1}} = 0
\]

for \( i = 1, 2, \ldots, N \). Thus, we have a total of \( N \) equations to satisfy. Since this number is less than the number of freedoms in \( Q \), we can conclude that a nonzero linear function \( Q \in F_m[Y_1, \ldots, Y_s] \) of the form (31) satisfying the interpolation conditions (31) can be found by solving a homogeneous linear system over \( \mathbb{F}_q^m \) with at most \( N \) constraints and at least \( s(D - g + 1) + D + k + g \) variables.

The following lemma gives the algebraic condition that the message functions \( f \in \mathcal{L}_m((k + 2g - 1)P_\infty) \) we are interested in list decoding must satisfy.

**Lemma 9.7.** If \( f \) is a function in \( \mathcal{L}_m((k + 2g - 1)P_\infty) \) whose encoding agrees with the received word \( y \) in at least \( t \) positions with \( t > D + k + 2g - 1 \), then

\[
Q(f, f^\sigma, \ldots, f^{\sigma^{s-1}}) = 0.
\]

**Proof.** The proof proceeds by comparing the number of zeros of the function \( Q(f, f^\sigma, \ldots, f^{\sigma^{s-1}}) = A_0 + A_1f + A_2f^\sigma + \cdots + A_sf^{\sigma^{s-1}} \) with \( D + k + 2g - 1 \). Note that \( Q(f, f^\sigma, \ldots, f^{\sigma^{s-1}}) \) is a function in \( \mathcal{L}_m((D + k + 2g - 1)P_\infty) \). If position \( i \) of the encoding of \( f \) agrees with \( y \), then

\[
0 = A_0(P_i) + A_1(P_i)y_i + A_2(P_i)y_i^\sigma + \cdots + A_s(P_i)y_i^{\sigma^{s-1}}
\]

\[
= A_0(P_i) + A_1(P_i)f(P_i) + A_2(P_i)(f(P_i))^\sigma + \cdots + A_s(P_i)(f(P_i))^{\sigma^{s-1}}
\]

\[
= A_0(P_i) + A_1(P_i)f(P_i) + A_2(P_i)f^\sigma(P_i) + \cdots + A_s(P_i)f^{\sigma^{s-1}}(P_i)
\]

\[
= (A_0 + A_1f + A_2f^\sigma + \cdots + A_sf^{\sigma^{s-1}})(P_i)
\]

i.e., \( P_i \) is a zero of \( Q(f, f^\sigma, \ldots, f^{\sigma^{s-1}}) \). Thus, there are at least \( t \) zeros for all the agreeing positions. Hence, \( Q(f, f^\sigma, \ldots, f^{\sigma^{s-1}}) \) must be the zero function when \( t > D + k + 2g - 1 \). \( \square \)

Let \( P \) be a rational place in \( F \) and let \( T \in F \) be a local parameter of \( P \). Then \( T^\sigma = T \). Here, we have two scenarios, i.e., \( P = P_\infty \) or \( P \neq P_\infty \). In Subsection 9.3 we will consider the case where \( P = P_\infty \) for the Garcia-Stichtenoth tower. While in this subsection, we only discuss the case where \( P \neq P_\infty \). This was the case with Reed-Solomon codes with message polynomials in \( \mathbb{F}_q[X] \) in Subsection 9.1 where \( P_\infty \) was the pole of \( X \), and \( P \) the zero of \( X \).

Assume that a function \( f \in \mathcal{L}_m((k + 2g - 1)P_\infty) \) has a local expansion at \( P \)

\[
f = \sum_{j=0}^{\infty} f_jT^j
\]
for some \( f_j \in \mathbb{F}_{q^m} \). Then \( f \) is uniquely determined by \((f_0, f_1, \ldots, f_{k+2g-1})\) since \( f \) has the pole degree at most \( k + 2g - 1 \).

**Lemma 9.8.** The set of solutions \( f \in L_m((k + 2g - 1)P_\infty) \) to the equation (34)

\[
A_0 + A_1 f + A_2 f^q + \cdots + A_s f^{q^s-1} = 0
\]

when at least one \( A_i \) is nonzero has size at most \( q^{(s-1)(k+2g-1)} \). Further, the possible coefficients \((f_0, f_1, \ldots, f_{k+2g-1})\) of \( f \)'s local expansion at \( P \) belong to an \((s-1, m)\)-ultra periodic affine subspace of \( \mathbb{F}_q^{(k+2g-1)m} \).

**Proof.** The argument is very similar to Lemma 9.4. Let \( u = \min\{\nu_{P_\infty}(A_i) : i = 1, 2, \ldots, s\} \). Then it is clear that \( u \geq 0 \) and \( \nu_P(A_0) \geq u \). Each \( A_i \) has a local expansion at \( P \):

\[
A_i = T^u \sum_{j=0}^{\infty} a_{i,j} T^j
\]

for \( i = 0, 1, \ldots, s \).

Assume that at \( P \), the function \( f \) has a local expansion (36). Then \( f^{q^i} \) has a local expansion at \( P \) as follows

\[
f^{q^i} = \sum_{j=0}^{\infty} f_j^q T^j.
\]

For \( l = 0, 1, \ldots \), define the linearized polynomial

\[
B_l(X) := a_{l,1}X + a_{l,2}X^q + \cdots + a_{l,s}X^{q^s-1}
\]

From the definition of \( u \), one knows that \( B_0(X) \) is nonzero. Equating the coefficient of \( T^{d+u} \) in \( A_0 + A_1 f + A_2 f^q + \cdots + A_s f^{q^s-1} \) to equal 0 gives us the condition

\[
a_{0,d} + B_d(f_0) + B_{d-1}(f_1) + \cdots + B_0(f_d) = 0.
\]

Let \( W = \{ \alpha \in \mathbb{F}_{q^m} : B_0(\alpha) = 0 \} \). Then \( W \) is an \( \mathbb{F}_q \)-subspace of \( \mathbb{F}_{q^m} \) of dimension at most \( s - 1 \), since \( B_0 \) is a nonzero linearized polynomial of \( q \)-degree at most \( s - 1 \). As in Lemma 9.4, for each fixed \( f_0, f_1, \ldots, f_{d-1} \), the coefficient \( f_d \) must belong to a coset of the subspace \( W \). This implies that the coefficients \((f_0, f_1, \ldots, f_{k+2g-1})\) belong to an \((s-1, m, k + 2g - 1)\)-periodic subspace of \( \mathbb{F}_q^{m(k+2g-1)} \). In particular, there are at most \( q^{(s-1)(k+2g-1)} \) solutions \( f \in L_m((k + 2g - 1)P_\infty) \) to (34).

The equation (37) also shows that each group of \( \ell \) successive coefficients \( f_{d-\ell+1}, f_{d-\ell+2}, \ldots, f_d \) belong to cosets of the same underlying \( \ell(s-1) \) dimensional subspace of \( \mathbb{F}_q^{m\ell} \). This implies that \((f_0, f_1, \ldots, f_{k+2g-1})\) in fact belong to an \((s-1, m)\)-ultra periodic subspace. \(\square\)

**Restricting message functions using local expansions.** Using Lemma 9.8 we will recover the message in terms of the coefficients of its local expansion at \( P \). In order to prune the subspace of possible solutions, we will pick a subcode that corresponds to restricting the coefficients to a carefully constructed subset of all possibilities. This requires us to index message functions in terms of the local expansion coefficients. However, not all \((k + 2g - 1)\) tuples over \( \mathbb{F}_{q^m} \) arise in the local expansion of functions in the \( k \)-dimensional subspace \( L_m((k + 2g - 1)P_\infty) \). Below we show that we can find a \( k \)-dimensional subspace

---

8This ultra-periodicity was also true for the Reed-Solomon case in Lemma 9.4, but we did not state it there as we will not make use of this extra property for picking a subcode in the case of Reed-Solomon codes.
of $\mathcal{L}_m((k+2g-1)P_\infty)$ such that their top $k$ local expansion coefficients give rise to all $k$-tuples over $\mathbb{F}_q^n$.

**Lemma 9.9.** There exist a set of functions $\{g_1, g_2, \ldots, g_k\}$ in $\mathcal{L}_m((k+2g-1)P_\infty)$ such that the $k \times k$ matrix $A$ formed by taking the $i$th row of $A$ to be the first $k$ coefficients in the local expansion \[\text{(35)}\] for $g_i$ at $P$ is non-singular.

**Proof.** Let $\{\psi_1, \psi_2, \ldots, \psi_g\}$ be a basis of $\mathcal{L}_m((k+2g-1)P_\infty - kP)$. Extend this basis to a basis $\{\psi_1, \psi_2, \ldots, \psi_g, g_1, g_2, \ldots, g_k\}$ of $\mathcal{L}_m((k+2g-1)P_\infty)$. We claim that the functions $\{g_1, g_2, \ldots, g_k\}$ are our desired functions.

Suppose that the matrix $A$ is obtained from expansion of functions $g_i$ and it is singular. This implies that there exists elements $\{\lambda_i\}_{i=1}^k$ such that the function $\sum_{i=1}^k \lambda_i g_i$ has expansion $\sum_{i=k}^\infty a_i T_i$ at $P$ for some $a_i \in \mathbb{F}_q^n$. Therefore, the function $\sum_{i=1}^k \lambda_i g_i$ belongs to the space $\mathcal{L}_m((k+2g-1)P_\infty - kP)$, i.e., $\sum_{i=1}^k \lambda_i g_i$ is a linear combination of $\psi_1, \psi_2, \ldots, \psi_g$. This forces that all $\lambda_i$ are equal to 0 since $\{\psi_1, \ldots, \psi_g, g_1, g_2, \ldots, g_k\}$ is linearly independent. This completes the proof. \[
\square
\]

With the above lemma in place, we now describe our AG code in a manner convenient for pruning the possible local expansion coefficients.

**Encoding.** Assume that we have found a set of functions $\{g_1, g_2, \ldots, g_k\}$ of $\mathcal{L}_m((k+2g-1)P_\infty)$ as in Lemma 9.9. After elementary row operations on the matrix $A$ defined in Lemma 9.9, we may assume that $A$ is the $k \times k$ identity matrix, i.e., we assume that, for $1 \leq i \leq k$, the function $g_i$ has expansion $T_i^{k-1} + \sum_{j=k}^\infty \lambda_{ij} T_j$ for some $\lambda_{ij} \in \mathbb{F}_q^n$.

Now for any subset $M \subseteq \mathbb{F}_q^n$, we may assume that our messages belong to $M$, and encode each message $(a_1, a_2, \ldots, a_k) \in M$ to the codeword $(f(P_1), f(P_2), \ldots, f(P_N))$, where $f = \sum_{i=1}^k a_i g_i$. Thus, our actual code is a subcode of $C(m; k+2g-1)$ given by \[\text{(38)}\]

$$C(m; k+2g-1 \mid M) \overset{\text{def}}{=} \{(f(P_1), f(P_2), \ldots, f(P_N)) : f = \sum_{i=1}^k a_i g_i, (a_1, a_2, \ldots, a_k) \in M\}.$$  

**Decoding.** To decode, we first establish the equation \[\text{(35)}\] and solve this equation to find the subspace of possible first $k$ coefficients $f_0, f_1, \ldots, f_{k-1}$ in the local expansion of the function $f = \sum_{i=1}^k a_i g_i$ at $P$. The following claim implies that the message tuple $(a_1, a_2, \ldots, a_k)$ belongs to this subspace.

**Lemma 9.10.** The first $k$ coefficients $f_0, f_1, \ldots, f_{k-1}$ of the local expansion of $f = \sum_{i=1}^k a_i g_i$ at $P$ equal $a_1, a_2, \ldots, a_k$.

**Proof.** Since $g_i$ has local expansion $T_i^{k-1} + \sum_{j=k}^\infty \lambda_{ij} T_j$, it is clear that the local expansion of $f$ is $\sum_{i=0}^{k-1} a_{i+1} T_i + \sum_{j=k}^{\infty} a_{k+1} T_j$ for some $a_{k+1}, a_{k+2}, \ldots$ in $\mathbb{F}_q^n$. Thus the first $k$ coefficients of the local expansion of $f$ are $a_1, a_2, \ldots, a_k$. \[
\square
\]

Combining Lemmas 9.7, 9.8 and 9.10 and recalling the choice of $D$ in \[\text{(32)}\], we get the following.

**Corollary 9.11.** For the code $C(m; k+2g-1 \mid \mathbb{F}_q^n)$, we can find an $(s-1, m)$-ultra periodic subspace of $\mathbb{F}_q^{mk}$ that includes all messages whose encoding differs from a received word $y \in \mathbb{F}_q^N$ in at most $\frac{s}{s+1}(N-k) - \frac{3s+1}{s+1}g$ positions.
9.3. Decoding the codes from the Garcia-Stichtenoth tower. Let $r$ be a prime power and let $q = r^2$. For $e \geq 2$, let $K_e$ be the function field $\mathbb{F}_q(x_1, x_2, \ldots, x_e)$ given by Garcia-Stichtenoth tower (10).

Put $F = K_e$ and $F_m = \mathbb{F}_{q^m} \cdot K_e$. The encoding and decoding is almost identical to the algebraic geometric code described in the previous section except here we use $P_\infty$ for local expansion.

**Encoding.** As in Lemma 9.9 we can find a set of functions $\{h_1, h_2, \ldots, h_k\}$ of $\mathcal{L}_m((k + 2g_e - 1)P_\infty)$ such that the $k \times k$ matrix $A$ formed by taking the $i$th row of $A$ to be the first $k$ coefficients in the expansion (40) for $h_i$ at $P_\infty$ is non-singular (note that in this case, the local expansion starts from $T^{-(k+2g_e-1)}$, while in the previous subsection the local expansion starts from $T^0$). Furthermore, after some elementary row operations on $A$, we may assume that, for $1 \leq i \leq k$, the function $h_i$ has expansion $T^{-(k+2g_e-1)} \left(T^{i-1} + \sum_{j=0}^{\infty} \lambda_{ij} T^j\right)$ for some $\lambda_{ij} \in \mathbb{F}_q$.

We encode a message $k$-tuple $(a_1, a_2, \ldots, a_k) \in \mathbb{F}_{q^m}$ by the codeword $(f(P_1), f(P_2), \ldots, f(P_N))$ where $f = \sum_{i=1}^{k} a_i h_i$, and $P_1, P_2, \ldots, P_N$ are arbitrary $\mathbb{F}_q$-rational points (other than $P_\infty$) in the function field. The block length $N$ can be any integer satisfying $k \leq N \leq r^{e-1}(r^2 - r)$. As in Section 9.2 for any subset $M \subseteq \mathbb{F}_{q^m}$, we can consider the subcode obtained by only encoding tuples in $M$:

$$C_{GS}(m; k+2g_e-1 \mid M) \overset{\text{def}}{=} \{(f(P_1), f(P_2), \ldots, f(P_N)) : f = \sum_{i=1}^{k} a_i h_i, (a_1, a_2, \ldots, a_k) \in M\}.$$

**Computing the code.** Note that an explicit specification of the code simply requires the evaluations of the basis functions $h_1, h_2, \ldots, h_k$ at the $N$ rational points. One can find a basis of $\mathcal{L}_m(lP_\infty)$ along with its evaluations at the rational points using poly($N, l, m$) operations over $\mathbb{F}_q$ (see also 10 Sec. 7). We can also compute the first $l$ coefficients of the local expansion of the basis functions at $P_\infty$ using poly($l, m$) operations over $\mathbb{F}_q$ as described in Section 4. The computation of the $h_i$’s following the method of Lemma 9.9 only requires elementary matrix operations, so we can compute its evaluations at the rational points also in polynomial time.

**List decoding.** In order to list decode, we can find a functional equation $A_0 + A_1 f + A_2 f^\sigma + \cdots + A_s f^{\sigma^{s-1}}$ exactly as in Lemma 9.7. To solve for $f$ from this equation, we consider the local expansions of the message functions $f$ at $P_\infty$. Let $T \in K_e$ be a local expansion of $P_\infty$ and suppose that a function $f \in \mathcal{L}_m((k+2g_e-1)P_\infty)$ has a local expansion at $P_\infty$

$$f = T^{-(k+2g_e-1)} \sum_{j=0}^{\infty} f_j T^j$$

for some $f_j \in \mathbb{F}_{q^m}$. As in Lemma 9.10 if $f = \sum_{i=1}^{k} a_i h_i$, then the top $k$ coefficients $f_0, f_1, \ldots, f_{k-1}$ in the above local expansion equal $a_1, a_2, \ldots, a_k$. Thus we can determine such $f$ uniquely by finding $f_0, f_1, \ldots, f_{k-1}$. The following lemma is similar to Lemma 9.8 and shows that the coefficients belong to an ultra-periodic subspace.

**Lemma 9.12.** Suppose $f \in \mathcal{L}_m((k+2g_e-1)P_\infty)$ satisfies the equation

$$A_0 + A_1 f + A_2 f^\sigma + \cdots + A_s f^{\sigma^{s-1}} = 0$$
when at least one $A_i$ is nonzero. Then the possible first $k$ coefficients $(f_0, f_1, \ldots, f_{k-1})$ of $f$’s local expansion \[ \sigma \] at $P_\infty$ belong to an $(s-1,m)$-ultra periodic affine subspace of $\mathbb{F}_q^{km}$.

Proof. Let $u = \min \{ \nu_{P_\infty}(A_i) : i = 1, 2, \ldots, s \}$ (so that $u$ is the maximum number of poles any $A_i$, $1 \leq i \leq s$, has at $P_\infty$). Then it is clear that $u \geq -D$ and $\nu_{P_\infty}(A_0) \geq u - (k + 2g_e - 1)$. Each $A_i$ has a local expansion at $P_\infty$:

$$A_0 = T^{u-(k+2g_e-1)} \sum_{j=0}^{\infty} a_{0,j} T^j; \quad \text{and} \quad A_i = T^u \sum_{j=0}^{\infty} a_{i,j} T^j \text{ for } i = 1, 2, \ldots, s.$$  

Assume that at $P_\infty$, the function $f$ has a local expansion \[ \sigma \]. Then $f^\sigma$ has a local expansion at $P$ as follows

$$f^\sigma = \sum_{j=0}^{\infty} f^{\sigma_j} T^j.$$  

For $l = 0, 1, \ldots$, define the linearized polynomial

$$B_l(X) := a_{1,l} X + a_{2,l} X^q + \cdots + a_{s,l} X^{q^{s-1}}.$$  

From the definition of $u$, one knows that $B_0(X)$ is nonzero. Equating the coefficient of $T^{d+u-(k+2g_e-1)}$ in $A_0 + A_1 f + A_2 f^q + \cdots + A_s f^{q^{s-1}}$ to equal 0 gives us the condition

$$a_{0,d} + B_d(f_0) + B_{d-1}(f_1) + \cdots + B_0(f_d) = 0.$$  

Arguing as in Lemma \[ \sigma \], this constrains $(f_0, f_1, \ldots, f_{k-1})$ to belong to an $(s-1,m)$-ultra periodic subspaces of $\mathbb{F}_q^{mk}$. \[ \square \]

Similar to Corollary \[ \sigma \], we can now conclude the following:

Corollary 9.13. The code $C_{\text{GS}}(m; k + 2g_e - 1 \mid \mathbb{F}_q^m)$ can be list decoded from up to $\frac{s}{s+1}(N-k) - \frac{3s+1}{s+1} g_e$ errors, pinning down the messages to an $(s-1,m)$-ultra periodic subspace of $\mathbb{F}_q^{mk}$.

We conclude the section by incorporating the trade-off between $g_e$ and $N$, and stating the rate vs. list decoding radius trade-off offered by these codes, in a form convenient for improvements to the list size using subspace evasive sets and subspace designs (see Section \[ \sigma \]). The claim about the number of possible solution subspaces follows since the subspace is determined by $A_0, A_1, \ldots, A_s$, and for our choice of parameter $D$, there are at most $q^{O(mN)}$ choices of those.

Theorem 9.14. Let $q$ be the even power of a prime. Let $1 \leq s \leq m$ be integers, and let $R \in (0, 1)$. Then for infinitely many $N$ (all integers of the form $q^{\ell/2}(\sqrt{q} - 1)$), there is a deterministic polynomial time construction of an $\mathbb{F}_q^m$-linear code $\text{GS}^{(q,m)}[N,k]$ of block length $N$ and dimension $k = R \cdot N$ that can be list decoded in $\text{poly}(N,m,\log q)$ time from $\frac{s}{s+1}(N-k) - \frac{3N}{\sqrt{q}-1}$ errors, pinning down the messages to one of $q^{O(mN)}$ possible $(s-1,m)$-ultra periodic $\mathbb{F}_q$-affine subspaces of $\mathbb{F}_q^{mk}$.

10. Subspace designs

The linear-algebraic list decoder discussed in the previous sections pins down the coefficients of the message to a periodic subspace. This subspace has linear dimension, so we need to restrict the coefficients further so that the subspace can be pruned to a small list of solutions. We already saw, in Section \[ \sigma \], an approach using h.s.e. sets to prune the
periodic subspace to a small list. In this section, we will develop an alternate approach based on a special collection of subspaces, which we call a *subspace design*, for pruning the periodic subspaces. This will lead to explicit constructions with somewhat larger list size.

We begin with the definition of the central object of study in this section, subspace designs.

**Definition 15.** Let \( \Lambda \) be a positive integer, and \( q \) a prime power. For positive integers \( r < \Lambda \) and \( d \), an \((r,d)\)-subspace design in \( \mathbb{F}_q^\Lambda \) is a collection of subspaces of \( \mathbb{F}_q^\Lambda \) such that for every \( r \)-dimensional subspace \( W \subset \mathbb{F}_q^\Lambda \), we have

\[
\sum_{H \in \mathcal{H}} \dim(W \cap H) \leq d. 
\]

The cardinality of a subspace design \( \mathcal{H} \) is the number of subspaces in its collection, i.e., \( |\mathcal{H}| \). If all subspaces in \( \mathcal{H} \) have the same dimension \( t \), then we refer to \( t \) as the dimension of the subspace design \( \mathcal{H} \).

Note that the condition (41) in particular implies for every \( r \)-dimensional subspace \( W \), at most \( d \) of the subspaces in an \((r,d)\)-subspace design non-trivially intersect it. This weaker property is called a “weak subspace design” in [8] which gave explicit constructions of subspace designs following our original definition in [17]. For our list decoding application, the stronger property (41) is required. In particular, the usefulness of subspace designs defined above, in the context of pruning periodic subspaces, is captured by the following key lemma.

**Lemma 10.1 (Periodic subspaces intersected with a subspace design).** Suppose \( H_1, H_2, \ldots, H_b \) are subspaces in an \((r,d)\)-subspace design in \( \mathbb{F}_q^\Lambda \), and \( T \) is a \((r,\Lambda,b)\)-periodic affine subspace of \( \mathbb{F}_q^{\Lambda b} \) with underlying subspace \( S \). Then the set

\[
T = \{(f_1, f_2, \ldots, f_b) \in T \mid f_j \in H_j \text{ for } j = 1, 2, \ldots, b\}
\]

is an affine subspace of \( \mathbb{F}_q^\Lambda \) of dimension at most \( d \). Also, the underlying subspace of \( T \) is contained in \( S \equiv S \cap (H_1 \times H_2 \times \cdots \times H_b) \).

**Proof.** It is clear that \( T \) is an affine subspace, since its elements are restricted by the set of linear constraints defining \( T \) and the \( H_j \)'s. Also, the difference of two elements in \( T \) is contained in both the subspaces \( S \) and \( (H_1 \times H_2 \times \cdots \times H_b) \), which implies that the underlying subspace of \( T \) is contained in \( S \).

We will prove the bound on dimension by proving that \( |T| \leq q^d \). To prove this, we will imagine the elements of \( T \) as the leaves of a tree of depth \( b \), with the nodes at level \( j \) representing the possible projections of \( T \) onto the first \( j \) blocks. The root of this tree has as children the elements of the affine space \( \text{proj}_{[1,\Lambda]}(T) \cap H_1 \). Let \( W \) be the subspace of \( \mathbb{F}_q^\Lambda \) of dimension at most \( r \) associated with the periodic subspace \( T \) (in the sense of Definition 11). Note that the underlying subspace of the affine space \( \text{proj}_{[1,\Lambda]}(T) \cap H_1 \) is contained in the subspace \( W \cap H_1 \).

Continuing this argument, the children of an element \( a \in \mathbb{F}_q^\Lambda \) at level \( j \) will be \( a \) followed by the possible extensions of \( a \) to the \((j + 1)\)’th block, given by

\[
\{\text{proj}_{[j+1,(j+1)\Delta]}(x) \mid x \in T \text{ and } \text{proj}_{j\Delta}(x) = a\} \cap H_{j+1}. 
\]

The periodic property of \( T \) and the fact that \( H_{j+1} \) is a subspace implies that the possible extensions of \( a \) are given by a coset of a subspace of \( W \cap H_{j+1} \). Thus the nodes at level \( j \) have degree at most \( q^{\dim(W \cap H_{j+1})} \) for \( j = 0, 1, \ldots, b - 1 \). Since the \( H_j \)'s belong to an
(r, d)-subspace design we have \( \sum_{j=1}^{b} \dim(W \cap H_j) \leq d \). Therefore, the tree has at most \( q^d \) leaves, which is also an upper bound on \( |T| \). \( \square \)

10.1. Constructing subspace designs. We now turn to the construction of subspace designs of large size and dimension. We first analyze the performance of a random collection of subspaces.

**Lemma 10.2.** Let \( \eta > 0 \) and \( q \) be a prime power. Let \( r, \Lambda \) be integers \( \Lambda \geq 8/\eta \) and \( r \leq \eta \Lambda/2 \). Consider a collection \( \mathcal{H} \) of subspaces of \( \mathbb{F}_q^\Lambda \) obtained by picking, independently at random, \( q^{\eta \Lambda/8} \) subspaces of \( \mathbb{F}_q^\Lambda \) of dimension \((1 - \eta)\Lambda\) each. Then, with probability at least \( 1 - q^{-\Lambda r} \), \( \mathcal{H} \) is an \((r, 8r/\eta)\)-subspace design.

**Proof.** Let \( \ell = 8r/\eta \), and let \( M = q^{\eta \Lambda/8} \) denote the number of randomly chosen subspaces. Let \( H_1, H_2, \ldots, H_M \) be the subspaces in the collection \( \mathcal{H} \). Fix a subspace \( W \) of \( \mathbb{F}_q^\Lambda \) of dimension \( r \). Fix a tuple of non-negative integers \((a_1, a_2, \ldots, a_M)\) summing up to \( \ell \). For each \( j \in \{1, 2, \ldots, M\} \), the probability that \( \dim(W \cap H_j) \geq a_j \) is at most \( q^{\eta \Lambda} q^{-\Lambda a_j} \). Since the choice of the different \( H_j \)'s are independent, the probability that \( \dim(W \cap H_j) \geq a_j \) for every \( j \) is at most \( q^{(r - \eta \Lambda)\ell} \leq q^{-\Lambda \ell/2} \) (the last step uses \( r \leq \eta \Lambda/2 \)).

A union bound over the at most \( q^{\Lambda r} \) subspaces \( W \subset \mathbb{F}_q^\Lambda \) of dimension \( r \), and the at most \( (\ell + M) \leq (M + \ell) \leq M^{2\ell} \) choices of the tuples \((a_1, a_2, \ldots, a_M)\), we get the probability that \( \mathcal{H} \) is not an \((r, \ell)\)-subspace design is at most

\[
q^{\Lambda r} \cdot q^{-\Lambda \ell/2} \cdot (q^{\eta \Lambda/8})^{2\ell} = q^{\Lambda r} \cdot q^{-\Lambda \ell/4} \leq q^{-\Lambda r}
\]

where the last step uses \( \ell \geq 8r/\eta \). \( \square \)

Note that given a collection \( \mathcal{H} \) of subspaces, one can deterministically check if it is an \((r, d)\)-subspace design in \( \mathbb{F}_q^\Lambda \) in \( q^{O(\Lambda r)|\mathcal{H}|} \) time by doing a brute-force check of all \( r \)-dimensional subspaces \( W \) of \( \mathbb{F}_q^\Lambda \), and for each computing \( \sum_{H \in \mathcal{H}} \dim(W \cap H) \) using \( |\mathcal{H}|^{O(1)} \) operations over \( \mathbb{F}_q \). Thus the above lemma gives a \( q^{O(\Lambda r)} \) time Las Vegas construction of an \((r, d)\)-subspace design with many subspaces each of large dimension \((1 - \eta)\Lambda\).

**Lemma 10.3.** For parameters \( \eta, r, \Lambda \) as in Lemma 10.2, for any \( b \leq q^{\eta \Lambda/8} \), one can compute an \((r, 8r/\eta)\)-subspace design in \( \mathbb{F}_q^\Lambda \) of dimension \((1 - \eta)\Lambda\) and cardinality \( b \) in \( q^{O(\Lambda r)} \) Las Vegas time.

As noted in the conference version [17] of this paper, the construction can be derandomized using the method of conditional expectations to successively find good subspaces \( H_i \) to add to the subspace design. However, as each step involves searching over all \((1 - \eta)\Lambda\)-dimensional subspaces of \( \mathbb{F}_q^\Lambda \), the construction time would be \( q^{O(\Lambda^2)} \) even for constructing subspace designs with few subspaces. For our application to reducing the list size for long algebraic-geometric codes (either folded or with rational points in a subfield), we will need subspace designs for ambient dimension \( \Lambda \) growing at least logarithmically in the code length. The \( q^{O(\Lambda^2)} \) complexity will thus lead to a quasi-polynomial code construction time, as claimed in the conference version [17]. In fact, even the Las Vegas construction time of \( q^{O(\Lambda r)} \) will be super-polynomial for the parameters used in the construction.

The question of explicit (polynomial time) constructions of subspace designs naturally arose following [17] and was addressed in the follow-up work by Guruswami and Kopparty [8], who proved the following.

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9For simplicity, we ignore the floor and ceiling signs in defining integers; these can be easily incorporated.
**Theorem 10.4 (Explicit subspace designs [8]).** For every \( \eta > 0 \), integers \( r, \Lambda \) with \( r \leq \eta \Lambda / 4 \), and prime powers \( q \) satisfying \( q^{\eta \Lambda / (2r)} > 2r/\eta \), for any \( b \leq q^{\eta \Lambda / (4r)} \), there exists an explicit \((r, r^2/\eta)\)-subspace design of cardinality \( b \) and dimension \((1 - \eta)\Lambda\), that can be constructed deterministically in time \( \text{poly}(b,q) \). In the case when \( q > \Lambda \), one can explicitly construct an \((r, 2r/\eta)\)-subspace design with the same parameters.

We note a couple of senses in which the parameters offered by the explicit construction are weaker than those guaranteed by the probabilistic construction. First, the total intersection dimension \( (11) \) is \( r^2/\eta \) rather than \( O(r/\eta) \) (except when \( q \) is large). This is because, for small fields, their construction yields only a weak subspace design, incurring a factor \( r \) loss when passing to a subspace design. Second, the number of subspaces in the design is smaller, roughly \( q^\Omega(\eta \Lambda / r) \) instead of \( q^\Omega(\eta \Lambda) \). Finally, there is a modest restriction the field size \( q \), and we need to pick \( r, \Lambda \) suitably to allow for fixed \( q \). Fortunately, all these restrictions can be accommodated for our application. We remark that a recent construction of subspace designs based on cyclotomic function fields [18] gives an analog of Theorem 10.4 over any field \( \mathbb{F}_q \) with an \((r, O(r \log_q \Lambda / \eta))\)-subspace design; for our application, however, the \( r^2/\eta \) bound is more useful as \( r \ll \Lambda \), and we can’t afford the dependence on \( \Lambda \) in the bound.

Let us now record a construction of subspaces with low-dimensional intersection with every periodic subspace based on the above subspace designs. This form will be convenient for later use in pre-coding Reed-Solomon codes.

**Theorem 10.5.** Let \( \eta \in (0, 1) \) and \( q \) be a prime power, and \( r, \Lambda, b \) be integers such that \( r \leq \eta \Lambda / 4 \) and \( b < q \). Then, one can construct a subspace \( V \) of \( \mathbb{F}_q^{\Lambda} \) of dimension at least \((1 - \eta)b\Lambda \) in either deterministic \( q^{O(\Lambda)} \) time: For every \((r, \Lambda, b)\)-periodic subspace \( T \subset \mathbb{F}_q^{\Lambda} \), \( V \cap T \) is an \( \mathbb{F}_q \)-affine subspace of dimension at most \( 2r/\eta \).

**Proof.** We will take \( V = H_1 \times H_2 \times \cdots H_b \) where the \( H_i \)'s belong to a \((r, 2r/\eta)\)-subspace design in \( \mathbb{F}_q^{\Lambda} \) of cardinality \( b \) and dimension at least \((1 - \eta)\Lambda \) as guaranteed by Theorem 10.4 when \( q > \Lambda \). Clearly \( \dim(V) \geq (1 - \eta)b\Lambda \) since each \( H_i \) has dimension at least \((1 - \eta)\Lambda \). The claim now follows using Lemma 10.1. \( \square \)

### 10.2. Cascaded subspace designs

In preparation for our results about algebraic-geometric codes, whose block length \( \gg q^m \) is much larger than the possible size of subspace designs in \( \mathbb{F}_q^m \), we now formalize a notion that combines several “levels” of subspace designs. The definition might seem somewhat technical, but it has a natural use in our application to list-size reduction for AG codes. Note that there is no “consistency” requirement between subspace designs at different levels other than the lengths and cardinalities matching.

**Definition 16** (Subspace designs of increasing length). Let \( l \) be a positive integer. For positive integers \( r_0 \leq r_1 \leq \cdots \leq r_l \) and \( m_0 \leq m_1 \leq \cdots \leq m_l \) such that \( m_{i-1} / m_i \) for \( 1 \leq i \leq l \), an \((r_0, r_1, \ldots, r_l)\)-cascaded subspace design with length-vector \((m_0, m_1, \ldots, m_l)\) and dimension vector \((d_0, d_1, \ldots, d_{l-1})\) is a collection of \( l \) subspace designs, specifically an \((r_{i-1}, r_i)\)-subspace design in \( \mathbb{F}_q^{m_{i-1}} \) of cardinality \( m_i / m_{i-1} \) and dimension \( d_{i-1} \) for each \( i = 1, 2, \ldots, l \).

Note that the \( l = 1 \) case of the above definition corresponds to an \((r_0, r_1)\)-subspace design in \( \mathbb{F}_q^{m_0} \) of dimension \( d_0 \) and cardinality \( m_1 / m_0 \). In Lemma 10.1, we used the subspace \( H_1 \times H_2 \times \cdots \times H_l \) based on a subspace design consisting of the \( H_i \)'s to prune a periodic subspace. Generalizing this, we now define a subspace associated with a cascaded subspace design based on the subspace designs comprising it.
Definition 17 (Canonical subspace). Let $\mathcal{M}$ be a cascaded subspace design with length-vector $(m_0, m_1, \ldots, m_l)$ such that the $t$'th subspace design in $\mathcal{M}$ has subspaces

$$H_1^{(i)}, H_2^{(i)}, \ldots, H_{m_i/m_{i-1}}^{(i)} \subset \mathbb{F}_q^{m_{i-1}} \text{ for } 1 \leq t \leq l.$$ 

The canonical subspace associated with such a cascaded subspace design, denoted $U(\mathcal{M})$, is a subspace of $\mathbb{F}_q^{m_l}$ defined as follows:

A vector $x \in \mathbb{F}_q^{m_l}$ belongs to $U(\mathcal{M})$ if and only if for every $t \in \{1, 2, \ldots, l\}$, each of the $m_i$-sized blocks of $x$ given $\text{proj}_{m_{i-1} \cap \{1, (j+1)m_i\}}(x)$ for $0 \leq j < m_i/m_{i-1}$ belongs $H_1^{(i)} \times H_2^{(i)} \times \cdots \times H_{m_i/m_{i-1}}^{(i)}$.

In other words, we apply the construction of Lemma 10.1 for (disjoint) intervals of length $m_i$ at each level $t \in \{1, 2, \ldots, l\}$.

The following simple fact, which follows by counting number of linear constraints imposed, gives a lower bound on the dimension of a canonical subspace.

Observation 10.6. For a cascaded subspace design $\mathcal{M}$ as above, if the $t$'th subspace design has dimension at least $(1 - \xi_{t-1})m_{t-1}$ for $1 \leq t \leq l$, then the dimension of the canonical subspace $U(\mathcal{M})$ is at least $(1 - (\xi_0 + \xi_1 + \cdots + \xi_{t-1}))m_l$.

The following is the crucial claim about pruning ultra-periodic subspaces using (the canonical subspace of) a cascaded subspace design. It generalizes Lemma 10.1 which corresponds to the $l = 1$ case.

Lemma 10.7. Suppose $\mathcal{M}$ is a $(r_0, r_1, \ldots, r_l)$-cascaded subspace design with length-vector $(m_0, m_1, \ldots, m_l)$. Let $T$ be a $(r_0, m_0)$-ultra periodic affine subspace of $\mathbb{F}_q^{m_0}$. Then the dimension of the affine space $T \cap U(\mathcal{M})$ is at most $r_l$.

Proof. The idea will be to apply Lemma 10.1 inductively, for increasing periods $m_0, m_1, \ldots, m_{l-1}$. Since $T$ is $(r_0, m_0)$-ultra periodic, it is $(r_0, m_0)$-periodic and $(m_1/m_0, r_0, m_1)$-periodic. Using this together with Lemma 10.1 it follows that

$$T \cap \{x \in \mathbb{F}_q^{m_0} | \text{proj}_{m_{i-1} \cap \{1, (j+1)m_i\}}(x) \in H_1^{(i)} \times H_2^{(i)} \times \cdots \times H_{m_i/m_{i-1}}^{(i)} \text{ for } 0 \leq j < m_i/m_{i-1}\}$$

is an affine subspace that is $(r_1, m_1)$-periodic. Continuing this argument, the affine subspace of $T$ formed by restricting each $m_i$-block to belong to $H_1^{(i)} \times H_2^{(i)} \times \cdots \times H_{m_i/m_{i-1}}^{(i)}$ for $1 \leq t \leq j$ is $(r_j, m_j)$-periodic. For $j = l$, we get the intersection $T \cap U(\mathcal{M}) \subset \mathbb{F}_q^{m_l}$ will be $(r_l, m_l)$-periodic, which simply means that it is an $r_l$-dimensional affine subspace of $\mathbb{F}_q^{m_l}$.

We conclude this section by constructing a canonical subspace that has low-dimensional intersection with ultra-periodic subspaces based on the explicit subspace designs of Theorem 10.4. This statement will be used in Section 11.2 for pre-coding algebraic-geometric codes based on the Garcia-Stichtenoth tower.

Theorem 10.8. Let $q \geq 4$ be a prime power. Let $\eta \in (0, 1)$ and integers $\Lambda, r \geq 2$ satisfy $\Lambda \geq cr^{-1} \log (r/\eta)$ for a large enough (absolute) constant $c > 0$. For all large enough multiples $\kappa$ of $\Lambda$, we can construct a subspace $U$ of $\mathbb{F}_q^{\kappa}$ of dimension at least $(1 - \eta)\kappa$ such that for every $(r, \Lambda)$-ultra periodic affine subspace $T \subset \mathbb{F}_q^{\kappa}$, the dimension of the affine subspace $U \cap T$ is at most $(r/\eta)^{O(\log \kappa)}$. The subspace $U$ can be constructed in deterministically in $\text{poly}(\kappa, q)$ time.
Proof. We will take $U$ to the canonical subspace $U(M)$ of an appropriate cascaded subspace design $M$. To this end, given our work so far, the main remaining task is to pick the parameters of $M$ carefully. Let $\eta_{t} = \frac{q}{4^{2t}}$ for $t = 0, 1, 2, \ldots$

Let $m_{0} = \Lambda$, $m_{1} = m_{0} \cdot \lceil (r/\eta)^{c/4} \rceil$, and for $i \geq 0$, $m_{i+1} = m_{i} \cdot q^{\sqrt{m_{i}}}$. Let $r_{0} = r$, and for $i \geq 0$, $r_{i+1} = \lceil r_{i}^{2}/\eta_{i} \rceil$. For this choice of parameters, one can verify that (i) $r_{i} \leq \eta_{i} m_{i}/4$, and (ii) $q^{\eta_{i} m_{i}/(4 r_{i})} \geq m_{i+1}/m_{i}$ for all $i \geq 0$. Indeed, to verify the first condition by induction, one only needs to check that $m_{i+1} \geq m_{i}^{2}$, which is true for $i = 0$ for a large enough choice of $r$, and for $i \geq 1$, $m_{i+1}$ in fact grows exponentially in $\sqrt{m_{i}}$. For the second condition, for $i = 0$ it follows from our assumption that $\Lambda \geq c r \eta^{-1} \log(r/\eta)$. For $i \geq 1$, it is implied by $r_{i}/\eta_{i} \ll \sqrt{m_{i}/4}$, which is true for $i = 1$ for large enough $c$, and for $i > 1$ by induction since $r_{i}/\eta_{i}$ grows quadratically in each step, whereas $m_{i}$ grows exponentially.

We can therefore conclude by Theorem 10.4 that we can construct a $(r_{i}, r_{i+1})$-subspace design of cardinality $m_{i+1}/m_{i}$ in $\mathbb{F}_{q}^{m_{i}}$ of dimension $(1 - \eta_{i}) m_{i}$.

Pick $l$ to the smallest integer so that $m_{l-1} \geq (\log_{q} \kappa)^{2}$. Since $m_{0} = \Lambda \geq 2$ and $m_{l} \geq q^{\sqrt{m_{l}}}$ for $1 \leq i \leq l$, it is easy to see that $l \leq O(\log^{*} \kappa)$. Redefine $m_{l-1}$ to equal $m_{l-1}'$ which is the smallest multiple of $m_{l-2}$ that is at least $(\log_{q} \kappa)^{2}$. Since $m_{l-2} < (\log_{q} \kappa)^{2}$, we have $(\log_{q} \kappa)^{2} \leq m_{l-1}' < 2(\log_{q} \kappa)^{2}$. We also redefine $m_{l}$ to equal the largest multiple $m_{l}'$ of $m_{l-1}'$ that is at most $\kappa$. This implies $\kappa - m_{l}' < m_{l-1}'$. Note that $m_{l-1}' \leq m_{l-2} q^{\sqrt{m_{l-2}}} \leq \sqrt{m_{l-2}^{2}}$ and $m_{l}' \leq q^{\sqrt{m_{l-1}}}$. For notational simplicity, let us re-denote $m_{l-1}'$ and $m_{l}'$ by $m_{l-1}$ and $m_{l}$.

Thus for these parameters, we can construct an $(r_{0}, r_{1}, \ldots, r_{l})$-cascaded subspace design $M_l$ with length-vector $(m_{0}, m_{1}, \ldots, m_{l})$ and dimension-vector $(d_{0}, d_{1}, \ldots, d_{l-1})$ where $d_{c} \geq (1 - \eta/2^{c+2}) m_{c}$.

The construction time for subspace designs guaranteed by Theorem 10.4 implies that $M_{l}$ can be constructed in $\text{poly}(m_{l}, q)$ time. We define the desired subspace $U \subset \mathbb{F}_{q}^{\kappa}$ as $U(M_{l}) \times 0^{\kappa - m_{l}}$, i.e., $U$ consists of the vectors in the canonical subspace $U(M_{l}) \subset \mathbb{F}_{q}^{m_{l}}$ padded with $\kappa - m_{l}$ zeroes at the end. By Observation 10.6, the dimension of $U$ is at least

$$\left(1 - \sum_{i=0}^{l-1} \frac{\eta}{4 \cdot 2^{i}}\right) m_{l} \geq (1 - \eta/2) m_{l} > (1 - \eta/2)(\kappa - m_{l-1})$$

$$> (1 - \eta/2) \kappa - 2(\log_{q} \kappa)^{2} > (1 - \eta) \kappa$$

for large enough $\kappa$. This proves that the subspace $U$ has dimension at least $(1 - \eta) \kappa$, and can be constructed deterministically in $\text{poly}(q, \kappa)$ time.

It remains to prove the claimed intersection property with ultra-periodic subspaces. Let $T$ be an arbitrary $(r, \Lambda)$-ultra periodic affine subspace of $\mathbb{F}_{q}^{\kappa}$. By Lemma 10.7, $\text{proj}_{m_{l}}(T) \cap U(M)$ is an affine subspace of $\mathbb{F}_{q}^{m_{l}}$ of dimension at most $r_{l}$. Clearly, the same dimension bound also holds for $T \cap U$ since the last $\kappa - m_{l}$ coordinates for vectors in $U$ are set to 0. The proof is complete by noting that for our choice of parameters, $r_{l} \leq (2r/\eta)^{2}$ and $l \leq O(\log^{*} \kappa)$.

11. Good list decodable subcodes of RS and AG codes

We now combine our code constructions in Section 9 with a pre-coding step that restricts coefficients to belong to a subspace design, and thereby obtain subcodes that are list decodable with smaller list-size in polynomial time.
11.1. Reed-Solomon codes. We begin with the case of Reed-Solomon codes. For a finite field \( \mathbb{F}_q \), constant \( \varepsilon > 0 \), integers \( n, k, m, s \) satisfying \( 1 \leq k < n \leq q \) and \( 1 \leq s \leq cm/12 \), we will define subcodes of RS\((q,m)n,k\). Below for a polynomial \( f \in \mathbb{F}_q[n] \) with \( k \) coefficients \( f_0, f_1, \ldots, f_{k-1} \), we denote by \( \hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{k-1} \) the representation of these coefficients as vectors in \( \mathbb{F}_q^n \) by fixing some \( \mathbb{F}_q \)-basis of \( \mathbb{F}_q^n \).

Define the subcode \( \hat{\text{RS}} \) of RS\((q,m)n,k\) consisting of the encodings of \( f \in \mathbb{F}_q[n] \) such that \( (\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{k-1}) \in V \) for a subspace \( V \subseteq \mathbb{F}_q^{mk} \) guaranteed by Theorem 11.1 when applied with the parameter choices
\[
\Lambda = m; \quad b = k; \quad r = s - 1; \quad \eta = \varepsilon.
\]
Note that \( \hat{\text{RS}} \) is an \( \mathbb{F}_q \)-linear code over the alphabet \( \mathbb{F}_q^n \) of rate \( (1 - \varepsilon)k/n \), and it can be constructed in deterministic \( O(s^2) \) time, or Las Vegas \( q^{O(m^2)} \) time.

**Theorem 11.1.** Given an input string \( \mathbf{y} \in \mathbb{F}_q^n \), a basis of an affine subspace of dimension at most \( O(s/\varepsilon) \) that includes all codewords of the above subcode within Hamming distance \( \frac{s}{s+1}(n - k) \) from \( \mathbf{y} \) can be found in deterministic \( \text{poly}(n, \log q, m) \) time.

**Proof.** By Lemma 9.4 we can compute the \((s - 1, m, k)\)-periodic subspace \( T \) of messages whose Reed-Solomon encodings can be within Hamming distance \( \frac{s}{s+1}(n - k) \) from \( \mathbf{y} \). By Theorem 10.5 the intersection \( T \cap V \) is an affine subspace over \( \mathbb{F}_q \) of dimension \( d = O(s/\varepsilon) \). Since both steps involve only basic linear algebra, they can be accomplished using \( \text{poly}(n, m) \) operations over \( \mathbb{F}_q \). \( \square \)

By picking \( s = \Theta(1/\varepsilon) \) and \( m = \Theta(1/\varepsilon^2) \) in the above construction, we can conclude the following.

**Corollary 11.2.** For every \( R \in (0,1) \) and \( \varepsilon > 0 \), and all large enough integers \( n < q \) with \( q \) a prime power, one can construct a rate \( R \) \( \mathbb{F}_q \)-linear subcode of a Reed-Solomon code of length \( n \) over \( \mathbb{F}_q^n \), such that the code can be (i) encoded in \( (n/\varepsilon)^{O(1)} \) time and (ii) list decoded from a fraction \( (1 - \varepsilon)(1 - R) \) of errors in \( (n/\varepsilon)^{O(1)} \) time, outputting a subspace over \( \mathbb{F}_q \) of dimension \( O(1/\varepsilon^2) \) including all close-by codewords. The code can be constructed deterministically in poly\((q)\) time.

We note that the above list decoding guarantee is in fact weaker than what is achieved for folded Reed-Solomon codes in [14], where the codewords were pinned down to a dimension \( O(1/\varepsilon) \) subspace. We can improve the list size above to poly\((1/\varepsilon)\) using pseudorandom subspace-evasive sets as in [14], or to \( \exp(-O(1)) \) using the explicit subspace-evasive sets from [1]. The main point of the above result is not the parameters but that an explicit subcode of RS codes has optimal list decoding radius with polynomial complexity.

11.2. Subcodes of Garcia-Stichtenoth codes. We now pre-code the codes constructed in Section 9.3. For a finite field \( \mathbb{F}_q \), constant \( \varepsilon > 0 \), and integers \( s, m \) satisfying \( 1 \leq s \leq O(cm/\log(1/\varepsilon)) \) and \( m \geq \Omega(1/\varepsilon^2) \), we will define subcodes of GS\((q,m)n,k\) guaranteed by Theorem 9.4.4. Note that messages space of this code can be identified with \( \mathbb{F}_q^{mk} \).

Define the subcode \( \hat{\text{GS}} \) of GS\((q,m)n,k\) consisting of the encodings of a subspace \( U \subseteq \mathbb{F}_q^{mk} \) guaranteed by Theorem 10.8 when applied with the parameter choices
\[
(42) \quad \eta = \varepsilon; \quad r = s - 1; \quad \Lambda = m; \quad \kappa = km.
\]

\(^{10}\)It can also be constructed in Monte Carlo \( (q/\varepsilon)^{O(1)} \) time by randomly picking subspaces for the subspace design used to construct \( V \) in Theorem 10.2.
Note that $\widehat{GS}$ is an $F_q$-linear code over the alphabet $F_q^n$ of rate $(1 - \varepsilon)k/N$. Also, it can be constructed in $\text{poly}(k, m, q)$ time by virtue of the construction complexity of $U$.

**Lemma 11.3.** Given an input string $y \in F_q^N$, a basis of an affine subspace of dimension at most
\[(s/\varepsilon)^{2O(s^* (km))}\]
that includes all codewords of the above subcode within Hamming distance $\frac{s}{s+1}(N - k) - 3N/(\sqrt{q} - 1)$ from $y$ can be found in deterministic $\text{poly}(n, \log q, m)$ time.

**Proof.** By Theorem 9.11, we can compute the $(s - 1, m)$-ultra periodic subspace $T$ of messages whose encodings can be within Hamming distance $\frac{s}{s+1}(N - k) - 3N/(\sqrt{q} - 1)$ from $y$. By Theorem 10.8, for the above choice of parameters, the intersection $T \cap U$ is is an affine subspace over $F_q$ of dimension $(s/\varepsilon)^{2O(s^* (km))}$. Since both steps involve only basic linear algebra, they can be accomplished using $\text{poly}(N, m)$ operations over $F_q$. \qed

By taking $q = \Theta(1/\varepsilon^2)$, and choosing $s = \Theta(1/\varepsilon)$ and $m = \Theta(\varepsilon^{-2} \log(1/\varepsilon))$ in the above lemma, we conclude the following main result concerning the explicit construction of codes list decodable up to the Singleton bound.

**Theorem 11.4** (Main deterministic code construction). For every $R \in (0, 1)$ and $\varepsilon > 0$, there is a deterministic polynomial time constructible family of error-correcting codes of rate $R$ over an alphabet of size $\exp(O(\varepsilon^{-2} \log^2(1/\varepsilon)))$ that can be list decoded in polynomial time from a fraction $(1 - R - \varepsilon)$ of errors, outputting a list of size at most $\exp_{1/\varepsilon}(\exp_{1/\varepsilon}(O(\log^* N)))$, where $N$ is block length of the code.

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Computer Science Department, Carnegie Mellon University, Pittsburgh, USA.

E-mail address: venkatg@cs.cmu.edu

Division of Mathematical Sciences, School of Physical & Mathematical Sciences, Nanyang Technological University, Singapore.

E-mail address: xingcp@ntu.edu.sg