The lattice $SU(2)$ confining string as an Abrikosov vortex

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Abstract

Numerical data [10] for the $SU(2)$ confining string in the maximal abelian projection are analysed. The distribution of the electric flux and monopole currents are perfectly described by the classical equations of motion for the dual Abelian Higgs model. The mass of the vector boson is equal to the mass of the monopole (Higgs particle) within numerical errors. The classical energy per unit length of the Abrikosov vortex reproduces 94% of the full non-Abelian string tension.

1. The monopole confinement mechanism [1] in lattice gluodynamics is confirmed by many lattice numerical calculations (see reviews [2] and references therein). The simplest effective dynamical picture corresponding to this mechanism is the dual Abelian Higgs model (AHM) [3] with the Lagrangian:

\[ \mathcal{L}_{AHM} = \frac{1}{4g^2} F_{\mu\nu}(B) + \frac{1}{2} |(\partial_{\mu} - iB_{\mu})\Phi|^2 + \lambda (|\Phi|^2 - \eta^2)^2, \]

where $B_{\mu}$ is the dual gauge field interacting with the monopole field $\Phi$. Electrically charged “quarks” can be introduced [3] using the Zwanziger formalism [4]. The confinement of electric charges is due to the dual Abrikosov string and exists already at the classical level. The first fact in favor of an effective Lagrangian like (1) was the numerical proof of the existence of the monopole condensate in the confining phase of lattice gluodynamics [1]. The further numerical investigation allowed to derive an effective monopole Lagrangian [3].
which is in agreement with (1). There are various attempts to determine the parameters of the effective dual theory (1) from phenomenological analysis and from calculations on the lattice (e.g. [3, 4, 8]). The results can be expressed through the gauge boson mass \( m_V = g \eta \) and Higgs (monopole) mass \( m_H = 2 \sqrt{2} \lambda \eta^2 \). In most cases, the analyses have led to the result \( m_V \approx m_H \). This is the so-called Bogomolnyi limit [9] of the AHM, where several exact results can be obtained. In particular the classical Abrikosov string tension is:

\[
\sqrt{\sigma} = \sqrt{\frac{\pi}{g^2}} m^2, \quad m = m_H = m_V .
\] (2)

Of course the simple model (1) cannot describe all properties of QCD, and it is important to find the region of applicability of this effective theory. An interesting example was found by Bali, Schlichter and Schilling [10] who have presented results on the profile of the flux tube between heavy quarks in SU(2) lattice gauge theory. Working within the maximal Abelian projection on a 32 \( \times \) 32 lattice at \( \beta = 2.5115 \), they have studied the correlators of the Abelian electric field \( \vec{E} \) and of the density of Abelian monopole currents \( \vec{k} \) with Abelian Wilson loops. Such correlators, divided by the expectation values of the Wilson loops themselves, describe the electric field and monopole currents induced by the presence of a static chromoelectric string. For Wilson loop in the \( zt \) plane only the \( z \) component \( E_z(\rho) \) of \( \vec{E} \) and the azimuthal component \( k_\theta(\rho) \) of \( \vec{k} \) were found to be non-vanishing (here \( \rho = \sqrt{x^2 + y^2} \) is the distance from the center of the flux tube and the azimuthal angle \( \theta \) is defined as usual \( \tan \theta = y/x \)). The authors of [10] fit the numerical data using specific parameterization of \( E_z(\rho) \) and \( k_\theta(\rho) \) which respect the boundary conditions for the continuum dual Abrikosov string.

2. Below we continue the analytical analysis of the numerical data for the profile of the confining string. We will avoid to use specific ansätze. Instead, we work directly with the Ginzburg–Landau equations, fitting the parameters of the dual AHM in such a way that the dual Abrikosov string reproduces the numerical data for \( E_z(\rho) \). Our experiments revealed that the usual \( |\Phi|^4 \) form of the Higgs field potential (1) is sufficient. The inclusion of higher order term \( c_6 |\Phi|^6 \) does not change the results and gives \( c_6 = 0 \). In the cylindrical coordinate system \( t, z, \rho, \rho^2 = x_a x_a \) (\( a = 1, 2 \)), for the unitary gauge \( \Im m \Phi = 0 \), the static axially symmetric and infinitely long ANO vortex can be parameterized by two functions \( b(\rho) \) and \( f(\rho) \):

\[
B_a = \varepsilon_{ab} \frac{x_b}{\rho} \left( \frac{1}{\rho} - b(\rho) \right) , \quad \Phi = \eta f(\rho) , \quad B_0 = B_3 = 0 .
\] (3)

The energy per unit length (i.e. the string tension \( \sigma \)) can be easily derived from (1):

\[
\sigma = \frac{2\pi}{g^2} \int_0^\infty \rho \rho d\rho \left\{ \frac{1}{2} \left( \frac{1}{\rho} b^2 \right)^2 + \frac{1}{2} m_V^2 (f')^2 + \frac{1}{2} m_V^2 f^2 \left( \frac{1}{\rho} - b \right)^2 + \frac{1}{8} m_V^2 m_H^2 (f^2 - 1)^2 \right\} .
\] (4)
The variational principle gives rise to the following equations of motion:

\[ \frac{1}{\rho} \left[ \rho \frac{d}{d\rho} \right] b' - \frac{1}{\rho^2} b + m_V^2 f^2 \left( \frac{1}{\rho} - b \right) = 0, \quad (5) \]

\[ \frac{1}{\rho} \left[ \rho \frac{d}{d\rho} \right] f' = f \left( \frac{1}{\rho} - b \right)^2 + \frac{1}{2} m_H^2 f (f^2 - 1), \quad (6) \]

which should be supplied with the boundary conditions

\[ b(0) = f(0) = 0, \quad \lim_{\rho \to \infty} b(\rho) = \frac{1}{\rho}, \quad \lim_{\rho \to \infty} f(\rho) = 1. \quad (7) \]

Note that (5) is nothing but the dual Ampere law, \( \vec{k} = \text{curl} \vec{E} \), in cylindrical coordinates. We have numerically solved the classical equations (5,6) fitting the data for \( E_z(\rho) \) of Ref. [10]. The fitting curves are plotted in Fig. 1. The numerical values of the parameters of the fit are the following:

\[ g/2\pi = 0.9274 \pm 0.0066, \]
\[ m_V = 0.5733 \pm 0.0383 = (1.3123 \pm 0.0771) \text{ GeV}, \quad (8) \]
\[ m_H = 0.5945 \pm 0.0063 = (1.3614 \pm 0.0143) \text{ GeV}. \]

The vector and scalar masses are equal to each other within the numerical errors. The physical meaning of this equality deserves a separate discussion although, as we mentioned, several models predict it. From a practical point of view we can estimate the string tension using eq. (2):

\[ \sqrt{\sigma} = 0.1744 \pm 0.0231 \approx (400.1 \pm 53.0) \text{ MeV} \approx 0.91 \sqrt{\sigma}_{SU(2)}. \quad (9) \]

Here and in (3) the dimensionless quantities correspond to the lattice spacing \( a = 1 \), to get the dimensional quantities we use the lattice results [10] corresponding to \( \beta = 2.5115 \): \( a = 0.086 \text{ fm}, \sqrt{\sigma}_{SU(2)} = 440 \text{ MeV} \).

3. One sees from Fig. 1 that, in spite of the good approximation to \( E_z(\rho) \), the magnetic current distribution is not perfectly well reproduced by our model in the region \( 1 < \rho < 3 \). Since the Ampere law (3) is an integral part of the dual AHM, we conclude that either the dual description is invalid in this region or the numerical data are not reliable here due to finite lattice spacing effects. Ideally, to check the validity of dual AHM one should perform the same lattice measurements for larger values of \( \beta \) when lattice configurations are closer to the continuum. Such an enterprise is certainly worthwhile although is impossible practically at the moment. Instead, we try to estimate the effect...
of hypercubic discretization by repeating our procedure on a coarse lattice similar to the one used in Ref. [10]. The discretized action corresponding to (1) is

$$S = \frac{1}{4g^2} \sum_{x,\mu\nu} F_{x,\mu\nu}^2 + \frac{1}{2} \sum_{x,\mu} |\Phi_x - e^{iB_{x,\mu}} \Phi_{x+\mu}|^2 + \lambda \sum_x (|\Phi_x|^2 - \eta^2)^2,$$

where $F_{x,\mu\nu} = B_{x,\mu} + B_{x+\hat{\mu},\nu} - B_{x+\hat{\nu},\mu} - B_{x,\nu}$, and $a = 1$. Since we consider an infinitely long ANO vortex the problem becomes essentially two-dimensional. In the unitary gauge $\Re e \Phi_x = \eta f_x$, $\Im m \Phi_x = 0$, the string tension is

$$\sigma = \frac{1}{2g^2} \left\{ \sum_x (F_{x,12})^2 + m_V^2 \sum_{x,\mu=1,2} |f_x - e^{iB_{x,\mu}} f_{x+\mu}|^2 + \frac{1}{4} m_H^2 m_V^2 \sum_x (f_x^2 - 1)^2 \right\},$$

Contrary to the continuum considerations above there are no a priori ansatzes for the ANO string solution on the lattice. Nevertheless, it is known how to introduce a single vortex to the system [11]: the recipe is to make a shift $F_{x,12} \rightarrow F_{x,12} + 2\pi \delta_{x,x_0}$ in the action (11). Of course, periodic boundary conditions are not allowed anymore. We have solved the corresponding equations of motion on the square lattices $32^2 \div 70^2$ and with free boundary conditions. We did not find any finite volume effects. Note that a $32^2$ lattice in our analysis corresponds to the $32^4$ lattice of ref.[10]. We found that the numerical data for $E_z$ and $k_\theta$ are strikingly well reproduced by a classical solution of the model (11). The discrepancy with the observed magnetic currents distribution mentioned above disappears completely. Fig. 2 represents the results of the fit. The corresponding values of the parameters are:

$$g/2\pi = 0.9519 \pm 0.0041,$$
$$m_V = 0.4522 \pm 0.0206 = (1.0351 \pm 0.0472) \text{ GeV},$$
$$m_H = 0.4747 \pm 0.0600 = (1.0866 \pm 0.1373) \text{ GeV}.$$  

It appears that vector and scalar masses are equal to each other again although they are now $\sim 23\%$ smaller than in [8]. Thus the seeming discrepancy between $SU(2)$ gluodynamics and classical effective dual AHM is recognized as merely a discretization artifact. The string tension (11) turns out to be

$$\sqrt{\sigma} = 0.1808 \pm 0.0213 \approx (414.8 \pm 48.9) \text{ MeV} \approx 0.94 \sqrt{\sigma}_{SU(2)}.$$  

4. To summarize, we have shown that the Abelian electric field profile and the magnetic current distribution around the static confining string in $SU(2)$ gluodynamics may be well described as a classical ANO vortex. Moreover, we found that the classical energy per unit length of the ANO vortex accounts for 94% of the quantum $SU(2)$ string tension.
The effective dual Abelian Higgs Model appears to lie on the border between type-I and type-II superconductivity. The magnetic charge of the Higgs field corresponds to an electric charge of the quark $q \approx 0.95$ (the charge of the quark in $SU(2)$ Wilson loop is unity). This approximate correspondence of classical AHM and quantum $SU(2)$ gluodynamics is dynamically more specific than just Abelian dominance. The detailed study of this correspondence implies investigation of the Abrikosov vortex in quantum AHM, which is in progress now.

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**Figure Captions.**

Fig. 1 The fit of the data of Ref. [10] by a continuum solution.

Fig. 2 The same as in Fig. 1 but using a classical solution on the two-dimensional lattice.
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Electric field distribution, $E_z(\rho)$
Magnetic currents, $k_\theta(\rho)$

Fig. 1

Electric field distribution, $E_z(\rho)$
Magnetic currents, $k_\theta(\rho)$

Fig. 2