Dominating Ideals and Closed Neighborhood Ideals of Graphs

Mehrdad Nasernejad, Ayesha Asloob Qureshi, Somayeh Bandari and Aslı Musapașaoglu

Abstract. We study the closed neighborhood ideals and the dominating ideals of graphs, in particular, some classes of trees and cycles. We prove that the closed neighborhood ideals and the dominating ideals of some classes of trees are normally torsion free. The closed neighborhood ideals and the dominating ideals of cycles fail to be normally torsion free. However, we prove that the closed neighborhood ideals of cycles admit the (strong) persistence property and the dominating ideals of cycles are nearly normally torsion free.

Mathematics Subject Classification. 13B25, 13F20, 05C25, 05E40.

Keywords. Closed neighborhood ideals, dominating ideals, trees, cycles, normality, normally torsion-freeness.

Introduction

Square-free monomial ideals have been connected to several combinatorial structures to facilitate the study of their algebraic and homological properties. Common examples of these combinatorial structures include graphs, hypergraphs, simplicial complexes and matroids. In particular, every square-free monomial ideal generated in degree two can be viewed as an edge ideal of a simple graph. Edge ideals were introduced by Villarreal in [23] and, since their first appearance, they have been a central topic of many articles. One of the interesting properties of the edge ideals is that their minimal primes correspond to the minimal vertex covers of their underlying graphs. In other words, the Alexander dual of the edge ideal of a graph $G$ is the cover ideal of $G$, that is, a square-free monomial ideal whose minimal generators correspond to the minimal vertex covers of the underlying graph. Inspired by this relation, the closed neighborhood ideals and the dominating ideals of

Mehrdad Nasernejad was in part supported by a grant from IPM (No. 14001300118). Ayesha Asloob Qureshi and Aslı Musapaşaoğlu were supported by The Scientific and Technological Research Council of Turkey - TÜBİTAK (Grant No: 118F169).

Published online: 17 June 2022
graphs were recently introduced in [20]. Let $G$ be a simple graph. The closed neighborhood ideal $NI(G)$ of $G$ is generated by square-free monomials that correspond to the closed neighborhoods of the vertices of $G$, whereas, the dominating ideal $DI(G)$ of $G$ is generated by the monomials that correspond to the dominating sets of $G$ (see Sect. 1 for the formal definitions). As shown in [20], $NI(G)$ and $DI(G)$ are the Alexander dual of each other, a similar relation that exists between edge ideals and cover ideals of $G$.

Domination in graphs was mathematically formulated by Berge and Ore in the 1960s and has been widely studied by many researchers due to its enormous and growing applications in various fields including computer sciences, operations research, linear algebra and optimization. Let $G$ be a simple graph with vertex set $V(G)$. A set $S \subseteq V(G)$ is known as dominating set of $G$ if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in $S$. We refer to [7] for further concepts related to the domination in graphs. In this paper, our main goal is to further extend the study on closed neighborhood ideals and dominating ideals. In particular, we focus on normally torsion-freeness and certain stability property of these ideals.

A breakdown of the contents of this paper is as follows: in Sect. 1, we provide all the needed definitions and notions. In Sect. 2, we focus on the closed neighborhood ideals and the dominating ideals of star graphs. Any square-free monomial ideal can be visualized as an edge ideal of a hypergraph. A hypergraph $H$ is called Mengerian if it satisfies a certain min–max equation, which is known as the Mengerian property in hypergraph theory or as the max-flow min-cut property in integer programming. Algebraically, it is equivalent to $I(H)$ being normally torsion free; see [9, Corollary 10.3.15], [24, Theorem 14.3.6]. This fact enhances the importance of normally torsion-free ideals. In Sect. 2, our main goal is to establish the normally torsion-freeness of the closed neighborhood ideals and the dominating ideals of star graphs, which is achieved in Corollaries 2.6 and 2.19. To do this, we first prove some results of general nature, which provide certain inductive and recursive techniques to create new normally torsion-free ideals based on the existing ones; see Theorem 2.3 and Lemma 2.5. We apply these techniques to study the normally torsion-freeness of the closed neighborhood ideals and the dominating ideals of cone graph and whisker graph of a given graph; see Corollary 2.8 and Lemma 2.10. In addition, in Corollary 2.4, we prove that three-path ideals of path graphs are normally torsion free.

In Sect. 3, we turn our attention to the closed neighborhood ideals and the dominating ideals of cycles. The edge ideals and the cover ideals of cycles are well studied in the context of normally torsion-freeness. It is a well known fact that the edge ideals and cover ideals of even cycles are normally torsion free, and odd cycles fails to have this property in general. Given a cycle $C_n$ of length $n$, it is natural to expect somewhat similar behaviour for $NI(C_n)$ and $DI(C_n)$, but we observe in Sect. 3 that this is not the case. Normally, torsion-freeness is not maintained by $NI(C_n)$, but we prove in Theorem 3.2 that they admit strong persistence property, and therefore, the persistence property. This facilitates to study the behavior of depth of powers of $NI(C_n)$.
in Corollary 3.7. As a final result, we prove in Theorem 3.9 that \( DI(C_n) \) are nearly normally torsion free.

1. Preliminaries

Let \( G \) be a finite simple undirected graph with vertex set \( V(G) \) and edge set \( E(G) \). The degree of a vertex \( v \in V(G) \) is denoted by \( \text{deg}(v) \) and it represents the number of vertices adjacent to \( v \). We briefly recall some well-known notions from graph theory. A set \( T \subseteq V(G) \) is called a vertex cover of \( G \) if it intersects every edge of \( G \) non-trivially. A vertex cover is called minimal if it does not properly contain any other vertex cover of \( G \).

For each vertex \( v \in V(G) \), the closed neighborhood of \( v \) in \( G \) is defined as follows:

\[
N_G[v] = \{ u \in V(G) : \{u, v\} \in E(G) \} \cup \{v\}.
\]

When there is no confusion about the underlying graph, we will denote \( N_G[v] \) simply by \( N[v] \). A subset \( S \subseteq V(G) \) is called a dominating set of \( G \) if \( S \cap N[v] \neq \emptyset \), for all \( v \in V(G) \). A dominating set is called minimal if it does not properly contain any other dominating set of \( G \). A minimum dominating set of \( G \) is a minimal dominating set with the smallest size. The dominating number of \( G \), denoted by \( \gamma(G) \), is the size of its minimum dominating set, that is,

\[
\gamma(G) = \min\{|S| : S \text{ is a minimal dominating set of } G\}.
\]

The dominating sets and domination numbers of graphs are well-studied topics in graph theory. We refer to [7] for further information.

Let \( R = K[x_1, \ldots, x_n] \) be a polynomial ring over a field \( K \). An ideal \( I \subset R \) is called a monomial ideal if it admits a generating set consisting of monomials. Furthermore, \( I \) is said to be square-free monomial ideal if it is generated by square-free monomials. Throughout the following text, the unique minimal generating set of a monomial ideal \( I \) will be denoted by \( \mathcal{G}(I) \).

The support of a monomial \( u \), denoted by \( \text{supp}(u) \), is the set of variables that divide \( u \). Moreover, for a monomial ideal \( I \), we set \( \text{supp}(I) = \bigcup_{u \in \mathcal{G}(I)} \text{supp}(u) \).

Let \( t \) be a fixed positive integer. The \( t \)-path ideal of \( G \), denoted by \( I_t(G) \), is defined as

\[
I_t(G) = (x_{i_1}x_{i_2} \cdots x_{i_t} : \{i_1, \ldots, i_t\} \text{ is a path of length } t-1 \text{ in } G).
\]

The notion of path ideals is a generalization of edge ideals. Indeed, we have \( I(G) = I_2(G) \).
In [20], the *closed neighborhood ideal* of $G$ has been introduced as

$$NI(G) = \left( \prod_{j \in N[i]} x_j : i \in V(G) \right).$$

Moreover, in [20], the dominating ideal of $G$ is defined as

$$DI(G) = \left( \prod_{i \in S} x_i : S \text{ is a minimal dominating set of } G \right).$$

Given a square-free monomial ideal $I \subset R$, the Alexander dual of $I$, denoted by $I^\vee$, is given by

$$I^\vee = \bigcap_{u \in G(I)} (x_i : x_i \in \text{supp}(u)).$$

It is a well-recognized fact that $J(G)$ is the Alexander dual of $I(G)$; for example, see [9, Lemma 9.1.4]. It is shown in [20, Lemma 2.2] that $DI(G)$ is the Alexander dual of $NI(G)$.

Next, we recall some notions related to hypergraphs. A finite hypergraph $\mathcal{H}$ on a vertex set $[n] = \{1, 2, \ldots, n\}$ is a collection of edges $\{E_1, \ldots, E_m\}$ with $E_i \subseteq [n]$, for all $i = 1, \ldots, m$. The vertex set $[n]$ of $\mathcal{H}$ is denoted by $V(\mathcal{H})$, and the edge set of $\mathcal{H}$ is denoted by $E(\mathcal{H})$. The *edge ideal* of $\mathcal{H}$ is given by

$$I(\mathcal{H}) = \left( \prod_{j \in E_i} x_j : E_i \in E(\mathcal{H}) \right).$$

A subset $W \subseteq V_\mathcal{H}$ is a *vertex cover* of $\mathcal{H}$ if $W \cap E_i \neq \emptyset$ for all $i = 1, \ldots, m$. A vertex cover $W$ is *minimal* if no proper subset of $W$ is a vertex cover of $\mathcal{H}$. The cover ideal of the hypergraph $\mathcal{H}$, denoted by $J(\mathcal{H})$, is given by

$$J(\mathcal{H}) = \left( \prod_{i \in W} x_i : W \text{ is a minimal vertex cover of } \mathcal{H} \right).$$

Similar to the case of edge ideal of graphs, the cover ideal $J(\mathcal{H})$ is the Alexander dual of $I(\mathcal{H})$, that is, $J(\mathcal{H}) = I(\mathcal{H})^\vee$, for example, see [24, Theorem 6.3.39].

Next, we recall some definitions and notions from commutative algebra. Let $R$ be a commutative Noetherian ring and $I$ be an ideal of $R$. A prime ideal $p \subset R$ is an *associated prime* of $I$ if there exists an element $v \in R$ such that $p = (I :_R v)$, where $(I :_R v) = \{ r \in R | rv \in I \}$. The set of associated primes of $I$, denoted by $\text{Ass}_R(R/I)$, is the set of all prime ideals associated with $I$. The minimal members of $\text{Ass}_R(R/I)$ are called the *minimal* primes of $I$, and $\text{Min}(I)$ denotes the set of minimal prime ideals of $I$. Moreover, the associated primes of $I$ which are not minimal are called the *embedded* primes of $I$. If $I$ is a square-free monomial ideal, then $\text{Ass}_R(R/I) = \text{Min}(I)$; for example see [9, Corollary 1.3.6]. Let $I$ be an ideal of $R$ and $p_1, \ldots, p_r$ be the minimal primes of $I$. When there is no confusion about the underlying ring,
we will denote the set of associated primes of \( I \) simply by \( \text{Ass}(R/I) \). Given an integer \( n \geq 1 \), the \( n \)-th symbolic power of \( I \) is defined to be the ideal 
\[
I^{(n)} = q_1 \cap \cdots \cap q_r,
\]
where \( q_i \) is the primary component of \( I^n \) corresponding to \( p_i \).

**Definition 1.1.** An ideal \( I \) is called \emph{normally torsion free} if \( \text{Ass}(R/I^k) \subseteq \text{Ass}(R/I) \), for all \( k \geq 1 \). If \( I \) is a square-free monomial ideal, then \( I \) is normally torsion free if and only if \( I^k = I^{(k)} \), for all \( k \geq 1 \); see [9, Theorem 1.4.6]. The concept of normally torsion-free ideals is generalized in [3] as follows: a monomial ideal \( I \) in a polynomial ring \( R = K[x_1, \ldots, x_n] \) over a field \( K \) is called \emph{nearly normally torsion free} if there exist a positive integer \( k \) and a monomial prime ideal \( p \) such that \( \text{Ass}(R/I^m) = \text{Min}(I) \) for all \( 1 \leq m \leq k \), and \( \text{Ass}(R/I^m) \subseteq \text{Min}(I) \cup \{p\} \) for all \( m \geq k + 1 \). In [16], several classes of nearly normally torsion-free ideals arising from graphs and hypergraphs are discussed. By Gitler et al. [5], it is well known that the cover ideals of bipartite graphs are normally torsion free. Furthermore, normally torsion-free square-free monomial ideals have been studied in Ha and Morey [8], and Sullivant [22]. On the other hand, little is known for the normally torsion-free monomial ideals which are not square free.

**Definition 1.2.** The ideal \( I \subset R \) is said to have the \emph{persistence property} if \( \text{Ass}(R/I^k) \subseteq \text{Ass}(R/I^{k+1}) \) for all positive integers \( k \). Moreover, an ideal \( I \) satisfies the \emph{strong persistence property} if \( (I^{k+1} : I) = I^k \) for all positive integers \( k \); for more details refer to [10,13]. The strong persistence property implies the persistence property, however the converse is not true, as noted in [10]. Furthermore, we say that \( I \) has the \emph{symbolic strong persistence property} if \( (I^{(k+1)} : I^{(1)}) = I^{(k)} \) for all \( k \), where \( I^{(k)} \) denotes the \( k \)-th symbolic power of \( I \).

Let \( R \) be a unitary commutative ring and \( I \) an ideal in \( R \). An element \( f \in R \) is \emph{integral} over \( I \), if there exists an equation 
\[
f^k + c_1f^{k-1} + \cdots + c_{k-1}f + c_k = 0 \quad \text{with} \quad c_i \in I^i.
\]
The set of elements \( I \) in \( R \) which are integral over \( I \) is the \emph{integral closure} of \( I \). The ideal \( I \) is \emph{integrally closed}, if \( I = \overline{I} \), and \( I \) is \emph{normal} if all powers of \( I \) are integrally closed; refer to [9] for more information. The notion of integrality for a monomial ideal \( I \) can be described in a simpler way as follows: a monomial \( u \in R = K[x_1, \ldots, x_n] \) is integral over \( I \subset R \) if and only if there exists an integer \( k \) such that \( u^k \in I^k \); see [9, Theorem 1.4.2].

2. On the Closed Neighborhood Ideals and Dominating Ideals of Some Classes of Trees

In this section, our main goal is to establish that the closed neighborhood ideals and the dominating ideals of star graphs are normally torsion free. To do this, we will first prove several results of general nature. The next proposition is a well-known result, but we re-prove it by a new proof.
Proposition 2.1. Let $I$ be an ideal in a commutative Noetherian ring $R$ such that it satisfies the strong persistence property. Then, $I$ has the persistence property.

Proof. Fix $k \geq 1$, and choose an arbitrary element $p \in \text{Ass}_S(S/I^k)$. This implies that $p = (I^k :_S h)$ for some $h \in S$. Since $I$ satisfies the strong persistence property, we have $(I^{k+1} :_S I) = I^k$, and so $p = ((I^{k+1} :_S I) :_S h)$. Let $G(I) = \{u_1, \ldots, u_m\}$. Hence, one obtains $p = (I^{k+1} :_S h \sum_{i=1}^m u_i S) = \cap_{i=1}^m (I^{k+1} :_S h u_i)$. Accordingly, we get $p = (I^{k+1} :_S h u_i)$ for some $1 \leq i \leq m$. Therefore, $p \in \text{Ass}_S(S/I^{k+1})$. This means that $I$ has the persistence property, as claimed.

To prove Theorem 2.3, we need the following result. We state it here for ease of reference.

Theorem 2.2. [19, Theorem 3.7] Let $I$ be a square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$ and $m = (x_1, \ldots, x_n)$. If there exists a square-free monomial $v \in I$ such that $v \in p \setminus p^2$ for any $p \in \text{Min}(I)$, and $m \setminus x_i \notin \text{Ass}(R/(I \setminus x_i)^s)$ for all $s$ and $x_i \in \text{supp}(v)$, then the following statements hold:

(i) $I$ is normally torsion free.

(ii) $I$ is normal.

(iii) $I$ has strong persistence property.

(iv) $I$ has persistence property.

(v) $I$ has the symbolic strong persistence property.

The next theorem will be used frequently to formulate proofs of some main results of this paper. It provides a way to create new normally torsion-free ideals based on the existing ones.

Theorem 2.3. Let $I$ be a normally torsion-free square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ and $h$ be a square-free monomial in $R$. Let there exist two variables $x_r$ and $x_s$ with $1 \leq r \neq s \leq n$ such that $\gcd(h, u) = 1$ or $\gcd(h, u) = x_r$ or $\gcd(h, u) = x_r x_s$ for all $u \in G(I)$. Then the following statements hold:

(i) $I + hR$ is normally torsion free.

(ii) $I + hR$ is nearly normally torsion free.

(iii) $I + hR$ is normal.

(iv) $I + hR$ has strong persistence property.

(v) $I + hR$ has persistence property.

(vi) $I + hR$ has the symbolic strong persistence property.

Proof. (i) For convenience of notation, put $L := I + hR$. If $L \setminus x_k = m \setminus x_k$ for some $1 \leq k \leq n$, then one can write $L = x_k J + m \setminus x_k$. If $J = R$, then $L = m$, and there is nothing to prove. Let $J \neq R$, and take an arbitrary element $v \in G(J)$. If $x_k \mid v$ for some $\ell \in \{1, \ldots, n\} \setminus \{k\}$, then $v \in m \setminus x_k$, and so $J \subseteq m \setminus x_k$. This implies that $L = m \setminus x_k$, and hence the assertion holds. We thus assume that $L \setminus x_k \neq m \setminus x_k$ for all $k = 1, \ldots, n$. We claim that $h \notin p \setminus p^2$ for any $p \in \text{Min}(L)$. Take an arbitrary element $p \in \text{Min}(L)$. Since
\[ h \in L \text{ and } L \subseteq p, \text{ one has } h \in p. \] Suppose, on the contrary, that \( h \in p^2. \) Due to \( h \) being square free, this gives that \(|\text{supp}(h) \cap \text{supp}(p)| \geq 2.\) We observe the following:

(i) If \( x_s \in \text{supp}(u) \) for some \( u \in G(I), \) then \( x_r \in \text{supp}(u) \) as well. It is due to the assumption on gcd\((h, u)\) with \( u \in G(I).\)

(ii) At most, one of \( x_r \) and \( x_s \) can be in \( \text{supp}(p). \) Indeed, if both \( x_r, x_s \in \text{supp}(p), \) then \( x_r, x_s \in \text{supp}(h) \cap \text{supp}(p). \) From (i), we see that \( u \in p \setminus \{x_s\} \) for all \( u \in G(I). \) Also, \( h \in p \setminus \{x_s\}. \) Hence, \( L \subset p \setminus \{x_s\}, \) a contradiction to the minimality of \( p. \)

To establish our claim, we have the following cases to discuss:

**Case 1.** \( x_r \in p. \) Take any \( x_i \in \text{supp}(h) \cap \text{supp}(p) \) such that \( x_r \neq x_i. \) Then \( x_s \neq x_i \) due to (ii). From the assumption on gcd\((h, u)\) with \( u \in G(I), \) it follows that \( x_i \notin \text{supp}(I). \) Therefore, \( I \subset p \setminus \{x_i\}. \) Since \( h \in p \setminus \{x_i\}, \) we conclude that \( L \subset p \setminus \{x_i\}, \) a contradiction to the minimality of \( p. \)

**Case 2.** \( x_s \in p. \) By mimicking the same argument as in Case 1, we again obtain a contradiction to the minimality of \( p. \)

**Case 3.** \( x_r \notin p \) and \( x_s \notin p. \) Take any \( x_i, x_j \in \text{supp}(h) \cap \text{supp}(p). \) Then \( x_i, x_j \notin \text{supp}(I), \) due to the assumption on gcd\((h, u)\) with \( u \in G(I). \) It yields that \( I \subset p \setminus \{x_i\}. \) Since \( h \in p \setminus \{x_i\}, \) we conclude that \( L \subset p \setminus \{x_i\}, \) again a contradiction to the minimality of \( p. \)

This shows that our claim holds true. To complete the proof, note that for all \( x_k \in \text{supp}(h), \) one has \( L \setminus x_k = I \setminus x_k. \) Based on [18, Theorem 3.21], we gain \( I \setminus x_k \) is normally torsion free as well. This leads to \( L \setminus x_k \) is normally torsion free. Fix \( s \geq 1. \) Suppose, on the contrary, that \( m \setminus x_k \in \text{Ass}(R/(L \setminus x_k)^s) \) for some \( k. \) Because \( \text{Ass}(R/(L \setminus x_k)^s) = \text{Min}(L \setminus x_k), \) we get \( m \setminus x_k \in \text{Min}(L \setminus x_k), \) and so \( L \setminus x_k = m \setminus x_k, \) which is a contradiction. Therefore, \( m \setminus x_i \notin \text{Ass}(R/(I \setminus x_i)^s) \) for all \( s \) and \( x_i \in \text{supp}(h). \) Consequently, the assertion can be concluded readily from Theorem 2.2.

(ii) It is well known, by [16] that normally torsion-freeness implies nearly normally torsion-freeness.

(iii) In view of [9, Theorem 1.4.6], every normally torsion-free square-free monomial ideal is normal. Hence, the claim can be deduced from (i).

(iv) According to [17, Theorem 6.2], every normal monomial ideal has strong persistence property. Thus, the assertion follows readily from (iii).

(v) By Proposition 2.1, the strong persistence property implies the persistence property. Therefore, we can conclude the claim from (iv).

(vi) According to [15, Theorem 5.1], every square-free monomial ideal has the symbolic strong persistence property, and so the assertion holds. \( \square \)

As an immediate consequence of Theorem 2.3, we give the following corollary.

**Corollary 2.4.** The path ideals corresponding to path graphs of length two are normally torsion free.

**Proof.** Let \( P = (V(P), E(P)) \) denote a path graph with the vertex set \( V(P) = \{x_1, \ldots, x_n\} \) and the edge set \( E(P) = \{\{x_i, x_{i+1}\} : i = 1, \ldots, n - 1\}. \) Thus, the assertion follows readily from (iii).
1} \cup \{\{x_n, x_1\}\}. Hence, the path ideal corresponding to the path graph \(P\) of length two is given by
\[
L := (x_ix_{i+1}x_{i+2} : i = 1, \ldots, n - 2).
\]
We proceed by induction on \(n\). If \(n = 3\), then \(L = (x_1x_2x_3)\), and there is nothing to prove. Let \(n > 3\) and the claim has been proven for \(n - 1\). Set \(h := x_{n-2}x_{n-1}x_n\) and \(I := (x_{i}x_{i+1}x_{i+2} : i = 1, \ldots, n - 3)\). One can easily check that, for each \(u \in G(I)\), we have \(\gcd(h, u) = 1\) or \(\gcd(h, u) = x_{n-2}\) or \(\gcd(h, u) = x_{n-2}x_{n-1}\). It follows from the induction hypothesis that \(I\) is normally torsion free. Since \(L = I + hR\), where \(R = K[x_1, \ldots, x_n]\), we can derive the assertion from Theorem 2.3. \(\square\)

As an application of Theorem 2.3, we give the following lemma.

**Lemma 2.5.** Let \(G = (V(G), E(G))\) and \(H = (V(H), E(H))\) be two finite simple graphs such that \(V(H) = V(G) \cup \{w\}\) with \(w \notin V(G)\), and \(E(H) = E(G) \cup \{\{v, w\}\}\) for some vertex \(v \in V(G)\). If \(NI(G)\) is normally torsion free and \(\prod_{j \in N_G(v)} x_j \notin G(NI(G))\), then \(NI(H)\) is normally torsion free.

**Proof.** Let \(NI(G)\) be normally torsion free. It is routine to check that \(NI(H) = NI(G) + (x_vx_w)R\), where \(R = K[x_\alpha : \alpha \in V(H)]\). In addition, one can easily see that either \(\gcd(x_vx_w, u) = 1\) or \(\gcd(x_vx_w, u) = x_v\) for all \(u \in G(NI(G))\). Therefore, the claim follows immediately from Theorem 2.3. \(\square\)

We are ready to state the first main result of this paper as an immediate corollary of Theorem 2.3 and Lemma 2.5.

**Corollary 2.6.** The closed neighborhood ideals of star graphs are normally torsion free.

**Proof.** Proceed by induction on the number of vertices and use Lemma 2.5. \(\square\)

In what follows, we investigate the closed neighborhood ideals related to the whisker graph and cone of a graph.

**Definition 2.7.** [24, Definition 7.3.10] Let \(G_0\) be a graph on the vertex set \(Y = \{y_1, \ldots, y_n\}\) and take a new set of variables \(X = \{x_1, \ldots, x_n\}\). The whisker graph or suspension of \(G_0\), denoted by \(G_0 \cup W(Y)\), is the graph obtained from \(G_0\) by attaching to each vertex \(y_i\) a new vertex \(x_i\) and the edge \(\{x_i, y_i\}\). The edge \(\{x_i, y_i\}\) is called a whisker.

**Question 2.8.** (i) Can we conclude that the closed neighborhood ideals of trees are normally torsion free?

(ii) Let \(G_0\) be a graph and let \(H := G_0 \cup W(Y)\) be its whisker graph. If \(NI(G_0)\) is normally torsion free, then can we deduce that \(NI(H)\) is normally torsion free?

**Definition 2.9.** [24, Definition 10.5.4] The cone \(C(G)\), over the graph \(G\), is obtained by adding a new vertex \(t\) to \(G\) and joining every vertex of \(G\) to \(t\).

**Lemma 2.10.** Let \(G\) be a graph and let \(H := C(G)\) be its cone. Then the following statements hold:
(i) $NI(G)$ is normally torsion free if and only if $NI(H)$ is normally torsion free.
(ii) $NI(G)$ is nearly normally torsion free if and only if $NI(H)$ is nearly normally torsion free.
(iii) $NI(G)$ is normal if and only if $NI(H)$ is normal.
(iv) $NI(G)$ has strong persistence property if and only if $NI(H)$ has strong persistence property.
(v) $NI(G)$ has persistence property if and only if $NI(H)$ has persistence property.
(vi) Both $NI(G)$ and $NI(H)$ have the symbolic strong persistence property.

Proof. Assume that the cone $H = C(G)$ is obtained by adding the new vertex $w$ to $G$ and joining every vertex of $G$ to $w$. Then one can easily see that

$$NI(H) = x_wNI(G) + \left(x_w \prod_{i \in V(G)} x_i\right).$$

Since $\prod_{i \in V(G)} x_i \in NI(G)$, this implies that $NI(H) = x_wNI(G)$.

(i) This claim can be deduced from [18, Lemma 3.12].
(ii) On account of [16, Lemma 3.6], one can derive this claim.
(iii) We can conclude this assertion by virtue of [2, Remark 1.2].
(iv) This claim is an immediate consequence of [17, Lemma 4.5].
(v) Due to [12, Theorem 5.2], we can deduce this assertion.
(vi) This claim follows readily from [15, Theorem 5.1].

We recall the following definition which will be used in the proof of Lemma 2.12.

Definition 2.11. [13, Definition 2.1] Let $I \subset R = K[x_1, \ldots, x_n]$ be a monomial ideal with $G(I) = \{u_1, \ldots, u_m\}$. Then $I$ is said to be unisplit, if there exists $u_i \in G(I)$ such that $\gcd(u_i, u_j) = 1$ for all $u_j \in G(I)$ with $i \neq j$.

Lemma 2.12. Let $G$ be a graph and let $H := C(G)$ be its cone. Then the following statements hold:

(i) $DI(G)$ is normally torsion free if and only if $DI(H)$ is normally torsion free.
(ii) $DI(G)$ is nearly normally torsion free if and only if $DI(H)$ is nearly normally torsion free.
(iii) $DI(G)$ is normal if and only if $DI(H)$ is normal.
(iv) $DI(H)$ has strong persistence property.
(v) $DI(H)$ has persistence property.
(vi) Both $DI(G)$ and $DI(H)$ have the symbolic strong persistence property.

Proof. Suppose that the cone $H = C(G)$ is obtained by adding the new vertex $w$ to $G$ and joining every vertex of $G$ to $w$. Using [20, Lemma 2.2] yields that

$$DI(H) = DI(G) + (x_w).$$

It follows now from [8, Lemma 3.4] that, for all $s$,

$$\text{Ass}(DI(H)^s) = \{(p, x_w) : p \in \text{Ass}(DI(G)^s)\}. \quad (1)$$
(i) Let $DI(G)$ be normally torsion free. Then the claim can be deduced from [18, Theorem 2.5]. Conversely, let $DI(H)$ be normally torsion free. By using [18, Theorem 3.21], we obtain that $DI(G)$ is normally torsion free.

(ii) Based on (1), and by considering the fact that
\[ \text{Min}(DI(H)) = \{(p, x_w) : p \in \text{Min}(DI(G))\}, \]
one can easily show this claim.

(iii) One concludes this assertion by [1, Theorem 3.12].

(iv) By [13, Definition 2.1], $DI(H)$ is a unisplit monomial ideal. Hence, [13, Theorems 2.10 and 3.1] imply that $DI(H)$ has the strong persistence property.

(v) Proposition 2.1 together with (iv) yields that $DI(H)$ has the persistence property.

(vi) This assertion follows promptly from [15, Theorem 5.1]. □

Our next goal is to show that the dominating ideals of star graphs are normally torsion free. To do this, we first prove some results of general nature. We recall some definitions from [4], which are necessary to establish Theorem 2.16. Let $H = (V(H), E(H))$ be a hypergraph with $V(H) = \{x_1, \ldots, x_n\}$.

**Definition 2.13.** (see [4, Definition 2.7]) A $d$-coloring of $H$ is any partition of $V(H) = C_1 \cup \cdots \cup C_d$ into $d$ disjoint sets such that for every $E \in E(H)$, we have $E \not\subseteq C_i$ for all $i = 1, \ldots, d$. (In the case of a graph $G$, this simply means that any two vertices connected by an edge receive different colors.) The $C_i$s are called the color classes of $H$. Each color class $C_i$ is an independent set, meaning that $C_i$ does not contain any edge of the hypergraph. The chromatic number of $H$, denoted by $\chi(H)$, is the minimal $d$ such that $H$ has a $d$-coloring.

**Definition 2.14.** (see [4, Definition 2.8]) The hypergraph $H$ is called critically $d$-chromatic if $\chi(H) = d$, but for every vertex $x \in V(H)$, $\chi(H \setminus \{x\}) < d$, where $H \setminus \{x\}$ denotes the hypergraph $H$ with $x$ and all edges containing $x$ removed.

**Definition 2.15.** (see [4, Definition 4.2]) For each $s$, the $s$-th expansion of $H$ is defined to be the hypergraph obtained by replacing each vertex $x_i \in V(H)$ by a collection $\{x_{ij} \mid j = 1, \ldots, s\}$, and replacing $E(H)$ by the edge set that consists of edges $\{x_{i_1, l_1}, \ldots, x_{i_r, l_r}\}$ whenever $\{x_{i_1}, \ldots, x_{i_r}\} \in E(H)$ and edges $\{x_{il}, x_{ik}\}$ for $l \neq k$. We denote this hypergraph by $H^s$. The new variables $x_{ij}$ are called the shadows of $x_i$. The process of setting $x_{il}$ to equal to $x_i$ for all $i$ and $l$ is called the depolarization.

The following result is a slight generalized form of [16, Theorem 4.9].

**Theorem 2.16.** Assume that $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are two finite simple hypergraphs such that $V(H) = V(G) \cup \{w_1, \ldots, w_t\}$ with $w_i \not\in V(G)$ for each $i = 1, \ldots, t$, and $E(H) = E(G) \cup \{(v, w_1, \ldots, w_t)\}$ for some vertex $v \in V(G)$. Then,
\[ \text{Ass}_R(R'/J(H)^s) = \text{Ass}_R(R/J(G)^s) \cup \{(x_v, x_{w_1}, \ldots, x_{w_t})\}, \]
for all $s$, where $R = K[x_\alpha : \alpha \in V(G)]$ and $R' = K[x_\alpha : \alpha \in V(H)]$. 
Proof. For convenience of notation, set \( I := J(\mathcal{G}) \) and \( J := J(\mathcal{H}) \). We first prove that \( \text{Ass}_R(R/I) \cup \{(x_v, x_{w_1}, \ldots, x_{w_s})\} \subseteq \text{Ass}_R(R'/J^s) \) for all \( s \). Fix \( s \geq 1 \), and assume that \( \mathcal{H}_p = (x_{i_1}, \ldots, x_{i_t}) \) is an arbitrary element of \( \text{Ass}_R(R/I^s) \). According to [4, Lemma 2.11], we get \( \mathcal{H}_p \in \text{Ass}(K[p]/J(\mathcal{G}_p)^s) \), where \( K[p] = K[x_{i_1}, \ldots, x_{i_t}] \) and \( \mathcal{G}_p \) is the induced subhypergraph of \( \mathcal{G} \) on the vertex set \( \{i_1, \ldots, i_r\} \subseteq V(\mathcal{G}) \). Since \( \mathcal{G}_p = \mathcal{H}_p \), we have \( \mathcal{H}_p \in \text{Ass}(K[p]/J(H_p^s)) \). This yields that \( \mathcal{H}_p \in \text{Ass}_R(R'/J^s) \). On account of \( (x_v, x_{w_1}, \ldots, x_{w_s}) \in \text{Ass}_R(R'/J^s) \), one derives

\[
\text{Ass}_R(R/I^s) \cup \{(x_v, x_{w_1}, \ldots, x_{w_s})\} \subseteq \text{Ass}_R(R'/J^s).
\]

To complete the proof, it is enough for us to show the reverse inclusion. Assume that \( \mathcal{H}_p = (x_{i_1}, \ldots, x_{i_t}) \) is an arbitrary element of \( \text{Ass}_R(R'/J^s) \) with \( \{i_1, \ldots, i_r\} \subseteq V(\mathcal{H}) \). If \( \{i_1, \ldots, i_r\} \subseteq V(\mathcal{G}) \), then [4, Lemma 2.11] implies that \( \mathcal{H}_p \in \text{Ass}_R(R/I^s) \), and the proof is done. Thus, let \( \{w_1, \ldots, w_s\} \cap \{i_1, \ldots, i_r\} \neq \emptyset \). It follows from [4, Corollary 4.5] that the associated primes of \( J(\mathcal{H})^s \) will correspond to critical chromatic subhypergraphs of size \( s + 1 \) in the \( s \)-th expansion of \( \mathcal{H} \). This means that one can take the induced subhypergraph on the vertex set \( \{i_1, \ldots, i_r\} \), and then form the \( s \)-th expansion on this induced subhypergraph, and within this new hypergraph find a critical \((s + 1)\)-chromatic hypergraph. Notice that since this expansion cannot have any critical chromatic subgraphs, this implies that \( \mathcal{H}_p \) must be connected. Without loss of generality, one may assume that \( i_1 = v \) and \( i_2 = w_1, i_3 = w_2, \ldots, i_{t+1} = w_t \). Thanks to \( w_1, \ldots, w_t \) are connected to \( v \) in the hypergraph \( \mathcal{H} \), and because this induced subhypergraph is critical, if we remove any vertex \( w_k \) for some \( 1 \leq k \leq t \), one can color the resulting hypergraph with at least \( s \) colors. This leads to that \( w_k \) has to be adjacent to at least \( s \) vertices. But the only things \( w_k \) is adjacent to are the shadows of \( w_i \) for each \( i = 1, \ldots, t \), and the shadows of \( v \), and so one has a clique among these vertices. Accordingly, \( w_k \) and its neighbors will form a clique of size \( s + 1 \). Since a clique is a critical graph, it follows that we do not need any element of \( \{i_{t+2}, \ldots, i_r\} \) or their shadows when making the critical \((s + 1)\)-chromatic hypergraph. Hence, we obtain \( \mathcal{H}_p = (x_v, x_{w_1}, \ldots, x_{w_s}) \). This finishes the proof. \( \square \)

Lemma 2.17. Let \( I \) be a normally torsion-free square-free monomial ideal in a polynomial ring \( R = K[x_1, \ldots, x_n] \) with \( \mathcal{G}(I) \subset R \). Then the ideal

\[
L := IS \cap (x_n, x_{n+1}, x_{n+2}, \ldots, x_m) \subset S = R[x_{n+1}, x_{n+2}, \ldots, x_m],
\]

is normally torsion free.

Proof. It is well known that one can view the square-free monomial ideal \( I \) as the cover ideal of a simple hypergraph \( \mathcal{H} \) such that the hypergraph \( \mathcal{H} \) corresponds to \( I^\vee \), where \( I^\vee \) denotes the Alexander dual of \( I \). Then we have \( I = J(\mathcal{H}) \), where \( J(\mathcal{H}) \) denotes the cover ideal of the hypergraph \( \mathcal{H} \). Fix \( k \geq 1 \). On account of Theorem 2.16, we get the following equality:

\[
\text{Ass}_S(S/L^k) = \text{Ass}_R(R/J(\mathcal{H})^k) \cup \{(x_n, x_{n+1}, x_{n+2}, \ldots, x_m)\}.
\]

Because \( I \) is normally torsion free, one derives that \( \text{Ass}_R(R/J(\mathcal{H})^k) = \text{Min}(J(\mathcal{H})) \), and hence \( \text{Ass}_S(S/L^k) = \text{Min}(J(\mathcal{H})) \cup \{(x_n, x_{n+1}, x_{n+2}, \ldots, x_m)\} \).
This gives rise to \( \text{Ass}_S(S/L^k) = \text{Min}(L) \). Therefore, \( L \) is normally torsion free, as claimed. \( \square \)

**Lemma 2.18.** Let \( G = (V(G), E(G)) \) and \( H = (V(H), E(H)) \) be two finite simple graphs such that \( V(H) = V(G) \cup \{w\} \) with \( w \notin V(G) \), and \( E(H) = E(G) \cup \{\{v, w\}\} \) for some vertex \( v \in V(G) \) and \( \prod_{j \in N_G[v]} x_j \notin G(NI(G)) \). If \( DI(G) \) is normally torsion free, then \( DI(H) \) is normally torsion free.

**Proof.** Let \( DI(G) \) be normally torsion free. It follows from [20, Lemma 2.2] that \( DI(H) = DI(G) \cap (x_v, x_w)R \), where \( R = K[x_\alpha : \alpha \in V(H)] \). Now, we can conclude the assertion from Lemma 2.17. \( \square \)

We are in a position to give the second main result of this paper in the following corollary, which is related to dominating ideals of star graphs.

**Corollary 2.19.** The dominating ideals of star graphs are normally torsion free.

**Proof.** We use the induction on the number of vertices together with Lemma 2.18. \( \square \)

**Question 2.20.** (i) Can we conclude that the dominating ideals of trees are normally torsion free?

(ii) Let \( G_0 \) be a graph and let \( H := G_0 \cup W(Y) \) be its whisker graph. Then if \( DI(G_0) \) is normally torsion free, then can we deduce that \( DI(H) \) is normally torsion free?

### 3. On the Closed Neighborhood Ideals and Dominating Ideals of Cycles

As stated in the introduction, the edge ideals and the cover ideals of bipartite graphs are known to be normally torsion free; see [5,21]. In particular, the edge ideals and the cover ideals of even cycles are normally torsion free. However, this behaviour changes when we consider the odd cycles. The cover ideals of odd cycles happen to be nearly normally torsion free; see [14], but edge ideals of odd cycles do not admit such tamed behavior for the set of their associated primes. Given these facts, it is natural to expect some irregularities for the closed neighborhood ideals and dominating ideals of even and odd cycles. It can be verified by using Macaulay2 [6] that, in general, the closed neighborhood ideals of cycles, regardless of the parity of their lengths, are neither normally torsion free or nearly normally torsion free. However, in this section, we will show that the closed neighborhood ideals of cycles admit strong persistence property. On the other side, as another main result of this section, we will show that the dominating ideals of cycles are nearly normally torsion free.

To establish the above-mentioned results, we begin by proving the following theorem which gives an inductive way to study the normality of an ideal.
Theorem 3.1. Let $I$ and $H$ be two normal square-free monomial ideals in a polynomial ring $R = K[x_1, \ldots, x_n]$ such that $I + H$ is normal. Let $x_c \in \{x_1, \ldots, x_n\}$ be a variable with $\gcd(v, x_c) = 1$ for all $v \in G(I) \cup G(H)$. Then $L := I + x_cH$ is normal.

Proof. Let $G(I) = \{u_1, \ldots, u_s\}$ and $G(H) = \{h_1, \ldots, h_r\}$. Since $\gcd(v, x_c) = 1$ for all $v \in G(I) \cup G(H)$; without loss of generality, one may assume that $x_c = x_1 \in K[x_1]$ and

$$G(I) \cup G(H) = \{u_1, \ldots, u_s, h_1, \ldots, h_r\} \subseteq K[x_2, \ldots, x_n].$$

We must show that $L^t = L^t$ for all integers $t \geq 1$. For this purpose, it is enough to prove that $L^{(t)} \subseteq L^t$. Let $\alpha$ be a monomial in $L^{(t)}$ and write $\alpha = \chi_{\delta}^{\beta}$ with $\chi_1 \nmid \beta$ and $\beta \in R$. On account of [9, Theorem 1.4.2], $\alpha^k \in L^{tk}$ for some integer $k \geq 1$. Write

$$\alpha^k = x_1^{\beta_k} \delta_k = \prod_{i=1}^{s} u_{i}^{p_i} x_1^{q_i + \varepsilon} \prod_{j=1}^{r} h_{j}^{q_j} \beta,$$

with $\sum_{i=1}^{s} p_i = p$, $\sum_{j=1}^{r} q_j = q$, $p + q = tk$, $\varepsilon \geq 0$, and $\beta$ is some monomial in $R$ such that $x_1 \nmid \beta$. Because $x_1 \nmid \beta$, $x_1 \nmid \delta$, and $\gcd(v, x_1) = 1$ for all $v \in G(I) \cup G(H)$, one can conclude that $\beta = q + \varepsilon$. Accordingly, by virtue of (2), we obtain

$$\delta^k = \prod_{i=1}^{s} u_{i}^{p_i} \prod_{j=1}^{r} h_{j}^{q_j} \beta \in (I + H)^{tk}.$$

This leads to $\delta \in (I + H)^{t}$. Thanks to $I + H$ is normal, we deduce that $(I + H)^{t} = (I + H)^{t}$, and so $\delta \in (I + H)^{t}$. Therefore, one can write

$$\delta = \prod_{i=1}^{s} u_{i}^{p_i} \prod_{j=1}^{r} h_{j}^{z_j} \gamma,$$

with $\sum_{i=1}^{s} l_i = l$, $\sum_{j=1}^{r} z_j = z$, $l + z = t$, and $\gamma$ is some monomial in $R$. Note that $x_1 \nmid \gamma$ as $x_1 \nmid \delta$. Due to $x_1^{bk} \delta^k \in L^{tk}$, it follows immediately from (3) that

$$\prod_{i=1}^{s} u_{i}^{l_i} x_1^{bk} \prod_{j=1}^{r} h_{j}^{z_j} \gamma^k \in L^{tk} = (I + x_1H)^{tk}.$$

Consequently, we conclude that $bk \geq zk$, that is, $b \geq z$. This gives rise to

$$x_1^{b} \delta = \prod_{i=1}^{s} u_{i}^{l_i} x_1^{b} \prod_{j=1}^{r} h_{j}^{z_j} \gamma \in (I + x_1H)^{t},$$

and the proof is over. \qed

We state the third main result of this paper in the next theorem, which is related to the closed neighborhood ideals of cycles.

Theorem 3.2. Let $C_n$ be a cycle graph of order $n$. Then the following statements hold:

(i) $NI(C_n)$ is normal.
(ii) $NI(C_n)$ has strong persistence property.
(iii) $NI(C_n)$ has persistence property.

Proof. (i) Let $C_n = (V(C_n), E(C_n))$ be a cycle graph of order $n$ with $V(C_n) = \{x_1, \ldots, x_n\}$ and $E(C_n) = \{(x_i, x_{i+1}) : i = 1, \ldots, n-1\} \cup \{(x_n, x_1)\}$. Then the closed neighborhood ideal of $C_n$ is given by

$$NI(C_n) = (x_ix_{i+1}x_{i+2} : i = 1, \ldots, n) \subset R = K[x_1, \ldots, x_n],$$

where $x_{n+1}$ (respectively, $x_{n+2}$) represents $x_1$ (respectively, $x_2$). If $n = 3$, then $NI(C_3) = (x_1x_2x_3)$, and so there is nothing to prove. Thus, let $n \geq 4$. Put $H := (x_2x_3, x_{n-1}x_n, x_2x_n)$ and $I := (x_ix_{i+1}x_{i+2} : i = 2, \ldots, n-2)$. One can easily see that $NI(C_n) = I + x_1H$. Our strategy is to use Theorem 3.1 to complete the proof. To do this, we first show that $I$, $H$, and $I + H$ are normal.

Assume that $G$ is a path graph with $V(G) = \{x_2, x_3, x_{n-1}, x_n\}$ and $E(G) = \{(x_2, x_3), (x_{n-1}, x_n), (x_2, x_n)\}$. It is routine to check that $I(G) = H$, where $I(G)$ denotes the edge ideal of $G$. Since, by [5, Corollary 2.6], the edge ideal of any path graph is normally torsion free, and by remembering this fact that every normally torsion-free square-free monomial ideal is normal, we deduce that $H$ is a normal square-free monomial ideal. Now, assume that $P$ is a path graph with $V(P) = \{x_2, x_3, \ldots, x_{n-1}, x_n\}$ and $E(P) = \{(x_i, x_{i+1}) : i = 2, \ldots, n-1\}$. It is not hard to check that $I(P) = I_3(P)$, where $I_3(P)$ denotes the path ideal of length 2 of $P$. It follows readily from Corollary 2.4 that $I = I_3(P)$ is normally torsion free, and so is normal. To complete the proof, we show that $I + H$ is normal. To accomplish this, we note that

$$I + H = (x_2x_3, x_{n-1}x_n, x_2x_n, x_ix_{i+1}x_{i+2} : i = 3, \ldots, n-3).$$

Set $A := (x_3, x_n)$ and $B := (x_{n-1}x_n, x_ix_{i+1}x_{i+2} : i = 3, \ldots, n-3)$. Notice that $I + H = B + x_2A$. It is clear that $A$ is a normal ideal. Furthermore, it follows from Corollary 2.4 and Theorem 2.3 that $B$ is normally torsion free, and so is normal. In addition, we have

$$B + A = (x_3, x_n, x_ix_{i+1}x_{i+2} : i = 4, \ldots, n-3).$$

One can easily conclude from Corollary 2.4 and Theorem 2.3 that $B + A$ is normally torsion free, and hence is normal. By virtue of Theorem 3.1, we deduce that $B + x_2A$ is normal, and so $I + H$ is normal as well. Finally, note that $\gcd(v, x_1) = 1$ for all $v \in G(I) \cup G(H)$. This finishes the proof.

The claims (ii) and (iii) can be proven similar to parts (iv) and (v) in Theorem 2.3. □

The neighborhood ideals of cycles are particularly good because they are generated by monomial of the same degree. This fact together with Theorem 3.2 enables us to study the depth of powers of $NI(C_n)$. For this purpose, we first recall the following definition and result from [10].

Definition 3.3. Let $I \subset R$ be a monomial ideal with $G(I) = \{u_1, \ldots, u_m\}$. The linear relation graph $\Gamma_I$ of $I$ is the graph with the edge set

$$E(\Gamma_I) = \{(x_i, x_j) : \text{there exist } u_k, u_l \in G(I) \text{ such that } x_iu_k = x_ju_l\},$$

and the vertex set $V(\Gamma_I) = \bigcup_{\{x_i, x_j\} \in E(\Gamma)} \{i, j\}$. 

Theorem 3.4. [10, Theorem 3.3] Let $I \subset R = K[x_1, \ldots, x_n]$ be a monomial ideal generated in a single degree whose linear relation graph has $r$ vertices and $s$ connected components. Then

$$\text{depth}(R/I^t) \leq n - t - 1 \text{ for } t = 1, \ldots, r - s.$$ 

To apply above theorem, we first analyze the linear relation graph of $NI(C_n)$. Let $V(C_n) = \{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}, \{n, 1\}$. We set the following notations.

1. $u_i = \prod_{j \in N[i]} x_j$. In simple words, $u_i$ is the monomial that corresponds to the closed neighborhood of the vertex $i$.

2. Note that $u_i = x_{i-1}x_i x_{i+1}$, for all $i = 2, \ldots, n - 1$ and $u_1 = x_n x_1 x_2, u_n = x_{n-1} x_n x_1$. To synchronize this notation for all $i$, if $i > n$ then we read $i$ as $i \pmod{n}$. In this way, we can write $u_i = x_{i-1} x_i x_{i+1}$, for all $i = 1, \ldots, n$.

Remark 3.5. Let $i \neq j$. Note that each variable $x_i$ appears in exactly three monomials in $G(NI(C_n))$, and these monomials are $u_{i-1} = x_{i-2} x_{i-1} x_i, u_i = x_{i-1} x_i x_{i+1}$ and $u_{i+1} = x_i x_{i+1} x_{i+2}$. From this observation, we conclude that $\{x_i, x_j\} \in E(\Gamma)$ if and only if there exists a path of length three from $i$ to $j$ in $C_n$. Here a path $P$ of length $n$ is defined on $n + 1$ vertices and $n$ edges.

Remark 3.6. Let $n \geq 4$, and set $I_n := NI(C_n)$. Remark 3.5 leads us to the following:

1. $|V(\Gamma_{I_n})| = n$. This can be easily verified because for every $i$, we can find another vertex $j$ such that there is a path of length three from $i$ to $j$ in $C_n$.

2. $\Gamma_{I_n}$ has one connected component if $n \neq 3k$, for all $k \geq 2$. Indeed, if $n = 1 \pmod{3}$, that is, $n = 3k + 1$ for some $k \geq 1$, then we have

$$E(\Gamma_{I_n}) = \{\{x_1, x_4\}, \{x_4, x_7\}, \ldots, \{x_{3k-2}, x_{3k+1}\}, \{x_{3k+1}, x_3\}, \{x_3, x_6\}, \ldots, \{x_{3k-3}, x_{3k}\}, (x_{3k}, x_2), \{x_2, x_5\}, \ldots, \{x_{3k-2}, x_1\}\}.$$ 

If $n = 2 \pmod{3}$, that is, $n = 3k + 2$ for some $k \geq 1$, then we have

$$E(\Gamma_{I_n}) = \{\{x_1, x_4\}, \{x_4, x_7\}, \ldots, \{x_{3k-2}, x_{3k+1}\}, \{x_{3k+1}, x_2\}, \{x_2, x_5\}, \ldots, \{x_{3k-1}, x_{3k+2}\}, \{x_{3k+2}, x_3\}, \{x_3, x_6\}, \ldots, \{x_{3k}, x_1\}\}.$$ 

3. $\Gamma_{I_n}$ has three connected components if $n = 3k$, for some $k \geq 2$. Set $V(\Gamma_1) = \{x_1, x_4, \ldots, x_{3k-2}\}$, and

$$E(\Gamma_1) = \{\{x_1, x_4\}, \{x_4, x_7\}, \ldots, \{x_{3k-2}, x_{3k}\}\}.$$ 

Set $V(\Gamma_2) = \{x_2, x_5, \ldots, x_{3k-1}\}$, and

$$E(\Gamma_1) = \{\{x_2, x_5\}, \{x_5, x_8\}, \ldots, \{x_{3k-1}, x_2\}\}.$$ 

Set $V(\Gamma_3) = \{x_3, x_6, \ldots, x_{3k}\}$, and

$$E(\Gamma_1) = \{\{x_3, x_6\}, \{x_6, x_9\}, \ldots, \{x_{3k}, x_3\}\}.$$
It can be easily verified that $\Gamma_{I_n}$ is the disjoint union of $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$.

Theorem 3.4 together with Remark 3.6 leads to the following corollary:

**Corollary 3.7.** Let $n \not\equiv 0 \pmod{3}$. Set $I_n = NI(C_n) \subset R = K[x_1, \ldots, x_n]$. Then $\text{depth}(R/I_n^{n-1}) = 0$. In particular, $m \in \text{Ass}(R/I_n^{n-1})$ and $\lim_{k \to \infty} \text{depth} R/I_n^k = 0$.

We provide the fourth main result of this paper in the subsequent theorem, which is related to the dominating ideals of cycles. We will use the following result to establish our proof.

**Corollary 3.8.** [16, Corollary 3.3] Let $I$ be a square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field $K$. Let $I(m \{x_i\})$ be normally torsion free for all $i = 1, \ldots, n$. Then $I$ is nearly normally torsion free.

Now, we state the next main result.

**Theorem 3.9.** The dominating ideals of cycles are nearly normally torsion free.

**Proof.** Let $C_n$ denote a cycle graph of order $n$ with $V(C_n) = \{x_1, \ldots, x_n\}$ and $E(C_n) = \{(x_i, x_{i+1}) : i = 1, \ldots, n-1\} \cup \{(x_n, x_1)\}$. In the light of [20, Lemma 2.2], the dominating ideal of $C_n$ is given by

$$DI(C_n) = \bigcap_{i=1}^{n} (x_i, x_{i+1}, x_{i+2}) \subset R = [x_1, \ldots, x_n],$$

where $x_{n+1}$ (respectively, $x_{n+2}$) represents $x_1$ (respectively, $x_2$). Set $I := DI(C_n)$. Our strategy is to use Corollary 3.8. To do this, we must show that $I(m \{x_i\})$ is normally torsion free for all $i = 1, \ldots, n$, where $m = (x_1, \ldots, x_n)$. Without loss of generality, it is sufficient for us to prove that $I(m \{x_i\})$ is normally torsion free. To simplify notation, set $F := \bigcap_{i=2}^{n-2} (x_i, x_{i+1}, x_{i+2})$. By virtue of Corollary 3.8, one has to show that the ideal $F = I(m \{x_i\})$ is normally torsion free. To do this, let $T = (V(T), E(T))$ be the rooted tree with the root 2, the vertex set $V(T) = \{x_2, \ldots, x_n\}$, and the edge set $E(T) = \{(x_i, x_{i+1}) : i = 2, \ldots, n-1\}$, where $(x_i, x_{i+1})$ denotes the directed edge from the vertex $x_i$ to the vertex $x_{i+1}$ for all $i = 2, \ldots, n-1$. It is not hard to check that $F$ is the Alexander dual of the path ideal generated by all paths of length 2 in the rooted tree $T$. Now, one can deduce from [11, Theorem 3.2] that $F = I(m \{x_i\})$ is normally torsion free. This completes the proof. \qed

**Acknowledgements**

We would like to thank Professor Adam Van Tuyl for his valuable comments in the preparation of Theorem 2.16. Moreover, Mehrdad Nasernejad was in part supported by a grant from IPM (No. 14001300118). Ayesha Asloob Qureshi and Asli Musapaşaoğlu were supported by The Scientific and Technological Research Council of Turkey - TÜBİTAK (Grant No: 118F169).

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
References

[1] Al-Ayyoub, I., Jaradat, I., Al-Zoubi, K.: On the normality of a class of monomial ideals via the Newton polyhedron. Mediterr. J. Math. 16(3), 16 (2019)

[2] Al-Ayyoub, I., Nasernejad, M., Roberts, L.: Normality of cover ideals of graphs and normality under some operations. Results Math. 74(4), 26 (2019)

[3] Andrei-Ciobanu, C.: Nearly normally torsionfree ideals. Combinatorial Structures in Algebra and Geometry, 1–13, NSA 26, Constanta, Romania, August 26-September 1, (2018)

[4] Francisco, C.A., Hà, H.T., Van Tuyl, A.: Colorings of hypergraphs, perfect graphs and associated primes of powers of monomial ideals. J. Algebra 331, 224–242 (2011)

[5] Gitler, I., Reyes, E., Villarreal, R.H.: Blowup algebras of ideals of vertex covers of bipartite graphs. Contemp. Math. 376, 273–279 (2005)

[6] Grayson, D.R., Stillman, M.E.: Macaulay2, a software system for research in algebraic geometry, http://www.math.uiuc.edu/Macaulay2/

[7] Haynes, T.W., Hedetniemi, S.T., Slater, P.J.: Fundamentals of Domination in Graphs. Marcel Dekker, New York (1998)

[8] Hà, H.T., Morey, S.: Embedded associated primes of powers of square-free monomial ideals. J. Pure Appl. Algebra 214, 301–308 (2010)

[9] Herzog, J., Hibi, T.: Monomial Ideals. Graduate Texts in Mathematics, vol. 260. Springer-Verlag (2011)

[10] Herzog, J., Qureshi, A.A.: Persistence and stability properties of powers of ideals. J. Pure Appl. Algebra 219, 530–542 (2015)

[11] Khashyarmanesh, K., Nasernejad, M.: A note on the Alexander dual of path ideals of rooted trees. Comm. Algebra 46, 283–289 (2018)

[12] Khashyarmanesh, K., Nasernejad, M.: Some results on the associated primes of monomial ideals. Southeast Asian Bull. Math. 39, 439–451 (2015)

[13] Nasernejad, M.: Persistence property for some classes of monomial ideals of a polynomial ring. J. Algebra Appl. 16(5), 17 (2017)

[14] Nasernejad, M., Khashyarmanesh, K., Al-Ayyoub, I.: Associated primes of powers of cover ideals under graph operations. Comm. Algebra 47(5), 1985–1996 (2019)

[15] Nasernejad, M., Khashyarmanesh, K., Roberts, L.G., Toledo, J.: The strong persistence property and symbolic strong persistence property. Czechoslovak Math. J., (2021), to appear

[16] Nasernejad, M., Qureshi, A.A., Khashyarmanesh, K., Roberts, L.G.: Classes of normally and nearly normally torsion-free monomial ideals. https://doi.org/10.1080/00927872.2022.2042546

[17] Rajaee, S., Nasernejad, M., Al-Ayyoub, I.: Superficial ideals for monomial ideals. J. Algebra Appl. 16(2), 28 (2018)

[18] Sayedsadeghi, M., Nasernejad, M.: Normally torsion-freeness of monomial ideals under monomial operators. Comm. Algebra 46(12), 5447–5459 (2018)

[19] Sayedsadeghi, M., Nasernejad, M., Qureshi, A.A.: On the embedded associated primes of monomial ideals. Rocky Mountain J. Math., (2021), to appear

[20] Sharifan, L., Moradi, S.: Closed neighborhood ideal of a graph. Rocky Mountain J. Math. 50(3), 1097–1107 (2020)
[21] Simis, A., Vasconcelos, W., Villarreal, R.: On the ideal theory of graphs. J. Algebra 167, 389–416 (1994)

[22] Sullivant, S.: Combinatorial symbolic powers. J. Algebra. 319(1), 115142 (2018)

[23] Villarreal, R.H.: Cohen-Macaulay graphs. Manuscripta Math. 66, 277–293 (1990)

[24] Villarreal, R.H.: Monomial Algebras. 2nd. Edition, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, (2015)

Mehrdad Nasernejad
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746
Tehran
Iran
e-mail: m_nasernejad@yahoo.com

Ayesha Asloob Qureshi and Aslı Musapaşaoğlu
Sabancı University, Faculty of Engineering and Natural Sciences, Orta Mahalle Tuzla 34956
Istanbul
Turkey
e-mail: aquareshi@sabanciuniv.edu

Ash Musapaşaoğlu
e-mail: atmusapasaoglu@sabanciuniv.edu

Somayeh Bandari
Department of Mathematics
Buein Zahra Technical University
Buein Zahra
Qazvin
Iran
e-mail: s.bandari@bzte.ac.ir

Received: July 9, 2021.
Revised: December 2, 2021.
Accepted: May 25, 2022.