A comparison of control strategies applied to a pricing problem in retail

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Abstract

When sales of a product are affected by randomness in demand, retailers can use dynamic pricing strategies to maximise their profits. In this article the pricing problem is formulated as a stochastic optimal control problem, where the optimal policy can be found by solving the associated Bellman equation. The aim is to investigate Approximate Dynamic Programming algorithms for this problem. For realistic retail applications, modelling the problem and solving it to optimality is intractable. Thus practitioners make simplifying assumptions and design suboptimal policies, but a thorough investigation of the relative performance of these policies is lacking.

To better understand such assumptions, we simulate the performance of two algorithms on a one-product system. It is found that for more than half of the realisations of the random disturbance, the often-used, but approximate, Certainty Equivalent Control policy yields larger profits than an optimal, maximum expected-value policy. This approximate algorithm, however, performs significantly worse in the remaining realisations, which colloquially can be interpreted as a more risk-seeking attitude by the retailer. Another policy, Open-Loop Feedback Control, is shown to work well as a compromise between the Certainty Equivalent Control and the optimal policy.

Subject areas: Applied mathematics, Revenue management

Keywords: Dynamic pricing, Stochastic control, Approximate dynamic programming

1 Introduction

The process of pricing products in order to control demand and maximise revenues has been undertaken for centuries. In recent decades, data- and model-driven approaches have become increasingly popular in order to advise on and automate the process for companies. There are several success stories from early adopters, for example in the airline industry. American Airlines estimated in 1992 that the introduction of revenue management software had, over the preceding three years, contributed $500 million of additional revenue per year, and would continue to do so in the future [1]. In an example from Chilean retail, the authors of [2] report an expected revenue improvement of 7%-16% from implementing model-driven strategies. For retailers with billions of pounds in revenues, small improvements to their revenue-management processes can be worth millions of pounds. In addition unsold items add up to thousands of tonnes of waste per year, so improving the control of demand for products is advantageous for both retailers and the environment.

We are interested in strategies to set the prices of products dynamically, and thus formulate the problem in a stochastic optimal control framework. Let $t$ denote equally spaced, discrete, dimensionless time points in $\{0, 1, \ldots, T\}$. Product stock levels $\hat{S} = (\hat{S}_t)_{t=0}^T$ are controlled by a pricing process $\hat{\alpha} = (\hat{\alpha}_t)_{t=0}^{T-1}$, and evolve according to a transition function $\hat{f}$ with random disturbance $\hat{W} = (\hat{W}_t)_{t=1}^T$,

$$\hat{S}_{t+1} = \hat{f}(t, \hat{S}_t, \hat{\alpha}_t, \hat{W}_{t+1}).$$

(1)
The goal is to find an $\hat{\alpha}$ which maximises the expected net revenues over a given time horizon $T$, 
\[
\max_{\hat{\alpha}} \mathbb{E}_{\hat{W}} \left[ \sum_{t=0}^{T} \hat{U}_t(S_t, \hat{\alpha}_t, \hat{W}_{t+1}) - \hat{U}(S_T) \right],
\]
where $\hat{U}_t$ denotes the revenue at time $t$, and $\hat{U}$ the cost of unsold stock. We use the subscript on $\mathbb{E}_{\hat{W}}$ to emphasise that the expectation is taken with respect to the random variables $\hat{W}_t$. The solution to this problem can be found by solving the associated Bellman equation, defined in (11).

In practical applications in retail and other domains, it is almost always intractable to solve the control problem to optimality [3]. The modelling of a real system and decision process can become very complicated, and we must create a very large state space in order to make use of the Bellman equation. We often have to take into account unobservable state variables, such as estimated parameters, and constraints that may depend on history. Decision-makers in business may change their mind about the objective over the course of the decision period, which should be incorporated in the modelling as parameters with corresponding, estimated probability distributions. From a software implementation perspective, writing code that can solve the Bellman equation efficiently can be much more complicated than other methods. Finally, the dimensions of the state, policy and exogenous information spaces, can quickly make a numerical solution to the Bellman equation intractable. An increase in the dimension can happen very quickly, even for one-product problems. In [4], the authors model a simple one-product demand function with uncertainty in the function parameters, and present an eight-dimensional dynamic programming solution to the problem. Much of the focus in the research community has therefore been on developing tractable algorithms with comparable average performance to the optimal policies, see for example [5] or [6] for an overview. One can either create explicit policy functions off-line, at the start of the decision process, or implicitly through an automated search for the best policy on-line for each decision point. An advantage of the pre-calculated, explicit policy functions is the speed at which we can make our decisions in the future. In the classical pricing problems, the dynamics of the underlying system do not require instant pricing decisions. Thus, suboptimal decision rules that are created as they are needed, are often used instead [7]. Estimation of the system and optimisation of prices are normally separated, and constraints are more easily updated at each decision point.

There are several proposed approximations in the literature, although for revenue management applications, much of the domain knowledge is kept within respective commercial organisations [7, Ch. 9]. Many of the suboptimal pricing algorithms are justified on practical grounds [8], or from asymptotic results [9]. The aim of this article is to highlight the implications that suboptimal policy choices have on the distribution of the relevant objective, and not only marginalised quantities such as the expected value. In particular, we consider a one-product pricing problem and investigate the optimal Bellman policy, an often-used suboptimal policy known as the Certainty Equivalent Control (CEC) policy, and a compromise between optimality and practicality known as the Open-Loop Feedback Control (OLFC) policy. Rather surprisingly, we find that, more than half the time, using the suboptimal CEC policy leads to a higher profit than using the Bellman policy, for a wide range of products. Of course, the Bellman policy performs better on average, but this is due to the CEC policy generating a larger, lower tail on the distribution of profits than the Bellman policy. Heuristically, one can say that the CEC policy has a higher risk associated to it. We wish to emphasise that this indicates that a choice of approximate control policy is implicitly a statement of risk attitude. Thus, we propose that it should be addressed at the same level as a description of a decision-maker’s risk attitude as expressed with utility functions or risk measures.

The article is organised as follows: In Section 2 we formulate the pricing problem mathematically, and describe an algorithm to solve the problem via the associated Bellman equation. We also give an example of a problem with one product, and present the optimal pricing rules. A discussion of the CEC and OLFC policies follows in Section 3 where we compare its performance to that of the optimal policy. Finally, we conclude and propose further work in Section 4.

2 The optimal control problem

We now consider a particular retail pricing problem. Given an initial amount of stock of a product and a future termination time $T$, the pricing problem is to set the price of the product dynamically, in order
to both maximise the revenue and minimise the cost of unsold stock at the termination time. In this section, we define the optimal control pricing problem, describe the optimality conditions given by the Bellman equation and solve it numerically for an example system.

Consider a system over discrete, equispaced, dimensionless time points \( t = 0, 1, \ldots, T \), with state \( \hat{S}_t \) and pricing process \( \hat{a} \) such that the \( \hat{a}_t \) that take values in a closed interval \( \hat{A} = [\hat{a}_{\min}, \hat{a}_{\max}] \) of prices. The state \( \hat{S}_t \) is the stock of a product at time \( t \), and \( \hat{a}_t \) the price for the product in the time period from \( t \) to \( t+1 \). We model the amount of product sold over each time period according to a forecast demand function \( \hat{q} : \hat{A} \to \mathbb{R}_+ \), which is bounded, continuous and decreasing. Exogenous influences on demand are considered as randomness in the system, and are modelled in a multiplicative and dimensionless fashion by a stochastic process \( \hat{W} = (W_1, \ldots, W_T) \) taking non-negative values. See, for example, [7, Ch. 7] for a discussion of demand models and the modelling of uncertainty.

For a given pricing process \( \hat{a} \), the system evolves from some initial state \( \hat{S}_0 > 0 \), according to the recursion

\[
\hat{S}_{t+1}^\hat{a} = \hat{S}_t^\hat{a} - \min(\hat{S}_t^\hat{a}, \hat{q}(\hat{a}_t)W_{t+1}), \quad t = 0, \ldots, T-1.
\]

The function \( \hat{q}(\hat{s}, \hat{a}, w) = \min(\hat{s}, \hat{q}(\hat{a})w) \) denotes the unit sales over a period at price \( \hat{a} \), starting with stock \( \hat{s} \), and with exogenous influences characterised by \( W \). The minimum operator is used ensure that the amount of product sold over the time period does not exceed the current stock level.

The revenue accrued over period \( t \to t+1 \) is \( \hat{a}_t \hat{q}(\hat{S}_t^\hat{a}, \hat{a}_t, W_{t+1}) \). The cost of remaining stock at time \( T \) is modelled by a cost per unit stock \( \hat{C} \geq 0 \). Let \( \hat{A} \) denote the set of feasible processes \( \hat{a} \) that take values in \( \hat{A} \). Define the value of having stock \( \hat{s} \) at time \( t \leq T \) by the value function \( \hat{v} \), such that

\[
\hat{v}(t, \hat{s}) = \max_{\hat{a} \in \hat{A}} \hat{J}(t, \hat{s}, \hat{a}), \quad \text{where}
\]

\[
\hat{J}(t, \hat{s}, \hat{a}) = \mathbb{E}_W \left[ \sum_{\tau=t}^{T-1} \hat{a}_\tau \hat{q}(\hat{S}_\tau^\hat{a}, \hat{a}_\tau, W_{\tau+1}) - \hat{C} \hat{S}_T^\hat{a} \mid \hat{S}_t^\hat{a} = \hat{s} \right].
\]

This leads us to the following mathematical formulation of the pricing problem:

**Definition 2.1.** Given an initial amount of stock \( \hat{S}_0 > 0 \) and a cost per unit unsold stock \( \hat{C} \geq 0 \), the **pricing problem** is to find a pricing process \( \hat{a}^* \in \hat{A} \) such that

\[
\hat{J}(0, \hat{S}_0, \hat{a}^*) = \hat{v}(0, \hat{S}_0),
\]

that is, \( \hat{a}^* = \arg \max_{\hat{a} \in \hat{A}} \hat{J}(0, \hat{S}_0, \hat{a}) \).

We choose to maximise the expected profit over the period, which assumes a risk-neutral decision-maker. It is still important, however, to understand the distribution of profits for a given pricing policy \( \hat{a} \). Therefore, we simulate the distribution when we investigate the performance of algorithms in Section 3. If we assume that the random variables \( W_t \) are independent, this stochastic optimal control problem can be solved by considering the optimality conditions that arise from the Dynamic Programming principle, also known as the Bellman equation.

### 2.1 Non-dimensionalisation of the system

We will now consider a non-dimensionalised representation of the system. The units of stock will be scaled with respect to the initial stock \( \hat{S}_0 \), and units of money will be scaled with respect to the upper bound on price, \( \hat{a}_{\max} \). Let the corresponding dimensionless quantities be defined without hats. Then we set

\[
s = \frac{\hat{s}}{\hat{S}_0}, \quad a = \frac{\hat{a}}{\hat{a}_{\max}}, \quad C = \frac{\hat{C}}{\hat{a}_{\max}}. \tag{7}
\]
The dimensionless functions $q(a)$ and $Q(s, a, w)$, for forecasted demand and realised sales respectively, are

$$q(a) = \frac{\hat{q}(a \cdot \hat{a}_{\text{max}})}{S_0}, \quad Q(s, a, w) = \min(s, q(a)w).$$  \hspace{1cm} (8)

The collection of pricing policies $A$ contains all the processes $\hat{\alpha}_t/\hat{a}_{\text{max}}$, where $\hat{\alpha} \in \hat{A}$. Now we can define the dimensionless value function

$$v(t, s) = \max_{\alpha \in A} J(t, s, \alpha), \quad \text{where}$$

$$J(t, s, \alpha) = \mathbb{E}_W \left[ \sum_{\tau=t}^{T-1} \alpha_\tau Q(S_\tau, \alpha_\tau, W_{\tau+1}) - CS_\tau \right| S_t = s].$$  \hspace{1cm} (10)

Finally, the dimensionless optimal control problem is to find $\alpha^* \in A$, such that $J(0, 1, \alpha^*) = v(0, 1)$. For the remainder of this article, we work with the non-dimensionalised system.

### 2.2 The Bellman equation

We choose $A$ to be the set of Markovian policies for the problem. This means that for each $\alpha \in A$, there exists a measurable function $a: [0, T) \times \mathbb{R}_+ \rightarrow A$, such that for each possible outcome $\omega$, the process $\alpha$ is given by $\alpha_t(\omega) = a(t, S_\alpha_t(\omega))$. Then, an approach to finding the value function above is to use the Dynamic Programming principle [10], which states that $v$ can be defined recursively by

$$v(t, s) = \max_{a \in A} \mathbb{E}_W \left[ aQ(s, a, W_{t+1}) + v(t+1, s - Q(s, a, W_{t+1})) \right].$$  \hspace{1cm} (11)

Thus, the value function is the solution to the backwards-in-time recursive relation (11) with terminal value $v(T, s) = -Cs$, and the optimal policy function $a(t, s)$ is given by the argmax for each $(t, s)$. The recursion (11) is called the Bellman equation, and a discussion of its validity can be found, for example, in [10]. Analytical solutions to Bellman equations are only available in very rare cases, and numerical approaches are normally needed to approximate the solutions.

We implement the following algorithm to solve the optimal control problem, using the Bellman equation:

1. Create a grid of equispaced points $0 = s_1 < s_2 < \cdots < s_K = S_0$, and arrays $v^K \in \mathbb{R}^{K \times (T+1)}$, $\alpha^K \in \mathbb{R}^{K \times T}$.

2. Set $I[v^K](s) = -Cs$.

3. Set $v^K[i, T] = I[v^K](s_i)$ for $i = 1, \ldots, K$.

4. For $t = T - 1, \ldots, 0$:
   
   (a) Set $v^K[i, t] = \max_{a \in A} \mathbb{E}_W \left[ aQ(s_i, a, W_{t+1}) + I[v^K](s_i - Q(s, a, W_{t+1})) \right]$ for $i = 1, \ldots, K$.
   
   (b) Set $\alpha^K[i, t]$ to the maximiser above.
   
   (c) Set $I[v^K](s) = \text{Interpolate}(s, (s_i)_i, (v^K[i, t])_i)$.

5. Return $v^K, \alpha^K$.

The expectation above is approximated using Monte–Carlo simulation, for which the variance of the approximation error decreases by one over the number of samples used [11]. In this article we use 1000 samples for the approximation of the expectation in step (iv)(a). We choose to use piecewise linear interpolation for $I[v^K](s)$ in step (iv)(c).
2.3 Example system

In order to investigate the optimal pricing of a specific system, we choose to look at a family of demand functions of the form \( q(a) = q_1 e^{-q_2 a} \), where \( q_1, q_2 > 0 \). For a discussion about their properties and usage in modelling demand, see \[7, Ch. 7\]. All the numerical experiments in this article use this family of exponential demand functions, with price constraints \( A = [0, 1] \). In the current section, and in Figures 1 to 4, we consider a particular choice of \( q_1, q_2 \) so that the demand function is given by \( q(a) = \frac{1}{3} e^{2 - 3a} \). In Figure 5 and Table 1 a larger range of values for \( q_1, q_2 \) are considered.

We assume the exogenous disturbance process is a sequence of shifted, independent and identically Beta-distributed random variables with mean 1 and variance \( \gamma^2 \). That is, we define \( W_t \sim 1 + X \), where \( X \sim \text{Beta}(\mu, \nu) \) and \( \mu = \nu = \frac{1}{87} - \frac{1}{2} \). To ensure that \( X \) is unimodal, we require that \( \gamma^2 < 1/12 \).

Set \( \gamma = 5 \times 10^{-2} \), \( C = 1 \) and \( T = 3 \). The numerical solution to the Bellman equation, and the corresponding optimal pricing policy, is shown in Figure 1. Let us now investigate the behaviour of the optimal pricing policy \( \alpha \) and the outcome of following this policy. Define a random variable \( P(\alpha) \), which for each realisation represents the total profit, is

\[
P(\alpha) = \sum_{t=0}^{T-1} \alpha_t Q(S_t^\alpha, \alpha_t, W_{t+1}) - CS_T^\alpha.\] (12)

By sampling from the stochastic process \( W = (W_1, \ldots, W_T) \), we can estimate the random variables \( \alpha_t \) and \( P(\alpha) \). The plots in Figure 2 show the results of simulating the system 10,000 times.

3 Suboptimal approximations

In this section we will look at two suboptimal policies among the class of algorithms that calculate decisions on-line. Whenever a decision must be made, these methods simulate the future behaviour of the system and optimise the current decision based on these simulations. We consider two special cases known as the Certainty Equivalent Control policy, which uses a point estimate of the system, and the Open-Loop Feedback Control policy, which uses more information about the future behaviour of the system [10, Ch. 6]. For the example system in this article, it turns out that we can find the CEC policy analytically. Numerical comparison experiments between these policies and the optimal Bellman

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4 A Beta(\( \mu, \nu \)) random variable has probability density function on \([0, 1]\), given by \( p(x) = \frac{x^{\mu-1}(1-x)^{\nu-1}}{B(\mu, \nu)} \). The function \( B(\mu, \nu) \) is the normalising factor, and we require that \( \mu, \nu > 1 \).

5 Preliminary investigations indicated that 1000 simulations is sufficient to obtain an accurate representation of the distributions in this article. However, we have chosen to use more when this is computationally convenient.
policy are made using the example pricing problem introduced in Subsection 3.3. We also extend the comparison to cover a much larger range of parameters in the problem, to show that the same results hold for a wide range of retail products.

3.1 The Certainty Equivalent Control policy

An often-used, tractable, algorithm for solving stochastic optimal control problems is the Certainty Equivalent Control (CEC) policy, as it is named in [10]. It is also known as (deterministic) Model Predictive Control in the engineering community. The algorithm is particularly practical, because the optimisation problem is reduced to a standard deterministic problem that can be solved with existing, commercial or open-source solvers which handle very large systems. At each decision time, a deterministic optimisation problem for the remaining decision horizon is solved. Only the decision for the current time step is used, whilst the subsequent decisions are discarded. For each decision time \( t = 0, 1, \ldots, T - 1 \), the CEC policy for the pricing problem calculates the current price using the following steps:

1. Observe the state \( s \).
2. Take a best estimate \( w_{t+1}, \ldots, w_T \) of \( (W_\tau)_0^T \).
3. Solve the optimisation problem
   \[
   \max_{a \in A^{T-t}} \left\{ \sum_{\tau=t}^{T-1} a_\tau Q(S^a_{\tau}, a_\tau, w_{\tau+1}) - C S^a_T \right\}, \quad \text{s.t.} \quad S^a_t = s, \tag{13}
   \]
   where \( A^{T-t} \) is the \( T-t \) times Cartesian product \( A \times \cdots \times A \).
4. Set the price corresponding to the element \( a_t \in A \) of a maximiser to (13).

For the one-product pricing problem, we can simplify the optimisation problem and in some cases obtain analytical solutions for the policy function. First, let us consider the expected value estimate \( w_\tau = \mathbb{E}[W_\tau] = 1 \) for \( \tau = t+1, \ldots, T \). Then we can rewrite the maximisation problem to find the certainty equivalent value function,

\[
\tilde{v}(t, s) = \max_{a \in A^{T-t}} \left\{ \sum_{\tau=t}^{T-1} (a_\tau + C) \min \left( s - \sum_{r=t}^{T-1} q(a_r), q(a_\tau) \right) - C s \right\}. \tag{14}
\]
The optimal choice here is to let \( a_\tau = a^* \in A \) for each \( \tau \), such that the same amount of stock is forecast to be sold in each period. Then, \( a^* = a^C(t, s) \) is given by the policy function

\[
a^C(t, s) = \arg \max_{a \in A} \left\{ (a + C) \min \left( \frac{s}{T - t}, q(a) \right) \right\}.
\]

(15)

Let \( P_A \) be the projection operator from \( \mathbb{R} \) onto the interval \( A \). Let us consider two families of demand functions that are popular in the literature, see for example [7, Ch. 7]. In the case when \( q(a) = q_1 e^{-q_2 a} \), the policy function is

\[
a^C(t, s) = P_A \left[ \max \left( \frac{1}{q_2} \log \left( \frac{q_1 (T - t)}{s} \right), \frac{1}{q_2} - C \right) \right].
\]

(16)

We start with investigating policy functions for a single combination of the system parameters, and then provide a comparison between the Bellman policy and CEC policy for a larger range of parameters in Subsection 3.2.

3.1.1 Example system

![Policy functions](image)

Figure 3: Comparison of the CEC and Bellman policy functions \( a^C \) and \( a^B \). The CEC function sets higher prices than the Bellman function for large \( s \).

The example system in Subsection 3.2 has a demand function of the form \( q(a) = q_1 e^{-q_2 a} \), where \( q_1 = e^2 / 3 \) and \( q_2 = 3 \). We can therefore compare the optimal Bellman policy with the pricing policy that a CEC algorithm would imply. Denote the Bellman policy function by \( a^B \), and consider \( a^B(t, s) - a^C(t, s) \) for each \( t = 0, \ldots, T - 1 \). The plot in Figure 3 shows this difference. Both of the policy functions reach the upper bound in \( A \) for small values of \( s \), but most of the time, the Bellman policy is pricing the products lower than the CEC policy.

What is more important than how the policy function works, is how it impacts the goal of the decision-process. Thus, we would like to see how the two policy processes \( a^B \) and \( a^C \) perform. One way to evaluate their performance is to look at the distribution of the profits \( P(a^B) \) and \( P(a^C) \). We remind the reader that \( P(a) = \sum_{t=0}^{T-1} \alpha_t Q(S_t^a, a_t, W_{t+1}) - CS_T^a \). From the optimality of the Bellman policy function, we must have that \( \mathbb{E}_W[P(a^B)] \geq \mathbb{E}_W[P(a^C)] \). Violations of this result can happen due to numerical errors in approximating \( a^B \) and the expectations. Marginalising a random variable with the expectation operator, however, loses a lot of information which can be of interest. An approximation of the distributions of \( P(a^B) \), \( P(a^C) \) and their difference \( P(a^B) - P(a^C) \), based on 10,000 realisations of the underlying \( W \), can be seen in Figure 4. Indeed in the experiment, the average value of following the Bellman policy is higher than following the CEC policy. However, from the bottom figure we see...
that in more than half of the cases, the CEC policy outperforms the optimal policy $\alpha^B$. What we can take from this experiment is that the CEC policy induces a more risk-seeking pricing strategy than $\alpha^B$: It results in slightly larger profits for a majority of the realisations of $W$, but at a cost of taking a more significant reduction in profits in the remaining realisations.

Figure 4: This shows the distributions from 10,000 samples of the profits of following the Bellman and CEC policies. The sample mean $\mathbb{E}_W[P(\alpha^B) - P(\alpha^C)] \approx 3.8 \times 10^{-3}$, confirms that Bellman is better on average, as it should be. Importantly, however, in more than half the samples the suboptimal policy outperforms the Bellman policy. The distribution of $P(\alpha^B) - P(\alpha^C)$ appears to be bimodal.

3.2 Bellman and CEC parameter comparisons

In the previous sections, our numerical experiments have only shown results for a fixed combination of the five parameters termination time $T$, unsold items cost $C$, uncertainty $\gamma$, and demand function parameters $q_1$ and $q_2$. In this section we explore the differences between the Bellman policy and the CEC policy for a larger parameter range. The formula for $a^C$ in (16) indicates that the initial price is largely determined by the relationship between $Tq_1$ and $q_2$, and hence we choose to keep $T = 3$ fixed whilst varying $q_1$ and $q_2$. We are interested in the difference between the profit following a Bellman policy $\alpha^B$ and a CEC policy $\alpha^C$. The policy $\alpha^B$ is computed numerically, and $\alpha^C$ is obtained using the function $a^C$ from the formula in Equation (16).

In particular, the difference between the two policies is measured using the $L^2$-norm with respect to the probability distribution induced by the disturbance ($W_1, W_2, \ldots, W_T$), that is

$$\|P(\alpha^B) - P(\alpha^C)\|_2 = \sqrt{\mathbb{E}_W \left[ (P(\alpha^B) - P(\alpha^C))^2 \right]}.$$  

(17)
To reduce the number of combinations of parameters, we choose only four combinations of \((C, \gamma) \in \{(0.5, 0.05), (0.5, 0.1), (1, 0.05), (1, 0.1)\}\). Then, for each combination of \((C, \gamma)\), we can create a heatmap of the difference between the two policy functions by varying the parameters \(q_1, q_2\). The relative \(L^2\) distance between the CEC and Bellman policy outcomes is shown in Figure 5. The norms were approximated with 1000 samples from \((W_1, W_2, \ldots, W_T)\). Comparing the left and right column, we see that the relative difference doubles as the standard deviation \(\gamma\) is doubled. The cost \(C\) plays a large role, both in terms of the shape of the difference surface and its magnitude. The black region at the bottom right of the plots corresponds to popular, low-elasticity products where both policies suggest to sell at the maximum price \(a = 1\).

The parameters used to generate Figure 4 were \(C = 1, \gamma = 0.05, q_1 = e^2/3 \approx 2.5,\) and \(q_2 = 3\). This corresponds to a point in the region where the relative difference is around 0.016 — see the white dot in the bottom left frame in Figure 5. This difference is in the middle of that seen for all the combinations of parameters, so the conclusions made in the article can be considered as relevant for a wider range of systems. Table 1 provides further data to underscore the claim that the CEC policy outperforms the Bellman policy for the majority of events, at the expense of a stronger underperformance for the remainder of the events. Notably, the CEC policy is better more than 50% of the time for all the parameter combinations considered in this article. We can also see from the table that the implicit risk-seeking attitude of the CEC policy increases with the relative \(L^2\) distance, as its profit distribution widens compared to the Bellman policy. In particular, the frequency at which the CEC policy outperforms the Bellman policy increases, at the expense of a larger underperformance, or tail loss, in the remaining realisations. The values in Table 1 were approximated using 10,000 samples from \(W\), and their corresponding parameter combinations are shown as grey dots in Figure 5.

| \(C\)  | \(\gamma\) | \(q_1\) | \(q_2\) | \(Q_{0.05}\) | Median | \(Q_{0.95}\) | \(L^2\) |
|-------|-------|-------|-------|----------|-------|----------|-------|
| 0.25  | 0.05  | 2.0   | 4.0   | -0.4     | -0.3  | 0.6      | 0.4   |
| 0.25  | 0.1   | 1.33  | 2.67  | -0.5     | -0.0  | 0.6      | 0.3   |
| 0.5   | 0.05  | 2.67  | 4.0   | -0.6     | -0.6  | 1.9      | 1.1   |
| 0.5   | 0.1   | 2.0   | 2.67  | -0.9     | -0.6  | 1.9      | 1.2   |
| 1.0   | 0.05  | 1.33  | 4.0   | -1.2     | -1.1  | 5.3      | 2.7   |
| 1.0   | 0.1   | 2.67  | 2.67  | -1.5     | -1.3  | 5.8      | 2.9   |

\(\times 10^{-2}\) \(\times 10^{-2}\) \(\times 10^{-2}\) \(\times 10^{-2}\)

Table 1: Statistics comparing profits \(P(\alpha^B)\) and \(P(\alpha^C)\) for six different parameter combinations. The columns \(Q_s\) represent the the \(s^{th}\) quantile of the relative difference \(1 - P(\alpha^C)/P(\alpha^B)\). The values in the column \(L^2\) refers to the relative \(L^2\) difference \(\|P(\alpha^B) - P(\alpha^C)\|_2/\|P(\alpha^B)\|_2\) from Figure 5. Each of the parameter combinations correspond to a grey dot in Figure 5.

3.3 Open-Loop Feedback Control policy

We conclude this article with the example of the Open-Loop Feedback Control (OLFC) policy [10, Ch. 6], also known as Stochastic Model Predictive Control [3]. This is another suboptimal policy, similar to the CEC policy, but which better takes into account the uncertainty in the system. The OLFC policy works as follows: At each decision time, a stochastic optimisation problem for the remaining decision horizon is solved, based on the most recent quantification of the uncertainty in the system. Only the decision for the current time step is used, whilst the subsequent decisions are discarded. For each decision time \(t = 0, 1, \ldots, T - 1\), the OLFC policy for the pricing problem calculates the current price using the following steps:

1. Observe the state \(s\).

2. Solve the optimisation problem

\[
\max_{a_t \in A^{s_t}} E_W \left[ \sum_{\tau=t}^{T-1} a_\tau Q(S_\tau, a_\tau, w_{\tau+1}) - CS_\tau^s \right].
\]

3. Set the price corresponding to the element \(a_t \in A\) of a maximiser to \((18)\).
Figure 5: The relative $L^2$ difference $\|P(\alpha^B) - P(\alpha^C)\|_2 / \|P(\alpha^B)\|_2$ representing how different the CEC policy is from the optimal Bellman policy. Each plot defines a combination of $(C, \gamma)$, whilst the axes vary the parameters $q_1, q_2$ of the demand function $q(a) = q_1 e^{-q_2 a}$. The white dot on the bottom, left plot corresponds to the parameters chosen for the experiments in this article. The gray dots correspond to the parameters in Table 1.
For this article, we approximate the expectation operator with Monte–Carlo using 1000 samples, in the same way that the expectation in the Bellman policy is computed.

As was done for the CEC policy, we compare the performance of following the OLFC policy $\alpha^O$ with the performance of following the Bellman policy $\alpha^B$. With the parameters from the example system in Subsection 22.3, the empirical distribution of $P(\alpha^B) - P(\alpha^O)$ is shown in Figure 6. The empirical distribution is generated using 10,000 samples of $W$. There are two notable differences in this distribution from the distribution of $P(\alpha^B) - P(\alpha^C)$ shown in Figure 4. First, the distribution appears to be unimodal and more concentrated around zero. Second, the values are an order of magnitude smaller. This supports the claim that the OLFC policy better approximates the Bellman policy.

![Simulations of Bellman and OLFC policies](image)

Figure 6: This shows the distributions from 10,000 samples of the profits of following the Bellman and OLFC policies. The OLFC policy generates profits an order of magnitude closer to the Bellman policy than the CEC policy (Figure 4). This distribution appears to be unimodal.

For completeness, Table 2 includes the statistics of the relative difference $1 - P(\alpha^O)/P(\alpha^B)$ for the same parameters that were used to compare the CEC and Bellman policies in Table 1. The distributions of the relative difference indicate a more symmetric distribution around the median, with values of an order of magnitude smaller than in the CEC comparison.

| $C$ | $\gamma$ | $q_1$ | $q_2$ | $Q_{0.05}$ | Median | $Q_{0.95}$ | $L^2$ |
|-----|---------|-------|-------|------------|---------|------------|-------|
| 0.25 | 0.05    | 2.0   | 4.0   | -0.6       | 0.1     | 1.2        | 0.6   |
| 0.25 | 0.1     | 1.33  | 2.67  | -2.5       | 0.4     | 4.2        | 2.1   |
| 0.5  | 0.05    | 2.67  | 4.0   | -0.5       | 0.1     | 1.2        | 0.6   |
| 0.5  | 0.1     | 2.0   | 2.67  | -2.3       | 0.3     | 4.6        | 2.1   |
| 1.0  | 0.05    | 1.33  | 4.0   | -0.8       | 0.0     | 2.8        | 1.1   |
| 1.0  | 0.1     | 2.67  | 2.67  | -2.2       | 0.4     | 5.8        | 2.4   |

Table 2: Statistics comparing the relative profits of $P(\alpha^O)$ against $P(\alpha^B)$ for six different parameter combinations. Note that the values are an order of magnitude smaller than in the comparison between Bellman and CEC shown in Table 1.

Note that the CEC policy is a special case of the OLFC policy, where the expectation is approximated with a zeroth-order expansion around the estimate $w_{t+1}, \ldots, w_T$. The OLFC policy is thus much more costly than the CEC policy, but will also better approximate the optimal Bellman policy. In practice, the decision maker can balance the cost-versus-optimality by how accurately they approximate the expectation.
4 Conclusion

In this article we have looked at a mathematical formulation of a retail pricing problem for profit maximisation, and have investigated the performance of two algorithms that balance practicality with degree of suboptimality. The motivation is to better understand how well the suboptimal policies approximate the returns from the optimal policy. Pricing problems are often formulated as an expected value maximisation, but different algorithms may induce different distributions of the profits. Even though the expected values of suboptimal policies are not better than the optimal policy, they may have a higher profit for many realisations of the underlying probability distribution. We found in Section 3 that there are a large number of reasonable system parameters for which the suboptimal Certainty Equivalent Control policy resulted in a higher profit than the optimal policy in more than half of the realisations. In the remaining realisations, however, the CEC policy resulted in much smaller profits. We interpret these results as an indication that the suboptimal policy is more risk-seeking than the optimal policy, in a colloquial sense of the term. The results in this article underscore the importance of looking at the impact of different suboptimal algorithms have on the distribution of an objective, and not only the impact of marginalised statistics such as the expected value.

The model problem in this article is fairly simple, and we propose two specific lines for future research that take this analysis closer to practical models. First, to investigate multi-product problems where the demand and availability of one product depends on the other products. Second, to introduce a state dependence in the uncertainty in the system, that is, to allow \( W_{t+1} \) to depend explicitly on the values of \( S_t \) and \( \alpha_t \). We hope that these two extensions will lead to a better understanding of whether the effects seen in this article will be stronger or diluted in real-life problems.

Authors’ contributions. JND and CLF contributed to specifying the problems and choosing the types of algorithms to study. ANR devised the method for comparing algorithms, performed all the calculations, and drafted the first version of the paper. All authors were responsible for the final preparation of the manuscript.

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