Stability of estimates for fundamental solutions under Feynman-Kac perturbations for symmetric Markov processes

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Abstract

In this paper, when a given symmetric Markov process $X$ satisfies the stability of global heat kernel two-sided (upper) estimates by Markov perturbations (See Definition 1.2), we give a necessary and sufficient condition on the stability of global two-sided (upper) estimates for fundamental solution of Feynman-Kac semigroup of $X$. As a corollary, under the same assumptions, a weak type global two-sided (upper) estimates holds for the fundamental solution of Feynman-Kac semigroup with (extended) Kato class conditions for measures. This generalizes all known results on the stability of global integral kernel estimates by symmetric Feynman-Kac perturbations with Kato class conditions in the framework of symmetric Markov processes.

Keywords: Feynman–Kac perturbation, symmetric Markov processes, Dirichlet forms, heat kernel, spectral function, continuous additive functional of zero energy, Kato class, Green-tight measures, conditionally Green-tight measures

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1 Introduction and results

Let $(E, d)$ be a locally compact separable metric space and $m$ a Radon measure on $E$ with full topological support. Denote by $E_\partial$ the one point compactification of $E$. Let $X = (\Omega, F_\infty, F_t, X_t, \zeta, P_x, x \in E_\partial, t \in [0, +\infty[)$ be an $m$-symmetric Markov process on $E_\partial$ with the life time $\zeta := \inf\{t > 0 \mid X_t = \partial\}$. Let $(\mathcal{E}, \mathcal{F})$ be a symmetric regular Dirichlet form on $L^2(E; m)$ associated to $X$.

Throughout the paper, we assume that $(\mathcal{E}, \mathcal{F})$ is irreducible ((I) in short), that is, any $(T_t)$-invariant set is $m$-trivial (see [17, §1.6]), $X$ has no killing inside $E$, that is, $P_t(x, \zeta < \infty) = 0$ for any $x \in E$, and that $X$ satisfies the absolute continuity condition ((AC) in short), that is, $P_t(x, dy) := P_{x}(X_t \in dy) \ll m(dy)$ for any $t > 0$ and $x \in E$.

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Let \( p_t(x,y) \) be a function of \((t,x,y) \in [0, +\infty [ \times E \times E\). Throughout this paper, we say that \( p_t(x,y) \) is a heat kernel of the form \((\mathcal{E}, \mathcal{F})\) if it satisfies the following (\cite{23} Section 3.3)):

1. for any \( t > 0 \), \( p_t(x,y) \) is \( \mathfrak{m} \times \mathfrak{m} \)-measurable in \((x,y) \in E \times E\);
2. for any \( t > 0 \), \( p_t(x,y) \geq 0 \) for \( \mathfrak{m} \)-a.e. \( x, y \in E \), and \( f \in \mathcal{B}(E) \)

\[
\int_E p_t(x,y)f(y)d\mathfrak{m}(y) = \int_E f(y)\mathbf{P}_x(X_t \in dy) =: \mathbf{E}_x[f(X_t)],
\]

for \( \mathfrak{m} \)-a.e. \( x \in E \);
3. for any \( t > 0 \), \( p_t(x,y) = p_t(y,x) \) \( \mathfrak{m} \)-a.e. \( x, y \in E \);
4. for all \( t, s > 0 \),

\[
p_{t+s}(x,y) = \int_E p_t(x,z)p_s(z,y)d\mathfrak{m}(z)
\]

for \( \mathfrak{m} \)-a.e. \( x, y \in E \).

Moreover, we say that \( p_t(x,y) \) is a heat kernel in the strict sense associated with the form \((\mathcal{E}, \mathcal{F})\) if it is Borel measurable in \((t,x,y) \in [0, +\infty [ \times E \times E\) and (2)–(4) above hold for all \( x, y \in E \). Under \((\mathbf{AC})\), \( X \) admits a heat kernel \( p_t(x,y) \) in the strict sense (see \cite{37}, (2.1), (2.2)), \cite{49} Theorem 2).

Let \((\mathcal{E}, \mathcal{F}_e)\) be the extended Dirichlet space of \((\mathcal{E}, \mathcal{F})\). Note that any element \( f \in \mathcal{F}_e \) admits a strictly \( \mathcal{E} \)-quasi continuous version \( \tilde{f} \) with \( \tilde{f}(\partial) = 0 \) (\cite{33} Lemma 2.1). Throughout this paper, we always take a strictly \( \mathcal{E} \)-quasi continuous version of the element of \( \mathcal{F}_e \), that is, we omit tildes from \( \tilde{f} \) for \( f \in \mathcal{F}_e \). Since \( X \) has no killing inside \( E \), by the Beurling-Deny formula (\cite{17} Theorem 3.2.1)), \( \mathcal{E} \) can be decomposed as follows: for \( f, g \in \mathcal{F} \),

\[
\mathcal{E}(f,g) = \mathcal{E}^{(c)}(f,g) + \mathcal{E}^{(j)}(f,g)
\]

\[
:= \mathcal{E}^{(c)}(f,g) + \int_{E \times E \setminus \text{diag}} (f(x) - f(y))(g(x) - g(y))J(dx dy)
\]

where \( \mathcal{E}^{(c)} \) is the strongly local part of \((\mathcal{E}, \mathcal{F})\) and \( \text{diag} := \{(x,x) \mid x \in E\} \) is the diagonal set of \( E \times E \).

This contains the particular case that \( \mathcal{E}^{(c)} = 0 \), i.e., \( X \) is a pure jump process, or that \( J = 0 \), i.e., \( X \) is a diffusion process.

In what follows, we consider \( k \geq 0 \) and a subset \( T \) of \([0, +\infty[^2 := [0, +\infty [ \times [0, +\infty [\). For measurable functions \( f \) on \([0, +\infty [ \times E \) and \( g \) on \([0, +\infty [^2\), we consider the function \( \Psi \) on \([0, +\infty [ \times E \times E \) by

\[
\Psi_t(x,y) := f(t,x)g(t,d(x,y)).
\]

(1.1)

Let \( \Phi \) be another measurable function defined on \([0, +\infty [ \times E \times E \). We then write \( \Phi_t(x,y) \geq_k \Psi_t(x,y) \) (resp. \( \Phi_t(x,y) \geq_k \Psi_t(x,y) \)) for \( \mathfrak{m} \)-a.e. \( x, y \) with \( d(x,y) \in T_t \) if there exist \( c_2, C_2 > 0 \) (resp. \( c_1, C_1 > 0 \)) independent of \( t \geq 0 \) such that

\[
\Phi_t(x,y) \leq C_2 e^{kt}f(t,x)g(t,c_2 d(x,y))
\]

(resp. \( C_1 e^{-kt}f(t,x)g(t,c_1 d(x,y)) \leq \Phi_t(x,y) \))

for \( \mathfrak{m} \)-a.e. \( x, y \in E \) with \( d(x,y) \in T_t \). Here \( T_t := \{s \in [0, +\infty [ \mid (t,s) \in T\} \) is the section of \( T \) at \( t \in [0, +\infty [ \). Moreover, we write \( \Phi_t(x,y) \asymp_k \Psi_t(x,y) \) for \( \mathfrak{m} \)-a.e. \( d(x,y) \in T \) if \( \Phi_t(x,y) \asymp_k \Psi_t(x,y) \) for \( \mathfrak{m} \)-a.e. \( d(x,y) \in T \) and \( \Phi_t(x,y) \asymp_k \Psi_t(x,y) \) for \( \mathfrak{m} \)-a.e. \( d(x,y) \in T \).
Similarly, \( \Phi_t(x,y) \asymp_k \Psi_t(x,y) \) for \( d(x,y) \in \mathbb{T}_t \) can be defined. If we omit "m-a.e.", it means that each statement holds for all \( x, y \in E \) with \( d(x,y) \in \mathbb{T}_t \).

When \( k = 0 \), we simply write \( \Phi_t(x,y) \asymp \Psi_t(x,y) \), \( \Phi_t(x,y) \asymp \Psi_t(x,y) \) and \( \Phi_t(x,y) \asymp \Psi_t(x,y) \) instead of \( \Phi_t(x,y) \asymp_0 \Psi_t(x,y) \), \( \Phi_t(x,y) \asymp_0 \Psi_t(x,y) \) and \( \Phi_t(x,y) \asymp_0 \Psi_t(x,y) \), respectively.

For measurable functions \( G \) on \( E \times E \) and \( H \) on \([0, +\infty[\), we write \( G(x,y) \lesssim H(d(x,y)) \) (resp. \( H(d(x,y)) \lesssim G(x,y) \)) \( m \)-a.e. \( x, y \in E \) if there exist \( c_2, C \) such that

\[
G(x,y) \leq C_2 H(c_2 d(x,y)) \quad (\text{resp. } C_1 H(c_1 d(x,y)) \leq G(x,y)) \quad m \text{-a.e. } x, y \in E.
\]

We write \( G(x,y) \asymp H(d(x,y)) \) \( m \)-a.e. \( x, y \in E \) if \( G(x,y) \lesssim H(d(x,y)) \) \( m \)-a.e. \( x, y \in E \) and \( G(x,y) \gtrsim H(d(x,y)) \) \( m \)-a.e. \( x, y \in E \).

Let \( \phi, \psi \) be measurable functions defined on \([0, +\infty[ \times E \times E\). We then write \( \phi_t(x,y) \lesssim_k \psi_t(x,y) \) (resp. \( \phi_t(x,y) \gtrsim_k \psi_t(x,y) \)) if there exists \( C \) such that \( \phi_t(x,y) \leq C \psi_t(x,y) \) for all \( t > 0 \) independent of \( x, y \in E \).

We write \( \phi_t(x,y) \asymp_k \psi_t(x,y) \) if both \( \phi_t(x,y) \lesssim_k \psi_t(x,y) \) and \( \phi_t(x,y) \gtrsim_k \psi_t(x,y) \) hold. When \( k = 0 \), we simply write \( \phi_t(x,y) \lesssim \psi_t(x,y) \) (resp. \( \phi_t(x,y) \gtrsim \psi_t(x,y) \)) instead of \( \phi_t(x,y) \lesssim_0 \psi_t(x,y) \) (resp. \( \phi_t(x,y) \gtrsim_0 \psi_t(x,y) \)). The reader should be aware of the difference between \( \lesssim_0 \) and \( \lesssim_k \).

We will consider the following assumptions on the stability of the two-sided (upper) estimates for heat kernel \( p_t(x,y) \) of \( X \). For this, let \( (\mathcal{E}, \mathcal{F}) \) be another symmetric Dirichlet form on \( L^2(E; \tilde{m}) \) with the same domain \( \mathcal{F} \). We prepare the following conditions: There exists a constant \( C_E > 0 \) such that

\[
C_E^{-1} \mathbb{I}_{\tilde{m}} \leq \tilde{m} \leq C_E \mathbb{I}_{\tilde{m}},
\]

and, there exist measurable functions \( J \) and \( \tilde{J} \) on \( E \times E \) such that \( J(dxdy) = J(x,y)\mathbb{I}_{\tilde{m}}(dx)\mathbb{I}_{\tilde{m}}(dy) \) and \( \tilde{J}(dxdy) = \tilde{J}(x,y)\mathbb{I}_{\tilde{m}}(dx)\mathbb{I}_{\tilde{m}}(dy) \) satisfying

\[
C_E^{-1} J(x,y) \leq \tilde{J}(x,y) \leq C_E J(x,y) \quad m \text{-a.e. } x, y \in E. \quad (1.4)
\]

We further remark the following:

- \([13]\) is equivalent to \( C_E^{-1} \mu_c^{(f)} \leq \tilde{\mu}_c^{(f)} \leq C_E \mu_c^{(f)} \) for \( f \in \mathcal{F} \) where \( \mu_c^{(f)} \) (resp. \( \tilde{\mu}_c^{(f)} \)) is the energy measure of continuous part for \( f \in \mathcal{F} \) with respect to \( (\mathcal{E}, \mathcal{F}) \) (resp. \( (\tilde{\mathcal{E}}, \tilde{\mathcal{F}}) \)) (see \([35]\) Proposition 1.5.5(b)], \([38]\)).

- \([14]\) implies that

\[
J(x,y) \asymp J^0(d(x,y)) \quad m \text{-a.e. } x, y \in E \implies \tilde{J}(x,y) \asymp J^0(d(x,y)) \quad m \text{-a.e. } x, y \in E. \quad (1.5)
\]

In fact, all results in this paper hold true under \([15]\) instead of \([14]\).

Throughout this paper, we assume the existence of densities \( J \) and \( \tilde{J} \). Further, we make the following assumptions.
(A.1) (with $T$) Suppose that the heat kernel $p_t(x,y)$ on $]0, +\infty[ \times E \times E$ in the strict sense associated with $(\mathcal{E}, \mathcal{F})$ satisfies the following: there exists a function $\phi_2$ of the type (1.1) whose second factor defined on a subset $T$ of $[0, +\infty[^2$ such that

$$p_t(x,y) \preceq \phi_2(t,x,y) \quad \text{m-a.e.} \quad d(x,y) \in T_t.$$  

(1.6)

Then any Dirichlet form $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ satisfying (1.2), (1.3) and (1.4), also admits a heat kernel $\bar{p}_t(x,y)$ on $]0, +\infty[ \times E \times E$ such that

$$\bar{p}_t(x,y) \preceq \phi_2(t,x,y) \quad \text{m-a.e.} \quad d(x,y) \in T_t.$$  

(1.7)

(A.2) (with $T$) Suppose that the heat kernel $p_t(x,y)$ on $]0, +\infty[ \times E \times E$ in the strict sense associated with $(\mathcal{E}, \mathcal{F})$ satisfies the following: there exist functions $\phi_1, \phi_2$ of the type (1.1) whose second factor defined on a subset $T$ of $[0, +\infty[^2$ such that

$$\phi_1(t,x,y) \preceq p_t(x,y) \preceq \phi_2(t,x,y) \quad \text{for} \quad d(x,y) \in T_t.$$  

(1.8)

Then any Dirichlet form $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ satistyng (1.2), (1.3) and (1.4) also admits a heat kernel $\bar{p}_t(x,y)$ on $]0, +\infty[ \times E \times E$ in the strict sense such that

$$\phi_1(t,x,y) \preceq \bar{p}_t(x,y) \preceq \phi_2(t,x,y) \quad \text{for} \quad d(x,y) \in T_t.$$  

(1.9)

**Remark 1.1** For example, if $T = \{(t,s) \mid s \leq t\}$ (resp. $T = \{(t,s) \mid s \geq t\}$), then $T_t = [0, t]$ (resp. $T_t = [t, +\infty[\}$. Note here that $T_t = [0, +\infty[\}$ for any $t \in [0, +\infty[\}$ provided $T = [0, +\infty[^2$, in this case, $d(x,y) \in T_t$ automatically holds.

**Definition 1.2** We say that the stability of heat kernel estimate (resp. heat kernel upper estimate) of $(\mathcal{E}, \mathcal{F})$ (or the associated process $X$) globally holds by Markov perturbations provided (A.2) (resp. (A.1)) is satisfied with $T = [0, +\infty[^2$.

(A.1) and (A.2) are satisfied for quite wide class of symmetric Markov processes. For instance, diffusion processes on smooth complete Riemannian manifold with non-negative Ricci curvature, diffusion processes on fractals, symmetric stable-like processes and so on (see Section 3). Both (A.1) and (A.2) are stable under a Girsanov transformation (see after Theorem 3.1 below). This property plays a key role in the proof of main results.

Let $\mu_1$ (resp. $\mu_2$) be a smooth measure in the strict sense corresponding to a positive continuous additive functional (PCAF in abbreviation) $A_{\mu_1}$ (resp. $A_{\mu_2}$) in the strict sense with respect to $X$ and let $\mu := \mu_1 - \mu_2$, which is a signed smooth measure in the strict sense. We denote the corresponding continuous additive functional of $\mu = \mu_1 - \mu_2$ by $A^\mu := A_{\mu_1} - A_{\mu_2}$ to emphasize the correspondence between $\mu$ and $A$. Let $F_i$ ($i = 1, 2$) be a non-negative bounded function on $E \times E_0$ symmetric on $E \times E$ which is extended to a function defined on $E_0 \times E_0$ vanishing on the diagonal set $\text{diag}$ of $E_0 \times E_0$. We set $F := F_1 - F_2$. Then $A^F_i := \sum_{0 < s \leq t} F(X_s, X_s)$ (whenever it is summable) is an additive functional of $X$. Let $\mathcal{F}_{\text{loc}}$ be the space of functions locally in $\mathcal{F}$ in the ordinary sense, $QC(E_0)$ the space of strictly $\mathcal{E}$-quasi continuous functions on $E_0$ (See Section 2 for the definition) and $S^1_{\text{D}}(X)$ the class of smooth measures of Dynkin class with respect to $X$ (see §2 for definitions).
We consider a bounded finely continuous (nearly) Borel function \( u \in \mathcal{R}_{\text{loc}} \cap QC(E_0) \) satisfying \( \mu_{(u)} \in S^+_0(X) \). In [31 Theorem 6.2(2)], we proved that the additive functional \( u(X_t) - u(X_0) \) admits the following strict decomposition: for all \( t \in [0, +\infty) \)

\[
   u(X_t) - u(X_0) = M^u_t + N^u_t \quad \text{P}_x\text{-a.s. for all } x \in E,
\]

where \( M^u \) is a square integrable martingale additive functional in the strict sense, and \( N^u \) is a continuous additive functional (CAF in abbreviation) in the strict sense which is locally of zero energy. We note that \( N^u \) is not a process of finite variation in general. \( M^u \) can be decomposed as

\[
   M^u_t = M^{u,c}_t + M^{u,j}_t \quad (1.11)
\]

where \( M^{u,c}_t \) and \( M^{u,j}_t \) are the jumping and continuous part of \( M^u \) respectively. Those are defined \( \text{P}_x\text{-a.s. for all } x \in E \) by [31 Theorem 6.2(2)]. The strict decompositions (1.10) and (1.11) on \([0, +\infty[\) guarantee the extension of the supermartingale multiplicative functional on \([0, \zeta]\) up to \([0, +\infty[\) (see [29 Proposition 3.1]). Let \( \mu_{(u)}^c, \mu_{(u)}^j \) and \( \mu_{(u)}^l \) be the smooth Revuz measures in the strict sense associated with the quadratic variational processes (or the sharp bracket PCAFs in the strict sense) \( \langle M^u \rangle, \langle M^{u,c} \rangle \) and \( \langle M^{u,j} \rangle \) respectively. Then

\[
   \mu_{(u)}(dx) = \mu_{(u)}^c(dx) + \mu_{(u)}^j(dx)
\]

We now consider the transforms by the additive functionals \( A_t := N^u_t + A^u_t + A^F_t \) of the form

\[
   e_{A}(t) := \exp(A_t), \quad t \geq 0. \quad (1.12)
\]

The transform (1.12) defines a semigroup, namely, the generalized Feynman-Kac semigroup

\[
   P^Af(x) := \mathbb{E}_x[e_{A}(t)f(X_t)], \quad f \in \mathcal{B}_b(E), \quad t \geq 0. \quad (1.13)
\]

Under some conditions, \( (P^A_t)_{t \geq 0} \) forms a strongly continuous semigroup on \( L^2(E; m) \) (see Lemma 3.2 below). Since \( N^u_t \) is time reversible, and \( F_1 \) and \( F_2 \) are symmetric on \( E \times E \), \( (P^A_t)_{t > 0} \) is symmetric in the sense that

\[
   (P^A_t f, g)_m = (f, P^A_t g)_m \quad \text{for } f, g \in \mathcal{B}_+(E)
\]

where \( (f, g)_m := \int_E f(x)g(y)m(dx) \). We say that \( (P^A_t)_{t > 0} \) admits an integral kernel \( p^A_t(x, y) \) on \( [0, +\infty[ \times E \times E \) if it satisfies the following:

1. for any \( t > 0 \), \( p^A_t(x, y) \) is \( m \times m \)-measurable in \( (x, y) \in E \times E \);
2. for any \( t > 0 \), \( p^A_t(x, y) \geq 0 \) for \( m\)-a.e. \( x, y \in E \), and for \( f \in \mathcal{B}_b(E) \)

\[
   \int_E p^A_t(x, y)f(y)m(dy) = \mathbb{E}_x[e_{A}(t)f(X_t)],
\]

for \( m\)-a.e. \( x \in E \);
3. for any \( t > 0 \), \( p^A_t(x, y) = p^A_t(y, x) \) \( m\)-a.e. \( x, y \in E \);
4. for all \( t, s > 0 \),

\[
   p^A_{t+s}(x, y) = \int_E p^A_t(x, z)p^A_s(z, y)m(dz)
\]

for \( m\)-a.e. \( x, y \in E \).
We say that \( p^A_t(x,y) \) is an integral kernel of \((P^A_t)_{t \geq 0}\) in the strict sense if it is Borel measurable in \((t,x,y)\) and (2)–(4) above hold for all \( x,y \in E \).

Stability of the short or global time estimate for an integral kernel of Feynman-Kac semigroup of various Markov processes has been studied by many authors ([10,28,10,14,15,17,48]). Takeda [45] studied the global stability of Li-Yau estimate for the integral kernel under local Feynman-Kac perturbations and gave a necessary and sufficient condition for the global stability of Li-Yau estimate in an analytic way. More precisely, let \( B = (B_t, \mathbf{P}_x) \) be a Brownian motion on a complete non-compact Riemannian manifold \((M,g)\) and assume the heat kernel \( p_t(x,y) \) associated with \( \frac{1}{2} \Delta_M \), the half of Laplace-Beltrami operator on \((M,g), \) satisfies the following Li-Yau estimate (cf. [21])

\[
\frac{C_1}{m(B(x,\sqrt{t}))} \exp \left( -c_1 \frac{d^2(x,y)}{t} \right) \leq p_t(x,y) \leq \frac{C_2}{m(B(x,\sqrt{t}))} \exp \left( -c_2 \frac{d^2(x,y)}{t} \right),
\]

where \( C_1, C_2 \) and \( c_2 \) are positive constants and \( m(B(x,r)) \) denotes the volume of the geodesic ball of radius \( r \) centered at \( x \in M \). For a signed smooth measure \( \mu = \mu_+ - \mu_- \), let \( p^\mu_t(x,y) \) be the integral kernel associated with the Schrödinger operator \( \frac{1}{2} \Delta_M + \mu \). Then the necessary and sufficient condition that \( p^\mu_t(x,y) \) also satisfies (1.14) is \( \lambda(\mu) > 1 \) where

\[
\lambda(\mu) = \inf \left\{ \frac{1}{2} \int_M (\nabla f, \nabla f) \, dm + \int_M f^2 \, d\mu^- \left| f \in C_c^\infty(M), \int_M f^2 \, d\mu^+ = 1 \right. \right\},
\]

provided \( \mu_+ \in S^1_{CS,\infty}(B) \) and \( \mu_- \in S^1_{JSp_0}(B) \). Here \( S^1_{CS,\infty}(B) \) (resp. \( S^1_{JSp_0}(B) \)) denotes the class of conditionally Green-tight smooth measures of Kato class (resp. the class of conditionally Green bounded measures) with respect to \( B \) (see [27], Definition 6.1]). Note that the analytic condition such as (1.15) is very useful in many concrete cases for confirming the global stability of an integral kernel under Feynman-Kac perturbations.

The purpose of this paper is to give the analytic condition on \( u, \mu \) and \( F \) under which the integral kernel \( p^A_t(x,y) \) of the Feynman-Kac semigroup \((P^A_t)_{t \geq 0}\) satisfies (1.7) (resp. (1.9)) under (A.1) (resp. (A.2) ). Before stating our main results, let us introduce a spectral function determined by a quadratic form and a measure. Let \( \mathcal{F}_b := \mathcal{F} \cap L^\infty(E; m) \). For given \( u, \mu \) and \( F \) in the above definition of the additive functionals \( A_t = N_1^u + A_t^\mu + A_t^F \), let \( (\mathcal{D}, \mathcal{F}_b) \) be a quadratic form on \( L^2(E; m) \) defined by

\[
\mathcal{D}(f,g) := \mathcal{E}(f,g) + \mathcal{E}(u,f)g - \mathcal{H}(f,g), \quad f,g \in \mathcal{F}_b,
\]

where

\[
\mathcal{H}(f,g) := \int_E f(x)g(x)\mu(dx) + \int_E \int_E f(x)g(y) \left( e^{F(x,y)} - 1 \right) N(x,dy)\mu_H(dx).
\]

It is known that \((\mathcal{D}, \mathcal{F}_b)\) is well-defined under \( \mu_0 + \mu_1 + \mu_2 + N(F_1 + F_2)\mu_H \in S^1_{JSp}(X) \) ( [29] ). Set \( \mathcal{D}_\alpha(f,g) := \mathcal{D}(f,g) + \alpha(f,g)m \) for \( \alpha \geq 0 \) and \( f,g \in \mathcal{F}_b \). Clearly, \( \mathcal{D}(f,g) = \mathcal{D}_0(f,g) \). For a measure \( \nu \in S^1_{JSp}(X) \), define the spectral function by

\[
\lambda^{\mathcal{D}_\alpha}(\nu) := \inf \left\{ \mathcal{D}_\alpha(f,f) \left| f \in \mathcal{C}, \int_E f^2 \, d\nu = 1 \right. \right\}.
\]

Here \( \mathcal{C} \) stands for the special standard core of \((\mathcal{E}, \mathcal{F})\) ( [17] ). Note that \((\mathcal{D}_\alpha, \mathcal{F}_b)\) is not necessarily non-negative definite and \( \nu \mapsto \lambda^{\mathcal{D}}(\nu) \) is non-increasing in the sense that \( \nu_1 \leq \nu_2 \) implies \( \lambda^{\mathcal{D}}(\nu_1) \geq \lambda^{\mathcal{D}}(\nu_2) \).
\( \lambda^2 (\nu_2) \). Define the signed measures \( \mathcal{P} := \mathcal{P}_1 - \mathcal{P}_2 \) and \( \mathcal{P}^* := \mathcal{P}_1^* - \mathcal{P}_2^* \) by

\[
\mathcal{P}_1 := N \left( (e^{F + [u]} - (F + [u]) - 1 + F_1) \mu_H + \mu_1 + \frac{1}{2} \mu_{e_1}^\epsilon \right), \quad \mathcal{P}_2 := N(F_2) \mu_H + \mu_2, \quad (1.18)
\]

\[
\mathcal{P}_1^* := N(e^{F_1 + [u]} - (F_1 + [u]) - 1 + F_1) \mu_H + \mu_1 + \frac{1}{2} \mu_{e_1}^\epsilon , \quad \mathcal{P}_2^* := \mu_2, \quad (1.19)
\]

where \( [u](x, y) := u(x) - u(y) \). Note that \( \mathcal{P}_1^* = \mu_1 + N(e^{[u]}(e^{F_1} - 1)) \mu_H + N(e^{[u]} - [u] - 1) \mu_H + \frac{1}{2} \mu_{e_1}^\epsilon \).

Let us denote by \( S^1_{\text{NK}}(X) \) the class of natural semi-Green-tight smooth measures of extended Kato class, \( S^1_{\text{NK}}(X) \) the class of natural Green-tight smooth measures of Kato class, and \( S^1_{\text{Bd}}(X) \) the class of Green-bounded smooth measures in the strict sense (see Definitions 2.1 and 2.4 for precise definitions of these).

Now, we are ready to state our results:

**Theorem 1.3** Suppose that \( X \) is transient. Let \( u \in \mathcal{F}_{\text{loc}} \cap QC(E_0) \) be a bounded finely continuous (nearly) Borel function on \( E \). Assume that \( \mu_1 + N(e^{F_1} - 1) \mu_H \in S^1_{\text{NK}}(X) \), \( \mu_u \in S^1_{\text{NK}}(X) \) and \( \mu_2 + N(F_2) \mu_H \in S^1_{\text{Bd}}(X) \) hold. Then we have the following:

1. If \( \lambda^2 (\mathcal{P}_1) > 0 \) and (A.1) holds with \( T \), then there exists an integral kernel \( p^A_1(x, y) \) of the Feynman-Kac semigroup \((P^A_1)_t \geq 0 \) such that \( p^A_1(x, y) \lesssim \phi_2(t, x, y) \) \( m \)-a.e. \( d(x, y) \in \mathcal{T}_t \).

2. If \( \lambda^2 (\mathcal{P}_1) > 0 \) and (A.2) holds with \( T \), then there exists an integral kernel \( p^A_2(x, y) \) of the Feynman-Kac semigroup \((P^A_2)_t \geq 0 \) in the strict sense such that

\[
\phi_1(t, x, y) \lesssim p^A_2(x, y) \lesssim \phi_2(t, x, y), \quad d(x, y) \in \mathcal{T}_t. \quad (1.20)
\]

3. If there exist an integral kernel \( p^A_1(x, y) \) of the Feynman-Kac semigroup \((P^A_1)_t \geq 0 \) in the strict sense and a heat kernel \( p_1(x, y) \) of \( X \) in the strict sense such that \( p^A_1(x, y) \lesssim p_1(x, y) \), then \( \lambda^2 (\mathcal{P}_1) > 0 \), in particular, under (A.2) with \( T = [0, +\infty]^2 \) and \( \phi_1(t, x, y) \simeq \phi_2(t, x, y) \), \( p^A_1(x, y) \lesssim \phi_2(t, x, y) \) implies \( \lambda^2 (\mathcal{P}_1) > 0 \).

**Remark 1.4** (1) Under \( \mu_u \in S^1_{\text{NK},\infty}((X) \), the condition \( \mu_1 + N(e^{F_1} - 1) \mu_H \in S^1_{\text{NK}}((X) \) in Theorem 1.3 is equivalent to \( \mathcal{P}_1^* \in S^1_{\text{NK}}((X) \) (see the proof of [20] Lemma 3.1]), consequently, this is also equivalent to \( \mu_1 + N(e^{[u]}(e^{F_1} - 1)) \mu_H \in S^1_{\text{NK}}((X) \), because \( \mathcal{P}_1^* = \mu_1 + N(e^{[u]}(e^{F_1} - 1)) \mu_H + N(e^{[u]} - [u] - 1) \mu_H + \frac{1}{2} \mu_{e_1}^\epsilon \).

Theorem 1.3 extends the result on the stability of the Li-Yau estimate for the heat kernel of a Brownian motion on Riemannian manifold under Feynman-Kac perturbations proved by Takeda [19]. The first main contribution in our result is to add the perturbation by continuous additive functional of locally of zero energy. The global integral kernel estimate under such a perturbation was firstly discussed by Glover-Rao-Song [19] Theorem 2.9] (see also Glover-Rao-Šikić-Song [20] Proposition 1.4]) in the framework of Brownian motion. But the stability of global heat kernel estimate has not been treated in this direction. Indeed, the global estimates shown in [19] Theorem 2.9] is weaker than the usual Gaussian estimate like Brownian motion. The second main contribution in our result is the relaxation of the class of measures in the creation part. In Theorem 1.3(1), we adopt the class of natural semi-Green-tight measures of extended Kato class in creation part instead of the Green-tight measures of Kato class. So Theorem 1.3(1) under (A.1) (more precisely Theorem 5.7 with Remark 5.9] also covers the result by Devyver [15].
Theorem 4.1]. The third main contribution is the relaxation from the conditionally (semi-)Green-tight measures of (extended) Kato class (resp. the class of conditionally Green bounded measures) into (semi-)Green-tight measures of (extended) Kato class (resp. the class of Green bounded measures) in Theorem 1.3. Our Theorem 1.3 improves Takeda’s result [15, Theorem 2] so that \( p_t^\mu(x,y) \) satisfies Li-Yau estimate if and only if (1.15) holds for \( \mu_+ \in S^1_{NK_1}(X) \) and \( \mu_- \in S^1_{DB}(X) \), which is weaker than the assertion in Devyver [15, Theorem 4.1].

The proof of Theorem 1.3 (1) (2) is constituted of several steps: First, by using a Girsanov transform in terms of \( \mu \), we reduce the proof to the case \( u = 0 \). Second, by using another Girsanov transform in terms of \( F_1, F_2 \), we reduce the proof to the local perturbation under the transformed process \( Y \).

The proof of Theorem 1.3 (3) is based on Theorem 3.3 below, the equivalences among gauge-ability, positivity of the bottom of spectrum and subcriticality. These equivalences had been regarded to hold by way of conditional gaugeability for Feynman-Kac functionals (see [5, 43]). In [27, 29], we prove that these equivalences hold without showing the conditional gaugeability, but showing the semi-conditional gaugeability under conditional (semi-)Green-tightness of related measures. Theorem 3.3 does not require the conditional (semi-)Green-tightness of measures and it is based on the characterization of the gaugeability under the Green-boundedness of underlying symmetrizing measure ([27 Lemma 4.3(5)]) and a time change method.

The analogous statements for Theorem 1.3 without assuming the transience of \( \mu \) in Corollary 1.5 is equivalent to

\[ \lambda^{2\alpha}(\overline{\mu}) > 0 \text{ and } (A.1) \]

holds with \( \mathbb{T} \), then there exists an integral kernel \( p^A_t(x,y) \) of the Feynman-Kac semigroup \( (P^A_t)_{t>0} \) such that \( p^A_t(x,y) \lesssim_k \phi_2(t,x,y) \) m.a.e. \( d(x,y) \in \mathbb{T} \) for some constant \( k := k(\alpha, \gamma) \geq 0 \) depending on \( \alpha \).

(2) If \( \lambda^{2\alpha}(\overline{\mu}) > 0 \) and (A.2) holds with \( \mathbb{T} \), then there exists an integral kernel \( p^A_t(x,y) \) of the Feynman-Kac semigroup \( (P^A_t)_{t>0} \) in the strict sense such that

\[ \phi_1(t,x,y) \lesssim_k p^A_t(x,y) \lesssim_k \phi_2(t,x,y), \quad d(x,y) \in \mathbb{T} \]

for some constant \( k := k(\alpha) \geq 0 \) depending on \( \alpha \).

(3) If there exist a kernel \( p^A_t(x,y) \) of the Feynman-Kac semigroup \( (P^A_t)_{t>0} \) in the strict sense and a heat kernel \( p_t(x,y) \) of \( X \) in the strict sense such that \( p^A_t(x,y) \lesssim_k p_t(x,y) \), then \( \lambda^{2\alpha}(\overline{\mu}) > 0 \) holds for \( \alpha > k \), in particular, under (A.2) with \( \mathbb{T} = [0, +\infty[^2 \) and \( \phi_1(t,x,y) \approx_\ell \phi_2(t,x,y) \) for some \( \ell \geq 0 \), \( p^A_t(x,y) \lesssim_k \phi_2(t,x,y) \) implies \( \lambda^{2\alpha}(\overline{\mu}) > 0 \) for \( \alpha > k + \ell \).

Remark 1.6 (1) Under \( \mu(\omega) \in S^1_{NK_\infty}(X^{(\alpha)}) \), the condition \( \mu_+ + N(e^{F_1} - 1)\mu_H \in S^1_{NK_1}(X^{(\alpha)}) \) in Corollary 1.5 is equivalent to \( \overline{\mu}_1 \in S^1_{NK_1}(X^{(\alpha)}) \) (see the proof of [30, Lemma 3.1]), consequently, this is also equivalent to \( \mu_+ + N(e^{|u|}(e^{F_1} - 1))\mu_H \in S^1_{NK_1}(X^{(\alpha)}) \), because \( \overline{\mu}_1 = \mu_+ + N(e^{|u|}(e^{F_1} - 1))\mu_H + N(e^{|u|} - |u| - 1)\mu_H + \frac{1}{2} \mu^c(\omega) \).

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The proof of Corollary 1.3(1) (resp. Corollary 1.3(2)) is quite similar to the proof of Theorem 1.3(1) (resp. Theorem 1.3(2)). However, the calculation in the proof of Corollary 1.3(1) contains extra factors yielding exponential growth and decay (see (4.14)).

We remark that from any Green-bounded measures a natural semi-Green-tight measure of extended Kato class for α-subprocess can be constructed by scaling (see Lemma 4.4 below). From this observation, we can obtain the following result as a consequence of Corollary 1.5. See Definition 2.1 for the definitions of \( S^{1}_{K}(X) \) and \( S^{1}_{K}(X) \).

**Theorem 1.7** Let \( u \in \mathcal{F}_{loc} \cap QC(E_{0}) \) be a bounded finely continuous (nearly) Borel function on \( E \). Assume \( \mu_{1} + N(e^{F_{1}} - 1) \mu_{H} \in S^{1}_{EK}(X) \), \( \mu_{(u)} \in S^{1}_{K}(X) \) and \( \mu_{2} + N(F_{2}) \mu_{H} \in S^{1}_{D}(X) \). Then we have the following:

1. Under (A.1) with \( T \), there exists an integral kernel \( p_{t} \) of the Feynman-Kac semigroup \( (P_{t})_{t>0} \) such that \( p_{t} \geq k \phi_{t}(x,y) \) m.e. \( d(x,y) \in \mathbb{T}_{t} \) for some \( k \geq 0 \).

2. Under (A.2) with \( T \), there exists an integral kernel \( p_{t} \) of the Feynman-Kac semigroup \( (P_{t})_{t>0} \) in the strict sense such that

\[
\phi_{1}(t,x,y) \leq k \phi_{1}(t,x,y) \leq k \phi_{2}(t,x,y), \quad d(x,y) \in \mathbb{T}_{t}
\]

for some \( k \geq 0 \).

**Remark 1.8** (1) Under \( \mu_{(u)} \in S^{1}_{K}(X) \), the condition \( \mu_{1} + N(e^{F_{1}} - 1) \mu_{H} \in S^{1}_{EK}(X) \) in Theorem 1.7 is equivalent to \( \mathcal{F}_{t} \in S^{1}_{EK}(X(\alpha)) \) (see the proof of [30] Lemma 3.1), consequently, this is also equivalent to \( \mu_{1} + N(e^{[u]}(e^{F_{1}} - 1)) \mu_{H} \in S^{1}_{EK}(X) \), because \( \mathcal{T}_{t} \) is equal to \( \mu_{1} + N(e^{[u]}(e^{F_{1}} - 1)) \mu_{H} + N(e^{[u]} - [u] - 1) \mu_{H} + \frac{1}{2} \mu_{c}^{e} \).

Theorem 1.7 also extends the previous known results on the integral kernel estimates under perturbation by measures of Kato classes in the framework of symmetric Markov processes. The conditions for measures in Theorem 1.7 are milder than those for known results (cf. [40] and reference therein).

The rest of the paper is organized as follows.

In Section 2 we collect basic terminologies and fundamental facts, and define several classes on Green-tight measures of Kato class. In Section 3 we summarize and develop the results in 27, 29 in the framework of this paper. Section 4 is devoted to prove our main results. The main ingredients used in the proofs are in the front of this section. In Section 5 we look our results more closely in the cases of symmetric diffusion process and symmetric jump process.

In this paper, we use \( c \) and \( C \) as positive constants which may be different at different occurrences. In this paper, we use the following notations: For \( a, b \in \mathbb{R} \), \( a \vee b := \max\{a, b\} \), \( a \wedge b := \min\{a, b\} \). \( E_{0} \) is the one point compactification of \( E \) where \( \partial \) is the cemetery point for processes. We use \( QC(E_{0}) \) to denote the family of all strictly \( \mathcal{E} \)-quasi continuous functions on \( E_{0} \), \( C_{0}(E) \) (resp. \( C_{\infty}(E) \)) to denote the family of continuous functions on \( E \) with compact support (resp. vanishing at infinity). \( \mathcal{B}_{+}(E) \) (resp. \( \mathcal{B}_{b}(E) \)) to denote the space of non-negative (resp. bounded) Borel functions on \( E \), and \( C_{b}(E) \) to denote the space of bounded continuous functions on \( E \). For a non-negative Borel measure \( \nu \) and a kernel \( K \) on \( E \), we write \( K \nu(x) := \int_{E} K(x,y) \nu(dy) \). We also use the notation \( \int_{E} f(x) \nu(dx) = f(x)dx \) for any \( f \in \mathcal{B}_{+}(E) \) or \( f \in \mathcal{B}_{b}(E) \). For a Borel subset \( F \) of \( E \), \( \tau_{F} := \inf\{t > 0 \mid X_{t} \notin F\} \) (resp. \( \sigma_{F} := \inf\{t > 0 \mid X_{t} \in F\} \)) is the first exit time of \( X_{t} \) from \( F \) (resp. hitting time of \( X_{t} \) to \( F \)).
Under (AC), for $\alpha > 0$, we can define the $\alpha$-order resolvent kernel $R_\alpha(x,y)(< \infty)$ for $x,y \in E$ (see [17, Lemma 4.2.4]). We can also define 0-order resolvent kernel $R(x,y) := R_0(x,y) := \lim_{n \to 0} R_n(x,y) \leq \infty$ for $x,y \in E$, which is also called the Green kernel of $X$ provided $X$ is transient. Recall that we use notations $R_\alpha\nu(x) = \int_E R_\alpha(x,y)\nu(dy)$ for $\alpha > 0$ and $R\nu(x) = \int_E R(x,y)\nu(dy)$ where $\nu$ is a Borel measure on $E$.

An increasing sequence $\{F_n\}$ of closed sets is said to be an $\mathcal{E}$-nest (resp. strict $\mathcal{E}$-nest) if $\mathbf{P}_x(\lim_{k \to \infty} \tau_{F_n} = \zeta) = 1$ (resp. $\mathbf{P}_x(\lim_{k \to \infty} \sigma_{F_n} = \infty) = 1$) m.a.e. $x \in E$. It is shown in [36] that for an increasing sequence $\{F_n\}$ of closed sets, $\{F_n\}$ is an $\mathcal{E}$-nest if and only if $\bigcup_{n=1}^{\infty} \mathcal{F}_{F_n}$ is $\mathcal{E}_{1/2}$-dense in $\mathcal{F}$ where $\mathcal{E}_{1/2}(\cdot,\cdot) := \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)_m$, equivalently $\lim_{n \to \infty} \text{Cap}(K \setminus F_n) = 0$ for any compact set $K$ in view of [17, Lemma 5.1.6]. The regularity of the given Dirichlet form $(\mathcal{E},\mathcal{F})$ for $X$ tells us that there always exists an $\mathcal{E}$-nest of compact sets ([36] Chapter IV 4 (a), Chapter V, Proposition 2.12)). Hence any $\mathcal{E}$-nest can be taken to be a sequence of compact sets. A function $f$ defined on $E$ (resp. $E_0$) is said to be $\mathcal{E}$-quasi continuous (resp. strictly $\mathcal{E}$-quasi continuous) if there exists an $\mathcal{E}$-nest (resp. a strict $\mathcal{E}$-nest) $\{F_n\}$ of closed sets such that $f|_{F_n}$ (resp. $f|_{F_n\cap D}$) is continuous for each $k \in \mathbb{N}$. Recall that $QC(E_0)$ is the family of all strictly $\mathcal{E}$-quasi continuous functions on $E_0$.

A positive Radon measure $\nu$ on $E$ is said to be of finite energy integral if there exists $C > 0$ such that

$$\int_E |\nu(x)|\nu(dx) \leq C \sqrt{\mathcal{E}_1(v,v')} \quad \text{for all } v \in \mathcal{F} \cap C_0(E), \quad (2.1)$$

Denote by $S_0(X)$ the family of measures of finite energy integrals and $S_00(X) := \{\nu \in S_0(X) | \nu(E) < \infty, |R_1|\nu|_{00} < \infty\}$. A positive Radon measure $\nu$ on $E$ is said to be of 0-order finite energy integral if $X$ is transient and there exists $C > 0$ satisfying (2.1) with $\mathcal{E}(\cdot,\cdot)$ in place of $\mathcal{E}_1(\cdot,\cdot)$, and we denote by $S_0^0(X)$ the family of measures of 0-order finite energy integrals. A positive Borel measure $\nu$ on $E$ is said to be smooth if $\nu$ charges no exceptional set and there exists an $\mathcal{E}$-nest $\{F_n\}_{n \geq 1}$ of compact sets satisfying $\nu(F_n) < \infty$ for each $n \geq 1$. It is known that $\nu$ is smooth if and only if $\nu$ charges no exceptional set and there exists an $\mathcal{E}$-nest of compact sets $\{F_n\}$ such that $1_{F_n \cap E_0} \in S_0(X)$ for each $n \geq 1$ ([17, cf. Theorem 2.2.4]). We denote by $S(X)$ the family of all smooth measures. $S(X)$ contains all positive Radon measures on $E$ charging no set of zero capacity. Let $S_1(X)$ be the family of positive smooth measures in the strict sense ([17, Page 238]).

**Definition 2.1** (1) A measure $\nu \in S_1(X)$ is said to be of Dynkin class (resp. Green-bounded) with respect to $X$ if $\sup_{x \in E} R\nu(x) < \infty$ for some $\beta > 0$ (resp. $\sup_{x \in E} R\nu(x) < \infty$). Denote by $S^1_D(X)$ (resp. $S^1_D(D)$) the family of measures of Dynkin class (resp. of Green-bounded measures).

(2) A measure $\nu \in S_1(X)$ is said to be of Kato class (resp. of extended Kato class) with respect to $X$ if $\lim_{\beta \to \infty} \sup_{x \in E} R\beta\nu(x) = 0$ (resp. $\lim_{\beta \to \infty} \sup_{x \in E} R\beta\nu(x) < 1$). Denote by $S^1_K(X)$ (resp. $S^1_{DK}(X)$) the family of measures of Kato class (resp. of extended Kato class).

When $X$ is transient, denote by $S^{00}_0(X) := \{\nu \in S^{00}_0(X) | \nu(E) < \infty \text{ and } U \nu \in L^\infty(E;\mathfrak{m})\}$, the family of finite measures of 0-order energy integrals with bounded 0-order potentials. Here $R\nu \in \mathcal{F}$ is the 0-order potential for $\nu \in S^{00}_0(X)$ defined by $\mathcal{E}(R\nu,v) = \int_E v d\nu$ for $v \in \mathcal{F} \cap C_0(E)$. By definition, we see $S^{00}_0(X) \subset S^{00}_D(X)$ and $S_00(X) \subset S^{00}_D(X)$ ([17, cf. Theorems 5.1.6 and 5.1.7]). Note that any measure $\nu \in S^{01}_D(X)$ is a positive Radon measure in view of Stollmann-Voigt’s inequality: $\int_E u^2 d\nu \leq \|R_\alpha\nu\|_{\infty} \mathcal{E}_\alpha(u,u)$, $u \in \mathcal{F}, \alpha \geq 0$ ([13, Theorem 3.1]). Conversely,
any positive Radon measure $\nu$ satisfying $\sup_{x \in E} R_\alpha \nu(x) < \infty$ for some $\alpha > 0$ always belongs to $S_1(X)$ in view of [29 Proposition 3.1].

We say that a positive continuous additive functional (PCAF in abbreviation) in the strict sense $A^\nu$ of $X$ and a positive measure $\nu \in S_1(X)$ are in the Revuz correspondence if they satisfy for any bounded $f \in \mathcal{B}_+(E)$,

$$
\int_E f(x)\nu(dx) = \lim_{t \downarrow 0} \frac{1}{t} \int_E \mathbb{E}_x \left[ \int_0^t f(X_s) dA^\nu_s \right] m(dx).
$$

It is known that the family of equivalence classes of the set of PCAFs in the strict sense and the family of positive measures belonging to $S_1(X)$ are in one to one correspondence under the Revuz correspondence ( [17, Theorem 5.1.4]).

A function $f$ on $E$ is said to be locally in $\mathcal{F}$ in the broad sense (denoted as $f \in \mathcal{F}_{\text{loc}}$) if there is an increasing sequence of finely open Borel sets $\{E_n\}$ with $\bigcup_{n=1}^{\infty} E_n = E$ q.e. and for every $n \geq 1$, there is $f_n \in \mathcal{F}$ such that $f = f_n$ m.a.e. on $E_n$. A function $f$ on $E$ is said to be locally in $\mathcal{F}$ in the ordinary sense (denoted as $f \in \mathcal{F}_{\text{loc}}$) if for any relatively compact open set $G$, there exists an element $f_G \in \mathcal{F}$ such that $f = f_G$ m.a.e. on $G$. Clearly $\mathcal{F}_{\text{loc}} \subset \mathcal{F}_{\text{loc}}$. It is shown in [32, Theorem 4.1], $\mathcal{F}_e \subset \mathcal{F}_{\text{loc}}$.

For a signed measure $\nu = \nu_1 - \nu_2$ with $\nu_i \in S^1_D(X)$ (resp. $\nu_i \in S^1_{D_0}(X)$) $(i = 1, 2)$, $R_\alpha \nu$ (resp. $R \nu$) is a difference of bounded $\alpha$-excessive (resp. excessive) functions belonging to $\mathcal{F}_{\text{loc}}$. Let $N^R_\nu$ be the CAF locally of zero energy appeared in the generalized Fukushima decomposition (see [31, Theorem 6.1]). The following lemma is a variant of [17, Lemma 5.4.1]. See [17, Exercise 2.2.4].

**Lemma 2.2** For a signed measure $\nu = \nu_1 - \nu_2$ with $\nu_i \in S^1_D(X)$ $(i = 1, 2)$, we have

$$
N^R_\nu = \alpha \int_0^t R_\alpha \nu(X_s) ds - A^\nu_t, \quad t \in [0, \zeta]
$$

$P_x$-a.s. for q.e. $x \in E$, and in the transient case with $\nu_i \in S^1_{D_0}(X)$ $(i = 1, 2)$, we have

$$
N^R_\nu = -A^\nu_t, \quad t \in [0, \zeta]
$$

$P_x$-a.s. for q.e. $x \in E$.

Next lemma is needed for the proof of main theorems.

**Lemma 2.3** Suppose that $X$ is transient and $\nu \in S(X)$. Then there exists an $\mathcal{E}$-nest $\{F_n\}$ of compact sets such that $1_{F_n} \nu \in S^0_{D_0}(X)$ for each $n \in \mathbb{N}$.

**Proof.** The proof follows from [17, Exercise 2.2.4]. We omit the details. \hfill \Box

Now, let us introduce the notions of natural Green-tight measures of (extended) Kato class in the strict sense ( [29]). First, we explain the notion of weighted capacity of the Dirichlet form associated with the time changed process:

Let $\nu \in S_1(X)$ and denote by $A^\nu_t$ the PCAF in the strict sense associated to $\nu$ in Revuz correspondence. Denote by $S_\nu^c$ the support of $A^\nu$ defined by $S_\nu^c := \{x \in E \mid P_x(R = 0) = 1\}$, where $R(\omega) := \inf\{t > 0 \mid A^\nu_t(\omega) > 0\}$. $S_\nu^c$ is nothing but the fine support of $\nu$, i.e., the topological support of $\nu$ with respect to the fine topology of $X$. Let $(X, \nu)$ be the time changed process of $X$ by $A^\nu_t$ and $(\mathcal{E}, \mathcal{F})$ the associated Dirichlet form on $L^2(S^\nu; \nu)$, where $S^\nu$ is the support
of $\nu$. It is known that $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$ is a regular Dirichlet form having $\mathcal{C}^0|_{S^\nu}$ as its core and $S^\nu \setminus S^\nu_0$ is $\mathcal{E}$-polar, i.e., 1-capacity 0 set with respect to $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$. The life time of $(X, \nu)$ is given by $A^\nu_t$. Let $\hat{\mathcal{E}}_t(f, f) := \hat{\mathcal{E}}(f, f) + \int_E f^2 d\nu$, and $\hat{R}_t\varphi(x) := E_x[\int_0^\infty e^{-\tau \varphi(X_t)} dA^\nu_t]$, the 1-order resolvent of a $\nu$-a.e. strictly positive bounded function $\varphi \in L^1(E; \nu)$ under $(X, \nu)$. Let $C^\nu : 2^E \to [0, +\infty]$ be the weighted 1-capacity with respect to $(\hat{\mathcal{E}}, \hat{\mathcal{F}})$, i.e., for an open subset $G$ of $E$, we define

\[ C^\nu(G) := \inf \{ \hat{\mathcal{E}}_1(f, f) \mid f \in \hat{\mathcal{F}}, \ f \geq \hat{R}_1\varphi \ \nu\text{-a.e. on } G \} \]

and for arbitrary subset $A$ of $E$

\[ C^\nu(A) := \inf \{ C^\nu(G) \mid A \subset G, \ G \text{ is an open subset of } E \}. \]

We emphasize that $C^\nu$ is defined to be an outer capacity on $E$. By definition, $C^\nu(E \setminus S^\nu) = 0$.

Note that $C^\nu(E) \leq \hat{\mathcal{E}}_1(\hat{R}_1\varphi, \hat{R}_1\varphi) = \int_E \varphi(x)\hat{R}_1\varphi(x) \nu(dx) < \infty$ always holds. Note also that $C^\nu$ is tight in the sense that there exists an increasing sequence $\{K_n\}$ of compact subsets of $S^\nu$ such that $\lim_{n \to \infty} C^\nu(S^\nu \setminus K_n) = 0$ equivalently $\lim_{n \to \infty} C^\nu(E \setminus K_n) = 0$.

**Definition 2.4** Let $\nu \in S_1(X)$.

1. $\nu$ is said to be a **natural Green-tight measure of Kato class with respect to** $X$ if $\nu \in S^1_{D_0}(X)$ and for any $\varepsilon > 0$ there exist a closed subset $K = K(\varepsilon)$ of $E$ and a constant $\delta > 0$ such that for all Borel set $B \subset K$ with $C^\nu(B) < \delta$,

\[ \sup_{x \in E} E_x [A^\nu_{TB, K^c}] < \varepsilon. \]

Denote by $S^1_{NK_\infty}(X)$ the family of natural Green-tight measures of Kato class with respect to $X$.

2. $\nu$ is said to be a **natural semi-Green-tight measure of extended Kato class with respect to** $X$ if $\nu \in S^1_{D_0}(X)$ and there exist a closed subset $K$ of $E$ and a constant $\delta > 0$ such that for all Borel set $B \subset K$ with $C^\nu(B) < \delta$,

\[ \sup_{x \in E} E_x [A^\nu_{TB, K^c}] < 1. \]

Denote by $S^1_{NK_1}(X)$ the family of natural semi-Green-tight measures of extended Kato class with respect to $X$.

We note that the closed set $K$ appeared in Definition 2.4 can be taken to be compact, because the weighted 1-capacity $C^\nu$ is tight.

**Remark 2.5**

1. Let us denote by $S^1_{CK_\infty}(X)$ (resp. $S^1_{CK_1}(X)$) the family of Green-tight Kato class measures (resp. the family of semi-Green-tight Kato class measures) in the sense of Chen with respect to $X$ (see [3, 29] for precise definitions). It is proved in [29] Lemma 4.4 that $S^1_{CK_1}(X) \subset S^1_{NK_1}(X) \subset S^1_{EK}(X) \cap S^1_{D_0}(X)$ and $S^1_{CK_\infty}(X) \subset S^1_{NK_\infty}(X) \subset S^1_{K}(X) \cap S^1_{D_0}(X)$.

2. It is proved in [27] Proposition 4.1 that the class $S^1_{NK_\infty}(X)$ is not so wide in the following sense: $S^1_{CK_\infty}(X) = S^1_{NK_\infty}(X)$ whenever $X$ has the doubly Feller property of resolvent.
(3) The advantage of the natural versions of semi-Green-tight measure is that they are stable under some Girsanov transform (cf. [29, Corollary 5.1 and Corollary 5.2]).

If \( X \) is transient, we can consider the Green kernel \( R(x,y) \) for \( x,y \in E \) under \((AC)\). We set \( d := \{(x,y) \mid R(x,y) = 0 \text{ or } +\infty\} \) and \( E^z := \{x \in E \mid (x,z) \in E \times E \setminus d\} \). Since \( R(x,y) > 0 \) for all \( x,y \in E \) (see [29, Lemma 6.1]), we see \( d := \{(x,y) \mid R(x,y) = +\infty\} \). By Getoor [18], there exists \( g \in L^1(E;m) \) with \( 0 < g \leq 1 \) m.a.e. such that \( Rg \in \mathcal{B}_b(E) \), we see \( m(E \setminus E^z) = 0 \) for all \( z \in E \). More strongly we have the following:

**Proposition 2.6** Suppose that \( X \) is transient. Then \( E \setminus E^z \) is exceptional for all \( z \in E \).

**Proof.** For any \( \nu \in S_{00}^{(0)}(X) \), \( R\nu \in \mathcal{B}_b(E) \). This implies that for given \( z \in E \), \( \nu(E \setminus E^z) = 0 \) for all \( \nu \in S_{00}^{(0)}(X) \). By Lemma 2.3 for given \( z \in E \), \( \nu(E \setminus E^z) = 0 \) for any smooth measure \( \nu \). Therefore we obtain the assertion. \( \square \)

3 Analytic characterization of gaugeability and subcriticality

Recall that \( F = F_1 - F_2 \) where \( F_1 \) and \( F_2 \) are non-negative bounded functions on \( E \times E_\partial \) symmetric on \( E \times E \) which is extended to a function defined on \( E_\partial \times E_\partial \) vanishing on the diagonal set \( \text{diag of } E_\partial \times E_\partial \). Let

\[
A^F_i = A^{F_1}_i - A^{F_2}_i, \quad A^F_i := \sum_{0<s\leq t} F_1(X_{s-},X_s) \quad (i = 1, 2).
\]

Note that if \( N(F_1 + F_2)\mu H \in \mathcal{S}_1(X) \), then \( N(|F|)\mu H \in \mathcal{S}_1(X) \). In this case, \( A^F \) can be defined as an additive functional in the strict sense. Note that \( A^F_i = \sum_{0<s\leq t} 1_{\{s<s\}} F(X_{s-},X_s) \) provided \( F(x,\partial) = 0 \) for \( x \in E_\partial \). Hereafter, we always assume \( F(x,\partial) = 0 \), \( x \in E_\partial \). For a bounded finely continuous (nearly) Borel function \( u \in \mathcal{F}_{\text{loc}} \cap QC(E_\partial) \) satisfying \( \mu_{(u)} \in \mathcal{S}_1(X) \) and \( N(|F|)\mu H \in \mathcal{S}_1(X) \), we set

\[
F^u(x,y) := F(x,y) + u(x) - u(y) \quad \text{and} \quad G^u = e^{F^u} - 1
\]

with identifying \( F^0 = F \) and \( G^0 = eF - 1 \).

Let \( M^F_t = \sum_{0<s\leq t} F(X_{s-},X_s) - \int_0^t N(F)(X_s)dH_s \). By [29, (3.1)–(3.4)], there exist purely discontinuous locally square integrable local martingale additive functionals \( M^{F_u} \) and \( M^{G_u} \) on \([0,\zeta]\) such that \( \Delta M^{F_u}_t = F^u(X_{t-},X_t) \) and \( \Delta M^{G_u}_t = G^u(X_{t-},X_t) \), \( t \in [0,\zeta[ \) \( P_x \)-a.s. for all \( x \in E \). \( M^{F_u} \) and \( M^{G_u} \) are given by

\[
M^{F_u}_t = M^F_t + M^{u,j}_t + M^{u,\infty}_t, \quad t < \zeta \tag{3.1}
\]

and

\[
M^{G_u}_t = M^{F_u}_t + \sum_{0<s\leq t} (G^u - F^u)(X_{s-},X_s) - \int_0^t N(G^u - F^u)(X_s)dH_s, \quad t < \zeta. \tag{3.2}
\]

Let \( Y_t := \exp(M^{G_u} + M^{u,c}_t) \) be the Doléans-Dade exponential of \( M^{G_u} + M^{u,c}_t \), that is, \( Y_t \) is the unique solution of

\[
Y_t = 1 + \int_0^t Y_{s-}d(M^{G_u}_s + M^{u,c}_s), \quad t < \zeta, \quad P_x \text{-a.s.} \tag{3.3}
\]
It then follows from Doléans-Dade formula and (3.2) that
\[
Y_t = \exp \left( M_t^{F_{u}} + M_t^{-u,c} - \int_0^t N(G^u - F^u)(X_s)\,dH_s - \frac{1}{2} (M_t^{u,c})_t \right) \quad (3.4)
\]
(29 Theorem 3.1). Note that \( Y_t \) is a positive and local martingale, therefore supermartingale on \([0, \zeta]\). It is inconvenient to treat additive functionals on \([0, \zeta]\) for our purpose. We see that \( Y_t \) can be extended for all \( t \in [0, +\infty[ \) provided \( \mu_{(u)} + N(F_1 + F_2)\mu_H \in S^1_D(X) \) (29 Proposition 3.1).

Let us denote by \( Y = (X_t, \mathbf{P}_Y^t) \) the transformed process of \( X \) by \( Y_t \). The transition semigroup \((\mathbf{P}_t^Y)_{t \geq 0}\) of \( Y \) is defined by
\[
\mathbf{P}_t^Y f(x) := \mathbf{E}_x[Y_t f(X_t)].
\]

**Theorem 3.1** (29 Theorem 3.2) Assume that a bounded function \( u \in \mathcal{F}_{loc} \cap QC(E_\partial) \) admits a Fukushima’s decomposition holding up to infinity under \( \mathbf{P}_x \) for q.e. \( x \in E \) and \( N(F_1 + F_2)\mu_H \in S^1_1(X) \). Let \((\mathcal{E}^Y, \mathcal{F}^Y)\) be the Dirichlet form of \( Y \) on \( L^2(E; e^{-2u}\mu) \). Then \( \mathcal{F} = \mathcal{F}^Y \) and for any \( f \in \mathcal{F}^Y \)
\[
\mathcal{E}^Y(f, f) = \frac{1}{2} \int_E e^{-2u(x)} \mu f^2(dx) + \int_{E \times E} (f(x) - f(y))^2 e^{F(x,y) - u(x) - u(y)} \,dJ(dx, dy).
\]

By Theorem 3.1 combined with the boundedness of \( u \) and \( F \), there exists a constant \( C_E > 0 \) such that
\[
C_E^{-1} \mathcal{E}^c(f, f) \leq \mathcal{E}^Y(f, f) \leq C_E \mathcal{E}^c(f, f),
\]
for \( f \in \mathcal{F}^Y \) and
\[
C_E^{-1} J(dx, dy) \leq J^Y(dx, dy) \leq C_E J(dx, dy).
\]

Here \( J^Y(dx, dy) := e^{F(x,y) - u(x) - u(y)} \,dJ(dx, dy) \) is the jumping measure of \((\mathcal{E}^Y, \mathcal{F}^Y)\). Therefore we see under (A.1) (resp. (A.2)) that \((\mathcal{E}^Y, \mathcal{F}^Y)\) admits a heat kernel \( p_t^Y(x,y) \) (resp. heat kernel \( p_t^Y(x,y) \) in the strict sense) on \([0, +\infty[ \times E \times E \) such that
\[
p_t^Y(x,y) \lesssim \phi_2(t,x,y) \quad \text{m-a.e.} \quad d(x,y) \in T_t \quad \text{(resp. } \phi_1(t,x,y) \lesssim p_t^Y(x,y) \lesssim \phi_2(t,x,y), \quad d(x,y) \in T_t) \quad \text{for } u \) and \( F \) appeared in Theorem 3.1.

Recall that \((\mathcal{O}, \mathcal{F}_h)\) is the quadratic form on \( L^2(E; \mu) \) defined in (1.16). Note that for bounded finely continuous (nearly) Borel function \( u \in \mathcal{F}_{loc} \cap QC(E_\partial) \) and \( f \in \mathcal{F} \cap C_0(E) \), we see that \( fe^u = f(e^u - 1) + f = f(e^{uO} - 1) + f \in \mathcal{F} \), where \( O \) is a relatively compact open set with \( \text{supp}[f] \subset O \) and \( uO \in \mathcal{F}_h \) satisfying \( u = uO \) m.a.e. on \( O \). Moreover, it follows from (3.4), Theorem 3.1 and the Feynman-Kac formula that for \( f \in \mathcal{F} \cap C_0(E) \),
\[
\mathcal{E}^Y(f, f) = \mathcal{O}(f e^{-u}, f e^{-u}) + \int_E f^2 e^{-2u} \,d\mu.
\]

Consider the non-local Feynman-Kac transforms by the additive functionals \( A_t := N^u_t + A^u_t + A^F_t \) of the form
\[
e_A(t) := \exp(A_t), \quad t \geq 0.
\]

We see that for \( \mu_{(u)} + N(F_1 + F_2)\mu_H \in S^1_D(X) \)
\[
e_A(t) = e^{u(X_t) - u(X_0)} Y_t \exp \left( A^F_t \right), \quad t \in [0, +\infty[.
\]
which implies that for $x \in E$ and $f \in \mathcal{B}_+(E)$,

$$P_t^A f(x) = e^{-u(x)} E_x^Y \left[ \exp \left( \frac{A t^r}{1} \right) (e^{u} f)(X_t) \right]. \quad (3.7)$$

We note that if $\nu \in S^1_D(X)$ (resp. $\nu \in S^1_{\partial_0}(X)$), then Stollmann-Voigt’s inequality (Theorem 3.1) tells us $\int_E f^2 d\nu \leq \|R_t \nu\|_{\infty} \mathcal{E}_1(f,f)$ for $f \in \mathcal{F}$ (resp. $\int_E f^2 d\nu \leq \|R \nu\|_{\infty} \mathcal{E}(f,f)$ for $f \in \mathcal{F}_e$), hence $\mathcal{F} \subset L^2(\mathcal{E};\nu)$ (resp. $\mathcal{F}_e \subset L^2(\mathcal{E};\nu)$). It is shown in [27] that under $\mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{R_2}(X)$, $\mu_{(u)} \in S^1_R(X)$ and $\mu_2 + N(F_2)\mu_H \in S^1_R(X)$, for all $x \in E$ and $y \rightarrow R_2(x,y)$ is finely continuous (see [29, Lemma 6.1]). We write $R_2(x,y) := R_2^0(x,y)$ for $x,y \in E$.

We consider the following conditions:

(A) $\mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{R_2}(X)$, $\mu_{(u)} \in S^1_R(X)$ and $\mu_2 + N(F_2)\mu_H \in S^1_R(X)$.

$$(A)^* \mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{R_2}(X^*)$, $\mu_{(u)} \in S^1_R(X^*)$.

Here $X^*$ is the subprocess killed by $e^{-A^2 - A^2}$. It is easy to see that $(A)$ implies $(A)^*$. Let us recall the quadratic form $(\mathcal{Q}, \mathcal{F})$ on $L^2(\mathcal{E};m)$ defined in [11].

Lemma 3.2 (cf. [30, Lemma 2.1]) Suppose $(A)^*$. Then for any sufficiently large $\alpha > 0$ there exists $C > 0$ such that

$$C^{-1} \mathcal{E}_1(f,f) \leq \mathcal{D}_\alpha(f,f) \leq C \mathcal{E}_1(f,f), \quad \text{for } f \in \mathcal{F}. \quad (3.8)$$

Consequently, $(P_t^A)_{t \geq 0}$ defines a strongly continuous semigroup on $L^2(\mathcal{E};m)$ associated with $(\mathcal{Q}, \mathcal{F})$ on $L^2(\mathcal{E};m)$.

Next theorem extends [27, Theorem 1.1] (see also [29, Theorem 1.1]).

Theorem 3.3 Suppose that $X$ is transient. Let $u \in \mathcal{F}_{\text{loc}} \cap QC(E_0)$ be a bounded finely continuous (nearly) Borel function on $E$. Assume $\mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{N_2}(X)$, $\mu_{(u)} \in S^1_{N_2}(X)$ and $\mu_2 + N(F_2)\mu_H \in S^1_{N_2}(X)$. Then the following are equivalent:

1. The functional $[5.5]$ is gaugeable, that is, $\sup_{x \in E} E_x[e_A(\xi)] < \infty$.

2. $\lambda^D(\overline{\nu}_1) > 0$.

3. For each $x \in E$, $R^A(x,y) < \infty$ for $m$-a.e. $y \in E$.

4. $R^A(x,y) < \infty$ for $m$-a.e. $x,y \in E$.

Proof. As noted in Remark [1.3(1)], the condition $\mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{N_2}(X)$ is equivalent to $\mu_1 + N(e^{[u]}(e^{F_1} - 1))\mu_H \in S^1_{N_2}(X)$ under $\mu_{(u)} \in S^1_{N_2}(X)$. The equivalence $\implies (2)$ is proved in [27, Theorem 1.1] and the implication $\implies (3)$ is easy. So it suffices to prove the implications $\implies (1)$ and the implication $\implies (2)$.

Recall that $[u](x,y) := u(x) - u(y)$. Let $U$ be the transformed process of $X$ by the supermartingale multiplicative functional $U_t := \text{Exp}(M^d - M^u)$, $t$. Let $\nu := \overline{\nu}_1 - \overline{\nu}_2$ be the measure given by $\overline{\nu}_1 = \mu_1 + N(e^{[u]} - [u] - 1)\mu_H + \frac{1}{2} \mu^c_{(u)}$ and $\overline{\nu}_2 = \mu_2$. Then we see
by [27] Corollary 5.1(1,5) that 
\[ e^{-2u}(\nu_1 + N(e^{[u]}(eF_1 - 1))\mu_H) \in S^1_{N_K}(U) \] and
\[ e^{-2u}(\nu_2 + N(e^{[u]}F_2))\mu_H \in S^1_{B_0}(U) \]. From these facts with [27] Lemma 4.4 and (4.10), one can easily show that (1) is equivalent to
\[ \sup_{x \in E} E_x^U[\exp(A^F + A^F)] < \infty \] and
\[ R^4(x, y) \] can be written by
\[ R^4(x, y) = e^{-u(x) - u(y)}(R^U)^{-u} \nu, F(x, y) \] where \( R^U \) is 0-order resolvent kernel of \( U \). In view of these, we can and do assume \( u = 0 \).

Since \( X \) is transient, by Getoor [18], there exists \( g \in L^1(E; m) \) with \( 0 < g \leq 1 \) such that \( Rg \in \mathcal{B}(E) \). In particular, \( gm \in S^1_{B_0}(X) \). Let \((X, gm)\) be the time changed process by \( A^g := \int_0^t g(X_s)ds \), i.e., \((X, gm) := (\Omega, X_{s}^{gm}, \mathbb{P}_x)_{x \in E}, \) where \( \tau^g := \inf\{s > 0 \mid A^g > t\} \) is the right continuous inverse of \( A^g \). Suppose that (1) holds. Then by [27] Lemma 5.2
\[ \sup_{x \in E} E_x[e_{A_{\cdot, gm}}(A^g_{\cdot})] = \sup_{x \in E} E_x[e_{A}(\zeta)] < \infty. \] (3.9)

Here \( A_{\tau, gm} \) is the CAF under \((X, gm)\) and \( A^g_{\cdot} \) is the life time of \((X, gm)\). Applying [27] Lemma 4.3 to the time changed process \((X, gm)\) with \( gm \in S^1_{B_0}(X, gm) \), (3.9) is equivalent to
\[ \sup_{x \in E} R^4 g(x) = \sup_{x \in E} E_x \left[ \int_0^\infty e_{A}(t)g(X_t)dt \right] = \sup_{x \in E} E_x \left[ \int_0^\infty e_{A_{\cdot, gm}}(\tau^g_{\cdot})dt \right] < \infty. \]

Then we have (1)\( \Rightarrow \) (3). Next suppose that (1) holds. Then there exists \( N \in \mathcal{B}(E) \) such that for \( x \in E \setminus N \), \( R^4(x, y) < \infty \) m-a.e. \( y \in E \). For each fixed \( x \in E \setminus N \), we set
\[ g_x(y) := \sum_{n=1}^{\infty} \frac{1}{2^n m(G_n)} 1_{E_n^x \cap E_n^{x-1}}(y), \]
where \( \{G_n\} \) is an increasing sequence of relatively compact open sets such that \( E = \bigcup_{n=1}^{\infty} G_n \) and \( m(G_n) \geq 1 \), and \( E_n^x := \{y \in G_n \mid R^4(x, y) \leq n\} \) for \( n \in \mathbb{N} \) with \( E_0^x := \emptyset \). It is easy to see that \( 0 < g_x \leq 1 \) m-a.e. and \( g_x \in \mathcal{B}(E) \), because \( R^4(x, y) < \infty \) m-a.e. \( y \in E \) yields \( m(\bigcap_{n=1}^{\infty} (E \setminus E_n^y)) = 0 \). Then for any \( z \in E \)
\[ R^4 g_x(z) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \] (3.10)

Take any \( g \in \mathcal{B}(E) \) with \( 0 < g \leq 1 \) on \( E \). Then \((g_x \wedge g) m \in S^1_{B_0}(X) \). Applying [27] Lemma 4.3(1)\( \iff \) (4)) to the time changed process \((X, (g_x \wedge g) m) \) with \((g_x \wedge g) m \in S^1_{B_0}(X, (g_x \wedge g) m) \), this is equivalent to the gaugeability of \( AF \) \( A_{(g_x \wedge g) m} \) under \((X, (g_x \wedge g) m) \) (see [27] Lemma 5.1(2)):
\[ \sup_{y \in E} E_y[e_{A}(\zeta)] = \sup_{y \in E} E_y[e_{A_{(g_x \wedge g) m}}(A^g_{\cdot})] < \infty. \]

Therefore we obtain (1). \( \square \)

4 PROOFS OF THEOREM 1.3, COROLLARY 1.5 AND THEOREM 1.7

Recall that \( \overline{\mu} := \overline{\mu}_1 - \overline{\mu}_2 \) and \( \overline{\mu}' := \overline{\mu}_1' - \overline{\mu}_2' \) are the signed measures defined in (1.13) and (1.19) respectively. In this section, we will prove our main results stated in Section 1. Before proving them, we give several lemmas on the measure \( \overline{\mu} \) and its gauge function \( h \) with respect to the process \( Y \). Throughout this section, \( u \) is a bounded finely continuous (nearly) Borel function in \( \mathcal{F}_{loc} \cap QC(E_0) \).
Lemma 4.1 Assume that $X$ is transient. Suppose that $\mu_1 + N(e^{F_1} - 1)\mu_H \in S_{\mathcal{NK}_1}^1(X)$ and $\mu_2 + N(F_2)\mu_H \in S_{\mathcal{D}_b}^1(X)$. Then, the following are equivalent:

1. The functional $\exp(A^\overline{\mu}_1)$ is gaugeable under $Y$, that is, $\sup_{x \in E} E^Y_x[\exp(A^\overline{\mu}_1)] < \infty$.
2. $\lambda^\overline{\mu}(\overline{\mu}_1) > 0$.

Proof. By assumption, we see $\mu_1 + N(F_1 + F_2)\mu_H \in S_{\mathcal{D}_b}^1(X)$. Then we have from Lemma 4.9(2) that

$$E^Y_x[\exp(A^\overline{\mu}_1)] = e^{u(x)}E_x[e^{-u(X_{\overline{\mu}_1})}e_{A(\overline{\mu})}].$$

(4.1)

Hence $\sup_{x \in E} E^Y_x[e_{A(\overline{\mu})}] < \infty$ is equivalent to $\sup_{x \in E} E^Y_x[\exp(A^\overline{\mu}_1)] < \infty$. Therefore the assertion follows from Theorem 3.3.

We define the gauge function $h$ with respect to $Y$ by

$$h(x) := E^Y_x[\exp(A^\overline{\mu}_1)], \quad x \in E_\partial.$$

Since any AF $C_t$ of $Y$ satisfies $P^Y_\overline{\partial}(C_t) = 1$, we see $h(\partial) = 1$. Then, by (4.1)

$$0 < e^{-\|R\overline{\mu}_2\|_{\infty}} = e^{-\sup_{y \in E} E^Y_y[A_{\overline{\mu}_1}^2 + A_{\overline{\mu}_2}^2]} \leq h(x) \leq \sup_{y \in E_\partial} E^Y_y[A^\overline{\mu}_1] < \infty$$

(4.2)

whenever $u = 0$ and $\lambda^\overline{\mu}(\overline{\mu}_1) > 0$. We have the following lemma in a similar way of Lemma 2:

Lemma 4.2 Assume that $X$ is transient and $u = 0$. Suppose that $\overline{\mu}_1 = \mu_1 + N(e^{F_1} - 1)\mu_H \in S_{\mathcal{NK}_1}^1(X)$ and $\overline{\mu}_2 = \mu_2 + N(F_2)\mu_H \in S_{\mathcal{D}_b}^1(X)$, and $\lambda^\overline{\mu}(\overline{\mu}_1) > 0$. Then we have $\overline{\mu}_2 \in S_{\mathcal{D}_b}^1(Y)$ and

$$h(x) = R^Y(h\overline{\mu})(x) + 1, \quad x \in E.$$  

(4.3)

In particular, we have $\overline{\mu}_1 = \mu_1 + N(e^{F_1 - F_2} + F_2 - 1)\mu_H \in S_{\mathcal{D}_b}^1(Y)$, hence $\overline{\mu}_1 \in S_{\mathcal{D}_b}^1(Y)$.

Proof. Since $\lambda^\overline{\mu}(\overline{\mu}_1) > 0$, we know $\sup_{x \in E_\partial} h(x) < \infty$ by Lemma 4.1. Recall that, since we assume that $u = 0$, we have $\overline{\nu} = (\overline{\mu}_1 - \overline{\mu}_2)$ with $\overline{\nu}_1 = \mu_1 + N(e^{F_1 - F_2} + F_2 - 1)\mu_H$ and $\overline{\nu}_2 = \mu_2 + N(F_2)\mu_H$. Thus, we see $\nu = (\overline{\mu}_1 - (\mu_2 + N(e^{F_1} - 1 - e^{-F_2})\mu_H)$. Thanks to the boundedness of $F_1$ and $F_2$, $(\overline{\nu}^Y, \overline{\mathcal{F}}^Y)$ is equivalent to $(\overline{\nu}, \overline{\mathcal{F}})$ so that $\nu \in S_{\mathcal{D}_b}^1(Y)$ is equivalent to $\nu \in S(X)$ is equivalent to $\nu \in S(Y)$. Since $\overline{\mu}_1 \in S(Y)$, there exists an $\overline{\nu}^Y$-nest $\{K_n\}$ of compact sets such that $1_{K_n}\overline{\mu}_1 \in S_{\mathcal{D}_b}^1(Y)$ for each $n \in \mathbb{N}$ by Lemma 2.3. Set $\overline{\nu}^n := 1_{K_n}\overline{\mu}_1 - (\mu_2 + N(e^{F_1} - F_2)\mu_H), M^n := \exp(A^\overline{\mu}_1^m)$ and $h_n(x) := E^Y_x[e^{\overline{\mu}^m_1}]$. Then

$$\hat{h}(x) := E^Y_x[\exp(-A_{\overline{\mu}_2}^m - A_{\overline{\mu}_1}^m + N(e^{F_1} - F_2)\mu_H)] \leq h_n(x).$$

In the same way of the proof of Lemma 4.9(2), we can deduce

$$\hat{h}(x) = E^Y_x[\exp(A_{\overline{\mu}_1}^m - A_{\overline{\mu}_2}^m - A_{\overline{\mu}_1}^m + N(e^{F_1} - F_2)\mu_H)] 
\geq E^Y_x[\exp(-A_{\overline{\mu}_2}^m - A_{\overline{\mu}_1}^m - A_{\overline{\mu}_1}^m + N(e^{F_1} - F_2)\mu_H)] 
\geq \exp(-\|R(\overline{\mu}_2 + N(e^{F_1} - 1)\mu_H)\|_{\infty}) > 0.$$
Hence
\[ 0 < \exp \left( -\|R(\mathbf{p}_2 + N(e^{F_1} - 1)\mu_H)\|_\infty \right) \leq h_n(x) \leq h(x) \leq \sup_{x \in \mathcal{E}_0} h(x) < \infty. \] (4.4)

The process \((h_n(X_t)M^n_t)_{t \geq 0}\) is a closed \(\mathbf{P}_x\)-martingale for all \(x \in \mathcal{E}\). Indeed, by the Markov property, we see
\[ \mathbf{E}_x^Y [M^n_t \mid \mathcal{F}_t] = \mathbf{E}_x^Y [M^n_t \cdot M^n_{\theta_t} \mid \mathcal{F}_t] = M^n_t \mathbf{E}_x^Y [M^n_{\theta_t}] = M^n_{h_n(X_t)}. \]

Thus, for any \((\mathcal{F}_t)\)-stopping time \(T\), we have
\[ \mathbf{E}_x^Y [h_n(X_T)1_{\{T < \infty\}}] = \mathbf{E}_x^Y [M^n_{\infty}(M^n_T)^{-1}1_{\{T < \infty\}}]. \]

Hence, applying [39, (1.13) Exercise p. 186], we have
\[
\begin{align*}
\mathbf{E}_x^Y \left[ \int_0^\infty h_n(X_s) \, dA^s_\theta \right] &= \mathbf{E}_x^Y \left[ e^{A^\theta_t s} \int_0^\infty e^{-A^\theta_t (dA^s_\theta)} \right] \\
&= \mathbf{E}_x^Y \left[ e^{A^\theta_t s} \left( 1 - e^{-A^\theta_t s} \right) \right] = h_n(x) - 1.
\end{align*}
\]

From this and the inequality \(e^{-\|F\|_\infty} F_2 \leq e^{F_2} (e^{F_2} - 1)\),
\[ e^{-\|F\|_\infty} R^Y (h_n \mathbf{p}_2) (x) \leq R^Y h_n(\mu_2 + N(1 - e^{-F_2})\mu_H)(x) \leq R^Y h_n(\mu_2 + N(e^{F_1} (1 - e^{-F_2})\mu_H)(x) = R^Y (h_n 1_{K_n} \mathbf{p}_2)(x) - h_n(x) + 1. \]

Thus, we see \(h_n \mathbf{p}_2 \in S^1_{D_0}(Y)\). By (4.4), we have \(\mathbf{p}_2 \in S^1_{D_0}(Y)\). By replacing \(M^n_t\) (resp. \(h_n\)) with \(M_t := \exp(\mathbf{A}^\theta_t)\) (resp. \(h\)), we can deduce (4.3) under \(\mathbf{p}_2 \in S^1_{D_0}(Y)\) by the same manner as shown above (see (4.1)).

Finally, we prove the last assertion. From (4.3),
\[ R^Y (h_\mathbf{p}_1) = R^Y (h_\mathbf{p}_2) + h - 1 \]
is bounded above. By (4.4), \(\mathbf{p}_1 \in S^1_{D_0}(Y)\) and \(N(F_2)\mu_H \leq \mathbf{p}_2 \in S^1_{D_0}(Y)\) implies
\[
\mathbf{p}_1^* = \mathbf{p}_1 + N(e^{F_1} - e^{-F_1} - F_2)\mu_H \leq \mathbf{p}_1 + N(e^{F_1} (1 - e^{-F_2})\mu_H \leq \mathbf{p}_1 + e^{\|F\|_\infty}N(F_2)\mu_H \in S^1_{D_0}(Y).
\]

The expression (4.3) yields the fine continuity of \(h\) on \(E\) with respect to \(Y\) (when \(u = 0\)). From this observation, \(h\) is \(\mathcal{E}_Y\)-quasi continuous on \(E\). Moreover, since \(R^Y (h_\mathbf{p})\) is a difference of bounded excessive functions on \(E\) with respect to \(Y\), we have
\[ h - 1 = R^Y (h_\mathbf{p}) \in \mathcal{R}^Y \]
by use of [17] Lemma 2.3.2]. Indeed, let \(\{G_n\}\) be an increasing sequence of relatively compact open sets and \(e^Y_{G_n}\) its 1-equilibrium potential with respect to \((\mathcal{E}_Y, \mathcal{F}_Y)\) on \(L^2(E; e^{2\mu}m)\). Take any bounded excessive function \(f\) with respect to \(Y\) and set \(f_n := f \wedge \|f\|_\infty e^Y_{G_n}\). Then \(f_n \leq \|f\|_\infty e^Y_{G_n}\) with [14] Lemma 2.3.2] yields \(f_n \in \mathcal{R}^Y\) and \(f = f_n\) on \(G_n\) for each \(n \in \mathbb{N}\). Thus we have \(f \in \mathcal{R}^Y\).
Lemma 4.3 Assume that $X$ is transient and $u = 0$. Suppose that $\overline{\mu}_1 = \mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{NK_1}(X)$ and $\overline{\mu}_2 = \mu_2 + N(F_2)\mu_H \in S^1_{D_0}(X)$ hold. Suppose further that $\lambda^2(\overline{\mu}_1) > 0$. Then the additive functional $h(X_t) - h(X_0)$ admits the following decomposition in the strict sense:

$$
\begin{align}
&\left\{ 
\begin{array}{l}
  h(X_t) - h(X_0) = M_t^h + N_t^h, \\
  N_t^h = -\int_0^t h(X_s)\,dA_s^\mu
\end{array}
\right.
\end{align}
$$

(4.6)

for all $t \in [0, +\infty[ \setminus P^Y_x$-a.s. for all $x \in E$, where $M_t^h$ (resp. $N_t^h$) is a square integrable martingale additive functional in the strict sense (resp. CAF in the strict sense which is locally of zero energy) under $Y$.

**Proof.** By Lemma 4.2 we already know $\overline{\mu}_1, \overline{\mu}_2 \in S^1_{D_0}(Y)$. Recall the $e^Y$-nest $(K_n)_{n \in \mathbb{N}}$ of compact sets satisfying $1_{K_n} \overline{\mu}_1 \in S^{(0)}(Y)$ and let $\overline{\mu}_a$ and $h_n(x)$ be as defined in the proof of Lemma 4.2. Note that $h_n$ increases to $h$ as $n \to \infty$. Then by (4.1) and the proof of Lemma 4.2 we have $h_n(x) = R^Y(h_n(x))(x) + 1$ whenever $\lambda^2(\overline{\mu}_1) > 0$. This expression also yields the fine continuity of $h_n$ on $E$ with respect to $Y$. Since $1_{K_n} \overline{\mu}_1 \in S^1_{D_0}(Y)$ and $\mu_2 + N(e^{F_1}(1 - e^{-F_2}))\mu_H \in S^1_{D_0}(Y)$, we see that for each $n \in \mathbb{N}$

$$
h_n - 1 = R^Y(h_n \overline{\mu}_1) \in \mathcal{F}^Y_{loc}
$$

is a difference of bounded excessive functions on $E$ with respect to $Y$. From this observation, $h_n$ is $e^Y$-quasi continuous on $E$. We easily see that $\{h_n\}$ is $e^Y$-quasi uniformly convergent to $h$ on $E$. Indeed, there exists an $e^Y$-nest $\{A_t\}$ of compact sets such that all $h_n$ and $h$ are continuous on each $A_t$. Here $e^Y$-nest $\{A_t\}$ implies $P^Y_x(\lim_{t \to \infty} \tau_{A_t} = \zeta) = 1$ q.e. $x \in E$. Since $h_n$ is increasing on $E_0$ and each $A_t$ is compact, $h_n$ uniformly converges to $h$ on $A_t \cup \{\partial\}$, because $h_n(\partial) = h(\partial) = 1$. In particular, for each $T \in ]0, +\infty[$

$$
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |h_n - h|(X_t) = 0 \quad P^Y_x\text{-a.s. on } \{T < \zeta\} \text{ for q.e. } x \in E.
$$

(4.7)

We apply the generalized Fukushima’s decomposition for bounded $h_n - 1 \in \mathcal{F}^Y_{loc}$ under $P^Y_x$ (see [33, Theorem 4.2]) and Lemma 2.2 (also [17, Lemma 5.4.1]) that

$$
\begin{align}
&\left\{ 
\begin{array}{l}
  h_n(X_t) - h_n(X_0) = M_t^{h_n} + N_t^{h_n}, \\
  N_t^{h_n} = -\int_0^t h_n(X_s)\,dA_s^\mu
\end{array}
\right.
\end{align}
$$

(4.8)

for all $t \in [0, \zeta[ \setminus P^Y_x$-a.s. for q.e. $x \in E$, where $M_t^{h_n}$ (resp. $N_t^{h_n}$) is the martingale additive functional locally of finite energy (resp. CAF locally of zero energy) under $Y$. We define $M_t^{h_n}, N_t^{h_n}$ for $t \geq \zeta$ under $P^Y_x$-a.s. for q.e. $x \in E$ by

$$
N_t^{h_n} := -\int_0^t h_n(X_s)\,dA_s^\mu, \quad M_t^{h_n} := h_n(X_t) - h_n(X_0) - N_t^{h_n}.
$$

Then $M_t^{h_n}$ is a martingale additive functional under $P^Y_x$ for q.e. $x \in E$. Indeed, $(M_t^{h_n})_{t \geq 0}$ is an AF, and for each $t > 0$ we see $E^Y_x[|M_t^{h_n}|] < \infty$ and $E^Y_x[M_t^{h_n}] = 0$ for q.e. $x \in E$ under $\overline{\mu}_1, \overline{\mu}_2 \in S^1_{D_0}(Y)$. The decomposition (4.8) also holds for all $t \in [0, +\infty[ \setminus P^Y_x$-a.s. for q.e. $x \in E$. We then see that for $m > n > 0$,

$$
N_t^{h_m} - N_t^{h_n} = -\int_0^t h_m(X_s)\,dA_s^\mu + \int_0^t h_n(X_s)\,dA_s^\mu
$$

$$
= -\int_0^t (h_m 1_{K_m} - h_n 1_{K_n})(X_s)\,dA_s^\mu + \int_0^t (h_m - h_n)(X_s)\,dA_s^{\mu_2 + N(e^{F_1}(1 - e^{-F_2}))\mu_H}.
$$
Hence
\[ |N_{t_{n}}^{h_{n}} - N_{t_{l}}^{h_{n}}| \leq \int_{0}^{t} (h - h_{n}1_{K_{n}})(X_{s})dA_{s}^{p_1} + \int_{0}^{t} (h - h_{n})(X_{s})dA_{s}^{p_2} + N(e^{F_1(1-e^{-F_2})})\mu_H \]
\[ \leq \int_{0}^{t} (h - h_{n}1_{K_{n}})(X_{s})dA_{s}^{p_1,2} + N(e^{F_1(1-e^{-F_2})})\mu_H . \]

Thus, using (4.7), we have
\[ \sup_{0 \leq t \leq T} |M_{t_{n}}^{h_{n}} - M_{t_{l}}^{h_{n}}| \leq \sup_{0 \leq t \leq T} |h_{m} - h_{n}|(X_{t}) + \sup_{0 \leq t \leq T} |N_{t_{n}}^{h_{m}} - N_{t_{l}}^{h_{n}}| \]
\[ \leq \sup_{0 \leq t \leq T} |h - h_{n}|(X_{t}) + \int_{0}^{T} |h - 1_{K_{n}}h_{n}|(X_{s})dA_{s}^{p_1,2} + e^{||F_1||\infty}N(\mu_H) \]
\[ \leq \sup_{0 \leq t \leq T} |h - h_{n}|(X_{t})(1 + A_{T_{1}}^{p_1,2} + e^{||F_1||\infty}N(\mu_H)) \]
\[ + ||h||_{\infty} \int_{0}^{T} 1_{K_{n}}(X_{s})dA_{s}^{p_1,2} + e^{||F_1||\infty}N(\mu_H) \]
\[ \rightarrow 0 \quad \text{as} \quad m, n \rightarrow \infty, \quad \text{P}_{x}^{Y} - \text{a.s. on} \quad \{ T < \zeta \} \quad \text{for q.e.} \quad x \in E. \]

Similarly, in view of the Lebesgue’s dominated convergence and \( \overline{p}_1, \overline{p}_2 \in S_{D_0}(Y) \), we can deduce
\[ |M_{t_{n}}^{h_{m}} - M_{t_{l}}^{h_{n}}| = |N_{t_{n}}^{h_{m}} - N_{t_{l}}^{h_{n}}| \]
\[ \leq \int_{0}^{\zeta} |h - 1_{K_{n}}h_{n}|(X_{s})dA_{s}^{p_1,2} + e^{||F_1||\infty}N(\mu_H) \rightarrow 0 \quad \text{as} \quad m, n \rightarrow \infty \]
\( \text{P}_{x}^{Y} - \text{a.s. for q.e.} \quad x \in E. \) Therefore, for each \( t \in [0, +\infty] \), \( \{ M_{t_{n}}^{h_{m}} \} \) and \( \{ N_{t_{n}}^{h_{n}} \} \) converge as \( n \rightarrow \infty \) under \( \text{P}_{x}^{Y} \) for q.e. \( x \in E \). Set \( M_{t}^{h} := \lim_{n \rightarrow \infty} M_{t_{n}}^{h_{n}} \) and \( N_{t}^{h} := \lim_{n \rightarrow \infty} N_{t_{n}}^{h_{n}} \). Then \( M^{h} \) (resp. \( N^{h} \)) satisfies that (1.0) holds for all \( t \in [0, +\infty] \) \( \text{P}_{x}^{Y} - \text{a.s. for q.e.} \quad x \in E \), and hence it is a square integrable martingale additive functional locally of finite energy (resp. CAF locally of zero energy) under \( \text{P}_{x}^{Y} \) for q.e. \( x \in E \). Moreover, from (4.8), we have for \( p \in \mathbb{N} \)
\[ \text{E}_{x}^{Y} \left[ |M_{t_{n}}^{h_{m}}|^{p} \right] \leq 2 \cdot 4^{p-1}\|h_{n}\|_{\infty}^{p} + 2^{p-1}\text{E}_{x}^{Y} \left[ \left( \int_{0}^{t} h_{n}(X_{s})dA_{s}^{p_1,2} + e^{||F_1||\infty}N(\mu_H) \right)^{p} \right] \]
\[ \leq 2 \cdot 4^{p-1}\|h\|_{\infty}^{p} + 2^{p-1}p! \left\| R^{Y}(h(\overline{p}_1 + \mu_2 + e^{||F_1||\infty}N(\mu_H))) \right\|_{\infty}^{p} =: C_{p} < \infty \]
for q.e. \( x \in E \), where we use [14] Lemma 2.2. So \( \{ M_{t_{n}}^{h_{m}} \}_{n \in \mathbb{N}} \) is uniformly \( \text{P}_{x}^{Y} - \text{integrable for q.e.} \quad x \in E \). In particular, we have
\[ R^{Y}(\mu_{(h_{n})})(x) = \text{E}_{x}^{Y} \left[ (M_{t_{n}}^{h_{m}})_{\infty} \right] = \text{E}_{x}^{Y} \left[ (M_{t_{\infty}}^{h_{m}})_{\infty}^{2} \right] \leq C_{2} \]
for q.e. \( x \in E \), consequently \( \sup_{x \in E} R^{Y}(\mu_{(h_{n})})(x) \leq C_{2} \) because of the fine continuity of \( x \mapsto R^{Y}(\mu_{(h_{n})})(x) \). Then we have \( \mu_{(h_{n})} \in S_{D_0}^{1}(Y) \). Therefore, we can redefine \( M_{t_{n}}^{h_{m}} \) (resp. \( N_{t_{n}}^{h_{n}} \)) as the martingale additive functional locally of finite energy (resp. CAF locally of zero energy) in the strict sense under \( Y \), and hence the following Fukushima’s decomposition in the strict sense holds (311 Theorem 6.2):
\[ h_{n}(X_{t}) - h_{n}(X_{0}) = M_{t_{n}}^{h_{n}} + N_{t_{n}}^{h_{n}} \]
for all \( t \in [0, +\infty] \) \( \text{P}_{x}^{Y} - \text{a.s.} \) for all \( x \in E \). Moreover, by combining the same way as above and Fatou’s lemma, we can obtain
\[ \sup_{x \in E} R^{Y}(\mu_{(h_{n})})(x) \leq 8\|h\|_{\infty}^{2} + 4 \left( \left\| R^{Y}(h(\overline{p}_1 + \mu_2 + e^{||F_1||\infty}N(\mu_H))) \right\|_{\infty}^{2} < \infty, \right) \]
that is, \( \mu(d\nu) \in S^1_{D_0}(Y) \). Using this fact, we can redefine both \( M^h \) and \( N^h \) in the strict sense under \( Y \), and which leads us to the conclusion. \( \square \)

Under \( \lambda^2(\mu_1) > 0 \), \( u = 0 \), \( \mu_1 = \mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{N^1} \) and \( \mu_2 = \mu_2 + N(F_2)\mu_H \in S^1_{D_0}(X) \), we define a martingale additive functional of \( Y \) by \( M_t := \int_0^t h(X_s)^{-1}dM^h_s \) and denote by \( L^h_t := \exp(M_t) \) the Doleans-Dade exponential of \( M_t \), that is, \( L^h_t \) is the unique solution of \( L^h_t = 1 + \int_0^t L^h_{s-}dM_s \) for \( t \in [0, \zeta], \) \( P^Y \)-a.s. for q.e. \( x \in E \). Then we see

\[
L^h_t = \exp \left( M_t - \frac{1}{2} \langle M^c \rangle_t \right) \prod_{0<s\leq t} \frac{h(X_s)}{h(X_{s-})} \exp \left( 1 - \frac{h(X_s)}{h(X_{s-})} \right),
\]

where \( M^c_t \) is the continuous part of \( M_t \). Note that \( L^h_t \) is also a multiplicative functional under \( Y \). By Lemma [1,3] and Itô’s formula applied to the semimartingale \( h(X_t) \) in (1.6) with the function \( \log x \), we see that \( L^h_t \) has the following expression:

\[
L^h_t = \frac{h(X_t)}{h(X_0)} \exp \left( A^h_T \right).
\]

Let \( Y^h := (\Omega, \mathcal{F}, \mathcal{F}_t, \tilde{X}_t, \mathcal{P}^Y, \zeta^h) \) the transformed process of \( Y \) by \( L^h_t \):

\[
\mathcal{E}^Y_{\tilde{X}}[f(X_t)] = \mathcal{E}^Y_{\tilde{X}}[L^h_t f(X_t)], \quad f \in \mathcal{B}_b(E).
\]

Set \( \ell := -\log h = -\log(R^Y(h\mu) + 1) \). Since \( R^Y(h\mu) \in \mathcal{F}^Y \), we have \( \ell \in \mathcal{F}^Y_{loc} \) and \( h = e^{-\ell} \). It is easy to see the fine continuity of \( \ell \) on \( E \) with respect to \( Y \). Then we can see that the transform \( L^h_t \) belongs to the class of Girsanov transforms considered in Section 3 (or in [13]) under \( Y \). In particular, \( Y^h \) is an \( h^2 \)-symmetric Hunt process on \( E \). From Theorem [5,1](cf. [29, Theorem 3.2]) the associated Dirichlet form \( (\mathcal{E}^{Y,h}, \mathcal{F}^{Y,h}) \) on \( L^2(E; h^2 \mathfrak{m}) \) is identified by

\[
\mathcal{E}^{Y,h}(f, f) = \frac{1}{2} \int_E h^2 \mu^c(f) + \int_{E \times E} (f(x) - f(y))^2 e^{F(x,y)} h(x) h(y) J(dx, dy)
\]

for \( f \in \mathcal{F} \). Let \( Y^h \) be the transformed process of \( Y \) by \( L^h_t \). Then \( (\mathcal{E}^{Y,h}, \mathcal{F}^{Y,h}) \) is nothing but the Dirichlet form on \( L^2(E; h^2 \mathfrak{m}) \) associated with \( Y^h \). Moreover we see that \( (\mathcal{E}^{Y,h}, \mathcal{F}^{Y,h}) \) is equivalent to \( (\mathcal{E}, \mathcal{F}) \) because of (1.2) and

\[
e^{-2\|R^Y\mathfrak{m}\|_\infty} e^{-\|F\|_\infty \mathcal{E}^{(c)}(f, f)} \leq (\mathcal{E}^{Y,h})^{(c)}(f, f) \leq \|h\|_{\infty}^2 e^{\|F\|_\infty \mathcal{E}^{(c)}(f, f)} \quad \text{ (4.10)}
\]

and

\[
e^{-2\|R^Y\mathfrak{m}\|_\infty} e^{-\|F\|_\infty J(dx, dy)} \leq J^{Y,h}(dx, dy) \leq \|h\|_{\infty}^2 e^{\|F\|_\infty J(dx, dy)}. \quad \text{ (4.11)}
\]

Here \( J^{Y,h}(dx, dy) := e^{F(x,y)} h(x) h(y) J(dx, dy) \) is the jumping measure of \( (\mathcal{E}^{Y,h}, \mathcal{F}^{Y,h}) \). Therefore, under (A.1) (resp. (A.2)), \( (\mathcal{E}^{Y,h}, \mathcal{F}^{Y,h}) \) admits a heat kernel \( p^{Y,h}_t(x, y) \) (resp. a heat kernel \( p^{Y,h}_t(x, y) \) in the strict sense) on \( 0, +\infty \times E \times E \) such that

\[
p^{Y,h}_t(x, y) \succneq \phi_2(t, x, y) \quad \text{m.a.e.} \quad d(x, y) \in T_t
\]

(resp. \( \phi_1(t, x, y) \succneq p^{Y,h}_t(x, y) \succneq \phi_2(t, x, y), \quad d(x, y) \in T_t \)).

Now, we give the proofs of Theorem [1,3] Corollary [1,5] and Theorem [1,7].
Thus we have

\[ m \exp(-u \mathbb{E}^1) > 0 \]  

(see \[29\] Lemma 5.1). Since \( \mu(\omega) \in S_{NK}^1(X) \), we have

\[
e^{-u N(e^{-u}(F_1 - 1)\mu_H)} = e^{-2u(\mathbb{E}_1 + N(e^{-u}(F_1 - 1)\mu_H))} = e^{-2u(\mathbb{E}_1 + N(e^{-u}F_2)\mu_H)}
\]

We apply Lemma 4.2 and (4.2). Recall that the additive functional \( A \) is given by \( A_t := A_t^0 + A_t^2 \) and \( A_t^0 = A_t^{F_1} - A_t^{F_2} \). Since

\[ p_t^A(t,x,y) = h(x)h(y)p_{t,Y}^{Y,h}(x), \tag{12.2} \]

we can conclude the assertion, because \( 0 < e^{-||Y,\mathbb{E}_1||} \leq h(x) \leq ||h|| \infty \) and \( p_t^{Y,h}(x,y) \leq \phi_2(t,x,y) \) m-a.e. \( d(x,y) \in T_t \) (resp. \( \phi_1(t,x,y) \leq p_t^{Y,h}(x,y) \leq \phi_2(t,x,y) \), \( d(x,y) \in T_t \) under (A.1) (resp. (A.2)). Therefore, we obtain the conclusion.

(3): Let \( d := \{(x,y) \in E \times E \mid R(x,y) = \infty \} \). Since \( p_t^A(t,x,y) \leq p_1(t,x,y) \), we see that there exists \( C > 0 \) such that

\[ R^A(x,y) \leq CR^A(x,y) < \infty \quad \text{for} \quad (x,y) \in E \times E \backslash d. \tag{13.3} \]

Thus we have \( R^A(x,y) < \infty \) for m-a.e. \( y \in E^x \) where \( E^x = \{ y \in E \mid (x,y) \notin d \} \). Since \( m(E \backslash E^x) = 0 \), \( R^A(x,y) < \infty \) m-a.e. \( y \in E \). By Theorem 1.3 we obtain \( \lambda^A(\mathbb{P}_t^1) > 0 \). \( \square \)

Next we prove Corollary 1.3. The proof is a mimic of the proof of Theorem 1.3 except the representation of CAF locally of zero energy part in (1.6).

**Proof of Corollary 1.3** (1.1, 2): As noted in the proof of Theorem 1.3, we may and do assume \( u = 0 \). First we assume \( \mathbb{P}_t = \mu_2 + N(F_2)\mu_H \in S_{NK}^1(X) \). Let \( Y^{(\alpha)} \) be the \( \alpha \)-subprocess of \( Y \) killed at rate \( \alpha \). Set the function \( h_\alpha(x) = \mathbb{E}_x^{Y^{(\alpha)}}[e^{\mathbb{E}^2}] \) for \( x \in E_0 \). Suppose \( \lambda^{\mathbb{P}_t^1}(\mathbb{P}_t^1) > 0 \), equivalently, \( \lambda^{\mathbb{P}_t^1}(\mathbb{P}_t^1) > 0 \) in view of \[29\] Lemma 5.1. Then \( h_\alpha \) satisfies

\[ 0 < e^{-||Y,\mathbb{E}_1||} \leq h_\alpha(x) \leq \sup_{x \in E} h_\alpha(x) < \infty \) and \( h_\alpha = R_1^{X}(h_\alpha Y) \) in \( \mathcal{F}_{\text{loc}}^Y \) by virtue of Lemma 4.2 and (1.5). By Lemma 4.2 \( \mathbb{P}_1 \in S_{FK}^1(Y) \) and \( \mathbb{P}_2 \in S_{FK}^2(Y) \). Using Lemma 4.2 by following the proof of Lemma 4.3 line by line, we have the following strict decomposition for \( h_\alpha - 1 \in \mathcal{F}_{\text{loc}}^Y \) under \( \mathbb{P}_x^Y \):

\[
\begin{aligned}
\{ h_\alpha(X_t) - h_\alpha(X_0) = M_t^{h_\alpha} + N_t^{h_\alpha}, \\
N_t^{h_\alpha} = \alpha \int_0^t (h_\alpha(X_s) - 1)(X_s)ds - \int_0^t h_\alpha(X_s)dM_t^{h_\alpha}
\end{aligned}
\]

for all \( t \in [0, +\infty] \) \( \mathbb{P}_x^Y \)-a.s. for all \( x \in E \). Let \( M_t^{(\alpha)} := \int_0^t h_\alpha(X_s) - 1dM_t^{h_\alpha} \) and denote by \( L_t^{h_\alpha} := \text{Exp}(M^{(\alpha)}_t) \) the Doléans-Dade exponential of \( M_t^{(\alpha)} \). By using Itô’s formula applied to \( h_\alpha(X_t) \) with the function \( \log x \), we then have

\[
L_t^{h_\alpha} = \text{Exp}\left(M_t^{(\alpha)} - \frac{1}{2}(M^{(\alpha)},c_t)\right) \prod_{0 < s \leq t} \frac{h_\alpha(X_s)}{h_\alpha(X_0)} \exp\left(1 - \frac{h_\alpha(X_s)}{h_\alpha(X_0)}\right)
\]

\[
=h_\alpha(X_t) \frac{h_\alpha(X_0)}{h_\alpha(X_0)} \exp\left(A_t^{\mathbb{E}} + \alpha \int_0^t \left(h_\alpha(X_s) - 1\right)(X_s)ds\right).
\]
By noting again that the transform $L^h_a$ belongs to the class of Girsanov transforms considered in Section 4, the transformed process $Y_t^h(x,y)$ by $L^h_a$ becomes an $h_a^2$-m-symmetric Hunt process on $E$ and the associated Dirichlet form $(\mathcal{E}^Y_h, \mathcal{F}^Y_h)$ is also equivalent to $(\mathcal{E}, \mathcal{F})$ in view of (4.10) and (1.11). Let $p_t^Y(x,y)$ be the heat kernel associated to $(\mathcal{E}^Y_h, \mathcal{F}^Y_h)$. By virtue of the above expression for $L^h_a$, we see

$$P_t^A f(x) = E^x_\mu [\exp\left( -\alpha \int_0^t (h_a^{-1} - 1)(s) ds \right) f(X_t)] = h_a(x) E_x^Y p_t^Y(x,y)$$

Then we have

$$e^{-k_1^2} h_a(x) h_a(y) p_t^Y(x,y) \leq p_t^A(x,y) \leq e^{k_2^2} h_a(x) h_a(y) p_t^Y(x,y),$$

where $k_1 = k_1(\alpha) := \alpha(\ell_1^2 - 1)$ and $k_2 = k_2(\alpha) := \alpha(1 - \ell_2^2)$ for $\ell_1 = \inf_{x \in E} h_a(x)$ and $\ell_2 = \sup_{x \in E} h_a(x)$. Since $p_t^Y(x,y) \lesssim \varphi_2(t,x,y)$ m-a.e. $d(x,y) \in T_t$ (resp. $\varphi_1(t,x,y) \lesssim p_t^Y(x,y)$ m-a.e. $d(x,y) \in T_t$) holds under (A.1) (resp. (A.2)), we see from (4.14) that $p_t^A(x,y) \lesssim_k \varphi_2(t,x,y)$ m-a.e. $d(x,y) \in T_t$ (resp. $\varphi_1(t,x,y) \lesssim_k p_t^A(x,y)$ m-a.e. $d(x,y) \in T_t$) (k := $\max\{k_1, k_2\}$ > 0) holds under (A.1) (resp. (A.2)).

Suppose that $p_t^A(x,y) \lesssim_k p_t(x,y)$ for $\alpha > k$. Then there exists $C > 0$ such that for each $x \in E$

$$R^A_\alpha(x,y) \leq CR_{\alpha-k}(x,y) < \infty \quad \text{for m-a.e. } y \in E.$$ 

Thus $R^A_\alpha(x,y) < \infty$ m.a.e. $y \in E$ for each $x \in E$. Applying Theorem 5.3 to $X^{(\alpha)}$, we have $\lambda^{2\alpha}(\mathbb{P}_1) > 0$. 

Finally, we give the proof of Theorem 1.7. To do this, we need the following lemma.

**Lemma 4.4** For each $\nu \in S_{EK}^1(X)$, there exists $\alpha > 0$ such that $\nu \in S_{NK1}^1(X^{(\alpha)})$.

**Proof.** By $\nu \in S_{EK}^1(X)$, we see that $\sup_{x \in E} R^\alpha_\nu(x) < 1$ for some large $\alpha > 0$. From this, we can see that $\sup_{x \in E} R^\alpha_\nu(x) < 1$ for any Borel set $K$ of $E$ with $\nu(K) < \infty$ and for any $\nu$-measurable set $B \subset K$ with $\nu(B) < \delta$ ($\delta > 0$), which implies that $\nu$ is in the family of semi-Green-tight Kato class measures in the sense of Chen with respect to $X^{(\alpha)}$. Therefore, $\nu \in S_{NK1}^1(X^{(\alpha)})$. (See Remark 2.1.)

**Proof of Theorem 1.7.** Since $\mu(\alpha) \in S_K^1(X)$, we have $e^{-2\alpha} \mu_1 = e^{-2\alpha} (\mu_1 + N(\nu^{[\alpha]} \nu F - 1) \mu_H) \in S_{EK}^1(U)$ and $e^{-2\alpha} \mu_2 + N F \mu_H \in S_{EK}^1(U)$ by [27, Lemma 4.1(3)]. By Lemma 4.1, there exists large $\alpha > 0$ such that $e^{-2\alpha} \mu_1 \in S_{NK1}^1(U^{(\alpha)})$. Applying Stollmann-Voigt’s inequality to the Girsanov transformed process $U$, we have that for large enough $\alpha > 0$

$$\lambda^{2\alpha}(\mathbb{P}_1) = \inf \left\{ \lambda^2(\mathbb{P}_1) \left| \int_E f^2 d\mathbb{P}_1 = 1 \right. \right\} = \inf \left\{ |\mathcal{E}^U(f_\alpha^u, f_\alpha^u) - \int_E f^2 d\mathbb{P} - \int_{E \times E} f(x)f(y)(e^{F(x,y)} - 1)N(x,dy)\mu_H(dx) | \left| f \in \mathcal{C}, \int_E f^2 d\mathbb{P}_1 = 1 \right. \right\}$$

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We will use the following notations in this section. Let $X$ for some $k > 0$ and $d$ kernel in the strict sense) which is equivalent to $\lambda = 0$. We say that $g$ satisfies $\mu(t)$, because $\mu(t)$, where $\mu(t) = \int_0^t N(e^{[u] - [u] - 1})(X_s)dH_s + \frac{1}{2}(M^{-u,e})t$. Applying Corollary (5.1) (2) to $U$, we can construct an integral kernel (resp. an integral kernel in the strict sense) $p_1(t,x,y) = (p_1^U)\mathcal{M}_r^F(x,y)$, such that $p_1(t,x,y) \lesssim_k \phi_2(t,x,y)$ m.a.e. $d(x,y) \in T$ (resp. $\phi_1(t,x,y) \lesssim_k p_1^U(t,x,y)$, $d(x,y) \in T$) under (A.1) (resp. (A.2)) for some $k > 0$. Therefore we obtain the conclusion.

5 Examples

We will use the following notations in this section.

**Definition 5.1** Let $g : [0, +\infty[ \rightarrow [0, +\infty]$, and $a \in ]0, +\infty]$, $\beta_1, \beta_2 > 0$, and $c, C > 0$.

1. We say that $g$ satisfies $L(\beta_1, c)$, the (global) lower scaling condition with index $\beta_1$, if

$$\frac{g(R)}{g(r)} \geq c \left( \frac{R}{r} \right)^{\beta_1} \text{ for all } r \leq R < \infty.$$  

2. We say that $g$ satisfies $U(\beta_2, C)$, the (global) upper scaling condition with index $\beta_2$, if

$$\frac{g(R)}{g(r)} \leq C \left( \frac{R}{r} \right)^{\beta_2} \text{ for all } r \leq R < \infty.$$  

5.1 Brownian Motion in $\mathbb{R}^d$

Let $X = (\Omega, X_t, P_x)$ be d-dimensional Brownian motion on $\mathbb{R}^d$. Note that $X$ satisfies (A.2) with $T = [0, +\infty[^2$. A signed Borel measure $\mu$ on $\mathbb{R}^d$ is said to be of Kato class if

$$\limsup_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| < r} \frac{|\mu|(dy)}{|x-y|^{d-2}} = 0 \text{ when } d \geq 3,$$

$$\limsup_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| < r} (\log |x-y|^{-1})|\mu|(dy) = 0 \text{ when } d = 2,$$

$$\sup_{x \in \mathbb{R}^d} |\mu|(B(x, 1)) < \infty \text{ when } d = 1.$$
Here $|\mu| := \mu_+ + \mu_-$ is the total variation measure of $\mu$. A signed Borel measure $\mu$ on $\mathbb{R}^d$ is said to be of local Kato class if $1_K \mu$ is of Kato class for every compact subset $K$ of $\mathbb{R}^d$. By definition, any measure $\mu$ of local Kato class is always a signed Radon measure. Denote by $K_d$ (resp. $K^{\text{loc}}_d$) the family of non-negative measures of Kato class (resp. local Kato class) on $\mathbb{R}^d$. It is essentially proved in [1] that a non-negative measure $\mu$ is in Kato class $K_d$ if and only if $\mu$ is a smooth measure in the strict sense and

$$\limsup_{t \to 0} \sup_{x \in \mathbb{R}^d} \left( \int_0^t p_s(x,y)ds \right) \mu(dy) = \limsup_{t \to 0} E_{\nu}[A^\mu_s] = 0,$$

where $A^\mu$ is a PCAF of $X$ admitting no exceptional set associated to $\mu$ under Revuz correspondence. That is, $K_d = S^1_K(X)$.

Take $\phi \in C^\infty(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$ with $|\nabla \phi|^2 dx \in K_d$. It is proved in [19] Theorem 2.8] that there exists a jointly continuous integral kernel $p^\phi_t(x,y)$ such that

$$E_x[e^{\int_t^\infty \phi(X_s)ds}] = \int_{\mathbb{R}^d} p^\phi_t(x,y)f(y)dy, \quad f \in \mathcal{B}_b(\mathbb{R}^d).$$

Our Theorem 1.7 with $\phi^i_t(x,y) = \frac{1}{(2\pi t)^{d/2}} \exp \left(-\frac{|x-y|^2}{2t}\right)$ $(i = 1, 2)$ tells us that there exist constants $C, C_1, C_2, k > 0$ with $C_1 \leq C_2$ such that

$$C^{-1}e^{-kt} \frac{e^{-\frac{|x-y|^2}{2t}}}{(2\pi t)^{d/2}} \leq p^\phi_t(x,y) \leq Ce^{kt} \frac{e^{-\frac{|x-y|^2}{2t}}}{(2\pi t)^{d/2}} \quad x,y \in \mathbb{R}^d, \quad t > 0,$$

which is blunter than the estimate in [19] Theorem 2.9]. However, Theorem 1.3 with $\phi_t(x,y) = \frac{1}{(2\pi t)^{d/2}} \exp \left(-\frac{|x-y|^2}{2t}\right)$ $(i = 1, 2)$ tells us that for $d \geq 3$ and $|\nabla \phi|^2 dx \in K_d$, $\lambda^\phi \left(\frac{1}{2} |\nabla \phi|^2 dx\right) > 0$ if and only if there exist $C, C_1, C_2 > 0$ with $C_1 \leq C_2$ such that (5.1) holds with $k = 0$. Note that for $\phi \in C^\infty(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$, $|\nabla \phi|^2 dx \in K_d$ is equivalent to $|\nabla \phi|^2 dx \in K^\infty_d$ by [14] Lemma 5.1(2) and [20] Proposition 1], where

$$K^\infty_d := \left\{ \nu \in K_d \mid \limsup_{R \to \infty} \sup_{x \in \mathbb{R}^d} \int_{|x| \geq R} \frac{\nu(dy)}{|x-y|^{d-2}} = 0 \right\}$$

is the class of Green-tight measures of Kato class in the sense of Zhao. It is easy to see $K^\infty_d = S^1_{K^\infty}(X)$, where

$$S^1_{K^\infty}(X) := \left\{ \nu \in S^1_K(X) \mid \forall \varepsilon > 0, \exists K : \text{compact such that } \sup_{x \in E} \int_{K^c} \frac{\nu(dy)}{|x-y|^{d-2}} < \varepsilon \right\}.$$
Hereafter, we focus on the case $d = 1$, that is, $X = (\Omega, X_t, P_x)$ is a 1-dimensional Brownian motion. Let $\beta > -\frac{3}{2}$ with $\beta \neq 1$. Define

$$H_\beta^t := \lim_{\varepsilon \to 0} \int_0^t |X_s|^\beta (\text{sgn}(X_s))1_{\{|X_s| \geq \varepsilon\}} ds. \quad (5.2)$$

It is known that $H_\beta^t$ is well-defined and continuous. $H_\beta^t$ is the so-called Hilbert transform of Brownian local time. For $\beta = -1$, $H_\beta^t$ is called the Cauchy principal value of Brownian local time. Set

$$\ell_\beta(x) := \begin{cases} x \log |x| - x, & \text{if } \beta = -1 \\ \frac{1}{\beta+1} \int_0^x |y|^\beta+1 dy, & \text{if } \beta \in [-\frac{3}{2}, +\infty \setminus \{ \pm 1 \}] \end{cases}$$

$\rho_1 := \ell_\beta \phi_R$ and $\rho_2 := \ell_\beta (1 - \phi_R)$ for sufficiently large $R > 0$. Here $\phi_R$ is any smooth function such that $\phi_R = 1$ on $B_R(0)$ and $\phi_R = 0$ on $\mathbb{R}^d \setminus B_{R+1}(0)$. It is known that $H_\beta^t$ is the (local) zero energy part $N^e_\beta$ in Fukushima’s decomposition for $\ell_\beta \in H^1(\mathbb{R})_{\text{loc}}$ (see [16]). Note that $|\rho_1'(x)|^2 dx$ is of Kato class and $|\rho'_2(x)|^2 dx$ is of local Kato class. If $\beta \leq 0$, then we can confirm that $\rho_2$ is bounded and

$$\ell_\beta''(x)(1 - \phi_R(x)) dx = \begin{cases} \frac{1}{\beta} (1 - \phi_R(x)) dx, & \text{if } \beta = -1 \\ |x|^\beta \text{sgn}(x)(1 - \phi_R(x)) dx, & \text{if } \beta \in [-\frac{3}{2}, -1 \cup ] -1, 0] \in S^1_K(X). \end{cases}$$

We then see that $H_\beta^t = N^e_\beta = N_1^{\rho_1} - A^\mu_1$ for $\mu(dx) := \frac{1}{\beta} \rho_2''(x) dx \in S^1_K(X) - S^1_K(X)$. Applying Theorem [17] again, we obtain that there exist $C, C_1, C_2, k > 0$ with $C_1 \leq C_2$ such that

$$C^{-1} e^{-kt} e^{-\frac{|x-y|^2}{Ct}} \leq p_\beta^t(x,y) \leq C e^{kt} e^{-\frac{|x-y|^2}{Ct}}, \quad x, y \in \mathbb{R}, \quad t > 0, \quad (5.3)$$

where $p_\beta^t(x,y)$ is the integral kernel corresponding to the Feynman-Kac semigroup $P_\beta^t f(x) := E_x[e^{H_\beta^t} f(X_t)]$ provided $\beta \in [-\frac{3}{2}, 0]$. 

**Remark 5.2** The condition $\alpha \leq 0$ in [14] Theorem 6.2] is incorrect. It should be corrected to be $\alpha < 0$ for $\ell''_\alpha(x)(1 - \phi_R(x)) dx = |x|^\alpha \text{sgn}(x)(1 - \phi_R(x)) dx \in S^1_K(X)$.

### 5.2 Symmetric diffusion processes

Throughout Sections 5.2 5.4, let $(E, d)$ be a locally compact separable metric space and $\mathfrak{m}$ be a positive Radon measure on $E$ with full support. We also assume that every ball in $(E, d)$ is relatively compact. Set $V(x, r) := \mathfrak{m}(B(x, r))$ for $B(x, r) = \{ y \in E \mid d(x,y) < r \}$. Throughout Sections 5.2 5.4, we assume that the volume doubling condition (VD): there exists a constant $C_D > 0$ such that $V(x, 2r) \leq C_D V(x, r)$ for all $x \in E$ and $r > 0$, and the reverse volume doubling condition (RVD): there exists constants $c_1 > 0$ and $d_1 > 0$ such that

$$\frac{V(x, r_2)}{V(x, r_1)} \geq c_1 \left( \frac{r_2}{r_1} \right)^{d_1} \quad \text{for all } x \in E, \quad 0 < r_1 \leq r_2 \leq 2 \text{diam}(E). \quad (5.4)$$

(cf. [23]). (VD) implies (RVD) if $E$ is connected. See [16] Proposition 2.1 and a paragraph before Remark 2.1].
In this section, we follow the terminology and conditions in [24]. Let $\phi$ be a fixed continuous increasing bijection on $]0, +\infty[$ satisfying $L(\beta, C^{-1}_\phi)$ and $U(\beta', C_\phi)$ for some $C_\phi \geq 1$ and $1 < \beta \leq \beta'$. That is,

$$C^{-1}_\phi \left( \frac{R}{r} \right)^\beta \leq \frac{\phi(R)}{\phi(r)} \leq C_\phi \left( \frac{R}{r} \right)^{\beta'} \quad \text{for all } 0 < r < R. \quad (5.5)$$

We consider the following condition: there exists a large $L > 0$ such that

$$\text{ess-} \sup_{r \geq L} \frac{r \phi'(r)}{\phi(r)} < \infty. \quad (5.6)$$

It is easy to see that (5.5) and (5.6) are satisfied for $\phi(r) = r^\beta$ with some $\beta > 1$. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local irreducible regular Dirichlet form on $L^2(E; m)$ and $X = (\Omega, X_t, P_x, x \in E_0)$ an $m$-symmetric diffusion process associated to $(\mathcal{E}, \mathcal{F})$. $X$ is said to be conservative, denoted by $(C)$ in short, if $P_x(X_t \in E$ for all $t > 0) = 1$ $m$-a.e. $x \in E$. We set

$$\Phi(s,t) = \Phi(\phi, s, t) := \sup_{r > 0} \left\{ \frac{s}{r} - \frac{t}{\phi(r)} \right\}. \quad (5.7)$$

**Definition 5.3 (Green kernel bounds)** We say that the condition $(G)_\phi$ holds if there exist constants $\kappa \in ]0, 1[$ and $C \in ]0, +\infty[$ such that, for any open ball $B = B_R(x_0)$, the Green kernel $R^B(x, y)$, $x, y \in B$ exists, is jointly continuous off diagonal, and satisfies

$$R^B(x_0, y) \leq C \int_{\kappa d(x_0, y)}^R \frac{\phi(s)ds}{sV(x_0, s)} \quad \text{for } 0 < d(x_0, y) < R,$$

$$R^B(x_0, y) \geq C^{-1} \int_{\kappa d(x_0, y)}^R \frac{\phi(s)ds}{sV(x_0, s)} \quad \text{for } 0 < d(x_0, y) < \kappa R.$$

**Definition 5.4 ((UE)$_\phi$, (LE)$_\phi$)** For $c \in ]0, 1[$ and $k \geq 0$, we say that $(UE)^{c,k}$ holds if the heat kernel $p_t(x, y)$ of $X$ exists and satisfies the following upper estimate

$$p_t(x, y) \leq \frac{Ce^{kt}}{V(x, \phi^{-1}(t))} \exp \left( -\frac{1}{2} \Phi(cd(x, y), t) \right) \quad (5.8)$$

for all $t > 0$ and $m$-a.e. $x, y \in E$, where $C > 0$ is a constant independent of $x, y, t$.

For $c \in [1, +\infty[$ and $k \geq 0$, we say that $(LE)^{c,k}$ holds if the heat kernel $p_t(x, y)$ of $X$ exists and satisfies the following lower estimate

$$p_t(x, y) \geq \frac{Ce^{-kt}}{V(x, \phi^{-1}(t))} \exp (-c \Phi(cd(x, y), t)) \quad (5.9)$$

for all $t > 0$ and $m$-a.e. $x, y \in E$, where $C > 0$ is a constant independent of $x, y, t$.

We say that $(UE)^k$ (resp. $(LE)^k$) holds if it satisfies $(UE)^{c,k}$ (resp. $(LE)^{c,k}$) for some $c \in ]0, 1]$ (resp. $c \in [1, +\infty[$). In particular, we say that $(UE)_\phi$ (resp. $(LE)_\phi$) holds if it satisfies $(UE)^k$ (resp. $(LE)^k$) with $k = 0$.

**Definition 5.5 ((NLE)$_\phi$)** We say that $(NLE)^k$ holds if the heat kernel $p_t(x, y)$ of $X$ exists and satisfies the following lower estimate

$$p_t(x, y) \geq \frac{Ce^{-kt}}{V(x, \phi^{-1}(t))} \quad (5.10)$$
for all $t > 0$ and $\mathfrak{m}$-a.e. $x,y \in E$, such that

$$d(x,y) \leq \varepsilon \Phi^{-1}(t),$$

(5.11)

where $C, \varepsilon > 0$ and $k \geq 0$ are constants independent of $x, y, t$. In particular, we say that $(NLE)_\phi$ holds if $(NLE)_\phi^k$ holds with $k = 0$.

Remark 5.6 (1) Clearly, $\Phi(s,t) = t \Phi(s/t,1)$. If $\phi(r) = Cr^\beta$ with some $C > 0$ and $\beta > 1$, then $\Phi(s,1) = cs^{\beta/\beta - 1}$. Consequently, under (5.6), we always have $\Phi(s,1) \geq cs^{\beta/\beta - 1}$ for some $c > 0$. In particular, $\Phi(s,t) \geq 0$ under (5.5).

(2) It is easy to see that under (5.11) the term $\Phi(d(x,y), t)$ in (5.8) is bounded by a constant, so that the upper bound $(UE)_\phi$ is consistent with $(NLE)_\phi$.

(3) It is known (cf. [4], [22]) that $(UE)_\phi$ implies that the heat kernel $p_t(x,y)$ admits a locally Hölder continuous in $x,y$ version, so that (5.8) and (5.10) are a posteriori true for all $x, y \in E$ under $(UE)_\phi + (NLE)_\phi$. It is known (cf. [24] Remark 6.12]) that $(LE)_\phi$ implies $(NLE)_\phi^k$. Hence (5.8) and (5.9) are a posteriori true for all $x, y \in E$ under $(UE)_\phi + (LE)_\phi$.

(4) By [24] Theorems 1.2 and 1.3, $(UE)_\phi + (NLE)_\phi$ (resp. $(UE)_\phi + (C)$) is stable in the following sense: if two strongly local regular Dirichlet forms $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ on $L^2(E; m^{(1)})$ and $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ on $L^2(E; m^{(2)})$ having common domain $\mathcal{F}^{(1)} = \mathcal{F}^{(2)}$ satisfy $C^{-1}m^{(1)} \leq m^{(2)} \leq Cm^{(1)}$, $C^{-1}\mathcal{E}^{(1)} \leq \mathcal{E}^{(2)} \leq C\mathcal{E}^{(1)}$ on $\mathcal{F}^{(1)} \times \mathcal{F}^{(1)}$, and $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ admits $(UE)_\phi + (NLE)_\phi$ (resp. $(UE)_\phi + (C)$) for some positive constants, then $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ also does for some constants. This shows that (A.1) (resp. (A.2)) with $T := [0, +\infty)$ holds under $(UE)_\phi + (C)$ (resp. $(UE)_\phi + (NLE)_\phi$) with

$$\phi_1(t,x,y) := 1_{\{d(x,y) \leq \Phi^{-1}(t)\}} \frac{1}{V(x, \Phi^{-1}(t))} \text{ and } \phi_2(t,x,y) := \frac{1}{V(x, \Phi^{-1}(t))} \exp\left(-\frac{1}{2} \Phi(d(x,y), t)\right).$$

(5) By [24] Theorem 1.2, $(UE)_\phi + (NLE)_\phi$ is equivalent to $(G)_\phi$. In this case, under the transience of $X$, we know the existence of global Green kernel $R(x,y)$ and it satisfies that there exist $\kappa \in [0, 1]$ and $C \in [0, +\infty)$ such that

$$C^{-1} \int_{\mathcal{R}(x,y)} \frac{\phi(s)ds}{sV(x,s)} \leq R(x,y) \leq C \int_{\mathcal{R}(x,y)} \frac{\phi(s)ds}{sV(x,s)}$$

for $x, y \in E$ with $x \neq y$.

To apply our main results in this framework, we need to assume (AC) for $X$. Let $P^A_t f(x) = E_x[\epsilon_A(t)f(X_t)]$ be the Feynman-Kac semigroup of $X$ defined by $\epsilon_A(t) = \exp(N^*_t + A^*_t - A^*_t^{1/2})$. Then we have the following:

Theorem 5.7 Suppose that $X$ satisfies (AC) and it is transient. Let $u \in \mathcal{F}_{loc} \cap QC(E_\theta)$ be a bounded finely continuous (nearly) Borel function on $E$. Assume $\mu_1 \in S_{\text{loc}}^1(X), \mu(u) \in S_{\text{loc}}^1(X)$ and $\mu_2 \in S_{\text{loc}}^1(X)$. Then we have the following:

(1) Suppose that $X$ is conservative. If $\lambda^2(\mathcal{F}_1) > 0$ and $(UE)_\phi$ holds, then there exists an integral kernel $p^A_t(x,y)$ of the Feynman-Kac semigroup $(P^A_t)_{t \geq 0}$ satisfying (5.8) with $k = 0$ and some $c \in [0, 1]$ (by replacing $p_t(x,y)$ with $p^A_t(x,y)$).

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(2) If \( \lambda^2(\mathcal{P}_1) > 0 \) and \((UE)_0 + (NLE)_0 \) (resp. \((UE)_0 + (LE)_0 \)) holds, then there exists an integral kernel \( p^A_t(x, y) \) of the Feynman-Kac semigroup \( (P^A_t)_{t>0} \) in the strict sense satisfying (5.8) and (5.10) (resp. (5.8) and (5.9)) for all \( x, y \in E \) with \( k = 0 \) and some adequate constants \( c > 0 \) (by replacing \( p_t(x, y) \) with \( p^A_t(x, y) \)).

(3) Assume that (5.6) holds. Suppose that \((UE)_0 + (NLE)_0 \) for all \( x, y \in E \), or \((LE)_0 \) for all \( x, y \in E \) holds. If there exists an integral kernel \( p^A_t(x, y) \) of the Feynman-Kac semigroup \( (P^A_t)_{t>0} \) in the strict sense satisfying (5.8) for all \( x, y \in E \) with \( k = 0 \) and \( c \in [0, 1] \) (by replacing \( p_t(x, y) \) with \( p^A_t(x, y) \)), then \( \lambda^2(\mathcal{P}_1) > 0 \).

Proof. (1): By use of [24, Theorem 1.3], \((UE)_0 + (C)\) is stable under the change of Dirichlet form satisfying (1.2) and (1.3) for some \( C_E > 0 \). So, (A.1) with \( T = [0, +\infty[^2 \) is satisfied for \( \phi_2(t, x, y) = \frac{1}{v(x, \phi^{-1}(t))} \exp \left( -\frac{1}{2} \Phi(d(x, y), t) \right) \). The conclusion follows from Theorem 1.3(1).

(2): It is known that the Poincaré inequality and the generalized capacity inequality are stable with respect to the quasi-isometry of Dirichlet forms (cf. [24]). From this and [24, Theorem 1.2], we see that \((UE)_0 + (NLE)_0 \) and \((UE)_0 + (LE)_0 \) of the heat kernel \( p_t(x, y) \) in the strict sense are stable under the change of Dirichlet form satisfying (1.2) and (1.3) for some \( C_E > 0 \). So, (A.2) with \( T = [0, +\infty[^2 \) is satisfied for \( \phi_2(t, x, y) = \frac{1}{v(x, \phi^{-1}(t))} \exp \left( -\frac{1}{2} \Phi(d(x, y), t) \right) \) and \( \phi_1(t, x, y) = \frac{1}{v(x, \phi^{-1}(t))} \mathbb{1}_{\{d(x, y) < \phi^{-1}(t)\}} \) (resp. \( \phi_1(t, x, y) = \frac{1}{v(x, \phi^{-1}(t))} \exp \left( -c_1 \Phi(d(x, y), t) \right) \)) for some \( c_1 \in [1, +\infty[ \) under \((UE)_0 + (NLE)_0 \) (resp. \((UE)_0 + (LE)_0 \)). The conclusion now follows from Theorem 1.3(2).

(3): Since \( \mathcal{P}_1 = \mathcal{P}_1 = \mu_1 + \frac{1}{2} \mu(\omega) \in S^1_{NK_1}(X) \), we have \( e^{-2m} t^1_1 \in S^1_{NK_1}(Y) \subset S^1_{EK}(Y) \) by [29, Corollaries 5.1(5) and 5.2(2)]. One can take small \( t_0 > 0 \) such that sup \( \{x \in E \} E^Y_x[A^1_{t_0}] < 1 \), whence sup \( \{x \in E \} E^Y_x[A^1_{t_0}] \leq \frac{1}{1 - \text{sup}_x \{E^Y_x[A^1_{t_0}] \}} < \infty \) by Khasminskii Lemma. Then for any \( f \in \mathcal{B}_E(E) \) and \( x \in E \), we have

\[
\int_E \left( \int_0^{t_0} p^A_t(x, y) dt \right) |f(y)| |m(dy)| = \int_0^{t_0} P^A_t|f|(x) dt \\
\leq e^{2||u||_{\infty}} \|f\|_{\infty} \int_0^{t_0} \sup_{z \in E} E^Y_z[e^{A^1_{t_0}}] dt \\
\leq e^{2||u||_{\infty}} \|f\|_{\infty} t_0 \sup_{z \in E} E^Y_z[e^{A^1_{t_0}}] < \infty,
\]

which implies that for each \( x \in E \), \( \int_0^{t_0} p^A_t(x, y) dt < \infty \) for m.a.e. \( y \in E \). Now suppose that \( p^A_t(x, y) \) satisfies \((UE)_0 \) for all \( x, y \in E \).

Assume first that (5.6) and \((UE)_0 + (NLE)_0 \) hold. By Remark 5.6(5) with the transience of \( X \), for each \( x \in E \), there exists a large \( L > 0 \) such that \( \int_L^{\infty} \frac{\phi(s)}{sV(x, s)} ds < \infty \). By (5.6),

\[
\int_L^{\infty} \frac{\phi'(s)}{V(x, s)} ds < \infty,
\]

hence \( \int_L^{\infty} \frac{dt}{V(x, \phi^{-1}(t))} < \infty \). Thus we have

\[
\int_0^{t_0} \frac{dt}{V(x, \phi^{-1}(t))} < \infty,
\]

because \( \phi \) is strictly increasing and \( r \mapsto V(x, r) \) is lower semi continuous. Then we can conclude \( \int_0^{t_0} p^A_t(x, y) dt < \infty \) for all \( x, y \in E \), since \( p^A_t(x, y) \) satisfies \((UE)_0 \) for all \( x, y \in E \). Thus we
obtain that for each \( x \in E \), \( R^A(x,y) < \infty \) for m.a.e. \( y \in E \). Applying Theorem 3.3 we have \( \lambda^2(\mathfrak{m}_1) > 0 \).

Next suppose that \((LE)_\phi\) holds for all \( x,y \in E \). We recall from the proof of Theorem 1.3(3) that \( d = \{(x,y) \in E \times E \mid R(x,y) = \infty \} \) and \( E^x = \{ y \in E \mid (x,y) \notin d \} \). Since for all \( (x,y) \notin d \)

\[
C^{-1}e^{-c\Phi(cd(x,y),t_0)} \int_{t_0}^\infty \frac{dt}{V(x,\phi^{-1}(t))} \leq C^{-1} \int_{t_0}^\infty \frac{1}{V(x,\phi^{-1}(t))}e^{-c\Phi(cd(x,y),t)}dt
\]

we can conclude \( \int_{t_0}^\infty p_t(x,y)dt < \infty \) holds for all \( (x,y) \notin d \), because \( p_t^A(x,y) \) satisfies \((UE)_\phi\) for all \( x,y \in E \). Hence we obtain that for each \( x \in E \), \( R^A(x,y) < \infty \) for m.a.e. \( y \in E^x \). Since \( \mathfrak{m}(E \setminus E^x) = 0 \), we have \( R^A(x,y) < \infty \) m.a.e. \( y \in E \). Applying Theorem 3.3 we obtain \( \lambda^2(\mathfrak{m}_1) > 0 \).

**Remark 5.8** Note that under \((UE)_\phi\) and \( d_1 > \beta \), \( X \) is transient. Indeed, by (5.1) and (5.3), for \( t \geq 1 \),

\[
\frac{V(x,\phi^{-1}(1))}{V(x,\phi^{-1}(t))} \leq c_1 \left( \frac{\phi^{-1}(1)}{\phi^{-1}(t)} \right)^{d_1} \leq \frac{c_1}{t^{d_1/\beta}}
\]

implies \( R(x,y) := \int_0^\infty p_t(x,y)dt < \infty \).

**Remark 5.9** Suppose that \( \phi(r) = r^2 \) and the intrinsic metric \( \rho(x,y) = \sup\{f(x) - f(y) \mid f \in \mathcal{F}_{\text{loc}} \cap C(E), \mu(f) \leq \mathfrak{m}\} \) is a non-degenerate complete metric compatible with the given topology. Then the (global) volume doubling condition \((VD)\) yields the conservativeness of \( X \) by Sturm [32, Theorem 4]. In this case, \((UE)_\phi+(LE)_\phi\), a localized version of \((UE)_\phi+(LE)_\phi\), is equivalent to \((PI)_\phi\), a localized version of (weak) Poincaré inequality \((PI)_\phi\), which is also equivalent to \((PHI)_\phi\), a localized versions of parabolic Harnack inequality \((PHI)_\phi\). Here \((UE)_\phi\) (resp. \((LE)_\phi\)) means that for any relatively compact open set \( G \), \( (5.8) \) (resp. \( (5.9) \)) with \( k = 0 \) and some \( c \in [0,1] \) (resp. \( c \in [1, \infty[ \) holds for all \( t > 0 \) and for m.a.e. \( x,y \in G \). It is easy to see that \((PI)_\phi+(VD)\) (consequently \((UE)_\phi+(LE)_\phi\)) is stable under the change of Dirichlet form satisfying \((1.2)\) and \((1.3)\) for some \( C_E > 0 \) in view of the remark after \((1.4)\). Moreover, \((PHI)_\phi\) yields the existence of local Hölder continuous heat kernel \( p_t(x,y) \). So \((UE)_\phi\) for all \( x,y \in E \) is a posteriori true under \((UE)_\phi+(LE)_\phi\). Then we can strengthen the assertion of Theorem 5.7(1) so that \((5.8)\) holds for all \( x,y \in E \) with \( k = 0 \) by replacing \( p_t(x,y) \) with \( p_t^A(x,y) \) provided \( \lambda^2(\mathfrak{m}_1) > 0 \) and \((UE)_\phi+(LE)_\phi\) holds. This statement extends Devyver [15, Theorem 4.1], because any complete smooth Riemannian manifold always satisfies \((UE)_\phi+(LE)_\phi\) with \( \phi(r) = r^2 \).

For \( \alpha > 0 \), let \( X^{(\alpha)} \) be the \( \alpha \)-subprocess killed at rate \( \alpha \). By virtue of Theorem 5.7 we obtain the next corollary without assuming the transience of \( X \).

**Corollary 5.10** Suppose that \( X \) satisfies \((AC)\). Let \( u \in \mathcal{F}_{\text{loc}} \cap QC(E_\partial) \) be a bounded finely continuous (nearly) Borel function on \( E \). Assume \( \mu_1 \in S^1_{NK_1}(X^{(\alpha)}) \), \( \mu(u) \in S^1_{NK_\infty}(X^{(\alpha)}) \) and \( \mu_2 \in S^2_{\mathfrak{j}}(X) \). Then we have the following:

1. Suppose that \( X \) is conservative. If \( \lambda^{2\alpha}(\mathfrak{m}_1) > 0 \) and \((UE)_\phi\) holds, then there exists an integral kernel \( p_t^A(x,y) \) of the Feynman-Kac semigroup \((P_t^A)_{t>0}\) satisfying \((5.8)\) with some constant \( k := k(\alpha) \geq 0 \) depending on \( \alpha \) and some \( c \in [0,1] \) (by replacing \( p_t(x,y) \) with \( p_t^A(x,y) \)).

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Thus we obtain that the proof of Theorem 5.7(3), we can take a small $t$ together with Theorem 5.7(2) (resp. Theorem 5.7(1)). We shall prove (3). By the same way of the proof of Theorem 5.7(2) (resp. (1)) is similar to that of Corollary 1.5(2) (resp. Corollary 1.5(1)).

**Proof.** The proof of (2) (resp. (1)) is similar to that of Corollary 1.5(2) (resp. Corollary 1.5(1)) together with Theorem 5.7(2) (resp. Theorem 5.7(1)). We shall prove (3). By the same way of the proof of Theorem 5.7(2), we can take a small $t_0 > 0$ such that $\int_{t_0}^{\infty} e^{-\alpha t} p_t^A(x,y) dt < \infty$ for $\text{m-a.e. } y \in E$ because $\mathbb{T}_1 \in S^1_{EK}(Y^{(a)})$. On the other hand, since $p_t^A(x,y)$ satisfies (5.8) for all $x, y \in E$ with some $k \geq 0$ and $c \in ]0, 1]$}

$$
\int_{t_0}^{\infty} e^{-\alpha t} p_t^A(x,y) dt \leq \int_{t_0}^{\infty} \frac{e^{-(a-k)t}}{V(x, \phi^{-1}(t))} \exp \left( -\frac{1}{2} \Phi(cd(x,y), t) \right) dt 
\leq \frac{1}{V(x, \phi^{-1}(t_0))} \int_{t_0}^{\infty} e^{-(a-k)t} dt < \infty.
$$

Thus we obtain that $R^A_{\alpha}(x,y) < \infty$ for $\text{m-a.e. } y \in E$, equivalently $\lambda^{\alpha_\alpha_\alpha} (\mathbb{T}_1) > 0$ by applying Theorem 3.3 to the case $(X^{(a)}, \mu_{(a)}, \mu_1, \mu_2)$.

The following theorem can be proved based on Lemma 4.4 and Corollary 5.10(2) in the same way of the proof of Theorem 5.7. We omit its proof.

**Theorem 5.11** Suppose that $X$ satisfies (AC). Let $u \in \mathcal{F}_{loc} \cap QC(E_0)$ be a bounded finely continuous (nearly) Borel function on $E$. Assume $\mu_1 \in S^1_{EK}(X)$, $\mu_{(a)} \in S^1_{EK}(X)$ and $\mu_2 \in S^1_{Y^a}(X)$.

(1) If $(UE)_{\phi}+(NLE)_{\phi}$ (resp. $(UE)_{\phi}+(LE)_{\phi}$) holds, then there exists an integral kernel $p_t^A(x,y)$ of the Feynman-Kac semigroup $(P_t^A)_{t \geq 0}$ in the strict sense satisfying (5.8) and (5.10) (resp. (5.8) and (5.9)) for all $x, y \in E$ with some constant $k := k(\alpha) \geq 0$ depending on $\alpha$ and adequate constants $c > 0$ (by replacing $p_t(x,y)$ with $p_t^A(x,y)$).

(2) If $(UES)_{\phi}$ holds, then there exists an integral kernel $p_t^A(x,y)$ of the Feynman-Kac semigroup $(P_t^A)_{t \geq 0}$ satisfying (5.8) with some constant $k := k(\alpha) \geq 0$ depending on $\alpha$ and an adequate constant $c > 0$ (by replacing $p_t(x,y)$ with $p_t^A(x,y)$).

5.3 Symmetric Jump Processes on Metric Measure Spaces

Recall that $(E,d)$ is a locally compact separable metric space with a positive Radon measure $m$ on $E$ with full support and every ball in $(E,d)$ is relatively compact. Recall $V(x,r) = m(B(x,r))$.

We follow the terminology and conditions in [11][12]. We assume diam$(E) = +\infty$ for simplicity. Since we have assumed (VD) and (RVD), we have that there exist $c_1, c_2 > 0$, $d_2 \geq d_1 > 0$ such that

$$
c_1 \left( \frac{R}{r} \right)^{d_1} \leq \frac{V(x,R)}{V(x,r)} \leq c_2 \left( \frac{R}{r} \right)^{d_2} \text{ for all } 0 < r < R < \infty, \quad x \in E. \quad (5.12)
$$
Let \((\mathcal{E}, \mathcal{F})\) be a regular Dirichlet form on \(L^2(E; m)\) with no killing part: i.e.,
\[
\mathcal{E}(f, g) = \mathcal{E}^{(c)}(f, g) + \int_{E \times E \setminus \text{diag}} (f(x) - f(y))(g(x) - g(y))J(dx, dy), \quad \text{for } f, g \in \mathcal{F},
\]
where \(\mathcal{E}^{(c)}\) is the strongly local part of \((\mathcal{E}, \mathcal{F})\). Let \(\mu^f\) be the energy measure of \((\mathcal{E}, \mathcal{F})\) defined by \(\mu^f(dx) = 2 \int E(f(x) - f(y))^2J(dx, dy)\) for \(f \in \mathcal{F}\). Let \(X\) be the associated \(m\)-symmetric jump process. Assume further that the Lévy system defined by \(\mu\) and \(\langle 0, HK \rangle\) is a non-decreasing function satisfying \(L(\beta_1, C_L)\) and \(U(\beta_2, C_U)\) for some \(0 < \beta_1 < \beta_2\). That is,
\[
C_L \left( \frac{R}{r} \right)^{\beta_1} \leq \frac{\psi(R)}{\psi(r)} \leq C_U \left( \frac{R}{r} \right)^{\beta_2} \quad \text{for all } 0 < r < R < \infty. \tag{5.13}
\]
Note that, since \(\psi\) satisfies \(L(\beta_1, C_L)\), we have \(\lim_{t \to 0} \psi(t) = 0\).

Recall that \(\Phi\) is defined in \([5.7]\) through \(\phi\).

**Definition 5.12 (HK(\(\phi, \varphi, \psi\), UHK(\(\phi, \varphi, \psi\), SHK(\(\phi, \varphi, \psi\)))** Suppose that \(\phi : [0, +\infty [ \to [0, +\infty [\) is a non-decreasing function satisfying \(L(\alpha_1, c_L)\) and \(U(\alpha_2, c_U)\) with some \(0 < \alpha_1 \leq \alpha_2\) and \(c_L, c_U > 0\). Suppose that \(\varphi : [0, +\infty [ \to [0, +\infty [\) is a non-decreasing function satisfying that for some \(c > 0\),
\[
\varphi(r) \leq cr, \quad \text{for all } r > 0. \tag{5.14}
\]

(i) We say that \(HK(\phi, \varphi, \psi)^*\) holds if there exists a heat kernel \(p_t(x, y)\) of \(X\) in the strict sense which has the following estimates: there exist \(k \geq 0, \eta, a_0 > 0\) and \(c \geq 1\) such that for all \(t > 0\) and \(x, y \in E\),
\[
ce^{-1}e^{-kt}\left(\frac{1}{V(x, \varphi^{-1}(t))}\frac{1}{t}1_{d(x,y) \leq \eta \varphi^{-1}(t)} + \frac{t}{V(x, d(x,y))}\frac{1}{V(x, d(x,y))}\right) \\
\leq p_t(x, y) \\
\leq c e^{kt}\left(\frac{1}{V(x, \varphi^{-1}(t))}\frac{1}{t}1_{d(x,y) \leq \eta \varphi^{-1}(t)} + \frac{1}{V(x, \phi^{-1}(t))}e^{-a_0 \Phi(d(x,y), t)}\right). \tag{5.15}
\]
We say that \(HK(\phi, \varphi, \psi)\) holds if \(HK(\phi, \varphi, \psi)^*\) holds with \(k = 0\).

(ii) We say that \(UHK(\phi, \varphi, \psi)^*\) holds if there exist \(k \geq 0\) and a heat kernel \(p_t(x, y)\) of \(X\) such that for all \(t > 0\) and \(m\)-a.e. \(x, y \in E\),
\[
p_t(x, y) \lesssim_k \frac{1}{V(x, \varphi^{-1}(t))}\frac{1}{t}1_{d(x,y) \leq \eta \varphi^{-1}(t)} + \frac{1}{V(x, \phi^{-1}(t))}e^{-a_0 \Phi(d(x,y), t)}. \tag{5.16}
\]
We say \(UHK(\phi, \varphi, \psi)\) holds if \(UHK(\phi, \varphi, \psi)^*\) holds with \(k = 0\).

(iii) We say that \(SHK(\phi, \varphi, \psi)^*\) holds if there exists a heat kernel \(p_t(x, y)\) of \(X\) in the strict sense such that for all \(t > 0\) and \(x, y \in E\),
\[
p(t, x, y) \lesssim_k \frac{1}{V(x, \varphi^{-1}(t))}\frac{1}{t}1_{d(x,y) \leq \eta \varphi^{-1}(t)} + \frac{1}{V(x, \phi^{-1}(t))}e^{-\Phi(d(x,y), t)}. \tag{5.17}
\]
We say that \(SHK(\phi, \varphi, \psi)\) holds if \(SHK(\phi, \varphi, \psi)^*\) holds with \(k = 0\).
Remark 5.13 \((HK(\phi))\) When \(\psi \simeq \phi \simeq \varphi\), we have
\[
\frac{1}{V(x, \varphi^{-1}(t))} - \frac{t}{V(x, d(x,y))}\psi(d(x,y)) + \frac{1}{V(x, \varphi^{-1}(t))} \phi(d(x,y)) + \frac{1}{V(x, \varphi^{-1}(t))} \Phi(d(x,y), t)
\]
Thus, when \(\psi \simeq \phi \simeq \varphi\), our notions \(HK(\phi, \varphi, \psi)\) and \(UHK(\phi, \varphi, \psi)\) are the same as \(HK(\phi)\) and \(UHK(\phi)\) in \([11]\).

Definition 5.14 We say that a metric space \((M, d)\) satisfies the chain condition \(Ch(A)\) if there exists a constant \(A \geq 1\) such that, for any \(n \in \mathbb{N}\) and \(x, y \in M\), there is a sequence \(\{z_k\}_{k=0}^n\) of points in \(M\) such that \(z_0 = x, z_n = y\) and
\[
d(z_{k-1}, z_k) \leq A \frac{d(x, y)}{n} \quad \text{for all} \quad k = 1, \ldots, n.
\]

5.3.1 Symmetric pure jump processes

In this subsection, we assume \(g^{(c)} = 0\) so that \((\mathcal{E}, \mathcal{F})\) is a regular Dirichlet form on \(L^2(E; m)\) of pure jump type.

It is established recently in \([11]\) and \([3]\) Theorem 2.14, Corollary 2.15, Theorem 2.17, Corollary 2.18 that, if either \(\phi \simeq \psi\) or the lower scaling index \(\alpha_1\) for \(\phi\) is strictly greater than 1, then \(HK(\phi, \psi)\) (resp. \(UHK(\phi, \psi)\)) is stable under the change of Dirichlet form satisfying \([12]\) and \([13]\) for some \(C_E > 0\). Hence, \((A.2)\) (resp. \((A.1)\)) is satisfied under \(HK(\phi, \psi)\) (resp. \(UHK(\phi, \psi)\)) for \(\mathcal{E}, \mathcal{F}\). Consequently, we can apply Theorem \([1.3]\) Corollary \([1.5]\) and Theorem \([1.7]\). We state the following only:

Theorem 5.15 Suppose that \(X\) satisfies (AC) and it is transient. We assume that either \(\phi \simeq \psi\) or the lower scaling index \(\alpha_1\) for \(\phi\) is strictly greater than 1.

Let \(u \in \mathcal{F}_{loc} \cap QC(E_0)\) be a bounded finely continuous (nearly) Borel function on \(E\). Assume \(\mu_1 + N(e^{F_1} - 1) \mu_H \in S_{NK1}^1(X), \mu_{(u)} \in S_{NK\infty}^1(X)\) and \(\mu_2 + N(F_2) \mu_H \in S_{D_0}^1(X)\). Then we have the following:

1. Suppose \(UHK(\phi, \psi)\) and (C) hold. If \(\lambda^2(\overline{\mu_1}) > 0\), then there exists an integral kernel \(p_k^A(x, y)\) of the Feynman-Kac semigroup \((P_k^A)_{t>0}\) satisfying \([5.10]\) with \(k = 0\) and \(\varphi = \phi\) (by replacing \(p_1^A(x, y)\) with \(p_k^A(x, y)\)).

2. Suppose \(HK(\phi, \psi)\) holds. If \(\lambda^2(\overline{\mu_1}) > 0\), then there exists an integral kernel \(p_0^A(x, y)\) of the Feynman-Kac semigroup \((P_0^A)_{t>0}\) in the strict sense satisfying \([5.15]\) with \(k = 0\) and \(\varphi = \phi\) (by replacing \(p_1^A(x, y)\) with \(p_k^A(x, y)\)).

3. If there exists an integral kernel \(p_k^A(x, y)\) of the Feynman-Kac semigroup \((P_k^A)_{t>0}\) in the strict sense satisfying \([5.15]\) with \(k = 0\) and \(\varphi = \phi\) (by replacing \(p_1^A(x, y)\) with \(p_k^A(x, y)\)), then \(\lambda^2(\overline{\mu_1}) > 0\).

Theorem 5.16 Suppose that \(X\) satisfies (AC). Let \(u \in \mathcal{F}_{loc} \cap QC(E_0)\) be a bounded finely continuous (nearly) Borel function on \(E\). Suppose that \(\mu_1 + N(e^{F_1} - 1) \mu_H \in S_{EK}^1(X), \mu_{(u)} \in S_{K}^1(X)\) and \(\mu_2 + N(F_2) \mu_H \in S_{D}^1(X)\) hold. If \(HK(\phi, \psi)\) holds, then there exists an integral kernel \(p_k^A(x, y)\) of the Feynman-Kac semigroup \((P_k^A)_{t>0}\) in the strict sense satisfying \([5.15]\) with \(\varphi = \phi\) for some constant \(k \geq 0\) depending on \(\beta_1, \beta_2, C_L, C_U, \alpha_1, \alpha_2, c_L, c_U\) (by replacing \(p_1^A(x, y)\) with \(p_k^A(x, y)\)).
Remark 5.17 Theorem 5.15 extends [44] and [28]. Theorem 5.16 also partially extends [48], which treats the non-symmetric α-stable like process \(X\) with non-symmetric bounded functions \(F_1\) and \(F_2\).

Remark 5.18 If our metric space \((E, d)\) also satisfy the chain condition \(\text{Ch}(A)\), then by [3] Theorem 2.17 \(HK(\phi, \varphi, \psi)\) is equivalent to \(\text{SHK}(\phi, \varphi, \psi)\). Thus, under the chain condition \(\text{Ch}(A)\), \(HK(\phi, \varphi, \psi)\) can be replaced by \(\text{SHK}(\phi, \varphi, \psi)\) in Theorems 5.15 and 5.16 above.

Definition 5.19 (J/\(\psi\)) We say that \(J/\psi\) holds if for \(m\)-a.e. \(x \in E\), \(N(x, dy) \ll m(dy)\) and for \(m\)-a.e. \(x, y \in E\), the Radon-Nikodym derivative \(J(x, y)\) satisfies

\[
\frac{C_1}{V(x, d(x, y))\psi(d(x, y))} \leq J(x, y) \leq \frac{C_2}{V(x, d(x, y))\psi(d(x, y))} \quad \text{for \(m\)-a.e. } x, y \in E.
\]

We now assume that our metric measure space \((E, d)\) allows a conservative diffusion process \(Z = (Z_t)_{t \geq 0}\) which has the transition density \(q(t, x, y)\) with respect to \(m\) satisfying \((UE)_{f, 0}^a\) and \((NLE)_{\psi}^\alpha\) in Definitions 5.2 and 5.3 that there exist constants \(C \geq 1\) and \(a_0 > 0\) such that for all \(t > 0\) and \(x, y \in E\),

\[
\frac{C^{-1}}{V(x, f^{-1}(t))} q(t, x, y) \leq \frac{C}{V(x, f^{-1}(t))} \exp \left(-a_0 F(d(x, y), t)\right), \tag{5.17}
\]

where the function \(F\) is defined as

\[
F(r, t) := F(f, r, t) = \sup_{s > 0} \left[ r - \frac{t}{f(s)} \right].
\]

Recall that \(\psi : [0, +\infty] \rightarrow [0, +\infty]\) is a non-increasing function which satisfies \(L(\beta_1, C_L), U(\beta_2, C_U)\). For the rest of this subsection, we assume that \(f\) is a strictly increasing function satisfying \(L(\gamma_1, c_F^{-1})\) and \(U(\gamma_2, c_F)\) with some \(1 < \gamma_1 \leq \gamma_2\), and \(\psi\) and \(f\) satisfy

\[
\int_0^1 \frac{df(s)}{\psi(s)} < \infty. \tag{5.18}
\]

We now define \(\phi\) as

\[
\phi(r) := \frac{f(r)}{\int_0^r \frac{df(s)}{\psi(s)}} , \quad r > 0.
\]

Then \(\phi\) is strictly increasing function satisfying (5.18), \(U(\alpha_2, c_U)\) and \(L(\alpha_1, c_L)\) for some \(\alpha_2 \geq \alpha_1 > 0\) and \(c_U, c_L > 0\) (see [3] Section 3).

Using [3] Theorem 2.19(iii)] and our Theorem 1.3 Corollary 1.5 and Theorem 1.7 we have the following.

Theorem 5.20 Suppose that \(X\) satisfies (AC). We assume that the lower scaling index \(\alpha_1\) for \(\phi\) is strictly greater than 1. Suppose \((E, d)\) satisfies \(\text{Ch}(A)\) and allows a conservative diffusion process whose transition density satisfies (5.17) with a strictly increasing function \(f\) satisfying (5.18) and \(L(\gamma_1, c_F^{-1})\) and \(U(\gamma_2, c_F)\) with some \(1 < \gamma_1 \leq \gamma_2\). Let \(u \in \mathcal{F}_{\text{koc}} \cap QC(E_0)\) be a bounded finely continuous (nearly) Borel function on \(E\).

1. Suppose that \(X\) is transient. Assume \(\mu_1 + N(e^{F_1} - 1)\mu_H \in S_{\text{Nk}}^1(X), \mu_2 + N(e^{F_2} - 1)\mu_H \in S_{\text{Nk}}^1(X)\) and \(\mu_2 + N(e^{F_2} - 1)\mu_H \in S_{\text{Nk}}^1(X)\). If \(\lambda^2(\mathcal{P}_1) > 0\) and \(J/\psi\) holds, then there exists an integral kernel \(p_t^A(x, y)\) of the Feynman-Kac semigroup \((P_t^A)_{t \geq 0}\) in the strict sense satisfying \(\text{SHK}(\phi, \varphi, \psi)\) (by replacing \(p_t(x, y)\) with \(p_t^A(x, y)\)).
(2) Suppose that \( \mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{EK}(X), \mu_1^* \in S^1_K(X) \) and \( \mu_2 + N(F_2)\mu_H \in S^1_D(X) \) hold. If \( J_x \) holds, then there exists an integral kernel \( p^1_t(x,y) \) of the Feynman-Kac semigroup \( (P^1_t)_{t \geq 0} \) in the strict sense satisfying \( SHK(\phi,\psi)^* \) for some constant \( k \geq 0 \) (by replacing \( p_t(x,y) \) with \( p^1_t(x,y) \)).

If \( \Phi \) satisfies \( U(\alpha_2,C_U) \) with \( \alpha_2 < d_1 \) where \( d_1 \) is the constant in then \( J_x \) implies that Green function \( G(x,y) \) satisfies

\[
G(x,y) \leq \Phi(d(x,y)) \leq \Phi(d(x,y)) \quad \text{for any } x,y \in E.
\]

(5.19)

5.3.2 Symmetric diffusion with jump

In this subsection, we assume that the lower scaling index \( \alpha_1 \) for \( \phi \) is strictly greater than 1, and

\( \phi(r) \leq \psi(r) \quad \text{for } r \in [0,1] \quad \text{and} \quad \phi(r) \geq \psi(r) \quad \text{for } r \in [1, +\infty[. \)

Very recently it has been established in [12] Theorems 1.13 and 1.14 that, under the above assumptions, \( HK(\phi,\phi \wedge \psi, \psi) \) (resp. \( UHK(\phi,\phi \wedge \psi, (+C)) \)) is stable under the change of Dirichlet form satisfying [12] and [13] for some \( C_E > 0 \). Hence, \( (A.2) \) (resp. \( (A.1) \)) is satisfied under \( HK(\phi,\phi \wedge \psi, \psi) \) (resp. \( UHK(\phi,\phi \wedge \psi, (+C)) \)) for \( (\mathcal{E}, \mathcal{F}) \). Consequently, we can also apply Theorem [13] Corollary [13] and Theorem [17].

**Theorem 5.21** Suppose that \( X \) satisfies (AC) and it is transient. Let \( u \in S_{loc} \cap QC(E_0) \) be a bounded finely continuous (nearly) Borel function on \( E \). Assume \( \mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{EK}(X), \mu_1^* \in S^1_K(X) \) and \( \mu_2 + N(F_2)\mu_H \in S^1_D(X) \). Then we have the following:

1. Suppose \( UHK(\phi,\phi \wedge \psi, \psi) \) and (C) hold. If \( \lambda^2(\mathcal{D}_1) > 0 \), then there exists an integral kernel \( p^1_t(x,y) \) of the Feynman-Kac semigroup \( (P^1_t)_{t \geq 0} \) satisfying \( (5.16) \) with \( k = 0 \) and \( \varphi = \phi \wedge \psi \) (by replacing \( p_t(x,y) \) with \( p^1_t(x,y) \)).

2. Suppose \( HK(\phi,\phi \wedge \psi, \psi) \) holds. If \( \lambda^2(\mathcal{D}_1) > 0 \), then there exists an integral kernel \( p^1_t(x,y) \) of the Feynman-Kac semigroup \( (P^1_t)_{t \geq 0} \) in the strict sense satisfying \( (5.15) \) with \( k = 0 \) and \( \varphi = \phi \wedge \psi \) (by replacing \( p_t(x,y) \) with \( p^1_t(x,y) \)).

3. If there exists an integral kernel \( p^1_t(x,y) \) of the Feynman-Kac semigroup \( (P^1_t)_{t \geq 0} \) in the strict sense satisfying \( (5.15) \) with \( k = 0 \) and \( \varphi = \phi \wedge \psi \) (by replacing \( p_t(x,y) \) with \( p^1_t(x,y) \)), then \( \lambda^2(\mathcal{D}_1) > 0 \).

**Theorem 5.22** Suppose that \( X \) satisfies (AC). Let \( u \in S_{loc} \cap QC(E_0) \) be a bounded finely continuous (nearly) Borel function on \( E \). Suppose that \( \mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{EK}(X), \mu_1^* \in S^1_K(X) \) and \( \mu_2 + N(F_2)\mu_H \in S^1_D(X) \) hold. If \( HK(\phi,\phi \wedge \psi, \psi) \) holds, then there exists an integral kernel \( p^1_t(x,y) \) of the Feynman-Kac semigroup \( (P^1_t)_{t \geq 0} \) in the strict sense satisfying \( (5.15) \) with \( \varphi = \phi \wedge \psi \) for some constant \( k \geq 0 \) depending on \( \beta_1, \beta_2, C_L, C_U, \alpha_1, \alpha_2, c_L, c_U \) (by replacing \( p_t(x,y) \) with \( p^1_t(x,y) \)).

**Remark 5.23** If our metric space \( (E,d) \) is connected and also satisfy the chain condition \( Ch(A) \), then by [12] Theorem 1.13 \( HK(\phi,\phi \wedge \psi, \psi) \) is equivalent to \( SHK(\phi,\phi \wedge \psi, \psi) \). Thus, under the chain condition \( Ch(A) \), \( HK(\phi,\phi \wedge \psi, \psi) \) can be replaced by \( SHK(\phi,\phi \wedge \psi, \psi) \) in Theorems 5.21 and 5.22 above.
Recall that $E$ is a locally compact separable metric space, and $m$ a $\sigma$-finite Radon measure with full support on $E$. Let $(\mathcal{E}^0, \mathcal{F}^0)$ be a strongly local regular Dirichlet form on $L^2(E; m)$, and $\mu^0_{(w)}$ be the $(\mathcal{E}^0)$-energy measure of $u \in \mathcal{F}^0$ so that $\mathcal{E}^0(u, u) = \frac{1}{2} \mu^0_{(w)}(E)$. We assume that the intrinsic metric $\rho$ of $(\mathcal{E}^0, \mathcal{F}^0)$ defined by $\rho(x, y) = \sup \{ f(x) - f(y) \mid f \in \mathcal{F}^0 \cap C_c(E) \text{ with } \mu^0_{(f)}(dz) \leq m(dz) \}$ is finite for any $x, y \in E$ and induces the original topology on $E$, and that $(E, \rho)$ is a complete metric space.

We assume that the Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ is Harnack-type Dirichlet space in [26, Chapter 2] so that Aronson-type heat kernel estimates for the diffusion process $Z^0$ associated with $(\mathcal{E}^0, \mathcal{F}^0)$ hold, that is $(UE)^c_0$ and $(LE)^c_g$ in Definition 5.4 hold with $g(r) = r^2$.

For a connected open subset $D$ of $(E, \rho)$, define for $x, y \in D$,

$$\rho_D(x, y) = \inf \{ \text{length}(\gamma) \mid \text{a continuous curve } \gamma \text{ in } D \text{ with } \gamma(0) = x \text{ and } \gamma(1) = y \}.$$ 

Denote by $\bar{D}$ the completion of $D$ under the metric $\rho_D$. We extend the definition of $m|_D$ to $\bar{D}$ by setting $m|_D(\bar{D} \setminus D) = 0$. For notational simplicity, we continue use $m$ to denote this measure $m|_\bar{D}$.

We assume that $D$ is an unbounded inner uniform subdomain of $E$. That is, there are constants $C_1, C_2 \in [0, +\infty]$ such that, for any $x, y \in D$, there exists a continuous curve $\gamma_{x,y} : [0, 1] \to D$ with $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$ whose length is at most $C_1 \rho_D(x, y)$, and for any $z \in \gamma_{x,y}([0, 1])$, $\rho(z, \partial D) := \inf_{w \in \partial D} \rho(z, w) \geq C_2 \rho_D(z, y)/\rho_D(x, y)$.

Let $(\mathcal{E}^0, \mathcal{F}^0_D)$ be the part Dirichlet form of $(\mathcal{E}^0, \mathcal{F}^0)$ on $D$. By [26, Proposition 2.13], we have that for $x, y \in D$,

$$\rho_D(x, y) = \sup \left\{ f(x) - f(y) \mid f \in \mathcal{F}^0_{D, \text{loc}} \cap C_c(D) \text{ with } \mu^0_{(f)}(dz) \leq m(dz) \right\}.$$ 

Denote $B_D(x, r) := \{ y \in D \mid \rho_D(x, y) < r \}$ and $m(B_D(x, r))$ by $V_D(x, r)$. Define $\mathcal{F}^0_{D, \text{ref}} := \{ f \in \mathcal{F}^0_{D, \text{loc}} \mid \mu^0_{(f)}(D) < \infty \}$ and

$$\mathcal{E}^0_{D, \text{ref}}(f, f) := \frac{1}{2} \mu^0_{(f)}(D) \quad \text{for } f \in \mathcal{F}^0_{D, \text{ref}}.$$ 

In this section we consider the Markov process $X$ on $\bar{D}$ associated with the following type of non-local symmetric Dirichlet forms $(\mathcal{E}, \mathcal{F})$ on $L^2(D; m)$:

$$\mathcal{F} = \mathcal{F}^0_{D, \text{ref}} \cap L^2(D; m),$$

and, for $u \in \mathcal{F}$,

$$\mathcal{E}(u, u) = \mathcal{E}^0_{D, \text{ref}}(u, u) + \frac{1}{2} \int_{D^2} (u(x) - u(y))^2 J(x, y) m \otimes m(dxdy),$$

where $J(x, y)$ is a non-negative symmetric measurable function on $D \times D \setminus \text{diag}$ satisfying certain conditions to be specified below.

**Definition 5.24** Let $\beta \in [0, +\infty]$ and $\psi$ be a strictly increasing function on $[0, +\infty]$ with $\psi(0) = 0$ and $\psi(1) = 1$ that satisfies $L(\beta_1, C_L)$ and $U(\beta_2, C_U)$ for some $0 < \beta_1 \leq \beta_2 < 2$. For a non-negative symmetric measurable function $J(x, y)$ on $D \times D \setminus \text{diag}$, we say
Define for 

\[ H(x, y) = \frac{\kappa_1}{V_D(x, \rho_D(x, y))\psi(\rho_D(x, y))} \exp(\kappa_2\rho_D(x, y)^\beta) \] 

for \((x, y) \in D \times D \setminus \text{diag};\)

(ii) \((J_{\psi, \beta, \geq})\) holds if there are positive constants \(\kappa_3\) and \(\kappa_4\) so that

\[ J(x, y) \geq \frac{\kappa_3}{V_D(x, \rho_D(x, y))\psi(\rho_D(x, y))} \exp(\kappa_4\rho_D(x, y)^\beta) \] 

for \((x, y) \in D \times D \setminus \text{diag};\)

(iii) \((J_{\psi, \beta})\) holds if both \((J_{\psi, \beta, \leq})\) and \((J_{\psi, \beta, \geq})\) hold with possibly different constants \(\kappa_i\) in the upper and lower bounds;

(iv) \((J_{\psi, 0+, \leq})\) holds if there are positive constants \(\kappa_5\) and \(\kappa_6\) so that

\[
\begin{cases}
J(x, y) \leq \frac{\kappa_5}{V_D(x, \rho_D(x, y))\psi(\rho_D(x, y))} & \text{for } (x, y) \in D \times D \setminus \text{diag},

\sup_{x \in D} \int_{\{y \in D \mid \rho_D(x, y) > 1\}} \rho_D(x, y)^2 J(x, y) m(dy) \leq \kappa_6 < \infty,
\end{cases}
\]

where

\[ \psi_*(r) := \psi(r) 1_{\{r \leq 1\}} + r^2 1_{\{r > 1\}} \] 

for \(r \geq 0;\) 

(5.20)

Define for \(\beta \in [0, +\infty], x \in \bar{D}, t > 0\) and \(r \geq 0,

\[ p_{\psi, \beta}(t, x, r) := \frac{1}{V_D(x, \sqrt{t})} \exp(-r^2/t) + \left( \frac{1}{V_D(x, \psi^{-1}(t))} \right) \wedge \left( \frac{t}{V(x, r)\psi(\psi^{-1}(r))} \right). \]

Define for \(\beta \in ]0, 1[, \)

\[ H_{\psi, \beta}(t, x, r) := \begin{cases} 
\frac{1}{V_D(x, \sqrt{t})} \wedge p_{\psi, \beta}(t, x, r) & \text{if } t \in ]0, 1], \\
\frac{1}{V_D(x, \sqrt{t})} \exp(-r^2/t) & \text{if } t \in ]1, +\infty[;
\end{cases} \]

for \(\beta \in ]1, +\infty[, \)

\[ H_{\psi, \beta}(t, x, r) := \begin{cases} 
\frac{1}{V_D(x, \sqrt{t})} \wedge p_{\psi, \beta}(t, x, r) & \text{if } t \in ]0, 1] \text{ and } r \leq 1, \\
t \frac{1}{V_D(x, \sqrt{t})} \exp(-r(1 + \log^+(r/t))(\beta - 1)/\beta) \wedge r^\beta & \text{if } t \in ]0, 1] \text{ and } r > 1, \\
\frac{1}{V_D(x, \sqrt{t})} \exp(-r(1 + \log^+(r/t))(\beta - 1)/\beta) \wedge (r^2/t) & \text{if } t \in ]1, +\infty[;
\end{cases} \]

where \(\log^+(x) := \log(x \vee 1),\) and \(H_{\psi, \infty}(t, x, r) := \lim_{\beta \to \infty} H_{\psi, \beta}(t, x, r)\) for \(\beta = \infty,\) that is,

\[ H_{\psi, \infty}(t, x, r) := \begin{cases} 
\frac{1}{V_D(x, \sqrt{t})} p_{\psi, \beta}(t, x, r) & \text{if } t \in ]0, 1] \text{ and } r \leq 1, \\
t \frac{1}{V_D(x, \sqrt{t})} \exp(-r(1 + \log^+(r/t))) & \text{if } t \in ]0, 1] \text{ and } r > 1, \\
\frac{1}{V_D(x, \sqrt{t})} \exp(-r(1 + \log^+(r/t)) \wedge (r^2/t)) & \text{if } t \in ]1, +\infty[.
\]

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We further define for \( x \in \bar{D}, t > 0 \) and \( r \geq 0 \),
\[
H_{\psi,0_+}(t,x,r) := \frac{1}{V_D(x,\sqrt{t})} \land p_{\psi,0}(t,x,r),
\]
where \( \psi \) is given by (5.20).

Combining [3, Theorems 1.5 and 1.6] and our Theorem 5.3 Corollary 1.5 and Theorem 1.7, we also obtain the stability result on estimates for fundamental solutions under Feynman-Kac perturbations for this process.

**Theorem 5.25** Suppose that \( X \) satisfies (AC) and it is transient. Let \( u \in \mathcal{F}_{\text{loc}} \cap QC(D_0) \) be a bounded finely continuous (nearly) Borel function on \( D \). Assume \( \mu_1 + N(e^{F_1} - 1) \mu_H \in S^{1}_{DK}(X) \), \( \mu_{(u)} \in S^{1}_{NK}(X) \) and \( \mu_2 + N(F_2) \mu_H \in S^{1}_{D_0}(X) \). Then we have the following:

1. Suppose \( (J_{\psi,0_+}, \leq) \) holds. If \( \lambda^2(\overline{\mathcal{P}_1}) > 0 \), then there exists an integral kernel \( p_t^A(x,y) \) of the Feynman-Kac semigroup \( (P_t^A)_{t>0} \) satisfying
\[
p_t^A(x,y) \leq c_1 H_{\psi,0_+}(t,x,c_2 \rho_D(x,y)) \quad \text{for all } x \in \bar{D} \text{ and } t > 0.
\]

2. Suppose \( J(x,y) \) satisfies \( (J_{\psi_1,\beta_1, \leq}) \) and \( (J_{\psi_2,\beta_2, \geq}) \) for some strictly increasing functions \( \psi_1, \psi_2 \) satisfying \( \psi_i(0) = 0 \), \( \psi_i(1) = 1 \) and \( L(\beta_i^1, C_i^1_L) \) and \( U(\beta_i^2, C_i^2_U) \) for some \( 0 < \beta_i^1 \leq \beta_i^2 < 2 \) for \( i = 1, 2 \), and for \( \beta_1 \leq \beta^* \in \{ 0 \} \cup [0, +\infty) \) excluding \( \beta_1 = \beta^* = 0_+ \). If \( \lambda^2(\overline{\mathcal{P}_1}) > 0 \), then there exists an integral kernel \( p_t^A(x,y) \) of the Feynman-Kac semigroup \( (P_t^A)_{t>0} \) in the strict sense satisfying
\[
c_1 H_{\psi_2,\beta^*}(t,x,c_2 \rho_D(x,y)) \leq p_t^A(x,y) \leq c_3 H_{\psi_1,\beta_1}(t,x,c_4 \rho_D(x,y)), \tag{5.21}
\]
for all \( x \in \bar{D} \) and \( t > 0 \).

3. If there exists an integral kernel \( p_t^A(x,y) \) of the Feynman-Kac semigroup \( (P_t^A)_{t>0} \) in the strict sense satisfying (5.21) then \( \lambda^2(\overline{\mathcal{P}_1}) > 0 \).

**Theorem 5.26** Suppose that \( X \) satisfies (AC). Let \( u \in \mathcal{F}_{\text{loc}} \cap QC(D_0) \) be a bounded finely continuous (nearly) Borel function on \( D \). Suppose that \( \mu_1 + N(e^{F_1} - 1) \mu_H \in S^{1}_{DK}(X), \mu_{(u)} \in S^{1}_{NK}(X) \) and \( \mu_2 + N(F_2) \mu_H \in S^{1}_{D_0}(X) \) hold.

Suppose \( J(x,y) \) satisfies \( (J_{\psi_1,\beta_1, \leq}) \) and \( (J_{\psi_2,\beta_2, \geq}) \) for some strictly increasing functions \( \psi_1, \psi_2 \) satisfying \( \psi_i(0) = 0 \), \( \psi_i(1) = 1 \) and \( L(\beta_i^1, C_i^1_L) \) and \( U(\beta_i^2, C_i^2_U) \) for some \( 0 < \beta_i^1 \leq \beta_i^2 < 2 \) for \( i = 1, 2 \), and for \( \beta_1 \leq \beta^* \in \{ 0 \} \cup [0, +\infty) \) excluding \( \beta_1 = \beta^* = 0_+ \). Then there exists an integral kernel \( p_t^A(x,y) \) of the Feynman-Kac semigroup \( (P_t^A)_{t>0} \) in the strict sense satisfying
\[
c_1 e^{-kt} H_{\psi_2,\beta^*}(t,x,c_2 \rho_D(x,y)) \leq p_t^A(x,y) \leq c_3 e^{kt} H_{\psi_1,\beta_1}(t,x,c_4 \rho_D(x,y)),
\]
for some \( k \geq 0 \).

**5.5 Symmetric jump processes with tempered jumps at infinity in \( \mathbb{R}^d \)**

Let \( (\mathcal{E}, \mathcal{F}) \) be a pure jump type regular Dirichlet form on \( L^2(\mathbb{R}^d) \) defined by
\[
\mathcal{E}(f,g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(x) - f(y))(g(x) - g(y))J(x,y)dx dy, \quad f, g \in \mathcal{F} = C_0(\mathbb{R}^d)^{\mathcal{E}_1/2},
\]
where \( J(x, y) \) is a symmetric Borel function on \( \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag} \) satisfying UJS condition, that is, there is a constant \( c > 0 \) such that for a.e \( x, y \in \mathbb{R}^d \),
\[
J(x, y) \leq \frac{c}{r^d} \int_{B(x, r)} J(z, y) \, dz \quad \text{whenever } r \leq |x - y|/2.
\]
Clearly, if \( J(x, y) \) satisfies UJS condition, so does \( \tilde{J}(x, y) \) satisfying (1.1).

We also assume that the jump kernel \( J \) has the following estimates;
\[
J(x, y) \asymp \frac{1}{|x - y|^d \phi(|x - y|) e^{\beta |x - y|^d}},
\]
where \( \beta \in ]0, +\infty[ \) and \( \phi : ]0, +\infty[ \to ]0, +\infty[ \) is a strictly increasing function satisfying \( L(\alpha_1, c_L) \) and \( U(\alpha_2, c_U) \) with some \( 0 < \alpha_1 \leq \alpha_2 < 2 \). When \( \beta = \infty \), (5.22) is equivalent to
\[
J(x, y) \asymp \frac{1}{|x - y|^d \phi(|x - y|)} 1_{\{|x - y| \leq 1\}}.
\]
Let \( X = (X_t, \mathcal{P}_x) \) be the symmetric Hunt process on \( \mathbb{R}^d \) associated to \( (\beta, \mathcal{F}) \). It is known in [7,8] that \( X \) is a conservative Feller process and admits a locally jointly Hölder continuous transition density, a heat kernel \( p_t(x, y) \) on \( ]0, +\infty[ \times \mathbb{R}^d \times \mathbb{R}^d \), which has the following estimates:
\[
p_t(x, y) \asymp_k q_\beta(t, |x - y|) \quad \text{for } (t, x, y) \in ]0, +\infty[ \times \mathbb{R}^d \times \mathbb{R}^d \quad (5.23)
\]
with \( k = 0 \), where for \( \beta \in ]0, 1] \),
\[
q_\beta(t, r) := \begin{cases} 
\frac{t}{r^{d/2} \exp \left( -\frac{1}{\beta} - \log^+(r^2/t) \right)} & \text{if } t \leq 1, \\
\frac{t}{r^{d/2}} & \text{if } t > 1;
\end{cases}
\]
and for \( \beta \in ]1, +\infty[ \),
\[
q_\beta(t, r) := \begin{cases} 
\frac{1}{r^{d/2} \exp \left( -\frac{1}{\beta} - \log^+(r^2/t) \right)} & \text{if } t \leq 1, r < 1, \\
\frac{t}{r^{d/2}} & \text{if } t \leq 1, r \geq 1, \\
\frac{t}{r^{d/2} \exp \left( -\frac{1}{\beta} - \log^+(r^2/t) \right)} & \text{if } t > 1.
\end{cases}
\]
Note that \( X \) is transient if and only if \( d \geq 3 \). The rotationally symmetric relativistic \( \alpha \)-stable process \( X \) on \( \mathbb{R}^d \) with mass \( m > 0 \) (see [7] Introduction for definition) is a typical example of this subsection.

By Theorem 1.32 (see also the proof of [47] Proposition 2.3), we can obtain the following:

**Theorem 5.27** Suppose that \( X \) satisfies (AC). Suppose \( d \geq 3 \). Let \( u \in \mathcal{F}_{\text{loc}} \cap \text{QC}(E_0) \) be a bounded finely continuous (nearly) Borel function on \( E \). Assume \( \mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{\alpha}(\mathbb{R}^d) \), \( \mu_{(u)} \in S^1_{\alpha}(\mathbb{R}^d) \) and \( \mu_2 + N(F_2)\mu_H \in S^1_{\alpha}(\mathbb{R}^d) \). If \( \lambda^\delta_{(\mu_{(u)})} > 0 \), then there exists an integral kernel \( p^A_t(x, y) \) of the Feynman-Kac semigroup \( (P^A_t)_{t \geq 0} \) in the strict sense satisfying (5.22) with \( k = 0 \).

The conclusion of Theorem 5.27 extends [47] Theorem 1.1, which treats the case that \( X \) is the rotationally symmetric relativistic \( \alpha \)-stable process with \( \mu_1 \in S^1_{\alpha}(\mathbb{R}^d) \cap S^0_{\alpha}(\mathbb{R}^d) \) and \( u = \mu_2 = F_1 = F_2 = 0 \).

The next corollary follows from Corollary 1.3.2 (2).
Corollary 5.28 Suppose that $X$ satisfies (AC). Let $u \in \mathcal{F}_{loc} \cap QC(E_0)$ be a bounded finely continuous (nearly) Borel function on $E$. Assume $\mu_1 + N(e^{F_1} - 1)\mu_H \in S_{NK_1}^1(X^{(\alpha)})$, $\mu_2 + N(F_2)\mu_H \in S_D^1(X)$ for a fixed $\alpha > 0$. If $\lambda^{2\alpha}(\mathcal{M}_1) > 0$, then there exists an integral kernel $p^A_t(x, y)$ of the Feynman-Kac semigroup $(P^A_t)_{t>0}$ in the strict sense satisfying (5.23) with some constant $k \geq 0$ depending on $\beta, \alpha_1, c_L, \alpha_2, c_U$.

The following theorem can be proved based on Corollary 5.28 in the same way of the proof of Theorem 1.7. We omit its proof.

Theorem 5.29 Suppose that $X$ satisfies (AC). Let $u \in \mathcal{F}_{loc} \cap QC(E_0)$ be a bounded finely continuous (nearly) Borel function on $E$. Suppose that $\mu_1 + N(e^{F_1} - 1)\mu_H \in S_{EK}^1(X)$, $\mu_2 + N(F_2)\mu_H \in S_D^1(X)$ hold. Then there exists an integral kernel $p^A_t(x, y)$ of the Feynman-Kac semigroup $(P^A_t)_{t>0}$ in the strict sense satisfying (5.23) for some constant $k := k(\alpha) \geq 0$ depending on $\beta, \alpha_1, c_L, \alpha_2, c_U$.

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