DYNAMICS OF LOOP QUANTUM GRAVITY AND SPIN FOAM MODELS IN THREE DIMENSIONS

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We present a rigorous regularization of Rovelli’s generalized projection operator in canonical 2+1 gravity. This work establishes a clear-cut connection between loop quantum gravity and the spin foam approach in this simplified setting. The point of view adopted here provides a new perspective to tackle the problem of dynamics in the physically relevant 3+1 case.

1. Introduction

The goal of the spin foam approach\(^1\) is to construct a mathematically well defined notion of path integral for non-perturbative quantum GR as a devise for computing the ‘dynamics’ of the theory. By ‘dynamics’ here we mean the characterization of the kernel of the quantum constraints given by the representation of the classical constraints in connection variables, schematically, \(G_i = D a E^a_i\), \(V_b = E^a_i F_{ab}\), and \(S = E^a_i E^b_j F_{ab}(A)\). Spin foam models have been studied as an attempt to give an explicit construction of the generalized projection operator \(P\) from the kinematical Hilbert space \(H_{\text{kin}}\), where the constraints above are defined as operators, into their kernel \(H_{\text{phys}}\). A tentative regularization of the formal projector into the kernel of \(S(x)\) and \(V^a(x)\)

\[
P = \prod_{x \in \Sigma} \delta(\hat{V}^a(x)) \delta(\hat{S}(x))^\eta = \int D[N_\mu] \exp(i \int_{\Sigma} N_0 \hat{S} + N_a \hat{V}^a) \tag{1}
\]

was presented in\(^2\). Given two spin network states \(s, s'\) the physical scalar product \(<s, s'>_{\text{phys}} := <Ps, s'>\) can be formally defined by

\(^{a}\)The field \(A\) is an \(SU(2)\) connection and \(E\) is its canonical momentum represented by the sensitized triad both defined on a 3-manifold \(\Sigma\) defining space. Space-time is assume to be of the topology \(\Sigma \times R\).

\(^{b}\)One can also define the notion of path integral for gravity as a lattice discretization of
\[ <s, s'>_{\text{phys}} = \langle Ps, s' \rangle = \int D[N] \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[ \int N(x) \hat{S}(x) \right]^n s, s' >, \quad (2) \]

where the exponential in (1) has been expanded in powers. From early on, it was realized that smooth loop states are naturally annihilated (independently of any regularization ambiguity) by \( \hat{S} \). Consequently, \( \hat{S} \) acts only on spin network nodes. Generically, it does so by creating new links and nodes modifying in this way the underlying graph of the spin network states. The action of \( \hat{S} \) can be visualized as an ‘interaction vertex’ in the time evolution of the node (see diagram on the right of Figure 1). Therefore, each term in the sum (2) represents a series of transitions–given by the local action of \( \hat{S} \) at spin network nodes–through different spin network states interpolating the boundary states \( s \) and \( s' \) respectively. They can in fact be expressed as a sum over ‘histories’ of spin network that can be pictured as a system of ‘colored’ branching surfaces described by a 2-complex labeled by spins. Every such history is called a spin foam.

Before even considering the issue of convergence of (2), the problem with this definition is evident: every single term in the sum is a divergent integral! Therefore, this way of presenting spin foams has to be considered as formal until a well defined regularization of (1) is provided. Possible regularization schemes are discussed in the literature although they have not been implemented in concrete examples.

Although many spin foam models for 4-dimensional gravity have been proposed, the rigorous connection with the well developed canonical theory of loop quantum gravity is still under investigation. Here we show that a clear-cut connection between the loop approach and spin foams can be established in three dimensions.

Pure gravity in three dimensions is a well studied example of integrable system that can be rigorously quantized. The reduced phase space of the theory is finite dimensional and there exist different quantization schemes that make use of this property.

From our perspective, 3-dimensional gravity is taken as an toy model for the formal path integral for GR in first order variables

\[ P = \int D[e] D[A] \mu[A, e] \exp \left[ i S_{\text{GR}}(e, A) \right] \]

where the formal measure \( \mu[A, e] \) must be determined by the Hamiltonian analysis of the theory.
the application of quantization techniques that are expected to be useful in four dimensions. In this sense we want to quantize the theory according to Dirac prescription which implies having to deal with the infinitely many degrees of freedom of a field theory at the kinematical level, i.e., we want to quantize first and then reduce at the quantum level. This is precisely the avenue that is explored by loop quantum gravity in four dimensions where the reduced phase space approach seems hopeless.

2. Canonical three dimensional gravity

Assuming the topology of space time to be of the form \( M = \Sigma \times \mathbb{R} \), where \( \Sigma \) is an arbitrary Riemann surface, the phase space of 3-dimensional gravity is parametrized the 2-dimensional connection \( A^i_a \) and the triad field \( E^b_j \) where \( a = 1, 2 \) are \( \Sigma \)-coordinate indices and \( i, j = 1, 2, 3 \) are \( su(2) \) indices. The symplectic structure is defined by \( \{ A^i_a(x), E^b_j(y) \} = \delta^b_a \delta^i_j \delta(x, y) \). Local symmetries of the theory are generated by the Gauss and curvature constraints, \( D_b E^b_j = 0 \) and \( F_{ab}(A) = 0 \), respectively.

As in 4d, the kinematical Hilbert space, \( H_{kin} \), is given by a certain set of \( SU(2) \) gauge invariant functionals of the connection \( \Psi[A] \) which are square integrable with respect to a natural diffeomorphism invariant measure, the Ashtekar-Lewandowski (AL) measure \(^5,6^\). As in standard gauge theories, the basic gauge invariant functional of \( A \) is given by the Wilson loop; namely, the trace of the holonomy of \( A \) around a close loop on the fundamental representation of \( SU(2) \). Any state in \( H_{kin} \) can be expressed in terms of linear combinations of products of Wilson loops. Natural orthonormal basis of \( H_{kin} \) can be constructed in terms of eigenstates of geometric operators of quantum geometry representing area and length, given by the so-called spin network states \(^7,8,9^\). Spin network states \( \Psi_{\gamma,\{j_\ell\},\{\iota_n\}}[A] \) are defined by a graph \( \gamma \) in \( \Sigma \), a collection of spins \( \{j_\ell\} \) associated with links \( \ell \in \gamma \), and a collection of \( SU(2) \) invariant tensors (called intertwiners) \( \{\iota_n\} \) associated to nodes \( n \in \gamma \) (see left diagram on Figure 1).

3. Dynamics and spin foams

In this section we introduce the regularization of the generalized projection operator \( P \) and state the results that are studied in detail in \(^10^\).

We start with the formal expression

\[
P = \left( \prod_{x \in \Sigma} \delta(\tilde{S}(x)) \right) = \int D[N] \exp(i \int_{\Sigma} \text{Tr}[N \tilde{F}(A)])
\]  (3)
For the moment we assume the genus of $\Sigma$ to be greater or equal than one (the sphere case is trivial). In this case, the Riemann surface admits a global (fiducial) flat metric $q^0_{ab}$. We introduce a square cellular decomposition of $\Sigma$ denoted $\Sigma_\epsilon$ and we require each square to have area $\epsilon^2$ with respect to $q^0_{ab}$. We define the set of continuous finite graphs in $\Sigma$ by $\Gamma$. The completion in the AL norm of the set of cylindrical functions based on $\Gamma_\epsilon$, defined by the set of continuous graphs contained in the 1-skeleton of the square cellular decomposition $\Sigma_\epsilon$, defines a subspace $H_\epsilon$ of the kinematical Hilbert space $H_{kin}$. In the limit $\epsilon \to 0$ we have $H_\epsilon \to H_{kin}$.

Based on the fiducial square cellular decomposition $\Sigma_\epsilon$ we can now define $P$ by introducing a regularization of the right hand side of (3). The integral in the exponential can be written as

$$\int_\Sigma \text{Tr}[NF(A)] = \lim_{\epsilon \to 0} \sum_{p^i} \epsilon^2 \text{Tr}[N_{p^i} F_{p^i}],$$

where $p^i$ labels the $i$th plaquette and $N_{p^i}$ and $F_{p^i}$ the value of $F(A)$ and $N$ at some interior point of the plaquette respectively. The basic observation is that the holonomy $U_{p^i} \in SU(2)$ around the plaquette $p^i$ can be written as

$$U_{p^i}[A] = \mathbb{1} + \epsilon^2 F_{p^i}(A) + O(\epsilon^2)$$

which implies

$$\int_\Sigma \text{Tr}[NF(A)] = \lim_{\epsilon \to 0} \sum_{p^i} \text{Tr}[N_{p^i} U_{p^i}[A]].$$

Notice that the explicit dependence on the regulator $\epsilon$ has dropped out of the sum on the right hand side, a sign that we should be able to remove the regulator upon quantization. With all this we can define the generalized...
projection operator as

$$\hat{P} := \lim_{\epsilon \to 0} \prod_{p} \int dN_{p,\epsilon} \exp(i\text{Tr}[N_{p,\epsilon}\hat{U}_{p,\epsilon}]) = \lim_{\epsilon \to 0} \prod_{p} \delta(U_{p,\epsilon}),$$

(4)

where the last equality follows from direct integration over $N_{p,\epsilon}$ at the classical level and $\delta(U)$ is the $SU(2)$ $\delta$-distribution defined on $L^2(SU(2))$. We promote $\delta(U)$ to an operator by using its expansion in $SU(2)$ irreducible representations, namely $\delta(U_{p,\epsilon}) = \sum_{j}(2j + 1) \chi_j(U_{p,\epsilon})$; $\chi_j(U)$ is the character of the $j$-representation matrix of $U_{p,\epsilon} \in SU(2)$. In contrast to the formal example in 4d, the previous expansion has a precise meaning in the quantum theory as each term in the sum can be promoted to a well defined self-adjoint operator in $H_{\text{kin}}$: the Wilson loop operator in the $j$-representation.

Now we state the results that are proved in detail in 10.

1 - The physical inner product $<s, s'_{\text{phys}} := <s P, s'_{\text{phys}}$ admits a spin foam representation, i.e., it can be expressed as a sum over spin network ‘histories’ interpolating $s$ and $s'$ respectively. The spin foams correspond to continuous 2-complexes defined independently from any background structure in the limit $\epsilon \to 0$.

2 - The regularization (4) is well defined. Lattice definitions of 3-dimensional gravity such as the Ponzano-Regge model are plagued with divergences. These divergences do not appear here however they can be traced to the presence of redundant $\delta$-distributions in (4) (see 11 for a treatment in the lattice context).

3 - The physical Hilbert space $H_{\text{phys}}$ defined by the equivalence classes of states in $H_{\text{kin}}$ up to physically null states (i.e. states $\psi$ for which $<\psi, \psi>_{\text{phys}} = 0$) is isomorphic to the Hilbert space obtained in other methods12. In addition, the spin foam representation allows to explicitly show how spin networks in the same homotopy class are physically equivalent and to easily prove skein relations.

4 - The spin foam representation provides a natural basis for $H_{\text{phys}}$ for $\Sigma$ of arbitrary genus. These states are eigenstates of a complete set of commuting

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This is obtained by the insertion of the resolution of the identity in $H_{\text{kin}}$

$$\mathbb{1} = \sum_{\gamma, \{j\}} |\gamma, \{j\}><\gamma, \{j\}|$$

(where the sum is over all continuous spin network states) between $\delta$-distributions in (4) in analogy with Feynman’s original derivation of the path integral representation of dynamics in QM.
quantum geometry operators (provided by the formalism). This completes
the quantization of 2+1 gravity in the spin foam approach and establishes
and establishes a clear-cut connection with the canonical picture.
5 - Similarly point particles can be systematically coupled to gravity\textsuperscript{13}.

We hope that the spin foam perspective, fully realized here in 2+1
gravity, can bring new breath to the problem of dynamics in 3+1 gravity.
This is an issue that will be studied further.

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