Mutually Unbiased Bases, Generalized Spin Matrices and Separability

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Abstract

A collection of orthonormal bases for a complex $d$-dimensional Hilbert space is called mutually unbiased (MUB) if for any two vectors $v$ and $w$ from different bases the square of the inner product equals $1/d$: $|\langle v, w \rangle|^2 = \frac{1}{d}$. The MUB problem is to prove or disprove the existence of a maximal set of $d+1$ bases. It has been shown in W. K. Wootters and B. D. Fields (1989, Annals of Physics, 191, 363) that such a collection exists if $d$ is a power of a prime number $p$. We revisit this problem and use $d \times d$ generalizations of the Pauli spin matrices to give a constructive proof of this result. Specifically we give explicit representations of commuting families of unitary matrices whose eigenvectors solve the MUB problem. Additionally we give formulas from which the orthogonal bases can be readily computed. We show how the techniques developed here provide a natural way to analyze the separability of the bases. The techniques used require properties of algebraic field extensions, and the relevant part of that theory is included in an Appendix.

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1 Introduction.

Let $H$ denote a complex $d$-dimensional Hilbert space and $\rho$ a density matrix modeling a $d$-level quantum system. Then $\rho$ is a positive semidefinite, trace one
matrix and as such is Hermitian and is determined by $d^2 - 1$ real numbers. A laboratory device that measures $\rho$ is represented by a Hermitian matrix $A = \sum_{k=1}^d \lambda_k P_k$, where $\{P_k: 1 \leq k \leq n\}$ is a set of rank one mutually orthogonal projections. (In Dirac notation $P_k$ denotes the outer product $|v_k\rangle \langle v_k|$.)

If the eigenvalues are distinct, $A$ is called non-degenerate, and the non-negative values $p_k(\rho, A) = \text{Tr}[\rho P_k]$ can be estimated by repeated experiments. Since $\sum_k p_k(\rho, A) = 1$, one obtains $d - 1$ independent pieces of information, and a minimum of $d + 1$ such well designed experiments would be required to recover the density $\rho$.

The problem of mutually unbiased bases (MUB) refers to the theoretical possibility of defining $d + 1$ such bases with the additional property that $\text{Tr}(P_r P_s) = \frac{1}{d}$ for any pair of projections associated with different experimental configurations, labeled by $r$ and $s$. Such a collection of bases provides an optimal way of estimating $\rho$, and we refer to [15] for a discussion of that feature.

As an example, for a two-level system there is such a set of bases that can be represented in terms of the usual Pauli matrices,

$$
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

The three sets of projections $\{ \frac{1}{d} (\sigma_0 \pm \sigma_x) \}, \{ \frac{1}{d} (\sigma_0 \pm \sigma_y) \},$ and $\{ \frac{1}{d} (\sigma_0 \pm \sigma_z) \}$ correspond to measurements along the three spin axes of a two-level system. The existence of such bases for $d = p$, $p$ a prime, was first established in [5] and was extended to $d = p^n$ in [15]. Recent papers on the subject include [1, 6], that discuss the general case, and [8], that works in the context of $d = 2^n$. To the best of our knowledge, there are no definitive results for other values of $d$.

While writing up our results, we attended a talk by Bill Wootters, who outlined a different approach to the problem of mutually unbiased bases and who brought [16] to our attention. Although the motivations of the two approaches appear to be quite different, they require the same mathematical tools and appear to lead to the same results. An interesting question is the relationship between the two approaches.

Our interest in this problem was stimulated by the following result in [1].

**Theorem 1.1** ([1] Thm 3.2) Suppose that one has $d^2$ unitary matrices orthogonal in the Frobenius or trace inner product, one of which is the identity matrix. Suppose further that these matrices can be grouped into $d + 1$ classes of $d$ commuting matrices and that the only matrix common to two different classes is the identity. Then there is a set of $d + 1$ mutually unbiased bases.

Motivated by the observation that the Pauli spin matrices can be derived as a Hadamard transform of certain basis matrices, we defined in [10] a family of $d^2$ matrices that are orthogonal with respect to the trace inner product. Accordingly we refer to them as (generalized) spin matrices. Although that approach seems to have been novel, these matrices have appeared earlier in the literature, for example in [2] and [3] and references therein. They were also used in [1].
In addition to providing an algorithm for deriving explicit solutions to the MUB problem for \( d = p^n \), a major goal of this paper is to emphasize the utility of the indexing of the generalized spin matrices. In fact, by interpreting the indices as vectors we are able to put the MUB problem into the context of a vector space over a finite field. Moreover, we can also use the indexing and results in [10] to write each mutually unbiased basis defined by a set of commuting matrices as a weighted sum of those matrices.

In Section 2 we define the generalized spin matrices and record a number of the properties given in [10]. In Section 3 we use the notation of the generalized spin matrices to facilitate a detailed solution of the mutually unbiased bases problem when \( d = p \) is an odd prime. A basic idea used in that solution reappears in the next two sections. In Section 4 we show how the use of (algebraic) field extensions produces a solution for \( d = p^2 \) and set the stage for Section 5, in which we give a constructive algorithm for solving the MUB problem explicitly in the general case of \( d = p^n \). In Section 6 we define the notion of separability of a basis and show how the separability of the derived bases is related to the index notation. To improve the readability of the paper, we have deferred many of the technicalities to the end of the paper. Thus the Appendices provide the details for computing the projections associated with a class of commuting spin matrices, the formal mathematics underlying the results in Section 4, and the theoretical foundation for the algorithm illustrated in Section 5.

It is important to emphasize that our methodology gives a specific solution of the MUB problem for \( d = p^n \). Once such a solution is in hand, there are many ways to construct other mutually unbiased bases, such as using conjugation by a unitary matrix.

Finally a word about notation. Throughout the paper we use the letters \( j, k, a, b \) to denote the elements of \( \mathbb{Z}_d \), the integers modulo \( d \). The letters \( u, v, \) and \( z \) denote vectors in \( V_2(F) \), the two dimensional vector space over a field \( F \), and \( w \) denotes a vector in \( V_{2n}(\mathbb{Z}_p) \), the \( 2n \)-dimensional vector space over \( \mathbb{Z}_p \), where \( p \) is a prime. The Greek letters \( \alpha, \beta \) are reserved for elements of the Galois field \( GF(p^n) \).

2 Generalized spin matrices

In what follows \( d \) denotes the dimension of the finite dimensional complex-Hilbert space \( H \), and the unitary matrices acting on \( H \) are indexed by subscripts \( u = (j, k) \), with the two forms of indices used interchangeably. Let \( \{|j\}, j = 0, \ldots, d-1 \} \) be a fixed orthonormal basis of \( H \). We will have occasion to use vector addition of indices, and such addition will be addition modulo \( d \). \( \eta \) denotes the complex number \( \exp(2\pi i/d) \), and it is easy to confirm that for integers \( b \) such that \( \eta^b \neq 1 \)

\[
\sum_{k=0}^{d-1} (\eta^b)^k = 0.
\]
Definition 2.1 Let \(0 \leq j, k < d\). Then \(S_{j,k} \equiv \sum_{m=0}^{d-1} \eta^{mj} |m\rangle \langle m+k|\).

It is easy to confirm that \(\text{Tr}(S_{j,k}) = 0\) unless \(S_{j,k} = S_{0,0}\), the \(d \times d\) identity matrix. A key property is that this set of matrices is closed under multiplication, up to scalar multiples of powers of \(\eta\).

Lemma 2.2 \(S_{j,k}S_{a,b} = \eta^{ka} S_{j+a,k+b}\). Thus, \(S_{j,k}\) and \(S_{a,b}\) commute if and only if \(ka = jb\) up to an additive multiple of \(d\).

Proof: Using the obvious notation,

\[ S_{j,k}S_{a,b} = \sum_{m=0}^{d-1} \sum_{n=0}^{d-1} \eta^{mj+n\alpha} \delta(m+k,n) |m\rangle \langle n+b|. \]

If \(m + k \leq d - 1\), \(n = m + k\) gives the only non-zero factor. If \(m + k \geq d\), \(n = m + k - d\) gives the only non-zero factor. Since \(\eta^d = 1\), we have \(S_{j,k}S_{a,b} = \eta^{ka} \sum_{m=0}^{d-1} \eta^{m(j+a)} |m\rangle \langle m+k+b|.\)

Some useful relations follow immediately, with \((iii)\) established by induction.

\[(S_0,1)S_{1,0} = \eta S_{1,1} = \eta S_{1,0}S_{0,1}\]

(iii) \((S_{j,k})^m = \eta^{jk \binom{m}{2}} S_{mj, mk}\) \hspace{1cm} (2.4)

where \(\binom{m}{2} \equiv 0\) for \(m = 0\) or 1.

We next establish that these matrices are unitary and are also orthogonal to one another with respect to the Frobenius inner product on the space of \(d \times d\) complex matrices, \(\langle A, B \rangle = \text{tr}(A^\dagger B)\), where \(A^\dagger\) is the Hermitian conjugate of \(A\).

Lemma 2.5 \((S_{j,k})^\dagger = \eta^{kj} S_{-j,-k}\). For each \(u\), \(S_u\) is unitary, and \(\text{Tr} \left[ (S_u)^\dagger S_v \right] = 0\) if \(u \neq v\).

Proof: \((S_{j,k})^\dagger = \sum_{m=0}^{d-1} \eta^{-mj} |m+k\rangle \langle m| = \eta^{jk} \sum_{n=k}^{d+k-1} \eta^{-nj} |n\rangle \langle n-k| = \eta^{jk} S_{-j,-k}\).

Let \(u = (j, k), v = (a, b)\); then

\[(S_u)^\dagger S_v = \eta^{jk} S_{-j,-k} S_{a,b} = \eta^{k(j-a)} S_{a-j,b-k}.\]

This has trace zero if \(u \neq v\), and if \(u = v\), we get the identity, so that \(S_u\) is unitary. \(\Box\)
It follows that \( \{ S_u : u = (j, k) \} \) is a set of \( d^2 \) unitary matrices that forms an orthogonal basis for the space of \( d \times d \) matrices and is closed under multiplication, up to multiples of powers of \( \eta \). Thus they can be regarded as analogues of the Pauli spin matrices, hence the terminology \emph{generalized spin matrices}.

One doesn’t quite recover the Pauli matrices through this procedure. In fact when \( d = 2 \), one has \( S_{0,1} = \sigma_x \), \( S_{1,0} = \sigma_z \), but \( S_{1,1} = i\sigma_y \) in order to fit into the general framework. The missing factor of \( i = (-1)^{1/2} \) reappears when we define the projections associated with these unitary matrices.

Such orthogonal families of unitary matrices play a key role in quantum in-
formation theory, as elaborated in [14], and, as established in Theorem 1.1, they are closely related to solutions of the MUB problem. The proof of Theorem 1.1 uses the fact that commuting unitary matrices can be simultaneously diagonal-
ized, and the bases related to the different classes have the MUB property. The
orthogonality of the unitary matrices is crucial to the analysis, and thus the
connection to the generalized spin matrices is immediate. Our problem then reduces to finding commuting classes, and the characterization of commutativ-
ity in terms of the indices enables us to rephrase the problem as a vector space
problem over a finite (algebraic) field. By using this specific class of orthogonal
unitary matrices, we are also able to give explicit formulas for the projections
defined by the basis vectors.

3 Spin matrices and the MUB problem for \( d \) prime

We begin with the case when \( d = p \) is a prime. As we have seen, \( S_{j,k} \) and \( S_{a,b} \) commute if and only if \( ka = jb \mod p \). We recast this condition in the context of a vector space over the finite field \( \mathbb{Z}_p \), the integers modulo the prime \( p \). Let \( V_2(\mathbb{Z}_p) = \{(j, k) : j, k \in \mathbb{Z}_p\} \), and define a symplectic
product:

\[
\mathbf{u} \cdot \mathbf{u}' \equiv kj' - jk' \mod p
\]

where \( \mathbf{u} = (j, k) \) and \( \mathbf{u}' = (j', k') \). Thus, \( S_u \) and \( S_v \) commute if and only if the symplectic product of their vector indices equals zero.

Once we have the classes of commuting matrices, we can make a direct computation (or invoke Theorem 1.1) to argue the existence of a complete set of mutually unbiased bases. We can construct these bases explicitly in terms of the spin matrices as follows.

**Proposition 3.2** Let \( a \in \mathbb{Z}_p \) and define

\[
C_a = \{ b (1, 0) + ba (0, 1) = b (1, a) : b \in \mathbb{Z}_p \}
\]

\[
C_\infty = \{ b (0, 1) : b \in \mathbb{Z}_p \}.
\]

There are \( p \) vectors in each of these \( p + 1 \) classes and \( C_r \cap C_s = \{(0, 0)\} \) for all \( r \neq s \) in \( I \equiv \{0, 1, \ldots, p - 1, \infty\} \). If \( u, v \) are in \( C_r \), then \( u \cdot v = 0 \).
Proof: The vectors \( e = (1, 0) \) and \( f = (0, 1) \) are linearly independent with \( f \circ e = 1 \) and \( e \circ e = f \circ f = 0 \). If \( b (1, a) = b' (1, a') \), then \( b = b' \) and if \( b \neq 0 \), \( a = a' \). This proves the first assertion for the \( C_a \) classes. Using the linearity of the symplectic product,

\[
[b (1, a)] \circ [c (1, a)] = bc (1, a) \circ (1, a) = 0.
\]

The same arguments work for \( C_\infty \). \( \square \)

The \( C_t \) can be thought of as lines in a two-dimensional space. In addition the vectors in \( C_t \) can be written as a multiple of a single vector \( u_t = (j_t, k_t) \), and \( C_t \) is an additive subgroup of \( V_2(\mathbb{Z}_p) \). The matrices associated with \( C_t \) are \( \{ S_{nu_t}, 0 \leq n < p \} \); they commute but do not form a multiplicative subgroup of the unitary matrices by virtue of Corollary 2.3 \( (iii) \). We nonetheless consider \( S_{nu_t} \) to be the “generator” of \( \{ S_{nu_t}, 0 \leq n < p \} \) with the understanding that it is \( S_{nu_t} \), not \( (S_{nu_t})^n = \eta^{j_tk_t(n/2)} S_{nu_t} \) that is in the class.

Theorem 1.1 guarantees that the orthonormal eigenvectors for each class solve the MUB problem, and we can use the indicial notation to express the associated orthogonal projections explicitly in terms of the unitary matrices \( \{ S_{nu_t}, 0 \leq n < p \} \). We begin with a definition that is valid for all \( d \) and is required to handle the computations in general.

**Definition 3.3** Let \( 0 \leq j, k < d \) and \( u = (j, k) \). If \( d \) is even and both \( j \) and \( k \) are odd, set \( \alpha_u = -\exp(\pi i/d) = -\eta^{j/2} \). Otherwise set \( \alpha_u = 1 \).

For example, for \( d = 2 \) and \( j = k = 1 \), \( \alpha_u = -i \). In general, for \( d \geq 2 \), \( \alpha_u^d \eta^{j_k/2} = 1 \).

**Definition 3.4** For each \( u = (j, k) \neq (0, 0) \) and \( 0 \leq r < d \), define

\[
P_u (r) = \frac{1}{d} \sum_{m=0}^{d-1} (\alpha_u \eta^r S_u)^m ,
\]

where \( (\alpha_u \eta^r S_u)^0 \equiv S_{0,0} \).

**Proposition 3.6** For \( d \) a prime, \( \{ P_u (r) : 0 \leq r < d \} \) is a complete set of mutually orthogonal projections.

It is easy to check that \( P_u (r) \) has trace one and that

\[
(\alpha_u \eta^r S_u)^d = \sum_{m=0}^{d-1} \eta^{-mt} P_u (m + r) ,
\]

(\([10]\), equation (13)). We need to confirm that the \( P_u (r) \)'s constitute a set of \( d \) orthogonal, one-dimensional projections, and we provide the details in Appendix A.

As just noted, the indices of members of a commuting class are multiples of a vector \( u_t \). Thus if \( u = bu_t \), then \( P_u (r) \) should be \( P_{u_t} (s) \) for some \( s \), and we confirm that fact next.
Corollary 3.8  If \( p > 2 \) is prime and \( u = bu_t = b(j_t,k_t) \) with \( 2 \leq b < p \), then \( P_u(r) = P_{u_t}(s) \), where \( s = b^{-1} \left( r - j_t k_t \right) \) and \( b^{-1} \) is the multiplicative inverse of \( b \) modulo \( p \).

Proof: From (iii) \( \Box \), it follows that \((Su)^m = \eta^{-mj_t k_t(b)}(Su_t)^{bm} \). Hence

\[
P_u(r) = \frac{1}{d} \sum_{m=0}^{d-1} \eta^{m(r-j_t k_t(b))} S_{u_t}^{bm} = \frac{1}{d} \sum_{n=0}^{d-1} \eta^{nb^{-1}(r-j_t k_t(b))} S_{u_t}^{n} = P_{u_t}(s),
\]

where we made the substitution \( n = bm \mod p \). \( \Box \)

We now show that \( Tr[P_u(r) P_{u'}(s)] = 1/d \), where it suffices to take \( u = (1,a) \) and \( u' = (1,a') \) as representatives of different classes \( C_a \). In general

\[
P_u(r) P_{u'}(s) = \frac{1}{p^2} \sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \alpha_u^m \eta^m a(s^n) \alpha_u^n \eta^n a'(s^n) S_{m+n} S_{m-n},
\]

and we see that the only contribution to the trace is for \( mu + nu' = (0,0) \mod p \). (Again, \( \binom{m}{n} \) is taken to be zero if \( m = 0 \) or \( 1 \).) This means that \( m \) and \( n \) satisfy

\[
\begin{pmatrix}
1 & 1 \\
1 & a' \\
\end{pmatrix}
\begin{pmatrix}
m \\
n \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\end{pmatrix}
\mod p.
\]

Since \( a \neq a' \), only \( m = n = 0 \) satisfy the equation. Hence \( Tr[P_{ba}(r) P_{b'a'}(s)] = 1/d \) as required. The details for \( C_\infty \) are similar. We now have proved the following theorem that recaptures the basic result of [5].

Theorem 3.9  If \( p \) is prime, there is a complete set of \( p + 1 \) mutually unbiased bases \( B_a, 0 \leq a < p \), and \( B_\infty \) that are the normalized eigenvectors of the corresponding sets of commuting spin matrices \( \{S_{b,ba}: b \in Z_p\} \leftrightarrow C_a \) and \( \{S_{0,b}: b \in Z_p\} \leftrightarrow C_\infty \). These bases can be computed from the projections in eq. (3.3).

Example: The classes for \( d = 2 \) are \( \{S_{0,0}, S_{1,0}\}, \{S_{0,0}, S_{1,1}\}, \) and \( \{S_{0,0}, S_{0,1}\} \), where \( S_{1,0} = \sigma_z, S_{0,1} = \sigma_x, \) and \( S_{1,1} = i \sigma_y \). The MUB’s are determined by the projectors \( \frac{1}{2} (\sigma_0 \pm \sigma_z), \frac{1}{2} (\sigma_0 \pm \sigma_y), \) and \( \frac{1}{2} (\sigma_0 \pm \sigma_x) \) from (3.3). The factor \( \alpha_{1,1} = -i \) is needed to recover the projections \( \frac{1}{2} (\sigma_0 \pm \sigma_y) \) from the general formula.

We obtain four classes of commuting spin matrices for \( d = 3 \) and can represent them in a \( 3 \times 3 \) table, where the row index denotes \( j \) and the column index \( k \) in \( S_{j,k} \). Similar tables can be constructed for larger values of \( p \), and in a finite geometry interpretation the classes \( C_r \) determine lines intersecting only at the origin.

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & C_\infty & C_\infty \\
1 & C_0 & C_1 & C_2 \\
2 & C_0 & C_2 & C_1 \\
\end{array}
\]
An additional feature of the spin matrices allows one to express estimates of the components of a density $\rho$ in the original fixed basis in terms of measurements in the MUB bases. We sketch the idea. Assume $d = p$ and express the density matrix as

$$\rho = \frac{1}{p} \left[ \sum_{j, k=0}^{p-1} s_{j,k} S_{j,k} \right] = \frac{1}{p} \left[ S_{00} + \sum_{t \in I} \sum_{u \in C_t - \{(0,0)} s_u S_u \right],$$

where $I \equiv \{0, 1, \ldots, p - 1, \infty\}$. From the orthogonality of the spin matrices and their representation in terms of the projections of their commuting class, we know that

$$s_u = Tr \left( S_u^\dagger \rho \right) = \alpha_u \sum_{m=0}^{d-1} \eta^m p_u(m), \quad (3.10)$$

where $p_u(m) \equiv Tr \left( P_u(m) \rho \right)$. A measuring device $M_u$ may be characterized by $\{P_u(m), 0 \leq m < p\}$. If the system is in a state modeled by the density $\rho$, $M_u$ determines the probability, $p_u(m)$, of the outcome $m$. The experimental results of measurements over an ensemble of systems give estimates for these probabilities and, by (3.10), estimates for all of the spin coefficients with indices in that commuting class. Since the spin coefficients themselves are Fourier transforms of entries of $\rho$ in the original basis ([10], equation (11)), it follows that an estimate of $\rho$ in this basis can be expressed explicitly in terms of measurements in the MUB bases. For a more complete discussion of the estimation problem see [15].

### 4 The MUB problem for $d = p^2$, $p$ an odd prime

It was shown in [15] that the MUB problem can be solved for powers of primes. We give a concrete construction based on algebraic techniques and motivated by the results in the preceding section and Theorem 1.1. This requires a certain amount of abstract algebra, and we present the special case of $d = p^2$ to illustrate the results and the ideas. (The case $p = 2$ requires a modification of the approach used here and is discussed in the next section.) However, the basic strategy is the same as before. We use the indices of the spin matrices to encode commutativity and techniques of vector spaces over finite fields to define the appropriate classes. The actual MUB bases can then be recovered from the classes of commuting spin matrices.

We are working with tensor products of the form $S_u \otimes S_v$, where commutativity is again encoded by the indices so that $S_{u_1} \otimes S_{v_1}$ commutes with $S_{u_2} \otimes S_{v_2}$ if and only if

$$u_1 \circ u_2 + v_1 \circ v_2 = 0 \text{ mod } p,$$

where $u = (j, k)$ and $v = (a, b)$. It is now useful to consider vectors in a four dimensional vector space over $Z_p$, $V_4(Z_p) = \{w = (j, k, a, b) = (u, v)\}$, and to define the symplectic product on the four dimensional space as

$$w_1 \circ w_2 \equiv u_1 \circ u_2 + v_1 \circ v_2. \quad (4.1)$$
The first two indices in \( w \) correspond to the indices in the first factor and the second two indices correspond to the second factor in the tensor product \( S_0 \otimes S_1 \).

The solution to the problem of finding the commuting classes of spin matrices now reduces to finding the classes of vectors \( w \) that satisfy \( w_1 \circ w_2 = 0 \). A technology for doing this is discussed in Appendix C. Here we simply give the results.

For \( p \) an odd prime, the procedure to define classes of four-vectors with symplectic products equal to zero requires a particular non-zero integer \( D \) in \( \mathbb{Z}_p \). \( D \) is defined by the requirement that \( D \neq k^2 \mod p \) for all \( k \) in \( \mathbb{Z}_p \), i.e. \( D \) is not a quadratic residue of \( p \).

**Theorem 4.2** Let \( p \) be an odd prime. Then commuting classes of spin matrices are indexed by the following subsets of \( V_4(\mathbb{Z}_p) \):

\[
C_{a_0, a_1} = \{(2b_0, a_0b_0 + a_1b_1D, 2b_1D, a_0b_1 + a_1b_0) : b_0, b_1 \in \mathbb{Z}_p\}
\]

\[
C_\infty = \{(0, b_0, 0, b_1) : b_0, b_1 \in \mathbb{Z}_p\},
\]

where \( a_0, a_1 \in \mathbb{Z}_p \) and \((j_1, k_1, j_2, k_2)\) corresponds to \( S_{j_1,k_1} \otimes S_{j_2,k_2} \). \( C_{a_0, a_1} \) is a subspace of \( V_4(\mathbb{Z}_p) \) with basis

\[
G_{a_0, a_1} = \{(2, a_0, 0, a_1), (0, a_1D, 2D, a_0)\}
\]

and \( C_\infty \) has the basis \( G_\infty = \{(0, 1, 0, 0), (0, 0, 0, 1)\} \).

The structure of \( C_{a_0, a_1} \) is hardly an intuitive result, but we take it as given and confirm the desired properties. There are \( p^2 + 1 \) such classes. We claim that each class has \( p^2 \) members, that \( w_1 \circ w_2 = 0 \) for vectors in the same class, and that the only vector common to any pair of classes is \((0, 0, 0, 0)\). If so, then the classes partition \( V_4(\mathbb{Z}_p) - \{(0, 0, 0, 0)\} \) as required.

The verification of these three properties is quite easy, and we leave the details to the reader. We should note, however, that in checking the last property we are led to the equations

\[
a_0b_0 + a_1b_1D = a'_0b_0 + a'_1b_1D
\]

\[
a_0b_1 + a_1b_0 = a'_0b_1 + a'_1b_0,
\]

where \( a_0, a_1 \) and \( a'_0, a'_1 \) denote indices of the first type of class and \( b_0 \neq 0 \neq b_1 \). This system can be rewritten as a matrix equation

\[
\begin{pmatrix}
    b_0 & b_1D \\
    b_1 & b_0
\end{pmatrix}
\begin{pmatrix}
    a_0 - a'_0 \\
    a_1 - a'_1
\end{pmatrix}
= \begin{pmatrix}
    0 \\
    0
\end{pmatrix}
\]

that has only the trivial solution provided \( b_0^2D \neq b_1^2 \mod p \). Since \( x^2 = D \) is not solvable in \( \mathbb{Z}_p \), all of the properties hold and we have classes of commuting spin matrices of the form \( S_{2b_0, a_0b_0 + a_1b_1D} \otimes S_{2b_1D, a_0b_1 + a_1b_0} \) indexed by \( a_0 \) and \( a_1 \). The matrices associated with \( C_\infty \) have the form \( S_{0, b_0} \otimes S_{0, b_1} \).
We can always find such values $D$. For example, if $p = 3$, $D = 2$; if $p = 5$, $D$ can be 2 or 3; and if $p = 7$, $D$ can be chosen to be one of 3, 5, or 6. The reason for this is clear. The square of $x$ and of its additive inverse $p - x$ are equal in $Z_p$. It then follows that there are $(p - 1)/2$ choices for $D$. This argument fails when $p = 2$, and we need to modify the methodology to handle that case.

The analysis can be illustrated in $V_4 (Z_p)$. For example, if $p = 3$ a complete set of mutually unbiased bases corresponds to the 10 classes of commuting spin matrices defined by the recipe above. We represent the result in a grid whose row label is $j_1j_2$ and whose column label is $k_1k_2$. The entries are $C_{a_0a_1}$.

\[
\begin{array}{cccccccccccc}
00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \\
00 & C_\infty & C_\infty & C_\infty & C_\infty & C_\infty & C_\infty & C_\infty \\
01 & C_{00} & C_{10} & C_{20} & C_{02} & C_{12} & C_{22} & C_{01} & C_{11} & C_{21} \\
02 & C_{00} & C_{20} & C_{10} & C_{01} & C_{21} & C_{11} & C_{02} & C_{22} & C_{12} \\
10 & C_{00} & C_{02} & C_{01} & C_{20} & C_{22} & C_{21} & C_{10} & C_{12} & C_{11} \\
11 & C_{00} & C_{21} & C_{12} & C_{11} & C_{02} & C_{20} & C_{22} & C_{10} & C_{01} \\
12 & C_{00} & C_{11} & C_{22} & C_{12} & C_{20} & C_{01} & C_{21} & C_{02} & C_{10} \\
20 & C_{00} & C_{01} & C_{02} & C_{10} & C_{11} & C_{12} & C_{20} & C_{21} & C_{22} \\
21 & C_{00} & C_{22} & C_{11} & C_{21} & C_{10} & C_{02} & C_{12} & C_{01} & C_{20} \\
22 & C_{00} & C_{12} & C_{21} & C_{22} & C_{01} & C_{10} & C_{11} & C_{20} & C_{02}
\end{array}
\]

The identity $S_{0,0} \otimes S_{0,0}$ lies in all the classes and each of the remaining $9^2 - 1$ tensor products is in exactly one class. If this grid of 81 points is considered as a plane, then the set of points corresponding to two classes can be thought of as lines that intersect at only one point, the origin. This representation gives some indication of the finite geometry implicit in the analysis. (In particular, a set of translations of a fixed class partitions the entire grid.)

We used properties of finite fields to obtain the commuting classes described in Theorem 4.2 and in Appendix C we define the methodology for $d = p^2$ that generalizes to the case when $d = p^a$. There are two basic ideas. The first is to use the form of the construction of the classes when $d = p$ but over an extension of the field $Z_p$, the Galois field $GF (p^2)$. This produces commuting classes $C_{a_0}$ of $V_2 (GF (p^2))$, where $a \in GF (p^2)$. The second idea is to map these classes isomorphically to $V_4 (Z_p)$ in such a way that the symplectic product of the two-dimensional vector space over the extended field is related to the symplectic product of the four-dimensional vector space over the smaller field.

5 The MUB problem for $d = p^n$, $p$ prime

The MUB problem for $d = p^n$ can be solved in a way similar to that used in the special case treated above using suitable generalizations of the methodology. A complication is that one cannot write down an explicit form of a function $f (x)$ that plays the role of $x^2 - D$ when $n = 2$ and works in all cases when $p > 2$. Instead, we must take as given $f (x)$ with the properties summarized in Appendix D and compute it in specific cases.
Specifically, we are guaranteed the existence of a finite field \( GF(p^n) \) that contains \( \mathbb{Z}_p \) and whose elements can be represented with the help of a polynomial \( f(x) \) of degree \( n \) that is irreducible over \( \mathbb{Z}_p \) and has \( n \) distinct roots in \( GF(p^n) \). The first step is the analogue of Proposition C.1, and the proof follows the reasoning used in the proof of Proposition 3.2.

Let \( V_2(GF(p^n)) = \{ u = (\alpha, \beta) : \alpha, \beta \in GF(p^n) \} \) and define the symplectic product:

\[ u \circ u' \equiv \beta \alpha' - \alpha \beta'. \]

**Proposition 5.1** Let \( \alpha \in GF(p^n) \) and define subsets of the vector space \( V_2(GF(p^n)) \):

\[
C_\alpha = \{ \beta (1, 0) + \beta \alpha (0, 1) : \beta \in GF(p^n) \} \\
C_\infty = \{ \beta (0, 1) : \beta \in GF(p^n) \}.
\]

Then these are \( p^n + 1 \) sets, each of which has \( p^n \) vectors with only \((0, 0)\) common to any two sets. If \( u \) and \( v \) are in the same set, \( u \circ v = 0 \).

In Appendix D we provide the technical structure that justifies the following theorem. The general argument follows the proof in the \( d = p^2 \) case, and we omit the details.

**Theorem 5.2** The elements of \( V_2(GF(p^n)) \) can be written as vectors in a \( 2n \)-dimensional vector space over \( \mathbb{Z}_p \). Let \( \{ e_j, f_j : 0 \leq j < n \} \) denote the \( 2n \) linearly independent vectors defined in Appendix D, which satisfy \( \text{Tr} (e_j \circ f_k) = \delta(j,k) \). The symplectic product in \( V_2(GF(p^n)) \) is denoted by “\( \circ \)”, and \( \text{Tr} \) is the trace operation. Using indexing beginning at \( 0 \), let \( M \) denote the linear mapping that maps \( e_j \) to the \( 2n \)-vector in \( V_{2n}(\mathbb{Z}_p) \) with a 1 in position \( 2j \) and zeroes elsewhere and maps \( f_j \) to the vector with a 1 in position \( 2j + 1 \) and zeroes elsewhere.

Then for every vector \( u \in V_2(GF(p^n)) \) we have \( w = M(u) \in V_{2n}(\mathbb{Z}_p) \), and the symplectic products are related by

\[ w_1 \circ w_2 = \text{Tr}(u_1 \circ u_2). \]

Commuting classes of vectors \( C_\alpha \) in \( V_2(GF(p^n)) \) map to commuting classes of vectors in \( V_{2n}(\mathbb{Z}_p) \), and, consequently, define commuting classes of tensor products of spin matrices.

Here is the way to apply this theorem in specific cases, given \( p, n \), and an irreducible polynomial \( f \) without multiple roots that generates \( GF(p^n) \):

**Step 1:** Given a (symbolic) root \( \lambda \) of

\[ f(\lambda) = \lambda^n + \sum_{k=0}^{n-1} c_k \lambda^k = 0, \]

find all \( n \) roots in terms of \( \lambda \). (If \( f \) is a primitive polynomial, the theory guarantees that the roots have the form \( \lambda P^t, 0 \leq t \leq n - 1 \).)
Step 2: Compute a set of coefficients \( d_k(\lambda) \) from
\[
f(\lambda) = (x - \lambda) \left( d_{n-1} x^{n-1} + \cdots + d_1 x + d_0 \right).
\]
The \( d_k(\lambda) \) can be written as symmetric functions of the roots and \( d_{n-1} = 1 \).

Step 3: Compute the inverse of \( f'(\lambda) \) as an element in \( GF(p^n) \).

Step 4: Define the bases \( f_k = \lambda^k (0, 1) \) and its dual \( e_k = d_k(\lambda) \left( f'(\lambda) \right)^{-1} (1, 0) \).

Step 5: For each \( \alpha = a_0 + a_1 \lambda + \cdots + a_{n-1} \lambda^{n-1} \) in \( GF(p^n) \), express vectors in \( C_\alpha \) as a linear combination of the \( e_j \)'s and \( f_k \)'s with coefficients in \( \mathbb{Z}_p \):
\[
\sum_{j=0}^{n-1} b_j \lambda^j \left( (1, 0) + \sum_{j=0}^{n-1} a_j \lambda^j (0, 1) \right) = \sum_{j=0}^{n-1} (x_j e_j + y_j f_j).
\]

Step 6: The class corresponding to \( C_\alpha \) and the corresponding set of commuting spin matrices are
\[
C_{a_0 \cdots a_{n-1}} = \{(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1})\}
\]
\[
S_{a_0 \cdots a_{n-1}} = \{S_{x_0, y_0} \otimes \cdots \otimes S_{x_{n-1}, y_{n-1}}\}. \tag{5.3}
\]
The associated projections can be computed using the methodology described in Appendix B.

To illustrate these theoretical results and the algorithm described, we first show that the machinery used in the case \( d = p^2 \) is indeed a special case of the general result. Since \( f(x) = x^2 - D = (x - \lambda)(x + \lambda) \), \( d_0 = \lambda \) and \( d_1 = 1 \).

From \( f'(\lambda) = 2\lambda \) and \( (2\lambda)^{-1} = \lambda(2D)^{-1} \), we have \( e_0 = 2^{-1}(1, 0) \) and \( e_1 = \lambda(2D)^{-1}(1, 0) \). As usual \( f_0 = (0, 1) \) and \( f_1 = \lambda(0, 1) \). This is the structure used in Appendix C to derive Theorem 12.

Example 1: For two qubits, \( p = n = 2 \), an appropriate polynomial is \( f(x) = x^2 + x + 1 \). Then \( f'(x) = 1 \). If \( f(\lambda) = 0 \), then \( \lambda^2 = \lambda + 1 \) is the second root, giving \( d_1 = 1 \) and \( d_0 = \lambda^2 \), since \( x^2 + x + 1 = (x - \lambda)(x - (\lambda + 1)) \).

Then
\[
e_0 = \lambda^2 (1, 0) \quad e_1 = (1, 0) \quad f_0 = (0, 1) \quad f_1 = \lambda(0, 1).
\]
The five classes of vectors in \( V_2 \left( GF \left( 2^2 \right) \right) \) indexed by \( \alpha = a_0 + a_1 \lambda \) are:
\[
C_0 = \{(0, 0), (1, 0), (\lambda, 0), (\lambda^2, 0)\} = \{0, e_1, e_0 + e_1, e_0\}.
\]

In the remaining classes we omit the 0 vector.
\[
C_1 = \{(1, 1), (\lambda, \lambda), (\lambda^2, \lambda^2)\} = \{e_1 + f_0, e_0 + e_1 + f_1, e_0 + f_0 + f_1\}
\]
\[
C_\lambda = \{(1, \lambda), (\lambda, \lambda^2), (\lambda^2, 1)\} = \{e_1 + f_1, e_0 + e_1 + f_0 + f_1, e_0 + f_0\}
\]
\[
C_{\lambda^2} = \{(1, \lambda^2), (\lambda, 1), (\lambda^2, \lambda)\} = \{e_1 + f_0 + f_1, e_0 + e_1 + f_0 + f_1, e_0 + f_1\}
\]
\[
C_\infty = \{(0, 1), (0, \lambda), (0, \lambda^2)\} = \{f_1, f_0 + f_1, f_0\}.
\]

If one plots each of the \( C_\alpha \) as four points in \( V_2 \left( GF \left( 2^2 \right) \right) \), using as coordinates the elements of \( GF \left( 2^2 \right) \), one obtains the left hand plots in [10, Figure 6].
remaining plots are obtained by translation and the result is a partition of the plane since “parallel” lines don’t intersect. Under the mapping $M$,

$$C_0 \rightarrow C_{0,0} = \{(0000),(0010),(1010),(1000)\},$$

$$C_1 \rightarrow C_{1,0} = \{(0000),(0110),(1011),(1101)\},$$

$$C_{\lambda} \rightarrow C_{0,1} = \{(0000),(0011),(1111),(1100)\},$$

$$C_{\lambda^2} \rightarrow C_{1,1} = \{(0000),(0111),(1110),(1001)\},$$

$$C_{\infty} \rightarrow C_{\infty} = \{(0000),(0100),(0001),(0010)\},$$

where we abuse the notation in the last set. We can write these in terms of the spin matrices, but it looks more familiar using Pauli matrices. Omitting the identity $\sigma_0 \otimes \sigma_0$, the classes are

$$C_{0,0} \leftrightarrow \{\sigma_0 \otimes \sigma_z, \sigma_z \otimes \sigma_z, \sigma_z \otimes \sigma_0\} \quad C_{1,0} \leftrightarrow \{\sigma_x \otimes \sigma_z, \sigma_z \otimes i \sigma_y, i \sigma_y \otimes \sigma_z\}$$

$$C_{0,1} \leftrightarrow \{\sigma_0 \otimes i \sigma_y, i \sigma_y \otimes i \sigma_y, i \sigma_y \otimes \sigma_0\} \quad C_{1,1} \leftrightarrow \{\sigma_x \otimes i \sigma_y, i \sigma_y \otimes \sigma_z, \sigma_z \otimes \sigma_z\}$$

$$C_{\infty} \leftrightarrow \{\sigma_x \otimes \sigma_0, \sigma_0 \otimes \sigma_x, \sigma_x \otimes \sigma_x\}.$$

We discuss the associated projections in the next section.

**Example 2:** For three qubits, $p = 2$ and $n = 3$, there are two primitive polynomials. We take $f(x) = x^3 + x + 1$. If $\lambda$ is a root, so are $\lambda^2$ and $\lambda^4 = \lambda + \lambda^2$. $f'((\lambda)) = \lambda^2 + 1$ and $((\lambda^2 + 1)^{-1} = \lambda$. From $x^3 + x + 1 = (x - \lambda)(x^2 + \lambda x + \lambda^2 + 1)$, we get

$$e_0 = (1, 0) \quad e_1 = \lambda^2 (1, 0) \quad e_2 = \lambda (1, 0).$$

We can summarize the subsequent analysis by writing out the classes $C_{a_{0,1},a_2}$ or the sets of associated spin matrices. (5.3) A more compact summary follows from the observation that each class $C_{a_{0,1},a_2}$ is a subspace of $V_6(Z_2)$ with a basis of three vectors defined by setting one of the $x_j = 1$ and the other $x’$s to zero. The basis for $C_{\infty}$ is obtained by setting one of the $y_j = 1$ and the others to zero. Denoting the bases by $G_{a_{0,1},a_2}$ we obtain:

$$G_{000} = \{(100000),(001000),(000010)\}$$

$$G_{100} = \{(110000),(000110),(001001)\}$$

$$G_{010} = \{(100100),(000011),(011000)\}$$

$$G_{110} = \{(110100),(000111),(011110)\}$$

$$G_{001} = \{(100001),(010110),(001101)\}$$

$$G_{101} = \{(110001),(010010),(001100)\}$$

$$G_{011} = \{(100101),(010111),(011001)\}$$

$$G_{111} = \{(110101),(010011),(011100)\}$$

$$G_{\infty} = \{(010000),(000100),(000001)\}.$$
$G_{010}$ is
\[
\{\sigma_0 \otimes \sigma_0 \otimes \sigma_0, \sigma_z \otimes \sigma_0 \otimes \sigma_0, \sigma_0 \otimes i\sigma_y \otimes \sigma_0, i\sigma_y \otimes \sigma_z \otimes \sigma_0, \\
\sigma_0 \otimes \sigma_0 \otimes i\sigma_y, \sigma_z \otimes \sigma_0 \otimes i\sigma_y, i\sigma_y \otimes \sigma_z \otimes i\sigma_y, \sigma_z \otimes i\sigma_y \otimes i\sigma_y \}\.
\]

Again we defer the discussion of the associated projectors to the next section.

6 Separable measurements

If $d = p^n$, the basic Hilbert space $H$ can be represented as an $n$-fold tensor product $H_1 \otimes \cdots \otimes H_n$ and each factor can be associated with a distinct subsystem. If a projection $P$ factors as $P_1 \otimes \cdots \otimes P_n$ compatible with the representation of $H$, then measurements can be made by coordinating local measurements at the $n$ different sites. One calls such a projection completely separable. The generalization of this idea is that

\[ P = P(I_1) \otimes \cdots \otimes P(I_m) \]

where the $I_k$ are disjoint sets of indices such that $I_1 \cup \cdots \cup I_m = \{1, \ldots, n\}$. A projection factoring this way is called $(I_1, \ldots, I_m)$ separable. In this case the $m$ subsystems can be measured separately without loss of information. If $P$ has no such factorization, we say it is completely inseparable. Separability properties of bases were discussed in some of the earlier work, [8] for example. The notation here facilitates a systematic analysis. Just as the commutativity of the spin matrices is encoded in the indices, the nature of separability of the mutually unbiased bases is also encoded in the indices. For example, let $n = 2$ and let $p$ be odd and consider the set $C_{a_0,0} = \{(2b_0, a_0b_0, 2b_1D, a_0b_1)\}$ of indices from Section 4. In the notation of Appendix B, $u_1 = (2, a_0), u_2 = (0, 0), v_1 = (0, 0)$, and $v_2 = (2D, a_0)$. The associated projections computed from Appendix B are

\[
P_{u,v}(r) = \frac{1}{p} \sum_{m_1} (\eta_{r_1} S_{u_1} \otimes S_{0,0})^{m_1} \left(\frac{1}{p} \sum_{m_2} (\eta_{r_2} S_{0,0} \otimes S_{v_2})^{m_2}\right),
\]

a tensor product of projections. Hence the projections associated with $C_{a_0,0}$ are completely separable.

The $G_{010}$ in Example 2 of Section 5 illustrates partial separability. Using 010 as a subscript in place of $u, v$, $P_{010}(r_1 r_2 r_3)$ can be written as

\[
\left(\frac{1}{4} \sum_{m_1} (\sigma_z \otimes \sigma_x)^{m_1} \right) \otimes \left(\frac{1}{2} \sum_{m_3} ((-1)^{r_3} \sigma_y)^{m_3}\right).
\]

We describe this as $(12) \ (3)$ separability. An examination of the remaining cases shows that $G_{\infty}$ and $G_{000}$ are completely separable, $G_{100}$ and $G_{101}$ are $(1) \ (23)$
and (13)(2) separable, respectively, and the remaining cases are completely inseparable.

These separability properties are also apparent in the basis vectors. For example, in Theorem 4.2 the subspace $C_{a_0,0}$ of $V_4(Z_p)$ can be written as a direct sum of two subspaces:

$$C_{a_0,0} = \text{span} \{(2, a_0, 0, 0)\} \oplus \text{span} \{(0, 0, 2D, a_0)\}.$$ 

In Example 2 of section 5 the subspace $C_{010}$ of $V_6(Z_2)$ can be written as

$$C_{010} = \text{span} \{(100100), (011100)\} \oplus \text{span} \{(000011)\}.$$ 

The general case is the obvious extension to more indices and different varieties of separability. We limit ourselves to a bipartite factorization for simplicity, and we omit the proof.

**Theorem 6.1** Let $I_1$ denote the indices of a subset of factors in $H_1 \otimes \cdots \otimes H_n$ and let $I_2$ denote the complementary factors. Suppose

$$C_{a_0 \ldots a_{n-1}} = C_{a_0 \ldots a_{n-1}}(I_1) \oplus C_{a_0 \ldots a_{n-1}}(I_2),$$

where the vectors in $C_{a_0 \ldots a_{n-1}}(I_k)$ have zero entries in the pairs of indices not indexed by $I_k$. Then the associated projections $P_{a_0 \ldots a_{n-1}}(r)$ are $(I_1, I_2)$ separable and

$$P_{a_0 \ldots a_{n-1}} = P_{a_0 \ldots a_{n-1}}(r (I_1)) \otimes P_{a_0 \ldots a_{n-1}}(r (I_2)),$$

where $r (I_k)$ has non-zero components only in positions indexed by $I_k$.

Finally, if

$$C_{a_0 \ldots a_{n-1}} = \bigoplus_{k=1}^m C_{a_0 \ldots a_{n-1}}(I_k),$$

then the vectors in $C_{a_0 \ldots a_{n-1}}(I_k)$ have symplectic product zero and hence the associated spin matrices commute. The formal verification is easy, and we leave it to the reader to confirm that property for the examples described above.

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**A Projections of generalized spin matrices**

Here are the details for the projections associated with the $S_u$. We recall the Definitions 3.3 and 3.4 and prove Proposition 3.6.
Proposition A.1 When $d$ is prime, $\{P_u(r) : 0 \leq r < d\}$ is a complete set of mutually orthogonal projections.

Proof: We have

$$P_u(r) P_u(s) = \frac{1}{d^2} \sum_{m=0}^{d-1} \left( \sum_{n=0}^{d-1} (\alpha_u)^{m+n} \eta^{m+sn} S_u^{m+n} \right).$$

Consider two cases. Suppose $0 \leq n \leq d - m - 1$. Define $t$ by $m \leq t \equiv m+n \leq d-1$ and replace this part of the $n$-summation by the corresponding $t$-summation. If $d - m \leq n < d$, $0 \leq t \equiv m+n - d < m$, and we have altogether

$$P_u(r) P_u(s) = \frac{1}{d^2} \sum_{m=0}^{d-1} \left[ \sum_{t=m}^{d-1} \eta^{m(r-s)} \alpha_u^t \eta^{ts} S_u^{t} + \sum_{t=0}^{m-1} \eta^{m(r-s)} \alpha_u^{t+d} \eta^{ts} S_u^{t+d} \right].$$

Now $\alpha_u^{t+d} S_u^{t+d} = (\alpha_u S_u)^t \alpha_u^d \eta^{jk(\frac{d}{2})}$. By virtue of the definition of $\alpha_u$, $\alpha_u^d \eta^{jk(\frac{d}{2})} = 1$, and it is precisely for this reason that we chose the specific form of $\alpha_u$. It follows that

$$P_u(r) P_u(s) = \frac{1}{d^2} \sum_{m=0}^{d-1} (\alpha_u \eta S_u)^t \sum_{m=0}^{d-1} \eta^{m(r-s)}.$$

When $r \neq s$, $\sum_{m=0}^{d-1} \eta^{m(r-s)} = 0$. When $r = s$, the second summation equals $d$, and thus $P_u(r) P_u(s) = \delta(r, s) P_u(r)$.

It remains to show that $(P_u(r))^\dagger = P_u(r)$, and again we need $\alpha_u^d \eta^{jk(\frac{d}{2})} = 1$.

$$(P_u(r))^\dagger = \frac{1}{d} \sum_{m=0}^{d-1} \alpha_u^{-m} \eta^{-mr} \left( \eta^{jk(\frac{m}{2})} S_{m,j,mk} \right)^\dagger = \frac{1}{d} \sum_{m=0}^{d-1} \alpha_u^{-m} \eta^{-mr} \eta^{-m^2j-k(jk(\frac{m}{2}))} S_{m,j,-mk}$$

where we use $m^2 - \binom{m}{2} = \binom{m+1}{2}$ and the substitution $n = d - m$ for $1 \leq m < d$. From the properties of the spin matrices, we obtain

$$(P_u(r))^\dagger = \frac{1}{d} \left[ S_{0,0} + \sum_{n=1}^{d-1} \alpha_u^n \eta^{nr} \eta^{jk(\frac{m}{2})} S_{n,j,nk} \alpha_u^{-d} \eta^{jk(\frac{d}{2})} \right] = P_u(r).$$

B Projections of tensor products of generalized spin matrices

In Theorem 4.2 which solves the MUB problem for the bipartite case, we obtained classes of matrices of the form $S_{2b_0,a_0b_0+a_1b_1D} \otimes S_{2b_D,a_0b_0+a_1b_1D}$ where $a_0$ and $a_1$ are fixed, and the $b_k$’s vary over $\mathbb{Z}_p$. Following the ideas used above, we
want to show how the projections for each class can be computed from the spin matrices in the class. From Lemma 2.2

\[ S_{2b_0, a_0 b_0 + a_1 b_1} = S_{b_0(2, a_0)} S_{b_1(0, a_1, a_1, D)} \]

so that, up to powers of \( \eta \), matrices in this class are of the form

\[ (S_{b_0(2, a_0) \otimes S_{b_0(0, a_1)}) \cdot (S_{b_1(0, a_1, D) \otimes S_{b_1(2D, a_0)}) \eta^{-b_0 b_1} 2 Da_1,} \]

Accordingly, set \( u_1 = (2, a_0), u_2 = (0, a_1 D), v_1 = (0, a_1), v_2 = (2D, a_0) \). For simplicity let \( u \) denote \((u_1, u_2)\), let \( v \) denote \((v_1, v_2)\), and let \( r = (r_1, r_2) \). Up to the factor \( \eta^{-b_0 b_1 2 Da_1} \) the matrices in the commuting class \( C_{a_0, a_1} \) have the form

\[ (S_{b_0 u_1} S_{b_1 u_2}) \otimes (S_{b_0 v_1} S_{b_1 v_2}) = (S_{b_0 u_1} \otimes S_{b_0 v_1}) (S_{b_1 u_2} \otimes S_{b_1 v_2}), \]

and this motivates the definition

\[ P_{u,v} (r) \equiv \frac{1}{d^2} \sum_{m_1} \sum_{m_2} (\eta^{r_1} S_{u_1} \otimes S_{v_1})^{m_1} (\eta^{r_2} S_{u_2} \otimes S_{v_2})^{m_2}. \]

**Proposition B.1** \( B_{a_0, a_1} = \{ P_{u,v} (r) : r_1, r_2 \in Z_p \} \) is the set of orthogonal projections generated by the commuting unitary matrices indexed by \( C_{a_0, a_1} \).

**Proof:** Expand \( P_{u,v} (r) P_{u,v} (s) \) using \( m \) and \( n \) for the summation variables. Then check that

\[ (S_{u_2} \otimes S_{v_2})^{m_2} (S_{u_1} \otimes S_{v_1})^{n_1} = (S_{u_2} \otimes S_{v_2})^{m_2} (S_{u_1} \otimes S_{v_1})^{n_1} \]

since \( u_1 \circ u_2 + v_1 \circ v_2 = 0 \). Hence, \( P_{u,v} (r) P_{u,v} (s) \) can be written as

\[ \frac{1}{d^2} \sum_{k_1} \sum_{k_2} \eta^{s_1 k_1 + s_2 k_2} (S_{u_1} \otimes S_{v_1})^{k_1} (S_{u_2} \otimes S_{v_2})^{k_2} \]

multiplied by \( \sum_{m_1} \sum_{m_2} \eta^{m_1 (r_1 - s_1) + m_2 (r_2 - s_2)} \). It follows that the product is \( P_{u,v} (r) \) if \( r = s \), and 0 otherwise. Clearly \( P_{u,v} (r) \) has trace 1 since only the \( m_1 = m_2 = 0 \) term contributes to the trace. We need to prove that \( P_{u,v} (r) = P_{u,v} (r) \). This can be verified using the same techniques illustrated above and we omit the details. Finally it is easy to check that

\[ (S_{u_1} \otimes S_{v_1})^{t_1} (S_{u_2} \otimes S_{v_2})^{t_2} = \sum_{n_1} \sum_{n_2} \eta^{-n_1 t_1 - n_2 t_2} P_{u,v} (n), \]

where \( n = (n_1, n_2) \).

Analogous results can be extended to the case of multiple tensor products using the same kind of reasoning. Since the only complication is notational, we omit the statements and proofs.
C Methodology for \( d = p^2 \), \( p \) an odd prime

Anticipating step 1 of Section 5, define the polynomial \( f(x) = x^2 - D \), where \( D \) is chosen so that \( f(x) \) does not have a root in \( \mathbb{Z}_p \). Now let \( \lambda \) denote a root of \( f(x) \) in \( \text{GF}(p^2) \). (The analogue is the introduction of the symbol \( i \) to denote a root of \( f(x) = x^2 + 1 \), which does not have a root in the real numbers.) Following \([9, 13]\) define the Galois field

\[
\text{GF}(p^2) = \{ j + k\lambda : j, k \in \mathbb{Z}_p \}
\]

with coordinate-wise addition and multiplication mod \( p \) defined by

\[
\begin{align*}
(j + k\lambda) + (a + b\lambda) &= (j + a) + (k + b)\lambda \\
(j + k\lambda)(a + b\lambda) &= ja + Dkb + \lambda(jb + ka).
\end{align*}
\]

In analogy with the definition of multiplication of complex numbers, \( \lambda^2 = D \). In \( \text{GF}(p^2) \) there are two distinct solutions of \( f(x) = 0 : \lambda \) and \( (p - 1)\lambda \) where we need \( p > 2 \) to guarantee that these are indeed distinct elements in \( \text{GF}(p^2) \). The remaining exercise is to convince oneself that this produces a field of \( p^2 \) elements. For example, \( (j - k\lambda)(j^2 - Dk^2)^{-1} \) is the multiplicative inverse of \( j + k\lambda \), and one sees the importance of the choice of \( D \) to guarantee that \( j^2 - Dk^2 \neq 0 \).

Let \( V_2 \left( \text{GF}(p^2) \right) = \{ u = (\alpha, \beta) : \alpha, \beta \in \text{GF}(p^2) \} \) and define the symplectic product:

\[
u \circ u' \equiv \beta \alpha' - \alpha \beta'.
\]

**Proposition C.1** Define subsets of \( V_2 \left( \text{GF}(p^2) \right) \) for each \( \alpha \) in \( \text{GF}(p^2) \)

\[
\begin{align*}
C_\alpha &= \{ \beta(1,0) + \beta \alpha(0,1) = \beta(1,\alpha) : \beta \in \text{GF}(p^2) \} \\
C_\infty &= \{ \beta(0,1) : \beta \in \text{GF}(p^2) \}.
\end{align*}
\]

Then these are \( p^2 + 1 \) sets, each of which has \( p^2 \) vectors and only \((0,0)\) is common to any two sets. If \( u \) and \( v \) are in the same set, \( u \circ v = 0 \).

The proofs of the assertions above are exactly the same as those in Proposition \([8, 2]\). Although we are using a different field, the arguments involving linear spaces are identical.

Now for the second idea. \( V_2 \left( \text{GF}(p^2) \right) \) is a two-dimensional vector space over the extended field. \( \text{GF}(p^2) \) can be thought of as a two-dimensional space over \( \mathbb{Z}_p \). Specifically, if \( \alpha = j_1 + j_2\lambda \) and \( \beta = k_1 + k_2\lambda \), then \( u = (\alpha, \beta) = \alpha(1,0) + \beta(0,1) \) can be written as

\[
\begin{align*}
u &= (j_1 + j_2\lambda)(1,0) + (k_1 + k_2\lambda)(0,1) \\
&= j_1(1,0) + j_2\lambda(1,0) + k_1(0,1) + k_2\lambda(0,1).
\end{align*}
\]

which motivates the representation of \( V_2(\text{GF}(p^2)) \) as a four-dimensional vector space over \( \mathbb{Z}_p \). However, to relate the symplectic product in \( V_2(\text{GF}(p^2)) \) to the
vector symplectic product in $\mathbb{H}$, we take special basis vectors. Specifically, we define

$$e_0 = 2^{-1}(1, 0), \ e_1 = (2D)^{-1}\lambda(1, 0), \ f_0 = (0, 1), \ f_1 = \lambda(0, 1)$$

and use these so that

$$(\alpha, \beta) = 2j_1e_0 + 2Dj_2e_1 + k_1f_0 + k_2f_1.$$ 

**Proposition C.2** Let $M$ be the linear mapping from $V_2(GF(p^2))$ to $V_4(Z_p)$ defined by its action on $e_r$ and $f_r$: $M(e_0) = (1, 0, 0, 0), \ M(e_1) = (0, 0, 1, 0), \ M(f_0) = (0, 1, 0, 0), \ M(f_1) = (0, 0, 0, 1)$. Then $M$ is a $Z_p$ isomorphism — a one-to-one, onto mapping that preserves the linear structure. Using the notation above, $w = M((\alpha, \beta)) = (2j_1, k_1, 2Dj_2, k_2)$.

We are now ready to relate the symplectic structures of $V_2(GF(p^2))$ and $V_4(Z_p)$. The point, of course, is that we want to define the classes $C_{a_0, a_1}$ of Theorem 4.1 in terms of the classes $C_{a}$ of Proposition C.1. To do this, we need the idea of the trace of a field extension. This gets us into the details of finite field theory, but for the specific case at hand we can simply define it as follows. The two solutions of $f(x) = 0$ are by definition $\lambda_1 = \lambda$ and $\lambda_2 = (p - 1)\lambda$, and the latter is just the additive inverse $-\lambda$. Then define the linear function $Tr$ as follows.

**Definition C.3** $Tr(j + \lambda k) \equiv \sum_{r=1}^2 (j + \lambda, k) = 2j$.

We now have all of the machinery we need for the case $d = p^2$. Furthermore, the same ingredients, suitably modified, work for $d = p^n$.

**Theorem C.4** Let $z = (\alpha, \beta) \in V_2(GF(p^2))$ and $w = M(z)$. Then

$$w_1 \circ w_2 = Tr(z_1 \circ z_2).$$

In particular, the class $C_{a}$ in $V_2(GF(p^2))$ maps to the class $C_{a_0, a_1}$ in $V_4(Z_p)$.

**Proof:** If $z = (\alpha, \beta)$ in the notation above, then $z_1 = 2j_1e_0 + 2Dj_2e_1 + k_1f_0 + k_2f_1$. Correspondingly, let $z_2 = 2r_1e_0 + 2Dr_2e_1 + s_1f_0 + s_2f_1$. We can compute $z_1 \circ z_2$ in terms of the $e_j$’s and $f_k$’s. Now $e_j \circ e_k = f_j \circ f_k = 0$ and $f_0 \circ e_0 = 2^{-1} = f_1 \circ e_1$, since $\lambda^2(2D)^{-1} = 2^{-1}$. Finally, $f_0 \circ e_1 = \lambda 2^{-1}$ and $f_1 \circ e_0 = \lambda(2D)^{-1}$. Since $Tr(2^{-1}) = 1$ and $Tr(\lambda) = \lambda + (-\lambda) = 0$, we have

$$Tr(z_1 \circ z_2) = (k_12r_1 - 2j_1s_1 + (k_22Dr_2 - 2Dj_2s_2) = (2j_1, k_1) \circ (2r_1, s_1) + (2Dj_2, k_2) \circ (2Dr_2, s_2) = (2j_1, k_1, 2Dj_2, k_2) \circ (2r_1, s_1, 2Dr_2, s_2),$$

which is $w_1 \circ w_2$ in $V_4(Z_p)$ as required. $\square$

The definition of the $e$’s and $f$’s gives $Tr(f_j \circ e_k) = \delta(j, k)$, and that was the point of defining the weights above. All of these techniques generalize, and details are outlined in Appendix D.
D  Finite fields for \( d = p^n, \ p \) prime

We summarize the theory of finite field extensions without proofs. For details see [9, 13]. \( GF(p^n) \) denotes a finite field with \( p^n \) elements that contains the field \( Z_p \) as a subfield. Up to isomorphisms, \( GF(p^n) \) is unique and is defined using a polynomial

\[
f(x) = c_0 + \cdots + c_{n-1}x^{n-1} + x^n
\]

(D.1)

that is irreducible over the field \( Z_p \). One can also assume that \( f \) factors into a product \( \prod_{k=1}^{n} (x - \lambda_k) \) with \( n \) distinct roots \( \lambda_k \) in \( GF(p^n) \). Using \( \lambda \) to denote one of these roots, the theory guarantees that elements of \( GF(p^n) \) can be written as

\[
\alpha = a_0 + a_1 \lambda + \cdots + a_{n-1} \lambda^{n-1} : a_k \in Z_p.
\]

Addition in \( GF(p^n) \) is coordinate-wise and in multiplication, one makes use of \( \lambda^n = - (c_0 + c_1 \lambda + \cdots + c_{n-1} \lambda^{n-1}) \). Then the fact that \( f(x) \) has no roots in \( Z_p \) is used to show \( GF(p^n) \) is a field.

As an example, for \( d = 2^2 \) it can be shown that \( f(x) = x^2 + x + 1 \) is the correct polynomial, since in \( Z_2 \) \( f(0) = 1 \) and \( f(1) = 1 \). Then

\[
GF(2^2) = \{ 0, 1, \lambda, \lambda^2 = \lambda + 1 \}.
\]

It is easy to check that \( x^2 + x + 1 = (x + \lambda)(x + (\lambda + 1)) \).

Different irreducible polynomials can generate the same finite field, but their solutions may have different properties. For example, if \( p = 3 \) and \( n = 2 \), the polynomial \( \tilde{f}(x) = x^2 + 2x + 2 \) can be used instead of \( f(x) = x^2 - D \) with \( D = 2 \). If \( \alpha \) is a root of \( \tilde{f}(x) \) in \( GF(3^2) \), then \( \lambda = \alpha^2 \) is a root of \( f(\lambda) = \lambda^2 - 2 \).

As an exercise in the notation, one can confirm that \( \alpha \) is a primitive root in the sense that all of the non-zero elements of \( GF(3^2) \) can be written as powers of \( \alpha \). The theory guarantees primitive polynomials for finite fields, but we do not assume any properties of the generating irreducible polynomials beyond those set forth in the first paragraph of this section.

The trace operation generalizes in the following way.

**Definition D.2** For each \( \alpha = \alpha(\lambda) = a_0 + a_1 \lambda + \cdots + a_{n-1} \lambda^{n-1}, \)

\[
Tr(\alpha) \equiv \sum_{r=1}^{n} \alpha(\lambda_r),
\]

where the \( \lambda_r \) are the distinct roots of \( f(x) \) in \( GF(p^n) \).

For example, take \( GF(2^2) \). Then \( Tr(1) = 0, Tr(\lambda) = \lambda + (\lambda + 1) = 1, \) and \( Tr(\lambda + 1) = 1 \).

From the representation of elements of \( GF(p^n) \), \( GF(p^n) \) can be considered as an \( n \) dimensional space over \( Z_p \). Then \( V_2(GF(p^n)) \) can be written as a \( 2n \)-dimensional space over \( Z_p \). We define \( n \) of the basis vectors as \( f_k = \lambda^k (0, 1), 0 \leq k \leq n - 1 \), as before, and we want a dual basis consisting of vectors

\[
\{ e_j = g_j(\lambda)(1, 0) : 0 \leq j \leq n - 1 \}
\]
that are linearly independent over $\mathbb{Z}_p$ and satisfy

$$Tr(e_j \circ f_k) = Tr\left(g_j(\lambda) \lambda^k\right) = \sum_{r=1}^{\ell} g_j(\lambda_r) \lambda_r^k = \delta(j,k).$$

The remainder of this Appendix is devoted to deriving the form of $g_j(\lambda)$. Examples in Section 5 illustrate the use of this machinery, and we follow the presentation in [7]. For an alternative method to compute the dual basis based on primitive polynomials see [9].

Since $f(x)$ does not have multiple roots, $f(x)$ and $f'(x)$ have no common non-constant factors and, in addition, $f'(\lambda) \neq 0$. From $f(x) = \prod_{j=1}^{\ell} (x - \lambda_j)$, $f'(\lambda_r) = \prod_{j \neq r} (\lambda_r - \lambda_j)$. With $\lambda$ denoting a generic root, one can check that there are values $d_k = d_k(\lambda)$ such that

$$f(x) = d_0 + d_1 x + \cdots + d_{n-1} x^{n-1}.$$ 

Combining these results, we define

$$F_k(x) \equiv \sum_{r=1}^{\ell} \frac{f(x)}{x - \lambda_r} \frac{\lambda_r^k}{f'(\lambda_r)} = \sum_{r=1}^{\ell} x^r \sum_{j=1}^{n} \frac{d_j(\lambda_r)}{f'(\lambda_r)} \lambda_r^k,$$

Now if we set $\lambda = \lambda_t$ for each of the $n$ distinct roots, only the $r = t$ term survives in the middle expression, so that $F_k(\lambda_t) = \lambda_t^k$. By the general theory of polynomials over finite fields $F_k(x)$ must then equal $x^k$. Thus

$$\delta(j,k) = \sum_{r=1}^{\ell} \frac{d_j(\lambda_r)}{f'(\lambda_r)} \lambda_r^k = Tr\left(\frac{d_j(\lambda)}{f'(\lambda)} \lambda^k\right),$$

and we have a key result.

**Proposition D.3** If $e_j = g_j(\lambda)(1,0)$, where $g_j(\lambda) = d_j(\lambda) / f'(\lambda)$, and $f_k = \lambda^k(0,1)$, then

$$Tr[f_k \circ e_j] = \delta(j,r),$$

and the set $\{e_j, f_k\}$ is linearly independent over $\mathbb{Z}_p$.

It remains to show how to compute $d_k(\lambda)$. From (D.1) and $f(x) = (x - \lambda)(d_0 + d_1 x + \cdots + d_{n-1} x^{n-1})$, $d_{n-1} = c_n = 1$. It follows for $1 \leq r \leq n$ that

$$d_{n-r} = \sum_{j=0}^{r-1} \lambda^j c_{n+j+1-r}.$$

The highest order term of $d_{n-r}$ is $\lambda^{r-1}$. 
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