Simple Gradecast Based Algorithms

Michael Ben-Or†    Danny Dolev†    Ezra N. Hoch†

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Abstract

Gradecast is a simple three-round algorithm presented by Feldman and Micali. The current work presents a very simple synchronous algorithm that utilized Gradecast to achieve Byzantine agreement. Two small variations of the presented algorithm lead to improved algorithms for solving the Approximate agreement problem and the Multi-consensus problem.

An optimal approximate agreement algorithm was presented by Fekete, which supports up to \( \frac{1}{4}n \) Byzantine nodes and has message complexity of \( O(n^k) \), where \( n \) is the number of nodes and \( k \) is the number of rounds. Our solution to the approximate agreement problem is optimal, simple and reduces the message complexity to \( O(k \cdot n^3) \), while supporting up to \( \frac{1}{3}n \) Byzantine nodes.

Multi consensus was first presented by Bar-Noy et al. It consists of consecutive executions of \( \ell \) Byzantine consensuses. Bar-Noy et al., show an optimal amortized solution to this problem, assuming that all nodes start each consensus instance at the same time, a property that cannot be guaranteed with early stopping. Our solution is simpler, preserves round complexity optimality, allows early stopping and does not require synchronized starts of the consensus instances.

1 Introduction

Byzantine consensus [12] is one of the fundamental problems in the field of distributed algorithm. Since its appearance it has been the focus of much research and many variations of the Byzantine consensus problem have been suggested (see [13, 2]). In the current work we are interested in the Byzantine consensus problem and two such variations: multi consensus and approximate agreement.

In the Byzantine consensus problem each node \( p \) has an input value \( v_p \), and all non-faulty nodes are required to reach the same output value \( v \) ("agreement"), s.t. if all non-faulty nodes have the same input value \( v' \) then the output value is \( v' \), i.e., \( v = v' \) ("validity").

Approximate agreement [4] aims at reaching an agreement on a value from the Real domain, s.t. the output values of non-faulty nodes are at most \( \epsilon \) apart; and are within the range of non-faulty nodes’ inputs. The multi consensus problem [3] consists of sequentially executing \( \ell \) Byzantine consensuses one after the other.

The first two problems can be solved in a way that overcomes the \( O(t) \) round complexity lower bound of Byzantine consensus [8], where \( t \) is the number of faulty nodes. Approximate agreement overcomes the lower bound by relaxing the "validity" property. Regarding multi consensus, it is reasonable to think that the \( O(t) \) lower bound leads to an \( O(\ell \cdot t) \) lower bound for multi consensus. However, [3] shows how to solve \( \ell \) sequential Byzantine consensuses in \( O(\ell + t) \) rounds, assuming synchronized starts of the different consensuses instances.

†The Hebrew University of Jerusalem, Jerusalem, Israel
In all three problems it is interesting to compare the round-complexity when there are \( f < t \) failures. That is, it is known that \( t \) must be \( < \frac{1}{3}n \) (\( n \) is the number of nodes in the system). However, what if in a specific run there are only \( f < t \) failures? Can the Byzantine consensus / approximate agreement / multi consensus problem be solved quicker? The answer is “yes” on all three accounts. The property of terminating in accordance to the actual number of failures \( f \) is termed “early-stopping”. The three solutions presented in this paper all have the early-stopping property.

The solutions presented herein all use Gradecast as a building block. Gradecast was first presented in [7], and has been used in many papers since (for example, [9]). In Gradecast a single node gradecasts its value \( v \), and each non-faulty node \( p \) has a pair of output values: a value \( v_p \) and a confidence \( c_p \in \{0, 1, 2\} \). The confidence \( c_p \) provides information regarding the \( v_q \) values obtained at other non-faulty nodes, \( q \) (see more in Section 2.1), and thus allows \( p \) to reason about \( q \)’s output value.

Specifically, we use Gradecast to detect faulty nodes and ignore them in future rounds. The idea to try and identify faulty nodes and ignore them in future rounds (not necessarily using Gradecast) has been around for some time (for example [1, 3, 5, 6, 16, 10]). [3] uses it to achieve efficient sequential composition of Byzantine consensus. In [6] a similar notion of “identifying” faulty nodes and ignoring them is used to efficiently solve the approximate agreement problem. In essence, our usage of Gradecast “transforms” Byzantine failures into crash failures.

Gradecast provides a simplification of the above notion. By using Gradecast we ensure that either a Byzantine node \( z \) discloses its faultiness, or all non-faulty nodes see the same message from \( z \). By using a very simple iterative algorithm we solve Byzantine consensus problem, multi consensus and approximate agreement. All solutions are simple, optimal in their resiliency (\( t < \frac{1}{3}n \)), stop-early and optimal in their running time (up to a constant factor induced by using Gradecast in each iteration). Moreover, for the approximate agreement and multi consensus, our solutions improve upon previously known solutions.

1.1 Related Work

Approximate Agreement: Approximate agreement was presented in [4]. The synchronous solution provided in [4] supports \( n > 3t \) and the convergence rate is \( \frac{n-2t}{n-2t+1} \) per round, which asymptotically is \( \sim \left( \frac{t}{n-2t} \right)^k \) after \( k \) rounds. To easily compare the different algorithms, we consider the number of rounds it takes to reach convergence of \( \frac{1}{n} \). For [4], within \( O(\log n) \) rounds the algorithm ensures all non-faulty nodes have converged to \( \frac{1}{n} \). The message complexity of [4] is \( O(n^2) \) per each round of the \( k \) rounds.

In [6] several results are presented. First, for Byzantine failures there is a solution that tolerates \( n > 4t \) and converges to \( \frac{1}{n} \) within \( O \left( \frac{\log n}{\log \log n} \right) \) rounds. For crash failures, [6] provides a solution tolerating \( n > 3t \) that converges to \( \frac{1}{n} \) within \( O \left( \frac{\log n}{\log \log n} \right) \) rounds. The message complexity of both algorithms is \( O(n^k) \). Moreover, [6] shows a lower bound for the Byzantine case of \( O \left( \frac{\log n}{\log \log n} \right) \) rounds to reach \( \frac{1}{n} \) convergence.

Using failure-transformers, the crash resistant algorithm from [6] can be transformed into a Byzantine resistant algorithm (for example [14]). Such a translation has a constant multiplicative overhead in the round complexity. The transformed algorithm is tolerant to \( n > 3t \) and has the original convergence rate up to a constant factor.

[17] solves the approximate agreement problem while tolerating \( n > 3t \) Byzantine failures; it
Table 1: Comparison of different approximation algorithms

| Algorithm                                      | Rounds            | Resiliency | Message comp.               | Early-stopping? |
|------------------------------------------------|-------------------|------------|-----------------------------|-----------------|
| [4]’s approximate agreement                    | \(O(\log n)\)    | \(n > 3 \cdot t\) | \(O(k \cdot n^2)\)            | ✓               |
| [6]’s “direct” algorithm                       | \(O\left(\frac{\log n}{\log \log n}\right)\) | \(n > 4 \cdot t\) | \(O(n^k)\)                    |                 |
| [6]’s “indirect” algorithm (crash-failure + transformation) | \(O\left(\frac{\log n}{\log \log n}\right)\) | \(n > 3 \cdot t\) | \(O(n^k)\)                    |                 |
| [17]’s approximate algorithm                   | \(O\left(\frac{\log n}{\log \log n}\right)\) | \(n > 3 \cdot t\) | \(O(n)\) as \(n \to \infty\) |                 |
| current                                        | \(O\left(\frac{\log n}{\log \log n}\right)\) | \(n > 3 \cdot t\) | \(O(k \cdot n^3)\)            | ✓               |

converges to \(\frac{1}{n}\) within \(O\left(\frac{\log n}{\log \log n}\right)\) rounds. Moreover, [17] presents algorithms with short messages for small ratios of Byzantine nodes \((\frac{n}{f} \to \infty)\), but when \(n > 4t\), it requires exponential message size.

The solution presented in the current paper has a better convergence rate than that of [4]; it has a higher Byzantine tolerance ratio than that of [6, 17] \((i.e., n > 3t\) instead of \(n > 4t\)) and also has an exponential improvement in the message complexity over that of [17] and [6] \((from O(n^k)\) to \(O(k \cdot n^3)\)). Moreover, the presented solution is simple and has a shorter presentation and much simpler proofs than the solutions of [6, 17].

**Multi Consensus:** The algorithm Multi-Consensus presented in [3] solves \(\ell\) sequential Byzantine consensuses within \(O(t + \ell)\) rounds and is resilient to \(n > 3t\). However, [3] assumes that the starts of the different \(\ell\) consensuses are synchronized, a property that cannot be ensured when a consensus stops early. In the current paper we show how to adapt ideas from [11] such that our solution does not require synchronized starts of the different consensuses.

In summary, a main contribution of this work is its simplicity. Using gradecast as a building block we present a very simple basic algorithm that solves the Byzantine consensus problem and two small variations of it that solve multi consensus and approximate agreement. All three algorithms support \(n > 3t\), have the early-stopping property and are asymptotically optimal in their running time (up to a constant multiplicative factor). Aside from the simplicity, following are the properties of the presented algorithms:

1. The basic algorithm solves the Byzantine consensus problem and terminates within \(3 \cdot \min\{f + 2, t + 1\}\) rounds.
2. The first variation solves the approximate agreement problem, with convergence rate of \(\left(\frac{t}{n - 2t}\right)^k \cdot \frac{1}{k^e}\), per \(3 \cdot k\) rounds \((i.e., within O(\frac{\log n}{\log \log n})\) rounds it converges to \(\frac{1}{n}\)). The message complexity is \(O(k \cdot n^3)\) per \(k\) rounds, as opposed to \(O(n^k)\) of the previous best known results. Moreover, the solution dynamically adapts to the number of failures at each round.
3. The second variation solves \(\ell\) sequential Byzantine consensuses within \(O(t + \ell)\) rounds, and efficiently overcomes the requirement of synchronized starts of the consensus instances \((a requirement assumed by [3])\).

We start with Section 2 that presents the assumed model. In Section 3 the basic algorithm is presented and is proved to solve the Byzantine consensus problem. The proofs are straightforward and are used as an intuitive introduction to the basic Gradecast schema. Section 4 presents a variation of the algorithm that solves the approximate agreement. Section 5 describes how to solve
Algorithm GRADECAST \((q, \text{IGNORE}_p)\)

/* Initialization */ /* executed on node \(p\) with leader node \(q\)*/
1: set ignore all messages being received below from nodes in \(\text{IGNORE}_p\);
2: if \(p = q\) then \(v = \text{‘the input value’};\)

/* Dissemination */
3: round 1 The leader \(q\) sends \(v\) to all;
4: round 2 \(p\) sends the value received from \(q\) to all;

/* Notations */
5: let \((j, v_j)\) represent that \(p\) received \(v_j\) from \(j\);
6: let \(maj\) be a value received the most among such values;
7: let \(#maj\) be the number of occurrences of \(maj\);

/* Support */
8: round 3 if \(#maj \geq n - t\) then \(p\) sends \(maj\) to all;

/* Notations */
9: let \((j, v'_j)\) represent that \(p\) received \(v'_j\) from \(j\);
10: let \(maj'\) be a value received the most among such values;
11: let \(#maj'\) be the number of occurrences of \(maj'\);

/* Grading */
12: if \(#maj' \geq n - t\) set \(v_p := maj'\) and \(c_p := 2;\)
13: otherwise, if \(#maj' \geq t + 1\) set \(v_p := maj'\) and \(c_p := 1;\)
14: otherwise set \(v_p = \bot\) and \(c_p = 0;\)
15: return \((q, v_p, c_p);\)

Figure 1: GRADECAST: The Gradecast protocol

the multi consensus problem. Lastly, Section 6 summarizes and concludes the work.

2 Model

The system consists of \(n\) nodes, out of which up to \(t < \frac{1}{3}n\) may be Byzantine, i.e., behave arbitrarily and collude together. Denote by \(f \leq t\) the actual number of faulty nodes in a given run. Communication is assumed to be synchronous and is done via message passing. The communication graph is complete graph.

The Byzantine consensus problem consists of each node \(p\) having an input value \(v_p\) from a finite set \(\mathcal{V}\) (i.e., \(v_p \in \mathcal{V}\)). Each node \(p\) also has an output value \(o_p \in \mathcal{V}\). Two properties should hold:

1. “agreement”: \(o_p = o_q\) for any two non-faulty nodes \(p, q\) (thus we can talk about the output value of the algorithm);
2. “validity”: if all non-faulty nodes start with the same input value \(v\), then the output value of the algorithm is \(v\).

2.1 Gradecast

Gradecast [7] is a distributed algorithm that ensures some properties that are similar to those of broadcast. Specifically, in Gradecast there is a sender node \(p\) that sends a value \(v\) to all other
nodes. Each node $q$’s output is a pair $\langle v_q, c_q \rangle$ where $v_q$ is the value $q$ thinks $p$ has sent and $c_q$ is $q$’s confidence in this value. The Gradecast properties ensure that:

1. if $p$ is non-faulty then $v_q = v$ and $c_q = 2$, for every non-faulty $q$;
2. for every non-faulty nodes $q, q'$: if $c_q > 0$ and $c_{q'} > 0$ then $v_q = v_{q'}$;
3. $|c_q - c_{q'}| \leq 1$ for every non-faulty nodes $q, q'$.

The protocol in Figure 1 is basically the original protocol presented in [7] with explicit handling of boycotting messages coming from nodes known to be faulty.

**Theorem 1** There is a 3 round Gradecast algorithm.

**Proof:** Figure 1 presents such an algorithm. Assuming that at initiation $\text {IGNORE}_p$, for every non-faulty node $p$, contains only faulty nodes, then the proof of [7] holds.

The implementation in Figure 1 implies the following claim.

**Claim 1** If the leader $q, q \in \text {IGNORE}_p$ for every non-faulty $p$ then following the completion of GRADECAST, $c_p = 0$ for every non-faulty $p$.

**Proof:** Every non-faulty node ignores all messages send by $q$ and as a result every non-faulty node will return $c_p = 0$.

3 Simple Byzantine Consensus

The idea behind the \text {BYZCONSUS} algorithm (Figure 2) is to use gradecast as a means of forcing the Byzantine nodes to “lie” at the expense of being expelled from the algorithm. That is, at each iteration a node $p$ will gradecast its own value, and then consider the values it received: a) any node that gradecasted a value with confidence $\leq 1$ will be marked as faulty, and will be ignored for the rest of the algorithm; b) any value with confidence $\geq 1$ will be considered, and $p$ will update its own value to be the majority of values with confidence $\geq 1$. Moreover, this mechanism ensures that for a faulty node $z$, if different non-faulty nodes consider different values for $z$’s gradecast, then at least one of them should obtain the value with zero confidence. For example, one considers $z$ gradecasted “0” with confidence 1, and the other considers $z$ gradecast’s confidence to be 0. The result of such a case is that all non-faulty nodes will mark $z$ to be faulty, and will remove it from the algorithm. In other words, a Byzantine node can produce, using GRADECAST, contradicting values to non-faulty nodes at most once.

Denote by $\cup \text {BAD}_r$ the union of all $\text {BAD}$ variables for non-faulty nodes at the beginning of iteration $r$. Similarly, denote by $\cap \text {BAD}_r$ the intersection of all $\text {BAD}$ variables for non-faulty nodes at the beginning of iteration $r$.

**Claim 2** If $\cup \text {BAD}_r$ contains only faulty nodes, then the properties of gradecast, following its execution in Line 3, hold.

**Proof:** Ignoring messages of faulty nodes does not affect the properties of gradecast, since gradecast works properly no matter what the faulty nodes do. Specifically, ignoring messages from faulty nodes is equivalent to the faulty nodes not sending those messages.

**Claim 3** If $\cup \text {BAD}_r$ contains only faulty nodes, then $\cup \text {BAD}_{r+1}$ contains only faulty nodes.
Algorithm ByzConsensus

/* Initialization */ /* executed on node p */
1: set BAD := ∅;

/* Main loop */
2: for r := 1 to t + 1 do:
3:     gradecast(p, BAD) with input value v;

/* Notations */
4: let ⟨q, v, c⟩ represent that q gradecasted v with confidence c;
5: let maj be the value received the most among values with confidence ≥ 1;
   (if there is more than one such value, take the lowest)
6: let #maj be the number of occurrences of maj with confidence 2;

/* Updates */
7: set v := maj;
8: set BAD := BAD ∪ {q | received ⟨q, *, c⟩ with c ≤ 1};
9: if #maj ≥ n − t then break loop;
10: end for
11: if executed for < t + 1 iterations then participate in one more iteration;
12: return v;

Figure 2: ByzConsensus: a simple Byzantine consensus algorithm

Proof: Consider a non-faulty node q. By Claim 2, q’s gradecast confidence is 2 at all non-faulty nodes. Thus, no non-faulty node adds q to BAD in the current iteration. Therefore, ∪BADr+1 contains only faulty nodes.

Corollary 1 The gradecast invoked in Line 3 satisfies the gradecast properties, and ∪BADr never contains non-faulty nodes.

Proof: By iteratively applying Claim 2 and Claim 3.

Claim 4 If at the beginning of some iteration all non-faulty nodes have the same value v, then all non-faulty nodes that are still in the main loop exit the loop and update their value to v.

Proof: All non-faulty nodes see at least n − f copies of v with confidence 2. Thus, by Line 5,7 they all update their value to v, and (if they are still in the loop) by Line 9 they all exit it.

Claim 5 If non-faulty nodes p, q have different values of maj at iteration r, then | ∩ BADr+1 | > | ∩ BADr |.

Proof: If p has a different value of maj than q, then (w.l.o.g.) by the definition of maj (Line 5) there is some Byzantine node z such that p received ⟨z, u, *⟩ from z’s gradecast, and q received ⟨z, u’, *⟩, s.t. u ≠ u’. By the properties of gradecast, all non-faulty nodes have confidence of at most 1 for z’s gradecast. Therefore, by Line 8, all non-faulty nodes add z to BAD. That is, z ∈ ∩BADr+1.

To conclude the proof, we need to show that z ∉ ∩BADr. Since p and q see different confidence for z’s gradecast, we conclude that some non-faulty node didn’t ignore z’s messages. (Otherwise, by Claim 1, z gradecast confidence would have been 0 at all non-faulty nodes.) Therefore, we conclude that z ∉ ∩BADr.
Claim 6 If all non-faulty nodes have the same value of $\text{maj}$ at iteration $r$, then all non-faulty nodes end iteration $r$ with the same value $v$.

Proof: Immediate from Line 7. \qed

Claim 7 If some node $p$ breaks the main loop due to Line 9 during iteration $r$, then all non-faulty nodes end iteration $r$ with the same value $v$.

Proof: For $p$ to pass the condition of Line 9, $\#\text{maj}$ must be at least $n-t$. That is, $p$ sees at least $n-t$ gradecast values equal to $\text{maj}$ with confidence 2. From the properties of gradecast, all other non-faulty nodes see $n-t$ gradecast values equal to $\text{maj}$ with confidence $\geq 1$. By Line 5, all they all update their value to be that same value. \qed

Theorem 2 ByzConsensus solves the Byzantine consensus problem.

Proof: From Claim 4 it is clear that “validity” holds. To show that “agreement” holds we consider two different cases. First, if a non-faulty node passes the condition of Line 9 in the first $t$ iterations, then by Claim 7 and Claim 4 “agreement” holds.

Second, if no non-faulty node ever passes the condition of Line 9 in the first $t$ iterations, then all non-faulty nodes perform the main loop of ByzConsensus $t+1$ times. By Claim 6 and Claim 4 this means that in every iteration of the first $f$ iterations there is some pair of non-faulty nodes that have different values of $\text{maj}$. By Claim 5 $|\cap \text{BAD}_{t+1}| > |\cap \text{BAD}_t| > \cdots > |\cap \text{BAD}_1| = 0$. Thus, $|\cap \text{BAD}_{t+1}| \geq t$. Therefore, in iteration $t+1$ all non-faulty nodes ignore all Byzantine nodes’ messages. Therefore, all non-faulty nodes see the same set of gradecast messages (all with confidence 2) and thus they all agree on the value of $\text{maj}$. By Lemma 6 all non-faulty nodes end iteration $t+1$ with the same value of $v$. \qed

Remark 3.1 Notice that the above proof also proves the “early stopping” property of ByzConsensus. More specifically, if there are $f \leq t$ actual failures, then ByzConsensus terminates within $\min \{f+2, t+1\}$ iterations (each iteration takes 3 rounds).

4 Approximate Agreement

In this section we are interested in an algorithm that solves the approximate agreement problem [4]. Approximate agreement is somewhat different from Byzantine agreement. Specifically, each node $p$ has a real input value $v_p \in \mathbb{R}$ and a real output value $o_p \in \mathbb{R}$. Denote by $L$ ($H$ resp.) the lowest (highest resp.) input values of non-faulty nodes. Given a constant $\epsilon$ the approximate agreement problem requires that:

1. “agreement”: $|o_p - o_q| \leq \epsilon$ for any two non-faulty nodes $p, q$;
2. “validity”: $o_p \in [L, H]$ for every non-faulty node $p$.

The algorithm ApproxAgree in Figure 3 has the following iterative structure: a) gradecast $v$ to everyone; b) collect all values received into a multi-set; c) perform some averaging method (denote it by AVG) on the multi-set, and use that as the input of the next iteration. AVG removes the $t$ lower and higher values, then computes the average of the remaining set.

For $\epsilon = \frac{H-L}{n}$ the algorithm in Figure 3 requires $O\left(\frac{\log n}{\log \log n}\right)$ iterations. The best previous approximate agreement that has an early-stopping property, polynomial message size and supports $n > 3t$ Byzantine nodes (see [4]), requires $O(\log n)$ iterations.
Algorithm \textsc{ApproxAgree}(\epsilon)

/* Initialization */ /* executed on node \( p \) */
1: set \( \text{BAD} := \emptyset \);

/* Main loop */
2: while true do:
3: gradecast \((p, \text{BAD})\) with input value \( v \);

/* Notations */
4: let \( q, v, c \) represent that \( q \) gradecasted \( v \) with confidence \( c \);
5: let \( \text{values} \) be the multiset of received values with confidence \( \geq 1 \), and add “0” until \( \text{values} \) contains \( n \) items;
6: let \( \text{values}' \) be the multiset of received values with confidence \( 2 \);

/* Updates */
7: set \( v := \text{AVG}(\text{values}) \);
8: set \( \text{BAD} := \text{BAD} \cup \{q \mid \text{received} \langle q, *, c \rangle \text{ with } c \leq 1 \} \);
9: if there are \( n - t \) items in \( \text{values}' \) that are at most \( \epsilon \) apart, then break loop;
10: end while
11: participate in one more iteration;
12: return \( v \).

Figure 3: \textsc{ApproxAgree}: an efficient approximate agreement algorithm

In a similar manner to \textsc{ByzConsensus} (Section 3) only Byzantine nodes can be added to the \( \text{BAD} \) set of any non-faulty node, and a given Byzantine node \( z \)'s value can be viewed differently by different non-faulty nodes at most once. That is, each Byzantine node can “lie” at most once.

Denote by \( V_r \) the multi-set containing the values \( v \) of all non-faulty nodes at the beginning of iteration \( r \). Denote by \( L(M) \) the lowest value in \( M \) and by \( H(M) \) the highest value in \( M \). \( M^t \) is the multi-set \( M \) after the \( t \) lowest and \( t \) highest values have been removed. Using these notations, the \textsc{AVG} method is defined as:

\[
\text{AVG}(M) := \frac{\sum_{e \in M^t} e}{|M^t|}.
\]

Remark 4.1 The \textsc{AVG} method has the following properties:

1. \( \text{AVG}(M) \in [L(M^t), H(M^t)] \);
2. if \( M \) contains \( n - t \) values in the range \([v, v + \epsilon]\) then \( \text{AVG}(M) \in [v, v + \epsilon] \);
3. for \( x \leq t \): if \( M \) is a multi-set of \( n - x \) values, and \( M_1 \) and \( M_2 \) contain \( M \) and additional \( x \) items (i.e., \( M_1, M_2 \) differ by at most \( x \) values) then

\[
|\text{AVG}(M_1) - \text{AVG}(M_2)| \leq (H(M) - L(M)) \cdot \frac{x}{n - 2t}.
\]

Proof:

1. \( \text{AVG}(M) \) is the average of the set \( M^t \), which is clearly between the lowest value in \( M^t \), and the highest value in \( M^t \).
2. Since \( M \) contains \( n - t \) values in the range \([v, v + \epsilon]\), then by removing the \( t \) highest values we remain with values that are all at most \( v + \epsilon \); that is, \( H(M^t) \leq v + \epsilon \). Similarly \( L(M^t) \geq v \). Thus, the average of \( M^t \) is in the range \([v, v + \epsilon]\).
3. Since $|M_1| = |M_2| = n - 2t$, we need to evaluate the difference between $|\sum_{e \in M_1} e - \sum_{e \in M_2} e|$. Since $M_1$ and $M_2$ differ by at most $x$ values, $M_1$, $M_2$ differ by at most $x$ values as well. Since $x \leq t$, each of these values is in the range $[L(M), H(M)]$; therefore, $|\sum_{e \in M_1} e - \sum_{e \in M_2} e| \leq (H(M) - L(M)) \cdot x$. By dividing both sides by $|M_1|$ we have that $|AVG(M_1) - AVG(M_2)| \leq (H(M) - L(M)) \cdot \frac{x}{n - 2t}$.

\[\Box\]

Remark 4.2 Notice that for any pair of non-faulty nodes $p, q$ it holds that the set ‘values’ of $p$ contains at least $n - f$ exact same values as ‘values’ of $q$. Moreover, ‘values’ of $p$ is contained in ‘values’ of $q$.

Claim 8 For non-faulty $p$, at the end of iteration $r$, the value $v$ is in the range $[L(V_{r-1}), H(V_{r-1})]$.

Proof: Immediate from the first property of AVG, and the fact that all non-faulty nodes’ values are in the set values of $p$ (which stems from Gradecast’s properties).

Claim 9 If a non-faulty node $p$ exits the main loop in Line 9 in iteration $r$ then $H(V_r) - L(V_r) \leq \epsilon$.

Proof: Due to the properties of gradecast, values of node $q$ contains the set values’ of node $p$. Thus, if $p$ passes the condition in Line 9 then values of $q$ contains $n - t$ values in the range $[v, v + \epsilon]$ (for some $v$). Therefore, by the second property of AVG, node $q$’s computed value will be in the range $[v, v + \epsilon]$. This claim holds for every non-faulty $q$, for the same value of $v$. That is, all non-faulty nodes compute their new value to be in the range $[v, v + \epsilon]$; I.e., within $\epsilon$ of each other.

\[\Box\]

Claim 10 If $H(V_r) - L(V_r) \leq \epsilon$ for some iteration $r$, then every non-faulty node $p$ that is still in the main loop (Line 9) during iteration $r$.

Proof: In iteration $r$ every non-faulty node $p$ sees $n - f$ values with confidence 2 that are within $\epsilon$ of each other and thus (if $p$ is still in the main loop) passes the condition of Line 9.

Denote by $NEW_r := |\cap BAD_{r+1}| - |\cap BAD_r|$; I.e., $NEW_r$ is the number of Byzantine nodes detected as faulty (by all non-faulty nodes) during iteration $r$.

Claim 11 For every iteration $r$ it holds that $H(V_{r+1}) - L(V_{r+1}) \leq (H(V_r) - L(V_r)) \cdot \frac{NEW_r}{n - 2t}$.

Proof: We consider two cases. First, if $NEW_r = 0$ then no new Byzantine node is added to $\cap BAD_{r+1}$. I.e., every Byzantine node $z \notin \cap BAD_r$ is seen by some non-faulty node $p$ as having gradecast value with confidence 2. Thus, every non-faulty node sees the same gradecast value of $z$. For Byzantine $z \in \cap BAD_r$ all non-faulty nodes ignore $z$’s messages. Therefore, all non-faulty nodes have the same value of the set values and thus they all update $v$ in the same manner. I.e., $H(V_{r+1}) - L(V_{r+1}) = 0$.

Continue with the case where $NEW_r > 0$. In a similar manner to the first case, for every node $z \in \cap BAD_r$ and for every node $z \notin \cap BAD_{r+1}$ the non-faulty nodes have the same gradecast value of $z$. Thus, for two non-faulty nodes $p, q$ there are at most $|\cap BAD_{r+1}| - |\cap BAD_r| = NEW_r$ different values in their values sets. By the third property of the AVG method, AVG(values) of $p$ and AVG(values) of $q$ are at most $(H(V_r) - L(V_r)) \cdot \frac{NEW_r}{n - 2t}$ apart.
Claim 12 Assume APPROXAEE$\epsilon$ runs for $k$ iterations, and no non-faulty node has exited the main loop. Then, $H(V_{k+1}) - L(V_{k+1}) \leq \frac{H-L}{k^3} \cdot \left(\frac{t}{n-2t}\right)^k$.

Proof: At each iteration $r$, by Claim 11 $H(V_{r+1}) - L(V_{r+1}) \leq (H(V_r) - L(V_r)) \cdot \frac{NEW_r}{n-2t}$. That is, after the $k$'s iteration we have that $H(V_{k+1}) - L(V_{k+1}) \leq (H-L) \cdot \prod_{i=1}^{k} \frac{NEW_i}{n-2t}$. The worst value of $\prod_{i=1}^{k} \frac{NEW_i}{n-2t}$ is reached when $NEW_i = NEW_j$ for every $i, j \in \{1, \ldots, k\}$. I.e., when $NEW_i = \frac{t}{k}$.

From the above we have that $H(V_{k+1}) - L(V_{k+1}) \leq (H-L) \cdot \left(\frac{t}{n-2t}\right)^k \cdot \frac{1}{k^3}$.

\[\Box\]

Theorem 3 APPROXAEE$\epsilon$ solves the approximate agreement problem, and for $\epsilon = \frac{H-L}{n}$ APPROXAEE$\epsilon$ converges within at most $O\left(\frac{\log n}{\log \log n}\right)$ rounds.

Proof: “validity” holds by iteratively applying Claim 8.

If some node terminates, then by Claim 9 and Claim 10 all nodes terminate within one iteration. Moreover, by Claim 10 and applying Claim 12 for large enough values of $k$ we have that eventually some node terminates; thus proving “agreement”.

Assume towards contradiction that APPROXAEE$\left(\frac{H-L}{n}\right)$ runs for $r = \left\lceil \frac{\log n}{\log \log n} \right\rceil$ iterations and has not terminated yet. By Claim 12 we have that $H(V_{r+1}) - L(V_{r+1}) \leq \frac{H-L}{r^3}$. Consider

$$\log \left(\frac{\log n}{\log \log n}\right) = \frac{\log n}{\log \log n} \cdot \log \left(\frac{\log n}{\log \log n}\right) = \frac{\log n}{\log \log n} \cdot (\log \log n - \log \log \log n).$$

Since $\log \log n - \log \log \log n \geq \frac{1}{2} \log \log n$ (for large values of $n$) we have that $\log \left(\frac{\log n}{\log \log n}\right) \geq \frac{1}{2} \log n$. Concluding that $\frac{\log n}{\log \log n} \cdot \log \left(\frac{\log n}{\log \log n}\right) \geq \sqrt{n}$. Therefore, $H(V_{r+1}) - L(V_{r+1}) \leq \frac{H-L}{\sqrt{n}}$. By running the algorithm for $2r = 2\left\lceil \sqrt{n} \right\rceil$ we have that $H(V_{2r+1}) - L(V_{2r+1}) \leq \frac{H-L}{n} = \epsilon$.

By Claim 10 every non-faulty node terminates by the end of iteration $2r + 2$. Each iteration is composed of a constant number of rounds, hence APPROXAEE$\left(\frac{H-L}{n}\right)$ terminates within $O\left(\frac{\log n}{\log \log n}\right)$ rounds. \[\Box\]

Remark 4.3 Notice that APPROXAEE has the early stopping property in two senses. First, if there are $f \leq t$ failures then the convergence rate for $k$ iterations is $\left(\frac{t}{n-2t}\right)^k \cdot \frac{1}{k^3}$. Second, in each iteration, the convergence of APPROXAEE depends solely on the number of discovered failures; which leads to a quick termination of the algorithm if in some iteration no new failures are discovered (or if few new failures are discovered).

5 Sequential Executions of Consensuses

In [3] the authors investigate the Multi-consensus problem: suppose we want to execute a sequence of Byzantine consensuses, such that each consensus possibly depends on the output of the previous consensus. That is, we must execute the consensuses sequentially and not concurrently. (The concurrent version is termed “interactive consistency”, initially stated in [15].)

The first solution that comes to mind is to simply execute $\ell$ instances of Byzantine consensus one after the other. However, due to the $O(f)$ round lower bound on Byzantine consensus [8], a
naive sequential execution will lead to a running time of \( O(\ell \cdot f) \). In [3] they give a solution that has total running time of \( O(f + \ell) \).

However, [3] assumes that at each consensus instance all correct nodes start the consensus at once; that is, they are always synchronized. This implies that for every one of the \( \ell \) consensuses, all nodes know exactly when the consensus starts. This assumption is problematic, since due to the early-stopping nature of the algorithm used in [3], the different nodes may terminate each invocation of a consensus at different rounds, leading to a problem with the synchronized starts assumption. Assuming synchronized starts, the algorithm ByzConsensus can easily (almost without any change) produce the same result; i.e., for \( \ell \) consecutive consensuses, ByzConsensus’s running time will be \( O(f + \ell) \).

In this section we consider two results. First, we analyze ByzConsensus assuming synchronized starts of each consensus, and compare it to the results of [3]. Second, we augment ByzConsensus with ideas from [11] such that ByzConsensus can solve \( \ell \) consensuses within \( O(f + \ell) \) rounds even if the initiations of each consensus are not synchronized among the non-faulty nodes.

5.1 Synchronized Starts

We start with the assumption that all \( \ell \) consensuses have synchronized starts. That is, all non-faulty nodes start the \( i \)th (out of \( \ell \)) instance of Byzantine consensus in the same round. With this assumption in place, ByzConsensus can be used almost as-is. The single modification required is to perform the initialization state (Line 1) only once for the entire sequence of \( \ell \) consensuses (and not once per consensus).

It is easy to see that the following two statements hold also for the modified algorithm:

1. Each Byzantine node \( z \) can cause non-faulty nodes to disagree on \( z \)’s gradecast value at most once throughout the sequence of \( \ell \) consensuses;
2. The number of iterations of a given consensus instance \( i \) is \( \min\{f + 2, t + 1\} \), where \( f \) is the actual number of Byzantine nodes for which there are non-faulty nodes that do not agree on their gradecast values during instance \( i \).

Using the above statements it is easy to conclude that for a sequence of \( \ell \) sequential consensuses, the number of iterations of the modified ByzConsensus is at most \( t + 2 \cdot \ell \), since each consensus takes at least 2 iterations to complete, and at most \( t \) more iterations are required (depending on the Byzantine nodes’ behavior).

Since each iteration is composed of executing gradecast, the total round complexity is \( 3 \cdot t + 6 \cdot \ell \). Notice that in each iteration there are at most \( n \) gradecasts, each requiring \( O(n \cdot t) \) messages. Therefore, the total bit complexity of \( \ell \) consensuses is \( O \left((t + \ell) \cdot n^2 \cdot t\right) = O \left(n^2 \cdot t^2 + \ell \cdot n^2 \cdot t\right) \).

By a simple optimization which defines a subset of \( 3 \cdot t + 1 \) as performing the gradecast (and the rest of the nodes just “listen” to messages), the total bit complexity is reduced to \( O \left(n \cdot t^3 + \ell \cdot n \cdot t^2\right) \).

Note that ByzConsensus is optimal with respect to its Byzantine resiliency and the total running time ([3] is also optimal with respect to amortized message size and total bits). Lastly, Multi-Consensus is not a complicated algorithm, but ByzConsensus is even simpler.

5.2 Unsynchronized Starts

In this subsection we remove the assumption of synchronized starts of the different \( \ell \) consensuses. This is done by implementing ideas of [11] in a way that is consistent with sequential executions of ByzConsensus. In [11] the authors show how to sequentially execute consensuses even if the consensuses are started at different times and terminate at different times. The reason we cannot use [11]’s results as-is in our case, is because [11] assumes that the running time of the algorithm
does not change over time; which does not hold for our solution. Specifically, ByzConsensus’s $k$th execution’s running time depends on what occurred in the $k-1$ instances that preceded it.

The required addition to ByzConsensus is as follows:

1. If $p$ wants to terminate according to ByzConsensus, it first sends “done” to every one, and waits;
2. In addition, if node $p$ receives $t+1$ “done” messages, then $p$ sends all nodes a “done” message as well;
3. Lastly, if node $p$ received $2 \cdot t + 1$ “done” messages, then $p$ completes the current instance of ByzConsensus;

Notice that if $i$ is the first round in which some non-faulty node $p$ halts, then all other non-faulty nodes either halt in round $i$, or in round $i+1$. That is, non-faulty nodes terminate within one round of each other (no matter what difference there was between their starting times).

Another addition required is to increase the length of each of ByzConsensus’s rounds according to the difference between different non-faulty nodes’ starting times. If the different starting times of each consensus instances is at most $\Delta$ rounds at different nodes, then each iteration of ByzConsensus needs to be increased by a factor of $\Delta$ so that messages of the $i$-th round are received (and considered as messages of the $i$-th round) by all other nodes.

Remark 5.1 The increased length of each iteration is only with respect to the original ByzConsensus. The additions regarding the “done” messages are left as is. Thus, we ensure that non-faulty nodes still terminate within one round of each other.

To see that the addition does not harm the correctness of ByzConsensus, consider the following. Suppose $p$ starts running at some round, and $q$ starts running $\Delta$ rounds afterwards. By the correctness proof of ByzConsensus, if $p$ terminates, then one iteration afterwards $q$ will also terminate. But actually, there is an even stronger property: if $p$ terminates then all other non-faulty nodes already have the same value as $p$ (see Claim 7). I.e., if $q$ terminates when $p$ terminates, then it will terminate with the same value.

To conclude, notice that the above addition ensures that a non-faulty node halts only if some non-faulty node has terminated in the execution of ByzConsensus. Thus, from that round onward, the sequence of non-faulty nodes’ termination is valid.

The above discussion leads to the following theorem:

**Theorem 4** The updated ByzConsensus algorithm solves $\ell$ sequential Byzantine consensus within $O(\Delta \cdot (\ell + t))$ rounds, while not requiring synchronized starts of the different consensus instances.

### 6 Conclusion

We presented a simple algorithm ByzConsensus that uses Gradecast as a building block, and solves the Byzantine consensus problem within $3 \cdot \min\{f+2, t+1\}$ rounds.

Two variations of ByzConsensus are given. The first one optimally solves the approximate agreement problem and reduces the message complexity of the best known optimal algorithm from $O(n^k)$ to $O(k \cdot n^3)$. The second variant of ByzConsensus optimally solves the multi consensus problem and efficiently supports unsynchronized starts of the consensus instances.

All three algorithms have optimal resiliency, optimal running time (up to a constant multiplicative factor) and have the early stopping property. Aside from their improved simplicity, the two variants also improve (in different aspects) upon previously best known solutions.
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