NORMAL HAUSDORFF SPECTRA OF PRO-$p$ GROUPS

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Abstract. Using wreath products, we construct a finitely generated pro-$p$ group $G$ with infinite normal Hausdorff spectrum

$$\text{hspec}_\mathcal{S}^p(G) = \{\text{hdim}_\mathcal{S}^p(G)(H) \mid H \leq_c G\};$$

here $\text{hdim}_\mathcal{S}^p(G) : \{X \mid X \subseteq G\} \to [0, 1]$ denotes the Hausdorff dimension function associated to the $p$-power series $\mathcal{S}^p : G^p$, $i \in \mathbb{N}_0$. More precisely, we show that $\text{hspec}_\mathcal{S}^p(G) = [0, 1/3] \cup \{1\}$ contains an infinite interval; this settles a question of Shalev. Furthermore, we prove that the normal Hausdorff spectra $\text{hspec}_\mathcal{S}^p(G)$ with respect to other filtration series $\mathcal{S}$ have a similar shape. In particular, our analysis applies to standard filtration series such as the Frattini series, the lower $p$-series and the modular dimension subgroup series.

Lastly, we pin down the ordinary Hausdorff spectra $\text{hspec}_\mathcal{S}^\mathcal{L}(G)$ with respect to the standard filtration series $\mathcal{S}$. The spectrum $\text{hspec}_\mathcal{L}(G)$ for the lower $p$-series $\mathcal{L}$ displays surprising new features.

1. Introduction

The concept of Hausdorff dimension has led to interesting applications in the context of profinite groups; see [4] and the references given therein. Let $G$ be a countably based infinite profinite group and consider a filtration series $\mathcal{S}$ of $G$, that is, a descending chain $G = G_0 \supseteq G_1 \supseteq \ldots$ of open normal subgroups $G_i \triangleright_o G$ such that $\bigcap_i G_i = 1$. These open normal subgroups form a base of neighbourhoods of the identity and induce a translation-invariant metric on $G$ given by $d^\mathcal{S}(x, y) = \inf \{|G : G_i|^{-1} \mid x \equiv y \pmod{G_i}\}$, for $x, y \in G$. This, in turn, supplies the Hausdorff dimension $\text{hdim}_\mathcal{S}^\mathcal{L}(U) \in [0, 1]$ of any subset $U \subseteq G$, with respect to the filtration series $\mathcal{S}$.

Barnea and Shalev [1] established the following ‘group-theoretic’ interpretation of the Hausdorff dimension of a closed subgroup $H$ of $G$ as a logarithmic density:

$$\text{hdim}_\mathcal{S}^\mathcal{L}(H) = \lim_{i \to \infty} \frac{\log |H G_i : G_i|}{\log |G : G_i|}.$$ 

The Hausdorff spectrum of $G$, with respect to $\mathcal{S}$, is

$$\text{hspec}_\mathcal{S}^\mathcal{L}(G) = \{\text{hdim}_\mathcal{S}^\mathcal{L}(G)(H) \mid H \leq_c G\} \subseteq [0, 1],$$

where $H$ runs through all closed subgroups of $G$. As indicated by Shalev in [7, §4.7], it is also natural to consider the normal Hausdorff spectrum of $G$, with

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respect to $S$, namely
\[
\text{hspec}^S(G) = \{\text{hdim}^S_H(H) \mid H \trianglelefteq_c G\}
\]
which reflects the range of Hausdorff dimensions of closed normal subgroups. Apart from the observations in [7, §4.7], very little appears to be known about normal Hausdorff spectra of profinite groups.

Throughout we will be concerned with pro-$p$ groups, where $p$ denotes an odd prime; in Appendix A we indicate how our results extend to $p = 2$. We recall that even for well structured groups, such as $p$-adic analytic pro-$p$ groups $G$, the Hausdorff dimension function and the Hausdorff spectrum of $G$ are known to be sensitive to the choice of $S$; compare [4]. However, for a finitely generated pro-$p$ group $G$ there are natural choices for $S$, such as the $p$-power series $P$, the Frattini series $F$, the lower $p$-series $L$ and the modular dimension subgroup series $D$; see Section 2.

In this paper, we are interested in a particular group $G$ constructed as follows. The pro-$p$ wreath product $W = C_p \wr \widehat{Z}_p$ is the inverse limit $\lim_{\leftarrow k \in \mathbb{N}} C_p \wr C_{p^k}$ of the finite standard wreath products of cyclic groups with respect to the natural projections; clearly, $W$ is 2-generated as a topological group. Let $F$ be the free pro-$p$ group on two generators and let $R \trianglelefteq_c F$ be the kernel of a presentation $\pi: F \to W$. We are interested in the pro-$p$ group $G = F/N$, where $N = [R, F]R^p \trianglelefteq_c F$.

Up to isomorphism, the group $G$ does not depend on the particular choice of $\pi$, as can be verified using Gaschütz’ Lemma; see [6, Prop. 2.2]. Indeed, $G$ can be described as the universal 2-generated covering group for central extensions of elementary abelian pro-$p$ groups by $W$.

**Theorem 1.1.** For $p > 2$, the normal Hausdorff spectra of the pro-$p$ group $G$ constructed above, with respect to the standard filtration series $\mathcal{P}$, $\mathcal{D}$, $\mathcal{F}$ and $\mathcal{L}$ respectively, satisfy:

\[
\text{hspec}^\mathcal{P}_c(G) = \text{hspec}^\mathcal{D}_c(G) = [0, 1/3] \cup \{1\},
\]
\[
\text{hspec}^\mathcal{F}_c(G) = [0, 1/(1+p)] \cup \{1\},
\]
\[
\text{hspec}^\mathcal{L}_c(G) = [0, 1/5] \cup \{3/5\} \cup \{1\}.
\]

In particular, they each contain an infinite real interval.

This solves a problem posed by Shalev [7, Problem 16]. We observe that the normal Hausdorff spectrum of $G$ is sensitive to changes in filtration and that the normal Hausdorff spectrum of $G$ with respect to the Frattini series varies with $p$.

In Section 4 we show that finite direct powers $G \times \ldots \times G$ of the group $G$ provide examples of normal Hausdorff spectra consisting of multiple intervals. Furthermore, the sequence $G \times \ldots \times G$, $m \in \mathbb{N}$, has normal Hausdorff spectra ‘converging’ to $[0, 1]$; compare Corollary 4.5. We highlight three natural problems.

**Problem 1.2.** Does there exist a finitely generated pro-$p$ group $H$

(a) with countably infinite normal Hausdorff spectrum $\text{hspec}^S_c(H)$,

(b) with full normal Hausdorff spectrum $\text{hspec}^S_c(H) = [0, 1]$,

(c) such that 1 is not an isolated point in $\text{hspec}^S_c(H)$,

for one or several of the standard series $S \in \{\mathcal{P}, \mathcal{D}, \mathcal{F}, \mathcal{L}\}?$
We also compute the entire Hausdorff spectra of $G$ with respect to the four standard filtration series, answering en route a question raised in [3, VIII.7.2].

**Theorem 1.3.** For $p > 2$, the Hausdorff spectra of the pro-$p$ group $G$ constructed above, with respect to the standard filtration series, satisfy:

\[
\text{hspec}^P(G) = \text{hspec}^\varnothing(G) = \text{hspec}^F(G) = [0, 1],
\]

\[
\text{hspec}^C(G) = [0, 4/5) \cup \{3/5 + 2m/5p^n \mid m, n \in \mathbb{N}_0 \text{ with } m^p/n < m \leq p^n\}.
\]

The qualitative shape of the spectrum $\text{hspec}^C(G)$, i.e., its decomposition into a continuous and a non-continuous, but dense part, is unprecedented and of considerable interest; in Corollary 2.11 we show that already the wreath product $W = C_p \wr \mathbb{Z}_p$ has a similar Hausdorff spectrum with respect to the lower $p$-series.

**Organisation.** Section 2 contains preliminary results. In Section 3 we give an explicit presentation of the pro-$p$ group $G$ and describe a series of finite quotients $G_k, k \in \mathbb{N}$, such that $G = \lim \leftarrow G_k$. In Section 4 we provide a general description of the normal Hausdorff spectrum of $G$ and, with respect to certain induced filtration series, we generalise this to finite direct powers of $G$. In Section 5 we compute the normal Hausdorff spectrum of $G$ with respect to the $p$-power filtration series $P$, and in Section 6 we compute the normal Hausdorff spectra of $G$ with respect to the other three standard filtration series $D, F, L$. In Section 7 we compute the entire Hausdorff spectra of $G$. Finally, in Appendix A we indicate how our results extend to the case $p = 2$.

**Notation.** Throughout, $p$ denotes an odd prime, although some results hold also for $p = 2$, possibly with minor modifications; only in Appendix A we discuss the analogous pro-2 groups. We denote by $\lim_{i \to \infty} a_i$ the lower limit (limes inferior) of a sequence $(a_i)_{i \in \mathbb{N}}$ in $\mathbb{R} \cup \{\pm \infty\}$. Tacitly, subgroups of profinite groups are generally understood to be closed subgroups. Subscripts are used to emphasise that a subgroup is closed respectively open, as in $H \leq_c G$ respectively $H \leq_o G$. We use left-normed commutators, e.g., $[x, y, z] = [[x, y], z]$.

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2. Preliminaries

2.1. Let $G$ be a finitely generated pro-$p$ group. We consider four natural filtration series on $G$. The $p$-power series of $G$ is given by

\[P: G^{p^i} = \langle x^{p^i} \mid x \in G \rangle, \quad i \in \mathbb{N}_0.\]

The lower $p$-series (or lower $p$-central series) of $G$ is given recursively by

\[L: P_1(G) = G, \quad \text{and} \quad P_i(G) = P_{i-1}(G)^p [P_{i-1}(G), G] \quad \text{for } i \geq 2,\]

while the Frattini series of $G$ is given recursively by

\[F: \Phi_0(G) = G, \quad \text{and} \quad \Phi_i(G) = \Phi_{i-1}(G)^p [\Phi_{i-1}(G), \Phi_{i-1}(G)] \quad \text{for } i \geq 1.\]
The (modular) dimension subgroup series (or Jennings series or Zassenhaus series) of $G$ can be defined recursively by

$$D_1(G) = G, \quad \text{and} \quad D_i(G) = D_{i/p}(G)^p \prod_{1 \leq j < i} [D_j(G), D_{i-j}(G)] \quad \text{for } i \geq 2.$$ 

As a default we set $P_0(G) = D_0(G) = G$.

2.2. Next, we collect auxiliary results to detect Hausdorff dimensions of closed subgroups of pro-$p$ groups. For a countably based infinite pro-$p$ group $G$, equipped with a filtration series $S: G = G_0 \supseteq G_1 \supseteq \ldots$, and a closed subgroup $H \leq \pi G$ we say that $H$ has strong Hausdorff dimension in $G$ with respect to $S$ if

$$\text{dim}^S_G(H) = \lim_{i \to \infty} \frac{\log_p |HG_i : G_i|}{\log_p |G : G_i|}$$

is given by a proper limit.

The first lemma is an easy variation of [4, Lem. 5.3] and we omit the proof.

**Lemma 2.1.** Let $G$ be a countably based infinite pro-$p$ group with closed subgroups $K \leq \pi G$. Let $S: G = G_0 \supseteq G_1 \supseteq \ldots$ be a filtration series of $G$ and write $S\mid_H: H = H_0 \supseteq H_1 \supseteq \ldots$, with $H_i = H \cap G_i$ for $i \in \mathbb{N}_0$, for the induced filtration series of $H$. If $K$ has strong Hausdorff dimension in $H$ with respect to $S\mid_H$, then

$$\text{dim}^S_G(K) = \text{dim}^S_G(H) \cdot \text{dim}^S_H(K).$$

**Lemma 2.2.** Let $G$ be a countably based infinite pro-$p$ group with closed subgroups $N \leq \pi G$ and $H \leq \pi G$. Let $S: G = G_0 \supseteq G_1 \supseteq \ldots$ be a filtration series of $G$, and consider the induced filtration series of $N$ and $G/N$ defined by

$$S\mid_N: G_i \cap N, \quad i \in \mathbb{N}_0, \quad \text{and} \quad S\mid_{G/N}: G_iN/N, \quad i \in \mathbb{N}_0.$$ 

Suppose that $N$ has strong Hausdorff dimension $\xi = \text{dim}^S_G(N)$ in $G$, with respect to $S$. Then we have

$$\text{dim}^S_G(H) \geq (1 - \xi) \text{dim}^S_{G/N} (HN/N) + \xi \lim_{i \to \infty} \frac{\log_p |HG_i \cap N : G_i \cap N|}{\log_p |G_i \cap N|} \quad (\ast)$$

$$\geq (1 - \xi) \text{dim}^S_{G/N} (HN/N) + \xi \text{dim}^S_N (H \cap N). \quad (\ast\ast)$$

Moreover, equality holds in $(\ast\ast)$, if $HN/N$ has strong Hausdorff dimension in $G/N$ with respect to $S\mid_{G/N}$ or if the lower limit on the right-hand side is actually a limit.

Similarly, equality holds in $(\ast\ast)$ if

(i) $H \cap N \leq \pi N$ is an open subgroup or

(ii) $G_iN = (G_i \cap H)N$, for all sufficiently large $i \in \mathbb{N}$.

**Proof.** We observe that

$$\text{dim}^S_G(H) = \lim_{i \to \infty} \left( \frac{\log_p |G : NG_i|}{\log_p |G : G_i|} \cdot \log_p |HG_iN : G_iN| \to 1 - \xi \text{ as } i \to \infty \right)$$

$$+ \log_p |NG_i : G_i| \cdot \log_p |HG_i \cap NG_i : G_i| \to \xi \text{ as } i \to \infty \right)$$
and that, for each $i \in \mathbb{N}$,\[
\log_p [H G_i \cap N G_i : G_i] = \frac{\log_p [H G_i \cap N : G_i \cap N]}{\log_p [N G_i : G_i]}.
\]
Finally,\[
\log_p [H G_i \cap N : G_i \cap N] \geq \log_p [(H \cap N) (G_i \cap N) : G_i \cap N]
\]
and, if condition (i) or (ii) holds, the difference between the two terms is bounded by a constant that is independent of $i \in \mathbb{N}$.

**Lemma 2.3.** Let $Z \cong C_p^{\aleph_0}$ be a countably based infinite elementary abelian pro-$p$ group, equipped with a filtration series $S$. Then, for every $\eta \in [0, 1]$, there exists a closed subgroup $K \leq_c Z$ with strong Hausdorff dimension $\eta$ in $Z$ with respect to $S$.

**Proof.** Write $S : Z = Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \ldots$ and let $\eta \in [0, 1]$. For $i \in \mathbb{N}$, we have $Z_{i-1} / Z_i \cong C_p^{d_i}$ for non-negative integers $d_i$.

**Claim:** There exist non-negative integers $e_1, e_2, \ldots$ such that, for each $i \in \mathbb{N}$, we have $0 \leq e_i \leq d_i$ and
\[
e_1 + \ldots + e_i = \lceil \eta (d_1 + \ldots + d_i) \rceil.
\]
Indeed, with $e_1 = \lceil \eta d_1 \rceil$ the statement holds true for $i = 1$. Now, let $i \geq 2$ and suppose that $e_1 + \ldots + e_{i-1} = \lceil \eta (d_1 + \ldots + d_{i-1}) \rceil$. Then\[
\lceil \eta (d_1 + \ldots + d_{i-1}) \rceil \leq \lceil \eta (d_1 + \ldots + d_i) \rceil \leq \lceil \eta (d_1 + \ldots + d_{i-1}) \rceil + d_i
\]
and thus we may set\[
e_i = \lceil \eta (d_1 + \ldots + d_i) \rceil - (e_1 + \ldots + e_{i-1}),
\]
to satisfy the statement for $i$. The claim is proved.

For all sufficiently large $i \in \mathbb{N}$ we have $d_1 + \ldots + d_i > 0$ and\[
\eta \leq \frac{e_1 + \ldots + e_i}{d_1 + \ldots + d_i} \leq \eta + \frac{1}{d_1 + \ldots + d_i}.
\]
With these preparations, it suffices to display a subgroup $K \leq_c Z$ such that\[
\log_p [K Z_i : Z_i] = e_1 + \ldots + e_i.
\]
For this purpose, we write\[
Z = \langle z_{1,1}, \ldots, z_{1,d_1}, z_{2,1}, \ldots, z_{2,d_2}, \ldots, z_{i,1}, \ldots, z_{i,d_i}, \ldots \rangle
\]
such that $Z_{i-1} = \langle z_{i,1}, \ldots, z_{i,d_i} \rangle Z_i$ for each $i \in \mathbb{N}$. Then we set\[
K = \langle z_{1,1}, \ldots, z_{1,e_1}, z_{2,1}, \ldots, z_{2,e_2}, \ldots, z_{i,1}, \ldots, z_{i,e_i}, \ldots \rangle.
\]

**Corollary 2.4.** Let $G$ be a countably based pro-$p$ group, equipped with a filtration series $S$, and let $N \leq_c H \leq_c G$ such that $H / N \cong C_p^{\aleph_0}$. Set $\xi = \text{hdim}_G^S(N)$ and $\eta = \text{hdim}_G^S(H)$. If $N$ or $H$ has strong Hausdorff dimension in $G$ with respect to $S$, then $[\xi, \eta] \leq \text{hspec}^S(G)$.

**Proof.** If $N$ has strong Hausdorff dimension, we apply Lemmata 2.1, 2.2 and 2.3. If $H$ has strong Hausdorff dimension the claim follows from [4, Thm. 5.4].\[\square\]
2.3. For convenience we recall two standard commutator collection formulae.

**Proposition 2.5.** Let $G = \langle a, b \rangle$ be a finite $p$-group, and let $r \in \mathbb{N}$. For $u, v \in G$ let $K(u, v)$ denote the normal closure in $G$ of (i) all commutators in $\{u, v\}$ of weight at least $p^r$ that have weight at least 2 in $v$, together with (ii) the $p^{r+s+1}$th powers of all commutators in $\{u, v\}$ of weight less than $p^s$ and of weight at least 2 in $v$ for $1 \leq s \leq r$. Then

$$ (ab)^{p^r} \equiv K(a, b) a^{p^r} b^{p^r} [b, a]^{(p^r_1)} \cdots [b, a, p^{r-2}, a]^{(p^r_{r-1})} [b, a, p^{r-1}, a], \quad (2.1) $$

$$ [a^{p^r}, b] \equiv K(a, b) [a, b]^{p^r} [a, b, a]^{(p^r_1)} \cdots [a, b, a, p^{r-2}, a]^{(p^r_{r-1})} a, \quad (2.2) $$

**Remark.** Under the standing assumption $p \geq 3$ and the extra assumptions

$$ \gamma_2(G)^p = 1 \quad \text{and} \quad [\gamma_2(G), \gamma_2(G)] \subseteq Z(G), $$

the congruences (2.1) and (2.2) simplify to

$$ (ab)^{p^r} \equiv L(a, b) a^{p^r} b^{p^r} [a, b, a]^{(p^r_1)} \quad \text{and} \quad [a^{p^r}, b] \equiv M(a, b) [a, b, a]^{p^r} $$

where $L(a, b)$ denotes the normal closure in $G$ of all commutators in $\{a, b\}$ of weight at least $p^r$ that have weight at least 2 in $a$ and $M(a, b)$ denotes the normal closure in $G$ of all commutators $[[a, b, \ldots, a], [b, a, \ldots, a]]$ with $i + j \geq p^r$.

The general result is recorded (in a slightly stronger form) in [5, Prop. 1.1.32]; we remark that (2.2) follows directly from (2.1), due to the identity $[a^{p^r}, b] = a^{-p^r} (a[a, b])^{p^r}$. The first congruence in (2.3) follows directly from (2.1); the second congruence in (2.3) is derived from (2.2) by standard commutator manipulations.

2.4. Now we describe, for $k \in \mathbb{N}$, the lower central series, the lower $p$-series and the Frattini series of the finite wreath product

$$ W_k = \langle x, y \rangle = \langle x \rangle \times \langle y, y^x, \ldots, y^{x^{p^k-1}} \rangle \cong C_p \wr C_p^k $$

with top group $\langle x \rangle \cong C_p^k$ and base group $\langle y, y^x, \ldots, y^{x^{p^k-1}} \rangle \cong C_p^k$.

**Proposition 2.6.** For $k \in \mathbb{N}$, the finite wreath product $W_k$ defined above is nilpotent of class $p^k$ and $W_k^{p^k} = \langle y^{p^k} y^{p^k x} \cdots y^{p^k x^{p^k-1}} \rangle \cong C_p$.

1. The lower central series of $W_k$ satisfies

$$ W_k = \gamma_1(W_k) = \langle x, y \rangle \gamma_2(W_k) \quad \text{with} \quad W_k/\gamma_2(W_k) \cong C_p^k \times C_p, $$

$$ \gamma_i(W_k) = \langle [y, x, \ldots, x] \rangle \gamma_{i+1}(W_k) \quad \text{with} \quad \gamma_i(W_k)/\gamma_{i+1}(W_k) \cong C_p \quad \text{for} \quad 2 \leq i \leq p^k. $$

2. The lower $p$-series of $W_k$ has length $p^k$; it satisfies, for $1 \leq i \leq k$,

$$ P_i(W_k) = \langle x^{p^i}, [y, x, \ldots, x] \rangle \quad \text{with} \quad P_i(W_k)/P_{i+1}(W_k) \cong C_p \times C_p $$

and, for $k < i \leq p^k$,

$$ P_i(W_k) = \langle [y, x, \ldots, x] \rangle \quad \text{with} \quad P_i(W_k)/P_{i+1}(W_k) \cong C_p. $$

3. The Frattini series of $W_k$ has length $k + 1$; it satisfies, for $0 \leq i < k$,

$$ \Phi_i(W_k) = \langle x^{p^i} \rangle \gamma_{p^i+1}(W_k) \quad \text{with} \quad \Phi_i(W_k)/\Phi_{i+1}(W_k) \cong C_p \times C_{p^{i+1}} \times C_p, $$

$$ \Phi_k(W_k) = \gamma_{p^k+1}(W_k) \quad \text{with} \quad \Phi_k(W_k)/\Phi_{k+1}(W_k) \cong C_p \times C_{p^{k+1}} \times C_p. $$
(4) The dimension subgroup series of \( W_k \) has length \( p^k \); in particular, it satisfies, for \( p^k - 1 + 1 \leq i \leq p^k 
abla
abla
amponents over the finite field \( \mathbb{F}_p \): here \( y^z \) corresponds to \((1 + t)^i \) modulo \( t^{p^k} \mathbb{F}_p[t] \), and it is easy to describe all normal subgroups. In particular the normal subgroups of \( W_k \) contained in the base group form a descending chain, corresponding to the groups \( t^{i-1} \mathbb{F}_p[t]/t^{p^k} \mathbb{F}_p[t] \), \( 1 \leq i \leq p^k + 1 \).

For \( 0 \leq m < k \) and \( z \in \langle y, y^2, \ldots, y^{p^k-1} \rangle \) the element
\[
(x^p)^m z^{p^k} = (x^p)^m z^{(p^k-1)p^m} \cdots z^{p^m} z = z^{x^{(p^k-1)p^m}} \cdots z^{p^m} z
\]
corresponds in \( \mathbb{F}_p[t]/t^{p^k} \mathbb{F}_p[t] \) to a multiple of
\[
\sum_{i=0}^{p^k-1} (1 + t)^{ip^m} = \sum_{i=0}^{p^k-1} (1 + t^{p^m})^i = \frac{(1 + t^{p^m})^{p^k} - 1}{(1 + t^{p^m}) - 1} = t^{(p^k-1)p^m};
\]
this shows that \( W_k^{p^k} = \langle y^z y^{x^2} \cdots y^{x^{p^k-1}} \rangle \cong \mathbb{C}_p \).

Clearly, \( \gamma_i(W_k) = W_k \). For \( 2 \leq i \leq p^k + 1 \), the group \( \gamma_i(W_k) \) corresponds to the subgroup \( t^{i-1} \mathbb{F}_p[t]/t^{p^k} \mathbb{F}_p[t] \) of the base group. In particular, \( W_k \) has nilpotency class \( p^k \). For \( 1 \leq i \leq k \), we have \( P_i(W_k) = \langle x^{p^i-1} \rangle \gamma_i(W_k) \), while for \( k < i \leq p^k \) we get \( P_i(W_k) = \gamma_i(W_k) \). For \( 0 \leq i \leq k \) a simple induction shows that the group \( \Phi_i(W_k) \) is the normal closure in \( W_k \) of the two elements
\[
x^{p^i} \quad \text{and} \quad [y, x, x^p, x^{p^2}, \ldots, x^{p^i-1}] = [y, x, \ldots, x];
\]
the intersection of \( \Phi_i(W_k) \) with the base group corresponds to \( t^{p^i} \mathbb{F}_p[t]/t^{p^k} \mathbb{F}_p[t] \). Thus \( \Phi_i(W_k) = \langle x^{p^i} \rangle \gamma_i_{p^i-1}+1(W_k) \). In particular, \( \Phi_k(W_k) \) is elementary abelian and \( \Phi_i(W_k) = 1 \) for \( i > k \). Finally, for \( i \geq p^k - 1 + 1 \), we use \([2, \text{Thm. 11.2}]\) to deduce that \( D_i(W_k) = \gamma_i(W_k) \). \( \square \)

The structural results for the finite wreath products \( W_k \) transfer naturally to the inverse limit \( W \cong \varprojlim_k W_k \), i.e., the pro-\( p \) wreath product
\[
W = \langle x, y \rangle = \langle x \rangle \ltimes B \cong \mathbb{C}_p \hat{\otimes} \mathbb{Z}_p
\]
with top group \( \langle x \rangle \cong \mathbb{Z}_p \) and base group \( B = \prod_{i \in \mathbb{Z}} \langle y^i \rangle \cong \mathbb{C}_p^{\mathbb{N}_0} \). Compatible with \([2, \text{Thm. 2.1}]\), the group \( W \) has a concrete realisation as a semidirect product
\[
W \cong \langle 1 + t \rangle \ltimes \mathbb{F}_p[t]^*, \quad \text{where} \quad (1 + t) \leq \mathbb{F}_p[t]^*,
\]
in terms of formal power series over the finite field \( \mathbb{F}_p \). We record the following lemma on closed normal subgroups of \( W \).

**Lemma 2.7.** Let \( W = \langle x, y \rangle \cong \mathbb{C}_p \hat{\otimes} \mathbb{Z}_p \) with base group \( B \) as above, and let \( 1 \neq K \subseteq W \) be a non-trivial closed normal subgroup. Then either \( K \) is open in \( W \) or \( K \) is open in \( B \); in particular, \( K \cap B \leq_o B \) and \( |K \cap B : [K \cap B, W]| = p. \)
Proof. The lower central series of \( W \) is well known and easy to compute: \( \gamma_1(W) = W \) and \( \gamma_i(W) = B_{i-1} \) for \( i \geq 2 \), where \( B = B_0 \supseteq B_1 \supseteq B_2 \supseteq \ldots \) with \( B_{i-1} = \langle \{y, x, \ldots, x\} B_i \rangle \) and \( |B_{i-1} : B_i| = p \); in other words, \( \langle x \rangle \) acts uniserially on \( B \); compare Proposition 2.6.

It follows that \( 1 \neq K \cap B = B_i \) for some non-negative integer \( i \), hence \( K \cap B \subseteq B_i \) and \( |K \cap B : [K \cap B, W]| = |B_i : B_{i+1}| = p \). Suppose now that \( K \not\subseteq B \). Then there exists \( x^n z \in K \) with \( m \in \mathbb{N} \) and \( z \in B \). We may assume that \( m = p^k \) is a \( p \)-power. Then \( \langle x^n z \rangle B = \langle x \rangle \times (B \times p^k \times B) \), where \( x^n z \) maps to \( x \) and, on the right-hand side, \( x \) acts diagonally and in each coordinate according to the original action in \( W \). Hence we may assume that \( x \in K \). Now the description of the lower central series of \( W \) yields \( \langle x \rangle B_1 \leq K \) and thus \( K \leq B \). \( \square \)

From Proposition 2.6 and Lemma 2.7 we deduce the following; cf. [3, Ch. VIII.7].

**Corollary 2.8.** The normal Hausdorff spectrum of the pro-\( p \) group \( W = C_p \wr \mathbb{Z}_p \) with respect to the standard filtration series \( \mathcal{P}, \mathcal{D}, \mathcal{F}, \mathcal{L} \) respectively, satisfies:

\[
\text{hspec}^p_3(W) = \text{hspec}^p_2(W) = \text{hspec}^p_1(W) = \{0, 1\} \quad \text{and} \quad \text{hspec}^c_2(W) = \{0, 1/2, 1\}.
\]

The next result is well known (and not difficult to prove directly); compare [9, Cor. 12.5.10]. It gives a first indication that Theorem 1.3 is at least plausible.

**Proposition 2.9.** The pro-\( p \) group \( W = C_p \wr \mathbb{Z}_p \) is not finitely presented.

The final result in this section concerns the finitely generated Hausdorff spectrum of the pro-\( p \) group \( W = C_p \wr \mathbb{Z}_p \), with respect to a standard filtration series \( \mathcal{S} \); it is defined as

\[
\text{hspec}^\mathcal{S}_\mathcal{S}(W) = \{\text{hdim}^\mathcal{S}_W(H) \mid H \leq_c W \text{ and } H \text{ finitely generated}\}
\]

and reflects the range of Hausdorff dimensions of (topologically) finitely generated subgroups; compare [7, §4.7].

**Theorem 2.10.** With respect to the standard filtration series \( \mathcal{P}, \mathcal{D}, \mathcal{F}, \mathcal{L} \) respectively, the pro-\( p \) group \( W = C_p \wr \mathbb{Z}_p \) satisfies:

\[
\text{hspec}^p_\mathcal{S}(W) = \text{hspec}^p_\mathcal{D}(W) = \text{hspec}^p_\mathcal{F}(W) = \{0 | m/p^n \mid n \in \mathbb{N}_0, 0 \leq m \leq p^n\},
\]

\[
\text{hspec}^c_\mathcal{F}(W) = \{0\} \cup \{1/2 + m/2^n \mid n \in \mathbb{N}_0, 0 \leq m \leq p^n\}.
\]

**Proof.** As above, let \( B \) denote the base group of the wreath product \( W = \langle x, y \rangle \). Let \( \mathcal{S} \in \{\mathcal{P}, \mathcal{D}, \mathcal{F}, \mathcal{L}\} \), and let \( K \) be a finitely generated subgroup of \( W \).

If \( K \subseteq B \) then \( K \) is finite and \( \text{hdim}^\mathcal{S}_W(K) = 0 \). Now suppose that \( K \not\subseteq B \); in the proof below we will no longer use that \( K \) is finitely generated, but it will become clear that this is automatically so. Write \( K = \langle x^n z \rangle M \), where \( n \in \mathbb{N}_0 \), \( z \in B \) and \( M = K \cap B \). Let \( B = B_0 \supseteq B_1 \supseteq \ldots \) be the filtration corresponding to \( \mathbb{F}_p[t] \supseteq t\mathbb{F}_p[t] \supseteq \ldots \) under (2.6), as in the proof of Lemma 2.7. We set

\[
J = \{j \in \mathbb{N}_0 \mid (M \cap B_j) \not\subseteq B_{j+1}\} \quad \text{and} \quad J_0 = \{j + p^n \mathbb{Z} \mid j \in J\} \subseteq \mathbb{Z}/p^n \mathbb{Z}.
\]

Under the isomorphism (2.6), we may regard \( M \) as an \( \mathbb{F}_p[t^{p^n}] \)-submodule of \( \mathbb{F}_p[t] \). Hence \( J + p^n \mathbb{N}_0 \subseteq J \) and

\[
\lim_{i \to \infty} \frac{\log_p(K \cap B) B_i : B_i}{\log_p B : B_i} = \frac{|J_0|}{p^n}.
\]

From Proposition 2.6 it is easily seen that \( B \) has strong Hausdorff dimension

\[
\text{hdim}^p_W(B) = \text{hdim}^D_W(B) = \text{hdim}^F_W(B) = 1 \quad \text{and} \quad \text{hdim}^c_W(B) = 1/2;
\]
compare Corollary 2.8. Using Lemma 2.2 we deduce that
\[ \text{hdim}^S_W(K) = (1 - \text{hdim}^S_W(B)) + \frac{|J_0|}{p^n} \]
lies in the desired range; in fact, the argument even shows that \( K \) has strong Hausdorff dimension.

Conversely, our analysis above shows that, for \( n \in \mathbb{N}_0 \) and \( 0 \leq m \leq p^n \), the subgroup \( K_{n,m} = \langle x^{p^n}, [y, x], [y, x, x], \ldots, [y, x, \ldots, x] \rangle \) has Hausdorff dimension \n\[ \text{hdim}^S_W(K_{n,m}) = \begin{cases} \frac{m}{p^n} & \text{if } S \in \{\mathcal{P}, \mathcal{D}, \mathcal{F}\}, \\ \frac{1}{2} + \frac{m}{2p^n} & \text{if } S = \mathcal{L}. \end{cases} \]
\hfill \square

The next corollary answers a question raised in [3, VIII.7.2]; it was shown there that \([0, \frac{1}{2}] \subseteq \text{hspm}^C(W)\), while \((\frac{1}{2}, 1) \cap \text{hspm}^C(W)\) remained undetermined.

Corollary 2.11. The Hausdorff spectrum of the pro-p group \( W = \hat{C}_p \wr \mathbb{Z}_p \) with respect to the lower p-series \( L \) is
\[ \text{hspm}^C(W) = [0, \frac{1}{2}] \cup \left\{ \frac{1}{2} + \frac{m}{2p^n} \mid n \in \mathbb{N}_0, 1 \leq m \leq p^n - 1 \right\} \cup \{1\}. \]
Furthermore, every subgroup \( K \subseteq W \) with \( \text{hdim}^C_W(K) > \frac{1}{2} \) has strong Hausdorff dimension in \( W \), with respect to \( L \).

Proof. The subgroups contained in the base group \( B \) of \( W \) yield \([0, \frac{1}{2}]\) as part of the Hausdorff spectrum; cf. Lemma 2.3. The proof of Theorem 2.10 shows that the subgroups not contained in \( B \) yield the remaining part of the claimed spectrum and that each of them has strong Hausdorff dimension in \( W \). \hfill \square

3. AN EXPLICIT PRESENTATION FOR THE PRO-P GROUP \( G \) AND A DESCRIPTION OF ITS FINITE QUOTIETS \( G_k \) FOR \( k \in \mathbb{N} \)

Recall that \( p \) is an odd prime. As indicated in the paragraph before Theorem 1.1 we consider the pro-p group \( G = F/N \), where
- \( F = \langle x, y \rangle \) is a free pro-p group and
- \( N = [R, F]^{p^{\infty}} \subseteq F \) for the kernel \( R \subseteq F \) of the presentation \( \pi : F \to W \) sending \( x, y \) to the generators of the same name in (2.5).

By producing generators for \( R \) and \( N \) as closed normal subgroups of \( F \) we obtain explicit presentations for the pro-p groups \( W \) and \( G \).

It is convenient to write \( y_i = y^{x^i} \) for \( i \in \mathbb{Z} \). Setting
\[ R_k = \langle \{x^{p^k}, y^p\} \cup \{[y_0, y_i] \mid 1 \leq i \leq \frac{p^k - 1}{2}\} \rangle^{F \subseteq_o} F \] (3.1)
for \( k \in \mathbb{N} \), we obtain a descending chain of open normal subgroups
\[ F \supseteq R_1 \supseteq R_2 \supseteq \ldots \] (3.2)
with quotient groups \( F/R_k \cong W_k \cong C_p \wr C_{p^k} \). Writing
\[ R = \bigcap_{k \in \mathbb{N}} R_k = \langle \{y^p\} \cup \{[y_0, y_i] \mid i \in \mathbb{N}\} \rangle^{F \subseteq_o} F, \]
we obtain \( F/R \cong W \cong \hat{C}_p \wr \mathbb{Z}_p \). With hindsight there is no harm in taking \( W_k = F/R_k \) for \( k \in \mathbb{N} \) and \( W = F/R \).
Setting $N_k = [R_k, F]R_k^p$ for $k \in \mathbb{N}$, we observe that

$$N_k = \langle x^{p+1}, y^{p^2}, [x^p, y], [y^p, x] \rangle \cup \{[y_0, y_i]^p | 1 \leq i \leq \frac{p-1}{2}\} \cup \{[y_0, y_i, x] | 1 \leq i \leq \frac{p^2-1}{2}\} \cup \{[y_0, y_i, y] | 1 \leq i \leq \frac{p^2-1}{2}\}\rangle F \trianglelefteq_o F,$$

and as in (3.2) we obtain a descending chain $F \supseteq N_1 \supseteq N_2 \supseteq \ldots$ of open normal subgroups. Moreover, it follows that $\bigcap_{k \in \mathbb{N}} N_k \supseteq [R, F]R_k^p = N$. On the other hand, if $z \not\in N$ then there exists an open normal subgroup $K \trianglelefteq_o F$ and $k \in \mathbb{N}$ such that $z \not\in NK = [R_k, F]R_k^p K$, hence $z \not\in N_k$. Thus we conclude that

$$\bigcap_{k \in \mathbb{N}} N_k = [R, F]R^p = N.$$

Consequently, $G = F/N \cong \lim\limits_{\rightarrow} G_k$, where

$$G_k = F/N_k \cong \langle x, y | x^{p+1}, y^{p^2}, [x^p, y], [y^p, x]; [y_0, y_i]^p, [y_0, y_i, x], [y_0, y_i, y] \rangle \text{ for } 1 \leq i \leq \frac{p^2-1}{2}$$

(3.3)

for $k \in \mathbb{N}$, and

$$G \cong \langle x, y | y^{p^2}, [y_0, y_i]^p, [y_0, y_i, x], [y_0, y_i, y] \rangle \text{ for } i \in \mathbb{N}$$

(3.4)

is a presentation of $G$ as a pro-$p$ group. To facilitate later use, we have underlined the two relations in (3.3) that do not yet occur in (3.4).

To summarise and supplement some of the notation introduced above, we define

$$Y = \langle y_i | i \in \mathbb{Z} \rangle R \trianglelefteq_o F, \quad H = Y/N \trianglelefteq_o G, \quad Z = R/N \trianglelefteq_o G.$$

Similarly for $k \in \mathbb{N}$ we set

$$Y_k = \langle y_i | i \in \mathbb{Z} \rangle R_k \trianglelefteq_o F, \quad H_k = Y_k/N_k \trianglelefteq G_k, \quad Z_k = R_k/N_k \trianglelefteq G_k.$$

Diagrammatically, we have:

$$
\begin{array}{c}
F \rightarrow G \\
| \\
Y \rightarrow H \\
| \\
R \rightarrow Z \\
| \\
N \rightarrow 1
\end{array}
\quad
\begin{array}{c}
G/Z \cong W \\
\quad W_k \cong G_k/Z_k \\
\quad \langle G_k \leftrightarrow F \rangle \\
\quad \langle H_k \leftrightarrow Y_k \rangle \\
\quad \langle Z_k \leftrightarrow R_k \rangle \\
\quad 1 \leftrightarrow N_k
\end{array}
$$

Lemma 3.1. The centre of $G$ is $Z(G) = Z$, and $Z_k \trianglelefteq Z(G_k)$ for $k \in \mathbb{N}$.

Proof. By construction, $Z \subseteq Z(G)$ and $Z_k \subseteq Z(G_k)$ for $k \in \mathbb{N}$. From (2.6) we see that $G/Z \cong W$ has trivial centre. Therefore $Z = Z(G)$.

In fact, $Z_k \trianglelefteq Z(G_k)$ for $k \in \mathbb{N}$; see Lemma 5.3 below.

4. General Description of the Normal Hausdorff Spectrum of the Pro-$p$ Group $G$ and its Finite Direct Powers

We continue to use the notation set up in Section 3 to study the pro-$p$ group $G$ and its finite direct powers.

Proposition 4.1. Let $K \trianglelefteq_o G$ be a closed normal subgroup such that $K \not\leq Z$. Then either $K$ is open in $H$ or $K$ is open in $G$; in particular, $K \cap H \leq_o H$. Furthermore, $[K \cap H, G] \leq_o H$. 

where $\xi$ finite direct powers $G$ Hausdorff spectrum of $G$.

For a standard filtration series $S$ the filtration series'. For any filtration series $S$ general shape of the normal Hausdorff spectrum of $Z$

$|\hat{\eta}|$ for each $j$.

Thus $K \cap Z$ contains $\hat{y}_i^m$ and $[\hat{y}_i, \hat{y}_j]$ for all $i, j \in \mathbb{N}$ with $i > m$. Hence the finite set

$$\{\hat{y}_i^m | 1 \leq i \leq m\} \cup \{[\hat{y}_i, \hat{y}_j] | 1 \leq i \leq j \leq m\}$$

generates the elementary abelian group $Z$ modulo $K \cap Z$, and $K \cap Z \leq_o Z$.

Finally, Lemma 2.7 implies that $[K \cap H, G] \not\subseteq Z$. Hence $[K \cap H, G] \leq_o H$. □

From Proposition 4.1 Lemma 3.1 and Lemmata 2.1 and 2.3 we deduce the general shape of the normal Hausdorff spectrum of $G$.

**Corollary 4.2.** Let $S$ be an arbitrary filtration series of $G$. Then the normal Hausdorff spectrum of $G$ has the form

$$\text{hspec}_{S}(G) = [0, \xi] \cup \eta \cup \{1\},$$

where $\xi = \text{hdim}_{S}(Z)$ and $\eta = \text{hdim}_{S}(H)$.

More generally we obtain a description of the normal Hausdorff spectrum of finite direct powers $G^{(m)} = G \times \ldots \times G$ of $G$, with respect to suitable 'product filtration series'. For any filtration series $S$: $G = S_0 \supseteq S_1 \ldots$ of $G$ we consider the naturally induced product filtration series on $G^{(m)}$ given by

$$S^{(m)}: G^{(m)} = G \times \ldots \times G \supseteq S_1 \times \ldots \times S_1 \supseteq S_2 \times \ldots \times S_2 \supseteq \ldots .$$

For a standard filtration series $S \in \{P, Q, T, D\}$ on $G$ the product filtration series $S^{(m)}$ is actually the corresponding standard filtration series on $G^{(m)}$.

**Corollary 4.3.** Let $m \in \mathbb{N}$, and let $K \leq_o G^{(m)}$. For $1 \leq j \leq m$, let $\pi_j: G^{(m)} \to G$ be the canonical projection onto the $j$th factor and set

$$\overline{K}(j) = \begin{cases} Z & \text{if } K\pi_j \subseteq Z, \\ G & \text{otherwise,} \end{cases} \quad \text{and} \quad K(j) = \begin{cases} 1 & \text{if } K\pi_j \subseteq Z, \\ H & \text{otherwise.} \end{cases}$$

Then $K = \prod_{j=1}^m \overline{K}(j)$ and $K$ contains an open normal subgroup of $\prod_{j=1}^m K(j)$.

Proof. Observe that

$$[K\pi_1, G] \times \ldots \times [K\pi_m, G] = [K, G^{(m)}] \leq K \leq K\pi_1 \times \ldots \times K\pi_m.$$ 

Thus $K$ is contained in $\prod_{j=1}^m \overline{K}(j)$, and it suffices to show that $[K\pi_j \cap H, G] \leq_o H$ for each $j$ with $K\pi_j \not\subseteq Z$. This follows by Proposition 4.1. □

**Corollary 4.4.** Let $m \in \mathbb{N}$, and let $S$ be a filtration series of $G$ such that $\text{hdim}_{S}(H) = 1$. Then the normal Hausdorff spectrum of $G^{(m)}$ has the form

$$\text{hspec}_{S}^{(m)}(G^{(m)}) = [0, \xi] \cup \bigcup_{1 \leq l \leq m-1} \left[\gamma/m, l+(m-l)\xi/m\right] \cup \{1\},$$

where $\xi = \text{hdim}_{S}(Z)$.
Proof. First let $K \leq c G^{(m)}$, and define $\overline{K}(j), \overline{K}(j)$ for $1 \leq j \leq m$ as in Corollary 4.3. From $\text{hdim}_{G}^{(m)}(H) = 1$ we deduce that

$$l/m = \text{hdim}_{G}^{(m)}\left(\prod_{j=1}^{m} \overline{K}(j)\right) \leq \text{hdim}_{G}^{(m)}(K) \leq \text{hdim}_{G}^{(m)}\left(\prod_{j=1}^{m} \overline{K}(j)\right) = l/m + m-l/m \xi,$$

where $l = \#\{j \mid 1 \leq j \leq m \text{ and } \overline{K}(j) = G\}$.

Conversely, for every $l \in \{0, 1, \ldots, m\}$ and $\beta \in \left[\frac{l}{m}, \frac{l+(m-l)\xi}{m}\right]$ there is a normal subgroup

$$K_{\beta} = G \times \ldots \times G \times U \times m-l \times U \leq c G^{(m)},$$

where $U \leq c Z$ for $l < m$ has $\text{hdim}_{G}^{(m)}(U) = m/m-1(\beta - l/m) \in [0, \xi]$; compare Corollary 4.2. This yields $\beta = \text{hdim}_{G}^{(m)}(K_{\beta}) \in \text{hspec}_{G}^{(m)}(G^{(m)})$. □

Corollary 4.4 shows that, once $\text{hdim}_{G}^{(m)}(H) = 1$, the general shape (e.g. the number of connected components) of the normal Hausdorff spectrum $\text{hspec}_{G}^{(m)}(G^{(m)})$ depends only on the parameters $\xi = \text{hdim}_{G}^{(m)}(Z)$ and $m \in \mathbb{N}$. For instance, if $\xi < 1/m$, then $\text{hspec}_{G}^{(m)}(G^{(m)})$ is the union of $m + 1$ disjoint intervals, whereas for $\xi \geq 1/2$ we obtain $\text{hspec}_{G}^{(m)}(G^{(m)}) = [0, 1 - (1-\xi)/m] \cup \{1\}$.

The proof of Theorem 1.1 in Sections 5 and 6 will give $\text{hdim}_{G}^{(m)}(H) = 1$ for the standard filtrations $S \in \{P, D, F\}$ and $\xi = \text{hdim}_{G}^{(m)}(Z) = \text{hdim}_{C}^{(m)}(Z) = 1/3$ respectively $\xi = \text{hdim}_{G}^{(m)}(Z) = 1/p+1$; the assertion for $H$ is already a consequence of [3] Prop. 4.2. We formulate a taylor-made corollary for these situations.

**Corollary 4.5.** Let $m, n \in \mathbb{N}$ with $m \geq \max\{2, n-1\}$ and $n \geq 2$. Let $S$ be a filtration series of $G$ such that $\text{hdim}_{G}^{(m)}(H) = 1$ and $\text{hdim}_{G}^{(m)}(Z) = 1/n$. Then

$$\text{hspec}_{G}^{(m)}(G^{(m)}) = [0, \frac{mn-(n-1)^{2}}{mn}] \cup \bigcup_{m-n+2 \leq l \leq m-1} \left[\frac{l}{m}, \frac{m+l(n-1)/mn}{m}\right] \cup \{1\}$$

consists of $n$ disjoint intervals.

**Proof.** From Corollary 4.2 we have

$$\text{hspec}_{G}^{(m)}(G^{(m)}) = \left[0, \frac{1}{n}\right] \cup \bigcup_{1 \leq l \leq m-1} \left[\frac{l}{m}, \frac{m+l(n-1)/mn}{m}\right] \cup \{1\}.$$

For $m - n + 1 \leq l \leq m - 1$ it is easy to verify that

$$\frac{m + l(n-1)}{mn} < \frac{l+1}{m}.$$

Hence it suffices to show that

$$\left[0, \frac{1}{n}\right] \cup \bigcup_{1 \leq l \leq m-n+1} \left[\frac{l}{m}, \frac{m+l(n-1)/mn}{m}\right] = [0, \frac{mn-(n-1)^{2}}{mn}].$$

For $m = n - 1$ this reduces to $[0, \frac{1}{n}] = [0, \frac{mn-(n-1)^{2}}{mn}]$. Now suppose that $m \geq n$. Then the claim follows from

$$\frac{1}{m} \leq \frac{1}{n} \quad \text{and} \quad \frac{l+1}{m} \leq \frac{m+l(n-1)/mn}{m} \quad \text{for } 1 \leq l \leq m - n. \quad \square$$
3. Let $k$ be an abelian quotient $\mathbb{Z}/hdimG$. First we compute the order of $G$.

Observe from Lemma 5.1. In view of Corollary 4.2 this proves Theorem 1.1 for the $p$-abelian quotient $\mathbb{Z}/hdimG$. This induces a permutation action on our chosen basis for the elementary abelian $\Phi(M)$. The orbits are given by $[\tilde{y}_i, \tilde{y}_j] \equiv X [\tilde{y}_j, \tilde{y}_{j'}] \leftarrow j - i \equiv p^k j' - i'$ and $\tilde{y}_i^p \equiv X \ldots \equiv X \tilde{y}_{p^k-1}$.

4. The normal Hausdorff spectrum of $G$ with respect to the $p$-power series

We continue to use the notation set up in Section 3 and establish that $\xi = hdim^p_G(Z) = 1/3$ and $\eta = hdim^p_G(H) = 1$, with respect to the $p$-power series $P$. First we compute the order of $G$.

In view of Corollary 4.2 this proves Theorem 1.1 for the $p$-power series. Indeed, $hdim^p_G(H) = 1$ is already a consequence of [4, Prop. 4.2]. It remains to show that

$$hdim^p_G(Z) = \lim_{i \to \infty} \frac{\log_p |ZG^p : G^p|}{\log_p |G : G^p|} = 1/3. \tag{5.1}$$

It is convenient to work with the finite quotients $G_k$, $k \in \mathbb{N}$, introduced in Section 3. Let $k \in \mathbb{N}$. From (3.3) and (3.4) we observe that

$$|G : G^p| = |G_k : G_k^p|.$$ 

First we compute the order of $G_k$, using the notation from Section 3.

**Lemma 5.1.** The logarithmic order of $G_k$ is

$$\log_p |G_k| = \frac{1}{3}(3p^k + 2k + 3).$$

In particular,

$$Z_k = R_k/N_k = \{ \{x_i^p, y_i^p\} \cup \{[y_0, y_i] \mid 1 \leq i \leq \frac{p^k-1}{2}\} \}/N_k \cong C_p \times \ldots \times C_p.$$

**Proof.** Observe from $F/R_k \cong W_k \cong C_p \wr C_p$ that

$$\log_p |G_k| = \log_p |F : R_k| + \log_p |R_k : N_k| = k + p^k + \log_p |R_k : N_k|.$$ 

By construction, $R_k/N_k$ is elementary abelian of exponent $p$. Moreover, (3.1) shows that $\{x_i^p, y_i^p\} \cup \{[y_0, y_i] \mid 1 \leq i \leq (p^k - 1)/2\}$ generates $R_k$ modulo $N_k$. In order to prove that the generators are independent, we construct a factor group $\tilde{G}_k$ of $G_k$ that has the maximal possible logarithmic order $\log_p |\tilde{G}_k| = p^k + k + 2 + (p^k - 1)/2$.

Consider the finite $p$-group

$$M = \langle \tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_{p^k-1} \rangle = E/[\Phi(E), E] \Phi(E)^p,$$

where $E$ is a free pro-$p$ group on $p^k$ generators with Frattini subgroup $\Phi(E) = [E, E] E^p$. Then the images of $\tilde{y}_0, \ldots, \tilde{y}_{p^k-1}$ generate independently the elementary abelian quotient $M/\Phi(M)$ and the commutators $[\tilde{y}_i, \tilde{y}_j]$, for $0 \leq i < j \leq p^k - 1$, together with the $p$th powers $\tilde{y}_i^p, \ldots, \tilde{y}_{p^k-1}^p$ generate independently the elementary abelian group $\Phi(M)$. The latter can be checked by considering homomorphisms from $M$ onto groups of the form $C_p^{p-1} \times C_p$ and $C_p^{p-2} \times \text{Heis}(F_p)$, where $\text{Heis}(F_p)$ denotes the group of upper unitriangular $3 \times 3$ matrices over $F_p$. Next consider the action of the cyclic group $X = \langle \tilde{x} \rangle \cong C_{p+1}$, with kernel $\langle \tilde{x}^p \rangle \cong C_p$, on $M$ that is induced by

$$\tilde{y}_i^\tilde{x} = \begin{cases} \tilde{y}_{i+1} & \text{if } 0 \leq i \leq p^k - 2, \\ \tilde{y}_0 & \text{if } i = p^k - 1. \end{cases}$$

This induces a permutation action on our chosen basis for the elementary abelian group $\Phi(M)$; the orbits are given by

$$[\tilde{y}_i, \tilde{y}_j] \equiv X [\tilde{y}_j, \tilde{y}_{j'}] \leftrightarrow j - i \equiv p^k j' - i' \quad \text{and} \quad \tilde{y}_0^p \equiv X \ldots \equiv X \tilde{y}_{p^k-1}^p.$$
We define \( \tilde{M} = M/\Phi(M), X\) and, for simplicity, continue to write \( \tilde{y}_0, \ldots, \tilde{y}_{p^k-1}\) for the images of these elements in \( \tilde{M} \). Then

- the images of \( \tilde{y}_0, \ldots, \tilde{y}_{p^k-1}\) generate independently the elementary abelian quotient \( \tilde{M}/\Phi(\tilde{M}) \) and
- the elements \( [\tilde{y}_0, \tilde{y}_i] \), for \( 1 \leq i \leq (p^k - 1)/2 \), together with \( \tilde{y}_0^p \) generate independently the elementary abelian group \( \Phi(M) \).

In particular, this yields \( \log_p |\tilde{M}| = p^k + (p^k - 1)/2 + 1 \).

Finally, we put \( \tilde{y} = \tilde{y}_0 \) and form the semidirect product

\[ \tilde{G}_k = (\tilde{x}, \tilde{y}) = X \rtimes \tilde{M} \]

with the induced action. Upon replacing \( x, y \) by \( \tilde{x}, \tilde{y} \), we see that all the defining relations of \( G_k \) in (3.3) are valid in \( \tilde{G}_k \). Since \( \log_p |G_k| \leq p^k + k + 2 + (p^k - 1)/2 = \log_p |\tilde{G}_k| \), we conclude that \( G_k \cong \tilde{G}_k \). \( \square \)

Our next aim is to prove the following structural result.

**Proposition 5.2.** In the set-up from Section 3 for \( k \geq 2 \), the subgroup \( G_k^{p^k} \subseteq G_k \) is elementary abelian and central in \( G_k \); it is generated independently by \( x^{p^k} \), \( w = y_{p^k-1} \cdots y_1 y_0 \) and \( v = w \cdot y_{p^k-1} \cdots y_1^{-1} y_0^{-1} \).

Consequently

\[ G_k^{p^k} \cong C_p \times C_p \times C_p \quad \text{and} \quad \log_p |G_k : G_k^{p^k}| = \log_p |G_k| - 3 \]

and

\[ G_k/G_k^{p^k} \cong \langle x, y \ | \ x^{p^k}, y^{p^k}, [y^p, x], w(x, y), v(x, y) \rangle \]

\[ [y_0, y_i]^{p^k}, [y_0, y_i, x], [y_0, y_i, y] \quad \text{for} \ 1 \leq i \leq \frac{p^k - 1}{2} \). \]

The proof requires a series of lemmata.

**Lemma 5.3.** The elements

\[ w = y_{p^k-1} \cdots y_1 y_0 \quad \text{and} \quad w' = y_{p^k-1}^{-1} \cdots y_1^{-1} y_0^{-1} \]

are of order \( p \) in \( G_k \) and lie in \( G_k^{p^k} \cap Z(G_k) \).

**Proof.** Recall that \( H_k = \langle y_0, y_1, \ldots, y_{p^k-1} \rangle \) \( Z_k \subseteq G_k \) and observe that \( [H_k, H_k] \) is a central subgroup of exponent \( p \) in \( G_k \). Furthermore, \( [y^p, x] = 1 \) implies \( y_{p^k-1}^p = \ldots = y_0^p \) in \( G_k \). Thus (2.1) yields

\[ w^p = y_{p^k-1}^p \cdots y_1^p y_0^p = y_{p^k+1}^p = 1. \]

As \( w \neq 1 \) we deduce that \( w \) has order \( p \). Likewise one shows that \( w' \) has order \( p \).

Clearly, \( w = x^{-p^k} (xy)^p \) and \( w' = x^{-p^k} (xy^{-1})^{p^k} \) lie in \( G_k^{p^k} \). In order to prove that \( w \) is central, it suffices to check that \( w \) commutes with the generators \( x \) and \( y \) of \( G_k \). First we observe that, for \( 1 \leq i \leq p^k - 1 \), the relation \( [y_0, y_i, x] = 1 \) implies

\[ [y_0, y_{p^k-i}]^{-1} = [y_{p^k-i}, y_0] = [y_0, y_i]^{-1} = [y_0, y_i] \quad \text{in} \ G_k. \]
Since \([H_k, H_k]\) is central in \(G_k\), we deduce inductively that
\[
[w, x] = (y_{p^k-1} \cdots y_1y_0)^{-1}(y_{p^k-1} \cdots y_1y_0)^{x}
\]
\[
= y_0^{-1}y_1^{-1} \cdots y_{p^k-2}^{-1} y_{p^k-1}^{-1} y_0 y_{p^k-1} \cdots y_{p^k-2} y_1
\]
\[
= y_0^{-1}y_1^{-1} \cdots y_{p^k-2}^{-1} y_0 y_{p^k-1} \cdots y_{p^k-2} y_1
\]
\[
= [y_0, y_1][y_0, y_2] \cdots [y_0, y_{p^k-2}][y_0, y_{p^k-1}]
\]
\[
= 1
\]
by (5.2).
Likewise, using the relation \([y_0, y_1, y_2] = 1\) and (5.2), we obtain
\[
[w, y] = [y_{p^k-1} \cdots y_1y_0] = [y_{p^k-1}, y_0][y_{p^k-2}, y_0] \cdots [y_1, y_0] = 1.
\]
A similar computation can be carried out for \(w'\).

**Lemma 5.4.** Putting
\[
v = w w' = y_{p^k-1} \cdots y_1 y_0 \cdot y_{p^k-1}^{-1} \cdots y_1^{-1} y_0^{-1},
\]
the subgroup \(\langle x^k, w, v \rangle \leq G_k\) is isomorphic to \(C_p \times C_p \times C_p\) and lies in \(G_k^{p^k} \cap Z(G_k)\).

**Proof.** From the presentation (3.3) and from Lemma 5.3 it is clear that the subgroup \(\langle x^k, w, v \rangle \leq G_k\) is elementary abelian and lies in \(G_k^{p^k} \cap Z(G_k)\). Furthermore, in order to prove that \(\langle x^k, w, v \rangle \cong C_p \times C_p \times C_p\), it suffices to establish that \(v \neq 1\).

Upon a similar rearrangement and cancellation as in the proof of Lemma 5.3, we obtain
\[
v = \prod_{i=0}^{p^k-2} [y_i, y_{p^k-1}][y_i, y_{p^k-2}] \cdots [y_i, y_{p^k-i}].
\]
Recall that all commutators appearing in the above product are central in \(G_k\). In particular, we have \([y_0, y_{p^k-j}] = [y_0, y_{p^k-j}][y_1, y_{p^k-j+1}]; for 1 \leq j \leq p^k - 1 and 1 \leq i \leq j - 1. This gives
\[
v = [y_0, y_{p^k-1}][y_0, y_{p^k-2}]^2 \cdots [y_0, y_{p^k-1}][y_0, y_1]^2 \prod_{i=0}^{p^k-1} [y_0, y_{(p^k-1)/2}]^{p^k-1} \quad \cdot [y_0, y_{(p^k-1)/2}]^{p^k-1} \cdots [y_0, y_2]^2 [y_0, y_1]
\]
\[
= [y_0, y_1]^2 [y_0, y_2]^4 \cdots [y_0, y_{(p^k-1)/2}]^{p^k-1}.
\]
Taking note of the second statement in Lemma 5.1 it follows that \(v \neq 1\).

**Lemma 5.5.** The group \(\gamma_2(G_k) \leq G_k\) has exponent \(p\).

**Proof.** Recall that \(H_k = \langle y_0, y_1, \ldots, y_{p^k-1} \rangle Z_k \leq G_k\) satisfies: \([H_k, H_k]\) is a central subgroup of exponent \(p\) in \(G_k\). Since \(p\) is odd, (2.1) shows that it suffices to prove that \([y, x]\) has order \(p\). But \([y, x] = y_0^{-1}y_1\); thus (2.1) and \(y_0^p = x^{-1}y_0^p x = y_1^p\) imply \([y, x]^p = y_0^p y_1^p = 1\).

**Lemma 5.6.** The group \(G_k\) has nilpotency class \(p^k\), and \(\gamma_m(G_k)/\gamma_{m+1}(G_k)\) is elementary abelian of rank at most \(2\) for \(2 \leq m \leq p^k\).
2. Again by Proposition 2.6, the nilpotency class of \( G \) has nilpotency class precisely \( p^k \).

**Lemma 5.7.** The group \( G_k \) satisfies
\[
G_k^p \subseteq \langle x^p, y^p \rangle \gamma_p(G_k) \quad \text{and} \quad G_k^{p^j} \subseteq \langle x^{p^j} \rangle \gamma_{p^j}(G_k) \quad \text{for } j \geq 2.
\]

**Proof.** Recall that \( H_k = \langle y_0, y_1, \ldots, y_{p^{k-1}} \rangle Z_k \leq G_k \) has exponent \( p^2 \), and observe that Proposition 2.3 together with Lemma 5.5 yields \( H_k^p = \langle y^p \rangle \). Every element \( g \in G \) is of the form \( g = x^m h \), with \( 0 \leq m < p^{k+1} \) and \( h \in H_k \). Using (2.3), based on Proposition 2.3 and Lemma 5.5, we conclude that
\[
g^p = (x^m h)^p = x^{mp} h^p \in \langle x^p, y^p \rangle \mod \gamma_p(G_k),
\]
and for \( j \geq 2 \),
\[
g^{p^j} = (x^m h)^{p^j} = x^{mp^j} h^{p^j} \in \langle x^{p^j} \rangle \mod \gamma_{p^j}(G_k). \quad \square
\]

**Proof of Proposition 5.7** Apply Lemmata 5.3, 5.6 and 5.7. \( \square \)

From Lemma 5.1 and Proposition 5.2 we deduce that
\[
\log_p|G : G^{p^k}| = \log_p|G_k : G_k^{p^k}| = \frac{1}{2}(3p^k + 2k - 3).
\]
On the other hand, we observe from Proposition 2.6 that
\[
\log_p|G : ZG^{p^k}| = \log_p|W_k : W_k^{p^k}| = p^k + k - 1,
\]
and
\[
\log_p|ZG^{p^k}| : G^{p^k}| = \frac{1}{2}(3p^k + 2k - 3) - (p^k + k - 1) = \frac{1}{2}(p^k - 1).
\]

Thus (5.1) follows from
\[
\lim_{i \to \infty} \frac{\log_p|ZG^{p^i}| : G^{p^i}|}{\log_p|G : G^{p^i}|} = \lim_{i \to \infty} \frac{\frac{1}{2}(p^i - 1)}{\frac{1}{2}(3p^i + 2i - 3)} = 1/3. \tag{5.3}
\]

**Remark 5.8.** In the literature, one sometimes encounters a variant of the \( p \)-power series, the *iterated \( p \)-power series* of \( G \) which is recursively given by
\[
\mathcal{J}: I_0(G) = G, \quad \text{and} \quad I_j(G) = I_{j-1}(G)^p \quad \text{for } j \geq 1.
\]
By a small modification of the proof of Lemma 5.7 we obtain inductively
\[
I_j(G_k) \subseteq \langle \langle x^{p^{j-1}} \rangle \gamma_{p^{j-1}}(G_k) \rangle \rangle \subseteq \langle x^{p^j} \rangle \gamma_{p^j}(G_k) \quad \text{for } j \geq 2,
\]
based on the commutator identities (2.3) for \( r = 1 \). With Proposition 5.2 and Lemma 5.6 this yields \( G_k^{p^k} \subseteq I_k(G_k) \subseteq \langle x^{p^k} \rangle \gamma_{p^k}(G_k) = G_k^{p^k} \). We conclude that the \( p \)-power series \( \mathcal{P} \) and the iterated \( p \)-power series \( \mathcal{J} \) of \( G \) coincide.

One may further note another natural filtration series \( N: N_i, i \in \mathbb{N}_0 \), of \( G \), consisting of the open normal subgroups defined in Section 3, where we set \( N_0 = G \). As \( N_i \leq G^{p^i} \) with \( \log_p|G^{p^i}| : N_i| \leq 4 \) for all \( i \in \mathbb{N}_0 \), we see that the filtration series \( \mathcal{P} \) and \( \mathcal{N} \) induce the same Hausdorff dimension function on \( G \).
6. The normal Hausdorff spectra of $G$ with respect to the lower $p$-series, the dimension subgroup series and the Frattini series

We continue to use the notation set up in Section 3 and work with the finite quotients $G_k$, $k \in \mathbb{N}$, of the pro-$p$ group $G$. Our aim is to pin down the lower central series, the lower $p$-series, the dimension subgroup series and the Frattini series of $G_k$. Subsequently, it will be easy to complete the proof of Theorem 1.4.

**Proposition 6.1.** The group $G_k$ is nilpotent of class $p^k$; its lower central series satisfies

$$G_k = \gamma_1(G_k) = \langle x, y \rangle, \quad \gamma_2(G_k) \quad \text{with} \quad G_k/\gamma_2(G_k) \cong C_{p^{k+1}} \times C_{p^2}$$

and, for $1 \leq i \leq (p^k - 1)/2$,

$$\gamma_{2i}(G_k) = \langle [y, x, 2i^{-1}, x] \rangle \quad \gamma_{2i+1}(G_k), \quad \gamma_{2i+1}(G_k) = \langle [y, x, 2i, x], [y, x, 2i^{-1}, x, y] \rangle \quad \gamma_{2i+2}(G_k)$$

with

$$\gamma_{2i}(G_k)/\gamma_{2i+1}(G_k) \cong C_p \quad \text{and} \quad \gamma_{2i+1}(G_k)/\gamma_{2i+2}(G_k) \cong C_p \times C_p.$$

**Proof.** By Lemma 5.6 the nilpotency class of $G_k$ is $p^k$. From $G_k = \langle x, y \rangle$ it is clear that $\gamma_2(G_k) = \langle [x, y] \rangle \gamma_3(G_k)$, and (3.3) gives $G_k/\gamma_2(G_k) \cong C_{p^{k+1}} \times C_{p^2}$. From Lemma 5.1 we know that

$$\log_p |G_k| = (3p^k + 2k + 3)/2 = ((k + 1) + 2) + \frac{p^k - 1}{2} (1 + 2),$$

and the proof of Lemma 5.6 shows that

$$\gamma_m(G_k) = \langle [y, x, m^{-1}, x], [y, x, m^{-2}, x, y] \rangle \quad \gamma_{m+1}(G_k) \quad \text{for} \quad 2 \leq m \leq p^k.$$

Consequently, it suffices to prove that $[y, x, m^{-2}, x, y] \in \gamma_{m+1}(G_k)$ whenever $m$ is even. More generally, we consider the elements

$$b_{j,m} = [[y, x, m^{-2}, x]^{x_j}, y] \quad \text{for} \quad 2 \leq m \leq p^k \quad \text{and} \quad j \in \mathbb{N}_0.$$

Writing $e_i = [y_0, y_i] \in Z_k \subseteq Z(G_k)$ for $i \in \mathbb{Z}$, we recall from Lemma 5.1 that

$$b_{j, m} \in [H_k, H_k] = \langle e_i \mid 1 \leq i \leq \frac{p^k - 1}{2} \rangle \cong C_p \times \frac{p^k - 2}{2} \times C_p.$$ 

Induction on $m$ shows that

$$[y, x, m^{-2}, x] \equiv \prod_{i=0}^{m-2} y_i (-1)^{m+i} (m^{-2}) \quad \text{modulo} \quad Z_k \subseteq Z(G_k),$$

and we deduce that

$$b_{j,m} = \prod_{i=0}^{m-2} y_{j+i} (-1)^{m+i} (m^{-2}) = \prod_{i=0}^{m-2} e_{j+i} (-1)^{m+i+1} (m^{-2}). \quad (6.1)$$

The identities

$$\binom{m-2}{i} - 2 \binom{m-1}{i} + \binom{m}{i} = \binom{m-2}{i-2} \quad (6.1)$$

imply that

$$b_{j,m} \equiv b_{j,m} b_{j,m+1} b_{j,m+2} = b_{j+2, m} \quad \text{modulo} \quad \gamma_{m+1}(G_k). \quad (6.2)$$

Now suppose that $m$ is even, and recall that $p \neq 2$. From (6.2) we obtain inductively $[y, x, m^{-2}, x, y] = b_{0, m} \equiv b_{j_0, m}$ modulo $\gamma_{m+1}(G_k)$ for

$$j_0 = \begin{cases} \frac{p^k + 1}{2} - \frac{m}{2} & \text{if } p^k + 1 - m \equiv 0, \\ \frac{p^k + 3}{2} - \frac{m}{2} & \text{if } p^k + 1 - m \equiv 2. \end{cases}$$
Hence, it suffices to prove that $b_{j_0,m} \in \gamma_{m+1}(G_k)$. First suppose that $p^k + 1 \equiv m \mod 4$ and hence $j_0 = \frac{p^k + 1}{2} - \frac{m}{2}$. From (6.1) and (5.2) we see that

\[
\begin{align*}
b_{j_0,m} &= \prod_{i=0}^{m/2-1} e_{j_0+i} (-1)^{i+1} \prod_{i=m/2}^{m-2} e_{j_0-i} (-1)^i (m-i-1) \\
&= \prod_{i=0}^{m/2-1} e_{j_0+i} (-1)^{i+1} \prod_{i=m/2}^{m-2} e_{j_0+i} (-1)^i (m-i-1) \\
&= \prod_{i=0}^{m/2-1} e_{j_0+i} (-1)^{i+1} \prod_{i'=1}^{m/2-1} e_{j_0+i'} (-1)^i (m-i') \\
&= \prod_{i=0}^{m/2-1} e_{j_0+i} (-1)^{i+1} (m-i)
\end{align*}
\]

and similarly

\[
\begin{align*}
b_{j_0,m+1}^{-1} &= \prod_{i=0}^{m/2-1} e_{j_0+i} (-1)^{i+1} \prod_{i=m/2}^{m-1} e_{j_0-i} (-1)^i (m-i-1) \\
&= \prod_{i=0}^{m/2-1} e_{j_0+i} (-1)^{i+1} \prod_{i=m/2}^{m-1} e_{j_0+i} (-1)^i (m-i-1) \\
&= \prod_{i=0}^{m/2-1} e_{j_0+i} (-1)^{i+1} \prod_{i'=0}^{m/2-1} e_{j_0+i'} (-1)^i (m-i') \\
&= \left( \prod_{i=0}^{m/2-1} e_{j_0+i} (-1)^{i+1} (m-i) \right)^2
\end{align*}
\]

Hence $b_{j_0,m} = b_{j_0,m+1}^{-1} \in \gamma_{m+1}(G_k)$, and $p \neq 2$ implies $b_{j_0,m} \in \gamma_{m+1}(G_k)$.

In the remaining case $p^k + 1 \equiv m + 2 \mod 4$ we have $j_0 = \frac{p^k + 3}{2} - \frac{m}{2}$, and a slight variation of the argument above shows that $b_{j_0+m} = b_{j_0-1,m+1}$, hence $b_{j_0,m} \in \gamma_{m+1}(G_k)$. □

**Corollary 6.2.** For $2 \leq m \leq p^k$ and $\nu(m) = \left\lfloor \frac{1}{2}(p^k - m + 2) \right\rfloor$, we have

\[
\gamma_m(G_k) \cap Z_k = \langle [y, x, 2, \ldots, 1, x, y] \mid \frac{m}{2} \leq j \leq (p^k - 1)/2 \rangle \cong C_p^{\nu(m)}
\]

and $\gamma_m(G_k) \cap Z(G_k) = \langle [y, x, p^k-1, x] \rangle \times (\gamma_m(G_k) \cap Z_k) \cong C_p^{\nu(m)+1}$. In particular, $\gamma_m(G_k) \cap Z(G_k) = \langle [y, x, 2, \ldots, 1, x, y] \mid m/2 \leq j \leq (p^k - 1)/2 \rangle$ for $m \equiv 0$.

**Proof.** Clearly, all non-trivial elements of the form $[y, x, \ldots, x, y]$ are central and of order $p$. By Proposition 6.1 and Lemma 5.7 also $[y, x, p^k-1, x]$ is central and of order $p$. Moreover, Proposition 6.1 shows that every $g \in \gamma_2(G_k)$ can be written as

\[
g = \prod_{i=1}^{p^k-1} [y, x, \ldots, x, y]^{\alpha(i)} \prod_{j=1}^{(p^k-1)/2} [y, x, 2, \ldots, 1, x, y]^{\beta(j)},
\]

where $\alpha(i), \beta(j) \in \{0, 1, \ldots, p-1\}$ are uniquely determined by $g$. Furthermore, $g$ is central if and only if $\alpha(i) = 0$ for $1 \leq i \leq p^k - 2$, and $g \in Z_k$ if and only if $\alpha(i) = 0$ for $1 \leq i \leq p^k - 1$. □

**Corollary 6.3.** The lower $p$-series of $G_k$ has length $p^k$ and satisfies:

\[
\begin{align*}
G_k = & P_1(G_k) = \langle x, y \rangle P_2(G_k) \quad \text{with } G_k/P_2(G_k) \cong C_p \times C_p, \\
P_2(G_k) = & \langle x^p, y^p, [y, x] \rangle P_3(G_k) \quad \text{with } P_2(G_k)/P_3(G_k) \cong C_p \times C_p \times C_p,
\end{align*}
\]

and, for $3 \leq i \leq p^k$, the ith term is $P_i(G_k) = \langle x^{p^{i-1}} \rangle \gamma_i(G_k)$ so that

\[
P_i(G_k) =
\begin{cases}
\langle x^{p^{i-1}}, [y, x, i-1, x] \rangle P_{i+1}(G_k) & \text{if } i \equiv 0 \mod 2 \text{ and } i \leq k + 1, \\
\langle x^{p^{i-1}}, [y, x, i-1, x], [y, x, i-2, x, y] \rangle P_{i+1}(G_k) & \text{if } i \equiv 1 \mod 2 \text{ and } i \leq k + 1, \\
\langle [y, x, i-1, x] \rangle P_{i+1}(G_k) & \text{if } i \equiv 0 \mod 2 \text{ and } i > k + 1, \\
\langle [y, x, i-1, x], [y, x, i-2, x, y] \rangle P_{i+1}(G_k) & \text{if } i \equiv 1 \mod 2 \text{ and } i > k + 1.
\end{cases}
\]
with
\[
P_i(G_k)/P_{i+1}(G_k) \cong \begin{cases} 
C_p \times C_p & \text{if } i \equiv 0 \mod k + 1, \\
C_p \times C_p \times C_p & \text{if } i \equiv 1 \mod k + 1, \\
C_p & \text{if } i \equiv 0 \mod k + 1, \\
C_p \times C_p & \text{if } i \equiv 1 \mod k + 1.
\end{cases}
\]

Proof. The descriptions of $G_k/P_2(G_k)$ and $P_2(G_k)/P_3(G_k)$ are straightforward. Let $i \geq 3$. Clearly, $P_i(G_k) \supseteq (x^{p^{i-1}})\gamma_i(G_k)$. In view of Proposition 6.1, it suffices to prove that $x^{p^{i-1}}$ is central modulo $\gamma_{i+1}(G_k)$. Indeed, from Lemma 5.5 and Proposition 2.5 (recall that $p > 2$) we obtain
\[
[x^{p^{i-1}}, y] \equiv [x, y]^{p^{i-1}} = 1 \pmod{\gamma_{p^{i-1}+1}(G_k)}.
\]

Corollary 6.4. The dimension subgroup series of $G_k$ has length $p^k$. For $1 \leq i \leq p^k$, the $i$th term is $D_i(G_k) = G_k^{p^i} \gamma_i(G_k)$, where $l(i) = \lceil \log_p i \rceil$.

Furthermore, if $i$ is not a power of $p$, equivalently if $l(i + 1) = l(i)$, then $D_i(G_k)/D_{i+1}(G_k) \cong \gamma_i(G_k)/\gamma_{i+1}(G_k)$ so that
\[
D_i(G_k) = \begin{cases} 
\langle [y, x, \ddots, x] \rangle_{D_{i+1}(G_k)} & \text{if } i \equiv 0, \\
\langle [y, x, \ddots, x], [y, x, \ddots, x, y] \rangle_{D_{i+1}(G_k)} & \text{if } i \equiv 1,
\end{cases}
\]

with
\[
D_i(G_k)/D_{i+1}(G_k) \cong \begin{cases} 
C_p & \text{if } i \equiv 0, \\
C_p \times C_p & \text{if } i \equiv 1.
\end{cases}
\]

whereas if $i = p^l$ is a power of $p$, equivalently if $l(i + 1) = l(i) + 1$ for $l = l(i)$, then $D_i(G_k)/D_{i+1}(G_k) \cong \langle x^{p^l} \rangle/\langle x^{p^{l+1}} \rangle \times \langle y^{p^l} \rangle/\langle y^{p^{l+1}} \rangle \times \gamma_i(G_k)/\gamma_{i+1}(G_k)$ so that
\[
D_i(G_k) = \langle x, y \rangle D_2(G_k), \\
D_p(G_k) = \langle x^{p^l}, y^{p^l}, [y, x, \ddots, x], [y, x, \ddots, x, y] \rangle D_{p+1}(G_k), \\
D_i(G_k) = \langle x^{p^l}, [y, x, \ddots, x], [y, x, \ddots, x, y] \rangle D_{i+1}(G_k)
\]

with
\[
D_i(G_k)/D_{i+1}(G_k) \cong \begin{cases} 
C_p \times C_p & \text{if } i = 1, \text{ equivalently if } l = 0, \\
C_p \times C_p \times C_p \times C_p & \text{if } i = p, \text{ equivalently if } l = 1, \\
C_p \times C_p \times C_p & \text{if } i = p^l \text{ with } 2 \leq l \leq k.
\end{cases}
\]

In particular, for $p^{k-1} + 1 \leq i \leq p^k$ and thus $l(i) = k$,
\[
D_i(G_k) = G_k^{p^k} \gamma_i(G_k) = \langle x^{p^k} \rangle \gamma_i(G_k),
\]

so that
\[
\log_p |D_i(G_k)| = \log_p |\gamma_i(G_k)| + 1.
\]

Proof. For $i \in \mathbb{N}$ write $l(i) = \lceil \log_p i \rceil$. From [2] Thm. 11.2 and Lemma 5.5, we obtain $D_i(G_k) = G_k^{p^l} \gamma_i(G_k)$. In particular, $D_i(G_k) = 1$ for $i > p^k$, by Proposition 6.1 and Corollary 6.3.
Now suppose that $1 \leq i \leq p^k$ and put $l = l(i)$. From Lemma 5.7 we observe that $G^p_k \cap \gamma_i(G_k) \subseteq \gamma_i(G_k)$, If $l(i + 1) = l$ then $\gamma_i(G_k) \subseteq \gamma_{i+1}(G_k)$, and hence

\[
D_i(G_k)/D_{i+1}(G_k) = G_k^p \gamma_i(G_k)/G_k^{p+1} \gamma_{i+1}(G_k)
\]

\[
\cong \gamma_i(G_k)/(G_k^p \cap \gamma_i(G_k)) \gamma_{i+1}(G_k)
\]

\[
\cong \gamma_i(G_k)/\gamma_{i+1}(G_k).
\]

Now suppose that $l(i + 1) = l + 1$, equivalently $i = p^l$. We observe that, modulo $H_k$, the $i$th factor of the dimension subgroup series is

\[
D_i(G_k)H_k/D_{i+1}(G_k)H_k = \langle x^{p^l} \rangle / \langle x^{p^l+1} \rangle \cong C_{p^l}.
\]

Comparing with the overall order of $G_k$, conveniently implicit in Corollary 6.3 we deduce that

\[
D_i(G_k)/D_{i+1}(G_k) = G_k^p \gamma_i(G_k)/G_k^{p+1} \gamma_{i+1}(G_k)
\]

\[
= \langle x^{p^i}, y^{p^l} \rangle \gamma_i(G_k)/\langle x^{p^{i+1}}, y^{p^{l+1}} \rangle \gamma_{i+1}(G_k)
\]

\[
\cong \langle x^{p^i} \rangle / \langle x^{p^{i+1}} \rangle \times \langle y^{p^l} \rangle / \langle y^{p^{l+1}} \rangle \times \gamma_i(G_k)/\gamma_{i+1}(G_k).
\]

All remaining assertions follow readily from Proposition 6.1. \qed

**Proposition 6.5.** The Frattini series of $G_k$ has length $k + 2$ and satisfies:

\[ G_k = \Phi_0(G_k) = \langle x, y \rangle \Phi_1(G_k) \] with $G_k/\Phi_1(G_k) \cong C_p \times C_p$,

\[ \Phi_1(G_k) = \langle x^{p^i}, y^{p^l}, [y, x, x], [y, x, x, x], \ldots, [y, x, \ldots, x], [y, x, y] \rangle \Phi_2(G_k) \]

with \[ \Phi_1(G_k)/\Phi_2(G_k) \cong C_{p^{i+3}} \]

and, for $2 \leq i \leq k$, the $i$th term is

\[ \Phi_i(G_k) = \langle x^{p^i}, [y, x, \ldots, x], [y, x, \ldots, x, x], \ldots, [y, x, \ldots, x, x, x], [y, x, x] \rangle \Phi_{i+1}(G_k) \]

with \[ \Phi_i(G_k)/\Phi_{i+1}(G_k) \cong \begin{cases} 
C_{p^{i+3}} & \text{for } i \neq k, \\
C_{p^{i+3}} & \text{for } i = k.
\end{cases} \]

where

\[ \nu(j) = \min \left\{ \left( \frac{p^{j-1}}{p-1} \right), p^k \right\} \]

for $1 \leq j \leq k$,

and

\[ \nu(j) = \begin{cases} 
(\frac{p^{j-1}}{p-1}) & \text{for } j = k + 1;
\end{cases} \]

Lastly,

\[ \Phi_{k+1}(G_k) = \langle [y, x, \ldots, x], [y, x, \ldots, x, x], \ldots, [y, x, p^k, x, y] \rangle \]

with \[ \Phi_{k+1}(G_k) \cong C_{p^{k+3}} \]

**Proof.** For ease of notation we set $c_1 = y$ and, for $i \geq 2$,

\[ c_i = [y, x, \ldots, x] \quad \text{and} \quad z_i = [c_{i-1}, y] = [y, x, \ldots, x, y]. \]

From Lemma 5.5 we observe that $c_i^p = z_i^p = 1$ for $i \geq 2$; furthermore, the elements $z_i \in [H_k, H_k] \subseteq Z_k$ are central in $G_k$. We claim that

\[ [c_i, c_j] = z_{i+j}^{(-1)^{j-1}} \mod \gamma_{i+j+1}(G_k) \quad \text{for } i > j \geq 1. \]

(6.3)

Indeed, $[c_i, c_1] = [c_i, y] = z_{i+1}$, and, modulo $\gamma_{i+j+1}(G_k)$, the Hall–Witt identity gives

\[ 1 = [c_i, c_{j-1}, x][c_{j-1}, x, c_i][x, c_i, c_{j-1}] = [c_j, c_i][c_{i+1}, c_j-1]^{-1}, \]
hence \([c_i, c_j] = [c_{i+1}, c_{j-1}]^{-1}\) from which the result follows by induction.

We use the generators specified in the statement of the proposition to define an ascending chain \(1 = L_{k+2} \leq L_{k+1} \leq \ldots \leq L_1 \leq L_0 = G_k\) so that each \(L_i\) is the desired candidate for \(\Phi_i(G_k)\). For \(1 \leq i \leq k + 1\) we deduce from Proposition 6.1 and Corollary 6.2 that

\[ L_i = \langle x^{p^i} \rangle M_i \quad \text{with} \quad M_i = \langle c_{\nu(i)+1} \rangle \gamma_{\nu(i)+2}(G_k) C_i \leq G_k, \]

where \(C_i = \langle y^{p^i} \rangle \times \langle z_j \mid 2 \nu(i-1) + 3 \leq j \leq p^k \text{ and } j \equiv 2 \rangle\) is central in \(G_k\). (Note that the factor \(\langle y^{p^i} \rangle\) vanishes if \(i \geq 2\).) Applying (2.3), based on Proposition 2.5 and Lemma 5.5, we see that \([x^{p^i}, G_k] = [x^{p^i}, H_k] \leq \gamma_{p^i+1}(G_k)\), hence \(L_i \leq G_k\) for \(1 \leq i \leq k + 1\). Using also (6.3), we see that the factor groups \(L_i/L_{i+1}\) are elementary abelian for \(0 \leq i \leq k + 1\). In particular, this shows that \(\Phi_i(G_k) \subseteq L_i\) for \(1 \leq i \leq k + 2\).

Clearly, for each \(i \in \{0, \ldots, k + 1\}\), the value of \(\log_p |L_i/L_{i+1}| = d(L_i/L_{i+1})\) is bounded by the number of explicit generators used to define \(L_i\) modulo \(L_{i+1}\); these numbers are specified in the statement of the proposition and a routine summation shows that they add up to the logarithmic order \(\log_p |G_k|\), as given in Lemma 5.1. Therefore each \(L_i/L_{i+1}\) has the expected rank and it suffices to show that \(\Phi_i(G_k) \supseteq L_i\) for \(1 \leq i \leq k + 1\).

Let \(i \in \{1, \ldots, k + 1\}\). It is enough to show that the following elements which generate \(L_i\) as a normal subgroup belong to \(\Phi_i(G_k)\):

\[ x^{p^i}, \quad c_{\nu(i)+1}, \quad \text{and} \quad z_j \quad \text{for} \quad 2 \nu(i-1) + 3 \leq j \leq p^k \text{ with } j \equiv 2. \]

Clearly, \(x^{p^i} \in \Phi_i(G_k)\) and, applying (2.3), based on Proposition 2.5 and Lemma 5.5, we see by induction on \(i\) that

\[ c_{\nu(i)+1} = [y, x, x^{p^i}, x] \equiv_{\Phi_i(G_k)} [y, x, x^{p^i}, \ldots, x^{p^i-1}] \equiv_{\Phi_i(G_k)} 1. \]

Now let \(2 \nu(i-1) + 3 \leq j \leq p^k \text{ with } j \equiv 2\). By Corollary 6.2 and reverse induction on \(j\) it suffices to show that \(z_j\) is contained in \(\Phi_i(G_k)\) modulo \(\gamma_{j+1}(G_k)\). This follows from (6.3) and the fact that \(c_{\nu(i-1)+1}, c_{j-\nu(i-1)-1} \in \Phi_{i-1}(G_k)\) by induction on \(i\).}

Using Corollary 4.2 we can now complete the proof of Theorem 1.1: it suffices to compute \(\text{hdim}_{G}^{S}(Z)\) and \(\text{hdim}_{G}^{S}(H)\) for the standard filtration series \(S \in \{L, D, F\}\).

Corollary 6.3 implies

\[ \text{hdim}_{G}^{S}(Z) = \lim_{i \to \infty} \frac{\log_p |ZP_i(G) : P_i(G)|}{\log_p |G : P_i(G)|} = \lim_{i \to \infty} \frac{i/2}{5i/2} = 1/5, \]

\[ \text{hdim}_{G}^{S}(H) = \lim_{i \to \infty} \frac{\log_p |HP_i(G) : P_i(G)|}{\log_p |G : P_i(G)|} = \lim_{i \to \infty} \frac{3i/2}{5i/2} = 3/5. \]

Corollary 6.4 implies

\[ \text{hdim}_{G}^{S}(Z) = \lim_{i \to \infty} \frac{\log_p |ZD_i(G) : D_i(G)|}{\log_p |G : D_i(G)|} = \lim_{i \to \infty} \frac{i/2}{3i/2} = 1/3, \]

\[ \text{hdim}_{G}^{S}(H) = \lim_{i \to \infty} \frac{\log_p |HD_i(G) : D_i(G)|}{\log_p |G : D_i(G)|} = \lim_{i \to \infty} \frac{3i/2}{3i/2} = 1. \]
Lastly, Proposition 6.5 implies
\[\text{hdim}_G^p(Z) = \lim_{i \to \infty} \frac{\log_p |Z\Phi_i(G) : \Phi_i(G)|}{\log_p |G : \Phi_i(G)|} = \lim_{i \to \infty} \frac{\sum_{j=1}^{i-1} p^{j-1}}{\sum_{j=1}^{i-1} (p^j + p^{j-1} + 1)} = \frac{1}{p+1},\]  
(6.7)
\[\text{hdim}_G^p(H) = \lim_{i \to \infty} \frac{\log_p |H\Phi_i(G) : \Phi_i(G)|}{\log_p |G : \Phi_i(G)|} = \lim_{i \to \infty} \frac{\sum_{j=1}^{i-1} (p^j + p^{j-1})}{\sum_{j=1}^{i-1} (p^j + p^{j-1} + 1)} = 1.\]

**Remark 6.6.** From (5.3), (6.4), (6.5), (6.6), (6.7) and the fact that subgroups of Hausdorff dimension 1 automatically have strong Hausdorff dimension we conclude that \(Z\) and \(H\) have strong Hausdorff dimension in \(G\) with respect to all standard filtration series \(\mathcal{P}, \mathcal{D}, \mathcal{F}\) and \(\mathcal{L}\).

### 7. The entire Hausdorff spectra of \(G\) with respect to the standard filtration series

We continue to use the notation set up in Section 3 to study and determine the entire Hausdorff spectra of the pro-\(p\) group \(G\), with respect to the standard filtration series \(\mathcal{P}, \mathcal{D}, \mathcal{F}, \mathcal{L}\).

**Proof of Theorem 1.3.** As in Sections 2 and 3, we write \(W = G/Z \cong C_p \wr \mathbb{Z}_p\), and we denote by \(\pi : G \to W\) the canonical projection with \(\ker \pi = Z\).

First suppose that \(S\) is one of the filtration series \(\mathcal{P}, \mathcal{D}, \mathcal{F}\) on \(G\). By Remark 6.6, the group \(H\) has strong Hausdorff dimension 1 in \(G\) with respect to \(S\). As every finitely generated subgroup of \(H\) is finite, it follows from [4, Thm. 5.4] that \(\text{hspec}^S(G) = [0, 1]\).

It remains to pin down the Hausdorff spectrum of \(G\) with respect to the lower \(p\)-series \(\mathcal{L} : P_i(G), i \in \mathbb{N}, \text{ on } G\). By Remark 6.6, the normal subgroups \(Z, H \subseteq G\) have strong Hausdorff dimensions \(\text{hdim}_G^\mathcal{L}(Z) = \frac{1}{5}\) and \(\text{hdim}_G^\mathcal{L}(H) = \frac{3}{5}\). From Corollary 2.4, Lemma 2.2 and Corollary 2.11 we deduce that \(\text{hspec}^\mathcal{L}(G)\) contains
\[S = [0, \frac{3}{5}] \cup \left\{\frac{3}{5} + \frac{2m}{5n} \mid m, n \in N_0 \text{ with } n^2 < m \leq n^3\right\}.\]
Thus it suffices to show that
\[\left(\frac{3}{5}, \frac{4}{5}\right) \subseteq \text{hspec}^\mathcal{L}(G) \subseteq \left(\frac{3}{5}, \frac{4}{5}\right) \cup S.\]  
(7.1)

First we prove the second inclusion. Let \(K \leq C\) be any closed subgroup with \(\text{hdim}_G^\mathcal{L}(K) > \frac{3}{5}\). In particular, this implies \(K \not\subseteq H\) and hence \(KH \leq G\).

We denote by \(\mathcal{L}_H\) and \(\mathcal{L}_{H\pi}\) the filtration series induced by \(\mathcal{L}\) on \(H\), via intersection, and on \(H\pi = HZ/Z\), via subsequent reduction modulo \(Z\). We write \(\mathcal{L}\) for the filtration series \(\mathcal{L}_W\) induced on \(W = G/Z\), as it coincides with the lower \(p\)-series of the quotient group. Using Corollary 2.11 and Lemma 2.2, we see that \((K \cap H)\pi\) has strong Hausdorff dimension
\[\alpha = \text{hdim}_G^\mathcal{L}_{H\pi}(K \cap H) = 2 \text{hdim}_H^\mathcal{L}(K) - 1 \in [0, 1]\]
in \(H\pi\) with respect to \(\mathcal{L}_{H\pi}\). Applying Lemma 2.2 twice, we deduce that
\[\text{hdim}_G^\mathcal{L}(K) = \frac{2}{5} + \frac{2}{5} \text{hdim}_H^\mathcal{L}(K \cap H)\]
\[= \frac{2}{5} + \frac{2}{5} \left(\frac{2}{3} \text{hdim}_H^\mathcal{L}(K \cap H) + [0, \frac{1}{3}]\right)\]
\[= \frac{2}{5}(1 + \alpha) + [0, \frac{1}{5}].\]  
(7.2)
For \( \alpha < 1/2 \) we obtain \( \text{hdim}^G_\xi(K) < 4/5 \) and there is nothing further to prove. Now suppose that \( \alpha \geq 1/2 \). It suffices to show that \( K \cap Z \leq_o Z \) and hence \( \text{hdim}^G_\xi(K \cap Z) = 1/5 \). With this extra information we can refine the analysis in (7.2) and use Corollary 2.11 once more to deduce that
\[
\text{hdim}^G_\xi(K) = \frac{2}{5}(1 + \alpha) + \frac{1}{5} = \frac{4}{5} \text{hdim}^G_\xi(K\pi) + \frac{1}{5} \in S.
\]
Let us prove that \( K \cap Z \leq_o Z \). As \( KH \leq_o G \), we have \( KH = \langle x^{p^n} \rangle H \), where \( n = \log_p |G : KH| \in \mathbb{N}_0 \). Using Lemma 2.2 we deduce from \( \alpha \geq 1/2 \) that
\[
\text{hdim}^G_\xi((K\cap H)\pi) \geq 1/4 = \frac{1}{2} \text{hdim}^G_\xi(H\pi).
\]
At this point it is useful to recall our analysis of \( \text{hspec}^G_\xi(W) \) in the proof of Theorem 2.10 and also the computations carried out in the proof of Proposition 6.5, involving the elements \( c_i = [y, x, i^{-1}, x] \) and \( z_i = [c_{i-1}, y] \). In particular, for \( i \in \mathbb{N} \) with \( i \geq 3 \) we have
\[
(\langle P_i(G) \cap H \rangle \pi) = \langle c_j | j \geq i \rangle \pi \quad \text{and} \quad P_i(G) \cap Z = \langle z_j | j \geq i \text{ and } j \equiv_2 1 \rangle;
\]
and the proof of Theorem 2.10 we deduce that, subject to replacing \( K \) by a suitable open subgroup \( \bar{K} = K \cap \langle x^{p^n} \rangle H \) with \( \bar{n} \geq n \) if necessary, we find \( m \geq (p^n + 1)/2 \) and \( a_1, \ldots, a_m \in K \cap H \) so that
\[
(K \cap H)M/M = \langle a_1, \ldots, a_m \rangle M/M \cong C_p^m, \quad \text{where } M = (P_{p^n+1}(G) \cap H)Z, \quad \text{and the numbers}
\]
\[
d(j) = \max\{i \in \mathbb{N} | a_j \in (P_i(G) \cap H)Z\}, \quad 1 \leq j \leq m,
\]
form a strictly increasing sequence \( 1 \leq d(1) < \ldots < d(m) < p^n \). Commuting \( a_1, \ldots, a_m \) repeatedly with \( x^{p^n} \), we see as in the proof of Theorem 2.10 that
\[
\{d(1), \ldots, d(m)\} + p^n\mathbb{N}_0 \subseteq \{i \in \mathbb{N} | \exists g \in K \cap H : g \equiv_{P_{p^n+1}(G)Z} c_i\}.
\]
For every \( k \in \mathbb{N} \) with \( k > p^n \) and \( k \equiv_2 1 \), the pigeonhole principle (Dirichlet’s ‘Schubfachprinzip’) yields \( i, j \in \mathbb{N} \) with \( i > j \geq 1 \) and \( i + j = k \), and we find \( g_i, g_j \in K \cap H \) with \( g_i \equiv_{P_{p^n+1}(G)Z} c_i \) and \( g_j \equiv_{P_{p^n+1}(G)Z} c_j \) so that (6.3) gives
\[
z_k \equiv_{P_{p^n+1}(G)} [c_i, c_j]^{(-1)^{i-j}} \equiv_{P_{p^n+1}(G)} [g_i, g_j]^{(-1)^{i-j}} \in K \cap Z.
\]
But this implies \( K \cap Z \supseteq \langle z_j | j > p^n \text{ and } j \equiv_2 1 \rangle = P_{p^n+1}(G) \cap Z \) and thus \( K \cap Z \leq_o Z \). This concludes the proof of the second inclusion in (7.1).

Finally we prove the first inclusion in (7.1). Let \( \xi \in (2, 4/5) \). Choose \( m, n \in \mathbb{N} \) such that \( 1 \leq m < p^n/2 \) and
\[
\frac{1}{5} (2 + (4m-1)/p^n) \leq \xi \leq \frac{1}{5} (3 + 2m/p^n).
\]
Consider the group \( K = \langle x^{p^n}, y_0, y_1, \ldots, y_{m-1} \rangle \). Using the proof of Theorem 2.10 and Lemma 2.2 we show below that \( K \) has Hausdorff dimension
\[
\text{hdim}^G_\xi(K) = \frac{4}{5} \text{hdim}^G_\xi(K\pi) + \frac{1}{5} \text{hdim}^G_\xi(K \cap Z) = \left( \frac{2}{5} + \frac{4}{5} \frac{m}{p^n} \right) + \frac{1}{5} = \frac{1}{5} (2 + (4m-1)/p^n).
\]
In a similar, but much more straightforward way, we see that \( ZK \) has strong Hausdorff dimension
\[
\text{hdim}^G_\xi(ZK) = \left( \frac{2}{5} + \frac{2m}{5p^n} \right) + \frac{1}{5} = \frac{1}{5} (3 + 2m/p^n).
\]
An application of [4, Thm. 5.4] yields \( L \leq_c G \) with \( K \leq L \leq ZK \) such that \( \text{hdim}_G(L) = \xi \).

The key to (7.4) consists in showing that
\[
\lim_{i \to \infty} \frac{\log_p |K P_i(G) \cap Z : P_i(G) \cap Z|}{\log_p |Z : P_i(G) \cap Z|} = \text{hdim}_Z(K \cap Z) = (2m - 1)/p^n. \tag{7.5}
\]

First we examine the lower limit on the left-hand side, restricting to indices of the form \( i = p^k + 1, \ k \in \mathbb{N} \). Let \( i = p^k + 1 \), where \( k \geq n \). Recall that \( G_k = G/\langle x^{p^k+1}, [x^{p^k}, y]\rangle \mathbb{C} \) and consider the canonical projection \( g_k : G \to G_k, \ g \mapsto \overline{g} \). As before, we write \( H_k = H g_k \). Furthermore, we observe that \( Z_k = \langle \overline{x^{p^k}} \rangle Z \overline{g}_k \) with \( |Z_k : Z g_k| = p \). By Corollary 6.3 we have
\[
|H_k : H_k \cap P_i(G_k)| = |H_k| = |H : H \cap P_i(G)|
\]
and hence
\[
\frac{\log_p |K P_i(G) \cap Z : P_i(G) \cap Z|}{\log_p |Z : P_i(G) \cap Z|} = \frac{\log_p |K g_k \cap Z g_k|}{\log_p |Z g_k|}.
\]

Observe that
\[
K g_k \cap H_k = \langle \overline{f^p} \rangle \cup \{\overline{f_0}, \overline{y_j} \mid 0 \leq j < p^k \text{ with } j \equiv p^n 0, 1, \ldots, m - 1\}.\]

From Lemma 5.1 we see that \( Z \overline{g}_k \cong C_p^{(p + 1)/2} \) and further we deduce that
\[
K \overline{g}_k \cap Z \overline{g}_k = \langle \{ \overline{f^p} \} \cup \{[\overline{f_0}, \overline{y_j}] \mid 0 \leq j < p^k, \ j \equiv p^n 0, \pm 1, \ldots, \pm (m - 1), \ j \equiv 0 \} \rangle 
\cong C_p^{((2m - 1)p^{k - n} + 1)/2}.
\]

This yields
\[
\lim_{i \to \infty} \frac{\log_p |K P_i(G) \cap Z : P_i(G) \cap Z|}{\log_p |Z : P_i(G) \cap Z|} \leq \lim_{k \to \infty} \frac{\log_p |K g_k \cap Z g_k|}{\log_p |Z g_k|} = \lim_{k \to \infty} \frac{(2m - 1)p^{k - n} + 1}{p^{k + 1}} = (2m - 1)/p^n.
\]

In order to establish (7.5) it now suffices to prove that
\[
\lim_{i \to \infty} \frac{\log_p |(K \cap Z)(P_i(G) \cap Z) : P_i(G) \cap Z|}{\log_p |Z : P_i(G) \cap Z|} \geq (2m - 1)/p^n. \tag{7.6}
\]

Our analysis above yields
\[
K \cap Z = \langle \{ f^p \} \cup \{[y_0, y_j] \mid j \in \mathbb{N} \text{ with } j \equiv p^n 0, \pm 1, \ldots, \pm (m - 1)\} \rangle.
\]

Setting
\[
L = \langle y_j \mid j \in \mathbb{N}_0 \text{ with } j \equiv p^n 0, \pm 1, \ldots, \pm (m - 1) \rangle Z,
\]
and recalling the notation \( c_1 = y = y_0 \), we conclude that
\[
K \cap Z \supseteq \{[g, c_1] \mid g \in L\}.
\]

Next we consider the set
\[
D = \{ j \in \mathbb{N} \mid \exists g \in L : g \equiv P_{j+1}(G) Z c_j \}.
\]
With regards to Lemma 5.3, the elements $G|((2G, is a presentation of $C$ and the elementary identity $(1+t)^{j+p^n} = (1+t)^{(1+t)^p}$ in $\mathbb{F}_p[t]$ translates to

$$y_j^{-1}y_j^{p^n} = y^{-x^j}g^{x^j+p^n} \equiv z \prod_{k=0}^{i} c_{k+1}^{j} \quad \text{for all } j \in \mathbb{N};$$

compare (2.4). Inductively, we obtain

$$D = D_0 + p^nN_0 \quad \text{for } D_0 = D \cap \{1, \ldots, p^n\}.$$ Observe that $|D_0| = 2m - 1$ and that, for each $k \in \mathbb{N}_0$, the set $(2kp^n + D_0) \cup ((2k + 1)p^n + D_0)$ consists of $2m - 1$ odd and $2m - 1$ even numbers.

For each $j \in D$ with $j \equiv 0$ there exists $g_j \in L$ with $g_j \equiv_{P_j(G)z} c_j$ and we deduce that

$$z_j+1 = [c_j, c_1] \equiv_{P_j(G)} [g_j, c_1] \in K \cap Z.$$ For $i = 2p^nq + r \in \mathbb{N}$, where $q, r \in \mathbb{N}_0$ with $0 \leq r < 2p^n$, the count

$$|\{j \in D \mid j \equiv 0 \text{ and } j < i - 1\}| \geq q(2m - 1) - 1$$

yields

$$\log_p(K \cap Z)(P_i(G) \cap Z) : P_i(G) \cap Z \geq q(2m - 1) - 1.$$ From Corollary 6.3 we observe that, for $i \geq 3$,

$$\log_p|Z : P_i(G) \cap Z| = \lfloor i/2 \rfloor \leq qp^n + p^n.$$ These estimates show that (7.6) holds. \hfill \Box

APPENDIX A. THE CASE $p = 2$

When $p$ is even, Theorems 1.1 and 1.3, and all the results of Sections 2 and 4, hold with corresponding proofs. The structural results of Sections 5 and 6 however are slightly different and we now sketch these differences below; for complete details, we refer the reader to the supplement [8].

Firstly, for $p = 2$,

$$G_k = F/N_k \cong \langle x, y \mid x^{2k+1}, y^4, [x^{2k}, y], [y^2, x]; [y_0, y_i]^2, [y_0, y_i, x], [y_0, y_i, y] \text{ for } 1 \leq i \leq 2^{k-1} \rangle \quad \text{(A.1)}$$

for $k \in \mathbb{N}$, and

$$G \cong \langle x, y \mid y^4, [y^2, x]; [y_0, y_i]^2, [y_0, y_i, x], [y_0, y_i, y] \text{ for } i \in \mathbb{N} \rangle \quad \text{(A.2)}$$

is a presentation of $G$ as a pro-$2$ group.

Next, we have $\log_2 |G_k| = 2^k + 2^{k-1} + k + 2$ and the exponent of $\gamma_2(G_k)$ is 4. With regards to Lemma 5.3, the elements

$$w = y_{2^{k-1}} \cdots y_1 y_0 \quad \text{and} \quad [w, x] = [w, y] = [y_0, y_{2^{k-1}}]$$

are of order 2 in $G_k$ and lie in $G_k^{2k}$. In particular the subgroup $\langle x^{2^k}, w, [w, x] \rangle$ is isomorphic to $C_2 \times C_2 \times C_2$ and lies in $G_k^{2k}$. Hence, for $k \geq 2$,

$$G_k^{2k} = \langle x^{2^k}, w, [w, x] \rangle \cong C_2 \times C_2 \times C_2, \quad \log_2 |G_k : G_k^{2k}| = \log_2 |G_k| - 3.$$
and
\[ G_k/G_k^{2^k} \cong \langle x, y \mid x^{2^k}, y^4, [y^2, x], w(x, y), [y_0, y_2, \ldots] \rangle; \]
\[ [y_0, y_1]^2, [y_0, y_1, x], [y_0, y_1, y] \text{ for } 1 \leq i < 2^{k-1}. \]

Lemma 5.7 is slightly different; here the group \( G \) satisfies \( G^2 \subseteq \langle x^2, y^2 \rangle \) and
\[ G_2^2 \subseteq \langle x^2, [y, x], [y, x, x, y] \rangle \gamma_2(G_k) \subseteq \langle x^2 \rangle \gamma_{2j-1}(G_k) \text{ for } j \geq 2. \]

The proof is similar, but one needs the fact
\[ [y, x, \ldots, x] \equiv [y, x, \ldots, x]^x \text{ for } i \geq 1, \]
which is proved by induction, using
\[ [y, x, \ldots, x] = ([y, x, \ldots, x]^x, [y, x, \ldots, x]^x] \text{ for } i \geq 2. \]

Furthermore, if \( j \leq m \), \( G_{2^j} \equiv [y, x, \ldots, x, y] \gamma_{2j}(G_k) \subseteq \langle y, x, \ldots, x, y \rangle \text{ for } j \geq 2. \]

The group \( G_k \) is nilpotent of class \( 2^k + 1 \); its lower central series satisfies
\[ G_k = \gamma_1(G_k) = \langle x, y \rangle \gamma_2(G_k) \text{ with } G_k/\gamma_2(G_k) \cong C_{2^{k+1}} \times C_4 \]
and, for \( 1 \leq i \leq 2^{k-1} \),
\[ \gamma_{2i+1}(G_k) = \left\langle [y, x, \ldots, x^i] \right\rangle \gamma_{2i+1}(G_k), \]
\[ \gamma_{2i+1}(G_k) = \left\{ \begin{array}{ll}
\langle [y, x, \ldots, x^i], [y, x, \ldots, x^i, y] \rangle \gamma_{2i+2}(G_k) & \text{for } i \neq 2^{k-1} \\
\langle [y, x, \ldots, x^i] \rangle \gamma_{2i+2}(G_k) & \text{for } i = 2^{k-1}
\end{array} \right. \]
with
\[ \gamma_{2i}(G_k)/\gamma_{2i+1}(G_k) \cong C_2 \text{ and } \gamma_{2i+1}(G_k)/\gamma_{2i+2}(G_k) \cong \left\{ \begin{array}{ll}
C_{2} \times C_{2} & \text{for } i \neq 2^{k-1} \\
C_{2} & \text{for } i = 2^{k-1}.
\end{array} \right. \]

The proof of the above is similar to that for the odd prime case, however here one takes
\[ j_0 = \begin{cases} 
2^{k-1} - \frac{m}{2} & \text{if } m \equiv_4 0, \\
2^{k-1} + 1 - \frac{m}{2} & \text{if } m \equiv_4 2.
\end{cases} \]

For the \( m \equiv_4 0 \) case, noting that \( e_2k-1 = [w, x] \in \gamma_{2k+1}(G_k) \), we have \( b_{j_0,m} \equiv b_{j_0,m+1} \) modulo \( \gamma_{m+1}(G_k) \). The \( m \equiv_4 2 \) case is similar.

The lower 2-series of \( G_k \) has length \( 2^{k+1} + 1 \) and satisfies the corresponding form, based on the lower central series of \( G_k \) above.

The dimension subgroup series of \( G_k \) has length \( 2^{k+1} + 1 \) and satisfies the corresponding form, based on the lower central series of \( G_k \) above.

The \( i \)-th term is \( D_i(G_k) = C_{2^{i(l(i))}} \gamma_{\lceil\frac{i}{2}\rceil}(G_k)^2 \gamma_i(G_k) \), where \( l(i) = \lfloor \log_2 i \rfloor \).

Furthermore, if \( i \) is not a power of 2, equivalently if \( l(i + 1) = l(i) \), then
\[ D_i(G_k)/D_{i+1}(G_k) \cong \gamma_{\lceil\frac{i}{2}\rceil}(G_k)^2 \gamma_i(G_k)/\gamma_{\lceil\frac{i}{2}\rceil}(G_k)^2 \gamma_{i+1}(G_k) \]
so that
\[ D_i(G_k) = \left\{ \begin{array}{ll}
\langle [y, x, \ldots, x] \rangle D_{i+1}(G_k) & \text{if } i \equiv_2 1, \\
\langle [y, x, \ldots, x, y, [y, x, \ldots, x] \rangle D_{i+1}(G_k) & \text{if } i \equiv_2 0,
\end{array} \right. \]
with
\[ D_i(G_k)/D_{i+1}(G_k) \cong \left\{ \begin{array}{ll}
C_{2} & \text{if } i \equiv_2 1 \text{ and } i < 2^k, \\
C_{2} \times C_{2} & \text{if } i \equiv_2 0 \text{ and } i < 2^k, \\
1 & \text{if } i = 2^k + 1, \\
C_{2} & \text{if } i = 2^k + 2.
\end{array} \right. \]
whereas if $i = 2^l$ is a power of 2, equivalently if $l(i + 1) = l + 1$ for $l = l(i)$, then

$D_i(G_k)/D_{i+1}(G_k) \cong \langle x^{2^l} \rangle/\langle x^{2^{l+1}} \rangle \times \langle y^{2^l} \rangle/\langle y^{2^{l+1}} \rangle \times \langle [y, x, i^{-3}, x, y] \rangle \gamma_i(G_k)/\gamma_{i+1}(G_k)$

so that

$D_1(G_k) = \langle x, y \rangle D_2(G_k)$,
$D_2(G_k) = \langle x^2, y^2, [y, x] \rangle D_3(G_k)$,
$D_i(G_k) = \langle x^{2^l}, [y, x, i^{-3}, x, y], [y, x, i^{-1}, x] \rangle D_{i+1}(G_k)$

with

$D_i(G_k)/D_{i+1}(G_k) \cong \begin{cases} C_2 \times C_2 & \text{if } i = 1, \text{ equivalently if } l = 0, \\ C_2 \times C_2 \times C_2 & \text{if } i = 2, \text{ equivalently if } l = 1, \\ C_2 \times C_2 \times C_2 & \text{if } i = 2^l \text{ with } 2 \leq l \leq k. \end{cases}$

In particular, for $2^{k-1} + 1 \leq i \leq 2^k$ and thus $l(i) = k$,

$D_i(G_k) = G_k^{2^k} \gamma_i(G_k) = \langle x^{2^k}, [y, x, \cdots, \cdots, x, y] \rangle \gamma_i(G_k)$,

so that

$\log_2 |D_i(G_k)| = \log_2 |\gamma_i(G_k)| + 1$.

Lastly, the Frattini series of $G_k$ has the corresponding form, though it has length $k + 1$.

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