BICATEGORIES OF SPANS AS GENERIC BICATEGORIES

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Abstract. In a bicategory of spans (an example of a “generic bicategory”) the factorization of a span \((s, t)\) as the span \((s, 1)\) followed by \((1, t)\) satisfies a simple universal property with respect to all factorizations in terms of the generic bicategory structure. Here we show that this universal property can in fact be used to characterize bicategories of spans.

This characterization of spans is very different from the others in that it does not mention any adjointness conditions within the bicategory.

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1 Introduction

Bicategories of spans were introduced by Bénabou [1] and have since become one of the fundamental constructions in 2-dimensional category theory. Given a category \(\mathcal{E}\) with pullbacks, its bicategory of spans, denoted \(\text{Span}(\mathcal{E})\), contains the same objects as those of \(\mathcal{E}\), has 1-cells given as diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
& s \searrow & t \\
\end{array}
\]

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called spans, 2-cells between such spans given as commuting diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{t} & Y \\
\downarrow{s} & & \downarrow{p} \\
T & \xleftarrow{q} & S
\end{array}
\]

and composition of 1-cells is defined by pullback, inducing a natural definition of horizontal composition of 2-cells. Whilst an explicit description as above is useful, one can gain a better understanding of where these bicategories of spans come from by giving more abstract characterizations.

The currently known characterizations all rely heavily on the adjoint properties of spans, and the Beck conditions that these adjoints satisfy. For example the universal property of spans (which may be viewed as a characterization) says giving a pseudofunctor $\text{Span}(\mathcal{E}) \to \mathcal{C}$ is equivalent to giving a pseudofunctor $\mathcal{E} \to \mathcal{C}$ which sends morphisms to left adjoints and satisfies the Beck condition [2].

There is also another more interesting characterization of bicategories of spans due to Lack, Walters, and Wood [6]. They show that bicategories of spans are precisely those bicategories which are Cartesian and satisfy the two axioms (1) every comonad has an Eilenberg-Moore object; and (2) every left adjoint arrow is comonadic.

In this paper we will give another (very different) characterization of bicategories of spans. Unlike the aforementioned characterization, which heavily relies on the adjoint properties of spans (even the definition of Cartesian bicategory involves adjoints within the bicategory) ours makes no mention of adjunctions; instead these adjoint properties of spans are seen as direct consequences of its underlying “generic bicategory” structure.

**Generic bicategories** were introduced by the author in order to better understand the universal properties bicategories of spans of polynomials [7], and can be defined as bicategories $\mathcal{C}$ such that for each 1-cell $c: X \to Z$ and object $Y$ the presheaf

\[
\mathcal{C}_{X,Z}(c, - \circ -) : \mathcal{C}_{Y,Z} \times \mathcal{C}_{X,Y} \to \text{Set}
\]

is a coproduct of representables. For a fixed $X,Y,Z$ and $c$, the “generic 2-cells out of $c$” may be defined as the initial objects within the connected components of the category of elements of this presheaf; that is the category whose objects are 2-cells from $c$ into a composite of 1-cells through $Y$, written (with composition in diagrammatic order) $c \Rightarrow a; b$, and whose morphisms from an object $\alpha: c \Rightarrow a; b$ to an object $\beta: c \Rightarrow a'; b'$ are pairs of 2-cells $a \Rightarrow a'$ and $b \Rightarrow b'$ which when pasted with $\alpha$ give $\beta$. A generic 2-cell is then an object $\delta: c \Rightarrow l;r$ which is initial in its connected component in this category.

In the case of spans, we have for each span $(s,t): X \to Z$ with vertex $T$, and each object $Y$, natural isomorphisms\(^2\)

\[
\text{Span}(\mathcal{E})_{X,Z}((s,t), - \circ -) \cong \sum_{h: T \to Y} \text{Span}(\mathcal{E})_{X,Y}( (s,h), - ) \cdot \text{Span}(\mathcal{E})_{Y,Z}( (h,t), - )
\]

\(^1\)Note that this characterization considers spans over a category with both pullbacks and a terminal object (a finitely complete category).

\(^2\)In general, the indexing set of a multi-adjoint is the set of connected components of the category of elements of the relevant presheaf. Since each connected component has a unique equivalence class of initial objects, this is also the set of equivalence classes of generics.
so that $\text{Span}(\mathcal{E})$ is a generic bicategory, and the generic 2-cells (which are recovered by substituting a pair of identity 2-cells on the right above) are the morphisms of spans $(s, t) \Rightarrow (s, h) ; (h, t)$ given as diagrams

\begin{align}
(1.1) & \\
\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (Y) at (2,2) {$Y$};
  \node (Z) at (4,0) {$Z$};
  \node (T) at (2,0) {$T$};

  \draw[->] (X) to node[above]{$s$} (T);
  \draw[->] (T) to node[above]{$t$} (Z);
  \draw[->] (T) to node[below]{$h$} (Y);
  \draw[->] (X) to node[below]{$s$} (Y);
  \draw[->] (Y) to node[below]{$h$} (Z);

  \draw[->] (X) to node[below]{$\pi_1$} (T);
  \draw[->] (Z) to node[below]{$\pi_2$} (T);

  \draw[->] (X) to node[above]{$\delta$} (Y);
  \draw[->] (Y) to node[above]{$\eta$} (Z);

\end{tikzpicture}
\end{align}

such that $\pi_1 \delta = \pi_2 \delta = 1_T$. We now observe a property unusual to generic bicategories, we have a “best generic 2-cell” (which we will call an initial generic), given by taking $h$ to be the identity. So for each span $(s, t)$, we have an initial generic $(s, t) \Rightarrow (s, 1) ; (1, t)$, and a general generic 2-cell $(s, t) \Rightarrow (s, h) ; (h, t)$ may be recovered by pasting this initial generic with the generic 2-cell $(1, 1) \Rightarrow (1, h) ; (h, 1)$.

As any 2-cell $(s, t) \Rightarrow (a, b) ; (c, d)$ factors through an essentially unique generic 2-cell $(s, t) \Rightarrow (s, h) ; (h, t)$, and this itself factors through the initial generic $(s, t) \Rightarrow (s, 1) ; (1, t)$, we see that bicategories of spans satisfy the existence part of the following condition.

\textbf{Axiom 1} (Every 1-cell has an initial generic 2-cell). For every 1-cell $c : X \to Z$ in $\mathcal{E}$, there exists an invertible generic 2-cell $\delta : c \Rightarrow l ; r$ (through some object $Y$) which is universal in that given any 2-cell $\gamma : c \Rightarrow a ; b$ (through another object $Y'$) there exists a generic 2-cell $\eta : 1_Y \Rightarrow h ; k$ and 2-cells $\alpha$ and $\beta$ as below factoring $\gamma$ as

\begin{align}
(1.2) & \\
\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (Y) at (2,2) {$Y$};
  \node (Z) at (4,0) {$Z$};
  \node (Y') at (2,-2) {$Y'$};

  \draw[->] (X) to node[above]{$\delta$} (Y);
  \draw[->] (Y) to node[above]{$\eta$} (Z);
  \draw[->] (Y) to node[below]{$\delta'$} (Y');
  \draw[->] (Z) to node[below]{$\eta'$} (Y');
  \draw[->] (X) to node[below]{$a$} (Y');
  \draw[->] (Y') to node[below]{$b$} (Z);

  \draw[->] (X) to node[below]{$\alpha$} (Y);
  \draw[->] (Z) to node[below]{$\beta$} (Y');

\end{tikzpicture}
\end{align}

Moreover, such a factorization is unique in that given another factorization

\begin{align}
\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (Y) at (2,2) {$Y$};
  \node (Z) at (4,0) {$Z$};
  \node (Y') at (2,-2) {$Y'$};

  \draw[->] (X) to node[above]{$\delta$} (Y);
  \draw[->] (Y) to node[above]{$\eta''$} (Z);
  \draw[->] (Y) to node[below]{$\delta'$} (Y');
  \draw[->] (Z) to node[below]{$\eta''$} (Y');
  \draw[->] (X) to node[below]{$a$} (Y');
  \draw[->] (Y') to node[below]{$b$} (Z);

  \draw[->] (X) to node[below]{$\alpha'$} (Y);
  \draw[->] (Z) to node[below]{$\beta'$} (Y');

\end{tikzpicture}
\end{align}

with $\eta' : 1_Y \Rightarrow h' ; k'$ generic, we have induced comparison isomorphisms $h \cong h'$ and $k \cong k'$ coherent with $\eta$ and $\eta'$ (necessarily unique by genericity of $\eta$) and $\alpha, \alpha', \beta, \beta'$.

To verify the uniqueness condition mentioned above is satisfied, it is easiest to first consider the case where $\eta'$ is a representative generic 2-cell, that is a 2-cell of the form $\eta' : 1 \Rightarrow (1, h') ; (h', 1)$. In this case we have that $\gamma$ factors through the generic 2-cell $(s, t) \Rightarrow (s, h) ; (h, t)$ as well as the generic 2-cell $(s, t) \Rightarrow (s, h') ; (h', t)$, but since $\text{Span}(\mathcal{E})_{X,Z}[(s, t) , (a, b) ; (c, d)]$ is isomorphic to

$$
\sum_{h : T \to Y} \text{Span}(\mathcal{E})_{X,Y}((s, h) , (a, b)) \cdot \text{Span}(\mathcal{E})_{Y,Z}((h, t) , (c, d))
$$
and both the triples \((h, \alpha, \beta)\) and \((h', \alpha', \beta')\) correspond to \(\gamma\) under this bijection, we must conclude the two triples are equal. Thus we have strict uniqueness when one requires \(\eta'\) be a representative generic 2-cell. Given a factorization of \(\gamma\) as on the left below (with \(\eta'\) not assumed representative)

\[
\begin{array}{ccc}
X & \xrightarrow{l} & Y \\
\downarrow_{\alpha} & \text{\text{id}} & \downarrow_{\beta} \\
Y & \overset{\delta}{\xrightarrow{c}} & Z
\end{array}
\]

we may factor \(\eta'\) through a representative generic \(\sigma\) as on the right above (the induced comparisons being of course invertible). The strict uniqueness that holds with representatives then ensures the comparisons are coherent with \(\alpha, \alpha'\) and \(\beta, \beta'\).

Finally, there is one more crucial observation we must make. Given any initial generic pasted with a generic out of an identity \((s, t) \Rightarrow (s, 1); (1, h); (h, 1); (1, t)\) we have that the identities \((s, h) \Rightarrow (s, 1); (1, h)\) and \((h, t) \Rightarrow (h, 1); (1, t)\) are initial generics, so that bicategories of spans also satisfy the following condition.

**Axiom 2** (Initial factorizations yield initial generics). For every such factorization \((1.2)\) the identities \((l; h) \Rightarrow l; h\) and \((k; r) \Rightarrow k; r\) are initial generics.

The purpose of this paper is to address the question: what really are bicategories of spans? To which we propose the answer that bicategories of spans are precisely the “perfect” generic bicategories; meaning those generic bicategories for which we have not only generic 2-cells, but “best generic 2-cells” (Axiom 1), and these best generics induce other best generics (Axiom 2).

### 2. Characterizing bicategories of spans

The goal of this paper is to show that just Axioms 1 and 2 are enough to characterize bicategories of spans. However, at least in the authors opinion, this is not an obvious fact. Whilst each 1-cell \(c\) having an initial generic \(c \cong l; r\) is suggestive of a mapping into spans, there are many points that are not clear. For example we must deduce that 2-cells are morphisms of spans, and deduce that composition is by pullback. Moreover, such facts will rely on one the more fundamental properties of the span construction used extensively in its characterization in terms of Cartesian bicategories [6]; that any 2-cell between left adjoints is both unique and invertible, and even this fact is not clear.

#### 2.1. Preliminaries.

The purpose of this subsection is to mention some properties of generic bicategories that will be needed. The properties mentioned here do not rely on axioms 1 or 2.

The following two lemmata are recalled directly from [7]. The following recalls that the 3-dimensional version of generic cells (3-ary generics) are constructed by composing two 2-ary generics.

**Lemma 2.1.1.** [7, Lemma 13] *Suppose \(C\) is a generic bicategory. A 2-cell \(\delta: c \Rightarrow l; m; r\) is 3-ary generic\(^3\) if and only if \(\delta\) can be factored as a composite of two 2-ary generic cells.*

\(^3\)That is initial in a connected component of the category whose objects are 2-cells \(c \Rightarrow x; y; z\) and whose morphisms are coherent triples of 2-cells \(x \Rightarrow x', y \Rightarrow y', z \Rightarrow z'\).
generics as on the left below if and only if $\delta$ can be factored as a composite of two 2-ary generics as on the right below

$$c \xrightarrow{\sigma_2} l; q \xrightarrow{\sigma_2} l; m; r$$

$$c \xrightarrow{\sigma_3} p; r \xrightarrow{\sigma_4 r} l; m; r$$

The following fact will only be used to check that $1_X \Rightarrow 1_X; 1_X$ is always an initial generic; a property we should definitely expect.

**Lemma 2.1.2.** [7, Lemma 11] Suppose $\mathcal{C}$ is a generic bicategory. If a left unitor $c \Rightarrow 1; c$ in $\mathcal{C}$ factors through a generic $c \Rightarrow l; r$ then the induced $r \Rightarrow c$ is invertible.

The desired property then follows almost immediately.

**Corollary 2.1.3.** Suppose $\mathcal{C}$ is a generic bicategory. Then the unitor $1_X \Rightarrow 1_X; 1_X$ is always an initial generic.

**Proof.** We first factor $1_X \Rightarrow 1_X; 1_X$ through a generic 2-cell. By Lemma 2.1.2, and the fact this both a left and right unitor, both induced comparison 2-cells are invertible, so that $1_X \Rightarrow 1_X; 1_X$ must itself be generic. It is then completely trivial to directly verify that $1_X \Rightarrow 1_X; 1_X$ satisfies the universal property of an initial generic 2-cell. □

Whilst our characterization makes no mention of adjunctions, the proof of the characterization will rely on them heavily. It will therefore be prudent to recall the so called “mates correspondence” [4].

**Proposition 2.1.4** (Mates correspondence). Suppose $f_1 \dashv g_1$ and $f_2 \dashv g_2$ are adjoint pairs in a bicategory. Then for any 1-cells $p$ and $q$ as below, pasting with units and counits of these adjunctions defines a natural bijection

$$A \xrightarrow{\alpha} B \xleftarrow{\beta} C$$

between 2-cells $\alpha$ and 2-cells $\beta$ as above.

In particular, we will need the fact that 2-cells of left adjoints are unique; as one would expect as this is a fundamental property of the span construction.

**Proposition 2.1.5.** Suppose $\mathcal{C}$ is a generic bicategory. A 2-cell of left adjoints in $\mathcal{C}$ (and similarly a 2-cell of right adjoints) is unique.

**Proof.** Recall that in a generic bicategory, identity 1-cells are subterminal [7, Prop. 9]. It then follows immediately from the mates correspondence that left adjoints and right adjoints are also subterminal. □

2.2. **Initial generics are a right followed by a left adjoint.** In this paper we do not view adjointness as one of the defining properties of spans (which is why our characterization makes no mention of adjunctions); instead the adjoint properties of spans are seen as a consequence of the underlying generic bicategory structure. The following results justify this viewpoint.

**Lemma 2.2.1.** Assume $\mathcal{C}$ is a generic bicategory. Then if a right unitor $\delta: l \Rightarrow l; 1$ is an initial generic 2-cell, $l$ must be a right adjoint. Dually, if a left unitor $\delta: r \Rightarrow 1; r$ is an initial generic 2-cell, $r$ must be a left adjoint.
Proof. Given that the unitor \( \delta : l \Rightarrow l \cdot 1 \) is an initial generic, we have an induced triple into \( 1 \cdot l \) as below

\[
\begin{array}{c}
X \xrightarrow{l} Y \\
\downarrow \delta \\
\downarrow \downarrow \alpha \\
\downarrow \downarrow \beta
\end{array}
\]

We now need to check that \( \alpha : hl \Rightarrow 1 \) and \( \beta h \cdot \eta : 1 \Rightarrow lh \) define the counit and unit of an adjunction \( h \dashv l \). As the above induced triple of cells is necessarily the identity (up to unitors) we already have one of the triangle identities.

Let us now make the observation that the pasting of \( \delta \) with \( \eta \) as below is a 3-ary generic cell by Lemma 2.1

\[
\begin{array}{c}
X \xrightarrow{l} Y \\
\downarrow \delta \\
\downarrow \downarrow \eta \\
\downarrow \downarrow \alpha \\
\downarrow \downarrow \beta
\end{array}
\]

and that the above pasted with

\[
\begin{array}{c}
X \xrightarrow{l} Y \\
\downarrow \delta \\
\downarrow \downarrow \eta \\
\downarrow \downarrow \alpha \\
\downarrow \downarrow \beta
\end{array}
\]

is equal to, by the previous triangle identity,

\[
\begin{array}{c}
X \xrightarrow{l} Y \\
\downarrow \delta \\
\downarrow \downarrow \eta \\
\downarrow \downarrow \alpha \\
\downarrow \downarrow \beta
\end{array}
\]

Thus the other triangle identity is recovered from 3-ary genericity of (2.1). The proof of the dual property is similar. \( \square \)

Since we may paste any initial generic \( \delta : c \Rightarrow l ; r \) with the generic \( 1 \Rightarrow 1 ; 1 \) and then apply Axiom 2, we may deduce the following.

Corollary 2.2.2. Assume \( \mathcal{C} \) is a generic bicategory satisfying axioms 1 and 2. Then for any initial generic \( \delta : c \Rightarrow l ; r \) we have that \( l \) is a right adjoint and \( r \) is a left adjoint.

2.3. Generics out of identities are precisely units of adjunctions. In a bicategory of spans the (representative) generics out an identity are of the form \( (1, 1) \Rightarrow (1, h) ; (h, 1) \) and coincide with the units of the adjunctions. Here we show that this fact about bicategories of spans follows from Axioms 1 and 2.

Lemma 2.3.1. Assume \( \mathcal{C} \) is a generic bicategory satisfying axioms 1 and 2. Then every generic 2-cell out of an identity \( \delta : 1 \Rightarrow h ; k \) is the unit of an adjunction \( h \dashv k \).
Proof. By Corollary 2.1.3 and Axiom 2 we that know both \(1; h \) and \(1; k \) are initial, and so \(h \) and \(k \) are left and right adjoints respectively by Lemma 2.2.1. Thus the 2-cell \(\delta: 1 \Rightarrow kh\) has a mate \(h^* \Rightarrow k\) where \(h^*\) is a right adjoint to \(h\). However, as \(Yh \downarrow \downarrow Y\), is equal to \(\delta\), both \(\delta\) and \(\eta\) lie in the same connected component. Hence \(\eta\) factors through \(\delta\) (which is initial in its connected component by definition), yielding comparisons \(h \Rightarrow h\) and \(k \Rightarrow h^*\). By Proposition 2.1.5, we must then conclude the 2-cell \(h^* \Rightarrow k\) is an isomorphism. \(\square\)

We now use the above lemma to prove its converse.

**Lemma 2.3.2.** Assume \(\mathcal{C}\) is a generic bicategory satisfying axioms 1 and 2. Then every unit of an adjunction \(\nu: 1 \Rightarrow f; g\) is a generic 2-cell out of an identity.

**Proof.** We again recall Corollary 2.1.3, and then factor a unit \(\nu: 1 \Rightarrow f; g\) as

\[
\begin{array}{c}
Y \\
\downarrow \eta \\
Y' \\
\downarrow \delta \\
1 \\
\end{array}
\]

is equal to \(\delta\), both \(\delta\) and \(\eta\) lie in the same connected component. Hence \(\eta\) factors through \(\delta\) (which is initial in its connected component by definition), yielding comparisons \(h \Rightarrow h\) and \(k \Rightarrow h^*\). By Proposition 2.1.5, we must then conclude the 2-cell \(h^* \Rightarrow k\) is an isomorphism. \(\square\)

**2.4. Adjoint composites are initial.** Whilst we know by Corollary 2.2.2 that initial generics \(c \cong l; r\) yield factorizations of any 1-cell \(c\) as a right adjoint followed by a left adjoint, thus giving a way of mapping 1-cells into spans, we would like this assignation to be essentially surjective. The following lemma ensures this property.

**Lemma 2.4.1.** Assume \(\mathcal{C}\) is a generic bicategory satisfying axioms 1 and 2. Then for any composable right adjoint \(s^*\) and left adjoint \(t\), the composite \(s^*; t\) is initial.

**Proof.** We first note that for any left adjoint \(t\), the composite \(1; t\) is initial, which is seen by recalling Corollary 2.1.3 and applying Axiom 2 to the diagram

\[
\begin{array}{c}
T \\
\downarrow \eta \\
X \\
\end{array}
\]

By then applying Axiom 2 to the diagram

\[
\begin{array}{c}
T \\
\downarrow \eta \\
X \\
\end{array}
\]

we see that \(s^*; t\) is initial. \(\square\)
2.5. **The left adjoints form a 1-category.** Given a bicategory of spans $\text{Span}(\mathcal{E})$, one may recover $\mathcal{E}$ as its category of left adjoints. However, the left adjoints only form a 1-category since any 2-cell between left adjoints is both unique and invertible. We have already partially verified this property, showing uniqueness of such 2-cells in Proposition 2.1.5, thus it just remains to check they are invertible.

**Lemma 2.5.1.** Assume $\mathcal{C}$ is a generic bicategory satisfying axioms 1 and 2. Then any 2-cell $\alpha: f_1 \Rightarrow f_2$ between left adjoints $f_1$ and $f_2$ is invertible.

**Proof.** Suppose $g_1$ and $g_2$ are respective right adjoints to $f_1$ and $f_2$. We then have an equality

\[
\begin{array}{ccc}
X & \xrightarrow{g_2} & Y \\
\downarrow & \xRightarrow{f_1} & \downarrow \\
X & \rightarrow & X
\end{array} = \begin{array}{ccc}
X & \xrightarrow{g_2} & Y \\
\downarrow & \xRightarrow{f_1} & \downarrow \\
X & \rightarrow & X
\end{array}
\]

yielding, as $g_2; f_1$ is initial by Lemma 2.4.1, an isomorphism $f_1 \cong f_2$ from uniqueness of such factorizations, thus showing $\alpha$ is invertible by Proposition 2.1.5. □

Since any 2-cell between left adjoints is both unique by Proposition 2.1.5 and invertible by Lemma 2.5.1 we have the following.

**Corollary 2.5.2.** Assume $\mathcal{C}$ is a generic bicategory satisfying axioms 1 and 2. Then the sub-bicategory of left adjoint 1-cells in $\mathcal{C}$ is equivalent to a locally discrete 2-category $\mathcal{E}$ whose morphisms are equivalence classes of left adjoints.

2.6. **Initial generics paste with generics to give generics.** The following shows that any initial generic $c \Rightarrow l ; r$ pasted with a generic out an identity $1 \Rightarrow h ; k$ yields a generic $c \Rightarrow (l ; h) ; (k ; r)$. With this proven it will then follow that when a 2-cell $\gamma: c \Rightarrow a ; b$ is factored through an initial generic as in (1,2), the induced comparisons $\alpha$ and $\beta$ must be invertible whenever $\gamma$ is generic.

**Lemma 2.6.1.** Suppose $\mathcal{C}$ is a generic bicategory, and that a 1-cell $c$ admits an initial generic $c \cong l ; r$ through an object $Y$. Then for any generic out an identity $1_Y \Rightarrow h ; k$ the pasting $c \Rightarrow (l ; h) ; (k ; r)$ is a 2-ary generic.

**Proof.** Consider a 1-cell $c$ with initial generic $l ; r$ pasted with a generic 2-cell $\eta: 1 \Rightarrow h ; k$ as on the left below. There is no loss in generality assuming $\eta$ is a representative generic.

We check that $c \Rightarrow (l ; h) ; (k ; r)$ is generic. To see this, we first factor this pasting through a generic 2-cell $\xi$ as on the right above. It remains to check the induced comparison maps $\theta$ and $\varphi$ are invertible.
We proceed by factoring \( \xi \) through the initial generic as

\[
\begin{array}{c}
X \\
\downarrow^c \\
Y \\
\downarrow \delta \\
Y' \\
\downarrow \\downarrow \\downarrow \\
Z
\end{array}
\]

where we chose \( \eta' \) to be a representative generic. This gives the equality

\[
\begin{array}{c}
X \\
\downarrow^c \\
Y \\
\downarrow \delta \\
Y' \\
\downarrow \\downarrow \\downarrow \\
Z
\end{array}
= \begin{array}{c}
X \\
\downarrow^c \\
Y \\
\downarrow \delta \\
Y' \\
\downarrow \\downarrow \\downarrow \\
Z
\end{array}
\]

so that by uniqueness \((h, k, \eta)\) and \((h', k', \eta')\) are equal (using that Axiom 1 gives strict uniqueness when one requires \( \eta \) and \( \eta' \) to be representative) and \( \theta \alpha \) and \( \varphi \beta \) are identities. Pasting both sides above with \( \alpha \) and \( \beta \) gives the equality

\[
\begin{array}{c}
X \\
\downarrow^c \\
Y \\
\downarrow \delta \\
Y' \\
\downarrow \\downarrow \\downarrow \\
Z
\end{array}
= \begin{array}{c}
X \\
\downarrow^c \\
Y \\
\downarrow \delta \\
Y' \\
\downarrow \\downarrow \\downarrow \\
Z
\end{array}
\]

so that \( \alpha \theta \) and \( \beta \varphi \) are also identities by genericity of \( \xi \). □

2.7. **Generics are indexed by classes of left adjoint 1-cells.** One of the more interesting properties of a generic bicategory \( \mathcal{C} \) is that the 2-cells in \( \mathcal{C} \) determine composition in \( \mathcal{C} \). More explicitly, one can define for each triple of objects \( X, Y, Z \) and 1-cell \( c: X \to Z \) in \( \mathcal{C} \), the set \( \mathcal{M}^{X,Y,Z}_c \) of equivalence classes of generic 2-cells \( \delta: c \Rightarrow l; r \) factoring through \( Y \). It is then not hard to see that this defines a presheaf \( \mathcal{M}^{X,Y,Z}_c: \mathcal{C}_{X,Z} \to \text{Set} \). Moreover, given an element of this presheaf (that is a generic 2-cell \( \delta: c \Rightarrow l; r \)) we have the canonical projections \( \delta \mapsto (l, r) \) and \( \delta \mapsto c \) yielding functors as below

\[
\begin{array}{c}
\mathcal{C}_{X,Y} \times \mathcal{C}_{Y,Z} \Rightarrow \mathcal{C}_{X,Z} \\
\text{el} \mathcal{M}^{X,Y,Z}_c
\end{array}
\]

It then a fact about generic bicategories that the composition functor defines an absolute left extension above\(^4\). Thus if we have a description of the generic 2-cells, or at least a way of keeping track of them, we may deduce the composition operation within our bicategory. It will therefore be useful to find a way of indexing our generic 2-cells, in the hope of later using this indexing to show composition in \( \mathcal{C} \) is by pullback.

\(^4\)Actually, this is an instance of a more general fact which is true for any familial functor.
**Lemma 2.7.1.** Assume \( \mathcal{C} \) is a generic bicategory satisfying axioms 1 and 2. Suppose \( c \) is a 1-cell with initial generic \( \delta : c \Rightarrow l; r \). Then a 2-cell \( c \Rightarrow a; b \) in \( \mathcal{C} \) is generic, if and only if it is (isomorphic to) a whiskering of a unit \( \eta \) of an adjunction by \( l \) and \( r \).

**Proof.** (\( \Rightarrow \)) : It is clear that any generic 2-cell \( c \Rightarrow a; b \) is a whiskering of a unit by \( l \) and \( r \), since we may factor such a 2-cell through the initial generic as

\[
\begin{array}{c}
X \\
\downarrow \delta \\
\downarrow \gamma \\
\downarrow \alpha \\
\downarrow \beta \\
Y \\
\downarrow \eta \\
\downarrow \theta \\
\downarrow \phi \\
\downarrow \psi \\
Z
\end{array}
\]

and note the induced \( \alpha \) and \( \beta \) are invertible, as \( c \Rightarrow (l; h); (k; r) \) is generic by Lemma 2.6.1.

(\( \Leftarrow \)) : Suppose now we are given a unit \( \eta \) whiskered by \( l \) and \( r \) giving a diagram as below

\[
\begin{array}{c}
X \\
\downarrow \delta \\
\downarrow \gamma \\
\downarrow \alpha \\
\downarrow \beta \\
Y \\
\downarrow \eta \\
\downarrow \theta \\
\downarrow \phi \\
\downarrow \psi \\
Z
\end{array}
\]

We factor the above through this initial generic, yielding the same diagram. We then note that \( (l; r) \Rightarrow (l; h); (k; r) \) is generic by Lemma 2.6.1.

We can now apply the above lemma to give a simple description of our indexing sets \( \mathfrak{M}^{X,Y',Z}_c \) of generic 2-cells.

**Lemma 2.7.2.** Assume \( \mathcal{C} \) is a generic bicategory satisfying axioms 1 and 2. Then for any 1-cell \( c : X \to Z \) (with a chosen initial generic \( l; r \) through an object \( Y \)) and object \( Y' \), the set of equivalence classes of generic 2-cells out of \( c \) and through \( Y' \), denoted \( \mathfrak{M}^{X,Y',Z}_c \), is the set of equivalence classes of left adjoints \( h : Y \to Y' \).

**Proof.** Suppose we are given a 1-cell \( c : X \to Z \) with a chosen initial generic \( l; r \) through an object \( Y \). We regard \( l \) and \( r \) as being fixed.

We see that from any generic 2-cell out of \( c \), we recover by Lemma 2.7.1, a 1-cell \( h : Y \to Y' \), a right adjoint \( k : Y' \to Y \) and a unit \( \eta : 1 \Rightarrow kh \). Conversely, given such a triple \( (h, k, \eta) \) we recover a generic 2-cell out of \( c \) by whiskering.

Let us suppose we have two isomorphic generic 2-cells out of \( c \), then one is isomorphic to a whiskering of data \( (h, k, \eta) \) and the other a whiskering of \( (h', k', \eta') \), and thus we have isomorphisms \( \alpha \) and \( \beta \) yielding an equality as below

\[
\begin{array}{c}
X \\
\downarrow \delta \\
\downarrow \gamma \\
\downarrow \alpha \\
\downarrow \beta \\
Y \\
\downarrow \eta \\
\downarrow \theta \\
\downarrow \phi \\
\downarrow \psi \\
Z
\end{array}
\]

\[
\begin{array}{c}
X \\
\downarrow \delta \\
\downarrow \gamma \\
\downarrow \alpha \\
\downarrow \beta \\
Y \\
\downarrow \eta \\
\downarrow \theta \\
\downarrow \phi \\
\downarrow \psi \\
Z
\end{array}
\]

By uniqueness, this induces comparison isomorphisms \( h \cong h' \) and \( k \cong k' \) compatible with \( \eta \). Indeed, denoting the induced isomorphism \( h \Rightarrow h' \) by \( \theta \) (and forgetting the
induced $k \Rightarrow k')$ we may take $\xi: k' \Rightarrow k$ to be the composite

$$k' \xrightarrow{\eta k'} k h k' \xrightarrow{kh'k'} k h' k' \xrightarrow{\xi} k$$

and one can verify that $\eta'$ pasted with $\theta^{-1}$ and $\xi$ is $\eta$, so that from an isomorphism $h \Rightarrow h'$ we recover an isomorphism of generic 2-cells $(h, k, \eta) \cong (h', k', \eta')$. It follows the assignment sending a generic 2-cell to its representative triple then to its left adjoint component

generic 2-cell $\Rightarrow (h, k, \eta) \Rightarrow h$

reflects and preserves isomorphisms. Thus we are to identify two generic 2-cells precisely when the left adjoint components are isomorphic (that is equal in their equivalence class). □

Remark 2.7.3. Note that a representative generic corresponding to a $h: Y \Rightarrow Y'$ has the form $c \Rightarrow (l; h); (k; r)$, so that the left and right projections are $l; h$ and $k; r$ respectively. This then defines the projections from $\mathcal{M}_c^{X,Y',Z}$ to $\mathcal{C}_{X,Y'}$ and $\mathcal{C}_{Y',Z}$.

2.8. The main theorem. We now have all of the necessary ingredients to prove our characterization of spans. We will first show that for a generic $\mathcal{C}$ satisfying axioms 1 and 2 the hom-categories must be equivalent to hom-categories of spans, and then proceed by using the generic bicategory structure (and our indexing of the generics) to deduce that composition must be given by pullback.

Theorem 2.8.1. The following are equivalent for any given bicategory $\mathcal{C}$:

(1) $\mathcal{C}$ is equivalent to a bicategory $\text{Span}(E)$ for a category $E$ with pullbacks;
(2) $\mathcal{C}$ is generic and satisfies axioms 1 and 2.

Proof. The implication $(1) \Rightarrow (2)$ was shown in the introduction, so it remains to check $(2) \Rightarrow (1)$. Let $X$ and $Z$ be given objects. We take $E$ to be the 1-category of representative left adjoints given by Corollary 2.5.2, and define the functor $\mathcal{C}_{X,Z} \rightarrow \text{Span}(E)_{X,Z}$ by the assignment sending a 1-cell $c$ with chosen initial generic $c \Rightarrow l; r$ to the span

$$
\begin{array}{ccc}
X & \xrightarrow{\eta} & Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\phi} & R
\end{array}
$$

where $l_*$ is a representative left adjoint of $l$. To assign a 2-cell $\alpha: c \Rightarrow c'$ to a morphism of spans, we first factor $\alpha$ through the initial generic $c \Rightarrow l; r$ giving a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\eta} & Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\phi} & R
\end{array}
$$

(2.2)

We then note that we have (from pasting $\theta$ with $\eta$, and $\eta$ with $\varphi$ respectively) 2-cells $l \Rightarrow l'; k$ and $r \Rightarrow h; r'$. As morphisms of left adjoints correspond to morphisms of right adjoints in the opposite direction by the mates correspondence, we also have 2-cells $h; l_* \Rightarrow l_*$ and $r'^*; k \Rightarrow r^*$ respectively. In particular, as the 2-cells of left
adjoints \( r \Rightarrow h; r' \) and \( h; l'_* \Rightarrow l_* \) collapse to identities in \( \mathcal{E} \), we have a commuting diagram

\[
\begin{array}{ccc}
X & \xleftarrow{r} & Y \\
\downarrow{l} & & \downarrow{r} \\
Y' & \xleftarrow{r'} & Z
\end{array}
\]

Moreover, this assignment is fully faithful since given a morphism of spans as in (2.3), we may recover a diagram (2.2) by taking \( \varphi \) as the mate of \( r \cong h; r' \) under the adjunction \( h \dashv k \), and \( \theta \) as the mate of \( h; l'_* \cong l_* \) under the adjunctions \( l_* \dashv l \) and \( l'_* \dashv l' \). It is clear this application of the mates correspondence defines a bijection.

To see that this assignment is essentially surjective, suppose we are given a span

\[
\begin{array}{ccc}
X & \xleftarrow{s} & T \\
\downarrow{l} & & \downarrow{t} \\
Y & \xleftarrow{r} & Z
\end{array}
\]

and note that we have the initial generic \( (s^*; t) \Rightarrow (s^*; t) \) by Lemma 2.4.1. By universality of the chosen initial generic \( l; r \) we have an induced comparison into \( s^*; t \) as on the top half below

\[
\begin{array}{ccc}
x & \xleftarrow{l} & y \\
\downarrow{\phi} & & \downarrow{\eta} \\
y & \xleftarrow{r} & z
\end{array}
\]

and conversely by universality of \( s^*; t \) we have an induced comparison into \( l; r \) as on the bottom half below. By uniqueness (and the fact \( 1_Y \Rightarrow 1_Y; 1_Y \) is a generic 2-cell) it follows that \( h_1; h_2 \cong 1_Y \) in \( \mathcal{C} \), meaning that \( h_1; h_2 = 1_Y \) in \( \mathcal{E} \). Clearly, we also have the dual so that \( h_2; h_1 = 1_T \), and thus applying our functor to the top half of (2.4) yields an isomorphism of spans \( (l, r) \cong (s, t) \), proving essential surjectivity.

We now note that the bicategory structure on \( \mathcal{E} \) transports through the equivalences \( \mathcal{C}_{X, Z} \rightarrow \text{Span}(\mathcal{E})_{X, Z} \) by Doctrinal adjunction [3] (viewing bicategories as the objects of the 2-category of bicategories, pseudofunctors, and icons [5]), and thus the family of hom-categories \( \text{Span}(\mathcal{E})_{X, Y} \) admits the structure of a bicategory with composition given by

\[
\text{Span}(\mathcal{E})_{X, Y} \times \text{Span}(\mathcal{E})_{Y, Z} \longrightarrow \mathcal{C}_{X, Y} \times \mathcal{C}_{Y, Z} \longrightarrow \mathcal{C}_{X, Z} \longrightarrow \text{Span}(\mathcal{E})_{X, Z}
\]

\[
(a, b), (c, d) \mapsto (a^*; b), (c^*; d) \mapsto l^*; r \mapsto (l, r)
\]

where \( l^*; r \) is a chosen initial generic for \( (a^*; b), (c^*; d) \), and \( l \) is a representative for the left adjoints to \( l^* \). We know composition in \( \mathcal{E} \) satisfies, for any initial generic \( s^*; t : X \rightarrow T \rightarrow Z \) and initial generics \( a^*; b : X \rightarrow Y \) and \( c^*; d : Y \rightarrow Z \), the natural bijection

\[
\mathcal{C}_{X, Z}[(s^*; t), (a^*; b), (c^*; d)] \cong \sum_{m \in \text{Mor}_{\mathcal{C}}^{X, Z}} \mathcal{C}_{X, Y}(l_m, (a^*; b)) \times \mathcal{C}_{X, Y}(r_m, (c^*; d))
\]
so that by Lemma 2.7.2
\[ C_{X,Z}[(s^*;t),(l^*;r)] \cong \sum_{h:T \to Y} C_{X,Y}((s^*;h),(a^*;b)) \times C_{X,Y}((h^*;t),(c^*;d)) \]
which says for all \( s \) and \( t \) in \( E \), giving a morphism of spans (that is a morphism into the vertex of \((l,r)\) as on the left below) is the same as giving a cone

\[
\begin{array}{c}
X \\
\downarrow s \\
T \\
\downarrow h \\
M \\
\downarrow t \\
Z \\
\end{array}
\quad
\begin{array}{c}
X \\
\downarrow s \\
T \\
\downarrow h \\
P \\
\downarrow c \\
Y \\
\downarrow b \\
Q \\
\downarrow d \\
Z \\
\end{array}
\]

as on the right above, thus showing that composition is by pullback (after taking \((s,t)\) to be \((l,r)\) in order to recover the limiting cone). \( \square \)

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