NAKAI-MOISHEZON CRITERIONS FOR COMPLEX HESSIAN EQUATIONS

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ABSTRACT. The $J$-equation proposed by Donaldson is a complex Hessian quotient equation on Kähler manifolds. The solvability of the $J$-equation is proved by Song-Weinkove to be equivalent to the existence of a subsolution. It is also conjectured by Lejmi-Szekelyhidi to be equivalent to a stability condition in terms of holomorphic intersection numbers as an analogue of the Nakai-Moishezon criterion in algebraic geometry. The conjecture is recently proved by Chen under a stronger uniform stability condition. In this paper, we establish a Nakai-Moishezon type criterion for pairs of Kähler classes on analytic Kähler varieties. As a consequence, we prove Lejmi-Szekelyhidi’s original conjecture for the $J$-equation. We also apply such a criterion to obtain a family of constant scalar curvature Kähler metrics on smooth minimal models.

1. INTRODUCTION

The $J$-equation is the critical equation of the $J$-flow introduced by Donaldson [11] in the framework of moment maps. Let $X$ be an $n$-dimensional compact Kähler manifold with two Kähler classes $\alpha$ and $\beta$ satisfying the normalization condition

$$\alpha^n = \alpha^{n-1} \cdot \beta,$$

or equivalently

$$\int_X \alpha^n = \int_X \alpha^{n-1} \wedge \beta,$$

where the integral on the right hand side of (1.2) is calculated by choosing any Kähler forms in $\alpha$ and $\beta$. For a given Kähler form $\chi \in \beta$, the $J$-equation is defined by

$$\omega^n = \omega^{n-1} \wedge \chi,$$

where $\omega \in \alpha$ is the desired Kähler form. The $J$-equation (1.3) can also be expressed as the complex Hessian quotient equation

$$tr_\omega(\chi) = \frac{n \omega^{n-1} \wedge \chi}{\omega^n} = n.$$

At any given point $p \in X$, we can diagonalize both $\chi$ and $\omega$ such that $\chi$ is an identity matrix of size $n \times n$ and $\omega$ has positive eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Then equation (1.3) becomes

$$\sigma_{n-1}(\Lambda) \sigma_n(\Lambda) = n,$$

where $\Lambda = \{\lambda_j\}_{j=1}^n$ and $\sigma_k$ is the $k$-th elementary symmetric polynomial of $\Lambda$ of degree $k$.

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Both the $J$-equation and the $J$-flow are extensively studied in [3, 4, 22, 23, 19, 13, 20, 16, 7, 21, 12, 6]. The convergence of the $J$-flow is first proved by Weinkove in [22] to be equivalent to the class condition $2\alpha - \beta > 0$ for Kähler surfaces. The general case is proved by Song-Weinkove in [19] by the $J$-flow that (1.3) admits a smooth solution $\omega$ if and only if there exists a Kähler form $\Omega \in \alpha$ such that

$$n\Omega^n - (n - 1)\Omega^{n-1} \wedge \chi > 0$$

as a positive $(n - 1, n - 1)$-form on $X$. The analytic positive condition (1.4) is later interpreted in [21] as a subsolution of the $J$-equation. This immediately implies that there is obstruction to solve the global $J$-equation. There are many well-known examples such as certain pairs of Kähler classes on Hirzebruch surfaces for which the $J$-equation does not admit smooth solutions (c.f. [12]).

The pointwise condition (1.4) is however difficult to verify, and one would like to replace it by a holomorphic/topological condition. Of course, (1.4) always holds whenever $n\alpha - (n - 1)\beta > 0$ is a Kähler class [23]. This leads to following conjecture proposed by Lejmi-Szekelyhidi [16].

**Conjecture 1.1.** Under the normalization condition (1.1), the $J$-equation (1.3) admits a unique smooth solution $\omega$ if and only for any $m$-dimensional analytic subvariety $Z$ of $X$ with $1 \leq m \leq n - 1$,

$$\left(n\alpha^m - m\alpha^{m-1} \cdot \beta\right) \cdot Z = \int_Z \left(n\alpha^m - m\alpha^{m-1} \wedge \beta\right) > 0.$$  

Conjecture 1.1 is proved for toric Kähler manifolds in [7]. Recently, a major progress is made toward this conjecture by Chen [6], proving Conjecture 1.1 under a slightly stronger topological condition (uniform $J$-positivity, see Definition 1.1 (3)). Chen’s result is generalized by Datar-Pingali [8] to a more general family of complex Hessian equations. Conjecture 1.1 is also proved in [8] for projective Kähler manifolds and positive line bundles.

The main goal of this paper is to prove Conjecture 1.1 and it suffices to construct a subsolution satisfying the analytic positive condition (1.4). It is crucial to find the connection between (1.4) and (1.5) as an analogue of the Nakai-Moishezon criterion for a Kähler class on a Kähler manifold established by Demailly-Paun [9]. This leads us to prove a Nakai-Moishezon type criterion for the $J$-equation. The following is the main result of the paper.

**Theorem 1.1.** Let $X$ be an $n$-dimensional compact analytic variety embedded in a Kähler manifold $\mathcal{M}$. Let $\alpha$ and $\beta$ be two Kähler classes in an open neighborhood of $X$ in $\mathcal{M}$ satisfying

$$\left.\frac{\alpha^{n-1} \cdot \beta}{\alpha^n}\right|_Y = \frac{\int_Y \alpha^{n-1} \wedge \beta}{\int_Y \alpha^n} \leq 1$$

for any component $Y$ of $X$. Then the following two conditions are equivalent.
(1) For any $m$-dimensional analytic subvariety $Z$ of $X$,
\[
\left( n\alpha^m - m\alpha^{m-1} \cdot \beta \right) \cdot Z = \int_Z \left( n\alpha^m - m\alpha^{m-1} \wedge \beta \right) > 0,
\]
where $1 \leq m < n$.

(2) For any smooth Kähler form $\chi \in \beta$ in an open neighborhood $U$ of $X$, there exists a smooth Kähler form $\omega \in \alpha$ in some open neighborhood $V \subset U$ of $X$ such that in $V$,
\[
n\omega^{n-1} - (n-1)\omega^{n-2} \wedge \chi > 0.
\]

In Theorem 1.1, $\mathcal{M}$ can be open or incomplete. In particular, we can let $\mathcal{M} = X$ if $X$ is a Kähler manifold, and the following corollary is immediate after suitable rescaling.

**Corollary 1.1.** Let $X$ be an $n$-dimensional compact Kähler manifold and let $\alpha$ and $\beta$ be two Kähler classes on $X$. If for any $m$-dimensional analytic subvariety $Z$ of $X$ with $1 \leq m \leq n-1$,
\[
m \frac{\alpha^{m-1} \cdot \beta}{\alpha^m} \bigg|_Z < n \frac{\alpha^{n-1} \cdot \beta}{\alpha^n},
\]
then for any smooth Kähler form $\chi \in \beta$, there exists a smooth Kähler form $\omega \in \alpha$ such that at any point $p \in X$ and $m$-dimensional subspace $W$ of $T_pX$
\[
m \frac{\omega^{m-1} \wedge \chi}{\omega^m} \bigg|_W < n \frac{\alpha^{n-1} \cdot \beta}{\alpha^n}.
\]

We remark that
\[
tr_{\omega|_W}(\chi|_W) = m \frac{\omega^{m-1} \wedge \chi}{\omega^m} \bigg|_W
\]
for $1 \leq m \leq n-1$ and (1.7) is equivalent to
\[
n \frac{\alpha^{n-1} \cdot \beta}{\alpha^n} \omega^{n-1} - (n-1)\omega^{n-2} \wedge \chi > 0
\]
as an $(n-1,n-1)$-form on $X$.

The condition (1.6) is analogous to the topological slope condition for Hermitian vector bundles over Kähler manifolds, while condition (1.7) is the corresponding pointwise slope condition. Therefore, we define the following positive conditions as an analogue of the Nakai-Moishezon Criterion.

**Definition 1.1.** Let $X$ be an $n$-dimensional compact Kähler manifold with two Kähler classes $\alpha$ and $\beta$.

(1) The pair $(\alpha, \beta)$ is said to be $J$-positive if for any $m$-dimensional analytic subvariety $Z$ of $X$,
\[
m \frac{\alpha^{m-1} \cdot \beta}{\alpha^m} \bigg|_Z < n \frac{\alpha^{n-1} \cdot \beta}{\alpha^n}.
\]
(2) The pair is said to be $J$-nef if for any $m$-dimensional subvariety $Z$ of $X$,

$$m \frac{\alpha^{m-1} \cdot \beta}{\alpha^m} \bigg|_Z \leq n \frac{\alpha^{n-1} \cdot \beta}{\alpha^n}.$$ 

(3) The pair is said to be uniformly $J$-positive if there exists $\varepsilon > 0$ such that for any $m$-dimensional analytic subvariety $Z$ of $X$,

$$m \frac{\alpha^{m-1} \cdot \beta}{\alpha^m} \bigg|_Z \leq (n - \varepsilon) \frac{\alpha^{n-1} \cdot \beta}{\alpha^n}.$$

Obviously, (3) $\Rightarrow$ (1) $\Rightarrow$ (2) and Definition 1.1 can be generalized to analytic Kähler varieties. The uniform $J$-positive condition (3) is introduced and proved by Chen [6] to be equivalent to the solvability of the $J$-equation. Our next result settles the original conjecture of Lejmi-Szekelyhidi (Conjecture 1.1).

**Corollary 1.2.** Let $X$ be an $n$-dimensional Kähler manifold with two Kähler classes $\alpha$ and $\beta$. For any Kähler form $\chi \in \beta$, the $J$-equation

$$(1.8) \quad \frac{\omega^{n-1} \wedge \chi}{\omega^n} = \frac{\alpha^{n-1} \cdot \beta}{\alpha^n}$$

admits a unique smooth solution $\omega \in \alpha$ if and only if the pair $(\alpha, \beta)$ is $J$-positive as in Definition 1.1.

The proof of Theorem 1.1 follows the road map in [6] based on the techniques from [9]. The key idea of [6] is to apply Blocki-Kolodziej’s gluing trick by local regularization [1] instead of the global regularization in [9]. The approach of [6] relies on the induction argument for smooth subvarieties of $X$. Our approach removes the smoothness assumption and has to work on the degenerate $J$-equation instead. We expect that Theorem 1.1 can be generalized to a family of complex Hessian quotient equations by applying the work in [8].

We would also like to extend Conjecture 1.1 to the singular case.

**Conjecture 1.2.** Let $X$ be an $n$-dimensional normal Kähler variety with two Kähler classes $\alpha$ and $\beta$. If $(\alpha, \beta)$ is $J$-positive, then for any Kähler form $\chi \in \beta$, the $J$-equation

$$\omega^{n-1} \wedge \chi = \left(\frac{\alpha^{n-1} \cdot \beta}{\alpha^n}\right) \omega^n$$

admits a unique solution $\omega$ as a Kähler current with bounded local potentials.

We can further extend the above conjecture to any $J$-nef pair $(\alpha, \beta)$ on $X$. We expect that the solution $\omega$ should be a Kähler current with vanishing Lelong number similar to degenerate complex Monge-Ampere equations studied in [2]. A viscosity approach in [10] might also be helpful to understand weak solutions of the degenerate global $J$-equation. We would like to remark that Theorem 1.1 and Definition 1.1 can indeed be generalized for certain special degenerate case. For example, if $\beta$ is a
semi-positive class (i.e., there exists a smooth closed semi-positive (1, 1)-form in $\beta$) satisfying $\alpha^{n-1} \cdot \beta > 0$, then Theorem 1.1 should still hold.

The more interesting question is what canonical solutions one expects for the $J$-equation if the $J$-positive condition fails. The natural approach is to apply the $J$-flow, especially for projective manifolds with large symmetry, as suggested by the author and investigated by Fang-Lai [12]. The interesting examples constructed in [12] suggest a natural analogy between the $J$-flow and the Yang-Mills flow in terms of formation of singularities. Let $\omega \in \alpha$ and $\chi \in \beta$ be two Kähler forms on an $n$-dimensional compact Kähler manifold $X$. Then the $J$-flow is defined as below.

\[
\frac{\partial \varphi}{\partial t} = c - \frac{(\omega + \sqrt{-1} \partial \overline{\partial} \varphi)^{n-1} \wedge \chi}{(\omega + \sqrt{-1} \partial \overline{\partial} \varphi)^n}, \quad \varphi(0) = 0,
\]

where $c = \frac{\alpha^{n-1} \cdot \beta}{\alpha^n}$. We conjecture that the $J$-flow always converges smoothly outside an analytic subvariety $Z$ of $X$. The limiting solution $\omega_\infty$ on $X \setminus Z$ satisfies a $J$-equation

\[
\frac{\omega_\infty^{n-1} \wedge \chi}{\omega_\infty^n} = c' > 0,
\]

where $c'$ is not necessarily equal to $c$ unless the pair $(\alpha, \beta)$ is $J$-nef. Furthermore, the limiting solution $\omega_\infty$ uniquely extends to a global Kähler current on $X$. A suitable restriction of this Kähler current on each component of $Z$ would still satisfy a $J$-equation, possibly degenerate. Such a phenomena is possibly related to the slope stability through a filtration of analytic subvarieties of $X$.

Finally, we extend the result of [23, 19, 14] for constant scalar curvature Kähler metrics on smooth minimal models.

**Theorem 1.2.** Let $X$ be an $n$-dimensional smooth minimal model, i.e., $X$ is a compact Kähler manifold with nef $K_X$. Let $\gamma$ be a Kähler class and $\alpha(X, \gamma)$ be the $\alpha$-invariant (c.f. (8.1)) for $\gamma$ on $X$. If there exists $\epsilon \in \left[0, \frac{n+1}{n} \alpha(X, \gamma)\right]$ such that for any $m$-dimensional analytic subvariety $Z$ of $X$

\[
m \frac{\gamma^{m-1} \cdot K_X}{\gamma^m} \bigg|_Z \leq (n + (n-m)\epsilon) \frac{\gamma^{n-1} \cdot K_X}{\gamma^n},
\]

then there exists a unique cscK metric (constant scalar curvature Kähler metric) in $\gamma$.

Theorem 1.2 immediately implies the result of [14, 18] on the existence of cscK metrics near the canonical class on a smooth minimal model because any sufficiently small perturbation of $K_X$ by a Kähler class will satisfy the assumption in Theorem 1.2.

**Corollary 1.3.** Let $X$ be an $n$-dimensional smooth minimal model. Then for any Kähler class $\gamma$, there exists a cscK metric in $K_X + \epsilon \gamma$ for any sufficiently small $\epsilon > 0$.

Let us give a brief outline of the paper. We first give a direct proof of Conjecture 1.1 for dimension 3 in §2 by combining the work of Chen [6] and solutions to degenerate
complex Monge-Ampere equations. In §3, we will start the proof of Theorem 1.1 with basic set-up. The main ingredients of the proof for Theorem 1.1 are established in §4, §5 and §6. We complete the proof for the main results of the paper in §7. Finally, we prove Theorem 1.2 in §8.

2. Lejmi-Szekelyhidi’s conjecture for dim = 3

In this section, we will directly prove Corollary 1.2 for dimension 3 without using Theorem 1.1. The proof will be different from the general case. Our approach here is to solve degenerate $J$-equations with prescribed singularities on Kähler surfaces. Then the argument of Chen ([6]) can be directly extend to the case of $J$-positivity from uniformly $J$-positivity and Lejmi-Szekelyhidi’s original conjecture immediately follows. We rephrase Corollary 1.2 in the case of dim $X = 3$ as below.

**Theorem 2.1.** Let $X$ be a 3-dimensional Kähler manifold and let $\alpha$ and $\beta$ be two Kähler classes on $X$ with $\alpha^2 \cdot \beta = \alpha^3$. If the pair $(\alpha, \beta)$ is $J$-positive (i.e., for any proper subvariety $Z$ of $X$,

$$(3\alpha - \beta) \cdot Z > 0$$

if dim $Z = 1$ and

$$(3\alpha^2 - 2\alpha \cdot \beta) \cdot Z > 0$$

if dim $Z = 2$), then for any Kähler form $\chi \in \beta$, there exists a Kähler form $\omega \in \alpha$ satisfying the $J$-equation

$$\omega^2 \wedge \chi = \omega^3$$

on $X$.

Our goal is to construct a subsolution of the $J$-equation (2.1) in an open neighborhood of any subvariety $Z$ of $X$. Let $\omega_0 \in \alpha$ and $\chi \in \beta$ be two fixed Kähler forms on $X$.

**Lemma 2.1.** Let $Z$ be an analytic subvariety of $X$ with dim $Z \leq 1$. Then there exists an open neighborhood $U$ of $Z$ and a smooth Kähler metric $\omega_U \in [\omega]|_U$ such that

$$3(\omega_U)^2 - 2\omega_U \wedge \chi > 0$$

in $U$.

**Proof.** If dim $Z = 0$, the lemma is obvious since every Kähler class in a sufficiently small neighborhood of a point is trivial.

If dim $Z = 1$, any two Kähler classes on $Z$ are proportional to each other and by the extension theorem (c.f. Proposition 3.3 [9]), there exist a neighborhood $U_Z$ of $Z$ and a Kähler form $\omega_Z \in \alpha|_{U_Z}$ such that

$$3\omega_Z - (1 + 2\epsilon)\chi > 0$$

for some $\epsilon > 0$. Let $S_Z$ be the singular set of $Z$. Then there exist a sufficiently small $\delta > 0$ and $\varphi \in C^\infty(U_Z \setminus S_Z) \cap \text{PSH}(U_Z, \omega)$ such that

$$3(\omega_Z + \sqrt{-1} \partial \overline{\partial} \varphi) - (1 + \epsilon)\chi > 0$$
and \( \varphi \) has Lelong number greater than \( \delta \) at \( S_Z \).

For simplicity, we assume \( S_Z \) is a single point \( p \) of \( Z \) and components of \( Z^o = Z \setminus \{ p \} \) are smooth open curves in a neighborhood \( U_p \) of \( p \) in \( X \). We assume there exists \( \varphi_p \in C^\infty(U_p) \) with

\[
3(\omega_p)^2 > 2\omega_p \wedge \chi, \ \omega_p = \omega_Z + \sqrt{-1}\partial \bar{\partial} \varphi_p
\]

where \( U_p \) is a sufficiently small neighborhood of \( p \). By subtracting a sufficiently large number from \( \varphi_p \), we can assume

\[
\varphi_p < \varphi - 2
\]

on \( U_p \setminus V_p \) for some sufficiently small open neighborhood \( V_p \subset U_p \) of \( p \). Since \( \varphi \to -\infty \) at \( p \), there exists a neighborhood \( W_p \subset V_p \) of \( p \) such that

\[
\varphi_p > \varphi + 2
\]

in \( W_p \). Let \( Z_1, Z_2, ..., Z_k \) be all the components of \( Z \) in \( U_p \). We replace \( \varphi \) in \( U_Z \setminus W_p \) by

\[
\tilde{\varphi} = \varphi + Ad^2,
\]

where \( d \) is the distance function to \( Z_1, ..., Z_k \) with respect to \( \omega_Z \). For sufficiently large \( A > 0 \), \( \tilde{\omega}_Z = \omega_Z + \sqrt{-1}\partial \bar{\partial} \tilde{\varphi} \) is a Kähler form and

\[
3\tilde{\omega}^2 > (2 + \epsilon)\tilde{\omega} \wedge \chi
\]

in some sufficiently small open neighborhood \( \tilde{U} \) of \( (Z_1 \cup ... \cup Z_k) \cap (U_p \setminus W_p) \). We can take the regularized maximum of \( \tilde{\varphi} \) and \( \varphi_p \) in \( \tilde{U} \) and denote it by \( \varphi_Z \). The lemma is then proved by choosing \( \omega_U = \omega + \sqrt{-1}\partial \bar{\partial} \varphi_Z \) for some sufficiently small open neighborhood \( U \) of \( Z \).

\( \square \)

Now let \( Z \) be an analytic subvariety of \( X \) with \( \dim Z = 2 \). By the assumption of the \( J \)-positive condition,

\[
3\alpha - \beta > 0
\]

is a Kähler class on \( Z \) and

\[
(3\alpha^2 - 2\alpha \cdot \beta) \cdot Z' > 0
\]

for any irreducible component \( Z' \) of \( Z \). Let \( Z' \) be an irreducible component of \( Z \) and \( S_Z \) be the singular set of \( Z \). Let \( \Phi : X' \to X \) be the resolution of singularities for \( Z \). We can assume the exceptional locus of \( \Phi \) is a union of smooth curves and it coincides with \( \Phi^{-1}(S_Z) \). We also let \( \hat{Z} \) be the strict transform of \( Z' \) by \( \Phi \).

For conveniences, we identify \( \Phi^*\omega_0 \) and \( \Phi^*\chi \) with \( \omega_0 \) and \( \chi \). We also identify \( \Phi^*\alpha \) and \( \Phi^*\beta \) with \( \alpha \) and \( \beta \).

There exists an effective \( \mathbb{Q} \)-divisor \( E \) whose support coincides with the exceptional locus of \( \Phi \) such that \( \alpha - [E] \) is a Kähler class on \( X' \). Let \( \sigma_E \) and \( h_E \) be the defining
section and hermitian metric on the line bundle associated to $[E]$ such that for any sufficiently small $\epsilon > 0$
\[3\omega_0 - \chi + \epsilon \sqrt{-1} \partial \bar{\partial} \log h_E\]
is Kähler on $\hat{Z}$. There exists $\varphi_\epsilon \in C^\infty(\hat{Z})$ such that
\[(3\omega_0 - \chi + \epsilon \sqrt{-1} \partial \bar{\partial} \log h_E + 3\sqrt{-1} \partial \bar{\partial} \varphi_\epsilon)^2 = c_\epsilon \chi^2,
\]
where the constant $c_\epsilon > 0$ is given by
\[c_\epsilon \beta^2 = ((3\alpha - \beta) - \epsilon[E])^2.
\]
By choosing sufficiently small $\epsilon > 0$, we can assume that $c_\epsilon > 1$
\[(3\alpha - \beta)^2 - \beta^2 = 3(3\alpha^2 - 2\beta \cdot \alpha) > 0
\]
on $\hat{Z}$. By letting
\[(2.4) \quad \varphi_Z = \varphi_\epsilon + \log |\sigma_E|_{h_E}^2, \quad \omega_Z = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_Z,
\]
we have on $\hat{Z} \setminus \Phi^{-1}(S_Z)$,
\[(2.5) \quad 3\omega_Z^2 - 2\omega_Z \wedge \chi = \frac{(3\omega_Z - \chi)^2 - \chi^2}{3} = \frac{c_\epsilon - 1}{3} \chi^2 > 0.
\]
Furthermore, $\omega_Z$ has positive Lelong number along $E$.

**Lemma 2.2.** Suppose $Z$ is an analytic subvariety of $X$ of dimension 2. Then there exists an open neighborhood $U$ of $Z$ and a smooth Kähler metric $\omega_U \in \alpha|_U$ such that $3(\omega_U)^2 - 2\omega_U \wedge \chi > 0$ on $U$.

**Proof.** Let $Z_1, ..., Z_k$ be the irreducible components of $Z \setminus S_Z$. For simplicity, we assume $k = 1$ and $S_Z$ is connected. By Lemma 2.1, there exists an open neighborhood $V$ of $S_Z$ and $\omega_V = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_V \in \alpha|_V$ for some $\varphi_V \in C^\infty(V) \cap \text{PSH}(V, \omega)$ such that $\omega_V > \chi$ in $V$ and so
\[3(\omega_V)^2 - 2\omega_V \wedge \chi > 0.
\]
We will then extend $\varphi_Z$ given in (2.4) using the square of the distance function as in the proof of Lemma 2.1. The rest of the proof is similar to that of Lemma 2.1. We let $\varphi_Z$ be the regularized maximum of $\varphi_V$ and the extension of $\varphi_Z$. The lemma is proved by letting $\omega_Z = \omega + \sqrt{-1} \partial \bar{\partial} \Phi(\varphi_Z)$ since $\omega_Z$ coincides with $\omega_V$ in some open neighborhood of $S_Z$. \qed
Now we can directly apply the result of Chen [6] to prove Theorem 2.1. The induction argument in [6] is based on the assumption of Lemma 2.1 and Lemma 2.2 if the subvariety $Z$ is smooth, where the uniform $J$-positivity is used instead of the $J$-positivity as in Definition 1.1. As long as Lemma 2.1 and Lemma 2.2 hold, the argument in section 4 of [6] immediately implies Theorem 2.1.

3. Set-up

In this section, we will lay out the proof of Theorem 1.1. For conveniences, we will focus on the case when $X$ is irreducible. The proof can be easily generalized for reducible situations and we will explain in §7. The rest of this section together with §4, §5, §6 and §7 are devoted to the proof of the following theorem.

**Theorem 3.1.** Let $X$ be an irreducible $n$-dimensional compact analytic subvariety of an ambient Kähler manifold $\mathcal{M}$. Let $\alpha$ and $\beta$ be two Kähler classes on $X$ satisfying

\begin{equation}
\frac{\alpha^{n-1} \cdot \beta \cdot X}{\alpha^n \cdot X} \leq 1.
\end{equation}

Suppose for any $m$-dimensional subvariety $Z$ of $X$ with $1 \leq m \leq n-1$,

\begin{equation}
(n\alpha^m - m\alpha^{m-1} \cdot \beta) \cdot Z > 0.
\end{equation}

Then for any smooth Kähler form $\chi \in \beta$ in an open neighborhood of $X$, there exists a smooth Kähler form $\omega \in \alpha$ such that in some open neighborhood of $X \subset \mathcal{M}$, we have

\begin{equation}
n\omega^{n-1} - (n-1)\omega^{n-2} \wedge \chi > 0.
\end{equation}

The following lemma is obvious by direct calculations.

**Lemma 3.1.** Suppose $(\alpha, \beta)$ are a pair of Kähler classes on $X$ satisfying the assumptions (3.1) and (3.2) in Theorem 3.1. Let $\alpha(t) = (1 + t)\alpha$ for $t \geq 0$. Then the pair $(\alpha(t), \beta)$ also satisfy (3.1) and (3.2). Furthermore, for any smooth Kähler form $\chi \in \beta$, there exists sufficiently large $T > 0$ and a Kähler form $\omega(T) \in \alpha(T)$ satisfying (3.3).

Lemma 3.1 implies that for any Kähler class $\beta$ of $X$, the set $\mathcal{K}_\beta$ defined on $X$ by

\[ \mathcal{K}_\beta = \{ \alpha \mid \alpha \text{ is Kähler and } (\alpha, \beta) \text{ satisfies (3.1) and (3.2)} \} \]

is connected.

Before we start our proof, we will first introduce some notations and elementary lemmas. The following notion introduced in [6] can be used to verify a subsolution for the $J$-equation.

**Definition 3.1.** For any $N \times N$ positive definite hermitian matrix $A$ and $B$,

\begin{equation}
P_B(A) = \max_{V \subset \mathbb{C}^N} \text{tr}_{A|_V}(B|_V),
\end{equation}

where $V$ is any hyperplane of $\mathbb{C}^N$. 
It is obvious $\mathcal{P}_B(A)$ is invariant under unitary transformation and so we can always make $B$ an diagonal or identity matrix. If $B$ is the identity matrix and the eigenvalues of $A$ are given by $\lambda_1, \ldots, \lambda_N$, then

$$\mathcal{P}_B(A) = \max_{k=1,\ldots,N} \left( \sum_{j=1, j\neq k}^{N} \lambda_j^{-1} \right).$$

**Lemma 3.2.** Let $A$, $B$ and $C$ be $N \times N$ positive definite hermitian matrices. Then for any $t \in [0,1]$, we have

$$\mathcal{P}_C(tA + (1-t)B) \leq t\mathcal{P}_C(A) + (1-t)\mathcal{P}_C(B).$$

**Proof.** We can assume $A$ is identity matrix and $B = \text{diag}(\lambda_1, \ldots, \lambda_N)$. Let $W$ be the hyperplane of $\mathbb{C}^N$ maximizing $\mathcal{P}_C(tA + (1-t)B)$. Let $\mu_1, \ldots, \mu_N$ be the diagonal elements of $C$. We can assume $W$ is spanned by $e_1, \ldots, e_{N-1}$ and

$$\mathcal{P}_C(tA + (1-t)B) = \sum_{j=1}^{N-1} \mu_j(t + (1-t)\lambda_j)^{-1}.$$

By convexity of the function $x^{-1}$, we have

$$\mathcal{P}_C(tA + (1-t)B) \leq \sum_{j=1}^{N-1} \mu_j(t + (1-t)(\lambda_j)^{-1}) \leq t\mathcal{P}_C(A) + (1-t)\mathcal{P}_C(B).$$

□

Lemma 3.2 shows that $\mathcal{P}_B(A)$ is convex in $B$. We can now extend the definition $\mathcal{P}_B(A)$ to

$$\mathcal{P}_\chi(\omega)$$

for any two Kähler forms $\omega$ and $\chi$ on a Kähler manifold. The following lemma is due to Chen ([6] Lemma 3.5).

**Lemma 3.3.** Suppose $A$, $B$ and $C$ are $N \times N$ complex-valued matrices such that

$$\begin{bmatrix} A & C \\ C^T & B \end{bmatrix}$$

is a positive definite hermitian matrix. Then

$$\mathcal{P}_{I_N}(A - CB^{-1}C^T) + \text{tr}_B(I_N) \leq \mathcal{P}_{I_{2N}} \left( \begin{bmatrix} A & C \\ C^T & B \end{bmatrix} \right),$$

where $I_N$ and $I_{2N}$ are identity matrices of rank $N$ and $2N$ respectively.

The following lemma shows that the operator $\mathcal{P}$ preserves the upper bound by taking the maximum for plurisubharmonic functions.
Lemma 3.4. Let \( u \) and \( v \) be two smooth plurisubharmonic functions on an open domain \( U \subset \mathbb{C}^n \) such that
\[
P_{\omega^C_n} (\sqrt{-1} \overline{\partial} u) < c, \quad P_{\omega^C_n} (\sqrt{-1} \overline{\partial} v) < c
\]
for some \( c > 0 \), where \( \omega^C_n \) is the standard Euclidean metric on \( \mathbb{C}^n \). Then
\[
P_{\omega^C_n} (\sqrt{-1} \overline{\partial} \max(u,v)) < c
\]
in the sense that
\[
nc (\sqrt{-1} \overline{\partial} \max(u,v))^{n-1} > (n-1) (\sqrt{-1} \overline{\partial} \max(u,v))^{n-2} \wedge \omega^C_n
\]
as currents.

Proof. Let \( \omega = \sqrt{-1} \overline{\partial} \max(u,v) \) be a positive closed current. Then
\[
\omega = \sqrt{-1} \overline{\partial} \max(u,v) \geq \chi_{\{u \geq v\}} \sqrt{-1} \overline{\partial} u + \chi_{\{u < v\}} \sqrt{-1} \overline{\partial} v
\]
in the sense of currents (c.f. [15]). Since \( u \) and \( v \) are smooth, we can define \( \eta = \chi_{\{u \geq v\}} \sqrt{-1} \overline{\partial} u + \chi_{\{u < v\}} \sqrt{-1} \overline{\partial} v \) to be a bounded hermitian form. Then \( P_{\omega^C_n} (\eta) < c \).

The local regularization of a positive closed current \( \Theta \) is defined by
\[
\Theta^{(r)}(p) = \int_{B_r(0)} r^{-2n} \rho \left( \frac{|y|}{r} \right) \Theta(p - y) d\Vol_{\mathbb{C}^n}(y)
\]
where \( r > 0 \) and \( \rho(t) \) is a smooth non-negative function with support in \([0, 1]\) satisfying
\[
\int_{B_1(0)} \rho(|y|) d\Vol_{\mathbb{C}^n}(y) = 1.
\]
For any \( V \subset U \), there exists \( \varepsilon > 0 \) such that in \( V \),
\[
P_{\omega^C_n} (\omega^{(r)}) \leq (P_{\omega^C_n} (\eta))^{(r)} < c - \varepsilon
\]
by convexity of the operator \( P \), or equivalently,
\[
n(c - \varepsilon)((\omega^{(r)})^{n-1} > (n-1)(\omega^{(r)})^{n-2} \wedge \omega^C_n,
\]
for any sufficiently small \( r > 0 \). The lemma is proved by letting \( r \to 0 \).

Lemma 3.4 also holds if the plurisubharmonic functions \( u \) and \( v \) are continuous. It also immediately implies that \( P_{\omega^C_n} (\sqrt{-1} \overline{\partial} \max(u,v)) < c \), where \( \max(u,v) \) is the regularized maximum of \( u \) and \( v \).

We will prove Theorem 3.1 by induction on dimension \( m = \max_i \dim Z_i \) of the subvariety \( Z \) of \( X \), where \( Z_i \) are components of \( Z \).

Lemma 3.5. Under the assumption of Theorem 3.1, for any subvariety \( Z \) of \( X \) with \( \dim Z \leq 1 \), there exist an open neighborhood \( U \) of \( Z \) in \( \mathcal{M} \) and a Kähler form \( \omega_Z \in \alpha|_U \) such that in \( U \),
\[
n(\omega_Z)^{n-1} - (n-1)(\omega_Z)^{n-2} \wedge \chi > 0.
\]

Proof. The proof is identical to that of Lemma 2.1.
4. The twisted $J$-equation

We first state the following PDE theorem due to Chen (Theorem 1.14 [6]) for the twisted $J$-equation.

**Theorem 4.1.** Let $M$ be an $n$-dimensional compact Kähler manifold equipped with two Kähler forms $\omega_0$ and $\chi$. Suppose $c \in \mathbb{R}^+$ and $F \in C^\infty(M)$ satisfy
\[
F > -\frac{1}{2n^{n+1}c^{n-1}}, \quad \int_M F\chi^n = c \int_M \omega_0^n - \int_M \chi \wedge \omega_0^{n-1} > 0.
\]
If
\[
nc\omega_0^{n-1} - (n-1)\chi \wedge \omega_0^{n-2} > 0,
\]
then there exists a unique Kähler form $\omega \in \omega_0$ satisfying the twisted $J$-equation
\[
c\omega^n = \chi \wedge \omega^{n-1} + F\chi^n
\]
and the inequality
\[
nc\omega^{n-1} - (n-1)\chi \wedge \omega^{n-2} > 0.
\]

Equation (4.1) coincides with the $J$-equation if we choose $F = 0$ and $c = 1$ with normalization $\alpha^n = \alpha^{n-1} \cdot \beta$. The advantage of Theorem (4.1) is that $F$ is allowed to be negative.

We continue our proof of Theorem 3.1 by induction of dimension. We will fix the Kähler metric
\[
\chi \in \beta, \ \omega_0 \in \alpha.
\]
in some open neighborhood of $X$ from the assumption of Theorem 3.1. Without loss of generality, we can assume that
\[
\lambda^{-1}\chi < \omega_0 < \lambda\chi
\]
for some sufficiently large $\lambda \geq 1$.

By induction, we can assume that for any subvariety $Z$ of $X$ with $\dim Z \leq m-1 < n-1$, there exists a smooth Kähler form $\omega_Z \in \alpha$ such that in some open neighborhood of $Z$, we have
\[
n(\omega_Z)^{n-1} - (n-1)(\omega_Z)^{n-2} \wedge \chi > 0
\]
or
\[
P_\chi(\omega_Z) < n.
\]

Now we let $Z$ be an $m$-dimensional analytic subvariety of $X$. We apply the resolution of singularities for $Z$ by
\[
\Phi : \mathcal{M}' \to \mathcal{M}
\]
such that $Z'$, the strict transform of all $m$-dimensional components $Z$ by $\Phi$, is a disjoint union of smooth $m$-dimensional submanifolds $\mathcal{M}'$. In fact, we can choose $\Phi$ to be successive blow-ups along smooth centers. For conveniences, we still use $\alpha$ and $\beta$ for $\Phi^*\alpha$ and $\Phi^*\beta$. Obviously, the pair $(\alpha, \beta)$ is $J$-nef on $X'$, although they are big and semi-positive.
We will perturb the Kähler class $\alpha$ to obtain strict positivity of $(\alpha, \beta)$ on $Z'$. Let $\theta$ be a Kähler form on $\mathcal{M}'$ and so $\theta$ restricted to $X'$ or $Z'$ is also a Kähler form. We let
\begin{equation}
\hat{\alpha} = \hat{\alpha}(t, \epsilon) = (1 + 2\lambda t)\alpha + \epsilon t[\theta], \quad \hat{\beta} = \hat{\beta}(t, \epsilon) = (1 + t)\beta + \epsilon^2 t \min(1, t)[\theta].
\end{equation}

**Lemma 4.1.** There exists $\epsilon_Z > 0$ such that for any $t > 0$, $0 < \epsilon < \epsilon_Z$ and any $k$-dimensional subvariety $V'$ of $Z'$ with $1 \leq k \leq m$, we have
\begin{equation}
\left( n\hat{\alpha}^k - k\hat{\alpha}^{k-1} \cdot \hat{\beta} \right) \cdot V' > 0,
\end{equation}
where $\hat{\alpha} = \hat{\beta} = \hat{\alpha}(t, \epsilon)$ and $\hat{\beta} = \hat{\beta}(t, \epsilon)$ are defined as in (4.4).

**Proof.** We first verify the case when $k = m$. Straightforward calculations show that there exists $C = C(Z, \alpha, \beta) > 0$ such that for any $t > 0$
\begin{align*}
\left( n\hat{\alpha}^m - m\hat{\alpha}^{m-1} \cdot \hat{\beta} \right) \cdot Z' & = (n(1 + 2\lambda t)\alpha - m(1 + t)\beta + \epsilon t (n - \epsilon m \min(1, t)) \theta) \cdot \hat{\alpha}^{m-1} \cdot Z' \\
& \geq (1 + t)^{m-1} (n\alpha^m - m\alpha^{m-1} \cdot \beta) \cdot Z' - C \sum_{k=1}^{m-2} (1 + t)^k (\epsilon t)^{m-k-1} \alpha^k \cdot [\theta]^{m-k-1} \cdot Z' \\
& > 2^{-1} (n\alpha^m - m\alpha^{m-1} \cdot \beta) \cdot Z' \\
& > 0.
\end{align*}
by choosing sufficiently small $\epsilon = \epsilon(Z, \alpha, \beta) > 0$.

For $1 \leq k < m$, we can assume $V'$ is irreducible and let $W = \Phi(V')$. Then $W$ is a subvariety of $X$ of dim $W \leq k < m$. By the assumption of the induction argument, we can assume that there exists a Kähler form $\omega_W \in \alpha$ in an open neighborhood of $W$ in $\mathcal{M}$ satisfying
\begin{equation*}
n(\omega_W)^{n-1} - (n-1)(\omega_W)^{n-2} \wedge \chi > 0.
\end{equation*}
It immediately implies that for any $l \leq n-1$,
\begin{equation*}
n(\omega_W)^l - l(\omega_W)^{l-1} \wedge \chi > 0.
\end{equation*}

We let $\omega_{V'} = \Phi^* \omega_W$. Then
\begin{equation*}
\left( n(\omega_{V'})^l - l(\omega_{V'})^{l-1} \wedge \chi \right) \wedge \theta^{k-l} \geq 0
\end{equation*}
in $V'$ for $l = 1, 2, \ldots, k$. We let
\begin{equation*}
\eta(t, \epsilon) = (1 + 2\lambda t)\omega_{V'} + \epsilon \theta, \quad \tilde{\chi}(t, \epsilon) = (1 + t)\chi + \epsilon^2 t \min(1, t)\theta.
\end{equation*}
For conveniences, we write $\hat{\alpha}, \hat{\beta}, \eta$ and $\hat{\chi}$ for $\hat{\alpha}(t, \epsilon), \hat{\beta}(t, \epsilon), \eta(t, \epsilon)$ and $\chi(t, \epsilon)$. Then for $t > 0$ and sufficiently small $\epsilon > 0$, we have

$$\left( n\hat{\alpha}^k - k\hat{\alpha}^{k-1} \cdot \hat{\beta} \right) \cdot V' = \int_{V'} (n\eta^k - k\eta^{k-1} \wedge \hat{\chi})$$

$$= \sum_{l=0}^{k} n \binom{k}{l} (1 + 2\lambda t)^l (\epsilon t)^{k-l} \int_{V'} (\omega_{V'})^l \wedge \theta^{k-l}$$

$$- \sum_{l=1}^{k} k \binom{k-1}{l-1} (1 + t)(1 + 2\lambda t)^{l-1}(\epsilon t)^{k-l} \int_{V'} (\omega_{V'})^{l-1} \wedge \chi \wedge \theta^{k-l}$$

$$- \sum_{l=0}^{k-1} k \binom{k-1}{l} (1 + 2\lambda t)^l (\epsilon t)^{k-l} \epsilon \min(1, t) \int_{V'} (\omega_{V'})^l \wedge \theta^{k-l}$$

$$= \sum_{l=1}^{k} \binom{k}{l} (1 + 2\lambda t)^{l-1}(\epsilon t)^{k-l} (n(1 + 2\lambda t) - n(1 + t) - (k - l)(1 + 2\lambda t)\epsilon \min(1, t)) \int_{V'} (\omega_{V'})^l \wedge \theta^{k-l}$$

$$+ \sum_{l=1}^{k} \binom{k}{l} (1 + t)(1 + 2\lambda t)^{l-1}(\epsilon t)^{k-l} \int_{V'} (n\omega_{V'} - l\chi) \wedge (\omega_{V'})^{l-1} \wedge \theta^{k-l}$$

$$+ (\epsilon t)^k (n - k\epsilon \min(1, t)) \int_{V'} \theta^k$$

$$> 0$$

by the choice of $\eta$ and by choosing sufficiently small $\epsilon = \epsilon(k, \lambda) > 0$.

□

Since $Z'$ is a union of disjoint smooth submanifolds of $\mathcal{M}'$ (or $X'$), for convenience, we let $\tilde{Z}$ be a fixed component of $Z'$. We let $S_Z$ be the singular set of $Z$ and $S_{\tilde{Z}} = \tilde{Z} \cap \Phi^{-1}(S_Z)$. Then $S_{\tilde{Z}}$ coincides with the exceptional locus of $\Phi$ on $\tilde{Z}$. We will consider the following twisted $J$-equation on $\tilde{Z}$

$$n(\hat{\omega})^m = m(\hat{\omega})^{m-1} \wedge \hat{\chi} + c_{t, \epsilon}(\hat{\chi})^m,$$

for $t > 0$ and $\epsilon \in (0, \epsilon_Z)$, where $\epsilon_Z > 0$ is defined in Lemma 4.1,

$$\hat{\omega} = \hat{\omega}(t, \epsilon) \in \hat{\alpha} = \hat{\alpha}(t, \epsilon)$$

is a Kähler form and $c_{t, \epsilon}$ is the normalization constant defined by

$$n(\hat{\alpha})^m = m(\hat{\alpha})^{m-1} \cdot \hat{\beta} + c_{t, \epsilon}(\hat{\beta})^m.$$

By Lemma 4.1, for any $\epsilon \in (0, \epsilon_Z)$ and $t > 0$, we have

$$c_{t, \epsilon} > 0.$$
Equation (4.6)) can always be solved for $t = 1$ for any $\epsilon \in (0, \epsilon_Z)$. In fact, if we let $\eta = \eta(t, \epsilon) = (1 + 2\lambda t)\omega_0 + \epsilon t\theta$, then at $t = 1$,

$$n\eta^{m-1} - (m-1)\eta^{m-2} \wedge \hat{\chi}$$

$$= ((1 + 2\lambda)\omega_0 - 2\chi + \epsilon(n - (m-1)\epsilon)\theta) \wedge \eta^{m-2}$$

$$> 0$$

by the assumption (4.3). Therefore we can apply Theorem 4.1 to solve equation (4.6).

Let

$$\mathcal{T}_\epsilon = \{t \in (0, 1) \mid (4.6) \text{ has a smooth solution at } t \text{ for } \epsilon \in (0, \epsilon_Z)\}$$

and

$$t_\epsilon = \inf \mathcal{T}_\epsilon.$$ 

Obviously, $\mathcal{T}_\epsilon$ is open and by applying the continuity method, for each $\epsilon \in (0, \epsilon_Z)$, equation (4.6) can be solved for all $t \in (t_\epsilon, 1]$.

**Lemma 4.2.** For any $\epsilon \in (0, \epsilon_Z)$, $t_\epsilon = 0$.

We will assume Lemma 4.2 in §5 and §6. The proof of Lemma 4.2 will be given in §7, by going through the same argument in §5 and §6. Indeed if $t_\epsilon > 0$ for some $\epsilon \in (0, \epsilon_Z)$, equation (4.6) is a non-degenerate at $t_\epsilon$ and one can repeat the same argument for $t_\epsilon = 0$ to obtain a smooth subsolution for (4.6) and then a smooth solution.

5. A mass concentration in the degenerate case

In this section, we will prove a mass concentration result similarly as in [6] based on the techniques in [9]. We will keep the notations as in §4 and assume Lemma 4.2.

Before stating the main result of this section. We will define a local regularization for global positive currents.

**Definition 5.1.** Let $M$ be an $n$-dimensional Kähler manifold and $\Theta$ be a closed $(1,1)$-positive current. Then the local regularization $\Theta^{(r)} = \{\Theta_j^{(r)}\}_{j \in J}$ of $\Theta$ with respect to a finite partition $\{B_j\}_{j \in J}$ of $M$ and a scale $r$ is defined as follows.

1. $\{B_j\}_j$ is a finite open covering of $M$. Each $B_j$ is biholomorphic to a Euclidean unit ball $B_1(0)$ in $\mathbb{C}^n$ equipped with a standard Euclidean metric $g_j$. We also require that $\{2B_j\}_j$ is also a covering of $M$, where $2B_j$ is biholomorphic to $B_2(0)$ with respect to $g_j$.

2. $\Theta_j^{(r)}(x)$ is the standard regularization in $B_j$ for $x \in B_j$, defined by the following convolution

$$\Theta_j^{(r)}(x) = \int_{B_j} r^{-2n} \rho \left( \frac{|y|}{r} \right) \Theta(x - y) d\text{Vol}_{g_j}(y)$$

(5.1)
for \( r \in (0,1) \), where \( \rho(t) \) is a smooth non-negative function with support in \([0,1]\) satisfying
\[
\int_{B_1(0)} \rho(|y|) d\text{Vol}_{g_j}(y) = 1.
\]

For simplicity we can assume that \( \rho(t) \) is a decreasing function in \( t \) and compactly supported on \([0,1]\). Also it is constant on \([0,1/2]\). The main advantage of applying the local regularization is that it does not lose positivity while the global regularization might. The other advantage is that if \( \Theta = \sqrt{-1\partial\bar\partial \varphi} \) locally for some plurisubharmonic function \( \varphi \), then \( \Theta^{(r)} = \sqrt{-1\partial\bar\partial \varphi^{(r)}} \), where \( \varphi^{(r)} \) is the local regularization defined by the same way as in (5.1).

Now we can state the main result of this section. First, we have to pick a partition for \( \hat{Z} \). Unlike the nondegenerate case in \([6]\), \( \hat{\chi} \) become degenerate near the exceptional locus of \( \Phi \) as \( t, \epsilon \to 0 \). Even after local regularization, the positivity can not be maintained uniformly as it depends on the scale of regularization \( r \). For any small \( \hat{\epsilon} > 0 \), we can choose a sufficiently fine open covering \( \{B_j\}_{j \in J} \) equipped a local Euclidean metric \( g_j \) as in Definition 5.1 so that in each \( B_j \), there exists Kähler metric \( \hat{\chi}_j \) with constant coefficients such that in each \( 2B_j \)
\[
(5.2) \quad \hat{\chi}_j \leq \chi \leq \hat{\chi}_j + \hat{\epsilon} g_j
\]
for some fixed small \( \hat{\epsilon} > 0 \) to be chosen later. In fact, for any given \( \epsilon > 0 \), by picking sufficiently small \( t \) and \( \epsilon > 0 \) dependent on \( \epsilon \), we will choose \( \hat{\epsilon} = \hat{\epsilon}(\epsilon) < (2m)^{-1} \delta_0 \epsilon \), where \( \delta_0 \) is defined as in Lemma 5.4.

We can always assume \( g_j \) is quasi-equivalent to a fixed Kähler metric on \( \hat{Z} \) independent of the choice \( \{B_j\}_j \) because one can fix a partition and make it into a finer partition. For convenience, let \( \theta \) be a fixed Kähler metric on \( \hat{Z} \) and we assume that in each \( 2B_j \)
\[
(5.3) \quad 2^{-1/100} g_j \leq \theta \leq 2^{1/100} g_j
\]
by choosing finer partitions. We define for \( t \in (0,1) \) and \( \epsilon \in (0, \epsilon_Z) \)
\[
(5.4) \quad \hat{\chi} = \chi(t, \epsilon) = (1 + t) \chi + \epsilon^2 t \min(1, t) \theta \in \hat{\alpha}, \quad \hat{\omega}_0 = (1 + 2\lambda t) \omega_0 + \epsilon t \theta \in \beta,
\]
where \( \epsilon_Z \) is given in Lemma 4.1 and \( \lambda \) in (4.3).

In fact, we will choose \( \theta \) as in (5.6). We also assume that for any \( t \in (0, \epsilon_Z) \) and \( \epsilon \in (0, \epsilon_Z), \)
\[
(5.5) \quad \hat{\chi} \leq \lambda \hat{\omega}_0
\]
for some fixed \( \lambda > 1 \).

**Theorem 5.1.** There exist \( \delta > 0 \), a finite Euclidean partition \( \{B_j\}_{j \in J} \) of \( \hat{Z}, \epsilon > 0, \)
\( r_0 > 0 \) and a Kähler current \( \Omega \in (1 - \delta) \alpha \), such that for all \( 0 < r < r_0, j \in J \), we have in \( B_j, \)
\[
\mathcal{P}_\chi (\Omega^{(r)}) < n - \epsilon.
\]
Furthermore, \( \Omega \) has positive Lelong number along \( S_\hat{Z} \).
The rest of the section is devoted to the proof of Theorem 5.1 using ideas from [6, 9]. We let
\[ Z = \hat{Z} \times \hat{Z} \]
and
\[ \Delta = \{(p, p) \in Z \mid p \in Z\} \]
be the diagonal submanifold of \( Z \). Suppose \( \Delta \) is locally defined by holomorphic functions \( \{f_{j,k}\}_{j,k} \) on finitely many domains \( \{U_j\}_j \) covering \( Z \). We define
\[ \psi_s = \log \left( \sum_j \rho_j \sum_k |f_{j,k}|^2 + s \right) \]
for \( s \in (0, \epsilon_Z) \), where \( \{\rho_j\}_j \) is a partition of unity for \( \{U_j\}_j \).

Let \( \pi_1 \) and \( \pi_2 \) be the projections maps from \( Z \) to \( \hat{Z} \) and let
\[ \chi_Z = \chi_Z(t, \epsilon) = \pi_1^* \hat{\chi} + \pi_2^* \hat{\chi}, \]
for \( t, \epsilon > 0 \), where \( \hat{\chi} = \hat{\chi}(t, \epsilon) \) is defined as in §4.

Since we assume the resolution \( \Phi \) is a successive blow-up along smooth center, there exists \( \phi \in \text{PSH}(\hat{Z}, \chi) \) such that \( \phi \) is smooth away from \( \hat{S}_Z \) and on \( \hat{Z} \setminus \hat{S}_Z \), We can choose the Kähler form \( \theta \) (as in (5.3)) on \( \hat{Z} \) such that
\[ \lambda^{-1} \theta \leq \omega_0 + \sqrt{-1} \overline{\partial \phi} \leq \lambda \theta \]
on \( \hat{Z} \setminus \hat{S}_Z \) for sufficiently large \( \lambda \geq 1 \). In particular, \( \phi \) has positive Lelong number along \( \hat{S}_Z \). If we let
\[ \tilde{\phi} = \pi_1^* \phi + \pi_2^* \phi, \]
then \( (\chi_Z + \sqrt{-1} \overline{\partial \tilde{\phi}})|_{Z \setminus (\pi_1^{-1}(\hat{S}_Z) \cup \pi_2^{-1}(\hat{S}_Z))} \) extends to a Kähler metric on \( Z \). We also define
\[ \chi_{Z,s} = \chi_{Z,s}(t, \epsilon) = \chi_Z + s \sqrt{-1} \overline{\partial \phi} + s^2 \sqrt{-1} \overline{\partial \psi_s}, \]
for a fixed sufficiently small \( s > 0 \) so that \( \chi_{Z,s} \) is a Kähler current and
\[ F_{Z,s} = F_{Z,s}(t, \epsilon) = \frac{(\chi_{Z,s})^{2m}}{(\chi_Z)^{2m}} - (1 + c_{t,\epsilon,s}) + \frac{c_{t,\epsilon}}{m+n}, \]
where \( c_{t,\epsilon} \) is defined in equation (4.7) and \( c_{t,\epsilon,s} \) is defined by the normalization
\[ \int_Z (\chi_{Z,s})^{2m} = (1 + c_{t,\epsilon,s}) \int_Z (\chi_Z)^{2m}. \]
Note that \( \lim_{s \to 0} c_{t, \epsilon, s} = 0 \). We then consider the following twisted \( J \)-equation on \( Z \) by
\[
(\omega_{Z, s})^{2m} = (\omega_{Z, s})^{2m-1} \wedge \left( \frac{2m}{m+n} \right) \chi_Z + F_{Z, s}(\chi_Z)^{2m},
\]
where
\[
(5.10) \quad \omega_{Z, s} \in \pi_1^* \hat{\alpha} + \pi_2^* \hat{\beta}
\]
and
\[
(5.11) \quad \int_Z F_{Z, s}(\chi_Z)^{2m} = \frac{c_{t, \epsilon}}{m+n} \int_Z (\chi_Z)^{2m} > 0
\]
Equation (5.9) can also be written as
\[
(5.12) \quad tr_{\omega_{Z, s}}(\chi_Z) + (m+n) F_{Z, s}(\omega_{Z, s})^{2m} = m + n.
\]

**Lemma 5.1.** There exists a sufficiently small \( s > 0 \), such that for any \( t \in (0, 1) \), \( \epsilon \in (0, \epsilon_Z) \) and \( s \in (0, 1) \),
\[
\inf_Z F_{Z, s} > -\frac{1}{2(2m)^{2m+1}(m+n)^{2m}}.
\]

**Proof.** Direct local calculations show that there exists \( A > 0 \) such that for \( s \in (0, \epsilon_Z) \).
\[
A\theta + \sqrt{-1} \partial \bar{\partial} \log \psi_s > 0.
\]
Therefore by definition of \( \hat{\phi} \) and \( \chi_{Z, s} \), for any \( \epsilon > 0 \), there exists sufficiently small \( s > 0 \) such that for any \( s \in (0, \epsilon_Z) \), \( t \in (0, \epsilon_Z) \) and \( \epsilon \in (0, \epsilon_Z) \)
\[
\chi_{Z, s} \geq (1 - \epsilon) \chi_Z
\]
and
\[
F_{Z, s} \geq (1 - \epsilon)^n - 1.
\]
The lemma then immediately follows by choosing sufficiently small \( \epsilon > 0 \).

We will fix \( s > 0 \) once for all from Lemma 5.1.

**Lemma 5.2.** For any \( s \in (0, 1) \), \( \epsilon \in (0, \epsilon_Z) \) and \( t \in (0, 1) \), there exists a unique smooth Kähler form \( \omega_{Z, s} = \omega_{Z, s}(t, \epsilon) \) that solves equation (5.9) and satisfies
\[
(5.13) \quad (m+n)(\omega_{Z, s})^{2m-1} - (2m-1)(\omega_{Z, s})^{2m-2} \wedge \chi_Z > 0.
\]

**Proof.** First, we note that equation (5.9) is well-defined because
\[
\int_Z (\omega_{Z, s})^{2m} = \binom{2m}{m} \int_Z \hat{\alpha}^m \int_Z \hat{\beta}^m
\]
and by (4.7),
\[
\int_{\mathcal{Z}} (\omega_{\mathcal{Z},s})^{2m-1} \land \left( \frac{2m}{m+n} \right)^2 \chi_{\mathcal{Z}} + F_{\mathcal{Z},s}(\chi_{\mathcal{Z}})^{2m} \]
\[
= \frac{2m}{m+n} \left( \begin{array}{c} 2m-1 \\ m \end{array} \right) \left( \int_{\hat{\mathcal{Z}}} \hat{\alpha}^m + \int_{\hat{\mathcal{Z}}} \hat{\alpha}^{m-1} \cdot \hat{\beta} \right) \int_{\hat{\mathcal{Z}}} \hat{\beta}^m + \frac{ct_\epsilon}{m+n} \left( \frac{2m}{m} \right) \left( \int_{\hat{\mathcal{Z}}} \hat{\beta}^m \right)^2 
\]
\[
= (m+n)^{-1} \left( \begin{array}{c} 2m \\ m \end{array} \right) \left( m \int_{\mathcal{Z}} \hat{\alpha}^m + m \int_{\mathcal{Z}} \hat{\alpha}^{m-1} \cdot \hat{\beta} + c t_\epsilon \int_{\mathcal{Z}} \hat{\beta}^m \right) \int_{\hat{\mathcal{Z}}} \hat{\beta}^m 
\]
\[
= \left( \frac{2m}{m} \right) \int_{\hat{\mathcal{Z}}} \hat{\alpha}^m \int_{\hat{\mathcal{Z}}} \hat{\beta}^m. 
\]

Since we assume Lemma 4.2, equation (4.6) can be solved for all \( t \in (0, 1) \) and \( \epsilon \in (0, \epsilon_Z) \). We let \( \hat{\omega} \) be the solution of (4.6) for \( t \in (0, 1) \) and \( \epsilon \in (0, \epsilon_Z) \). Then on \( \hat{\mathcal{Z}} \), we have
\[
n\hat{\omega}^{m-1} - (m-1)\hat{\omega}^{m-2} \land \hat{\chi} > 0
\]
and
\[
tr_{\hat{\omega}}(\hat{\chi}) < n.
\]

We define
\[
\hat{\omega}_{\mathcal{Z}} = \pi_1^* \hat{\omega} + \pi_2^* \hat{\chi}.
\]

Then
\[
\mathcal{P}_{\hat{\omega}_{\mathcal{Z}}}(\chi_{\mathcal{Z}}) 
\leq \max \left( \mathcal{P}_{\hat{\omega}}(\chi_{\mathcal{Z}})|_{\hat{\mathcal{Z}}} + tr_{\hat{\omega}}(\hat{\chi})|_{\hat{\mathcal{Z}}} + m - 1 \right) 
\leq n + m
\]
or equivalently
\[
(m+n)(\hat{\omega}_{\mathcal{Z}})^{2m-1} - (2m-1)(\hat{\omega}_{\mathcal{Z}})^{2m-2} \land \chi_{\mathcal{Z}} > 0.
\]

Therefore \( \hat{\omega}_{\mathcal{Z}} \) is a subsolution for equation (5.9) and we can now directly apply Theorem 4.1 for the twisted \( J \)-equation (5.9) by combining Lemma 5.1 and (5.14) to obtain a unique solution of equation (5.9) satisfying (5.13). This completes the proof of the lemma.

We define \( \omega_s = \omega_s(t, \epsilon) \) as the push-forward of \( (\omega_{\mathcal{Z},s})^m \land \pi_2^* \hat{\chi} \) by \( \pi_1 \) as the following for any \( z \in \hat{\mathcal{Z}} \)
\[
\omega_s(z) = V^{-1}(\pi_1)_* ((\omega_{\mathcal{Z},s})^m \land \pi_2^* \hat{\chi})(z) 
= V^{-1} \int_{\pi_1^{-1}(z)} (\omega_{\mathcal{Z},s})^m \land \pi_2^* \hat{\chi},
\]
where
\[
V = V(t, \epsilon) = m \hat{\beta}^m \cdot \hat{\mathcal{Z}}.
\]
Lemma 5.3. For any $s \in (0, 1)$, $t \in (0, 1)$ and $\epsilon \in (0, \epsilon_Z)$,
\[ \omega_s \in \hat{\alpha}. \]

Proof. It suffices to calculate the push-forward of the Kähler class as follows
\[
\omega_s = V^{-1} \int_{\pi_1^{-1}(z)} (\omega_{Z,s})^m \wedge \pi_2^* \hat{\chi}
\]
\[ \in V^{-1} \int_{\pi_1^{-1}(z)} (\pi_1^* \hat{\alpha} + \pi_2^* \hat{\beta})^m \wedge \pi_2^* \hat{\beta} = \hat{\alpha}. \]

Since $(\omega_{Z,s})^m$ is a positive closed $(m, m)$-form, $\omega_s$ is a Kähler form on $\hat{Z}$ and $(\omega_s)^m$ always converges weakly as $s \to 0$, after passing to a sequence. We then define
\[ \Theta = \Theta(t, \epsilon) = \lim_{s \to 0} (\omega_{Z,s})^m \]
for the corresponding convergent subsequence.

Lemma 5.4. There exists $\delta_0 > 0$ such that for any $t \in (0, 1)$ and $\epsilon \in (0, \epsilon_Z)$,
\[ \Theta > 100 V\delta_0 [\Delta], \]
where $[\Delta]$ is the current of integration along $\Delta$.

Proof. Since $\omega_{Z,s}$ is the solution of $(5.9)$,
\[ (n + m)\omega_{Z,s} > \chi_Z \]
by Theorem 4.1. Then for some fixed sufficiently small $s > 0$
\[ (\omega_{Z,s})^{2m} \geq \frac{2m}{(m + n)^{2m}} \chi_Z^{2m} + (\chi_{Z,s})^{2m} - (1 + c_{t,\epsilon,s} - c_{t,\epsilon})(\chi_Z)^{2m} \]
\[ \geq \frac{m}{(m + n)^{2m}} \chi_{Z,s}^{2m} \]
since $\lim_{s \to 0} c_{t,\epsilon,s} = 0$. $\chi_Z$ is bounded above by a multiple of $\chi_{Z,s}$. The proof follows by the same argument in the proof of Proposition 2.6 in [9] by the fact that
\[
\square
\]

We remark that although $\Theta$ might depend on the choice of sequence $s_j$, its lower bound is independent of such a sequence and $t \in (0, 1)$, $\epsilon \in (0, \epsilon_Z)$.

Locally we let $z = (z_1, ..., z_m)$ be the coordinates for $\hat{Z}$ from $\pi_1$ and $w = (w_1, ..., w_m)$ for $\hat{Z}$ from $\pi_2$. Then we write
\[ (5.17) \]
\[ \omega_{Z,s} = \omega_H + \omega_M + \omega_{M^r} + \omega_V, \]
where $\omega_H$ is the horizontal component, i.e., in $\sqrt{-1} dz_i \wedge d\bar{z}_j$, $\omega_V$ is the vertical component in $\sqrt{-1} dw_i \wedge d\bar{w}_j$, $\omega_M$ and $\omega_{M^r}$ are the off-diagonal or the mixed components in $\sqrt{-1} dz_i \wedge dw_j$ or $\sqrt{-1} dw_i \wedge d\bar{z}_j$. We also let $\chi_H = \pi_1^* \hat{\chi}$ and $\chi_V = \pi_2^* \hat{\chi}$. Our goal is to calculate
\[ P(\chi_s). \]
Lemma 5.5. At any point \( z \in \dot{Z} \), for any \( t \in (0, 1), \epsilon \in (0, \epsilon_Z) \) and \( s \in (0, 1) \), we have

\[
(5.18) \quad \mathcal{P}_s(\omega_s) < n.
\]

Proof. Direct calculations show that

\[
\omega_s = V^{-1}(\pi_1)_* ((\omega_H + \omega_V + \omega_M + \omega_{M^T})^m \wedge \chi_V)
\]

\[
= V^{-1}(\pi_1)_* (m\omega_H \wedge \omega_V^{m-1} \wedge \chi_V + m(m-1)\omega_M \wedge \omega_{M^T} \wedge \omega_V^{m-2} \wedge \chi_V)
\]

\[
= V^{-1}(\pi_1)_* \{ (tr_{\omega_V}(\chi_V)\omega_H + tr_{\omega_V}(\chi_V \wedge \omega_M \wedge \omega_{M^T})) \wedge \omega_V^m \}.
\]

At any point \( p \in \pi_1^{-1}(z) \) for \( z \in \dot{Z} \), we can assume

\[
\chi_V = \sqrt{-1} \sum_{i,j=1}^{m} \delta_{ij} dw_i \wedge d\bar{w}_j, \quad \omega_V = \sqrt{-1} \sum_{i,j=1}^{m} \delta_{ij} \lambda_j dw_i \wedge d\bar{w}_j,
\]

\[
\chi_H = \sqrt{-1} \sum_{i,j=1}^{m} \delta_{ij} dz_i \wedge d\bar{z}_j, \quad \omega_H = \sqrt{-1} \sum_{i,j=1}^{m} \delta_{ij} \mu_j dz_i \wedge d\bar{z}_j
\]

and

\[
\omega_M = \sqrt{-1} \sum_{i,j=1}^{m} \eta_{ij} dz_i \wedge d\bar{w}_j,
\]

where \((\delta_{ij})\) is the identity matrix. Then \((1, 1)\)-form

\[(5.19) \quad \Gamma = tr_{\omega_V}(\chi_V)\omega_H + tr_{\omega_V}(\chi_V \wedge \omega_M \wedge \omega_{M^T}) \]

corresponds to a positive hermitian matrix of size \( m \times m \) and the coefficients for \( \sqrt{-1} dz_i \wedge d\bar{z}_j \) is given by

\[
\Gamma_{ij} = \left( \sum_{k=1}^{m} \lambda_k^{-1} \right) \delta_{ij} \mu_i - \sum_{k,l=1, l \neq k}^{m} (\lambda_k \lambda_l)^{-1} \eta_{il} \eta_{lj}.
\]

Then the matrix \( \Gamma \) of size \( m \times m \) is bounded below by the following calculations.

\[
\Gamma = \left[ \left( \sum_{k=1}^{m} \lambda_k^{-1} \right) \delta_{ij} \mu_i - \sum_{k,l=1}^{m} (\lambda_k \lambda_l)^{-1} \eta_{il} \eta_{lj} \right]_{i,j} + \left[ \sum_{l=1}^{m} (\lambda_l^2)^{-1} \eta_{il} \eta_{lj} \right]_{i,j}
\]

\[
\geq \left[ \left( \sum_{k=1}^{m} \lambda_k^{-1} \right) \delta_{ij} \mu_i - \sum_{k,l=1}^{m} (\lambda_k \lambda_l)^{-1} \eta_{il} \eta_{lj} \right]_{i,j}
\]

\[
= \left( \sum_{k=1}^{m} \lambda_k^{-1} \right) \left( \delta_{ij} \mu_i - \sum_{l=1}^{m} \lambda_l^{-1} \eta_{il} \eta_{lj} \right).
\]
We can apply Lemma 3.3 and
\[
\mathcal{P}_{\tilde{\chi}}((tr_{\omega_V}(\chi_V))^{-1}\Gamma)
\leq \mathcal{P}_{\chi}(\omega_{Z,s}) - tr_{\omega_V}(\chi_V)
\leq (m + n) - tr_{\omega_V}(\chi_V),
\]
where \(\mathcal{P}_{\tilde{\chi}}(\Gamma)\) is taken on the horizontal \(m\)-dimensional space \(\tilde{Z}\) for components in \(\sqrt{-1}dz_i \wedge d\bar{z}_j\). Since
\[
\omega_s = V^{-1}(\pi_1)_*(\Gamma \wedge \omega_V^m), \quad V^{-1}\int_{\pi_1^{-1}(z)} tr_{\omega_V}(\chi_V)\omega_V^m = 1,
\]
by the convexity of \(\mathcal{P}\) operator for fixed \(z \in \tilde{Z}\), we have
\[
\mathcal{P}_{\tilde{\chi}}(\omega_s)
= \mathcal{P}_{\tilde{\chi}}\left( V^{-1}\int_{\pi_1^{-1}(z)} \Gamma \wedge \omega_V^m \right)
\leq V^{-1}\int_{\pi_1^{-1}(z)} tr_{\omega_V}(\chi_V)\mathcal{P}_{\tilde{\chi}}( (tr_{\omega_V}(\chi_V))^{-1}\Gamma) \wedge \omega_V^m
\leq (m + n) - V^{-1}\left( \int_{\pi_1^{-1}(z)} tr_{\omega_V}(\chi_V)\omega_V^m - \int_{\pi_1^{-1}(z)} (tr_{\omega_V}(\chi_V))^2 \omega_V^m \right)
\leq (m + n) - V^{-1}\left( \int_{\pi_1^{-1}(z)} tr_{\omega_V}(\chi_V)\omega_V^m \right)^2
\leq n.
\]

Let \(\Delta_{\eta}\) be the \(\eta\)-neighborhood of \(\Delta\) in \(Z\). We define the following \((1,1)\)-forms on \(\tilde{Z}\)
\[(5.20) \quad \omega_{s,\eta}(z) = V^{-1}\int_{\pi_1^{-1}(z) \cap \Delta_{\eta}} (\omega_{Z,s})^m \wedge \chi_V,
\]
\[(5.21) \quad \omega'_{s,\eta}(z) = V^{-1}\int_{\pi_1^{-1}(z) \cap \Delta_{\eta}} \chi_Z \wedge (\omega_V)^{m-1} \wedge \chi_V
\]
and
\[
\Omega_{s,\eta}(z) = \omega_s - \omega_{s,\eta} + \omega'_{s,\eta}
\]
\[(5.22) \quad = V^{-1}\left( \int_{\pi_1^{-1}(z) \setminus \Delta_{\eta}} (\omega_{Z,s})^m \wedge \chi_V + \int_{\pi_1^{-1}(z) \cap \Delta_{\eta}} \chi_Z \wedge (\omega_V)^{m-1} \wedge \chi_V \right).\]
Lemma 5.6. Under the same assumption of Lemma 5.5, we have

\begin{equation}
\mathcal{P}_\chi(\Omega_{s,\eta}) \leq n + m V^{-1} \int_{\pi_1(z) \cap \Delta_\eta} tr_{\omega_V}(\chi_V)(\omega_V)^m.
\end{equation}

Proof. By definition of \( \Omega_{s,\eta} \) and \( \Gamma \), we have

\[ \Omega_{s,\eta} = \mathcal{V}^{-1} \int_{\pi_1(z) \cap \Delta_\eta} \Gamma \wedge (\omega_V)^m + \mathcal{V}^{-1} \int_{\pi_1(z) \cap \Delta_\eta} \chi_H \wedge (\omega_V)^m \wedge \chi_V. \]

By Jensen’s inequality and the calculations in the proof of Lemma 5.5, we have

\[ \mathcal{P}_\chi(\Omega_{s,\eta}) \leq \mathcal{V}^{-1} \int_{\pi_1(z) \cap \Delta_\eta} \mathcal{P}_{\chi_H}(tr_{\omega_V}(\chi_V)^{-1}\Gamma) tr_{\omega_V}(\chi_V)(\omega_V)^m 
+ m \mathcal{V}^{-1} \int_{\pi_1(z) \cap \Delta_\eta} \mathcal{P}_{\chi_H}(m^{-1}\chi_H) \wedge (\omega_V)^m \wedge \chi_V \]

\[ \leq \mathcal{V}^{-1} \left( (m + n) \int_{\pi_1(z) \cap \Delta_\eta} tr_{\omega_V}(\chi_V)(\omega_V)^m - \int_{\pi_1(z) \cap \Delta_\eta} (tr_{\omega_V}(\chi_V))^2(\omega_V)^m \right) 
+ m \mathcal{V}^{-1} \int_{\pi_1(z) \cap \Delta_\eta} tr_{\omega_V}(\chi_V)(\omega_V)^m \]

\[ \leq n + \mathcal{V}^{-1} \int_{\pi_1(z) \cap \Delta_\eta} (tr_{\omega_V}(\chi_V))^2(\omega_V)^m + (m - m - n) \mathcal{V}^{-1} \int_{\pi_1(z) \cap \Delta_\eta} tr_{\omega_V}(\chi_V)(\omega_V)^m \]

\[ \leq n + m \mathcal{V}^{-1} \int_{\pi_1(z) \cap \Delta_\eta} tr_{\omega_V}(\chi_V)(\omega_V)^m. \]

The last inequality follows from the fact that \( tr_{\omega_V}(\chi_V) \leq tr_{\omega_V}(\chi_V) \leq m + n \).

Recall \( \theta \) is a Kähler metric on \( \hat{Z} \). Then we have the following lemma.

Lemma 5.7. For any \( \varepsilon > 0, t \in (0,1) \) and \( \epsilon \in (0,\epsilon_Z) \), there exist sufficiently small \( \eta = \eta(\varepsilon,t,\epsilon) > 0 \) and \( s_0 = s_0(\varepsilon,t,\epsilon,\eta) \in (0,1) \) such that for all \( s \in (0,s_0) \),

\begin{equation}
\int_{\Delta_\eta} (\omega_V)^{m-1} \wedge \chi_V \wedge \pi_1^*(\theta^m) < \varepsilon.
\end{equation}

Proof. We prove by contradiction. Suppose there exist \( \varepsilon' > 0, t \in (0,1), \epsilon \in (0,\epsilon_Z) \) and sequences \( \eta_j \to 0, s_j \to 0 \) such that for each \( j \) and \( \omega_V = \omega_V(t,\epsilon,s_j) \), we have

\[ \int_{\Delta_{\eta_j}} (\omega_V)^{m-1} \wedge \chi_V \wedge \pi_1^*(\theta^m) \geq \varepsilon'. \]

We note that

\[ \int_{\Delta_\eta} (\omega_V)^{m-1} \wedge \chi_V \wedge \pi_1^*(\theta^m) = \int_{\Delta_\eta} (\omega_{Z,s})^{m-1} \wedge \chi_V \wedge \pi_1^*(\theta^m). \]
Suppose \((\omega_{Z,s_j})^{m-1}\) converges to a closed positive \((m-1,m-1)\)-current \(T\) on \(Z\) as \(s_j \to 0\), after passing to a sequence. By Siu’s decomposition theorem, we have

\[
T = \sum_{i=1}^{\infty} a_i[E_i] + R
\]

where \(a_i > 0\), \(E_j\) is an analytic subvariety of \(Z\) with

\[
\dim E_i \leq m + 1
\]

and \(R\) is the residue current with vanishing Lelong number everywhere. After taking a subsequence, for any fixed \(\eta > 0\)

\[
\lim_{s_j \to 0} \int_{\Delta_{\eta}} (\omega_V)^{m-1} \wedge \chi_V \wedge \pi_1^*(\theta^m)
\]

\[
= \sum_{i=1, \dim E_i = m+1} a_i \int_{\Delta_{\eta} \cap E_i} \chi_V \wedge \pi_1^*(\theta^m) + \int_{\Delta_{\eta}} R \wedge \chi_V \wedge \pi_1^*(\theta^m).
\]

Since \(\int_{Z} (\omega_V)^{m-1} \wedge \chi_V \wedge \theta^m\) is uniformly bounded above, for any \(\epsilon > 0\), there exists \(J > 0\) such that

\[
\sum_{j > J} a_j \int_{E_j} \chi_V \wedge \pi_1^*(\theta^m) < \epsilon.
\]

Since \(R\) has vanishing Lelong number, by definition and by choosing \(\eta > 0\) sufficiently small,

\[
\int_{\Delta_{\eta}} R \wedge \chi_V \wedge \pi_1^*(\theta^m) < \epsilon.
\]

Since \(\chi_V \wedge \pi_1^*(\theta^m)\) is a fixed smooth form on \(Z\), by choosing \(\eta\) sufficiently small, we have

\[
a_i \int_{\Delta_{\eta} \cap E_i} \chi_V \wedge \pi_1^*(\theta^m) < \epsilon
\]

for all \(i\) with \(\dim E_i = m + 1\).

Combining the above estimates, we have

\[
\lim_{s_j \to 0} \int_{\Delta_{\eta}} (\omega_V)^{m-1} \wedge \chi_V \wedge \theta^m = \sum_i a_i \int_{\Delta_{\eta} \cap E_i} \chi_V \wedge \theta^m + \int_{\Delta_{\eta}} R \wedge \chi_V \wedge \theta^m < 3\epsilon.
\]

Then for sufficiently small \(s_j\), we have

\[
\int_{\Delta_{\eta}} (\omega_V)^{m-1} \wedge \chi_V \wedge \theta^m < 4\epsilon.
\]

This leads to contradiction by choosing \(4\epsilon < \epsilon'\) and the lemma is proved. \(\square\)

**Lemma 5.8.** There exists \(r_0 > 0\) such that for any \(\epsilon > 0\), \(t \in (0,1)\) and \(\epsilon \in (0,\varepsilon_Z)\), there exist sufficiently small \(\eta = \eta(r_0, \varepsilon, t, \epsilon), s_0 = s_0(r_0, \varepsilon, t, \epsilon, \eta) > 0\) such that for any \(s \in (0, s_0)\), \(r \in (0, r_0)\), \(j \in J\), we have in \(B_j\)

\[(5.25)\]

\[\mathcal{P}_{\chi_j}(\Omega_{s,\eta}^{(r)}) \leq n + \epsilon,\]
where $\hat{\chi}_j$ is defined in (5.2) and $\Omega^{(r)}_{s,\eta}$ is the regularization in $2B_j$.

**Proof.** Since $\hat{\chi}_j$ has constant coefficients, for any $z \in B_j$, by convexity of $\mathcal{P}_{\hat{\chi}_j}(\cdot)$ we have

$$
\mathcal{P}_{\hat{\chi}_j}(\Omega^{(r)}_{s,\eta})(z) = \mathcal{P}_{\hat{\chi}_j} \left( \int_{y \in B_r(z)} r^{-2m} \rho(r^{-1}|y|) \Omega_{s,\eta}(z + y) d\text{Vol}_m(y) \right)
$$

\[
\leq \int_{B_r(z)} r^{-2m} \rho(r^{-1}|y|) \mathcal{P}_{\hat{\chi}_j} \left( \Omega_{s,\eta}(z + y) \right) d\text{Vol}_m(y)
\]

\[
\leq \int_{B_r(z)} r^{-2m} \rho(r^{-1}|y|) \left( n + mV^{-1} \int_{\pi^{-1}_1(z+y) \cap \Delta_\eta} \text{tr}_{\omega_V}(\chi_V)(\omega_V)^m \right) d\text{Vol}_m(y)
\]

\[
\leq n + m2^mV^{-1} \int_{\Delta_\eta} (\omega_V)^{m-1} \wedge \chi_V \wedge \theta^m
\]

by choosing sufficiently small $\eta, s > 0$. Here the second inequality follows from Lemma 5.6 and the last inequality follows from Lemma 5.7. \qed

**Lemma 5.9.** There exists $r_0 > 0$ such that for any $t \in (0, 1)$, $\epsilon \in (0, \epsilon_Z)$ and $\eta \in (0, 1)$, there exists $s_0 \in (0, 1)$ so that for any $s \in (0, s_0)$ and $r \in (0, r_0)$, we have

(5.26) \[
(\omega^{(r)}_{s,\eta} + 100\delta_0\sqrt{-1} \partial \overline{\partial} \phi)^{(r)} > 10\delta_0 \theta,
\]

where $\phi$ is defined in (5.6).

**Proof.** For any $z \in \tilde{Z}$, for any sequence $s_j \to 0$, after passing to a subsequence, we have $(\omega_{Z,s_j})^m$ weakly converges to a closed positive current $\Theta$ and by Lemma 5.4,

\[
\lim_{s_j \to 0} \omega^{(r)}_{s_j,\eta}(z)
\]

\[
= \lim_{s_j \to 0} \int_{y \in B_r(z)} r^{-2m} \rho(r^{-1}|y|) \omega_{s_j,\eta}(z + y) d\text{Vol}_m(y)
\]

\[
= \mathcal{V}^{-1} \lim_{s_j \to 0} \int_{y \in B_r(z)} r^{-2m} \rho(r^{-1}|y|) \left( \int_{w \in \pi^{-1}_1(z+y) \cap \Delta_\eta} (\omega_{Z,s_j})^m(z + y, w) \wedge \chi_V(w) \right) d\text{Vol}_m(y)
\]

\[
= \mathcal{V}^{-1} \int_{(z',w) \in (B_r(z) \times Z) \cap \Delta_\eta} (r^{-2m} \rho(r^{-1}|z' - z|) \Theta(z', w) \wedge \chi_V(w)) d\text{Vol}_m(z')
\]

\[
\geq 100\delta_0 \int_{(z',w) \in (B_r(z) \times Z) \cap \Delta} (r^{-2m} \rho(r^{-1}|z' - z|) \chi_V(w)) d\text{Vol}_m(z').
\]

\[
= 100\delta_0 \int_{z' \in B_r(z)} r^{-2m} \rho(r^{-1}|z' - z|) \hat{\chi}(z') d\text{Vol}_m(z')
\]
because \( r^{-2m} \rho(r^{-1}|z' - z|)d\text{Vol}_{C^m}(z') \) is independent of \( s_j \). By definition of \( \phi \), we have
\[
\lim_{s_j \to 0} \left( \omega_{s_j, \eta} + 100\delta_0 \sqrt{-1 \partial \bar{\partial} \phi} \right)^{(r)}(z) \\
\geq 100 \int_{z \in B_r(z)} \left( r^{-2m} \rho(r^{-1}|z' - z|) (\delta_0 \chi + \delta_0 \sqrt{-1 \partial \bar{\partial} \phi})(z') \right) d\text{Vol}_{C^m}(z') \\
\geq 99\delta_0 \theta
\]
by choosing \( r \in (0, r_0) \) for some uniform small \( r_0 \).

We claim that for fixed \( t, \epsilon, \) any \( r \in (0, r_0) \) for some \( r_0 > 0 \) independent of \( t, \epsilon \) and for sufficiently small \( s > 0 \) (possibly dependent on \( t, \epsilon \)),
\[
(5.27) \quad (\omega_{s, \eta} + 100\delta_0 \sqrt{-1 \partial \bar{\partial} \phi})^{(r)} > 10\delta_0 \theta.
\]
Suppose (5.27) fails at \( s_j \) and \( z_j \) for a sequence \( s_j \to 0 \) and \( z_j \to z \). Then by passing to a subsequence and by similar calculation for \( \lim_{s_j \to 0} \omega_{s_j, \eta}^{(r)} \), we have
\[
\lim_{j \to \infty} \left( \omega_{s_j, \eta} + 100\delta_0 \sqrt{-1 \partial \bar{\partial} \phi} \right)^{(r)}(z_j) \\
= V^{-1} \lim_{j \to \infty} \int_{y \in B_r(0)} r^{-2m} \rho \left( \frac{|y|}{r} \right) \left( \int_{w \in \pi_1^{-1}(z_j + y, w) \cap \Delta_q} (\omega_{z_j, s_j})^{m}(z_j + y, w) \wedge \chi_{V}(w) \right) d\text{Vol}_{C^m}(y) \\
+ 20\delta_0 \lim_{j \to \infty} (\hat{\chi} + \sqrt{-1 \partial \bar{\partial} \phi})^{(r)}(z_j) - 20\delta_0 \lim_{j \to \infty} \chi^{(r)}(z_j) \\
\geq V^{-1} \lim_{j \to \infty} \int_{y \in B_r(0)} r^{-2m} \rho \left( \frac{|y + z_j - z|}{r} \right) \left( \int_{w \in \pi_1^{-1}(z_j + y, w) \cap \Delta_q} (\omega_{z, s_j})^{m}(z + y, w) \wedge \chi_{V}(w) \right) d\text{Vol}_{C^m}(y) \\
+ 20\delta_0 \theta^{(r)}(z) - 20\delta_0 (\hat{\chi})^{(r)}(z) \\
\geq V^{-1} \lim_{j \to \infty} \int_{y \in B_r(0)} r^{-2m} \rho \left( \frac{|y|}{r} \right) \left( \int_{w \in \pi_1^{-1}(z_j + y, w) \cap \Delta_q} (\omega_{z, s_j})^{m}(z + y, w) \wedge \chi_{V}(w) \right) d\text{Vol}_{C^m}(y) \\
- 2^{-m} V^{-1} \left( \lim_{j \to \infty} \sup_{y \in B_r(0)} r^{-2m} |\rho(r^{-1}|y|) - \rho(r^{-1}|y + z_j - z|)| \right) \left( \int_{z} (\omega_{z, s_j})^{m} \wedge \chi_{V} \wedge \theta^{m-1} \right) \theta(z) \\
+ 20\delta_0 \theta^{(r)}(z) - 20\delta_0 (\hat{\chi})^{(r)}(z) \\
\geq 15\delta_0 \theta(z)
\]
for sufficiently large \( j > 0 \) and \( r \in (0, r_0) \). This leads to contradiction and the lemma is proved.

\begin{lemma}
There exists \( r_0 > 0 \) such that for any \( \epsilon > 0, t \in (0, 1) \) and \( \epsilon \in (0, \epsilon_Z) \), there exist \( s_0 > 0 \) and \( \eta_0 > 0 \) such that for any \( s \in (0, s_0), \eta \in (0, \eta_0) \) and \( r \in (0, r_0) \),
\[
(5.28) \quad (\omega_{s, \eta}^{(r)}) < \epsilon \theta.
\]
\end{lemma}

\begin{proof}
The proof can be easily obtained by combining Lemma 5.7 and the argument in Lemma 5.9.
\end{proof}
For any small \(\varepsilon > 0\), we will choose sufficiently small \(t_0 = t_0(\varepsilon) \in (0, 1)\) and \(\varepsilon_0 = \varepsilon(\varepsilon) \in (0, \varepsilon Z)\), such that in each \(2B_j\) from the covering, we have
\[(5.29) \quad \hat{\chi}_j \leq \hat{\chi} \leq \hat{\chi}_j + 2\hat{\varepsilon}g_j, \quad \hat{\chi} < 2\theta, \quad 2^{-1}\theta \leq g_j \leq 2\theta
\]
for some fixed \(\hat{\varepsilon} > 0\) satisfying
\[(5.30) \quad \hat{\varepsilon} = \hat{\varepsilon}(\varepsilon) < m^{-1}\delta_0\varepsilon,
\]
where \(\delta_0\) is given in Lemma 5.4. We remark that \(\theta_0\) does not depend on the choice of \(\hat{\varepsilon} \in (0, 1)\). From now on, we fix \(\theta_0\) and for fixed \(\varepsilon > 0\) we will choose the covering \(\{B_j\}_{j \in J}\) and \(\hat{\varepsilon}\) satisfying and (5.29) and (5.30).

**Proposition 5.1.** There exists \(r_0 > 0\) such that for any \(\varepsilon > 0\), there exist \(t_0 = t_0(\varepsilon) > 0\) and \(\varepsilon_0 = \varepsilon_0(\varepsilon) > 0\), such that for any \(t \in (0, t_0)\) and \(\varepsilon \in (0, \varepsilon_0)\), there exist \(s_0 > 0\) so that for any \(s \in (0, s_0)\), we have on \(\hat{Z}\),
\[(5.31) \quad \mathcal{P}_{\hat{\chi} + \hat{\varepsilon}g_j} \left( (\omega_s - \lambda^{-1}\delta_0 \hat{\omega} + 100\delta_0\sqrt{-1\partial\bar{\partial}\phi}) \right)^{(r)} < n + 2\varepsilon.
\]

**Proof.** By Lemma 5.9 and Lemma 5.10, for any \(r \in (0, r_0)\), \(t \in (0, t_0)\) and \(\varepsilon \in (0, \varepsilon_0)\), there exists sufficiently small \(\eta > 0\) such that for any sufficiently small \(s > 0\), we have
\[(\omega_s - \lambda^{-1}\delta_0 \hat{\omega} + 100\delta_0\sqrt{-1\partial\bar{\partial}\phi})^{(r)} = (\Omega_{s, \eta} + (\omega_{s, \eta} + 100\delta_0\sqrt{-1\partial\bar{\partial}\phi}) - \lambda^{-1}\delta_0 \hat{\omega} - \omega'_{s, \eta})^{(r)} \geq \Omega^{(r)}_{s, \eta} + 10\delta_0\theta - 2\delta_0\theta - \delta_0\theta > 0.
\]
By the choice of \(\hat{\varepsilon}\) as in (5.30), we have
\[
\mathcal{P}_{\hat{\chi} + \hat{\varepsilon}g_j} \left( (\omega_s - \lambda^{-1}\delta_0 \hat{\omega} + 100\delta_0\sqrt{-1\partial\bar{\partial}\phi}) \right)^{(r)} \\
\leq \mathcal{P}_{\hat{\chi} + \hat{\varepsilon}g_j} \left( \Omega_{s, \eta}^{(r)} + 5\delta_0\theta \right) \\
\leq \mathcal{P}_{\hat{\chi} + \hat{\varepsilon}g_j} (\Omega_{s, \eta}^{(r)} + \mathcal{P}_{\hat{\varepsilon}g_j}(5\delta_0\theta)) \\
\leq n + \varepsilon + \frac{m\hat{\varepsilon}}{\delta_0} \\
\leq n + 2\varepsilon.
\]

Estimate (5.31) in Proposition 5.1 immediately implies that
\[(5.32) \quad \mathcal{P}_{\chi} \left( (\omega_s - 2K^{-1}\delta_0 \hat{\omega} + 100\delta_0\sqrt{-1\partial\bar{\partial}\phi}) \right)^{(r)} < n + 2\varepsilon.
\]
by the choice of \(\chi_j\). Now we can now complete the proof of Theorem 5.1.
Proof of Theorem 5.1. We fix a sufficiently small $\varepsilon > 0$, by Proposition 5.1, there exist $r_0 > 0$ such that for fixed $t \in (0, t_0(\varepsilon))$ and $\varepsilon \in (0, \varepsilon_0(\varepsilon))$, there exist a sequence $s_i \to 0$ such that for any $r \in (0, r_0)$,

$$P_{\chi + \theta}((\omega_{s_i} - \lambda^{-1}\delta_0\omega + 100\delta_0\sqrt{-1}\partial\bar{\partial}\phi)^{(r)}) \leq n + 2\varepsilon.$$  

After possibly passing to a subsequence, we let

$$\tilde{\omega} = \tilde{\omega}(t, \epsilon) = \lim_{s_i \to 0} (\omega_{s_i} - \lambda^{-1}\delta_0\omega + 100\delta_0\sqrt{-1}\partial\bar{\partial}\phi).$$

Obviously $\tilde{\omega}(t, \epsilon) \in (1 - \lambda^{-1}\delta_0)\alpha$ and by Lemma 5.9,

$$\tilde{\omega}(t, \epsilon) \geq 100\delta_0\tilde{\chi} - \lambda^{-1}\delta_0\omega + 100\delta_0\sqrt{-1}\partial\bar{\partial}\phi$$

$$\geq 10\delta_0\theta - 4\delta_0\theta$$

$$\geq 5\delta_0\theta.$$  

On each $B_j$, after passing to a subsequence, there exist plurisubharmonic functions $\psi_{s_i}$ and $\psi$ such that

$$\omega_{s_i} - \lambda^{-1}\delta_0\omega + 100\delta_0\sqrt{-1}\partial\bar{\partial}\phi = \sqrt{-1}\partial\bar{\partial}\psi_{s_i}, \sqrt{-1}\partial\bar{\partial}\psi = \tilde{\omega}$$

and

$$\lim_{s_i \to 0} \psi_{s_i} = \tilde{\psi}$$

in $L^1$ locally. Then for any $r \in (0, r_0)$, $\psi_{s_i}^{(r)}$ converges to $\tilde{\psi}^{(r)}$ uniformly in any compact subset of $B_j$. Since in $B_j$,

$$(n + 2\varepsilon) \left(\sqrt{-1}\partial\bar{\partial}\psi_{s_i}^{(r)}\right)^m \geq m \left(\sqrt{-1}\partial\bar{\partial}\psi_{s_i}^{(r)}\right)^{m-1} \wedge \tilde{\chi} \geq 0,$$

by letting $i \to \infty$, we have for any $t \in (0, t_0(\varepsilon), \epsilon \in (0, \varepsilon_0(\varepsilon))$ and $r \in (0, r_0)$,

$$(n + 2\varepsilon) \left(\tilde{\omega}^{(r)}\right)^m = n \left(\sqrt{-1}\partial\bar{\partial}\psi^{(r)}\right) \geq m \left(\sqrt{-1}\partial\bar{\partial}\tilde{\psi}^{(r)}\right)^{m-1} \wedge \tilde{\chi} = m \left(\tilde{\omega}^{(r)}\right)^{m-1} \wedge \tilde{\chi}. $$

Now by taking a sequence $t_k, \epsilon_k \to 0$, we can assume $\tilde{\omega}(t_k, \epsilon_k)$ converges to a closed positive current $\tilde{\omega}_0 \in (1 - \lambda^{-1}\delta_0)\alpha$. In particular, $\tilde{\omega}_0$ has strictly positive Lelong number along $S_2$ since $\phi$ has strictly positive Lelong number along $S_2$.

By the same argument above, we have for any $r \in (0, r_0)$,

$$(n + 2\varepsilon) \left(\tilde{\omega}_0^{(r)}\right)^m \geq m \left(\tilde{\omega}_0^{(r)}\right)^{m-1} \wedge \tilde{\chi}.$$  

We let $\Omega = (1 + \lambda^{-1}\delta_0)\tilde{\omega}_0$. Then

$$\Omega \in (1 - \lambda^{-2}\delta_0^2)\alpha, \quad P_\chi(\Omega) \leq \frac{n + 2\varepsilon}{(1 + \lambda^{-1}\delta_0)}.$$  

This completes the proof of Theorem 5.1 by choosing $\delta = \lambda^{-2}\delta_0^2$ and $\varepsilon << \lambda^{-1}\delta_0$ sufficiently small. \qed
6. Gluing Argument

In this section, we will follow the idea of Chen [6] for gluing the local potentials by induction assumption and the trick of Blocki-Kolodziej [1]. The main difference of our argument from [6] is that we do not assume the pluriclosed set of the modified plurisubharmonic function is a smooth subvariety and we also have to extend such a function.

We keep the same notations as in §3, §4 and §5. The following is the main result of this section.

Theorem 6.1. Under the same assumptions of Theorem 3.1, we let $Z$ be an $m$-dimensional analytic subvariety of $X$ and let $S_Z$ be the set of singular points of $Z$. Then there exists $\varphi_Z \in C^\infty(Z \setminus S_Z)$ such that

1. $\omega_Z = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_Z$ is a Kähler form on $Z \setminus S_Z$ satisfying
   
   $$n(\omega_Z)^m - m(\omega_Z)^{m-1} \wedge \chi > 0,$$

2. $\varphi_Z$ tends to $-\infty$ uniformly at $S_Z$.

In fact, $\varphi_Z$ in Theorem 6.1 has positive Lelong number along $S_Z$. We first recall the following well-known definition (c.f. [1]) for Lelong number.

Definition 6.1. Let $\varphi$ be a plurisubharmonic function on a domain $U \subset \mathbb{C}^m$. We define for any $p \in U$,

$$\nu_\varphi(p, r) = \frac{\varphi_R(p) - \varphi_r(p)}{\log R - \log r},$$

where $0 < r < R$, $B_R(p) \subset \subset U$ and $\varphi_r(p)$ is defined by

$$\varphi_r(p) = \max_{B_r(p)} \varphi.$$

As $r \to 0$, $\nu_\varphi(p, r)$ converges decreasingly to the Lelong number of $\varphi$ at $x$

$$\lim_{r \to 0} \nu_\varphi(p, r) = \nu_\varphi(p).$$

We let $\hat{Z}$ be a fixed component of the strict transform of $Z$ by $\Phi$ as in §4 and §5. We keep the same notations for $\chi$, $\omega_0$ and $\theta$ as before. For any small $\delta > 0$, we will fix a covering of $\hat{Z}$ by finitely many Euclidean balls $\{B_{i,4R} = B_{4R}(p_i)\}_{i \in I, p_i \in \hat{Z}}$ such that in each $B_{i,4R}$, we have

$$\theta = \sqrt{-1}\partial\bar{\partial}\phi_{i,\theta}, \ |\phi_{i,\theta} - r^2| \leq \tau R^2, \ r = |z|^2,$$

$$\omega_0 = \sqrt{-1}\partial\bar{\partial}\phi_{i,\omega_0}, \ |\nabla\phi_{i,\omega_0}| \leq K R, \phi_{i,\omega_0}(p_i) = 0,$$

$$\chi = \sqrt{-1}\partial\bar{\partial}\phi_{i,\chi}, \ |\nabla\phi_{i,\chi}| \leq K R, \phi_{i,\chi}(p_i) = 0$$

for some fixed sufficiently large $K > 0$ and sufficiently small $\tau > 0$ independent of $R$, where $\{p_i\}_{i \in I}$ is a set of finitely many points on $\hat{Z}$ and $z$ is the local holomorphic coordinates in $B_{i,4R}$. Without loss of generality, we can always assume $\{B_{i,4R}\}_i$ are
finer coverings of \( \{ B_j \}_{j \in J} \) in §5. Furthermore, we can require \( 4R < r_0 \), where \( r_0 > 0 \) is given in Theorem 5.1.

We choose \( \Omega \) from Theorem 5.1 and we let
\[
\Omega = (1 - \delta)\omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi,
\]
for some \( \delta > 0 \) and \( \varphi \in \text{PSH}(\hat{\mathcal{Z}}, (1 - \delta)\omega_0) \). Then for some \( \varepsilon > 0 \),
\[
\mathcal{P}_\chi(\Omega^{(r)}) < n - \varepsilon
\]
for all \( 0 < r < R < r_0 \).

We also define the following local plurisubharmonic functions at any \( p \in B_{i,3R} \) for any \( r < R \),
\[
\varphi_i(p) = \varphi(p) + (1 - \delta) \phi_{i,\omega_0}(p)
\]
and
\[
\varphi_{i,r}(p) = \sup_{B_{i,r}(p)} (\varphi + (1 - \delta) \phi_{i,\omega_0}),
\]
where \( B_{i,r}(p) \) is the Euclidean ball in \( B_{i,4R} \) since \( p \in B_{i,3R} \) and \( r < R \). Then by the choice of \( \Omega \) above from Theorem 5.1, we immediately have the following lemma.

**Lemma 6.1.** There exists \( \varepsilon > 0 \) such that in each \( B_{i,3R} \), we have
\[
\mathcal{P}_\chi \left( \sqrt{-1} \partial \overline{\partial} \varphi_i^{(r)} \right) < n - \varepsilon
\]
for any \( 0 < r < R \), where \( \varphi_i^{(r)} \) is the local regularization of \( \varphi_i \) in \( B_{i,4R} \).

We note that \( \nu_{\varphi}(p) = \nu_{\varphi_i}(p) \) for any \( p \in B_{i,3R} \) since \( \phi_{i,\omega_0} \) is smooth. The following lemma from [6] (Lemma 4.2) gives some basic properties of \( \nu_{i,\varphi}(p, r) \) in each \( B_{i,3R} \). It can be proved using log convexity and Poisson kernel as in [1].

**Lemma 6.2.** For \( r < R \) and \( p \in B_{i,3R} \), the following estimates hold.

1. \( 0 \leq \varphi_{i,r}(p) - \varphi_{i,\hat{\mathcal{Z}}}(p) \leq (\log 2) \nu_{i,\varphi_i}(p, r) \).
2. \( 0 \leq \varphi_{i,r}(p) - \varphi_i^{(r)}(p) \leq \eta \nu_{i,\varphi_i}(p, r) \),

where \( \nu_{i,\varphi_i}(p, r) \) is defined for \( \varphi \) in \( B_{i,3R} \) and \( \eta > 0 \) is defined by
\[
\eta = 2^{2m} - \text{Vol}(\partial B_1(0)) \int_0^1 t^{2m-1}(\log t) \rho(t) dt.
\]

We let \( \mathcal{S}_\varphi \) be the analytic subvariety of \( \hat{\mathcal{Z}} \) defined by
\[
\mathcal{S}_\varphi = \{ p \in \hat{\mathcal{Z}} \mid \nu_{\varphi}(p) \geq \tilde{\varepsilon} \}
\]
for a fixed \( \tilde{\varepsilon} > 0 \) satisfying
\[
\tilde{\varepsilon} < \min \left( \frac{\delta^3 R^2}{6 + 4\eta}, \inf_{p \in \Phi^{-1}(\mathcal{S}_\varphi)} \frac{\nu_{\varphi}(p)}{4} \right).
\]
By Siu’s decomposition theorem, $S$ is an analytic subvariety of $\hat{Z}$ containing $S_{\hat{Z}}$.

By induction of dimensions, there exist an open neighborhood $U$ of $\Phi(S)$ in $\mathcal{M}$ and a smooth Kähler metric $\omega_U \in \alpha|_U$ such that
\[
n(\omega_U)^{n-1} - (n-1)(\omega_U)^{n-2} \wedge \chi > 0
\]
in $U$. We let $\hat{U} = \Phi^{-1}(U)$ and
\[
\omega_{\hat{U}} = \Phi^*\omega_{\hat{U}} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{\hat{U}}
\]
for some uniformly bounded and smooth $\varphi_{\hat{U}}$ (after possibly shrinking $U$). Then immediately, we have
\[
n(\omega_{\hat{U}})^{n-1} - (n-1)(\omega_{\hat{U}})^{n-2} \wedge \chi \geq 0
\]
in $\hat{U}$ and the strict inequality holds in $\hat{Z} \setminus S_{\hat{Z}}$.

We define $\tilde{\varphi}_{i,r}$ on $B_{i,3R}$ by
\[
\tilde{\varphi}_{i,r}(p) = \varphi^{(r)}_i(p) - (1 - \delta)\varphi_i(p_{\omega_0}) - \delta^3 \varphi_{i,\delta}(p) + \delta^2 \varphi
\]
Since we can choose a fixed sufficiently small $\delta$, we will assume that
\[
\delta^2 \theta < \omega_0 + \delta \sqrt{-1} \partial \bar{\partial} \varphi.
\]
Immediately we have
\[
\omega_0 + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_{i,r} \geq \Omega^{(r)} - \delta^3 \theta + \delta(\omega_0 + \delta \sqrt{-1} \partial \bar{\partial} \varphi) > \Omega^{(r)}
\]
and
\[
\mathcal{P}_\chi(\omega_0 + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_{i,r}) < n - \varepsilon
\]
in each $B_{i,4R}$ if we choose sufficiently small $\delta > 0$.

Let $r_0$ be defined in Theorem 5.1. The following lemma corresponds to Proposition 4.1 in [6] and provides the key estimates in this section.

**Lemma 6.3.** There exist $0 < r_1 < \min(r_0, R/2)$ and an open neighborhood $\hat{V} \subset \subset \hat{U}$ of $S$ such that the following estimates hold for any $r < r_1$.

1. If $p \in \hat{Z} \setminus \hat{V}$,
   \[
   \max_{\{i \in \mathcal{I} \mid p \in B_{i,3R}\}} \nu_{i,\varphi_i}(p, r) \leq 2\tilde{\varepsilon},
   \]
   and
   \[
   \max_{\{i \in \mathcal{I} \mid p \in B_{i,3R}\}} \tilde{\varphi}_{i,r}(p) > \sup_{\hat{U}} \varphi_{\hat{U}} + 3\tilde{\varepsilon} \log r + 1.
   \]

2. If
   \[
   \max_{\{i \in \mathcal{I} \mid p \in B_{i,3R}\}} \nu_{i,\varphi_i}(p, r) \geq 4\tilde{\varepsilon},
   \]
   then
   \[
   \max_{\{i \in \mathcal{I} \mid p \in B_{i,3R}\}} \tilde{\varphi}_{i,r}(p) \leq \inf_{\hat{U}} \varphi_{\hat{U}} + 3\tilde{\varepsilon} \log r - 1.
   \]
(3) If
\[ \max_{\{i \in I \mid x \in B_{i,3R}\}} \nu_{i,\varphi_i}(p, r) \leq 4\tilde{\epsilon}, \]
then
\[ \max_{\{j \in J \mid p \in B_{j,3R} \setminus B_{j,2R}\}} \tilde{\varphi}_{j,r}(p) \leq \max_{\{i \in I \mid x \in B_{i,r}\}} \tilde{\varphi}_{i,r}(p) - 2\tilde{\epsilon}. \]

**Proof.** We follow the argument in [6] and prove the estimates respectively.

(1). First we fix any open $\hat{V} \subset \subset \hat{U}$. Then we can pick $r_1$ sufficiently small so that (6.14) holds since the Lelong number of $\varphi$ at any point in $\hat{Z} \setminus \hat{V}$ is less than $\tilde{\epsilon}$ and $\nu_{\varphi}(\cdot, r)$ is decreasing in $r > 0$ and upper semi-continuous for any fixed $r > 0$.

By assumption, there exists $i \in I$ such that $\nu_{i,\varphi_i}(p, r) \leq 2\tilde{\epsilon}$ for $p \in \hat{Z} \setminus \hat{V} \cap B_{i,3R}$, and by Definition 6.1 there exists $C_1 = C_1(R, \varphi) > 0$ such that
\[ \varphi_{i,r}(p) \geq 2\tilde{\epsilon} \log r - C_1 \]
for $0 < r < r_1$. By Lemma 6.2 (2),
\[ \varphi_{i(r)}(p) \geq 2\tilde{\epsilon} \log r - \eta \nu_{i,\varphi_i}(p, r) - C_1. \]

Therefore
\[
\tilde{\varphi}_{i,r}(p) - \sup_{\hat{U}} \varphi_{\hat{U}} - 3\tilde{\epsilon} \log r - 1
= \varphi_{i}(p) - (1 - \delta)\phi_{i,\omega_0}(p) - \delta^3 \phi_{i,\theta} + \delta^2 \phi(p) - \sup_{\hat{U}} \varphi_{\hat{U}} - 3\tilde{\epsilon} \log r - 1
\geq -\tilde{\epsilon} \log r - \eta \nu_{i,\varphi_i}(p, r) - \sup_{B_{i,AR}} \phi_{i,\omega_0} - \delta^3 \sup_{B_{i,AR}} \phi_{i,\theta} + \delta^2 \inf_{\hat{Z} \setminus \hat{V}} \phi - \sup_{\hat{U}} \varphi_{\hat{U}} - 1 - C_1
\geq -\tilde{\epsilon} \log r - C_2
\]
for some fixed $C_2 > 0$ only depending on $R$, $\{B_{i,AR}\}_{i \in I}$, $\phi_{i,\omega_0}$, $\phi_{i,\theta}$, $K$, $\varphi_{\hat{U}}$ and $\phi_{\hat{Z} \setminus \hat{V}}$. Then (6.15) follows by choosing $0 < r < r_1 < e^{-C_2/\tilde{\epsilon}}$.

(2). By (2) in Lemma 6.2, there exists $C_3 = C_3(R, \varphi) > 0$ such that
\[ \varphi_{i}(p) \leq \varphi_{i,r}(p) \leq 4\tilde{\epsilon} \log r + C_3 \]
for some $i \in I$. Therefore there exists $C_4 = C_4(\{B_{i,AR}\}_{i \in I}, \phi_{i,\omega_0}, \phi_{i,\theta}, \phi, \phi_{\hat{U}}) > 0$
\[
\tilde{\varphi}_{i,r}(p) - \inf_{\hat{U}} \varphi_{\hat{U}} - 3\tilde{\epsilon} \log r + 1
= \varphi_{i}(p) - (1 - \delta)\phi_{i,\omega_0}(p) - \delta^3 \phi_{i,\theta}(p) + \delta^2 \phi(p) - \varphi_{\hat{U}}(p) - 3\tilde{\epsilon} \log r + 1
\leq \tilde{\epsilon} \log r - (1 - \delta) \inf_{B_{i,AR}} \phi_{i,\omega_0} - \delta^3 \inf_{B_{i,AR}} \phi_{i,\theta} + \delta^2 \sup_{\hat{Z}} \phi - \inf_{\hat{U}} \varphi_{\hat{U}} + 1 + C_3
\leq \tilde{\epsilon} \log r + C_4.
\]
Then (6.16) follows by choosing $0 < r < r_1 < e^{-C_4/\tilde{\epsilon}}$. 
(3). Suppose 

\[ p \in (B_{i,3R} \setminus B_{i,2R}) \cap B_{i,R} \]

and 

\[ \{v' \mid p \in B_{i',3R} \} \]

for \( r < \min(r_0, R/2) \). Then at \( p \),

\[
\tilde{\varphi}_{j,r}(p) = \varphi_{j,r}(p) - (1 - \delta)\phi_{j,\omega_0}(p) - \delta^3\phi_{j,\theta}(p) + \delta^2\phi(p) \\
\leq \varphi_{j,r}(p) + \nu_{j,\varphi}(p,r) \log 2 - (1 - \delta)\phi_{j,\omega_0}(p) - \delta^3\phi_{j,\theta}(p) + \delta^2\phi(p) \\
\leq \sup_{B_{j,2\rho}(p)} \varphi + (1 - \delta) \left( \sup_{B_{j,2\rho}(p)} \phi_{j,\omega_0} - \phi_{j,\omega_0}(p) \right) + 4\tilde{\epsilon} - \delta^3\phi_{i,\theta}(p) - \delta^3(\phi_{j,\theta}(p) - \phi_{i,\theta}(p)) + \delta^2\phi(p) \\
\leq \sup_{B_{i,r}(p) \subset B_{i,2R}} \varphi + 4\tilde{\epsilon} + (1 - \delta)KrR - \delta^3\phi_{i,\theta}(p) - \delta^3(4R^2 - R^2) + \delta^2\phi(p) \\
\leq \varphi_{i,r}(p) - (1 - \delta)\phi_{i,\omega_0}(p) - \delta^3\phi_{i,\theta}(p) + \delta^2\phi(p) + (1 - \delta) \sup_{B_{i,r}(p)} (\phi_{i,\omega_0}(p) - \phi_{i,\omega_0}) \\
+ (1 - \delta)KrR + 4\tilde{\epsilon} - 3\delta^3R^2 \\
\leq \tilde{\varphi}_{i,r}(p) + (4 + 4\eta)\tilde{\epsilon} + 2KrR - 3\delta^3R^2 \\
\leq \tilde{\varphi}_{i,r}(p) - 2\tilde{\epsilon}
\]

by the choice \( \tilde{\epsilon} \) in (6.10) and choosing \( 0 < r < r_1 < \delta^3R \). 

We define

\[
\tilde{\varphi}_\epsilon = \max_{i \in \mathcal{I}} \{ \tilde{\varphi}_{i,r}, \varphi_{\tilde{U}} + 3\tilde{\epsilon}\log r \} \\
= \int \int_{\mathbb{R}^{I+1}} (\epsilon)^{I+1} \prod_{i=0}^{I} \rho \left( \frac{s_i}{\epsilon} \right) \max_{i \in \mathcal{I}} (\tilde{\varphi}_{i,r} + s_i, \varphi_{\tilde{U}} + 3\tilde{\epsilon}\log r + s_0) \; ds_0 ds_1 ... ds_I
\]

as the regularized maximum of \( \tilde{\varphi}_{i,r} \) and \( (1 - \delta)\varphi_{\tilde{U}} \) for sufficiently small \( \epsilon > 0 \) by assuming \( \tilde{\varphi}_{i,r} \equiv 0 \) outside \( B_{i,3R} \) and \( \varphi_{\tilde{U}} + 3\tilde{\epsilon}\log r \equiv 0 \) outside \( \tilde{U} \). By Lemma 6.3, \( \tilde{\varphi}_\epsilon \in C^\infty(\tilde{Z}) \cap \text{PSH}(\tilde{Z}, \omega_0) \) is well-defined for sufficiently small \( r \) and \( \epsilon \). Furthermore, \( \tilde{\varphi} \) coincides with \( \varphi_{\tilde{U}} \) in an open neighborhood of \( S_\sharp \) by the choice \( \tilde{\epsilon} \).

Let \( \Omega_1 \in \alpha \) be the closed \((1,1)\)-current defined by 

\[ \Omega_1 = \omega_0 + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_\epsilon \]

for fixed sufficiently small \( \epsilon > 0 \). Then by construction, \( \Omega_1 \in \alpha \) is smooth on \( \hat{Z} \)

\[ \mathcal{P}_\chi(\Omega_1) < n. \]

We can assume that \( \mathcal{P}_\chi(\Omega_1) < (1 - \epsilon')n \) on \( \hat{Z} \) for some \( \epsilon' > 0 \). We then let 

\[ \Omega_2 = (1 - \epsilon')\Omega_1 + \epsilon'(\omega_0 + \delta\sqrt{-1} \partial \bar{\partial} \phi) \].
Then immediately we have the following lemma.

**Lemma 6.4.** The Kähler current $\Omega_2 \in \alpha$ is smooth on $\hat{Z} \setminus S_Z$ and

$$P_\chi(\Omega_2) < n$$

or equivalently,

$$n\Omega_2^m - m\Omega_2^{m-1} \wedge \chi > 0$$

on $\hat{Z} \setminus S_Z$. Furthermore, $\tilde{\Omega}$ has positive Lelong number along $S_Z$.

**Proof of Theorem 6.1.** If $Z$ is irreducible, Theorem 6.1 follows immediately from Lemma 6.4 since $Z \setminus S_Z = \Phi(\hat{Z} \setminus S_Z)$ and $\varphi_U$ is the pullback of a $\omega_0$-PSH function on an open neighborhood of $U$. If $Z$ is not irreducible, we apply the construction of $\Phi : M' \to M$ by resolving singularities of $Z$ as in §4. Then the strict transform of $Z$ by $\Phi$ is a union of disjoint smooth $m$-dimensional submanifolds of $X$. Then Theorem 6.1 again follows by applying Lemma 6.4 to each component.

□

7. **Proof of Theorem 3.1**

In this section, we will complete the proof of Theorem 3.1 and its corollaries.

**Proof of Theorem 3.1.** First we assume Lemma 4.2. Let $Z$ be an $m$-dimensional analytic subvariety of $X$ and $S_Z$ be the singular set of $Z$ (including lower dimensional components of $Z$). By the induction assumption, there exists an open neighborhood $U$ of $S_Z$ in $M$ such that there exists $\varphi_U \in C^\infty(U) \cap \text{PSH}(U, \omega_0)$ satisfying

$$n(\omega_U)^{n-1} - (n-1)(\omega_U)^{n-2} \wedge \chi > 0, \quad \omega_U = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_U$$

in $U$.

We choose $\varphi_Z$ as in Theorem 6.1. Then there exist $A > 0$ and open neighborhoods $U_0 \subset \subset U_1 \subset \subset U_2 \subset \subset U$ of $S_Z$ in $M$ such that

1. Both $Z \setminus U_1$ and $Z \setminus U_2$ are a union of finitely many $m$-dimensional smooth open analytic varieties of $Z$,
2. $\varphi_Z < (\varphi_U - A) - 2$ in $Z \cap U_1$,
3. $\varphi_Z > (\varphi_U - A) + 2$ in $Z \cap (U \setminus U_2)$,

since $\varphi_Z$ is smooth on $Z \setminus S_Z$ and tends to $-\infty$ uniformly along $S_Z$.

Let $\varphi''_Z$ be a smooth extension of $\varphi_Z$ in $Z \setminus U_1$ and let

$$\varphi''_Z(p) = \varphi_Z(p) + Bd^2(p)$$

for $B > 1$, where $d$ is the distance function from $p$ to $Z$ with respect to any fixed Kähler metric on $M$. By choosing sufficiently large $B$, $\omega''_Z = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi''_Z$ is a Kähler form and

$$n(\omega''_Z)^{n-1} - (n-1)(\omega''_Z)^{n-2} \wedge \chi > 0$$
in an open neighborhood of $Z \setminus U_0$ in $X$. Since $\varphi''_Z$ is also a continuous extension of $\varphi'_Z$ and $\varphi_U$ is continuous, there exists a sufficiently small open neighborhood $U_3$ of $Z \setminus U_1$ in $\mathcal{M}$ such that

1. $\varphi''_Z < (\varphi_U - A) - 1$ in $U_3 \cap U_1$
2. $\varphi''_Z > (\varphi_U - A) + 1$ in $U_3 \setminus U_2$.

We define $\bar{\varphi}$ to be the regularized maximum of $\varphi''_Z$ and $\varphi_U$. Then

$\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \bar{\varphi}$

is a Kähler form in an open neighborhood of $X$ and it satisfies

\[ n\omega^{n-1} - (n-1)\omega^{n-2} \wedge \chi > 0 \]

in an open neighborhood of $Z$. This proves Theorem 3.1 by assuming Lemma 4.2.

To prove Lemma 4.2, for any $\varepsilon \in (0, \varepsilon_Z)$, we let $T_\varepsilon = \{ t \in [0, T] \mid (4.6) \text{ is solvable} \}$, $t_\varepsilon = \inf T_\varepsilon$.

Obviously $T_\varepsilon$ is open in $[0, T]$ and it suffices to show $T_\varepsilon$ is closed or $t_\varepsilon = 0$. Suppose $t_\varepsilon > 0$, then we can apply the same argument in §4, §5 and §6 with suitable small changes and let $t \to t_\varepsilon$ decreasingly to obtain a Kähler form $\omega \in \bar{\alpha} = \bar{\alpha}(t_\varepsilon, \varepsilon)$ satisfying

\[ n(\omega')^n - (n-1)(\omega')^{n-2} \wedge \chi > 0 \]

in an open neighborhood of any given $m$-dimensional subvariety $Z$ of $X$. But such a subsolution $\omega'$ implies that equation (4.6) can be solved at $t_\varepsilon$ and this leads to contradiction.

\[ \square \]

Theorem 3.1 is a special case of Theorem 1.1 by assuming $X$ is irreducible. However, Theorem 1.1 can be proved by the same argument for Theorem 3.1. Corollary 1.1 is an immediate consequence of Theorem 1.1 and Corollary 1.2 follows by combining Corollary 1.1 and the result of [19].

8. Applications

In this section, we will prove Theorem 1.2 by combining the ideas from [3, 23, 19, 17, 14, 5].

Let $X$ be a Kähler manifold of dim $X = n$. Tian’s $\alpha$-invariant for a Kähler class $\gamma$ on $X$ is defined by

\[ (8.1) \quad \alpha(X, \gamma) = \sup \{ \kappa > 0 \mid \int_X e^{-\kappa(\varphi - \sup \varphi)} \omega^n \leq C_\kappa, \forall \varphi \in PSH(X, \omega) \} \]

for any given Kähler form $\omega \in \gamma$. We remark that the $\alpha$-invariant does not depend on the choice of $\omega \in \alpha$ and it is always positive.

The following lemma is proved in [14] (Lemma 2.1) after applying the important work of Chen-Cheng [5] relating properness of the Mabuchi $K$-energy and existence of constant scalar curvature Kähler metrics.
Lemma 8.1. Let $X$ be an $n$-dimensional compact Kähler manifold and let $\gamma$ be a Kähler class on $X$ satisfying
\[ \gamma^n = \gamma^{n-1} \cdot K_X. \]
If there exist $\epsilon \in (0, \frac{n+1}{n} \alpha(X, \gamma))$, a Kähler from $\omega' \in [\omega]$ and a closed $(1,1)$-form $\eta \in [K_X]$ such that
\[ \eta + \epsilon \omega' > 0 \]
and
\[ ((n+\epsilon)\omega' - (n-1)\eta) \wedge (\omega')^{n-2} > 0 \]
everywhere on $X$, then the Mabuchi $K$-energy is proper on $\text{PSH}(X, \omega') \cap C^\infty(X)$. In particular, there exists a unique cscK metric in $\gamma$.

Now we can apply Theorem 1.1 and Lemma 8.1 to prove the following theorem (Theorem 1.2).

Proof of Theorem 1.2. If $K_X \cdot \gamma^{n-1} > 0$, we can assume that $K_X \cdot \gamma^{n-1} = \gamma^n$ by rescaling $\gamma$. Then
\[ \frac{(K_X + \epsilon \gamma) \cdot \gamma^{n-1}}{\gamma^n} = 1 + \epsilon. \]
We let
\[ \gamma_\epsilon = (1 + \epsilon) \gamma, \quad \beta_\epsilon = K_X + \epsilon \gamma. \]
Then both $\gamma_\epsilon$ and $\beta_\epsilon$ are Kähler for any $\epsilon \in (0, \alpha(X, \gamma))$ and $\gamma_\epsilon^n = \gamma_\epsilon^{n-1} \cdot \beta_\epsilon$. Furthermore, for any $m$-dimensional analytic subvariety $Z$ of $X$
\[ (n\gamma_\epsilon^m - m\gamma_\epsilon^{m-1} \cdot \beta_\epsilon) \cdot Z = (1 + \epsilon)^{m-1} ((n + (n-m)\epsilon) \gamma - mK_X) \cdot \gamma^{m-1} \cdot Z > 0 \]
by the assumption of Theorem 1.2. We can now apply Theorem 1.1 or Corollary 1.1 and so there exist Kähler forms $\omega_\epsilon \in \gamma_\epsilon$ and $\chi_\epsilon \in \beta_\epsilon$ such that
\[ (n\omega_\epsilon - (n-1)\chi_\epsilon) \wedge \omega_\epsilon^{n-2} > 0. \]
We let
\[ \omega'_\epsilon = (1 + \epsilon)^{-1} \omega_\epsilon \in \gamma, \quad \eta_\epsilon = \chi_\epsilon - \epsilon \omega'_\epsilon \in [K_X]. \]
Then $\eta_\epsilon$ is Kähler and (8.2) is equivalent to
\[ ((n+\epsilon)\omega'_\epsilon - (n-1)\eta_\epsilon) \wedge (\omega'_\epsilon)^{n-2} > 0. \]
Then we can apply Lemma 8.1 and so there exists a unique cscK metric in $\gamma$.

If $K_X \cdot \gamma^{n-1} = 0$, then we claim that $K_X \cdot \gamma^{m-1} \cdot Z = 0$ for any $m$-dimensional analytic subvariety $Z$ of $X$ by the following argument. By Poincare duality, there exists a smooth closed real $(n-m, n-m)$-form $\Omega$ such that
\[ K_X \cdot \gamma^{m-1} \cdot Z = \int_X \eta \wedge \theta^{m-1} \wedge \Omega \geq 0 \]
since $K_X$ is nef, where $\eta$ is smooth closed (1, 1)-form in $[K_X]$ and $\theta$ is a Kähler form in $\gamma$. Without loss of generality, we can assume $\Omega \leq C\theta^n$ for some fixed $C > 0$. For any $\epsilon > 0$, we choose $\eta_\epsilon$ to be a Kähler from in $K_X + \epsilon \gamma$. Then

$$K_X \cdot \gamma^{n-1} \cdot Z = \lim_{\epsilon \to 0} \int_X \eta_\epsilon \wedge \theta^n \wedge \Omega \leq C \lim_{\epsilon \to 0} \int_X \eta_\epsilon \wedge \theta^{n-1} = CK_X \cdot \gamma^{n-1} = 0.$$ 

Now we let $\beta = K_X + \gamma$. Then $\gamma^n = \gamma^{n-1} \cdot \beta$ and for any $m$-dimensional analytic subvariety $Z$ of $X$

$$(n\gamma^m - m\gamma^{m-1} \cdot \beta) \cdot Z$$

$$= ((n-m)\gamma - mK_X) \cdot \gamma^{m-1} \cdot Z$$

$$= (n-m)\gamma^m \cdot Z$$

$$> 0.$$ 

As we argued before, there exist Kähler forms $\omega \in \gamma$ and $\chi \in \beta$ satisfying

$$n\omega^{n-1} - (n-1)\omega^{n-2} \wedge \chi.$$ 

By letting $\eta = \chi - \omega$, we have

$$(\omega - (n-1)\eta) \wedge \omega^{n-2} > 0.$$ 

We can again apply Lemma 8.1 and complete the proof of Theorem 1.2.

\[ \square \]

Proof of Corollary 1.3. Let $\gamma$ be a Kähler class on $X$ and $\gamma_\epsilon = K_X + \epsilon \gamma$ for $\epsilon > 0$.

We first assume that $\gamma^{n-1} \cdot K_X > 0$. Then for any $m$-dimensional analytic subvariety $Z$ with $m < n$,

$$m \frac{\gamma^{m-1 \cdot K_X}}{\gamma^m} \bigg|_Z$$

$$= m \frac{\sum_{l=1}^{m} \frac{m-1}{l-1} \epsilon^{m-l} \gamma^m \cdot (K_X)^l}{\sum_{l=0}^{m} \frac{m}{l} \epsilon^{m-l} \gamma^m \cdot (K_X)^l} \bigg|_Z$$

$$\leq m \max_{l=1, \ldots, m} \frac{m-1}{l-1} \frac{m}{l}$$

$$= \max_{l=1, \ldots, m} l$$

$$= m.$$
On the other hand, if we let \( \nu = \max\{l \geq 0 \mid \gamma^{n-l} \cdot (K_X)^l > 0\} \) be the numerical dimension of \( K_X \), then
\[
\sum_{l=1}^{\nu} \frac{n^{\nu-1} \cdot K_X}{\gamma^n} = \frac{\sum_{l=0}^{\nu} \binom{n-1}{l-1} e^{n-l} \gamma^{n-l} \cdot (K_X)^l}{\sum_{l=0}^{\nu} \binom{n}{l} e^{n-l} \gamma^{n-l} \cdot (K_X)^l} = n + O(\epsilon).
\]
Then for sufficiently small \( \epsilon \), \( \gamma_\epsilon \) satisfies the estimate (1.10) in Theorem 1.2 for any analytic subvariety \( Z \) of \( X \).

If \( \gamma^{n-1} \cdot K_X = 0 \), then \( K_X \cdot \gamma^{m-1} \cdot Z = 0 \) for any \( m \)-dimensional analytic subvariety \( Z \) of \( X \). The corollary immediately follows from Theorem 1.2.

\[\square\]

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