EXTREMAL FUNCTIONS FOR A CLASS OF TRACE TRUDINGER-MOSER INEQUALITIES ON A COMPACT RIEMANN SURFACE WITH SMOOTH BOUNDARY

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Abstract. In this paper, we establish several trace Trudinger-Moser inequalities and obtain the corresponding extremals on a compact Riemann surface \((\Sigma, g)\) with smooth boundary \(\partial \Sigma\). To be exact, let \(\lambda_1(\partial \Sigma)\) denotes the first eigenvalue of the Laplace-Beltrami operator \(\Delta_g\) on \(\partial \Sigma\). Moreover, for any \(0 \leq \alpha < \lambda_1(\partial \Sigma)\), we set
\[
H = \{u \in W^{1,2}(\Sigma, g) : (\int_{\Sigma} |\nabla_g u|^2 dv_g - \alpha \int_{\partial \Sigma} u^2 ds_g)^{1/2} \leq 1 \text{ and } \int_{\partial \Sigma} u ds_g = 0\},
\]
where \(W^{1,2}(\Sigma, g)\) is the usual Sobolev space. By the method of blow-up analysis, we first prove the supremum
\[
\sup_{u \in H} \int_{\partial \Sigma} e^{\pi u^2} ds_g
\]
is attained by some function \(u_\alpha \in H \cap C^\infty(\Sigma)\). Further, we extend the result to the case of higher order eigenvalues. The results generalize those of Li-Liu [9] and Yang [19, 20].

1. Introduction. Let \(\Omega \subseteq \mathbb{R}^2\) be a smooth bounded domain and \(W^{1,2}_0(\Omega)\) be the completion of \(C^\infty_0(\Omega)\) under the Sobolev norm \(\|\nabla_{\mathbb{R}^2} u\|_2^2 = \int_\Omega |\nabla_{\mathbb{R}^2} u|^2 dx\), where \(\nabla_{\mathbb{R}^2}\) is the gradient operator on \(\mathbb{R}^2\) and \(\|\cdot\|_2\) denotes the standard \(L^2\)-norm. The classical Trudinger-Moser inequality \([22, 16, 15, 17, 13]\), as the limit case of the Sobolev embedding, says
\[
\sup_{u \in W^{1,2}_0(\Omega), \|\nabla_{\mathbb{R}^2} u\|_2 \leq 1} \int_\Omega e^{\beta u^2} dx < +\infty, \quad \forall \beta \leq 4\pi. \tag{1.1}
\]
Moreover, \(4\pi\) is called the best constant for this inequality in the sense that when \(\beta > 4\pi\), all integrals in (1.1) are still finite, but the supremum is infinite.

Trudinger-Moser inequalities were introduced on Riemannian manifolds by Aubin [2], Cherrier [5], Fontana [7] and others. In particular, let \((\Sigma, g)\) be a compact Riemann surface with smooth boundary \(\partial \Sigma\) and \(W^{1,2}(\Sigma, g)\) be the completion of \(C^\infty(\Sigma)\) under the norm
\[
\|u\|_{W^{1,2}(\Sigma, g)}^2 = \int_\Sigma \left( u^2 + |\nabla_g u|^2 \right) dv_g,
\]
and

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where $\nabla_g$ stands for the gradient operator on $(\Sigma, g)$. Liu [11] derived a trace Trudinger-Moser inequality in his doctoral thesis from the result of Osgood-Phillips-Sarnak [14]: for all functions $u \in W^{1,2}(\Sigma, g)$, there holds
\[
\log \int_{\partial \Sigma} e^{u} \, ds_g \leq \frac{1}{4\pi} \int_{\Sigma} |\nabla_g u|^2 \, dv_g + \int_{\partial \Sigma} u \, ds_g + C \tag{1.2}
\]
for some constant $C$ depending only on $(\Sigma, g)$. Later Li-Liu [9] obtained a strong version of (1.2), namely
\[
\sup_{u \in W^{1,2}(\Sigma, g), \int_{\Sigma} |\nabla_g u|^2 \, dv_g = 1, \int_{\partial \Sigma} u \, ds_g = 0} \int_{\partial \Sigma} e^{\gamma u^2} \, ds_g < +\infty \tag{1.3}
\]
for any $\gamma \leq \pi$. This inequality is sharp in the sense that all integrals in (1.3) are still finite when $\gamma > \pi$, but the supremum is infinite. Moreover, for any $\gamma \leq \pi$, the supremum is attained.

Another form of (1.3) was established by Yang [18], say
\[
\sup_{u \in W^{1,2}(\Sigma, g), \int_{\Sigma} |\nabla_g u|^2 \, dv_g = 1, \int_{\partial \Sigma} u \, ds_g = 0} \int_{\partial \Sigma} e^{\pi u^2} \, ds_g < +\infty.
\]
Also, an improvement of (1.3) was obtained by Yang [19] as follows:
\[
\sup_{u \in W^{1,2}(\Sigma, g), \int_{\Sigma} |\nabla_g u|^2 \, dv_g = 1, \int_{\partial \Sigma} u \, ds_g = 0} \int_{\partial \Sigma} e^{\pi u^2(1+\alpha\|u\|^2_{L^2(\partial \Sigma)})} \, ds_g < +\infty \tag{1.4}
\]
for all $0 \leq \alpha < \lambda_1(\partial \Sigma)$, where
\[
\lambda_1(\partial \Sigma) = \inf_{u \in W^{1,2}(\Sigma, g), \int_{\Sigma} u \, dv_g = 0} \frac{\int_{\Sigma} |\nabla_g u|^2 \, dv_g}{\int_{\partial \Sigma} u^2 \, ds_g} \tag{1.5}
\]
is the first eigenvalue of a fractional power of the Laplace-Beltrami operator $\Delta_g$ on the boundary $\partial \Sigma$. This inequality is sharp in the sense that all integrals in (1.4) are still finite when $\alpha \geq \lambda_1(\partial \Sigma)$, but the supremum is infinite. Moreover, for sufficiently small $\alpha > 0$, the supremum in (1.4) can be attained.

A different form of (1.4) was also derived by Yang [20], namely
\[
\sup_{u \in W^{1,2}(\Sigma, g), \int_{\Sigma} (|\nabla_g u|^2 - \alpha u^2) \, dv_g \leq 1, \int_{\Sigma} u \, dv_g = 0} \int_{\Sigma} e^{4\pi u^2} \, dv_g < +\infty \tag{1.6}
\]
for all $0 \leq \alpha < \lambda_1(\Sigma)$, where
\[
\lambda_1(\Sigma) = \inf_{u \in W^{1,2}(\Sigma, g), \int_{\Sigma} u \, dv_g = 0, u \neq 0} \frac{\int_{\Sigma} |\nabla_g u|^2 \, dv_g}{\int_{\Sigma} u^2 \, dv_g}.
\]
Further, he extended (1.6) to the case of higher order eigenvalues. For simplicity, we denote
\[
E^\perp = \left\{ u \in W^{1,2}(\Sigma, g) : \int_{\Sigma} uv \, dv_g = 0, \forall v \in E \right\},
\]
where $E \subset W^{1,2}(\Sigma, g)$ is a function space. For any positive integer $k$, we set
\[
E_{\lambda_k}(\Sigma) = \left\{ u \in W^{1,2}(\Sigma, g) : \Delta_g u = \lambda_k(\Sigma) u \right\},
\]
\[
\lambda_{k+1}(\Sigma) = \inf_{u \in E^\perp_{\lambda_k}(\Sigma), \int_{\Sigma} u \, dv_g = 0, u \neq 0} \frac{\int_{\Sigma} |\nabla_g u|^2 \, dv_g}{\int_{\Sigma} u^2 \, dv_g}
\]
and
\[
E_k(\Sigma) = E_{\lambda_1}(\Sigma) \oplus E_{\lambda_2}(\Sigma) \oplus \cdots \oplus E_{\lambda_k}(\Sigma).
\]
Then the supremum
\[ \sup_{u \in E_k^1(\Sigma), \int_{\Sigma}(|\nabla_g u|^2 - \alpha u^2)dv_g \leq 1, \int_{\Sigma} u dv_g = 0} \int_{\Sigma} e^{4\pi u^2} dv_g < +\infty \] (1.7)
for all $0 \leq \alpha < \lambda_{k+1}(\Sigma)$; moreover the above supremum can be attained by some function $u_\alpha \in E_k^1(\Sigma)$.

In this paper, we will establish two new trace Trudinger-Moser inequalities, which are extensions of (1.6) and (1.7) respectively. Precisely we first have the following:

**Theorem 1.1.** Let $(\Sigma, g)$ be a compact Riemannian surface with smooth boundary $\partial \Sigma$ and $\lambda_1(\partial \Sigma)$ be defined by (1.5). For any $0 \leq \alpha < \lambda_1(\partial \Sigma)$, we let
\[ H = \left\{ u \in W^{1,2}(\Sigma, g) \colon \|u\|_{1,\alpha} \leq 1 \text{ and } \int_{\partial \Sigma} u ds_g = 0 \right\}, \]
where
\[ \|u\|_{1,\alpha} = \left( \int_{\Sigma} |\nabla_g u|^2 dv_g - \alpha \int_{\partial \Sigma} u^2 ds_g \right)^{1/2}. \] (1.8)
Then the supremum
\[ \sup_{u \in H} \int_{\partial \Sigma} e^{\pi u^2} ds_g \] (1.9)
is attained by some function $u_\alpha \in H \cap C^\infty(\Sigma)$.

Moreover, we extend Theorem 1.1 to the case of higher order eigenvalues. Let us introduce some notations. For any positive integer $k$, we set
\[ E_{\lambda_k}(\partial \Sigma) = \left\{ u \in W^{1,2}(\Sigma, g) \colon \Delta_g u = 0 \text{ in } (\Sigma, g) \text{ and } \frac{\partial u}{\partial n} = \lambda_k(\partial \Sigma) u \text{ on } \partial \Sigma \right\}, \]
where $n$ denotes the outward unit normal vector on $\partial \Sigma$,
\[ \lambda_{k+1}(\partial \Sigma) = \inf_{u \in E_{\lambda_k}^1(\partial \Sigma), \int_{\partial \Sigma} u dv_g = 0, u \neq 0} \frac{\int_{\Sigma} |\nabla_g u|^2 dv_g}{\int_{\partial \Sigma} u^2 ds_g} \] (1.10)
and
\[ E_k(\partial \Sigma) = E_{\lambda_1}(\partial \Sigma) \oplus E_{\lambda_2}(\partial \Sigma) \oplus \cdots \oplus E_{\lambda_k}(\partial \Sigma). \] (1.11)

**Theorem 1.2.** Let $(\Sigma, g)$ be a compact Riemannian surface with smooth boundary $\partial \Sigma$, $k$ be an positive integer and $\lambda_{k+1}(\partial \Sigma)$ be defined by (1.10). For any $0 \leq \alpha < \lambda_{k+1}(\partial \Sigma)$, we let
\[ S = \left\{ u \in E_k^1(\partial \Sigma) \colon \|u\|_{1,\alpha} \leq 1 \text{ and } \int_{\partial \Sigma} u ds_g = 0 \right\}, \] (1.12)
where $\|u\|_{1,\alpha}$ is defined as in (1.8). Then the supremum
\[ \sup_{u \in S} \int_{\partial \Sigma} e^{\pi u^2} ds_g \]
is attained by some function $u_\alpha \in S \cap C^\infty(\Sigma)$.

Clearly Theorem 1.1 generalizes (1.3) and Theorem 1.2 extends (1.7) to the trace Trudinger-Moser inequality. For theirs proofs, we employ the method of blow-up analysis, which was originally used by Carleson-Chang [4], Ding-Jost-Li-Wang [6], Adimurthi-Struwe [1], Li [8], Liu [11], Li-Liu [9], Yang [18, 19] and Mancini-Martinaazzi [12]. In the remaining part of this paper, we prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3 respectively.
2. The first eigenvalue case. In this section, we will prove Theorem 1.1 in four steps: firstly, we consider the existence of maximizers for subcritical functionals and give the corresponding Euler-Lagrange equation; secondly, we deal with the asymptotic behavior of the maximizers through blow-up analysis; thirdly, we deduce an upper bound of the supremum 
\[
\sup_{u \in \mathcal{H}} \int_{\partial \Sigma} e^{\pi u^2} \, ds_g
\]
under the assumption that blow-up occurs; finally, we construct a sequence of functions to show Theorem 1.1 holds. Here and in the sequel, we do not distinguish sequence and subsequence.

2.1. Existence of maximizers for subcritical functionals. Let \(0 < \alpha < \lambda_1(\partial \Sigma)\).

Analogous to ([18], Lemma 4.1) and ([19], Lemma 3.2), we have the following:

**Lemma 2.1.** For any \(0 < \epsilon < \pi\), the supremum 
\[
\sup_{u \in \mathcal{H}} \int_{\partial \Sigma} e^{(\pi - \epsilon)u^2} \, ds_g
\]
is attained by some function \(u_\epsilon \in \mathcal{H}\).

Moreover, it is not difficult to check that \(u_\epsilon\) satisfies the Euler-Lagrange equation

\[
\left\{ \begin{array}{l}
\Delta_g u_\epsilon = 0 \text{ in } \Sigma, \\
\frac{\partial u_\epsilon}{\partial n} = \frac{1}{\lambda_\epsilon} u_\epsilon e^{(\pi - \epsilon)u^2} + \alpha u_\epsilon - \frac{\mu_\epsilon}{\lambda_\epsilon} \text{ on } \partial \Sigma, \\
\lambda_\epsilon = \int_{\partial \Sigma} u_\epsilon^2 e^{(\pi - \epsilon)u^2} \, ds_g, \\
\mu_\epsilon = \frac{1}{\ell(\partial \Sigma)} \int_{\partial \Sigma} u_\epsilon e^{(\pi - \epsilon)u^2} \, ds_g,
\end{array} \right. \tag{2.1}
\]

where \(\Delta_g\) denotes the Laplace-Beltrami operator, \(n\) denotes the outward unit normal vector on \(\partial \Sigma\) and \(\ell(\partial \Sigma)\) denotes the length of \(\partial \Sigma\). Applying elliptic estimates and maximum principle to (2.1) respectively, we have \(u_\epsilon \in \mathcal{H} \cap C^\infty(\Sigma)\) and \(u_\epsilon \not\equiv 0\) on \(\partial \Sigma\).

In addition, we have

\[
\lim_{\epsilon \to 0} \int_{\partial \Sigma} e^{(\pi - \epsilon)u^2} \, ds_g = \sup_{u \in \mathcal{H}} \int_{\partial \Sigma} e^{u^2} \, ds_g \tag{2.2}
\]
from Lebesgue’s dominated convergence theorem. It follows from (2.3) and the fact of \(e^t \leq 1 + te^t\) for any \(t \geq 0\) that

\[
\lambda_\epsilon \geq \frac{1}{\pi - \epsilon} \int_{\partial \Sigma} (e^{(\pi - \epsilon)u^2} - 1) \, ds_g,
\]

which gives

\[
\liminf_{\epsilon \to 0} \lambda_\epsilon > 0. \tag{2.3}
\]

From (2.3), one gets \(\mu_\epsilon/\lambda_\epsilon\) is bounded with respect to \(\epsilon\).

2.2. Blow-up analysis. We now perform the blow-up analysis. Without loss of generality, we set \(c_\epsilon = |u_\epsilon(x_\epsilon)| = \max_{\Sigma} u_\epsilon\). We first assume that \(c_\epsilon\) is bounded, which together with elliptic estimates completes the proof of Theorem 1.1. In the remainder of Section 2, we assume

\[
\lim_{\epsilon \to 0} c_\epsilon = \lim_{\epsilon \to 0} u_\epsilon(x_\epsilon) = +\infty
\]
and \(x_\epsilon \to p\) as \(\epsilon \to 0\). Applying maximum principle to (2.1), we have \(p \in \partial \Sigma\).

**Lemma 2.2.** There hold \(u_\epsilon \rightharpoonup 0\) weakly in \(W^{1,2}(\Sigma, g)\) and \(u_\epsilon \to 0\) strongly in \(L^2(\partial \Sigma, g)\) as \(\epsilon \to 0\). Furthermore, \(|\nabla_g u_\epsilon|^2 \, dv_g \rightharpoonup \delta_p\) in sense of measure, where \(\delta_p\) is the usual Dirac measure centered at \(p\).
Proof. Since \( u_\varepsilon \) is bounded in \( W^{1,2}(\Sigma, g) \), there exists some function \( u_0 \) such that \( u_\varepsilon \rightharpoonup u_0 \) weakly in \( W^{1,2}(\Sigma, g) \) and \( u_\varepsilon \rightarrow u_0 \) strongly in \( L^2(\partial \Sigma, g) \) as \( \varepsilon \rightarrow 0 \). Then there holds \( \int_{\partial \Sigma} |\nabla_g u_0|^2 dv_g = 0 \) and \( \|u_0\|_{1,\alpha} \leq 1 \).

Suppose \( u_0 \neq 0 \), we can see that \( \int_{\Sigma} |\nabla_g u_0|^2 dv_g > 0 \) and

\[
1 \geq \|u_0\|_{1,\alpha} \geq \left( 1 - \frac{\alpha}{\lambda_1(\partial \Sigma)} \right) \int_{\Sigma} |\nabla_g u_0|^2 dv_g > 0.
\]

Then there holds \( \lim_{\varepsilon \rightarrow 0} \|\nabla_g (u_\varepsilon - u_0)\|_{L^2(\Sigma)}^2 = 1 - \|u_0\|_{1,\alpha} := \zeta \) and \( 0 \leq \zeta < 1 \). From the Hölder inequality, for sufficiently small \( \varepsilon \), there holds

\[
\int_{\partial \Sigma} e^{q(\pi - \varepsilon)u^2} ds_g \leq \int_{\partial \Sigma} e^{q(\pi - \varepsilon)(1 + \frac{1}{2})u^2} ds_g \leq \left( \int_{\partial \Sigma} e^{q(\pi - \varepsilon)(1 + \frac{1}{2})u^2} ds_g \right)^{1/2} \left( \int_{\partial \Sigma} e^{q(\pi - \varepsilon)(1 + \delta)u^2} ds_g \right)^{1/2},
\]

where \( \delta > 0 \), \( q \), \( s > 1 \) satisfying \( 1/r + 1/s = 1 \) and \( sq(1 + \delta)(\zeta + 1)/2 \leq 1 \). By the inequality (1.3), we get \( e^{(\pi - \varepsilon)u^2} \) is bounded in \( L^q(\partial \Sigma, g) \). Applying the elliptic estimate to (2.1), one gets \( u_\varepsilon \) is uniformly bounded, which contradicts \( c_\varepsilon \rightarrow +\infty \). Therefore, the assumption is not established.

Suppose \( |\nabla_g u_\varepsilon|^2 dv_g \rightarrow \mu \neq \delta_g \) in sense of measure. Then there exists some constant \( r_0 > 0 \) such that \( \lim_{\varepsilon \rightarrow 0} \int_{B_{r_0}(p)} |\nabla_g u_\varepsilon|^2 dv_g = \eta < 1 \), where \( B_{r_0}(p) \) is a geodesic ball centered at \( p \) with radius \( r_0 \). We can see that \( \int_{B_{r_0}(p)} |\nabla_g u_\varepsilon|^2 dv_g \leq \eta + 1)/2 < 1 \) for sufficiently small \( \varepsilon \). Then we choose a cut-off function \( \rho \in C^0_c(\overline{B_{r_0}(p)}) \) which is equal to 1 in \( B_{r_0/2}(p) \) such that \( \int_{B_{r_0}(p)} |\nabla_g(\rho u_\varepsilon)|^2 dv_g \leq (\eta + 1)/4 < 1 \) for sufficiently small \( \varepsilon \). Hence we obtain

\[
\int_{B_{r_0/2}(p) \cap \partial \Sigma} e^{q(\pi - \varepsilon)u^2} ds_g \leq \int_{B_{r_0}(p) \cap \partial \Sigma} e^{q(\pi - \varepsilon)(\rho u_\varepsilon)^2} ds_g \leq \int_{B_{r_0}(p) \cap \partial \Sigma} \frac{e^{q(\pi - \varepsilon)(\rho u_\varepsilon)^2}}{\|\nabla_g (\rho u_\varepsilon)|^2 dv_g} ds_g.
\]

By the inequality (1.3), \( e^{(\pi - \varepsilon)u^2} \) is bounded in \( L^q(B_{r_0/2}(p) \cap \partial \Sigma, g) \) for some \( q > 1 \). Applying the elliptic estimate to (2.1), we get \( u_\varepsilon \) is uniformly bounded in \( B_{r_0/4}(p) \cap \partial \Sigma \), which contradicts \( c_\varepsilon \rightarrow +\infty \). Therefore, Lemma 2.2 follows.

Now we analyse the asymptotic behavior of \( u_\varepsilon \) near the concentration point \( p \).

Let

\[
r_\varepsilon = \frac{\lambda_\varepsilon}{\varepsilon^2 e^{(\pi - \varepsilon)u^2}}.
\]

Following ([21], Lemma 4), we can take an isothermal coordinate system \((U, \phi)\) near \( x_0 \), such that \( \phi(x_0) = 0 \), \( \phi(U) = \mathbb{R}^2_+ \) and \( \phi(U \cap \partial \Sigma) = \partial \mathbb{R}^2_+ \cap \mathbb{R}_r \) for some fixed \( r > 0 \), where \( \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq r^2, \ x_2 > 0\} \) and \( \mathbb{R}_r = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\} \). In such coordinates, the metric \( g \) has the representation
\[ g = e^{2f}(dx_1^2 + dx_2^2) \] and \( f \) is a smooth function with \( f(0) = 0 \). Denote \( \bar{\pi}_\epsilon = \phi(x_\epsilon) \), \( U_\epsilon = \{ x \in \mathbb{R}^2 : \bar{\pi}_\epsilon + r_\epsilon x \in \phi(U) \} \),

\[ \bar{\pi}_\epsilon(x) = \begin{cases} u_\epsilon \circ \phi^{-1}(x_1, x_2), & x_2 \geq 0, \\ u_\epsilon \circ \phi^{-1}(x_1, -x_2), & x_2 < 0, \end{cases} \]

and

\[ \bar{f}(x) = \begin{cases} f(x_1, x_2), & x_2 \geq 0, \\ f(x_1, -x_2), & x_2 < 0, \end{cases} \]
on \( \mathbb{B}_r \). Define two blowing up functions in \( U_\epsilon \)

\[ \psi_\epsilon(x) = \frac{\bar{\pi}_\epsilon(U_\epsilon + r_\epsilon x)}{c_\epsilon} \] (2.5)

and

\[ \varphi_\epsilon(x) = c_\epsilon (\bar{\pi}_\epsilon(U_\epsilon + r_\epsilon x) - c_\epsilon). \] (2.6)

From (2.1) and (2.4)-(2.6), a direct computation shows

\[ \begin{cases} \Delta_{\mathbb{R}^2} \psi_\epsilon = 0 \\ \frac{\partial \psi_\epsilon}{\partial \nu} = -e^{f(\pi_\epsilon + r_\epsilon \pi_\nu)} e^{(\pi_\epsilon + 1)\varphi_\epsilon} + \alpha r_\epsilon \psi_\epsilon - \frac{r_\epsilon}{c_\epsilon} \psi_\epsilon \end{cases} \text{ in } \mathbb{B}_R^+(0) \]

and

\[ \begin{cases} \Delta_{\mathbb{R}^2} \varphi_\epsilon = 0 \\ \frac{\partial \varphi_\epsilon}{\partial \nu} = -e^{f(\pi_\epsilon + r_\epsilon \pi_\nu)} (\psi_\epsilon e^{(\pi_\epsilon + 1)\varphi_\epsilon} + \alpha c_\epsilon^2 r_\epsilon \pi_\epsilon - \frac{r_\epsilon}{c_\epsilon} \pi_\epsilon) \end{cases} \text{ on } \partial \mathbb{R}^2_+ \cap \mathbb{B}_R(0), \] (2.7)

where \( R > 0 \) is fixed, \( \Delta_{\mathbb{R}^2} \) denotes the Laplace operator on \( \mathbb{R}^2 \) and \( \nu \) denotes the outward unit normal vector on \( \partial \mathbb{R}^2_+ \). Noticing (2.7), (2.8) and using the same argument as ([19], Lemma 4.5 and Lemma 4.6), we obtain

\[ \lim_{\epsilon \to 0} \psi_\epsilon = 1 \text{ in } C^1_{\text{loc}}(\overline{\mathbb{B}_R^+}) \] (2.9)

and

\[ \lim_{\epsilon \to 0} \varphi_\epsilon = \varphi \text{ in } C^1_{\text{loc}}(\overline{\mathbb{B}_R^+}), \] (2.10)

where \( \varphi \) satisfies

\[ \begin{cases} \Delta_{\mathbb{R}^2} \varphi = 0, \\ \frac{\partial \varphi}{\partial \nu} = e^{2\pi \varphi}, \\ \varphi(0) = \sup \varphi = 0. \end{cases} \]

By a result of Li-Zhu [10], we have

\[ \varphi(x) = -\frac{1}{2\pi} \log \left( \pi^2 x_1^2 + (1 + \pi x_2)^2 \right). \] (2.11)

A direct calculation gives

\[ \int_{\partial \mathbb{B}_R^+} e^{2\pi \varphi} dx_1 = 1. \] (2.12)

Next we discuss the convergence behavior of \( u_\epsilon \) away from \( p \). Denote \( u_{\epsilon, \beta} = \min \{ \beta c_\epsilon, u_\epsilon \} \in W^{1,2}(\Sigma, g) \) for any real number \( 0 < \beta < 1 \). Following ([19], Lemma 4.7), we get

\[ \lim_{\epsilon \to 0} \| \nabla g u_{\epsilon, \beta} \|_2^2 = \beta. \] (2.13)
Lemma 2.3. Letting $\lambda_\epsilon$ be defined by (2.1), we obtain

(i) $\limsup_{\epsilon \to 0} \int_{\partial \Sigma} e^{(\pi-\epsilon)u^2} ds_g = \ell(\partial \Sigma) + \lim_{\epsilon \to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}$

(ii) $\lim_{\epsilon \to 0} \frac{\lambda_\epsilon}{c_\epsilon} = \lim_{R \to +\infty} \lim_{\epsilon \to 0} \int_{\phi^{-1}(B_{Rr}(x)) \cap \partial \Sigma} e^{(\pi-\epsilon)u^2} ds_g.$

Proof. Recalling (2.1) and (2.13), one gets for any real number $0 < \beta < 1,$

$$\int_{\partial \Sigma} e^{(\pi-\epsilon)u^2} ds_g - \ell(\partial \Sigma)$$

$$= \int_{\{x \in \partial \Sigma : u_x \leq \beta c_\epsilon\}} (e^{(\pi-\epsilon)u_x^2} - 1) ds_g + \int_{\{x \in \partial \Sigma : u_x > \beta c_\epsilon\}} (e^{(\pi-\epsilon)u_x^2} - 1) ds_g$$

$$\leq \int_{\partial \Sigma} (e^{(\pi-\epsilon)u_x^2} - 1) ds_g + \frac{1}{\beta^2 c_\epsilon^2} \int_{\{x \in \partial \Sigma : u_x > \beta c_\epsilon\}} u_x^2 e^{(\pi-\epsilon)u_x^2} ds_g$$

$$\leq \int_{\partial \Sigma} (e^{r(\pi-\epsilon)u_x^2} - 1) ds_g + \frac{\lambda_\epsilon}{\beta^2 c_\epsilon^2} + \frac{\lambda_\epsilon}{\beta^2 c_\epsilon^2} e^{(\pi-\epsilon)u_x^2} ds_g$$

where $r, s > 1$ and $1/r + 1/s = 1.$ By (1.3) and (2.13), $e^{(\pi-\epsilon)u_x^2}$ is bounded in $L^r(\partial \Sigma, g)$ for some $r > 1.$ Letting $\epsilon \to 0$ first and then $\beta \to 1,$ we obtain

$$\limsup_{\epsilon \to 0} \int_{\partial \Sigma} e^{(\pi-\epsilon)u^2} ds_g - \ell(\partial \Sigma) \leq \lim_{\epsilon \to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}. \quad (2.14)$$

According to $c_\epsilon = \max_{\Sigma} u_x,$ (2.1) and Lemma 2.2, we have

$$\int_{\partial \Sigma} e^{(\pi-\epsilon)u_x^2} ds_g - \ell(\partial \Sigma) \geq \int_{\partial \Sigma} u_x^2 e^{(\pi-\epsilon)u_x^2} ds_g - \frac{1}{c_\epsilon^2} \int_{\partial \Sigma} u_x^2 ds_g,$$

that is to say

$$\limsup_{\epsilon \to 0} \int_{\partial \Sigma} e^{(\pi-\epsilon)u^2} ds_g - \ell(\partial \Sigma) \geq \lim_{\epsilon \to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}. \quad (2.15)$$

Combining (2.14) and (2.15), one gets (i).

Applying (2.1) and (2.4)-(2.6), we have

$$\int_{\phi^{-1}(B_{Rr}(x)) \cap \partial \Sigma} e^{(\pi-\epsilon)u_x^2} ds_g = \int_{B_{Rr} \cap \partial \Sigma^*} r_\epsilon e^{(\pi-\epsilon)u_x} e^{(\pi-\epsilon)(\psi_x + 1)} \varphi_x e^{f(x_0 + r, x)} dx_1$$

$$= \int_{B_{Rr} \cap \partial \Sigma^*} \frac{\lambda_\epsilon}{\beta^2 c_\epsilon^2} e^{(\pi-\epsilon)(\psi_x + 1)} \varphi_x e^{f(x_0 + r, x)} dx_1.$$

From (2.9)-(2.12), (ii) holds. Summarizing, we have the lemma. \qed

Next we consider the properties of $c_\epsilon u_x.$ Combining Lemma 2.3 and ([19], Lemma 4.9), we obtain

$$\frac{1}{\lambda_\epsilon} c_\epsilon u_x e^{(\pi-\epsilon)u_x^2} ds_g \rightarrow \delta_p. \quad (2.16)$$

Furthermore, one has
Lemma 2.4. There holds
\[
\begin{cases}
c_\epsilon u_\epsilon \to G \text{ weakly in } W^{1, q}(\Sigma, g), \ \forall 1 < q < 2, \\
c_\epsilon u_\epsilon \to G \text{ strongly in } L^2(\partial \Sigma, g), \\
c_\epsilon u_\epsilon \to G \text{ in } C^1_{loc}(\overline{\Sigma \setminus \{p\}}),
\end{cases}
\]
where $G$ is a Green function satisfying
\[
\begin{cases}
\Delta_p G = \delta_p \text{ in } \Sigma, \\
\frac{\partial G}{\partial n} = \alpha G - \frac{1}{\ell(\partial \Sigma)} \text{ on } \partial \Sigma \setminus \{p\}, \\
\int_{\partial \Sigma} G ds_g = 0.
\end{cases}
\] (2.17)

Proof. From (2.1), there holds
\[
\begin{cases}
\Delta_p (c_\epsilon u_\epsilon) = 0 \text{ in } \Sigma, \\
\frac{\partial (c_\epsilon u_\epsilon)}{\partial n} = \frac{1}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{(\pi - \epsilon) u_\epsilon^2} + \alpha c_\epsilon u_\epsilon - c_\epsilon \frac{\mu_\epsilon}{\lambda_\epsilon} \text{ on } \partial \Sigma \\
\int_{\partial \Sigma} c_\epsilon u_\epsilon ds_g = 0.
\end{cases}
\] (2.18)

Integrating both side of (2.18), we obtain
\[
\lim_{\epsilon \to 0} \frac{c_\epsilon \mu_\epsilon}{\lambda_\epsilon} = \frac{1}{\ell(\partial \Sigma)}. \tag{2.19}
\]

From the Hölder inequality and the Sobolev embedding theorem, one gets
\[
\int_{\partial \Sigma} |c_\epsilon u_\epsilon| ds_g \leq \ell(\partial \Sigma)^{\frac{1}{p}} \|c_\epsilon u_\epsilon\|_{L^p(\partial \Sigma)} \leq C \|\nabla_g (c_\epsilon u_\epsilon)\|_{L^q(\Sigma)}. \tag{2.20}
\]

It is well known (see for example [9], Proposition 3.5) that
\[
\int_{\Sigma} |\nabla_g (c_\epsilon u_\epsilon)|^q dv_g \leq \sup_{\|\Phi\|_{W^{1,q}}} \int_{\Sigma} \nabla_g \Phi \nabla_g (c_\epsilon u_\epsilon) dv_g, \tag{2.21}
\]
where $1/q + 1/q' = 1$. For any $1 < q < 2$, the Sobolev embedding theorem implies that $\|\Phi\|_{C^0(\overline{\Sigma})} \leq C$, where $C$ is a constant depending only on $(\Sigma, g)$. Using (2.16), (2.18)-(2.21) and the divergence theorem, we have
\[
\|\nabla_g (c_\epsilon u_\epsilon)\|_{L^q(\Sigma)}^q \leq \int_{\partial \Sigma} \Phi \frac{1}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{(\pi - \epsilon) u_\epsilon^2} ds_g + \alpha \int_{\partial \Sigma} \Phi c_\epsilon u_\epsilon ds_g - c_\epsilon \frac{\mu_\epsilon}{\lambda_\epsilon} \int_{\partial \Sigma} \Phi ds_g \\
\leq \Phi(p) + \alpha(1) + C \|\nabla_g (c_\epsilon u_\epsilon)\|_{L^q(\Sigma)} + C,
\]
which gives $\|\nabla_g (c_\epsilon u_\epsilon)\|_{L^q(\Sigma)} \leq C$. The Poincaré inequality implies that $c_\epsilon u_\epsilon$ is bounded in $W^{1, q}(\Sigma, g)$ for any $1 < q < 2$. Hence there exists some function $G$ such that $c_\epsilon u_\epsilon \to G$ weakly in $W^{1, q}(\Sigma, g)$ and $c_\epsilon u_\epsilon \to G$ strongly in $L^2(\partial \Sigma, g)$ as $\epsilon \to 0$. Testing (2.18) by $\Phi \in C^\infty(\overline{\Sigma})$, we have (2.17).

For any fixed $\delta > 0$, we choose a cut-off function $\eta \in C^\infty(\overline{\Sigma})$ such that $\eta \equiv 0$ on $\overline{B_3(p)}$ and $\eta \equiv 1$ on $\Sigma \setminus B_{2\delta}(p)$. By Lemma 2.2, we have $\lim_{\epsilon \to 0} \|\nabla_g (\eta u_\epsilon)\|_2 = 0$. Hence $e^{(\pi - \epsilon) u_\epsilon^2}$ is bounded in $L^q(\Sigma \setminus B_{2\delta}(p))$ for any $q > 1$. It follows from (2.18) that $\partial (c_\epsilon u_\epsilon)/\partial n \in L^p(\Sigma \setminus B_{2\delta}(p))$ for some $g_0 > 2$. Applying the elliptic estimate to (2.18), we get $c_\epsilon u_\epsilon$ is bounded in $C^1(\overline{\Sigma \setminus B_{4\delta}(p)})$. This completes the proof of the lemma.

Applying the elliptic estimate to (2.17), we can decompose $G$ near $p$ as the form
\[
G = -\frac{1}{\pi} \log r + A_p + O(r), \tag{2.22}
\]
where $r = \text{dist}(x, p)$ and $A_p$ is a constant depending on $\alpha, p$ and $(\Sigma, g)$.

2.3. Upper bound estimate. To derive an upper bound of $\sup_{u \in \mathcal{H}} \int_{\partial \Sigma} e^{\pi u^2} ds_g$, we use the capacity estimate, which was first used by Li [8] in this topic and also used by Li-Liu [9].

Lemma 2.5. Under the hypotheses $c_\epsilon \to +\infty$ and $x_\epsilon \to p \in \partial \Sigma$ as $\epsilon \to 0$, there holds

$$\sup_{u \in \mathcal{H}} \int_{\partial \Sigma} e^{\pi u^2} ds_g \leq \ell(\partial \Sigma) + 2\pi e^{\pi A_p}. \quad (2.23)$$

Proof. We take an isothermal coordinate system $(U, \phi)$ near $p$ such that $\phi(p) = 0$, $\phi$ maps $U$ to $\mathbb{R}^2_+$, and $\phi(U \cap \partial \Sigma) \subset \partial \mathbb{R}^2_+$. In such coordinates, the metric $g$ has the representation $g = e^{2f}(dx_1^2 + dx_2^2)$ and $f$ is a smooth function with $f(0) = 0$. Denote $\overline{u} = u \circ \phi^{-1}$. We claim that

$$\lim_{\epsilon \to 0} \frac{\lambda_\epsilon}{c_\epsilon^2} \leq 2\pi e^{\pi A_p}. \quad (2.24)$$

To confirm this claim, we set $a = \sup_{\partial B_R \cap \mathbb{R}^2_+ \bar{u}_\epsilon}$ and $b = \inf_{\partial B_R \cap \mathbb{R}^2_+ \bar{u}_\epsilon}$ for sufficiently small $\delta > 0$ and some fixed $R > 0$, where $\bar{u}_\epsilon = u_\epsilon \circ \phi^{-1}$. According to (2.10), (2.11), (2.22) and Lemma 2.4, one gets

$$a = \frac{1}{c_\epsilon} \left( \frac{1}{\pi} \log \frac{1}{\delta} + A_p + o_\delta(1) + o_\epsilon(1) \right)$$

and

$$b = c_\epsilon + \frac{1}{c_\epsilon} \left( -\frac{1}{2\pi} \log (1 + \pi^2 R^2) + o_\epsilon(1) \right),$$

where $o_\delta(1) \to 0, o_\epsilon(1) \to 0$ as $\epsilon \to 0$. From a direct computation, there holds

$$\pi(a - b)^2 = \pi c_\epsilon^2 + 2 \log \delta - 2\pi A_p - \log(1 + \pi^2 R^2) + o_\delta(1) + o_\epsilon(1). \quad (2.25)$$

Define

$$W_{a, b} = \left\{ \overline{u} \in W^{1,2}(\overline{\mathbb{B}^\epsilon_+ \setminus \mathbb{B}^+_{R_\epsilon}}) : \sup_{\partial \mathbb{B}^\epsilon_+ \setminus \mathbb{B}^+_{R_\epsilon}} \overline{u} = a, \inf_{\partial \mathbb{B}^\epsilon_+ \setminus \mathbb{B}^+_{R_\epsilon}} \overline{u} = b, \right. \left. \frac{\partial \overline{u}}{\partial v} |_{\partial \mathbb{B}^\epsilon_+ \cap (\mathbb{B}^\epsilon_+ \setminus \mathbb{B}^+_{R_\epsilon})} = 0 \right\}$$

The direct method of variation implies that $\inf_{u \in W_{a, b}} \int_{\mathbb{B}^\epsilon_+ \setminus \mathbb{B}^+_{R_\epsilon}} \|
abla \mathbb{R}_2 u\|^2 dx$ can be attained by some function $m(x) \in W_{a, b}$ with $\Delta_{\mathbb{R}_2} m(x) = 0$. We can check that

$$m(x) = \frac{a (\log |x| - \log(R_\epsilon)) + b (\log \delta - \log |x|)}{\log \delta - \log(R_\epsilon)}$$

and

$$\int_{\mathbb{B}^\epsilon_+ \setminus \mathbb{B}^+_{R_\epsilon}} \|
abla \mathbb{R}_2 m(x)\|^2 dx = \frac{\pi(a - b)^2}{\log \delta - \log(R_\epsilon)}. \quad (2.26)$$

Recalling (2.1) and (2.4), we have

$$\log \delta - \log(R_\epsilon) = \log \delta - \log R - \log \frac{\lambda_\epsilon}{c_\epsilon} + (\pi - \epsilon)c_\epsilon^2. \quad (2.27)$$
Letting $u^*_e = \max\{a, \min\{b, \overline{u}_e\}\} \in W_{a,b}$, we obtain $|\nabla u^*_e| \leq |\nabla \overline{u}_e|$ in $\mathbb{B}^+_e \setminus B^+_r$ for sufficiently small $\epsilon$. These and $\|u_e\|_{1,\alpha} = 1$ lead to

\[
\int_{\mathbb{B}^+_e \setminus B^+_r} |\nabla u^*_e|^2 \, dx
\leq 1 + \alpha \int_{\partial \Sigma} u^*_e \, ds_g - \int_{\Sigma \setminus \phi^{-1}(\mathbb{B}^+_e)} |\nabla u_e|^2 \, dv_g - \int_{\phi^{-1}(\mathbb{B}^+_r)} |\nabla u_e|^2 \, dv_g. \tag{2.28}
\]

Now we compute $\int_{\Sigma \setminus \phi^{-1}(\mathbb{B}^+_e)} |\nabla u_e|^2 \, dv_g$ and $\int_{\phi^{-1}(\mathbb{B}^+_r)} |\nabla u_e|^2 \, dv_g$. In view of (2.22), we obtain

\[
\int_{\Sigma \setminus \phi^{-1}(\mathbb{B}^+_e)} |\nabla u_e|^2 \, dv_g = \frac{1}{c^2} \left( \frac{1}{\pi} \log \Delta + A_p + \alpha \|G\|_{L^2(\partial \Sigma)}^2 + o_r(1) + o(1) \right). \tag{2.29}
\]

According to (2.6), (2.10) and (2.11), one gets

\[
\int_{\phi^{-1}(\mathbb{B}^+_r)} |\nabla u_e|^2 \, dv_g = \frac{1}{c^2} \left( \frac{1}{\pi} \log R + \frac{1}{\pi} \log \frac{\pi}{2} + o_r(1) + o(1) \right), \tag{2.30}
\]

where $o_R(1) \to 0$ as $R \to +\infty$. In view of (2.25)-(2.30), we obtain

\[
\log \frac{\lambda}{c^2} \leq \log (2\pi) + \pi A_p + o(1),
\]

where $o(1) \to 0$ as $\epsilon \to 0$ first, then $R \to +\infty$ and $\delta \to 0$. Hence the claim (2.24) is confirmed. Combining (2.2), (2.24) and Lemma 2.3, we finish the proof of the lemma. \hfill \Box

### 2.4. Existence result.

The content in this section is carried out under the hypothesis $0 \leq \alpha < \lambda_1(\partial \Sigma)$. We take an isothermal coordinate system $(U, \phi)$ near $p$ such that $\phi(p) = 0$, $\phi$ maps $U$ to $\mathbb{R}^2$, and $\phi(U \cap \partial \Sigma) \subset \partial \mathbb{R}^2$. In such coordinates, the metric $g$ has the representation $g = e^{2f}(dx_1^2 + dx_2^2)$ and $f$ is a smooth function with $f(0) = 0$. Set a cut-off function $\xi \in C^\infty_0(B_{2R_0}(p))$ with $\xi = 1$ on $B_{R_0}(p)$ and $\|\nabla \xi\|_{L^\infty} = O(1/(R_0))$. Denote $\beta = G + 1/\pi \log r - A_p$, where $G$ is defined as in (2.22). Let $R = \log^2 \epsilon$, then $R \to +\infty$ and $R \to 0$ as $\epsilon \to 0$. We construct a blow-up sequence of functions

\[
v_e = \begin{cases}
  \left( \frac{c - \frac{1}{2\pi c} \log \frac{\pi^2 x_1^2 + (\pi x_2 + \epsilon)^2}{c^2} + B}{c} \right) \phi, & x \in B_{R_0}(p), \\
  G - \xi \beta, & x \in B_{2R_0}(p) \setminus B_{R_0}(p), \\
  \frac{G}{c}, & x \in \Sigma \setminus B_{2R_0}(p),
\end{cases}
\tag{2.31}
\]

for some constants $B$, $c$ to be determined later, such that

\[
\int_{\Sigma} |\nabla v_e|^2 dv_g - \alpha \int_{\partial \Sigma} (v_e - \overline{v}_e)^2 \, ds_g = 1 \tag{2.32}
\]
and \( v_\epsilon - \overline{v}_\epsilon \in \mathcal{H} \), where \( \overline{v}_\epsilon = \int_{\partial \Sigma} v_\epsilon ds_g / \ell(\partial \Sigma) \). Note that \( \int_{\partial \Sigma} G ds_g = 0 \), one has \( \overline{v}_\epsilon = O(Re \log(Re)) \), and then

\[
\int_{\partial \Sigma} |v_\epsilon - \overline{v}_\epsilon|^2 ds_g = \frac{\|G\|_2^2}{c^2} + O \left( Re \log^2(Re) \right). \tag{2.33}
\]

In order to assure that \( v_\epsilon \in W^{1,2}(\Sigma, g) \), we obtain

\[
c^2 - \frac{1}{2\pi} \log (\pi^2 R^2) + B + O \left( \frac{1}{R} \right) = -\frac{1}{2\pi} \log(Re) + A_p,
\]

which is equivalent to

\[
c^2 = \frac{1}{\pi} \log \pi - \frac{1}{\pi} \log \epsilon - B + A_p + O \left( \frac{1}{R} \right). \tag{2.34}
\]

A delicate calculation shows

\[
\int_{B_{R\epsilon}(p)} \left| \nabla_g v_\epsilon \right|^2 dv_g = \frac{1}{4\pi^2 c^2} \int_{Q(R)} \left| \nabla_{\mathbb{R}^2} \log \left( \pi^2 x_1^2 + \pi^2 x_2^2 \right) \right|^2 dx_1 dx_2
\]

\[
= \frac{1}{\pi^2 c^2} \left( \log(\pi R) + \int_0^\pi \log \sin \theta \ d\theta - 2 \int_0^{\arcsin \frac{1}{\sqrt{2}}} \log \sin \theta \ d\theta + O \left( \frac{\log R}{R} \right) \right)
\]

\[
= \frac{1}{\pi^2 c^2} \left( \log R + \log \frac{\pi}{2} + O \left( \frac{\log R}{R} \right) \right),
\]

where \( Q(R) = \{(x_1, x_2) : (x_1, x_2 - 1/\pi) \in \mathbb{B}_R^+, \ x_2 \geq 1/\pi \} \). According to (2.31) and (2.29), one has

\[
\int_{\Sigma \setminus B_{R\epsilon}(p)} \left| \nabla_g v_\epsilon \right|^2 dv_g = \frac{1}{c^2} \left( A_p + \alpha \|G\|_2^2(\partial \Sigma) - \frac{1}{\pi} \log(Re) + O \left( Re \log^2(Re) \right) \right).
\]

Then we get

\[
\int_{\Sigma} \left| \nabla_g v_\epsilon \right|^2 dv_g = \frac{1}{c^2} \left( A_p + \alpha \|G\|_2^2(\partial \Sigma) + \frac{1}{\pi} \log \left( \frac{\pi}{2\epsilon} \right) + O \left( \frac{\log R}{R} \right) + O \left( Re \log^2(Re) \right) \right).
\]

In view of (2.32), (2.33), (2.35), there holds

\[
c^2 = A_p + \frac{1}{\pi} \log \left( \frac{\pi}{2\epsilon} \right) + O \left( \frac{\log R}{R} \right) + O \left( Re \log^2(Re) \right). \tag{2.36}
\]

According to (2.34) and (2.36), one gets

\[
B = \frac{1}{\pi} \log 2 + O \left( \frac{\log R}{R} \right) + O \left( Re \log^2(Re) \right).
\]

It follows that in \( \partial \Sigma \cap B_{Re}(p) \),

\[
\pi(v_\epsilon - \overline{v}_\epsilon)^2 \geq \log(2\pi) + \pi A_p - \log \left( \frac{c^2 + \pi^2 x_1^2}{\epsilon} \right) + O \left( \frac{\log R}{R} \right) + O \left( Re \log^2(Re) \right).
\]

Hence

\[
\int_{\partial \Sigma \cap B_{Re}(p)} e^{\pi(v_\epsilon - \overline{v}_\epsilon)^2} ds_g \geq 2\pi e^{\pi A_p} + O \left( \frac{\log R}{R} \right) + O \left( Re \log^2(Re) \right). \tag{2.37}
\]
On the other hand, from the fact $e^t \geq t + 1$ for any $t > 0$ and (2.31), we get
\[
\int_{\partial \Sigma \setminus B_R(p)} e^{\pi(v_\epsilon - \pi_\epsilon)^2} \, ds_g \geq \int_{\partial \Sigma \setminus B_R(p)} (1 + \pi(v_\epsilon - \pi_\epsilon)^2) \, ds_g \geq \ell(\partial \Sigma) + \frac{\pi \|G\|^2_{L^2(\partial \Sigma)}}{c^2} + O\left(R \epsilon \log^2(Re)\right). \tag{2.38}
\]
From (2.37) and (2.38), there holds
\[
\int_{\partial \Sigma} e^{\pi(v_\epsilon - \pi_\epsilon)^2} \, ds_g \geq \ell(\partial \Sigma) + 2\pi e^{\pi \Lambda_\epsilon} + \frac{\pi \|G\|^2_{L^2(\partial \Sigma)}}{c^2} + O\left(\frac{\log R}{R}\right) + O\left(R \epsilon \log^2(Re)\right).
\]
Since $R = \log^2 \epsilon$, we obtain
\[
\int_{\partial \Sigma} e^{\pi(v_\epsilon - \pi_\epsilon)^2} \, ds_g > \ell(\partial \Sigma) + 2\pi e^{\pi \Lambda_\epsilon} \tag{2.39}
\]
for sufficiently small $\epsilon > 0$. The contradiction between (2.23) and (2.39) indicates that $c_\epsilon$ must be bounded, which together with elliptic estimates completes the proof of Theorem 1.1.

3. Higher order eigenvalue cases. In this section, we will prove Theorem 1.2 involving higher order eigenvalues through blow-up analysis. Let $k$ be a positive integer and $E_k(\partial \Sigma)$ be defined by (1.11). Denote the dimension of $E_k(\partial \Sigma)$ is $s_k$. From ([3], Theorem 9.31), it is known that $s_k$ is a finite constant depending only on $k$. Then we can find a set of normal orthogonal basis \( \{e_i \in C^\infty(\Sigma), 1 \leq i \leq s_k\} \) of $E_k(\partial \Sigma)$ satisfying
\[
\begin{align*}
\int_{\partial \Sigma} e_i \, ds_g &= 0, \\
\Delta_g e_i &= 0 \quad \text{in } \Sigma, \\
\frac{\partial e_i}{\partial n} &= \lambda_k(\partial \Sigma)e_i \quad \text{on } \partial \Sigma,
\end{align*}
\tag{3.1}
\]
where $k_0 \leq k$ is a positive integer.

3.1. Blow-up analysis. Let $\lambda_{k+1}(\partial \Sigma)$ and $S$ be defined by (1.10) and (1.12). In view of Lemma 2.1 and (3.1), we have

Lemma 3.1. Let $0 < \alpha < \lambda_{k+1}(\partial \Sigma)$ be fixed. For any $0 < \epsilon < \pi$, the supremum $\sup_{u \in S} \int_{\partial \Sigma} e^{(\pi - \epsilon)u^2} \, ds_g$ is attained by some function $u_\epsilon \in S \cap C^\infty(\Sigma)$. Moreover, the Euler-Lagrange equation of $u_\epsilon$ is
\[
\begin{align*}
\Delta_g u_\epsilon &= 0 \quad \text{in } \Sigma, \\
\frac{\partial u_\epsilon}{\partial n} &= \frac{1}{\lambda_\epsilon} u_\epsilon e^{(\pi - \epsilon)u^2} + \alpha u_\epsilon - \sum_{i=1}^{s_k} \beta_{\epsilon,i} e_i \quad \text{on } \partial \Sigma, \\
\lambda_\epsilon &= \int_{\partial \Sigma} u_\epsilon^2 e^{(\pi - \epsilon)u^2} \, ds_g, \\
\mu_\epsilon &= \frac{\ell(\partial \Sigma)}{\lambda_\epsilon} \int_{\partial \Sigma} u_\epsilon e^{(\pi - \epsilon)u^2} \, ds_g, \\
\beta_{\epsilon,i} &= \int_{\partial \Sigma} u_\epsilon e^{(\pi - \epsilon)u^2} e_i \, ds_g.
\end{align*}
\tag{3.2}
\]

We now perform the blow-up analysis. Let $c_\epsilon = |u_\epsilon(x_\epsilon)| = \max_{\Sigma} |u_\epsilon|$. Let us assume that $c_\epsilon = u_\epsilon(x_\epsilon) \to +\infty$ and $x_\epsilon \to p$ as $\epsilon \to 0$. Latter we see that $c_\epsilon$ is bounded. Applying maximum principle to (3.2), we have $p \in \partial \Sigma$. Analogous to Lemma 2.4, we get
Lemma 3.2. There holds $c_\epsilon u_\epsilon \to G$ weakly in $W^{1,q}(\Sigma,g)$ ($\forall 1 < q < 2$), $c_\epsilon u_\epsilon \to G$ strongly in $L^2(\partial \Sigma, g)$ and $c_\epsilon u_\epsilon \to G$ in $C^1_{loc}(\Sigma \setminus \{p\})$ as $\epsilon \to 0$, where $G$ is a Green function satisfying

$$\begin{cases}
\Delta_g G = \delta_p & \text{in } \Sigma, \\
\frac{\partial G}{\partial n} = \alpha G - \frac{1}{\ell(\partial \Sigma)} - \sum_{i=1}^{n} e_i c_i(p) & \text{on } \partial \Sigma \setminus \{p\}, \\
\int_{\partial \Sigma} G ds = 0.
\end{cases}$$

Moreover, $G$ near $p$ can be decomposed into

$$G = -\frac{1}{\pi} \log r + A_p + O(r),$$

where $r = \text{dist}(x,p)$ and $A_p$ is a constant depending on $\alpha, p$ and $(\Sigma, g)$. Analogous to Lemma 2.5, using the capacity estimate, we derive an upper bound of the supremum (1.9):

$$\sup_{u \in \mathcal{S}} \int_{\partial \Sigma} e^{\pi u^2} ds_g \leq \ell(\partial \Sigma) + 2\pi e^{\pi A_p}. \quad (3.4)$$

3.2. Existence result. The content is carried out under the hypothesis $0 \leq \alpha < \lambda_{k+1}(\partial \Sigma)$. We take an isothermal coordinate system $(U, \phi)$ near $p$ such that $\phi(p) = 0$, $\phi$ maps $U$ to $\mathbb{R}^2_+$, and $\phi(U \cap \partial \Sigma) \subset \partial \mathbb{R}^2_+$. In such coordinates, the metric $g$ has the representation $g = e^{2f}(dx_1^2 + dx_2^2)$ and $f$ is a smooth function with $f(0) = 0$. Set a cut-off function $\xi \in C_0^\infty(B_{2R}(p))$ with $\xi = 1$ on $B_{R}(p)$ and $\|\nabla_g \xi\|_{L^\infty} = O(1/(2Re))$. Denote $\beta = G + (1/\pi) \log r - A_p$, where $G$ is defined as in (3.3). Let $R = \log^2 \epsilon$, then $R \to +\infty$ and $R \epsilon \to 0$ as $\epsilon \to 0$. We construct a blow-up sequence of functions

$$v_\epsilon = \begin{cases}
\left(c - \frac{1}{2\pi c} \log \frac{\pi^2 x_1^2 + (\pi x_2 + \epsilon)^2}{\epsilon^2} + \frac{B}{c}\right) \circ \phi, & x \in B_{2Re}, \\
\frac{G - \epsilon \beta}{c}, & x \in B_{2Re} \setminus B_{Re}, \\
\frac{G}{c}, & x \in \Sigma \setminus B_{2Re},
\end{cases}$$

for some constants $B, c$ to be determined later, such that

$$\int_{\Sigma} |\nabla g v_\epsilon|^2 dv_g - \alpha \int_{\partial \Sigma} (v_\epsilon - \bar{v}_\epsilon)^2 ds_g = 1$$

and $v_\epsilon - \bar{v}_\epsilon \in \mathcal{S}$, where $\bar{v}_\epsilon = \int_{\partial \Sigma} v_\epsilon ds_g / \ell(\partial \Sigma)$. Similar to the subsection 2.4, we determine the constants

$$B = \frac{1}{\pi} \log 2 + O\left(\frac{\log R}{R}\right) + O\left(\Re \log^2(Re)\right)$$

and

$$c^2 = A_p + \frac{1}{\pi} \log \left(\frac{\pi}{2\epsilon}\right) + O\left(\frac{\log R}{R}\right) + O\left(\Re \log^2(Re)\right).$$

Then we get

$$\int_{\partial \Sigma} e^{\pi (v_\epsilon - \bar{v}_\epsilon)^2} ds_g \geq 2\pi e^{\pi A_p} + \ell(\partial \Sigma) + \frac{\pi \|G\|^2_{L^2(\partial \Sigma)}}{c^2} + O\left(\Re \log^2(Re)\right) + O\left(\frac{\log R}{R}\right). \quad (3.5)$$
Following Yang [20], we set
\[
v^{∗}_\epsilon = (v_\epsilon - \nabla_\epsilon) - \sum_{i=1}^{s_\epsilon} e_i \int_{\partial \Sigma} (v_\epsilon - \nabla_\epsilon) e_i \, ds_g \in E^\perp_k,
\]
which gives
\[
\begin{cases}
v^{∗}_\epsilon = (v_\epsilon - \nabla_\epsilon) + O \left( \frac{1}{R^2} \right), \\
\|v^{∗}_\epsilon\|_{1,\alpha}^2 = 1 + O \left( \frac{1}{R^2} \right), \\
\int_{\partial \Sigma} v^{∗}_\epsilon \, ds_g = 0.
\end{cases}
\]
It is easy to verify \( V = v^{∗}_\epsilon / \|v^{∗}_\epsilon\|_{1,\alpha}^2 \in S \). According to this and (3.5), we have
\[
\int_{\partial \Sigma} e^{\pi V^2} \, ds_g \geq \left( 1 + O \left( \frac{1}{R^2} \right) \right) \int_{\partial \Sigma} e^{\pi (v_\epsilon - \nabla_\epsilon)^2} \, ds_g \\
\geq 2\pi e^{\pi A_\epsilon} + \ell(\partial \Sigma) + \pi \|G\|_{L^2(\partial \Sigma)}^2 + O \left( R\epsilon \log^2 R \right) + O \left( \frac{\log R}{R} \right).
\]
Since \( R = \log^2 \epsilon \), we obtain
\[
\int_{\partial \Sigma} e^{\pi V^2} \, ds_g > 2\pi e^{\pi A_\epsilon} + \ell(\partial \Sigma) \quad (3.6)
\]
for sufficiently small \( \epsilon > 0 \). The contradiction between (3.4) and (3.6) indicates that \( c_\epsilon \) must be bounded, which together with elliptic estimates completes the proof of Theorem 1.2.

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