A Gauss curvature flow related to the Orlicz-Aleksandrov problem *

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Abstract: In this paper we first obtain the existence of smooth solutions to Orlicz-Aleksandrov problem via a Gauss-like curvature flow.

Keywords: Gauss curvature flow; Orlicz-Aleksandrov problem; Monge-Ampère equation

2010 Mathematics Subject Classification: 53E99 52A20 35K96.

1 Introduction

The curvature measure of convex bodies is one of the basic principles in convex geometry analysis. In particular, it plays key role in the Brunn-Minkowski theory of convex bodies. The most studied of the curvature measures is the Aleksandrov’s integral curvature (also called integral Gauss curvature) defined and studied by Aleksandrov [1] using a topological argument. Moreover, the famous Aleksandrov problem with respect to integral curvatures is an important cornerstone of the Brunn-Minkowski theory.

The integral curvature, $J(K, \omega)$ on the unit sphere $S^{n-1}$, of $K \in \mathcal{K}_o^n$ (the set of convex bodies containing the origin in their interiors) is defined by

$$J(K, \omega) = \mathcal{H}^{n-1}(\mathcal{R}_K(\omega))$$

for each Borel set $\omega \subset S^{n-1}$. Here $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, and $\mathcal{R}_K$ is the radial Gauss image.

The classical Aleksandrov problem, roughly speaking, asks for necessary and sufficient conditions for a given Borel measure $\mu$ on the unit sphere so that the measure is the

*Research is supported in part by the Natural Science Foundation of China (No.11871275; No.11371194).
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integral curvature of a convex body in $\mathbb{R}^n$. Namely, this problem is to find a convex body $K \subset \mathbb{R}^n$ such that

$$dJ(K, \cdot) = d\mu \text{ on } S^{n-1}. \quad (1.1)$$

The PDE associated with classical Aleksandrov problem for integral curvature asks (see [29] or [30]): If the given measure $\mu$ has a density $g$, then the (1.1) is equivalent to solving the following Monge-Ampère type equation

$$\frac{h}{(|\nabla h|^2 + h^2)^{\frac{n}{2}}} \det(\nabla^2 h + hI) = g,$$

where $h$ is the support function of convex body $K$, $\nabla h$ is the gradient of $h$, while $\nabla^2 h$ is the Hessian matrix of $h$ with respect to an orthonormal frame on $S^{n-1}$, and $I$ is the identity matrix of order $n - 1$.

In the Huang et al’s work [18], for all real $p$, there is a geometrically natural $L^p$-extension of integral curvature (also called $L^p$-integral curvature). To state the $L^p$-Aleksandrov problem with respect to $L^p$-integral curvature, in full generality, it is necessary to introduce the entropy functional

$$E(K) = -\int_{S^{n-1}} \log h_K(v) dv$$

for $K \in \mathcal{K}_o^n$. Here the integration is with respect to spherical Lebesgue measure.

For $p \neq 0$, the $L^p$-integral curvature, $\mathcal{J}_p(K, \cdot)$, of $K \in \mathcal{K}_o^n$, as a Borel measure on $S^{n-1}$ is defined by the variational formula

$$\left. \frac{d}{dt} E(K + pt \cdot Q) \right|_{t=0} = \frac{1}{p} \int_{S^{n-1}} \rho_Q^p(u) d\mathcal{J}_p(K, u) \quad (1.2)$$

which holds for each $Q \in \mathcal{K}_o^n$, where $\rho_Q$ is the radial function and $K + pt \cdot Q$ is the harmonic $L^p$-combination (see [18]). It turns out that for each $K \in \mathcal{K}_o^n$,

$$d\mathcal{J}_p(K, \cdot) = \rho^p_K d\mathcal{J}(K, \cdot).$$

When $p = 0$ in the harmonic $L^p$-combination, the integral curvature, $\mathcal{J}(K, \cdot)$, of $K \in \mathcal{K}_o^n$ is a Borel measure on $S^{n-1}$ that can be defined by

$$\left. \frac{d}{dt} E(K + t \cdot Q) \right|_{t=0} = -\int_{S^{n-1}} \log \rho_Q(u) d\mathcal{J}(K, u), \quad (1.3)$$

which holds for $Q \in \mathcal{K}_o^n$. It should be emphasized here that the variational formula (1.3) is not Aleksandrov’s definition of classical integral curvatures.

**The $L^p$-Aleksandrov problem** For a fixed $p \in \mathbb{R}$, and a given Borel measure $\mu$ on $S^{n-1}$, what are the necessary and sufficient conditions so that

$$d\mathcal{J}_p(K, \cdot) = d\mu \quad (1.4)$$
of a convex body $K \in \mathcal{K}_o^n$?

Moreover, the PDE associated with the $L_p$-Aleksandrov problem asks (see [18]): If the given measure $\mu$ has a density $g$, then the (1.4) is equivalent to solving the following Monge-Ampère type equation

$$\frac{h^{1-p}}{(|\nabla h|^2 + h^2)^{\frac{p}{2}}} \det(\nabla^2 h + hI) = g.$$  \hspace{1cm} (1.5)

For the $L_p$ Aleksandrov problem (1.4), when $p > 0$, the existence of solutions (measure solutions) has been completely solved (see [18], Theorem 7.1); When $p = 0$, the $L_p$ Aleksandrov problem (1.4) is exactly classical Aleksandrov problem; When $p < 0$ and the measure is even, the sufficient condition for the existence of the solution to the $L_p$ Aleksandrov problem (1.4) is given (see [18], Theorem 7.3). However, there are still many important problems, such as non-even solution, smoothness of the solution, etc., has not been solved.

Recently, the concept of the Orlicz-integral curvature $\mathcal{J}_\phi(K, \cdot)$ of convex body $K \in \mathcal{K}_o^n$ was defined by the following variational formula (see [12] in detail): Let $\Omega \subset S^{n-1}$ be a closed set not contained in any closed hemisphere of $S^{n-1}$. Denote $C(\Omega)$ by the set of continuous functions on $\Omega$. For $f \in C(\Omega)$ and $\rho_K \in C^+(\Omega)$. If $\phi$ is of continuously differentiable and strictly monotonic function on $(0, \infty)$, then

$$\frac{d}{dt}E(\langle \rho_t \rangle) \bigg|_{t=0} = \int_{\Omega} f(u)d\mathcal{J}_\phi(K, u),$$

where $\rho_t(u) = \phi^{-1}(\phi(\rho_K(u)) + tf(u))$ and $\langle \rho_t \rangle$ denotes the convex hull. It turns out that for $K \in \mathcal{K}_o^n$

$$d\mathcal{J}_\phi(K, \cdot) = \phi(\rho_K)d\mathcal{J}(K, \cdot).$$  \hspace{1cm} (1.6)

When $\phi(t) = t^p$ in (1.6), it is just the $L_p$ integral curvature.

The following Orlicz-Aleksandrov problem was proposed in [12]:

**The Orlicz-Aleksandrov problem** For a suitable continuous function $\phi : (0, \infty) \to (0, \infty)$, and a non-zero finite Borel measure $\mu$ on $S^{n-1}$, do there exists a constant $\lambda > 0$ and a convex body $K \in \mathcal{K}_o^n$ such that

$$\lambda d\mathcal{J}_\phi(K, \cdot) = d\mu \text{ on } S^{n-1}?$$  \hspace{1cm} (1.7)

For the Orlicz-Aleksandrov problem, when the given measure is even, was first solved in two situations via the variational method, see, e.g., [12].
We note that when the given measure $\mu$ has a density $\varphi^1$, then the Orlicz-Aleksandrov problem (1.7) is equivalent to solving the following Monge-Ampère type equation (see Section 3 in detail):

$$\frac{\lambda h\varphi(1/h)}{(|\nabla h|^2 + h^2)^{n/2}} \det(\nabla^2 h + hI) = \varphi^1,$$

(1.8)

We know that the Orlicz Aleksandrov problem is a generalization of the classical Aleksandrov problem [1, 29, 30]. When $\varphi(t) = t^p$ with $t \in \mathbb{R}$, Eq. (1.8) corresponds to the $L_p$ Aleksandrov problem [18, 35].

Moreover, the smoothness of solutions to Aleksandrov type problems (or Minkowski type problems) and the non-even smooth solution of the Orlicz Aleksandrov problems (as well as related Monge-Ampère equation) are open.

Recently, the argument of the smoothness of the even-solutions of Minkowski type problems via the geometric flow method has been made great progress (see [8, 9, 10, 25, 26] for details).

Motivated by the above statements, we first in this paper study the existence of smooth non-even solution to the Eq. (1.8) with $\lambda = 1$. The following theorem shows the existence of the smooth solution to the Orlicz-Aleksandrov problem.

**Theorem 1.1** Suppose $\varphi : (0, +\infty) \to (0, +\infty)$ is a continuous function. For any given positive smooth function $\varphi^1$ on $S^{n-1}$ satisfying

$$\limsup_{s \to +\infty} \varphi(s) < \varphi^1 < \liminf_{s \to 0^+} \varphi(s),$$

(1.9)

then the equation (1.8) has a smooth solution $h$ with $\lambda = 1$.

In order to obtain the Theorem 1.1, our main idea is reflected in the following two folds: I). Find a suitable anisotropic Gauss-like curvature flow; II). Find a monotone functional of the solution to the flow, which is the key to prove the existence of a solution to Eq. (1.8).

Let $M_0$ be a closed, smooth and strictly convex hypersurface in $\mathbb{R}^n$ enclosing the origin and given by a smooth embedding $X_0 : S^{n-1} \to \mathbb{R}^n$. In this paper we consider a family of closed hypersurfaces $M_t$ given by smooth maps $X : S^{n-1} \times [0, T) \to \mathbb{R}^n$ satisfying the initial value problem:

$$\begin{cases}
\frac{\partial X(x, t)}{\partial t} = -g(v) \frac{r^n}{\varphi^1(r)} \mathcal{K} v + X(x, t), \\
X(x, 0) = X_0(x),
\end{cases}$$

(1.10)

where $g$ is a given positive smooth function on $S^{n-1}$, $r = |X|$ is the distance from $X$ to the origin, $\varphi : (0, +\infty) \to (0, +\infty)$ is a positive smooth function, $\mathcal{K}$ is the Gauss curvature of
the hypersurface $M_t$ parametrized by $X(x, t)$, $\nu$ is the unit outer normal vector at $X(x, t)$, and $T$ is the maximal time for which the solution exists.

The Gauss curvature flow was introduced by Firey [11] to model the shape change of worn stones. Since then, various Gauss curvature flows have been extensively studied, see, e.g. [2, 3, 6, 7, 8, 9, 10, 13, 16, 22, 23, 25, 26, 33]. In addition, the method of geometric flow to solve some famous geometric inequalities has also attracted the attention of many scholars, see, e.g. [5, 15, 19, 20, 21, 27, 28, 34].

We will show that the flow (1.10) has a long-time solution, and derive that the support function of limiting hypersurface of this flow provides a smooth solution to Eq. (1.8). The following functional related to the flow (1.10) plays an important role in our argument,

$$F(M_t) = \int_{S^{n-1}} \log h(x, t) dx - \int_{S^{n-1}} \frac{\varphi(r(\xi, t))}{g(x)} d\xi, \quad (\cdot, t) \in S^{n-1} \times [0, T),$$

where $h, r$ are the support function and radial function of $M_t$, and

$$\varphi(t) = \int_0^t \frac{s}{s} ds.$$

Now, we obtain the long-time existence and convergence of the flow (1.10).

**Theorem 1.2** Let $M_0$ be a closed, smooth, and uniformly convex hypersurface in $\mathbb{R}^n$. Assume functions $g$ and $\varphi$ satisfy the assumptions of Theorem 1.1, then the flow (1.10) has a smooth solution $M_t$, which exists for any time $t \in [0, \infty)$. Moreover, when $t \to \infty$, the support function $h_t$ of $M_t$ converges in $C^\infty$ to a smooth solution $h_\infty$ to (1.8) with $\lambda = 1$, which is the support function of a smooth, closed, and uniformly convex hypersurface $M_\infty$.

The organization is as follows. The corresponding background materials and some results are introduced in Section 2. In Section 3, we establish the Gauss-like curvature flow and related functional. In Section 4, we obtain the long-time existence of the flow (1.10). In section 5, we prove the Theorem 1.2, and provide a special uniqueness of Orlicz-Aleksandrov problem.

## 2 Preliminaries

### 2.1 Convex body and Orlicz integral curvature

The basic facts in this subsection can be found in Gardner and Schneider’s book [14, 31], which are the standard references regarding convex bodies, and references [17, 18]. Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. The unit sphere in $\mathbb{R}^n$ is denoted by $S^{n-1}$. A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior. Denote by $\mathcal{K}_0^n$ the
class of convex bodies in \( \mathbb{R}^n \) that contain the origin in their interiors. Let \( K \in \mathcal{K}_n^o \), the radial function \( \rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) is defined by

\[
\rho_K(x) = \max\{\lambda : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

For \( u \in S^{n-1} \), there is \( \rho_K(u)u \in \partial K \).

The support function, \( h_K : S^{n-1} \to \mathbb{R} \), of a convex body \( K \) in \( \mathbb{R}^n \) is defined by

\[
h_K(u) = \max\{u \cdot x : x \in K\}
\]

for \( u \in S^{n-1} \), where \( u \cdot x \) is the standard inner product of \( u \) and \( x \) in \( \mathbb{R}^n \).

The radial function and the support function are related,

\[
h_K(u) = \frac{1}{\rho_K(u)} \quad \text{and} \quad \rho_K(x) = \frac{1}{h_K(x)}.
\]

For a convex body \( K \in \mathcal{K}_n^o \), the polar body \( K^* \) of \( K \) is

\[
K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \text{ for all } y \in K\}.
\]

The support function and radial function of the convex body and its polar are related in the following way,

\[
h_K(x) = \frac{1}{\rho_{K^*}(x)}, \quad \rho_K(x) = \frac{1}{h_{K^*}(x)}.
\]

(2.1)

The integral curvature, \( J(K, \cdot) \), of \( K \in \mathcal{K}_n^o \) is a Borel measure on \( S^{n-1} \) defined by

\[
J(K, \omega) = \mathcal{H}(\mathcal{R}_K(\omega))
\]

for each Borel set \( \omega \subset S^{n-1} \), where radial Gauss image \( \mathcal{R}_K(\omega) \) of \( \omega \) given by

\[
\mathcal{R}_K(\omega) = \{u \in S^{n-1} : \rho_K(v)u \in H_K(u) \text{ for some } v \in \omega\},
\]

where \( H_K \) is the supporting hyperplane of \( K \) with the outer unit normal \( u \).

If the convex body \( K \) is \( C^2 \) smooth with positive Gauss curvature, then the integral curvature has a continuous density (see, e.g., [18, 29]),

\[
\frac{h}{(||\nabla h||^2 + h^2)^{\frac{n}{2}}} \det(\nabla^2 h + I),
\]

(2.2)

where \( h = \rho_K^{-1} \), while \( \nabla h \) and \( \nabla^2 h \) are the gradient and the Hessian matrix of \( h \), and \( I \) is the identity matrix, with respect to an orthonormal frame on \( S^{n-1} \).
For $K \in \mathcal{K}_0^n$ and $\phi : (0, \infty) \to (0, \infty)$ is a continuous function. The Orlicz-integral curvature was defined by (see [12])
\[
\mathcal{J}_\phi(K, \omega) = \int_{\mathbb{R}_K(\omega)} \phi(\rho_K(\alpha^*_K(u)))du
\]
for each Borel set $\omega \subset S^{n-1}$. Here $\alpha^*_K(u)$ is the reverse radial Gauss map of $K$. Moreover, the Orlicz-integral curvature is absolutely continuous with respect to the classical integral curvature $\mathcal{J}(K, \cdot)$, namely
\[
d\mathcal{J}_\phi(K, \cdot) = \phi(\rho_K)d\mathcal{J}(K, \cdot). \quad (2.3)
\]
Obviously, when $\phi(t) = t^p$ ($p \in \mathbb{R}$), the Orlicz-integral curvature is just the $L_p$-integral curvature introduced by Huang et al [18]
\[
\mathcal{J}_p(K, \omega) = \int_{\mathbb{R}_K(\omega)} \rho^p_K(\alpha^*_K(u))du.
\]

2.2 Convex hypersurfaces

Let $M$ be a closed, smooth, uniformly convex hypersurfaces in $\mathbb{R}^n$. Assume that $M$ is parametrized by the inverse Gauss map
\[
X = v^{-1}_M : S^{n-1} \to M.
\]
The support function $h : S^{n-1} \to \mathbb{R}$ of $M$ is defined by
\[
h(x) = \max\{\langle x, y \rangle, \; y \in M\}, \quad x \in S^{n-1}.
\]
The supremum is attained at a point $y$ such that $x$ is the outer normal of $M$ at $X$. It is easy to check that
\[
X(x) = h(x)x + \nabla h(x), \quad (2.4)
\]
where $\nabla$ is the covariant derivative with respect to the standard metric $\delta_{ij}$ of the sphere $S^{n-1}$. Denote the radial function of $M_1$ by $\rho(u, t)$. From (2.4), $u$ and $x$ are related by
\[
\rho(u)u = h(x)x + \nabla h(x), \quad (2.5)
\]
then there is
\[
x = \frac{\rho(u)u - \nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}.
\]
From the definitions of radial function and $r$, then
\[
r = |X| = \left(|\nabla h|^2 + r^2\right)^{\frac{1}{2}} \quad (2.6)
\]
and

\[ h = \frac{r^2}{\sqrt{r^2 + |\nabla r|^2}}. \]  

(2.7)

The second fundamental form \( A_{ij} \) of \( M \) can be computed in terms of the support function (see e.g., [4, 32])

\[ A_{ij} = \nabla_i \nabla_j h + h \delta_{ij}, \]  

(2.8)

where \( \nabla_{ij} = \nabla_i \nabla_j \) denotes the second order covariant derivative with respect to \( e_{ij} \). The induced metric matrix \( g_{ij} \) of \( M \) can be derived by Weingarten’s formula,

\[ e_{ij} = \langle \nabla_i x, \nabla_j x \rangle = A_{ik} A_{lj} g^{kl}. \]  

(2.9)

It follows from (2.8) and (2.9) that the principal radii of curvature of \( M \), under a smooth local orthonormal frame on \( S^{n-1} \), are the eigenvalues of the matrix

\[ b_{ij} = \nabla_i \nabla_j h + h \delta_{ij}. \]

We will use \( b^{ij} \) to denote the inverse matrix of \( b_{ij} \). In particular, the Gauss curvature is given by

\[ K(x) = (\det(\nabla_i \nabla_j h + h \delta_{ij}))^{-1} = S_n^{-1}(\nabla_i \nabla_j h + h \delta_{ij}), \]  

(2.10)

where

\[ S_k = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \]

denotes the \( k \)-th elementary symmetric polynomial.

Let \( r, \alpha \) and \( \alpha^* \) be the radial function, radial Gauss mapping and reverse radial Gauss mapping of \( M \). It is well-known that the determinants of the Jacobian of radial Gauss mapping \( \alpha \) and reverse radial Gauss mapping of \( M \) are given by, see e.g. [17],

\[ |J\alpha|(\xi) = \frac{r^{n}(\xi)K(r(\xi))}{h(\alpha(\xi))}, \]  

(2.11)

and

\[ |J\alpha^*|(x) = \frac{h(x)}{r^{n}(\alpha^*(x))K(v^{-1}_M(x))}, \]  

(2.12)
3 Gauss curvature flow and its associated functional

In this section, we shall introduce an anisotropic Gauss-like curvature flow and its associated functional for solving the Orlicz-Aleksandrov problem.

First, we need to show that Orlicz-Aleksandrov problem is equivalent to solving a Monge-Ampère type equation. From (2.3), we know that the Orlicz-integral curvature $J_\phi(K, \cdot)$ is absolutely continuous with respect to the integral curvature $J(K, \cdot)$, namely,

$$dJ_\phi(K, \cdot) = \phi(\rho_K)dJ(K, \cdot).$$

(3.1)

If the convex body $K$ is $C^2$ smooth with positive Gauss curvature, then it follows from (2.2) and (3.1) that the $J_\phi(K, \cdot)$ has a continuous density, given by

$$h_K\phi(1/h_K)\det(\nabla^2 h_K + h_K I).$$

(3.2)

From (3.1) and (3.2), if the given measure $\mu$ on $S^{n-1}$ has a density $g$, then the equation (1.7) is reduced into

$$\lambda h\phi(1/h)\det(\nabla^2 h + h I) = g \text{ on } S^{n-1}.$$  

(3.3)

Let $M_0$ be a closed, smooth and strictly convex hypersurface in $\mathbb{R}^n$ enclosing the origin, and $\phi : (0, +\infty) \to (0, +\infty)$ be a positive smooth function. Consider the following anisotropic Gauss-like curvature flow

$$\begin{cases}
\frac{\partial X(x, t)}{\partial t} = -g(\nu)\frac{\phi'}{\phi}K\nu + X(x, t), \\
X(x, 0) = X_0(x).
\end{cases}$$

(3.4)

By the definition of support function, it is easy for us to know $h(x, t) = \langle x, X(x, t) \rangle$. From the evolution equation of $X(x, t)$ in (3.4), we derive the evolution equation of the corresponding support function $h(x, t)$

$$\frac{\partial h(x, t)}{\partial t} = -g(\nu)\frac{\phi''}{\phi}K + h(x, t) \text{ on } S^{n-1} \times [0, T).$$

(3.5)

Denote the radial function of $M_0$ by $\rho(u, t)$. For each $t$, let $u$ and $x$ be related through the following equality:

$$\rho(u, t)u = \nabla h(x, t) + h(x, t)x.$$  

(3.6)

Thus, $x$ can be expressed as $x = x(u, t)$, by (3.6), we get

$$\log \rho(u, t) = \log h(x, t) - \log(x, u).$$
Differentiating the above identity, it is easy to see
\[
\frac{1}{\rho(u,t)} \frac{\partial \rho(u,t)}{\partial t} = \frac{1}{h(x,t)} \frac{\partial h(x,t)}{\partial t} \tag{3.7}
\]
Therefore, by the definitions of radial function and \( r \), the normalised flow (3.4) can be also described by the following scalar equation for \( r(\cdot,t) \),
\[
\frac{\partial r}{\partial t}(\xi,t) = -\frac{1}{g_1(x)} \frac{r^{n+1}}{\phi(r)} K + r(\xi,t) \text{ on } S^{n-1} \times [0,T), \tag{3.8}
\]
where \( K \) denotes the Gauss curvature at \( r(\xi,t) \xi \in M_t \).

It can be checked that flow (3.4) is the gradient flow of the functional given by
\[
\mathcal{F}(M_t) = \int_{S^{n-1}} \log h(x,t) dx - \int_{S^{n-1}} \frac{\varphi(r(\xi,t))}{g(x)} d\xi, \quad (\cdot,t) \in S^{n-1} \times [0,T),
\]
where
\[
\varphi(t) = \int_0^t \frac{\phi(s)}{s} ds,
\]
\( h \) and \( r \) are the support function and radial function of \( M_t \) respectively.

**Lemma 3.1** Let \( M_t \) be a strictly convex solution to the flow (3.4). Then the functional \( \mathcal{F}(M_t) \) is non-increasing along the flow (3.4). That is
\[
\frac{\partial}{\partial t} \mathcal{F}(M_t) \leq 0.
\]
with equalities if and only if the support function of \( M_t \) satisfies the elliptic equation (3.3).

**Proof.** From (3.5) and (3.8). By the fact that \( r^{\alpha} d\xi = \frac{1}{K} dx \), we have
\[
\frac{\partial}{\partial t} \mathcal{F}(M_t) = \int_{S^{n-1}} \frac{\partial h}{h(x,t)} \frac{1}{\partial t} dx - \int_{S^{n-1}} \frac{\varphi(r(\xi,t))}{g(x)} \frac{\partial r}{\partial t} \frac{1}{\partial t} d\xi
\]
\[
= \int_{S^{n-1}} \frac{\partial h}{h} \frac{1}{\partial t} dx - \int_{S^{n-1}} \frac{\varphi(r)}{g(x)} \frac{1}{\partial t} \frac{\partial h}{\partial t} d\xi
\]
\[
= \int_{S^{n-1}} \left( \frac{g(x) r^{n+1} K - \phi(r) h}{g(x) h r^{n+1} K} \right) \frac{\partial h}{\partial t} d\xi
\]
\[
= \int_{S^{n-1}} \left( \frac{g(x) r^{n+1} K / \phi(r) - h}{g(x) h r^{n+1} K / \phi(r)} \right) \frac{\partial h}{\partial t} d\xi
\]
\[
= - \int_{S^{n-1}} \left( \frac{g(x) r^{n+1} K}{\phi(r)} - h \right)^2 \frac{1}{g(x) h r^{n+1} K / \phi(r)} dx
\]
\[
\leq 0.
\]
Clearly \( \frac{\partial}{\partial t} \mathcal{F}(M_t) = 0 \) holds if and only if
\[
\frac{g(x) r^{n+1} K}{\phi(r)} = h.
\]
From (2.6) and the concept of radial function, the above equation implies equation (3.3) with \( \lambda = 1 \).
4 The long-time existence of the flow

In this section, we will obtain the long-time existence of the flow (3.4). It is equivalent to obtain the long-time existence of the evolution equation (3.5). The main work is to obtain the $C^0$, $C^1$ and $C^2$-estimates for the (3.5).

4.1 $C^0$, $C^1$-Estimates

The following lemma obtains the $C^0$-estimate.

**Lemma 4.1** Let $h$ be a smooth solution of (3.5), and $g$ be a positive, smooth function on $S^{n-1}$ satisfying (1.9), then there is a positive constant $C$ independent of $t$ such that

\[
\frac{1}{C} \leq h(x, t) \leq C, \tag{4.1}
\]

and

\[
\frac{1}{C} \leq r(\xi, t) \leq C \tag{4.2}
\]

for $\forall (\cdot, t) \in S^{n-1} \times (0,T]$.

**Proof.** Since (4.1) and (4.2) are equivalent, hence, for upper bound (or lower bound) we only need to establish (4.1) or (4.2). Suppose that $h(x, t)$ is maximized at point $x_1 \in S^{n-1}$, then at $x_1$, we get

\[
\nabla h = 0, \quad \nabla^2 h \leq 0 \quad \text{and} \quad r = h.
\]

From (3.5), at $x_1$, we have

\[
\frac{\partial h}{\partial t} = -g(x)K \frac{h^n}{\phi(h)} + h \leq -g(x) \frac{h}{\phi(h)} + h = \frac{h}{\phi(h)} \left( \phi(h) - g(x) \right).
\]

Taking $\Lambda = \limsup_{s \to +\infty} \phi(s)$. By (1.9), $\epsilon = \frac{1}{2}(\min_{S^{n-1}} g(x) - \Lambda)$ is positive and there exists some positive constant $C_1 > 0$ such that

\[
\phi(h) < \Lambda + \epsilon
\]

for $h < C_1$. This together with (1.9)

\[
\phi(h) - g(x) < \Lambda + \epsilon - \min_{S^{n-1}} g(x) < 0,
\]

which implies that at maximal point

\[
\frac{\partial h}{\partial t} < 0
\]

Therefore

\[
h \leq \max\{C_1, \max_{S^{n-1}} h(x, 0)\}.\]
Similarly, we can estimate $\min_{S^{n-1}} h(x, t)$. Suppose that $h(x, t)$ is minimized at point $x_2 \in S^{n-1}$, then at $x_2$, we get

$$\nabla h = 0, \quad \nabla^2 h \geq 0 \quad \text{and} \quad r = h.$$  

From (3.5), at $x_2$, we have

$$\frac{\partial h}{\partial t} = -g(x)K \frac{h}{\phi(h)} + h \geq -g(x) \frac{h}{\phi(h)} + h = \frac{h}{\phi(h)} \left( \phi(h) - g(x) \right).$$

Taking $\Delta = \liminf_{s \to 0^+} \phi(s)$. By (1.9), $\varepsilon = \frac{1}{2}(\Delta - \max_{S^{n-1}} g(x))$ is positive and there exists some positive constant $C_2 > 0$ such that

$$\phi(h) \geq \Delta - \varepsilon$$

for $h < C_2$. This together with (1.9)

$$\phi(h) - g(x) \geq \Delta - \varepsilon - \max_{S^{n-1}} g(x) > 0,$$

which implies that at minimal point

$$\frac{\partial h}{\partial t} > 0$$

Therefore

$$h \geq \min(C_2, \min_{S^{n-1}} h(x, 0)).$$

The proof of the lemma is completed. \qed

Since the convexity of $M_t$, combining with Lemma 4.1, we can obtain the $C^1$-estimates as follows.

**Lemma 4.2** Under the assumption of Lemma 4.1, we have

$$|\nabla h(x, t)| \leq C, \quad \text{and} \quad |\nabla r(\xi, t)| \leq C,$$

for $\forall (\cdot, t) \in S^{n-1} \times (0, T]$. Here $C$ is a positive constant depending only on the constant in Lemma 4.1.

**Proof.** By virtue of the (2.4), (2.5) and (2.6), there is

$$r^2 = h^2 + |\nabla h|^2,$$

which implies that

$$|\nabla h| \leq r.$$
Then from (2.7), we have

\[ h = \frac{r^2}{\sqrt{r^2 + |\nabla r|^2}}, \]

which implies that

\[ |\nabla r| = \frac{|\nabla h|}{h} r \leq \frac{r^2}{h}. \]

From the Lemma 4.1, we directly obtain the estimate of this lemma. \( \square \)

4.2 \( C^2 \)-Estimate

In this subsection, we will establish the upper and lower bound of principal curvatures. This estimates can be obtained by considering proper auxiliary functions; see, e.g., \[9, 25\] for similar techniques. We take a local orthonormal frame \( \{e_1, ..., e_{n-1}\} \) on \( S^{n-1} \) such that the standard metric on \( S^{n-1} \) is \( \{\delta_{ij}\} \). We first derive an upper bound for the Gauss curvature.

**Lemma 4.3** Let \( h \) be a smooth solution of (3.5), and \( g_1 \) be a positive, smooth function on \( S^{n-1} \) satisfying (1.9), then there is a positive constant \( C \) independent of \( t \) such that

\[ K(x, t) \leq C, \]

for \( \forall (x, t) \in S^{n-1} \times [0, T) \).

**Proof.** Let us consider the following auxiliary function

\[ \Theta(x, t) = -\frac{\partial_t h + h}{h - \varepsilon_0} = g(x) \frac{r^n}{\phi(r)} \frac{K}{h - \varepsilon_0} \] (4.3)

where \( \varepsilon_0 \) is a positive constant satisfying \( \varepsilon_0 = \frac{1}{2} \inf h(x, t), \forall (x, t) \in S^{n-1} \times [0, T) \).

From (4.3), the upper bound of \( K \) follows from \( \Theta(x, t) \). Hence we only need to derive the upper bound of \( \Theta(x, t) \). At any maximum of \( \Theta \) at \( x_0 \) we have

\[ 0 = \nabla_j \Theta = \frac{-\partial_t h_i + h_i}{h - \varepsilon_0} + \frac{(\partial_t h - h)h_i}{(h - \varepsilon_0)^2}, \] (4.4)

and using the (4.4), then

\[ 0 \geq \nabla_{ij} \Theta = \frac{-\partial_t h_{ij} + h_{ij}}{h - \varepsilon_0} + \frac{(\partial_t h - h)h_{ij}}{(h - \varepsilon_0)^2}, \] (4.5)

where \( \nabla_{ij} \Theta \leq 0 \) should be understood in the sense of negative semi-definite matrix. As in the background metrial, we know the fact \( b_{ij} = h_{ij} + h \delta_{ij} \), and \( \delta^i_{ij} \) its inverse matrix, which together with the (4.5), we can get

\( \partial_t b_{ij} = \partial_t h_{ij} + \partial_t h \delta_{ij} \)

\[ \geq h_{ij} + \frac{(\partial_t h - h)h_{ij}}{h - \varepsilon_0} + \partial_t h \delta_{ij} \]
\[
\begin{align*}
&= b_{ij} + \frac{(\partial_i h - h)h_{ij}}{h - \varepsilon_0} + (\partial_i h - h)\delta_{ij} \\
&= b_{ij} + \frac{\partial_i h - h}{h - \varepsilon_0} (h_{ij} + h\delta_{ij} - \varepsilon_0\delta_{ij}) \\
&= b_{ij} - \Theta(b_{ij} - \varepsilon_0\delta_{ij}).
\end{align*}
\]

By the fact (2.10), we obtain
\[
\partial_t K = -Kb_{ij} \partial_t b_{ij} \\
\leq -Kb_{ij}[b_{ij} - \Theta(b_{ij} - \varepsilon_0\delta_{ij})] \\
= -K[(n-1)(1 - \Theta) + \Theta\varepsilon_0 H],
\]

where \(H\) denotes the mean curvature of \(X(\cdot, t)\).

From the (4.3) and Lemma 4.1, there exists a constant \(C_1\) such that
\[
\frac{1}{C_1} \Theta(x, t) \leq K(x, t) \leq C_1 \Theta(x, t),
\]

where \(C_1\) is a positive constant. Noting
\[
\frac{1}{n-1} H \geq \frac{1}{n} H, \\
\]

and combining the inequalities (4.6), we obtain
\[
\partial_t K \leq (n-1)K\Theta - (n-1)\varepsilon_0 \Theta K^{\frac{n}{n-1}}.
\]

Now we estimate \(\partial_t \Theta\). From (4.3), we have
\[
\partial_t \Theta = \partial_t \left( \frac{r^n}{(h - \varepsilon_0)\phi(r)} \right) g(x) K + \frac{g(x) r^n}{(h - \varepsilon_0)\phi(r)} \partial_t K, \\
\]

where
\[
\partial_t h = h - (h - \varepsilon_0)\Theta,
\]

From (4.8) and (4.7), we have at \(x_0\)
\[
\partial_t \Theta \leq (h - \varepsilon_0) \left( \frac{r^n}{h\phi(r)} - \frac{n}{h} + \frac{1}{h - \varepsilon_0} \right) \Theta^2 + \frac{g(x) r^n}{(h - \varepsilon_0)\phi(r)} \left( (n-1)K\Theta - (n-1)\varepsilon_0 \Theta K^{\frac{n}{n-1}} \right) \\
\leq C_2 \Theta^2 + \frac{g(x) r^n}{(h - \varepsilon_0)\phi(r)} \left( (n-1)K\Theta - (n-1)\varepsilon_0 \Theta K^{\frac{n}{n-1}} \right) \\
\leq C_3 \Theta^2 \left( C_4 - \varepsilon_0 \Theta^{\frac{1}{n-1}} \right),
\]

where \(C_2, C_3, C_4\) are positive constants.
where $C_2, C_3$ and $C_4$ are positive constant depending only on the constant $C$ in Lemma 4.1, and the upper and lower bounds of $g$ on $S^{n-1}$ and $\phi$ on $[1/C, C]$.

Now one can see that whenever $\Theta > \left(\frac{C_4}{\varepsilon_0}\right)^{n-1}$ which is independent of $t$, 

$$\partial_t \Theta < 0,$$

which implies that $\Theta$ has a uniform upper bound.

For any $(x, t)$

$$K(x, t) = \frac{(h - \varepsilon_0)\phi(r)\Theta(x, t)}{g(x)r^n} \leq \frac{(h - \varepsilon_0)\phi(r)\Theta(x_0, t)}{g(x)r^n} \leq C,$$

namely, $K$ has a uniform upper bound. \hfill \Box

Now, we estimate the principal curvatures are bounded from below along the flow \eqref{eq:3.4}. To obtain the positive lower bound for the principal curvatures of $M_t$, we will study an expanding flow by Gauss curvature for the dual hypersurface of $M_t$.

**Lemma 4.4** Under the conditions of Lemma 4.3, then the principal curvature $k_i$ for $i = 1, \ldots, n - 1$ satisfies

$$\frac{1}{C} \leq k_i \leq C,$$

where $C$ is a positive constant independent of $t$.

**Proof.** To prove the lower bound of $k_i$, we employ the dual flow of \eqref{eq:3.4}, and establish an upper bound of principal curvature for the dual flow. This together with Lemma 4.3, also implies the upper bound of $k_i$.

We denote by $M_t^*$ the polar set of $M_t = X(S^{n-1}, t)$. From the definition of polar set, if $r(\cdot, t)$ is the radial function of $M_t$, then

$$r(\xi, t) = \frac{1}{h^*(\xi, t)},$$

where $h^*(\xi, t)$ denotes the upper function of $M_t^*$. It is well-know that $\alpha_{M_t} = \alpha_{M_t^*}$, see e.g. \cite{17}. This implies $|\text{Jac}\alpha_{M_t}||\text{Jac}\alpha_{M_t^*}| = 1$, thus by \eqref{eq:2.11} and \eqref{eq:2.12} we have, under a local orthonormal frame on $S^{n-1}$

$$\frac{(h^*(\xi, t))^{n+1}h^{n+1}(x, t)}{K(p^*)K(p)} = 1,$$

where $p \in M_t$, $p^* \in M_t^*$ are two points satisfying $p \cdot p^* = 1$, and $x, \xi$ are the unit outer normals of $M_t$ and $M_t^*$ at $p$ and $p^*$. Therefore by equation \eqref{eq:3.8}, we obtain the equation for $h^*$,

$$\partial_t h^*(\xi, t) = g(\xi)\frac{(h^*(\xi, t))^2}{\phi(r^*)r^{n+1}} - h^*(\xi, t), \quad \forall (\cdot, t) \in S^{n-1} \times (0, T], \tag{4.12}$$
where $\mathcal{K}^* = (\det(\nabla^2 h^* + h^* I))^{-1}$ is the Gauss curvature of $M^*_t$ at the point $p^* = \nabla h^*(\xi, t) + h^*(\xi, t)\xi$, and

$$r^* = |p^*| = \sqrt{\nabla h^*|^2 + (h^*|^2}$$

is the distance from $p^*$ to the origin. Note that $g$ takes value at

$$x = \frac{p^*}{|p^*|} = \frac{\nabla h^* + h^* \xi}{\sqrt{|\nabla h^*|^2 + (h^*)^2}} \in S^{n-1}.$$

By (4.10), $\frac{1}{C} \leq h^* \leq C$ and $|\nabla h^*| \leq C$ for some $C$ only depending on $\max_{S^{n-1} \times (0, T]} h$ and $\min_{S^{n-1} \times (0, T]} h$.

Let $b_{ij}^* = h_{ij}^* + h^* \delta_{ij}$, and $b_{ij}^{**}$ be the inverse matrix of $b_{ij}^*$. As discussed in Section 2, the eigenvalues of $b_{ij}^*$ and $b_{ij}^{**}$ are respectively the principal radii and principal curvature of $M^*_t$. Consider the following function

$$W(\xi, t, \tau) = \log b_{\tau\tau}^{**} - \beta \log h^* + A(r^*)^2,$$

where $\tau$ is a unit vector in the tangential space of $S^{n-1}$, while $\beta$ and $A = A(\beta)$ are large constants to be specified later on. Assume $w$ attain its maximum at $(\xi_0, t_0)$, along the direction $\tau = e_1$. By rotation, we also assume $b_{ij}^{**}$ and $b_{ij}$ are diagonal at this point.

It is direct to see, at the point where $W$ attains its maximum

$$0 = \nabla_i W = -b_{i1}^{**} b_{1j}^* - \beta \frac{\partial}{\partial r^*} + A(r^*)^2,$$

and

$$0 \geq \nabla_{ij} W = -b_{i1}^{**} b_{1j}^* - (b_{11}^{**})^2 b_{1j}^* b_{1i}^* - \beta \left( \frac{h_{ij}^*}{h^*} - \frac{h^* h_{ij}^*}{(h^*)^2} \right) + A(r^* r_{ij}^* + r_i^* r_j^*),$$

where $\nabla_{ij} W \leq 0$ should be understood in the sense of negative semi-definite matrix. Note that $b_{ijk}$ is symmetric in all indices. Without loss of generality, if we assume $t_0 > 0$, then at $(\xi_0, t_0)$, we also have

$$0 \leq \partial_i W = b_{i1}^* \partial_i b_{11}^* - \beta \frac{\partial h^*}{h^*} + A r^* \partial_i r^*$$

$$= -b_{i1}^{**} \partial_i b_{11}^* - \beta \frac{\partial h^*}{h^*} + A r^* \partial_i r^*.$$

We can rewrite the flow (4.12) as

$$\log(\partial_i h^* + h^*) = \log S_n + \alpha(\xi, t),$$

where

$$\alpha(\xi, t) = \log \left( g(\xi) \frac{(h^*)^2}{\phi(r^*)(r^*)^n} \right).$$
Differentiating (4.17) gives

\[
\frac{\partial h^*_{k} + h^*_{k}}{\partial h^* + h^*} = b^i_j b^*_{ij,k} + \nabla_k \alpha,
\] (4.18)

and

\[
\frac{\partial h^*_{11} + h^*_{11}}{\partial h^* + h^*} = \frac{(\partial h^*_{11} + h^*_{11})^2}{(\partial h^* + h^*)^2} + b^i_j b^*_{ij,11} - b^i_j b^*_{ij,11} + \nabla_{11} \alpha.
\] (4.19)

Dividing (4.16) by \(\partial h^* + h^*\) and using (4.19), we have

\[
0 \leq -b^i_j \left( \frac{\partial h^*_{11} + h^*_{11}}{\partial h^* + h^*} - \frac{b^*_{11}}{\partial h^* + h^*} + 1 \right) - \frac{\beta \partial h^*}{h^*(\partial h^* + h^*)} + A \frac{r^* \partial r^*}{\partial h^* + h^*}
\]

\[
= -b^i_j \frac{\partial h^*_{11} + h^*_{11}}{\partial h^* + h^*} - b^i_j \frac{1 + \beta}{\partial h^* + h^*} - \frac{\beta \partial h^*}{h^*(\partial h^* + h^*)} + A \frac{r^* \partial r^*}{\partial h^* + h^*}
\]

\[
\leq -b^i_j b^*_{ij,11} + b^i_j b^*_{ij,11} (b^*_{ij,11})^2 - b^i_j \nabla_{11} \alpha + \frac{1 + \beta}{\partial h^* + h^*} + A \frac{r^* \partial r^*}{\partial h^* + h^*}.
\] (4.20)

By the Ricci identity, we have

\[
b^*_{ij,11} = b^*_{11,ij} - \delta_i b^*_{11} + \delta_1 b^*_{11} - \delta_1 b^*_{11} + \delta_1 b^*_{11}.
\]

Plugging this identity in (4.20), and employing (4.15), we obtain

\[
0 \leq -b^i_j \left( b^i_j (b^*_{11,ij})^2 - b^i_j b^*_{ij,11} (b^*_{ij,11})^2 \right) + (H^* - (n - 1) b^i_j)
\]

\[
- \beta H^* + C \beta - \frac{h^* h^*}{(h^*)^2} - b^i_j \nabla_{11} \alpha + \frac{1 + \beta}{\partial h^* + h^*}
\]

\[
+ A \frac{r^* \partial r^*}{\partial h^* + h^*} - Ab^i_j (r^* r^* + r^* r^*)
\]

\[
\leq -\beta H^* + C \beta - b^i_j \nabla_{11} \alpha + \frac{1 + \beta}{\partial h^* + h^*} + A \frac{r^* \partial r^*}{\partial h^* + h^*} - Ab^i_j (r^* r^* + r^* r^*),
\] (4.21)

where \(H^* = \sum b^i_j\) is the mean curvature of \(M^*_i\).

It is direct to calculate

\[
r^*_i = \frac{h^* \partial h^* + \sum h^*_k \partial h^*_k}{r^*},
\]

\[
r^*_i = \frac{h^*_i + \sum h^*_k h^*_k}{r^*} = \frac{h^*_i b^*_i}{r^*},
\]

\[
r^*_i = \frac{h^* h^*_i + h^*_i h^*_i + \sum h^*_k h^*_k + \sum h^*_k h^*_k h^*_k}{r^*} - \frac{h^*_i b^*_i b^*_i b^*_i}{(r^*)^3}.
\] (4.22)

Hence, by (4.18)

\[
\frac{r^* r^*_i}{h^*_i + h^*} - b^i_j (r^* r^*_i + r^* r^*_i) = \frac{h^* \partial h^*}{\partial h^* + h^*} - h^* b^*_{ij} - b^*_{ij} (h^*_i)^2
\]
\[-\frac{|
abla h|^2}{\partial_t h^* + h^*} + \sum h_k^* \nabla_k \alpha.\]

Since
\[\frac{h^* \partial_t h^*}{\partial_t h^* + h^*} - \frac{|
abla h|^2}{\partial_t h^* + h^*} = h^* - \frac{(r)^2}{\partial_t h^* + h^*},\]
and
\[-h^* b_i^j h_i^j - b_i^j (h_i^j)^2 = -h^* b_i^j (h_i^j - h^* \delta_{ij}) - b_i^j (b_i^j - h^* \delta_{ij})^2 = (n - 1)h^* - \sum b_i^j,\]
we further deduce
\[\frac{r^* \partial_t r^*}{\partial_t h^* + h^*} - b_i^j (r^* r_i^j + r_i^j) \leq C - \frac{(r)^2}{\partial_t h^* + h^*} + \sum h_k^* \nabla_k \alpha. \tag{4.23}\]

Plugging (4.23) in (4.21), we get
\[0 \leq -\beta \mathcal{H}^* + C \beta + CA - b_i^{11} \nabla_{11} \alpha + \frac{1 + \beta - A(r)^2}{\partial_t h^* + h^*} + A \sum h_k^* \nabla_k \alpha\]
and
\[\leq -\beta \mathcal{H}^* + C \beta + CA - b_i^{11} \nabla_{11} \alpha + A \sum h_k^* \nabla_k \alpha, \tag{4.24}\]
provided \(A > 2(1 + \beta) / \min_{S^{n-1} \times (0, T]} (r^*)^2 \geq C(1 + \beta)\) for some \(C > 0\) only depending on \(\max_{S^{n-1} \times (0, T]} h.\)

By (4.14) and (4.22), we have
\[-b_i^{11} \nabla_{11} \alpha + A \sum h_k^* \nabla_k \alpha \leq C b_i^{11} (1 + (h_i^{11})^2) + CA - b_i^{11} \sum \alpha_k^i h_k^* + A \sum \alpha_k^i h_k^* \]
\[\leq C b_i^{11} + \frac{C}{b_i^{11}} + CA + C\beta.\]

Hence (4.24) can be further estimated as
\[0 \leq -\beta \mathcal{H}^* + C b_i^{11} + C \beta + CA \leq (-\beta + C) b_i^{11} + C \beta + CA,\]
by choosing \(\beta\) large. This inequality tells us the principal curvature of \(M^*\) are bounded from above, namely
\[\max_{\xi \in S^{n-1}} k_i^*(\xi, t) \leq C, \quad \forall t \in (0, T], \quad i = 1, 2, \ldots, n - 1.\]

By the Lemma 4.3 and (4.11), we have \(\mathcal{K}^{*}(\cdot, t) \geq \frac{1}{C}.\) Therefore
\[\frac{1}{C} \leq k_i^*(\cdot, t) \leq C, \quad \forall (\cdot, t) \in S^{n-1} \times (0, T], \quad i = 1, 2, \ldots, n - 1.\]
By duality, Lemma 4.4 follows. □

As a consequence of the above a priori estimates, one sees that the convexity of the hypersurface $M_t$ is preserved under the flow (3.4) and the solution is uniformly convex.

Now we have proved that the principal curvatures of $M_t$ have uniform positive upper and lower bounds, this together with Lemmas 4.1 and 4.2 implies that the evolution equation (3.5) is uniformly parabolic on any finite time interval. Thus, the result of [24] and the standard parabolic theory show that the smooth solution of (3.5) exists for all time, namely, flow (3.4) has a long-time solution. And by these estimates again, a subsequence of $M_t$ converges in $C^\infty$ to a positive, smooth, uniformly convex hypersurface $M^\infty$ in $\mathbb{R}^n$.

5 Existence of the solutions to the Monge-Ampère equation

In this section, we complete proof of Theorem 1.2, namely we will prove the support function $\tilde{h}$ of $M^\infty$ satisfies the following Monge-Ampère equation:

$$
\frac{h\phi(1/h)}{(|\nabla h|^2 + h^2)^{\frac{n}{2}}} \text{det}(\nabla^2 h + hI) = g,
$$

(5.1)

Recalling the functional $F(M_t)$ we defined in Section 3

$$
F(M_t) = \int_{S^{n-1}} \log h(x, t) dx - \int_{S^{n-1}} \frac{\phi(r(\xi, t))}{g(x)} d\xi, \quad t \in [0, T].
$$

From the Lemma 3.1, there exists a positive constant $C$ which is independent of $t$, such that

$$
F(M_t) \leq C.
$$

(5.2)

Since $F(M_t)$ is non-increasing for any $t > 0$. From

$$
\int_0^t \left(- \frac{\partial}{\partial t} F(M_t)\right) dt = F(M_0) - F(M_t) \leq F(M_0),
$$

we have

$$
\int_0^\infty \left(- \frac{\partial}{\partial t} F(M_t)\right) dt \leq F(M_0),
$$

this implies that there exists a subsequence of times $t_j \to \infty$ such that

$$
- \frac{\partial}{\partial t} F(M_{t_j}) \to 0 \quad \text{as} \quad t_j \to \infty.
$$

Recalling Lemma 3.1

$$
\frac{\partial F(M_t)}{\partial t} = - \int_{S^{n-1}} \left( \frac{g(x)\phi'^2 - hI}{g(x)\phi(x)\phi'} \right)^2 d\xi.
$$
Since \( h, r \) and \( K \) have uniform positive upper and lower bounds, by passing to the limit, we obtain

\[
\frac{g(x) r^n \tilde{K}}{\phi(r)} = \tilde{h},
\]

where \( \tilde{h} \) and \( \tilde{K} \) are the support function and Gauss curvature of \( M_\infty \). Namely

\[
g(x) \frac{\left( \sqrt{\vert \nabla \tilde{h} \vert^2 + \tilde{h}^2} \right)^n}{\phi(1/\tilde{h})} \tilde{K} = \tilde{h} \quad \text{on} \quad S^{n-1},
\]

which is just equation (5.1). The proof of Theorem 1.2 is now completed.

At the same time, for Theorem 1.1, we have showed that for smooth \( \phi \) and \( g \), there exists a smooth solution \( h \) to (1.7) with \( \lambda = 1 \).

Finally, we provide a special uniqueness of the Orlicz-Aleksandrov problem under the appropriate condition.

**Theorem 5.1** Assume \( \phi \) is a positive, continuous function. If whenever

\[
\phi(cs^{-1}) \leq \phi(s^{-1})
\]

hold for some positive \( c, s \), there must be \( c \geq 1 \). Then the solution to the

\[
\frac{h \phi(1/h)}{(\vert \nabla h \vert^2 + h^2)^{\frac{n}{2}}} \det(\nabla^2 h + hI) = g.
\]

is unique.

**Proof of Theorem 5.1.** Let \( h_1 \) and \( h_2 \) are two solutions of (5.4). Assume \( \frac{h_1}{h_2} \) attain its maximum at \( x_0 \in S^{n-1} \). Taking \( Q = \log \frac{h_1}{h_2} \), then at \( x_0 \)

\[
0 = \nabla Q = \frac{\nabla h_1}{h_1} - \frac{\nabla h_2}{h_2},
\]

and

\[
0 \geq \nabla^2 Q = \frac{\nabla^2 h_1}{h_1} - \frac{\nabla^2 h_2}{h_2}.
\]

By the equation (5.4), we have at \( x_0 \)

\[
1 = \frac{\det(\nabla^2 h_2 + h_2 I) \left(\vert \nabla h_2 \vert^2 + h_2^2\right)^{-\frac{n}{2} \phi(h_2^{-1})} h_2}{\det(\nabla^2 h_1 + h_1 I) \left(\vert \nabla h_1 \vert^2 + h_1^2\right)^{-\frac{n}{2} \phi(h_1^{-1})} h_1}
\]

\[
= \frac{h_2^n \det(\nabla^2 h_2 + I) \left(\frac{\vert \nabla h_2 \vert^2}{h_2^2} + 1\right)^{-\frac{n}{2} \phi(h_2^{-1})}}{h_1^n \det(\nabla^2 h_1 + I) \left(\frac{\vert \nabla h_1 \vert^2}{h_1^2} + 1\right)^{-\frac{n}{2} \phi(h_1^{-1})}}
\]

(5.5)
Write $h_2(x_0) = ch_1(x_0)$, then the above inequality reads

$$\phi(h_2^{-1}) \geq \frac{\phi(h_1^{-1})}{\phi(h_1^{-1})}.$$

By our assumption (5.3), we have $c \geq 1$. Namely $h_1(x_0) \geq h_2(x_0)$.

Interchanging $h_1$ and $h_2$, then $h_2(x_0) \geq h_1(x_0)$. Therefore, we have $h_1 \equiv h_2$. $\square$

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