CLASSIFICATION OF 3-DIMENSIONAL CONFORMALLY FLAT QUASI-PARA-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study 3-dimensional conformally flat quasi-Para-Sasakian manifolds. First, the necessary and sufficient conditions are provided for 3-dimensional quasi-Para-Sasakian manifolds to be conformally flat. Next, a characterization of 3-dimensional conformally flat quasi-Para-Sasakian manifold with $\beta=\text{const.}$ is given.

1. Introduction

Almost paracontact metric structures are the natural odd-dimensional analogue to almost paraHermitian structures, just like almost contact metric structures correspond to the almost Hermitian ones. The study of almost paracontact geometry was introduced by Kaneyuki and Williams in [8] and then it was continued by many other authors. A systematic study of almost paracontact metric manifolds was carried out in paper of Zamkovoy, [18]. Comparing with the huge literature in almost contact geometry, it seems that there are necessary new studies in almost paracontact geometry. Therefore, paracontact metric manifolds have been studied in recent years by many authors, emphasizing similarities and differences with respect to the most well known contact case. Interesting papers connecting these fields are (see, for example, [5], [4], [16], [18], and references therein).

Z. Olszak studied normal almost contact metric manifolds of dimension 3 [13]. He derive certain necessary and sufficient conditions for an almost contact metric structure on manifold to be normal and curvature properties of such structures and normal almost contact metric structures on a manifold of constant curvature are studied. Recently, J. Welyczko studied curvature and torsion of Frenet-Legendre curves in 3-dimensional normal almost paracontact metric manifolds [17] and then normal almost paracontact metric manifolds are studied by [1], [9], [10].

The notion of quasi-Sasakian manifolds, introduced by D. E. Blair in [2], unifies Sasakian and cosymplectic manifolds. By definition, a quasi-Sasakian manifold is a normal almost contact metric manifold whose fundamental 2-form $\Phi := g(\cdot, \phi \cdot)$ is closed. Quasi-Sasakian manifolds can be viewed as an odd-dimensional counterpart of Kaeahler structures. These manifolds studied by several authors(e.g. [7], [12], [14], [15]).

Although quasi-Sasakian manifolds were studied by several different authors and are considered a well-established topic in contact Riemannian geometry, To the authors knowledge, there do not exist any study about quasi-Para-Sasakian manifolds.
Motivated by these considerations, in [11], the author makes the first contribution to investigate basic properties and general curvature identities of quasi-Para-Sasakian manifolds.

In this paper, we study 3-dimensional conformally flat quasi-Para-Sasakian manifolds. The paper is organized as follows:

Section 2 is preliminary section, where we recall the definition of almost paracontact metric manifold and quasi-Para-Sasakian manifolds.

In Section 3, we mainly proved that for a 3-dimensional quasi-Para-Sasakian manifold $M$ with $\beta =$const., the followings are equivalent.

1) $M$ is locally symmetric.
2) $M$ is conformally flat and its scalar curvature $\tau$ is const.,
3) $M$ is conformally flat and $\beta$ is const.,
4) If $\beta = 0$, then $M$ is a paracosymplectic manifold which is locally a product of the real line $R$ and a 2-dimensional para-Kaehlerian manifold or

If $\beta \neq 0$, then $M$ is of constant negative curvature and the quasi-para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure.

2. Preliminaries

Let $M$ be a $(2n + 1)$-dimensional differentiable manifold and $\phi$ is a $(1, 1)$ tensor field, $\xi$ is a vector field and $\eta$ is a one-form on $M$. Then $(\phi, \xi, \eta)$ is called an almost paracontact structure on $M$ if

(i) $\phi^2 = Id - \eta \otimes \xi$, $\eta(\xi) = 1$,
(ii) the tensor field $\phi$ induces an almost paracomplex structure on the distribution $D = \ker \eta$, that is the eigendistributions $D^\pm$, corresponding to the eigenvalues $\pm 1$, have equal dimensions, $\dim D^+ = \dim D^- = n$.

The manifold $M$ is said to be an almost paracontact manifold if it is endowed with an almost paracontact structure [18].

Let $M$ be an almost paracontact manifold. $M$ will be called an almost paracontact metric manifold if it is additionally endowed with a pseudo-Riemannian metric $g$ of a signature $(n + 1, n)$, i.e.

\begin{equation}
(2.1) \quad g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y).
\end{equation}

For such manifold, we have

\begin{equation}
(2.2) \quad \eta(X) = g(X, \xi), \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0.
\end{equation}

Moreover, we can define a skew-symmetric tensor field (a 2-form) $\Phi$ by

\begin{equation}
(2.3) \quad \Phi(X, Y) = g(X, \phi Y),
\end{equation}

usually called fundamental form.

For an almost paracontact manifold, there exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ such that $g(X_i, X_j) = \delta_{ij}$, $g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \phi X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a $\phi$-basis.

On an almost paracontact manifold, one defines the $(1, 2)$-tensor field $N^{(1)}$ by

\begin{equation}
(2.4) \quad N^{(1)}(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi,
\end{equation}

where $[\phi, \phi]$ is the Nijenhuis torsion of $\phi$

\begin{equation}
[\phi, \phi](X, Y) = \phi^2 [X, Y] + [\phi X, \phi Y] - \phi [\phi X, Y] - \phi [X, \phi Y].
\end{equation}
If $N^{(1)}$ vanishes identically, then the almost paracontact manifold (structure) is said to be normal [18]. The normality condition says that the almost paracomplex structure $J$ defined on $M \times \mathbb{R}$

$$J(X, \lambda \frac{dt}{dt}) = (\phi X + \lambda \xi, \eta(X) \frac{dt}{dt}),$$

is integrable.

If $d\eta(X, Y) = g(X, \phi Y)$, then $(M, \phi, \xi, \eta, g)$ is said to be paracontact metric manifold. In a paracontact metric manifold one defines a symmetric, trace-free operator $h = \frac{1}{2} L_\xi \phi$, where $L_\xi$, denotes the Lie derivative. It is known [18] that $h$ anti-commutes with $\phi$ and satisfies $h\xi = 0$, $\text{tr} h = \text{tr} h \phi = 0$ and $\nabla \xi = -\phi + \phi h$, where $\nabla$ is the Levi-Civita connection of the pseudo-Riemannian manifold $(M, g)$.

Moreover $h = 0$ if and only if $\xi$ is Killing vector field. In this case $(M, \phi, \xi, \eta, g)$ is said to be $K$-paracontact manifold. Similarly as in the class of almost contact metric manifolds [3], a normal almost paracontact metric manifold will be called para-Sasakian if $\Phi = d\eta$ [6].

Now, we will give some results about 3-dimensional quasi-para-Sasakian manifolds that we will use next section.

**Proposition 1.** [17] For a 3-dimensional almost paracontact metric manifold $M$ the following three conditions are mutually equivalent

(a) $M$ is normal,
(b) there exist functions $\alpha, \beta$ on $M$ such that

$$\nabla X \phi Y = \beta(g(X, Y)\xi - \eta(Y)X) + \alpha(g(\phi X, Y)\xi - \eta(Y)\phi X),$$

(c) there exist functions $\alpha, \beta$ on $M$ such that

$$\nabla X \xi = \alpha(X - \eta(X)\xi) + \beta \phi X.$$

**Corollary 1.** [9] For a normal almost paracontact metric structure $(\phi, \xi, \eta, g)$ on $M$, we have $\nabla \xi \eta = 0$ and $d\eta = -\beta \Phi$. The functions $\alpha, \beta$ realizing (2.5) as well as (2.6) are given by [17]

$$2\alpha = \text{Trace} \{X \rightarrow \nabla X \xi\}, \quad 2\beta = \text{Trace} \{X \rightarrow \phi \nabla X \xi\}.$$

**Proposition 2.** [17] For a 3-dimensional almost paracontact metric manifold $M$, the following three conditions are mutually equivalent

(a) $M$ is quasi-para-Sasakian,
(b) there exists a function $\beta$ on $M$ such that

$$\nabla X \phi Y = \beta(g(X, Y)\xi - \eta(Y)X),$$

(c) there exists a function $\beta$ on $M$ such that

$$\nabla X \xi = \beta \phi X.$$

A 3-dimensional normal almost paracontact metric manifold is

- paracosymplectic if $\alpha = \beta = 0$ [6],
- quasi-para-Sasakian if and only if $\alpha = 0$ and $\beta \neq 0$ [6], [17],
- $\beta$-para-Sasakian if and only if $\alpha = 0$ and $\beta \neq 0$ and $\beta$ is constant, in particular, para-Sasakian if $\beta = -1$ [17], [18],
- $\alpha$-para-Kenmotsu if $\alpha \neq 0$ and $\alpha$ is constant and $\beta = 0$.

Obviously, the class of para-Sasakian manifolds is contained in the class of quasi-para-Sasakian manifolds. The converse does not hold in general. Also in this context
the para-Sasakian condition implies the $K$-paracontact condition and the converse holds only in dimension 3. A paracontact metric manifold will be called paracosymplectic if $d\Phi = 0$, $d\eta = 0$, obviously, the class of paracosymplectic manifolds is contained in the class of quasi-para-Sasakian manifolds.

**Theorem 1.** Let $(M, \phi, \xi, \eta, g)$ be a 3-dimensional normal almost paracontact metric manifold. Then the following curvature identities hold

\[
R(X, Y)Z = (2(\xi(\alpha) + \alpha^2 + \beta^2) + \frac{1}{2} \tau)(g(Y, Z)X - g(X, Z)Y)
\]

\[
- (\xi(\alpha) + 3(\alpha^2 + \beta^2) + \frac{1}{2} \tau)((g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi)
\]

\[
+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + (\phi Z(\beta) - Z(\alpha))(\eta(Y)X - \eta(X)Y)
\]

\[
(\phi Y(\beta) - Y(\alpha))(\eta(Z)X - g(X, Z)\xi)
\]

\[
- (\phi X(\beta) - X(\alpha))(\eta(Z)Y - g(Y, Z)\xi)
\]

\[
+ (\phi \text{grad} \beta + \text{grad} \alpha)(\eta(Y)g(X, Z) - \eta(X)g(Y, Z)).
\]

(2.10)

\[
S(Y, Z) = - (\xi(\alpha) + \alpha^2 + \beta^2 + \frac{1}{2} \tau)g(\phi Y, \phi Z)
\]

\[
+ \eta(Z)(\phi Y(\beta) - Y(\alpha))
\]

\[
+ \eta(Y)(\phi Z(\beta) - Z(\alpha)) - 2(\alpha^2 + \beta^2)\eta(Y)\eta(Z),
\]

where $R$, $S$ and $\tau$ are resp. Riemannian curvature, Ricci tensor and scalar curvature of $M$.

If we take $\alpha = 0$ in Theorem [1] we get following

**Theorem 2.** Let $(M, \phi, \xi, \eta, g)$ be a 3-dimensional quasi-Para-Sasakian manifold. Then the following curvature identities hold

\[
R(X, Y)Z = (2\beta^2 + \frac{1}{2} \tau)(g(Y, Z)X - g(X, Z)Y)
\]

\[
- (3\beta^2 + \frac{1}{2} \tau)((g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi)
\]

\[
+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + (\phi Z(\beta) - Z(\alpha))(\eta(Y)X - \eta(X)Y)
\]

\[
+ \phi Y(\beta)(\eta(Z)X - g(X, Z)\xi)
\]

\[
- \phi X(\beta)(\eta(Z)Y - g(Y, Z)\xi)
\]

\[
+ (\phi \text{grad} \beta)(\eta(Y)g(X, Z) - \eta(X)g(Y, Z)).
\]

(2.12)

\[
S(Y, Z) = (\beta^2 + \frac{1}{2} \tau)g(Y, Z) - (3\beta^2 + \frac{1}{2} \tau)\eta(Y)\eta(Z)
\]

\[
+ \eta(Y)\phi Z(\beta) + \eta(Z)\phi Y(\beta).
\]

(2.13)

where $R$, $S$ and $\tau$ are resp. Riemannian curvature, Ricci tensor and scalar curvature of $M$.

**Remark 1.** In the proof of Theorem [1] the author showed that $\xi(\beta) + 2\alpha \beta = 0$. Namely, for 3-dimensional quasi-Para-Sasakian manifolds,

\[
(2.14)
\]

\[
\xi(\beta) = 0.
\]
Theorem 3. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a quasi-parasasakian manifold of constant curvature $K$. Then $K \leq 0$. Furthermore,

- If $K = 0$, the manifold is paracosymplectic,
- If $K < 0$, the structure $(\phi, \xi, \eta, g)$ is obtained by a homothetic deformation of a parasasakian structure on $M^{2n+1}$.

3. 3-dimensional conformally flat quasi-Para-Sasakian manifolds

For the conformal flatness, we will use linear $(1,1)$-tensor field $L$ defined by

\[ L = Q - \left( \frac{\tau}{4} \right), \]

where $S(X,Y) = g(QX, Y)$.

From now on, we will use the notion $df(X)$ instead of $g(\text{grad} f, X)$.

Lemma 1. The linear operator $L$ of a 3-dimensional quasi-Para-Sasakian manifold is given by

\[ LY = \left( \frac{\tau}{4} + \beta^2 \right) Y - \left( 3\beta^2 + \frac{\tau}{2} \right) \eta(Y)\xi - \eta(Y)\phi \text{grad} \beta + d\beta(\phi Y)\xi. \]

Proof. By (2.13), we obtain

\[ QY = \left( \frac{\tau}{2} + \beta^2 \right) Y - \left( 3\beta^2 + \frac{\tau}{2} \right) \eta(Y)\xi - \eta(Y)\phi \text{grad} \beta + d\beta(\phi Y)\xi. \]

The requested equation comes from combining (3.1) and the above last equation. □

From (3.2), we have

\[ (\nabla_X L)Y = \nabla_X LY - L\nabla_X Y \]

\[ = \left( \frac{d\tau(X)}{4} + 2\beta d\beta(X) \right) Y - \left( 6\beta d\beta(X) + \frac{d\tau(X)}{2} \right) \eta(Y)\xi \]

\[ - \left( 3\beta^2 + \frac{\tau}{2} \right) (\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi \]

\[ - (\nabla_X Y)\phi \text{grad} \beta - \eta(Y)\phi \text{grad} \beta + \eta(\phi Y)\nabla_X \xi. \]

If we use (2.8), (2.9) and (2.14) in the last equation, we can state following:

Lemma 2. The covariant derivative of the linear operator $L$ of a 3-dimensional quasi-Para-Sasakian manifold is given by

\[ (\nabla_X L)Y = \left( \frac{d\tau(X)}{4} + 2\beta d\beta(X) \right) Y - \left( 6\beta d\beta(X) + \frac{d\tau(X)}{2} \right) \eta(Y)\xi \]

\[ - \beta \left( 3\beta^2 + \frac{\tau}{2} \right) (g(\phi X, Y)\xi + \eta(Y)\phi X) - \beta g(\phi X, Y)\phi \text{grad} \beta \]

\[ - \beta d\beta(X)\eta(Y)\xi - \eta(Y)\phi \text{grad} \beta + (\nabla_X d\beta)(\phi Y)\xi \]

\[ - \beta \eta(Y) d\beta(X)\xi + \beta d\beta(\phi Y)\phi X. \]

Lemma 3. For the the function $\beta$ of 3-dimensional quasi-Para-Sasakian manifold, the following equality holds

\[ \nabla_\xi \text{grad} \beta = \beta \phi \text{grad} \beta. \]
Proof. By virtue of (2.14), we have
\[
[\xi, X](\beta) = \xi(X(\beta)) - X(\xi(\beta)) = g(\nabla_\xi \text{grad} \beta, X) + g(\text{grad} \beta, \nabla_\xi X).
\]
By (2.9), we get
\[
[\xi, X](\beta) = g(\text{grad} \beta, [\xi, X]) = g(\text{grad} \beta, \nabla_\xi X) + \beta g(\phi \text{grad} \beta, X).
\]
The proof comes from (3.6) and (3.7).

For any point \( p \in U \subset M \), there exists a local orthonormal \( \phi \)-basis \( \{ e_1 = \phi e_2, \ e_2 = \phi e_1, \ e_3 = \xi \} \), where \( g(e_1, e_1) = -g(e_2, e_2) = g(e_3, e_3) = 1 \).

For the sake of shortness, we will give followings
\[
\begin{align*}
\tau_i &= d\tau(e_i), \\
\beta_i &= d\beta(e_i), \\
\beta_{ij} &= (\nabla_{e_i} d\beta)(e_j), \\
L_{ij} &= (\nabla_{e_i} L)e_j, \\
\text{grad} \beta &= \beta_{11} e_1 + \beta_{22} e_2 + \beta_{33} e_3, \\
\nabla_{e_i} \text{grad} \beta &= \beta_{1i} e_1 + \beta_{2i} e_2 + \beta_{3i} e_3
\end{align*}
\]
for \( 1 \leq i, j \leq 3 \), where \( \tau_i, \beta_i, \beta_{ij} \) are the functions and \( L_{ij} \) are the vector fields on \( U \).

Also, one can easily see that \( \beta_{ij} = \beta_{ji} \). If we use (3.3), (2.9) and (2.13) in the following well known formula for semi-Riemannian manifolds
\[
\text{trace} \{ Y \to (\nabla_Y Q)X \} = 1/2 \nabla_X \tau
\]
we obtain
\[
(3.8) \quad \xi(\tau) = 0.
\]
From (2.14) and (3.8), we obtain
\[
(3.9) \quad \beta_3 = 0, \ \tau_3 = 0.
\]
(3.5) implies that
\[
(3.10) \quad \beta_{13} = \beta_{31} = -\beta_{12}, \ \beta_{23} = \beta_{32} = -\beta_{21}, \ \beta_{33} = 0.
\]

Lemma 4. For a 3-dimensional quasi-Para-Sasakian manifold, we have
\[
L_{ij} - L_{ji} = 0 \quad \text{for} \quad 1 \leq i, j \leq 3 \iff
\]
\[
(3.11) \quad \tau_1 = -20\beta_1, \ \tau_2 = -20\beta_2, \ \beta_{12} = \beta_{21} = 0, \ \beta_{22} = -\beta_{11} = \beta \left( 3\beta^2 + \frac{\tau}{2} \right).
\]

Proof. By direct computations, using (3.4), (3.5), (3.9) and (3.10), we derive
\[
\begin{align*}
L_{12} - L_{21} &= -\left( \frac{T_2}{4} + 5\beta_2 \right) e_1 + \left( \frac{T_1}{4} + 5\beta_1 \right) e_2 \\
&\quad + \left( \beta_{11} - \beta_{22} + \beta(\tau + 6\beta^2) \right) \xi, \\
L_{13} - L_{31} &= \beta_{12} e_1 + \left( -\beta_{11} - \beta \left( 3\beta^2 + \frac{\tau}{2} \right) \right) e_2 \\
&\quad + \left( -\frac{\tau_1}{4} - 5\beta_1 \right) \xi, \\
L_{23} - L_{32} &= \left( \beta_{22} - \beta \left( 3\beta^2 + \frac{\tau}{2} \right) \right) e_1 - \beta_{12} e_2 \\
&\quad + \left( -\frac{\tau_2}{4} - 5\beta_2 \right) \xi.
\end{align*}
\]
(3.12)
The proof follows from (3.12).
We know that a semi-Riemannian manifold is conformally flat $\iff (\nabla_X L)Y - (\nabla_Y L)X = 0$, for any vector fields $X$ and $Y$. Hence, we can say that a 3-dimensional quasi-Para-Sasakian manifold is conformally flat if and only if (3.11) holds. By (3.11), we can give following result.

**Theorem 4.** A 3-dimensional quasi-Para-Sasakian manifold is conformally flat if and only if the function $\beta$ satisfies the followings

\[
\tau + 10\beta^2 = \text{const.},
\]

\[
(\nabla_X d\beta)(Y) = -\beta \left(3\beta^2 + \frac{\tau}{2}\right)(g(X, Y) - \eta(X)\eta(Y)) - \beta\eta(X)d\beta(\phi Y) - \beta\eta(Y)d\beta(\phi X).
\]

(3.13)

**Theorem 5.** For a 3-dimensional quasi-Para-Sasakian manifold $M$ with $\beta = \text{const.}$, the following assertions are equivalent to each other:

i) $M$ is locally symmetric.

ii) $M$ is conformally flat and its scalar curvature $\tau$ is const.

iii) $M$ is conformally flat and $\beta$ is const.

iv) If $\beta = 0$, then $M$ is a paracosymplectic manifold which is locally a product of the real line $R$ and a 2-dimensional para-Kaehlerian manifold or

\[\begin{align*}
\bullet & \text{ If } \beta \neq 0, \text{ then } M \text{ is of constant negative curvature and the quasi-Para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure.}
\end{align*}\]

Proof. First of all, (i) implies (ii) because of the dim$M = 3$. From (3.13), one can see (ii) $\iff$ (iii). Now, we will show (iii) implies (iv). If $\beta$ is const., using (3.13), we get $\beta \left(3\beta^2 + \frac{\tau}{2}\right) = 0$ and $\tau$ is const. Now there are two possibilities. If $\beta = 0$, then $M$ is a paracosymplectic manifold which is locally a product of the real line $R$ and a 2-dimensional para-Kaehlerian manifold. If $\beta \neq 0$, then $\tau = -6\beta^2$, namely $M$ has constant negative curvature. By using $\tau = -6\beta^2$ in (3.13), we get $M$ is Einstein since $S = \frac{5}{2}g$. Using Theorem 3, one can say that the quasi-para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure. One can easily deduce that (iv) $\Rightarrow$ (i).}

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