QUANTUM INDICES AND REFINED ENUMERATION OF REAL PLANE CURVES

GRIGORY MIKHALKIN

Abstract. We associate a half-integer number, called the quantum index, to algebraic curves in the real plane satisfying certain conditions. The area encompassed by the logarithmic image of such curves is equal to \(\pi^2\) times the quantum index of the curve and thus has a discrete spectrum of values. We use the quantum index to refine enumeration of real rational curves in a way consistent with the Block-Göttsche invariants from tropical enumerative geometry.

1. Introduction.

1.1. Quantum index. Geometry of real algebraic curves in the plane is one of the most classical subjects in Algebraic Geometry.

It is easy to see that the logarithmic image \(\text{Log}(\mathbb{R}C^o) \subset \mathbb{R}^2\) of any real algebraic curve \(\mathbb{R}C^o \subset (\mathbb{R}^\times)^2 \subset \mathbb{R}^2\) under the map \(\text{Log}\) bounds a region of finite area in \(\mathbb{R}^2\) (see Figures 1, 2, 3 for some examples of \(\text{Log}(\mathbb{R}C^o)\) in degrees 1 and 2). Furthermore, this area is universally bounded from above for all curves of a given degree by the Passare-Rullgård inequality \cite{23} for the area of amoebas.

E.g. if \(\mathbb{R}C^o\) is a circle contained in the positive quadrant \((\mathbb{R}_{>0})^2\) then it bounds a disk \(D \subset (\mathbb{R}_{>0})^2\), \(\partial D = \mathbb{R}C^o\). The area of the disk \(D\) is

\[
\int_D dx dy = \pi r^2,
\]

where \(r\) is its radius. Clearly, \(\text{Area}(D)\) may be arbitrarily large. In the same time it can be proved that the area of \(\text{Log}D\) is

\[
\int_D \frac{dx}{x} \frac{dy}{y} < \pi^2.
\]

The inequality can be established either through direct computation or as a corollary of the Passare-Rullgård upper bound on the area of amoeba \cite{23}. Thus this logarithmic area of \(D\) stays bounded no matter how large is the radius \(r\). In the same time it is clear that \(\text{Area}(\text{Log}(D))\) can assume any value between 0 and \(\pi^2\).
In this paper we impose the following conditions on an algebraic curve $R \subset \mathbb{RP}^2$ (in the main body of the paper it is also formulated for other toric surfaces in place of $\mathbb{RP}^2$) so that such continuous behavior of the logarithmic area is no longer possible.

Namely, we assume that $R \subset \mathbb{RP}^2$ is an irreducible curve of type I (see subsection 2.1). Then according to [24] $R$ comes with a canonical orientation (defined up to simultaneous reversal in all components of $R$). This enables us to consider the signed area (with multiplicities) $\text{Area}_{\log}(R)$ bounded by $\log(R \circ) \subset \mathbb{R}^2$. Unless one of the two possible complex orientations of $R$ is chosen, $\text{Area}_{\log}(R)$ is only well-defined up to sign.

The curve $R \subset \mathbb{RP}^2$ is the zero set of an irreducible homogeneous polynomial $f(x_0, x_1, x_2)$. For simplicity in the introduction we assume that $R$ is disjoint from the points $(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)$. The restriction of $f$ to $x_j = 0, j = 0, 1, 2$, is a homogeneous polynomial $f_j$ in two variables responsible for the intersection of $R$ with the three coordinate axes of $\mathbb{RP}^2$. We say that $R$ has real or purely imaginary coordinate intersections if for any (complex) zero $(z_a : z_b)$ of $f_j$ we have $(z_b/z_a)^2 \in \mathbb{R}$. Theorem 1 asserts that in this case $\text{Area}_{\log}(R)$ must be divisible by $\pi^2$ and therefore cannot vary continuously. The number $k = \frac{\text{Area}_{\log}(R)}{\pi^2}$ is thus a half-integer number naturally associated to the curve. We call it the quantum index of $R$.

**Theorem 1** (special case for $\mathbb{RP}^2$). Let $R \subset \mathbb{RP}^2$ be a real curve of degree $d$ and type I enhanced with a complex orientation. If $R$ has real or purely imaginary coordinate intersection then

$$\text{Area}_{\log}(R) = k\pi^2$$

with $k \in \frac{1}{2}\mathbb{Z}$ and $-\frac{d^2}{2} \leq k \leq \frac{d^2}{2}$.

To our knowledge this classical-looking result is new even in the case $d = 2$. Meanwhile the special case of $d = 1$ is well-known. The identity $|\text{Area}_{\log}(R)| = \frac{\pi^2}{2}$ in the case of lines was used by Mikael Passare [22] in his elegant new proof of Euler’s formula $\zeta(2) = \frac{\pi^2}{6}$. Another known special case of Theorem 1 is the case of the so-called simple Harnack curves introduced in [16]. As it was shown in [20] these curves have the maximal possible value of $|\text{Area}_{\log}(R)|$ for their degree (equal to $\frac{d^2}{2}\pi^2$). Simple Harnack curves have many geometric properties [16]. By now these curves have appeared in a number of situations outside of real algebraic geometry, in particular in random perfect matchings of bipartite doubly periodic planar graphs of Richard Kenyon, Andrei Okounkov and Scott Sheffield [12]. The quantum index of Theorem 1
can be interpreted as a measure of proximity of a real curve to a simple Harnack curve.

Half-integrality of the quantum index $k$ may be explained through appearance of $2k$ as the degree of some map as exhibited in Proposition 3. In accordance with this interpretation Theorem 2 computes the quantum index through the degree of the real logarithmic Gauß map of $\mathbb{R}C$.

Theorem 3 studies the quantum index in the special case when $\mathbb{R}C$ is not only of type I, but also of toric type I (Definition 7). This condition implies that all coordinate intersections of $\mathbb{R}C$ are real. In this case the quantum index may be refined to the index diagram (Definition 9), a closed broken lattice curve $\Sigma \subset \mathbb{R}^2$ well-defined up to a translation by $2\mathbb{Z}^2$.

The broken curve $\Sigma$ is an immersed multicomponent curve with each component corresponding to a component of the compactification $\mathbb{R}C^*$ of $\mathbb{R}C^\circ$ defined by its Newton polygon $\Delta$. The complex orientation of $\mathbb{R}C$ induces an orientation of the closed broken curve $\Sigma$ so that we may compute the signed area $\text{Area} \Sigma$ inside $\Sigma$ which is a half-integer number as the vertices of $\Sigma$ are integer.

**Theorem 3 (simplified version).** If $\mathbb{R}C \subset \mathbb{R}P^2$ is a real algebraic curve of toric type I enhanced with a choice of its complex orientation then its quantum index $k$ coincides with $\text{Area} \Sigma$.

Each edge of $\Sigma$ corresponds to an intersection of $\mathbb{R}\bar{C}$ with a toric divisor of the toric variety $\mathbb{R}\Delta$ corresponding to the Newton polygon $\Delta$ and thus to a side $E \subset \partial \Delta$. If this intersection is transversal then the corresponding oriented edge of $\Sigma$ is given by the primitive integer outer normal vector $\vec{n}(E)$. More generally it is given by $\vec{n}(E)$ times the multiplicity of the intersection. This makes finding the index diagram $\Sigma$ and thus the quantum index $k$ especially easy at least in the case of rational curves with real coordinate intersections (cf. e.g. Figures 2 and 4).

The index diagram $\Sigma$ can be viewed as a non-commutative version of the Newton polygon $\Delta$: it is made from the same elements (the vectors $\vec{n}(E)$ taken $\#(E \cap \mathbb{Z}^2) - 1$ times) as $\partial \Delta$, but the real structure on $\mathbb{R}\bar{C}$ gives those pieces a cyclic order (in the case of connected $\mathbb{R}\bar{C}$) or divides these elements into several cyclically ordered subsets.

Recall that Mikael Forsberg, Mikael Passare and August Tsikh in [6] have defined the amoeba index map which is a locally constant map on $\mathbb{R}^2 \setminus \mathcal{A}$, the complement of the amoeba $\mathcal{A} = \text{Log}(\mathbb{C}C^\circ)$ of the complexification $\mathbb{C}C^\circ$ of $\mathbb{R}C^\circ$. Each connected component of $\mathbb{R}^2 \setminus \mathcal{A}$ is associated a lattice point of the Newton polygon $\Delta$. 
For toric type I curves the formula (6) defines the real index map so that each connected component of the normalization $\mathbb{R}\tilde{C}^o$ or a solitary real singularity of $\mathbb{R}C^o$ acquires a real index which is a lattice point of the convex hull of the index diagram $\Sigma$. Theorem 4 computes the amoeba index map $\text{ind}$ in terms of the linking number with the curve $\mathbb{R}C^o$ enhanced with the real indices.

1.2. **Refined real enumerative geometry in the plane.** The second part of the paper is devoted to applications of the quantum index of real curves introduced in this paper to enumerative geometry over complex and real numbers. The space of planar projective rational curves of degree $d$ is $(3d - 1)$-dimensional. Thus given a generic configuration $\mathcal{P}$ of $3d - 1$ points in the projective plane we expect a finite set $\mathcal{S}_d$ of such curves. What we can do next with this set depends on our choice of ground field.

Our two main choices are the fields $\mathbb{C}$ and $\mathbb{R}$ of complex and real numbers. For both of these cases we choose $\mathcal{P} \subset \mathbb{RP}^2$ generically and denote with $\mathcal{S}^C_d$ (resp., $\mathcal{S}^R_d$) the finite set of all planar projective rational curves of degree $d$ defined over $\mathbb{C}$ (resp., over $\mathbb{R}$) passing through $\mathcal{P}$. It is easy to see that the cardinality $N^C_d = \#(\mathcal{S}^C_d)$ does not depend on the choice of $\mathcal{P}$ (even if $\mathcal{P}$ is chosen generically in $\mathbb{CP}^2$ rather than $\mathbb{RP}^2$). In the same time the cardinality $\#(\mathcal{S}^R_d)$ depends on the choice of generic configuration $\mathcal{P}$ and a priori only the parity of this set remains invariant.

According to the seminal result of Welschinger [26] the curves $\mathbb{R}C \in \mathcal{S}^R_d$ come with natural signs $w(\mathbb{R}C) = \pm 1$ so that the integer number

$$N^R_d = \sum_{\mathbb{R}C \in \mathcal{S}^R_d} w(\mathbb{R}C)$$

is invariant of the choice of $\mathcal{P}$. The number $N^R_d$ is thus known as the Welschinger invariant and is the fundamental notion of real enumerative geometry. Itenberg, Kharlamov and Shustin in [9] have established non-trivial lower bounds on $\#(\mathcal{S}^R_d)$ with the help of $N^R_d$.

Both integer numbers $N^C_d$ and $N^R_d$ were simultaneously computed with the help of passing to the tropical limit in [17]. Namely, $N^C_d$ and $N^R_d$ can be presented as sums of multiplicities of corresponding tropical curves passing through a generic configuration of points in the tropical plane. The tropical curves are the same in both cases, however the rules for defining their $\mathbb{C}$ and $\mathbb{R}$ multiplicities are different, so the sums $N^C_d$ and $N^R_d$ are different as well.

With the help of this presentation Block and Göttsche in [1] have proposed combining the numbers $N^C_d$ and $N^R_d$ into a single number
$N_{d}^{\text{trop}}$, which is no longer an integer number, but an integer $q$-number (a Laurent polynomial in $q$ with positive integer coefficients invariant under the substitution $q \mapsto \frac{1}{q}$). The value at $q = 1$ is capable to recover the number of complex curves while the value at $q = -1$ should be capable to recover the number of real curves in the same enumerative problem. E.g. there are $q + 10 + q^{-1}$ many of rational cubic curves passing through 8 generic points in $\mathbb{R}P^2$. In the same time there are 12 curves over $\mathbb{C}$ and 8 curves over $\mathbb{R}$ (if we count real curves with the Welschinger sign [26]).

Conjecturally (see [7]) the $q$-refinement $N_{d}^{\text{trop}}$ of the integer number $N_{d}^{\text{C}}$ agrees with the $\chi_y$-genus refinement of Severi degrees proposed by Göttsche and Shende in [7]. Also this refinement looks to be at least vaguely resemblant of the refinements of Donaldson-Thomas invariants considered by Kontsevich and Soibelman [13] and Nekrasov and Okounkov [21] in some other frameworks (in particular, for 3-folds).

The quantum index allows us to obtain a refined enumeration of planar curves entirely within classical real algebraic geometry of the plane with the help of Theorem 5. Once again for simplicity we discuss only the special case of the projective plane here in the introduction while in the main body of the paper the theorem is formulated for other toric surfaces as well.

Recall that the space of rational curves of degree $d$ in $\mathbb{R}P^2$ is $(3d-1)$-dimensional. Thus we expect a finite number of such curves if we impose on them $3d-1$ conditions. Let us choose a generic configuration of $3d-1$ points on the three coordinate axes of $\mathbb{R}P^2$ (the $x$-axis, the $y$-axis and the $\infty$-axis) so that each axis contains no more than $d$ points: e.g. there are $d$ generic points on the $x$- and $y$-axis and $3d-1$ generic points on the $\infty$-axis. The elementary generalization of the classical Menelaus theorem (see Figure 7) found already by Carnot [4] (later further generalized as the Weil reciprocity law) ensures that there is a unique $3d$th point on the $\infty$-axis such that any irreducible curve of degree $d$ passing through our $3d-1$ points also passes through the $3d$th point. The resulting configuration of $3d$ points on the union of three coordinate axes varies in a $(3d-1)$-dimensional family of Menelaus configurations.

We define the square map $\text{Sq} : \mathbb{C}P^2 \to \mathbb{C}P^2$ by $\text{Sq}(z_0 : z_1 : z_2) = (z_2^2 : z_1^2 : z_0^2)$. An irreducible rational curve $\mathbb{R}C \subset \mathbb{R}P^2$ such that $\text{Sq}(\mathbb{C}C)$ passes through a Menelaus configuration $\mathcal{P}$ is of type I and has real or purely imaginary coordinate intersection. Thus the quantum index $k$ is well-defined.
In (21) we define $R_{d,k}(\mathcal{P})$ (here we write $d$ instead of $\Delta$ since we restrict ourselves to the special case of $\mathbb{RP}^2$ in the introduction) as one quarter of the number of irreducible oriented rational curves $\mathbb{RC} \subset \mathbb{RP}^2$ of degree $d$ and quantum index $k$ such that $\text{Sq}(\mathbb{CC})$ passes through $\mathcal{P}$. Each curve $\mathbb{RC}$ here is taken with the sign (19) which is a modification of the Welschinger sign [26]. Note that such curves come in quadruples thanks to the action of the deck transformations of the 4-1 covering $\text{Sq}|_{(\mathbb{R}^\times)^2} : (\mathbb{R}^\times)^2 \to (\mathbb{R}^\times)^2$. This is the reason for including $\frac{1}{4}$ in the definition of $R_{d,k}(\mathcal{P})$. The points of $\mathcal{P}$ contained in the closure of the positive quadrant $(\mathbb{R}_{>0})^2$ (positive points) correspond to real coordinate axes intersections of $\mathbb{RC}$, other (negative) points correspond to purely imaginary coordinate axes intersections.

The image of each component of $\mathbb{CC} \setminus \mathbb{RC}$ under $\text{Sq}$ may be viewed as an open holomorphic disk $F$ in $\mathbb{CP}^2$ with the boundary contained in the closure $L = (\mathbb{R}_{>0})^2$ of the positive quadrant. The subspace $L \subset \mathbb{CP}^2$ is a Lagrangian submanifold with boundary. The positive points of $\mathcal{P}$ correspond to tangencies of $\partial F$ and $\partial L$ while the negative points of $\mathcal{P}$ correspond to intersections of the open disk $F$ with the coordinate axes of $\mathbb{RP}^2$ away from $\partial L$. From this viewpoint $R_{d,k}(\mathcal{P})$ is the number of certain holomorphic disks whose boundary is contained in $L$, a framework widely used in symplectic geometry. An unconventional feature is presence of boundary in the contractible (and thus orientable) Lagrangian surface $L$. Positive points of $\mathcal{P}$ are contained in the boundary $\partial L$. The holomorphic disks are tangent to $\partial L$ at these points. Negative points of $\mathcal{P}$ are disjoint from $L$ and and thus from the boundaries of the holomorphic disks.

The number of negative points on three coordinate axes is given by $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_j \leq d$.

**Theorem 5.** The number $R_{d,k,\lambda} = R_{d,k}(\mathcal{P})$ is invariant of the choice of $\mathcal{P}$ and depends only on $d$, $k$ and $\lambda$.

In particular, $R_{d,k}(\mathcal{P})$ depends only on $d$ and $k$ when all points of $\mathcal{P}$ are positive.

For a positive point $p \in \mathcal{P}$ the inverse image $\text{Sq}^{-1}(p)$ consists of two points: a positive point $p_+ \in \partial L$ and a negative point $p_- \notin \partial L$. The condition $\text{Sq}(\mathbb{CC}) \ni p$ is equivalent to the condition that $\mathbb{RC}$ passes through $p_+$ or $p_-$. Note that the invariance claimed in Theorem 5 relies on including into $R_{d,k}(\mathcal{P})$ both of these possibilities. If we leave out only the curves passing through $p_+$ (or $p_-$) as in (22) then the resulting sum $\hat{R}_{d,k}(\mathcal{P})$ is no longer invariant under deformations of $\mathcal{P}$. Nevertheless, a partial invariance result for $\hat{R}_{d,k}(\mathcal{P})$ is provided by Theorem 6.
The generating function \( R_d(\lambda) = \sum_k R_{d,k,\lambda} q^k \) defined in (25) is a Laurent polynomial in \( q^{\frac{1}{2}} \). As such it can be compared with the modification \( N^\partial,\text{trop}_d \) of the Block-Göttsche refined tropical invariants \( N^\partial_{\text{trop}}_d \) where we take for \( P \) a generic Menelaus configuration of points in the boundary \( \partial TP^2 = TP^2 \setminus \mathbb{R}^2 \) rather than a generic configuration of points in \( \mathbb{R}^2 \). Namely, the number \( N^\partial,\text{trop}_d \) is given by (47), where \( \Delta \) is a triangle with vertices \((0,0), (d,0)\) and \((0,d)\). The last theorem of the paper is an identity between \( R_d = R_d(0,0,0) \) and \( N^\partial,\text{trop}_d \).

**Theorem 7** (special case of \( \mathbb{RP}^2 \)).

\[
R_d = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{3d-2} N^\partial,\text{trop}_d.
\]

This theorem has a surprising corollary. As the number \( N^\partial,\text{C}_d \) of irreducible rational complex curves \( \mathbb{C}C \subset \mathbb{CP}^2 \) of degree \( d \) passing through \( P \) coincides with \( N^\partial,\text{trop}_d(1) \), this number is completely determined by \( R_d \), the number accounting only for curves defined over \( \mathbb{R} \). Note that for this purpose it is crucial to use the quantum refinement by \( q^k \) since for \( q = 1 \) we would have to divide by 0 (the value of \( (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{3d-2} \) at \( q^{\frac{1}{2}} = 1 \)) to recover \( N^\partial,\text{trop}_d(1) \).

2. Conventions and notations

2.1. **Real curves of type I and their complex orientation.** A real curve \( \mathbb{R}C \subset \mathbb{RP}^2 \) is given by a single homogeneous polynomial equation \( F(z_0, z_1, z_2) = \sum_{j,k,l} a_{j,k,l} z_0^j z_1^k z_2^l = 0, \ j + k + l = d, \ a_{j,k,l} \in \mathbb{R} \). The locus \( \mathbb{C}C \subset \mathbb{CP}^2 \) of complex solutions of \( F = 0 \) is called the complexification of \( \mathbb{R}C \). We assume \( F \) to be irreducible over \( \mathbb{C} \) and such that \( \mathbb{C}C \) does not coincide with a coordinate axis \( \{z_j = 0\}, \ j = 0, 1, 2 \). The normalization

\[
\nu : \mathbb{C}\tilde{C} \to \mathbb{C}C
\]

defines a parameterization of \( \mathbb{C}C \) by a Riemann surface \( \mathbb{C}\tilde{C} \). The antiholomorphic involution of complex conjugation \( \text{conj} \) acts on \( \mathbb{C}C \) in an orientation-reversing way so that its fixed point locus is \( \mathbb{R}C \). The restriction of \( \text{conj} \) to the smooth locus of \( \mathbb{C}C \) lifts to an antiholomorphic involution \( \tilde{\text{conj}} : \mathbb{C}\tilde{C} \to \mathbb{C}\tilde{C} \) on the normalization. We denote with \( \mathbb{R}\tilde{C} \) the fixed point locus of \( \text{conj} \). Clearly, \( \nu(\mathbb{R}\tilde{C}) \subset \mathbb{R}C \). Irreducibility of \( \mathbb{C}C \) is equivalent to connectedness of \( \mathbb{C}\tilde{C} \).

Following Felix Klein we say that \( \mathbb{R}C \) is of type I if \( \mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C} \) is disconnected. In such case it consists of two connected components \( S \) and \( S' = \text{conj}(S) \) which are naturally oriented by the complex orientation.
of the Riemann surface $C \tilde{C}$. We have $R \tilde{C} = \partial S = \partial S'$, so a choice of one of these components, say $S$, induces the boundary orientation on $R \tilde{C}$. The resulting orientation is called a complex orientation of $R \tilde{C}$ and is subject to Rokhlin's complex orientation formula [24]. If we choose $S'$ instead of $S$ then the orientations of all components of $R \tilde{C}$ will reverse simultaneously. Thus any orientation of a component of $R \tilde{C}$ determines a component of $C \tilde{C} \setminus R \tilde{C}$.

2.2. Toric viewpoint and reality of coordinate intersections. The projective plane $\mathbb{CP}^2$ can be thought of as the toric compactification of the torus $(\mathbb{C}^\times)^2$. The curve $C \tilde{C} \subset \mathbb{CP}^2$ is the closure of its toric part $C C^\circ = C C \cap (\mathbb{C}^\times)^2$. The complement $\partial \mathbb{CP}^2 = \mathbb{CP}^2 \setminus (\mathbb{C}^\times)^2$ is the union of three axes: the $x$-axis, the $y$-axis and the $\infty$-axis. These axes intersect pairwise at the points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1) \in \mathbb{RP}^2$. If the coefficients $a_{0,0}, a_{d,0}, a_{0,d}$ are non-zero then $C \tilde{C}$ is disjoint from the intersection points of the axes. In the general case it is reasonable to consider other toric surfaces compactifying $(\mathbb{C}^\times)^2$, so that the closure of $C C^\circ$ is disjoint from pairwise intersections of toric divisors.

Let us consider the (non-homogeneous) polynomial $f(x, y) = F(1, x, y)$ and its Newton polygon

$$\Delta = \text{ConvexHull}\{(j, k) \in \mathbb{R}^2 \mid a_{j,k} \neq 0\}.$$ 

If $\Delta$ has non-empty interior then the dual fan to $\Delta$ defines a toric compactification $C \Delta \supset (\mathbb{C}^\times)^2$. The toric divisors of $C \Delta$ correspond to the sides of $\Delta$. Their pairwise intersections correspond to the vertices of $\Delta$ and are disjoint from the compactification $C \tilde{C}$ of the curve $C C^\circ$. We denote with $\partial C \Delta \subset C \Delta$ the union of toric divisors. Accordingly, we denote with $R \Delta$ (resp. $\partial R \Delta, R C^\circ, R C$) the real part of $C \Delta$ (resp. $\partial C \Delta, C C^\circ, C C$). E.g. we have $\mathbb{RP}^2 = R \Delta$ for the triangle $\Delta = \text{ConvexHull}\{(0,0), (1,0), (0,1)\}$ or a positive integer multiple of this triangle.

Let $\text{Sq} : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2$ be the map defined by $\text{Sq}(x, y) = (x^2, y^2)$. This map extends to a map $\text{Sq}^\Delta : C \Delta \to C \Delta$.

We call a point $p \in C \Delta$ real or purely imaginary if $\text{Sq}^\Delta(p) \in R \Delta$. We say that a curve $R C \subset \mathbb{RP}^2$ has real or purely imaginary coordinate intersection if every point of $C C \cap \partial C \Delta$ is real or purely imaginary.

2.3. Logarithmic area and other numbers associated to a real curve of type I. Let $R C$ be a real curve of type I enhanced with a choice of a complex orientation. Consider the image $\text{Log}(R C^\circ) \subset \mathbb{R}^2$, where $\text{Log} : (\mathbb{C}^\times)^2 \to \mathbb{R}^2$ the map defined by

$$\text{Log}(x, y) = (\log |x|, \log |y|).$$
For a point \( p \in \mathbb{R}^2 \setminus \text{Log}(\mathbb{R}^C) \) we define \( \text{ind}(p) \in \mathbb{Z} \) as the intersection number of an oriented ray \( R \subset \mathbb{R}^2 \) emanating from \( x \) in a generic direction and the oriented curve \( \text{Log}(\mathbb{R}^C) \) (this number can be considered as the linking number of \( p \) and \( \text{Log}(\mathbb{R}^C) \)).

**Definition 1.** The integral

\[
\text{Area}_{\text{Log}}(\mathbb{R}^C) = \int_{\mathbb{R}^2} \text{ind}_{\mathbb{R}^C}(x) \, dx
\]

is called the *logarithmic area* of \( \mathbb{R}^C \).

This is the signed area encompassed by \( \text{Log}(\mathbb{R}^C) \), where the area of each region of \( \mathbb{R}^2 \setminus \text{Log}(\mathbb{R}^C) \) is taken with the multiplicity equal to the linking number of \( \text{Log}(\mathbb{R}^C) \).

Let \( S \subset \mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C} \) be the component corresponding to the chosen complex orientation of \( \mathbb{R}\tilde{C} \). The intersection points \( \nu(S) \cap (\mathbb{R}^\times)^2 \) are the so-called *solitary real singularities* of \( \mathbb{R}^C \). The *multiplicity* of a solitary real singularity \( p \in \nu(S) \cap (\mathbb{R}^\times)^2 \) is the local intersection number of \( S \) and \( (\mathbb{R}^\times)^2 \) at \( p \). Here the orientation of \( S \) is induced by the inclusion \( S \subset \mathbb{C}\tilde{C} \), while the orientation of \( (\mathbb{R}^\times)^2 \) is induced by the covering \( \text{Log} \mid_{(\mathbb{R}^\times)^2} : (\mathbb{R}^\times)^2 \to \mathbb{R}^2 \). In other words, the quadrants \( \mathbb{R}^2_{>0} \) and \( \mathbb{R}^2_{<0} \) are counterclockwise-oriented while the quadrants \( \mathbb{R}_{>0} \times \mathbb{R}_{<0} \) and \( \mathbb{R}_{<0} \times \mathbb{R}_{>0} \) are clockwise-oriented. The *toric solitary singularities number* \( E(\mathbb{R}^C) \in \mathbb{Z} \) is the sum of multiplicities over all solitary real singularities of \( \mathbb{R}^C \), i.e. the total intersection number of \( S \) and \( (\mathbb{R}^\times)^2 \) (enhanced with our choice of orientation).

The logarithmic Gauß map sends a smooth point of \( \mathbb{R}^C \) to the tangent direction of the corresponding point on \( \text{Log}(\mathbb{R}^C) \subset \mathbb{R}^2 \). This map uniquely extends to a map

\[
\gamma : \mathbb{R}\tilde{C} \to \mathbb{R}P^1,
\]

cf. [11], [16]. *The logarithmic rotation number* \( \text{Rot}_{\text{Log}}(\mathbb{R}^C) \in \mathbb{Z} \) is the degree of \( \gamma \).

3. Quantum indices of real curves.

**Theorem 1.** Let \( \mathbb{R}^C \subset \mathbb{R}P^2 \) be a real curve of type I enhanced with a complex orientation. If \( \mathbb{R}^C \) has real or purely imaginary coordinate intersection then

\[
\text{Area}_{\text{Log}}(\mathbb{R}^C) = k\pi^2
\]

where \( k \in \frac{1}{2} \mathbb{Z} \),

\[-\text{Area}(\Delta) \leq k \leq \text{Area}(\Delta)\]

and \( k \equiv \text{Area}(\Delta) \pmod{1} \).
Note that as $\Delta \subset \mathbb{R}^2$ is a lattice polynomial, its area is a half-integer number.

**Definition 2.** We say that $k(\mathbb{R}C) = \frac{1}{\pi} \text{Area}_{\text{Log}}(\mathbb{R}C)$ is the quantum index of $\mathbb{R}C$.

If $\mathbb{R}C$ is an irreducible real curve of type I with real or purely imaginary coordinate intersection, but the complex orientation of $\mathbb{R}C$ is not fixed then its quantum index is well-defined up to sign.

The quantum index $k(\mathbb{R}C)$ can also be expressed without computing the logarithmic area.

**Proposition 3.** The integer number $2k(\mathbb{R}C)$ coincides with the degree of the map

$$2 \text{Arg} : \mathbb{C}C^o \setminus \mathbb{R}C^o \to (\mathbb{R}/\pi\mathbb{Z})^2,$$

i.e. with the number of inverse images at a generic point of the torus $(\mathbb{R}/\pi\mathbb{Z})^2$ counted with the sign according to the orientation. (In particular, this number does not depend on the choice of a point in $(\mathbb{R}/\pi\mathbb{Z})^2$ as long as this choice is generic.) Here the orientation of $\mathbb{C}C^o \setminus \mathbb{R}C^o$ is defined by the condition that it coincides with the complex orientation of $\mathbb{C}C$ on the component $S \subset \mathbb{C}C^o \setminus \mathbb{R}C^o$ determined by the orientation of $\mathbb{R}C$ and is opposite to the complex orientation of $\mathbb{C}C$ on the component $\text{conj}(S) \subset \mathbb{C}C^o \setminus \mathbb{R}C^o$. The map $2\text{Arg}$ is defined by $2\text{Arg}(x, y) = (2\text{arg}(x), 2\text{arg}(y))$.

We say that $\mathbb{R}\tilde{C}$ is transversal to $\partial\mathbb{R}\Delta$ if for any $p \in \mathbb{R}C \cap \partial\mathbb{R}\Delta$ we have $\nu^{-1}(p) \subset \mathbb{R}\tilde{C}$ and the composition $\mathbb{R}\tilde{C} \to \mathbb{R}\tilde{C} \subset \mathbb{R}\Delta$ is an immersion near $\nu^{-1}(p) \subset \mathbb{R}\tilde{C}$, and this immersion is transversal to $\partial\mathbb{R}\Delta$.

**Theorem 2.** Let $\mathbb{R}C$ be a curve of type I with real or purely imaginary coordinate intersections such that $\mathbb{R}\tilde{C}$ is transversal to $\partial\mathbb{R}\Delta$. Then

$$k(\mathbb{R}C) = -\frac{1}{2} \text{Rot}_{\text{Log}}(\mathbb{R}C) + E(\mathbb{R}C).$$

If $\mathbb{R}\tilde{C}$ is not transversal to $\partial\mathbb{R}\Delta$ then an adjustment of the right-hand side according to the order of tangency and the orientation of $\mathbb{R}C$ should be added to the formula of Theorem 2.

**Example 4 (Simple Harnack curves).** If $\mathbb{R}C^o \subset (\mathbb{R}^\times)^2$ is a simple Harnack curve (see [16]) then $k(\mathbb{R}C) = \pm \text{Area}(\Delta)$. Vice versa, if $k(\mathbb{R}C) = \pm \text{Area}(\Delta)$ then $\mathbb{R}C^o$ is a simple Harnack curve, see [20]. This characterizes real curves of the highest and lowest quantum index.
Example 5 (Quantum indices of real lines). Any real line is a curve of type I and has real coordinate intersections. The quantum index of a real line in \( \mathbb{RP}^2 \) disjoint from the points \((1:0:0),(0:1:0),(0:0:1)\) is \(\pm\frac{1}{2}\) (depending on its orientation), see Figure 1. The quantum index of a line passing through exactly one of these points is 0.

\[ k = -1 \quad k = 0 \quad k = +1 \]

**Figure 1.** Oriented lines, their logarithmic images and quantum indices.

Example 6 (Quantum indices of real conics). A smooth nonempty real conic is a curve of type I. Figure 2 depicts real conics in \( \mathbb{RP}^2 \) that intersect the coordinates axes in 6 real points.

\[ k = \pm 2 \quad k = \pm 1 \quad k = 0 \quad k = \mp 1 \]

**Figure 2.** Projective hyperbolas, their logarithmic images and quantum indices.

Note that a circle in \( \mathbb{R}^2 \) intersects the infinite axis of \( \mathbb{RP}^2 \) at the points \((0:1:\pm i)\). Thus a circle intersecting the coordinate axes of \( \mathbb{R}^2 \)
in 4 real points has real or purely imaginary coordinate intersection, see Figure 3. A circle passing through the origin in \( \mathbb{R}^2 \) has quantum index \( \pm \frac{1}{2} \). Otherwise, the quantum index of a circle is 0 or \( \pm 1 \).

\[ k = \pm 1 \quad k = \pm \frac{1}{2} \quad k = 0 \]

Figure 3. Circles, their logarithmic images and quantum indices.

4. Toric type I curves: quantum indices and diagrams

4.1. Definition of toric type I curves and their index diagrams.

Denote with \( \mathbb{C}\tilde{C}^\circ \subset \mathbb{C}C^\circ \) the normalization of an algebraic curve \( \mathbb{C}C^\circ \subset (\mathbb{C}^\times)^2 \) and with \( \mathbb{R}\tilde{C}^\circ \) its real part. The composition of the normalization and the inclusion map induces a map \( \mathbb{C}\tilde{C}^\circ \setminus \mathbb{R}\tilde{C}^\circ \to (\mathbb{C}^\times)^2 \).

Definition 7. We say that an irreducible real algebraic curve \( \mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2 \) has toric type I if \( \mathbb{R}C \) is of type I (see Section 2.1) and the induced homomorphism

\[ H_1(\mathbb{C}\tilde{C}^\circ \setminus \mathbb{R}\tilde{C}^\circ) \to H_1((\mathbb{C}^\times)^2) = \mathbb{Z}^2 \]

is trivial.

Each side \( E \subset \Delta \) is dual to a unique primitive integer vector \( \vec{n}(E) \subset \mathbb{Z}^2 \) (which sits in the space dual to the vector space containing the Newton polygon \( \Delta \)) oriented away from \( \Delta \). We refer to \( \vec{n}(E) \) as the normal vector to \( E \subset \partial \Delta \).

Proposition 8. If \( \mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2 \) is of toric type I then

\[ \mathbb{C}\tilde{C} \cap \partial \mathbb{C}\Delta \subset \mathbb{R}\tilde{C} \subset \mathbb{R}\Delta. \]

In other words \( \mathbb{R}C \) has real coordinate intersection. Thus it has a well-defined quantum index for any of its two complex orientation.
Proof. The homology class in $H_1((\mathbb{C}^*)^2) = \mathbb{Z}^2$ of a small loop in $\mathbb{C}C^o$ around a point of $\mathbb{C}C \cap \partial \Delta$ is a positive multiple of $\vec{n}(E)$ for a side $E \subset \Delta$. Therefore this class is non-zero. Thus such loop must intersect $\mathbb{R}C^o$ if $\mathbb{R}C^o$ is of toric type I. \hfill \square

Definition 9. A continuous map $a : \Sigma \to \mathbb{R}^2$ from a graph $\Sigma$ is called the index diagram of the curve $\mathbb{R}C^o$ of toric type I enhanced with a choice of the complex orientation corresponding to $S \subset \tilde{\mathbb{C}} \setminus \tilde{\mathbb{R}C}$ if the following conditions hold.

- The vertices of the graph $\Sigma$ are parameterized by the connected components $K^o \subset \mathbb{R}C^o$. We denote the corresponding vertex with $v(K^o) \in \Sigma$.
- The image $a(v(K^o)) = (a, b) \in \mathbb{Z}^2$ is a lattice point in $\mathbb{R}^2$ such that $K^o$ is contained in the $((-1)^a, (-1)^b)$-quadrant of $(\mathbb{R}^*)^2$.
- Vertices $v(K^o_1), v(K^o_2) \in \Sigma$ are connected with an oriented edge $e$ (which we identify with the straight oriented interval $[0, 1]$) if and only if $K^o_1$ and $K^o_2$ are adjacent at a point $p_e \in \mathbb{R}C$ in the order defined by the complex orientation of $\mathbb{R}C$. (Clearly both $K^o_1$ and $K^o_2$ are non-compact in such case.)
- The restriction $a|_e : e \approx [0, 1] \to \mathbb{R}^2$ is an affine map with
  \begin{equation}
  a(v(K^o_2)) - a(v(K^o_1)) = m_e \vec{n}(E).
  \end{equation}
  Here $m_e$ is the local intersection number of $\tilde{\mathbb{C}}$ and $\partial \Delta$ at $p_e$.
- There exists a continuous map
  \begin{equation}
  \tilde{l} : \tilde{S} = (S \setminus \partial \Delta) \cup \mathbb{R}C^o \to \mathbb{C}^2
  \end{equation}
  holomorphic on $\tilde{S} \setminus \mathbb{R}C^o$ such that $e^{\pi \tilde{l}}$ coincides with the tautological map $\tilde{S} \to (\mathbb{C}^*)^2$ while for every connected component $K^o \subset \mathbb{R}C^o$ we have
  \begin{equation}
  \Im \tilde{l}(K^o) = a(v(K^o)).
  \end{equation}
  Here both the exponent $e^{\pi \tilde{l}}$ and the imaginary part $\Im \tilde{l}(K^o)$ are understood coordinatewise.

Topologically the graph $\Sigma$ is the disjoint union of $n$ circles and $m$ points, where $n$ is the number of components of $\mathbb{R}C$ intersecting $\partial \Delta$, and $m$ is the number of compact components of $\mathbb{R}C^o$.

Denote with $\tilde{\Sigma} \subset \mathbb{R}^2$ the convex hull of $a(\Sigma)$. The map
\begin{equation}
\alpha : K^o \mapsto \Im \tilde{l}(K^o) = a(v(K^o)) \in \tilde{\Sigma} \cap \mathbb{Z}^2
\end{equation}
defined on the components of $\mathbb{R}C^o$ is called the real index map. Since the map $\tilde{l}$ is holomorphic its imaginary part $\Im \tilde{l}$ is harmonic, and thus $\Im \tilde{l}(\tilde{S}) \subset \tilde{\Sigma}$. 

Proposition 10. Any curve $\mathbb{R}C^o \subset (\mathbb{R}^*)^2$ of toric type I admits an index diagram $\Sigma(\mathbb{R}C) \subset \mathbb{R}^2$ which is unique up to a translation by $2\mathbb{Z}^2$ in $\mathbb{R}^2$.

Proof. Since $\mathbb{R}C$ is of toric type I the surface $\tilde{S} \subset (\mathbb{C}^*)^2$ lifts under the exponent map $\mathbb{C}^2 \to (\mathbb{C}^*)^2$. Translating the lift by integer multiples of $\pi$ if needed ensures that $(a, b) + 2i\mathbb{Z}^2 \subset \mathbb{C}^2$ corresponds to the lift of the $((-1)^a, (-1)^b)$-quadrant in $(\mathbb{C}^*)^2$. Denote this lift with $\tilde{l}$, and define the map on the vertices of $\Sigma$ by (6). An edge $e \subset \Sigma$ is mapped to the image of the accumulation set at the end of $\tilde{S}$ corresponding to $e$. To check the condition (4) we change coordinates in $(\mathbb{C}^*)^2$ multiplicatively so that that the toric divisor corresponding to $e$ is the $x$-axis. Then $\tilde{l}$ maps the accumulation set at the $e$-end of $\tilde{S}$ to the vertical interval of length $2m_e$. Reversing the coordinate change we recover an interval parallel to $\vec{n}(E)$. □

Note that for each connected component $K \subset \mathbb{R}\tilde{C}$ (which is necessarily closed) with $K \cap \partial\mathbb{R}\Delta \neq \emptyset$ the formula (4) already determines the part $a(K) : \Sigma(K) \to \mathbb{R}^2$ corresponding to $K$ of the index diagram $a : \Sigma \to \mathbb{R}^2$ up to a translation in $\mathbb{R}^2$. Indeed it suffices to choose arbitrarily $\alpha(K^o)$ of an arc $K^o \subset K \setminus \partial\mathbb{R}\Delta$ and proceed inductively.

Proposition 11. If $\mathbb{R}C^o$ is a curve of toric type I then the broken line $a(\Sigma(K))$ resulting from inductive application of (4) is closed for any connected component $K$ of $\mathbb{R}\tilde{C}$ with $K \cap \partial\mathbb{R}\Delta \neq \emptyset$.

Conversely, suppose that $\mathbb{R}\tilde{C}$ is an M-curve (i.e. the number of its components is one plus the genus of $\mathbb{C}\tilde{C}$) with $(\mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}) \cap \partial\mathbb{C}\Delta = \emptyset$ such that the broken line defined inductively by (4) for every connected components $K \subset \mathbb{R}\tilde{C}$ is closed. Then $\mathbb{R}C^o$ has toric type I.

Proof. The first part of the statement is a corollary of Proposition 10. Conversely, for an M-curve $\mathbb{R}\tilde{C}$ each component of $\mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$ is a sphere with punctures corresponding to the components of $\mathbb{R}\tilde{C}$. The homology class of a loop for each component is determined inductively by (4). It is zero by our hypothesis. □

4.2. Quantum index and toric complex orientation formula.

Let $\mathbb{R}C^o \subset (\mathbb{R}^*)^2$ be a curve of toric type I enhanced with the complex orientation corresponding to a half $S \subset \mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$. Denote with

$$\text{Area } \Sigma \in \frac{1}{2}\mathbb{Z}$$

the signed area (with multiplicities) enclosed by $a(\Sigma)$ in $\mathbb{R}^2$.
Let $p \in \mathbb{R}^2$ be a point, let $R_\epsilon \subset \mathbb{R}^2$ be the oriented ray emanating from $p$ in a generic direction $\epsilon$ in $\mathbb{R}^2$. Define $lk_\epsilon(p, \Sigma)$ as the intersection number of the image $a(\Sigma)$ and $R_\epsilon$ in points other than $p$. If $p \notin a(\Sigma)$ then this number is the index of $p$ with respect to $a(\Sigma)$ (considered in Section 2.3), and does not depend on the choice of $\epsilon$. Otherwise, $lk_\epsilon(p, \Sigma)$ depends on $\epsilon$.

For each quadrant $Q = ((-1)^a, (-1)^b)_{\mathbb{R}^2_{>0}}$ we define

$$
(8) \quad lk_\epsilon(Q, \Sigma) = \sum_{k_a, k_b \in \mathbb{Z}} lk_\epsilon((a + 2k_a, b + 2k_b), \Sigma) \in \mathbb{Z}.
$$

Any connected component $K \subset \mathbb{R}\tilde{C}$ disjoint from $\partial \mathbb{R} \triangle$ is contained in a single quadrant $Q$. The image $\text{Log}(K)$ is a closed oriented curve in $\mathbb{R}^2$. Let $\lambda(K) \in \mathbb{Z}$ be the rotation number of $\text{Log}(K)$, i.e. the degree of the logarithmic Gauß map of $K \subset \mathbb{R}^2$. (E.g. if $K \subset \mathbb{R}^2$ is a positively oriented embedded circle contained in the $(+, +)$- or $(-, -)$-quadrant (resp. in the $(+, -)$- or $(-, +)$-quadrant) then $\lambda(K) = 1$ (resp. $\lambda(K) = -1$).) Any point of $S \cap Q$ is a real isolated singular point $p \in \mathbb{R}C^\circ$. We denote with $\lambda(p) \in \mathbb{Z}$ the intersection number of $S$ and $Q$. Recall that the orientation of $(\mathbb{R}^x)^2$ (and thus, of $Q$) is defined in Section 2.3 as the pull-back of the standard orientation of $\mathbb{R}^2$ by $\text{Log}_\mid Q$.

If $K^\circ \subset \mathbb{R}\tilde{C}^\circ$ is a connected component (not necessarily compact) then the local degree of the oriented logarithmic Gauß map $\tilde{\gamma}_\mid K : K \to \mathbb{R} \tilde{\mathbb{P}}^1$ at a point $\epsilon \in \mathbb{R} \tilde{\mathbb{P}}^1$ may depend on the choice of $\epsilon$. For $(a, b) \in \mathbb{Z}^2$ and $\epsilon \in \mathbb{R} \tilde{\mathbb{P}}^1 \setminus Q \tilde{\mathbb{P}}^1$ we set

$$
(9) \quad \lambda_\epsilon(a, b) = -\sum_K \lambda(K) + \sum_p \lambda(p),
$$

where the sums are taken over all components $K^\circ \subset \mathbb{R} \tilde{C}^\circ$ with $\alpha(K^\circ) = (a, b)$ and all isolated singular points $p$ of $\mathbb{R}C^\circ$ with $\text{Im} \tilde{l}(p) = (a, b)$. The following statement is straightforward (with the help of the maximum principle for $\text{Im} \tilde{l}$).

**Proposition 12.** The number $\lambda_\epsilon(a, b)$ does not depend on $\epsilon$ if $(a, b) \notin a(\Sigma)$. If $(a, b) \notin \Sigma$ then $\lambda_\epsilon(a, b) = 0$.

For each quadrant $Q = ((-1)^a, (-1)^b)_{\mathbb{R}^2_{>0}} \subset (\mathbb{R}^x)^2$ we may take the sum

$$
(10) \quad \lambda_\epsilon(Q) = \sum_{k_a, k_b \in \mathbb{Z}} \lambda_\epsilon(a + 2k_a, b + 2k_b).
$$

The result is independent on the translation ambiguity in the definition of the real index map.
Theorem 3. If $\mathbb{R}C^o \subset (\mathbb{R}^x)^2$ is a real algebraic curve of toric type I enhanced with a choice of its complex orientation then

(11) \[ k(\mathbb{R}C) = \text{Area } \Sigma(\mathbb{R}C). \]

For each $(a, b) \in \mathbb{Z}^2$ and $\epsilon \in \mathbb{RP}^1 \setminus \mathbb{QP}^1$ we have

(12) \[ \lambda_{\epsilon}(a, b) = \text{lk}_{\epsilon}((a, b), \Sigma). \]

Corollary 13. For a curve of toric type I with the index diagram $\Sigma$ we have

(13) \[ \lambda_{\epsilon}(Q) = \text{lk}_{\epsilon}(Q, \Sigma) \]

for each quadrant $Q \subset (\mathbb{R}^x)^2$.

The equality (13) may be viewed as the toric complex orientation formula for toric type I curves.

Corollary 14. The total number of closed components of a curve $\mathbb{R}\tilde{C}^o$ of toric type I and its solitary real singularities is not less than the number of lattice points $(a, b) \in \mathbb{Z}^2 \setminus \Sigma$ with $\text{lk}((a, b), \Sigma) \neq 0$.

Proof. If $\text{lk}((a, b), \Sigma) \neq 0$ then by (12) $\lambda(a, b) \neq 0$ and thus there exists a closed component or a solitary real singularity of $\mathbb{R}C^o$ of real index $(a, b)$. \hfill \Box

Example 15. All real rational curves which intersect $\partial \mathbb{R}\Delta$ in $\#(\partial \Delta \cap \mathbb{Z}^2)$ points (counted with multiplicity) have toric type I as $\mathbb{C}\tilde{C}^o \setminus \mathbb{R}\tilde{C}^o$ is the disjoint union of two open disks. Therefore we may compute the quantum index of such curves with the help of Theorem 3.

Figure 4. Squares of the real conics from Figure 2 and their diagrams $\Sigma$ for both possible orientations.

Figure 4 depicts the images of the real conics from Figure 2 under $\text{Sq}^\Delta$ (reparameterized with the help of the moment map). Each of these conics may be oriented in two different ways producing two different
diagrams. For one of these conics the diagrams for the two opposite orientations coincide. For the other three conics they are different.

Note that in the case of $\mathbb{C}\Delta = \mathbb{CP}^2$ the orientation can be uniquely recovered from the diagram as the edges correspond to the normals to $\Delta$. E.g. the vertical edges are always directed downwards.

**Remark 16.** The diagram $\Sigma$ may be viewed as a non-commutative version of the polygon $\Delta$. Here the set of normal vectors is given a cyclic order.

Note that $-\text{Area } \Delta \leq \text{Area } \Sigma \leq \text{Area } \Delta$ for any (possibly multicomponent) closed broken curve $\Sigma$ whose oriented edges are normal vectors to $\Delta$ so that each side $E \subset \Delta$ contribute to $\#(E \cap \mathbb{Z}^2) - 1$ normal vectors (counted with multiplicity). Furthermore, we have $\text{Area } \Sigma = \pm \text{Area } \Delta$ if and only if $\Sigma \subset \mathbb{R}^2$ is a single-component broken curve coinciding with the polygon $\Delta$ itself rotated by 90 degrees (as we can represent the primitive normal vector to a vector $(a, b) \in \mathbb{Z}^2$ by $(-b, a)$ identifying the vector space $\mathbb{R}^2$ with its dual).

Recall the notion of cyclically maximal position of $\mathbb{R}\tilde{C} \subset \mathbb{R}\Delta$ in $\mathbb{R}\Delta$ from Definition 2 of [16]. It can be rephrased that $\mathbb{R}\tilde{C}$ has a connected component $K$ intersecting $\partial \mathbb{R}\Delta$ in $m = \#(\partial \Delta \cap \mathbb{Z}^2)$ points counted with multiplicity, and the cyclic order of the intersection points on $K$ agrees with the cyclic order of the corresponding normal vectors to $\partial \Delta$. This condition is equivalent to the equality [14]. It was proved in [16] that for each $\Delta$ the topological type of the triad $(\mathbb{R}\Delta; \mathbb{R}\tilde{C}_\Delta, \partial \mathbb{R}\Delta)$ is unique if a curve $\mathbb{R}\tilde{C}$ with the Newton polygon $\Delta$ has cyclically maximal position and is transversal to $\partial \mathbb{R}\Delta$. In this case $(\mathbb{R}\Delta; \mathbb{R}\tilde{C}_\Delta, \partial \mathbb{R}\Delta)$ is called the simple Harnack $\Delta$-triad.

The number of points in $(\Delta \setminus \partial \Delta) \cap \mathbb{Z}^2$ is equal to the arithmetic genus $g$ of $\mathbb{R}\tilde{C}$. On the other hand, Corollary [14] implies that the total number of closed connected components of $\mathbb{R}C^\circ$ and its isolated real singular points is at least $g$. Thus all closed components of $\mathbb{R}C^\circ$ are smooth ovals and all singular points of $\mathbb{R}C^\circ$ are solitary double points, and the curve $\mathbb{C}C^\circ$ is a nodal $M$-curve. We get the following statement.

**Corollary 17.** If $\mathbb{R}C^\circ \subset (\mathbb{R}^\times)^2$ is a curve of toric type I with

$$\text{(14) } \text{Area } \Sigma(\mathbb{R}C) = \pm \text{Area } \Delta$$

then it is an $M$-curve whose only singularities are solitary real nodes.

Furthermore, the topological type of $(\mathbb{R}\Delta; \mathbb{R}\tilde{C}, \partial \mathbb{R}\Delta)$ is obtained from the simple Harnack $\Delta$-triad $(\mathbb{R}\Delta; \mathbb{R}\tilde{C}_\Delta, \partial \mathbb{R}\Delta)$ by contracting some of the ovals of $\mathbb{R}C^\circ_\Delta$ to solitary double points and replacing some $n$-tuples of consecutive transversal intersection points of $\mathbb{R}\tilde{C}_\Delta$ and $\partial \mathbb{R}\Delta$ (sitting on the same toric divisor) with points of $n$th order of tangency.
Proof. The curve $\text{Sq}^\Delta(\mathbb{C}C^\circ)$ also has toric type I. Its diagram is obtained from $\Sigma(\mathbb{R}C)$ by scaling both coordinates by 2, so the quantum index of $\text{Sq}^\Delta(\mathbb{C}C^\circ)$ is equal to $\pm \text{Area}(2\Delta)$. Corollary 14 implies that the only singularities of $\text{Sq}^\Delta(\mathbb{C}C^\circ)$ are solitary real nodes, so that $\text{Sq}^\Delta(\mathbb{R}C)$ does not have self-intersections. Therefore for each toric divisor $\mathbb{R}E$ the order of intersection points on $\mathbb{R}E$ and that on the component $K \subset \mathbb{R}C$ agree.

Let us look at the compact components of $\mathbb{R}C^\circ$. Their number and distribution among the quadrants of $(\mathbb{R}^\times)^2$ is determined by the lattice points of $\Sigma(\mathbb{R}C)$ and thus by $\Delta$. Furthermore, Corollary 14 implies that in each quadrant of $(\mathbb{R}^\times)^2$ all ovals and solitary real nodes of $\mathbb{R}C^\circ$ have the same orientation. The complex orientation formula [24] ensures that these components cannot be nested among themselves and that they are arranged with respect to $K$ so that their complex orientation is coherent with the complex orientation of $K$. □

Remark 18. The proof of Corollary 14 is applicable also for pseudoholomorphic, and even the so-called flexible (see [25]) real curves of toric type I. Thus Corollary 17 may be considered as a further generalization of the topological uniqueness theorem for simple Harnack curves [16] from its version for pseudoholomorphic curves [2] recently found by Erwan Brugallé.

![Figure 5](image-url)  

**Figure 5.** A quartic curve of type I, but not of toric type I.

**Example 19.** The curve sketched on Figure 5 is isotopic to a smooth real quartic curve of type I. Namely, it can be obtained as a perturbation of a union of 4 lines. However, it is not an $M$-curve while its diagram coincides with the diagram of the simple Harnack curve of the same degree (i.e. the triangle with vertices $(0,0), (0,-4), (-4,-4)$ for one of the orientations). By Corollary 17 this curve is not of toric
type I. In other words there is a cycle in $\mathbb{C}C^o \setminus \mathbb{R}C^o$ that is homologically non-trivial in $H_1((\mathbb{C}^x)^2)$. Also we can deduce this from the toric complex orientation formula (13).

4.3. The real index map vs. the amoeba index map. To advance the viewpoint of the index diagram $\Sigma$ as a non-commutative version of the Newton polygon $\Delta$ it is interesting to compare the real index map (6) for toric type I curves and the amoeba index map

\[ \text{ind} : \mathbb{R}^2 \setminus \text{Log}(\mathbb{C}C^o) \to \Delta \cap \mathbb{Z}^2 \]

of Forsberg-Passare-Tsikh [6]. The map (15) is locally constant and thus it indexes the components $K$ of the amoeba complement $\mathbb{R}^2 \setminus \text{Log}(\mathbb{C}C^o)$ by lattice points of $\Delta$.

One obvious distinction between $\text{ind}$ and $\alpha$ is that they take values in dual spaces: the Newton polygon $\Delta$ belongs to the dual vector space to $\mathbb{R}^2 = \text{Log}(\mathbb{C}^x)^2$. But thanks to the symplectic form $\omega((a,b),(c,d)) = ad - bc$, $a,b,c,d \in \mathbb{R}$ we have a preferred isomorphism between these spaces. Denote with $(a,b)^* = (b,-a)$ the corresponding identification.

As usual, we fix a complex orientation on $\mathbb{R}\tilde{C}$ and consider the corresponding component $S^o \subset \mathbb{C}\tilde{C}^o \setminus \mathbb{R}\tilde{C}^o$. Let

$p, p' \in \mathbb{R}^2 \setminus \text{Log}(\mathbb{C}C^o)$

and $\gamma \subset \mathbb{R}^2$ be a smooth path between $p$ and $p'$. We assume $\gamma$ to be in general position with respect to $\mathbb{R}C^o$. The intersection number $\#(\gamma, \text{Log}R) \in \mathbb{Z}$ is well-defined as $\text{Log}(\mathbb{R}C) \subset \mathbb{R}^2$ is proper.

**Proposition 20.** We have $\#(\gamma, \text{Log} \mathbb{R}C) = 0$.

**Proof.** The local degree of the map $\text{Log}|_{S^o} : S^o \to \mathbb{R}^2$ changes along $\gamma$ according to the intersection with $\mathbb{R}C^o$. Since the local degree at the endpoints of $\gamma$ vanishes we have $\#(\gamma, \text{Log} \mathbb{R}C) = 0$. \qed

Using the real index map (6) we may refine the intersection number $\#(\gamma, \text{Log} \mathbb{R}C) = 0$ to

\[ \#_\alpha(\gamma, \text{Log} \mathbb{R}C^o) = \sum_{q \in \text{Log}^{-1}(\gamma) \cap \mathbb{R}C^o} \#_q(\gamma, \text{Log} \mathbb{R}C^o) \alpha(q) \in \mathbb{Z}^2 \]

Here $\#_q(\gamma, \text{Log} \mathbb{R}C^o) = \pm 1$ is the local intersection number between $\gamma$ and $\text{Log} \mathbb{R}C^o$ and $\alpha(q) \in \mathbb{Z}^2$ is the real index of the component of $\mathbb{R}\tilde{C}^o$ containing the point $q$.

**Theorem 4.** Let $\mathbb{R}C^o \subset (\mathbb{C}^x)^2$ be an algebraic curve of toric type I. For any two points $p, p' \in \mathbb{R}^2 \setminus S^o \cup \mathbb{R}C^o$ and a generic smooth path $\gamma \subset \mathbb{R}^2$ connecting $p$ and $p'$ we have

\[ (\text{ind}(p') - \text{ind}(p))^* = \#_\alpha(\gamma, \text{Log} \mathbb{R}C^o). \]
Proof. Consider the 1-dimensional submanifold
\[ M = (\log |\mathbb{C}^\circ|)^{-1}(\gamma) \subset \mathbb{C}^\circ. \]
Its orientation is induced by that of \( \gamma \) through the pull-back map with the help of the orientations of the ambient spaces: the standard orientation \( \mathbb{R}^2 \supset \gamma \) and the complex orientation of \( \mathbb{C}^\circ \supset M \). Since \( \gamma \) is chosen generically, the 1-submanifold \( M \) is smooth.

Any component of \( M \) disjoint from \( \mathbb{R} \mathbb{C}^\circ \) is null-homologous in \( \mathbb{C}^\circ \times (\mathbb{C}^\circ)^2 \) as \( \mathbb{R} \mathbb{C}^\circ \) is a toric type I curve. A component \( L \subset M \) intersecting \( \mathbb{R} \mathbb{C}^\circ \) consists of two arcs interchanged by conj. Let \( q \in \mathbb{R} \mathbb{C}^\circ \) be the source and \( q' \in \mathbb{R} \mathbb{C}^\circ \) be the target of the arc \( \delta = L \cap S^\circ \) with the orientation induced from \( M \). We have
\[ \text{Im} \tilde{l}(q') - \text{Im} \tilde{l}(q) = \alpha(q') - \alpha(q) \]
by (6) and therefore \( [L] = \alpha(q') - \alpha(q) \in \mathbb{Z}^2 = H_1((\mathbb{C}^\times)^2) \) so that \( [M] \in H_1((\mathbb{C}^\times)^2) \) is given by the right-hand side of (17).

By [16] we may interpret \( \text{ind}(p) \) as the linear functional on \( H_1(\log^{-1}(p)) = \mathbb{Z}^2 \) that associates to each oriented loop \( N \subset \log^{-1}(p) \) the linking number of \( N \) and the closure of the surface \( \mathbb{C}^\circ \) in \( \mathbb{C}^2 \). Suppose that \( N' \subset \log^{-1}(p') \) is a loop homologous to \( N \) in \( \mathbb{C}^\circ \times (\mathbb{C}^\circ)^2 \) so that \( N' - N = \partial P \) for a surface \( P \subset (\mathbb{C}^\times)^2 \). Then the difference of the linking numbers of \( N' \) and \( N \) coincides with the intersection number of \( P \) and \( \mathbb{C}^\circ \). Choosing the membrane \( P \) to be contained in \( \log^{-1}(\gamma) \) we identify the difference with the intersection number of \( [N] \) and \( [M] \) in
\[ \mathbb{Z}^2 = H_1(\log^{-1}(p)) = H_1((\mathbb{C}^\times)^2) = H_1(S^1 \times S^1). \]

\[ \square \]

5. Refined real enumerative geometry

5.1. Invariance of real refined enumeration. Let \( \Delta \subset \mathbb{R}^2 \) be a lattice polygon with non-empty interior. Let \( E_j \subset \partial \Delta, j = 1, \ldots, n \), be its sides of integer length \( m_j = \#(E_j \cap \mathbb{Z}^2) - 1 \) enumerated counterclockwise. We denote with \( v_j \in \Delta \) the vertices of \( \Delta \) enumerated so that \( E_j \) connects \( v_{j-1} \) and \( v_j \) (using the convention \( v_0 = v_n \)). Let \( \mathbb{C}E_j \) be the corresponding toric divisors. Let \( \mathcal{P} = \{ p_l \}_{l=1}^m \) be a configuration of \( m = \#(\partial \Delta \cap \mathbb{Z}^2) \) points on \( \partial \mathbb{C} \Delta \) such that we have exactly \( m_j \) points on the toric divisor \( \mathbb{C}E_j \) (in particular, \( \mathcal{P} \cap \bigcup_{j=1}^n \{ v_j \} = \emptyset \)).

The divisor \( \mathbb{C}E_j \ni v_{j-1}, v_j \) is isomorphic to \( \mathbb{C} \mathbb{P}^1 \). The outward normal vector \( \vec{n}(E_j) = (a_j, b_j) \) defines the multiplicative-linear map
\[ \pi_j : (\mathbb{C}^\times)^2 \to \mathbb{C}^\times = CE_j \setminus \{v_{j-1}, v_j\} \] by \( \pi_j(z, w) = a_j z + b_j w \) (note that this map extends to a continuous map on \((\mathbb{C}^\times)^2 \cup (CE_j \setminus \{v_{j-1}, v_j\})\)). Let \( u_j = \pi_j(1, 1) \). Consider the isomorphisms \( CE_j \to \mathbb{P}^1 \) mapping \( v_{j-1}, v_j, u_j \) to \( 0, \infty, 1 \). Taken together they define the map \( \rho : \bigcup_{j=1}^n CE_j \setminus \{0, \infty_j\} \to \mathbb{C}^\times \).

**Condition 21** (Menelaus condition on \( \mathcal{P} \)).

\[ \prod_{l=1}^m \rho(p_l) = (-1)^m. \]

The following proposition is known as the *Menelaus theorem* in the case of \( \mathbb{R}\Delta = \mathbb{R}^2 \) and \( m = 3 \) (lines), and generalized by Carnot [3] to higher degree curves. It is also known as the *Weil reciprocity law*, see e.g. [8].

**Proposition 22.** There exists a curve \( \overline{C} \subset C \Delta \) transverse to \( \partial C \Delta \) and such that \( \overline{C} \cap \partial C \Delta = \mathcal{P} \) if and only if (18) holds.

**Proof.** The torus part \( \mathbb{C}C^\circ = \overline{C} \cap (\mathbb{C}^\times)^2 \) is defined by a polynomial \( f(z, w) = \sum a_{(\iota_1, \iota_2)} z^{\iota_1} w^{\iota_2} \). Note that the condition \( \overline{C} \cap CE_j = \mathcal{P} \cap CE_j \) implies that the Newton polygon of \( f \) coincides with \( \Delta \) (up translation in \( \mathbb{R}^2 \)). Furthermore, the intersection \( \overline{C} \cap CE_j \) is determined by \( a_{(\iota_1, \iota_2)} \) with \((\iota_1, \iota_2) \in E_j^\circ\). Suppose that (18) holds. The set \( \pi_j^{-1}(\mathcal{P} \cap CE_j) \) is the zero locus of a polynomial \( f_j \) with Newton polygon \( \Delta f_j \) is a translate of the side \( E_j \). Multiplying \( f_j \) by an appropriate monomial we ensure that \( \Delta f_j = E_j \), \( f_j = \sum a_{(\iota_1, \iota_2)}^{(j)} z^{\iota_1} w^{\iota_2} \). Furthermore, \( f_j : (\mathbb{C}^\times)^2 \to \mathbb{C} \) is the only polynomial with the Newton polygon \( E_j \), and such that the closure of its zero set in \( \mathbb{C} \Delta \) contains \( \mathcal{P} \cap CE_j \). We have

\[ a_{v_{j-1}}^{(j)} / a_{v_j}^{(j)} = (-1)^{m_j} \prod_{p_l \in CE_j} \rho(p_l) \]

by the Vieta theorem. Therefore we can choose \( f_j \) in such a way that \( a_{v_j}^{(j)} = a_{v_j}^{(j+1)} \) (using the convention \( a_{v_n}^{(n+1)} = a_{v_n}^{(0)} \)) if and only if (18) holds.

In other words, (18) means that the linear system defined by the divisor \( \mathcal{P} \) on the (singular) elliptic curve \( \partial C \Delta \) is \( O(\Delta) \), i.e. any curve with the Newton polygon \( \Delta \) passing through the point \( \{p_j\}_{j=1}^{m-1} \) also passes through \( p_m \). By Proposition 22 for any subset of \( m-1 \) points on \( \mathcal{P} \) there exists a unique remaining point with this condition. We
assume that $\mathcal{P}$ is a generic Menelaus configuration of $m$ points on $\partial \mathbb{R} \Delta$, i.e. the first $m - 1$ points of $\mathcal{P}$ are chosen generically on $\partial \mathbb{R} \Delta$.

The configuration $(\text{Sq}^\Delta)^{-1}(\mathcal{P})$ consists of real or purely imaginary points. Thus any oriented real rational curve $\mathbb{R} \tilde{C} \subset \mathbb{R} \Delta$ with the Newton polygon $\Delta$ such that $\text{Sq}^\Delta(\mathbb{C} \tilde{C})$ passes through $\mathcal{P}$ has the quantum index $k(\mathbb{R} \tilde{C}) \in \frac{1}{2} \mathbb{Z}$. (Note that rationality over the real numbers implies, in particular, that $\mathbb{R} \tilde{C}$ is non-empty and that $\mathbb{R} \tilde{C}$ is of type I.)

Define
\begin{equation}
\sigma(\mathbb{R} C) = (-1)^{\frac{m - \text{Rot}_{\text{Log}}(\mathbb{R} C)}{2}}.
\end{equation}
Since the parities of $\text{Rot}_{\text{Log}}(\mathbb{R} C)$ and $m$ coincides we have $\sigma(\mathbb{R} C) = \pm 1$.

**Remark 23.** Note that if $\mathbb{R} C^o$ is nodal then its toric solitary singularities number $E(\mathbb{R} C)$ has the same parity as the number of solitary nodes of $\mathbb{R} C^o$. Thus the Welschinger sign $w(\mathbb{R} C)$ (see [26]) coincides with $(-1)^{E(\mathbb{R} C)}$. Since the curve $\mathbb{R} \tilde{C}$ intersects the union $\partial \mathbb{R} \Delta$ of toric divisors in $m$ distinct points and $\frac{m}{2} \equiv \text{Area}(\Delta) \pmod{1}$ by Pick’s formula, we have
\begin{equation}
\sigma(\mathbb{R} C) = (-1)^{\text{Area}(\Delta) - k(\mathbb{R} C)} w(\mathbb{R} C)
\end{equation}
by Theorem 2.

We define
\begin{equation}
R_{\Delta, k}(\mathcal{P}) = \frac{1}{4} \sum_{\mathbb{R} \tilde{C}} \sigma(\mathbb{R} C),
\end{equation}
where the sum is taken over all oriented real rational curves $\mathbb{R} \tilde{C}$ (in particular, irreducible over $\mathbb{C}$) with the Newton polygon $\Delta$ and $k(\mathbb{R} \tilde{C}) = k$ such that $\text{Sq}^\Delta(\mathbb{C} \tilde{C}) \supset \mathcal{P}$. We have the coefficient $\frac{1}{4}$ in the right-hand side of (21) as the group of the deck transformations of $\text{Sq}^\Delta : \mathbb{C} \Delta \to \mathbb{C} \Delta$ is $\mathbb{Z}_2^2$, so each curve $\mathbb{R} \tilde{C}$ comes in four copies with the same image $\text{Sq}^\Delta(\mathbb{C} \tilde{C})$. (Alternatively, we can take a sum over different oriented images $\text{Sq}^\Delta(\mathbb{R} \tilde{C})$ without the coefficient $\frac{1}{4}$.)

Recall that we call a point in $\partial \mathbb{R} \Delta$ positive if it is adjacent to the quadrant $(\mathbb{R}_{>0})^2$ and negative otherwise. Note that $(\text{Sq}^\Delta)^{-1}(p)$ consists of real points if $p$ is positive and of purely imaginary points if it is negative. Let $\lambda_j$ be the number of negative points in $\mathcal{P} \cap \mathbb{R} E_j$. Denote $\lambda = (\lambda_j)_{j=1}^n$.

**Theorem 5.** The number $R_{\Delta, k}(\lambda) = R_{\Delta, k}(\mathcal{P})$ is invariant of the choice of $\mathcal{P}$ and depends only on $\Delta$, $k$ and $\lambda$.

If all points of $\mathcal{P}$ are negative (i.e. $m_j = \lambda_j$) the number $R_{\Delta, k}(\lambda)$ is the number of oriented real rational curves $\mathbb{R} \tilde{C}$ of quantum index $k$. 

contained in the positive quadrant $\mathbb{R}_{>0}^2 \subset (\mathbb{R}^\times)^2$ and passing through all points of the purely imaginary configuration $(\text{Sq}^\Delta)^{-1}(\mathcal{P})$. For a positive point $p \in \mathcal{P}$ a curve $\mathbb{R}C$ should pass through one of the two real points in $(\text{Sq}^\Delta)^{-1}(p)$.

Define
\begin{equation}
\tilde{R}_{\Delta,k}(\mathcal{P}) = \sum_{\mathbb{R}C} \sigma(\mathbb{R}C)
\end{equation}
where the sum is taken over all oriented real rational curves $\mathbb{R}C$ of quantum index $k$ with the Newton polygon $\Delta$ and $k(\mathbb{R}C) = k$ passing through $\mathcal{P}$. Let
\begin{equation}
\Delta_d = \text{ConvexHull}\{(0, 0), (d, 0), (0, d)\}.
\end{equation}
We have $\mathbb{C}\Delta_d = \mathbb{CP}^2$.

**Example 24.** The curves $\mathbb{R}C$ with the Newton polygon $\Delta_2$ are projective conics. In this case $n = 3$, $m_1 = m_2 = m_3 = 2$, $m = 6$, and for any generic Menelaus configuration $\mathcal{P} \subset \partial \mathbb{R}\mathbb{P}^2$ we have a unique conic through $\mathcal{P}$. This gives us two oriented curves through $\mathcal{P}$ of opposite quantum index.

We may assume (applying the reflections in $x$ and $y$ axes if needed) that $\mathcal{P}$ contains a positive point in the $x$-axis and a positive point in the $y$-axis. As the positivity of the last point of $\mathcal{P}$ will be determined by the Menelaus condition we have 3 possibility for the non-decreasing sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. The possible values of $k(\mathbb{R}C)$ are listed in the following table, cf. Figure 2. In particular in this case $\tilde{R}_{\Delta_2,k}(\mathcal{P})$ is different for different configurations with the same $\lambda$ for two last rows of Table 1. Thus the numbers $\tilde{R}_{\Delta,k}(\mathcal{P})$ may vary when we deform $\mathcal{P}$.

Define
\begin{equation}
\tilde{R}_{\Delta,\text{even}}(\mathcal{P}) = \sum_{k \in \frac{m}{2} + 2\mathbb{Z}} \tilde{R}_{\Delta,k}(\mathcal{P}); \quad \tilde{R}_{\Delta,\text{odd}}(\mathcal{P}) = \sum_{k \in \frac{m}{2} + 1 + 2\mathbb{Z}} \tilde{R}_{\Delta,k}(\mathcal{P}).
\end{equation}

**Theorem 6.** The numbers $\tilde{R}_{\Delta,\text{even}}(\mathcal{P})$ and $\tilde{R}_{\Delta,\text{odd}}(\mathcal{P})$ do not depend of $\mathcal{P}$ as long as $d$ is even and all the points of $\mathcal{P}$ are positive (i.e. $\lambda = (0, 0, 0)$).
5.2. **Refined real and refined tropical enumerative geometry.**

We return to the study of the invariant $R_{\Delta,k}$ from Theorem 5.

**Definition 25.** The sum

$$R_\Delta(\lambda) = \sum_{k=-\text{Area}(\Delta)}^{\text{Area}(\Delta)} R_{\Delta,k}(\lambda) q^k$$

is called the **real refined enumerative invariant** of $(\mathbb{R}^2)^2$ in degree $\Delta$.

If $\lambda = 0$ (i.e. all $\lambda_j = 0$) then all points of $(\text{Sq}^\Delta)^{-1}(\mathcal{P})$ are real. In such case we use notations $R_{\Delta,k} = R_{\Delta,k}(0)$ and $R_\Delta = R_\Delta(0)$.

Recall that the Block-Göttsche invariant $[1]$ is a symmetric (with respect to the substitution $q \mapsto q^{-1}$) Laurent polynomial with positive integer coefficients. This polynomial is responsible for the enumeration of the tropical curves with the Newton polygon $\Delta$ of genus $g$, passing through a generic collection of points in $\mathbb{R}^2$, see [10]. The expression $N_{\Delta}^{\partial,\text{trop}}$ defined by (47) may be viewed as the counterpart of $N_{\text{trop}}^{\Delta,0}$ defined in [1] for the case when the tropical curves pass through a collection of $m$ points on the boundary of the toric tropical surface $\mathbb{T}\Delta$ which are generic among those satisfying to the tropical Menelaus condition [42].

**Theorem 7.**

$$R_\Delta = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{m-2} N_{\Delta}^{\partial,\text{trop}}.$$

**Corollary 26.** The number $N_{\Delta}^{\partial,\text{C}}$ of complex rational curves in $\mathbb{C}\Delta$ with the Newton polygon $\Delta$ passing through $\mathcal{P}$ is determined by $R_\Delta$.

**Proof.** By [17] the number $N_{\Delta}^{\partial,\text{C}}$ coincides with the value of $N_{\Delta}^{\partial,\text{trop}}$ at $q = 1$. \hfill \Box

Let us reiterate that $R_\Delta$ accounts only for curves in $(\mathbb{C}^\times)^2$ defined over $\mathbb{R}$.

5.3. **Holomorphic disk interpretation.** Recall that an orientation of a real rational curve $\mathbb{R}C$ defines a connected component $S \subset \mathbb{C}\tilde{C} \setminus \mathbb{R}\tilde{C}$. Let $D$ be the topological closure of the image of $S$ in $\mathbb{C}\Delta$. The disk $D \subset \mathbb{C}\Delta$ is a holomorphic disk whose boundary $\partial D = \mathbb{R}C$ is contained in the Lagrangian subvariety $\mathbb{R}\Delta \subset \mathbb{C}\Delta$.

Let $L$ be the topological closure of the quadrant $\mathbb{R}^2_{>0}$ in $\mathbb{C}\Delta$. Note that $L$ is a Lagrangian subvariety of $\mathbb{C}\Delta$ with boundary. The image $\text{Sq}^\Delta(D)$ is a holomorphic disk whose boundary is contained in $L$.

Thus the expression (25) may also be interpreted as a refined enumeration of holomorphic disks with boundary in $L$, passing through $\mathcal{P}$, and tangent to $\partial \mathbb{R}\Delta$ at the points of $\mathcal{P}$. 

These disks are images under $\text{Sq}^\Delta$ of disks $D$ with boundary in $\mathbb{R}\Delta$ and
\begin{equation}
\text{Sq}^\Delta(D) \cap \partial C\Delta = \mathcal{P} \subset \partial \mathbb{R}\Delta.
\end{equation}
Let $\mathcal{C}\Delta$ be the result of blowup of the toric variety $C\Delta$ at $\mathcal{P}$. Let $\hat{L} = (\mathbb{R}^\times)^2 \setminus \mathbb{R}\Delta$ where $(\mathbb{R}^\times)^2$ is the topological closure of $(\mathbb{R}^\times)^2$ in $\mathcal{C}\Delta$ and $\mathbb{R}\Delta$ is the proper transform of $\partial \mathbb{R}\Delta$ in $\mathcal{C}\Delta$. Then a holomorphic disk $D$ lifts to a holomorphic disk $\hat{D}$ with the boundary in the non-compact Lagrangian subvariety $\hat{L} \subset \mathcal{C}\Delta$ without boundary. Furthermore, the Maslov index of $\hat{D}$ is 0.

6. Proofs

6.1. **Proof of Proposition 3 and Theorems 1, 2 and 3.** Consider the map $\text{Arg} : (\mathbb{C}^\times)^2 \rightarrow (\mathbb{R}/2\pi \mathbb{Z})^2$ defined by
\begin{equation}
\text{Arg}(z, w) = (\text{arg}(z), \text{arg}(w))
\end{equation}
and the map $2\text{Arg} : (\mathbb{C}^\times)^2 \rightarrow (\mathbb{R}/\pi \mathbb{Z})^2$ obtained by multiplication of $\text{Arg}$ by 2, in other words a composition of $\text{Arg}$ with with the quotient map $(\mathbb{R}/2\pi \mathbb{Z})^2 \rightarrow (\mathbb{R}/\pi \mathbb{Z})^2$. The involution of complex conjugation in $(\mathbb{C}^\times)^2$ descends to $(\mathbb{R}/\pi \mathbb{Z})^2$ as the involution $\sigma : (a, b) \mapsto (-a, -b)$. Denote with
\begin{equation}
P = (\mathbb{R}/\pi \mathbb{Z})^2/\sigma
\end{equation}
the quotient space. The orbifold $P$ is the so-called pillowcase. The projections of the four points $(0, 0), (\frac{\pi}{2}, 0), (0, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{\pi}{2})$ form the $\mathbb{Z}_2$-orbifold locus of $P$ (the corners of the pillowcase). All other points are smooth. We denote with $0 \in P$ the origin of $P$, i.e. the projection of $(0, 0)$. Note that $(2\text{Arg})^{-1}(0, 0) = (\mathbb{R}^\times)^2$. The product volume form on $(\mathbb{R}/\pi \mathbb{Z})^2$ defines the volume form $d\text{Vol}_P$ on the smooth points of the orbifold $P$ since the involution $\sigma$ is orientation-preserving.

Let $\mathbb{R}C$ be a real curve of type I with real or purely imaginary coordinate intersection. Consider the surface $S^\circ = S \setminus \nu^{-1}(\partial C\Delta)$, where $S$ is the component of $\mathcal{C}\Delta \setminus \mathbb{R}\Delta$ corresponding to the orientation of $\mathbb{R}C$ and $\nu$ is the normalization map $[1]$. Denote with
\begin{equation}
\beta : S^\circ \rightarrow P
\end{equation}
the composition of the map $2\text{Arg}|_{S^\circ} : S^\circ \rightarrow (\mathbb{R}/\pi \mathbb{Z})^2$ and the projection $(\mathbb{R}/\pi \mathbb{Z})^2 \rightarrow P$.

Let $p \in P$ be a regular point of $\beta$. A point $q \in \beta^{-1}(p)$ is called positive (resp. negative) if locally near $q$ the map $\beta$ is an orientation-preserving (resp. orientation-reversing) open embedding. The difference between the number of positive and negative points in $\beta^{-1}(p)$ is
called the degree at $p$. A priori, since $\beta$ is a non-proper map, the degree at different points could be different.

**Lemma 27.** We have

$$\text{Area}_{\text{Log}}(\mathbb{R}C) = \int_{S^\circ} \beta^* \text{Vol}_P.$$  

Furthermore, the degree of $\beta$ at a generic point of $P$ is $2k(\mathbb{R}C)$.

**Proof.** Consider the form

$$\frac{dx}{x} \wedge \frac{dy}{y} = (d\log |x| + id\arg(x)) \wedge (d\log |y| + id\arg(y)) =$$

$$d\log |x| \wedge d\log |y| - d\arg(x) \wedge d\arg(y) +$$

$$i(d\log |x| \wedge d\arg(y) + d\arg(x) \wedge d\log |y|)$$

on $(\mathbb{C}^\times)^2$. As it is a holomorphic 2-form, it must restrict to the zero form on any holomorphic curve in $(\mathbb{C}^\times)^2$. In particular, the real part of this form must vanish everywhere on $S^\circ$, so that $d\log |x| \wedge d\log |y| = d\arg(x) \wedge d\arg(y)$ on $S^\circ$, and thus (30) holds, cf. [15].

The smooth map $\beta : S^\circ \to P$ extends to a continuous map

$$\tilde{\beta} : \bar{S} \to P$$

for a surface with boundary $\bar{S} \supset S^\circ$ such that $S^\circ = \bar{S} \setminus \partial \bar{S}$. Each $p \in \bar{C} \cap \mathbb{C}E_j$ corresponds to a geodesic in $(\mathbb{R}/\pi\mathbb{Z})^2$ in the direction parallel to $\bar{n}(E_j)$ for a side $E_j \subset \partial \Delta$, cf. [19]. Since $\text{Sq}^2(p) \in \mathbb{R}\Delta$ the corresponding geodesic passes through two of the points $(0, 0), (\frac{\pi}{2}, 0), (0, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{\pi}{2})$. The image of this circle in $P$ is a geodesic segment connecting the corresponding $\mathbb{Z}_2$-orbifold points of $P$.

Thus $\tilde{\beta}(S)$ is a 2-cycle in $P$ and it covers a generic point $l$ times (counted algebraically), where $l$ is a number independent on the choice of a generic point. But then $\int_{S^\circ} \beta^* \text{Vol}_P = l\text{Area}(P) = l\pi^2/2$. Since we have already proved (30) we get $l = 2\frac{\text{Area}(\mathbb{R}C)}{\pi^2} = 2k(\mathbb{R}C)$ (the last equality is the definition of $k(\mathbb{R}C)$).

Note that this lemma implies Proposition 3.

**Proof of Theorem 1.** We have $k(\mathbb{R}C) \in \frac{1}{2}\mathbb{Z}$ since $2k(\mathbb{R}C)$ is the degree of $\beta$ at a generic point of $P$ by Lemma 27. Let $\tilde{a} \in (\mathbb{R}/\pi\mathbb{Z})^2$ be a generic point and $a \in P$ be the point corresponding to $\tilde{a}$. The inverse image $\beta^{-1}(a)$ consists of points of $S^\circ$ mapped to $\tilde{a}$ or $\sigma(\tilde{a})$. If $2\text{Arg}(p) = -\tilde{a}$ for $p \in S^\circ$ then $2\text{Arg}(\text{conj}(p)) = \tilde{a}$, where $\text{conj}(p) \in \text{conj}(S^\circ)$. Thus we have a 1-1 correspondence between sets $\beta^{-1}(a)$ and $R = (2\text{Arg})^{-1}(\tilde{a}) \cap \mathbb{C}C^\circ$. 


Consider the continuous involution $\text{conj}_{\tilde{a}} : \mathbb{C}\Delta \to \mathbb{C}\Delta$ extending the involution of $(\mathbb{C}^\times)^2$ defined by $z \mapsto e^{i\tilde{a}} \text{conj}(e^{i\tilde{a}}(z))$. Note that the fixed point locus of this involution in $(\mathbb{C}^\times)^2$ coincides with $(2 \text{Arg})^{-1}(\tilde{a})$, cf. \[15\]. Note that

\begin{equation}
R \subset \mathbb{C}C^\circ \cap \text{conj}_{\tilde{a}}(\mathbb{C}C^\circ)
\end{equation}

while $R \setminus (\mathbb{C}C^\circ \cap \text{conj}_{\tilde{a}}(\mathbb{C}C^\circ))$ consists of pairs of points interchanged by the involution $\text{conj}_{\tilde{a}}$. For generic $\tilde{a}$ the curve $\text{conj}_{\tilde{a}}(\mathbb{C}C^\circ)$ is transverse to $\mathbb{C}C^\circ$, while $\text{conj}_{\tilde{a}}(\mathbb{C}C^\circ) \cap \mathbb{C}C^\circ \cap \partial \mathbb{C}\Delta = \emptyset$.

Thus the number of points in $R$ is not greater than $\#(\mathbb{C}C^\circ \cap \text{conj}_{\tilde{a}}(\mathbb{C}C^\circ))$, while we have $\#(\mathbb{C}C^\circ \cap \text{conj}_{\tilde{a}}(\mathbb{C}C^\circ)) = 2 \text{Area}(\Delta)$ by the Kouchnirenko-Bernstein theorem \[14\]. Thus the degree of $\beta$ takes values between $-2 \text{Area}(\Delta)$ and $2 \text{Area}(\Delta)$. Also $\#(R) = 2 \text{Area}(\Delta)$. \hfill $\square$

**Proof of Theorem 2.** Let us compute the degree of the map \[32\] at a generic point $a \in P$ close to the origin $0 \in P$. The set $\beta^{-1}(0) \cap S^\circ$ contributes $2E$ to the degree of $\tilde{\beta}$ as the intersection number gets doubled when we pass from $(\mathbb{R}/\pi\mathbb{Z})^2$ to $P$.

Note that the set $S^R_\beta = \beta^{-1}(0) \cap \tilde{\partial}S$ can be thought of as the topological closure of $\mathbb{R}C^\circ$ in $\tilde{S}$ by our assumption of transversality to $\partial\mathbb{R}\Delta$. Consider a non-vanishing tangent vector field $\gamma$ on $\tilde{\mathbb{C}}C^\circ$ such that it extends to a tangent vector field on $\mathbb{R}\mathbb{C} \setminus \mathbb{C}C^\circ$. Our condition on zeroes of $\gamma$ implies that $\pm i\gamma$ is consistent with a trivialized regular neighborhood $U \approx S^R_\beta \times [0, 1)$ (we take $i\gamma$ on the components of $S^R_\beta$ where $\gamma$ agrees with the complex orientation of $\mathbb{R}C^\circ$ and $-i\gamma$ otherwise). The lift $\tilde{\beta}_\epsilon$ of $\beta|_{S^R_\epsilon \times \{\epsilon\}}$ to $\mathbb{R}^2/\{(x, y) \sim (-x, -y)\}$ is approximated (to the first order by $\epsilon$) by $\epsilon\gamma$ for small $\epsilon > 0$. Thus the linking number of the image of $\tilde{\beta}_\epsilon$ and $(0, 0) \in \mathbb{R}^2/\{(x, y) \sim (-x, -y)\}$ is $\text{Rot}_{\text{Log}}(\mathbb{R}C)$. Thus $S^R_\beta$ contributes $-\text{Rot}_{\text{Log}}(\mathbb{R}C)$ to the degree of $\tilde{\beta}$. We have the appearance of the negative sign since the basis composed of vectors $v_1, v_2, iv_1, iv_2$ is negatively-oriented in $\mathbb{C}^2$ whenever vectors $v_1, v_2$ are linearly independent over $\mathbb{C}$. Thus a positive rotation in $(i\mathbb{R})^2$ (and therefore also in $P$) corresponds to a negative contribution to the degree of $\tilde{\beta}$. \hfill $\square$

**Proof of Theorem 3.** The map \[33\] gives the lift of $\beta|_{S^\circ}$ to the universal covering $\mathbb{C}^2$ of $(\mathbb{C}^\times)^2$ after rescaling each coordinate by $\pi$. Thus the signed area of $\beta(S^\circ)$ coincides with $\pi^2 \text{Area}(\Sigma(\mathbb{R}C))$. Lemma \[27\] now implies that $k(\mathbb{R}C) = \text{Area}(\Sigma(\mathbb{R}C))$. For $(a, b) \in \mathbb{Z}^2$ and $\epsilon \in \mathbb{R}^P \setminus \mathbb{Q}P^1$ we consider a point $p_\epsilon$ obtained by a small translation of $(a, b)$ in the direction of $\epsilon$. A point of $S^\circ$ mapped to $p_\epsilon$ by the lift of $\beta$ must correspond to a point of $\mathbb{R}C^\circ$ of real index $(a, b)$ which is either singular
or has $\epsilon$ as the image of its logarithmic Gauß map. Summing up the contributions of all such points we get \cite{12}.

6.2. Invariance of the numbers $R_{\Delta,k}(P)$.

Proof of Theorem 5. First we compute the dimension of the space of rational curves $\mathbb{R}C^o \subset (\mathbb{R}^\times)^2$ with the Newton polygon $\Delta$. Two coordinate functions on $(\mathbb{C}^\times)^2$ define two meromorphic functions on the Riemann surface $\mathbb{C}\bar{C}$. The set of zeros and poles of these functions are $\partial\mathbb{C}\bar{C} = \bar{C} \cap \partial\mathbb{C}\Delta$. The order of each of these zeroes and poles is determined by the multiplicity of the corresponding intersection points of $\mathbb{C}\bar{C}$ with $\mathbb{C}E_j$ as well as by the slope of $E_j \subset \Delta$. We may consider $\partial\mathbb{C}\bar{C}$ as an $m$-tuple of points in $\mathbb{C}\bar{C}$. As $\mathbb{C}\bar{C}$ is rational, we may freely deform this $m$-tuple while each such deformation extends to the deformation of the coordinate functions. The group $PSL_2(\mathbb{R})$ of symmetries of $\mathbb{R}\bar{C}$ is 3-dimensional, so the space of real deformations of the $m$-tuple $\partial\mathbb{C}\bar{C}$ in $\mathbb{C}\bar{C}$ is $(\#(\partial\mathbb{C}\bar{C}) - 3)$-dimensional. The resulting curve is well-defined up to the multiplicative translation in $(\mathbb{R}^\times)^2$. Altogether, the dimension of the space of rational curves in $(\mathbb{R}^\times)^2$ is $m - 1$, which coincides with the dimension of the space of configurations $\mathcal{P}$.

Note that a deformation of a single point $p$ to $p'$ in the $m$-tuple $\partial\mathbb{C}\bar{C}$ corresponds to adding rational functions with zeroes and poles only at $p$ and $p'$ to the corresponding coordinate functions. A non-immersed point of $\mathbb{C}\bar{C}^o$ corresponds to a common zero of the differentials of the coordinate functions. But the differential of each coordinate function can be perturbed separately by addition of a rational function as above. Thus the set $\mathcal{R}_{\Delta,k}(\mathcal{P})$ of curves $\mathbb{R}C^o \subset (\mathbb{R}^\times)^2$ of the Newton polygon $\Delta$, quantum index $k$ satisfying to $\text{Sq}\Delta(\mathbb{C}\bar{C}) \supset \mathcal{P}$ is finite, while each curve from $\mathcal{R}_{\Delta,k}(\mathcal{P})$ is immersed for generic $\mathcal{P}$.

It is convenient to consider not only conventional irreducible rational curves, but also the so-called stable rational curves (cf. \cite{3}) in $\mathbb{C}\Delta$. These are pairs $(\mathcal{F}, \Gamma)$ consisting of a possibly disconnected Riemann surfaces $\mathcal{F}$ considered together with a holomorphic map to $\mathbb{C}\Delta$, and a tree $\Gamma$ whose vertices are the components of $\mathcal{F}$ so that each edge of $\Gamma$ corresponds to an intersection point of the images of the corresponding components. Conventional rational curves are stable rational curves with the tree $\Gamma$ consisting of a single vertex. Any sequence of (stable) rational curves in $(\mathbb{R}^\times)^2$ with the Newton polygon $\Delta$ admits a subsequence converging to a stable rational curve $\mathbb{R}\bar{D} \subset (\mathbb{R}^\times)^2$ with a Newton polygon $\Delta' \subset \Delta$. Note that if $\mathbb{C}\bar{D} \cap \partial\mathbb{C}\Delta$ is disjoint from the points of intersection of toric divisors $\mathbb{C}E_j$ then $\Delta' = \Delta$. Therefore,
the map

$$\text{ev} : \mathcal{M}_\Delta \to \mathcal{M}_{\partial \Delta}$$

is proper. Here the source $\mathcal{M}_\Delta$ is the space of stable oriented real rational curves with the Newton polygon $\Delta$. The target $\mathcal{M}_{\partial \Delta}$ is the space of conj-invariant Menelaus configurations of $m$ points in $\partial \mathbb{C} \Delta$ with $m_j$ points in $\mathbb{C} \Sigma_j$, $m = \sum m_j = \#(\partial \Delta \cap \mathbb{Z}^2)$. Note that the space $\mathcal{M}_{\partial \Delta}$ is smooth as its complexification can be presented as a product of symmetric powers of $\mathbb{C}^\times$. The map ev sends a curve $\mathbb{R} C^\circ$ to the configuration $\text{Sq}^\Delta (\mathbb{C} \bar{C}) \cap \partial \mathbb{C} \Delta \subset \partial \mathbb{C} \Delta$.

We refer to stable rational curves with disconnected $F$ as reducible rational curves in $\mathbb{C} \Delta$. Note that the dimension of the space of deformation of such curves is equal to the sum of the space of deformation of their irreducible components, i.e. to $m$ minus the number of components. Thus for a generic choice of $\mathcal{P}$ there are no reducible rational curves of Newton polygon $\Delta$ passing through $\mathcal{P}$.

Consider the space $\mathcal{M}_{\partial \Delta, \lambda} \subset \mathcal{M}_{\partial \Delta}$ of real configurations $P \subset \mathbb{R} \partial \Delta$ with $\lambda_j$ points in $\mathbb{R} E_j \setminus ((\mathbb{R} > 0)^2)$. Any curve in $\text{ev}^{-1}(\mathcal{M}_{\partial \Delta, \lambda})$ satisfies the hypothesis of Theorem 1, thus its quantum index is well-defined.

Let $P, P' \in \mathcal{M}_{\partial \Delta, \lambda}$ and $\gamma = \{P_t\}_{t \in [0,1]} \in \mathcal{M}_{\partial \Delta, \lambda}$ be a smooth generic path connecting two such configurations $\mathcal{P} = P_0$ and $\mathcal{P}' = P_1$.

All but finitely many values of $t$ correspond to $P_t$ such that $\mathcal{R}_{\Delta, k}(P_t)$ consists of irreducible curves. Suppose that there are no reducible curves in $\mathcal{R}_{\Delta, k}(P_t)$ for all $t$. In this case an orientation of a curve uniquely determines the orientations of all curves in a connected component of $\text{ev}^{-1}(\gamma)$. Thus, $\text{ev}^{-1}(\gamma)$ splits to a disjoint union of components according to the quantum index $k$. Furthermore, we may deduce that $R_{\Delta, k}(\mathcal{P}) = R_{\Delta, k}(\mathcal{P}')$ from the Welschinger theorem [26]. Namely, in the case when all curves of $\mathcal{R}_{\Delta, k}(P_t)$ are irreducible we deduce from [26] an even stronger statement (used also in the proof of Theorem 6): for each connected component $C$ of the family $\mathcal{R}_{\Delta, k}(P_t)$ of curves from $\mathcal{M}_\Delta$ we have

$$\sum_{R \in \mathcal{R}_{\Delta, k}(P_t) \cap C} \sigma(R C) = \sum_{R \in \mathcal{R}_{\Delta, k}(P_0) \cap C} \sigma(R C).$$

For such a deduction we use the Welschinger theorem only in the case of real curves of bidegree $(a, b)$, $a, b \in \mathbb{Z}_{>0}$, in $\mathbb{R} \mathbb{P}^1 \times \mathbb{R} \mathbb{P}^1$. In this case Theorem 2.1 of [26] asserts that if $\mathcal{P}_{ab}$ and $\mathcal{P}'_{ab}$ are two generic configurations of $2a + 2b - 1$ points in $\mathbb{R} \mathbb{P}^1 \times \mathbb{R} \mathbb{P}^1$, and $\mathcal{R}_{ab}(\mathcal{P}_{ab})$ (resp. $\mathcal{R}_{ab}(\mathcal{P}'_{ab})$) is the set of real rational curves of bidegree $(a, b)$ passing
through $P_{ab}$ (resp. $P'_{ab}$) then $R_{ab}(P_{ab})$ (resp. $R_{ab}(P'_{ab})$) is finite, consists of immersed irreducible nodal curves, and

$$
(36) \sum_{R \tilde{C}_{ab} \in R_{ab}(P_{ab})} w(R \tilde{C}_{ab}) = \sum_{R \tilde{C}_{ab} \in R_{ab}(P'_{ab})} w(R \tilde{C}_{ab}).
$$

Here $w(R \tilde{C}_{ab}) = \pm 1$ is positive if the number of isolated real points of $R \tilde{C}_{ab}$ is even and negative otherwise. Furthermore, the proof of Theorem 2.1 of [20] implies that for any generic path $\gamma_{ab}$ connecting $P_{ab}$ and $P'_{ab}$ in the space of configurations of $2a + 2b - 1$ points in $\mathbb{RP}^1 \times \mathbb{RP}^1$ the subspace of rational curves the space $R_{ab}(\gamma_{ab})$ consisting of pairs $(R \tilde{C}_{ab}, Q_{ab})$ such that $Q_{ab} \in \gamma_{ab}$ and $R \tilde{C}_{ab}$ is a stable oriented real rational curves of bidegree $(a, b)$ in $\mathbb{RP}^1 \times \mathbb{RP}^1$ passing through $Q_{ab}$, is an oriented 1-cobordism from $R_{ab}(P_{ab})$ to $R_{ab}(P'_{ab})$. Here the orientation of $R_{ab}(\gamma_{ab})$ is induced from $\gamma$ by the forgetting map $ev_{\gamma_{ab}} : R_{ab}(\gamma_{ab}) \to \gamma_{ab}$ near an immersed rational curve $R \tilde{C}_{ab}$ with $w(R \tilde{C}_{ab}) = 1$, and is opposite to the orientation induced in this way near an immersed rational curve $R \tilde{C}_{ab}$ with $w(R \tilde{C}_{ab}) = -1$. In other words, the law of conservation of the signed number of real curves of bidegree $(a, b)$ in $\mathbb{RP}^1 \times \mathbb{RP}^1$ passing through $P_{ab}$ is local.

Consider the curve $R \tilde{C} \subset \mathbb{RP}^1 \times \mathbb{RP}^1$ obtained as the topological closure of $R C^o$ for $R C \in R_{ab}(P_{t}) \cap C$. The Newton polygon $\Delta_{ab}$ of $R \tilde{C}$ is a rectangle circumscribing $\Delta$. Up to translation we have $\Delta_{ab} = [0, a] \times [0, b]$. Branches of $R \tilde{C}$ near $\partial(\mathbb{RP}^1 \times \mathbb{RP}^1)$ are described by the sides of $\Delta$. Points of $R \tilde{C} \cap \{(0, \infty) \times \mathbb{R}^\times \}$ (resp. of $R \tilde{C} \cap (\mathbb{R}^\times \times \{0, \infty\})$) correspond to vertical (resp. horizontal) sides of $\partial \Delta$ while sides of $\partial \Delta$ that are neither horizontal nor vertical correspond to branches of $R \tilde{C}$ passing through a point of $\{0, \infty\} \times \{0, \infty\}$.

We deduce [35] from [36] inductively reducing the size of the complement $\Delta_{ab} \setminus \Delta$ which can be measured numerically as $2a + 2b - m$. The base of the induction is the case $\Delta = \Delta_{ab}$, i.e. $m = 2a + 2b$. Our component $C$ determines a Menelaus configuration $\tilde{P}_t \subset \partial \mathbb{RP} \Delta$ of $m$ points such that $Sq^1(\tilde{P}_t) = P_t$ and curves from $C \cap R_{ab}(P_t)$ pass through $\tilde{P}_t$ for each $t$. Let $\gamma_{ab}$ be a path of configurations of $2a + 2b - 1$ points in $\mathbb{RP}^1 \times \mathbb{RP}^1$ obtained by removing one point from the family of Menelaus configurations $\tilde{P}_t$, and perturbing it slightly to a generic configuration in $\mathbb{RP}^1 \times \mathbb{RP}^1$ (no longer contained in $\partial(\mathbb{RP}^1 \times \mathbb{RP}^1)$). The path $\gamma_{ab}$ connects generic configurations $P_{ab}$ and $P'_{ab}$ close to $P$ and $P'$ (after adding the $m$th point determined by the Menelaus condition). The component $C$ gives a component of the cobordism connecting $R_{ab}(P_{ab})$
to \( \mathcal{R}_{ab}(\mathcal{P}_t^{\Delta'}) \) that consists of irreducible curves close to \( \mathcal{C} \) (in particular, far from the curves of bidegree \((a,b)\) containing components of \( \partial(\mathbb{RP}^1 \times \mathbb{RP}^1) \). Thus (35) follows from (36) with the help of (20).

Let \( \Delta \neq \Delta_{ab} \). Inclusion \( \Delta \subset \Delta_{ab} \) yields a birational transformation (37)

\[
\tau : \mathbb{C}\Delta \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1
\]

regular near all points of \( \mathcal{P} \). Consider a family of points \( p(t) \in \mathcal{P}_t \) that belong to a divisor corresponding neither vertical nor horizontal side of \( \partial \Delta \), we have \( \tau(p(t)) \in \{0, \infty\} \times \{0, \infty\} \). Without loss of generality (using symmetries of \( \mathbb{RP}^1 \times \mathbb{RP}^1 \)) we may assume that \( \tau(p(t)) = (0, 0) \).

The restriction of the first (resp. second) coordinate of \( \mathbb{RP}^1 \times \mathbb{RP}^1 \) to the normalization of any curve in the component \( \mathcal{C} \) passing through \( p(t) \) has a zero of order \( n_x \) (resp. \( n_y \)) at the corresponding point \( \tilde{p}(t) \in \partial \mathcal{C} \), where \((-n_y, n_x)\) is a primitive integer vector parallel to the corresponding side of \( \partial \Delta \), \( n_x, n_y > 0 \). Inside \( \mathbb{RP}^1 \times \mathbb{RP}^1 \) we may perturb \( \mathbb{R}\tilde{C} \) to a curve \( \mathbb{R}\tilde{C}' \) perturbing slightly all points of \( \partial \mathcal{C} \) except for \( \tilde{p}(t) \) and splitting \( \tilde{p}(t) \) to two points \( \tilde{p}(1,0)(t) \) and \( \tilde{p}(n_x-1,n_y)(t) \). The first coordinate is required to have a simple zero at \( \tilde{p}(1,0)(t) \) and a zero of order \( n_x-1 \) at \( \tilde{p}(n_x-1,n_y)(t) \). The second coordinate is required to have a non-zero value at \( \tilde{p}(1,0)(t) \) and a zero of order \( n_y \) at \( \tilde{p}(n_x-1,n_y)(t) \). The rest of the points of \( \partial \mathcal{C} \) keep the orders of zeroes of coordinates unchanged.

We have \( m' = m+1 \) for the Newton polygon \( \Delta' \) of \( \mathbb{R}\tilde{C}' \). Furthermore, the birational transformation \( r = (\tau')^{-1} \circ \tau \) between \( \mathbb{C}\Delta \) and \( \mathbb{C}\Delta' \) (see (37)) is biregular near \( \mathcal{P}_t \setminus \{p(t)\} \). Consider a family of Menelaus configurations \( \mathcal{P}_t^{\Delta'} \subset \partial\mathbb{R}\Delta' \) obtained from \( r(\mathcal{P}_t \setminus \{p(t)\}) \) by adding to it a point \( p(1,0) \) such that \( \tau'(p(1,0)) \in \mathbb{R}^\times \times \{0\} \subset \mathbb{RP}^1 \times \mathbb{RP}^1 \) close to \((0,0)\) while the remaining point \( p(n_x-1,n_y)(t) \) is determined by the Menelaus condition. Reversing the roles of the first and the second coordinate (i.e., of \( n_x \) and \( n_y \)) if needed we may assume that \( GCD(n_x-1,n_y) \) is not divisible by two, and thus the resulting configuration is necessarily real. Also it is convenient to assume that \( n_x > 1 \) unless \( n_x = n_y = 1 \). By induction we may assume that (35) holds for \( \Delta' \). (if \( n_x-1, n_y \) is not primitive the curves from \( \mathcal{R}_{\Delta',k}(\mathcal{P}_t^{\Delta'}) \) have a tangency of order \( GCD(n_x-1,n_y) \) to \( \partial\mathbb{R}\Delta' \) at \( p(n_x-1,n_y)(t) \)). We have a 1-1 correspondence between curves from \( \mathcal{R}_{\Delta,k}(\mathcal{P}_t) \cap \mathcal{C} \) and a part of \( \mathcal{R}_{\Delta',k}(\mathcal{P}_t^{\Delta'}) \) close to \( \mathcal{R}_{\Delta,k}(\mathcal{P}_t) \cap \mathcal{C} \). Corresponding curves have the same number of elliptic nodes as the ends of \( \mathbb{R}\tilde{C}^o \) are either real (and thus their real intersection points are hyperbolic) or intersect the boundary divisors in purely imaginary points. Thus (35) holds for \( \Delta \).
Suppose now that

\[ R_\Delta(P_t) = \bigcup_{k' = -\text{Area}(\Delta)}^{\text{Area}(\Delta)} R_{\Delta,k'}(P_t) \]

contains a reducible curve \( \mathbb{R} \tilde{D} \) for \( t = t_0 \).

As the dimension of the space of deformation of each component is equal to the number of points in its intersection with \( \partial C \Delta \) minus 1, for the generic path \( \gamma \) (in the non-singular space \( \mathcal{M}_\Delta \)) the curve \( \text{Sq}^\Delta(\mathbb{R} \tilde{D}_j) \), \( j = 1, 2 \). Also, the presence of multiplicative translations in \( (\mathbb{R}^*)^2 \) implies that \( \text{Sq}^\Delta(\mathbb{R} \tilde{D}_1) \) and \( \text{Sq}^\Delta(\mathbb{R} \tilde{D}_2) \) intersect transversally.

We may assume that all points \( \text{Sq}^\Delta(P_t) \) except for two points \( p_j(t) \in \mathbb{R} \tilde{D}_j \), \( j = 1, 2 \), remain independent of \( t \in [t_0 - \epsilon, t_0 + \epsilon] \) for a small \( \epsilon > 0 \). For \( t \in [t_0 - \epsilon, t_0 + \epsilon] \) the deformation \( p_1(t) \in \partial \mathbb{R} \Delta \) determines the deformation \( p_2(t) \) by the Menelaus condition. The points \( p_1(t_0) \) and \( p_2(t_0) \) must belong to two different components of \( \mathbb{R} \tilde{D} \). Indeed, otherwise one of the component of \( \mathbb{R} \tilde{D} \) contains only points of \( P_t \) invariant of \( t \) and then \( R_\Delta(P_t) \) contains reducible curves for all \( t \in [t_0 - \epsilon, t_0 + \epsilon] \) which contradicts to our assumption. Denote with \( R^\Delta_{\tilde{D}}(P_t) \) for \( t_0 - \epsilon \leq t \leq t_0 + \epsilon \) the curves whose images under \( \text{Sq}^\Delta \) is close to \( \text{Sq}^\Delta(\mathbb{R} \tilde{D}) \).

Let us choose some orientations of \( \mathbb{R} \tilde{D}_1 \) and \( \mathbb{R} \tilde{D}_2 \). Then the intersection points of

\[ I = \text{Sq}^\Delta(\mathbb{R} \tilde{D}_1) \cap \text{Sq}^\Delta(\mathbb{R} \tilde{D}_2) \]

come with the intersection sign in \( \mathbb{R}_0^{>0} \). The set of positive points \( I_+ \subset I \) has the same cardinality as the set of negative points \( I_- \subset I \), \( I = I_+ \cap I_- \) as \( \mathbb{R}_0^{>0} \) is contractible.

The curves in \( R^\Delta_{\tilde{D}}(P_{t_0+\epsilon}) \) are obtained by smoothing a nodal point \( q \in I \) in one of the two ways, one that agrees with our choice of orientation and one that does not. Without loss of generality (changing the direction of the path \( \gamma \) if needed) we may assume that the orientation-preserving smoothing in a point \( q \in I_+ \) corresponds to a curve \( \mathbb{R} \tilde{D}_q+, \in R_\Delta(P_{t_0+\epsilon}) \) and thus the orientation-reversing smoothing at the same point corresponds to a curve \( \mathbb{R} \tilde{D}_q-, \in R_\Delta(P_{t_0-\epsilon}) \). The following lemma determines the situation at all the other points of \( I \).

**Lemma 28.** A curve obtained by the smoothing of a node from \( I_+ \) in the orientation-preserving way, or by the smoothing of nodes from \( I_- \) in the orientation-reversing way belongs to \( R_\Delta(P_{t_0+\epsilon}) \), \( \epsilon > 0 \).
Accordingly, a curve obtained by the smoothing of a node from $I_-$ in the orientation-preserving way, or by the smoothing of nodes from $I_+$ in the orientation-reversing way belongs to $\mathcal{R}_\Delta(P_{t_0-\epsilon})$.

**Proof.** Let $\mathbb{R}D_{q',s} \in \mathcal{R}_\Delta(P_{t_0+\epsilon})$, $s = \pm 1$, be the curve obtained by smoothing $\mathbb{R}D$ at a point $q' \in I$ according to the sign $s$. Note that $\text{Sq}^\Delta(\mathbb{R}D_{q,+})$ and $\text{Sq}^\Delta(\mathbb{R}D_{q',s})$ are tangent to each other at $m$ points of $P_{t_0+\epsilon}$ and must intersect each other at pairs of points close to the nodes of $I \setminus \{q, q'\}$. Tangencies contribute to $2m$ of the intersection, this number is the integer perimeter of the polygon $2\Delta$. The number of nodes in $I \setminus \{q\}$ is equal to the number of lattice points inside the polygon $2\Delta$ by the genus formula since $\text{Sq}^\Delta(\mathbb{R}D_{q,+})$ is a rational curve and $2\Delta$ is its Newton polygon. By Pick’s formula the total number of intersection points resulting from the nodes and tangencies as above equals twice the area of $2\Delta$. By the Kouchnirenko-Bernstein formula, these curves do not have any other intersection points which implies that $s = +1$ if $q' \in I_+$, see Figure 6.

![Figure 6](image-url) The signs of intersection points of two components of $\mathbb{R}D$ and the corresponding direction of smoothing.

Note that any curve in $\mathcal{R}_{\Delta}^{\mathbb{R}D}(P_{t_0\pm\epsilon})$ is obtained by smoothing $\mathbb{R}D$ at a point $q \in I$. The quantum index of the result is $\pm k(\mathbb{R}D_1) \pm k(\mathbb{R}D_2)$, where the signs are determined by the agreement or disagreement of the orientation of the resulting curve with the chosen orientations of $\mathbb{R}D_j$. Since $\#(I_+) = \#(I_-)$, Lemma 28 implies that $\mathcal{R}_{\Delta}^{\mathbb{R}D}(P_{t_0+\epsilon})$ and $\mathcal{R}_{\Delta}^{\mathbb{R}D}(P_{t_0-\epsilon})$ have the same number of curves of each quantum index. □

**Proof of Theorem 6.** By the same reason as above we have $\tilde{R}_{\Delta,k}(\mathcal{P}) = \tilde{R}_{\Delta,k}(\mathcal{P}')$ if there are no reducible curves with the Newton polygon $\Delta_d$ that pass through $\mathcal{P}_t$. Also we may assume that if $\mathbb{R}D$ is a reducible
curve with the Newton polygon $\Delta_d$ passing through $P_{t_0}$ then it consists of two components $R_{D_1}$ and $R_{D_2}$ that intersect transversely at a finite set $I$. Note that the degree of both components, $R_{D_1}$ and $R_{D_2}$, must be even, as a real curve of odd degree must intersect $\partial \mathbb{R}P^2$ in a negative point as the boundary of the positive quadrant is null-homologous.

We have two smoothings of $\mathbb{R}D$ at $q \in I$ that pass through $P_{t_0-\epsilon}$ and $P_{t_0+\epsilon}$. One can be oriented in accordance with the orientations of $R_{D_1}$ and $R_{D_2}$ and the other in accordance with the orientation of $R_{D_1}$, but opposite to the orientation of $R_{D_2}$. The corresponding quantum indices are different by $2k(R_{D_2})$. The index $k(R_{D_2})$ is integer since the degree of $R_{D_2}$ is even. □

6.3. Indices of real phase-tropical curves. We start by recalling the basic notions of tropical geometry (cf. [17], [18]) specializing to the case of plane curves. Recall that a metric graph is a topological space homeomorphic to $\Gamma^o = \Gamma \setminus \partial \Gamma$ enhanced with a complete inner metric. Here $\Gamma$ is a finite graph and $\partial \Gamma$ is the set of its 1-valent vertices. The metric graph is also sometimes called a tropical curve (while in some other instances the term tropical curve is reserved for the equivalence class of metric graphs with respect to tropical modifications). In this paper we require the graph $\Gamma$ to be connected so that $\Gamma^o$ is irreducible as a tropical curve. We assume that $\Gamma$ has a vertex of valence at least three, and that $\Gamma$ does not have 2-valent vertices. The half-open edges of $\Gamma^o$, obtained from the closed edges of $\Gamma$ adjacent to $\partial \Gamma$ are called leaves.

A plane tropical curve is a proper continuous map $h : \Gamma^o \to \mathbb{R}^2$ such that $h|_E$ is smooth for every edge $E \subset \Gamma$ with $dh(u) \in \mathbb{Z}^2$ for a unit tangent vector $u$ at any point of $E$. In addition we require the following balancing condition at every vertex $v \in \Gamma$

$$\sum_E dh(u(E)) = 0,$$

where $u(E)$ is the unit tangent vector in the outgoing direction with respect to $v$ and the sum is taken over all edges $E$ adjacent to $v$.

The collection of vectors $\{dh(u_v)\}_{v \in \partial \Gamma}$ where $u_v$ is a unit vector tangent to the leaf adjacent to $v$ (and directed towards $v$) is called the (toric) degree of $h : \Gamma^o \to \mathbb{R}^2$. The identity (40) implies that the sum of all vectors in this collection is zero. Therefore this collection is dual to a lattice polygon $\Delta \in \mathbb{Z}^2$ which is well-defined up to translations in $\mathbb{Z}^2$. The polygon $\Delta$ is determined by $h(\Gamma^o)$. We call $\Delta$ the Newton polygon of $h : \Gamma^o \to \mathbb{R}^2$. 
Tropical curves appear as limits of scaled sequences of complex curves in the plane. Let $A$ be any set and $\alpha \to t_\alpha \in \mathbb{R}$ be a function unbounded from above (this function is called the tropical scaling sequence). Let $\mathbb{C}C_\alpha \subset (\mathbb{C}^\times)^2$, $\alpha \in A$, be a family of complex curves with the Newton polygon $\Delta$.

**Definition 29.** We say that a family $\mathbb{C}C_\alpha$ has a phase-tropical limit with respect to $t_\alpha$ if for every $p \in \mathbb{R}^2$ we have

$$\lim_{t_\alpha \to +\infty} t_\alpha^{-p}\mathbb{C}C_\alpha = \Phi_p$$

for a (possibly empty) algebraic curve $\Phi(p) \subset (\mathbb{C}^\times)^2$. Here $t_\alpha^{-p}\mathbb{C}C_\alpha$ is the multiplicative translation of the curve $\mathbb{C}C_\alpha$ by $t_\alpha^{-p} \in (\mathbb{C}^\times)^2$. The coefficients of the polynomials defining $t_\alpha^{-p}\mathbb{C}C_\alpha$ represent a point in the projective space of dimension $\#(\Delta \cap \mathbb{Z}^2) - 1$. The limit is understood in the sense of topology of this projective space. The curve $\Phi_p \subset (\mathbb{C}^\times)^2$ may be reducible and even non-reduced.

We say that $h : \Gamma^o \to \mathbb{R}^2$ is the tropical limit of $\mathbb{C}C_\alpha$ with respect to $t_\alpha$ if for a sufficiently small open convex neighborhood $p \in U \subset \mathbb{R}^2$ the irreducible components $\Psi \subset \Phi(p) \subset (\mathbb{C}^\times)^2$ correspond to the connected components $\psi \subset h^{-1}(U)$ so that the lattice polygon $\Delta_\psi$ determined by the ends of the open graph $\psi$ coincides with the Newton polygon $\Delta_\Psi$ of the irreducible component $\Psi$ taken with some multiplicity. The same component $\Psi$ may correspond to several components of $h^{-1}(U)$ so that the sum of all resulting multiplicities is equal to the multiplicity of $\Psi$ in $\Phi(p)$. Each connected component of $h^{-1}(U)$ corresponds to a unique component of $\Phi(p)$.

If $h$ does not contract any edge of $\Gamma^o$ to a point then the open set $\psi \subset \Gamma^o$ may contain at most one vertex. If $v \in \Gamma$ is such a vertex then we call $\Psi$ the phase $\Phi_v$ of the vertex $v$. If $\psi$ is contained in an edge $E$ then we call $\Psi$ the phase $\Phi_E$ of the edge $E$. The phases $\Phi(E) \subset (\mathbb{C}^\times)^2$ do not depend on the choice of a point $p \in h(E)$ and are well-defined up to multiplicative translations by $(\mathbb{R}^\times)^2$. The curve $h : \Gamma^o \to \mathbb{R}^2$ enhanced with the phases $\Phi_v$ and $\Phi_E$ for its vertices and edges is called the phase-tropical limit of $\mathbb{C}C_\alpha$ with respect to the scaling sequence $t_\alpha \to +\infty$.

We consider the phases in $(\mathbb{C}^\times)^2$ that are different by multiplicative translation by vectors from $(\mathbb{R}_{>0})^2$ equivalent.

Note that the Newton polygon of the phase $\Phi_E$ of an edge $E$ is an interval. Thus after a suitable change of coordinates in $(\mathbb{C}^\times)^2$ the (irreducible) curve $\Phi_E$ is given by a linear equation in one variable. Therefore, $\Phi_E$ is a multiplicative translation of a subtorus $S^1 \approx T_E \subset S^1 \times S^1$ in the direction parallel to $h(E)$. 
Let us orient $E$. Then $T_E$ as well as the quotient space $B_E = (S^1 \times S^1)/T_E$ also acquire an orientation. The image $\text{Arg}(\Phi_E)$ coincides with $\pi_E^{-1}(\sigma_E)$ for some $\sigma_E \in B_E$, where $\pi_E : S^1 \times S^1 \to B_E$ is the projection. Since $B_E$ is isomorphic to $S^1$ and oriented, we have a canonical isomorphism $B_E = \mathbb{Z}/2\pi \mathbb{Z}$. Thus, a phase $\Phi_E$ of an oriented edge $E$ of a planar tropical curve is determined by a single argument $\sigma(E) \in \mathbb{Z}/2\pi \mathbb{Z}$. The change of the orientation of $E$ results in the change of sign of $\sigma(E)$.

Let $v \in \Gamma^\circ$ be a vertex and $E_j$ be the edges adjacent to $v$. Orient $E_j$ outwards from $v$. The oriented edges $E_j$ can be associated a momentum $\mu(E_j)$ with respect to the origin $0 \in \mathbb{R}^2$. This is the wedge product of the vector connecting the origin with a point of $E_j$ and the unit tangent vector $u(E_j)$ coherent with the orientation. Clearly, it does not depend on the choice of the point in $E_j$.

Recall that the vertex $v$ is dual to the lattice polygon $\Delta_v$ determined by the integer vectors $dh(u(E_j))$. The multiplicity is defined as $m(v) = 2 \text{Area } \Delta_v$, cf. [17].

**Proposition 30** (tropical Menelaus theorem). For any tropical curve $h : \Gamma^\circ \to \mathbb{R}^2$ and a vertex $v \in \Gamma^\circ$ the momenta $\mu(E_j)$ of the edges adjacent to $v$ and oriented outwards from $v$ satisfy to the equality

$$
\sum_j \mu(E_j) = 0. \tag{42}
$$

If $\sigma(E_j) \in \mathbb{Z}/2\pi \mathbb{Z}$ are phases of the oriented edges $E_j$ then

$$
\sum_j w(E_j)\sigma(E_j) = \pi m(v) \tag{43}
$$

(assuming that $\sigma(E_j)$ appear in the phase-tropical limit of a family $\mathbb{C}C_\alpha \subset (\mathbb{C}^\times)^2$ of complex curves).

This statement can be viewed as a counterpart of the ancient Menelaus theorem (before its generalizations by Carnot and Weil) stating that three points $D, E, F$ on the extensions of three sides of a planar triangle $ABC$ are collinear if and only if

$$
\frac{|AD|}{|DB|} \frac{|BE|}{|EC|} \frac{|CF|}{|FA|} = -1. \tag{44}
$$

Here the length is taken with the minus sign if the direction of an interval (e.g. $|CF|$) is opposite to the orientation of the triangle, see Figure [7]

**Proof.** The wedge product of the balancing condition (40) with the vector connecting 0 and $v$ gives (42). To deduce (43) we consider
the polynomial $f_v$ (whose Newton polygon is $\Delta_v$) defining the phase $\Phi_v \subset (\mathbb{C}^\times)^2$. By Vieta’s theorem, the product of the roots cut by $f_v$ on a divisor of $\mathbb{C}\Delta_v$ corresponding to an oriented side $F \subset \Delta_v$ is $(-1)^{\#(F \cap \mathbb{Z}^2)}$ times the ratio of the coefficients at the endpoints of $F$. Therefore the sum of the phases of the edges of $\Gamma$ corresponding to $F$ is the argument of this ratio plus $\#(F \cap \mathbb{Z}^2)\pi$. Since by Pick’s formula the parity of $\#(\partial \Delta \cap \mathbb{Z}^2)$ coincides with that of $m(v) = 2\text{Area}({\Delta_v})$ we recover (43).

**Corollary 31.** We have $\sum_E \mu(E) = 0$, where the sum is taken over all leaves of $h : \Gamma^\circ \to \mathbb{R}^2$ oriented in the outwards direction.

*Proof.* Take the sum of the expression (42) over all vertices of $\Gamma^\circ$. The momenta of all bounded edges will enter twice with the opposite signs. □

If all curves $\mathbb{C}C_\alpha$ are defined over $\mathbb{R}$ then the phases $\Phi(p)$ must be real for all points $p \in \mathbb{R}^2$. Note, however that in general, the phase $\Phi_v$ for a vertex $v \in \Gamma^\circ$ does not have to be real as the involution of complex conjugation may exchange it with $\Phi_{v'}$ for another vertex $v' \in \Gamma$ with $h(v) = h(v')$. We say that a vertex $v$ is real if $\Phi_v$ is defined over $\mathbb{R}$.

Let $\mathbb{R}C_\alpha$ be a scaled sequence of type I curves enhanced with a complex orientation, so that a component $S_\alpha \subset \mathbb{C}C_\alpha \setminus \mathbb{R}C_\alpha$ is fixed for all $\alpha$. Suppose that $\mathbb{C}C_\alpha$ has a phase-tropical limit, and the orientations of $\mathbb{R}C_\alpha$ agree with some complex orientations of the real part $\mathbb{R}\Phi(p)$ of the phases $\Phi(p)$. The quantum index of $\mathbb{R}C_\alpha$ is well-defined if it has real or purely imaginary coordinate intersection. Similarly, the phase $\mathbb{R}\Phi_v$ of a real vertex $v$ of the tropical limit has a well-defined quantum index if $\sigma(E) \equiv 0 \pmod{\pi}$ for any edge $E$ adjacent to $v$. 

![Figure 7. The Menelaus theorem.](image-url)
Proposition 32. For large $t_\alpha$ we have
\begin{equation}
(45) \quad k(\mathbb{R}C_\alpha) = \sum_v k(\mathbb{R}\Phi_v),
\end{equation}
where the sum is taken over all real vertices whenever all quantum indices in \((45)\) are well-defined.

Proof. Additivity of the quantum index with respect to the phases $\Phi_v$ follows from Theorem 1 through additivity of the degree of the map $2 \text{Arg}$ restricted to $S \cap (\mathbb{C}^*)^2$. Non-real vertices have zero contribution to $k(\mathbb{R}C_\alpha)$ as the signed area of the amoeba of the whole complex curve is zero. \qed

Proof of Theorem 7. Recall the definition of the (tropical) Block-Göttche invariants, see [10], which refine tropical enumerative invariants of [17]. Namely, to any 3-valent (open) tropical immersed curve $h : \Gamma^o \to \mathbb{R}^2$ we may associate the Laurent polynomial
\begin{equation}
(46) \quad n_q(h(\Gamma^o)) = \prod_v q^{\frac{m(v)}{2}} - q^{-\frac{m(v)}{2}},
\end{equation}
where $v$ runs over all vertices $v \in \Gamma$ and $m(v)$ is the multiplicity of the vertex $v$. The genus of a (connected) tropical curve $\Gamma^o$ is the first Betti number of $\Gamma^o$. In particular, a rational tropical curve is a tree.

Let us fix a collection $\mu = \{m_j\}_{j=1}^m$, $m = \#(\partial \Delta \cap \mathbb{Z}^2)$, of generic real numbers subject to the condition $\sum_{j=1}^m \mu_j = 0$. This means that $\mu_j$, $j = 1, \ldots, m - 1$ are chosen generically, and $\mu_m$ is determined from our condition.

If $h : \Gamma^o \to \mathbb{R}^2$ is a tropical curve with the Newton polygon $\Delta$ then we number its leaves so that the first $m_1$ leaves are dual to the side $E_1 \subset \partial \Delta$, the second $m_2$ to the side $E_2 \subset \partial \Delta$ and so on with the last $m_n$ leaves dual to $E_n$. We say that $h : \Gamma \to \mathbb{R}^2$ passes through the $\partial \mathbb{T} \Delta$-points determined by $\mu$ if the $j$th unbounded edge of $\Gamma$ has the momentum $\mu_j$. Note that a leaf $E \subset \Gamma^o$ must have the momentum $\mu(E)$ if it passes through a point $p_E$ on the oriented line parallel to the vector $(dh)u(E)$ with the momentum $\mu(E)$. Thus a generic choice of the momenta ensures that $h : \Gamma^o \to \mathbb{R}^2$ passes through a generic collection of $m - 1$ points in $\mathbb{R}^2$. Thus we have only finitely many rational tropical curves with the Newton polygon $\Delta$ passing through the $\partial \mathbb{T} \Delta$ points determined by $\mu$ by Lemma 4.22 of [17] (as the number of combinatorial types of tropical curves with the given Newton polygon $\Delta$ is finite). By Proposition 4.11 of [17] all these tropical curves are simple in the sense of Definition 4.2 of [17].
The Block-Göttsche number associated to \( \mu \) is

\[
N_{\Delta}^{\partial, \text{trop}} = N_{\Delta}^{\partial, \text{trop}}(\mu) = \sum_{h: \Gamma^\circ \to \mathbb{R}^2} n_q(h(\Gamma^\circ)),
\]

where the sum is taken over all \( h : \Gamma^\circ \to \mathbb{R}^2 \) passing through the \( \partial \mathbb{T} \Delta \) points determined by \( \mu \). Independence of \( N_{\Delta}^{\partial, \text{trop}} \) from \( \mu \) can be proved in the same way as in \([10]\). Also it follows from Theorem 5 once we prove coincidence of \( R_{\Delta} \) and \( N_{\Delta}^{\partial, \text{trop}}(\mu) \).

A toric divisor \( CE_j \subset \mathbb{C} \Delta \) is the compactification of the torus \( \mathbb{C} \times \) obtained by taking the quotient group of \( (\mathbb{C}^\times)^2 \) by the subgroup defined by the side \( E_j \subset \Delta \). Thus a configuration \( \mathcal{P} = \{p_j\}_{j=1}^m \subset \partial \mathbb{C} \Delta \) is given by a collection of \( m \) nonzero complex numbers as well as an attribution of the points to the toric divisors. This collection is real if the corresponding numbers are real and positive if these number are positive.

We set \( \mathcal{P}^t = \{p_{t1}, \ldots, p_{tm}\} \subset \partial \mathbb{R} \Delta \) be the configuration of points with the same toric divisor attribution as \( \mathcal{P} \), and given by the positive numbers \( \{t^{2\mu_j}\}, \ t > 1 \). By Proposition 8.7 of \([17]\) the amoebas of rational complex curves with the Newton polygon \( \Delta \) passing through \( (\text{Sq}^\Delta)^{-1}(\mathcal{P}_t) \) converge when \( t \to +\infty \) to tropical curves passing through the \( \partial \mathbb{T} \Delta \)-points determined by \( \mu \). Proposition 8.23 of \([17]\) determines the number of complex curves with amoeba in a small neighborhood of a rational tropical curve \( h : \Gamma^\circ \to \mathbb{R}^2 \) passing through any choice of points \( \tilde{p}_j^t \in (\text{Sq}^\Delta)^{-1}(p_j^t), \ j = 1, \ldots, m - 1 \), for large \( t \), while Remark 8.25 of \([17]\) determines the number of the corresponding real curves. E.g. if the weights of all edges of \( \Gamma^\circ \) are odd we have a single real curve for any choice of \( \mathcal{P}^t = \{\tilde{p}_j^t\} \). In general, some choices of \( \mathcal{P}_t \) may correspond to no real solutions, while others may correspond to multiple solutions. We claim that nevertheless there are \( 2^{m-1} \) different real curves whose amoeba is close to \( h : \Gamma^\circ \to \mathbb{R}^2 \) with the image under \( \text{Sq}^\Delta \) passing through \( \mathcal{P}^t \) for large \( t \). Thus we have \( 2^m \) different oriented curves. We show this by induction on \( m \) as follows.

If \( \Gamma^\circ \) has a single vertex \( v \) (so that \( m = 3 \)) then there are 4 different real rational phase \( \Phi_v \), which differ by the deck transformations of the map \( \text{Sq}^\Delta \). Thus we have 8 different oriented real rational phases in this case. The positive logarithmic rotation number for half of them is positive, for the other half is negative. Adding each new 3-valent vertex \( v' \) to the tree \( \Gamma \) doubles the number of oriented real phases as there are two ways to attach the phase for \( v' \): so that the logarithmic rotation number of the resulting real curve will increase by one and so that it will decrease by one. Inductively we get 4 real oriented curves for each of the \( 2^{m-2} \) possible distribution of signs for the vertices of \( \Gamma^\circ \).
For each vertex $v$ the real phase $\Re \Phi_v$ is the image of a line by a multiplicative-linear map of determinant $m(v)$ by Corollary 8.20 of [17]. Therefore $k(\Re \Phi_v) = \pm \frac{m(v)}{2}$, where the sign is determined by the degree of the logarithmic Gauß map. According to our sign convention [19] each oriented real curve comes with the sign equal to the number of negative vertices. Thus by Proposition 32 the contribution of $h : \Gamma \to \mathbb{R}^2$ to $R_\Delta(P^t)$ for large $t$ is $\prod_v (q^{\frac m2} - q^{-\frac m2})$ which coincides with the numerator of the Block-Göttsche multiplicity (46).

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Université de Genève, Mathématiques, Batelle Villa, 1227 Carouge, Suisse