Shock Formation of 3D Euler-Poisson System for Electron Fluid with Steady Ion Background

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Abstract

We prove shock formation for 3D Euler-Poisson system for electron fluid in plasma. The shock solution we construct is of large initial data, also compactly supported during the lifespan. In addition, the blowup time and location can be computed explicitly.

1 Introduction

We consider the 3D repulsive Euler-Poisson system for electron fluid in plasma

\[
\begin{aligned}
    n_e m_e (\partial_t u + u \cdot \nabla_x u) + \nabla_x p_e &= -n_e e \nabla_x \phi, \\
    \partial_t n_e + \nabla_x \cdot (n_e u) &= 0, \\
    \Delta_x \phi &= 4\pi e (n_+ - n_e).
\end{aligned}
\]

(1.1)

Here, we denote charge of electron by e, mass of electron by \(m_e\), electron density by \(n_e\) : \(\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}\), velocity by \(u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3\) and pressure by \(p_e = p(n_e) = \frac{1}{\gamma} n_e^\gamma\) with \(\gamma > 1\). The electric potential \(\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}\) depends on both electrons fluid \(n_e\) and ion background \(n_+\). In addition, it satisfies \(\phi(x) \to 0\) as \(|x| \to \infty\).

The Euler-Poisson system arises in various physical backgrounds involving compressible fluids interacting with a self-consistent potential. In stellar dynamics, Euler-Poisson system is used to describe the evolution of self-gravitational gaseous stars in which the potential results from the attractive gravitational interaction. In plasma physics, Euler-Poisson system is a simplified version of the so-called “two-fluid” model which describes plasma dynamics for ion and electron fluids. In such case, the potential comes from repulsive Coulomb interaction. Because the ratio of the electron mass and the ion mass is very small, the heavier ions are treated as motionless in some cases and the density of ions can be regarded as a time independent function.

There have been a lot of works devoted to the study of Euler-Poisson system. For two-dimensional Euler-Poisson system \([14]\), the global existence of smooth irrotational flows with small initial amplitude was proved in \([21]\), \([23]\), \([25]\) and \([26]\). In three dimensional case, Guo constructed the global irrotational flows with small velocity for the electron fluid in \([12]\). There are also many results for other types of Euler-Poisson system in plasma. In \([18]\), Guo and Pausader established the global existence and uniqueness for ion fluid with Boltzmann statistics, but whether the shock waves can develop in 2D remains open. What is more, for full Euler-Poisson system concerning plasma namely two-fluid model, the global existence for small solutions was established by Guo, Ionescu and Pausader in \([17]\) and \([15]\) via the method of space-time resonance.

The situation is different for attractive Euler-Poisson system in gaseous stars. In the star-formation model, the range of index \(\gamma\) in the pressure law \(p(\gamma) = \rho^\gamma\) is essential. An important question is the dynamical stability of Lane-Emden stars solutions. In \([34]\), Rein showed the existence of global weak solutions by variational methods and obtained stability of Lane-Emden stars.

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for $\gamma > \frac{4}{3}$. On the contrary, unstability was proved by Jang for $\gamma = \frac{5}{6}$ in [22], and for $\frac{6}{5} < \gamma < \frac{4}{3}$ in [20] that the expanding solutions are asymptotically stable when $\gamma = \frac{4}{3}$. In addition, Liu [28] studied the viscous effect in expanding configuration. The gravitational collapse is another fascinating issue. The key is to find the self-similar solutions, and they have been constructed in [9], [11] and [30]. Besides, in [13], gravitational collapse was shown by Guo, Hadzić and Jang based on pressureless model. Furthermore, a very recent work [14] showed the existence of smooth radially symmetric self-similar solutions to the gravitational Euler-Poisson system when $1 < \gamma < \frac{4}{3}$.

However, the blowup mechanism for Euler-Poisson system needs further investigation. In fact, there has been a lot of remarkable results for compressible Euler in recent years. For 3D Euler system, Sideris argued by contradiction and proved in [35] that $C^1$ regular solutions of Euler have a finite lifespan. The first proof of shock formation was given by Christodoulou for relativistic fluid in [6] and non-relativistic fluid by Christodoulou and Miao in [7], both of them concerns irrotational flows. Luk and Speck then proved for shock formation for 2D fluid flows with vorticity in [24]. Buckmaster, Shkoller and Vical constructed the stable asymptotically self-similar type blowup solutions to 2D and 3D compressible Euler in [31]. They also analyzed the shock formation in [34] for the 3d non-isentropic Euler system, in which sounds waves interact with entropy waves to produce vorticity. Unstable shock solutions are constructed by Buckmaster and Iyer for 2D Euler system in [2] and Buckmaster, Drivas, Shkoller and Vicol proved that a discontinuous shock instantaneously develops after the pre-shock in [4]. In addition, Merle, Raphael, Rodnianski and Szeftel constructed smooth spherical blowup solutions for compressible Euler and Navier-Stokes system by analyzing phase portrait in [27] and [30]. For Euler-Poisson system, the study of shock formation is of great interests. On one hand, it has been shown in [12], [15], [21], [26] that smooth solutions with small amplitude to Euler–Poisson system for electrons persist forever with no shock formation. On the other hand, Makino and Perthame proved in [31] that smooth solutions with gravitational force blow up in finite time if the initial data are spherically symmetric and have a compact support. In addition, Guo and Tahvildar-Zadeh proved in [19] that shock waves do develop for “large” perturbations of the constant state equilibrium of $n_e \equiv 0, u \equiv 0$. Moreover, in [36], Wang showed that for a larger class of initial data, with no matter repulsive forces or attractive, the finite time blowup take place. The assumption on the compact support was removed by Li and Wang in [27].

In this article we study the shock formation for Euler-Poisson system (1.1) for large initial data. Our main results can be stated roughly as:

**Theorem 1.1.** The system (1.1) admits solution $(u, n_e)$ developing shock in finite time. More precisely, the minimum negative slopes of $u_1$ and $n_e$ go to $-\infty$ in finite time, while themselves remain bounded.

The idea is mainly inspired by the pioneering work [1] for 3D compressible Euler system. After Riemann transformation, we get the system (2.8) about $(w, z, u_\nu)$ see Section 2, which is regarded as a perturbation of 3D Burgers equation

\[ \partial_t w + w \cdot \nabla w = 0. \]  

(1.2)

It is widely acknowledged that (1.2) admits a family of self-similar solutions, which are the prototypes of various shock formation problems and universally appear in the different circumstances of fluid dynamics, see [10]. Here we still expect that the singular solutions we construct to (2.8) is of self-similar type, and behave just like that of (1.2), namely $\bar{W}$ defined in (2.10). In order to verify the assertion rigorously we need to perform our argument in self-similar variables $(y, s)$ and investigate the difference $\tilde{W} = W - \bar{W}$, where $W$ is the singular solution we construct in self-similar variable $(y, s)$.

However, due to the presence of the electric potential, (1.1) does not enjoy finite speed of propagation as compressible Euler system in [1]. Indeed, the absence of finite speed of propagation is usually caused by nonlocal term in various fluid dynamics, such as Biot-Savart law in incompressible flow, or nonlocal drift. As a result, when estimating nonlinear terms or external forces one
has to take the nonlocal effect into consideration. In [37], Yang gave a good example analyzing external forces with nonlocal terms.

Fortunately, when it comes to Euler-Poisson system, the situation changes, because the integrand in nonlocal term is just the electron density $n_e$. On one hand, the transport of $n_e$ relies heavily on the velocity, which stays bounded in $L^\infty$ (see Lemma 4.7) during the whole lifespan. That means if $n_{e,0}$ is supported in compact set $x_0$ at initial time, then for any $t > -\frac{x_0}{\tau_{\infty}}$, $n_e(x, t)$ is supported in compact set $\psi(x_0)$, where $\psi$ is Lagrangian trajectory generated by velocity field. On the other hand, the velocity field $u$ is also compactly supported for all time, because nontrivial velocity cannot be attached to vacuum. As a result, we can perform estimates just in the compact set $\mathcal{A}(s)$ in self-similar variables.

In our construction, due to the various symmetries of the system, some unstable modes need to be considered. They are parameters $\kappa$, $\tau$, $\xi_1$, $\xi_2$, $\xi_3$, corresponding to Galilean transform, time translation and spatial translation. In order to give precise description of blowup solutions, we impose some constrains on $\kappa$, $\tau$, $\xi_1$, $\xi_2$, $\xi_3$, as well as $\phi_{22}$, $\phi_{23}$ and $\phi_{33}$ concerning the rotation symmetry and second fundamental form of wave front, forcing the amount $\nabla W(0, s) = \nabla^2 W(0, s) = 0$. In this paper, in order to avoid the over-repeating parts of [1] and focus on the differences, we impose the ansatz that $u$ and $n_{e,0}$ are even about $x_2$ and $x_3$ in (2.2) and $\nabla^2(u_{1,0} + \frac{n_{e,0}^2}{\omega^2})|_{t= -\frac{x_0}{\tau_{\infty}}} = 0$ in (8.6) so that these five modes are excluded. As a consequence, there are more unstable modes than that of [1] for our blowup solutions. Of course, if we involve these five modes in our argument, similar conclusions can be obtained systematically as in [1].

As the end of the introduction, we claim that the similar shock formation result for 2D Euler-Poisson system can also be obtained by following the work [3].

The remainder of the paper is organized as follows: In Section 2, we perform a series of transforms from sound speed form (2.6) to self-similar form (2.12), then we apply higher order derivatives and get (2.15). Also, similar process is taken to the perturbation $\bar{W} = W - \overline{W}$. In section 3, we make some assumptions on the initial data and give the main theorem, both of which are stated in self-similar variables. In section 4, we make bootstrap assumptions for no more than 4 order derivatives, and give the $\bar{H}^k$ estimate in Proposition 4.2 for $k \geq 18$. Also, we analyze the modulation variables. In section 5, we estimate the bounds of transport terms and external forces under the bootstrap assumptions. In section 6, we close the bootstrap assumptions made in Proposition 4.1. In section 7, we prove the energy estimate Proposition 4.2. In section 8, we state the main theorem in physical variables.

Notations.

For a vector $A = (a_1, a_2, a_3) \in \mathbb{R}^3$, we always use $\hat{a}$ to denote anyone of $a_2$ and $a_3$, as well as $|\hat{\alpha}| = (a_2^2 + a_3^2)^{\frac{1}{2}}$, and for multi-index $\gamma = (\gamma_1, \gamma_2)$ is similar. We let the subscript $i, j, k \in \{1, 2, 3\}$, and Greek subscript $\mu, \nu \in \{2, 3\}$.

The inequality $A \lesssim B$ means $A \leq CB$, where $C$ is a constant depending on $\alpha$ and $\kappa$, but independent of $M$, $\varepsilon$, $l$ and $L$. Concerning the various constants we suppose $0 < \varepsilon \ll l \leq \frac{1}{10} < 1 \ll M \ll L = \varepsilon^{- \frac{1}{\delta_1}}$, and for any $\delta_1$, $\delta_2 > 0$, there exists $\delta_3 > 0$, such that $\varepsilon^{\delta_1} M^{\delta_2} \ll \varepsilon^{\delta_3}$, where $\delta_3 < \delta_1$.

2 The Reformulation of the Main Problem

We assume that $n_{e,0} = n_e(x, -\frac{x_2}{1+\alpha}, \varepsilon)$ and $u_0 = u(x, -\frac{x_2}{1+\alpha}, \varepsilon)$ are supported in the set

$$X = \{ |x_1| \leq \varepsilon^{\frac{1}{\tau}}, |\hat{x}| \leq \varepsilon^{\frac{1}{\tau}} \}$$

(2.1)

and $n_{e,0} > 0$ and assume that

$$u_0(x_1, x_2, x_3) = u_0(x_1, -x_2, x_3) = u_0(x_1, x_2, -x_3)$$

(2.2a)
\begin{equation}
ne_0(x_1, x_2, x_3) = ne_0(x_1, -x_2, x_3) = ne_0(x_1, x_2, -x_3) \tag{2.2b}
\end{equation}

Also, for background \(n_+\) we further assume that \(n_+\) is supported in
\[x_+ = \{ |x| \leq 1 \}\] (2.3)
and lies in Sobolev space \(W^{4, \infty}\) with
\[\|n_+\|_{W^{4, \infty}} \leq 1.\] (2.4)

Besides, the total charges satisfy neutrality
\[\int_{X_+} \left( n_+ - ne \right) = 0. \]
The vorticity \(\omega_e = \nabla \times u\), so the specific vorticity \(\zeta_e = \omega_e / ne\) satisfies
\[\partial_t \zeta_e + (\zeta_e \cdot \nabla) \zeta_e - (\zeta_e \cdot \nabla) u_e = 0. \] (2.5)

If we let sound speed \(\sigma_e = \frac{1}{\alpha} n_e^\alpha\), where \(\alpha = \frac{\gamma - 1}{2}\), \(m_e = e = 1\), then we have the system about \(u\) and \(\sigma_e\)
\[
\begin{cases}
\partial_t u + (u \cdot \nabla) u + \sigma_e \nabla \sigma_e = -\nabla \phi, \\
\partial_t \sigma_e + (u \cdot \nabla) \sigma_e + \alpha \sigma_e \nabla \cdot u = 0.
\end{cases} \tag{2.6}
\]

### 2.1 A Series of Transforms

In order to investigate the structure of equation in depth, various transforms performed on the origin system (2.6) are needed. Next we will see the blow up solution we construct are close to the stable self-similar solution of 3D Burgers, while the blowup location is related to the amplitude of velocity field. To this end, we need three steps to obtain the final form of the system we study. The key point to figure out the different forms of electric potential produced by charges.

**Step 1. Translation**
At first, we define
\[t = \frac{1 + \alpha}{2} t\]
and assume the blowup take place at the space-time point \((x, t) = (\xi(T_*), T_*)\), where \(\xi(t)\) is a function depending only on \(t\). In fact, \(\xi(t)\) is used to track the location of the most steepened point of the gradient, which we assume initially satisfies \(\xi(-\varepsilon) = 0\). Let
\[x = x - \xi(t), \quad \tilde{u}(x, t) = u(x, t), \quad \tilde{\sigma}(x, t) = \sigma_e(x, t), \quad \tilde{n}_e(x, t) = n_e(x, t), \quad \tilde{n}_+(x, t) = n_+(x, t), \quad \tilde{\zeta}(x, t) = \zeta_e(x, t).\]

Then the potential function \(\phi\) is translation invariant, that is
\[\phi(x) = \int \frac{1}{|\xi|} (n_+ - n_e)(x - \xi) d\xi = \int \frac{1}{|\xi + \xi|} (n_+ - n_e)(x + \xi - \xi) d\xi = \tilde{\phi}(x).\]

Therefore, the system (2.6) is reformulated as
\[
\begin{cases}
\frac{1+\alpha}{2} \partial_t \tilde{u} + ((\tilde{u} - \tilde{\xi}) \cdot \nabla) \tilde{u} + \alpha \tilde{\sigma} \nabla \tilde{\sigma} = -\nabla \tilde{\phi}, \\
\frac{1+\alpha}{2} \partial_t \tilde{\sigma} + ((\tilde{u} - \tilde{\xi}) \cdot \nabla) \tilde{\sigma} + \alpha \tilde{\sigma} \nabla \cdot \tilde{u} = 0. \tag{2.7}
\end{cases}
\]

**Step 2. Riemann Transform**
We define the constants
\[ \beta_1 = \frac{1}{\alpha + 1}, \quad \beta_2 = \frac{1 - \alpha}{1 + \alpha}, \quad \beta_3 = \frac{\alpha}{1 + \alpha}, \]
and introduce the Riemann variables
\[ w = \tilde{u}_1 + \tilde{s}, \quad z = \tilde{u}_1 - \tilde{s}, \quad \mu = 2, 3. \]

Then the system (2.7) is rewritten as
\begin{align*}
\partial_t w + (w + \beta_2 z - 2\beta_1 \xi_1) \partial_t w + 2\beta_1 (\tilde{u}_1 - \tilde{\xi}_1) \partial_t \tilde{u}_1 w &= -2\beta_3 \tilde{s} \partial_{\nu} \tilde{u}_1 - 2\beta_1 \partial_1 \tilde{\phi}, \\
\partial_t z + (\beta_2 w + z - 2\beta_1 \xi_1) \partial_t \tilde{u}_1 z &= 2\beta_3 \tilde{s} \partial_{\nu} \tilde{u}_1 - 2\beta_1 \partial_1 \tilde{\phi}, \\
\partial_t \tilde{u}_1 + (\beta_1 w + \beta_1 z - 2\beta_1 \xi_1) \partial_t \tilde{u}_1 + 2\beta_1 (\tilde{u}_1 - \tilde{\xi}_1) &= 2\beta_3 \tilde{s} \partial_{\nu} \tilde{u}_1 - 2\beta_1 \partial_1 \tilde{\phi}. \quad (2.8) \end{align*}

**Step 3. Self-Similar Transform**

Self-similar transform is widely used in the study of blowup phenomenon in PDE theory. By zooming in the area near the blowup point corresponding to blowup speed, we can give the precise description for the profile, asymptotic behavior, stability properties.

In this problem, we make the ansatz that the shock accumulation of 3D Euler-Poisson system appears like 3D Burgers equation, so the blowup speed inherits from the shock formation of Burgers equation.

We define the self-similar variables
\[ s = -\log(\tau(t) - t), \quad y_1 = e^{\tilde{z}} x_1, \quad y_\nu = e^{\tilde{z}} x_\nu, \quad \beta_\tau = \frac{1}{1 - \tilde{\tau}} \]
and set
\[ w(x,t) = e^{-\tilde{z}} W(y, s) + \kappa(t), \quad z(x,t) = Z(y, s), \quad \tilde{u}_i(x,t) = U_i(y, s), \]
as well as
\[ \tilde{\sigma}(x,t) = S(y, s), \quad \tilde{\eta}_a(x,t) = R_a(y, s), \quad \tilde{\eta}_a(x,t) = R_a(y, s), \quad \tilde{\zeta}(x,t) = \Omega(y, s), \]
where we assume initially
\[ \tau(-\varepsilon) = 0, \quad \kappa(-\varepsilon) = \kappa_0 > 1. \]

Meanwhile the electric potential $\Phi$ in the self-similar variables $(y, s)$ is
\[ \Phi(y, s) = \int \frac{(R_+ - R_-)(y - z)e^{-\tilde{z}}}{(e^{-3\tilde{z}}|z|^2 + e^{-s}|z|^2)\tilde{z}} dz. \quad (2.9) \]

Now by (2.3) and (2.4), we have
\[ \text{supp} R_+ \subset \mathcal{A}_+ = \{|y_1| \leq e^{\tilde{z}}, \ |y| \leq e^{\tilde{z}}\} \]
\[ \|\partial^\gamma R_+\|_{L^\infty} \leq e^{\frac{5\gamma}{2} - \frac{5}{2}}, \ |\gamma| \leq 4 \]
Therefore, we obtain from (2.3) that
\begin{align*}
(\partial_\tau - \frac{1}{2}) W + (gw + \frac{3}{2}y_1) \partial_{y_1} W + (h\nu + \frac{1}{2} y_\nu) \partial_{y_\nu} W &= FW - \beta_\tau e^{-\tilde{z}} \kappa, \\
\partial_\tau Z + (gz + \frac{3}{2}y_1) \partial_{y_1} Z + (h\nu + \frac{1}{2} y_\nu) \partial_{y_\nu} Z &= FZ, \\
\partial_\tau U_i + (gu + \frac{3}{2}y_1) \partial_{y_1} U_i + (h\nu + \frac{1}{2} y_\nu) \partial_{y_\nu} U_i &= FU_i, \\
\partial_\tau S + (gu + \frac{3}{2}y_1) \partial_{y_1} S + (h\nu + \frac{1}{2} y_\nu) \partial_{y_\nu} S &= FS, \quad (2.12) \end{align*}
where the transport velocity fields are
\[
g_W = \beta_r W + G_W, G_W = \beta_r e^{\frac{\tau}{2}}(\kappa + \beta_2 Z - 2\beta_1 \xi_1),
\]
(2.13a)
\[
g_Z = \beta_2 \beta_r W + G_Z, G_Z = \beta_r e^{\frac{\tau}{2}}(\beta_2 \kappa + Z - 2\beta_1 \xi_1),
\]
(2.13b)
\[
g_U = \beta_1 \beta_r W + G_U, G_U = \beta_r e^{\frac{\tau}{2}}(\beta_1 \kappa + \beta_1 Z - 2\beta_1 \xi_1),
\]
(2.13c)
\[h^\nu = \beta_r e^{\frac{\tau}{2}}(2\beta_1 U_\mu - 2\beta_1 \xi_\mu),
\]
(2.13d)

and the external forces are
\[
F_W = -2\beta_3 \beta_r S(\partial_t U_\nu) - 2\beta_1 \beta_r e^{\frac{\tau}{2}} \partial_1 \Phi,
\]
(2.14a)
\[
F_Z = 2\beta_3 \beta_r e^{\frac{\tau}{2}} S(\partial_t U_\nu) - 2\beta_1 \beta_r e^{\frac{\tau}{2}} \partial_1 \Phi,
\]
(2.14b)
\[
F_{U_\nu} = -2\beta_3 \beta_r e^{\frac{\tau}{2}} S \partial_\nu S - 2\beta_1 \beta_r e^{\frac{\tau}{2}} \partial_\nu \Phi,
\]
(2.14c)
\[
F_{U_1} = -2\beta_3 \beta_r e^{\frac{\tau}{2}} \partial_1 S - 2\beta_1 \beta_r e^{\frac{\tau}{2}} \partial_1 \Phi,
\]
(2.14d)
\[
F_Z = -2\beta_3 \beta_r S(e^{\frac{\tau}{2}} \partial_\nu U_1 + e^{\frac{\tau}{2}} \partial_\nu U_\nu).
\]
(2.14e)

2.2 The Evolution of Higher Order Derivatives

In the study of quasilinear system, it is necessary to consider the evolution of higher order derivatives. Apply \(\partial^\gamma\) to the equations of \((W, Z, U_1)\), with \(|\gamma| \geq 1\), and \(\gamma = (\gamma_1, \gamma_2)\), we get
\[
(\partial_s + \frac{3\gamma_1 + \gamma_2 + \gamma_3 - 1}{2} + \beta_r(1 + \gamma_1 \mathbb{1}_{\gamma_1 \geq 2}) \partial_1 W) \partial^\gamma W + (V_W \cdot \nabla) \partial^\gamma W = F_W^{(\gamma)},
\]
(2.15a)
\[
(\partial_s + \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2} + \beta_2 \beta_r \gamma_1 \partial_1 W) \partial^\gamma Z + (V_Z \cdot \nabla) \partial^\gamma Z = F_Z^{(\gamma)},
\]
(2.15b)
\[
(\partial_s + \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2} + \beta_1 \beta_r \gamma_1 \partial_1 W) \partial^\gamma U_\nu + (V_Z \cdot \nabla) \partial^\gamma U_\nu = F_{U_\nu}^{(\gamma)},
\]
(2.15c)

where
\[
V_W = (g_W + \frac{3}{2} y_1, h^2 + \frac{1}{2} y_2, h^3 + \frac{1}{2} y_3),
\]
(2.16a)
\[
V_Z = (g_Z + \frac{3}{2} y_1, h^2 + \frac{1}{2} y_2, h^3 + \frac{1}{2} y_3),
\]
(2.16b)
\[
V_U = (g_U + \frac{3}{2} y_1, h^2 + \frac{1}{2} y_2, h^3 + \frac{1}{2} y_3).
\]
(2.16c)

And the external forces are given by
\[
F_W^{(\gamma)} = \partial^\gamma F_W - \sum_{0 \leq \beta < \gamma} \left(\gamma_{\beta}\right) \left(\partial^{\gamma - \beta} W \partial_1 \partial^{\beta} W + \partial^{\gamma - \beta} h^\mu \partial_\mu \partial^{\beta} W\right)
\]
\[- \mathbb{1}_{|\gamma| \geq 2} \beta_r \sum_{|\beta| = |\gamma| - 1} \partial^{\gamma - \beta} W \partial_1 \partial^{\beta} W - \mathbb{1}_{|\gamma| \geq 3} \beta_r \sum_{\beta \leq \gamma - 2} \left(\gamma_{\beta}\right) \partial^{\gamma - \beta} W \partial_1 \partial^{\beta} W,
\]
(2.17a)
\[
F_Z^{(\gamma)} = \partial^\gamma F_Z - \sum_{0 \leq \beta < \gamma} \left(\gamma_{\beta}\right) \left(\partial^{\gamma - \beta} Z \partial_1 \partial^{\beta} Z + \partial^{\gamma - \beta} h^\mu \partial_\mu \partial^{\beta} Z\right)
\]
\[- \beta_2 \beta_r \sum_{|\beta| = |\gamma| - 1} \partial^{\gamma - \beta} W \partial_1 \partial^{\beta} Z - \mathbb{1}_{|\gamma| \geq 2} \beta_2 \beta_r \sum_{\beta \leq \gamma - 2} \left(\gamma_{\beta}\right) \partial^{\gamma - \beta} W \partial_1 \partial^{\beta} Z,
\]
(2.17b)
\[
F_{U_\nu}^{(\gamma)} = \partial^\gamma F_{U_\nu} - \sum_{0 \leq \beta < \gamma} \left(\gamma_{\beta}\right) \left(\partial^{\gamma - \beta} U_1 \partial_1 \partial^{\beta} U_1 + \partial^{\gamma - \beta} h^\mu \partial_\mu \partial^{\beta} U_1\right)
\]
\[ -\beta_1\beta_\tau \sum_{|\gamma| = |\gamma| - 1 \gamma_1 = \delta_1} \partial^{\gamma - \delta} W \partial^\delta \partial_1 U_i - \mathbb{1}_{|\gamma| \geq 2} \beta_1\beta_\tau \sum_{\beta \leq \gamma - 2} \left( \frac{\gamma}{\beta} \right) \partial^{\gamma - \beta} W \partial^\beta \partial_1 U_i. \]  

(2.17c)

### 2.3 The Perturbation around the Burgers Profile

In order to describe the asymptotic profile for \( W(s, y) \), a solution of (2.12a), we rely heavily on the the stable self-similar solutions to 3D Burgers equation

\[ -\frac{1}{2} \frac{\partial W}{\partial t} + \left( \frac{3}{2} y_1 + W \right) \partial_1 W + \frac{1}{2} y_\mu \partial_\mu W = 0. \]  

(2.18)

An example of such solutions is

\[ \frac{\partial W}{\partial t} = \left( 1 + |y| \right)^{2} \left( 1 + |y| \right)^{-\frac{1}{2}} y_1, \]  

(2.19)

where \( W^* \) solves 1D Burgers self-similar equation

\[ -\frac{1}{2} W^* + (\frac{3}{2} y + W^*) \partial_y W^* = 0. \]

Moreover, Buckmaster, Shkoller and Vicol in [11] gave the asymptotic estimates of \( W \), by means of the weighed function \( \eta(y) = (1 + |y|^2 + |y|^6) \)

\[ \| \eta^{-\frac{1}{2}} W \|_{L^\infty} \leq 1, \| \eta^{\frac{1}{2}} \partial_1 W \|_{L^\infty} \leq 1, \| \nabla^2 W \|_{L^\infty} \leq 1, \| \eta^{\frac{1}{2}} \nabla W \|_{L^\infty} \leq 1, \| \eta^{\frac{1}{2}} \nabla^2 W \|_{L^\infty} \leq 1. \]  

(2.20)

In fact, there are more solutions to (2.18) than just (2.19). In [6], Christodoulou defined the non-degeneracy condition to classify somehow genuine 3d shock formation in the quasilinear system. In [11], the authors introduced the equivalent notion called "genericity", by checking whether the Hessian matrix \( \partial_1 \nabla^2 W \) is positive definite. The family of stable generic solutions of (2.18) can be indexed by a 3-tensor \( A_\alpha \), where \( \alpha \) is a multi-index with \( |\alpha| = 3 \). In particular, we have

**Proposition 2.1.** ([11]) Let \( A \) be symmetric a 3-tensor such that \( A_{ij} = \mathcal{M}_{jk} \) with \( \mathcal{M} \) is a positive definite symmetric matrix. Then there exists a \( C^\infty \) solution \( \frac{\partial W}{\partial A} \) to

\[ -\frac{1}{2} \frac{\partial W}{\partial A} + \left( \frac{3}{2} y_1 + \frac{\partial W}{\partial A} \right) + \frac{\dot{y}}{2} \cdot \nabla \frac{\partial W}{\partial A} = 0, \]

which satisfies

- \( \frac{\partial W}{\partial A} (0) = 0, \partial_1 \frac{\partial W}{\partial A} (0) = -1, \nabla \frac{\partial W}{\partial A} (0) = 0, \)
- \( \partial^\gamma \frac{\partial W}{\partial A} (0) \) for \( |\gamma| \) even,
- \( \partial^\gamma \frac{\partial W}{\partial A} (0) = A_\alpha \) for \( |\alpha| = 3 \).

In view of the above proposition, (2.19) is indeed a generic solution because

\[ \nabla^2 \partial_1 W = \text{diag}\{6, 2, 2\} > 0. \]

Now we ignore the differences between each \( \frac{\partial W}{\partial A} \) and compute the evolution of perturbation. Let \( \widetilde{W} = W - \frac{\partial W}{\partial A} \), it follows from (2.12a) and (2.18) that

\[ \partial_\tau \widetilde{W} + (\beta_\tau \partial_1 \widetilde{W} - \frac{1}{2}) \widetilde{W} + (\mathcal{V}_W \cdot \nabla) \widetilde{W} = F_W - e^{-\tau} \beta_\tau k + ((\beta_\tau - 1) \mathcal{V}_W - G_W) \partial_1 W - h^\mu \partial_\mu W =: F_{\tau \widetilde{W}}. \]

Furthermore, the higher derivatives satisfies

\[ \left( \partial_\tau + \frac{3\gamma_1 + \gamma_2 + \gamma_3 - 1}{2} + \beta_\tau (\partial_1 \mathcal{V}_W + \gamma_1 \partial_1 W) \right) \partial^{\gamma} \mathcal{V}_W + (\mathcal{V}_W \cdot \nabla) \partial^{\gamma} \mathcal{V}_W = F_{\tau \mathcal{V}_W}^{(\gamma)}. \]  

(2.21)
where $\tilde{F}_W^{(\gamma)}$ is

$$
\tilde{F}_W^{(\gamma)} = \partial^\gamma \tilde{F}_W - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} \left( \partial^{\gamma - \beta} G_W \partial_\beta \partial^\beta \tilde{W} + \partial^{\gamma - \beta \gamma} h^0 \partial_\gamma \partial^\beta \tilde{W} + \beta_1 \partial^{\gamma - \beta} (\partial_1 \tilde{W}) \partial^\beta \tilde{W} \right)
- \beta \tau \sum_{1 \leq |\beta| \leq \gamma} \binom{\gamma}{\beta} \partial^{\gamma - \beta} W \partial_\beta \partial^\beta \tilde{W} - \beta \tau \sum_{|\beta| = \gamma - 1, \beta_1 = \gamma_1} \binom{\gamma}{\beta} \partial^{\gamma - \beta} W \partial_\beta \partial^\beta \tilde{W}.
$$

(2.22)

3 Main Theorem

3.1 Assumptions on the Initial Data

For the modulation variables, we have assumed in physical variables

$$
\kappa_{|r=-\varepsilon} = \kappa_0 > 0, \quad \tau_{|r=-\varepsilon} = 0, \quad \xi_{|r=-\varepsilon} = 0
$$

(3.1)
in Section 2. Next we mainly state the assumptions on the initial data in self-similar variables. Note that (2.1) implies that the initial data for $(\hat{W}, \hat{Z}, \hat{U})$ are supported in the set

$$
\mathcal{A}_0 = \left\{ |y_1| \leq \varepsilon^{-1}, |y| \leq \varepsilon^{-\frac{1}{2}} \right\}.
$$

(3.2)

Then in order to match $\hat{W}$ and $\tilde{W}$ at $y = 0$, we assume

$$
W(0, \log \varepsilon) = 0, \quad \partial_r W(0, -\log \varepsilon) = -1, \quad \tilde{\nabla}W(0, \log \varepsilon) = 0, \quad \nabla^2 W(0, -\log \varepsilon) = 0.
$$

(3.3)

We define $\tilde{W} = W - \tilde{W}$. Then we get

$$
\tilde{W}(y, -\log \varepsilon) = W(y, -\log \varepsilon) - \tilde{W}(y).
$$

(3.4)

Since $\tilde{W}$ does not decay at infinity and $W(y, -\log \varepsilon)$ is supported compactly, $\tilde{W} = W - \tilde{W}$ can only be expected to stay close to zero in a smaller region $\{ |y| \leq L \ll \varepsilon^{-\frac{1}{2}} \}$. Therefore, the assumptions on the initial data and the bootstrap assumptions for $\hat{W}$ in the following should be restricted in the region $\{ |y| \leq L \}$. Furthermore, if $y$ is very close to the origin, $|y| \leq l$, then it is convenient to use Taylor expansion to give the estimates. Therefore, the assumptions on $\hat{W}$ are made in three different pieces:

$$
|y| \leq l, \quad l < |y| \leq L, \quad |y| > L,
$$

where

$$
l = M^{-\frac{1}{4}}, \quad L = \varepsilon^{-\frac{1}{4}}.
$$

(3.5)

We assume the initial data of $W, Z, U$, specific vorticity $\Omega$, and their $\hat{H}^k$ norm as following

i). For $|y| \leq L$, we assume that

$$
\eta^{-\frac{1}{2}}(y) |\tilde{W}(y, -\log \varepsilon)| \leq \varepsilon^{\frac{1}{2}}, \quad \eta^{\frac{1}{2}}(y) |\partial_r \tilde{W}(y, -\log \varepsilon)| \leq \varepsilon^{\frac{1}{2}}, \quad |\tilde{\nabla} \tilde{W}(y, -\log \varepsilon)| \leq \varepsilon^{\frac{1}{2}}.
$$

(3.6a)

(3.6b)

(3.6c)

ii). For $|y| \leq l \ll 1$, we assume

$$
|\partial^\gamma \tilde{W}(y, -\log \varepsilon)| \leq M \varepsilon^{\frac{1}{2}}, \quad |\partial^\gamma \tilde{W}(0, -\log \varepsilon)| \leq M \varepsilon^{\frac{1}{2}}, \quad |\gamma| = 4.
$$

(3.7a)

(3.7b)
iii). Because we have (3.9), (2.20) and \( W = W + \tilde{W} \) when \( |y| \leq L \), so we just make assumptions for \( \partial^n W \) with \( |\gamma| \leq 1 \) in the region \( \{y \geq L\} \cap X_0 \).

\[
\eta^{-\frac{2}{3}}(y)|W(y, -\log \varepsilon)| \leq 1 + \varepsilon^\frac{1}{18},
\]

(3.8a)

\[
\eta^{\frac{2}{3}}(y)|\partial_l W(y, -\log \varepsilon)| \leq 1 + \varepsilon^\frac{1}{18},
\]

(3.8b)

\[
|\nabla W(y, -\log \varepsilon)| \leq 1 + \varepsilon^\frac{1}{18}.
\]

(3.8c)

iv). For the second derivatives of \( W \), for all \( y \in X_0 \), we have

\[
\eta^{\frac{2}{3}}(y)|\partial^n W(y, -\log \varepsilon)| \leq 1, \quad \gamma_1 \geq 1, \quad |\gamma| = 2,
\]

(3.9a)

\[
\eta^{\frac{2}{3}}(y)|\nabla^2 W(y, -\log \varepsilon)| \leq 1.
\]

(3.9b)

v). For \( Z \), we assume

\[
|\partial^n Z(y, -\log \varepsilon)| \leq \begin{cases} \varepsilon^{\frac{2}{3}}, & \gamma_1 \geq 1 \wedge |\gamma| = 1, 2 \\ \varepsilon, & \gamma_1 = 0 \wedge |\gamma| = 0, 1, 2 \end{cases}
\]

(3.10)

vi). For \( U_\nu \), we assume

\[
|\partial^n U_\nu(y, -\log \varepsilon)| \leq \begin{cases} \varepsilon^{\frac{2}{3}}, & \gamma_1 = 1 \wedge |\gamma| = 0 \\ \varepsilon, & \gamma_1 = 0 \wedge |\gamma| = 0, 1, 2 \end{cases}
\]

(3.11)

vii). For the initial specific vorticity, we assume that

\[
\|\Omega_0\| \leq 1.
\]

(3.12)

viii). At last, we assume the homogenous Sobolev norm for initial data of \( W, Z, U_\nu \) and \( R_\nu \) satisfy

\[
\varepsilon\|W(\cdot, -\log \varepsilon)\|_{H^k}^2 + \|Z(\cdot, -\log \varepsilon)\|_{H^k}^2 + \|U_\nu(\cdot, -\log \varepsilon)\|_{H^k}^2 \leq \varepsilon
\]

(3.13)

and

\[
\|R_\nu\|_{H^k}^2 \leq \varepsilon
\]

(3.14)

for all \( k \geq 18 \).

### 3.2 The Statement of Main Theorem

**Theorem 3.1.** (Shock formation in self-similar variables) Let \( \nu = 2, 3, \gamma > 1, \alpha = \frac{\gamma - 1}{\gamma} > 1, \) \( \kappa_0 = \kappa_0(\alpha) > 1 \). We set the initial time is \( s = -\log \varepsilon \) and consider the system (2.12)-(2.12) of \((W, Z, U_\nu)\). Assume the initial data \((W, Z, U_\nu)_{s = -\log \varepsilon}\) satisfy the assumptions (3.9) - (3.11). Then there exist \( M(\alpha, \kappa_0) \gg 1 \) and \( 0 < \varepsilon(\alpha, \kappa_0, M) \ll 1 \) such that the system (2.12)-(2.12) admits a unique solution \((W, Z, U_\nu) \in C([-\log \varepsilon, \infty); H^k)\) defined in \( \mathcal{X}(s) \) (4.14), with \( k \geq 18 \). Furthermore, it holds that

\[
\|W(\cdot, s)\|_{H^k}^2 + \varepsilon\|Z(\cdot, s)\|_{H^k}^2 + \|U_\nu(\cdot, s)\|_{H^k}^2 \leq \lambda^{-k}e^{-s-\log \varepsilon} + (1 - e^{-s-\log \varepsilon})M^{4k}
\]

(3.15)

Besides, \( W \) behave similarly to the stable self-similar profile \( W \) of 3D Burgers. More precisely, if \( W = W - \tilde{W} \), then \( W \) satisfies (1.2)-(1.3), as well as \( \partial^n W(0, s) = 0 \) for \( |\gamma| \geq 2 \). In addition, \( W \) satisfies (1.1)-(1.3). In fact, the limit function \( W_{\mathcal{X}}(y) = \lim_{s \to \infty} W(y, s) \) are uniquely determined by \( \partial^n W(0, s) \) for \( |\alpha| = 3 \).

The size of \( Z \) and \( U_\nu \) stay in \( O(\varepsilon) \) for \( s \geq -\log \varepsilon \). In particular, they satisfy (1.9)-(1.10) and (4.11).
The sound speed $S(y,s)$ satisfies

$$
\|S(-,s) - \frac{K_0}{2}\|_{L^\infty} \leq \varepsilon^\frac{1}{2}.
$$

**Remark 3.2.** Compared with [1], not only $(W,Z,U_\nu)|_{-\log \varepsilon}$ but also $R|_{-\log \varepsilon}$ and $S|_{-\log \varepsilon} = \frac{1}{n} R^n|_{-\log \varepsilon}$ are required to be compactly supported. Because only if electron density is compactly supported, can system still keep the finite speed of propagation, then the electric field effect caused by charges in compact set is under control in this framework.

**Remark 3.3.** Theorem 3.1 holds for all $\gamma > 1$, as long as taking appropriate $\kappa_0$ in (6.4) and $\varepsilon$ sufficiently small. Unlike the cases where the range of $\gamma$ plays an essential role in Euler-Poisson system, Theorem 3.1 suggests that the dynamics of shock formation dominates the pressure and repulsive force due to the electric potential, as long as the gradient of initial data is steep enough.

**Remark 3.4.** In fact the evolution of specific vorticity $\zeta$ has similar form to that in Euler system, since $\nabla \times \nabla \phi = 0$. Combined with the similar bounds in the bootstrap assumption, there holds the same conclusion obtained in [1]. That is there exists $C > 0$, such that $\frac{1}{C}(\Omega(y_0))^{\frac{3}{2}} < \Omega(\Psi_U^0(s),s) < C(\Omega(y_0))^{\frac{3}{2}}$ for all $s \geq -\log \varepsilon$, where $\Psi_U^0(s)$ is the Lagrangian trajectory generated by velocity $\nu_U$.

**Proof of Theorem 3.1.** The condition of support set (4.1) will be proved in Lemma 4.6. (6.4) will be proved in Proposition 4.4. The bounds of $\partial^\gamma W$, $\partial^\gamma \tilde{W}$, $\partial^\gamma Z$ and $\partial^\gamma U_\nu$ will be proved in Section 7.

The proof of the convergence $\lim_{s \to -\infty} W(y,s) = \nabla_\lambda$ is very similar to that in [1]. The only difference here is due to the electric potential. However, it plays an ignorable role in external forces.

### 4 Bootstrap Assumptions

#### 4.1 Assumptions on Derivatives

In this section, we make bootstrap bounds on derivatives of $W, \tilde{W}, Z$ and $U_\nu$, as well as the support. Note that the bootstrap constants we choose in the following differ from order to order, from variable to variable, even vary by several orders of magnitude. There is a profound connection between these constants and the structure of equation. The appropriate choice of these constants is essential and provides technical convenient in the proof.

We should point out that the following three subsections and the external force estimates as their corollary in the next section are parallel, so the cross-reference is safety as long as the bootstrap constant can be narrowed in the argument.

**Proposition 4.1.** (Main bootstrap assumption) We have the following bounds for $\partial^\gamma W$, $\partial^\gamma \tilde{W}$, $\partial^\gamma Z$ and $\partial^\gamma U_\nu$:

1. $W$ **bootstrap**.

   First, we postulate the following derivative estimates of $W$:

   $$
   |\partial^\gamma W| \leq \begin{cases} 
   2\eta^\dag(y), & |\gamma| = 0 \\
   2\eta^{-\dag}(y), & \gamma_1 = 1, |\gamma| = 0 \\
   2, & \gamma_1 = 0, |\gamma| = 1 \\
   2M \eta^{-\dag}(y), & \gamma_1 \geq 1, |\gamma| = 2 \\
   2M \eta^{-\dag}(y), & \gamma_1 = 0, |\gamma| = 2
   \end{cases}
   \quad (4.1)
   $$

   Next, we assume that for $|y| \leq L$, the following bounds hold

   $$
   |\tilde{W}(y,s)| \leq \varepsilon^{\frac{3}{2}} \eta^\dag(y), \quad |y| \leq L
   \quad (4.2a)
   $$
\[ |\partial_1 W(y, s)| \leq \epsilon \frac{\eta}{\sqrt{s}} (y), \quad |y| \leq L \]  
(4.2b)

\[ |\nabla W(y, s)| \leq \epsilon \frac{\eta}{\sqrt{s}}, \quad |y| \leq L. \]  
(4.2c)

Furthermore, for \(|y| \leq l\) we assume that
\[ |\partial^\gamma \tilde{W}| \leq 2 M \epsilon^{\frac{1}{20}} l^{1-|\gamma|}, \quad |\gamma| = 3 \]  
(4.3a)

\[ |\partial^\gamma \tilde{W}| \leq 2 M \epsilon^{\frac{1}{20}}, \quad |\gamma| = 4 \]  
(4.3b)

while for \(y = 0\), we have
\[ |\partial^\gamma \tilde{W}(0, s)| \leq \epsilon \frac{1}{l}, \quad |\gamma| = 3 \]  
(4.4)

ii). \(Z\) bootstrap.

We postulate the following derivative estimates of \(Z\):
\[ |\partial^\gamma Z| \leq M \begin{cases} e^{-\frac{s}{2}}, & \gamma_1 \geq 1, |\gamma| = 1, 2 \\ \epsilon^\frac{s}{2}, & |\gamma| = 0 \\ \epsilon^\frac{s}{2} e^{-\frac{s}{2}}, & \gamma_1 = 0, |\gamma| = 1 \\ e^{-s}, & \gamma_1 = 0, |\gamma| = 2 \end{cases} \]  
(4.5)

iii). \(U\) bootstrap.

We postulate the following derivative estimates of \(U\):
\[ |\partial^\gamma U| \leq M \begin{cases} e^{-\frac{s}{2}}, & \gamma_1 = 1, |\gamma| = 0, \\ \epsilon^\frac{s}{2}, & |\gamma| = 0, |\gamma| = 1 \\ \epsilon^\frac{s}{2} e^{-\frac{s}{2}}, & \gamma_1 = 0, |\gamma| = 1 \\ e^{-s}, & \gamma_1 = 0, |\gamma| = 2 \end{cases} \]  
(4.6)

In fact, from the second inequality of (4.1) and (4.2b), we can get
\[ |\partial_1 W| \leq 1 + \epsilon \frac{1}{\sqrt{s}}. \]  
(4.7)

Indeed, when \(|y| \geq L, 2 \eta^{-\frac{s}{4}} \ll 1\). When \(l \leq |y| \leq L\), by (4.2b) and (4.2b), \(|\partial_1 W| \leq (1 + \epsilon \frac{1}{\sqrt{s}}) \eta^{-\frac{s}{4}} \leq 1 + \epsilon \frac{1}{\sqrt{s}}

The \(L^\infty\) type bounds are not enough to close the bootstrap argument because the higher order derivatives are present in nonlinear terms. So we need to perform \(H^k\) estimates and use Sobolev interpolation to compensate the loss of derivatives.

**Proposition 4.2.** For integer \(k \geq 18\), we have
\[ \|Z\|_{H^k}^2 + \|U\|_{H^k}^2 \leq 2 \lambda^{-k} e^{-s} + e^{-s} (1 - e^{-s} \epsilon^{-1}) M^{4k}, \]  
(4.8a)

\[ \|W\|_{H^k}^2 \leq 2 \lambda^{-k} e^{-s} \epsilon^{-1} + (1 - e^{-s} \epsilon^{-1}) M^{4k}, \]  
(4.8b)

**Proof.** We will prove Proposition 4.2 in Section 7.

**Proposition 1.2** yields the higher order derivatives estimates as follows:

**Corollary 4.3.** There hold that
\[ |\partial^\gamma W(y, s)| \lesssim \begin{cases} e^{\frac{s}{20}} \eta^{-\frac{s}{4}} (y), & \gamma_1 \neq 0, |\gamma| = 3, 4 \\ e^{\frac{s}{20}} \eta^{-\frac{s}{4}} (y), & \gamma_1 = 0, |\gamma| = 3, 4. \end{cases} \]  
(4.9)
\begin{equation}
|\partial^\gamma Z(y,s)| \lesssim \begin{cases} e^{-\left(\frac{\gamma}{\alpha} - \frac{s}{M}\right)s}, & \gamma_1 \geq 1, |\gamma| = 3, \\ e^{-\left(1 - \frac{|\gamma|}{\alpha}\right)s}, & |\gamma| = 3, 4, 5, \end{cases}
\end{equation}

\begin{equation}
|\partial^\gamma U_\nu(y,s)| \lesssim \begin{cases} e^{-\left(\frac{\gamma}{\alpha} - \frac{s}{M}\right)s}, & \gamma_1 \geq 1, |\gamma| = 2, 3, \\ e^{-\left(1 - \frac{|\gamma|}{\alpha}\right)s}, & |\gamma| = 3, 4, 5, \end{cases}
\end{equation}

Proof. The proof is just repeteation of that in [] and we omit the details.

Since \( U_1 = \frac{1}{\alpha}(e^{-\frac{s}{\alpha}W + \kappa} + Z) \), \( S = \frac{1}{\alpha}(e^{-\frac{s}{\alpha}W + \kappa} - Z) \) and \( S = (\alpha R_\nu)^\frac{1}{\alpha} \), the estimates of \( U_1, S \) and \( R_\nu \) can be seen as a corollary of Proposition 4.1.

Corollary 4.4. For \( y \in \mathcal{X}(s) \) we have

\begin{equation}
|\partial^\gamma U_1| + |\partial^\gamma S| \lesssim \begin{cases} 1, & |\gamma| = 0, \\ e^{-\frac{s}{\alpha}y - \frac{1}{\alpha}(y),} & \gamma = (1, 0, 0), \\ e^{-\frac{s}{\alpha}}, & \gamma_1 = 0, |\gamma| = 1, \\ Me^{-\frac{s}{\alpha}y - \frac{1}{\alpha}(y),} & \gamma_1 \geq 1, |\gamma| = 2, \\ Me^{-\frac{s}{\alpha}y - \frac{1}{\alpha}(y),} & \gamma_1 = 0, |\gamma| = 2, \\ e^{-\frac{s}{\alpha} + \frac{1}{\alpha}(y,)} & \gamma_1 \neq 0, |\gamma| = 3, 4, \\ e^{-\frac{s}{\alpha} + \frac{1}{\alpha}(y,)} & \gamma_1 = 0, |\gamma| = 3, 4 \end{cases}
\end{equation}

while for \( |y| \leq 1 \) and \( |\gamma| = 4 \) we have

\begin{equation}
|\partial^\gamma U_1| + |\partial^\gamma S| \lesssim e^{-\frac{s}{\alpha}}.
\end{equation}

Proof. Note that

\begin{equation}
|\partial^\gamma U_1| + |\partial^\gamma S| \lesssim e^{-\frac{s}{\alpha}}|\partial^\gamma W + \kappa 1_{|\gamma|=0}| + |\partial^\gamma Z|
\end{equation}

and

\begin{equation}
|\partial^\gamma R_\nu| \lesssim |S^{\frac{1}{\alpha} - 1}| |\partial^\gamma S| \lesssim M^{\frac{1}{\alpha} - 1} |\partial^\gamma S|.
\end{equation}

Then (4.13) and (4.12) follow from Proposition 4.1.

\section{4.2 The Evolution of Support Set}

We assume that \((W, Z, U_\nu, R_\nu)\) have compact support

\begin{equation}
\mathcal{X}(s) = \left\{ |y_1| \leq 2\varepsilon^{\frac{1}{\alpha}}e^{\frac{s}{\alpha}}, |y| \leq 2\varepsilon^{\frac{1}{\alpha}}e^{\frac{s}{\alpha}} \right\}, \quad s \geq -\log \varepsilon.
\end{equation}

Then if \( y \in \mathcal{X}(s) \), the weighed function \( \eta(y) \) satisfies

\begin{equation}
\eta^{\frac{1}{\alpha}}(y) \leq 4\varepsilon^{\frac{1}{\alpha}}e^{s}.
\end{equation}

Since the evolution of \( n_\nu \) does not involve nonlocal term, \( n_\nu \) is expected to be compactly supported for all time once it is assumed at the initial time.
Lemma 4.5. For any $s \geq -\log \varepsilon$, $S$ is supported in the set $X(s)$, as long as $n_{e,0}$ is supported in $x$.

Proof. From $x = x - \xi(t)$, $\tilde{\sigma}(x, t) = \sigma_e(x, t) = \frac{1}{n_e}(u - \xi)$, we know that if $n_{e,0}$ is supported in $x$, then $\tilde{\sigma}(x, -\varepsilon)$ is supported in $x = \{|x| \leq \varepsilon, |\bar{x}| \leq \varepsilon^{1 \over 2}\}$. Suppose $\varphi(x, t)$ is the Lagrangian trajectories generated by velocity field $(2.15a)$ and $(2.17a)$, then evaluating at $y$ implies that trajectories generated by velocity field $\tilde{\sigma}(x, t)$ and $\varphi(x, t)$ is supported in $x = x_{0} \subset X$, from $(2.7)$

$$\frac{1 + \alpha}{2} \partial_{\tilde{\sigma}} + ((\tilde{u} - \xi) \cdot \nabla)\tilde{\sigma} = -\alpha \tilde{\sigma} \nabla \tilde{\sigma},$$

we get that

$$\tilde{\sigma}(\varphi(x, t), t) = \tilde{\sigma}(x(0), 0) \exp \left(-\frac{2\alpha}{1 + \alpha} \int_{-\varepsilon}^{t} (\nabla \cdot \tilde{u}) \circ \varphi(x, \delta) d\delta \right),$$

which implies that if $\tilde{\sigma}(\varphi(x, t), t) \neq 0$, then $\tilde{\sigma}(x(0), 0) \neq 0$. Namely, for any $x_{0}$ with $\tilde{\sigma}(x_{0}, t) \neq 0$, there exists $x_{0} \in X$, such that $x_{0} = \varphi(x_{0}, t)$. But $(4.22)$ and the bootstrap bounds of $U_{1}$ and $U_{\nu}$ imply that

$$|x_{0} - x_{0}| = |\varphi(x_{0}, t) - \varphi(x_{0}, 0)| < \|\tilde{u} - \xi\|_{L_{\infty}T_{*}} < \varepsilon^{1 \over 2},$$

which means $x_{0} \notin \{|x| \leq 1\} = x$. Thus we reach a contradiction.

The definitions of $X_{+}$ in $(2.10)$ and $X(s)$ in $(4.14)$ imply that $X(s) \subset X_{+}$ for any $s \geq -\log \varepsilon$.

4.3 Analysis of Modulation Variables

In this section we will impose the following constraints on derivatives of $W$ at $y = 0$ by choosing appropriate $\kappa$, $\tau$ and $\xi$. In order to do this, we first make some preparations. Plugging

$$W(0, s) = 0, \partial_{1}W(0, s) = -1, \tilde{\nabla}W(0, s) = 0, \nabla^{2}W(0, s) = 0$$

into $(2.15a)$ and $(2.17a)$, then evaluating at $y = 0$ we get

$$-G_{W}^{0} = F_{W}^{0} - \beta_{r} e^{-\hat{\tau}_{r}}, \quad |\gamma| = 0$$

and

$$F_{W}^{(\gamma)} = \partial_{\tau} F_{W}^{0} + \partial_{\nu} G_{W}^{0}, \quad |\gamma| = 1, 2.$$

By the evolution of $\partial_{1}W$ at $y = 0$, we have

$$-(1 - \beta_{r}) = \partial_{1}F_{W}^{0} + \partial_{1}G_{W}^{0},$$

which implies that

$$\hat{\tau} = \frac{1}{\beta_{r}}(\partial_{1}F_{W}^{0} + \partial_{1}G_{W}^{0}).$$

Next, by the evolution of $\partial_{1}\nabla W$ at $y = 0$, it holds

$$F_{W}^{(2, 0, 0)} = \partial_{11}F_{W} - \partial_{11}G_{W} \partial_{1}W - \partial_{11}h^{\mu} \partial_{\mu}W - 2\partial_{1}G_{W} \partial_{1}W - 2\partial_{1}h^{\nu} \partial_{\nu}W.$$

For $i = 1, 2, 3$, we have

$$G_{W}^{0} \partial_{11}W^{0} + h^{\mu} \partial_{10\mu}W^{0} = \partial_{11}F_{W}^{0} + \partial_{11}G_{W}^{0}.$$

Due to $(4.18)$, we have $\mathcal{H}^{0} = \partial_{1} \nabla^{2}W^{0} = \partial_{1} \tilde{\nabla}^{2}W^{0} + \partial_{1} \nabla^{2}W^{0} = \text{diag}(6, 2) + \varepsilon^{1 \over 2}$, so $\mathcal{H}^{0}$ is invertible and

$$(\mathcal{H}^{0})^{-1} \leq 1.$$
Thus,
\[
|G_W^0| + |h^\mu,^0| \leq (H^0)^{-1}(|\partial_1 \nabla G_W^0| + |\partial_1 \nabla F_W^0|) \\
\leq |\partial_1 \nabla G_W^0| + |\partial_1 \nabla F_W^0|.
\] (4.20)

**Lemma 4.6.** For \( s \geq -\log \varepsilon \), there holds
\[
|G_W^0(s)| + |h^\mu,^0(s)| \leq Me^{-s},
\] (4.21)

**Proof.** By (4.20), it suffices to estimate \( \partial_1 \nabla G_W^0 \) and \( \partial_1 \nabla F_W^0 \). Indeed, (5.1), (4.5) and (5.8) imply
\[
|\partial_1 \nabla G_W^0| \leq e^{s/2} |\partial_1 \nabla Z| \leq Me^{-s}
\] and
\[
|\partial_1 \nabla F_W^0| \leq e^{-s}\eta^{-\frac{1}{12}} + \frac{2\epsilon}{\beta} < e^{-s}.
\]

This lemma plays a vital important role in our argument. On one hand, \( \text{(4.21)} \) holds under the constrains \( \text{(4.16)} \), which are guaranteed by the choosing modulation variables appropriately. On the other hand, \( \text{(4.21)} \) implies the transport effect in the evolution of \( \partial^\tau W \) is ignorable, compared with the linear damping term, as we will see in the following sections.

Next we will perform the bootstrap argument to the modulation variables, by using bootstrap bounds assumed in Section 4.1 and their direct corollary: bounds of external forces, whose proof will be postponed in the next section.

**Proposition 4.7.** Under the bootstrap assumptions, the modulation variables satisfy the estimates
\[
|\dot{\tau}| \leq 2Me^{-s}, \quad |\dot{\kappa}| \leq M, \quad |\dot{\xi}| \leq M\tilde{\tau}, \quad |T_\gamma| \leq 3M\varepsilon^2.
\] (4.22)

Moreover, it holds that
\[
\frac{1}{2}\kappa_0 \leq |\kappa| \leq 2\kappa_0 \leq M, \quad |\xi| \leq M\varepsilon
\] (4.23)

and
\[
|1 - \beta_\tau| = \frac{|\dot{\tau}|}{1 - \tau} \leq 2Me^{-s} \leq 2Me.
\] (4.24)

**Proof.** We estimates \( \dot{\xi} \) first. Due to \( \text{(2.13a)} \) and \( \text{(2.13d)} \), we have
\[
2\beta_1 \dot{\xi}_1 = -\frac{1}{\beta_\tau} G_W^0 e^{-\tilde{\tau}} + \kappa + \beta_2 Z^0
\]
and
\[
2\beta_1 \dot{\xi}_\nu = -\frac{\tilde{\tau}}{\beta_\tau} h^\mu,^0 + 2\beta_1 U_\mu.
\]

Hence by \( \text{(4.21)} \), \( \text{(4.5)} \) and \( \text{(4.6)} \), it follows that
\[
|\dot{\xi}_1| + |\dot{\xi}_\mu| \leq Me^{-\frac{3\epsilon}{2}} + \kappa_0 + M\varepsilon \tilde{\tau} + Me^{-\frac{2\epsilon}{3}} \leq 2\kappa_0 \leq \frac{1}{2} M\tilde{\tau}
\]
by taking \( M \) large enough, which improves the bound in \( \text{(4.22)} \).

Next we turn to the estimate of \( \dot{\kappa} \). By \( \text{(4.12)} \), \( \text{(4.13)} \) and \( \text{(4.6)} \), we have
\[
|F_W^0| \leq 2\beta_3 \beta_\tau |S^0(\partial_\mu U_\mu^0)| + 2\beta_1 \beta_\tau e^s |(\partial_1 \Phi)^0| \\
\leq Me^{\frac{3\epsilon}{2}} e^{-\tilde{\tau}} + e^{-\tilde{\tau}} \\
\leq 2e^{-\tilde{\tau}}.
\]
which, together with (4.17) and (4.21), yields

$$|\dot{\kappa}| \leq \beta \tau e^{\frac{2}{s}}(|F_0^0| + |G_0^0|) \leq (1 + M\varepsilon)(Me^{-\frac{2}{s}} + 2) \leq \frac{1}{2} M,$$

improving the bound in (4.22).

Now we are left for the estimate of $\dot{\tau}$. (4.5), (4.19), (5.1) and (5.8) imply that

$$|\dot{\tau}| \leq |\partial_1 F_0^0| + |\partial_1 G_0^0| \lesssim e^{-s} + Me^{-s} \leq \frac{3}{2} Me^{-s}.$$

Thus the bound in (4.22) is improved.

Now it remains to estimate $T_*$. From the relation $-\log(\tau(t) - t) = s$ and $\tau(T_*) = T_*$, we have

$$\int_{-\varepsilon}^{T_*} (1 - \dot{\tau}(t))dt = \varepsilon,$$

which implies that

$$(T_* + \varepsilon) \leq \varepsilon + \|\dot{\tau}\|_{L^\infty}(T_* + \varepsilon).$$

Therefore, we get from the estimate of $\dot{\tau}$ in (4.22) that

$$T_* \leq 2M\varepsilon T_* + 2M\varepsilon^2.$$

Thus $T_* \leq 2M\varepsilon^2$ as long as $\varepsilon$ is taken small enough, which improves the bound in (4.22).

Finally, we are left with (4.23) and (4.24). Combining initial data (3.1), integration $\dot{\kappa}$ and $\dot{\xi}$ from $-\varepsilon$ to $T_*$ in (4.22) gives (4.23). And (4.24) follows from (4.23) directly.

5 Estimates for Transport Fields and External Forces

In this section, we compute the bounds of transport fields and external forces.

5.1 Transport Term

**Lemma 5.1.** For $s \geq -\log \varepsilon$ and $y \in X(s)$, we have

$$|\partial^\gamma G_W| + |\partial^\gamma G_Z| + |\partial^\gamma G_U| \leq e^{\frac{2}{s}}|\partial^\gamma Z|,$$  \hspace{1cm}  (5.1)

and

$$|\partial^\gamma h^\mu| \leq e^{-\frac{2}{s}}|\partial^\gamma U^\mu|,$$ \hspace{1cm} (5.2)

Meanwhile, when $|\gamma| = 0$, a sharper bound for $G_W$ can be obtained for any $y \in X(s)$

$$|G_W| \lesssim M(e^{-\frac{2}{s}} + |y_1|e^{-s} + e^{\frac{2}{s}}|y|).$$ \hspace{1cm} (5.3)

**Proof.** (5.1) and (5.2) follow from (2.13). It remains to prove (5.3). By mean value theorem and (5.1),

$$|G_W| \lesssim |G_0^0_W| + |y_1|e^{\frac{2}{s}}\|\partial_1 Z\|_{L^\infty} + |\dot{y}|e^{\frac{2}{s}}\|\nabla Z\|_{L^\infty}$$

$$\lesssim M(e^{-s} + |y_1|e^{-s} + e^{\frac{2}{s}}|y|),$$

where in the last inequality we have used (4.21) and (3.5).
5.2 External Forces with Electric Potential

Now we estimate the derivatives of electric potential. In the proof the compact support of $R_e$ and $R_+$ is of vital importance, because the Poisson kernel in 3D is not integrable in whole space.

**Lemma 5.2.** For any $y \in \mathcal{X}(s)$, and $|\gamma| \leq 4$, we have
\begin{equation}
|\partial^n \partial_\gamma \Phi(y)| \lesssim e^{-\frac{7}{2}s} (\|\partial^n R_e\|_{L^\infty} + \|\partial^n R_+\|_{L^\infty}) \tag{5.4}
\end{equation}
and
\begin{equation}
|\partial^n \partial_\nu \Phi(y)| \lesssim e^{-\frac{7}{2}s} (\|\partial^n R_e\|_{L^\infty} + \|\partial^n R_+\|_{L^\infty}). \tag{5.5}
\end{equation}

**Proof.** By (2.9), we have
\begin{equation}
\partial_\nu \Phi = \int_{R^3} -\frac{e^{-\frac{7}{2}s} e^{-s} e^{-s/2} (R_e - R_+)(y - z) dz}{(e^{-3s}|z|^2 + e^{-s}|z|^2)\frac{s}{2}}.
\end{equation}

Combining the bootstrap bounds and the above estimates of electric potential, we can obtain the $L^\infty$ bounds of external forces as follows.

**Lemma 5.3.** For $y \in \mathcal{X}(s)$ we have various bounds on the external forces:
\begin{equation}
|\partial^n F_W| + e^{\frac{7}{2}s}|\partial^n F_Z| \lesssim \begin{cases}
e^{-\frac{7}{2}s}, & |\gamma| = 0 \\
e^{-s} e^{-\frac{7}{2}s} \frac{e^{-s/2}|\gamma|}{(e^{-3s}|\gamma|^2 + e^{-s}|\gamma|^2)\frac{s}{2}}, & \gamma_1 \geq 1, |\gamma| = 1, 2 \\
M e^{-s}, & \gamma_1 = 0, |\gamma| = 1 \\
e^{-(1-\frac{7}{2}s)} e^{-s} \frac{e^{-s/2}|\gamma|}{(e^{-3s}|\gamma|^2 + e^{-s}|\gamma|^2)\frac{s}{2}}, & \gamma_1 = 0, |\gamma| = 2 \\
e^{-(1-\frac{7}{2}s)} e^{-s} \frac{e^{-s/2}|\gamma|}{(e^{-3s}|\gamma|^2 + e^{-s}|\gamma|^2)\frac{s}{2}}, & |\gamma| = 3 \\
e^{-(1-\frac{7}{2}s)} e^{-s} \frac{e^{-s/2}|\gamma|}{(e^{-3s}|\gamma|^2 + e^{-s}|\gamma|^2)\frac{s}{2}}, & |\gamma| = 4
\end{cases} \tag{5.8}
\end{equation}
and
\[ |\partial^\gamma F_{\nu}| \lesssim \begin{cases} e^{-s}, & |\gamma| = 0 \\ M e^{-s} \eta^{-\frac{1}{2}}, & \gamma_1 = 0, \ |\gamma| = 1 \\ e^{-s + \frac{2|\gamma| - 1}{24s}} \eta^{-\frac{1}{2}}, & \gamma_1 = 0, \ |\gamma| = 2 \end{cases} \] (5.9)

Proof. We divide the proof into two steps.

Step 1. Proof of the bound on \( \partial^\gamma F_W \) and \( \partial^\gamma F_Z \).

Recall (2.14a),
\[ F_W = -2\beta_2 \beta_5 S(\partial_\nu U_\nu) - 2\beta_1 \beta_6 e^s \partial_\nu \Phi. \]

Then by (5.3), we have
\[ |\partial^\gamma F_W| \lesssim |\partial^\gamma (S\partial_\nu U_\nu)| + e^s |\partial^\gamma \partial_\nu \Phi| \lesssim |\partial^\gamma (S\partial_\nu U_\nu)| + e^{-\frac{5}{2}} (\|\partial^\gamma R_\nu\|_{L^\infty} + \|\partial^\gamma R_+\|_{L^\infty}). \] (5.10)

When \(|\gamma| = 0\), then by (1.12), (2.11), (4.11) and (4.6), (5.10) yields
\[ |F_W| \lesssim M e^s e^{-\frac{5}{2}} + e^{-\frac{5}{2}} \lesssim e^{-\frac{5}{2}}. \]

When \(|\gamma| \leq 2\) and \(\gamma_1 \geq 1\), using the Leibniz rule, we get
\[ |\partial^\gamma (S\partial_\nu U_\nu)| \lesssim |\partial^\gamma (\partial_\nu U_\nu)| + |\partial^\gamma S\partial_\nu U_\nu|. \]

By (4.12), (4.6) and (4.11), we have
\[ |S\partial^\gamma (\partial_\nu U_\nu)| \lesssim e^{-\frac{5}{2}} e^{\frac{2|\gamma| - 1}{24s}} e^{-s} \lesssim e^{-s} e^{-\frac{5}{2} + \frac{2|\gamma| - 1}{24s}}. \]

Then (4.15) gives
\[ |S\partial^\gamma (\partial_\nu U_\nu)| \lesssim e^{\frac{5}{2}} e^{-\frac{2|\gamma| - 1}{24s}} e^{-s} \lesssim e^{-s} e^{-\frac{5}{2} + \frac{2|\gamma| - 1}{24s}}. \]

Similarly we also have
\[ |\partial^\gamma S\partial_\nu U_\nu| \lesssim M e^{-s} e^{\frac{5}{2}} \eta^{-\frac{1}{2}}. \]

Meanwhile, (4.13) and (2.11) yield
\[ e^{-\frac{5}{2}} (\|\partial^\gamma R_\nu\|_{L^\infty} + \|\partial^\gamma R_+\|_{L^\infty}) \lesssim M e^{-s} \eta^{-\frac{1}{2}} + e^{-2s}. \]

Therefore, from (5.10) we obtain
\[ |\partial^\gamma F_W| \lesssim e^{-s} \eta^{-\frac{1}{2}} e^{\frac{2|\gamma| - 1}{24s}}. \]

Other cases are similar and we just give the sketch of the proof. When \(|\gamma| = |\gamma| = 1\), (4.12), (4.13) and (4.6) give
\[ |\nabla (S\partial_\nu U_\nu)| \lesssim |S\nabla^2 U_\nu| \lesssim M e^{-s}. \]

(4.13) and (2.11) imply that
\[ e^{-\frac{5}{2}} (\|\partial^\gamma R_\nu\|_{L^\infty} + \|\partial^\gamma R_+\|_{L^\infty}) \lesssim e^{-s}. \]

Therefore, by (5.10) we get
\[ |\partial_\nu F_W| \lesssim M e^{-s}. \]

When \(|\gamma| = |\gamma| = 2\), (4.12), (4.13), (4.6) and (4.11) yield
\[ |\nabla^2 (S\partial_\nu U_\nu)| \lesssim |S\nabla^3 U_\nu| \lesssim e^{-(1 - \frac{24s}{2|\gamma| - 1})s}. \]
Meanwhile, \((4.13)\) and \((2.11)\) imply that
\[
e^{-\frac{2}{3}t}(\|\partial^\gamma R_e\|_{L^\infty} + \|\partial^\gamma R_+\|_{L^\infty}) \lesssim M^{\frac{1}{3}} e^{-s\eta^{-\frac{1}{2}}}.
\]

Hence, by \((5.10)\) we get
\[
|\partial^\gamma F_W| \lesssim e^{-(1 - \frac{3}{4}\gamma^2)s}.
\]

Finally, when \(|\gamma| = 3, 4\), we have
\[
|\partial^\gamma F_W| \lesssim e^{-(1 - \frac{3}{4}\gamma^2)s}.
\]

The fact that \(F_W = e^{\frac{2}{5}t}F_Z\), combined with the various bounds on \(F_W\), gives \((5.8)\).

**Step 2.** Proof of the bound on \(\partial^\gamma F_{U_c}\).

Recall \((2.12)\),
\[
F_{U_c} = -2\beta_\alpha\beta_\gamma e^{-\frac{2}{3}t} S \partial_\nu S - 2\beta_\beta_\gamma e^{-\frac{2}{3}t} \partial_\nu \Phi,
\]
then by \((4.12)\) and \((5.5)\), we have
\[
|\partial^\gamma F_{U_c}| \lesssim e^{-\frac{2}{3}t}|\partial^\gamma (S \partial_\nu S)| + e^{-s}(\|\partial^\gamma R_e\|_{L^\infty} + \|\partial^\gamma R_+\|_{L^\infty}).
\]

When \(|\gamma| = 0\), by \((4.12)\), \((4.13)\) and \((4.5)\), we have
\[
|F_{U_c}| \lesssim e^{-s} + e^{-s} \lesssim e^{-s}.
\]

When \(\gamma_1 = 0, |\gamma| = 1\), by \((4.12)\), \((5.5)\) and \((4.13)\), we get
\[
e^{-\frac{2}{3}t}|\nabla (S \partial_\nu S)| \lesssim e^{-\frac{2}{3}t}|\nabla^2 S| \lesssim M e^{-s\eta^{-\frac{1}{2}}}.
\]

Also, \((4.13)\) and \((2.11)\) yield
\[
e^{-s}(\|\partial^\gamma R_e\|_{L^\infty} + \|\partial^\gamma R_+\|_{L^\infty}) \lesssim e^{-\frac{2}{3}s}.
\]

Therefore, we have
\[
|\nabla F_{U_c}| \lesssim M e^{-s\eta^{-\frac{1}{2}}}.
\]

When \(\gamma_1 = 0, |\gamma| = 2\), from \((4.12)\), \((5.5)\) and \((4.13)\) we have
\[
e^{-\frac{2}{3}t}|\nabla^2 (S \partial_\nu S)| \lesssim e^{-\frac{2}{3}t}|\nabla^3 S| \lesssim e^{-s + \frac{4s}{3} + \frac{6s}{3}} e^{-\frac{2}{3}t} e^{-\frac{2}{3}s} \eta^{-\frac{1}{2}}
\]

Meanwhile,
\[
e^{-s}(\|\partial^\gamma R_e\|_{L^\infty} + \|\partial^\gamma R_+\|_{L^\infty}) \lesssim M^{\frac{1}{2}} e^{-\frac{2}{3}s} \eta^{-\frac{1}{2}}.
\]

Thus we get
\[
|\partial^\gamma F_{U_c}| \lesssim e^{-s + \frac{4s}{3} + \frac{6s}{3}} e^{-\frac{2}{3}s} \eta^{-\frac{1}{2}}.
\]

\[\square\]

**Corollary 5.4.** For \(y \in \mathcal{X}(s)\) and \(k \geq 18\), we have
\[
|F_W^{(\gamma)}| \lesssim \begin{cases} 
e^{-\frac{2}{3}t}, & |\gamma| = 0 \\
\eta^{\frac{1}{2} + \frac{4s}{3} + \frac{6s}{3}}(y), & \gamma = (1, 0, 0) \\
M^{\frac{1}{2}} \eta^{-\frac{1}{2}}(y), & \gamma_1 \geq 1, |\gamma| = 2 \\
M^{\frac{1}{2}} \eta^{-\frac{1}{2}}, & \gamma_1 = 0, |\gamma| = 1 \\
\eta^{-\frac{1}{2}}(y), & \gamma_1 = 0, |\gamma| = 2 \\
\end{cases}
\]

(5.11)
We just need to consider the cases $|\gamma| \geq 1$.

**Step 1.** Proof of the bound on $F_W^{(\gamma)}$.

Recall (2.17a),

$$F_W^{(\gamma)} = \partial^\gamma F_W - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} (\partial^{\gamma-\beta} G_W \partial_\beta \partial^3 W + \partial^{\gamma-\beta} h^\mu \partial_\beta \partial^3 W)$$

$$- \mathbb{1}_{|\gamma| \geq 2 \beta} \sum_{|\beta| = |\gamma| - 1} \begin{cases} \binom{\gamma}{\beta} \partial^{\gamma-\beta} W \partial^3 \partial_\beta W \end{cases} \sum_{\beta \leq \gamma - 2} \begin{cases} \binom{\gamma}{\beta} \partial^{\gamma-\beta} W \partial^3 \partial_\beta W. \end{cases}$$

In the following we will utilize the estimates (4.4), (4.5), (5.1), (5.2), (5.8) and (4.15) repeatedly. When $\gamma = (1,0,0)$, we have

$$|F_W^{(1,0,0)}| \lesssim |\partial_1 F_W| + |\partial_1 G_W||\partial_1 W| + |\partial_1 h^\mu||\partial_\mu W|$$

$$\lesssim M e^{-s} \eta^{-\frac{1}{3} + \frac{3|\gamma|}{3|\gamma|}} + M e^{-s} \eta^{-\frac{1}{3} + Me^{-2s}}$$

$$\lesssim M e^{-s} \eta^{-\frac{1}{3} + \frac{3|\gamma|}{3|\gamma|}}.$$

When $\gamma = (2,0,0)$, similarly we have

$$|F_W^{(2,0,0)}| \lesssim |\partial_1 F_W| + |\partial_1 G_W||\partial_1 W| + |\partial_1 h^\mu||\partial_\mu W| + |\partial_1 h^\mu||\partial_\mu W| + |\partial_1 W||\partial_1 W|$$

$$\lesssim e^{-s} \eta^{-\frac{1}{3} + \frac{3|\gamma|}{3|\gamma|}} + M e^{-s} \eta^{-\frac{1}{3} + Me^{-2s}} M \eta^{-\frac{1}{3} + e^{-2s} M \eta^{-\frac{1}{3}}} + M \eta^{-\frac{1}{3}}.$$
\[ \lesssim \eta^{-\frac{1}{2}}. \]

When \( \gamma = (0, 2, 0) \) and \( k \geq 18 \), it holds that

\[
|F^{(0,2,0)}| \lesssim |\nabla^2 F| + |\nabla^2 G||\partial_1 W| + |\nabla^2 W||\partial_1 \nabla Z| + |\nabla^2 h'\eta'| + |\nabla h'| \lesssim |\nabla^2 F| + |\nabla^2 G||\partial_1 W| + |\nabla^2 W||\partial_1 \nabla W|
\]

\[ \lesssim e^{-(s - \frac{20}{3})\eta} + M e^{-\frac{5}{2} s} \eta^{-\frac{1}{2}} + M e^{-\frac{1}{2} s} + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} \]

\[ \lesssim e^{-\frac{1}{2} \eta}. \]

**Step 2.** Proof of the bounds on \( F^{(\gamma)}_Z \).

Recall from (4.44) that

\[
F^{(\gamma)}_Z = \partial^\gamma F\nabla Z - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} \partial^{\gamma-\beta} G_Z \partial_1 \partial^\beta Z + \partial^{\gamma-\beta} h' \partial_\mu \partial^\beta Z
\]

\[ - \beta_2 \beta_\tau \sum_{|\beta| = |\gamma| - 1} \partial^{\gamma-\beta} W \partial^\beta \partial_1 Z - \sum_{|\beta| = |\gamma| - 1} \binom{\gamma}{\beta} \partial^{\gamma-\beta} W \partial^\beta \partial_1 Z. \]

We keep utilizing the estimates (4.41), (4.45), (5.1), (5.2) and (5.8), repeatedly.

When \( \gamma = (1, 0, 0) \), we have

\[
|F^{(1,0,0)}_Z| \lesssim |\partial_1 F| + |\partial_1 G_Z||\partial_1 Z| + |\partial_1 h'\eta'| + |\partial_1 W||\partial_1 Z|
\]

\[ \lesssim e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} + M e^{-\frac{5}{2} s} + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} \]

\[ \lesssim e^{-\frac{1}{2} s}. \]

When \( |\gamma| = 2 \), it holds that

\[
|F^{(\gamma)}_Z| \lesssim |\partial_1 \nabla F| + |\partial_1 \nabla G_Z||\partial_1 Z| + |\partial_1 G_Z||\partial_1 \nabla Z| + |\partial_1 \nabla Z||\partial_1 W|
\]

\[ + |\partial_1 h'\eta'| + |\partial_1 W||\partial_1 Z| + |\partial_1 h'| + |\partial_1 \nabla h'| + |\partial_1 W||\partial_1 Z|
\]

\[ \lesssim e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} + M e^{-\frac{5}{2} s} + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} \]

\[ + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} \]

\[ \lesssim e^{-\frac{1}{2} s}. \]

When \( \gamma = (0, 1, 0) \), we have

\[
|F^{(0,1,0)}_Z| \lesssim |\nabla F| + |\nabla G_Z||\partial_1 Z| + |\partial_1 G_Z||\partial_1 W| + |\partial_1 \nabla W||\partial_1 Z|
\]

\[ \lesssim M e^{-\frac{1}{2} s} + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} + M e^{-\frac{1}{2} s} \]

\[ \lesssim e^{-\frac{1}{2} s}. \]

When \( \gamma = (0, 0, 0) \), we get

\[
|F^{(0,0,0)}_Z| \lesssim |\nabla^2 F| + |\nabla^2 G_Z||\partial_1 Z| + |\nabla^2 G_Z||\partial_1 \nabla Z| + |\nabla^2 h'\eta'| + |\nabla^2 \nabla h'| + |\nabla^2 W||\partial_1 \nabla Z| + |\nabla^2 W||\partial_1 Z|
\]

\[ \lesssim e^{-\frac{1}{2} s} + M e^{-\frac{1}{2} s} + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} + M e^{-\frac{1}{2} s} \eta^{-\frac{1}{2}} \]

\[ \lesssim e^{-\frac{1}{2} s}. \]

**Step 3.** Proof of the bounds on \( F^{(\gamma)}_{\nu} \).
The proof of bounds for $F_{U^\gamma}$ are very similar to those of $F_{W^\gamma}$ and $F_{Z^\gamma}$, and we left it to readers.

**Lemma 5.5.** For $\partial^\gamma \tilde{F}_W$ and $|y| \leq L$, we have

$$|\partial^\gamma \tilde{F}_W| \lesssim \varepsilon^{\frac{1}{10}} \begin{cases} \eta^{-\frac{1}{2}}(y), & |\gamma| = 0 \\ e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}} + \frac{3(\beta - 1)}{4}\varepsilon(\beta - 1)}(y), & \gamma = (1, 0, 0) \\ \eta^{-\frac{1}{2}}(y), & \gamma_1 = 0, \ |\gamma| = 1 \\ 1 & |\gamma| = 4, \ |y| \leq 1 \end{cases}$$

(5.14)

Furthermore, for $y = 0$ and $|\gamma| = 3$ we have

$$|\langle \partial^\gamma \tilde{F}_W \rangle^0| \lesssim e^{-(\frac{1}{2} - \frac{3}{10}\gamma)\varepsilon}.$$  

(5.15)

**Proof.** Due to (5.6), we have

$$|G_{W^\gamma}| \lesssim M(e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + L e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}}) \leq \varepsilon^{\frac{1}{5}}.$$  

(5.16)

for $|y| \leq L$. Recall from (2.22), we have

$$\tilde{F}_W = F_W - e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + ((\beta - 1)W - G_W) \partial_1 W - h^\mu \partial_\mu W.$$  

When $|\gamma| = 0$, by (1.24) and (2.20) we get

$$|\tilde{F}_W| \lesssim |F_W| + M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + |((\beta - 1)W + G_W)| |\partial_1 W| + |h^\mu| |\partial_\mu W|

\lesssim e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + (M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + e^{\varepsilon^\gamma \eta^{-\frac{1}{2}}})\eta^{-\frac{1}{2}} + M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}} + M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}}}.$$  

(5.17)

When $\gamma = (1, 0, 0)$, (1.24) gives

$$|\partial_1 \tilde{F}_W| \lesssim |\partial_1 F_W| + (\beta - 1)|\partial_1 W|^2 + |\partial_1 G_W||\partial_1 W| + |(\beta - 1)W + G_W||\partial_1 W| + |h^\mu| |\partial_\mu W|

\lesssim e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + (M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + e^{\varepsilon^\gamma \eta^{-\frac{1}{2}}})\eta^{-\frac{1}{2}} + M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}} + M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}}}.$$  

(5.18)

When $\gamma = (0, 1, 0)$, it holds from (2.20) and (5.1) that

$$|\partial_\beta \tilde{F}_W| \lesssim |\partial_\beta F_W| + (\beta - 1)|\partial_\beta W||\partial_1 W| + |\partial_\beta G_W||\partial_1 W| + |(\beta - 1)W + G_W||\partial_1 W| + |h^\mu| |\partial_\mu W|

\lesssim M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + (M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + e^{\varepsilon^\gamma \eta^{-\frac{1}{2}}})\eta^{-\frac{1}{2}} + M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}} + M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}}}.$$  

(5.19)

When $|\gamma| = 4$, it holds for $|\beta| \leq 4$ that

$$|\partial^\beta \partial_1 (W)^2| \lesssim 1, \quad |\partial^\beta \partial_1 W| \lesssim \eta^{-\frac{1}{2}}, \quad |\partial^\beta \partial_\mu W| \lesssim 1.$$  

Hence, from (2.20), (5.1) and (5.2) we get

$$|\nabla^4 \tilde{F}_W| \lesssim |\nabla^4 F_W| + |\nabla^4 ((\beta - 1)W - G_W)\partial_1 W)| + |\nabla^4 (h^\mu \partial_\mu W)|

\lesssim e^{-(1 - \frac{3}{10}\gamma)\varepsilon} + M e^{-\varepsilon^\gamma \eta^{-\frac{1}{2}}} + \sum_{\beta \leq \gamma} |\partial^{\gamma - \beta} G_W||\partial_\beta \partial_1 W| + \sum_{\beta \leq \gamma} |\partial^{\gamma - \beta} h^\mu||\partial_\beta \partial_\mu W|

\lesssim e^{\frac{1}{10}} + (e^{\varepsilon^\gamma \eta^{-\frac{1}{2}}} + M e^{\varepsilon^\gamma \eta^{-\frac{1}{2}}})\eta^{-\frac{1}{2}} + M e^{\varepsilon^\gamma \eta^{-\frac{1}{2}}}.  

(5.20)
When $|\gamma| = 3$, we note that the even order derivatives of $W$ and $\nabla W$ vanish at $y = 0$, as well as $\partial_1 W(0) = -1$. Therefore, from the expression of $\tilde{F}_W$, according to Leibniz rule, combined with (5.8), (2.20), (5.1) and (5.2) we have

$$\lesssim e^{\frac{s}{3}}.$$ 

Moreover, for fixed $|\gamma| > \delta > 0$ and $M$ large enough it holds that

$$\int_{s_0}^{s} (1 + \Psi_{W}(s'))^{-\delta} ds' \lesssim \delta \log \frac{1}{l} \quad l \leq |y_0| \leq L$$

(6.2a)

$$\int_{s_0}^{s} (1 + \Psi_{W}(s'))^{-\delta} ds' \lesssim \frac{5}{23} L^{-2\delta} = \frac{5}{23} e^{4} \quad |y_0| \geq L$$

(6.2b)

**Proof.** The proof of (6.1) is similar to that in [1]. It suffices to show

$$y \cdot \nabla W(y) \geq \frac{1}{5} |y|^2, \quad |y| \geq l,$$

(6.3)

which follows from (1.1), (4.2a) and (5.3).

---

**Corollary 5.6.** For $\tilde{F}_{W}^{(\gamma)}$ and $|y| \leq L$ we have

$$|\tilde{F}_{W}^{(\gamma)}| \lesssim e^{\frac{s}{3}} \begin{cases} 
\eta^{-\frac{1}{2} + \frac{3-|\gamma|}{2|\gamma|}}(y) & |\gamma| = 1, 0, 0, |y| \leq L \\
\eta^{-\frac{1}{2}}(y) & \gamma_1 = 0, |\gamma| = 1, |y| \leq L \\
1 & |\gamma| = 4, |y| \leq l 
\end{cases}$$ \quad (5.17)

Furthermore, we have

$$|\tilde{F}_{W}^{(\gamma)}|^0 \lesssim e^{-(\frac{1}{2} + \frac{3-|\gamma|}{2|\gamma|})s} \quad |\gamma| = 3$$ \quad (5.18)

The proof is similar to that of Lemma 5.5 and we omit it.

---

**6 Proof of Proposition 4.1**

**6.1 Velocity Fields in Terms of Lagrangian Trajectories**

We define $\Psi_{W}^{0}$ to be the Lagrangian flow associated with the velocity field $V_W$:

$$\begin{cases}
\partial_s \Psi_{W}^{0}(s) = V_{W}(\Psi_{W}^{0}(y, s), s), \\
\Psi_{W}^{0}(s_0) = y_0.
\end{cases}$$

We also define $\Psi_{U}^{0}$ and $\Psi_{U}^{0}$ similarly. In order to close the bootstrap argument, we need to control the integration $\int \eta^{-\delta}$ for $\Psi$. As in [1], it’s necessary to analyze the lowest speed of velocity fields $(\tilde{V}_W, \nabla Z, \tilde{V}_U)$ escaping from the origin.

**Lemma 6.1.** Let $|y_0| \geq 1$, $s \geq - \log \varepsilon$, then the Lagrangian flow $\Psi_{W}^{0}$ escapes from the origin at an exponential speed, that is

$$|\Psi_{W}^{0}(s)| \geq |y_0| e^{\frac{-|y_0|^2}{5}}.$$ \quad (6.1)

Moreover, for fixed $\delta > 0$ and $M$ large enough it holds that

$$\int_{s_0}^{s} (1 + \Psi_{W}^{0}(s'))^{-\delta} ds' \lesssim \delta \log \frac{1}{l} \quad l \leq |y_0| \leq L$$

(6.2a)

$$\int_{s_0}^{s} (1 + \Psi_{W}^{0}(s'))^{-\delta} ds' \lesssim \frac{5}{23} L^{-2\delta} = \frac{5}{23} e^{4} \quad |y_0| \geq L$$

(6.2b)

**Proof.** The proof of (6.1) is similar to that in [1]. It suffices to show

$$y \cdot \nabla W(y) \geq \frac{1}{5} |y|^2, \quad |y| \geq l,$$

(6.3)

which follows from (1.1), (4.2a) and (5.3).
For (6.2a), since $2\eta(y) \geq (1 + |y|^2)$, (6.3) implies that $le^\frac{\tau}{s} = p,$
\[
\int_{s_0}^s (1 + \Psi_W(s'))^{-\delta} ds' \leq \int_{s_0}^\infty (1 + t^2 e^{\frac{\tau}{s}}(s'-s_0))^{-\delta} ds'.
\]
Write $le^\frac{\tau}{s} = p$, then we have
\[
\int_{s_0}^s (1 + \Psi_W(s'))^{-\delta} ds' \leq \int_1^\infty \frac{5dp}{(1 + p^2)^p} + \int_1^\infty \frac{5dp}{(1 + p^2)^p} \leq 5 \log \frac{1}{l} + \frac{5}{2\delta} \leq 6 \log \frac{1}{l},
\]
where we take $M$ large enough in the last inequality and note that $l = M^{-\frac{1}{2\delta}}$.

The bound (6.2b) can be established similarly. In fact, from (6.1) and $\eta$ we get
\[
\int_{s_0}^\infty (1 + L^2 e^{\frac{\tau}{s}}(s'-s_0))^{-\delta} ds' \leq \frac{5}{2\delta} L^{-2\delta} = \frac{5}{2\delta} e^{\delta^2}.
\]

\begin{lemma}
Let $0 \leq \sigma_1 < \frac{1}{2}$, $2\sigma_1 < \sigma_2$. Let $\Psi(s) = (\Psi_1(s), \Psi_2(s))$ denote either $\Psi_W(s)$ or $\Psi_U(s)$.
Suppose
\[
\kappa_0 \geq \frac{3}{1 - \max\{\beta_1, \beta_2\}},
\]
then for any $y_0 \in X_0$
\[
\int_{-\log \varepsilon}^\infty e^{\sigma_1 s'} (1 + |\Psi_1(s')|)^{-\sigma_2} ds' \leq C_{\sigma_1, \sigma_2}.
\]
In particular,
\[
\sup_{y_0 \in X_0} \int_{-\log \varepsilon}^s |\partial_1 W| \circ \Psi_W(s') ds' \lesssim 1.
\]

\begin{proof}
We want to show if $\Psi(s) = \Psi_W(s)$ or $\Psi_U(s)$, then
\[
\frac{d}{ds} \Psi_1(s) \leq -\frac{1}{2} e^{\frac{\tau}{\sigma_1}} \text{ if } \Psi_1 \leq e^{\frac{\tau}{\sigma_1}}, \ s \geq -\log \varepsilon.
\]
We just verify (6.7) for $\Psi(s) = \Psi_W(s)$ and the other case is similar. By definition, it holds that
\[
\frac{d}{ds} \Psi_1 = \frac{3}{2} \Psi_1 + \beta_2 \beta_1 W \circ \Psi + G_U \circ \Psi.
\]
Since $\beta_2 \leq 1$, by (4.1) and taking $\varepsilon$ sufficiently small, under the assumption $\Psi_1(s) \leq e^{\frac{\tau}{\sigma_1}},$ we have
\[
\frac{d}{ds} \Psi_1 \leq \frac{3}{2} e^{\frac{\tau}{\sigma_1}} + 2\eta W(\Psi) - (1 - \beta_2)\kappa_0 e^{\frac{\tau}{\sigma_1}} + \varepsilon e^{\frac{\tau}{\sigma_1}}
\leq \frac{3}{2} e^{\frac{\tau}{\sigma_1}} - (1 - \beta_2)\kappa_0 e^{\frac{\tau}{\sigma_1}} + \varepsilon e^{\frac{\tau}{\sigma_1}}
\leq -\frac{1}{2} e^{\frac{\tau}{\sigma_1}},
\]
where the last inequality we have used $1 - \beta_2 > 0$ and (6.4).

Next, the following proof is similar to [11], which we just sketch it.
Now we claim that
\[
|\Psi_{\beta}(s)| \geq \min\{||e^\frac{x}{2} - e^\frac{y}{2}|, e^\frac{x}{2}\}
\] (6.8)
for all \(y_1\) starting from \(|y_1| \leq \varepsilon^{-1}\).

Then to prove (6.5), we first note that since \(\int_{-\log \varepsilon}^{\infty} e^{(\sigma_1 - \frac{T}{2})s'} ds' \leq 1\), it suffice to prove
\[
\int_{-\log \varepsilon}^{\infty} e^{\sigma_1 s'} (1 + |e^x - e^y|)^{-\sigma_2} ds' \leq C,
\]
which can be obtained by change of variable \(r = e^{\frac{x}{2}}\).

\[\square\]

### 6.2 \(L^\infty\) Estimates for Specific Vorticity \(\Omega\)

**Lemma 6.3.** The sound speed for \(\Omega(y,s)\)
\[
\|S(\cdot, s) - \frac{\kappa_0}{2}\|_{L^\infty(y)} \leq \varepsilon^\frac{1}{2}
\] (6.9)
for all \(s \geq -\log \varepsilon\).

**Proof.** Since \(S = \frac{1}{2}(e^{-\frac{x}{2}} W + \kappa - Z)\) and \(y \in X(s)\), \(1.1\) and \(1.5\) yield
\[
|S - \frac{\kappa_0}{2}| \leq \frac{\kappa - \kappa_0}{2} + \frac{e^{-\frac{x}{2}}}{2} |W| + |Z| \\leq \frac{1}{2} \int_{-\varepsilon}^{\infty} |\kappa| + 2e^{\frac{x}{2}} \eta^{-\frac{1}{2}} \eta + \frac{1}{2} M e^\frac{x}{2}
\]
\[
\leq \varepsilon^\frac{1}{2},
\]
where in the last inequality we have utilized \(1.22\), \(1.14\) and \(T \leq 3M\varepsilon^2\).

\[\square\]

**Lemma 6.4.** For specific vorticity \(\tilde{\zeta}(x,t) = \Omega(y,s)\), we have
\[
\|\tilde{\zeta}(\cdot, t)\|_{L^\infty} = \|\Omega(\cdot, s)\|_{L^\infty} \leq 2
\] (6.10)

**Proof.** From (2.13), \(x = \xi(t)\), \(t = \frac{1 + \alpha}{2} t\), \(\tilde{\zeta}(x,t) = \zeta(x,t)\) we have
\[
\frac{1 + \alpha}{2} \partial_\xi \tilde{\zeta} + ((\vec{u} - \xi) \cdot \nabla) \tilde{\zeta} - (\zeta \cdot \nabla) \vec{u} = 0.
\] (6.11)

We estimate \(\tilde{\xi}_1\) first. \(4.6\) gives
\[
|\tilde{n}_c \tilde{\xi}_1| = |(\nabla \times \vec{u})_1| = |\partial_{x_3} \vec{u}_1 - \partial_{x_2} \vec{u}_2| \leq M \varepsilon^\frac{1}{4}.
\]
Meanwhile, \(\tilde{n}_c = (x_0 \tilde{x})\frac{1}{2}\) and \(1.9\), we have \(|n_c| \geq \kappa_0^\frac{1}{2}\). Therefore, we have
\[
|\tilde{\xi}_1| \leq \varepsilon^\frac{1}{4}.
\] (6.12)

In order to estimate \(\tilde{\xi}_\nu\), we rewrite the equation \(6.11\),
\[
\partial_t \tilde{\xi}_\nu + 2\beta_1 ((\vec{u} - \zeta) \cdot \nabla) \tilde{\xi}_\nu = 2\beta_1 (\tilde{\xi}_1 \partial_{x_3} \vec{u}_\nu + \tilde{\xi}_2 \partial_{x_2} \vec{u}_\nu + \tilde{\xi}_3 \partial_{x_1} \vec{u}_\nu).
\] (6.13)

Now for \(x_0 \in X\) we define the trajectory \(\psi^{x_0}\) by \(\partial_t \psi^{x_0} = 2\beta_1 (\vec{u} - \zeta) \circ \psi^{x_0}\) for \(t > -\varepsilon\) and \(\psi^{x_0}(-\varepsilon) = x_0\). Let \(Q_j = \tilde{\zeta}_j \circ \psi^{x_0}\), then \(6.13\) becomes
\[
\partial_t Q_j = 2\beta_1 (\partial_{x_3} \vec{u}_j \circ \psi^{x_0}) Q_j.
\]
Since (4.10) and (6.12) lead to $|2\beta_1(\partial_\gamma\tilde{\Psi}) \phi^{\text{ex}}| + |2\beta_1\partial_{\gamma}^n\tilde{\Psi}| \leq M \varepsilon^\frac{1}{2}$, we have
\[
\frac{1}{2}\frac{d}{dt}(Q_2 + Q_3) \leq M \varepsilon^\frac{1}{2}(Q_2 + Q_3) + M \varepsilon^\frac{1}{4}.
\]
Integration from $-\varepsilon$ to $t \in (-\varepsilon, T_*)$, by (6.6) and Grönewall inequality, we have
\[
Q_2^2 + Q_3^2 \leq \frac{3}{2},
\]
and (6.10) follows.

### 6.3 Closure of $L^\infty$ Type Bootstrap on $Z$

**Lemma 6.5.** Let $\Psi = \Psi_Z^{y_0}$, then we have
\[
|\tilde{\Psi}(s)| \lesssim \varepsilon^\frac{1}{2},
\]
\[
e^{-\mu}\partial_\gamma Z \circ \tilde{\Psi}(s) \lesssim 1, \quad \gamma_1 \geq 1, |\gamma| = 1, 2
\]
\[
e^{-\mu}\nabla Z \circ \tilde{\Psi}(s) \lesssim \varepsilon^\frac{1}{2},
\]
\[
e^{-\mu}\nabla^2 Z \circ \tilde{\Psi}(s) \lesssim 1.
\]

**Proof.** For $\mu \leq \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2}$, multiplying $e^{\mu s}$ on both sides of (2.15b), we get
\[
\partial_s(e^{\mu s}\partial_\gamma Z) + D_Z^{(\gamma,\mu)}(e^{\mu s}\partial_\gamma Z) + (V_Z \cdot \nabla)(e^{\mu s}\partial_\gamma Z) = e^{\mu s}P^{(\gamma)}_Z,
\]
where
\[
D_Z^{(\gamma,\mu)} = -\mu + \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2} + \beta_2\beta_1\gamma_1 \partial_1 W.
\]
Composing the Lagrangian trajectories $\Psi_Z$, by Grönewall inequality, we have
\[
e^{-\mu}\partial_\gamma Z \circ \Psi_Z^{y_0}(s) \leq e^{-\mu}\partial_\gamma Z(y_0, -\log \varepsilon) + \int_{-\log \varepsilon}^s e^{\mu s'}|P^{(\gamma)}_Z(s')| \exp \left(-\int_{-\log \varepsilon}^{s'} D_Z^{(\gamma,\mu)} \circ \Psi_Z^{y_0}(s'')ds''\right)
\]
\[
+ \int_{-\log \varepsilon}^s e^{\mu s'}|P^{(\gamma)}_Z(s')| \exp \left(-\int_{s'}^s D_Z^{(\gamma,\mu)} \circ \Psi_Z^{y_0}(s'')ds''\right)\]
\[
\lesssim e^{-\mu}\partial_\gamma Z(y_0, -\log \varepsilon) + \int_{-\log \varepsilon}^s D_Z^{(\gamma,\mu)} \circ \Psi_Z^{y_0}(s')ds' + \mu + \int_{-\log \varepsilon}^s |\partial_1 W|ds,
\]
so by (6.11) and $\mu \leq \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2}$, we have
\[
\exp \left(-\int_{-\log \varepsilon}^s D_Z^{(\gamma,\mu)} \circ \Psi_Z^{y_0}(s')ds'\right) \lesssim \exp \left(\int_{-\log \varepsilon}^s \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2} + \mu\right) \lesssim 1.
\]
Combining (6.15) and (6.16), we get
\[
e^{-\mu}\partial_\gamma Z \circ \Psi_Z^{y_0}(s) \leq e^{-\mu}\partial_\gamma Z(y_0, -\log \varepsilon) + \int_{-\log \varepsilon}^s e^{\mu s'}|P^{(\gamma)}_Z(s')|.
\]

Next we plug initial data (4.3) and external forces bounds (5.12) into (6.17) for different $\gamma$.  

25
When \( |\gamma| = 0 \), we let \( \mu = 0 \), and then (6.17) becomes
\[
|Z \circ \Psi_{Z}^{\infty}(s)| \lesssim \varepsilon + \int_{-\log \varepsilon}^{s} e^{-s'} ds' \lesssim 2\varepsilon \lesssim \varepsilon^{\frac{1}{2}}.
\]
When \( \gamma = (1, 0, 0) \), we let \( \mu = \frac{3}{2} \). Then (6.17) yields
\[
|\partial Z \circ \Psi^{\infty}_{Z}(s)| \lesssim e^{-\frac{3}{2} s} |\partial Z(y_{0}, - \log \varepsilon)| + \int_{-\log \varepsilon}^{s} e^{\frac{3}{2} s'} e^{-\frac{3}{2} s'} e^{-\frac{3}{2} (s-s')} ds' \lesssim 1 + \int_{-\log \varepsilon}^{s} (1 + |\Psi_{1}(s')|^{2}) e^{-\frac{3}{2} (s-s')} ds' \lesssim 1,
\]
where in the last inequality we have used (6.5) for \( \sigma_{1} = 0, \sigma_{2} = \frac{1}{\sqrt{2(\gamma-1)}} \).

When \( \gamma_{1} \geq 1, |\gamma| = 2 \), we let \( \mu = \frac{3}{2} \). Then
\[
\mu - \frac{3\gamma_{1} + \gamma_{2} + \gamma_{3}}{2} \leq -\frac{1}{2}
\]
So we have
\[
e^{-\frac{3}{2} s} |\partial Z \circ \Psi^{\infty}_{Z}| \lesssim e^{-\frac{3}{2} s} |\partial Z(y_{0}, - \log \beta)| + \int_{-\log \varepsilon}^{s} e^{-\frac{3}{2} s'} e^{-\frac{3}{2} s'} e^{-\frac{3}{2} (s-s')} ds' \lesssim 1.
\]
At last when \( |\gamma| = |\gamma| = 1 \) or 2, after setting \( \mu = \frac{3}{2} \) and 1, we have
\[
e^{-\frac{3}{2} s} |\nabla Z \circ \Psi| \lesssim e^{-\frac{3}{2} s} |\nabla Z(y_{0}, - \log \varepsilon)| + M^{2} \int_{-\log \varepsilon}^{s} e^{-s'} ds' \lesssim \varepsilon^{\frac{1}{2}} + M^{2} \varepsilon \lesssim \varepsilon^{\frac{1}{2}}
\]
and
\[
e^{-\frac{3}{2} s} |\nabla^{2} Z \circ \Psi| \lesssim e^{-\frac{3}{2} s} |\nabla^{2} Z(y_{0}, - \log \varepsilon)| + M \int_{-\log \varepsilon}^{s} e^{-\frac{3}{2} s} + \frac{3}{2} s' ds' \lesssim 1,
\]
for \( k \geq 18 \). \( \square \)

### 6.4 Closure of \( L^{\infty} \) Type Bootstrap on \( U_{\nu} \)

**Lemma 6.6**. For \( \Psi = \Psi_{U}^{\infty} \), we have
\[
|U_{\nu} \circ \Psi(s)| \lesssim \varepsilon^{\frac{1}{2}}, \quad (6.18a)
\]
\[
\varepsilon^{\frac{3}{2}} |\nabla U_{\nu} \circ \Psi(s)| \lesssim \varepsilon^{\frac{1}{2}}, \quad (6.18b)
\]
\[
\varepsilon^{k} |\nabla^{2} U_{\nu} \circ \Psi(s)| \lesssim 1. \quad (6.18c)
\]


Besides, it holds that
\[
\varepsilon^{\frac{3}{2}} |\partial_{1} U_{\nu}| \lesssim 1. \quad (6.19)
\]

**Proof**. The proof of (6.18) is very similar to Lemma 6.4 by using (3.11), (4.6) and (5.9) so we omit it.

For estimating \( \partial_{1} U_{\nu} \), we use the specific vorticity, \( \nabla \times u = \rho \zeta = (\alpha \sigma) W \zeta \). Rewrite it to self-similar variables we get
\[
|e^{\frac{3}{2} s} \partial_{1} U_{2} - e^{\frac{3}{2} s} \partial_{2} U_{1}| \lesssim |(\alpha \sigma) W \zeta|.
\]
Then by (4.12) and (5.10) we have
\[
\varepsilon^{\frac{3}{2}} |\partial_{1} U_{2}| \lesssim \kappa_{0}^{\frac{1}{2}} + 1 \lesssim 1.
\]
And it is similar to establish $|\partial_t U_3| \lesssim e^{-\frac{1}{2}t}$.

\[ \square \]

6.5 Closure of $L^\infty$ Type Bootstrap on $\tilde{W}$

Lemma 6.7. We have

\begin{align}
&|\partial^\gamma \tilde{W}| \leq \frac{1}{2} M \varepsilon \frac{1}{3} l^{1-|\gamma|} & |\gamma| \leq 3, \ |y| \leq l \\
&|\partial^\gamma \tilde{W}| \leq \frac{1}{2} M \varepsilon \frac{1}{3} & |\gamma| = 4, \ |y| \leq l
\end{align}

(6.20a)

(6.20b)

Proof. Case 1. $|\gamma| = 4, \ |y| \leq l$

First recall (2.21)

\begin{equation}
\left( \partial_s + \frac{3\gamma_1 + \gamma_2 + \gamma_3 - 1}{2} + \beta_t (\partial_t \tilde{W} + \gamma_1 \partial_1 W) \right) \partial^\gamma \tilde{W} + (V_W \cdot \nabla) \partial^\gamma \tilde{W} = \tilde{F}^{(\gamma)},
\end{equation}

(6.21)

In this case, the damping term are bounded from below

\begin{align}
D^{(\gamma)}_{\tilde{W}} &= \frac{3\gamma_1 + \gamma_2 + \gamma_3 - 1}{2} + \beta_t (\partial_t \tilde{W} + \gamma_1 \partial_1 W) \\
&= \frac{3}{2} + \gamma_1 + \beta_t (\partial_t \tilde{W} + \gamma_1 \partial_1 W) \\
&\geq \frac{3}{2} + \gamma_1 - (1 + 2M^2\varepsilon)(1 + \gamma_1 (1 + \varepsilon^{\frac{1}{2}})) \\
&\geq \frac{1}{3},
\end{align}

where we have used (2.20), (4.21) and (4.24) in the first inequality.

Note that if $\Psi^{(n)}_W(s)$ denotes the flow generated by velocity field $V_W$, then we have

\begin{equation}
\frac{d}{ds} \left( \partial^\gamma \tilde{W} \circ \Psi^{(n)}_W \right) + \left( D^{(\gamma)}_{\tilde{W}} \circ \Psi^{(n)}_W \right) \left( \partial^\gamma \tilde{W} \circ \Psi^{(n)}_W \right) = \tilde{F}^{(\gamma)} \circ \Psi^{(n)}_W.
\end{equation}

Then applying Grönwall’s inequality as well as (5.11), we get

\begin{equation}
|\partial^\gamma \tilde{W} \circ \Psi^{(n)}_W| \lesssim e^{\frac{1}{3}} + \|\partial^\gamma \tilde{W}(y_0, -\log \varepsilon)\| \lesssim e^{\frac{1}{3}} + \varepsilon^{\frac{1}{2}} \lesssim \frac{1}{2} M \varepsilon \frac{1}{3}.
\end{equation}

(6.22)

Case 2. $|\gamma| \leq 3, \ |y| \leq l$

By (4.16), $\tilde{W} = W - \tilde{W}$ and the explicit expression of $\tilde{W}$, we have

\begin{equation}
\tilde{W}(0, s) = \tilde{W}(0, 0) = \nabla^2 \tilde{W}(0, 0) = 0.
\end{equation}

(6.23)

By (4.21), when $y = 0$ there holds

\begin{equation}
\partial_s (\partial^\gamma \tilde{W})^0 = \tilde{F}^{(\gamma), 0} - G_W^0 (\partial_t \partial^\gamma \tilde{W})^0 - h^{n, 0} (\partial_n \partial^\gamma \tilde{W})^0 - (1 + \gamma_1) (1 - \beta_t) (\partial^\gamma \tilde{W})^0.
\end{equation}

Then by (4.18), (4.19) and (4.21), we have

\begin{equation}
|\partial_s (\partial^\gamma \tilde{W})^0| \lesssim e^{-\left(\frac{1}{2} - \frac{1}{10}\right)s} + M e^{-s} e^{\frac{1}{3}} + M e^{-s} e^{\frac{1}{4}} + M e^{-s} e^{\frac{1}{5}} \lesssim e^{-\left(\frac{1}{2} - \frac{1}{10}\right)s}.
\end{equation}

Thus, it follows from Newton-Leibniz formula that

\begin{equation}
|\partial^\gamma \tilde{W}(0, s)| \leq |\partial^\gamma \tilde{W}(0, -\log \varepsilon)| + \int_{-\log \varepsilon}^{s} |\partial_s (\partial^\gamma \tilde{W})^0(s')| ds' \leq \frac{1}{10} e^{\frac{1}{3}}.
\end{equation}

27
Thus, we get from (6.23) that
\[ |\partial^\gamma \tilde{W}(0, s)| \leq \frac{1}{10} \varepsilon^{\frac{1}{2}}, \quad |\gamma| \leq 3, \quad (6.24) \]

At last, when \(|y| \leq l\), according to Taylor expansion, it is easy to get \(|\partial^\gamma \tilde{W}(y, s)| \leq \frac{1}{2} M \varepsilon \frac{1}{2} l^{1-|\gamma|}\) by integrating from 0 to \(y\).

**Lemma 6.8.** For \(|y| \leq L\), we have
\[
|\tilde{W} \circ \Psi_W^y(s)| \leq \frac{1}{2} \varepsilon^{\frac{1}{2}} \eta^\frac{1}{3},
\]
\[
|\partial_1 \tilde{W} \circ \Psi_W^y(s)| \leq \frac{1}{2} \varepsilon^{\frac{1}{2}} \eta^\frac{1}{3},
\]
\[
|\nabla \tilde{W} \circ \Psi_W^y(s)| \leq \frac{1}{2} \varepsilon^{\frac{1}{2}}.
\]

**Proof.** The strategy is to perform weighed estimates under Lagrangian trajectories regime.

**Step 1.** General weighed estimate for \(|l| \leq |y| \leq L\)

Let \(|\mu| \leq \frac{1}{\varepsilon}, q = \eta^\mu \partial^\gamma \tilde{W}\), recall (2.21) for the evolution of \(\partial^\gamma \tilde{W}\),
\[
(\partial_1 + D_W^{(1)}) \partial^\gamma \tilde{W} + (V_W \cdot \nabla) \partial^\gamma \tilde{W} = F^{(1)}_W,
\]  
then we have
\[
\partial_1 q + D_W q + V_W \cdot \nabla q = \eta^\mu F^{(1)}_W,
\]  
where
\[
D_W = D_W^{(1)} - 3\mu + 3\mu \eta^{-1} - 2\mu \eta^{-1} (y_1(\beta_1 W + G_W) + 3h^0 y_0 |\hat{y}|^4)
\]
\[
= D_W^{(1)} - 3\mu + 3\mu \eta^{-1} - 2\mu \eta^{-1}.
\]  
First we estimate \(D_\eta\). By (1.11), (5.3), (5.16) and (5.2), we have
\[
|D_\eta| \leq \eta^{-1} \left(2|y_1| \eta^\frac{1}{2} + |y_1||G_W| + 3|h^0||y_0||\hat{y}|^4 \right)
\]
\[
\leq 2\eta^{-\frac{1}{2}} + \eta^{-1}|y_1| \left(M_{e^{-\frac{1}{2}}} |y_1| e^{-\frac{1}{2}} + |y_1| e^{-\frac{1}{2}} + |\hat{y}||y_1|^\frac{1}{2} \right) + 3\eta^{-\frac{1}{2}} M_{e^{-\frac{1}{2}}} e^{-\frac{1}{2}}
\]
\[
\leq 5\eta^{-\frac{1}{2}} + e^{-\frac{1}{2}},
\]
as long as \(\varepsilon\) is taken small enough.

When \(|y_0| \geq l\), since \(s_0 \geq \log \varepsilon, l \ll 1\), by (6.29), \(D_\eta\) can be controlled as follows
\[
2\mu \int_{s_0}^s |D_\eta \circ \Psi_W^y(s')| ds' \leq \int_{s_0}^s 10 \left(1 + l^2 e^{\frac{1}{2} (s'-s_0)} \right)^{-\frac{1}{2}} + e^{-\frac{1}{2}} ds' \leq 60 \log \frac{1}{l} + e^{-\frac{1}{2}} \leq 70 \log \frac{1}{l}, \quad (6.29)
\]
for all \(|\mu| \geq \frac{1}{\varepsilon} 2\). Hence by Grönwall’s inequality, we have
\[
|q \circ \Psi_W^y(s)| \leq l^{-70}|q(y_0)| \exp \left(\int_{s_0}^s (3\mu - D_W^{(1)} - 3\mu \eta^{-1}) \circ \Psi_W^y(s') ds'\right)
\]
\[
+ l^{-70} \int_{s_0}^s \eta^\mu F^{(1)}_W \circ \Psi_W^y(s') \exp \left(\int_{s_0}^{s'} (3\mu - D_W^{(1)} - 3\mu \eta^{-1}) \circ \Psi_W^y(s'') ds''\right) ds'.
\]  
(6.30)

**Step 2.** Estimate of \(\tilde{W}(y, s)\) for \(l \leq |y| \leq L\).

Taking \(\mu = -\frac{1}{6}\), then the damping exponent is \(3\mu - D_W^{(1)} - 3\mu \eta^{-1} = -\beta_1 \partial_1 \tilde{W} + \frac{1}{2} \eta^{-1}\). By (2.20)
and (6.2a), we have
\[
\int_{s_0}^s \left( \beta_r |\partial_1 \tilde{W}| + \frac{1}{2} \eta^{-1} \right) \circ \Psi_{\tilde{W}}^0(s') ds' \leq 5 \int_{s_0}^s \eta^{-\frac{3}{2}} \circ \Psi_{\tilde{W}}^0(s') ds' \leq 30 \log \frac{1}{l} (6.31)
\]
for all \( s \geq s_0 \geq - \log \varepsilon \). Next, we apply (5.14) with \( |\gamma| = 0 \) on the forcing term and obtain
\[
\int_{s_0}^s [\eta^{-\frac{3}{2}} \tilde{F}_W | \circ \Psi_{\tilde{W}}^0(s') ds' \leq \varepsilon \frac{1}{l} \int_{s_0}^s [\eta^{-\frac{3}{2}} \circ \Psi_{\tilde{W}}^0(s') ds' \leq \varepsilon \frac{1}{l} \log \frac{1}{l} (6.32)
\]
for all \( s \geq s_0 \geq - \log \varepsilon \). Plugging the above two bounds into (6.30), we have
\[
[\eta^{-\frac{3}{2}} \tilde{W} | \circ \Psi_{\tilde{W}}^0(s)] \leq l^{-100} \eta^{-\frac{3}{2}} (y_0) | \tilde{W} (y_0, s_0) | + M \varepsilon^{\frac{1}{2}} l^{-100} \log \frac{1}{l}, (6.33)
\]
where the implicit constant of force estimate is absorbed in \( M \). Then we use initial data (3.9) at \( s_0 = - \log \varepsilon \) and bootstrap assumption (4.32) for \( s \geq - \log \varepsilon \) to get
\[
[\eta^{-\frac{3}{2}} \tilde{W} (y, s)] \leq l^{-100} \max \{M \varepsilon^{\frac{1}{2}} l^{1}, \varepsilon^{\frac{3}{2}}\} + M \varepsilon^{\frac{3}{4}} l^{-100} \log \frac{1}{l} \leq \frac{1}{10} \varepsilon^{\frac{3}{4}}. (6.34)
\]

**Step 3.** Estimate of \( \partial_1 \tilde{W} (y, s) \) for \( l \leq |y| \leq L \).

We adapt the same argument as in **Step 2.** First, we take \( \mu = \frac{1}{4} \) and the damping exponent is \( 3\mu - D_W^{(1,0,0)} - 3\mu \eta^{-1} = - \beta_r (|\partial_1 \tilde{W} + \partial_1 \tilde{W}| - \eta^{-1} \). Then it holds that
\[
\int_{s_0}^s \beta_r (|\partial_1 \tilde{W} + \partial_1 \tilde{W}| + \eta^{-1}) \circ \Psi_{\tilde{W}}^0(s') ds' \leq 80 \log \frac{1}{l} (6.35)
\]
Due to (5.11), the forcing term is bounded by
\[
\int_{s_0}^s [\eta^{-\frac{3}{2}} \tilde{F}_W^{(1,0,0)} | \circ \Psi_{\tilde{W}}^0(s') ds' \leq \varepsilon \frac{1}{l} \log \frac{1}{l} (6.36)
\]
Therefore, (3.9), combined with the above two estimate, gives
\[
[\eta^{\frac{3}{2}} (y) \partial_1 \tilde{W} (y, s)] \leq l^{-140} \max \{M \varepsilon^{\frac{3}{2}} l^{3}, \varepsilon^{\frac{3}{2}}\} + M \varepsilon^{\frac{3}{4}} l^{-140} \log \frac{1}{l} \leq \frac{1}{10} \varepsilon^{\frac{3}{2}}. (6.37)
\]
**Step 4.** Estimate of \( \tilde{\nabla} \tilde{W} (y, s) \) for \( l \leq |y| \leq L \).

Without loss of generality we assume \( \gamma = (0, 1, 0) \). We take \( \mu = 0 \) and the damping exponent is \( 3\mu - D_W^{(0,1,0)} - 3\mu \eta^{-1} = - \beta_r (|\partial_1 \tilde{W} + \partial_1 \tilde{W}| - \eta^{-1} \). Then from (5.11), (6.2a), we have
\[
[\tilde{\nabla} \tilde{W} (y, s)] \leq l^{-100} \max \{M \varepsilon^{\frac{3}{2}} l^{3}, \varepsilon^{\frac{3}{2}}\} + M \varepsilon^{\frac{3}{4}} l^{-100} \leq \frac{1}{10} \varepsilon^{\frac{3}{2}}. (6.38)
\]
**Step 5.** Estimate of \( \tilde{W} (y, s) \) for \( |y| \leq l \).

When \( |y| \leq l \) the bounds can be obtained by Taylor expansion of \( \tilde{W} \) and \( \tilde{\nabla} \tilde{W} \) at \( y = 0 \) with \( \tilde{W} (0, s) = \nabla \tilde{W} (0, s) = 0 \).

\[
\square
\]

6.6 Closure of \( L^\infty \) Type Bootstrap on \( W \)

**Lemma 6.9.** If set \( \Psi = \Psi_{\tilde{W}}^0 \), we have
\[
[\eta^{-\frac{3}{2}} W \circ \Psi (s)] \leq \frac{3}{2} \varepsilon^{\frac{3}{2}} (6.37a)
\]
\[
[\eta^{\frac{3}{2}} \partial_1 W \circ \Psi (s)] \leq \frac{3}{2} \varepsilon^{\frac{3}{2}} \] (6.37b)
where \(D^{(\gamma)}_W = \frac{3n+1+\gamma\gamma}{2} + \beta_\gamma \gamma \partial_1 W\).

Let \(q = \eta^\mu \partial^\gamma W\), then when \(y_0 > 1\), we have

\[
|q \circ \Psi^{\eta^\mu}_{W}(s')| \leq M^{-70}|q(y_0)| \exp \left( \int_{s_0}^{s} (3\mu - D^{(\gamma)}_W - 3\mu^2) \circ \Psi^{\eta^\mu}_{W}(s') ds' \right)
+ M^{-70} \int_{s_0}^{s} |\eta^\mu F^{(\gamma)}_W \circ \Psi^{\eta^\mu}_{W}(s')| \exp \left( \int_{s_0}^{s} (3\mu - D^{(\gamma)}_W - 3\mu^2) \circ \Psi^{\eta^\mu}_{W}(s'') ds'' \right) ds'.
\]  \hfill (6.39)

When \(|y_0| \geq L\), by (6.2b) with \(\delta = \frac{1}{5}\), we get

\[
2\mu \int_{s_0}^{s} |D_{\eta} \circ \Psi^{\eta^\mu}_{W}(s')| ds' \leq \int_{s_0}^{\infty} \left( 1 + L^2 e^{\frac{\mu}{2} (s' - s_0)} \right)^{-\frac{1}{4}} e^{-\frac{\mu}{2} s'} ds' \lesssim L^{-\frac{1}{4}} + e^{\frac{\mu}{2}} \lesssim e^{\frac{\mu}{2}},
\]  \hfill (6.40)

which yields

\[
|q \circ \Psi^{\eta^\mu}_{W}(s')| \leq e^{\frac{\mu}{2}} |q(y_0)| \exp \left( \int_{s_0}^{s} (3\mu - D^{(\gamma)}_W - 3\mu^2) \circ \Psi^{\eta^\mu}_{W}(s') ds' \right)
+ e^{\frac{\mu}{2}} \int_{s_0}^{s} |\eta^\mu F^{(\gamma)}_W \circ \Psi^{\eta^\mu}_{W}(s')| \exp \left( \int_{s_0}^{s} (3\mu - D^{(\gamma)}_W - 3\mu^2) \circ \Psi^{\eta^\mu}_{W}(s'') ds'' \right) ds'.
\]  \hfill (6.41)

Now we note that (6.37a)-(6.37c) hold due to (2.20) and \(W = W + W\). So we only need to prove (6.37d)-(6.37c) for \(|y| \geq L\) and (6.37d)-(6.37c) for \(|y| \geq 1\).

**Step 1.** Estimate of \(W(y, s)\) for \(|y| \geq L\).

We take \(\mu = \frac{1}{2}\), then the damping term is \(3\mu - D^{(\gamma)}_W - 3\mu^2 = \frac{1}{2} \eta^{-1}\) and the forcing term is \(\eta^\mu (F_W - e^{-\frac{\mu}{2} \beta_\gamma \kappa})\). By (6.2b) with \(\delta = 1\), it holds that

\[
\int_{s_0}^{s} \frac{1}{2} \eta^{-1} \circ \Psi^{\eta^\mu}_{W}(s') ds' \leq \int_{s_0}^{\infty} \left( 1 + L^2 e^{\frac{\mu}{2} (s' - s_0)} \right)^{-\frac{1}{4}} e^{-\frac{\mu}{2} s'} ds' \lesssim L^{-1} \lesssim e^{\frac{\mu}{4}},
\]

as well as

\[
\int_{s_0}^{s} \left| (F_W - e^{-\frac{\mu}{2} \beta_\gamma \kappa}) \circ \Psi^{\eta^\mu}_{W}(s') \right| \lesssim \int_{s_0}^{s} e^{-\frac{\mu}{4} s'} \lesssim e^{\frac{\mu}{4}}.
\]

Then (6.41) implies that

\[
|\eta^\mu W \circ \Psi^{\eta^\mu}_{W}(s')| \leq e^{\frac{3\mu}{4}} \left( |\eta^\mu W(y_0, s_0)| + e^{\frac{\mu}{4}} \right).
\]

When \(s_0 > -\log \varepsilon, |y_0| = L\), (4.2b) gives \(|\eta^\mu W(y, 0)| \leq 1 + e^{\frac{\mu}{4}}\). If \(s_0 = -\log \varepsilon\), then by initial assumption (3.6), there still holds \(|\eta^\mu W(y, 0)| \leq 1 + e^{\frac{\mu}{4}}\). In conclusion, for any \(|y| \geq L,\)

\[
|\eta^\mu W \circ \Psi^{\eta^\mu}_{W}(s)| \leq e^{\frac{3\mu}{4}} \left( 1 + e^{\frac{\mu}{4}} + e^{\frac{\mu}{4}} \right) \lesssim \frac{3}{2}.
\]

**Step 2.** Estimate of \(\partial_1 W(y, s)\) for \(|y| \geq L\).
We take $\mu = \frac{1}{4}$, then the damping term is $|3\mu - D_W^{(\gamma)} - 3\mu \eta^{-1}| \leq |\beta \partial_1 W| + |\eta^{-1}| \leq 3\eta^{-\frac{3}{4}}$. For $|y_0| \geq L$, by (6.21) with $\delta = \frac{1}{4}$ we have
\[
\int_{s_0}^{s} (3\mu - D_W^{(\gamma)} - 3\mu \eta^{-1}) \circ \Psi_W^{y_0}(s') ds' \leq 3 \int_{s_0}^{s} \eta^{-\frac{3}{4}} \circ \Psi_W^{y_0}(s') ds' \lesssim L^{-\frac{3}{4}} \lesssim \varepsilon^\frac{1}{4}.
\]
Meanwhile, (5.11) yields
\[
\int_{s_0}^{s} |\eta^{\frac{3}{4}} F_W^{(1,0,0)} \circ \Psi_W^{y_0}(s')| ds' \lesssim \varepsilon^\frac{1}{4} \int_{s_0}^{s} \eta^{\frac{3}{4}} \circ \Psi_W^{y_0}(s') \lesssim \varepsilon^\frac{1}{4}
\]
for $|y_0| \geq L$. Gather these estimate, by (6.41), we have
\[
|\eta^{\frac{3}{4}} \partial_1 W \circ \Psi_W^{y_0}(s)| \leq e^{2\epsilon^\frac{1}{4} + (|\eta^{\frac{3}{4}} \partial_1 W(y_0, s_0)| + \epsilon^\frac{1}{4})}.
\]
When $s_0 > -\log \epsilon$ we let $|y_0| = L$ and then $|\eta^{\frac{3}{4}} \partial_1 W| \leq |\eta^{\frac{3}{4}} | \partial_1 W| + |\eta^{\frac{3}{4}} | \partial_1 \tilde{W}| \leq 1 + e^\frac{1}{4}$, and bootstrap bound (4.1) for $s_0 = -\log \epsilon$, such that we have $|\partial_1 W(y_0, s_0)| \leq L$. For $|y| \geq L$ and $s \geq -\log \epsilon$.

**Step 3.** $\nabla W(y, s)$ for $|y| \geq L$

We take $\mu = 0$ and the damping term is given by $3\mu - D_W^{(\gamma)} - 3\mu \eta^{-1} = -\beta \partial_1 W$, which we will appeal same estimate as in previous proof. For the forcing term we have
\[
\int_{s_0}^{s} |F_W^{(0,1,0)} \circ \Psi_W^{y_0}(s')| ds' \leq M \epsilon^{\frac{3}{4}} \eta^{-\frac{3}{4}} \lesssim \epsilon^\frac{1}{4}.
\]
For $s_0 > -\log \epsilon$ we still use $W = \tilde{W} + \overline{W}$ while for $s_0 = -\log \epsilon$ we use initial data assumption. So at last
\[
|\nabla W(y, s)| \leq e^{2\epsilon^\frac{1}{4} + (1 + \epsilon^\frac{1}{4})} \leq \frac{3}{2}
\]
holds for all $|y| \geq L$ and $s \geq -\log \epsilon$.

**Step 4.** $\partial_1 W(y, s)$ with $|\gamma| = 2$ for $|y| \geq l$

We take $\mu = \frac{1}{8}$ for $|\gamma| = 2$, $\gamma_1 \geq 0$ and $\mu = \frac{1}{6}$ for $|\gamma| = 2$, $\gamma_1 = 0$, respectively.

Case 1. $\gamma_1 = 0$, $|\gamma| = 2$. The damping term becomes $3\mu - D_W^{(\gamma)} - 3\mu \eta^{-1} = -\beta \partial_1 W$, so
\[
\int_{s_0}^{s} \beta \partial_1 W \circ \Psi_W^{y_0}(s') ds' \leq 40 \log \frac{1}{l},
\]
and forcing term can be bounded as
\[
\int_{s_0}^{s} |\eta^{\frac{3}{4}} F_W^{(\gamma)} \circ \Psi_W^{y_0}(s')| ds' \leq M^\frac{3}{4} \eta^{-\frac{3}{4}} \lesssim 6 M^\frac{3}{4} \log \frac{1}{l}.
\]
Then (6.39), together with (6.42) and (6.43), gives
\[
\eta^{\frac{1}{4}} |(y) \nabla^2 W(y, s)| \leq l^{-100} \eta^{\frac{1}{4}} (y_0) |\nabla^2 W(y_0, s_0)| + M^\frac{3}{4} l^{-100} \log \frac{1}{l}
\]
\[
\leq l^{-100} \max \{1, M \epsilon^\frac{1}{10} \} + M^\frac{3}{4} l^{-102}
\]
\[
\leq \frac{3}{2} M,
\]
where the second inequality we use initial data (6.58) for $s_0 = -\log \epsilon$ and bootstrap bound (4.1)
for $s_0 > -\log \varepsilon$.

Case 2. $\gamma_1 \geq 1$, $|\gamma| = 2$. The damping term $3\mu - D^{(\gamma)}_W - 3\mu \eta^{-1} = -\frac{2\gamma_1 - 1}{2} - (2\gamma_1 - 1)\beta_1 \partial_1 W - \eta^{-1}$, whose exponential integral is bounded by

$$\exp \left( \int_{s_0}^{s} (3\mu - D^{(\gamma)}_W \circ \Psi^{(\gamma)}_W(s'',ds'')) \right) \leq l^{-120\varepsilon \gamma}.$$ 

On the other hand the forcing term

$$\int_{s_0}^{s} \eta \frac{\partial^{(\gamma)}_W}{S} e^{\varepsilon \gamma} ds' \leq 2M^{\frac{s}{2}}.$$ 

So at last

$$\eta^{\frac{1}{2}}(y) \partial^{(\gamma)} W(y,s) \leq l^{-180} \eta^{\frac{1}{2}}(y_0)|\partial^{(\gamma)} W(y_0,s_0)| + 2M^{\frac{s}{2}} l^{-180} \leq l^{-180} \max \{1, 2M \varepsilon \gamma \} + 2M^{\frac{s}{2}} l^{-180} \leq \frac{3}{2} M^{\frac{s}{2}}.$$ 

**Step 5.** $W(y,s), \nabla W(y,s)$ for $|y| \leq L$ and $\nabla^2 W(y,s)$ for $|y| \leq l$

The first two bounds can be obtained by the sum of (4.2.12a) and (2.20), while the last results from Taylor expansion of $\nabla^2 W(y,s)$ at $y = 0$, in view of $\nabla^2 W(0,s) = \nabla^2 W(0,s) = 0$ and (4.3a).

### 7 Energy Estimates

In this section we will perform high order energy estimates to the system about $U_i$ and $S$. We use the homogeneous Sobolev space instead of inhomogeneous one because $\|U\|_{L^2}, \|S\|_{L^2}$ can not be taken enough small at initial data.

Recall from (2.12a) and (2.12b), $U_i$ and $S$ satisfy the equations

$$\partial_s U_i + (V_0 \cdot \nabla) U_i = F_{U_i}, \quad \partial_s S + (V_0 \cdot \nabla) S = F_S,$$

where the velocity fields are

$$V_0 = \left( \frac{3}{2} y_1 + gy_1, \frac{1}{2} gy_2 + k_2 \right),$$

and the external forces are

$$F_{U_i} = -2\beta_3 \beta_2 e^{\frac{s}{2}} \delta^{\frac{1}{2}} S \partial_1 S - 2\beta_3 \beta_2 e^{\frac{s}{2}} \delta^{\frac{1}{2}} \partial_1 \Phi - 2\beta_3 \beta_2 e^{-\frac{s}{2}} \delta^{\frac{3}{2}} \partial_2 S - 2\beta_3 \beta_2 e^{-\frac{s}{2}} \delta^{\frac{3}{2}} \partial_2 \Phi,$$

$$F_S = -2\beta_3 \beta_2 S(e^{\frac{s}{2}} \partial_1 U_1 + e^{-\frac{s}{2}} \partial_2 U_2).$$

Now we apply $\partial^{(\gamma)}$ to both sides of above equations with $|\gamma| = k$ to get

**Lemma 7.1.** For $|\gamma| = k$, it holds that

$$\partial_s (\partial^{(\gamma)}_i U_i) + (V_0 \cdot \nabla) (\partial^{(\gamma)}_i U_i) + \mathcal{D}_s \partial^{(\gamma)}_i U_i + 2\beta_3 \beta_2 S(e^{\frac{s}{2}} \delta^{\frac{1}{2}} \partial_1 (\partial^{(\gamma)} S) + e^{-\frac{s}{2}} \delta^{\frac{3}{2}} \partial_2 (\partial^{(\gamma)} S)] + \beta_3 (1 + \gamma_1) \partial_1 W \partial^{(\gamma)} S - 2\beta_3 \beta_2 S(e^{\frac{s}{2}} \delta^{\frac{1}{2}} \partial_1 (\partial^{(\gamma)} S) + e^{-\frac{s}{2}} \delta^{\frac{3}{2}} \partial_2 (\partial^{(\gamma)} S)] + \mathcal{F}_{U_i}^{(\gamma)}, \quad (7.1)$$

and

$$\partial_s (\partial^{(\gamma)} S) + (V_0 \cdot \nabla) (\partial^{(\gamma)} S) + \mathcal{D}_s \partial^{(\gamma)} S + 2\beta_3 \beta_2 S[e^{\frac{s}{2}} \partial_1 (\partial^{(\gamma)} U_1) + e^{-\frac{s}{2}} \partial_2 (\partial^{(\gamma)} U_2)] + \beta_3 (1 + \gamma_1) \partial^{(\gamma)} U_1 \partial_1 W = \mathcal{F}_{S}^{(\gamma)}. \quad (7.2)$$
Here, the damping term is
\[ D_\gamma = \gamma_1 (1 + \partial_t g_U) + \frac{1}{2} |\gamma|. \]

And the external forces terms are
\[ F^{(\gamma)}_{U_i} = P_1 + P_2 + P_3 + P_4, \]
\[ F^{(\gamma)}_S = Q_1 + Q_2 + Q_3 + Q_4. \]

with
\[ P_1 = - \sum_{|\beta| = |\gamma| - 1} \left( \frac{\gamma}{\beta} \right) \partial^{\gamma - \beta} g_U \partial^3 \partial_1 U_i - \sum_{|\beta| = |\gamma| - 1} \left( \frac{\gamma}{\beta} \right) \partial^{\gamma - \beta} h^\nu \partial^3 \partial_1 U_i, \]
\[ P_2 = - \sum_{1 \leq |\beta| \leq |\gamma| - 2} \left( \frac{\gamma}{\beta} \right) (\partial^{\gamma - \beta} g_U \partial^3 \partial_1 U_i + \partial^{\gamma - \beta} h^\nu \partial^3 \partial_1 U_i), \]
\[ P_3 = - 2 \beta_r \beta_3 e^{-\frac{1}{2} \delta^s \partial_1 Z \partial^s S} - 2 \beta_r \beta_3 \sum_{|\beta| = |\gamma| - 1} \left( \frac{\gamma}{\beta} \right) e^{-\frac{1}{2} \delta^s \partial^\gamma S} \partial^3 \partial_1 U_i + \beta_r \beta_3 \sum_{|\beta| = |\gamma| - 1} \left( \frac{\gamma}{\beta} \right) e^{-\frac{1}{2} \delta^s \partial^\gamma S} \partial^3 \partial_1 U_i, \]
\[ P_4 = - 2 \beta_r \beta_3 \sum_{1 \leq |\beta| \leq |\gamma| - 2} \left( \frac{\gamma}{\beta} \right) (e^{-\frac{1}{2} \delta^s \partial^\gamma S} \partial^3 \partial_1 U_i + e^{-\frac{1}{2} \delta^s \partial^\gamma S} \partial^3 \partial_1 U_i), \]

and
\[ Q_1 = \sum_{|\beta| = |\gamma| - 1} \left( \frac{\gamma}{\beta} \right) \partial^{\gamma - \beta} g_U \partial^3 \partial_1 S - \sum_{|\beta| = |\gamma| - 1} \left( \frac{\gamma}{\beta} \right) \partial^{\gamma - \beta} h^\nu \partial^3 \partial_1 S, \]
\[ Q_2 = - 2 \beta_r \beta_1 e^{-\frac{1}{2} \delta^s \partial_1 Z \partial^s U_\nu} - 2 \beta_r \beta_3 \sum_{|\beta| = |\gamma| - 1} \left( \frac{\gamma}{\beta} \right) e^{-\frac{1}{2} \delta^s \partial^\gamma S} \partial^3 \partial_1 U_\nu + \beta_r (\beta_1 + \beta_3 \gamma_1) e^{-\frac{1}{2} \delta^s \partial_1 Z \partial^s U_1} - 2 \beta_r \beta_3 \sum_{|\beta| = |\gamma| - 1} \left( \frac{\gamma}{\beta} \right) e^{-\frac{1}{2} \delta^s \partial^\gamma S} \partial^3 \partial_1 U_1, \]
\[ Q_3 = - \sum_{1 \leq |\beta| \leq |\gamma| - 2} \left( \frac{\gamma}{\beta} \right) (\partial^{\gamma - \beta} g_U \partial^3 \partial_1 S + \partial^{\gamma - \beta} h^\nu \partial^3 \partial_1 S)
- 2 \beta_r \beta_3 \sum_{1 \leq |\beta| \leq |\gamma| - 2} \left( \frac{\gamma}{\beta} \right) (e^{-\frac{1}{2} \delta^s \partial^\gamma S} \partial^3 \partial_1 U_1 + e^{-\frac{1}{2} \delta^s \partial^\gamma S} \partial^3 \partial_1 U_\nu). \]

Proof. (7.1) follows from direct computation, while the (7.2) is similar.

The main focus is on the third term of $P_3$. Since $\partial_1 S = \frac{1}{2} (e^{-\frac{1}{2} \delta^s \partial_1 W - \partial_1 Z}$, we have
\[ \partial^\gamma (-2 \beta_r \beta_3 e^{-\frac{1}{2} \delta^s \partial_1 S} \partial_1 S) \]
Thus (7.1) follows.

Proof. By Hölder inequality, we have

$$\text{(7.3)}$$

and the second and the third term on the right hand side can be further rewritten as

$$-2\beta_1 \beta_2 e^2 \delta S \partial_i S \partial_1 S - 2\beta_1 \beta_2 e^2 \delta S \partial_1 S = -2\beta_1 \beta_2 e^2 (1 + \gamma_1) e^2 \delta S \partial_1 S$$

$$= -\beta_1 \beta_2 (1 + \gamma_1) \delta^1 \partial^2 S (\partial_1 W + e^2 \partial_1 Z).$$

Thus (7.1) follows.

Now we define the energy $E_k(s)$ by refined $H^k$ norm for $k \geq 18$

$$E_k^2(s) = \sum_{|\gamma| = k} \lambda^{\gamma} \left( ||\partial^\gamma U(\cdot, s)||^2 + ||\partial^\gamma S(\cdot, s)||^2 \right), \quad (7.3)$$

where $\lambda = \lambda(k) = \frac{\delta}{12k^2}$ and $\delta \in (0, \frac{1}{32}]$ is to be determined. We can easily see that $E_k(s)$ is equivalent to $H^k$ norm despite the factor $\lambda^{\gamma}$. The introduction of the factor is just a technical issue.

**Lemma 7.2.** For $k \geq 18$ and $|\gamma| = k$, the estimate for external force $F^{(\gamma)}_{U_i}$ and $F^{(\gamma)}_S$ are

$$2 \sum_{|\gamma| = k} \lambda^{\gamma} \int_{\mathbb{R}^3} |F^{(\gamma)}_{U_i}| \lesssim (2 + 8\delta) E_k^2 + e^{-s} M^{4k-1},$$

$$2 \sum_{|\gamma| = k} \lambda^{\gamma} \int_{\mathbb{R}^3} |F^{(\gamma)}_S| \lesssim (2 + 8\delta) E_k^2 + e^{-s} M^{4k-1}. \quad (7.5a)$$

**Proof.** The proof is fairly similar to that of the Lemma 12.2 in \cite{1} and we omit it. $\Box$

**Lemma 7.3.** For the electric potential terms, we have

$$\int e^2 (\partial^\gamma U_i) \partial_i (\partial^\gamma \Phi) \leq e^{-s} \|U\|_{H^k} (\|S\|_{H^k} + \|R_\Gamma\|_{H^k}), \quad (7.5a)$$

$$\int e^2 (\partial^\gamma U_i) \partial_\nu (\partial^\gamma \Phi) \leq e^{-s} \|U\|_{H^k} (\|S\|_{H^k} + \|R_\Gamma\|_{H^k}). \quad (7.5b)$$

**Proof.** By Hölder inequality, we have

$$\int e^2 (\partial^\gamma U_i) \partial_i (\partial^\gamma \Phi) \leq e^2 \|\partial^\gamma U\|_{L^2} \|\partial^\gamma \partial_i \Phi\|_{L^2}.$$

Take $\partial_1$ to (2.3), we get

$$\partial_1 \Phi = \int_{\mathbb{R}^3} \frac{-e^{-1} e^{-3z} z_1 (R_+ - R_e) (y - z)}{(e^{-3z} |z|^2 + e^{-s} |z|^2)^{\frac{1}{2}}}.$$

then (5.6) implies that

$$\|\partial^\gamma \partial_1 \Phi\|_{L^2} \leq e^{-\frac{1}{2} s} \|P_1 \ast \partial^\gamma (R_+ - R_e)\|_{L^2} \lesssim e^{-\frac{1}{2} s} (\|R_\Gamma\|_{H^k} + \|R_\Gamma\|_{H^k}).$$

At last, it suffices to show that

$$\|R_e\|_{H^k} \lesssim \|\partial^\gamma S\|_{L^2} \lesssim \|S\|_{H^k}.$$
Indeed, for $|\gamma| = k$, write $\partial^\gamma(S^\frac{k}{2})$ by using Faà di Bruno formula (see [S]), for $|\gamma| = k$,

$$|\partial^\gamma(S^\frac{k}{2})| \lesssim \sum_{|\gamma|=1}^k S^\frac{k}{2} - |\nu| \sum_{P(\beta,\nu)} \prod_{l=1}^k |\partial^\beta_l S|^{\nu_l} \nu_l! (|\beta|)!^{\nu_l}.$$  

Then we taking $\frac{1}{2} = \sum_l \frac{1}{\nu_l}$ and $\sum_l \beta_l \nu_l = k$; $\sum_l \nu_l = |\nu|$, by Hölder inequality and Sobolev interpolation,

$$\sum_{|\gamma|=k} \|\partial^\gamma(S^\frac{k}{2})\|_{L^2} \lesssim \prod_{l=1}^k \|S^\frac{k}{2} - |\nu| \|_{L^\infty} \|\partial^\beta_l S\|_{L^\infty}^{\nu_l} \|\nu_l\|_{L^\nu_l}$$

$$\lesssim \prod_{l=1}^k \|S^\frac{k}{2} - |\nu| \|_{L^\infty} \|\partial^\beta_l S\|_{L^\infty}^{\nu_l} \|\nu_l\|_{L^\nu_l}$$

$$\lesssim \prod_{l=1}^k \|S^\frac{k}{2} - |\nu| \|_{L^\infty} \|\partial^\beta_l S\|_{L^\infty}^{\nu_l} \|\nu_l\|_{L^\nu_l}$$

$$\lesssim \|S^\frac{k}{2} - |\nu| \|_{L^\infty} \|\nu_l\|_{L^\nu_l}.$$  

**Proposition 7.4.** For $k \geq 18$, we have

$$E_k^2(s) \leq e^{-2s} \epsilon^{-1} + 2\epsilon^{-s} M^{4k-1}(1 - \epsilon^{-1} e^{-s}). \quad (7.6)$$

**Proof.** Taking $L^2$ inner product on (7.1) and (7.2) with $\lambda^{\nu} \partial^\gamma U_i$ and $\lambda^{\nu} \partial^\gamma S$, respectively. Then we use integration by parts:

$$\partial_s \int \lambda^{\nu} (|\partial^\gamma U|^2 + |\partial^\gamma S|^2) + \lambda^{\nu} \int (2D_\gamma - \nabla \cdot V_U)(|\partial^\gamma U|^2 + |\partial^\gamma S|^2)$$

$$+ 2\beta_1 \lambda^{\nu} \int (\beta_1 + \beta_3 + 2\beta_3 \gamma_1) \partial_l W \partial^\gamma S \partial^\gamma U_1$$

$$= 4\beta_1 \beta_3 \lambda^{\nu} \int [e^{\frac{\nu}{2}} (\partial^\gamma U_1) (e^{\frac{\nu}{2}} (\partial^\gamma S)) (\partial^\gamma S)]$$

$$- 4\beta_1 \beta_3 \lambda^{\nu} \int [e^{\frac{\nu}{2}} (\partial^\gamma U_1) \partial_l (e^{\frac{\nu}{2}} (\partial^\gamma U_2) \partial_l (e^{\frac{\nu}{2}} (\partial^\gamma \Phi))]$$

$$+ 2\lambda^{\nu} \int F^\nu U_1 \partial^\gamma U_1 + F^\nu S \partial^\gamma S.$$  

Then summing over all $|\gamma| = k$ we get

$$\partial_s E_k^2(s) + A_1 + A_2 = B_1 + B_2 + B_3. \quad (7.7)$$

We first estimate the damping term $A_1$ as

$$2D_\gamma - \nabla \cdot V_U$$

$$= 2\gamma_1 (1 + \partial_1 g_U) + |\gamma| - \frac{5}{2} - (\partial_1 g_U + \partial_2 h^2 + \partial_3 h^3)$$

$$= (2\gamma_1 - 1) (\beta_1 \beta_3 \partial_1 W + \partial_1 G_U) + 2\gamma_1 + |\gamma| - \frac{5}{2} - \partial_2 h^2 - \partial_3 h^3$$

$$\geq |\gamma| - \frac{5}{2} + 2\gamma_1 - \beta_1 \beta_3 (1 + \epsilon^\nu) (2\gamma_1 - 1) - \epsilon^\nu,$$

where we have used (1.27), (5.1) (5.2) and $\beta_1 + \beta_3 = 1$. $A_2$ also has lower bound as following,

$$2\beta_1 \lambda^{\nu} (\beta_1 + \beta_3 + 2\beta_3 \gamma_1) \partial_1 W \partial^\gamma S \partial^\gamma U_1.$$
\[ \geq -\beta_r(\beta_1 + \beta_3 + 2\beta_3\gamma_1)|\partial_1 W|(|\partial^\gamma S|^2 + |\partial^\gamma U|^2) \]
\[ \geq -\beta_r(1 + 2\beta_3\gamma_1)(|\partial^\gamma S|^2 + |\partial^\gamma U|^2). \]

So \( A_1 + A_2 \) is bounded from below as

\[
\sum_{|\gamma|=k} \left[ |\gamma| - \frac{5}{2} + 2\gamma_1 - \beta_r\beta_1(1 + \varepsilon\frac{1}{2}) (2\gamma_1 - 1) - \varepsilon\frac{1}{2} - \beta_r(1 + 2\beta_3\gamma_1) \right] \int_{\mathbb{R}^3} \lambda^{\gamma}(|\partial^\gamma S|^2 + |\partial^\gamma U|^2) 
\geq \sum_{|\gamma|=k} \left[ |\gamma| - \frac{5}{2} + 2\gamma_1(1 - \beta_r) + \beta_r\beta_1 - \beta_r - \varepsilon\frac{1}{2} \right] \int_{\mathbb{R}^3} \lambda^{\gamma}(|\partial^\gamma S|^2 + |\partial^\gamma U|^2) 
\geq (k - 5) \sum_{|\gamma|=k} \int_{\mathbb{R}^3} \lambda^{\gamma}(|\partial^\gamma S|^2 + |\partial^\gamma U|^2).
\]

For \( B_1 \), \ref{eq:41} and \ref{eq:45} give

\[
B_1 \leq 4\beta_r\beta_3 \sum_{|\gamma|=k} \lambda^{\gamma}(\|\partial_1 W\|_{L^\infty} + e^{\frac{1}{2}}\|\partial_1 Z\|_{L^\infty} + e^{-s}\|\nabla W\|_{L^\infty} + e^{-\frac{1}{2}}\|\nabla Z\|_{L^\infty})\|\partial^\gamma U\|_{L^2}\|\partial^\gamma S\|_{L^2} 
\leq 2(1 + M\varepsilon)(1 + e^{\frac{1}{2}})E_k^2(s) 
\leq (2 + e^{\frac{1}{2}})E_k^2(s).
\]

Furthermore, by Lemma 8.1, Lemma 8.2 and \ref{eq:814}, we get

\[ B_2 + B_3 \leq 2(2 + 8\delta)E_k^2(s) + 2e^{-s}M^{4k-1} + e^{-s}E_k^2(s), \]

where the factor \( \varepsilon \) can be absorbed in \( M^{4k-1} \). Therefore, from \( k \geq 18 \) and \( \delta \leq \frac{1}{12} \) and \ref{eq:77}, we finally get

\[
\frac{d}{ds}E_k^2(s) + 2E_k^2(s) \leq 2e^{-s}M^{4k-1}, 
\]

so

\[ E_k^2(s) \leq e^{-2(s_0-s)}E_k^2(s_0) + 2e^{-s}M^{4k-1}(1 - e^{-(s-s_0)}). \]

**Proof of Proposition 4.4.** First by \ref{eq:813}, we have \( E_k^2(-\log \varepsilon) \leq \varepsilon \), so

\[
\|U\|_{H^k}^2 + \|S\|_{H^k}^2 \leq \lambda^{-k}(\varepsilon^{-1}e^{-2s} + 2e^{-s}M^{4k-1}(1 - \varepsilon^{-1}e^{-s})) 
\leq \lambda^{-k}\varepsilon^{-1}e^{-2s} + 2e^{-s}\lambda^{-k}M^{4k-1}(1 - \varepsilon^{-1}e^{-s}).
\]

Then,

\[
\|Z\|_{H^k}^2 \leq 2\left( \|U_1\|_{H^k}^2 + \|S\|_{H^k}^2 \right) 
\leq 2\lambda^{-k}e^{-s} + e^{-s}(1 - e^{-s}\varepsilon^{-1})M^{4k},
\]

where we use \( \varepsilon^{-1} \leq e^{s} \) and \( 2\lambda^{-k} < M \). And for \( \|W\|_{H^k} \) is similar

\[
\|W\|_{H^k}^2 \leq 2e^{s}\left( \|U_1\|_{H^k}^2 + \|W\|_{H^k}^2 \right) 
\leq 2\lambda^{-k}e^{-s}\varepsilon^{-1} + (1 - e^{-s}\varepsilon^{-1})M^{4k}.
\]

\[ \square \]

36
8 The Main Theorem in Physical Variable

For the completion of the article, it necessary to state the main theorem in physical variable. In fact, the initial data set in \((x, t)\) in the following is equivalent to that of in \((y, s)\) assumed in Section 3.

We set the initial time to be \(t_0 = -\frac{2}{\kappa_0} \varepsilon\), and set

\[
\kappa_0 := \kappa(t_0), \quad \tau_0 := \tau(t_0), \quad \xi_0 := \xi(t_0)
\]

and

\[
((u_1)_0(x), (u_2)_0(x), (u_3)_0(x)) = u_0(x) := u(x, t_0), \quad n_{e, 0}(x) := n_e(x, t_0), \quad \sigma_0 := \frac{\kappa_0^2}{a}.
\]

We assume the initial data of \(n_{e, 0}\) satisfies neutrality

\[
\int_{X^+} (n_+ - n_{e, 0}) = 0.
\]

We also assume that \(u_0\) and \(n_{e, 0}\) satisfy even condition:

\[
u_0(x_1, x_2, x_3) = u_0(x_1, -x_2, -x_3) = u_0(x_1, x_2, -x_3) \quad (8.3a)
\]

\[
\nu_{e, 0}(x_1, x_2, x_3) = n_{e, 0}(x_1, -x_2, -x_3) = n_{e, 0}(x_1, x_2, -x_3) \quad (8.3b)
\]

We introduce the Riemann type variables

\[
\tilde{w}_0(x) := (u_1)_0(x) + \sigma_0(x), \quad \tilde{z}_0(x) := (u_1)_0(x) - \sigma_0(x),
\]

and assume the initial data \((\tilde{w}_0 - \kappa_0, \tilde{z}_0, (\tilde{w}_0)_0, n_{e, 0})\) is supported in the set \(\mathbb{P}\),

\[
x = \{|x_1| \leq \varepsilon^{\frac{1}{2}}, |x| \leq \varepsilon^{\frac{3}{4}}\}.
\]

We choose \(\tilde{w}_0(x)\) such that

- the minimum negative slope in the \(e_1\) direction (8.5a)
- \(\tilde{w}_0(x)\) attains its global minimum at \(x = 0\) (8.5b)

\[
\tilde{w}_0(0) = \kappa_0, \quad \tilde{w}_0(x) = -\frac{1}{\varepsilon}, \quad \tilde{w}_0(0) = 0, \quad \tilde{w}_0^{(2)}(x) > 0.
\]

Set

\[
\tilde{w}_0 := \tilde{w}_0(x) - \tilde{w}_0(x - \xi(t)) = w_0(x) - \tilde{w}_0(x) = \varepsilon^{\frac{1}{4}} \tilde{W}(y, -\log \varepsilon) + \kappa_0.
\]

We assume that \((\varepsilon^{\frac{1}{2}} x_1, \varepsilon^{-\frac{1}{2}} x)\) \(\leq 2 \varepsilon^{-\frac{1}{4}}\), according to \(3.6\)

- \(\tilde{w}_0(x) - \kappa_0) \leq \varepsilon^{\frac{1}{4}} (e^3 + x_1^2 + |x|^6)^{\frac{1}{4}}, \quad (8.7a)
- \(|\tilde{w}_0(x)| \leq \varepsilon^{\frac{1}{4}} (e^3 + x_1^2 + |x|^6)^{-\frac{1}{4}}, \quad (8.7b)
- \(|\tilde{w}_0(x)| \leq \varepsilon^{\frac{1}{4}}. \quad (8.7c)
\]

Furthermore, for \(\varepsilon^{-\frac{1}{2}} x_1, \varepsilon^{-\frac{1}{2}} x) \leq 1\), we assume that

- \(\tilde{w}_0(x) | \leq \varepsilon^{-\frac{1}{4}} (e^{\gamma_1 + \gamma_2 + \gamma_3}), \quad |\gamma| = 4 \quad (8.8)
- \(\tilde{w}_0(x)| \leq \varepsilon^{-\frac{1}{4}} (e^{\gamma_1 + \gamma_2 + \gamma_3}), \quad |\gamma| = 3 \quad (8.9)
\]

while \(x = 0\), we assume
For \( \tilde{w}_0 \) and \( x \in \mathcal{X} \) but \( |(e^{-\frac{\gamma}{2}x_1}, e^{-\frac{\gamma}{4}x})| \geq \frac{1}{2} e^{-\frac{\gamma}{2}} \), we assume that

\begin{align}
|\tilde{w}_0(x) - \kappa_0| &\leq (1 + \varepsilon^{4})(\varepsilon^{3} + x^2 + |x|^6)^rac{1}{2}, \\
|\partial_x \tilde{w}_0(x)| &\leq (1 + \varepsilon^{4})(\varepsilon^{3} + x^2 + |x|^6)^{-\frac{1}{2}}, \\
|\nabla_x \tilde{w}_0(x)| &\leq 1.
\end{align}

(8.10a) (8.10b) (8.10c)

For \( x \in \mathcal{X} \), we assume the \( \nabla^2 \tilde{w}_0 \) satisfy

\begin{align}
|\partial^2_x \tilde{w}_0(x)| &\leq \varepsilon^{-\frac{3}{2}}(\varepsilon^{3} + x^2 + |x|^6)^{-\frac{1}{2}}, \\
|\partial_x \nabla_x \tilde{w}_0(x)| &\leq \varepsilon^{-\frac{3}{2}}(\varepsilon^{3} + x^2 + |x|^6)^{-\frac{1}{2}}, \\
|\nabla^2_x \tilde{w}_0(x)| &\leq (\varepsilon^{3} + x^2 + |x|^6)^{-\frac{1}{2}}.
\end{align}

(8.11a) (8.11b) (8.11c)

For \( \tilde{z}_0 \) and \( (\tilde{u}_\nu)_0 \), we assume that

\begin{align}
|\tilde{z}_0(x)| &\leq \varepsilon, \quad |\partial_x \tilde{z}_0(x)| \leq 1, \quad |\nabla \tilde{z}_0(x)| \leq \varepsilon^{\frac{1}{2}}, \\
|\partial^2_x \tilde{z}_0(x)| &\leq \varepsilon^{-\frac{1}{2}}, \quad |\partial_x \nabla_x \tilde{z}_0(x)| \leq \varepsilon^{-\frac{1}{2}}, \quad |\nabla^2_x \tilde{z}_0(x)| \leq 1.
\end{align}

(8.12)

and

\begin{align}
|\tilde{u}_\nu(0)(x)| &\leq \varepsilon, \quad |\partial_x \tilde{u}_\nu(0)(x)| \leq 1, \quad |\nabla \tilde{u}_\nu(0)(x)| \leq \varepsilon^{\frac{1}{2}}, \quad |\nabla^2 \tilde{u}_\nu(0)(x)| \leq 1.
\end{align}

(8.13)

For the initial specific vorticity, we assume that

\[
\left\| \frac{\nabla_x \times \tilde{u}_0(x)}{n_{\nu,0}} \right\|_{L^\infty} \leq 1.
\]

(8.14)

For the Sobolev norm of initial condition we assume for a fixed \( k \geq 18 \)

\[
\sum_{|\gamma| = k} \varepsilon^2 \| \partial^\gamma \tilde{w}_0 \|_{L^2_x} + \| \partial^2_x \tilde{w}_0 \|_{L^2_x} + \| \partial^\gamma \tilde{u}_\nu \|_{L^2_x} \leq \varepsilon^{\frac{1}{2} - (2\gamma + |\gamma|)}.
\]

(8.15)

At last recall the transform

\[
(x, t) = (x - \varepsilon(\frac{1 + \alpha}{2} t), \frac{1 + \alpha}{2} t), \quad \tilde{u}(x, t) = u(x, t), \quad \tilde{z}(x, t) = \zeta(x, t) = \frac{\nabla \times u}{n_{\nu}}, \quad \tilde{\sigma}(x, t) = \sigma(x, t) = \frac{\tilde{w}_0}{n_{\nu}}.
\]

(8.16)

**Theorem 8.1.** Let \( \gamma > 1, \alpha = \frac{1}{2} - \frac{1}{2}, T_\ast > 0, u_0, n_{\nu,0}, \sigma_0, \tilde{w}_0, \tilde{z}_0 \) are defined above, where \( n_{\nu,0} \) satisfying the neutrality [8.2] and even condition [8.3]. The modulation variables \( \kappa, \tau, \xi \) have initial conditions given by [8.1]. We also assume the initial data \((\tilde{w}_0 - \kappa_0, \tilde{z}_0, (\tilde{u}_\nu)_0, n_{\nu,0})\) is supported in the set [8.3] and satisfy the condition [8.3] and [8.14].

Then there exist \( \kappa_0 > 1, \varepsilon \ll 1, T_\ast = O(\varepsilon) \) and a unique solution

\[
(u, n_{\nu}) \in C \left([-\frac{2}{1 + \alpha} \varepsilon, -\frac{2}{1 + \alpha} T_\ast]; H^k \right) \cap C^1 \left([-\frac{2}{1 + \alpha} \varepsilon, -\frac{2}{1 + \alpha} T_\ast]; H^{k-1} \right)
\]

(8.17)

to (1.1) which blows up in an asymptotically self-similar type at time \( T_\ast \), at the single point \( \xi_\ast \in \mathbb{R}^3 \).

Also, if use the variable \((x, t), \tilde{u}(x, t), \tilde{\sigma}(x, t)\) defined in (8.16), the following result holds:

- \( T_\ast = O(\varepsilon^2), \xi_\ast = O(\varepsilon), |\kappa_\ast - \kappa_0| = O(\varepsilon) \), where \( T_\ast \) is defined by \( \int_{\varepsilon}^{T_\ast} (1 - \tau) dt = \varepsilon, \xi_\ast = \lim_{t \to T_\ast} \xi(t), \kappa_\ast = \lim_{t \to T_\ast} \kappa(t) \).
- \( \text{We have sup}_{t \in [-\varepsilon, T_\ast]} \| \tilde{u}_1 \|_{L^\infty} + \| \tilde{u}_\nu \|_{L^\infty} + \| \tilde{\sigma} - \frac{1}{2} \kappa_0 \|_{L^\infty} \leq 1 \).
- \( \text{There holds} \lim_{t \to T_\ast} |\partial_x \tilde{w}(\xi(t), t)|_{L^\infty} \leq \frac{2}{T_\ast - T} \) as \( T \to T_\ast \).
• The only blowup first order derivatives are $\partial_{x_1} \tilde{u}$ and $\partial_{x_1} \tilde{n}_e$, the other first derivatives remain bounded uniformly in $t$:

$$
\lim_{t \to T^*} \partial_{x_1} u_1(\xi(t), t) = \lim_{t \to T^*} \partial_{x_1} \tilde{n}_e(\xi(t), t) = -\infty, \quad (8.17a)
$$

$$
\sup_{t \in [-\varepsilon, T_\ast)} \|\tilde{n}_e(\cdot, t)\|_{L^\infty} + \|\tilde{u}_1(\cdot, t)\|_{L^\infty} + \|\nabla u_1(\cdot, t)\|_{L^\infty} \lesssim 1. \quad (8.17b)
$$

• The electron density is uniformly bounded from below in the support set of $n_e$, especially in the set $X$:

$$
\sup_{t \in [-\varepsilon, T_\ast)} \|\tilde{n}_e(\cdot, t) - \frac{\alpha}{2} u_1(\cdot, t)\|_{L^\infty_X} \leq \varepsilon^{\frac{1}{2}}. \quad (18.22)
$$

Proof. The proof is easy to obtained from bootstrap argument performed in above sections, which we left it to readers.

\[\square\]

Remark 8.2. The assumption (8.3) is made to avoid the over repeating parts with [1]. In fact, similar conclusions to [1] can be obtained if such assumption is removed.

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