Gravitational scalar field coupled directly to the Maxwell field and its effect to solar-system experiments

Yasunori Fujii$^1$ and Misao Sasaki$^2$

$^1$Advanced Research Institute for Science and Engineering, Waseda University, Tokyo, 169-8555 Japan
$^2$Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto, 606-8502 Japan

The effect of the massless gravitational scalar field assumed to couple directly to the Maxwell field to the solar-system experiments is estimated. We start with discussing the theoretical significances of this coupling. Rather disappointingly, however, we find that the scalar-field parameters never affect the observation in the limit of the geometric optics, indicating a marked difference from the well-known contribution through the spacetime metric.

I. INTRODUCTION

The light-ray passing near the Sun is subject to the time-delay of propagation proportional to $(1 + \gamma)a$, where $\gamma$ is one of Eddington’s parameters, while $a$ is Sun’s Schwarzschild radius. The most stringent test of General Relativity (GR) has been made by using the Cassini spacecraft with the measured result

$$\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}. \quad (1.1)$$

This is compared with the prediction from the Brans-Dicke model of the scalar-tensor theory (STT):

$$\gamma = 1 - 4\xi^2, \quad (1.2)$$

where

$$\xi^{-2} = 6 + \epsilon \xi^{-1}, \quad \text{with} \quad \epsilon = \pm 1, \quad (1.3)$$

or also given by the Jordan-Brans-Dicke parameter $\omega_0 = \epsilon(4\xi)^{-1}$ and $\epsilon = \text{Sign}(\omega_0)$. The result barely allows a negative region of $\gamma - 1$ extending to $-0.2 \times 10^{-5}$. This can be consistent with if $4\xi^2 \lesssim 0.2 \times 10^{-5}$, translated into $\epsilon = +1$ and

$$\xi \lesssim 0.5 \times 10^{-6}, \quad \text{or} \quad \omega_0 \gtrsim 5 \times 10^5. \quad (1.4)$$

We might complain that the obtained size of $\xi$ is unnaturally small to the extent that it appears as if STT itself were nearly dying.

A possible way out is to assume the scalar field to mediate the force of a finite range, refusing to reach the distance relevant to the solar-system experiment in such a way that, as discussed in [8], the experiment of the kind of is no longer sensitive to the coupling strength between the matter and the scalar field.

In the present study instead we continue to assume the massless field, but focus upon another type of the scalar-field interaction, the direct coupling to the electromagnetic field. We emphasize that this coupling, if there is any, is outside the BD model mentioned above, hence not included in the estimate [12]. Also the effect of this coupling to the solar-system experiment has never been explored seriously, as far as we know, yet appears tractable. In spite of these alluring features, we eventually reach a negative conclusion of no chance to detect the effect of the scalar field, as long as we cling to the geometric optics approximation. We still believe the analysis to contain interesting ingredients worth presenting in some detail.
In order to highlight the theoretical background of the direct coupling, we start with summarizing how the result $\square$ was derived. Brans and Dicke assumed the validity of Weak Equivalence Principle (WEP) implemented by the decoupling of the scalar field from the matter Lagrangian $L_{\text{matt}}$, resulting in the geodesic equation for any matter particle, including the photon $\square$. Stated in a more rigorous language, we assume the existence of a unique conformal frame (called BDCF) with $L_{\text{matt}}$ endowed with the above decoupling property. In other conformal frames (CFs), the geodesic equation acquires a nonzero right-hand side, which turns out to contain the scalar field in the way common to all the particles independently of any specific properties of individual particles. For this reason the Universal Free-Fall, an expression of WEP, is maintained intact. Their model thus provides with the “metric theory” implying that the effects of the scalar field occur only through the spacetime metric. The weak-field approximation in BDCF yields the result $\square$.

On the other hand, there have been some arguments on the possible presence of direct coupling of the scalar field to the Maxwell field, described by a gauge-invariant Lagrangian given by

$$L_{\text{smx}} = -\sqrt{-g} \frac{1}{4} \Phi g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \quad (1.5)$$

where $\Phi$ stands for some combination of the scalar field, and $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ $\square$. Since the electromagnetic field is part of the matter in the sense of GR, this term violates WEP through the terms involving the scalar field, against one of BD’s requirements. In fact, the matrix elements of $(1.5)$ estimated for the matter fields depend generally not only on the scalar field but also on the electric charge, obviously bringing the scalar field, against one of BD’s requirements. In fact, the matrix elements of $(1.5)$ estimated for the matter fields depend generally not only on the scalar field but also on the electric charge, obviously bringing about the contributions of the scalar field depending on whether a falling object contains electrically charged constituents or not, for example. In this context the effect of $\square$ is not represented through the spacetime metric, requiring separate analysis for the physical effects.

As another aspect the field $\Phi^{1/2}$ can be absorbed into the modified electromagnetic field $\tilde{A}_\mu$ defined by $\Phi^{1/2}A_\mu$ in such a way that the terms of the highest derivatives agree with the standard form $\square$ but without $\Phi$-dependence. This rescaling combined with the coupling with other more conventional current of charged fields, like the electron, for example, will cause the rescaling of the electric charge $e$, thus the fine-structure constant depending on the spacetime coordinates through $\Phi$. This motivated Bekenstein to propose his theoretical model $\square$. Many developments have followed in this direction also taking the cosmological evolution of the scalar field into account $\square$. It does not appear, however, that serious efforts have been made studying how the proposal is related to the more fundamental STT. The interaction $(1.5)$ might be only an effective coupling derived from a deeper origin. We point out that the spacetime-dependent fine-structure constant may follow also from STT, not necessarily due to the coupling like $(1.5)$, as discussed in Chapter 6.1 of $\square$ and $\square$.

Another important suggestion for the coupling as in $(1.5)$ comes from string theory. The field equations of the bosonic closed string sector are derived from the Lagrangian in 26 dimensions, as shown in $\square$, particularly in Eq. (3.4.58) of $\square$:

$$L_{\text{str}} = \sqrt{-g} e^{-2\phi} \left( \frac{1}{2} R + 2 g^{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right)$$

$$= \sqrt{-g} \left( \frac{1}{2} \xi \phi^2 R + \frac{1}{2} \xi g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{12} \xi \phi^2 H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right), \quad (1.6)$$

where $\phi = 2e^{-\phi}$ has been introduced in the second line with $\epsilon = -1$ and $\xi^{-1} = 4$, or $\omega_0 = -1$. Remarkably enough, the first two terms in the second line look the same as in STT in 4 dimensions. It even seems as if STT had been prepared for string theory invented decades later.

This also suggests that string theory is formulated in what is called the string CF, which plays the same role as BDCF. The last term in the parenthesis in the second line of $(1.5)$ then indicates that the gauge-field Lagrangian in 4 dimensions is likely to be multiplied by the scalar field, basically in the same way as in
Although the conclusion depends on the yet-to-be-established details in the theoretical transition to the physical 4-dimensional spacetime, the coupling of this type and hence WEP violation appear rather generic.

As a concomitant aspect of the approach we notice the unmistakable sign \( \epsilon = -1 \) in (1.6), implying a negative kinetic energy of \( \phi \) according to our sign convention, though the mixing coupling between \( \phi \) and the spinless portion of the metric tensor caused by the nonminimal coupling term recovers eventually the right sign corresponding to positive energies under the condition \( \zeta^2 > 0 \), in agreement with the unitarity requirement imposed in deriving the field equations. We add that this positive-energy condition is met by the choice \( \epsilon \xi^{-1} = -4 \) both in 4 and 26 dimensions. We point out that the same result \( \epsilon \xi^{-1} = -4 \) is shared by the dimensional compactification \textit{a la} Kaluza-Klein (KK), as demonstrated in Eq. (1.23) of [3], and is also favored by the cosmological equations in the presence of the cosmological constant, as elaborated in Chapter 4.4 of [3]. Further noteworthy is the likely occurrence of the direct interaction of the type (1.5) in the KK compactification, as well.

From (1.3) we further derive \( 0 < \zeta^2 \lesssim 1/6 \) for \( \epsilon = \pm 1 \) [10]. In particular the scalar-field-matter decoupling assumed to play the central role in the Least Coupling Principle [11] is realized only for \( \epsilon = +1 \), contrary to the simple-minded interpretation of string theory, also to the KK approach, as well as to the cosmological constraint mentioned above. We also note in this connection that no argument has been offered in [11] for the suspected sign reversal in the string-theoretical process of descending to 4 dimensions.

Given the theoretical significances as we have learned above, observational consequences of (1.5) appear to deserve further scrutiny whatever the origin. We here focus upon the possible effect to the solar-system experiments. The result may turn out to be as large as (1.2). We may even hope that the two effects, from (1.2) and (1.5), conspire to nearly cancel each other leaving a small deviation of \( \gamma \) from unity, as given by (1.4), still allowing much larger and hence more natural value of \( \xi \) than indicated by (1.4), likely with \( \epsilon = -1 \).

With this wishful anticipation, we derive the field equations in Section 2, and develop the geometric optics approximation in Section 3 to be applied to light-rays passing near the Sun. By studying the solution in the limit of geometric optics, however, we find the result which is independent of the scalar-field parameters. Section 4 is devoted to concluding remarks.

Appendix A accommodates details of simple but lengthy calculations to derive (3.10).

### II. FIELD EQUATIONS

In order to have an idea on what the scalar field is expected to be like, we start with BD Lagrangian

\[
\mathcal{L}_{\text{att}} = \sqrt{-g} \left( \frac{1}{2} \xi \phi^2 R - \frac{1}{2} \epsilon g^\mu\nu \partial_\mu \phi \partial_\nu \phi + L_{\text{matter}} \right),
\]

where we use the reduced Planckian unit system with \( c = \hbar = M_P \left( = \sqrt{\hbar/8\pi G} \right) = 1 \).

We introduce the weak-field \( \sigma \) for the scalar field by

\[
\phi = \xi^{-1/2} (1 + \zeta \sigma),
\]

where \( \zeta \) is given by (1.3). We have chosen the parameters in such a way that the nonminimal coupling term, the first term in the parenthesis of (2.1), reduces to the standard Einstein-Hilbert term as \( \sigma \rightarrow 0 \). After the process of diagonalization, we arrive at the field equation

\[
\Box \sigma = \zeta T,
\]

where \( T \) is the trace of the matter energy-momentum tensor.
By adhering to the massless theory at this moment, the static field around the Sun is given by

\[ \zeta \sigma \approx \frac{c^2}{4\pi r} = a^2 \frac{1}{r}, \]

where the Schwarzschild radius of the Sun has been defined by

\[ a = \frac{2M_\odot}{8\pi} = \frac{M_\odot}{4\pi}. \]

We note that (2.3) may not necessarily follow if \( \sigma \) is interpreted as the scalar field in [4]. As a result the coefficients in (2.4) and (2.5) can be different. We nevertheless use the results here, the same in deriving (1.2), because the final result does not depend on such details, as far as the scalar force is assumed to be long-range, and the coupling strength is chosen to be of the gravitational size.

We also assume that (1.3) reduces to the free Maxwell Lagrangian in the limit \( \sigma \to 0 \). In accordance with this we choose \( \Phi \) to be given by

\[ \Phi = \left( \frac{\zeta}{2} \phi \right)^\kappa = (1 + \zeta \sigma)^\kappa \approx 1 + \zeta \kappa \sigma, \]

where \( \kappa \) is a constant. Accepting the scalar field as given by (2.4)-(2.6) we now vary (1.5) with respect to \( A_\mu \) to derive

\[ \nabla_\mu (\Phi F^{\mu\nu}) = 0. \]

Ignoring contributions from other ordinary charged fields for the moment, it might be convenient to put (2.7) into the form

\[ \nabla_\mu F^{\mu\nu} = -j^\nu = -\Psi_\mu F^{\mu\nu}, \quad \text{with} \quad \Psi_\mu = \frac{\partial_\mu \Phi}{\Phi}. \]

By the repeated use of this equation, we readily derive the conservation law

\[ \nabla_\nu j^\nu = 0. \]

Basically in the same way as in the conventional electrodynamics, this conservation law allows us to impose the gauge condition

\[ \chi = \nabla_\mu (\Phi^n A^\mu) = 0, \]

where \( n \) is an arbitrary real number. Consistency of this condition with the field equation is assured by the linear differential equation for \( \chi \),

\[ \Box \chi + n \Psi_\mu (\nabla^\mu \chi) = 0, \]

derived also from the field equation, where \( \Box = \nabla_\mu \nabla^\mu \).

By imposing (2.10), the field equation (2.7) is put into

\[ \Box A^\mu + \Psi_\nu \left( \nabla^\mu A^\nu - (1-n) \nabla^\nu A^\mu \right) + n (\nabla^\mu \Psi^\nu) A_\nu + R^{\mu\nu} A_\nu = 0, \]

where the last term comes from the failure of commutativity between \( \nabla_\mu \) and \( \nabla_\nu \) which we encountered in re-arranging terms in the far-left-hand side of (2.8).
III. GEOMETRIC OPTICS

Let us first consider the geometric optics for the conventional free Maxwell field with $\Phi = 1$; 

$$\nabla_\mu F^{\mu \nu} = 0. \quad (3.1)$$

We put 

$$A^\mu = a^\mu e^{iS}, \quad (3.2)$$

in which the phase $S$ is assumed to vary much faster than the amplitude $a^\mu$ does. We also introduce the scalar amplitude $A$ and the normalized polarization vector $\epsilon^\mu$ in such a way 

$$a^\mu = A \epsilon^\mu, \quad \text{and} \quad \epsilon_\mu \epsilon^\mu = 1. \quad (3.3)$$

We find 

$$F^{\mu \nu} = \nabla_\mu (a^\nu e^{iS}) - \nabla_\nu (a^\mu e^{iS}) = [i (k^\mu a^\nu - k^\nu a^\mu) + (\nabla^\mu a^\nu - \nabla^\nu a^\mu)] e^{iS}, \quad (3.4)$$

where 

$$k^\mu = \nabla^\mu S, \quad (3.5)$$

representing the normal to the surface of a constant phase, $S$, hence to be called a wave-vector. The first set of the two terms in the last line of (3.4) comes from differentiating the phase $S$, hence to be called the terms of rank 1. The two terms in the second set, on the other hand, are called the terms of rank 0 because no differentiation of $S$ is involved. The rank $r$ expresses how many times $e^{iS/\epsilon}$ is differentiated, corresponding to expanding $e^{iS/\epsilon}$ into the series of $\epsilon^{-r}$ in many of conventional calculations. A decrease of $r$ by every unit implies a factor as small as $\sim \lambda/\ell \sim 10^{-12}$, where $\lambda$ and $\ell$ are the wavelength of radiation and the typical size of the solar system, respectively.

With the help of $k^\mu$, we impose a gauge condition 

$$k_\mu \epsilon^\mu = 0, \quad (3.6)$$

corresponding to the choice $n = 0$ in (2.10). Other choices of $n$ will be discussed later.

Now we substitute (3.4) into (3.1), obtaining 

$$\nabla_\mu F^{\mu \nu} = \left(i k_\mu [i (k^\mu a^\nu - k^\nu a^\mu) + (\nabla^\mu a^\nu - \nabla^\nu a^\mu)] + \nabla_\mu [i (k^\mu a^\nu - k^\nu a^\mu) + (\nabla^\mu a^\nu - \nabla^\nu a^\mu)]\right) e^{iS}$$

$$= \left(-k_\mu k^\mu a^\nu + k^\nu k_\mu a^\mu + iB^\nu + C^\nu\right) e^{iS}, \quad (3.7)$$

where (3.6) applies to the second term in the last line, and 

$$B^\nu = a^\nu \nabla_\mu k^\mu - a^\mu \nabla_\mu k^\nu + k^\mu \nabla_\mu a^\nu - k^\nu \nabla_\mu a^\mu + k_\mu \nabla^\mu a^\nu - k_\nu \nabla^\nu a^\mu, \quad (3.8)$$

$$C^\nu = \nabla_\mu (\nabla^\mu a^\nu - \nabla^\nu a^\mu) + R_\mu^\nu a^\mu. \quad (3.9)$$

Note that the first and the second terms in the last line of (3.7) are of rank 2 (bilinear in $k$), while $B^\nu$ and $C^\nu$ have rank 1 and 0 (linear in $k$ and constant), respectively. We have used (3.5), also understanding that $\nabla_\mu$, for example, no longer operates to $e^{iS}$, though $\nabla_\mu (k^\mu a^\nu) = (\nabla_\mu k^\mu) a^\nu + k^\mu (\nabla_\mu a^\nu)$, for example.

We multiply (3.8) by $\epsilon_\nu$. Analyzing each term separately, as shown explicitly in Appendix A, we obtain 

$$\epsilon_\nu B^\nu = A (\nabla_\mu k^\mu) + 2k^\mu (\nabla_\mu A). \quad (3.10)$$
In order to include the effect of the scalar field, we now go back to (2.7) in its complete expression. By multiplying by $\epsilon_\nu$ we find

$$\Phi \epsilon_\nu \nabla_\mu F^{\mu\nu} + (\nabla_\mu \Phi) \epsilon_\nu F^{\mu\nu} = 0.$$  \hspace{1cm} (3.11)

For the first term we use (3.7) and (3.10) obtaining

$$\Phi \epsilon_\nu \left( -k_\mu k^\mu a^\nu + iB^\nu + C^\nu \right) e^{iS} = \Phi \left( -k_\mu k^\mu A + i [A (\nabla_\mu k^\mu) + 2k^\mu \nabla_\mu A] + \epsilon_\nu C^\nu \right) e^{iS}.$$  \hspace{1cm} (3.12)

For the second term in (3.11) we use (3.4) finding

$$\left( \nabla_\mu \Phi \right) \epsilon_\nu \left( i \left( k^\mu a^\nu - k^\nu a^\mu \right) + \left( \nabla^\mu a^\nu - \nabla^\nu a^\mu \right) \right) e^{iS} = \left( i \left( \nabla_\mu \Phi \right) k^\mu A - \epsilon_\nu \left( \nabla_\mu \Phi \right) \left( \nabla^\nu a^\mu \right) \right) e^{iS},$$

where we have used

$$\epsilon_\nu \nabla^\mu a^\nu = \nabla^\mu \left( \epsilon_\nu a^\nu \right) - \left( \nabla_\nu \epsilon_\nu \right) a^\nu = 0,$$  \hspace{1cm} (3.14)

due to (3.6) and (A.16).

Summing (3.12) and (3.13), we finally obtain

$$- (\Phi A k_\mu k^\mu - 2iF - 2G) e^{iS} = 0,$$  \hspace{1cm} (3.15)

where

$$F = \Phi k_\mu \nabla_\mu A + \frac{1}{2} A \nabla_\mu (\Phi k^\mu),$$  \hspace{1cm} (3.16)
$$G = \Phi \epsilon_\nu C^\nu - \left( \nabla_\nu \Phi \right) \epsilon_\nu \nabla^\nu a^\mu,$$  \hspace{1cm} (3.17)

which are real-valued, carrying the rank 1 and rank 0, respectively.

The complex-valued equation (3.15) represents two real-valued equations

$$k_\mu k^\mu = 0,$$  \hspace{1cm} (3.18)
$$F = 0,$$  \hspace{1cm} (3.19)

where we have dropped $2G/(\Phi A)$ on the right-hand side of (3.18) because it has rank $r = 0$.

The first one (3.18) implies a geodesic equation, based on the standard relation

$$\nabla_\nu (k_\mu k^\mu) = 2k^\mu \nabla_\nu k_\mu = 2k^\mu \nabla_\nu k_\mu = 2k^\mu \nabla_\nu k_\mu = 2 \frac{dx^\mu}{d\lambda} (\nabla_\nu k_\mu) = 2 \frac{Dk_\mu}{D\lambda},$$

where $\lambda$ is the distance measured along a ray defined by

$$k^\mu = \frac{dx^\mu}{d\lambda}.$$  \hspace{1cm} (3.20)

By accepting (3.18), the far-left-hand side of (3.20) vanishes, and so does the far-right-hand side to result in

$$\frac{Dk_\mu}{D\lambda} = 0.$$  \hspace{1cm} (3.21)

This tells us simply that the light-ray propagates exactly along the same geodesic as the one without the scalar field included. In other words, the solar-system experiments using the light-rays fail to constrain the scalar-field parameters which describe how it couples to the Maxwell field through the direct interaction as in (1.5). This might sound rather disappointing because the interaction breaks WEP, indicating the occurrence of the inhomogeneous term on the right-hand side of the geodesic equation. We might be content with an interpretation that the effect fails to show up in the limit of the geometric optics.
In fact on the right-hand side of (3.18), we could have retained the term of $G/(ΦA)$, which depends on the derivative of $Φ$ according to (3.17), thus providing the inhomogeneous term we had expected. Including such terms of the lower rank is hardly promising, however, in the realistic situation where the phenomena are described successfully by the geometric optics.

On the other hand, (3.19) tells us how the amplitude of the ray is affected by the presence of the scalar field. In this connection we first notice that this equation in the absence of the scalar field, $Φ = 1$, reduces to

$$F_0(A) = k^\mu \nabla_\mu A + \frac{1}{2} A \nabla_\mu (k^\mu) = 0,$$

which is known to entail the conservation of photon flux $[12]$, $A$ falling off like $(distance)^{-1}$ in its propagation in a spherically symmetric flat spacetime, as shown toward the end of Appendix A. In the presence of the scalar field, we find that (3.19) given by (3.16) is re-expressed as

$$F = Φ^{-1/2} F_0(A_*) = 0,$$

in terms of the modified amplitude $A_*$ defined by

$$A_* = Φ^{-1/2} A.$$

We could first calculate $A_*$ as the photon-flux-conserving amplitude in Schwarzschild spacetime, and then use (3.25) to obtain $A$, representing how $Φ$ affects the amplitudes, though no observational result is available at present.

**IV. CONCLUDING REMARKS**

According to the result in the preceding section, the scalar-field parameters, typically $κ$ in (2.6), remain largely unconstrained. Probably we may look for other phenomena in which the effects will show up, like the one suggested in 6.5 of [3]. At this moment, in particular, we admit disappointingly that we no longer have a reasonable candidate for producing $γ > 1$ as indicated by the measurement [1]. The strong argument for this result [13] may call for something entirely new. Even in more general terms we failed to offer a successful scenario of conspiracy to save the simple-minded STT, by which $γ - 1 = -4ζ^2$ in (1.2) is nearly canceled by the contribution from (1.5), thus allowing much larger and more natural value of $ζ^2 > 1/6$ hence of $ξ > 1/6$ with $ε = -1$ [10]. We might be inclined to support the idea of a massive scalar field [3]. See [14] for the detailed argument on the consistency between the massive and the massless behaviors of the scalar field in the local and the cosmological environments, respectively.

As already mentioned in Section 1, there is another way to describe the Maxwell field by absorbing $Φ$ into the electromagnetic field, which should be important to discuss possible spacetime-dependent fine-structure constant. In fact (1.5) is re-expressed as

$$L_{smx} = -\sqrt{-g} \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} \tilde{F}_{\mu\nu} \tilde{F}_{\rho\sigma},$$

where

$$\tilde{F}_{\mu\nu} = \hat{F}_{\mu\nu} - \frac{1}{2} \left( Ψ_\mu A_\nu - Ψ_\nu A_\mu \right),$$

with

$$A_\mu = Φ^{-1/2} \tilde{A}_\mu, \quad \hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$
The field equation with respect to $\tilde{A}_\mu$ as an independent variable will be somewhat complicated because $\tilde{F}_{\rho\sigma}$ now depends on the un-differentiated $\tilde{A}_\rho$ even in flat spacetime as illustrated by
\[
\frac{\partial \tilde{F}_{\rho\sigma}}{\partial \tilde{A}_\mu} = -\frac{1}{2} \left( \delta^\mu_{\rho} \Psi_\sigma - (\rho \leftrightarrow \sigma) \right).
\] (4.4)

Let us present the following explicit calculation only in flat spacetime, for the moment, with understanding the “comma-goes-to-semicolon” rule [12], except for the additional term of $R^\nu_{\mu\lambda} A^\lambda$, as we encountered in (2.19) and (3.9). From (4.1) we then derive
\[
\left( \partial_\nu + \frac{1}{2} \Psi_\nu \right) \tilde{F}^{\nu\mu} = 0.
\] (4.5)

We impose the gauge condition
\[
\tilde{\chi} = \partial_\mu \left( \Phi \tilde{n} \tilde{A}^\mu \right) = 0,
\] (4.6)

which corresponds to choosing $n = \tilde{n} + 1/2$ in (2.10). Due to this condition we have
\[
\partial_\mu \tilde{A}^\mu = -\tilde{n} \Psi_\nu \tilde{A}^\nu.
\] (4.7)

Substituting this into (4.5) together with (4.2) yields
\[
\Box \tilde{A}^\mu + \left( \tilde{n} - \frac{1}{2} \right) \Psi_\nu \left( \partial^\nu \tilde{A}^\mu \right) + \left( \text{terms of } r = 0 \right) = 0.
\] (4.8)

Remarkably, the explicit scalar-field dependence appears only in the terms of rank $r = 0$ if $\tilde{n} = 1/2$.

We also note that the corresponding field equation for $A_\mu$ given by (2.12) reduces to
\[
\Box A^\mu + \left( \text{terms of } r = 0 \right) = 0,
\] (4.9)

if no scalar field is present. We thus find that the field equation with respect to $\tilde{A}_\mu$ with the scalar field included but with the special choice $\tilde{n} = 1/2$ is equivalent to the field equation with respect to $A_\mu$ without the scalar field, as far as we confine ourselves to the terms of $r = 2$ and $r = 1$. In view of the gauge invariance of any of the physical observables, we come to conclude that the description in terms of $\tilde{A}_\mu$ is essentially the scalar-field-free description in terms of $A_\mu$ for any sensible physical situation in which geometric optics applies including the amplitude variation corresponding to $r = 1$.

We close the paper by adding another comment: We carried out our analysis exclusively in the CF identified with BDCF, though we are outside the pure BD model by introducing the WEP violating interaction. As emphasized in 4.4.3 of [3], however, this CF is most likely different from the physical CF in which we should expect time-independent masses of particles, also with acceptable cosmological evolution in the presence of a cosmological constant. In moving to the physical CF, we apply a conformal transformation with the function basically behaving like $\Omega \sim t^\eta$, with $\eta$ a constant of the order one.

We find, on the other hand, that the physically observed time-delay is obtained by spatially integrating the ratio
\[
\frac{dt}{dr} = \frac{dt/d\lambda}{dr/d\lambda} = \frac{k^t}{k^r}.
\] (4.10)

It also follows that this ratio will be transformed roughly in the same manner as for $k^r$, resulting in a multiplicative factor $f = 1 + \eta(\Delta t/t_0)$, where $t_0 \sim 10^{10}y \sim 10^{17}$sec while $\Delta t \sim 10^8$sec for the approximate travel time of radiation in the solar system, yielding $f \sim 1$ to the accuracy of $\sim 10^{-14}$.

The same argument applies also to the deflection of light, for which the ratio (4.10) is replaced by $d\phi/dr = k^\phi/k^r$. 
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APPENDIX A: DERIVING (3.10)

Consider (3.8),

\[ B^\nu = a^\nu \nabla_\mu k^\mu - a^\mu \nabla_\mu k^\nu + k^\nu \nabla_\mu a^\mu - k^\mu \nabla_\mu a^\nu + k_\mu (\nabla^\mu a^\nu - \nabla^\nu a^\mu). \]  (A.1)

Multiply with \( \epsilon_\nu \) obtaining

\[ \epsilon_\nu B^\nu = (\epsilon \cdot a) \nabla_\mu k^\mu \]  (A.2)

\[ -a^\mu \epsilon_\nu \nabla_\mu k^\nu \]  (A.3)

\[ + k^\mu \epsilon_\nu \nabla_\mu a^\nu \]  (A.4)

\[ - (\epsilon \cdot k) \nabla_\mu a^\mu \]  (A.5)

\[ + k_\mu \epsilon_\nu \nabla^\mu a^\nu \]  (A.6)

\[ - k_\mu \epsilon_\nu \nabla^\nu a^\mu. \]  (A.7)

We find immediately

\[ (A.2) = A \nabla_\mu k^\mu, \]  (A.8)

\[ (A.3) = 0, \]  (A.9)

also with

\[ (A.10) = -a^\mu [\nabla_\mu (\epsilon \cdot k) - k^\nu (\nabla_\nu \epsilon_\nu)] = a^\mu k^\nu (\nabla_\mu \epsilon_\nu), \]  (A.10)

\[ (A.11) = k^\mu [\nabla_\mu (\epsilon \cdot a) - a^\nu (\nabla_\nu \epsilon_\nu)] = k^\mu \nabla_\mu A - k^\mu a^\nu (\nabla_\mu \epsilon_\nu), \]  (A.11)

\[ (A.12) = k_\mu [\nabla^\mu (\epsilon \cdot a) - a^\nu \nabla^\mu (\epsilon_\nu)] = k_\mu \nabla_\mu A - k^\mu a^\nu (\nabla^\mu \epsilon_\nu), \]  (A.12)

\[ (A.13) = -k_\mu \epsilon_\mu (\nabla^\mu a^\nu) = -k_\mu \epsilon_\mu [([\nabla^\mu \epsilon^\nu] \cdot A + \epsilon^\nu (\nabla^\mu A)], \]  (A.13)

Collecting them we find

\[ \epsilon_\nu B^\nu = A (\nabla_\mu k^\mu) + 2k^\mu (\nabla_\mu A) + \mathcal{R}, \]  (A.14)

where the remainder is given by

\[ \mathcal{R} = (A.10) + (A.11) + (A.12) + (A.13) = -2k^\nu a^\mu (\nabla_\mu \epsilon_\nu), \]  (A.15)

which vanishes because

\[ (A.16) = a^\nu (\nabla_\mu \epsilon_\nu) = A \epsilon^\nu (\nabla_\mu \epsilon_\nu) = \frac{A}{2} (\nabla_\mu (\epsilon \cdot \epsilon)) = \frac{A}{2} \partial_\mu (1) = 0. \]  (A.16)

We thus obtain \( (A.10) \):

\[ \epsilon_\nu B^\nu = A (\nabla_\mu k^\mu) + 2k^\mu (\nabla_\mu A). \]  (A.17)
We add that (A.17) allows a natural solution of $A \propto r^{-1}$. Consider spherically symmetric 3-space, for which we have

$$(\nabla_\mu k^\mu) = (\partial_\theta + \cot \theta) k^\theta + \partial_\phi k^\theta + \left( \partial_r + \frac{2}{r} \right) k^r.$$  \hspace{1cm} (A.18)

Make simplifying assumptions

$$k^\theta = k^\phi = 0, \quad \text{and} \quad \partial_r k^r \approx 0,$$  \hspace{1cm} (A.19)

to put (A.17) into

$$A \frac{2}{r} k^r + 2k^r \frac{dA}{dr} \approx 0,$$  \hspace{1cm} (A.20)

finding

$$A(r) \propto r^{-1}.$$  \hspace{1cm} (A.21)