Coupled dynamics of atoms and radiation pressure driven interferometers

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We consider the motion of the end mirror of a cavity in whose standing wave mode pattern atoms are trapped. The atoms and the light field strongly couple to each other because the atoms form a distributed Bragg mirror with a reflectivity that can be fairly high. We analyze how the dipole potential in which the atoms move is modified due to this backaction of the atoms. We show that the position of the atoms can become bistable. These results are of a more general nature and can be applied to any situation where atoms are trapped in an optical lattice inside a cavity and where the backaction of the atoms on the light field cannot be neglected. We analyze the dynamics of the coupled system in the adiabatic limit where the light field adjusts to the position of the atoms and the light field instantaneously and where the atoms move much faster than the mirror. We calculate the side band spectrum of the light transmitted through the cavity and show that these spectra can be used to detect the coupled motion of the atoms and the mirror.

I. INTRODUCTION

In recent years the coupling of the light field inside an optical resonator to the mechanical motion of the end mirrors [1, 2, 3] has received renewed attention due to several developments. The improved capabilities to micromachine sub-micrometer sized mechanical structures with well defined mechanical and optical properties enables the creation of moveable mirrors with small damping whose motion is strongly affected by the radiation pressure due to the light field inside an optical cavity. The coupled nonlinear dynamics found in those systems is very rich [4, 5] and is relevant for systems as large as gravitation wave antennas [6, 7, 8] down to micromachined cavities [9]. The radiation pressure force can be used to isolate the mechanical motion of the end mirror from the seismic background [2]. Recently, schemes for active cooling of mirrors with radiation pressure have been proposed [9] and implemented [10, 11, 12]. It is expected that with these methods cooling of the mirror motion to its quantum mechanical ground state will be possible and that the quantized motion of a macroscopic object, including non-classical states of motion and squeezing, can be studied this way [13, 14, 15]. Also, the mirror could become entangled with the light field [16], and questions related to position measurements near the quantum limit can be addressed. In the classical regime the mirror motion can exhibit instabilities, self-sustained oscillations [4] and chaos due to the retardation of the electromagnetic field in certain cases.

From a theoretical point of view, radiation pressure driven cavities offer many challenges and opportunities. These systems provide excellent testing grounds for theories of systems with delay [8, 17]. Furthermore quantizing the light field is a nontrivial question in principle for time-dependent boundary conditions [18].

In this paper we extend the study of radiation pressure driven interferometers to the case where atoms are trapped in the optical dipole potential generated by the standing-wave light field inside the cavity, see Fig. 1. In this system the atoms, light field and mirror interact with each other on an equal footing in the most general case. We derive the equations governing this system from first principles. Treating the motion of the mirror as well as the light field classically, we show that in many cases of interest the motion of the atoms can also be described classically. We restrict ourselves to those situations in this paper. Our key result is that, surprisingly if one considers the complexity of the problem, it is possible to develop an effective theory that captures many of the aspects of the system and is amenable to a fairly simple analysis under moderate assumptions. The key observation is that the effect of the atoms on the light field can be represented by an effective mirror whose reflection and transmission coefficient can be calculated from the self-consistently determined atomic density distribution. As we will discuss the atoms naturally arrange themselves in a configuration corresponding to a Bragg mirror of maximum reflectivity for a given number of atoms [14]. Thus, all degrees of freedom of the atoms collapse into a single coordinate describing the position of this fictitious mirror.

The collective back-action of the atoms, as described by what we may call the “atom mirror”, strongly affects the light field. Thus the atoms qualitatively change the potential in which they move themselves. Furthermore, because of the modifications of the radiation pressure force, they also influence the mirror motion. As a consequence of the back-action on the light field the potential wells in which the atoms sit become narrower and for certain mirror positions exhibit two local minima per half optical wavelength, i.e. the atomic position becomes bistable. The character of the bifurcation...
points where the single potential well branches off into two valleys can be modified by changing the cavity parameters, such as e.g. the reflectivities of the two end mirrors. In that way it is possible to choose which of the two valleys is the deeper one and accordingly in which direction the atoms will move. The two side valleys can also be made equally deep, in which case the branching point acts like an atomic beam splitter. We also show that the atomic motion can exhibit hysteresis, where the atoms move along different paths depending on the direction of motion of the mirror, giving rise to additional instabilities of the mirror motion.

In this paper we consider only the adiabatic limit where the mirror moves slowly compared with the atoms, which in turn have a much slower response time than the intracavity light field. Even in that limit the coupled system exhibits a remarkably rich phenomenology that we illustrate by analyzing and discussing the modified dipole potential felt by the atoms as well as the modified motion of the end mirror.

We propose to measure the sideband spectrum of the light transmitted through the cavity as a quantitative indicator of the coupled atom-mirror motion, as the coupling of the mirror motion to the atoms gives rise to unambiguous spectral signatures that should be within easy experimental reach.

This article is organized as follows: In Section II we describe the model for the coupled atom-mirror-field system and derive the basic dynamical equations. In section III we consider the dynamics of the mirror with field and atoms following its motion adiabatically, and discuss several aspects of the atomic motion. In section IV we calculate the spectrum of the light transmitted through the cavity and show that it can serve as a convenient probe of the coupled mirror-atom motion. Finally, section V is a conclusion and outlook.

II. MODEL

The system under consideration is shown schematically in Fig. I. Atoms are trapped in the standing-wave light field of a Fabry-Pérot cavity with one of its end mirrors allowed to move under the effect of radiation pressure and subject to a harmonic restoring force, $\xi$ being the displacement of the mirror from its rest position. The cavity length offset $L$ only gives rise to a shift of the resonances and we will set it to zero in all that follows.

We measure all lengths in units of the vacuum wavelength of the light and all times in units of $\Omega_P^{-1}$, where $\Omega_P$ is the oscillator frequency of the unperturbed moving mirror. The electrical fields are described in terms of their intensity integrated over the beam profile, i.e. in terms of power.

A. Atoms

Using $\hbar \Omega_P$ as the unit of energy and $\hbar k_0 = \hbar 2\pi/\lambda_0$ as the unit of momentum the free hamiltonian of the atoms takes the form

$$\hat{H} = \int d^3x \hat{\psi}^\dagger(x) \left[ -\frac{E_R}{4} \nabla^2 + \frac{U_0}{2} \hat{\psi}^\dagger(x) \hat{\psi}(x) \right] \hat{\psi}(x), \quad (1)$$

where $\hat{\psi}$ and $\hat{\psi}^\dagger$ are the atomic field annihilation and creation operators. Our theory can easily be adapted to describe fermions or bosons although at the level of our present theory there are no significant effects due to the statistics of the atoms. For the sake of definiteness we nonetheless assume that the field operators in Eq. (1) satisfy bosonic commutation relations. The energy

$$E_R = \frac{(2\hbar k_0)^2}{2M\hbar \Omega_P},$$

with $M$ the mass of the atoms is the two-photon recoil energy and

$$U_0 = \frac{E_R a}{\lambda_0 \pi},$$

with $a$ the s-wave scattering length is the interaction strength between the atoms describing s-wave collisions.

We assume that the atoms can be approximated as two-level atoms with transition frequency $\omega_\perp$ and we treat their interactions with the light field in the dipole and rotating wave approximations. We consider only one state of polarization of the light field and describe this component by a scalar field. Furthermore we assume that the intracavity light field of frequency $\omega_L$ is far detuned from the atomic transition, $|\Delta| \equiv |\omega_L - \omega_\perp| \gg \Omega_R, \gamma$. Here $\Omega_R$ is the Rabi frequency of the transition and $\gamma$ is its linewidth. Then we can adiabatically eliminate the upper state and, choosing the $z$ axis along the direction of propagation of the light field, the interaction between light field and atoms is given by the dipole potential

$$\hat{H}_{\text{int}} = g \int d^3x \hat{\psi}^\dagger(x) \hat{\psi}(x) |E(z)|^2 |u_\perp(x, y; z)|^2 \quad (2)$$

where

$$g = \frac{2}{\hbar^2 \epsilon \kappa \lambda_0^2 \Omega_P^2} \frac{\varphi^2}{\Delta}, \quad (3)$$

where $\varphi$ is the electric dipole moment along the polarization direction of the laser field and the electrical field has been split into a part varying along the cavity axis $z$ and a transverse profile $u_\perp(x, y; z)$ assumed to be normalized to unity,

$$\int dxdy |u_\perp(x, y; z)|^2 = 1. \quad (4)$$

for every $z$. For simplicity we assume in the following that the $z$-dependence of $u_\perp$ along the atomic cloud can
be neglected. We further assume that the light field is red detuned, $\Delta < 0$, so that the atoms are attracted to the intensity maxima.

The standing wave that forms in the cavity results in an optical lattice potential. We assume that its wells are very deep, i.e. $g|E_0|^2 \gg E_R$ where $E_0$ is the amplitude of the standing wave. In the language of condensed matter physics we assume that the atoms are deeply in the Mott insulator regime \cite{Mott}. It is then a good approximation to assume that every site has a specific number of atoms in the ground state of the harmonic oscillator potential representing the lattice at that site. Due to the back-action of the atoms on the light field the distance between the lattice sites is smaller than $\lambda_0/2$, a point to which we return shortly.

The interaction between atoms only gives rise to a constant energy shift. Its influence on the localized wave functions is irrelevant since all that will be required in the following is that the atomic density at every site is negligible. We further assume that the light field is an optical lattice potential. We assume that its wells are very deep, i.e. $\Delta E < 0$, so that the atoms are attracted to the intensity maxima.

The order of magnitude of the localization is given by the localized within much better than an optical wavelength.

The standing wave action of the atoms on the light field the distance between the lattice sites is smaller than $\lambda_0/2$, a point to which we return shortly.

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The standing wave action of the cavity results in a stack of pancake shaped discs.

For large detuning, optically the oscillator energy, we are justified in neglecting thermal excitations. If the atoms start in the lattice ground state, the atomic motion is then reduced to the motion of a Gaussian wavepacket in a slowly changing harmonic potential, a situation that is well described by the classical dynamics of the atomic center of mass.

Because atoms at different sites see the exact same potential at all times they will be at the same displacement from their respective local minima given that they started at the bottom of their respective potential wells. Therefore it is possible to describe the atomic motion in terms of a single coordinate $z_a$, the position of each atom modulo the lattice period.

\begin{equation}
\frac{\pi w^2 E_R}{4g|E_0|^2 k_0^2}^{1/4}
\end{equation}

along the cavity axis and

\begin{equation}
\frac{\pi w^2 E_R}{4g|E_0|^2}^{1/4}
\end{equation}

along the cavity axis and

\begin{equation}
\frac{\pi w^2 E_R}{4g|E_0|^2 k_0^2}^{1/4}
\end{equation}

in the transverse direction. For typical laser powers and detunings the oscillator length is roughly ten to a hundred times smaller than the optical wavelength. The oscillator frequencies are

\begin{equation}
\sqrt{\frac{g|E_0|^2 k_0^2 E_R}{\pi w^2}}
\end{equation}

for oscillations along the cavity axis and

\begin{equation}
\sqrt{\frac{g|E_0|^2 E_R}{\pi w^4}}
\end{equation}

for oscillations in the transverse direction. The parameters that we are interested in are such that the oscillator length in the longitudinal direction is much smaller than an optical wavelength and the oscillator length in the transverse direction is much smaller than the beam waist. Since the beam is typically much wider than an optical wavelength the atoms can be thought of as forming a stack of pancake shaped discs.

We consider an optical lattice with $N$ sites occupied by $n$ atoms each. Due to the phase shift suffered by the light upon propagation through each atomic sheet the lattice sites move closer together, the self-consistent period of the resulting lattice being determined later on. If the temperature of the atoms is far below the oscillator energy, we are justified in neglecting thermal excitations. If the atoms start in the lattice ground state, the atomic motion is then reduced to the motion of a Gaussian wavepacket in a slowly changing harmonic potential, a situation that is well described by the classical dynamics of the atomic center of mass.

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\begin{equation}
\alpha^{-1} = m_P \Omega_P^2 \lambda_0 c
\end{equation}

is a power characteristic of the mirror. The electromagnetic field is treated classically and from Fig. 2 we read off that

\begin{equation}
F_{\text{RP}} = \frac{1}{c} (|E_1|^2 + |E_3|^2 - |E_4|^2 - |E_2|^2) = \frac{2|R_2|^2 |E_1|^2}{c}
\end{equation}

where we have used that in our situation $E_2 \equiv 0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{Field geometry at movable mirror.}
\end{figure}

\subsection*{B. Moving mirror}

The mirror motion is described by the Newtonian equation of motion

\begin{equation}
\ddot{\xi} + \Gamma \dot{\xi} + \xi = \frac{F_{\text{RP}}}{m_P \Omega_P^2 \lambda_0} \equiv acF_{\text{RP}},
\end{equation}

where $c$ is the speed of light, $\Gamma^{-1}$ is the mechanical quality factor of the mirror, $F_{\text{RP}}$ is the radiation pressure force, and

\begin{equation}
\alpha^{-1} = m_P \Omega_P^2 \lambda_0 c
\end{equation}

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\begin{equation}
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\end{equation}

where we have used that in our situation $E_2 \equiv 0$.

\subsection*{C. Light field}

We now turn to the back-action of the atoms on the light field. The influence of the cavity mirrors will be considered in the next section.

It is well known \cite{Mott} that for large detuning, optically driving the atoms results in the polarization

\begin{equation}
P = \frac{-2\psi^2}{\hbar \Delta} \psi \psi^\dagger E u\perp
\end{equation}

where $E u\perp = E u_\perp$.
leading to the inhomogeneous optical field propagation equation
\[ \nabla^2 (E u_\perp) + k_0^2 (E u_\perp) = \chi |\psi|^2 (E u_\perp) \] (12)
where in our units the polarizability is given by
\[ \chi = \frac{2 \mu_0 \Omega_F}{h} \frac{\varphi^2 J_0^2}{\Delta} \] (13)

In principle, the shape of the transverse profile \( u_{\perp} \) is the result of the complicated dynamics resulting from the repeated propagation of the light field through the cavity and its interaction with the complicatedly structured dielectric constituted by the atoms. A complete description of these processes is beyond the scope of this paper. We instead adopt the point of view that, specifically, we assume that \( u_{\perp} \) varies slowly with \( z \), \( |\partial u_{\perp}/\partial z| \ll |k_0 u_{\perp}| \) so that we can neglect its derivatives with respect to \( z \). Then, after dividing by \( E u_{\perp} \) to separate the variables, we have from the wave equation
\[ \frac{\partial^2 E(z)}{\partial z^2} + k_0^2 E(z) = \beta^2(z) E(z) \] (14)
where \( \beta(z) \) is determined for every \( z \) by the eigenvalue problem
\[ \chi |\psi(x, y, z)|^2 u_{\perp}(x, y, z) - \nabla^2_{\perp} u_{\perp}(x, y, z) = \beta^2(z) u_{\perp}(x, y, z). \] (15)
In principle this formula solves our problem since it expresses the propagation equation for \( E \) in terms of \( \psi \) and \( u_{\perp} \) independently of how the latter was obtained. In order to obtain a more transparent formula we multiply Eq. (15) by \( u_{\perp}^* \), integrate over \( x \) and \( y \) and after a partial integration we find
\[ \\beta^2(z) = \chi \int dx dy |u_{\perp}(x, y, z)|^2 |\psi(x, y, z)|^2 + \int dx dy \left( \nabla_{\perp} u_{\perp}^* (x, y; z) \cdot \nabla_{\perp} u_{\perp} (x, y; z) \right) \]
\[ \equiv \chi |\psi(z)|^2 + k_{\perp}^2(z), \] (16)
ending up with the intuitively appealing equation
\[ \frac{\partial^2 E(z)}{\partial z^2} + (k_0^2 - k_{\perp}^2(z)) E(z) = \chi |\psi(z)|^2 E(z), \] (17)
where \( |\psi(z)|^2 \) can be interpreted as the density distribution averaged over the transverse beam profile. In the following we neglect the reduction of the wave vector \( k_{\perp} \) due to the finite extend of the beam in the transverse direction, an approximation valid if the beam is not too strongly focused.

For the situation at hand \( |\psi(z)|^2 \) is readily found, since the wave function of the harmonic oscillator factorizes into a \( z \)-dependent part and a transverse part. As discussed above the atomic density distribution is much narrower than the beam waist \( w \). Hence we find, assuming a Gaussian beam profile,
\[ |\psi(z)|^2 = \frac{1}{\pi w^2} |\psi_z(z)|^2, \] (18)
where \( \psi_z(z) \) is the \( z \)-dependent factor of the oscillator wave function of the atoms.

Equation (17) allows us to determine the effect of the atoms on the light field. The basic idea is to replace the entire configuration of atoms by a fictitious “atom mirror” at some position \( z_a \) and hence to describe them in terms of a pair of effective reflection and transmission coefficients. The period of the self-consistent lattice automatically fulfills the Bragg condition. Due to the self-consistency requirement this is even true for a lattice with nonuniform filling, see Ref. [10] for more details. (Parenthetically, this means that we can take into account an external trapping potential through an appropriate choice of effective \( N \) and \( n \). The thermal motion of the atoms will lower the reflectivity of the atom mirror and could be taken into account through a Debye-Waller factor.) All that remains to do to describe the effect of the atoms on the light field is to determine the transmission and reflection coefficient \( T \) and \( R \) of the atom mirror. A schematic of the situation is shown in Fig. 3.

Consider first a single atomic sheet. Making use of the fact that the atoms are localized in the \( z \)-direction to a much narrower region than an optical wavelength, we readily find the transmission and reflection coefficient
\[ T_s = \sqrt{ \frac{1}{1 + \Lambda^2} } e^{i \phi}, \] (19)
\[ R_s = \frac{\Lambda}{1 + \Lambda^2} e^{i \phi}, \] (20)
by matching boundary conditions at the atom sheet. Here
\[ \phi = - \arctan \Lambda \] (21)
is the phase shift suffered by light upon transmission through the sheet of atoms and we have introduced the...
The entire lattice of atoms can be considered as a periodically stratified dielectric medium. Its total reflection and transmission coefficients are readily found using standard methods of the optics of dielectric films \[19, 22\]. In essence, one can find the transfer matrix of a unit cell of the lattice from the transmission and reflection coefficient Eq. (20 \[19\]). The transfer matrix of the entire lattice is then the \(N\)th power of the transfer matrix of the unit cell. This can be done in closed form using Tchebychev polynomials since we neglect scattering losses and therefore the transfer matrix of the unit cell is unimodular. From the total transfer matrix the total reflection and transmission coefficient are extracted as \[25\].

\[
R = \frac{-i\Lambda e^{2i\phi}}{1 - i\Lambda} \\
T = \frac{e^{2i\phi \Lambda}}{1 - i\Lambda}.
\]

An example of the transmission and reflection coefficient of such a lattice of atoms is shown in Fig. 4 for the case of \(^{87}\text{Rb}\). From the figure it is apparent that rather high reflectivities can be obtained in principle. One of the main experimental difficulties will be to achieve fairly high filling factors over many sites. Also, the high reflectivities in Fig. 4 were obtained by using a small beam waist and a fairly small detuning of around 100 linewidths. It should be noted that already fairly moderate reflection coefficients of the atoms of around 10% give rise to interesting effects as we discuss later in this paper. S. Slama et al. have recently reported Bragg reflection coefficients of the order of 30% \[22\]. (Note that the assumption of a constant and small beam waist is not as bad as might be expected, as the Gaussian shaped atomic wavepackets at every site act like lenses that constantly refocus the beam. A more detailed analysis of the spatial mode structure of the light field inside the cavity will be presented in a future publication.)

In the following we consider the three representative cases of an empty cavity, a cavity containing atoms giving rise to an intermediate reflection coefficient of a few tens of a percent and atoms with a reflection coefficient close to one.

**III. ADIABATIC DYNAMICS**

We restrict ourselves to the limit where the motion of the moving mirror is extremely slow compared to the motion of the atoms which is in turn very slow compared to the dynamics of the light field. For this to be true we must have \(\Omega_P \ll \omega_{osc,z}\) and \(\omega_{osc,z} \ll \kappa\) where \(\kappa\) is the linewidth of the cavity. This is the most relevant case for the typical systems currently available. The light field then follows the atoms adiabatically, so that both light field and atoms adjust instantaneously to any position of the moving mirror. We can then eliminate the atoms and the light field and express the radiation pressure force acting on the mirror in terms of the mirror position alone.

To obtain the effective radiation pressure force in presence of the atoms we observe that, given positions of the mirror and of the atoms, the equilibrium light field is determined by the boundary conditions at the fictitious “atom mirror,” see Fig. 5. The steady-state boundary conditions are found to be

\[
\begin{align*}
\epsilon_1 &= T_1 e^{ik_0z} \sqrt{1 + e^{2ik_0z} R_1} \epsilon_2 \\
\epsilon_2 &= R \epsilon_1 + T \epsilon_3 \\
\epsilon_3 &= R_2 e^{2ik_0(L+\xi-z)} \epsilon_4 \\
\epsilon_4 &= T \epsilon_1 + R \epsilon_3.
\end{align*}
\]

from which we obtain, e.g. for \(\epsilon_4\),
\[ \epsilon_4 = \frac{-e^{ik_0 z_a} \sqrt{P_0^2 T_1^2}}{RR_1 e^{2ik_0 z_a} + RR_2 e^{2ik_0 (L + \xi - z_a)} - R_1 R_2 (R^2 - T^2) e^{2ik_0 (L + \xi)} - 1}. \] (30)

From these fields we find the adiabatic dipole potential felt by the atoms for a given displacement of the moving mirror as

\[ V(z_a; \xi) = g|\epsilon_4 + \epsilon_3|^2 = \frac{g P_0 |T_1|^2 (1 + |R_2|^2 - 2|R_2| \sin 2k_0 (L + \xi - z_a))}{|RR_1 e^{2ik_0 z_a} + RR_2 e^{2ik_0 (L + \xi - z_a)} - R_1 R_2 (R^2 - T^2) e^{2ik_0 (L + \xi)} - 1|^2}, \] (31)

a result that holds for every lattice site provided that \( R_1 \simeq R_2 \). Otherwise the depth of the potential wells is different at each lattice site. [20]

The potential \( V(z_a; \xi) \) is periodic with period \( \lambda_0/2 \) in both \( z_a \) and \( \xi \) direction. In the numerator we have, for \( |R_2| \) close to one, the well known \( \cos^2 k_0 (L + \xi - z) \)-type potential. Its minima are moving as the cavity end-mirror is moving.

The effect of the denominator, which gives rise to a series of resonances, is somewhat less intuitive. If the reflection coefficient of the atoms is small their back-action on the cavity field is weak and we see essentially the reflection coefficient of the atoms is high, see Fig 7. In this case the adiabatic motion of the atoms is particularly interesting if the fixed mirror has the higher reflectivity. We discuss it in detail for the case of small atomic reflectivity, but only for a very narrow range of \( \xi \) values near \( \xi \approx 0.25 \).

The relative depth of the two local minima is determined by the reflectivities of the two end mirrors. The potential well closer to the mirror of higher reflectivity is deeper. As illustrated in Fig. 8 the system shows behavior reminiscent of an “avoided crossing” where the two resonances corresponding to the two modes of motion cross: In positions where the atoms would be on resonance with both mirrors the potential depth has a fairly narrow local maximum. The width of the potential barrier arising this way becomes smaller as the reflectivity of the atomic mirror increases. If the moving mirror sweeps through such a resonance, the atoms have to decide which branch of the potential valley to follow on the other side of the resonance [28]. Since we can decide which branch of the potential well is deeper by choosing which branch of the potential, it is possible to manipulate the motion of the atoms by controlling the reflectivity of the end mirrors.

The adiabatic motion of the atoms is particularly interesting if the fixed mirror has the higher reflectivity. We discuss it in detail for the case of small atomic reflectivity. The trajectories of the local minima corresponding to the mirror moving in the positive, resp. negative \( \xi \) direction are drawn as a green dashed line and a blue solid line, respectively, in Fig. 9. If the moving mirror starts in a region far off resonance, e.g. at the green dot labeled “1” in Fig. 9 there is only one potential minimum corresponding to the maximum of the numerator in the dipole potential. Moving in the direction of growing \( \xi \) the po-
Figure 7: Dipole potential for higher atomic reflectivity corresponding to $T = 0.05$ and $T_1 = 0.2$, $T_2 = 0.1$. For this choice of reflectivities of the cavity end mirrors the potential well closer to the moving mirror is always deeper and, in contrast to Fig. 6, the atoms move on diagonals following the bottom of the potential well for both directions of the mirror motion.

Figure 8: Enlargement of the resonance region of Fig. 7, showing that the potential well corresponding to motion with the moving mirror is deeper than the one corresponding to the resonance with the fixed mirror, and explaining why the atomic motion is locked to the mirror motion. The figure also shows the avoided crossing of the dipole potential.

Figure 9: (Color Online) Radiation pressure force for the two directions of motion of the mirror. The atoms take the paths shown in Fig. 6 for the mirror moving in the positive and negative $\xi$-direction, respectively. The difference in area of the two curves is a measure of the work performed on the moving mirror. All parameters as in Fig. 6.

The atomic motion is totally different if the moving mirror moves in the negative $\xi$ direction. Then the branch of the potential closer to the moving mirror provides a local minimum all the way and the atoms never move to the left branch, see the blue line in Fig. 6.

The motion of the atoms is similar for high atomic reflectivity. If the mirror is moving in the positive $\xi$ direction the atoms turn left at every resonance into the vertical potential valley. Just before reaching the zero potential line they roll to the right and fall into the diagonal valley. If the mirror moves back in the negative $\xi$ direction the atoms also turn left at every resonance and move to the left into the diagonal potential valley before reaching the zero potential line.

Interestingly, the radiation pressure force is different for the two distinct paths the atoms take as the mirror is moving back and forth, see Fig. 6. Hence the light field performs work on the mirror in every round trip, and this work can be fairly large. It should be noted however that our model is too simplistic to study this feature quantitatively. First, the assumption of adiabaticity of the atomic motion is likely to break down near the zeros of the numerator, where the atoms take a sharp turn to the right. The potential well becomes extremely shallow in this region and when the atoms fall into the other branch of the potential, both situations for which the assumption of adiabaticity of the atomic motion may not hold. If the atoms have some inertia, they could overcome the potential wall that prevents them from going over the potential hill and continue their motion on the other side. Tunneling through the barrier is another possibility. For these reasons we concentrate in the following on the case where...
the atoms move with the moving mirror, i.e. where the reflectivity of the moving mirror is higher than that of the fixed mirror. For this type of motion the adiabaticity assumption holds at all times.

Once the position of the atom mirror $z_a(\xi)$ has been determined we can find the radiation pressure force by inserting $z_a(\xi)$ into the expression for $\epsilon_4$. Fig. 10 shows the three different cases of low atomic reflection coefficient, high atomic reflection coefficient as well as an empty cavity for comparison. A small atomic reflectivity broadens the resonance. Also, the peak radiation pressure force increases. As the atomic reflectivity increases the resonance becomes wider and develops a broad dip in the middle. For very high atomic reflectivities adjacent resonances “collide” and give rise to an avoided crossing behavior that results in the double peak structure of Fig. 10. The double resonances in this regime are much wider than the empty cavity resonances and the peaks are substantially higher. Also the radiation pressure force never falls off to values as low as for the empty cavity. The low intensity region between the double peaks are the limiting shape of the aforementioned dips that develop in the resonant lineshape.

From the radiation pressure force we find the total potential of the moving mirror by a simple integration and by adding the harmonic potential due to the restoring force. Figure 11 shows the total potentials, again for the three different cases considered for the radiation pressure force above. Every time the mirror moves through a resonance the potential makes a steep step down. The steps are highest if the atoms have intermediate reflectivity due to the broad and high resonances. The global minimum of the potential is near $\xi \approx 5$ in this case. They are smaller for the double peaks for high atomic reflectivity and smallest for the empty cavity.

IV. SIGNATURES OF THE COUPLED MOTION

The question of how to detect the coupled motion of the mirror and the atoms naturally arises. A possible observable is the spectrum of the light transmitted through the cavity. The amplitude and phase modulation of the optical field due to the mirror motion are detectable as sidebands in the spectrum of transmitted light.

To find the spectrum of the transmitted light we integrate the Newtonian equations of motion of the mirror subject to the modified radiation pressure force using the velocity Verlet algorithm. [29]

A. Large amplitude oscillations

First we analyze what happens as the mirror undergoes large amplitude oscillations such that in every period it sweeps over several resonances. [30] We consider again the three cases of atoms with high reflectivity, intermediate reflectivity and an empty cavity. The cavity and atomic parameters are the same as before and as for the quality factor of the cavity we choose $\Gamma^{-1} = 10^4$.

Figure 12 shows the resulting sideband spectra. For clarity each data point has been obtained by averaging over 1000 frequency bins. The spectrum of the empty cavity falls off nearly exponentially, a consequence of the near-Lorentzian lineshape of the resonances and the fact that the mirror moves through each resonance at nearly constant velocity.
The widths of the transmission spectra are comparable in all three cases, confirming that the widths of the resonances are of the same order. However the spectra acquire a peak near $\omega = 0$ for nonzero atomic reflectivities, a consequence of the larger resonance widths in these cases; in addition the light field never falls off to zero, in contrast to the empty cavity case where it does so to a good approximation for high-reflectivity mirrors. Especially for high atomic reflectivity the cavity always leaks a significant amount of light that gives rise to an enhanced DC component in the transmission spectrum.

The fringes that appear in the case of high atomic reflectivity can be understood in the following way: two light pulses exit the cavity whenever the mirror passes through a resonance, see Fig. 10, and they give rise to a double-slit diffraction pattern in frequency space. Their spacing corresponds to the inverse of the time it takes the mirror to travel from the first peak to the second peak of the double resonance.

The inset of Fig. 12 shows the details of the low frequency part of the spectrum for the case of high atomic reflectivity. (That low frequency part is similar in all cases.) It consists of a peak at the frequency corresponding to the time it takes the mirror to move from one resonance to the next and of a series of additional peaks at each harmonic. These strong components are a result of the highly nonlinear potential. The fact that the mirror moves at different speeds between different resonances gives rise to a broadening of the harmonics into bands whose width increases with growing harmonic number to finally overlap. Effects due to the damping of the mirror motion cannot be resolved because of this “inhomogeneous broadening”.

B. Small amplitude oscillations

In our units the average thermal energy of the mirror at room temperature is $T \approx 10^{-5}$. Comparing with the energy scale in Fig. 11 we see that the mirror can undergo many oscillations in each one of the sawtooth-shaped local potential minima before it hops to a different local minimum due to thermal activation or some other perturbation. Thus it is an interesting question to determine the oscillation frequency at every local minimum for small amplitude oscillations.

Figure 13 shows these frequencies for the three cases already discussed. For high atomic reflectivity, and in contrast to the intermediate reflectivity and empty cavity cases, the oscillation frequency near $\xi = 0$ is much lower than the restoring frequency of the mirror. This is because the resonances have wide wings in this case and, if added to the harmonic oscillator potential of the mirror, they can compensate the harmonic restoring force almost completely over a large range of displacements $\xi$ and give rise to an extremely flat potential near the local minimum. In the other two cases the onset of the radiation pressure force near the resonances is very abrupt and the frequency of the mirror motion can never be significantly smaller than the oscillator frequency. At higher displacements from the mirror rest positions the stronger resonances found with atoms inside the cavity give rise to stronger confinement and lead to resonance frequencies that eventually become higher than in the empty cavity case. Note that for the empty cavity cases we could find only a few more local minima beyond $\xi = 8$ and these could not be reasonably approximated by a harmonic potential.
oscillations and chaotic behavior. It will be interesting phenomena such as instabilities, self-sustained retardation of the light field gives rise to many interesting, and so is the case where the retardation of the light field is taken into account. Already in an empty cavity each other on an equal footing is particularly interesting, and especially of its fluctuations, is interesting because of the powerful methods to measure the fluctuations of the electromagnetic field. If they are linked to fluctuations of the atomic position or the mirror position, as must be expected, one could then use the light field as a powerful diagnostic tool for the study of these mechanical fluctuations.

Many open questions remain to be studied. First of all a more careful analysis of the spatial mode structure inside the cavity in the presence of the atoms is desirable. Such an analysis could start from the Maxwell-Bloch equations and should include scattering losses due to spontaneous emission from the atomic excited state. It would yield more accurate values for the effective reflection coefficient of the atoms and the modified dipole potential, especially if $R_1$ and $R_2$ are very different.

One should also consider different adiabaticity regimes, or abandon the assumption of adiabaticity altogether. The situation where the atomic motion is roughly as fast as the mirror motion and the atom and mirror influence each other on an equal footing is particularly interesting, and so is the case where the retardation of the light field is taken into account. Already in an empty cavity the retardation of the light field gives rise to many interesting phenomena such as instabilities, self-sustained oscillations and chaotic behavior. It will be interesting to see how all these effects are influenced by the presence of the atoms.

A related question of practical significance is what happens if the atoms are subject to a friction force. Atoms can be efficiently cooled to very low temperatures with laser cooling and one could hope that by means of the coupling to the mirror through the cavity field one should be able to also cool the mirror. Loosely speaking, if the atoms are subject to a friction force, they lag slightly behind the position at which they ought to be according to the position of the mirror. This retardation has been seen to lead to the damping of the mirror motion in preliminary results, but these calculations need to be carried out in much more detail, especially since the momentum kicks in laser cooling, and hence the fluctuations in the atomic position, are substantial and cannot be neglected. They can be taken into account e.g. using a master equations approach.

Finally, one needs to address the situation where all constituents of our model are quantized. Quantization of the atomic motion should allow one to study whether the resonances can be used as beam splitters, and the properties of the resulting interferometer. Also, a detailed understanding of the role of damping of the atomic motion requires a proper quantum treatment. The quantization of the mirror motion is called for in view of the experimental goal to cool it to its quantum mechanical ground state. Finally, a quantum treatment of the light field, and especially of its fluctuations, is interesting because of the powerful methods to measure the fluctuations of the electromagnetic field. If they are linked to fluctuations of the atomic position or the mirror position, as must be expected, one could then use the light field as a powerful diagnostic tool for the study of these mechanical fluctuations.

VI. ACKNOWLEDGEMENTS

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[25] Strictly speaking, Eq. 24 is the reflection coefficient for a wave propagating in the positive $z$ direction with the reflection coefficient in the negative $z$ direction differing by an additional phase $e^{4iN\phi}$. This phase is however immaterial for our discussion since it can be absorbed in a redefinition of $z_a$. The transmission coefficients for the two directions are exactly equal as they have to be for general reasons.

[26] This should bring about only minor quantitative changes that are irrelevant for the limit of the adiabatic motion that we are considering here: We assume from the outset that the atoms are adiabatically moving in deep potential wells, i.e. they are always near the bottom of their respective wells and never see the tops of the potential barriers separating them.

[27] The additional $\lambda_0/4$ offset comes from our particular choice of phase of the reflection and transmission coefficients.

[28] The resonances act like beam splitters for the atoms and a more exact analysis should treat the motion of the atoms near these branching points quantum-mechanically. Also one can easily convince oneself that atoms originally in neighboring potential wells can end up in the same potential well if they choose to go into different branches at a resonance. This provides one with the essential ingredients for an interferometer that can in principle operate in a massively parallel fashion. This problem will be studied in greater detail in a future publication.

[29] The canonical nature of this algorithm is necessary because we have to integrate the equations of motion over very long time intervals, typically several ten thousand cycles. At the same time we have to integrate with high temporal resolution because the radiation pressure force is so sharply peaked. The position of the atoms is determined at every time step using Brent’s method in an interval centered at the atomic position of the previous time step and with a width that depends on the velocity of the mirror. This way we ensure that the atoms follow a continuous path along a potential valley without jumps to different valleys. The transmitted field at every time is easily found from $\epsilon_4$. The side band spectrum is found by numerically fourier transforming the field.

[30] To avoid too large velocities at the bottom of the potential, which would make an accurate numerical integration more difficult, we choose the initial conditions such that the mirror swings through roughly eight resonances. This number is not exact since the turning point on the left is typically at a resonance.