THE CYCLIC HOPF $H \mod K$ THEOREM

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Abstract. The $H \mod K$ theorem gives all possible periodic solutions in a $\Gamma$–equivariant dynamical system, based on the group-theoretical aspects. In addition, it classifies the spatio temporal symmetries that are possible. By the contrary, the equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for each $C$–axial subgroup of $\Gamma \times S^1$. In this paper we identify which periodic solution types, whose existence is guaranteed by the $H \mod K$ theorem, are obtainable by Hopf bifurcation, when the group $\Gamma$ is finite cyclic.

1. Introduction

In the formalism of equivariant differential equations [1], [2] and [3] have been described two methods for obtaining periodic solutions: the $H \mod K$ theorem and the equivariant Hopf theorem. While the $H \mod K$ theorem offers the complete set of possible periodic solutions based exclusively on the structure of the group $\Gamma$ acting on the differential equation, the equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for all $C$–axial subgroups of $\Gamma \times S^1$.

Not always all solutions predicted by the $H \mod K$ theorem can be obtained by the generic Hopf bifurcation [3]. In [4] there are described which periodic solutions, whose existence is guaranteed by the $H \mod K$ theorem are obtainable by the Hopf bifurcation when the group $\Gamma$ is finite abelian. In this article, we pose a more specific question: what periodic solutions predicted by the $H \mod K$ theorem are obtainable by the Hopf bifurcation when the group $\Gamma$ is finite cyclic. We will answer this question by finding which additional constraints have to be added to the Abelian Hopf $H \mod K$ theorem [4] so that the periodic solutions predicted by the $H \mod K$ theorem coincide with the ones obtained by the equivariant Hopf theorem when the group $\Gamma$ is finite cyclic.

2. The $H \mod K$ theorem

We call $(\gamma, \theta) \in \Gamma \times S^1$ a spatio-temporal symmetry of the solution $x(t)$. A spatio-temporal symmetry of $x(t)$ for which $\theta = 0$ is called a spatial symmetry, since it fixes the point $x(t)$ at every moment of time. The group of all spatio-temporal symmetries of $x(t)$ is denoted

\[ \Sigma_{x(t)} \subseteq \Gamma \times S^1. \]

As shown in [3], the symmetry group $\Sigma_{x(t)}$ can be identified with a pair of subgroups $H$ and $K$ of $\Gamma$ and a homomorphism $\Theta : H \to S^1$ with kernel $K$. Define

\[ K = \{ \gamma \in \Gamma : \gamma x(t) = x(t) \ \forall t \} \]
\[ H = \{ \gamma \in \Gamma : \gamma x(t) = \{ x(t) \} \ \forall t \}. \]

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The subgroup \( K \subseteq \Sigma_{x(t)} \) is the group of spatial symmetries of \( x(t) \) and the subgroup \( H \) consists of those symmetries that preserve the trajectory of \( x(t) \), i.e., the spatial parts of the spatio-temporal symmetries of \( x(t) \). The groups \( H \subseteq \Gamma \) and \( \Sigma_{x(t)} \subseteq \Gamma \times S^1 \) are isomorphic; the isomorphism is in fact just the restriction to \( \Sigma_{x(t)} \) of the projection of \( \Gamma \times S^1 \) onto \( \Gamma \). Therefore the group \( \Sigma_{x(t)} \) can be written as

\[
\Sigma^\Theta = \{(h, \Theta(h)) : h \in H, \Theta(h) \in S^1\}.
\]

Moreover, we call \( \Sigma^\Theta \) a twisted subgroup of \( \Gamma \times S^1 \). In our case \( \Gamma \) is a finite cyclic group and the \( H \mod K \) theorem states necessary and sufficient conditions for the existence of a periodic solution to a \( \Gamma \)-equivariant system of ODEs with specified spatio-temporal symmetries \( K \subset H \subset \Gamma \). Recall that the isotropy subgroup \( \Sigma_x \) of a point \( x \in \mathbb{R}^n \) consists of group elements that fix \( x \); that is they satisfy

\[
\Sigma_x = \sigma \in \Gamma : \sigma x = x.
\]

Let \( N(H) \) be the normalizer of \( H \) in \( \Gamma \), satisfying \( N(H) = \{\gamma \in \Gamma : \gamma H = H \gamma\} \). Let also \( \text{Fix}(K) = \{x \in \mathbb{R}^n : kx = x \ \forall k \in K\} \).

**Definition 1.** Let \( K \subset \Gamma \) be an isotropy subgroup. The variety \( L_K \) is defined by

\[
L_K = \bigcup_{\gamma \notin K} \text{Fix}(\gamma) \cap \text{Fix}(K).
\]

**Theorem 1.** (\( H \mod K \) Theorem [3]) Let \( \Gamma \) be a finite group acting on \( \mathbb{R}^n \). There is a periodic solution to some \( \Gamma \)-equivariant system of ODEs on \( \mathbb{R}^n \) with spatial symmetries \( K \) and spatio-temporal symmetries \( H \) if and only if the following conditions hold:

(a) \( H/K \) is cyclic;
(b) \( K \) is an isotropy subgroup;
(c) \( \dim \text{Fix}(K) \geq 2 \). If \( \dim \text{Fix}(K) = 2 \), then either \( H = K \) or \( H = N(K) \);
(d) \( H \) fixes a connected component of \( \text{Fix}(K) \setminus L_K \), where \( L_K \) appears as in Definition 1 above;

Moreover, if (a) – (d) hold, the system can be chosen so that the periodic solution is stable.

**Definition 2.** The pair of subgroups \((H, K)\) is called admissible if the pair satisfies hypotheses (a) – (d) of Theorem 1, that is, if there exist periodic solutions to some \( \Gamma \)-equivariant system with \((H, K)\) symmetry.

3. Hopf Bifurcation with Cyclic Symmetries

In the following we recall two results from [4] needed later for the proof of the Theorem 2. Let \( x_0 \in \mathbb{R}^n \). Suppose that \( V \) is an \( \Sigma_{x_0} \)-invariant subspace of \( \mathbb{R}^n \). Let \( \hat{V} = x_0 + V \), and observe that \( \hat{V} \) is also \( \Sigma_{x_0} \)-invariant.

**Lemma 1.** Let \( g \) be an \( \Sigma_{x_0} \)-equivariant map on \( \hat{V} \) such that \( g(x_0) = 0 \). Then \( g \) extends to a \( \Gamma \)-equivariant mapping \( f \) on \( \mathbb{R}^n \) so that the center subspace of \((df)_{x_0}\) equals the center subspace of \((dg)_{x_0}\).

**Proof.** See [4]. \( \square \)

**Lemma 2.** Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be \( \Gamma \)-equivariant and let \( f(x_0) = 0 \). Let \( V \) be the center subspace of \((df)_{x_0}\). Then there exists a \( \Gamma \)-equivariant diffeomorphism \( \psi : \mathbb{R}^n \to \mathbb{R}^n \) such that \( \psi(x_0) = x_0 \) and the center manifold of the transformed vector field

\[
\psi_* f(x) \equiv (d\psi)^{-1}_{\psi(x)} f(\psi(x))
\]

is \( \hat{V} \).
Proof. See [4]. □

In order to state the Cyclic Hopf theorem, we need first the following lemma.

Lemma 3. The group $\Gamma$ is cyclic if and only if it is a homomorphic image of $\mathbb{Z}$.

Proof. To show that $\Gamma$ is cyclic if and only if it is a homomorphic image of $\mathbb{Z}$, let $\Gamma = \langle a \rangle$ then the map

$$\mathbb{Z} \to \Gamma, \ n \to a^n$$

is a homomorphism (since $a^{n+m} = a^na^m$ for all $n, m \in \mathbb{Z}$) whose image is $\Gamma$.

Conversely, if $f : \mathbb{Z} \to \Gamma$ is an epimorphism then let $a = f(1)$. Every $\gamma \in \Gamma$ takes the form $\gamma = f(n)$ for some $n \in \mathbb{Z}$. If $n \geq 0$ then

$$\gamma = f(1 + \ldots + 1) = f(1) \circ \ldots \circ f(1) = (f(1))^n = a^n.$$ 

The same formula holds if $n < 0$. Thus $\Gamma = \langle a \rangle$. □

Theorem 2. (cyclic Hopf theorem). In systems with finite cyclic symmetry, generically, Hopf bifurcation at a point $x_0$ occurs with simple eigenvalues, and there exists a unique branch of small-amplitude periodic solutions emanating from $x_0$. Moreover the spatio-temporal symmetries of the bifurcating periodic solutions are

(2) $H = \Sigma_{x_0}$, $H$ is cyclic

and

(3) $K = \ker V(H)$, $K$ is cyclic,

and $H$ acts $H$–simply on $V$. In addition let $\mathbb{Z}_k$ act on $\mathbb{R}^k$ by a cyclic permutation of coordinates. Let $\mathbb{Z}_q \subseteq \mathbb{Z}_n \subseteq \mathbb{Z}_k$. Then there is a $\mathbb{Z}_n$–simple representation with kernel $\mathbb{Z}_q$ with the single exception when $n = k$ is even and $q = k/2$.

Proof. The proof relies on the proof of the homologous Theorem in [3], with changes concerning the form of the subgroups $H$ and $K$. However, we will prefer to give the proof entirely, including the parts that coincide with the proof in [3], to easy the lecture of the paper. We begin as in [3], by showing that the equivariant Hopf bifurcation leads to a unique branch of small-amplitude periodic solutions emanating from $x_0$. From Lemma 1 it follows that the bifurcation point $x_0 = 0$ and therefore $\Gamma = \Sigma_{x_0}$. Moreover, from Lemma 2 it follows that if reducing to the center manifold, we may assume that $\mathbb{R}^n = V$ and therefore from [2] it follows that the center subspace $V$ at the Hopf bifurcation point is $\Gamma$–simple. This means that $V$ is either a direct sum of two absolutely irreducible representations or it is itself irreducible but not absolutely irreducible. Since the irreducible representations of abelian groups (and subsequently cyclic groups) are one-dimensional and absolutely irreducible or two-dimensional and non-absolutely irreducible, it follows that $V$ is two-dimensional and therefore the eigenvalues obtained at the linearization about the bifurcation point $x_0$ are simple. Now the standard Hopf bifurcation theorem applies to obtain a unique branch of periodic solutions.

Let $x(t, \lambda)$ be the unique branch of small-amplitude periodic solutions that emanate at the Hopf bifurcation point $x_0$. For each $t$,

$$x_0 = \lim_{\lambda \to 0} x(t, \lambda).$$
Let $H$ be the spatio-temporal symmetry subgroup of $x(\cdot, \lambda)$, and let $\Phi : H \to S^1$ be the homomorphism that associates a symmetry $h \in H$ with a phase shift $\Phi(h) \in S^1$. To prove that $H \subset \Sigma_{x_0}$ we have

$$hx_0 = \lim_{\lambda \to 0} hx(0, \lambda) \quad \text{by continuity of } h$$

$$= \lim_{\lambda \to 0} x(\Phi(h), \lambda) \quad \text{by definition of spatio–temporal symmetries}$$

$$= x_0$$

and therefore $h \in \Sigma_{x_0}$. In the following we proof that $\Sigma_{x_0} \subset H$. Let $\gamma \in \Sigma_{x_0} \subseteq \Gamma$; therefore $\gamma x(t, \lambda)$ is also a periodic solution. Since the periodic is unique (as shown above), we have

$$\gamma \{x(t, \lambda)\} = \{x(t, \lambda)\},$$

so $\gamma \in H$. Lemma 2 allows us to assume that the center manifold at $x_0$ is $\hat{V} = v + x_0$, which may be identified with $V$, and therefore $V$ is $H$–invariant. Therefore $V$ is $H$–simple since $\gamma$ is cyclic (and subsequently abelian). Since $\Gamma$ is cyclic, all its subgroups are cyclic, in particular $H$ and $K$.

The proof of the last condition is the proof of Proposition 6.2 in [4].

4. Constructing systems with cyclic symmetry near Hopf points

This section consists in recalling the results corresponding section 4 in [4] where the construction of systems with abelian symmetry near Hopf points has been carried out. When $\Gamma$ is finite cyclic, a key step in constructing $H \mod K$ periodic solutions from Hopf bifurcation at $x_0$ is the construction of a locally $\Sigma_{x_0}$–equivariant vector field. We first construct, for finite symmetry groups, a $\Gamma$–equivariant vector field that has a stable equilibrium, $x_0 \in \mathbb{R}^n$, with the desired isotropy. We will use

**Lemma 4.** For any finite set of distinct points $y_1, \ldots, y_l$, vectors $v_1, \ldots, v_l$ in $\mathbb{R}^n$ and matrices $A_1, \ldots, A_l \in \text{GL}(n)$, there exists a polynomial map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g(y_j) = v_j$ and $(dg)_{y_j} = A_j$.

*Proof.* See [5].

**Theorem 3.** Let $\Gamma$ be a finite cyclic group acting on $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. Then there exists a $\Gamma$–equivariant system of ODEs on $\mathbb{R}^n$ with a stable equilibrium $x_0$.

*Proof.* See [4].

In conclusion any point $x_0 \in \mathbb{R}^n$ can be a stable equilibrium for a $\Gamma$–equivariant vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. It is clear that $(df)_{x_0}$ must commute with the isotropy subgroup $\Sigma_{x_0}$ of $x_0$ [3]. The following result states that the linearization about the equilibrium $x_0$ can be any linear map that commutes with the isotropy subgroup.

**Theorem 4.** Let $x_0 \in \mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map that commutes with the isotropy subgroup $\Sigma_{x_0}$ of $x_0$. Then there exists a polynomial $\Gamma$–equivariant vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x_0) = 0$ and $(df)_{x_0} = A$.

*Proof.* See [4].

When constructing a Hopf bifurcation at points $x_0 \in \mathbb{R}^n$ we do not necessarily assume full isotropy. Genericity of $\Sigma_{x_0}$–simple subspaces at points of Hopf bifurcation is given by $\Gamma$–equivariant mappings as follows.
Lemma 5. Let \( \Gamma \) act on \( \mathbb{R}^n \) and fix \( x_0 \in \mathbb{R}^n \). Let \( V \) be a \( \Sigma_{x_0} \)-invariant neighborhood of \( x_0 \) such that \( \gamma V \cap V = \emptyset \) for any \( \gamma \in \Gamma \setminus \Sigma_{x_0} \). Let \( g : V \times \mathbb{R} \to \mathbb{R}^n \) be a smooth \( \Sigma_{x_0} \)-equivariant vector field. Then there exists an extension of \( g \) to a smooth \( \Gamma \)-equivariant vector field \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \).

Prove. See [4].

5. The cyclic Hopf \( H \) mod \( K \) theorem

Theorem 5. (cyclic Hopf \( H \) mod \( K \) theorem). Let \( \Gamma \) be a finite cyclic group acting on \( \mathbb{R}^n \). There is an \( H \) mod \( K \) periodic solution that arises by a generic Hopf bifurcation if and only if the following seven conditions hold: Theorem [1](a) – (d), \( H \) is a cyclic isotropy subgroup, there exists an \( H \)-simple subspace \( V \) such that \( K = \ker_V (H) \), \( K \) is cyclic and let \( \mathbb{Z}_k \) act on \( \mathbb{R}^k \) by a cyclic permutation of coordinates. Let \( \mathbb{Z}_q \subseteq \mathbb{Z}_n \subseteq \mathbb{Z}_k \). Then there is a \( \mathbb{Z}_n \)-simple representation with kernel \( \mathbb{Z}_q \) with the single exception when \( n = k \) is even and \( q = \frac{k}{2} \).

Proof. Necessity follows from the \( H \) mod \( K \) theorem (Theorem [1]) and the cyclic Hopf theorem (Theorem [2]). We’ll prove the sufficiency next. The idea of the proof will again, rely heavily on the proof of Abelian Hopf \( H \) mod \( K \) theorem in [4]. Let \( x_0 \in \mathbb{R}^n \) and let \( H \) be the isotropy subgroup of the point \( x_0 \), i.e. \( H = \Sigma_{x_0} \). Moreover, let \( W \) be a \( H \)-simple representation. Since \( \Gamma \) is cyclic (in particular, abelian), \( W \) is two-dimensional. Now we can define the linear maps \( A(\lambda) : W \to W \) by

\[
A(\lambda) = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}.
\]

Since \( W \) is two-dimensional it is easy to prove the commutativity with \( A \). We have

\[
A(\lambda) \cdot W = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} \cdot \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \lambda a - b & -\lambda b - a \\ a + \lambda b & a\lambda - b \end{bmatrix} = W \cdot A(\lambda).
\]

Next we can extend Theorem [4] to a bifurcation problem as in Lemma [5]. Let \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) be a \( \Gamma \)-equivariant polynomial such that for all \( \gamma \in \Gamma \), \( f(\gamma \cdot x_0, \lambda) = 0 \) and \( (df)_{x_0,\lambda} |_{W} = \gamma A(\lambda) \gamma^{-1} \). Moreover, let \( g = f |_{W + x_0} \). From the way \( f \) has been constructed, \( g \) is \( H \)-equivariant on \( W + x_0 \) and \( g(x_0) = 0 \), hence from Lemma [2] we have that \( W \) is the center subspace of \( (dg)_{x_0,0} \).

Next consider \( (dg)_{x_0,\lambda} |_{W} \); its eigenvalues are \( \sigma(\lambda) \pm i \rho(\lambda) \) with \( \sigma(0) = 0 \), \( \rho(0) = 1 \) and \( \sigma'(0) \neq 0 \). Then the equivariant Hopf theorem extended to a point \( x_0 \in \mathbb{R}^n \) implies the existence of small-amplitude periodic solutions emanating from \( x_0 \) with spatio-temporal symmetries \( H \) and spatial symmetries \( K \).

\[
\Box
\]

6. General considerations between the differences of the results in this article and [4]

In the first place it must be highlighted that one can start we the methodology used in [4] and add the restrictions presented in this paper to obtain the Cyclic Hopf \( H \) mod \( K \) Theorem, but not vice-versa. This is obvious, because any cyclic group is abelian but not any abelian group is cyclic.

In this section we use the Cyclic Hopf \( H \) mod \( K \) Theorem to exhibit symmetry pairs \((H,K)\) that are admissible by the Abelian Hopf \( H \) mod \( K \) Theorem but not admissible by the Cyclic Hopf \( H \) mod \( K \) Theorem. Let \( \mathbb{Z}_l \) act on \( \mathbb{R}^l \) by cyclic permutation of coordinates and \( \Gamma = \mathbb{Z}_l \times \mathbb{Z}_k \) act on \( \mathbb{R}^l \times \mathbb{R}^k \) by the diagonal action, where \( l,k > 1 \). We show Abelian...
Hopf $H \mod K$ admissible but not Cyclic Hopf $H \mod K$ admissible pairs for this action of $\Gamma$ by classifying in Theorem (6) all Cyclic Hopf $H \mod K$ admissible pairs $K \subset H \subset \Gamma$ and showing that there are admissible pairs that are not on the list.

**Theorem 6.** By applying the Cyclic Hopf $H \mod K$ Theorem, the $(H, K)$ Hopf-admissible pairs in $\Gamma$ are $(\mathbb{Z}_m \times \mathbb{Z}_n, \mathbb{Z}_m \times \mathbb{Z}_q)$ where $q$ divides $n$ except when $q = \frac{k}{2}$ and $n = k$, and $(\mathbb{Z}_m \times \mathbb{Z}_n, \mathbb{Z}_p \times \mathbb{Z}_n)$ where $p$ divides $m$ except when $p = \frac{k}{2}$ and $m = \frac{k}{2}$. Moreover, $m$ and $n$ are coprimes, $m$ and $q$ are coprimes with $m \neq q$, and $p$ and $n$ are coprimes, with $p \neq n$.

**Proof.** The proof is a restriction to the cases $m$ and $n$ are coprimes with $m \neq n$, and $m$ and $q$ are not coprimes. They are admissible by the Abelian Hopf $H \mod K$ by applying Theorem 6.1 in [4]. However, they are not admissible by the Cyclic Hopf $H \mod K$ Theorem because of the application of the the Fundamental Theorem of finitely generated abelian groups. Indeed, if, for example $m$ and $n$ are not coprimes then they have a common divisor integer $a \in \mathbb{R}_+$ that is prime, and in this case $m = ab$, $n = ac$ for some integers $b \in \mathbb{R}_+$, $c \in \mathbb{R}_+$ and the group $\mathbb{Z}_{ab} \times \mathbb{Z}_{ac}$ is not cyclic. A similar case applies for the group $K = \mathbb{Z}_m \times \mathbb{Z}_q$ if $m$ and $q$ are not coprimes with $m \neq q$, or the group $K = \mathbb{Z}_p \times \mathbb{Z}_n$ if $n$ and $p$ are not coprimes with $n \neq p$.

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