Modelling and processing of data from a fibre-optic sensor of vibrations

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Abstract. A new technique of vibration sensing, based on a polarimetric fibre-optic strain sensor, is presented; it is designed for localisation of multiple sources of disturbances in a broad spectrum without using fibre gratings. A mathematical model of the sensor is used for development of a variational method for estimation of amplitudes of component vibrations on the basis of noisy samples of the voltage at the output of the sensor.

1. Introduction
In this paper, a new technique of vibration sensing is presented. Its idea is based on a polarimetric fibre-optic strain sensor [1–2], but it also allows localising the source of disturbances in a wide spectrum without using fibre gratings [3–5]. Early solutions allowed only localising one disturbance point at a time [6]. Several improvements have been introduced since then [7–8]. The solution shown in Fig. 1, and studied hereinafter, can be used for strain mapping in large planar constructions, e.g. in aeroplane wings.

![Diagram of a sensor under study](image)

**Figure 1.** The diagram of a sensor under study

2. Formulation of the research problem
The output signal (voltage) of the sensor is modelled by the equation:

\[ u(t) = \frac{U_2 - U_1}{2} \left[ \sin(\phi(t)) + 1 \right] + U_i \]  

(1)

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where \( t \) is time, and:

\[
\phi(t) = \phi_0 + \sum_{k=2}^{K} B_k A_k e^{-\alpha_k t} \sin(\omega_k t + \varphi_k)
\]

(2)

is a model of an optical signal in the sensor. It is assumed that some measured values of the following parameters are known:

\[ U_1, U_2, \omega_k, a_k, \quad k = 2, \ldots, K \]

(3)

The values of the parameters \( A_k \) for \( k = 2, \ldots, K \) are to be estimated on the basis of the noisy samples of \( u(t) \). The value \( \phi_0 \in (0, 2\pi) \) and \( \varphi_k \) (for \( k = 2, \ldots, K \)) are assumed to be unknown, but constant during a cycle of measurements.

The physical justification for the above-formulated mathematical model of the sensor is as follows:

- Eq.(1) is modelling an essential property of polarimetric-type sensors whose output intensity signal maps the phase delay between two orthogonal linearly polarized modes propagating in a highly birefringent single mode fibre. As phases differing by multiplicities of \( \pi/2 \) are indistinguishable in the polarization states of quasi-monochromatic light (provided the degree of polarization is constant), the phase signal leaving the optical fibre and passing through the analyser has to be transformed by a periodic sine-type function.

- Eq.(2) is modelling the time evolution of the phase difference between the orthogonal modes in the optical fibre under an external mechanical distortion. It has been assumed that the distortion induced by each transducer has the form of relaxation oscillations whose frequency is \( \omega_k \) and amplitude \( A_k \). The output phase signal is a linear combination of signals induced by all transducers with the coefficients \( B_k \) being related to mechanical and optical parameters of the transducer. The quantity \( \phi_0 \) is an initial phase shift determining the operating point of the transfer characteristic modelled by Eq.(1); its value depends on many factors including the length of the fibre, the central wavelength of light and temperature fluctuations.

A sample experimental signal from the sensor is presented in Fig. 2, together with its mathematical model.

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**Figure 2.** A sample experimental signal from the sensor (black line) and its mathematical model (red line).
3. Proposed solution of the research problem

3.1. Mathematical model of the measurement data

The data representative of the signal \( u(t) \) may be modelled in the following way:

\[
\tilde{u}_n = u(t_n) + \Delta\tilde{u}_n \quad \text{for } n = 1, \ldots, N
\]

where \( u(t) \) is defined by Eq. (1), and \( \Delta\tilde{u}_n \) are realisations of random variables modelling both measurement errors and imperfections of signal modelling. For the sake of simplicity, the above model may be given the form:

\[
\tilde{u}_n = \frac{U_2 - U_1}{2}[\sin(\phi(t_n)) + 1] + U_1 + \Delta\tilde{u}_n \quad \text{for } n = 1, \ldots, N
\]

where:

\[
\phi(t_n) = \phi_0 + \sum_{k=2}^{K} A_k e^{-a_k t} \sin(\omega_k t_n + \varphi_k) = \phi_0 + \sum_{k=2}^{K} A_k e^{-a_k t}[\sin(\omega_k t_n)\cos(\varphi_k) + \cos(\omega_k t_n)\sin(\varphi_k)]
\]

or:

\[
\phi(t_n) = \phi_0 + \sum_{k=2}^{K} x_{n,k} A_k \cos(\varphi_k) + \sum_{k=2}^{K} x_{n,K+k-1} A_k \sin(\varphi_k)
\]

with:

\[
x_{n,k} \equiv B_k e^{-a_k t} \sin(\omega_k t_n) \quad \text{and} \quad x_{n,K+k-1} \equiv B_k e^{-a_k t} \cos(\omega_k t_n) \quad \text{for } k = 2, \ldots, K
\]

For the sake of numerical simplicity, the following normalisation of the data:

\[
\bar{y}_n = 2 \frac{\tilde{u}_n - U_1}{U_2 - U_1} - 1 \quad \text{for } n = 1, \ldots, N
\]

is applied. It transforms the model into the form:

\[
\bar{y}_n = \sin\left( p_1 + \sum_{k=2}^{K} x_{n,k} p_k + \sum_{k=2}^{K} x_{n,K+k-1} p_{K+k-1} \right) + \Delta\bar{y}_n \quad \text{for } n = 1, \ldots, N
\]

where:

\[
\Delta\bar{y}_n = \frac{2}{U_2 - U_1} \Delta\tilde{u}_n \quad \text{for } n = 1, \ldots, N
\]

\[
p_1 = \phi_0, \quad p_k = A_k \cos(\varphi_k) \quad \text{and} \quad p_{K+k-1} = A_k \sin(\varphi_k) \quad \text{for } k = 2, \ldots, K
\]

3.2. Methods for parameter estimation

The normalised model of the data, defined by Eq. (10), may be given the vector-matrix form:

\[
\tilde{y} = \sin(Xp) + \Delta\tilde{y} \quad \text{for } n = 1, \ldots, N
\]

where the operation \( \sin(\cdot) \) is to be applied to each element of the vector \( Xp \), and:

\[
\tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_N \end{bmatrix}, \quad \Delta\tilde{y} = \begin{bmatrix} \Delta\tilde{y}_1 \\ \vdots \\ \Delta\tilde{y}_N \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_{1,2} & \cdots & x_{1,2K-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N,2} & \cdots & x_{N,2K-1} \end{bmatrix}, \quad \text{and} \quad p = \begin{bmatrix} p_1 \\ \vdots \\ p_{2K-1} \end{bmatrix}
\]

A family of variational solutions may be defined by the following formula:

\[
\hat{p} = \arg_{p} \inf \left\{ f(p) \mid p \in P \right\}
\]

where:

\[
f(p) = \| \tilde{y} - \sin(Xp) \|_q \quad \text{with} \quad q = 1, 2 \quad \text{or} \quad \infty
\]
is the objective function, and $D = [-\pi, \pi] \times [-p_{1,\text{max}}, p_{1,\text{max}}] \times \cdots \times [-p_{2K-1,\text{max}}, p_{2K-1,\text{max}}]$ is the set of admissible solutions, resulting from physical constraints of the parameters to be estimated. Thus, the estimates $\hat{p}$ of \( p \) may be determined according to Eq.(15), using any procedure of constrained optimisation. Those estimates may be next used for obtaining the estimates of the amplitudes \( A_k \) in the following way:

$$A_k = \sqrt{p_k^2 + p_{K+k-1}^2} \quad \text{for} \ k = 2, \ldots, K$$  \hspace{1cm} (17)

### 3.3. Constrained least-squares algorithm for parameter estimation

The simplest numerical algorithm for determination of $\hat{p}$ may be designed for \( q = 2 \). Then the objective function $f(p)$ is differentiable, and any procedure of constrained nonlinear least squares may be applied for minimisation. In this case, it is convenient to use $\|\tilde{y} - \sin(Xp)\|_2$ rather than $\|\tilde{y} - \sin(Xp)\|_2$ as the objective function:

$$f_{\text{LS}}(p) = \|\tilde{y} - \sin(Xp)\|^2 \equiv \sum_{n=1}^{N} [\tilde{y}_n - \sin(x_n^T p)]^2$$  \hspace{1cm} (18)

where $x_n^T$ is the \( n \)th row of the matrix $X$. The gradient vector of such an objective function has the form:

$$g_{\text{LS}}(p) = -\frac{\partial f_{\text{LS}}(p)}{\partial p} = 2 \sum_{n=1}^{N} [\sin(x_n^T p) - \tilde{y}_n] \cos(x_n^T p) x_n = \sum_{n=1}^{N} [\sin(2x_n^T p) - 2\tilde{y}_n \cos(x_n^T p)] x_n$$  \hspace{1cm} (19)

The corresponding estimation algorithm has the form:

(1) Normalise the acquired measurement data according to Eq.(9) and form the vector $\tilde{y}$.

(2) Compute the matrix $X$ according to Eq.(8) and Eq.(14).

(3) Define the functions $f_{\text{LS}}(p)$ and $g_{\text{LS}}(p)$.

(4) Compute the LS estimate $\hat{p}_{\text{LS}}$ of $p$ by means a procedure for constrained nonlinear optimisation using the functions $f_{\text{LS}}(p)$ and $g_{\text{LS}}(p)$.

### 4. Methodology of study based on synthetic data

Synthetic data, necessary for evaluation of estimation uncertainty, have been generated after Eq.(5), Eq.(6) and Eq.(8). The values of the parameters of transducers, used for this purpose, are shown in Table 1. The other parameters are as follows: $U_1 = 0.94 \text{ V}$, $U_2 = 4.15 \text{ V}$ and $\phi_0 = 0.58 \text{ rad}$.

| \( k \) | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| \( A_k \) [mm] | 0.0751 | 0.0792 | 0.0738 | 0.0674 |
| \( B_k \) [mm\(^{-1}\)] | 5.84 | 5.12 | 4.58 | 4.25 |
| \( a_k \) [s\(^{-1}\)] | 1.61 | 2.83 | 4.77 | 4.77 |
| \( \omega_k / 2\pi \) [Hz] | 22.77 | 27.35 | 30.44 | 36.62 |
| \( \phi_k \) [rad] | -0.533 | -1.28 | -1.19 | -1.02 |

The computations have been completed for \( N = 1701 \), for error corrupted data. The random errors $\Delta u_n$ have been simulated by means of uncorrelated pseudorandom numbers following the zero-mean normal distribution with the standard deviation $\sigma_u$, truncated outside of the interval $[-3\sigma_u, +3\sigma_u]$. 

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following the normal zero-mean distribution with the standard deviation $\sigma_u$. The estimates $\hat{A}_k(r)$ of $A_k$ ($k = 2, \ldots, K$), obtained for $R$ realisations of data errors ($r = 1, \ldots, R$), have been averaged and the following indicators of their uncertainty have been determined: the relative mean error (= relative bias) of the estimates:

$$\hat{b}_k = \frac{\overline{A}_k - A_k}{A_k} \text{ for } k = 2, \ldots, K \quad (20)$$

and the relative standard error (= relative standard deviation) of the estimates:

$$\hat{s}_k = \frac{1}{A_k} \sqrt{ \frac{1}{R - 1} \sum_{r=1}^{R} \left( \hat{A}_k(r) - \overline{A}_k \right)^2 } \text{ for } k = 2, \ldots, K \quad (21)$$

where $\overline{A}_k = \frac{1}{R} \sum_{r=1}^{R} \hat{A}_k(r)$ for $k = 2, \ldots, K$.

The expanded uncertainty of estimation has been characterised by three indicators:

$$\hat{u}_k^* = \inf \left\{ \frac{\hat{A}_k(r) - A_k}{A_k} \right\}_{r=1}^{R} \text{ and } \hat{u}_k^* = \sup \left\{ \frac{\hat{A}_k(r) - A_k}{A_k} \right\}_{r=1}^{R} \text{ for } k = 2, \ldots, K \quad (22)$$

5. Preliminary results of study based on synthetic data

Since the value of $\sigma_u$, estimated on the basis of currently available real-world measurement data, is 40 mV, the first numerical experiment has been programmed for $\sigma_u = 40$ mV. Since the target value of $\sigma_u$, which may be attained by reduction of noise in electronic circuits, is 10 mV, the first numerical experiment has been programmed for $\sigma_u = 10$ mV. The results of both experiments, obtained using the method described in Section 3.2 and a low-pass rectangular filter for pre-processing the data, are shown in Table 2 and Table 3.

The practical applicability of the developed method has been also confirmed in an experiment with a set of real-world data (for $N = 3400$ points); it has ended with the following results:

$$\left| \frac{\hat{A}_k - A_k}{A_k} \right| \leq 7.32 \% \text{ for } k = 2, \ldots, K \quad (23)$$

Table 2. The results obtained for 100 sets of error-corrupted data ($\sigma_u = 40$ mV, $R = 100$)

| $k$ | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|
| $\hat{b}_k$ | $-2.58 \%$ | $-4.18 \%$ | $-10.48 \%$ | $-7.03 \%$ |
| $\hat{s}_k$ | $1.49 \%$ | $4.16 \%$ | $7.54 \%$ | $7.57 \%$ |
| $\hat{u}_k^*$ | $1.58 \%$ | $11.13 \%$ | $22.18 \%$ | $26.88 \%$ |
| $\hat{u}_k^*$ | $5.25 \%$ | $3.68 \%$ | $5.26 \%$ | $6.22 \%$ |

Table 3. The results obtained for 100 sets of error-corrupted data ($\sigma_u = 10$ mV, $R = 100$)

| $k$ | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|
| $\hat{b}_k$ | $-0.07 \%$ | $-1.68 \%$ | $-2.86 \%$ | $-0.45 \%$ |
| $\hat{s}_k$ | $0.62 \%$ | $1.02 \%$ | $1.73 \%$ | $1.60 \%$ |
| $\hat{u}_k^*$ | $1.68 \%$ | $0.86 \%$ | $0.75 \%$ | $5.93 \%$ |
| $\hat{u}_k^*$ | $0.96 \%$ | $3.10 \%$ | $5.90 \%$ | $2.23 \%$ |
6. Conclusion
A new sensor of vibration, designed for localisation of multiple sources of disturbances in a broad spectrum, has been presented. Its mathematical grey-box model has been used for development of a variational method for estimation of amplitudes of component vibrations on the basis of noisy samples of the voltage at the output of the sensor. The relative uncertainty of 10–20 % has been already attained, and prospects for its reduction to several percent have been shown.

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