Some aspects of the functor $K_2$ of fields

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We review connections between the group $K_2$ of a field and universal central extensions, quadratic forms, central simple algebras, differential forms, abelian extensions, abelian coverings, explicit reciprocity laws, special values of zeta functions, and special values of L-functions. No proofs, minimal bibliography.

1. The birth of $K_2$ [11], [15], [31].

It was in 1956 that A. Grothendieck had the idea of assigning to every scheme $X$ a group $K(X)$; the letter $K$ stands for *Klassen*, and indeed the elements of $K(X)$ are equivalence classes of certain objects associated to $X$. The need for such a group arose in his generalisation of the Riemann-Roch theorem of F. Hirzebruch from complex analytic geometry to a relative version in abstract algebraic geometry.

Let us recall the defintion of this group when $X = \text{Spec}(A)$ is an affine scheme, i.e. the spectrum of a commutative ring $A$. Isomorphism classes of finitely generated projective $A$-modules form a commutative monoid $M$ whose law of composition is given by taking the the direct sum. There arise in practice a number of morphisms from $M$ into commutative groups; Grothendieck defined $K(X)$ — also denoted $K(A)$ in this affine case — as the universal object amongst them. Thus, $K(A)$ is simply the group of differences of the monoid $M$, just as $\mathbb{Z}$ is the group of difference of the monoid $\mathbb{N}$; it is a powerful invariant of the ring $A$. For example, when $A$ is the ring of integers in a finite extension of $\mathbb{Q}$, the torsion subgroup of $K(A)$ is the group of ideal classes of $A$. When $A$ is a field $F$, the rank or dimension gives an isomorphism $K(F) \to \mathbb{Z}$.

H. Bass realised that a pre-Grothendieck construction of J. H. C. Whitehead — the quotient of $\text{GL}(A) = \lim \text{GL}_n(A)$ by its derived subgroup — should be the right definition of the $K_1$ of a ring $A$. When $A$ is a field $F$, the determinant gives an isomorphism $K_1(F) \to F^\times$.

The next step was taken by J. Milnor who defined the group $K_2(A)$ and a cup-product map $K_1(A) \times K_1(A) \to K_2(A)$ for every ring $A$; it turned out to be the homology group $H_2(\text{D}(\text{GL}(A)), \mathbb{Z})$, where $\text{D}(\ )$ is the derived subgroup. A deep theorem of H. Matsumoto gave a presentation of the $K_2$ of a field $F$. We have chosen to take this presentation as our starting point to recount a few places where the group $K_2$ makes its appearance.

We have had to leave out more than we have been able to include.
2. Definition of $K_2$ [2], [20], [29].

Let $F$ be a (commutative) field and let $A$ be a commutative group. An $A$-valued symbol on $F$ is a $\mathbb{Z}$-bilinear map $s : F^\times \times F^\times \rightarrow A$ satisfying

$$s(x,y) = 0 \quad \text{for all } (x,y) \in F^\times \times F^\times \text{ such that } x + y = 1.$$ 

A symbol $s$ is said to be trivial if $s(x,y) = 0$ for all $x, y \in F^\times$.

Clearly, if $s$ is an $A$-valued symbol on $F$, if $f : F' \rightarrow F$ is a homomorphism of fields and if $a : A \rightarrow A'$ is a homomorphism of commutative groups, then $a \circ s \circ f$ is an $A'$-valued symbol on $F'$.

**Example 1.** Take $F = \mathbb{R}$, $A = \{1, -1\}$ and set $s_{\infty}(x,y) = -1$ if and only if $x < 0$ and $y < 0$. Then $s_{\infty}$ is a symbol. Note that $s_{\infty}(x,y) = 1$ if and only if the conic

$$xR^2 + yS^2 = 1$$

has a solution $(R,S)$ over $\mathbb{R}$. Note also that the symbol $s_{\infty}$ is continuous.

**Example 2.** Keep $A = \{1, -1\}$ and define an $A$-valued continuous symbol on $\mathbb{Q}_2^\times$ bilinearly by

$$s_2(x,y) = (-1)^{x-1} \frac{y-1}{2}, \quad s_2(x,2) = (-1)^{\frac{x^2-1}{4}} (x, y \in \mathbb{Z}_2^\times).$$

One has $s_2(x,y) = 1$ if and only if the conic (1) has a solution over $\mathbb{Q}_2$.

**Example 3.** Let $p$ be an odd prime number and take $F = \mathbb{Q}_p$, $A = F_p^\times$. Denoting by $v : \mathbb{Q}_p^\times \rightarrow \mathbb{Z}$ the valuation, one sees that the element

$$(-1)^{v(x)v(y)} x^{v(y)} y^{-v(x)}$$

belongs to $\mathbb{Z}_p^\times$ for every $x, y \in \mathbb{Q}_p^\times$; let $s_p(x,y)$ be its image in $F_p^\times$. Then $s_p$ is a continuous symbol. Further, $h_p(x,y) = s_p(x,y)^{\frac{p-1}{2}}$ is also a symbol, with values in $\{1, -1\}$. As an exercise, show that $h_p(x,y) = 1$ if and only if the conic (1) has a solution $(R,S)$ over $\mathbb{Q}_p$.

As an aside, let us mention the local-to-global principle for $\mathbb{Q}$-conics (i.e, curves of genus 0): for $x, y \in \mathbb{Q}^\times$, the conic (1) has a solution $(R,S)$ over $\mathbb{Q}$ if (and of course only if) $h_p(x,y) = 1$ for every odd prime $p$, $s_2(x,y) = 1$, and $s_{\infty}(x,y) = 1$. It so happens that if all but one of these local conditions are satisfied, then so is the recalcitrant local condition (cf. th. 10).

(Let $F$ be any field and let $x, y \in F^\times$. The conic (1) has a solution over $F$ if and only if there exists a homomorphism $F[R,S]/(xR^2 + yS^2) \rightarrow F$ of
F-algebras. *Examples* 1–3 thus provide a criterion for the existence of such a homomorphism when $F = \mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_p$ ($p$ odd prime), respectively.)

*Example* 4. Let $k$ be any field, take $F = k((T))$, $A = k^\times$ and denote by $v : F^\times \to \mathbb{Z}$ the valuation. For every pair $x, y \in F^\times$, the element $(2)$ belongs to $k[[T]]^\times$; let $s(x, y)$ be its image in $k^\times$. Then $s$ is a symbol.

*Example* 5. More generally, let $F$ be any field endowed with a discrete valuation $v : F^\times \to \mathbb{Z}$, and denote the residue field by $k$. We get a $k^\times$-valued symbol $s$ on $F$ by defining $s(x, y)$ to be the image in $k^\times$ of the element $(2)$; $s$ is called the *tame* symbol associated to $v$. *Examples* 3 and 4 are particular cases; *Examples* 1 and 2 are not.

Does there exist a universal symbol $u : F^\times \times F^\times \to U_F$ on $F$? In other words, does there exist a commutative group $U_F$ and a symbol $u$ on $F$ with values in $U_F$ such that, given any $A$-valued symbol $s$ on $F$ (A being a commutative group), there exists a unique homomorphism $f : U_F \to A$ of groups such that $s = f \circ u$? Clearly, if such a universal symbol exists, it is unique, up to unique isomorphism.

Nor is the existence of a universal symbol hard to see: just take $U_F$ to be the quotient of $F^\times \otimes_\mathbb{Z} F^\times$ by the subgroup generated by those elements $x \otimes y (x, y \in F^\times)$ for which $x + y = 1$.

**Definition 1.** — Let $F$ be a field. The universal symbol on $F$ is denoted $\{\cdot, \cdot\}_F : F^\times \times F^\times \to K_2(F)$, or more precisely by $\{\cdot, \cdot\}_F$.

The risk of confusing the element $\{x, y\} \in K_2(F)$ with the subset $\{x, y\} \subset F^\times$ is minor.

*Example* 6. Let $k$ be a field, take $F = k((T))$ and $A = K_2(k)$. Write $x \in F^\times$ as $x = \frac{a_m T^m + \ldots + a_0}{b_n T^n + \ldots + b_0}$ ($a_m, b_n \in k^\times$) and define $c(x) = \frac{a_m}{b_0}$, which does not depend on the way $x$ is written as a quotient of two polynomials. Then $(x, y) \mapsto \{c(x), c(y)\}_k$ is a $K_2(k)$-valued symbol on $F$. Hence there is a unique homomorphism $K_2(F) \to K_2(k)$ such that $\{x, y\}_F = \{c(x), c(y)\}_k$ for all $x, y \in F^\times$. One can show that it is a retraction of the canonical homomorphism $K_2(k) \to K_2(F)$.

In fact, Milnor defines a graded ring $K(F) = \bigoplus_n K_n(F)$, functorially in the field $F$, as being the quotient of the $\mathbb{N}$-graded tensor algebra of the $\mathbb{Z}$-module $F^\times$:

$$T(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes F^\times) \oplus \ldots$$

by the homogenous ideal generated by the elements $x \otimes y (x, y \in F^\times)$ such that $x + y = 1$. Thus, $K_0(F) = \mathbb{Z}$ and $K_1(F)$ is generated by $\{x\} (x \in F^\times)$, subject to the only conditions that $\{xy\} = \{x\} + \{y\}$; the map $x \mapsto \{x\}$
is an isomorphism $F^\times \to K_1(F)$. We shall say very little about the groups $K_n(F)$ for $n > 2$.

If we are given a graded ring $A = \bigoplus_n A_n$ and a homomorphism of groups $\varphi : F^\times \to A_1$ such that $(x, y) \mapsto \varphi(x)\varphi(y)$ is an $A_2$-valued symbol on $F$, there exists a unique homomorphism of graded rings $K(F) \to A$ extending $\varphi$. Sometimes, this homomorphism or its components are also called *symbols*, by abuse of language. People go to the extent of calling elements of $K(F)$ or their images in $A$ symbols.

Quillen has defined higher $K$-groups for arbitrary schemes; they do not agree for fields with those of Milnor for $n > 2$. In a certain precise sense, Milnor's theory is the “simplest part” of Quillen’s theory.

**Proposition 1.** — Let $F$ be a field. In the group $K_2(F)$, one has:

1. $\{x, -x\} = 0$ \quad $(x \in F^\times)$,
2. $\{x, y\} + \{y, x\} = 0$ \quad $(x, y \in F^\times)$,
3. $\{x, x\} = \{-1, x\}$ \quad $(x \in F^\times)$.

**Proof**: i). This is clear by bilinearity if $x = 1$. If $x \neq 1$, write $-x = (1 - x)(1 - x^{-1})^{-1}$ and compute

$$\{x, -x\} = \{x, 1 - x\} - \{x, 1 - x^{-1}\} = \{x^{-1}, 1 - x^{-1}\} = 0.$$  

Using i), we have

$$0 = \{xy, -xy\} = \{x, -x\} + \{x, y\} + \{y, x\} + \{y, -y\} = \{x, y\} + \{y, x\}$$

so ii) holds. For iii), we have $\{x, x\} = \{-1, x\} + \{-x, x\} = \{-1, x\}$, again using i). Note: ii) implies that the projection $T(F^\times) \to K(F)$ factors through the exterior algebra $\wedge(F^\times)$ if we tensor all three with $\mathbb{Z}[\frac{1}{2}]$. 

3. **Computations of $K_2$** [17], [20], [3].

The fact that the group $K_2(F)$ vanishes for every finite field $F$ will have many consequences when $K_2$ will be related to other objects, like central extensions, central simple algebras, quadratic forms, etc.

**Theorem 1** (Steinberg). — For every finite field $F$, the group $K_2(F)$ is trivial. Consequently, all the higher $K$-groups of a finite field are trivial.

**Proof**: The multiplicative group $F^\times$ of a finite field is always cyclic; let $\zeta$ be a generator. It is sufficient to show that $\{\zeta, \zeta\} = 0$. 

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One has $2\{ζ, ζ\} = 0$ by Prop. 1ii). This implies that $2\{x, y\} = 0$ for any $x, y \in F^\times$ : writing $x = ζ^m$, $y = ζ^n$, we have $2\{x, y\} = 2\{ζ^m, ζ^n\} = 2mn \{ζ, ζ\} = 0$. So $K_2(F)$ is killed by 2.

If $F$ is of characteristic 2, then $ζ = −ζ$ and $\{ζ, ζ\} = \{ζ, −ζ\} = 0$ by Prop. 1i), so the proof is complete in this case.

Suppose that 2 is invertible in $F$. A counting argument shows that the equation $ζx^2 + ζy^2 = 1$ has a solution $(x, y) \in F^\times \times F^\times$ : the number of elements $ζx^2 (x \in F)$, plus the number of elements $1 − ζy^2 (y \in F)$, is $> \text{Card } F$, so there is a pair $x, y \in F$ such that $ζx^2 + ζy^2 = 1$. Necessarily $x, y \in F^\times$, as $ζ$ is not a square.

We then have $0 = \{ζx^2, ζy^2\} = \{ζ, ζ\}$ since, as we have seen, $K_2(F)$ is killed by 2. This completes the proof.

The symbols of Examples 1, 2, 3 are the only continuous symbols on $R$, $Q_2$, $Q_p$ ($p$ odd) respectively (Moore). For more information about the $K_2$ of local fields, see the section on the uniqueness of reciprocity laws.

Let us state the result of Tate’s computation of $K_2(Q)$. He was inspired by the first proof of the law of quadratic reciprocity given by the young Gauss.

Example 2 provides a homomorphism $s_2 : K_2(Q) \to \{1, −1\}$ and, similarly, Example 3 provides $s_p : K_2(Q) \to F_p^\times$ for every odd prime $p$. Given $x, y \in Q^\times$, one has $s_p(x, y) = 1$ for almost all $p$. So we get a map into the direct sum

$$K_2(Q) \longrightarrow \{1, −1\} \oplus (Z/3Z)^{\times} \oplus (Z/5Z)^{\times} \oplus (Z/7Z)^{\times} \oplus \ldots$$

**Theorem 2 (Tate). — This map is an isomorphism of groups.**

It is easily seen that the law of quadratic reciprocity is a consequence of Tate’s calculation : we have another symbol on $Q$, namely the symbol $s_∞$ of Example 1; expressing it in terms of the symbols $s_2, s_3, s_5, \ldots$ (it turns out to be the product $s_2s_3s_5 \ldots$, cf. Examples 2, 3) leads to the result.

When $k$ is a field and $F = k(T)$, the group $K_2(F)$ is canonically isomorphic to the direct sum of $K_2(k)$ and the various groups $(k[T]/p)^{\times}$, where $p$ runs through the maximal ideals of $k[T]$ (cf. Example 4). Thus we get an exact sequence

$$0 \rightarrow K_2(k) \rightarrow K_2(F) \rightarrow \bigoplus_p (k[T]/p)^{\times} \rightarrow 1$$

which admits a retraction (cf. Example 6). Applying this result to “the place at infinity” of $F|k$, one is lead to Weil’s reciprocity law.
Let \( A = \mathbb{F}_2[T] \) be the polynomial ring in one indeterminate over the field \( \mathbb{F}_2 \), and let \( A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots \) be its gradation. Consider the homomorphism \( \varphi : \mathbb{R}^x \to A_1 \) which sends \( x \in \mathbb{R}^x \) to \( T \) if and only if \( x < 0 \). Then \( (x, y) \mapsto \varphi(x)\varphi(y) \) \((x, y \in \mathbb{R}^x)\) is the symbol \( s_\infty \) of Example 1; it can be shown that the associated homomorphism \( K(\mathbb{R})/2K(\mathbb{R}) \to \mathbb{F}_2[T] \) is an isomorphism of graded rings.

Let \( F \) be a global field. It can be shown (Bass-Tate) that the real places of \( F \) — suppose there are \( r_1 \) of them — induce an isomorphism \( K_n(F) \to (\mathbb{Z}/2\mathbb{Z})^{r_1} \) \((n > 2)\).

### 4. \( K_2 \) and universal central extensions [28], [20].

Recall that the derived subgroup \( D(G) \) of a group \( G \) is the subgroup of \( G \) generated by all the commutators \( [x, y] = xyx^{-1}y^{-1} \) \((x, y \in G)\); it is normal in \( G \) and \( G/D(G) \) is the largest commutative quotient of \( G \). A group \( G \) is said to be perfect if \( D(G) = G \).

Recall also that an extension of a group \( G \) is a pair \((G', p)\) consisting of a group \( G' \) and a surjective homomorphism \( p : G' \to G \); it is said to be central if the subgroup \( \text{Ker}(p) \subset G' \) is central. A section of an extension is a homomorphism \( s : G \to G' \) such that \( p \circ s = \text{Id}_G \); an extension is said to split if it admits a section. A morphism from an extension \((G', p)\) to an extension \((G'', p')\) of \( G \) is a homomorphism of groups \( f : G' \to G'' \) such that \( p = p' \circ f \). If an extension \((G', p)\) of \( G \) splits, it is isomorphic to \((\text{Ker}(p) \times G, \text{pr}_G)\).

A central extension \( U \) of a group \( G \) is said to be universal if, given any central extension \( G' \) of \( G \), there exists a unique morphism \( f : U \to G' \). The universal central extension, if it exists, is unique, up to unique isomorphism.

With these definitions, let us recall a few facts. A group \( G \) admits a universal central extension if and only if \( G \) is perfect. A central extension \( U \) of a group \( G \) is universal if and only if \( U \) is perfect and every central extension of \( U \) splits.

The universal central extension \( \tilde{G} \) of a perfect group \( G \) can be identified with the homology group \( H_2(G, \mathbb{Z}) \) (whose definition we do not recall); \( \text{Ker}(\tilde{G} \to G) \) is the central subgroup of \( \tilde{G} \). Under the identification \( H^2(G, H_2(G, \mathbb{Z})) \to \text{End}_\mathbb{Z}(H_2(G, \mathbb{Z})) \), the extension \( \tilde{G} \) corresponds to \( \text{Id} : H_2(G, \mathbb{Z}) \to H_2(G, \mathbb{Z}) \). Also, a central extension \((G', p')\) of \( G \) is universal if and only if \( H_1(G', \mathbb{Z}) = \{0\} \) and \( H_2(G', \mathbb{Z}) = \{0\} \).

Some people think of \( G/D(G) = H_1(G, \mathbb{Z}) \) as the \( \pi_0(G) \) of a discrete group \( G \) and call it “connected” if \( \pi_0(G) = \{1\} \); \( \tilde{G} \) is then its “universal cover” and the kernel of \( \tilde{G} \to G \) is the “fundamental group” \( \pi_1(G) \) of \( G \).
Now let $F$ be a field and $n > 2$ an integer. The group $\text{SL}_n(F)$ is perfect — with three exceptions: when $(n, \text{Card } F)$ equals $(3, 2)$, $(3, 4)$ or $(4, 2)$; we do not consider them in what follows. Let $(\tilde{G}, p)$ be its universal central extension. For $x, y \in F^\times$, choose $x_{12}, y_{13} \in \tilde{G}$ such that

$$p(x_{12}) = \text{diag}(x, x^{-1}, 1, 1, \ldots) \quad \text{and} \quad p(y_{13}) = \text{diag}(y, 1, y^{-1}, 1, \ldots)$$

in $\text{SL}_n(F)$. Then the commutator $(x_{12}, y_{13})$ — which depends only on $x$ and $y$ — belongs to $\text{Ker}(p) = H_2(\text{SL}_n(F), \mathbb{Z})$ and defines a bilinear map

$$(4) \quad (x, y) \mapsto (x_{12}, y_{13}) : F^\times \times F^\times \to H_2(\text{SL}_n(F), \mathbb{Z}).$$

**Theorem 3** (Matsumoto, Steinberg). — *Apart from the three exceptional $(n, F)$ mentioned above, the map $(4)$ is a symbol inducing an isomorphism $K_2(F) \to H_2(\text{SL}_n(F), \mathbb{Z})$.*

Note that, in view of the triviality of the $K_2$ of a finite field, this theorem implies that the *discrete* groups $\text{SL}_n(F)$ ($n > 2$, $F$ finite) are “simply connected” — leaving aside the three exceptions noted above, which are not “connected”.

In fact, $K_2(F)$ is the “fundamental group” of $G(F)$ for every simple, simply connected split algebraic $F$-group $G$, not just $G = \text{SL}_n(F)$. The only exceptions are $\text{Sp}_n(F)$, for which $K_2(F)$ is a quotient of the “fundamental group”, with kernel $\mathbb{Z}$.

5. **$K_2$ and quadratic forms** [19], [13], [22].

Let $F$ be a field in which 2 is invertible. Recall that a (regular) quadratic space over $F$ is a (finite-dimensional) vector $F$-space $V$ endowed with a regular symmetric bilinear form $b$. It is possible to choose a basis for $V$ and $a_1, \ldots, a_n \in F^\times$ such that one has

$$(5) \quad b(\xi, \xi) = a_1\xi_1^2 + a_2\xi_2^2 + \cdots + a_n\xi_n^2$$

for all $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ ($n = \dim V$) in $V$. The isometry classes of quadratic $F$-spaces form a monoid with orthogonal direct sum as the law of addition; this monoid is integral (Witt). The corresponding group of differences $\tilde{W}(F)$ is called the Grothendieck group of $F$. The class of the form $(5)$ in $\tilde{W}(F)$ is denoted $(a_1, a_2, \ldots, a_n) = \langle a_1 \rangle + \langle a_2 \rangle + \cdots + \langle a_n \rangle$. Tensor product over $F$ makes $\tilde{W}(F)$ into a ring whose multiplication is characterised by $\langle x \rangle \langle y \rangle = \langle xy \rangle$ and for which $\langle 1 \rangle$ is the neutral element. Taking dimensions gives a canonical surjection of rings $\dim : \tilde{W}(F) \to \mathbb{Z}$. Denote by $\hat{1}$ the kernel (the augmentation ideal).
Let \( h = \langle 1, -1 \rangle = \langle 1 \rangle + \langle -1 \rangle \) be the class of the hyperbolic plane; it corresponds to the quadratic space \( L \oplus \text{Hom}_F(L, F), (\xi, \xi^*) \mapsto \xi^*(\xi) \), where \( L \) is a vector \( F \)-line. The subgroup \( H \) generated by \( h \) is an ideal in \( \hat{W}(F) \) (a quadratic space \( \langle a_1, a_2, \ldots, a_n \rangle \) “represents 0” if and only if \( \langle a_1, a_2, \ldots, a_n \rangle = \langle b_1, b_2, \ldots, b_{n-2} \rangle + h \) for some quadratic space \( \langle b_1, b_2, \ldots, b_{n-1} \rangle \)); one has \( H \cap \hat{I} = \{ 0 \} \). The quotient \( W(F) = \hat{W}(F) / H \) is called the Witt ring of \( F \); the image \( I = \hat{I} / (H \cap \hat{I}) = \hat{I} \) of the ideal \( \hat{I} \) is maximal in \( W(F) \) with \( \mathbb{Z}/2\mathbb{Z} \) as the quotient. We will be mainly interested in the graded \( \mathbb{F}_2 \)-algebra associated to the filtered ring

\[ \ldots \subset I^3 \subset I^2 \subset I \subset W(F). \]

There is a homomorphism \( s_1 : F^\times \to I/I^2 \) given by \( s_1(x) = \langle x \rangle - \langle 1 \rangle \).

**Lemma 1** (Milnor, 1970). — The map \( s_2(x, y) = s_1(x)s_1(y) \) is an \((I^2/I^3)\)-valued symbol on \( F \); the corresponding homomorphism from \( K_2(F) \) is trivial on \( 2K_2(F) \).

There is therefore a unique homomorphism of graded \( \mathbb{F}_2 \)-algebras

\[ s : K(F)/2K(F) \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots \]

extending \( s_1 \) and \( s_2 \). It is easy to see that the discriminant provides an inverse \( w_1 : I/I^2 \to F^\times/F^\times 2 \) of \( s_1 \) and the Hasse invariant an inverse \( w_2 : I^2/I^3 \to K_2(F)/2K_2(F) \) of \( s_2 \).

Milnor showed that the map (6) is always surjective and conjectured (1970) that it is bijective for all fields \( F \) (in which 2 is invertible). He proved the bijectivity for global fields.

The conjecture was finally proved by Orlov, Vishik & Voevodsky, in a preprint available on the arXiv.

**Theorem 4** (Orlov, Vishik, Voevodsky, 1996). — The map (6) is an isomorphism of graded \( \mathbb{F}_2 \)-algebras.

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Continuing to assume that 2 is invertible in the field \( F \), let \( \overline{F} \) be a separable algebraic closure of \( F \) and \( \Gamma = \text{Gal}(\overline{F}|F) \) the (profinite) group of \( F \)-automorphism of \( \overline{F} \). Consider the exact sequence

\[ \{ 1 \} \to \{ 1, -1 \} \to \overline{F}^\times \xrightarrow{\delta} \overline{F}^\times \to \{ 1 \} \]

of discrete \( \Gamma \)-modules. The associated long cohomology sequence furnishes — upon identifying the \( \Gamma \)-module \( \{ 1, -1 \} = 2\overline{F}^\times \) with \( \mathbb{Z}/2\mathbb{Z} \) — an injection \( \delta_1 : F^\times/F^\times 2 \to H^1(\Gamma, \mathbb{Z}/2\mathbb{Z}) \) which is an isomorphism by Hilbert’s theorem 90: \( H^1(\Gamma, \overline{F}^\times) = \{ 0 \} \).
Lemma 2 (Tate, 1970). — The map $\delta_2(x, y) = \delta_1(x) \bowtie \delta_1(y)$ is an $H^2(\Gamma, \mathbb{Z}/2\mathbb{Z})$-valued symbol on $F$; the corresponding homomorphism from $K_2(F)$ is trivial on $2K_2(F)$.

There is therefore a unique homomorphism of graded $F_2$-algebras

$$
(7) \quad \delta : K(F)/2K(F) \rightarrow \mathbb{Z}/2\mathbb{Z} \oplus H^1(\Gamma, \mathbb{Z}/2\mathbb{Z}) \oplus H^2(\Gamma, \mathbb{Z}/2\mathbb{Z}) \oplus \cdots
$$

extending $\delta_1$ and $\delta_2$. We have seen that $\delta_1$ is an isomorphism; Merkurjev (1981) proved that $\delta_2$ is an isomorphism. Milnor (1970) proved that $\delta$ is an isomorphism when $F$ is a finite, a local, a global or a maximally ordered field. His conjecture — that this is an isomorphism for all fields $F$ (in which 2 is invertible) — was finally proved by Voevodsky in 1996 and got him a medal. Nearly a decade earlier, Rost and Merkurjev & Suslin had proved that $\delta_3$ is an isomorphism.

Theorem 5 (Voevodsky, 1996). — The map (7) is an isomorphism of graded $F_2$-algebras.

As a consequence of theorems 4 and 5, the graded ring associated to the filtered ring $W(F)$ is canonically isomorphic to the cohomology ring of the $\Gamma$-module $\mathbb{Z}/2\mathbb{Z}$. To see how far this result goes, note that previous attempts to construct maps between the two — let alone showing that they are isomorphisms — had been successful only in low degrees.

6. $K_2$ and central simple algebras [20], [27], [30].

Let $F$ be a (commutative) field. Recall that a (finite-dimensional, associative, unital) $F$-algebra $A$ is called simple if the only bilateral ideals of $A$ are $\{0\}$ and $A$. An $F$-algebra $A$ is called central if $F$ is precisely the centre of $A$. Every central simple $F$-algebra $A$ is isomorphic to the matrix algebra $M_n(D)$ of a (skew) field $D$ over $F$ (Wedderburn); the pair $(n, D)$ is uniquely determined by $A$, up to isomorphism. Two such algebras $A, A'$ are called similar if the corresponding (skew) fields are isomorphic. Similarity classes of central simple $F$-algebras form a group $B(F)$ (the Brauer group of $F$) with tensor product as the law of multiplication. It is a torsion group, as a restriction-corestriction argument shows.

The group $B(F)$ can also be viewed as the group of $F$-isomorphism classes of $F$-algebras which become isomorphic, over a separable algebraic closure $\bar{F}$ of $F$, to $M_n(\bar{F})$ (some $n$), with tensor product of algebras providing the group law. It can also be viewed as the group of $F$-varieties which are $\bar{F}$-isomorphic to $P_n$ (some $n$).
Now let $n > 0$ be an integer and assume that $F$ contains a primitive $n^{th}$ root $\zeta$ of $1$ (i.e. $\zeta \in F^\times$ is of order $n$; thus $n$ is invertible in $F$). For $a, b \in F^\times$, consider the $F$-algebra $A_\zeta(a, b)$ with the presentation

$$x^n = a; \quad y^n = b; \quad xy = \zeta yx.$$ 

It is a central simple $F$-algebra — called a cyclic algebra in general and a quaternion algebra when $n = 2$, since it was so called in the case $F = \mathbb{R}$, $\zeta = -1$, $a = -1$, $b = -1$ by its discoverer Hamilton — whose class $s_\zeta(a, b) \in \mathcal{B}(F)$ is killed by $n$, i.e. lies in the $n$-torsion $\mathcal{B}_n(F)$.

(Before going on, observe that a quaternion $\mathbb{Q}_p$-algebra defined by $a, b \in \mathbb{Q}_p^\times$ is the matrix algebra precisely when $\chi_p(a, b) = 1$ (Example 3) for an odd prime $p$, when $s_2(a, b) = 1$ (Example 2) for $p = 2$, and when $s_\infty(a, b) = 1$ (Example 1) for $\mathbb{Q}_p = \mathbb{R}$. The local-to-global principle for quaternion $\mathbb{Q}$-algebras $A$, essentially the same as the one for conics, says that $A$ is a matrix algebra if (and of course only if) $A \otimes \mathbb{Q}_p$ is a matrix algebra for every place $p$ of $\mathbb{Q}$. As before, it is in fact sufficient to demand that they be matrix algebras at all places, with one possible exception; the “exception” is then not an exception. There is a local-to-global principle for central simple algebras over every global field, i.e. over finite extensions of $\mathbb{Q}$ or of $\mathbb{F}_p(T)$ ($p$ prime).)

**Lemma 3** (Tate, 1970). — *The map $(a, b) \mapsto s_\zeta(a, b)$ is an $n\mathcal{B}(F)$-valued symbol on $F$.*

**Theorem 6** (Merkurjev & Suslin, 1982). — *The associated map is an isomorphism $K_2(F)/nK_2(F) \to n\mathcal{B}(F)$.*

Thus every central simple algebra whose class is killed by $n$ is similar to a product of cyclic algebras in the presence of $n^{th}$ roots of $1$.

The choice $\zeta$ of a primitive $n^{th}$ root of $1$ in $F$ allows us to identify $n\mathcal{B}(F)$ with $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$. The lemma implies that there is a unique homomorphism of graded $(\mathbb{Z}/n\mathbb{Z})$-algebras

$$\delta_\zeta : K(F)/nK(F) \longrightarrow \mathbb{Z}/n\mathbb{Z} \oplus H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \oplus H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}) \oplus \cdots$$

which restricts to the symbol $s_\zeta$ on $F^\times \times F^\times$. But let turn to what happens when a primitive $n^{th}$ root of $1$ may not be available in $F$.

***

More generally, without assuming the existence of a primitive $n^{th}$ root of $1$ in $F$ but merely that $n$ is invertible in $F$, we have an exact sequence

$$\{1\} \to \mathbb{Z}/n\mathbb{Z}(1) \to F^\times \xrightarrow{(\cdot)^n} F^\times \to \{1\}$$

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of discrete $\Gamma$-modules, where $\mathbb{Z}/n\mathbb{Z}(1) = \overline{F}^\times$ is the group of $n^{th}$ roots of 1 in $\overline{F}^\times$ with its natural $\Gamma$-action. The associated long exact cohomology sequence and Hilbert’s theorem 90 furnish — as we have seen before in the case $n = 2$ — an isomorphism $\delta_1 : F^\times/F^\times n \to H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}(1))$. (The next few terms of the same cohomology sequence furnish a canonical isomorphism $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}(1)) \to nB(F)$, a group which also classifies $F$-varieties potentially isomorphic to $P_{n-1}$.) Cup product on cohomology

$\cup : H^r(\Gamma, \mathbb{Z}/n\mathbb{Z}(r)) \times H^s(\Gamma, \mathbb{Z}/n\mathbb{Z}(s)) \to H^{r+s}(\Gamma, \mathbb{Z}/n\mathbb{Z}(r+s))$

then provides a bilinear map $\delta_2 : F^\times/F^\times n \times F^\times/F^\times n \to H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}(2))$.

Lemma 4 (Tate, 1970). — The map $\delta_2(x, y) = \delta_1(x) \cup \delta_1(y)$ is a symbol on $F$.

Choosing a primitive $n^{th}$ root $\zeta$ of 1 when there is one in $F$, and using it to identify the groups $nB(F)$, $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z})$ and $H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}(2))$, the symbol $\delta_2$ turns out to be the same as the symbol $s_\zeta$ of lemma 3. The map (8) associated to $s_\zeta$ is thus the same in this case as the one in the :

Conjecture (Bloch & Kato, 1986). — The associated homomorphism of graded $(\mathbb{Z}/n\mathbb{Z})$-algebras

$\delta : K_2(F)/nK_2(F) \to \mathbb{Z}/n\mathbb{Z} \oplus H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}(1)) \oplus H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}(2)) \oplus \cdots$

is an isomorphism for all fields $F$ in which $n$ is invertible.

The main theorem (cf. th. 6) of Merkurjev & Suslin (1982) says that the map $\delta_2 : K_2(F)/nK_2(F) \to H^2(\Gamma, \mathbb{Z}/n\mathbb{Z}(2))$ is always an isomorphism; Tate had proved this earlier (1976) for global fields. Bloch, Gabber & Kato prove this conjecture when $F$ is a field of characteristic 0 endowed with a henselian discrete valuation of residual characteristic $p \neq 0$ and $n$ is a power of $p$.

The Bloch-Kato conjecture — which remains a major open problem in its full generality — makes the remarkable prediction that the graded algebra $\oplus H^r(\Gamma, \mathbb{Z}/n\mathbb{Z}(r))$ is generated by elements of degree 1. Galois groups should thus be very special among profinite groups in this respect.

7. $K_2$ and differential forms [10], [8].

Let $A$ be a commutative ring and $B$ a commutative $A$-algebra. Recall that an $A$-derivation on $B$ is a pair $(\varphi, M)$ consisting of a $B$-module $M$ and an $A$-linear map $\varphi : B \to M$ such that

$\varphi(xy) = x\varphi(y) + \varphi(x)y$ for all $x, y \in B$. 

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A morphism of A-derivations \((\varphi, M), (\varphi', M')\) on B is an B-linear map \(f : M \to M'\) such that \(\varphi' = f \circ \varphi\). Does there exist a universal A-derivation \((d_B, \Omega^1_{B|A})\) on B? Clearly it would then be unique, up to unique isomorphism. As for the existence, we just need to take the B-module generated by \(d_B(x) (x \in B)\) subject to the relations

\[
i \ d_B(\alpha x) = \alpha d_B(x) \text{ for all } \alpha \in A, x \in B, \]
\[
ii \ d_B(x + y) = d_B(x) + d_B(y) \text{ for all } x, y \in B, \]
\[
iii \ d_B(xy) = d_B(x) y + x d_B(y) \text{ for all } x, y \in B. \]

Write \(\Omega^0_{B|A} = B\) and put \(\Omega^n_{B|A} = \bigwedge^n_{B} \Omega^1_{B|A}\) for \(n > 0\). Recall that there is a unique system of A-linear maps \(d_B : \Omega^n_{B|A} \to \Omega^{n+1}_{B|A}\) extending \(d_B : B \to \Omega^1_{B|A}\), verifying \(d_B \circ d_B = 0\), and such that

\[d_B(\omega \wedge \omega') = d_B(\omega) \wedge \omega' + (-1)^{r + r'} \omega \wedge d_B(\omega') \quad (\omega \in \Omega^r_{B|A}, \omega' \in \Omega^s_{B|A}).\]

Let F be a field. In what follows, we shall take \(A = \mathbb{Z}, B = F\) and simplify notations to \(d, \Omega^n_F\). The groups \(\Omega^n_F\) are generated by the elements

\[
\frac{x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n}}{(x \in F, y_1, \ldots, y_n \in F^\times)}. \tag{10}
\]

i.e. the map \(F \otimes F^\times \otimes \cdots \otimes F^\times \to \Omega^n_F\) which sends \(x \otimes y_1 \otimes \cdots \otimes y_n\) to (10) is surjective. We have a homomorphism \(d\log_1 : F^\times \to \Omega^1_F\), \(d\log_1(x) = \frac{dx}{x}\), of groups.

**Lemma 5** (Tate, 1970). — The map \(d\log_2 : (x, y) = \frac{dx}{x} \wedge \frac{dy}{y}\) is an \(\Omega^2_F\)-valued symbol on F.

There is therefore a unique map \(d\log : K(F) \to \bigoplus_n \Omega^n_F\) of graded rings which restricts to \(d\log_2\) on \(K_2(F)\); it restricts to \(d\log_1\) on \(F^\times\).

Now suppose that F is of characteristic \(p \neq 0\) and denote by \(F^p\) the subfield of F which is the image of the homomorphism \((\cdot)^p : F \to F\). Denote by \(\mathbb{Z}_p \subset \Omega^p_F\) the kernel of the differential \(d : \Omega^p_F \to \Omega^{p+1}_F\) and by \(B^p_F \subset \mathbb{Z}_p\) the image of the differential \(d : \Omega^{n-1}_F \to \Omega^n_F\).

As \(d(\alpha^p \omega) = \alpha^p d(\omega)\) for all \(\alpha \in F, \omega \in \Omega^n_F\), the differential \(d\) is \(F^p\)-linear. Thus, \(B^p_F\) and \(\mathbb{Z}_p\) are vector sub-\(F^p\)-spaces. We shall consider them as vector F-spaces via the isomorphism \((\cdot)^p : F \to F^p\).

Recall that there is a (unique) system of isomorphisms \(\gamma : \Omega^n_F \to \mathbb{Z}_p^F/B^p_F\) of vector F-spaces such that

\[
\gamma \left( \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n} \right) = x^p \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n} + B^p_F.
\]

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for the system of generators (10) of $\Omega^g_F$ (Cartier). We have an $F_p$-linear map
$$\varphi = \gamma - \text{Id} : \Omega^g_F \to \Omega^g_F / B^n_F \text{ defined by } \varphi(\omega) = \gamma(\omega) - \omega + B^n_F;$$
in degree 0, it is the endomorphism $x \mapsto x^p - x$ of the additive group $F$. Explicitly, for
the system of generators (10) of the group $\Omega^n_F$, one has
$$\varphi \left( x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n} \right) = (x^p - x) \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n} + B^n_F.$$  
Let $\nu(n)_F$ be kernel of this map. It is easy to see that $\varphi \circ \text{dlog}_n = 0$, i.e. $\text{dlog}_n$ maps $K_n(F)$ into $\nu(n)_F = \text{Ker}(\varphi)$. As $\text{dlog}_n(pK_n(F)) = \{0\}$, we have a map of graded $F_p$-algebras
$$\text{dlog} : K(F)/pK(F) \to \nu(0)_F \oplus \nu(1)_F \oplus \nu(2)_F \oplus \cdots$$

**Theorem 7** (Bloch, Gabber & Kato, 1986). — The map (11) is an isomorphism for every field $F$ of characteristic $p \neq 0$.

In degree 0 we get the isomorphism $\mathbb{Z}/p\mathbb{Z} \to F_p = \text{Ker}(\varphi : F \to F)$. There is also an interpretation of the quotients $K(F)/p^nK(F)$ but it is more complicated to state.

***

Let us briefly mention the case $p = 2$ where the notions of (regular) bilinear form and (regular) quadratic space are distinct, so we get two objects: a filtered ring $W(F)$ and a $W(F)$-module $Q(F)$. A theorem of Kato (1982) says that the kernel $\nu(n)_F$ of $\varphi : \Omega^n_F \to \Omega^n_F / B^n_F$ is isomorphic to $I^n/I^{n+1}$ and the cokernel to $I^nQ(F)/I^{n+1}Q(F)$.

8. $K_2$ and abelian extensions [23].

Let $F$ be a field, $\tilde{F}$ the maximal abelian extension of $F$ and $\Gamma = \text{Gal}(\tilde{F}|F)$ the (profinite, commutative) group of $F$-automorphisms of $\tilde{F}$.

Assume that $F$ is finite. We know that then there is a natural map $\rho : K_0(F) = \mathbb{Z} \to \Gamma$, namely $n \mapsto \varphi^n$ where $\varphi$ is the automorphism $x \mapsto x^q$, $q = \text{Card } F$. The image of $\rho$ is dense in $\Gamma$.

Next, let $F$ be a local field, i.e. a finite extension of $\mathbb{Q}_p$ ($p$ prime) or a field isomorphic to $k((T))$ where $k$ is a finite field. Then there is a natural $\rho : K_1(F) = F^\times \to \Gamma$ — which is less easy to describe — whose image is dense in $\Gamma$. This map and its properties form the essential content of the theory of abelian extensions of local fields.

What we have just seen are the 0- and 1-dimensional versions of a general theory of $n$-dimensional local fields. Such a field $F$ ($n > 1$) is complete with respect to a discrete valuation whose residue field is an $(n-1)$-dimensional local field.
THEOREM 8 (Kato, S. Saito, 1986). — Let $F$ be an $n$-dimensional local field. There is a natural homomorphism $\rho : K_n(F) \to \Gamma = \text{Gal}(\bar{F}|F)$ whose image is dense.

In addition, this “reciprocity map” is compatible with norms from finite abelian extensions, a notion which we do not pursue here.

Thus for a 2-dimensional local field $F$, the group $\text{Gal}(\bar{F}|F)$ is generated — as a profinite commutative group — by elements of the form $\rho(\{x, y\})$ $(x, y \in F^\times)$, and all relations among these elements are consequences of bilinearity and “$\{x, y\} = 1$ whenever $x + y = 1$”. Examples of such fields include $\mathbb{Q}_p((T))$ ($p$ prime) and $k((X))((Y))$, where $k$ is a finite field.

9. $K_2$ and abelian coverings of curves [5], [25], [32].

Let $F$ be a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((T))$, i.e. a field complete with respect to a discrete valuation with finite residue field, and let $C$ be a smooth proper absolutely connected $F$-curve. For $x \in C^{(1)}$ a closed point of $C$, we have the tame symbol $K_2(F(C)) \to F(x)^\times$ at the place $x$ of the function field $F(C)$ (Example 5). Define

$$V_1(C) = \text{Coker}(K_2(F(C)) \to \bigoplus_{x \in C^{(1)}} F(x)^\times).$$

For every closed point $x \in C^{(1)}$, there is a norm map $F(x)^\times \to F^\times$, inducing a map $V_1(C) \to F^\times$. Let $V_1(C)_0 = \text{Ker}(V_1(C) \to F^\times)$, a group introduced by S. Bloch.

Let $\varpi_1(C)$ be the profinite group classifying abelian étale coverings of $C$; it is the maximal commutative quotient of the étale fundamental group $\pi_1(C)$ of $C$. There is a natural surjection $\varpi_1(C) \to \varpi_1(F)$; denote by $\varpi_1(C)_0$ the kernel, called the “the geometric part”. By general results (i.e. valid for any smooth proper absolutely connected $F$-variety), the torsion subgroup of $\varpi_1(C)_0$ is finite, and the quotient modulo the torsion subgroup is $\hat{\mathbb{Z}}^r$, where $r$ is the toric rank of the jacobian variety (of the albanese variety, in general) $J$ of $C$, i.e. the dimension of the maximal split torus in the closed fibre of the Néron model of $J$.

Generalising from the classical case of reciprocity maps for 1-dimensional schemes like (the spectrum of) the ring of integers in a number field, one constructs “reciprocity maps”, $\sigma : V_1(C) \to \varpi_1(C)$, $\sigma_0 : V_1(C)_0 \to \varpi_1(C)_0$. 

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We have a commutative diagram with exact rows

\[
\begin{array}{cccccc}
\{0\} & \rightarrow & V_1(C)_0 & \rightarrow & V_1(C) & \rightarrow & F^\times \\
\sigma_0 & \downarrow & \sigma & \downarrow & \rho & \downarrow & \\
\{0\} & \rightarrow & \varpi_1(C)_0 & \rightarrow & \varpi_1(C) & \rightarrow & \varpi_1(F) & \rightarrow & \{0\}
\end{array}
\]

in which \(\rho\) is the reciprocity map for the local field \(F\), inducing an isomorphism upon completion of \(F^\times\).

**Theorem 9** (Bloch 1981, S. Saito 1985, Yoshida 2003). — The kernel of the reciprocity map \(\sigma_0\) is the maximal divisible subgroup of \(V_1(C)_0\), and the image is the torsion subgroup of \(\varpi_1(C)_0\). The kernel of \(\sigma\) is the maximal divisible subgroup of \(V_1(C)\), and the quotient of \(\varpi_1(C)\) by the closure of \(\text{Im}(\sigma)\) is \(\hat{\mathbb{Z}}^r\), where \(r\) is the toric rank of the jacobian of \(C\).

The idea is that we have described the group \(\varpi_1(C)_0\) purely in terms of the group \(V_1(C)_0\) constructed from the function field \(F(C)\), just as we describe the group \(\text{Gal}(H|K)\) of \(K\)-automorphisms of the maximal unramified abelian extension \(H\) of a number field \(K\) in terms of a group constructed from \(K\), namely the ideal class group.

10. \(K_2\) and uniqueness of reciprocity laws [21], [9].

Let us briefly recall the definition of the norm residue symbol (sometimes called the Hilbert symbol). Let \(F\) be a local field other than \(\mathbb{C}\) and let \(\mu(F) = \{1\}\) be the group of roots of 1 in \(F\); it is cyclic, and its order \(m\) is invertible in \(F\). Consider the extension \(L = F(\sqrt[m]{F^\times})\) of \(F\) — it is the maximal abelian extension of exponent \(m\). It is of finite degree as the closed subgroup \(F^\times/m \subset F^\times\) is of finite index. (Exercise: compute this index.)

The quotient \(F^\times/F^\times m\) admits two descriptions. On the one hand, by the theory of abelian extensions of local fields, it is canonically isomorphic to the group \(G = \text{Gal}(L|F)\) of \(F\)-automorphisms of \(L\). On the other hand, we get an isomorphism \(\delta : F^\times/F^\times m \rightarrow H^1(G, \mu(F))\) from the short exact sequence

\[
\{1\} \rightarrow \mu(F) \rightarrow L^\times \xrightarrow{(0)^m} L^\times \rightarrow \{1\}
\]

of discrete \(G\)-modules. As the \(G\)-action on \(\mu(F)\) is trivial, we have \(H^1(G, \mu(F)) = \text{Hom}(G, \mu(F))\). Thus we get an isomorphism

\[
F^\times/F^\times m \rightarrow \text{Hom}(F^\times/F^\times m, \mu(F)),
\]
i.e. a perfect duality of $\mathbb{Z}/m\mathbb{Z}$-modules. The corresponding bilinear map 
\[
\left(\frac{\cdot}{F}\right) : F^\times \times F^\times \to \mu(F)
\] 
happens to be a symbol — the norm residue symbol. It is the universal continuous symbol on $F$ (Moore). Further, it can be shown that the natural surjective map $K_2(F) \to \mu(F)$ admits a section and that its kernel is a uniquely divisible group (Tate, Merkurjev). Of course, for any divisor $d$ of $m$ we get a continuous $\mu(F)$-valued symbol on $F$ by raising the norm residue symbol to the power $n/d$.

When $F = \mathbb{R}$ (resp. $\mathbb{Q}_2$), we have $m = 2$ and the symbol $s_\infty$ (resp. $s_2$) of Example 1 (resp. Example 2) is the norm residue symbol. When $F = \mathbb{Q}_p$ ($p$ odd), we have $m = p - 1$ and the symbol $s_p$ of Example 3 can be viewed as the norm residue symbol — up to the canonical isomorphism $\mu(F) \to F^\times$. Example 4 includes the case of local function fields.

Now let $F$ be a global field, $\mu(F) \subset F^\times$ the group of roots of 1 in $F$ and, for every real or ultrametric place $v$ of $F$, let $\mu(F_v)$ be the group of roots of 1 in the local field $F_v$. Put $m = \text{Card } \mu(F)$, $m_v = \text{Card } \mu(F_v)$; $m$ divides $m_v$ for every $v$. (Imaginary places of $F$ play no role in what follows : $\mathbb{C}^\times$ is connected and therefore the universal continuous symbol on $\mathbb{C}$ is trivial).

Consider $\bigoplus_v \mu(F_v)$ (where $v$ runs over the real or ultrametric places of $F$). We have a natural homomorphism from this direct sum to $\mu(F)$ which on the $v$-th component is the map $\zeta \mapsto \zeta^{m_v/m}$. Also, for $x, y \in F^\times$, we have 
\[
\left(\frac{a, b}{F_v}\right) = 1
\] 
for almost all $v$. We thus get a sequence
\[
K_2(F) \xrightarrow{\lambda} \bigoplus_v \mu(F_v) \xrightarrow{-} \mu(F) \to \{1\}.
\]
The “explicit” reciprocity law says that this sequence is a complex. Tate’s computation (th. 2) amounts to saying that (12) is exact for $F = \mathbb{Q}$ and that $\text{Ker}(\lambda) = \{0\}$. We have seen that, in the notation of Examples 1–3, Hilbert’s product formula $s_\infty s_2 h_3 h_5 \ldots = 1$, which is equivalent to the law of quadratic reciprocity, is a particular case. In general, one has “uniqueness of reciprocity laws” in the following sense:

**Theorem 10** (Moore, 1968). — *The sequence* (12) *is exact for every global field* $F$.

When $F$ is a function field of characteristic $p$, Tate proved that $\text{Ker}(\lambda)$ is a finite group of order prime to $p$. For number fields, the finiteness of $\text{Ker}(\lambda)$ was proved by Brumer in the totally real abelian case and by Garland in general, as a consequence of his theorem — whose proof uses riemannian geometry and harmonic forms — about the vanishing of $H^2(SL_n(\mathfrak{o}), \mathbb{R})$ ($n > 6$) for the ring of integers $\mathfrak{o}$ of $F$. Now these finiteness results are corollaries of the general results of Quillen.
11. $K_2$ and special values of $\zeta$-functions [2], [16], [18], [24].

Let $F$ be a global function field over a (finite) field $k$. Let us rewrite the sequence (12) in terms of the multiplicative groups $k_v^\times$ of the residue fields $k_v$ at the various places $v$ — necessarily ultrametric — of $F$. At every $v$, there is a canonical isomorphism $\mu(F_v) \to k_v^\times$; there is also an isomorphism $\mu(F) \to k^\times$. So the exact sequence (12) becomes

\[(0) \to \text{Ker}(\lambda) \to K_2(F) \xrightarrow{\lambda} \bigoplus_v k_v^\times \to k^\times \to \{1\}, \tag{13}\]

(cf. (3) in the case $F = \mathbb{k}(T)$) which can also be derived more directly by looking at the (tame) symbols corresponding to the various discrete valuations of $F$.

**Theorem 11 (Tate, 1970).** — *For a function field $F$ (in one variable) over a finite field of $q$ elements, we have*

\[\text{Card Ker}(\lambda) = (q^2 - 1)\zeta_F(-1) \text{ Card Coker}(\lambda), \tag{14}\]

*where $\zeta_F$ is the zeta function of $F$.*

Let us give a brief sketch of the proof. Let $\overline{k}$ be an algebraic closure of $k$ and put $F_\infty = \overline{F}$; denote the groups of automorphisms of these $\mathbf{Z}$-extensions by $\Gamma = \text{Gal}(\overline{k}|k) = \text{Gal}(F_\infty|F)$. Let $D$ be the free commutative group on the places of $F_\infty$ (i.e. on the rational points of the $\overline{k}$-curve corresponding to $F_\infty$, or, equivalently, on the set of discrete valuations of $F_\infty$). The maps $\text{div} : F_\infty^\times \to D$ “divisor of a function” and $\text{deg} : D \to \mathbb{Z}$ “degree of a divisor” induce the exact sequences

\[\{1\} \to k^\times \to F_\infty^\times \to D \to C \to \{0\}, \quad \{0\} \to J(\overline{k}) \to C \to \mathbb{Z} \to \{0\}, \tag{15}\]

of $\Gamma$-modules, where $J$ is the jacobian of $F_\infty$. Tensoring the first one with $k^\times$ yields the exact sequence

\[(0) \to \text{Tor}(k^\times, C) \to \overline{k}^\times \otimes F_\infty^\times \to \overline{k}^\times \otimes D \to \overline{k}^\times \to \{1\}. \tag{16}\]

($J(\overline{k}) \otimes k^\times = \{0\}$ since $J(\overline{k})$ is a divisible and $\overline{k}^\times$ a torsion group). On the other hand, taking tame symbols (cf. *Example 5*) at the various discrete valuations of $F_\infty$ provides an exact sequence of $\Gamma$-modules

\[\{0\} \to \text{Ker}(\lambda_\infty) \to K_2(F_\infty) \xrightarrow{\lambda_\infty} \overline{k}^\times \otimes D \to \overline{k}^\times \to \{1\}. \tag{17}\]

The map $e : \overline{k}^\times \otimes F_\infty^\times \to K_2(F_\infty)$ which sends $w \otimes f$ to $\{w, f\}_{F_\infty}$ induces a $\Gamma$-equivariant map of sequences (15)→(16) which can be shown to be an
isomorphism. It can be shown further that the map $(13) \rightarrow (16)^{\Gamma}$ induced by the inclusion $F \subset F_{\infty}$ is also an isomorphism. Thus Ker$(\lambda)$ is isomorphic to Ker$((\lambda_{\infty})^{\Gamma})$, and hence to Tor$(k_{\infty}^{\times}, C)^{\Gamma}$, which is the kernel of $1 - \sigma$ acting on Tor$(k_{\infty}^{\times}, C)$, where $\sigma \in \Gamma$ is the (topological) generator $t \mapsto t^q$. The order of this kernel is the same as that of the kernel of $1 - q\sigma$ acting on $C$, or, what comes to the same, $1 - q\pi$ acting on the jacobian $J$ of $F_{\infty}$, where $\pi$ is the Frobenius endomorphism. To conclude the proof of $(14)$, it suffices to remark that

$$\deg(1 - q\pi) = (q^2 - 1)\zeta_F(-1)(q - 1).$$

What is the analogue of $(14)$ for a number field $F$? The first difficulty is that the zeta function $\zeta_F$ has a zero of order $r_2$ (= the number of imaginary places of $F$) at $s = -1$, but this can be overcome by assuming that $F$ is totally real ($r_2 = 0$). Next, one has to interpret the number $q^2 - 1$ in this context.

So let $F$ be a number field. Let us restrict ourselves to ultrametric places, as in $(13)$, and consider only the universal continuous tame symbols at these places, again as in $(13)$. At each place $v$, denote by $k_v$ the residue field; there is a canonical surjection $\mu(F_v) \rightarrow k_v^{\times}$, which is an isomorphism for almost all $v$ (the norm residue symbols are tame at almost all places of $F$). From $(12)$, we deduce an exact sequence (th. 9)

$$(17) \quad K_2(F) \xrightarrow{\rho} \bigoplus_v k_v^{\times} \rightarrow \{1\}$$

which will play the role — in the totally real case — played by $(13)$ in the function field case. If we had introduced the $K_2$ of rings, we could have interpreted Ker$(\rho)$ as $K_2(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers of $F$.

What should be the analogue of the number $q^2 - 1$? Birch and Tate, with the help of numerical computations of Atkin, interpreted it as the largest integer $m$ such that $\text{Gal}((\overline{F}|F)$ acts trivially on $\mathbb{Z}/m\mathbb{Z}(2)$; it can also be defined as Card$H^0(F, \mathbb{Q}/\mathbb{Z}(2))$. They were thus led to the

**Conjecture** (Birch-Tate, 1969). — *For a totally real number field $F$,*

$$\text{Card Ker}(\rho) = w_2(F)|\zeta_F(-1)|,$$

where $w_2(F)$ is the largest $m$ for which $\text{Gal}((\overline{F}|F)$ acts trivially on $\mathbb{Z}/m\mathbb{Z}(2)$.

The sign of $\zeta_F(-1)$ is $-1$ if and only if $F$ has an odd number of real places, as follows from the functional equation for $\zeta_F$. Let us note that Siegel has proved (1969) that $\zeta_F(-1)$ is a rational number and Serre has proved (1971) that $w_2(F)\zeta_F(-1)$ is an integer.
In the special case $F = \mathbb{Q}$, one has $\zeta_{\mathbb{Q}}(-1) = -1/12$, Card Ker$(\rho) = 2$ and $w_2(\mathbb{Q}) = 24$.

Tate’s result that $K_2(F) \to H^1(F, \mathbb{Q}/\mathbb{Z}(2))$ is an isomorphism for a totally real $F$ led Lichtenbaum to formulate his more general conjectures about $\zeta_F(-m)$, for every odd integer $m > 0$, in terms of the $\text{Gal}(\overline{F}|F)$-module $\mathbb{Q}/\mathbb{Z}(m+1)$.

**Theorem 12 (Wiles, 1990).** — The Birch-Tate conjecture is true up to powers of 2 for every totally real number field $F$, and indeed exactly true if moreover $F$ is abelian (over $\mathbb{Q}$).

The proof rests on the main conjecture of Iwasawa theory for all totally real number fields, which has been proved by Wiles. However, for (totally real) abelian number fields, it had been proved earlier by Mazur & Wiles. In this abelian case, the work of Kolyvagin, Rubin and Thaine, when combined with the earlier work of Iwasawa, gives a different and simpler proof of the main conjecture.

I thank Prof. John Coates and R. Sujatha for their help in getting these remarks right, and for their critical reading of an earlier version of this report up to this point.

**12. K_2 and special values of L-functions [4], [6], [7], [12].**

Let $C$ be a smooth, projective, absolutely connected curve over $\mathbb{Q}$ of genus $g > 1$. Let $N$ be the conductor and $L(C, s)$ the $L$-function associated to the (continuous, linear) $\text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$-representation $H^1(\overline{C}_{\mathbb{Q}}, \mathbb{Q}_l)$ (for any prime number $l$); $L(C, s)$ converges when the real part of $s$ is $> \frac{3}{2}$.

**Conjecture (Hasse-Weil).** — The function

$$
\Lambda(C, s) = \frac{N^\frac{g}{2}}{(2\pi)^gs} \Gamma(s)^g L(C, s)
$$

admits an analytic continuation to the whole of $C$ and satisfies the functional equation $\Lambda(C, s) = w \Lambda(C, s - 2)$, with $w = +1$ or $w = -1$.

It would follow that $\Lambda(C, 0) \in \mathbb{R}^\times$. The conjecture is known to be true for modular curves. It is also true for curves of genus 1, as a result of the seminal work of Wiles and others showing that all elliptic $\mathbb{Q}$-curves are quotients of (the jacobians of) modular curves. We are interested in the special value $\Lambda(C, 0)$.

Let $F = \mathbb{Q}(C)$ be the field of functions of $C$. It is also the field of functions of any regular, proper, flat $\mathbb{Z}$-scheme whose generic fibre is $C$; such "integral
models” of $C$ are known to exist. Fix such a scheme $\Sigma$. Every codimension-1 point $P$ of $\Sigma$ gives rise to a discrete valuation $v_P$ of $F$ and hence (Example 5) to the “tame symbol” homomorphism $h_P : K_2(F) \to k(P)\times$, where $k(P)$ is the residue field at $P$. Denote by $K_2(C, \mathbb{Z})$ the intersection of the kernels of all these homomorphisms; it is independent of the choice of $\Sigma$. Put $K_2(C, \mathbb{Q}) = K_2(C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

**Conjecture (Beilinson).** — The vector space $K_2(C, \mathbb{Q})$ is $g$-dimensional.

Let $X = C(C)$ be complex analytic curve deduced from $C$. It is a compact connected orientable surface of genus $g$ and comes equipped with a real-analytic involution, induced by complex conjugation.

For $(f, g) \in F^\times \times F^\times$ (recall that $F = \mathbb{Q}(C)$), consider the (real-analytic) 1-form

$$\eta_{f,g} = \log |f| d\text{Arg}(g) - \log |g| d\text{Arg}(f)$$

on the complement in $X$ of the divisors of $f$ and $g$; it is bilinear in $f$ and $g$. This 1-form is closed, as $d\eta_{f,g}$ is the imaginary part of $d \log(f) \wedge d \log(g)$, which vanishes.

Let $S$ be a finite subset of $X$ and $\omega$ a closed (smooth) 1-form on $X - S$. For any oriented smooth loop $\gamma$ in $X - S$, we have the number

$$(\gamma, \omega)_{X,S} = \frac{1}{2\pi} \int_{\gamma} \omega$$

which depends only on the class of $\gamma$ in $H_1(X - S, \mathbb{Z})$.

Let $f, g \in F^\times$ be such that $f + g = 1$ and take $S$ to be complement in $X$ of the divisors of $f, g$. We have $(\gamma, \eta_{f,g})_{X,S} = 0$, since $\eta_{f,g} = dD(f)$, where $D$ is the dilogarithm function, a real-analytic function of $z \neq 0, 1$ in $\mathbb{C}$:

$$D(z) = \text{Arg}(1 - z) \log |z| - \text{Im} \left( \int_0^z \log(1 - t) \frac{dt}{t} \right).$$

Further, for $s \in S$, let $\gamma_s$ be a smooth loop around $s$ in $X$, and let $f, g \in F^\times$. It can be checked that

$$(\gamma_s, \eta_{f,g})_{X,S} = \log |t_s(\{f, g\})|,$$

where $t_s$ is the tame symbol at the place of $F$ determined by $s$. Thus, if $\{f, g\}$ happens to lie in $K_2(C, \mathbb{Z})$, then $(\gamma_s, \eta_{f,g})_{X,S} = 0$ for every $s \in S$. Thus we get a pairing $\langle \ , \ \rangle : H_1(X, \mathbb{Z}) \times K_2(C, \mathbb{Z}) \to \mathbb{R}$. 

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The invariants $H_1(X, Z)^+$ under complex conjugation lie in the left kernel of $\langle \ , \rangle$. It is the restriction to the anti-invariants which gives the “regulator map”:

\[
\langle \ , \rangle : H_1(X, \mathbb{Q})^- \times K_2(C, \mathbb{Q}) \to \mathbb{R},
\]

coming from integrating 1-forms against 1-cycles, as we have seen. Note that $\dim H_1(X, \mathbb{Q})^- = g$.

**Conjecture (Beilinson).** — *The pairing $\langle \ , \rangle$ (19) is perfect.*

Choosing $\mathbb{Q}$-bases for $H_1(X, \mathbb{Q})^-$ and $K_2(C, \mathbb{Q})$, the pairing (19) gives a matrix in $\text{GL}_g(\mathbb{R})$. The class of its determinant in $\mathbb{R}^\times/\mathbb{Q}^\times$ is independent of the choice of the $\mathbb{Q}$-bases.

**Conjecture (Beilinson).** — *The determinant of (19) equals $\Lambda(C, 0)$ in $\mathbb{R}^\times/\mathbb{Q}^\times$.***

A weak version has been proved for elliptic curves having complex multiplications. Similar conjectures have been advanced for curves over any number field. Numerical verification has been carried out in some cases.

I would like to thank Ramesh Sreekantan and Tim Dokchitser for a careful reading of this section.

13. **Beyond the multiplicative group and the point [26], [1]**

$K_2$ is related to numerous other things, and it has been generalised to numerous other functors. There is the $K_2$ of a ring. There are Milnor’s higher $K$-groups (for fields), which we have furtively encountered. There are Quillen’s $K$-groups, which make sense for any scheme, not just for rings and fields. These groups can be sheafified, and the cohomology of these sheaves is of interest.

We have chosen to close by touching upon a recent generalisation due to Kato, Somekawa and Akhtar. We shall be even more cryptic here than elsewhere in this report, as we merely wish to illustrate — by one example among many — how $K_2$ continues to play a central rôle in current research.

The idea is to replace the multiplicative group $F^\times$ of a field $F$ with an algebraic $F$-group $G$ which is an extension

\[
\{0\} \to T \to G \to A \to \{0\}
\]

of an abelian $F$-variety $A$ by an $F$-torus $T$, i.e. $G$ is a semi-abelian variety over $F$. 

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Let $G_1, G_2, \ldots, G_s$ be semi-abelian $F$-varieties. Inspired by the presentation of $K_2(F)$ by generators and relators, the group $K(F; G_1, \ldots, G_s)$ is defined by a presentation a typical generator for which is an element of

$$G_1(E) \otimes \cdots \otimes G_s(E)$$

for a variable finite extension $E$ of $F$; they are added as elements of this group and are subject to two sets of relators, coming from the projection formula for $G_i(E) \to G_i(E')$ for a morphism $E \to E'$ of finite extensions, and from a certain reciprocity law for $G_i(K) \to \bigoplus_v G_i(E_v)$ for the function field $K$ of an $F$-curve, with $v$ running over the places of $K$ and $E_v$ denoting the residue field at $v$. When each $G_i$ ($1 \leq i \leq s$) is the multiplicative group, one recovers Milnor’s $K_s(F)$.

For every $n > 1$ invertible in $F$, there is a functorial homomorphism $c$ which makes the diagram

$$\begin{array}{ccc}
G_1(E) \otimes \cdots \otimes G_s(E) & \longrightarrow & H^1(E, nG_1) \otimes \cdots \otimes H^1(E, nG_s) \\
\downarrow & & \downarrow \\
K(F; G_1, \ldots, G_s) \otimes \mathbb{Z}/n\mathbb{Z} & \longrightarrow & H^s(F, nG_1 \otimes \cdots \otimes nG_s)
\end{array}$$

commute for every finite extension $E$ of $F$; the map $(g)$ is a particular case. It is conjectured that this $c$ is always injective; this follows from the Bloch-Kato conjecture in the case when each $G_i = F^\times$; in this case, the result is known, as we have seen, if $s = 2$ (Merkurjev-Suslin) or if $n = 2^a$ (Voevodsky).

Take $F$ to be a number field, $s = 2$, and $G_2 = F^\times$. There is a certain local-to-global exact sequence for $K(F; G_1, F^\times)$ of which Moore’s exact sequence (12) is the particular case $G_1 = F^\times$; the case when $G_1$ is the jacobian of a smooth projective $F$-curve having a 0-cycle of degree 1 reduces to a theorem of Bloch and Kato-Saito.

Already Akhtar has further generalised these $K$-groups to “mixed $K$-groups”. His construction can be seen as a passage from $\text{Spec} F$ to a certain number of smooth quasiprojective $F$-varieties $X_1, \ldots, X_r$. A typical generator of the group $K(F; X_1, \ldots, X_r; G_1, \ldots, G_s)$ is an element of

$$A_0((X_1)_E) \otimes \cdots \otimes A_0((X_r)_E) \otimes G_1(E) \otimes \cdots \otimes G_s(E)$$

involving the Chow groups $A_0((X_i)_E)$ (modulo rational equivalence) of 0-cycles on $(X_i)_E = X_i \times_F E$, and the groups of rational points $G_j(E)$, for all
finite extensions $E$ of $F$; the relators come essentially, as before, from the projection formula and the reciprocity law for local symbols. The case $r = 0$ gives the $K$-groups of Kato-Somekawa. At the other extreme, when $s = 0$ and the $X_i$ are projective, one retrieves the group $A_0(X_1 \times_F \cdots \times_F X_r)$ of 0-cycles modulo rational equivalence. The groups $K(F; X; G_1, \ldots, G_s)$, where $X$ is a smooth projective $F$-variety and each $G_i$ is the multiplicative group $F^\times$, turn out to be the “higher Chow groups” of $X$ as defined by Bloch. Interesting computations have been made over finite fields and local fields, but much remains to be done.

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