Neuro-adaptive fixed-time non-singular fast terminal sliding mode control design for a class of under-actuated nonlinear systems

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ABSTRACT
This paper presents a fixed-time neuro-adaptive control design for a class of uncertain under-actuated nonlinear systems (UNS) using a non-singular fast terminal sliding mode control (TSMC) with a radial basis function (RBF)-based estimator to achieve the convergence and robustness against the uncertainties. The mathematical model of the considered class is reduced into an equivalent regular form. A fast TSMC is designed for the transformed form to improve the control performance and annihilate the associated singularity problem of the conventional TSMC. Lyapunov stability theory ensures the steering of the sliding manifold and system states in fixed time. RBF neural networks are adaptively estimate the nonlinear drift functions. The theoretical design, analysis, and simulations of cart-pendulum and quadcopter demonstrate the feasibility and benefits of the regular form transformation and the designed control design. Comparing the proposed synthesis with the standard literature presents the attractive nature of proposed method for such a class.

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1. Introduction

under-actuated systems are mechanical systems with less dimension of space spanned by the applied stabilising inputs than space spanned by the configuration variables (Fantoni & Lozano, 2002; Ullah, Khan, et al., 2020; Ullah, Mehmood, et al., 2020). Because of the low probability of system failure, low power consumption, and low cost and weight, robust control of under-actuated mechanical systems is becoming increasingly important in today’s research community, and they are now regularly used in several applications such as robotics, aerospace vehicles, and marine vehicles. This class also has many potential applications in locomotive systems, underwater vehicles, surface vessels, robots, satellites, and unmanned aerial vehicles. The stabilisation of this class, however, is a difficult task due to high-order nonholonomic constraints, the lack of actuators for certain configuration variables, and coupling effects (M. Zhang & Tarn, 2002). A large number of researchers synthesised various advanced nonlinear control algorithms to provide appealing results and closed-loop stability. Some of these efforts are detailed in this article.

Backstepping, a nonlinear control algorithm, is used to transform the $n$th order structure into a new recursive form with $n$ number of subsystems (each one of relative degree one). This control scheme has generally been used in recent years for the global stabilisation of UNS, such as surface vessels (Ghomam et al., 2006), spacecrafts (X. Huang & Yan, 2017) and unmanned aerial vehicles (Gruszka et al., 2011). Unfortunately, as the degree of freedom of the aforementioned class increases, such a control strategy’s design procedure becomes incredibly hard. Furthermore, many physical systems are sensitive to many unknown disturbances, such as external perturbation and parametric variations, making these approaches challenging to apply practically. An adaptive fuzzy-based output feedback controller is presented for a class of Euler–Lagrange mechatronic systems to address the issues mentioned earlier (Yang et al., 2021a, 2021b). Nevertheless, the optimal tracking performance cannot be obtained because fuzzy rules are all designed based on experience (Chen et al., 2021).

In the sense of robust nonlinear control algorithms, the sliding mode control (SMC) has received much interest (Ji & Liu, 2021). In the presence of disturbances, this strategy will drive the dynamics of UNS to meet the desired target, like the double-pendulum crane in Wu et al. (2021), unmanned aerial vehicle (UAV) in Mechal et al. (2021), cart-pendulum in Riachy et al. (2008), satellite in Ye et al. (2020) and ball and beam system in Din et al. (2017) and Zaare and Soltanpour (2021). However, the chattering phenomenon has become more prevalent in UNS because of the high dynamics coupling and nonlinear solid terms. Various higher-order sliding mode techniques have been used to address this shortcoming (see, for instance, Din et al., 2017; ud Din & Khan, 2018). In controlling UNS dynamics, the control strategies developed in Din et al. (2017) and ud Din and Khan (2018) produce some very impressive findings. These methods were fascinating, but the steady-state error resulted in less accuracy due to their asymptotic convergence. To eliminate the chattering problem and get high precision, Khan
et al. (2017) presented a fast TSMC law. Compared to conventional linear SMC, this method achieves finite-time convergence and delivers excellent robustness with high precision. However, the negative fractional powers in its sliding manifold, this scheme may cause the singularity issue (D. Zhao et al., 2015). There have been significant research efforts on the finite-time control method for applications that need a time response constraint, such as robotic manipulators (T. Li & Zhao, 2017), guidance systems (Pan et al., 2019), and spacecraft (B. Li et al., 2019).

This technique not only guarantees finite-time enforcement of the system states to the origin, but it also assures high precision (Tian et al., 2018). As a result, an accurate prior estimate of the state components, leading to a longer convergence duration (B. Tian et al., 2018). As a result, an accurate prior estimate of the settling time cannot be made (Zuo, 2015). As the finite-time system states converge, this scheme is closely related to the states’ initial condition; therefore, this law cannot be used in practical application if the prior initial state conditions are uncertain. However, the finite-time TSMC’s convergence speed is determined by the values of initial system states, which leads to a longer stabilisation time when the values of the initial system state are high.

The literature (Polyakov, 2011) developed fixed-time convergence theory, an extension of the finite-time convergence theory, to tackle this problem. In comparison to finite-time stability, fixed-time stability possessed optimal stability within bounded time without knowing the initial conditions. This algorithm has been widely used in analysis over the last two decades due to its outstanding characteristics. A fast fixed-time non-singular TSMC law was introduced in Ni et al. (2016) to compensate for the chaos in power systems. The control researchers in Y. Huang and Jia (2017) proposed a new non-singular fixed-time fast TSMC scheme for the control of second-order multi-agent systems. The literature (Y. Zhang et al., 2018) developed a novel fixed-time non-singular TSMC approach to control manoeuvring objectives. A fixed-time robust non-singular TSMC law was proposed in the literature (Corradini & Cristofaro, 2018) for the control of uncertain nonlinear systems. For the stability of a single cart-pendulum system, another non-singular fixed-time TSMC technique was introduced in Y. Tian et al. (2020).

Nevertheless, these approaches were either limited in their applicability or had theoretical problems such as chattering, step life, and poor robustness in the entire operation of system dynamics (Fessi et al., 2017). Several neural networks are used in the controller architecture to increase robustness by estimating the external disturbances and uncertain nonlinearities (see Talebi et al., 2008 and the references therein).

In this paper, a new non-singular fast TSMC strategy is proposed for the fixed-time control of uncertain under-actuated nonlinear systems and eliminating the associated singularity problem associated with the classical TSMC scheme. The main contributions of this work are three-fold. The first one is transforming the generalised dynamical model of the understudy class of n degree of freedom into its equivalent control convenient regular form. The transformed regular form divides the overall dynamical model into two blocks named directly driven block and indirectly driven block by the applied control actuator. The second contribution is the employment of RBF-based neural networks to estimate the highly nonlinear drift functions, which can suppress the strong influence of uncertain disturbances. The last contribution is developing a fixed-time TSMC framework for the transformed regular form to ensure significant control characteristics with fast convergence and high precision. The proposed control synthesis annihilates the singularity issue and ensures minimum settling time for any initial state conditions. The Lyapunov stability function presents the step-by-step proof of fixed-time stabilisation of both system states and sliding manifolds. The theoretical analysis and MATLAB simulations of two benchmark examples (cart-pendulum and quadcopter systems) demonstrate the feasibility and benefits of the non-singular coordinate transformation, the proposed control synthesis, and RBF-based functions estimator. The simulations of these benchmarks are compared with the standard literature to show the proposed control scheme’s attractive nature for the understudy class. This paper proceeds as follows: The dynamical model description of the understudy problem and its regular form transformation are introduced in Section 2. In Section 3, the proposed control technique’s design procedure is given to control the understudy class. Two benchmark examples of cart-pendulum and quadcopter, along with detailed discussions of simulation results, are demonstrated in Section 4. The concluding remarks are summarised in the last section.

2. Problem formulation

Consider the following motion equation for any mechanical system in vector form

\[
\ddot{\mathbf{q}} + \mathbf{F}(\dot{\mathbf{q}}) + \mathbf{R}(\mathbf{q}) = \mathbf{U},
\]

where \( q, \dot{q} \in R^n \) and \( U \in R \) are the system states and applied control input, respectively, \( F(\dot{q}) \) are the frictional forces, \( R(q) \) are gravitational forces and \( \mathbf{U} \) points to coriolis force or centrifugal force (see Olfati-Saber, 2001 for more details). The system in Equation (1) can be written in the following family of \( n \) under-actuated systems:

\[
\begin{align*}
\ddot{q}_1 &= F_1 (q, \dot{q}) + G_1 (q) \mathbf{U} \\
\ddot{q}_2 &= F_2 (q, \dot{q}) + G_2 (q) \mathbf{U} \\
&\vdots \\
\ddot{q}_n &= F_n (q, \dot{q}) + G_n (q) \mathbf{U}
\end{align*}
\]

Note that a signal control input \( U \) is used to control all configuration variables. For simplicity and without loss of generality, (2) can be presented in the following comprehensive style:

\[
\begin{align*}
\ddot{q}_i &= F_i (q, \dot{q}) + G_i (q) \mathbf{U} \\
\ddot{q}_n &= F_n (q, \dot{q}) + G_n (q) \mathbf{U} \\
\end{align*}
\]

Assumption 2.1: According to the controllability condition, the drift function \( G_n \) is assumed to be a function of the non-zero value for \( q \) and \( t \) (Utkin et al., 2009).

Remark 2.1: Since dynamical model (3) is coupled both in inputs and states. Therefore, the following non-singular coordinate transformation is presented to reduce the dynamical model
into a control convenient regular form:
\[ y_i = q_i - \psi_i(q_n, t) \quad \text{and} \quad z_n = q_n, \]
where \( q_i = \psi_i(q_n, t) \) is the solution of \( \frac{dq_i}{dt} = \frac{\partial}{\partial q_i} \psi_i(q_n, t) \frac{dq_n}{dt} = \frac{G_i}{G_n} \). Consequently, the following regular form is achieved:
\[
\begin{aligned}
\dot{y}_i &= \dot{q}_i - \frac{\partial}{\partial q_i} \psi_i(q_n, t) \dot{q}_n = F_i - \frac{G_i}{G_n} F_n \\
\dot{z}_n &= \dot{q}_n \\
\dot{y}_i &= \ddot{F}_i (y_i, y_i, z_n, \dot{z}_n) \\
\dot{z}_n &= \ddot{G}_n (y_i, z_n) + \ddot{G}_n (y_i, z_n) U
\end{aligned}
\]
Transformed dynamic model (4) can be expressed in the following state-space form:
\[
\begin{aligned}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= \ddot{F}_i (y_i, z_n, y_2, z_2), \\
\dot{z}_n &= z_2, \\
\dot{z}_2 &= \ddot{G}_n (y_i, z_n, y_2, z_2) + \ddot{G}_n (y_i, z_n) U + \Delta_n (y_i, z_n, t),
\end{aligned}
\]
where \( y_1 = y_i, y_2 = \dot{y}_i, z_1 = z_n \) and \( z_2 = \dot{z}_n \). The \( y_i \) dynamics are known as the internal dynamics (which are not directly depend on the applied control input \( U \)) and the \( z_n \) dynamics are named as the visible dynamics (that are directly controlled by \( U \)). The term \( \Delta_n (y_i, z_n, t) \) (uncertainty of matched/unmatched nature) is assumed to be less or equal to a positive constant.

**Remark 2.2:** The main objective of this work is to investigate a fixed-time control algorithm that will track the \( y \)-dynamics on the desired trajectories while regulating the \( z \)-dynamics to the equilibrium points.

The step-by-step design procedure of the proposed method for the proposed control problem will be investigated in the next section.

### 3. Control law design

In this section, the step-by-step control design of RBF neural network estimator-based non-singular TSMC synthesis is presented to enforce both the internal dynamics \( y_i \) and visible dynamics \( z_n \) of the considered class of under-actuated electromechanical systems to their equilibrium points in fixed time (see Figure 1).

#### 3.1 Fixed-time TSM control law design

For this purpose, the following error variable is presented to track the actual internal dynamics at its desired trajectory \( y_i^* \):
\[
\xi_i = y_i - y_i^*.
\]

The time derivatives of the error variables \( \xi_i \) can be evaluated as follow:
\[
\begin{aligned}
\dot{\xi}_i &= y_{i2} - y_i^*, \\
\ddot{\xi}_i &= \ddot{F}_i (y_i, z_n) - y_i^* - \ddot{G}_n (y_i, z_n) + \ddot{G}_n (y_i, z_n) U \\
\dddot{\xi}_i &= \dddot{F}_i (y_i, z_n, y_2, z_2) - \dddot{G}_n (y_i, z_n) U
\end{aligned}
\]

The fixed-time convergence of \( y_i \) dynamics can be guaranteed by defining the following non-singular fast terminal sliding manifold:
\[
\mathcal{S}_i = \dot{\xi}_i + \alpha_i |\xi_i|^{\mu_i} \text{sign}(\xi_i) + \beta_i \chi_i(\xi_i)
\]
with
\[
\chi_i(\xi_i) = \begin{cases} 
|\xi_i|^{\nu_i} \text{sign}(\xi_i), & \xi_i \geq \rho_i, \\
\chi_i \sin(\xi_i) + \varsigma_i |\xi_i|^{\nu_i} \text{sign}(\xi_i), & |\xi_i| < \rho_i,
\end{cases}
\]
where \( \alpha_i > 0, \beta_i > 0, \kappa_i = 0.5(\mu_i + 1) + 0.5(\mu_i - 1) \text{sign}(\|\xi_i\| - 1), \gamma_i = 0.5(\mu_i + \eta_i) + 0.5(\mu_i - \eta_i) \text{sign}(\|\xi_i\| - 1), \mu_i > 1, \chi_i = \frac{\eta_i \sin(\xi_i) - \eta_i \sin(\xi_i)}{\epsilon \alpha_i}, \varsigma_i = \frac{\eta_i \sin(\xi_i)}{\epsilon \alpha_i}, 0 < \rho_i < 1, 0 < \eta_i < 1 \text{ and } 1 < \epsilon_i < 2.

Proposed sliding surface (8) can be expanded to the following three parts:
\[
\mathcal{S}_i = \begin{cases} 
\dot{\xi}_i + \alpha_i |\xi_i|^{\mu_i} \text{sign}(\xi_i), & |\xi_i| > 1, \\
\dot{\xi}_i + \alpha_i \xi_i + \beta_i |\xi_i|^{\nu_i} \text{sign}(\xi_i), & \rho_i < |\xi_i| \leq 1, \\
\dot{\xi}_i + \alpha_i \xi_i + \beta_i (\chi_i |\xi_i| \text{sign}(\xi_i)), & |\xi_i| \leq \rho_i.
\end{cases}
\]

**Remark 3.1:** Equation (10) ensures the fast convergence rate either the system states are far from or close to the desired equilibria. In addition, the elimination of singularity issues associated with conventional TSM is also claimed by the proposed sliding surface.

Now, the time differentiation of sliding manifold (9) can be evaluated as
\[
\dot{\xi}_i = \dddot{\xi}_i + \alpha_i \kappa_i |\xi_i|^{\nu_i - 1} \dddot{\xi}_i + \beta_i \chi_i(\xi_i)
\]
with
\[
\chi_i(\xi_i) = \begin{cases} 
\gamma_i |\xi_i|^{\nu_i - 1} \dddot{\xi}_i, & |\xi_i| \geq \rho_i, \\
\chi_i \cos(\xi_i) \dddot{\xi}_i + \varsigma_i |\xi_i|^{\nu_i - 1} \dddot{\xi}_i, & |\xi_i| < \rho_i.
\end{cases}
\]
Note that the achievement of $\bar{\xi}_i = -\bar{s}_i$, in the sliding phase, is the primary requirement of this work. Therefore, along the following terminal attractor, the steering of the sliding mode is essential to meet this requirement:

$$\bar{\gamma}_i = \bar{\xi}_i + \alpha_i|\xi_i|^{\kappa_1}\text{sign}(\xi_i) + \beta_i\chi_n(\bar{\gamma}_i) = 0. \quad (14)$$

In this case, the sliding variable $\bar{\gamma}_i$, treated as a virtual output in visible dynamics block, and the drift function $\bar{F}_i$ is acted as a virtual input to the internal dynamics block in (5).

**Remark 3.2:** If the system relative degree is more significant than one (i.e. the nonlinear distribution function $\bar{F}_i$ in (5) does not include $\dot{\bar{z}}$), then another fixed-time terminal attractor will be defined to increase its relative degree and consequently, ensure the control of dynamics expressed in (14):

$$\bar{\gamma}_i = \bar{\gamma}_i + \alpha_i|\gamma_i|^{\kappa_2}\text{sign}(\gamma_i) + \beta_n\chi_n(\bar{\gamma}_i) \quad (15)$$

with

$$\chi_n(\bar{\gamma}_i) = \begin{cases} |\bar{\gamma}_i|^{\kappa_2}\text{sign}(|\bar{\gamma}_i|), & |\bar{\gamma}_i| \geq \rho_n, \\ \bar{\chi}_n \sin(|\bar{\gamma}_i|) + \bar{s}_n|\gamma_i|^{\kappa_2}\text{sign}(\gamma_i), & |\gamma_i| < \rho_n. \end{cases} \quad (16)$$

**Remark 3.3:** The following reaching law is presented to tackle the associated chattering issue with conventional SMC:

$$\dot{\bar{\gamma}}_i = -\bar{\alpha}_i|\gamma_i|^{\bar{\kappa}_1}\text{sign}(\gamma_i) - \bar{\beta}_i|\gamma_i|^{\bar{\kappa}_2}\text{sign}(\gamma_i), \quad (17)$$

where $\bar{\alpha}_i$, $\bar{\kappa}_1$, $\bar{\beta}_i$ and $\bar{\gamma}_i$ are positive controller gains.

The forthcoming theorem details the fixed-time convergence of the under-study class of UNS.

**Theorem 3.1:** Given the dynamical model of UNS as defined in (5) with the fast terminal attractor expressed in (15) and the reachability law selected in (17). Then, the following control scheme ensures the fast convergence of the sliding mode, in fixed time, against the terminal attractor:

$$U = \bar{G}_n^{-1}\left(\bar{\alpha}_i|\gamma_i|^{\bar{\kappa}_1}\text{sign}(|\gamma_i|) \right.$$  

$$- \bar{\beta}_i|\gamma_i|^{\bar{\kappa}_2}\text{sign}(|\gamma_i|) - \frac{\partial \bar{F}_i}{\partial \bar{\gamma}_i}\frac{1}{\gamma_i} - \frac{\partial \bar{F}_i}{\partial \gamma_i}\dot{\gamma}_i - \frac{\partial \bar{F}_i}{\partial \dot{\gamma}_i}\ddot{\gamma}_i - \bar{\xi}_i \bigr) \quad (18)$$

Consequently, the system states will be enforced to their desired equilibria.

**Proof:** To prove this theorem, differentiate the given terminal attractor in (13) along the dynamics of system (5):

$$\ddot{\bar{\gamma}}_i = \frac{\partial \bar{F}_i}{\partial \gamma_i}\frac{1}{\gamma_i} + \frac{\partial \bar{F}_i}{\partial \gamma_i}\dot{\gamma}_i + \frac{\partial \bar{F}_i}{\partial \dot{\gamma}_i}\ddot{\gamma}_i - \bar{\xi}_i + \bar{s}_i. \quad (19)$$

The time differentiation of Lyapunov function $\Lambda_i = \frac{1}{2}\bar{\gamma}_i^2$ along (19) can be evaluated as follows:

$$\dot{\Lambda}_i = \gamma_i\left(-\bar{\alpha}_i|\gamma_i|^{\bar{\kappa}_1}\text{sign}(\gamma_i) - \bar{\beta}_i|\gamma_i|^{\bar{\kappa}_2}\text{sign}(\gamma_i) + \frac{\partial \bar{F}_i}{\partial \gamma_i}\frac{1}{\gamma_i} \right). \quad (21)$$

Considering the identity $|\gamma_i| = \gamma_i\text{sign}(\gamma_i)$, (21) can be re-written as follows:

$$\dot{\Lambda}_i = -\bar{\alpha}_i|\gamma_i|^{\bar{\kappa}_1} + \bar{\alpha}_i|\gamma_i|^{\bar{\kappa}_1} + \bar{\beta}_i|\gamma_i|^{\bar{\kappa}_2} \leq -\bar{\alpha}_i|\gamma_i|^{\bar{\kappa}_1} + \bar{\beta}_i|\gamma_i|^{\bar{\kappa}_2} \quad (22)$$

where $\bar{\Delta} = \frac{1}{|\gamma_i|^{\kappa_1}}|\gamma_i|^{\kappa_1}$. It is worth noting that the above inequality (22) remains true subject to $\bar{\beta}_i \leq \bar{\beta}_i - |\Delta_i|$. It may also be written as

$$\dot{\Lambda} = -\bar{\alpha}_i|\gamma_i|^{\bar{\kappa}_1} - \bar{\beta}_i|\gamma_i|^{\bar{\kappa}_2}, \quad (23)$$

where $\bar{\alpha}_i = \sqrt{2}\bar{\alpha}_i$ and $\bar{\beta}_i = \sqrt{2}\bar{\beta}_i$.

Thus, the sliding mode establishment, against the fixed-time terminal attractor, has been proved. The following differential equation for system (17) is presented to find its upper bound of convergence time:

$$\bar{\gamma}_i = \begin{cases} -\bar{\alpha}_i|\gamma_i|^{\bar{\kappa}_1}\text{sign}(\gamma_i) - \bar{\beta}_i|\gamma_i|^{\bar{\kappa}_2}\text{sign}(\gamma_i), & |\gamma_i| > 1, \\ -\bar{\alpha}_i|\gamma_i|^{\bar{\kappa}_1}\text{sign}(\gamma_i), & |\gamma_i| \leq 1. \end{cases} \quad (24)$$

By solving (24), the upper bound of convergence time can be evaluated as

$$t_r = \lim_{\gamma_i(0) \to \infty} \left(\int_{0}^{\gamma_i(0)} \frac{1}{\bar{\beta}_i|\gamma_i|^{\bar{\kappa}_1}\text{sign}(\gamma_i)} \text{d}\gamma_i \right.$$  

$$+ \int_{0}^{\gamma_i(0)} \frac{1}{\bar{\alpha}_i\gamma_i + \bar{\beta}_i|\gamma_i|^{\bar{\kappa}_2}\text{sign}(\gamma_i)} \text{d}\gamma_i \bigr) \leq \frac{1}{(\bar{\alpha}_i + \bar{\beta}_i)(\mu_i - 1) + \bar{\beta}_i(1 - \eta_i)} \ln\left(1 + \frac{\alpha_i}{\bar{\beta}_i}\right). \quad (25)$$
Having enforced the sliding mode, the convergence of system states can be analysed by considering (8):

\[
\dot{\xi}_i = -\alpha_i |\xi_i|^{\gamma_i} \text{sign}(\xi_i) - \beta_i \chi_i(\xi_i).
\] (26)

The time differentiation of the augmented Lyapunov function \( \Lambda_{11} = \frac{1}{2} \xi^T_i \) can be calculated as

\[
\dot{\Lambda}_{11} = \dot{\xi}_i \left(-\alpha_i |\xi_i|^{\gamma_i} \text{sign}(\xi_i) - \beta_i |\xi_i|^{\gamma_i} \text{sign}(\xi_i)\right)
= -\alpha_i |\xi_i|^{\gamma_i+1} - \beta_i |\xi_i|^{\gamma_i+1}
= -\alpha_i |\Lambda_{2i}|^{\gamma_i+1} - \beta_i |\Lambda_{2i}|^{\gamma_i+1},
\] (27)

where \( \alpha_i = \sqrt{2} \alpha_i \) and \( \beta_i = \sqrt{2} \beta_i \). From (27), the convergence time \( t_c \) of the system states, required to drive to their equilibrium points, is bounded by

\[
t_c \leq \frac{1}{(\alpha_i + \beta_i)(\mu_j - 1)} + \frac{1}{\beta_i(1 - \eta_i)} \ln \left(1 + \frac{\dot{\xi}_i}{\beta_i}\right).
\] (28)

Thus, the proof is completed. In this study, the designed fixed-time control law (18) is used to establish a sliding mode against the proposed fixed-time terminal attractor (15). Having confirmed sliding mode, \( Y_i \to 0 \) will be satisfied. Consequently, it will confirm the tracking of system outputs on the desired trajectory in fixed time (28), i.e., \( y_i \to y_i^* \). The total convergence time is the algebraic sum of \( t_c \) and \( t_r \).

Some nonlinear drift functions are estimated via an advanced neural network method in the following subsection.

### 3.2 Neuro-adaptive fixed-time TSM control law design

The control input \( U \), developed in Theorem 3.1, requires the estimates of the highly nonlinear drift functions in feedback. Therefore, in this subsection, a radial basis function-based neural network is proposed to estimate these uncertain functions adaptively. Consequently, it ensures chattering elimination and robust performance. This algorithm adjusts the network weight matrices according to some adaptive control laws developed via the Lyapunov stability function. Because of these network weights, the estimated functions adapt and ensure the target data at the running condition. Whenever there is a new situation in the system, the network will improve its knowledge to handle this situation. The system states \( x = [y_{11i}, y_{22i}, z_{1n1}, z_{2n2}] \in \mathbb{R}^n \) are the network input vector and the estimated functions \( \hat{F}_n, \hat{G}_n \) are the network targets. The network output is expressed as follows:

\[
\hat{F}_n = \hat{W}_j h_j(x),
\]

\[
\hat{G}_n = \hat{W}_j h_j(x),
\] (29)

where \( \hat{W}_j = [\hat{W}_{j1}, \hat{W}_{j2}, \ldots, \hat{W}_{jn}]^T \) is the network weight matrix and \( h_j(x) = [h_{1j}(x), h_{2j}(x), \ldots, h_{nj}(x)]^T \) is the Gaussian function vector, which is introduced as follows:

\[
h_j(x) = \exp \left(-\frac{(x - c_{jk})^T(x - c_{jk})}{2b_j^2}\right),
\] (30)

where \( b_j \) is the width of the Gaussian function, \( k \) is the network input number, \( c_{jk} = [c_{j1}, c_{j2}, \ldots, c_{jn}]^T \) is the centre vector and \( j \)

is the number of hidden layer network nodes. For any \( e > 0 \), the following inequality holds:

\[
\hat{F}_n = W_{j} h_j(x) + e_F(x),
\]

\[
\hat{G}_n = W_{j} h_j(x) + e_G(x),
\] (31)

where

\[
\hat{F}_n = \hat{F}_n - \hat{F}_n = \hat{W}_j h_j(x) - e_F(x),
\]

\[
\hat{G}_n = \hat{G}_n - \hat{G}_n = \hat{W}_j h_j(x) - e_G(x),
\]

\[
\hat{W}_{j} = \hat{W}_j - W_{j} = \hat{W}_j h_j(x) - e_F(x),
\]

\[
\hat{W}_{j} = \hat{W}_j - W_{j} = \hat{W}_j h_j(x) - e_G(x),
\] (32)

and \( e_F, e_G \) are network approximation errors which are bounded over a compact set, i.e., \( |e(x)| \leq \xi \) with \( \xi > 0 \) an unknown constant.

Having invoked the proposed network, the control input in (18) can be re-designed in the following form:

\[
U = \hat{G}_n^{-1} \left(-\hat{F}_n - \frac{\partial \hat{F}_n}{\partial z_{n2}} \left(\alpha_i |\hat{Y}_i|^{\gamma_i} \text{sign}(\hat{Y}_i)\right)
+ \hat{\beta}_i |\hat{Y}_i|^{\gamma_i} \text{sign}(\hat{Y}_i)
+ \frac{\partial \hat{F}_i}{\partial y_{11}} \dot{y}_{11} + \frac{\partial \hat{F}_i}{\partial z_{n1}} \dot{z}_{n1} + \frac{\partial \hat{F}_i}{\partial y_{22}} \dot{y}_{22} + \dot{\hat{Y}}_i + \hat{\xi}_i + Z\right),
\] (33)

where

\[
Z = -\frac{\alpha_i}{2} \left(\gamma_1^{-1} \dot{\hat{W}}_{j1}^{2} \right) - \frac{\alpha_i}{2} \left(\gamma_2^{-1} \dot{\hat{W}}_{j2}^{2} \right) - \frac{\alpha_i}{2} \left(\gamma_2^{-1} \dot{\hat{W}}_{j2}^{2} \right).
\]

**Lemma 3.2:** Consider the positive constants \( a_1, a_2, \ldots, a_n \) and \( 0 < \rho_1 < 1 \), then the following inequality holds (Z. Y. Zhao et al., 2022):

\[
(a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2)^{2\rho_1}
\leq (a_1^{2\rho_1} + a_2^{2\rho_1} + a_3^{2\rho_1} + \cdots + a_n^{2\rho_1})^2.
\] (34)

**Theorem 3.3:** Consider the dynamical model of under-study class (5) along with fixed-time terminal sliding manifold (15) and reachability law (17). Then, the neuro-adaptive control input developed in (33) guarantees the establishment of sliding mode in fixed time.

**Proof:** The following Lyapunov function is presented to approximate the network weights:

\[
\Lambda = \frac{1}{2} \gamma_1^2 + \frac{1}{2} \gamma_2^2 + \frac{1}{2} \gamma_1^2 \hat{W}_{j1}^T \hat{W}_{j1} + \frac{1}{2} \gamma_2^2 \hat{W}_{j2}^T \hat{W}_{j2}. \] (35)

The time differentiation of (35) along (19) can be evaluated as follows:

\[
\dot{\Lambda} = \gamma_1 \left(\frac{\partial \hat{F}_i}{\partial y_{11}} \dot{y}_{11} + \frac{\partial \hat{F}_i}{\partial z_{n1}} \dot{z}_{n1} + \frac{\partial \hat{F}_i}{\partial y_{22}} \dot{y}_{22} + \frac{\partial \hat{F}_i}{\partial z_{n2}} \dot{z}_{n2} \right) \hat{F}_n + \hat{G}_n U.
\]
Aftersubstituting (40), one can get

$$\dot{\lambda} = \frac{1}{\gamma_1} \dot{W}_{jF}^T \dot{W}_{jF} + \frac{1}{\gamma_2} \dot{W}_{jG}^T \dot{W}_{jG}. \quad (36)$$

Invoking the right side of $U(t)$, one can get

$$\dot{\lambda} = \gamma_1 \left( - \frac{\partial F_i}{\partial z_{n_2}} - \frac{\partial F_i}{\partial z_{n_2}} (G_n U + \Delta) - \dot{y}_i \gamma_i \right)$$

Taking (31) and (32) into account, (37) becomes

$$\dot{\lambda} = \gamma_1 \left( - \frac{\partial F_i}{\partial z_{n_2}} - \frac{\partial F_i}{\partial z_{n_2}} (G_n U + \Delta) - \dot{y}_i \gamma_i \right) + \frac{1}{\gamma_1} \dot{W}_{jF}^T \dot{W}_{jF} + \frac{1}{\gamma_2} \dot{W}_{jG}^T \dot{W}_{jG}. \quad (38)$$

After considering (29) and (31), one can get

$$\dot{\lambda} = -\alpha_i |\gamma_i|^r + |\dot{\gamma}_i|^r + Z$$

From (39), the estimated weights are chosen as

$$\dot{W}_{jF} = \gamma_1 \gamma_1 \left( - \frac{\partial F_i}{\partial z_{n_2}} (h_j) \right) \gamma_j$$

After substituting (40), one can get

$$\dot{\lambda} = -\alpha_i |\gamma_i|^r + |\dot{\gamma}_i|^r + Z$$

By using Lemma 3.2 and setting $\alpha_i$ and $\beta_i$, the following inequality can be obtained:

$$\dot{\lambda} \leq -\alpha_i (\gamma_i^2 + \frac{1}{2} \gamma_i^{-1} \dot{W}_{jF}^2 + \frac{1}{2} \gamma_i^{-1} \dot{W}_{jG}^2) \frac{\gamma_i}{\gamma_i^2}$$

Consequently, it will ensure the fast fixed-time convergence of sliding surface and system state $z$ to zero and equilibrium point, respectively.

In the next section, the feasibility of the proposed synthesis is verified with the help of two illustrative examples in the MATLAB environment.

4. Benchmark examples

The brief introduction of the two highly nonlinear under-actuated systems (cart-pendulum and quadcopter UAV) as benchmark examples and their fixed-time TSM control laws are designed and simulated in this section.

4.1 Cart-pendulum system

The cart-pendulum system is presented as a benchmark example of the under-study class of UNS, which has two configuration variables needed to control by only one control input. The pendulum’s pole freely moves around the pivot point, while the cart can only travel in the horizontal plane (see Figure 2).
\[ \beta \]

\[ \beta \]

\[ l \]

\[ (\text{system:} \]

\[ \text{Figure 2. Closed-loop block diagram of a cart-pendulum system.} \]

\[ 4.1.1 \text{ Dynamical model description} \]

Consider the following uncertain dynamics of the aforesaid system:

\[
\begin{align*}
\dot{x} &= -\frac{1}{\varrho(\theta)} \left( m g \cos \theta \sin \theta - \frac{4}{3} \mathcal{U} \right), \\
\dot{\theta} &= \frac{1}{l_0(\theta)} \left( (M + m) g \sin \theta - \cos \theta \mathcal{U} \right),
\end{align*}
\]

(44)

where \( \mathcal{U} = u + m \dot{\theta}^2 \sin \theta, \varrho(\theta) = \frac{1}{3} (M + m) - m \cos^2 \theta \). The parameter values of the cart-pendulum system are rod length \( l = 0.36 \text{ m} \), cart mass \( M = 2.5 \text{ kg} \), gravitational acceleration \( g = 9.81 \text{ m/s}^2 \), rod mass \( m = 0.23 \text{ kg} \) and the moment of inertia \( J = 0.099 \text{ kg.m}^2 \). By introducing the following non-singular coordinate

\[ y = \dot{x} + \frac{4l}{3 \cos \theta} \dot{\theta} \Rightarrow \dot{y} = \ddot{x} + \frac{4l}{3 \cos^2 \theta} \dot{\theta}^2 + \frac{4l}{3 \cos \theta} \ddot{\theta} \]

(45)

and invoking \( \ddot{x} \) and \( \dot{\theta} \) given in (44), the dynamical system becomes

\[
\begin{align*}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= \bar{F}_1(\theta_1, \theta_2) \tan \theta_1, \\
\dot{\theta}_1 &= \theta_2, \\
\dot{\theta}_2 &= \tilde{G}_2(\theta_1) + \bar{G}_2(\theta_1) \mathcal{U},
\end{align*}
\]

(46)

where

\[ \bar{F}_1 = \frac{g}{\varrho} \left( \left( \frac{4}{3} - \cos^2 \theta_1 \right) m + \frac{4}{3} M \right) + \frac{4l}{3} \cos \theta_1, \]

\[ \bar{F}_2 = \frac{(M + m) g \sin \theta_1}{l_0}, \]

and \[ \tilde{G}_2 = -\frac{\cos \theta_1}{l_0}. \]

Now, transformed model (46) is ready for the proposed control design, which is presented in the next subsection.

\[ 4.1.2 \text{ Control design} \]

Now, the control design \( \mathcal{U} \) will be designed to enforce the pendulum’s pole at a vertically upright unstable position \( \theta = 0^\circ \) while steering the translational cart position at the origin. This law can be developed by defining the mismatch between the cart’s actual and a reference position:

\[ \xi_1 = y_1 - y_d. \]

(47)

The time differentiation of \( \xi_1 \) is evaluated as follows:

\[
\begin{align*}
\dot{\xi}_1 &= y_2 - y_d, \\
\ddot{\xi}_1 &= \bar{F}_1(\theta_1, \theta_2) \tan \theta_1 - \ddot{y}_d, \\
\end{align*}
\]

(48)

where \( \bar{F}_1(\theta, \dot{\theta}) = \frac{4}{3} \frac{\varrho(\theta)}{\cos \theta} \tan \theta \) and \( \ddot{y}_d = \frac{d}{d \theta} (\bar{F}_1(\theta, \dot{\theta}) \tan \theta + \bar{F}_1(\theta, \dot{\theta}) \sec^2 \theta - \frac{d}{d \theta} (\bar{F}_1(\theta, \dot{\theta}) = \frac{2}{\varrho(\theta)} m g \cos \theta \sin \theta (\frac{4}{3} - \cos^2 \theta) m + \frac{4}{3} M - \varrho(\theta) - \frac{4l}{3 \cos \theta} \dot{\theta}). \]

The fast stabilisation of \( y \) dynamics can be ensured by defining the following sliding surface:

\[ s_1 = \ddot{\xi}_1 + \alpha_1 |\xi_1|^{\kappa_1} \sin (\xi_1) + \beta_1 \chi_1(\xi_1). \]

(49)

Here the nonlinear term \( \bar{F}_1 \) has a positive value for all \(-\pi/2 \leq \theta \leq \pi/2\); therefore, the virtual input \( \tan \theta \) is used to control the internal dynamics. The following combination is offered to get this intermediate control

\[ \tan \theta - \ddot{y}_d = -s_1. \]

(50)

This can be accomplished by assuming the sliding attractor of the following form:

\[ Y_1 = \tan \theta - \ddot{y}_d + s_1 \Rightarrow \dot{Y}_1 = \sec^2 \theta \dot{\theta} - \ddot{y}_d + \ddot{s}_1. \]

(51)

Since control input \( \mathcal{U} \) does not appear in (51), therefore, another terminal attractor is presented as follows:

\[ Y_2 = \dot{Y}_1 + \alpha_2 |Y_1|^{\kappa_2} \sin (Y_1) + \beta_2 \chi_2(Y_1). \]

(52)

The reachability law appears as follows:

\[ \dot{Y}_2 = -\tilde{\alpha}_2 |Y_2|^{\kappa_2} \sin (Y_2) - \tilde{\beta}_2 |Y_2|^{\kappa_2} \sin (Y_2), \]

(53)

where \( \tilde{\alpha}_2, \tilde{\beta}_2, \tilde{k}_2 = 0.5(\tilde{\mu}_2 + 1) + 0.5(\tilde{\mu}_2 - 1) \sin(|Y_2| - 1), \gamma_2 = 0.5(\tilde{\mu}_2 + \tilde{\eta}_2) + 0.5(\tilde{\mu}_2 - \tilde{\eta}_2) \sin(|Y_2| - 1), \tilde{\mu}_2 > 1 \) and \( 0 < \tilde{\eta}_2 < 1 \) are positive constants.

The final control input can be developed from (52) and (53) as follows:

\[
\mathcal{U} = \frac{1}{\bar{G}_2(\theta_1)} \left( \frac{1}{\sec^2 \theta} \left( -2 \sec^2 \theta \tan \theta + \ddot{y}_d - \ddot{s}_1 \right) - \alpha_2 \kappa_2 |Y_1|^{\kappa_2 - 1} \dot{Y}_1 - \beta_2 \chi_2(\dot{Y}_1) - \tilde{\alpha}_2 |Y_2|^{\kappa_2} \sin (Y_2) - \tilde{\beta}_2 |Y_2|^{\kappa_2} \sin (Y_2) \right) - \tilde{\bar{F}}_2(\theta_1),
\]

(54)

where

\[ \tilde{\bar{F}}_2 = -\alpha_2 \kappa_2 |Y_1|^{\kappa_2 - 1} \dot{Y}_1 - \beta_2 \chi_2(\dot{Y}_1), \]

\[ \tilde{\bar{F}}_2 = -\alpha_2 \kappa_2 |Y_2|^{\kappa_2 - 1} \dot{Y}_1 - \beta_2 \chi_2(\dot{Y}_1). \]

Consequently, this will drive the system states to the equilibrium points in fixed time.

\[ \text{Remark 4.1: Note that the approximation of nonlinear drift terms is accomplished via feed-forward neural networks. In simulation, the words } \bar{F}_1 \text{ and } \bar{F}_2 \text{ should be replaced by respective estimates } \tilde{\bar{F}}_1 \text{ and } \tilde{\bar{F}}_2. \]
4.1.3 Result discussion
Now, the designed control law $U$ (54) is simulated in MATLAB/Simulink environment to verify the applicability of the developed control algorithm.

Figure 3 shows the estimation of highly nonlinear functions $\bar{F}_1$ and $\bar{F}_2$ via the proposed neural network.

Figure 4 shows the simulation outcome for the cart location $y_1$. When compared with the results of Riachy et al. (2008), the proposed control method stabilises the cart position at equilibrium in a short time. Figure 5 depicts the pendulum angle $\theta_1$ convergence to its target value. It is apparent from the figure that as compared to the standard result of Riachy et al. (2008), the pendulum angle is quickly stabilised at the origin via the proposed control technique and then stays there without any oscillation. In Figure 6, the regulated input is shown. It is worth mentioning that the control feedback has a virtually chatter-free structure when executing the primary tasks. This technique achieves excellent stabilisation with reduced steady-state error and improved robustness in each control design stage. The overall calculations were carried out in the presence of matched uncertainties to verify the robustness of the proposed control scheme. Furthermore, some unmatched complexities were inserted into the system, which is tackled in each stage through the virtual control input framework. It is worth mentioning that the newly designed control law is free from high-frequency chattering, which is highly risky in realistic implementations. In other words, the proposed algorithm has significantly decreased the unwanted chattering effects.

4.2 Quadcopter UAV
Now, a fixed-time control law is developed for the under-actuated quadcopter system with the goal of achieving complete flight trajectory control, which is further subdivided into two subsystems: fully actuated and under-actuated (see Figure 7).

4.2.1 Dynamical model description
The quadcopter system is typically developed via either Euler–Newton or Euler–Lagrange equations of motion. The following dynamical model of the aforesaid system, having 12
configuration variables, is adopted from Bouabdallah and Siegwart (2005):

$$\dot{x} = b_m \varepsilon_x U_1, \quad \dot{y} = b_m \varepsilon_y U_1, \quad \ddot{z} = b_m \varepsilon_z U_1 - g, \quad \dot{\phi} = J_3 \ddot{\phi} + J_3 \dot{\theta} \psi + J_3 U_2, \quad \dot{\theta} = J_3 \dot{\phi} \dot{\phi} + J_3 \dot{\theta} \psi + J_3 U_3, \quad \ddot{\psi} = J_2 \dot{\phi} \dot{\theta} + J_2 U_4,$$

where $\varepsilon_x = \sin \psi \sin \varphi \cos \vartheta \cos \psi, \quad \varepsilon_y = \cos \psi \sin \varphi \cos \vartheta, \quad \varepsilon_z = \cos \psi \cos \varphi \cos \vartheta.$ The control inputs are $U_1 = \frac{b_m}{J_x}, \quad U_2 = \frac{b_m}{J_y}, \quad U_3 = \frac{b_m}{J_z},$ and $U_4 = \frac{b_m}{\sum_{i=1}^{4} \psi}$. The typical model parameters are the inertia constants ($I_x, I_y, I_z$), position states $(x, y, z, \psi, \varphi, \theta)$, thrust coefficient ($b$), arm's length ($l$), quadcopter mass ($m$) and drag factor ($d$). The control inputs are $U_1 = \frac{b_m}{I_x}, \quad U_2 = \frac{b_m}{I_y}, \quad U_3 = \frac{b_m}{I_z}, \quad U_4 = \frac{b_m}{\sum_{i=1}^{4} \psi}$. It is evident that control inputs $U_2, U_3$ and $U_4$ derive $\psi, \varphi, \theta$ and $\psi$ angles, respectively, whereas the control input $U_1$ controls three configuration variables $(x, y, z)$, which is not an easy task. Therefore, the following coordinates conversion will transform the original model to a control convenient regular form:

$$\nu = x - \Gamma_x, \quad \varphi = y - \Gamma_y.$$

Note that $\Gamma_x = (\varepsilon_x/\varepsilon_z) z$ and $\Gamma_y = (\varepsilon_y/\varepsilon_z) z$ are the solutions of $d(\Gamma_x)/dz = \varepsilon_x/\varepsilon_z$ and $d(\Gamma_y)/dz = \varepsilon_y/\varepsilon_z$.

The time differentiation of (56) can be evaluated in the following form:

$$\dot{\nu} = \dot{x} - \frac{\varepsilon_x}{\varepsilon_z} \dot{z}, \quad \ddot{\nu} = \ddot{x} - \frac{\varepsilon_x}{\varepsilon_z} \ddot{z}, \quad \dot{\varphi} = \dot{\theta} - \frac{\varepsilon_y}{\varepsilon_z} \dot{z}, \quad \ddot{\varphi} = \ddot{\theta} - \frac{\varepsilon_y}{\varepsilon_z} \ddot{z}.$$

After invoking $\dot{x}, \dot{y}$ and $\ddot{z}$ in (57), the regular form of quadcopter system (55) can be achieved as follows:

$$\begin{align*}
\dot{z}_1 &= z_2, \quad \dot{z}_2 = -g + b_m \varepsilon_z U_1, \\
\dot{\psi}_1 &= \psi_2, \quad \dot{\psi}_2 = J_3 \dot{\theta} \psi_2 + J_3 U_4, \\
\dot{\varphi}_1 &= \varphi_2, \quad \dot{\varphi}_2 = J_3 \dot{\phi} \psi_2 + J_3 \dot{\phi} \psi_2 + J_3 U_2, \\
\dot{\theta}_1 &= \theta_2, \quad \dot{\theta}_2 = J_3 \dot{\phi} \dot{\psi}_2 + J_3 \dot{\phi} \dot{\psi}_2 + J_3 U_3.
\end{align*}$$

It is evident that system (58) has two fully actuated subsystems $(z, \psi)$ and two under-actuated subsystems $(\varphi - \varphi, \nu - \varphi)$. Since the internal dynamics ($x$ and $y$) are interconnected with $\varphi$ and $\varphi$-dynamics in a practical scenario. Therefore, these dynamics are indirectly controlled by control inputs $U_2$ and $U_3$.

Now, system 58 is ready to design any control system for its stability.

### 4.2.2 Control design

Now, the complete flight control of the quadcopter UAV (58) is developed in two parts: fully actuated and under-actuated. However, for simplicity’s purpose, the detail control design of $\psi$-dynamics (in the case of the fully actuated subsystem) and $(q - \varphi)$-dynamics (in case of the under-actuated subsystem) is presented.

Now, to control the $\psi$-dynamics, the following mismatch is defined:

$$\varepsilon_\psi = \psi_1 - \psi_d.$$

The sliding mode can be enforced against the following fast fixed-time terminal attractor to get the main tracking objective:

$$\Gamma_\psi = \dot{\varepsilon}_\psi + \alpha_\psi |\varepsilon_\psi|^k \text{sign}(\varepsilon_\psi) + \beta_\psi \chi_\psi(\varepsilon_\psi).$$

The reachability law can be defined as follows:

$$\Gamma_\psi = -\tilde{\alpha}_\psi |Y_\psi|^{k'} \text{sign}(Y_\psi) - \tilde{\beta}_\psi |Y_\psi|^{k''} \text{sign}(Y_\psi).$$

In the light of (60) and (61), the control input $U_4$ can be developed as follows:

$$U_4 = \frac{1}{J_3} \left( -J_2 \dot{\varphi} \dot{\psi}_2 - \tilde{\alpha}_\psi Y_\psi \xi_\psi \text{sign}(Y_\psi) - \tilde{\beta}_\psi Y_\psi \dot{Y}_\psi \text{sign}(Y_\psi) + \dot{\varepsilon}_d 
- \alpha_\psi k' |\xi_\psi |^{k'-1} \tilde{\varepsilon}_\psi - \beta_\psi \chi_\psi(\xi_\psi) \right).$$

Note that control law (62) along sliding surface (60) will ensure the fast convergence of $\psi$-dynamics in fixed time. Consequently, the mismatch $\xi_\psi$ will approach to zero in fixed time (Khan et al., 2017).

Similarly, the proposed terminal sliding manifold is used to develop the following control law to control $z$-dynamics:

$$U_1 = \frac{1}{b_m \cos \varphi_1 \cos \psi_1} \left( g - \tilde{\alpha}_z \gamma_z \xi_\psi \text{sign}(\gamma_z) - \tilde{\beta}_z \gamma_z \dot{\gamma}_z \text{sign}(\gamma_z) - \xi_z \right).$$
+ \ddot{z}_d - \alpha_2 \kappa_2 |\xi_2|^{\kappa_2 - 1} \xi_2 - \beta_2 \dot{\chi}_2 (\xi_2),
\end{equation}
where $\mathcal{Z}_z = -\frac{\beta}{2} (\mathcal{W}_{1}^{-1}{\mathcal{W}_{1}}^{T} z_1)$.

Thus, the control design is completed for the fixed-time stabilisation of fully actuated dynamics. Now, before proceeding to controller design for the under-actuated subsystem, the following remark is presented.

**Remark 4.2**: Here, the main objective of the control design is to track $\nu(t)$ on the desired trajectory $\nu_d(t)$ in fixed time while steering $\vartheta$ at the origin. The terminal attractor can be established in the following form to meet this objective:

\begin{equation}
S_\nu = \ddot{\xi}_\nu + \alpha_1 |\xi_\nu|^{\kappa_1} \operatorname{sign} (\xi_\nu) + \beta_1 \chi_1 (\xi_\nu),
\end{equation}

along with

\begin{equation}
g \dot{\xi}_\nu - \ddot{\nu}_d = -S_\nu,
\end{equation}

and

\begin{equation}
\mathcal{Y}_1 = g \dot{\xi}_\nu - \ddot{\nu}_d + S_\nu.
\end{equation}

Since $\dot{\vartheta}$ does not appear in the first derivative of (66), therefore, the system relative degree can be increased by introducing the following hierarchical manifold:

\begin{equation}
\mathcal{Y}_2 = \ddot{\mathcal{Y}}_1 + \alpha_2 |\mathcal{Y}_1|^{\kappa_2} \operatorname{sign} (\mathcal{Y}_1) + \beta_2 \dot{\chi}_2 (\mathcal{Y}_1),
\end{equation}

According to Remark 3.3, the reachability law is presented as follows:

\begin{equation}
\ddot{\mathcal{Y}}_2 = -\bar{\alpha}_2 |\mathcal{Y}_2|^{\bar{\kappa}_2} \operatorname{sign} (\mathcal{Y}_2) - \bar{\beta}_2 |\mathcal{Y}_2|^{\bar{\gamma}_2} \operatorname{sign} (\mathcal{Y}_2),
\end{equation}

where $\bar{\kappa}_1$ and $\bar{\kappa}_2$ are positive design constants. Following Theorem 3.1, the following control input can ensure the sliding mode establishment against the proposed fixed-time terminal attractor (67):

\begin{equation}
U_3 = \frac{1}{jy^3} \left( -\frac{1}{g} \frac{\partial \mathcal{E}_u}{\partial \varphi} \dot{\varphi} + \frac{\partial \mathcal{E}_u}{\partial \psi} \dot{\psi} + \frac{d}{dt} \left( \frac{\partial \mathcal{E}_u}{\partial \vartheta} \right) \dot{\vartheta} 
+ \frac{d}{dt} \left( \frac{\partial \mathcal{E}_u}{\partial \varphi} \right) \dot{\varphi} + \frac{d}{dt} \left( \frac{\partial \mathcal{E}_u}{\partial \psi} \right) \dot{\psi} 
- \ddot{\mathcal{Y}}_2 + S_\nu + \alpha_2 |\mathcal{Y}_1|^{\kappa_2 - 1} \mathcal{Y}_1 
+ \beta_2 \dot{\chi}_2 (\mathcal{Y}_1) + \bar{\alpha}_2 |\mathcal{Y}_2|^{\bar{\kappa}_2} \operatorname{sign} (\mathcal{Y}_2) 
+ \bar{\beta}_2 |\mathcal{Y}_2|^{\bar{\gamma}_2} \operatorname{sign} (\mathcal{Y}_2) + \mathcal{Z}_\vartheta 
- (Jy \psi \varphi_2 + Jy \varphi \varphi_2) \right),
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{Feed-forward neural network performance in estimation of nonlinear functions.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{The tracking performance of translational subsystem via the proposed control law in comparison with standard literature (Zheng et al., 2014).}
\end{figure}
where \( Z_{\varphi} = -\frac{\gamma}{2}(\gamma^{-1}\hat{W}_{\varphi}^2)^{\frac{\tau+1}{2}} - \frac{\gamma}{2}(\gamma^{-1}\hat{W}_{\varphi}^2)^{\frac{\tau+1}{2}} \). When the sliding mode is established, then \( \Upsilon_2 \to 0 \) eventually leads to assure the tracking error \( \xi_{\varphi} \) enforcement to zero in fixed time.

In a similar fashion, the following control scheme can be designed to control the other under-actuated subsystem (of \( y \) and \( \varphi \) dynamics):

\[
U_2 = \frac{1}{J_3} \left( g \left( \frac{\partial \xi_{\varphi}}{\partial \varphi} \right) \dot{\varphi} + \frac{d}{dt} \left( \frac{\partial \xi_{\varphi}}{\partial \varphi} \right) \dot{\varphi} \right) + \frac{d}{dt} \left( \frac{\partial \xi_{\varphi}}{\partial \varphi} \right) \dot{\varphi} + \alpha_2(\kappa_2)\left| \Upsilon_1 \right|^{\kappa_2-1} \Upsilon_1 + \beta_2 \xi_2(\Upsilon_1) + \beta_2 |\Upsilon_2|^{\kappa_2} \text{sign}(\Upsilon_2) - \left( J_2 \psi_2 \dot{\varphi}_2 + J_3 \dot{\varphi}_2 \right),
\]

(70)

where \( Z_{\varphi} = -\frac{\gamma}{2}(\gamma^{-1}\hat{W}_{\varphi}^2)^{\frac{\tau+1}{2}} - \frac{\gamma}{2}(\gamma^{-1}\hat{W}_{\varphi}^2)^{\frac{\tau+1}{2}} \).

**Remark 4.3:** It is worthy of mentioning that the approximation of nonlinear drift terms is accomplished via feed-forward neural networks. In the simulation, the words \( G_z = b_m E_z, F_\varphi = \) \( J_1 \dot{\varphi}_1 + J_2 \dot{\varphi}_2 \) and \( F_\theta = J_1 \psi_1 \dot{\varphi}_1 + J_2 \psi_2 \dot{\varphi}_2 \) should be replaced by respective estimates \( \hat{G}_z, \hat{F}_\varphi \) and \( \hat{F}_\theta \).

Thus, the discussion on the control of under-actuated subsystems is finished.

### 4.2.3 Result discussion

Numerical simulations in the Matlab/Simulink environment are used to validate the efficacy and supremacy of the proposed control algorithm in terms of robustness against unknown disturbances \( \Delta = K \sin(2\pi t) \) and trajectory tracking in high precision. In addition, the RBF-based neural network is used to estimate the nonlinear drift functions of the under-actuated subsystem. The developed control algorithm simulations are compared to those of a standard fractional-order SMC scheme (designed in Zheng et al., 2014). The model parameters for the quadcopter that have been used in the simulation are as follows: the thrust coefficient is \( b = 3.13 \times 10^{-5} \text{N.s}^2 \), the arm length \( l = 0.23 \text{m} \), the quadcopter’s mass is \( m = 0.650 \text{ kg} \), the inertia coefficients are \( J_x = J_y = J_z = 7.5 \times 10^{-3} \text{ kg.m}^2 \), the drag...
coefficients are \( d = 7.5 \times 10^{-7} \) N.m.s\(^2\), and the rotor inertia coefficient is \( J_r \). For simulation tests, \([0 \ 0 \ 0]\) m and \([0 \ 0 \ 0]\) rad are the initial values of translational positions and angular orientations. Meanwhile, the trial and error method is used to select controller gains.

Figure 8 shows the estimation of highly nonlinear functions \( G_z, F_\psi \) and \( F_\theta \) via the proposed neural network.

Figure 9 shows the tracking performance of the quadcopter system under the newly configured control algorithm. The proposed controller enforces the tracking error to zero with a slightly faster-tracking speed than Zheng et al. (2014). Meanwhile, Figure 10 depicts the stabilisation of roll, pitch, and yaw angles at origin, which further assures that the configured flight controller produces a significantly faster converging response with less overshoot than its counterpart presented in Zheng et al. (2014). In addition, using the designed control method eliminates the problem of high-frequency chattering in conventional sliding modes. In addition, the time histories of the applied control inputs \( \omega_1, \omega_2, \omega_3 \) and \( \omega_4 \) under the proposed control protocol are depicted in Figures 11, from which it is clear that the control inputs are substantially smooth and behave in a very feasible manner. Furthermore, the configured control scheme with applied control inputs suppressed unwanted external uncertainties, resulting in the output tracking error being reduced to zero in a fixed period of time (see Figure 11).

In order to demonstrate the better efficacy of the proposed work, the simulation results are compared with the results of Hou et al. (2015). In Figure 12, the proposed flight controller provides better tracking performance of linear positions on the desired trajectory. This controller also offers robustness against matched uncertainty. Similarly, Figure 13 demonstrates the better regulatory response of the proposed control algorithm (with and without RBF-NN) as compared to Hou et al. (2015). It is worth mentioning that, even though the desired translational and rotational variables change all the time, the newly developed algorithm can derive the quadcopter’s actual positions and orientations on the desired equilibrium points. Moreover, Table 1 provides a more accurate and detailed presentation-based quantitative analysis that includes the settling time, maximum percent overshoot, and steady-state errors.

Table 1. Quantitative analysis of simulation results with Hou et al. (2015).

| Specifications | Subsystem   | 2-SMC (Hou et al., 2015) | FTNT-SMC | AN-FTNT-SMC |
|---------------|-------------|------------------------|----------|-------------|
| Settling time (s) \( \psi \) | 0.0072 | 0.0031 | 0.0086 |
| z | 17.0146 | 3.2325 | 3.4364 |
| y | 17.26899884 | 7.3469 | 7.2365 |
| x | 17.2365 | 7.1258 | 7.2456 |
| Steady state error \( \psi \) | 0.0152 | 2.3 \times 10^{-6} | 2.6 \times 10^{-6} |
| z | 0.0025 | 1.6 \times 10^{-4} | 2.9 \times 10^{-4} |
| y | 0.0045 | 5.6 \times 10^{-4} | 6.5 \times 10^{-4} |
| x | 0.0014 | 3.7 \times 10^{-4} | 4.1 \times 10^{-4} |
| Maximum overshoot \( \psi \) | 0.0517 | 0.0145 | 0.0145 |
| z | 0.0086 | 0.0013 | 0.0024 |
| y | 0.0451 | 0.3532 | 0.3253 |
| x | 0.0124 | 0.0265 | 0.0258 |
The overall theoretical and numerical study of both cart-pendulum and quadcopter UAV concludes that the newly proposed control method is one of the best candidates for controlling the considered class (which can be transformed in its equivalent regular form).

5. Conclusion

In this research work, a non-singular fast terminal sliding mode-based neuro-adaptive control synthesis is developed for the UNS class. Some non-singular coordinates are presented to transform the generalised mathematical model of the considered class to control convenient regular form to pursue the design. The drift function estimations via RBF neural network are introduced to improve the newly configured control system’s robustness. A non-singular fast TSMC synthesis is devised for the formulated problem, which is free from the singularity problem and has minimum settling time without knowing the initial conditions. The fixed-time stability of both system states and sliding manifolds is ensured with the help of Lyapunov stability theory. The simulation results of quadcopter and cart-pendulum systems under the proposed control algorithm are compared with their respective standard literature (Riachi et al., 2008; Zheng et al., 2014) to comprehensively demonstrate the feasibility and benefits of the proposed synthesis over the existing standard literature.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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References

Bouabdallah, S., & Siegwart, R. (2005). Backstepping and sliding-mode techniques applied to an indoor micro quadrotor. In IEEE International Conference on Robotics and Automation (pp. 2247–2252). IEEE.
Chen, Y., Wang, C., Zeng, W., & Wu, Y. (2021). Horizontal nonlinear path following guidance law for a small UAV with parameter optimized by NMPC. IEEE Access, 9, 127102–127116. https://doi.org/10.1109/ACCESS.2021.3111101
Corradini, M. L., & Cristofaro, A. (2018). Nonsingular terminal sliding-mode control of nonlinear planar systems with global fixed-time stability guarantees. Automatica, 95(8), 561–565. https://doi.org/10.1016/j.automatica.2018.06.032
Din, S. U., Khan, Q., Rehman, F., & Akmeliawanti, R. (2017). A comparative experimental study of robust sliding mode control strategies for underactuated systems. IEEE Access, 5, 10068–10080. https://doi.org/10.1109/ACCESS.2017.271216
Fantoni, I., & Lozano, R. (2002). Non-linear control for underactuated mechanical systems. Springer Science & Business Media.
Fessi, R., Bouallègue, S., Haggège, J., & Vaidyanathan, S. (2017). Terminal sliding mode controller design for a quadrotor unmanned aerial vehicle. In Applications of sliding mode control in science and engineering (pp. 81–98). Springer.
Ghommam, J., Mnif, F., Benali, A., & Derbel, N. (2006). Asymptotic backstepping stabilization of an underactuated surface vessel. IEEE Transactions on Control Systems Technology, 14(6), 1150–1157. https://doi.org/10.1109/TCST.2006.880220
Gruszka, A., Malisof, M., & Mazenc, F. (2011). On tracking for the PVTOL model with bounded feedbacks. In American Control Conference (ACC), 2011 (pp. 1428–1433). IEEE.
Hou, Y., Zuo, Z., & Shi, Z. (2015). Fixed-time terminal sliding mode trajectory tracking control of quadrotor helicopter. In Proceedings of the 34th Chinese Control Conference (pp. 4361–4366). IEEE.
Huang, X., & Yan, Y. (2017). Saturated backstepping control of underactuated spacecraft hovering for formation flights. IEEE Transactions on Aerospace and Electronic Systems, 53(4), 1988–2000. https://doi.org/10.1109/TAES.2017.2679838
Huang, Y., & Jia, Y. (2017). Fixed-time consensus tracking control for second-order multi-agent systems with bounded input uncertainties via NFFTS. IET Control Theory & Applications, 11(16), 2900–2909. https://doi.org/10.1049/iet ct.2016.0017
Ji, N., & Liu, J. (2021). Sliding mode control for underactuated system with input constraint based on RBF neural network and Hurwitz stability analysis. Asian Journal of Control. https://doi.org/10.1002/asjc.2692
Khan, Q., Akmeliawanti, R., Bhatti, A. I., & Ashraf, M. (2017). Robust stabilization of underactuated nonlinear systems: A fast terminal sliding mode approach. ISA Transactions, 66(4), 241–248. https://doi.org/10.1016/j.isatra.2016.10.017
Li, B., Qin, K., Xiao, B., & Yang, Y. (2019). Finite-time extended state observer based fault tolerant output feedback control for attitude stabilization. ISA Transactions, 91(12), 11–20. https://doi.org/10.1016/j.isatra.2019.01.039
Li, T., & Zhao, H. (2017). Global finite-time adaptive control for uncalibrated robot manipulator based on visual servoing. ISAS Transactions, 68(4), 402–411. https://doi.org/10.1616/is chemical engineering systems I: Local stabilization with application to an inverted pendulum. IEEE Transactions on Automatic Control, 57(8), 2106–2110. https://doi.org/10.1109/TAC.2011.2179869
Riachi, S., Orlov, Y., Floquet, T., Santiesteban, R., & Richard, J. P. (2008). Second-order sliding mode control of underactuated mechanical systems I: Local stabilization with application to an inverted pendulum. International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal, 18(4–5), 529–543. https://doi.org/10.1002/1099-1239
Taleb, H. A., Khorasani, K., & Tafazoli, S. (2008). A recurrent neural-network-based sensor and actuator fault detection and isolation for nonlinear systems with application to the satellite’s attitude control subsystem. IEEE Transactions on Neural Networks, 20(1), 45–60. https://doi.org/10.1109/TNN.2008.2004373
Tian, B., Lu, H., Zuo, Z., & Wang, H. (2018). Fixed-time stabilization of high-order integrator systems with mismatched disturbances. Nonlinear Dynamics, 94(4), 2889–2899. https://doi.org/10.1007/s11071-018-4532-3
Tian, Y., Cai, Y., & Deng, Y. (2020). A fast nonsingular terminal sliding mode control method for nonlinear systems with fixed-time stability guarantees. IEEE Access, 8, 60444–60454. https://doi.org/10.1109/ACCESS.2020.2980044
ud Din, S., & Khan, Q. (2018). Smooth super-twisting sliding mode control for the class of underactuated systems. PloS One, 13(10), e0203667. https://doi.org/10.1371/journal.pone.0203667
Ullah, S., Khan, Q., Mehmoord, A., & Bhatti, A. I. (2020). Robust backstepping sliding mode control design for a class of underactuated electro-mechanical nonlinear systems. *Journal of Electrical Engineering & Technology, 15*(4), 1821–1828. https://doi.org/10.1007/s42835-020-00436-3

Ullah, S., Khan, Q., Mehmood, A., Kirmani, S. A. M., & Mechali, O. (2021). Neuro-adaptive fast integral terminal sliding mode control design with variable gain robust exact differentiator for under-actuated quadcopter UAV. *ISA Transactions.* https://doi.org/10.1016/j.isatra.2021.02.045

Ullah, S., Mehmood, A., Khan, Q., Rehman, S., & Iqbal, J. (2020). Robust integral sliding mode control design for stability enhancement of under-actuated quadcopter. *International Journal of Control, Automation and Systems, 18*(7), 1671–1678. https://doi.org/10.1007/s12555-019-0302-3

Utkin, V., Guldner, J., & Shi, J. (2009). *Sliding mode control in electro-mechanical systems.* CRC Press.

Wu, Q., Wang, X., Hua, L., & Xia, M. (2021). Modeling and nonlinear sliding mode controls of double pendulum cranes considering distributed mass beams, varying roped length and external disturbances. *Mechanical Systems and Signal Processing, 158*(9), Article 107756. https://doi.org/10.1016/j.ymssp.2021.107756

Yang, T., Sun, N., & Fang, Y. (2021a). Adaptive fuzzy control for a class of MIMO underactuated systems with plant uncertainties and actuator deadzones: Design and experiments. *IEEE Transactions on Cybernetics, 1–14.* https://doi.org/10.1109/TCYB.2021.3050475

Yang, T., Sun, N., & Fang, Y. (2021b). Adaptive fuzzy control for uncertain mechatronic systems with state estimation and input nonlinearities. *IEEE Transactions on Industrial Informatics, 18*(3), 1770–1780. https://doi.org/10.1109/TII.2021.3089143

Ye, D., Zhang, H., Tian, Y., Zhao, Y., & Sun, Z. (2020). Fuzzy sliding mode control of nonparallel-ground-track imaging satellite with high precision. *International Journal of Control, Automation and Systems, 18*(6), 1617–1628. https://doi.org/10.1007/s12555-018-0369-2

Zaare, S., & Soltanpour, M. R. (2021). The position control of the ball and beam system using state-disturbance observe-based adaptive fuzzy sliding mode control in presence of matched and mismatched uncertainties. *Mechanical Systems and Signal Processing, 150,* Article 107243. https://doi.org/10.1016/j.ymssp.2020.107243

Zhang, M., & Tarn, T. J. (2002). Hybrid control of the pendubot. *IEEE/ASME Transactions on Mechatronics, 7*(1), 79–86. https://doi.org/10.1109/3516.990890

Zhao, D., Zhu, Q., & Dubbeldam, J. (2015). Terminal sliding mode control for continuous stirred tank reactor. *Chemical Engineering Research and Design, 94*(9), 266–274. https://doi.org/10.1016/j.cherd.2014.08.005

Zhao, Z. Y., Jin, X. Z., Wu, X. M., Wang, H., & Chi, J. (2022). Neural network-based fixed-time sliding mode control for a class of nonlinear Euler-Lagrange systems. *Applied Mathematics and Computation, 415*(3), Article 126718. https://doi.org/10.1016/j.amc.2021.126718

Zheng, E. H., Xiong, J. J., & Luo, J. L. (2014). Second order sliding mode control for a quadrotor UAV. *ISA Transactions, 53*(4), 1350–1356. https://doi.org/10.1016/j.isatra.2014.03.010

Zuo, Z. (2015). Nonsingular fixed-time consensus tracking for second-order multi-agent networks. *Automatica, 54*(3), 305–309. https://doi.org/10.1016/j.automatica.2015.01.021