Toward Optimal Coupon Allocation in Social Networks: An Approximate Submodular Optimization Approach

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Abstract
CMO Council reports that 71% of internet users in the U.S. were influenced by coupons and discounts when making their purchase decisions. It has also been shown that offering coupons to a small fraction of users (called seed users) may affect the purchase decisions of many other users in a social network. This motivates us to study the optimal coupon allocation problem, and our objective is to allocate coupons to a set of users so as to maximize the expected cascade. Different from existing studies on influence maximization (IM), our framework allows a general utility function and a more complex set of constraints. In particular, we formulate our problem as an approximate submodular maximization problem subject to matroid and knapsack constraints. Existing techniques relying on the submodularity of the utility function, such as greedy algorithm, can not work directly on a non-submodular function. We use \( \epsilon \) to measure the difference between our function and its closest submodular function and propose a novel approximate algorithm with approximation ratio \( \beta(\epsilon) \) with \( \lim_{\epsilon \to 0} \beta(\epsilon) = 1 - 1/e \). This is the best approximation guarantee for approximate submodular maximization subject to a partition matroid and knapsack constraints, our results apply to a broad range of optimization problems that can be formulated as an approximate submodular maximization problem.

1 Introduction
It has been reported that one of the most effective ways of running promotions on social media is the use of coupon campaigns on the largest social media site such as Facebook and Twitter. Different from conventional online advertising, the offer of a coupon on social network could trigger a large amount of likes and shares on your product and build brand awareness rapidly. This motivates us to study the coupon allocation problem, e.g., allocating coupons to a few users so as to maximize the expected cascade. More specifically, we assume that the company would like to promote a product through offering coupons with different values to a subset of users. Whether a user would accept the coupon and further become the seed node is probabilistic, which depends on the value of the coupon. Our goal is to select a set of users and allocate one proper coupon to each of them, such that the expected cascade is maximized.

We notice that our problem is closely related to influence maximization problem (IM). Although IM has been extensively studied in the literature [Kempe et al., 2003; Chen et al., 2013; Leskovec et al., 2007; Cohen et al., 2014; Chalermsook et al., 2015], most of existing studies assume a “budgeted, deterministic and submodular” setting, where there is a predefined budget for selecting seed nodes, any node is guaranteed to be activated as a seed node once it is selected, and the utility function is assumed to be submodular in terms of seed nodes. However, these assumptions may not always hold in reality. Firstly, the probability that a user is activated as a seed node depends on many factors, including the actual amount of rewards received from the company [Yang et al., 2016]. Secondly, the utility function may not always satisfy the submodular property, e.g., some user can only be influenced if majority of her friends are activated [Kuhnle et al., 2017].

In this paper, we study the coupon allocation problem that allows a general utility function. In particular, we assume an \( \epsilon \)-approximate submodular function where \( \epsilon \) measures the distance between a given function and its closest submodular function (a detailed definition can be found in Section 3.1). We formulate the coupon allocation problem as an approximate submodular maximization problem subject to a partition matroid and knapsack constraints. Existing techniques relying on the submodularity of the utility function, such as greedy algorithm, can not work directly on a non-submodular function. We propose a novel approximate algorithm with approximation ratio \( \beta(\epsilon) \) with \( \lim_{\epsilon \to 0} \beta(\epsilon) = 1 - 1/e \). This is, to the best of our knowledge, the strongest theoretical result available for approximate submodular maximization problem subject to a partition matroid and knapsack constraints. Although we restrict our attention to coupon allocation in this paper, our results apply to a broad range of non-submodular maximization problems.

The contributions of this paper can be summarized as follows: (1) We are the first to study the coupon allocation problem that allows a general utility function. Our objective is to determine the best coupon allocation so as to maximize
the expected cascade subject to (partition) matroid and knapsack constraints. (2) We develop an efficient algorithm with approximation ratio that depends on $\epsilon$. When $\epsilon = 0$ (the utility function is submodular), our result converges to $1 - 1/e$ which is the best possible for submodular maximization subject to matroid constraints. This research contributes fundamentally to the development of approximate solutions for any problems that fall into the family of approximate submodular maximization.

Some important notations are listed in Table 1.

| Notation | Meaning |
|----------|---------|
| $S$ | the ground set that contains all u-c pairs |
| $S_v$ | all u-c pairs whose user is $v$ |
| $f(S)$ | the expected cascade of $S$ |
| $g$ | a submodular function used to bound $f$ |
| $a \oplus b$ | the coordinate wise maximum of $a$ and $b$ |
| $F_x(\{vd\})$ | $F(x) = F(x \oplus 1_{vd}) - F(x)$ |

Table 1: Symbol table.

2 Related Work

IM has been extensively studied in the literature [Kempe et al., 2003; Chen et al., 2013; Leskovec et al., 2007; Cohen et al., 2014; Chalermsook et al., 2015], their objective is to find a set of influential customers so as to maximize the expected cascade. However, our work differ from all existing studies in several major aspects. Traditional IM assumes any node is guaranteed to be activated once it is selected, we relax this assumption by allowing users to respond differently to different coupon values. Recently, (Yang et al., 2016) study discount allocation problem in social networks. However, they assume a submodular utility function and continuous setting of discount value, our model allows a general utility function. We formulate our problem as an approximate submodular maximization problem subject to matroid and knapsack constraints. Existing approaches, such as greedy algorithm [Horel and Singer, 2016], can not apply directly to matroid and knapsack constraints. We propose a novel algorithm that provides the first bounded approximate solutions to this problem. It was worth noting that our approximation ratio converges to $1 - 1/e$ when $\epsilon = 0$. This is the best theoretical result available for approximate submodular maximization subject to a partition matroid constraint.

3 Preliminaries

3.1 Submodular Function and Its Continuous Extensions

A submodular function is a set function $h : 2^\Omega \to \mathbb{R}$, where $2^\Omega$ denotes the power set of $\Omega$, which satisfies a natural “diminishing returns” property: the marginal gain from adding an element to a set $X$ is at least as high as the marginal gain from adding the same element to a superset of $X$. Formally, a submodular function satisfies the follow property: For every $X, Y \subseteq \Omega$ with $X \subseteq Y$ and every $x \in \Omega \setminus Y$, we have that $h(X \cup \{x\}) - h(X) \geq h(Y \cup \{x\}) - h(Y)$ We say a submodular function $f$ is monotone if $h(\{1\}) \leq h(\emptyset)$ whenever $X \subseteq Y$.

Continuous Extensions

Consider any vector $x = \{x_1, x_2, ..., x_n\}$ such that each $0 \leq x_i \leq 1$. The multilinear extension of $h$ is defined as $H(x) = \sum_{S \subseteq \Omega} h(S) \prod_{i \in S} x_i \prod_{i \notin S}(1 - x_i)$, and the concave extension of $h$ is defined as $h^+(x) = \max\{\sum_{X \subseteq \Omega} \alpha_X f(X) | \alpha_X \geq 0, \sum_{X \subseteq \Omega} \alpha_X \leq 1, \sum_X \alpha_X X \leq x\}$.

$\epsilon$-approximate Submodular Function

An $\epsilon$-approximate submodular function is a set function $f : 2^\Omega \to \mathbb{R}$ that satisfies the following condition: there exists a submodular function $h$ such that for any $X \subseteq \Omega$, we have $(1 - \epsilon)h(X) \leq f(X) \leq (1 + \epsilon)h(X)$.

3.2 Coupon Adoption and Propagation Model

Assume there are $n$ users $V = \{1, 2, ..., n\}$ and $m$ types of coupons $D = \{1, 2, ..., m\}$. For simplicity of presentation, we will directly use $d$ to denote the coupon value of $d$. Given the set of users $V$ and possible coupon values $D$, define $S \subseteq V \times D$ as the solution space, adding a user-coupon (u-c) pair $[vd] \in S$ to our solution translates to offering coupon $d \in D$ to user $v \in V$. When any coupon $d \in D$ is allocated to any user $v \in V$, we assume that with probability $p_v(d) \in (0, 1)$, $v$ adopts $d$ and becomes a seed node, thus incurring a cost $d$. We further assume that coupons cannot be combined, if a user received multiple coupons, her adoption decision only depends on the coupon with the highest value.

After a set of users become seed nodes, they start to influence other users. Assume $U$ is the seed set, the expected cascade of $U$, which is the expected number of influenced users given seed set $U$, is denoted as $\gamma(U)$. In this work, we assume that $\gamma(U)$ is a $\epsilon$-approximate submodular function: there exists a submodular function $g$ such that for any $U \subseteq \Omega$, we have $(1 - \epsilon)g(U) \leq \gamma(U) \leq (1 + \epsilon)g(U)$. Our propagation model subsumes many classic propagation models, including Independent Cascade Model and Linear Threshold Model [Kempe et al., 2003] as special cases, as the propagation functions defined in their models are submodular and monotone, i.e., $\epsilon = 0$.

4 Problem Statement

Our objective is to identify the best coupon allocation policy, not necessarily deterministic, to maximize the expected cascade. Given a coupon allocation $S \subseteq S$, we use $d_S(v)$ to denote the largest coupon value allocated to user $v$ under $S$, then the probability that a subset of users $U \subseteq V$ successfully become the seed set is

$$\Pr(U; S) = \prod_{u \in U} p_u(d_S(u)) \prod_{v \in V \setminus U} (1 - p_v(d_S(v)))$$

As introduced earlier, we use $\gamma(U)$ to denote the expected cascade under seed set $U$, then the expected cascade under allocation $S$ is $f(S) = \sum_{U \subseteq \Omega} \Pr(U; S) \gamma(U)$.

We first prove that $f$ is also $\epsilon$-approximate submodular.
Proof. Based on a similar proof provided in [Soma and Yoshida, 2015], we can show that if $q(U)$ is submodular of $U$, $\sum_{U \in 2^V} \Pr(U; S)q(U)$ is submodular of $S$. Because $(1 - \epsilon)q(U) \leq \gamma(U) \leq (1 + \epsilon)q(U)$ we have $(1 - \epsilon)\sum_{U \in 2^V} \Pr(U; S)q(U) \leq f(S) \leq (1 + \epsilon)\sum_{U \in 2^V} \Pr(U; S)q(U)$. We finish the proof by defining $g(S) = \sum_{U \in 2^V} \Pr(U; S)q(U)$.

The expected cost of a coupon allocation $S$ is

$$
c(S) = \sum_{U \in 2^V} \left( \Pr(U; S) \sum_{u \in U} d_S(u) \right)
$$

**Coupon Allocation Policy** We denote by $\Theta = \{\theta_S : S \subseteq S\}$ a coupon allocation policy, where $\theta_S$ is the probability that $S$ is adopted. Then the expected cascade under $\Theta$ is $f(\Theta) = \sum_{S \subseteq S} \theta_S f(S)$. The expected cost of $\Theta$ is $c(\Theta) = \sum_{S \subseteq S} \theta_S c(S)$.

**Definition 1** (Feasible Coupon Allocation Policy). We say a policy $\Theta$ is feasible if and only the following conditions are satisfied:

- **(Attention Constraint)** For every user $v \in V$, we denote $S_v = \{|vd| : d \in D\}$. Let $\forall S_v > 0, \forall S_v : |S_v \cap S| \leq 1$, e.g., every user receives at most one coupon.

- **(Budget Constraint)** $\sum_{S \subseteq S} \theta_S c(S) \leq B$, e.g., the expected value of the coupons redeemed is at most $B$.

Our objective is to identify a feasible coupon allocation policy that maximizes the expected cascade. We present the formal definition of our problem in P.A.

**P.A**

$$\text{maximize } \sum_{S \subseteq S} \theta_S f(S)$$

subject to:

$$\begin{align*}
\forall S_v > 0, \forall S_v & : |S_v \cap S| \leq 1 \\
\sum_{S \subseteq S} \theta_S c(S) & \leq B \\
\sum_{S \subseteq S} \theta_S & = 1
\end{align*}$$

We extend our model in Section 6 by incorporating one more constraint, e.g., a feasible policy can not allocate coupons to more than $K$ users. This constraint captures the fact that the company often has limited budgeted on coupon producing and distribution.

5 An Approximate Solution

We first introduce a new problem P.B as follows.

**P.B**

Maximize $f^+(y)$

subject to:

$$\begin{align*}
\forall v \in V & : \sum_{d \in D} y_{vd} \leq 1 \quad (C1) \\
\sum_{y_{vd} \in S} y_{vd} p_v(d) d & \leq B \quad (C2) \\
\forall [vd] & \in S : y_{vd} \in [0, 1] \quad (C3)
\end{align*}$$

In the above formulation, $y$ is a $n \times m$ decision matrix and $f^+(y)$ is a concave extension of $f$.

$$f^+(y) = \max \left\{ \sum_{S \subseteq S} \alpha_S f(S) : \begin{array}{l}
\alpha_S \geq 0; \\
\forall v : \sum_{s \subseteq S} \alpha_S \leq 1; \\
\forall [vd] : \sum_{s \subseteq S} \alpha_S \leq y_{vd}
\end{array} \right\}$$

We first prove that P.B is a relaxed version of P.A.

**Lemma 2**. Assume $\Theta^*$ is the optimal solution to P.A and $y^+$ is the optimal solution to P.B, we have $f^+(y^+) \geq \sum_{S \subseteq S} \theta_S^* f(S)$.

Proof. Given the optimal policy $\Theta^*$, for every $u-c$ pair $[vd] \in S$, we define $y^*_{vd}$ as the probability that $[vd]$ is offered by $\Theta^*$, e.g., $y^*_{vd} = \sum_{S \subseteq S} \theta_S^*$. We first prove that $y^+$ is a feasible solution to P.B. Because $\Theta^*$ is a feasible policy, we have

1. $\forall \theta_S > 0 : |S_v \cap S| \leq 1$, e.g., every user receives at most one coupon under $\Theta^*$. It follows that $\forall v \in V : \forall S_v > 0, \forall S_v : |S_v \cap S| \leq 1$, e.g., every user receives at most one coupon.

2. $\sum_{S \subseteq S} \theta_S c(S) \leq B$, e.g., the expected cost of any coupon allocation under $\Theta^*$ is no larger than $B$. Because $\forall [vd] \in S : y^*_{vd} p_v(d) d$ is the expected cost of $\Theta^*$, we have $\forall [vd] \in S : y^*_{vd} p_v(d) d \leq B$, thus condition (C1) is satisfied.

3. $\sum_{S \subseteq S} \theta_S = 1$. Because $\forall [vd] \in S : y_{vd}$ is the probability that $[vd]$ is offered by $\Theta^*$, we have $\forall [vd] \in S : y_{vd} \leq 1$, thus condition (C2) is satisfied.

On the other hand, by setting $\alpha_S = \theta_S$ for every $S \in S$ in [1], we have $f^+(y^+) \geq \sum_{S \subseteq S} \theta_S^* f(S)$.

Assume $g^+$ is the concave extension of $g$, we next prove that $g^+(y)$ is an approximate of $f^+(y)$.

**Lemma 3**. $f^+(y) \leq (1 + \epsilon)g^+(y)$

Proof. Given any $y$, assume the value of $f^+(y)$ is achieved at $\{a_A : A \subseteq V\}$, we have $\sum_{A \subseteq V} a_A f(A) \leq \sum_{A \subseteq V} a_A (1 + \epsilon)g(A) = (1 + \epsilon)\sum_{A \subseteq V} a_A g(A) \leq (1 + \epsilon)g^+(y)$.

5.1 Algorithm Design

In this section, we present a greedy algorithm that achieves a bounded approximation ratio. Our general idea is to first find a fractional solution with a bounded approximation ratio and then round it to an integral solution.

**Continuous Greedy**

In [Vondrák, 2008] they develop a continuous greedy algorithm based on the multilinear extension in order to maximize a submodular monotone function over a matroid constraint. We extend their results to non-submodular maximization subject to a partition matroid and knapsack constraints. We use $1_{vd}$ to denote a $n \times m$ matrix with entry $(v, d)$ one and all other entries zero. Given two matrices $a$ and $b$, let $a \oplus b$ denote the coordinate-wise maximum. Define $F_k([vd]) = F(x \oplus 1_{vd}) - F(x)$. A detailed description of our algorithm is listed in Algorithm 1.
Algorithm 1 Continuous Greedy
1. Set $\delta = 1/(nm)^2$, $t = 0$, $f(\emptyset) = 0$.
2. while $t < 1$
3. Let $R(t)$ contain each $[vd]$ independent with probability $y_{vd}(t)$.
4. For each $[vd] \in S$, estimate
   \[\omega_{vd} = E[f(R(t) \cup \{[vd]\})] - E[[f(R(t))]]\]
5. Solve the following linear programming problem and obtain the optimal solution $\bar{y}$
6. \[\text{P.C: Maximize } \sum_{[vd] \in S} \omega_{vd} y_{vd}\]
   subject to: Conditions (C1)–(C3)
7. Let $y_{vd}(t + \delta) = y_{vd}(t) + \bar{y}_{vd}$.
8. Increment $t = t + \delta$;
9. end while

Lemma 4. Let $y(\frac{1}{3})$ denote the solution returned from Algorithm 1; we have $F_y(y(\frac{1}{3})) \geq \frac{1 - e^{-\frac{(1+2\epsilon)}{1+e}}}{{1+e}} f^+(y^+)$.

Proof. As proved in [Calinescu et al., 2011], if $g$ is a submodular function, $g^*(y) \leq \min_{S \subseteq S} g(S) + \sum_{[vd] \in S} y_{vd}g([vd])$. Let $y^+$ denote the optimal solution to problem P.B, assume $y(t)$ is our solution at round $t$, then for every round $t$, we have

$g^+(y^+) \leq \min_{S \subseteq S} g(S) + \sum_{[vd] \in S} y_{vd}g([vd])$

$\leq G(y(t)) + \sum_{[vd] \in S} y_{vd}G_y(t([vd]))$

$\leq \frac{1}{1 - \epsilon} F_y(y(t))$

$\leq F_y(y(t)) + \sum_{[vd] \in S} y_{vd} \left( F_y(y(t) \oplus 1_{vd}) - \frac{1 - \epsilon}{1 + \epsilon} F_y(y(t)) \right)$

It follows that

$\frac{1 - \epsilon}{1 + \epsilon} f^+(y^+) \leq (1 - \epsilon) g^+(y^+)$

$\leq F_y(y(t)) + \sum_{[vd] \in S} y_{vd} \left( F_y(y(t) \oplus 1_{vd}) - \frac{1 - \epsilon}{1 + \epsilon} F_y(y(t)) \right)$

$= F_y(y(t)) + \sum_{[vd] \in S} y_{vd} \left( F_y(y(t) \oplus 1_{vd}) - F_y(y(t)) \right)$

$\leq \sum_{[vd] \in S} y_{vd} F_y(t([vd]) \leq \sum_{[vd] \in S} \bar{y} F_y(t([vd])$ (3)

The first inequality is due to Lemma 3. Let $\bar{y}$ denote the optimal solution to problem P.C in Algorithm 2 at round $t$, then we have

$\frac{1 - \epsilon}{1 + \epsilon} f^+(y^+) \leq (1 + \sum_{[vd] \in S} \frac{2\epsilon}{1 + \epsilon}) F_y(y(t))$

It follows that

$\frac{1 - \epsilon}{1 + \epsilon} f^+(y^+) \leq \sum_{[vd] \in S} \bar{y} F_y(t([vd])$ (3)

The first inequality is due to Lemma 3 and the second inequality is because $y$ is the optimal solution to problem P.C. According to Line 7, the increased value of our solution at round $t + \delta$ is at least

$F_y(y(t + \delta)) - F_y(y(t))$

$= \sum_{[vd] \in S} \bar{y} \prod_{v \in [vd]} (1 - \bar{y} y_{vd}) F_y(t([vd])$

$\geq \sum_{[vd] \in S} \bar{y} \prod_{v \in [vd]} (1 - \bar{y} y_{vd}) F_y(t([vd])$

$= \delta (1 - \bar{y})^{m-1} \sum_{[vd] \in S} \bar{y} F_y(t([vd])$

$\geq \delta (1 - \bar{y})^{m-1} \sum_{[vd] \in S} \bar{y} F_y(t([vd])$

$= \delta (1 - \frac{1}{m}) \sum_{[vd] \in S} \bar{y} F_y(t([vd])$ (4)

The last inequality is due to $\delta = \frac{1}{(nm)^2}$. If we take (5) and (4) together, we get

$\frac{1}{1 - \epsilon} F_y(y(t + \delta) - F_y(y(t))$

$\geq \delta (1 - \frac{1}{m}) \left( \frac{1 - \epsilon}{1 + \epsilon} f^+(y^+) - (1 + \sum_{[vd] \in S} 2y_{vd}) F_y(t([vd])\right)$

$\geq \delta (1 - \frac{1}{m}) \left( \frac{1 - \epsilon}{1 + \epsilon} f^+(y^+) - (1 + \frac{2\epsilon n}{1 + \epsilon}) F_y(t([vd])\right)$

$\geq \delta (1 - \frac{1}{m}) \left( \frac{1 - \epsilon}{1 + \epsilon} f^+(y^+) - (1 + \frac{2\epsilon n}{1 + \epsilon}) F_y(t([vd])\right)$ (5)

Inequality (5) is due to $\sum_{[vd] \in S} y_{vd} \leq n$. Define $\Delta_t = \left(\frac{1 - \frac{2\epsilon n}{1 + \epsilon}}{1 + \epsilon} F_y(y^+) - (1 + \frac{2\epsilon n}{1 + \epsilon}) F_y(t([vd]))\right)$, according to (6), we have $\Delta_{t+\delta} = (1 - \delta) \Delta_t$, thus

$\Delta_{t+\delta} = (1 - \delta) (1 + \frac{2\epsilon n}{1 + \epsilon}) \Delta_0 = e^{-\frac{(1+2\epsilon)}{1+e}} \Delta_0$

It follows that

$F_y(y(t + \delta)) \geq (1 - e^{-\frac{(1+2\epsilon)}{1+e}})(1 - \frac{1}{m}) \frac{1 - \epsilon}{1 + \epsilon} f^+(y^+) - (1 - e^{-\frac{(1+2\epsilon)}{1+e}})(1 - \frac{1}{m}) \frac{1 - \epsilon}{1 + \epsilon} f^+(y^+)$

$(o(1)$ can be removed when choosing small enough $\delta$

Vondrák, 2008) \Rightarrow F_y(y(\frac{1}{3})) \geq \frac{(1 - e^{-\frac{(1+2\epsilon)}{1+e}})(1 - \frac{1}{m}) \frac{1 - \epsilon}{1 + \epsilon} f^+(y^+)}{(1 - \frac{1}{m}) \frac{1 - \epsilon}{1 + \epsilon} f^+(y^+)}$

Rounding

In the rest of this paper, we use $y$ to denote $y(\frac{1}{3})$ for short.

(2) After obtaining $y$, we next round the fractional solution to integral solutions. Notice that classic rounding approach relying on the submodularity of the utility function, such as page rounding [Calinescu et al., 2011], can not work directly on a general function.

We first introduce a dummy coupon with value 0 and define $y_{vd} = 1 - \sum_{d \in D} y_{vd}$. For each user $\nu \in \mathcal{V}$, we assign exactly one discount to her, discount $d \in D$ or $d = 0$ to user $\nu$ with probability $y_{vd}$. The returned solution is denoted as $T$. We next prove that $T$ is feasible and its expected cascade $E[f(T)]]$ is close to the optimal solution.
Theorem 1. Our rounding approach returns a feasible solution $T$ to problem P.A, and

$$
\mathbb{E}[f(T)] \geq \beta f^+(y^+)
$$

where $\beta = \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^n \frac{(1 - (1 + \frac{2\epsilon n}{1 + (2n + 1)\epsilon}) (1 - \epsilon)}{1 + (2n + 1)\epsilon}.
$$

Proof. We first prove the feasibility of our rounding approach. It is easy to verify that our solution satisfied C1, this is because each user receives at most one coupon. Because each u-c pair is selected with probability $y_{ud}$ and $y$ is a feasible solution to problem P.B, the expected cost of $T$ is

$$\sum_{d \in D} y_{ud} p_v(d) \leq B \text{ (C2 is satisfied).}
$$

We next prove that $\mathbb{E}[f(T)] \geq \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^n f(y)$ by induction on $n$. First assume that $n = 1$ and $S = S_1$, e.g., there is only one user. Let $R$ be a set in which each u-c pair $[vd]$ appears with probability $y_{ud}$, thus $\mathbb{E}[f(R)] = f(y)$. Then,

$$
\mathbb{E}[f(R)] = \sum_{R \subseteq S} \text{Pr}[R] f(R) \leq \sum_{R \subseteq S} (1 + \epsilon) \text{Pr}[R] g(R)
$$

(due to submodularity of $g$)

$$
\leq \sum_{R \subseteq S} \frac{1 + \epsilon}{1 - \epsilon} \text{Pr}[R] \sum_{[vd] \in R} f([vd]) = \left( \frac{1 + \epsilon}{1 - \epsilon} \right) \text{Pr}(vd \in R) f([vd])
$$

$$
= \left( \frac{1 + \epsilon}{1 - \epsilon} \right) \mathbb{E}[f(T)]
$$

Now consider $n > 1$. Define $T' = T \cap (S_1 \cup S_2 \cdots \cup S_{n-1})$, $T'' = T \cap S_k$ and $R' = R \cap (S_1 \cup S_2 \cdots \cup S_{k-1})$, $R'' = R \cap S_k$. For a given $T'$, define $f^{T'}(S) = f(T' \cup S)$. We have $(1 - \epsilon) g^{T''}(S) \leq f^{T''}(S) \leq (1 + \epsilon) g^{T'}(S)$, because $g^{T'}(S)$ is a submodular function of $S$, $f(T' \cup S)$ is an $\epsilon$-approximate submodular function of $S$.

$$
\mathbb{E}[f(T)] = \mathbb{E}[f(T' \cup T'')] = \sum_{T'} \text{Pr}[T'] \mathbb{E}[f^{T'}(T'')]
$$

$$
\geq \left( \frac{1 - \epsilon}{1 + \epsilon} \right) \sum_{T'} \text{Pr}[T'] \mathbb{E}[f^{T'}(T'')]
$$

Similarly, for a given $R''$, $f^{R''}(S)$ is an approximate submodular function of $S$.

$$
\mathbb{E}[f(T' \cup R'')] = \sum_{R''} \text{Pr}[R''] \mathbb{E}[f^{R''}(T')]
$$

$$
\geq \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^{n-1} \sum_{R''} \text{Pr}[R''] \mathbb{E}[f^{R''}(R'')]
$$

$$
= \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^{n-1} \mathbb{E}[f(R' \cup R'')] \quad \text{(7)}
$$

Therefore, we have

$$
\mathbb{E}[f(T)] \geq \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^n \mathbb{E}[f(R)]
$$

$$
\geq \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^n (1 - e^{-\frac{2\epsilon n}{1 + (2n + 1)\epsilon}})(1 - \epsilon) f^+(y^+)
$$

The first inequality is due to (7) and (8), and the second inequality is due to Lemma 4.

\hspace{1cm} \Box

6 Extension: Incorporating Distribution Cost

We now discuss one extension of our analysis thus far. The difference between the extended model and the previous model is that in the new model, we add one more constraint to Definition 1. In particular, we assume a feasible policy cannot allocate coupons to more than $K$ users. This additional constraint models the fact that the company only have limited resource to produce and distribute coupons to different users.

- (Coupon Distribution Budget Constraint) $\forall S \geq K$, e.g., at most $K$ users receive a coupon.

Similar to the previous model, we formulate our problem $\text{P.A}^+$ as follows.

$$
\text{P.A}^+ \max \sum_{S \subseteq S} \theta_S f(S)
$$

subject to:

$$
\begin{align*}
\forall \theta_S \geq 0, \forall v : |S_v \cap S| &\leq 1 \\
\sum_{S \subseteq S} \theta_S c(S) &\leq B \\
\sum_{S \subseteq S} \theta_S &\leq 1 \\
\forall \theta_S > 0 : |S_v| &\leq K \quad \text{(Coupon Distribution Budget)}
\end{align*}
$$

We follow a similar idea used in the previous section to design our algorithm, e.g., compute a fractional solution first and then round it to integral solution. However, to tackle the new challenge brought by the additional constraint, we need a brand new design of our rounding approach to ensure the feasibility of the final solution. We first introduce a new problem $\text{P.B}^+$ as follows.

$$
\text{P.B}^+ \max f^+(y)
$$

subject to:

$$
\begin{align*}
(C1) &\sim (C3) \\
\sum_{[vd] \in S} y_{vd} &\leq K \quad \text{(C4)}
\end{align*}
$$

Continuous Greedy Algorithm

We present the continuous greedy algorithm in Algorithm 2.

The only difference between Algorithm 2 and Algorithm 1 is that in Algorithm 2, we redefine problem $\text{P.C}^+$ to incorporate constraint (C4).

Algorithm 2 Continuous Greedy

1: Replace $\text{P.C}$ in Algorithm 1 by $\text{P.C}^+$

$$
\text{P.C}^+ \max \sum_{[vd] \in S} \omega_{vd} y_{vd}
$$

subject to: Conditions (C1)~(C4)
Based on a similar proof of Lemma 4, we can prove the following lemma. This bound is slightly better than Lemma 4, e.g., \( n \) is replaced by \( K \).

**Lemma 5.** Let \( y \) denote the solution returned from Algorithm 2 and \( y^+ \) denote the optimal solution to problem \( PB^+ \), we have \( F(y) \geq \frac{1}{1+(2K+1)\epsilon} f^+(y^+) \).

**Rounding**

We first notice that directly applying the previous rounding approach may violate constraint (C4), e.g., the number of users who receive coupons could be larger than \( K \). We next design a brand new rounding approach that satisfies all constraints. The general idea of our new approach can be described as follows: partition all users into two groups, then apply a simple randomized rounding to each group, and the one with larger utility is returned as the final solution.

Let \( Z_v = \sum_{e \in\{1, 2, \ldots, v\}} \sum_{d \in \mathcal{D}} y_{vd} \). We first partition all users into two groups \( G_1 \) and \( G_2 \), \( G_1 = \{v[Z_v]−\{Z_v\} = 1\} \) and \( G_2 = V \setminus G_1 \). Then we apply a simple randomized rounding to each group as follows:

- **Rounding \( G_1 \):** The rounding approach used for \( G_1 \) is similar to the one used in the previous section. We first introduce a dummy coupon with value 0 and define \( y_{v0} = 1 − \sum_{d \in \mathcal{D}} y_{vd} \). For each user in \( G_1 \), we assign exactly one discount to her, discount \( d \) to user \( v \) with probability \( y_{vd} \). The returned solution is denoted as \( T_1 \).

- **Rounding \( G_2 \):** Given \( G_1 \), we can naturally partition \( G_2 \) into \(|G_1|\) subgroups: \( \{G_i^l | l = 1, 2, \ldots, |G_1|\} \). The \( l \)-th subgroup of \( G_2 \) is defined as \( G_l^i = \{v: u_l < v < u_{l+1}\} \) where \( u_l \) denotes the index of the \( l \)-th user in \( G_1 \). For every subgroup \( G_l^i \), we introduce a dummy u-c pair with an arbitrary user \( v \in G_l^i \) (resp. \( G_l^i \)) and coupon value 0, and define \( y_{v0} = 1 − \sum_{u \in G_l^i} \sum_{d \in \mathcal{D}} y_{ud} \). We select exactly one u-c pair from each subgroup, discount \( d \) to user \( v \) with probability \( y_{vd} \). The returned solution is denoted as \( T_2 \).

- We compare the expected cascade from the above two solutions and choose the one with larger value as the final output.

Recall that we use \( R \) to denote a set in which each u-c pair \( [vd] \) appears with probability \( y_{vd} \). We partition \( R \) into two subsets \( R_1 \) and \( R_2 \). \( R_1 \) (resp. \( R_2 \)) contains all u-c pairs whose user is from \( G_1 \) (resp. \( G_2 \)). Similar to the proof of 5, we can prove the following lemma.

**Lemma 6.** Both \( T_1 \) and \( T_2 \) are feasible solutions to \( PA^+ \), and \( \mathbb{E}[f(T_1)] \geq \left(1−\frac{1}{1+\epsilon}\right)^n \mathbb{E}[f(R_1)] \) and \( \mathbb{E}[f(T_2)] \geq \left(1−\frac{1}{1+\epsilon}\right)^n \mathbb{E}[f(R_2)] \).

We next provide the performance bound of our solution.

**Theorem 2.** The expected cascade of our solution is lower bounded by \( \max\{\mathbb{E}[f(T_1)], \mathbb{E}[f(T_2)]\} \geq \beta' f^+(y^+) \) where \( \beta' = \frac{1}{2} \left(1−\frac{1}{1+\epsilon}\right)^n \left(1−\frac{1}{1+\epsilon}\right)^n \mathbb{E}[f(R_2)] \).

**Proof.** Because \( f \) is an \( \epsilon \)-approximate submodular function, we have \( f(R_1) + f(R_2) \geq (1−\epsilon)(g(R_1) + g(R_2)) \geq (1−\epsilon)g(R) \geq \frac{1}{1+\epsilon} f(R) \).

As proved in Lemma 6, we have \( \mathbb{E}[f(T_1)] \geq \left(1−\frac{1}{1+\epsilon}\right)^n \mathbb{E}[f(R_1)] \) and \( \mathbb{E}[f(T_2)] \geq \left(1−\frac{1}{1+\epsilon}\right)^n \mathbb{E}[f(R_2)] \).

Thus \( \max\{\mathbb{E}[f(T_1)], \mathbb{E}[f(T_2)]\} \geq \frac{1}{2} \left(1−\frac{1}{1+\epsilon}\right)^n \mathbb{E}[f(R)] \). The last inequality is due to Lemma 5.

**7 Performance Evaluation**

We use Independent Cascade (IC) model as the information diffusion model. We set the propagation probability of each directed edge \( uv \) to be \( \kappa/N_{in}(v) \) where \( N_{in}(v) \) is the indegree of \( v \). We vary the value of \( \kappa \) and examine its effect on the quality of the solutions. Coupon values normally range from 0 to 1 in steps of 0.1. Regarding the coupon adoption probabilities, we randomly select 75% nodes to assign \( p_v(d) = \sqrt{d} \), and 20% nodes to assign \( p_v(d) = d \), and 5% nodes to assign \( p_v(d) = d^2 \) as their adoption probability functions. We compare our results with two benchmark solutions: Greedy and Random. Greedy starts with an empty seed set, in each iteration, it selects the u-c pair with the largest benefit cost ratio (the ratio of the increased expected cascade of the expected cost) and add it to the seed set. Random randomly selects a u-c pair in each iteration and add it to the seed set. This process is repeated until the budget is exhausted.

Figure 1 shows the expected cascade yielded by our algorithm and baselines when budget \( B \) ranges from 10 to 100. We test the scenario with different influence probabilities \( \kappa \in \{0.8, 1\} \). We observe that our solution (CG) consistently outperforms the runner-up Greedy. The gap between CG and Greedy becomes even larger as \( B \) increases.

**8 Conclusion**

In this paper, we study coupon allocation problem in social networks. Our framework allows a general utility function and more complicated constraints. Therefore, existing techniques relying on the submodularity of the utility function can not apply to our problem directly. We propose a novel approximate algorithm with approximation ratio depending on \( \epsilon \). Although we limit our attention to coupon allocation problem in this paper, our results apply to a broad range of approximate submodular maximization problems.
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