Transonic Black Hole Accretion as Analogue System

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Abstract

Classical black hole analogues (alternatively, the analogue systems) are fluid dynamical analogue of general relativistic black holes. Such analogue effects may be observed when acoustic perturbations (sound waves) propagate through a classical dissipation-less transonic fluid. The acoustic horizon, which resembles the actual black hole event horizon in many ways, may be generated at the transonic point in the fluid flow. Acoustic horizon emits quasi thermal phonon spectra, which is analogous to the actual Hawking radiation, and possesses the temperature referred as the analogue Hawking temperature, or simply, the analogue temperature.

Transonic accretion onto astrophysical black holes is a very interesting example of classical analogue system found naturally in the Universe. An accreting black holes system as a classical analogue is unique in the sense that only for such a system, both kind of horizons, the electromagnetic and the acoustic (generated due to transonicity of accreting fluid) are simultaneously present in the same system. Hence an accreting astrophysical black hole is the ideal-most candidate to theoretically study and to compare the properties of these two different kind of horizons. Also such system is unique in the aspect that general relativistic spherical accretion onto the Schwarzschild black hole represents the only classical analogue system found in the nature so far, where the analogue Hawking temperature may be higher than the actual Hawking temperature.

1 Black Holes

Black holes are vacuum solutions of Einstein’s field equations in general relativity. Classically, these objects are conceived as singularities in space time, censored from the rest of the Universe by mathematically defined one way surfaces, the event horizons. The space time metric defining the vacuum exterior of a classical black hole, and the black hole itself, is characterized by only three parameters, the mass of the black hole $M_{BH}$, the rotation (spin) $J$ and charge $q$. For $J = q = 0$, one obtains a Schwarzschild black hole, and for $q = 0$ one obtains a Kerr black hole. These two kind of black holes are important in astrophysics. In astrophysics, black holes are the end point of gravitational collapse of massive celestial objects. Astrophysical black holes may be broadly classified into two categories, the stellar mass ($M_{BH} \sim a few \ M_\odot$, where $M_\odot$ is the mass of the Sun), and super massive ($M_{BH} \geq 10^6 \ M_\odot$) black holes (SMBH). While the birth history of the stellar mass black holes is theoretically known with almost absolute certainty (they are the endpoint of the gravitational collapse of massive stars), the formation scenario of the supermassive black hole is not unanimously understood. A SMBH may form through the monolithic collapse of early proto-spheroid gaseous mass originated at the time of galaxy formation. Or a number of stellar/intermediate mass black holes may merge to form it. Also the runaway growth of a seed black hole by accretion in a specially favoured high-density environment may lead to the formation of SMBH. However, it is yet to be well understood exactly which of the above processes routes towards the SMBH formation, see, e.g. Rees 2002 for a comprehensive review on the formation and evolution of SMBH. Both kind of astrophysical black holes, the stellar mass and SMBH, however, accrete matter from the surrounding. Depending on the intrinsic angular momentum content of accreting material, either spherically symmetric (for zero angular momentum flow), or axisymmetric (for flow with non-zero finite angular momentum) flow geometry is invoked to study an accreting black hole system (Frank, King & Raine 1992).
2 Black hole thermodynamics

Within purely classical framework, black holes in any diffeomorphism covariant theory of gravity (where the field equations directly follow from the diffeomorphism covariant Lagrangian) and in general relativity, mathematically resembles some aspects of classical thermodynamic systems (Wald 1984, 1994, 2001, Kiefer 1998, Brown 1995 and references therein). In early seventies, a series of influential works (Bekenstein 1972, 1972a, 1973, 1975, Israel 1976, Bardeen, Carter & Hawking 1977, see also Bekenstein 1980 for a review) revealed the idea that classical black holes in general relativity, obey certain laws which bear remarkable analogy to the ordinary laws of classical thermodynamics. Such analogy between black hole mechanics and ordinary thermodynamics (‘the generalized Second Law’, as it is customarily called) leads to the idea of the ‘surface gravity’ of black hole, $\kappa$, which can be obtained by computing the norm of the gradient of the norms of the Killing fields evaluated at the stationary black hole horizon, and is constant on the horizon (analogous to the constancy of temperature $T$ on a body in thermal equilibrium - the Zeroth Law of classical thermodynamics).

Also, $\kappa=0$ can not be reached by performing finite number of operations (analogous to the ‘weak version’ of the third law of classical thermodynamics where temperature of a system cannot be made to reach at absolute zero, see discussions at Kiefer 1998). It was found by analogy via black hole uniqueness theorem (see. e.g. Heusler 1996 and references therein) that the role of entropy in classical thermodynamic system is played by a constant multiple of the surface area of a classical black hole.

3 Hawking Radiation

The resemblance between the laws of ordinary thermodynamics to those of black hole mechanics were, however, initially regarded as purely formal. This is because, the physical temperature of a black hole is absolute zero (see, e.g. Wald 2001). Hence physical relationship between the surface gravity of the black hole, and the temperature of a classical thermodynamic system can not be conceived. This further indicates that a classical black hole can never radiate. However, introduction of quantum effects might bring a radical change to the situation. In an epoch making paper published in 1975, Hawking (1975) used quantum field theoretic calculation on curved spacetime to show that the physical temperature and entropy of black hole does have finite non-zero value (see Page 2004 for an excellent review of black hole thermodynamics and Hawking radiation). A classical space time describing gravitational collapse leading to the Schwarzschild black hole was assumed to be the dynamical back ground, and a linear quantum field, initially in it’s vacuum state prior to the collapse, was considered to propagate against this background. The vacuum expectation value of the energy momentum tensor of this field turned out to be negative near the horizon. This phenomenon leads to the flux of negative energy into the hole. Such negative energy flux would decrease the mass of the black hole and would lead to the fact that the quantum state of the outgoing mode of the field would contain particles. The expected number of such particles would correspond to radiation from a perfect black body of finite size. Hence the spectrum of such radiation is thermal in nature, and the temperature of such radiation, the Hawking temperature $T_H$ from a Schwarzschild black hole, can be computed as

$$T_H = \frac{\hbar c^3}{8\pi k_b G M_{BH}}$$  \hspace{1cm} (1)

where $c$ and $k_b$ are the velocity of light in vacuum and the Boltzmann’s constant, respectively. The semi classical description for Hawking radiation treats the gravitational field classically and the quan-

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1The surface gravity may be defined as the acceleration measured by red-shifts of light rays passing close to the horizon (Helfer 2003)

2For a lucid description of the physical interpretation of Hawking radiation, see, e.g., Wald 1994, Kiefer 1998, Helfer 2003.
ized radiation field satisfies the d’Alembert equation. At any time, black hole evaporation is an
adiabatic process if the residual mass of the hole at that time is larger than the Planck mass.

4 Towards an analogy of Hawking effect: The motivation

Substituting the values of the fundamental constant in eq. (1), one can rewrite $T_H$ for a Schwarzschild
black hole as:

$$T_H \sim 6.2 \times 10^{-8} \left( \frac{M_\odot}{M_{BH}} \right) ^0 K$$  \hspace{1cm} (2)

It is evident from the above equation that for one solar mass black hole, the value of the Hawking
temperature would be too small to be experimentally detected. A rough estimate shows that $T_H$ for
stellar mass black holes would be around $10^7$ times colder than the cosmic microwave background
radiation. The situation for super massive black hole will be much more worse, as $T_H \propto \frac{1}{M_{BH}}$. Hence
$T_H$ would be a measurable quantity for primordial black holes with very small size and mass, if such
black holes really exist, and if instruments can be fabricated to detect them. The lower bound of mass
for such black holes may be estimated analytically. The time-scale $T$ over which the mass of the black
hole changes significantly due to the Hawking’s process may be obtained as (Helfer 2003)

$$T \sim \left( \frac{M_{BH}}{M_\odot} \right)^3 10^{65}$$  \hspace{1cm} (3)

As the above time scale is a measure of the lifetime of the hole itself, the lower bound for a primordial
hole may be obtained by setting $T$ equal to the present age of the Universe. Hence the lower bound
for the mass of the primordial black holes comes out to be around $10^{15}$ gm. The size of such a black
hole would be of the order of $10^{-13}$ cm and the corresponding $T_H$ would be about $10^{110} K$, which is
comparable, as we will see in §10, with the macroscopic fluid temperature of the freely falling matter
(spherically symmetric accretion) onto an one solar mass isolated Schwarzschild black hole. However,
present day instrumental technique is far from efficient to detect these primordial black holes with such an extremely small dimension, if such holes exist at all in first place. Hence, the observational manifestation of Hawking radiation seems to be practically impossible.

On the other hand, due to the infinite redshift caused by the event horizon, the initial configuration
of the emergent Hawking Quanta is supposed to possess trans-Planckian frequencies and it’s wave
lengths are beyond the Planck scale; thus low energy effective theories cannot self consistently deal
with the Hawking radiation (Parentani 2002). Also, the nature of the fundamental degrees of freedom
and the physics of such ultra short distance is yet to be well understood. Hence the fundamental issues like the statistical meaning of the black hole entropy, or the exact physical origin of the outgoing
mode of the quantum field, remains unresolved (Wald 2001).

Perhaps the above mentioned issues served as the principal motivations to launch a theory, analogous
to the Hawking’s one, whose effects would be possible to comprehend through relatively more
perceivable physical systems. The theory of analogue Hawking radiation opens up the possibility to
experimentally verify some basic features of black hole physics by creating the sonic horizons in the
laboratory. A number of works have been carried out to formulate the condensed matter or optical
analogs of event horizons. It is also expected that analogue Hawking radiation may find important

\[^{3}\text{Literatures on study of analogue systems in condensed matter or optics are quite large in numbers. Condensed matter or optical analogue systems deserve the right to be discussed as separate review articles on its own. In this article, we, by no means, are able to provide the complete list of references for theoretical or experimental works on such systems. However, to have an idea on the analogue effects in condensed matter or optical systems, readers are refereed to the book by Novello, Visser & Volovik 2002 for review, and to some of the representative papers like Leonhard 2002, 2003, Garay, Anglin, Cirac, & Zoller 2000, 2001, Jacobson & Volovik 1998, Volovik 1999, 2000, 2001, Brevik & Halnes 2002, Schützhold, Günter & Gerhard 2002, Schützhold & Unruh 2002, de Lorenci, Klippert & Obukhov 2003, Reznik 2000, Novello, Perez Bergliaffa, Salim, DeLorenci & Klippert 2003.} \]
uses in the fields of FRW cosmology (Barcelo, Liberati & Visser 2003), inflationary models, quantum gravity and sub-Planckian models of string theory (Parentani 2002).

For space limitation, in this article, we will, however, mainly describe the formalism behind the classical analogue systems. By ‘classical analogue systems’ we refer to the examples where the analogue effects are studied in classical fluids, and not in quantum fluids. In the following section, we narrate the basic features of a classical analogue system. Hereafter, we shall use $T_{AH}$ to denote the analogue Hawking temperature, and $T_H$ to denote the the actual Hawking temperature as defined in eq. (1). We shall also use the words ‘analogue’, ‘acoustic’ and ‘sonic’ synonymously in describing the horizons or black holes. Also the phrases ‘analogue Hawking radiation/effect/temperature’ should be taken as identical in meaning with the phrase ‘analogue radiation/effect/temperature’. A system manifesting the effects of analogue radiation, will be termed as analogue system.

5 Theory of analogue radiation

In a pioneering work, Unruh (1981) showed that a classical system, relatively more clearly perceivable than a quantum black hole system, does exist, which resembles the black hole as far as the quantum thermal radiation is concerned. The behaviour of a linear quantum field in a classical gravitational field was simulated by the propagation of acoustic disturbance in a convergent fluid flow. In such a system, it is possible to study the effect of the reaction of the quantum field on it’s own mode of propagation and to contemplate the experimental investigation of the thermal emission mechanism. If one considers the equation of motion for a transonic barotropic irrotational fluid, it can be shown (Unruh 1981) that the scaler field representing the acoustic perturbation (i.e, the propagation of sound wave) satisfies a differential equation which is analogous to the equation of a massless scaler field propagating in a metric. Such a metric closely resembles the Schwarzschild metric near the horizon. Thus acoustic propagation through a supersonic fluid forms an analogue of event horizon, as the ‘acoustic horizon’ at the transonic point. The behaviour of the normal modes near the acoustic horizon indicates that the acoustic wave with a quasi-thermal spectrum will be emitted from the acoustic horizon and the temperature of such acoustic emission may be calculated as:

$$T_{AH} = \frac{\hbar}{4\pi k_b} \left[ \frac{1}{a_s} \frac{\partial u_\perp^2}{\partial n} \right]_{\text{acoustic horizon}}$$

Where $a_s$ is the sound speed, $u_\perp$ is the component of the dynamical flow velocity normal to the acoustic horizon, and $\frac{\partial}{\partial n}$ represents the normal derivative. Equation (4) has clear resemblance with eq. (1) and hence $T_{AH}$ is designated as analogue Hawking Temperature and such quasi-thermal radiation form acoustic (analogue) black hole is known as the Analogue Hawking radiation. Note that the sound speed in Unruh’s original treatment (the above equation) was assumed to be constant in space, i.e., equation of state used was isothermal.

Unruh’s work was followed by other important papers (Jacobson 1991, Unruh, 1995, Visser, 1998, Jacobson 1999, Bilić 1999). A generalized treatment for classical analogue radiation for Newtonian fluid may be obtained in Visser (1998). Visser (1998) showed that for barotropic, inviscid, hydrodynamic fluid, the equation of motion for the velocity potential $\psi$ describing an acoustic disturbance is identical to the d’Alembertian equation of motion for a minimally coupled massless scaler field propagating in a (3+1) Lorenzian geometry as:

$$\Delta \psi \equiv \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \psi \right) = 0$$
The acoustic propagation is described by the following metric, which is algebraically dependent on the density and dynamical as well as sonic velocity of the flow:

\[ G_{\mu\nu}(t, \vec{x}) = \frac{\rho}{c} \left[ \begin{array}{ccc}
-\left(a_s^2 - u^2\right) & \ldots & -\vec{u} \\
\ldots & \ldots & \ldots \\
-\vec{u} & \ldots & I
\end{array} \right] \] (6)

The above acoustic metric for a point sink, is conformally related to the Painlevé-Gullstrand-Lemaître form of Schwarzschild metric. The conformal factor may be neglected in connection to the calculation of the analogue Hawking temperature. The surface gravity \( \kappa \) for above mentioned acoustic metric can be calculated as:

\[ \kappa = \frac{1}{2} \frac{\partial}{\partial n} \left( a_s^2 - u^2 \right) \] (7)

From eq (7), it is straightforward to calculate the expression for analogue temperature as

\[ T_{AH} = \frac{\hbar}{4\pi k_B} \left[ \frac{1}{a_s} \frac{\partial}{\partial n} \left( a_s^2 - u^2 \right) \right]_{\text{acoustics horizon}} \] (8)

The above equation shows that one needs to know the exact location of the acoustic horizon, the exact values of the dynamical and sound velocities, and their space gradients on the acoustic horizon; to determine the analogue Hawking temperature of a classical analogue system. Note that the only difference between eq (8) and eq. (4) is that, in Unruh’s original expression, positional constancy of sound speed was assumed, whereas such assumption has not been made in eq. (8).

For analogue systems discussed above, the fluid particles are coupled to the flat metric of Mankowski’s space (because the governing equation for fluid dynamics in the above treatment is completely Newtonian), whereas the sound wave propagating through the non-relativistic fluid is coupled to the curved pseudo-Riemannian metric. Phonons (quanta of acoustic perturbations) are the null geodesics, which generate the null surface, i.e., the acoustic horizon. Introduction of viscosity may destroy the Lorentzian invariance and hence the acoustic analogue is best observed in a vorticity free completely dissipationless fluid. That is why, the Fermi superfluids and the Bose-Einstein condensates are ideal to simulate the analogue effects. The most important issue emerging out of the above discussions is the following:

Even if the governing equation for fluid flow is completely non-relativistic (Newtonian), the acoustic fluctuations embedded into it are described by a curved pseudo-Riemannian geometry. This information is useful to portray the immense importance of the study of the acoustic black holes, i.e. the black hole analogues, or simply, the analogue systems.

In summary, analogue (acoustic) black holes (or systems) are fluid-dynamic analogue of general relativistic black holes. Analogue black holes possess analogue (acoustic) event horizons at local transonic points. Analogue black holes (alternatively, analogue systems) emit analogue Hawking radiation, the temperature of which is termed as analogue Hawking temperature, and it’s expression is formulated using Newtonian description of fluid flow. Black hole analogues are important to study because it may be possible to create them experimentally in laboratories to study some properties of the black hole event horizon, and to study the experimental manifestation of Hawking radiation. In the subsequent sections, we will describe how accretion processes onto astrophysical black holes may constitute a concrete example of analogue system naturally found in the Universe. We will also demonstrate that accreting black holes are unique as analogue systems because they are the only system found in nature so far, where the analogue temperature may exceed the actual Hawking temperature.
6 Transonic black hole accretion in astrophysics

Gravitational capture of surrounding fluid by massive astrophysical objects is known as accretion. There remains a major difference between black hole accretion and accretion onto other cosmic objects including neutron stars and white dwarfs. For celestial bodies other than black holes, infall of matter terminates either by a direct collision with the hard surface of the accretor or with the outer boundary of the magneto-sphere, resulting the luminosity through energy release from the surface. Whereas for black hole accretion, matter ultimately dives through the event horizon from where radiation is prohibited to escape according to the rule of classical general relativity and the emergence of luminosity occurs on the way towards the black hole event horizon. The efficiency of accretion process may be thought as a measure of the fractional conversion of gravitational binding energy of matter to the emergent radiation and is considerably high for black hole accretion compared to accretion onto any other astrophysical objects. Hence accretion onto classical astrophysical black holes has been recognized as a fundamental phenomena of increasing importance in relativistic and high energy astrophysics. The extraction of gravitational energy from the black hole accretion is believed to power the energy generation mechanism of X-ray binaries and of the most luminous objects of the Universe, the Quasars and active galactic nuclei (Frank, King & Raine 1992). The black hole accretion is, thus, the most appealing way through which the all pervading power of gravity is explicitly manifested.

If the instantaneous dynamical velocity and local acoustic velocity of the accreting fluid, moving along a space curve parameterized by \( r \), are \( u(r) \) and \( a_s(r) \) respectively, then the local Mach number \( M(r) \) of the fluid can be defined as \( M(r) = \frac{u(r)}{a_s(r)} \). The flow will be locally subsonic or supersonic according to \( M(r) < 1 \) or \( > 1 \), i.e., according to \( u(r) < a_s(r) \) or \( u(r) > a_s(r) \). The flow is transonic if at any moment it crosses \( M = 1 \). This happens when a subsonic to supersonic or supersonic to subsonic transition takes place either continuously or discontinuously. The point(s) where such crossing takes place continuously is (are) called sonic point(s), and where such crossing takes place discontinuously are called shocks or discontinuities. In order to satisfy the inner boundary conditions imposed by the event horizon, accretion onto black holes exhibit transonic properties in general.

Investigation of accretion processes onto celestial objects was initiated by Hoyle & Littleton (1939) by computing the rate at which pressure-less matter would be captured by a moving star. Subsequently, theory of stationary, spherically symmetric and transonic hydrodynamic accretion of adiabatic fluid onto a gravitating astrophysical object at rest was formulated in a seminal paper by Bondi (1952) using purely Newtonian potential and by including the pressure effect of the accreting material. Later on, Michel (1972) discussed fully general relativistic polytropic accretion on to a Schwarzschild black hole by formulating the governing equations for steady spherical flow of perfect fluid in Schwarzschild metric. Following Michel’s relativistic generalization of Bondi’s treatment, Begelman (1978) and Moncrief (1980) discussed some aspects of the critical points of the flow for such an accretion. Spherical accretion and wind in general relativity have also been considered using equations of state other than the polytropic one and by incorporating various radiative processes (Shapiro, 1973a,b, Blumenthal & Mathews 1976, Brinkmann 1980). Malec (1999) provided the solution for general relativistic spherical accretion with and without back reaction, and showed that relativistic effects enhance mass accretion when back reaction is neglected. The exact values of dynamical and thermodynamic accretion variables on the sonic surface, and at extreme close vicinity of the black hole event horizons, have recently been calculated using complete general relativistic (Das 2002) as well as pseudo general relativistic (Das & Sarkar 2001) treatments.

For flow of matter with non-zero angular momentum density, spherically symmetry is broken and accretion phenomena is studied employing axisymmetric configuration. Accreting matter is thrown into circular orbits around the central accretor, leading to the formation of the accretion discs around...
the galactic and extra-galactic black holes. For certain values of the intrinsic angular momentum density of accreting material, the number of sonic point, unlike spherical accretion, may exceed one, and accretion is called 'multi-transonic'. Study of such multi-transonicity was initiated by Abramowicz & Zurek (1981). Subsequently, multi-transonic accretion disc in general relativity has been studied in a number of works (Fukue 1987, Chakrabarti 1990, Kafatos & Yang 1994, Yang & Kafatos 1995, Pariev 1996, Peitz & Appl 1997, Lu, Yu, Yuan & Young 1997, Das 2004, Barai, Das & Wiita 2004, hereafter BDW). All the above works, except BDW, usually deal with low angular momentum sub-Keplerian prograde flow. BDW studied the retrograde flows as well and showed that a higher angular momentum (as high as Keplerian) retrograde flow can also produce multi-transonicity. Sub-Keplerian weakly rotating flows are exhibited in various physical situations, such as detached binary systems fed by accretion from OB stellar winds (Illarionov & Sunyaev 1975; Liang & Nolan 1984), semi-detached low-mass non-magnetic binaries (Bisikalo et al. 1998), and super-massive BHs fed by accretion from slowly rotating central stellar clusters (Illarionov 1988; Ho 1999 and references therein). Even for a standard Keplerian accretion disk, turbulence may produce such low angular momentum flow (e.g., Igumenshchev & Abramowicz 1999, and references therein).

7 Motivation to study the analogue behaviour of transonic black hole accretion

Since the publication of the seminal paper by Bondi in 1952, the transonic behaviour of accreting fluid onto compact astrophysical objects has been extensively studied in the astrophysics community, and the pioneering work by Unruh in 1981 initiated a substantial number of works in the theory of analogue Hawking radiation with diverse fields of application stated in §4. It is surprising that no attempt was made to bridge these two categories of research, astrophysical black hole accretion and the theory of analogue Hawking radiation, by providing a self-consistent study of analogue Hawking radiation for real astrophysical fluid flows, i.e., by establishing the fact that accreting black holes can be considered as a natural example of analogue system. Since both the theory of transonic astrophysical accretion and the theory of analogue Hawking radiation stem from almost exactly the same physics, the propagation of a transonic fluid with acoustic disturbances embedded into it, it is important to study analogue Hawking radiation for transonic accretion onto astrophysical black holes and to compute $T_{AH}$ for such accretion.

In the following sections, we will describe the details of the transonic accretion and will show how the accreting black hole system can be considered as a classical analogue system. Hereafter, we define gravitational radius $r_g = \frac{2GM_{BH}}{c^2}$, where $M_{BH}$ is the mass of the black hole, $G$ is the universal gravitational constant and $c$ is the velocity of light. The radial distances and velocities are scaled in units of $r_g$ and $c$ respectively and all other derived quantities are scaled accordingly; $G = c = M_{BH} = 1$ is used.

8 Newtonian and semi-Newtonian spherical accretion

In this section we will follow the methodology developed in Das & Sarkar 2001 to describe the simplest possible example of transonic black hole accretion. If the gravitational potential is defined as $\Phi$, then the equation of motion for spherically accreting matter onto the black hole can be written as:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\partial \Phi}{\partial r} = 0$$ (9)

where symbols have their usual meaning. The first term of eqn. (9) is the Eulerian time derivative of the dynamical velocity at a given $r$, the second term is the ‘advective’ term, the third term is
the momentum deposition due to pressure gradient and the final term is due to the gravitational acceleration. Note that $\Phi$ may be purely Newtonian, or may represent any of the following semi-Newtonian pseudo-Schwarzschild black hole potentials:

$$\Phi_1 = -\frac{1}{2(r-1)}, \quad \Phi_2 = -\frac{1}{2r} \left[ 1 - \frac{3}{2r} + 12 \left( \frac{1}{2r} \right)^2 \right]$$

$$\Phi_3 = -1 + \left( 1 - \frac{1}{r} \right)^{\frac{1}{2}}, \quad \Phi_4 = \frac{1}{2} \ln \left( 1 - \frac{1}{r} \right)$$

(10)

$\Phi_1$ is proposed by Paczyński and Wiita (1980), $\Phi_2$ by Nowak and Wagoner (1991), $\Phi_3$ and $\Phi_4$ by Artemova, Björnsson & Novikov (1996). Introduction of such potentials allows one to investigate the complicated physical processes taking place in accretion in a semi-Newtonian framework by avoiding pure general relativistic calculations. Most of the features of space-time around a compact object are retained and some crucial properties of the analogous relativistic accretion solutions could be reproduced with high accuracy. Hence, those potentials might be designated as ‘pseudo-Schwarzschild’ potentials, see Das & Sarkar 2001 and Das 2002 for details.

The continuity equation can be written as

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( \rho ur^2 \right) = 0$$

(11)

For a polytropic equation of state, the steady state solutions of eqn. (9) and eqn. (11) provides:

1) Conservation of specific energy $E$ of the flow:

$$E = \frac{u^2}{2} + \frac{a_s^2}{\gamma - 1} + \Phi$$

(12)

where $\gamma (=c_p/c_v)$ is the adiabatic index of the accreting material.

2) Conservation of Baryon number (or accretion rate $\dot{M}$):

$$\dot{M} = 4\pi \rho ur^2$$

(13)

Defining the entropy accretion rate $\Xi = \dot{M} \gamma^{\frac{1}{\gamma - 1}} K^{\frac{1}{\gamma - 1}}$, we have:

$$\Xi = 4\pi a_s^{\frac{1}{\gamma - 1}} ur^2$$

(14)

The space gradient of dynamical velocity is obtained as:

$$\left( \frac{du}{dr} \right) = \frac{2a_s^2}{r} - \Phi' \left\{ \frac{r_h}{r} \right\}$$

(15)

where $\Phi'$ is equal to $d\Phi/dr$. Since the flow is assumed to be smooth everywhere, if the denominator of eqn. (15) vanishes at any radial distance $r$, the numerator must also vanish there to maintain the continuity of the flow. One thus calculates the sonic point quantities as

$$u|_{(r=r_h)} = a_s|_{(r=r_h)} = \sqrt{\frac{r_h \Phi'}{2}}$$

(16)

where $r_h$ is the sonic point, a family of which forms the spherical acoustic horizon. The location of the acoustic horizon, i.e., $r_h$, can be obtained by algebraically solving the following equation:

$$E - \frac{1}{2} \left( \frac{\gamma + 1}{\gamma - 1} \right) r_h \Phi' \Bigg|_{(r=r_h)} - \Phi \Bigg|_{(r=r_h)} = 0$$

(17)
at its corresponding sonic point can be obtained by solving the following quadratic equation:

\[
(1 + \gamma) \left( \frac{du}{dr} \right)_{(r=r_h)}^2 + 2.829 (\gamma - 1) \sqrt{\Phi' \bigg|_{(r=r_h)} \left( \frac{du}{dr} \right)_{(r=r_h)} - \Phi'' \bigg|_{(r=r_h)}} + (2\gamma - 1) \left( \frac{du}{dr} \right)_{(r=r_h)} + \Phi' \bigg|_{(r=r_h)} = 0 \tag{18}
\]

From the above discussion it is evident that for completely Newtonian flow \((\Phi = -\frac{1}{r})\), as well as for semi-Newtonian flow, one obtains the exact location of the transonic point, which is the location of the acoustic horizon because of the stationarity approximation. One can also calculate the exact value of \([u, a_s, du/dr, da_s/dr]\) at the acoustic horizon. From eq. (8), it is obvious that all the required quantities to calculate the value of analogue Hawking temperature may be obtained from first principle from the calculations presented above, if the gravitational potential is purely Newtonian in nature. Hence, Newtonian spherical accretion onto astrophysical black holes is the example of analogue system where the exact value of \(T_{AH}\) can be calculated using only two measurable parameters, namely, \(E\) and \(\gamma\). Also to be noted that the above formulation was carried out using position dependent sound speed. Unruh’s original calculation of \(T_{AH}\) (see eq. (4)) assumes the positional constancy of sound speed, which may not always be a real physical situation. This was modified to include the position dependent sound speed by Visser (1998) for flat space and by Bilić (1999) for curved space. However, none of these works provides the exact calculation of \(r_h\) and acoustic horizon related quantities from first principles, and hence the numerical value of \(T_{AH}\) in any existing literature (even for flat space, let alone curved space) was obtained using approximate, order of magnitude, calculations only.

The most important point to mention in this context is that, the accreting astrophysical black holes are the only real physical candidates for which both the horizons, actual (electromagnetic) and analogue (sonic), may exist together simultaneously. Hence our application of analogue Hawking radiation to the theory of transonic astrophysical accretion may be useful to compare some properties of these two kind of horizons, by theoretically studying and comparing the behaviour of the same flow close to these horizons. Spherically accreting black hole system is, therefore, not only a concrete example of analogue system naturally found in the Universe, it is also most important among all the existing analogue models, because study of black hole accretion as black hole analogue may bridge the gap between two apparently different school of thoughts, the theory of transonic astrophysical accretion, and the theory of analogue Hawking radiation, sharing intrinsically the same physical origin.

If \(\Phi\) represents any of the pseudo-Schwarzschild potentials defined by eq. (10), the analogue Hawking temperature can still be determined from the knowledge of eq. (16 - 18). One only needs to modify eq. (8) by incorporating such potentials instead of the purely Newtonian one, while calculating the acoustic metric and other related quantities. It is not difficult to show that (Das & Dasgupta 2004, in preparation) for purely Newtonian as well as for semi-Newtonian black hole potentials, \(T_{AH}\) can never exceed \(T_H\), which is not true for general relativistic accretion, as we will see in §10.

9 General relativistic black hole accretion

In this section, we will describe fully general relativistic astrophysical accretion, both spherically symmetric and axisymmetric, onto astrophysical black holes. The mass of the accreting fluid is assumed to be much less compared to \(M_{BH}\) (which is usually reality for astrophysical black hole accretion), so that the gravity field is controlled essentially by \(M_{BH}\) only. Calculations presented in the following sections are mostly adopted from Das 2004, Das 2004a and BDW.
9.1 Mono-transonic spherical accretion

Accretion is \( \theta \) and \( \phi \) symmetric and possesses only radial inflow velocity. Stationary solutions are considered. Accretion is governed by the radial part of the general relativistic time independent Euler and continuity equations in Schwarzschild metric. The conserved specific flow energy \( \mathcal{E} \) (the relativistic analogue of Bernoulli’s constant) along each stream line reads \( \mathcal{E} = h u_t \) (Anderson 1989) where \( h \) and \( u_\mu \) are the specific enthalpy and the four velocity, which can be re-cast in terms of the radial three velocity \( u \) and the polytropic sound speed \( a_s \) to obtain:

\[
\mathcal{E} = \left[ \frac{\gamma - 1}{\gamma - (1 + a_s^2)} \right] \frac{1 - \frac{r}{r_h}}{1 - u^2}, \tag{19}
\]

We concentrate on positive Bernoulli constant solutions. The mass accretion rate \( \dot{M} \) may be obtained by integrating the continuity equation:

\[
\dot{M} = 4\pi \rho u r^2 \sqrt{\frac{r - 1}{r (1 - u^2)}}, \tag{20}
\]

where \( \rho \) is the proper mass density. A polytropic equation of state, \( p = K \rho^\gamma \), is assumed (this is common in astrophysics to describe relativistic accretion) to define \( \Xi \) as:

\[
\Xi = K \frac{1}{\gamma - 1} \dot{M} = 4\pi \rho u r^2 \sqrt{\frac{r - 1}{r (1 - u^2)}} \frac{a_s^2 (\gamma - 1)}{\gamma - (1 + a_s^2)}. \tag{21}
\]

Simultaneous solution of eq. (19 - 21) provides the dynamical three velocity gradient at any radial distance \( r \):

\[
\frac{d u}{d r} = \frac{u (1 - u^2) [a_s^2 (4r - 3) - 1]}{2r (r - 1) (u^2 - a_s^2)} \tag{22}
\]

The sonic point conditions are obtained as:

\[
u_{|r=r_h} = a_s_{|r=r_h} = \sqrt{\frac{1}{4r_h - 3}}. \tag{23}\]

Substitution of \( u_{|r=r_h} \) and \( a_s_{|r=r_h} \) into eq. (19) for \( r = r_h \) provides:

\[
r_h^3 + r_h^2 \Gamma_1 + r_h \Gamma_2 + \Gamma_3 = 0, \tag{24}\]

where

\[
\Gamma_1 = \left[ \frac{2\mathcal{E}^2 (2 - 3\gamma) + 9 (\gamma - 1)}{4 (\gamma - 1) (\mathcal{E}^2 - 1)} \right], \quad \Gamma_2 = \left[ \frac{\mathcal{E}^2 (3\gamma - 2)^2 - 27 (\gamma - 1)^2}{32 (\mathcal{E}^2 - 1) (\gamma - 1)^2} \right], \quad \Gamma_3 = \frac{27}{64 (\mathcal{E}^2 - 1)}. \tag{25}\]

Solution of eq. (24) provides the location of the acoustic horizon in terms of only two accretion parameters \( \{\mathcal{E}, \gamma\} \), which is the two parameter input set to study the flow. Eq. (24) can be solved completely analytically by employing the Cardano-Tartaglia-del Ferro technique. One defines:

\[
\Sigma_1 = \frac{9 \Gamma_2 - \Gamma_1^2}{9}, \quad \Sigma_2 = \frac{9 \Gamma_1 \Gamma_2 - 27 \Gamma_3 - 2 \Gamma_1^3}{54}, \quad \Psi = \Sigma_1^3 + \Sigma_2^2, \quad \Theta = \cos^{-1} \left( \frac{\Sigma_2}{\sqrt{-\Sigma_1^3}} \right), \quad \Omega_1 = \frac{3}{\sqrt{\Sigma_2 + \sqrt{\Sigma_2^2 + \Sigma_1^3}}}, \tag{26}
\]

10
\[ \Omega_2 = \sqrt[3]{\Sigma_2 - \sqrt{\Sigma_2^2 + \Sigma_1^3}}, \quad \Omega_\pm = (\Omega_1 \pm \Omega_2) \]  

so that the three roots for \( r_h \) come out to be:

\[ \begin{align*}
1_{r_h} &= -\frac{\Gamma_1}{3} + \Omega_+, \quad 2_{r_h} = -\frac{\Gamma_1}{3} - \frac{1}{2} \left( \Omega_+ - i\sqrt{3\Omega_-} \right), \\
3_{r_h} &= -\frac{\Gamma_1}{3} - \frac{1}{2} \left( \Omega_- - i\sqrt{3\Omega_-} \right). 
\end{align*} \]  

However, note that not all \( \psi_1 \{i = 1, 2, 3\} \) would be real for all \( \{E, \gamma\} \). It is easy to show that if \( \Psi > 0 \), only one root is real; if \( \Psi = 0 \), all roots are real and at least two of them are identical; and if \( \Psi < 0 \), all roots are real and distinct. Selection of the real physical \( (r_h \) has to be greater than unity) roots requires the following discussion.

Although one can analytically calculate \( r_h \) and other variables at \( r_h \), there is no way that one can analytically calculate the flow variables at any arbitrary \( r \). One needs to integrate eq. (24) numerically to obtain the transonic profile of the flow for all range of \( r \), starting from infinity and ending on to the actual event horizon. To do so, one must set the appropriate limits on \( \{E, \gamma\} \) to model the realistic situations encountered in astrophysics. As \( E \) is scaled in terms of the rest mass energy and includes the rest mass energy, \( E < 1 \) corresponds to the negative energy accretion state where radiative extraction of rest mass energy from the fluid is required. For such extraction to be made possible, the accreting fluid has to possess viscosity or other dissipative mechanisms, which may violate the Lorenzian invariance. On the other hand, although almost any \( E > 1 \) is mathematically allowed, large values of \( E \) represents flows starting from infinity with extremely high thermal energy, and \( E > 2 \) accretion represents enormously hot flow configurations at very large distance from the black hole, which are not properly conceivable in realistic astrophysical situations. Hence one sets \( 1 \lesssim E \lesssim 2 \). Now, \( \gamma = 1 \) corresponds to isothermal accretion where accreting fluid remains optically thin. This is the physical lower limit for \( \gamma \); \( \gamma < 1 \) is not realistic in accretion astrophysics. On the other hand, \( \gamma > 2 \) is possible only for superdense matter with substantially large magnetic field (which requires the accreting material to be governed by general relativistic magneto-hydrodynamic equations, dealing with which is beyond the scope of this article) and direction dependent anisotropic pressure. One thus sets \( 1 \lesssim \gamma \lesssim 2 \) as well, so \( \{E, \gamma\} \) has the boundaries \( 1 \lesssim \{E, \gamma\} \lesssim 2 \). However, one should note that the most preferred values of \( \gamma \) for realistic black hole accretion ranges from \( 4/3 \) to \( 5/3 \) (Frank et al. 1992).

Coming back to the solution for \( r_h \), one finds that for the preferred range of \( \{E, \gamma\} \), one always obtains \( \Psi < 0 \). Hence the roots are always real and three real unequal roots can be computed as:

\[ \begin{align*}
1_{r_h} &= 2\sqrt{-\Sigma_1\cos \left( \frac{\Theta}{3} \right)} - \frac{\Gamma_1}{3}, \\
2_{r_h} &= 2\sqrt{-\Sigma_1\cos \left( \frac{\Theta + 2\pi}{3} \right)} - \frac{\Gamma_1}{3}, \\
3_{r_h} &= 2\sqrt{-\Sigma_1\cos \left( \frac{\Theta + 4\pi}{3} \right)} - \frac{\Gamma_1}{3}.
\end{align*} \]  

One finds that for all \( 1 \lesssim \{E, \gamma\} \lesssim 2 \), \( 3_{r_h} \) becomes negative. It is observed that \( \{r_h, 3_{r_h}\} > 1 \) for most values of the astrophysically tuned \( \{E, \gamma\} \). However, it is also found that \( 3_{r_h} \) does not allow steady physical flows to pass through it; either \( u_s \) or \( a_s \), or both, becomes superluminal before the flow reaches the actual event horizon, or the Mach number profile shows intrinsic fluctuations for \( r < r_h \). This information is obtained by numerically integrating the complete flow profile passing through \( 3_{r_h} \). Hence it turns out that one needs to concentrate only on \( 1_{r_h} \) for realistic astrophysical black hole accretion. Both large \( E \) and large \( \gamma \) enhance the thermal energy of the flow so that the accreting fluid acquires the large radial velocity to overcome \( a_s \) only in the close vicinity of the black hole. Hence \( r_h \).
anti-correlates with \{E, \gamma\}. To obtain \(\frac{dv}{dr}\) and \(\frac{da_s}{dr}\) on the acoustic horizon, L’ Hospital’s rule is applied to eq. (22) to have:

\[
\left(\frac{du}{dr}\right)_{r=r_h} = \Phi_{12} - \Phi_{123},
\]

\[
\left(\frac{da_s}{dr}\right)_{r=r_h} = \Phi_4 \left(\frac{1}{\sqrt{4r_h - 3}} + \frac{\Phi_{12} - \Phi_{123}}{2}\right),
\]

where

\[
\Phi_{12} = -\Phi_2/2\Phi_1, \quad \Phi_{123} = \sqrt{\Phi_2^2 - 4\Phi_1 \Phi_3}/2\Phi_1,
\]

\[
\Phi_1 = \frac{6r_h (r_h - 1)}{\sqrt{4r_h - 3}}, \quad \Phi_2 = \frac{2}{4r_h - 3} [4r_h (\gamma - 1) - (3\gamma - 2)]
\]

\[
\Phi_3 = \frac{8 (r_h - 1)}{(4r_h - 3)^2} [r_h^2 (\gamma - 1)^2 - r_h (10\gamma^2 - 19\gamma + 9)] + (6\gamma^2 - 11\gamma + 3)], \quad \Phi_4 = \frac{2 (2r_h - 1) - \gamma (4r_h - 3)}{4 (r_h - 1)}
\]

\[
(30)
\]

9.2 Multi-transonic disc accretion

In this section, we follow the treatment provided by Das 2004 and BDW because only in those works a consistent methodology has been developed which can handle the transonic solutions upto the extreme close vicinity of the black hole event horizon (down to a distance upto around 1.001r_g). The Boyer–Lindquist coordinates with signature +++ is used, and an azimuthally Lorentz boosted orthonormal tetrad basis co-rotating with the accreting fluid. \(\lambda\) is defined to be the specific angular momentum of the flow and any gravo-magneto-viscous non-alignment between \(\lambda\) and the Kerr parameter \(a\), is neglected. Stationary axisymmetric solution of the following equations is considered:

\[
\nabla^\mu \mathcal{E}^{\mu\nu} = 0, \quad (\rho \nu^\mu)_{;\mu} = 0,
\]

where \(\mathcal{E}^{\mu\nu}\), \(v^\mu\), and \(\nu\) are the energy momentum tensor, the four velocity and the rest mass density of the accreting fluid, respectively. The semicolon denotes the covariant derivative. The ‘stationarity’ condition implies the vanishing of the temporal derivative of any scalar field in the disc or the vanishing of the Lie Derivative of any vector or tensor field along the Killing vector \(\partial_\mu\). This implies that \(\partial_\mu \mathcal{E}^{\mu\nu} = 0\) if \(\mathcal{E}^{\mu\nu}\) is a scalar field, and \(L_{\partial_\mu} \mathcal{E}^{\mu\nu} = 0\) if \(\mathcal{E}^{\mu\nu}\) represents a vector or a tensor field. The ‘axisymmetry’ property is endowed with the space like Killing field \(\left(\frac{\partial}{\partial \phi}\right)^\mu\).

The radial momentum balance condition may be obtained from \((v^\mu v_\mu + g^\mu_\nu) \mathcal{E}^{\mu\nu} = 0\). Exact solutions of these conservation equations require knowledge of the accretion geometry and the introduction of a suitable equation of state. Polytropic accretion is considered for which \(p = K \rho^\gamma\), where \(K\) and \(\gamma\) are the monotonic and continuous functions of the specific entropy density, and the constant adiabatic index of the flow, respectively. The specific proper flow enthalpy is taken to be

\[
h = (\gamma - 1) \left(\gamma - (1 + a_s^2)^{-1}\right),
\]

where \(a_s\) is the polytropic sound speed defined as \(a_s = (\partial p/\partial \epsilon)_S^{1/2} = \Psi_1(T(r), \gamma) = \Psi_2(p, \rho, \gamma)\); here \(T(r)\) is the local flow temperature, \(\epsilon\) is the mass-energy density, and \{\(\Psi_1, \Psi_2\)\} are known functions. The subscript \(S\) indicates that the derivative is taken at constant specific entropy. Hence, \(a_s^2(r) = \gamma \kappa T(r)/(\mu m_H) = \Theta^2 T(r)\), where \(\Theta = [\gamma \kappa/(\mu m_H)]^{1/2}\), \(\mu\) is the mean molecular weight, \(m_H\) is the mass of the hydrogen atom, and \(\kappa\) is Boltzmann’s constant.

The temporal component of the first part of eq. (31) leads to the conservation of specific flow energy \(E\). The Kerr metric in the equatorial plane of the black hole may be written as (e.g., Novikov & Thorne 1973)

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\frac{r^2 \Delta}{A} dt^2 + \frac{A}{r^2} (d\phi - \omega dt)^2 + \frac{r^2}{\Delta} dr^2 + dz^2,
\]

(32)
where \( \Delta = r^2 - 2ar + a^2 \), \( A = r^4 + r^2a^2 + 2ra^2 \), and \( \omega = 2ar/A \). The angular velocity \( \Omega \) is

\[
\Omega = -\frac{(g_{t\phi} + \lambda g_{tt})}{(g_{\phi\phi} + \lambda g_{t\phi})} = \frac{4a}{r} - \frac{\lambda (4a^2 - r^2 \Delta)}{A} \left( \frac{A}{r^2} - \frac{4\lambda r}{r} \right).
\]  (33)

The normalization relation \( v_\mu v^\mu = -1 \), along with the above value of \( \Omega \), provides:

\[
v_t = \left[ \frac{A r^2 \Delta}{(1 - u^2) \{ A^2 - 8\lambda r A + \lambda^2 r^2 (4a^2 - r^2 \Delta) \}} \right]^{\frac{1}{2}},
\]  (34)

where \( u \) is the radial three velocity in the co-rotating fluid frame. Hence the conserved specific energy (which includes the rest mass energy) can be re-written as:

\[
E = \frac{(\gamma - 1)}{[\gamma - (1 + \Theta^2 T)]} \left[ \frac{A r^2 \Delta}{(1 - u^2) \{ A^2 - 8\lambda r A + \lambda^2 r^2 (4a^2 - r^2 \Delta) \}} \right]^{\frac{1}{2}}.
\]  (35)

For a suitably chosen disc geometry (see BDW for details about the disk structure in Kerr metric, also see Das 2004, for the details of relativistic accretion in a different disc geometry in Schwarzschild metric) The mass and entropy accretion rate comes out to be:

\[
\dot{M} = 4\pi r \Theta r \sqrt{T} M r^2 \left[ \frac{2\Delta (\gamma - 1) \Theta^2 T}{\gamma (1 - \Theta^2 TM^2) \{ \gamma - (1 + \Theta^2 T) \}} \right]^{\frac{1}{2}};
\]  (36)

\[
\dot{\Xi} = 4\sqrt{2} \pi r^2 u \sqrt{\frac{\Delta}{(1 - u^2) \psi}} \left[ \frac{a^2_s (\gamma - 1)}{\gamma \{ \gamma - (1 + a^2) \}} \right]^{\frac{1}{2}}.
\]  (37)

The velocity gradient can be calculated as:

\[
\frac{du}{dr} = \frac{2a^2}{(\gamma + 1)} \left[ \frac{r - 1}{\Delta} + \frac{2}{\gamma} - \frac{u \sigma}{\psi} \right] - \frac{\chi}{2},
\]  (38)

where

\[
\sigma = 2\lambda v_t - a^2, \quad \chi = \frac{1}{\Delta} \frac{d\Delta}{dr} + \frac{\lambda}{(1 - \Omega \lambda)} \frac{d\Omega}{dr} - \frac{(g_{\phi\phi} / dr + \lambda g_{t\phi} / dr)}{(g_{\phi\phi} + \lambda g_{t\phi})}.
\]  (39)

The sonic point conditions can be obtained as:

\[
a_s (r = r_h) = \left[ \frac{u^2 (\gamma + 1) \psi}{2\psi - u^2 v_t \sigma} \right] (r = r_h)^{\frac{1}{2}}, \quad u (r = r_h) = \left[ \frac{\chi \Delta r}{2r (r - 1) + 4\Delta} \right] (r = r_h)^{\frac{1}{2}},
\]  (40)

For any value of \([E, \lambda, \gamma, \sigma]\), substitution of the values of \(u (r = r_h)\) and \(a_s (r = r_h)\) in terms of \(r_h\) in the expression for \(E\) (see eq. (35)), provides a polynomial in \(r_h\), the solution of which determines the location of the sonic point(s) \(r_h\). After lengthy algebraic manipulations, the following quadratic equation is formed, which can be solved to obtain \((du/dr) (r = r_h)\):

\[
\alpha (\frac{du}{dr})^2 (r = r_h) + \beta (\frac{du}{dr}) (r = r_h) + \zeta = 0,
\]  (41)

where the coefficients are
Putting the Kerr parameter equals to zero in the corresponding equations above, one can obtain the value of sonic point and

1) The sonic point is obtained by solving the following equation:

\[ \frac{2}{u} + \frac{w_v \delta_3}{1 - u^2} = \frac{3u^2 - 1}{u(1 - u^2)} - \frac{u(\gamma - 1 - a_s^2)}{a_s^2(1 - u^2)}, \]

\[ \delta_6 = \frac{(\gamma - 1 - a_s^2)\chi}{2a_s^2} + \frac{\delta_2 \delta_3 \chi v_t}{2(1 - \delta_2)}, \]

\[ \tau_1 = \frac{r - 1}{\Delta} \frac{2}{r} - \frac{\sigma v_t \chi}{4\psi}, \tau_2 = \frac{(4\lambda^2 v_t - a_s^2)\psi - v_t \sigma^2}{\sigma \psi}, \]

\[ \tau_3 = \frac{\sigma \tau_2 \chi}{4\psi}, \tau_4 = \frac{1}{\Delta} - \frac{2(r - 1)^2}{\Delta^2} - \frac{2}{r^2} - \frac{v_t \sigma d\chi}{4\psi \, dr}, \]

\[ \tau_5 = \frac{2}{\gamma + 1} \left[ a_s^2 \tau_4 - \left\{ (\gamma - 1 - a_s^2) \tau_1 + v_t a_s^2 \tau_3 \right\} \frac{\chi}{2} \right] - \frac{1}{2} \frac{d\chi}{dr}, \]

\[ \tau_6 = \frac{2v_t u}{(\gamma + 1)(1 - u^2)} \left[ \frac{\tau_1}{v_t} \left( \gamma - 1 - a_s^2 \right) + a_s^2 \tau_3 \right]. \] (42)

Note that all the above quantities are evaluated at \( r_h \).

Putting the Kerr parameter equals to zero in the corresponding equations above, one can obtain the value of sonic point and \([u, a_s, du/dr, da_s/dr]\) at the sonic point, for accretion onto Schwarzschild black holes.

Adopting a different disc geometry, the corresponding quantities in Schwarzschild metric can also directly be computed as the following (Das 2004):

1) The sonic point is obtained by solving the following equation:

\[ E^2 \left[ r_h^3 + \lambda^2 (1 - r_h) \right] - \frac{r_h - 1}{1 - \Psi(r_h, \lambda)} \left[ \frac{r_h(\gamma - 1)}{\gamma - \eta(r_h, \lambda)} \right]^2 = 0 \] (43)

where

\[ \eta(r_h, \lambda) = \left[ 1 + \frac{\gamma + 1}{2} \Psi(r_h, \lambda) \right] \]

and

\[ \Psi(r_h, \lambda) = \left[ \frac{1}{2r_h} \frac{2r_h^3 - \lambda^2}{r_h^2 + \lambda^2 (1 - r_h)} - \frac{2r_h - 1}{2r_h (r_h - 1)} \right] \]

\[ - \frac{1 - 2r_h}{2(r_h - 1)} + \frac{\lambda^2 - 3r_h^2}{2(r_h - 1)} \right] \]

2) The sonic velocities can be found as:

\[ u(r = r_h) = \sqrt{\frac{2}{\gamma + 1} a_s(r = r_h)} = \sqrt{\left[ \frac{1}{2r_h} \frac{2r_h^3 - \lambda^2}{r_h^2 + \lambda^2 (1 - r_h)} - \frac{2r_h - 1}{2r_h (r_h - 1)} \right]} \]

\[ \frac{1 - 2r_h}{2(r_h - 1)} + \frac{\lambda^2 - 3r_h^2}{2(r_h - 1)} \right] (r = r_h) \] (44)

3) The velocity gradient can be obtained by solving the following equation:

\[ \frac{2(2\gamma - 3a_s^2)}{(\gamma + 1)(u_h^2 - 1)^2} \left( \frac{du}{dr} \right)_s^2 + 4\xi(r_h, \lambda) \left[ \frac{\gamma - (1 + a_h^2)}{u_h^2 - 1} \right] \left( \frac{du}{dr} \right)_h \]
\[ + \frac{2}{\gamma + 1} a_s^2 \xi (r_h, \lambda) \left[ 2 \xi (r_h, \lambda) \left[ \frac{\gamma - (1 + a_s^2)}{\gamma + 1} \right] - \frac{2r_h - 1}{r_h (r_h - 1)} - \frac{3r_h^2 - \lambda^2}{r_h^2 (1 - r_h)} \right] + \frac{20r_h^3 - 12r_h^2 - 2\lambda^2 (3r_h - 2)}{5r_h^4 - 4r_h^3 - \lambda^2 (3r_h^2 - 4r_h + 1)} = 0 \] (45)

where

\[ \xi (r_h, \lambda) = \left[ \frac{1 - 2r}{2r (r - 1)} + \frac{\lambda^2 - 3r^2}{2[r^3 + \lambda^2 (1 - r)]} \right] \]

In the above two equations, \( u_h \) and \( a_h \) indicates the value of \( u \) and \( a_s \) on the acoustic horizon (on \( r = r_h \)), and the subscript \( h \) in general indicates that the quantities are measured on the acoustic horizon. The relation between the dynamical velocity gradient and acoustic velocity gradient at any point (including at the sonic point) can be obtained as:

\[ \frac{d a_s (r)}{d r} = \frac{a_s}{\gamma + 1} \left\{ \gamma - (1 + a_s^2) \right\} \left[ \frac{1}{u (u^2 - 1)} \frac{du}{dr} + f (r, \lambda) \right] \] (46)

where:

\[ f (r, \lambda) = \frac{1 - 2r}{2r (r - 1)} + \frac{\lambda^2 - 3r^2}{2[r^3 + \lambda^2 r + \lambda^2]} \] (47)

Note that unlike spherical accretion, one finds three sonic points in disc accretion for some values of \([E, \lambda, \gamma]\) for accretion onto the Schwarzschild as well as the Kerr black hole, among which the largest and the smallest values correspond to the \( X \) type outer, \( r_o \), and inner, \( r_i \), sonic points, respectively. The \( O \) type middle sonic point, \( r_m \), which is unphysical in the sense that no steady transonic solution passes through it, lies between \( r_i \) and \( r_o \). If the accretion through \( r_o \) can be perturbed in a way so that it produces an amount of entropy exactly equal to \([\Xi (r_i) - \Xi (r_o)]\), supersonic flow through \( r_o \) can join the subsonic flow through \( r_i \) by developing a standing shock. The exact location of such a shock, as well as the details of the post-shock flow, may be obtained by formulating and solving the general relativistic Rankine-Hugoniot condition for the flow geometry described above. Note that as a flow through \( r_i \) does not connect the black hole event horizon with infinity (such a flow folds back onto itself, see, e.g., Fig. 1 of BDW), solutions through \( r_i \) do not have independent physical existence, and can only be accessed if the supersonic flow through \( r_o \) undergoes a shock, so that it generates extra entropy, becomes subsonic, and produces the physical segment of the solution through \( r_i \).

10 Analogue Hawking temperature for general relativistic accretion

10.1 Relativistic sonic geometry

As mentioned in §8, it is not quite difficult to calculate \( T_{AH} \) for Newtonian, and also, perhaps, for semi-Newtonian black hole accretion. For general relativistic flow, however, non-trivial effort has to be made to accomplish such goals. This is because, treatments in Unruh (1981) and Visser (1998), and in related works, are based on the distinct fact that the background fluid is governed by Newtonian spacetime, whereas for general relativistic accretion presented in §9.1 & 9.2 in this paper, the background fluid is characterized by purely Schwarzschild metric. Hence a relativistic version of acoustic geometry is necessary to calculate \( T_{AH} \) for this purpose. The work by Bilic (1999) is the first (and perhaps, the only one in the literature as far as our knowledge is concerned) which calculates the generalized expression for acoustic analogue of surface gravity for propagation of relativistic transonic fluid. The relativistic acoustic metric \( G_{\mu \nu} \) comes out to be (Moncrief 1980, Bilic 1999):

\[ G_{\mu \nu} = N \frac{h a_s}{h a_s} \left[ g_{\mu \nu} - \left( 1 - a_s^2 \right) u_\mu u_\nu \right] \] (48)
where $h$ is the relativistic enthalpy density and $N$ is the particle number density. The corresponding surface gravity may be calculated as:

$$\kappa = \frac{\sqrt{\xi^\mu \xi_\mu}}{1 - a_s^2} \left. \frac{\partial}{\partial n} (u - a_s) \right|_{\text{acoustic horizon}}$$  \hspace{1cm} (49)$$

Where $\xi^\mu$ is the stationary Killing field and $\partial/\partial n$ is the normal derivative.

### 10.2 Analogue temperature for spherical accretion

For spherically symmetric general relativistic flow onto Schwarzschild black holes described in §9.1, one can evaluate the exact value of the Killing fields and Killing vectors to calculate the surface gravity for that geometry. The analogue Hawking temperature for such geometry comes out to be (Das 2004a):

$$T_{AH} = \frac{\hbar c^3}{4\pi k_B G M_{BH}} \left[ \frac{r_h^2 (r_h - 0.75)}{(r_h - 1)^\frac{3}{2}} \right] \left| \frac{d}{dr} (a_s - u) \right|_{r=r_h},$$  \hspace{1cm} (50)$$

where the values of $r_h$, $(du/dr)_h$ and $(da_s/dr)_h$ are obtained using the system of units and scaling used in this article.

It is evident from the above formula that the exact value of $T_{AH}$ can be analytically calculated from the results obtained in §9.1. While eq. (27) provides the location of the acoustic horizon ($r_h$), the value of $\left| \frac{d}{dr} (a - u) \right|_{r=r_h}$ is obtained from eq. (29) as a function of $\varepsilon$ and $\gamma$, both of which are real, physical, measurable quantities. Note again, that, since $r_h$ and other quantities appearing in eq. (50) are analytically calculated as a function of $\{\varepsilon, \gamma\}$, eq. (50) provides an exact analytical value of the general relativistic analogue Hawking temperature for all possible solutions of an spherically accreting astrophysical black hole system, something which has never been done in the literature before. If $\sqrt{3r_h - 3(1/2 - 1/\Phi_4)(\Phi_{12} - \Phi_{123})} > 1$, one always obtains $(da_s/dr < du/dr)_h$ from eq. (29), which indicates the presence of the acoustic white holes at $r_h$. This inequality holds good for certain astrophysically relevant range of $\{\varepsilon, \gamma\}$; see following discussions.

For a particular value of $\{\varepsilon, \gamma\}$, one can define the quantity $\tau$ to be the ratio of $T_{AH}$ and $T_H$ as:

$$\tau = T_{AH}/T_H.$$  \hspace{1cm} (51)$$

This ratio $\tau$ comes out to be independent of the mass of the black hole, which enables us to compare the properties of two kind of horizons (actual and acoustic) for an spherically accreting black hole with any mass, starting from the primordial holes to the super massive black holes at galactic centers.

Note, however, that the analogue has been applied to describe the classical perturbation of the fluid in terms of a field satisfying the wave equation on a curved effective geometry. Main motivation of the methodology described in this section is not a rigorous demonstration of how the phonon field generated in this system could be made quantized. To accomplish that task, one can show that the effective action for the acoustic perturbation is equivalent to the field theoretical action in curved space, and the corresponding commutation and dispersion relations (see, e.g., Unruh & Schützhold 2003) may directly follow from there.

In the accompanying figure (taken from Das 2004a), the complete $\{\varepsilon, \gamma\}$space is classified according to the value of $\tau$. For accretion with $\{\varepsilon, \gamma\}$taken from the lightly shaded regions marked by $0 < T_{AH} < T_H$, the Hawking temperature dominates over its analogue counterpart. For high $\gamma$, high $\varepsilon$ flow (‘hot’ accretion), analogue Hawking radiation becomes the dominant process compared to the actual Hawking radiation ($T_{AH} > T_H$). For low $\varepsilon$ and intermediate/high $\gamma$, acoustic white holes...
Figure 1: Parameter space classification for \( \{E, \gamma\} \) for four different sets of values of \( T_{AH} \).

appear (white region marked by WH, where \( da_s/dr < du/dr \) at \( r_h \)). For \( \{E, \gamma\} \) belonging to this region, one obtains outflow (outgoing solutions with Mach number increasing with increase of \( r \)) only. This has also been verified by obtaining the complete flow profile by integrating eq. (24). The dark shaded region (lower right corner of the figure) represents \( \{E, \gamma\} \) for which \( r_h \) comes out to be physical (\( r_h > 1 \)) but \( \Phi_{123}^2 \) becomes negative, hence \( (du/dr)_h \) and \( (da_s/dr)_h \) are not real and \( T_{AH} \) becomes imaginary. Note that both \( T_{AH} > T_H \) and white hole regions are obtained even for \( 4/3 < \gamma < 5/3 \), which is the range of values of \( \gamma \) for most realistic flows of matter around astrophysical black holes. Hence the domination of the analogue Hawking temperature over the actual Hawking temperature and the emergence of analogue white holes are real astrophysical phenomena.

Note one important point that the accreting spherical black holes in general relativity is the only analogue system found till date, where the analogue temperature may be higher than the actual Hawking temperature. Newtonian or semi-Newtonian accreting system does not show this behaviour. Also note that both \( T_H \) and \( T_{AH} \) comes out to be quite less compared to the macroscopic classical fluid temperature of accreting matter\(^4\). Hence black hole accretion system may not be a good candidate to allow any observational test for detecting the analogue radiation.

10.3 Analogue effects in accretion disc

Calculation of \( T_{AH} \) for disc accretion (as described in §9.2) is also possible. One needs to calculate the acoustic metric and the Killing vectors for axisymmetric space time. Then one needs to use those results to calculate the surface gravity and analogue temperature. Initial attempts (Das & Bilic, 2004, in preparation) shows that although the geometry would be different (axisymmetric) than the spherical case, \( T_{AH} \) will still be a function of \([u_s, du/dr, da_s/dr]\) at the sonic point, with different functional form as that of eq. (50). Hence for disc accretion also, the exact value of \( T_{AH} \) can directly be calculated from the results obtained in §9.2. However, as we discussed before, for axisymmetric accretion, more than one sonic point is formed, hence the question now may arise that which sonic points are to be taken, to form the acoustic horizon. The answer seems to be quite straight forward. Firstly, all the middle sonic points would be excluded because no stable solutions pass through that. Secondly, as accretion can pass through the inner sonic point only if a shock forms (see §9.2), one should exclude the inner sonic points as well. This is because, shock production is dissipative and may violate the Lorenzian invariance. Hence, if the accretion is multi-transonic, which is the situation for certain values of \([E, \lambda, \gamma, a]\), only the flow passing through the outer sonic point will resemble an analogue

\(^4\)The value of the flow temperature for non-radiative spherical accretion is found to be of the order of \(10^{10} - 10^{11} \text{K} \).
system. For other values of $[\mathcal{E}, \lambda, \gamma, a]$, disc accretion is mono-transonic and hence there will not be any confusion about the choice of the sonic points for producing the acoustic horizon.

However, although an observer at positive infinity will see only one acoustic black hole, there may be one alternative possibility for an observer situated between the two sonic points. An observer in between two sonic points (the inner sonic point and the outer sonic point) may perhaps see a white hole and a black hole or two white holes (see Barcelo et. al. 2004 for discussion about somewhat similar situation). Detail study of this issue is beyond the scope of this article and will be presented elsewhere.

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