Loop Quantum Cosmology II: Volume Operators

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Abstract

Volume operators measuring the total volume of space in a loop quantum theory of cosmological models are constructed. In the case of models with rotational symmetry an investigation of the Higgs constraint imposed on the reduced connection variables is necessary, a complete solution of which is given for isotropic models; in this case the volume spectrum can be calculated explicitly. It is observed that the stronger the symmetry conditions are the smaller is the volume spectrum, which can be interpreted as level splitting due to broken symmetries. Some implications for quantum cosmology are presented.

1 Introduction

In this second part we continue the investigation of quantum symmetry reduction for cosmological models started in the first part [1]. There we presented kinematical properties: The general framework of quantum symmetry reduction [2] was specialized to transitive symmetry groups by means of which homogeneous models can be described. Furthermore, we quantized and solved the Gauß and diffeomorphism constraints for all these models. The treated models are, in order of increasing symmetry, Bianchi class A models (anisotropic), locally rotationally symmetric (LRS, [3]) models, and isotropic models. For models with a nontrivial isotropy subgroup, LRS and isotropic models here, there is a further kinematical constraint, the Higgs constraint, which emerges in the context of symmetry reduction. A complete solution of this constraint has not been given, neither in the general framework of reference [2] nor in the special cases of reference [1]. In the present paper we deal with this constraint in detail for isotropic models, in which case we present a complete solution, thereby determining all kinematical states. This task is complicated by the fact that the quantum configuration space is not a group implying that kinematical quantum states are not given by ordinary spin networks. For LRS models the treatment is analogous. All these models serve as examples for a solution of the Higgs constraint in the general framework.

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Moreover, we will use here these kinematical Hilbert spaces to quantize operators measuring the total volume of space and to investigate their spectra. As a first application of quantum symmetry reduction it has been observed in reference [4] that the area spectrum in spherically symmetric sectors of loop quantum gravity is only a small subset of the full spectrum. The huge spectrum of the non-symmetric operator was interpreted as consequence of a level splitting caused by broken spherical symmetry. The same phenomenon will be observed here for the volume spectrum. Its phenomenology is richer in this case, because we can relax symmetry conditions in steps: Starting with isotropic models we can first proceed to locally rotationally symmetric models with only one axis, followed by anisotropic but still homogeneous Bianchi models, and finally break the symmetry completely to reach the full theory. In each step a part of the maximal symmetry group is broken, and in each step the volume spectrum is enlarged by new eigenvalues and possibly a shift in the old ones. Whereas the spectrum of the full theory is very complex – the eigenvalues can be given explicitly only in simple cases [4] – the volume spectrum for isotropic models can be calculated explicitly. In view of the important role which the volume operator plays also for dynamics [6] this simplification of its spectrum, besides a geometric simplification of isotropic spin network states, shows that isotropic models can be good test models to understand the Wheeler-DeWitt equation in quantum gravity [4]. Contrary to most former treatments of quantum cosmological models this Wheeler-DeWitt equation of loop quantum cosmology is a discrete equation, not a differential equation, e.g. in the scale factor of the universe, as in minisuperspace quantizations. This is a manifestation of the discrete structure of space revealed in loop quantum gravity.

In the next section we will recall the kinematical properties of the models treated in reference [1]. In Section 3 the volume of Bianchi class A models will be quantized and compared with the volume operator of the full theory. In this case the isotropy subgroup is trivial, and therefore the Higgs constraint is empty. But in case of LRS and isotropic models we have to solve the Higgs constraint in quantum theory, which will be done in Section 4 for isotropic models in detail. The results are used to quantize the volume operator and to calculate its spectrum. For LRS models the treatment will mainly be analogous to isotropic models but not given completely in this paper. Finally, in Section 5 we present some applications, e.g. construction of weave states and some cosmological implications.

2 Bianchi, LRS and Isotropic Models

The setting for implementing a (quantum) symmetry reduction is a symmetry group $S$ acting on a principal fiber bundle $P(\Sigma, G, \pi)$ over the space manifold $\Sigma$ which is here assumed to be compact (this is only for ease of presentation, otherwise the framework has to be adapted appropriately). The structure group is $G = SU(2)$ for gravity in the real Ashtekar formulation [3, 4]. A classical symmetry reduction can be done, in the most general framework, by using the classification of invariant connections [10], which shows that for a transitive symmetry group each invariant connection can be expressed by some scalar fields (collectively called Higgs field) subject to a Higgs constraint. This constraint
is empty for a free action of the group $S$, and depends on a homomorphism $\lambda: F \to G$ (more precisely, its conjugacy class) if the isotropy subgroup $F$ (for a fixed but arbitrary base point $x_0$ in $\Sigma$) of $S$ is nontrivial. The space manifold $\Sigma$ can be identified with $S/F$ or an appropriate compactification thereof. This framework is specialized to cosmological models in reference [1], and its results will now be recalled briefly.

The models of interest are Bianchi class A models with a freely acting symmetry group, i.e. $F = \{1\}$, and LRS and isotropic models, for which the symmetry group can be written as a semidirect product $S = N \rtimes \rho F$ with the translation subgroup $N$ and the isotropy subgroup $F$. The representation $\rho: F \to \text{Aut} N$ describes how the isotropy subgroup acts on the tangent space $\mathcal{L} N$ of a point in $\Sigma$. For LRS models we have $F = U(1)$ and for isotropic models $F = SU(2)$, $\rho$ acting in both cases by rotations. An invariant connection can always be written as $A = \phi^i_j \omega^j \tau_i$, where $\tau_i = -\frac{i}{2} \sigma_i$ (using the Pauli matrices $\sigma_i$) are generators of $G = SU(2)$ and $\omega^j$ left invariant one-forms on $N$ (for Bianchi models we set $N := S$). The components $\phi^i_j$ of a linear map $\phi: \mathcal{L} N \to \mathcal{L} G$ are collectively denoted as Higgs field. For Bianchi models these components are unrestricted, whereas they are restricted by the Higgs constraint to be of the form

$$\phi^i_1 = 2^{-\frac{1}{2}} (a\Lambda^i_1 + b\Lambda^i_2) \quad , \quad \phi^i_2 = 2^{-\frac{1}{2}} (-b\Lambda^i_1 + a\Lambda^i_2) \quad , \quad \phi^i_3 = c\Lambda^i_3$$

for LRS models and $\phi^i_j = c\Lambda^i_j$ for isotropic models, respectively, with a fixed but arbitrary dreibein $\Lambda$. The dreibein $\Lambda$ depends on the homomorphism $\lambda: F \to G$ chosen in its conjugacy class. Fixing such a homomorphism and, therefore, $\Lambda$ amounts to a partial gauge fixing which will be undone in the quantum theory. Without gauge fixing $\Lambda$ is arbitrary but pure gauge.

The momenta conjugate to the connections above are invariant (with respect to the $S$-action) density-weighted dreibeine given by $E^a_i = \sqrt{g_0} p^i_a X^a_i$ in terms of left invariant vector fields $X^a_i$ obeying $\omega^j (X^a_i) = \delta^j_1$. For Bianchi models the $p^i_a$ are arbitrary and conjugate to $\phi^i_j$, whereas they are restricted to be of the form

$$p^1_i = 2^{-\frac{1}{2}} (p_a \Lambda^i_1 + p_b \Lambda^i_2) \quad , \quad p^2_i = 2^{-\frac{1}{2}} (-p_b \Lambda^i_1 + p_a \Lambda^i_2) \quad , \quad p^3_i = p_c \Lambda^i_3$$

for LRS, and $p^i_a = p\Lambda^i_a$ for isotropic models. The density weight is provided by the determinant $g_0$ of the left invariant metric on $\Sigma$ defined by $\omega^1 \wedge \omega^2 \wedge \omega^3 = \sqrt{g_0} d^3 x$.

From these momenta expressions for the volume are built as follows:

$$V = \int_\Sigma d^3 x \sqrt{\frac{1}{6} \left| \epsilon^{ijk} \epsilon_{abc} E^a_i E^b_j E^c_k \right|} = \int_\Sigma d^3 x \sqrt{\frac{1}{6} g_0^{\frac{3}{2}} \left| \epsilon^{ijk} \epsilon_{abc} p^a_i p^b_j p^c_k X^a_i X^b_j X^c_k \right|}$$

$$= \int_\Sigma d^3 x \sqrt{\frac{1}{6} g_0^{\frac{3}{2}} \left| \epsilon^{ijk} \epsilon_{IJK} p^a_i p^b_j p^c_k \det (X^a_i) \right|} = V_0 \sqrt{\frac{1}{6} \left| \epsilon^{ijk} \epsilon_{IJK} p^a_i p^b_j p^c_k \right|}$$

(1)

using $\det X^a_i = (\det \omega^i_a)^{-1} = g_0^{-\frac{1}{2}}$ and $V_0 := \int_\Sigma d^3 x \sqrt{g_0}$. For LRS models this leads to

$$V = V_0 \sqrt{\frac{1}{2} (p^2_a + p^2_b) |p_c|} ,$$

(2)
and for isotropic models to
\[ V = V_0 |p|^{\frac{3}{2}}. \] (3)

The basic ingredient for a quantum symmetry reduction \[2\] is a pull back map from the space of functions on the space of connections on \(\Sigma\), which is the auxiliary Hilbert space of the full theory, to a space of functions on the space of fields classifying invariant connections, i.e. to functions on spaces of Higgs fields. In quantum theory one uses certain extensions of the spaces of connections and Higgs fields, which can in the case of Higgs fields best be described in terms of point holonomies \[11\]. For Bianchi models there are three Higgs ‘fields’ \(\phi_I^i\), \(1 \leq I \leq 3\) in a single point \(x = 0\) (strictly speaking, they are no longer fields in a homogeneous context) leading to three point holonomies. These can be extended to ordinary holonomies by reintroducing an auxiliary manifold \(\overline{S/F}\) in which the point holonomies are written as holonomies associated with three edges \(e_I\) parallel to the invariant vector fields \(X_I\) (the auxiliary manifold should be compactified such that the edges are closed curves). Then the auxiliary Hilbert space \(H_{\text{aux}} = L^2(SU(2)^3, d\mu_H^3)\) (\(d\mu_H\) is the Haar measure on \(SU(2)\)) is spanned by spin networks associated with graphs containing three closed edges meeting in the 6-vertex \(x_0\) which is the base point chosen in \(\Sigma\). The Gauß constraint enforces gauge invariance of those spin networks, i.e. the six edge representations (each edge is incoming and outgoing) are to be contracted to the trivial representation in \(x_0\). This auxiliary Hilbert space illustrates the reduction of degrees of freedom to finitely many ones by the symmetry reduction.

Up to now all Bianchi class A models are presented on the same auxiliary Hilbert space. Differences are introduced already at the kinematical level by the diffeomorphism constraint: It enforces invariance under inner automorphisms acting on \(\overline{S/F}\), which can be interpreted as independence of choosing the base point \(x_0\) \[1\]. Inner automorphisms are certainly sensitive to the algebraic structure of the symmetry group. E.g., for the Bianchi I model with \(S = \mathbb{R}^3\) they are all trivial, and therefore the diffeomorphism constraint is empty. For Bianchi IX with \(S = SU(2)\), however, the group of inner automorphisms is isomorphic to \(SO(3)\) acting on \(\overline{S/F} = S^3\) by rotations. Therefore, all rotated spin networks are equivalent leading after group averaging to linear combinations of spin network states which are invariant under permutation of the edge spins. This reduces the number of allowed spin networks and affects the volume spectrum (but only slightly), as we will see below. These two Bianchi class A models are most interesting for our purposes, because they can be reduced further to isotropic models.

In models with a nontrivial isotropy subgroup, LRS and isotropic models, the situation is more complicated. Here we have the Higgs constraint, which is easy to solve classically, but which implies that in the quantum theory we will no longer have functions on a group \((SU(2)^3\) above) but on a certain union of conjugacy classes which is not a subgroup. For use in quantum theory the Higgs constraint can advantageously be written as
\[ h(\rho(f)(e_I)) = \exp(\lambda(f))h(e_I)\exp(-\lambda(f)) \] (4)
where \(\rho\) is the action of the isotropy subgroup \(F\) on \(N \cong S/F\) (or \(\overline{S/F}\) after compactification), and \(\lambda: F \to G\) the homomorphism introduced above. In the rotationally symmetric
models $\lambda$ will embed $F = U(1)$ as a subgroup of $G = SU(2)$ for LRS models, or be the identity for isotropic models. With $h(e_I)$ we denote the holonomy associated to the edge $e_I$ in the auxiliary manifold for a fixed Higgs field. The Higgs constraint is thus interpreted geometrically as saying that holonomies to edges which are rotated by elements of $F$ are gauge equivalent. Therefore, one would expect that this constraint can be solved by using a special class of spin networks: For LRS models two of the three holonomies of homogeneous spin networks are gauge equivalent, and for isotropic models all three holonomies. Spin networks for LRS models should then consist of only two edges, an axial one and a transversal one representing the two equivalent edges, and for isotropic models of only one edge. However, this consideration takes into account only the edges, not the vertex contractor which is an additional labeling. A reduction of this contractor is not obvious from the constraint (4). Indeed we will see in Section 4 that there is an insertion in the single edge of isotropic spin networks, which can be seen as a remnant of the vertex contractor. This insertion enables a non-vanishing volume, which shows its necessity from another viewpoint because the volume operators need vertices (more than 3-valent for gauge invariant vertices) to act on non-trivially.

These isotropic spin networks will be found by studying functions on the quantum configuration space

$$U_{\text{iso}}^{[\lambda]} = \{ (\exp(c\Lambda_1^i \tau_i), \exp(c\Lambda_2^i \tau_i), \exp(c\Lambda_3^i \tau_i)) \}$$

which is obtained by exponentiating the classical solution space of the Higgs constraint. It is a union of conjugacy classes in $SU(2)^3$ labeled by $c$, and the gauge group $G = SU(2)$ acts on it by diagonal conjugation: $g(h_1, h_2, h_3)g^{-1} = (gh_1g^{-1}, gh_2g^{-1}, gh_3g^{-1})$. This shows that the dreibein $\Lambda$ is pure gauge, but it is needed to undo gauge fixing. Relaxing the gauge fixing is necessary to be able to further on use $SU(2)$-spin networks and point holonomies. The fact that $U_{\text{iso}}^{[\lambda]}$ is no longer a group implies that quantum states, i.e. functions thereon, are no longer ordinary spin networks. These are usually obtained by making use of the Peter–Weyl theorem which determines all functions on a group. It can now no longer be used, and we will have to determine all gauge invariant functions by hand. This leads to the possibility of insertions mentioned above, which do not appear in ordinary spin networks.

### 3 Volume Operator for Bianchi Class A Models

Acting on functions in $H_{\text{aux}}$ the momentum operators are represented as

$$\hat{p}'_I = \frac{\iota' l_P^2}{2} \left( J_i^{(R)}(h_I) + J_i^{(L)}(h_I) \right)$$

where $J_i^{(R)}(h_I) := -iX_i^{(R)}(h_I)$ and $J_i^{(L)}(h_I) := -iX_i^{(L)}(h_I)$ are right and left invariant selfadjoint angular momentum operators defined via the right and left invariant vector fields acting on the copy of $SU(2)$ associated with the edge $e_I$. Furthermore, $\iota$ is the Immirzi parameter, $\iota' := \iota V_0$, and $l_P$ the Planck length. The appearance of both right and
left invariant vector fields is due to the fact that each of the edges $e_I$ is both incoming and outgoing in the vertex $x_0$.

These operators can now be inserted in the classical expression (1) to obtain the volume operator

$$
\hat{V} = V_0 \sqrt{\frac{1}{6} |\epsilon_{ijk}\epsilon_{IJK}\hat{p}_i^I\hat{p}_j^J\hat{p}_k^K|} = V_0 (|l_P^2|^{3/2} \sqrt{|\hat{q}|})
$$

(7)

with the operator

$$
\hat{q} = \frac{1}{48} \sum_{I,J,K=1}^3 \epsilon_{ijk}\epsilon_{IJK} \left( J_i^{(R)}(h_I) + J_i^{(L)}(h_I) \right) \left( J_j^{(R)}(h_J) + J_j^{(L)}(h_J) \right) \left( J_k^{(R)}(h_K) + J_k^{(L)}(h_K) \right)
$$

(8)

It is to be compared with the contribution to the volume operator of the full theory [12] in a single vertex. Here we have the 6-vertex with (after cutting each of the closed edges $e_I$ in two pieces) three incoming and three outgoing edges, each edge contributing either by a left or right invariant vector field. If we expand the product in $\hat{q}$ of the three terms containing the derivative operators, we obtain a sum of terms each being a gauge invariant product of angular momentum operators of the form $\epsilon_{ijk}\epsilon_{IJK}J_i^{(R/L)}(h_I)J_j^{R/L}(h_J)J_k^{R/L}(h_K)$. These correspond to all non-planar sets of three edges incident in $x_0$, and the factor $\epsilon_{IJK}$ introduces the correct sign for the dreibein of the associated three edges. Thus, we see that the operator in the single vertex here equals exactly a vertex contribution of the full operator.

The scale factor $V_0$ is different from the full operator (and arbitrary, for we could choose another metric $g_0$), but note that the operator of reference [12] also contains an arbitrary scale factor, called $\kappa_0$ there, as a relic of the regularization. The only important difference between the symmetric and the non-symmetric operators is the missing vertex sum for the non-symmetric one. This is analogous to the area operator in spherically symmetric sectors [4]. Note that we are lead naturally to the operator of reference [12] by using the quantization (6) of the $p_i^I$ which is forced on us by the general treatment of point holonomies. The alternative operator of reference [13], however, cannot be obtained in the present context (it contains absolute squares for each triple product of angular momentum operators not just for the sum; for a comparison of the operators see reference [14]).

Although the vertices appearing here are at most 6-valent leading to a slight simplification of the volume operator, it is impossible to calculate all eigenvalues explicitly. The vertices appearing here can, however, all be found also in a lattice formulation of loop quantum gravity [15]. Hence, the techniques developed in reference [16] by using the octagonal group can be employed to determine the volume spectrum of Bianchi models.

The operator (7) is valid for all Bianchi class A models irrespective of the particular type. However, the volume spectrum depends on the type, because the diffeomorphism constraint selects special linear combinations of spin networks. The greater the group of inner automorphisms of $S$ the smaller is the volume spectrum (see the remarks in the preceding section, and reference [1] for more details). These are only minor changes of the spectra as compared to the changes introduced by symmetry conditions, which we will study now.
4 Solving the Higgs Constraint and Volume Operator for Isotropic Models

As a consequence of the Higgs constraint (4), not all three holonomies are independent if there is a nontrivial isotropy subgroup leading to the following relations between invariant vector fields: \( h_I = g h_J g^{-1} \) implies

\[
X_i^{(R)}(h_I) = \text{tr} \left[ (\tau_i h_I)^T \frac{\partial}{\partial h_I} \right] = \text{tr} \left[ (\tau_i g h_J g^{-1})^T g^{-T} \frac{\partial}{\partial h_J} g^T \right]
\]

\[
= \text{tr} \left[ (g^{-1} \tau_i g h_J)^T \frac{\partial}{\partial h_J} \right] = \text{Ad}(g^{-1})_{ij} X_j^{(R)}(h_J)
\] (9)

with the matrix elements \( \text{Ad}(g)_{ij} \) defined by \( g \tau_i g^{-1} =: \text{Ad}(g)_{ij} \tau^j \). This implies

\[
X_i^{(R)}(h_I) \tau^i = \text{Ad}(g^{-1})_{ij} X_j^{(R)}(h_J) \tau^i = g X_j^{(R)}(h_J) \tau^j g^{-1}
\] (10)

and analogously for \( X_i^{(L)} \). This equation can be used to derive the volume operators for LRS and isotropic models from the operator (4) for Bianchi models.

The essential ingredient of equation (8) is \( \epsilon^{ijk} J^i J^j J^k \) (we define

\[
J_i^I := \frac{1}{2} \left( J_i^{(R)}(h_I) + J_i^{(L)}(h_I) \right)
\]

and later \( J_i := J_i^I \tau^i \) for ease of notation), which can be written as \(-4 \text{tr}(J_i^1 \tau^j J_i^2 \tau^j J_i^3 \tau^k)\).

For LRS models we insert

\[
h_2 = \exp \left( \frac{\tau}{2} A_3^1 \right) h_1 \exp \left( -\frac{\tau}{2} A_3^1 \right)
\]

yielding

\[
-4 \text{tr}(J_i^1 J_i^2 J_i^3) = -4 \text{tr} \left[ J_i^1 \tau^i \exp \left( \frac{\tau}{2} A_3^1 \right) J_i^2 \tau^j \exp \left( -\frac{\tau}{2} A_3^m \tau_m \right) J_i^3 \tau^k \right]
\]

\[
= -4 J_i^1 J_j^1 J_k^3 \text{tr} \left[ \tau^i (\epsilon_m n^j A_3^l + A_3^m \delta_m \tau^m) \tau^k \right]
\]

\[
= -4 J_i^1 J_j^1 J_k^3 (A_3^j \delta^i - \delta^i \delta^j + A_3^j \epsilon_{jkl})
\]

\[
= -A_3^i J_j^1 J_k^3 + A_3^i J_i^1 J_k^3 J_j^2 - A_3^i J_i^1 \epsilon_{jkl} J_j^1 J_k^3.
\] (11)

Analogously we obtain in case of isotropic models

\[
-4 \text{tr}(J_i^1 J_i^2 J_i^3) = -4 \text{tr} \left[ \exp \left( \frac{\tau}{2} A_3^i \right) J_i^2 \tau^i \exp \left( -\frac{\tau}{2} A_3^m \tau_m \right) J_i^3 \tau_j \exp \left( \frac{\tau}{2} A_3^o \tau_o \right) J_i^3 \tau_k \right]
\]

\[
= -4 J_i^1 J_j^2 J_k^3 \text{tr} \left[ (\epsilon_{ijk} \tau^p A_2^p + A_2^k \delta_m \tau_m) (\epsilon_{ijl} \tau^q A_1^l + A_2^i \delta_l \tau_l) \tau_k \right]
\]

\[
= J_i^1 J_j^2 J_k^3 (\epsilon_{ijk} A_3 \delta^k - \epsilon_{ijl} A_2^k \delta_l - \delta_{ik} A_2^k \delta_{jl} + A_2^i \delta^k - A_3^i \delta^k J_i^3 (J^3)^2
\] (12)
using $\Lambda^i_j \Lambda^k_l = \delta_{j,k}$ and $\Lambda^i_j J_i^3 \propto \delta_{i3}$ (the last relation will be established in Subsection 4.4).

We see in these preliminary expressions that only derivative operators for the independent holonomies $h_1, h_3$ for LRS and $h_3$ for isotropic models appear. But they contain operators of the form $\Lambda^i_j J_i^j$ whose action we do not know yet. Note that these operators are gauge invariant: by definition $\Lambda^i_j \tau_i$ transforms by conjugation under a gauge transformation. Therefore, $\Lambda^i_j J_i^j = -2 \text{tr}(\Lambda^i_j \tau_i J_j^i \tau^i)$ is gauge invariant. We have here already undone the partial gauge fixing, meaning that $\Lambda^i_j$ is not a fixed dreibein, but transforms under gauge transformations which change the gauge fixing. The unit vectors $\Lambda_I$ have to be regarded as functions as $\Lambda_I: SU(2)^3 \rightarrow S^2, \Lambda_I(g_1, g_2, g_3) = L(g_I)$ for $g_I \neq 1$ from the classical configuration space of Bianchi models to the unit sphere embedded in the Lie algebra of $SU(2)$. The function $L: SU(2) \setminus \{1\} \rightarrow S^2 = \{X \in LSU(2) : X^i X_i = 1\}$ is defined for $g \neq 1$ as $L(g) := [(\log g)^i (\log g)_j]^{\frac{1}{2}} \log g$ using the matrix logarithm which can be made unique by fixing a branch, e.g. by demanding that $\log g \in L\text{SU}(2)$ has minimum Cartan–Killing norm. As a consequence, the operators $\Lambda^i_j J_i^j$ are not symmetric, although $\Lambda^i_j$ are real functions and the $J^j_i$ are self-adjoint: $(\Lambda^i_j J_i^j)^* = J^j_i \Lambda^i_j \neq \Lambda^j_i J_i^j$ because of $[\Lambda^i_j, J^j_i] \neq 0$. The commutator is a complicated function on $SU(2)$ due to the logarithms in the definition of $\Lambda_3$. We also see that there are factor ordering ambiguities in the expressions (11) and (12) which we ignored above.

To understand the action of the operator in equation (12) we have to gain more knowledge about the quantum states of isotropic models. To compute the complete spectrum of the volume operator we have to know all these states.

### 4.1 Quantum States for LRS and Isotropic Models

In the course of quantum symmetry reduction quantum states are defined as functions on the spaces (ı) for isotropic models and

$$U^{[\Lambda]}_{\text{LRS}} = \left\{\left(\exp(2^{-\frac{\pi}{2}}(a\Lambda^i_1 + b\Lambda^i_2)\tau_i), \exp(2^{-\frac{\pi}{2}}(-b\Lambda^i_1 + a\Lambda^i_2)\tau_i), \exp(c\Lambda^i_3\tau_i)\right)\right\}$$  (13)

for LRS models. These spaces are obtained by exponentiating the solution spaces of the classical Higgs constraint, and their elements solve the Higgs constraint in the form (ı). They are parameterized by the parameters $\Lambda^i_j$, which are pure gauge but arbitrary after relaxing the gauge fixing, and $a, b, c$ for LRS models and $c$ for isotropic models, respectively. Furthermore, they are submanifolds of the configuration space $SU(2)^3$ of Bianchi models, but not subgroups. All functions on them can be generated by pull backs of spin network functions on $SU(2)^3$, but not all of these pull backs will be independent. Pull backs of gauge invariant spin networks only depend on $c$ for isotropic models, and on $A := \sqrt{a^2 + b^2}$ and $c$ for LRS models; they are automatically gauge invariant on the reduced configuration spaces. This implies that the parameterizations are highly redundant: Gauge invariant functions can be expressed as functions only in one $SU(2)$-element, which we choose as the third, for $U^{[\Lambda]}_{\text{iso}}$, and in the first (choosing this one of the first two) and third element for $U^{[\Lambda]}_{\text{LRS}}$. This corresponds to the fact that equation (ı) eliminates the holonomies $h_1$ and $h_2$ for isotropic models, and $h_2$ for LRS models. A special class of such functions is given
by spin network functions associated with graphs containing only one edge \(e_3\) for isotropic models, or two edges \(e_1, e_3\) for LRS models. But these do not suffice to generate all gauge invariant functions, neither for LRS nor for isotropic models. To show this we use a small lemma, which will also prove useful when calculating particular spin networks:

**Lemma 1** Let \(g := \exp(A\tau_i)\) and \(h := \exp(B\tau_j)\) with \(A, B \in \mathbb{R}\), \(i \neq j\) be matrices in the fundamental representation of \(SU(2)\). Then

\[
gh = hg + h^{-1}g + hg^{-1} - \text{tr}(gh).
\]

**Proof:** This can directly be proved by using \(\exp(A\tau_i) = \cos(A/2) + 2\sin(A/2)\tau_i\). \(\square\)

By means of this lemma we can express a product of arbitrary factors \(\exp(A_k\tau_i)\) as a sum of terms with at most three factors. To calculate the gauge invariant trace we then need only \(\text{tr}\exp(A\tau_i) = 2\cos(A/2),\) \(\text{tr}[\exp(A\tau_i)\exp(B\tau_j)] = 2\cos(A/2)\cos(B/2)\) for \(i \neq j\) and

\[
\text{tr}[\exp(A\tau_1)\exp(B\tau_2)\exp(C\tau_3)] = 2\cos(\frac{1}{2}A)\cos(\frac{1}{2}B)\cos(\frac{1}{2}C) - 2\sin(\frac{1}{2}A)\sin(\frac{1}{2}B)\sin(\frac{1}{2}C).
\]

We now show that any pull back of a gauge invariant function on \(SU(2)^3\) to a gauge invariant function on \(U^{[\lambda]}_{\text{LRS}}\) is invariant under the reflection \(A = \sqrt{a^2 + b^2} \mapsto -A\). An overcomplete set of such functions on \(SU(2)^3\) is given by \(\text{tr}(h_1^{m_1}h_2^{m_2}h_3^{m_3})\) with an arbitrary finite number of factors and arbitrary \(n_i\). By using the lemma these functions can be simplified to \(\text{tr}(h_1^{m_1}h_2^{m_2}h_3^{m_3})\) (up to factors of \(\cos(A/2)\) or \(\cos(c/2)\)). They are gauge invariant, and so we can choose the gauge \(h_1 = \exp(A\tau_1), h_2 = \exp(A\tau_2), h_3 = \exp(c\tau_3)\). Using a gauge transformation \(g = \exp(\pi\tau_3) = 2\tau_3\) (in general \(g = 2\Lambda^i_3\tau_i\)) with \(ghg^{-1} = h_3\), \(gh_1g^{-1} = h_1^{-1}\), \(gh_2g^{-1} = h_2^{-1}\) we see that all gauge invariant functions are invariant under \(h_1 \mapsto h_1^{-1}, h_2 \mapsto h_2^{-1}, h_3 \mapsto h_3\) which is equivalent to \(A \mapsto -A, c \mapsto c\). Of course, the gauge invariant spin networks with only two edges \(e_1, e_3\) are also invariant under this transformation. The key point is that there is a gauge transformation which fixes \(h_3\), which depends on \(c\), but inverts \(h_1\) and \(h_2\), which depend on \(A\).

The situation is different if we are interested in the transformation \(A \mapsto A, c \mapsto -c\): There is no gauge transformation fixing both \(h_1\) and \(h_2\), but inverting \(h_3\). Therefore, gauge invariant spin networks on the three edges \(e_1, e_2, e_3\) do not need to be invariant under \(A \mapsto A, c \mapsto -c\). A counterexample is provided by a spin network with three edge spins \(\frac{1}{2}\) and an appropriate gauge invariant vertex contractor such that it can be written as \(\text{tr}[\exp(A\tau_1)\exp(A\tau_2)\exp(c\tau_3)] = 2\cos^2(A/2)\cos(c/2) - 2\sin^2(A/2)\sin(c/2)\). In contrast, reduced spin networks with the two edges \(e_1, e_3\) give always rise to gauge invariant functions being invariant under \(A \mapsto A, c \mapsto -c\), which can be shown as above by using the gauge transformation \(g' = \exp(\pi\tau_1) = 2\tau_1\).

Thus, we see that the obvious candidates for functions on \(U^{[\lambda]}_{\text{LRS}}\), namely spin network functions with two edges, are not sufficient to generate all gauge invariant functions. This fact can be traced back to the twisting in equation \(\Box\) introduced by the non-trivial gauge transformation on the right hand side. A related fact is that \(U^{[\lambda]}_{\text{LRS}}\) is not a subgroup of
$SU(2)^3$, but only a subset being a union of conjugacy classes. The Peter–Weyl theorem does no longer apply in this situation; spin network functions with two edges could only be expected as a sufficient class of functions if the reduced configuration space would be a subgroup of $SU(2)^3$, e.g. $SU(2)^2$ which would be obtained in case of a trivial gauge transformation in equation (4). This sector, however, does not allow nontrivial Higgs fields $\mu$.

An analogous discussion applies for the isotropic models: Gauge invariant spin network functions with one edge are always invariant under $c \mapsto -c$, but this is not necessarily true for gauge invariant functions on $U_{\text{iso}}$.  

4.2 Insertsions

We now have to face the two problems of investigating the operator $\Lambda^i_j J^3_i$ (determining its action, a symmetric ordering and selfadjoint extensions) and of determining all quantum states. Luckily, these two problems are connected, and one problem will provide the solution for the other. This can easily be seen by applying $\Lambda^i_j X^i_j(h_3)$ to a gauge invariant function on two edges, which, as shown above, can always be written as a linear combination of terms $\text{tr}(h^{m_1}_1 h^{m_3}_3)$ being invariant under $A \mapsto A$, $c \mapsto -c$. Applying the operator

$$2\Lambda^i_j X^i_j(h_3) = 2(\Lambda^i_j \tau_i h_3)^A B \frac{\partial}{\partial (h_3)^A B}$$

effects a replacement of $h_3$ with $2\Lambda^i_j \tau_i h_3 = \exp(\pi \Lambda^i_j \tau_i) h_3$:

$$2\Lambda^i_j X^i_j(h_3) \text{tr}(h^{m_1}_1 h^{m_3}_3) = \text{tr}[h^{m_1}_1 \exp(\pi \Lambda^i_j \tau_i) h^{m_3}_3]$$

and the same for $\Lambda^i_j X^i_j(L)(h_3)$. This function is no longer invariant under $A \mapsto A$, $c \mapsto -c$: The gauge transformation $g' = \exp(\pi \tau_i)$ used above transforms $\exp(\pi \Lambda^i_j \tau_i) h_3$ into $-\exp(\pi \Lambda^i_j \tau_i) h_3^{-1}$, and therefore the above function changes sign under $A \mapsto A$, $c \mapsto -c$ for $m_3$ odd.

The operator $\Lambda^i_j J^3_i$, which will appear in the volume operators, does not fix the space of spin network functions with two edges. Therefore, we have to understand the remaining states in order to investigate the volume operators. We can visualize them as spin networks with two closed edges, but with an additional insertion in its 4-vertex, which is associated to the holonomy $h_3$. This symbolizes the insertion of $\exp(\pi \Lambda^i_j \tau_i)$, and it can be interpreted, together with the 4-vertex contractor, as a remnant of the 6-vertex contractor of a spin network on $SU(2)^3$ after reduction to local rotational symmetry. The kinematical Hilbert space thus splits into a subspace of ordinary spin network functions with two edges, and a subspace of spin network functions with insertion. The volume operator will fix neither of these subspaces.

An analogous discussion applies to the case of isotropic models. Here, there remains only one closed edge after reduction, and the insertion in its 2-vertex is the only remnant of the 6-vertex contractor, or of the 4-vertex contractor with the insertion for LRS models. In this case of isotropic models we will demonstrate in the next subsection that we now
have found all quantum states, and develop a calculus on the solution space of the Higgs
constraint enabling us to deal with operators like \( \Lambda_3^i J_i^3 \).

### 4.3 Kinematical Hilbert Space of Isotropic Models

Before discussing in more detail quantum states of isotropic models, we determine a mea-
sure on the space \( \mathcal{U}_{\text{iso}}^{[\lambda]} \), which can be regarded as the space of generalized isotropic
connections. This measure will be derived from the Ashtekar–Lewandowski measure along the
lines of quantum symmetry reduction.

Reducing first to homogeneous connections, the Ashtekar–Lewandowski measure of
the full theory is reduced to the finite-dimensional measure \( d\mu_H^3 \) on \( SU(2)^3 \). Here we
parameterize \( SU(2) \) as \( g = \exp(c\Lambda^i_3 \tau_i) \in SU(2) \) with \( \Lambda_3 \in S^2 \) leading to the normalized
Haar measure

\[
\mu_H(f) = \frac{1}{4\pi^2} \int_0^{4\pi} \int_0^\pi \int_0^\pi f(c, \Lambda_3) \sin^2 \frac{\varphi}{2} \, d^2\Lambda_3 \, dc
\]

with the solid angle measure \( d^2\Lambda_3 = \sin \vartheta \, d\vartheta \, d\varphi \) for \( \Lambda_3^i = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta) \in S^2 \).

To reduce further to isotropy we have to describe in more detail the space \( \mathcal{U}_{\text{iso}}^{[\lambda]} \). It can
be written as \( \mathcal{U}_{\text{iso}}^{[\lambda]} = \bigcup_{c \in U(1)} \Theta_c \) where \( \Theta_c \) are conjugacy classes in \( SU(2)^3 \) with respect to
the diagonal conjugation \( g(h_1, h_2, h_3)g^{-1} = (gh_1g^{-1}, gh_2g^{-1}, gh_3g^{-1}) \) of \( SU(2) \) on \( SU(2)^3 \).
The \( \Theta_c \) are labeled by an element \( c \in \mathbb{R}/(4\pi\mathbb{Z}) \cong U(1) \) and take the form

\[
\Theta_c = \{ g(\exp(c\tau_1), \exp(c\tau_2), \exp(c\tau_3))g^{-1} : g \in SU(2) \} = \{ (\exp(c\Lambda^i_1 \tau_i), \exp(c\Lambda^i_2 \tau_i), \exp(c\Lambda^i_3 \tau_i)) : \Lambda^i_1, \Lambda^i_2, \Lambda^i_3 \in SO(3) \} \subset SU(2)^3.
\]

The components \( \Lambda^i_j \) build a dreibein, which shows that \( \Theta_c \) for \( c \neq 0 \) is homeomorphic to
\( SO(3) \). It is however not a group, nor is \( \mathcal{U}_{\text{iso}}^{[\lambda]} \), which can e.g. be seen by multiplying the
elements \( \exp(c\tau_1), \exp(c\tau_2), \exp(c\tau_3) \) and \( \exp(-c\tau_1), \exp(-c\tau_2), \exp(c\tau_3) \), which are both
contained in \( \Theta_c \).

An invariant normalized measure on the conjugacy class \( \Theta_c \) is defined by

\[
\int_{\Theta_c} f(\Lambda) \, d\mu_{\Theta_c}(\Lambda) := \int_{SU(2)} f(g(\exp(c\tau_1), \exp(c\tau_2), \exp(c\tau_3))g^{-1}) \, d\mu_H(g) . \tag{14}
\]

On the right hand side the element \( (\exp(c\tau_1), \exp(c\tau_2), \exp(c\tau_3)) \) can be replaced by an
arbitrary element of \( \Theta_c \); the measure \( d\mu_{\Theta_c} \) is independent of this choice. The integration
is only over one copy of \( SU(2) \), because \( \Theta_c \) is defined as a conjugacy class with respect to
the diagonal conjugation of \( SU(2) \) on \( SU(2)^3 \) (and not conjugation in the group \( SU(2)^3 \)).

Deleting the point \( c = 0 \) we can represent \( \mathcal{U}_{\text{iso}}^{[\lambda]} \) as a fiber bundle with fibers \( U(1) \setminus \{1\} \)
and base homeomorphic to \( SO(3) \) represented by some \( \Theta_c \). We can then build the product
of Haar measure \( dc \) on \( U(1) \) weighted with the volume \( \text{Vol} \Theta_c = (2\pi)^{-1} \sin^2(c/2) \) of the
conjugacy classes and the invariant measure on \( \Theta_c \) to obtain a measure on \( \mathcal{U}_{\text{iso}}^{[\lambda]} \):

\[
(2\pi)^{-1} \int_{U(1)} \int_{\Theta_c} f(\exp(c\Lambda_1), \exp(c\Lambda_2), \exp(c\Lambda_3)) \, d\mu_{\Theta_c}(\Lambda) \sin^2 \frac{\varphi}{2} \, dc.
\]
The point \( c = 0 \) has no effect because it is of measure zero.

To make contact with reduced gauge invariant spin networks consisting of a single edge we will now restrict ourselves to gauge invariant functions \( f \). Those functions do not depend on the dreibein \( \Lambda^i_j \), but only on the parameter \( c \). This implies that any such function can be written as \( f(\exp(c\Lambda_1), \exp(c\Lambda_2), \exp(c\Lambda_3)) = F(\exp(c\Lambda_3)) \) for some function \( F \) on \( SU(2) \). Its integral over \( \mathcal{U}_{iso}^{[\lambda]} \) is

\[
(2\pi)^{-1} \int_{U(1)} \int_{\Theta_c} F(\exp(c\Lambda_3)) d\mu_{\Theta_c}(\Lambda) \sin^2 \frac{c}{2} dc =
(2\pi)^{-1} \int_{U(1)} \int_{SU(2)} F(g \exp(c\tau_3)g^{-1}) d\mu_H(g) \sin^2 \frac{c}{2} dc
\]

in which, parameterizing \( SU(2) \) with Euler angles \( g = \exp(\varphi\tau_3) \exp(\psi\tau_2) \exp(\psi\tau_3) \), the \( \psi \)-integration is trivial. After performing this integration the remaining part of the \( SU(2) \)-measure and the \( U(1) \)-measure recombine to Haar measure on \( SU(2) \) (this is Weyl’s integral formula for the group \( SU(2) \)) leading to the measure

\[
\int_{\mathcal{U}_{iso}^{[\lambda]}} f(h_1, h_2, h_3) d\mu(h_1, h_2, h_3) = \int_{SU(2)} F(h_3) d\mu_H(h_3)
\]

on \( \mathcal{U}_{iso}^{[\lambda]} \) for gauge invariant functions \( f \). (Heuristically, the measure \( d\mu_{\Theta_c} \) on \( \Theta_c \) is

\[
2\pi \sin^{-2} \frac{c}{2} \delta(c' - c)d\mu_H(g(c', \Lambda))
\]

and integrating the \( \delta \)-function replaces \( c' \) by \( c \).)

The measure just derived shows that the kinematical Hilbert space for isotropic models can be represented as a space of functions on \( SU(2) \) with the usual Haar measure. The spin networks with one edge correspond to the character functions

\[
\chi_j(c) = \frac{\sin(j + \frac{1}{2})c}{\sin \frac{c}{2}} \quad , \quad j \in \frac{1}{2} \mathbb{N}_0
\]

which are invariant under \( c \mapsto -c \). But we have seen that these functions do not suffice to generate all gauge invariant functions on \( \mathcal{U}_{iso}^{[\lambda]} \). (To avoid misunderstanding, we note that the \( \chi_j \) certainly span the space of class functions on \( SU(2) \). However, the gauge transformations on \( \mathcal{U}_{iso}^{[\lambda]} \) are not just conjugation on \( SU(2) \), but on a subspace of \( SU(2)^3 \). Therefore, there can be more gauge invariant functions which do not reduce to class functions after restricting to dependence on one edge only.) We are now going to determine the remaining class of functions.

To that end we recall that any gauge invariant function on \( \mathcal{U}_{iso}^{[\lambda]} \) can be written as a linear combination of functions \( \text{tr}[\exp(m_1 c\Lambda_1) \exp(m_2 c\Lambda_2) \exp(m_3 c\Lambda_3)] \) (up to irrelevant factors of \( \cos(c/2) \)) with some integers \( m_i \). If some of the \( m_i \) vanish the trace is invariant under \( c \mapsto -c \); otherwise it is easily evaluated to

\[
\text{tr}[\exp(m_1 c\Lambda_1) \exp(m_2 c\Lambda_2) \exp(m_3 c\Lambda_3)] = 2\cos(\frac{1}{2}m_1 c) \cos(\frac{1}{2}m_2 c) \cos(\frac{1}{2}m_3 c) - 2\sin(\frac{1}{2}m_1 c) \sin(\frac{1}{2}m_2 c) \sin(\frac{1}{2}m_3 c).
\]
The first term is invariant under $c \mapsto -c$ and can thus be expanded in the functions $\chi_j$. The second term, however, is antisymmetric and provides new functions
\[
\sin \left( \frac{1}{2} m_1 c \right) \sin \left( \frac{1}{2} m_2 c \right) \sin \left( \frac{1}{2} m_3 c \right) = -\frac{1}{4} \left[ \sin \left( \frac{1}{2} (m_1 + m_2 + m_3) c \right) + \sin \left( \frac{1}{2} (-m_1 - m_2 + m_3) c \right) + \sin \left( \frac{1}{2} (-m_1 + m_2 - m_3) c \right) \right].
\] (17)

All these functions with different sums $m_1 + m_2 + m_3$ are independent provided we choose $m_I > 0$.

**Lemma 2** All gauge invariant functions on $U^{\lambda}_{\text{iso}}$ can be generated by symmetric functions, which are spanned by the $\chi_j$, and the functions
\[
\sin(kc), \quad k \geq \frac{1}{2}.
\]

**Proof:** As the discussion above shows, all independent functions are given by $\chi_j$ and the functions $\sin(m_1 c/2) \sin(m_2 c/2) \sin(m_3 c/2)$ for all different values of $m := m_1 + m_2 + m_3$, $m_I > 0$. In all cases $m$ can be decomposed as $m_1 = m - 2$, $m_2 = m_3 = 1$. The expansion (17) then yields
\[
\sin \left( \frac{1}{2} (m - 2) c \right) \sin^2 \frac{1}{2} c = \frac{1}{4} \left[ \sin \left( \frac{1}{2} mc \right) - 2 \sin \left( \frac{1}{2} (m - 2) c \right) + \sin \left( \frac{1}{2} (m - 4) c \right) \right].
\]

Induction over $m \geq 3$ then shows that all independent antisymmetric functions are given by
\[
f_j := \sin(jc) - \frac{j}{j-1} \sin((j-1)c), \quad j \geq \frac{3}{2}.
\]

This set of functions can be simplified if we can generate the functions $\sin(c/2)$ and $\sin c$. This can indeed be achieved by using the sequence
\[
f_{\frac{3}{2}} + \frac{3}{5} f_{\frac{5}{2}} + \cdots + \frac{3}{5} \cdot \frac{5}{7} \cdots \frac{M-2}{M} f_{\frac{M}{2}} = -3 \sin(\frac{1}{2} c) + \frac{3}{M} \sin(\frac{1}{2} M c)
\]
for $M$ odd, and analogously for even $M$. In the Haar measure this sequence converges to $-3 \sin(c/2)$ (the norm of $\sin(Mc/2)$ is independent of $M$), whereas for even $M$ we can obtain $\sin c$.

Thus all antisymmetric functions are generated by $\sin(kc)$ with $k \geq \frac{1}{2}$. \hfill \Box

The kinematical Hilbert space is seen to be the linear span $\langle \chi_j, \sin kc : j \in \frac{1}{2} \mathbb{N}_0, k \in \frac{1}{2} \mathbb{N} \rangle$ completed in the measure $\langle \cdot \rangle$.

### 4.4 Derivative Operators

Instead of the functions $\sin(kc)$ we will use functions which appear more naturally when using derivative operators like $\Lambda^3_i X^3_i$. As already noted, this operator maps a function
\( \chi_j \) to a function being antisymmetric with respect to \( c \mapsto -c \). Writing the characters as \( \chi_j(c) = \text{tr} \pi^{(j)}(g) = g^{(A_1) \cdots (A_{\mathbf{2}})}_{A_1 \cdots A_{\mathbf{2}}} \), \( g = \exp(c\Lambda_3) \in SU(2) \) we calculate

\[
\xi_j(c) := \Lambda_3^i X_i^3 \chi_j(c) = 2j (\Lambda_3 g)^{(A_1) \cdots (A_{\mathbf{2}})}_{A_1 \cdots A_{\mathbf{2}}} \big|_{g=\exp(c\Lambda_3)} = \frac{d}{dc} \chi_j(\exp(c\Lambda_3)) = j \frac{\cos(j + \frac{1}{2})c}{\sin^2 \frac{c}{2}} - \frac{\sin j c}{2 \sin^2 \frac{c}{2}}
\]

noting that \( \Lambda_3 \) is a function on \( SU(2) \) defined by \( g = \exp(c\Lambda_3) \). Similarly, we can now justify the relation \( \Lambda_3^I J_i^3 = 0 \) for \( I \neq 3 \) used in simplifying the expression \((12)\): This operator leads to an insertion of \( \Lambda_3^i \tau_i \) into the trace of factors of \( h_3 = \exp(c\Lambda_3) \) (with or without insertion of \( \Lambda_3 \)) which vanishes upon tracing.

The component \( \Lambda_3^i X_i^3 \) of the invariant vector field is represented simply as derivative with respect to \( c \). But this derivative operator is, as already noted for \( \Lambda_3^i X_i^3 \), not symmetric with respect to the measure \((15)\). Therefore, we now compute the adjoint of \( \frac{d}{dc} \) and introduce new functions related to \( \xi_j \).

Due to

\[
\int_0^{4\pi} f(c) g'(c) \sin^2 \frac{c}{2} dc = -\int_0^{4\pi} f'(c) g(c) \sin \frac{c}{2} dc - \int_0^{4\pi} f(c) g(c) \sin \frac{c}{2} \cos \frac{c}{2} dc
\]

we get

\[
\left( \frac{d}{dc} \right)^* = -\frac{d}{dc} - \cot \frac{c}{2}
\]

as adjoint of \( \frac{d}{dc} \) with respect to Haar measure. Computing the commutator

\[
\left[ \frac{d}{dc}, \left( \frac{d}{dc} \right)^* \right] = -\frac{d}{dc} \cot \frac{c}{2} = (2 \sin^2 \frac{c}{2})^{-1}
\]

we see that \( \frac{d}{dc} \) is not a normal operator.

More important for what follows will be the anti-selfadjoint combination

\[
\frac{1}{2} \left( \frac{d}{dc} - \left( \frac{d}{dc} \right)^* \right) = \frac{d}{dc} + \frac{1}{2} \cot \frac{c}{2}.
\]

By means of this derivative operator we define our final class of functions

\[
\zeta_j(c) := (j + \frac{1}{2})^{-1} \left( \frac{d}{dc} + \frac{1}{2} \cot \frac{c}{2} \right) \chi_j(c) = \frac{\cos(j + \frac{1}{2})c}{\sin^2 \frac{c}{2}}, \quad j \in \frac{1}{2} \mathbb{N}_0
\]

and

\[
\zeta_{-\frac{3}{2}} := (\sqrt{2} \sin \frac{c}{2})^{-1}
\]

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which are antisymmetric in $c$. (The singularity in $c = 0$ is compensated in the Haar measure.) We now take a closer look on the action of $\frac{d}{dc} + \frac{1}{2} \cot \frac{c}{2}$. Acting on $\chi_j$ yields by definition

$$\left(\frac{d}{dc} + \frac{1}{2} \cot \frac{c}{2}\right) \chi_j(c) = \left(j + \frac{1}{2}\right) \zeta_j(c) \quad (20)$$

and acting on $\zeta_j$

$$\left(\frac{d}{dc} + \frac{1}{2} \cot \frac{c}{2}\right) \zeta_j(c) = -\left(j + \frac{1}{2}\right) \chi_j(c). \quad (21)$$

The functions $\zeta_j$ are easily seen to be orthonormal, which is also true for the functions $\chi_j$. Furthermore, each $\zeta_j$ is orthogonal to any $\chi_{j_2}$ because their product is antisymmetric in $c$. Thus, the set

$$\{\chi_{j_1}, \zeta_{j_2} : j_1 \in \frac{1}{2} \mathbb{N}_0, j_2 \in \frac{1}{2} \mathbb{N}_0 \cup \{-\frac{1}{2}\}\} \quad (22)$$

is an orthonormal basis of functions in $c$ with respect to the measure $(2\pi)^{-1} \sin^2 \frac{c}{2}$.

We can build another selfadjoint operator from $\frac{d}{dc}$, namely the selfadjoint quadratic differential operator

$$\left(\frac{d}{dc}\right)^* \frac{d}{dc} = -\frac{d^2}{dc^2} - \cot \frac{c}{2} \frac{d}{dc}, \quad (23)$$

which is the radial component of the Laplace operator on $SU(2)$, i.e. the part independent of $\vartheta, \varphi$ being the only non-vanishing contribution when acting on gauge (conjugation) invariant functions. Indeed, it is straightforward to show, that

$$\left(\frac{d}{dc}\right)^* \frac{d}{dc} \chi_j(c) = j(j+1)\chi_j(c)$$

for the class functions $\chi_j$. Our additional functions $\zeta_j$ are also eigenfunctions with the same eigenvalues:

$$\left(\frac{d}{dc}\right)^* \frac{d}{dc} \zeta_j(c) = j(j+1)\zeta_j(c),$$

which follows from the definition of $\zeta_j$ and

$$\left[\left(\frac{d}{dc}\right)^* \frac{d}{dc}, \frac{d}{dc}\right] = \left[\left(\frac{d}{dc}\right)^* \frac{d}{dc}, \left(\frac{d}{dc}\right)^*\right].$$

However, neither the functions $\xi_j$ defined earlier nor the functions $\sin j c$ are eigenfunctions of $\left(\frac{d}{dc}\right)^* \frac{d}{dc}$. We have, for instance,

$$\left(\frac{d}{dc}\right)^* \frac{d}{dc} \sin j c = j(j+1) \sin j c - j \frac{\cos(j - \frac{1}{2})c}{\sin \frac{c}{2}} = j(j+1) \sin j c - j\zeta_{j-1}(c)$$

showing that the function $\zeta_j$ can be obtained by acting with $\left(\frac{d}{dc}\right)^* \frac{d}{dc}$ on $\sin((j+1)c)$. Vice versa, we can reobtain the function $\sin j c$ by using the equations

$$\sin(jc) = \frac{1}{2} (\zeta_{j-1}(c) - \zeta_j(c)), \quad j > \frac{1}{2} \quad (24)$$
and

\[ \sin \frac{\xi}{2} = \frac{1}{2} \left( \sqrt{2} \zeta_{-\frac{1}{2}} - \zeta_{\frac{1}{2}} \right) \]

which follow from adding the two equations

\[ \sin \frac{\xi}{2} \sin(\mu c) = \cos \left((j + \frac{1}{2}) c \right) - \cos \frac{\xi}{2} \cos(\mu c) = \cos \frac{\xi}{2} \cos(\mu c) - \cos \left((j + \frac{1}{2}) c \right) \]

and dividing by \( \sin(\xi/2) \).

We now arrived at our final set of generating functions:

**Theorem 1** The set

\[ \{ \chi_j, \zeta_k : j \in \frac{1}{2} \mathbb{N}_0, k \in \frac{1}{2} \mathbb{N}_0 \cup \{-\frac{1}{2}\} \} \quad (25) \]

forms an orthonormal basis of the kinematical Hilbert space of isotropic models \( \mathcal{H}_{aux} = L^2(\mathcal{U}_{iso}^{[\lambda]}, d\mu_H) \).

**Proof:** According to Lemma 2 all antisymmetric functions on \( \mathcal{U}_{iso}^{[\lambda]} \) can be generated by the functions \( \sin \mu c \) for \( j \geq \frac{1}{2} \). With the preceding equations we see that this set of functions is equivalent to the set \( \{ \zeta_j : j \geq -\frac{1}{2} \} \). That all functions contained in the set of generating functions are orthonormal has already been shown above. \( \Box \)

### 4.5 Isotropic Volume

Finally, we have to translate the derivative operators \( \frac{d}{dc} \) back to invariant vector field operators \( X^3_i \). We were lead to \( \frac{d}{dc} \) by studying the action of \( \Lambda^3_i X^3_i \) on spin network functions, which is not selfadjoint. The selfadjoint operator \( \frac{i}{2} (\Lambda^3_i J^3_i + (\Lambda^3_i J^3_i)^*) \) is now identified with the derivative operator

\[ \frac{i}{2} \left( \frac{d}{dc} - \left( \frac{d}{dc} \right)^* \right) = \frac{i}{2} \frac{d}{dc} + \frac{i}{2} \cot \frac{\xi}{2}, \]

which we regard as quantization of the classical real phase space function \( \Lambda^3_i E_i^3 \). Its action on the quantum states \( \chi_j \) and \( \zeta_j \) for \( j \geq 0 \) is (using equations (20) and (21))

\[ \widehat{\Lambda^3_i E_i^3} \chi_j = i \left( j + \frac{1}{2} \right) \zeta_j, \quad (26) \]
\[ \widehat{\Lambda^3_i E_i^3} \zeta_j = -i \left( j + \frac{1}{2} \right) \chi_j, \quad (27) \]

whereas it annihilates \( \zeta_{-\frac{1}{2}} \).

For the volume operator we need the spectrum of

\[ \left| \widehat{\Lambda^3_i E_i^3} \right| := \sqrt{\Lambda^3_i E_i^3}^2, \]

which can be read off from the previous equations. It has the twofold degenerate eigenvalues \( j + \frac{1}{2} \) for \( j \geq 0 \) with eigenfunctions \( \chi_j \) and \( \zeta_j \) and the non-degenerate eigenvalue 0 with eigenfunction \( \zeta_{-\frac{1}{2}} \).
The operator \((J^3)^2\), which commutes with \(\Lambda^3_i E^3_i\), is to be identified with the quadratic derivative \(\left(\frac{d}{d\epsilon}\right)^* \frac{d}{d\epsilon}\) with eigenvalues \(j(j + 1)\) to the same eigenfunctions as above.

With these ingredients we can now quantize the volume (3) using equations (12) and (14) as

\[
\hat{V} = V_0 (\ell l_P^2)^{3/2} \sqrt{\Lambda^3_i E^3_i} \sqrt{(J^3)^2}.
\]

(28)

The spectrum is then easily obtained using the information presented above as

\[
\left\{ V_0 (\ell l_P^2)^{3/2} \sqrt{j \left(j + \frac{1}{2}\right) (j + 1)} : j \in \frac{1}{2}\mathbb{N}_0 \right\}
\]

(29)

which is twofold degenerate for \(j > 0\), whereas \(j = 0\) is triply degenerate.

### 4.6 Remarks on LRS Models

The general features of solving the Higgs constraint in the spin network context to arrive at the kinematical Hilbert space are illustrated by the example of isotropic models, which was considered in detail above. E.g., one has to determine the quantum states with its possible insertions and to carry over the spin network techniques. Conceptually, the situation for LRS models is the same, but it is complicated by the appearance of two edges and, in connection, the dependence of gauge invariant states on two variables. We showed also in this case the necessity of insertions. All other steps are to be done in analogy to isotropic models. We will not present them here because they do not bring in anything new.

### 5 Consequences and Discussion

In this final section we comment on some applications of the material contained in the present paper.

#### 5.1 Quantum Symmetry Reduction

The isotropic models considered in detail in Section 4 provide the first example of a symmetry reduction with nontrivial Higgs sector and non-empty Higgs constraint being carried out completely along the lines of quantum symmetry reduction \(\square\). They show a concrete illustration of how to solve the Higgs constraint in quantum theory by using the geometrical Higgs constraint \(\square\). Furthermore, the need for insertions and their interpretation as remnants of vertex contractors has shown up. The treatment proved that spin network techniques can be adapted to solution spaces of the Higgs constraint. An essential ingredient was to relax the partial gauge fixing, thereby restoring the full \(SU(2)\)-gauge invariance.

More complicated models are provided by locally rotationally symmetric systems which lead to spin networks with an axial and a transversal edge, and which can be treated
along the same lines. The transversal edge represents the information contained in the two edges which are equivalent upon solving the Higgs constraint. This is similar to the spherically symmetric sector of loop quantum gravity [2, 4], where instead of the axial edge (representing a point holonomy and not a real edge) we have a radial manifold on which Higgs vertices are lined up. These vertices also contain one edge (in an auxiliary manifold) which is obtained after reducing two transversal edges when solving the Higgs constraint. Therefore, LRS models are good toy models for determining the structure of spherically symmetric Higgs vertices. This was our main motivation for studying cosmological models, because single Higgs vertices can here be investigated on their own.

5.2 Level Splitting

In the volume spectra we can see a phenomenon first observed in case of the area spectrum in reference [4]. Starting from the full spectrum of reference [12], which is, however, not known explicitly, we obtain only a subset of this spectrum after reducing to homogeneous geometries. This is a consequence of the fact that there is only one point \(x_0\) in the reduced manifold \(B\), and therefore only one vertex. The vertex sum in the full volume spectrum then disappears, and the spectrum is reduced because the eigenvalues are in general irrational. We then can enhance the symmetry further to LRS and, finally, isotropic models, which have the simple volume spectrum (29).

Vice versa, starting from isotropic models the symmetry can be broken in steps to finally obtain an arbitrary anisotropic, inhomogeneous geometry. In each step the broken symmetry leads to a splitting of eigenvalues of the volume operator leading from the spectrum (29) to the full spectrum. Note that we can only compare the eigenvalues, not the degeneracies, because reduced models are represented in different Hilbert spaces which are not subspaces of the full Hilbert space. Alternatively, their states can be described by distributional states of the full theory as described in reference [2]. In particular, we cannot determine which eigenvalues of the full theory are related by this level splitting to a particular eigenvalue in the spectrum (29). The degeneracy of two for \(j > 0\) in this spectrum has nothing to do with such a degeneracy expected from level splitting.

Another feature of the high symmetry of isotropic models is that we can explicitly calculate the complete volume spectrum, a task which would be hopeless in the full theory.

5.3 Weaves

As made explicit by a spin network, the quantum nature of gravity breaks explicitly any continuous space symmetry. A nontrivial spin network (or a finite linear combination) cannot be invariant with respect to a transitive symmetry group. Therefore, our homogeneous or isotropic states are, when not regarded only as states of a reduced toy model, idealized states comparable to plane waves in quantum mechanics. Accordingly, they are represented as generalized states of the full theory [2], i.e. as elements of the topological dual \(\Phi'\) of the space \(\Phi\) of cylindrical functions. But using a nuclear topology of \(\Phi\), this space is dense in its dual in the weak topology [18]. Therefore, any distributional state can
be approximated weakly by certain combinations of spin network states. Although such an approximation may be very complicated to construct explicitly, this provides a simple existence proof for $S$-weave states. We define here those states as states which approximate a given generalized state being symmetric with respect to the symmetry group $S$. For instance, we can build isotropic $S$-weaves by approximating the states found in Section 4, regarded as distributional states of the full theory using the map $\sigma[\lambda]$ of reference [2].

We denote them as $S$-weaves to point out that they are not necessarily equivalent to the weaves defined in reference [19]. There states were defined as weaves which approximate a given classical metric at large distances as compared to the Planck scale. Such a geometrical condition is not contained in our definition of $S$-weaves, and the meaning of approximation is different in both cases. Note, however, that our definition and the existence proof are not trivial, because it is not obvious how to construct finite linear combinations of spin networks whose inner product with any other spin network is approximately independent of its position (already before solving the diffeomorphism constraint). But a connection between both concepts of weave states exists. For suppose that we solved the Hamiltonian constraint of the isotropic model associated with Bianchi I [7], and we found a distinguished solution representing the unique classical solution, namely Minkowski space-time, we can approximate this solution by an $S$-weave for its associated distributional state. This $S$-weave is then expected to contain, besides its approximate isotropy, geometrical information approximating the Euclidean metric of space.

However, the $S$-weaves are only approximately symmetric. They manifestly break the symmetry, and therefore applying the volume operator will not lead to a spectrum of the simple form (29), but rather of the form of the complicated spectrum in the full theory. Consequences of this fact will now be considered in a final subsection.

5.4 Cosmology

In cosmology one usually reduces a theory of gravity classically to homogeneous metrics reducing the degrees of freedom to finitely many. Dynamics is then encoded in the Wheeler–DeWitt equation which is a hyperbolic differential equation with respect to the scale factor [20]. To account for fluctuations which are necessary for structure formation, however, one has to disturb the homogeneous geometries and can treat the ensuing inhomogeneities as perturbations [21]. In an appropriate neighborhood of homogeneous models the Wheeler–DeWitt equation will remain hyperbolic in one variable [22].

But we have seen that in a theory using quantized geometries even small perturbations have drastic effects: We can approximate a symmetric state arbitrarily precisely by $S$-weaves, which represent slightly perturbed symmetric metrics, but the volume spectrum of these states will never reach the simple spectrum (29) for isotropy. This is so because $S$-weaves are ordinary, however complicated, spin network states which, when chosen as volume eigenstates, correspond to a volume eigenvalue with a large vertex sum. Thus, we see that homogeneous metrics are very special in a quantum theory, and in view of the present paper results obtained with minisuperspace models are unlikely to be reproduced in a full quantum theory of gravity. Even minor perturbations break the special features
of symmetric states, e.g. concerning the volume spectrum, to full extent.

Of course, up to now our discussion remained at the kinematical level, and the role of these kinematical properties after solving the Hamiltonian constraint has not been investigated yet. However, for dynamics the volume operator plays an important role, too, for it appears quite naturally in the quantized Hamiltonian constraint \[6, 7\]. Therefore, the kinematical volume spectrum is significant for dynamics, and its features associated with symmetry reduction should be expected to have a great impact on dynamics.

Finally, we note that the models discussed here may provide new insights into the issue of the Hamiltonian constraint. Already the simple geometries of reduced spin networks (for instance, only one edge and an insertion for isotropic models) simplify its action considerably. Maybe more important is the fact that the volume spectrum is simplified, and even completely known in case of isotropy, which facilitates determining the matrix elements of the constraint.

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