Non-Abelian Toda lattice and analogs of Painlevé III equation

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March 17, 2022

Abstract

In integrable models, stationary equations for higher symmetries serve as one of the main sources of reductions consistent with dynamics. We apply this method to the non-Abelian two-dimensional Toda lattice. It is shown that already the stationary equation of the simplest higher flow gives a non-trivial non-autonomous constraint that reduces the Toda lattice to a non-Abelian analog of the pumped Maxwell–Bloch equations. The Toda lattice itself is interpreted as an auto-Bäcklund transformation acting on the solutions of this system. Further self-similar reduction leads to non-Abelian analogs of the Painlevé III equation.

Keywords: non-Abelian Toda lattice, self-similar solution, Painlevé equation

1 Introduction

The non-Abelian Toda lattice [1]

\[(g_n x g_n^{-1})_y = g_{n+1} g_n^{-1} - g_n g_{n-1}^{-1}\]

is one of the fundamental three-dimensional models (two continuous independent variables \(x, y\) and one discrete variable \(n\)). If the field variables are scalar then the substitution \(g_n = e^{u_n}\) is possible which leads to the equation \(u_{n,xy} = e^{u_{n+1} - u_n} - e^{u_{n-1} - u_n}\) [2]. In the non-Abelian setting, we assume that \(g_n\) are elements of an arbitrary non-commutative algebra \(\mathcal{A}\) with identity element 1 and the operation of taking the inverse (for example, a matrix algebra). The polynomial form of the Toda lattice [3]

\[f_{n,y} = p_n - p_{n+1}, \quad p_{n,x} = f_n p_n - p_n f_{n-1}\]  

is obtained by the substitution \(p_n = -g_n g_n^{-1}\) and \(f_n = g_n x g_n^{-1}\). The aim of our work is to construct a reduction of this lattice equation to non-Abelian analogs of the Painlevé equation \(P_3\)

\[w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{1}{2}(\alpha w^2 + \beta) + \gamma w^3 + \delta w.\]  

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The reduction is carried out in two stages. First, we reduce (1) to a two-dimensional system. The easiest way to do this is related with the 2-periodicity condition, which leads to the non-Abelian sinh–Gordon equation [3, 4]. In the scalar case, other boundary conditions are also known, leading to exponential systems associated with Cartan matrices for simple Lie algebras [5]. The reduction we use is of a different type and, as far as we know, has not been considered before (although it leads to some well-known equations). It has the form of some non-autonomous constraint and is related to the stationary equation of the simplest higher symmetry from the Toda lattice hierarchy. This reduction is defined in Section 2. It leads to a two-dimensional system, which can be considered as a non-Abelian analog of the pumped Maxwell–Bloch equations [6]. In this case, the auxiliary linear equations for the Toda lattice are transformed into the zero curvature representation with variable spectral parameter for this system, and the shift in \( n \) is interpreted as a Bäcklund transformation, see Section 3.

At the next stage (sections 4 and 5), we apply self-similar reductions, which lead to third order ODE systems of Painlevé type. In the non-Abelian case, these systems have no global first integrals; however, the reduction of the order of equations is possible on some special invariant submanifolds. This brings to non-Abelian analogs of \( P_3 \) with full and degenerate sets of parameters. The corresponding isomonodromic Lax pairs and Bäcklund transformations are also derived. These results generalize the scalar reductions described in [7, 8, 9, 10].

It should be noted that analogs of the Painlevé equations under study are written not as a single second order equation, but as a coupled system of two first order equations. If the variables are scalar, then the exclusion of one of them and some additional transformations lead to \( P_3 \) in the standard form, but in the non-Abelian case this is not always possible (at least, if we restrict ourselves to the operations of addition, multiplication, and taking the inverse element). Non-Abelian analogs of the Painlevé equations are being actively studied, see for example [11, 12]. It is known that a scalar equation can admit several non-equivalent non-Abelian analogs. Some classification results based on the Kovalevskaya–Painlevé test and other approaches can be found in recent papers [13, 14, 15]. Here we do not try to reproduce all known analogs of \( P_3 \) and restrict ourselves to only those that are obtained as a result of our reduction.

## 2 Reduction of the Toda lattice

We start by defining a constraint that is compatible with the Toda lattice. The following assertion is central to our construction, so we present it with a detailed proof. We then explain the origin of this reduction and show how it extends to linear equations for \( \psi \)-functions.

**Theorem 1.** The non-Abelian Toda lattice (1) admits the constraint

\[
f_{n-1} + f_n = \mu_n p_n^{-1}, \quad \mu_n := \varepsilon n + \mu_0, \quad \varepsilon, \mu_0 \in \mathbb{C},
\]

(3)

**Proof.** 1) The case \( \varepsilon = \mu_0 = 0 \) amounts to the 2-periodicity condition (since the differentiation of the equality \( f_{n-1} + f_n = 0 \) with respect to \( y \) implies \( p_{n-1} = p_{n+1} \)). Suppose now that \( \mu_n \neq 0 \), then the equality \( \mu_n = 0 \) is possible for at most one value of \( n \).

2) Consider first the case \( \mu_n \neq 0 \) for all \( n \). Then the constraint (3) leads to a separation of variables and the Toda lattice reduces to the pair of two-dimensional equations

\[
f_{n,x} + f_{n+1,x} = f_n^2 - f_{n+1}^2,
\]

(4)

\[
f_{n,y} = \mu_n (f_{n-1} + f_n)^{-1} - \mu_{n+1} (f_n + f_{n+1})^{-1}.
\]

(5)
Indeed,
\[ f_{n-1,x} + f_{n,x} = \mu_n (p_n^{-1})_x = -\mu_n p_n^{-1} f_n + \mu_n f_{n-1} p_n^{-1} = -(f_{n-1} + f_n) f_n + f_{n-1} f_n = f_{n-1}^2 - f_n^2. \] (6)

Conversely, if the variables \( f_n \) satisfy the pair (4) and (5) then we obtain a solution of the Toda lattice by setting \( p_n = \mu_n (f_{n-1} - f_n)^{-1} \). To prove the theorem, it suffices to verify that equations (4) and (5) are consistent, which is expressed by the equality
\[ D_y (f_n^2 - f_{n+1}^2) = D_x (\mu_n (f_{n-1} + f_n)^{-1} - \mu_{n+2} (f_{n+1} + f_{n+2})^{-1}). \]

We write this as \( a_n - a_{n+1} = 0 \), where
\[ a_n = D_y (f_n^2) - D_x (\mu_n (f_{n-1} + f_n)^{-1} + \mu_{n+1} (f_n + f_{n+1})^{-1}). \]

First, we transform one term from the right side. From (6) it follows
\[ -D_x ((f_{n-1} + f_n)^{-1}) = f_{n-1} (f_{n-1} + f_n)^{-1} - (f_{n-1} + f_n)^{-1} f_n. \]

Then
\[
\begin{align*}
\mu_n (f_{n-1} + f_n)^{-1} - \mu_{n+1} & (f_n + f_{n+1})^{-1} \\
+ \mu_n (f_{n-1} + f_n)^{-1} & - \mu_{n+1} (f_n + f_{n+1})^{-1} f_n \\
+ \mu_n (f_{n-1} + f_n)^{-1} - (f_{n-1} + f_n)^{-1} f_n & \\
+ \mu_{n+1} (f_n + f_{n+1})^{-1} & - (f_n + f_{n+1})^{-1} f_{n+1}
\end{align*}
\]

(here and below we identify the number \( \mu \in \mathbb{C} \) with the element \( \mu 1 \in A \), where 1 is the unit element of the noncommutative algebra), therefore \( a_n - a_{n+1} = \mu_n - 2 \mu_{n+1} + \mu_{n+2} \), which is 0 for \( \mu_n = \varepsilon n + \mu_0 \).

3) If \( \mu_k = 0 \) for some \( k \) then \( \varepsilon \neq 0 \) and \( \mu_0 = -\varepsilon k \). We can set \( \mu_n = n \) without loss of generality, by scaling and changing \( n \to n - k \). Then it is easy to see that the solution satisfies
\[ f_{-n} = -f_{-n-1}, \quad p_{-n} = p_n, \quad n = 1, 2, \ldots \] (7)

with the lattice equations (5) and (4) restricted to the half-line \( n = 1, 2, \ldots \):
\[
\begin{align*}
f_{0,y} &= p_0 - (f_0 + f_1)^{-1}, \quad f_{n,y} = n (f_{n-1} + f_n)^{-1} - (n + 1) (f_n + f_{n+1})^{-1}, \\
p_{0,x} &= f_0 p_0 + p_0 f_0, \quad f_{n-1,x} + f_{n,x} = f_{n-1}^2 - f_n^2.
\end{align*}
\] (8)

The compatibility of these equations is proved in the same way as before, except that when checking the equation \( (f_0 + f_1)_x = (f_0 + f_1)_y \), the equality arises
\[ (p_0 - 2 (f_1 + f_2)^{-1})_x = (f_0^2 - f_1^2)_y. \]

It is easy to verify that it holds identically. \( \square \)

**Remark 1.** It is possible to write the constraint (3) in the form \( p_n = \mu_n (f_{n-1} - f_n)^{-1} \), but this is not quite equivalent, since in the case 3) for \( \mu_n = n \) it leads to the boundary condition \( p_0 = 0 \) instead of \( f_{-1} + f_0 = 0 \). Then the equation \( p_{0,x} = f_0 p_0 - p_0 f_{-1} \) holds automatically, while the reflection conditions (7) are not necessary. As a result, equations (5) and (4) turn into two independent subsystems, one for the variables \( f_0, f_1, \ldots \) and another for \( f_{-1}, f_{-2}, \ldots \). Each of these subsystems is equivalent to (8) with \( p_0 = 0 \). Therefore, the constraint in the form (3) is a bit more general. The equation \( (f_{-1} + f_0) p_0 = 0 \) may also have solutions with zero divisors, but we will not analyze the resulting boundary conditions.
Note that (4) and (5) are themselves well-known integrable lattice equations. Equation (4) defines Bäcklund transformations with zero parameters for the equation \( f_t = f_{xxx} - 3f^2f_x - 3f_xf^2 \) which is one of two non-Abelian versions of the modified KdV equation. The substitution \( v_n = p_n/\mu_n \) turns (5) into

\[
v_{n,y} = v_n(p_{n+1}v_{n+1} - \mu_n v_{n-1} - \mu_n v_n).
\]

For \( \mu_n = 1 \), this is the modified Volterra lattice (again, one of two non-Abelian versions), and for \( \mu_n = n \) this is its master-symmetry. Master-symmetries for scalar equations of Volterra lattice type were studied in [16, 17], some non-Abelian generalizations and reductions to the Painlevé equations were found in [18]. In the scalar case, the consistency of equations (4) and (5) was noted in [19] (for \( \mu_n = 1 \)) and in [20] (for \( \mu_n = n \)), but the fact that they are both embedded in the two-dimensional Toda lattice went unnoticed. Similar results are also known regarding the compatibility of differential-difference equations with discrete equations on a square lattice, see e.g. [21].

Let us demonstrate that the reduction (3) is related to the stationary equation for the simplest higher symmetry of the Toda lattice. Recall that this lattice itself serves as a compatibility condition for the linear equations

\[
\psi_{n,x} = \psi_{n+1} + f_n \psi_n, \quad \psi_{n,y} = p_n \psi_{n-1},
\]

and its symmetries are defined as compatibility conditions of (9) with equations of the form

\[
\psi_{n,t_k} = \psi_{n+k} + h_n^{(k,1)} \psi_{n+k-1} + \cdots + h_n^{(k,k)} \psi_n.
\]

The gauge \( h_n^{(k,1)} = f_n + \cdots + f_{n+k-1} \) can be chosen without loss of generality. In particular, for \( k = 2 \) we have the equation

\[
\psi_{n,t} = \psi_{n+2} + (f_n + f_{n+1}) \psi_{n+1} + h_n \psi_n
\]

and the conditions of its compatibility with (9) are of the form

\[
f_{n,x} + f_{n+1,x} = f_n^2 - f_{n+1}^2 - h_n + h_{n+1}, \quad h_n, x = f_n, h_n, \quad h_n, x = f_n, h_n, [f_n, h_n] = f_n, h_n, [f_n, h_n],
\]

\[
h_n, y = p_n (f_{n-1} + f_n) - (f_n + f_{n+1}) p_{n+1}, \quad p_n, t = h_n p_n - p_n h_{n-1}.
\]

It is easy to see that if all field variables are commutative, then the requirement that \( f_n \) and \( p_n \) be independent of \( t \) implies that \( h_n \) is independent of \( x \) and \( n \), and then equations are reduced to (4) and (3), up to a transformation of \( y \). Thus, in the scalar case the constraint (3) is equivalent to the stationary equation for the higher symmetry. In the non-Abelian setting, the stationary equation is more general, but it still holds true by virtue of the constraint (3), if we choose \( h_n = -\varepsilon y - \kappa \in \mathbb{C} \).

Equations (9) and the stationary equation (10)

\[
\psi_{n+2} + (f_{n+1} + f_n) \psi_{n+1} = (\varepsilon y + \kappa) \psi_n
\]

can be transformed into zero curvature representations for the reduction under study, with \( \kappa \) playing the role of a spectral parameter. Indeed, let us write these equations in the matrix form

\[
\Psi_{n,x} = U_n \Psi_n, \quad \Psi_{n,y} = V_n \Psi_n, \quad \Psi_{n+1} = W_n \Psi_n,
\]

where \( \Psi_n = (\psi_n, \psi_{n+1})^T \) and

\[
U_n = \begin{pmatrix} f_n & 1 \\ \varepsilon y + \kappa & -f_n \end{pmatrix}, \quad V_n = \begin{pmatrix} \mu_n & p_n \\ \varepsilon y + \kappa & \frac{p_n}{\mu_n+1} \end{pmatrix}, \quad W_n = \begin{pmatrix} 0 & 1 \\ \varepsilon y + \kappa & -f_n - f_{n+1} \end{pmatrix}.
\]
Then the compatibility conditions are
\[ W_{n,x} = U_{n+1}W_n - W_nU_n, \quad W_{n,y} = V_{n+1}W_n - W_nV_n, \quad U_{n,y} = V_{n,x} + [V_n, U_n] \]
and it is easy to check that these equations are equivalent to (1) and (3). The above matrices can be brought to a more symmetric form by introducing the variable spectral parameter \( \lambda^2 = \varepsilon y + \kappa \) (this leads to appearance of a term with \( \partial \lambda \) in the linear equations for \( \psi \)-functions) [6]. Simple calculations bring to the following representation.

**Proposition 2.** The Toda lattice equations (1) with the constraint (3) are equivalent to equations

\[ W_{n,x} = U_{n+1}W_n - W_nU_n, \quad W_{n,y} + \frac{\varepsilon}{2\lambda} W_{n,\lambda} = V_{n+1}W_n - W_nV_n, \]
\[ U_{n,y} + \frac{\varepsilon}{2\lambda} U_{n,\lambda} = V_{n,x} + [V_n, U_n] \]  \hspace{1cm} (13)

with the matrices

\[ U_n = \begin{pmatrix} f_n & \lambda \\ \lambda & -f_n \end{pmatrix}, \quad V_n = \lambda^{-1} \begin{pmatrix} \mu_n + \mu_{n+1} \\ 2\lambda p_{n+1} \\ p_n \end{pmatrix}, \quad W_n = \begin{pmatrix} 0 & \lambda \\ \lambda & -f_n - f_{n+1} \end{pmatrix} \]  \hspace{1cm} (14)

### 3 Partial differential systems

Due to the constraint (3), the Toda lattice (1) turns into a closed system for the variables \( f = f_n, p = p_n, q = p_{n+1} \) and parameters \( \mu = \mu_n, \nu = \mu_{n+1} \), for any \( n \):

\[ f_y = p - q, \quad p_x = fp + pf - \mu, \quad q_x = -fq - qf + \nu \]  \hspace{1cm} (15)

(recall that \( \mu \) and \( \nu \) are understood as scalars multiplied by 1 \( \in \mathbb{A} \)). Denoting the shift action \( n \mapsto n + 1 \) in the lattice with a tilde, we reformulate Theorem 1 and Proposition 2 as the following statements, which can also be verified by direct calculations.

**Proposition 3.** The Bäcklund transformation

\[ \tilde{p} = q, \quad \tilde{q} = p + \nu q^{-1} q p^{-1}, \quad \tilde{f} = -f + \nu q^{-1}, \quad \tilde{\mu} = \nu, \quad \tilde{\nu} = -\mu + 2\nu \]  \hspace{1cm} (16)

maps the solution \( f, p, q \) of the system (15) with parameters \( \mu, \nu \) into the solution \( \tilde{f}, \tilde{p}, \tilde{q} \) of the same system with parameters \( \tilde{\mu}, \tilde{\nu} \).

**Proposition 4.** The system (15) admits the representation

\[ U_y + \frac{\nu - \mu}{2\lambda} U_{\lambda} = V_x + [V, U], \]  \hspace{1cm} (17)

and its Bäcklund transformation (16) admits the representation

\[ W_x = \tilde{U} W - W U, \quad W_y + \frac{\nu - \mu}{2\lambda} W_{\lambda} = \tilde{V} W - W V, \]  \hspace{1cm} (18)

where

\[ U = \begin{pmatrix} f & \lambda \\ \lambda & -f \end{pmatrix}, \quad V = \frac{1}{\lambda} \begin{pmatrix} \mu + \nu \\ 2\lambda q \end{pmatrix}, \quad W = \begin{pmatrix} 0 & \lambda \\ \lambda & -\nu q^{-1} \end{pmatrix} \]  \hspace{1cm} (19)
In the commutative case, the elimination of \( p \) brings the system (15) to the equation

\[
ff_{xy} = f_x f_{xy} + 4f^3 f_y + (\mu + \nu) f_x + 2(\nu - \mu) f^2.
\]

For \( \nu = \mu \), this is the so-called “negative” symmetry of the mKdV equation \( f_t = f_{xxx} - 6f^2 f_x \). On the other hand, we can compare (15) with the pumped Maxwell–Bloch system introduced in [6]:

\[
E_y = \rho, \quad \rho_x = N E, \quad 2N_x = -\rho^* E - \rho E^* + 2c, \quad E, \rho \in \mathbb{C}, \quad N \in \mathbb{R},
\]

where \( c \) is the pumping parameter. For \( \rho, E \in \mathbb{R} \), these equations are simplified to the system

\[
E_y = \rho, \quad \rho_x = N E, \quad N_x = -\rho E + c \tag{20}
\]

studied in [7, 8, 9, 10] where it was shown that it admits a selfsimilar reduction to the \( P_3 \) equation. The system (20) and the scalar system (15) are related by the simple change

\[
2f = iE, \quad 4p = N + i\rho, \quad 4q = N - i\rho, \quad c = 4\mu = -4\nu.
\]

Thus, the system (15) can be viewed as a non-Abelian generalization both for the negative mKdV flow with an additional parameter \( \nu - \mu \) and for the real pumped Maxwell–Bloch system with an additional parameter \( \nu + \mu \).

The system (15) with \( \mu = \nu = 0 \)

\[
f_y = p - q, \quad p_x = fp + pf, \quad q_x = -f q - qf \tag{21}
\]

is a degenerate case corresponding to the constraint (3) with \( \mu_n = 0 \), that is, to the 2-periodic boundary condition

\[
f_{2n} = f, \quad f_{2n+1} = -f, \quad p_{2n} = p, \quad p_{2n+1} = q.
\]

**Remark 2.** The general 2-periodicity condition \( f_{n+2} = f_n, p_{n+2} = p_n \) is also equivalent to (21). Indeed, we have \( f_n + f_{n+1} y = p_n - p_{n+2} = 0 \), that is \( f_n + f_{n+1} = a(x) \), and it is possible to set \( a = 0 \) without loss of generality by the gauge transformation \( f_n = A f_n A^{-1} + A_{x} A^{-1}, \quad p_n = A p_n A^{-1} \) with \( 2A_{x} A^{-1} = a \).

For the system (21), an additional constraint \( pq = \beta(y) \in \mathbb{C} \) is possible due to the relation \( (pq)_x = [f, pq] \); moreover, one can set \( \beta = 1 \) by the change \( (p, q, \partial_y) \to \beta^{1/2} (p, q, \partial_y) \). This brings to a more special reduction

\[
f_{2n} = f, \quad f_{2n+1} = -f, \quad p_{2n} = p, \quad p_{2n+1} = p^{-1}
\]

and the non-Abelian sinh-Gordon equation [3]

\[
f_y = p - p^{-1}, \quad p_x = fp + pf. \tag{22}
\]

The mapping (16) in this case turns into the trivial change \( \tilde{f} = -f, \tilde{p} = p^{-1} \). The zero curvature representation for (22) is of the form

\[
U_y - V_x = [V, U], \quad U = \begin{pmatrix} f & \lambda \\ \lambda & -f \end{pmatrix}, \quad V = \lambda^{-1} \begin{pmatrix} 0 & p \\ p^{-1} & 0 \end{pmatrix}. \tag{23}
\]

Notice, that the systems (21) and (22) with scalar variables are equivalent since the relation \( pq = \beta(y) \) is the first integral. In the non-Abelian case, this is only a partial first integral, so the system (21) is more general than (22). Another difference is that the scalar system (22) easily reduces to the sinh-Gordon equation in rational form

\[
p_{xy} = p_x p_u + 2p^2 - 2, \tag{24}
\]

while in the non-Abelian case the elimination of \( f \) only by use of non-commutative algebra operations is impossible.
4 Sinh-Gordon equation and non-Abelian analog of $P_3^{(8)}$

A self-similar reduction of the general system (15) brings to a system of three first-order ODEs. In the following sections, we show that this system has a partial first integral (which did not exist before the reduction), which allows us to reduce the order and to obtain an analog of the $P_3$ equation with a full set of parameters. In this section, we start with a simpler case of the sinh-Gordon equation (22), which is already second order.

Obviously, the scalar equation (24) is invariant under the Lorentz group $(x, y) \mapsto (\varepsilon x, y/\varepsilon)$. Therefore, the self-similar substitution is possible

$$p(x, y) = p(z), \quad z = -2xy,$$

which brings to equation (2) with parameters $\alpha = 1, \beta = -1, \gamma = \delta = 0$ for the variable $p(z)$:

$$p'' = \frac{(p')^2}{p} - \frac{p'}{z} + \frac{p^2 - 1}{z}. \quad (25)$$

This is the simplest and most well-studied case of $P_3$ known as $P_3^{(8)}$ equation (the classification of different cases of $P_3$ is given, for example, in [22], see also [23, 24, 25]).

In the non-Abelian case, the reduction is essentially just as simple, but it can be generalized slightly by adding the conjugation by elements of the form

$$y^a := \exp(a \log y), \quad a \in A,$$

where $a$ is an arbitrary non-Abelian constant. An analog of equation (25) is the system (26) from the following Proposition.

**Proposition 5.** The non-Abelian sinh-Gordon equation (22) admits the self-similar reduction

$$p(x, y) = y^a p(z) y^{-a}, \quad f(x, y) = -2y^{1+a} f(z) y^{-a}, \quad z = -2xy, \quad a \in A,$$

where $f(z)$ and $p(z)$ satisfy the equations

$$zf' = \frac{1}{2} (p - p^{-1}) - f - [a, f], \quad p' = fp + pf. \quad (26)$$

This system admits the isomonodromic Lax pair $A' = B_\zeta + [B, A]$ with the matrices

$$A = \begin{pmatrix} (a + zf)/\zeta & p/\zeta^2 - z/2 \\
-p^{-1}/\zeta^2 - z/2 & (a - zf)/\zeta \end{pmatrix}, \quad B = \begin{pmatrix} f & -\zeta/2 \\
-\zeta/2 & -f \end{pmatrix}.$$

**Proof.** Equations (26) are obtained straightforwardly. In order to obtain the matrices $A$ and $B$, we apply an additional change of the spectral parameter $\lambda = y\zeta$, then the matrices (23) take the form

$$U = y^{1+a} \tilde{U} y^{-a}, \quad \tilde{U} = \begin{pmatrix} -2f/\zeta & \zeta \\
\zeta^2 & 2f \end{pmatrix}, \quad V = -\zeta^{-1} y^{-1+a} \tilde{V} y^{-a}, \quad \tilde{V} = \begin{pmatrix} 0 & p \\
p^{-1} & 0 \end{pmatrix},$$

and the derivatives are replaced according to the rule

$$\partial_x \rightarrow -2y \partial_z, \quad \partial_y \rightarrow \partial_y + \frac{z}{y} \partial_z - \frac{\zeta}{y} \partial_\zeta.$$

Then the dependence on $y$ in the equation $U_y = V_x + [V, U]$ is canceled out and it takes the form

$$(z\tilde{U} - 2\zeta^{-1} \tilde{V})' = \zeta \tilde{U}_\zeta + [\tilde{U}, \zeta^{-1} \tilde{V} + a].$$

The above Lax pair appears as a result of the changes $-2B = \tilde{U}$ and $A = \zeta^{-2} \tilde{V} + \zeta^{-1} (zB + a)$. \qed
5 Self-similar reduction of system (15)

5.1 Third order ODE system

The scaling group for the system (15) is different from the group for the sinh-Gordon equation: for (22) the homogeneity weights are \( \rho(\partial_x) = -\rho(\partial_y) = \rho(f) = 1 \) and \( \rho(p) = 0 \), while for (15), as it is easy to see,

\[
\rho(\partial_x) = 1, \quad \rho(\partial_y) = -2, \quad \rho(f) = 1, \quad \rho(p) = \rho(q) = -1.
\]

Accordingly, an independent self-similar variable should be \( z = xy^{1/2} \) rather than \( xy \) and we arrive at the self-similar substitution

\[
\begin{align*}
    f(x,y) &= y^{1/2-a} f(z) y^a, \quad p(x,y) = y^{-1/2-a} p(z) y^a, \quad q(x,y) = y^{-1/2-a} q(z) y^a, \quad z = xy^{1/2}, \quad a \in A. 
\end{align*}
\]

(27)

Proposition 6. The reduction (27) in equations (15) brings to the system

\[
(z f)' = 2p - 2q + 2[a,f], \quad p' = fp + pf - \mu, \quad q' = -fq - qf + \nu,
\]

(28)

which is invariant under the Bäcklund transformation

\[
\begin{align*}
    \tilde{p} &= q, \quad \tilde{q} = p - \frac{\nu z}{2} (fq^{-1} + q^{-1}f) - \frac{\nu^2 z}{2} q^{-2} + \nu[a,q^{-1}], \\
    \tilde{f} &= -f + \nu q^{-1}, \quad \tilde{\mu} = \nu, \quad \tilde{\nu} = -\mu + 2\nu.
\end{align*}
\]

(29)

Proof. Equations (28) are obtained straightforwardly. The transformation (16) preserves the homogeneity with respect to the above weights. Therefore, if a solution \( f(x,y), p(x,y), q(x,y) \) of the system (15) possesses the self-similar structure (27), then this is true also for the new solution \( \tilde{f}(x,y), \tilde{p}(x,y), \tilde{q}(x,y) \). Hence, it is also described by a system of the form (28) and we only have to rewrite equations (16) under the reduction (27). \( \square \)

Applying the reduction to equations (17) and (18), we obtain the following proposition. Note that the representations (30), (31) can be brought to the standard form with unit coefficient at \( \partial_\zeta \) by dividing \( A \) by \( \zeta^2 - \nu + \mu \).

Proposition 7. The system (28) admits the isomonodromic Lax pair

\[
A' = (\zeta^2 - \nu + \mu)B_\zeta + [B,A]
\]

(30)

and the transformation (29) is equivalent to the equations

\[
K' = \tilde{B}K - KB, \quad (\zeta^2 - \nu + \mu)K_\zeta = \tilde{A}K - KA
\]

(31)

with the matrices

\[
A = \begin{pmatrix}
    \zeta z f - 2\zeta a - \frac{\nu z}{2} \\
    \zeta^2 z - 2q
\end{pmatrix}
\begin{pmatrix}
    \zeta^2 z - 2p \\
    -\zeta z f - 2\zeta a + \zeta \kappa
\end{pmatrix},
\]

(32)

\[
B = \begin{pmatrix}
    f \\
    -f
\end{pmatrix}, \quad K = \begin{pmatrix}
    0 & \zeta \\
    \zeta & -\nu q^{-1}
\end{pmatrix},
\]

where \( \kappa \) in \( A \) is an additional scalar parameter such that \( \kappa = \kappa + 1 \).
Proof. We extend the substitution (27) with the relation \( \lambda = y^{1/2} \zeta \), then \( y \) in the matrices (19) is separated out:

\[
U = y^{1/2-a} B y^a, \quad V = y^{-1-a} C y^a, \quad W = y^{1/2-a} K y^a, \quad C = \frac{1}{\zeta} \left( \frac{\mu+\nu}{2q} \begin{pmatrix} p \\ 0 \end{pmatrix} \right).
\]

The derivatives are replaced according to the rule

\[
\partial_z \rightarrow y^{-1/2} \partial_z, \quad \partial_y \rightarrow \partial_y + \frac{z}{2y} \partial_z - \frac{\zeta}{2y} \partial_\zeta, \quad \partial_\lambda \rightarrow y^{-1/2} \partial_\zeta.
\]

As a result, equation (17) transforms to

\[
\zeta (zB - 2C)' = (\zeta^2 - \nu + \mu) B\zeta - [B, 2\zeta(C + a)].
\]

In order to bring this relation to the form (30) we only have to denote \( A = \zeta(zB - 2C - 2a + \kappa) \).

In the scalar case, the order of the system (28) can be reduced by use of a first integral. We will demonstrate that in the non-Abelian case this is possible due to a partial first integral, that is, equations (28) can be restricted to some invariant submanifold \( J = 0 \) which is also preserved under the Bäcklund transformation (29). This leads to a second order system, which is a non-Abelian analog of \( P_3 \). We study the cases of \( \nu = \mu \) and \( \nu \neq \mu \) separately (this corresponds to the constraint (3) with \( \epsilon = 0 \) and \( \epsilon \neq 0 \)).

5.2 The case \( \nu = \mu \): a non-Abelian analog of \( P_3^{(7)} \)

Proposition 8. The system (28) with \( \nu = \mu \neq 0 \) admits the invariant submanifold

\[
J(\kappa) = 2pq - \mu(zf - 2a - \kappa) = 0, \quad (33)
\]

which is mapped to the submanifold \( \tilde{J}(\tilde{\kappa}) = 0 \) with \( \tilde{\kappa} = \kappa + 1 \) under the transformation (29).

Proof. The partial first integral (33) is easily derived: we have

\[
(2pq)' = 2[f, pq] + 2\mu(p - q) = 2[f, pq] + \mu(zf)' + 2\mu[f, a] \Rightarrow J' = [f, J],
\]

which implies the invariance of the equation \( J = 0 \). In a similar way, it is easy to check that \( \tilde{J}(\tilde{\kappa}) = qJ(\kappa)q^{-1} \), which proves the invariance with respect to the Bäcklund transformation. \( \square \)

On the level set \( J = 0 \), we have \( 2q = \mu p^{-1}(zf - 2a - \kappa) \) and (28) turns into a second order system. A direct calculations bring to the following formulas.
Proposition 9. The system
\[(zf)' = 2p - \mu p^{-1}(zf - 2a - \kappa) + 2[a, f], \quad p' = fp + pf - \mu\] (34)
admits the Bäcklund transformation
\[\tilde{p} = \frac{\mu}{2} p^{-1}(zf - 2a - \kappa), \quad \tilde{f} = -f + \mu \tilde{p}^{-1}, \quad \tilde{\mu} = \mu, \quad \tilde{\kappa} = \kappa + 1\]
and is equivalent to the Lax representation \(A' = \zeta^2 B_\zeta + [B, A]\), where
\[A = \begin{pmatrix} \zeta zf - 2\zeta a - 2a & \zeta \kappa \\ \zeta^2 z - \mu p^{-1}(zf - 2a - \kappa) & -\zeta zf - 2\zeta a + \zeta \kappa \end{pmatrix}, \quad B = \begin{pmatrix} f & \zeta \\ \zeta & -f \end{pmatrix}\].

It is easy to check that if \(f, p\) and \(a\) are scalars then the elimination of \(f\) brings (34) to the equation
\[p'' = \frac{p'^2}{p} - \frac{p'}{z} + \frac{1}{z}(4p^2 + \mu(4a - 1 + 2\kappa)) - \frac{\mu^2}{p},\]
which is (2) with the values of parameters
\[\alpha = 4, \quad \beta = \mu(4a - 1 + 2\kappa), \quad \gamma = 0, \quad \delta = -\mu^2,\]
that is, the intermediate \(P_3^{(7)}\) equation.

5.3 The case \(\nu \neq \mu\): a non-Abelian analog of \(P_3^{(6)}\)

Proposition 10. The system (28) with \(\nu - \mu = \varepsilon \neq 0\) admits the invariant submanifold
\[J(\kappa) = 2q - \varepsilon z + \varepsilon(zf + 2a - \kappa)(2p - \varepsilon z)^{-1}(zf - 2a + \kappa - 2\mu/\varepsilon - 1) = 0,\] (35)
which is mapped to the submanifold \(\tilde{J}(\tilde{\kappa}) = 0\) with \(\tilde{\kappa} = \kappa + 1\) under the transformation (29).

Proof. In this case the partial first integral is less obvious, but we can find it with the help of the representation (30). For \(\zeta = \varepsilon^{1/2}\), the \(2 \times 2\) matrix \(A\) satisfies the Lax equation \(A' = [B, A]\). It is easy to prove that the quasideterminant \(|A|_{12} = a_{12} - a_{11} a_{21}^{-1} a_{22}\) is a partial first integral for such an equation and this gives the expression (35), up to a scalar factor.

Moreover, for \(\zeta = \varepsilon^{1/2}\) we have \(\tilde{A}K = KA\), where \(K = \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & -\nu q^{-1} \end{pmatrix}\), and from here it is easy to obtain the relation \(|A|_{21} = |A|_{12}\). Taking into account the homological relations for quasideterminants, this proves that the submanifold \(J = 0\) is preserved under the Bäcklund transformation. \(\Box\)

Remark 3. For the case \(\varepsilon = 0\) from Section 5.2, the above proof is not directly applicable, since \(A\) contains a term with \(\zeta^{-1}\). Nevertheless, the integral (35) can be defined also in this case by combining \(\varepsilon\) with the last factor and setting \(\varepsilon = 0\). This brings to the constraint
\[2q - (zf + 2a - \kappa)p^{-1}\mu = 0 \quad \Rightarrow \quad 2qp - \mu(zf + 2a - \kappa) = 0.\]
It is slightly different from (33), but it also defines an invariant submanifold for (28) with \(\nu = \mu\). In order to obtain (33), we can apply the change \(\kappa - 2\mu/\varepsilon = -\hat{\kappa}\) (35). Since \(\tilde{\mu} = \mu + \varepsilon\), the new parameter is also transformed according to the rule \(\hat{\kappa} = \tilde{\kappa} + 1\), then (35) takes the form
\[2q - \varepsilon z + \varepsilon(zf + 2a + \hat{\kappa} - 2\mu/\varepsilon)(2p - \varepsilon z)^{-1}(zf - 2a - \hat{\kappa}) = 0,\]
and passing to the limit \(\varepsilon \to 0\) in this expression, we obtain
\[2q - \mu p^{-1}(zf - 2a - \hat{\kappa}) = 0 \quad \Rightarrow \quad 2pq - \mu(zf - 2a - \hat{\kappa}) = 0.\]
Thus, the constraint (35) contains two constraints of type (33) as the limiting cases.
The elimination of \( q \) by use of (35) brings to the system (36) below. As before, its Lax representation is obtained from the general formulas (30) and (31) by replacing \( q \); the matrix \( A \) becomes rather unwieldy and we do not write it explicitly.

**Proposition 11.** The system

\[
(zf)' = 2p - \varepsilon z + \varepsilon(zf + 2a - \kappa)(2p - \varepsilon z)^{-1}(zf - 2a + \kappa - 2\mu/\varepsilon - 1) + 2[a, f], \\
p' = fp + pf - \mu
\]  

(36)
is invariant with respect to the B"acklund transformation

\[
\tilde{p} = \varepsilon z - \varepsilon(zf + 2a - \kappa)(2p - \varepsilon z)^{-1}(zf - 2a + \kappa - 2\mu/\varepsilon - 1), \\
\tilde{f} = -f + (\mu + \varepsilon)\tilde{p}^{-1}, \quad \tilde{\mu} = \mu + \varepsilon, \quad \tilde{\kappa} = \kappa + 1.
\]

This system gives an analog of \( P_3 \) equation with generic parameters. In order to demonstrate this, let us compare it with (2) assuming that all variables are scalar. In this case we can set \( a = 0 \) without loss of generality (this is equivalent to changing of \( \kappa \)), which gives the system

\[
(zf)' = 2p - \varepsilon z + \varepsilon(zf - \kappa)(zf + \kappa - 2\mu/\varepsilon - 1)/2p - \varepsilon z, \\
p' = 2fp - \mu.
\]

(37)

It is clear that the variable \( f \) can be eliminated, but the calculations here turn out to be more complicated than in the previous sections, where \( P_3 \) was obtained directly for the variable \( p \). Comparing the terms with derivatives with the canonical list of Painlevé equations, one can find a point change

\[
p(z) = \frac{\varepsilon z}{2(1 - v(Z))}, \quad Z = z^2,
\]

which leads to \( P_5 \) equation

\[
v'' = \left( \frac{1}{2v} + \frac{1}{v - 1} \right) (v')^2 - \frac{v'}{Z} + \frac{(v - 1)^2}{Z^2} \left( \alpha v + \frac{\beta}{v} \right) + \frac{v}{Z} + \frac{\delta}{v - 1} \frac{v(v + 1)}{v - 1},
\]

with the values of parameters

\[
\alpha = \frac{\mu^2}{2\varepsilon^2}, \quad \beta = -\frac{(2\mu + \varepsilon - 2\varepsilon \kappa)^2}{8\varepsilon^2}, \quad \gamma = \frac{\varepsilon}{2}, \quad \delta = 0.
\]

It is known that \( P_5 \) with \( \delta = 0 \) is a degenerate case which is related with \( P_3 \) (see [24]), therefore it is possible to find also a substitution

\[
w = \frac{2p(2p - \varepsilon z)}{zp' - 2\kappa p + \mu z}
\]

which brings (37) to the \( P_3^{(6)} \) equation (2) with parameters equal to

\[
\alpha = -2\kappa - 1, \quad \beta = \varepsilon(2\kappa - 1) - 4\mu, \quad \gamma = 1, \quad \delta = -\varepsilon^2.
\]

The inverse substitution is

\[
4p = z(w' + w^2 + \varepsilon) - 2\kappa w.
\]

**Acknowledgements**

The authors are grateful to V.V. Sokolov and I.A. Bobrova for many useful discussions.

The work was done at Ufa Institute of Mathematics with the support by the grant #21-11-00006 of the Russian Science Foundation, https://rscf.ru/project/21-11-00006/.
Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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