Multiobjective Optimum Design of a 3-RRR Spherical Parallel Manipulator with Kinematic and Dynamic Dexterties

Guanglei Wu

Abstract

This paper deals with the kinematic synthesis problem of a 3-RRR spherical parallel manipulator, based on the evaluation criteria of the kinematic, kinetostatic and dynamic performances of the manipulator. A multiobjective optimization problem is formulated to optimize the structural and geometric parameters of the spherical parallel manipulator. The proposed approach is illustrated with the optimum design of a special spherical parallel manipulator with unlimited rolling motion. The corresponding optimization problem aims to maximize the kinematic and dynamic dexterties over its regular shaped workspace.

Keywords: Spherical parallel manipulator, multiobjective optimization, Cartesian stiffness matrix, dexterity, Generalized Inertia Ellipsoid

1 Introduction

A three Degrees of Freedom (3-DOF) spherical parallel manipulator (SPM) is generally composed of two pyramid-shaped platforms, namely, a mobile platform (MP) and a fixed base that are connected together by three identical legs, each one consisting of two curved links and three revolute joints. The axes of all joints intersect at a common point, namely, the center of rotation. Such a spherical parallel manipulator provides a three degrees of freedom rotational motion. Most of the SPMs find their applications as orienting devices, such as camera orienting and medical instrument alignment (Gosselin and Hamel, 1994; Li and Payandeh, 2002; Cavallo and Michelini, 2004; Chaker et al., 2012). Besides, they can also be used to develop active spherical manipulators, i.e., wrist joint (Asada and Granito, 1985).

In designing parallel manipulators, a fundamental problem is that their performance heavily depends on their geometry (Hay and Snyman, 2004) and the mutual dependency of the performance measures. The manipulator performance depends on its dimensions while the mutual dependency among the performances is related to manipulator applications (Merlet, 2006b). The evaluation criteria for design optimization can be classified into two groups: one relates to the kinematic performance of the manipulator while the other relates to the kinetostatic/dynamic performance of the manipulator (Caro et al., 2011). In the kinematic considerations, a common concern is the workspace (Merlet, 2006a; Kong and Gosselin, 2004; Liu et al., 2000; Bonev and Gosselin, 2006). The size and shape of the workspace are of primary importance. Workspace based design optimization can usually be solved with two different formulations, the first formulation aiming to design a manipulator whose workspace contains a prescribed workspace (Hay and Snyman, 2004) and the second approach being to design a manipulator whose workspace is as large as possible (Lou et al., 2005). In Ref. (Bai, 2010), the SPM dexterity was optimized within a prescribed workspace by identifying...
the design space. It is known from (Gosselin and Angeles, 1989) that the orientation workspace of a SPM is a maximum when the geometric angles of the links are equal to 90°. However, maximizing the workspace may lead to a poor design with regard to the manipulator dexterity and manipulability (Stamper et al., 1997; Durand and Reboulet, 1997). This problem can be solved by properly defining the constraints on dexterity (Merlet, 2006a; Huang et al., 2003). For the optimum design of SPMs, a number of works focusing on the kinematic performance, mainly the dexterity and workspace, have been reported, whereas, the kinetostatic/dynamic aspects receive relatively less attention. In general, the design process simultaneously deals with the two previously mentioned groups, both of which include a number of performance measures that essentially vary throughout the workspace. On the kinetostatic aspect, the SPM stiffness is an important consideration (Liu et al., 2000) to characterize its elastostatic performance. When they are used to develop spherically actuated joint, not only the MP angular displacement but also the translational displacement of the rotation center should be evaluated from the Cartesian stiffness matrix of the manipulator and should be minimized. Moreover, the dynamic performance of the manipulator should be as high as possible.

Among the evaluation criteria for optimum geometric parameters design, an efficient approach is to solve a multiobjective optimization problem, which takes all or most of the evaluation criteria into account. As the objective functions are usually conflicting, no single solution can be achieved in this process. The solutions of such a problem are non-dominated solutions, also called Pareto-optimal solutions. Some multiobjective optimization problems of parallel manipulators (PMs) have been reported in the last few years. Hao and Merlet proposed a method different from the classical approaches to obtain all the possible design solutions that satisfy a set of compulsory design requirements, where the design space is identified via the interval analysis based approach (Hao and Merlet, 2005). Ceccarelli et al. focused on the workspace, singularity and stiffness properties to formulate a multi-criterion optimum design procedure for both parallel and serial manipulators (Ceccarelli et al., 2005). Stock and Miller formulated a weighted sum multi-criterion optimization problem with manipulability and workspace as two objective functions (Stock and Miller, 2003). Krefft and Hesselbach formulated a multi-criterion elastodynamic optimization problem for parallel mechanisms while considering workspace, velocity transmission, inertia, stiffness and the first natural frequency as optimization objectives (Krefft and Hesselbach, 2005). Altuzarra et al. dealt with the multiobjective optimum design of a parallel Schönflies motion generator, in which the manipulator workspace volume and dexterity were considered as objective functions (Altuzarra et al., 2009).

In this work, a multiobjective design optimization problem is formulated. The design optimization problem of the 3-DOF spherical parallel manipulator considers the kinematic performance, the accuracy and the dynamic dexterity of the mechanism under design. The performances of the mechanism are also optimized over a regular shaped workspace. The multiobjective de-

Figure 1: 3-RRR unlimited-roll SPM: (a) CAD model, (b) application as spherically actuated joint.
The multiobjective optimization problem is illustrated with a 3-RRR SPM shown in Figure 1, which can replace the serial chains based wrist mechanisms. The non-dominated solutions, also called Pareto-optimal solutions, of the multiobjective optimization problem are obtained with a genetic algorithm.

### 2 Manipulator Architecture

The spherical parallel manipulator under study is a novel robotic wrist with an unlimited roll motion (Bai, 2010; Bai et al., 2009), which only consists of three curved links connected to a mobile platform (MP). The mobile platform is supposed to be quite stiffer than the links, which is considered as a rigid body. The three links are driven by three actuators moving independently on a circular rail of model HCR 150 from THK via pinion and gear-ring transmissions. Thanks to the circular guide, the overall stiffness of the mechanism is increased. Moreover, such a design enables the SPM to generate an unlimited rolling motion, in addition to limited pitch and yaw rotations.

A general spherical parallel manipulator is shown in Figure 2(a) (Liu et al., 2000). Figure 2(b) represents the parameters associated with the $i$th leg of the SPM, $i = 1, 2, 3$. The SPM is composed of three legs that connect the mobile-platform to the base. Each leg is composed of three revolute joints. The axes of the revolute joints intersect and their unit vectors are denoted by $\mathbf{u}_i$, $\mathbf{w}_i$, and $\mathbf{v}_i$, $i = 1, 2, 3$. The arc angles of the three proximal curved links are the same and equal to $\alpha_1$. Likewise, the arc angles of the three distal curved links are the same and equal to $\alpha_2$. The radii of the link midcurves are the same and equal to $R$. Geometric angles $\beta$ and $\gamma$ define the geometry of the two pyramidal base and mobile platforms. The presented SPM in Figure 1(a) is a special case with $\gamma = 0$. The origin $O$ of the reference coordinate system $\mathcal{F}_a$ is located at the center of rotation.

### 3 Kinematic and Kinetostatic Modeling of the SPM

The kinematics of the SPMs has been well documented (Gosselin and Angeles, 1989), which is not repeated in detail here. Hereafter, the orientation of the mobile platform is described by the orientation representation of azimuth-tilt-torsion $(\phi - \theta - \sigma)$ (Bonev, 2008), for which the rotation matrix is expressed as

$$
Q = \begin{bmatrix}
c\phi c\theta c(\phi - \sigma) + s\phi s(\phi - \sigma) & c\phi c\theta s(\phi - \sigma) - \phi c(\phi - \sigma) & c\phi s\theta \\
c\phi c\theta c(\phi - \sigma) - c\phi s(\phi - \sigma) & s\phi c\theta s(\phi - \sigma) + c\phi c(\phi - \sigma) & s\phi s\theta \\
c\phi c\theta s(\phi - \sigma) - s\phi c(\phi - \sigma) & s\phi c\theta c(\phi - \sigma) + s\phi s(\phi - \sigma) & c\theta
\end{bmatrix}
$$

(1)
can be stated as

\[ u_i = [-\sin \eta_i \sin \gamma \cos \eta_i \sin \gamma - \cos \gamma]^T \]  

(2)

where \( \eta_i = 2(i-1)\pi/3, i = 1, 2, 3. \)

Unit vector \( w_i \) of the intermediate revolute joint axis in the ith leg is expressed in the base frame \( F_a \) as:

\[
\begin{bmatrix}
-s_\eta \gamma c_\alpha + (c_\eta s_\beta - s_\eta c_\gamma c_\beta) c_\alpha_1 \\
-c_\eta \gamma c_\alpha + (s_\eta s_\beta + c_\eta c_\gamma c_\beta) c_\alpha_1 \\
-\gamma c_\alpha + s_\gamma c_\beta c_\alpha_1
\end{bmatrix}
\]  

(3)

The unit vector \( v_i \) of the last revolute joint axis in the ith leg, is a function of the mobile-platform orientation, namely,

\[ v_i = Qv_i^* \]  

(4)

where \( v_i^* \) corresponds to the unit vector of the last revolute joint axis in the ith leg when the mobile platform is in its home configuration:

\[ v_i^* = [-\sin \eta_i \sin \beta \cos \eta_i \sin \beta \cos \beta]^T \]  

(5)

### 3.1 Kinematic Jacobian matrix

Let \( \omega \) denote the angular velocity of the mobile-platform, the screws velocity equation via the ith leg can be stated as

\[ \mathbf{s}_\omega = \begin{bmatrix} \omega \\ 0 \end{bmatrix} = \hat{\theta}_i \dot{\mathbf{s}}_A^i + \psi \dot{\mathbf{s}}_B^i + \xi \dot{\mathbf{s}}_C^i \]  

(6)

with the screws for the revolute joints at points \( A_i, B_i \) and \( C_i \) expressed as

\[ \dot{\mathbf{s}}_A^i = \begin{bmatrix} u_i \\ 0 \end{bmatrix}, \dot{\mathbf{s}}_B^i = \begin{bmatrix} w_i \\ 0 \end{bmatrix}, \dot{\mathbf{s}}_C^i = \begin{bmatrix} v_i \\ 0 \end{bmatrix} \]

Since the axes of the two passive revolute joints in each leg lie in the plane \( B_iOC_i \), the following screw is reciprocal to all the revolute joint screws of the ith leg and does not lie in its constraint wrench system:

\[ \dot{\mathbf{s}}_i^* = \begin{bmatrix} 0 \\ w_i \times v_i \end{bmatrix} \]  

(7)

Applying the orthogonal product (\( \circ \)) (Tsai, 1998) to both sides of Eqn. (6) yields

\[ \dot{\mathbf{s}}_i^* \circ \mathbf{s}_\omega = (w_i \times v_i)^T \omega = (u_i \times w_i) \cdot v_i \hat{\theta}_i \]  

(8)

As a consequence, the expression mapping from the mobile platform twist to the input angular velocities is stated as:

\[ A \omega = B \hat{\theta} \]  

(9)

with

\[ A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, a_i = w_i \times v_i \]  

(10a)

\[ B = \text{diag}[b_1, b_2, b_3], b_i = (u_i \times w_i) \cdot v_i \]  

(10b)

where \( \hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3]^T \). Matrices \( A \) and \( B \) are the forward and inverse Jacobian matrices of the manipulator, respectively. If \( B \) is nonsingular, the kinematic Jacobian matrix \( J \) is obtained as

\[ J = B^{-1} A \]  

(11)

### 3.2 Cartesian stiffness matrix

The stiffness model of the SPM under study is established with virtual spring approach (Pashkevich et al., 2009), by considering the actuation stiffness, link deformation and the influence of the passive joints. The flexible model of the ith leg is represented in Figure 3. Figure 3(b) illustrates the link deflections and variations in passive revolute joint angles.

Let the center of rotation be the reference point of the mobile platform. Analog to Eqn. (6), the small displacement screw of the mobile-platform can be expressed as:

\[ \mathbf{s}_\theta = \begin{bmatrix} \Delta \phi^T \\ \Delta p \end{bmatrix} = \Delta \hat{\theta}_i \dot{\mathbf{s}}_A^i + \Delta \psi \dot{\mathbf{s}}_B^i + \Delta \xi \dot{\mathbf{s}}_C^i \]  

(12)

where \( \Delta p = [\Delta x, \Delta y, \Delta z]^T \) is linear displacement of the rotation center and \( \Delta \phi = [\Delta \phi_x, \Delta \phi_y, \Delta \phi_z]^T \) is the MP orientation error. Note that this equation only includes the joint variations, while for the real manipulator, link deflections should be considered as well.

The screws associated with the link deflections are formulated as follows:

\[ \begin{bmatrix} \mathbf{s}_{u1}^i \\ \mathbf{s}_{u2}^i \\ \mathbf{s}_{u3}^i \end{bmatrix} = \begin{bmatrix} r_i \\ r_i \times r_i \end{bmatrix}, \mathbf{s}_{u4}^i = \mathbf{s}_{u5}^i = \mathbf{s}_{u6}^i = \begin{bmatrix} 0 \\ r_i \end{bmatrix} \]  

(13)

where \( n_i = w_i \times v_i \) is the normal vectors of plane \( B_iOC_i \), \( r_i = w_i \times n_i \), and \( r_i^C \) is the position vector of point \( C_i \) from \( O \). The directions of the vectors \( r_i \) and \( n_i \) are identical to \( \Delta u_{i1}^1 \) and \( \Delta u_{i6}^6 \), respectively.

By considering the link deflections \( \Delta u_{i1}^1, \ldots, \Delta u_{i6}^6 \) and variations in passive joint angles and adding all the deflection freedoms to Eqn. (12), the mobile platform deflection in the ith leg is stated as

\[ \mathbf{s}_{\theta_O} = \Delta \hat{\theta}_i \dot{\mathbf{s}}_A^i + \Delta \psi \dot{\mathbf{s}}_B^i + \Delta \xi \dot{\mathbf{s}}_C^i + \Delta u_{i1}^1 \dot{\mathbf{s}}_A^i + \Delta u_{i2}^1 \dot{\mathbf{s}}_B^i + \Delta u_{i3}^i \dot{\mathbf{s}}_A^i + \Delta u_{i4}^i \dot{\mathbf{s}}_B^i + \Delta u_{i5}^i \dot{\mathbf{s}}_C^i + \Delta u_{i6}^i \dot{\mathbf{s}}_C^i \]  

(14)

The previous equation can be written in a compact form by separating the terms related to the variations...
On the other hand, the wrench applied to the articulated joints in the $i$th leg being denoted by a vector $\tau$, the equilibrium condition for the system is written as,

$$J^i_\tau f_i = \tau_i, \quad J^i_\tau f_i = 0, \quad \Delta u_i = K^{-1}_{\theta} \tau_i$$  (18)

Combining Eqsns. (15), (17) and (18), the kinetostatic model of the $i$th leg can be reduced to a system of two matrix equations, namely,

$$\begin{bmatrix} S^i_\theta & J^i_q \end{bmatrix} \begin{bmatrix} f_i \end{bmatrix} = \begin{bmatrix} \hat{s}^i_O \end{bmatrix}$$  (19)

where the sub-matrix $S^i_\theta = J^i_\theta K^{-1}_{\theta} J^i_\tau$ describes the spring compliance relative to the center of rotation, and the sub-matrix $J^i_q$ takes into account the passive joint influence on the mobile platform motions. $K_{\theta}^{-1}$ is a $7 \times 7$ matrix, describing the compliance of the virtual springs and taking the form:

$$K_{\theta}^{-1} = \begin{bmatrix} K_{act}^{-1} & 0_{1 \times 6} \\ 0_{6 \times 1} & K_L^{-1} \end{bmatrix}$$  (20)

where $K_{act}$ corresponds to the stiffness of the $i$th actuator. $K_L$ of size $6 \times 6$ is the stiffness matrix of the curved link in the $i$th leg, which is calculated by means of the Euler-Bernoulli stiffness model of a cantilever. In Figure 3(b), $\Delta u_1$, $\Delta u_2$ and $\Delta u_3$ show the three force directions while $\Delta u_4$, $\Delta u_5$ and $\Delta u_6$ show the three moment directions, thus, using Castiglione's theorem (Hibbeler, 1997), the compliance matrix of the curved link takes the form:

$$K_L^{-1} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & C_{16} \\ C_{12} & C_{22} & 0 & 0 & 0 & C_{26} \\ 0 & 0 & C_{33} & C_{34} & C_{35} & 0 \\ 0 & 0 & C_{44} & C_{45} & C_{46} & 0 \\ 0 & 0 & C_{55} & C_{56} & C_{56} & 0 \\ C_{16} & C_{26} & 0 & 0 & 0 & C_{66} \end{bmatrix}$$  (21)

where the corresponding elements are given in Appendix A.

The matrix $J^i_q$ of size $6 \times 7$ is the Jacobian matrix related to the virtual springs and $J^i_\theta$ of size $6 \times 2$, the one related to revolute joints in the $i$th leg. The Cartesian stiffness matrix $K_i$ of the $i$th leg is obtained from Eqn. (19),

$$f_i = K_i \hat{s}^i_O$$  (22)

where $K_i$ is a $6 \times 6$ sub-matrix, which is extracted from the inverse of the $8 \times 8$ matrix on the left-hand side of Eqn. (19). From $f = \sum_{i=1}^3 f_i$, $S_O = \hat{s}^i_O$ and $f = K \hat{s}_O$, the Cartesian stiffness matrix $K$ of the system is found by simple addition, namely,

$$K = \sum_{i=1}^3 K_i$$  (23)

in the passive revolute joint angles and those related to the actuator and link deflections, namely,

$$\hat{s}^i_O = J^i_\theta \Delta u_i + J^i_q \Delta q_i$$  (15)

with

$$J^i_\theta = \begin{bmatrix} \hat{s}^i_A & \hat{s}^i_{u_1} & \hat{s}^i_{u_2} & \hat{s}^i_{u_3} & \hat{s}^i_{u_4} & \hat{s}^i_{u_5} & \hat{s}^i_{u_6} \end{bmatrix}$$  (16a)

$$J^i_q = \begin{bmatrix} \hat{s}^i_{\theta_1} \\ \hat{s}^i_{\theta_2} \\ \hat{s}^i_{\phi_1} \\ \hat{s}^i_{\phi_2} \end{bmatrix}$$  (16b)

$$\Delta u_i = [\Delta \theta_1 \ \Delta u_1^i \ \Delta u_2^i \ \Delta u_3^i \ \Delta u_4^i \ \Delta u_5^i \ \Delta u_6^i]^T$$  (16c)

$$\Delta q_i = [\Delta \psi_1 \ \Delta \xi_i]^T$$  (16d)

Let the external wrench applied to the end of the $i$th leg be denoted by $f_i$, the constitutive law of the $i$th leg can be expressed as

$$f_i = \begin{bmatrix} K^T_{\tau \tau} & K^T_{\tau \theta} \\ K^T_{\theta \tau} & K^T_{\theta \theta} \end{bmatrix} \begin{bmatrix} \Delta \phi \ \Delta \rho \end{bmatrix} \rightarrow f_i = K_i \hat{s}^i_O$$  (17)
3.3 Mass matrix

The mass in motion of the mechanism influences the dynamic performance, such as inertia, acceleration, etc., hence, formulating the mass matrix is one important procedure in the dynamic analysis. Mass matrix is the function of manipulator dimensions and material properties, i.e., link lengths, cross-sectional area, mass density. Generally, the manipulator mass matrix (inertia matrix) can be obtained on the basis of its kinetic energy. The total kinetic energy \( T \) includes the energy \( T_p \) of the mobile platform, \( T_i \) of the curved links and \( T_s \) of the slide units:

- The kinetic energy of the mobile platform is
  \[
  T_p = \frac{1}{2} m_p v_p^T v_p + \frac{1}{2} \omega^T I_p \omega \tag{24}
  \]
  with
  \[
  v_p = R \cos \beta p \times \omega, \quad I_p = \text{diag} \begin{bmatrix} I_{xx} & I_{yy} & I_{zz} \end{bmatrix} \tag{25}
  \]
  where \( m_p \) is the mass of the mobile-platform and \( I_{xx}, I_{yy}, I_{zz} \) are the mass moments of inertia of the mobile-platform about \( x-, y-, z- \) axes, respectively.

- The kinetic energy of the curved links is
  \[
  T_i = \frac{1}{2} \sum_{i=1}^{3} \left( m_i v_i^T v_i + I_i \omega^2 \right) \tag{26}
  \]
  with
  \[
  v_i^T = \frac{1}{2} R \left( \dot{R}_i \mathbf{w}_i \times \mathbf{u}_i + \mathbf{v}_i \times \omega \right) \tag{27a}
  \]
  \[
  I_i = \frac{1}{2} m_i R^2 \left( 1 - \frac{\sin \alpha_2 \cos \alpha_2}{\alpha_2} \right) \tag{27b}
  \]
  \[
  \omega_i = - \frac{(\mathbf{u}_i \times \mathbf{v}_i) \times \omega}{(\mathbf{u}_i \times \mathbf{w}_i) \times \mathbf{v}_i} = j_i \psi_i \cdot \omega \tag{27c}
  \]
  where \( m_i \) is the link mass and \( I_i \) is its mass moment of inertia about \( \mathbf{w}_i \).

- The kinetic energy of the slide units is
  \[
  T_s = \frac{1}{2} \left( I_g n_g^2 + m_s R_s^2 \right) \dot{\theta}^T \dot{\theta} \tag{28}
  \]
  where \( m_s \) is the mass of the slide unit and \( R_s \) is the distance from its mass center to z-axis. \( I_g \) is the mass moment of inertia of the pinion and \( n_g \) is the gear ratio.

Consequently, the SPM kinetic energy can be written in the following form

\[
T = T_p + T_i + T_s = \frac{1}{2} \dot{\theta}^T M \dot{\theta} \tag{29}
\]
360° full rotation, i.e., $\theta \geq 45^\circ$ and $\sigma \in (-180^\circ, 180^\circ)$, see Figure 4.

4.1 Design variables

Variables $\alpha_1$, $\alpha_2$, $\beta$ and $\gamma$ are part of the geometric parameters of a 3-RRR SPM and $\gamma = 0$ for the manipulator under study. Moreover, the radius $R$ of the link midcurve is another design variable and the cross section of the links is supposed to be a square of side length $a$. These variables are shown in Figure 5. As a consequence, the design variable vector is expressed as follows:

$$x = [\alpha_1, \alpha_2, \beta, a, R]$$  \hspace{1cm} (31)

4.2 Objective functions

The kinematic performance is one of the major concerns in the manipulator design, of which a criterion is the evaluation of the dexterity of SPMs. A commonly used criterion to evaluate this kinematic performance is the global conditioning index (GCI) (Gosselin and Angeles, 1991), which describes the isotropy of the kinematic performance. The GCI is defined over a workspace $\Omega$ as

$$GCI = \frac{\int_{\Omega} \frac{1}{\kappa(J)}dW}{\int_{\Omega} dW}$$  \hspace{1cm} (32)

where $\kappa(J)$ is the condition number of the kinematic Jacobian matrix (11). In practice, the GCI of a robotic manipulator is calculated through a discrete approach as

$$GCI = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\kappa_i(J)}$$  \hspace{1cm} (33)

where $n$ is the number of the discrete workspace points. As a result, the first objective function of the optimization problem is written as:

$$f_1(x) = GCI \rightarrow \max$$  \hspace{1cm} (34)

Referring to the kinematic dexterity, an important criterion to evaluate the dynamic performance is dynamic dexterity, which is made on the basis of the concept of Generalized Inertia Ellipsoid (GIE) (Asada, 1983). In order to enhance the dynamic performance and to make acceleration isotropic, the mass matrix (30) should be optimized to obtain a better dynamic dexterity. Similar to GCI, a global dynamic index (GDI) is used to evaluate the dynamic dexterity, namely,

$$GDI = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\kappa_i(M)}$$  \hspace{1cm} (35)

where $\kappa_i(M)$ is the condition number of the mass matrix of the $i$th workspace point. Thus, the second objective function of the optimization problem is written as:

$$f_2(x) = GDI \rightarrow \max$$  \hspace{1cm} (36)

4.3 Optimization constraints

In this section, the kinematic constraints, conditioning of the kinematic Jacobian matrix and accuracies due to the elastic deformation are considered. Constraining the conditioning of the Jacobian matrix aims to guarantee dexterous workspace free of singularity, whereas limits on accuracy consideration ensures that the mechanism is sufficiently stiff.

4.3.1 Kinematic constraints

According to the determination of design space reported in (Bai, 2010), the bounds of the parameter $\alpha_1$, $\alpha_2$, $\beta$ and $\gamma$ subject to the prescribed workspace are stated as:

$$45^\circ \leq \beta \leq 90^\circ, \ 45^\circ \leq \alpha_1, \alpha_2 \leq 135^\circ$$  \hspace{1cm} (37)

The sequence of the first, second and third slide units appearing on the circular guide counterclockwise is constant. In order to avoid collision, the angles $\theta_{ij}$ between the projections of vectors $w_i$ and $w_j$ in the $xy$ quadrant, $i, j = 1, 2, 3, i \neq j$, as shown in Figure 6, have the minimum value, say 10°. To avoid collision
and make the mechanism compact, the following constraints should be satisfied:

\[ \theta_{12}, \theta_{23}, \theta_{31} \geq \epsilon_0 = 10^\circ \]
\[ R_0 = 0.120 \text{ m} \leq R \sin \alpha_1 \leq R_s = 0.200 \text{ m} \]

Moreover, the SPM should not reach any singularity in its orientation workspace. Therefore, the following conditions should be satisfied:

\[ \det(A) \geq \epsilon, \quad b_i = (u_i \times w_i) \cdot v_i \leq -\epsilon \]  
(39)

where \( A \) is the forward Jacobian matrix of the manipulator defined in Eqn. (9) and \( \epsilon > 0 \) is a previously specified tolerance set to 0.001.

### 4.3.2 Conditioning number of the kinematic Jacobian matrix

Maximizing the GCI and constraining the kinematic Jacobian matrix cannot prevent the prescribed workspace away from ill-conditioned configurations. For the design optimization in order to achieve a dexterous workspace, the minimum of the inverse condition number of the kinematic Jacobian matrix \( \kappa^{-1}(J) \), based on 2-norm, should be higher than a prescribed value throughout the workspace, say 0.1, namely,

\[ \min(\kappa^{-1}(J)) \geq 0.1 \]  
(40)

### 4.3.3 Accuracy constraints

The accuracy constraints of the optimization problem for the SPM are related to the dimensions of the curved link and the maximum positional deflection of the rotation center and angular deflection of the moving-platform subject to a given wrench applied on the latter. The control loop stiffness is \( K_{act}^3 = 10^6 \text{ Nm/ rad} \). Let the static wrench capability be specified as the eight possible combinations of moments \( m = [\pm 10, \pm 10, \pm 10] \text{ Nm} \), while the allowable maximum positional and rotational errors for the workspace points are 1 mm and \( 2\alpha = 0.0349 \text{ rad} \), respectively, thus, the accuracy constraints can be written as:

\[ \|\Delta p\|_n = \sqrt{\Delta x_n^2 + \Delta y_n^2 + \Delta z_n^2} \leq \epsilon_p \]  
(41)
\[ \|\Delta \phi\|_n = \sqrt{\Delta \phi_{z,n}^2 + \Delta \phi_{y,n}^2 + \Delta \phi_{x,n}^2} \leq \epsilon_r \]

where the linear and angular displacements are computed from \( f_0 = \text{K}^{-1} f \) with the Cartesian stiffness matrix (23) and \( \epsilon_p = 1 \text{ mm}, \epsilon_r = 0.0349 \text{ rad} \).

### 4.4 Formulation of the multiobjective optimization problem

Mathematically, the multi-objective design optimization problem for the spherical parallel manipulator can be formulated as:

\[
\begin{align*}
\text{maximize} & \quad f_1(x) = \text{GCI} \\
\text{maximize} & \quad f_2(x) = \text{GDI} \\
\text{over} & \quad x = [\alpha_1, \alpha_2, \beta, a, R] \\
\text{subject to} & \quad g_1 : \theta \geq 45^\circ \\
& \quad g_2 : R_0 \leq R \sin \alpha_1 \leq R_s \\
& \quad g_3 : \theta_{12}, \theta_{23}, \theta_{31} \geq \epsilon_0 = 10^\circ \\
& \quad g_4 : \det(A) \geq \epsilon, \quad (u_i \times w_i) \cdot v_i \leq -\epsilon \\
& \quad g_5 : \min(\kappa^{-1}(J)) \geq 0.1 \\
& \quad g_6 : \sqrt{\Delta x_n^2 + \Delta y_n^2 + \Delta z_n^2} \leq \epsilon_p \\
& \quad g_7 : \sqrt{\Delta \phi_{z,n}^2 + \Delta \phi_{y,n}^2 + \Delta \phi_{x,n}^2} \leq \epsilon_r \\
& \quad x_0 \leq x \leq x_{ub} \\
& \quad i = 1, 2, 3
\end{align*}
\]

where \( x_0 \) and \( x_{ub} \), respectively, are the lower and upper bounds of the variables \( x \) given by Table 1.

![Circular rail diagram](image-url)

Figure 6: Slide unit configuration of the 3-DOF SPM.

| \( x_{lb} \) | \( x_{ub} \) | \( \alpha_1 \) [deg] | \( \alpha_2 \) [deg] | \( \beta \) [deg] | \( a \) [m] | \( R \) [m] |
|---|---|---|---|---|---|---|
| 45 | 45 | 45 | 0.005 | 0.120 |
| 135 | 135 | 90 | 0.030 | 0.300 |

Table 1: The lower and upper bounds of the design variables \( x \).
Table 2: Algorithm parameters of the implemented NSGA-II

| Population size | Generation Directional crossover | Crossover | Mutation | Distribution |
|-----------------|----------------------------------|-----------|-----------|--------------|
| 40              | 200                              | 0.5       | 0.9       | 0.1          | 20           |

Table 3: Three Pareto-optimal solutions

| ID | $\alpha_1$ [deg] | $\alpha_2$ [deg] | $\beta$ [deg] | $a$ [m] | $R$ [m] | GCI | GDI |
|----|------------------|------------------|--------------|--------|--------|-----|-----|
| I  | 56.2             | 81.0             | 89.8         | 0.0128 | 0.1445 | 0.366 | 0.711 |
| II | 51.6             | 84.3             | 89.9         | 0.0133 | 0.1533 | 0.453 | 0.665 |
| III| 47.2             | 90.8             | 89.2         | 0.0127 | 0.1641 | 0.536 | 0.625 |

4.5 Pareto-optimal solutions

For the proposed SPM, the actuation transmission mechanism is a combination of actuator of model RE35 GB and gearhead of model GP42 C from Maxon (Maxon, 2012) and a set of gear ring-pinion with ratio $n_g = 8$. Moreover, the components are supposed to be made of steel, thus, $E = 210$ Gpa, $\nu = 0.3$. Moreover, the moving platform is supposed to be a regular triangle, thus, the MP and link masses are given by

$$m_p = \frac{3\sqrt{3}}{4} \rho h R^2 \sin^2 \beta, m_l = \rho a^2 R \alpha_2$$  \hspace{1cm} (43)

where $\rho$ is the mass density and $h = 0.006$ m is the thickness of the moving platform. The total mass $m_s$ of each slide unit, including the mass of the actuator, gearhead, pinion and the manufactured components, is equal to $m_s = 2.1$ kg.

The previous formulated optimization problem (42) is solved by the genetic algorithm NSGA-II (Deb et al., 2002) with Matlab, of which the algorithm parameters are given in Table 2.

The Pareto front of the formulated optimization problem for the SPM is shown in Figure 7 and three optimal solutions, i.e., two extreme and one intermediate, are listed in Table 3.

Figure 8 illustrates the variational trends as well as the inter-dependency between the objective functions and design variables by means of a scatter matrix. The lower triangular part of the matrix represents the correlation coefficients whereas the upper one shows the corresponding scatter plots. The diagonal elements represent the probability density charts of each variable. The correlation coefficients vary from $-1$ to $1$. Two variables are strongly dependent when their correlation coefficient is close to $-1$ or $1$ and independent when the latter is null. Figure 8 shows:

- both objectives functions $GCI$ and $GDI$ are strongly dependent as their correlation coefficient is equal to $-0.975$;
- both objectives functions $GCI$ and $GDI$ are strongly dependent on all design variables as all of the corresponding correlation coefficients are greater than 0.6;
- $GCI$ is slightly more dependent than $GDI$ of the design variables as all the corresponding correlation coefficients of former are greater than those of latter;
- $GDI$ is less dependent on the design variables $\beta$ and $a$ than the other variables although the two former variables influence the SPM mass, this is due to the large portion of the slide unit mass in the total mechanism mass.
5 Conclusions

In this paper, the geometric synthesis of spherical parallel manipulators is discussed. A multiobjective design optimization problem based on the genetic algorithm was formulated in order to determine the mechanism optimum structural and geometric parameters. The objective functions were defined on the basis of the criteria of both kinematic and kinetostatic/dynamic performances. This approach is illustrated with the optimum design of an unlimited-roll spherical parallel manipulator, aiming at maximizing the kinematic and dynamic dexterities to achieve relatively better kinematic and dynamic performances simultaneously. It is found that the parameter $\beta$ being equal to 90° is a preferred structure for the SPM under study. Finally, the Pareto-front was obtained to show the approximation of the optimal solutions between the various (antagonistic) criteria, subject to the dependency of the performance. The future work will aim to maximize the orientation workspace and optimize the cross-section type of the curved links.

References

Altuzarra, O., Salgado, O., Hernandez, A., and Angeles, J. Multiobjective optimum design of a symmetric parallel schöning-s-motion generator.
ASME J. Mechanical Design, 2009. 131(3):031002. doi:10.1115/1.3066659.

Asada, H. A geometrical representation of manipulator dynamics and its application to arm design. ASME J. Dynamic Systems, Measurement and Control, 1983. 105(3):131–142. doi:10.1115/1.3140644.

Asada, H. and Granito, J. Kinematic and static characterization of wrist joints and their optimal design. In IEEE International Conference on Robotics and Automation, pages 244–250, 1985. doi:10.1109/ROBOT.1985.1087324.

Bai, S. Optimum design of spherical parallel manipulator for a prescribed workspace. Mechanism and Machine Theory, 2010. 45(2):200–211. doi:10.1016/j.mechmachtheory.2009.06.007.

Bai, S., Hansen, M. R., and Andersen, T. O. Modelling of a special class of spherical parallel manipulators with Euler parameters. Robotica, 2009. 27(2):161–170. doi:10.1017/S0263574708004402.

Bonev, I. A. Direct kinematics of zero-torsion parallel mechanisms. In IEEE International Conference on Robotics and Automation, Pasadena, California, USA, pages 3851–3856, 2008. doi:10.1109/ROBOT.2008.4543802.

Bonev, I. A. and Gosselin, C. M. Analytical determination of the workspace of symmetrical spherical parallel mechanisms. IEEE Transactions on Robotics, 2006. 22(5):1011–1017. doi:10.1109/TRO.2006.878983.

Caro, S., Chablat, D., Ur-Rehman, R., and Wenger, P. Multiobjective design optimization of 3-PRR planar parallel manipulators. In Global Product Development, pages 373–383. Springer-Verlag Berlin Heidelberg, 2011. doi:10.1007/978-3-642-15973-2_37.

Cavallo, E. and Michelini, R. C. A robotic equipment for the guidance of a vectored thruster. In 35th International Symposium on Robotics. Paris, France, 2004.

Ceccarelli, M., Carbone, G., and Ottaviano, E. Multi criteria optimum design of manipulators. In Bulletin of the Polish Academy of Technical Sciences, 2005. 53(1):9–18.

Chaker, A., Mlika, A., Laribi, M. A., Romdhane, L., and Zeghloul, S. Synthesis of spherical parallel manipulator for dexterous medical task. Frontiers of Mechanical Engineering, 2012. 7(2):150–162. doi:10.1007/s11465-012-0325-4.

Deb, K., Pratap, A., Agarwal, S., and Meyarivan, T. A fast and elitist multiobjective genetic algorithm: NSGA-II. IEEE Trans. Evolutionary Computation, 2002. 6(2):182–197. doi:10.1109/4235.996017.

Durand, S. L. and Rebourlet, C. Optimal design of a redundant spherical parallel manipulator. Robotica, 1997. 15(4):399–405. doi:10.1017/S0263574797000490.

Gosselin, C. M. and Angeles, J. The optimum kinematic design of a spherical three-degree-of-freedom parallel manipulator. ASME J. Mechanisms, Transmissions, and Automation in Design, 1989. 111:202–207. doi:10.1115/1.3258984.

Gosselin, C. M. and Angeles, J. A global performance index for the kinematic optimization of robotic manipulators. ASME J. Mechanical Design, 1991. 113(3):220–226. doi:10.1115/1.2912772.

Gosselin, C. M. and Hamel, J. F. The Agile Eye: a high-performance three-degree-of-freedom camera-orienting device. In IEEE International Conference on Robotics and Automation, San Diego, CA, pages 781–786, 1994. doi:10.1109/ROBOT.1994.351393.

Hao, F. and Merlet, J.-P. Multi-criteria optimal design of parallel manipulators based on interval analysis. Mechanism and Machine Theory, 2005. 40(2):157–171. doi:10.1016/j.mechmachtheory.2004.07.002.

Hay, A. M. and Snyman, J. A. Methodologies for the optimal design of parallel manipulators. Inter. J. Numerical Methods in Engineering, 2004. 59(11):131–152. doi:10.1002/nme.871.

Hibbeler, R. C. Mechanics of Materials. Prentice Hall, 1997.

Huang, T., Gosselin, C. M., Whitehouse, D. J., and Chetwynd, D. G. Analytic approach for optimal design of a type of spherical parallel manipulators using dexterous performance indices. I MechE. J. Mechan. Eng. Sci., 2003. 217(4):447–455. doi:10.1243/09544060321509720.

Kong, K. and Gosselin, C. M. Type synthesis of three-degree-of-freedom spherical parallel manipulators. Inter. J. Robotics Research, 2004. 23(3):237–245. doi:10.1177/0278364904041562.

Kreft, M. and Hesselbach, J. Elastodynamic optimization of parallel kinematics. In Proceedings of the IEEE International Conference on Automation Science and Engineering. Edmonton, AB, Canada, pages 357–362, 2005. doi:10.1109/COASE.2005.1506795.
Appendix A

The elements of the compliance matrix (21) for the curved beam

\[
C_{11} = \frac{R}{2} \left( \frac{s_1}{GI_x} + \frac{s_2}{EI_y} \right) \\
C_{12} = \frac{s_8 R}{2} \left( \frac{1}{GI_x} - \frac{1}{EI_y} \right) \\
C_{16} = \frac{R^2}{2} \left( \frac{s_2}{EI_y} - \frac{s_7}{GI_x} \right) \\
C_{22} = \frac{R}{2} \left( \frac{s_2}{GI_x} + \frac{s_1}{EI_y} \right) \\
C_{26} = \frac{R^2}{2} \left( \frac{s_4}{GI_x} - \frac{s_2}{EI_y} \right) \\
C_{33} = \frac{Ro_2}{EI_z} \\
C_{34} = \frac{s_6 R^2}{EI_z} \\
C_{35} = \frac{s_6 R^2}{EI_z} \\
C_{44} = \frac{Ro_2}{GA} + \frac{R^3}{2} \left( \frac{s_3}{GI_x} + \frac{s_2}{EI_y} \right)
\]

with

\[
s_1 = \alpha_2 + \sin \alpha_2 \cos \alpha_2 \\
s_2 = \alpha_2 - \sin \alpha_2 \cos \alpha_2 \\
s_3 = 3\alpha_2 + \sin \alpha_2 \cos \alpha_2 / 2 - 4 \sin \alpha_2 \\
s_4 = 1 - \cos \alpha_2 - \sin^2 \alpha_2 / 2 \\
s_5 = \sin \alpha_2 - \alpha_2 \\
s_6 = \cos \alpha_2 - 1 \\
s_7 = 2 \sin \alpha_2 - \alpha_2 - \sin \alpha_2 \cos \alpha_2 \\
s_8 = - \sin^2 \alpha_2
\]

where \( E \) is the Young’s modulus and \( G = E/(1 + \nu) \) is the shear modulus with the Poisson’s ratio \( \nu \). \( I_x \), \( I_y \) and \( I_z \) are the moments of inertia, respectively. \( A \) is the area of the cross-section.

Li, T. and Payandeh, S. Design of spherical parallel mechanisms for application to laparoscopic surgery. *Robotica*, 2002. 20(2):133–138. doi:10.1017/S0263574701003873.

Liu, X. J., Jin, Z. L., and Gao, F. Optimum design of 3-dof spherical parallel manipulators with respect to the conditioning and stiffness indices. *Mechanism and Machine Theory*, 2000. 35(9):1257–1267. doi:10.1016/S0094-114X(99)00072-5.

Lou, Y., Liu, G., Chen, N., and Li, Z. Optimal design of parallel manipulators for maximum effective regular workspace. In *Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems*. Alberta, pages 795–800, 2005. doi:10.1109/ROBOT.2005.1545144.

Maxon. *Maxon Motor and Gearhead products catalog*. 2012. URL http://www.maxonmotor.com/maxon/view/catalog/.

Merlet, J.-P. Jacobian, manipulability, condition number, and accuracy of parallel robots. *ASME J. Mechanical Design*, 2006a. 128(1):199–206. doi:10.1115/1.2121740.

Merlet, J.-P. *Parallel Robots*. Kluwer, Norwell, 2006b.

Pashkevich, A., Chablat, D., and Wenger, P. Stiffness analysis of overconstrained parallel manipulators. *Mechanism and Machine Theory*, 2009. 44(5):966–982. doi:10.1016/j.mechmachtheory.2009.08.017.

Stamper, R. E., Tsai, L.-W., and Walsh, G. C. Optimization of a three-dof translational platform for well-conditioned workspace. In *Proceedings of the IEEE International Conference on Robotics and Automation*. Albuquerque, NM, pages 3250–3255, 1997. doi:10.1109/ROBOT.1997.606784.

Stock, M. and Miller, K. Optimal kinematic design of spatial parallel manipulators: Application of linear delta robot. *ASME J. Mechanical Design*, 2003. 125(2):292–301. doi:10.1115/1.1563632.

Tsai, L.-W. The Jacobian analysis of parallel manipulators using reciprocal screws. In J. Lenarčič and M. L. Husty, editors, *Advances in Robot Kinematics: Analysis and Control*, pages 327–336. Kluwer Academic Publishers, 1998.