RECONSTRUCTION OF GROUPOIDS AND $C^\ast$-RIGIDITY OF DYNAMICAL SYSTEMS

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Abstract. We show how to construct a graded locally compact Hausdorff étale groupoid from a $C^\ast$-algebra carrying a coaction of a discrete group, together with a suitable abelian subalgebra. We call this groupoid the extended Weyl groupoid. When the coaction is trivial and the subalgebra is Cartan, our groupoid agrees with Renault’s Weyl groupoid. We prove that if $G$ is a second-countable locally compact étale groupoid carrying a grading of a discrete group, and if the interior of the trivially graded isotropy is abelian and torsion free, then the extended Weyl groupoid of its reduced $C^\ast$-algebra is isomorphic as a graded groupoid to $G$. In particular, two such groupoids are isomorphic as graded groupoids if and only if there is an equivariant diagonal-preserving isomorphism of their reduced $C^\ast$-algebras. We introduce graded equivalence of groupoids, and establish that two graded groupoids in which the trivially graded isotropy has torsion-free abelian interior are graded equivalent if and only if there is an equivariant diagonal-preserving Morita equivalence between their reduced $C^\ast$-algebras. We use these results to establish rigidity results for a number of classes of dynamical systems, including all actions of the natural numbers by local homeomorphisms of locally compact Hausdorff spaces.

Introduction

Background. The use of operator algebras to encode dynamics goes all the way back to the foundational results of Murray and von Neumann on the group von Neumann algebra construction [38]. Crossed-product algebras and their generalisations have played a crucial role in both von Neumann algebra theory and $C^\ast$-algebra theory ever since. Recently, particularly since the work of Cuntz and Krieger [17] on operator-algebraic representations of shifts of finite type, and connections with Bowen–Franks theory [3], significant strides have been made in the direction of $C^\ast$-rigidity of dynamical systems. In broad terms this is the principle that dynamical systems can be recovered, up to a suitable notion of equivalence, from associated $C^\ast$-algebraic data.

A seminal result in this direction was Krieger’s celebrated theorem [30] showing that nonsingular ergodic actions of $\mathbb{Z}$ are classified up to orbit equivalence by isomorphism of the associated von Neumann factors. This was soon followed by Cuntz and Krieger’s construction [17] of $C^\ast$-algebras from irreducible shifts of finite type and Rørdam’s proof [45] that stable isomorphism of Cuntz–Krieger algebras classifies irreducible shifts of finite type.

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type up to equivalence via the combination of flow equivalence and the so-called Cuntz splice on directed graphs. Later, building on work of Boyle [5], Giordano–Putnam–Skau [25] proved the remarkable result that for minimal homeomorphisms of the Cantor set, flip conjugacy, continuous orbit equivalence, and diagonal-preserving isomorphism of the associated crossed-product $C^*$-algebras are equivalent. Tomiyama [50] and Boyle–Tomiyama [6] subsequently proved that topologically free homeomorphisms of the Cantor set (minimal or not), are continuously orbit equivalent if and only if they each decompose as a disjoint union of two subsystems one pair of which are conjugate and the other pair of which are flip-conjugate. These results introduced, in particular, diagonal-preserving $C^*$-isomorphisms as a key ingredient in $C^*$-algebraic rigidity of topological dynamics.

The importance of diagonal-preserving isomorphism in operator algebras associated to groupoids goes back further. Feldman and Moore [21, 22, 23] proved that a Borel equivalence relation $R$ can be reconstructed from the pair consisting of its associated von Neumann algebra $M$ and the canonical Cartan subalgebra $D \subseteq M$, and that every Cartan pair of von Neumann algebras arises from such an $R$ and a Borel 2-cocycle on $R$. In his thesis [42], Renault introduced a notion of a Cartan subalgebra of a $C^*$-algebra, and proved that a topologically principal étale groupoid $G$ and a continuous cocycle $c$ on $G$ can be recovered from the Cartan pair $(C^*(G), C_0(G(0)))$, and that every Cartan pair arises this way. Subsequently Kumjian [31] refined Renault’s notion of a twisted groupoid $C^*$-algebra and showed that Renault’s theorem extended to these more-general twists, in the setting of principal étale groupoids. Later in [13], Renault further extended Kumjian’s results to topologically principal groupoids. Renault’s machinery, and techniques from groupoid homology, underpinned Matsumoto and Matui’s remarkable recent results [34] that irreducible (two-sided) shifts of finite type are flow equivalent if and only if there is a diagonal-preserving isomorphism of the stabilisations of the associated Cuntz–Krieger algebras, and that the corresponding one-sided shifts are continuously orbit equivalent if and only if the Cuntz–Krieger algebras are isomorphic in a diagonal-preserving way.

These results all require topologically freeness of actions, or topologically principal groupoids. While seemingly fairly natural, these conditions are not generic. General $C^*$-algebraic rigidity theorems for homeomorphisms, local homeomorphisms, and more general group actions, require a version of Renault’s theory for non-topologically-principal groupoids. Ad hoc results in this direction have been achieved recently for graph $C^*$-algebras [10, 12], but there is no general theory available.

**Our results.** A version of Renault’s theory is impossible for general groupoids: for example, there is no way to distinguish the groupoids $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ using their $C^*$-algebras. Our key observation, inspired by techniques developed in [10, 13] is that this obstruction disappears if we insist that the fibres of the interior of the isotropy bundle of the groupoid are torsion-free and abelian; in the 1-unit case, we can then use [26, Theorem 8.57] to recover the groupoid as the quotient of the unitary group of its $C^*$-algebra by the connected component of the identity. Groupoids of this sort include all groupoids arising from actions of $\mathbb{Z}^k$ by homeomorphisms or of $\mathbb{N}^k$ by local homeomorphisms; this includes all Cantor systems, all groupoids associated to graphs and $k$-graphs and their topological analogues, and many other natural examples.

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1 Throughout this paper, we often use the phrase “diagonal-preserving” to describe homomorphisms interwining distinguished abelian subalgebras of $C^*$-algebras; we do not necessarily mean $C^*$-diagonals in the technical sense of Kumjian [31].
Our key results, following the idea introduced in [1], deal with groupoids $G$ graded by cocycles $c$ into discrete groups $\Gamma$. The reduced $C^\ast$-algebra $C^\ast_r(G)$ then carries a natural coaction $\delta_c$ of $\Gamma$. We prove that if the interior of the isotropy in $c^{-1}(\text{id}_\Gamma)$ (where $\text{id}_\Gamma$ is the identity element of $\Gamma$) is torsion-free and abelian, then $G$ and $c$ can be reconstructed from $C^\ast_r(G)$, the subalgebra $C_0(G^{(0)})$ and the coaction $\delta_c$. We also obtain a $C^\ast$-algebraic characterisation of groupoid equivariances that respect gradings in an appropriate sense.

Incorporating gradings and coactions has significant advantages [12], but the reader may, in the first instance, wish to keep in mind the case where $\Gamma$ is the trivial group, so $c$ is trivial: our results in this situation are special cases of our general theorems, but the statements are simpler, and still have substantial new content (we summarise our results in this setting in Section 3).

We detail the consequences of our results for dynamical systems, demonstrating the breadth of their applicability: we develop $C^\ast$-algebraic characterisations of appropriate notions of stabiliser-preserving orbit equivalence and topological conjugacy for group actions whose essential stabilisers are torsion-free and abelian, generalising Li’s continuous-orbit-equivalence-rigidity theorem [33, Theorem 1.2]; we characterise stabiliser-preserving continuous orbit equivalence and eventual conjugacy both of local homeomorphisms and of the associated stabilised systems in the same terms, generalising [10, Theorem 5.1], [11, Corollary 6.3], [12, Theorem 4.1 and Theorem 5.1], [34, Theorem 2.3], and [50, Theorem 2]; and we generalise Boyle and Tomiyama’s theorem [6, Theorem 3.6] to arbitrary homeomorphisms of second-countable compact Hausdorff spaces. Our results have many other potential applications, particularly to topological graphs, to actions of $Z^k$ on locally compact spaces, and to actions of $N^k$ by local homeomorphisms. In fact, our results have recently been applied to shift spaces in [7], to $k$-graphs in [14], and to self-similar groups in [52].

Précis. The paper is laid out as follows. We give some very brief background in Section 1 and recall some facts about normalisers in Section 2. Our main results are about $C^\ast$-algebras $A$ carrying coactions $\delta$ of discrete groups and an abelian $C^\ast$-subalgebra $D$ containing an approximate unit of the generalised fixed-point algebra $A^\delta$, but in Section 3 we state these results as they apply to ungraded groupoids and $C^\ast$-algebras. In Section 4 we show how to construct a locally compact Hausdorff étale groupoid $\mathcal{H}(A, D, \delta)$ from a separable $C^\ast$-algebra $A$, a coaction $\delta$ of a discrete group on $A$ and an abelian $C^\ast$-subalgebra $D$ of the generalised fixed-point algebra $A^\delta$. Our construction builds on those of Kumjian [31] and Renault [42, 43], but extends them by incorporating the structure of the unitary groups of the fibres of the relative commutant $D^\delta_A$ in $A^\delta$. Our main application is to groupoid $C^\ast$-algebras, but this general construction yields an interesting invariant for general systems $(A, D, \delta)$. In Section 5 we prove that if $G$ is an étale groupoid in which the interior of the isotropy is abelian, then the $C^\ast$-algebra of the interior of the isotropy is a maximal abelian subalgebra of $C^\ast_r(G)$; this is needed in Section 6 but also answers a question left open in [9].

In Section 6 we prove our main theorem: if $(G, c)$ is a graded second-countable locally compact Hausdorff étale groupoid and the interior of the isotropy in $c^{-1}(\text{id}_\Gamma)$ is torsion-free and abelian, then $\mathcal{H}(C^\ast_r(G), C_0(G^{(0)}), \delta_c) \cong G$ via an isomorphism that intertwines gradings. Consequently, two such graded groupoids $(G_1, c_1)$ and $(G_2, c_2)$ are isomorphic if and only if there is an isomorphism between the corresponding triples $(C^\ast_r(G_1), C_0(G_1^{(0)}), \delta_{c_1})$ and $(C^\ast_r(G_2), C_0(G_2^{(0)}), \delta_{c_2})$. Sections 7–9 detail the consequences of Section 6 for group
actions, for local homeomorphisms, and for homeomorphisms, including extensions of Li’s rigidity theorem, Matsumoto and Matui’s theorem about continuous orbit equivalence, and Boyle and Tomiyama’s theorem.

The final two sections deal with Morita equivalence. In Section 10 we introduce equivariant Morita equivalence of triples \((A_i, D_i, \delta_i)\) as above, and prove that such a Morita equivalence induces a graded equivalence of extended Weyl groupoids. In Section 11 we apply this result to triples \((C^*_r(G_i), C_0(G_i(0)), \delta_c)\) corresponding to graded groupoids \((G_i, c_i)\) such that the interior of the isotropy in each \(c_i^{-1}(\text{id})\) is torsion-free and abelian. We prove that \((G_1, c_1)\) and \((G_2, c_2)\) are graded equivalent if and only if the associated triples \((C^*_r(G_i), C_0(G_i(0)), \delta_c)\) are equivariantly Morita equivalent. Restricting attention to ample groupoids, we use the results of 15 to relate these notions to versions of graded Kakutani equivalence and to graded stable isomorphism of groupoids. We finish by detailing our applications to surjective local homeomorphisms and their dilations.

Acknowledgments. In a preliminary version of this paper, we introduced the notion of a weakly Cartan subalgebra (which was different from the notion introduced in 20). In this version, we have added Lemma 4.1 which shows that what we had previously called a weakly Cartan subalgebra of \(A^\delta\) is simply an abelian \(C^*\)-subalgebra of \(A^\delta\) containing an approximate unit for of \(A^\delta\). So we have eliminated the phrase “weakly Cartan subalgebra” in this version. We thank Bartosz Kwaśniewski and Ralf Meyer for showing us (1) and (2) in Lemma 4.1. We would also like to thank the two referees for valuable comments and suggestions.

1. Background

We establish some brief background and notational conventions for étale groupoids and their reduced \(C^*\)-algebras, \(C_0(X)\)-algebras, coactions on \(C^*\)-algebras, and \(C^*\)-algebraic Morita equivalence. For more details see 17, 51, Appendix C, 18, and 41.

When \(\Gamma\) is a group, then we use \(\text{id}_\Gamma\) to denote its identity element, and when \(X\) is a set, then we let \((e_x)_{x \in X}\) be an orthonormal basis for the Hilbert space \(l^2(X)\).

1.1. Étale groupoids. A groupoid \(G\) is the set of morphisms of a small category with inverses; the space of identity morphisms is called the unit space and denoted \(G(0)\), and the set of composable pairs of morphisms is denoted \(G(2)\). A locally compact Hausdorff groupoid is a groupoid \(G\) with a locally compact Hausdorff topology under which the inverse and multiplication maps are continuous. A map \(c\) from \(G\) to a discrete group \(\Gamma\) is a cocycle if \(c(\eta_1 \eta_2) = c(\eta_1)c(\eta_2)\) for \((\eta_1, \eta_2) \in G(2)\) (this forces \(c(G(0)) = \{\text{id}_\Gamma\}\) and \(c(\eta^{-1}) = c(\eta)^{-1}\)). A graded groupoid is a pair \((G, c)\) consisting of a groupoid \(G\) and a cocycle \(c : G \to \Gamma\). We say that \((G, c)\) is trivially graded if \(\Gamma\) is the trivial group.

We write \(r, s\) for the range and source maps \(r(\eta) = \eta\eta^{-1}\) and \(s(\eta) = \eta^{-1}\eta\) from \(G\) to \(G(0)\). A subset \(X \subseteq G(0)\) is full if \(\{r(\eta) : s(\eta) \in X\} = G(0)\). The isotropy of \(G\) is \(\text{Iso}(G) := \{\eta \in G : r(\eta) = s(\eta)\}\). We say that \(G\) is étale if \(r\) (equivalently \(s\)) is a local homeomorphism from \(G\) to \(G(0)\), and that it is ample if it is étale and \(G(0)\) is totally disconnected. For \(u \in G(0)\), we write \(G_u := s^{-1}(u)\) and \(G^u := r^{-1}(u)\). Using that \(r, s\) are local homeomorphisms and that \(G\) is Hausdorff, one can check that \(G(0)\) is clopen in \(C^*_r(G)\), and also that \(G\) has a basis of open sets \(U\) such that \(r|_U\) is a homeomorphism of \(U\) onto \(r(U)\) and \(s|_U\) is a homeomorphism of \(U\) onto \(s(U)\); we call such sets bisections.
Given a locally compact Hausdorff étale groupoid $G$, the space $C_c(G)$ is a $*$-algebra under the operations $f^*(\gamma) = \overline{f(\gamma)}$ and $(fg)(\gamma) = \sum_{a \in G(\gamma)} f(a) g(a^{-1}\gamma)$. For each $u \in G(0)$, there is a $*$-representation $\rho_u$ of $C_c(G)$ on $\ell^2(G_u)$ defined by $\rho_u(f) e_\gamma = \sum_{a \in G(\gamma)} f(a)e_{a\gamma}$. We call $\rho_u$ the regular representation of $C_c(G)$ at $u$. The reduced $C^*$-algebra $C^*_r(G)$ is the completion of $C_c(G)$ with respect to the norm $\|f\| = \sup_{u \in G(0)} \|\rho_u(f)\|$. This norm agrees with the supremum norm on functions $f$ supported on bisections. Since $G(0)$ is clopen, $C_c(G(0))$ includes in $C_c(G)$ in the canonical way, and this extends to an injection $C_0(G(0)) \hookrightarrow C^*_r(G)$. The representations $\rho_u$ extend to representations $\rho_u : C^*_r(G) \rightarrow \mathcal{B}(\ell^2(G_u))$. So there is a norm-decreasing map $a \mapsto \rho_a$ from $C^*_r(G)$ to $C_0(G)$ given by

\[ f_a(\gamma) = (\rho_a(\gamma)(a)e_{s(\gamma)} | e_{t(\gamma)}) \quad \text{for all } a \in C^*_r(G) \text{ and } \gamma \in G, \]

and $a \mapsto \rho_a$ restricts to the identity map on $C_c(G)$.

1.2. $C_0(X)$-algebras. Let $X$ be a locally compact Hausdorff space. A $C_0(X)$-algebra is a $C^*$-algebra $A$ together with a nondegenerate inclusion $\iota : C_0(X) \rightarrow ZM(A)$ of $C_0(X)$ into the centre of the multiplier algebra of $A$. We obtain a family of ideals $I_x := \iota(\{f \in C_0(X) : f(x) = 0\})A$ of $A$ (these subsets are automatically linear subspaces), and then a bundle of $C^*$-algebras $\{A_x : x \in X\}$ over $X$ given by $A_x := A/I_x$. Each $a \in A$ determines a section $f_a : X \rightarrow A = \bigsqcup_{x \in X} A_x$ such that $f_a(x) = a + I_x \in A_x$. There is a unique topology on $A$ under which these sections are all continuous. With respect to this topology, $A$ is an upper-semicontinuous bundle of $C^*$-algebras in the sense that $b \mapsto \|b\|$ is upper semicontinuous from $A$ to $[0, \infty)$.

Given any upper-semicontinuous bundle $\mathcal{A}$ of $C^*$-algebras over $X$, the space $\Gamma_0(X,\mathcal{A})$ of continuous sections of $\mathcal{A}$ that vanish at infinity is a $C^*$-algebra under pointwise operations and the supremum norm, and it becomes a $C_0(X)$-algebra with respect to the map $\iota : C_0(X) \rightarrow ZM(\Gamma_0(X,\mathcal{A}))$ given by $\iota(f)\xi(x) = f(x)\xi(x)$ for $f \in C_0(X)$ and $\xi \in \Gamma_0(X,\mathcal{A})$. If $\mathcal{A}$ is the bundle coming from a $C_0(X)$-algebra $A$ as above, the map $a \mapsto \rho_a$ is an isomorphism $A \cong \Gamma_0(X,\mathcal{A})$.

1.3. Coactions. Given a discrete group $\Gamma$, we write $\lambda_g$ for the image of $g \in \Gamma$ in the left regular representation of $\Gamma$ on $\ell^2(\Gamma)$. We write $\delta_\Gamma : C^*_r(\Gamma) \rightarrow C^*_r(\Gamma) \otimes C^*_r(\Gamma)$ (we use the minimal tensor product) for the comultiplication such that $\delta_\Gamma(\lambda_g) = \lambda_g \otimes \lambda_g$ for $g \in \Gamma$. Given a $C^*$-algebra $A$, a coaction of $\Gamma$ on $A$ is a nondegenerate homomorphism $\delta : A \rightarrow A \otimes C^*_r(\Gamma)$ satisfying the coaction identity $(\delta \otimes 1) \circ \delta = (1 \otimes \delta_\Gamma) \circ \delta$. The spectral subspaces of $A$ are the spaces $A_g := \{a \in A : \delta(a) = a \otimes \lambda_g\}$; we write $A^\delta$ for the neutral spectral subspace $A_{id_\Gamma}$, and call it the generalised fixed-point algebra for $\delta$. Since we are dealing with reduced coactions, they automatically satisfy $A = \overline{\text{span}(\bigcup_{g \neq 1} A_g)}$.

For each $g \in \Gamma$ there is a norm-decreasing linear map $\Phi_g : A \rightarrow A_g$ that fixes $A_g$ pointwise and annihilates $A_h$ for $h \neq g$. Specifically, writing $\text{Tr}$ for the canonical trace on $C^*_r(\Gamma)$, the map $\Phi_g$ is given by $\Phi_g(a) = (\text{id}_A \otimes \text{Tr})(\delta(a)(1_A \otimes \lambda_{g^{-1}}))$ for $a \in A$. In particular $\Phi^{\delta} := \Phi_{id_\Gamma} : A \rightarrow A^{\delta}$ is a conditional expectation.

We say that the coaction $\delta$ is trivial if $\Gamma$ is the trivial group.

1.4. Morita equivalence. Throughout the paper, we say that an element or a subset of a $C^*$-algebra $A$ is $A$-full (or just full) if it generates $A$ as an ideal. Given $C^*$-algebras $A$ and $B$, an $A$–$B$-imprimitivity bimodule is an $A$–$B$-bimodule $X$ carrying a left $A$-linear $A$-valued inner-product $A \langle \cdot , \cdot \rangle$ and a right $B$-linear $B$-valued inner-product $\langle \cdot , \cdot \rangle_B$ such that $A(x, y \cdot b) = A(x \cdot b^*, y)$ and $\langle a \cdot x, y \rangle_B = \langle x, a^* \cdot y \rangle_B$ for all $x, y \in X$, $a \in A$ and $b \in B$, respectively. This provides a way to relate different $C^*$-algebras through their representations on Hilbert spaces.
and such that $x \cdot \langle y, z \rangle_B = A\langle x, y \rangle \cdot z$ for all $x, y, z \in X$. We say that $C^*$-algebras $A$ and $B$ are Morita equivalent if there exists an $A$–$B$-imprimitivity bimodule. For any $C^*$-algebra $A$ and any positive element (in particular, any projection) $a \in M(A)$, the space $Aa$ is an imprimitivity bimodule from the ideal $AaA := \text{span}\{bac : b, c \in A\}$ generated by $a$ to the hereditary subalgebra $\overline{aAa}$ under $AaA \langle x, y \rangle = xy^* \text{ and } \langle x, y \rangle_aAa = x^*y$. So if $a$ is $A$-full, then $A$ is Morita equivalent to $\overline{aAa}$. If $X$ is an $A$–$B$-imprimitivity bimodule, then the conjugate module $X^*$ is a $B$–$A$-imprimitivity bimodule, and $L := A \oplus X \oplus X^* \oplus B$ becomes a $C^*$-algebra called the linking algebra, under the natural operations obtained by regarding elements $(a, x, y^*, b)$ as $2 \times 2$-matrices $(\begin{pmatrix} a & x \\ y^* & b \end{pmatrix})$: products of the form $xy^*$ and $y^*x$ are given by $A\langle x, y \rangle$ and $\langle x, y \rangle_B$ respectively. The projections $P = 1_{M(A)}$ and $Q = 1_{M(B)}$ are complementary full multiplier projections such that $PLP \cong A$, $QLQ \cong B$, and $PLQ \cong X$. The Brown–Green–Rieffel theorem implies that if $A$ and $B$ admit countable approximate units, then they are Morita equivalent if and only if they are stably isomorphic.

2. Normalisers

In this section we recall the notion of normalisers, and prove some fundamental results that we will need later (mainly in Section 4).

**Definition 2.1** (See [31][32][33]). Given a $C^*$-algebra $A$ and a $C^*$-subalgebra $D$ of $A$ containing an approximate unit for $A$, a normaliser $n$ of $D$ in $A$ is an element $n \in A$ such that $nDn^* \cup n^*Dn \subseteq D$. We write $N_A(D)$, or just $N(D)$ if $A$ is clear from context, for the set of normalisers of $D$ in $A$.

**Notation 2.2.** For the remainder of the section, $A$ is a $C^*$-algebra and $D \subseteq A$ is an abelian $C^*$-subalgebra containing an approximate unit for $A$. We write $D'_A$ for the relative commutant $D'_A = \{a \in A : ad = da \text{ for all } d \in D\}$ and $\hat{D}$ for the set of characters of $D$.

For $d \in D$, let $\text{supp}^o(d) := \{\phi \in \hat{D} : \phi(d) \neq 0\}$ and $I(d) := \{d' \in D : \text{supp}^o(d') \subseteq \text{supp}^o(d)\}$. So $\text{supp}^o(d)$ is an open subset of $\hat{D}$, and $I(d)$ is an ideal of $D$.

We establish some basic properties of normalisers, several of which also appear in [31].

**Lemma 2.3** (Kumjian, Renault). Let $A$ be a separable $C^*$-algebra and $D$ an abelian $C^*$-subalgebra containing an approximate unit for $A$. For $m, n \in N(D)$,

1. $n^*n, nn^* \in D$;
2. there is a unique homeomorphism $\alpha_n : \text{supp}^o(n^*n) \to \text{supp}^o(nn^*)$ such that $\phi(n^*n)\alpha_n(\phi)(d) = \phi(n^*dn)$ for $\phi \in \text{supp}^o(n^*n)$ and $d \in D$;
3. there is a unique isomorphism $\alpha^\#: I(nn^*) \to I(n^*n)$ such that $\phi(\alpha^\#(d)) = \alpha_n(\phi)(d)$ for $\phi \in \text{supp}^o(n^*n)$ and $d \in I(nn^*)$;
4. if $d \in I(nn^*)$, then $dn = \alpha_n(\phi)(d)$;
5. $mn \in N(D)$, $\text{supp}^o((mn)^*(mn)) = \alpha_n^{-1}(\text{supp}^o(m^*m) \cap \text{supp}^o(nn^*))$, and on this domain, $\alpha_m \circ \alpha_n = \alpha_{mn}$;
6. $\alpha_n^* = \alpha_n^{-1}$; and
7. if $U \subseteq \text{supp}^o(n^*n) \cap \text{supp}^o(m^*m) \subseteq \hat{D}$ is open and satisfies $\alpha_n|U = \alpha_m|U$, then $dn^*m = n^*md$ for all $d \in D$ with $\text{supp}^o(d) \subseteq U$, and $\phi(\alpha_{n^*m^*})(d) = \phi(m^*\phi(n^*n))$ for $\phi \in U$.

**Proof.** Let $(u_j)$ be an approximate unit for $A$ in $D$. Then each $n^*u_jn, nu_jn^* \in D$ because $n \in N(D)$. So $nn^* = \lim_j n^*u_jn = D$ and $nn^* = \lim_j nu_jn^* \in D$. 

This is proved in [31 Proposition 1.6]. We summarise the points of the proof that we need for the remaining statements. Let \( n = \|n\| \) be the polar decomposition of \( n \) in \( A^* \). Then \( vv^*d = dv^* = d \) for \( d \in I(nn^*) \), and \( v^*vd = dv^*v = d \) for \( d \in I(n^*n) \). Kumjian shows that \( vI(n^*n)v^* \subseteq I(nn^*) \) and \( v^*I(nn^*)v \subseteq I(n^*n) \). So \( d \mapsto v^*dv \) defines an isomorphism \( \alpha_n^* : I(nn^*) \rightarrow I(n^*n) \), and there is a homeomorphism \( \alpha_n : \text{supp}^\circ(n^*n) \rightarrow \text{supp}^\circ(nn^*) \) such that \( \alpha_n(\phi)(d) = \phi(\alpha_n^*(d)) \) for all \( d \in I(nn^*) \).

That each of \( \hat{D} \) and \( D \) separates elements of the other gives uniqueness of \( \alpha_n \) and \( \alpha_n^* \).

As above, let \( n = \|n\| \) be the polar decomposition of \( n \). By (1), we have \( |n| = (n^*n)^{1/2} \in D \). It follows that if \( d \in I(nn^*) \), then
\[
dn = dv|n| = vv^*dv|n| = v\alpha_n^*(d)|n| = |n|\alpha_n^*(d) = n\alpha_n^*(d).
\]

Fix \( d \in D \) and \( m, n \in N(D) \). Then \( (mn)^*d(mm^*) = n^*(m^*dm)n^* \in n^*Dn \subseteq D \); likewise \( (mn)d(mm^*) \in D \), so \( mn \in N(D) \). We have \( n^*m^*mn^* = n^*\alpha_n^*(m^*mn^*) \) by (1). So \( \text{supp}^\circ((mn)^*(mm^*)) = \alpha_n^{-1}(\text{supp}^\circ(m^*m) \cap \text{supp}^\circ(nn^*)) \). For \( \phi \in \text{supp}^\circ((mn)^*(mm^*)) \),
\[
\phi(n^*m^*mn)\alpha_{mn}(\phi)(d) = \phi(n^*m^*nm) = \phi(n^*n)\alpha_n(\phi)(dm^*m) \\
= \phi(n^*n)\alpha_n(m^*m)\alpha_m(\alpha_n(\phi))(d) = \phi(n^*m^*mn)\alpha_m(\alpha_n(\phi))(d).
\]

This shows that \( \alpha_{mn}(\phi) = \alpha_m(\alpha_n(\phi)) \).

Suppose \( \phi \in \text{supp}^\circ((n^*n)^*(n^*n)) \) and \( d \in D \). Then
\[
\phi((n^*n)^*(n^*n))\alpha_{n^*n}(\phi)(d) = \phi((n^*n)^*d(n^*n)) = \phi(n^*n)\phi(d)n^*(n^*n) = \phi(n^*n(n^*n))\phi(d).
\]
So \( \alpha_{n^*n} = \text{id}_{\text{supp}^\circ(n^*n)} \), and then (5) gives \( \alpha_n \circ \alpha_n = \text{id}_{\text{supp}^\circ(n^*n)} \). Thus, \( \alpha_n = \alpha_n^{-1}. \)

Fix \( d \in D \) with \( \text{supp}^\circ(d) \subseteq U \). As \( \alpha_n|_U = \alpha_m|_U \), statement (3) and (1) give \( \alpha_n^*(d) = \alpha_m^*(d) \). Two applications of (1) then give \( dn^*m = n^*\alpha_n^*(d)m = n^*md \). The definition of \( \alpha_n^* \) shows that \( \alpha_n^*(nn^*) = n^*n \), so \( \alpha_n(\phi)(mn^*) = \phi(n^*n) \) for all \( \phi \). So for \( \phi \in U \), we have \( \phi(m^*m^*m) = \phi(m^*m)\alpha_m(\phi)(mn^*) = \phi(m^*m)\alpha_n(\phi) = \phi(m^*m)\phi(n^*n) \).

3. RESULTS FOR UNGRADED GROUPOIDS AND \( C^*\)-ALGEBRAS

To maximise their generality, we formulate our key results later in the paper for graded groupoids and \( C^*\)-algebras carrying coactions, under suitable hypotheses on the trivially-graded subgroupoids and generalised fixed-point subalgebras. In this section, we summarise the consequences of our main results for trivial gradings and coactions.

Lemma 4.5 below applied to a separable \( C^*\)-algebra \( A \), an abelian \( C^*\)-subalgebra \( D \subseteq A \) containing an approximate unit for \( A \), and the trivial coaction on \( A \) yields an equivalence relation \( \sim \) on \( \{(n, \phi) : n \in N_A(D), \phi \in \text{supp}^\circ(n^*n)\} \) as follows: given \( \phi \in \hat{D} \), we write \( J_\phi \) for the ideal of \( D_A \) generated by \( \ker(\phi) \); and then \( (n, \phi) \sim (m, \psi) \) if and only if \( \phi = \psi, \alpha_n \) and \( \alpha_m \) agree on a neighbourhood \( U \) of \( \phi \), and there exists \( d \in D \) with \( \text{supp}^\circ(d) \subseteq U \) and \( \phi(d) = 1 \) such that \( \phi(m^*m)^{-\epsilon}n^*d^*md + J_\phi \) is homotopic to the identity in the unitary group of \( D_A/J_\phi \). We write \([n, \phi] \) for the equivalence class of \((n, \phi)\) with respect to this equivalence relation.

Our first main theorem says that the quotient space can be made into an étale groupoid.

**Theorem 3.1** (see Theorem 4.9). Let \( A \) be a separable \( C^*\)-algebra and \( D \) an abelian \( C^*\)-subalgebra of \( A \) that contains an approximate unit for \( A \). Then
\[
\mathcal{H}(A, D) := \{[n, \phi] : n \in N_A(D), \phi \in \text{supp}^\circ(n^*n)\}
\]
is a second-countable locally compact locally Hausdorff étale groupoid with composable pairs $\mathcal{H}(A, D)^{(2)} = \{[[m, \phi], [n, \psi]] : \phi = \alpha_n(\psi)\}$, multiplication and inverses given by $[m, \alpha_n(\psi)][n, \psi] = [mn, \psi]$ and $[n, \psi]^{-1} = [n^*, \alpha_n(\psi)]$, and topology with basic open sets $Z(n, U) = \{[n, \phi] : \phi \in U\}$ indexed by $n \in N(D)$ and open $U \subseteq \text{supp}^\#(n^*n) \subseteq \hat{D}$.

If $D \subseteq A$ is Cartan as in [13], then $\mathcal{H}(A, D)$ is the Weyl groupoid of [13]. If $A = C^*(E)$ and $D = \mathcal{D}(E)$ where $E$ is a countable graph, then $\mathcal{H}(A, D)$ is the extended Weyl groupoid $\mathcal{G}((C^*(E), \mathcal{D}(E)))$ of [10]. If $A = C^*(\Gamma)$ where $\Gamma$ is a torsion-free and abelian discrete group, and $D = C_1 A$, then $N_A(D)$ consists of the nonzero scalar multiples of unitaries in $C^*(\Gamma)$, and it then follows from [26, Theorem 8.57] that $\mathcal{H}(A, D)$ is isomorphic to $\Gamma$ by an isomorphism that takes $[n, \phi]$ to the class of the unitary $(n^*n)^{-\frac{1}{2}}$ in $K_1(C^*(\Gamma)) \cong \Gamma$. The following generalises this.

**Proposition 3.2** (see Lemma 6.4 and Proposition 6.5). Let $G$ be a second-countable locally compact Hausdorff étale groupoid. Then $C_0(G^{(0)})$ is an abelian $C^*$-subalgebra of $C^*_r(G)$ and contains an approximate unit for $C^*_r(G)$. If $\text{Iso}(G)^\circ$ is torsion-free and abelian, then there is an isomorphism $G \cong \mathcal{H}(C^*_r(G), C_0(G^{(0)}))$ that carries $\gamma \in G$ to $[f, s(\gamma)]$ for any $f \in C_c(G)$ supported on a bisection with $f(\gamma) \neq 0$.

As an almost immediate consequence of Proposition 3.2 we obtain the following.

**Theorem 3.3** (see Theorem 6.2). Let $G_1, G_2$ be second-countable locally compact Hausdorff étale groupoids such that each $\text{Iso}(G_i)^\circ$ is torsion-free and abelian.

1. Any isomorphism $\kappa : G_2 \to G_1$ induces an isomorphism $\phi : C^*_r(G_1) \to C^*_r(G_2)$ such that $\phi(f) = f \circ \kappa$ for $f \in C_c(G_1)$, and in particular $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$.
2. Any isomorphism $\phi : C^*_r(G_1) \to C^*_r(G_2)$ satisfying $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$ induces an isomorphism $\kappa : G_2 \to G_1$ such that $f \circ \kappa = \phi(f)$ for $f \in C_0(G_1^{(0)})$.

If $D_i \subseteq A_i$ is a nested pair of $C^*$-algebras for $i = 1, 2$, we say that $(A_1, D_1)$ and $(A_2, D_2)$ are Morita equivalent if there is an $A_1$-$A_2$-imprimitivity bimodule $X$ such that $X = \text{span}\{x \in X : \langle x, D_1 \cdot x \rangle_{A_2} \subseteq D_2 \text{ and } A_1 \langle x \cdot D_2, x \rangle \subseteq D_1\}$.

**Theorem 3.4** (see Theorem 10.6). Let $A_1, A_2$ be separable $C^*$-algebras. Suppose, for $i = 1, 2$, that $D_i$ is an abelian $C^*$-subalgebra of $A_i$ containing an approximate unit for $A_i$. Suppose that $X$ is a Morita equivalence between $(A_1, D_1)$ and $(A_2, D_2)$. Let $A$ be the linking algebra of $X$ and let $D := D_1 \oplus D_2 \subseteq A$. Then $D$ is a $C^*$-algebra of $A$ that contains an approximate unit for $A$. The groupoid $\mathcal{H} := \mathcal{H}(A, D)$ contains $\hat{D}_1$ and $\hat{D}_2$ as complementary full clopen subsets of $H^{(0)}$, $\hat{D}_1 \mathcal{H} \hat{D}_2 \cong \mathcal{H}(A_i, D_i)$ for $i = 1, 2$, and $\hat{D}_1 \mathcal{H} \hat{D}_2$ is an equivalence from $\mathcal{H}(A_1, D_1)$ to $\mathcal{H}(A_2, D_2)$.

**Theorem 3.5** (see Theorem 11.1). Let $G_1, G_2$ be second-countable locally compact Hausdorff étale groupoids such that each $\text{Iso}(G_i)^\circ$ is torsion-free and abelian. The following are equivalent:

1. $G_1$ and $G_2$ are equivalent;
and compact operators on Corollary 3.6 as defined in [35]. Combining our results with [15], we obtain the following.

groupoids such that each conditions of Theorem 3.5 are also equivalent to the following:

δ

\[ A \]

\[ \text{commutant} \]

\[ G \]

\[ A \]

\[ \text{coaction of a discrete group } \Gamma \text{ on } \]

As in [15], we let \[ H = \mathbb{N} \times \mathbb{N} \] regarded as a discrete groupoid; we have \[ C_\tau(\mathbb{N}) \cong K \], the compact operators on \( \ell^2(\mathbb{N}) \), with canonical diagonal subalgebra \( \mathcal{C} \). As in [15], we say \( G_1 \) and \( G_2 \) are weakly \( \text{Kakutani equivalent} \) if there are full open subsets \( U_i \subseteq G_i^{(0)} \) such that \( U_1 G_1 U_1 \cong U_2 G_2 U_2 \). If \( U_1 \) and \( U_2 \) are compact open, \( G_1 \) and \( G_2 \) are \( \text{Kakutani equivalent} \) as defined in [35]. Combining our results with [15], we obtain the following.

Corollary 3.6 (see Corollary 11.3). Let \( G_1, G_2 \) be second-countable ample Hausdorff groupoids such that each \( \text{Iso}(G)^0 \) is torsion-free and abelian. Then the equivalent conditions of Theorem 3.5 are also equivalent to the following:

\[ G_1 \times \mathcal{R} \cong G_2 \times \mathcal{R}; \]

\[ (G_1) \]

\[ (G_2) \]

\[ (G_1) \]

\[ (G_2) \]

\[ (G_1) \]

\[ (G_2) \]

\[ (G_1) \]

\[ (G_2) \]

\[ (G_1) \] for each \( \mathcal{R} = \mathbb{N} \times \mathbb{N} \), with canonical diagonal subalgebra \( \mathcal{C} \). As in [15], we say \( G_1 \)

\[ \text{and } G_2 \]

\[ \text{are weakly } \text{Kakutani equivalent} \] if there are full open subsets \( U_i \subseteq G_i^{(0)} \) such that \( U_1 G_1 U_1 \cong U_2 G_2 U_2 \). If \( U_1 \) and \( U_2 \) are compact open, \( G_1 \) and \( G_2 \) are \( \text{Kakutani equivalent} \) as defined in [35]. Combining our results with [15], we obtain the following.

\[ \text{Corollary 3.6 (see Corollary 11.3). Let } G_1, G_2 \text{ be second-countable ample Hausdorff groupoids such that each } \text{Iso}(G)^0 \text{ is torsion-free and abelian. Then the equivalent conditions of Theorem 3.5 are also equivalent to the following:} \]

(1) \[ G_1 \times \mathcal{R} \cong G_2 \times \mathcal{R}; \]

(2) \[ G_1 \text{ and } G_2 \text{ are } \text{Kakutani equivalent}; \]

(3) \[ G_1 \text{ and } G_2 \text{ are weakly } \text{Kakutani equivalent}; \]

(4) \[ \text{there is an isomorphism } \phi : C_{\tau}(G_1) \otimes K \rightarrow C_{\tau}(G_2) \otimes K \text{ satisfying } \phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)}); \]

(5) \[ \text{there exist } C_{\tau}(G_i) \text{-full projections } p_i \in M(C_0(G_i^{(0)})) \text{ and an isomorphism } \phi \text{ of } p_1 C_{\tau}(G_1) p_1 \text{ onto } p_2 C_{\tau}(G_2) p_2 \text{ such that } \phi(p_1 C_0(G_1^{(0)})) = p_2 C_0(G_1^{(0)}); \]

(6) \[ \text{there exist } C_{\tau}(G_i) \text{-full projections } p_i \in M(C_0(G_i^{(0)})) \text{ and an isomorphism } \phi \text{ of } p_1 C_{\tau}(G_1) p_1 \text{ onto } p_2 C_{\tau}(G_2) p_2 \text{ such that } \phi(p_1 C_0(G_1^{(0)})) = p_2 C_0(G_1^{(0)}); \]

(7) \[ \text{there exist } C_{\tau}(G_i) \text{-full projections } p_i \in M(C_0(G_i^{(0)})) \text{ and an isomorphism } \phi \text{ of } p_1 C_{\tau}(G_1) p_1 \text{ onto } p_2 C_{\tau}(G_2) p_2 \text{ such that } \phi(p_1 C_0(G_1^{(0)})) = p_2 C_0(G_1^{(0)}); \]

4. The extended Weyl groupoid

We construct a graded groupoid \((\mathcal{H}(A,D,\delta),\mathcal{C})\) from a separable \( \text{C}^* \)-algebra \( A \), a coaction of a discrete group \( \Gamma \) on \( A \), and an abelian \( \text{C}^* \)-subalgebra \( D \subseteq A^\delta \) containing an approximate unit of \( A^\delta \). The reader may wish to keep in mind the case where \( \delta \) is trivial, so \( A^\delta = A \); in this case, if \( D \) is a Cartan subalgebra of \( A \), then \( \mathcal{H}(A,D,\delta) \) agrees with Renault’s Weyl groupoid [33]. Given a countable directed graph, then \( \mathcal{H}(\mathcal{C}^*(E),\mathcal{C}_0(\partial E),\delta) \) is the extended Weyl groupoid \( \mathcal{G}_{(\mathcal{C}^*(E),\mathcal{C}_0(\partial E),\delta)} \) of [10], both when \( \delta \) is the trivial coaction, and when \( \delta \) is the coaction of \( Z \) dual to the gauge action.

Given a \( \text{C}^* \)-algebra \( A \) and an abelian subalgebra \( D \) of \( A \), we write \( D_A^\prime \) for the relative commutant \( D_A^\prime = \{ a \in A : ad = da \text{ for all } d \in D \} \) and \( \hat{D} \) for the set of characters of \( D \).

Lemma 4.1. Let \( A \) be a separable \( \text{C}^* \)-algebra and \( D \) an abelian \( \text{C}^* \)-subalgebra of \( A \) containing an approximate unit for \( A \). Then

(1) for each \( \phi \in \hat{D} \), the quotient \( D_A^\prime/J_\phi \) by the ideal \( J_\phi := \ker(\phi)D_A^\prime \) of \( D_A^\prime \) is unital,

(2) for each \( \phi \in \hat{D} \), there exist \( d \in D \) and an open neighbourhood \( U \) of \( \phi \) such that \( d + J_\psi = 1_{D_A^\prime/J_\phi} \) for all \( \psi \in U \), and

(3) for each \( \phi \in \hat{D} \) and for each \( d \in D \), we have \( d + J_\phi = \phi(d)1_{D_A^\prime/J_\phi} \).
Proof. We first prove (2), and then (1) follows directly. So fix φ ∈ ˆD. Since D is abelian, we have D ≅ C0( ˆD), so there exists an open neighbourhood U of φ and an element d ∈ D such that ψ(d) = 1 for all ψ ∈ U. We claim that d + Jφ is a unit for D′A/Jφ for each ψ ∈ U. To establish the claim, fix ψ ∈ U. It suffices to show that d + Jφ is a unit for D′A/Jφ. For this, fix a ∈ D′A so that a + Jφ is a typical element of D′A/Jφ. Let (uj)j be an approximate unit for A such that each uj ∈ D. Since ψ(d) = 1, we have ψ(duj − uj) = 0 for all j, giving duj − uj ∈ Jφ for all j. Hence duj − uj ∈ Jφ for all j. Since uj → a, we deduce that da − a ∈ Jφ. Hence (d + Jφ)(a + Jφ) = a + Jφ. Thus d + Jφ is a unit for D′A/Jφ as required.

To see (3), fix φ ∈ ˆD and d ∈ D. Choose do ∈ D and an open U ⊃ φ such that d0 + Jφ = 1D′A/Jφ for all ψ in U. Since (d0 + Jφ)2 = 1D′A/Jφ = d0 + Jφ, we have φ(d0) = φ(d0). Thus, φ(d0) = 1 and φ(φ(d)d0) = φ(d). Consequently, d − φ(d)d0 ∈ Jφ, giving
\[
d + Jφ = φ(d)d0 + Jφ = φ(d)(d0 + Jφ) = φ(d)1D′A/Jφ.
\]

Throughout the remaining part of this section, A is a separable C∗-algebra, Γ is a discrete group, δ is a coaction of Γ on A, and D is a an abelian C∗-subalgebra of Aδ that contains an approximate unit for Aδ. We write
\[
D′A δ := \{a ∈ A δ : ad = da \text{ for all } d ∈ D\},
\]
for the relative commutant of D in Aδ, Jφ for the ideal ker(φ)D′A δ, and
\[
π φ : D′A δ → D′A δ/Jφ
\]
for the canonical quotient map.

Following [1], we say that a normaliser n of D is a homogeneous normaliser if n ∈ Aδ for some g ∈ Γ. We write Ng(D) := N(D) ∩ Aδ, and we write N∗(D) := g∈Γ Ng(D). The groupoid H(A, D, δ) consists of equivalence classes [n, φ] where n ∈ N∗(D) and φ ∈ suppδ(n∗n). To define the appropriate equivalence relation, we first need two lemmas.

Lemma 4.2. Let A be a separable C∗-algebra, δ a coaction of a discrete group Γ on A, and D an abelian C∗-subalgebra of Aδ containing an approximate unit for Aδ. Let n, m ∈ N(D) and φ ∈ suppδ(n∗n) ∩ suppδ(m∗m), and suppose that there is an open neighbourhood U of φ such that U ⊆ suppδ(n∗n) ∩ suppδ(m∗m) and αn|U = αm|U. Fix d ∈ D with suppδ(d) ⊆ U and φ(d) = 1, and let
\[
(4.1) \quad w := φ(m∗m)^{−1/2}φ(n∗n)^{−1/2}dn∗md.
\]
Then w ∈ D′A δ and πφ(w) is unitary in D′A δ/Jφ. We have
\[
π φ(φ(n∗n)^{−1/2}φ(m∗m)^{−1/2}dm∗nd) = π φ(w∗),
\]
and πφ(w) is independent of the choices of U and d.

Proof. We have w ∈ D′A δ by Lemma 2.3[7]. By Lemma 4.1 and Lemma 2.3 and since suppδ(d) ⊆ U, φ(d) = 1, and αm|U = αn|U, we have
\[
φ(m∗m)φ(n∗n)π φ(w∗) = π φ( (dn∗md)^{−1/2}(dn∗md) ) = π φ( d^{−1/2}dm∗nd ) = φ(m∗m)^{−1/2}md∗nd = π φ(w∗).
\]
so πφ(w∗w) = 1D′A δ/Jφ. Switching the roles of m, n gives π(w∗w) = 1D′A δ/Jφ. We clearly have πφ(φ(m∗m)^{−1/2}φ(n∗n)^{−1/2}dm∗nd) = π φ(w∗).
Now fix open $U_1, U_2 \subseteq \hat{D}$ and $d_1, d_2 \in D$ with $\phi \in U_1 \cap U_2$, $\alpha_n | U_i = \alpha_m | U_i$, sup$\phi^\circ(d_i) \subseteq U_i$ and $\phi(d_i) = 1$. It follows from Lemma 2.3.1 that $n^*md_1d_2 = d_1d_2n^*m$. Thus
\[
\pi_\phi(d_1n^*md_1d_2n^*md_2) = \pi_\phi(d_1d_2n^*mm^*n) = \phi(n^*mm^*n)\big|_{D^\delta/J_\phi},
\]
and we deduce that
\[
\pi_\phi(\phi(m^*m)^{-1/2}\phi(n^*n)^{-1/2}d_1n^*md_1) = \pi_\phi(\phi(m^*m)^{-1/2}\phi(n^*n)^{-1/2}d_2n^*md_2).
\]

**Notation 4.3.** Let $A$ be a separable $C^*$-algebra, $\delta$ a coaction of a discrete group $\Gamma$ on $A$, and $D$ an abelian $C^*$-subalgebra of $A^\delta$ containing an approximate unit for $A^\delta$. Suppose that $n, m \in N(D)$, $\phi \in \text{supp}^\circ(n^*n) \cap \text{supp}^\circ(m^*m)$, and that there is an open $U \ni \phi$ such that $\alpha_m | U = \alpha_n | U$. We write
\[
U^\phi_{n^*m} := \pi_\phi(w)
\]
for any $w$ of the form $(1.1)$. If $\phi$ is clear from context, we just write $U_{n^*m}$ for $U^\phi_{n^*m}$

**Lemma 4.4.** Let $A$ be a separable $C^*$-algebra, $\delta$ a coaction of a discrete group $\Gamma$ on $A$, and $D$ an abelian $C^*$-subalgebra of $A^\delta$ containing an approximate unit for $A^\delta$. Take $n_1, n_2, m \in N(D)$, $\phi \in \text{supp}^\circ(n_1^*n_1) \cap \text{supp}^\circ(n_2^*n_2) \cap \text{supp}^\circ(m^*m)$, and open $U \ni \phi$ with $\alpha_m | U = \alpha_n | U$. Then

(1) $U^*_{n_1^*n_1} = 1_{D^\delta/J_\phi}$,
(2) $U^*_{n_1^*n_2} = U_{mn_1^*}$, and
(3) $U^*_{n_1^*m}U^*_{m^*n_2} = U_{n_1^*n_2}$.

**Proof.** By normalising, we can assume that $\phi(n_1^*n_1) = \phi(m^*m) = 1$. For (1), just calculate:
\[
\pi_\phi(d_1n_1^*nd_2) = \phi(d_2\phi(n_1^*n_1) \cdot 1_{D^\delta/J_\phi} = 1_{D^\delta/J_\phi}.
\]
Statement (2) follows from Lemma 1.2 because $\pi_\phi$ is a *-homomorphism. For (3), take $d_1$ supported on $U_i$ with $\phi(d_i) = 1$. A quick calculation using that $\alpha_n(\phi)(m^*m) = \phi(m^*m)$ and that $\alpha_n^*m(\phi) = \phi$ gives
\[
U^*_{n_1^*m}U^*_{m^*n_2} = \phi(m^*m)\phi(d_1d_2)\pi_\phi(d_1d_2n_1^*n_2d_2) = U_{n_1^*n_2}.
\]

We are now ready to describe the elements of $\mathcal{H}(A, D, \delta)$.

**Lemma 4.5.** Let $A$ be a separable $C^*$-algebra, $\delta$ a coaction of a discrete group $\Gamma$ on $A$, and $D$ an abelian $C^*$-subalgebra of $A^\delta$ containing an approximate unit for $A^\delta$. Define $\sim$ on $(n, \phi) : n \in N_\delta(D), \phi \in \text{supp}^\circ(n^*n)$ by $(n, \phi) \sim (m, \psi)$ if and only if

(R1) $\phi = \psi$,
(R2) $n^*m \in A^\delta$,
(R3) there exists an open neighbourhood $U$ of $\phi$ in $\hat{D}$ such that $U \subseteq \text{supp}^\circ(n^*n) \cap \text{supp}^\circ(m^*m)$ and $\alpha_m | U = \alpha_n | U$, and
(R4) the unitary $U^\phi_{n^*m}$ of Notation 4.3 belongs to the connected component $U_0(D^\delta_{\phi}/J_\phi)$ of the identity in the unitary group of $D^\delta_{\phi}/J_\phi$.

Then $\sim$ is an equivalence relation.

**Proof.** Reflexivity follows from Lemma 4.4.11. Symmetry follows from Lemma 4.4.2 and that if $U^\phi_{n^*m} \in U_0(D^\delta_{\phi}/J_\phi)$, then $(U^\phi_{n^*m})^* \in U_0(D^\delta_{\phi}/J_\phi)$.

For transitivity, suppose $(n_1, \phi) \sim (m, \psi) \sim (n_2, \omega)$. Then $\phi = \psi = \omega, n_1^*m, m^*n_2 \in A^\delta$, there are open sets $\phi \in U_i \subseteq \text{supp}^\circ(n_1^*n_1) \cap \text{supp}^\circ(m^*m)$ such that $\alpha_n | U_i = \alpha_m | U_i$, and $U^*_{n_1^*m}U^*_{m^*n_2} \in U(D^\delta_{\phi}/J_\phi)$. The set $V := U_1 \cap U_2 \subseteq \text{supp}^\circ(n_1^*n_1) \cap \text{supp}^\circ(n_2^*n_2)$ is open, contains $\phi$ and satisfies $\alpha_{n_1} | V = \alpha_{n_2} | V$. We have $n_1 \in A_{g_1}$, $m \in A_h$ and $n_2 \in A_{g_2}$ for some $g_1, h, g_2 \in$
Let \( \phi(n, \phi) \) denote the equivalence class of \( \phi \) under \( \sim \). Let \( \alpha \in \text{supp}^\circ(n^*m) \) satisfy \( \alpha \cdot \phi(n, \phi) = \phi(n, \phi) \cdot \alpha \). Then there is a cocycle \( c_\alpha : \mathcal{H}(A, D, \delta) \to \Gamma \) such that \( c_\alpha([n, \phi]) = g \) if and only if \( n \in A_g \).

We now construct our extended Weyl groupoid.

**Proposition 4.7.** Let \( A \) be a separable \( C^* \)-algebra, \( \delta \) a coaction of a discrete group \( \Gamma \) on \( A \), and \( D \) an abelian \( C^* \)-subalgebra of \( A^\delta \) containing an approximate unit for \( A^\delta \). Let \( \sim \) be the equivalence relation of Lemma 4.5, and for \( n \in N_s(D) \) and \( \phi \in \text{supp}^\circ(n^*n) \), let \( n, \phi \) denote the equivalence class of \((n, \phi)\) under \( \sim \). Define
\[
\mathcal{H}(A, D, \delta) := \{[n, \phi] : n \in N_s(D), \phi \in \text{supp}^\circ(n^*n)\}.
\]
There are maps
\[
r, s : \mathcal{H}(A, D, \delta) \to \tilde{D},
\]
\[
M : \mathcal{H}(A, D, \delta) \times \mathcal{H}(A, D, \delta) \to \mathcal{H}(A, D, \delta), \quad \text{and} \quad I : \mathcal{H}(A, D, \delta) \to \mathcal{H}(A, D, \delta)
\]
such that
\[
r([n, \phi]) = \alpha_n(\phi), \quad s([n, \phi]) = \phi,
\]
\[
M([n, \phi], [m, \psi]) = [nm, \psi], \quad \text{and} \quad I([n, \phi]) = [n^*, \alpha_n(\phi)].
\]
Moreover, \( \mathcal{H}(A, D, \delta) \) is a groupoid under these operations, and there is a cocycle \( c_\delta : \mathcal{H}(A, D, \delta) \to \Gamma \) such that \( c_\delta([n, \phi]) = g \) if and only if \( n \in A_g \).

We need the following lemma for the proof of Proposition 4.7.

**Lemma 4.8.** Let \( A \) be a separable \( C^* \)-algebra, \( \delta \) a coaction of a discrete group \( \Gamma \) on \( A \), and \( D \) an abelian \( C^* \)-subalgebra of \( A^\delta \) containing an approximate unit for \( A^\delta \). Suppose \( m \in N(D), \phi \in \text{supp}^\circ(m^*m) \), and \( \phi(m^*m) = 1 \). Then there is an isomorphism \( \iota_m : D_{A^\delta}/J_{\alpha_m(\phi)} \to D_{A^\delta}/J_\phi \) such that \( \iota_m(\pi_{\alpha_m(\phi)}(a)) = \pi_\phi(m^*a m) \) for \( a \in D_{A^\delta} \).

**Proof.** By Lemma 2.3 \( a \in J_{\alpha_m(\phi)} \implies m^*a m \in J_\phi \). Thus \( a \mapsto m^*a m \) descends to a linear \( * \)-preserving map \( \iota_m : D_{A^\delta}/J_{\alpha_m(\phi)} \to D_{A^\delta}/J_\phi \). For \( a_1, a_2 \in D_{A^\delta} \),
\[
\pi_\phi(m^*a_1 m) \pi_\phi(m^*a_2 m) = \pi_\phi(m^*a_1 m m^*a_2 m) = \pi_\phi(m^*m^*a_1 a_2 m)
\]
so \( \iota_m \) is multiplicative and hence a \( * \)-homomorphism. Symmetry gives a \( * \)-homomorphism \( \iota_m^* : D_{A^\delta}/J_\phi \to D_{A^\delta}/J_{\alpha_m(\phi)} \) such that \( \iota_m^*(\pi_\phi(\alpha)) = \pi_{\alpha_m(\phi)}(m a m^*) \) for \( a \in D_{A^\delta} \). It is easy to check that \( \iota_m^* \) is an inverse to \( \iota_m \).

**Proof of Proposition 4.7.** If \( (n, \phi) \sim (m, \psi) \), then \( \phi = \psi \), and since \( \alpha_n = \alpha_m \) on a neighbourhood of \( \phi \), we have \( \alpha_n(\phi) = \alpha_m(\phi) \); so \( r \) and \( s \) are well defined. Suppose that \( [n, \phi] = [n', \phi] \) and \( r([m, \psi]) = s([n, \phi]) \). Then \( m, n^\prime m \in N_s(D) \). We claim that
(nm, ψ) ∼ (n'm, ψ). Indeed, (R1) is clear, and (R2) is immediate because (n, φ) ∼ (n', φ) forces (nm)*n'm = m*n*n'm ∈ A^δ. Take an open U with φ ∈ U ⊆ supp^φ(ν*n') ∩ ((n')*n') and α_n/U = α_n'/U. Fix d supported on U with φ(d) = 1. Since (n, φ) ∼ (n', φ), we have U^φ_{nm'} ∈ U_d(D_A'/I_φ). Now V := α_n^{-1}(U ∩ supp^φ(mm')) is a neighbourhood of ψ, and Lemma 2.3 gives α_{nm}|V = α_{n'm}|V, giving (R3). For (R4), we may assume that ψ(m*nm) = 1. The argument for (R3) is very similar to that in the preceding paragraph. For (R4), we may assume that ψ(m*nm) = φ(n*n') = 1. Choose V ⊃ ψ open such that V ⊆ supp^φ((nm)*nm) ∩ supp^φ((nm')*nm'), and d ∈ D with supp^φ(d) ⊆ V and ψ(d) = 1. Then

\[ U^\psi_{(nm)*nm} = \pi_\psi(\alpha^\#_m(\alpha^\#_{mn'}(d)n'n'm'd)) = \phi(n*n') \pi_\psi(dm*m'd) = \pi_\psi(dm*m'd) = U^\psi_{m*nm'} \in U_0(D_A'/I_\psi). \]

Thus, (nm, ψ) ∼ (n'm, ψ). Hence

(4.2) if [n, φ] = [n', φ] and \( r([n, φ]) = r([m, ψ]) \), then [nm, ψ] = [n'm, ψ].

By (4.2) and (4.3), if [n, φ] = [n', φ], [m, ψ] = [m', ψ], and r([m, ψ]) = r([n, φ]), then [nm, ψ] = [n'm, ψ], so M is well-defined.

To see that I is well-defined, suppose that [n, φ] = [m, φ]. We claim that (n*, α_n(φ)) ∼ (m*, α_m(φ)). Again (R1) and (R2) are clear, and (R3) is routine because α_n* = α_n^{-1} and similarly for α_m*. For (R4), we may assume that ψ(m*nm) = φ(n*n') = 1. For an open U ⊃ φ with \( α_n|U = α_m|U \) and d supported on U with φ(d) = 1,

\[ U^\alpha_{nm*} = \pi_{\alpha_n(φ)}((ndn^*)nm*(ndn^*)) = t_n(\pi_\phi(dm*m'd)) = t_n(\pi_\phi(dm*m'd)) = U_0(D_A'/I_{\alpha_n(φ)}). \]

So (R4) is also satisfied, and (n*, ψ) ∼ (m*, ψ). Thus I is well-defined.

The multiplication defined by M is associative because multiplication in A is associative. By construction, we have [n, φ]^{-1}[n, φ] = [n, φ] = [d, φ] for any d ∈ D with φ(d) > 0. Similarly, [n, φ][n, φ]^{-1} = [c, α_n(φ)] for any c ∈ D with α_n(φ)(c) > 0. Since \( \pi_\phi(d) = \phi(d)1_{D_A'/I_\phi} \) for d ∈ D, it is routine to check using the definition of M that

\[ [c, α_m(φ)][m, φ] = [m, φ][d, φ] \]

for any c, d ∈ D such that α_m(ψ)(c) > 0 and ψ(d) > 0. So \( H(A, D, δ) \) is a groupoid.

Finally, the formula \( c_n([n, φ]) = g \) if \( n ∈ N_g(D) \) is well defined by (R2), and it is multiplicative because n ∈ A_g and m ∈ A_h implies nm ∈ A_{g+h}.

We now show how to make \( (H(A, D, δ), c_δ) \) into a graded Hausdorff étale groupoid.

**Theorem 4.9.** Let A be a separable C^*-algebra, δ a coaction of a countable discrete group Γ on A, and D an abelian C^*-subalgebra of A^δ containing an approximate unit for A^δ.
Let $\mathcal{H}(A, D, \delta)$ be the groupoid of Proposition 4.4. For $n \in N_*(D)$ and an open set $X \subseteq \hat{D}$ contained in $\text{supp}^\circ(n^*n)$ let

$$Z(n, X) := \{ [n, \phi] : \phi \in \mathcal{H}(A, D, \delta) \}.$$ 

Then

$$(4.4) \quad \{ Z(n, X) : n \in N_*(D), \ X \subseteq \hat{D} \text{ is open and } X \subseteq \text{supp}^\circ(n^*n) \}$$

constitutes a countable basis for a locally compact locally Hausdorff étale topology on $\mathcal{H}(A, D, \delta)$, and $c_5$ is continuous with respect to this topology.

To prove the theorem, we need some preliminary lemmas.

**Lemma 4.10.** Let $A$ be a separable $C^*$-algebra, $\delta$ a coaction of a countable discrete group $\Gamma$ on $A$, and $D$ an abelian $C^*$-subalgebra of $A^\delta$ containing an approximate unit for $A^\delta$. Let $\mathcal{H}(A, D, \delta)$ be the groupoid of Proposition 4.4. If $[n, \phi], [m, \phi] \in \mathcal{H}(A, D, \delta)$ and $[n, \phi] = [m, \phi]$, then there is an open $U \subseteq \hat{D}$ with $\phi \in U \subseteq \text{supp}^\circ(m^*m) \cap \text{supp}^\circ(n^*n)$ such that $[n, \psi] = [m, \psi]$ for all $\psi \in U$.

**Proof.** Since $[n, \phi] = [m, \phi]$, there is an open $U'$ with $\phi \in U' \subseteq \text{supp}^\circ(n^*n) \cap \text{supp}^\circ(m^*m)$ such that $\alpha_{n, \psi} = \alpha_{m, \psi}$. By scaling $n$ and $m$ by appropriate elements of $D$, we may assume that $\psi(m^*m) = \psi(n^*n) = 1$ for all $\psi \in U'$. Since $[n, \phi] = [m, \phi]$ we have $n^*m \in A^\delta$, so (R1), (R2) and (R3) are satisfied for $(n, \psi)$ and $(m, \psi)$ for each $\psi \in U'$.

Since $U^\psi_{n^*m} = \pi_\psi(\text{dn}^*\text{md}) \subseteq U_0(D'_{A_*}/J_\phi)$, there is a path of unitaries in $D'_{A_*}/J_\phi$ from $\pi_\psi(\text{dn}^*\text{md})$ to $1_{D'_{A_*}/J_\phi}$. Choose $0 = t(0) < t(1) < \cdots < t(k) = 1$ such that $\| w_{t(j)} - w_{t(j+1)} \| < \frac{1}{4}$ for all $j$. Let $a_0 = \text{dn}^*\text{md}$ and $a_k = d$, and choose $a_j \in \pi_\psi^{-1}(w_{t(j)}), 0 < j < k$. Since $\psi \mapsto \| \pi_\psi(a) \|$ is upper semicontinuous, there is an open $U \ni \phi$ such that

$$\| \pi_\psi(a_ja_j^* - a_k) \| < \frac{1}{4} \quad \text{for all } j \in U \text{ and } j < k.$$  

We claim that $(n, \psi) \sim (m, \psi)$ for $\psi \in U$.

Fix $\psi \in U$. We have already seen that (R1)–(R3) are satisfied for $(n, \psi)$ and $(m, \psi)$. The first two properties of the $a_j$ ensure that $\pi_\psi(a_ja_j^*)$, $\pi_\psi(\pi_\psi(a_ja_j^* - a_k))$, and $\pi_\psi(a_ja_{j+1})$ are invertible, and that $\| \pi_\psi(a_ja_j^* - a_k) \| < 4/3$. So $\| \pi_\psi(a_j - a_{j+1}) \| < 3/8 < \| \pi_\psi(a_ja_j^* - a_k) \|^{-1}$. So [46 Proposition 2.1.11] shows that each $\pi_\psi(a_j) \sim_h \pi_\psi(a_{j+1})$ in $(D'_{A_*}/J_\phi)^{-1}$. Hence $U_{n^*m}^\psi \sim_h \pi_\psi(a_k) = 1_{D'_{A_*}/J_\phi}$ in $(D'_{A_*}/J_\phi)^{-1}$, and then $U_n^\psi$ [46 Proposition 2.1.8] gives $U_{n^*m}^\psi \sim_h 1_{D'_{A_*}/J_\phi}$ in the unitary group of $D'_{A_*}/J_\phi$. So (R4) is satisfied, and hence $(n, \psi) \sim (m, \psi)$. □

**Lemma 4.11.** Let $A$ be a separable $C^*$-algebra, $\delta$ a coaction of a countable discrete group $\Gamma$ on $A$, and $D$ an abelian $C^*$-subalgebra of $A^\delta$ containing an approximate unit for $A^\delta$. Suppose that $n, m \in N(D)$ satisfy $\| n - m \| < \| n \|/5$. Then $\alpha_n(\phi) = \alpha_m(\phi)$ for all $\phi \in \hat{D}$ such that $\phi(n^*n) > \| n \|^2/2$.

**Proof.** Since $\alpha_n = \alpha_n[\| n \|]$ and $\alpha_m = \alpha_m[\| n \|]$, it suffices to prove the result for $\| n \| = 1$. So, by assumption, $\| n - m \| < 1/5$ giving $\| m \| \leq 6/5$. For any $d \in D$ with $\| d \| \leq 1$, we have

$$(4.5) \quad \| n^*dn - m^*dm \| \leq \| n^*dn - n^*dm \| + \| n^*dm - m^*dm \| < \frac{1}{5} + \frac{6}{25} < \frac{1}{2}.$$  

Fix $\phi \in \hat{D}$ with $\phi(n^*n) > \frac{1}{2}$, and $d \in D$ with $0 \leq d \leq 1$ and $\phi(d) = 1$. Then $\alpha_n(\phi)(\text{dn}^*) = \phi(d)\alpha_n(\phi)(\text{nn}^*) = \phi(n^*n) > \frac{1}{2}$, so (4.5) gives $\alpha_n(\phi)(\text{mdn}^*) > \alpha_n(\phi)(\text{dn}^*) - \frac{1}{2} > 0$. Hence
0 < \alpha_n(\phi)(mdm^*) = \alpha_m^{-1}(\alpha_n(\phi))(d)\alpha_n(\phi)(mm^*), giving \alpha_m^{-1}(\alpha_n(\phi))(d) \neq 0. So if 0 \leq d \leq 1 and \phi(d) = 1, then \alpha_m^{-1}(\alpha_n(\phi))(d) > 0. So Urysohn's lemma forces \alpha_m^{-1}(\alpha_n(\phi)) = \phi. □

Proof of Theorem 4.4. To see that the Z(n, X) are a basis for a topology, suppose that [n, \phi] \in Z(n, X) \cap Z(n', Y). Then \phi \in X \cap Y, and [n, \phi] = [n', \phi]. By Lemma 4.10 there is an open W with \phi \in W \subseteq X \cap Y such that [n, \psi] = [n', \psi] for all \psi \in W. Hence Z(n, W) \subseteq Z(n, X) \cap Z(n', Y).

To see that this topology is second countable, we first claim that each N_\delta(D) has a countable dense subset. For this, fix a dense sequence (a_i)_{i=1}^\infty in A_g. For i, j \in \mathbb{N} fix n_{i,j} \in B(a_i, 1/j) \cap N_\delta(D) if this set is nonempty, and otherwise let n_{i,j} = 0 \in N_\delta(D). Fix n \in N_\delta(D). Choose i_0 with \|a_{i_0} - n\| < \frac{1}{3k}. Then n_{i_0, 2k} \in B(a_{i_0}, 1/2k), forcing \|n_{i_0, 2k} - n\| < \frac{1}{3}. So n \in \{n_{i,j} : i, j \in \mathbb{N}\}, proving the claim. Now since \Gamma is countable, N_\delta(D) has a countable dense sequence (n_i)_{i=1}^\infty. Choose a countable dense basis \{U_j\} for \hat{D}. We claim that for n \in N_\delta(D), an open subset X \subseteq supp(\phi(n^*n)), and \phi, there are i, j \in \mathbb{N} such that [n, \phi] \in Z(n_{i,j}) \subseteq Z(n, X). Since [n, \phi] = [nd, \phi] \in Z(nd, X \cap supp(\phi)(d)) \subseteq Z(n, X) for d \in D with 0 \leq d \leq 1 and \phi(d) = 1, we may assume that \phi(n^*n) = \|n\|^2. Choose j so that \phi \in U_j \subseteq X and \phi^*(\phi(n^*n)) \geq \|n\|^2/2 for all \psi \in U_j. Fix a subsequence (n_{i_k})_{k=1}^\infty of (n_i)_{i=1}^\infty converging to n with n_{i_k} \in A_\delta and \|n_{i_k} - n\| < \|n\|/5. So, Lemma 4.11 gives \alpha_n|_{U_i} = \alpha_{n_{i_k}}|_{U_j}. Since sup_{\psi \in U_j} \|\pi_\psi(d_{\psi}n_{i_k}d_{\psi} - \pi_\psi(n_{i_k}^*n_{i_k}))\| \rightarrow 0 where d_{\psi} \in D with \|d_{\psi}\| = 1, supp(\phi(n_{i_k}^*n_{i_k})) \subseteq U_j, and \psi(d_{\psi}) = 1, we have sup_{\psi \in U_j} \|U_{\psi}\} - U_{n_{i_k}}\| \rightarrow 0. Since U_{n_{i_k}}^* = 1_{D_{A_{\delta}/J_\delta}}, and since unitaries in B(1_{D_{A_{\delta}/J_\delta}}, 2) are homotopic to 1_{D_{A_{\delta}/J_\delta}}, for large k, we have U_{n_{i_k}}^* \sim_{h} 1_{D_{A_{\delta}/J_\delta}} for \psi \in U_j. Thus [n, \phi] \in Z(n_{i_k}, U_j) \subseteq Z(n, X). So the Z(n_{i_k}, U_j) form a countable basis for the topology.

To see that \mathcal{H}(A, D, \delta) is locally Hausdorff and étale, fix a basic open set Z(n, X). The source map [n, \psi] \mapsto \psi is a homeomorphism h : Z(n, X) \rightarrow X: it is bijective by definition of \sim, continuous as h^{-1}(Y) = Z(n, Y) for Y \subseteq X, and open because each open subset of Z(n, X) is a union of sets of the form Z(n, Y) with Y \subseteq X open, and each h(Z(n, Y)) = Y is open. Similarly, the range map is a homeomorphism Z(n, X) \rightarrow \alpha_n(X). Thus, since \hat{D} is Hausdorff, \mathcal{H}(A, D, \delta) is locally Hausdorff and étale.

The map I is a homeomorphism because I(Z(n, X)) = Z(n^*, \alpha_n(X)) and \alpha_n is a homeomorphism on supp(\phi(n^*n)). To see that M is continuous, suppose that \{[n_i, \phi_i] \rightarrow [n, \phi], that [m_i, \psi_i] \rightarrow [m, \psi], and that each \phi_i = \alpha_m(\psi_i). Then the preceding paragraph gives s([n, \phi]) = r([m, \psi]), and then M([n, \phi], [m, \psi]) = [nm, \psi]. Fix an open V with \psi \in V \subseteq supp(\phi(n^*n) \cap \alpha_m^{-1}(supp(\phi(n^*n))). Then Z(m, V) \ni [m, \psi] and Z(n, \alpha_m(V)) \ni [n, \phi] are open, giving [n_i, \phi_i] \in Z(n, \alpha_m(V)) and [m_i, \psi_i] \in Z(m, V) for large i. Since Z(m, V) and Z(n, \alpha_m(V)) are bisections, for large i we have [n_i, \phi_i] = [n, \phi_i] and [m_i, \psi_i] = [m, \psi_i], so M([n_i, \phi_i], [m_i, \psi_i]) = [nm, \psi_i]. In particular, \psi_i \rightarrow \psi, and as s is a homeomorphism on Z(m, V), we obtain [nm, \psi_i] \rightarrow [nm, \psi]. So M is continuous.

For local compactness, fix \gamma \in \mathcal{H}(A, D, \delta) and an open W \ni \gamma. Choose n \in N_\delta(D) and X open with \gamma \in Z(n, X). Since \hat{D} is locally compact, there is a compact neighbourhood K of s(\gamma) in X. Now \{[n, \phi] : \phi \in K\} is the inverse image of K under the homeomorphism s|_{Z(n, X)}, and hence a compact neighbourhood of \gamma. Finally, \psi_{\delta} is continuous because it is constant on basic open sets.

Our results so far do not require that \text{span} N_\delta(D) is all of A; but our next lemma indicates that this is the situation of greatest interest.
Lemma 4.12. Let $\delta$ be a coaction of a discrete group $\Gamma$ on a separable $C^*$-algebra $A$, and $D$ an abelian $C^*$-subalgebra of $A^\delta$ containing an approximate unit for $A^\delta$. Let $A_N = \text{span} N(D) \subseteq A$. Then $A_N$ is a $C^*$-algebra, $\delta_N := \delta|_{A_N}$ is a coaction, $D$ is an abelian $C^*$-subalgebra of $A_N^\delta$ and contains an approximate unit for $A_N^\delta$, and $\mathcal{H}(A, D, \delta) \cong \mathcal{H}(A_N, D, \delta_N)$.

Proof. As $N(D)$ is closed under multiplication and adjoints, $A_N$ is a $C^*$-algebra. Since $D \subseteq N(D)$, we have $D \subseteq A_N$, and $D$ clearly contains an approximate unit for $A_N^\delta$. Since $A_N = \text{span} \bigcup_g A_g$, the restriction of $\delta_N := \delta|_{A_N}$ takes values in $A_N \otimes C^*_r(G)$. It is nondegenerate because $D \subseteq A_N^\delta$ contains an approximate unit. Finally, it is easy to see that $[n, \phi] \mapsto [n, \phi]$ is an isomorphism from $\mathcal{H}(A_N, D, \delta_N)$ to $\mathcal{H}(A, D, \delta)$.

5. THE INTERIOR OF THE ISOORTY IN AN ÉTALE GROUPOID

This section contains some technical results that we need in order to prove our reconstruction results in the next section. Our proof of the first, Lemma 5.1, is largely due to Becky Armstrong; the key elements, in the more general situation of twisted groupoid $C^*$-algebras, will appear in her PhD thesis.

Lemma 5.1 (Armstrong). Let $G$ be a locally compact Hausdorff étale groupoid such that $\text{Iso}(G)^0$ is abelian. Then $C^*(\text{Iso}(G)^0) = C^*_r(\text{Iso}(G)^0)$. The inclusion $C_0(G^{(0)}) \hookrightarrow C^*(\text{Iso}(G)^0)$ makes $C^*(\text{Iso}(G)^0)$ into a $C_0(G^{(0)})$-algebra. The fibre homomorphisms $\pi_u : C^*(\text{Iso}(G)^0) \to C^*(\text{Iso}(G)^0)_u$ have the property that $u \mapsto \|\pi_u(a)\|$ is continuous for $a \in C^*(\text{Iso}(G)^0)$. For each $u \in G^{(0)}$ there is an isomorphism $C^*(\text{Iso}(G)^0)_u \cong C^*(\text{Iso}(G)^0)$ that takes $\pi_u(d)$ to $d(u)1_{C^*(\text{Iso}(G)^0)_u}$ for $d \in C_0(G^{(0)})$.

Proof. Theorem 3.5 of [11] shows that $C^*(\text{Iso}(G)^0) = C^*_r(\text{Iso}(G)^0)$. For $f \in C_c(\text{Iso}(G)^0)$, $d \in C_0(G^{(0)})$ and $\gamma \in \text{Iso}(G)^0$, we have that $(df)(\gamma) = d(r(\gamma))f(\gamma) = f(\gamma)d(s(\gamma)) = (fd)(\gamma)$. So $C_0(G^{(0)})$ is central in $C^*(\text{Iso}(G)^0)$ by continuity, and any approximate unit for $C_0(G^{(0)})$ is an approximate unit for $C^*(\text{Iso}(G)^0)$. Thus $C^*(\text{Iso}(G)^0)$ is a $C_0(G^{(0)})$-algebra.

Fix $u \in G^{(0)}$. For each $\gamma \in \text{Iso}(G)^0_u$, choose $a_\gamma \in C_c(\text{Iso}(G)^0)$ supported on a bisection with $a_\gamma(\gamma) = 1$. Then $\gamma \mapsto \pi_u(a_\gamma)$ is a unitary representation of $\text{Iso}(G)^0_u$ in $C^*(\text{Iso}(G)^0)_u$, and thus determines a homomorphism $\tilde{\pi}_u : C^*(\text{Iso}(G)^0)_u \to C^*(\text{Iso}(G)^0)_u$.

The regular representation $\rho_u : C^*_r(\text{Iso}(G)^0) \to \mathcal{B}(\ell^2(\text{Iso}(G)^0_u))$ satisfies $\rho_u(f) = 0$ for $f \in C_0(G^{(0)}) \setminus \{u\}$, and hence descends to a representation

$$\tilde{\rho}_u : C^*(\text{Iso}(G)^0)_u \to \mathcal{B}(\ell^2(\text{Iso}(G)^0_u)).$$

The representation $\tilde{\rho}_u \circ \tilde{\pi}_u$ is precisely the regular representation $C^*(\text{Iso}(G)^0)_u$ and hence faithful since $\text{Iso}(G)^0_u$ is abelian. Identifying $C^*_r(\text{Iso}(G)^0_u)$ with $C^*(\text{Iso}(G)^0)_u$,

$$\tilde{\pi}_u \circ \tilde{\rho}_u(\pi_u(a_\gamma)) = \pi_u(a_\gamma)$$

for $\gamma \in \text{Iso}(G)^0_u$, and so $\rho_u \circ \tilde{\rho}_u = \text{id}_{C^*(\text{Iso}(G)^0)_u}$. Since $\tilde{\rho}_u(\pi_u(d)) = d(u)1_{C^*(\text{Iso}(G)^0)_u}$, this $\tilde{\rho}_u : C^*(\text{Iso}(G)^0)_u \to C^*(\text{Iso}(G)^0)_u$ is the desired isomorphism.

Fix $a \in C_c(\text{Iso}(G)^0)$ supported on a bisection $U$. Write $\sigma : s(U) \to U$ for the inverse of the source map. Then $\|\pi_u(a)\| = |a(\sigma(u))|$, so $u \mapsto \|\pi_u(a)\|$ is continuous. An $\varepsilon$-argument now shows that $u \mapsto \|\pi_u(a)\|$ is continuous for all $a \in C^*(\text{Iso}(G)^0)$. \qed
The continuity of the map \( u \mapsto \|\pi_u(a)\| \) above also follows from \[32\] Corollary 5.6. For our next lemma, recall from Section 11 that there is a norm-decreasing injection \( a \mapsto f_a \) from \( C^*_r(G) \) to \( C_0(G) \) given by \( f_a(\gamma) = (\rho_{\gamma}(a)e_{\gamma}(\gamma) | e_{\gamma}) \). We show that \( \{a \in C^*_r(G) : f_a \) is supported on \( \text{Iso}(G)^o \} \) is a \( C_0(G^{(0)}) \)-algebra.

**Lemma 5.2.** Let \( G \) be a locally compact Hausdorff étale groupoid such that \( \text{Iso}(G)^o \) is abelian. Let \( D := C_0(G^{(0)}) \subseteq C^*_r(G) \). Let \( A := \{a \in C^*_r(G) : \text{supp}^\circ(f_a) \subseteq \text{Iso}(G)^o\} \). Then \( A \) is a \( C_0(G^{(0)}) \)-algebra with respect to \( D \mapsto A \). For each \( u \in G^{(0)} \), let \( J_u := \{f \in D : f(u) = 0\}A \), the ideal of \( A \) generated by \( \{f \in D : f(u) = 0\} \), and let \( A_u := A/J_u \). Then there is an isomorphism \( \tilde{\Phi}_u : A_u \rightarrow C^*(\text{Iso}(G)^o_u) \) such that

\[
\tilde{\Phi}_u(f) = \sum_{\gamma \in \text{Iso}(G)^o_u} f(\gamma)\lambda_\gamma \quad \text{for all } f \in C_c(G) \cap A,
\]

where \( \lambda_\gamma \) is the image of \( \gamma \) in the left regular representation of \( \text{Iso}(G)^o_u \) in \( C^*(\text{Iso}(G)^o_u) \).

**Proof.** For \( b \in C^*_r(G) \) and \( g \in D \), we have \( f_{gb}(\gamma) = g(r(\gamma))f_b(\gamma) \) and \( f_{gb}(\gamma) = f_b(\gamma)g(s(\gamma)) \), and so \( D \mapsto A \) is a central inclusion. Since \( D \) contains an approximate unit for \( C^*_r(G) \), we deduce that \( A \) is a \( C_0(G^{(0)}) \)-algebra. So we fix \( u \in G^{(0)} \) and show that there is an isomorphism \( \tilde{\Phi}_u : A_u \rightarrow C^*(\text{Iso}(G)^o_u) \).

Let \( P \in \mathcal{B}(\ell^2(\text{Iso}(G)^o_u)) \) be the orthogonal projection onto \( \ell^2(\text{Iso}(G)^o_u) \). Define \( \Phi_u : C^*_r(G) \rightarrow \mathcal{B}(\ell^2(\text{Iso}(G)^o_u)) \) by \( \Phi_u(a) := P\rho_u(a)P \). Then for \( f \in C_c(G) \) we have

\[
\Phi_u(f) = \sum_{\gamma \in \text{Iso}(G)^o_u} f(\gamma)\lambda_\gamma \in C^*(\text{Iso}(G)^o_u),
\]

and hence \( \tilde{\Phi}_u(C^*_r(G)) \subseteq C^*(\text{Iso}(G)^o_u) \) by continuity.

If \( a \in \{f \in D : f(u) = 0\}A \), then \( \Phi_u(a) = 0 \). Hence \( \Phi_u \) induces a map \( \tilde{\Phi}_u : A_u \rightarrow C^*(\text{Iso}(G)^o_u) \). Proposition 4.2 of \[12\] shows that \( f_{ab}(\gamma) = \sum_{\alpha\beta = \gamma} f_a(\alpha)f_b(\beta) = (f_a * f_b)(\gamma) \) and \( f_{a*}(\gamma) = f_a(\gamma^{-1}) = f_{a^*}(\gamma) \), so \( \Phi_u \) is a *-homomorphism. This \( \tilde{\Phi}_u \) is surjective because its image contains the canonical generators of \( C^*(\text{Iso}(G)^o_u) \).

To see that \( \tilde{\Phi}_u \) is injective, we first claim that \( J_u = \{a \in A : f_a|_{\text{Iso}(G)^o_u} = 0\} \). For \( a \in \text{Iso}(G)^o_u \), observe that \( \{a \in A : f_a|_{\text{Iso}(G)^o_u} = 0\} \) is an ideal containing \( \{f \in D : f(u) = 0\} \). For the reverse containment, suppose that \( f_a|_{\text{Iso}(G)^o_u} = 0 \) and \( \|a\| \leq 1 \). Since \( f_a|_{\text{Iso}(G)^o_u} = 0 \), we have \( \|\pi_u(a)\| = 0 \). The map \( G^{(0)} \ni v \mapsto \|\pi_u(a)\| \) is upper semicontinuous by \[51\] Proposition C.10(a)]. So for \( n \in \mathbb{N} \) the set \( U_n := \{v \in G^{(0)} : \|\pi_u(a)\| < 1/n\} \) is an open neighbourhood of \( u \), and \( G^{(0)} \setminus U_n \) is compact. So there exists \( g_n \in C_0(G^{(0)})_+ \) with \( g_n \leq 1 \), \( g_n(u) = 0 \), and \( g_n \equiv 1 \) on \( G^{(0)} \setminus U_n \). Now \( \|a - g_na\| < \frac{1}{n} \). Since each \( g_n \) is in \( J_u \), we deduce that \( a = \lim_n g_na \in J_u \). This proves the claim. Now suppose \( b \in A \setminus J_u \). Then \( f_b|_{\text{Iso}(G)^o_u} \neq 0 \), so \( f_{b*}(u) \neq 0 \). Hence \( (\Phi_u(b^*b)\delta_u | \delta_u) = f_{b*}(u) \neq 0 \), forcing \( \tilde{\Phi}_u(b^*b) \neq 0 \). Thus \( \Phi_u(b) \neq 0 \) because \( \tilde{\Phi}_u \) is a homomorphism. Hence \( \tilde{\Phi}_u \) is injective.

For the next result, recall from \[39\] Proposition 1.9 that if \( G \) is an étale groupoid then the canonical inclusion \( \iota : C^*_r(\text{Iso}(G)^o) \hookrightarrow C^*_r(G) \) extends to an injective homomorphism \( \iota : C^*_r(\text{Iso}(G)^o) \hookrightarrow C^*_r(G) \).

**Corollary 5.3.** Let \( G \) be a locally compact Hausdorff étale groupoid such that \( \text{Iso}(G)^o \) is abelian. Let \( A := \{a \in C^*_r(G) : \text{supp}^\circ(f_a) \subseteq \text{Iso}(G)^o\} \). Then \( A = C_0(G^{(0)})C^*_r(G) = \iota(C^*(\text{Iso}(G)^o)) \).
Proof. Fix $a \in A$. For $d \in C_0(G(0))$, we have $f_a = d$ and [12, Proposition 4.2] gives $f_{ad}(\gamma) = (f_a * f_d)(\gamma) = f_a(\gamma)d(s(\gamma))$ and similarly $f_{da}(\gamma) = d(r(\gamma))f_a(\gamma)$. Since $a \in A$, we have $f_a(\gamma) \neq 0$ only if $r(\gamma) = s(\gamma)$, so $f_{ad} = f_{da}$. Since $b \mapsto f_b$ is injective, it follows that $A \subseteq C_0(G(0))'/C_r(G)$. Now fix $a \notin A$. Since $f_a$ is continuous, there is then $\gamma \notin \text{Iso}(G)$ such that $f_a(\gamma) \neq 0$. Fix $d \in C_0(G(0))$ with $d(r(\gamma)) = 1$ and $d(s(\gamma)) = 0$. Then $f_{da}(\gamma) = d(r(\gamma))f_a(\gamma) = f_a(\gamma) \neq 0$, and $f_{ad}(\gamma) = f_a(\gamma)d(s(\gamma)) = 0$. So $da \neq ad$. Hence $C_0(G(0))'/C_r(G) \subseteq A$, and thus $C_0(G(0))'/C_r(G) = A$.

If $f \in C_c(\text{Iso}(G)^\circ)$ and $d \in C_0(G(0))$, then $(df)(\gamma) = d(r(\gamma))f(\gamma)$ and $(fd)(\gamma) = f(\gamma)d(s(\gamma))$ for every $\gamma \in G$. Since $\text{supp}(f) \subseteq \text{Iso}(G)^\circ$, we obtain $df = fd$. Thus $\iota(C^*(\text{Iso}(G)^\circ)) \subseteq A$.

Both $A$ and $\iota(C^*(\text{Iso}(G)^\circ))$ are $C_0(G(0))$-algebras with respect to the inclusion of $D = C_0(G(0))$ in both. Write $J_u$ for the ideal of $A$ generated by $C_0(G(0) \setminus \{u\}) \subseteq D$ and $K_u$ for the ideal of $\iota(C^*(\text{Iso}(G)^\circ))$ generated by $C_0(G(0) \setminus \{u\}) \subseteq D$, so $A_u = A/J_u$ and $\iota(C^*(\text{Iso}(G)^\circ))_u = \iota(C^*(\text{Iso}(G)^\circ))/K_u$.

Lemma 5.2 gives isomorphisms $\hat{\Phi}_u^{-1} : \iota(C^*(\text{Iso}(G)^\circ))_u \to A_u$ such that $\hat{\Phi}_u^{-1}(\iota(f) + K_u) = f + K_u$ for $f \in C_c(\text{Iso}(G)^\circ)$. Thus $\{m_f : u \mapsto f + J_u \mid f \in C_c(\text{Iso}(G)^\circ)\}$ is a fibrewise dense vector space of continuous sections of $A := \bigcup_u A_u$ that are the images of a fibrewise dense vector space of continuous sections $k_f$ of $\mathcal{I} := \bigcup_u \iota(C^*(\text{Iso}(G)^\circ))_u$. So [24, Proposition 1.6] gives $A \cong \mathcal{I}$ as bundles, and hence there is an isomorphism $\Gamma_0(A) \cong \Gamma_0(\mathcal{I})$ carrying $k_f$ to $m_f$. Since $f \mapsto k_f$ is an isomorphism $C^*(\text{Iso}(G)^\circ) \to \Gamma_0(\mathcal{I})$ and $f \mapsto m_f$ is an isomorphism $A \to \Gamma_0(A)$, we obtain an isomorphism $\iota(C^*(\text{Iso}(G)^\circ)) \cong A$ extending $\iota(C^*(\text{Iso}(G)^\circ))$. So $A = C_c(\text{Iso}(G)^\circ) = \iota(C^*(\text{Iso}(G)^\circ))$. □

We take the opportunity to resolve a loose end from [9].

Corollary 5.4 (cf. [9, Theorem 4.3]). Let $G$ be a locally compact Hausdorff étale groupoid such that $\text{Iso}(G)^\circ$ is abelian. Then $\iota(C^*(\text{Iso}(G)^\circ)) \subseteq C^r(G)$ is maximal abelian.

Proof. Lemma 4.4 of [9] says that $\iota(C^*(\text{Iso}(G)^\circ)) \subseteq C^r(G)$ is maximal abelian if $\{a \in C^r(G) : \text{supp}(f_a) \subseteq \text{Iso}(G)^\circ\} \subseteq \iota(C^*(\text{Iso}(G)^\circ))$, which follows from Corollary 5.3. □

6. Reconstruction of groupoids

Let $\Gamma$ be a discrete group. If $G$ is an étale groupoid, and $c : G \to \Gamma$ is a continuous cocycle, we call $(G, c)$ a $\Gamma$-graded groupoid. To state our main theorem, we first show that $c$ induces a coaction on $C^r(G)$. Recall that, for $g \in \Gamma$, we write $\lambda_g \in \mathcal{B}(L^2(\Gamma))$ for the image of $g$ in the left regular representation of $\Gamma$.

Lemma 6.1. Let $G$ be a locally compact Hausdorff étale groupoid. Suppose that $c : G \to \Gamma$ is a continuous cocycle. Then there is a coaction $\delta_c : C^*(G) \to C^r(G) \otimes C^*(\Gamma)$ such that $\delta_c(f) = f \otimes \lambda_g$ whenever $g \in \Gamma$ and $f \in C_c(G)$ satisfy $\text{supp}(f) \subseteq c^{-1}(g)$.

Proof. Let $\mathcal{H}$ be the Hilbert $C^*(G(0))$-module completion of $C_c(G)$ under $\langle f, g \rangle_{C(G(0))} = \langle f^*g \rangle_{G(0)}$. For $g \in \Gamma$, write $C_c(G)_g := \text{C}^c(e^{-1}(g)) \subseteq C_c(G)$, and let $\mathcal{H}_g = C_c(G)_g \subseteq \mathcal{H}$. The $\mathcal{H}_g$ are mutually orthogonal because $G(0) \subseteq c^{-1}(e)$, so a calculation using inner product shows that there are isometries $V_h : \mathcal{H} \to \mathcal{H} \otimes L^2(\Gamma)$ such that $V_h(\xi) = \xi \otimes e_{gh}$ for $\xi \in \mathcal{H}_g$. As in [27, Appendix A] there is a faithful representation $\pi : C^r(G) \to \mathcal{L}(\mathcal{H})$ extending left multiplication. So $\bigoplus_h \text{Ad} V_h \circ \pi : C^r(G) \to \mathcal{L}(\mathcal{H} \otimes L^2(\Gamma))$ is faithful.
A routine calculation shows that for $f \in C_c(G)_g$ and $\xi \in C_c(G)_{hl}$ and $h, l \in \Gamma$, we have $
bigoplus_h (\text{Ad} V_h \circ \pi)(f)(\xi \otimes e_l) = e_{hl}(\pi(f) \otimes \lambda_g)(\xi \otimes \delta_l)$. So $\delta_c := (\pi^{-1} \otimes \text{id}) \circ (\bigoplus_h \text{Ad} V_h \circ \pi)$ satisfies $\delta_c(f) = f \otimes \lambda_g$. It is routine to check that this is a coaction. 

**Theorem 6.2.** Fix a discrete group $\Gamma$ and $\Gamma$-graded second-countable locally compact Hausdorff étale groupoids $(G_1, c_1), (G_2, c_2)$ with $\text{Iso}(c_1^{-1}(\text{id}_\Gamma))$ torsion-free and abelian.

(1) Suppose that $\kappa : G_2 \to G_1$ is an isomorphism satisfying $c_1 \circ \kappa = c_2$. Then there is an isomorphism $\phi : C^*_{\Gamma}(G_1) \to C^*_{\Gamma}(G_2)$ such that $\phi(f) = f \circ \kappa$ for $f \in C_c(G_1)$. We have $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$ and $\delta_{c_2} \circ \phi = (\phi \otimes \text{id}) \circ \delta_{c_1}$.

(2) Suppose that $\phi : C^*_{\Gamma}(G_1) \to C^*_{\Gamma}(G_2)$ is an isomorphism satisfying $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$ and $\delta_{c_2} \circ \phi = (\phi \otimes \text{id}) \circ \delta_{c_1}$. Then there is an isomorphism $\kappa : G_2 \to G_1$ such that $\kappa|_{c_2^{(0)}}$ is the homeomorphism induced by $\phi|_{C_0(G_1^{(0)})}$ and $c_1 \circ \kappa = c_2$.

The first step in proving Theorem 6.2 is showing that $C^*_{\Gamma}(c_1^{-1}(\text{id}_\Gamma)) \cong C^*_{\Gamma}(G)^{\delta_c} \subset C^*_{\Gamma}(G)$.

**Lemma 6.3.** Let $(G, c)$ be a $\Gamma$-graded locally compact Hausdorff étale groupoid. The canonical inclusion $\iota : C_c(c_1^{-1}(\text{id}_\Gamma)) \to C_c(G)$ extends to an isomorphism $C^*_{\Gamma}(c_1^{-1}(\text{id}_\Gamma)) \cong C^*_{\Gamma}(G)^{\delta_c}$.

**Proof.** Proposition 1.9 of [39] shows that $\iota$ extends to an injective homomorphism from $C^*_{\Gamma}(c_1^{-1}(\text{id}_\Gamma))$ to $C^*_{\Gamma}(G)$. Clearly $\iota(C^*_{\Gamma}(c_1^{-1}(\text{id}_\Gamma))) \subset C^*_{\Gamma}(G)^{\delta_c}$. For $g \in \Gamma$, if $f \in C_c(G)_g$, then $\Phi^g(\iota(f)) = \delta_{c_1}(g) \iota(f)$, so $C^*_{\Gamma}(G)^{\delta_c} = \text{span} \bigcup_g C_c(G)_g$. It follows that $\iota(C^*_{\Gamma}(c_1^{-1}(\text{id}_\Gamma))) = \Phi^G(\iota(C^*_{\Gamma}(G))) = C^*_{\Gamma}(G)^{\delta_c}$ by linearity and continuity. 

We now show that graded groupoids determine triples as in Theorem 4.9

**Lemma 6.4.** Let $\Gamma$ be a locally compact group, let $G$ be a second-countable locally compact Hausdorff étale groupoid and let $c : G \to \Gamma$ be a continuous cocycle. Then $C^*_{\Gamma}(G)^{\delta_c}$ is separable $C^*$-algebra and $C_0(G^{(0)})$ is an abelian $C^*$-subalgebra of the generalised fixed-point algebra $C^*_{\Gamma}(G)^{\delta_c}$ and contains an approximate unit for $C^*_{\Gamma}(G)^{\delta_c}$.

**Proof.** Since $G$ is second-countable, $C^*_{\Gamma}(G)^{\delta_c}$ is separable. Clearly $C_0(G^{(0)})$ is abelian, contains an approximate unit for $C^*_{\Gamma}(G)^{\delta_c}$, and is contained in $C^*_{\Gamma}(G)^{\delta_c}$. 

To prove Theorem 6.2, we show that $(\mathcal{H}(C^*_{\Gamma}(G), C_0(G^{(0)}), \delta_c, c_\delta)) \cong (G, c)$ (cf. [13] Proposition 4.13(ii), [10] Proposition 4.8, [11] Corollary 3.11 and [13] Proposition 3.6). For $u \in G^{(0)}$, we write $\hat{u} : C_0(G^{(0)}) \to \mathbb{C}$ for evaluation at $u$.

**Proposition 6.5.** Let $\Gamma$ be a discrete group, and $(G, c)$ a $\Gamma$-graded second-countable locally compact Hausdorff étale groupoid with $\text{Iso}(c_1^{-1}(\text{id}_\Gamma))$ torsion-free and abelian. There is an isomorphism $\theta : (G, c) \to (\mathcal{H}(C^*_{\Gamma}(G), C_0(G^{(0)}), \delta_c, c_\delta))$ such that for $\gamma \in G$ and $n \in C_c(G)$ supported on a bisection $U \subset c^{-1}(c(\gamma))$ with $n(\gamma) > 0$, we have $\theta(\gamma) = [n, s(\gamma)]$.

To prove proposition 6.5 we need two lemmas. We implicitly identify $C_0(G^{(0)})^\gamma$ with $G^{(0)}$ via $\hat{u} \mapsto u$. Henceforth in this section, we identify $C_0(G^{(0)})^\gamma C^*_{\Gamma}(c_1^{-1}(\text{id}_\Gamma))$ with $C^*_{\Gamma}(\text{Iso}(G)^\circ)$ using Corollary 5.3 and Lemma 5.1. For $u \in G^{(0)}$, we identify $(C_0(G^{(0)})^\gamma C^*_{\Gamma}(c_1^{-1}(\text{id}_\Gamma)))_u$ with $C^*_{\Gamma}(\text{Iso}(c_1^{-1}(\text{id}_\Gamma))^\circ)$, and $\pi_u : C_0(G^{(0)})^\gamma C^*_{\Gamma}(c_1^{-1}(\text{id}_\Gamma)) \to C_0(G^{(0)})^\gamma C^*_{\Gamma}(c_1^{-1}(\text{id}_\Gamma))/J_u$ with the regular representation $\rho_u : C^*_{\Gamma}(\text{Iso}(c_1^{-1}(\text{id}_\Gamma))^\circ) \to \mathcal{B}(l^2(\text{Iso}(c_1^{-1}(\text{id}_\Gamma))^\circ))$. 


Lemma 6.6 (cf. [13] Proposition 4.8, [10] Lemma 4.10(ii)). Let $G$ be a second-countable locally compact Hausdorff étale groupoid. Let $D := C_0(G^{(0)}) \subseteq C^*(G)$. If $n \in N_{C^*_r(G)}(D)$ and $f_n$ as in (1.1) satisfies $f_n(\gamma) \neq 0$, then $r(\gamma) = \alpha_n(s(\gamma))$. 

Proof. Suppose for contradiction that $r(\gamma) \neq \alpha_n(s(\gamma))$. Since $nn^*(r(\gamma)) \geq |f_n(\gamma)|^2 > 0$, there exist orthogonal $d, d' \in C_c(supp^o(nn^*))$ such that $d(r(\gamma)) = 1 = d'(\alpha_n(s(\gamma)))$. So

$$0 = (dd^*nn^*)(r(\gamma)) = (dn(d' \circ \alpha_n)n^*)(r(\gamma))$$

$$\geq d(r(\gamma))|n\sqrt{d' \circ \alpha_n(\gamma)}|^2 = d(r(\gamma))|f_n(\gamma)\sqrt{d'(\alpha_n(s(\gamma)))}|^2 = |f_n(\gamma)|^2 > 0. \quad \square$$

Lemma 6.7. Let $G$ be a second-countable locally compact Hausdorff étale groupoid. Let $D := C_0(G^{(0)}) \subseteq C^*(G)$. Suppose that $n \in C_0(G)$ is supported on a bisection. Then $n \in N_{C^*_r(G)}(D)$, and $\alpha_n(s(\gamma)) = r(\gamma)$ for $\gamma \in supp^o(n)$. If $c : G \to \Gamma$ is a grading, then $C^*_r(G) = \overline{\text{span}}N_*(D)$. 

Proof. That $n \in N_{C^*_r(G)}(D)$ and $\alpha_n(s(\gamma)) = r(\gamma)$ for $\gamma \in supp^o(n)$ follow from [13] Proposition 4.8. The Stone–Weierstrass theorem gives $C_c(c^{-1}(g)) = \overline{\text{span}}\{f \in C_c(c^{-1}(g)) : supp(f) \text{ is contained in a bisection}\}$ for $g \in \Gamma$. Since $C_c(G) = \overline{\text{span}}\bigcup_{g \in \Gamma} C_c(c^{-1}(g))$, it follows that $C^*_r(G) = \overline{\text{span}}N_*(C_0(G^{(0)}))$. \quad \square$

Proof of Proposition 6.3. Since $c$ is continuous, $\Gamma$ is discrete, and $G$ is étale, there exists a bisection $U \subseteq c^{-1}(c(\gamma))$ containing $\Gamma$, and then an element $n \in C_c(U) \subseteq C_0(G)$ with $n(\gamma) = 1$. Fix $m, n \in C_c(G)$ supported on bisections contained in $c^{-1}(c(\gamma))$ with $n(\gamma) = m(\gamma) = 1$. Choose an open $\gamma \in U \subseteq supp^o(m) \cap supp^o(n)$. Lemma 6.7 shows that $\alpha_n = \alpha_m$ on $s(U)$. Let $u := s(\gamma)$. For $d \in C_0(s(U))$ with $d(u) = 1$, we have $supp^o(nd)$, $supp^o(dm) \subseteq U$, so $(dn \cdot md)|_{\text{iso}(c^{-1}(\gamma))}^\circ$ is just the point-mass $\delta_u$. Hence $U_{n \cdot m} = \pi_u(dn \cdot md) = 1_{C^*_r(\text{iso}(G))}$. Since $\alpha_{nd} = \alpha_n = \alpha_m = \alpha_{md}$ on $supp^o(d) \subseteq U$, we have $[n, u] = [m, u]$. So $\theta$ is well-defined.

To see that $\theta$ is injective, fix $\gamma \neq \eta \in G$. Choose $n, m$ supported on open bisections containing $\gamma$ and $\eta$ respectively, so $\theta(\gamma) = [n, \overline{s(\gamma)}]$ and $\theta(\eta) = [m, \overline{s(\eta)}]$. If $s(\gamma) \neq s(\eta)$, then clearly $\theta(\gamma) \neq \theta(\eta)$, so suppose that $s(\gamma) = s(\eta) =: u$. If $r(\gamma) \neq r(\eta)$, then $\alpha_n(\hat{u}) = r(\gamma) = r(\eta) = \alpha_m(\hat{u})$ by Lemma 6.7 so again $\theta(\gamma) \neq \theta(\eta)$. So suppose that $r(\gamma) = r(\eta) := v$. Fix $d \in C_0(G^{(0)})$ with $d(u) = 1$, and put $w = dn \cdot md$. Then $w|_{\text{iso}(c^{-1}(\eta))}^\circ = \overline{\delta}_u \cdot \overline{\delta}_{\eta} = \overline{\delta}_{\eta} \cdot \overline{\delta}_{\eta} = \overline{\delta}_{\gamma}$. Hence $U_{n \cdot m} = \pi_u(w) = U_{n \cdot \eta} \subseteq C^*_r(\text{iso}(c^{-1}(\eta))_u^\circ)$. Since $\text{iso}(c^{-1}(\eta))_u^\circ$ is a discrete torsion-free abelian group, [13] Theorem 8.57 gives $\text{iso}(c^{-1}(\eta))_u^\circ \cong \mathcal{U}(C^*_r(\text{iso}(c^{-1}(\eta))_u^\circ) \cap U_\gamma(C^*_r(\text{iso}(c^{-1}(\eta))_u^\circ)))$ via the map $\gamma \mapsto U_\gamma U_\gamma(C^*_r(\text{iso}(c^{-1}(\eta))_u^\circ))$; so $u_\gamma \cdot u_\eta \not\in U_\gamma(C^*_r(\text{iso}(c^{-1}(\eta))_u^\circ))$, and $\theta(\gamma) \neq \theta(\eta)$.

To see that $\theta$ is surjective, fix $[n, \hat{u}] \in \mathcal{H}(C^*_r(G), C_0(G^{(0)}), \delta_c)$, and let $\hat{v} := \alpha_n(\hat{u})$. Regard $n$ as an element of $C_0(G)$ using (1.1). Since $0 < n^*n(u) = mn^*(\alpha_m(u)) = mn^*(v) = \sum_{\gamma \in G^*} |n(\gamma)|^2$, we have $n(\gamma) \neq 0$ for some $\gamma \in G^*$. So Lemma 6.6 gives $\alpha_n(s(\gamma)) = r(\gamma) = \hat{v} = \alpha_n(\hat{u})$. Since $\alpha_n$ is bijective, $s(\gamma) = u$. Fix an open bisection $B \ni \gamma$ with $f_n$ nonzero on $B$. Fix $m \in C_0(B)$ with $m$ identically 1 on a neighbourhood of $\gamma$. Lemma 6.6 gives $\alpha_n = \alpha_m$ on $s(supp^o(m))$, so Lemma 2.3 yields $\alpha_{n \cdot m} = id$ on a neighbourhood of $\hat{u}$. Thus $n^*m$ is supported on $\text{iso}(c^{-1}(\eta))_u$ by Lemma 6.6. By [13] Theorem 8.57, $\text{iso}(c^{-1}(\eta))_u^\circ \cong \mathcal{U}(C^*_r(\text{iso}(c^{-1}(\eta))_u^\circ) \cap U_\gamma(C^*_r(\text{iso}(c^{-1}(\eta))_u^\circ)))$, so $U_{n \cdot m} \sim U_\gamma$ for some $\eta \in \text{iso}(c^{-1}(\eta))_u^\circ$. A bisection neighbourhood $W$ of $\eta^{-1}$ and $h \in C_c(W)$ with $h(\eta^{-1}) = 1$. Then $\pi_u(w_{n \cdot m}h) = \pi_u(w_{n \cdot m})\pi_u(h) = \pi_u(w_{n \cdot m})\delta_{\eta^{-1}} \cdot h \cdot 1_{C^*_r(\text{iso}(c^{-1}(\eta))_u^\circ)}$. Since $h = id = \alpha_{n \cdot m}$, we have $\alpha_n = \alpha_{mh}$ on a neighbourhood of $u$, so $\overline{[n, \hat{u}] = [mh, \overline{\hat{u}}]} = \theta(\gamma \eta^{-1})$. \quad \square
To see that \( \theta \) is open, recall that \( G \) is a normal space, so has a basis of open bisections \( U \) with closure contained in precompact open bisections \( V \). For such \( U, V \), fix \( n \in C_n(V) \) with \( n|_{V} = 1 \). Then \( \theta(\gamma) = [n, s(\gamma)] \) for every \( \gamma \in U \), so \( \theta(U) = Z(n, U) \) is open.

Finally, to see that \( \theta \) is continuous, fix \( n \in N_\theta(D) \) and an open set \( U \subseteq \text{supp}^\circ(n^*n) \). Fix \( u \in U \). Since \( \theta \) is surjective, \( [n, u] = \theta(\gamma) = [f, v] \) for some \( f \in C_n(G) \) supported on a bisection \( B \subseteq c^{-1}(y) \) containing \( \gamma \). So Lemma \ref{lem1} yields an open neighbourhood \( V \) of \( u \) such that \( [n, v] = [f, v] \) for all \( v \in V \). Then \( \gamma \in BV \subseteq \theta^{-1}(Z(n, U)) \). \( \square \)

**Proof of Theorem \ref{thm6.2}** Statement (1) is clear. For (2) let \( \gamma \in \text{Iso}(G \rtimes \Gamma) \) be included \( \phi(\gamma) \in \text{Iso}(G_1 \rtimes \Gamma) \). The formula \( \phi^*([n, x]) := [\phi^{-1}(n), h(x)] \) defines a graded isomorphism \( \phi^*: \mathcal{H}(C^*_r(G_2), C_0(G_1^{(0)}), \delta_{c_2}) \rightarrow \mathcal{H}(C^*_r(G), C_0(G_0^{(0)}), \delta_{c_1}) \).

Proposition \ref{prop5.3} gives graded isomorphisms \( \theta_1 : G_1 \rightarrow \mathcal{H}(C^*_r(G_1), C_0(G_0^{(0)}), \delta_{c_1}) \). So \( \kappa := \theta_1^{-1} \circ \phi^* \circ \theta_2 : G_2 \rightarrow G_1 \) is a graded isomorphism. For \( x \in G_2^{(0)} \) and \( a \in C_0(G_2^{(0)}) \) with \( a(x) > 0 \), Proposition \ref{prop5.3} gives \( \kappa(x) = \theta_1^{-1} \circ \phi^*([a, x]) = \theta_1^{-1}(a \circ h^{-1}, h(x)) = h(x) \). \( \square \)

7. **Group actions**

In this section we consider transformation groupoids of continuous actions of countable discrete groups on topological spaces. We characterise isomorphism of transformation groupoids in terms of continuous orbit equivalence of the actions. If in each group, the subgroup \( \{ \gamma \in \text{Iso}(G \rtimes \Gamma) \} \) acting on the right of a second-countable locally compact Hausdorff space \( X \). We write \( x\gamma \) for the action of \( \gamma \) on \( x \).

Define \( X \rtimes \Gamma := X \times \Gamma \)

under the product topology, let \( (X \times \Gamma)^{(2)} := \{(x_1, \gamma_1), (x_2, \gamma_2) \} : x_1 = x_1\gamma_1 \), and define \( (x_1, \gamma_1)(x_1\gamma_1, \gamma_2) = (x_1, \gamma_1\gamma_2) \), and \( (x, \gamma)^{-1} = (x\gamma, \gamma^{-1}) \). Then \( X \rtimes \Gamma \) is a second-countable locally compact Hausdorff étale groupoid. Its unit space is \( X \times \{\text{id}_\Gamma\} \), which we identify with \( X \), so \( r(x, \gamma) = x \) and \( s(x, \gamma) = x\gamma \). We have \( C^*_r(X \rtimes \Gamma) \cong C_0(X) \rtimes \Gamma \) via an isomorphism that carries \( C_0((X \rtimes \Gamma)^{(0)}) \) to \( C_0(X) \subseteq C_0(X) \rtimes \Gamma \) (see \[17 \], Example 3.2.8).

For \( x \in X \), we write \( \text{Stab}(x) := \{ \gamma \in \Gamma : x\gamma = x \} \) for the stabiliser subgroup of \( x \) in \( \Gamma \); observe that then \( (X \rtimes \Gamma)^x = \{x\} \times \text{Stab}(x) \). We also consider the essential stabiliser subgroup \( \text{Stab}^{ess}(x) := \{ \gamma \in \Gamma : \gamma \in \text{Stab}(y) \text{ for all } y \text{ in some neighbourhood } U \text{ of } x \} \).

Observe that \( \text{Iso}(X \rtimes \Gamma)^{\circ} = \bigcup_{x \in X} \{x\} \times \text{Stab}^{ess}(x) \). We say \( (X, \Gamma) \) is topologically free if each \( \text{Stab}^{ess}(x) = \{\text{id}_\Gamma\} \); a Baire-category argument shows that \( (X, \Gamma) \) is topologically free if and only if \( x \in X \) : \( \text{Stab}(x) = \{\text{id}_\Gamma\} \) = \( X \).

**Definition 7.1.** Let \( \Gamma \rhd X \) and \( \Lambda \rhd Y \) be actions of countable discrete groups on locally compact Hausdorff spaces. A continuous orbit equivalence \( (h, \pi, \eta) \) from \( (X, \Gamma) \) to \( (Y, \Lambda) \) consists of a homeomorphism \( h : X \rightarrow Y \) and continuous maps \( \phi : X \times \Gamma \rightarrow \Lambda \) and \( \eta : Y \times \Lambda \rightarrow \Gamma \) such that \( h(x\gamma) = h(x)\phi(x, \gamma) \) for all \( x, \gamma \) and \( h^{-1}(y\lambda) = h^{-1}(y)\eta(y, \lambda) \) for all \( y, \lambda \). We call \( h \) the underlying homeomorphism of \((h, \phi, \eta)\).
For topologically free systems, the intertwining condition appearing in Definition 7.1 has some important consequences.

**Lemma 7.2.** Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be topologically free actions of countable discrete groups on locally compact Hausdorff spaces. Let $(h, \phi, \eta)$ be a continuous orbit equivalence from $(X, \Gamma)$ to $(Y, \Lambda)$. Fix $x \in X$. We have

$$\phi(x, \gamma \gamma') = \phi(x, \gamma) \phi(x \gamma, \gamma')$$

for all $\gamma, \gamma' \in \Gamma$, and each $\theta_x := \phi(x, \cdot) : \Gamma \rightarrow \Lambda$ is a bijection that carries $\text{id}_{\Gamma}$ to $\text{id}_\Lambda$ and restricts to bijections $\text{Stab}(x) \rightarrow \text{Stab}(h(x))$ and $\text{Stab}^{\text{ess}}(x) \rightarrow \text{Stab}^{\text{ess}}(h(x))$.

**Proof.** The first part is proved in the same way as [33, Lemma 2.8], the only difference is that the actions considered in [33] are left actions and here we consider right actions.

For the second, by [33, Corollary 2.11], for all $x \in X$ with $\text{Stab}(x)$ trivial, $\theta_x$ is a bijection. For arbitrary $x \in X$, take $x_n \rightarrow x$ such that $\text{Stab}(x_n)$ is trivial. Since $\phi$ is continuous and $\Lambda$ is a discrete group, $\theta_x = \theta_{x_n}$ for large $n$. So $\theta_x$ is a bijection. That $\theta_x(\text{id}_{\Gamma}) = \text{id}_\Lambda$ follows from the first statement. The intertwining condition in Definition 7.1 and that $h$ is a homeomorphism gives $\theta_x(\text{Stab}(x)) = \text{Stab}(h(x))$; and $\theta_x(\text{Stab}^{\text{ess}}(x)) = \text{Stab}^{\text{ess}}(h(x))$ because both are trivial. \hfill $\Box$

Lemma 7.2 prompts the following definition.

**Definition 7.3.** Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be actions of countable discrete groups on locally compact Hausdorff spaces. Consider a map $\phi : X \times \Gamma \rightarrow \Lambda$.

1. We call $\phi$ a cocycle if $\phi(x, \gamma \gamma') = \phi(x, \gamma) \phi(x \gamma, \gamma')$ for all $x, \gamma, \gamma'$.

2. Let $h : X \rightarrow Y$ be a homeomorphism. We say that $(h, \phi)$ preserves stabilisers if $\phi(x, \cdot)$ restricts to bijections $\text{Stab}(x) \rightarrow \text{Stab}(h(x))$, and that $(h, \phi)$ preserves essential stabilisers if $\phi(x, \cdot)$ restricts to bijections $\text{Stab}^{\text{ess}}(x) \rightarrow \text{Stab}^{\text{ess}}(h(x))$.

**Proposition 7.4.** Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be actions of countable discrete groups on locally compact Hausdorff spaces. Suppose that $h : X \rightarrow Y$ is a homeomorphism and $\phi : X \times \Gamma \rightarrow \Lambda$ is continuous. The following are equivalent:

1. there is an isomorphism $\Theta : X \times \Gamma \rightarrow Y \times \Lambda$ such that $\Theta(x, \text{id}_{\Gamma}) = (h(x), \text{id}_{\Lambda})$ and $\Theta(x, \gamma) = (h(x), \phi(x, \gamma))$ for all $x \in X$ and $\gamma \in \Gamma$;

2. $\phi$ is a cocycle, $(h, \phi)$ preserves stabilisers, and there is a map $\eta : Y \times \Lambda \rightarrow \Gamma$ such that $(h, \phi, \eta)$ is a continuous orbit equivalence; and

3. $\phi$ is a cocycle, $(h, \phi)$ preserves essential stabilisers, and there is a map $\eta : Y \times \Lambda \rightarrow \Gamma$ such that $(h, \phi, \eta)$ is a continuous orbit equivalence.

**Proof.** (1) $\implies$ (2): Define $\eta : Y \times \Lambda \rightarrow \Gamma$ by $\Theta^{-1}(y, \lambda) = (h^{-1}(y), \eta(y, \lambda))$. Then $(h, \phi, \eta)$ is a continuous orbit equivalence and $\phi, \eta$ are cocycles. The pair $(h, \phi)$ preserves stabilisers because $\Theta\{(x, \gamma) : \gamma \in \Gamma_x\} = \{(h(x), \lambda) : \lambda \in \Lambda_{h(x)}\}$.

(2) $\implies$ (3): It suffices to show that $\phi(x, \gamma) \in \text{Stab}^{\text{ess}}(h(x))$ if and only if $\gamma \in \text{Stab}^{\text{ess}}(x)$. First suppose that $\gamma \in \text{Stab}^{\text{ess}}(x)$. Fix an open neighbourhood $U \ni x$ such that $x' \gamma = x'$ for all $x' \in U$. Since $\phi$ is continuous and $\Lambda$ is discrete, we can assume that $\phi(x', \gamma) = \phi(x, \gamma) = \lambda$ for all $x' \in U$. Then $U' := h(U)$ is an open neighbourhood of $h(x)$ such that for $u \in U'$,

$$y \lambda = h(h^{-1}(y)) \lambda = h(h^{-1}(y)) \phi(h^{-1}(y), \gamma) = h(h^{-1}(y) \gamma) = h(h^{-1}(y)) = y.$$ 

Hence $\phi(x, \gamma) \in \text{Stab}^{\text{ess}}(h(x))$. 


Now suppose that $\phi(x, \gamma) \in \text{Stab}^{\text{ess}}(h(x))$. There is an open $U' \ni h(x)$ such that $\lambda := \phi(x, \gamma)$ satisfies $y\lambda = y$ for all $y \in U'$. By continuity of $h$ and $\phi$, there is an open $U \ni x$ such that $h(U) \subseteq U'$ and $\phi(x', \gamma) = \lambda$ for all $x' \in U$. So $x'\gamma = x'$ for all $x' \in U$, giving $\gamma \in \text{Stab}^{\text{ess}}(x)$.

(3) $\implies$ (1): Define $\Theta : X \rtimes \Gamma \to Y \rtimes \Lambda$ by $\Theta(x, \gamma) = (h(x), \phi(x, \gamma))$. Then $\Theta$ is open, continuous, and compatible with the action of $\Gamma$ on $X$. By continuity of $h$ and $\phi$, there is an open neighbourhood $U \ni x$ such that $h(U) \subseteq U'$ and $\phi(x', \gamma) = \lambda$ for all $x' \in U$. So $x'\gamma = x'$ for all $x' \in U$, giving $\gamma \in \text{Stab}^{\text{ess}}(x)$.

This gives the following generalisation of Li's rigidity theorem [33, Theorem 1.2].

**Corollary 7.5.** Let $\Gamma \acts X$ and $\Lambda \acts Y$ be actions of countable discrete groups on second-countable locally compact Hausdorff spaces. Suppose that $\text{Stab}^{\text{ess}}(x)$ and $\text{Stab}^{\text{ess}}(y)$ are torsion-free and abelian for all $x \in X$ and all $y \in Y$, and that $h$ is a homeomorphism from $X$ to $Y$. Then the following are equivalent:

1. There exist cocycles $\phi : X \rtimes \Gamma \to \Lambda$ and $\eta : Y \rtimes \Lambda \to \Gamma$ such that $(h, \phi, \eta)$ is a continuous orbit equivalence from $(X, \Gamma)$ to $(Y, \Lambda)$ and $(h, \phi)$ and $(h^{-1}, \eta)$ preserve essential stabilisers;
2. There is an isomorphism $\Theta : X \rtimes \Gamma \to Y \rtimes \Lambda$ such that $\Theta(x, \gamma) = (h(x), \text{id}_\Lambda)$ for all $x \in X$; and
3. There is an isomorphism $\phi : C_0(X) \rtimes \Gamma \to C_0(Y) \rtimes \Lambda$ such that $\phi(C_0(X)) = C_0(Y)$ and $\phi(f) = f \circ h^{-1}$ for $f \in C_0(X)$.

**Proof.** This follows directly from Theorem 6.2 and Proposition 7.4.

**Remark 7.6.** Let $\Gamma \acts X$ and $\Gamma \acts Y$ be actions of a countable discrete group on second-countable locally compact Hausdorff spaces, and consider the reduced crossed-products $C_0(X) \rtimes \Gamma$ and $C_0(Y) \rtimes \Gamma$, with dual coactions $\delta_X$ and $\delta_Y$ of $\Gamma$. As $(C_0(X) \rtimes \Gamma)^\delta_X = C_0(X)$ and likewise for $Y$, any equivariant isomorphism $C_0(X) \rtimes \Gamma \to C_0(Y) \rtimes \Gamma$ restricts to an isomorphism $C_0(X) \to C_0(Y)$. Theorem 6.2 therefore shows that $C_0(X) \rtimes \Gamma$ and $C_0(Y) \rtimes \Gamma$ are equivariantly isomorphic if and only if $X \rtimes \Gamma \cong Y \rtimes \Gamma$ as graded groupoids. We have $X \rtimes \Gamma \cong Y \rtimes \Gamma$ as graded groupoids if and only if there is a homeomorphism $h : X \to Y$ such that $h(x)\gamma = h(x\gamma)$ for all $x \in X$ and $\gamma \in \Gamma$, so if and only if $(X, \Gamma)$ and $(Y, \Gamma)$ are topologically conjugate. So we recover [29, Proposition 4.3].

8. Local homeomorphisms of locally compact Hausdorff spaces

In this section we adapt the ideas of Section 7 to actions of $\mathbb{N}$ by local homeomorphisms. We first characterise an appropriate notion of continuous orbit equivalence in terms of...
diagonal-preserving isomorphism of $C^*$-algebras. We then characterise eventual conjugacy in terms of gauge-equivariant diagonal-preserving isomorphisms.

A Deaconu–Renault system is a pair $(X, \sigma)$ consisting of a locally compact Hausdorff space $X$ and a local homeomorphism $\sigma : \text{dom}(\sigma) \to \text{ran}(\sigma)$ from an open set $\text{dom}(\sigma) \subseteq X$ to an open set $\text{ran}(\sigma) \subseteq X$. Inductively define $\text{dom}(\sigma^n) := \sigma^{-1}(\text{dom}(\sigma^{n-1}))$, so each $\sigma^n : \text{dom}(\sigma^n) \to \text{ran}(\sigma^n)$ is a local homeomorphism and $\sigma^m \circ \sigma^n = \sigma^{m+n}$ on $\text{dom}(\sigma^{m+n})$. We write $D_n := \text{dom}(\sigma^n)$ and $\sigma^0 := \text{id}_X$. For $x \in X$ we define the stabiliser group at $x$ by

$$\text{Stab}(x) := \{ m - n : m, n \in \mathbb{N}, x \in D_m \cap D_n, \text{ and } \sigma^n(x) = \sigma^m(x) \} \subseteq \mathbb{Z},$$

and we define the essential stabiliser group at $x$ by

$$\text{Stab}^{\text{ess}}(x) := \{ m - n : m, n \in \mathbb{N} \text{ and there is an open neighbourhood } U \subseteq D_m \cap D_n \text{ of } x \text{ such that } \sigma^n|_U = \sigma^m|_U \} \subseteq \text{Stab}(x).$$

With the convention that $\min(\emptyset) = \infty$, we define the minimal stabiliser of $x$ to be

$$\text{Stab}_{\text{min}}(x) := \min\{ n \in \text{Stab}(x) : n \geq 1 \},$$

and the minimal essential stabiliser at $x$ to be

$$\text{Stab}_{\text{min}}^{\text{ess}}(x) := \min\{ n \in \text{Stab}^{\text{ess}}(x) : n \geq 1 \}.$$

The Deaconu–Renault groupoid of $(X, \sigma)$ is

$$G = G(X, \sigma) = \bigcup_{n,m \in \mathbb{N}} \{(x, n-m, y) \in D_n \times \{n-m\} \times D_m : \sigma^n(x) = \sigma^m(y)\},$$

under the topology with basic open sets $Z(U, n, m, V) := \{(x, n-m, y) : x \in U, y \in V, \text{ and } \sigma^n(x) = \sigma^m(y)\}$ indexed by quadruples $(U, n, m, V)$ where $n, m \in \mathbb{N}$, $U \subseteq D_n$ and $V \subseteq D_m$ are open, and $\sigma^n|_U$ and $\sigma^m|_V$ are homeomorphisms. Each $Z(U, n, m, V)$ can be written as $Z(U', n, m, V')$ with $\sigma^n(U') = \sigma^m(V')$ (put $U' = U \cap (\sigma^n)^{-1}(V)$ and $V' = V \cap (\sigma^m)^{-1}(U)$). This $G$ is a locally compact Hausdorff étale groupoid, with $G^{(0)} = \{(x, 0, x) : x \in X\}$ identified with $X$. It is also amenable by the argument of [19] Lemma 3.5, so $C^*_r(G) = C^*(G)$. The isotropy subgroupoid of $G$ is $\{(x, n, x) : x \in X, n \in \text{Stab}(x)\}$, and the interior of the isotropy is $\text{Iso}(G)^{\circ} = \{(x, n, x) : x \in X, n \in \text{Stab}^{\text{ess}}(x)\}$. So $\text{Iso}(G)^{\circ}$ is torsion-free and abelian. Taking $\Gamma = \{ e \}$ and $c : G \to \Gamma$ the trivial cocycle, we obtain a (trivially) graded groupoid $G$.

8.1. Continuous orbit equivalence. We show that stabiliser-preserving continuous orbit equivalence of Deaconu–Renault systems characterises isomorphism of their groupoids.

**Definition 8.1.** Let $(X, \sigma)$ and $(Y, \tau)$ be Deaconu–Renault systems. We say that $(X, \sigma)$ and $(Y, \tau)$ are continuous orbit equivalent if there exist a homeomorphism $h : X \to Y$ and continuous maps $k, l : \text{dom}(\sigma) \to \mathbb{N}$ and $k', l' : \text{dom}(\tau) \to \mathbb{N}$ such that

$$\tau^{l(x)}(h(x)) = \tau^{k(x)}(h(\sigma(x))) \quad \text{and} \quad \sigma^{l'(y)}(h^{-1}(y)) = \sigma^{k'(y)}(h^{-1}(\tau(y)))$$

for all $x, y$. We call $(h, l, k, l', k')$ a continuous orbit equivalence and we call $h$ the underlying homeomorphism. We say that $(h, l, k, l', k')$ preserves stabilisers if $\text{Stab}_{\text{min}}(h(x)) < $
and $(Y, \tau)$ this section generalises [10, Theorem 5.1], [34, Theorem 2.3], and [50, Theorem 2]:

Our definition of continuous orbit equivalence boils down to the usual notion for homeomorphisms (see for instance [6], [25], and [50]), and to orbit equivalence of graphs if $(X, \sigma)$ and $(Y, \tau)$ are the shifts on their boundary-path spaces (see [10]). Our main theorem in this section generalises [10, Theorem 5.1], [34, Theorem 2.3], and [50, Theorem 2]:

**Theorem 8.2.** Let $(X, \sigma)$ and $(Y, \tau)$ be Deaconu–Renault systems with $X, Y$ second countable, and suppose that $h : X \to Y$ is a homeomorphism. Then the following are equivalent:

1. there is a stabiliser-preserving continuous orbit equivalence from $(X, \sigma)$ to $(Y, \tau)$ with underlying homeomorphism $h$;
2. there is a groupoid isomorphism $\Theta : G(X, \sigma) \to G(Y, \tau)$ such that $\Theta|_X = h$; and
3. there is an isomorphism $\phi : C^*(G(X, \sigma)) \to C^*(G(Y, \tau))$ such that $\phi(C_0(X)) = C_0(Y)$ with $\phi(f) = f \circ h^{-1}$ for $f \in C_0(Y)$.

To prove Theorem 8.2 we need to relate isomorphism of Deaconu–Renault groupoids to continuous orbit equivalence. Arklin, Eilers, and Ruiz [2] (see also [16]) proved that isomorphism of graph groupoids (and hence diagonal-preserving isomorphism of graph $C^*$-algebras) is characterised by continuous orbit equivalence of the shift maps on their boundary path spaces with underlying homeomorphism $h$ satisfying $\text{Stab}^{\text{ess}}(h(x)) = \{0\} \iff \text{Stab}^{\text{ess}}(x) = \{0\}$. The following is the analogous result for Deaconu–Renault systems.

**Proposition 8.3.** Let $(X, \sigma)$ and $(Y, \tau)$ be Deaconu–Renault systems. Let $h : X \to Y$ be a homeomorphism and let $l, k : \text{dom}(\sigma) \to \mathbb{N}$ and $l', k' : \text{dom}(\tau) \to \mathbb{N}$ be continuous maps such that $h(x) \in \text{dom}(\tau^{l(x)})$ and $h(\sigma(x)) \in \text{dom}(\tau^{k(x)})$ for $x \in \text{dom}(\sigma)$, and $h^{-1}(y) \in \text{dom}(\sigma^{l'(y)})$ and $h^{-1}(\tau(y)) \in \text{dom}(\sigma^{k'(y)})$ for $y \in \text{dom}(\tau)$. The following are equivalent:

1. there are groupoid isomorphisms $\Theta : G(X, \sigma) \to G(Y, \tau)$ and $\Theta' : G(Y, \tau) \to G(X, \sigma)$ such that $\Theta|_X = h$, $\Theta'|_Y = h^{-1}$, $\Theta(x, 1, \sigma(x)) = (h(x), l(x) - k(x), h(\sigma(x)))$ for $x \in \text{dom}(\sigma)$, and $\Theta'(y, 1, \tau(y)) = (h^{-1}(y), l'(y) - k'(y), h^{-1}(\tau(y)))$ for $y \in \text{dom}(\tau)$;
2. $(h, l, k, l', k')$ is an essential-stabiliser-preserving continuous orbit equivalence; and
Lemma 8.4. Let \((X, \sigma)\) be a Deaconu–Renault system. The function \(l_X : G(X, \sigma) \to \mathbb{N}\) given by \(l_X(x, n, y) := \min\{l \in \mathbb{N} : l \geq n \text{ and } \sigma^l(x) = \sigma^l(y)\}\) is continuous.

Proof. Suppose that \((x_i, n_i, y_i) \to (x, n, y)\) in \(G(X, \sigma)\). Then \(n_i = n\) for large \(i\), so we can assume that \(n_i = n\) for all \(i\). We first show that \(l_X(x_i, n_i, y_i) \leq l_X(x, n, y)\) for large \(i\). To see this, fix a basic open neighbourhood \(Z(U, p, p - n, V)\) of \((x, n, y)\); so \(\sigma^p(U) = \sigma^{p-n}(V)\), and \(\sigma^p|_L\) and \(\sigma^{p-n}|_V\) are homeomorphisms. Since \((x_i, n_i, y_i) \to (x, n, y)\) we have \((x_i, n_i, y_i) \in Z(U, p, p - n, V)\) for large \(i\). So \(\sigma^p(x_i) = \sigma^{p-n}(y_i)\) for large \(i\). Let \(l := l_X(x, n, y)\). Then \(l \leq p\), say \(p = l + q\). Hence \(\sigma^l(x_i) = \sigma^p(x_i) = \sigma^{p-n}(y_i) = \sigma^q(\sigma^{l-q}(y_i))\) for large \(i\).

Since \(\sigma^p\) is locally injective and \(\lim \sigma^i(x_i) = \sigma^i(x) = \sigma^{i-n}(y) = \lim \sigma^{i-n}(y_i)\), we deduce that \(\sigma^i(x_i) = \sigma^{i-n}(y_i)\) for large \(i\). So \(l_X(x_i, n_i, y_i) \leq l\) for large \(i\).

It now suffices to show that \(l_X(x_i, n_i, y_i) \geq l\) for large \(i\); equivalently, if \(l_X(x_i, n_i, y_i) = k\) for infinitely many \(i\), then \(k \geq l\). Suppose that \(l(x_i, n_i, y_i) = k\) for all \(j\). Then \(k \geq n\) by definition of \(l_X(x, n, y)\). Since \(\sigma^k(x_i) = \sigma^{k-n}(y_i)\) for all \(j\) and \(x_{ij} \to x\) and \(y_{ij} \to y\), continuity forces \(\sigma^k(x) = \sigma^{k-n}(y)\). So \(k \geq l\).

Given a Deaconu–Renault system \((X, \sigma)\), define \(c_X : G(X, \sigma) \to \mathbb{Z}\) by \(c_X(x, n, y) := n\).

Lemma 8.5. Let \((X, \sigma)\) and \((Y, \tau)\) be Deaconu–Renault systems. Suppose that \(\Theta : G(X, \sigma) \to G(Y, \tau)\) is an isomorphism of groupoids. Let \(h : X \to Y\) be the restriction of \(\Theta\) to \(G(X, \sigma)^{(0)}\). For \(p \in \mathbb{N}\), the functions \(l_p, k_p : D_p \to \mathbb{N}\) given by

\[
\begin{align*}
l_p(x) &:= \min\{l \in \mathbb{N} : \tau^l(h(x)) = \tau^{l-c_y(\Theta(x, p, \sigma^p(x)))(h(\sigma^p(x)))}\}, \\
k_p(x) &:= l_p(x) - c_y(\Theta(x, p, \sigma^p(x)))
\end{align*}
\]

are continuous, and \(\tau_{p}(x)(h(x)) = \tau_{p}(x)(h(\sigma^p(x)))\) for all \(x \in D_p\). For \(p \in \mathbb{N}\) and \(x \in D_p\),

\[
\sum_{n=0}^{p-1} (l_1(\sigma^n(x)) - k_1(\sigma^n(x))) = l_p(x) - k_p(x) = c_y(\Theta(x, p, \sigma^p(x))).
\]

Proof. Since \(l_p(x) = l_Y(\Theta(x, p, \sigma^p(x)))\) and \(\Theta\) is continuous, \(l_p\) is continuous by Lemma 8.4. Now \(k_p\) is continuous because \(l_p\) and \(c_y\) are. We have \(\tau_{p}(x)(h(x)) = \tau_{p}(x)(h(\sigma^p(x)))\) and \(l_p(x) - k_p(x) = c_y(\Theta(x, p, \sigma^p(x)))\) by definition of \(k_p\). So \(\sum_{n=0}^{p-1} (l_1(\sigma^n(x)) - k_1(\sigma^n(x))) = \sum_{n=0}^{p-1} c_y(\Theta(\sigma^n(x), 1, \sigma^{n+1}(x))) = c_y(\Theta(x, p, \sigma^p(x)))\).

Lemma 8.6. Let \((X, \sigma)\) and \((Y, \tau)\) be Deaconu–Renault systems as above. Suppose that \(\Theta : G(X, \sigma) \to G(Y, \tau)\) is an isomorphism. Let \(h, k, l_p\) be as in Lemma 8.5 and let \(x \in X\). Then \(\text{Stab}^\text{ess}_{\text{min}}(x) < \infty\) if and only if \(\text{Stab}^\text{ess}_{\text{min}}(h(x)) < \infty\), and if \(\text{Stab}^\text{ess}_{\text{min}}(x) < \infty\) and \(\sigma^\text{Stab}^\text{ess}_{\text{min}}(x)(x) = x\), then \(\lvert l_{\text{Stab}^\text{ess}_{\text{min}}(h(x))}(x) - k_{\text{Stab}^\text{ess}_{\text{min}}(x)}(x)\rvert = \text{Stab}^\text{ess}_{\text{min}}(h(x))\).

Proof. For any Deaconu–Renault system \((Z, \beta)\) and \(a \in Z\), \(\text{Stab}^\text{ess}_{\text{min}}(a) = \infty\) if and only if \(\text{Iso}(G(Z, \beta))^o = \{(a, 0, a)\}\). Thus \(\text{Stab}^\text{ess}_{\text{min}}(x) = \infty \iff \text{Stab}^\text{ess}_{\text{min}}(h(x)) = \infty\) as \(\Theta\) restricts to an isomorphism \(\text{Iso}(G(X, \sigma))^o \to \text{Iso}(G(Y, \tau))^o\) mapping \(G(X, \sigma)^{(0)}\) onto \(G(Y, \tau)^{(0)}\).

Suppose that \(\text{Stab}^\text{ess}_{\text{min}}(h(x)) < \infty\). Then \(\Theta(x, \text{Stab}^\text{ess}_{\text{min}}(x), x) = (h(x), q, h(x))\) for some \(q \in Z\). Since \((x, \text{Stab}^\text{ess}_{\text{min}}(x), x)\) generates \(\text{Iso}(G(X, \sigma))^o\), we deduce that \((h(x), q, h(x))\)
generates $\text{Iso}(G(Y, \tau))_h(x)$. Thus $q = \pm \text{Stab}_{\text{min}}^{\text{ess}}(h(x))$. So if $\sigma_{\text{Stab}_{\text{min}}^{\text{ess}}(x)}(x) = x$, then
\[
|l_{\text{Stab}_{\text{min}}^{\text{ess}}(x)}(x) - k_{\text{Stab}_{\text{min}}^{\text{ess}}(x)}(x)| = |c_Y(\Theta(x, \text{Stab}_{\text{min}}^{\text{ess}}(x), x))| = \text{Stab}_{\text{min}}^{\text{ess}}(h(x)).
\]

Given a Deaconu–Renault system $(X, \sigma)$ and $l : \text{dom}(\sigma) \to \mathbb{N}$, we inductively define $l_m : \text{dom}(\sigma^m) \to \mathbb{N}, m \geq 1$ by $l_1 = l$ and $l_{m+1}(x) = l(x) + l_m(\sigma(x))$. For $m, n \geq 1$ we have
\[
(8.1) \quad l_m(x) = \sum_{i=0}^{m-1} l(\sigma^i(x)) \quad \text{and} \quad l_{m+n}(x) = l_m(x) + n(\sigma^m(x)).
\]

**Lemma 8.7.** Let $(X, \sigma)$ and $(Y, \tau)$ be Deaconu–Renault systems and let $(h, l, k, l', k')$ be a continuous orbit equivalence from $(X, \sigma)$ to $(Y, \tau)$. Then there is a continuous cocycle $c_{(h, l, k, l', k')} : G(X, \sigma) \to \mathbb{Z}$ such that $c_{(h, l, k, l', k')}(m-n, x') = l_m(x) - k_m(x) - l_n(x') + k_n(x')$.

**Proof.** Suppose that $x \in D_{m+1}, x' \in D_{n+1}$, and $\sigma^m(x) = \sigma^n(x')$. A computation shows that $l_m(x) - k_m(x) - l_n(x') + k_n(x') = l_{m+1}(x) - k_{m+1}(x) - l_{n+1}(x') + k_{n+1}(x')$. Therefore, $c_{(h, l, k, l', k')} : G(X, \sigma) \to \mathbb{Z}$ is a well-defined map. It is easy to check that this map is a cocycle. For continuity, suppose that $\sigma^m(x) = \sigma^n(x')$. Fix open subneighbourhoods $U \ni x$ and $V \ni x'$ of $D_m$ and $D_n$ such that $\sigma^m|_U$ and $\sigma^n|_V$ are homeomorphisms, $l, k, \ldots, l \circ \sigma^m, k \circ \sigma^m$ are constant on $U$, and $l, k, \ldots, l \circ \sigma^m, k \circ \sigma^m$ are constant on $V$. Then $c_{(h, l, k, l', k')}$ is constant on $Z(U, m, n, V)$.

**Lemma 8.8.** Let $(X, \sigma)$ and $(Y, \tau)$ be Deaconu–Renault systems, and let $(h, l, k, l', l')$ be a continuous orbit equivalence from $(X, \sigma)$ to $(Y, \tau)$. Then there is a continuous groupoid homomorphism $\Theta_{k,l} : G(X, \sigma) \to G(Y, \tau)$ such that $\Theta_{k,l}(m-n, x') = (h(x), l_m(x) - k_m(x) - l_n(x') + k_n(x'), h(x'))$ whenever $\sigma^m(x) = \sigma^n(x')$. For each $x \in X$ there is a group homomorphism $\pi_x : \text{Stab}(h(x)) \to \text{Stab}(h(x))$ such that
\[
(8.2) \quad \pi_x(m-n) = l_m(x) - k_m(x) - l_n(x') + k_n(x) \quad \text{whenever} \quad \sigma^m(x) = \sigma^n(x).
\]

For $x \in X$ and $m, n \in \mathbb{N}$, we have $\text{Stab}(\sigma^m(x)) = \text{Stab}(\sigma^n(x)), \text{Stab}(h(\sigma^m(x))) = \text{Stab}(h(\sigma^n(x)))$ and $\pi_{\sigma^m(x)}(x) = \pi_{\sigma^n(x)}$.

**Proof.** Lemma 8.7 yields a continuous homomorphism $\Theta_{k,l} : G(X, \sigma) \to G(Y, \tau)$ such that $\Theta_{k,l}(m-n, x') = (h(x), l_m(x) - k_m(x) - l_n(x') + k_n(x'), h(x'))$ whenever $\sigma^m(x) = \sigma^n(x')$.

For $x \in X$ the map $\pi_x : \text{Stab}(h(x)) \to \text{Stab}(h(x))$ defined by $(h(x), \pi_x(p), h(x)) = \Theta(x, p, x)$ is a homomorphism satisfying (8.2). That each $\text{Stab}(x) = \text{Stab}(\sigma(x))$ follows from the definition of $\text{Stab}$, and then induction gives $\text{Stab}(\sigma^m(x)) = \text{Stab}(\sigma^n(x))$ for all $x$. Since $h$ intertwines $\sigma$-orbits and $\tau$-orbits, it follows immediately that $\text{Stab}(h(\sigma^m(x))) = \text{Stab}(h(\sigma^n(x)))$ for all $x$. For the final statement, let $p := l_m(\sigma^m(x)) - k_m(\sigma^m(x)) - l_n(\sigma^m(x)) + k_n(\sigma^m(x))$ and calculate:
\[
(h(\sigma^m(x)), \pi_{\sigma^m(x)}(q), h(\sigma^m(x))) = \Theta(\sigma^m(x), q, \sigma^m(x))
\]
\[
= \Theta(\sigma^m(x), n-m, \sigma^n(x))\Theta(\sigma^n(x), q, \sigma^n(x))\Theta(\sigma^n(x), m-n, \sigma^m(x))
\]
\[
= (h(\sigma^m(x), -p, h(\sigma^n(x))))(h(\sigma^n(x)), \pi_{\sigma^n(x)}(q), h(\sigma^n(x)))(h(\sigma^m(x)), p, h(\sigma^n(x)))
\]
\[
= (h(\sigma^m(x)), \pi_{\sigma^m(x)}(q), h(\sigma^m(x))).
\]

**Proof of Proposition 8.3 (1) $\implies$ (2):** Let $k_p, l_p$ be as in Lemma 8.5. Then $l - k = l_1 - k_1$ on $\text{dom}(\sigma)$. Likewise, if $k_{p'} l_{p'} : \text{dom}(\tau^n) \to \mathbb{N}$ are the functions obtained from Lemma 8.5 for $\Theta^{-1}$, then $l' - k' = l_1' - k_1'$ on $\text{dom}(\tau)$. So Lemmas 8.5 and 8.6 show that $(h, l, k, l', k')$ is an essential-stabiliser-preserving continuous orbit equivalence.
For both (2) \iff (3) and (3) \iff (1), fix a continuous orbit equivalence \((h,k,l,k',l')\) from \((X,\sigma)\) to \((Y,\tau)\). Let \[
\Theta : G(X,\sigma) \to G(Y,\tau), \quad \text{and} \quad \Theta' : G(Y,\tau) \to G(X,\sigma)
\]
be the homomorphisms of Lemma 8.8 for \((h,k,l,k')\) and for \((h^{-1},k',l',k,l)\) respectively, and for each \(x \in X\), let \[
\pi_x : \text{Stab}(x) \to \text{Stab}(h(x))
\]
be the homomorphism (8.2).

(2) \iff (3). Using that, by Lemma 8.8 Stab(\cdot), Stab(h(\cdot)) and \(x \mapsto \pi_x\) are constant on orbits, and that \((h,l,k,l',k')\) preserves essential stabilisers, it is easy to check that \(\pi_x(\text{Stab}^{\text{ess}}(x)) = \text{Stab}^{\text{ess}}(h(x))\) for all \(x\). Fix \(x \in X\) and \(n \in \text{Stab}(h(x))\). Since \(\Theta \circ \Theta'\) is continuous and \(\Theta(\Theta'(h(x),n,h(x))) = (h(x),m,h(x))\) for some \(m \in \text{Stab}(h(x))\), there exist \(p,q \in \mathbb{N}\) with \(p - q = n\), and open neighbourhoods \(U,V\) of \(h(x)\) such that \(\tau^p|_U = \tau^q|_V\) are homeomorphisms, \(\tau^p(U) = \tau^q(V)\), and \(\Theta(\Theta'(y,n,y')) = (y,m,y')\) for \(y \in U\), \(y' \in V\), with \(\tau^p(y) = \tau^q(y')\). So \((y,n-m,y) = (y,n,y')(y,m,y')^{-1} \in G(Y,\tau)\) for all \(y \in U\), giving \(n - m \in \text{Stab}^{\text{ess}}(h(x))\). Hence \(\pi_x(r) = n - m\) for some \(r \in \text{Stab}^{\text{ess}}(x)\). Thus \(\pi_x(r + s) = n\) where \(s = c_X(\Theta'(h(x),n,h(x)))\). So \(\text{Stab}(h(x)) < \infty \iff \text{Stab}(x) < \infty\), and symmetry gives the reverse implication. Suppose that \(\text{Stab}(h(x)) < \infty\) and that \(\sigma^{\text{Stab}(h(x))(x)} = x\). Since \(\text{Stab}(h(x))\) generates \(\text{Stab}(h(x))\) and \(\text{Stab}(x)\) generates \(\text{Stab}(x)\), we have \(h_{\text{Stab}(h(x))}(x) = \pi_x(\text{Stab}(x)) = \pm \text{Stab}(h(x))\). Hence \((h, l, k, l', k')\) preserves stabilisers.

(3) \iff (1): Let \(\Theta := \Theta_{k,l} : G(X,\sigma) \to G(Y,\tau)\) and \(\Theta' := \Theta_{k',l'} : G(Y,\tau) \to G(X,\sigma)\) be as in Lemma 8.8. Then \(\Theta\) and \(\Theta'\) are continuous groupoid homomorphisms. We show that \(\Theta\) is bijective and \(\Theta^{-1}\) is continuous. For injectivity, suppose \(\Theta(x_1,n_1,x_1') = \Theta(x_2,n_2,x_2')\). As \(h\) is a homeomorphism, \(x_1 = x_2\) and \(x_1' = x_2'\). So \(\Theta(x_1,n_1-x_2,n_2,x_2') = \Theta(x_1,n_1,x_1') \Theta(x_1,n_2,x_2')^{-1} = (h(x_1),0,h(x_1))\), giving \(\pi_x(n_1-n_2) = 0\). As \((h,k,l,k',l')\) preserves stabilisers and \(\text{Stab}(\cdot)\), \(\text{Stab}(h(\cdot))\) and \(x \mapsto \pi_x\) are constant on orbits, each \(\pi_x : \text{Stab}(x) \to \text{Stab}(h(x))\) is bijective. Thus \((x_1,n_1,x_1') = (x_2,n_2,x_2')\).

For surjectivity, fix \((y,n,y') \in G(Y,\tau)\). We have \(\Theta(\Theta'(y,n,y')) = (y,m,y')\) for some \(m \in \mathbb{Z}\), so \(n - m \in \text{Stab}(y)\). Since \(\pi_{h^{-1}(y)}\) is bijective, \(n - m = \pi_{h^{-1}(y)}(p)\) for some \(p \in \text{Stab}(h^{-1}(y))\). So \(\Theta(h^{-1}(y),p+c_X(\Theta'(y,n,y'))), h^{-1}(y')) = (y,m,y')\).

To see that \(\Theta^{-1}\) is continuous, suppose \((y_n,m_n,y_n') \to (y,m,y')\) in \(G(Y,\tau)\). Fix \(p,q \in \mathbb{N}\) and open \(U' \ni h^{-1}(y)\) and \(V' \ni h^{-1}(y')\) such that \(\sigma^p|_U\) and \(\sigma^q|_V\) are homeomorphisms, \(\sigma^p(h^{-1}(y)) = \sigma^q(h^{-1}(y'))\), and \(\Theta^{-1}(y,m,y') = (h^{-1}(y),p-q,h^{-1}(y'))\). Choose open subneighbourhoods \(U' \ni h^{-1}(y)\) and \(V' \ni h^{-1}(y')\) of \(U,V\) such that \(\Theta(x,p-q,x') = (h(x),m,h(x'))\) whenever \(x \in U', x' \in V'\), and \(\sigma^p(x) = \sigma^q(x')\). Fix \(N\) such that \(y_n \in h(U'), y'_n \in h(V')\), and \(m_n = m\) for \(n \geq N\). Then \(\Theta^{-1}(y_n,m_n,y_n') \in Z(U',p,q,V') \subseteq Z(U,p,q,V)\) for \(n \geq N\). So \(\Theta^{-1}(y_n,m_n,y_n') \to \Theta^{-1}(y,m,y')\).

We thus have that \(\Theta\) is an isomorphism. A similar argument shows that \(\Theta'\) is also an isomorphism. 

\(\square\)

\textbf{Proof of Theorem 8.2.} The equivalence (1) \iff (2) follows from Proposition 8.3 and Lemma 8.8, and the equivalence (2) \iff (3) follows from Theorem 8.2. 

\(\square\)

\textbf{8.2. Eventual conjugacy.} Here we generalise [12, Theorem 4.1] by showing that the isomorphism of \(C^*\)-algebras in Theorem 8.2 is gauge-equivariant if and only if the groupoid isomorphism is cocycle-preserving, which is if and only if the continuous orbit equivalence is an eventual conjugacy.
**Definition 8.9.** Let \((X, \sigma)\) and \((Y, \tau)\) be Deaconu–Renault systems. We say that \((X, \sigma)\) and \((Y, \tau)\) are *eventually conjugate* if there is a stabiliser-preserving continuous orbit equivalence \((h, l, k, l', k')\) from \((X, \sigma)\) to \((Y, \tau)\) such that \(l(x) = k(x) + 1\) for all \(x \in X\).

Given \((X, \sigma)\), there is an action \(\gamma^X : \mathbb{T} \to \text{Aut}(C^*(G(X, \sigma)))\) such that \(\gamma^X_z(f)(x, n, x') = z^nf(x, n, x')\) for all \(z \in \mathbb{T}, (x, n, x') \in G(X, \sigma)\) and \(f \in C_c(G(X, \sigma))\).

**Theorem 9.1.** Suppose that \(\sigma\) is such that either \(h\) or \(\gamma^X\) is an eventual conjugacy. We combine the results and techniques developed in Sections 7 and 8 to obtain a generalisation of Boyle and Tomiyama’s theorem [6, Theorem 3.6]. If \(\sigma\) is an eventual conjugacy, then the formula for \(k_1\) in Lemma 8.5 gives \(l_1(x) = k_1(x) = 1\), so the continuous orbit equivalence constructed in the proof of \((2) \implies (1)\) in Theorem 8.2 is an eventual conjugacy.

**Theorem 9.10.** Let \((X, \sigma)\) and \((Y, \tau)\) be Deaconu–Renault systems and let \(h : X \to Y\) be a homeomorphism. Then the following are equivalent:

1. there is an eventual conjugacy from \((X, \sigma)\) to \((Y, \tau)\) with underlying homeomorphism \(h\);
2. there is an isomorphism \(\Theta : G(X, \sigma) \to G(Y, \tau)\) such that \(\Theta|_X = h\) and \(c_X = c_Y \circ \Theta\);
3. there is an isomorphism \(\phi : C^*(G(X, \sigma)) \to C^*(G(Y, \tau))\) such that \(\phi(C_0(X)) = C_0(Y)\), with \(\phi(f) = f \circ h^{-1}\) for \(f \in C_0(X)\), and \(\phi \circ \gamma^X_z = \gamma^Y_z \circ \phi\).

**Proof.** Theorem 8.2 applied to the \(Z\)-coactions dual to \(\gamma^X\) and \(\gamma^Y\) gives \((2) \iff (3)\).

\((2) \implies (1)\). If \(c_Y = c_Y \circ \Theta\), then the formula for \(k_1\) in Lemma 8.5 gives \(l_1(x) = k_1(x) = 1\), so the continuous orbit equivalence constructed in the proof of \((2) \implies (1)\) in Theorem 8.2 is an eventual conjugacy.

\((1) \implies (2)\). Suppose that \((h, l, k, l', k')\) is an eventual conjugacy from \((X, \sigma)\) to \((Y, \tau)\). The formula 8.1 gives \(l_p(x) - k_p(x) = p\) for all \(x \in X\). Thus, \(\Theta_{k,l}\) of Lemma 8.8 satisfies \(c_Y \circ \Theta_{k,l} = c_X\). As in the proof of \((1) \implies (2)\) in Theorem 8.2, \(\Theta_{k,l}\) an isomorphism.

9. HOMEOMORPHISMS OF COMPACT HAUSDORFF SPACES

We now specialise to the case where \(X\) is a compact Hausdorff space and \(\sigma : X \to X\) is a homeomorphism. We combine the results and techniques developed in Sections 7 and 8 to obtain a generalisation of Boyle and Tomiyama’s theorem [6, Theorem 3.6]. If \(\sigma : X \to X\) is a homeomorphism, then \(\alpha : G(X, \sigma) \to X \times \mathbb{Z}\), \(\alpha(x, n, y) \mapsto (x, n)\) is an isomorphism, so induced an isomorphism \(\phi : C^*(G(X, \sigma)) \cong C(X) \times_\sigma \mathbb{Z}\) with \(\phi(C(G(X, \sigma)[0])) = C(X)\).

Using Theorem 8.2 we can prove the following generalisation of [5, Theorem 3.6] (and thus of [25, Theorem 2.4] and [50, Corollary]). Following [5], we say that homeomorphisms \(\sigma : X \to X\) and \(\tau : Y \to Y\) are flip conjugate if there is a homeomorphism \(h : X \to Y\) such that either \(h \circ \sigma = \tau \circ h\) or \(h \circ \sigma = \tau^{-1} \circ h\).

**Theorem 9.11.** Suppose that \(\sigma : X \to X\) and \(\tau : Y \to Y\) are homeomorphisms of second-countable compact Hausdorff spaces. The following are equivalent:

1. \(G(X, \sigma)\) and \(G(Y, \tau)\) are isomorphic;
2. \(C(X) \times_\sigma \mathbb{Z} \cong C(Y) \times_\tau \mathbb{Z}\) via an isomorphism that maps \(C(X)\) to \(C(Y)\); and
3. there exist decompositions \(X = X_1 \sqcup X_2\) and \(Y = Y_1 \sqcup Y_2\) into disjoint open invariant sets such that \(\sigma|_{X_1}\) is conjugate to \(\tau|_{Y_1}\) and \(\sigma|_{X_2}\) is conjugate to \(\tau^{-1}|_{Y_2}\).

If \(\sigma\) and \(\tau\) are topologically transitive or \(X\) and \(Y\) are connected, then these conditions hold if and only \(\sigma\) and \(\tau\) are flip-conjugate.

Our proof of \((1) \implies (3)\) closely follows [5], and requires some preliminary results. Take \(X, Y, \sigma, \tau\) as in Theorem 9.11 and an isomorphism \(\Theta : G(X, \sigma) \to G(Y, \tau)\). Define \(h : X \to Y\) by \(\theta(x, 0, x) = (h(x), 0, h(x))\). Let \(c_X : G(X, \sigma) \to \mathbb{Z}\) and \(c_Y : G(Y, \tau) \to \mathbb{Z}\) be the canonical cocycles. Define \(f : X \times \mathbb{Z} \to \mathbb{Z}\) by \(f(n, x) := c_Y(\theta(x, n, \sigma^n(x)))\). Then
\[
(9.1) \quad f(m + n, x) = f(m, x) + f(n, \sigma^m(x)) \quad \text{for all } m, n \in \mathbb{Z} \text{ and } x \in X.
\]
For $x \in X$, $f(\cdot, x)$ is a bijection of $\mathbb{Z}$ with inverse $n \mapsto c_X(\theta^{-1}(h(x)), n, \tau^n(h(x)))$.

For $m, n \in \mathbb{Z}$, we let $[m, n] := \{k \in \mathbb{Z} : m \leq k \leq n\}$.

**Lemma 9.2.** For each $M \in \mathbb{N}$ there exists $\overline{M} \in \mathbb{N}$ such that

\begin{equation}
[-M, M] \subseteq \{f(n, x) : n \in [-M, M]\} \quad \text{for all } x \in X.
\end{equation}

**Proof.** Let $x \in X$. Fix $M_x \in \mathbb{N}$ such that $[-M, M] \subseteq \{f(n, x) : n \in [-M_x, M_x]\}$. Continuity of the map $f(n, \cdot)$ for each $n$ implies $x$ has an open neighbourhood $U_x$ such that $[-M, M] \subseteq \{f(n, x') : n \in [-M_x, M_x]\}$ for $x' \in U_x$. Compactness of $X$ gives a finite $F \subseteq X$ such that $\bigcup_{x \in F} U_x = X$. So $\overline{M} := \max\{M_x : x \in F\}$ satisfies (9.2). \qed

**Lemma 9.3.** There is a positive integer $N$ such that

$X_1 := \{x \in X : f(n, x) > 0 \text{ and } f(-n, x) < 0 \text{ for } n > N\}$ and

$X_2 := \{x \in X : f(n, x) < 0 \text{ and } f(-n, x) > 0 \text{ for } n > N\}$

are clopen $\sigma$-invariant subsets such that $X = X_1 \cup X_2$.

**Proof.** Compactness of $X$ implies that $M := \max\{|f(1, x)| : x \in X\}$ is finite. Lemma 9.2 gives $N \in \mathbb{N}$ with $[-M, M] \subseteq \{f(n, x) : n \in [-N, N]\}$ for all $x$.

To see that $X = X_1 \cup X_2$, fix $x \in X$. Choose $m > N$. Since $n \mapsto f(n, x)$ is bijective and $[-M, M] \subseteq \{f(n, x) : n \in [-N, N]\}$, we have $|f(m, x)| > M$. Since $|f(m + 1, x) - f(m, x)| = |f(1, \sigma^m(x))| \leq M$, it follows that $f(m + 1, x)$ and $f(m, x)$ have the same sign. Similarly, $f(-m + 1, x)$ and $f(-m, x)$ have the same sign. Since $n \mapsto f(n, x)$ is bijective, $x \in X_1 \cup X_2$. Continuity of $f(n, \cdot)$ imply $X_i$ are clopen.

We show that $\sigma(X_1) \subseteq X_1$ (a similar argument gives $\sigma(X_2) \subseteq X_2$). Fix $x \in X_1$. Choose $m > N$. Then $f(m + 1, x) > M$. Since $|f(m + 1, x) - f(m, \sigma(x))| = |f(1, x)| \leq M$, we have $\sigma(x) \notin X_2$; so $\sigma(x) \in X_1$. \qed

**Lemma 9.4.** There are continuous functions $a : X_1 \to \mathbb{Z}$ and $b : X_2 \to \mathbb{Z}$ such that $f(1, x) = a(x) - a(\sigma(x)) + 1$ for $x \in X_1$, and $f(1, x) = b(x) - b(\sigma(x)) - 1$ for $x \in X_2$.

**Proof.** Define $n(x) := f(1, x)$ for all $x \in X$. Fix $x \in X_1$. Since $f(n, x) > 0$ and $f(-n, x) < 0$ for $n > N$, both $|\{n \geq 0 : f(n, x) < 0\}|$ and $|\{m < 0 : f(m, x) \geq 0\}|$ are finite. Let

$a(x) := |\{m < 0 : f(m, x) \geq 0\}| - |\{n \geq 0 : f(n, x) < 0\}|$.

Continuity of $f(n, \cdot)$ implies that $a$ is continuous.

Take $x \in X_1$. Using (9.1) at the third equality, we calculate

\[
\begin{align*}
a(x) + 1 &= |\{m < 0 : f(m, x) \geq 0\}| - |\{n \geq 0 : f(n, x) < 0\}| + 1 \\
&= |\{m < 1 : f(m, x) \geq 0\}| - |\{n \geq 1 : f(n, x) < 0\}| \\
&= |\{m < 1 : f(m - 1, \sigma(x)) \geq -n(x)\}| - |\{n \geq 1 : f(n - 1, \sigma(x)) < -n(x)\}| \\
&= |\{m < 0 : f(m, \sigma(x)) \geq -n(x)\}| - |\{n \geq 0 : f(n, \sigma(x)) < -n(x)\}|.
\end{align*}
\]

Suppose now that $n(x) \geq 0$. Then

\[
\begin{align*}
|\{m < 0 : f(m, \sigma(x)) \geq -n(x)\}| - |\{m < 0 : f(m, \sigma(x)) \geq 0\}| \\
&= |\{m < 0 : f(m, \sigma(x)) \geq -n(x)\}|, \text{ and} \\
|\{n \geq 0 : f(n, \sigma(x)) < 0\}| - |\{n \geq 0 : f(n, \sigma(x)) < -n(x)\}| \\
&= |\{n \geq 0 : f(n, \sigma(x)) \geq -n(x)\}|.
\end{align*}
\]
Since \( \{f(m, \sigma(x)) : m \in \mathbb{Z}\} = \mathbb{Z} \), we have
\[
\left| \{m < 0 : 0 > f(m, \sigma(x)) \geq -n(x)\} \right| + \left| \{n \geq 0 : 0 > f(n, \sigma(x)) \geq -n(x)\} \right| = n(x).
\]
Hence
\[
a(x) = \left| \{m < 0 : f(m, \sigma(x)) \geq -n(x)\} \right| - \left| \{n \geq 0 : f(n, \sigma(x)) < -n(x)\} \right| - 1
\]
\[
= \left| \{m < 0 : f(m, \sigma(x)) \geq 0\} \right| - \left| \{n \geq 0 : f(n, \sigma(x)) < 0\} \right| + n(x) - 1
\]
so \( n(x) = a(x) - a(\sigma(x)) + 1 \). A similar argument applies for \( n(x) < 0 \).

Similarly, \( b(x) := \left| \{m < 0 : f(m, x) \leq 0\} \right| - \left| \{n \geq 0 : f(n, x) > 0\} \right| \) defines a continuous function such that \( n(x) = b(x) - b(\sigma(x)) = 1 \) for \( x \in X_2 \).

**Proof of Theorem 5.4** The equivalence of (1) and (2) follows directly from Theorem 8.2

(3) \(\Rightarrow\) (1): Suppose \( h_1 : X_1 \to Y_1 \) and \( h_2 : X_2 \to Y_2 \) are homeomorphisms such that \( h_1(\sigma(x)) = \tau(h_1(x)) \) for \( x \in X_1 \) and \( h_2(\sigma(y)) = \tau^{-1}(h_2(y)) \) for \( y \in X_2 \).

Then \( \theta(x, y, z) = \begin{cases} (h_1(x), n, h_1(y)) & \text{if } x, y \in X_1, \\ (h_2(x), -n, h_2(y)) & \text{if } x, y \in X_2, \end{cases} \)
defines an isomorphism \( \theta : G(X, \sigma) \to G(Y, \tau) \).

(1) \(\Rightarrow\) (3): Let \( X_1 \) and \( X_2 \) be as in Lemma 9.3 and \( a \) and \( b \) be as in Lemma 9.4. Let \( Y_1 := h(X_1) \) and \( Y_2 := h(X_2) \). Define \( h_1 : X_1 \to Y_1 \) by \( h_1(x) = \tau^{a(x)}(h(x)) \) and \( h_2 : X_2 \to Y_2 \) by \( h_2(x) = \tau^{b(x)}(h(x)) \). Then \( h_1(\sigma(x)) = \tau^{a(\sigma(x))}(h(\sigma(x))) = \tau^{a(x)-f(1,x)+1}(h(\sigma(x))) = \tau^{a(\sigma(x))}(h(x)) = \tau(h_1(x)) \) for \( x \in X_1 \) because \( \tau^{f(1,x)}(h(x)) = h(\sigma(x)) \), and \( h_2(\sigma(x)) = \tau^{b(\sigma(x))}(h(\sigma(x))) = \tau^{b(x)-f(1,x)-1}(h(\sigma(x))) = \tau^{b(x)-1}(h(x)) = \tau^{-1}(h_1(x)) \) for \( x \in X_2 \) because \( \tau^{f(1,x)}(h(x)) = h(\sigma(x)) \).

10. **Equivariant Morita equivalence**

In this section we define equivalence of graded groupoids and equivariant Morita equivalence of nested pairs of \( C^* \)-algebras. We show that given coactions \( \delta_i : A_i \to A_i \otimes C^*_r(\Gamma) \) and abelian \( C^* \)-subalgebras \( D_i \subseteq A_i \) with \( D_i \) containing an approximate unit for \( A_i \), the pairs \((A_i, D_i) \) are equivariantly Morita equivalent if and only if their extended Weyl groupoids are graded equivalent.

Groupoids \( G_1 \) and \( G_2 \) are equivalent if there is a topological space \( Z \) carrying commuting free and proper actions of \( G_1 \) and \( G_2 \) on the left and right respectively such that \( r : Z \to G_1^{(0)} \) and \( s : Z \to G_2^{(0)} \) induce homeomorphisms \( G_1 \setminus Z \cong G_2^{(0)} \) and \( Z/G_2 \cong G_1^{(0)} \). The associated linking groupoid \( L(G_1, G_2) \) is
\[
L := L(G_1, G_2) = G_1 \sqcup Z \sqcup Z^{\text{op}} \sqcup G_2
\]
with \( L^{(0)} = G_1^{(0)} \cup G_2^{(0)} \), the obvious range and source maps, and multiplication determined by multiplication in \( G_1 \) and \( G_2 \), the actions of the \( G_i \) on \( Z \) and \( Z^{\text{op}} \), and the maps \( G_1^{[\cdot, \cdot]} : \{(z, y^{op}) \in Z \times Z^{op} : s(z) = r(y^{op})\} \to G_1 \) and \( [\cdot, \cdot]_{G_2} : \{(y^{op}, z) \in Z^{op} \times Z : s(y^{op}) = r(z)\} \to G_2 \) determined by \( G_i[z, y^{op}] : y = z \) and \( y^{[\cdot, \cdot]}_{G_2} : z \). Conversely, if \( G \) is a groupoid and \( K_1, K_2 \) are complementary \( G \)-full open subsets of \( G^{(0)} \), then \( Z := K_1 L K_2 \) is a \( K_1 G K_1 - K_2 G K_2 \)-equivalence under the actions given by multiplication in \( G \).
Definition 10.1. Let \( c_i : G_i \to \Gamma, i = 1, 2 \) be gradings of locally compact groupoids. A graded \((G_1, c_1)-(G_2, c_2)\) equivalence consists of a \( G_1-G_2\) equivalence \( Z \) and a continuous map \( c_Z : Z \to \Gamma \) satisfying \( c_Z(\gamma \cdot z \cdot \eta) = c_1(\gamma) c_Z(z) c_2(\eta) \) for all \( \gamma, z, \eta \).

Lemma 10.2. Let \( \Gamma \) be a discrete group. Then graded equivalence as described in Definition \[10.1\] is an equivalence relation on \( \Gamma \)-graded groupoids.

Proof. Suppose that \((G_i, c_i)\) is a \( \Gamma \)-graded groupoid for \( i = 1, 2, 3 \) and that \((Z, c_Z)\) is a \((G_i, c_i)-(G_{i+1}, c_{i+1})\)-equivalence for \( i = 1, 2 \). Define \( \sim \) on \( Z_1 \times s \cdot Z_2 := \{(z_1, z_2) \in Z_1 \times Z_2 : s(z_1) = r(z_2)\} \) by \((z_1 \cdot \gamma, \gamma_{-1} \cdot z_2) \sim (z_1, z_2) \) for \( \gamma \in (G_2)^{s(z_1)} \). By [36, page 6], \( Z_1 \ast_{G_2} Z_2 := (Z_1 \times s \cdot Z_2) / \sim \) is a \( G_1-G_3\) equivalence with \( \gamma_1 \cdot [z_1, z_2] = [\gamma_1 \cdot z_1, z_2] \) and \([z_1, z_2] \cdot \gamma_3 = [z_1, z_2 \cdot \gamma_3]\). For \([z_1, z_2] \in Z_1 \ast_{G_2} Z_2 \) and \( \gamma \in G_2^{s(z_1)} \), we have
\[
c_{Z_1}(z_1 \cdot \gamma)c_{Z_2}((\gamma_{-1} \cdot z_2) = c_{Z_1}(z_1)c_{Z_2}(\gamma)c_{Z_3}(z_2) = c_{Z_1}(z_1)c_{Z_2}(z_2),
\]
so there is a map \( \tilde{c} : Z_1 \ast_{G_2} Z_2 \to \Gamma \) such that \( \tilde{c}([z_1, z_2]) = c_{Z_1}(z_1)c_{Z_2}(z_2) \). So for \( \gamma_1 \in G_1, [z_1, z_2] \in Z_1 \ast_{G_2} Z_2 \), and \( \gamma_3 \in G_3 \) with \( s(\gamma_1) = r([z_1, z_2]) \) and \( s([z_1, z_2]) = r(\gamma_3) \), we have \( \tilde{c}(\gamma_1 \cdot [z_1, z_2] \cdot \gamma_3) = c_{Z_1}(\gamma_1 \cdot z_1)c_{Z_2}(z_2 \cdot \gamma_3) = c_{Z_1}(\gamma_1)c([z_1, z_2])c(\gamma_3). \) So \((Z_1 \ast_{G_2} Z_2, \tilde{c})\) is a \((G_1, c_1)-(G_3, c_3)\)-equivalence. \( \square \)

Lemma 10.3. Let \( \Gamma \) be a discrete group and let \((G_1, c_1), (G_2, c_2)\) be \( \Gamma \)-graded locally compact Hausdorff étale groupoids with each \( \text{Iso}(c_i^{-1}(\text{id}_\Gamma))^0 \) torsion-free and abelian. Suppose that \((Z, c_Z)\) is a graded equivalence from \((G_1, c_1)\) to \((G_2, c_2)\). Let \( G = L(G_1, G_2) \), and define \( c : G \to \Gamma \) by \( c|_{G_1} = c_1, c|_{G_2} = c_2 \) and \( c(z^n) = c_Z(z)^{-1} \) for \( z \in Z \). Then \((G, c)\) is a \( \Gamma \)-graded groupoid, \( \text{Iso}(c^{-1}(\text{id}_\Gamma))^0 \) is torsion-free and abelian, and \( (G^{(i)}_{G_1}G^{(i)}_{G_2}, c) \cong (G_1, c_1) \) for each \( i \). Conversely, given a \( \Gamma \)-graded groupoid \((G, c)\) such that \( \text{Iso}(c^{-1}(\text{id}_\Gamma))^0 \) is torsion-free and abelian, and given complementary open \( G \)-full sets \( K_1, K_2 \subseteq G^{(i)} \) such that each \( K_1 G K_i, c \cong (G_1, c_1) \), the pair \( (K_1 G K_2, c|_{K_1 G K_2}) \) is a \((G_1, c_1)-(G_2, c_2)\) equivalence under the left and right actions given by multiplication in \( G \).

Proof. Each \( c_Z(\gamma \cdot z \cdot \eta) = c_1(\gamma)c_Z(z)c_2(\eta) \) so \( c \) is a cocycle; and \( \text{Iso}(G)^0 \cap c^{-1}(\text{id}_\Gamma) = (\text{Iso}(G_1)^0 \cap c_1^{-1}(\text{id}_\Gamma)) \cup (\text{Iso}(G_2)^0 \cap c_2^{-1}(\text{id}_\Gamma)) \) is abelian and torsion free. As \((G^{(i)}_{G_1}G^{(i)}_{G_2}, c) \cong (G_1, c_1)\), the first statement follows. For the second, we saw that \( K_1 G K_2 \) is a \( G_1-G_2 \)-equivalence; and each \( c_Z(\gamma \cdot z \cdot \eta) = c(\gamma z \eta) = c(\gamma)c(z)c(\eta) = c_1(\gamma)c_Z(z)c_2(\eta) \). \( \square \)

We now turn to Morita equivalence of pairs \((A, D)\) where \( A \) is a \( C^* \)-algebra and \( D \subseteq A \) is a \( C^* \)-subalgebra. As in Section \[11\] if \( X \) is an \( A_1-A_2 \) imprimitivity bimodule, then \( X^* \) is its adjoint module and the linking algebra \( A = A_1 \oplus X \oplus X^* \oplus A_2 \) contains \( A_1, A_2 \) as complementary full corners. Writing \( P_i \) for \( 1_{M(A_i)} \in M(A) \), the multiplier module of \( X \) is \( M(X) := P_1 M(A) P_2 \) (see \[19\]); so \( M(X^*) \cong P_2 M(A) P_1 = M(X)^* \), and the adjoint operation in \( M(A) \) determines an extension to multiplier modules of the map \( \xi \mapsto \xi^* \) from \( X \) to \( X^* \).

Definition 10.4. Suppose that for \( i = 1, 2 \), \( A_i \) is a \( C^* \)-algebra carrying a coaction \( \delta_i \) of a discrete group \( \Gamma \), and \( D_i \subseteq A_i^{\delta_i} \) is a \( C^* \)-subalgebra. We say that \((A_1, D_1)\) and \((A_2, D_2)\) are equivariantly Morita equivalent if there are an \( A_1-A_2 \) imprimitivity bimodule \( X \) and a linear map \( \zeta : X \to M(X \otimes C^*_\Gamma(\Gamma)) \) such that: \( \zeta \) is a right-Hilbert bimodule morphism in the sense that \( \zeta(a \cdot x \cdot b) = \delta_1(a) \cdot \zeta(x) \cdot \delta_2(b) \) for all \( a, b \); we have \( \zeta(\text{id}_X) \circ \zeta = (\text{id}_X \otimes \delta_\Gamma) \circ \zeta \); and for each \( g \in \Gamma \), the subspace \( X_g := \{ x \in X : \zeta(x) = x \otimes \lambda_g \} \) satisfies
\[
(10.1) \quad X_g = \text{span}\{ \xi \in X_g : A_1 \langle \xi, \xi \cdot D_2 \rangle \leq D_1 \text{ and } \langle D_1 \cdot \xi, \xi \rangle A_2 \leq D_2 \}. 
\]
If $\Gamma = \{0\}$, we say that $(A_1, D_1)$ and $(A_2, D_2)$ are Morita equivalent.

The following is well known.

**Lemma 10.5.** Let $A_1, A_2$ be separable $C^*$-algebras and let $X$ be an $A_1$-$A_2$-imprimitivity bimodule. Then $X$ is separable.

**Proof.** Since $\langle \cdot, \cdot \rangle_{A_2}$ is full, and $A_2$ is separable there are sequences $(y_n), (z_n)$ in $X$ such that $(\langle y_n, z_n \rangle_{A_2})_n$ is dense in $A_2$. Cohen factorisation and the imprimitivity condition give $X = \text{span} \{ x \cdot \langle y_n, z_n \rangle_{A_2} : x \in X, n \in \mathbb{N} \} = \text{span} \{ A_1 \langle x, y_n \rangle \cdot z_n : x \in X, n \in \mathbb{N} \} \subseteq \text{span} \{ A_1 \cdot z_n : n \in \mathbb{N} \}$, which is separable because $A_1$ is.

We now state the main result of the section; the proof occupies the rest of the section.

**Theorem 10.6.** Let $\Gamma$ be a discrete group, and let $A_1, A_2$ be separable $C^*$-algebras. Suppose, for $i = 1, 2$, that $\delta_i$ is a coaction of $\Gamma$ on $A_i$ and $D_i \subseteq A_i^\delta$ is an abelian $C^*$-subalgebra containing an approximate unit for $A_i^\delta$. Suppose that $(X, \zeta)$ is an equivariant Morita equivalence between $(A_1, D_1)$ and $(A_2, D_2)$. Let $A$ be the linking algebra of $X$, let $D := D_1 \oplus D_2 \subseteq A$ and let $\delta : A \to A \otimes C^*_\Gamma(\Gamma)$ be the map that restricts to $\delta_i$ on each $A_i$, to $\zeta$ on $X$ and to $x^* \mapsto \zeta(x)^*$ on $X^*$. Then $\delta$ is a coaction and $D \subseteq A^\delta$ is an abelian $C^*$-subalgebra that contains an approximate unit for $A^\delta$. The sets $\hat{D}_i$ are complementary full clopen subsets of the unit space of $\mathcal{H} := \mathcal{H}(A, D, \delta)$, we have $(\hat{D}_i, \mathcal{H}\hat{D}_i, c_{\delta_i}) \cong (\mathcal{H}(A_i, D_i, \delta_i), c_{\delta_i})$ for $i = 1, 2$, and $(\hat{D}_1 \mathcal{H}\hat{D}_2, \delta)$ is an equivalence from $(\mathcal{H}(A_1, D_1, \delta_1), c_{\delta_1})$ to $(\mathcal{H}(A_2, D_2, \delta_2), c_{\delta_2})$.

**Lemma 10.7.** For $i = 1, 2$, let $A_i$ be a separable $C^*$-algebra and $D_i \subseteq A_i$ an abelian $C^*$-subalgebra containing an approximate unit for $A_i$. Let $X$ be an $A_1$-$A_2$-imprimitivity bimodule such that

$$X = \text{span} \{ x \in X : \langle D_1 \cdot x, x \rangle \leq A_2 \subseteq D_2 \text{ and } A_1 \langle x, x \cdot D_2 \rangle \subseteq D_1 \}.$$  

Let $A$ be the linking algebra of $X$. Then $D := D_1 \oplus D_2$ is an abelian $C^*$-subalgebra of $A$ and contains an approximate unit for $A$. The spaces $N_{A}(D_i) \subseteq A_i$ and

$$\{ \xi \in X : A_1 \langle \xi, \xi \cdot D_2 \rangle \subseteq D_1 \text{ and } A_1 \langle \xi, D_1 \cdot \xi \rangle \subseteq D_2 \}$$

are all contained in $N_{A}(D)$.

**Proof.** Fix approximate units $(u_j^1)_j$ for $A_i$ in $D_i$. Then $(u_j^1 \oplus u_j^2)_j$ is an approximate unit for $A$ in $D$, which is clearly an abelian $C^*$-subalgebra of $A$. Clearly each $N_{A_i}(D_i) \subseteq N_{A}(D)$.

If $A_1 \langle \xi, \xi \cdot D_2 \rangle \subseteq D_1$ and $A_1 \langle D_1 \cdot \xi, \xi \rangle \subseteq D_2$, then for $d_i \in D_i$, we have

$$\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} 0 & \xi^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \xi \cdot d_2 \\ 0 & 0 \end{pmatrix} \in D_1 \oplus D_2,$$

and a similar computation shows that

$$\begin{pmatrix} 0 & \xi^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \langle d_1^* \cdot \xi, \xi \rangle_{A_2} \end{pmatrix} \in D.$$

**Lemma 10.8.** Let $\Gamma$ be a discrete group. For $i = 1, 2$, let $\delta_i$ be a coaction of $\Gamma$ on a separable $C^*$-algebra $A_i$, and let $D_i \subseteq A_i^\delta$ be an abelian $C^*$-subalgebra containing an approximate unit for $A_i^\delta$. Suppose that $(X, \zeta)$ is an equivariant Morita equivalence between
Let $A$ be the linking algebra of $X$, let $D = D_1 \oplus D_2$, and define $\delta : A \to A \otimes C^*_\Gamma(\Gamma)$ by

$$\delta \left( \begin{array}{c} a_1 \\ \eta^* \\ a_2 \end{array} \right) = \left( \begin{array}{c} \delta_1(a_1) \\ \zeta(\eta^*) \\ \delta_2(a_2) \end{array} \right).$$

Then $\delta$ is a coaction of $\Gamma$ on $A$, $D$ is an abelian $C^*$-subalgebra of $A^\delta$ and contains an approximate unit for $A^\delta$, and $D_{A^\delta} = (D_1)^{\delta_1} A_1^{\delta_1} \oplus (D_2)^{\delta_2} A_2^{\delta_2}$. For $i = 1, 2$, the pair $(P_i AP_1, P_i D_i)$ is equivariantly isomorphic to $(A_i, D_i)$. We have $P_i AP_2 = \text{span}\{P_n P_2 : n \in N_*(D)\}$.

**Proof.** It is routine to check that $\delta$ is a coaction and that $D$ is an abelian $C^*$-subalgebra of $A^\delta$ and contains an approximate unit for $A^\delta$. Clearly $(D_1)^{\delta_1} \oplus (D_2)^{\delta_2} \subseteq D^\prime_{A^\delta}$. For the reverse, fix $(a_1 \delta \xi \cdot a_2) \in D^\prime_{A^\delta}$. Then

$$\left( \begin{array}{c} a_1 \\ \eta^* \\ a_2 \end{array} \right) \otimes 1_{C^*_\Gamma(\Gamma)} = \left( \begin{array}{c} a_1 \\ \eta^* \\ a_2 \end{array} \right) \otimes 1_{C^*_\Gamma(\Gamma)} = \delta \left( \begin{array}{c} a_1 \\ \eta^* \\ a_2 \end{array} \right) = \left( \begin{array}{c} \delta_1(a_1) \\ \zeta(\eta^*) \\ \delta_2(a_2) \end{array} \right),$$

so each $a_i \in A_{i}^{\delta_i}$, and for $d_i \in D_i$,

$$\left( \begin{array}{c} a_1 d_1 \\ \eta^* \cdot d_1 \\ a_2 d_2 \end{array} \right) = \left( \begin{array}{c} a_1 \xi \\ \eta^* \cdot d_2 \\ a_2 \end{array} \right) \left( \begin{array}{c} d_1 \\ 0 \\ d_2 \end{array} \right) = \left( \begin{array}{c} d_1 a_1 \\ d_2 \eta^* \cdot d_2 a_2 \end{array} \right) \left( \begin{array}{c} d_1 \xi \\ d_2 \eta^* \cdot d_2 a_2 \end{array} \right),$$

so each $a_i \in (D_i)^{\delta_i}$. Moreover, $d_1 \cdot \xi = 0$ for all $d_1 \in D_1$, so $\eta^* = 0$ by Cohen factorisation, and similarly $\eta^* = 0$. So $(a_1 \delta \xi \cdot a_2) = (0 \delta \eta^* \cdot a_2) \in (D_1)^{\delta_1} \oplus (D_2)^{\delta_2}$. Each $(P_i AP_1, P_i D_i, \delta_{P_i AP_1}) \cong (A_i, D_i, \delta_i)$ by construction, and Lemma 10.7 gives

$$\{\xi \in X_g : A_i(\xi, \cdot, D_2) \subseteq D_1, (D_1 \cdot \xi, \cdot) A_2 \subseteq D_2 \} \subseteq N_*(D)$$

for all $g \in G$.

Each $X_g \subseteq X = P_1 AP_2$, so $P_1 AP_2 = \text{span}\{P_n P_2 : n \in N_*(D)\}$ by (10.1). □

We call the system $(A, D, \delta)$ of Lemma 10.8 the linking system for $(X, \zeta)$. The elements $P_i := 1_{M(D_i)} = 1_{M(A_i)}$ are complementary full multiplier projections of $A$ and $(P_i AP_1, P_i D_i, \delta_{P_i AP_1}) \cong (A_i, D_i, \delta_i)$ for $i = 1, 2$. So the $(A_i, D_i, \delta_i)$ are complementary full subsystems of their linking system. The converse requires additional hypotheses.

**Lemma 10.9.** Let $A$ be a separable $C^*$-algebra, $\delta$ a coaction of a discrete group $\Gamma$ on $A$, and $D \subseteq A$ an abelian $C^*$-subalgebra containing an approximate unit for $A^\delta$. Suppose that $P_1, P_2 \in M(D)$ are complementary full projections. Then for each $i$, we have that $\delta_i := \delta_{P_i AP_1}$ is a coaction of $\Gamma$ on $A_i := P_i AP_1$ and $D_i := P_i D$ is an abelian $C^*$-algebra of $A_i^{\delta_i}$ and contains an approximate unit for $A_i^{\delta_i}$. If $P_i AP_2 = \text{span}\{P_n P_2 : n \in N_*(D)\}$, then $(P_i AP_1, P_i D_i, \delta_{P_i AP_1})$ is an equivariant Morita equivalence from $(A_i, D_i)$ to $(A_2, D_2)$.

**Proof.** Let $A_i^{\delta_i} := P_i A^\delta P_i$. Fix an approximate unit $(u_i)_{j_i}$ for $A$ in $D$. It is obvious that $D_i$ is a $\ast$-subalgebra of $A_i^{\delta_i}$, that $(P_i u_j)_{j_i}$ is an approximate unit for $A_i^{\delta_i}$ in $D_i$, and that $\delta_i$ is a $\ast$-homomorphism. Since $P_i \in M(D_i) \subseteq M(A_i^{\delta_i})$, the extension $\tilde{\delta}$ of $\delta$ to $M(A_i^{\delta_i})$ satisfies $\tilde{\delta}(P_i) = P_i \otimes 1_{C^*_\Gamma(\Gamma)}$, so for $g \in \Gamma$ and $a \in A_i$, we have $\delta(P_i a P_i) = \tilde{\delta}(P_i)(a \otimes \lambda_g) \tilde{\delta}(P_i) = P_i a P_i \otimes \lambda_g \in A_i \otimes C^*_\Gamma(\Gamma)$. It follows that $\delta_i$ is a coaction of $\Gamma$ on $A_i$ with generalised fixed-point algebra $A_i^{\delta_i}$.

It is well known that $P_i AP_2$ is an $A_1-A_2$-imprimitivity bimodule. By definition of the inner products, $P_i AP_2 = \text{span}\{P_n P_2 : n \in N_*(D)\}$ if and only if

$$(P_i AP_2)_g = \text{span}\{\xi \in (P_i AP_2)_g : A_i(\xi, \cdot, D_2) \subseteq D_1, (D_1 \cdot \xi, \cdot) A_2 \subseteq D_2\}$$
for each $g \in \Gamma$. That $\delta|_{P_1AP_1}(a \cdot \xi \cdot b) = \delta_1(a) \cdot \delta|_{P_1AP_2}(\xi) \cdot \delta_2(b)$ for all $\xi, a, b$ is just the homomorphism property of $\delta$. That $(\delta|_{P_1AP_2} \otimes \text{id}_\Gamma) \circ \delta|_{P_1AP_2} = (\text{id}_{P_1AP_2} \otimes \delta_\Gamma) \circ \delta|_{P_1AP_2}$ follows from the coaction identity for $\delta$. \hfill $\Box$

**Corollary 10.10.** Let $\Gamma$ be a discrete group. For $i = 1, 2$, let $\delta_i$ be a coaction of $\Gamma$ on a separable $C^*$-algebra $A_i$, and let $D_i \subseteq A_i^\delta$ be an abelian $C^*$-subalgebra containing an approximate unit for $A_i^\delta$ such that $\overline{\text{span}}_{\mathbb{N}}(D_i) = A_i$. Then $(A_1, D_1)$ and $(A_2, D_2)$ are equivariantly Morita equivalent if and only if there exist a separable $C^*$-algebra $A$, a coaction $\delta$ of $\Gamma$ on $A$, an abelian $C^*$-subalgebra $D \subseteq A^\delta$ that contains an approximate unit for $A^\delta$ such that $\overline{\text{span}}_{\mathbb{N}}(D) = A$, a pair of complementary full projections $P_1, P_2 \in M(D)$, and isomorphisms $\phi_i: P_iAP_i \to A_i$ such that

$$\phi_i(P_iDP_i) = D_i \quad \text{and} \quad \delta_i \circ \phi_i = (\phi_i \otimes \text{id}_{\mathcal{C}_0(\Gamma)}) \circ \delta|_{P_iAP_i} \quad \text{for} \quad i = 1, 2.$$ 

**Proof.** We have $P_2AP_1 = (P_1AP_2)^* \subseteq N_i(D)$, and each $P_iAP_i \subseteq A \subseteq \overline{\text{span}}_{\mathbb{N}}(D_i)$. So Lemma [10.8] gives “only if,” Lemma [10.9] gives “if” because $A = \overline{\text{span}}_{\mathbb{N}}(D)$ forces $P_iAP_i = \overline{\text{span}}\{P_i n P_i : n \in N_i(D)\} \subseteq N_i(D_i)$. \hfill $\Box$

**Lemma 10.11.** Let $\Gamma$ be a discrete group. For $i = 1, 2$, let $\delta_i$ be a coaction of $\Gamma$ on a separable $C^*$-algebra $A_i$, and let $D_i \subseteq A_i^\delta$ be an abelian $C^*$-subalgebra containing an approximate unit for $A_i^\delta$. Suppose that $(X, \xi)$ is an equivariant $(A_1, D_1) - (A_2, D_2)$-equivalence. Let $(A, D, \delta)$ be the linking system. Fix $g \in \Gamma$ and $n = (a_1, a_2) \in N_g(D)$. Then $a_1^* \cdot \xi = \xi \cdot a_2^* = a_1 \cdot \eta = \eta \cdot a_2 = 0$. Moreover, $a_1 \in N_{g^*}(D_1)$, $a_2 \in N_g(D_2)$, and $(D_1 \cdot \xi, \xi), A_2 \cup (D_1 \cdot \eta, \eta), A_2 \subseteq D_2$, and $A_1(\xi \cdot D_2, \xi) \cup A_1(\eta \cdot D_2, \eta) \subseteq D_2$.

**Proof.** Let $d_1 \in D_1$ and let $d_2 \in D_2$. Then

$$n^*(d_1 + 0)n = \begin{pmatrix} a_1^*d_1a_1 & a_1^*d_1 \cdot \xi \\ \xi^* \cdot d_1a_1 & \langle \xi^*d_1, \xi, A_2 \rangle \end{pmatrix}, \quad n^*(0 + d_2)n = \begin{pmatrix} A_1(\eta, \eta, d_2) & \eta \cdot d_2a_2 \\ a^*_2d_2 \cdot \eta^* & a^*_2d_2a_2 \end{pmatrix},$$

$$n(d_1 + 0)n^* = \begin{pmatrix} a_1^*d_1a_1^* & a_1^*d_1 \cdot \eta \\ \eta^* \cdot d_1^*a_1^* & \langle \eta^*d_1^*, \eta, A^*_2 \rangle \end{pmatrix}, \quad n(0 + d_2)n^* = \begin{pmatrix} A_1(\xi, \xi, d_2) & \xi \cdot d_2a_2^* \\ a_2^*d_2 \cdot \xi^* & a_2^*d_2a_2^* \end{pmatrix}.$$

That $n^*(d_1 + 0)n, n(d_1 + 0)n^* \in D$, gives $a_1 \in N(D_1)$ and $(a_1^*d_1) \cdot \xi = (\xi d_2) \cdot a_2 = 0$. Taking the limit as the $d_i$ range over approximate identities for the $A_i$ proves the first assertion. Since $n \in N_g(D)$ we have

$$\begin{pmatrix} a_1 \otimes \lambda_g & \xi \otimes \lambda_g \\ \eta^* \otimes \lambda_g & a_2 \otimes \lambda_g \end{pmatrix} = n \otimes \lambda_g = \delta(n) = \begin{pmatrix} \delta_1(a_1) & \zeta(\xi) \\ \zeta(\eta) & \delta_2(a_2) \end{pmatrix},$$

so $a_1 \in N_{g^*}(D_1)$. Since $n^*(d_1 + 0)n, n(d_1 + 0)n^* \in D_1 + D_2$, we have $\langle d_1^* \cdot \xi, \xi, A_2 \rangle, \langle d_1^* \eta, \eta, A_2 \rangle \subseteq D_2$ and $A_1(\eta, \eta, d_2), A_1(\xi, \xi, d_2) \subseteq D_1$, which proves the second assertion. \hfill $\Box$

In what follows, we sometimes deal with Weyl groupoids for multiple triples $(A, D, \delta)$. If $\mathcal{H} = \mathcal{H}(A, D, \delta)$ is a Weyl groupoid, and if $n \in N_{\mathbb{N}}(D) \subseteq A$ and $\phi \in \text{supp}^\circ(n^*n) \subseteq \overline{D}$, we write $[n, \phi]_\mathcal{H}$ for the corresponding element of $\mathcal{H}$.

**Lemma 10.12.** Let $\Gamma$ be a discrete group. For $i = 1, 2$, let $\delta_i$ be a coaction of $\Gamma$ on a separable $C^*$-algebra $A_i$, and let $D_i \subseteq A_i^\delta$ be an abelian $C^*$-subalgebra containing an approximate unit for $A_i^\delta$. Suppose that $(X, \xi)$ is an equivariant $(A_1, D_1) - (A_2, D_2)$-equivalence. Let $(A, D, \delta)$ be the linking system. Let $\mathcal{H} = \mathcal{H}(A, D, \delta)$. Suppose that

$$n = \begin{pmatrix} n_1 & \xi \\ \eta^* & n_2 \end{pmatrix} \in N_g(D) \quad \text{where} \quad n_1 \in A_1, n_2 \in A_2, \text{and} \xi, \eta \in X.$$
Fix \( i \in \{1, 2\} \), and suppose that \( \phi \in \text{supp}^g(n^*n) \) satisfies \([n, \phi]_H \in \tilde{D}_i \mathcal{H} \tilde{D}_i\). Then \( \phi \in \text{supp}^g(n^*n_i) \), and \([n, \psi]_H = [n, \psi]_H \) for all \( \psi \in \text{supp}^g(n^*n_i) \).

**Proof.** By symmetry, it suffices to prove the result for \( i = 1 \). By Lemma \[10.11\], we have \( n_1 \in N_g(D_1) \) and \( n_2 \in N_g(D_2) \). Since \([n, \phi]_H \in \tilde{D}_1 \mathcal{H} \tilde{D}_1\), we have \( \phi, \alpha_n(\phi) \in \tilde{D}_1 \).

To see that \( \phi \in \text{supp}^g(n^*n_1) \), fix \( d_1 \in D_1 \) with \( \alpha_n(\phi)(d_1) = 1 \). Then

\[
0 < \phi(n^*n) = \alpha_n(\phi)(d_1)\phi(n^*n) = \phi(n^*d_1n)
\]

\[
\phi\left( n^*d_1n_1 \frac{\xi^* \cdot d_1 \xi}{\langle \xi, d_1 \xi \rangle_{\mathcal{A}_2}} \right) = \phi(n^*d_1n_1) = \alpha_n(\phi)(d_1)\phi(n^*n_1).
\]

Since Lemma \[10.11\] gives \( n_1 \in N_g(D_1) \), we have \( n^*n \in A^\delta \). Let \( U := \text{supp}^g(n_1^*n_1) \subseteq \text{supp}^g(n^*n) \) and \( \psi \in U \). Then \( \psi \in \tilde{D}_1 \), and therefore \( \psi \) vanishes on \( D_2 \). So for \( d = (d_1, 0) \in \tilde{D}_+ \) with \( \alpha_n(\psi)(d) = 1 \),

\[
\alpha_n(\psi)(d)\psi(n^*n) = \psi(n^*d) = \psi\left( n^*d_1n_1 + A_1(\eta \cdot d_2, \eta) \right) = \psi(n^*d_1n_1 + A_1(\eta \cdot d_2, \eta)) \geq 0.
\]

Since \( \psi(n^*n) = \psi(n^*n_1) \) and \( D_+ \) separates points in \( \tilde{D} \), this gives \( \alpha_n(\psi) = \alpha_n(\psi) \), so \( \alpha_n \) and \( \alpha_n \) agree on \( U \). To see that \([n, \psi]_H = [n_1, \psi]_H \) it remains only to establish (R4). We have \( n^*n_1 = \left( n^*n_1, 0 \right) \left( n^*n_1, 0 \right) \). Thus for \( d \in D \) such that \( \psi(d) = 1 \) and \( \text{supp}^g(d) \subseteq U \), we have \( \pi_n((\psi(n^*n_1)^{-1/2}(\psi(n^*n_1)))^{-1/2}(dn^*n_1d)) = 1_{A_1^\delta} \). Hence, \([n, \psi]_H = [n_1, \psi]_H \). □

**Proof of Theorem \[10.3\]**. We have that \( \delta \) is a coaction and \( D \subseteq A^\delta \) is an abelian \( C^* \)-subalgebra containing an approximate unit for \( A^\delta \) by Lemma \[10.8\]. Clearly \( \tilde{D}_i \) are complementary open subsets of \( \tilde{D} \). We show that \( \tilde{D}_1 \) is full \( (D_2 \) is full by a symmetric argument). Fix \( \phi \in \tilde{D}_2 \). Since \( X \) is full in \( A_2 \), there exists \( \xi \in X \) such that \( \langle \xi, \xi \rangle_{\mathcal{A}_2} \in D_2 \) and \( \phi(\langle \xi, \xi \rangle_{\mathcal{A}_2}) \neq 0 \). Lemma \[10.8\] shows that \( \delta \) is a coaction, so as in Section \[1\] we have \( A = \text{span} \bigcup_{g \in G} A_g \). Let \( P_i = 1_{M(D_i)} \in M(A) \). Then \( X = P_1 P_2 = \text{span} \bigcup_{g \in G} X_g \). Hence \[10.11\] shows that

\[
X = \text{span} \bigcup_{g \in G} \{ \eta \in X_g : A_1(\eta \cdot D_2 \mathcal{A}_2) \subseteq D_1 \text{ and } \langle D_1 \cdot \eta, \eta \rangle_{\mathcal{A}_2} \subseteq D_2 \}.
\]

In particular, we can approximate \( \xi \) by an element \( \sum_j \eta_j \) with each \( \eta_j \in N_*(D) \) satisfying \( \eta_j^* D_2 \eta_j \subseteq D_1 \). Since \( \phi(\langle \xi, \xi \rangle_{\mathcal{A}_2}) \neq 0 \), we may assume that there exist \( j, k \) such that \( \phi(\langle \eta_j, \eta_k \rangle_{\mathcal{A}_2}) \neq 0 \). Using that \( X \) is an imprimitivity bimodule at the last step, we calculate:

\[
0 \neq \phi(\langle \eta_j, \eta_k \rangle_{\mathcal{A}_2})^2 = \phi(\langle \eta_j, \eta_k \rangle_{\mathcal{A}_2} \langle \eta_k, \eta_j \rangle_{\mathcal{A}_2}) = \phi(\langle \eta_j, A_1 \langle \eta_j, \eta_k \rangle_{\mathcal{A}_2} \rangle_{\mathcal{A}_2})
\]

Rewriting this in terms of multiplication in \( A \), we obtain \( \phi(\eta_j^* \eta_k^* \eta_k^* \eta_j) \neq 0 \), and therefore \( \eta := \eta_j \) satisfies \( \phi(\eta \eta^* \alpha_n(\phi)(\eta \eta^*) \neq 0 \). In particular, \( \eta \in N_*(D) \) and \( \phi \in \text{dom}(\alpha_n) \). Since \( \eta D_2 \eta \subseteq D_1 \), we have \( r(\eta, \phi) = \alpha_n(\phi) \in \tilde{D}_1 \). So \( \phi = s(\eta, \phi) \in s(\tilde{D}_1 \mathcal{H}) \).

We must show that each \( \tilde{D}_i \mathcal{H} \tilde{D}_i \cong \mathcal{H}(A_1, D_1, \delta_i) \). It suffices to do this for \( i = 1 \). Let \( j_1 : A_1 \rightarrow A \) be the inclusion map. For \( \phi \in \tilde{D}_1 \), let \( \overline{\phi} \) be the extension of \( \phi \) to \( D \) given by \( \overline{\phi}(d_1) = \phi(d_1) \) for \( d_1 \in D_1 \) and \( \overline{\phi}(d_2) = 0 \) for \( d_2 \in D_2 \).

We claim that there is a map \( \Theta : \mathcal{H}(A_1, D_1, \delta_1) \rightarrow \tilde{D}_1 \mathcal{H}(A, D, \delta) \tilde{D}_1 \) such that

\[
\Theta([n, \phi]_{\mathcal{H}_1}) = [j_1(n), \overline{\phi}]_{\mathcal{H}} \quad \text{for all } [n, \phi] \in \mathcal{H}(A, D, \delta).
\]
For this, suppose that \([n, \phi]_{H_1} = [m, \psi]_{H_1}\). Then \(\phi = \psi\) by definition of \(\sim\) in \((A_1, D_1, \delta_1)\), so \((j_1(n), \overline{\phi})\) and \((j_1(m), \overline{\psi})\) satisfy (R1). Since \(n^*m \in A_1^\delta\), we have \(n, m \in N_\delta(D_1)\) for some \(g\). Hence \(\delta(j_1(n)) = j_1(n) \otimes \lambda_g\) and \(\delta(j_1(m)) = j_1(m) \otimes \lambda_g\), so \(j_1(n)^*j_1(m) \in A_1^\delta\), giving (R2). Since \((n, \phi), (m, \psi)\) satisfy (R3), there is a neighbourhood \(U \subseteq \overline{D_1}\) of \(\phi\) such that \(U \subseteq \text{supp}^\delta(n^*m) \cap \text{supp}^\delta(m^*n)\), and \(\alpha_n|_U = \alpha_m|_U\). Since \(\widehat{D}_1\) is open in \(\overline{D}_1\), the set \(\overline{U} := \{\psi : \psi \in U\}\) is a neighbourhood of \(\overline{\phi}\) in \(\overline{\Theta}\). Lemma 10.12 gives \(\alpha_{j_1(n)}|_{\overline{U}} = \alpha_m|_{\overline{U}}\). So \((j_1(n), \overline{\phi})\) and \((j_1(m), \overline{\psi})\) satisfy (R3). Lemma 10.8 gives \(D_{\phi}^j = (D_1)_A^{j_2} \circ (D_2)_A^{j_2}.\) Since \(\overline{\delta} \circ \overline{\psi} = (j_1 \otimes \text{id}_{C_1(\Gamma)}|J_1) \circ \delta_1\), the corresponding ideal \(J_{\phi} \subseteq (D_1)_A^{j_2}\) satisfies \(J_{\phi} = j_1(J_{\phi}) \subseteq (D_2)_A^{j_2}\). Hence the projection map \(p_1 : D_{\phi}^j \to (D_1)_A^{j_2}\) is the element of \(\overline{U}\) that \(\overline{U}\) is a groupoid homomorphism. Thus \(\Theta\) is a groupoid homomorphism.

11. Equivalence of graded groupoids and equivariant Morita equivalence

In this section we show how equivalence of graded groupoids relates to equivariant Morita equivalence of pairs \((A, D)\). Our main result in this direction is the following.

**Theorem 11.1.** Let \(\Gamma\) be a discrete group and let \((G_1, c_1), (G_2, c_2)\) be \(\Gamma\)-graded second-countable locally compact Hausdorff étale groupoids, and suppose that each \(\text{Iso}(c_1^{-1}(\text{id}_{\Gamma}))\) is torsion-free and abelian. The following are equivalent:

- \((G_1, c_1)\) and \((G_2, c_2)\) are Morita equivalent.
- \((G_1, c_1)\) and \((G_2, c_2)\) are 

**Proof:**

...
(1) the graded groupoids \((G_1, c_1)\) and \((G_2, c_2)\) are graded equivalent;
(2) there exist a second-countable locally compact Hausdorff étale groupoid \(G\), a grading \(c\) of \(G\) by \(\Gamma\) such that \(\text{Iso}(c^{-1}(\text{id}_\Gamma))^\circ\) is torsion-free and abelian, a pair of complementary open \(G\)-full subsets \(K_1, K_2 \subseteq G^{(0)}\), and isomorphisms \(\kappa_1 : K_1 G K_1 \to G_1\) and \(\kappa_2 : K_2 G K_2 \to G_2\) such that \(c_i \circ \kappa_i = c|_{K_i G K_i}\);
(3) \((C_r^*(G_1), C_0(G_1^{(0)}))\) and \((C_r^*(G_2), C_0(G_2^{(0)}))\) are equivariantly Morita equivalent; and
(4) there exist a separable \(C^*\)-algebra \(A\), a coaction \(\delta\) of \(\Gamma\) on \(A\), an abelian \(C^*\)-subalgebra \(D \subseteq A^\delta\) containing an approximate unit for \(A^\delta\) such that \(\text{supp}(N)(D) = A\), a pair of complementary full projections \(P_1, P_2 \in M(D) \subseteq M(A)\), and isomorphisms \(\phi_i : P_i AP_i \to C^*_r(G_i)\) such that \(\phi_i(P_i DP_i) = D_i\) and \(\delta_i \circ \phi_i = (\phi_i \otimes \text{id}_{C^*_r(\Gamma)}) \circ \delta|_{P_i AP_i}\).

**Proof.** Lemma \[10.3\] gives \((1) \iff (2)\), Corollary \[10.10\] gives \((3) \iff (4)\), and Theorem \[11.10\] gives \((2) \implies (1)\). Let \((G, c)\), \(K_1\) and \(\kappa_i\) be as in \((2)\). Lemma \[6.1\] shows that \(C_0(G^{(0)})\) is an abelian \(C^*\)-subalgebra of \(C^*_r(G)^{\delta(c)}\) and contains an approximate unit for \(C^*_r(G)^{\delta(c)}\). Theorem 4.1 of [18] shows that \(P_i := 1_{K_i} \in M(C_0(G^{(0)}))\) defines complementary full projections, and the inclusions \(C_c(G_i) \hookrightarrow C_c(G)\) induce isomorphisms \(\phi_i : P_i C_c^*(G)P_i \to C_c^*(G_i)\) that carry \(P_i C_0(G^{(0)})\) to \(C_0(G^{(0)})\). We have \(P_i C_c^*(G)P_2 = C_c(G_1^{(0)}G_2^{(0)}) \subseteq C_c^*(G)\). Since the \(G_i\) are étale, the Haar system of [18, Lemma 2.2] consists of counting measures, so \(G\) is also étale. Hence Lemma \[6.7\] gives \(P_i C_c^*(G)P_2 = \text{supp}(P_i n P_2) : n \in N_\epsilon(C_0(G^{(0)}))\).Fix \(i \in \{1, 2\}\). For \(g \in \Gamma\) and \(f \in C_c(G_i)\), \(\text{supp}(\phi_i(f)) = \text{supp}(f) \subseteq c_i^{-1}(g)\), and so \(\delta_i(\phi_i(f)) = \phi_i(f) \otimes \lambda_g = (\phi_i \otimes \text{id}_{C_c^*(\Gamma)})(f \otimes \lambda_g) = (\phi_i \otimes \text{id}_{C_c^*(G_i)})(\delta_i(f))\). Since \(P_i C_c^*(G)P_i = \text{supp} \bigcup_g C_c(G_i)\), we obtain \(\delta_i \circ \phi_i = (\phi_i \otimes \text{id}_{C_c^*(G_i)}) \circ \delta|_{P_i AP_i}\). ✷

In some situations the following stronger form of equivalence than the one considered in Theorem \[11.1\] is interesting (see Corollaries \[11.3\] and \[11.5\]). If \(\Gamma\) is the trivial group, then Theorems \[11.1\] and \[11.2\] both reduce to Theorem \[8.3\].

**Theorem 11.2.** Let \(\Gamma\) be a locally compact group, let \(G_1\) and \(G_2\) be second-countable locally compact Hausdorff étale groupoids, and let \(c_i : G_i \to \Gamma\) be continuous cocycles such that \(\text{Iso}(c_i^{-1}(\text{id}_\Gamma))^\circ\) is torsion-free and abelian. The following are equivalent:

1. there is a graded \((G_1, c_1)\)–\((G_2, c_2)\)-equivalence \((Z, c_Z)\) such that \(c_Z^{-1}(e) = c_1^{-1}(e) = c_2^{-1}(e)\)-equivalence;
2. there is a second-countable locally compact Hausdorff étale groupoid \(G\), a grading \(c\) of \(G\) by \(\Gamma\) such that \(\text{Iso}(c^{-1}(\text{id}_\Gamma))^\circ\) is torsion-free and abelian, a pair of complementary open \(c^{-1}(\text{id}_\Gamma)\)-full subsets \(K_1, K_2 \subseteq G^{(0)}\), and isomorphisms \(\kappa_1 : K_1 G K_1 \to G_1\) and \(\kappa_2 : K_2 G K_2 \to G_2\) such that \(c_i \circ \kappa_i = c|_{K_i G K_i}\);
3. there exists an equivariant \((C^*_r(G_1), C_0(G_1^{(0)}))\)–\((C^*_r(G_2), C_0(G_2^{(0)}))\)-imprimitivity bimodule \((X, \varsigma)\) such that \(X_{\text{id}_\Gamma} = C^*_r(G_1)^{\delta(c_1)} - C^*_r(G_2)^{\delta(c_2)}\)-imprimitivity bimodule; and
4. there exists a separable \(C^*\)-algebra \(A\), a coaction \(\delta\) of \(\Gamma\) on \(A\), an abelian \(C^*\)-subalgebra \(D \subseteq A^\delta\) containing an approximate unit for \(A^\delta\) such that \(\text{supp}(N)(D) = A\), a pair of complementary \(A^\delta\)-full projections \(P_1, P_2 \in M(D)\), and isomorphisms \(\phi_i : P_i AP_i \to C_r^*(G_i)\) such that \(\phi_i(P_i DP_i) = D_i\) and \(\delta_i \circ \phi_i = (\phi_i \otimes \text{id}_{C^*_r(\Gamma)}) \circ \delta|_{P_i AP_i}\).

**Proof.** \((1) \implies (2)\). By Lemma \[10.3\] \(G := L(G_1, G_2)\) and \(c : G \to \Gamma\) given by \(c|_{G_1} = c_1\), \(c|_{G_2} = c_2\) and \(c(z^{op}) = c_2(z)^{-1}\) for \(z \in Z\) constitute a graded groupoid with \(\text{Iso}(c^{-1}(\text{id}_\Gamma))^\circ \cong \text{Iso}(c_1^{-1}(\text{id}_\Gamma))^\circ \sqcup \text{Iso}(c_2^{-1}(\text{id}_\Gamma))^\circ\) torsion-free and abelian, and each
The proof of Corollary 11.1.10 gives (3) $\iff (4)$, and Theorem 11.1.6 gives (3) $\implies (4)$. For (2) $\implies (1)$, the proof of (2) $\implies (1)$ in Theorem 11.1.4 shows that $P_i := 1_{K_i}$ defines full multiplier projections in $M(C^*_r(G))$ such that $P_i C^*_r(G) P_i$ is equivariantly isomorphic to $C^*_r(G_i)$. To see that $P_1$ is $C^*_r(G)$-full, we show that $C_{c_1}r(G_1) \subseteq C_{r}(G)\delta 1 P_i C^*_r(G)\delta 1$. It suffices to show that for each $x \in G$ there exists $a \in C^*_r(G)\delta 1 P_i C^*_r(G)\delta 1 \cap C_0(G_0)$ such that $a(x) > 0$. This is clear for $x \in K_1$, so fix $x \in K_2$. Since $K_1$ is $c^1$(id$_\Gamma$)-full there is an open bisection $U \subseteq c^1$(id$_\Gamma$) with $x \in r(U)$ and $s(U) \subseteq K_1$. Write $U \cap r^{-1}(x) = \{\gamma\}$. Fix $f \in C_c(U)$ with $f(\gamma) = 1$. Then $f f^* = f P f^* \in C_0(K_2) \cap C^*_r(G)\delta 1 P_i C^*_r(G)\delta 1$, and $f f^*(x) = 1$. Symmetry shows that $P_2$ is also full.

Next we specialise to ample graded groupoids. Recall that $\mathcal{R}$ is the discrete groupoid $\mathbb{N} \times \mathbb{N}$, and $\mathcal{C}$ denotes the canonical diagonal in $C^*(\mathcal{R}) \cong \mathcal{K}$. Given a grading $c : G \to \Gamma$ of a groupoid $G$, we define $\bar{c} : G \times \mathcal{R} \to \Gamma$ by $\bar{c}(\eta_1, \eta_2) = c(\eta_1)$.

As in [15], we say $G_1$ and $G_2$ are weakly Kakutani $c_1$–$c_2$ equivalent if there are open $c^1$(id$_\Gamma$)-full subsets $X_i \subseteq G_i$ and an isomorphism $\kappa : X_1 G_1 X_1 \to X_2 G_2 X_2$ such that $c_2 \circ \kappa = c_1$ on $X_1 G_1 X_1$. As in [35], if $X_1, X_2$ are clopen, we say $G_1$ and $G_2$ are Kakutani $c_1$–$c_2$ equivalent. Theorem 11.2.2 combined with the results of [15] gives the following.

**Corollary 11.3.** Let $\Gamma$ be a discrete group, and let $(G_1, c_1), (G_2, c_2)$ be second-countable $\Gamma$-graded ample Hausdorff groupoids such that each $\text{Iso}(c^1_{c_1}(\text{id}_\Gamma))^0$ is torsion-free and abelian. The following are equivalent:

1. there exists a graded $(G_1, c_1)$–$(G_2, c_2)$-equivalence $(Z, c_2)$ such that $c^2_{c_1}(\text{id}_\Gamma)$ is a $c^1_{c_1}(\text{id}_\Gamma)$–$c^2_{c_1}(\text{id}_\Gamma)$-equivalence;
2. there exist $\chi : G \to Z$ an $\epsilon$-equivalence of $\text{Iso}(c_1^{-1}(\text{id}_\Gamma))^{-v}$ is torsion-free and abelian, a pair of complementary clopen $c^1(\text{id}_\Gamma)$-full subsets $X_1, X_2 \subseteq G$, and isomorphisms $\kappa_i : X_i G X_i \to G_i$ such that $X_1 \cup X_2 = \mathcal{C}$, and each $c_2 \circ \kappa_i |_{X_i G X_i} = c|_{X_i G X_i}$;
3. there is an isomorphism $\kappa : G_1 \times \mathcal{R} \to G_2 \times \mathcal{R}$ such that $\bar{c}_2 \circ \kappa = \bar{c}_1$;
4. $G_1$ and $G_2$ are Kakutani $c_1$–$c_2$ equivalent;
5. $G_1$ and $G_2$ are weakly Kakutani $c_1$–$c_2$ equivalent;
6. there is an equivariant $(C^*_r(G_1), C_0(G_1)^0, C^*_r(G_2), C^*_r(G_2)^0))$-imprimitivity bimodule $(X, \zeta)$ such that $X_{\text{id}_\Gamma}$ is a $c^1_{c_1}(\text{id}_\Gamma)$–$c^2_{c_1}(\text{id}_\Gamma)$-imprimitivity bimodule;
7. there exist a separable $C^*$-algebra $A$, a coaction $\delta$ of $\Gamma$ on $A$, an abelian $C^*$-subalgebra $D \subseteq A$ containing an approximate unit for $\mathcal{A}$, a pair of complementary $A$-$\delta$-full projections $P_1, P_2 \in M(D)$, and isomorphisms $\phi_i : P_1 A P_1 \to C^*_r(G_i)$ such that $P_1 A P_2 = \min\{P_{n} P_2 : n \in \mathbb{N}(\mathcal{D})\}$ and such that $\phi_i(P_1 A P_i) = D_i$ and $\delta_i \circ \phi_i = (\phi_i \otimes \text{id}_{C^*_r(\Gamma)}) \circ \delta_i |_{P_1 A P_i}$;
8. there is an isomorphism $\phi : C^*_r(G_1) \otimes \mathcal{K} \to C^*_r(G_2) \otimes \mathcal{K}$ satisfying $\phi(C_0(G_1)^0) \otimes \mathcal{C} = C_0(G_2)^0$ and $(\delta_{c_2} \otimes \text{id}_{\mathcal{K}}) \circ \phi = (\phi \otimes \text{id}_{C^*_r(\Gamma)}) \circ (\delta_{c_1} \otimes \text{id}_{\mathcal{K}})$;
9. there are $C^*_r(c_{c_1}^{-1}(\epsilon))$-full projections $p_i \in M(C_0(G_1)^0)$ and an isomorphism $\phi : p_1 C^*_r(G_1) p_1 \to p_2 C^*_r(G_2) p_2$ such that $\phi(p_1 C_0(G_1)^0) = p_2 C_0(G_1)^0$, and $\delta_{c_2} \circ \phi = (\phi \otimes \text{id}_{C^*_r(\Gamma)}) \circ \delta_{c_1}$ on $p_1 C^*_r(G_1) p_1$, and
There are \( C^*_r(\mathfrak{c}_1^{-1}(\text{id}_\Gamma)) \)-full ideals \( I \subseteq C_0(G^{(0)}_1) \) and an isomorphism of hereditary subalgebras \( \phi : I_1C^*_r(G_1)I_1 \rightarrow I_2C^*_r(G_2)I_2 \) such that \( \phi(I_1) = I_2 \) and \( \delta_{c_2} \circ \phi = (\phi \otimes \text{id}_{C^*_r(\Gamma)}) \circ \delta_{c_1} \) on \( I_1C^*_r(G_1)I_1 \).

**Proof.** Theorem \[11.2\] gives (1) \( \iff \) (2) \( \iff \) (6) \( \iff \) (7). For equivalence of (1) and (3)–(5), we just summarise the modifications needed to the arguments of [[15]]. The proof of (3) \( \implies \) (1) follows the second paragraph of the proof of \[15\] Theorem 2.1: if \( \kappa : G_1 \times \mathcal{R} \rightarrow G_2 \times \mathcal{R} \) is an isomorphism satisfying \( \bar{c}_2 \circ \kappa = \bar{c}_1 \), then each \( G_i \times \{1\} \times \mathbb{N} \) is a \((G_i,c_i)-(G_i \times \mathcal{R},\bar{c}_i)\)-equivalence, so Lemma \[10.2\] shows that \((G_1,c_1)\) and \((G_2,c_2)\) are equivalent. The argument of \[15\] Theorem 2.1 gives (1) \( \implies \) (3) once we prove that given \((G,c)\) and a clopen \( c^{-1}(\text{id}_\Gamma) \)-full \( K \subseteq C^0(\mathfrak{c}) \) there is an isomorphism \( G \times \mathcal{R} \cong KGK \times \mathcal{R} \) that is equivariant for \( \mathfrak{c} \) and \( \mathfrak{c}|_{KGK \times \mathcal{R}} \). For this, apply \[15\] Lemma 2.4 to \( c^{-1}(\text{id}_\Gamma) \) and \( K \) to obtain a bisection \( Y \subseteq c^{-1}(\text{id}_\Gamma) \times \mathcal{R} \) with range \( K \times \mathcal{N} \) and source \( G^{(0)} \times \mathcal{N} \); then conjugation by this \( Y \) implements a graded isomorphism \( G \times \mathcal{R} \cong KGK \times \mathcal{R} \).

For (1) \( \iff \) (4) \( \iff \) (5), we follow the proof of \[15\] Theorem 3.2. The implications (4) \( \iff \) (5) \( \implies \) (1) follow the first paragraph of that proof using Lemma \[10.2\] for (5) \( \implies \) (1). For (1) \( \iff \) (4), we follow the proof of \[15\] Theorem 3.2 observing that since \( r(c_{z}^{-1}(\text{id}_\Gamma)) = G_1^{(0)} \) and \( s(c_{z}^{-1}(\text{id}_\Gamma)) = G_2^{(0)} \), we can choose the bisections \( V_i \) to belong to \( c_{z}^{-1}(\text{id}_\Gamma) \). Thus the bisection \( Y \) obtained in the penultimate paragraph of the proof satisfies \( Y \subseteq c^{-1}(\text{id}_\Gamma) \). As in the proof of \[15\] Theorem 3.2, conjugation by \( Y \) is an isomorphism \( r(Y)G_1r(Y) \cong s(Y)G_2s(Y) \), and it is graded because \( Y \subseteq c^{-1}(\text{id}_\Gamma) \).

Theorem \[6.2\] gives (3) \( \iff \) (8), so it suffices to prove (4) \( \implies \) (9) \( \implies \) (10) \( \implies \) (5). For (4) \( \implies \) (9), suppose that \( G_1 \) and \( G_2 \) are Kakutani \( c_1-c_2 \)-equivalent with respect to \( X_i \subseteq G_i^{(0)} \) and \( \kappa : X_iG_1X_1 \rightarrow X_2G_2X_2 \). Then \( p_i := 1_{X_i} \in M(C^*(G_i)) \) defines \( C^*_r(G_1) \)-full projections such that \( p_iC^*_r(G_1)p_i \cong C^*_r(X_iG_1X_i) \), and \( \kappa \) induces a \( \delta_{c_1}-\delta_{c_2} \)-equivariant isomorphism \( C^*_r(X_iG_1X_i) \rightarrow C^*_r(X_iG_1X_i) \).

For (9) \( \implies \) (10), take \( I_i = p_iC_0(G_i^{(0)}) \). For (10) \( \implies \) (5), fix \( I_i \subseteq C_0(G_i^{(0)}) \) for some open \( U_i \subseteq G_i^{(0)} \). Since each \( I_i \) is \( C^*_r(c_1^{-1}(\text{id}_\Gamma)) \)-full, each \( U_i \) is \( c_1^{-1}(\text{id}_\Gamma) \)-full by \[17\] Theorem 4.3.3. As in the proof of \[17\] Theorem 3.4.4, restriction of functions induces isomorphisms \( R_i : I_1C^*_r(G_1)I_1 \rightarrow C^*_r(U_1G_1U_1) \). Now \( \psi := R_2 \circ \phi \circ R_1^{-1} : C^*_r(U_1G_1U_1) \rightarrow C^*_r(U_2G_2U_2) \) is an equivariant isomorphism mapping \( C_0(U_1) \) onto \( C_0(U_1) \). Hence (2) \( \implies \) (1) in Theorem \[6.2\] yields a graded isomorphism \( \kappa : U_1G_1U_1 \rightarrow U_2G_2U_2 \).

**11.1. Stable continuous orbit equivalence.** Let \( X \) be a locally compact Hausdorff space, and \( \sigma : X \rightarrow X \) a local homeomorphism. Let \( \tilde{X} := X \times \mathbb{N} \) with the product topology and define a (surjective) local homeomorphism \( \tilde{\sigma} : \tilde{X} \rightarrow \tilde{X} \) by \( \tilde{\sigma}(x,0) = (\sigma(x),0) \) and \( \tilde{\sigma}(x,n+1) = (x,n) \). We call \((\tilde{X},\tilde{\sigma})\) the stabilisation of \((X,\sigma)\). Then \( G(\tilde{X},\tilde{\sigma}) \cong G(X,\sigma) \times \mathcal{R} \) via \( ((x,m),p,(y,n)) \mapsto ((x,p-m+n+y),(m,n)) \). We have the following partial generalisation of \[11\] Corollary 6.3.

**Corollary 11.4.** Let \( \sigma : X \rightarrow X \) and \( \tau : Y \rightarrow Y \), be local homeomorphisms of second-countable locally compact totally disconnected Hausdorff spaces. The following are equivalent:

1. there is a stabiliser-preserving continuous orbit equivalence from \((\tilde{X},\tilde{\sigma})\) to \((\tilde{Y},\tilde{\tau})\);
2. \( G(X,\sigma) \) and \( G(Y,\tau) \) are equivalent;
3. \((C^*(G(X,\sigma)),C_0(X))\) and \((C^*(G(Y,\tau)),C_0(Y))\) are Morita equivalent; and
(4) there is an isomorphism $C^*(G(X, \sigma)) \otimes K \to C^*(G(Y, \tau)) \otimes K$ that carries $C_0(X) \otimes C$ to $C_0(Y) \otimes C$.

Proof. Theorem 8.2 shows that (1) holds if and only if $G(\tilde{X}, \tilde{\sigma}) \cong G(\tilde{Y}, \tilde{\tau})$, so the discussion preceding the corollary shows that (1) holds if and only if $G(X, \sigma) \times R \cong G(Y, \tau) \times R$. Now the result follows from (1) $\iff$ (3) $\iff$ (6) $\iff$ (8) in Corollary 11.3 applied to the trivial cocycles on $G_1 = G(X, \sigma)$ and $G_2 = G(Y, \tau)$. 

Let $\sigma : X \to X$ be a surjective local homeomorphism of a compact Hausdorff space. Let $X := \{\xi \in X^Z : \sigma(\xi_n) = \xi_{n+1} \text{ for every } n \in \mathbb{Z}\}$, and define $\tilde{\sigma} : X \to X$ by $\tilde{\sigma}(\xi_n) = \sigma(\xi_n)$. We call $\sigma$ expansive if there is a "metricisation" $(X, d)$ of $X$ and an $\epsilon > 0$ such that $\sup_n d(\sigma^n(x), \sigma^n(x')) < \epsilon \implies x = x'$. We call $\epsilon$ an expansive constant for $(X, d, \sigma)$. The following generalises [12, Theorem 5.1].

**Corollary 11.5.** Let $X, Y$ be second-countable locally compact totally disconnected Hausdorff spaces and let $\sigma : \text{dom}(\sigma) \to \text{ran}(\sigma)$ and $\tau : \text{dom}(\tau) \to \text{ran}(\tau)$ be local homeomorphisms between open subsets of $X, Y$. The following are equivalent:

1. there are continuous, open maps $f : X \to Y$ and $f' : Y \to X$, and continuous maps $a : X \to \mathbb{N}$, $k : \text{dom}(\sigma) \to \mathbb{N}$, $a' : Y \to \mathbb{N}$, and $k' : \text{dom}(\tau) \to \mathbb{N}$ such that $\sigma^{a(x)}(f'(f(x))) = \sigma^{a(x)}(x)$ for $x \in X$, $\tau^{k(y)}(f(\tau(y))) = \tau^{k(y)+1}(f(y))$ for $y \in \text{dom}(\tau)$; $\tau^{a'(y)}(f'(f(y))) = \tau^{a'(y)}(y)$ for $y \in Y$, and $\sigma^{k'(y)}(f'(\tau(y))) = \sigma^{k'(y)+1}(f'(y))$ for $y \in \text{dom}(\tau)$;
2. there is a graded $(G(X, \sigma), c_X) - (G(Y, \tau), c_Y)$-equivalence $(Z, c_Z)$ such that $c_Z^{-1}(0)$ is a $c_X^{-1}(0)-c_Y^{-1}(0)$-equivalence;
3. there is an isomorphism $\kappa : G(X, \sigma) \times R \to G(Y, \tau) \times R$ such that $\tau_Y \circ \kappa = \tau_X$;
4. there is a $\mathbb{T}$-equivariant Morita equivalence between $(C^*(G(X, \sigma)), C_0(X))$ and $(C^*(G(Y, \tau)), C_0(Y))$ with respect to the gauge actions $\gamma^X$ and $\gamma^Y$ whose fixed-point submodule is a $C^*(G(X, \sigma))-C^*(G(Y, \tau))$-imprimitivity bimodule; and
5. there is an isomorphism $C^*(G(X, \sigma)) \otimes K(\mathbb{N}) \to C^*(G(Y, \tau)) \otimes K(\mathbb{N})$ that carries $C_0(X) \otimes C$ to $C_0(Y) \otimes C$, and intertwines the actions $\gamma^X \otimes \text{id}$ and $\gamma^Y \otimes \text{id}$.

If $X$ and $Y$ are compact, $\text{dom}(\sigma) = \text{ran}(\sigma) = X$, and $\text{dom}(\tau) = \text{ran}(\tau) = Y$, then each of the above five conditions implies

6. $(X, \sigma)$ and $(Y, \tau)$ are conjugate.

If in addition $\sigma$ and $\tau$ are expansive, then all six conditions are equivalent.

**Remark 11.6.** If $(X, d)$ is a totally disconnected metric space and $\sigma : X \to X$ is a surjective expansive local homeomorphism, then $(X, \sigma)$ is conjugate to the edge shift of a finite graph with no sinks or sources (see [11, Section 1] and [28, Theorem 1]), so the equivalence of (2), (5) and (6) follows from Theorem 8.11 and [12, Theorem 5.1].

**Proof of Corollary 11.3** Equivalence of (2)–(5) follows from equivalence of statements (1), (3), (6) and (8) in Corollary 11.3

(1) $\implies$ (2). For each $n \in \mathbb{N}$, fix a countable family $I_n$ of mutually disjoint compact open subsets of $X$ such that $\sigma^n|_U$ is a homeomorphism for each $U \in I_n$ and $\text{dom}(\sigma^n) = \bigcup_{U \in I_n} U$. Fix an injection $i : \bigcup_n \{n\} \times I_n \to \mathbb{N}$. For $x \in X$, let $U_x$ be the element of $I_{i(x)}$ containing $x$, and let $b(x) = i(a(x), U_x)$. Let $Z := \{(f(x), b(x)) : x \in X\} \subseteq Y \times \mathbb{N}$. Since $a$ is continuous and $f$ is open, $Z$ is open.
Fix $y \in Y$. Since $\tau^{\alpha(y)}(f(f'(y))) = \tau^{\alpha(y)}(y)$, we see that $(y, 0, f(f'(y))) \in G(Y, \tau)$. Thus $Z$ is $\xi_Y^{-1}(0)$-full in $(G(Y, \tau) \times \mathcal{R})(0)$. Corollary 3.3 yields a graded $G(Y, \tau) \times Z(G(Y, \tau) \times \mathcal{R})$-$\mathcal{Z}$-equivalence $(W, c_W)$ such that $c_W^{-1}(0)$ is a $\xi_Y^{-1}(0) - \xi_Y^{-1}(0)$-equivalence.

Since $\sigma^{\alpha(x)}(f'(f(x))) = \sigma^{\alpha(x)}(x)$ for $x \in X$, the formula $h(x) := (f(x), b(x))$ gives a homeomorphism $h : X \to Z$. Since each $\tau^{k(x)}(f(\sigma(x))) = \tau^{k(x)+1}(f(x))$, we have $(f(x), n, f(x')) \in G(Y, \tau)$ for all $(x, n, x') \in G(X, \sigma)$. There is an injective homomorphism $\phi : G(X, \sigma) \to G(Y, \tau) \times \mathcal{R}$ given by $\phi(x, n, x') := ((f(x), n, f(x')), (b(x), b(x')))$. We have $\phi(G(X, \sigma)) \subset Z(G(Y, \tau) \times \mathcal{R})Z$. Since $h$ is a homeomorphism and $\sigma^{k(y)}(f'(\tau(y))) = \sigma^{k(y)+1}(f'(y))$ for $y \in Y$, we see that $\phi(G(X, \sigma)) = Z(G(Y, \tau) \times \mathcal{R})Z$, giving (2).

(3) $\implies$ (1). Write $\pi_X : G(X, \sigma) \times \mathcal{R} \to G(X, \sigma)$ and $\pi_Y : G(Y, \tau) \times \mathcal{R} \to G(Y, \tau)$ for the projection maps. Define $f : X \to Y$ and $f' : Y \to X$ by $\pi_Y(\kappa((x, 0, x), (0, 0))) = (f(x), 0, f(x))$ and $\pi_X(\kappa^{-1}((y, 0, y), (0, 0))) = (f'(y), 0, f'(y))$. Clearly $f$ and $f'$ are continuous and open. Define $l_X : G(X, \sigma) \to N$ and $l_Y : G(Y, \tau) \to N$ as in Lemma 8.3. Fix $x \in X$. Then $\kappa((x, 0, x), (0, 0)) = ((f(x), 0, f(x)), (n, n))$ for some $n \in N$. Since $\tau_Y \circ \kappa = \xi_Y$, it follows that $\pi_X(\kappa^{-1}((f(x), 0, f(x)), (n, 0))) = (x, n, f'(f(x)))$. Let $a(x) := l_X(x, 0, f'(f(x)))$. Then $a : X \to N$ is continuous, and $\sigma^{a(x)}(f'(f(x))) = \sigma^{a(x)}(x)$ for all $x$. Similarly, $a' : Y \to N$ defined by $a'(y) := l_Y(y, 0, f(f'(y)))$ is continuous, and $\tau^{\alpha(y)}(f(f'(y))) = \tau^{\alpha(y)}(y)$ for all $y$.

Fix $x \in \text{dom}(\sigma)$. Then $\pi_Y(\kappa((1, 1, \sigma(x)), (0, 0))) = (f(x), 1, f(\sigma(x)))$ because $\xi_Y \circ \kappa = \xi_X$. Let $k(x) := l_Y(f(x), 1, f(\sigma(x)))$. So $k : \text{dom}(\sigma) \to N$ is continuous, and $\tau^{k(x)}(f(\sigma(x))) = \tau^{k(x)+1}(f(x))$ for $x \in \text{dom}(\sigma)$. Similarly, $k'(y) := l_Y(f(y), 1, f'(y))$ defines a continuous $k' : \text{dom}(\tau) \to N$ such that $\sigma^{k(y)}(f'(\tau(y))) = \sigma^{k(y)+1}(f'(y))$.

(1) $\implies$ (6). Since $X$ is compact, $m := \sup_{x \in X} k(x)$ is finite. We have $\tau^m(f(\sigma(x))) = \tau^{m-k(x)}(\tau^{k(x)+1}(f(x))) = \tau^{m-k(x)}(f(x)) = \tau^m(f(x))$ for all $x$. So $\phi := \tau^{m} \circ f : X \to Y$ is continuous and satisfies $\phi(\sigma(x)) = \tau^m(f(\sigma(x))) = \tau^m(f(x)) = \tau(\phi(x))$ for all $x$. Thus $\overline{\phi}(\xi_n) := (\phi(\xi_n))_{n \in \mathbb{N}}$ defines a continuous map $\overline{\phi} : \overline{X} \to \overline{Y}$. By definition, $\overline{\tau} \circ \overline{\phi} = \overline{\phi} \circ \overline{\sigma}$. We will show that $\overline{\phi}$ is bijective and hence a conjugacy.

For injectivity, suppose that $\overline{\phi}(\xi) = \overline{\phi}(\xi')$. Then each $\tau^m(f(\xi_n)) = \phi(\xi_n) = \phi(\xi_n') = \tau^m(f(\xi_n'))$. Let $p := \max\{\sup_{x \in X} a(z), \sup_{y \in Y} b'(k(y))\}$. Then $\sigma^p(x) = \sigma^p(f'(f(x)))$ for $x \in X$, and $\sigma^{p+j}(f'(y)) = \sigma^p(\tau^j(f'(y)))$ for $y \in Y$, $j \in \mathbb{N}$. So for $n \in \mathbb{Z}$,

$$\xi_{n+p+m} = \sigma^{p+m}(\xi_n) = \sigma^p(f'(\tau^m(f(\xi_n)))) = \sigma^p(f'(\tau^m(\xi_n))) = \xi_{n+p+m}. $$

Hence $\xi = \xi'$, and we deduce that $\overline{\phi}$ is injective.

For surjectivity, fix $\eta \in \overline{Y}$. Let $q := \sup_{z \in Y} \max\{a'(z), k'(z)\}$, and put $\eta' := \tau^{-q}(\eta)$. For $y \in Y$, $\sigma^q(f'(\tau^q(y))) = \sigma^{q-k(y)}(\sigma^{k(y)}(f'(\tau(y)))) = \sigma^{q+1}(f'(y))$. Thus $(\sigma^q(f'(\eta_n')))_{n \in \mathbb{N}} \in \overline{X}$. Since $\tau^{m-j}(f(x)) = \tau^m(f(\sigma^j(x)))$ for all $x, j$, and since $\tau^{m-j}(f'(y)) = \tau^q(y)$ for $y \in Y$, $\phi(\sigma^q(f'(\eta_n'))) = \tau^m(f(\sigma^q(f'(\eta_n')))) = \tau^{m+q}(f'(f(\eta_n'))) = \tau^{m+q}(\eta_n')$ for all $n \in \mathbb{Z}$. So $\overline{\phi}(\sigma^q(f'(\eta_n')))_{n \in \mathbb{N}} = \tau^{m+q}(\eta_n') = \eta$.

(6) $\implies$ (1). Suppose that $(X, d)$ and $(Y, d')$ are metrisations of $X, Y$ and that $\epsilon, \epsilon'$ are expansive constants for $(X, D, \sigma)$ and $(Y, d', \tau)$. Suppose that $\psi : \overline{X} \to \overline{Y}$ is a conjugacy. Fix $\delta > 0$ such that $d(\xi, \xi') < \delta \implies d(\psi(\xi), \psi(\xi')) < \epsilon$. Let $M := \sup_{x, x' \in X} d(x, x') < \infty$, and fix $N$ with $2^{-N+1}M < \delta$. Then $d(\xi, \xi') < \delta$ whenever $\xi_n = \xi_n'$ for all $n > -N$. So given $\xi, \xi' \in \overline{X}$, and putting $\eta = \psi(\xi)$ and $\eta' = \psi(\xi')$, if $\xi_n = \xi_n'$ for all $n > -N$, then $d'(\eta_m, \eta_n') \leq d(\tau^m(\eta), \tau^m(\eta')) < \epsilon$ for all $m$, giving $\eta_n = \eta_n'$. So we can define $f : X \to Y$ by $\psi(\xi) = (f(\xi_n-N))_{n \in \mathbb{Z}}$, and then $f \circ \sigma = \tau \circ f$. 


To see that $f$ is continuous, fix $x \in X$ and $\gamma > 0$. Choose $\xi \in \overline{X}$ such that $x = \xi_{-N}$. Choose an open neighbourhood $U$ of $\xi$ in $\overline{X}$ such that $d(p(\xi), p(\xi')) < \gamma$ for every $\xi' \in U$. Since $\sigma$ is surjective and open, there is an open $V \ni x$ such that $V \subseteq \{\xi'_{-N} : \xi' \in U\}$. So $d(f(x), f(x')) \leq \sup_{\xi' \in U} d(p(\xi), p(\xi')) < \gamma$ for all $x' \in V$.

To see that $f$ is open, fix $V \subseteq X$ open. Then $U := \{\xi \in \overline{X} : \xi_{-N} \in V\}$ is open, so $\psi(U)$ is open. The coordinate projections $\overline{Y} \to Y$ are open maps, so $f(V)$ is open.

So $f$ is continuous and open, $\tau \circ f = f \circ \sigma$, and $\psi(\xi) = (f(\xi_{-N}))_{n \in \mathbb{Z}}$ for $\xi \in \overline{X}$. Symmetry gives a continuous open function $f' : (Y, \tau) \to (X, \sigma)$ with $\sigma \circ f' = f' \circ \tau$ and $\psi^{-1}(\eta) = (f'(\eta_{n-N}))_{n \in \mathbb{Z}}$ for $\eta \in \overline{Y}$. We have $f \circ f' = \sigma^{N+N'}$ and $f \circ f' = \tau^{N+N'}$, giving (1). \hfill \Box

**Remark 11.7.** We deduce an interesting “stable isomorphism implies isomorphism” statement. Let $\sigma : X \to X$ and $\tau : Y \to Y$ be homeomorphisms of second-countable totally disconnected compact Hausdorff spaces. If $(C(X) \times_{\sigma} \mathbb{Z}) \otimes K$ and $(C(Y) \times_{\tau} \mathbb{Z}) \otimes K$ are $(\gamma^X \otimes \text{id})-(\gamma^Y \otimes \text{id})$-equivariantly isomorphic, then there is a $\gamma^X-\gamma^Y$-equivariant (and diagonal-preserving by Remark 7.6) isomorphism $C(X) \times_{\sigma} \mathbb{Z} \to C(Y) \times_{\tau} \mathbb{Z}$.

To prove this, first observe that $(X, \sigma) \cong (\overline{X}, \sigma)$ and $(Y, \tau) \cong (\overline{Y}, \tau)$ and then apply (5) $\Rightarrow$ (6) of Corollary 11.5 to see that if there is a diagonal-preserving equivariant isomorphism $(C(X) \times_{\sigma} \mathbb{Z}) \otimes K \cong (C(Y) \times_{\tau} \mathbb{Z}) \otimes K$, then there is an equivariant isomorphism $C(X) \times_{\sigma} \mathbb{Z} \cong C(Y) \times_{\tau} \mathbb{Z}$. Next note that $((C(X) \times_{\sigma} \mathbb{Z}) \otimes K)^{\gamma^X \otimes \text{id}} = C(X) \otimes K$ and likewise for $(Y, \tau)$; so any equivariant isomorphism $\phi : C(X) \times_{\sigma} \mathbb{Z} \to C(Y) \times_{\tau} \mathbb{Z}$ carries $C(X) \otimes \mathcal{C}$ to a maximal abelian subalgebra of $C(Y) \otimes \mathcal{K}$; that is, to $C(Y) \otimes D$ for some maximal abelian $D \subseteq \mathcal{K}$. Fix a unitary $U \in M(K)$ that conjugates $D$ to $\mathcal{D}$. Then $\phi' := (\text{id} \otimes \text{Ad}_U) \circ \phi$ is a diagonal-preserving equivariant isomorphism.

This result could also be obtained without recourse to groupoids using the techniques of [29] Proposition 4.3 (it is at least implicitly contained in that result).

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