RIGHT CIRCULANT MATRICES WITH GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS AND CODING THEORY

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Abstract. In this paper, we give two new coding algorithms by means of right circulant matrices with elements generalized Fibonacci and Lucas polynomials. For this purpose, we study basic properties of right circulant matrices using generalized Fibonacci polynomials $F_{p,q,n}(x)$, generalized Lucas polynomials $L_{p,q,n}(x)$ and geometric sequences.

1. Introduction

There are many studies for the coding theory using Fibonacci numbers, Lucas numbers, Lucas $p$ numbers, Pell numbers, generalized Pell $(p,i)$-numbers in the literature (for more details see [16], [20], [21], [22], [24] and references therein). Here we consider two classes of right circulant matrices whose entries are generalized Fibonacci and Lucas polynomials to obtain new coding/decoding algorithms.

Let $n > 0$ be an integer. From [6], we know that a $g$-circulant matrix is a square matrix of order $n$ in the following form

$$A_{g,n} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_{n-g+1} & a_{n-g+2} & \cdots & a_{n-g} \\ a_{n-2g+1} & a_{n-2g+2} & \cdots & a_{n-2g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g+1} & a_{g+2} & \cdots & a_g \end{pmatrix},$$

where $g$ is a nonnegative integer and each of subscripts is understood to be reduced modulo $n$. The first rows of $A_{g,n}$ is $(a_1, a_2, ..., a_n)$ and its $(j+1)$th row is obtained by giving its $j$th row a right circulant shift by $g$ positions.

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It is clear that \( g = 1 \) or \( g = n + 1 \) yields the standard right circulant matrix, or simply, circulant matrix. Thus a right circulant matrix is written as

\[
RCirc(a_1, a_2, ..., a_n) = \begin{pmatrix}
a_1 & a_2 & \cdots & a_n \\
an & a_1 & \cdots & a_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_2 & a_3 & \cdots & a_1
\end{pmatrix}.
\]

A geometric sequence is known to be a sequence \( \{a_k\}_{k=1}^{\infty} \) such that each term is given by a multiple of the previous one.

In [6], it was given a \( g \)-circulant, right circulant and left circulant matrices whose entries are \( h(x) \)-Fibonacci polynomials and presented the determinants of these matrices. In [9], it was introduced the right circulant matrices with ratio of the elements of Fibonacci and a geometric sequence and given eigenvalues, determinants, Euclidean norms and inverses of these matrices.

In [9], it has been dealt with circulant matrices with the Jacobsthal and Jacobsthal-Lucas numbers, studied the invertibility of these circulant matrices and presented the determinant and the inverse matrix. Similarly, in [18], it has been studied inverses and determinants of the circulant matrices related to Fibonacci and Lucas numbers. Furthermore, there are many applications of circulant matrices in the literature. For example, these matrices has been studied related to models and several differential equations such as fractional order models for nonlocal epidemics, differential delay equations (for more details one can see [1], [15], [17], [25] and the references therein).

Recently, \( h(x) \)-Fibonacci polynomials are defined by \( F_{h,0}(x) = 0, F_{h,1}(x) = 1 \) and \( F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x) \) for \( n \geq 1 \), and then it was given some properties of them in [14].

Let \( p(x) \) and \( q(x) \) be polynomials with real coefficients, \( p(x) \neq 0, q(x) \neq 0 \) and \( p^2(x) + 4q(x) > 0 \). In [12], it was defined generalized Fibonacci polynomials \( F_{p,q,n}(x) \) by

\[
F_{p,q,n+1}(x) = p(x)F_{p,q,n}(x) + q(x)F_{p,q,n-1}(x), \quad n \geq 1 \tag{1.1}
\]

with the initial values \( F_{p,q,0}(x) = 0, F_{p,q,1}(x) = 1 \) and generalized Lucas polynomials \( L_{p,q,n}(x) \) were defined by

\[
L_{p,q,n+1}(x) = p(x)L_{p,q,n}(x) + q(x)L_{p,q,n-1}(x), \quad n \geq 1 \tag{1.2}
\]
with the initial values $L_{p,q,0}(x) = 2, L_{p,q,1}(x) = p(x)$. For $p(x) = x$ and $q(x) = 1$ we have Catalan’s Fibonacci polynomials $F_n(x)$; for $p(x) = 2x$ and $q(x) = 1$ we have Byrd’s polynomials $\varphi_n(x)$; for $p(x) = k$ and $q(x) = t$ we have generalized Fibonacci numbers $U_n$; for $p(x) = k$ and $q(x) = 1$ we have $k$-Fibonacci numbers $F_{k,n}$; for $p(x) = q(x) = 1$ we have classical Fibonacci numbers $F_n$ (for more details see [5], [7], [11], [13], [19] and [23]).

For $p(x) = x$ and $q(x) = 1$ we have Lucas polynomials $L_n(x)$; for $p(x) = k$ and $q(x) = t$ we have generalized Lucas numbers $V_n$; for $p(x) = k$ and $q(x) = 1$ we have $k$-Lucas numbers $L_{k,n}$; for $p(x) = q(x) = 1$ we have classical Lucas numbers $L_n$ (for more details see [8], [10], [13], [19] and [23]).

Let $\alpha(x)$ and $\beta(x)$ be the roots of the characteristic equation

$$v^2 - vp(x) - q(x) = 0$$

of the recurrence relation (1.1). From [12], we know that Binet formulas for generalized Fibonacci polynomials $F_{p,q,n}(x)$ and generalized Lucas polynomials $L_{p,q,n}(x)$ are of the following forms, respectively:

$$F_{p,q,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \text{ for } n \geq 0$$

and

$$L_{p,q,n}(x) = \alpha^n(x) + \beta^n(x), \text{ for } n \geq 0,$$

where

$$\alpha(x) = \frac{p(x) + \sqrt{p^2(x) + 4q(x)}}{2} \quad \text{and} \quad \beta(x) = \frac{p(x) - \sqrt{p^2(x) + 4q(x)}}{2}.$$ 

It is clear that $\alpha(x) + \beta(x) = p(x), \alpha(x) \beta(x) = -q(x)$ and $\alpha(x) - \beta(x) = \sqrt{p^2(x) + 4q(x)}$.

In this study, we investigate right circulant matrices using generalized Fibonacci polynomials, generalized Lucas polynomials and geometric sequences. In Section 2 we give the eigenvalues and determinants of the right circulant matrices whose entries are the ratio of the elements of generalized Fibonacci sequence and some geometric sequences. In Section 3 we give right circulant matrices whose entries are the generalized Fibonacci and Lucas polynomials and calculate the determinants of these matrices. In Section 4 we give some applications of right circulant matrices to coding theory.

From now on, we shortly denote $\alpha(x)$ by $\alpha$, $\beta(x)$ by $\beta$, $p(x)$ by $p$ and $q(x)$ by $q$. 

2. Right Circulant Matrices with Generalized Fibonacci Polynomials and Geometric Sequences

Let \( \{f_n\}_{n=1}^\infty \) be the sequence of the form

\[
f_n = \frac{F_{p,q,n}(x)}{ar^n},
\]

where \( F_{p,q,n}(x) \) is the \( n \)-th generalized Fibonacci polynomial and \( ar^n \) is the \( n \)-th element of any geometric sequence.

Using these types of sequences, we consider the following right circulant matrix \( \mathcal{F}_n \):

\[
\mathcal{F}_n = \begin{pmatrix}
  f_0 & f_1 & f_2 & \cdots & f_{n-2} & f_{n-1} \\
  f_{n-1} & f_0 & f_1 & \cdots & f_{n-3} & f_{n-2} \\
  f_{n-2} & f_{n-1} & f_0 & \cdots & f_{n-4} & f_{n-3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  f_2 & f_3 & f_4 & \cdots & f_0 & f_1 \\
  f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_0
\end{pmatrix}.
\]

**Theorem 2.1.** The eigenvalues of the matrix \( \mathcal{F}_n \) are as follows:

\[
\lambda_m = \frac{-r F_{p,q,n}(x) - w^{-m} (q F_{p,q,n-1}(x) - r^n)}{ar^{n-1} (r - \alpha w^{-m}) (r - \beta w^{-m})},
\]

where \( m = 0, 1, ..., n-1, \) \( \alpha = \frac{p+\sqrt{p^2+4q}}{2}, \beta = \frac{p-\sqrt{p^2+4q}}{2} \) and \( w = e^{\frac{2\pi i}{n}} \).

**Proof.** From [2], we know that the eigenvalues of a right circulant matrix \( \mathcal{F}_n \) are

\[
\lambda_m = \sum_{k=0}^{n-1} f_k w^{-mk}, \tag{2.1}
\]

where \( m = e^{\frac{2\pi i}{n}} \) and \( m = 0, 1, 2, ..., n-1 \). Using the equation (2.1) and the Binet’s formula for the generalized Fibonacci polynomials \( F_{p,q,n}(x) \), we get

\[
\lambda_m = \sum_{k=0}^{n-1} \frac{F_{p,q,k}(x)}{ar^k} w^{-mk} = \sum_{k=0}^{n-1} \frac{\alpha^k - \beta^k}{(\alpha - \beta) ar^k} w^{-mk},
\]
where \( \alpha = \frac{p + \sqrt{p^2 + 4q}}{2} \), \( \beta = \frac{p - \sqrt{p^2 + 4q}}{2} \). Then using the properties of the geometric series, we obtain

\[
\lambda_m = \frac{1}{a(\alpha - \beta)} \left( \frac{1 - (\alpha/r)^n}{1 - \alpha w^{-m}/r} - \frac{1 - (\beta/r)^n}{1 - \beta w^{-m}/r} \right)
\]

\[
= \frac{1}{ar^{n-1}(\alpha - \beta)} \left( \frac{r^n - \alpha^n}{r - \alpha w^{-m}} - \frac{r^n - \beta^n}{r - \beta w^{-m}} \right)
\]

\[
= \frac{1}{ar^{n-1}(\alpha - \beta)} \left( \frac{(r^n - \alpha^n)(r - \beta w^{-m}) - (r - \alpha w^{-m})(r^n - \beta^n)}{(r - \alpha w^{-m})(r - \beta w^{-m})} \right).
\]

By the fact \( \alpha \beta = -q \), we find

\[
\lambda_m = \frac{-r(r^n - \beta^n) + r^n w^{-m}(\alpha - \beta) - w^{-m}(-\beta \alpha^n + \alpha \beta^n)}{ar^{n-1}(\alpha - \beta)(r - \alpha w^{-m})(r - \beta w^{-m})}
\]

\[
= \frac{-r(r^n - \beta^n) + r^n w^{-m}(\alpha - \beta) - w^{-m}q(\alpha^{n-1} - \beta^{n-1})}{ar^{n-1}(\alpha - \beta)(r - \alpha w^{-m})(r - \beta w^{-m})}
\]

\[
= \frac{-rF_{p,q,n}(x) + r^n w^{-m}F_{p,q,1}(x) - w^{-m}qF_{p,q,n-1}(x)}{ar^{n-1}(r - \alpha w^{-m})(r - \beta w^{-m})}
\]

and so

\[
\lambda_m = \frac{-rF_{p,q,n}(x) - w^{-m}(qF_{p,q,n-1}(x) - r^n)}{ar^{n-1}(r - \alpha w^{-m})(r - \beta w^{-m})}.
\]

Now we give the following theorem.

**Theorem 2.2.** The determinant of the matrix \( \mathcal{F}_n \) is

\[
\det(\mathcal{F}_n) = \frac{(-1)^n r^n F_{p,q,n}(x) - (qF_{p,q,n-1}(x) - r^n)^n}{a^n r^{n(n-1)} - (r^{2n} - r^n L_{p,q,n}(x) + (-q)^n)}.
\]

**Proof.** Since the determinant of a matrix is the product of its eigenvalues, by Theorem 2.1 we obtain

\[
\det(\mathcal{F}_n) = \prod_{m=0}^{n-1} \frac{-rF_{p,q,n}(x) - w^{-m}(qF_{p,q,n-1}(x) - r^n)}{ar^{n-1}(r - \alpha w^{-m})(r - \beta w^{-m})}.
\]

From the complex analysis, we know

\[
\prod_{m=0}^{n-1} (x - yw^{-m}) = x^n - y^n
\]
Applying the equation \((2.3)\) to the equation \((2.2)\), we find
\[
\det(F_n) = \frac{(-1)^n r^n F_{p,q,n}^n(x) - (q F_{p,q,n-1}(x) - r^n)^n}{\alpha_n r^{n(n-1)} (r^n - (r^n - \alpha^n)) (r^n - \beta^n)}
\]
\[
= \frac{(-1)^n r^n F_{p,q,n}^n(x) - (q F_{p,q,n-1}(x) - r^n)^n}{\alpha_n r^{n(n-1)} (r^{2n} - r^n (\alpha^n + \beta^n) + (-q^n))}.
\]

Using the Binet’s formulas for the generalized Lucas polynomials \(L_{p,q,n}(x)\), we get
\[
\det(F_n) = \frac{(-1)^n r^n F_{p,q,n}^n(x) - (q F_{p,q,n-1}(x) - r^n)^n}{\alpha_n r^{n(n-1)} (r^{2n} - r^n L_{p,q,n}(x) + (-q^n))}.
\]

Notice that for \(p = x\) and \(q = 1\); for \(p = 2x\) and \(q = 1\); \(p = k\) and \(q = t\); \(p = k\) and \(q = 1\); \(p = q = 1\) in Theorem 2.1 and Theorem 2.2 we have similar theorems for Catalan’s Fibonacci polynomials \(F_n(x)\), Byrd’s polynomials \(\varphi_n(x)\), generalized Fibonacci numbers \(U_n\), \(k\)-Fibonacci numbers \(F_{k,n}\), classical Fibonacci numbers \(F_n\), respectively. Also, in [3], the right circulant matrix with Fibonacci sequence was defined and given eigenvalues, Euclidean norm of this matrix.

3. Right Circulant Matrices with Generalized Fibonacci and Lucas Polynomials

In this section we give the determinant of a right circulant matrix whose elements are generalized Fibonacci polynomials \(F_{p,q,n}(x)\) and generalized Lucas polynomials \(L_{p,q,n}(x)\).

**Theorem 3.1.** Let \(G_n\) be a right circulant matrix of the following form:

\[
G_n = \begin{pmatrix}
F_{p,q,1}(x) & F_{p,q,2}(x) & F_{p,q,3}(x) & \cdots & F_{p,q,n-1}(x) & F_{p,q,n}(x) \\
F_{p,q,n}(x) & F_{p,q,1}(x) & F_{p,q,2}(x) & \cdots & F_{p,q,n-2}(x) & F_{p,q,n-1}(x) \\
F_{p,q,n-1}(x) & F_{p,q,n}(x) & F_{p,q,1}(x) & \cdots & F_{p,q,n-3}(x) & F_{p,q,n-2}(x) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
F_{p,q,4}(x) & F_{p,q,5}(x) & F_{p,q,6}(x) & \cdots & F_{p,q,2}(x) & F_{p,q,3}(x) \\
F_{p,q,3}(x) & F_{p,q,4}(x) & F_{p,q,5}(x) & \cdots & F_{p,q,1}(x) & F_{p,q,2}(x) \\
F_{p,q,2}(x) & F_{p,q,3}(x) & F_{p,q,4}(x) & \cdots & F_{p,q,n}(x) & F_{p,q,1}(x)
\end{pmatrix}
\]

(3.1)
Then we have
\[
\det(G_n) = (1 - F_{p,q,n+1}(x))^{n-1} + (qF_{p,q,n}(x))^{n-2} \sum_{k=1}^{n-1} \left( \frac{1 - F_{p,q,n+1}(x)}{qF_{p,q,n}(x)} \right)^{k-1} qF_{p,q,k}(x).
\] (3.2)

Proof. For \( n = 1 \), \( \det(G_1) = 1 \) satisfies the equation (3.2). Let us consider the case \( n \geq 2 \). Consider the following matrices:

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-p & 0 & 0 & 0 & \cdots & 0 & 1 & \vdots \\
-q & 0 & 0 & 0 & \cdots & 0 & 1 & -p \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & -p & \cdots & 0 & 0 & 0 \\
0 & 1 & -p & -q & \cdots & 0 & 0 & 0 
\end{pmatrix}_{n \times n}
\] (3.3)

and

\[
B = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \left( \frac{qF_{p,q,n}(x)}{1-F_{p,q,n+1}(x)} \right)^{n-2} & 0 & \cdots & 0 & 1 \\
0 & 0 & \left( \frac{qF_{p,q,n}(x)}{1-F_{p,q,n+1}(x)} \right)^{n-3} & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \left( \frac{qF_{p,q,n}(x)}{1-F_{p,q,n+1}(x)} \right) & 1 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 
\end{pmatrix}_{n \times n}
\]

Notice that
\[
det(A) = det(B) = (-1)^{(n-1)(n-2)/2}.
\]

If we multiply the matrices \( A, G_n \) and \( B \) we have the following product matrices:

\[
AG_nB = \begin{pmatrix}
\alpha_n & F_{p,q,1}(x) & F_{p,q,n-1}(x) & \cdots & F_{p,q,3}(x) & F_{p,q,2}(x) \\
0 & \beta_n & F_{p,q,n-2}(x) & \cdots & qF_{p,q,2}(x) & qF_{p,q,1}(x) \\
0 & 0 & F_{p,q,1}(x) - F_{p,q,n+1}(x) & \cdots & \vdots & \vdots \\
0 & 0 & -qF_{p,q,n}(x) & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -qF_{p,q,n}(x) & F_{p,q,1}(x) - F_{p,q,n+1}(x) \\
0 & 0 & 0 & \cdots & 0 & 0 
\end{pmatrix}
\]
Since \( \det(AG) \), we have

\[
\alpha_{n,p,q} = \sum_{k=1}^{n-1} \left( \frac{qF_{p,q,n}(x)}{F_{p,q,1}(x) - F_{p,q,n+1}(x)} \right)^{n-(k+1)} F_{p,q,k+1}(x) \tag{3.4}
\]

and

\[
\beta_{n,p,q} = (1 - F_{p,q,n+1}(x)) + \sum_{k=1}^{n-1} \left( \frac{qF_{p,q,n}(x)}{F_{p,q,1}(x) - F_{p,q,n+1}(x)} \right)^{n-(k+1)} qF_{p,q,k}(x). \tag{3.5}
\]

Then we have

\[
\det(AG_nB) = F_{p,q,1}(x) \beta_{n,p,q} (F_{p,q,1}(x) - F_{p,q,n+1}(x))^{n-2}.
\]

Using the equation (3.5), we get

\[
det(AG_nB) = (1 - F_{p,q,n+1}(x))^{n-1} \\
+ (qF_{p,q,n}(x))^{n-2} \sum_{k=1}^{n-1} \left( \frac{1 - F_{p,q,n+1}(x)}{qF_{p,q,n}(x)} \right)^{k-1} qF_{p,q,k}(x).
\]

Since \( \det(AG_nB) = \det(G_n) \), we find

\[
det(G_n) = (1 - F_{p,q,n+1}(x))^{n-1} \\
+ (qF_{p,q,n}(x))^{n-2} \sum_{k=1}^{n-1} \left( \frac{1 - F_{p,q,n+1}(x)}{qF_{p,q,n}(x)} \right)^{k-1} qF_{p,q,k}(x).
\]

Now we give the following theorem for generalized Lucas polynomials \( L_{p,q,n}(x) \).

**Theorem 3.2.** Let \( H_n \) be a right circulant matrix of the following form:

\[
H_n = \begin{pmatrix}
L_{p,q,1}(x) & L_{p,q,2}(x) & L_{p,q,3}(x) & \cdots & L_{p,q,n-1}(x) & L_{p,q,n}(x) \\
L_{p,q,n}(x) & L_{p,q,1}(x) & L_{p,q,2}(x) & \cdots & L_{p,q,n-2}(x) & L_{p,q,n-1}(x) \\
L_{p,q,n-1}(x) & L_{p,q,n}(x) & L_{p,q,1}(x) & \cdots & L_{p,q,n-3}(x) & L_{p,q,n-2}(x) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
L_{p,q,4}(x) & L_{p,q,5}(x) & L_{p,q,6}(x) & \cdots & L_{p,q,2}(x) & L_{p,q,3}(x) \\
L_{p,q,3}(x) & L_{p,q,4}(x) & L_{p,q,5}(x) & \cdots & L_{p,q,1}(x) & L_{p,q,2}(x) \\
L_{p,q,2}(x) & L_{p,q,3}(x) & L_{p,q,4}(x) & \cdots & L_{p,q,n}(x) & L_{p,q,1}(x)
\end{pmatrix} \tag{3.6}
\]
Then we have
\[
\det(H_n) = L_{p,q,1}(x) (L_{p,q,1}(x) - L_{p,q,n+1}(x))^{n-1} \\
+ L_{p,q,1}(x) q^{n-1} (L_{p,q,n}(x) - 2)^{n-2} \sum_{k=1}^{n-3} \left( \frac{L_{p,q,1}(x) - L_{p,q,n+1}(x)}{q L_{p,q,n}(x) - 2q} \right) \left( \frac{q L_{p,q,n}(x) - 2q}{L_{p,q,1}(x) - L_{p,q,n+1}(x)} \right)^{k-1} L_{p,q,k}(x) \\
- 2q^{n-1} (L_{p,q,n}(x) - 2)^{n-2} \sum_{k=1}^{n-3} \left( \frac{L_{p,q,1}(x) - L_{p,q,n+1}(x)}{q L_{p,q,n}(x) - 2q} \right) \left( q L_{p,q,n}(x) - 2q \right) \left( L_{p,q,n}(x) - 2q \right)^{k-1} L_{p,q,k+1}(x).
\]

Proof. For \( n = 1 \), \( \det(H_1) = p \) satisfies the equation (3.7). Let us consider the case \( n \geq 2 \). Let \( A \) be a matrix of the form given in (3.3) and \( D \) be a matrix of the following form:

\[
D = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \left( \frac{q L_{p,q,n}(x) - 2q}{L_{p,q,1}(x) - L_{p,q,n+1}(x)} \right)^{n-2} & 0 & \cdots & 0 & 1 \\
0 & \left( \frac{q L_{p,q,n}(x) - 2q}{L_{p,q,1}(x) - L_{p,q,n+1}(x)} \right)^{n-3} & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \left( \frac{q L_{p,q,n}(x) - 2q}{L_{p,q,1}(x) - L_{p,q,n+1}(x)} \right) & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0
\end{pmatrix}_{n \times n}.
\]

Using the properties of determinants and multiplying these matrices \( A, H_n \) and \( D \) the proof can be completed by similar arguments used in the proof of the above theorem.

\[ \square \]

4. An Application to Coding Theory

In this section, we give two new coding/decoding methods using the right circulant matrices \( G_n \) and \( H_n \) for \( p = q = 1 \). At first, we give an algorithm by means of the generalized Fibonacci polynomials. Following the notations in [21], we give generalized Fibonacci and Lucas blocking algorithms with transformations

\[
M \times G_n = E, \quad M \times H_n = E
\]

and

\[
E \times (G_n)^{-1} = M, \quad E \times (H_n)^{-1} = M,
\]

where \( M \) is nonsingular square message matrix, \( E \) is a code matrix, \( G_n \) is coding matrix and the inverse matrix \((G_n)^{-1}\) is decoding matrix.

We put our message in a matrix adding zero between two words and end of the message until we obtain the size of the message matrix is \( 3m \). Dividing the
message square matrix $M$ into the block matrices, named $B_i$ ($1 \leq i \leq m^2$), of size $3 \times 3$, from left to right, we can construct a new coding method.

Now we explain the symbols of our coding method. Suppose that matrices $B_i$ and $E_i$ are of the following forms:

$$B_i = \begin{bmatrix} b_{i1}^1 & b_{i2}^1 & b_{i3}^1 \\ b_{i4}^2 & b_{i5}^2 & b_{i6}^2 \\ b_{i7}^3 & b_{i8}^3 & b_{i9}^3 \end{bmatrix} \quad \text{and} \quad E_i = \begin{bmatrix} e_{i1}^1 & e_{i2}^1 & e_{i3}^1 \\ e_{i4}^2 & e_{i5}^2 & e_{i6}^2 \\ e_{i7}^3 & e_{i8}^3 & e_{i9}^3 \end{bmatrix}.$$ 

We use the matrix $G_n$ given in (3.1) for $p = q = 1$ and rewrite the elements of this matrix as $G_n = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \\ g_5 & g_6 \end{bmatrix}$. The number of the block matrices $B_i$ is denoted by $b$. In accordance with $b$, we choose the number $n$ as follows:

$$n = \begin{cases} 3, & b = 1 \\ 3b, & b \neq 1 \end{cases}.$$ 

Using the chosen $n$, we write the following character table according to $mod27$ (this table can be enlarged according to the used characters in the message matrix). We begin the “$n$” for the first character.

| A  | B  | C  | D  | E  | F  | G  | H  | I  |
|----|----|----|----|----|----|----|----|----|
| $n$| $n+1$| $n+2$| $n+3$| $n+4$| $n+5$| $n+6$| $n+7$| $n+8$|
| J  | K  | L  | M  | N  | O  | P  | Q  | R  |
| $n+9$| $n+10$| $n+11$| $n+12$| $n+13$| $n+14$| $n+15$| $n+16$| $n+17$|
| S  | T  | U  | V  | W  | X  | Y  | Z  | $0$ |
| $n+18$| $n+19$| $n+20$| $n+21$| $n+22$| $n+23$| $n+24$| $n+25$| $n+26$|

(4.1)

**Generalized Fibonacci Blocking Algorithm**

**Coding Algorithm**

**Step 1.** Divide the matrix $M$ into blocks $B_i$ ($1 \leq i \leq m^2$).

**Step 2.** Choose $n$.

**Step 3.** Determine $b_j^i$ ($1 \leq j \leq 9$).

**Step 4.** Compute $\det(B_i) \rightarrow d_i$.

**Step 5.** Construct $K = [d_i, b_j^k]_{k \in \{1,2,3,4,6,7,8,9\}}$.

**Step 6.** End of algorithm.

**Decoding Algorithm**
Step 1. Compute $G_n$.

Step 2. Determine $g_j$ (1 ≤ j ≤ 3).

Step 3. Compute $g_1 b_1^i + g_3 b_2^i + g_2 b_3^i \to e_1^i$, (1 ≤ i ≤ m).

\[
\begin{align*}
  g_2 b_1^i + g_1 b_2^i + g_3 b_3^i & \to e_2^i, \\
  g_3 b_1^i + g_2 b_2^i + g_1 b_3^i & \to e_3^i, \\
  g_1 b_1^i + g_3 b_2^i + g_2 b_3^i & \to e_7^i, \\
  g_2 b_1^i + g_1 b_2^i + g_3 b_3^i & \to e_8^i, \\
  g_3 b_1^i + g_2 b_2^i + g_1 b_3^i & \to e_9^i.
\end{align*}
\]

Step 4. Solve

\[
\det (G_3) \times d_i = e_1^i e_3^i (g_2 b_4^i + g_1 x_i + g_3 b_6^i) + e_3^i e_4^i (g_1 b_4^i + g_3 b_1^i + g_2 b_6^i) \\
+ e_2^i e_4^i (g_3 b_4^i + g_2 x_i + g_1 b_6^i) - (e_3^i e_7^i (g_2 b_4^i + g_1 x_i + g_3 b_6^i) \\
+ e_8^i e_7^i (g_3 b_4^i + g_2 x_i + g_1 b_6^i) + e_8^i e_2^i (g_1 b_4^i + g_3 x_i + g_2 b_6^i).
\]

Step 5. Substitute for $x_i = b_5^i$.

Step 6. Construct $B_i$.

Step 7. Construct $M$.

Step 8. End of algorithm.

We give an application of the above generalized Fibonacci blocking algorithm in the following example for $b = 1$.

**Example 4.1.** Let us consider the message matrix for the following message text:

"SUMEYRA"

Using the message text, we get the following message matrix $M$:

\[
M = \begin{bmatrix}
S & U & M \\
E & Y & R \\
A & 0 & 0
\end{bmatrix}_{3 \times 3}.
\]

**Coding Algorithm:**

**Step 1.** We construct the message text $M$ of size $3 \times 3$, named $B_1$:

\[
B_1 = \begin{bmatrix}
S & U & M \\
E & Y & R \\
A & 0 & 0
\end{bmatrix}.
\]
Step 2. Since $b = 1$, we calculate $n = 3$. For $n = 3$, we use the following “letter table” for the message matrix $M$:

| S | U | M | E | Y | R | A | 0 |
|---|---|---|---|---|---|---|---|
| 21 | 23 | 15 | 7 | 27 | 20 | 3 | 2 |

Step 3. We have the elements of the block $B_1$ as follows:

| $b_1^1 = 21$ | $b_2^1 = 23$ | $b_3^1 = 15$ |
| $b_4^1 = 7$   | $b_5^1 = 27$  | $b_6^1 = 20$  |
| $b_7^1 = 3$   | $b_8^1 = 2$   | $b_9^1 = 2$   |

Step 4. Now we calculate the determinant $d_1$ of the block $B_1$:

$$d_1 = \det(B_1) = 347$$

Step 5. Using Step 3 and Step 4, we obtain the following matrix $K$:

$$K = \begin{bmatrix} 347 & 21 & 23 & 15 & 7 & 20 & 3 & 2 & 2 \end{bmatrix}.$$  

Step 6. End of algorithm.

Decoding algorithm:

Step 1. By (3.7), we know that

$$G_3 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$  

Step 2. The elements of $G_3$ are denoted by

$$g_1 = 1, g_2 = 1 \text{ and } g_3 = 2.$$  

Step 3. We compute the elements $e_1^1, e_2^1, e_3^1, e_4^1, e_7^1, e_8^1, e_9^1$ to construct the matrix $E_1$:

$$e_1^1 = 82, e_2^1 = 74, e_3^1 = 80, e_7^1 = 9, e_8^1 = 9 \text{ and } e_9^1 = 10.$$  

Step 4. We calculate the elements $x_1$:

$$4 \times 347 = 80624 + 2926x_1 - 78912 - 2938x_1$$

$$\Rightarrow x_1 = 27.$$  

Step 5. We rename $x_1$ as follows:

$$x_1 = b_5^1 = 27.$$
**Step 6.** We construct the block matrix $B_1$:

$$B_1 = \begin{bmatrix} 21 & 23 & 15 \\ 7 & 27 & 20 \\ 3 & 2 & 2 \end{bmatrix}.$$

**Step 7.** We obtain the message matrix $M$:

$$M = \begin{bmatrix} 21 & 23 & 15 \\ 7 & 27 & 20 \\ 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} S & U & M \\ E & Y & R \\ A & 0 & 0 \end{bmatrix}.$$

**Step 8.** End of algorithm.

Now, we give another blocking algorithm by means of the generalized Lucas polynomials $L_{p,q,n}(x)$. Let’s suppose

$$B_i = \begin{bmatrix} b_{i1}^j \\ b_{i2}^j \\ b_{i3}^j \\ b_{i4}^j \end{bmatrix} \quad \text{and} \quad E_i = \begin{bmatrix} e_{i1}^j \\ e_{i2}^j \\ e_{i3}^j \\ e_{i4}^j \end{bmatrix}.$$

We use the matrix $H_n$ given in (3.6) for $p = q = 1$ and we rewrite the elements of this matrix as $H_n = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix}$. Similarly, the number of the block matrices $B_i$ is denoted by $b$. According to $b$, we choose the number $n$ as follows:

$$n = \begin{cases} 2, & b = 1 \\ 2b, & b \neq 1 \end{cases}.$$

Using the chosen $n$, we write the character table given in (4.1) according to $mod27$ or we can differently array this table. For example, we begin the ”$n$” for the first, second, central, last character etc.

**Generalized Lucas Blocking Algorithm**

**Coding Algorithm**

**Step 1.** Divide the matrix $M$ into blocks $B_i \ (1 \leq i \leq m^2)$.

**Step 2.** Choose $n$.

**Step 3.** Determine $b_j^j \ (1 \leq j \leq 4)$.

**Step 4.** Compute $\det(B_i) \rightarrow d_i$.

**Step 5.** Construct $K = [d_i, b_k^j]_{k \in \{1,3,4\}}$.

**Step 6.** End of algorithm.

**Decoding Algorithm**

**Step 1.** Compute $H_n$. 
Step 2. Determine $h_j$ ($1 \leq j \leq 2$).

Step 3. Compute $h_1b_3^i + h_2b_4^i \rightarrow e_3^i$, ($1 \leq i \leq m^2$).
$h_2b_3^i + h_1b_4^i \rightarrow e_4^i$.

Step 4. Solve $\det (H_2) \times d_i = e_4^i(h_1b_1^i + h_2x_i) - e_3^i(h_2b_1^i + h_1x_i)$.

Step 5. Substitute for $x_i = b_2^i$.

Step 6. Construct $B_1$.

Step 7. Construct $M$.

Step 8. End of algorithm.

We give following example as an application of the generalized Lucas blocking algorithm for $b = 1$.

Example 4.2. Let us consider the message matrix for the following message text:

“GOOD”

Using the message text, we get the following message matrix $M$:

$$M = \begin{bmatrix} G & O \\ O & D \end{bmatrix}_{2 \times 2}.$$  

Coding Algorithm:

Step 1. We construct the message text $M$ of size $2 \times 2$, named $B_1$:

$$B_1 = \begin{bmatrix} G & O \\ O & D \end{bmatrix}. $$

Step 2. Since $b = 1$, we calculate $n = 2$. For $n = 2$, we use the following “letter table” for the message matrix $M$:

$$\begin{array}{|c|c|c|c|}
\hline
G & O & O & D \\
8 & 16 & 16 & 5 \\
\hline
\end{array}$$

Step 3. We have the elements of the block $B_1$ as follows:

$$b_1^1 = 8 \quad b_2^1 = 16 \quad b_3^1 = 16 \quad b_4^1 = 5$$

Step 4. Now we calculate the determinants $d_1$ of the block $B_1$:

$$d_1 = \det(B_1) = -216$$

Step 5. Using Step 3 and Step 4 we obtain the following matrix $K$:

$$K = \begin{bmatrix} -216 & 8 & 16 & 5 \end{bmatrix}.$$
Step 6. End of algorithm.

Decoding algorithm:

Step 1. By (3.6), we know that

\[
H_2 = \begin{bmatrix}
1 & 3 \\
3 & 1 \\
\end{bmatrix}.
\]

Step 2. The elements of \( H_2 \) are denoted by

\[
h_1 = 1 \text{ and } h_2 = 3.
\]

Step 3. We compute the elements \( e_3^1, e_4^1 \) to construct the matrix \( E_1 \):

\[
e_3^1 = 31, \ e_4^1 = 53.
\]

Step 4. We calculate the elements \( x_1 \):

\[
(-8) \times (-216) = 424 + 159x_1 - 744 - 31x_1 \\
\Rightarrow \ x_1 = 16.
\]

Step 5. We rename \( x_1 \) as follows:

\[
x_1 = b_2^1 = 16.
\]

Step 6. We construct the block matrix \( B_1 \):

\[
B_1 = \begin{bmatrix}
8 & 16 \\
16 & 5 \\
\end{bmatrix}.
\]

Step 7. We obtain the message matrix \( M \):

\[
M = \begin{bmatrix}
8 & 16 \\
16 & 5 \\
\end{bmatrix} = \begin{bmatrix}
G & O \\
O & D \\
\end{bmatrix}.
\]

Step 8. End of algorithm.

5. Conclusions

We have presented two new coding/decoding algorithms by means of the blocks of sizes \( 3 \times 3 \) and \( 2 \times 2 \). Since the determinant of the matrix \( G_2 \) is 0, we study the matrix \( G_n \) for \( n \geq 3 \) in the generalized Fibonacci blocking algorithm, although we can study with the matrix \( H_n \) for \( n \geq 2 \) in the generalized Lucas blocking algorithm.

By differently taking \( p \) and \( q \), we can obtain different algorithms. Furthermore, it can be mixed the above new blocking methods with the previous methods given
It is possible to produce new blocking methods similar to minesweeper algorithm given in [24].

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