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Original articles

Impact of optimal vaccination and social distancing on COVID-19 pandemic

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Abstract

The first COVID-19 case was reported at Wuhan in China at the end of December 2019 but till today the virus has caused millions of deaths worldwide. Governments of each country, observing the severity, took non-pharmaceutical interventions from the very beginning to break the chain of higher transmission. Fortunately, vaccines are available now in most countries and people are asked to take recommended vaccines as precautionary measures. In this work, an epidemiological model on COVID-19 is proposed where people from the susceptible and asymptomatically infected phase move to the vaccinated class after a full two-dose vaccination. The overall analysis says that the disease transmission rate from symptomatically infected people is most sensitive on the disease prevalence. Moreover, better disease control can be achieved by vaccination of the susceptible class. In the later part of the work, a corresponding optimal control problem is considered where maintaining social distancing and vaccination procedure change with time. The result says that even in absence of social distancing, only the vaccination to people can significantly reduce the overall infected population. From the analysis, it is observed that maintaining physical distancing and taking vaccines at an early stage decreases the infection level significantly in the environment by reducing the probability of becoming infected.

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1. Introduction

The spread of coronavirus first started from Wuhan in China in the middle of December 2019 \cite{8,9,25}. The first case was reported near Hunan seafood market where live animals are traded \cite{20}. But within few weeks it spreads all over the Chinese province and in few months, the whole world was affected by this virus. Within three months, WHO declared COVID-19 as a pandemic observing its severity. In order to reduce the higher disease transmission, the Government of almost every country took some non-pharmaceutical interventions (such as maintaining physical distancing, using face-mask and alcohol-based hand sanitizers) from the very beginning. In the case of COVID-19, the respiratory system mainly becomes affected. In many cases, people have mild symptoms and recover on their

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different countries worldwide. The COVID-19 vaccines act as protective shields against the disease by developing tested up to July 22. There are several vaccines (at least 13) for COVID-19 available now which are administered in vaccination drive [41]. Also, according to the Indian Council of Medical Research, 45,29,39,545 samples have been 16th January 2021. Till 22nd July 42,34,17,030 vaccine doses have been administered so far under the nationwide the country stands at 4,19,470 as per the health ministry's update. India has started COVID-19 vaccination drive on 38,470 new recoveries. The active cases have come below 5 lakh with 4,05,513 cases. Total COVID-19 deaths in the number of daily deaths has also gone below 500 marks as 483 new deaths are reported on that day with less than one lakh new cases are reported for the last 45-days. There have been less than one lakh new cases (causes 5,115,893 number of confirmed COVID-19 cases) on 16th September 2020. After that, the graph of per-day reported cases showed a declination with time. But again started to increase from February 2021 and the second wave struck the country. The per-day reported cases reached up to 4,14,433 cases (total 21,485,285 confirmed cases) on 17th March 2021, was 11,474,302 which increased at a very higher rate and reached at number 29,761,964 on 17th June 2021. It means almost 18,287,662 newly infected cases were reported within three months only. In 2020, the first wave has attacked the country, and the highest peak in terms of newly reported cases in a single day reached 97,859 (causes 5,115,893 number of confirmed COVID-19 cases) on 16th September 2020. The continued rise of COVID-19 positivity rate has indicated that more people are carrying the virus with time. The government in India, from the very beginning, has announced some preventive measures to reduce the high transmission. One of the precautionary measures is maintaining social distancing to reduce the number of times people come into close contact with each other. The other preventive measures include adopting a self-quarantine strategy when slight symptoms are shown, using of face-masks and alcohol-based hand sanitizer, etc. When the number of infected cases reached 500, the Government of India announced for a 14-hours ‘Janta Curfew’ on 22nd March 2020 and called for nationwide lockdown on 25th March 2020. The lockdown initially was for a fortnight, but as the situation became worse, it was expanded up to May 2020 through different phases. Then the unlockdown period started but with a long list of restrictions. State-wise lockdown was imposed a few times later depending on the severity of the infection. According to the data of PIB, Government of India, the active cases declined to 7,29,243 on 20th June 2021 and less than 60,000 daily cases were reported (after 81 days) with 58,419 new cases [34]. Till that date, a total 2,87,66,009 number of people were recovered. As per the data, daily recoveries continuously outnumbered the daily new cases for 38 consecutive days. Total 39.10 crores of sample tests had been conducted till the date and 27.66 crores of vaccine doses were administered. From the data of Worldometer, the confirmed cases in India on 17th March 2021, was 11,474,302 which increased at a very higher rate and reached at number 29,761,964 on 17th June 2021. It means almost 18,287,662 newly infected cases were reported within three months only. In 2020, the first wave has attacked the country, and the highest peak in terms of newly reported cases in a single day reached 97,859 (causes 5,115,893 number of confirmed COVID-19 cases) on 16th September 2020. After that, the graph of per-day reported cases showed a declination with time. But again started to increase from February 2021 and the second wave struck the country. The per-day reported cases reached up to 4,14,433 cases (total 21,485,285 confirmed cases) on 6th May 2021 which is the highest peak till that date in terms of single-day infection. According to the experts, the first wave of virus attacked most of the elders but, in the second wave, more youth are affected. It is predicted that the third wave can be dangerous for children.

The worldwide update on 22nd July 2021 has revealed that the USA has the highest COVID-19 cases with 35,215,590 number of confirmed cases. India is in the next position with 31,291,704 cases and Brazil has 19,586,983 number of confirmed cases till that date [42]. India is in the top position in terms of recovery of the people with 30,460,308 number of recovered cases till 22nd July 2021. The USA is in the second position with 29,478,173 recovery reports. According to the health ministry’s update, India has reported a new 35,342 number of COVID-19 cases on 22nd July 2021 [34]. There have been less than one lakh new cases are reported for the last 45-days. The number of daily deaths has also gone below 500 marks as 483 new deaths are reported on that day with 38,470 new recoveries. The active cases have come below 5 lakh with 4,05,513 cases. Total COVID-19 deaths in the country stands at 4,19,470 as per the health ministry’s update. India has started COVID-19 vaccination drive on 16th January 2021. Till 22nd July 42,34,17,030 vaccine doses have been administered so far under the nationwide vaccination drive [41]. Also, according to the Indian Council of Medical Research, 45,29,39,545 samples have been tested up to July 22. There are several vaccines (at least 13) for COVID-19 available now which are administered in different countries worldwide. The COVID-19 vaccines act as protective shields against the disease by developing
an immune response to the virus. Increasing immunity power by vaccination reduces the risk of developing the illness. A COVID-19 vaccine protects a person from serious illness and death but it is still unknown up to which extent these vaccines restrain the transmission and keep people safe from being infected. So, at the current stage, non-pharmaceutical interventions are equally important as vaccines. In India, there are three approved vaccines for people such as Covishield, Covaxin, and Sputnik V. Recently the Indian Government has approved Indian pharma company Cipla to import Moderna COVID-19 vaccine which has shown almost 95% efficiency [17]. The impact of these vaccines on the pandemic will depend on the effectiveness mainly. The government of the country, under the nationwide vaccination drive, has supported the states by providing the vaccines free of cost. The vaccination procedure is considered to be an integral pillar of the comprehensive strategy of the Government to control the pandemic along with testing, live tracking treatment, and adopting behavioral changes.

Some works have been published analyzing the transmission of coronavirus but the impact of vaccines is not considered there [12,26–29,32,35–38]. Bhopal and Bhopal (2020) in their work have presented the significance of the epidemiological data on COVID-19 which are arranged by sex and age group [4]. However, there are some literature available discussing the effect of underreporting infections and the impact of vaccination on COVID-19 transmission [1,2]. In this work, we have emphasized how the non-pharmaceutical intervention (social distancing) as well as the pharmaceutical intervention (vaccination) reduce the chance of becoming infected by the virus. It means how the precautionary measures decrease the contact rate of people with infected ones and people become less infected because of the immunity developed by vaccination. A compartmental epidemic model on COVID-19 is proposed with a separated compartment of vaccinated people. It is considered here that people from susceptible and asymptomatic states take vaccines to develop the immunity to protect themselves. Section 2 contains the proposed model on COVID-19 with non-negative initial conditions; Section 3 shows that the system is biologically well-defined. In Section 4, the basic reproduction number ($R_0$) is derived along with the endemic equilibrium point. In Section 5 it is observed how some of the system parameters are sensitive to $R_0$ and affect the disease transmission. The stability criteria of the equilibrium points are obtained in Section 6, and Section 7 shows the change of stability of disease-free equilibrium point through transcritical bifurcation. The consequent section contains the numerical simulation of the proposed model without any optimal control interventions. In the later part, an optimal control problem is formulated in Section 9 to reduce the overall infected population in the system and the next section contains the corresponding numerical scenarios to support the analytical part. The work ends with a brief conclusion.

2. Formulation of mathematical model

The novel beta coronavirus causes a pandemic situation worldwide from 2020. Different models have been proposed in order to curb the high transmission rate of this virus. In this section, we have proposed a compartmental SIRV (Susceptible–Infected–Recovered–Vaccinated) model with a separate compartment of vaccinated people. In India, the first vaccination started on 16th January 2021. All the people, if eligible, were requested from the very first day to take two doses of vaccine maintaining a certain time interval. The total population ($N$) is divided into the following subpopulations: susceptible population ($S$), asymptotically or pre-symptomatically infected population who are exposed to coronavirus without showing any symptoms ($I_1$), symptomatically infected and so quarantined population ($I_2$), hospitalized population ($H$), recovered population ($R$) and vaccinated population ($V$). In a susceptible environment, people become infected and move to an asymptomatic state when they come in contact with asymptotically infected class ($I_1$) and symptomatically infected class ($I_2$) with rates $\beta_1$ and $\beta_2$ respectively. The recruitment rate in the susceptible class ($\Lambda$) is assumed to be constant. It denotes the new susceptible people, who are coming by birth or immigration. The parameter $d$ denotes the natural death rate which is incorporated in each class. It is considered that $q_1$ portion of susceptible and $q_2$ portion of asymptotically infected people move to the vaccinated compartment after completing a full two-dose vaccination procedure. A person, whether is COVID-19 positive, is detected mostly by RT–PCR test. The swab from a person’s throat or nose is used in this test. Besides, there are several tests like TrueNAT, antigen testing, etc. to detect COVID-19 in the human body. But these tests do not show appropriate results every time and may result in false-negative tests. So, a person who tests negative through the tests still may have COVID-19. Moreover, in some cases, the symptoms develop after one or two weeks and so, a person turns out COVID positive even after two or three tests. So, people from asymptotically infected class move to symptomatically infected class with rate $\kappa$ when symptoms are shown and even when one has COVID-19 positive report. Further, deterioration of health condition indicates hospitalization of
an infected person whereas people may recover by natural immunity also. So, people from symptomatically infected class move to either hospitals with rate $\phi_1$ for regular observation or to recovered class with rate $\phi_2$. Moreover, asymptomatically infected people can also move to the recovered class with a rate of $\alpha$ if they recuperate by natural immunity. Also, hospitalized people move to recovered class (after proper medical treatment) with rate $\psi$. There are some reports stated that the recovery from the disease does not guarantee permanent recovery and so some of the recovered people move back to susceptible class further with constant rate $\eta$ [37]. Lastly, the parameters $\mu_1$ and $\mu_2$ denote the disease-related death rates at symptomatically infected class and hospitalized compartment respectively.

So, the model is proposed in system (1) as follows:

$$\frac{dS}{dt} = \Lambda - (\beta_1 I_1 + \beta_2 I_2)S - dS - q_1 S + \eta R, \quad S(0) > 0,$$

$$\frac{dI_1}{dt} = (\beta_1 I_1 + \beta_2 I_2)S - q_2 I_1 - \kappa I_1 - \alpha I_1 - d I_1, \quad I_1(0) \geq 0,$$

$$\frac{dI_2}{dt} = \kappa I_1 - (\phi_1 + \phi_2) I_2 - (d + \mu_1) I_2, \quad I_2(0) \geq 0,$$

$$\frac{dH}{dt} = \phi_1 I_2 - \psi H - (d + \mu_2) H, \quad H(0) \geq 0,$$

$$\frac{dR}{dt} = \alpha I_1 + \phi_2 I_2 + \psi H - d R - \eta R, \quad R(0) \geq 0,$$

$$\frac{dV}{dt} = q_1 S + q_2 I_1 - d V, \quad V(0) > 0,$$

Fig. 1 contains a schematic diagram of the model system for clear understanding.

3. Positivity and boundedness

The following two theorems in this section show that the variables in system (1) are positive and bounded with time and so, the proposed system is biologically well-posed.
Theorem 3.1. Solutions of system (1) in $\mathbb{R}_+^6$ are positive for $t > 0$.

Proof. As the right side functions of system (1) is continuous and locally Lipschitzian, so, there exists an unique solution of the system on $[0, \tau)$ with $0 < \tau \leq +\infty$ [16]. Let us show, $S(t) > 0$, $\forall t \in [0, \tau)$. If the statement is not true, then $\exists \ t_1 \in (0, \tau)$ such that $S(t_1) = 0$, $\dot{S}(t_1) \leq 0$ and $S(t) > 0$, $\forall t \in [0, t_1)$. Then we have $I_1(t) \geq 0$, $\forall t \in [0, t_1)$. If it does not hold, then $\exists \ t_2 \in (0, t_1)$ such that $I_1(t_2) = 0$, $\dot{I}_1(t_2) < 0$ and $I_1(t) \geq 0$, $\forall t \in [0, t_2)$. Our claim is $V(t) > 0$, $\forall t \in [0, t_2)$. If it does not hold, then $\exists \ t_3 \in (0, t_2)$ such that $V(t_3) = 0$, $\dot{V}(t_3) \leq 0$ and $V(t) > 0$, $\forall t \in [0, t_3)$. The last equation gives

$$\frac{dV}{dt} |_{t=t_3} = q_1S(t_3) + q_2I_1(t_3) - dV(t_3) = q_1S(t_3) + q_2I_1(t_3) > 0,$$

which is a contradiction to $\dot{V}(t_3) \leq 0$. So, $V(t) > 0$, $\forall t \in [0, t_2)$.

Next we claim $I_2(t) \geq 0$, $\forall t \in [0, t_2)$. Suppose it is not true, then $\exists \ t_4 \in (0, t_2)$ such that $I_2(t_4) = 0$, $\dot{I}_2(t_4) < 0$ and $I_2(t) \geq 0$, $\forall t \in [0, t_4)$. The third equation gives

$$\frac{dI_2}{dt} |_{t=t_4} = \kappa I_1(t_4) - (\phi_1 + \phi_2 + d + \mu_1)I_2(t_4) = \kappa I_1(t_4) \geq 0,$$

which contradicts $\dot{I}_2(t_4) < 0$. So, $I_2(t) \geq 0$, $\forall t \in [0, t_2)$. Similarly, we can show that $H(t) \geq 0$ and $R(t) \geq 0$, $\forall t \in [0, t_2)$. From the second equation of (1), we have

$$\frac{dI_1}{dt} |_{t=t_2} = (\beta_1 I_1(t_2) + \beta_2 I_2(t_2))S(t_2) - (q_2 + \kappa + \alpha + d)I_1(t_2) = \beta_2 I_2(t_2)S(t_2) \geq 0$$

which contradicts $\dot{I}_1(t_2) < 0$. Hence, $I_1(t) \geq 0$, $\forall t \in [0, t_1)$. By the above steps, we have $V(t) > 0$, $I_2(t) \geq 0$, $H(t) \geq 0$, $R(t) \geq 0$, $\forall t \in [0, t_1)$. From the first equation of (1) we get

$$\frac{dS}{dt} |_{t=t_1} = A - (\beta_1 I_1(t_1) + \beta_2 I_2(t_1))S(t_1) - (d + q_1)S(t_1) + \eta R(t_1) = A + \eta R(t_1) > 0,$$

which contradicts $\dot{S}(t_1) \leq 0$. Hence we have, $S(t) > 0$, $\forall t \in [0, \tau)$ with $0 < \tau \leq +\infty$. Following the previous steps we get $I_1(t) \geq 0$, $I_2(t) \geq 0$, $H(t) \geq 0$, $R(t) \geq 0$ and $V(t) > 0$, $\forall t \in [0, \tau)$ with $0 < \tau \leq +\infty$. \[□\]

Theorem 3.2. Solutions of system (1) starting from $\mathbb{R}_+^6$ are bounded with time.

Proof.

Let, $N(t) = S(t) + I_1(t) + I_2(t) + H(t) + R(t) + V(t)$

$$\therefore \frac{dN}{dt} = A - d(S + I_1 + I_2 + H + R + V) - \mu_1 I_2 - \mu_2 H \leq A - dN$$

$$\Rightarrow 0 < N(t) \leq \frac{A}{d} + \left( N(0) - \frac{A}{d} \right) e^{-dt}$$

Here $N(0)$ is total population size at initial time.

So, $0 < \lim_{t \to \infty} N(t) \leq \frac{A}{d} + \epsilon$, for any $\epsilon > 0$. The solutions of the system remain in the region: $\Omega \equiv \{ (S, I_1, I_2, H, R, V) \in \mathbb{R}_+^6 : 0 < N(t) \leq \frac{A}{d} + \epsilon, \text{ for any } \epsilon > 0 \}$. \[□\]
4. Equilibrium analysis

System (1) has a disease-free equilibrium point (DFE) $E_0(S_0, 0, 0, 0, V_0)$, where $S_0 = \frac{A}{(d + q_1)}, \quad V_0 = \frac{q_1 S_0}{d} = \frac{q_1 A}{d(d + q_1)}$ and an endemic equilibrium point $E^*(S^*, I^*_1, I^*_2, H^*, R^*, V^*)$.

4.1. Basic reproduction number ($R_0$)

Basic reproduction number $R_0$ is obtained by the process developed by van den Driessche and Watmough [39]. Consider, $x \equiv (I_1, I_2)$. Denote, $p_0 = d + q_1, \quad p_1 = q_2 + \kappa + \alpha + d, \quad p_2 = \phi_1 + \phi_2 + d + \mu_1, \quad p_3 = \psi + d + \mu_2$ and $p_4 = d + \eta$. Then we have:

$$\frac{dx}{dt} = \bar{F}(x) - \nu(x),$$

$$\bar{F}(x) = \begin{pmatrix} (\beta_1 I_1 + \beta_2 I_2) S \\ 0 \end{pmatrix}, \quad \nu(x) = \begin{pmatrix} p_1 I_1 \\ -\kappa I_1 + p_2 I_2 \end{pmatrix},$$

where $\bar{F}(x)$ and $\nu(x)$ contain the compartment containing new infection term and other terms respectively. So, at the disease-free equilibrium $E_0 = (S_0, 0, 0, 0, V_0)$ we have

$$F = (D\bar{F}(x))_{E_0} = \begin{pmatrix} \beta_1 S_0 & \beta_2 S_0 \\ 0 & 0 \end{pmatrix}; \quad V = (D\nu(x))_{E_0} = \begin{pmatrix} p_1 & 0 \\ -\kappa & p_2 \end{pmatrix}.$$

The spectral radius of the next generation matrix $FV^{-1}$ is $R_0$ and is given by:

$$R_0 = \frac{(\beta_1 p_2 + \beta_2 \kappa) S_0}{p_1 p_2} > 0 \quad (2)$$

Endemic equilibrium point $E^*(S^*, I^*_1, I^*_2, H^*, R^*, V^*)$

Consider, $p_0 = d + q_1, \quad p_1 = q_2 + \kappa + \alpha + d, \quad p_2 = \phi_1 + \phi_2 + d + \mu_1, \quad p_3 = \psi + d + \mu_2$ and $p_4 = d + \eta$. Then we have

$$\Lambda - (\beta_1 I^*_1 + \beta_2 I^*_2)S^* - p_0 S^* + \eta R^* = 0,$$

$$(\beta_1 I^*_1 + \beta_2 I^*_2)S^* - p_1 I^*_1 = 0,$$

$$\kappa I^*_1 - p_2 I^*_2 = 0,$$

$$\phi_1 I^*_s - p_3 H^* = 0,$$

$$\alpha I^*_s + \phi_2 I^*_s + \psi H^* - p_4 R^* = 0,$$

$$q_1 S^* + q_2 I^*_1 - d V^* = 0.$$

Solving these equations, we get $S^* = \frac{S_0}{R_0}, \quad I^*_1 = \frac{p_2 I^*_2}{\kappa}, \quad H^* = \frac{\phi_1 I^*_2}{p_3}, \quad R^* = \frac{(\alpha p_2 p_3 + \kappa \phi_2 p_3 + \kappa \psi \phi_1) I^*_2}{\kappa p_3 p_4}, \quad V^* = \frac{(\kappa q_1 S_0 + q_2 p_2 R_0 I^*_2)}{\kappa d R_0}$ and $I^*_2 = \frac{\kappa S_0 p_0 p_3 p_4}{A} \left(1 - \frac{1}{R_0}\right)$, where $A = d p_1 p_2 p_3 + \eta (d + q_2) p_2 p_3 + \kappa \eta [(d + \mu_1) p_3 + \phi_1 (d + \mu_2)]$. So, $I^*_2 > 0$ when $R_0 > 1$. Hence, from the calculation we get the following theorem as

**Theorem 4.1.** System (1) contains one unique

(i) disease-free equilibrium (DFE) $E_0 \left(\frac{A}{p_0}, 0, 0, 0, \frac{q_1 S_0}{d}\right)$ for any parametric values and

(ii) endemic equilibrium point $E^*(S^*, I^*_1, I^*_2, H^*, R^*, V^*)$ for $R_0 > 1$.

5. Sensitivity analysis

From the expression of basic reproduction number it is observed that $R_0$ depends on recruitment rate ($\Lambda$), natural death rate ($d$), disease transmission rates ($\beta_1$, $\beta_2$), disease related death rate ($\mu_1$), vaccination rates ($q_1$, $q_2$), moving rate of asymptotically infected people to symptomatically infected and recovered classes ($\kappa$, $\alpha$) and moving
rates of symptomatically infected people into hospitalized and recovered classes ($\phi_1$, $\phi_2$). It is shown below how $\beta_1$, $\beta_2$, $\phi_1$, $q_1$, $q_2$ affect on the transmission of the disease.

Now, $R_0 = \frac{(\beta_1 p_2 + \beta_2 \kappa) S_0}{p_1 p_2}$ where $S_0 = \frac{\Lambda}{p_0}$, $V_0 = \frac{q_1 S_0}{d}$, $p_0 = (d + q_1)$, $p_1 = q_2 + \kappa + \alpha + d$, $p_2 = \phi_1 + \phi_2 + d + \mu_1$. So, we have:

$$\frac{\partial R_0}{\partial \beta_1} = \frac{S_0}{p_1} > 0, \quad \frac{\partial R_0}{\partial \beta_2} = \frac{\kappa S_0}{p_1 p_2} > 0,$$

$$\frac{\partial R_0}{\partial \phi_1} = -\frac{\kappa \beta_2 S_0}{p_1 p_2^2} < 0, \quad \frac{\partial R_0}{\partial q_1} = -\frac{(\beta_1 p_2 + \kappa \beta_2) S_0}{p_0 p_1 p_2} < 0, \quad \frac{\partial R_0}{\partial q_2} = -\frac{(\beta_1 p_2 + \kappa \beta_2) S_0}{p_1^2 p_2} < 0.$$

Computing the normalized forward sensitivity index for the parameters $\beta_1$, $\beta_2$, $\phi_1$, $q_1$ and $q_2$ by the method of Arriola and Hyman, we have [3]:

$$\Gamma_{\beta_1} = \left[ \frac{\frac{\partial R_0}{\partial \beta_1}}{R_0} \right] = \left[ \frac{\beta_1}{\beta_1 p_2 + \kappa \beta_2} < 1 \right]$$

$$\Gamma_{\beta_2} = \left[ \frac{\frac{\partial R_0}{\partial \beta_2}}{R_0} \right] = \left[ \frac{\kappa \beta_2}{\beta_1 p_2 + \kappa \beta_2} < 1 \right]$$

$$\Gamma_{\phi_1} = \left[ \frac{\frac{\partial R_0}{\partial \phi_1}}{R_0} \right] = \left[ \frac{-\phi_1 \kappa \beta_2}{p_2(\beta_1 p_2 + \kappa \beta_2)} \right]$$

$$\Gamma_{q_1} = \left[ \frac{\frac{\partial R_0}{\partial q_1}}{R_0} \right] = \left[ \frac{-q_1}{p_0} \right]$$

$$\Gamma_{q_2} = \left[ \frac{\frac{\partial R_0}{\partial q_2}}{R_0} \right] = \left[ \frac{-q_2}{p_1} \right]$$

From the expression of $R_0$ and also from the calculation it is observed that the virus transmission rates ($\beta_1$, $\beta_2$) maintain a directly proportional relation with $R_0$. It means increasing $\beta_i$ ($i = 1, 2$) escalates the basic reproduction number resulting in the occurrence of an epidemic situation in the system. It is evident that if people from the susceptible class come in contact with infected people (both asymptotically and symptomatically) frequently without any precautionary measures, then the disease invades the population easily, and even at a larger rate. On the other hand, the rate at which symptomatically infected people move to hospitals, if increases, the prevalence can be reduced to some certain extent with time, i.e., the hospitalization rate is inversely proportional with $R_0$. If more people get admitted to the hospitals for clinical treatment without ignoring the slightest symptoms, then the chance of an epidemic or pandemic outbreak reduces. Moreover, the vaccination rates ($q_1$, $q_2$) are inversely proportional with $R_0$ which means basic reproduction number decreases with increase of $q_1$ and $q_2$. It is biologically relevant because if more people are provided with vaccines at susceptible and asymptomatic stages, then the chances of becoming infected reduce which lessens the higher disease transmission. From the sensitivity index, it is observed that the transmission rate from symptomatically infected is most sensitive among all the parameters to reduce the disease prevalence, and vaccination reduces the count of the symptomatically infected population. Hence, lowering the virus transmission through social distancing and vaccination along with other precautionary measures would help to handle this pandemic situation with time.

6. Stability analysis

Local stability of $E_0$ and $E^*$: We discuss the local stability conditions for the disease-free equilibrium point as well as endemic equilibrium point in this section. Let, $p_0 = d + q_1$, $p_1 = q_2 + \kappa + \alpha + d$, $p_2 = \phi_1 + \phi_2 + d + \mu_1$, $p_3 = \psi + d + \mu_2$ and $p_4 = d + \eta$. 291
The Jacobian matrix of system (1) is given as:

\[
\mathcal{J} = \begin{pmatrix}
-(\beta_1I_1 + \beta_2I_2) - p_0 & -\beta_1S & -\beta_2S & 0 & \eta & 0 \\
(\beta_1I_1 + \beta_2I_2) & \beta_1S - p_1 & \beta_2S & 0 & 0 & 0 \\
0 & \kappa & -p_2 & 0 & 0 & 0 \\
0 & 0 & \phi_1 & -p_3 & 0 & 0 \\
0 & \alpha & \phi_2 & \psi & -p_4 & 0 \\
q_1 & q_2 & 0 & 0 & 0 & -d
\end{pmatrix}
\]

(4)

**Theorem 6.1.** Disease-free equilibrium \((E_0)\) of the proposed system is locally asymptotically stable (LAS) for \(R_0 < 1\) when \(P_i > 0\) for \(i = 1, 2, \ldots, 5\).

**Proof.** Jacobian matrix corresponding to DFE \(E_0 = \left(\frac{A}{p_0}, 0, 0, 0, \frac{q_1S_0}{d}\right)\) is given as follows:

\[
\mathcal{J}|_{E_0} = \begin{pmatrix}
-p_0 & -\beta_1S_0 & -\beta_2S_0 & 0 & \eta & 0 \\
0 & \beta_1S_0 - p_1 & \beta_2S_0 & 0 & 0 & 0 \\
0 & \kappa & -p_2 & 0 & 0 & 0 \\
0 & 0 & \phi_1 & -p_3 & 0 & 0 \\
0 & \alpha & \phi_2 & \psi & -p_4 & 0 \\
q_1 & q_2 & 0 & 0 & 0 & -d
\end{pmatrix}
\]

The characteristic equation of the corresponding Jacobian matrix is \(\lambda^6 + P_1\lambda^5 + P_2\lambda^4 + P_3\lambda^3 + P_4\lambda^2 + P_5\lambda + P_6 = 0\), where, \(P_1 = p_0 + p_2 + p_3 + p_4 + d - (\beta_1S_0 - p_1)\), \(P_2 = p_0(p_2 + p_3 + p_4) + p_3p_4 + d[p_0(p_2 + p_3 + p_4) + p_3p_4 + p_1p_2(1-R_0) - (\beta_1S_0 - p_1)(d + p_0 + p_3 + p_4)],\)

\(P_3 = p_0p_2(p_3 + p_4) + (p_0 + p_2)p_3p_4 + d[p_0(p_2 + p_3 + p_4) + p_2(p_3 + p_4) + p_3p_4 + p_1p_2(d + p_0 + p_3 + p_4)(1 - R_0) - (\beta_1S_0 - p_1)(d + p_3 + p_4) + p_0(p_3 + p_4) + p_3p_4),\)

\(P_4 = p_0p_2p_3p_4 + d[p_0p_2(p_3 + p_4) + p_2p_3p_4] + p_1p_2[d(p_0 + p_3 + p_4) + p_0(p_3 + p_4) + p_3p_4(1 - R_0)],\)

\(P_5 = dp_0p_2p_3p_4 + p_1p_2(dp_0(p_3 + p_4) + dp_3p_4 + p_0p_3p_4)(1 - R_0) - (\beta_1S_0 - p_1)p_0p_3p_4\) and \(P_6 = dp_0p_1p_2p_3p_4(1 - R_0)\).

So, \(P_6 > 0\) when \(R_0 < 1\). Also, \(P_i > 0\) for \(i = 1, 2, \ldots, 5\) if \(\beta_1S_0 < p_1\) along with \(R_0 < 1\).

**Theorem 6.2.** The endemic equilibrium point \(E^*\) of system (1) is LAS for \(R_0 > 1\) when the conditions (i) and (ii), as mentioned in the proof, are satisfied.

**Proof.** The Jacobian matrix at the endemic equilibrium point \(E^*\) is given as:

\[
\mathcal{J}|_{E^*} = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & 0 & a_{15} & 0 \\
a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\
0 & a_{32} & a_{33} & 0 & 0 & 0 \\
0 & 0 & a_{43} & a_{44} & 0 & 0 \\
0 & a_{52} & a_{53} & a_{54} & a_{55} & 0 \\
a_{61} & a_{62} & 0 & 0 & 0 & a_{66}
\end{pmatrix}
\]

where \(a_{11} = -(\beta_1I_1 + \beta_2I_2) - p_0, a_{12} = -\beta_1S, a_{13} = -\beta_2S, a_{15} = \eta, a_{21} = (\beta_1I_1 + \beta_2I_2), a_{22} = \beta_1S - p_1, a_{23} = \beta_2S, a_{32} = \kappa, a_{33} = -p_2, a_{43} = \phi_1, a_{44} = -p_3, a_{52} = \alpha, a_{53} = \phi_2, a_{54} = \psi, a_{55} = -p_4, a_{61} = q_1, a_{62} = q_2, a_{66} = -d.\)
Characteristic equation of $\mathcal{J}_{\mathcal{E}^*}$ is $\lambda^6 + Q_1 \lambda^5 + Q_2 \lambda^4 + Q_3 \lambda^3 + Q_4 \lambda^2 + Q_5 \lambda + Q_6 = 0$, where

$Q_1 = -(a_{11} + a_{22} + a_{33} + a_{44} + a_{55} + a_{66}),$

$Q_2 = a_{11}(a_{22} + a_{33} + a_{44} + a_{55} + a_{66}) + a_{22}(a_{33} + a_{44} + a_{55} + a_{66}) + a_{33}(a_{44} + a_{55} + a_{66})$

$+ a_{44}(a_{55} + a_{66}) + a_{55}a_{66} - a_{12}a_{21} - a_{23}a_{32},$

$Q_3 = -(a_{11}a_{22} + a_{12}a_{21} + a_{22}a_{33} + a_{44} + a_{55} + a_{66}) - (a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32})(a_{44} + a_{55} + a_{66})$

$- a_{44}(a_{11} + a_{22} + a_{33})(a_{55} + a_{66}) - (a_{11} + a_{22} + a_{33} + a_{44})a_{55}a_{66} + a_{11}a_{23}a_{32}$

$- a_{13}a_{23}a_{32} - a_{52}a_{21}a_{15},$

$Q_4 = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32})a_{44} + a_{55} + a_{66}) - a_{53}a_{23}a_{21}a_{15}$

$+ a_{44}(a_{55} + a_{66})a_{11}(a_{22} + a_{33} - a_{23}a_{32} - a_{23}a_{32}) + a_{15}a_{23}a_{21}(a_{33} + a_{44} + a_{66})$

$+ a_{55}a_{66}a_{11}(a_{22} + a_{33} + a_{44}) + a_{22}(a_{33} + a_{44}) + a_{33}a_{44} - a_{12}a_{21} - a_{23}a_{32},$

$Q_5 = a_{44}(a_{55} + a_{66})a_{11}(a_{22}a_{33} - a_{23}a_{32} + a_{21}(a_{13}a_{32} - a_{12}a_{33})$

$- a_{55}a_{66}(a_{33} + a_{44})(a_{11}a_{22} - a_{12}a_{21}) + a_{15}a_{23}a_{21}(a_{33} + a_{44} + a_{66})(a_{33}a_{32} - a_{33}a_{52})$

$+ a_{56}a_{66}(a_{11}a_{23}a_{32} - (a_{11} + a_{22}a_{33} + a_{32}a_{32} - a_{13}a_{21})) - a_{13}a_{21}(a_{44}a_{33}a_{43}a_{32} + a_{44}a_{33}a_{52}),$

$Q_6 = a_{66}(a_{11}a_{44}a_{55}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}a_{44}a_{55}(a_{21}a_{33} - a_{13}a_{32}) + a_{54}a_{43}a_{32}a_{21}a_{15}$

$- a_{15}a_{21}a_{44}(a_{32}a_{53} - a_{33}a_{52}).$

Let us consider:

$$\Omega_1 = Q_1, \quad \Omega_2 = \begin{vmatrix} Q_1 & 1 & 0 & 0 \\ Q_3 & Q_2 & Q_1 \\ Q_5 & Q_4 & Q_3 \\ 0 & Q_6 & Q_5 \end{vmatrix}, \quad \Omega_3 = \begin{vmatrix} Q_1 & 0 & 0 \\ Q_3 & Q_2 & Q_1 \\ Q_5 & Q_4 & Q_3 \\ 0 & Q_6 & Q_5 \end{vmatrix}, \quad \Omega_4 = \begin{vmatrix} Q_1 & 0 & 0 \\ Q_3 & Q_2 & Q_1 \\ Q_5 & Q_4 & Q_3 \\ 0 & Q_6 & Q_5 \end{vmatrix}, \quad \Omega_5 = \begin{vmatrix} Q_1 & 1 & 0 & 0 & 0 \\ Q_3 & Q_2 & Q_1 & 1 & 0 \\ Q_5 & Q_4 & Q_3 & Q_2 & Q_1 \\ 0 & Q_6 & Q_4 & Q_3 & Q_2 \\ 0 & 0 & 0 & Q_6 & Q_5 \end{vmatrix}, \quad \Omega_6 = \text{Det}(\mathcal{J}_{\mathcal{E}^*}) = Q_6.$$

According to Routh–Hurwitz criterion [31], $\mathcal{E}^*$ is locally asymptotically stable (LAS) when $\Omega_i > 0$ for $i = 1, 2, 3, 4, 5, 6$, i.e., (i) $Q_1 > 0$ for $i = 1, 6$; (ii) $\Omega_i > 0$ for $i = 2, 3, 4, 5$. □

**Global stability of $E_0$:** Now we show the global stability of the disease-free equilibrium point with the help of Lyapunov function.

**Theorem 6.3.** The disease-free equilibrium $E_0$ of system (1) is globally asymptotically stable (GAS) if $S \leq S_0$ and $R_0 < 1$, where $S_0 = \frac{1}{(d + q_1)}$.

**Proof.** Let us consider the Lyapunov function $V_1 = p_2 I_1 + \beta_2 S_0 I_2$, where $S_0 = \frac{1}{(d + q_1)}$, $p_0 = (d + q_1)$, $p_1 = q_2 + \kappa + \alpha + d$ ans $p_2 = \phi_1 + \phi_2 + d + \mu_1$. Here $V_1$ is a positive definite function for all points except DFE. Then, the time derivative of $V_1$ computed along the solutions of system (1) is as follows:

$$\frac{dV_1}{dt} \leq I_1(\beta_1 I_1 + \beta_2 I_2)S - p_1 I_1] + \beta_2 S_0[\kappa I_1 - p_2 I_2]$$

$\leq I_1(\beta_1 S_0 p_2 + \kappa \beta_2 S_0 - p_1 p_2)$

$= p_1 p_2 (R_0 - 1)I_1$

$< 0$ (if $R_0 < 1$)

Furthermore, $\frac{dV_1}{dt} = 0$ at $E_0$. Hence, by LaSalle’s invariance principle [24], $E_0$ is globally asymptotically stable when $S \leq S_0$ with $R_0 < 1$. □

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7. Bifurcation analysis at $R_0 = 1$

The result of the central manifold theory, discussed by Castillo-Chavéz and Song [6], is stated in the following theorem:

**Theorem 7.1.** Consider the following system of ODEs with a parameter $\Phi$:

$$\frac{dX}{dt} = f(X, \Phi), \quad f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \quad \text{and} \quad f \in C^2(\mathbb{R}^n \times \mathbb{R}).$$

Let $O$ be taken as an equilibrium point of the mentioned system with $f(O, \Phi) = 0$ for all $\Phi$. Let us further assume

(I) $B = D_X f(O, 0) = (\frac{\partial f}{\partial x_j}(O, 0))$ be the linearization matrix of the mentioned system at the equilibrium $O$ and $\Phi$ evaluated at 0. $B$ has a simple zero eigenvalue and other eigenvalues of the matrix have negative real parts.

(II) $B$ contains a right eigenvector $w$ which is non-negative and also a left eigenvector $v$ corresponding to the zero eigenvalue.

If $f_k$ is considered to be the $k$th component of $f$ and

$$a = \sum_{k,i,j=1}^n v_kw_iw_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(O, 0), \quad b = \sum_{k,i=1}^n v_kw_i \frac{\partial^2 f_k}{\partial x_i \partial \Phi}(O, 0),$$

then the local dynamics of the system around $O$ is determined by the sign of $a$ and $b$.

1. $a > 0, b > 0$: (i) $O$ is locally asymptotically stable and there exists a positive unstable equilibrium for $\Phi < 0$ and $|\Phi| \ll 1$. (ii) Further $O$ is unstable and there exists a negative and locally asymptotically stable equilibrium for $0 < \Phi \ll 1$.

2. $a < 0, b < 0$: (i) $O$ is unstable for $\Phi < 0$ and $|\Phi| \ll 1$. (ii) Further $O$ is locally asymptotically stable, and there exists a positive unstable equilibrium for $0 < \Phi \ll 1$.

3. $a > 0, b < 0$: (i) $O$ is unstable, and there exists a locally asymptotically stable negative equilibrium for $\Phi < 0$ and $|\Phi| \ll 1$. (ii) Further $O$ is stable, and a positive unstable equilibrium appears for $0 < \Phi \ll 1$.

4. $a < 0, b > 0$: $O$ changes its stability from stable to unstable when $\Phi$ changes its sign from negative to positive. As a result, a negative unstable equilibrium point turns into positive and locally asymptotically stable equilibrium point.

The components of the right eigenvector $w$ may not be non-negative and it depends on the positivity of corresponding component of equilibrium (Remark 1 in [6]).

Even if some components of $w$ become negative, then also the theorem can be applied, though on that case one needs to compare $w$ with the equilibrium. The comparison is necessary as the general parameterization of the center manifold theory before changing the coordinate is

$W^{\theta} = \{X_0 + \theta(t)y + k(\theta, \Phi), |\theta| \leq \theta_0, \theta(0) = 0\}$ provided that $X_0$ is a non-negative equilibrium of the system (usually $X_0$ is the DFE). Hence, $X_0 - 2\frac{b\Phi}{a} > 0$ requires that $w(j) > 0$ whenever $X_0(j) = 0$. If $X_0(j) > 0$, then $w(j)$ need not be positive.

Let us redefine $S = x_1, \quad I_1 = x_2, \quad I_2 = x_3, \quad H = x_4, \quad R = x_5$ and $V = x_6$, then the system (1) can be rewritten as:

$$\begin{align*}
\frac{dx_1}{dt} &= \Lambda - (\beta_1 x_2 + \beta_2 x_3)x_1 - p_0 x_1 + \eta x_5 \equiv f_1, \\
\frac{dx_2}{dt} &= (\beta_1 x_2 + \beta_2 x_3)x_1 - p_1 x_2 \equiv f_2, \\
\frac{dx_3}{dt} &= \kappa x_2 - p_2 x_3 \equiv f_3, \\
\frac{dx_4}{dt} &= \phi_1 x_3 - p_3 x_4 \equiv f_4, \\
\frac{dx_5}{dt} &= \alpha x_2 + \phi_2 x_3 + \psi x_4 - p_4 x_5 \equiv f_5, \\
\frac{dx_6}{dt} &= q_1 x_1 + q_2 x_2 - dx_6 \equiv f_6.
\end{align*}$$

(5)
We have considered $\Phi = \beta_2$ as bifurcation parameter for $R_0 = 1$. Thus at $\Phi = \Phi^* = \beta_2^*$, $R_0 = 1$ gives $\beta_2^* = \frac{p_2(p_1 - \beta_1 S_0)}{\kappa S_0}$, where $S_0 = \frac{\Lambda}{p_0}$, $p_0 = d + q_1$, $p_1 = q_2 + \kappa + \alpha + d$, $p_2 = \phi_1 + \phi_2 + d + \mu_1$, $p_3 = \psi + d + \mu_2$ and $p_4 = d + \eta$. The linearized matrix of the model system (5) at $E_0 \left( \frac{\Lambda}{p_0}, 0, 0, 0, \frac{q_1 S_0}{d} \right)$ with bifurcation parameter $\beta_2 = \beta_2^*$ is given by

$$
\mathcal{J}_{E_0} = \begin{pmatrix}
-p_0 & -\beta_1 S_0 & -\beta_2 S_0 & 0 & \eta & 0 \\
0 & \beta_1 S_0 - p_1 & \beta_2 S_0 & 0 & 0 & 0 \\
0 & \kappa & -p_2 & 0 & 0 & 0 \\
0 & 0 & \phi_1 & -p_3 & 0 & 0 \\
0 & \alpha & \phi_2 & \psi & -p_4 & 0 \\
q_1 & q_2 & 0 & 0 & 0 & -d \\
\end{pmatrix}
$$

The characteristic equation of the corresponding Jacobian matrix is $\lambda^6 + P_1 \lambda^5 + P_2 \lambda^4 + P_3 \lambda^3 + P_4 \lambda^2 + P_5 \lambda + P_6 = 0$, where, $P_1 = p_0 + p_2 + p_3 + p_4 + d - (\beta_1 S_0 - p_1)$,

$P_2 = p_0(p_2 + p_3 + p_4) + p_2(p_1 + p_4) + p_3 p_4 + d(p_0 + p_2 + p_3 + p_4) + p_1 p_2(1 - R_0) - (\beta_1 S_0 - p_1)(d + p_0 + p_3 + p_4)$,

$P_3 = p_0 p_2(p_3 + p_4) + (p_0 + p_2)p_3 p_4 + d[p_0(p_2 + p_3 + p_4) + p_2(p_3 + p_4) + p_3 p_4] + p_1 p_2(d + p_0 + p_3 + p_4)(1 - R_0) - (\beta_1 S_0 - p_1) p_0 p_3 p_4 + P_6$.

So, $\mathcal{J}_{E_0}(\beta_2^*)$ has a zero eigenvalue at $R_0 = 1$ as $P_6|_{R_0=1} = 0$.

The right eigenvector corresponding to the zero eigenvalue of $\mathcal{J}_{E_0}(\beta_2^*)$ is denoted by $w = (w_1, w_2, w_3, w_4, w_5, w_6)^T$, where $w_1 = d[\eta (\alpha p_2 p_3 + \phi_2 \kappa p_3 + \psi \kappa \phi_1) - p_1 p_2 p_3 p_4]$, $w_2 = d p_0 p_2 p_3 p_4$, $w_3 = \kappa d p_0 p_3 p_4$, $w_4 = \phi_1 d k p_0 p_4$, $w_5 = d p_0 (p_2 p_3 + \phi_2 k p_3 + \psi k \phi_1) \phi_1$ and $w_6 = d q_1 p_2 p_3 p_4 + q_1 \eta (\alpha p_2 p_3 + \phi_2 k p_3 + \psi k \phi_1) - p_1 p_2 p_3 p_4$.

Also, the left eigenvector of $\mathcal{J}_{E_0}(\beta_2^*)$ corresponding to zero eigenvalue is $v = (v_1, v_2, v_3, v_4, v_5, v_6)^T$, where $v_1 = 0$, $v_2 = \kappa$, $v_3 = p_1 - \beta_1 S_0$, $v_4 = 0$, $v_5 = 0$ and $v_6 = 0$. Hence

$$a = \sum_{k,i,j=1}^n v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(E_0) = 2v_2 w_1 [\beta_1 w_2 + \beta_2 w_3] = 0, \quad (as \quad w_1 < 0, \quad w_2 > 0, \quad w_3 > 0)$$

$$b = \sum_{k,i=1}^n v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta_2}(E_0) = v_2 w_3 S_0 > 0$$

So, we have the following theorem:

**Theorem 7.2.** System (1) undergoes a transcritical (forward) bifurcation around the disease-free equilibrium $E_0$ at $R_0 = 1$ taking $\beta_2$ as the bifurcation parameter.

8. Numerical results without implementing control strategy

This section contains the numerical figures to support the analytical findings of the proposed model when no control strategies are implemented. India is considered to be the second most populated country across the world next to China and India population almost contribute 17.7% of total World population. According to the data of 20th June, 2021, India has almost 1.39 billion people. As the current birth rate here is 17.377 births per 1000 people, so, the recruitment rate ($\Lambda$) is taken approximately 65,000. We have done unit conversion from year to day to estimate the parameter values here. The total population in India is estimated 1.38 billion people in 2020, i.e., $S(0)$ is taken as $1.38 \times 10^8$. Also, current death rate is 7.344 people per 1000 people which gives $d = 0.00005$ (approx.). To calculate the parameter values we have chosen the data from 20th June, 2020 to 20th June, 2021. According to Worldometers, there was total 411,727 number of Covid cases in India on 20th June, 2020 among which active cases, death cases and recovered cases were 170, 230; 13, 316; 228, 181 respectively [11,42]. But at 20th June, 2021, total Covid cases has increased up to 29, 934, 361. Let $\beta$ be the overall disease transmission rate such that the new infected cases per unit time be $\beta S I \equiv (\beta_1 I_1 + \beta_2 I_2) S$. Total documented cases ($I$) up to 20th June, 2020 is 411,727; population of India ($S$) is currently $1.39 \times 10^9$ and new human cases from 20th June, 2020
Table 1: Parameter values used for numerical simulation of system (1).

| Parametric values | Value |
|-------------------|-------|
| \( \Lambda \)     | \( 6.5 \times 10^4 \) |
| \( \beta_1 \)     | \( 10^{-10} \) |
| \( \beta_2 \)     | \( 0.413 \times 10^{-10} \) |
| \( \eta \)        | \( 0.1 \) |
| \( q_1 \)         | \( 0.0006 \) |
| \( q_2 \)         | \( 0.0006 \) |
| \( \mu_1 \)       | \( 1.78 \times 10^{-5} \) |
| \( \mu_2 \)       | \( 1.78 \times 10^{-5} \) |
| \( \phi_1 \)      | \( 6.49 \times 10^{-5} \) |
| \( \phi_2 \)      | \( 0.001 \) |
| \( \alpha \)      | \( 0.005 \) |
| \( \psi \)        | \( 0.0026 \) |

Fig. 2: Stability of the system around DFE \( E_0 \).

to 20th June, 2021 (\( \beta SI \)) is 29,533,634 (which implies total 29,934,361 confirmed cases till that date). Then by unit conversion from year to day gives \( \beta \approx 1.413 \times 10^{-10} \). From the data of 20th June, 2021, among the total confirmed cases, there are 709,116; 388,716 and 28,836, 529 number of active cases, death cases and recovered cases respectively. Hence, we get \( \phi_1 \) as \( 6.49 \times 10^{-5} \); \( \mu_1 + \mu_2 \) as \( 3.56 \times 10^5 \) and \( \psi \) as 0.0026. In India, vaccination procedure has started on 16th January, 2021 and a total of 261,740,273 vaccine doses have been administered till 21th June, 2021. Then we get \( q_1 + q_2 \) as 0.0012 by the unit conversion. Table 1 contains the parametric values used in numerical simulation. Let us consider \( I_1(0) = 3 \times 10^5 \), \( I_2(0) = 1.8 \times 10^5 \), \( H(0) = 2 \times 10^5 \), \( R(0) = 5 \times 10^5 \) and \( V(0) = 10^2 \).

Fig. 2 shows that a trajectory starting from the mentioned initial point converges to the DFE \( E_0 \) for \( \beta_2 = 1 \times 10^{-11} \) along with other parametric values from Table 1, and we get \( R_0 \) as 0.949 < 1 here. So, the basic reproduction number, when lies below unity, we get an infection-free system.

Fig. 3 depicts that for the parametric values of Table 1, the system converges to the endemic equilibrium point \( E^*(2.85 \times 10^7, 9.33 \times 10^6, 5.77 \times 10^6, 1.40 \times 10^7, 6.60 \times 10^6, 4.54 \times 10^8) \). Here we get \( R_0 \) = 3.506 which exceeds unity. So, taking \( \beta_2 \) as the regulating parameter it is observed that the system undergoes a forward (transcritical) bifurcation at \( \beta_2 = \beta_{2[T C]} = 1.0623 \times 10^{-11} \) and \( E_0 \) becomes unstable for \( \beta_2 > \beta_{2[T C]} \) (see Fig. (4.a)). Also from Fig. (4.b), it is observed that \( E_0 \) is stable for \( R_0 < 1 \) and becomes unstable for \( R_0 > 1 \). Also, a stable branch of endemic equilibrium evolves from \( R_0 = 1 \). Fig. 5 depicts how some of the parameters affect the disease transmission in the system. It is observed that the increasing value of \( \beta_1 \) and \( \beta_2 \) increases the chances of the population become infected with coronavirus as these parameters are directly proportional with \( R_0 \). If the virus transmits among the susceptible population at a higher rate, then the disease fatality will only increase with time. On the other hand, if more people move to hospitals without ignoring their physical conditions, then the disease prevalence can be controlled to some certain extent.
It is the reason $\phi_1$ maintains an inversely proportional relation with $R_0$. Moreover, if people from susceptible and asymptotically infected classes start to take vaccines at the early stages with a higher rate, then the chances of becoming infected become lower. It ultimately reduces the fatality of the current situation.

The tornado plot of the sensitivity index of the parameters is shown in Fig. 6. For the parametric values of Table 1, the calculated sensitivity index are as follows: $\Gamma_{\beta_1} = 0.0377$, $\Gamma_{\beta_2} = 0.9622$, $\Gamma_{\phi_1} = -0.0551$, $\Gamma_{q_1} = -0.9231$ and $\Gamma_{q_2} = -0.0079$. So, it is observed that $\beta_2$ and $q_1$ are more sensitive than others. Vaccination to susceptible people help to control the disease transmission to a greater extent in fact.

In Fig. 7, the impact of vaccination on disease propagation is illustrated. Both Fig. (7.a) and (7.b) reveal that the count of asymptomatically infected people decreases with the increase of vaccination rates ($q_1$, $q_2$). It means the infection level and fatality starts to decrease if more people take the recommended vaccines as a precautionary measure before getting severely infected.

In Fig. 8, the asymptptomatically infected population is plotted with increasing virus transmission rates ($\beta_1$, $\beta_2$) and it is observed that the count of infected individuals increases more for higher value of $\beta_2$ than $\beta_1$. It means frequent contact with symptomatically infected people than the people who are in the pre-symptomatic state actually increases the infection in the system.
Fig. 5. Plots of basic reproduction number $R_0$ with the variation of $\beta_1$, $\beta_2$, $\phi_1$, $q_1$ and $q_2$.

Fig. 6. Sensitivity index of the parameters $\beta_1$, $\beta_2$, $\phi_1$, $q_1$ and $q_2$ for $R_0$.

Fig. 7. The effect of (a) $q_1$ and (b) $q_2$ on the asymptotically infected population ($I_1$).

9. Optimal control problem

The model system is reintroduced in this section by implementing some control interventions which can reduce the disease burden. Maintaining social distances and proper hygiene is one of the important precautionary measures
which is advised to be followed by each and every one. Besides it, there are different vaccines are available now and people are asked to take the proper dosage of available vaccines to avoid further infection. So, these strategies are incorporated into this system to reduce the rapid transmission. Worldwide high disease transmission ensures that there must be Covid cases without showing any kind of symptoms and this asymptomatic transmission of COVID-19 has made the situation worse in terms of controlling the spread. So, maintaining a safe distancing in population as well as vaccination to all the people are considered to be the control policies. The analysis is performed to observe the impact of the control policies to reduce the incidence of transmission of disease and also to obtain the optimal cost burden. Let us first describe the control strategies one by one.

**Increase the awareness of social distancing and maintaining hygiene:** People can be aware of a disease and its prevalence when they are provided with the necessary information. It helps them to bring behavioral changes and instigate to take precautionary measures for not becoming infected. Day-to-day updates on different news portals and live tracking sites help to increase the cautiousness among the population. Now people become infected when they come in contact with any of the infected classes (asymptomatic and symptomatic). So, maintaining a proper physical distance is one of the main ways to curb the higher disease transmission. In this work, it is considered that the $u_1$ portion of the susceptible population maintains that social distancing and takes other precautionary measures (using the face masks, maintaining enough hygiene, etc.). So, only $(1 - u_1)S$ of susceptible individuals move to the pre-symptomatic or asymptomatic stage after contact with infected people. In system (7), $u_1$ denotes the intensity of maintaining physical distancing with $0 \leq u_1 \leq 1$, where $u_1 = 0$ means not maintaining the distancing at all and $u_1 = 1$ means full maintenance of distancing. As the awareness depends on the infectivity and disease fatality, so, $u_1(t)$ is taken as one control intervention.

**Increasing the vaccination rates of population:** Vaccination is another strategy that reduces the rate of infection only if people go through the procedure as early as possible. Fortunately, there are many vaccines available now for COVID-19. The vaccination programs, firstly, may take some time for implementation, and also some time is required for individuals to develop immunity after inoculation. Governments of almost every country have conducted several awareness programs to make people to understand the importance of the recommended vaccines and requested people to be vaccinated to avoid the infection further. Taking vaccines at an early stage can decrease the disease burden. So, instead of constant values, time-dependent vaccination rate functions $q_1(t)$ and $q_2(t)$ are considered here with the restrictions $0 \leq q_1(t) \leq 1$ and $0 \leq q_2(t) \leq 1$. By implementing these control policies, the overall chances of becoming infected with coronavirus would be lessened. Here, 1 denotes when all people take vaccines as an important precautionary measure and 0 denotes the case when no person becomes vaccinated.

The main work is to determine optimal control strategies with minimum implemented cost. So, the region for the control interventions $u_1(t)$, $q_1(t)$ and $q_2(t)$ is given as:

$$
\Psi = \{(u_1(t), q_1(t), q_2(t)) \mid (u_1(t), q_1(t), q_2(t)) \in [0, 1] \times [0, 1] \times [0, 1], t \in [0, T_f]\},
$$
where $T_f$ is the final time up to which the control policies are executed, and also $u_1(t)$, $q_i(t)$ for $i = 1, 2$ are measurable and bounded functions.

9.1. Deduction of total cost which needs to be minimized

(i) **Cost incurred in maintaining social distancing and proper hygiene**: The total cost incurred maintaining social distancing and other precautionary measures is given by:

$$
\int_0^{T_f} \left[ w_2 u_1^2(t) \right] dt
$$

The integrand term $w_2 u_1^2(t)$ represents the cost of spreading awareness regarding social distancing and maintaining hygiene. This cost is comparatively higher because it considers the associated efforts for convincing people. There is some literature revealing the cost incurred for some mitigation strategies like self-protective measures etc. with second-order nonlinearity term $[5, 21]$. This work analyzes how the optimal control strategy representing social distancing reduces the overall count of the infective population in the system.

(ii) **Cost incurred in vaccination**: Total cost associated with the vaccination of susceptible and asymptomatically infected individuals is:

$$
\int_0^{T_f} \left[ w_1 I_2(t) + w_3 q_1^2(t) + w_4 q_2^2(t) \right] dt
$$

Here $w_1 I_2(t)$ denotes the cost associated with symptomatically infected population for losing manpower $[14, 19, 21]$. The terms $w_3 q_1^2(t)$ and $w_4 q_2^2(t)$ denote the expenditure of vaccination procedure provided to susceptible and asymptomatically infected people respectively. These two terms also include the opportunity losses in terms of productivity loss due to the overall vaccination procedure. The control policies $q_1(t)$ and $q_2(t)$ are considered up to second-order non-linearity terms $[14, 19, 21]$.

The following control problem is considered based on previous discussions along with the cost functional $J$ to be minimized:

$$
J[u_1(t), q_1(t), q_2(t)] = \int_0^{T_f} \left[ w_1 I_2(t) + w_2 u_1^2(t) + w_3 q_1^2(t) + w_4 q_2^2(t) \right] dt
$$

subject to the model system:

\begin{align*}
\frac{dS}{dt} &= \Lambda - (1 - u_1(t))(\beta_1 I_1 + \beta_2 I_2)S - dS - q_1(t)S + \eta R, \\
\frac{dI_1}{dt} &= (1 - u_1(t))(\beta_1 I_1 + \beta_2 I_2)S - q_2(t)I_1 - \kappa I_1 - \alpha I_1 - dI_1, \\
\frac{dI_2}{dt} &= \kappa I_1 - (\phi_1 + \phi_2)I_2 - (d + \mu_1)I_2, \\
\frac{dH}{dt} &= \phi_1 I_2 - \psi H - (d + \mu_2)H, \\
\frac{dR}{dt} &= \alpha I_1 + \phi_2 I_2 + \psi H - dR - \eta R, \\
\frac{dV}{dt} &= q_1(t)S + q_2(t)I_1 - dV,
\end{align*}

with initial conditions $S(0) > 0$, $I_1(0) \geq 0$, $I_2(0) \geq 0$, $H(0) \geq 0$, $R(0) \geq 0$ and $V(0) > 0$. We have already considered $p_2 = (\phi_1 + \phi_2 + d + \mu_1)$, $p_3 = (\psi + d + \mu_2)$ and $p_4 = (d + \eta)$. The functional $J$ denotes the total incurred cost as stated and the integrand $L(S, I_1, I_2, H, R, V, u_1(t), q_1(t), q_2(t)) = w_1 I_2(t) + w_2 u_1^2(t) + w_3 q_1^2(t) + w_4 q_2^2(t)$ denotes the cost at time $t$. Positive parameters $w_1$, $w_2$, $w_3$ and $w_4$ are weight constants balancing the units of the integrand $[14,21]$. The optimal control interventions $u_1^*$, $q_1^*$ and $q_2^*$ exist in $\psi$, mainly minimize the cost functional $J$. 

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Theorem 9.1. The optimal control interventions $u_i^*$, $q_i^*$ and $q_i^*$ in $\Psi$ of the control system (6)–(7) exist such that $J(u_1^*, q_1^*, q_2^*) = \min \{ J(u_1, q_1, q_2) \}$.

Proof. Proof is given in Appendix. □

Theorem 9.2. If the optimal controls $u_i^*$ for $i = 1, 2$ and corresponding optimal states $(S^*, I_1^*, I_2^*, H^*, R^*, V^*)$ exist for the control system, then we have adjoint variables $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_6) \in \mathbb{R}^6$ satisfying the canonical equations:

\[
\begin{align*}
\frac{d\lambda_1}{dt} &= \lambda_1[(1 - u_1)(\beta_1 I_1 + \beta_2 I_2) + d + q_1] - \lambda_2[(1 - u_1)(\beta_1 I_1 + \beta_2 I_2)] - \lambda_6(q_1) \\
\frac{d\lambda_2}{dt} &= \lambda_1[(1 - u_1)\beta_1 S] - \lambda_2[(1 - u_1)\beta_1 S - q_2 - (\kappa + \alpha + d)] - \lambda_3(\kappa) - \lambda_5(\alpha) - \lambda_6(q_2) \\
\frac{d\lambda_3}{dt} &= -w_1 + \lambda_1[(1 - u_1)\beta_2 S] - \lambda_2[(1 - u_1)\beta_2 S + \lambda_3(p_2) - \lambda_4(\phi_1) - \lambda_5(\phi_2)] \\
\frac{d\lambda_4}{dt} &= \lambda_4(p_3) - \lambda_5(\psi) \\
\frac{d\lambda_5}{dt} &= -\lambda_1(\eta) + \lambda_5(p_4) \\
\frac{d\lambda_6}{dt} &= \lambda_6(d)
\end{align*}
\]  

with transversality conditions $\lambda_i(T_f) = 0$ for $i = 1, 2, \ldots, 6$. The corresponding optimal controls $u_i^*$, $q_i^*$ and $q_i^*$ are given as:

\[
\begin{align*}
\frac{d\lambda_1}{dt} &= \lambda_1[(1 - u_1)(\beta_1 I_1 + \beta_2 I_2) + d + q_1] - \lambda_2[(1 - u_1)(\beta_1 I_1 + \beta_2 I_2)] - \lambda_6(q_1) \\
\frac{d\lambda_2}{dt} &= \lambda_1[(1 - u_1)\beta_1 S] - \lambda_2[(1 - u_1)\beta_1 S - q_2 - (\kappa + \alpha + d)] - \lambda_3(\kappa) - \lambda_5(\alpha) - \lambda_6(q_2) \\
\frac{d\lambda_3}{dt} &= -w_1 + \lambda_1[(1 - u_1)\beta_2 S] - \lambda_2[(1 - u_1)\beta_2 S + \lambda_3(p_2) - \lambda_4(\phi_1) - \lambda_5(\phi_2)] \\
\frac{d\lambda_4}{dt} &= \lambda_4(p_3) - \lambda_5(\psi) \\
\frac{d\lambda_5}{dt} &= -\lambda_1(\eta) + \lambda_5(p_4) \\
\frac{d\lambda_6}{dt} &= \lambda_6(d)
\end{align*}
\]

Proof. Proof is given in Appendix. □

10. Numerical simulation of the optimal control problem

In the proposed model, mainly two types of control strategies are implemented in order to reduce the disease burden and to minimize the cost incurred for the implementation of the control intervention. Maintaining social distancing, proper hygiene, and vaccination at the early stage — these are the main precautionary measures one should maintain to avoid being infected with coronavirus. We consider $u_1$ and $q_i$ for $i = 1, 2$ as the control variables, where $u_1$ fraction of people in the susceptible environment maintain proper distancing and other precautionary measures, and $q_1$, $q_2$ are the vaccination rates of people in susceptible and pre-symptomatic stages respectively. All the parametric values and positive weight constants, which are used to perform the numerical simulation here, are listed in Table 2. The initial population size is considered as follows: $S(0) = 1.38 \times 10^6$, $I_1(0) = 3 \times 10^5$, $I_2(0) = 180$, $H(0) = 2 \times 10^2$, $R(0) = 500$ and $V(0) = 10^2$ to solve the control system in Eqs. (6)–(7). The numerical simulations are performed in MATLAB using forward–backward sweep method for control interventions [22]. It is assumed that the control interventions are implemented for two month, i.e., $T_f = 60$ days.

Fig. 9 depicts the dynamics of model (7) when no time-dependent control policies are implemented, i.e., when $u_1 = 0$ and $q_i = 0.0006$ for $i = 1, 2$. At $T_f = 60$, the population becomes $(5160981, 397778.6960, 406.8855, 3452.8266, 120750, 1001)$. The count of the asymptotically infected population decreases at a slower rate, whereas the symptomatically infected population increases throughout the time. It is observed that the number of overall infected (both asymptotically and symptomatically) individuals remains significantly higher in this case.
Table 2
Parametric values for numerical simulation of model (7).

| Parametric values | Value       |
|-------------------|-------------|
| \( \Lambda \)     | \( 6.5 \times 10^2 \) |
| \( \beta_1 \)     | \( 1.413 \times 10^{-10} \) |
| \( \beta_2 \)     | \( 0.433 \times 10^{-9} \) |
| \( \eta \)        | 0.1         |
| \( q_1 \)        | 0.0006      |
| \( \mu_1 \)      | \( 1.78 \times 10^{-5} \) |
| \( \phi_1 \)     | \( 6.49 \times 10^{-5} \) |
| \( \alpha \)     | 0.005       |
| \( \psi \)       | 0.0026      |
| \( w_1 \)        | 40          |
| \( w_2 \)        | 1.5         |
| \( w_3 \)        | 50          |
| \( w_4 \)        | 200         |

Fig. 9. Profiles of populations in absence of control policies.

Next, we consider the cases when people maintain physical distancing to avoid further infection and the vaccines are provided at constant rates. People in hospitals are already under strict restrictions and so, we are not considering any extra control policy for them. Fig. 10 depicts the population profiles when \( u_1 = u_1^* \) and \( q_1 = q_2 = 0.0006 \). At \( T_f = 60 \), the population becomes (5181165.1330, 3307.8807, 260453.4143, 394.8829, 3078.1103, 120908.0058).

When only \( u_1 \) is implemented, the susceptible population increases as only a fraction of susceptible maintain physical distancing and rest move to asymptomatically infected class \( (I_1) \). The overall count of infected people decreases by implementing social distancing as a control strategy. The count of recovered people also decreases as a lesser number of the population becomes infected. The corresponding graph of optimal control intervention is depicted in Fig. 11. The control variable works with the highest intensity almost throughout the period.

Next, let us consider the case when vaccination provided to susceptible class depends on the severity of virus transmission, and hence, is considered to be time-dependent. Fig. 12 shows the dynamics of model (7) for \( q_1 = q_1^* \) and \( u_1 = 0, \quad q_2 = 0.0006 \). At \( T_f = 60 \), the population becomes (653010.0872, 3302.9429, 260779.2258, 395.1730, 3085.3708, 5236443.2060). In this case, the count of susceptible population decreases as these people, after vaccination, move to the vaccinated compartment. So, the count of vaccinated people increases at a higher rate. Moreover, a decrease in the susceptible population leads to a declination of the infected population as well as the recovered population. Fig. 13 depicts corresponding graph of optimal control intervention of \( q_1^* \) when \( u_1 = 0, \quad q_2 = 0.0006 \). From this figure, it is observed that this control strategy works with the highest intensity immediately after implementation and remains at its highest throughout the whole time period.
Fig. 10. Diagrams of the population in presence of optimal control $u_1^*$ only and $q_1 = q_2 = 0.0006$.

Now, we consider that situation when susceptible people are given the vaccines at a constant rate, but people who are in the asymptomatic or pre-symptomatic state are given vaccines depending on the severity (so, time-dependent). Fig. 14 shows the dynamics of model system (7) for $q_2 = q_2^*$ and $u_1 = 0$, $q_1 = 0.0006$. At $T_f = 60$, the population becomes $(5153294.3107, 94.9403, 18546.0253, 185.9842, 198.6641, 397705.2411)$. Here the count of asymptptomatically infected people significantly decreases than the case when the vaccination rate is constant. As these people, after full vaccination, move to the vaccinated compartment, so, the count of vaccinated people automatically increases. The graphs reveal that the overall infected population falls off at a significant level. Fig. 15 depicts corresponding graph of optimal control policy $q_2^*$ when $u_1 = 0$, $q_1 = 0.0006$. From this figure, it is observed that $q_2$ works with the highest intensity almost for the whole time before decreasing in the last week.

Now, we consider the case when overall vaccination strategy changes with time depending on the infection level in absence of social distancing ($u_1 = 0$, $q_i = q_i^*$ for $i = 1, 2$). Fig. 16 depicts the dynamics of model system (7) for these control policies and at $T_f = 60$, population becomes $(65015.9027, 3.0362, 18446.8118, 185.8953, 196.0166, 5486177.5950)$. The declination of the susceptible population and asymptomatically infected population in fact reduces the level of overall infection in the system as well as increases the count of the vaccinated population to a higher extent. As a larger number of people take vaccines at the pre-symptomatic state or even before getting infected, so, the count of infected population as well as recovered population decrease. Fig. 17 depicts the optimal...
graphs of control policies $q_1^*$, $q_2^*$ when $u_1 = 0$. The control strategy denoting $q_1$ works with the highest intensity after one or days of implementation and remains at its highest value throughout the time period. On the other hand, $q_2$ works with the highest intensity for almost two weeks and then decreases with time.

Implementation of all control strategies works better to control the disease burden than the case when a single control policy or only two control policies are applied. Now, we consider the case where a part of susceptible people maintains physical distancing and takes precautionary measures, and both susceptible and asymptomatically infected people are given vaccines according to the severity to curb the high transmission. Fig. 18 depicts the population trajectories in presence of all the control policies and at $T_f = 60$, population becomes $(65015.8899, 2.6375, 18438.7653, 185.8883, 195.8845, 5486186.2138)$. As only a part of the susceptible moves to the asymptomatic or pre-symptomatic phase, it reduces the count of the infected population. Moreover, the available vaccination procedure also decreases the overall infection in the system as people are advised to take vaccines as early as possible for self-protection. Fig. 19 depicts the optimal graphs of all the control policies. The intensity of $u_1$ remains at the highest almost all the time and then decreases in the last week of the duration of control implementation. The control $q_1$ also works with the highest intensity throughout the time period but $q_2$ works for two weeks after implementation and then decreases with time.
Fig. 20 describes the cost design analysis \((J)\) and count of symptomatically infected population \((I_2)\) of the control system in absence and presence of time-dependent control interventions \(u_1^*, q_1^*\) and \(q_2^*\). In Fig. (20.a) optimal cost profiles are shown which reveals that in absence of control strategies the cost occurred due to productivity loss and it is quite higher as the number of the infected population is higher in this case.

In Fig. 21, the count of symptomatically infected population \((I_2)\) and vaccinated population \((V)\) have been plotted. We have not considered the situation with social distancing and compare the graphs when vaccination rates are constant with time-depending vaccination rates. It is observed that in presence of these optimal control strategies, the count of the infected population decreases significantly even in absence of social distancing. Also, the number of people in the vaccinated compartment increases at a higher rate when the overall vaccination depends on the severity of disease transmission and changes with time. From the figures, it is observed that the vaccination at the early stages itself decreases the level of infectivity significantly which reduces the disease burden. So, the proposed control strategies are useful to reduce the number of the overall infected population in the system.

From Fig. 22 it is observed that \(q_2(t)\) works with the highest intensity for the longest time when it is applied alone. When all the control interventions are applied, the highest intensity of \(q_2\) remains for almost 15 days and then decreases with a steepness. It means maintaining social distancing \((u_1(t))\) and vaccination to susceptible \((q_1(t))\) actually decrease the higher need of vaccination of symptomatically infected people. On the other hand, when a
vaccine is provided to both susceptible and asymptomatically infected people, even in absence of social distancing, the intensity of $q_2^*$ decreases slowly with time after working with the highest intensity for a fortnight.

11. Conclusion

Coronavirus was first reported in Wuhan, China, and then spreads worldwide within few months. Observing the severity, WHO declared the disease as a pandemic that affects not only the health of people but social and economical balance have also been disturbed. The Governments in each country called for a full or partial lockdown state-wise or in the whole country observing the fatality. From the very beginning, every people were suggested to follow some non-pharmaceutical intervention measures such as maintaining social distancing, proper hygiene, etc. According to the data of 20th June, 2021, there are total 178,488,817 confirmed COVID-19 cases worldwide with 303,883 number of newly reported cases [11,42]. Just after USA, India is in the second position in terms of confirmed cases with 29,934,361 number of total cases among which the active cases, death cases and recovered cases are 709,116; 388,716 and 28,836,529 respectively [11,18,30,42]. Fortunately, after several trials, there are some vaccines available now for COVID-19. All people are requested by the Government to take a full vaccination
of two-dose once their turn comes. There are three approved vaccines are available in India named Covishield, Covaxin, and Sputnik V. Recently, the Indian government has given approval to Indian pharma company Cipla to import Moderna vaccine which has shown nearly 95% efficiency against COVID-19 [17]. According to the report of 21st June 2021, a total of 2,41,43,47,324 vaccine doses have been administered globally. In India, the vaccination procedure started on 16th January 2021. Till 21st June, total 261,740,273 doses are administered [41].

KWOK et al. in their work have suggested increased vaccination to gain immunity that leads to herd immunity in a country and may stop the spread of COVID-19 eventually [23]. But there are still some uncertainties regarding the effectiveness of the available vaccines and so, achieving herd immunities is not assured for a whole population. Hence, the importance of non-pharmaceutical interventions is stated there. Though we have not dealt with herd immunity in this work, a few remarks can be made here. The novel betacoronavirus is a mutant RNA virus. So, if the reinfection occurs, the severity of the disease will be less if a population gains herd immunity. It is true that achieving herd immunity for COVID-19 is a bit time taking because of vaccine hesitancy, uneven vaccine roll-out, etc. Also, this herd immunity may not prevent infection as such, especially when a mutated strain arises,
Fig. 20. Graph of (a) cost distribution and (b) symptomatically infected population ($I_2$) in the absence and presence of implemented time-dependent control interventions.

Fig. 21. In absence of social distancing ($u_1 = 0$), graph of (a) symptomatically infected population ($I_2$) and (b) vaccinated population ($V$) in the presence of optimal vaccination strategy ($q_1^*, q_2^*$) and constant vaccination rates ($q_1 = q_2 = 0.0006$).

Fig. 22. Graph of optimal vaccination strategy given to asymptomatically infected people ($q_2^*$) in presence and absence of other control interventions.
but it evidently helps to reduce the higher possibility of spread of disease and thus the chance of infection. For example, the mass vaccination strategy for polio vaccine and rotavirus vaccine reduce the growth of these viruses and consequently the spread of diseases even if these are RNA viruses.

In this work, we have proposed a compartmental epidemic model to analyze the transmission of coronavirus. A separate compartment for vaccinated people is considered here, where people from susceptible and asymptomatic states move after a complete two-dose vaccine. The proposed system is biologically well-posed and an endemic equilibrium point exists when the basic reproduction number exceeds unity. It is observed that vaccination of susceptible population is more effective to reduce the overall infected people in the system. Also, the probability of becoming infected increases when a person frequently comes in contact with symptomatically infected people. The second part of the work contains a corresponding optimal control problem. It is considered that the chance of disease transmission is reduced when people adopt some behavioral changes in terms of maintaining social distancing and proper hygiene, and in system (7), the disease transmits into \((1 - u_1)S\) amount of susceptible people only. As these changes depend on the severity and vary with time, so it is considered as a control intervention. Moreover, the vaccination procedure also changes with time and is considered to be another control strategy to reduce the disease burden in the system. When only one control policy is implemented in the system, it is observed that the policy works with the highest intensity for a larger time. On the other hand, when all the control strategies work simultaneously, the vaccination to susceptible works with the highest intensity throughout the period. The control which denotes social distancing also works with the highest intensity for quite a long time but decreases in the last few days. And, vaccination to asymptotically infected people works with the highest intensity for almost two weeks and then decreases with time. It is also observed these control policies reduce the count of the overall infected population significantly, when implemented altogether. The number of people in vaccinated compartment increases at a higher rate here, resulting in reducing the chance of becoming infected. Thus, the vaccination have a significant impact on mitigating COVID-19 outbreaks and the non-pharmaceutical interventions are also equally essential to decease the transmission.

There are many factors present in the environment. But, we try to avoid some of the factors while formulating a mathematical model to reduce the complexity. For example, in this work, we have considered that the people who undergo a full vaccination process cannot be infected in near future. But the effect of the vaccine may start to fade after a certain time, and a portion of vaccinated people move to the susceptible phase again. Here, we try to analyze the dynamics of the system when the vaccinated people become susceptible at a rate \(\xi\) due to the waning effect of vaccination. Then the susceptible and vaccinated classes of the proposed system become

\[
\frac{dS}{dt} = \Lambda - (\beta_1 I_1 + \beta_2 I_2)S - dS - q_1 S + \eta R + \xi V,
\]

\[
\frac{dV}{dt} = q_1 S + q_2 I_1 - dV - \xi V
\]

while the other compartments remain unchanged. For this model, we get the DFE as \(E_{00}(S_{00}, 0, 0, 0, V_{00})\), where \(S_{00} = \frac{\Lambda}{d + q_1 + \xi}\) and \(V_{00} = \frac{q_2}{d + q_1 + \xi}\). Also, the basic reproduction number becomes \(R_{00} = \frac{S_{00}(\beta_1 p_1 + \beta_2 p_2)}{d + q_1 + \xi}\), and we get \(R_{00} > R_0\) as \(S_{00} > S_0\). It means the basic reproduction number takes a higher value if we incorporate the waning effect of vaccines as we get a larger number of susceptibles in this case. It leads to a situation where the chances of getting infected increases significantly.

Also, the scenario in Fig. 23 reveals that the waning effect increases the count of overall infected people in the system. From Fig. (24.a), it is observed that the infection level rises sharply for a smaller waning rate, whereas the steepness reduces for an increasing value of \(\xi\). In Fig. (24.b), the recovered population also first increases with a higher rate, but with the increase of waning rate, the count tends to a saturation level. So, it can be concluded that the waning effect of vaccination though increases the infected population, but ultimately it leads to higher recovery.

There remain some limitations that need to be stated while forming an epidemic model. In a population, it is assumed that each person is moving and has an equal chance of contact with each other. But the mixing of people in a large population is not homogeneous. Also, it is considered that the virus transmission rate maintains a constant value throughout the period of a disease outbreak (pandemic). But, immigration and emigration of people in a population increase the chance of infection and reinfection. Moreover, the proposed model does not account for age structures in the population. As for COVID-19, the older generation was severely affected when the second wave started. Moreover, most of the system parameters are taken as an average basis, i.e., immunity, susceptibility, recovery, etc. are taken to be the same for all people of the population. It may happen that the contact rate becomes higher for a portion of people only. Henceforth, the model system is suitable to describe the pandemic for a large period of time but it is not that fruitful for details in a very small period of time.
Fig. 23. Trajectory profile of Symptomatically infected population ($I_2$) for different vaccination waning rate ($\xi$). The parametric values are taken from Table 1.

Fig. 24. Change in (24.a) Symptomatically infected population ($I_2$), and (24.b) Recovered population ($R$) with the increase in waning rate ($\xi$). The parametric values are taken from Table 1.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Some data used in this work have been obtained from the official sources as stated in the references [11,18,30,42]. The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Appendix

A.1. Existence of optimal control functions

Now we derive the conditions under which optimal control policies exist with minimized cost function in a finite time span. To establish the existence, we shall take the help of results proved in [13,14].

Proof of Theorem 9.1. For existence of optimal controls, the following conditions have to be satisfied:

(i) Set of solutions of the system (7) with control variables in \( \Psi \neq \phi \).

(ii) \( \Psi \) is closed, convex and state system can be expressed as a linear function of control variables where the coefficients will depend on time and state variables.

(iii) The integrand \( L \) of Eq. (6) is convex on \( \Psi \) and \( L(S, I_1, I_2, H, R, V, u_1, q_1, q_2) \geq g(u_1, q_1, q_2) \), where \( g(u_1, q_1, q_2) \) is continuous and \( \|(u_1, q_1, q_2)\|^{-1} g(u_1, q_1, q_2) \to \infty \) whenever \( \|(u_1, q_1, q_2)\| \to \infty \), here \( \|\cdot\| \) represents the \( L^1(0, T_f) \) norm.

In optimal system (7), let us consider \( N = S + I_1 + I_2 + H + R + V \).

So, \( \frac{dN}{dt} = \lambda - dN - u_1 I_1 - u_2 H \)

\( \Rightarrow 0 < N(t) \leq \frac{\lambda}{d} + \left( N(0) - \frac{\lambda}{d} \right) e^{-dt}, \)

where \( N(0) \) is the total population size at the initial stage.

As \( t \to \infty \), \( 0 < N(t) \leq \frac{\lambda}{d} + \epsilon \), for any \( \epsilon > 0 \).

Now the solution of (7) is bounded and also the right hand side of the system are locally Lipschitzian functions in presence of the control variables in \( \Psi \). Therefore, by Picard–Lindelöf theorem condition, the solution of the optimal system along with implemented optimal control strategies exist in \( \Psi \) and is non-zero [10].

Also, the set \( \Psi \), in which control variables are defined, is closed and convex. Again each of the equations of the stated system can be written as a linear equation in terms of \( u_1, q_1 \) and \( q_2 \) along with coefficients depending on state variables and so, condition (ii) is also satisfied. The integrand function \( L(S, I_1, I_2, H, R, V, u_1, q_1, q_2) \) is a convex function on \( \Psi \) as the control variables are of order two.

Now, \( L(S, I_1, I_2, H, R, V, u_1, q_1, q_2) = w_1 I_2 + w_2 u_1^2 + w_3 q_1^2 + w_4 q_2^2 \geq w_2 u_1^2 + w_3 q_1^2 + w_4 q_2^2 \)

Let, \( \overline{w} = \min(w_2, w_3, w_4) > 0 \) and \( k(u_1, q_1, q_2) = \overline{w}(u_1^2 + q_1^2 + q_2^2) \) which is a continuous function.

Then \( L(S, I_1, I_2, H, R, V, u_1, q_1, q_2) \geq k(u_1, q_1, q_2) \).

Here \( k \) is continuous and \( \|(u_1, q_1, q_2)\|^{-1} k(u_1, q_1, q_2) \to \infty \) whenever \( \|(u_1, q_1, q_2)\| \to \infty \). So, condition (iii) is also fulfilled which implies the existence of control variables \( u_1^*, q_1^* \) and \( q_2^* \) with \( J[u_1^*, q_1^*, q_2^*] = \min[J[u_1, q_1, q_2]] \). \( \square \)

A.2. Characterization of control interventions

The optimal control functions in the optimal system is obtained with the help of Pontryagin’s Maximum Principle [13,33]. Consider the following Hamiltonian function:

\[
\overline{H}(S, I_1, I_2, H, R, V, u_1, q_1, q_2, \lambda) = L(S, I_1, I_2, H, R, V, u_1, q_1, q_2) + \lambda_1 \frac{dS}{dt} + \lambda_2 \frac{dI_1}{dt} + \lambda_3 \frac{dI_2}{dt} + \lambda_4 \frac{dH}{dt} + \lambda_5 \frac{dR}{dt} + \lambda_6 \frac{dV}{dt}
\]
So, \( \bar{H} = w_1 I + w_2 u_2^2 + w_3 q_1^2 + w_4 q_2^2 + \lambda_1[A - (1 - u_1(t))(\beta_1 I_1 + \beta_2 I_2)S - dS - q_1(t)S + \eta R] \\
+ \lambda_2[(1 - u_1(t))(\beta_1 I_1 + \beta_2 I_2)S - q_2(t)I_1 - (\kappa + \alpha + d)I_1] + \lambda_3[\xi I_1 - p_2 I_2] \\
+ \lambda_4[\phi_1 I_2 - p_3 H] + \lambda_5[\phi_1 I_2 - p_3 H] \\
+ \lambda_6[q_1(t)S + q_2(t)I_1 - dV] \)

(11)

Here the adjoint variables are denoted by \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \). In order to minimize the cost function, the Hamiltonian function needs to be minimized by Pontryagin’s Maximum Principle.

**Proof of Theorem 9.2.** For the control system (7), let us consider \( u^*_1, q^*_i \) (for \( i = 1, 2 \)) are the applied optimal control interventions along with optimal state variables \( S^*, I^*_1, I^*_2, H^*, R^*, V^* \). Then there exist adjoint variables \( \lambda_i \), for \( i = 1, 2, \ldots, 6 \), satisfying the canonical equations:

\[
\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial I_1}, \quad \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I_2}, \quad \frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial H}, \quad \frac{d\lambda_5}{dt} = -\frac{\partial H}{\partial R}, \quad \frac{d\lambda_6}{dt} = -\frac{\partial H}{\partial V}. \]

So, we have

\[
dl_1 \frac{dt}{dt} = \lambda_1[(1 - u_1)(\beta_1 I_1 + \beta_2 I_2) + d + q_1] - \lambda_2[(1 - u_1)(\beta_1 I_1 + \beta_2 I_2)] - \lambda_6(q_1) \\
\frac{d\lambda_2}{dt} = \lambda_1[(1 - u_1)\beta_1 S] - \lambda_2[(1 - u_1)\beta_1 S - q_2 - (\kappa + \alpha + d)] - \lambda_3(\kappa) - \lambda_5(\alpha) - \lambda_6(q_2) \\
\frac{d\lambda_3}{dt} = -w_1 + \lambda_1[(1 - u_1)\beta_2 S] - \lambda_2[(1 - u_1)\beta_2 S] + \lambda_3(p_2) - \lambda_4(\phi_1) - \lambda_5(\phi_2) \\
\frac{d\lambda_4}{dt} = \lambda_4(p_3) - \lambda_5(\psi) \\
\frac{d\lambda_5}{dt} = -\lambda_1(\eta) + \lambda_5(p_4) \\
\frac{d\lambda_6}{dt} = \lambda_6(d) \]

(12)

with the transversality conditions \( \lambda_i(T_f) = 0 \), for \( i = 1, 2, 3, 4, 5, 6 \).

From optimality conditions : \( \frac{\partial H}{\partial u_1} \bigg|_{u_1 = u^*_1} = 0, \quad \frac{\partial H}{\partial q_1} \bigg|_{q_1 = q^*_1} = 0 \) and \( \frac{\partial H}{\partial q_2} \bigg|_{q_2 = q^*_2} = 0 \).

So, \( u^*_1 = \frac{(\beta_1 I^*_1 + \beta_2 I^*_2)S^*}{2w_2} (\lambda_2 - \lambda_1) \), \( q^*_1 = S^* \left( \frac{1}{2w_3} (\lambda_1 - \lambda_6) \right) \) and \( q^*_2 = \frac{I^*_1}{2w_4} (\lambda_2 - \lambda_6) \).

So, in \( \Psi \), we have

\[
u^*_1 = \begin{cases} 0, & \text{if } \frac{(\beta_1 I^*_1 + \beta_2 I^*_2)S^*}{2w_2} (\lambda_2 - \lambda_1) < 0 \\
\frac{(\beta_1 I^*_1 + \beta_2 I^*_2)S^*}{2w_2} (\lambda_2 - \lambda_1), & \text{if } 0 \leq \frac{(\beta_1 I^*_1 + \beta_2 I^*_2)S^*}{2w_2} (\lambda_2 - \lambda_1) \leq 1 \\
1, & \text{if } \frac{(\beta_1 I^*_1 + \beta_2 I^*_2)S^*}{2w_2} (\lambda_2 - \lambda_1) > 1 \end{cases} \]

\[
q^*_1 = \frac{S^*}{2w_3} (\lambda_1 - \lambda_6), \quad \text{if } 0 \leq \frac{S^*}{2w_3} (\lambda_1 - \lambda_6) \leq 1 \\
0, & \text{if } \frac{S^*}{2w_3} (\lambda_1 - \lambda_6) < 0 \\
1, & \text{if } \frac{S^*}{2w_3} (\lambda_1 - \lambda_6) > 1 \]
\[ q_2^* = \begin{cases} 0, & \text{if } \frac{I^*}{2u_4}(\lambda_2 - \lambda_6) < 0 \\ \frac{I^*}{2u_4}(\lambda_2 - \lambda_6), & \text{if } 0 \leq \frac{I^*}{2u_4}(\lambda_2 - \lambda_6) \leq 1 \\ 1, & \text{if } \frac{I^*}{2u_4}(\lambda_2 - \lambda_6) > 1 \end{cases} \]

which is equivalent as (9). \(\square\)

A.3. Optimal system

The optimal system involving optimal control variables \(u_1^*, q_1^*, q_2^*\) along with minimized Hamiltonian \(\bar{H}\) at \((S^*, I_1^*, I_2^*, H^*, R^*, V^*, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)\) is given as follows:

\[
\begin{align*}
\frac{dS^*}{dt} &= \lambda - (1 - u^*_1)(\beta_1 I_1^* + \beta_2 I_2^*)S^* - dS^* - q_1^*S^* + \eta R^* \\
\frac{dI_1^*}{dt} &= (1 - u_1^*)(\beta_1 I_1^* + \beta_2 I_2^*)S^* - q_2^*I_1^* - (\kappa + \alpha + d)I_1^* \\
\frac{dI_2^*}{dt} &= \kappa I_1^* - p_2 I_2^*, \\
\frac{dH^*}{dt} &= \phi_1 I_2^* - p_3 H^*, \\
\frac{dR^*}{dt} &= \alpha I_1^* + \phi_2 I_2^* + \psi H^* - p_4 R^*, \\
\frac{dV^*}{dt} &= q_1^*S^* + q_2^* I_1^* - dV^*,
\end{align*}
\]

(13)

with non-negative initial conditions and corresponding adjoint system is:

\[
\begin{align*}
\frac{d\lambda_1}{dt} &= \lambda_1[(1 - u^*_1)(\beta_1 I_1^* + \beta_2 I_2^*) + d + q_1^*] - \lambda_2[(1 - u^*_1)(\beta_1 I_1^* + \beta_2 I_2^*)] - \lambda_6(q_1^*) \\
\frac{d\lambda_2}{dt} &= \lambda_1[(1 - u^*_1)\beta_1 S^*] - \lambda_2[(1 - u^*_1)\beta_1 S^* - q_2^* - (\kappa + \alpha + d)] - \lambda_3(\kappa) - \lambda_5(\alpha) - \lambda_6(q_2^*) \\
\frac{d\lambda_3}{dt} &= -u_1 + \lambda_1[(1 - u^*_1)\beta_2 S^*] - \lambda_2[(1 - u^*_1)\beta_2 S^*] + \lambda_3(p_2) - \lambda_4(\phi_1) - \lambda_5(\phi_2) \\
\frac{d\lambda_4}{dt} &= \lambda_4(p_3) - \lambda_5(\psi) \\
\frac{d\lambda_5}{dt} &= -\lambda_1(\eta) + \lambda_5(p_4) \\
\frac{d\lambda_6}{dt} &= \lambda_6(d),
\end{align*}
\]

(14)

with transversality conditions \(\lambda_i(T_f) = 0, \text{ for } i = 1, 2, \ldots, 6\), and the control strategies \(u_1^*, q_i^*\) (for \(i = 1, 2\)) are same as in (9).

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