ON TAME $\rho$-QUATERNIONIC MANIFOLDS

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Abstract. We introduce the notion of tame $\rho$-quaternionic manifold that permits the construction of a finite family of $\rho$-connections, significant for the geometry involved. This provides, for example, the following:

- A new simple global characterisation of flat (complex-)quaternionic manifolds.
- A new simple construction of the metric and the corresponding Levi-Civita connection of a quaternionic-Kähler manifold by starting from its twistor space; moreover, our method provides a natural generalization of this correspondence.

Introduction

A (complex) $\rho$-quaternionic manifold $M$ is the parameter space of a locally complete family $\mathcal{S}$ of Riemann spheres embedded into a complex manifold $Z$, the twistor space of $M$, with nonnegative normal bundles $[15]$.

In this paper, we introduce the notion of ‘tame’ $\rho$-quaternionic manifold (see Definition 2.2, below) that permits the construction, through the Ward transformation, of a finite family of $\rho$-connections, significant for the geometry of $M$ (determined by $Z$). For example, if $M$ is quaternionic then only one such ‘fundamental monopole’ exists (with respect to some line bundle over $Z$) and its flatness is equivalent to the flatness of $M$ (Theorem 3.1). If, further, $Z$ is endowed with a contact structure, our approach leads to a quick simple proof (Theorem 3.3) of the fact $[9]$ that then $M$ is quaternionic-Kähler. Moreover, our method provides a natural generalization of this fact (Theorem 3.4; compare [1]). Finally, this is related to the properties of twistorial of harmonic morphisms with one-dimensional fibres which we discuss in Section 4.

1. $\rho$-QUATERNIONIC MANIFOLDS

For simplicity, excepting the last section, we work in the complex analytic category. Also, the manifolds are assumed connected, unless otherwise stated. A linear $\rho$-quaternionic structure $[15]$ on a (complex) vector space $U$ is given by a (holomorphic) embedding of the Riemann sphere $Y$ into $\text{Gr}(U)$ such that the corresponding tautological exact sequence of vector bundles

\begin{equation}
0 \longrightarrow \mathcal{F} \longrightarrow Y \times U \longrightarrow \mathcal{U} \longrightarrow 0
\end{equation}

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induces an isomorphism between $U$ and the space of sections of $U$. Consequently, $H^0(\mathcal{F})$ and $H^1(\mathcal{F})$ are trivial and, thus, $\mathcal{F} = L^* \otimes V$, where $L$ is any line bundle of Chern number 1 over $Y$. It follows (see [17, Proposition 2.5]) that

$$V = H^1((L^2)^*) \otimes H^0(L^* \otimes U).$$

Further, let $E$ be the dual of the space of sections of $F^*$. Then

$$E = H^0(L) \otimes H^0(L^* \otimes U)$$

is the space of sections of $L \otimes H^0(L^* \otimes U)$; in particular, also, $E$ is a $\rho$-quaternionic vector space. Moreover, dualizing (1.1), passing to the cohomology exact sequence and then dualizing again, we obtain a linear map $\rho : E \to U$ that intertwines the embeddings of the Riemann sphere giving the linear $\rho$-quaternionic structures on $E$ and $U$; thus, the latter is determined by the pair $(E, \rho)$ (compare [10], [12]).

More generally, a $\rho$-quaternionic manifold [15] is (locally) given by a diagram

$$\begin{array}{ccc}
\psi & \longrightarrow & Y \\
\downarrow & & \downarrow \pi \\
Z & \longrightarrow & M
\end{array}$$

where $\pi : Y \to M$ is a Riemann sphere bundle and $\psi$ is a surjective submersion such that, for any $x \in M$, the restriction of $\psi$ to $\pi^{-1}(x)$ is a diffeomorphism onto its image $t_x$, and the corresponding (surjective) morphism between the normal bundles of $\pi^{-1}(x)$ in $Y$ and $t_x$ in $Z$ induces an isomorphism between their spaces of sections (compare [8]).

As the normal bundle of $\pi^{-1}(x)$ in $Y$ can be identified with the trivial vector bundle $\pi^{-1}(x) \times T_x M$, the obtained isomorphism between $T_x M$ and the space of sections of the normal bundle of $t_x$ in $Z$ gives a linear $\rho$-quaternionic structure on $T_x M$, for any $x \in M$.

Note that, $d\pi$ induces an embedding of $\ker d\psi$ as a vector subbundle of $\pi^*(TM)$.

We denote by $TM$ the vector bundle over $Y$ which is the quotient of $\pi^*(TM)$ through $\ker d\psi$. Then $TM$ is the direct image by $\pi$ of $TM$.

The usual terminology is that $Z$ is the twistor space of (the $\rho$-quaternionic manifold) $M$ and $t_x$ is the twistor sphere corresponding to $x \in M$. Note that, from [8] it follows that, in a neighbourhood of any of its points, $M$ is determined by the embedding of the corresponding twistor sphere in $Z$.

**Definition 1.1.** 1) We say that $M$ is of **constant type** if all the normal bundles of the twistor spheres have the same isomorphism type.

2) If $Y$ is trivial such that the corresponding projection from $Y$ onto the Riemann sphere factorises into $\psi$ followed by a submersion from $Z$ onto the Riemann sphere, we say that $M$ is $\rho$-**hypercomplex** (compare [10], [12], [18], and the references therein).

By passing to an open neighbourhood of any point of $M$, we may suppose that there exists a line bundle $L$ over $Z$ such that, for any $x \in M$, the restriction of $L$ to $t_x$ has
Chern number 1. Let $H$ be the direct image by $\pi$ of $\psi^*L$. Thus, for any $x \in M$, the fibre of $H$ over $x$ is the space of sections of $L|_{\pi^{-1}(x)}$, and, hence, we have $Y = P(H^*)$.

From (1.2), it follows that the dual $E$ of the direct image by $\pi$ of $(\ker d\psi)^*$ is $H \otimes F$ for some vector bundle $F$ (locally defined) over $M$. Note, however, that $E$ does not depend on $L$. Furthermore, similarly to the $\rho$-quaternionic vector spaces case, we have a morphism $\rho : E \rightarrow TM$ characterising, up to integrability, the $\rho$-quaternionic structure on $M$.

Remark 1.2. In [15], it is shown that, locally, there exists a $\rho$-connection on $Y$ compatible with $\psi$. By this we mean that there exists a morphism of vector bundles $c : \pi^*(E) \rightarrow TY$ which when composed with the morphism $TY \rightarrow \pi^*(TM)$, induced by $d\pi$, gives $\pi^*\rho$, and such that $c\{e \otimes f | f \in F\} = (\ker d\psi)\text{Ann}e$, for any $e \in H \setminus 0$, where $\text{Ann}e$ denotes the annihilator of $e$ in $H^*$.

Furthermore, the compatible (local) $\rho$-connections on $Y$ form an affine space over the space of sections of $E^*$.

Also, the choice of $L$ such that $H$ is the direct image by $\pi$ of $\psi^*L$, and a compatible $\rho$-connection $c$ determines a $\rho$-connection $\nabla^H$ on $H$ whose projectivisation is $c$ (compare [24]). This is related, through the Ward transformation (see [19]), to the fact that any two $\rho$-connections on $H$ inducing the same compatible $\rho$-connection on $Y$ ‘differ’ by an anti-self-dual $\rho$-connection on the trivial line bundle over $M$ (recall [19] that, anti-self-duality for a $\rho$-connection means that its ‘restrictions’ to the projections by $\pi$ of the fibres of $\psi$ are flat connections).

Proposition 1.3. $H$ is a $\rho$-quaternionic manifold whose twistor space is $L$.

Proof. This follows from the following (commutative) diagram

(1.3)

where $H + Y$ is the pull back by $\pi$ of $H$, and $\tilde{\psi}$ is induced by the projection $H + Y \rightarrow \psi^*L$ given by the fact that $H$ is the direct image by $\pi$ of $\psi^*L$.

Corollary 1.4 (compare [18]). Let $P$ be the frame bundle of $H$. Then $P$ is endowed with a GL(2)-invariant $\rho$-hypercomplex structure whose twistor space is $(L \oplus L) \setminus 0$.

Proof. A quick consequence of Proposition 1.3 is that $\text{Hom}(\mathbb{C}^2, H)$ is endowed with a GL(2)-invariant $\rho$-quaternionic structure whose twistor space is $\text{Hom}(\mathbb{C}^2, L)$. Now, we may embed $P$ into $\text{Hom}(\mathbb{C}^2, H)$ as the open subset which is the union of the free orbits of the action of GL(2). Consequently, $P$ is endowed with a GL(2)-invariant
\(\rho\)-quaternionic structure whose twistor space is \((L \oplus L) \setminus 0\).

To complete the proof, note that, as \(Y\) is a bundle associated to \(P\) its pull back by the projection \(\pi_P : P \to M\) is trivial. Furthermore, the projection from \(\pi_P^* Y = P \times \mathbb{C}P^1\) to \(\mathbb{C}P^1\) factorises as the restriction to \(\pi_P^* Y(= \pi^* P)\) of the morphism of vector bundles \(\pi^* (\text{Hom}(\mathbb{C}^2, H)) = \text{Hom}(\mathbb{C}^2, \pi^* H) \to \text{Hom}(\mathbb{C}^2, (\psi^* L))\), over \(Y\), followed by the projection \(\text{Hom}(\mathbb{C}^2, (\psi^* L)) \setminus 0 \to P(\text{Hom}(\mathbb{C}^2, (\psi^* L))) = Y \times \mathbb{C}P^1\), followed by \(Y \times \mathbb{C}P^1 \to \mathbb{C}P^1\). The proof quickly follows. \(\square\)

**Remark 1.5.** With the same notations as in Corollary [14], the pull back of \(L\) through the projection from the twistor space of \(P\) to \(Z\) is canonically isomorphic to the pull back through the projection \((L \oplus L) \setminus 0 \to \mathbb{C}P^1\) of the dual of the tautological line bundle. Indeed, this can be seen by restricting to any twistor sphere in \(Z\).

**Proposition 1.6.** Let \(L\) and \(L'\) be line bundles over \(Z\) whose restrictions to each twistor sphere have Chern numbers 1.

If there exists \(k \in \mathbb{N} \setminus \{0\}\) such that \(L^k\) and \((L')^k\) are isomorphic then there exists a brackets preserving isomorphism between \(TP/GL(2)\) and \(TP'/GL(2)\), where \(P\) and \(P'\) are the frame bundles of the direct images by \(\pi\) of \(\psi^* L\) and \(\psi^* L'\), respectively.

**Proof.** Let \(H\) and \(H'\) be the direct images by \(\pi\) of \(\psi^* L\) and \(\psi^* L'\), respectively. Then \(H' = \mathcal{L} \otimes H\), where \(\mathcal{L}\) is the line bundle over \(M\) corresponding to \(L' \otimes L^*\) through the Ward transformation.

If there exists \(k \in \mathbb{N} \setminus \{0\}\) such that \(L^k\) and \((L')^k\) are isomorphic then \(\mathcal{L}^k\) admits a nowhere zero section. Consequently, with respect to suitable open coverings, \(\mathcal{L}\) is given by locally constant cocycles. The proof quickly follows. \(\square\)

**Theorem 1.7.** Let \(M\) be a \(\rho\)-quaternionic manifold, and let \(\mathcal{L}\) be any line bundle (globally defined) on the twistor space \(Z\) of \(M\) whose restriction to some (and, hence, any) twistor sphere is nontrivial; denote \(\mathcal{T}Z = T(\mathcal{L} \setminus 0)/\langle \mathbb{C} \setminus \{0\} \rangle\).

Then, for any twistor sphere \(t \subseteq Z\), there exists an exact sequence

\[0 \to (L \oplus L)|_t \to (\mathcal{T}Z)|_t \to Nt \to 0\]

where \(Nt\) is the normal bundle of \(t\) in \(Z\) and \(L\) is any line bundle defined in some open neighbourhood of \(t\) such that \(L|_t\) has Chern number 1 and \(\mathcal{L} = L^k\), for some \(k \in \mathbb{Z} \setminus \{0\}\).

Moreover, the direct image by \(\pi\) of \(\psi^* (\mathcal{T}Z)\) is \(TP/GL(2)\), where \(P\) is the frame bundle of the direct image by \(\pi\) of \(\psi^* L\).

(Note that, we can take for \(\mathcal{L}\) the anticanonical line bundle of \(Z\) and, as the obstruction to the existence of \(L\) is topological, by passing to an open neighbourhood of each point of \(M\), we can always find a line bundle \(L\), as in the statement of Theorem [17].)

**Proof.** Note that, \(\mathcal{T}Z = T(L \setminus 0)/\langle \mathbb{C} \setminus \{0\} \rangle\). Also, the action of \(GL(2)\) on \(P\) is twistorial, corresponding to its obvious right action on the twistor space \(Z_P = (L \oplus L) \setminus 0\) of
Consequently, for the second statement, it is sufficient to prove that $\psi^*(\mathcal{F}Z) = TP/GL(2)$.

As $Z_P = (L \oplus L) \setminus 0$ we have a projection from it onto $Z \times \mathbb{C}P^1$ (the projectivisation of $L \oplus L$). Let $\pi_1$ and $\pi_2$ be the projections onto the corresponding factors of $Z \times \mathbb{C}P^1$. Then $Z_P = (\pi_1^* L) \otimes \pi_2^*(\mathcal{O}(-1))$, where $\mathcal{O}(-1)$ is the tautological line bundle over $\mathbb{C}P^1$. Therefore, for any $\ell \in \mathbb{C}P^1$, the fibre of the projection from $Z_P$ onto $\mathbb{C}P^1$ is $(L \otimes \ell) \setminus 0$. But this is the same with the image of the corresponding map from $P$ to $Z_P$ (the existence of these maps is a consequence of Corollary 1.4). Denote by $\psi_\ell : P \to (L \otimes \ell) \setminus 0$ the obtained map, and, note that, for any $a \in GL(2)$, we have $\psi_\ell \circ R_a = R_a \circ \psi_a(\ell)$, where $R_a$ denotes the (right) translation by $a$ on $P$ and $Z_P$.

Now, note that, $TP$ is the bundle over $P \times \mathbb{C}P^1$ whose restriction to $P \times \{\ell\}$, for any $\ell \in \mathbb{C}P^1$, is the quotient of $TP$ through $\ker d\psi_\ell$ (where we have identified $P$ and $P \times \{\ell\}$). Equivalently, $TP$ is characterised by the fact that its restriction to $P \times \{\ell\}$, for any $\ell \in \mathbb{C}P^1$, is the pull back by $\psi_\ell$ of the tangent bundle of $(L \otimes \ell) \setminus \{0\}$. It follows that $TP/GL(2) = \psi^*(T(L \setminus 0)/\mathbb{C} \setminus \{0\})$ and the proof is complete. □

Remark 1.8. 1) With the same notations as in Theorem 1.7, we, also, have that $TP/GL(2)$ is a $\rho$-quaternionic manifold and its twistor space is $\mathcal{F}Z$ (compare [16] and the references therein).

2) Let $\rho_P : E_P \to TP$ be the morphism of vector bundles giving (up to integrability) the $\rho$-quaternionic structure of $P$. The $GL(2)$-invariance of the $\rho$-hypercomplex structure of $P$ implies that we can pass to quotients, thus, obtaining a morphism $\tilde{E} \to \tilde{T}M$ of vector bundles over $M$, where $\tilde{E} = E_P/GL(2)$ and $\tilde{T}M = TP/GL(2)$. Furthermore, $\tilde{E} = H \otimes \tilde{F}$ for some vector bundle $\tilde{F}$, and we denote by $\rho_M$ the canonical morphism of vector bundles from $\tilde{T}M$ onto $TM$ (with kernel $AdP$). Consequently, we have the following diagram

$$
\begin{array}{ccccc}
0 & \longrightarrow & \text{End}H & \longrightarrow & \tilde{E} \\
& & \downarrow & & \downarrow \rho \\
0 & \longrightarrow & \text{Ad}P & \longrightarrow & \tilde{T}M \\
& & \rho_M & & \longrightarrow \rho_M \\
& & \longrightarrow & & \longrightarrow 0 \\
\end{array}
$$

(1.4)

where recall that $E = H \otimes F$, with $\rho : E \to TM$ giving the $\rho$-quaternionic structure of $M$. Furthermore, we have a canonical exact sequence

$$
0 \longrightarrow H^* \longrightarrow \tilde{F} \longrightarrow F \longrightarrow 0, 
$$

(1.5)

which when tensorised with $H$ gives the first row of (1.4).

3) Facts contained in this section, up to now (in particular, Theorem 1.7), admit more general formulations. The degree of generality, chosen by us, is due to the purposes of this paper.
2. ON TAME $\rho$-QUATERNIONIC MANIFOLDS

In this section, the notations are as in Section 1. We start with a useful result which is, also, interesting in itself, as it can be seen as an extension of both the Ward transformation and the filtration given by the Birkhoff–Grothendieck theorem.

**Proposition 2.1.** Let $M$ be a $\rho$-quaternionic manifold with twistor space $Z$. Let $F$ be a vector bundle over $Z$ whose restriction to each twistor sphere has the same isomorphism type $\bigoplus_{j=1}^k a_j O(n_j)$, for some $k \in \mathbb{N} \setminus \{0\}$, and $a_j \in \mathbb{N} \setminus \{0\}$, $n_j \in \mathbb{Z}$, $j = 1, \ldots, k$, with $n_j \geq n_{j+1} + 2$, for any $j = 1, \ldots, k - 1$.

Then there exists a unique filtration $F^1 \subseteq \cdots \subseteq F^k = F$ whose restriction to each twistor sphere $t \subseteq Z$ is the filtration of $F|_t$ given by the Birkhoff–Grothendieck theorem.

**Proof.** As it is sufficient to prove that this holds in an open neighbourhood of each twistor sphere, we may suppose that there exists a line bundle $L$ over $Z$ whose restriction to each twistor sphere has Chern number 1. Further, by tensorising $F$ with a power of $L$, if necessary, we may suppose $n_1 = 0$, and, thus, $n_j \leq -2$, for any $j = 2, \ldots, k$.

Now, $\psi^* F$ is endowed with a flat partial connection over $\ker d\psi$, and admits a filtration $\widetilde{F}^1 \subseteq \cdots \subseteq \widetilde{F}^k = \psi^* F$ whose restriction to each fibre of $\pi$ is the filtration of the restriction of $\psi^* F$ to that fibre, given by the Birkhoff–Grothendieck theorem.

As $n_j \leq -2$, for any $j = 2, \ldots, k$, the direct image by $\pi$ of the partial connection, over $\ker d\psi$, on $\psi^* F$ gives an anti-self-dual $\rho$-connection on the direct image by $\pi$ of $\widetilde{F}^1$ which corresponds, through the Ward transformation, with a vector subbundle $F^1$ of $F$. The proof quickly follows, inductively. $\Box$

**Definition 2.2.** Let $M$ be a $\rho$-quaternionic manifold and let $L$ be any line bundle on the twistor space $Z$ of $M$ whose restriction to some twistor sphere is nontrivial; denote $\mathcal{T} Z = T(\mathcal{L} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$.

We say that $M$ is tame, with respect to $L$, if there exists a filtration $\mathcal{T}^1 Z \subseteq \cdots \subseteq \mathcal{T}^k Z = \mathcal{T} Z$ whose restriction to each twistor sphere $t \subseteq Z$ is the filtration of $\mathcal{T} Z|_t$ given by the Birkhoff–Grothendieck theorem.

Let $M$ be a tame $\rho$-quaternionic manifold, with respect to some line bundle $L$ over $Z$. By Theorem [17] the filtration of $\mathcal{T} Z$ corresponds to a filtration (compare [10]) $0 = \widetilde{T}^0 M \subseteq \cdots \subseteq \widetilde{T}^l M = \widetilde{T} M$, for some $l \in \mathbb{N}$, where recall that $\widetilde{T} M = TP/GL(2)$, with $P$ is the frame bundle of the direct image by $\pi$ of $\psi^* L$, where $L$ is any line bundle, locally defined over $Z$, whose restrictions to the twistor spheres have Chern number 1 and such that $L = L^p$, for some $p \in \mathbb{Z} \setminus \{0\}$. In particular, $P$ is of constant type.

Furthermore, there exists a unique decreasing sequence of natural numbers $p_j$, $j = 1, \ldots, l$, such that $\widetilde{T}^j M/\widetilde{T}^{j-1} M = (\odot^{p_j} H) \otimes \widetilde{F}_j$, for some vector bundles $\widetilde{F}_j$, $j = 1, \ldots, l$, where $\otimes$ denotes the symmetric power.
Definition 2.3. Let $M$ be a $\rho$-quaternionic manifold which is tame with respect to some line bundle $\mathcal{L}$ over its twistor space $Z$.

The anti-self-dual $\rho$-connections on $\tilde{F}_j$ obtained by applying the Ward transformation to $L^{-\rho_j} \otimes (\mathcal{T}^j Z/\mathcal{T}^{j-1} Z)$, $j = 1, \ldots, l$, are called the fundamental monopoles of $M$, with respect to $\mathcal{L}$.

Suppose now that $M$ is a $\rho$-quaternionic manifold of constant type. The canonical filtration given by the Birkhoff–Grothendieck theorem gives the (increasing) canonical filtration of the tangent bundle of $M$: $0 = T^0 M \subseteq \cdots \subseteq T^k M = TM$, where $k \in \mathbb{N}$. Accordingly, we have a canonical filtration $0 = E^0 \subseteq \cdots \subseteq E^k = E$ such that $\rho(E^j) \subseteq T^j M$, for $j = 0, \ldots, k$. Note that, the canonical filtration of $E$ is increasing if and only if $\rho : E \to TM$ is surjective; equivalently, the Birkhoff–Grothendieck decomposition of the normal bundles of the twistor spheres contains no trivial term (more precisely, only positive Chern numbers are appearing in that decomposition).

With $L$ a line bundle (locally defined) on $Z$ whose restriction to each twistor sphere has Chern number $1$, and assuming $k \in \mathbb{N}\setminus\{0\}$ (equivalently, $M$ of positive dimension), there exists a unique decreasing sequence of natural numbers $n_j$, $j = 1, \ldots, k$, such that $T^j M/T^{j-1} M = (\odot^{n_j} H) \otimes F_j$, for some vector bundles $F_j$, $j = 1, \ldots, k$.

If the normal exact sequences of the twistor spheres split then $\tilde{T}M$ admits an increasing filtration with $k$ or $k+1$ terms, according to whether or not there exists $j_0 \in \{k-1, k\}$ such that $n_{j_0} = 1$. Moreover, for any $j \in \{1, \ldots, k\} \setminus \{j_0\}$ the corresponding term of the filtration of $\tilde{T}M$ is given by $T^j M$; in particular, $p_j = n_j$ and $F_j = F_j$. Note that, if $0 \leq \{n_j\}_{j=1,\ldots,k}$ then $\tilde{T}^{k+1} M/\tilde{T}^k M = T^k M/T^{k-1} M$.

Remark 2.4. 1) The canonical filtrations of $TM$ and $E$, and, if $k$ is nonzero, the natural numbers $n_j$, $j = 1, \ldots, k$, are independent of $L$.

2) By Proposition 1.6 and Theorem 1.7 we may, for example, assume $TP_j/\text{GL}(r_j)$ globally defined on $M$ where $P_j$ is the frame bundle of $F_j$ and $r_j = \text{rank} F_j$. Therefore, by a slight abuse of terminology, we may speak of $\rho$-connections on $F_j$ (although $F_j$ may be defined only locally on $M$).

3) If there exists $j_0 \in \{k-1, k\}$ such that $n_{j_0} = 1$ then $\tilde{T}^{j_0} M/\tilde{T}^{j_0-1} M = H \otimes \tilde{F}_{j_0}$ for some vector bundle $\tilde{F}_{j_0}$ and we have an exact sequence

$$0 \to H^* \to \tilde{F}_{j_0} \to F_{j_0} \to 0$$

which splits if and only if $H$ admits a connection over $M$.

Corollary 2.5. Let $M$ be a $\rho$-quaternionic manifold of constant type, and let $\mathcal{L}$ be any line bundle on the twistor space $Z$ of $M$ whose restriction to some twistor sphere is nontrivial; denote $\mathcal{T} Z = T(\mathcal{L} \setminus 0)/(\mathbb{C} \setminus \{0\})$.

Suppose that $M$ is tame, with respect to $\mathcal{L}$, and the normal exact sequences of the twistor spheres split.

(i) If there exists $j_0 \in \{k-1, k\}$ such that $n_{j_0} = 1$ then $\tilde{F}_{j_0}$ and $F_j$, for any
follows that for any tame \( \rho \) dual and the anti-self-dual \( \rho \) obtained by building, firstly, a suitable bracket (see [15] and the references therein) considered in [5]. With our approach, the relevant Weyl connection can be quickly considered in [5]. With our approach, the relevant Weyl connection can be quickly considered in [5]. With our approach, the relevant Weyl connection can be quickly considered in [5]. With our approach, the relevant Weyl connection can be quickly considered in [5]. With our approach, the relevant Weyl connection can be quickly considered in [5]. With our approach, the relevant Weyl connection can be quickly considered in [5]. With our approach, the relevant Weyl connection can be quickly considered in [5].

Proof. This is a consequence of the Ward transformation and Theorem [17]. 

2.1. On \( \rho \)-hypercomplex manifolds of constant type. In this subsection \( M \) is \( \rho \)-hypercomplex. Then \( Y = M \times \mathbb{C}P^1 \) such that the projection from \( Y \) onto \( \mathbb{C}P^1 \) factorises into \( \psi \) followed by a surjective submersion \( \chi : Z \to \mathbb{C}P^1 \). For any \( z \in \mathbb{C}P^1 \), we denote by \( \psi_z \) the restriction of \( \psi \) to \( M \times \{ z \} \). Thus, for any \( z \in \mathbb{C}P^1 \), on identifying \( M \) with \( M \times \{ z \} \), we have that \( \psi_z : M \to N_z \) is a surjective submersion, where \( N_z = \chi^{-1}(z) \) (compare the proof of Theorem [17]). Let \( L \) be the pull back by \( \chi \) of the dual of the tautological bundle over \( \mathbb{C}P^1 \), and denote \( \mathcal{Z} = T(L \setminus 0)/((\mathbb{C} \setminus \{ 0 \})\).

**Corollary 2.6.** Suppose that, for some \( k \in \mathbb{N} \{ 0 \} \), there exist positive natural numbers \( n_1 > \ldots > n_k \) and an increasing filtration \( 0 = N^0 \subseteq \cdots \subseteq N^k = \ker d\chi \) over \( Z \) such that \( L^{-n_j} \otimes (N^j/N^{j-1}) \) restricted to any twistor sphere is trivial, \( j = 1, \ldots, k \).

Then \( M \) is tame, with respect to \( L \), and its fundamental monopoles are (essentially) obtained by applying the Ward transformation to \( L^{-n_j} \otimes (N^j/N^{j-1}) \), \( j = 1, \ldots, k \).

**Proof.** Note that, the normal bundle of each twistor sphere \( t \subseteq Z \) is isomorphic to \((\ker d\chi)_t\). Also, we have an exact sequence \( 0 \to \ker d\chi \to \mathcal{Z} \to \chi^*(L \otimes L) \to 0 \), induced by \( d\chi \). Consequently, the filtration of \( \ker d\chi \) determines a filtration of \( \mathcal{Z} \) as required.

**Remark 2.7.**

1) If there exists \( j_0 \in \{ k - 1, k \} \) such that \( n_{j_0} = 1 \) then \( \widetilde{F}_{j_0} = H^* \oplus F_{j_0} \) (because \( H \) admits the trivial flat connection) and its anti-self-dual \( \rho \)-connection is, accordingly, a direct sum.

2) The filtration of \( \ker d\psi \) corresponds to increasing filtrations

\[ 0 = T^0N_z \subseteq T^1N_z \subseteq \cdots \subseteq T^kN_z = TN_z \]

and, consequently, to isomorphisms \( F_j = \psi_z^*(T^jN_z/T^{j-1}N_z) \) under which the anti-self-dual \( \rho \)-connections of \( F_j \), \( j = 1, \ldots, k \), are induced by the partial Bott connections of the foliations determined by \( \psi_z, z \in \mathbb{C}P^1 \) (compare [19]).

3) If \( k = 1 \) then \( TM = (\oplus^nH) \otimes F_1 \) and the tensor product of the trivial connection and the anti-self-dual \( \rho \)-connection of \( F_1 \) is just the Obata \( \rho \)-connection from [19].

4) With the same notations as in Corollary [14], from Remark [13] and Corollary [2.6] it follows that for any tame \( \rho \)-quaternionic manifold whose twistor spheres have positive normal bundles, also, the \( \rho \)-hypercomplex manifold \( P \) is tame.

5) If \( Z \) is surface then obviously \( M \) is of constant type. The case \( n_1 = 2 \) was considered in [15]. With our approach, the relevant Weyl connection can be quickly obtained by building, firstly, a suitable bracket (see [15] and the references therein) on \( E \).

If \( n_1 \in \mathbb{N} \) and \( M \) is \( \rho \)-hypercomplex then \( M \) is tame with one possibly nontrivial
fundamental monopole, determining the Obata $\rho$-connection \cite{19} of $M$; the case $n_1 = 2$ was considered in \cite{4}.

3. QUATERNIONIC MANIFOLDS

The quaternionic manifolds are characterised, among the $\rho$-quaternionic manifolds, by the fact that the Birkhoff–Grothendieck decompositions of the normal bundles of the twistor spheres contain only terms of Chern number 1. Consequently, they are tame $\rho$-quaternionic manifolds with only one fundamental monopole. Also, note that, the dimension of any quaternionic manifold is even. Furthermore, for any line bundle, over the twistor space of a quaternionic manifold, whose restriction to some twistor sphere is nontrivial, the corresponding fundamental monopole is a (classical) connection.

The ‘flat model’ is the Grassmannian $\text{Gr}_2(n+2)$ with twistor space $Z = \mathbb{C}P^{n+1}$ and $Y$ the flag manifold $F_{1,2}(n+2)$. In this case, the fundamental monopole, with respect to any nontrivial line bundle $L$ on $Z$, (essentially) is the trivial connection on $\text{Gr}_2(n+2) \times \mathbb{C}^{n+2}$.

**Theorem 3.1** (compare \cite{23}, \cite{20}). Let $M$ be a quaternionic manifold, dim $M = 2n$, and let $Z$ be its twistor space. Then the following assertions are equivalent:

(i) There exists a line bundle $\mathcal{L}$ over $Z$ whose restriction to some twistor sphere is nontrivial and such that the fundamental monopole of $M$, with respect to $\mathcal{L}$, is flat.

(ii) There exists a twistorial local diffeomorphism from a covering space of $M$ to $\text{Gr}_2(n+2)$.

**Proof.** Let $\mathcal{L}$ be a line bundle over $Z$ whose restriction to some twistor sphere is nontrivial. With the same notations as in (1.5) and Corollary 2.5, we have $j_0 = k = 1$ and $\tilde{F} = F_1$. Also, although $\tilde{F}$ may exist only locally, $\text{Gr}_2(\tilde{F})$ is globally defined over $M$, and the locally defined embeddings $H^* \subseteq \tilde{F}$, given by (1.5), define a (global) section $\sigma$ of $\text{Gr}_2(\tilde{F})$.

The fundamental monopole of $M$, with respect to $\mathcal{L}$, induces a connection on $\text{Gr}_2(\tilde{F})$ which, if (i) holds, is flat. Thus, assuming (i), by passing to a covering space of $M$, if necessary, we have $\text{Gr}_2(\tilde{F}) = M \times \text{Gr}_2(V)$, where $V$ is a vector space of dimension $n+2$. Hence, $\sigma_x = (x, \varphi(x))$, for any $x \in M$, for some map $\varphi : M \to \text{Gr}_2(V)$.

To complete the proof we have to show that $\varphi$ is a twistorial local diffeomorphism. For this, by passing to an open neighbourhood of each point of $M$, if necessary, we may suppose that $\mathcal{L}$ restricted to any twistor sphere has Chern number 1. Then the flatness of the fundamental monopole, with respect to $\mathcal{L}$, is equivalent to the existence of an isomorphism of vector bundles $\alpha : \mathcal{T}Z \to \mathcal{L} \otimes V$, where $\mathcal{T}Z = T(\mathcal{L}^* \setminus 0)/\mathbb{C} \setminus 0$.

As $\mathcal{L}$ has rank 1, the adjoint bundle of $\mathcal{L}^* \setminus 0$ is trivial. Thus, we have an embedding $M \times \mathbb{C} \subseteq \mathcal{T}Z$. Denote by $\mathbf{1}$ the section of $\mathcal{T}Z$ induced through this embedding by the section $x \mapsto (x, 1)$ of $M \times \mathbb{C}$.

Then $s = \alpha(\mathbf{1})$ is a nowhere zero section of $\mathcal{L} \otimes V$ which induces a section of $P(\mathcal{L} \otimes V) = Z \times PV$, obviously, given by $z \mapsto (z, \varphi_Z(z))$, for some map $\varphi_Z : Z \to PV$. 
We claim that $\varphi_Z$ is a local diffeomorphism. Indeed, the differential of $\varphi_Z$ is determined by $\nabla s$ (see [20]), where $\nabla$ is the tensor product of the canonical $\rho_Z$-connection on $\mathcal{L}$ and the trivial connection on $Z \times V$, where $\rho_Z : \mathcal{I}Z \to T\mathcal{I}Z$ is the projection. Thus, also, denoting by $\nabla$ the $\rho_Z$-connection on $\mathcal{I}Z$ with respect to which $\alpha$ is covariantly constant, we have to show that $\nabla \mathcal{I}$ is an isomorphism, at each point.

The existence of $\alpha$ shows that the frame bundle of $\mathcal{I}Z$ admits a reduction to $\mathbb{C} \setminus \{0\}$, embedded into $GL(V)$ through $\lambda \mapsto \lambda \text{Id}_V$. Furthermore, $\nabla$ is the flat $\rho_Z$-connection corresponding to this reduction. Hence, $\nabla \mathcal{I} = \text{Id}_{\mathcal{I}Z}$ which implies that $\varphi_Z$ is a local diffeomorphism. Moreover, as $\mathcal{L}$ restricted to each twistor sphere has Chern number 1, we have that $\varphi_Z$ maps each twistor sphere diffeomorphically onto a projective line of $PV$, and the proof quickly follows. □

Let $M$ be a quaternionic manifold with twistor space $Z$ given by $\psi : Y \to Z$ and $\pi : Y \to Z$. By passing to an open neighbourhood of each point, if necessary, let $L$ be a line over $Z$ whose restriction to some twistor sphere has Chern number 1 and let $\nabla^H$ be the connection on the direct image $H$ by $\pi$ of $\psi^* L$, determined by $L$ and a connection on $Y$ compatible with $\psi$ (recall Remark 1.2).

Denote by $\nabla^F$ the connection on $F$ given by the fundamental monopole with respect to $L$ and the decomposition $\tilde{F} = H^* \oplus F$ corresponding to $\nabla^H$, where recall that $F$ is the vector bundle over $M$ such that $TM = H \otimes F$.

**Theorem 3.2** (compare [23], [2]). The connection $\nabla^H \otimes \nabla^F$ is torsion free. Conversely, any torsion free connection on $M$, compatible with the underlying almost quaternionic structure of $M$, is obtained this way.

**Proof.** Let $P$ be the frame bundle of $H$ and let $\pi_P : P \to M$ be its projection. Then $P$ is a hypercomplex manifold and its Obata connection $\nabla^P$ is the tensor product of the trivial connection on $P \times \mathbb{C}^2$ and the pull back by $\pi_P$ of the fundamental monopole of $M$ with respect to $L$.

After passing to $GL(2)$-quotients, the inclusion into $TP$ of the ‘horizontal’ distribution on $P$, corresponding to $\nabla^H$, gives the inclusion $TM = H \otimes F \subseteq H \otimes \tilde{F} = \tilde{T}M$ (see Remark 1.8(2)).

From the fact that $\pi_P$ is twistorial and from Remark 1.8, we deduce that the torsion of $\nabla^H \otimes \nabla^F$ is given by the horizontal part of the torsion of $\nabla^P$ evaluated on horizontal vector fields. But $\nabla^P$ is torsion free and, hence, also, $\nabla^H \otimes \nabla^F$ is torsion free.

Now, the converse statement is an immediate consequence of the fact that both the compatible connections on $Y$ and the torsion free connections on $M$, compatible with the underlying almost quaternionic structure of $M$, are affine spaces over the space of 1-forms on $M$. □

### 3.1. Hypercomplex and hyper-Kähler manifolds.

In Theorem 3.1 and with the same notations as in (1.5), if $M$ is a hypercomplex manifold, assertion (i) is equivalent to the fact that the connection on $F$, given by the Ward transformation applied to
\( \chi^*(\mathcal{O}(-1)) \otimes (\ker d\chi) \), is flat. Indeed, from the exact sequence appearing in the proof of Corollary 2.6, it follows that (1.5) splits, the fundamental monopole of \( M \) (with respect to \( \chi^*(\mathcal{O}(1)) \)) induces a connection on \( F \), and \( H^* = \tilde{F}/F \) is endowed with a flat connection. (Consequently, the connection on \( H \) induced by the fundamental monopole of \( M \) and the projection \( \tilde{F} = H^* \oplus F \to H^* \) is flat.) Therefore the connection of \( \tilde{F} \) is flat if and only if the connection of \( F \) is flat; equivalently, the Obata connection of \( M \) is flat. The flat model is \( \mathbb{C}^{2n} \) identified with the space of projective lines in \( \mathbb{C}P^{n+1} \) which are disjoint from a fixed projective subspace of codimension 2.

A manifold is hyper-Kähler if and only if it is hypercomplex and endowed with a Riemannian metric preserved by the Obata connection. If \( M \) is hypercomplex then any hyper-Kähler metric on it corresponds, under the Ward transformation, with a nondegenerate section of \( \Lambda^2(\chi^*(\mathcal{O}(-1)) \otimes (\ker d\chi))^* \); in particular, the dimension of \( M \) is divisible by 4. From the fact that the Obata connection is torsion free, we deduce that any such section restricts to give symplectic structures on the fibres of \( \chi \), a well-known fact [6].

3.2. Quaternionic-Kähler manifolds. With notations as in Theorem 3.1 (but without any flatness assumption) a contact structure on \( Z \) can be defined as a nonzero section \( \theta \) of \( \mathcal{T}^*Z \otimes \mathcal{L} \) such that \( \theta(1) = 0 \) and \( d\nabla \theta \) is nondegenerate, where \( \nabla \) is the canonical \( \rho_Z \)-connection on \( \mathcal{L} \), with \( \rho_Z : \mathcal{T}Z \to TZ \) the projection. (This, obviously, does not require \( Z \) to be a twistor space.)

We shall assume, by passing to an open subset of \( M \), that \( \theta \) restricted to any twistor sphere \( t \subseteq Z \) induces an isomorphism between \( Tt \) and \( \mathcal{L}|_t \). Equivalently, we assume the twistor spheres transversal to the contact distribution \( \mathcal{H} = \rho_Z(\ker \theta) \).

Similarly as before, by passing to an open neighbourhood of each point of \( M \) we may find a line bundle \( L \) on \( Z \) such that \( L^2 = \mathcal{L} \); in particular, \( L \) restricted to any twistor sphere has Chern number 1. Then \( d\nabla \theta \) defines a linear symplectic structure on \( L^* \otimes \mathcal{T}Z \), and therefore the fundamental monopole preserves a linear symplectic structure on \( \tilde{F} \). Moreover, as \( \theta \) induces a contact structure on each twistor sphere (the corresponding contact distributions are just the trivial zero distributions), also, \( H^* \) is endowed with a linear symplectic structure and the embedding \( H^* \subseteq \tilde{F} \) preserves the linear symplectic structures. Therefore (1.5) splits, as we may identify \( F \) with the symplectic orthogonal complement of \( H^* \) in \( \tilde{F} \).

Now, just recall that \( TM = H \otimes F \) and, similarly to the hyper-Kähler case, we obtain the Riemannian metric \( g_0 \) on \( M \). To describe the Levi-Civita connection of \( g_0 \) we, again, proceed similarly to the hyper-Kähler case: endow \( F \) with the connection \( \nabla^F \) induced from \( L^* \otimes \mathcal{H} \) by the Ward transformation, and recall that the splitting of (1.5) corresponds to a connection \( \nabla^H \) on \( H \) which, obviously, preserves the linear symplectic structure of \( H^* \).

**Theorem 3.3** (compare [24], [9]). The connection \( \nabla^H \otimes \nabla^F \) is the Levi-Civita connection of \( g_0 \), and, thus, \((M, g_0)\) is quaternionic-Kähler.
Proof. Note that, we may also describe the linear symplectic structure of $F$ as follows. Firstly, at each point, $\rho^{-1}_Z(\mathcal{H})$ is equal to the symplectic orthogonal complement of $1$ in $\mathcal{T}Z$. Hence, the restriction of $d\bar{\nabla}_1$ to $\rho^{-1}_Z(\mathcal{H})$ descends to a linear symplectic structure on $L^* \otimes \mathcal{H}$ which quite clearly corresponds to the linear symplectic structure of $F$. In particular, $\nabla^F$ preserves the linear symplectic structure of $F$.

The second description of the linear symplectic structure of $F$ also shows that the connection induced by $\nabla^H$ on $Y(P(H^*))$ is (given by) $(d\psi)^{-1}(\mathcal{H})$, where recall that $\psi : Y \to Z$ is the submersion giving the twistor space $Z$. Furthermore, on denoting by $\theta$ the $L^2$-valued 1-form on $Z$ such that $\theta = \theta \circ \rho_Z$, we have that $\theta$ lifts to $Y$ as the composition of the projection $TY \to \ker d\pi$, with kernel $(d\psi)^{-1}(\mathcal{H})$, followed by an isomorphism from $\ker d\pi$ onto $\psi^*(L^2)$. It follows quickly that $\nabla^H$ is the connection determined by $L$ and the connection it induces on $Y$.

Thus, $\nabla^H \otimes \nabla^F$ preserves $g_\theta$ and, together with Theorem 3.2, this completes the proof. \hfill \Box

The infinitesimal automorphisms of a contact structure on $Z$, given by $\theta : \mathcal{T}Z \to \mathcal{L}$, can be characterised as sections $u$ of $\mathcal{T}Z$ such that $\mathcal{L}^u \theta = 0$, where $\nabla$ is the canonical $\rho_Z$-connection of $\mathcal{L}$; equivalently (as it is more familiar), the local flow of $\rho_Z(u)$ preserves the contact distribution $\mathcal{H}(= \rho(\ker \theta))$. Note that, if $u$ is an infinitesimal automorphism of $\theta$ then

\begin{equation}
(3.1) \quad d\nabla^u = -\nabla^\theta (u),
\end{equation}

for any $v \in \mathcal{T}Z$.

Let $V$ be a (finite dimensional) vector space of infinitesimal automorphisms of $\theta$. Let $s_{\theta,V}$ be the section of $\mathcal{L} \otimes V^*$ given by $s_{\theta,V}(z,u) = \theta(u_z)$, for any $z \in Z$ and $u \in V$. Then the zero set of $s_{\theta,V}$ is the base locus of $\mathcal{L}$, with respect to $\theta(V)$, and in its complement the differential of the map $\varphi_{\theta,V} : Z \to P(V^*)$ induced by $s_{\theta,V}$ is given by $\nabla s_{\theta,V} : \mathcal{T}Z \to \mathcal{L} \otimes V^*$ (here, $\nabla$, also, denotes the tensor product of the canonical $\rho_Z$-connection on $\mathcal{L}$ and the trivial connection on $Z \times V^*$). Thus, from (3.1) we deduce that, at each $z \in Z$, the kernel of $d\varphi_{\theta,V}$ is the image by $\rho_Z$ of the symplectic orthogonal complement of $\{u_z \mid u \in V\}$ (essentially, a well known fact).

Suppose, now, that $Z$ is the twistor space of the quaternionic-Kähler manifold $(M,g_\theta)$, then we may suppose that each $u \in V$ is such that the local flow of $\rho_Z(u)$ maps twistor spheres to twistor spheres and thus it corresponds to a Killing vector field $X^u$ on $(M,g_\theta)$. In fact, by applying Theorem 1.7 we may obtain $X^u$ as the image through $\tilde{\rho}$ of the direct image by $\pi$ of $\psi^*u$.

Similarly, $s_{\theta,V}$ corresponds to a section $S_{\theta,V}$ of $(\text{End}_0H) \otimes V^*$ obtained as follows, where $\text{End}_0$ denotes the space of trace free endomorphisms. Firstly, recall that $P$ admits a reduction $P_0$ to $\text{SL}(2)$ corresponding to the symplectic structure of $H$ induced by $\theta$ (and $L$). Also, we have a decomposition $TP_0/\text{SL}(2) = (\text{End}_0H) \oplus TM$ corresponding to $\nabla^H$ of Theorem 3.3. Further, for any $u \in V$, the direct image by $\pi$ of $\psi^*u$ restricts
to give a section $\tilde{X}^u$ of $TP_0/\text{SL}(2)$. Then $S_{\theta,V}(x,u)$ is the projection to $\text{End}_0(H_x)$ of $\tilde{X}^u_x$.

Now, the base locus of $S_{\theta,V}$ is formed of the points $x \in M$ where $S_{\theta,V}(x)$ seen as an element of $\text{Hom}(\text{End}_0(H_x), V^*)$ has rank less than 3. Thus, outside its base locus, $S_{\theta,V}$ determines a map $\Phi_{\theta,V}: M \to \text{Gr}_3(V^*)$ associating to any $x \in M$ the image of $S_{\theta,V}(x)$ seen as linear map from $\text{End}_0(H_x)$ to $V^*$. Furthermore, similarly to the line bundles case, the differential of $\Phi_{\theta,V}$ is determined by $\nabla_{S_{\theta,V}}: TP_0/\text{SL}(2) \to \text{Hom}(\text{End}_0(H_x), V^*) \otimes V^*$, where $\nabla$ is the $\tilde{\rho}$-connection on $\text{End}_0(H_x)$ induced by the canonical $\tilde{\rho}$-connection of $H$, tensorised with the trivial connection on $M \times V^*$. (Note that, these ideas, also, provide an alternative proof for Theorem 3.1, without involving the twistor space of $M$.)

Theorem 3.4 (compare [22], [9], [1]). Let $M$ be a quaternionic manifold with twistor space $Z$, given by $\psi: Y \to Z$. If the fibres of $\psi$ are connected then there exists a natural correspondence between the following:

(i) Pairs $(g, \nabla)$ formed of a section of $\otimes^2 T^* M$ and a torsion free connection on $M$ both compatible with its quaternionic structure and such that $\nabla g = 0$.

(ii) Distributions $\mathcal{H}$ on $Z$ of corank 1 and transversal to the twistor spheres.

Moreover, for any such $(g, \nabla)$ and $\mathcal{H}$, with nontrivial $g$ (equivalently, nonintegrable $\mathcal{H}$), the kernel of $g$ gives a foliation on $M$ locally defined by twistorial submersions $\varphi$ onto quaternionic-Kähler manifolds $M_\varphi$ such that:

1. $\varphi$ preserves, by pull back and an obvious Lie groups morphism, the metrics and the corresponding connections;

2. the differential of the (local) submersion between the twistor spaces, corresponding to $\varphi$, maps $\mathcal{H}$ onto the contact distribution of the twistor space of $M_\varphi$.

Proof. A pair as in (i) gives a distribution as in (ii) after a straightforward argument by involving the first Bianchi identity.

How to pass from (ii) to (i), and the second statement follows quickly from the proof of Theorem 3.3.

In Theorem 3.4, the connectedness assumption on the fibres of $\psi$ is necessary only when passing from (i) to (ii). Another such sufficient assumption is $M$ be the complexification of a ‘real’ quaternionic manifold.

Examples of twistorial submersions as in Theorem 3.4 can be found in [7]. Also, in [12, §5] a construction of distributions as in Theorem 3.4 can be found.

4. On twistorial harmonic morphisms with one-dimensional fibres

In this section, although some of the results hold in more generality, for simplicity, $M$ will denote a real(-analytic) quaternionic-Kähler or hyper-Kähler manifold. To unify the notations of this and the previous sections, one just have to replace in the latter $M$ with $M^\mathbb{C}$. Further, as, now, $\psi$ restricted to $\pi^{-1}(M)$ is a diffeomorphism, the diagram giving the twistor space simplifies, as it is well known, to a fibration whose total space and
projection we will denote by $Y$ and $\pi$, respectively (instead of $\pi^{-1}(M)$ and $\pi|_{\pi^{-1}(M)}$, respectively). Note that, $Y$ embeds into the complex Grassmannian of $TM$ such that any point of $Y$ is an isotropic space of dimension $2k$ tangent to $M$, where $\dim M = 4k$, with $k \in \mathbb{N} \setminus \{0\}$.

The main general reference for harmonic morphisms is [3]; see, also, [14] and [11] for more recent results.

**Proposition 4.1.** Let $\varphi : M \to N$ be a harmonic morphisms of warped-product type, with one-dimensional fibres, and, locally, let $N \subseteq M$ be a horizontal section. Denote by $\rho$ the restriction to $TM|_N$ of $d\varphi$.

Then $(N, TM|_N, \rho)$ is a $\rho$-quaternionic manifold, where $TM|_N$ is endowed with the restriction of the Levi-Civita connection of $M$.

**Proof.** For any $y \in Y|_N$ we have that $y \cap T^CN$ is an isotropic space of dimension $2k-1$. Consequently, $y^\perp \subseteq T^CN$ is coisotropic. Therefore $(TM|_N, \rho)$ is an almost $\rho$-quaternionic structure on $N$, and, from the integrability result of [16] and by applying [13 Lemma 5.1] (see [14 Appendix A.2] and [3 Chapter 11]), we obtain that this is integrable. \hfill $\Box$

Presumably, the following result is not new. We omit the proof.

**Proposition 4.2.** Any Killing vector field on a quaternionic-Kähler manifold $M$ is quaternionic; consequently, it corresponds to a holomorphic vector field on the twistor space of $M$ preserving the contact distribution.

We end with the following result.

**Theorem 4.3.** Let $\varphi : M \to N$ be a twistorial harmonic morphism with one-dimensional fibres, and let $\mathcal{H}$ be the holomorphic distribution on the twistor space of $M$ given by its metric.

Then $\varphi$ corresponds to a holomorphic vector field on the twistor space of $M$ which if transversal to $\mathcal{H}$ it is an infinitesimal contact transformation.

**Proof.** This follows from Proposition 4.1 and [21] (see [14 Chapter 3]). \hfill $\Box$

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