Geometrical structure and nature of cylindrical space associated with point particles

M. Honda∗

Abstract

A unific helical field as geometric structure in a cylindrical space is proposed which is capable of determining attributes inherent to point-like particles. An operational discussion on transition between our observational space and the warped infinitesimal space is expanded. It is seen that the rotational eigenvalue equation, which is satisfied by vector field equivalent to a Gromeka-Beltrami flow subject to fluid and plasma physics, provides a spatio-unifier that sustains complex orthogonal coupling between a rotational, internal coordinate space and angular momentum space. Self-consistent normalization of the rotational coordinate owing to the unifier is shown to be responsible for the renormalization in quantum electrodynamics (QED). For the demonstration, a numerical value corresponding to the fine-structure constant is derived from theoretical analysis involving the rotational eigenvalue that charged leptons should refer to. It is found that the eigenstates of the fields having helical mirror-asymmetry are reflected in parity violation in β-decay, and also, the chiral eigenstates of the lowest order even mode exhibit affinity for gravitational interaction. This study is essential for going beyond the standard model and elucidating origin of space-time.

1 Introduction

Reducing elements of matters has continuously been the cutting edge issue on natural philosophy and science. The quest is currently directed toward revealing structure of the leptons, and quarks as unextractable constituents of hadrons. Knowledge of the smallest elements will be available for clarifying physical law in the highest energy region, and imperative for constructing a theory of everything. Let us focus on, particularly, the most stable charged lepton solely existing, namely, electron. As the matter now stands, one has none of the observational evidence of the element-divided substructure; upper limit of the radius is reported $10^{-20}$ cm [1]. Unless the electron is

∗Plasma Astrophysics Laboratory, Institute for Global Science, Mie, Japan; Email: honda-mitsuru- gg@alumni.osaka-u.ac.jp
made out of any solid ingredient, we would have no choice but to consider that the
electric charge $e$ and spin $1/2$ as the attributes observed without individuality reflect
structure of another space expanded somewhere in a virtually infinitesimal region. Ac-
cording to the observational fact that magnetic monopole is never yet discovered, it is
anticipated, that the mechanism that generates the magnetic dipole moment coupled
with the spin angular momentum, which is characterized by $\hbar = h/2\pi$ ($h$ is the Planck
constant [2]), is in an asymmetric relation with the mechanism that generates $e$ as the
electric attribute, while being in a complementary relation with it.

When we embark on elucidation of the spatial structure that engenders those at-
tributes, the worthwhile first step is to find an analogy in structure of a nucleus, which
has the same spin $1/2$, and the charge and magnetic moment distribution as large as
the classical electron radius of about $10^{-13}$ cm. It can be presumed that the nucleus is
constantly emitting and absorbing $\pi$-mesons, to have the boson cloud (e.g., Ref. [3]).
We recall that this concrete image is similar to a picture of electron in QED such
that virtual photons dress it. As for orbital angular momentum $L$, the meson cloud
is required to have the eigenvalue of $l = 1$ for which to be compatible with the rele-
vant experimental results; this means that the $\pi$-mesons are really rotating on an orbit
(e.g., Ref. [4]). With this in view, it might be better to naively suppose this kind of
image for the structure driving the spin of electron. Anyway, no longer in doubt is
validity of imposing the commutation relation equivalent to that for the operator of
$L = R \times P$, on spin angular momentum $S$ [5], where $R$ and $P$ are position vector and
momentum, respectively. Accordingly, it would be an innocent attempt to envisage,
for the generation principle of $S$, a rotational coordinate of the space expanded in the
infinitesimal region of $|R| \to 0$. However, a primitive question of like what coordinate
must be rotated has hitherto remained unanswered, despite its seriousness.

It is not completely absurd to look for, further in macroscopic subjects, a clue to
the puzzle, since in fact the asymmetry of the electromagnetic attributes is being cast
to predominance of magnetic fields up to cosmological scales. It is Abelian plasma
physics that accounts for the dynamics of many-body system of charged particles in-
teracting with classical electromagnetic fields. In the context, it will be better to seek
out, in magnetohydrodynamical features of plasma, an intuitive image for the $S$ gen-
eration. Now, we closer look at the astrophysical jets, which are launched from active
galactic nuclei including black holes, to extend up to million light years. Intriguingly,
in the jets we often recognize signatures of helical motion as well as helical structure
of magnetic field [6]. Such structure could appear as a result of the turbulent pro-
cess of plasmas, in which the magnetic field configuration with the minimum energy
is self-organized under conservation of the magnetic helicity [7, 8]. Practically, helical
magnetic field structure observed in, for example, interplanetary magnetic clouds [9]
and plasma ejecta in the solar corona [10] has been interpreted as this kind of relaxed
state. Let us therefore suppose the structure formation to be a universal magneto-
hydrodynamical phenomenon. In this aspect, invoking that the $\pi$-mesons could be
regarded as the lowest energy excitation state in vacuum [4], we conjecture that the
helical structure might reflect the geometrical structure that generates the rotational,
inner coordinates connecting to $S$. In regard to this, comparing the galactic nucleus to one neutron $n^0$, we consider the nuclear reaction of $n^0 \rightarrow p^+ + \pi^-$, where $p^+$ and $\pi^-$ are proton and $\pi^-$-meson, respectively. We then make a spinning black hole [11] and accretion disk [12] having angular momentum correspond, respectively, to $p^+$ and $\pi^-$ cloud surrounding it. This leads to the correspondence between a bipolar jet and a lepton pair emitted in decay of $\pi^-$. Altogether, the activity of galactic nuclei involving angular momentum transfer is seen as though visual projection of the $\beta$-decay. In particular, the lepton–jet correspondence conforming to the foregoing conjecture seems to signify that the structure in the infinitesimal space and outer space are described by common geometry. — This intuition is original motivation of the present study.

The crucial matter, that the divergence difficulty in QED can be successfully removed by the conventional renormalization, implies latency of a fundamental operation related to transition to the infinitesimal space. If nature is essentially inventive, geometry of the space should be able to provide, in a self-consistent manner, the physical meaning of charge renormalization, which has been obscure [13]. Meanwhile, polarization picture of QED vacuum itself suggests that one could by no means reach generation principle of the observed charge $e$ within the theoretical framework that postulates vacuum dielectric (and magnetic) permittivity $\varepsilon_0$ ($\mu_0$) to be constant. This originates from the logical structure of gauge symmetry based on causality, such that in space-time concept, existence of the photon, which has the speed of $c := 1/\sqrt{\varepsilon_0\mu_0}$, results necessarily from charge conservation. By taking a hint from this impasse per se, an attempt might be made to find out a form of the concerned operation. We here anticipate slight vestige of that form in the macroscopic dielectric distribution such that phase velocity of light deviates from $c$. When contrasted with the magnetohydrodynamical analog of the spin generation, the guideline is obtained in which one should also have an insight into collective phenomena of plasma as dielectric medium, particularly, dispersion of light propagating through the plasma.

Of course, it is impossible from plasma physics to literally derive the concerned mechanics that assigns "point" the intrinsic attributes. However, when specific description of plasmas as the magnetic fluid and dielectric is appropriately generalized to be sublimated into an abstracted form, we may have a fortuitous chance to access to a unified generation principle of $\hbar$ and $e$. Should the theory built up gives a basic framework of space-time and vacuum, it would encompass generation mechanism of the fine-structure constant including $c$ [14], $\alpha := e^2/4\pi\hbar c$, and its relation to the other interactions. Laws of the infinitesimal space must reproduce observable phenomena in the infinite limit. Thus, a priority task is to write out the operation of spatial transformation, responsible for the reproducibility.

In the present paper, by virtue of this new approach I explore the geometrical structure of point-like particles including the electron, without beginning with a modeled ingredient having its own scale scarcely measurable. Making reference to plasma dielectric dispersion theory, I propose a convenient form of the warp transformation that describes transition to the space expanded in the infinitesimal region, thereby, manifesting the meaning of "point-like" in particle physics. I show that the space is
the cylindrical one, which is, in association with symmetry breaking, spanned by a unific rotational field equivalent to the called Gromeka-Beltrami flow (e.g., Ref. [15]). The field obeys the rotational eigenvalue equation, which is found to describe generation of angular momentum space orthogonal to coordinate space. We figure out that spin precession of charged leptons is appearance of the generation mechanism of time as one degree of freedom, which owes to complementation of those isotopic spaces under the rotation. A radical process is revealed through which quantities corresponding to the observed charge $e$ and $g$-factor including radiative corrections [16] come out. The determination of $\alpha$ involves a left-handed rotational eigenmode, which meets the fact, that the leptons that take part in the current weak interaction are only of the left-handedness [17]. This eigenmode regulates helical structure of the rotational field. The geometry is equivalent to that to describe the helical structure of a relaxed plasma, whereby we will reconfirm significance as to the lepton–astrophysical jet correspondence. I show up this theory qualifies as the principle one of forces and matters, exhibiting its compatibility with the standard model.

This paper is organized as follows: §2 is devoted to a preparation for operationally expanding the discussions. By investigating spatial distribution of photon states in the dielectric medium (§2.1), we arrange a rule of wavenumber transformation of light (§2.2). Taking the abstracted form into account, made is the spatial transformation for transition to the infinitesimal space (§2.3). In §3, we turn to quark-antiquark potential inside and outside mesons for revealing a signature of cylindrical space expanded in the infinitesimal region. By availing complex analysis, I give a reason for quark confinement (§3.1), and reconstruct nuclear potential [18] (§3.2), so as to verify a proposed form of the spatial transformation. In addition, we consider physical meaning of a factor involved in the transformation (§3.3). In §4, I provide mathematical description for the structure in the cylindrical space, which unifies forces and matters in a low energy region. The rotational eigenvalue equation is proposed which should be satisfied by vector field spanned in the space (§4.1). The equation is solved as a boundary value problem, to yield the eigenstate that regulates internal space of lepton pairs (§4.2) in response to mirror-asymmetry of the helical structure of the vector fields (§4.3). Also, general property of the eigenmodes is noted (§4.4). In §5, we address major issues concerning observation of the rotational field in the cylindrical space of one electron. We see relations of the field with states of the spin and orbital motion (§5.1). From the rotational eigenvalue equation governing the field, we derive a mathematical symbol which generates the spin precession (§5.2) and $\alpha$ (§5.3). Then, briefly given is QED interpretation of the charge generation mechanism (§5.4). For the sake of reinforcing the theory, §6 is added where its consistency with foundation of the standard model [19, 20] is tested. Introducing a transformation form of cylindrical function that describes transition from strong to electroweak interaction (§6.1), first, I give an analytical meaning of appearance of quark charge (§6.2). Next, I examine the compatibility with the conventional vacuum model [21], and explain a notion of imprint of lepton mass (§6.3) and charge (§6.4) in the electroweak interaction. It is also remarked that the theory potentially covers gravitational interaction (§6.5). Finally, §7 is devoted to summary.
and conclusion.

2 Construction of spatial transformation based on plasma analogy

For mass state of photons in plasma, we see how the wavenumber transforms between two adjacent regions with different plasma densities. The transformation is arranged in a form associated with elementary excitation of particles. The abstracted form is incorporated in spatial transformation involved in potential transformation.

2.1 Brief review of dielectric dispersion theory

We consider a discharged gas distributing in vacuum, namely, plasma, which comprises freely moving electrons and the charge compensating ions. Light propagating through it induces currents carried mainly by electrons having the small inertia, whereupon this effect is fed back to the light. The interaction results in giving rise to optical dispersion. When describing this phenomenon theoretically, one begins with electromagnetic field equation in vacuum [22], taking the free currents into account. For simplicity, provided the background ions are immovable, the current formation is considered which involves a simple harmonic oscillation of the nonrelativistic electrons that experience a single force $-eE$, where $E$ is the self-consistent electric field. Within the framework of this model, one has the following type of governing equation for transverse electromagnetic fields $F: \{E, B\}$ [23]:

$$\left( \frac{\partial^2}{\partial t^2} - c^2 \Delta \right) F = -(ne^2/m_e)F, \quad (1)$$

where $t$ is time on the ion rest frame, $\Delta = \nabla^2$, and $n$ and $m_e$ are the plasma density and the electron rest mass, respectively. When supposing the plasma and plane wave to be infinitely pervading the vacuum, Eq. (1) immediately leads to the dispersion relation of the electromagnetic wave having the angular frequency $\omega$ and wavenumber $k$:

$$\omega^2 = c^2k^2 + \omega_p^2, \quad (2)$$

where $\omega_p$ is the plasma frequency that satisfies $\omega_p^2 = ne^2/m_e$ [24]. The appearance of $\omega_p$ signifies that dielectric response of the plasma is collective. For Eq. (2), the refractive index is given by $ck/\omega = \sqrt{1 - (\omega_p^2/\omega^2)}$, where $k = |k|$.

On both sides of Eq. (2), we multiply square of $\hbar$ essential for wave-particle duality, and introduce $E = h\omega$ and $p = h\mathbf{k}$, respectively, as energy and momentum eigenvalue of the light quantum. Then, the relativistic relation comes out, to give [25]

$$E^2 = c^2p^2 + m^2c^4, \quad (3)$$

where $m$ is defined by

$$m = \hbar\omega_p/c^2, \quad (4)$$
so that it has dimension of mass. This suggests that the photons behave as if they have the mass, as similarly seen in superconductors. It is claimed that the optical dispersion model describing the basic, collective phenomenon of plasma exhibits affinity for both quantum mechanics and relativity, even though excluding these effects on plasma particles.

It should be noted that real position vector \( \mathbf{R} \) of the harmonic oscillator, which assigns the photons \( m \), is parallel to \( \mathbf{E} \) (resp. orthogonal to magnetic field \( \mathbf{B} \)), on account of \( \mathbf{R} = \left( e / k_s \right) \mathbf{E} \) (resp. \( \perp \mathbf{B} \)), where \( k_s = m_e \omega^2 \) is equivalent to the spring constant. This means, when regarding \( k_s / e \) as an arbitrary constant \( \Lambda \) such that \( \Lambda \to 0 \) reflects \( E (= \hbar \omega) \to 0 \), Eq. (1) with the following replacement results in describing the coordinate wave that traces position of the electrons:

\[
\mathbf{E} \to \Lambda \mathbf{R}, \quad \mathbf{B} \to i\Lambda \mathbf{R},
\]

(5)

where \( i = \sqrt{-1} \). Equation (5) seems to expose a hidden projective form relating virtual photons to rotational coordinate, in accordant with the intuitive image of spinning electron. Besides, a view of the coordinate wave is reminiscent of general coordinate transformation of space-time [26]. These stuffs imply that the electron will be with a symmetry between photon and graviton, characterized by \( c \), thereby, including the origin of \( \alpha \) (cf., also Ref. [27]).

### 2.2 Wavenumber transformation of light

#### 2.2.1 Elementary excitation representation of propagative photon with mass

The nonuniformity of plasma is taken into consideration, which varies phase velocity of the light. For instance, as shown in Fig. 1(a), we set two distinct regions: the high density region I specified by the plasma frequency of \( \omega_{p1} \) and the low density region II by that of \( \omega_p (< \omega_{p1}) \) such that on an infinite boundary surface, these come in contact with one another. A monochromatic light with \( \omega (> \omega_{p1}) \) propagating through the region I in the direction normal to the contact surface is transmitted to the region II across the surface. Provided \( \omega \) is unchanged in the entire process, the dispersion relations in the region I and II can be written as \( \omega^2 = c^2 k^2 + \omega_{p1}^2 \) and \( \omega^2 = c^2 k_2^2 + \omega_p^2 \), respectively. By combining them, we obtain, for positive real wavenumber transformation of \( k \to k_2 \), the factorized representation of \( k_2 \):

\[
k_2 = \sqrt{\left( k^2 + \frac{\omega_{p1}^2}{c^2} \right) \left( 1 - \frac{\omega_p^2}{\omega^2} \right)}.
\]

(6)

Note that the second factor of the right-hand side (RHS) is just the refractive index: \( \tilde{n} = ck_2/\omega \) in the region II.

Here, the situation is supposed in which keeping the ratio of \( \omega_p / \omega \) a constant, \( \omega \downarrow \omega_{p1} \) is taken to give \( k \to 0 \). Introducing the definitions of \( \mu = k_2 \mid_{k=0}, \tilde{\mu} = \sqrt{\omega_{p1}^2 / c^2 + k^2} \mid_{k=0}, \)
Figure 1: Schematics of the wavenumber transformation $k \rightarrow k_2$, which comes about when light (solid curves) propagates from plasma region I to II across the boundary (dashed lines): a surface of discontinuity of the density $n$ (solid lines). Assuming angular frequency of the light, $\omega$, to be invariant for the transformation, shown are the cases of $\omega_p < \omega_p 1 < \omega$ (a) and $\omega_p 1 < \omega < \omega_p$ (b), where $\omega_p 1$ and $\omega_p$ stand for the plasma frequency of the region I and II, respectively.

and $\delta = \frac{\omega^2_p}{\omega^2} \big|_{\omega=\omega_p 1}$, the wavenumber transformation can be expressed as

$$k (= 0) \rightarrow \mu = \bar{\mu} \sqrt{1 - \delta}.$$  \hspace{1cm} (7)

Equation (7) can be regarded as representing creation of a propagative photon having the mass of $m = \hbar \bar{\mu} / c$. Particularly, in the case for which $\delta \ll 1$, the phase velocity in the region II, $c/\bar{n}$, is approximately given by $c(1 + \delta/2)$, so that it indicates small deviation from $c$. This is of the property desired in terms of a clue to the spatial transformation. Expected is the relation of Eq. (7) with the lowest energy excitation of real particles.

2.2.2 Elementary excitation representation of non-propagative photon with mass

The density profile is inverted with respect to the propagation direction of the light. As shown in Fig. 1(b), we configure the low density region I with the plasma frequency of $\omega_p 1$ and the high density region II with $\omega_p (> \omega_p 1)$ such that an infinite contact surface separates them. A light with $\omega (> \omega_p 1)$ propagating through the region I normally incidents on the surface to be transmitted to the region II. Here, the situation is considered in which the transmitted light decays because of $\omega < \omega_p$. For the dispersion relations same as given above, we obtain, for $k \rightarrow k_2$, the following representation of $k_2$:

$$k_2 = -i \frac{\omega_p}{c} \sqrt{1 - \frac{\omega^2}{\omega_p^2}},$$  \hspace{1cm} (8)

making the negative sign have the physical meaning that a damping solution has been chosen. Again keeping the ratio of $\omega/\omega_p$, $\omega_p 1 \uparrow \omega$ is taken to give $k \rightarrow 0$, and then, the
wavenumber transformation can be expressed as

\[ k (= 0) \to \mu = -i\bar{\mu}^*\sqrt{1 - \delta}, \]  

(9)

where \( \bar{\mu}^* = \omega_p/c, \delta = \omega^2/\omega_p^2|_{\omega=\omega_p}, \) and \( m^* = \hbar\bar{\mu}^*/c, \) which signifies the mass of non-propagative photon. Expected is the relation of Eq. (9) with the lowest energy excitation of virtual particles.

### 2.3 The application to spatial transformation

The wavenumber transformation of the photon having the mass is coupled with coordinate of the propagation direction, to generate phase transformation of the plane wave between the spatial region I and II. In reference to this specific example, we are tempted to straightforwardly make out transformation between an internal space of particles and the \( \mathbb{R}^3 \) real space from which observer looks into it. In inseparable relation of the observation with coordinates, we focus on the fact that, detecting one particle owes to presence of interaction potential expanded, by the particle itself, in \( \mathbb{R}^3 \). We provide a generic form of static, three-dimensional isotropic potential, \( V(\xi') \), centered at coordinate origin \((X,Y,Z) = (0,0,0)\) in the flat \( \mathbb{R}^3 \). Here, as a definition of the dimensionless variable, given is \( \xi' = \mu|\mathbf{R}| = \mu R \in (0, +\infty) \). For an equipotential surface of \( R = \sqrt{X^2 + Y^2 + Z^2} = \xi' / \mu \), equation of circle having the dimensional radius \( R \) on the equatorial plane \( Z = 0 \) is given by

\[ X^2 + Y^2 = \xi'^2 / \mu^2. \]  

(10)

This locus is described by the blowup that includes another dimensionless parameter \( \theta' \):

\[ \tilde{X} = \mu R \cos \theta' = \xi' \cos \theta', \quad \tilde{Y} = \mu R \sin \theta' = \xi' \sin \theta'. \]  

(11)

Use of these variables \((\tilde{X}, \tilde{Y})\) leads Eq. (10) readily to the dimensionless equation of

\[ \tilde{X}^2 + \tilde{Y}^2 = \xi'^2. \]  

(12)

For \( V(R) \), we set a domain of definition in the region of \( R \neq 0 \). Introducing the minimum allowable Euclidean radius of the equipotential sphere \( \epsilon \), supposed is that in the domain of \( R \geq \epsilon, \mu R \) indicates a well-defined real value of \( \xi' \). This means that as long as \( R \geq \epsilon \) is satisfied, the indeterminate form of \( \mu R \) for \( R \to 0 \) and \( \mu \to \infty \) is merely of the apparent one. On the other hand, in the domain of \( R < \epsilon \) where the indetermination led by \( R \to 0 \) essentially sets in, we require discontinuous transition from Eq. (12) to

\[ \tilde{X}^2 + \tilde{Y}^2 = 0, \]  

(13)

as a plausible expression of null radius, daringly for \(|\tilde{X}| \to 0\) and \(|\tilde{Y}| \to 0\). Although this appears to be in contradiction with the classical Pythagorean theorem, there is a physical reasoning by which to escape from this dilemma:
We reconsider a non-propagative photon characterized by the real mass \( m^* \neq 0 \), the real energy \( E(< m^*c^2) \), and the imaginary momentum \( \mathbf{p} \). We then make the null vector transition from the virtual particle state having a real \( p^* \) that satisfies \((p^*)^2 = -p_i^2\), into massless state:

\[
(E/c)^2 + (p^*)^2 = (m^*c)^2 \longrightarrow (E/c)^2 + (p^*)^2 = 0. \tag{14}
\]

This is made to correspond to Eq. (12) \( \rightarrow \) (13). The transition of Eq. (14) with \( E/c \rightarrow 0 \) and \( p^* \rightarrow 0 \) actually allows for replacement such as \( E/c \rightarrow p \) and \( p^* \rightarrow i\mathbf{p} = \mathbf{q} \), with \( \mathbf{p} \) being a real three-dimensional vector. The massless particle having the momentum \( \mathbf{p} \) whose magnitude indicates \( E/c \), and transverse polarization \( \mathbf{e}_\perp \) in \( \mathbf{q} \), is definitely an entity observed as photon in vacuum. Hence, Eq. (14) accommodated by the replacement is a physically possible process as spatial transfer from the medium to the vacuum.

The singular region characterized by Eq. (13) is referred to as infinitesimal region. In correspondence to what observer can never jump on internal coordinate of the massless photon, \( \mathbb{R}^3 \) spanned by \( V(R) \) cannot exist in the infinitesimal region, because of the definition. Nevertheless, in analog of the replacement by which to access to \((\mathbf{p}, \mathbf{e}_\perp)\) as an eigenstate of the photon, it will be allowed to construct a sort of spatial transformation to access to internal coordinate inherent to particles. That is, we can admit that for Eq. (12) \( \rightarrow \) (13), the following replacement invoking a basis vector \( \hat{\mathbf{x}} \) holds:

\[
\hat{X} \sim \cos \theta' \longrightarrow \hat{x}, \quad \hat{Y} \sim \sin \theta' \longrightarrow i\hat{y}. \tag{15}
\]

When introducing the orthogonal basis vector \( \hat{y} \) that satisfies \( \hat{y} = i\hat{x} \), we obtain \( \hat{Y} \rightarrow \hat{y} \). As is, it turns out that Eq. (15) prompts a leap from the real \( XY \)-plane to Wessel-Argand-Gauß plane, the so-called complex plane, settling \( \hat{x} \) and \( \hat{y} \) as the axes. This plane exists in \( \mathbb{R}^3 \times \mathbb{R}^3 \) host space as a null complex three-dimensional space subject to \( \hat{x} + i\hat{y} = 0 \), sustaining orthogonality of the two \( \mathbb{R}^3 \)-real spaces. On the plane we set \( \zeta = \hat{x} + i\hat{y} \) with \( \hat{x} \) and \( \hat{y} \) real, to give \( \hat{(x, y)} = (\xi \cos \theta, \xi \sin \theta) \) along Eq. (11).

It is a matter of finding a linear transformation of \((\theta', \xi') \rightarrow (\theta, \xi)\) responsible for \((\hat{X}, \hat{Y}) \rightarrow (\hat{x}, \hat{y})\), such as \( \theta'/g_0 \rightarrow \theta \) and \( \xi'/g_0 \rightarrow \xi \) with \( g_0 = 2 \), in consideration of the multiplicity of \( \mathbb{R}^3 \) and circumference continuity at \( R = \epsilon \). Putting \( \xi = kr \in (0, \infty) \) for \( \zeta \in \mathbb{C}_r \subseteq \{ \hat{z} \in \mathbb{C} \mid \hat{z} \neq 0 \} \), \( \xi' \rightarrow g_0 \xi \) is cast to the expression of

\[
\mu R (= \infty \cdot 0) \longrightarrow g_0 kr. \tag{16}
\]

Here, the left-hand side (LHS) including the round bracket represents that for \( R \rightarrow 0 \) and \( \mu \rightarrow \infty \), \( \mu R \) is in essentially indeterminate region, manifesting extension of the number system \( \mathbb{R}_* := \{ \xi', -\xi' \} \). Equations (15) and (16) can be understood as a feasible form of transformation from \( \mathbb{R}^3 \) to the infinitesimal \( \mathbb{R}^3 \times \mathbb{R}^3 \) space, along \( \mathbb{R}_* \cup \{ \pm \infty \} \supset \{ \pm \xi' \} = \{ 0 \} \rightarrow \mathbb{C}_r \) or \( \mathbb{R}_* \cup \{ \pm 0 \} \supset \{ 0 \} \rightarrow \mathbb{C}_r \), to be involved in potential transformation of \( V(\xi') \rightarrow \eta(\xi) \).

Let us now consider the inverse transition: \( \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) in order to get a potential transformation of which the output is \( V(\xi') \). For the inverse of Eq. (15), we read the
following replacement, invoking a basis vector $\mathbf{X}$:

$$\tilde{x} \sim \cos \theta \rightarrow \mathbf{X}, \quad i\tilde{y} \sim i\sin \theta \rightarrow -\mathbf{X}. \quad (17)$$

As the inverse transformation of Eq. (16) in which Eq. (17) is involved, a possible form is

$$kr \left(= 0 \cdot \infty \right) \rightarrow g_0^{-1} \mu R. \quad (18)$$

Here, the LHS including the round bracket represents that for $k \to 0$ and $r \to \infty$, $kr$ is in essentially indeterminate region, manifesting extension of $C_\star$. Equations (17) and (18) suggest that in the infinite distance of $r^3$, exists $R^3$, along $C_\star \cup \{0\} \supset \{\pm\infty\} \rightarrow \mathbb{R}$ or $C_\star \cup \{\infty\} \supset \{\pm\infty\} \rightarrow \mathbb{R}$. The limit of $k \to 0$, cooperating with $r \to \infty$, may be interpreted as a projection linked to the foregoing setting of $Z = 0$ in $\mathbb{R}^3$. As the projection, feasible is that of cylindrical function onto the base as we will see later.

Apparently, Eq. (18) is of a form that enables us to make $k \left(= 0\right)$ and $\mu$ correspond to those in Eqs. (7) and (9), and $r = \infty$ to a bulk condition for the plane wave to have the diverging wavelength. Therefore, also for $\mu$ concerned here, the same form of factorization is provided which consists of $\bar{\mu}$, $\bar{\mu}^*$, and dimensionless parameter $\delta$ (typically $\ll 1$), though these are at the moment to be regarded as the abstracted symbols. Here, we pay attention to what in Eq. (7) $k = 0$ was required for fixing $\bar{\mu}$, whereas in Eq. (9) $\bar{\mu}^*$ was free of $k$. Besides, since $\delta$ therein could be handled as a constant inherent to the dispersible media, the corresponding symbolic parameter as well is supposed a constant inherent to elementary excitation. Taking all this into consideration, we put forth, as a hypothesis, the $r^3 \times r^3 \rightarrow \mathbb{R}^3$ transformation related to $V(R)$ generation reflecting the lowest energy excitation of real and virtual particle, respectively, as represented in the form of

$$kr \left[= 0^{(l)} \cdot \infty \right] \rightarrow g^{-1} \bar{\mu} R^*, \quad (19a)$$

$$kr \left[= 0 \cdot \infty^{(l)} \right] \rightarrow g^{-1} \left(-i\bar{\mu}^* \right) R^* = g^{-1} \bar{\mu}^* R, \quad (19b)$$

where definition of the reciprocal of the transformation factor $g$ is given by

$$g^{-1} = g_0^{-1} \sqrt{1 - \delta}. \quad (20)$$

The primes in Eqs. (19a) and (19b) denote that $k \to 0$ and $r \to \infty$ take the lead in the essential indetermination along $C_\star \cup \{0\}$ and $C_\star \cup \{\infty\}$, respectively. When regarding $\bar{\mu}$ and $\bar{\mu}^*$ after the transformations as $\bar{\mu}(m)$ and $\bar{\mu}^*(m^*)$, respectively, $m$ and $m^*$ could be interpreted as observable mass of real and virtual particle, respectively.

Following the indeterminate expressions of Eq. (19), the LHS of Eq. (16) should be rewritten as $\mu R \left[= \infty \cdot 0^{(l)} \right]$. The argument would virtually postulate the smaller mass of $\mu \ll \ell_P^{-1}$, viz., $R_S \ll \ell_P$, where $\ell_P$ and $R_S$ denote the Planck length and Schwarzschild radius [28], respectively. On the other hand, for $\mu \gg \ell_P^{-1}$, a singularity could arise such that it is represented by another form of indetermination: $\mu R \left[= \infty^{(l)} \cdot 0 \right]$ designating the lead of $\mu \rightarrow \infty$. This realm might be responsible for a black hole expanding far-field, Newtonian potential in $\xi' = \mu R \gg 1$. 


3 Cylindrical space as an infinitesimal space

Prior to application of the spatial transformation to point-like leptons, its feasibility is confirmed by the direct use for reproduction of familiar phenomenological potentials. At the same time, it is revealed that cylindrical space expands in the infinitesimal region. We discuss physical meaning of the transformation factor involving $\delta$.

3.1 Complex analytical consideration of quark-antiquark potential

It is speculated that structure of leptons would contain a vestige of the mechanism whereby they are created. At the outset, we recall the decay reaction of $\pi^-$-meson:

$$\pi^- \longrightarrow \begin{cases} 
\mu^- + \bar{\nu}_\mu, \\
e^- + \bar{\nu}_e,
\end{cases}$$

(21)

which has been compared to the galactic nuclear activity. Combining the second equation with $n^0 \longrightarrow p^+ + \pi^-$ leads to $n^0 \longrightarrow p^+ + e^- + \bar{\nu}_e$, i.e., the expression of the $\beta^-$-decay that emits electron and electron antineutrino. Within the framework of the quark model [20], the elementary process can be expressed as $\nu_e + d \longrightarrow e^- + u$. For convenience, here, we note quark and antiquark as $q$ and $\bar{q}$, respectively, and lepton and antilepton as $\ell$ and $\bar{\ell}$, respectively, to give the somewhat generic expression:

$$q_L + (\bar{q})_R \longrightarrow \ell_L + (\bar{\ell})_R,$$

(22)

where the subscripts L and R denote the left- and right-handed state, respectively.

We aim at associating $k \rightarrow 0$ followed by Eq. (19), with elementary excitation of quarks. For the moment, this is intended for a $q\bar{q}$ pair whose motion is nonrelativistic so that $|P| \rightarrow 0$ (retaining $|P| \neq 0$), though practically the limit is supposed, for heavier mesons, a good approximation. We call a basic form of the $q\bar{q}$ potential that has been confirmed for the relevant mesons: the so-called Cornell potential of

$$V(R) = C_1 R - C_{-1} R^{-1},$$

(23)

with $C_1$ and $C_{-1}$ real constants [29]. A special attention is paid to the fact that this type of function mathematically satisfies the ordinary differential equation of

$$\left( \frac{d^2}{dR^2} + \frac{1}{R} \frac{d}{dR} - \frac{1}{R^2} \right) V(R) = 0.$$ 

(24)

Let us see the large bracket of Eq. (24) as an operator that works for $V(R)$. Then, it seems as if exposing a part of the Laplacian for cylindrical coordinate system. On the other hand, the equipotential surface of $V(R)$ is, in fact, not cylindrical. Therefore, the cylindrical system is, if any, an entity that ought to be distinguished from $\mathbb{R}^3$. At this juncture, we virtually introduce an $r^3$ real space to configure the cylindrical
system. Thereupon, we presume the harmonic function \( \phi(\mathbf{r}) \) spanned in the space, which satisfies the Laplace equation: \( \Delta \phi(r, \theta, z) = 0 \). Concerning the solution of the form of \( \phi = \eta(r)e^{i(m\theta-kz)} \), we look at the following equation for \( \eta \):

\[
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - k^2 \right) \eta(r) = 0.
\]  

(25)

For the large bracket set to \( |m| = 1 \), taking \( k \to 0 \) compared to \( |P| \to 0 \), and simultaneously carrying out the replacement of \( r \to R \), bring about the operator of Eq. (24). Evidently, this procedure entails the function transformation of \( \eta(r) \to V(R) \). Note here that \( k \neq 0 \) is necessary for ensuring \( \mathbb{R}^3 \neq \mathbb{R}^3 \).

In Eq. (25) divided by \( k^2 \), we put \( \xi = kr \in (0, \infty) \). In the general case for which \( m \) is an integer, two linearly independent solutions to this equation are the modified Bessel functions of order \( m \), denoted as \( I_m(\xi) \) and \( K_m(\xi) \) [in the case of \( m \neq \text{integer} \), \( I_m(\xi) \) and \( I_{-m}(\xi) \)] \(^1\). Hence, \( I_{\pm 1}(\xi) \) and \( K_{\pm 1}(\xi) \) are in on the current issue. Moreover, their asymptotic forms for the small \( \xi \), in which \( k \to 0 \) takes the lead, should be considered. Then, \( k \to 0 \) signifies a projection onto cylindrical base as expected before. The scalar function \( \phi \) as linear combination of the asymptotic forms, \( \phi(k \to 0) \), constitutes a complex function, which is denoted by \( w = f(\zeta) \). Here, \( \zeta = \xi e^{i\theta} \) and \( (\xi \cos \theta, \xi \sin \theta) = (\tilde{x}, \tilde{y}) \), along the notation given in §2.3.

Letting \( u(\xi, \theta) \) and \( v(\xi, \theta) \) be real functions, we reasonably postulate the complex function of \( w = u + iv \) to be analytic. Then, \( u \) and \( v \) satisfy the Cauchy-Riemann equation of

\[
\frac{\partial u}{\partial \xi} = \xi^{-1}\frac{\partial v}{\partial \theta}, \quad \xi^{-1}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial \xi},
\]  

(26)

to be the harmonic conjugates on the \( \tilde{x}\tilde{y} \)-plane. The function \( w = \phi(k \to 0) \) that satisfies Eq. (26) can be written as

\[
\lim_{k \to 0} \left[ c_1 I_1(\xi)e^{i(\theta-kz)} - c_{-1} K_{-1}(\xi)e^{i(-\theta-kz)} \right],
\]  

(27)

where \( c_1 \) and \( c_{-1} \) are constants. It is found that \( \mathbb{R}^3 \times \mathbb{R}^3 \) spaces are required for spanning the two modes of \( m = \pm 1 \), to be intertwined via the complex plane. Incidentally, \( K_1 (= K_{-1}) \) and \( I_{-1} (= I_1) \) are called later. One can check that the expressions of

\[
u = \left( \frac{c_1}{2} \xi - c_{-1} \frac{1}{\xi} \right) \cos \theta, \quad v = \left( \frac{c_1}{2} \xi + c_{-1} \frac{1}{\xi} \right) \sin \theta,
\]  

(28)

derived from Eq. (27), satisfy Eq. (26). We represent the variable \( \xi \), which has a non-zero but small value due principally to \( k \to 0 \), as \( kr \in (0^{(\infty)} \cdot \infty) \), letting the prime have the same meaning as that in Eq. (19a).

Of importance is to clarify a domain of \( \xi \) allowed for Eq. (28), that is, an upper limit of \( \xi \) by which \( r \to \infty \) is well constrained. The complex function \( f(\zeta) \) provides the following form of linear map from the coordinate plane \( \xi \) to \( w \):

\[
w = (c_1/2) \zeta - c_{-1} \zeta^{-1}.
\]  

(29)

\(^1\)Throughout this paper, notation of special functions conforms to Ref. [30].
With respect to magnitude of $|\zeta| = \xi = kr$, the transformation $f: (\tilde{x}, \tilde{y}) \rightarrow (u, v)$ involves a critical value of $\tilde{k} = ka = \sqrt{2c_{-1}/c_1}$ corresponding to $r = a$. For the coordinate transformation to be bijective conformal map, $\xi < \tilde{k}$ must be satisfied, so that one can regard $\tilde{k}$ as the upper limit of $\xi$. In particular, circular rotation on $\xi = \tilde{k}$ is, in the case of $c_1 c_{-1} > 0$, transformed to oscillation on line segment of imaginary axis connecting $(u, v) = (0, -\sqrt{2c_1 c_{-1}} i)$ to $(0, \sqrt{2c_1 c_{-1}} i)$ on the $w$-plane. On the other hand, in the case of $c_1 c_{-1} < 0$ (including the Joukowski transformation for $c_1 = 2, c_{-1} = -1$ as a special case), circular rotation on $\xi = \sqrt{-2c_1 c_{-1}}$ is transformed to oscillation on line segment of real axis connecting $(-\sqrt{-2c_1 c_{-1}}, 0)$ to $(\sqrt{-2c_1 c_{-1}}, 0)$. In other words, two-valuedness arises on the real axis of the $w$-plane. It turns out that such a finite domain prohibited for real $u$ does not appear in the former case; therein we can set $\mathbb{C}_*$ to as $\{\zeta \mid 0 < \xi \leq \tilde{k}\}$.

In the case for which $c_1 c_{-1} > 0$, the real function of $u = \Re(w) = \Re[\phi (k \rightarrow 0)]$ for $\xi < \tilde{k}$ could appear as Eq. (23). It is natural to identify the concerned $\mathbb{R}^3$ cylindrical space with the $\mathbb{R}^3$ space that has been introduced in §2.3. In the transformation of $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ for generation of $V(R)$, the lead by $k \rightarrow 0$ and $\mathbb{C}_* \cup \{0\}$ should be involved. According to Eq. (17), $u$ in Eq. (28) could be rewritten as $\eta(\xi) \tilde{X}$, where $\eta$ stands for a real amplitude of the intertwined modes. Now, the transformation is applied to $\eta(\xi)$, of which the type is the same as Eq. (19a):

$$\eta := kr \left[= 0^{ (+)} \cdot \infty \right] \rightarrow g^{-1} \tilde{R}^u.$$ (30)

The function of $R^u$ generated is then found to show the form of Eq. (23). It appears that $R^u$ with the asterisk indicates the coordinate of the closed $\mathbb{R}^3$-space confining colors. It is remarked that $|m| = 1$ would be responsible for magnitude of spin of gauge bosons mediating the strong force.

### 3.2 Reconstruction of the Yukawa potential

If one prohibits the two-valuedness on the $w$-plane, $\xi$ cannot exceed $\tilde{k}$, as long as $k \rightarrow 0$ takes the lead. This we could recognize physically as principle of quark confinement. In order to move to a region of larger $\xi$, we shall prepare $\phi (r \rightarrow \infty)$ owing to the lead of $r \rightarrow \infty$, instead of Eq. (27), i.e.,

$$\lim_{r \rightarrow \infty} [c_1 I_1(\xi)e^{i(\theta - k z)} - c_{-1} K_{-1}(\xi)e^{i(-\theta - k z)}].$$ (31)

In contrast to the indeterminate form denoted in §3.1, given herein is $\xi := kr \left[= 0 \cdot \infty^{(0)} \right]$. This manifests that $k \rightarrow 0$ is taken to the extent that the large $\xi$ is well maintained. In that region, Eq. (31) leads to

$$\frac{c_1 e^\xi - \pi c_{-1} e^{-\xi}}{\sqrt{2\pi \xi}} \cos \theta + \frac{i c_1 e^\xi + \pi c_{-1} e^{-\xi}}{\sqrt{2\pi \xi}} \sin \theta.$$ (32)

When postulating the relation of $c_{-1}/c_1 = 1/\pi$ corresponding to $\tilde{k}^2/2$, Eq. (32) can be cast to the linear combination form for $m \neq \text{integer}$:

$$c_1 \left[I_{1/2}(\xi) \cos \theta + i I_{-1/2}(\xi) \sin \theta \right].$$ (33)
This suggests that the system spanning \( \phi (r \to \infty) \) could be accommodated by \( \theta \to \theta'/g_0 \). As is, we again apply Eq. (17) to (33), in terms of \( \mathbb{C}_* \cup \{\infty\} \), where \( \mathbb{C}_* = \{\zeta \mid k < \xi < \infty\} \). The resulting quantity is written as \( \eta(\xi)\hat{X} \) as before, and then, the function \( \eta \) is given by

\[
\eta(\xi) = \frac{2}{\pi} c_0 K_{\pm 1/2}(\xi) = c_0 \sqrt{\frac{2}{\pi \xi}} e^{-\xi},
\]

where \( c_1 \to -c_0 \) has been taken into account. We interpret \( \eta \) as an amplitude reflected in virtual displacement along \( \hat{X} \). Then, \( \eta^2 \) is associated with potential energy of internal harmonic oscillation. The spatial transformation is applied to it, of which the type is the same as Eq. (19b). This results in yielding the energy transformation of

\[
\eta^2(\xi) \longrightarrow V(R) = G^2 e^{-\bar{\mu}^* \sqrt{1-\delta} R},
\]

where \( G^2 (= \text{const.}) \sim c_0^3 / (\bar{\mu}^* \sqrt{1-\delta}) \).

According to the asymptotic property of \( K_{\pm 1}(kr) \xrightarrow{r \to \infty} K_{\pm 1/2}(kr) \), the product \( \eta^2 \sim c_0^2 K_{-1/2} K_{1/2} \) can be understood as far-field interference between \( K_{-1} \) and its counterpart \( K_1 \). Actually, formation and interference of the equivalent fields can be seen in organization of vortices as two-dimensional structure of plasma turbulence [31] and interaction of the vortices [32]. The attractive force representation of \( V(R) \) can be obtained by setting \( c_0 \) to imaginary as is orthogonal to \( c_1 \). The resulting function exposes a form of the coupling potential for nuclei [18]. Accordingly, it will be appropriate to make the harmonic oscillator correspond to \( \pi \)-meson having spin 0.

The thing that inertia of the constituent quarks is so small as to degrade nonrelativistic approximation for the motion, is thought of as reflecting the constraint on \( k \to 0 \), contrast to the previous constraint on \( r \to \infty \). Relating to this, \( \xi \gtrsim O(1) \) can be reflected in the \( \mathbb{R}^3 \) region for which the Yukawa potential is allowed: \( R \gtrsim 1/\bar{\mu}^* \). What \( k = 0 \) is, in any case, prohibited in the system subject to Eq. (25) suggests that quarks could never be in rest state.

### 3.3 Physical meaning of the transformation factor \( g \)

The function of \( \Psi(R) = -V(R)/(4\pi G) \) is a solution of the following equation:

\[
\left[ \frac{1}{R^2} \frac{d}{dR} \left( R^2 \frac{d}{dR} \right) - (\bar{\mu}^*)^2 (1 - \delta) \right] \Psi(R) = G\delta(R),
\]

where \( \delta(R) \) in the RHS is the Dirac delta function. Equation (36) is no more than the next equation a spherically symmetric solution of \( \Psi(R) \) obeys:

\[
[\Delta - (\bar{\mu}^*)^2] \Psi(R) = G\delta(R).
\]

Here, the definition of real parameter, \( \mu^* = \bar{\mu}^* \sqrt{1-\delta} \), has been introduced. Noted is that, when using this, the transformation corresponding to Eq. (9) can be expressed
as \( k (= 0) \rightarrow -i \mu^* \). Since \( G \delta (\mathbf{R}) \) physically stands for a steady source, \( \Psi \) can be interpreted as a steady-state solution of wave from the source at \( \mathbb{R}^3 \) origin.

We contemplate the regular region of \( R \neq 0 \) in which the RHS of Eq. (37) vanishes. The homogeneous equation is compared with the Klein-Gordon equation for \( E^2 = c^2 \mathbf{P}^2 + m_{\pi}^2 c^4 \), where \( m_{\pi} \) is the pion mass we observe in our laboratory. Making \( \mu^* \) correspond to \( m_{\pi} c / \hbar \), then, we obtain the correspondence of

\[
\delta \leftrightarrow \left[ E / (m_{\pi} c^2) \right]^2,
\]

in harmony with \( \delta = \omega^2 / \omega_p^2 \big|_{\omega=\omega_{p1}} \) of Eq. (9). According to Eq. (38), \( \Psi \) describing the steady state of \( \partial_t = 0 \) might appear to be obtained by putting \( \mu^* (\delta = 0) \) in Eq. (37). Meanwhile, \( E \rightarrow 0 \) meets zero-point energy. In the plasma analog of \( \delta \) is thought of as symbolizing discreet appearance of time in harmony with \( \piN \). From Eq. (38), it can be understood that, in the lowest energy excitation of the Yukawa-type boson, \( \delta \lesssim (10^{-2})^2 \) is reflected in a quite small \( \partial_t^2 \). In parallel with this, when considering excitation of charged leptons, we anticipate \( \delta \) is reflected in \( \partial_t, \delta \lesssim 10^{-2} \), and its correspondence to energy of virtual photon (these are validated later). For the factor \( g \) that consists of \( \delta (\ll 1) \) unfixed at the moment and the common principal factor \( g_0 = 2 \), we prepare the expression of

\[
g = g_0 (1 + \delta g),
\]

\[\text{(40)}\]
where \( \delta_g = (1 - \delta)^{-1/2} - 1 = \delta/2 + 3\delta^2/8 + \cdots \). Later use is made of Eq. (40), when a cylindrical function associated with leptons is raised to the power of \( g \) in the form of Eq. (39).

4 Rotational field in the cylindrical space

On the basis of the preliminary study, an equation of rotational field is proposed which describes the \( \ell \bar{\ell} \)-pair created in the \( \beta \)-decay. Providing a cylindrical boundary condition, we find the possible eigenstates and eigenvalues. We reveal asymmetry of the eigenmode, and argue the relation with left-handed selectivity in the charged current weak interaction.

4.1 Equation of the rotational field

Equation for the function \( \eta(kr) \), which generates the potential of strong force inside and outside meson, is \( (k^2\nabla_\perp^2 - 1) \eta = 0 \) for \( |m| = 1 \) and \( 1/2 \), respectively. This means that separation of \( q\bar{q} \) pair, accompanied by creation of the meson, can be understood as the system transformation involving \( |m| = 1 \to 1/2 \). Now, we consider in the cylindrical space the transformation of \( k \) responsible for the \( q - \bar{q} \) scattering process given by Eq. (22). The internal \( k \)-transformation is to have the form of \( k \to -i\mu^* \) (cf. §3.3). Such a scattering-like transformation is applied to \( \eta \sim I_{\pm 1}(kr) \) potentially coexisting with the internal function that generates \( \Psi(R) \) in the detectable region of \( r \to \infty \). Then, we have \( I_{\pm 1}(kr) \to \mp iJ_{\pm 1}(\mu^*r) \) (the double signs correspond). As a matter of course, the equation that \( \eta(\mu^*r) \) obeys is \( [(\mu^*)^{-2}\nabla_\perp^2 + 1]_{m=\pm 1} \eta = 0 \), where \( \mu^* \neq 0 \). Although \( \mu^* \) herein corresponds to that in Eq. (37), we should distinguish them, because the former is of the quantity coupled with the internal coordinate, albeit being detectable. The function \( \eta \) is included in the scalar function \( \phi' = \eta(\mu^*r)e^{i(\pm \theta - kz)} \), which is of solution of the following equation in the cylindrical system:

\[
[\Delta + k^2 + (\mu^*)^2]_{m=\pm 1} \phi'(r, \theta, z) = 0. \tag{41}
\]

Here, \( k \) is newly introduced of which the value can be taken arbitrarily. In particular, for \( k = 0 \) permitted here, \( \phi' \) represents single-valued amplitude of vibrations of a circular membrane on \( z \)-constant slices. In the current context, regarding \( \eta^2(\mu^*r) \) as the internal potential that works for \( \ell \bar{\ell} \) pair, it is appropriate to interpret \( \phi' \) as scalar field of another virtual boson, namely, the weak. And, we envisage that \( \mu^* \) and \( |m| = 1 \) are reflected in the mass and magnitude of the spin, respectively. The comprehensive description including transition of gauge fields is given in §6. For the moment, we focus on making out an internal transformation reflected in the particle change of Eq. (21).}

Since the \( \ell \bar{\ell} \) pair is observed as matter particles separated from one another, the system transformation \( |m| = 1 \to 1/2 \) is applied to Eq. (41). Then, the equation of \( \phi' \) describing the \( \ell \bar{\ell} \) pair is given by

\[
(\Delta + \kappa^2)_{m=\pm 1/2} \phi' = 0, \tag{42a}
\]
\[ \kappa^2 = k^2 + \mu^2, \]  

(42b)

with \( k \) arbitrary and \( \mu \neq 0 \). Natural is to suppose that \( |m| = 1/2 \) is reflected in magnitude of spin of \( \ell \) and \( \bar{\ell} \). This is based on what \( \phi' \) exhibits two-valuedness for \( |k| = k = 0 \). On the other hand, the field spanned in the most fundamental real space ought to be a single-valued function. A necessary condition for this demand is \( k \neq 0 \) to sustain the three-dimensional structure of \( \phi' \). The former \( k = 0 \) is compared to rest state of \( \ell \) and \( \bar{\ell} \), whereas the latter \( k \neq 0 \) to a moving state of them. In addition, in the transformation from Eq. (41) to (42), involved is the following replacement on the cylindrical base:

\[ \mu^* \longrightarrow \mu. \]  

(43)

This signifies mass change in a process through which the virtual gauge boson couples with the leptons.

The appearance of \( \phi' \) having the finite \( k \) without the restriction of \( k \rightarrow 0 \) is indicative of the onset of spatial direction \( k/k = \hat{z} \), which could be associated with spontaneous symmetry breaking. In this aspect, we introduce the following vector field related to the scalar field:

\[ \Phi(r) = \mu^{-2} [\nabla \times \nabla \times (\phi'\hat{z}) + \kappa \nabla \times (\phi'\hat{z})]. \]  

(44)

When postulating \( \nabla \cdot (\phi'\hat{z}) = 0 \), \( \Phi \) is found to be a Gromeka-Beltrami vector field that satisfies the equation of

\[ \nabla \times \Phi = \kappa \Phi, \]  

(45)

or \( (\nabla \times \Phi) \times \Phi = 0 \) [33]. It should be pointed out that reducing the second-order differential Eq. (42) to the first-order Eq. (45) can be compared to factorization of the Klein-Gordon equation [34]. Equation (45) can be recognized as the rotational eigenvalue equation in \( \mathbb{R}^3 \) space, so that \( \Phi \) is referred to as rotational field. It follows that, in Eq. (42), \( \phi' \) reflects eigenstates of the rotational field, and at the same time, \( \kappa(k, \mu) \) indicates the corresponding eigenvalues. An analogy to generation of \( \Phi \) can be found in transition of the turbulent state of plasma that has been referred in §3.2 (e.g., see Ref. [8]). There is a consensus that, when the plasma attains an energy relaxed state conserving the magnetic helicity, the magnetic field self-organized macroscopically in \( \mathbb{R}^3 \) space would take a configuration such that the vector field \( B \) satisfies an equation of the same form as Eq. (45) [7].

In the generic case for which \( m \) is arbitrary, \( \Phi \) is written out below. For Eq. (42), the solution concerned is \( \phi' = \phi_m J_m(\mu r) e^{i(m\theta-kz)} \), where \( \phi_m \) is constant. Substituting this into Eq. (44), one can obtain a possible expression of \( \Phi = (\Phi_r, \Phi_\theta, \Phi_z) \), where

\[
\begin{align*}
\Phi_r &= i \frac{\phi_m}{\mu} \left[ \frac{m \kappa}{\mu r} J_m(\mu r) - \frac{k}{\mu r} \frac{d J_m(\mu r)}{d(\mu r)} \right] e^{i(m\theta-kz)}, \\
\Phi_\theta &= \frac{\phi_m}{\mu} \left[ \frac{m \kappa}{\mu r} J_m(\mu r) - \frac{k}{\mu r} \frac{d J_m(\mu r)}{d(\mu r)} \right] e^{i(m\theta-kz)}, \\
\Phi_z &= \phi_m J_m(\mu r) e^{i(m\theta-kz)}
\end{align*}
\]  

(46)

and \( \mu r \in (0, \infty) \). Note the relation of \( \Phi_z = \phi' \), i.e., \( \Phi \) contains \( \phi' \) of Eq. (42a).
4.2 Eigenstates and eigenvalues of the rotational field

The helical pattern of $\Phi$ indicating rotational and translational symmetry is characterized by $m$ as a modal number, and $k$. These can be simultaneously reflected in spin state inherent to particles and their motional state in $\mathbb{R}^3$ (e.g., see Fig. 6.11 in Ref. [35]), respectively. Worth considering is a plausible situation in which $\Phi(m,k)$ is confined within a cylinder settled in $\mathbb{R}^3$. In analogy with the conducting wall condition for plasma magnetic fields that retains gauge invariance, we impose the cylindrical boundary condition of $\Phi_r(r=a) = 0$:

$$\left. \frac{m\kappa}{\mu a} J_m(\mu a) - \kappa \frac{d J_m(\mu r)}{d(\mu r)} \right|_{r=a} = 0. \quad (47)$$

Expected is that this condition regulates $\Phi(m = \pm 1/2, k)$ as internal structure of spin $1/2$ point particles, to determine susceptibility to the concerned interaction. Of particular interest is the regulation for the case in which $k \neq 0$, most likely responsible for the dynamical interaction. In the following, we thus investigate the rotational eigenstates of $\Phi(m, k \neq 0)$ involving the eigenvalues derived from Eq. (47). Henceforth, we set to $k > 0$ without loss of generality; then, $m \gtrless 0$ is responsible for positive and negative helicity state, respectively.

4.2.1 The case of mode $m = +\frac{1}{2}$

A moving, spin $1/2$ lepton coupled with virtual photon field is thought of as referring an allowable mode of the helix specified by $|m| = 1/2$ and $k \neq 0$. In this light, the eigenstates of $\Phi(m = \pm 1/2, k \neq 0)$ are examined for each $m$. Equation (47) for $m = +1/2$ can be expressed as

$$\tilde{\kappa} + \tilde{k} \left( 1 - 2\tilde{\xi} \cot \tilde{\xi} \right) = 0. \quad (48)$$

Here, $\tilde{\kappa} = \kappa a$, $\tilde{k} = ka (> 0)$, and $\tilde{\xi} = \mu a (> 0)$ are real, and among them, we have the relation of $\tilde{\kappa} = \pm \sqrt{\tilde{k}^2 + \tilde{\xi}^2}$ in response to Eq. (42b). Hence, the case analysis for the sign of $\tilde{\kappa}$ each is carried out below.

(A) The case of $\tilde{\kappa} > 0$. It is reasonable to consider that the lowest energy excitation of a "spinning point" reflects a rotational eigenstate accompanied by minimal magnitude of $\kappa$. We express $\tilde{\kappa}$ as a function of $\tilde{\xi}$, and write it as $\kappa_{R}^{(+)} (> 0)$; that is,

$$\kappa_{R}^{(+)} = \frac{\tilde{\xi} \left( 2\tilde{\xi} \cot \tilde{\xi} - 1 \right)}{\sqrt{\left( 2\tilde{\xi} \cot \tilde{\xi} - 1 \right)^2 - 1}}, \quad \text{for } \tilde{\xi} \cot \tilde{\xi} > 1. \quad (49)$$

Here, the subscript R stands for right-handedness of the helix ($m > 0$) and sign in the superscript indicates the one of $\tilde{\kappa}$. In the domain of definition, i.e., the discrete regions of $\tilde{\xi}$ satisfying the condition of $\tilde{\xi} \cot \tilde{\xi} > 1$, investigated is existence of the local
minimum of $\kappa_R^{(+)}$. If there exists, we evaluate $\kappa_R^{(+)} = \tilde{\kappa}$, $\tilde{\xi}$, and $\hat{\kappa} \left( = \sqrt{\tilde{\kappa}^2 - \tilde{\xi}^2} \right)$ at the stable point: these constitute a set of the eigenvalues for the eigenstate of $\Phi$ for $m = +1/2$. Along this guideline, $\kappa_R^{(+)}$ is plotted in Fig. 2(a). One can see that there is no local minimum in the discrete domains. This therefore leads to the conclusion that for $m = +1/2$ and $\tilde{\kappa} > 0$ exists no eigenstate of $\Phi$.

(B) The case of $\tilde{\kappa} < 0$. For Eq. (48), $-\tilde{\kappa}$ is expressed as a function of $\tilde{\xi}$, to be denoted as $\kappa_R^{(-)} (> 0)$, where the notations are the same as before. This is explicitly written as

$$
\kappa_R^{(-)} = -\kappa_R^{(+)}, \quad \text{for } \cot \tilde{\xi} < 0. 
$$

In Fig. 2(b), $\kappa_R^{(-)}$ is plotted; now we find the local minima in the discrete domains wherein $\cot \tilde{\xi} < 0$ is satisfied. Setting $d\kappa_R^{(-)}(\tilde{\xi})/d\tilde{\xi} = 0$ yields the eigenvalue equation for $\tilde{\xi}$, which can be expressed as

$$
(\tau - 4\upsilon^{-1})(\tau - \upsilon^{-1})\upsilon^{-1} + \upsilon = 0, \quad \text{for } \upsilon^{-1} < 0,
$$

where

$$
\tau = \tilde{\xi}^{-1}, \quad \upsilon^{-1} = \cot \tilde{\xi}.
$$

The transcendental Eq. (51) is numerically solved, and the solutions are numbered such as $n = 1, 2, 3, \ldots$ in order from that indicating smaller value, to be denoted as $\xi_{1/2,n}$ for the mode concerned here. It follows from this that the well-defined $\kappa_{1/2,n} = \kappa_R^{(-)}(\tilde{\xi} = \xi_{1/2,n})$ and $k_{1/2,n} = \sqrt{\kappa_{1/2,n}^2 - \xi_{1/2,n}^2}$ are evaluated. In particular, the eigenvalues for $n = 1$ indicating a rotational ground state are listed in the upper row of Table 1.
Table 1: A set of the discrete eigenvalues $\xi_{m,n}$, $\kappa_{m,n}$, and $k_{m,n}$ for $|m| = 1/2$ and $n = 1$.

| $m$ | $\xi_{m,1}$ | $\kappa_{m,1}$ | $k_{m,1}$ |
|-----|-------------|---------------|-----------|
| 1/2 | 1.891       | 2.110         | 0.9363    |
| -1/2| 3.445       | 3.632         | 1.152     |

4.2.2 The case of mode $m = -\frac{1}{2}$

Equation (47) for $m = -1/2$ can be expressed as

$$\tilde{\kappa} - \tilde{k} \left(1 + 2\tilde{\xi} \tan \tilde{\xi}\right) = 0.$$  \hspace{1cm} (52)

By the method given above, we look for the eigenvalues Eq. (52) contains.

(A) The case of $\tilde{\kappa} > 0$. When $\tilde{\kappa}$ of Eq. (52) as a function of $\tilde{\xi}$ is denoted as $\kappa_{L}^{(+)} (> 0)$, this can be expressed as

$$\kappa_{L}^{(+)} = \frac{\tilde{\xi} \left(2\tilde{\xi} \tan \tilde{\xi} + 1\right)}{\sqrt{\left(2\tilde{\xi} \tan \tilde{\xi} + 1\right)^2 - 1}}, \quad \text{for } \tan \tilde{\xi} > 0,$$  \hspace{1cm} (53)

where the subscript $L$ stands for left-handedness of the helix ($m < 0$). If there exists the local minimum of $\kappa_{L}^{(+)}$ in the domains wherein $\tan \tilde{\xi} > 0$ is satisfied, $\kappa_{L}^{(+)} = \tilde{\kappa}$, $\tilde{\xi}$, and $\tilde{k}$ are evaluated at the stable point. In Fig. 3(a), the function $\kappa_{L}^{(+)}$ is plotted, to show up the profile similar to that seen in Fig. 2(b), and existence of the local minima in the domains. Setting $d\kappa_{L}^{(+)}(\tilde{\xi})/d\tilde{\xi} = 0$ yields

$$\left(\tau + 4\upsilon\right)\left(\tau + \upsilon\right)\upsilon - \tau = 0, \quad \text{for } \upsilon > 0.$$  \hspace{1cm} (54)

Equation (54) provides the solutions, $\tilde{\xi} = \xi_{-1/2,n}$, thereby determining the values of $\kappa_{-1/2,n} = \kappa_{L}^{(+)}(\tilde{\xi} = \xi_{-1/2,n})$ and $k_{-1/2,n} = \sqrt{\kappa_{-1/2,n}^2 - \xi_{-1/2,n}^2}$. Especially for $n = 1$, these values are listed in the bottom row of Table 1.

(B) The case of $\tilde{\kappa} < 0$. For Eq. (52), $-\tilde{\kappa}$ is expressed as a function of $\tilde{\xi}$, to be denoted as $\kappa_{L}^{(-)} (> 0)$. The function $\kappa_{L}^{(-)} = -\kappa_{L}^{(+)}$, which is valid for $\tilde{\xi} \tan \tilde{\xi} < -1$, is plotted in Fig. 3(b). As would be expected, there exists no local minimum in the domains. It is thus concluded that we have no eigenstate in this case.

4.3 On mirror-asymmetry of the rotational field

In light of the allowed combination of $(m, \kappa) = (\pm, \pm)$, the rotational eigenvalues for $|m| = 1/2$ are summarized in Table 2. We can claim that the eigenstates do exist for $m\kappa < 0$, whereas do not for $m\kappa > 0$. Remarkable is the relation of $\kappa_{-1/2,n} \neq \kappa_{1/2,n}$, and mirror-asymmetry of left- and right-handed helix representing the eigenstates.
Figure 3: Plots of $\kappa^{(+)}_L(\tilde{\xi})$ [Eq. (53)] (a) and $\kappa^{(-)}_L(\tilde{\xi})$ (b). The functions positive in the discrete domains are shown by solid curves. Both figures have a common horizontal axis. In (a), the points at which $\kappa^{(+)}_L$ takes the local minima are indicated by filled circles. Note $\tilde{\xi} \neq 0$.

Table 2: Summary of the rotational eigenvalues.

| $m$   | $\kappa(> 0)$ | $\kappa(< 0)$ |
|-------|---------------|---------------|
| $+1/2 \ (R)$ | $-\kappa_{1/2,n}$ |               |
| $-1/2 \ (L)$ | $\kappa_{-1/2,n}$ |               |

By making $\Phi$ correspond to the force-free magnetic field $B$ that satisfies $\nabla \times B = \kappa B$ (in $\mathbb{R}^3$ coordinate), given below is a magnetohydrodynamical picture on the asymmetry. For current density of the plasma, we have $J_c = \kappa B$ in accordance with the Ampère-Maxwell law: $\nabla \times B = J_c/c$. Hence, $\kappa > 0$ (resp. $< 0$) corresponds to the picture in which $B$ is parallel (resp. anti-parallel) to $J_c$, which is signified as $h_{12} = B \cdot J_c > 0$ (resp. $< 0$). Relating to the vector potential $A$, in general, $B = \nabla \times A$ holds, so that the force-free field can be expressed as $B = \kappa A + \nabla \chi$ with $\chi(R)$ a scalar function, though the second term of the RHS is ignored here. This approximation states that Ohmic current correction is taken into no account. Self-interaction Hamiltonian in free-space compatible with this, i.e., $\mathcal{H} = -J_c \cdot A/c$, can be cast to the form of $-\kappa h_{01} = -\kappa^2 |A|^2 (< 0)$, where $h_{01} = A \cdot B$ is, so to say, density of the magnetic helicity. In parallel with the relation of $\kappa$ to $h_{12}$, we have $\kappa > 0 \Leftrightarrow h_{01} > 0$ as well as $\kappa < 0 \Leftrightarrow h_{01} < 0$, to see that for both of them $\mathcal{H} < 0$ holds. Selecting the left- and right-handed helix respectively for the positive and negative $\kappa$ is amenable to a basic property of diamagnetic response of plasmas. Let us imagine the two distinct systems, which indicate $\kappa \gtrless 0$ each, while having a common distribution of $|A|^2$. It then turns out that, when magnitude of the eigenvalue of $\kappa$ concomitant with the systems depends
on its sign, the larger absolute value leads to the lower level of $\mathcal{H}$.

Recalling Table 2, it reads as follows: in the concerned elementary excitation of a spinning point involving a sort of currents, the left-handedness must be preferably self-selected due to the relation of $\kappa_{-1/2,n} > \kappa_{1/2,n}$. This meets the fact that $\ell_L$ and $q_L$ are selectively involved in the charged current weak interaction [17].

This consequence encourages us to put forth the following mathematical function transformation that describes the particle change of Eq. (21):

$$K_{-1/2}(kr)K_{1/2}(kr) \rightarrow \Phi (m = \mp 1/2, k). \quad (55)$$

The LHS involves $k \neq 0$ and $k \to 0$, while the RHS postulates $\mu \neq 0$. The rotational ground states of $\Phi$, specified by $\kappa_{-1/2,1}$ and $\kappa_{1/2,1}$, could be interpreted as being reflected in the spin (and momentum) state of $(\ell)_L$ and $(\bar{\ell})_R$, respectively.

### 4.4 Cases of the general mode number

Also for $|m| \neq 1/2$, the cylinder condition Eq. (47) determines the eigenstate of $\Phi (m, k)$ at least mathematically. Except for $m = 0$ that renders spectrum of $\tilde{\kappa}$ continuous, one can calculate the discrete eigenvalues according to the procedure same as explained in §4.2. This is not without physical meaning, in that $\Phi (m, k)$ might be related to gauge bosons with spin $m$. Thus, we here outline the results for the case in which $m$ is non-zero integers.

In general, on the pattern of existence of the eigenstates, it is found that $\kappa_R^- (= -\tilde{\kappa} > 0)$ and $\kappa_L^+(= \tilde{\kappa} > 0)$, respectively, for $(m, \kappa) = (+, -)$ and $(-, +)$, i.e., for $mk < 0$, have locally stable points in the domains. As is, a triad of the well-defined eigenvalues $\xi_{m,n}$, $\kappa_{m,n}$, and $k_{m,n}$ can be obtained as before. For the modes in which $m$ is an even number, we have $\xi_{m,n} = \xi_{-m,n}$, $\kappa_{m,n} = \kappa_{-m,n}$, and $k_{m,n} = k_{-m,n}$, whereas the odd modes lead to $\xi_{|m|,n} < \xi_{-|m|,n}$, $\kappa_{|m|,n} < \kappa_{-|m|,n}$, and $k_{|m|,n} < k_{-|m|,n}$ (like for $|m| = 1/2$), violating mirror-symmetry. This modal parity stems from the property of the Bessel function: $J_{-m} = (-1)^m J_m$.

As an example, the somewhat detailed explanation is given for $|m| = 1$ as one of the key modes. The equation determining the stable points is commonly ascribed to

$$J_0^2(\xi) \left[ \xi J_0(\xi) - 2J_1(\xi) \right] + J_1^2(\xi) = 0, \quad (56)$$

regardless of combination of $(m, \kappa) = (\pm, \pm)$. Noticed is that the allowed $\xi$-regions are expressed as $j_{0,n} < \xi < j_{1,n}$ and $j_{1,n} < \xi < j_{2,n}$ for $(m, \kappa) = (\pm, \mp)$ and $(\pm, \pm)$ (the double signs correspond), respectively, where $j_{m,n}$ denotes the $n$-th zeros of $J_m(\xi)$. However, one can ascertain that, for the latter, there exists no solution of Eq. (56). It should be mentioned that the function of $\xi$ for the case of $(+, -)$, shown in Fig. 4(a):

$$\kappa_R^- = -\frac{\xi \left[ \xi J_0(\xi) - J_1(\xi) \right]}{\sqrt{\xi J_0(\xi) \left[ \xi J_0(\xi) - 2J_1(\xi) \right]}}, \quad (57)$$
Figure 4: Plots of $\kappa_R^-(\tilde{\xi})$ [Eq. (57)] (a) and $\kappa_L^+(\tilde{\xi}) = -\kappa_R^-(\tilde{\xi})$ (b). These functions having, respectively, the domain of $j_{0,n'} < \tilde{\xi} < j_{1,n'}$ for odd and even number of $n'$ ($= 1, 2, 3, \ldots$) are designated by solid curves. Both figures have a common horizontal axis, on which the positions of $j_{0,n'}$ are indicated, in place of a scale of the linear axis. The points at which $\kappa_R^-$ and $\kappa_L^+$ take the local minima in their domains are indicated by open and filled circles, respectively. The broken curves in (a) and (b) correspond to $\kappa_L^-$ and $\kappa_R^+$ for $j_{1,n'} < \tilde{\xi} < j_{2,n'}$ with $n'$ odd and even, respectively (see text).

Table 3: A set of the discrete eigenvalues $\xi_{m,n}$, $\kappa_{m,n}$, and $k_{m,n}$ for $|m| = 1, 2$ and $n = 1$.

| $m$ | $\xi_{m,1}$ | $\kappa_{m,1}$ | $k_{m,1}$ |
|-----|-------------|----------------|----------|
| 1   | 2.857       | 3.112          | 1.234    |
| -1  | 5.937       | 6.162          | 1.652    |
| ±2  | 4.461       | 4.707          | 1.504    |

is related to the function for the $(-, +)$ case shown in (b), as $-\kappa_R^- = \kappa_L^+$. The domains of $\kappa_R^-$ and $\kappa_L^+$ are found to be $j_{0,n'} < \tilde{\xi} < j_{1,n'}$ with $n'$ odd and even number, respectively. As seen in Fig. 4, the values of $\tilde{\xi}$ at which the positive functions of $\kappa_R^-$ and $\kappa_L^+$ take the local minima in each domains, namely, the solutions of Eq. (56), denoted as $\xi_{\pm 1,n}$, appear such as $\xi_{1,1}$, $\xi_{-1,1}$, $\xi_{1,2}$, $\xi_{-1,2}$, $\ldots$ in order from the smaller one. It turns out that the $n$-state eigenvalues of $\xi_{1,n} = \xi_{1,(n'+1)/2}$ for $n'$ odd and $\xi_{-1,n} = \xi_{-1,n'/2}$ for $n'$ even alternately appear.

For convenience, the eigenvalues for $n = 1$ are exemplified in Table 3. In addition, the triad for $(|m|, n) = (2, 1)$ as the lowest order even mode is given in the bottom row.
5. Observation of the rotational field

We deal with an observational issue on the rotational field, mainly, of $\Phi(m = -1/2, k \neq 0)$. We see that mechanics owing to the geometric structure describes a charged, spin 1/2 point particle moving in an orbit. Specifically, it is shown that the structure sets up rotational coordinates of complex space, generating the spin, and furthermore, self-renormalization of the coordinates is responsible for observable precession of the spin. The dynamical mechanism is linked to the charge renormalization.

5.1 Relation of $\Phi$ with low energy lepton states

The proviso of $k \neq 0$ in $\Phi$ is responsible for $|K| = K \neq 0$ incorporated in the finite momentum of $P = \hbar K$ in $\mathbb{R}^3$. And, we do not have such a constraint on $k \rightarrow 0$ that is required for conformal map to be bijective, so as to represent the lowest energy excitation of $\pi$-meson. This circumstance resembles the situation in which Eq. (30) is applicable to reproduction of the Cornell potential for $q\bar{q}$ moving nonrelativistically. It can be, therefore, considered that derived from $\Phi$ for $k \rightarrow 0$ is kinematics of the charged lepton moving nonrelativistically (i.e., $K \rightarrow 0$ and $K \neq 0$). When $k \rightarrow 0$ is taken into account for $k \neq 0$ prerequisite to the regulation of $\Phi$, the operation corresponds to a projection of the helical structure onto the cylindrical base. We refer to it as $k \rightarrow 0$ correspondence, and for a spatial transformation involving it, presume the form of

$$\kappa r \rightarrow g^{-1}\mu \rho,$$  \hspace{1cm}(58)$$

where $\rho$ represents a radial coordinate vector (its initial point = origin “o”) on a complex plane, and satisfies $\mu r = \mu|\rho|$. The detectability owing to $r \rightarrow \infty$ can then be related to setting a stereographic projection point on a sphere so as to cover $\mathbb{C}_s \cup \{\infty\}$, where $\mathbb{C}_s = \{\mu \rho \in \mathbb{C} | 0 < \mu|\rho| \leq \xi_{m,n}\}$; its planar shape is the same as that for the Cornell regime. The sphere is incomplete in relation to $\mathbb{C}_s \not\subset \mathbb{C}$.

In view of kinematic states of a spinning point, we reveal the geometric mechanics that induces fundamental representation of $SU(2)$, viz., $S = (\hbar/2)\sigma$, where $\sigma$ stands for the vector of Pauli matrices: $(\sigma_x, \sigma_y, \sigma_z)$ [5]. For this purpose, the charged leptons are represented simply by electron. Foremost, for the special case in which $k = 0$, we describe the two-valuedness of the spin in the rest state. Setting to $k = 0$ gives, e.g., $\Phi_z(m = \mp 1/2, k = 0) \sim J_{\mp 1/2}(\mu r)e^{\mp i\sigma_z/2}$. Hence, rotation of $\theta = 0 \rightarrow 2\pi$ inverts sign of the initial function, and it is recovered first by the rotation of $4\pi$. Meanwhile, one could render $o$ as the rotation center identical with the $\mathbb{R}^3$-coordinate origin $O$ at which the electron is at rest. When making the rotation angle of either $-\theta$ or $\theta$ correspond to $\mathbb{R}^3$-rotation angle $\Theta$, therefore, $\Phi_z \hat{z} = \Theta \hat{z}$ embodies such a basic property of $S$ that rotational operator for wave functions generates exp $[i\Theta \cdot S/\hbar] |\Theta| = 2\pi = -I$, where $|\Theta| = \Theta$, and $I = \sigma_x^2 = \sigma_y^2 = \sigma_z^2$ is the unit matrix.

As for $k \neq 0$, we take account of the motion of spin and circular orbit of a single electron affected by uniform magnetic field $B$, as shown in Fig.5(a) (cf. Fig.3-3 in Ref. [36], and the relevant explanation therein). To expose $\Phi(m = -1/2, k \neq 0)$ of
Figure 5: Cyclotron motion of a single electron having the spin $S$ and its longitudinally polarized state (a), and the corresponding rotation of unit radial vector $\hat{r} = R/|R|$ and change of the polarization for the one orbital turn: $\sigma \rightarrow \sigma'(b)$. In (a), the angular momentum coupling is depicted.

the electron, we recall the potential theory on $q\bar{q}$ harmonic oscillation, which has been developed in §3. Here, we focus on phase term of $\phi'$ concomitant with $\Phi$. Applied to it is the transformation that has the form of $kz \rightarrow g^{-1}k\zeta$ in conjunction with Eq. (58), whereupon the transformed one is raised to the power of $g$ along Eq. (39). This procedure enables one to observe the motion of the "point". On the resulting $e^{-i[(1+\delta g)\theta+k\zeta]}$, we impose the lifting shift of $(\theta,\zeta) = (0,0) \rightarrow (-2\pi,2\pi k\zeta^{-1})$, which advances the phase. Henceforth, let negative sign of rotational angles be clockwise. When we write the advancing phase as $\Delta\theta$, to get

$$\delta_g = \Delta\theta/(2\pi),$$

Eq. (40) can be compared to the expression of $g$-factor: $2[1 + \Delta\Theta/(2\pi\Gamma)]$. Here, $\Delta\Theta$ and $\Gamma$ are the observed advancing angle of the spin precession and the Lorentz factor, respectively, and conform to $\Delta\Theta = \Gamma\Delta\Theta(\Gamma \rightarrow 1)$. For anti-clockwise, $\Theta(-= -\theta) = 2\pi$ rotation of unit coordinate vector $\hat{r}$ as shown in Fig. 5(b), we find the correspondence between $\Delta\Theta/\Gamma$ and $\Delta\theta$:

$$\Delta\theta \longleftrightarrow \Delta\Theta(K \rightarrow 0).$$

5.2 Mechanism advancing spin precession frequency

5.2.1 Coordinate mechanics of spin $\frac{1}{2}$ particle

The internal mechanics that determines the value of $\Delta\theta$ must describe the observed kinematics of the electron. Full information of the kinematic state is contained in the total angular momentum as the coupling of the orbital angular momentum $L$ and $S$:

$$J = L + S,$$

(61a)
as shown in Fig. 5(a). We pay attention to the formalism such that for $K \to 0$, Eq. (61a) reduces to

$$J \longrightarrow S.$$  \hspace{1cm}  \text{(61b)}

Reminding this, we construct for heuristic the rotational internal coordinate that is allied with $\hat{r}$ and capable of generating $\sigma$. By combining the remarked correspondence of $\Phi \leftrightarrow B$ with Eq. (5), we prepare the replacement of $\Phi \to i\lambda r$ with $\lambda$ constant, allowing for its application to Eq. (45). Taking this replacement into account, we introduce the covering vector for $J/\sqrt{3}$, defined by

$$\Lambda := [\nabla \times \Phi]_{\Phi \to i\lambda r} = i\lambda \kappa r.$$  \hspace{1cm}  \text{(62)}

This reads as follows: the internal quantity $\Lambda$ is, for $\lambda \to \hbar$, reflected in the expressions of the orbital angular momentum $-P \times R$ and $P = -i\hbar \nabla$ in $\mathbb{R}^3$. To $\Lambda$ as a primordial stuff unifying them, applied is Eq. (58) for the $k \to 0$ correspondence. This results in yielding the following projective form just in parallel with (61b):

$$\Lambda \longrightarrow \Lambda_0,$$  \hspace{1cm}  \text{(63a)}

where

$$\Lambda_0 = ig^{-1}\lambda \bar{\mu} \rho.$$  \hspace{1cm}  \text{(63b)}

In the context, the quantity $\Lambda_0$ ought to directly describe $S/\sqrt{3}$ of the electron moving nonrelativistically. It is now a matter of normalizing Eq. (63b), to provide the following symbolic master equation that spin half particles would refer to:

$$\dot{\hat{\sigma}} = i\xi \hat{\rho},$$  \hspace{1cm}  \text{(64a)}

where $\xi = \mu |\rho|$, and

$$\dot{\hat{\sigma}} = \Lambda_0/\lambda g_0, \quad \dot{\hat{\rho}} = \rho/|\rho|.$$  \hspace{1cm}  \text{(64b)}

Note that the definition of $\xi$ differs from the previous one for $kr$. Equation (64) sustains orthogonality of the base spaces $(\rho, \Lambda_0)$, serving as a spatio-unifier.

We delve into rotation of $(\rho, \Lambda_0)$, in terms of a self-consistent procedure of the normalization. First, it is supposed that $\hat{\rho}$ is a basis vector normalized when $\rho$ gets on real axis of the complex plane. Then, trivially $|\hat{\rho}| = 1$ is established on the real axis, and the real basis vector is to come out in $\hat{r}$. The vector configuration of Eq. (64a) on this plane is illustrated in Fig. 6(a), providing a value of $\xi$ larger than unity. Second, supposed is that $\hat{\sigma}$ is another independent basis vector normalized when $\rho$ rotates by $-\pi/2$ to be purely imaginary and $\Lambda_0$ gets on the real axis. Recalling $\lambda \to \hbar$, we identify $|\Lambda_0| = \lambda g_0$ with the detectable $|S|/\sqrt{3} = \hbar \sqrt{s(s+1)}/\sqrt{3} = \hbar/2$ for $s = 1/2$. Then, $|\hat{\sigma}| = 1$ is established on the real axis; the vector configuration of Eq. (64a) on the second plane is shown in Fig. 6(b).
Besides, the vector configuration of \((\hat{\rho}, \hat{\sigma})\) [Fig. 5(b)] is recast to the configuration of \((\hat{\rho}, \hat{\sigma})\) on a normalized complex plane, which is shown in Fig. 6(c). This can be regarded as superposition of \(\hat{\rho}\) on the first plane (a) and \(\hat{\sigma}\) put on the imaginary axis by the rotation of \(\pi/2\) on the second plane (b). For \(\Theta\) and \(\theta\) as rotation angles of \(\hat{\rho}\) and \(\hat{\sigma}\), respectively, we have \(\Theta = -\theta\), and therefore, \(\hat{\rho}\) rotated by \(-2\pi\) is to coincide with the initial one, as is compatible with the nature of \(\hat{\rho}\). On the other hand, \(\sigma \rightarrow \sigma'\) incidental to the one turn of \(\hat{\rho}\) should be described by \(\hat{\sigma} \rightarrow \hat{\sigma}'\) along with \(\Delta \theta\), where invariant of the magnitude: \(|\hat{\sigma}| = |\hat{\sigma}'| = 1\) is imposed. Then, \(\hat{\sigma}\) is responsible for \(\sigma/\sqrt{3}\), on account of \(\hat{\sigma}^2 \rightarrow \sigma^2/3 = (\sigma_x^2 + \sigma_y^2 + \sigma_z^2)/3 = I\). It turns out that \(\Lambda_0 = (\lambda/g_0)\hat{\sigma}\) is, for \(\lambda \hat{\sigma} \rightarrow \hbar \sigma/\sqrt{3}\), reflected in \(S/\sqrt{3}\).

5.2.2 Concept of coordinate self-renormalization

It can be claimed that \(\hat{\rho}\) represents \((n_d - 1)\)-dimensional projective plane of \(\mathbb{R}^{n_d}\) space having the dimensions of \(n_d = 3\), and \(\hat{\sigma}\) does likewise. In addition, \(\xi\) involving \(\rho\) takes on one degree of freedom that mediates between those two spaces. It thus follows that degrees of freedom of the internal projective space are \(2n_d - 1\). Although \(\hat{\rho}\) and \(\hat{\sigma}\) are distinguished from one another, \(\hat{\sigma}\) in Fig. 6(b) embodies \(\hat{\rho}\) in (a) on the real axis, so that observer has no way to distinguish both the basis vectors for the comparison. This suggests such an isotopic relation of \(\hat{\sigma}\) to \(\hat{\rho}\) that after the aforementioned clockwise quarter-turn of \(\rho\), \(\hat{\sigma}\) maintains gauge of the real axis that connects to \(\mathbb{R}^3\), on behalf of \(\hat{\rho}\) that was there before the turn. This instant, \(\hat{\rho}\) is on the imaginary axis, having the magnitude of \(1/\xi\). That is, as seen in Figs. 6(a,b) for \(\xi > 1\), the turn leads to the shrinkage of \(\hat{\rho}\), which is undetectable at this moment. Especially for the one clockwise turn, the vector shrunk up gets again on the real axis, to be expressed as \(\Delta \hat{\rho} = (1/\xi^4)\hat{\rho}\). This must be an observable portion. Meanwhile, for the one turn of \(\hat{\rho}\) in Fig. 6(c),
postulated is invariant of $|\hat{\rho}|$. In order to reconcile the both, we should consider that the appearance of $\hat{\rho}$ having $|\hat{\rho}| = 1$ results from superposition of $\Delta \hat{\rho}$ on a coordinative vector tentatively normalized on the real axis in another way. The scalar $\xi$ seems to play a key rôle in the radical process, referred to as coordinate self-renormalization, or CSR in short, hereafter.

Concerning the pre-normalization, it will be reasonable to recall the critical value of $\xi$, i.e., $\tilde{\xi} = \mu a$, and introduce $\hat{\rho}_a = \rho/a$ having the magnitude of $\xi/\tilde{\xi}$. By employing them, one can rewrite $\xi \hat{\rho}$ as $\tilde{\xi} \hat{\rho}_a$ in Eq. (64a). Noted is $|\hat{\rho}_a| < |\hat{\rho}|$ because of $\xi < \tilde{\xi}$. Taking all this into account, we make the following transformation that undertakes CSR:

$$\hat{\rho}_a \rightarrow \hat{\rho} = \hat{\rho}_a + \Delta \hat{\rho}, \quad \text{on } \Re.$$  \hspace{1cm} (65)

For $\Delta \hat{\rho}$ to be observable would require the complementation between $\Delta \hat{\rho}$ and $\Delta \theta$ related to displacement of $\hat{\sigma}$, in harmony with the isotopic relation between $\hat{\rho}$ and $\hat{\sigma}$. That is, it is supposed that the transformation of

$$\hat{\sigma} \rightarrow \hat{\sigma}' = \hat{\sigma} + \Delta \hat{\sigma}, \quad \text{on } \Re.$$  \hspace{1cm} (66)

is self-consistently implemented such that the following relation is satisfied:

$$\Delta \hat{\rho} = \Re (\Delta \hat{\sigma}), \quad \text{on } \Re.$$  \hspace{1cm} (67)

as displayed in Fig. 7. After all, Eq. (65) with $\Delta \hat{\rho}$ gives an interpretation of the observational fact of advance of spin precession frequency. The CSR process brings about clock in effect.
5.3 Generation of numerical value for the fine-structure constant

It appears that the CSR connotes the origin of elementary charge $e$, giving the physical meaning of QED renormalization. From Eq. (67), we have $|\Delta \hat{\rho}| = |\hat{\sigma}'| \sin \Delta \theta$. Equating this to the relation of $|\Delta \hat{\rho}| = 1/\xi^4$ yields the first equation that connects $\xi$ and $\Delta \theta$:

$$\xi^{-4} = \sin \Delta \theta. \quad (68)$$

Another equation can be derived from $|\xi| \hat{\rho} = |\tilde{\xi}| \hat{\rho}_a|$. Noted here is that the vector Eq. (65) operates on the real axis, thereby giving the relation of the magnitude, $|\hat{\rho}_a| = 1 - |\Delta \hat{\rho}|$. Hence, we have the second equation expressed as

$$\xi = \xi_{-1/2,n} \left(1 - \xi^{-4}\right), \quad (69)$$

where $\tilde{\xi}$ has been set to the eigenvalue of $\Phi(m = -1/2, k \neq 0)$. This gauge setup must properly connect $\rho$ with $R$. Equations (68) and (69) give a set of solutions of $\xi$ and $\Delta \theta$.

In order to highlight the physical correspondence, we approximately solve them, assuming $\Delta \theta \ll 1$. When ignoring $\sim (\Delta \theta)^3$ and the higher order terms, Eq. (68) reduces to

$$\xi^{-4} \approx \Delta \theta. \quad (70)$$

Meanwhile, it is found that within the common approximation, Eq. (69) allows the expression of

$$\xi^{-4} \approx \xi_{-1/2,n}^{-4}/(1 - 4\xi_{-1/2,n}^{-4}). \quad (71)$$

Thus, if only the numerical value of $\xi_{-1/2,n}$ is given, one can algebraically calculate the values of $\xi$ and $\Delta \theta$ from Eqs. (70) and (71). Making use of Eq. (70), Eq. (60) is cast to

$$\sim \xi^{-4} \leftrightarrow \Delta \Theta(K \to 0). \quad (72)$$

This means that, at least, the scalar quantity $\xi^{-4}$ shall indicate a value comparable to the dimensionless physical constant $\alpha$ [16]. To fix the value of $\xi_{-1/2,n}$, we call for the rotational ground state specified by $n = 1$, which minimizes the cylinder confining $\Phi$. Substituting $\xi_{-1/2,1}^{-4} \approx 141$ into Eq. (71) results in yielding

$$\xi^4 \approx 137, \quad (73)$$

indicative of $\alpha^{-1}$. Certainly, the assumption of $\Delta \theta \ll 1$ is justified. Combining Eqs. (59), (70), and (73), an approximate value of Eq. (40) can be estimated as

$$g \approx 2.00233. \quad (74)$$

This is also reconciled with experimental value of the $g$-factor for charged leptons.

The results and the argument on clock generation are both consistent with the preliminary consideration in §3.3. Thus, $\delta_g$ can be understood as manifestation of
one-dimension of time as the degree of freedom of $\xi$, which is released through the
dynamical mechanism that properly connects two $\mathbb{R}^{n_d}$ spaces (a complex $n_d$-dimensional
space) with $\mathbb{R}^{n_d}$ space. This means that conceivable dimensions of space-time are
$3n_d + 1 = 10$, and total degrees of freedom amount to 11. Logical necessity of the
extra-dimensions implies that a principle of unification of forces may be self-contained
within this theoretical framework [37].

5.4 The QED interpretation

The scale of $\sim \xi^{-4}$ can be interpreted as origin of electric energy accompanying the
point charge excited in vacuum. Then, inequality of $\xi^{-4} > \xi^{-4}_{-1/2,1}$ is amenable to notion
of vacuum polarization: it sustains the QED vacuum picture in which the effective
charge decreases outward due to electrostatic shielding effect by the polarization of
virtual electron-positron pairs. Thickness of the imperfect shielding shell reflects $\Delta \xi = \xi_{-1/2,1} - \xi$.

The virtual pair creation is supposed to generally occur in electromagnetic in-
teraction. A feasible form of $\alpha$ as the coupling constant can be found, e.g., in the
electron-electron scattering that exchanges virtual photons, giving rise to such a kind
of pair creation. It is a matter of summing up the Feynman diagrams of the possible
propagation processes as to the scattering amplitude (e.g., Ref. [13]). Letting the ob-
served elementary charge and theoretical charge be denoted as $e$ and $e_{\text{th}}$, respectively,
$e = e_{\text{th}}/\sqrt{1 - \varrho}$ with $\varrho$ constant is derived from the integration. Consequently, QED
holds out

$$e^2 = e_{\text{th}}^2/(1 - \varrho).$$

(75)

Its correspondence to Eq. (71) is pronounced. We find that the finite $\varrho$, by which a
logarithmically diverging term has been replaced by means of the conventional renor-
malization, reflects the quantity of $\sim 4\xi^{-4}_{-1/2,1}$, to be expressed as $4e_{\text{th}}^2$ in a good approx-
imation. Physical meaning of the charge renormalization can be attributed to CSR as
the fundamental mechanism that embeds particles into space, creating time.

6 Internal structure governing $\beta$-decay

The outcomes in the previous section strongly suggest entity of the rotational field of the
mode $|m| = 1/2$, concomitant with spin half point particles. If Eq. (55) is truly reflec-
ted in Eq. (21), the current theory must be capable of describing the weak interaction as
well. To complete the theory, we reveal its compatibility with the standard model that
can well account for the relevant experimental facts.

6.1 Function transformation reflected in gauge field transition

Within the conventional framework [19], the unified theory of electromagnetic and weak
force is constructed such that covariant derivative for $q$ and $\ell$ satisfies requirement
for $SU(2) \times U(1)$ gauge transformation to be locally invariant. The crucial point is that the operators for $(qL, \ell_L)$ and $(qR, \ell_R)$ exhibit asymmetry between them. This LR asymmetry relies on the unquestionable fact that the weak interaction involving neutral current works for the both, whereas the charged current interaction does only for the former. We cope with this asymmetry, based on the mirror-asymmetry of the eigenstates of $\Phi$.

To begin with, we return to Eq. (25) with $|m| = 1$, and reconsider the region of $r \to \infty$ so that the solution can be expressed as $\eta(kr) = c_± I_{±1}(kr)$, where $c_+ = c_1$ (cf. §4.1). This means that, by largely separating $q\bar{q}$ pair without violating a color superconductivity phase, we enhance strong force to isolate it, though in reality the separation ends up with newly creating a meson. The field of it having the spin-parity of $J^P = 0^-$ reflects the function of the LHS of Eq. (55). In the development of the field in $\mathbb{R}^3$ space, induced is the internal transformation of $k \to -i\mu^*$, which is recast to the form of

\[ kr \longrightarrow \varphi := -i\xi, \quad (76) \]

where the definition of $\xi = \mu^* r$ is newly introduced. Although this scalar quantity should be distinguished from $\mu r$ in §5, the same notation $\xi$ is used by a reason explained later. The process in which $W^{±0}$ and $B^0$ particles appear in place of gluons is considered to be basically sustained by the function transformation owing to Eq. (76): $I_{±1}(kr) \to I_{±1}(\varphi) = \mp iJ_{±1}(\xi)$. The energy transformation of internal harmonic potential, parallel to Eq. (35), can then be written in the form of

\[ \eta^2(kr) \longrightarrow U_±(\xi) = -\phi_{±1}^2 J_{±1}^2(\xi), \quad (77) \]

where the double signs correspond, and $\phi_{±1}$ is real constant (along with $c_{±}$). It is considered that Eq. (77) represents generation of the internal potential that works for $\ell\bar{\ell}$ pair separated, in contrast to Eq. (35) that represents generation of the external potential of $q\bar{q}$ pair confined. The function transformation of $I_{±1}(kr) = I_{±1}[(k/\mu^*)\xi] \to -J_{±1}(\xi)$ is shown in Fig. 8.

Since $\phi_{±1} J_{±1}(\xi)$ provides $\eta(\xi)$ in $\phi'$, i.e., the solution of Eq. (41), it constitutes $\Phi_±$ for $m = ±1$, but set to as $\mu = \mu^*$. It is expected, as the $\ell\bar{\ell}$ pair is represented by $\Phi(\mu; m = ±1, k ≠ 0)$, that the generated gauge bosons are represented by $\Phi(\mu^*; m = ±1, k ≠ 0)$. Henceforth, we refer to the former and latter eigenstate as $\Phi_{±1/2}$ and $\Phi_{±1}$, respectively.

### 6.2 Generation of quark elementary charge

We focus on the second equation of Eq. (21), which can be decomposed into the elementary processes of $d_{(-1/2)} \longrightarrow u_{(1/2)} + W_{(-1)}$ and $W_{(-1)} + \nu_{e(1/2)} \longrightarrow e_{(-1/2)}$. Here, the values in the brackets indicate the third component of the weak isospin, $T_z$. Extending the correspondence between $e_{(-1/2)}$ and $\Phi_{-1/2}$, it is natural to suppose the correspondence between $T_z$ and $m$ (in $\Phi_m$), at least, for particle species endowed with
negative $T_z$ and negative charge. The foregoing Yukawa interaction is rewritten in the
next form:

$$[d_L + (\bar{u})_R] \rightarrow W^-, \quad W^- \rightarrow e_L^- + (\bar{\nu}_e)_R.$$  (78)

The square bracket of the first equation emphasizes the fact that presence of $d\bar{u}$ pair
is restricted to a closed space of the interior of $\pi^-$-meson.

In common with $e^-$ and $\bar{\nu}_e$ in the RHS of the second equation, $d$ and $\bar{u}$ behave as
spin 1/2 point particles each. We read that among them, the left-handed $d_L$, which
acts commonly with $e_L^-$ on the charged current interaction, refers CSR subject to the
master Eq. (64), as basic internal mechanics of $T_z = -1/2$. Respecting the rotational
motion of $e_L^-$, the CSR to which the gauge of cylindrical base of $\Phi_{-1/2}$ contributes is
found to set up a proper rotational coordinate in the $\mathbb{R}^3$ space that allows for the orbit.
This setup is none other than development of electromagnetic potential. On the basis
of this idea, we reconfigure the charged current interaction of $d_L$ undergoing rotational
motion in the closed space, with $W^-$. Expected is that the CSR to which the base
gauge of $\Phi_{-1}$ contributes is called for setting up a proper coordinate, i.e., developing
the interaction potential, in the closed space of $J^P = 0^-$ particle. Not only in $\Phi_{+1/2}$,
but in $\Phi_{\pm 1}$, the mirror symmetry is breaking; the selection of the negative mode is
found on $\kappa_{-1,n} > \kappa_{1,n}$. In particular, the CSR involving rotational ground state of
$\Phi_{-1}$ is expected to determine charge of $d_L$, in analog of the generation of $e$ in which
the ground state of $\Phi_{-1/2}$ is involved.

From this notion follows: the $\xi^{-4}$ scale of Eq. (71) of which $\xi_{-1/2,1}$ is replaced by
$\xi_{-1,1}$ is responsible for square of the $d$ ($s, b$) quark charge $e_q$. This replacement owes
to base coupling of $\Phi_{-1/2}$ and $\Phi_{-1}$; the underlying principle is eventually clarified. As
a consequence, the following approximate expression is responsible for $|e_q|/e = 1/3$
related to $SU(3)$ symmetry [20]:

$$\sqrt{(\xi_{-1/2,1}^4 - 4)/(\xi_{-1,1}^4 - 4)} \approx 0.333,$$  (79)
where \( \xi_{-1,1} \approx 1240 \) has been employed with reference to Table 3.

### 6.3 The compatibility with a standard vacuum model

#### 6.3.1 The Higgs potential

We further examine Eq. (77), aiming at revealing analytic meaning of the mass change \( m_W \rightarrow m_{\ell \bar{\ell}} \) in \( W^- \rightarrow \ell + \bar{\ell} \). While there exist innumerable local minimum points of the function \(-J_{\pm 1}^2(\xi)\), the point that indicates the minimum value is only one. Hence, by the transformation of Eq. (77), the value of \( \xi \) at which the potential indicates the minimum changes from 0 to the well-defined \( \xi_w \neq 0 \). This feature can be seen in Fig. 8. The value of \( \xi_w \) could regulate the quantity \( \mu^* \) that characterizes the Yukawa interaction (cf. §3.3). It follows that \( \xi_w \) must be reflected in \( m_W \).

For real \( \xi \), expansion of \( J_{\pm 1}^2(\xi) \) is given by

\[
J_{\pm 1}^2(\xi) = \sum_{m'=0}^{\infty} (-1)^{m'} \frac{(2m'+1)!! \xi^{2(m'+1)}}{m'! (m'+2)! (2m'+2)!!}.
\] (80)

Note that Eq. (80) can be expressed as a function of \( \xi^2 \). Meanwhile, for Eq. (76) with \( \xi = \mu^*r \) real, we have \( \xi^2 = \varphi^*\varphi \), where the asterisk in this form denotes the complex conjugate. Hence, \( U_\pm(\xi) \) can be expressed as a function of \( \varphi^*\varphi \) of which the expansion is written as

\[
U_\pm(\varphi^*\varphi) = -\frac{\phi_{\pm 1}^2}{4} \left[ \varphi^*\varphi - \frac{1}{4} (\varphi^*\varphi)^2 + \frac{5}{192} (\varphi^*\varphi)^3 - \cdots \right].
\] (81)

In this aspect, we make the comparison with a standard vacuum model involving spontaneous symmetry breaking. As a rule, one assumes condensate of the complex scalar field \( \psi \) in vacuum. For the \( \psi \)-particles themselves, the interaction potential is introduced of which the approximate form can be expressed as

\[
U(\psi^*\psi) = \mu_m^2 \psi^*\psi + \varsigma (\psi^*\psi)^2,
\] (82)

with \( \mu_m \) and \( \varsigma \) constants. The theory for \( \mu_m^2 > 0 \) describes \( \psi \) associated with the mass of \( \mu_m \). Within this framework, the vacuum state can be represented by the expected value of \( \psi \), such as \( \langle \psi \rangle_0 = 0 \) for \( \mu_m^2 > 0 \), that is, the value of \( \sqrt{\psi^*\psi} \) at which \( U \) indicates the minimum. On the other hand, for \( \mu_m^2 < 0 \), as clearly seen in the rearranged form of Eq. (82):

\[
U(\psi^*\psi) = \varsigma \left( \psi^*\psi + \frac{\mu_m^2}{2\varsigma} \right)^2 - \frac{\mu_m^2}{4\varsigma},
\] (83)

\( U \) indicates the minimum at \( \psi^*\psi = -\mu_m^2/2\varsigma (> 0) \), so that the expected value is given by

\[
\langle \psi \rangle_0 = \sqrt{-\mu_m^2/2\varsigma}.
\] (84)
This finite $\langle \psi \rangle_0$ is involved in assigning the gauge boson the mass on-mass-shell via coupling constant, to linearly couple with the mass. Customarily, $U$ is called Higgs potential [21]; $\psi$ can be expressed as $\psi = \langle \psi \rangle_0 + H/\sqrt{2}$, where $H$ the Higgs field.

It is obvious that $U_\pm(\varphi^* \varphi)$ of Eq. (81) provides an analytic function representation for $U(\psi^* \psi)$ of Eq. (83). Approving the correspondence of $\varphi^* \varphi \leftrightarrow \psi^* \psi$, we have

$$\sqrt{\varphi^* \varphi} = \xi_w \leftrightarrow \langle \psi \rangle_0.$$  \hfill (85)

This indeed supports the relation of $\xi_w \propto m_W$. The key Eq. (76) that brings about $\xi_w$ can be regarded as a trigger of the spontaneous symmetry breaking.

Relating to the broken symmetry, $\bar{\nu}_e$ having $T_z = 0$ must refer the ground state of $\Phi_{1/2}$. It is inferred from this, that a spin 1 neutral boson that can interact with the right-handed particles, namely, $B^0$ having $T_z = 0$, will reflect the ground state of $\Phi_1$. As for eigenvalues of $\Phi_{1/2}$, recalling that $\xi_{-1,1}$ regulates $e_q$ in $J^P = 0^-$ particle accompanied by $\Psi_x$, we conjecture that the counterpart, $\xi_{1,1}$, regulates mass (denoted as $m_1$) in $0^+$ particle accompanied by $\psi$. The point is that, provided the positive and negative mode of $\Phi_{1/2}$ share the same base coordinate, the ratio of $\xi_{1,1}/\xi_w$ is to indicate the mass ratio $m_1/m_W$. That is, we have

$$m_1 = (\xi_{1,1}/\xi_w) m_W.$$  \hfill (86)

Note $\xi_{1,1} \cong 2.86$ from Table 3, and $\xi_w \cong 1.84$. Concerning $m_W$, one may employ the value of 80.4 GeV/$c^2$ [38]. Substituting these values into (86), we obtain

$$m_1 \cong 125 \text{ GeV}/c^2.$$  \hfill (87)

This accords with the mass of Higgs boson that has been measured [39]. The result corroborates validity of the correspondence between Eq. (81) and (83), as well as, Eq. (85).

### 6.3.2 Imprint of lepton mass

The mass change $m_W \rightarrow m_{\ell, \ell}$ is considered to reflect $\mu^* \rightarrow \mu$ [Eq. (43)] cooperating with the system transformation of $|m| = 1 \rightarrow 1/2$. The replacement means the coupling of $\xi = \mu^* \tau$ in Eq. (76) with $\xi = \mu \tau$ sustaining $\Phi_{\pm 1/2}$ on the base (that is why for the both, the same notation $\xi$ has been used). The relation of $\varphi^* \varphi = \xi^2$ can be, therefore, regarded as the scalar coupling of the $\varphi$-field and leptons, which could be associated with $m_1 \rightarrow m_{\ell, \ell}$. The coupling would enforce internal rotation of $\ell \bar{\ell}$ acting on $U_\pm(\varphi^* \varphi) = U_\pm(\xi)$. It is reasonable to suppose that $\xi_w \rightarrow \xi_{1,1}$ corresponds to $\langle \psi \rangle_0 \rightarrow \psi = \langle \psi \rangle_0 + H/\sqrt{2}$, and $m_1 \rightarrow m_{\ell, \ell}$ reflects $\xi_{1,1} \rightarrow \xi_{\mp 1/2}$. It follows that the latter symbolically represents a process in which inertial mass of $m_{\ell, \ell}$ is imprinted in vacuum. An acceptable scenario is that $\xi = \xi_{-1/2,1} - \Delta \xi \rightarrow \xi_{-1/2,1}$ corresponds to mass renormalization of a charged lepton so that $\xi \cong \xi_{-1/2,1}$ signifies complementation between the mass and charge renormalization.
Let us relate the imprint of $m_\ell$ and $m_\ell$ to excitation of the $\varphi$-field in $U_-$ and $U_+$, respectively. The excitation energy denoted as $\omega_{-1/2}$ and $\omega_{1/2}$ are given by

$$\omega_{-1/2} = U_- (\xi_{-1/2,1}) - U_- (\xi_w), \quad \omega_{1/2} = U_+ (\xi_{1/2,1}) - U_+ (\xi_w).$$

(88)

It should be noticed that if $\phi_1^2/\phi_{-1}^2 \sim \mathcal{O}(1)$, we have $\omega_{1/2} \ll \omega_{-1/2}$, because of $U_+ (\xi_w) \approx U_+ (\xi_{1/2,1})$ stemming from $\xi_w \approx \xi_{1/2,1} (\approx 1.89; \text{Table 1})$. There is a possibility that Eq. (88) corresponds to mass of harmonic oscillator analogous to $m_\pi$. From the point of view, invoked is a relation indicating that $m_n^2$ should be proportional to mass of quarks constituting the meson $[4, 40]$. In this analog, we preliminarily read the theoretical ratio of $m_\ell/m_\ell$ as $(\omega_{1/2}/\omega_{-1/2})^2 \approx 3.53 \times 10^{-6} (\phi_1^2/\phi_{-1}^2)^2$. The evaluation is attempted later.

### 6.4 An analytic form unifying forces and matters

It is crucially important to find an analytic relation between strength of the weak interaction $(g, g')$ and $e$. Here, the notations are standard: $g$ (resp. $g'$) denotes the strength of coupling of $W$ (resp. $B^0$) boson with the weak isospin (resp. weak hypercharge) of the left-handed (resp. both-handed) particles. In light of orthogonal decomposition of discrete $\kappa_{m,1}$-spectrum: $(\xi_{m,1}, k_{m,1})$, we recall the mass imprinting process of $m_1 \rightarrow m_{1,\ell}$. A point worth noting is that the process goes for $\xi_{m,1}$ with $m = 1 \rightarrow \mp 1/2$. From this naturally inferred is that $(g, g') \rightarrow e$ as a charge imprinting process goes for $k_{m,1}$ with $m = \mp 1 \rightarrow -1/2$. Taking the covariant derivative form into consideration, the mode-mode coupling among $\Phi_{-1}$, $\Phi_1$, and $\Phi_{-1/2}$ through the finite $\hat{k}$ is expected to embody a basic principle of the electroweak interaction involving $g$, $g'$, and $e$.

To the modal coupling, a nice similarity can be found in Eq. (27), that is, the coupling of cylindrical functions of order $m = \pm 1$, which utilizes the common finite $k$. An attention should be paid to the ratio of $c_{-1}^2/c_1^2 \sim \hat{k}^4$ self-contained in Eq. (27), which plays a critical rôle in quark confinement. Notice that confinement of $\Phi_m$ inside a cylinder compared to perfectly conducting wall resembles the color confinement. Hence, retaining such a form of internal scaling, we should describe the multi-mode coupling among $\Phi_m$ that utilizes the common, one degree of freedom of $\hat{z}$-direction. First, a priori introduced is the hyper coupling constant defined by $\phi_0^2 = (\xi_{-1/2,1,1}) - 4$, reflected in $\sim \hbar c$, i.e., a scale of square of the Planck charge. Second, regarding the mode intensity $|\phi_m|^2$ as constant of the mode coupling, we hypothesize that a group of ground states of $\Phi_m$ is self-adjusted such that their intensities satisfy $|\phi_m|^2/\phi_0^2 = k_{m,1}^4$ each$^2$. Then, we have a chain rule for $m = -1/2$, $-1$, and $1$, which can be expressed as

$$\phi_0^2 = \frac{|\phi_{-1/2}|^2}{k_{-1/2,1}^4} = \frac{|\phi_{-1}|^2}{k_{-1,1}^4} = \frac{|\phi_1|^2}{k_{1,1}^4}. \quad (89)$$

$^2$This can be compared to the expression of $\langle 0 | \hat{E} \cdot \hat{E} | 0 \rangle \sim \hbar c \cdot k^4$ for the vacuum fluctuations of square of the averaged electric field operator $\hat{E}$, as a quantum effect that appears in the region in which density of photons having the wavenumber $k$ is of the classical limit of $\sim k^3$ and less $[36]$. Here, the average means that taken over the volume of $k^{-3}$ in $\mathbb{R}^3$. 

35
We read that the concerned multi-mode coupling is subject to this rule. With reference to Tables 1 and 3, then, we obtain the key ratios evaluated as follows:

\[ \frac{|\phi_{-1}|^2}{|\phi_{-1}|^2} = \left( \frac{k_{-1/2,1}/k_{-1,1}}{k_{1/2,1}/k_{1,1}} \right)^4 \simeq 0.24, \]  
\[ \frac{|\phi_{-1}|^2}{|\phi_{1}|^2} = \left( \frac{k_{-1/2,1}/k_{-1,1}}{k_{1/2,1}/k_{1,1}} \right)^4 \simeq 0.76. \]  

Equation (90) can be compared to an input parameter of the Glashow-Weinberg-Salam model [19]. That is, for the Weinberg angle \( \theta_W \), a comparison of the value of Eq. (90a) can be made with experimental values for \( \sin^2 \theta_W = \frac{e^2}{g_1^2} \) [Eq. (90b) with \( \cos^2 \theta_W = \frac{e^2}{g_1'^2} \)]. The theoretical Eq. (90) is indicative of the mathematical constants reflected in the experimental values in a lower energy region (e.g., Ref. [38], and references therein).

In terms of a mixing state of \( \Phi_{\pm 1} \), massless photon can be specified by \( \mu^* \to 0 \) in the concerned \( r \to \infty \), along \( \{0\} \subset \mathbb{C}_s(\pm 1) \cup \{\infty\} \), where \( \mathbb{C}_s(m) = \{0 < \xi \leq \xi_{m,1}\} \). In connection with the mass imprint, the theoretical value for \( m_\ell/m_\ell \) works out, by letting \( \phi_{2}^2/\phi_{-2}^2 = (k_{1,1}/k_{-1,1})^4 \) in the foregoing expression of \( (\omega_{1/2}/\omega_{-1/2})^2 \), at \( 3.4 \times 10^{-7} \). Making correspondence of this to the mass ratio of \( \bar{\nu}_e \) to \( e^- \), the mass of \( \bar{\nu}_e \) is predicted to be \( 0.17 \text{eV}/c^2 \), as it is consistent with an experimental upper limit on electron-based \( \bar{\nu}_e \) mass [41].

The outcomes totally support the reasoning that the \( P \)-violation observed in \( \beta \)-decay will be ascribed to the mirror-asymmetry of the helices of \( \Phi_m \). To summarize, the transition from the strong to electroweak force can be represented by the potential transformation of

\[ I_{\pm 1} \longrightarrow -J_{\pm 1}^2 (\leftrightarrow \text{Higgs type}), \]  

indicative of the onset of \( \Phi_{\pm 1} \), while the transformation reflected in \( \pi \longrightarrow \ell + \bar{\ell} \) is given by

\[ K_{-1/2}K_{1/2} (\leftrightarrow \text{Yukawa type}) \longrightarrow \Phi_{\pm 1/2}. \]

In the unified picture of gauge and matter particles, we come down to the idea that their ingredients are ultimately reduced to the mathematical functions of cylindrical system.

### 6.5 On the coupling with a higher mode

The mechanism determining \( m_W \) has been identified with the Higgs mechanism that assigns the gauge particles the inertial mass. If the unified concept is accountable for equivalence principle [26], the mass imprinting mechanism must be related to gravitational interaction under presence of gravitational field. From this perspective, we describe onset of the helical field \( \Phi \) having \( |m| = 2 \), and characteristic of the eigenstates \( \Phi_{\pm 2} \), which are supposed to be in on graviton having spin 2, and add consideration as to the coupling with \( \Phi_{\pm 1} \).

Origin of the cylindrical mode of \( |m| = 2 \) can be ascribed to the modified Bessel functions \( I_m \) indicating the property of modal degeneracy in the region of \( k \to \infty \),
i.e., \( I_m(kr) \xrightarrow{k \to \infty} e^{kr}/\sqrt{2\pi kr} \). For one can suppose, for \( k \to 0 \) as the inverse, \( I_{\pm 2} \) separating from \( I_{\pm 1} \) that engenders spin 1 gauge bosons. This circumstance is compared to separation of gravitational force from the other forces. Equation (76) executes \( I_{\pm 2}(kr) \to \mp J_{\pm 2}(\xi) \) in parallel to the transformation for the \(|m| = 1\) mode, resulting in establishing \( \Phi \) for \(|m| = 2\). Accordingly, it turns out that \( \Phi_{\pm 1} \) could coexist with, at least, \( \Phi_{\pm 2} \), sharing the same base coordinate.

Concerning discrete eigenvalues for \( \Phi_{\pm 2} \), we have the symmetric relations of \( \xi_{2,n} = \xi_{-2,n} \), \( \kappa_{2,n} = \kappa_{-2,n} \), and \( k_{2,n} = k_{-2,n} \), in contrast to the asymmetric ones for \(|m| = 1/2\) and 1. This suggests that the interaction that involves \( \Phi_{\pm 2} \) having \(|\phi_2| = |\phi_{-2}|\) is free of currents and handedness, in other words, independent of charges and spin. The symmetry of the lowest order even mode is responsible for universality of gravitational interaction. Altogether, \( \Phi \) can be regarded as a fundamental ingredient capable of representing gravitational force, in addition to the electroweak force and spin 1/2 point particles.

In the regime in which \( \phi_{\pm 2} \) is set to imaginary as orthogonal to \( \phi_{\pm 1} \), the potential of \( \sim -J_{\pm 2}(\xi) \) having a single minimum point is generated along Eq. (77); Yukawa-type interaction could appear. The value of \( \xi = \sqrt{\phi^* \phi} \) at which the potential indicates the minimum, denoted as \( \xi_0(\approx 3.05) \), is considered an alternative representation of vacuum state, being arguably reflected in the inertial mass of a hypothetical particle \( m_G \). This particle might behave as if a virtual mediator of gravitational force, like that \( \pi \)-meson mediates strong nuclear force. Remarked is that the massless graviton state can be specified by the manner by which the massless photon state has been specified, but instead of \( \mathbb{C}_{s(\pm 1)} \), extended is \( \mathbb{C}_{s(\pm 2)} \). The internal coupling with the \(|m| = 1\) mode via the scalar field \( \phi \) is to conform to the relation of \( \xi_0/\xi_w = m_G/m_W \). Resultantly, for the weakly interacting particle, the mass is predicted to be

\[
m_G = (\xi_0/\xi_w) m_W \approx 133 \text{ GeV}/c^2.
\]

Furthermore, there is a possibility of excitation of the scalar field, similar to that characterized by \( m_1 \) compared to the Higgs boson mass. In the current context, the excitation is to refer \( \xi_{2,1}(\approx 4.46; \text{ Table 3}) \) regulating the mass of \( m_2 = (\xi_{2,1}/\xi_0) m_G \). When using Eqs. (86), (87), and (92), the mass expression can be recast to

\[
m_2 = (\xi_{2,1}/\xi_{1,1}) m_1 \approx 195 \text{ GeV}/c^2.
\]

The principle process, in which gravity refers the inertial mass of \( m_{\ell,\bar{\ell}} \) imprinted by \( m_1 \to m_{\ell,\bar{\ell}} \), can be represented by \( m_2 \to m_{\ell,\bar{\ell}} \).

It is mentioned that \( m_G c^2 \) agrees with the energy at which a sign of spectral excess of gamma-rays from the Galactic center region was found [42], although the finding has been controversial. The particle will qualify as a tentative candidate for dark matter.

7 Conclusion

We have developed a methodology to gain access to the cylindrical space that couples with geometrical structure of particles having no extractable substructure. Referring
spatial change of mass state of photons, we have constructed a spatial transformation responsible for the potential generation indicative of elementary excitation of particles. Taking a hint from the form of the Cornell potential, we have introduced harmonic function in cylindrical system, and clarified entity of cylindrical space in an infinitesimal region with large extra-dimensions. It has been confirmed that the Yukawa potential is reproducible by applying the spatial transformation to the harmonic function. For quark confinement, the analytical reason based on conformal map has been given. We have seen that the principal factor of $g_0 = 2$ involved in the transformation between the cylindrical and observational space could be reflected in the $g$-factor of the Dirac regime.

Considering the internal transformation of the harmonic function, which is reflected in transition from the strong to electroweak force, we have introduced a membrane oscillatory field to describe the weak and leptons, and the rotational field $\Phi$ including it as the cylindrical axial ($z$) component. The field $\Phi$ as a reduced ingredient is the Gromeka-Beltrami field that satisfies the rotational eigenvalue equation, and has helical structure due to the phase term of $e^{i(m\theta-kz)}$. It has been clarified that the cylindrical boundary condition depending on the mode number $m$ well regulates the structure, and the discrete eigenstates of $\Phi$ determined by the condition, $\Phi_m$, can have mirror-symmetry of the helices.

By studying correspondence of $\Phi$ with $m = -1/2$ to one electron undergoing the cyclotron motion and spin precession, we found out an isotopic relation between the internal coordinate space and angular momentum space, and a mechanical sequence maintaining their basis vectors. It has been suggested that the coordinate self-renormalization subject to the mechanics is responsible for QED renormalization. We discovered that the coordinate setup involving the eigenvalue $\xi_{-1/2,1}$ for the rotational ground state of $\Phi_{-1/2}$ generates a numerical value corresponding to Sommerfeld’s constant: 1/137.

The mirror-asymmetry of $\Phi_{\pm 1/2, \pm 1}$ has been associated with $P$-violation in $\beta$-decay. The quantities have been revealed which correspond to the scalar field conventionally assumed to be condensed in vacuum, and its expected value. We have seen that the eigenvalues $\xi_{-1,1}$ and $\xi_{1,1}$ for the ground states of $\Phi_{\pm 1}$ are involved in determination of a quark electric charge and the Higgs boson mass, respectively. The imprint mechanism of elementary charge, which is reconciled with the Weinberg angle, and that of lepton mass have been proposed. We have argued onset of the higher mode $\Phi_{\pm 2}$ and the coupling with $\Phi_{\pm 1}$. It has been pointed out that the even mode having mirror-symmetry of the helices could represent graviton.

In conclusion, $\Phi$ with $m = \mp 1/2$ referring the rotational eigenvalue of $\kappa_{\mp 1/2,1}$ is interpreted as existence itself of a charged and neutral lepton, respectively. A pair of the fields are of the helical geometry that describes the classically indescribable two-valuedness ("klassisch nicht beschreibbare Art von Zweideutigkeit" [43]) by the single-valued functions. The theory supports a common image of the cosmic helices not only for the structure of spin half point particles, but for the unified structure including gauge bosons. This means that it seems as if collective dynamics of plasma evolving
toward an energy relaxed state refers the internal structure of particles constituting the macroscopic plasma. I hope the idea will shed light on a variety of fields.

References

[1] H. Dehmelt, Physica Scripta T 22, 102 (1988).
[2] M. Planck, Ann. d. Phys. 4, 553 (1901).
[3] R. P. Feynman, ”The Feynman lectures on physics, volume III,” Addison-Wesley, Reading (1965).
[4] G. Takeda, ”Soryushi,” Shokabo, Tokyo (1986).
[5] W. Pauli, Jr., Zeit. f. Phys. 43, 601 (1927).
[6] A. E. Broderick and A. Loeb, Astrophys. J. 703, L104 (2009).
[7] L. Woltjer, Proc. Nat. Acad. Sci. 44, 489 (1958).
[8] J. B. Taylor, Rev. Mod. Phys. 58, 741 (1986).
[9] L. F. Burlaga, J. Geophys. Res. 93, 7217 (1988).
[10] S. P. Plunkett et al., Solar Phys. 194, 371 (2000).
[11] R. P. Kerr, Phys. Rev. Lett. 11, 237 (1963); K. Akiyama et al. (The Event Horizon Telescope Collaboration), Astrophys. J. 875, L1 (2019).
[12] H. C. Ford et al., Astrophys. J. 435, L27 (1994).
[13] R. P. Feynman, ”The theory of fundamental processes,” Addison-Wesley, Reading (1961).
[14] A. Sommerfeld, Ann. d. Phys. 51, 1 (1916).
[15] E. Beltrami, in Rendiconti del Reale Instituto Lombardo, Series II, 22 (1889); trans. by G. Filipponi, Int. J. Fusion Energy 3(3), 53 (1985).
[16] J. Schwinger, Phys. Rev. 73, 416 (1948).
[17] T. D. Lee and C. N. Yang, Phys. Rev. 104, 254 (1956); C. S. Wu et al., Phys. Rev. Lett. 105, 1413 (1957).
[18] H. Yukawa, Proc. Phys. Math. Soc. Japan 17, 48 (1935).
[19] S. L. Glashow, Nucl. Phys. 22, 579 (1961); S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in Elementary Particle Theory, ed. by N. Svartholm, Almqvist & Wiksell, Stockholm, 367 (1968).
[20] M. Gell-Mann, Phys. Lett. 8, 214 (1964).
[21] P. W. Higgs, Phys. Rev. Lett. 13, 508 (1964).
[22] J. C. Maxwell, Phil. Trans. R. Soc. Lond. 155, 459 (1865).
[23] Sir. J. J. Thomson, Proc. Phys. Soc. 40, 79 (1927).
[24] L. Tonks and I. Langmuir, Phys. Rev. 33, 195 (1929).
[25] A. Einstein, Ann. d. Phys. 17, 891 (1905).
[26] A. Einstein, Ann. d. Phys. 49, 769 (1916).
[27] W. Heisenberg, ”Wandlungen in den Grundlagen der Naturwissenschaft,” S. Hirzel Verlag, Stuttgart (1958).
[28] K. Schwarzschild, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 7, 189 (1916).
[29] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T.-M. Yan, Phys. Rev. D 17, 3090 (1978); Erratum ibid. 21, 313 (1980).
[30] G. N. Watson, ”A treatise on the theory of Bessel functions,” Cambridge Univ. Press, London (1958).
[31] A. Hasegawa and K. Mima, Phys. Rev. Lett. 39, 205 (1977).
[32] M. Honda, J. Meyer-ter-Vehn, and A. Pukhov, Phys. Rev. Lett. 85, 2128 (2000).
[33] S. Chandrasekhar and P. C. Kendall, Astrophys. J. 126, 457 (1957).
[34] P. A. M. Dirac, Proc. R. Soc. Lond. A 117, 610 (1928).
[35] R. Penrose, ”The emperor’s new mind: Concerning computers, minds, and the laws of physics,” Oxford Univ. Press, Oxford (1989).
[36] J. J. Sakurai, ”Advanced quantum mechanics,” Addison-Wesley, Reading (1967).
[37] Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.), 966 (1921); O. Klein, Zeit. f. Phys. 37, 895 (1926).
[38] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018).
[39] G. Aad et al. (The ATLAS and CMS Collaborations), Phys. Rev. Lett. 114, 191803 (2015).
[40] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).
[41] Ch. Kraus et al., Eur. Phys. J. C 40, 447 (2005); V. N. Aseev et al., Phys. Rev. D 84, 112003 (2011).

[42] T. Bringmann, X. Huang, A. Ibarra, S. Vogl, and C. Weniger, J. Cosmol. Astropart. Phys. 07, 054 (2012); E. Tempel, A. Hektor, and M. Raidal, ibid. 09, 032 (2012); M. Ackermann et al. (Fermi-LAT Collaboration), Phys. Rev. D 88, 082002 (2013).

[43] W. Pauli, Jr., Zeit. f. Phys. 31, 373 (1925); ibid. 765 (1925).