Abstract. In this paper, we prove a motivic enhancement of the theorem of the fixed part in
Hodge theory due to Deligne. In the pure motivic case, this was done for the first time by André in
[And96]. Our main result is an extension to the mixed case, which strengthens a result by Arapura
[Ara13] and also provides an alternative and simpler proof, in the framework of Nori motives, of a
result by Ayoub [Ayo14].

Key-words. Periods, Nori motives, algebraic monodromy group, Weil generic points.

1 Introduction

Let $k \hookrightarrow \mathbb{C}$ be a subfield. Recall that the category $\text{MM}(k)$ of Nori motives is constructed out
of quadruples $(X, Y, n, i)$ where $X$ is a $k$-variety, $Y$ is a closed subvariety of $X$, $i$ is a non-negative
integer and $j$ an integer. In [Ara13], Arapura has constructed a relative version (i.e., relative to a
fix base scheme $S$ over $k$) which is a tannakian category, we denote $\text{MM}(S)$. He also showed that
[Ara13, Theorem 6.4.1] the category of relative pure Nori motives is equivalent to the category of
André’s motives defined in terms of motivated cycles in [And96]. For $M \in \text{MM}(S)$, one can consider the tannakian subcategory $\langle M \rangle$ generated by $M$. The Betti
realization in this setting is a tensor functor $H_B : \text{MM}(S) \to \text{LS}(S^{\text{an}})$, where $\text{LS}(S^{\text{an}})$ is the
tannakian category of local systems over the analytic variety $S^{\text{an}}$. For any $s \in S$, taking the fiber
at $s$ provides a fiber functor, and by tannakian duality, one associates to $\langle M \rangle$ (resp. $\langle H_B(M) \rangle$) the
algebraic group $G_{\text{mot},s}(M)$ (resp. $G_{\text{mono},s}(M)$). This way, $H_B$ provides a morphism of algebraic
groups

$G_{\text{mono},s}(M) \to G_{\text{mot},s}(M)$.

Indeed, this map is an embedding because the monodromy group $G_{\text{mono},s}(M)$ is the Zariski closure
of the image of the monodromy representation $\pi_1(S^{\text{an}}, s) \to \text{GL}(H_B(M)_s)$, and thus a subgroup of
$G_{\text{mot},s}(M)$.

There is yet another algebraic group onstage, corresponding to the largest constant submotive
of $M$, which we denote by $G_{\text{mot},0,s}(M)$. One has a quotient map

$G_{\text{mot},s}(M) \to G_{\text{mot},0,s}(M)$.

Patching this two maps together gives a short exact sequence of algebraic groups:

Theorem. The sequence of algebraic groups

$0 \to G_{\text{mono},s}(M) \to G_{\text{mot},s}(M) \to G_{\text{mot},0,s}(M) \to 0$ (*)&

is exact.
This theorem is due to Nori, Jossen and Ayoub independently and in different contexts (only Ayoub, in [Ayo14], has published details, but his motives are defined differently; nevertheless, by a more recent and quite non-trivial theorem of Choudhury-Gallauer [CG14], his category is equivalent to Nori’s category). Thus, this theorem is equivalent to Ayoub’s result [Ayo14, Proposition 37]. We give a simpler proof in this article.

To prove the exactness, we proceed in two steps: first we show that the group $G_{\text{mono,}s}(M)$ is a normal subgroup of $G_{\text{mot,}s}(M)$, and then the induced map $G_{\text{mot,}s}(M)/G_{\text{mono,}s}(M) \to G_{\text{mot,}0,s}(M)$ is surjective.

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2 Preliminaries

We shall use Nori’s construction of a $\mathbb{Q}$-linear tannakian category $\mathbb{M}(\text{Spec } k)$ of mixed motives over a field $k$ of characteristic zero. Forgetting the tensor product (which is constructed using Nori’s subtle basic lemma about cellular decompositions), this abelian category is characterized by certain universal property with respect to functorial maps from a diagram category $\Delta$ consisting of quadruples $(X, Y, i, j)$ where $X$ is a $k$-variety, $Y$ is a closed subvariety of $X$, $i$ is a non-negative integer and $j$ an integer, to abelian categories. The corresponding object in $\mathbb{M}(\text{Spec } k)$ is denoted by $h^i(X, Y)(j)$.

To classical cohomology theories (Betti, de Rham, étale,...), there correspond tensor functors, called “realizations”, to categories of finite dimensional vector spaces over appropriate fields. For instance, de Rham cohomology gives rise to the de Rham realization $H_{dR} : \mathbb{M}(\text{Spec } k) \to \text{Vec}_\mathbb{Q}$, Betti cohomology (which depends on a fixed embedding of $k$ in $\mathbb{C}$, assumed to exist) gives rise to Betti realization $H_B : \mathbb{M}(\text{Spec } k) \to \text{Vec}_\mathbb{Q}$. There is a canonical comparison isomorphism, called the “period isomorphism”, between the complexification of the two realizations:

$$H_{dR} \otimes_k \mathbb{C} \cong H_B \otimes_k \mathbb{C}.$$

For brevity, one writes $H_{dR}^i(X, Y)(j)$ instead of $H_{dR}(h^i(X, Y)(j))$, and $H_B^i(X, Y)(j)$ instead of $H_B(h^i(X, Y)(j))$ where we view $i$ as cohomological degree and $j$ as Tate twist.

There exists an analogue in the relative context, due to Arapura [Ara13], that is, over a smooth base $S$, which corresponds intuitively to the notion of “family of Nori motives over $S$”. It is constructed from pairs $(X, Y)$, but in the relative context, that is, over $S$. We write $h_i((X, Y)/S)(j)$ for the relative analogues of $h^i(X, Y)(j)$, associated to the quadruples $(X \xrightarrow{f} S, Y, i, j)$ consisting of a quasiprojective family $f : X \to S$ so that $f$ can be completed to a smooth projective map, a subvariety $Y \subset X$ so that $Y$ together with the boundary is a divisor with normal crossings, and indices $i \in \mathbb{N}$, $j \in \mathbb{Z}$. This is a tannakian category such that its image under a suitable Betti realization lies in the category of locally constant sheaves of $\mathbb{Q}$-vector spaces. We denote this category by $\mathbb{M}(S)$.

Now, let us start from a Nori motive $M$ on $\text{Spec } k(S)$ with $k$ an algebraically closed subfield of $\mathbb{C}$. One can write $M$ as a subobject of $h^i(X, Y)(j)$. Then up to replacing $S$ by a dense open subset, one may assume that it extends to a pair $(X, Y)$ satisfying the same condition but relative to $S$. This process is called “spreading out”: from a motive $M$ on the generic point, one gets a family
of motives $M$ over $S$, that is, an object of $\text{MM}(S)$. In this relative situation, there is an analog of $H_{\text{dR}} : \text{MM}(S) \to \text{MIC}(S)$, where $\text{MIC}(S)$ is the tannakian category of vector bundles over $S$ endowed with an integrable connection. Thus for any $M \in \text{MM}(S)$, $H_{\text{dR}}(M)$ consists of a vector bundle (the de Rham cohomology of the family $M$ of Nori motives) together with an integrable connection $\nabla_M$, called the “Gauss-Manin connection”.

Similarly, if $S$ is defined over an algebraically closed subfield $k$ of $\mathbb{C}$, there is an analogue of Betti realization, $H_B : \text{MM}(S) \to \text{LS}(S^{an})$, where $\text{LS}(S^{an})$ is the tannakian category of local systems over the analytic variety $S^{an}$.

There is also a comparison isomorphism (a special case of the “Riemann-Hilbert correspondence”), which goes as follows: if one considers the associated analytic vector bundle $H_{\text{dR}}(M)^{an}$ with connection $\nabla_M^{an}$, its local system of “horizontal sections” $\ker \nabla_M^{an}$ is precisely $H_B(M)$, and

$$H_{\text{dR}}(M)^{an} \cong H_B(M) \otimes \mathcal{O}_{S^{an}}$$

as an isomorphism of sheaves on $S^{an}$.

Now consider the tannakian subcategory $\langle M \rangle$ of $\text{MM}(S)$ generated by $M$. Because $M$ is determined by $M$, taking the generic point $\langle M \rangle \to \langle M \rangle$ is an equivalence of tannakian categories. After spreading out, one can pick a closed point $s \in S$ and consider the Betti realization of the fiber at $s$, which gives rise to a fiber functor $\omega_s$, and thus provides a tannakian group $G_{\text{mot},s}(M)$ such that the category $\langle M \rangle \cong \langle M \rangle$ is equivalent to $\text{Rep}_Q G_{\text{mot},s}(M)$ via $\omega_s$.

On the other hand, one can look at $\langle H_{\text{dR}}(M) \rangle, \langle H_{\text{dR}}(M)^{an} \rangle, \langle H_B(M) \rangle$, the tannakian subcategories of $\text{MIC}(S)$, $\text{MIC}(S^{an})$, $\text{LS}(S^{an})$ generated by $H_{\text{dR}}(M), H_{\text{dR}}(M)^{an}, H_B(M)$ respectively.

Taking the fiber at $s$ as a fiber functor, tannakian duality shows that these are equivalent to categories of representations of appropriate linear algebraic groups over $\mathbb{Q}$, which we denote by $G_{\text{diff},s}(M), G_{\text{diff},s}(M^{an}), G_{\text{mono},s}(M^{an})$ respectively. Because the connection $\nabla_M$ underlying $H_{\text{dR}}(M)$ is regular, the analytification functor $\langle H_{\text{dR}}(M) \rangle \to \langle H_{\text{dR}}(M)^{an} \rangle$ is an equivalence, hence $G_{\text{diff},s}(M) = G_{\text{diff},s}(M^{an})$. On the other hand, by the comparison isomorphism, $\langle H_{\text{dR}}(M)^{an} \rangle \to \langle H_B(M) \rangle$ is an equivalence as well, so that $G_{\text{diff},s}(M^{an}) = G_{\text{mono},s}(M)$.

There is one more tannakian category in this context: the tannakian subcategory $\langle M \rangle_0$ of $\langle M \rangle$ generated by “constant motives”, i.e. motives of the form $M \otimes_k S$. More precisely, motives in the essential image of the base change functor

$$\text{MM}(\text{Spec } k) \hookrightarrow \text{MM}(S).$$

Applying tannakian duality, one has the quotient

$$G_{\text{mot},s}(M) \twoheadrightarrow G_{\text{mot},0,s}(M).$$

3 The motivic theorem of the fixed part

Let us discuss the main ingredient in the proof of the theorem.

The classical theorem of the fixed part, a.k.a. the global invariant cycle theorem, is a fundamental result in the topology/homology of families of algebraic varieties. In its simplest form, it states that for a projective smooth morphism $X \to S$ over $\mathbb{C}$ and a base point $s \in S$, $H^i(X_s)^{\pi_1(S,s)} = \text{Im}(i_*^*: H^i(X) \to H^i(X_s))$. Here $i : X_s \subset X$ is the canonical inclusion of the fibre in the total space. Stated like this, this is just a description of the monodromy invariant subspace of cohomology at the level of vector spaces (no extra structure involved). But this can be upgraded in the abelian
category of Hodge structures: indeed the right hand side is a sub-Hodge structure of $H^i(X_s)$, and hence so is $H^i(X_s)_{\pi_1(S,s)}$. This can be upgraded in a similar way in the abelian category of pure motives, see [And96].

We now turn to the theorem of the fixed part in the mixed motivic setting. Let us first recall that by definition of $\text{MM}(\text{Spec } k)$, any object of $\text{MM}(\text{Spec } k)$ is a subquotient of an object of the form $h^i(X,Y)(j)$. There are two very useful (and quite non-trivial) results which improve on this:

(a) any object of $\text{MM}(\text{Spec } k)$ is a subobject, and not only a subquotient, of an object of the form $h^i(X,Y)(j)$ [FJ18, Theorem 6.13].

(b) one may assume that $X$ is a complement of a divisor $D$ in a projective smooth $k$-variety $\overline{X}$, and that $Y$ is the restriction to $X$ of a normal crossings divisor $\overline{Y}$ in $\overline{X}$ containing $D$ [Ara13, Theorem 6.1.2].

The theorem also works in the context of pairs $(X,Y)$ as in (b) above, in the relative case (over $S$):

$$H_B(h^i((X,Y)/S)(j))_{\pi_1(S,s)} = H^i((X_s,Y_s)(j))_{\pi_1(S,s)} = \text{Im}(\iota^*_s : H^i(X,Y)(j) \to H^i(X_s,Y_s)(j)).$$

As explained in [Ara13, Lemma 7.2.2], this implies that $H_B(h^i((X,Y)/S)(j))_{\pi_1(S,s)}$ is the Betti realization of a constant submotive of $h^i((X,Y)/S)(j)$.

In tannakian terms this means the following. We start with a family $M \in \text{MM}(S)$ of motives over $S$ and build $\langle M \rangle$. We consider an object $N \in \langle M \rangle$, the local system $H_B(N)$ on $S^{an}$, and its fibre at $s$, $H_B(N)_s$, which is a finite dimensional $\mathbb{Q}$-vector space underlying a representation of $\pi_1(S,s)$. We are interested in the invariant space $H_B(N)^{\pi_1(S,s)}_s$. Note that $H_B(N)_s$ is a representation of $G_{\text{mono},s}(M)$. Thus for any family of Nori motives $N \in \langle M \rangle$ of the form $h^i((X,Y)/S)(j)$, $H_B(N)^{\pi_1(S,s)}_s$ is stable under $G_{\text{mot},s}(M)$, hence is fiber at $s$ of the Betti realization of a submotive of $N$ over $S$ which is constant, that is, belongs to $\langle M \rangle_0$.

Let us now pass from motives of the form $h^i((X,Y)/S)(j)$ to the general case. It is rather straightforward, using the fact that $\text{MM}(S)$ is an abelian category, to treat the case of a submotive of such an object $h^i((X,Y)/S)(j)$ but there is a problem with quotients: invariants in a quotient are duals of coinvariants of the dual, and one cannot reduce to invariants except if the monodromy is semisimple. Fortunately, (a refinement of) point (a) above allows to ignore subquotients and to deal only with subobjects. In conclusion:

Theorem (Motivic theorem of the fixed part). For any family of Nori motives $N \in \langle M \rangle$, $H_B(N)^{\pi_1(S,s)}_s$ is stable under $G_{\text{mot},s}(M)$, hence is fiber at $s$ of the Betti realization of a submotive of $N$ over $S$ which is constant, that is, belongs to $\langle M \rangle_0$.

4 Conclusion of the proof of the main theorem

Let us go back to the sequence $(\ast)$. The problem of its exactness splits into two (successive) subproblems:

1) The group $G_{\text{mono},s}(M)$ is a normal subgroup of $G_{\text{mot},s}(M)$.

2) The induced map $G_{\text{mot},s}(M)/G_{\text{mono},s}(M) \to G_{\text{mot},0,s}(M)$ is surjective (and hence an isomorphism).
For 1), one can use the normality criterion of [And92, lemme 1]: it suffices to show that for any character $\chi : G_{\text{mono},s}(M) \to \mathbb{G}_m$, the semi-invariant space $(H_B(N)^{\pi_1(S,s)})^\chi$ is stable under $G_{\text{mot},s}(M)$. We recall $(H_B(N)^{\pi_1(S,s)})^\chi$ is the space of elements $v \in H_B(N)^{\pi_1(S,s)}$ such that $h(v) = \chi(h).v$ for every $h \in G_{\text{mono},s}(M)$.

Now if all characters of $G_{\text{mono},s}(M)$ are trivial, then via the tannakian dictionary, the desired properties 1) and 2) become nothing but the motivic theorem of the fixed part. Therefore, exactness of $(*)$ holds if all characters of $G_{\text{mono},s}(M)$ are trivial. However, in general, $\mathbb{Q}$-characters of $G_{\text{mono},s}(M)$ may be non-trivial; “but not so much”: given a character $\chi : G_{\text{mono},s}(M) \to \mathbb{Q}^\times$ that appears in a representation of $G_{\text{mot},s}(M)$, the target of $\chi$ is $\mathbb{Z}^\times = \{\pm 1\}$, i.e. $\chi$ is either trivial or of order 2. See [And17b, Lemma C.9]. Thus we assume that $\chi$ is of order exactly 2. By passing to a double étale covering $S'$ of $S$, we further assume that it is trivial. In this case, the new sequence $(*)$ is exact again by above discussion. But how can one descend from $S'$ to $S$?

5 Weil generic points

One can start from a motive $M$ over $k(S)$, where $k$ is an algebraically closed subfield of $\mathbb{C}$, which spreads out as an object $M$ of $\mathcal{M}(S)$. Without loss of generality, one may assume that $k$ is countable (since a motive is defined in terms of pairs of varieties which themselves involve only finitely many equations, and the same for $S$). This allows to find an embedding of $k$-extensions $k(S) \hookrightarrow \mathbb{C}$. Such an embedding corresponds to a complex point $s \in S(\mathbb{C})$ and complex points of this type (i.e. such that $\text{Spec}(\mathbb{C}) \to S$ maps to the generic point of $S$) are called Weil generic points. The point in using Weil generic points is that the realizations at a Weil generic point $s$ make sense both for $\langle M \rangle$ and for $\langle M \rangle$, and moreover they coincide. In particular, $G_{\text{mot},s}(M) \cong G_{\text{mot},s}(M)$. Therefore, one may choose a Weil generic point $s$ of $S$ in order to define the Betti realization $\omega_s$ using $s$ (instead of $H_B(N)^{\pi_1(S,s)}$).

One can extend the Weil generic point $s$ to an embedding of the algebraic closure $\overline{k(S)} \hookrightarrow \mathbb{C}$, which thus defines a Weil generic point of every finite covering of $S$, still denoted by $s$.

6 An exact sequence for monodromy groups

Let $G_{\text{mono},s}(M)$ be again the algebraic monodromy group, i.e. the Zariski closure of the image of the monodromy representation $\pi_1(S,s) \to \text{GL}(\omega_s(M))$. This is a subgroup of $G_{\text{mot},s}(M)$. By the motivic theorem of the fixed part, the space of $G_{\text{mono},s}(M)$-invariants in $\omega_s(M)$ is the image by $\omega_s$ of a constant subobject of $M$; in particular, it is stable under $G_{\text{mot},s}(M)$.

The same remains true of course if one replace $M$ by any object of $\langle M \rangle$ and $S$ by any double étale covering $S'$.

One has an exact sequence

$$1 \to \pi_1(S',s) \to \pi(S,s) \to \text{Gal}(\overline{k(S')}/k(S)) = \mathbb{Z}/2 \to 1,$$

which induces a similar exact sequence for the images of $\pi_1(S,s)$ and $\pi_1(S',s)$ in $\text{GL}(\omega_s(M)) = \text{GL}(\omega_s(M'))$, and also for their Zariski closures

$$1 \to G_{\text{mono},s}(M_{S'}) \to G_{\text{mono},s}(M) \to \text{Gal}(\overline{k(S')}/k(S)) = \mathbb{Z}/2 \to 1.$$
7 An exact sequence for motivic Galois groups

Our aim is to deduce from the motivic theorem of the fixed part the following little variation: the space $\omega_{s}(M)^{x}$ of elements on which $G_{\text{mono},s}(M)$ acts through $\chi$ is stable under $G_{\text{mot},s}(M)$. For this, we may enlarge $G_{\text{mot},s}(M)$, for instance replace it by the motivic Galois group $G'_{\text{mot},s}(M)$ of the tannakian category generated by $M$ and by the Artin motives defined by the finite extension $k(S')$ of $k(S)$ (a posteriori, one can see that this is no enlargement). The subcategory of Artin motives inside this category gives rise to a morphism $G'_{\text{mot},s}(M) \to \text{Gal}(k(S')/k(S))$, and there is an exact sequence

$$1 \to G'_{\text{mot},s}(M_{S'}) = G_{\text{mot},s}(M_{S'}) \to G'_{\text{mot},s}(M) \to \text{Gal}(k(S')/k(S)) \to 1,$$

by the work of Jossen [Jos16, Theorem 10.7].

8 A commutative diagram of groups

Now the embedding of monodromy groups in motivic Galois groups is compatible with the last two exact sequences, so that they match into a commutative diagram of exact sequences. Since $G_{\text{mot},s}(M_{S'})$ is a normal subgroup of $G'_{\text{mot},s}(M)$, it makes sense to form the compositum $G_{\text{mot},s}(M_{S'}).G_{\text{mono},s}(M)$. Using the commutative diagram, this is the full group $G'_{\text{mot},s}(M)$.

Let $W = \omega_{s}(M)^{x} \oplus \omega_{s}(M)^{G_{\text{mono},s}(M)} = \omega_{s}(M)^{G_{\text{mono},s}(M_{S'})}$. Then $W$ is $G'_{\text{mot},s}(M_{S'})$-stable by the motivic theorem of the fixed part. It is also stable by $G_{\text{mono},s}(M)$. So $W$ is stable under $G'_{\text{mot},s}(M)$. So we may substitute $W$ to the original representation $\omega_{s}(M)$; in other words, we may assume that $G_{\text{mono},s}(M)$ acts by $\mathbb{Z}/2$ on $\omega_{s}(M)$, and denoting by a subscript $\pm$ the $\pm 1$-eigenspaces. We have to show that $\omega_{s}(M)^{-}$ is stable under $G'_{\text{mot},s}(M)$. Since $\text{Hom}_{G_{\text{mono},s}(M)}(\omega_{s}(M)^{+},\omega_{s}(M)^{-}) = \text{Hom}_{G_{\text{mono},s}(M)}(\omega_{s}(M)^{-},\omega_{s}(M)^{+}) = 0$, the exact sequence

$$0 \to \omega_{s}(M)^{+} \to \omega_{s}(M) \to \omega_{s}(M)^{-} \to 0 \quad (\ast)$$

has a $\mathbb{Z}/2$-equivariant splitting, and we have to show that this splitting is $G'_{\text{mot},s}(M)$-equivariant. For this, we apply $(\ast)$ to the dual representation $\omega_{s}(M)^{\vee}$, and compare this exact sequence to the $(G'_{\text{mot},s}(M)$-equivariant) dual exact sequence of $(\ast)$ for $\omega_{s}(M)$; their $\mathbb{Z}/2$-equivariant splittings both correspond to the decomposition $\omega_{s}(M)^{\vee} = (\omega_{s}(M)^{\vee})^{+} \oplus (\omega_{s}(M)^{\vee})^{-}$. Therefore the morphisms

$$(\omega_{s}(M)^{\vee})^{+} \to (\omega_{s}(M)^{+})^{\vee}, \quad (\omega_{s}(M)/\omega_{s}(M)^{+})^{\vee} \to \omega_{s}(M)^{\vee}/(\omega_{s}(M)^{\vee})^{+}$$

are bijective and we get the claim for $\omega_{s}(M)^{\vee}$, and hence for $\omega_{s}(M)$.

This proves that the first factor $\omega_{s}(M)^{x}$ is also stable under $G'_{\text{mot},s}(M)$, and thus under $G_{\text{mot},s}(M)$, as desired.

Remark. One can consider the category of (relative) exponential motives, along the same construction of category of Nori motives, by introducing potentials associated to varieties over complex numbers. Typical objects are of the form $(X \to S, Y, n, f, i)$ where $X \to S$ is a morphism of quasi-projective varieties, $Y$ a closed subvariety of $X$, $f \in \mathcal{O}(X)$, and the integers $n \geq 0$, $i \in \mathbb{Z}$. In the case, $S = \text{Spec}(k)$ for a subfield $k \hookrightarrow \mathbb{C}$, there are analogues of Betti and de Rham realization functors. There is also a version of Nori’s Basic Lemma. This yields a tannakian category of
exponential motives, which we denote by $M^\exp(k)$. See [FJ20] for details of constructions. This construction may extend to the relative case, leading to the tannakian category $M^\exp(S)$ of relative exponential Nori motives. One can ask if the motivic theorem of the fixed part can be formulated in terms of the category $M^\exp(S)$. A first obstruction is that there is no known theorem of the fixed part in the exponential category $M^\exp(k)$ to start with. Because the (exponential) Gauss-Manin connections are no longer regular, one may replace the monodromy group by the differential Galois group $G_{\text{diff}, s}(M)$ associated to a relative exponential Nori motive $M/S$. Note that, in general, the characters of $G_{\text{diff}, s}(M)$ are no longer of finite order. One may ask whether, nevertheless, the natural sequence of algebraic groups is exact:

$$G_{\text{diff}, s}(M) \to G_{\text{mot}, s}(M) \to G_{\text{mot, 0}, s}(M) \to 0.$$


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