Abstract  Let $R$ be a rational function. The iterations $(R^n)_n$ of $R$ gives a complex dynamical system on the Riemann sphere. We associate a $C^*$-algebra and study a relation between the $C^*$-algebra and the original complex dynamical system. In this short note, we recover the number of $n$th backward orbits counted without multiplicity starting at branched points in terms of associated $C^*$-algebras with gauge actions. In particular, we can partially imagine how a branched point is moved to another branched point under the iteration of $R$. We use KMS states and a Perron–Frobenius type operator on the space of traces to show it.

Keywords  Complex dynamical system · $C^*$-algebra · Backward orbit · Branched point · $C^*$-correspondences

Mathematics Subject Classification  46L08 · 46L55

1 Introduction

Iteration of a rational function $R$ gives a complex dynamical system on the Riemann sphere $\hat{\mathbb{C}}$. Since rational functions in general have branched points (i.e. critical points),
$R$ is not a local homeomorphism any more. Hence we are not able to introduce an étale groupoid and to associate a groupoid $C^*$-algebra like Renault in [14] in the usual way. For a branched covering $\pi : M \to M$, Deaconu and Muhly [1] introduced a $C^*$-algebra $C^*(M, \pi)$ as the $C^*$-algebra of the étale groupoid by subtracting the branched points.

In [7], we introduced slightly different $C^*$-algebras using the Cuntz–Pimsner construction to include branched points. Since the Riemann sphere $\hat{\mathbb{C}}$ is decomposed into the union of the Julia set $J_R$ and Fatou set $F_R$, we associated three $C^*$-algebras $\mathcal{O}_R(\hat{\mathbb{C}})$, $\mathcal{O}_R(J_R)$ and $\mathcal{O}_R(F_R)$ by considering $R$ as dynamical systems on $\hat{\mathbb{C}}$, $J_R$ and $F_R$, respectively. We have studied how properties of $R$ as complex dynamical systems are related with the structure of the associated $C^*$-algebras and their K-groups [3, 4, 7, 17]. One of our aims is to analyze the singularity structure of the branched points in terms of operator algebras. For example, in [3], we showed that the extreme KMS states are parameterized by the branched points. Recently Thomsen introduces and studies another convolution $C^*$-algebra of the transformation groupoid adding local transfers for a rational function in [15, 16].

In this short note, we study backward orbit structure in terms of operator algebras. In particular, we recover the number of $n$th backward orbits counted without multiplicity starting at branched points in terms of associated $C^*$-algebras with gauge actions. If there exists a branched point in the backward orbits, then the number decreases at the branched point, because we do not count the multiplicity. In particular, we can partially imagine how a branched point is moved to another branched point under the iteration of $R$. We use KMS states and a Perron–Frobenius type operator to show it. We should mention that Kumjian and Renault [10] study the existence and uniqueness of KMS states associated to general expansive maps which are local homeomorphisms.

On the other hand, Nekrashevych [12] studies the Cuntz–Pimsner algebras for self-similar groups like iterated monodromy groups of expanding dynamical systems. Surprisingly, he reconstructed the complex dynamical system on the Julia set of a hyperbolic rational function from the Cuntz–Pimsner algebra and the gauge action on it. Nekrashevych’s work does not include the case that the Julia set contains branched points, thus our results contain a new fact on this case.

Let $R$ be a rational function of the form $R(z) = \frac{P(z)}{Q(z)}$ with relatively prime polynomials $P$ and $Q$. The degree of $R$ is denoted by $N = \deg R := \max\{\deg P, \deg Q\}$. We regard a rational function $R$ as a $N$-fold branched covering map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The sequence $(R^n)_n$ of iterations of $R$ gives a complex dynamical system on $\hat{\mathbb{C}}$. The Fatou set $F_R$ of $R$ is the maximal open subset of $\hat{\mathbb{C}}$ on which $(R^n)_n$ is equicontinuous (or a normal family), and the Julia set $J_R$ of $R$ is the complement of the Fatou set in $\hat{\mathbb{C}}$. The Fatou set $F_R$ is a stable part and the Julia set $J_R$ is an unstable part.

Recall that a branched point (or critical point) of $R$ is a point $z_0$ at which $R$ is not locally one to one. It is a zero of $R'$ or a pole of $R$ of order two or higher. The image $w_0 = R(z_0)$ is called a branch value (or critical value) of $R$. Using appropriate local charts, if $R(z) = w_0 + c(z - z_0)^n + \text{(higher terms)}$ with $n \geq 1$ and $c \neq 0$ on some neighborhood of $z_0$, then the integer $n = e(z_0) = e_R(z_0)$ is called the branch index of $R$ at $z_0$. Thus $e(z_0) \geq 2$ if $z_0$ is a branched point, and $e(z_0) = 1$ if $z_0$ is not. Therefore $R$ is an $e(z_0) : 1$ map in a punctured neighborhood
of \( z_0 \). By the Riemann-Hurwitz formula, there exist \( 2N - 2 \) branched points counted with multiplicity, that is, \( \sum_{z \in \hat{\mathbb{C}}} (e(z) - 1) = 2 \deg R - 2 \). Furthermore for each \( w \in \hat{\mathbb{C}} \), we have \( \sum_{z \in R^{-1}(w)} e(z) = \deg R \). Let \( B_R \) be the set of branched points of \( R \) and \( C_R := R(B_R) \) be the set of the critical values of \( R \). Then the restriction \( R : \hat{\mathbb{C}} \setminus R^{-1}(C_R) \to \hat{\mathbb{C}} \setminus C_R \) is a \( N : 1 \) regular covering, where \( N = \deg R \). This means that any point \( y \in \hat{\mathbb{C}} \setminus C_R \) has an open neighborhood \( V \) such that \( R^{-1}(V) \) has \( N \) connected components \( U_1, \ldots, U_N \) and the restriction \( R|_{U_k} : U_k \to V \) is a homeomorphism for \( k = 1, \ldots, N \). Thus \( R \) has \( N \) analytic local cross sections \( S_k = (R|_{U_k})^{-1} \). But if \( y \) is in \( C_R \), then there exist no such open neighborhood \( V \). This fact causes many difficulties to analyze the associated \( C^* \)-algebra, since we include the branched points to construct the \( C^* \)-correspondence. If we will construct the associated groupoid naively, the étaleness (or r-discreteness) is not satisfied in general. This is the reason why we associated our \( C^* \)-algebras using the Cuntz–Pimsner construction in [7]. One of our aims is to analyze the singularity structure of the branched points in terms of operator algebras.

In our previous paper [3], we study KMS-states for the gauge action based on the work of Laca and Neshveyev [11]. The gauge action has a phase transition at \( \beta = \log \deg R \). We can recover the degree of \( R \), the number of branched points, the number of exceptional points and the orbits of exceptional points from the structure of the KMS states. But we could not know anything about how branched points are related to each other under the iteration of \( R \). In this note we study the orbit structure of branched points under iteration. The proof depends on the fact that extreme KMS states are parameterized by the branched points as described in [3].

2 Construction of the Associated \( C^* \)-Algebras

Since a rational function \( R \) of degree at least two is not a homeomorphism, we can not use the crossed product construction. We replace the crossed product construction by the Cuntz–Pimsner construction to obtain the associated \( C^* \)-algebra.

We recall Cuntz–Pimsner algebras [13]. Let \( A \) be a \( C^* \)-algebra and \( X \) be a Hilbert right \( A \)-module. We denote by \( L(X) \) the algebra of the adjointable bounded operators on \( X \). For \( \xi, \eta \in X \), the “rank one” operator \( \theta_{\xi, \eta} \) is defined by \( \theta_{\xi, \eta}(\zeta) = \xi(\eta| \zeta) \) for \( \zeta \in X \). The closure of the linear span of rank one operators is denoted by \( K(X) \).

A family \( (u_i)_{i \in I} \) in \( X \) is called a basis (see [4,5]), (or a standard normalized tight frame more precisely as in [2]) of \( X \) if

\[
x = \sum_{i \in I} u_i (u_i|x)_A \quad \text{for any } x \in X,
\]

where the sum is taken as unconditional norm convergence, that is, for the directed set \( \Lambda := \{ F \mid F \subset I \text{ is a finite subset} \} \),

\[
x = \lim_{F \in \Lambda} \sum_{i \in F} u_i (u_i|x)_A
\]
Furthermore, \((u_i)_{i \in I}\) is called a finite basis if \((u_i)_{i \in I}\) is a finite set. If a Hilbert \(C^*\)-module is countably generated, then there exists a countable basis (that is, finite or a countably infinite basis) of \(X\) and written as \([u_i]_{i=1}^{\infty}\), where some \(u_i\) may be zero. If \(A\) has a unit and \(X\) has a finite basis, then \(X\) is algebraically finitely generated and projective over \(A\) and \(K(X) = L(X)\).

We say that \(X\) is a Hilbert \(C^*\)-bimodule (or \(C^*\)-correspondence) over \(A\) if \(X\) is a Hilbert right \(A\)-module with a homomorphism \(\phi : A \to L(X)\). In this note, we assume that \(X\) is full and \(\phi\) is injective. Let \(F(X) = \bigoplus_{n=0}^{\infty} X^\otimes n\) be the full Fock module of \(X\) with the convention \(X^\otimes 0 = A\). For \(x \in X\), the creation operator \(T_x \in L(F(X))\) is defined by

\[
T_x(a) = xa \quad \text{and} \quad T_x(x_1 \otimes \cdots \otimes x_n) = x \otimes x_1 \otimes \cdots \otimes x_n.
\]

We define \(i_{F(X)} : A \to L(F(X))\) by

\[
i_{F(X)}(a)(b) = ab \quad \text{and} \quad i_{F(X)}(a)(x_1 \otimes \cdots \otimes x_n) = (\phi(a)x_1) \otimes \cdots \otimes x_n
\]

for \(a, b \in A\). The Cuntz-Toeplitz algebra \(T_X\) is the \(C^*\)-subalgebra of \(L(F(X))\) generated by \(i_{F(X)}(a)\) with \(a \in A\) and \(T_x\) with \(x \in X\). Let \(j_K : K(X) \to T_Y\) be the homomorphism defined by \(j_K(\theta_{x,y}) = T_x T_y^*\). We consider the ideal \(I_X = \phi^{-1}(K(X))\) of \(A\). Let \(\mathcal{J}_X\) be the ideal of \(T_X\) generated by \([i_{F(X)}(a) - (j_K \circ \phi)(a) ; a \in I_X]\). Then the Cuntz–Pimsner algebra \(O_X\) is the quotient \(T_X/\mathcal{J}_X\). Let \(\pi : T_X \to O_X\) be the quotient map. Put \(S_x = \pi(T_x)\) and \(i(a) = \pi(i_{F(X)}(a))\). Let \(j_K : K(X) \to O_X\) be the homomorphism defined by \(j_K(\theta_{x,y}) = S_x S_y^*\). Then \(\pi(i(j_K \circ \phi)(a)) = (i_K \circ \phi)(a)\) for \(a \in I_X\).

We note that the Cuntz–Pimsner algebra \(O_X\) is the universal \(C^*\)-algebra generated by \(i(a)\) with \(a \in A\) and \(S_x\) with \(x \in X\) satisfying that \(i(a)S_x = S_{\phi(a)x}\), \(S_x i(a) = S_{xa}\), \(S_x^*S_y = i((x|y)_A)\) for \(a \in A, x, y \in X\) and \(i(a) = (i_K \circ \phi)(a)\) for \(a \in I_X\). We usually identify \(i(a)\) with \(a\) in \(A\). If \(X\) has a countable basis \(\{u_1, u_2, \ldots\}\), then the last condition should be replaced by \(i(a) = \lim_{n \to \infty} \sum_{k=1}^{n} i(a)S_{u_k}S_{u_k}^*\) under the operator norm convergence for any \(a \in I_X\). Since \(\phi(a) \in K(X)\), we automatically have \(\phi(a) = \lim_{n \to \infty} \sum_{k=1}^{n} \phi(a)\theta_{u_k, u_k}\) under the operator norm convergence, because \(\sum_{k=1}^{n} \phi(a)\theta_{u_k, u_k}\) is an approximate unit for \(K(X)\).

There exists an action \(\gamma : \mathbb{R} \to \text{Aut} O_X\) with \(\gamma_t(S_x) = e^{it} S_x\), which is called the gauge action. Since we assume that \(\phi : A \to L(X)\) is isometric, there is an embedding \(\phi_n : L(X^\otimes n) \to L(X^\otimes n+1)\) with \(\phi_n(T) = T \otimes id_X\) for \(T \in L(X^\otimes n)\) with the convention \(\phi_0 = \phi : A \to L(X)\). We denote by \(\mathcal{F}_X\) the \(C^*\)-algebra generated by all \(K(X^\otimes n)\), \(n \geq 0\) in the inductive limit algebra \(\lim_{\longrightarrow} L(X^\otimes n)\). Let \(\mathcal{F}_n\) be the \(C^*\)-subalgebra of \(\mathcal{F}_X\) generated by \(K(X^\otimes k)\), \(k = 0, 1, \ldots, n\), with the convention \(\mathcal{F}_0 = A = K(X^\otimes 0)\). Then \(\mathcal{F}_X = \lim_{\longrightarrow} \mathcal{F}_n\). Consult [6,13] for a general Cuntz–Pimsner algebra.

Let \(A = C(\hat{\mathbb{C}})\) and \(X = C(\text{graph } R)\) be the set of continuous functions on \(\hat{\mathbb{C}}\) and on graph \(R\), respectively, where graph \(R = \{(x, y) \in \hat{\mathbb{C}}^2 ; y = R(x)\}\) is the graph of \(R\). Then \(X\) is an \(A-A\) bimodule by

\[
(a \cdot \xi \cdot b)(x, y) = a(x)\xi(x, y)b(y), \quad a, b \in A, \ \xi \in X.
\]
We define an \( A \)-valued inner product \(( \cdot | \cdot )_A\) on \( X\) by
\[
(\xi | \eta)_A(y) = \sum_{x \in R^{-1}(y)} e(x)\overline{\xi(x,y)}\eta(x,y), \quad \xi, \eta \in X, \ y \in \hat{A}.
\]

Thanks to the branch index \( e(x) \), the inner product above gives a continuous function and \( X \) is a full Hilbert bimodule over \( A \) without completion. The left action of \( A \) is unital and faithful.

Since the Julia set \( J_R \) is completely invariant under \( R \), i.e., \( R(J_R) = J_R = R^{-1}(J_R) \), we can consider the restriction \( R|_{J_R} : J_R \to J_R \), which will be often denoted by the same letter \( R \). Let \( \text{graph} \ R|_{J_R} = \{(x, y) \in J_R \times J_R : y = R(x)\} \) be the graph of the restriction map \( R|_{J_R} \). In the same way as above, \( X(J_R) \) is a full Hilbert bimodule over \( C(J_R) \). Since the Fatou set \( F_R \) is also completely invariant, \( X(F_R) := C_0(\text{graph} \ R|_{F_R}) \) is a full Hilbert bimodule over \( C_0(F_R) \).

**Definition** (\( C^* \)-algebra associated with a complex dynamical system) Let \( R \) be a rational function with \( \text{deg} \ R \geq 2 \). The \( C^* \)-algebra \( \mathcal{O}_R(\hat{\mathbb{C}}) \) is defined as the Cuntz–Pimsner algebra of the Hilbert bimodule \( X = C(\text{graph} \ R) \) over \( A = C(\hat{\mathbb{C}}) \). When the Julia set \( J_R \) is not empty (for example \( \text{deg} \ R \geq 2 \)), we define the \( C^* \)-algebra \( \mathcal{O}_R(J_R) \) as the Cuntz–Pimsner algebra of the Hilbert bimodule \( X = C(\text{graph} \ R|_{J_R}) \) over \( A = C(J_R) \). When the Fatou set \( F_R \) is not empty, the \( C^* \)-algebra \( \mathcal{O}_R(F_R) \) is defined similarly.

### 3 Perron–Frobenius Operator

We shall introduce a Perron–Frobenius operator associated with a bimodule on the space of traces. Let \( A \) be a unital \( C^* \)-algebra. We denote by \( \text{Trace}(A) \) the set of bounded tracial functionals on \( A \), \( \text{Trace}^+ (A) \) the set of bounded tracial positive functionals on \( A \) and \( \text{Trace}^+_1(A) \) the set of tracial states on \( A \). We assume that \( \text{Trace}(A) \) is not empty. Let \( X \) be a countably generated Hilbert \( A \)-module and \( \{u_i\}_{i=1}^\infty \) a countable basis of \( X \). For a tracial state \( \tau \) on \( A \), \( \sup_n \sum_{i=1}^n \tau((u_i|u_i)_A) \in [0, \infty] \) does not depend on the choice of basis \( \{u_i\}_{i=1}^\infty \) as in \([8,9]\). We put \( d_\tau = \sup_n \sum_{i=1}^n \tau((u_i|u_i)_A) \). We call that \( X \) is of finite degree type if \( \sup_{\tau \in \text{Trace}^+_1(A)} d_\tau < \infty \) \([5,8,9]\). For example, let \( X \) be the Hilbert bimodule associated with a rational function \( R \). Then \( X_A \) is of finite degree type and \( \sup_{\tau \in \text{Trace}^+_1(A)} d_\tau = \text{degree} \ R \).

**Definition** (Perron–Frobenius operator) Let \( A \) be a unital \( C^* \)-algebra and \( X \) a countably generated (right) full Hilbert module over \( A \). Let \( \phi : A \to L(X) \) be a unital faithful homomorphism so that \( X \) is a bimodule over \( A \). Let \( \{u_i\}_{i=1}^\infty \) be a basis of \( X \). If \( X \) is of finite degree type, then there exists a bounded linear operator \( F_X : \text{Trace}(A) \to \text{Trace}(A) \) such that for \( \tau \in \text{Trace}(A) \),
\[
F_X(\tau)(a) = \sum_{i=1}^\infty \tau((u_i | \phi(a)u_i)_A).
\]
Then $F_X$ does not depend on the choice of basis. We call $F_X$ a Perron–Frobenius operator associated with a bimodule $X$ of finite degree type. See [3,5,8,9] for example.

Example Let $R$ be a rational function with $\deg R \geq 2$. Consider the $C^*$-algebra $O_R(\hat{\mathbb{C}})$ associated with a complex dynamical system $(\mathbb{R}^n)_n$ on the Riemann sphere. The $C^*$-algebra $O_R(\hat{\mathbb{C}})$ is defined as the Cuntz–Pimsner algebra of the Hilbert bimodule $X = C(\text{graph } R)$ over $A = C(\hat{\mathbb{C}})$. Then we shall show that the Perron–Frobenius operator $F_X$ associated with a bimodule $X$ is described as follows: For a finite Borel measure $\mu$ and $a \in A = C(\hat{\mathbb{C}})$, we have that

$$(F_X(\mu))(a) = \mu(\tilde{a}).$$

where a Borel function $\tilde{a}$ is defined by $\tilde{a}(y) = \sum_{x \in R^{-1}(y)} a(x)$ and we identify the finite Borel measure $\mu$ on $\hat{\mathbb{C}}$ with the associated finite trace on $C(\hat{\mathbb{C}})$ by the same symbol $\mu$. In particular, we have

$$F_X(\delta_y) = \sum_{x \in R^{-1}(y)} \delta_x,$$

where $\delta_y$ is the Dirac measure on $y$. It is crucial that the sum on $x \in R^{-1}(y)$ should be taken without multiplicity in these formulae.

Let $\{u_i\}_{i=1}^{\infty}$ be a countable basis of $X$. For $f \in X = C(\text{graph } R)$, we have $|f(x, y)| \leq \|f\|_2$ and the identity $\sum_{i=1}^{\infty} u_i(u_i|f)_A = f$ converges in norm $\| \cdot \|_2$. Hence the left side converges also pointwisely. For each fixed $y \in \hat{\mathbb{C}}$ and $x \in R^{-1}(y)$, we consider the value of $\sum_{i=1}^{\infty} u_i(u_i|f)_A = f$ at $(x, y) = (x, R(x))$:

$$\lim_{n \to \infty} \left( \sum_{i=1}^{n} u_i(x, y)(u_i|f)_A(y) \right) = \lim_{n \to \infty} \left( \sum_{i=1}^{n} u_i(x, y) \sum_{z \in R^{-1}(x)} e_R(z)u_i(z, y)f(z, y) \right) = f(x, y).$$

We take $f \in X$ such that $f(x, y) = 1$ and $f(x', y) = 0$ for $x' \in R^{-1}(x)$ with $x' \neq x$. Then we have

$$\lim_{n \to \infty} \left( \sum_{i=1}^{n} u_i(x, y) \sum_{z \in R^{-1}(y)} e_R(z)u_i(z, y)f(z, y) \right) = \lim_{n \to \infty} \left( \sum_{i=1}^{n} e_R(x)u_i(x, y)u_i(x, y) \right) = \sum_{i=1}^{\infty} e_R(x)|u_i(x, y)|^2 = 1.$$
For \(a \in A\), we have
\[
\sum_{i=1}^{\infty} (u_i | \phi(a)u_i)_A(y) = \sum_{i=1}^{\infty} \sum_{x \in R^{-1}(y)} e_R(x) \overline{u_i(x, y)} a(x) u_i(x, y)
\]
\[
= \sum_{i=1}^{\infty} \sum_{x \in R^{-1}(y)} e_R(x) |u_i(x, y)|^2
\]
\[
= \sum_{x \in R^{-1}(y)} a(x) \left( \sum_{i=1}^{\infty} e_R(x) |u_i(x, y)|^2 \right)
\]
\[
= \sum_{x \in R^{-1}(y)} a(x) = \tilde{a}(y).
\]

Therefore we have
\[
(F_X(\mu))(a) = \mu \left( \sum_{i=1}^{\infty} (u_i | \phi(a)u_i)_A \right) = \mu(\tilde{a}).
\]

**Proposition 3.1** Let \(A\) be a unital \(C^*\)-algebra and \(X\) a full Hilbert bi-module over \(A\) with a unital faithful left action. Let \(O_X\) be the Cuntz–Pimsner algebra for \(X\) with a gauge action \(\gamma\). Let \(B = O_X^\gamma\) be the fixed point algebra under \(\gamma\). Let
\[
Y = \{ y \in O_X \mid \gamma_z(y) = zy \ (z \in \mathbb{T}) \}
\]
be the 1-spectral subspace. Then we have the following:

1. \(Y\) is a Hilbert bi-module over \(B\) under a natural action \(b \cdot y \cdot c = byc\) with a \(B\)-valued inner product \((y \mid w)_B = y^*w\) for \(b, c \in B\) and \(y, w \in Y\). Moreover the linear span of \(\{ S_x b \mid b \in B, x \in X \}\) is dense in \(Y\).

2. If \([u_i]_{i=1}^{\infty}\) is a basis of \(X\), then \([S_{u_i}]_{i=1}^{\infty}\) is a basis of \(Y\).

3. Let \(F_Y : \text{Trace}(B) \to \text{Trace}(B)\) be the Perron–Frobenius operator associated with \(Y\). Then \((F_Y(\tau))(a) = (F_X(\tau|_A))(a)\) for any trace \(\tau \in \text{Trace}(B)\) and its restriction \(\tau|_A\) to \(A\) and \(a \in A\).

**Proof** (1) Since \(y, w \in Y\) are in the 1-spectral subspace, \(y^*w\) is in the fixed point algebra \(B = O_X^\gamma\) under \(\gamma\). Since \(\|(y \mid y)_B\| = \|y^*y\|\), \(Y\) is complete. The others are also easily checked.

(2) Let \([u_i]_{i=1}^{\infty}\) be a basis of \(X\). For any \(x \in X, b \in B\), we have
\[
\sum_i S_{u_i}(S_{u_i} | S_x b)_B = \sum_i S_{u_i} S_{u_i}^* S_x b = S_{\sum_i u_i(x)} b = S_x b.
\]

since \(\|S_x\| = \|x\|\).
(3) For any trace $\tau \in \text{Trace}(B)$ and for any $a \in A$,

$$
(F_Y(\tau))(a) = \sum_{i=1}^{\infty} \tau((S_{u_i} \ | \ aS_{u_i})_B) = \sum_{i=1}^{\infty} \tau(S^*_{u_i}saS_{u_i})
$$

$$
= \sum_{i=1}^{\infty} \tau((u_i \ | \ a(u_i))_A) = (F_X(\tau \ | \ A))(a).
$$

\textbf{Remark} In the above, the Perron–Frobenius operator $F_X$ associated with a bimodule $X$ depends on the choice of the bimodule by definition. Since the bimodule $Y$ is defined only by the $C^*$-algebra $O_X$ with the gauge action $\gamma$, the Perron–Frobenius operator $F_Y$ associated with the bimodule $Y$ is an invariant of the $C^*$-algebra $O_X$ with the gauge action $\gamma$ up to conjugacy and does not depend on the original bi-module $X$. Moreover (3) of the above proposition shows that, for a fixed $\beta$, the set $\text{Ext}(KMS_\beta(O_X, \gamma))$ of extreme $\beta$–KMS states for the gauge action $\gamma$ on $O_X$ and the set

$$
\{(F_Y(\tau \ | \ B))(1) \ | \ \tau \in \text{Ext}(KMS_\beta(O_X, \gamma))\}.
$$

are invariants of $C^*$-algebra $O_X$ with the gauge action $\gamma$. We do study the number $(F_Y(\tau \ | \ B))(1) = (F_X(\tau \ | \ A))(1)$ and more generally a sequence $((F^n_Y(\tau \ | \ B))(1))_n = ((F^n_X(\tau \ | \ A))(1))_n$ in the next section.

\section{4 Orbit Structure of Branched Points}

Since a rational function is analytic as a map of $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ and has a rigid nature, its behavior on the singularities determines the main property of the rational function. Therefore it is important to study the orbit structure of the branched points of a rational function.

\textbf{Definition} Let $R$ be a rational function with $N = \text{deg}R \geq 2$. We denote by $b_n(z) = \#(R^{-n}(z))$ the number of the $n$th backward orbit $R^{-n}(z)$ counted without multiplicity starting at $z \in \hat{\mathbb{C}}$. We define the associated sequence $b(z) := (b_n(z))_{n=0}^{\infty}$. If the backward orbit $\bigcup_{n=1}^{\infty} R^{-n}(z)$ has no intersection with the set $B_R$ of the branched points, then

$$
b(z) = (1, N, N^2, N^3, \ldots, N^n, \ldots).
$$

In general the sequence $b(z)$ measures the existence of branched points in the backward orbit $\bigcup_{n=1}^{\infty} R^{-n}(z)$ starting at $z$.

\textbf{Example} Let $R(z) = z^2$. Then $B_R = \{0, \infty\}$. Since $R^{-1}(0) = \{0\}$ and $R^{-1}(\infty) = \{\infty\}$, we have

$$
b(0) = (1, 1, 1, 1, 1, \ldots), \quad b(\infty) = (1, 1, 1, 1, 1, \ldots).
$$
Example  Let \( R(z) = z^2 + 1 \). Then \( B_R = \{ 0, \infty \} \). Since there exist no branched point in the backward orbit \( \bigcup_{n=1}^{\infty} R^{-n}(0) \) starting at 0 and \( R^{-1}(\infty) = \{ \infty \} \),
\[
b(0) = (1, 2, 4, 8, 16, \ldots, 2^n, \ldots), \quad b(\infty) = (1, 1, 1, 1, \ldots)
\]

Example  Let \( R(z) = z^2 - 1 \). Then \( B_R = \{ 0, \infty \} \). Since \( R(0) = -1, R(-1) = 0 \) and \( R^{-1}(\infty) = \{ \infty \} \), we have that
\[
b(0) = (1, 2, 3, 6, 11, \ldots) = (b_n(0))_n, \quad \text{where} \quad b_{2n}(0) = \frac{1 + 2^{2n+1}}{3},
\]
\[
b_{2n+1}(0) = \frac{2 + 2^{2n+2}}{3}, \quad b(\infty) = (1, 1, 1, 1, \ldots)
\]

Example  There exists a constant \( c \) with \( R(z) = z^2 + c \) such that \( R^3(0) = 0 \) and \( R(0) \neq 0, R^2(0) \neq 0 \). Then \( B_R = \{ 0, \infty \} \). We have that
\[
b(0) = (1, 2, 4, \ldots, 2^{m-1}, 2^m - 1, \ldots), \quad b(\infty) = (1, 1, 1, 1, \ldots)
\]

In fact, consider a sequence \((f_m)_m\) of real functions defined by
\[
f_{m+1}(x) = xf_m(x)^2 + 1, \quad f_1(x) = 1, \quad f_2(x) = x + 1
\]
Then \( f_m \) is a polynomial of degree \( 2^{m-1} - 1 \) and has a real root. Let
\[
c_m := \min\{ x \in \mathbb{R} \mid f_m(x) = 0 \}
\]
Then \( f_m(c_m) = 0 \). We shall show that
\[
c_2 = -1 > c_3 > \cdots > c_m > c_{m+1} > \cdots
\]
Since \( f_{m+1}(c_m) = c_m f_m(c_m)^2 + 1 = 1 > 0 \) and \( f_{m+1}(x) \to -\infty \) as \( x \to -\infty \), we have \( c_{m+1} < c_m \). For \( k = 1, 2, \ldots, m-1 \), we have \( f_k(c_m) \neq 0 \), because \( c_m < c_k \) and \( c_k := \min\{ x \in \mathbb{R} \mid f_k(x) = 0 \} \).

Define \( R(z) = z^2 + c \). Let \( g_n(c) \) be the constant term of \( n \)th iteration \( R^n \) of \( R \). Then \( g_{n+1}(c) = g_n(c)^2 + c \) and \( g_1(c) = c \). Then we have \( g_n(c) = c f_n(c) \) by induction. Fix a natural number \( m \geq 2 \) and let \( R(z) = z^2 + c_m \) in particular. Since \( g_n(c_m) \) is the constant term of \( n \)th iteration \( R^n \) of \( R \), \( R^n(0) = g_n(c_m) \). Then \( R^m(0) = g_m(c_m) = c_m f_m(c_m) = 0 \). But for \( k = 1, 2, \ldots, m-1 \), we have \( R^k(0) = g_k(c_m) = c_m f_k(c_m) \neq 0 \).
Remark The main theorem below shows that we can distinguish these examples of quadratic polynomials in terms of $C^*$-algebras with gauge action, which could not be distinguished in our previous paper [3] where we counted only the numbers of extreme $\beta$-KMS states.

**Theorem 4.1** Let $Q$ and $R$ be rational functions with the degrees at least two. Suppose that there exists an isomorphism $h : \mathcal{O}_Q(\hat{\mathbb{C}}) \to \mathcal{O}_R(\hat{\mathbb{C}})$ such that $\gamma_R = h \gamma_Q h^{-1}$, where $\gamma_Q$ and $\gamma_R$ are the associated gauge actions. Then their backward orbit structures given by the number of $n$th backward orbit starting at the branched points are the same, that is,

$$\{b(z) \mid z \in B_Q\} = \{b(z) \mid z \in B_R\}$$

**Proof** Suppose that there exists an isomorphism $h : \mathcal{O}_Q(\hat{\mathbb{C}}) \to \mathcal{O}_R(\hat{\mathbb{C}})$ such that $\gamma_R = h \gamma_Q h^{-1}$. Then the fixed point algebras by the gauge actions are isomorphic, which will be denoted by $B$. Moreover the 1-spectral subspaces are isomorphic as Hilbert bi-module over $B$, which are denoted by $Y$. We should be careful that the Hilbert bi-modules $X$ over the coefficient algebra $A = C(\hat{\mathbb{C}})$ are not necessarily isomorphic. Therefore we should investigate invariants in terms of the Hilbert bi-module $Y$ over $B$.

We also note that $\deg R = \deg Q =: N$, since the number of extreme $\beta$-KMS states $\text{Ext}(KMS_\beta(\mathcal{O}_X, \gamma))$ for the gauge action $\gamma$ on $\mathcal{O}_X$ is exactly $N = \deg R$ for $\beta > \deg R$, as in Theorem A in [3].

Fix $\beta > N = \deg R$. Put $F_{X,\beta} = e^{-\beta} F_X$. For a branched point $z \in B_R$, let $\delta_z$ be the Dirac measure on $\hat{\mathbb{C}}$ corresponding to one point $z$. Define a trace $\tau_{\beta,z}$ on $A = C(\hat{\mathbb{C}})$, by

$$\tau_{\beta,z} = m_{\beta,z} \sum_{k=0}^{\infty} F_{X,\beta}^k(\delta_z) = m_{\beta,z} \sum_{k=0}^{\infty} \frac{1}{e^{k\beta}} \sum_{x \in \mathbb{R}^{-k}(z)} \delta_x,$$

where $m_{\beta,z}$ is the normalized constant and given by

$$m_{\beta,z} = \left(\sum_{k=0}^{\infty} \frac{1}{e^{k\beta}} \sum_{x \in \mathbb{R}^{-k}(z)} 1\right)^{-1} = \left(\sum_{k=0}^{\infty} \frac{1}{e^{k\beta}} b_k(z)\right)^{-1}.$$

Let $E : \mathcal{O}_R(\hat{\mathbb{C}}) \to \mathcal{O}_R(\hat{\mathbb{C}})^\gamma$ be the conditional expectation on to the fixed point algebra $\mathcal{O}_R(\hat{\mathbb{C}})^\gamma$ by the gauge action defined by $E(T) = \int_T \gamma_z(T) dz$. By Theorem A in [3], there exists a unique $\beta$-KMS state $\varphi_{\beta,z}$ on $\mathcal{O}_R(\hat{\mathbb{C}})$ such that its restriction to $A = C(\hat{\mathbb{C}})$ is exactly $\tau_{\beta,z}$. Moreover the set of extreme $\beta$-KMS states has a bijective correspondence to the set $B_R$ of the branched points under the correspondence between $\varphi_{\beta,z}$ and $z \in B_R$. The state satisfies that $\varphi_{\beta,z} = \varphi_{\beta,z} \circ E$ and

$$\varphi_{\beta,z}(S_{x_1} S_{x_2} \ldots S_{x_n} S_{y_1} \ldots S_{y_2} S_{y_1}^*) = e^{-\beta} \tau_{\beta,z}((y_1 \otimes y_2 \ldots \otimes y_n \mid x_1 \otimes x_2 \ldots \otimes x_n)_A).$$
Consider the fixed point algebra $B = \mathcal{O}_{\mathbb{R}}(\mathbb{C})^\gamma$ by the gauge action $\gamma$. Let $F_Y : \text{Trace}(B) \to \text{Trace}(B)$ be the Perron–Frobenius operator associated with the 1-spectral subspace

$$Y = \{ y \in \mathcal{O}_X \mid \gamma_t(y) = ty \ (t \in \mathbb{T}) \}.$$

Define a sequence $c(z) = (c_n(z))_n$ by

$$c_n(z) = (F^n_{X,\beta}(\phi_{\beta,z}|B))(1) = (F^n_{X,\beta}(\tau_{\beta,z}))(1),$$

which depends only on $(\mathcal{O}_{\mathbb{R}}(\mathbb{C}), \mathbb{T}, \gamma_{\mathbb{R}})$ and $\phi_{\beta,z}$. Therefore the family $\{c(z) \mid z \in B_R\}$ of such sequences depends only on $(\mathcal{O}_{\mathbb{R}}(\mathbb{C}), \mathbb{T}, \gamma_{\mathbb{R}})$ up to conjugacy. To make the proof finished, it is enough to show that the number $b_n(z)$ of the $n$th backward orbit $R^{-n}(z)$ starting at $z \in B_R$ is described in terms of the sequence $c(z) = (c_n(z))_n$.

Since

$$\tau_{\beta,z} - F_{X,\beta}(\tau_{\beta,z}) = m_{\beta,z} \delta_z,$$

we have

$$1 - c_1(z) = (\tau_{\beta,z} - F_{X,\beta}(\tau_{\beta,z}))(1) = m_{\beta,z} \delta_z(1) = m_{\beta,z}.$$ 

In general, since

$$F^n_{X,\beta}(\tau_{\beta,z}) - F^{n+1}_{X,\beta}(\tau_{\beta,z}) = m_{\beta,z} F^n_{X,\beta}(\delta_z),$$

we have

$$c_n(z) - c_{n+1}(z) = (F^n_{X,\beta}(\tau_{\beta,z}) - F^{n+1}_{X,\beta}(\tau_{\beta,z}))(1) = m_{\beta,z} F^n_{X,\beta}(\delta_z)(1) = m_{\beta,z} b_n(z).$$

Hence

$$b_n(z) = \frac{c_n(z) - c_{n+1}(z)}{1 - c_1(z)}.$$

By a similar argument we have a theorem on the $C^*$-algebra $\mathcal{O}_{\mathbb{R}}(J_R)$ associated with a complex dynamical system $(R^n)_n$ restricted to the Julia set $J_R$.

**Theorem 4.2** Let $Q$ and $R$ be rational functions with the degrees at least two. Suppose that there exists an isomorphism $h : \mathcal{O}_Q(J_Q) \to \mathcal{O}_{\mathbb{R}}(J_R)$ such that $\gamma_R = h_\gamma Q h^{-1}$, where $\gamma_Q$ and $\gamma_R$ are the associated gauge actions. Then their backward orbit structures given by the number of $n$th backward orbit starting at the branched points on the Julia sets are the same, that is,

$$\{b(z) \mid z \in B_Q \cap J_Q\} = \{b(z) \mid z \in B_R \cap J_R\}.$$
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