A maximum principle related to volume growth and applications

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Abstract
In this paper, we derive a new form of maximum principle for smooth functions on a complete noncompact Riemannian manifold \( M \) for which there exists a bounded vector field \( X \) such that \( \langle \nabla f, X \rangle \geq 0 \) on \( M \) and \( \text{div} X \geq af \) outside a suitable compact subset of \( M \), for some constant \( a > 0 \), under the assumption that \( M \) has either polynomial or exponential volume growth. We then use it to obtain some Bernstein-type results for hypersurfaces immersed into a Riemannian manifold endowed with a Killing vector field, as well as to some results on the existence and size of minimal submanifolds immersed into a Riemannian manifold endowed with a conformal vector field.

Keywords Maximum principle · Riemannian manifolds · Volume growth · Bernstein-type results · Constant mean curvature hypersurfaces · Minimal submanifolds

Mathematics Subject Classification Primary 53C42 · Secondary 53B30 · 53C50 · 53Z05 · 83C99

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1 Introduction

Maximum principles appear naturally in differential geometry, due to the fact that many different geometric situations are analytically modeled by certain linear or quasilinear elliptic partial differential operators, for which several versions of maximum principles play a key role in the theory. See, for instance, [3] or [9] for two recent monographies on the topic corroborating this fact. In a recent paper of us [1], we derived a new form of maximum principle which is appropriate for controlling the behavior of a smooth vector field with nonnegative divergence on a complete noncompact Riemannian manifold, and which is the analogue of the simple fact that, on such a manifold, a nonnegative subharmonic function that vanishes at infinity actually vanishes identically (Theorem 2.2 in [1]).

In this paper, we derive a maximum principle for smooth functions $f$ on a complete non-compact Riemannian manifold $M$, assuming that there exists a bounded vector field $X$ on $M$ such that $(\nabla f, X) \geq 0$ on $M$ and $\text{div} X \geq af$ outside a suitable compact subset $K$ of $M$, for some constant $a > 0$, under the assumption that $M$ has either polynomial or exponential volume growth (see Theorem 2.1 below for the precise statement of the result). In Sect. 3, we present some first interesting applications of our maximum principle to Bernstein-type results for hypersurfaces immersed into a Riemannian manifold endowed with a Killing vector field, allowing us to extend some of the results of [1] to the case of bounded second fundamental form, replacing the behavior of the Gauss map of the hypersurface at infinity by an estimate on the size of the support function on $M$. See, for instance, Theorem 3.1, and its Corollaries 3.2 and 3.3, for the case of constant mean curvature hypersurfaces, and Theorem 3.5 for its generalization to the case of constant higher order mean curvature.

Finally, in Sect. 4 we apply our maximum principle to some results on the existence and size of minimal submanifolds immersed into a Riemannian manifold endowed with a conformal vector field (Theorem 4.2) and, in particular, into Riemannian warped products (Corollary 4.3). Yet more particularly, we prove, among other results, a generalization to warped product ambient spaces of the fact that there exists no complete noncompact minimal submanifold with image contained in an Euclidean ball and having polynomial volume growth (see item (a) in Corollary 4.4), and a generalization of the fact that the same happens for complete noncompact minimal submanifolds into the hyperbolic space with image contained in the open half-space bounded by a horosphere (see item (a) in Corollary 4.5).

2 The maximum principle

Let $M$ be a connected, oriented, complete noncompact Riemannian manifold. We denote by $B(p, t)$ the geodesic ball centered at $p$ and with radius $t$.

Given a continuous function $\sigma : (0, +\infty) \to (0, +\infty)$, we say that $M$ has volume growth like $\sigma(t)$ if there exists $p \in M$ such that

$$\text{vol}(B(p, t)) = O(\sigma(t))$$

as $t \to +\infty$, where vol denotes the Riemannian volume.

If $p, q \in M$ are at distance $d$ from each other, it is straightforward to check that

$$\frac{\text{vol}(B(p, t))}{\sigma(t)} \geq \frac{\text{vol}(B(q, t-d))}{\sigma(t-d)} \cdot \frac{\sigma(t-d)}{\sigma(t)}.$$

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Hence, the choice of \( p \) in the notion of volume growth is immaterial, so that, henceforth, we shall simply say that \( M \) has polynomial (resp. exponential) volume growth, according to the case.

For the statement of the coming result, we also recall that a vector field \( X \) on a complete Riemannian manifold \( M \) is complete provided its flow \( \{ \psi_t \} \) is globally defined, and that this is always the case if \( X \) is bounded on \( M \). Moreover, a subset \( \Omega \) of \( M \) is stable under the flow of \( X \) if \( \psi_t(\Omega) \subseteq \Omega \) for every \( t \geq 0 \). In particular, this also holds for \( \Omega = M \).

**Theorem 2.1**  Let \( M \) be a connected, oriented, complete noncompact Riemannian manifold, let \( X \in \mathfrak{X}(M) \) be a bounded vector field on \( M \), with \(|X| \leq c < +\infty \), and \( K \) be a (possibly empty) compact subset of \( M \) such that \( M \setminus K \) is stable under the flow of \( X \). Assume that \( f \in C^\infty(M) \) is such that \( (\nabla f, X) \geq 0 \) on \( M \) and \( \text{div} X \geq af \) on \( M \setminus K \), for some \( a > 0 \).

\( (a) \)  If \( M \) has polynomial volume growth, then \( f \leq 0 \) on \( M \setminus K \).
\( (b) \)  If \( M \) has exponential volume growth, say like \( e^B \), then \( f \leq \frac{cB}{a} \) on \( M \setminus K \).

**Proof**  Suppose that there is a \( p \in M \setminus K \) such that \( f(p) > 0 \), and choose \( \alpha \) and \( r \) satisfying \( 0 < \alpha < f(p) \) and \( B = B(p, r) \subseteq A_\alpha = \{ x \in M \setminus K ; f(x) > \alpha \} \).

Since \( |X| \) is bounded, the flow \( \psi_t \) of \( X \) is defined for every \( t \in \mathbb{R} \), whence we can define the smooth function \( \varphi : [0, +\infty) \to (0, +\infty) \) by letting

\[
\varphi(t) = \text{vol}(\psi_t(B)) = \int_{\psi_t(B)} dM = \int_B \psi_t^* dM.
\]

Since \( \overline{B} \) is compact, we can differentiate under the integral sign to obtain

\[
\varphi'(0) = \frac{d}{dt} \int_{t=0} \int_{\psi_t(B)} dM = \frac{d}{dt} \int_{t=0} \int_{\psi_t(B)} \psi_t^* (dM) = \int_{\psi_0(B)} \frac{d}{dt} \int_{t=0} \psi_t^* (dM) = \int_{\psi_0(B)} \text{div} X \, dM.
\]

(2.1)

Now,

\[
\frac{d}{dt} f(\psi_t(x)) = (\nabla f, X)_{\psi_t(x)} \geq 0,
\]

so that, for \( x \in A_\alpha \), we have

\[
f(\psi_t(x)) \geq f(\psi_0(x)) = f(x) > \alpha, \quad \forall \ t \geq 0.
\]

Since \( M \setminus K \) is stable under the flow of \( X \), we thus get \( \psi_t(A_\alpha) \subseteq A_\alpha \), for all \( t \geq 0 \).

The inequality \( \text{div} X \geq af \) on \( M \setminus K \), together with (2.1) and the fact that \( \psi_t(B) \subseteq \psi_t(A_\alpha) \subseteq A_\alpha \subseteq M \setminus K \), then give

\[
\varphi'(t) \geq \int_{\psi_t(B)} af \, dM > aa \int_{\psi_t(B)} dM = aa \varphi(t) \quad \text{(2.2)}
\]

for all \( t \geq 0 \). In particular, \( \varphi'(t) > 0 \) for all \( t \geq 0 \), whence \( \varphi(t) \geq \varphi(0) = \text{vol}(B) > 0 \) for all \( t \geq 0 \). Integrating the inequality \( \frac{\varphi'(t)}{\varphi(t)} > aa \) along the interval \([0, t]\), we obtain
\[ \varphi(t) > \text{vol}(B)e^{\alpha t}, \quad \forall \ t > 0. \tag{2.3} \]

Let \( d(x, \psi_t(x)) \) denote the Riemannian distance between \( x \) and \( \psi_t(x) \). Since \( |X| \leq c \), we get

\[ d(x, \psi_t(x)) \leq \int_0^t \left| \frac{d}{ds} \psi_s(x) \right| ds = \int_0^t |X(\psi_s(x))| ds \leq ct. \tag{2.4} \]

On the other hand, for every \( x \in B \) we have

\[ d(p, \psi_t(x)) \leq d(x, \psi_t(x)) + d(p, x) < ct + r. \]

In turn, this means that \( \psi_t(B) \subset B(p, ct + r) \) for every \( t \geq 0 \), and it follows from (2.3) that

\[ \text{vol}(B(p, ct + r)) \geq \text{vol}(\psi_t(B)) = \varphi(t) > \text{vol}(B)e^{\alpha t}, \quad \forall \ t \geq 0. \]

A linear change of variables thus gives some constant \( C > 0 \) such that

\[ \text{vol}(B(p, t)) > Ce^{\frac{\alpha}{r}t}, \quad \forall \ t \geq r. \tag{2.5} \]

If \( M \) has polynomial volume growth, then (2.5) cannot be true for every \( t \geq r \). Thus, our initial supposition that \( f(p) > 0 \) for some \( p \in M \setminus K \) leads to a contradiction, whence \( f \leq 0 \) on \( M \setminus K \).

On the other hand, if \( M \) has exponential volume growth, say like \( e^{\Theta t} \), and there exists \( p \in M \setminus K \) such that \( f(p) > \Theta \), then we start the previous reasoning by choosing the real number \( a \) satisfying \( \Theta < a < \dot{f}(p) \). Therefore, (2.5) cannot be true for every \( t \geq r \), which is a contradiction. Hence, \( f \leq \frac{\Theta}{a} \) on \( M \setminus K \). \( \square \)

**Remark 2.2** The hypothesis on the stability of \( M \setminus K \) under the flow of \( X \) is necessary. Indeed, with \( M = \mathbb{R}^2 \) one could take \( X = \nabla f \) with a smooth \( f \) such that \( f(x) = K_n(|x|) \) on \( |x| > 1 \), where \( K_n \) is the \( n \)-th modified Bessel function of the second kind, \( n = 0, 1, \ldots \). This function \( f \) is strictly positive, bounded, with bounded gradient and satisfies the inequality \( \Delta f = \text{div}X \geq f \) on \( \mathbb{R}^2 \setminus K \), with \( K = \overline{B(0;1)} \). Indeed, by a direct computation one has

\[ \nabla f(x) = K_n'(|x|) \frac{x}{|x|} \quad \text{and} \quad \Delta f(x) = K_n''(|x|) + \frac{K_n'(|x|)}{|x|} \]

outside \( B(0; 1) \), where the function \( K_n(t) > 0 \) satisfies the modified Bessel differential equation

\[ t^2y''(t) + ty'(t) - (t^2 + n^2)y(t) = 0. \]

It easily follows from here that, outside \( B(0; 1) \),

\[ \text{div}X(x) = \Delta f(x) = \left( 1 + \frac{n^2}{|x|^2} \right) K_n(|x|) \geq f(x). \]

Since \( \nabla f(x) = K_n'(|x|) \frac{x}{|x|} \) outside \( B(0; 1) \) and \( K_n' < 0 \), we note that \( \mathbb{R}^2 \setminus \overline{B(0;1)} \) is not stable under the flow of \( X = \nabla f \). We would like to thank professors S. Pigola and A. Setti for calling our attention to this example.
3 Bernstein-type results for hypersurfaces

Let \( \overline{M}^{n+1} \) be an oriented Riemannian manifold endowed with a Killing vector field \( Y \). Let also \( \varphi : M^n \to \overline{M}^{n+1} \) be an isometric immersion of a connected, orientable, complete noncompact Riemannian manifold \( M^n \) into \( \overline{M} \), and orient \( M \) by the choice of a globally defined unit normal vector field \( N \).

In this section, we will apply item (a) of Theorem 2.1 to study the behavior of \( \varphi \). Our aim is to extend some of the results of [1] to the case of bounded second fundamental form, replacing the behavior of \( N \) at infinity by a suitable estimate on the size of the support function \( \eta = \langle N, Y \rangle \) on \( M \).

We start by computing the gradient \( \nabla \eta \) of \( \eta \). To this end, we let \( A(\cdot) = -\nabla(\cdot)N \) stand for the Weingarten operator of \( \varphi \) with respect to \( N \), fix a point \( p \) on \( M \) and a vector \( v \in T_p M \). Then, Killing’s equation, together with the symmetry of \( A \), give at \( p \)

\[
\langle \nabla \eta, v \rangle = v(\eta) = \langle \nabla_v N, Y \rangle + \langle N, \nabla_v Y \rangle = -\langle A_p v, Y \rangle - \langle \nabla_N Y, v \rangle = \langle -A_p Y^\top - \nabla_N Y, v \rangle, \tag{3.1}
\]

where \( Y^\top \) denotes the orthogonal projection of \( Y|_M \) onto \( TM \). We now observe, thanks again to Killing’s equation, that \( \langle \nabla_N Y, N \rangle = 0 \). Therefore, \( \nabla_N Y \) is tangent to \( M \), and the above computation gives

\[
\nabla \eta = -A Y^\top - \nabla_N Y. \tag{3.2}
\]

If the Killing vector field \( Y \) has unit norm, then Cauchy-Schwarz inequality shows that \( \eta \leq 1 \). Moreover, equality holds on all of \( M \) if and only if \( N = Y \) along \( M \), in which case \( M \) is a leaf of the distribution \( \langle Y \rangle^\perp \). Also in this case, for \( p \in M \) and \( u, v \in T_p M \), the Killing condition of \( Y \) allows us to compute, at \( p \),

\[
\langle A_p u, v \rangle = -\langle \nabla_u N, v \rangle = -\langle \nabla_u Y, v \rangle = \langle \nabla_v Y, u \rangle = -\langle A_p v, u \rangle.
\]

Since \( A \) is symmetric, this implies \( A_p = 0 \) and, since \( p \) was arbitrarily chosen, \( M \) is totally geodesic in \( \overline{M} \).

As a final preliminary, we say (cf. [5] or [6], for instance) that \( Y \) is a canonical direction for \( \varphi \) (or for \( M \), whenever \( \varphi \) is clear from the context) if \( Y^\top \) is a principal direction of \( A \).

We can state and prove our first result, which goes as follows.

\textbf{Theorem 3.1} Let \( \overline{M}^{n+1} \) be an oriented Einstein Riemannian manifold endowed with a Killing vector field \( Y \) of unit norm. Let \( \varphi : M^n \to \overline{M}^{n+1} \) be an isometric immersion of a connected, orientable, complete noncompact Riemannian manifold \( M^n \) into \( \overline{M} \). Orient \( M \) by the choice of a globally defined unit normal vector field \( N \), and assume that \( M \) has cmc and bounded second fundamental form \( A \). If \( M \) has polynomial volume growth, \( Y \) is either parallel or a canonical direction for \( \varphi \) and the support function \( \eta = \langle N, Y \rangle \) satisfies

\[
\eta \geq \frac{1}{|A|^2 + 1} \tag{3.2}
\]

on \( M \), then \( M \) is a leaf of the distribution \( \langle Y \rangle^\perp \). In particular, \( M \) is totally geodesic in \( \overline{M} \).
Proof As before, we let $\eta = \langle N, Y \rangle$ be the support function with respect to $Y$, let $Y^\top$ denote the orthogonal projection of $Y|_M$ onto $TM$ and set $X = AY^\top$ and $f = 1 - \eta$.

If $Y$ is parallel, then (3.1) readily gives $\nabla \eta = -AY^\top = -X$. If $Y$ is a canonical direction for $\varphi$, say with $AY^\top = \lambda Y^\top$, then (3.1), together with the fact that $\overline{\nabla}_N Y$ has no orthogonal component and $|Y| = 1$, give

$$\langle \nabla \eta, X \rangle = \langle X - \overline{\nabla}_N Y, X \rangle = |X|^2 - \langle \overline{\nabla}_N Y, \lambda Y^\top \rangle = -|X|^2 - \lambda \langle \overline{\nabla}_N Y, Y \rangle = -|X|^2.$$

Thus, in each of the cases above, we have

$$\langle \nabla f, X \rangle = -\langle \nabla \eta, X \rangle = |X|^2 \geq 0.$$

On the other hand, as computed in [1] for any Killing vector field $Y$,

$$\text{div}_M(X) = -\text{Ric}_M(Y^\top, N) + Y^\top(nH) + \eta|A|^2,$$

where $H = \frac{1}{n} \text{tr}(A)$ stands for the mean curvature of $\varphi$ and $\text{Ric}_M$ for the Ricci tensor of $\overline{M}$. Nevertheless, since $H$ is constant and $\overline{M}$ is Einstein, we get

$$\text{div}_M(X) = \eta|A|^2.$$

Hence, (3.2) is equivalent to $\text{div}_M(X) \geq f$ on $M$.

The boundedness of $A$ and the fact that $|Y| = 1$ imply the boundedness of $X$. Since $M$ has polynomial volume growth, Theorem 2.1 implies $f \leq 0$ on $M$. However, since $f \geq 0$, we conclude that $f \equiv 0$ on $M$; hence, $N \equiv Y$ on $M$, which is, then, a leaf of $(Y)^\perp$. \qed

For the following corollaries, recall that a Riemannian group is a Lie group $G$ endowed with a biinvariant metric. In this case, it is a well-known fact that the elements of the Lie algebra $\mathfrak{g}$ of $G$ are Killing vector fields of constant norm, and those in the center of $\mathfrak{g}$ are parallel. If the biinvariant metric of $G$ is Einstein, then of course we can let $\overline{M} = G$ in the previous result, thus getting the following

Corollary 3.2 Let $G^{n+1}$ be an Einstein Riemannian Lie group with Lie algebra $\mathfrak{g}$, and $\varphi : M^n \to G^{n+1}$ be a connected, orientable, complete noncompact hypersurface of $G$, oriented by the choice of a unit normal vector field $N$. Assume that $M$ is of cmc and that its second fundamental form $A$ with respect to $N$ is bounded. Assume further that there exists a nontrivial element $Y \in \mathfrak{g}$ which is either in the center of $\mathfrak{g}$ or is a canonical direction for $\varphi$. If $M$ has polynomial volume growth and the support function $(N, Y)$ satisfies (3.2) on $M$, then $M$ is a lateral class of a codimension one Lie subgroup of $G$.

Proof The previous result assures that $M$ is a leaf of $(Y)^\perp$ and, as such, is totally geodesic in $G$. Since the distribution $(Y)^\perp$ is generated by left invariant vector fields and $M$ is a leaf of it, we conclude that $(Y)^\perp$ is integrable, hence, a codimension one Lie subalgebra of $\mathfrak{g}$. Therefore, the connectedness of $M$ guarantees that it coincides with a lateral class of a Lie subgroup of $G$. \qed

The next corollary specializes the former to cmc hypersurfaces of $\mathbb{R}^{n+1}$. It can be seen as a partial extension of a famous result of Schoen, Simon and Yau (cf. [11] or [12]) to the case of bounded scalar curvature.
Corollary 3.3 Let $M^n$ be a connected, orientable, complete noncompact Riemannian manifold of bounded scalar curvature and polynomial volume growth. Assume that $\varphi : M \to \mathbb{R}^{n+1}$ is a cmc immersion, and let $N$ be a unit normal vector field along $M$. If there exists a unit vector $Y \in \mathbb{R}^{n+1}$ such that the support function $\langle N, Y \rangle$ satisfies (3.2) on $M$, then $\varphi(M)$ is a hyperplane orthogonal to $Y$.

Proof If $A$ stands for the Weingarten operator relative to $N$, then Gauss’ equation gives $|A|^2 = n^2 H^2 - n(n - 1) R$, where $R$ is the scalar curvature of $M$ and $H$ is the mean curvature of $\varphi$ with respect to $N$. Therefore, the boundedness of $R$ implies that of $A$, and it suffices to apply the previous corollary to $\varphi(M)$. 

We now extend Theorem 3.1 to the case of higher-order mean curvatures, and to this end we need to recall a few facts concerning these objects.

In the sequel, $\varphi : M^n \to M^{n+1}$ stands for an isometric immersion from a connected, orientable Riemannian manifold $M^n$ into an oriented Riemannian manifold $\overline{M}$. We orient $M$ by the choice of a globally defined unit normal vector field $N$, and let $A$ denote the corresponding second fundamental form.

Following section 3 of [2], one defines the $r$-th Newton transformation $T_r : \mathfrak{X}(M) \to \mathfrak{X}(M)$ recursively by letting

$$T_0 = I \quad \text{and} \quad T_r = S_r I - AT_{r-1}, \quad 1 \leq r \leq n,$$

where $I$ denotes the identity in $\mathfrak{X}(M)$ and $S_r(p)$ the $r$-th elementary symmetric sum of the eigenvalues of $A_p$, for every $p \in M$.

An easy induction shows that each

$$T_r = S_r I - S_{r-1} A + \cdots + (-1)^{r-1} S_1 A^{r-1} + (-1)^r A^r.$$

In particular, $T_n = p_A(A)$, where $p_A$ is the characteristic polynomial of $A$; hence, $T_n = 0$ by Cayley-Hamilton theorem.

Since $A$ is self-adjoint and $T_r$ is a polynomial in $A$, every base which diagonalizes $A$ also diagonalizes $T_r$, and using this fact one can establish, for $1 \leq r \leq n$, the standard formulas

$$\begin{align*}
\text{tr}(T_r) &= (n - r) S_r, \\
\text{tr}(AT_r) &= (r + 1) S_{r+1}, \\
\text{tr}(A^2 T_{r-1}) &= S_1 S_r - (r + 1) S_{r+1},
\end{align*}$$

(3.3)

where $\text{tr}(\cdot)$ stands for the trace of the linear operator within parentheses. In particular, $r = 1$ in the first and third formulas above yields

$$\begin{align*}
\text{tr}(T_1) &= (n - 1) S_1 = n(n - 1) H \quad \text{and} \quad |A|^2 = \text{tr}(A^2) = S_1^2 - 2 S_2.
\end{align*}$$

Given a Killing vector field $Y$ on $\overline{M}$, and letting (as before) $Y^\perp$ denote the orthogonal projection of $Y|_M$ onto $TM$, one can compute for $0 \leq r \leq n$ (cf. formula (8.4) of [2])

$$\text{div}_M(T_r Y^\perp) = \langle \text{div}_M T_r, Y \rangle + \text{tr}(AT_r) \langle N, Y \rangle.$$

(3.4)

Here, $\text{div}_M T_r$, the divergence of $T_r$, is the vector field on $M$ defined by

$$\text{div}_M T_r = \text{tr}(\nabla T_r).$$
One can show (cf. Lemma 3.1 of [2], for instance) that, if \{e_1, \ldots, e_n\} is a local orthonormal frame on \(M\) and \(V \in \mathfrak{X}(M)\), then

\[
\langle \text{div}_M T_r, V \rangle = \sum_{j=1}^r \sum_{i=1}^n \langle \tilde{R}(N, T_{r-j}e_i)e_i, A^{r-1}V \rangle,
\]

where \(\tilde{R}\) is the curvature operator of \(\tilde{M}\). In particular, if \(\tilde{M}\) has constant sectional curvature, then this formula readily shows that \(\text{div}_M T_r = 0\) on \(M\).

We now need the following

**Lemma 3.4** In the notations above, if \(\tilde{M}\) has constant sectional curvature and \(Y\) is a Killing vector field on \(\tilde{M}\), then, for \(1 \leq r \leq n\), we have

\[
\text{div}_M(AT_{r-1}Y^\top) = \langle \nabla S_r, Y^\top \rangle + \text{tr}(A^2T_{r-1})(N, Y).
\]

**Proof** Since \(AT_{r-1} = S_rI - T_r\), we can compute

\[
\text{div}_M(AT_{r-1}Y^\top) = \text{div}_M(S_rY^\top) - \text{div}_M(T_rY^\top) = \langle \nabla S_r, Y^\top \rangle + S_r\text{div}_M(Y^\top) - \text{div}_M(T_rY^\top).
\]

Now, taking into account that \(\text{div}_M T_r = 0\) and substituting (3.4) and (3.3), we obtain

\[
\text{div}_M(AT_{r-1}Y^\top) = \langle \nabla S_r, Y^\top \rangle + S_r\text{tr}(A)\langle N, Y \rangle - \text{tr}(AT_r)\langle N, Y \rangle = \langle \nabla S_r, Y^\top \rangle + (S_rS_r - (r + 1)S_{r+1})\langle N, Y \rangle = \langle \nabla S_r, Y^\top \rangle + \text{tr}(A^2T_{r-1})\langle N, Y \rangle.
\]

\(\square\)

We are now ready to generalize Theorem 3.1.

**Theorem 3.5** Let \(\tilde{M}^{n+1}\) be an oriented Riemannian manifold with constant sectional curvature, endowed with a Killing vector field \(Y\) of unit norm. Let \(\varphi : M^n \to \tilde{M}^{n+1}\) be an isometric immersion of a connected, orientable, complete noncompact Riemannian manifold \(M^n\) into \(\tilde{M}\). Orient \(M\) by the choice of a globally defined unit normal vector field \(N\), and assume that the corresponding second fundamental form \(A\) is bounded. Assume further that \(T_{r-1}\) is nonnegative and \(\text{tr}(T_r)\) is constant on \(M\), for some \(1 \leq r < n\). If \(M\) has polynomial volume growth, \(Y\) is either parallel or a canonical direction for \(\varphi\) and the support function \(\eta = \langle N, Y \rangle\) satisfies

\[
\eta \geq \frac{1}{\text{tr}(A^2T_{r-1}) + 1}
\]

on \(M\), then \(M\) is a leaf of the distribution \(\langle Y \rangle\). In particular, \(M\) is totally geodesic in \(\tilde{M}\).

**Proof** Once again we let \(\eta = \langle N, Y \rangle\) and \(f = 1 - \eta\), so that \(f \geq 0\), with equality if and only if \(N = Y\) along \(M\). We also set \(X = AT_{r-1}Y^\top\).

As in the proof of Theorem 3.1, if \(Y\) is parallel, then \(\nabla \eta = -AY^\top\). If \(Y\) is a canonical direction for \(\varphi\), then the fact that \(T_{r-1}\) is a polynomial in \(A\) assures that \(AT_{r-1}Y^\top = \mu Y^\top\) for some function \(\mu\) on \(M\); then (3.1) yields...
\[ \langle \nabla \eta, X \rangle = \langle -AY^T - \nabla_N Y, X \rangle \]
\[ = -\langle AY^T, A T_{r-1} Y^T \rangle - \langle \nabla_N Y, \mu Y^T \rangle \]
\[ = -\langle A^2 T_{r-1} Y^T, Y^T \rangle - \mu \langle \nabla_N Y, Y^T \rangle \]
\[ = -\langle A^2 T_{r-1} Y^T, Y^T \rangle, \]

where, in the last equality above, we used the fact that \( \nabla N Y \) has no orthogonal component and \( |Y| = 1 \). Thus, in each of the cases above, we have

\[ \langle \nabla f, X \rangle = -\langle \nabla \eta, X \rangle = \langle A^2 T_{r-1} Y^T, Y^T \rangle. \]

Since \( T_{r-1} \) is nonnegative and self-adjoint, it has a square root \( Q_{r-1} \) which also commutes with \( A \). Hence, \( Q_{r-1} \) is self-adjoint and the last computation above provides

\[ \langle \nabla f, X \rangle = \langle (AQ_{r-1})^2 Y^T, Y^T \rangle = |AQ_{r-1} Y^T|^2 \geq 0. \]

On the other hand, since \( S_r \) is constant, the previous lemma gives

\[ \text{div}_M(X) = \text{tr}(A^2 T_{r-1})\langle N, Y \rangle. \]

Hence, (3.5) is equivalent to \( \text{div}_M(X) \geq f \) on \( M \).

The boundedness of \( A \) on \( M \) implies that of \( A T_{r-1} \); this, together with the fact that \( |Y| = 1 \), give the boundedness of \( X \) on \( M \). Since \( M \) has polynomial volume growth, Theorem 2.1 implies \( f \leq 0 \) on \( M \). The rest of the proof thus goes as in the proof of Theorem 3.1. \( \square \)

### 4 On the existence and size of minimal submanifolds

Along all of this section, unless stated otherwise, \( \overline{M}^m \) stands for a Riemannian manifold with metric tensor \( \overline{g} = \langle \cdot, \cdot \rangle \) and Levi-Civita connection \( \nabla \).

We recall that a vector field \( Y \in \mathfrak{X}(\overline{M}) \) is conformal with conformal factor \( \phi \in C^\infty(\overline{M}) \) provided \( \mathcal{L}_Y \overline{g} = 2\phi \overline{g} \), where \( \mathcal{L}_Y \) stands for the Lie derivative in the direction of \( Y \). If this is so, it is straightforward to verify that \( \langle \nabla_{\overline{Z}} Y, Z \rangle = \phi \langle Z, Z \rangle \), for all \( Z \in \mathfrak{X}(\overline{M}) \). Then, the divergence of \( Y \) on \( \overline{M} \) is given by \( \text{div}_\overline{M}(Y) = m\phi \).

We shall need the following

**Lemma 4.1** Let \( \overline{M}^m \) be a Riemannian manifold with metric tensor \( \overline{g} = \langle \cdot, \cdot \rangle \) and \( Y \in \mathfrak{X}(\overline{M}) \) be a conformal vector field with conformal factor \( \phi \). If \( \varphi : M^n \to \overline{M}^m \) is an isometric immersion and \( X = (Y_{\mid M})^T \) is the orthogonal projection of \( Y_{\mid M} \) into \( TM \), then

\[ \text{div}_M(X) = n(\varphi_{\mid M} + \langle Y_{\mid M}, \overline{H} \rangle), \]

(4.1)

where \( \overline{H} \) stands for the mean curvature vector of \( \varphi \).

**Proof** We fix a local orthonormal frame field \( (e_1, \ldots, e_n) \) on an open set \( U \subset M \). Setting \( l = m - n \) and shrinking \( U \), if necessary, we can also consider an orthonormal frame field \( (N_1, \ldots, N_l) \) for \( TU^\perp \).
Writing $Y$ instead of $Y|_M$ for the sake of simplicity, we have $X = Y - \sum_{j=1}^{l} \langle Y, N_j \rangle N_j$ on $U$. Letting $V$ stand for the Levi-Civita connection of $M$, we get on $U$ that
\[
\text{div}_M(X) = \sum_{i=1}^{n} \langle \nabla_{e_i} X, e_i \rangle = \sum_{i=1}^{n} \langle \nabla_{e_i} Y, e_i \rangle
\]
\[
= \sum_{i=1}^{n} \langle \nabla_{e_i} Y, e_i \rangle - \sum_{i=1}^{n} \sum_{j=1}^{l} \langle \nabla_{e_i} \langle Y, N_j \rangle N_j, e_i \rangle
\]
\[
= n\phi - \sum_{j=1}^{l} \langle Y, N_j \rangle \sum_{i=1}^{n} \langle \nabla_{e_i} N_j, e_i \rangle.
\]

Denoting by $A_j$ the Weingarten operator of $\varphi$ in the direction of $N_j$, we can continue the computation above by writing
\[
\text{div}_M(X) = n\phi + \sum_{j=1}^{l} \langle Y, N_j \rangle \sum_{i=1}^{n} \langle A_j e_i, e_i \rangle
\]
\[
= n\phi + \sum_{j=1}^{l} \langle Y, N_j \rangle \text{tr}(A_j),
\]
where $\text{tr}(\cdot)$ stands for the trace of the operator within parentheses. Therefore,
\[
\text{div}_M(X) = n\phi + \left( \sum_{j=1}^{l} \text{tr}(A_j) N_j \right) = n\phi + \langle Y, n\overline{H} \rangle,
\]
as wished. \qed

In the sequel, if $B^k$ is another Riemannian manifold, $\pi : \overline{M}^m \to B^k$ is a Riemannian submersion and $Z \in \mathfrak{X}(B)$, we let $\tilde{Z}$ denote the horizontal lift of $Z$ to $\overline{M}$. Also, given a smooth function $h : B \to \mathbb{R}$, we shall write $\tilde{h}$ to denote the composition $\tilde{h} = h \circ \pi : \overline{M} \to \mathbb{R}$. Letting $Dh$ and $\nabla \tilde{h}$ denote the gradients of $h$ (on $B$) and $\tilde{h}$ (on $\overline{M}$), respectively, we have $\nabla \tilde{h} = \overline{D} h$.

If, in addition, $\varphi : M^m \to \overline{M}^m$ is an isometric immersion, then $\varphi$ can be locally seen as the inclusion. Therefore, if $h$ is as above and there is no danger of confusion, we set $f = \tilde{h} \circ \varphi := \tilde{h}|_M$ (look at the diagram below).

If $\nabla f$ denotes the gradient of $f$ on $M$, then
\[
\nabla f = (\nabla \tilde{h})^\top = (\overline{D} h)^\top,
\]
(4.2)
the orthogonal projection, onto \( TM \), of the restriction of \( \nabla h \) to \( T_M \).

We now assume that there exist a smooth function \( g : B \to [0, +\infty) \) and a vector field \( Y \in \mathfrak{X}(M) \) such that \( \nabla h = \bar{g} Y \). Then, with \( f \) as above and \( X = (Y|_M)^T \), (4.2) gives \( \nabla f = \bar{g}|_M X \), whence

\[
\langle \nabla f, X \rangle = \bar{g}|_M |X|^2 \geq 0. \tag{4.3}
\]

Moreover, if \( Y \) is conformal with conformal factor \( \phi \) and \( a \) is a positive constant, then it follows from (4.1) that

\[
\text{div}_M(X) \geq af \iff n(\phi|_M + \langle Y|_M, \bar{H} \rangle) \geq a\bar{h}|_M. \tag{4.4}
\]

In particular, this is automatically true if \( \varphi \) is minimal and \( n\varphi \geq \bar{h} \) along \( \varphi(M) \).

We can now state and prove our main results.

**Theorem 4.2** Let \( \pi : \bar{M}^m \to B^k \) be a Riemannian submersion and \( Y \in \mathfrak{X}(\bar{M}) \) be conformal with conformal factor \( \phi \). Assume that there exist smooth functions \( g : B \to \mathbb{R} \) and \( h : B \to (0, +\infty) \) such that \( \nabla h = \bar{g} Y \). Let \( \varphi : M^m \to \bar{M}^m \) be an isometric immersion from an oriented, complete noncompact Riemannian manifold \( M \) into \( \bar{M} \), such that \( \bar{g} \geq 0 \), \( n\varphi \geq \bar{h} \) and \(|Y| \leq c \) on \( \varphi(M) \), for some positive constants \( a \) and \( c \).

(a) If \( M \) has polynomial volume growth, then \( \varphi \) cannot be minimal.

(b) If \( M \) has exponential volume growth, say like \( e^{\beta t} \), and \( \varphi \) is minimal, then \( \bar{h} \leq \frac{\beta}{a} \) on \( \varphi(M) \).

**Proof** We have done almost all of the work along the previous discussion: with \( f = \bar{h}|_M \) and \( X = (Y|_M)^T \), we have \(|X| \leq |Y| \leq c \) and, from (4.3), \( \langle \nabla f, X \rangle \geq 0 \) on \( M \). Moreover, if \( \varphi \) is minimal, then (4.4) and our hypotheses give \( \text{div}_M(X) \geq af \) on all of \( M \). We now consider cases (a) and (b) separately:

(a) Theorem 2.1 ascertains that \( f \leq 0 \) along \( M \), which is a contradiction, for \( h \) is positive on \( B \).

(b) This follows immediately from Theorem 2.1.

\( \square \)

We now specialize the previous result in the following way: we let \( \Sigma^{m-1} \) be a Riemannian manifold with metric \( d\sigma^2 \) and \( I \subset \mathbb{R} \) be an open interval with its standard metric \( dt^2 \). We set \( \bar{M}^m = \Sigma^{m-1} \times I \) and let \( \pi_\Sigma : \bar{M} \to \Sigma \) and \( \pi_I : \bar{M} \to I \) denote the standard projections. If \( h : I \to (0, +\infty) \) is a smooth function and \( \bar{h} = h \circ \pi_I : \bar{M} \to (0, +\infty) \), then \( \bar{g} = \bar{h}^2 \pi_\Sigma^* d\sigma^2 + \pi_I^* dt^2 \) is a metric tensor on \( \bar{M} \), with respect to which \( \bar{M} \) is said to be the warped product of \( \Sigma \) and \( I \), with warping function \( h \). We summarize the above by writing \( \bar{M} = \Sigma \times_h I \), and note that \( \pi_I : \bar{M} \to I \) is a Riemannian submersion.

Let \( \partial \bar{g} \) denote the canonical vector field on \( I \) and \( \bar{\partial} \) be its horizontal lift to \( \bar{M} \). It is a standard fact that \( Y = \bar{h} \bar{\partial} \) is a conformal vector field with conformal factor \( \phi = \bar{h} \phi \). Moreover, if \( g = \frac{\bar{h}}{h} \), then

\[
\nabla \bar{h} = \bar{h} \bar{\partial} \bar{g} \bar{h} = \bar{g} Y. \tag{4.5}
\]
with $\tilde{g} = g \circ \pi_1 : \overline{M} \to \mathbb{R}$.

We are thus left with the following

**Corollary 4.3** Let $\overline{M} = \Sigma \times h$ be a warped product as above. Let $\varphi : M^n \to \overline{M}^m$ be an isometric immersion from an oriented, complete noncompact Riemannian manifold $M$ into $\overline{M}$, such that $\tilde{h} \leq \frac{a}{\alpha} \tilde{h}'$ and $\tilde{h} \leq c$ on $\varphi(M)$, for some positive constants $a$ and $c$.

(a) If $M$ has polynomial volume growth, then $\varphi$ cannot be minimal.
(b) If $M$ has exponential volume growth, say like $e^{\beta t}$, and $\varphi$ is minimal, then $\beta \geq a$.

**Proof** Setting $Y = \tilde{h} \partial_t$ and $g = \frac{\tilde{h}'}{\tilde{h}}$, we already know that $Y$ is conformal, with conformal factor $\phi = \tilde{h}'$, and (4.5) gives $\nabla \tilde{h} = \tilde{g} Y$. Moreover, our hypotheses assure that $\tilde{g} \geq \frac{a}{n} > 0$, $n\phi - a \tilde{h} \geq 0$ and $|Y| \leq c$ along $\varphi(M)$. Item (a) is now a particular case of the previous result.

As for item (b), we conclude from the previous result that $\tilde{h} \leq \frac{a}{\alpha} \tilde{h}$ on $\varphi(M)$. Since $\tilde{h} > 0$, one has to have $\beta > 0$. We can then apply item (b) again, this time with $\frac{\beta}{\alpha}$ in place of $c$, to conclude that $\tilde{h} \leq c \left( \frac{\beta}{\alpha} \right)^l$ on $\varphi(M)$. By iterating this reasoning, we therefore conclude that $\tilde{h} \leq c \left( \frac{\beta}{\alpha} \right)^l$ on $\varphi(M)$, for every $l \geq 1$. If $0 < \beta < a$, then $0 < \frac{\beta}{\alpha} < 1$ and, letting $l \to +\infty$, we conclude that $\tilde{h} \leq 0$ on $\varphi(M)$, which is impossible. Hence, $\beta \geq a$. \hfill $\square$

We close this section with the following interesting particular cases of the previous result.

**Corollary 4.4** Let $(\Sigma, d\sigma^2)$ be an $(m-1)$-dimensional Riemannian manifold and let $\overline{M} = \Sigma \times t, (0, +\infty)$ be the corresponding warped product, with warping function $h(t) = t$. Let $\varphi : M^n \to \overline{M}$ be an isometric immersion from an oriented, complete noncompact Riemannian manifold $M$ into $\overline{M}$, such that $\varphi(M) \subset \Sigma \times (0, R)$, for some $R > 0$.

(a) If $M$ has polynomial volume growth, then $\varphi$ cannot be minimal.
(b) If $M$ has exponential volume growth, say like $e^{\beta t}$, and $\varphi$ is minimal, then $R \geq \frac{a}{\beta}$.

**Proof** Observe that, in the notations of the statement of Corollary 4.3, we have $\tilde{h} = t$, $\tilde{h}' = 1$, so that $\tilde{h} \leq \frac{a}{\alpha} \tilde{h}'$ and $\tilde{h} \leq c$ on $\varphi(M)$, with $a = \frac{n}{R}$ and $c = R$. Thus, the former corollary gives item (a), as well as, in item (b), $\beta \geq \frac{a}{R}$, which is the same as $R \geq \frac{a}{\beta}$. \hfill $\square$

Corollary 4.4 applies in particular to the case of bounded isometric immersions into the Euclidean space. Actually, let $\varphi : M^n \to \mathbb{H}^m$ be an isometric immersion from an oriented, complete noncompact Riemannian manifold $M$ into the Euclidean $m$-space, such that $\varphi(M) \subset B^{\mathbb{R}^m}(0, R)$, for some $R > 0$. Taking any $S > R$ and composing $\varphi$ with a translation (which depends on the chosen $S$), we can assume that

$$\varphi(M) \subset B^{\mathbb{R}^m}(0, S)$$

and $0 \not\in \varphi(M)$. We now look at $\mathbb{R}^m \setminus \{0\}$ as the warped product

$$\mathbb{R}^m \setminus \{0\} = \mathbb{S}^{m-1} \times_\tau (0, +\infty),$$

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and observe that the isometric immersion

\[ \varphi : M^n \to \mathbb{R}^m \setminus \{0\} = \mathbb{S}^{m-1} \times \{(0, +\infty)\} \]

satisfies \( \varphi(M) \subset \mathbb{S}^{m-1} \times (0, S) \). Therefore, by Corollary 4.4 with \( (\Sigma, d\sigma^2) \) the unit Euclidean \((m - 1)\)-sphere, we know that if \( M \) has polynomial volume growth then \( \varphi \) cannot be minimal, whereas if \( M \) has exponential volume growth, say like \( e^{\beta t} \), and \( \varphi \) is minimal, then \( S \geq \frac{n}{\beta} \). And since this holds for every \( S > R \), we finally get \( R \geq \frac{n}{\beta} \).

In this context, lower mean curvature estimates for bounded isometric immersions are very well known and valid under the assumption of stochastic completeness. See for instance [3, Proposition 5.2]. At this point, it is worth pointing out that stochastic completeness holds on complete manifolds with appropriate bounds on the volume growth. For instance \([7]\) (see also \[8, Theorem 9.1\]) Grigor’yan proved that stochastic completeness holds on every complete manifold \( M \) satisfying

\[ \int_0^\infty \frac{t}{\log V(t)} \, dt = \infty, \]

where \( V(t) \) denotes the volume of a geodesic ball of radius \( t \) with a fixed center. For example, this condition is satisfied if \( M \) has quadratic exponential volume growth, say like \( e^{\theta t^2} \), so much more general than what is assumed in our Corollary 4.4. On the other side, one of the advantages of our result is that it applies to the case of general warped product ambient spaces with warping function \( h(t) = t \) and with no geometric assumptions on the Riemannian fiber \( (\Sigma, d\sigma^2) \).

In this direction, when the warping function is \( h(t) = e^t \) we obtain the following.

**Corollary 4.5** Let \( (\Sigma, d\sigma^2) \) be an \((m - 1)\)-dimensional Riemannian manifold and let \( \tilde{M} = \Sigma \times e \mathbb{R} \) be the corresponding warped product, with warping function \( h(t) = e^t \). Let \( \varphi : M^n \to \tilde{M} \) be an isometric immersion from an oriented, complete noncompact Riemannian manifold \( M \) into \( \tilde{M} \), such that \( \varphi(M) \subset \Sigma \times (-\infty, b) \), for some \( b \in \mathbb{R} \).

(a) If \( M \) has polynomial volume growth, then \( \varphi \) cannot be minimal.

(b) If \( M \) has exponential volume growth, say like \( e^{\beta t} \), and \( \varphi \) is minimal, then \( \beta \geq n \).

**Proof** In this case \( \tilde{h}(t) = \tilde{h}'(t) = e^t \) and, in the notations of the statement of the Corollary 4.3, we have \( \tilde{h} \leq n \tilde{h}' \) on \( \tilde{M} \) with \( a = n \). Moreover, the condition that \( \varphi(M) \) is contained in the open half-space \( \Sigma \times (-\infty, b) \) for some \( b \in \mathbb{R} \) is equivalent to the fact that \( \tilde{h} \leq c \) on \( \varphi(M) \) for certain positive \( c \). Thus, the result follows directly from Corollary 4.3. \( \square \)

In particular, by choosing \((\Sigma, d\sigma^2)\) as the Euclidean \((m - 1)\)-space, we can look at the hyperbolic \( m \)-space as the warped product

\[ \mathbb{H}^m = \mathbb{R}^{m-1} \times e \mathbb{R}, \]

where \( t \) is the standard coordinate function on \( \mathbb{R} \) and the \( t \)-slices \( \mathbb{R}^{m-1} \times \{t\} \) are horospheres. Thus, Corollary 4.5 holds in particular for isometric immersions inside horoballs of \( \mathbb{H}^m \). Similarly to the previous result, lower mean curvature estimates under properness or stochastic completeness are known also for isometric immersions inside horoballs (see for instance \[10\] and, more recently, \[4\]). Once again, one of the advantages of our result is
that it applies not only to isometric immersions into the hyperbolic space but, more gener-
ally, to isometric immersions into general warped product ambient spaces with warping
function $h(t) = e^t$ and with no geometric assumptions on the Riemannian fiber $(\Sigma, d\sigma^2)$.

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