MULTIPLICITY RESULTS FOR DISCRETE ANISOTROPIC EQUATIONS

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Abstract. In this article we continue the study of discrete anisotropic equations and we will provide a new multiplicity results of the solutions for a discrete anisotropic equation. The procedure viewed here is according to variational methods and critical point theory. In fact, using a consequence of the local minimum theorem due Bonanno and mountain pass theorem we look into the existence results for our problem under algebraic conditions with the classical Ambrosetti-Rabinowitz (AR) condition on the nonlinear term. Furthermore, for mingling two algebraic conditions on the nonlinear term employing two consequences of the local minimum theorem due Bonanno we guarantee the existence of two solutions, applying the mountain pass theorem given by Pucci and Serrin we establish the existence of third solution for our problem.

1. Introduction. In this paper we are mainly concerned with existence and multiplicity results for the following discrete problem

\[
\begin{aligned}
&\Delta(\alpha(k)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = \lambda f(k, u(k)), \quad k \in [1, T], \\
u(0) = u(T + 1) = 0
\end{aligned}
\]  

\( (P_f^\lambda) \)

where \( T \geq 2 \) is a fixed positive integer number, \([1, T]\) is the discrete interval \( \{1, \ldots, T\} \subset \mathbb{N} \), \( u(k) \in \mathbb{R} \) for all \( k \in [1, T] \), \( \Delta u(k-1) = u(k) - u(k - 1) \) is the forward difference operator, \( \alpha : [1, T + 1] \rightarrow (0, +\infty) \) and \( p : [0, T] \rightarrow (1, +\infty) \) are some fixed functions; \( f : [1, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function. Let

\[
p^- = \min_{k \in [0,T]} p(k) , \quad p^+ = \max_{k \in [0,T]} p(k)
\]

and

\[
\alpha^- = \min_{k \in [1,T+1]} \alpha(k) , \quad \alpha^+ = \max_{k \in [1,T+1]} \alpha(k).
\]

Discrete boundary value problems and nonlinear difference equations emerge from real world problems and are claimed to be employed as handy means for the description of the processes which are endowed with discrete intervals. Indeed,
common among many fields of research such as computer science, mechanical engineering, control systems, artificial or biological neural networks, and economics, is the fact that the mathematical modelling of fundamental questions is usually tended towards considering discrete boundary value problems and nonlinear difference equations. Regarding these issues, a thoroughgoing overview has been given in, as an example, the monograph [28] and the reference therein. On the other hand, in recent years some researchers have studied the existence and multiplicity of solutions for equations involving the discrete $p$-Laplacian operator by using various fixed point theorems, lower and upper solutions method, critical point theory and variational methods, Morse theory and the mountain-pass theorem. For background and recent results, we refer the reader to [2, 3, 8, 10, 11, 12, 24, 27, 29, 30, 31, 36, 43, 45] and the references therein.

Anisotropic operators appear in several places in the literature. Recent relevant applications include models in physics [6, 14, 22, 23], biology [4, 5] and image processing (see, for instance, the monograph by Weickert [44]). We also refer to Fragala, Gazzola and Kawohl [16] and El Hamidi and Vétois [15] as basic references in the treatment of nonlinear anisotropic problems. Moreover, note that Mihailescu et al. (see [33, 34]) were the first authors to study anisotropic elliptic problems with variable exponent, and after that, in recent years, a great deal of work has been done in the study of the existence of solutions for discrete anisotropic boundary value problems (BVPs). For background and recent results, we refer the reader to [18, 20, 21, 35, 37, 42] and the references therein.

Our approach is variational method and the main tools are a local minimum theorem for differentiable functionals due Bonanno [7] and Mountain Pass Theorem. Two of the consequences of the local minimum theorem are here applied (see Theorems 2.1 and 2.2). Indeed, we investigate existence results for the problem $(P_{\lambda}^f)$ under some algebraic conditions with the classical Ambrosetti-Rabinowitz (AR) condition on the nonlinear term (see [1]). Moreover, by combining two algebraic conditions on the nonlinear term employing two consequences of the local minimum theorem due Bonanno we guarantee the existence of two solutions, applying the mountain pass theorem given by Pucci and Serrin ([38]) we establish the existence of third solution for the problem $(P_{\lambda}^f)$. For some related results we would also like to mention [7].

Here, we state two special cases of our results.

**Theorem 1.1.** Let $p > 2$, $g : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function and $g(0) \neq 0$. Assume that $\lim_{\xi \to 0^+} \frac{g(\xi)}{\xi^{1/p}} = +\infty$ and

$(AR)$ there exist constants $\nu > p$ and $R > 0$ such that, for all $\xi \geq R$,

$$0 < \nu \int_0^\xi g(\zeta) d\zeta \leq \xi g(\xi).$$

Then, for each $\lambda \in \left(0, p^2 T^{p-1} \sup_{\gamma > 0} \gamma^p \int_0^\gamma \frac{g(\zeta)}{g(\xi)} d\zeta \right)$, the problem

$$\begin{cases}
-\Delta(|\Delta u(k-1)|^{p-2} \Delta u(k-1)) = \lambda g(u(k)), & k \in [1, T], \\
u(0) = u(T) = 0
\end{cases}$$

admits at least two positive solutions.
Theorem 1.2. Let \( p > 2, g : \mathbb{R} \to \mathbb{R} \) be a non-negative continuous function and \( g(0) \neq 0 \). Assume that \( \lim_{\xi \to 0^+} \frac{g(\xi)}{\xi^{p-1}} = +\infty, \lim_{\xi \to +\infty} \frac{g(\xi)}{\xi^{p-1}} = 0 \) and

\[
\int_0^1 g(t)dt < \frac{1}{2pT^2} \int_0^2 g(t)dt.
\]

Then for each \( \lambda \in \left( \frac{2pT^{p-1}}{\int_0^1 g(t)dt}, \frac{1}{pT^{p-1}} \int_0^1 g(x)dx \right) \), the problem (1) admits at least three positive solutions.

For a through research on the subject, we also refer the reader to [13, 25, 26].

2. Preliminaries. Our main tools include the following theorems, consequences of the local minimum theorem due Bonanno [7, Theorem 3.1], which is in turn motivated by Ricceri’s variational principle (see [41]), and is related to the celebrated three critical point theorem of Pucci and Serrin [38, 39].

For a given non-empty set \( X \), and two functionals \( \Phi, \Psi : X \to \mathbb{R} \), we define the following functions

\[
\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}(r_1, r_2)} \sup_{u \in \Phi^{-1}(r_1, r_2)} \frac{\Psi(u) - \Psi(v)}{r_2 - \Phi(v)},
\]

\[
\rho_1(r_1, r_2) = \sup_{v \in \Phi^{-1}(r_1, r_2)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(-\infty, r_1]} \Psi(u)}{\Phi(v) - r_1}
\]

for all \( r_1, r_2 \in \mathbb{R}, r_1 < r_2 \), and

\[
\rho_2(r) = \sup_{v \in \Phi^{-1}(r, +\infty)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(-\infty, r]} \Psi(u)}{\Phi(v) - r}
\]

for all \( r \in \mathbb{R} \). We would like to provide some finite dimensional counterparts of abstract results from [7]. These read as follows.

**Theorem 2.1** ([7, Theorem 5.1]). Let \( X \) be a real finite dimensional Banach space; \( \Phi : X \to \mathbb{R} \) be \( C^1 \) functional whose derivative admits a continuous inverse on \( X^* \), \( \Psi : X \to \mathbb{R} \) be a \( C^1 \) functional. Assume that there are \( r_1, r_2 \in \mathbb{R}, r_1 < r_2 \), such that \( \beta(r_1, r_2) < \rho_1(r_1, r_2) \). Then, setting \( I_\lambda := \Phi - \lambda \Psi \), for each \( \lambda \in \left( \frac{1}{\rho_1(r_1, r_2)}, \frac{1}{\rho_2(r_1, r_2)} \right) \) there is \( u_{0, \lambda} \in \Phi^{-1}(r_1, r_2) \) such that \( I_\lambda(u_{0, \lambda}) \leq I_\lambda(u) \) \( \forall u \in \Phi^{-1}(r_1, r_2) \) and \( I_\lambda'(u_{0, \lambda}) = 0 \).

**Theorem 2.2** ([7, Theorem 5.3]). Let \( X \) be a real finite dimensional Banach space; \( \Phi : X \to \mathbb{R} \) be \( C^1 \) functional whose derivative admits a continuous inverse on \( X^* \), \( \Psi : X \to \mathbb{R} \) be a \( C^1 \) functional. Fix \( \inf_X \Phi < r < \sup_X \Phi \) and assume that \( \rho_2(r) > 0 \), and for each \( \lambda > \frac{1}{\rho_2(r)} \), the functional \( I_\lambda := \Phi - \lambda \Psi \) is coercive. Then, for each \( \lambda \in \left( \frac{1}{\rho_2(r)}, +\infty \right) \) there is \( u_{0, \lambda} \in \Phi^{-1}(r, +\infty) \) such that \( I_\lambda(u_{0, \lambda}) \leq I_\lambda(u) \) \( \forall u \in \Phi^{-1}(r, +\infty) \) and \( I_\lambda'(u_{0, \lambda}) = 0 \).

Here and in the sequel, we take the \( T \)-dimensional Hilbert space

\[ X := \{ u : [0, T+1] \to \mathbb{R} : u(0) = u(T+1) = 0 \} \]

endowed with the norm \( \| u \| := \left( \sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{\frac{1}{2}} \).

Put \( F(k, t) := \int_0^t f(k, \xi) d\xi \) for all \( (k, t) \in [1, T] \times \mathbb{R} \).

**Remark 1.** We recall that a map \( f : [1, T] \times \mathbb{R} \to \mathbb{R} \) is continuous if for each \( k \in [1, T] \) function \( t \to f(k, t) \) is continuous.
We say that a vector \( u \in X \) is a solution of the problem \((P^I_X)\) if and only if
\[
\sum_{k=1}^{T+1} \alpha(k)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1) - \lambda \sum_{k=1}^{T} f(k, u(k))v(k) = 0
\]
holds for all \( v \in X \).

Now for every \( u \in X \), we define
\[
\Phi(u) = \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} \quad (2)
\]
and
\[
\Psi(u) = \sum_{k=1}^{T} F(k, u(k)). \quad (3)
\]

Standard arguments show that \( \Phi \) and \( \Psi \) are \( C^1 \) functionals whose derivatives at the point \( u \in X \) are given by \( \Phi'(u)(v) = \sum_{k=1}^{T+1} \alpha(k)\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1) \) and \( \Psi'(u)(v) = \sum_{k=1}^{T} f(k, u(k))v(k) \) for all \( u, v \in X \), respectively. Hence, a critical point of the functional \( \Phi - \lambda \Psi \), gives us a solution of \((P^I_X)\).

We need the following proposition in the proofs of our main results.

**Proposition 1.** Let \( J : X \to X^* \) be the operator defined by
\[
J(u)(v) = \sum_{k=1}^{T+1} \alpha(k)|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1)
\]
for every \( u, v \in X \). Then, \( J \) admits a continuous inverse on \( X^* \).

**Proof.** Assume that \( ||u|| > 1 \), by [19, Section 2(A.1)] we have
\[
J(u)(u) = \sum_{k=1}^{T+1} \alpha(k)|\Delta u(k-1)|^{p(k-1)} \geq \alpha^+ \left(T^{-\frac{p^*}{p^*}}||u||^{p^*} - 1\right).
\]
Since \( p^- > 1 \), this follows that \( J \) is coercive. Following Lemma 4.2 in [32] we obtain that if \( p \geq 2 \), then for \( c_p = \frac{2}{p(p^* - 1)} \)
\[
(|x|^{p^*} - |y|^{p^*})^2 \geq c_p |x - y|^2 \text{ for all } x, y \in \mathbb{R}.
\]
Therefore
\[
\langle J(u) - J(v), u - v \rangle \geq \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)(2p(k-1)-1)}|\Delta u(k-1) - \Delta v(k-1)|^{p(k-1)} > 0
\]
for every \( u, v \in X, u \neq v \), which means that \( J \) is strictly monotone. Since it is also coercive and continuous and the setting is finite dimensional Theorem 2.2 from [17] states that \( J \) is invertible with continuous inverse. \( \square \)

**3. Main results.** In this section, we formulate our basic conclusions. For a non-negative constant \( \gamma \) and a constant \( \delta > 1 \) with \( \gamma p^- \neq \frac{2\alpha^+p^+T^{p^-}\delta^{p^-}}{\alpha^-p^-} \), we set
\[
a_{\gamma}(\delta) := \frac{\sum_{k=1}^{T} \sup_{t \leq \gamma} F(k, t) - \sum_{k=1}^{T} F(k, \delta)}{\alpha^-p^-\gamma p^- - 2\alpha^+p^+T^{p^-}\delta^{p^-}}.
\]
Theorem 3.1. Assume that $p^+ \geq 2$, $f(k,0) \neq 0$ for all $k \in [1,T]$ and suppose that there exist a non-negative constant $\gamma_1$ and two positive constants $\gamma_2$ and $\delta$, with

$$\frac{\gamma_1}{T^\frac{p^-}{p^+} \sqrt{2}} < \delta < \frac{p^+ - \alpha^-}{2 \alpha^+ p^+ T^\frac{p^-}{p^+} \gamma_2^+}$$

such that

1. $a_{\gamma_2}(\delta) < a_{\gamma_1}(\delta)$;
2. there exist $\nu > 2$ and $R > 0$ such that

$$0 < \nu F(k,\xi) \leq \xi f(k,\xi)$$

for all $|\xi| \geq R$ and for all $k \in [1,T]$.

Then, for each $\lambda \in \left( \frac{1}{p^+ - p^-}, \frac{1}{p^+ - p^-} \right)$, the problem $(P'_\lambda)$ admits at least two non-trivial solutions $u_1$ and $u_2$ in $X$, such that

$$\frac{\alpha^-}{p^+ - p^-} \gamma_1^- \leq \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)}|\Delta u_1(k-1)|^{p(k-1)} \leq \frac{\alpha^-}{p^+ - p^-} \gamma_2^-.$$

Moreover, there exists at least one another solution $u_3$.

Proof. Put $I_{\lambda} = \Phi - \lambda \Psi$, where $\Phi$ and $\Psi$ are given as in (2) and (3), respectively. Of course $\Phi$ is $C^1$ functional while Proposition 1 gives that its Gâteaux derivative admits a continuous inverse on $X^*$. The functional $\Psi : X \to \mathbb{R}$ is well defined and is continuously Gâteaux differentiable whose Gâteaux derivative is compact. Choose $r_1 = \frac{\alpha^-}{p^+ - p^-} \gamma_1^-$, $r_2 = \frac{\alpha^-}{p^+ - p^-} \gamma_2^-$ and for $\delta > 1$, we define $w_\delta \in X$ by

$$w_\delta(k) = \begin{cases} \delta & \text{for } k \in [1,T] \\ 0 & \text{for } k = 0, T + 1. \end{cases}$$

Thus $\|w_\delta\|^2 = \sum_{k=1}^{T+1} |\Delta w_\delta(k-1)|^2 = 2\delta^2$ and $\frac{2\alpha^-}{p^+ - p^-} \delta^{p^-} \leq \Phi(w_\delta) \leq 2\alpha^- \delta^{p^+}$. From the condition (4), we obtain $r_1 < \Phi(w_\delta) < r_2$. Moreover, for all $u \in X$ with $\Phi(u) \leq r_1$, from the definition of $\Phi$ one has

$$\Phi^{-1}(-\infty, r_1) = \{u \in X ; \Phi(u) < r_1\} \subseteq \{u \in X ; |u(k)| \leq \gamma_1, \text{ for all } k \in [0, T + 1]\}.$$

Indeed, if $u \in X$ and $\Phi(u) < r_1$, one has $\sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} < r_1$. Then, $|\Delta u(k-1)| < \left( \frac{p(k-1)}{\alpha(k)} r_1^{1/p(k-1)} \right)^{1/p(k-1)} \leq \left( \frac{p^+}{\alpha^- r_1} \right)^{1/p^-}$ for every $k \in [1,T + 1]$. Consequently, since $u \in X$ we deduce by easy induction that

$$|u(k)| \leq |\Delta u(k-1)| + |u(k-1)| < \left( \frac{p^+}{\alpha^- r_1} \right)^{1/p^-} + |u(k-1)| \leq k \left( \frac{p^+}{\alpha^- r_1} \right)^{1/p^-} \leq T \left( \frac{p^+}{\alpha^- r_1} \right)^{1/p^-} = \gamma_1$$

for every $k \in [1,T]$. From this, $\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u) \leq \sum_{k=1}^{T} \sup_{|t| \leq \gamma_1} F(k,t)$. Now, arguing as above, we obtain $\sup_{u \in \Phi^{-1}(-\infty, r_2)} \Psi(u) \leq \sum_{k=1}^{T} \sup_{|t| \leq \gamma_2} F(k,t)$. 


Therefore, by \((A_1)\) one has
\[
\beta(r_1, r_2) \leq \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2)} \Psi(u) - \Psi(w_\delta)}{r_2 - \Phi(w_\delta)}
\]
\[
\leq \frac{1}{p^+ p^- T^p} \left( \sum_{k=1}^{\infty} \sup_{|t| \leq \gamma_2} F(k, t) - \sum_{k=1}^{T} F(k, w_\delta(k)) \right)
\leq \frac{1}{p^+ p^- T^p} \left( \sum_{k=1}^{\infty} \sup_{|t| \leq \gamma_2} F(k, t) - \sum_{k=1}^{T} F(k, \delta) \right)
\leq \frac{1}{p^+ p^- T^p} a_{\gamma_2}(\delta).
\]

On the other hand, one has
\[
\rho_1(r_1, r_2) \geq \frac{\Psi(w_\delta) - \sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}{\Phi(w_\delta) - r_1}
\]
\[
\geq \frac{1}{p^+ p^- T^p} \left( \sum_{k=1}^{\infty} F(k, w_\delta(k)) - \sum_{k=1}^{T} F(k, t) \right)
\geq \frac{1}{p^+ p^- T^p} \left( \sum_{k=1}^{\infty} F(k, \delta) - \sum_{k=1}^{T} F(k, t) \right)
\geq \frac{1}{p^+ p^- T^p} a_{\gamma_1}(\delta).
\]

Hence, from \((A_1)\), one has \(\beta(r_1, r_2) < \rho_2(r_1, r_2)\). Therefore, from Theorem 2.1, for each \(\lambda \in \left( \frac{1}{p^+ p^- T^p a_{\gamma_1}(\delta)}, \frac{1}{p^+ p^- T^p a_{\gamma_2}(\delta)} \right)\), the functional \(I_\lambda\) admits at least one non-trivial critical point \(u_1\) such that \(r_1 < \Phi(u_1) < r_2\), that is, \(\frac{\alpha_a p^+}{p^+ p^-} \gamma_1 \leq \sum_{k=1}^{T+1} a(k) T^{p(k-1)} |\Delta u_1(k-1)|^{p(k-1)} \leq \frac{\alpha_a p^+}{p^+ p^-} \gamma_2\). Now, we prove the existence of the second local minimum distinct from the first one. To this goal, we verify the hypotheses of the mountain-pass theorem for the functional \(I_\lambda\). Clearly, the functional \(I_\lambda\) is of class \(C^1\) and \(I_\lambda(0) = 0\). The first part of the proof guarantees that \(u_1 \in X\) is a local nontrivial local minimum for \(I_\lambda\) in \(X\). We can assume that \(u_1\) is a strict local minimum for \(I_\lambda\) in \(X\). Therefore, there is \(\rho > 0\) such that \(\inf_{\|u-u_1\| = \rho} I_\lambda(u) > I_\lambda(u_1)\), so the condition [40, (I_1), Theorem 2.2] is verified. By integrating the condition (5), there exist constants \(a_1, a_2 > 0\) such that \(F(k, t) \geq a_1 |t|^{p^+} - a_2\) for all \(k \in [1, T]\) and \(t \in \mathbb{R}\). Now, choosing any \(u \in X \setminus \{0\}\), by the assumption \(p^+ > 2\) and [19, Section 2(A.4)] one has
\[
I_\lambda(u) = (\Phi - \lambda \Psi)(u)
\]
\[
\leq \frac{2p^+ (T+1) \alpha}{p^-} \left( C_{p^+} \|u\|^{p^+} + 1 \right) - \lambda \sum_{k=1}^{T} F(k, u(k))
\]
\[
\leq \frac{2p^+ (T+1) \alpha}{p^-} \left( C_{p^+} \|u\|^{p^+} + 1 \right) - \lambda \alpha_1 \zeta \|u\|^{p^+} + \lambda a_2 \rightarrow -\infty
\]
as \(\|u\| \rightarrow +\infty\), so condition [40, (I_2), Theorem 2.2] is satisfied; here \(\zeta\) is some positive constant that for any \(u \in X\), \(\sqrt{T \sum_{k=1}^{T} |u(k)|^{p^+}} \geq \zeta \|u\|\). Thus, the functional \(I_\lambda\) satisfies the mountain pass geometry. Moreover, \(I_\lambda\) satisfies the Palais-Smale condition since it is anti-coercive. Hence, by the classical theorem of Ambrosetti.
and Rabinowitz we establish a critical point \( u_2 \) of \( I_\lambda \) such that \( I_\lambda(u_2) > I_\lambda(u_1) \). Since \( f(k,0) \neq 0 \) for all \( k \in [1,T] \), \( u_1 \) and \( u_2 \) are two distinct non-trivial solutions of \( (P^3_\lambda) \). Moreover, by direct maximization we obtain the third solution \( u_3 \) which is an argument of a maximum. Not that \( I_\lambda(u_3) \geq I_\lambda(u_2) > I_\lambda(u_1) \). If equality \( I_\lambda(u_3) = I_\lambda(u_2) \) holds, then we obtain in fact infinitely many solutions on the level \( I_\lambda(u_3) = I_\lambda(u_2) \). The proof is complete. \( \square \)

**Remark 2.** In Theorem 3.1 we have guaranteed the existence of at least two nontrivial weak solutions for \( (P^1_\lambda) \). One of these solutions has been achieved in relation with the classical Ambrosetti-Rabinowitz condition on the data by taking \( f(k,0) \neq 0 \) for all \( k \in [1,T] \). If the condition \( f(k,0) \neq 0 \) for all \( k \in [1,T] \) does not hold, the second solution \( u_2 \) of the problem \( (P^1_\lambda) \) may be trivial.

Now, we point out an immediate consequence of Theorem 3.1.

**Theorem 3.2.** Assume that \( p^+ > 2 \), \( f(k,0) \neq 0 \) for all \( k \in [1,T] \) and there exist two positive constants \( \delta \) and \( \gamma \), with \( \delta < \sqrt{\frac{p^- - \alpha^+}{2\alpha^+ p^+ T^{p^-}}} \) such that the assumption \((A_2)\) in Theorem 3.1 holds. Furthermore, suppose that

\[
\sum_{k=1}^{T} \sup_{|t| \leq \gamma} F(k,t) < \frac{\alpha^+}{\alpha^- T^{p^-}} \sum_{k=1}^{T} F(k,\delta).
\]

Then, for each \( \lambda \in \left( \frac{2\alpha^+ \delta^+}{p^- \sum_{k=1}^{T} F(k,\delta)}, \frac{\alpha^- \gamma^-}{\sum_{k=1}^{T} \sup_{|t| \leq \gamma} F(k,t)} \right) \), the problem \( (P^3_\lambda) \) admits at least two non-trivial solutions \( u_1 \) and \( u_2 \) in \( X \) such that

\[
0 < \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u_1(k-1)|^{p(k-1)} \leq \frac{\alpha^-}{T^{p^-} p^-} \gamma^-.
\]

Moreover, there exists at least one another solution \( u_3 \).

**Proof.** The conclusion follows from Theorem 3.1, by taking \( \gamma_1 = 0 \) and \( \gamma_2 = \gamma \). Indeed, owing to the inequality (7), one has

\[
a_\gamma(\delta) = \sum_{k=1}^{T} \sup_{|t| \leq \gamma} F(k,t) - \frac{\sum_{k=1}^{T} F(k,\delta)}{\alpha^+ T^{p^-} \delta^+} < \frac{\sum_{k=1}^{T} F(k,t)}{\alpha^- T^{p^-} \delta^+} = a_0(\delta).
\]

In particular, one has \( a_\gamma(\delta) < \frac{\sum_{k=1}^{T} \sup_{|t| \leq \gamma} F(k,t)}{\alpha^- T^{p^-} \delta^+} \), which follows

\[
\frac{\alpha^-}{p^+ T^{p^-} \sum_{k=1}^{T} \sup_{|t| \leq \gamma} F(k,t)} < \frac{1}{p^+ T^{p^-}} \frac{1}{a_{\gamma_2}(\delta)}.
\]

Hence, Theorem 3.1 ensures the conclusion. \( \square \)

Now we illustrate Theorem 3.2 by presenting the following example.

**Example 1.** Let \( T = 10, p(k) = k^2 + 10 \) for all \( k \in [0,10] \), \( \alpha(k) = 1 + \frac{1}{k} \) for all \( k \in [1,11] \), \( f(k,t) = e^k (1 + \tanh^2(t)) \) for all \( (k,t) \in [1,10] \times \mathbb{R} \). Thus, \( p^+ = 110, p^- = 10, \alpha^- = 1 \) and \( \alpha^+ = 2 \). Moreover, \( f \) is a continuous and a non-negative function, \( f(k,0) \neq 0 \) for all \( k \in [1,10] \), \( F(k,t) = e^k (2t - \tanh t) \) for all \( (k,t) \in [1,10] \times \mathbb{R} \).
Theorem 2.2. Moreover, for \( I \) to function \( \Phi \) and \( \Psi \) fulfill, by choosing \( \delta \) and \( \gamma \) with \( \delta < \sqrt{\frac{p^-}{2\alpha^+ p^+ T^p}} \) and \( \gamma <\sum_{k=1}^{T} \sup_{|t| \leq \gamma} F(k, t) < \sum_{k=1}^{T} F(k, \delta) \) and

\[
\limsup_{|\xi| \rightarrow +\infty} \frac{F(k, \xi)}{|\xi|^p} \leq 0 \quad \text{uniformly in } \mathbb{R}. \tag{8}
\]

Then, for each \( \lambda > \tilde{\lambda} \), where \( \tilde{\lambda} := \frac{1}{p^+ p^- T^p} \sum_{k=1}^{T} F(k, 1) \sum_{k=1}^{T} \sup_{|t| \leq \gamma} F(k, t) \), the problem \( (P^I_{\lambda}) \) admits at least one non-trivial solution \( u_1 \in X \) such that

\[
\sum_{k=1}^{T+1} \frac{\alpha(k)}{\rho(k-1)} |\Delta u_1(k-1)|^{p(k-1)} > \frac{\alpha^-}{T p^+ p^-} \gamma p^-. \]

Proof. Take the real Banach space \( X \) as defined in Section 2, and put \( I_{\lambda} = \Phi - \lambda \Psi \) where \( \Phi \) and \( \Psi \) are given as in (2) and (3), respectively. Our goal is to apply Theorem 2.2 to function \( I_{\lambda} \). The functionals \( \Phi \) and \( \Psi \) satisfy all assumptions requested in Theorem 2.2. Moreover, for \( \lambda > 0 \), the functional \( I_{\lambda} \) is coercive. Indeed, fix \( 0 < \varepsilon < \frac{\alpha^-}{p^+ (T+1)^p \sqrt{T^p}} \). From (8) there is a function \( \rho_\varepsilon : [1, T] \rightarrow \mathbb{R} \) such that

\[
\sum_{k=1}^{T} \rho_\varepsilon(k) \leq \infty \quad \text{and} \quad F(k, t) \leq \varepsilon t^p + \rho_\varepsilon(k), \quad \text{for every } k \in [1, T] \text{ and } t \in \mathbb{R}. \]

Taking \( p^- \geq 2 \), parts (A.1) and (A.3) in [19, Section 2] into account, it follows that, for each \( u \in X \) with \( \|u\| \geq 1 \),

\[
\Phi(u) - \lambda \Psi(u) \geq \frac{\alpha^-}{p^+} \left( T^{2 - \frac{p^-}{p^+}} \|u\|^{p^-} - T \right) - \lambda \varepsilon T (T + 1)^p \|u\|^{p^-} - \lambda \sum_{k=1}^{T} \rho_\varepsilon(k) \geq \left( \frac{T^{2 - \frac{p^-}{p^+}}}{p^+} - \lambda T (T+1)^p \varepsilon \right) \|u\|^{p^-} - \frac{\alpha^- T}{p^+} - \lambda \sum_{k=1}^{T} \rho_\varepsilon(k),
\]

and thus \( \lim_{\|u\| \rightarrow +\infty} (\Phi(u) - \lambda \Psi(u)) = +\infty \), which means the functional \( I_{\lambda} = \Phi - \lambda \Psi \) is coercive. Put \( \bar{\rho} = \frac{\alpha^-}{T p^+ p^-} \gamma p^- \). Arguing as in the proof of Theorem 3.1, we obtain
that \( \rho_2(\mathcal{F}) \geq \frac{1}{p^+ p^- r^+} \sum_{k=1}^{T+1} \sup_{a, b} F(k, t) - \frac{1}{2a^+ p^+ T^+ \delta^+} \). Hence, from our assumption it follows that \( \rho_2(\mathcal{F}) > 0 \). Therefore, from Theorem 2.2 for each \( \lambda > \hat{\lambda} \), the functional \( I_\lambda \) admits at least one local minimum \( \bar{u}_1 \) such that

\[
\sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta \bar{u}_1(k-1)|^{p(k-1)} > \frac{\alpha^-}{T^+ p^- \gamma^+}.
\]

The conclusion is achieved. \( \square \)

**Remark 3.** If \( f \) is non-negative then the solutions ensured in Theorems 3.1, 3.2 and 3.3 is non-negative. Here we reason exactly as in [20] supposing the solution is not positive and arriving at contradiction.

Now, we point out some results in which the function \( f \) has separated variables. To be precise, consider the following problem

\[
\begin{cases}
-\Delta(\alpha(k)|u(k-1)|^{p(k-1)-2}u(k-1)) = \lambda \theta_k g(u(k)), & k \in [1, T], \\
u(0) = u(T + 1) = 0
\end{cases}
\]

(\( P^\varphi_\lambda \))

where \( \theta : [1, T] \to \mathbb{R} \) is a non-negative and non-zero function and \( \theta_k := \theta(k) \) for all \( k \in [1, T] \) such that \( \sum_{k=1}^{T} \theta_k < \infty \) and \( g : \mathbb{R} \to \mathbb{R} \) is a non-negative continuous function.

Put \( G(x) = \int_{0}^{x} g(\xi) d\xi \) for all \( x \in \mathbb{R} \), Set \( f(k, t) = \theta(k) g(t) \) for every \( (k, t) \in [1, T] \times \mathbb{R} \) and put \( \Theta := \sum_{k=1}^{T} \theta_k \).

The following existence results are consequences of Theorems 3.1–3.3, respectively.

**Theorem 3.4.** Assume that \( p^+ > 2 \), \( g(0) \neq 0 \) and there exist a non-negative constant \( \gamma_1 \) and two positive constants \( \gamma_2 \) and \( \delta \), with \( \frac{\gamma_1}{T^+ \sqrt{2}} < \delta < \frac{p^- \alpha^-}{2 (T^+ p^+ T^- \gamma^+)} \), such that \( \frac{G(\gamma_1) - G(\delta)}{\alpha^- p^- \gamma^+ - 2a^+ p^+ T^+ \delta^+} < \frac{G(\gamma_2) - G(\delta)}{\alpha^- p^- \gamma^+ - 2a^+ p^+ T^+ \delta^+} \). Furthermore, suppose that there exist constants \( \nu > p^+ \) and \( R > 0 \) such that for all \( \xi \geq R \)

\[
0 < \nu G(\xi) \leq \xi g(\xi).
\]

(9)

Then, for each \( \lambda \in [\lambda_1, \lambda_2] \), where \( \lambda_1 = \frac{1}{p^+ p^- r^+} \frac{\alpha^- p^- \gamma^+ - 2a^+ p^+ T^+ \delta^+}{G(\gamma_1) - G(\delta)} \) and \( \lambda_2 = \frac{1}{p^+ p^- r^+} \frac{\alpha^- p^- \gamma^+ - 2a^+ p^+ T^+ \delta^+}{G(\gamma_2) - G(\delta)} \), the problem \( (P^\varphi_\lambda) \) admits at least two positive solutions \( u_1 \) and \( u_2 \) in \( X \) such that

\[
\frac{\alpha^-}{T^+ p^- \gamma_1^+} \leq \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u_1(k-1)|^{p(k-1)} \leq \frac{\alpha^-}{T^+ p^- \gamma_2^+}.
\]

**Theorem 3.5.** Assume that \( p^+ > 2 \), \( g(0) \neq 0 \) and there exist two positive constants \( \delta \) and \( \gamma \), with \( \delta < \sqrt{\frac{p^- \alpha^-}{2a^+ p^+ T^+ \gamma^+}} \), such that

\[
G(\gamma) > \frac{\alpha^- p^- G(\delta)}{2a^+ p^+ T^+ \delta^+}.
\]

(10)

Furthermore, suppose that the assumption (9) holds. Then, for every \( \lambda \in \left( \frac{2a^+ \delta^+}{p^- \Theta G(\delta)}, \frac{\alpha^- \gamma^+}{p^+ T^+ \Theta G(\gamma)} \right) \),
the problem \((P_{\lambda}^{p,\theta})\) admits at least two positive solutions \(u_1\) and \(u_2\) in \(X\) such that
\[
\frac{\alpha^-}{P^+} \gamma^-_1 \leq \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u_1(k-1)|^{p(k-1)} \leq \frac{\alpha^-}{P^+} \gamma^-_2 .
\]

**Theorem 3.6.** Assume that \(p^- > 2\), there exist two positive constants \(\bar{\gamma}\) and \(\bar{\delta}\) with
\[
\bar{\delta} < \sqrt[p^-]{\frac{p^- \alpha^-}{2\alpha^+ P^+ P^+}} \bar{\gamma}^- \tag{11}
\]
such that
\[
G(\bar{\gamma}) < G(\bar{\delta}) \tag{12}
\]
and
\[
\limsup_{|\xi| \to +\infty} \frac{G(\xi)}{|\xi|^p} \leq 0 . \tag{13}
\]
Then, for each \(\lambda > \bar{\lambda}\), where \(\bar{\lambda} := \frac{1}{p^+ p^- \Theta G(\bar{\gamma}) - G(\bar{\delta})} \), the problem \((P_{\lambda}^{p,\theta})\) admits at least one positive solution \(\bar{u}_1 \in X\) such that
\[
\sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta \bar{u}_1(k-1)|^{p(k-1)} > \frac{\alpha^-}{P^+} \bar{\gamma}^0 .
\]

Now we illustrate Theorem 3.6 by presenting the following example.

**Example 2.** Let \(T = 3\), \(p(k) = k + 2\) for all \(k \in [0,3]\), \(\alpha(k) = \frac{1}{k}\) for all \(k \in [1,4]\), \(\theta(k) = 1\) for all \(k \in [1,4]\) and
\[
g(t) = \begin{cases} 
1 - t^2, & |t| \leq 1, \\
0, & |t| > 1.
\end{cases}
\]
Thus, \(p^- = 2\), \(p^+ = 5\), \(\alpha^- = \frac{1}{4}\), \(\alpha^+ = 1\) and \(\Theta = 4\). Moreover, \(g\) is a non-negative continuous function and \(g(0) \neq 0\). By the expression of \(g\) we have
\[
G(t) = \begin{cases} 
t - \frac{t^3}{3}, & |t| \leq 1, \\
0, & |t| > 1.
\end{cases}
\]
So, \(\limsup_{|\xi| \to +\infty} \frac{G(\xi)}{|\xi|^p} = \limsup_{|\xi| \to +\infty} \frac{G(\xi)}{|\xi|^p} = 0\). Now, by choosing \(\bar{\delta} = \frac{1}{2}\) and \(\bar{\gamma} = 6\sqrt{5}\) we clearly observe that (11) and (12) are satisfied. Indeed,
\[
\bar{\delta} = \frac{1}{2} < 1 = \sqrt[p^-]{\frac{p^- \alpha^-}{2\alpha^+ P^+ P^+}} \bar{\gamma}^- \tag{11}
\]
and \(G(\bar{\gamma}) = 0 < \frac{11}{24} = G(\bar{\delta})\). Therefore, by Theorem 3.6 for every \(\lambda > \frac{04}{176}\), the problem
\[
\begin{cases} 
-\Delta(\epsilon^k|\Delta u(k-1)|^{\ln(k+\epsilon^{-1})-1} \Delta u(k-1)) = \lambda \epsilon^k g(k), & k \in [1,3], \\
u(0) = u(4) = 0
\end{cases}
\]
has at least one positive solution \(\bar{u}_1\) in the space \(X_3 := \{u : [0,4] \to \mathbb{R} : u(0) = u(4) = 0\}\) such that \(\sum_{k=1}^{T+1} \frac{1}{k(k+1)} |\Delta \bar{u}_1(k-1)|^{k+1} > 1\).

A further consequence of Theorem 3.1 is the following existence result.
**Theorem 3.7.** Assume that $p^+ > 2$, $g(0) \neq 0$ and

$$\lim_{\xi \to 0^+} \frac{g(\xi)}{\xi^{p^- - 1}} = +\infty. \quad (14)$$

Furthermore, suppose that the assumption (9) holds. Then, for every $\lambda \in (0, \lambda^*_\gamma)$, where $\lambda^*_\gamma := \frac{\alpha^-}{p^+ - \Theta} \sup_{\gamma > 0} \frac{\gamma^{p^+}}{G(\gamma)}$, the problem $(P^{\Phi, \Psi}_X)$ admits at least two positive solutions in $X$.

**Proof.** Fix $\lambda \in [0, \lambda^*_\gamma]$. Then there is $\gamma > 0$ such that $\lambda < \frac{\alpha^-}{p^+ - \Theta} \sup_{\gamma > 0} \frac{\gamma^{p^+}}{G(\gamma)}$.

From (14) there exists a positive constant $\delta$ with $\delta < \frac{p^+ - \alpha^-}{\sqrt{2\alpha^+ + p^+ - \gamma_2^+}}$ such that $\frac{1}{\lambda} < \frac{p^- \Theta G(\delta)}{2\alpha^+ \delta p^+}$.

Therefore, we can use Theorem 3.2 and so the proof is complete. \( \square \)

**Remark 4.** Theorem 1.1 immediately follows from Theorem 3.7.

Now we illustrate Theorem 3.7 by presenting the following example.

**Example 3.** Let $T = 5$, $p(k) = 1 + \ln(k + e)$ for all $k \in [1, 5]$, $\alpha(k) = e^k$ for all $k \in [1, 6]$, $\theta(k) = k^2$ for all $k \in [1, 5]$ and

$$g(t) = \begin{cases} 20 + t^{20}, & |t| \geq 1, \\ 22 - t^2, & |t| < 1. \end{cases}$$

Direct calculations give $p^- = 2$, $p^+ = 1 + \ln(5 + e) > 2$, $\alpha^- = e$, $\Theta = 14$, $\sup_{\gamma > 0} \frac{\gamma^{p^+}}{G(\gamma)} = \frac{1}{2^p}$ and $\lim_{\xi \to 0^+} \frac{g(\xi)}{\xi^{p^- - 1}} = \lim_{\xi \to 0^+} \frac{g(\xi)}{\xi} = +\infty$. In this case, choosing $\nu = 5 > 2$ and $R = 5 > 1$, the assumption (9) is fulfilled. Hence, by applying Theorem 3.7, for every $\lambda \in (0, \frac{\alpha^-}{p^+ - \Theta} \sup_{\gamma > 0} \frac{\gamma^{p^+}}{G(\gamma)}) = (0, \frac{e}{1575(1 + \ln(5 + e))})$ the problem

$$\begin{cases} -\Delta(e^k |\Delta u(k - 1)|^{\ln(k + e - 1) - 1} \Delta u(k - 1)) = \lambda e^k g(k), \quad k \in [1, 5], \\ u(0) = u(6) = 0 \end{cases}$$

has at least two positive solutions in the space $X_5 := \{u : [0, 6] \to \mathbb{R} : u(0) = u(6) = 0\}$.

**Remark 5.** If in Theorem 3.7, $f(k, 0) \neq 0$ for all $k \in [1, T]$, then the ensured solution is obviously non-trivial. On the other hand, the non-triviality of the solution can be achieved also in the case $f(k, 0) = 0$ for all $k \in [1, T]$ requiring the extra condition at zero, that is there are discrete intervals $[1, T_1] \subseteq [1, T]$ and $[1, T_2] \subset [1, T]$ where $T_1, T_2 \geq 2$, such that

$$\limsup_{\xi \to 0^+} \inf_{k \in [1, T_1]} \frac{F(k, \xi)}{|\xi|^{p^-}} = +\infty$$

and

$$\liminf_{\xi \to 0^+} \inf_{k \in [1, T_2]} \frac{F(k, \xi)}{|\xi|^{p^-}} > -\infty.$$

Indeed, let $0 < \tilde{\lambda} < \lambda^*$ where $\lambda^* = \frac{\alpha^-}{p^+ - \Theta} \sup_{\gamma > 0} \frac{\gamma^{p^+}}{\sum_{k=1}^{T} \max_{|t| \leq \gamma} F(k, t)}$. Then, there exists $\tilde{\gamma} > 0$ such that $\lambda < \frac{\alpha^-}{p^+ - \Theta} \sup_{\gamma > 0} \frac{\gamma^{p^+}}{\sum_{k=1}^{T} \max_{|t| \leq \gamma} F(k, t)}$. Let $\Phi$ and $\Psi$ be as given in (2) and (3), respectively. Due to Theorem 3.7, for every $\lambda \in (0, \tilde{\lambda})$ there exists a critical point of $I_{\lambda} = \Phi - \lambda \Psi$ such that $u_{\lambda} \in \Phi^{-1}(-\infty, r_{\lambda})$ where $r_{\lambda} = \frac{\alpha^-}{p^+ - \tilde{\gamma}p^+}$. 


In particular, $u_\lambda$ is a global minimum of the restriction of $I_\lambda$ to $\Phi^{-1}(-\infty, r_\lambda)$. We will prove that the function $u_\lambda$ cannot be trivial.

Let us show that
\[
\lim_{\|u\| \to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty.
\] (15)

Thanks to our assumptions at zero, we can fix a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants $\varepsilon, \kappa$ (with $\varepsilon > 0$) such that for every $\xi \in [0, \varepsilon]$
\[
\lim_{n \to +\infty} \inf_{k \in [1, T_3]} F(k, \xi_n) \geq \kappa |\xi|^{p^-}.
\]

Now, let us consider a discrete interval $[1, T_3] \subset [1, T_2]$ where $T_3 \geq 2$. Further, let a function $v \in E$ such that

- $(k_1)$ $v(k) \in [0, 1]$ for every $k \in [1, T]$,
- $(k_2)$ $v(k) = 1$ for every $k \in [1, T_3]$,
- $(k_3)$ $v(k) = 0$ for every $k \in [T_1 + 1, T]$.

Finally, fix $M > 0$ and consider a real positive number $\eta$ with
\[
M < \frac{\eta T_3 + \kappa \sum_{k=T_3+1}^{T_1} |v(k)|^{p^-}}{\frac{\alpha^+ 2^{p^+} (T+1)}{p^-} (C_{p^+} \|v\|^{p^+} + 1)}.
\]

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \varepsilon$ and $\inf_{k \in [1, T_3]} F(k, \xi_n) \geq \eta |\xi_n|^{p^-}$ for every $n > n_0$. At this point, for every $n > n_0$, and bearing in mind the properties of the function $v$ (that is $0 \leq \xi_n v(k) < \varepsilon$ for every $k > n$ large enough), by using [19, Section 2(A.4)], we obtain
\[
\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = \frac{\sum_{k=1}^{T_3} F(k, \xi_n) + \sum_{k=T_3+1}^{T_1} F(k, \xi_n v(k))}{\Phi(\xi_n v)}
\]
\[
> \frac{\eta T_3 + \kappa \sum_{k=T_3+1}^{T_1} |v(k)|^{p^-}}{\frac{\alpha^+ 2^{p^+} (T+1)}{p^-} (C_{p^+} \|v\|^{p^+} + 1)} > M.
\]

Since $M$ can be chosen arbitrarily large, it follows that $\lim_{n \to \infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty$, from which (15) clearly follows. Hence, there exists a sequence $\{w_n\} \subset X$ converging to zero such that, for $n$ large enough, $w_n \in \Phi^{-1}(-\infty, r)$ and
\[
I_\lambda(w_n) = \Phi(w_n) - \lambda \Psi(w_n) < 0.
\]

Since $u_\lambda$ is a global minimum of the restriction of $I_\lambda$ to $\Phi^{-1}(-\infty, r)$, we conclude that $I_\lambda(u_\lambda) < 0$, so that $u_\lambda$ is not trivial.

Next, as a consequence of Theorems 3.5 and 3.6, the following theorem of the existence of three solutions is obtained.

**Theorem 3.8.** Suppose that $p^- \geq 2$, $g(0) \neq 0$ and (13) holds. Moreover, assume that there exist four positive constants $\gamma$, $\delta$, $\bar{\gamma}$ and $\bar{\delta}$ with
\[
\delta < \sqrt[p^-]{\frac{p^- \alpha^-}{2 \alpha^+ p^+ T^p}} \gamma < \sqrt[p^-]{\frac{p^- \alpha^-}{2 \alpha^+ p^+ T^p}} \bar{\gamma} < \delta.
\]
such that (10) and (12) hold, and
\[
\frac{G(\gamma)}{\gamma^2} < \frac{G(\delta) - G(\bar{\gamma})}{\alpha^- p^- \bar{\gamma}^- - 2\alpha^+ p^+ T^{p^+} \delta^{p^+}}
\]  
(16)
are satisfied. Then for each \( \lambda \in \Lambda = \left( \max \left\{ \hat{\lambda}, \frac{2\alpha^+ p^+ T^{p^+} \delta^{p^+}}{G(\delta)}, \frac{\alpha^-}{T^{p^-} \bar{p}^- + G(\gamma)} \right\} \right) \), the problem \((P_{\lambda}^{g, \theta})\) admits at least three positive solutions \( u_1, \bar{u}_1 \) and \( u_3 \) such that
\[
\sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u_1(k-1)|^{p(k-1)} < \frac{\alpha^-}{T^{p^-} \bar{p}^-} \bar{\gamma}^{p^-},
\]
and
\[
\sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta \bar{u}_1(k-1)|^{p(k-1)} > \frac{\alpha^-}{T^{p^-} \bar{p}^-} \bar{\gamma}^{p^-}.
\]
Proof. First, in view of (16), we have \( \Lambda \neq \emptyset \). Next, fix \( \lambda \in \Lambda \). Employing Theorem 3.5, there is a positive weak solution \( \bar{u}_1 \) such that
\[
\sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u_1(k-1)|^{p(k-1)} < \frac{\alpha^-}{T^{p^-} \bar{p}^-} \bar{\gamma}^{p^-},
\]
which is a local minimum for the associated functional \( I_{\lambda} \), while Theorem 3.6 ensures a positive solution \( \bar{u}_1 \) such that
\[
\sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta \bar{u}_1(k-1)|^{p(k-1)} > \frac{\alpha^-}{T^{p^-} \bar{p}^-} \bar{\gamma}^{p^-},
\]
which is a local minimum for \( I_{\lambda} \). Arguing as in the proof of Theorem 3.3, from the condition (13) we observe that the functional \( I_{\lambda} \) is coercive, then it satisfies the (PS) condition. Hence, the conclusion follows from the mountain pass theorem as given by Pucci and Serrin (see [38]).

The following existence result is a consequence of Theorem 3.8.

**Theorem 3.9.** Assume that \( p^- > 2 \), \( g(0) \neq 0 \),
\[
\limsup_{\xi \to 0^+} \frac{G(\xi)}{\xi^{p^-}} = +\infty
\]  
(17)
and
\[
\limsup_{\xi \to +\infty} \frac{G(\xi)}{\xi^{p^-}} = 0.
\]  
(18)
Furthermore, suppose that there exist two positive constants \( \bar{\gamma} \) and \( \bar{\delta} \) with
\[
\sqrt{\frac{p^- \alpha^-}{2\alpha^+ p^+ T^{p^+} \bar{\gamma}^{p^-}}} < \bar{\delta}
\]  
(19)
such that
\[
\frac{G(\bar{\gamma})}{\bar{\gamma}^{p^-}} < \frac{\alpha^-}{2\alpha^+ p^+ T^{p^+} \bar{\gamma}^{p^-}} \frac{G(\bar{\delta})}{\bar{\delta}^{p^-}}.
\]  
(20)
Then for each \( \lambda \in \left( \frac{2\alpha^+ p^+ T^{p^+} \delta^{p^+}}{G(\delta)}, \frac{\alpha^-}{T^{p^-} \bar{p}^-} G(\gamma) \right) \), the problem \((P_{\lambda}^{g, \theta})\) admits at least three positive solutions.
Moreover, by a continuous function, calculations give $p$ and $\gamma$. Theorem 1.2 immediately follows from Theorem 3.9.

**Remark 6.** Theorem 1.2 immediately follows from Theorem 3.9.

Finally, we present one application of Theorem 3.9 as follows.

**Example 4.** Let $T = 99$, $p(k) = e^{k+1}$ for all $k \in [1, 99]$, $\alpha(k) = 1 + \sin k$ for all $k \in [1, 100]$, $\theta(k) = k$ for all $k \in [1, 99]$, $g(t) = 1 + \frac{t^2}{1+t^2}$ for all $t \in \mathbb{R}$. Direct calculations give $p^- = e$, $p^+ = e^{100}$, $\alpha^- = 1$, $\alpha^+ = 2$, $\Theta = 4950$, $g$ is a non-negative continuous function, $g(0) \neq 0$, $G(t) = 2t - \arctan t$ for all $t \in \mathbb{R}$,

$$\lim_{\xi \to 0^+} \frac{G(\xi)}{\xi^{p^-}} = \lim_{\xi \to 0^+} \frac{2\xi - \arctan \xi}{\xi^e} = +\infty$$

and

$$\lim_{\xi \to +\infty} \frac{G(\xi)}{\xi^{p^-}} = \lim_{\xi \to +\infty} \frac{2\xi - \arctan \xi}{\xi^e} = 0.$$  
Moreover, by $\gamma = 99 \times e^{100}$ and $\delta = \frac{1}{2}$, we see that (19) and (20) are satisfied. Hence, by applying Theorem 3.9, for every

$$\lambda \in \left(\frac{99 \times e^{100}}{2475 \times 2e^{100}-1(1-\arctan(\frac{1}{2}))}, \frac{(99 \times e^{100})e}{2(99 \times e^{100})-\arctan(99 \times e^{100})}\right),$$

the problem

$$\begin{cases} -\Delta((1+\sin k)|\Delta u(k-1)|^{e-2}\Delta u(k-1)) = \lambda k(1 + \frac{k^2}{1+k^2}), & k \in [1, 99], \\ u(0) \equiv u(100) = 0 \end{cases}$$

has at least three positive solutions in the space

$$X_{99} := \{u : [0, 100] \to \mathbb{R} : u(0) = u(100) = 0\}.$$

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