Thermodynamics and Cosmological Constant of Non-Local Field Theories from p-Adic Strings

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Abstract: We develop the thermodynamics of field theories characterized by non-local propagators. We analyze the partition function and main thermodynamic properties arising from perturbative thermal loops. We focus on the p-adic models associated with the tachyon phenomenology in string theories. We reproduce well known features of these theories, but also obtain many new results. In particular, we explain how to maintain consistency of such non-local theories by avoiding the appearance of ghosts at finite temperature. As a consequence of this fact, the vacuum energy in p-adic theories becomes positive. It is also hierarchically suppressed, and we explore the parameter space where it is consistent with the observed value of the cosmological constant.

Keywords: Cosmological constant, non-local field theory, p-adic, finite temperature field theory.
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1. Introduction

In this paper we study a special class of field theories which have an infinite number of higher derivative terms in the form of an exponential. Such non-local structure of quantum field theories is recurrent in many stringy models. For instance, this is the case for tachyonic actions in string field theory \[1\]-\[10\] (for a review see \[11\]), bulk fields localized on codimension-2 branes \[12\] and various toy models of string theory such as $p$-adic strings \[13, 14\], zeta strings \[15\], and strings quantized on a random lattice \[16\]-\[18\]. More general non-local theories, in which the derivatives do not necessarily appear in the combination $\square = -\partial^2_t + \nabla^2$, arise in noncommutative field theories \[19\], field theories with a minimal length scale \[20\] (such as doubly special relativity), fluid dynamics \[21, 22\] and quantum algebras \[21\].

In addition, like most higher-derivative theories, these theories have better ultra-violet (UV) convergence. Unlike finite-order higher derivative theories, by virtue of having an infinite set of higher derivative terms, they have been conjectured to be free of ghosts\(^1\) and to have a well-posed initial value problem \[24, 14, 25, 26\] making them phenomenologically interesting to study. In particular, these models have been found to provide novel cosmological properties such as non-slow-roll inflation \[27\], crossing of the phantom-divide in the context of dark energy \[28\], non-singular bouncing solutions \[29\] (also see \[30\] for similar work with non-local gravitational actions) among others \[31\]. However, most of these analysis have largely been classical, and little attention has been paid to quantum loop calculations, but see \[17, 32\] for an exception.

Our goal here is to analyze the thermodynamic properties of this type of theory. There are three main motivations for this study. Firstly, as was explained in \[33\], some of the results in the $p$-adic theory closely parallel the ones that comes from purely stringy calculations\(^2\). The computations involved are relatively simple but we can nevertheless probe both high and low temperature regimes (with respect to the Hagedorn temperature), and therefore hope to clarify some aspects of string thermodynamics.

Secondly, it is well documented that the thermal history of our universe plays a crucial role in defining our cosmology. In particular there have been recent studies involving the Hagedorn phase in the early universe \[35\]-\[37\], especially to see whether cosmological perturbations can have a thermal origin \[38, 39\]. Our results may prove useful in this context.

Thirdly, higher derivative alternatives to supersymmetric extensions of the Standard Model have recently been proposed \[40\]. In brief, these theories attempt to upgrade the mass scale in a Pauli-Villars regularization scheme to a physical parameter with observational consequences, possibly detectable at the Large Hadron Collider (LHC). Since the non-local modifications to the propagator in $p$-adic or string field theory models make the loop integrals finite, one could possibly treat the

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\(^1\)There are no perturbative states or poles in the propagator. There are instabilities causing oscillations to grow unboundedly \[14\], but this is due to the presence of the tachyon \[23\].

\(^2\)For a different field theoretic approach aiming to capture the physics of the Hagedorn phase, see \[34\].
mass scale associated with the non-locality as a regularization parameter. On the other hand one can also try to explore whether similar consistent, both theoretical and phenomenological, extensions of the Standard Model exists if we believe the non-locality to be physical and possibly at the TeV scale; see [41] in this direction. The tools that we have developed to deal with quantum corrections should prove helpful in this context. As an interesting by-product, while exploring quantum consistency of these models, we found that suitable insertions of counter-terms make the vacuum energy positive. Further, this energy can be hierarchically suppressed if the string scale small compared to the Planck scale, and therefore lends itself to the phenomenology of dark energy.

In this paper our focus will be on the \( p \)-adic string theories, whose action is given by \([12,13]\)

\[
S = \frac{m_s^D}{g_p^2} \int d^D x \left[ -\frac{1}{2} \varphi e^{-\Box/M^2} \varphi + \frac{1}{p+1} \varphi^{p+1} \right],
\]

(1.1)

where \( \Box = -\partial_t^2 + \nabla_{D-1}^2 \) in flat space, and we have defined

\[
\frac{1}{g_p^2} \equiv \frac{1}{g_o^2} \frac{p^2}{p-1} \quad \text{and} \quad M^2(p) \equiv \frac{2m_s^2}{\ln p}.
\]

(1.2)

The dimensionless scalar field \( \varphi(x) \) describes the open string tachyon, \( m_s \) is the string mass scale, and \( g_o \) is the open string coupling constant. Though the action (1.1) was originally derived for \( p \) a prime number, it appears that it can be continued to any positive integer and even makes sense in the limit \( p \to 1 \) \([14]\). Setting \( \Box = 0 \) in the action, the resulting potential takes the form

\[
U = \frac{m_s^D}{g_p^2} (\frac{1}{2} \varphi^2 - \frac{1}{p+1} \varphi^{p+1}).
\]

Its shape is shown in Figure 1.

Figure 1: The potential of the \( p \)-adic tachyon for \( p = 3, 7 \) and \( p \to \infty \).
The action (1.1) is a simplified model of the bosonic string which only qualitatively reproduces some aspects of a more realistic theory; that being said, there are several nontrivial similarities between $p$-adic string theory and the full string theory. For example, near the true vacuum of the theory ($\varphi = 0$) the field naively has no particle-like excitations since its mass squared goes to infinity\(^3\). This is the $p$-adic version of the statement that there are no open string excitations of the tachyon vacuum. A second similarity is the existence of lump-like soliton solutions representing $p$-adic D-branes [43]. The theory of small fluctuations about these lump solutions has a spectrum of equally spaced masses-squared for the modes [46], as in the case of normal bosonic string theory. It is worth pointing out that one obtains a very similar action with exponential kinetic operators (and usually assumed to have a cubic or quartic potential) while quantizing strings on a random lattice [16]. These field theories are also known to reproduce several features, such as the Regge behavior [17], of their stringy duals. Although our analysis focuses on the specific $p$-adic action, it can easily be applied to such theories as well.

In this present paper we develop the field theoretic tools to compute thermodynamic quantities for such non-local models. In [33] we had already explained how these models reproduce some of the results of string theory. Here we provide details of these calculations. Second, we point out how quantum consistency of such models (that the theory remains ghost free) can be maintained by inserting suitable counter-terms. Surprisingly, we find that our prescription leads to some interesting consequences for the cosmological constant. The cosmological constant can not only be positive but it can be hierarchically suppressed if the string scale is small compared to the Planck scale. Finally, we discuss non-perturbative effects. This includes summing over classes of infinite numbers of diagrams which gives rise to singularities at non-zero coupling constants. In [47] we will study finite temperature solitonic solutions leading to contributions which are divergent as the coupling vanishes.

The paper is organized as follows. In the next section we start with the finite temperature formulation of these non-local field theories with emphasis on the free theory. In section 3 we focus on $\varphi^4$ interactions corresponding to $p = 3$ and $D = 4$. We compute the 2-loop partition function and briefly discuss the connections to string theory. In section 4 we focus on consistency of the model with respect to quantum loop corrections, and provide a prescription for counter-terms which needs to be added to address this issue. We also discuss the implication of our prescription for the cosmological constant. In section 5 we focus on contributions to the free energy at arbitrary higher loops and comment on the validity of the perturbative analysis. In section 6 we generalize our results to arbitrary values of $p$, while in section 7, we study $p$-adic theories in arbitrary dimensions. The main aim in this section is to discuss its relevance for the cosmological constant. Finally, in section 8, we conclude with a discussion on unresolved issues and allude to future research directions that may be able to shed some light on them.

\(^3\)Reference [14] found anharmonic oscillations around the vacuum by numerically solving the full nonlinear equation of motion. However, these solutions do not correspond to conventional particle states.
2. Free Energy at Zero Order: No Particle Degrees of Freedom

Consider the action for $D = 4$ space-time dimensions and $p = 3$. The finite temperature action corresponding to (1.1) is

$$ S = \int_0^\beta d\tau \int d^3x \left[ -\frac{1}{2} \phi(x, \tau) e^{-\left(\beta^2 / \partial^2 + \nabla^2 \right) / M^2} \phi(x, \tau) - \lambda \phi^4(x, \tau) \right], \quad (2.1) $$

where we have performed the rescaling

$$ \phi \equiv \frac{m^2 s g^3 \phi}{\lambda} \equiv -\frac{1}{18} g^2 o m^4 s. \quad (2.2) $$

Notice that $\phi$ has dimension of energy-squared. We could multiply the kinetic term by some positive parameter with dimension of energy-squared so that $\phi$ has dimension of energy, but it is actually more convenient to write it this way. Note that $\lambda$ has dimension of energy to the minus fourth power. Analysis of this model follows very closely that of the usual scalar theory at finite temperature. See Chapters 2 and 3 of Ref. [48].

To perform the functional integral we use the Fourier transform

$$ \phi(x, \tau) = \frac{1}{\sqrt{\beta V}} \sum_n \sum_k e^{i(k \cdot x + \omega_n \tau)} \phi_n(k). \quad (2.3) $$

The Fourier amplitude $\phi_n(k)$ is dimensionless, which is very convenient for performing functional integrals. The Matsubara frequency is $\omega_n = 2\pi n T$. After integration over space and imaginary time we get the free action

$$ S_0 = -\frac{1}{2} \sum_n \sum_k D_0^{-1}(\omega_n, k) \phi_n^*(k) \phi_n(k), \quad (2.4) $$

where we have used $\phi_n^*(k) = \phi_{-n}(-k)$; this is a consequence of the reality of the scalar field. This action defines the free propagator

$$ D_0(\omega_n, k) = e^{-(\omega_n^2 + k^2) / M^2}. \quad (2.5) $$

The partition function to zero order in $\lambda$ is

$$ Z_0 = N' \prod_n \prod_k \left[ \int_{-\infty}^{\infty} dA_n(k) e^{-\frac{1}{2} D_0^{-1}(\omega_n, k) A_n^2(k)} \right] = N' \prod_n \prod_k [2\pi D_0(\omega_n, k)]^{1/2}. \quad (2.6) $$

Here $N'$ is an as yet arbitrary constant. Integration is over the amplitudes of the Fourier components since the phase of $\phi_n(k)$ drops out of the action. Taking the logarithm

$$ \ln Z_0 = \ln N' + \frac{1}{2} \ln(2\pi) \sum_n \sum_k + \frac{1}{2} \sum_n \sum_k \ln[D_0(\omega_n, k)]. \quad (2.7) $$
As usual one chooses $N'$ such that the sum of the first two terms cancel. Multiplication of the partition function by a temperature independent constant does not change the thermodynamics. Thus

$$\ln Z_0 = -\frac{1}{2} \sum_n \sum_k \frac{\omega_n^2 + k^2}{M^2}. \quad (2.8)$$

We will now use two different approaches to show that this expression is zero.

In the first approach we express the sum as a contour integral. The general formula is

$$T \sum_n f(k_0 = i\omega_n) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} dk_0 \left(f(k_0) + f(-k_0)\right)$$

$$+ \frac{1}{2\pi i} \int_{-\infty+\epsilon}^{\infty+\epsilon} dk_0 \left(f(k_0) + f(-k_0)\right) \frac{1}{e^{\beta k_0} - 1}, \quad (2.9)$$

under the assumption that the function $f(k_0)$ has no singularities on the imaginary axis. The first integral on the right side is referred to as a vacuum contribution. It can be converted to Euclidean space using $k_0 = i k_4$. Then the expression

$$\int \frac{dk_1 d^3k}{(2\pi)^4} \left(\frac{k_1^2 + k^2}{M^2}\right)$$

is zero by the usual regularization procedure. The second integral on the right side is referred to as a finite temperature contribution. Since $f(k_0) = (-k_0^2 + k^2)/M^2$ is analytic in the right half plane, the contour can be pushed to infinity, showing that it is zero.

In the second approach we write the logarithm of the partition function as

$$\ln Z_0 = -\frac{V}{2M^2} \sum_n \int d^3k \left(k^2 + (2\pi n T)^2\right) \quad (2.10)$$

We use dimensional regularization to compute the above integral, and introduce two small parameters, $\epsilon$ and $\delta$.

$$I_n(\epsilon, \delta) = \int \frac{d^{3+\epsilon}k}{(k^2 + (2\pi n T)^2)^{-1+\delta}} \quad (2.11)$$

We then use the standard formula

$$\int \frac{d^Dp}{(2\pi)^D} \frac{k^A}{(k^2 + m^2)^B} = \frac{\Gamma(B - A - D/2)\Gamma(A + D/2)m^{2A-2B+D}}{(4\pi)^D/2\Gamma(B)\Gamma(D/2)}. \quad (2.12)$$

Substituting $B = -1 + \delta$, $D = 3 + \epsilon$, $A = 0$ and $m = 2\pi n T$ we have

$$I_n(\epsilon, \delta) = \frac{\Gamma(-5/2 + \delta - \epsilon/2)\Gamma(3/2 + \epsilon/2)(2\pi n T)^{5-2\delta+\epsilon}}{(4\pi)^{3/2+\epsilon/2}\Gamma(-1+\delta)\Gamma(3/2+\epsilon/2)} \frac{\Gamma(-5/2 + \delta - \epsilon/2)(2\pi n T)^{5-2\delta+\epsilon}}{(4\pi)^{3/2+\epsilon/2}\Gamma(-1+\delta)}. \quad (2.13)$$
Figure 2: This diagram constitutes the sole contribution to first order in $\lambda$ for the case $p = 3$.

We thus have

$$\ln Z_0 = -\frac{V}{2M^2} \sum_n I_n = -\frac{V \Gamma(-5/2 + \delta - \epsilon/2)(2\pi T)^{5-2\delta+\epsilon}}{2M^2(4\pi)^{3/2+\epsilon/2} \Gamma(-1 + \delta)} \sum_n n^{5-2\delta+\epsilon}$$

$$= -\frac{V \Gamma(-5/2 + \delta - \epsilon/2)(2\pi T)^{5-2\delta+\epsilon}}{M^2(4\pi)^{3/2+\epsilon/2} \Gamma(-1 + \delta)} \zeta(5 - 2\delta + \epsilon)$$

where $\zeta$ is the Reimann zeta function. We now observe that as $\epsilon$ and $\delta \rightarrow 0$ all the terms except $\Gamma(-1 + \delta)$ are regular. Thus using the Laurant series expansion

$$\Gamma(-n + \epsilon) = (-1)^n \frac{1}{n!} \left[ \frac{1}{\epsilon} - \gamma + \psi(n) + O(\epsilon) \right]$$

(2.14)

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant, and

$$\psi(n) = \sum_{i=1}^{n} \frac{1}{i}$$

(2.15)

we find

$$\ln Z_0 = V \left[ \frac{2\pi^{7/2} \Gamma(-5/2) \zeta(5) T^5}{M^2} \right] \delta + O(\delta^2, \epsilon^2, \delta \epsilon)$$

(2.16)

Once again we find that $\ln Z_0$ vanishes.

The fact that the free theory gives no contribution to the pressure, energy density or entropy density should not be a surprise. The free propagator has no poles and therefore no thermal excitations. This is consistent with the fact that free $p$-adic theories are trivial in the sense that they contain no physical particle-like degrees of freedom.

3. Free Energy at First Order and Thermal Duality

For interacting non-local theories such as (2.1), the Feynman rules are identical to those of the usual scalar theory with the use of the appropriate propagator. The only diagram at first order in the coupling $\lambda$ is shown in Figure 2. There is a combinatoric factor of 3 and a factor $(-\lambda)$ associated with the vertex. It leads to

$$\ln Z_1 = 3(-\lambda)\beta V \left[ T \sum_n \int \frac{d^3 k}{(2\pi)^3} D_0(\omega_n, k) \right]^2.$$  

(3.1)
Due to the exponential nature of the bare propagator the loop diagrams are expected to be convergent in both the IR and UV. A useful formula is

$$\sum_n \int \frac{d^3k}{(2\pi)^3} D_0^N(\omega_n, k) = \left( \frac{M}{2\sqrt{N\pi}} \right)^3 \zeta\left( \frac{2\sqrt{N\pi T}}{M} \right),$$  \hspace{1cm} (3.2)

where

$$\zeta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2x^2} = \theta_3(0, e^{-x^2}),$$  \hspace{1cm} (3.3)

the third Jacobi elliptic theta function. The pressure at this order is therefore given by

$$P_1 = -3\lambda \left( \frac{M^6T^2}{2^6\pi^3} \right) \zeta^2\left( \frac{2\pi T}{M} \right).$$  \hspace{1cm} (3.4)

Remarkably, one can write $\zeta(x)$ as:

$$\zeta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2x^2} = \frac{\sqrt{\pi}}{x} \sum_{m=-\infty}^{\infty} e^{-\frac{m^2x^2}{x^2}} = \frac{\sqrt{\pi}}{x} \zeta\left( \frac{\pi}{x} \right).$$  \hspace{1cm} (3.5)

The first equality shows explicitly the contribution of the $n^{th}$ thermal mode. In contrast to a standard quantum field theory, one can see that the higher thermal modes are strongly suppressed at high temperatures. When $x \to \infty$, the leading
Figure 4: Partition function of the p-adic string model for \( p = 3 \) in 4 space-time dimensions (arbitrary units). The symmetry of \( Z_1(T) \) with respect to \( T_c \) shows the exact realization of the thermal duality at this order in \( \lambda \).

...
where $\Lambda \equiv 3\lambda T_c^8$ is the cosmological constant. This is precisely the kind of asymptotic behavior that has been suggested in the stringy literature [50, 53].

In [33] we discussed the thermodynamic implications of this result (3.4). We found that at low temperatures, in addition to a negative cosmological constant, the thermal properties of the $p$-adic fluid resembles that of a pressureless dust. In this low density phase the temperature changes very slowly (logarithmically) with the change of energy density. Qualitatively, this is also what has been argued from the string theory side [54, 35, 36]; in the Hagedorn phase the energy density is dominated by the most massive string states and hence behaves as a pressureless fluid, with the temperature remaining almost a constant. However, unlike [54], our partition function does not suffer from the problem of having a negative heat capacity, and gives way to a $P/\rho = \omega \approx 1$ stiff fluid phase at high temperature.

In this context, we note that if $g_o^2 \ll 1$, we can describe the physics near the Hagedorn temperature $T_c$ by using the partition function at first order in $\lambda$. We will see this explicitly when we compare (3.1) with the order $\lambda^2$ contributions in section 4. From Eq. (3.4) and Figure 3 we conclude that this is a smooth transition and not a first order phase transition, as has been conjectured from string theory [50]. More recently, the possibility of such a smooth transition has also been suggested [53, 54], although in [53] a different higher temperature was identified to correspond to much milder phase transitions in some supersymmetric string models. Finally, we point out that to this order, since $\lambda$ is defined negative for the $p$-adic model, the entropy density is positive, and that makes it consistent. On the other hand, the vacuum energy density or cosmological constant is negative, which is not what is observed in nature.

4. Self-Energy, Ghosts and a Positive Cosmological Constant

Higher derivative theories are typically plagued with ghosts. For instance, a fourth order theory has a propagator of the form $[\rho^4 + B\rho^2 + C]^{-1}$ where $B$ and $C$ are some constants. In general, such a propagator has two poles corresponding to two different physical degrees of freedom. It is easy to check that at least one of them is ghost-like; it has the wrong sign for the residue at the pole. One of the virtues of $p$-adic type models is that the modified higher derivative propagator has no poles, i.e. there are no perturbative states, ghosts or otherwise. However, this statement is true only at the tree-level. Do quantum corrections spoil this property?

The lowest order non-zero contribution to the partition function gives rise to an order $\lambda$ contribution to the self-energy $\Pi$, defined by

$$D^{-1} = D_0^{-1} + \Pi,$$

4There can be a double pole, but such theories are also not consistent [56].
where $D$ is the full propagator. The standard computation gives

$$
\Pi_1 = 12\lambda T \sum_n \int \frac{d^3k}{(2\pi)^3} D_0(\omega_n, k)
= 12\lambda T \left( \frac{M}{2\sqrt{\pi}} \right)^3 \varsigma \left( \frac{2\pi T}{M} \right). 
$$

(4.2)

This is dimensionless, just as the propagator is. However, we note the reappearance of a pole: apparently the ghost comes back to haunt us! In [32] a similar result was also obtained. One possible interpretation of the pole is in terms of massive closed string states [32], but here we adopt a different point of view. We realize that the appearance of a pole in the propagator in the complex $p_0$ plane at $T = 0$ can be avoided by adding a counter-term to the Lagrangian of the form $-\frac{1}{2} \gamma \phi^2$ and adjusting $\gamma$ to cancel the self-energy contribution. This leads to

$$
\gamma = -\frac{3\lambda M^4}{4\pi^2}. 
$$

(4.3)

The coefficient $\gamma$ must be adjusted order by order to cancel the self-energy contributions. Thus, our prescription provides a unique way of extending the consistency of the tree-level $p$-adic theory to all orders in quantum loops.

In addressing the problem of ghosts, we discover a rather remarkable consequence: the cosmological constant becomes positive. This is because the counter-term also contributes to the pressure at order $\lambda$. The contribution is

$$
-\frac{1}{2} \gamma T \sum_n \int \frac{d^3k}{(2\pi)^3} D_0(\omega_n, k) = \frac{3\lambda M^4}{8\pi^2 T} \left( \frac{M}{2\sqrt{\pi}} \right)^3 \varsigma \left( \frac{2\pi T}{M} \right),
$$

(4.4)

which results in a total pressure of

$$
P_1 = -3\lambda \left( \frac{M^2}{4\pi} \right)^4 \frac{2\sqrt{\pi T}}{M} \varsigma \left( \frac{2\pi T}{M} \right) \left[ \frac{2\sqrt{\pi T}}{M} \varsigma \left( \frac{2\pi T}{M} \right) - 2 \right].
$$

(4.5)

This is written in such a way as to make clear both the zero and high temperature limits. A negative value of $\lambda$ leads to a positive vacuum energy or cosmological constant.

$$
\Lambda = -3\lambda \left( \frac{M^2}{4\pi} \right)^4
$$

(4.6)

In addition, the entropy density is a monotonically increasing function of temperature, making the approximation in this theory thermodynamically self-consistent.

5. Higher Order Diagrams and Limits to Perturbation Theory

In this section we explore contributions to higher order in $\lambda$ than previously considered. First we compute the diagrams which are of order $\lambda^2$. Then we sum the infinite set of ring diagrams and the infinite set of necklace diagrams.
5.1 Free energy at second order

It has been argued that the duality relation (3.6) must be broken when nonperturbative effects are included [50, 52]. We find that in our case it is broken at the next order in $\lambda$. Let us compute the contribution that is second order in $\lambda$. One such contribution arises from the necklace diagram shown in Figure 5.

$$P_{2,\text{necklace}} = 36\lambda^2 \left[ T \left( \frac{M}{2\sqrt{\pi}} \right)^3 \zeta \left( \frac{2\pi T}{M} \right) \right]^2 \left[ T \left( \frac{M}{2\sqrt{2\pi}} \right)^3 \zeta \left( \frac{2\sqrt{2\pi}T}{M} \right) \right]$$

(5.1)

One can see already under what conditions the perturbative expansion would be meaningful. For $T \gg M, \varsigma \to 1$, and therefore $P_1 \sim (M^3T)(\lambda M^3T)$ while $P_{2,\text{necklace}} \sim (M^3T)(\lambda M^3T)^2$. (5.2)

We will show later in this section that this argument can be extended to arbitrary order. Thus, the perturbative expansion is only valid when $\lambda M^3T \ll 1$. At the other temperature extreme, $T \ll M, \varsigma \to M/T$, and now

$$P_1 \sim M^4(\lambda M^4) \text{ while } P_{2,\text{necklace}} \sim M^4(\lambda M^4)^2.$$ 

(5.3)

Thus, at low temperature the perturbative expansion parameter is $\lambda M^4 \sim g_o^2$. It is clear already that if $g_o^2$ is small enough we can trust the order $\lambda$ result all the way up to a temperature over the critical one $T_c$.

Returning to the problem of calculating the partition function to second order in $\lambda$, we recognize that there is a topologically different type of diagram at the same order in $\lambda$. This diagram contains two vertices, and every leg of one vertex is connected to a leg of the other vertex. This is oftentimes called a sunset diagram. Apart from a combinatoric factor, what needs evaluation is

$$\left[ T \sum_{n_1} \int \frac{d^3k_1}{(2\pi)^3} D_0(\omega_{n_1}, k_1) \right] \cdots \left[ T \sum_{n_4} \int \frac{d^3k_4}{(2\pi)^3} D_0(\omega_{n_4}, k_4) \right]$$

\[ \times (2\pi)^3 \delta(k_1 + \cdots + k_4) \beta \delta_{n_1+\cdots+n_4,0}. \]

The momentum conserving delta function can be expressed in integral form. Then the momentum integrals factorize and the remaining integral can be evaluated with the result

$$P_{\text{sunset}} = \frac{3}{2} \lambda^2 \left( \frac{M}{2\sqrt{\pi}} \right)^9 \chi(T, M).$$

(5.4)
Here the combinatoric factor has been included and
\[
\chi(T, M) = \left[ T \sum_{n_1} e^{-x^2n_1^2} \right] \cdots \left[ T \sum_{n_4} e^{-x^2n_4^2} \right] \beta \delta_{n_1+\cdots+n_4,0}. \tag{5.5}
\]

It does not seem possible to obtain the function \( \chi \) in closed form. However, it can be rewritten with the aid of the theta function of the third kind.
\[
\theta_3(u, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2uni} \tag{5.6}
\]

The sum \( \varsigma(x) \) encountered so often is just \( \varsigma(x) = \theta_3(0, e^{-x^2}) \). The theta function can also be written as an infinite product.
\[
\theta_3(u, q) = \prod_{n=1}^{\infty} \left[ 1 + q^{2n-1} \cos(2u) + q^{2(2n-1)} \right] \left( 1 - q^{2n} \right). \tag{5.7}
\]

The usefulness arises when we represent the frequency conserving delta function as an integral.
\[
\delta_{n_1+\cdots+n_4,0} = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{i(n_1+\cdots+n_4)\phi}. \tag{5.8}
\]

Then we can write
\[
\chi(T, M) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \left[ \theta_3 \left( \frac{i}{2} \phi, e^{-x^2} \right) \right]^4. \tag{5.9}
\]

We see that the thermal duality relation Eq. (3.6) is only verified at the leading order in \( \lambda \), but is broken at order \( \lambda^2 \). However, the fact that we can write all the results in terms of \( \theta_3(u, e^{-x^2}) \), which obeys
\[
\theta_3(u, e^{-x^2}) = \frac{\sqrt{\pi}}{x} e^{-u^2/x^2} \theta_3 \left( \frac{i}{x} u, e^{-x^2/x^2} \right), \tag{5.10}
\]

allows an alternative interpretation in terms of inverse modes, but they need to be weighted in a different way.

The low and high temperature limits of the sunset contribution are readily obtained. The high temperature limit is dominated by all \( n_i = 0 \). Hence
\[
\chi(T \gg M, M) = T^3. \tag{5.11}
\]

In the low temperature limit the sums can be replaced by integrals and the limits on \( \phi \) extended to \( \pm \infty \).
\[
\chi(T \ll M, M) = \frac{1}{2} \left( \frac{M}{2\sqrt{\pi}} \right)^3. \tag{5.12}
\]

Therefore
\[
P_{\text{sunset}}(T \gg M) = \frac{3}{2} \lambda^2 \left( \frac{M}{2\sqrt{\pi}} \right)^9 T^3. \tag{5.13}
\]
and

\[ P_{\text{sunset}}(T \ll M) = \frac{3}{4} \lambda^2 \left( \frac{M^2}{4\pi} \right)^6. \]  

(5.14)

Note that both the necklace and the sunset contribution grows at high temperature as \( T^3 \) compared to the first order in \( \lambda \) contribution which grows as \( T^2 \). The ratio is

\[ P_{\text{2,necklace}}/P_1 \sim P_{\text{sunset}}/P_1 \sim \lambda M^3 T \sim g_0^2 T/m_s. \]  

(5.15)

This ratio is small and only approaches one when the temperature reaches \( T \sim m_s/g_0^2 \).

5.2 Ring diagrams

In ordinary field theories with massless particles one generally finds a certain class of diagrams that are infrared divergent, and that the divergence becomes more severe with increasing number of loops. These are the ring diagrams. Their contribution to the pressure is

\[
P_{\text{ring}} = \frac{1}{2} T \sum_n \int \frac{d^3k}{(2\pi)^3} \sum_{l=2}^{\infty} \frac{1}{l} \left[ -\Pi_1 D_0(\omega_n, k) \right]^l
\]
\[
= -\frac{1}{2} T \sum_n \int \frac{d^3k}{(2\pi)^3} \left[ \ln (1 + \Pi_1 D_0) - \Pi_1 D_0 \right].
\]  

(5.16)

In ordinary massless \( \phi^4 \) theory the propagator is \( D_0 = 1/(\omega_n^2 + k^2) \) and \( \Pi_1 \) is frequency and momentum independent. The sum of these ring diagrams is proportional to \( \lambda^{3/2} \), nonanalytic in the coupling constant. This nonanalytic contribution arises from the \( n = 0 \) term in the Matsubara summation. In the present non-local theory the individual diagrams are convergent and there is no need to sum the series. If one chooses to sum the series anyway there is a limit to its convergence. The largest value of \( D_0 \) is 1. For the series to converge to the logarithm one therefore needs \( -1 < \Pi_1 \leq 1 \). For given \( \lambda \) and \( M \) there is a maximum \( T \) for which the series converges to the logarithm.

In fact, it is possible to show that there is no divergence problem from the set of ring diagrams. Each diagram in the series is convergent. We can evaluate them by using Eq. (3.2). In the limit in which \( l \to \infty \), they go as \( 1/l^{3/2} \). Hence at large \( l \), the terms go as \( 1/l^{5/2} \). This series converges much more rapidly than does a logarithm.

5.3 Necklace diagrams

There is another set of diagrams that are easily summed which are similar to the set of ladder diagrams in scattering theory. Start with the Figure 5 diagram, and keep adding loops to make a string. The first and last loops are not connected to each other. This looks like an open necklace for \( p = 3 \), see Figure 6. Including the appropriate combinatoric factors, we find:

\[
P_{\text{necklace}} = 3 \sum_{l=0}^{\infty} (-\lambda)^{l+1} 12^l \left[ T \sum_n \int \frac{d^3k}{(2\pi)^3} D_0(\omega_n, k) \right]^2.
\]
Figure 6: A necklace diagram which is fourth order in $\lambda$ for the case $p = 3$.

\[
\times \left[ T \sum_n \int \frac{d^3k}{(2\pi)^3} D_0^2(\omega_n, k) \right]^l
\]

\[
= -3\lambda \left[ T \left( \frac{M}{2\sqrt{\pi}} \right)^3 \varsigma \left( \frac{2\pi T}{M} \right) \right]^2 \sum_{l=0}^{\infty} \left[ -12\lambda T \left( \frac{M}{2\sqrt{2\pi}} \right)^3 \varsigma \left( \frac{2\sqrt{2}\pi T}{M} \right) \right]^l
\]

\[
= \frac{-3\lambda \left[ T \left( \frac{M}{2\pi} \right)^3 \varsigma \left( \frac{2\pi T}{M} \right) \right]^2}{1 + 12\lambda T \left( \frac{M}{2\sqrt{2\pi}} \right)^3 \varsigma \left( \frac{2\sqrt{2}\pi T}{M} \right)}.
\]

Notice that the numerator is just the lowest order result but without the mass counter-term. To include the latter one just makes a self-energy insertion on either end of the string. (Making an insertion on both ends is one higher order in $\lambda$ and is just one contribution to the set of super ring diagrams.) The result is

\[
P_{\text{necklace}} = \frac{-3\lambda \left( \frac{M^2}{4\pi} \right)^4 \frac{2\sqrt{2}\pi T}{M} \varsigma \left( \frac{2\pi T}{M} \right) \left[ \frac{2\sqrt{2}\pi T}{M} \varsigma \left( \frac{2\pi T}{M} \right) - 2 \right]}{1 + 12\lambda T \left( \frac{M}{2\sqrt{2\pi}} \right)^3 \varsigma \left( \frac{2\sqrt{2}\pi T}{M} \right)}.
\]

This expression has some very interesting properties when $\lambda < 0$. As before, the vacuum energy density is positive. The entropy density is positive at all temperatures. In addition, there is a maximum temperature determined by the vanishing of the denominator. We can try to interpret this fact as arising due to the potential being unbounded from below for large values of $\phi$. Indeed, the potential is

\[
U(\phi) = \frac{1}{2}\phi^2 + \lambda\phi^4.
\]

ignoring the correction from $\gamma\phi^2$. The maximum height of the potential is

\[
U_{\text{max}} = -\frac{1}{16\lambda}.
\]

For the sake of estimation, use the lowest order expression for the energy density in the limit $T > M$. It is

\[
\epsilon_1 = -3\lambda \left( \frac{M^2}{4\pi} \right)^3 T^2.
\]

When the temperature is high enough the states will spill over the top of the potential. One can estimate this temperature by equating $U_{\text{max}}$ and $\epsilon_1$. This gives a so-called critical temperature of

\[
T^2 = \frac{1}{6\lambda^2} \left( \frac{2\pi}{M^2} \right)^3.
\]
On the other hand we can find the temperature for which the denominator of Eq. (5.18) vanishes. In the same high $T$ approximation we get

$$T^2 = \frac{4}{9\lambda^2} \left( \frac{2\pi}{M^2} \right)^3.$$  

(5.23)

Parametrically these two estimates are identical and equivalent to a limiting temperature of order $T \sim m_s/g_o^2$. At this temperature there is no longer an identifiable expansion parameter. If one wants to consider temperatures higher than the limiting temperature one finds a negative entropy density indicating that the system is unstable ($\phi = 0$ no longer being a stable vacuum).

Our loop calculations suggest some general power counting arguments. For low temperature an $l$-loop graph is suppressed as $g_o^{2(l-1)}$, while at high temperature the expansion parameter is $(g_o^2 T/m_s)^{l-1}$. What is left undetermined of course is the coefficient in front of the power, and one would need to sum all the diagrams at a given loop level to determine this. For the necklace diagrams this coefficient remained a constant with increasing loops leading to the singular expression for the pressure (5.18). However, it is possible that when one accounts for all the diagrams, the coefficients may decrease faster than $1/l$ with increasing loops and therefore lead to no divergences. However, in the light of the physical argument we have presented which relates the existence of the tachyon with the occurrence of the singular behavior, this seems rather unlikely. In a realistic string model one does not expect real tachyons to be present. Accordingly, we expect that either the presence of supersymmetry and/or additional interactions will ameliorate the above singularity. In fact, soliton contributions become important at the same temperature [47].

6. Arbitrary Even Powered Potentials

The above analysis can easily be extended to an interaction term of the form $-\lambda\phi^{2N}$ where $N = (p+1)/2$. (The previous analysis corresponds to $N = 2$.) Note that $\lambda$ has energy dimension $-4(N-1)$.

6.1 Lowest order diagrams

Consider the partition function to first order in $\lambda$. It involves a diagram with one vertex and $2N$ legs which can be connected into $N$ loops. See Figure 5. Taking into account the combinatoric factor gives

$$\ln Z_1 = (2N - 1)!!(-\lambda)\beta V \left[ T \sum_n \int \frac{d^3k}{(2\pi)^3} D_0(\omega_n, k) \right]^N$$

(6.1)

Similarly, the first order contribution to the self-energy is

$$\Pi_1 = 2N(2N - 1)!!\lambda \left[ T \sum_n \int \frac{d^3k}{(2\pi)^3} D_0(\omega_n, k) \right]^{N-1}$$

(6.2)
Figure 7: This four loop diagram is the leading order in $\lambda$ for the case $p = 7$.

We add the counter-term $-\frac{1}{2} \gamma \phi^2$ and adjust the coefficient to cancel the self-energy contribution at $T = 0$:

$$
\gamma = -2N(2N - 1)!! \lambda \left( \frac{M^2}{4\pi} \right)^{2(N-1)}.
$$

This leads to the total pressure at first order

$$
P_1 = -(2N - 1)!! \lambda \left( \frac{M^2}{4\pi} \right) \sqrt{2\pi T} \left( \frac{2\pi T}{M} \right) \left( \frac{2\pi T}{M} \right)^{N-1} - N.
$$

The vacuum energy density is

$$
\epsilon_{\text{vac}} = -(N - 1)(2N - 1)!! \lambda \left( \frac{M^2}{4\pi} \right)^{2N},
$$

and the high temperature limit of the pressure is

$$
P_1 = -(2N - 1)!! \lambda \left( \frac{M}{2\sqrt{\pi}} \right)^{3N} T^N,
$$

which is notable for its proportionality to $T^N$.

### 6.2 Necklace diagrams

The necklace diagrams are obtained by connecting each vertex with two legs. When $N = 2$ the end vertices have one closed loop attached to them while the interior vertices have none. When $N > 2$ the end vertices have $N - 1$ closed loops attached to them while the interior vertices have $N - 2$ closed loops attached (see Figure 8). The pressure contribution of the set of such diagrams is

$$
P_{\text{necklace}} = (-\lambda)(2N - 1)!! \left[ T \sum_n \int \frac{d^3k}{(2\pi)^3} D_0(\omega_n, k) \right]^N \times
$$
Figure 8: The necklace diagram of fourth order in $\lambda$ for the case $p = 7$.

$$\sum_{l=0}^{\infty} \left\{ (-\lambda)4(2N-1)!! \left[ T \sum_n \int \frac{d^3k}{(2\pi)^3} D_0(\omega_n, k) \right]^{N-2} T \sum_n \int \frac{d^3k}{(2\pi)^3} D_0^2(\omega_n, k) \right\}^l$$

$$= \frac{- (2N-1)!! \lambda \left[ T \left( \frac{M}{2\sqrt{\pi}} \right)^3 \varsigma \left( \frac{2\pi T}{M} \right) \right]^N}{1 + 4(2N-1)!! \lambda T \left( \frac{M}{2\sqrt{2\pi}} \right)^3 \varsigma \left( \frac{2\sqrt{2\pi} T}{M} \right) \left[ T \left( \frac{M}{2\sqrt{\pi}} \right)^3 \varsigma \left( \frac{2\pi T}{M} \right) \right]^{N-2}} \quad (6.7)$$

After taking into account self-energy corrections on the loops attached to the end vertices we finally get

$$P_{\text{necklace}} = \frac{- (2N-1)!! \lambda \left( \frac{M^2}{4\pi} \right)^{2N} \frac{2\sqrt{\pi} T}{M} \varsigma \left( \frac{2\sqrt{2\pi} T}{M} \right) \left[ \left( \frac{2\sqrt{2\pi} T}{M} \varsigma \left( \frac{2\pi T}{M} \right) \right)^{N-1} - N \right]}{1 + 4(2N-1)!! \lambda T \left( \frac{M}{2\sqrt{2\pi}} \right)^3 \varsigma \left( \frac{2\sqrt{2\pi} T}{M} \right) \left[ T \left( \frac{M}{2\sqrt{\pi}} \right)^3 \varsigma \left( \frac{2\pi T}{M} \right) \right]^{N-2}} \quad (6.8)$$

As before, there is a maximum temperature determined by the vanishing of the denominator. As with the 3-adic case, we can ascribe this singular behavior to the presence of the tachyon in the potential. The important point is that the potential is unbounded from below for large values of $\phi$. The potential is

$$U(\phi) = \frac{1}{2} \phi^2 + \lambda \phi^{2N} \quad (6.9)$$

ignoring the correction from $\gamma \phi^2$. The maximum height of the potential is

$$U_{\text{max}} = \frac{N - 1}{2N} \left( \frac{-1}{2NL} \right)^{1/(N-1)} \quad (6.10)$$

For the sake of estimation, use the $N$-loop expression for the energy density in the limit $T > M$. It is

$$\epsilon_1 = -\lambda(N - 1)(2N - 1)!! \left( \frac{M}{2\sqrt{\pi}} \right)^{3N} T^N \quad (6.11)$$
When the temperature is high enough the states will spill over the top of the potential. One can estimate this temperature by equating $U_{\text{max}}$ and $\epsilon_1$. This gives a so-called critical temperature of

$$T = \left(\frac{1}{(2N-1)!!}\right)^{1/N} \left(\frac{-1}{2N\lambda}\right)^{1/(N-1)} \left(\frac{2\sqrt{\pi}}{M}\right)^3$$

(6.12)

On the other hand we can find the temperature for which the denominator of the necklace pressure vanishes. In the same high $T$ approximation we get

$$T = \left(\frac{-1}{\sqrt{2}(2N-1)!!\lambda}\right)^{1/(N-1)} \left(\frac{2\sqrt{\pi}}{M}\right)^3$$

(6.13)

Parametrically these two estimates are identical.

6.3 Sunset diagrams

The sunset diagram has two vertices, and every leg of one vertex is connected to a leg of the other vertex. This is the generalization of the usual sunset diagram from $\phi^4$ theory. See Figure 9. Apart from a combinatoric factor, what needs evaluation is

$$\left[ T\sum_{n_1} \int \frac{d^3k_1}{(2\pi)^3} D_0(\omega_{n_1},k_1) \right] \cdots \left[ T\sum_{n_{2N}} \int \frac{d^3k_{2N}}{(2\pi)^3} D_0(\omega_{n_{2N}},k_{2N}) \right]$$

$$\times (2\pi)^3 \delta(k_1 + \cdots + k_{2N}) \beta \delta_{n_1 + \cdots + n_{2N},0}.$$  

The momentum conserving delta function can be expressed in integral form. Then the momentum integrals factorize and the remaining integral evaluated with the result

$$P_{\text{sunset}} = \lambda^2 \frac{(2N)!}{2(2N)^{3/2}} \left(\frac{M}{2\sqrt{\pi}}\right)^{3(2N-1)} \chi(T,M).$$

(6.14)

Here the combinatoric factor has been included and

$$\chi(T,M) = \left[ T\sum_{n_1} e^{-x^2n_1^2} \right] \cdots \left[ T\sum_{n_{2N}} e^{-x^2n_{2N}^2} \right] \beta \delta_{n_1 + \cdots + n_{2N},0}.$$  

(6.15)
Analogous to the case \( p = 3 \) we can write
\[
\chi(T, M) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \left[ \theta_3 \left( \frac{1}{2} e^{-x^2} \right) \right]^{2N} \tag{6.16}
\]

The low and high temperature limits are readily obtained. The high temperature limit is dominated by all \( n_i = 0 \). Hence
\[
\chi(T \gg M) = T^{2N-1} \tag{6.17}
\]
In the low temperature limit the sums can be replaced by integrals and the limits on \( \phi \) extended to \( \pm \infty \).
\[
\chi(T \ll M) = \frac{1}{\sqrt{2N}} \left( \frac{M}{2\sqrt{\pi}} \right)^{2N-1} \tag{6.18}
\]
Therefore
\[
P_{\text{sunset}}(T \gg M) = \lambda^2 \frac{(2N)!}{2(2N)^{3/2}} \left( \frac{M}{2\sqrt{\pi}} \right)^{3(2N-1)} T^{2N-1} \tag{6.19}
\]
and
\[
P_{\text{sunset}}(T \ll M) = \lambda^2 \frac{(2N)!}{8N^2} \left( \frac{M^2}{4\pi} \right) ^{2(2N-1)}. \tag{6.20}
\]
These are precisely the same asymptotic behaviors that one obtains from the 3-loop necklace diagram.

In particular, both the sunset and necklace contributions grow at high temperature as \( T^{2N-1} \) compared to the lowest order in \( \lambda \) contribution which grows as \( T^N \). The ratio is
\[
P_{2, \text{necklace}}/P_1 \sim P_{\text{sunset}}/P_1 \sim \lambda (M^3T)^{N-1}. \tag{6.21}
\]
This ratio is small and only approaches one when the limiting temperature, as defined by the vanishing of the denominator in the formula for the necklace contribution, is reached. Hence these 3-loop contributions can be neglected since it is important only at unreachable temperatures where other contributions undoubtedly become important too. Also notice that the two contributions do not cancel each other because they both have positive coefficients.

### 7. D dimensional \( p \)-adic Theory and the Vacuum Energy

In this section our aim is to calculate the cosmological constant that one obtains in \( p \)-adic string theories for any general value of \( p \), and in particular to see how it depends on the different scales and couplings. First of all, we note that \( p \)-adic strings can be formulated in arbitrary space time dimension; however, what we observe today is the effective four dimensional cosmological constant. Thus we proceed in two steps. In the first step, we compute the \( D \) dimensional cosmological constant for arbitrary \( p \). For this purpose we are going to assume that the extra \( d = D - 4 \) dimensions are all compactified on a circle of radius \( R \). In the second step, we obtain the dimensionally reduced cosmological constants. In particular, the results
of the computation simplifies considerably in two opposite limits: (a) \(RM \ll 1\), which corresponds to the usual dimensional reduction where one only includes the zero-modes, and (b) \(RM \gg 1\), the t-dual limit which is interesting for TeV scale phenomenology. We therefore can obtain estimates of the cosmological constant in terms of the physical parameters in these two limits.

### 7.1 3-adic in arbitrary dimensions

#### 7.1.1 Higher dimensional vacuum energy

We will first illustrate our method for 3-adic theory and later generalize to the \(p\)-adic case. Starting from an arbitrary \(D\) dimensional theory we are led to the action

\[
S = \int_0^\beta d\tau \int_0^{2\pi R} dy \int d^D x \left[ -\frac{i}{2} \phi e^{-\left(\partial^2/\partial\tau^2 + \nabla_y^2 + \nabla_x^2\right)/M^2} \right],
\]

where here \(\phi = \phi(x, y, \tau)\). Notice that \(\phi\) now has mass dimension \(D/2\), while

\[
\lambda = -\frac{m_s - D g_0^2}{18}
\]

has mass dimension \(-D\). All the calculations that were done for four dimensions now go through except that instead of one compact dimension (temperature) we now have \(d + 1\) of them. Accordingly, we have to replace

\[
\beta \to \beta (2\pi R)^d
\]

and

\[
\sum_n \int \frac{d^D k}{(2\pi)^D} e^{-(\omega_n^2 + k^2)/M^2} \to \sum_n \sum_{\{n_i\}} \int \frac{d^D k}{(2\pi)^D} e^{-(\omega_n^2 + \sum_i \omega_{n_i}^2 + k^2)/M^2}
\]

\[
\equiv \sum_n \sum_{\{n_i\}} \int \frac{d^D k}{(2\pi)^D} D_d(\{\omega_n\}, k)
\]

where

\[
\omega_{n_i} = \frac{n_i}{R}.
\]

According to the generalization, the \(D\) dimension partition function at order \(\lambda\) reads

\[
\ln Z_1 = 3(-\lambda) \beta V (2\pi R)^d \left[ T (2\pi R)^{-d} \sum_n \int \frac{d^D k}{(2\pi)^D} D_d(\omega_n, k) \right]^2.
\]

It is useful to define

\[
\sum_n \int \frac{d^D k}{(2\pi)^D} D_d(\omega_n, k) = \left( \frac{M}{2\sqrt{\pi}} \right)^3 \varsigma_d \left( \frac{2\pi T}{M}, \frac{1}{RM} \right),
\]

analogous to (3.2) valid for four dimensions. It is easy to check that

\[
\varsigma_d(x, y) = \varsigma(x) \varsigma^d(y),
\]
One can now follow all the steps leading to the positive cosmological constant. The pressure in $D$ dimensions, coming from the partition function is

$$P_1 = 3(-\lambda) \left( \frac{M}{2\sqrt{\pi}} \right)^6 \left[ T(2\pi R)^{-d} \varsigma_d \left( \frac{2\pi T}{M}, \frac{1}{RM} \right) \right]^2 . \tag{7.9}$$

while the self-energy is given by

$$\Pi_1 = 12\lambda T(2\pi R)^{-d} \sum_n \int \frac{d^3 k}{(2\pi)^3} D_d(\{\omega_n\}, k)$$

$$= 12\lambda T(2\pi R)^{-d} \left( \frac{M}{2\sqrt{\pi}} \right)^3 \varsigma_d \left( \frac{2\pi T}{M}, \frac{1}{RM} \right) . \tag{7.10}$$

Again, the appearance of a pole can be avoided by adding a counter-term to the Lagrangian of the form $-\frac{1}{2}\gamma \phi^2$ and adjusting $\gamma$ to cancel the self-energy contribution. This leads to

$$\gamma = -\frac{3\lambda M^4}{4\pi^2} (2\pi R)^{-d} \varsigma_d \left( \frac{1}{RM} \right) . \tag{7.11}$$

The contribution of the counter-term to the pressure is given by

$$P_c = -\frac{1}{2} \frac{\gamma T}{(2\pi R)^d} \sum_n \int \frac{d^3 k}{(2\pi)^3} D_d(\{\omega_n\}, k)$$

$$= \frac{3\lambda M^4 T}{8\pi^2 (2\pi R)^d} \left( \frac{M}{2\sqrt{\pi}} \right)^3 \varsigma_d \left( \frac{2\pi T}{M}, \frac{1}{RM} \right) . \tag{7.12}$$

The total two loop pressure then reads

$$P_1 = -3\lambda \left( \frac{M^2}{4\pi} \right)^4 2\sqrt{\pi} T \frac{2\pi T}{M} \varsigma^{2d} \left( \frac{1}{RM} \right) \frac{2\sqrt{\pi} T}{M} \varsigma \left( \frac{2\pi T}{M} \right) - 2 \right] . \tag{7.13}$$

The $D$ dimensional cosmological constant is thus given by

$$\Lambda_D = -3\lambda \left( \frac{M^2}{4\pi} \right)^4 \varsigma^{2d} \left( \frac{1}{RM} \right) . \tag{7.14}$$

### 7.1.2 Dimensional reduction

We start by noting that $D$ and four dimensional pressures and energy densities are just related via

$$P_4 \text{ dimensions} = (2\pi R)^d P_D \text{ dimensions}$$

$$\Lambda_4 \text{ dimensions} = (2\pi R)^d \Lambda_D \text{ dimensions} . \tag{7.15}$$

We conclude that

$$\Lambda = -\frac{3\lambda \varsigma^{2d}}{(2\pi R)^d} \left( \frac{M^2}{4\pi} \right)^4 . \tag{7.16}$$
It is interesting to look at the two opposite limits. For $MR \ll 1$, $\zeta(1/RM)$ is approximately 1, and we have

$$\Lambda = -\frac{3\lambda}{(2\pi R)^d} \left(\frac{M^2}{4\pi}\right)^4 \frac{g_o^2}{6m_s^4(2\pi R)^d} \left(\frac{M^2}{4\pi}\right)^4 \quad \text{for } R \ll M^{-1} \quad (7.17)$$

We would have got exactly the same expression by first performing a dimensional reduction of the $D$ dimensional 3-adic action, and then realizing that the $\lambda$ in (2.1) has to be rescaled by

$$\lambda \rightarrow \frac{\lambda}{(2\pi R)^d}$$

On the other hand when $MR \gg 1$, using the approximation $\zeta = (\sqrt{\pi MR})^d$ we have

$$\Lambda = -\frac{3\lambda M^{2D}}{(4\pi)^d} \left(\frac{R}{2}\right)^d \frac{g_o^2 M^{2D}}{6m_s^4(4\pi)^d} \left(\frac{R}{2}\right)^d \quad \text{for } R \gg M^{-1} \quad (7.18)$$

Physically it is more clarifying to re-express the cosmological constant in terms of the Planck and string scales. To do this we need to compactify the $D$ dimensional gravitational action. Again considering $d$ compact toroidal extra dimensions, it is possible to perform a straightforward dimensional reduction of the 10 dimensional supergravity action.

$$S_{\text{sugra}} = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} e^{-2\Phi} R_{10} + \cdots \quad (7.19)$$

Here $\Phi$ is the dilaton which we assume to be stabilized and as usual identified with the string coupling constant, $g_o = e^\Phi$, and $G_{10} = (2\pi^2)^3\alpha'^4$ is the 10 dimensional gravitational constant. Performing the standard dimensional reduction we find

$$S_{\text{sugra}} = \frac{V_6}{16\pi(2\pi^2)^3g_o^2\alpha'^4} \int d^4x \sqrt{-g} R_4 = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R_4 \quad (7.20)$$

where $G_N$ is Newton’s constant and $V_6$ is the volume of the compactified space, in this case $(2\pi R)^6$. Thus one finds the well known relation between the 10 dimensional string scale and the Planck mass.

$$M_P^2 = \frac{1}{G_N} = \frac{V_6}{(2\pi^2)^3g_o^2\alpha'^4} = \frac{2V_6m_s^8}{\pi^6g_o^2} \quad (7.21)$$

For the two limits the cosmological constant reads

$$\Lambda \sim \begin{cases} (m_s/M_P)^6M_P^4 & \text{for } R \ll M^{-1} \\ (m_sR)^{12}(m_s/M_P)^6M_P^4 & \text{for } R \gg M^{-1} \end{cases} \quad (7.22)$$

An intriguing feature of the above expressions is that the cosmological constant is suppressed by some power of the ratio of the string scale to the Planck scale. In particular there is considerable phenomenological interest when $m_s \sim \text{TeV}$, but we see by power counting that for the 3-adic case, the suppression is not sufficient.
7.2 *Arbitrary* $N$

We are now ready to compute the cosmological constant for arbitrary values of $N$. The analysis with $N = 2$ can easily be extended to an interaction term of the form

$$-\lambda \phi^{2N} \text{ with } \lambda \equiv -\frac{m_s^{-D(N-1)} g_p}{2N}$$

(7.23)

that arises from the $p$-adic action (7.1). We will focus only on the case when $p$ is an odd positive integer. We note that $\lambda$ has energy dimension $-D(N - 1)$.

Consider the partition function to first order in $\lambda$. It involves a diagram with one vertex and $2N$ legs which can be connected into $N$ loops. Taking into account the combinatoric factor gives

$$\ln Z_1 = (2N - 1)!!(-\lambda)\beta(2\pi R)^dV \left[ T(2\pi R)^{-d} \sum_{\{n_i\}} \int \frac{d^3k}{(2\pi)^3} D_d(\{\omega_{n_i}\}, k) \right]^N$$

(7.24)

leading to a $D$ dimensional pressure

$$P_1 = (2N - 1)!!(-\lambda) \left[ \left( \frac{M}{2\sqrt{\pi}} \right)^3 T\varsigma \left( \frac{2\pi T}{M} \right) \left( \frac{\varsigma(\frac{1}{M R})}{2\pi R} \right)^d \right]^N$$

(7.25)

As before one can also calculate the first order contribution to the self-energy:

$$\Pi_1 = 2N(2N - 1)!!\lambda \left[ T(2\pi R)^{-d} \sum_{\{n_i\}} \int \frac{d^3k}{(2\pi)^3} D_d(\{\omega_{n_i}\}, k) \right]^{N-1}$$

(7.26)

Again, we can add the counter-term $-\frac{1}{2} \gamma \phi^2$ and adjust the coefficient to cancel the self-energy contribution at $T = 0$.

$$\gamma = -2N(2N - 1)!!\lambda \left[ \left( \frac{M}{2\sqrt{\pi}} \right)^4 \left( \frac{\varsigma(\frac{1}{M R})}{2\pi R} \right)^d \right]^{N-1}$$

(7.27)

This leads to the total pressure

$$P_1 = -(2N - 1)!!\lambda \left( \frac{M^2}{4\pi} \right)^{2N} \left( \frac{\varsigma(\frac{1}{M R})}{2\pi R} \right)^d \left( \frac{2\sqrt{\pi}T}{M} \right)^{2N} \left( \frac{2\pi T}{M} \right)^{N-1}$$

(7.28)
at first order. In particular, the high temperature limit of the pressure is
\[ P_1 = -(2N - 1)!! \lambda \left( \frac{M}{2\sqrt{\pi}} \right)^{3N} \left( \frac{1}{M R} \right)^{dN} T^N \tag{7.29} \]
which is notable for its proportionality to \( T^N \).

Finally, by looking at the \( T \to 0 \) limit we find that the \( D \) dimensional vacuum energy density is
\[ \Lambda_{\text{vac}} = -(N - 1)(2N - 1)!! \lambda \left( \frac{\frac{1}{M R}}{2\pi R} \right)^{dN} \left( \frac{M^2}{4\pi} \right)^{2N}. \tag{7.30} \]
After dimensional reduction we have
\[ \Lambda_4 = -(N - 1)(2N - 1)!! \lambda (2\pi R)^d \left( \frac{\frac{1}{M R}}{2\pi R} \right)^{dN} \left( \frac{M^2}{4\pi} \right)^{2N}. \tag{7.31} \]
We can again look into the two limiting cases.
\[ \Lambda = \begin{cases} -(N - 1)(2N - 1)!! \lambda (2\pi R)^{-d(N-1)} \left( \frac{M^2}{4\pi} \right)^{2N} \sim \left( \frac{m_s}{M_p} \right)^{2(N+1)} M_p^4 & \text{for } R \ll M^{-1} \\ -(N - 1)(2N - 1)!! \lambda (2\pi R)^d \left( \frac{M}{2\sqrt{\pi}} \right)^D \sim g_0^{2N} \left( \frac{m_s}{M_p} \right)^2 M_p^4 & \text{for } R \gg M^{-1} \end{cases} \tag{7.32} \]

### 7.3 Numerical Results for the Cosmological Constant

We can estimate what parameter choices would reproduce the observed cosmological constant \( \Lambda = (2.3 \text{ meV})^4 \). We will specialize to the case when \( D = 6 \ i.e., \ we have six compact extra dimensions. The Planck mass is related to Newton’s constant, the open string coupling, the Regge slope, and the volume of compactified space by
\[ M_P^2 = \frac{1}{G_N} = \frac{V_6}{(2\pi^2)^3 g_s^2 \alpha' ^4}, \tag{7.33} \]
with \( m_s^2 = 1/2\alpha' \). Rewriting in terms of the string scale and the radii of the compactified dimensions we have
\[ M_P^2 = \frac{(2\pi R)^6 m_s^{6} 16}{(2\pi^2)^3 g_s^2} = \frac{2^7 R^6 m_s^8}{g_s^2} \tag{7.34} \]
This imposes a constraint on the three independent parameters in the theory, \( m_s, g_0 \) and \( R \). In terms of the same variables, the coupling constant is given by
\[ \lambda = -\frac{g_p^{2(N-1)}}{2N m_s^{10(N-1)}}. \tag{7.35} \]
We are now ready to estimate the cosmological constant in \( p \)-adic theories. Let us first look at the \( MR \ll 1 \) limit. In this case we have
\[ \Lambda = -\lambda \frac{(N - 1)(2N - 1)!!}{(2\pi R)^{6(N-1)}} \left( \frac{M^2}{4\pi} \right)^{2N}. \tag{7.36} \]
Curiously, one observes that in both the expression for $M_p$ and $\Lambda$, the same combination of $g_o/R^3$ appears, and hence one can eliminate them to obtain the value of the string scale required to obtain the correct hierarchy between the observed cosmological constant and the Planck scale:

$$\frac{\Lambda}{M_p^4} = Q_\prec \left( \frac{m_s}{M_p} \right)^{2(N+1)} \quad \text{with} \quad Q_\prec \equiv \frac{(N-1)(2N-1)!}{2^{N+2}N^8(2N-1)^{2N}(\ln(2N-1))^{2N}} \frac{2(2N-1)}{(2N-1)^2}^{N-1}.$$  \hfill (7.37)

For example, by using $M_P = 1.22 \times 10^{19}$ GeV, the numerical values for the string scale are $m_s = 0.55$ GeV, 1820 TeV, and 385 PeV for $N = 2$, 3, and 4, respectively.

Rewriting the constraint (7.34) we have

$$g_o^2 = 2^7 (m_s R)^6 \left( \frac{m_s}{M_p} \right)^2 < \left( \frac{m_s}{M_p} \right)^2.$$  \hfill (7.38)

Thus we find that for $N = 3, 4$ the value of the string coupling can at most be $10^{-15}$ and $10^{-12}$ respectively. For larger values of $N$, the values of the string coupling can be larger. For example, $N = 9$ and $m_s = 10^{15}$ GeV, allows us to have $g_o \sim 10^{-4}$. A coupling of order $g_o \sim 10^{-2}$ can be achieved for $N > 11$, for instance with $N = 12$, and $m_s = 10^{17}$ GeV. One can see the typical dependence with temperature in Figure 10, where we plot the energy density. In Figure 11, one can see the entropy density versus temperature.

Let us now look at the opposite case when $MR \gg 1$. In this case

$$\Lambda = -\lambda(N-1)(2N-1)!!(2\pi R)^6 \left( \frac{M}{2\sqrt{\pi}} \right)^{10N}.$$  \hfill (7.39)

As compared to the $MR \ll 1$ case, it is important to note that in this case, $g_o$ and $R$ appears in different combinations in the expression of $\Lambda$ and $M_p$. This enables

![Figure 10](image10.png)  \hfill Figure 10: Energy density ($\rho$) as a function of the temperature ($T$), for $N = 4$, $m_s = 385$ PeV and small coupling $g_o \ll 1$. 

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Figure 11: Entropy density \( s \) as a function of the temperature \( T \), for \( N = 4, m_s = 385 \) TeV and small coupling \( g_0 \ll 1 \).

Let us to be able to choose the value of \( m_s \) freely. As before we can eliminate the compactification radius in favor or \( M_p \):

\[
\frac{\Lambda}{M_p^4} = Q > \left( \frac{m_s}{M_p} \right)^2 g_o^{2N} \quad \text{with} \quad Q > \equiv \frac{(N - 1)(2N - 1)!!}{2^N N (\ln(2N - 1))^3 N (2\pi)^{5N - 6}} \left( \frac{2(N - 1)}{2N - 1} \right)^{N-1}.
\]

(7.40)

Let us choose \( m_s \sim 10 \) TeV. Ignoring order one factors, we find the following estimates: \( g_o \sim 10^{-22}, 10^{-14}, 10^{-9}, \) and \( 10^{-2} \) for \( p = 2, 3, 4, \) and 9 respectively. Again, as we increase the values of \( N \), the string coupling required for producing the hierarchy becomes larger.

However, we should check that these values are consistent with (7.34). It is easy to see that we must now satisfy

\[
g_o^2 > \left( \frac{m_s}{M_P} \right)^2 \sim 10^{-30}
\]

(7.41)

Thus we again find that while the \( N = 2 \) case is excluded, \( N = 3 \) and higher are consistent.

8. Discussion and Conclusion

We have analyzed the main thermodynamic properties of p-adic string models that describe the tachyon phenomenology of string theory. We have reproduced qualitatively many results of string theories at finite temperature already discussed in the literature; such as thermal duality at leading order for \( p = 3 \), or the temperature dependence of radiative corrections.

On the other hand, p-adic models constitute a motivated example of field theories characterized by non-local propagators. The finite temperature computations we
have developed constitute a basic approach that may be generalized to other tachyonic field models from string theory reduction, non-commutative field theories, or quantum algebras. We do not mean that all the technical details and specific results will apply, but rather the algorithms for the perturbative treatment we developed should be. In any case, a particular property of the p-adic case is the absence of poles in the propagator. Non-local field theories with real states at finite temperature are expected to have a contribution from the free theory similar to standard thermodynamics.

One can contemplate non-local generalizations of the Standard Model. From the field theory perspective, we have provided a unique prescription of adding counter-terms loop-by-loop which ensures that the theory remains ghost free and hence consistent. The counter-terms are of fundamental importance since they determine the thermodynamic properties of the model at low temperatures. In the p-adic case, it is able to change the sign of the vacuum energy, giving a positive cosmological constant that is suppressed by the ratio of the string to the Planck scale. In fact, we are even able to reproduce the observed value of the cosmological constant for a range of relevant parameters. However, we need a small string coupling to achieve this.

Our study unfolded as follows. First, we studied the finite temperature contribution of the free theory and determined that its vanishing contribution is consistent with the physical degrees of freedom of the theory. Specifically, the p-adic model gives a zero contribution due to the non-existence of poles in the p-adic propagator. The interacting theory is not trivial, and the first non-vanishing contribution to the partition function comes from the diagram with one vertex. For example, for \( p = 3 \), this contribution is given by a 2-loop diagram that satisfies the well-known thermal duality of string theory, namely, a symmetry between the behaviors at low and high temperatures. However, we found that this result does not generalize to arbitrary \( p \) and breaks down at higher order in the coupling, i.e. high temperatures. The symmetry is also broken if one takes into account the self-energy counter-term.

We then computed some of the typical higher order finite temperature loop corrections, such as ring, necklace and sunset diagrams. It is interesting to note that the non-local theories have a better UV behavior as compared to the local theories, and for the p-adic case, none of these individual diagrams are divergent. Also, we found that ring diagrams provide a series with good convergence properties, in contrast with the infrared divergences that are usually found in the local counter-parts. On the other hand, the infinite series arising from the necklace diagrams diverges at a finite temperature. This behavior may be related to the fact that the p-adic potential is not bounded from below.

In general, the picture that has emerged from our study is that one can trust perturbation theory as long as \( T < m_s/g_o^2 \). For small string coupling this temperature can be quite high, and therefore we can probe the Hagedorn transition. However, once we reach \( T \sim m_s/g_o^2 \), one can no longer trust perturbation theory, all the higher order terms become important, and the partition function seems to diverge. In addition, at this temperature, contributions from non-perturbative states start to be important\[47\]. What happens at even higher temperatures is an open question.
Appendix: Dimensional reduction of the vacuum energy

In this appendix we perform the usual dimensional reduction procedure, valid as long as $MR \ll 1$, to obtain the effective four dimensional vacuum energy. We will see that this is identical to what we derived in section 7. Let us start with the higher dimensional $p$-adic action (1.1):

$$S = \frac{m_s^D}{g_p^2} \int d^D x \left[ -\frac{1}{2} \phi e^{-\Box/M^2} \phi + \frac{1}{2N} \phi^{2N} \right],$$

where to maintain clarity we have introduced $\bar{g}_p$ to denote the $D$ dimensional coupling constant. When $MR \ll 1$, one can perform the usual dimensional reduction where one only keeps the zero modes. This leads rather straightforwardly to the effective four dimensional action

$$S = \frac{m_s^D (2\pi R)^d}{g_p^2} \int d^4 x \left[ -\frac{1}{2} \phi e^{-\Box/M^2} \phi + \frac{1}{2N} \phi^{2N} \right],$$

(1)

where now the $\Box$ operator just represents the four dimensional D'Alembertian. This is exactly of the same form as the 4 dimensional $p$-adic action with the following identification

$$g_p^2 = \frac{\bar{g}_p^2}{(2\pi RM_s)^d}. \quad (2)$$

One can now simply read off the 2-loop four dimensional cosmological constant from the result (6.5) derived in section 7:

$$\epsilon_{\text{vac}} = -(N - 1)(2N - 1)!! \lambda \left( \frac{M^2}{4\pi} \right)^{2N}$$

$$= \frac{(N - 1)(2N - 1)!!}{2N} m_s^{4(N-1)} \left( \frac{\bar{g}_p^2}{(2\pi RM_s)^d} \right)^{2(N-1)} \left( \frac{M^2}{4\pi} \right)^{2N}. \quad (3)$$

This is precisely the same result. Let us try to estimate this in the general case of $d$ extra dimensions. For this purpose we note that the dimensional reduction of the gravitational action leads to a relation between the effective four-dimensional Planck's constant, the string scale and the internal radius, of the form

$$M_p^2 \sim \frac{(2\pi R m_s)^d m_s^2}{g_o^2}. \quad (4)$$

Rewriting the relation we find

$$\frac{g_o^2}{(2\pi R m_s)^d} \sim \left( \frac{m_s}{M_p} \right)^2. \quad (5)$$

Now perimetrically, the vacuum energy is given by

$$\epsilon_{\text{vac}} \sim m_s^4 \left[ \frac{\bar{g}_p^2}{(2\pi RM_s)^d} \right]^{2(N-1)} \sim m_s^4 \left[ \frac{m_s}{M_p} \right]^{4(N-1)}, \quad (6)$$

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We can draw a very important conclusion from the above result, namely that the hierarchial suppression of the vacuum energy in $p$-adic models is independent of the number of extra dimensions, at least in the limit of small radius, $MR \ll 1$.

This approach also allows us to estimate the effect of higher loops on the value of the cosmological constant. As we argued in section 5, at low temperatures, higher loop corrections are suppressed as $g_o^{-1}$, where $g_o$ represents the effective four-dimensional open string coupling. Thus, as long as we have $g_o \ll 1$, we can trust our 2-loop result for the cosmological constant. For the higher dimensional theory, this means

$$\frac{\bar{g}_o^2}{(2\pi R M_s)^d} \sim \frac{m_s^2}{M_P^2} \ll 1.$$  (7)

To be specific, for the ten dimensional supergravity we have

$$\frac{\bar{g}_o^2}{(2\pi R M_s)^d} \sim \frac{2m_s^2}{\pi^6 M_P^2} \ll 1.$$  (8)

Since $m_s < M_p$, this condition can easily be met.

Acknowledgements

We thank Neil Barnaby, Gianluca Calcagni, Debashish Ghoshal and Bala Sathia-panal for useful and interesting conversations on the subject. This work was supported by the U.S. DOE Grant Nos. DE-FG02-87ER40328 and DOE/DE-FG02-94ER40823, the FPA 2008-00592 (DGICYT, Spain), the CAM/UCM 910309, and MICINN Consolider-Ingenio MULTIDARK CSD2009-00064.

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