The Reliability Function of Lossy Source-Channel Coding of Variable-Length Codes with Feedback

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Abstract

We consider transmission of discrete memoryless sources (DMSes) across discrete memoryless channels (DMCs) using variable-length lossy source-channel codes with feedback. The reliability function (optimum error exponent) is shown to be equal to \( \max \{0, B(1 - R(D)/C)\} \), where \( R(D) \) is the rate-distortion function of the source, \( B \) is the maximum relative entropy between output distributions of the DMC, and \( C \) is the Shannon capacity of the channel. We show that, in this setting and in this asymptotic regime, separate source-channel coding is, in fact, optimal.

Index Terms

Variable-length codes, Source-channel coding, Feedback, Reliability function.

I. INTRODUCTION

The communication model for discrete memoryless channel (DMCs) with feedback in which the blocklength \( \tau \in \mathbb{N} \) is a random variable whose expectation is over bounded by some positive real number \( N \in \mathbb{R}_+ \) was first proposed by Burnashev in a seminal work [1]. He demonstrated that the reliability function or optimal error exponent for DMCs with feedback improves dramatically over the no feedback case and the case where the blocklength is deterministic. This class of codes is known as variable-length codes with feedback. The reliability function of a DMC with variable-length feedback admits a particularly simple expression \( E_{\text{Burn}}(R) = B(1 - R/C) \) for all rates \( 0 \leq R \leq C \), where \( C \) is the capacity of the DMC and \( B \) (usually written as \( C_1 \) in the literature) is the relative entropy between conditional output distributions of the two most “most distinguishable” channel input symbols [1]. In this paper, we consider variable-length transmission of a discrete memoryless source (DMS) over a DMC with feedback under an excess-distortion constraint. Different from the recent elegant work of Kostina, Polyanskiy, and Verdú [2] which considers the minimum expected delay (length) of such variable-length joint source-channel codes with feedback under a non-vanishing excess-distortion probability, we are interested in finding the optimal excess-distortion exponent (reliability function) of such codes.

A. Related Works

Source-channel codes with fixed (non-random) source and channel block lengths were first introduced by Shannon [3] in 1959. By Shannon’s fundamental source and channel coding theorems, transmission with vanishing probability of error is possible whenever the source entropy (or rate-distortion function in the lossy case) is less than the channel capacity. Gallager [4] and Jelinek [5] indicated that direct (not separate) source-channel coding can lead to a larger error exponent, which means that the separation rule for joint source-channel coding does not hold from the error exponent perspective. For \( R \geq R_{cr} \) (where \( R_{cr} \) is the critical rate of the DMC), Csiszár [6] later proved that the optimal error exponent for lossless joint source-channel coding is equal to \( \min_{R}(e(R) + E(R)) \), where \( e(R) \) and \( E(R) \) are reliability functions of the DMS and DMC, respectively. The achievable joint source-channel coding scheme which was proposed in [6] is a universal code, i.e., the coding scheme does not depend on knowledge of the DMS or the DMC.

Wang, Ingber, and Kochman [7] recently proved that the no-excess-distortion probability has an exponential behavior for any lossy joint source-channel codes with fixed-length joint source-channel coding under the condition that \( R(D) > C \) (assuming that the source and channel blocklengths are the same). Furthermore, they showed that the best exponential behavior (or strong converse exponent) is attainable by a separation-based scheme. The separation rule for no excess-distortion exponent holds can be explained by observing the fact that the probability of no-excess-distortion can be approximated by product of the probability of no-excess-distortion in source coding and the probability of correct decoding in the channel coding phase under the condition that \( R(D) > C \). Their achievable separation-based joint source-channel code is a universal joint source-channel code which is based on [8].

By assuming side information available at decoder, the celebrated results of Shannon [9] and Slepian and Wolf [10] imply that almost lossless communication is possible using separate source and channel codes if \( H(U|V) < C \) for a source \( U \) with side information \( V \) and a channel with capacity \( C \). On the other hand, Shamai and Verdú [11] proved that codes with \( H(U|V) > C \) cannot exist even if joint source-channel coding is employed. Hence, for the problem of transmitting a DMS

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over a DMC with side information available at decoder, separate source and channel coding is asymptotically optimal when a vanishingly small probability of error is allowed. Using properties of the Lovász theta function, Nayak, Tuncel, and Rose [12] later showed that separate source and channel coding is asymptotically suboptimal in general for the problem of designing codes for zero-error transmission of a source through a channel when the receiver has side information about the source. They also derived conditions on sources and channels for the optimality of separate source-channel coding.

Recently, Kostina, Polyanskiy, and Verdú [2] quantified the minimal average delay (code length) attainable by lossy source-channel codes with feedback and concluded that such codes lead to a significant improvement in the fundamental delay-distortion tradeoff. They showed that separate source-channel coding fails to achieve these minimal average delays if a non-vanishing distortion probability is allowed. In addition, the authors also investigated the minimum energy required to reproduce relative entropy between conditional output distributions of the two most “most distinguishable” channel input symbols [1]. Our technical contributions are twofold.

1) Our first contribution, the direct part, is to judiciously modify Yamamoto-Itoh’s coding scheme [13] by combining it with known error exponent results in lossy source coding [14] so that it becomes a joint source-channel code for the DMC with feedback. We ensure that the so constructed code achieves the excess-distortion exponent $B(1 - R(D)/C)$. 

2) Our second and main contribution, the converse part, consists in providing several new and novel analytical arguments (e.g., Lemma 5) to upper bound the excess-distortion error exponents of variable-length source-channel codes with feedback. Our proof techniques are based partly on Berlin et al.’s [15] simplified converse proof of Burnashev’s exponent [1]. The most interesting contribution for this part is the introduction and analysis of a new (and optimal) decoding rule that is amenable to lossy joint source-channel coding problems. This rule is called the distortion-MAP rule. The well-known MAP decoding rule can be considered as a special case of the the distortion-MAP rule when the permitted distortion is equal to zero.

C. Organization of the Paper

The rest of this paper is structured as follows. In Section II, we provide a precise problem statement for variable-length source-channel coding with feedback. We state the main result in Section III. The achievability proof is provided in Section IV, and the converse proof is provided in Section V. Technical derivations are relegated to the appendices.

II. Problem Setting

A. Notational Conventions

We use information-theoretic notation [16] in the standard manner. Asymptotic notation such as $O(\cdot)$ are also used in the standard manner. We use $\ln x$ to denote the natural logarithm so information units throughout are in nats. The binary entropy function is defined as $h(x) := -x \ln x - (1 - x) \ln(1 - x)$ for $x \in [0, 1]$. The minimum of two numbers $a$ and $b$ is denoted interchangeably as $\min\{a, b\}$ and $a \wedge b$. As is usual in information theory, $Z_i^n$ denotes the vector $(Z_i, Z_{i+1}, \ldots, Z_j)$. In this paper, we also define $\alpha/0 = \infty$ for all $\alpha \geq 0$ and $0 \times \infty = 0$.

B. Basic Definitions

Throughout, we let $\{V_n\}_{n=1}^\infty$ be DMS with distribution $F_Y$ and taking values in a finite set $\mathcal{V}$.

**Definition 1.** A $(|\mathcal{V}|^N, N)$-variable-length joint source-channel code with feedback for a DMC $P_{Y|X}$, where $N$ is a positive integer, is defined by

- A sequence of encoders $f_n : \mathcal{V}^n \times \mathcal{Y}^{n-1} \rightarrow \mathcal{X}$, $n \geq 1$, defining channel inputs

$$X_n = f_n(V^n, Y^{n-1}).$$

(1)

- A sequence of decoders $g_n : \mathcal{Y}^n \rightarrow \mathcal{V}^n$, $n \geq 1$, each providing an estimate $\hat{V}_n \in \mathcal{V}^n$ at time $n$ at the decoder.

- An integer-valued random variable $\tau_N$ which is a stopping time of the filtration $\{\sigma(Y^n)\}_{n=0}^\infty$.

The final decision at the decoder is computed at the stopping time $\tau_N$ as follows:

$$\hat{V}_{\tau_N} := g_{\tau_N}(Y^{\tau_N}).$$

(2)
The excess-distortion probability of the coding scheme specified above is defined as
\[ P_d(N, D) := \mathbb{P}(d(\hat{V}_N^N, V^N) > D), \]
for some distortion measure \( d : \mathcal{Y}^N \times \mathcal{Y}^N \to [0, +\infty) \) satisfying the following properties [16]
\[ d(v^N, \hat{v}^N) = \frac{1}{N} \sum_{i=1}^{N} d(v_i, \hat{v}_i), \]
\[ d_{\text{max}} = \max_{(v, \hat{v}) \in \mathcal{Y} \times \mathcal{Y}} d(v, \hat{v}) < \infty, \]
for any pair of sequences \( v^N \in \mathcal{Y}^N \) and \( \hat{v}^N \in \mathcal{Y}^N \).

**Definition 2.** \( E \in \mathbb{R}_+ \) is an achievable distortion exponent at distortion level \( D \) if there exists a sequence of \( (|\mathcal{Y}|^N, N) \)-variable-length joint source-channel codes with feedback indexed by \( N \in \mathbb{N} \) satisfying
\[ \limsup_{N \to \infty} \frac{E(\tau_N)}{N} \leq 1, \]
\[ \liminf_{N \to \infty} \frac{-\ln P_d(N, D)}{N} \geq E. \]
The excess-distortion reliability function of the DMC with the variable-length joint source-channel code with feedback is \( E^*(D) = \sup \{ E : \text{is an achievable distortion exponent at distortion level } D \} \).

**Definition 3.** For a DMC \( P_{Y|X} \), we define the channel parameters
\[ B := \max_{x, x' \in \mathcal{X}} D(P_{Y|X} \cdot | x) \| P_{Y|X} \cdot | x') \]
\[ \lambda := \min_{(x, y) \in \mathcal{X} \times \mathcal{Y}} P_{Y|X}(y|x) \]
\[ C := \max_{P_X} I(X; Y), \]
Note that if \( B < \infty \), \( \lambda \in (0, 1/2) \).

In addition, define the distortion ball and the rate-distortion function respectively as
\[ S_D(v^N) := \{ w^N \in \mathcal{Y}^N : d(w^N, v^N) \leq D \}, \forall v^N \in \mathcal{Y}^N, \]
\[ R(Q, D) := \min_{P_{XY}: P_X = Q, E_{P_{XY}} \{ d(X,Y) \leq D \}} I(X; Y). \]
If \( Q = P_V \), we write \( R(P_V, D) = R(D) \) for brevity.

### III. Main Results

**Theorem 1.** Assuming \( B < \infty \), the following holds:
\[ E^*(D) = \max \left\{ 0, B \left( 1 - \frac{R(D)}{C} \right) \right\}. \]

**Proof:** The proof is a combination of Propositions 1 and 2 in Sections IV and V, respectively.

Some remarks in order.

1) If \( D = 0 \), the problem reduces to (almost) lossless source coding and \( R(P_V, D) = H(P_V) \). If the source \( P_V \) is uniformly distributed over \( V \) and with \( R := \log |V| \), then the expression in (14) reduces to Burnashev’s exponent \( E_{\text{Burn}}(R) = B(1 - R/C) \) [1], where \( R \) represents the rate of the channel code.

2) Since the channel encoder can detect erroneous source sequences (by using feedback link) and retransmit these sequences (by allowing the use of variable-length codes), errors in the source code (of the joint source-channel code) only cause the retransmission probability (or the code length) to increase. Therefore, the design of the variable-length joint source-channel code with feedback is equivalent to the problem of designing (and subsequently concatenating) an error exponent-optimal channel code and an excess-distortion exponent-optimal lossy source code to minimize the retransmission probability. This intuitively means that the separation rule holds for the variable-length source-channel coding with feedback in the error exponents regime.

3) Indeed, from the proof of Theorem 1, we conclude that the separation is optimal for variable-length source-channel code with feedback for \( R(D) < C \) in the regime of interest. In contrast, Kostina, Polyanskiy, and Verdú [2] considered the non-vanishing error formalism for the same problem and concluded that separation is not optimal.

4) For fixed-length source-channel coding without feedback, Gallager [4] and Jelinek [5] indicated that joint source-channel coding leads to a larger error exponent than the separation scheme. More specifically, for \( R \geq R_{ct} \), Csiszár [6] proved...
that the optimal error exponent for lossless joint source-channel coding is equal to \( \min_R (e(R) + E(R)) \), where \( e(R) \) and \( E(R) \) are source and channel reliability functions (of the DMS and DMC respectively). This fact can be explained as follows. In Csiszár’s setup, the problem consists in designing a joint source-channel code which has as large an error exponent as possible. The end-to-end error probability can be expressed as \( P_e = P_{e,src} + P_{e,ch} - P_{e,src} \times P_{e,ch} \); this indicates that a separate scheme is likely to be suboptimal as the smallest exponent of the probabilities in this expression dominates. Indeed, for channels without feedback, there is no mechanism to detect errors.

5) In contrast, in the strong converse exponent regime (still without feedback), Wang, Inger, and Kochman [7] showed that separation is optimal. This is because the end-to-end correct decoding probability is \( P_e = P_{e,src} \times P_{e,ch} \), so the resultant exponent is the sum of the exponents of each probability. This hints at the fact that the source and channel coding can be designed independently and the overall system still performs optimally.

6) The fact that separation is optimal holds for the vanishing error probability formalism does not mean that the same is true for zero-error communication. Zero-error coding can be considered as the problem of designing a coding scheme which has infinite error exponent. Hence, joint source-channel coding is usually better than separate source-channel coding. For example, for fixed-length joint source-channel code with no feedback, Nayak, Tuncel, and Rose [12] showed that separate source and channel coding is asymptotically suboptimal in the zero-error case when the receiver has side information about the source.

IV. ACHIEVABILITY PROOF

**Definition 4.** [17, Chapter 2] Given a DMS which produces an i.i.d. sequence \( V_1, V_2, \ldots, V_N \sim P_V \), a \((|V|^N, N)\)-block code \((\tilde{f}_N, \tilde{g}_N)\) for \( V_N \) consists of

- An encoding function \( f_N : \mathcal{V}^N \rightarrow \mathcal{R}(\tilde{f}_N) \), where \( \mathcal{R}(\tilde{f}_N) \), the range of the encoding function \( \tilde{f}_N \), is some finite set;
- A decoding function \( \tilde{g}_N : \mathcal{R}(\tilde{f}_N) \rightarrow \mathcal{V}^N \).

Define Marton’s exponent [14]

\[
F(P_V, R, D) := \inf_{Q : R(Q,D) > R} D(Q \| P_V). 
\]  

**Lemma 1.** [17, Chapter 9] For any \( \varepsilon > 0 \), there exists a sequence of a sequence of \((|V|^N, N)\)-block codes \( \{(\tilde{f}_N, \tilde{g}_N)\}_{N \geq 1} \) for the source \( V_N \sim P_N^N \) such that

\[
|\mathcal{R}(\tilde{f}_N)| \leq \exp (N(R(D) + 2\varepsilon))
\]

holds and the probability of excess-distortion satisfies

\[
P(V_N \in \mathcal{L}(N)) \leq \exp \left( -\frac{N}{2} F(P_V, R(D) + \varepsilon, D) \right)
\]

for \( N \) sufficiently large, where

\[
\mathcal{L}(N) := \{ v_N \in \mathcal{V}^N : d(v_N, \tilde{g}_N(\tilde{f}_N(v_N))) > D \}.
\]

**Proposition 1.** The following inequality holds:

\[
E^*(D) \geq \max \left\{ 0, B \left( 1 - \frac{R(D)}{C} \right) \right\}.
\]  

**Proof of Theorem 1:** We only need to analyze the case in which \( R(D) < C \) since (19) trivially holds for \( R(D) \geq C \). Take a sufficiently small \( \varepsilon > 0 \) such that \( R(D) + 3\varepsilon < C \). Recall the \((|V|^N, N)\)-block code for source \((\tilde{f}_N, \tilde{g}_N)\) in Lemma 1. Let \( c, e \) be the two control messages in the Yamamoto-Itoh coding scheme [13]. We modify the Yamamoto-Itoh coding scheme of length \( N \) [13], which consists of two sub-blocks (modes) of length \( \gamma N \) (message mode) and \((1 - \gamma)N \) (control mode) for some \( \gamma \in (0, 1) \), as follows:

- If a sequence \( v_N \in \mathcal{L}(N) \) where \( \mathcal{L}(N) \) is defined in (18) in Lemma 1, the encoder assumes that the decoded message in the message mode is wrong, so it always sends message \( c \) in the control mode.
- If a sequence \( v_N \notin \mathcal{L}(N) \), the encoder sends \( \tilde{f}_N(v_N) \in \{1, \ldots, N(R(D) + 2\varepsilon)\} \) over the feedback channel, and first uses the same two-mode (phase) coding block as in the Yamamoto-Itoh coding scheme \( (\varphi_N, \phi_N) \) of length \( N \) [13]. Then it maps the decoded message \( \tilde{f}_N(v_N) \) to \( \tilde{g}_N(\tilde{f}_N(v_N)) \in \mathcal{V}^N \). Let \( P_{1e|e} \) denote the average error probability in the message mode for this subset of messages.

Let \( P_{2ec} \) and \( P_{2ce} \) respectively denote the transition error probabilities for \( c \rightarrow e \) and \( e \rightarrow c \) in the control mode for the modified Yamamoto-Itoh block. Choose a pair of input symbols \((x_0, x'_0) \in \mathcal{X}^2\) such that

\[
(x_0, x'_0) = \arg \max_{(x,x') \in \mathcal{X}^2} D(P_{Y|X}(\cdot|x) \| P_{Y|X}(\cdot|x'))
\]  

(20)
and assign the two codewords $x_e = (x_0, x_0, \cdots, x_0)$ and $x_e = (x_0', x_0', \cdots, x_0')$ of length $(1 - \gamma)N$ to control signals $c$ and $e$, respectively. Using the same arguments as Yamamoto-Itoh [13], the average error probability $P_E$ of this modified coding block is bounded by
\begin{equation}
P_E \leq \left( P_{1Te} \mathbb{P}(V^N \notin \mathcal{L}^{(N)}) + \mathbb{P}(V^N \in \mathcal{L}^{(N)}) \right) P_{2ec}.
\end{equation}
\begin{equation}
\leq P_{2ec}.
\end{equation}

Similarly, by using the same arguments as Yamamoto-Itoh [13], the retransmission probability $P_X$ of this modified coding block is bounded by
\begin{equation}
P_X \leq \left( P_{1Te} + \mathbb{P}(V^N \in \mathcal{L}^{(N)}) \right) (1 - P_{2ec}) + (1 - P_{1Te})P_{2ec}.
\end{equation}

Now since $|R(f_N)| = \exp(N(R(D) + 2\varepsilon))$, by Gallager’s [18] error exponent analysis\(^1\) we know that if
\begin{equation}
\frac{R(D) + 2\varepsilon}{\gamma} < C,
\end{equation}
then\(^2\)
\begin{equation}
P_{1Te} \leq \exp \left( - \gamma N \hat{F}(R(D), \varepsilon, \gamma) \right),
\end{equation}
where $\hat{F}(R(D), \varepsilon, \gamma) > 0$ is related to the random coding error exponent [18].

Besides, using the same decoding strategy for the control mode as in the Yamamoto-Itoh coding scheme [13], we have
\begin{equation}
P_{2ec} \leq \exp \left( - N \beta \right),
\end{equation}
for some $\beta > 0$.

In addition, by Stein’s lemma [16] and the choice of $(x_0, x_0')$ in (20), we also have
\begin{equation}
\lim_{N \to \infty} \frac{\ln P_{2ec}}{(1 - \gamma)N} = B.
\end{equation}
It follows that
\begin{equation}
\liminf_{N \to \infty} \frac{\ln P_E}{N} \geq (1 - \gamma) \liminf_{N \to \infty} \frac{\ln P_{2ec}}{(1 - \gamma)N} = (1 - \gamma)B.
\end{equation}
By choosing
\begin{equation}
\gamma = \frac{R(D) + 3\varepsilon}{C} \in (0, 1),
\end{equation}
to satisfy (24), we have from (29) that
\begin{equation}
\liminf_{N \to \infty} \frac{\ln P_E}{N} \geq B \left( 1 - \frac{R(D) + 3\varepsilon}{C} \right).
\end{equation}
Note that with this choice of $\gamma$ from (23), we have
\begin{equation}
P_X \leq P_{1Te} + \mathbb{P}(V^N \notin \mathcal{L}^{(N)}) + P_{2ec}
\end{equation}
\begin{equation}
\leq \exp(-\alpha N),
\end{equation}
where $\alpha := \min\{\beta, \gamma \hat{F}(R(D), \varepsilon, \gamma), F(P_Y, R(D) + \varepsilon, D)/2\} > 0$.

Now, in order to form a variable-length source-channel feedback code from the modified Yamamoto-Itoh block, we repeat this block multiple times and also define a stopping time and the final decision for the variable-length joint source-channel feedback code. More specifically, the desired code is created by repeating the length-$N$ transmission at times $n \in \{\mu N : \mu = 1, 2, 3, \ldots\}$ and using the same decoding algorithm as in [13]. The decoder defines a stopping time $\tau_N$ as follows:

1) If $n = \mu N, \mu \in \{2, 3, 4, \ldots\}$, we define
\begin{equation}
1\{\tau_N = n\} = \prod_{t=1}^{\mu-1} \mathbb{1} \left\{ \phi_N(Y^{(t-1)N+N}_{(t-1)N+\gamma N+1}) = e \right\} \mathbb{1} \left\{ \phi_N(Y^n_{(\mu-1)N+\gamma N+1}) = e \right\},
\end{equation}
where $\phi_N$ is the decoder of the Yamamoto-Itoh coding scheme.

\(^1\)The random coding bound for DMCs holds for any distribution of messages in the message set.
\(^2\)We use the notation $a_n \leq b_n$ to mean $\limsup_{n \to \infty} \frac{1}{n} \log \frac{a_n}{b_n} \leq 0$ [16].
2) If \( n = N \), we define
\[
\mathbf{1}\{\tau_N = n\} = \mathbf{1}\{\phi_N(Y_N^{\tau_N}) = c\}; \quad (35)
\]
3) Otherwise,
\[
\mathbf{1}\{\tau_N = n\} = \mathbf{1}\{\emptyset\}. \quad (36)
\]
In addition, the estimated sequence of the transmitted sequence \( v^N \) at the stopping time \( \tau_N \) has the following form:
\[
\hat{v}_N^{\tau_N} := \hat{g}_N(\phi_N(Y_N^{\tau_N-(1-\gamma)N})). \quad (37)
\]
Note that \( \phi_N(Y_N^{\tau_N-(1-\gamma)N}) \) is the estimated message corresponding to \( \hat{f}_N(v^N) \) at the stopping time \( \tau_N \).

Since \( \mathcal{Y} \) is finite, for each fixed \( n \in \mathbb{Z}_+ \), all the decoding regions at each decoder are finite sets, which are Borel sets in \( \mathbb{R}^n \). Combining this fact with the definition of \( \tau_N \), we have \( \{\tau_N = n\} \in \sigma(Y^n) \) for all \( n \in \mathbb{N} \). Furthermore, by the proposed transmission scheme, given \( \{V^N = v^N\} \) we have that \( Y_{(t-1)N+N} \) for \( t \in \mathbb{N} \) are independently identically distributed vectors since we restart a new modified Yamamoto-Itoh coding block at each time \( tN \) for \( t \in \mathbb{N} \) and that all these modified Yamamoto-Itoh coding blocks have the same encoding and decoding algorithms. Therefore, we have
\[
\mathbb{P}\left( \phi_N(Y_{(t-1)N+N}^{(t-1)N+(\gamma+1)N}) = e \right) = \mathbb{P}_X, \quad t \in \mathbb{N}, \quad (38)
\]
where \( \mathbb{P}_X \) is the retransmission probability which can be upper bounded as in (33).

It follows from (34) and (38) that
\[
\mathbb{P}(\tau_N = n) = \begin{cases} 
\mathbb{P}_X^{\mu-1}(1 - \mathbb{P}_X), & \text{if } n \in \{\mu N : \mu \in \mathbb{N}\} \\
0, & \text{otherwise}
\end{cases} \quad (39)
\]
Hence, we have
\[
\sum_{n=0}^{\infty} \mathbb{P}(\tau_N = n) = \sum_{\mu=1}^{\infty} (\mathbb{P}_X)_{\mu-1}(1 - \mathbb{P}_X) = 1. \quad (40)
\]
Thus, \( \tau_N \) is indeed a stopping time with respect to \( \{\sigma(Y^n)\}_{n=0}^{\infty} \).

Observe that for this code, we have
\[
\mathbb{E}(\tau_N) = \sum_{n=0}^{\infty} n \mathbb{P}(\tau_N = n) \quad (41)
\]
\[
= \sum_{\mu=1}^{\infty} \mu N \mathbb{P}_X^{\mu-1}(1 - \mathbb{P}_X) \quad (42)
\]
\[
= \frac{N}{1 - \mathbb{P}_X} \quad (43)
\]
\[
\leq N + o(1), \quad (44)
\]
where (44) follows from (33). Similarly, we also have
\[
\mathbb{P}(d(\hat{V}_{\tau_N}^N, V^N) > D) = \frac{\mathbb{P}_E}{1 - \mathbb{P}_X} \to 0, \quad (45)
\]
as \( N \to \infty \). Therefore, the resultant excess-distortion exponent of the modified Yamamoto-Itoh coding block is
\[
\liminf_{N \to \infty} \frac{-\ln \mathbb{P}(d(\hat{V}_{\tau_N}^N, V^N) > D)}{N} = \liminf_{N \to \infty} -\frac{\ln \mathbb{P}_E}{N} + \frac{\ln(1 - \mathbb{P}_X)}{N} \quad (46)
\]
\[
\geq B \left( 1 - \frac{R(D) + 3\varepsilon}{C} \right) + \lim_{N \to \infty} \frac{\ln(1 - \exp(-\alpha N))}{N} \quad (47)
\]
\[
\geq B \left( 1 - \frac{R(D) + 3\varepsilon}{C} \right). \quad (48)
\]
\[
\text{Since } \varepsilon > 0 \text{ can be made arbitrarily small, we obtain}
\]
\[
E^*(D) \geq B \left( 1 - \frac{R(D) + 3\varepsilon}{C} \right). \quad (49)
\]
This concludes the proof for the achievability part.
V. CONVERSE PROOF

Fix a \(|\mathcal{V}|^N, N\)-variable-length joint source-channel code with feedback in as Definition 1. This specifies the excess-distortion probability \( P_d(N, D) \). Define the posterior distribution \( P_{V^N|Y^n}(v^N|y^n) \) as

\[
P_{V^N|Y^n}(v^N|y^n) := \frac{\prod_{k=1}^n P_{Y|X}(y_k|f_k(v^N, y^{k-1})) \prod_{j=1}^n P_v(v_j)}{\sum_v \prod_{k=1}^n P_{Y|X}(y_k|f_k(v^N, y^{k-1})) \prod_{j=1}^n P_v(v_j)}.
\]

(51)

Define the random stopping times

\[
\tau^*_N := \inf \left\{ n : \min_{v^n \in \mathcal{V}^N} \sum_{w^n \in \mathcal{V}^N : d(w^n, v^n) > D} P_{V^N|Y^n}(w^n|y^n) \leq \delta_N \right\},
\]

(52)

\[
\tau'_N := \inf \left\{ n : \min_{v^n \in \mathcal{V}^N} \sum_{w^n \in \mathcal{V}^N : d(w^n, v^n) > D} P_{V^N|Y^n}(w^n|y^n) \leq P_d(N, D) \right\},
\]

(53)

for some \( \delta_N \geq P_d(N, D) \) to be determined later.

Lemma 2. For any fixed sequence of encoders \( \{f_n\}_{n=1}^\infty \) of a \(|\mathcal{V}|^N, N\)-variable-length joint source-channel code with feedback (Definition 1), there exists a decoding strategy called the distortion-MAP decoding rule that achieves the highest distortion exponent among all decoding rules. In addition, any variable-length joint-source channel code with the distortion-MAP decoding rule satisfies the following fact:

\[
\mathbb{P}(d(\hat{V}_n^N, V^N) > D|Y^n) = \min_{v^n \in \mathcal{V}^N} \sum_{w^n \in \mathcal{V}^N : d(w^n, v^n) > D} P_{V^N|Y^n}(w^n|y^n) \quad \text{a.s.}
\]

(54)

Proof: Observe that for each \( n \in \mathbb{N} \), we have

\[
\mathbb{P}(d(\hat{V}_n^N, V^N) \leq D) = \sum_{w^n \in \mathcal{V}^N} P_{V^N}(w^n) \sum_{y^n \in \mathcal{Y}^N} P_{Y^n|V^n}(y^n|w^n) \mathbb{I}\{d(g_n(y^n), w^n) \leq D\}
\]

(55)

\[
= \sum_{y^n \in \mathcal{Y}^N} P_{Y^n}(y^n) \sum_{w^n \in \mathcal{W}_D(g_n(y^n))} P_{V^N|Y^n}(w^n|y^n)
\]

(56)

\[
\leq \sum_{y^n \in \mathcal{Y}^N} P_{Y^n}(y^n) \max_{v^n \in \mathcal{V}^N} \sum_{w^n \in \mathcal{W}_D(v^n)} P_{V^N|Y^n}(w^n|y^n)
\]

(57)

\[
= \sum_{y^n \in \mathcal{Y}^N} P_{Y^n}(y^n) \max_{w^n \in \mathcal{W}_D(v^n)} \sum_{w^n \in \mathcal{V}^N : d(w^n, v^n) \leq D} P_{V^N|Y^n}(w^n|y^n).
\]

(58)

Here, (57) follows from the fact that \( g_n(y^n) \in \mathcal{V}^N \). One important note is that the inequality (57) becomes an equality if we choose the decoding function \( g_n(y^n) = v_0^n \), where \( v_0^n \) is in the set

\[
\left\{ v_0^n \in \mathcal{V}^N : \sum_{w^n \in \mathcal{V}^N : d(w^n, v_0^n) \leq D} P_{V^N|Y^n}(w^n|y^n) = \max_{v^n \in \mathcal{V}^N} \sum_{w^n \in \mathcal{V}^N : d(w^n, v^n) \leq D} P_{V^N|Y^n}(w^n|y^n) \right\}.
\]

(59)

Using this so-called distortion-MAP decoding rule, it is easy to see from (58) that

\[
\mathbb{P}(d(\hat{V}_n^N, V^N) > D|Y^n) = 1 - \max_{v^n \in \mathcal{V}^N} \sum_{w^n \in \mathcal{V}^N : d(w^n, v^n) \leq D} P_{V^N|Y^n}(w^n|y^n)
\]

(60)

\[
= \min_{w^n \in \mathcal{V}^N} \sum_{w^n \in \mathcal{V}^N : d(w^n, v^n) > D} P_{V^N|Y^n}(w^n|y^n).
\]

(61)

This means that joint source-channel codes with the distortion-MAP decoding rule have the largest achievable exponent if we use the same sequence of encoders \( \{f_n\}_{n=1}^\infty \) and the same stopping time rule \( \tau_N \). This concludes the proof of Lemma 2. ■

Remark 1. From Lemma 2, for the purpose of finding upper bound on the distortion reliability function of \(|\mathcal{V}|^N, N\)-variable-length joint source-channel codes with feedback, it is enough to consider codes that use the distortion-MAP decoding rule, and hence (54) holds. For the rest of this converse proof, we assume that all codes use the distortion-MAP decoding rule.

Lemma 3. For all \(|\mathcal{V}|^N, N\)-variable-length joint source-channel codes with feedback, the following statements hold:

\[
\mathbb{P}(d(\hat{V}_{\tau_N}^N, V^N) > D|Y^{\tau_N}) \leq \delta_N, \quad \text{a.s.}
\]

(62)

\[
\mathbb{P}(d(\hat{V}_{\tau'_N}^N, V^N) > D) \leq \delta_N,
\]

(63)

\[
\mathbb{P}(d(\hat{V}_{\tau_N}^N, V^N) > D) \leq P_d(N, D),
\]

(64)

\[
\mathbb{E}(\tau_N) \leq \mathbb{E}(\tau'_N) \leq \mathbb{E}(\tau_N).
\]

(65)
\textbf{Proof:} We know from (52) and (54) that for all \( k \leq \tau_N^* - 1, \)
\[ P(d(\hat{V}_N^k, V^N) > D) = E \left[ P(d(\hat{V}_N^k, V^N) > D | Y^k) \right] = E \left[ \min_{w^N \in V^N} \sum_{w^N \in V^N; d(w^N, V^N) > D} P_{V^N | Y^k}(w^N | Y^k) \right] \]
\[ > \delta_N. \] 
(66)

Moreover, from (52) and (54) we also know that
\[ P(d(\hat{V}_N^{\tau_N}, V^N) > D | Y^{\tau_N}) = \min_{w^N \in V^N} \sum_{w^N \in V^N; d(w^N, V^N) > D} P_{V^N | Y^{\tau_N}}(w^N | Y^{\tau_N}) \leq \delta_N, \]
(69)
so (62) follows from (70). The bound in (63) follows from (62) by taking expectations on both sides of (62).

Similarly, from the definition of \( \tau_N^* \), we also have (64). Furthermore, for all \( k \leq \tau_N^* - 1 \)
\[ P(d(\hat{V}_N^k, V^N) > D) > P_d(N, D). \]
(71)
Since \( P(d(\hat{V}_N^k, V^N) > D) \leq P_d(N, D) \), from (71) we have \( \tau_N \geq \tau_N^* \). In addition, from (52) and (53) we obtain \( \tau_N^* \geq \tau_N^* \) since \( \delta_N \geq P_d(N, D) \). Hence, we obtain (65). This concludes the proof of Lemma 3.

\textbf{Lemma 4.} We have that for \( N \rightarrow \infty, \)
\[ E[\tau_N^*]C \geq (1 - \delta_N) NR(D) + O(\sqrt{N}). \]
(72)

\textbf{Proof:} Let \( Q^{-1}(\cdot) \) be the inverse of the complementary cumulative distribution function of a standard Gaussian. Since \( \tau_N^* \) is a stopping time of the joint source-channel coding scheme, from (63) and [2, Theorem 5] we have
\[ E[\tau_N^*]C \geq (1 - P(d(\hat{V}_N^{\tau_N}, V^N) > D)) NR(D) - \frac{\sqrt{N\nu(D)}}{2 \pi} \exp \left(-\frac{[Q^{-1}(P(d(\hat{V}_N^{\tau_N}, V^N) > D))]^2}{2}\right) + O(\ln N) \geq (1 - \delta_N) NR(D) + O(\sqrt{N}). \]
(74)
Note that \( \nu(D) \), which is immaterial for the discussions to follows, is defined in [2, Eq. (60)]. The bound in (74) holds because the exponential term in (73) is upper bounded by 1 and \( \nu(D) \) is finite.

\textbf{Lemma 5.} For any \((|V|^N, N) \) variable-length joint source-channel code with feedback, for \( N \) sufficiently large and if \( \lambda \delta_N \geq P_d(N, D) \), the following holds
\[ E[\tau_N^* - \tau_N^*] \geq -\frac{\ln P_d(N, D)}{B} + \ln \left[ \min \{\lambda \delta_N, 1 - \delta_N\} - 2 \right]. \]
(75)

\textbf{Proof:} Preliminaries for the proof are provided in Appendix A. The actual proof can be found in Appendix B.

\textbf{Lemma 6.} [15, Proposition 2] If \( B < \infty \), then for all \( v^N \in \mathcal{V}^N \) and \( y^n \in \mathcal{Y}^n \),
\[ P_{V^N | Y^n}(v^N | y^n) \geq \lambda P_{V^N | Y^{n-1}}(v^N | y^{n-1}). \]
(76)

\textbf{Proposition 2.} The following holds
\[ E^*(D) \leq \max \left\{ 0, B \left( 1 - \frac{R(D)}{C} \right) \right\}. \]
(77)

\textbf{Proof of Proposition 2:} Define
\[ \beta := \liminf_{N \rightarrow \infty} -\frac{\ln P_d(N, D)}{N} \]
(78)
We will consider two cases \( \beta > 0 \) and \( \beta = 0 \). For the case \( \beta = 0 \), we also have that \( E(D) = 0 \). Therefore, we only need to consider the case \( \beta > 0 \). Define
\[ \xi := \limsup_{N \rightarrow \infty} -\frac{\ln P_d(N, D)}{N} \geq \beta > 0. \]
(79)
Observe that from (78) and (79) we also have that for \( N \) sufficiently large,
\[ e^{-2N\xi} < P_d(N, D) < e^{-\beta N/2}. \]
(80)
Now, by choosing \( \lambda \delta_N := 1/(−\ln P_d(N, D)) \geq P_d(N, D) \), from the upper bound in (80), we have that
\[
\lim_{N \to \infty} \delta_N = 0. \tag{81}
\]
On the other hand, observe that
\[
0 \geq \limsup_{N \to \infty} \frac{\ln(\lambda \delta_N)}{N} \geq \liminf_{N \to \infty} \frac{\ln(\lambda \delta_N)}{N} = \liminf_{N \to \infty} -\ln(\ln P_d(N, D)) \tag{82}
\]
\[
= \liminf_{N \to \infty} -\frac{\ln(2\xi N)}{N} \geq \liminf_{N \to \infty} -\frac{\ln(2\xi N)}{N} = 0, \tag{83}
\]
where (84) follows from the lower bound in (80) and (85) follows from the assumption that \( \xi > 0 \) [cf. (79)]. Therefore,
\[
\lim_{N \to \infty} \frac{\ln(\lambda \delta_N)}{N} = 0. \tag{86}
\]
It follows from Lemmas 3, 4, and 5 that for any \((|Y|^N, N)\)-variable-length joint source-channel code,
\[
\mathbb{E}(\tau_N) \geq \mathbb{E}[\tau_N] = \mathbb{E}[\tau_N^\prime] + \mathbb{E}[\tau_N - \tau_N^\prime] \geq \frac{(1 - \delta_N)NR(D) + O(\sqrt{N}) - \ln P_d(N, D)}{C} + \frac{\ln(\lambda \delta_N) - 2}{B} \tag{87}
\]
\[
\geq \frac{(1 - \delta_N)NR(D) + O(\sqrt{N}) - \ln P_d(N, D)}{C} + \frac{\ln(\lambda \delta_N) - 2}{B} \tag{88}
\]
Hence, we obtain
\[
\liminf_{N \to \infty} -\frac{\ln P_d(N, D)}{N} \leq B \left(1 - \frac{R(D)}{C}\right) \tag{89}
\]
where (89) follows from (6), (81), (86), and (88). This implies that \( E^*(D) \) is upper bounded by the right-hand-side of (89). This concludes the proof of Proposition 2.

\section*{APPENDIX A}

\subsection*{PRELIMINARIES FOR THE PROOF OF LEMMA 5}

In all proofs of this section, we use the following notations for simplicity of presentation:
\[
\mathfrak{S}_D(\hat{V}_m^N) := \mathcal{V}^N \setminus \mathcal{S}_D(\hat{V}_m^N), \quad \mathcal{K}_m := \mathcal{S}_D(\hat{V}_m^N) \times \mathfrak{S}_D(\hat{V}_m^N), \tag{90}
\]
and for each given \( Y^m = y^m, V^N = v^N \), define \( Q_n(v^N, y^m) \) and \( \bar{Q}_n(v^N, y^m) \) as
\[
Q_n(v^N, y^m) := \{y_{m+1}^n \in \mathcal{Y}_m^N : g_n(y^n) \in \mathcal{S}_D(v^N)\}, \tag{92}
\]
\[
\bar{Q}_n(v^N, y^m) := \mathcal{Y}_m^N \setminus Q_n(v^N, y^m). \tag{93}
\]
Note that \( Q_n(v^N, y^m) \) and \( \bar{Q}_n(v^N, y^m) \) are deterministic subsets of \( \mathcal{Y}_m^N \) for each given \( (v^N, y^m) \in \mathcal{V}^N \times \mathcal{Y}_m \). Similarly, for a fixed \( y^m, \mathcal{S}_D(\hat{V}_m^N) \subset \mathcal{V}^N \) and \( \mathcal{K}_m \subset \mathcal{V}^N \times \mathcal{V}^N \) are also deterministic subsets. Now define the probabilities
\[
T_{n,m}(v^N) := P(Y_{m+1}^n \in Q_n(v^N, y^m)|V^N = v^N, Y^m = y^m), \tag{94}
\]
\[
\bar{T}_{n,m}(v^N) := 1 - T_{n,m}(v^N), \tag{95}
\]
\[
L_{n,m}^{(1)} := P(Y_{m+1}^n \in Q_n(v^N, y^m)|V^N \in \mathcal{S}_D(\hat{V}_m^N), Y^m = y^m), \tag{96}
\]
\[
L_{n,m}^{(2)} := P(Y_{m+1}^n \in Q_n(v^N, y^m)|V^N \notin \mathcal{S}_D(\hat{V}_m^N), Y^m = y^m), \tag{97}
\]
\[
\bar{T}_{n,m}^{(1)} := 1 - L_{n,m}^{(1)}, \tag{98}
\]
\[
\bar{T}_{n,m}^{(2)} := 1 - L_{n,m}^{(2)}. \tag{99}
\]

\textbf{Lemma 7.} Fix \( m, n \in \mathbb{N} \) and \( n \geq m \). For the code we fixed in the converse proof, the following holds:
\[
P(Y_{m+1}^n \in \bar{Q}_n(V^N, y^m)|Y^m = y^m) \geq \left( \max \{\bar{T}_{n,m}^{(1)}, \bar{T}_{n,m}^{(2)}\} \right) \left( \min \{P_{V^N|Y^m}(\mathcal{S}_D(\hat{V}_m^N)|y^m), P_{V^N|Y^m}(\mathfrak{S}_D(\hat{V}_m^N)|y^m)\} \right). \tag{100}
\]
Proof: Indeed, for any \( n, m \in \mathbb{N}, n \geq m \) and each realization \( Y^m = y^m \) by the law of total probability we have

\[
\mathbb{P}(Y^m_{n+1} \in \mathcal{Q}_n(V^N, y^m)|Y^m = y^m) = \mathcal{T}^{(1)}_{n,m} P_{V^N|Y^m}(\mathcal{S}_D(V^N_m)|y^m) + \mathcal{T}^{(2)}_{n,m} P_{V^N|Y^m}(\mathcal{S}_D(V^N_m)|y^m) + \min \left\{ P_{V^N|Y^m}(\mathcal{S}_D(V^N_m)|y^m), P_{V^N|Y^m}(\mathcal{S}_D(V^N_m)|y^m) \right\} (T^{(1)}_{n,m} + T^{(2)}_{n,m}).
\]

By lower bounding \( T^{(1)}_{n,m} + T^{(2)}_{n,m} \) by \( \max \{ T^{(1)}_{n,m}, T^{(2)}_{n,m} \} \), we obtain (100) and this completes the proof of Lemma 7.

Lemma 8. Fix \( m, n \in \mathbb{N} \) and \( n \geq m \). For the code we fixed in the converse proof, the following holds almost surely:

\[
(n-m)B \geq -\mathbb{P}(d(\hat{V}^N_m, V^N) \leq D|Y^m) \ln \left( \frac{\mathbb{P}(d(\hat{V}^N_m, V^N) > D|Y^m)}{\min \left\{ \mathbb{P}(d(\hat{V}^N_m, V^N) > D|Y^m), \mathbb{P}(d(\hat{V}^N_m, V^N) \leq D|Y^m) \right\}} \right) - 1.
\]

Proof: For each fixed \( Y^m = y^m \), \( \mathcal{S}_D(\hat{V}^N_m) \subset V^N \) is a deterministic set since \( \hat{V}^N_m = g_m(y^m) \). Similarly, \( \mathcal{K}_m \subset V^N \times V^N \) is also a deterministic set for fixed \( Y^m = y^m \). It follows that

\[
L^{(1)}_{n,m} = \mathbb{P}\left( \left\{ Y^m_{n+1} \in \mathcal{Q}_n(V^N, y^m) \right\} \cap \left\{ V^N \in \mathcal{S}(\hat{V}^N_m) \right\}|Y^m = y^m\right)
\]

\[
= \sum_{v^N \in \mathcal{S}_D(\hat{V}^N_m)} \mathbb{P}\left( \left\{ Y^m_{n+1} \in \mathcal{Q}_n(V^N, y^m) \right\} \cap \left\{ V^N = v^N \right\}|Y^m = y^m\right)
\]

\[
= \sum_{v^N \in \mathcal{S}_D(\hat{V}^N_m)} \mathbb{P}\left( \left\{ Y^m_{n+1} \in \mathcal{Q}_n(v^N, y^m) \right\} \cap \left\{ V^N = v^N \right\}|Y^m = y^m\right)
\]

\[
= \frac{\sum_{v^N \in \mathcal{S}_D(\hat{V}^N_m)} T_{n,m}(v^N) P_{V^N|Y^m}(v^N)|y^m)}{\sum_{v^N \in \mathcal{S}_D(\hat{V}^N_m)} P_{V^N|Y^m}(v^N)|y^m}.
\]

where (107) follows from Bayes rule and the definition of \( T_{n,m}(\cdot) \). Similarly, we also have

\[
L^{(2)}_{n,m} = \frac{\sum_{v^N \in V^N \setminus \mathcal{S}_D(\hat{V}^N_m)} T_{n,m}(v^N) P_{V^N|Y^m}(v^N)|y^m)}{\sum_{v^N \in V^N \setminus \mathcal{S}_D(\hat{V}^N_m)} P_{V^N|Y^m}(v^N)|y^m}.
\]

It follows from (107) and (108) that

\[
\frac{L^{(1)}_{n,m}}{L^{(2)}_{n,m}} = \frac{\sum_{(v^N, v^N) \in \mathcal{K}_m} T_{n,m}(v^N) P_{V^N|Y^m}(v^N)|y^m) P_{V^N|Y^m}(v^N)|y^m)}{\sum_{(v^N, v^N) \in \mathcal{K}_m} P_{V^N|Y^m}(v^N)|y^m) P_{V^N|Y^m}(v^N)|y^m)}.
\]

Now, from (107), we have

\[
\frac{L^{(1)}_{n,m}}{L^{(2)}_{n,m}} = \frac{\left[ \sum_{v^N \in \mathcal{S}_D(\hat{V}^N_m)} T_{n,m}(v^N) P_{V^N|Y^m}(v^N)|y^m) \right]}{\left[ \sum_{v^N \in \mathcal{S}_D(\hat{V}^N_m)} P_{V^N|Y^m}(v^N)|y^m) \right]} \ln \frac{L^{(1)}_{n,m}}{L^{(2)}_{n,m}}.
\]

\[
\leq \frac{\sum_{(v^N, v^N) \in \mathcal{K}_m} P_{V^N|Y^m}(v^N)|y^m) P_{V^N|Y^m}(v^N)|y^m)}{\sum_{(v^N, v^N) \in \mathcal{K}_m} P_{V^N|Y^m}(v^N)|y^m) P_{V^N|Y^m}(v^N)|y^m)}.
\]

where (111) uses the notation \( \mathcal{K}_m \) in (91) and (112) follows the log-sum-inequality [16] and (109).

Similarly, we also have

\[
\frac{\mathcal{T}^{(1)}_{n,m}}{\mathcal{T}^{(2)}_{n,m}} \leq \frac{\sum_{(v^N, v^N) \in \mathcal{K}_m} P_{V^N|Y^m}(v^N)|y^m) P_{V^N|Y^m}(v^N)|y^m)}{\sum_{(v^N, v^N) \in \mathcal{K}_m} P_{V^N|Y^m}(v^N)|y^m) P_{V^N|Y^m}(v^N)|y^m)}.
\]
Now, observe that for every \((v_1^N, v_2^N) \in K_m\),
\[
T_{n,m}(v_1^N) \ln \frac{T_{n,m}(v_1^N)}{T_{n,m}(v_2^N)} + T_{n,m}(v_1^N) \ln \frac{T_{n,m}(v_2^N)}{T_{n,m}(v_2^N)} \leq \sum_{y_{m+1}^n \in Y^{n-m}} \mathbb{P}(Y_{m+1}^n = y_{m+1}^n | V^N = v_1^N, Y^m = y^m) \ln \frac{\mathbb{P}(Y_{m+1}^n = y_{m+1}^n | V^N = v_1^N, Y^m = y^m)}{\mathbb{P}(Y_{m+1}^n = y_{m+1}^n | V^N = v_2^N, Y^m = y^m)}
\]
\[
= \sum_{y_{m+1}^n \in Y^{n-m}} \mathbb{P}(Y_{m+1}^n = y_{m+1}^n | V^N = v_1^N, Y^m = y^m) \sum_{k=m+1}^n \ln \frac{\mathbb{P}(Y_k = y_k | V^N = v_1^N, Y^{k-1} = y^{k-1})}{\mathbb{P}(Y_k = y_k | V^N = v_2^N, Y^{k-1} = y^{k-1})}
\]
\[
= \sum_{k=m+1}^n \sum_{y_k \in Y} \mathbb{P}(Y_k = y_k | V^N = v_1^N, Y^{k-1} = y^{k-1}) \ln \frac{\mathbb{P}(Y_k = y_k | V^N = v_1^N, Y^{k-1} = y^{k-1})}{\mathbb{P}(Y_k = y_k | V^N = v_2^N, Y^{k-1} = y^{k-1})}
\]
\[
= \sum_{k=m+1}^n \sum_{y_k \in Y} P_{Y^k | X} (y_k | f_k(v_1^N, y^{k-1})) \ln \frac{P_{Y^k | X} (y_k | f_k(v_1^N, y^{k-1}))}{P_{Y^k | X} (y_k | f_k(v_2^N, y^{k-1}))}
\]
\[
\leq (n - m) B.
\]

Here, (114) follows from the log-sum inequality [16], (115) follows from the memoryless property of the DMC, (116) follows from \(X_k = f_k(V^N, Y^{k-1})\) [cf. (1)], and (119) follows from the definition of \(B\) in (9).

Combining (112), (113), and (119), we obtain
\[
L_{n,m}^{(1)} \ln \frac{L_{n,m}^{(1)}}{L_{n,m}^{(2)}} + L_{n,m}^{(1)} \ln \frac{T_{n,m}^{(2)}}{L_{n,m}^{(2)}} \leq (n - m) B.
\]

Similarly, we also have
\[
L_{n,m}^{(2)} \ln \frac{L_{n,m}^{(2)}}{L_{n,m}^{(1)}} + L_{n,m}^{(2)} \ln \frac{T_{n,m}^{(1)}}{L_{n,m}^{(1)}} \leq (n - m) B.
\]

Observe that
\[
\mathbb{P}(d(\hat{V}_m^N, V^N) \leq D | Y^m = y^m) = \mathbb{P}(Y_{m+1}^n \in Q_n(V^N, y^m) | Y^m = y^m) = L_{n,m}^{(1)} P_{V^N | Y^m} (S_D(\hat{V}_m^N) | y^m) + L_{n,m}^{(2)} P_{V^N | Y^m} (S_D(V^N) | Y^m = y^m),
\]

where (123) follows from the law of total probability. It follows from (123) that either \(L_{n,m}^{(1)} \leq 1/2\) or \(L_{n,m}^{(2)} \leq 1/2\). Hence, we have from (98) and (99) that either \(L_{n,m}^{(1)} \leq T_{n,m}^{(1)}\) or \(L_{n,m}^{(2)} \leq T_{n,m}^{(2)}\). Now, we consider two cases:

- **Case 1:** \(L_{n,m}^{(1)} \leq T_{n,m}^{(1)}\).

  It follows from (120) that
  \[
  (n - m) B \geq -L_{n,m}^{(1)} \ln T_{n,m}^{(1)} - h(T_{n,m}^{(1)})
  \]
  \[
  \geq -L_{n,m}^{(1)} \ln T_{n,m}^{(1)} - 1
  \]
  \[
  \geq -L_{n,m}^{(1)} \ln (\max\{T_{n,m}^{(1)}, L_{n,m}^{(2)}\}) - 1.
  \]

In addition, it follows from (121) and the assumption that \(L_{n,m}^{(1)} \leq T_{n,m}^{(2)}\) that
\[
(n - m) B \geq -L_{n,m}^{(2)} \ln L_{n,m}^{(2)} - h(L_{n,m}^{(2)})
\]
\[
\geq -L_{n,m}^{(2)} \ln L_{n,m}^{(2)} - 1
\]
\[
\geq -L_{n,m}^{(2)} \ln (\max\{T_{n,m}^{(1)}, L_{n,m}^{(2)}\}) - 1.
\]
From (126) and (129) we obtain
\[
(n-m)B \geq \left[-L_{n,m}^{(1)}P_{Y^m}|Y^m(\mathcal{S}_D(V^N)|y^m) + L_{n,m}^{(2)}P_{Y^m}|Y^m(\mathcal{S}_D(V^N)|y^m)\right] \ln(\max\{\mathcal{T}_{n,m}^{(1)}, \mathcal{T}_{n,m}^{(2)}\}) - 1
\]
(130)
\[
= -\mathbb{P}(d(V_n^N, V^N) \leq D|Y^m = y^m) \ln(\max(\mathcal{T}_{n,m}^{(1)}, \mathcal{T}_{n,m}^{(2)})) - 1
\geq -\mathbb{P}(d(V_n^N, V^N) \leq D|Y^m = y^m)
\times \ln \left[ \min \left\{ \frac{\mathbb{P}(d(V_n^N, V^N) > D|Y^m = y^m)}{\mathbb{P}(d(V_n^N, V^N) \leq D|Y^m = y^m)} \right\} \right] - 1,
\]
(132)
where (131) follows from (123), and (132) follows from Lemma 7.

- Case 2: $L_{n,m}^{(2)} \leq \mathcal{T}_{n,m}^{(2)}$.
  For this case, we have from (120) and (121) that
  \[
  (n-m)B \geq -L_{n,m}^{(1)} \ln L_{n,m}^{(2)} - 1
  \]
  (133)
  (n-m)B \geq -\mathcal{T}_{n,m}^{(2)} \ln \mathcal{T}_{n,m}^{(1)} - 1.
  (134)

Using the same arguments as (124)–(131) we also obtain (132). Since for both cases the bound in (132) holds for all $y^m$, we have that (103) holds almost surely. This concludes the proof of Lemma 8.

\section*{Appendix B}
\section*{Proof of Lemma 5}
\textbf{Proof:} The proof is partly based on the proof of \cite[Lemma 1]{15}. For brevity, for $l, m \in \mathbb{N} \cup \{0\}$, define
\[
G_{l,m} := \mathbb{P}(d(V_n^L, V^L) > D|Y^m)
\]
(135)
\[
\bar{G}_{l,m} := \mathbb{P}(d(V_n^L, V^L) \leq D|Y^m) = 1 - G_{l,m}
\]
(136)
\[
\lambda_m := -\ln \left[ \min \{G_{m,m}, \bar{G}_{m,m}\} \right].
\]
(137)
Note that if $m = 0$ in (135) or (136), we drop the conditioning on $Y^m$ in the probabilities and thus $G_{l,0}$ and $\bar{G}_{l,0}$ are deterministic. From the definitions of $\tau_N'$ and $\tau_N$ in (53) and (52) respectively, we have $\tau_N' \geq \tau_N$. Hence from Lemma 8 (with $n$ replaced by $\tau_N \land n$ and $m$ replaced by $\tau_N' \land n$), we have for all $n \in \mathbb{N}$,
\[
(\tau_N' \land n - \tau_N \land n)B \geq -\bar{G}_{\tau_N' \land n, \tau_N \land n} \ln \left[ \frac{G_{\tau_N' \land n, \tau_N \land n}}{\min \{G_{\tau_N' \land n, \tau_N \land n}, G_{\tau_N' \land n, \tau_N \land n}\}} \right] - 1, \quad \text{a.s.}
\]
(138)

On the other hand, from (65) we have
\[
\mathbb{P}(\tau_N^* < \infty) = \mathbb{P}(\tau_N' < \infty) = 1,
\]
(139)
hence, by the definitions of $\tau_N$ in (52), $\tau_N'$ in (53), and the fact in (139), the following inequalities hold almost surely:
\[
\min_{w^N \in V^N} \sum_{w^N \in V^N : d(w^N, v^N) > D} P_{V^N|Y^N} (w^N|Y^N) \leq \delta_N,
\]
(140)
\[
\min_{w^N \in V^N} \sum_{w^N \in V^N : d(w^N, v^N) > D} P_{V^N|Y^{N-1}} (w^N|Y^{N-1}) > \delta_N.
\]
(141)
It follows from (141) and Lemma 6 that
\[
\min_{w^N \in V^N} \sum_{w^N \in V^N : d(w^N, v^N) > D} P_{V^N|Y^N} (w^N|Y^N) > \lambda \delta_N,
\]
(142)
Since we use the distortion-MAP decoding at time $\tau_N^*$, by (54), (140), and (142),
\[
\delta_N \geq G_{\tau_N^*, \tau_N^*} > \lambda \delta_N, \quad \text{a.s.}
\]
(143)
Hence, we have
\[
\min \{G_{\tau_N^*, \tau_N^*}, \bar{G}_{\tau_N^*, \tau_N^*}\} \geq \min \{\lambda \delta_N, 1 - \delta_N\}, \quad \text{a.s.}
\]
(144)
Now, observe that

$$\lim_{n \to \infty} G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} = \lim_{n \to \infty} \min_{v_N \in V_N} \sum_{w_N \in V_N, d(w_N, v_N) > D} P_{V_N | Y \tau_N^+ \wedge n} (w_N | Y \tau_N^- \wedge n)$$

(145)

$$= \min_{v_N \in V_N} \lim_{n \to \infty} \sum_{w_N \in V_N, d(w_N, v_N) > D} P_{V_N | Y \tau_N^+ \wedge n} (w_N | Y \tau_N^- \wedge n)$$

(146)

$$= \min_{v_N \in V_N} \sum_{w_N \in V_N, d(w_N, v_N) > D} \lim_{n \to \infty} P_{V_N | Y \tau_N^+ \wedge n} (w_N | Y \tau_N^- \wedge n)$$

(147)

$$= \min_{v_N \in V_N} \sum_{w_N \in V_N, d(w_N, v_N) > D} P_{V_N | Y \tau_N^+} (w_N | Y \tau_N^-),$$

(148)

$$= G_{\tau_N^+, \tau_N^-}.$$  

(149)

Here, (146) follows from \( \lim_{n \to \infty} \min \{X_n, Y_n\} = \min \{X, Y\} \) a.s. if \( \lim_{n \to \infty} X_n = X \) a.s. and \( \lim_{n \to \infty} Y_n = Y \) a.s., (148) follows from Lévy’s zero-one law [19] since \( \sigma(Y \tau_N^- \wedge n) \) \( n \to \infty \) is a filtration and \( \sigma(Y \tau_N^-) \) is the maximal \( \sigma \)-algebra generated by \( \{\sigma(Y \tau_N^- \wedge n)\}_{n=1}^{\infty} \). It follows from (136) and (149) that

$$\lim_{n \to \infty} G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} = G_{\tau_N^+, \tau_N^-}.$$  

(150)

Hence by taking \( n \to \infty \) in (138) and applying (149) and (150), we have

$$\Gamma_N := \lim_{n \to \infty} \left[ G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} \Lambda_{\tau_N^+ \wedge n} \right.$$  

$$+ (\tau_N^+ \wedge n - \tau_N^- \wedge n) B + 1 + G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} \ln G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} \right]$$

(151)

$$\leq \lim_{n \to \infty} \sup \left[ G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} \Lambda_{\tau_N^+ \wedge n} \right.$$  

$$+ \lim_{n \to \infty} \left[ (\tau_N^+ \wedge n - \tau_N^- \wedge n) B + 1 + G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} \ln G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} \right]$$

(152)

$$\leq -\ln \left[ \min \{\lambda \delta N, 1 - \delta N\} \right]$$

(153)

Here, (152) follows from the fact that \( \lim_{n \to \infty} (x_n + y_n) \leq \limsup_{n \to \infty} x_n + \liminf_{n \to \infty} y_n \) for any two real sequences \( \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \) [20]. For (153), note that the first term in (152) can be upper bounded by \( -\ln \left[ \min \{\lambda \delta N, 1 - \delta N\} \right] \) due to the facts that \( G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} \leq 1 \) and \( \Lambda_{\tau_N^+ \wedge n} \geq 0 \) as well as (149), (150), and (144).

Now we record a simple fact that follows from the bounded convergence theorem [20]: We have

$$\lim_{n \to \infty} G_{\tau_N^+ \wedge n, 0} = \lim_{n \to \infty} \mathbb{P}(d(V_{\tau_N^+ \wedge n}^{N+}, V_N) > D)$$

(154)

$$= \lim_{n \to \infty} \mathbb{E}[\mathbb{1}\{d(V_{\tau_N^+ \wedge n}^{N+}, V_N) > D\}]$$

(155)

$$= \mathbb{E}[\lim_{n \to \infty} \mathbb{1}\{d(V_{\tau_N^+ \wedge n}^{N+}, V_N) > D\}]$$

(156)

$$= \mathbb{E}[\mathbb{1}\{d(V_{\tau_N^+}^{N+}, V_N) > D\}]$$

(157)

$$= \mathbb{P}(d(V_{\tau_N^+}^{N+}, V_N) > D) = G_{\tau_N^+, 0}.$$  

(158)

It follows by taking expectations on both sides of (153) that

$$\mathbb{E}[\Gamma_N] \leq -\ln \left[ \min \{\lambda \delta N, 1 - \delta N\} \right]$$

(159)

$$+ \mathbb{E} \left[ \lim_{n \to \infty} \left[ (\tau_N^+ \wedge n - \tau_N^- \wedge n) B + 1 + G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} \ln G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} \right] \right]$$

(160)

$$\leq -\ln \left[ \min \{\lambda \delta N, 1 - \delta N\} \right]$$

(161)

$$+ \mathbb{E} \left[ \lim_{n \to \infty} \left[ (\tau_N^+ \wedge n - \tau_N^- \wedge n) B \right] \right] + \mathbb{E} \left[ \lim_{n \to \infty} \left[ 1 + G_{\tau_N^+ \wedge n, \tau_N^- \wedge n} \ln (1 - G_{\tau_N^+ \wedge n, \tau_N^- \wedge n}) \right] \right]$$

(162)

$$\leq -\ln \left[ \min \{\lambda \delta N, 1 - \delta N\} \right] + \mathbb{E}[\tau_N^+ - \tau_N^-] B + \lim_{n \to \infty} \mathbb{E}[\tau_N^+ - \tau_N^-] B + 1 + (1 - G_{\tau_N^+, 0}) \ln (1 - G_{\tau_N^+, 0})$$

(163)

$$\leq -\ln \left[ \min \{\lambda \delta N, 1 - \delta N\} \right] + \mathbb{E}[\tau_N^+ - \tau_N^-] B + 1 + (1 - \mathbb{P}_d(N, D)) \ln \mathbb{P}_d(N, D).$$  

(164)
Here, (160) follows from the fact that \(\liminf_{n \to \infty} \ln(1 - x_n) = \lim_{n \to \infty} x_n\), and the increasing nature of the function \(\Gamma_N\) in (151), the fact that \(\Gamma_N \geq 0\) a.s. in (138), and the bound in (165), we have that

\[
0 \leq E[\Gamma_N] \leq E[\tau_N^* - \tau_N^\prime]B - \ln \left[\min\{1, 1 - \delta_N\}\right] + 1 + (1 - P_d(N, D)) \ln P_d(N, D).
\]

Because \(-P_d(N, D) \ln P_d(N, D) \leq 1\), it follows that

\[
E[\tau_N^* - \tau_N^\prime]B + \ln P_d(N, D) + 2 \geq \ln \left[\min\{1, 1 - \delta_N\}\right],
\]

or equivalently, (75). This concludes the proof of Lemma 5.

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