Enumerating moves in the optimal solution of the Tower of Hanoi

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Abstract

In the Tower of Hanoi problem, there is six types of moves between the three pegs. The main purpose of the present paper is to find out the number of each of these six elementary moves in the optimal sequence of moves. We present a recursive function based on indicator functions, which counts the number of each elementary move, we investigate some of its properties including combinatorial identities, recursive formulas and generating functions. Also we found an interesting sequence that is strongly related to counting each type of these elementary moves that we’ll establish some if its properties as well.

1 Introduction

The problem of the Tower of Hanoi is one of the most famous problems used to introduce the concept of mathematical induction. Since its invention in 1883 by the french mathematician Edward Lucas [9], this game has received a lot of attention from mathematicians, and this is due to the interesting mathematics hiding in and around this puzzle.

Recall that the Tower of Hanoi puzzle consists of \( n \) discs of different sizes, and three pegs \( i, j \) and \( k \). At the beginning all discs are stacked on one of the three pegs, in which no disc lies on top of a smaller disc. The goal is to transfer the whole tower of the \( n \) discs to another peg using the minimum number of moves, where a legal move is to move one disc at time and never put a disc on a smaller one. The Tower of Hanoi with \( n \) discs can be solved optimally in \( u_n = 2^n - 1 \) moves [4], using the following recursive procedure, first move the sub-tower of the first \( (n-1) \) discs from the source peg \( i \) to the middle peg \( k \) where \( j \) is the destination peg, then move the biggest disc to the destination peg \( j \), and finally move the sub-tower of the first \( (n-1) \) discs to the final peg. This recursive procedure satisfy the following recurrence relation

\[
u_n = 2u_{n-1} + 1.
\]

The optimal sequence of moves provided by this recursive procedure is of length \( 2^n - 1 \). We call an elementary move \( x \), a move of a disc from a peg to another peg. There is six elementary moves \( a = ij \) (which denotes a move of a topmost disc from peg \( i \) to peg \( j \)), \( b = jk \), \( c = ki \), \( \bar{a} = ji \), \( \bar{b} = kj \), and \( \bar{c} = ik \).
In this paper we introduce a recursive function that counts the number of each elementary move in the optimal sequence of moves. We denote this function by $f_{ij}^n(x)$ where $x \in \mathcal{A} = \{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$ is the elementary move to be counted in the optimal sequence of moves that transfer a tower of $n$ discs from peg $i$ to peg $j$.

$$f_{ij}^n: \left\{ \begin{array}{l} \mathcal{A} \to \mathbb{N}, \\ x \mapsto f_{ij}^n(x). \end{array} \right.$$ (2)

The function $f_{ij}^n(x)$ is based on indicator functions

$$1_A(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in A; \\ 0 & \text{otherwise}. \end{array} \right.$$ 

The function $f_{ij}^n(x)$ takes as input an elementary move $x \in \mathcal{A}$ and then it gives as output the number of this elementary move in the optimal sequences of moves. We introduce many combinatorial results around this function such as recurrent relations, explicit and implicit formulas, ordinary and exponential generating functions and more. We find that the number of each elementary move is related to some sequences in the OEIS. Studying this function have lead us to discover a very interesting sequence, which we denote by $\varphi_n$. This sequence appears in the number of each elementary move, which makes it worth to be studied well and it is the main of the next section.

We shall mention that many classical sequences are hidden in the Tower of Hanoi puzzle or in its variations. We mention here the Stern diatomic sequence [3], the Stirling numbers of the second kind [11], the second order Eulerian numbers, Lah numbers and Catalan numbers [8], Fibonacci numbers [7], the Anti-Ramsey numbers [2], but also Sierpinski gasket [6] and the Pascal triangle [3]. We mention also [1] for more counting on the Tower of Hanoi. For more informations about the Tower of Hanoi problem, we refer to the comprehensive monograph [4].

## 2 Presentation of an interesting sequence

In this section we present a new integer sequence that is strongly related to the Tower of Hanoi as it is described in the next section, we denote this sequence by $\varphi_n$. We show that in the optimal sequence of moves, the number of elementary move $a$ is $\varphi_n - 3\varphi_{n-2}$, while elementary moves $b$ and $c$ appears the same number of times which is $\varphi_{n-2}$, on the other side the number of elementary move $\overline{a}$ is $2\varphi_{n-3}$, and the number of elementary move $\overline{b}$ is the same as the number of elementary move $\overline{c}$ which equals to
ϕ_{n-1} - 2ϕ_{n-3}. As we can see the sequence ϕ_n appears in the number of each elementary move. Therefore, before we present the recursive function \( f_n^{ij}(x) \) mentioned in the introduction, we present some properties and combinatorial identities of the sequence ϕ_n.

**Definition 1.** For all integer \( n \geq 2 \), we define \( ϕ_n \) using the following recurrence relation

\[
ϕ_n = 4ϕ_{n-2} + \left\lceil \frac{n+1}{2} \right\rceil,
\]

with \( ϕ_0 = 0 \), \( ϕ_1 = 1 \).

**Lemma 1.** For all integer \( n \geq 4 \), we have

\[
ϕ_n = 5ϕ_{n-2} - 4ϕ_{n-4} + 1,
\]

with \( ϕ_0 = 0 \), \( ϕ_1 = ϕ_2 = 1 \) and \( ϕ_3 = 6 \).

**Proof.** We have \( ϕ_n = 4ϕ_{n-2} + \left\lceil \frac{n+1}{2} \right\rceil \) and \( ϕ_{n-2} = 4ϕ_{n-4} + \left\lceil \frac{n-1}{2} \right\rceil \), then

\[
ϕ_n - ϕ_{n-2} = 4ϕ_{n-2} - 4ϕ_{n-4} + \left\lceil \frac{n+1}{2} \right\rceil - \left\lceil \frac{n-1}{2} \right\rceil
\]

We have \( \left\lceil \frac{n+1}{2} \right\rceil - \left\lceil \frac{n-1}{2} \right\rceil = 1 \) for all \( n \geq 0 \), therefore \( ϕ_n = 5ϕ_{n-2} - 4ϕ_{n-4} + 1 \). □

**Theorem 1.** Let \( P(t) = \sum_{n \geq 0} ϕ_n t^n \) be the ordinary generating function of \( ϕ_n \), then

\[
P(t) = \frac{t}{(1 - 4t^2)(1 + t)(1 - t)^2}
\]

**Proof.** We have \( ϕ_n = 5ϕ_{n-2} - 4ϕ_{n-4} + 1 \) then

\[
\sum_{n \geq 4} ϕ_n t^{n-4} = 5 \sum_{n \geq 4} ϕ_{n-2} t^{n-4} - 4 \sum_{n \geq 4} ϕ_{n-4} t^{n-4} + \sum_{n \geq 4} t^{n-4},
\]

therefore

\[
t^{-4}(P(t) - ϕ_0 - tϕ_1 - t^2ϕ_2 - t^3ϕ_3) = 5t^{-2}(P(t) - ϕ_0 - tϕ_1) - 4P(t) + \frac{1}{1 - t}.
\]

Hence

\[
P(t)(1 - 5t^2 + 4t^4) = t + t^2 + t^3 + t^4 + \frac{t^4}{1 - t}.
\]

Finally

\[
P(t) = \frac{t}{(1 - 4t^2)(1 + t)(1 - t)^2}
\]

□

**Corollary 1.** For all integer \( n \geq 0 \), we have

\[
ϕ_n = \sum_{i+j+2k=n} (-1)^i j 4^k.
\]
Proof. We have

\[ P(t) = \sum_{n \geq 0} \varphi_n t^n \]

\[ = \sum_{i \geq 0} (-1)^i i \sum_{j \geq 0} j t^{j-1} \sum_{k \geq 0} 4^k t^{2k} \]

\[ = \sum_{n \geq 0} \left( \sum_{i+j+2k=n} (-1)^i j 4^k \right) t^n \]

It finally comes, by identification

\[ \varphi_n = \sum_{i+j+2k=n} (-1)^i j 4^k. \]

\[ \square \]

Corollary 2. For all integer \( n \geq 0 \), we have

\[ \varphi_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} 4^k \left\lfloor \frac{n - 2k + 1}{2} \right\rfloor \]

(7)

Proof. We have

\[ \varphi_n = \sum_{i+j+2k=n} (-1)^i j 4^k \]

\[ = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} 4^k \sum_{i+j=n-2k} (-1)^i j \]

\[ = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} 4^k \sum_{j=0}^{n-2k} (-1)^{n-2k-j} j \]

\[ = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{n-2k} 4^k \sum_{j=0}^{n-2k} (-1)^j j \]

We know that

\[ \sum_{j=0}^{n} (-1)^j j = (-1)^n \left\lfloor \frac{n+1}{2} \right\rfloor = \begin{cases} \frac{n}{2} & \text{if } n \text{ even}, \\ \frac{n+1}{2} & \text{otherwise.} \end{cases} \]

therefore

\[ \varphi_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{n-2k} 4^k \left\lfloor \frac{n - 2k + 1}{2} \right\rfloor \]

\[ = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} 4^k \left\lfloor \frac{n - 2k + 1}{2} \right\rfloor. \]

\[ \square \]
Lemma 2. For all \( n \geq 0 \), we have
\[
\sum_{k=0}^{n} k4^k = \frac{4}{9}(4^n(3n - 1) + 1). \tag{8}
\]

Proof. For \( n = 0 \), we have \( \sum_{k=0}^{0} k4^k = 0 = \frac{4}{9}(4^0(3(0) - 1) + 1) \). Let’s suppose that the lemma is true up to \( n - 1 \), and we will prove it for \( n \). We have
\[
\sum_{k=0}^{n} k4^k = \sum_{k=0}^{n-1} 4^k k + n4^n = \frac{4}{9}(4^{n-1}(3(n - 1) - 1) + 1) + n4^n = \frac{4}{9}(3n4^{n-1} - 4n + 9n4^{n-1}) = \frac{4}{9}(4^n(3n - 1) + 1).
\]

Corollary 3. For all \( n \geq 0 \), we have
\[
\varphi_n = \frac{1}{9}(4\left\lfloor \frac{n+1}{2} \right\rfloor + 1 - 3\left\lfloor \frac{n+1}{2} \right\rfloor - 4) = \begin{cases} 
\frac{1}{18}(2^{n+3} - 3n - 8) & \text{if } n \text{ even} \\
\frac{1}{18}(2^{n+4} - 3n - 11) & \text{otherwise}
\end{cases}
\tag{9}
\]
and \( \varphi_n = 0 \) for all \( n \leq 0 \).

Proof. We have \( \varphi_n = \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} 4^k\left\lfloor \frac{n+1}{2} \right\rfloor + 1 \), then
\[
\varphi_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} 4^k\left(\frac{n+1}{2} - k\right) = \frac{1}{2} \left( n \sum_{k=0}^{\frac{n}{2}} 4^k - 2 \sum_{k=0}^{\frac{n}{2}} 4^k \right) \quad \text{if } n \text{ is even.}
\]
\[
\varphi_n = \sum_{k=0}^{\frac{n+1}{2}} 4^k\left(\frac{n+1}{2} - k\right) = \frac{1}{2} \left( (n + 1) \sum_{k=0}^{\frac{n-1}{2}} 4^k - 2 \sum_{k=0}^{\frac{n-1}{2}} 4^k \right) \quad \text{if } n \text{ is odd.}
\]

Using the lemma 2 and some elementary calculations we can find the result. \( \Box \)

Corollary 4. For all \( n \geq 0 \), we have
\[
\varphi_n = \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor - 1} \left(\left\lfloor \frac{n+1}{2} \right\rfloor + 1\right)3^k \tag{10}
\]

Proof. We have
\[
\varphi_n = \frac{1}{9}(4\left\lfloor \frac{n+1}{2} \right\rfloor + 1 - 3\left\lfloor \frac{n+1}{2} \right\rfloor - 4) = \frac{1}{9}\left( \sum_{k=2}^{\left\lfloor \frac{n+1}{2} \right\rfloor + 1} \left( \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \right)3^k - 3\left\lfloor \frac{n+1}{2} \right\rfloor - 4 \right)
\]
\[
= \frac{1}{9}\left( \sum_{k=2}^{\left\lfloor \frac{n+1}{2} \right\rfloor + 1} \left( \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \right)3^k + 3\left(\left\lfloor \frac{n+1}{2} \right\rfloor + 1 \right) - 3\left\lfloor \frac{n+1}{2} \right\rfloor - 4 \right)
\]
\[
= \sum_{k=2}^{\left\lfloor \frac{n+1}{2} \right\rfloor + 1} \left( \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \right)3^k
\]
\[
= \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor - 1} \left( \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \right)3^k.
\]

\( \Box \)
Corollary 5. For all $n \geq 0$, we have

$$\varphi_n = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{i=0}^{2k} (-1)^{i+1} J_i J_{2k-i},$$

(11)

where $J_n$ is the known Jacobsthal sequence.

Proof. We know that $J_n = \frac{1}{3} (2^n - (-1)^n)$ [10], then

$$\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{i=0}^{2k} (-1)^{i+1} J_i J_{2k-i} = \frac{1}{9} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{i=0}^{2k} (-1)^{i+1} (2^i - (-1)^i) (2^{2k-i} - (-1)^{2k-i})$$

$$= \frac{1}{9} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{i=0}^{2k} (-1)^{i+1} (2^{2k} + (-1)^i 2^i + (-1)^i 2^{2k-i} + 1)$$

$$= \frac{1}{9} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{i=0}^{2k} (2^i + 2^{2k-i} + (-1)^i 2^{2k+1})$$

$$= \frac{1}{9} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{i=0}^{2k} 2^i + 2^{2k} \sum_{i=0}^{2k} \frac{1}{2^i} - (2^{2k} + 1) \sum_{i=0}^{2k} (-1)^i$$

$$= \frac{1}{9} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} (2^{2k+1} - 1 + 2^{2k+1} - 1 - 2^{2k} - 1)$$

$$= \frac{1}{9} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} (3 \times 4^k - 3)$$

$$= \frac{1}{9} (4^{\lfloor \frac{n+1}{2} \rfloor + 1} - 1 - 3(\lfloor \frac{n+1}{2} \rfloor + 1))$$

$$= \frac{1}{9} (4^{\lfloor \frac{n+1}{2} \rfloor + 1} - 3(\lfloor \frac{n+1}{2} \rfloor - 4))$$

$$= \varphi_n.$$

\[ \square \]

Corollary 6. Let $E(t) = \sum_{n \geq 0} \varphi_n \frac{t^n}{n!}$ be the exponential generating function of $\varphi_n$, then we have

$$E(t) = \frac{1}{18} (\exp(t)(8 \exp(t) - 3t - 1) + 8 \sinh(2t) - 3 \sinh(t)).$$

(12)
Proof. We have

$$E(t) = \sum_{n \geq 0} \varphi_n \frac{t^n}{n!}$$

$$= \sum_{n \geq 0} \varphi_{2n} \frac{t^{2n}}{(2n)!} + \sum_{n \geq 0} \varphi_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

$$= \sum_{n \geq 0} \frac{1}{18} (2^{2n+3} - 3(2n) - 8) \frac{t^{2n}}{(2n)!} + \sum_{n \geq 0} \frac{1}{18} (2^{2n+5} - 3(2n+1) - 11) \frac{t^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{18} \left( 8 \sum_{n \geq 0} \frac{(2t)^{2n}}{(2n)!} - 3t \sum_{n \geq 0} \frac{(t)^{2n-1}}{(2n-1)!} - 8 \sum_{n \geq 0} \frac{(t)^{2n}}{(2n)!} + 16 \sum_{n \geq 0} \frac{(2t)^{2n+1}}{(2n+1)!} - 3t \sum_{n \geq 0} \frac{(t)^{2n}}{(2n)!} - 11 \sum_{n \geq 0} \frac{(t)^{2n+1}}{(2n+1)!} \right)$$

$$= \frac{1}{18} (8 \cosh(2t) - 3t \sinh(t) - 8 \cosh(t) + 16 \sinh(2t) - 3t \cosh(t) - 11 \sinh(t))$$

$$= \frac{1}{18} (8(\cosh(2t) + 2 \sinh(2t) - \cosh(t) - \sinh(t)) - 3t \sinh(t) + t \cosh(t) + \sinh(t)))$$

$$= \frac{1}{18} (8(\exp(2t) - \exp(t) + \sinh(2t)) - 3t \exp(t) + \sinh(t))$$

$$= \frac{1}{18} (\exp(t)(8 \exp(t) - 3t - 1) + 8 \sinh(2t) - 3 \sinh(t))$$

\[\square\]

Property 1. For all \( n \geq 1 \), we have

$$\varphi_{2n-1} = \varphi_{2n}. \quad (13)$$

Proof. For \( n=1 \), we have \( \varphi_1 = \varphi_2 = 1 \). Now we suppose that the property is true up to \( 2n - 2 \), then we have

$$\varphi_{2n-1} = 4\varphi_{2n-3} + \left\lfloor \frac{2n}{2} \right\rfloor = \varphi_{2n-3} + n,$$

$$\varphi_{2n} = 4\varphi_{2n-2} + \left\lfloor \frac{2n+1}{2} \right\rfloor = \varphi_{2n-2} + n.$$

By the induction hypothesis we have \( \varphi_{2n-3} = \varphi_{2n-2} \). Hence the result. \[\square\]

Remark 1. For all \( n \geq 2 \), we have

$$\varphi_{2n-1} = \varphi_{2n} = A014825(n - 1).$$

3 Counting the number of each elementary move

In this section we present the recursive function

$$f_n^{ij} : \begin{cases} A \to \mathbb{N}, \\ x \mapsto f_n^{ij}(x). \end{cases}$$

Where \( x \in A = \{a, b, c, \overline{a}, \overline{b}, \overline{c}\} \) is an elementary move. \( f_n^{ij}(x) \) counts the number of appearance of move \( x \) in the optimal sequence of elementary moves that transfers a tower of \( n \) discs from peg \( i \) to peg \( j \) using peg \( k \) as an auxiliary peg.
Theorem 2. For all $n \geq 4$, $x \in \mathcal{A}$, $f^{ij}_n(x)$ satisfies the following recurrence relation

$$f^{ij}_n(x) = 5f^{ij}_{n-2}(x) - 4f^{ij}_{n-4}(x) + 2f^{ij}_{n-6}(x) + f^{ij}_{n-7}(x) - f^{ij}_{n-9}(x) - 2f^{ij}_{n-11}(x)$$

where $f^{ij}_0(x) = 0$, $f^{ij}_1(x) = 1_{\{a\}}(x)$, $f^{ij}_2(x) = 1_{\{a,\overline{n}\}}(x)$ and $f^{ij}_3(x) = 31_{\{a\}}(x) + 1_{\{\overline{n},\overline{n},\overline{n}\}}(x)$.

Proof. Let $S_n^{ij}$ be the optimal sequence of moves that transfer a tower of $n$ discs from peg $i$ to peg $j$, then as a direct result of the optimal procedure that solves the Tower of Hanoi problem which is described in the introduction, we have

$$S_n^{ij} = S_{n-1}^{ik}[i \to j]S_{n-1}^{kj},$$

where $[i \to j]$ denotes a single move from peg $i$ to $j$. We obtain

$$f^{ij}_{n-1}(x) = f^{ij}(x) - 2f^{ij}_{n-2}(x) + 1_{\{\overline{n}\}}(x) - 1_{\{\overline{a},\overline{n}\}}(x)$$

On the other hand we have

$$f^{ij}_{n}(x) = 2f^{ij}_{n-2}(x) + f^{ij}_{n-1}(x) - 1_{\{a\}}(x) + 1_{\{b,c\}}(x)$$

By replacing (15) in (16) we obtain

$$f^{ij}_{n+1}(x) - 2f^{ij}_{n-1}(x) + 1_{\{\overline{n}\}}(x) - 1_{\{\overline{a},\overline{n}\}}(x) = 2f^{ij}_{n-1}(x) - 2f^{ij}_{n-3}(x) + 1_{\{\overline{n}\}}(x) - 1_{\{\overline{a},\overline{n}\}}(x) - f^{ij}_{n-1}(x) - 1_{\{a\}}(x) + 1_{\{b,c\}}(x).$$

Thus

$$f^{ij}_{n+1}(x) - 5f^{ij}_{n-1}(x) + 4f^{ij}_{n-3}(x) = -2f^{ij}_{n-1}(x) - 1_{\{\overline{n}\}}(x) + 21_{\{\overline{n}\}}(x) + 1_{\{b,c\}}(x)$$

therefore,

$$f^{ij}_{n+1}(x) = 5f^{ij}_{n-1}(x) - 4f^{ij}_{n-3}(x) - 21_{\{a\}}(x) - 1_{\{\overline{n}\}}(x) + 21_{\{\overline{n}\}}(x) + 1_{\{b,c\}}(x)$$

For the initial conditions, $\forall x \in \mathcal{A}$ we have,

$$S^0_{ij} = \emptyset \implies f^{ij}_0(x) = 0,$$

$$S^1_{ij} = [i \to j] \implies f^{ij}_1(x) = 1_{\{a\}}(x),$$

$$S^2_{ij} = [i \to k, i \to j, k \to j] \implies f^{ij}_2(x) = 1_{\{\overline{n}\}}(x) + 1_{\{a\}}(x) + 1_{\{\overline{n}\}}(x) = 1_{\{a,\overline{n}\}}(x),$$

$$S^3_{ij} = [i \to j, i \to k, j \to k, i \to j, k \to i, k \to j, i \to j] \implies f^{ij}_3(x) = 31_{\{a\}}(x) + 1_{\{\overline{n},\overline{n},\overline{n}\}}(x).$$

\[\square\]

Corollary 7. For all $n \geq 5$, $x \in \mathcal{A}$, $f^{ij}_n(x)$ satisfies the following recurrence relation

$$f^{ij}_n(x) = f^{ij}_{n-1}(x) + 5f^{ij}_{n-2}(x) - 5f^{ij}_{n-3}(x) - 4f^{ij}_{n-4}(x) + 4f^{ij}_{n-5}(x).$$

Proof. We have

$$f^{ij}_{n}(x) = 5f^{ij}_{n-2}(x) - 4f^{ij}_{n-4}(x) + 21_{\{\overline{n}\}}(x) + 1_{\{b\}}(x) + 1_{\{c\}}(x) - 1_{\{\overline{n}\}}(x) - 1_{\{\overline{n}\}}(x) - 21_{\{a\}}(x)$$

$$f^{ij}_{n-1}(x) = 5f^{ij}_{n-3}(x) - 4f^{ij}_{n-5}(x) + 21_{\{\overline{n}\}}(x) + 1_{\{b\}}(x) + 1_{\{c\}}(x) - 1_{\{\overline{n}\}}(x) - 1_{\{\overline{n}\}}(x) - 21_{\{a\}}(x)$$

then,

$$f^{ij}_{n}(x) - f^{ij}_{n-1}(x) = 5f^{ij}_{n-2}(x) - 5f^{ij}_{n-3}(x) - 4f^{ij}_{n-4}(x) + 4f^{ij}_{n-5}(x).$$

Hence the result. \[\square\]
Corollary 8. For all \( n \geq 0, x \in A \), we have
\[
\sum_{x \in A} f_{n}^{ij}(x) = 2^n - 1.
\] (18)

Proof. It is easy to see. We can prove by combinatorial arguments, or algebraically by finding the explicit
formula of \( f_{n}^{ij}(x) \) using the coming results.

Corollary 9. For all integer \( n \geq 0, x \in A \), we have

\[
f_{n+1}^{ij}(a) = \begin{cases} 
0 & \text{if } n = 0; \\
1 & \text{if } n = 1, 2; \\
5f_{n-1}^{ij}(a) - 4f_{n-3}^{ij}(a) - 2 & \text{otherwise}. 
\end{cases}
\] (19)

\[
f_{n+1}^{ij}(b) = \begin{cases} 
0 & \text{if } n = 0, 1, 2; \\
5f_{n-1}^{ij}(b) - 4f_{n-3}^{ij}(b) + 1 & \text{otherwise}. 
\end{cases}
\] (20)

\[
f_{n+1}^{ij}(c) = \begin{cases} 
0 & \text{if } n = 0, 1, 2; \\
5f_{n-1}^{ij}(c) - 4f_{n-3}^{ij}(c) + 1 & \text{otherwise}. 
\end{cases}
\] (21)

\[
f_{n+1}^{ij}(\bar{a}) = \begin{cases} 
0 & \text{if } n = 0, 1, 2; \\
5f_{n-1}^{ij}(\bar{a}) - 4f_{n-3}^{ij}(\bar{a}) + 2 & \text{otherwise}. 
\end{cases}
\] (22)

\[
f_{n+1}^{ij}(\bar{b}) = \begin{cases} 
0 & \text{if } n = 0, 1; \\
1 & \text{if } n = 2; \\
5f_{n-1}^{ij}(\bar{b}) - 4f_{n-3}^{ij}(\bar{b}) - 1 & \text{otherwise}. 
\end{cases}
\] (23)

\[
f_{n+1}^{ij}(\bar{c}) = \begin{cases} 
0 & \text{if } n = 0, 1; \\
1 & \text{if } n = 2; \\
5f_{n-1}^{ij}(\bar{c}) - 4f_{n-3}^{ij}(\bar{c}) - 1 & \text{otherwise}. 
\end{cases}
\] (24)

Theorem 3. Let \( G_{x}^{ij}(t, x) = \sum_{n \geq 0} f_{n}^{ij}(x)t^n \) be the ordinary generating function of the sequence \( f_{n}^{ij}(x) \) for
all \( x \in A \), then
\[
G_{x}^{ij}(t, x) = \frac{1 \{a\}(x) + 1 \{\bar{a}\}(x)t + 2(1 \{b\}(x) - 31 \{a\}(x))t^3 + 2(1 \{\bar{c}\}(x) - 1 \{\bar{a}\}(x))t^4}{(1 - 4t^2)(1 + t)(1 - t)^2}.
\] (25)

Proof. We have
\[
f_{n}^{ij}(x) = 5f_{n-2}^{ij}(x) - 4f_{n-4}^{ij}(x) + \epsilon(x)
\]
where \( \epsilon(x) = 21 \{\bar{a}\}(x) + 1 \{b\}(x) + 1 \{\bar{c}\}(x) - 1 \{\bar{a}\}(x) - \bar{1} \{\bar{a}\}(x) - 21 \{a\}(x) \).
Therefore
\[
\sum_{n \geq 4} f_{n}^{ij}(x)t^{n-4} = 5 \sum_{n \geq 4} f_{n-2}^{ij}(x)t^{n-4} - 4 \sum_{n \geq 4} f_{n-4}^{ij}(x)t^{n-4} + \epsilon(x) \sum_{n \geq 4} t^{n-4},
\]
which gives
\[
t^{-4}(G_{x}^{ij}(t, x) - f_{1}^{ij}(x)t - f_{2}^{ij}(x)t^2 - f_{3}^{ij}(x)t^3) = 5t^2(G_{x}^{ij}(t, x) - f_{1}^{ij}(x)t) - 4G_{x}^{ij}(t, x) + \frac{\epsilon(x)}{1 - t}
\]
\[
G_{x}^{ij}(t, x)(t^{-4} - 5t^{-2} + 4) = t^{-4}(f_{1}^{ij}(x)t + f_{2}^{ij}(x)t^2 + f_{3}^{ij}(x)t^3) - 5t^{-2}(f_{1}^{ij}(x)t) + \frac{\epsilon(x)}{1 - t}
\]

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then
\[
G^{ij}(t, x)(1-5t^2+4t^4) = \frac{f_1^{ij}(x)t + (f_2^{ij}(x) - f_1^{ij}(x))t^2 + (f_3^{ij}(x) - f_2^{ij}(x) - 5f_1^{ij}(x))t^3 + (c(x) - f_3^{ij}(x) + 5f_1^{ij}(x))t^4}{1-t}
\]
Hence the result. \(\Box\)

**Corollary 10.** The ordinary generating functions of the sequences that counts the number of each elementary move in the set \(A\) are

\[
G^{ij}(t, a) = \frac{t - 3t^3}{(1+t)(1-t)^2(1-4t^2)},
\]

\[
G^{ij}(t, b) = \frac{t^3}{(1+t)(1-t)^2(1-4t^2)},
\]

\[
G^{ij}(t, c) = \frac{2t^4}{(1+t)(1-t)^2(1-4t^2)},
\]

\[
G^{ij}(t, \bar{a}) = \frac{t^2}{(1+t)(1-t)^2(1-4t^2)},
\]

\[
G^{ij}(t, \bar{b}) = \frac{t^2 - 2t^4}{(1+t)(1-t)^2(1-4t^2)},
\]

\[
G^{ij}(t, \bar{c}) = \frac{t^2 - 2t^4}{(1+t)(1-t)^2(1-4t^2)}.
\]

**Theorem 4.** For all \(n \geq 0, x \in A\) we have

\[
f_n^{ij}(x) = 1_{\{a\}}(x)\varphi_n + 1_{\{\bar{a}\}}(x)\varphi_{n-1} + (1_{\{b,c\}}(x) - 31_{\{a\}}(x))\varphi_{n-2} + (21_{\{\bar{a}\}}(x) - 21_{\{\bar{b}\}}(x))\varphi_{n-3}.
\]

**Proof.** We have

\[
G^{ij}(t, x) = \frac{1_{\{a\}}(x)t + 1_{\{\bar{a}\}}(x)t^2 + (1_{\{b,c\}}(x) - 31_{\{a\}}(x))t^3 + (21_{\{\bar{a}\}}(x) - 21_{\{\bar{b}\}}(x))t^4}{(1-4t^2)(1+t)(1-t)^2}
\]

\[
= (1_{\{a\}}(x) + 1_{\{\bar{a}\}}(x)t + (1_{\{b,c\}}(x) - 31_{\{a\}}(x))t^2 + (21_{\{\bar{a}\}}(x) - 21_{\{\bar{b}\}}(x))t^3)\frac{t}{(1-4t^2)(1+t)(1-t)^2}
\]

\[
= (1_{\{a\}}(x) + 1_{\{\bar{a}\}}(x)t + (1_{\{b,c\}}(x) - 31_{\{a\}}(x))t^2 + (21_{\{\bar{a}\}}(x) - 21_{\{\bar{b}\}}(x))t^3)\sum_{n \geq 0} \varphi_n t^n
\]

\[
= \sum_{n \geq 0} (1_{\{a\}}(x)\varphi_n + 1_{\{\bar{a}\}}(x)\varphi_{n-1} + (1_{\{b,c\}}(x) - 31_{\{a\}}(x))\varphi_{n-2} + (21_{\{\bar{a}\}}(x) - 21_{\{\bar{b}\}}(x))\varphi_{n-3})t^n
\]

\(\Box\)

**Corollary 11.** For all \(n \geq 0, x \in A\), we have

\[
f_n^{ij}(x) = (\varphi_n - 3\varphi_{n-2})1_{\{a\}}(x) + \varphi_{n-2}1_{\{b,c\}}(x) + 2\varphi_{n-3}1_{\{\bar{a}\}}(x) + (\varphi_{n-1} - 2\varphi_{n-3})1_{\{\bar{b}\}}(x).
\]

**Corollary 12.** For all \(n \geq 0, x \in A\), we have

\[
f_n^{ij}(x) = \begin{cases} 
1_{\{a\}}(x)\varphi_n + (1_{\{b,c\}}(x) + 21_{\{\bar{a}\}}(x) - 31_{\{a\}}(x) - 21_{\{b,c\}}(x))\varphi_{n-2} & \text{if } n \text{ even}, \\
1_{\{a\}}(x)\varphi_n + (1_{\{\bar{a},\bar{b},\bar{c}\}}(x) - 31_{\{a\}}(x))\varphi_{n-1} + 2(1_{\{\bar{a}\}}(x) - 1_{\{\bar{b}\}}(x))\varphi_{n-3} & \text{otherwise}. 
\end{cases}
\]
Proof. Use property \[\Box\]

**Corollary 13.** For all \(n \geq 0\), we have

\[
\begin{align*}
    f_n^{ij}(a) &= \varphi_n - 3\varphi_{n-2}, \quad (36) \\
    f_n^{ij}(b) &= \varphi_{n-2}, \quad (37) \\
    f_n^{ij}(c) &= \varphi_{n-2}, \quad (38) \\
    f_n^{ij}(\overline{a}) &= 2\varphi_{n-3}, \quad (39) \\
    f_n^{ij}(\overline{b}) &= \varphi_{n-1} - 2\varphi_{n-3} \quad (40) \\
    f_n^{ij}(\overline{c}) &= \varphi_{n-1} - 2\varphi_{n-3}. \quad (41)
\end{align*}
\]

**Remark 2.** We can find lemma 1 using equation 37.

**Corollary 14.** For all \(n \geq 0\) we have

\[
\begin{align*}
    f_n^{ij}(a) &= \frac{1}{9}(4^{\lfloor \frac{n+1}{2} \rfloor} + 6\lfloor \frac{n+1}{2} \rfloor - 1) = \begin{cases} 
        \frac{1}{9}(2^n + 3n - 1) & \text{if } n \text{ even,} \\
        \frac{1}{9}(2^n+1 + 3n + 2) & \text{otherwise,}
    \end{cases} \quad (43) \\
    f_n^{ij}(b) &= \frac{1}{9}(4^{\lfloor \frac{n+1}{2} \rfloor} + 3\lfloor \frac{n-1}{2} \rfloor - 4) = \begin{cases} 
        \frac{1}{18}(2^n+1 - 3n - 2) & \text{if } n \text{ even,} \\
        \frac{1}{18}(2^n-2 - 3n - 5) & \text{otherwise,}
    \end{cases} \quad (44) \\
    f_n^{ij}(c) &= \frac{1}{9}(4^{\lfloor \frac{n+1}{2} \rfloor} + 3\lfloor \frac{n-1}{2} \rfloor - 4) = \begin{cases} 
        \frac{1}{18}(2^n+1 - 3n - 2) & \text{if } n \text{ even,} \\
        \frac{1}{18}(2^n-2 - 3n - 5) & \text{otherwise,}
    \end{cases} \quad (45) \\
    f_n^{ij}(\overline{a}) &= \frac{2}{9}(4^{\lfloor \frac{n+2}{2} \rfloor} + 3\lfloor \frac{n-2}{2} \rfloor - 4) = \begin{cases} 
        \frac{1}{9}(2^n+1 - 3n - 2) & \text{if } n \text{ even,} \\
        \frac{1}{9}(2^n-3n + 1) & \text{otherwise,}
    \end{cases} \quad (46) \\
    f_n^{ij}(\overline{b}) &= \frac{1}{18}(4^{\lfloor \frac{n+2}{2} \rfloor} + 6\lfloor \frac{n+2}{2} \rfloor - 10) = \begin{cases} 
        \frac{1}{18}(2^n+2 + 3n - 4) & \text{if } n \text{ even,} \\
        \frac{1}{18}(2^n+1 + 3n - 7) & \text{otherwise,}
    \end{cases} \quad (47) \\
    f_n^{ij}(\overline{c}) &= \frac{1}{18}(4^{\lfloor \frac{n+2}{2} \rfloor} + 6\lfloor \frac{n+2}{2} \rfloor - 10) = \begin{cases} 
        \frac{1}{18}(2^n+2 + 3n - 4) & \text{if } n \text{ even,} \\
        \frac{1}{18}(2^n+1 + 3n - 7) & \text{otherwise.}
    \end{cases} \quad (48)
\end{align*}
\]

Proof. Use corollaries 3 and 13 \[\Box\]

**Property 2.** For all \(n \geq 0\), we have

\[
\begin{align*}
    f_n^{ij}(\overline{a}) &= f_n^{ij}(\overline{b}) \geq f_n^{ij}(\overline{c}) \geq f_n^{ij}(a) \geq f_n^{ij}(b) = f_n^{ij}(c) \quad \text{if } n \geq 6 \text{ even,} \quad (49) \\
    f_n^{ij}(a) &\geq f_n^{ij}(b) = f_n^{ij}(c) \geq f_n^{ij}(\overline{a}) = f_n^{ij}(\overline{b}) \geq f_n^{ij}(\overline{c}) \quad \text{if } n \geq 0 \text{ even.} \quad (50)
\end{align*}
\]

**Remark 3.** In relation 49, if \(0 \leq n \leq 5\) then we should have

\[
    f_n^{ij}(\overline{a}) = f_n^{ij}(\overline{b}) \geq f_n^{ij}(a) \geq f_n^{ij}(\overline{c}) \geq f_n^{ij}(b) = f_n^{ij}(c).
\]

**Proposition 1.** Here is some propositions which are true for all \(n \geq 0\).

\[
\begin{align*}
    f_n^{ij}(a) &\geq f_n^{ij}(b) = f_n^{ij}(c), \quad (51) \\
    f_n^{ij}(\overline{a}) &= f_n^{ij}(\overline{b}) \geq f_n^{ij}(\overline{c}) \quad (52) \\
    f_n^{ij}(a) &= f_n^{ij}(b) + \lfloor \frac{n+1}{2} \rfloor = f_n^{ij}(c) + \lfloor \frac{n+1}{2} \rfloor. \quad (53)
\end{align*}
\]
Proof. According to corollary \( \square \) we have

\[ f_n^{ij}(x) = \varphi_n - 3\varphi_{n-2} \mathbb{I}_{\{a\}}(x) + \varphi_{n-3} \mathbb{I}_{\{b,c\}}(x) + 2\varphi_{n-1} \mathbb{I}_{\{\pi\}}(x) + (\varphi_n - 2\varphi_{n-3}) \mathbb{I}_{\{\beta,\pi\}}(x) \]

\[ = (4\varphi_n - (\varphi_n - 3\varphi_{n-2}) - 12\varphi_{n-4} - 3[(\varphi_n - 3\varphi_{n-2}) - 2\varphi_{n-4} + (4\varphi_n - 4\varphi_{n-4} + (\varphi_n - 3\varphi_{n-2}) - 8\varphi_{n-5} - 2[(\varphi_n - 3\varphi_{n-2}) - 2[(\varphi_n - 3\varphi_{n-2}) - \varphi_{n-6} - 3\varphi_{n-8} - 3\varphi_{n-10}] - \varphi_{n-12}] - \varphi_{n-14}] - \varphi_{n-16}) \mathbb{I}_{\{\beta,\pi\}}(x) \]

\[ = 4f_n^{ij}(x) + (3 - 2[\frac{n+1}{2}]\mathbb{I}_{\{a\}}(x) + [\frac{n-1}{2}]\mathbb{I}_{\{b,c\}}(x) + 2[\frac{n-2}{2}]\mathbb{I}_{\{\pi\}}(x) + (2 - [\frac{n}{2}])\mathbb{I}_{\{\beta,\pi\}}(x) \]

\[ \square \]

Corollary 15. For all \( n \geq 2 \), we have

\[ f_n^{ij}(a) = 4f_n^{ij}(a) + 3 - 2[\frac{n+1}{2}], \quad f_n^{ij}(a) = 0, \quad f_n^{ij}(a) = 1. \]  

\[ f_n^{ij}(b) = 4f_n^{ij}(b) + [\frac{n-1}{2}], \quad f_n^{ij}(b) = f_n^{ij}(b) = 0. \]  

\[ f_n^{ij}(c) = 4f_n^{ij}(c) + [\frac{n-1}{2}], \quad f_n^{ij}(c) = f_n^{ij}(c) = 0. \]  

\[ f_n^{ij}(\pi) = 4f_n^{ij}(\pi) + 2[\frac{n-2}{2}], \quad f_n^{ij}(\pi) = f_n^{ij}(\pi) = 0. \]  

\[ f_n^{ij}(\beta) = 4f_n^{ij}(\beta) + 2 - [\frac{n}{2}], \quad f_n^{ij}(\beta) = f_n^{ij}(\beta) = 0. \]  

\[ f_n^{ij}(\tau) = 4f_n^{ij}(\tau) + 2 - [\frac{n}{2}], \quad f_n^{ij}(\tau) = f_n^{ij}(\tau) = 0. \]  

Lemma 3. For all \( n \geq 0, x \in A \), we have

\[ f_n^{ij}(x) = \sum_{y \in A} f_n^{ij}(y) \mathbb{I}_{\{y\}}(x). \]  

Corollary 16. For all \( n \geq 0, x \in A \), we have

\[ f_n^{ij}(x) = f_n^{ij}(a) - \frac{n+1}{2}\mathbb{I}_{\{b,c\}}(x) - \frac{1}{9}((-1)^{n+1}\mathbb{I}_{\{\pi,\beta,\pi\}}(x) + \frac{n}{2}\mathbb{I}_{\{\beta,\pi\}}(x). \]  

Proof. Use the facts in proposition \( \square \) and lemma \( \square \)  

Property 3. For all \( n \geq 1 \), we have

\[ f_n^{ij}(a) = f_n^{ij}(a), \]  

\[ f_n^{ij}(b) = f_n^{ij}(b), \]  

\[ f_n^{ij}(c) = f_n^{ij}(c). \]
and for all \( n \geq 0 \)

\[
\begin{align*}
\langle f^{ij}_{2n}(\pi) \rangle &= \langle f^{ij}_{2n+1}(\pi) \rangle \\
\langle f^{ij}_{2n}(\beta) \rangle &= \langle f^{ij}_{2n+1}(\beta) \rangle \\
\langle f^{ij}_{2n}(\tau) \rangle &= \langle f^{ij}_{2n+1}(\tau) \rangle
\end{align*}
\] (69)-(72)

**Remark 4.** Here are some relations between our sequences and some OEIS sequences.

\[
\begin{align*}
f^{ij}_{2n}(a) &= f^{ij}_{2n-1}(a) = A073724(n), \quad \forall n \geq 1. \\
f^{ij}_{2n}(b) &= f^{ij}_{2n-1}(b) = A014825(n-1), \quad \forall n \geq 2. \\
f^{ij}_{2n}(c) &= f^{ij}_{2n-1}(c) = A014825(n-1), \quad \forall n \geq 2. \\
f^{ij}_{2n}(\pi) &= f^{ij}_{2n+1}(\pi) = A145766(n-1), \quad \forall n \geq 1. \\
f^{ij}_{2n}(\beta) &= f^{ij}_{2n+1}(\beta) = A160156(n-1), \quad \forall n \geq 1. \\
f^{ij}_{2n}(\tau) &= f^{ij}_{2n+1}(\tau) = A160156(n-1), \quad \forall n \geq 1.
\end{align*}
\] (73)-(78)

**Proposition 2.** For reasons of symmetry of the Tower of Hanoi puzzle, for all \( n \geq 0 \), we have

\[
\begin{align*}
f^{ij}_n(a) &= f^{ik}_n(\pi) = f^{kj}_n(b) = f^{ji}_n(\bar{\pi}) = f^{ki}_n(c) = f^{jk}_n(b), \\
f^{ij}_n(b) &= f^{ki}_n(a) = f^{ji}_n(\pi) = f^{kj}_n(\bar{\pi}) = f^{ji}_n(c) = f^{jk}_n(b), \\
f^{ij}_n(c) &= f^{ki}_n(b) = f^{kj}_n(a) = f^{jk}_n(\pi) = f^{ji}_n(b) = f^{ik}_n(\pi), \\
f^{ij}_n(\pi) &= f^{ik}_n(c) = f^{ki}_n(b) = f^{ji}_n(a) = f^{ji}_n(\pi) = f^{ki}_n(b), \\
f^{ij}_n(\bar{\pi}) &= f^{ki}_n(\pi) = f^{ji}_n(c) = f^{kj}_n(b) = f^{ji}_n(b) = f^{ik}_n(\pi), \\
f^{ij}_n(\tau) &= f^{ki}_n(b) = f^{jk}_n(\pi) = f^{kj}_n(c) = f^{ji}_n(b) = f^{ik}_n(\pi).
\end{align*}
\] (79)-(85)

**Theorem 6.** Let \( E(t, x) = \sum_{n \geq 0} \frac{f^{ij}_n(x)}{n!} \) be the exponential generating function of \( f^{ij}_n(x) \), then

\[
E(t, x) = E(t, a)1_{(a)}(x) + E(t, b)1_{(b,c)}(x) + E(t, \pi)1_{(\pi)}(x) + E(t, \bar{\pi})1_{(\bar{\pi})}(x).
\] (86)

Where

\[
\begin{align*}
E(t, a) &= \frac{1}{9} [e^t(e^t + 3t - 1) + \sinh(2t) + 3\sinh(t)] \\
E(t, b) &= \frac{1}{18} [e^t(2e^t - 3t - 2) + 2\sinh(2t) - 3\sinh(t)] \\
E(t, c) &= \frac{1}{18} [e^t(2e^t - 3t - 2) + 2\sinh(2t) - 3\sinh(t)] \\
E(t, \pi) &= \frac{1}{9} [e^t(2e^t - 3t - 2) - \sinh(2t) + 3\sinh(t)] \\
E(t, \bar{\pi}) &= \frac{1}{18} [e^t(4e^t + 3t - 4) - 2\sinh(2t) - 3\sinh(t)] \\
E(t, \tau) &= \frac{1}{18} [e^t(4e^t + 3t - 4) - 2\sinh(2t) - 3\sinh(t)]
\end{align*}
\] (87)-(93)
Example 1. Let consider a Tower of Hanoi with \( n \) discs and three pegs \( i, j \) and \( k \), where the goal is to transfer the \( n \)-tower from peg \( i \) to peg \( j \) using peg \( k \) as auxiliary peg. The optimal sequences are as follow:

\[
\begin{align*}
    1 & : a \\
    2 & : \overline{a} b \\
    3 & : \overline{ab} \bar{c} a \bar{b} a \\
    4 & : \overline{ac} \bar{b} \bar{c} \bar{a} \bar{b} a \bar{c} a \bar{b} \bar{c} a \bar{b} a \\
    5 & : \overline{ac} \bar{b} \bar{c} \bar{a} \bar{b} \bar{c} \bar{a} \bar{b} \bar{c} \bar{a} \bar{b} a \bar{c} a \bar{b} \bar{c} a \bar{b} a \\
    6 & : \overline{ac} \bar{b} \bar{c} \bar{a} \bar{b} a \bar{c} a \bar{b} \bar{c} a \bar{b} \bar{c} a \bar{b} a \bar{c} a \bar{b} \bar{c} a \bar{b} a \bar{c} a \bar{b} \bar{c} a \bar{b} a \bar{c} a \bar{b} \bar{c} a \bar{b} a
\end{align*}
\]

Table 1: The optimal sequences of moves for the first values of \( n \)

The following table shows the number of elementary move in the optimal sequences above,

| \( n \) | \( f_{n}^{ij}(a) \) | \( f_{n}^{ij}(\bar{a}) \) | \( f_{n}^{ij}(b) \) | \( f_{n}^{ij}(\bar{b}) \) | \( f_{n}^{ij}(c) \) | \( f_{n}^{ij}(\bar{c}) \) |
|-------|------------------|------------------|------------------|------------------|------------------|------------------|
| 1     | 1                | 0                | 0                | 0                | 0                | 0                |
| 2     | 1                | 1                | 1                | 0                | 0                | 0                |
| 3     | 3                | 1                | 1                | 1                | 1                | 0                |
| 4     | 3                | 4                | 4                | 1                | 1                | 2                |
| 5     | 9                | 4                | 4                | 6                | 6                | 2                |
| 6     | 9                | 15               | 15               | 6                | 6                | 12               |
| 7     | 31               | 15               | 15               | 27               | 27               | 12               |
| 8     | 31               | 58               | 58               | 27               | 27               | 54               |
| 9     | 117              | 58               | 58               | 112              | 112              | 54               |
| 10    | 117              | 229              | 229              | 112              | 112              | 224              |

Figure 2: Evaluation of sequences
4 Counting the number of each elementary move that each disc make

In this section we extend our work to calculate not only the number of each elementary move in the optimal sequence of moves, but also the number of times that each disc \( d \in D \) \( = \{1, \ldots, n\} \) make the elementary move \( x \in A \) during the execution of the optimal solution.

We define the recursive function

\[
g_{n}^{ij} : \begin{cases} 
D \times A \to \mathbb{N}, \\
(d, x) \mapsto f_{n}^{ij}(d,x).
\end{cases}
\]

Where \( d \in D \) \( = \{1, \ldots, n\} \) is a disc and \( x \in A = \{a, b, c, i, j, k\} \) is an elementary move, we define it as the number of times that disc \( d \) makes the elementary move \( x \) in the sequence of optimal moves to solve the tower of Hanoi, where the goal is to transfer a tower of \( n \) discs from peg \( i \) to peg \( j \) using peg \( k \) as an auxiliary peg.

**Theorem 7.** For all \( n \geq 4, d \in D, x \in A \), we have \( g_{n}^{ij}(d,x) \) satisfies the following recurrence relation

\[
g_{n}^{ij}(d,x) = 5g_{n-2}^{ij}(d,x) - 4g_{n-4}^{ij}(d,x) + \mathbbm{1}_{\{a\}}(x)\mathbbm{1}_{\{n\}}(d) + \mathbbm{1}_{\{\overline{i}, \overline{j}\}}(x)\mathbbm{1}_{\{n-1\}}(d) \\
+ (\mathbbm{1}_{\{b,c\}}(x) - 3\mathbbm{1}_{\{a\}}(x))\mathbbm{1}_{\{n-2\}}(d) - 2(\mathbbm{1}_{\{\overline{i}, \overline{j}\}}(x) - \mathbbm{1}_{\{\overline{x}\}}(x))\mathbbm{1}_{\{n-3\}}(d)
\]

(94)

where \( g_{0}^{ij}(d,x) = 0, g_{1}^{ij}(d,x) = \mathbbm{1}_{\{a\}}(x)\mathbbm{1}_{\{1\}}(d), g_{2}^{ij}(d,x) = \mathbbm{1}_{\{a\}}(x)\mathbbm{1}_{\{2\}}(d) + \mathbbm{1}_{\{\overline{i}, \overline{j}\}}(x)\mathbbm{1}_{\{1\}}(d) \) and \( g_{3}^{ij}(d,x) = \mathbbm{1}_{\{a\}}(x)\mathbbm{1}_{\{3\}}(d) + \mathbbm{1}_{\{\overline{i}, \overline{j}\}}(x)\mathbbm{1}_{\{2\}}(d) + (\mathbbm{1}_{\{b,c\}}(x) + 2\mathbbm{1}_{\{a\}}(x))\mathbbm{1}_{\{1\}}(d). \)

**Proof.** We have

\[
S_{n}^{ij} = S_{n-1}^{ik}[i \to j]S_{n-1}^{kj},
\]

then we obtain

\[
g_{n}^{ij}(d,x) = g_{n-1}^{ij}(d,x) + \mathbbm{1}_{\{a\}}(x)\mathbbm{1}_{\{n\}}(d) + g_{n-1}^{kj}(d,x) \\
= g_{n-2}^{ij}(d,x) + \mathbbm{1}_{\{\overline{i}, \overline{j}\}}(x)\mathbbm{1}_{\{n-1\}}(d) + g_{n-2}^{jk}(d,x) + \mathbbm{1}_{\{a\}}(x)\mathbbm{1}_{\{n\}}(d) + g_{n-2}^{ji}(d,x) + \mathbbm{1}_{\{\overline{i}, \overline{j}\}}(x)\mathbbm{1}_{\{n-1\}}(d) + g_{n-2}^{ij}(d,x) \\
= 2g_{n-2}^{ij}(d,x) + g_{n-1}^{ji}(d,x) + \mathbbm{1}_{\{a\}}(x)\mathbbm{1}_{\{n\}}(d) + (\mathbbm{1}_{\{\overline{i}, \overline{j}\}}(x) - \mathbbm{1}_{\{\overline{x}\}}(x))\mathbbm{1}_{\{n-1\}}(d).
\]

Which implies that

\[
g_{n-1}^{ij}(d,x) = g_{n}^{ij}(d,x) - 2g_{n-2}^{ij}(d,x) - \mathbbm{1}_{\{a\}}(x)\mathbbm{1}_{\{n\}}(d) - (\mathbbm{1}_{\{\overline{i}, \overline{j}\}}(x) - \mathbbm{1}_{\{\overline{x}\}}(x))\mathbbm{1}_{\{n-1\}}(d)
\]

(95)

On the other hand we have

\[
g_{n}^{ji}(d,x) = 2g_{n-2}^{ji}(d,x) + g_{n-1}^{ji}(d,x) + \mathbbm{1}_{\{\overline{i}, \overline{j}\}}(x)\mathbbm{1}_{\{n\}}(d) + (\mathbbm{1}_{\{b,c\}}(x) - \mathbbm{1}_{\{a\}}(x))\mathbbm{1}_{\{n-1\}}(d)
\]

(96)

We obtain the result by replacing \( 95 \) in \( 96 \). \( \square \)

**Proposition 3.** For all \( n \geq 0, d \in D, x \in A \), we have

\[
\sum_{x \in A} g_{n}^{ij}(d,x) = 2^{n-d}.
\]

(97)

\[
\sum_{d \in D} g_{n}^{ij}(d,x) = f_{n}^{ij}(x).
\]

(98)

**Proof.** The first one can be proved by using the fact that disc \( d \in D \) moves exactly \( 2^{n-d} \) times during the execution of the optimal sequence of moves \( \mathcal{H} \). While the second proposition is a direct result of the definitions of \( g_{n}^{ij}(d,x) \) and \( f_{n}^{ij}(x) \). \( \square \)
Theorem 8. Let \( H_{ij}^{ij}(t, d, x) = \sum_{n \geq 0} g_{ij}^{ij}(d, x) t^n \) be the ordinary generating function of the sequence \( g_{ij}^{ij}(d, x) \) for all \( d \in D \) and \( x \in A \), then

\[
H_{ij}^{ij}(t, d, x) = \frac{1}{1-4t^2(1-t^2)} \sum_{n \geq 0} 1_{\{n\}}(d) t^n = t^d.
\]

Proof. Using the same technique as in the proof of Theorem 3, in addition to the following remark

\[
\sum_{n \geq 0} 1_{\{n\}}(d) t^n = t^d.
\]

Theorem 9. For all \( n \geq 0, d \in D, x \in A \), we have

\[
g_{ij}^{ij}(d, x) = 1_{\{a\}}(x) \varphi_{n-d+1} + (1_{\{\bar{a}\}}(x) - 1_{\{a\}}(x)) \varphi_{n-d} + (1_{\{b,c\}}(x) - 31_{\{a\}}(x) + 1_{\{\bar{a},\bar{b}\}}(x)) \varphi_{n-d-2} - 2(1_{\{\bar{a}\}}(x) - 1_{\{\bar{a},\bar{b}\}}(x)) \varphi_{n-d-3}. \tag{100}
\]

Proposition 4. For all \( n \geq 2, d \in D, x \in A \) we have

\[
g_{ij}^{ij}(d, x) = g_{n-1}^{ij}(d-1, x). \tag{101}
\]

This last result allow us to identify triangles for the sequence \( g_{ij}^{ij}(d, x) \) where the move \( x \) is fixed, we present here an example.

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 |   |   |   |   |   |   |   |   |   |   |
| 2 | 0 | 1 |   |   |   |   |   |   |   |   |   |
| 3 | 2 | 0 | 1 |   |   |   |   |   |   |   |   |
| 4 | 0 | 2 | 0 | 1 |   |   |   |   |   |   |   |
| 5 | 6 | 0 | 2 | 0 | 1 |   |   |   |   |   |   |
| 6 | 0 | 6 | 0 | 2 | 0 | 1 |   |   |   |   |   |
| 7 | 22 | 0 | 6 | 0 | 2 | 0 | 1 |   |   |   |   |
| 8 | 0 | 22 | 0 | 6 | 0 | 2 | 0 | 1 |   |   |   |
| 9 | 86 | 0 | 22 | 0 | 6 | 0 | 2 | 0 | 1 |   |   |
| 10 | 0 | 86 | 0 | 22 | 0 | 6 | 0 | 2 | 0 | 1 |   |
| ... |   |   |   |   |   |   |   |   |   |   |   |

Table 2: The triangle of \( g_{ij}^{ij}(d, a) \)

Proposition 5. For all \( n \geq 0, d \in D \), we have

\[
g_{ij}^{ij}(d, a) = g_{n}^{ij}(d, b) = g_{n}^{ij}(d, c) = 0, \quad (n-d) \text{ even}, \tag{102}
\]

\[
g_{ij}^{ij}(d, \bar{a}) = g_{n}^{ij}(d, \bar{b}) = g_{n}^{ij}(d, \bar{c}) = 0, \quad (n-d) \text{ odd}. \tag{103}
\]

Corollary 17. For all \( n \geq 0, d \in D \) we have

\[
g_{n}^{ij}(d, b) = g_{n}^{ij}(d, c), \tag{104}
\]

\[
g_{n}^{ij}(d, \bar{b}) = g_{n}^{ij}(d, \bar{c}). \tag{105}
\]
Corollary 18. For all $n \geq 0$, $d \in D$ we have

$$g_{n}^{ij}(d, a) = \varphi_{n-d+1} - \varphi_{n-d} - 3\varphi_{n-d-1} + 3\varphi_{n-d-2},$$  \hspace{1cm} (106)

$$g_{n}^{ij}(d, b) = g_{n}^{ij}(d, c) = \varphi_{n-d-1} - \varphi_{n-d-2},$$  \hspace{1cm} (107)

$$g_{n}^{ij}(d, \overline{a}) = 2\varphi_{n-d-2} - 2\varphi_{n-d-3},$$  \hspace{1cm} (108)

$$g_{n}^{ij}(d, \overline{b}) = g_{n}^{ij}(d, \overline{c}) = \varphi_{n-d} - \varphi_{n-d-1} - 2\varphi_{n-d-2} + 2\varphi_{n-d-3}.$$  \hspace{1cm} (109)

Corollary 19. For all $n \geq 0$, $d \in D$, we have

$$g_{n}^{ij}(d, a) = \begin{cases} \frac{1}{3}(2^{n-d} + 2) & (n - d) \text{ even,} \\ 0 & (n - d) \text{ odd.} \end{cases}$$  \hspace{1cm} (110)

$$g_{n}^{ij}(d, b) = g_{n}^{ij}(d, c) = \begin{cases} \frac{1}{3}(2^{n-d} - 1) & (n - d) \text{ even,} \\ 0 & (n - d) \text{ odd.} \end{cases}$$  \hspace{1cm} (111)

$$g_{n}^{ij}(d, \overline{a}) = \begin{cases} 0 & (n - d) \text{ even,} \\ \frac{1}{3}(2^{n-d} - 2) & (n - d) \text{ odd.} \end{cases}$$  \hspace{1cm} (112)

$$g_{n}^{ij}(d, \overline{b}) = g_{n}^{ij}(d, \overline{c}) = \begin{cases} 0 & (n - d) \text{ even,} \\ \frac{1}{3}(2^{n-d} + 1) & (n - d) \text{ odd.} \end{cases}$$  \hspace{1cm} (113)

5 Conclusion

The Tower of Hanoi puzzle still fascinating us by its richness of mathematical properties. However there is a lot to be discovered around this puzzle. The goal of this work was to find a way to calculate the number of each elementary move in the optimal sequence of moves, this goal was achieved by using the function $f_{n}^{ij}(x)$. It is known that disc $d$ moves exactly $2^{n-d}$ times during the optimal solution [4], however, with the results presented in this work, we can count the number of times disc $d$ moves the elementary move $x \in A$.

Interesting sequence $\varphi_{n}$ that relate all the six sequences that counts each one of the six elementary moves was discovered and many properties and combinatorial identities of $f_{n}^{ij}(x)$ and $\varphi_{n}$ were presented. We also found combinatorial interpretations for some sequences in the On-Line Encyclopedia of Integer Sequences [10] and new combinatorial interpretations for other sequences in the same encyclopedia. The same technique can be used to find the number of elementary moves in the optimal sequence of moves in other variants of the Tower of Hanoi problem such as the cyclic and the linear variants [3].

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