**Fully stable on Gamma Acts**

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**Abstract.**

Let \( M \) be a \( \Gamma \)-monoid and \( A \) a unitary right \( M_\Gamma \)-act. We have introduced and studied the notion of full stability on gamma acts. We say that \( A \) is fully stable if \( f(B) \subseteq B \), for each \( M_\Gamma \)-subact \( B \) of \( A \) and \( M_\Gamma \)-homomorphism \( f: B \rightarrow A \). This is equivalent to saying that \( f(a\Gamma M) \subseteq a\Gamma M \) for each element \( a \) in \( A \). Many properties and characterizations of this class of gamma acts have been considered. In fact we show that full stability of \( M_\Gamma \)-act \( A \) is equivalent to the following equivalent conditions

1. For each \( a \in A \), \( \left( \Gamma_A(R_M(a\Gamma M)) \right) = a\Gamma M \).
2. For all \( a, b \in A \) and \( R_M(a\Gamma M) \subseteq R_M(b\Gamma M) \) implies that \( b \in a\Gamma M \).
3. For each \( a \in A \), there is a \( \Gamma \)-compatible \( \rho \) on \( M \) such that \( \left( \Gamma_A(\rho) \right) = a\Gamma M \).
4. \([a\Gamma M : b\Gamma M ] = [R_M(b\Gamma M) : R_M(a\Gamma M)] \) for all \( a, b \in A \) where \( M \) is a commutative \( \Gamma \)-monoid.
5. Every \( M_\Gamma \)-homomorphism from any essential \( M_\Gamma \)-subact \( B \) of \( A \) into \( A \) satisfies \( f(B) \subseteq B \) and \( \text{Hom}_{M_\Gamma}(B, C) = \emptyset \), for any \( M_\Gamma \)-subacts \( B, C \) of \( A \) with zero intersection.

**1. Introduction**

Let \( M \) and \( \Gamma \) be nonempty sets. Then \( M \) is called a \( \Gamma \)-monoid, if there is a mapping \( \times \Gamma \rightarrow M \rightarrow M \) defined by \( (m, \alpha, n) \rightarrow \max \) which is satisfying the condition

1. For all \( m, n, m' \in M \) and \( \alpha, \beta \in \Gamma \), \( m(\alpha, n)(\beta, m') = m(\alpha, \beta, n) \).

(2) For each \( m \in M \) and \( \alpha \in \Gamma \) there is an element \( e \in M \) such that \( e\beta m = m \). Let \( M \) be a \( \Gamma \)-monoid. A nonempty set \( A \) is called right \( M_\Gamma \)-act, if there is a mapping \( \times \Gamma \rightarrow M \rightarrow A \) written \( (a, \alpha, m) \rightarrow a\alpha m \) such that the following condition is satisfied

\[ \alpha(\beta_1, \alpha, \beta_2) = (\alpha, \beta_1, \beta_2) \] for all \( m_1, m_2 \in M \), \( \alpha, \beta \in \Gamma \) and \( a \in A \). Let \( A \) and \( A' \) be two \( M_\Gamma \)-acts. A mapping \( g: A \rightarrow A' \) is called \( M_\Gamma \)-homomorphism if \( g(\alpha a\beta m) = g(\alpha a)\beta m \), for all \( m \in M \), \( \alpha, \beta \in \Gamma \) and \( a \in A \). Let \( A \) be an \( M_\Gamma \)-act and \( U \) a nonempty subset of \( A \). Then \( \left[ U \right]_A = \bigcup_{u \in U} u\Gamma M \) where \( u\Gamma M = \{u\alpha m \mid m \in M \text{ and } \alpha \in \Gamma \} \), Abbas \(^1\). A non-empty subset \( U \) of \( M_\Gamma \)-act \( A \) is said to be a set of generating elements or generating set of \( A \), if \( A = [U]_A \). We say that \( A \) is finitely generated if \( [U]_A = A \) for some subset \( U \) of \( A \) which \( |U| \)
<\infty$. And $A$ is a cyclic if $A = [\{u\}]_A$ for some $u \in A$. Let $A$ be an $M_\Gamma$-act. Then $A$ is called decomposable, if there are two gamma subacts $L$, $K$ of $A$ such that $A = L \cup K$, otherwise $A$ is call indecomposable. Every cyclic $M_\Gamma$-act is indecomposable. Let $A$ be an $M_\Gamma$-act. $A$ is called simple $M_\Gamma$-act, if it contains only trivial $M_\Gamma$-subacts, Abbas 1. Let $A$ be an $M_\Gamma$-act with $B$ an $M_\Gamma$-subact $A$. Define $[B: A]$ as follows $[B: A] = \{ m \in M | a \alpha m \in B \text{ for all } \alpha \in \Gamma \text{ and } a \in A \}$. If $[B: A]$ is a nonempty subset of $M$, then $[B: A]$ is a right ideal of a $\Gamma$-monoid $M$. Indeed, for $m \in M$, $\alpha \in \Gamma$ and $n \in [B: A]$, we have $n\beta(m \alpha m') = (n\beta m) \alpha m' \in [B: A]$ for all $m' \in M$ and $\beta \in \Gamma$, Abbas 2. Let $A$ be right $M_\Gamma$-act. An element $\theta$ in $A$ is called a zero of $A$, if $\theta \alpha m = \theta$, for all $m \in M$ and $\alpha \in \Gamma$. A left (right) $M_\Gamma$-act $A$ can have more than one zero elements, Abbas 1.

2. Fully Stable On Gamma Acts.

2.1. Definition : Let $A$ be an $M_\Gamma$-act. A relation $\rho$ on $A$ is called a $\Gamma$-compatible on $A$, if for all $(a_1, a_2) \in \rho$, implies that $(a_1 \alpha m, a_2 \beta m) \in \rho$ for all $m \in M$ and $\alpha, \beta \in \Gamma$. Let $A$ be an $M_\Gamma$-act. Then $\Delta_{A \times A} = \{ (a, a) | a \in A \}$ is a trivial $\Gamma$-compatible, $A \times A$ is a universal $\Gamma$-compatible on $A$.

2.2. Definition : For an $M_\Gamma$-act $A$ with $T = \text{End}_{M_\Gamma}(A)$, $H$ is a non-empty subset of $M$, $K$ is a non-empty subset of $A$, $J$ is a non-empty subset of $M$. Define $R_M(K) = \{(m_1, m_2) \in M \times M | k \alpha m_1 = k \beta m_2 \text{ for all } k \in K, \alpha, \beta \in \Gamma \}$. $L_A(J) = \{ a \in A | a \alpha m_1 = a \beta m_2 \text{ for all } (m_1, m_2) \in J, \alpha, \beta \in \Gamma \}$.

2.3. Proposition: If $A$ is an $M_\Gamma$-act with $B$ is a non-empty subset of $K$ is a non-empty subset of $A$, $I$ is a non-empty subset of $J$ is a non-empty subset of $M \times M$, then

1. $R_M(K)$ is a $\Gamma$-compatible on $M$.
2. $R_M(K)$ is a non-empty subset of $R_M(B)$ and $L_A(J)$ a non-empty subset of $L_A(I)$.
3. If $K$ is an $M_\Gamma$-act, then $R_M(K)$ is a $\Gamma$-congruence of $M$.
4. If $J$ is a left $\Gamma$-compatible, then $L_A(J)$ is an $M_\Gamma$-subact of $A$.
5. For each $M_\Gamma$-acts $L$ and $K$, then $R_M(L \cup K) = R_M(L) \cap R_M(K)$.
6. For each relations $I$ and $J$, then $L_A(I \cup J) = L_A(I) \cap L_A(J)$.

2.4. Definition: An $M_\Gamma$-act $A$ is said to be fully stable, if every $M_\Gamma$-subact of $A$ is stable. A $\Gamma$-monoid $M$ is called right (left) fully stable, if $M$ as a right (left) $M_\Gamma$-act is fully stable.

2.5. Proposition: The following are equivalent for an $M_\Gamma$-act $A$.

1. $A$ is fully stable.
2. Every indecomposable $M_\Gamma$-subact of $A$ is stable.
3. Every cyclic $M$ -subact of $A$ is stable.
For each \( \alpha \in A \) and an \( M_{T} \)-homomorphism \( f: \alpha \Gamma^{M} \rightarrow A \), there exist \( m \in M \) and \( \alpha \in \Gamma \) such that \( f(\alpha a) = a \alpha m \).

**Proof:** (1)\( \implies \) (2) It is clear. (2)\( \implies \) (3) Every cyclic \( M_{T} \)-act is indecomposable, Abbas\(^2\). (3)\( \implies \) (4) Let \( a \in A \) and an \( M_{T} \)-homomorphism \( f: \alpha \Gamma^{M} \rightarrow A \), by (3) \( f(\alpha M) \subseteq a \Gamma M \). Hence \( f(\alpha a) = f(\alpha \alpha 1) \in f(\alpha M) \subseteq a \Gamma M \), then there exist \( m \in M \) and \( \alpha \in \Gamma \) such that \( f(\alpha a) = a \alpha m \). (4)\( \implies \) (1) Let \( B \) be an \( M_{T} \)-subact \( A \) and \( M_{T} \)-homomorphism \( f: B \rightarrow A \). Let \( f(b) \in f(B) \), for each \( b \in B \subseteq A \), then, \( f(b) = f(b \alpha M) = b \alpha m \), by (4). We have \( f(b) = b \alpha m \in b \Gamma M \subseteq B \) thus \( f(B) \subseteq B \). Hence \( A \) is fully stable.

### 2.6. Examples and Remarks:

(1) Every fully stable \( M \)-act is a fully stable \( M_{T} \)-act. In particular the \( Z_{p} \)-act \( Z_{p}^{\infty} \) is fully stable, for each prime number \( p \), Abbas\(^2\).

(2) Every simple \( M_{T} \)-act is fully stable.

(3) A cyclic \( M_{T} \)-act may be not fully stable, in general \( Z \) as \( (Z_{H} \)-act) is not fully stable, for any \( H \)-submonoid \( H \) of \( Z \). For define \( g: 2HZ \rightarrow Z \) by \( g(2mn) = 3mn \), for each \( m \in H \) and \( n \in Z \). Then \( g(2HZ) \neq 2HZ \). By the same way, we can see that \( Q \) as \( (Z_{H} \)-act) is not fully stable.

(4) Every \( M_{T} \)-subact of \( A \) is fully stable, if and only if, \( A \) is fully stable \( M_{T} \)-act.

It suffices to consider stability over a very restricted to \( M_{T} \)-subact.

### 2.7. Proposition:

The following are equivalent for an \( M_{T} \)-act \( A \).

(1) \( A \) is fully stable. (2) Every 2-generating \( M_{T} \)-subact of \( A \) is fully stable.

**Proof:** (1)\( \implies \) (2) It is clear, by (2.6)-(4). (2)\( \implies \) (1) Let \( B \) be an \( M_{T} \)-subact of \( A \) and \( f: B \rightarrow A \) an \( M_{T} \)-homomorphism. For each element \( b \in B \). Consider the \( M_{T} \)-subact \( C = b \Gamma M \cup f(b) \Gamma M \). Then \( f(b \Gamma M) = b \Gamma M \rightarrow C \). By fully stability of \( C \), \( f(b \Gamma M) \subseteq b \Gamma M \). Hence \( f(b) \in B \), thus \( f(B) \subseteq B \).

The following theorem gives a characterization of fully stable gamma acts in terms of right annihilator of a cyclic \( M_{T} \)-subacts.

### 2.8. Theorem:

The following are equivalent for an \( M_{T} \)-act \( A \).

(1) \( A \) is fully stable (2) If \( L, K \) are an \( M_{T} \)-subacts of \( A \) with \( L \) is an \( M_{T} \)-homomorphic image of \( K \), then \( L \subseteq K \).

(3) For all \( a, b \in A \) and \( b \notin a \Gamma M \) implies that \( R_{M}(a \Gamma M) \not\subseteq R_{M}(b \Gamma M) \). (4) \( L_{A}(R_{M}(a \Gamma M)) = a \Gamma M \), for each \( a \in A \).
Proof: (1)⇒(2) Let L and K be $M_{\Gamma}$-subacts of A and f: $A \to L$ an $M_{\Gamma}$-epimorphism. By (1), L=f(K) \subseteq K. (2)⇒(3) Assume that there are a, b \in A with b \notin a\Gamma M and R_{M}(a\Gamma M) \subseteq R_{M}(b\Gamma M). Define the mapping $f: a\Gamma M \to b\Gamma M$ by $f(\alpha a_{M}) = b\alpha m$, for all $\alpha \in \Gamma$ and $a \in A$. If $\alpha a_{M_{1}} = \beta b_{M_{2}}$, then $(m_{1}, m_{2}) \in R_{M}(a\Gamma M)$, hence $(m_{1}, m_{2}) \in R_{M}(b\Gamma M)$, this implies that $b\alpha m_{1} = b\beta m_{2}$, f is a well-defined and it is easy matter to see that f is an $M_{\Gamma}$-epimorphism. By (2), $b\Gamma M \subseteq a\Gamma M$. Then b = b\alpha 1 \in b\Gamma M \subseteq a\Gamma M$ which is a contradiction. (3)⇒(4) Assume that there exists $b \in L_{A}(R_{M}(a\Gamma M))$ and $b \notin a\Gamma M$. By (3) $R_{M}(a\Gamma M) \subseteq R_{M}(b\Gamma M)$ and hence there is $(m_{1}, m_{2}) \in R_{M}(a\Gamma M)$ and $(m_{1}, m_{2}) \notin R_{M}(b\Gamma M)$ so $b\alpha m_{1} = b\beta m_{2}$, for all $\alpha, \beta \in \Gamma$ which is a contradiction. (4) ⇒ (1) Let f: $a\Gamma M \to A$ be an $M_{\Gamma}$-homomorphism and let $(m_{1}, m_{2}) \in R_{M}(a\Gamma M)$, $\alpha a_{m_{1}} = \beta b_{m_{2}}$, for all $m_{1}, m_{2} \in M$ and $\alpha, \beta \in \Gamma$, f(a)=f(\alpha a_{1})=f(\alpha a_{1}) \in f(a\Gamma M), f(\alpha) a_{m_{1}} = f(\alpha a_{m_{1}}) = f(\alpha b_{m_{2}}) = f(\alpha b_{m_{2}}) \in f(a\Gamma M), f(a) a_{m_{1}} = f(\alpha b_{m_{2}}) = f(\alpha b_{m_{2}})$, then f(a) $\in L_{A}(R_{M}(a\Gamma M) \subseteq a\Gamma M$. Hence f(a\Gamma M) $\subseteq a\Gamma M$.

2.9. Corollary: The following are equivalent for $\Gamma$-monoid $M$.

(1) $M$ is fully stable. (2) If $J, J'$ are $\Gamma$-ideals of $M$ with $J$ is an $M_{\Gamma}$-homomorphic image of $J'$, then $J \subseteq J'$. (3) For all $m, n \in M$ and $n \notin m\Gamma M$ implies that $R_{M}(m\Gamma M) \nsubseteq R_{M}(n\Gamma M)$. (4) $L_{M}(R_{M}(m\Gamma M)) = m\Gamma M$, for each $m \in M$.

The following proposition tells that the fully stable gamma act is an algebraic property.

2.10. Proposition: An $M_{\Gamma}$-isomorphic to fully stable $M_{\Gamma}$-act is fully stable.

Proof: Let $b_{1}, b_{2} \in B$ and $R_{M}(b_{1}\Gamma M) \subseteq R_{M}(b_{2}\Gamma M)$. Since f is $M_{\Gamma}$-isomorphism, f:$A \to B$, there are $a_{1}, a_{2} \in A$ such that f($a_{1}$) = $b_{1}$ and f($a_{2}$) = $b_{2}$. Let $(m_{1}, m_{2}) \in R_{M}(a_{1}\Gamma M)$, then $a_{1} a_{m_{1}} = a_{1} b_{m_{2}}$, for all $\alpha, \beta \in \Gamma$ and $m_{1}, m_{2} \in M$. Thus $(m_{1}, m_{2}) \in R_{M}(a_{1}\Gamma M) \subseteq R_{M}(b_{2}\Gamma M)$ implies that $b_{2} a_{m_{1}} = b_{2} b_{m_{2}}$, hence $f(a_{1} a_{m_{1}}) = f(a_{2} b_{m_{2}})$. Then $(m_{1}, m_{2}) \in R_{M}(a_{2}\Gamma M)$ implies that $a_{2}\Gamma M \subseteq a_{1}\Gamma M$, by (2.8),(3). Then $b_{2}\Gamma M = f(a_{2}) \Gamma M = f(a_{2}\Gamma M) \subseteq f(a_{1}\Gamma M) = f(a_{1})\Gamma M = b_{1}\Gamma M$. Hence $B$ is fully stable, by (2.8).

2.11. Lemma: The following statements are equivalent for an $M_{\Gamma}$-act $A$.

(1) A satisfies the condition (*); Any two $M_{\Gamma}$-isomorphic distinct $M_{\Gamma}$-subacts are equal. (2) $g(B) \subseteq B$, for each $M_{\Gamma}$-monomorphism g from any cyclic $M_{\Gamma}$-subacts $B$ of A into A. (3) $R_{M}(a\Gamma M) = R_{M}(b\Gamma M)$ implies that $a\Gamma M = b\Gamma M$ for each $a, b \in A$. (4) Any two $M_{\Gamma}$-isomorphic distinct cyclic $M_{\Gamma}$-subacts are equal.

Proof: (1)⇒(2) Assume A satisfies the condition(*) and there is a cyclic $M_{\Gamma}$-subact $B$ of A and an $M_{\Gamma}$-monomorphism g: $B \to A$ with g(B)\nsubseteq B. Then B and g(B) are distinct cyclic $M_{\Gamma}$-subacts of A. By (1) and g(B) is $M_{\Gamma}$-isomorphic to B, contradiction.(2)⇒(3) Assume that $R_{M}(a\Gamma M) = R_{M}(b\Gamma M)$ for each $a, b \in A$ and a well-defined and $M_{\Gamma}$-monomorphism g: $a\Gamma M \to A$ by g($a\alpha m$) = $b\beta m$ for each $m \in M$ and $\alpha \in \Gamma$. Then g($a\Gamma M$) $\subseteq a\Gamma M$ and hence $b\Gamma M = g(a\Gamma M) \subseteq a\Gamma M$, since $b\beta m \in b\Gamma M$, $b\alpha m =$
g(a\alpha m) \in a\Gamma M. By similarity we have a\Gamma M \subseteq b\Gamma M. (3)\Rightarrow(4) Assume that (3) and there are two distinct \( M_\Gamma \)-isomorphic cyclic \( M_\Gamma \)-subacts \( a\Gamma M \). Without loss of generality, there exists an element \( z \in a\Gamma M \) and \( z \not\in b\Gamma M \). Let \( f: a\Gamma M \to b\Gamma M \) be an \( M_\Gamma \)-isomorphism. \( z \not\in f(z) \), then \( R_M(z^\Gamma M) = R_M(f(z)^\Gamma M) \), but \( z^\Gamma M \neq f(z)^\Gamma M \) which contradicts with (3).

(4)\Rightarrow(1) Suppose that distinct \( M_\Gamma \)-subacts of \( A \) are \( M_\Gamma \)-isomorphic. Then we may choose \( M_\Gamma \)-isomorphic between \( L, K \) are two \( M_\Gamma \)-subacts of \( A \) with \( L \not\subseteq K \). Choose an \( M_\Gamma \)-isomorphism \( f:L \to K \) and an element \( \ell \in L \) and \( \ell \not\in K \). Then \( \ell^\Gamma M \) is \( M_\Gamma \)-isomorphic to \( f(\ell)^\Gamma M \) but \( \ell^\Gamma M \neq f(\ell)^\Gamma M \), contradiction.

2.12. Proposition: A fully stable \( M_\Gamma \)-act has the condition (*).

Proof: Let \( A \) be a fully stable \( M_\Gamma \)-act. By (2.8),(3), \( R_M(a^\Gamma M) = R_M(b^\Gamma M) \) implies that \( b^\Gamma M = a^\Gamma M \) for each \( a, b \in A \). Hence \( A \) has condition (*), by (2.11).

2.13. Corollary: Let \( A_1, A_2 \) be two \( M_\Gamma \)-subacts of fully stable \( M_\Gamma \)-act \( A \). Then \( A_1 \subseteq A_2 \), if and only if, there is an \( M_\Gamma \)-monomorphism \( f:A_1 \to A_2 \).

Proof: Let \( f:A_1 \to A_2 \) be an \( M_\Gamma \)-monomorphism. Then \( A_1 \cong f(A_1) \subseteq A_2 \). By (2.12) & (2.11), \( A_1 = f(A_1) \subseteq A_2 \).

2.14. Lemma: Let \( \delta \) be a \( \Gamma \)-congruence on a commutative \( \Gamma \)-monoid \( M \). Then \( L_A(\delta) \cong \text{Hom}_{M_\Gamma}(\frac{M}{\delta}, A) \) for any \( M_\Gamma \)-act \( A \).

Proof: Define \( g:L_A(\delta) \to \text{Hom}_{M_\Gamma}(\frac{M}{\delta}, A) \) as follows for \( a \in L_A(\delta) \) then \( a\alpha m = a\alpha m' \), for each \( (m, m') \in \delta \) and \( \alpha \in \Gamma \). Let \( [m]_\delta \in \frac{M}{\delta} \), put \( g(a)([m]_\delta) = a\alpha m \). Let \( m' \in M, \beta \in \Gamma \) and \( a \in L_A(\delta) \), then \( g(a\beta m')([m]_\delta) = (a\beta m') \alpha m = a\beta (m\alpha \alpha m') = a\beta (m\alpha \alpha m') = g(a)([m]_\delta) \beta m' \), this shows that \( g \) is an \( M_\Gamma \)-homomorphism. Also define \( h: \text{Hom}_{M_\Gamma}(\frac{M}{\delta}, A) \to L_A(\delta) \) by \( h(f) = f([1]_\delta), \) for each \( f \in \text{Hom}_{M_\Gamma}(\frac{M}{\delta}, A) \). It is clear, \( h \) is a well-defined. For each \( (m, m') \in \delta \), \( f([1]_\delta) \alpha m = f([1]_\delta) \alpha m' \) and this shows that \( h(f) \in L_A(\delta) \). \( h \) is an \( M_\Gamma \)-homomorphism, since \( h(f\alpha n) = f\alpha n([1]_\delta) = f(n\alpha [1]_\delta) = f([1]_\delta) \alpha n = f([1]_\delta) \alpha n = f([1]_\delta) \alpha n = f([1]_\delta) \alpha n = f([1]_\delta) \alpha n = f([1]_\delta) \alpha n \).

For each \( f \in \text{Hom}_{M_\Gamma}(\frac{M}{\delta}, A) \), we have \( (gh)(f)([m]_\delta) = g(h(f))([m]_\delta) = g(f([1]_\delta)([m]_\delta) = f([1]_\delta) \alpha m = f([1]_\delta) \alpha m \). That is, \( gh = 1_{\text{Hom}_{M_\Gamma}(\frac{M}{\delta}, A)} \). Also for each \( a \in L_A(\delta), \alpha \in \Gamma \), we have \( (hg)(a) = h(g(a)) = h(g(a)) = g(a)([1]_\delta) = a\alpha 1 = a. \) Then \( hg = 1_{L_A(\delta)} \). Hence \( h \) is \( M_\Gamma \)-isomorphism.

A \( \Gamma \)-monoid \( M \) is said to be commutative provided \( m_1\alpha m_2 = m_2\gamma m_1 \) for all \( m_1, m_2 \in M \) and \( \alpha, \gamma \in \Gamma \). This definition is similar as in Madhusudhana, since \( (m_1\alpha m_2)\gamma = m_2\gamma(1\alpha m_1) = (m_2\gamma 1)\alpha m_1 = m_2\alpha m_1 \) for all \( \alpha, \gamma \in \Gamma \) and \( m_1, m_2 \in M=S \) be the \( \Gamma \)-semigroup \( S \) with an identity adjoined usually denoted by the symbol 1).
2.15. Proposition: Let M be a commutative $\Gamma$-monoid. If A is fully stable $M_\Gamma$-act then $a\Gamma M \cong \text{Hom}_{M_\Gamma}(a\Gamma M, A)$ for each $a \in A$.

Proof: Since A is a fully stable $M_\Gamma$-act, then $L_A(R_M(a\Gamma M)) = a\Gamma M$, for each $a \in A$. (2.8). So (2.14), we have $a\Gamma M = L_A(R_M(a\Gamma M)) \cong \text{Hom}_{M_\Gamma}(\frac{M}{R_M(a\Gamma M)}, A) \cong \text{Hom}_{M_\Gamma}(a\Gamma M, A)$.

We have proved in (2.15) & (2.12) that if an $M_\Gamma$-act A is fully stable, then the following proposition we consider the converse under certain condition.

2.16. Proposition: Let M be a commutative $\Gamma$-monoid. Then A is fully stable $M_\Gamma$-act, if and only if, A has condition (*) and $a\Gamma M \cong \text{Hom}_{M_\Gamma}(a\Gamma M, A)$ for each $a \in A$.

Proof: Necessity. By (2.12) & (2.15).

Sufficiency. By using (2.8),(4) to prove A is fully stable. Let $a \in A$, $L_A(R_M(a\Gamma M)) \cong \text{Hom}_{M_\Gamma}(\frac{M}{R_M(a\Gamma M)}, A)$, by (2.14). Then $\text{Hom}_{M_\Gamma}(\frac{M}{R_M(a\Gamma M)}, A) \cong \text{Hom}_{M_\Gamma}(a\Gamma M, A)$. By hypothesis, $L_A(R_M(a\Gamma M)) = a\Gamma M$, for each $a \in A$, by (2.15). Hence A is fully stable.

The following proposition gives another characterization of fully stable gamma acts, but in terms of compatibility.

2.17. Proposition: The following are equivalent for an $M_\Gamma$-act A.

(1) A is fully stable. (2) For each $a \in A$, there is a $\Gamma$-compatible $\rho$ on M such that $L_A(\rho) = a\Gamma M$.

Proof: (1)$\Rightarrow$(2) Obvious. (2)$\Rightarrow$(1) Let $\rho$ be a $\Gamma$-compatible on M such that $L_A(\rho) = a\Gamma M$. Let $(m, n) \in \rho$, there is a $b \in L_A(\rho)$, that is, $atm = abn$ then $(m, n) \in R_M(a\Gamma M)$. Thus $\rho \subseteq R_M(a\Gamma M)$ and hence $L_AR_M(a\Gamma M) \subseteq L_A(\rho) = a\Gamma M$.

The following proposition guarantees each cyclic gamma subact contained in a left annihilator of certain compatible, there is another compatible that is bigger than the certain one such that the left annihilator of it is exactly the cyclic subact.

2.18. Proposition: The following are equivalent for an $M_\Gamma$-act A.

(1) A is fully stable. (2) For each $a \in A$, and a $\Gamma$-compatible $\rho$ on M with $a\Gamma M \subseteq L_A(\rho)$, there exists a $\Gamma$-compatible $\rho'$ on M such that $\rho \subseteq \rho'$ and $a\Gamma M = L_A(\rho')$. (3) For each $a \in A$ and a $\Gamma$-compatible $\rho$ on M with $a\Gamma M \subseteq L_A(\rho)$, there exists a $\Gamma$-compatible $\rho'$ on M such that $\rho \subseteq \rho'$ and $a\Gamma M \subseteq L_A(\rho')$.

Proof: (1)$\Rightarrow$(2) Let $a \in A$ and a $\Gamma$-compatible $\rho$ on M such that $a\Gamma M \subseteq L_A(\rho)$. Since A is fully stable, $a\Gamma M = L_A(R_M(a\Gamma M))$, by (2.8). We set $\rho' = R_M(a\Gamma M) \cup \rho$. Since $a\Gamma M = L_A(R_M(a\Gamma M)) \subseteq L_A(\rho)$, $R_M(a\Gamma M) \not\subseteq \rho$. Hence $\rho \subseteq \rho'$, we have $L_A(\rho') = L_A(R_M(a\Gamma M) \cup \rho) = L_A(R_M(a\Gamma M)) \cap L_A(\rho) = a\Gamma M$.

(2)$\Rightarrow$(3) Obvious.(3)$\Rightarrow$(1) Let $a \in A$ and $\mathcal{H} = \{ \rho : \rho$ is a $\Gamma$-compatible on M and $a\Gamma M \subseteq L_A(\rho) \}$. H is
non empty set, since \( \Delta_{M \times M} \in \mathcal{H} \). Let \( \{ \rho_i \}_{i \in I} \) be any chain of a \( \Gamma \)-compatible \( \rho_i \in \mathcal{H} \). By Zorn’s lemma, \( \mathcal{H} \) has a maximal member \( \rho \in \mathcal{H} \) so that \( a \Gamma M \subseteq L_A(\rho) \). Assume that \( a \Gamma M \neq L_A(\rho) \).

Then by (3), there exists a \( \Gamma \)-compatible \( \rho' \) with \( \rho \subseteq \rho' \) and \( a \Gamma M \subseteq L_A(\rho') \) which is a contradiction to the maximal of \( A \). Thus we have \( a \Gamma M = L_A(\rho) \). Hence, \( A \) is fully stable, (2.17).

### 2.19. Proposition:

Let \( M \) be a commutative \( \Gamma \)-monoid. If \( A \) is fully stable \( M_\Gamma \)-act, then \( \left[ R_M(A) : R_M(a \Gamma M) \right] \subseteq [a \Gamma M : A] \), for each \( a \in A \).

**Proof:** For each \( m' \in [R_M(A) : R_M(a \Gamma M)] \) and \( b \in A \). Define \( g: a \Gamma M \to A \) by \( g(\alpha m) = b \alpha m' \alpha m \), for each \( m \in M \) and \( \alpha \in \Gamma \). If \( \alpha \Gamma m_1 = \alpha \Gamma m_2 \), hence \( (m_1, m_2) \in R_M(a \Gamma M) \) then \( (m_1 \alpha m', m_2 \alpha m') \in R_M(A) \) thus \( b \beta (m_1 \alpha m') = b \beta (m_2 \alpha m') \). By a commutativity of \( M \), \( b \beta (m' \alpha m_1) = b \beta (m' \alpha m_2) \). Thus \( g \) is a well-defined and it is easily to see that \( g \) is \( M_\Gamma \)-homomorphism, full stability of \( A \) implies that there exist are element \( n \in M \) and \( \alpha \in \Gamma \) such that \( g(a) = a \alpha n \). Thus \( b \alpha m' = g(a) = a \alpha n \in a \Gamma M \) implies that \( m' \in [a \Gamma M : A] \).

### 2.20. Lemma:

Let \( A \) be a fully stable \( M_\Gamma \)-act with \( L_A(L \cap K) \subseteq L_A(L) \cup L_A(K) \) for each two \( \Gamma \)-compatibles \( L \) and \( K \) on \( M \). If an \( M_\Gamma \)-subact \( B \) of \( A \) satisfies the double annihilator condition, then so does \( B \cup a \Gamma M \) for all \( a \in A \) and \( a \in B \).

**Proof:** Let \( a \in A \). By the assumption \( B = L_A(R_M(B)) \), since \( A \) is fully stable, then \( L_A(R_M(a \Gamma M)) = a \Gamma M \), by (2.8). By hypothesis, we have that \( L_A(R_M(B) \cup a \Gamma M) = L_A(R_M(B) \cap R_M(a \Gamma M)) \subseteq L_A(R_M(B) \cup L_A(R_M(a \Gamma M))) = B \cup a \Gamma M \). In the other inclusion is clear, therefore \( L_A(R_M(B \cup a \Gamma M) = B \cup a \Gamma M) \).

### 2.21. Proposition:

Let \( A \) be an \( M_\Gamma \)-act with \( L_A(L \cap K) \subseteq L_A(L) \cup L_A(K) \) for each two \( \Gamma \)-compatibles \( L \) and \( K \) on \( M \). Then \( A \) is fully stable \( M_\Gamma \)-act, if and only if, every finitely \( \Gamma \)-generated \( M_\Gamma \)-subact \( B \) of \( A \) satisfies the double annihilator condition.

**Proof:** Necessity. Let \( B = \bigcup_{i=1}^{n} a_i \Gamma M \) be a finitely \( \Gamma \)-generated \( M_\Gamma \)-act for some \( a_i \in A \). We use induction on the number of generator of \( B \). For \( n=1 \), this is just the definition. Assume that, the condition holds for \( m \leq n-1 \). Then by (2.20), the double annihilator condition is satisfied for \( M_\Gamma \)-subacts \( \Gamma \)-generated by \( m+1 \) element. **Sufficiency.** By (2.8),(4).

### 3. Full Stability With Multiplication On Gamma Acts.

#### 3.1. Lemma:

Let \( M \) be a commutative \( \Gamma \)-monoid and \( A \) be a \( \Gamma \)-multiplication \( M_\Gamma \)-act. Then for each \( M_\Gamma \)-subact \( B \) of \( A \) and \( M_\Gamma \)-homomorphism \( g: B \to A \).

1. \( [g(B): A] \subseteq [R_M(A) : R_M(B)] \).
2. \( g(B) \subseteq A \Gamma [R_M(A) : R_M(B)] \).
Proof: (1) for each \( m \in [g(B): A] \) and \((m_1, m_2) \in R_M(B)\) then \((m_1, m_2) \in R_M(g(B))\). For each \( a \in A\) and there exists \( \alpha \in \Gamma \), we have \( \alpha \alpha m \in g(B)\), hence \((\alpha \alpha m)\beta m_1 = (\alpha \alpha m)\beta m_2\) for all \( \beta \in \Gamma, \alpha (m_\beta m_1) = \alpha \alpha (m_\beta m_2)\), by a commutative of \( M, \alpha (m_\beta m) = \alpha (m_\beta \alpha m)\), hence \((m_\beta m, m_\beta \alpha m) \in R_M(A)\). Thus \( m \in [R_M(A): R_M(B)]\). (2) \( g(B) \) is an \( M_\Gamma\)-subact of \( A\). Since \( A \) is \( \Gamma\)-multiplication, then \( g(B) = A\Gamma\) \( [g(B): A] \). By (1) we have \( g(B) \subseteq A\Gamma[R_M(A): R_M(B)]\).

The following, we show by example that the equality in (3.1), (1) may not be true, as \((Z \Gamma, \cdot)\)-act \( Z\). Then \( [2ZZ: Z] = 2ZZ\). While \( [R_Z(Z): R_Z(2ZZ)] = [\Delta_{xz}, \Delta_{xz}] = Z\).

It is well-known that \( Z\) as \( Z\Gamma\)-act is \( \Gamma\)-multiplication. But \( Z\) as \( Z\Gamma\)-act is not fully stable. The following theorem makes the conditions of (2.19) are equivalent. An \( M_\Gamma\)-act \( A \) is said to be a \( \Gamma\)-multiplication, if each \( M_\Gamma\)-subact of \( A \) is of the form \( A\Gamma I\), for some \( \Gamma\)-ideal \( I \) of \( M\).

3.2. Theorem: Let \( M \) be a commutative \( \Gamma\)-monoid and \( A \) an \( M_\Gamma\)-act. Consider the following condition.

(1) \( A \) is fully stable (2) For each \( a \in A, [R_M(A): R_M(a\Gamma M)] \subseteq [a\Gamma M : A]\). (3) \( g(a\Gamma M) \subseteq A\Gamma[a\Gamma M : A]\) for each \( a \in A \) and \( M_\Gamma\)-homomorphism \( g : a\Gamma M \rightarrow A\). Then (1) \( \Rightarrow \) (2), (3) \( \Rightarrow \) (1) and (2) \( \Rightarrow \) (3). If \( A \) is \( \Gamma\)-multiplication

Proof: (1) \( \Rightarrow \) (2): By (2.18). (2) \( \Rightarrow \) (3): Let \( a \in A \) and \( M_\Gamma\)-homomorphism \( g : a\Gamma M \rightarrow A \) by Lemma (2.1), \( g(a\Gamma M) \subseteq A\Gamma[R_M(A): R_M(a\Gamma M)]\) and by (2), \( g(a\Gamma M) \subseteq A\Gamma[a\Gamma M : A]\). (3) \( \Rightarrow \) (1): Let \( a \in A \) and \( M_\Gamma\)-homomorphism \( g : a\Gamma M \rightarrow A\), by (3), \( g(a\Gamma M) \subseteq A\Gamma[a\Gamma M : A]\). It is easy to show that \( A\Gamma[a\Gamma M : A] \subseteq a\Gamma M\), thus \( g(a\Gamma M) \subseteq a\Gamma M\).

3.3. Corollary: Let \( M \) be a commutative \( \Gamma\)-monoid. Then the following are equivalent for an \( M_\Gamma\)-act \( A\).

(1) \( A\) is fully stable (2) Every \( \Gamma\)-multiplication \( M_\Gamma\)-subact is stable.

If \( M \) is a \( \Gamma\)-monoid, then \((M \times M) \times \Gamma \times M \rightarrow M \times M\) by define \( (m_1, m_2)\alpha m = (m_1\alpha m, m_2\alpha m)\) for each \( m_1, m_2, m \in M \) and \( \alpha \in \Gamma\).

The following statement we give a characterization of fully stable gamma acts in terms of residual condition.

3.4. Proposition: Let \( M \) be a commutative \( \Gamma\)-monoid and \( A \) an \( M_\Gamma\)-act. Then the following are equivalent.

(1) \( A\) is fully stable. (2) \([a\Gamma M : b\Gamma M] = [R_M(b\Gamma M): R_M(a\Gamma M)]\), for all \( a, b \in A\).
Proof: (1)⇒(2) Let m ∈ [αΓM : βΓM]. Then b′αm ∈ αΓM for each b′ ∈ bΓM. By (2.8), b′αm ∈ αΓM = L_A (R_M (αΓM)). For each (m_1, m_2) ∈ R_M (αΓM), then (b′αm)βm_1 = (b′αm)βm_2. Thus b′α(mβm_1) = b′α(mβm_2) and hence (m_1, m_2)βm ∈ R_M (bΓM), since b′ ∈ bΓM = L_A (R_M (bΓM)), by (2.8). Then m ∈ [R_M (bΓM) : R_M (αΓM)]. For the other inclusion. Let m ∈ [R_M (bΓM) : R_M (αΓM)], then for each (m_1, m_2) ∈ R_M (αΓM), then (m_1, m_2)βm ∈ R_M (bΓM) implies that b′α(m_1βm) = b′α(m_2βm) for each b′ ∈ bΓM, hence (b′αm)βm = (b′αm)βm_2, since (m_1, m_2) ∈ R_M (αΓM). Then b′αm ∈ L_A (R_M (αΓM)) = αΓM, by (2.8). Thus m ∈ [αΓM : bΓM]. (2)⇒(1): Let a and b ∈ A such that R_M (αΓM) ⊆ R_M (bΓM). By (2), [R_M (bΓM) : R_M (αΓM)] = [αΓM : bΓM]. Hence bΓM ⊆ aΓM.

An M_Γ-act A is projective, if every M_Γ-epimorphism f: C → B and any M_Γ-homomorphism g: A → B, where C, B are two M_Γ-acts, there is an M_Γ-homomorphism h: A → C such that f h = g.

3.5. Proposition: Let M be a Γ-monoid and A a fully stable M_Γ-act, in which every proper M_Γ-subact of A is projective. Then every M_Γ-homomorphism image of A is a fully stable M_Γ-act.

Proof: Let A and B be two M_Γ-acts, f: A → B an M_Γ-epimorphism and let B = A^δ for some Γ-congruence δ on A. Let C = A^δ be an S_Γ-subact of A^δ for some Γ-congruence δ^c on C and δ^c ⊆ δ and g: C^δ^c → A^δ M_Γ-homomorphism. Since C is a projective M_Γ-act, there exists g': C → A such that π_δ g' = gπ_δ^c where π_δ: A^δ → A^δ and π_δ^c: C^δ^c → C^δ^c both are natural M_Γ-epimorphism. By full stability of A, then g'(C)^δ^c ⊆ C implies π_δ g'(C)^δ^c ⊆ π_δ(C)^δ^c, hence g(C)^δ^c = gπ_δ^c(C) = π_δ g'(C)^δ^c = π_δ(C)^δ^c.

A Γ-monoid M is called right (left) hereditary, if every right (left) Γ-ideal of M is projective. The following theorem, we characterize fully stable Γ-monoids in terms of their gamma acts.

3.6. Theorem: Let M be a right hereditary Γ-monoid. Then the following statements are equivalents.

(1) M is fully stable Γ-monoid. (2) Every cyclic M_Γ-act is fully stable.

Proof: (1)⇒(2) Let aΓM be a cyclic M_Γ-act. Then aΓM = M^δ for some Γ-congruence δ on M. M^δ is a fully stable M_Γ-act, by (2.10) & (1). Hence aΓM is an M_Γ-isomorphic to a fully stable M_Γ-act. Hence aΓM is fully stable.(2)⇒(1) Obvious.

It is a well-known that for each element a in fully stable M_Γ-act A and an M_Γ-homomorphism f: aΓM → A, there exist m ∈ M and α ∈ Γ such that f(a) = αam. The following statement makes full stability satisfies for each finite elements in A.
3.7. Proposition: Let A be an $M_\Gamma$-act such that $R_M(N \cap K) = R_M(N) \cup R_M(K)$ for every finitely $\Gamma$-generated $M_\Gamma$-subacts N and K of A. Then A is fully stable, if and only if, for every $M_\Gamma$-homomorphism $f : \bigcup_{i=1}^{n} a_i \Gamma M \rightarrow A$ where $a_i \in A$, there are $m \in M$ and $\alpha_i \in \Gamma$ such that $f(a_i) = a_i \alpha_i m$ for each $i = 1, 2, 3, \ldots, n$.

Proof: Necessity. Let $B = \bigcup_{i=1}^{n} a_i \Gamma M$ be a finitely generated $M_\Gamma$-act for some $a_i \in A (i = 1, 2, \ldots, n)$ and any $M_\Gamma$-homomorphism $f : B \rightarrow A$. We use induction on the number of generator of B. For $n = 1$, this is just the definition. Assume that, the condition holds for $m \leq n - 1$. Then $B = \bigcup_{i=1}^{n-1} a_i \Gamma M \cup a_n \Gamma M$, there are two elements $m, m' \in M$ and $\alpha, \beta \in \Gamma$ such that $f(b_2) = b_2 \alpha m$, for each $b_2 \in \bigcup_{i=1}^{n-1} a_i \Gamma M$ and $f(b_1) = b_1 \beta m'$, for each $b_1 \in a_n \Gamma M$. Now, for each $b \in B$, then $b = \bigcup_{i=1}^{n-1} a_i \Gamma M$ and $b \in a_n \Gamma M$. Then $f(b) = b \alpha m = b \beta m'$, hence $(m, m') \in R_M \bigcup_{i=1}^{n-1} a_i \Gamma M \cap a_n \Gamma M$. Thus $(m, m') \in R_M \bigcup_{i=1}^{n-1} a_i \Gamma M \cap a_n \Gamma M$. Thus $f(b) = b \alpha m = b \beta m'$, for each $b \in a_n \Gamma M$. Thus $f(b) = b \alpha m$ for each $b \in \bigcup_{i=1}^{n-1} a_i \Gamma M$.

Sufficiency. Obvious.

4. Full Stability With Uniform On Gamma Acts.

4.1. Definition: Let $B$ be $M_\Gamma$-subact of $M_\Gamma$-act A. A relative complement of B in A is any $M_\Gamma$-subact $L$ of A which is maximal with respect to the property $B \subseteq L = \emptyset$.

The following Lemma tells that a relative complement $M_\Gamma$-subact always exists.

4.2. Lemma: Let $L, K$ be $M_\Gamma$-subacts of $M_\Gamma$-act A with $L \cap K = \emptyset$. Then there is a complement $L'$ of $L$ in A with $K \subseteq L'$.

Proof: Let $\tau = \{ C | C$ is an $M_\Gamma$-subact of A and K is $M_\Gamma$-subact of C and $L \cap C = \emptyset \}$, then $\tau \neq \emptyset$, since K in $\tau$. Since the union of every totally ordered subset in $\tau$ lies evidently again in $\tau$, every totally ordered subset from $\tau$ has an upper bound in $\tau$. By Zorn's lemma, there exists a maximal element $L' \in \tau$.

4.3. Definition: Let A be an $M_\Gamma$-act and $B$ non-zero $M_\Gamma$-subact of A. Then $B$ is called an essential in A, if $B \cap L = \emptyset$, for each non zero $M_\Gamma$-subact $L$ of A. This is equivalent to saying that if for every $M_\Gamma$-subact $L$ of A and $B \cap L = \emptyset$ then $L = \emptyset$.

4.4. Proposition: Let $L$ be an $M_\Gamma$-subact of $M_\Gamma$-act A, then we have $L$ is an essential $M_\Gamma$-subact of A, if and only if, for each $a (\neq \emptyset) \in A$, there exist $m \in M$ and $\alpha \in \Gamma$ such that $a \alpha m \neq \emptyset$ and $a \alpha m \in L$.

Proof: Necessity. From $a (\neq \emptyset) \in A$, we have $a \Gamma M \neq \emptyset$ and so $L \cap a \Gamma M \neq \emptyset$, since L is an essential in A, thus $a \alpha m \in L$. Sufficiency. Let L be a non-zero $M_\Gamma$-subact of $M_\Gamma$-act A and let $L'$ any non-zero
M_Γ-subact of M_Γ-act A. Then there is a( ≠ Θ) ∈ L’. Then aαm ≠ Θ and aαm ∈ L, then Θ ≠ aαm ∈ L’ ∩ L and so L is an essential in A.

Note that: The Z is an essential N_N-subact of Q, since for each Θ ≠ q = z/n ∈ Q, for some Θ ≠ z, n ∈ Z, there exist n ∈ N and k ∈ N such that zkn/n ≠ Θ and zkn/n = zk ∈ Z. But Q is not an essential N_N-subact of R, since √θ ∈ R, there are not n∈N and k ∈ N such that √θ zk ∈ Q.

4.5. Proposition: Let C be an M_Γ-act and let B, B’ M_Γ-subacts of M_Γ-act A. Then

(1) B is an essential M_Γ-subact of B’ and B’ is an essential M_Γ-subact of A if and only if B is an essential M_Γ-subact of A.

(2) If B_i is an essential M_Γ-subact of A, i = 1, 2,…..n then ∩_{i=1}^n B_i is an essential M_Γ-subact of A.

(3) If f: C→A is an M_Γ-homomorphism and B is an essential M_Γ-subact of A, then f^{-1}(B) is an essential M_Γ-subact of C.

(4) If L is an essential M_Γ-subact of B and L is an essential M_Γ-subact of B’ then L is an essential M_Γ-subact of B ∪ B’.

(5) If B_i is an essential M_Γ-subact of A, i = 1, 2,…..n then ∪_{i=1}^n B_i is an essential M_Γ-subact of A.

4.6. Proposition: Let B be an M_Γ-subact of M_Γ-act A. If L is any relative complement of B in A, then L\dot{∩}B is essential in A.

Proof: Since B\dot{∩}L= Θ, we have L\dot{∪}B = L\dot{∪}B is an M_Γ-subact of A. Suppose that K is an M_Γ-subact of A with K\dot{∩}(L\dot{∪}B) = Θ. Then the union K\dot{∪}(L\dot{∪}B), whence B \dot{∩} (L \dot{∪} K) = Θ. By the maximal of L, we obtain L \dot{∪} K = L and thus K = Θ.

4.7. Definition: An M_Γ-act A is said to be uniform, if every non-zero M_Γ-subact of A is essential in A. The following proposition gives that every gamma homomorphism between any two disjoint gamma subacts of a fully stable gamma act is trivial.

4.8. Proposition: Let A= A_1 \dot{∪} A_2 be a fully stable M_Γ-act. Then Hom_{M_Γ}(A_1, A_2) = Hom_{M_Γ}(A_2, A_1) = Θ.

Proof: Let g: A_1 → A_2 be an M_Γ-homomorphism. Since full stability of A, g(A_1) ⊆ A_1 and by M_Γ-homomorphic of g, g(A_1) ⊆ A_2 implies that g(A_1) ⊆ A_1 \cap A_2 = Θ.
The $Z_\mathbb{Z}$-act $Z\mathbb{Z}$ is not fully stable $Z_\mathbb{Z}$-act. In fact, if $Z\mathbb{Z}$ is fully stable $Z_\mathbb{Z}$-act, then $\text{Hom}_{Z_\mathbb{Z}}(Z, Z)$ must be equal to $\Theta$ which is not true.

4.9. Corollary: Let $A$ be a fully stable $M_\mathbb{F}$-act. Then $\text{Hom}_{M_\mathbb{F}}(A_1, A_2) = \text{Hom}_{M_\mathbb{F}}(A_1, A_2) = \Theta$, for any $M_\mathbb{F}$-subacts $A_1, A_2$ of $A$ with zero intersection.

The following theorem discusses converse of (4.9).

4.10. Theorem: The following statements are equivalent for an $M_\mathbb{F}$-act $A$.

1. $A$ is fully stable. 2. Every $M_\mathbb{F}$-homomorphism from any essential $M_\mathbb{F}$-subact $B$ of $A$ into $A$ satisfies $f(B) \subseteq B$ and $\text{Hom}_{M_\mathbb{F}}(B, C) = \text{Hom}_{M_\mathbb{F}}(C, B) = \Theta$, for any $M_\mathbb{F}$-subacts $B, C$ of $A$ with zero intersection.

Proof: (1) $\Rightarrow$ (2) By (4.9). (2) $\Rightarrow$ (1) Let $B$ be an $M_\mathbb{F}$-subact of $A$ and a non-zero $M_\mathbb{F}$-homomorphism $f: B \to A$. Let $L$ be a complement $B$ in $A$, by (4.2). Thus $f$ can be extended to $g: L \cup B \to A$ by (4.6) & (2), $f(B) = g(B) \subseteq g(L \cup B) \subseteq L \cup B$. Hence $f(B) \subseteq L \cup B$ and $\pi: L \cup B \to B$. And if $\pi f \in \text{Hom}_{M_\mathbb{F}}(B, L)$ where $\pi: L \cup B \to L$ is the natural projection to $L$. Since $B \cap L = \Theta$, $\text{Hom}_{M_\mathbb{F}}(B, L) = \Theta$, by (2). Thus we can conclude.

4.11. Corollary: The following statements are equivalent for an uniform $M_\mathbb{F}$-act $A$.

1. $A$ is fully stable. 2. Every essential $M_\mathbb{F}$-subact $B$ of $A$ is stable.

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