ANATOMY OF A \( q \)-GENERALIZATION OF THE LAGUERRE/HERMITE ORTHOGONAL POLYNOMIALS

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Abstract. We study a \( q \)-generalization of the classical Laguerre/Hermite orthogonal polynomials. Explicit results include: the recursive coefficients, matrix elements of generators for the Heisenberg algebra, and the Hankel determinants. The power of quadratic relation is illustrated by comparing two ways of calculating recursive coefficients. Finally, we derive a \( q \)-deformed version of the Toda equations for both \( q \)-Laguerre/Hermite ensembles, and check the compatibility with the quadratic relation.

1. Introduction

One special property about the orthogonal polynomials \(^{1} \) is that the matrix representations of the Heisenberg algebra are usually very simple. For instance, the famous three-term recurrence relation \(^{2} \) \(^{3} \) \(^{4} \) among orthogonal polynomials

\[
x p_{n} = a_{n+1} p_{n+1} + b_{n} p_{n} + a_{n} p_{n-1} \tag{1.1}
\]

implies that the matrix form of the position operator, \( x \), with respect to the orthogonal polynomial basis, is tri-diagonal for all kinds of weight functions. In addition, for many classical weight functions, one can show that both translation \((\frac{d}{dx})\) and dilation \((x\frac{d}{dx})\) operators are also tri-diagonal, such as the Laguerre and Hermite weight functions \(^{5} \) \(^{6} \) \(^{7} \) \(^{8} \) \(^{9} \). In this paper, we shall extend these classical examples to a more general setting, which we call generalized \( q \)-Laguerre/Hermite ensembles, and give explicit results for the representation of the Heisenberg algebra. By \textit{generalized}, we mean that, in addition to the exponential functions, we also multiply the \textit{standard} Laguerre/Hermite weights by a polynomial prefactor. Or equivalently, we add a logarithmic term to the polynomial potential (see the definitions in Sec. 2).

Admittedly, the study of the Laguerre/Hermite ensembles might seem uninteresting for the apparent linearity features. However, from physical point of view, these two sets of orthogonal polynomials lie at the heart of fundamental physics. In particular, they are associated with the quantum systems of three dimensional hydrogen atom (associated Laguerre polynomials) and one dimensional simple harmonic oscillator (Hermite polynomials). Given the mysterious connection between these two systems, \(^{10} \), it might be worthwhile to examine

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the physical relevance of the quadratic relation [3] between these two ensembles (see Sec. 2). Furthermore, $q$-deformation are often considered as certain microscopic/quantum modifications of the existing physical models. Hence we would like to extend the classical scenarios to a possible $q$-generalizations.

To summarize, our new findings in this paper include:

- A systematic study of the generalized $q$-Laguerre and $q$-Hermite ensembles, including explicit solutions of the recursive coefficients, their orthogonal polynomial representations of the the Heisenberg algebra, and their corresponding Hankel determinants (at $\kappa = 1$).
- We elucidate the power of the quadratic relation in comparing two ways of solving recursive coefficients for the $q$-Hermite ensemble.
- We find $q$-generalized Toda equations for the recursive coefficients of the $q$-Laguerre/Hermite ensembles, and check their compatibility with the quadratic relation.

This paper is organized as follows: after briefly reviewing the basic ideas of the quadratic relations for the classical $q$-deformed Laguerre/Hermite ensembles in section 2, we give explicit solutions of the dilation matrix elements and the recursive coefficients of the $q$-Laguerre/Hermite ensembles in section 3. Section 4 is devoted to the computations of the Hankel determinants for both ensembles. In section 5, we propose possible $q$-deformed Toda equations for the $q$-Hermite and $q$-Laguerre ensembles, which are based on the $\kappa$-deformation of the $q$-derivative matrix elements and that of the recursive coefficients. A brief summary and future outlook conclude this paper at section 6.

2. The Quadratic Relations for the Laguerre/Hermite Orthogonal Polynomials

2.1. Review of the quadratic relation among recursive coefficients for the orthogonal polynomials associated with the classical Laguerre/Hermite weights.

Given the classical Laguerre weight defined as

$$v^{(\alpha)}(x; \kappa) := x^\alpha \exp(-\kappa x), \quad 0 \leq \kappa, \quad -1 < \alpha, \quad 0 \leq x,$$

we can compute the orthonormal polynomials $p^{(\alpha)}_n(x; \kappa)$ as

$$\int_0^{\infty} p^{(\alpha)}_m(x; \kappa)p^{(\alpha)}_n(x; \kappa)v^{(\alpha)}(x; \kappa)dx = \delta_{mn}$$

through Gram-Schmidt process.

Similarly, from the classical Hermite weight,

$$\omega^{(\alpha)}(x; \kappa) := |x|^{2\alpha+1} \exp(-\kappa^2 x^2), \quad 0 \leq \kappa, \quad x \in \mathbb{R},$$

we obtain associated orthonormal polynomials $P^{(\alpha)}_n(x, \kappa)$ as

$$\int_{-\infty}^{\infty} P^{(\alpha)}_m(x; \kappa)P^{(\alpha)}_n(x; \kappa)\omega^{(\alpha)}(x; \kappa)dx = \delta_{mn}.$$
The quadratic relation among these two sets of orthonormal polynomials is based on a simple connection between the Laguerre and Hermite weights. Namely,

$$\omega^{(\alpha)}(x; \kappa) = |x| v^{(\alpha)}(x^2; \kappa^2). \quad (2.5)$$

One immediate consequence of Eq. (2.5) is that, the orthonormal polynomials $P^{(\alpha)}_n(x; \kappa)$ can be expressed in terms of the orthogonal polynomials $p^{(\alpha)}_n(x; \kappa)$ as follows,

$$P^{(\alpha)}_{2n}(x; \kappa) = P^{(\alpha)}_n(x^2; \kappa^2), \quad P^{(\alpha)}_{2n+1}(x; \kappa) = x P^{(\alpha)}_n(x^2; \kappa^2). \quad (2.6)$$

Having established this fundamental relation, we can then derive many useful consequences. In this paper, we shall focus on quadratic relation for the recursive coefficients.

The set of orthonormal polynomials associated with any weight function can be viewed as a complete set of basis for the function space. Hence, it induces a natural realization of the Heisenberg algebra, $\left[\frac{d}{dx}, x\right] = 1$. In particular, the matrix elements of the position operator consist of the three-term recursive coefficients among orthonormal polynomials. In the case of the Laguerre weight, it is given as

$$xp^{(\alpha)}_n(x; \kappa) = a^{(\alpha)}_{n+1}(\kappa)p^{(\alpha)}_{n+1}(x; \kappa) + b^{(\alpha)}(\kappa)p^{(\alpha)}_n(x; \kappa) + a^{(\alpha)}_n(\kappa)p^{(\alpha)}_{n-1}(x; \kappa), \quad (2.7)$$

and in the case of the Hermite weight, we have

$$xp^{(\alpha)}_n(x; \kappa) = A^{(\alpha)}_{n+1}(\kappa)p^{(\alpha)}_{n+1}(x; \kappa) + A^{(\alpha)}_n(\kappa)p^{(\alpha)}_{n-1}(x; \kappa). \quad (2.8)$$

By computing $x^2p^{(\alpha)}_n(x, \kappa)$ in two ways, we obtain the quadratic relation among the two sets of recursive coefficients:

$$a^{(\alpha)}_{n}(\kappa^2) = A^{(\alpha)}_{2n}(\kappa)A^{(\alpha)}_{2n-1}(\kappa), \quad (2.9)$$

$$b^{(\alpha)}_{n}(\kappa^2) = \left(A^{(\alpha)}_{2n+1}(\kappa)\right)^2 + \left(A^{(\alpha)}_{2n}(\kappa)\right)^2. \quad (2.10)$$

2.2. On the quadratic relation between generalized $q$-Laguerre/Hermite ensembles.

In this paper, we take generalized $q$-Hermite and $q$-Laguerre ensembles as an illustrative example of the quadratic relation. We consider the generalized $q$-Laguerre weight $(0 < \kappa < \frac{1}{q})$

$$v^{(\alpha)}(x; \kappa, q) := |x|^{\alpha} (q \kappa x; q)_\infty = |x|^{\alpha} \prod_{l=0}^{\infty} (1 - q^{l+1} \kappa x), \quad (2.11)$$

and the generalized $q$-Hermite weight

$$\omega^{(\alpha)}(x; \kappa, q) = |x|^{2\alpha+1} (q^2 \kappa^2 x^2; q^2)_\infty = |x|v^{(\alpha)}(x^2; \kappa^2, q^2). \quad (2.12)$$
Given the orthonormal polynomials of the $q$-Laguerre ensemble $p_n^{(\alpha)}(x; \kappa, q)$, we can express the orthonormal polynomials of the $q$-Hermite ensemble $P_n^{(\alpha)}(x; \kappa, q)$ as follows:

\begin{align}
P_{2n}^{(\alpha)}(x; \kappa, q) &= \sqrt{\frac{1+q}{2}} p_n^{(\alpha)}(x^2; \kappa^2, q^2) \text{ (even, deg } = 2n), \tag{2.13}
P_{2n+1}^{(\alpha)}(x; \kappa, q) &= \sqrt{\frac{1+q}{2}} x p_{n+1}^{(\alpha)}(x^2; \kappa^2, q^2) \text{ (odd, deg } = 2n+1). \tag{2.14}
\end{align}

One can easily check that $P_{2n}^{(\alpha)}(x; \kappa, q)$ and $P_{2n+1}^{(\alpha)}(x; \kappa, q)$ satisfy the orthonormal conditions, for instance,

\begin{align}
\int_{-1}^{1} P_{2m}^{(\alpha)}(x; \kappa, q) P_{2n}^{(\alpha)}(x; \kappa, q) \omega^{(\alpha)}(x; \kappa, q) dx
&= 2(1-q) \sum_{k=0}^{\infty} P_{2m}^{(\alpha)}(q^k; \kappa, q) P_{2n}^{(\alpha)}(q^k; \kappa, q) \omega^{(\alpha)}(q^k; \kappa, q) q^k
&= 2(1-q) \left( \frac{1+q}{2} \right) \sum_{k=0}^{\infty} p_n^{(\alpha)}(q^{2k}; \kappa^2, q^2) p_n^{(\alpha)}(q^{2k}; \kappa^2, q^2) q^k v^{(\alpha)}(q^{2k}; \kappa^2, q^2) q^k
&= \int_{0}^{1} p_m^{(\alpha)}(x; \kappa^2, q^2) p_n^{(\alpha)}(x; \kappa^2, q^2) v^{(\alpha)}(x; \kappa^2, q^2) dx = \delta_{mn}. \tag{2.15}
\end{align}

\begin{align}
\int_{-1}^{1} P_{2m+1}^{(\alpha)}(x; \kappa, q) P_{2n+1}^{(\alpha)}(x; \kappa, q) \omega^{(\alpha)}(x; \kappa, q) dx
&= 2(1-q) \sum_{k=0}^{\infty} P_{2m+1}^{(\alpha)}(q^k; \kappa, q) P_{2n+1}^{(\alpha)}(q^k; \kappa, q) \omega^{(\alpha)}(q^k; \kappa, q) q^k
&= 2(1-q) \left( \frac{1+q}{2} \right) \sum_{k=0}^{\infty} q^k p_n^{(\alpha+1)}(q^{2k}; \kappa^2, q^2) q^k p_n^{(\alpha+1)}(q^{2k}; \kappa^2, q^2) q^k v^{(\alpha)}(q^{2k}; \kappa^2, q^2) q^k
&= (1-q^2) \sum_{k=0}^{\infty} p_m^{(\alpha+1)}(q^{2k}; \kappa^2, q^2) p_n^{(\alpha+1)}(q^{2k}; \kappa^2, q^2) v^{(\alpha+1)}(q^{2k}; \kappa^2, q^2) q^{2k}
&= \int_{0}^{1} p_m^{(\alpha+1)}(x; \kappa^2, q^2) p_n^{(\alpha+1)}(x; \kappa^2, q^2) v^{(\alpha+1)}(x; \kappa^2, q^2) dx = \delta_{mn}. \tag{2.16}
\end{align}

The $P_{2m}^{(\alpha)} P_{2m+1}^{(\alpha)}$ orthogonality is trivial due to the even parity of the generalized $q$-Hermite weight.

Similar to the classical cases Eqs. (2.9), (2.10), there exists a correspondence between recursive coefficients associated with the generalized $q$-Laguerre and $q$-Hermite ensembles.
Theorem 2.1. The recursive coefficients associated with $q$-generalized Laguerre and Hermite ensembles satisfying the following relations:

\begin{align}
    a_n^{(a)}(\kappa^2, q^2) &= A_{2n}^{(a)}(\kappa, q)A_{2n-1}^{(a)}(\kappa, q), \quad (2.17) \\
    b_n^{(a)}(\kappa^2, q^2) &= \left( A_{2n+1}^{(a)}(\kappa, q) \right)^2 + \left( A_{2n}^{(a)}(\kappa, q) \right)^2, \quad (2.18) \\
    a_n^{(a+1)}(\kappa^2, q^2) &= A_{2n+1}^{(a)}(\kappa, q)A_{2n}^{(a)}(\kappa, q), \quad (2.19) \\
    b_n^{(a+1)}(\kappa^2, q^2) &= \left[ A_{2n+2}^{(a)}(\kappa, q) \right]^2 + \left[ A_{2n+1}^{(a)}(\kappa, q) \right]^2. \quad (2.20)
\end{align}

Proof.

\[ x^2 P_{2n}^{(a)}(x; \kappa, q) \]

\begin{align*}
    &\quad = x \left[ A_{2n+1}^{(a)}(\kappa, q)P_{2n+1}^{(a)}(x; \kappa, q) + A_{2n}^{(a)}(\kappa, q)P_{2n-1}^{(a)}(x; \kappa, q) \right] \\
    &\quad = A_{2n+1}^{(a)}(\kappa, q) \left[ A_{2n+2}^{(a)}(\kappa, q)P_{2n+2}^{(a)}(x; \kappa, q) + A_{2n+1}^{(a)}(\kappa, q)P_{2n+1}^{(a)}(x; \kappa, q) \right] \\
    &\quad + A_{2n}^{(a)}(\kappa, q) \left[ A_{2n+1}^{(a)}(\kappa, q)P_{2n+1}^{(a)}(x; \kappa, q) + A_{2n+1}^{(a)}(\kappa, q)P_{2n}^{(a)}(x; \kappa, q) \right] \\
    &\quad = \left[ A_{2n+1}^{(a)}(\kappa, q)A_{2n+2}^{(a)}(\kappa, q) \right] P_{2n+2}^{(a)}(x; \kappa, q) + \left( A_{2n+1}^{(a)}(\kappa, q) \right)^2 P_{2n}^{(a)}(x; \kappa, q) \\
    &\quad + \left[ A_{2n}^{(a)}(\kappa, q)A_{2n+2}^{(a)}(\kappa, q) \right] P_{2n-2}^{(a)}(x; \kappa, q).
\end{align*}

On the other hand, using the expression of Eq. (2.13), we have

\[ x^2 P_{2n}^{(a)}(x; \kappa, q) \]

\begin{align*}
    &\quad = x^2 \sqrt{\frac{1+q}{2}} P_{n}^{(a)}(x^2; \kappa^2, q^2) \\
    &\quad = \sqrt{\frac{1+q}{2}} \left[ a_n^{(a)}(\kappa^2, q^2)P_{n+1}^{(a)}(x^2; \kappa^2, q^2) + b_n^{(a)}(\kappa^2, q^2)P_{n}^{(a)}(x^2; \kappa^2, q^2) + a_n^{(a)}(\kappa^2, q^2)P_{n-1}^{(a)}(x^2; \kappa^2, q^2) \right] \\
    &\quad = a_n^{(a)}(\kappa^2, q^2)P_{2n+2}^{(a)}(x; \kappa, q) + b_n^{(a)}(\kappa^2, q^2)P_{2n}^{(a)}(x; \kappa, q) + a_n^{(a)}(\kappa^2, q^2)P_{2n-2}^{(a)}(x; \kappa, q).
\end{align*}

By comparing the coefficients on both expressions, we get Eqs. (2.17), (2.18).

If we examine similar calculations for the odd $q$-Hermite orthonormal polynomials, we get

\[ x^2 P_{2n+1}^{(a)} = [A_{2n+3}^{(a)}A_{2n+2}^{(a)}]P_{2n+3}^{(a)} + \left( A_{2n+2}^{(a)} \right)^2 \left( A_{2n+1}^{(a)} \right)^2 P_{2n+1}^{(a)} + [A_{2n+2}^{(a)}A_{2n}^{(a)}]P_{2n-1}^{(a)}.\]
Alternatively,

\[ x^2 P_{2n+1}^{(\alpha)}(x; \kappa, q) \]

\[ = x^2 \sqrt{\frac{1 + q}{2}} p_{n}^{(\alpha+1)}(x^2; \kappa^2, q^2) \]

\[ = \sqrt{\frac{1 + q}{2}} \left[ a_{n+1}^{(\alpha+1)}(\kappa^2, q^2) p_{n+1}^{(\alpha+1)}(x^2; \kappa^2, q^2) + b_{n}^{(\alpha+1)}(\kappa^2, q^2) p_{n}^{(\alpha+1)}(x^2; \kappa^2, q^2) + a_{n}^{(\alpha+1)}(\kappa^2, q^2) p_{n-1}^{(\alpha+1)}(x^2; \kappa^2, q^2) \right] \]

\[ = a_{n+1}^{(\alpha+1)}(\kappa^2, q^2) P_{2n+3}^{(\alpha)}(x; \kappa, q) + b_{n}^{(\alpha+1)}(\kappa^2, q^2) P_{2n+1}^{(\alpha)}(x; \kappa, q) + a_{n}^{(\alpha+1)}(\kappa^2, q^2) P_{2n-1}^{(\alpha)}(x; \kappa, q). \]

Thus, we have shown Eqs. (2.19), (2.20).

Eliminating the recursive coefficients of the \(q\)-Laguerre orthonormal polynomials, \(a_n^{(\alpha)}\), \(b_n^{(\alpha)}\), in both sets of the equation, we obtain

\[ A_{2n}^{(\alpha)}(\kappa, q) A_{2n-2}^{(\alpha)}(\kappa, q) = A_{2n+1}^{(\alpha-1)}(\kappa, q) A_{2n-1}^{(\alpha-1)}(\kappa, q), \tag{2.21} \]

and

\[ \left( A_{2n+1}^{(\alpha)}(\kappa, q) \right)^2 + \left( A_{2n}^{(\alpha)}(\kappa, q) \right)^2 = \left( A_{2n+2}^{(\alpha-1)}(\kappa, q) \right)^2 + \left( A_{2n+1}^{(\alpha-1)}(\kappa, q) \right)^2. \tag{2.22} \]

We shall see how these quadratic relation will help us derive the explicit form of the recursive coefficients \(A_n^{(\alpha)}(1, q)\) from the (easier) solutions of \(a_n^{(\alpha)}(1, q^2), b_n^{(\alpha)}(1, q^2)\) in Sec. 3. For the general \(\kappa\) case, we check the compatibility between the quadratic relation and the evolution equations (w.r.t \(\kappa\)) in Sec. 5.

3. Explicit solutions of the Fourier and recursive coefficients for the \(q\)-Laguerre/Hermite ensembles

In this section, we derive the matrix representations of the \(q\)-dilation operator in the generalized \(q\)-Laguerre/Hermite ensembles, and use it to solve for recursive coefficients of the orthonormal polynomials. Since we shall focus on the special case \(\kappa = 1\), we shall suppress the \(\kappa\) (and \(q\)) dependences to make the equations simpler.

3.1. Solution of the recursive coefficients for the generalized \(q\)-Laguerre ensemble (at \(\kappa = 1\)).

**Theorem 3.1.** The matrix elements (Fourier coefficients) of the \(q\)-dilation operator with respect to the orthonormal \(q\)-Laguerre polynomials are given by

\[ xD_q p_n^{(\alpha)}(x) = \left( \frac{1 - q^n}{1 - q} \right) p_n^{(\alpha)}(x) + \left[ q^{-\alpha+1} (1 - q)^{-1} (1 - q^n)^{\frac{1}{2}} (1 - q^{n+\alpha})^{\frac{1}{2}} \right] p_{n-1}^{(\alpha)}(x). \tag{3.1} \]
Theorem 3.2. The recursive coefficients for the orthonormal $q$-Laguerre polynomial associated with weight $v^{(\alpha)}(x; 1, q)$ is given by

$$a_n^{(\alpha)} = q^{2n+\alpha-1}(1 - q^n)(1 - q^{n+\alpha}) \geq 0, \quad (3.2)$$

$$b_n^{(\alpha)} = -q^{2n+\alpha}(1 + q) + q^n(1 + q^\alpha). \quad (3.3)$$

Proof of Theorems 3.1 and 3.2. To compute the recursive coefficients, $a_n^{(\alpha)}(q), b_n^{(\alpha)}(q)$, we first study the Fourier coefficients of the action of dilation on the orthonormal $q$-Laguerre polynomials:

$$x^D_q p_n^{(\alpha)}(x) = \sum_{j=0}^{n} p_j^{(\alpha)}(x)c_{jn}^{(\alpha)}, \quad (3.4)$$

where projection formula gives

$$c_{jn}^{(\alpha)} = \int_{-1}^{1} p_j^{(\alpha)}(x)[x^D_q p_n^{(\alpha)}(x)]v^{(\alpha)}(x)d_qx$$

$$= \frac{-1}{1 - q} \int_{-1}^{1} p_j^{(\alpha)}(x)p_n^{(\alpha)}(qx)v^{(\alpha)}(x)d_qx$$

$$+ \frac{1}{1 - q} \int_{-1}^{1} p_j^{(\alpha)}(x)p_n^{(\alpha)}(x)v^{(\alpha)}(x)d_qx. \quad (3.5)$$

Using the Pearson relation for the generalized $q$-Laguerre weight,

$$v^{(\alpha)}\left(\frac{x}{q}\right) = q^{-\alpha}(1 - x)v^{(\alpha)}(x), \quad (3.6)$$

we get

$$c_{jn}^{(\alpha)} = -\frac{q^{-(\alpha+1)}}{1 - q} \int_{-1}^{1} p_j^{(\alpha)}\left(\frac{x}{q}\right)p_n^{(\alpha)}(x)v^{(\alpha)}(x)d_qx$$

$$+ \frac{q^{-(\alpha+1)}}{1 - q} \int_{-1}^{1} p_j^{(\alpha)}\left(\frac{x}{q}\right)p_n^{(\alpha)}(x)v^{(\alpha)}(x)d_qx$$

$$+ \frac{1}{1 - q} \delta_{jn}. \quad (3.7)$$

From this expression, it is clear that $c_{jn}^{(\alpha)} = 0$ if $j \leq n - 2$. Thus, we have

$$x^D_q p_n^{(\alpha)}(x) = p_n^{(\alpha)}(x)c_{nn}^{(\alpha)} + p_{n-1}^{(\alpha)}(x)c_{n-1,n}^{(\alpha)}. \quad (3.8)$$

From the coefficients of the orthonormal $q$-Laguerre polynomials:

$$p_n^{(\alpha)}(x) = \gamma_n^{(\alpha)}\left(x^n + \eta_n^{(\alpha)}x^{n-1} + \cdots\right), \quad (3.9)$$
we deduce the decomposition of the rescaled orthonormal polynomials,

\[
p_n^{(\alpha)} \left(\frac{x}{q}\right) = q^{-n} p_n^{(\alpha)}(x) + \frac{\eta_n^{(\alpha)}}{a_n^{(\alpha)}} q^{-n}(q-1)p_{n-1}^{(\alpha)}(x) + \cdots, \tag{3.10}
\]

\[
p_n^{(\alpha)}(qx) = q^n p_n^{(\alpha)}(x) + \frac{\eta_n^{(\alpha)}}{a_n^{(\alpha)}} q^{n-1}(1-q)p_{n-1}^{(\alpha)}(x) + \cdots. \tag{3.11}
\]

Substituting these results back to Eq. (3.5), we obtain

\[
c_n^{(\alpha)} = \frac{1 - q^n}{1 - q} \text{ (independent of } \alpha\text{),} \tag{3.12}
\]

\[
c_n^{(\alpha)} = -\frac{\eta_n^{(\alpha)}}{a_n^{(\alpha)}} q^{n-1}. \tag{3.13}
\]

On the other hand, we can relate the Fourier coefficients, \(c_n^{(\alpha)}\) and \(c_{n-1,n}^{(\alpha)}\) to the recursive coefficient \(a_n^{(\alpha)}\) using the second integral expression (3.7) together with

\[
x p_n^{(\alpha)} \left(\frac{x}{q}\right) = q^{-n} a_{n+1} p_{n+1}^{(\alpha)}(x) + (\eta_n^{(\alpha)} q^{-n-1} - \eta_{n+1}^{(\alpha)} q^{-n}) p_n^{(\alpha)}(x) + \cdots. \tag{3.14}
\]

The results are:

\[
c_n^{(\alpha)} = -\frac{q^{-(\alpha+n+1)}}{1 - q} + \frac{q^{-(\alpha+1)}}{1 - q} (\eta_n^{(\alpha)} q^{-n-1} - \eta_{n+1}^{(\alpha)} q^{-n}) + \frac{1}{1 - q}, \tag{3.15}
\]

\[
c_{n-1,n}^{(\alpha)} = \frac{q^{-(\alpha+n)}}{1 - q} a_n^{(\alpha)}. \tag{3.16}
\]

By comparing two expressions for \(c_n^{(\alpha)}\), Eqs. (3.12) and (3.16), we solve

\[
\eta_n^{(\alpha)} = \frac{(q^{n+\alpha} - 1)(1 - q^n)}{1 - q}. \tag{3.17}
\]

From the expressions for \(c_{n-1,n}^{(\alpha)}\), Eqs. (3.13) and (3.16), we solve

\[
(a_n^{(\alpha)})^2 = q^{2n+\alpha-1}(1 - q^n)(1 - q^{n+\alpha}) \geq 0. \tag{3.18}
\]

Finally, the middle recursive coefficient \(b_n^{(\alpha)}\) can be solved as

\[
b_n^{(\alpha)} = \eta_n^{(\alpha)} - \eta_{n+1}^{(\alpha)} = -q^{2n+\alpha}(1 + q) + q^{n+\alpha} + q^n. \tag{3.19}
\]
3.2. Solution of the recursive coefficients for the generalized $q$-Hermite ensemble (at $\kappa = 1$).

In this section, we calculate the recursive coefficients for the orthonormal polynomials associated with generalized $q$-Hermite ensembles, $A_n^{(\alpha)}$. In order to derive a master equation for the recursive coefficients $A_n^{(\alpha)}$, we first study the Fourier coefficients of the action of dilation on the orthonormal $q$-Hermite polynomials. In this section, we set $\kappa = 1$ and suppress the $q$ dependence.

**Theorem 3.3.** The matrix elements (Fourier coefficients) of the $q$-dilation operator with respect to the orthonormal $q$-Hermite polynomials are given in terms of the recursive coefficients $A_n^{(\alpha)}$ as

$$x D_q^x P_n^{(\alpha)}(x) = \sum_{j=0}^n P_j^{(\alpha)}(x) C_{jn}^{(\alpha)} = \frac{1 - q^n P_n^{(\alpha)}(x)}{1 - q} + \left[ \frac{q^{-2}(1 + q) A_n^{(\alpha)}}{A_{n-1}^{(\alpha)} A_n^{(\alpha)}} \sum_{l=0}^{n-1} (A_l^{(\alpha)})^2 \right] P_{n-2}^{(\alpha)}(x). \quad (3.20)$$

**Proof.** The Fourier coefficients of the $q$-dilation operator can be computed by using projection formula,

$$C_{jn}^{(\alpha)} = \int_{-1}^{1} P_j^{(\alpha)}(x) [x D_q^x P_n^{(\alpha)}(x)] \omega^{(\alpha)}(x) d_qx \quad (3.21)$$

Substituting the definition of the $q$-derivative $D_q^x P_n^{(\alpha)}$ (see (A.2)) and recalling the definition of the $q$-integration, we get

$$C_{jn}^{(\alpha)} = -2 \sum_{l=0}^{\infty} P_j^{(\alpha)}(q^l) P_n^{(\alpha)}(q^{l+1}) \omega^{(\alpha)}(q^l) q^l + 2 \sum_{l=0}^{\infty} P_j^{(\alpha)}(q^l) P_n^{(\alpha)}(q^{l+1}) \omega^{(\alpha)}(q^l) q^l = \frac{-1}{1 - q} \int_{-1}^{1} P_j^{(\alpha)}(x) P_n^{(\alpha)}(q x) \omega^{(\alpha)}(x) \, d_q x + \frac{1}{1 - q} \int_{-1}^{1} P_j^{(\alpha)}(x) P_n^{(\alpha)}(x) \omega^{(\alpha)}(x) \, d_q x. \quad (3.22)$$

Upon shifting the index $\kappa$ and using the Pearson relation for the generalized $q$-Hermite weight

$$\omega^{(\alpha)} \left( \frac{x}{q} \right) = q^{-(2\alpha+1)} (1 - x^2) \omega^{(\alpha)}(x), \quad (3.23)$$
we get

\[ C_{jn}^{(\alpha)} = \frac{q^{-(2\alpha + 2)}}{1 - q} \int_{-1}^{1} P_j^{(\alpha)} \left( \frac{x}{q} \right) P_n^{(\alpha)}(x) \omega^{(\alpha)}(x) d_q x \]

\[ + \frac{q^{-(2\alpha + 2)}}{1 - q} \int_{-1}^{1} P_j^{(\alpha)} \left( \frac{x}{q} \right) P_n^{(\alpha)}(x) \omega^{(\alpha)}(x) d_q x \]

\[ + \frac{1}{1 - q} \delta_{jn}. \quad (3.24) \]

From this expression, it is clear that \( C_{jn}^{(\alpha)} = 0 \) if \( j < n - 2 \). Thus, we have

\[ x D_x^q P_n^{(\alpha)}(x) = P_n^{(\alpha)}(x) C_{nn}^{(\alpha)} + P_{n-2}^{(\alpha)}(x) C_{n-2,n}^{(\alpha)}. \quad (3.25) \]

To facilitate the computations, we need to invoke the structure of the orthonormal coefficients. Let

\[ P_n^{(\alpha)}(x) = \gamma_n^{(\alpha)} \left( x^n + \zeta_n^{(\alpha)} x^{n-2} + \ldots \right), \]

we can decompose the rescaled orthonormal polynomials \( P_n^{(\alpha)} \left( \frac{x}{q} \right) \) as follows:

\[ P_n^{(\alpha)} \left( \frac{x}{q} \right) = \gamma_n^{(\alpha)} \left( q^{-n} x^n + \zeta_n^{(\alpha)} q^{-n+2} x^{n-2} + \ldots \right) \]

\[ = q^{-n} \gamma_n^{(\alpha)} \left[ (x^n + \zeta_n^{(\alpha)} x^{n-2} + \ldots) - \zeta_n^{(\alpha)} x^{n-2} + \zeta_n^{(\alpha)} q^2 x^{n-2} + \ldots \right] \]

\[ = q^{-n} P_n^{(\alpha)}(x) - \frac{q^{-n}(1 - q^2) \gamma_n^{(\alpha)}}{A_n^{(\alpha)} A_n^{(\alpha)}} P_{n-2}^{(\alpha)} + \ldots, \quad (3.26) \]

\[ P_n^{(\alpha)}(q x) = \gamma_n^{(\alpha)} \left( q^n x^n + \zeta_n^{(\alpha)} q^{n-2} x^{n-2} + \ldots \right) \]

\[ = q^n \gamma_n^{(\alpha)} \left[ (x^n + \zeta_n^{(\alpha)} x^{n-2} + \ldots) - \zeta_n^{(\alpha)} x^{n-2} + \zeta_n^{(\alpha)} q^{-2} x^{n-2} + \ldots \right] \]

\[ = q^n P_n^{(\alpha)}(x) + \frac{q^{n-2}(1 - q^2) \gamma_n^{(\alpha)}}{A_n^{(\alpha)} A_n^{(\alpha)}} P_{n-2}^{(\alpha)} + \ldots. \quad (3.27) \]

Substituting Eq. (3.27) back to Eq. (3.22), we obtain

\[ C_{nn}^{(\alpha)} = \frac{1 - q^n}{1 - q} \] (independent of \( \alpha \)), \quad (3.28)

\[ C_{n-2,n}^{(\alpha)} = \frac{-(1 + q) q^{n-2}}{A_{n-1}^{(\alpha)} A_n^{(\alpha)}} \zeta_n^{(\alpha)} . \quad (3.29) \]
Note that
\[
\zeta_n^{(\alpha)} = -\sum_{l=0}^{n-1} \left( A_l^{(\alpha)} \right)^2, \tag{3.30}
\]
and
\[
C_{n+1,n+1}^{(\alpha)} - C_{nn}^{(\alpha)} = q^n. \tag{3.31}
\]

It is worth emphasizing that, both Eqs. (3.12), (3.28) are just special cases of a universal feature of the \(q\)-dilation operator. We give a general proof without using Pearson relation in Appendix B.

Having derived the Fourier coefficients \(C_{nn}^{(\alpha)}\) and \(C_{n-2,n}^{(\alpha)}\) for \(x \mathcal{D}_q^n[P_n^{(\alpha)}(x)]\), we can use Eq. (3.24) to derive a master equation for the recursive coefficients \(A_n^{(\alpha)}\) defined in Eq. (2.8) with \(\kappa = 1\),
\[
xP_n^{(\alpha)}(x) = A_{n+1}^{(\alpha)}P_{n+1}(x) + A_n^{(\alpha)}P_{n-1}(x). \tag{3.32}
\]

Namely,
\[
xP_n^{(\alpha)} \left( \frac{x}{q} \right) = P_{n+1}^{(\alpha)}(x) - q^{-n}A_{n+1}^{(\alpha)} + P_{n-1}(x) \left[ q^{-n}A_n^{(\alpha)} - \frac{q^{-n}\zeta_n^{(\alpha)}(1-q^2)}{A_n^{(\alpha)}} \right] + \cdots, \tag{3.33}
\]
Eq. (3.24) implies
\[
1 - q^n = (1 - q)C_{nn}^{(\alpha)} = (1 - q^{-n-2\alpha-2}) + q^{-n-2\alpha-2} \left[ \left( A_{n+1}^{(\alpha)} \right)^2 + \left( A_n^{(\alpha)} \right)^2 - \zeta_n^{(\alpha)}(1-q^2) \right]. \tag{3.34}
\]
By taking a first difference on both sides of the equation, \((1-q)(C_{n+1,n+1}^{(\alpha)} - C_{nn}^{(\alpha)})\), and define a shifted difference
\[
B_k^{(\alpha)} := \left( A_k^{(\alpha)} \right)^2 - q \left( A_{k-1}^{(\alpha)} \right)^2 \iff \left( A_n^{(\alpha)} \right)^2 = \sum_{k=1}^{n} q^{-k}B_k^{(\alpha)}, \tag{3.35}
\]
with \(A_0^{(\alpha)} = 0, B_1^{(\alpha)} = \left( A_1^{(\alpha)} \right)^2\), we get
\[
(1-q)q^{2n+2\alpha+3} = (q-1) + B_{n+2}^{(\alpha)} + B_{n+1}^{(\alpha)} + (1-q^2) \sum_{k=1}^{n} B_k^{(\alpha)}. \tag{3.36}
\]
By taking another difference \((1-q)(q^{2n+2\alpha+3} - q^{2n+2\alpha+1})\), and defining another shifted difference \(C_n^{(\alpha)} := B_{n+1}^{(\alpha)} - qB_n^{(\alpha)}\). That is,
\[
B_n^{(\alpha)} = q^{n-1}B_1^{(\alpha)} + \sum_{k=1}^{n-1} q^{n-1-k}C_k, \tag{3.37}
\]
we get
\[ C_{n+1}^{(\alpha)} = -qC_n^{(\alpha)} - (1 - q)q^{2n+2\alpha+1}. \] (3.38)

The explicit form of \( C_n^{(\alpha)} \) can be solved by iteration, and is given by
\[ C_n^{(\alpha)} = (-q)^{n-1}C_1^{(\alpha)} - (1 - q)^2(-1)^n q^{n+2\alpha+1}[1 - (-q)^{n-1}], \] (3.39)
where \( C_1^{(\alpha)} = B_2^{(\alpha)} - qB_1^{(\alpha)} = (A_2^{(\alpha)})^2 - 2q(A_1^{(\alpha)})^2. \)

Substituting the explicit form of \( C_n^{(\alpha)} \), and performing the summation, Eq. (3.37), we get for \( m \geq 1, \)
\[ B_{2m}^{(\alpha)} = -q^{2m-1}(A_1^{(\alpha)})^2 + q^{2m-2}(A_2^{(\alpha)})^2 - q^{2m+2\alpha+1}(1 - q)(1 - q^{2m-2}), \] (3.40)
\[ B_{2m-1}^{(\alpha)} = q^{2m-2}(A_1^{(\alpha)})^2 - q^{2m+2\alpha-1}(1 - q)(1 - q^{2m-2}). \] (3.41)

Substituting the explicit forms of \( B_n^{(\alpha)} \), and performing the summation, (3.36), we get
\[ (A_{2m}^{(\alpha)})^2 = mq^{2m-2}(A_2^{(\alpha)})^2 + q^{2m+2\alpha}[(1 - q^{2m}) - m(1 - q^2)], \] (3.42)
\[ (A_{2m-1}^{(\alpha)})^2 = q^{2m-2}(A_1^{(\alpha)})^2 + (m - 1)q^{2m-3}(A_2^{(\alpha)})^2 \]
\[ + \frac{q^{2m+2\alpha}}{1 + q}[(1 - q^{2m-2}) - (m - 1)(1 - q)]. \] (3.43)

3.3. The power of the quadratic relation.

To show the power of the quadratic relation, we first use the solutions of the recursive coefficients of the \( q \)-Laguerre orthonormal polynomials \( a_n^{(\alpha)}, b_n^{(\alpha)} \), Eqs. (3.18), (3.19) to deduce the recursive coefficients of the \( q \)-Hermite polynomials.

**Theorem 3.4.** The recursive coefficients for the orthonormal polynomials of the \( q \)-Hermite ensemble are
\[ \left[A_{2n}^{(\alpha)}(1, q)\right]^2 = q^{2n+2\alpha}(1 - q^{2n}), \] (3.44)
\[ \left[A_{2n+1}^{(\alpha)}(1, q)\right]^2 = q^{2n}(1 - q^{2n+2\alpha+2}). \] (3.45)

**Proof.** First of all, we have learned that, using Eqs. (2.17), (2.18),
\[ (A_{2n}^{(\alpha)}(q))^2(A_{2n+1}^{(\alpha)}(q))^2 = (a_n^{(\alpha+1)}(q^2))^2 \]
\[ = q^{4n+2\alpha}(1 - q^{2n})(1 - q^{2n+2\alpha+2}), \]
and

\[
\left( A_{2n}^{(\alpha)}(q) \right)^2 + \left( A_{2n+1}^{(\alpha)}(q) \right)^2 = b_n^{(\alpha)}(q^2)
= -q^{4n+2\alpha}(1 + q^{2n}) + q^{2n}(q^{2\alpha} + 1).
\]

This implies that \( A_{2n}^{(\alpha)} \) and \( A_{2n+1}^{(\alpha)} \) can be solved from the quadratic equation,

\[
z^2 - b_n^{(\alpha)}(q^2)z + (a^{(\alpha+1)}(q^2))^2 = 0, \tag{3.46}
\]

and two roots are

\[
z_{\pm}^{(\alpha)}(q) = \frac{b_n^{(\alpha)}(q^2) \pm \sqrt{b_n^{(\alpha)}(q^2)^2 - 4(a^{(\alpha+1)}(q^2))^2}}{2}. \tag{3.47}
\]

Substituting the solutions for \( a^{(\alpha+1)}(q^2) \) and \( b_n^{(\alpha)}(q^2) \), Eqs. (3.18), (3.19), we get

\[
z_{\pm}^{(\alpha)}(q) = \frac{-q^{4n+2\alpha}(1 + q^{2n}) + q^{2n}(1 + q^{2\alpha}) \pm [q^{4n+2\alpha}(1 - q^{2n}) + q^{2n}(1 - q^{2\alpha})]}{2}
= \begin{cases} 
-q^{4n+2\alpha+2} + q^{2n} & (+) \\
-q^{4n+2\alpha} + q^{2n+2\alpha} & (-)
\end{cases}
\]

By taking \( \alpha \to -\frac{1}{2} \),

\[
z_{\pm}^{(-\frac{1}{2})}(q) \rightarrow \begin{cases} 
-q^{4n+1} + q^{2n} \rightarrow A_{2n}^{-\frac{1}{2}} \\
-q^{4n-1} + q^{2n-1} \rightarrow A_{2n}^{-\frac{1}{2}}
\end{cases} \tag{3.48}
\]

We deduce that

\[
\left( A_{2n}^{(\alpha)} \right)^2 = q^{2n+2\alpha}(1 - q^{2n}) \Rightarrow \left( A_{2}^{(\alpha)} \right)^2 = q^{2\alpha+2}(1 - q^2) \tag{3.49}
\]

\[
\left( A_{2n+1}^{(\alpha)} \right)^2 = q^{2n}(1 - q^{2n+2\alpha+2}) \Rightarrow \left( A_{1}^{(\alpha)} \right)^2 = 1 - q^{2\alpha+2}. \tag{3.50}
\]

\( \square \)

From this second approach, it is clear that not only we can easily derive the recursive coefficients of the generalized \( q \)-Hermite orthonormal polynomials, but also we get the initial values \( A_1^{(\alpha)}, A_2^{(\alpha)} \) for free. One can now check by substituting the initial values \( A_1^{(\alpha)}, A_2^{(\alpha)} \) into Eqs. (3.42), (3.43), which are solutions from (laborious) successive reduction, that we obtain consistent results as derived from the quadratic relation.
4. Hankel determinants and its \(q\)-generalization

4.1. Hankel determinant for the \(q\)-Laguerre ensemble (for \(\kappa = 1\)).

In this section, we shall compute the Hankel determinant associated with the generalized \(q\)-Laguerre ensemble, which is defined as the determinant of the moments associated with the weight function.

\[
\mu_k^{(\alpha)} := \int_0^1 x^k v^{(\alpha)}(x, q) dx,
\]

\[
\Delta_n^{(\alpha)} := \begin{vmatrix}
\mu_0^{(\alpha)} & \mu_1^{(\alpha)} & \cdots & \mu_{n-1}^{(\alpha)} \\
\mu_1^{(\alpha)} & \mu_2^{(\alpha)} & \cdots & \mu_{n}^{(\alpha)} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n-1}^{(\alpha)} & \mu_n^{(\alpha)} & \cdots & \mu_{2n-2}^{(\alpha)} \\
\end{vmatrix}, \quad \Delta_0^{(\alpha)} := 1.
\]

\[
\tilde{\Delta}_n^{(\alpha)} := \begin{vmatrix}
\mu_0^{(\alpha)} & \mu_1^{(\alpha)} & \cdots & \mu_{n-2}^{(\alpha)} & \mu_n^{(\alpha)} \\
\mu_1^{(\alpha)} & \mu_2^{(\alpha)} & \cdots & \mu_{n-1}^{(\alpha)} & \mu_{n+1}^{(\alpha)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mu_{n-2}^{(\alpha)} & \mu_{n-1}^{(\alpha)} & \cdots & \mu_{2n-3}^{(\alpha)} & \mu_{2n-1}^{(\alpha)} \\
\end{vmatrix}, \quad \tilde{\Delta}_0^{(\alpha)} := 1.
\]

In addition, the recursive coefficients for the orthonormal polynomials can be expressed in terms of the Hankel determinants as

\[
\left(a_n^{(\alpha)}\right)^2 = \frac{\Delta_n^{(\alpha)}\Delta_{n+1}^{(\alpha)}}{\left(\Delta_n^{(\alpha)}\right)^2}, \quad \eta_n^{(\alpha)} = \frac{\tilde{\Delta}_n^{(\alpha)}\Delta_{n+1}^{(\alpha)}}{\Delta_n^{(\alpha)}\tilde{\Delta}_{n+1}^{(\alpha)}} - \frac{\tilde{\Delta}_n^{(\alpha)}}{\Delta_n^{(\alpha)}}.
\]

Defining the ratio among the Hankel determinants of adjacent rank,

\[
r_n^{(\alpha)} := \frac{\Delta_n^{(\alpha)}}{\Delta_{n-1}^{(\alpha)}}, \quad r_1^{(\alpha)} = \frac{\Delta_1^{(\alpha)}}{\Delta_0^{(\alpha)}} = \mu_0^{(\alpha)} = \frac{1}{(\gamma_0^{(\alpha)})^2},
\]

we see that

\[
\left(a_n^{(\alpha)}\right)^2 = \frac{r_n^{(\alpha)}}{\Delta_n^{(\alpha)}}, \quad \text{and} \quad r_n^{(\alpha)} = r_1^{(\alpha)} \prod_{k=1}^{n-1} \left(a_k^{(\alpha)}\right)^2.
\]

Now we can substitute the solutions of the recursive coefficients, Eq.\((\text{3.18})\), for \(\left(a_k^{(\alpha)}\right)^2\) and we get

\[
r_n^{(\alpha)} = r_1^{(\alpha)} q^{\sum_{j=1}^{n-1}(2j+\alpha-1)} \left(\prod_{l=1}^{n-1} (1 - q^l)(1 - q^{l+\alpha})\right)
\]

\[
= r_1^{(\alpha)} q^{(\alpha-1)(n+\alpha-1)} \left(\prod_{l=1}^{n-1} (1 - q^l)(1 - q^{\alpha+l})\right).
\]
Next, we compute the Hankel determinant

\[
\Delta_n^{(\alpha)} = \Delta_1^{(\alpha)} \prod_{k=2}^{n} r_n^{(\alpha)}
\]

\[
= \Delta_1^{(\alpha)} \left( r_1^{(\alpha)} \right)^{n-1} q^{\sum_{j=2}^{n} (j-1)(j+\alpha-1)} \prod_{m=1}^{n-1} \prod_{l=1}^{m} \left( (1 - q^l)(1 - q^{l+\alpha}) \right)
\]

\[
= \left( \Delta_1^{(\alpha)} \right)^n q^{\frac{n(n-1)(2\alpha+3\alpha-1)}{6}} \prod_{j=1}^{n-1} (1 - q^j)^{n-j} (1 - q^{j+\alpha})^{n-j}.
\]

(4.8)

The value of zeroth moment, \(\mu_0^{(0)} = \Delta_1^{(0)}\), for the \(q\)-Laguerre weight is given in Appendix C.

4.2. Hankel determinant for the \(q\)-Hermite ensemble (for \(\kappa = 1\)).

Following the same definitions, we compute the Hankel determinant associated with the \(q\)-Hermite ensemble. We recall the solutions of the recursive coefficients, Eqs.(3.49),(3.50),

\[
\left( A_{2l}^{(\alpha)}(q) \right)^2 = q^{2(l+\alpha)}(1 - q^{2l}),
\]

\[
\left( A_{2l-1}^{(\alpha)}(q) \right)^2 = q^{2(l-1)}(1 - q^{2l+2\alpha}),
\]

and define the ratios among adjacent Hankel determinants, \(R_n^{(\alpha)} := \frac{\Delta_n^{(\alpha)}}{\Delta_{n-1}^{(\alpha)}}\), we get

\[
R_{2m}^{(\alpha)} = R_1^{(\alpha)} \prod_{l=1}^{2m-1} \left( A_l^{(\alpha)} \right)^2
\]

\[
= R_1^{(\alpha)} \left\{ \prod_{j=1}^{m-1} \left( A_{2j}^{(\alpha)} \right)^2 \right\} \left\{ \prod_{l=1}^{m} \left( A_{2l-1}^{(\alpha)} \right)^2 \right\}
\]

\[
= R_1^{(\alpha)} q^{2\sum_{j=1}^{m-1} (j+\alpha)} \left( \prod_{k=1}^{m-1} (1 - q^{2k}) \right) q^2 \sum_{i=1}^{m} (l-1) \left( \prod_{l=1}^{m} (1 - q^{2l+2\alpha}) \right)
\]

\[
= R_1^{(\alpha)} q^{2(m+\alpha)(m-1)} \prod_{k=1}^{m-1} (1 - q^{2k}) \prod_{l=1}^{m} (1 - q^{2l+2\alpha}).
\]

(4.9)
Similarly,

\[ R_{2m-1}^{(\alpha)} = R_1^{(\alpha)} \prod_{l=1}^{2m-2} \left( A_l^{(\alpha)} \right)^2 \]

\[ = R_1^{(\alpha)} \left\{ \prod_{j=1}^{m-1} \left( A_{2j}^{(\alpha)} \right)^2 \right\} \left\{ \prod_{l=1}^{m-1} \left( A_{2l-1}^{(\alpha)} \right)^2 \right\} \]

\[ = R_1^{(\alpha)} q^{(m+2\alpha)(m-1)} \left( \prod_{k=1}^{m-1} (1 - q^{2k}) \right) q^{(m-2)(m-1)} \left( \prod_{l=1}^{m-1} (1 - q^{2l+2\alpha}) \right) \]

\[ = R_1^{(\alpha)} q^{2(m-1)(m+\alpha-1)} \prod_{k=1}^{m-1} (1 - q^{2k})(1 - q^{2k+2\alpha}). \quad (4.10) \]

From the results of Eqs. (4.9), (4.10), we can then derive explicit expressions for the Hankel determinants as a finite product. For the even case,

\[ \Delta_{2j}^{(\alpha)} = \Delta_1^{(\alpha)} \prod_{k=2}^{2j} R_k^{(\alpha)} \]

\[ = \Delta_1^{(\alpha)} \left( \prod_{k=1}^{j} R_{2k}^{(\alpha)} \right) \left( \prod_{l=1}^{j-1} R_{2l+1}^{(\alpha)} \right) \]

\[ = \left( \Delta_1^{(\alpha)} \right)^{2j} q^{j(\alpha-1)(4j+6\alpha+1)} \left( \prod_{k=1}^{j-1} (1 - q^{2k})^{2j-2k} \right) \left( \prod_{l=1}^{j} (1 - q^{2l+2\alpha})^{2j-2l+1} \right). \quad (4.11) \]

The odd case can be easily computed

\[ \Delta_{2j-1}^{(\alpha)} = \frac{\Delta_{2j}^{(\alpha)}}{R_{2j}^{(\alpha)}} \]

\[ = \left( \Delta_1^{(\alpha)} \right)^{2j-1} q^{\frac{j-1}{3} [4j^2 + (6\alpha - 5)j - 6\alpha]} \left( \prod_{k=1}^{j-1} (1 - q^{2k})^{2j-2k-1} (1 - q^{2k+2\alpha})^{2j-2k} \right). \]

As a consistent check, we specialize our results to \( \alpha = 0 \) and \( \alpha = -\frac{1}{2} \).
\( \alpha = 0 : \)

\[
R_{2j}^{(0)} = R_1^{(0)} q^{2j(j-1)} (1 - q^{2j}) \prod_{k=1}^{j-1} (1 - q^{2k})^2,
\]

\( R_{2j-1}^{(0)} = R_1^{(0)} q^{2(j-1)} \prod_{k=1}^{j-1} (1 - q^{2k})^2, \) \hspace{1cm} (4.13)

\[
\Delta_{2j}^{(0)} = \left( \Delta_1^{(0)} \right)^{2j} q^{j(j-1)(j+1)} \prod_{k=1}^{j-1} ((1 - q^{2k})^{4j-4k+1})^{(1-q^{2j})}, \]

\[
\Delta_{2j-1}^{(0)} = \left( \Delta_1^{(0)} \right)^{2j-1} q^{j(j-1)(j-5)} \prod_{k=1}^{j-1} (1 - q^{2k})^{4j-4k-1}. \]

\( \alpha = -\frac{1}{2} : \)

\[
R_{2j}^{(-\frac{1}{2})} = R_1^{(-\frac{1}{2})} q^{(2j-1)(j-1)} \prod_{k=1}^{2j-1} (1 - q^{k}), \]

\( R_{2j-1}^{(-\frac{1}{2})} = R_1^{(-\frac{1}{2})} q^{(2j-3)(j-1)} \prod_{k=1}^{2j-2} (1 - q^{k}), \)

\[
\Delta_{2j}^{(-\frac{1}{2})} = \left( R_1^{(-\frac{1}{2})} \right)^{2j} q^{2j(j-1)(j+1)} \prod_{k=1}^{2j-1} ((1 - q^{k})^{2j-k}), \]

\[
\Delta_{2j-1}^{(-\frac{1}{2})} = \left( R_1^{(-\frac{1}{2})} \right)^{2j-1} q^{j(j-1)(j-3)(j-5)} \prod_{k=1}^{2j-2} (1 - q^{k})^{2j-k-1}. \]

The value of zeroth moment, \( \mu_0^{(-\frac{1}{2})} = \Delta_1^{(-\frac{1}{2})} \), for the \( q \)-Hermite weight is given in Appendix C.

5. \( q \)-Generalization of the Toda equations from \( \kappa \)-deformation of the \( q \)-Laguerre/Hermite ensembles

5.1. \( q \)-Difference equations for the recursive coefficients of the \( q \)-Laguerre polynomials.

In this section, we study the \( q \)-difference equations describing the \( \kappa \) dependence of the recursive coefficients \( a_n^{(\alpha)}(\kappa), b_n^{(\alpha)}(\kappa) \) associated with the generalized \( q \)-Laguerre ensemble. In the classical case, such equations correspond to the Lax pair formulation of the Toda equations [13]. Hence, our results provide a \( q \)-generalization of the classical Toda equation.
To achieve this, we introduce the Fourier expansion (w.r.t $\kappa$ variable) of the $q$-derivative on the $q$-Laguerre orthonormal polynomials,

$$D_q^\kappa p_n^{(a)}(x, \kappa) = \sum_{j=0}^{n} p_j^{(a)}(x, \kappa)\xi_{jn}^{(a)}(\kappa). \quad (5.1)$$

Recalling the definition of the $q$-derivative, we can also transform this expansion formula as a $q$-shifting relation ($\lambda := (1 - q)\kappa$):

$$p_n^{(a)}(x, q\kappa) = p_n^{(a)}(x, \kappa)[1 - \lambda\xi_{nn}^{(a)}(\kappa)] - \lambda \sum_{j=0}^{n-1} p_j^{(a)}(x, \kappa)\xi_{jn}^{(a)}(\kappa). \quad (5.2)$$

Next, we compute the $q$-derivative with respect to the $\kappa$ variable on the action of position operator, $xp_n^{(a)}(x, \kappa)$, in two ways:

We first compute the $q$-derivative with respect to $\kappa$ on the recursive relation,

$$D_q^\kappa[xp_n^{(a)}(x, \kappa)] = D_q^\kappa[a_n^{(a)}(\kappa)p_n^{(a)}(x, \kappa) + b_n^{(a)}(\kappa)p_n^{(a)}(x, \kappa) + a_n^{(a)}(\kappa)p_{n-1}^{(a)}(x, \kappa)]$$

= $[D_q^\kappa a_n^{(a)} + \xi_{n+1,n}^{(a)}b_{n+1}^{(a)}]p_{n+1}^{(a)} + [D_q^\kappa b_n^{(a)} + \xi_{n+1,n}^{(a)}a_{n+1}^{(a)} + \xi_{nn}^{(a)}\bar{b}_n^{(a)}]p_n^{(a)}$

+ $[D_q^\kappa a_n^{(a)} + \xi_{n-1,n}^{(a)}\bar{a}_n^{(a)} + \xi_{n-1,n}^{(a)}\bar{a}_n^{(a)}]p_{n-1}^{(a)} + [\xi_{n-1,n}^{(a)}a_{n-1}^{(a)}]p_{n-1}^{(a)} + [\xi_{n-2,n-1}^{(a)}\bar{a}_n^{(a)}]p_{n-2}^{(a)}. \quad (5.3)$

Here $\xi_{mn}^{(a)}$ are the Fourier coefficients (matrix elements) of Eq. (5.1),

$$\bar{a}_n^{(a)} := a_n^{(a)}(q\kappa), \quad \bar{b}_n^{(a)} := b_n^{(a)}(q\kappa)$$

are the rescaled recursive coefficients, and we suppress the dependence on $\kappa$ for simplicity.

On the other hand, since $D_q^\kappa$ commutes with the position operator $x$, we first compute the $q$-derivative (w.r.t $\kappa$ variable) of the orthonormal polynomials and then apply the position operator.

$$xD_q^\kappa[p_n^{(a)}(x, \kappa)]$$

= $[\xi_{mn}^{(a)}a_{n+1}^{(a)}]p_{n+1}^{(a)} + [\xi_{n-1,n}^{(a)}a_n^{(a)} + \xi_{nn}^{(a)}\bar{b}_n^{(a)}]p_n^{(a)}$

+ $[\xi_{n-1,n}^{(a)}a_n^{(a)} + \xi_{n-1,n}^{(a)}\bar{b}_n^{(a)}]p_{n-1}^{(a)} + [\xi_{n-1,n}^{(a)}a_{n-1}^{(a)}]p_{n-2}. \quad (5.4)$

By comparing the corresponding coefficients of each orthonormal polynomials as calculated in Eqs. (5.3), (5.4), we get the following set of relations:

$$D_q^\kappa a_n^{(a)} = \xi_{n-1,n-1}^{(a)}a_n^{(a)} - \xi_{mn}^{(a)}a_n^{(a)}, \quad (5.5)$$

$$D_q^\kappa b_n^{(a)} = \xi_{nn}^{(a)}b_n^{(a)} - \bar{b}_n^{(a)} + [\xi_{n-1,n}^{(a)}a_n^{(a)} - \xi_{n-1,n}^{(a)}\bar{a}_n^{(a)}], \quad (5.6)$$

$$D_q^\kappa a_n^{(a)} = \xi_{n-1,n}^{(a)}b_n^{(a)} - \bar{b}_n^{(a)} + [\xi_{n-1,n}^{(a)}a_n^{(a)} - \xi_{n-1,n}^{(a)}\bar{a}_n^{(a)}], \quad (5.7)$$

$$\xi_{n-2,n-1}^{(a)} = \xi_{n-1,n}^{(a)}. \quad (5.8)$$
Note that the last result allows us to replace the rescaled recursive coefficients \( \bar{a}_n^{(\alpha)} \) in terms of the Fourier coefficients and the unscaled recursive coefficients

\[
a_n^{(\alpha)}(q\kappa) = \frac{\xi_{n-1,n}^{(\alpha)}(\kappa)}{\xi_{n-2,n-1}^{(\alpha)}(\kappa)} a_{n-1}^{(\alpha)}(\kappa) = \frac{1 - \lambda \xi_{n-1,n-1}^{(\alpha)}}{1 - \lambda \xi_{nn}^{(\alpha)}} a_n^{(\alpha)}(\kappa),
\]

where the second equality of the above relation follows from Eq. (5.5). We have also checked that Eqs. (5.5), (5.7) are compatible.

Finally, after some manipulations, we get the coupled \( q \)-difference equations.

\[
\mathcal{D}_q^\kappa a_n^{(\alpha)}(\kappa) = \frac{\xi_{n-1,n-1}^{(\alpha)}(\kappa) - \xi_{nn}^{(\alpha)}(\kappa)}{1 - \lambda \xi_{nn}^{(\alpha)}(\kappa)} a_n^{(\alpha)}(\kappa)
\]

\[
\mathcal{D}_q^\kappa b_n^{(\alpha)}(\kappa) = \frac{q}{1 - q} \left[ \left( \frac{a_n^{(\alpha)}(\kappa)}{1 - \lambda \xi_{nn}^{(\alpha)}} \right)^2 - \left( \frac{a_{n+1}^{(\alpha)}(\kappa)}{1 - \lambda \xi_{n+1,n+1}^{(\alpha)}} \right)^2 \right].
\]

Our next task is to find an expression relating \( \xi_{mn}^{(\alpha)} \) in terms of the recursive coefficients \( a_n^{(\alpha)}, b_n^{(\alpha)} \). By taking \( q \)-derivative w.r.t \( \kappa \) variable on the orthonormal condition, and recalling the \( q \)-Leibniz rule, (A.6), we can derive a master equation among these Fourier coefficients.

\[
\mathcal{D}_q^\kappa \left[ \int_0^1 p_m^{(\alpha)}(x, \kappa) p_n^{(\alpha)}(x, \kappa) v^{(\alpha)}(x, \kappa) d_q x \right] = 0.
\]

This implies for \( m \leq n \),

\[
(1 - \lambda \xi_{mn}^{(\alpha)}) \xi_{mn}^{(\alpha)} - \lambda \sum_{j=0}^{m-1} \xi_{jm}^{(\alpha)} \xi_{jn}^{(\alpha)} + \delta_{mn} \xi_{mn}^{(\alpha)} = \delta_{m,n-1} \left[ \frac{q}{1 - q} \bar{a}_n^{(\alpha)} \right] + \delta_{mn} \left[ \frac{q}{1 - q} \bar{b}_n^{(\alpha)} \right].
\]

From these results, we can exact useful information by specifying the value of \( m \):

**Case 1**: \( m < n - 1 \iff m + 2 \leq n \)

We find, by induction, \( \xi_{mn}^{(\alpha)}(\kappa) = 0 \), if \( m \leq n - 2 \). Consequently, there are only two terms in the \( q \)-derivative (w.r.t \( \kappa \) variable) of the \( q \)-Laguerre orthonormal polynomials,

\[
\mathcal{D}_q^\kappa p_m^{(\alpha)}(x, \kappa) = p_m^{(\alpha)}(x, \kappa) \xi_{mn}^{(\alpha)}(\kappa) + p_{m-1}^{(\alpha)} \xi_{m-1,n}^{(\alpha)}(\kappa).
\]

**Case 2**: \( m = n - 1 \)

In this case, we relate the two Fourier coefficients as follows:

\[
\xi_{n-1,n}^{(\alpha)} = \frac{q}{1 - q} \bar{a}_n^{(\alpha)} = \left( \frac{q}{1 - q} \right) \frac{a_n^{(\alpha)}}{1 - \lambda \xi_{nn}^{(\alpha)}}.
\]

**Case 3**: \( m = n \)

By suitable rearrangements, we derive a recursive equation relating the diagonal
Fourier coefficients $\xi^{(\alpha)}_{nn}$ to the recursive coefficients $a^{(\alpha)}_n, b^{(\alpha)}_n$ as follows:

\[(1 - \lambda \xi^{(\alpha)}_{nn})^2 + \frac{(q\kappa b^{(\alpha)}_n)^2}{(1 - \lambda \xi^{(\alpha)}_{n-1,n-1})^2} = 1 - q\kappa \bar{b}^{(\alpha)}_n. \quad (5.16)\]

By using Eq.(5.19), we can rewrite Eq.(5.16) as a quadratic equation for $(1 - \lambda \xi^{(\alpha)}_{nn})^2$,

\[(1 - \lambda \xi^{(\alpha)}_{nn})^2 + \frac{(q\kappa a^{(\alpha)}_n)^2}{(1 - \lambda \xi^{(\alpha)}_{nn})^2} = 1 - q\kappa \bar{b}^{(\alpha)}_n. \quad (5.17)\]

From the solution of this equation, we then obtain an expression of $\xi^{(\alpha)}_{nn}$ in terms of $a^{(\alpha)}_n$ and $\bar{b}^{(\alpha)}_n$,

\[\xi^{(\alpha)}_{nn} = \frac{1}{2(1-q)\kappa} \left\{ 2 - \sqrt{(1-q\kappa \bar{b}^{(\alpha)}_n)} + \sqrt{1 - 2q\kappa \bar{b}^{(\alpha)}_n + 4(q\kappa)^2 \left[ (\bar{b}^{(\alpha)}_n)^2 - (a^{(\alpha)}_n)^2 \right]} \right\}. \quad (5.18)\]

Substituting this expression back to Eqs.(5.10), (5.11), we then obtain a set of closed $q$-difference equations for the recursive coefficients, $a^{(\alpha)}_n$ and $b^{(\alpha)}_n$, of the $q$-generalized Laguerre ensemble.

5.2. $q$-Difference equations for the recursive coefficients of the $q$-Hermite orthonormal polynomials.

In this section, we shall derive the $q$-difference equation for the recursive coefficients of the $q$-Hermite orthonormal polynomials. In order to achieve this, we need to introduce the Fourier coefficients of the $q$-derivative of the $q$-Hermite orthonormal polynomials with respect to parameter $\kappa$,

\[D^\kappa_P^{(\alpha)}(x; \kappa) = \sum_{j=0}^{n} P^{(\alpha)}_j(x; \kappa) \xi^{(\alpha)}_{jn}(\kappa), \quad (n - j \text{ is even}). \quad (5.19)\]

Note that, by recalling the definition of the $q$-derivative, (A.2), we can also transform the equation above into the Fourier expansion of the $q$-evolved $q$-Hermite orthonormal polynomials with respect to parameter $\kappa$,

\[P^{(\alpha)}_n(x; q\kappa) = [1 - \lambda \xi^{(\alpha)}_{nn}] P^{(\alpha)}_n(x; \kappa) - \lambda \sum_{j=0}^{n-1} P^{(\alpha)}_j(x; \kappa) \xi^{(\alpha)}_{jn}(\kappa). \quad (5.20)\]
Next, we compute the $D_q^\kappa$ derivative on the product $xP_n^{(\alpha)}(x; \kappa)$ in two ways.

\[
D_q^\kappa[xP_n^{(\alpha)}(x; \kappa)] = D_q^\kappa[A_{n+1}^{(\alpha)}(x; \kappa) + A_n^{(\alpha)}(x; \kappa)]P_n^{(\alpha)}(x; \kappa) + A_n^{(\alpha)}(x; \kappa)P_{n-1}^{(\alpha)}(x; \kappa) = x \sum_{j=0}^n P_j^{(\alpha)}(x; \kappa) \Xi_{jn}^{(\alpha)}(\kappa)
\]

\[
= \sum_{j=0}^n [A_{j+1}^{(\alpha)}(x; \kappa) + A_j^{(\alpha)}(x; \kappa)] P_{j-1}^{(\alpha)}(x; \kappa) \Xi_{jn}^{(\alpha)}(\kappa). \tag{5.21}
\]

By comparing the coefficients of $P_n^{(\alpha)}(x; \kappa)$ of the first and the third lines of the previous equation, we get the following results:

\[
D_q^\kappa A_{n+1}^{(\alpha)}(\kappa) = \Xi_{jn}^{(\alpha)}(\kappa) A_{n+1}^{(\alpha)}(\kappa) - \Xi_{n+1,n+1}^{(\alpha)}(\kappa) A_{n+1}^{(\alpha)}(q\kappa), \tag{5.22}
\]

which implies

\[
D_q^\kappa A_n^{(\alpha)}(\kappa) = \frac{\Xi_{n-1,n-1}^{(\alpha)}(\kappa) - \Xi_{nn}^{(\alpha)}(\kappa)}{1 + \kappa(q - 1)\Xi_{nn}^{(\alpha)}(\kappa)} A_n^{(\alpha)}(\kappa), \tag{5.23}
\]

and

\[
\frac{A_n^{(\alpha)}(q\kappa)}{1 - \lambda \Xi_{n-1,n-1}^{(\alpha)}(\kappa)} = \frac{A_n^{(\alpha)}(\kappa)}{1 - \lambda \Xi_{nn}^{(\alpha)}(\kappa)}. \tag{5.24}
\]

Our next task is to derive a set of algebraic equations for the Fourier coefficients $\Xi_{jn}^{(\alpha)}(\kappa)$. By taking the $q$-derivative (w.r.t $\kappa$) on the orthonormal condition

\[
D_q^\kappa \left( \int_{-1}^1 P_m^{(\alpha)}(x; \kappa) P_n^{(\alpha)}(x; \kappa) \omega^{(\alpha)}(x; \kappa) d_q x \right) = 0, \tag{5.25}
\]

we get (assuming $m \leq n$)

\[
\int_{-1}^1 \left(D_q^\kappa P_m^{(\alpha)}(x; \kappa)\right) P_n^{(\alpha)}(x; \kappa) \omega^{(\alpha)}(x; \kappa) d_q x
\]

\[
+ \int_{-1}^1 P_m^{(\alpha)}(x; q\kappa) \left(D_q^\kappa P_n^{(\alpha)}(x; \kappa)\right) \omega^{(\alpha)}(x; \kappa) d_q x
\]

\[
+ \int_{-1}^1 P_m^{(\alpha)}(x; q\kappa) P_n^{(\alpha)}(x; q\kappa) \left(D_q^\kappa \omega^{(\alpha)}(x; \kappa)\right) d_q x = 0. \tag{5.26}
\]
Substituting the Fourier expansions, Eqs. (5.19) and (5.20), into the first two terms of (5.26), and using the Pearson relation (in the $\kappa$ variable) for the $q$-Hermite weight, we get

$$
\begin{align*}
\delta_{mn} \Xi_m^{(a)} + (1 - \lambda \Xi_m^{(a)}) \Xi_m^{(a)} - \lambda \sum_{j=0}^{m-1} \Xi_j^{(a)} \Xi_j^{(a)} &+ \delta_{mn} \left( \frac{q^2 \kappa}{q - 1} \right) \left\{ \left( \bar{A}_{n+1}^{(a)} \right)^2 + \left( \bar{A}_n^{(a)} \right)^2 \right\} + \delta_{m,n-2} \left( \frac{q^2 \kappa}{q - 1} \right) \bar{A}_{n+1}^{(a)} \bar{A}_n^{(a)} = 0.
\end{align*}
$$

(5.27)

In order to illustrate the content of this equation, we consider the following specializations:

Case 1: $m < n - 2 \iff m + 3 \leq n$

In this case, the master equation reduces to

$$(1 - \lambda \Xi_{mn}^{(a)}) \Xi_{mn}^{(a)} - \lambda \sum_{j=0}^{m-1} \Xi_{jm}^{(a)} \Xi_{jn}^{(a)} = 0.$$ 

By the mathematical induction, we show that $\Xi_{mn}^{(a)} = 0$, if $3 \leq n - m$. Consequently, the Fourier expansion of the $q$-derivative (w.r.t $\kappa$ variable) on the $q$-Hermite orthonormal polynomials only consist of two terms:

$$
\mathcal{D}_\kappa P_n^{(a)}(x, \kappa) = \bar{P}_n^{(a)}(x, \kappa) \Xi_{mn}^{(a)}(\kappa) + \bar{P}_{n-2}(x, \kappa) \Xi_{mn-2,n}^{(a)}(\kappa).
$$

(5.28)

Case 2: $m = n - 2 \iff m - 1 = n - 3$

In this case, the master equation reduces to

$$
(1 - \lambda \Xi_{n-2,n-2}^{(a)} \Xi_{n-2,n}^{(a)} - \lambda \sum_{j=0}^{n-3} \Xi_{j,n-2}^{(a)} \Xi_{jn}^{(a)} + \frac{q^2 \kappa}{q - 1} \bar{A}_n^{(a)} \bar{A}_{n-1}^{(a)} = 0.
$$

Since we have showed that $\Xi_{jn}^{(a)} = 0$ for $j \leq n - 3$ in Case 1, we can use the equation above to express the off-diagonal Fourier coefficient in terms of the diagonal ones:

$$
\Xi_{n-1,n+1}^{(a)} = \left( \frac{q^2 \kappa}{1 - q} \right) \frac{\bar{A}_{n+1}^{(a)} \bar{A}_n^{(a)}}{1 - \lambda \Xi_{n-1,n-1}^{(a)}} = \left( \frac{q^2 \kappa}{1 - q} \right) \frac{\bar{A}_{n+1}^{(a)} \bar{A}_n^{(a)}}{1 - \lambda \Xi_{n-1,n+1}^{(a)}}.
$$

(5.29)

For the second equality of the equation above, we use Eq. (5.24) to replace $\bar{A}_n^{(a)}$ in terms of $A_n^{(a)}$.

Case 3: $m = n - 1, m - 1 = n - 2$

Due to the parity preserving property associated with the $q$-Hermite ensemble, $\Xi_{n-1,n}^{(a)} = 0$, we have no constraint in this case.

Case 4: $m = n$

In this case, we have

$$
\Xi_{nn}^{(a)} + (1 - \lambda \Xi_{nn}^{(a)}) \Xi_{nn}^{(a)} - \lambda (\Xi_{n-2,n}^{(a)})^2 = \left( \frac{q^2 \kappa}{1 - q} \right) \left[ \left( \bar{A}_{n+1}^{(a)} \right)^2 + \left( \bar{A}_n^{(a)} \right)^2 \right].
$$

(5.30)
After suitable rearrangement, we get

\[
(1 - \lambda \Xi^{(a)}_{nn})^2 + \frac{(q\kappa)^4 \left( \bar{A}_n^{(a)} \right)^2 \left( A_{n-1}^{(a)} \right)^2}{(1 - \lambda \Xi^{(a)}_{n-2,n-2})^2} = 1 - (q\kappa)^2 \left\{ \left( \bar{A}_{n+1}^{(a)} \right)^2 + (\bar{A}_n^{(a)})^2 \right\}. \tag{5.31}
\]

Similar to the case of the \(q\)-Laguerre ensemble Eq. (5.16), we can solve \((1 - \lambda \Xi^{(a)}_{nn})^2\) as a continued fraction in terms of the recursive coefficient \(\bar{A}_n^{(a)}\).

On the other hand, by replacing \(\bar{A}_n^{(a)}\) into \(A_n^{(a)}\), using Eq. (5.24), we can derive a quadratic equation for \((1 - \lambda \Xi^{(a)}_{nn})^2\),

\[
(1 - \lambda \Xi^{(a)}_{nn})^2 + \frac{(q\kappa)^4 \left( \bar{A}_n^{(a)} \right)^2 \left( A_{n-1}^{(a)} \right)^2}{(1 - \lambda \Xi^{(a)}_{n-2,n-2})^2} = 1 - (q\kappa)^2 \left\{ \left( \bar{A}_{n+1}^{(a)} \right)^2 + (\bar{A}_n^{(a)})^2 \right\}. \tag{5.32}
\]

Hence, \(\Xi^{(a)}_{nn}\) can be solved in terms of \(A_n^{(a)}\) and \(\bar{A}_n^{(a)}\), and substituting the solution of \(\Xi^{(a)}_{nn}\) for Eq. (5.32) back to Eq. (5.23), we get closed \(q\)-difference equations for the recursive coefficients of the generalized \(q\)-Hermite ensemble.

5.3. On the compatibility of the quadratic relation and \(q\)-Toda equations.

In this section, we check the compatibility between the quadratic relation Eqs. (2.17), (2.18), (2.19), (2.20) and the \(q\)-Toda equation Eqs. (5.10), (5.11), (5.23). To see this, we first example the Fourier coefficients of the \(q\)-derivative of the orthonormal \(q\)-Laguerre/Hermite polynomials (w.r.t \(\kappa\)).

**Theorem 5.1.** The Fourier coefficients of the \(q\)-derivative (w.r.t \(\kappa\)) of the orthonormal \(q\)-Laguerre/Hermite polynomials (Eqs. 5.1, 5.19) are related by the quadratic relations

\[
\Xi^{(a)}_{2n,2n}(\kappa, q) = (1 + q)\kappa \xi^{(a)}_{nn}(\kappa^2, q^2),
\]

\[
\Xi^{(a)}_{2n+1,2n+1}(\kappa, q) = (1 + q)\kappa \xi^{(a+1)}_{nn}(\kappa^2, q^2).
\]
Proof. We relate the $q$-derivative (w.r.t $\kappa$) of the orthonormal $q$-Hermite polynomials Eq. [2.20] to that of the $q$-Laguerre polynomials in two ways. First of all, for even polynomials

$$
\mathcal{D}_q^\kappa P^{(\alpha)}_{2n}(x; \kappa, q)
= \frac{P^{(\alpha)}_{2n}(x; \kappa, q) - P^{(\alpha)}_{2n}(x; q\kappa, q)}{(1 - q)\kappa}
= \frac{p^{(\alpha)}_n(x^2; \kappa^2, q^2) - p^{(\alpha)}_n(x^2; q^2\kappa^2, q^2)}{(1 - q^2)\kappa^2} \frac{1 + q}{\sqrt{2}}\kappa
= \frac{(1 + q)^{\frac{3}{2}}\kappa}{\sqrt{2}} \left[ \mathcal{D}^2_{q^2} p^{(\alpha)}_n(x^2; \kappa^2, q^2) \right]
= \frac{(1 + q)^{\frac{3}{2}}\kappa}{\sqrt{2}} \left[ p^{(\alpha)}_n(x^2; \kappa^2, q^2) \xi^{(\alpha)}_{nn}(\kappa^2, q^2) + p^{(\alpha)}_{n-1}(x^2; \kappa^2, q^2) \xi^{(\alpha)}_{n-1,n}(\kappa^2, q^2) \right]. \tag{5.33}
$$

On the other hand, if we write the Fourier expansion of the $q$-derivative (w.r.t $\kappa$) of the $q$-Hermite polynomials

$$
\mathcal{D}_q^\kappa P^{(\alpha)}_{2n+1}(x; \kappa, q)
= P^{(\alpha)}_{2n+1}(x; \kappa, q) \Xi^{(\alpha)}_{2n,2n+1}(\kappa, q) + P^{(\alpha)}_{2n}(x; \kappa, q) \Xi^{(\alpha)}_{2n-2,2n}(\kappa, q)
= \sqrt{\frac{1 + q}{2}} p^{(\alpha)}_n(x^2; \kappa^2, q^2) \Xi^{(\alpha)}_{2n,2n}(\kappa, q) + \sqrt{\frac{1 + q}{2}} p^{(\alpha)}_{n-1}(x^2; \kappa^2, q^2) \Xi^{(\alpha)}_{2n-2,2n}(\kappa, q).
\tag{5.34}
$$

By comparing the two results Eqs. (5.33), (5.34), we obtain

$$
\Xi^{(\alpha)}_{2n,2n}(\kappa, q) = (1 + q)\kappa \xi^{(\alpha)}_{nn}(\kappa^2, q^2),
\tag{5.35}
\Xi^{(\alpha)}_{2n-2,2n}(\kappa, q) = (1 + q)\kappa \xi^{(\alpha)}_{n-1,n}(\kappa^2, q^2). \tag{5.36}
$$

Next, we compare the odd $q$-Hermite polynomials.

$$
\mathcal{D}_q^\kappa P^{(\alpha)}_{2n+1}(x; \kappa, q)
= \frac{(1 + q)^{\frac{3}{2}}\kappa}{\sqrt{2}} \left[ \mathcal{D}^2_{q^2} P^{(\alpha+1)}_n(x^2; \kappa^2, q^2) \right]
= \frac{(1 + q)^{\frac{3}{2}}\kappa}{\sqrt{2}} \left[ p^{(\alpha+1)}_n(x^2; \kappa^2, q^2) \xi^{(\alpha+1)}_{nn}(\kappa^2, q^2) + p^{(\alpha+1)}_{n-1}(x^2; \kappa^2, q^2) \xi^{(\alpha+1)}_{n-1,n}(\kappa^2, q^2) \right]. \tag{5.37}
$$
and
\[
D_q^\kappa P_{2n+1}^{(\alpha)}(x; \kappa, q)
= P_{2n+1}^{(\alpha)}(x; \kappa, q) \Xi_{2n+1,2n+1}^{(\alpha)}(\kappa, q) + P_{2n-1}^{(\alpha)}(x; \kappa, q) \Xi_{2n-1,2n+1}^{(\alpha)}(\kappa, q)
= \sqrt{\frac{1 + q}{2}} x p_n^{(\alpha+1)}(x^2; \kappa^2, q^2) \Xi_{2n+1,2n+1}^{(\alpha)}(\kappa, q) + \sqrt{\frac{1 + q}{2}} x p_{n-1}^{(\alpha+1)}(x^2; \kappa^2, q^2) \Xi_{2n-1,2n+1}^{(\alpha)}(\kappa, q).
\]

(5.38)

By comparing the two results, Eqs. (5.37), (5.38), we obtain
\[
\Xi_{2n+1,2n+1}^{(\alpha)}(\kappa, q) = (1 + q) \kappa \xi_{nn}^{(\alpha+1)}(\kappa^2, q^2),
\]

(5.39)
\[
\Xi_{2n-1,2n+1}^{(\alpha)}(\kappa, q) = (1 + q) \kappa \xi_{n-1,n}^{(\alpha+1)}(\kappa^2, q^2).
\]

(5.40)

In fact, the quadratic relation among the Fourier coefficients of the \(q\)-Laguerre/Hermite polynomials are equivalent to the quadratic relation among the recursive coefficients of the \(q\)-Laguerre/Hermite polynomials. To see this, we rewrite the Eq. (5.29) (set \(n = m - 1\)) as
\[
A_{m}^{(\alpha)}(\kappa, q) A_{m-1}^{(\alpha)}(\kappa, q) = \frac{1 - q}{q^2 \kappa} \Xi_{m-2,m}^{(\alpha)}(\kappa, q) \left[ 1 - (1 - q) \kappa \Xi_{mm}^{(\alpha)}(\kappa, q) \right].
\]

(5.41)

For \(m = 2n\), after substituting Eqs. (5.35), (5.36) and using Eq. (5.15), we get
\[
A_{2n}^{(\alpha)}(\kappa, q) A_{2n-1}^{(\alpha)}(\kappa, q) = \frac{1 - q^2}{q^2} \xi_{n-1,n}^{(\alpha)}(\kappa^2, q^2) \left[ 1 - (1 - q^2) \kappa^2 \xi_{nn}^{(\alpha)}(\kappa^2, q^2) \right] = a_n^{(\alpha)}(\kappa^2, q^2).
\]

(5.42)

For \(m = 2n + 1\), after substituting Eqs. (5.39), (5.40) and using Eq. (5.15), we get
\[
A_{2n+1}^{(\alpha)}(\kappa, q) A_{2n}^{(\alpha)}(\kappa, q) = \frac{1 - q^2}{q^2} \xi_{n-1,n}^{(\alpha+1)}(\kappa^2, q^2) \left[ 1 - (1 - q^2) \kappa^2 \xi_{nn}^{(\alpha+1)}(\kappa^2, q^2) \right] = a_n^{(\alpha+1)}(\kappa^2, q^2).
\]

(5.43)

Similarly, rewriting Eq. (5.31),
\[
(q \kappa)^2 \left[ \left( A_{m+1}^{(\alpha)}(q \kappa, q) \right)^2 + \left( A_m^{(\alpha)}(q \kappa, q) \right)^2 \right]
= 1 - [1 - (1 - q) \kappa \Xi_{mm}^{(\alpha)}(\kappa, q)]^2 - \frac{(q \kappa)^4 \left[ A_m^{(\alpha)}(\kappa, q) \right]^2 \left[ A_{m-1}^{(\alpha)}(\kappa, q) \right]^2}{[1 - (1 - q) \kappa \Xi_{mm}^{(\alpha)}(\kappa, q)]^2},
\]

(5.44)

then by substituting Eqs. (2.17), (2.19), (5.16), (5.35), (5.36), for either \(m = 2n\) or \(m = 2n + 1\), we reproduce Eqs. (2.18), (2.20).
6. Summary and Conclusion

In this paper, we study fundamental properties of the generalized $q$-Laguerre/Hermite ensembles with a deformation parameter $\kappa$.

For the special value $\kappa = 1$, we give explicit solutions of their recursive coefficients for the orthonormal polynomial systems. In addition, to show that these ensembles provide ”good” basis for a realization of the Heisenberg algebra, we also calculate the matrix elements (Fourier coefficients) of the dilation operator with respect of orthonormal polynomial basis explicitly.

In view of the intimate connections between matrix models and orthogonal polynomial systems [11], we compute the Hankel determinants as products of the recursive coefficients explicitly. The physical consequences for identifying the Hankel determinants as partition functions of these matrix models with non-polynomial type potential may worth further explorations.

Finally, we examine the deformation of the $q$-Laguerre/Hermite systems in the form of $q$-difference equations dictating the $\kappa$ dependence of the recurrence coefficients and Fourier coefficients. This provides a $q$-generalization of the Toda equations in the Lax pair formulation [13].

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Appendix A. Some Basic Definitions and Relations for $q$-Analysis ($0 < q < 1$)

In this section, we collect some basic definitions and formulas which are relevant to our discussions.

The $q$-integral for a function $f(x)$ over the region $x \in [0, a]$ is defined as

$$F(a) := \int_0^a f(x) d_q x := a(1 - q) \sum_{k=0}^{\infty} f(aq^k)q^k. \quad (A.1)$$

This is compatible with the definition of the $q$-derivative

$$D_q f(x) := \frac{f(qx) - f(x)}{qx - x} = \frac{f(x) - f(qx)}{x(1 - q)} \quad (A.2)$$

in the following senses:

1) Fundamental theorem of the $q$-calculus

$$D_q F(a) = f(a), \quad (A.3)$$

$$\int_0^a [D_q f(x)]d_q x = f(a) - f(0). \quad (A.4)$$

2) The linear change of variables can be implemented in $q$-integral:

$$\int_0^a f(cx)d_q x = \frac{1}{c} \int_0^{ca} f(y)d_q y. \quad (A.5)$$

There are some subtleties associated with the $q$-derivative, in particular, the $q$-Leibnitz rule is given as

$$D_q[f(x)g(x)] = \frac{f(qx)g(qx) - f(x)g(x)}{(q - 1)x}$$

$$= f(qx)[D_q g(x)] + [D_q f(x)]g(x)$$

$$= [D_q f(x)]g(qx) + f(x)[D_q g(x)]. \quad (A.6)$$
Appendix B. On the q-deformation of the Heisenberg Algebra

In the classical case, the Heisenberg algebra is generated by two operators: $\mathcal{D} := \frac{d}{dx}$ (translation) and $x$ (position). They satisfy the commutation relation,

$$[\mathcal{D}, x] = \left[ \frac{d}{dx}, x \right] = 1. \quad (B.1)$$

We can also include the dilation operator $\mathcal{S} := x \frac{d}{dx}$, such that

$$[\mathcal{S}, x] = \left[ x \frac{d}{dx}, x \right] = x, \quad (B.2)$$

and

$$[\mathcal{S}, \mathcal{D}] = \left[ x \frac{d}{dx}, \frac{d}{dx} \right] = -\frac{d}{dx}. \quad (B.3)$$

In this way, we can view the position $(x)$ and translation $(\frac{d}{dx})$ operators as raising and lowering operators with respect to the eigenfunctions of dilation.

To study the $q$-deformation of the Heisenberg algebra, which is useful in our computations of the recursive coefficients for the $q$-Laguerre and $q$-Hermite ensembles, we compute the action of three commutators on a general function $f(x)$: $(\mathcal{S}_q := x\mathcal{D}_q)$

**Theorem B.1.** The $q$-difference realization of the $q$-Heisenberg algebra is given by

$$[\mathcal{D}_q, x] f(x) = f(qx) \quad (B.4)$$

$$[\mathcal{S}_q, x] f(x) = x f(qx) \quad (B.5)$$

$$[\mathcal{S}_q, \mathcal{D}_q] f(x) = -\mathcal{D}_q f(qx). \quad (B.6)$$

**Proof.**

$$[\mathcal{D}_q, x] f(x) = \mathcal{D}_q (x f(x)) - x (\mathcal{D}_q f(x))$$

$$= \frac{q x f(qx) - x f(x)}{(q-1)x} - \frac{f(qx) - f(x)}{q-1}$$

$$= f(qx).$$

$$[\mathcal{S}_q, x] f(x) = \mathcal{S}_q (x f(x)) - x (\mathcal{S}_q f(x))$$

$$= \frac{q x f(qx) - x f(x)}{q-1} - \frac{x f(qx) - f(x)}{q-1}$$

$$= x f(qx).$$
\[ [S_q, D_q] f(x) = S_q (D_q f(x)) - D_q (S_q f(x)) \]
\[ = S_q \left( \frac{f(qx) - f(x)}{(q - 1)x} \right) - D_q \left( \frac{f(qx) - f(x)}{q - 1} \right) \]
\[ = - \frac{f(q^2 x)}{q(q - 1)x} + \frac{f(qx)}{q(q - 1)x} \]
\[ = - D_q f(qx). \]  
(B.7)

**Theorem B.2.** The eigenfunctions of the q-dilation operator are monomial \( f_n(x) = x^n \) with eigenvalue \( \frac{1 - q^n}{1 - q} \).

**Proof.**

\[ S_q f_n(x) = x [D_q f_n(x)] = \frac{x^n - q^n x^n}{1 - q} = \left( \frac{1 - q^n}{1 - q} \right) x^n. \]

\[ \square \]

**Theorem B.3.** For any q-weight function, the diagonal matrix elements (or the Fourier coefficients) of the q-dilation operator in terms of orthonormal polynomial basis, defined by

\[ S_q p_n(x) = \sum_{j=0}^{n} p_j(x)c_{jn}, \]

have an universal expression:

\[ c_{nn} = \frac{1 - q^n}{1 - q}. \]  
(B.8)

**Proof.** By using \([D_q, x] p_n(x) = p_n(qx)\), we express

\[ LHS = \frac{xp_n(x) - (qx)p_n(qx)}{(1 - q)x} - (c_{nn} p_n + c_{n-1,n} p_{n-1} + \cdots) \]

and

\[ RHS = \left[ \gamma_n (qx)^n + \delta_n (qx)^{n-1} + \cdots \right]. \]

Then we compare the coefficient of \( x^n \) to get

\[ \frac{\gamma_n - q^{n+1} \gamma_n}{1 - q} - \gamma_n c_{nn} = q^n \gamma_n, \]

which implies

\[ c_{nn} = \frac{1 - q^n}{1 - q} - q^n = \frac{1 - q^n}{1 - q}. \]

\[ \square \]
Appendix C. A Short Note for the $q$-exponential Function

The $q$-deformed exponential function is defined either by an infinite product or as a power series in $x$,

$$e_q(x) := \frac{1}{\prod_{k=0}^{\infty} (1 - q^k x)} \quad (C.1)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{x^n}{\prod_{k=1}^{n} (1 - q^k)} \quad (C.2)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{x^n}{(1 - q)(1 - q^2) \cdots (1 - q^n)}. \quad (C.3)$$

Theorem C.1. The $q$-deformed exponential function satisfies the following $q$-difference equation:

$$D_q^x e_q(x) = \left( \frac{1}{1 - q} \right) e_q(x). \quad (C.4)$$

Proof.

$$D_q^x e_q(x)$$

$$= \frac{e_q(x) - e_q(qx)}{(1 - q)x}$$

$$= \frac{1}{(1 - q)x} \left[ x + \sum_{n=2}^{\infty} \frac{x^n}{(1 - q)(1 - q^2) \cdots (1 - q^n)} - qx - \sum_{n=2}^{\infty} \frac{q^n x^n}{(1 - q)(1 - q^2) \cdots (1 - q^n)} \right]$$

$$= \left( \frac{1}{1 - q} \right) \left[ 1 + \sum_{n=2}^{\infty} \frac{x^n - 1}{(1 - q)(1 - q^2) \cdots (1 - q^{n-1})} \right]$$

$$= \left( \frac{1}{1 - q} \right) e_q(x).$$

From this result, it is easy to see that, using the power series form, Eq. (C.4), as $\epsilon := 1 - q \to 0$, we can connect this $q$-exponential function to the classical exponential and Gaussian functions.

$$\lim_{\epsilon \to 0} e_{1-\epsilon}(\epsilon x) = \lim_{\epsilon \to 0} \left( 1 + \sum_{n=1}^{\infty} \frac{\epsilon^n x^n}{\epsilon \cdot 2 \epsilon \cdots n \epsilon} \right) = e^x.$$  \hspace{1cm} (C.5)

$$\lim_{q \to 1} e_q^2((1 - q^2)x^2) = \lim_{\epsilon \to 0} \left( 1 + \sum_{n=1}^{\infty} \frac{(2\epsilon)^n x^{2n}}{2 \epsilon \cdot 4 \epsilon \cdots 2n \epsilon} \right) = e^{x^2}.$$
**Theorem C.2.** The general eigenfunctions of the $q$-derivative operator are $e_q((1-q)\kappa x)$ (up to normalization) with eigenvalue $\kappa$.

**Proof.** We can solve the eigenvalue problem

$$\mathcal{D}_q f_\kappa(x) = \kappa f_\kappa(x)$$  \hspace{1cm} (C.6)

by using power series expansion, let

$$f_\kappa(x) = c_0 + \sum_{n=1}^{\infty} c_n x^n,$$

$$\Rightarrow f_\kappa(qx) = c_0 + \sum_{n=1}^{\infty} c_n q^n x^n,$$

$$\mathcal{D}_q f_\kappa(x) = \frac{f_\kappa(x) - f_\kappa(qx)}{(1-q)x}$$

$$= c_1 + \sum_{n=1}^{\infty} c_{n+1} \left( \frac{1-q^{n+1}}{1-q} \right) x^n.$$  \hspace{1cm} (C.8)

By comparing Eq. (C.8) with Eq. (C.6), we obtain a recursive relation among the coefficients $c_n$

$$c_n = \frac{(1-q)\kappa}{1-q^n} c_{n-1}.$$  \hspace{1cm} (C.9)

Telescoping leads to general solution

$$c_n = \frac{(1-q)^n \kappa^n}{\prod_{l=1}^{n}(1-q^l)} c_0,$$

and $f_\kappa(x) = c_0 e_q((1-q)\kappa x)$.

Equipped with these preliminaries, we can compute the zeroth moment of the $q$-Laguerre ensemble (with $\alpha = 0$).

**Theorem C.3.** The zeroth moment of the $q$-Laguerre ensemble is given by

$$\mu_{0,L}^{(0)} = \left( \frac{1-q}{q\kappa} \right) [1 - (\kappa; q)_\infty].$$  \hspace{1cm} (C.10)

**Proof.** In this paper, the $q$-Laguerre weight function is defined as the inverse of the $q$-Pochhammer symbol,

$$e_q(\kappa x) v^{(0)}(x; \frac{\kappa}{q}, q) = 1.$$  \hspace{1cm} (C.11)

By taking $q$-derivative on both sides of the equation above, we get

$$[\mathcal{D}_q^x e_q(\kappa x)] v^{(0)}(x; \frac{\kappa}{q}, q) + e_q(q\kappa x) \left[ \mathcal{D}_q^x v^{(0)}(x; \frac{\kappa}{q}, q) \right] = 0.$$  \hspace{1cm} (C.12)
Thus, the $\alpha = 0$ $q$-Laguerre weight can be written as a total $q$-derivative,

$$v^{(0)}(x; \kappa, q) = \frac{1}{e_q(q \kappa x)} = (q \kappa x; q)_{\infty}$$

$$= -\frac{1}{[D_q e_q(\kappa x)]} v^{(0)}(x; \frac{\kappa}{q}, q) D_q x v^{(0)}(x; \frac{\kappa}{q}, q)$$

$$= -\frac{(1 - q)}{\kappa} D_q (\kappa x; q)_{\infty}. \quad (C.12)$$

Using the fundamental theorem of the $q$-calculus, the zeroth moment can be calculated as

$$\mu^{(0)}_{0,L} = \int_0^1 v^{(0)}(x; \kappa, q) d_q x$$

$$= - \left( \frac{1 - q}{q \kappa} \right) \int_0^1 D_q (\kappa x; q)_{\infty} d_q x$$

$$= \left( \frac{1 - q}{q \kappa} \right) [1 - (\kappa; q)_{\infty}] . \quad (C.13)$$

Similarly, we can compute the zeroth moment of the $q$-Hermite ensemble.

**Theorem C.4.** The zeroth moment of the $q$-Hermite ensemble is given by the incomplete $q$-Gamma integral

$$\mu^{(-\frac{1}{2})}_{0,H} = \frac{2}{1 + q} \int_0^1 y^{-\frac{1}{2}} (q^2 \kappa^2 y; q^2)_{\infty} d_q y. \quad (C.14)$$

**Proof.** Using the formula of change of variables for $q$-integral ($y = x^2$)

$$\int_0^1 f(x^2) d_q x = \frac{1}{1 + q} \int_0^1 \frac{f(y)}{\sqrt{y}} d_q y, \quad (C.15)$$

we can compute the zeroth moment of the $q$-Hermite ensemble (with $\alpha = -\frac{1}{2}$)

$$\mu^{(-\frac{1}{2})}_{0,H}(\kappa, q) = 2 \int_0^1 (q^2 \kappa^2 x^2; q^2)_{\infty} d_q x$$

$$= \frac{2}{1 + q} \int_0^1 y^{-\frac{1}{2}} (q^2 \kappa^2 y; q^2)_{\infty} d_q y. \quad (C.16)$$

□
By comparing with the integral representation of the $q$-Gamma function [15],

$$
\Gamma_q(t) := (1-q)^{-t} \int_0^1 x^{t-1} (qx; q)_\infty dx,
$$
(C.17)

$$
\Gamma_{q^2}(\frac{1}{2}) = \frac{1}{(1-q^2)^{\frac{1}{2}}} \int_0^1 \frac{1}{\sqrt{x}} (q^2 x; q^2)_\infty d_{q^2}x
= \frac{\kappa}{(1-q^2)^{\frac{1}{2}}} \int_{\kappa^2}^{\infty} \frac{1}{\sqrt{y}} (q^2 \kappa^2 y; q^2)_\infty d_{q^2}y,
$$
(C.18)

we find that

$$
\mu^{(-\frac{1}{2})}_{0,H}(1, q) = 2 \sqrt{\frac{1-q}{1+q}} \Gamma_{q^2}(\frac{1}{2}).
$$
(C.19)

In addition, there exists a quadratic relation between the zeroth weights of $q$-Laguerre/Hermite ensembles

$$
\mu^{(-\frac{1}{2})}_{0,H}(\kappa, q) = \frac{2}{1+q} \mu^{(-\frac{1}{2})}_{0,L}(\kappa^2, q^2).
$$
(C.20)
APPENDIX D. MATRIX ELEMENTS OF THE TRANSLATION OPERATOR

In this appendix, we calculate the matrix elements of the translation operators in terms of recursive coefficients.

D.1. Matrix elements of the translation operator in the basis of Laguerre polynomials.

The matrix elements of the translation operator in the basis of Laguerre polynomials are defined as

\[
\mathcal{D}p_n(x) = \sum_{j=0}^{n-1} p_j(x) t_{jn}. \quad (D.1)
\]

By multiplying both sides of the equation by position operator, we get

\[
\mathcal{S}p_n(x) = x\mathcal{D}p_n(x) = \sum_{j=0}^{n-1} [xp_j(x)] t_{jn}. \quad (D.2)
\]

Using the result in Sec. 3.1, Eq. (3.8), we get

\[
c_{nn}p_n(x) + c_{n-1,n}p_{n-1}(x) = \sum_{j=0}^{n-1} [a_{j+1}p_{j+1}(x) + b_jp_j(x) + a_jp_{j-1}(x)] t_{jn}. \quad (D.3)
\]

Comparing the coefficients on both sides, we get a set of coupled equations for \( t_{mn} \):

\[
c_{nn} = a_n t_{n-1,n}, \\
 c_{n-1,n} = a_{n-1} t_{n-2,n} + b_{n-1} t_{n-1,n}, \\
 0 = c_{n-k,n} = a_{n-k} t_{n-k-1,n} + b_{n-k} t_{n-k,n} + a_{n-k+1} t_{n-k+1,n}. \quad (k \geq 2)
\]

The solution can be solved by iteration and are given by

\[
 t_{n-1,n} = \frac{c_{nn}}{a_n}, \\
 t_{n-2,n} = \frac{1}{a_{n-1}} \left[ c_{n-1,n} - \frac{b_{n-1} c_{nn}}{a_n} \right], \\
 t_{n-k-1,n} = -\frac{b_{n-k}}{a_{n-k}} t_{n-k,n} - \frac{a_{n-k+1}}{a_{n-k}} t_{n-k+1,n}. \quad (k \geq 2)
\]

D.2. Matrix elements of the translation operator in the basis of Hermite polynomials.

The matrix elements of the translation operator in the basis of Hermite polynomials are defined as

\[
\mathcal{D}P_n(x) = \sum_{j=0}^{n-1} P_j(x) T_{jn}. \quad n - j : \text{odd} \quad (D.4)
\]
By multiplying both sides of the equation by position operator, we get

\[
SP_n(x) = xDP_n(x) = \sum_{j=0}^{n-1} [xP_j(x)] T_jn. \quad \text{(D.5)}
\]

Using the result in Sec. 3.2, Eq. (3.25), we get

\[
C_{nn} P_n(x) + C_{n-2,n} P_{n-2}(x) = \sum_{j=0}^{n-1} [A_{j+1} P_{j+1}(x) + A_j P_{j-1}(x)] T_jn. \quad \text{(D.6)}
\]

Comparing the coefficients on both sides, we get a set of coupled equations for \( T_{mn} \):

\[
C_{nn} = A_n T_{n-1,n},
\]

\[
C_{n-2,n} = A_{n-2} T_{n-3,n} + A_{n-1} T_{n-1,n},
\]

\[
0 = C_{n-k,n} = A_{n-k} T_{n-k-1,n} + A_{n-k+1} T_{n-k+1,n}. \quad (k \geq 4)
\]

The solution can be solved by iteration and are given by

\[
T_{n-1,n} = \frac{C_{nn}}{A_n},
\]

\[
T_{n-(2k+1),n} = (-1)^k \frac{A_{n-2k+1} A_{n-2k-1} \cdots A_{n-1} C_{nn}}{A_{n-2k} A_{n-2k-2} \cdots A_{n-2} A_n}. \quad (k \geq 1)
\]

REFERENCES

[1] Ismail, Mourad E. H., *Orthogonal Polynomials, Their Recursions and Functional Equations*, Symmetries and Integrability of Difference Equations, Cambridge Univ. Press, London Math. Soc. Lecture Note Series: 381, 2011.

[2] Van Assche, Walter *Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials*, Difference equations, special functions and orthogonal polynomials, 687-725, World Soc. Publ., Hackensack, NJ, 2007.

[3] Boelen, Lies; Van Assche, Walter, *Discrete Painlevé equations for recurrence coefficients of semiclassical Laguerre polynomials*, Proc. Amer. Math. Soc. 138 (2010), no. 4, 1317-1331.

[4] Clarkson, Peter A., *Recurrence coefficients for discrete orthonormal polynomials and the Painlevé equations*, J. Phys. A 46 (2013), no. 18, 185205, 18 pp.

[5] Ismail, Mourad E. H.; Stanton, Dennis; Viennot, Gérard, *The combinatorics of q-Hermite polynomials and the Askey-Wilson integral*, European J. Combin. 8 (1987), no. 4, 379-392.

[6] Simion, R.; Stanton, D., *Specializations of generalized Laguerre polynomials*, SIAM J. Math. Anal. 25 (1994), no. 2, 712-719.

[7] Chen, Yang; Ismail, Mourad E. H., *Ladder operators for q-orthogonal polynomials*, J. Math. Anal. Appl. 345 (2008), no. 1, 1-10.

[8] Kasraoui, Anisse; Stanton, Dennis; Zeng, Jianguo, *The combinatorics of Al-Salam-Chihara q-Laguerre polynomials*, Adv. in Appl. Math. 47 (2011), no. 2, 216-239.

[9] Chen, Yang; Griffin, James, *Non linear difference equations arising from a deformation of the q-Laguerre weight*, Indag. Math. (N.S.) 26 (2015), no. 1, 266-279.

[10] K. Fujikawa, *Path integral of the hydrogen atom, Jacobi’s principle of least action and one-dimensional quantum gravity*, Nucl. Phys. B 484, 495 (1997).

[11] P. Di Francesco; P. H. Ginsparg; J. Zinn-Justin, *2-D Gravity and random matrices*, Phys. Rept. 254, 1 (1995).

[12] Chen, Yang; Its, Alexander, *Painlevé III and a singular linear statistics in Hermitian random matrix ensembles. I.*, J. Approx. Theory 162 (2010), no. 2, 270-297.
[13] H. Flaschka, *The Toda lattice. II. Existence of integrals*, Phys. Rev. B 9, p.1924-1925.

[14] H. Flaschka, *On the Toda lattice. II. Inverse-scattering solution*, Progr. Theoret. Phys. 51 (1974), 703-716.

[15] Alberto De Sole; Victor G. Kac, *On integral representations of q-gamma and q-beta functions*, Rend. Mat. Acc. Lincei s. 9, v. 16:11-29 (2005)