BLOW-UP FOR THE ONE DIMENSIONAL STOCHASTIC WAVE EQUATIONS

WEIJUN DENG†

Abstract. The paper is concerned with the problem of explosive solutions for a class of semi-linear stochastic wave equations. The challenging open problem ([17]) which is raised by C. Mueller and G. Richards is included in this problem. We develop an Ωδ-comparative approach. With the aid of new approach, under appropriate conditions on the initial data and the nonlinear multiplicative noise term \( (c_2 u + f(u)) W(t, x) \) with \( |f(u)| \geq \kappa |u|^r \), \( r > 1, \kappa > 0 \), we prove in Theorem 3.1 that the solutions to the stochastic wave equation will blow up in finite time with positive probability.

Key words. semilinear stochastic wave equation, finite time blow-up, comparative approach, 2-parameter white noise.

AMS subject classifications. 60H15, 60H30, 35L05, 35L15, 35R60.

1. Introduction. Consider the initial value problem for the nonlinear stochastic wave equation:

\[
\begin{align*}
\partial_t^2 u & = \partial_x^2 u + c_1 u + (c_2 u + f(u)) \dot{W}(t, x), t > 0, x \in D, \\
u(0, x) & = (J + T + 1)u_0(x), \partial_t u(0, x) = (J + T + 1)v_0(x).
\end{align*}
\]

(1.1)

Here \( f(u) \) is locally Lipschitz function and satisfy

\[ |f(u)| \geq \kappa |u|^r, r > 1, \kappa > 0, \]

c_1, c_2 \text{ are given constants}, D = (0, J) \subset R, \dot{W}(t, x) \text{ is 2-parameter white noise.}

In this article we want to study blow-up phenomena: does solutions to stochastic wave equation (1.1) finite time blow-up occur with positive probability? It is expected that such a white noise has a strong influence on the solutions which blow up. The challenging open problem ([17]) which is raised by C. Mueller and G. Richards is included in this problem.

For deterministic nonlinear partial differential equations, there is a very extensive literature on blow-up in finite time. Let us just mention a few: ([1, 2, 5, 6, 7, 8, 9, 10, 11, 12]), for example.

On the other hand, for stochastic partial differential equations (SPDE), there are very few papers about finite time blow-up. It is mathematically very difficult to consider space-time white noise, this is due to the lack of smoothing effect in the stochastic differential equation. We refer the reader to ([10, 15, 13, 11, 14]) for new developments.

Our strategy to study the blow-up is based on the \( \Omega_\delta \)-comparative approach. We divide our proof in five steps.

First, we introduce a blow-up lemma for one dimensional semilinear wave equations. Next, we establish a comparison lemma on semilinear wave equations. Another step in the proof is we need to verify the essential supremum of the solution of (1.1) over a subset \( \Omega - \Omega_\delta \) of the probability space, will blow up in finite time. We utilize the close relationship between stochastic partial differential equations and deterministic partial differential equations. Using reduction to absurdity method, suppose that on the contrary, the essential supremum of the solution of (1.1) exists for a long time over a subset \( \Omega - \Omega_\delta \) of the probability space. Consider the deterministic partial

†School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, P.R.China (weijundeng@csu.edu.cn).
differential equations:

\begin{equation}
\partial_t^2 u = \partial_x^2 u + \frac{\beta^2}{4} |u|^3 - \frac{\beta^2}{4} u, \quad t > 0, \quad x \in D,
\end{equation}

\begin{align}
&u(t, x) = 0, \quad \text{for} \quad x \in \partial D, \quad t \geq 0, \\
&u(0, x) = u_0(x), \quad \partial_t u(0, x) = v_0(x),
\end{align}

suppose that $|u|^r$ increases fast, in other words, $r > 1$. By the first step, under appropriate conditions on the initial data solutions of (1.2), will blow up in finite time. Then we construct a comparison between square moment of the solution of (1.1) over the subset $\Omega - \Omega_\delta$ and solution of (1.2), apply the previous comparison lemma to get conflicting results. The fourth steps to do in the proof is that the essential infimum of the blow-up time of the solution of (1.1) is bounded. Finally, we show that the solutions of (1.1) will blow up in bound time with positive probability.

With the aid of the $\Omega_\delta$-comparative approach, under appropriate conditions on the initial data and the nonlinear multiplicative noise term $(c_2 u + f(u))W(t, x)$ with $|f(u)| \geq \kappa |u|^r, r > 1, \kappa > 0$, we prove in Theorem 3.1 that the solution of the stochastic wave equation will blow up in bound time with positive probability.

The rest of this paper is organized as follows. We shall first give problem statement and preliminaries in Section 2. Then, in Section 3, we develop a comparative approach and prove the main theorem (Theorem 3.1).

\section{Problem statement and preliminaries}

\subsection{Problem statement}

Let $\Omega$ be an uncountable Polish space with the metric $\gamma$ and $\mathcal{B}(\Omega)$ be the topological $\sigma$-field. Suppose $W = \{W(t, x), t \in [0, +\infty), x \in \mathcal{D}\}$ is a 2-parameter white noise defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Let us discuss the rigorous meaning of solution to (1.1), and the definition of finite time blow-up. We regard (1.1) as short-hand for the following integral equation (2.1)

\begin{align}
&u(t, x) = \int_D K(t, x - y)(J + T + 1)(1 + u_0(y))dy + \int_D S(t, x - y)(J + T + 1)v_0(y)dy \\
&\quad + c_1(S * u)(t, x) + \int_0^t \int_D S(t - s, x - y)(c_2 u + f(u))W(ds, dy),
\end{align}

which may only be well-defined for a small time (see below). Here $*$ denotes convolution, i.e.

\begin{equation}
(S * u)(t, x) \triangleq \int_0^t \int_D S(t - s, x - y)u(s, y)dsdy.
\end{equation}

$K(t, x)$ and $S(t, x)$ are the wave kernels:

\begin{align}
K(t, x) &\triangleq \sum_{n \in \mathbb{Z}} \frac{1}{2} \delta(x + t + nJ) + \delta(x - t + nJ), \\
S(t, x) &\triangleq \sum_{n \in \mathbb{Z}} \frac{1}{2} I_{[-t,t]}(x + nJ),
\end{align}

where $\delta(\cdot)$ is the delta function.

The above formula for $K(t, x)$ should be interpreted in the sense of Schwartz distributions. One can define a solution to (1.1) in terms of distributions and then show that such a solution exists if and only if (2.1) holds. The integral in (2.1) involving $W(\cdot, \cdot)$ should be interpreted in the sense of Walsh’s theory of martingale measures (see [20], chapter 2). By standard arguments (e.g., see Theorem 3.2 and exercise 3.7 in Walsh [20]), (2.1) has a unique continuous solution $u(t, x)$ valid for $t < \sigma_L$, where

\begin{equation}
\sigma_L \triangleq \inf \{ t > 0 : \sup_{x \in \mathcal{D}} |u(t, x)| \geq L \}
\end{equation}

2
and the infimum of the empty set is taken to be $+\infty$. Letting $L \to +\infty$, we conclude that (1.1) has a unique solution for $t < \sigma$, where

\begin{equation}
\sigma \triangleq \lim_{L \to +\infty} \sigma_L.
\end{equation}

It follows that, if $\sigma < +\infty$, then $\lim_{t \to \sigma^-} \sup_{x \in \mathcal{D}} |u(t,x)| = +\infty$. With these definitions in place, if $P(\sigma < +\infty) > 0$, we say that solutions to (1.1) blow up in finite time with positive probability.

### 2.2. Preliminaries

We shall use the following Lemmas:

**Lemma 2.1.** Consider the following initial value problem for the nonlinear stochastic wave equation:

\begin{equation}
\begin{aligned}
\partial_t^2 u &= \partial_x^2 u + g_1(u) + f_1(u) \dot{W}(t,x), \ t \geq 0, \\
u(0,x) &= u_1(x), \ \partial_t u(0,x) = v_1(x),
\end{aligned}
\end{equation}

The function $g_1, f_1: \mathbb{R} \to \mathbb{R}$ are measurable and there exists a constant $L, K > 0$, such that for $\phi_1(u) = f_1(u), g_1(u),$

\begin{equation}
|\phi_1(u) - \phi_1(v)| \leq L|u - v|, \ |\phi_1(u)| \leq K(1 + |u|),
\end{equation}

for which $u_1(x), v_1(x) \in C^1(\mathbb{R})$. Then the equation (2.5) has a unique solution, which has a Hölder continuous version.

**Proof.** Existence, uniqueness and Hölder continuous of the solution to the nonlinear stochastic wave equation (2.5) is covered in ([20], p.323, Exercise 3.7). The proof is omit.

**Lemma 2.2.** Let $u(t,x): I \times D \to \mathbb{R}$ be a solution of the following initial-boundary value problem for the nonlinear wave equation:

\begin{equation}
\begin{aligned}
\partial_t^2 u &= \partial_x^2 u + \frac{\partial^2}{\partial x^2} |u|^r - \frac{\partial^2}{\partial x^2} u, \ t > 0, \ x \in D, \\
u(t,x) &= 0, \text{ for } x \in \partial D, \ t \geq 0, \\
u(0,x) &= u_0(x), \ \partial_t u(0,x) = v_0(x),
\end{aligned}
\end{equation}

for which $u_0(x), v_0(x) \in C^\infty(D), r > 1$. Then $u(t,x) \in C^\infty(I \times D)$.

**Proof.** Applying Proposition 3.1 of ([18], P433) and Corollary 1.6 of ([18], P421), we can easily prove the conclusion.

**Lemma 2.3.** ([7], P185, Lemma 1.1) Let $\phi(t) \in C^2$ satisfy

\[ \ddot{\phi} \geq h(\phi), \ t \geq 0, \]

with $\phi(t) = \alpha > 0, \dot{\phi}(0) = \beta > 0$. Suppose that $h(s) \geq 0$ for all $s \geq \alpha$. Then

1. $\dot{\phi}(t) > 0$ wherever $\phi(t)$ exists; and
2. the inequality

\[ t \leq \int_\alpha^{\phi(t)} \left[ \beta^2 + 2 \int_\alpha^s h(\xi) d\xi \right] d\xi \]

obtains.

We define the collection

\[ \{ \dot{W}(A) = \int_A \dot{W}(dsdx) |A \text{ be Borel subset of } [0,v] \times D \}, \]
as a centered Gaussian random field with covariance given by
\[ \mathbb{E}[\hat{W}(A)\hat{W}(B)] = \pi(A \cap B), \]
where \( \pi \) denotes the Lebesgue measure on \( \mathbb{R}^+ \times \mathcal{D} \).

We define for each \( t > 0 \) the \( \sigma \)-algebra
\[ \mathcal{G}_t = \sigma\{\hat{W}(A)|A \text{ is Borel subset of } [0, t] \times \overline{\mathcal{D}}\} \vee \mathcal{N}, \quad \mathcal{F}_t = \cap_{s \geq t} \mathcal{G}_s, \quad t \geq 0, \]
where \( \mathcal{N} \) are the totality of \( \mathbb{P} \)-null sets of \( \mathcal{F} = \mathcal{B}(\Omega) \). Then, it is clear that the filtered complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) satisfies usual hypotheses.

In order to express the idea of the proof clearly, let us define the following concept.

**Definition 2.1.** Let \( \mathcal{E}_\delta \) be a random variable defined on the complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \). \( \mathcal{E}_\delta \) be an open set, \( \Omega \supseteq \mathcal{E}_\delta, 0 < \mathbb{P}(\mathcal{E}_\delta) \leq \delta, \mathbb{E}_\delta \mathcal{E}_\delta = \int_{\Omega \setminus \mathcal{E}_\delta} \mathbb{E}(\omega)\mathbb{P}(d\omega) \) is called the partial expectation of \( \xi, \) if \( \int_{\Omega \setminus \mathcal{E}_\delta} |\xi(\omega)|\mathbb{P}(d\omega) < +\infty \).

The partial expectation operator \( \mathbb{E}_\delta \) has the following proposition:

**Proposition 2.2.** Let \( \hat{W} = \{\hat{W}(t, x), t \in [0, +\infty), x \in \mathcal{D}\} \) be a 2-parameter white noise, and \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) be a complete probability space, \( \{v(t, x; \cdot), (t, x) \in [0, +\infty) \times \mathcal{D}\} \) is predictable and \( \mathbb{E}_\delta \int_0^t \int_\mathcal{D} |v(s, y; \omega)|^2dsdy < +\infty, 0 \leq t \leq T \). Then, it follows that
\[ \mathbb{E}_\delta \int_0^t \int_\mathcal{D} v(s, y; \omega)\hat{W}(dsdy) = 0, 0 \leq t \leq T, \]
and
\[ \mathbb{E}_\delta (\int_0^t \int_\mathcal{D} v(s, y; \omega)\hat{W}(dsdy))^2 = \mathbb{E}_\delta \int_0^t \int_\mathcal{D} |v(s, y; \omega)|^2dsdy, 0 \leq t \leq T. \]
Moreover, Burkholder’s inequality and Kolmogorov Lemma on \( \mathbb{E}_\delta \)-version also hold.

Proofs of the above results are straightforward by the definitions of stochastic integral and the indicator of \( \Omega - \mathcal{E}_\delta \) be an \( \{\mathcal{F}_t\}_{t \geq 0}\)-predictable random process.

Let us introduce the following lemma that will be used later.

**Lemma 2.4.** (\[12\], P2, Proposition 2.1) Let \( \Omega \) be an Polish space, \( \mathcal{B}(\Omega) \) be the topological \( \sigma \)-field, and \( \mathbb{P} \) be a probability on \( (\Omega, \mathcal{B}(\Omega)) \). Then, for every \( B \in \mathcal{B}(\Omega) \),
\[ \mathbb{P}(B) = \inf_{G \subset B, G \text{ open}} \mathbb{P}(G). \]

### 3. Blow-up for initial data.
Let \( \psi(x) \) denote the first eigenfunction for the problem \( \left( \frac{\partial}{\partial t} \right)^2 \psi(x) + \mu \psi(x) = 0, x \in \mathcal{D}, \) under the Dirichlet condition \( \psi(x) = 0 \) on \( \partial \mathcal{D} \), and let \( \mu = \mu_1 \) be the corresponding first eigenvalue, i.e. \( \mu_1 = \left( \frac{\pi}{2} \right)^2, \psi(x) = \frac{x}{2\pi} \sin \frac{x}{2}. \)

We assume that

- H1) \( 0 \leq u_0(x) \in C^\infty(D), 0 \leq v_0(x) \in C^\infty(D), \) there exist \( x_0 \in \mathcal{D} \) such that \( v_0(x_0) > 0. \)
- H2) \( r > 1, \lambda_1 \triangleq \mu_1 + \frac{c_1^2 + c_2^2}{2}, \alpha \triangleq \int_\mathcal{D} \psi(x)u_0(x)dx \geq \left( \frac{4\lambda_1}{\alpha} \right)^{\frac{1}{r+1}}, \beta \triangleq \int_\mathcal{D} \psi(x)v_0(x)dx, \) and
\[ T \triangleq \int_{\alpha}^{\infty} \left[ \lambda_1 \alpha^2 + \beta^2 - \lambda_1 \alpha \frac{s^2}{2} + \frac{\kappa^2}{2 + 2r} (s^{(r+1)} - \alpha^{(r+1)}) \right]^\frac{1}{r} ds. \]

Consider the initial value problem for the nonlinear stochastic wave equation:
\[ \begin{aligned} \frac{\partial_t^2 u}{(2u_x^2 + c_1 u + (c_2 u + f(u))\hat{W}(t, x), t > 0, x \in \mathcal{D},} \\
\quad u(0, x) = (J + T + 1)(1 + u_0(x)), \quad \partial_t u(0, x) = (J + T + 1)v_0(x). \end{aligned} \]
Here \( f(u) \) is locally Lipschitz function and satisfies
\[ |f(u)| \geq \kappa|u|^r, r > 1, \kappa > 0, \]
c₁, c₂ are given constants, T is given by (3.1), D = (0, J) ⊂ R, W(t, x) is 2-parameter white noise.

The main result of this article is the following.

**Theorem 3.1.** The solution of (3.2), for which H₁ and H₂ are satisfied, will blow up in bounded time with positive probability, more precisely, for all ε > 0,

\[ \mathbb{P}(\sigma < T + \varepsilon) > 0, \]

where T is given by (3.1).

Before proving this theorem, the following lemmas are introduced.

**Lemma 3.2.** Let \( u(t, x) \) be a solution of the following initial-boundary value problem for the nonlinear wave equation:

\[
\begin{align*}
\partial_t^2 u &= \partial_x^2 u + \frac{k^2}{4} |u|^r - \frac{c_2^2 + c_2^2}{2} u, t > 0, x \in D, \\
u(t, x) &= 0, \text{ for } x \in \partial D, t \geq 0, \\
u(0, x) &= u_0(x), \partial_t u(0, x) = v_0(x),
\end{align*}
\]

for which H₁ and H₂ are satisfied. Then

\[
\lim_{t \to t_1^-} \sup_{x \in D} |u(t, x)| = +\infty
\]

for some finite \( t_1 \leq T \), where T is given by (3.1).

**Proof.** The solution \( u(t, x) \) satisfies the following nonlinear integral equation:

\[
u(t, x) = S \ast \left( \frac{k^2}{4} |u|^r - \frac{c_2^2 + c_2^2}{2} u \right)(t, x) + I(t, x),
\]

where \( I(t, x) = \int_D K(t, x - y)u_0(y)dy + \int_D S(t, x - y)v_0(y)dy \) and \( \ast \) denotes convolution, i.e.

\[
(S \ast \eta)(t, x) \triangleq \int_0^t \int_D S(t - s, x - y)\eta(s, y)dsdy.
\]

By Lemma 2.2, we have \( u(t, x) \in C^2 \). Let \( \phi(t) = \int_0^t \psi(x)u(t, x)dx \), multiply (3.3) by \( \psi \) and integrate over D, we obtain

\[
\int_0^J \psi u_{tx}dx = \frac{\rho}{\alpha} = \int_0^J \psi u_{xx}dx + \frac{k^2}{4} \int_0^J \psi |u|^r dx - \frac{c_2^2 + c_2^2}{2} \int_0^J \psi u dx.
\]

By Jensen’s inequality, we have \( \int_0^J \psi |u|^r dx \geq |\phi|^r \), since \( \int_0^J \psi(x)dx = 1, \psi(x) \geq 0, x \in D \). Using integration by parts and the boundary conditions satisfied by \( u \) and \( \psi \), we see that

\[
\int_0^J \psi u_{xx}dx = \int_0^J u\psi_{xx}dx = -\mu_1 \int_0^J u\psi dx = -\mu_1 \phi.
\]

Thus we arrive at

\[
\frac{\rho}{\alpha} + (\mu_1 + \frac{c_2^2 + c_2^2}{2}) \phi \geq \frac{k^2}{4} |\phi|^r
\]

with

\[
\phi(0) = \int_0^J \psi(x)u(0, x)dx = \alpha > 0, \hat{\phi}(0) = \int_0^J \psi(x)u_t(0, x)dx = \beta > 0.
\]
Hypothesis H2) implies that Lemma 3.2 is applicable with \( h(s) = \frac{\kappa^2}{4} s^r - \lambda_1 s \); therefore

\[
t \leq \int_{\alpha}^{\phi(t)} \left[ \lambda_1 \alpha^2 + \beta^2 - \lambda_1 \alpha^2 + \frac{\kappa^2}{2 + 2r} (s^{(r+1)} - \alpha^{(r+1)}) \right] \frac{1}{t} ds,
\]

and thus \( \phi(t) \) develops a singularity in a finite time \( t \leq T \), where

\[
T = \int_{\alpha}^{\infty} \left[ \lambda_1 \alpha^2 + \beta^2 - \lambda_1 \alpha^2 + \frac{\kappa^2}{2 + 2r} (s^{(r+1)} - \alpha^{(r+1)}) \right] \frac{1}{t} ds.
\]

Finally, since \( \phi(t) > 0 \), we have

\[
\phi(t) = |\phi(t)| = \left| \int_{0}^{J} \psi(x)u(t, x)dx \right| \leq \sup_{x \in D} |u(t, x)|,
\]

which proves the Lemma. \( \square \)

Now let us prove the following comparison Lemma which could be also of interest in itself.

**Lemma 3.3.** Let \( U(t, x) \) satisfy equation (3.3), \( 0 < t < t_1 \), define

\[
\mathcal{V} = \left\{ v(\cdot, \cdot) \in C([0, t_j] \times D) \left| v(t, x) > S \ast (\frac{\kappa^2}{4} |v|^r - \frac{\beta^2 + \gamma^2}{2} v)(t, x) + I(t, x) \right. \right\},
\]

where \( I(t, x) = \int_{D} K(t, x, y)u_0(y)dy + \int_{D} S(t, x - y)v_0(y)dy \) and \( t_1 \) is given by the Lemma 3.2 Then the set \( \mathcal{V} \) has the following properties:

1. \( \mathcal{V} \) is a nonempty convex set,

2. \( v(t, x) > U(t, x) \), for any \( v(t, x) \in \mathcal{V} \), \( (t, x) \in [0, t_j] \times D \).

**Proof.** (1) First of all, noting \( U(t, x) = S \ast (\frac{\kappa^2}{4} |v|^r - \frac{\beta^2 + \gamma^2}{2} v)(t, x) + I(t, x) \), if let \( M = \max_{(t, x) \in [0, t_j] \times D} U(t, x) \), \( f_0(t) = \exp \left\{ \frac{\kappa^2}{4} (M + 1)^{r-1} - \frac{\beta^2 + \gamma^2}{2} J(t - t_f) \right\}, 0 \leq t \leq t_f \), define \( v_1(t, x) = U(t, x) + f_0(t) \), then we have

\[
I(t, x) + S \ast (\frac{\kappa^2}{4} |v_1|^r - \frac{\beta^2 + \gamma^2}{2} v_1)(t, x)
\]

\[
\leq I(t, x) + \frac{\kappa^2}{4} \int_{D} S(t - s, x - y)\left(|U(s, y)|^r + r(M + 1)^{r-1} f_0(s)\right)dsdy - \frac{\beta^2 + \gamma^2}{2} \int_{0}^{t} f_0(s)dsdy
\]

\[
\leq U(t, x) + \left[ \frac{\kappa^2}{4} (M + 1)^{r-1} - \frac{\beta^2 + \gamma^2}{2} \right] J(t - t_f) f_0(s)ds
\]

\[
< U(t, x) + f_0(t) = v_1(t, x),
\]

for \( (t, x) \in [0, t_f] \times D \). Thus \( v_1(t, x) \in \mathcal{V} \), i.e. \( \mathcal{V} \) is a nonempty set. Next, by Jensen’s inequality, it is easy to see that \( \mathcal{V} \) is a convex set. In order to prove (2) we use (1), if (2) is false, then there exists \( v_2(t, x) \in \mathcal{V} \) and \( (t, x) \in [0, t_f] \times D \), such that \( v_2(t, x) < v_1(t, x) \), since \( U(t, x) \) and \( \mathcal{V} \) is non-intersect. Select \( v_1(t, x) \in \mathcal{V} \) as above, then there exists \( 0 < \theta < 1 \), such that \( U(t, x) = \theta v_1(t, x) + (1 - \theta)v_2(t, x) \), thus we obtain \( U(t, x) \) intersects with \( \mathcal{V} \), this is a contradiction, the proof is complete. \( \square \)

We present the following result that the essential supremum of the solution of (3.2) over a subset \( \Omega - \Omega_0 \) of the probability space, will blow up in finite time.

**Lemma 3.4.** Let \( u(t, x) \) is the solution of (3.2) for which H1) and H2) are satisfied. Then, for given an open set \( \Omega_0 \subset \Omega, 0 < \Psi(\Omega_0) \leq \delta \leq \frac{1}{3} \), it follows that

\[
\lim_{t \to t_0} \max_{x \in \Omega - \Omega_0} |u(t, x)| = +\infty,
\]
for some bounded time $t_0 = t_0(\Omega_\delta) \leq T$, where $T$ is given by (3.7).

Before proving Lemma 3.4, let us remark on some details of our approach.

If $\max_{[0,T] \times \overline{D}} \text{esssup}_{\Omega-\Omega_\delta} |u(t,x)| < +\infty$, by the Definition 2.1 of the partial expectation operator $E_\delta$, it is clear that

$$E_\delta \sup_{x \in \overline{D}} u(t,x)^2 \leq \max_{[0,T] \times \overline{D}} \text{esssup}_{\Omega-\Omega_\delta} |u(t,x)|^2 < +\infty, \quad t \in [0,T]$$

(3.5)

and

$$E_\delta \int_0^t \int_D |S(t-s,x-y)|f(u(s,y))|^q dsdy < +\infty,$$

(3.6)

for $(t,x) \in [0,T] \times \overline{D}$, $1 \leq q < +\infty$. It follows that the equation (3.2) exists a continuous local solution $u(t,x)$ on $[0,T] \times \overline{D}$, for $\omega \in (\Omega-\Omega_\delta)$, since (by Proposition 2.2) the Burkholder’s inequality and Kolmogorov Lemma on $E_\delta$-version hold. Moreover, by (3.5), using dominated convergence theorem, we can carry out

$$E_\delta u(t,x)^2 \in C([0,T] \times \overline{D}).$$

(3.7)

We now turn to the proof of Lemma 3.4.

Proof. Suppose that $\max_{[0,T] \times \overline{D}} \text{esssup}_{\Omega-\Omega_\delta} |u(t,x)| < +\infty$. If define $I(t,x) = \int_D K(t,x-y)u_0(y)dy + \int_D S(t,x-y)v_0(y)dy$, by (2.8), (2.9), (2.1) and (3.5) using Jensen’s inequality and Schwarz’s inequality, noting $(c+d)^2 \geq \frac{1}{2}c^2 - d^2, (c+d)^2 \geq \frac{1}{2}J(t,x+1)^2 - \frac{1}{(J+1)^2}$, $S(t,x)^2 \geq \frac{1}{2}S(t,x)$ and $\int_0^t \int_D S(t-s,x-y)dsdy \leq \frac{(J+1)^2}{2}$, then we have

$$E_\delta u(t,x)^2 = E_\delta [(J+T+1)I(t,x) + (J+T+1) + c_1S*u]^2$$

$$+ E_\delta \int_0^t \int_D [S(t-s,x-y)(c_2u(s,y) + f(u))^2 dsdy$$

$$\geq E_\delta [(J+T+1)I(t,x) + (J+T+1)^2/[2(J+T+1)^2]$$

$$- E_\delta (c_1S*u)^2/[2(J+T+1)^2]$$

$$+ E_\delta \int_0^t \int_D [S(t-s,x-y)(c_2u(s,y) + f(u))^2 dsdy$$

$$> (1-\delta)[I(t,x) + 1]^2/2 - \frac{c_1^2}{2} E_\delta \int_0^t \int_D S(t-s,x-y)u(s,y)^2 dsdy$$

$$+ \frac{1}{2}E_\delta \int_0^t \int_D S(t-s,x-y)(\frac{k^2}{2}|u(s,y)|^{2r} - (c_2u(s,y))^2)dsdy$$

$$> I(t,x) - \frac{c_1^2}{2}E_\delta \int_0^t \int_D S(t-s,x-y)u(s,y)^2 dsdy$$

$$+ \frac{1}{2}E_\delta \int_0^t \int_D S(t-s,x-y)(\frac{k^2}{2}|u(s,y)|^{2r} - (c_2u(s,y))^2)dsdy$$

$$\geq I(t,x) + \frac{k^2}{4} \int_0^t \int_D S(t-s,x-y)(E_\delta u(s,y)^2)^{r} dsdy$$

$$- \int_0^t \int_D S(t-s,x-y)\frac{c_1^2 + c_2^2}{2} E_\delta u(s,y)^2 dsdy$$

$$= I(t,x) + S \cdot \frac{k^2}{4} |E_\delta u|^2 - \frac{c_1^2 + c_2^2}{2} E_\delta u^2(t,x), (t,x) \in [0,T] \times \overline{D}. $$
Now, combining (3.7) and (3.8), using Lemma 3.3, we obtain

$$\mathbb{E}_{\delta} u(t, x)^2 > U(t, x), (t, x) \in [0, t_f] \times \overline{D}.$$  

Thus we arrive at

$$\mathbb{E}_{\delta} \sup_{x \in \overline{D}} u(t, x)^2 \geq \sup_{x \in \overline{D}} \mathbb{E}_{\delta} u(t, x)^2 \geq \sup_{x \in \overline{D}} U(t, x).$$

Let $t_f \to t_1^-$, by Lemma 3.2 we have

$$\lim_{t_1 \to t_f^-} \mathbb{E}_{\delta} \sup_{x \in \overline{D}} u(t, x)^2 = +\infty,$$

this contradicts with (3.5). Thus there exists some bounded time $K$ with (3.1). Then we have

$$E_{\tau}(3.14),$$

This leads to $K \delta$ for given an open set $\Omega$ of the solution of (1.1) is bounded:

$$\text{Thus we arrive at}$$

$$\text{Now, combining (3.7) and (3.8), using Lemma 3.3, we obtain}$$

$$\text{There exist}$$

$$\text{Moreover, noting that} \Omega \in \Omega \delta, \text{it follows that} \omega_f \in \Omega - \Omega \delta \text{and} \omega_f \in E_n. \text{If plug} \omega_f \text{back into (3.10), then we obtain}$$

$$\sigma_n(\omega_f) < t_0, \text{for all} n \in \mathbb{N}.$$  

Let $n \to +\infty$, it is then obvious that

$$\text{In addition, it is evident that, according to the conclusion of Lemma 3.4 and the definition of essinf,}$$

$$\tau = \sup_{E_0 \cap \Omega = \emptyset} \left( \inf_{\Omega - \Omega_\delta} \sigma \right) \leq \sup_{\Omega - E_0} \left( \inf_{\Omega - \Omega_\delta} \sigma \right) \leq \sup_{\Omega - \Omega_\delta} \left( \inf_{\Omega - \Omega_\delta} \sigma \right).$$

Combining (3.13) and (3.14), we get

$$\tau \leq \sup_{\Omega - \Omega_\delta} \left( \inf_{\Omega - \Omega_\delta} \sigma \right) \leq T.$$  

This completes the proof of Lemma 3.5. \qed
We are now in a position to prove Theorem 3.1.

Proof. Suppose that on the contrary, there exists \( \varepsilon_0 > 0 \), such that \( \mathbb{P}(\sigma < T + \varepsilon_0) = 0 \), then according to the definition of \( \text{essinf} \), we have

\[
\tau = \sup_{E_0 \subset \Omega, \mathbb{P}(E_0) = 0} \left( \inf_{\Omega - E_0} \sigma \right) \geq T + \varepsilon_0 > T.
\]

However, by Lemma 3.5, we have \( \tau \leq T \), this leads to a contradiction. Thus, for all \( \varepsilon > 0 \), we have \( \mathbb{P}(\sigma < T + \varepsilon) > 0 \), the proof is complete. \( \square \)

REFERENCES

[1] J.M. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, Quart. J. Math. Oxford Ser. (2) 28 (1977), No. 112, 473–486.
[2] L.A. Cakarelli and A. Friedman, The blow-up boundary for nonlinear wave equations, Trans. Amer. Math. Soc. 297 (1986), No. 1, 223–241.
[3] P.L. Chow, Nonlinear stochastic wave equations: blow-up of second moments in \( L^2 \)-norm, Ann. Appl. Probab. 19 (2009), No. 6, 2039–2046.
[4] M. Dozzi and J.A. Lopez-Mimbela, Finite time blowup and existence of global positive solutions of a semi-linear SPDEs, Stoch. Proc. Appl. 120 (2010), 767–776.
[5] H. Fujita, On the blowing up of solutions of the Cauchy problem for \( u_t = \Delta u + u^{1+\alpha} \), J. Fac. Sci. Univ. Tokyo Sect.I 13 (1966), 109–124.
[6] Y. Giga and R.V. Kohn, Asymptotically self-similar blow-up of semilinear heat equations, Comm. Pure Appl. Math. 38(3) (1985), 297–319.
[7] R.T. Glassey, Blow-up Theorems for Nonlinear Wave Equations, Math. Z. 132 (1973), 183–203.
[8] R.T. Glassey, On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations, J. Math. Phys. 18 (1977), No. 9, 1794–1797.
[9] T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations, Comm. Pure Appl. Math. 33 (1980), No. 4, 501–505.
[10] H.A. Levine, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form \( P u_t = -Au + F(u) \), Arch. Rational Mech. Anal. 51 (1973), 371–386.
[11] H.A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form \( P u_{tt} = -Au + F(u) \), Trans. Amer. Math. Soc. 192 (1974), 1–21.
[12] F. Merle, Solution of a nonlinear heat equation with arbitrarily given blow-up points, Comm. Pure Appl. Math. 45 (3) (1992), 263–300.
[13] C. Mueller, Long time existence for the heat equation with a noise term, Probab. Theory Related Fields 90 (1991), No. 4, 505–517.
[14] C. Mueller, Long time existence for the wave equation with a noise term, Ann. Probab. 25 (1997), No. 1, 133–151.
[15] C. Mueller, The critical parameter for the heat equation with a noise term to blow up in finite time, Ann. Probab. 28 (2000), No. 4, 1735–1746.
[16] C. Mueller and R. Sowers, Blowup for the heat equation with a noise term, Probab. Theory Related Fields 97 (1993), No. 3, 287–320.
[17] C. Mueller and G. Richards, Can solutions of the one-dimensional wave equation with nonlinear multiplicative noise blow up?, Open Problems in Mathematics, Volume 2 (2014).
[18] Michael E. Taylor, Partial Differential Equations III, Nonlinear Equations, Second Edition, Springer, 2010.
[19] Ikeda, N. and Watanabe, S., Stochastic Differential Equations and Diffusion Processes, Second Edition, North Holland-Kodansha, Amsterdam-Tokyo, 1989.
[20] J.B. Walsh, An introduction to stochastic partial differential equations, École d’été de probabilités de Saint-Flour, XIV, 265–439, Lecture Notes in Math., 1180, Springer, Berlin, 1986.