CONTINUOUS AND TWISTED \( L_\infty \) MORPHISMS

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Abstract. The purpose of this paper is to develop a suitable notion of continuous \( L_\infty \) morphism between DG Lie algebras, and to study twists of such morphisms.

0. Introduction

Let \( K \) be a field containing \( \mathbb{R} \). Consider two DG Lie algebras associated to the polynomial algebra \( K[[t]] := K[t_1, \ldots, t_n] \). The first is the algebra of poly derivations \( T_{\text{poly}}(K[[t]]) \), and the second is the algebra of poly differential operators \( D_{\text{poly}}(K[[t]]) \). A very important result of M. Kontsevich [Ko1], known as the Formality Theorem, gives an explicit formula for an \( L_\infty \) quasi-isomorphism
\[ U : T_{\text{poly}}(K[[t]]) \rightarrow D_{\text{poly}}(K[[t]]). \]

Here is the main result of our paper.

Theorem 0.1. Assume \( \mathbb{R} \subset K \). Let \( A = \bigoplus_{i \geq 0} A^i \) be a super-commutative associative unital complete DG algebra in \( \text{Dir Inv Mod } K \). Consider the induced continuous \( A \)-multilinear \( L_\infty \) morphism
\[ U_A : A \hat{\otimes} T_{\text{poly}}(K[[t]]) \rightarrow A \hat{\otimes} D_{\text{poly}}(K[[t]]). \]
Suppose \( \omega \in A^1 \hat{\otimes} T_{\text{poly}}(K[[t]]) \) is a solution of the Maurer-Cartan equation in \( A \hat{\otimes} T_{\text{poly}}(K[[t]]) \). Define \( \omega' := (\partial^1 U_A)(\omega) \in A^1 \hat{\otimes} D_{\text{poly}}(K[[t]]) \). Then \( \omega' \) is a solution of the Maurer-Cartan equation in \( A \hat{\otimes} D_{\text{poly}}(K[[t]]) \), and there is continuous \( A \)-multilinear \( L_\infty \) quasi-isomorphism
\[ U_{A,\omega} : (A \hat{\otimes} T_{\text{poly}}(K[[t]]) \omega) \rightarrow (A \hat{\otimes} D_{\text{poly}}(K[[t]]) \omega'), \]
whose Taylor coefficients are
\[ (\partial^j U_{A,\omega})(\alpha) := \sum_{k \geq 0} \frac{1}{(j+k)!} (\partial^j+k U_A)(\omega^k \wedge \alpha) \]
for \( \alpha \in \prod \{ A \hat{\otimes} T_{\text{poly}}(K[[t]]) \} \).

Below is an outline of the paper, in which we mention the various terms appearing in the theorem.

In Section 1 we develop the theory of dir-inv modules. A dir-inv structure on a \( K \)-module \( M \) is a generalization of an adic topology. The category of dir-inv modules and continuous homomorphisms is denoted by \( \text{Dir Inv Mod } K \). The concept of dir-inv module, and related complete tensor product \( \hat{\otimes} \), are quite flexible, and

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are particularly well-suited for infinitely generated modules. Among other things we introduce the notion of DG Lie algebra in Dir Inv Mod \( K \).

Section 2 concentrates on poly differential operators. The results here are mostly generalizations of material from [EGA IV].

In Section 3 we review the coalgebra approach to \( L_\infty \) morphisms. The notions of continuous, \( A \)-multilinear and twisted \( L_\infty \) morphism are defined. The main result of this section is Theorem 3.27.

In Section 4 we recall the Kontsevich Formality Theorem. By combining it with Theorem 3.27 we deduce Theorem 0.1 (repeated as Theorem 4.15). In Theorem 0.1 the DG Lie algebras \( A \otimes T_{\text{poly}}(K[[t]]) \) and \( A \otimes D_{\text{poly}}(K[[t]]) \) are the \( A \)-multilinear extensions of \( T_{\text{poly}}(K[[t]]) \) and \( D_{\text{poly}}(K[[t]]) \) respectively, and \( (A \otimes T_{\text{poly}}(K[[t]]))^{\omega} \) and \( (A \otimes D_{\text{poly}}(K[[t]]))^{\omega} \) are their twists. The \( L_\infty \) morphism \( \mathcal{U}_A \) is the continuous \( A \)-multilinear extension of \( \mathcal{U} \), and \( \mathcal{U}_A,\omega \) is its twist.

Theorem 0.1 is used in Ye2, in which we study deformation quantization of algebraic varieties.

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1. DIR-INV MODULES

We begin the paper with a generalization of the notion of adic topology. In this section \( K \) is a commutative base ring, and \( C \) is a commutative \( K \)-algebra. The category \( \text{Mod} C \) is abelian and has direct and inverse limits. Unless specified otherwise, all limits are taken in \( \text{Mod} C \).

**Definition 1.1.** (1) Let \( M \in \text{Mod} C \). An inv module structure on \( M \) is an inverse system \( \{F^i M\}_{i \in \mathbb{N}} \) of \( C \)-submodules of \( M \). The pair \((M, \{F^i M\}_{i \in \mathbb{N}})\) is called an inv \( C \)-module.

(2) Let \((M, \{F^i M\}_{i \in \mathbb{N}})\) and \((N, \{F^i N\}_{i \in \mathbb{N}})\) be two inv \( C \)-modules. A function \( \phi : M \rightarrow N \) (\( C \)-linear or not) is said to be continuous if for every \( i \in \mathbb{N} \) there exists \( i' \in \mathbb{N} \) such that \( \phi(F^{i'} M) \subset F^i N \).

(3) Define \( \text{Inv Mod} C \) to be the category whose objects are the inv \( C \)-modules, and whose morphisms are the continuous \( C \)-linear homomorphisms.

We do not assume that the canonical homomorphism \( M \rightarrow \lim_{i \rightarrow \infty} F^i M \) is surjective nor injective. There is a full embedding \( \text{Mod} C \hookrightarrow \text{Inv Mod} C \), \( M \mapsto (M, \{F^i M\}_{i \in \mathbb{N}}) \). If \((M, \{F^i M\}_{i \in \mathbb{N}})\) and \((N, \{F^i N\}_{i \in \mathbb{N}})\) are two inv \( C \)-modules then \( M \oplus N \) is an inv module, with inverse system of submodules \( F^i(M \oplus N) := F^i M \oplus F^i N \). Thus \( \text{Inv Mod} C \) is a \( C \)-linear additive category.

Let \((M, \{F^i M\}_{i \in \mathbb{N}})\) be an inv \( C \)-module, let \( M', M'' \) be two \( C \)-modules, and suppose \( \phi : M' \rightarrow M \) and \( \psi : M \rightarrow M'' \) are \( C \)-linear homomorphisms. We get induced inv module structures on \( M' \) and \( M'' \) by defining \( F^i M' := \phi^{-1}(F^i M) \) and \( F^i M'' := \psi(F^i M) \).

Recall that a directed set is a partially ordered set \( J \) with the property that for any \( j_1, j_2 \in J \) there exists \( j_3 \in J \) such that \( j_1, j_2 \leq j_3 \).

**Definition 1.2.** (1) Let \( M \in \text{Mod} C \). A dir-inv module structure on \( M \) is a direct system \( \{F_j M\}_{j \in J} \) of \( C \)-submodules of \( M \), indexed by a nonempty directed set \( J \), together with an inv module structure on each \( F_j M \), such that for every \( j_1 \leq j_2 \) the inclusion \( F_{j_1} M \hookrightarrow F_{j_2} M \) is continuous. The pair \((M, \{F_j M\}_{j \in J})\) is called a dir-inv \( C \)-module.
(2) Let \((M, \{F_j M\}_{j \in J})\) and \((N, \{F_k N\}_{k \in K})\) be two dir-inv \(C\)-modules. A function \(\phi : M \rightarrow N\) (C-linear or not) is said to be continuous if for every \(j \in J\) there exists \(k \in K\) such that \(\phi(F_j M) \subset F_k N\), and \(\phi : F_j M \rightarrow F_k N\) is a continuous function between these two \(C\)-modules.

(3) Define \(\text{Dir Inv Mod} \ C\) to be the category whose objects are the dir-inv \(C\)-modules, and whose morphisms are the continuous \(C\)-linear homomorphisms.

There is no requirement that the canonical homomorphism \(\lim_{j \rightarrow} F_j M \rightarrow M\) will be surjective. An inv \(C\)-module \(M\) is endowed with the dir-inv module structure \(\{F_j M\}_{j \in J}\), where \(J := \{0\}\) and \(F_0 M := M\). Thus we get a full embedding \(\text{Inv Mod} \ C \hookrightarrow \text{Dir Inv Mod} \ C\). Given two dir-inv \(C\)-modules \((M, \{F_j M\}_{j \in J})\) and \((N, \{F_k N\}_{k \in K})\), we make \(M \oplus N\) into a dir-inv module as follows. The directed set is \(J \times K\), with the component-wise partial order, and the direct system of inv modules is \(F_{(j,k)} (M \oplus N) := F_j M \oplus F_k N\). The condition \(J \neq \emptyset\) in part (1) of the definition ensures that the zero module \(0 \in \text{Mod} \ C\) is an initial object in \(\text{Dir Inv Mod} \ C\). So \(\text{Dir Inv Mod} \ C\) is a \(C\)-linear additive category.

Let \((M, \{F_j M\}_{j \in J})\) be a dir-inv \(C\)-module, let \(M', M''\) be two \(C\)-modules, and suppose \(\phi : M' \rightarrow M\) and \(\psi : M \rightarrow M''\) are \(C\)-linear homomorphisms. We get induced dir-inv module structures \(\{F_j M'\}_{j \in J}\) and \(\{F_j M''\}_{j \in J}\) on \(M'\) and \(M''\) as follows. Define \(F_j (M') := \phi^{-1}(F_j M)\) and \(F_j M'' := \psi(F_j M)\), which have induced inv module structures via the homomorphisms \(\phi : F_j M' \rightarrow F_j M\) and \(\psi : F_j M \rightarrow F_j M''\).

**Definition 1.3.**

(1) An inv \(C\)-module \((M, \{F^i M\}_{i \in \mathbb{N}})\) is called discrete if \(F^i M = 0\) for \(i \geq 0\).

(2) An inv \(C\)-module \((M, \{F^i M\}_{i \in \mathbb{N}})\) is called complete if the canonical homomorphism \(M \rightarrow \lim_{\rightarrow i} M/F^i M\) is bijective.

(3) A dir-inv \(C\)-module \(M\) is called complete (resp. discrete) if it isomorphic, in \(\text{Dir Inv Mod} \ C\), to a dir-inv module \((N, \{F_j N\}_{j \in J})\), where all the inv modules \(F_j N\) are complete (resp. discrete) as defined above, and the canonical homomorphism \(\lim_{j \rightarrow} F_j N \rightarrow N\) is bijective.

(4) A dir-inv \(C\)-module \(M\) is called trivial if it isomorphic, in \(\text{Dir Inv Mod} \ C\), to an object of \(\text{Mod} \ C\), via the embedding \(\text{Mod} \ C \hookrightarrow \text{Dir Inv Mod} \ C\).

Note that \(M\) is a trivial dir-inv module if it is isomorphic, in \(\text{Dir Inv Mod} \ C\), to a discrete inv module. There do exist discrete dir-inv modules that are not trivial dir-inv modules; see Example 1.10. It is easy to see that if \(M\) is a discrete dir-inv module then it is also complete.

The base ring \(\mathbb{K}\) is endowed with the inv structure \({\ldots, 0, 0}\), so it is a trivial dir-inv \(\mathbb{K}\)-module. But the \(\mathbb{K}\)-algebra \(C\) could have more interesting dir-inv structures (cf. Example 1.5).

If \(f^* : C \rightarrow C'\) is a homomorphism of \(\mathbb{K}\)-algebras, then there is a functor \(f_* : \text{Dir Inv Mod} \ C' \rightarrow \text{Dir Inv Mod} \ C\). In particular any dir-inv \(C\)-module is a dir-inv \(\mathbb{K}\)-module.

**Definition 1.4.**

(1) Given an inv \(C\)-module \((M, \{F^i M\}_{i \in \mathbb{N}})\) its completion is the inv \(C\)-module \(\widehat{M}, \{F^i \widehat{M}\}_{i \in \mathbb{N}}\), defined as follows: \(\widehat{M} := \lim_{\rightarrow i} M/F^i M\) and \(F^i \widehat{M} := \text{Ker}(\widehat{M} \rightarrow M/F^i M)\). Thus we obtain an additive endofunctor \(M \mapsto \widehat{M}\) of \(\text{Inv Mod} \ C\).
(2) Given a dir-inv C-module \((M, \{F_j M\}_{j \in J})\) its completion is the dir-inv C-module \((\hat{M}, \{\hat{F}_j M\}_{j \in J})\) defined as follows. For any \(j \in J\) let \(\hat{F}_j M\) be the completion of the inv C-module \(F_j M\) as defined above. Then let \(\hat{M} := \lim_{\to j} \hat{F}_j M\) and \(\hat{F}_j \hat{M} := \text{Im}(\hat{F}_j M \to \hat{M})\). Thus we obtain an additive endofunctor \(M \mapsto \hat{M}\) of \(\text{Dir Inv Mod} C\).

An inv C-module \(M\) is complete iff the functorial homomorphism \(M \to \hat{M}\) is an isomorphism; and of course \(\hat{M}\) is complete. Nonetheless:

**Proposition 1.5.** Suppose \(M \in \text{Dir Inv Mod} C\) is complete. Then there is an isomorphism \(M \cong \hat{M}\) in \(\text{Dir Inv Mod} C\). This isomorphism is functorial.

**Proof.** For any dir-inv module \((M, \{F_j M\}_{j \in J})\) let’s define \(M' := \lim_{\to j} F_j M\). So \((M', \{F'_j M\}_{j \in J})\) is a dir-inv module, and there are functorial morphisms \(M' \to M\) and \(M' \to \hat{M}\). If \(M\) is complete then both these morphisms are isomorphisms. \(\square\)

Suppose \(\{M_k\}_{k \in K}\) is a collection of dir-inv modules, indexed by a set \(K\). There is an induced dir-inv module structure on \(M := \bigoplus_{k \in K} M_k\), constructed as follows. For any \(k\) let us denote by \(\{F_j M_k\}_{j \in J_k}\) the dir-inv structure of \(M_k\); so that each \(F_j M_k\) is an inv module. For each finite subset \(K_0 \subset K\) let \(J_{K_0} := \prod_{k \in K_0} J_k\), made into a directed set by component-wise partial order. Define \(J := \coprod_{K_0} J_{K_0}\), where \(K_0\) runs over the finite subsets of \(K\). For two finite subsets \(K_0 \subset K_1\), and two elements \(j_0 = \{j_{0,k}\}_{k \in K_0} \in J_{K_0}\) and \(j_1 = \{j_{1,k}\}_{k \in K_1} \in J_{K_1}\), we declare that \(j_0 \leq j_1\) if \(j_{0,k} \leq j_{1,k}\) for all \(k \in K_0\). This makes \(J\) into a directed set. Now for any \(j = \{j_k\}_{k \in K_0} \in J\) let \(F_j M := \bigoplus_{k \in K_0} F_{j_k} M_k\), which is an inv module. The dir-inv structure on \(M\) is \(\{F_j M\}_{j \in J}\).

**Proposition 1.6.** Let \(\{M_k\}_{k \in K}\) be a collection of dir-inv C-modules, and let \(M := \bigoplus_{k \in K} M_k\), endowed with the induced dir-inv structure.

1. \(M\) is a coproduct of \(\{M_k\}_{k \in K}\) in the category \(\text{Dir Inv Mod} C\).
2. There is a functorial isomorphism \(\hat{M} \cong \bigoplus_{k \in K} \hat{M}_k\).

**Proof.** (1) is obvious. For (2) we note that both \(\hat{M}\) and \(\bigoplus_{k \in K} \hat{M}_k\) are direct limits for the direct system \(\{M_j\}_{j \in J}\). \(\square\)

Suppose \(\{M_k\}_{k \in \mathbb{N}}\) is a collection of inv C-modules. For each \(k\) let \(\{F^i M_k\}_{i \in \mathbb{N}}\) be the inv structure of \(M_k\). Then \(M := \prod_{k \in \mathbb{N}} M_k\) is an inv module, with inv structure \(F^i M := (\prod_{k > i} M_k) \times (\prod_{k \leq i} F^i M_k)\). Next let \(\{M_k\}_{k \in \mathbb{N}}\) be a collection of dir-inv C-modules, and for each \(k\) let \(\{F_j M_k\}_{j \in J_k}\) be the dir-inv structure of \(M_k\). Then there is an induced dir-inv structure on \(M := \prod_{k \in \mathbb{N}} M_k\). Define a directed set \(J := \prod_{k \in \mathbb{N}} J_k\), with component-wise partial order. For any \(j = \{j_k\}_{k \in \mathbb{N}} \in J\) define \(F_j M := \prod_{k \in \mathbb{N}} F_{j_k} M_k\), which is an inv C-module as explained above. The dir-inv structure on \(M\) is \(\{F_j M\}_{j \in J}\).

**Proposition 1.7.** Let \(\{M_k\}_{k \in \mathbb{N}}\) be a collection of dir-inv C-modules, and let \(M := \prod_{k \in \mathbb{N}} M_k\), endowed with the induced dir-inv structure. Then \(M\) is a product of \(\{M_k\}_{k \in \mathbb{N}}\) in \(\text{Dir Inv Mod} C\).
Proof. All we need to consider is continuity. First assume that all the \( M_k \) are inv \( C \)-modules. Let’s denote by \( \pi_k : M \to M_k \) the projection. For each \( k, i \in \mathbb{N} \) and \( i' \geq \max(i, k) \) we have \( \pi_k(F^i M) = F^k M_k \). This shows that the \( \pi_k \) are continuous. Suppose \( L \) is an inv \( C \)-module and \( \phi_k : L \to M_k \) are morphisms in \( \text{Inv Mod} C \). For any \( i \in \mathbb{N} \) there exists \( i' \in \mathbb{N} \) such that \( \phi_k(F^{i'} L) \subset F^i M_k \) for all \( k \leq i \). Therefore the homomorphism \( \phi : L \to M \) with components \( \phi_k \) is continuous.

Now let \( M_k \) be dir-inv \( C \)-modules, with dir-inv structures \( \{F_j M_k\}_{j \in J_k} \). For any \( j = \{j_k\} \in J \) one has \( \pi_k(F_j M) = F_{j_k} M_k \), and as shown above \( \pi_k : F_j M \to F_{j_k} M_k \) is continuous. Given a dir-inv module \( L \) and and morphisms \( \phi_k : L \to M_k \) in \( \text{Dir Inv Mod} C \), we have to prove that \( \phi : L \to M \) is continuous. Let \( \{F_j L\}_{j \in J_L} \) be the dir-inv structure of \( L \). Take any \( j \in J_L \). Since \( \phi_k \) is continuous, there exists some \( j_k \in J_k \) such that \( \phi_k(F_j L) \subset F_{j_k} M_k \). But then \( \phi(F_j L) \subset F_j M \) for \( j := \{j_k\}_{k \in \mathbb{N}} \), and by the previous paragraph \( \phi : F_j L \to F_j M \) is continuous. \( \square \)

The following examples should help to clarify the notion of dir-inv module.

Example 1.8. Let \( c \) be an ideal in \( C \). Then each finitely generated \( C \)-module \( M \) has an inv structure \( \{F^i M\}_{i \in \mathbb{N}} \), where we define the submodules \( F^i M := c^{i+1} M \). This is called the \( c \)-adic inv structure. Any \( C \)-module \( M \) has a dir-inv structure \( \{F_j M\}_{j \in J} \), which is the collection of finitely generated \( C \)-submodules of \( M \), directed by inclusion, and each \( F_j M \) is given the \( c \)-adic inv structure. We get a fully faithful functor \( \text{Mod} C \to \text{Dir Inv Mod} C \). This dir-inv module structure on \( M \) is called the \( c \)-adic dir-inv structure.

In case \( C \) is noetherian and \( c \)-adically complete, then the finitely generated modules are complete as inv \( C \)-modules, and hence all modules are complete as dir-inv modules.

Example 1.9. Suppose \( (M, \{F^i M\}_{i \in \mathbb{N}}) \) is an inv \( C \)-module, and \( \{i_k\}_{k \in \mathbb{N}} \) is a non-decreasing sequence in \( \mathbb{N} \) with \( \lim_{k \to \infty} i_k = \infty \). Then \( \{F^{i_k} M\}_{k \in \mathbb{N}} \) is a new inv structure on \( M \), yet the identity map \( (M, \{F^i M\}_{i \in \mathbb{N}}) \to (M, \{F^{i_k} M\}_{k \in \mathbb{N}}) \) is an isomorphism in \( \text{Inv Mod} C \).

A similar modification can be done for dir-inv modules. Suppose \( (M, \{F_j M\}_{j \in J}) \) is a dir-inv \( C \)-module, and \( J' \subset J \) is a subset that is cofinal in \( J \). Then \( \{F_j M\}_{j \in J'} \) is a new dir-inv structure on \( M \), yet the identity map \( (M, \{F_j M\}_{j \in J}) \to (M, \{F_j M\}_{j \in J'}) \) is an isomorphism in \( \text{Dir Inv Mod} C \).

Example 1.10. Let \( M \) be the free \( \mathbb{K} \)-module with basis \( \{e_p\}_{p \in \mathbb{N}} \); so \( M = \bigoplus_{p \in \mathbb{N}} \mathbb{K} e_p \) in \( \text{Mod} \mathbb{K} \). We put on \( M \) the inv module structure \( \{F^i M\}_{i \in \mathbb{N}} \) with \( F^i M := 0 \) for all \( i \). Let \( N \) be the same \( \mathbb{K} \)-module as \( M \), but put on it the inv module structure \( \{F^i N\}_{i \in \mathbb{N}} \) with \( F^i N := \bigoplus_{p=0}^\infty \mathbb{K} e_p \). Also let \( L \) be the \( \mathbb{K} \)-module \( M \), but put on it the dir-inv module structure \( \{F_j L\}_{j \in \mathbb{N}} \), with \( F_j L := \bigoplus_{p=0}^j \mathbb{K} e_p \) the discrete inv module whose inv structure is \( \{0, \ldots, 0\} \). Both \( L \) and \( M \) are discrete and complete as dir-inv \( \mathbb{K} \)-modules, and \( N \cong \prod_{p \in \mathbb{N}} \mathbb{K} e_p \). The dir-inv module \( M \) is trivial. \( L \) is not a trivial dir-inv \( \mathbb{K} \)-module, because it is not isomorphic in \( \text{Dir Inv Mod} \mathbb{K} \) to any inv module. The identity maps \( L \to M \to N \) are continuous. The only continuous \( \mathbb{K} \)-linear homomorphisms \( M \to L \) are those with finitely generated images.

Remark 1.11. In the situation of the previous example, suppose we put on the three modules \( L, M, N \) genuine \( \mathbb{K} \)-linear topologies, using the limiting processes and starting from the discrete topology. Namely \( M, N/F^i N \) and \( F_j L \) get the discrete
topologies; \( L \cong \lim_{j\to} F_j L \) gets the \( \lim_{j\to} \) topology; and \( N \subset \lim_{i\to} N/F^i N \) gets the \( \lim_{i\to} \) topology (as in \([Ye1\text{ Section 1.1]}\)). Then \( L \) and \( M \) become the same discrete topological module, and \( \hat{N} \) is the topological completion of \( N \). We see that the notion of a dir-inv structure is more subtle than that of a topology, even though a similar language is used.

**Example 1.12.** Suppose \( \mathbb{K} \) is a field, and let \( M := \mathbb{K} \), the free module of rank 1. Up to isomorphism in \( \text{Dir Inv Mod} \, \mathbb{K} \), \( M \) has three distinct dir-inv module structures. We can denote them by \( M_1, M_2, M_3 \) in such a way that the identity maps \( M_1 \to M_2 \to M_3 \) are continuous. The only continuous \( \mathbb{K} \)-linear homomorphisms \( M_i \to M_j \) with \( i > j \) are the zero homomorphisms. \( M_2 \) is the trivial dir-inv structure, and it is the only interesting one (the others are “pathological”).

**Example 1.13.** Suppose \( M = \bigoplus_{p \in \mathbb{Z}} M^p \) is a graded \( C \)-module. The grading induces a dir-inv structure on \( M \), with \( J := \mathbb{N} \), \( F_j M := \bigoplus_{p = -j} M^p \), and \( F^i F_j M := \bigoplus_{p = -j + i} M^p \). The completion satisfies \( \hat{M} \cong (\prod_{p \geq 0} M^p) \oplus (\bigoplus_{p < 0} M^p) \) in \( \text{Dir Inv Mod} \, C \), where each \( M^p \) has the trivial dir-inv module structure.

It makes sense to talk about convergence of sequences in a dir-inv module. Suppose \( \{M, \{F^i M\}_{i \in \mathbb{N}}\} \) is an inv \( C \)-module and \( \{m_i\}_{i \in \mathbb{N}} \) is a sequence in \( M \). We say that \( \lim_{i \to \infty} m_i = 0 \) if for every \( i_0 \) there is some \( i_1 \) such that \( \{m_i\}_{i \geq i_1} \subset F_{i_0} M \).

**Proposition 1.14.** Assume \( M \) is a complete dir-inv \( C \)-module. Then any Cauchy sequence in \( M \) has a unique limit.

**Proof.** Consider a Cauchy sequence \( \{m_i\}_{i \in \mathbb{N}} \) in \( M \). Convergence is an invariant of isomorphisms in \( \text{Dir Inv Mod} \, C \). By Definition 1.3 we may assume that in the dir-inv structure \( \{F_j M\}_{j \in J} \) of \( M \) each inv module \( F_j M \) is complete. By passing to the sequence \( \{m_i - m_{i_1}\}_{i \in \mathbb{N}} \) for suitable \( i_1 \), we can also assume the sequence is contained in one of the inv modules \( F_j M \). Thus we reduce to the case of convergence in a complete inv module, which is standard. \( \Box \)

Let \( \{M, \{F^i M\}_{i \in \mathbb{N}}\} \) and \( \{N, \{F^i N\}_{i \in \mathbb{N}}\} \) be two inv \( C \)-modules. We make \( M \otimes_C N \) into an inv module by defining

\[
F^i(M \otimes_C N) := \text{Im} \left( (M \otimes_C F^i N) \oplus (F^i M \otimes_C N) \to M \otimes_C N \right).
\]

For two dir-inv \( C \)-modules \( \{M, \{F_j M\}_{j \in J}\} \) and \( \{N, \{F_k N\}_{k \in K}\} \), we put on \( M \otimes_C N \) the dir-inv module structure \( \{F_{(j,k)}(M \otimes_C N)\}_{(j,k) \in J \times K} \), where

\[
F_{(j,k)}(M \otimes_C N) := \text{Im}(F_j M \otimes_C F_k N \to M \otimes_C N).
\]

**Definition 1.15.** Given \( M, N \in \text{Dir Inv Mod} \, C \) we define \( \hat{N} \otimes_C M \) to be the completion of the dir-inv \( C \)-module \( N \otimes_C M \).

**Example 1.16.** Let’s examine the behavior of the dir-inv modules \( L, M, N \) from Example 1.10 with respect to complete tensor product. There is an isomorphism \( L \otimes_{\mathbb{K}} N \cong \bigoplus_{p \in \mathbb{Z}} \hat{N} \) in \( \text{Dir Inv Mod} \, \mathbb{K} \), so according to Proposition 1.2 there is also an isomorphism \( L \otimes_{\mathbb{K}} N \cong \bigoplus_{p \in \mathbb{Z}} \hat{N} \) in \( \text{Dir Inv Mod} \, \mathbb{K} \). On the other hand \( M \otimes_{\mathbb{K}} N \)
is an inv $\mathbb{K}$-module with inv structure $F^i(M \otimes_{\mathbb{K}} N) = M \otimes_{\mathbb{K}} F^iN$, so $M \otimes_{\mathbb{K}} N \cong \prod_{p \in \mathbb{N}} M$ in $\text{Dir Inv Mod } \mathbb{K}$. The series $\sum_{p=0}^{\infty} e_p \otimes e_p$ converges in $M \otimes_{\mathbb{K}} N$, but not in $L \otimes_{\mathbb{K}} N$.

A graded object in $\text{Dir Inv Mod } C$, or a graded dir-inv $C$-module, is an object $M \in \text{Dir Inv Mod } C$ of the form $M = \bigoplus_{i \in \mathbb{Z}} M^i$, with $M^i \in \text{Dir Inv Mod } C$. According to Proposition 1.16 we have $\hat{M} \cong \bigoplus_{i \in \mathbb{Z}} \hat{M}^i$. Given two graded objects $M = \bigoplus_{i \in \mathbb{Z}} M^i$ and $N = \bigoplus_{i \in \mathbb{Z}} N^i$ in $\text{Dir Inv Mod } C$, the tensor product is also a graded object in $\text{Dir Inv Mod } C$, with

$$(M \otimes_C N)^i := \bigoplus_{p+q=i} M^p \otimes_C N^q.$$ 

In this paper “algebra” is taken in the weakest possible sense: by $C$-algebra we mean an $C$-module $A$ together with a $C$-bilinear function $\mu_A : A \times A \to A$. If $A$ is associative, or a Lie algebra, then we will specify that. However, “commutative algebra” will mean, by default, a commutative associative unital $C$-algebra. Another convention is that a homomorphism between unital algebras is a unital homomorphism, and a module over a unital algebra is a unital module.

**Definition 1.17.**  (1) An algebra in $\text{Dir Inv Mod } C$ is an object $A \in \text{Dir Inv Mod } C$, together with a continuous $C$-bilinear function $\mu_A : A \times A \to A$.

(2) A differential graded algebra in $\text{Dir Inv Mod } C$ is a graded object $A = \bigoplus_{i \in \mathbb{Z}} A^i$ in $\text{Dir Inv Mod } C$, together with continuous $C$-(bi)linear functions $\mu_A : A \times A \to A$ and $d_A : A \to A$, such that $A$ is a differential graded algebra, in the usual sense, with respect to the differential $d_A$ and the multiplication $\mu_A$.

(3) Let $A$ be an algebra in $\text{Dir Inv Mod } C$, with dir-inv structure $\{F_j A\}_{j \in J}$. We say that $A$ is a unital algebra in $\text{Dir Inv Mod } C$ if it has a unit element $1_A$ (in the usual sense), such that $1_A \in \bigcup_{j \in J} F_j A$.

The base ring $\mathbb{K}$, with its trivial dir-inv structure, is a unital algebra in $\text{Dir Inv Mod } \mathbb{K}$. In item (3) above, the condition $1_A \in \bigcup_{j \in J} F_j A$ is equivalent to the ring homomorphism $\mathbb{K} \to A$ being continuous.

We will use the common abbreviation “DG” for “differential graded”. An algebra in $\text{Dir Inv Mod } C$ can have further attributes, such as “Lie” or “associative”, which have their usual meanings. If $A \in \text{Inv Mod } C$ then we also say it is an algebra in $\text{Inv Mod } C$.

**Example 1.18.** In the situation of Example 1.15 the $\mathbb{K}$-adic inv structure makes $C$ and $\hat{C}$ into unital algebras in $\text{Inv Mod } C$.

Recall that a graded algebra $A$ is called super-commutative if $ba = (-1)^{|i|}ab$ and $c^2 = 0$ for all $a \in A^i$, $b \in A^j$, $c \in A^k$ and $k$ odd. There is no essential difference between left and right DG $A$-modules.

**Proposition 1.19.** Let $A$ and $g$ be DG algebras in $\text{Dir Inv Mod } C$.

(1) The completion $\hat{A}$ is a DG algebra in $\text{Dir Inv Mod } C$.

(2) If $A$ is complete, then the canonical isomorphism $A \cong \hat{A}$ of Proposition 1.16 is an isomorphism of DG algebras.

(3) The complete tensor product $A \hat{\otimes}_C g$ is a DG algebra in $\text{Dir Inv Mod } C$.

(4) If $A$ is a super-commutative associative unital algebra, then so is $\hat{A}$. 

(5) If $g$ is a DG Lie algebra and $A$ is a super-commutative associative unital algebra, then $A \hat{\otimes}_C g$ is a DG Lie algebra.

Proof. (1) This is a consequence of a slightly more general fact. Consider modules $M_1, \ldots, M_r, N \in \text{Dir Inv Mod} C$ and a continuous $C$-multilinear linear function $\phi : M_1 \times \cdots \times M_r \to N$. We claim that there is an induced continuous $C$-multilinear linear function $\hat{\phi} : \prod_k M_k \to \hat{N}$. This operation is functorial (w.r.t. morphisms $M_k \to M_k'$ and $N \to N'$), and monoidal (i.e. it respects composition in the $k$th argument with a continuous multilinear function $\psi : L_1 \times \cdots \times L_s \to M_k$).

First assume $M_1, \ldots, M_r, N \in \text{Inv Mod} C$, with inv structures $\{F^i M_j\}_{i \in \mathbb{N}}$ etc. For any $i \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that $\phi(\prod_k F^{i'} M_k) \subset F^i N$. Therefore there’s an induced continuous $C$-multilinear function $\hat{\phi} : \prod_k M_k \to \hat{N}$. It is easy to verify that $\phi \to \hat{\phi}$ is functorial and monoidal.

Next consider the general case, i.e. $M_1, \ldots, M_r, N \in \text{Dir Inv Mod} C$. Let $\{F_j M_k\}_{j \in J_k}$ be the dir-inv structure of $M_k$, and let $\{F_j N\}_{j \in J_N}$ be the dir-inv structure of $N$. By continuity of $\phi$, given $(j_1, \ldots, j_r) \in \prod_k J_k$ there exists $j' \in J_N$ such that $\phi(\prod_k F_{j_k} M_k) \subset F_{j'} N$, and $\hat{\phi} : \prod_k F_{j_k} M_k \to F_{j'} N$ is continuous. By the previous paragraph this extends to $\hat{\phi} : \prod_k F_{j_k} M_k \to \hat{\phi} F_{j'} N$. Passing to the direct limit in $(j_1, \ldots, j_r)$ we obtain $\hat{\phi} : \prod_k M_k \to \hat{N}$. Again this operation is functorial and monoidal.

(2) Let $A' \subset A$ be as in the proof of Proposition 1.19. This is a subalgebra. The arguments used in the proof of part (1) above show that $A' \to A$ and $A' \to \hat{A}$ are algebra homomorphisms.

(3) Let us write $\cdot_{A}$ and $\cdot_{g}$ for the two multiplications, and $d_{A}$ and $d_{g}$ for the differentials. Then $A \otimes_C g$ is a DG algebra with multiplication

\[(a_1 \otimes \gamma_1) \cdot (a_2 \otimes \gamma_2) := (-1)^{|i_2 j_1|} (a_1 \cdot_{A} a_2) \otimes (\gamma_1 \cdot_{g} \gamma_2)\]

and differential

\[d(a_1 \otimes \gamma_1) := d_{A}(a_1) \otimes \gamma_1 + (-1)^{|i_1|} a_1 \otimes d_{g}(\gamma_1)\]

for $a_k \in A^{i_k}$ and $\gamma_k \in g^{j_k}$. These operations are continuous, so $A \otimes_C g$ is a DG algebra in $\text{Dir Inv Mod} C$. Now use part (1).

(4, 5) The various identities (Lie etc.) are preserved by $\hat{\otimes}$. Definition 1.17(3) ensures that $\hat{A}$ has a unit element. \hfill $\square$

**Definition 1.20.** Suppose $A$ is a DG super-commutative associative unital algebra in $\text{Dir Inv Mod} C$.

(1) A **DG $A$-module** in $\text{Dir Inv Mod} C$ is a graded object $M \in \text{Dir Inv Mod} C$, together with a continuous $C$-bilinear homomorphism $A \times M \to M$, which makes $M$ into a DG $A$-module in the usual sense.

(2) A **DG $A$-module Lie algebra** in $\text{Dir Inv Mod} C$ is a DG Lie algebra $g \in \text{Dir Inv Mod} C$, together with a continuous $C$-bilinear homomorphism $A \times g \to g$, such that $g$ is a DG $A$-module, and

\[[a_1 \gamma_1, a_2 \gamma_2] = (-1)^{|i_2 j_1|} a_1 a_2 [\gamma_1, \gamma_2]\]

for all $a_k \in A^{i_k}$ and $\gamma_k \in g^{j_k}$. 


Example 1.21. If $A$ is a DG super-commutative associative unital algebra in $\text{Dir Inv Mod} C$, and $g$ is a DG Lie algebra in $\text{Dir Inv Mod} C$, then $A \otimes_C g$ is a DG $\hat{A}$-module Lie algebra in $\text{Dir Inv Mod} C$.

Let $A$ be a DG super-commutative associative unital algebra in $\text{Dir Inv Mod} C$, and let $M, N$ be two DG $A$-modules in $\text{Dir Inv Mod} C$. The tensor product $M \otimes_A N$ is a quotient of $M \otimes_C N$, and as such it has a dir-inv structure. Moreover, $M \otimes_A N$ is a DG $A$-module in $\text{Dir Inv Mod} C$, and we define $M \hat{\otimes}_A N$ to be its completion, which is a DG $\hat{A}$-module in $\text{Dir Inv Mod} C$.

Proposition 1.22. Let $A$ and $B$ be DG super-commutative associative unital algebras in $\text{Dir Inv Mod} C$, and let $A \to B$ be a continuous homomorphism of DG $C$-algebras.

1. Suppose $M$ is a DG $A$-module in $\text{Dir Inv Mod} C$. Then $B \hat{\otimes}_A M$ is a DG $B$-module in $\text{Dir Inv Mod} C$.

2. Suppose $g$ is a DG $A$-module Lie algebra in $\text{Dir Inv Mod} C$. Then $B \hat{\otimes}_A g$ is a DG $\hat{B}$-module Lie algebra in $\text{Dir Inv Mod} C$.

Proof. Like Proposition [1.21].

Example 1.23. In the situation of Example [1.21] one has

$$ \text{Hom}^{\text{cont}}_C(M, N) \cong \text{Hom}_{\text{Dir Inv Mod} C}(M, N), $$

i.e. the $C$-module of continuous $C$-linear homomorphisms. In general this module has no obvious structure. However, if $M$ is an inv $C$-module with inv structure $\{F^i M\}_{i \in \mathbb{N}}$, and $N$ is a discrete inv $C$-module, then

$$ \text{Hom}^{\text{cont}}_C(M, N) \cong \lim_{i \to} \text{Hom}_C(M/F^i M, N). $$

In this case we consider each

$$ F_i \text{Hom}^{\text{cont}}_C(M, N) := \text{Hom}_C(M/F^i M, N) $$

as a discrete inv module, and this endows $\text{Hom}^{\text{cont}}_C(M, N)$ with a dir-inv structure.

Example 1.24. This example is taken from [Ye1]. Assume $K$ is noetherian and $C$ a finitely generated $K$-algebra. For $q \in \mathbb{N}$ define $B_q(C) := B^{-q}(C) := C \otimes K \cdots \otimes K$, and define $B_q(C) := B^{-q}(C)$ to be the adic completion of $B_q(C)$ with respect to the ideal $\text{Ker}(B_q(C) \to C)$.

There is a $K$-algebra homomorphism $B^0(C) \to \hat{B}^{-q}(C)$, corresponding to the two extreme tensor factors, and in this way we view $\hat{B}^{-q}(C)$ as a complete inv $B^0(C)$-module. There is a continuous coboundary operator that makes $\hat{B}(C) := \bigoplus_{q \in \mathbb{N}} \hat{B}^{-q}(C)$ into a complex of $B^0(C)$-modules, and there is a quasi-isomorphism $\hat{B}(C) \to C$. We call $\hat{B}(C)$ the complete un-normalized bar complex of $C$. 
Next define $\hat{C}_q(C) = \hat{C}^{-q}(C) := C \otimes_{B^q(C)} \hat{B}^{-q}(C)$. This is a complete inv $C$-module. The complex $\hat{C}(C)$ is called the complete Hochschild chain complex of $C$. Finally let $c^{pq}_{cd}(C) := \text{Hom}^q_C(\hat{C}^{-q}(C), C)$. The complex $C_{cd}(C) := \bigoplus_{q \in \mathbb{N}} c^{pq}_{cd}(C)$ is called the continuous Hochschild cochain complex of $C$.

2. Poly Differential Operators

In this section $K$ is a commutative base ring, and $C$ is a commutative $K$-algebra. The symbol $\otimes$ means $\otimes_K$. We discuss some basic properties of poly differential operators, expanding results from [EGA IV].

Definition 2.1. Let $M_1, \ldots, M_p, N$ be $C$-modules. A $K$-multilinear function $\phi : M_1 \times \cdots \times M_p \to N$ is called a poly differential operator (over $C$ relative to $K$) if there exists some $d \in \mathbb{N}$ such that for any $(m_1, \ldots, m_p) \in \prod M_i$ and any $i \in \{1, \ldots, p\}$ the function $M_i \to N, m \mapsto \phi(m_1, \ldots, m_{i-1}, m, m_{i+1}, \ldots, m_p)$ is a differential operator of order $\leq d$, in the sense of [EGA IV, Section 16.8]. In this case we say that $\phi$ has order $\leq d$ in each argument.

We shall denote the set of poly differential operators $\prod M_i \to N$ over $C$ relative to $K$, of order $\leq d$ in all arguments, by $F_d \text{Diff}_\text{poly}(C; M_1, \ldots, M_p; N)$.

And we define $D \text{iff}_\text{poly}(C; M_1, \ldots, M_p; N) := \bigcup_{d \geq 0} F_d \text{Diff}_\text{poly}(C; M_1, \ldots, M_p; N)$, the union being inside the set of all $K$-multilinear functions $\prod M_i \to N$. By default we only consider poly differential operators relative to $K$.

For a natural number $p$ the $p$-th un-normalized $B_p(C)$ was defined in Example 1.24. Let $I_p(C)$ be the kernel of the ring homomorphism $B_p(C) \to C$. Define $C_p(C) := C \otimes_{B_0(C)} B_p(C)$, the $p$-th Hochschild chain module of $C$ (relative to $K$). For any $d \in \mathbb{N}$ define $B_{p,d}(C) := B_p(C) / I_p(C)^{d+1}$, $C_{p,d}(C) := C \otimes_{B_0(C)} B_{p,d}(C)$ and $C_{p,d}(C; M_1, \ldots, M_p) := C_{p,d}(C) \otimes_{B_{p,d}(C)} (M_1 \otimes \cdots \otimes M_p)$.

Let $\phi_{\text{uni}} : \prod_{i=1}^p M_i \to C_{p,d}(C; M_1, \ldots, M_p)$ be the $K$-multilinear function $\phi_{\text{uni}}(m_1, \ldots, m_p) := 1 \otimes (m_1 \otimes \cdots \otimes m_p)$.

Observe that for $p = 1$ we get $C_{1,d}(C) = \mathcal{P}^d(C)$, the module of principal parts of order $d$ (see [EGA IV]). In the same way that $\mathcal{P}^d(C)$ parameterizes differential operators, $C_{p,d}(C)$ parameterizes poly differential operators:

Lemma 2.2. The assignment $\psi \mapsto \psi \circ \phi_{\text{uni}}$ is a bijection $\text{Hom}_C(C_{p,d}(C; M_1, \ldots, M_p), N) \to F_d \text{Diff}_\text{poly}(C; M_1, \ldots, M_p; N)$. 


Proof: The same arguments used in [EGA IV, Section 16.8] also apply here. Cf. [Ye1, Section 1.4].

In case $M_1 = \cdots = M_p = N = C$ we see that
\[
\Diff_{\text{poly}}(C; C, \ldots, C; C) \cong \lim_{\longrightarrow} \Hom_C(C_{p,d}(C), C)
\]
\[
\cong \Hom_C^\cont(\tilde{C}_p(C), C) = C_{p,d}^\cont(C),
\]
with notation of Example [22].

Proposition 2.4. Suppose $C$ is a finitely generated $K$-algebra, with ideal $\mathfrak{c} \subset C$. Let $M_1, \ldots, M_p, N$ be $C$-modules, and let $\phi : \prod M_i \to N$ be a multi differential operator over $C$ relative to $K$. Then $\phi$ is continuous for the $\mathfrak{c}$-adic dir-inv structures on $M_1, \ldots, M_p, N$.

Proof. Suppose $\phi$ has order $\leq d$ in each of its arguments, and let $\psi : C_{p,d}(C; M_1, \ldots, M_p) \to N$
be the corresponding $C$-linear homomorphism. As in [Ye1, Proposition 1.4.3], since $C$ is a finitely generated $K$-algebra, it follows that $B_{p,d}(C)$ is a finitely generated module over $B_0(C)$; and hence $C_{p,d}(C)$ is a finitely generated $C$-module. Let’s denote by $\{F_j M_i\}_{j \in J_i}$ and $\{F_k N\}_{k \in K}$ the $\mathfrak{c}$-adic dir-inv structures on $M_i$ and $N$. For any $j_1, \ldots, j_p$ the $B_{p-2}(C)$-module $F_{j_1} M_1 \otimes \cdots \otimes F_{j_p} M_p$ is finitely generated, and hence the $C$-module $C_{p,d}(C; F_{j_1} M_1, \ldots, F_{j_p} M_p)$ is finitely generated. Therefore
\[
\psi(C_{p,d}(C; F_{j_1} M_1, \ldots, F_{j_p} M_p)) = F_k N
\]
for some $k \in K$.

It remains to prove that $\phi : \prod F_{j_i} M_i \to F_k N$ is continuous for the $\mathfrak{c}$-adic inv structures. But just like [Ye1, Proposition 1.4.6], for any $i$ and $l$ one has
\[
\phi(F_{j_1} M_1, \ldots, c^{i+ld} F_{j_l} M_1, \ldots, F_{j_p} M_p) \subset c^l F_k N.
\]

Suppose $C'$ is a commutative $C$-algebra with ideal $\mathfrak{c'} \subset C'$. One says that $C'$ is $\mathfrak{c}'$-adically formally étale over $C$ if the following condition holds. Let $D$ be a commutative $C$-algebra with nilpotent ideal $\mathfrak{d}$, and let $f : C' \to D/\mathfrak{d}$ be a $C$-algebra homomorphism such that $f(\mathfrak{c}'^i) = 0$ for $i \gg 0$. Then $f$ lifts uniquely to a $C$-algebra homomorphism $\tilde{f} : C' \to D$. The important instances are when $C \to C'$ is étale (and then $\mathfrak{c}' = 0$); and when $C'$ is the $\mathfrak{c}$-adic completion of $C$ for some ideal $\mathfrak{c} \subset A$ (and $\mathfrak{c}' = C' \mathfrak{c}$). In both these instances $C'$ is $\mathfrak{c}$-adically complete; and if $C$ is noetherian, then $C \to C'$ is also flat.

Lemma 2.6. Let $C'$ be a $\mathfrak{c}'$-adically formally étale $C$-algebra. Define $C'_j := C'/\mathfrak{c}'^{j+1}$. Consider $C'$ and $C_{p,d}(C)$ as inv $C$-modules, with the $\mathfrak{c}'$-adic and discrete inv structures respectively. Then the canonical homomorphism
\[
C' \otimes_C C_{p,d}(C) \to \lim_{\longrightarrow} C_{p,d}(C'_j)
\]
is bijective.
Proof. Define ideals
\[ \mathcal{I}'_p := \ker \left( C_p(C') \to C_p(C'_0) \right) \]
and
\[ J := \ker \left( C'_j \otimes_C C_{p,d}(C) \to C'_j \right). \]
By the transitivity and the base change properties of formally étale homomorphisms, the ring homomorphism
\[ C_p(C) \cong C \otimes \cdots \otimes C \to C'_j \otimes \cdots \otimes C' \cong C'_p(C') \]
is \( \mathcal{I}'_p \)-adically formally étale. Consider the commutative diagram of ring homomorphisms (with solid arrows)
\[
\begin{array}{cccccc}
C & \rightarrow & C_p(C) & \rightarrow & C'_j \otimes_C C_{p,d}(C) & \rightarrow & C_{p,d}(C'_j) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C' & \rightarrow & C_p(C') & \rightarrow & C'_j & \rightarrow & C'_j.
\end{array}
\]
The ideal \( J \) satisfies \( J^{d+1} = 0 \), and the ideal \( \ker \left( C_{p,d}(C'_j) \to C'_j \right) \) is nilpotent too. Due to the unique lifting property the dashed arrows exist and are unique, making the whole diagram commutative. Moreover \( g : C'_p(C') \to C'_p(C'_j) \) has to be the canonical surjection, and \( \bar{f} \) is surjective.
A little calculation shows that \( \bar{f}((P'_d(C'))^{d+1}) = 0 \), and hence \( \bar{f} \) induces a homomorphism
\[ \bar{f} : C_{p,d}(C') \to C' \otimes_C C_{p,d}(C). \]
Let
\[ \mathcal{I}'_{p,d} := \ker \left( C_{p,d}(C') \to C_{p,d}(C'_0) \right). \]
Another calculation shows that \( \bar{f}((\mathcal{I}'_{p,d})^{(d+1)}) = 0 \). The conclusion is that there are surjections
\[ C_{p,d}(C'_{d+1}) \xrightarrow{\bar{f}} C' \otimes_C C_{p,d}(C) \xrightarrow{\cdot} C_{p,d}(C'_j), \]
such that \( e \circ \bar{f} \) is the canonical surjection. Passing to the inverse limit we deduce that
\[ C' \otimes_C C_{p,d}(C) \to \lim_{\to} C_{p,d}(C'_j) \]
is bijective. \( \square \)

Proposition 2.7. Assume \( C \) is a noetherian finitely generated \( \mathbb{K} \)-algebra, and \( C' \) is a noetherian, \( \mathcal{C} \)-adically complete, flat, \( \mathcal{C} \)-adically formally étale \( C \)-algebra. Let \( M_1, \ldots, M_p, N \) be \( C \)-modules, and define \( M'_i := C' \otimes_C M_i \) and \( N' := C' \otimes_C N \).

1. Suppose \( \phi : \prod_{i=1}^p M_i \to N \) is a poly differential operator over \( C \). Then \( \phi \) extends uniquely to a poly differential operator \( \phi' : \prod_{i=1}^p M_i' \to N' \) over \( C' \).
   If \( \phi \) has order \( \leq d \) then so does \( \phi' \).

2. The homomorphism
   \[ C' \otimes_C F_d \text{Diff}_\text{poly}(C; M_1, \ldots, M_p; N) \]
   \[ \quad \to F_d \text{Diff}_\text{poly}(C'; M_1', \ldots, M_p'; N'), \]
   \( c' \otimes \phi \mapsto c' \phi \), is bijective.
Combining this with (2.8) we get
\[ k \]
Now for any \( C \)
\( \otimes \)
tensor algebra. Let us denote the multiplication in \( T_g \)
-module. Therefore
\[ \phi \]
Backtracking we see that
\[ φ \]
(2.9)
The conclusion is that
\[ \sum \]
3. \( L_\infty \) Morphisms and Their Twists
In this section we expand some results on \( L_\infty \) algebras and morphisms from [Kol] Section 4. Much of the material presented here is based on discussions with Vladimir Hinich. There is some overlap with Section 2.2 of [Fu], with Section 6.1 of [Le], and possibly with other accounts.
Let \( k \) be a field of characteristic 0. Given a graded \( k \)-module \( g = \bigoplus_{i \in \mathbb{Z}} g^i \) and a natural number \( j \) let \( T^j g := g \otimes \cdots \otimes g \). The direct sum \( T g := \bigoplus_{j \in \mathbb{N}} T^j g \) is the tensor algebra. Let us denote the multiplication in \( T g \) by \( \otimes \). (This is just another way of writing \( \otimes \), but it will be convenient to do so.)
The permutation group $\mathfrak{S}_j$ acts on $T^j\mathfrak{g}$ as follows. For any sequence of integers $d = (d_1, \ldots, d_j)$ there is a group homomorphism $\text{sgn}_d : \mathfrak{S}_j \to \{\pm 1\}$ such that on a transposition $\sigma = (p, p + 1)$ the value is $\text{sgn}_d(\sigma) = (-1)^{d_p d_{p+1}}$. The action of a permutation $\sigma \in \mathfrak{S}_j$ on $T^j\mathfrak{g}$ is then

$$\sigma(\gamma_1 \circ \cdots \circ \gamma_j) := \text{sgn}_d(\sigma)\gamma_{\sigma(1)} \circ \cdots \circ \gamma_{\sigma(j)}$$

for $\gamma_1 \in \mathfrak{g}^{d_1}, \ldots, \gamma_j \in \mathfrak{g}^{d_j}$. Define $\tilde{S}\mathfrak{g}$ to be the set of $\mathfrak{S}_j$-invariants inside $T^j\mathfrak{g}$, and $\tilde{\mathfrak{g}} := \bigoplus_{j \geq 0} \tilde{S}^j\mathfrak{g}$.

The $\mathbb{K}$-module $\tilde{\mathfrak{g}}$ is also a coalgebra, with coproduct $\tilde{\Delta} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$ given by the formula

$$\tilde{\Delta}(\gamma_1 \circ \cdots \circ \gamma_j) := \sum_{p=0}^j (\gamma_1 \circ \cdots \circ \gamma_p) \otimes (\gamma_{p+1} \circ \cdots \circ \gamma_j).$$

The submodule $\tilde{\mathfrak{g}} \subset T\mathfrak{g}$ is a sub-coalgebra (but not a subalgebra!).

The super-symmetric algebra $S\mathfrak{g} = \bigoplus_{j \geq 0} S^j\mathfrak{g}$ is defined to be the quotient of $T\mathfrak{g}$ by the ideal generated by the elements $\gamma_1 \circ \gamma_2 - (-1)^{d_1 d_2} \gamma_2 \circ \gamma_1$, for all $\gamma_1 \in \mathfrak{g}^{d_1}$ and $\gamma_2 \in \mathfrak{g}^{d_2}$. In other words, $S^j\mathfrak{g}$ is the set of coinvariants of $T^j\mathfrak{g}$ under the action of the group $\mathfrak{S}_j$. The product in the algebra $S\mathfrak{g}$ is denoted by $\circ$. The canonical projection is $\pi : T\mathfrak{g} \to S\mathfrak{g}$ is an algebra homomorphism: $\pi(\gamma_1 \circ \gamma_2) = \gamma_1 \cdot \gamma_2$.

In fact $S\mathfrak{g}$ is a commutative cocommutative Hopf algebra. The comultiplication

$$\Delta : S\mathfrak{g} \to S\mathfrak{g} \otimes S\mathfrak{g}$$

is the unique $\mathbb{K}$-algebra homomorphism such that

$$\Delta(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma$$

for all $\gamma \in \mathfrak{g}$. The antipode is $\gamma \mapsto -\gamma$. The projection $\pi : T\mathfrak{g} \to S\mathfrak{g}$ is not a coalgebra homomorphism. However:

**Lemma 3.1.** Let $\tau : S\mathfrak{g} \to T\mathfrak{g}$ be the $\mathbb{K}$-module homomorphism defined by

$$\tau(\gamma_1 \cdots \gamma_j) := \sum_{\sigma \in \mathfrak{S}_j} \text{sgn}_{(d_1, \ldots, d_j)}(\sigma)\gamma_{\sigma(1)} \circ \cdots \circ \gamma_{\sigma(j)}$$

for $\gamma_1 \in \mathfrak{g}^{d_1}, \ldots, \gamma_j \in \mathfrak{g}^{d_j}$. Then $\tau : S\mathfrak{g} \to \tilde{\mathfrak{g}}$ is a coalgebra isomorphism, where $S\mathfrak{g}$ has the comultiplication $\Delta$ and $\tilde{\mathfrak{g}}$ has the comultiplication $\tilde{\Delta}$.

**Proof.** Define $\tilde{\tau} : T\mathfrak{g} \to S\mathfrak{g}$ to be the $\mathbb{K}$-module homomorphism

$$\tilde{\tau}(\gamma_1 \circ \cdots \circ \gamma_j) := \frac{1}{j!}\tau(\gamma_1 \circ \cdots \circ \gamma_j) = \frac{1}{j!}\gamma_1 \cdots \gamma_j$$

for $\gamma_1, \ldots, \gamma_j \in \mathfrak{g}$. So $\tilde{\tau} \circ \tau$ is the identity map of $S\mathfrak{g}$, and $\tilde{\tau} : \tilde{\mathfrak{g}} \to S\mathfrak{g}$ is bijective. It suffices to prove that

$$((\tilde{\tau} \otimes \tilde{\tau}) \circ (\tau \otimes \tau) \circ \Delta = (\tilde{\tau} \otimes \tilde{\tau}) \circ \tilde{\Delta} \circ \tau.$$

Take any $\gamma_1 \in \mathfrak{g}^{d_1}, \ldots, \gamma_j \in \mathfrak{g}^{d_j}$ and write $d := (d_1, \ldots, d_j)$. Then

$$((\tilde{\tau} \otimes \tilde{\tau}) \circ \tilde{\Delta} \circ \tau)(\gamma_1 \cdots \gamma_j)$$

$$= \sum_{p=0}^j \sum_{\sigma \in \mathfrak{S}_j} \frac{1}{p!(j-p)!} \text{sgn}_d(\sigma)\gamma_{\sigma(1)} \circ \cdots \circ \gamma_{\sigma(p)} \otimes (\gamma_{\sigma(p+1)} \cdots \gamma_{\sigma(j)}).$$
On the other hand
\[ ((\hat{\pi} \otimes \hat{\pi}) \circ (\tau \otimes \tau) \circ \Delta)(\gamma_1 \cdots \gamma_j) = \Delta(\gamma_1 \cdots \gamma_j) = (1 \otimes \gamma_1 + \gamma_1 \otimes 1) \cdots (1 \otimes \gamma_j + \gamma_j \otimes 1) \]
\[ \sum_{p=0}^j \sum_{\sigma \in \mathfrak{S}_{p,j-p}} \text{sgn}_d(\sigma)(\gamma_{\sigma(1)} \cdots \gamma_{\sigma(p)}) \otimes (\gamma_{\sigma(p+1)} \cdots \gamma_{\sigma(j)}), \]
where \( \mathfrak{S}_{p,j-p} \) is the set of \( (p, j - p) \)-shuffles inside the group \( \mathfrak{S}_j \). Since the algebra \( Sg \) is super-commutative the two sums are equal. \( \square \)

The grading on \( g \) induces a grading on \( Sg \), which we call the degree. Thus for \( \gamma_i \in \mathfrak{g}^{d_i} \) the degree of \( \gamma_1 \cdots \gamma_j \in S^j g \) is \( d_1 + \cdots + d_j \) (unless \( \gamma_1 \cdots \gamma_j = 0 \)). We consider \( Sg \) as a graded algebra for this grading. Actually there is another grading on \( Sg \) by order, where we define the order of \( \gamma_1 \cdots \gamma_j \) to be \( j \) (again, unless this element is zero). But this grading will have a different role.

By definition the \( j \)-th super-exterior power of \( g \) is
\[
\bigwedge^j g := S^j(g[1])[-j],
\]
where \( g[1] \) is the shifted graded module whose degree \( i \) component is \( g[1]^i = g^{i+1} \).
When \( g \) is concentrated in degree 0 then these are the usual constructions of symmetric and exterior algebras, respectively.

We denote by \( \ln : Sg \to S^1 g = g \) the projection. So \( \ln(\gamma) \) is the 1-st order term of \( \gamma \in Sg \). (The expression “\( \ln \)” might stand for “linear” or “logarithm”.)

**Definition 3.3.** Let \( g \) and \( g' \) be two graded \( K \)-modules, and let \( \Psi : Sg \to Sg' \) be a \( K \)-linear homomorphism. For any \( j \geq 1 \) the \( j \)-th Taylor coefficient of \( \Psi \) is defined to be
\[
\partial^j \Psi := \ln \circ \Psi : S^j g \to g'.
\]

**Lemma 3.4.** Suppose we are given a sequence of \( K \)-linear homomorphisms \( \psi_j : S^j g \to g', j \geq 1 \), each of degree 0. Then there is a unique coalgebra homomorphism \( \Psi : Sg \to Sg' \), homogeneous of degree 0 and satisfying \( \Psi(1) = 1 \), whose Taylor coefficients are \( \partial^j \Psi = \psi_j \).

**Proof.** Let \( \tilde{\ln} : Sg' \to S^1 g' = g' \) be the projection for this coalgebra. Consider the exact sequence of coalgebras
\[
0 \to K \to \tilde{S}g \to \tilde{S}^{\geq 1} g \to 0.
\]
According to [Ko1, Section 4.1] (see also [Fu, Lemma 2.1.5]) the sequence \( \{\psi_j\}_{j \geq 1} \) uniquely determines a coalgebra homomorphism \( \tilde{\Psi} : \tilde{S}^{\geq 1} g \to \tilde{S}^{\geq 1} g' \) such that
\[
\ln \circ \Psi|_{Sg} = \psi_j \circ \tau^{-1}|_{Sg}
\]
for all \( j \geq 1 \). Here \( \tau : Sg \xrightarrow{\cong} \tilde{S}g \) is the coalgebra isomorphism of Lemma 3.1.
Using \( Sg \) we can lift \( \tilde{\Psi} \) uniquely to a coalgebra homomorphism \( \Psi : Sg \to Sg' \) by setting \( \Psi(1) := 1 \). Now define the coalgebra homomorphism \( \Psi : Sg \to Sg' \) to be \( \Psi := \tau^{-1} \circ \tilde{\Psi} \circ \tau \).

A \( K \)-linear map \( Q : Sg \to Sg \) is a coderivation if
\[
\Delta \circ Q = (Q \otimes 1 + 1 \otimes Q) \circ \Delta,
\]
where \( 1 := 1_{Sg} \), the identity map.
Lemma 3.6. Given a sequence of \(K\)-linear homomorphisms \(\psi_j : S^j g \to g\), each of degree 1, there is a unique coderivation \(Q\) of degree 1, such that \(Q(1) = 0\) and \(\partial^j Q = \psi_j\). Furthermore, one has \(Q(\gamma) \in S_{\gamma}^1 g\) for every \(\gamma \in S_g\).

Proof. According to [Ko1, Section 4.3] (see also [Bu, Lemma 2.1.2]) the sequence \(\{\psi_j\}_{j \geq 1}\) uniquely determines a coderivation \(\tilde{Q} : S^1 g \to S^1 g\), such that

\[
\tilde{Q} \circ \tau = \psi_j \circ \tau^{-1} \mid_{S^j g}
\]

for all \(j \geq 1\). Using (3.5) this can be lifted uniquely to a coderivation \(Q : S^1 g \to \tilde{S}^1 g\) by setting \(\tilde{Q}(1) := 0\). Now define the coderivation \(Q : S g \to S g\) to be \(Q := \tau^{-1} \circ \tilde{Q} \circ \tau\).

We will be mostly interested in the coalgebras \(S(g[1])\) and \(S(g'[1])\). Observe that if \(\Psi : S(g[1]) \to S(g'[1])\) is a homogeneous \(K\)-linear homomorphism of degree 1, then, using formula (3.2), each Taylor coefficient \(\partial^i \Psi\) may be viewed as a homogeneous \(K\)-linear homomorphism \(\partial^i \Psi : \Lambda^i g \to g\) of degree \(i + 1 - j\).

Definition 3.7. Let \(g\) be a graded \(K\)-module. An \(L_\infty\) algebra structure on \(g\) is a coderivation \(Q : S(g[1]) \to S(g[1])\) of degree 1, satisfying \(Q(1) = 0\) and \(Q \circ Q = 0\). We call the pair \((g, Q)\) an \(L_\infty\) algebra.

The notion of \(L_\infty\) algebra generalizes that of DG Lie algebra in the following sense:

Proposition 3.8 ([Ko1, Section 4.3]). Let \(Q : S(g[1]) \to S(g[1])\) be a coderivation of degree 1 with \(Q(1) = 0\). Then the following conditions are equivalent:

(i) \(\partial^j Q = 0\) for all \(j \geq 3\), and \(Q \circ Q = 0\).

(ii) \(\partial^j Q = 0\) for all \(j \geq 3\), and \(g\) is a DG Lie algebra with respect to the differential \(d := \partial^1 Q\) and the bracket \([-,-] := \partial^2 Q\).

In view of this, we shall say that \((g, Q)\) is a DG Lie algebra if the equivalent conditions of the proposition hold. An easy calculation shows that given an \(L_\infty\) algebra \((g, Q)\), the function \(\partial^j Q : g \to g\) is a differential, and \(\partial^j Q\) induces a graded Lie bracket on \(H(g, \partial^j Q)\). We shall denote this graded Lie algebra by \(H(g, Q)\).

Definition 3.9. Let \((g, Q)\) and \((g', Q')\) be \(L_\infty\) algebras. An \(L_\infty\) morphism \(\Psi : (g, Q) \to (g', Q')\) is a coalgebra homomorphism \(\Psi : S(g[1]) \to S(g'[1])\) of degree 0, satisfying \(\Psi(1) = 1\) and \(\Psi \circ Q = Q' \circ \Psi\).

Proposition 3.10 ([Ko1, Section 4.3]). Let \((g, Q)\) and \((g', Q')\) be DG Lie algebras, and let \(\Psi : S(g[1]) \to S(g'[1])\) be a coalgebra homomorphism of degree 0 such that \(\Psi(1) = 1\). Then \(\Psi\) is an \(L_\infty\) morphism (i.e. \(\Psi \circ Q = Q' \circ \Psi\)) iff the Taylor coefficients \(\psi_i := \partial^i \Psi : \Lambda^i g \to g'\) satisfy the following identity:

\[
d(\psi_i(\gamma_1 \wedge \cdots \wedge \gamma_i)) - \sum_{k=1}^i \pm \psi_i(\gamma_1 \wedge \cdots \wedge d(\gamma_k) \wedge \cdots \wedge \gamma_i) =
\]

\[
\frac{1}{2} \sum_{k,l \geq 1} \sum_{k+l=1} \sum_{\sigma \in S_{k+l}} \pm \psi_l(\gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(k)}) \psi_k(\gamma_{\sigma(k+1)} \wedge \cdots \wedge \gamma_{\sigma(i)})
\]

\[
+ \sum_{k<l} \pm \psi_{l-1}(\gamma_k \gamma_l \wedge \gamma_1 \wedge \cdots \gamma_k \wedge \cdots \wedge \gamma_l).
\]
Here \( \gamma_k \in g \) are homogeneous elements, \( \mathcal{S}_1 \) is the permutation group of \( \{1, \ldots, i\} \), and the signs depend only on the indices, the permutations and the degrees of the elements \( \gamma_k \). (See \cite{KGN} Section 6] or \cite{CTI} Theorem 3.1] for the explicit signs.)

The proposition shows that when \((g, Q)\) and \((g', Q')\) are DG Lie algebras and \(\partial^i \Psi = 0\) for all \(j \geq 2\), then \(\partial^i \Psi : g \to g'\) is a homomorphism of DG Lie algebras; and conversely. It also implies that for any \(L_\infty\) morphism \(\Psi : (g, Q) \to (g', Q')\), the map \(H(\Psi) : H(g, Q) \to H(g', Q')\) is a homomorphism of graded Lie algebras.

Given DG Lie algebras \(g\) and \(g'\) we consider them as \(L_\infty\) algebras \((g, Q)\) and \((g', Q')\), as explained in Proposition 3.11. If \(\Psi : (g, Q) \to (g', Q')\) is an \(L_\infty\) morphism, then we shall say (by slight abuse of notation) that \(\Psi : g \to g'\) is an \(L_\infty\) morphism.

From here until Theorem 3.21 (inclusive) \(C\) is a commutative \(K\)-algebra, and \(g, g'\) are graded \(C\)-modules. Suppose \((g, Q)\) is an \(L_\infty\) algebra structure on \(g\) such that the Taylor coefficients \(\partial^i Q : \bigwedge^i g \to g\) are all \(C\)-multilinear. Then we say \((g, Q)\) is a \(C\)-multilinear \(L_\infty\) algebra. Similarly one defines the notion of \(C\)-multilinear \(L_\infty\) morphism \(\Psi : (g, Q) \to (g', Q')\).

With \(C\) and \(g\) as above let \(S_C g\) be the super-symmetric associative unital free \(C\)-algebra over \(C\). Namely \(S_C g\) is the quotient of the tensor algebra \(T_C g = C \oplus g \oplus (g \otimes_C g) \oplus \cdots\) by the ideal generated by the super-commutativity relations. The algebra \(S_C g\) is a Hopf algebra over \(C\), with comultiplication \(\Delta_C : S_C g \to S_C g \otimes_C S_C g\).

The formulas are just as in the case \(C = K\). It will be useful to note that \(\Delta_C\) preserves the grading by order, namely

\[
\Delta_C(S_C^j g) \subset \bigoplus_{j+k=i} S_C^j g \otimes_C S_C^k g.
\]

**Lemma 3.11.**

1. Let \(g\) be a graded \(C\)-module. There is a canonical bijection \(Q \to Q_C\) between the set of \(C\)-multilinear \(L_\infty\) algebra structures \(Q\) on \(g\), and the set of coderivations \(Q_C : S_C(g[1]) \to S_C(g[1])\) over \(C\) of degree 1, such that \(Q_C(1) = 0\) and \(Q_C \circ Q_C = 0\).

2. Let \((g, Q)\) and \((g', Q')\) be two \(C\)-multilinear \(L_\infty\) algebras. The set of \(C\)-multilinear \(L_\infty\) morphisms \(\Psi : (g, Q) \to (g', Q')\) is canonically bijective to the set of coalgebra homomorphisms \(\Psi_C : S_C(g[1]) \to S_C(g'[1])\) over \(C\) of degree 0, such that \(\Psi_C(1) = 1\) and \(\Psi_C \circ Q_C = Q'_C \circ \Psi_C\).

**Proof.** The data for a coderivation \(Q_C : S_C(g[1]) \to S_C(g[1])\) over \(C\) is its sequence of \(C\)-linear Taylor coefficients \(\partial^i Q_C : \bigwedge^i g \to g\). But giving such a homomorphism \(\partial^i Q_C\) is the same as giving a \(C\)-multilinear homomorphism \(\partial^i Q : \bigwedge^i g \to g\), so there is a corresponding \(C\)-multilinear coderivation \(Q : S(g[1]) \to S(g[1])\). One checks that \(Q \circ Q = 0\) if \(Q_C \circ Q_C = 0\).

Similarly for coalgebra homomorphisms. \(\square\)

An element \(\gamma \in S_C(g[1])\) is called primitive if \(\Delta_C(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma\).

**Lemma 3.12.** The set of primitive elements of \(S_C(g[1])\) is precisely \(S_C^1(g[1]) = g[1]\).

**Proof.** By definition of the comultiplication any \(\gamma \in g[1]\) is primitive. For the converse, let us denote by \(\mu\) the multiplication in \(S_C(g[1])\). One checks that \((\mu \circ \Delta_C)(\gamma) = 2^2 \gamma\) for \(\gamma \in S_C^1(g[1])\). If \(\gamma\) is primitive then \((\mu \circ \Delta_C)(\gamma) = 2 \gamma\), so indeed \(\gamma \in S_C^1(g[1])\). \(\square\)
Now let’s assume that $C$ is a local ring, with nilpotent maximal ideal $m$. Suppose we are given two $C$-multilinear $L_{\infty}$ algebras $(g, Q)$ and $(g', Q')$, and a $C$-multilinear $L_{\infty}$ morphism $\Psi : (g, Q) \to (g', Q')$. Because the coderivation $Q$ is $C$-multilinear, the $C$-submodule $mg \subseteq g$ becomes a $C$-multilinear $L_{\infty}$ algebra $(mg, Q)$. Likewise for $mg'$, and $\Psi : (mg, Q) \to (mg', Q')$ is a $C$-multilinear $L_{\infty}$ morphism.

The fact that $m$ is nilpotent is essential for the next definition.

**Definition 3.13.** The Maurer-Cartan equation in $(mg, Q)$ is
\[
\sum_{i=1}^{\infty} \frac{1}{i!} (\partial^i Q)(\omega^i) = 0
\]
for $\omega \in (mg)^1 = (mg[1])^0$.

An element $e \in S_C(g[1])$ is called **group-like** if $\Delta_C(e) = e \otimes e$. For $\omega \in mg$ we define
\[
\exp(\omega) := \sum_{i \geq 0} \frac{1}{i!} \omega^i \in S_C(g[1]).
\]

**Lemma 3.14.** The function $\exp$ is a bijection from $mg[1]$ to the set of invertible group-like elements of $S_C(g[1])$, with inverse $\ln$.

**Proof.** Let $\omega \in mg[1]$ and $e := \exp(\omega)$. The element $e$ is invertible, with inverse $\exp^*(-\omega)$. Using the fact that $\Delta_C(\omega) = \omega \otimes 1 + 1 \otimes \omega$ it easily follows that $\Delta_C(e) = e \otimes e$. And trivially $\ln(e) = \omega$.

For the opposite direction, let $e$ be invertible and group-like. Write it as $e = \sum_i \gamma_i$, with $\gamma_i \in S_C(g[1])$. Since $e$ is invertible one must have $\gamma_i \in C - m$, and $\gamma_i \in mS_C(g[1])$ for all $i \geq 1$. The equation $\Delta_C(e) = e \otimes e$ implies that
\[
\Delta_C(\gamma_i) = \sum_{j+k=i} \gamma_j \otimes \gamma_k
\]
for all $i$. Hence
\[
(3.15) \quad 2^i \gamma_i = \mu(\Delta_C(\gamma_i)) = \sum_{j+k=i} \gamma_j \gamma_k.
\]

For $i = 0$ we get $\gamma_0 = \frac{1}{0!} \gamma_0^0$, and since $\gamma_0$ is invertible, it follows that $\gamma_0 = 1$. Let $\omega := \gamma_1 = \ln(e) \in mS_C^1(g[1]) = mg[1]$. Using induction and equation (3.15) we see that $\gamma_i = \frac{1}{i!} \omega^i$ for all $i$. Thus $e = \exp(\omega)$. $\square$

**Lemma 3.16.** Let $\omega \in (mg[1])^0 = mg$ and $e := \exp(\omega)$. Then $\omega$ is a solution of the MC equation iff $Q(e) = 0$.

**Proof.** Since $e$ is group-like and invertible (by Lemma 3.14), we have
\[
\Delta_C(Q(e)) = Q(e) \otimes e + e \otimes Q(e)
\]
and
\[
\Delta_C(e^{-1}Q(e)) = \Delta_C(e)^{-1} \Delta_C(Q(e)) = e^{-1}Q(e) \otimes 1 + 1 \otimes e^{-1}Q(e).
\]
So the element $e^{-1}Q(e)$ is primitive, and by Lemma 3.12 we get $e^{-1}Q(e) \in g[1]$. On the other hand hence $Q(e)$ has no 0-order term, and $Q(1) = 0$. Thus in the 1st
order term we get
\[ e^{-1}Q(e) = \ln(e^{-1}Q(e)) = \ln((1 - \omega + \frac{1}{2}\omega^2 \pm \cdots)Q(e)) = \ln(Q(e)) = \sum_{i=0}^{\infty} \frac{1}{i!} \ln(Q(\omega^i)) = \sum_{i=1}^{\infty} \frac{1}{i!}(\partial^iQ)(\omega^i). \] (3.17)

Since \( e \) is invertible we are done. \( \square \)

**Lemma 3.18.** Given an element \( \omega \in mg[1] \), define \( \omega' := \sum_{i=1}^{\infty} \frac{1}{i!}(\partial^i\Psi)(\omega^i) \in mg'[1] \), \( e := \exp(\omega) \) and \( e' := \exp(\omega') \). Then \( e' = \Psi(e) \).

**Proof.** From Lemma 3.14 we see that \( \Delta_C(e) = e \otimes e \), and therefore also \( \Delta_C(\Psi(e)) = \Psi(e) \otimes \Psi(e) \in S_C(g'[1]) \). Since \( \Psi \) is \( C \)-linear and \( \Psi(1) = 1 \) we get \( \Psi(e) \in 1 + mg'[1] \). Thus \( \Psi(e) \) is group-like and invertible. According to Lemma 3.14 it suffices to prove that \( \ln(e') = \ln(\Psi(e)) \). Now \( \ln(e') = \omega' \) by definition. Since \( \Psi(1) = 1 \) and \( \ln(1) = 0 \) it follows that
\[ \ln(\Psi(e)) = \ln(\Psi(\sum_{i=0}^{\infty} \frac{1}{i!}\omega^i)) = \sum_{i=0}^{\infty} \frac{1}{i!} \ln(\Psi(\omega^i)) = \sum_{i=1}^{\infty} \frac{1}{i!}(\partial^i\Psi)(\omega^i) = \omega'. \]

\( \square \)

**Proposition 3.19.** Suppose \( \omega \in mg[1] \) is a solution of the MC equation in \( (mg, Q) \). Define \( \omega' := \sum_{i=1}^{\infty} \frac{1}{i!}(\partial^i\Psi)(\omega^i) \in mg'[1] \). Then \( \omega' \) is a solution of the MC equation in \( (mg', Q') \).

**Proof.** Let \( e := \exp(\omega) \) and \( e' := \exp(\omega') \). By Lemma 3.16 we get \( Q(e) = 0 \). Hence \( Q'(\Psi(e)) = \Psi(Q(e)) = 0 \). According to Lemma 3.18 we have \( \Psi(e) = e' \), so \( Q'(e') = 0 \). Again by Lemma 3.16 we deduce that \( \omega' \) solves the MC equation. \( \square \)

**Definition 3.20.** Let \( \omega \in mg[1] \).

1. The coderivation \( Q_{\omega} \) of \( S_C(g[1]) \) over \( C \), with \( Q_{\omega}(1) := 0 \) and with Taylor coefficients
\[ (\partial^iQ_{\omega})(\gamma) := \sum_{j \geq 0} \frac{1}{j!}(\partial^{i+j}Q)(\omega^i\gamma) \]
for \( i \geq 1 \) and \( \gamma \in S_C(g[1]) \), is called the twist of \( Q \) by \( \omega \).

2. The coalgebra homomorphism \( \Psi_{\omega} : S_C(g[1]) \rightarrow S_C(g'[1]) \) over \( C \), with \( \Psi_{\omega}(1) := 1 \) and Taylor coefficients
\[ (\partial^i\Psi_{\omega})(\gamma) := \sum_{j \geq 0} \frac{1}{j!}(\partial^{i+j}\Psi)(\omega^i\gamma) \]
for \( i \geq 1 \) and \( \gamma \in S_C(g[1]) \), is called the twist of \( \Psi \) by \( \omega \).

**Theorem 3.21.** Let \( C \) be a commutative local \( K \)-algebra with nilpotent maximal ideal \( m \). Let \( (g, Q) \) and \( (g', Q') \) be \( C \)-multilinear \( L_\infty \) algebras and \( \Psi : (g, Q) \rightarrow C \)-multilinear \( L_\infty \) algebras and \( \Psi : (g, Q) \rightarrow \).
(g', Q') a C-multilinear $L_\infty$ morphism. Suppose $\omega \in \mathfrak{m}g^1$ a solution of the MC equation in $(\mathfrak{m}g, Q)$. Define

$$\omega' := \sum_{i=1}^{\infty} \frac{1}{i!} (\partial^i \Psi)(\omega^i) \in \mathfrak{m}g^1.$$ 

Then $(g, Q_\omega)$ and $(g', Q'_\omega)$ are $L_\infty$ algebras, and

$$\Psi_\omega : (g, Q_\omega) \to (g', Q'_\omega)$$

is an $L_\infty$ morphism.

Proof. Let $e := \exp(\omega)$. Define $\Phi_e : S_C(\mathfrak{g}[1]) \to S_C(\mathfrak{g}[1])$ to be $\Phi_e(\gamma) := e\gamma$. Since $e$ is group-like and invertible it follows that $\Phi_e$ is a coalgebra automorphism. Therefore $\tilde{Q}_\omega := \Phi_e^{-1} \circ Q_1 \circ \Phi_e$ is a degree 1 coderivation of $S_C(\mathfrak{g}[1])$, satisfying $\tilde{Q}_\omega \circ Q_\omega = 0$ and $\tilde{Q}_\omega(1) = e^{-1}Q(e) = 0$; cf. Lemma 3.10 So $(g, \tilde{Q}_\omega)$ is an $L_\infty$ algebra. Likewise we have a coalgebra automorphism $\tilde{\Phi}_e$ and a coderivation $\tilde{Q}'_\omega := \Phi_e^{-1} \circ Q' \circ \Phi_e$ of $S_C(\mathfrak{g}'[1])$, where $e' := \exp(\omega')$. The degree 0 coalgebra homomorphism $\tilde{\Psi}_\omega := \Phi_e^{-1} \circ \Psi \circ \Phi_e$ satisfies $\tilde{\Psi}_\omega \circ Q_\omega = \tilde{Q}'_\omega \circ \tilde{\Psi}_\omega$, and also $\tilde{\Psi}_\omega(1) = e^{-1}\Psi(e) = e^{-1}e' = 1$, by Lemma 3.18 Hence we have an $L_\infty$ morphism $\tilde{\Psi}_\omega : (g, \tilde{Q}_\omega) \to (g', \tilde{Q}'_\omega)$.

Let us calculate the Taylor coefficients of $\tilde{Q}_\omega$. For $\gamma \in S^1_C(\mathfrak{g}[1])$ one has

$$(\partial^i \tilde{Q}_\omega)(\gamma) = \ln(\tilde{Q}_\omega(\gamma)) = \ln(e^{-1}Q(e\gamma)).$$

But just as in 5.11, since $Q(e\gamma)$ has no zero order term, we obtain

$$\ln(e^{-1}Q(e\gamma)) = \ln(Q(e\gamma)).$$

And

$$\ln(Q(e\gamma)) = \ln(Q(\sum_{j \geq 0} \frac{1}{j!} \omega^j \gamma)) = \sum_{j \geq 0} \frac{1}{j!} \ln(Q(\omega^j \gamma))$$

$$= \sum_{j \geq 0} \frac{1}{j!} (\partial^j Q)(\omega^j \gamma) = (\partial^j Q_\omega)(\gamma).$$

Therefore $\tilde{Q}_\omega = Q_\omega$. Similarly we see that $\tilde{Q}'_\omega = Q'_\omega$ and $\tilde{\Psi}_\omega = \Psi_\omega$. \hfill \Box

Remark 3.23. The formulation of Theorem 3.21 as well as the idea for the proof, were suggested by Vladimir Hinich. An analogous result, for $A_\infty$ algebras, is in [Lc] Section 6.1.

If $(g, Q)$ is a DG Lie algebra then the sum occurring in Definition 3.20(1) is finite, so the coderivation $Q_\omega$ can be defined without a nilpotence assumption on the coefficients.

Lemma 3.24. Let $(g, Q)$ be a DG Lie algebra, and let $\omega \in \mathfrak{g}^1$ be a solution of the MC equation. Then the $L_\infty$ algebra $(g, Q_\omega)$ is also a DG Lie algebra. In fact, for $\gamma_1 \in \mathfrak{g}$ one has

$$(\partial^1 Q_\omega)(\gamma_1) = (\partial^1 Q)(\gamma_1) + (\partial^2 Q)(\omega \gamma_1) = d(\gamma_1) + [\omega, \gamma_1] = (d + ad(\omega))(\gamma_1),$$

$$(\partial^2 Q_\omega)(\gamma_1 \gamma_2) = (\partial^2 Q)(\gamma_1 \gamma_2) = [\gamma_1, \gamma_2],$$

and $\partial^i Q_\omega = 0$ for $j \geq 3$. 


Proof. Like equation \((3.22)\), with \(C := \mathbb{K}\) and \(e := 1\).

In the situation of the lemma, the twisted DG Lie algebra \((g, Q_\omega)\) will usually be denoted by \(g_\omega\).

Let \(A\) be a DG super-commutative associative unital DG algebra in \(\text{Dir Inv Mod} \mathbb{K}\). The notion of DG \(A\)-module Lie algebra in \(\text{Dir Inv Mod} \mathbb{K}\) was introduced in Definition 1.20.

Definition 3.25. Let \(A\) be a DG super-commutative associative unital DG algebra in \(\text{Dir Inv Mod} \mathbb{K}\), let \(g\) and \(g'\) be DG \(A\)-module Lie algebras in \(\text{Dir Inv Mod} \mathbb{K}\), and let \(\Psi : g \to g'\) be an \(L_\infty\) morphism.

1. If each Taylor coefficient \(\partial^j \Psi : \prod^j g \to g'\) is continuous then we say that \(\Psi\) is a continuous \(L_\infty\) morphism.
2. Assume each Taylor coefficient \(\partial^j \Psi : \prod^j g \to g'\) is \(A\)-multilinear, i.e.

\[
(\partial^j \Psi)(a_1 \gamma_1, \ldots, a_j \gamma_j) = \pm a_1 \cdots a_j \cdot (\partial^j \Psi)(\gamma_1, \ldots, \gamma_j)
\]

for all homogeneous elements \(a_k \in A\) and \(\gamma_k \in g\), with sign according to the Koszul rule, then we say that \(\Psi\) is an \(A\)-multilinear \(L_\infty\) morphism.

Proposition 3.26. Let \(A\) and \(B\) be DG super-commutative associative unital DG algebras in \(\text{Dir Inv Mod} \mathbb{K}\), and let \(g\) and \(g'\) be DG \(A\)-module Lie algebras in \(\text{Dir Inv Mod} \mathbb{K}\). Suppose \(A \to B\) is a continuous DG algebra homomorphism, and \(\Psi : g \to g'\) is a continuous \(A\)-multilinear \(L_\infty\) morphism. Let \(\partial^j \Psi_B : \prod^j (B \mathring{\otimes}_A g) \to B \mathring{\otimes}_A g'\) be the unique continuous \(B\)-multilinear homomorphism extending \(\partial^j \Psi\). Then the degree 0 coalgebra homomorphism

\[
\Psi_B : S(B \mathring{\otimes}_A g[1]) \to S(B \mathring{\otimes}_A g'[1]),
\]

with \(\Psi_B(1) := 1\) and with Taylor coefficients \(\partial^j \Psi_B\), is an \(L_\infty\) morphism

\[
\Psi_B : B \mathring{\otimes}_A g \to B \mathring{\otimes}_A g'.
\]

Proof. First consider the continuous \(B\)-multilinear homomorphisms \(\partial^j \Psi_B : \prod^j (B \mathring{\otimes}_A g) \to B \mathring{\otimes}_A g'\) extending \(\partial^j \Psi\). It is a straightforward calculation to verify that the \(L_\infty\) morphism identities of Proposition 3.10 hold for the sequence of operators \(\{\partial^j \Psi_B\}_{j \geq 1}\). The completion process respects these identities (cf. proof of Proposition 1.19).

Theorem 3.27. Let \(g\) and \(g'\) be DG Lie algebras in \(\text{Dir Inv Mod} \mathbb{K}\), and let \(\Psi : g \to g'\) be a continuous \(L_\infty\) morphism. Let \(A = \bigoplus_{i \in \mathbb{N}} A^i\) be a complete associative unital super-commutative DG algebra in \(\text{Dir Inv Mod} \mathbb{K}\). By Proposition 3.26 there is an induced continuous \(A\)-multilinear \(L_\infty\) morphism \(\Psi_A : A \mathring{\otimes} g \to A \mathring{\otimes} g'\). Let \(\omega \in A^1 \mathring{\otimes} g^0\) be a solution of the MC equation in \(A \mathring{\otimes} g\). Assume \(d_g = 0\), \((\partial^j \Psi_A)(\omega^j) = 0\) for all \(j \geq 2\), and also that \(g'\) is bounded below. Define \(\omega' := (\partial^1 \Psi_A)(\omega) \in A^1 \mathring{\otimes} g^0\). Then:

1. The element \(\omega'\) is a solution of the MC equation in \(A \mathring{\otimes} g'\).
2. Given \(c \in S^j (A \mathring{\otimes} g[1])\) there exists a natural number \(k_0\) such that \((\partial^{j+k} \Psi_A)(\omega^k c) = 0\) for all \(k > k_0\).
3. The degree 0 coalgebra homomorphism

\[
\Psi_{A,\omega} : S(A \mathring{\otimes} g[1]) \to S(A \mathring{\otimes} g'[1]),
\]
with $\Psi_{A,\omega}(1) := 1$ and Taylor coefficients

$$(\partial^i \Psi_{A,\omega})(c) := \sum_{k \geq 0} \frac{1}{(i+k)!} (\partial^i+k \Psi_A) (\omega^k c)$$

for $c \in S^i (A \otimes g[1])$, is a continuous $A$-multilinear $L_\infty$ morphism

$$\Psi_{A,\omega} : (A \otimes g) \to (A \otimes g')_{\omega}.$$  

Proof. We shall use a “deformation argument”. Consider the base field $K$ as a discrete inv $K$-module. The polynomial algebra $K[h]$ is endowed with the dir-inv $K$-module structure such that the homomorphism $\bigoplus_{i \in \mathbb{N}} K \to K[h]$, whose $i$-th component is multiplication by $h^i$, is an isomorphism in $\text{Dir Inv Mod} K$. Note that $K[h]$ is a discrete dir-inv module, but it is not trivial. We view $K[h]$ as a DG algebra concentrated in degree 0 (with zero differential).

For any $i \in \mathbb{N}$ let $A[h]^i := K[h] \otimes A^i$, and let $A[h] := \bigoplus_{i \in \mathbb{N}} A[h]^i$, which is a DG algebra in $\text{Dir Inv Mod} K$, with differential $d_{A[h]} := 1 \otimes d_A$. We will need a “twisted” version of $A[h]$, which we denote by $A[h]^{\sim}$. Let $A[h]^{\sim i} := h^i A[h]^i$, and define $A[h]^{\sim} := \bigoplus_{i \in \mathbb{N}} A[h]^{\sim i}$, which is a graded subalgebra of $A[h]$. The differential is $d_{A[h]^{\sim}} := h d_{A[h]}$. The dir-inv structure is such that the homomorphism $\bigoplus_{i,j \in \mathbb{N}} A^i \to A[h]^{\sim}$, whose $(i,j)$-th component is multiplication by $h^{i+j}$, is an isomorphism in $\text{Dir Inv Mod} K$. The specialization $h \mapsto 1$ is a continuous DG algebra homomorphism $A[h]^{\sim} \to A$. There is an induced continuous $A[h]^{\sim}$-multilinear $L_\infty$ morphism $\Psi_{A[h]^{\sim}} : A[h]^{\sim} \otimes g \to A[h]^{\sim} \otimes g'$.

We proceed in several steps.

Step 1. Say $r_0$ bounds $g'$ from below, i.e. $g'^r = 0$ for all $r < r_0$. Take some $j \geq 1$. For any $l \in \{1, \ldots, j\}$ choose $p_l, q_l \in \mathbb{Z}$, $\gamma_l \in g^{p_l}$ and $a_l \in A[h]^{\sim q_l}$. Also choose $\gamma_0 \in g^0$ and $a_0 \in A[h]^{\sim 1}$. Let $p := \sum_{l=1}^j p_l$ and $q := \sum_{l=1}^j q_l$. Because $\partial^{i+k} \Psi_{A[h]^{\sim}}$ is induced from $\partial^{i+k} \Psi$, and this is a homogeneous map of degree $1 - j - k$, we have

$$(\partial^{i+k} \Psi_{A[h]^{\sim}})((a_0 \otimes \gamma_0) \cdot (a_1 \otimes \gamma_1) \cdots (a_j \otimes \gamma_j))$$

$$= \pm d_{\partial^{i+k} \Psi}(a_0 \otimes \gamma_0) \cdot (a_1 \otimes \gamma_1) \cdots (a_j \otimes \gamma_j) \in A[h]^{\sim k+q} \otimes g^{p+1-j-k}.$$  

But $g^{p+1-j-k} = 0$ for all $k > p + 1 - j - r_0$.

Using multilinearity and continuity we conclude that given any $c \in S^i (A[h]^{\sim} \otimes g[1])$ there exists a natural number $k_0$ such that $(\partial^{i+k} \Psi_{A[h]^{\sim}})((h^\omega)^k c) = 0$ for all $k > k_0$.

Step 2. We are going to prove that $h \omega$ is a solution of the MC equation in $A[h]^{\sim} \otimes g$. It is given that $\omega$ is a solution of the MC equation in $A \otimes g$. Because $d_g = 0$, this means that

$$(d_A \otimes 1)(\omega) + \frac{1}{2} [\omega, \omega] = 0.$$  

Hence

$$d_{A[h]^{\sim}} \otimes g(h \omega) + \frac{1}{2} [h \omega, h \omega] = h^2 (d_A \otimes 1)(\omega) + \frac{1}{2} h^2 [\omega, \omega] = 0.$$  

So $h \omega$ solves the MC equation in $A[h]^{\sim} \otimes g$.

Step 3. Now we shall prove that $h \omega'$ solves the MC equation in $A[h]^{\sim} \otimes g'$. This will require an infinitesimal argument. For any natural number $m$ define $K[h]_m := K[h]/(h^{m+1})$ and $A[h]_m := K[h]_m \otimes A$. The latter is a DG algebra with differential

$$d_{A[h]_m} := 1 \otimes d_A.$$  

Let $A[h]_m^{\sim} := \bigoplus_{i=0}^m h^i A[h]^i$, which is a subalgebra of $A[h]_m$,
but its differential is $d_{A[h]_m} := h d_{A[h]_m}$. There is a surjective DG Lie algebra homomorphism $A[h]^{-} \otimes g' \to A[h]^{-} \otimes g'$, with kernel $(A[h]^{-} \cap h^{m+1} A[h]) \otimes g'$. Since $\bigcap_{m \geq 0} h^{m+1} A[h] = 0$, it suffices to prove that $h \omega'$ solves the MC equation in $A[h]^{-} \otimes g'$. 

Now $C := \mathbb{K}[h]_m$ is an artinian local ring with maximal ideal $m := (h)$. Define the DG Lie algebra $\mathfrak{h} := A[h]_m \otimes g$, with differential $d_{\mathfrak{h}} := h d_{A[h]_m} \otimes 1 + 1 \otimes d_g$; so $A[h]^{-} \otimes g \subset \mathfrak{h}$ as DG Lie algebras. Similarly define $\mathfrak{h}'$. There is a $C$-multilinear $L_\infty$ morphism $\Phi : \mathfrak{h} \to \mathfrak{h}'$ extending $\Psi_{A[h]_m} : A[h]^{-} \otimes g \to A[h]^{-} \otimes g'$. By step 2 the element $\nu := h \omega \in m\mathfrak{h}$ is a solution of the MC equation. According to Proposition 3.10 the element $\nu' := \sum_{k \geq 1} (\partial^k \Phi)(\nu^k)$ is a solution of the MC equation in $\mathfrak{h}'$. But $\nu' = h \omega'$.

Step 4. Pick a natural number $m$. Let $\mathfrak{h}, \mathfrak{h}', \Phi, \nu$ and $\nu'$ be as in step 3. According to Theorem 3.21 there is a twisted $L_\infty$ morphism $\Phi_{\nu} : \mathfrak{h}_{\nu} \to \mathfrak{h}'_{\nu'}$. Since $(A[h]^{-} \otimes g)_{\nu} \subset \mathfrak{h}_{\nu}$ and $(A[h]^{-} \otimes g')_{\nu'} \subset \mathfrak{h}'_{\nu'}$ as DG Lie algebras, and $\Phi_{\nu}$ extends $\Psi_{A[h]_m, \nu}$, it follows that $\Psi_{A[h]_m, \nu} : A[h]^{-} \otimes g \to A[h]^{-} \otimes g'$ is an $L_\infty$ morphism. This means that the Taylor coefficients

$$\partial^j \Psi_{A[h]_m, \nu} : \prod_{i=1}^j (A[h]^{-} \otimes g)_{\nu} \to (A[h]^{-} \otimes g')_{\nu'}$$

satisfy the identities of Proposition 3.10. As explained in step 3, this implies that

$$\partial^j \Psi_{A[h]^{-} \otimes g} : \prod_{i=1}^j (A[h]^{-} \otimes g)_{\nu} \to (A[h]^{-} \otimes g')_{\nu'}$$

also satisfy these identities. We conclude that $\Psi_{A[h], \nu}$ is an $L_\infty$ morphism.

Step 5. Specialization $h \mapsto 1$ induces surjective DG Lie algebra homomorphisms $A[h]^{-} \otimes g \to A \otimes g$ and $A[h]^{-} \otimes g' \to A \otimes g'$, sending $h \omega \mapsto \omega$, $h \omega' \mapsto \omega'$ and $\Psi_{A[h]^{-}, \nu} \mapsto \Psi_{A, \omega}$. Therefore assertions (1-3) of the theorem hold. □

4. The Universal $L_\infty$ Morphism of Kontsevich

In this section $\mathbb{K}$ is a field of characteristic $0$ and $C$ is a commutative $\mathbb{K}$-algebra. Recall that we denote by $\mathcal{T}_C = \mathcal{T}(C/\mathbb{K}) := \text{Der}_{\mathbb{K}}(C)$, the module of derivations of $C$ relative to $\mathbb{K}$. This is a Lie algebra over $\mathbb{K}$. Following [Ko1] we make the next definitions.

**Definition 4.1.** For $p \geq -1$ let

$$\mathcal{T}^p_{\text{poly}}(C) := \bigwedge_{C}^{p+1} \mathcal{T}_C,$$

the module of poly derivations (or poly tangents) of degree $p$ of $C$ relative to $\mathbb{K}$. Let

$$\mathcal{T}_{\text{poly}}(C) := \bigoplus_{p} \mathcal{T}^p_{\text{poly}}(C).$$

This is a DG Lie algebra, with zero differential, and with the Schouten-Nijenhuis bracket, which is determined by the formulas

$$[\alpha_1 \wedge \alpha_2, \alpha_3] = \alpha_1 \wedge [\alpha_2, \alpha_3] + (-1)^{(p_2+1)p_3}[\alpha_1, \alpha_3] \wedge \alpha_2$$

and

$$[\alpha_1, \alpha_2] = (-1)^{1+p_1p_2}[\alpha_2, \alpha_1]$$

for elements $\alpha_i \in \mathcal{T}^p_{\text{poly}}(C)$. 

Definition 4.2. For any \( p \geq -1 \) let \( \mathcal{D}^p_{\text{poly}}(C) \) be the set of \( \mathbb{K} \)-multilinear multi differential operators \( \phi : C^{p+1} \to C \) (see Definition 2.1). The direct sum
\[
\mathcal{D}_{\text{poly}}(C) := \bigoplus_p \mathcal{D}^p_{\text{poly}}(C)
\]
is a DG Lie algebra. The differential \( d_\mathcal{D} \) is the shifted Hochschild differential, and the Lie bracket is the Gerstenhaber bracket (see [Ko1, Section 3.4.2]). The elements of \( \mathcal{D}_{\text{poly}}(C) \) are called poly differential operators relative to \( \mathbb{K} \).

In the notation of Section 2 and Example 1.24 one has
\[
\mathcal{D}^p_{\text{poly}}(C) = \mathcal{D}_{\text{poly}}(C; C, \ldots, C) = \mathcal{C}^{p+1}_{\text{cd}}(C);
\]
see formula (2.3).

Observe that \( \mathcal{D}^p_{\text{poly}}(C) \subset \text{Hom}_\mathbb{K}(C^\otimes (p+1), C) \), and \( \mathcal{D}_{\text{poly}}(C) \) is a sub DG Lie algebra of the shifted Hochschild cochain complex of \( C \) relative to \( \mathbb{K} \). For \( p = -1, 0 \) we have \( \mathcal{D}^{-1}_{\text{poly}}(C) = C \) and \( \mathcal{D}^0_{\text{poly}}(C) = \mathcal{D}(C) \), the ring of differential operators. Note that \( \mathcal{D}^p_{\text{poly}}(C) \) is a left module over \( \mathcal{D}(C) \), by the formula \( D : \phi := D \circ \phi \); and in this way it is also a left \( C \)-module.

When \( C := \mathbb{K}[t] = \mathbb{K}[t_1, \ldots, t_n] \), the polynomial algebra in \( n \geq 1 \) variables, and \( p \geq 1 \), the following is true. The \( \mathbb{K}[t] \)-module \( \mathcal{T}^{-1}_{\text{poly}}(\mathbb{K}[t]) \) is free with finite basis \( \{\partial_{t_1}^j \cdots \partial_{t_p}^j \} \), indexed by the sequences \( 0 \leq i_1 < \cdots < i_p \leq n \). The \( \mathbb{K}[t] \)-module \( \mathcal{D}^{-1}_{\text{poly}}(\mathbb{K}[t]) \) is also free, with countable basis
\[
(4.3)
\{(\partial_{t_1}^j \cdots \partial_{t_p}^j) \mid j_1, \ldots, j_p \in \mathbb{N}^n, \}
\]
where for \( j = (j, k, j, k, \ldots) \in \mathbb{N}^n \) we write \( (\partial_{t_1}^j) \cdot \cdots \cdot (\partial_{t_p}^j) \).

For any \( p \geq -1 \) let \( F_m \mathcal{D}^p_{\text{poly}}(C) \) be the set of poly differential operators of order \( \leq m \) in each argument. This is \( C \)-submodule of \( \mathcal{D}^p_{\text{poly}}(C) \).

Lemma 4.4.
(1) For any \( m, p \) one has
\[
d_\mathcal{D}(F_m \mathcal{D}^p_{\text{poly}}(C)) \subset F_{m-1} \mathcal{D}^{p+1}_{\text{poly}}(C).
\]
(2) For any \( m, m' \), one has
\[
[F_m \mathcal{D}^p_{\text{poly}}(C), F_{m'} \mathcal{D}^{p'}_{\text{poly}}(C)] \subset F_{m+m'} \mathcal{D}^{p+p'}_{\text{poly}}(C);
\]
and
\[
[-, -] : F_m \mathcal{D}^p_{\text{poly}}(C) \times F_{m'} \mathcal{D}^{p'}_{\text{poly}}(C) \to \mathcal{D}^{p+p'}_{\text{poly}}(C)
\]
is a poly differential operator of order \( \leq m+m' \) in each of its two arguments.

Proof. These assertions follow easily from the definitions of the Hochschild differential and the Gerstenhaber bracket; cf. [Ko1, Section 3.4.2].

Lemma 4.5. Assume \( C \) is a finitely generated \( \mathbb{K} \)-algebra. Then \( \mathcal{T}^p_{\text{poly}}(C) \) and \( F_m \mathcal{D}^p_{\text{poly}}(C) \) are finitely generated \( C \)-modules.

Proof. One has
\[
\mathcal{T}^p_{\text{poly}}(C) \cong \text{Hom}_A(\Omega^{p+1}_C, A)
\]
and
\[
F_m \mathcal{D}^p_{\text{poly}}(C) \cong \text{Hom}_C(\mathcal{C}_{p+1, m}(C), C);
\]
see Lemma 2.2. The \( C \)-modules \( \Omega^{p+1}_C \) and \( \mathcal{C}_{p+1, m}(C) \) are finitely generated.
Proposition 4.6. Assume $C$ is a finitely generated $\mathbb{K}$-algebra, and $C'$ is a noetherian, $\mathfrak{c}'$-adically complete, flat, $\mathfrak{c}'$-adically formally étale $C$-algebra. Let’s write $\mathcal{G}$ for either $\mathcal{T}_{\text{poly}}$ or $\mathcal{D}_{\text{poly}}$. Then:

1. There is a DG Lie algebra homomorphism $\mathcal{G}(C) \to \mathcal{G}(C')$, which is functorial in $C \to C'$.
2. The induced $C'$-linear homomorphism $C' \otimes C \mathcal{G}(C) \to \mathcal{G}(C')$ is bijective.
3. For any $m$ the isomorphisms in (2), for $\mathcal{G} = \mathcal{D}_{\text{poly}}$, restrict to isomorphisms

$$C' \otimes C \mathcal{F}_m \mathcal{D}_{\text{poly}}^p(C) \cong \mathcal{F}_m \mathcal{D}_{\text{poly}}^p(C').$$

Proof. Consider $\mathcal{G} = \mathcal{D}_{\text{poly}}$. Let $\phi \in \mathcal{D}_{\text{poly}}^p(C)$. According to Proposition 2.7 applied to the case $M_1, \ldots, M_{p+1}, N := A$, there is a unique $\phi' \in \mathcal{D}_{\text{poly}}^p(C')$ extending $\phi$. From the definitions of the Gerstenhaber bracket and the Hochschild differential, it immediately follows that the function $\mathcal{D}_{\text{poly}}(C) \to \mathcal{D}_{\text{poly}}(C')$, $\phi \mapsto \phi'$, is a DG Lie algebra homomorphism. Parts (2,3) are also consequences of Proposition 2.7.

The case $\mathcal{G} = \mathcal{T}_{\text{poly}}$ is done similarly (and is well-known). □

Consider $C := \mathbb{K}[t]$ and $C' := \mathbb{K}[[t]] = \mathbb{K}[t_1, \ldots, t_n]$, the power series algebra. Since $\mathcal{T}_{\text{poly}}^p(\mathbb{K}[t]) \cong \mathbb{K}[t]$, $\mathcal{T}_{\text{poly}}^p(\mathbb{K}[[t]])$ is a finitely generated left $\mathbb{K}[[t]]$-module, it is an inv $\mathbb{K}[[t]]$-module with the $(t)$-adic inv structure; cf. Example 1.8. Likewise $\mathcal{D}_{\text{poly}}^p(\mathbb{K}[[t]])$ is a dir-inv $\mathbb{K}[[t]]$-module. By Proposition 1.6

$$\mathcal{F}_m \mathcal{D}_{\text{poly}}^p(\mathbb{K}[[t]]) \cong \mathbb{K}[[t]] \otimes_{\mathbb{K}[t]} \mathcal{F}_m \mathcal{D}_{\text{poly}}^p(\mathbb{K}[t]),$$

which is a finitely generated $\mathbb{K}[[t]]$-module. So according to Example 1.9 we may take $\{\mathcal{F}_m \mathcal{D}_{\text{poly}}^p(\mathbb{K}[[t]])\}_{m \in \mathbb{N}}$ as the dir-inv structure of $\mathcal{D}_{\text{poly}}^p(\mathbb{K}[[t]])$. Now forgetting the $\mathbb{K}[[t]]$-module structure, $\mathcal{T}_{\text{poly}}^p(\mathbb{K}[[t]])$ becomes an inv $\mathbb{K}$-module, and $\mathcal{D}_{\text{poly}}^p(\mathbb{K}[[t]])$ becomes a dir-inv $\mathbb{K}$-module.

Proposition 4.7. Let $\mathcal{G}$ stand either for $\mathcal{T}_{\text{poly}}$ or $\mathcal{D}_{\text{poly}}$. Then $\mathcal{G}(\mathbb{K}[[t]])$ is a complete DG Lie algebra in $\text{Dir Inv Mod} \mathbb{K}$.  

Proof. Use Proposition 2.7 and, for the case $\mathcal{G} = \mathcal{D}_{\text{poly}}$, also Lemma 4.4. □

Remark 4.8. One might prefer to view $\mathcal{T}_{\text{poly}}^p(\mathbb{K}[[t]])$ and $\mathcal{D}_{\text{poly}}^p(\mathbb{K}[[t]])$ as topological DG Lie algebras. This can certainly be done: put on $\mathcal{T}_{\text{poly}}^p(\mathbb{K}[[t]])$ and $\mathcal{F}_m \mathcal{D}_{\text{poly}}^p(\mathbb{K}[[t]])$ the $(t)$-adic topology, and put on $\mathcal{D}_{\text{poly}}^p(\mathbb{K}[[t]]) = \lim_{m \to \infty} \mathcal{F}_m \mathcal{D}_{\text{poly}}^p(\mathbb{K}[[t]])$ the direct limit topology (see [Ye1, Section 1.1]). However the dir-inv structure is better suited for our work.

Definition 4.9. For $p \geq 0$ let $\mathcal{D}_{\text{poly}}^{\text{nor},p}(C)$ be the submodule of $\mathcal{D}_{\text{poly}}^p(C)$ consisting of poly differential operators $\phi$ such that $\phi(c_1, \ldots, c_{p+1}) = 0$ if $c_i = 1$ for some $i$. For $p = -1$ we let $\mathcal{D}_{\text{poly}}^{\text{nor},-1}(C) := C$. Define $\mathcal{D}_{\text{poly}}^{\text{nor}}(C) := \bigoplus_{p \geq -1} \mathcal{D}_{\text{poly}}^{\text{nor},p}(C)$. We call $\mathcal{D}_{\text{poly}}^{\text{nor}}(C)$ the algebra of normalized poly differential operators.

From the formulas for the Gerstenhaber bracket and the Hochschild differential (see [Ko1, Section 3.4.2]) it immediately follows that $\mathcal{D}_{\text{poly}}^{\text{nor}}(C)$ is a sub DG Lie algebra of $\mathcal{D}_{\text{poly}}(C)$.

For any integer $p \geq 1$ there is a $C$-linear homomorphism

$$\mathcal{U}_t : \mathcal{T}_{\text{poly}}^{p-1}(C) \to \mathcal{D}_{\text{poly}}^{\text{nor},p-1}(C)$$
with formula

\[(4.10) \quad U_t(\xi_1 \wedge \cdots \wedge \xi_p)(c_1, \ldots, c_p) := \frac{1}{p!} \sum_{\sigma \in S_p} \text{sgn}(\sigma)\xi_{\sigma(1)}(c_1) \cdots \xi_{\sigma(p)}(c_p)\]

for elements \(\xi_1, \ldots, \xi_p \in \mathcal{T}_C\) and \(c_1, \ldots, c_p \in C\). For \(p = 0\) the map \(U_t : T_{\text{poly}}^{-1}(C) \to D_{\text{poly}}^{-1}(C)\) is the identity (of \(C\)).

Suppose \(M\) and \(N\) are complexes in \(\text{Dir Inv Mod} C\) and \(\phi, \phi' : M \to N\) are morphisms of complexes in \(\text{Dir Inv Mod} C\) (i.e. all maps are continuous for the dir-inv structures). We say \(\phi\) and \(\phi'\) are homotopic if there is a degree \(-1\) homomorphism of graded dir-inv modules \(\eta : M \to N\) such that \(d_N \circ \eta + \eta \circ d_M = \phi - \phi'\). We say that \(\phi : M \to N\) is a homotopy equivalence in \(\text{Dir Inv Mod} C\) if there is a morphism of complexes \(\psi : N \to M\) in \(\text{Dir Inv Mod} C\) such that \(\psi \circ \phi\) is homotopic to \(1_M\) and \(\phi \circ \psi\) is homotopic to \(1_N\).

**Theorem 4.11.** Let \(C\) be a commutative \(\mathbb{K}\)-algebra with ideal \(\mathfrak{c}\). Assume \(C\) is noetherian and \(\mathfrak{c}\)-adically complete. Also assume there is a \(\mathbb{K}\)-algebra homomorphism \(\mathbb{K}[t_1, \ldots, t_n] \to C\) which is flat and \(\mathfrak{c}\)-adically formally etale. Then the homomorphism \(U_t : T_{\text{poly}}(C) \to D_{\text{poly}}(C)\) and the inclusion \(D_{\text{poly}}(C) \to D_{\text{poly}}(C)\) are both homotopy equivalences in \(\text{Dir Inv Mod} C\).

**Proof.** Recall that \(B_q(C) = B_0^{-q}(C) := C^\otimes(q+2)\), and this is a \(B_0(C)\)-algebra via the extreme factors. So \(B_q(C) \cong B_0(C) \otimes C^\otimes q\) as \(B_0(C)\)-modules. Let \(C : = C/\mathbb{K}\), the quotient \(\mathbb{K}\)-module, and define \(B_q^{\text{nor}}(C) = B_0^{\text{nor}, -q}(C) := B_0(C) \otimes \mathbb{K}^\otimes q\), the \(q\)-th normalized bar module of \(C\). According to [ML, Section X.2], \(B^{\text{nor}}(C) := \bigoplus_q B^{\text{nor}, -q}(C)\) has a coboundary operator such that the obvious surjection \(\phi : B(C) \to B^{\text{nor}}(C)\) is a quasi-isomorphism of complexes of \(B_0(C)\)-modules.

Define

\[C_q^{\text{nor}}(C) = C^{\text{nor}, -q}(C) := C \otimes B_0(C) B_q^{\text{nor}}(C) \cong C \otimes \mathbb{K}^\otimes q.\]

Because the complexes \(B(C)\) and \(B^{\text{nor}}(C)\) are bounded above and consist of free \(B_0(C)\)-modules, it follows that \(\phi : C(C) \to C^{\text{nor}}(C)\) is a quasi-isomorphism of complexes of \(C\)-modules. Let \(\hat{\Omega}^0_C\) be the \(\mathfrak{c}\)-adic completion of \(\Omega^0_C\), so that \(\hat{\Omega}^0_C \cong C \otimes_{\mathbb{K}[\mathfrak{c}]} \Omega^0_{\mathbb{K}[\mathfrak{c}]}\). There is a \(\mathbb{C}\)-linear homomorphism \(\psi : C^{\text{nor}}(C) \to \Omega^0_C\) with formula

\[\psi(1 \otimes (c_1 \otimes \cdots \otimes c_q)) := d(c_1) \wedge \cdots \wedge d(c_q).\]

Consider the polynomial algebra \(\mathbb{K}[t] = \mathbb{K}[t_1, \ldots, t_n]\). For \(i \in \{1, \ldots, n\}\) and \(j \in \{1, \ldots, q\}\) let

\[\tilde{d}_j(t_i) := 1 \otimes \cdots \otimes 1 \otimes (t_i \otimes 1 - 1 \otimes t_i) \otimes 1 \otimes \cdots \otimes 1 \in B_q(\mathbb{K}[t]),\]

and use the same expression to denote the image of this element in \(C_q(\mathbb{K}[t])\). It is easy to verify that \(C_q(\mathbb{K}[t])\) is a polynomial algebra over \(\mathbb{K}[t]\) in the set of generators \(\{\tilde{d}_j(t_i)\}\). Another easy calculation shows that \(\text{Ker} (\phi : C_q(\mathbb{K}[t]) \to C_q^{\text{nor}}(\mathbb{K}[t]))\) is generated as \(\mathbb{K}[t]\)-module by monomials in elements of the set \(\{\tilde{d}_j(t_i)\}\).

Let’s introduce a grading on \(C_q(\mathbb{K}[t])\) by \(\text{deg}(\tilde{d}_j(t_i)) := 1\) and \(\text{deg}(t_i) := 0\). The coboundary operator of \(C(\mathbb{K}[t])\) has degree \(0\) in this grading. The grading is inherited by \(C_q^{\text{nor}}(\mathbb{K}[t])\), and hence \(\phi : C(\mathbb{K}[t]) \to C_q^{\text{nor}}(\mathbb{K}[t])\) is a quasi-isomorphism of complexes in \(\text{Gr Mod} \mathbb{K}[t]\), the category of graded \(\mathbb{K}[t]\)-modules. Also let’s put a grading on \(\Omega^0_{\mathbb{K}[t]}\) with \(\text{deg}(d(t_i)) := 1\). By [Ye2, Lemma 4.3], \(\psi : C^{\text{nor}}(\mathbb{K}[t]) \to\)
$\bigoplus_q \Omega^q_{K[t]}[q]$ is a quasi-isomorphism in $GrMod \, K[t]$. Because we are dealing with bounded above complexes of free graded $K[t]$-modules it follows that both $\phi$ and $\psi$ are homotopy equivalences in $GrMod \, K[t]$.

Now let's go back to the formally étale homomorphism $K[t] \to C$. We get homotopy equivalences

$$C \otimes_{K[t]} C(K[t]) \xrightarrow{\phi} C \otimes_{K[t]} C^{nor}(K[t]) \xrightarrow{\psi} \bigoplus_q \hat{\Omega}^q_{C}[q]$$

in $GrMod \, C$. We know that $\hat{C}_q(C)$ is a power series algebra in the set of generators $\{d_j(t_i)\}$; see [Ye2, Lemma 2.6]. Therefore $\hat{C}_q(C)$ is isomorphic to the completion of $C \otimes_{K[t]} C_q(K[t])$ with respect to the grading (see Example 4.13). Define $\hat{C}^{nor}_q(C)$ to be the completion of $C \otimes_{K[t]} C^{nor}_q(K[t])$ with respect to the grading. We then have a homotopy equivalence of complexes in $Inv \, Mod$:

$$\hat{C}(C) \to \hat{C}^{nor}(C) \to \bigoplus_q \hat{\Omega}^q_{C}[q].$$

Applying $Hom^{cont}_C(-, C)$ we arrive at quasi-isomorphisms

$$\bigoplus_q (\Lambda^q_C T_C)[-q] \to C^{nor}_{cd}(C) \to C_{cd}(C),$$

where by definition $C^{nor}_{cd}(C)$ is the continuous dual of $\hat{C}^{nor}(C)$. An easy calculation shows that $C^{nor}_{cd}(C) = D^{nor}_{poly}(C).$ \hfill $\square$

One instance to which this theorem applies is $C := K[[t_1, \ldots, t_n]]$. Here is another:

**Corollary 4.12.** Suppose $C$ is a smooth $K$-algebra. Then the homomorphism $U_1 : T_{poly}(C) \to D_{poly}^{nor}(C)$ and the inclusion $D_{poly}^{nor}(C) \to D_{poly}(C)$ are both quasi-isomorphisms.

**Proof.** There is an open covering $Spec \, C = \bigcup Spec \, C_i$ such that for every $i$ there is an étale homomorphism $K[t_1, \ldots, t_n] \to C_i$. Now use Theorem 4.11 \hfill $\square$

Here is a slight variation of the celebrated result of Kontsevich, known as the *Formality Theorem* [Ko1, Theorem 6.4].

**Theorem 4.13.** Let $K[t] = K[t_1, \ldots, t_n]$ be the polynomial algebra in $n$ variables, and assume that $\mathbb{R} \subset K$. There is a collection of $K$-linear homomorphisms $U_j : \wedge^j T_{poly}(K[t]) \to D_{poly}(K[t])$, indexed by $j \in \{1, 2, \ldots\}$, satisfying the following conditions.

(i) The sequence $U = \{U_j\}$ is an $L_\infty$-morphism $T_{poly}(K[t]) \to D_{poly}(K[t]).$

(ii) Each $U_j$ is a poly differential operator of $K[t]$-modules.

(iii) Each $U_j$ is equivariant for the standard action of $GL_n(K)$ on $K[t].$

(iv) The homomorphism $U_1$ is given by equation (4.10).

(v) For any $j \geq 2$ and $\alpha_1, \ldots, \alpha_j \in T^0_{poly}(K[t])$ one has $U_j(\alpha_1 \wedge \cdots \wedge \alpha_j) = 0.$

(vi) For any $j \geq 2$, $\alpha_1 \in gl_n(K) \subset T^0_{poly}(K[t])$ and $\alpha_2, \ldots, \alpha_j \in T^0_{poly}(K[t])$ one has $U_j(\alpha_1 \wedge \cdots \wedge \alpha_j) = 0.$

**Proof.** First let's assume that $K = \mathbb{R}$. Theorem 6.4 in [Ko1] talks about the differentiable manifold $\mathbb{R}^n$, and considers $C^\infty$ functions on it, rather than polynomial
functions. However, by construction the operators \( U_j \) are multi differential operators with polynomial coefficients (see [Ko1 Section 6.3]). Therefore they descend to operators

\[
U_j : \wedge^j T_{\text{poly}}(\mathbb{R}[t]) \to D_{\text{poly}}(\mathbb{R}[t]),
\]

and conditions (i) and (ii) hold. Conditions (iii), (v) and (vi) are properties P3, P4 and P5 respectively in [Ko1 Section 7]. For condition (iv) see [Ko1 Sections 4.6.1-2].

For a field extension \( \mathbb{R} \subset \mathbb{K} \) use base change. □

**Remark 4.14.** It is likely that the operator \( U_j \) sends \( \wedge^j T_{\text{poly}}(\mathbb{K}[t]) \) into \( D_{\text{poly}}^\text{nor}(\mathbb{K}[t]) \). This is clear for \( j = 1 \), where \( U_1(T_{\text{poly}}(\mathbb{K}[t])) = F_1D_{\text{poly}}^\text{nor}(\mathbb{K}[t]) \); but this requires checking for \( j \geq 2 \).

In the next theorem \( T_{\text{poly}}(\mathbb{K}[t]) \) and \( D_{\text{poly}}(\mathbb{K}[t]) \) are considered as DG Lie algebras in \( \text{Dir Inv Mod} \mathbb{K} \), with their \( t \)-adic dir-inv structures. Recall the notions of twisted DG Lie algebra (Lemma 3.24) and multilinear extensions of \( L_\infty \) morphisms (Proposition 3.26).

**Theorem 4.15.** Assume \( \mathbb{R} \subset \mathbb{K} \). Let \( A = \bigoplus_{i \geq 0} A^i \) be a complete super-commutative associative unital DG algebra in \( \text{Dir Inv Mod} \mathbb{K} \). Consider the induced continuous \( A \)-multilinear \( L_\infty \) morphism

\[
U_A : A \hat{\otimes} T_{\text{poly}}(\mathbb{K}[t]) \to A \hat{\otimes} D_{\text{poly}}(\mathbb{K}[t]).
\]

Suppose \( \omega \in A^1 \hat{\otimes} T^0_{\text{poly}}(\mathbb{K}[t]) \) is a solution of the Maurer-Cartan equation in \( A \hat{\otimes} T_{\text{poly}}(\mathbb{K}[t]) \). Define \( \omega' := (\partial_1 U_A)(\omega) \in A^1 \hat{\otimes} D^0_{\text{poly}}(\mathbb{K}[t]) \). Then \( \omega' \) is a solution of the the Maurer-Cartan equation in \( A \hat{\otimes} D_{\text{poly}}(\mathbb{K}[t]) \), and there is continuous \( A \)-multilinear \( L_\infty \) quasi-isomorphism

\[
U_{A,\omega} : (A \hat{\otimes} T_{\text{poly}}(\mathbb{K}[t]))_\omega \to (A \hat{\otimes} D_{\text{poly}}(\mathbb{K}[t]))_{\omega'}
\]

whose Taylor coefficients are

\[
(\partial^n U_{A,\omega})(\alpha) := \sum_{k \geq 0} \frac{1}{n!k!}(\partial^{j+k} U_A)(\omega^k \land \alpha)
\]

for \( \alpha \in \prod_{j}(A \hat{\otimes} T_{\text{poly}}(\mathbb{K}[t]))_j \).

**Proof.** By condition (ii) of Theorem 4.13 and by Proposition 3.24 each operator \( \partial U := U_j \) is continuous for the \( t \)-adic dir-inv structures on \( T_{\text{poly}}(\mathbb{K}[t]) \) and \( D_{\text{poly}}(\mathbb{K}[t]) \). Therefore there is a unique continuous \( A \)-multilinear extension \( \partial U_A \). Condition (v) of Theorem 4.13 implies that \( \partial U_A(\omega^j) = 0 \) for \( j \geq 2 \). By Theorem 3.27 we get an \( L_\infty \) morphism \( U_{A,\omega} \).

It remains to prove that \( \partial U_A \) is a quasi-isomorphism. According to Theorem 4.11 for every \( i \) the \( \mathbb{K} \)-linear homomorphism

\[
\partial U_A : A^i \hat{\otimes} T_{\text{poly}}(\mathbb{K}[t]) \to A^i \hat{\otimes} T_{\text{poly}}(\mathbb{K}[t])
\]

is a quasi-isomorphism. Since we are looking at bounded below complexes, a spectral sequence argument implies that

\[
\partial U_A : A \hat{\otimes} T_{\text{poly}}(\mathbb{K}[t]) \to A \hat{\otimes} T_{\text{poly}}(\mathbb{K}[t])
\]

is a quasi-isomorphism. □
CONTINUOUS AND TWISTED $L_\infty$ MORPHISMS

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