ON THE IRREGULARITY OF ENSO: HIDDEN AND UNSTABLE PERIODIC ORBITS AS A RESULT OF HOMOCLINIC BIFURCATIONS IN THE SUAREZ-SCHOPF DELAYED OSCILLATOR

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Abstract. In dynamical systems, a hidden attractor is an attracting set, which cannot be localized by a trajectory starting from a small neighbourhood of any equilibrium. Without appealing to special analytical-numerical procedures and the intuition behind them, hidden attractors in multiparameter families of systems (especially infinite-dimensional ones) can be only discovered by an accident. In this paper we follow the feedback gain intuition and its justification via the generalized Poincaré-Bendixson theory for infinite-dimensional systems. As a result, we discover hidden periodic orbits in the Suarez-Schopf delayed oscillator for El Niño-Southern Oscillation (ENSO) with parameters from the linear stability region. Moreover, from our developments on inertial (slow) manifolds theory we show that hidden oscillations coexist with single unstable periodic orbits and that self-excited oscillations must coexist with pairs of unstable periodic orbits. The corresponding phase portraits transit into each other through a bifurcation of a homoclinic “figure eight”. We propose a description of the dynamics in the linear stability region by dividing it to subregions corresponding to self-excited oscillations, hidden oscillations and the non-oscillatory part. For different parameters from the hidden oscillations region there is a variety of periods, which may correspond to real ENSO events. Multistability in the hidden oscillations region may correspond to the stochastic irregularity of ENSO. On the other hand, a homoclinic “figure eight” may lead to reach (chaotic) dynamics under a small amplitude periodic forcing. Thus, the simple Suarez-Schopf model may become a basis for both theories of irregularity. We also propose a smaller non-oscillatory region (motivated by dimension estimates), which can be analytically described.

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1. Introduction

In dynamical systems, a hidden attractor is an attracting set, which cannot be localized by a trajectory starting from a small neighbourhood of any equilibrium. This concept is motivated by the standard approach for numerical studying of nonlinear systems based on the local analysis of equilibria, where trajectories from neighbourhoods of unstable equilibria are traced to localize self-excited attractors in the system. The notion of a hidden attractor was suggested by G. A. Leonov and N. V. Kuznetsov, who also justified its significance for applied and theoretical problems [15]. Since then, this area has attracted more and more attention. Hidden attractors were discovered in many applied models (besides [15], see the review of D. Dudkowski et al [8] and N. V. Kuznetsov [14]), where their presence may lead to a sudden switch to an unpredicted behaviour and disastrous consequences. This indicates that one should be very careful when analyzing nonlinear systems. Here we are interested in hidden oscillations, which are hidden attractors given by periodic orbits.

When parameters leading to a hidden oscillation are found, it can be localized by different methods, including the guessing of an initial point from its basin of attraction as a result of a more careful treatment. Often, in a region of the system natural parameters there may be parameters corresponding to self-excited attractors, which evolves into the hidden orbit under changes of parameters (this is also the case we encounter in our analysis below). But the main question is how to find a region in the space of parameters, where one may expect the presence of hidden attractors and which should be given more attention?

The first general ingredient of many results concerned with the localization of hidden attractors is the continuation by parameter procedure (another approach is concerned with the so-called perpetual points [8]). Here one considers a family of parametrized, say by \( \varepsilon \in [0, 1] \), vector fields such that the corresponding to \( \varepsilon = 0 \) system has an easily localized (for example, self-excited) periodic orbit. Then one traces the evolution of this orbit under the change of \( \varepsilon \) with the hope that a hidden attractor of the system at \( \varepsilon = 1 \), which corresponds to the original system, will be revealed.

The second ingredient should be, of course, the choice of the family of vectors field. To the best of our knowledge, there is only one approach that has repeatedly proven its effectiveness for applied models and which leads to discovery of interesting regions in the space of parameters. We call it the feedback gain approach. Here one applies a linear feedback with gain parameters to the system linearized at a given asymptotically stable equilibrium. Outside of a sufficiently large neighbourhood of the equilibrium this feedback may be saturated to guarantee, for example, the uniqueness of the stationary state and dissipativity of the system. The gain parameters are taken to guarantee required spectral properties, which allow to analytically justify the existence of an easily localized periodic orbit under some additional assumptions. Verification of these additional assumptions leads to the discovery of interesting regions in the parameters space.

The first success (namely, the discovery of a hidden chaotic attractor in the Chua system) in the field is concerned with the justification via the describing function method proposed by G. A. Leonov and N. V. Kuznetsov [15]. This method is strictly restricted to ODEs and is not applicable (at least for the moment) for infinite-dimensional systems.

Another method (which is more appropriate for us) was suggested by I. M. Burkin [6] also in the case of ODEs. It is based on the generalized Poincaré-Bendixson theory developed by R. A. Smith [20]. To apply this theory, the gain parameters are chosen so that the equilibrium become unstable with a two-dimensional unstable manifold. Then the feedback is saturated preserving the uniqueness of the equilibrium and providing the dissipativity
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for the system (already at this point one can obtain an interesting region of parameters). The final condition is a frequency-domain condition, which in this case has the form of an inequality containing the transfer function of the linear part and the gain parameters. If this condition is satisfied then there is a two-dimensional inertial (slow) manifold and the Poincaré-Bendixson trichotomy holds. A bit delicate study of the inertial manifold properties allows to show the existence of a periodic orbit, which attracts typical points from a neighbourhood of the equilibrium.

In our works [3, 5, 4] it is shown that these ideas has a natural geometric generalization that extends the area of applications. In particular, this geometric theory can be applied for delay equations due to the recent progress on the Frequency Theorem [2] and semigroups in Hilbert spaces [1] done by the present author. Of course, the experimentalist must be convinced by examples, some of which are already given in [6]. Here we will follow this approach to study the delayed action oscillator proposed by M. J. Suarez and P. S. Schopf in 1988 [21] as a simple model for El Niño–Southern Oscillation (ENSO).

It may seem surprising that neither M. J. Suarez and P. S. Schopf [21] nor I. Boutle, R. H. S. Taylor and R. A. Römer [7], who conducted independent simulations of the model, found oscillations in the linear stability region. Apparently, they, misled by local linear analysis, were not interested in this region. A bit more careful analysis shows that there are many self-excited oscillations (periodic orbits) for parameters close to the neutral curve, which determines the linear stability region. As we move towards the true stability region, these oscillations become hidden and then disappear. There is also a variety of periods for these oscillations, which may agree with real ENSO events (see Section 5).

It turns out that a somewhat complete picture can be described. Namely, using numerical estimates to bound the global attractor, we establish the existence of a two-dimensional inertial (slow) manifold with the aid of our above mentioned developments on the theory [3] (see Section 3). From this we can conclude that discovered self-excited oscillations must coexist with a pair of unstable periodic orbits (see Fig. 5) and hidden oscillations must coexist with a single unstable periodic orbit (see Fig. 2). Moreover, between these phase portraits there must exist a homoclinic “figure eight” (see Fig. 1), which bifurcates into the single unstable orbit or the pair of unstable orbits. Hidden periodic orbits are located close to the true stability region and, as we move towards this region, the single unstable periodic orbit tends to the hidden periodic orbit and then they both collapse on collision that leads to a gradient-like (convergent) behaviour. We conjecture that this scenario completely describes dynamics in the linear stability region (see Remark 5).

Note that discovered phase portraits indicate the irregularity of ENSO (see R. Kleeman [12], the monograph of M. J. McPhaden, A. Santoso and W. Cai [18] or H. A. Dijkstra [9] for many discussions) in a more natural manner than the dynamics corresponding to the linear instability region, where all typical transient processes lead only to an attracting periodic orbit.

Namely, multistability and the presence of hidden attractors described at Fig. 2 may correspond to the stochastic irregularity of ENSO [12]. In this theory, the irregularity is believed to be caused by external forces (noise), which act on smaller time and space scales. Such noise may force the current system state to different basins of attraction. Note that Fig. 2 describes the dynamics only schematically. In the Suarez-Schopf model the single unstable periodic orbit is flattened to zero (that agrees with the homoclinic bifurcation scenario) and at some of its parts (especially corresponding to the warm and cold phases) can be located
Figure 1. A self-excited attracting periodic orbit (cyan) coexists with a homoclinic “figure eight” orbit (pink), which bifurcates into a single unstable cycle (Fig. 2) or a pair of unstable cycles (Fig. 3).

close to the hidden periodic orbit as well as to the unstable separatrices (see Remark 4). This only promotes the role of noise.

On the other hand, it is well-known that a homoclinic “figure eight” (as at Fig. 1) may lead to reach (chaotic) dynamics under the influence of a small periodic forcing (see, for example, the paper of S. V. Gonchenko, C. Simó and A. Vieiro [10]). This foretastes the phenomena observed in a bit complex delay model studied by E. Tziperman et al. [19], who laid the foundation for the irregularity theory based on the low-dimensional chaos caused by a small amplitude periodic forcing [18] [9].

Thus, our study shows that the Suarez-Schopf model can be a basis for both theories of irregularity or may even reconcile them. Discovered patterns and relative simplicity of the model open a perspective of studying via more delicate analytical techniques. It also shows that not only engineering system, but also models from climate dynamics must be studied more carefully.

Note that our methods are rather general and can be applied for many other models. Thus, they should be of interest for a wide range of experimentalists. Despite that the mathematical background behind [3] [1] [2] may seem too large for not mathematicians, in applications one follows straightforward computations. In fact, our theory justifies some kind of “advanced non-local linear analysis” that may help to discover nonlinear phenomena unreachable for the standard local analysis.

This paper is organized as follows. In Section 2 we state basic rigorous facts about dynamics of the Suarez-Schopf model and apply the feedback gain approach to localize hidden periodic orbits. In Section 3 we show the existence of two-dimensional inertial (slow)
manifolds and unstable periodic orbits for certain parameters. In Section 4 by the same approach we study the existence of homoclinic orbits. In Section 5 we propose a description of the dynamics in the linear stability region based on results of our analytical-numerical approach and discuss periods of hidden periodic orbits. In Section 6 we propose an analytical region and state mathematical conjectures on its non-oscillatory nature.

2. Hidden oscillations in the Suarez-Schopf model

Now we consider the Suarez-Schopf model \[21\] given by the scalar delay equation

\[
\dot{x}(t) = x(t) - \alpha x(t - \tau) - x^3(t), \tag{2.1}
\]

where \(\alpha \in (0, 1)\) and \(\tau > 0\) are dimensionless parameters. Here \(x(\cdot)\) represents the sea temperature anomaly. It is easy to see that the central inversion \(x \mapsto -x\) maps solutions into (possibly different) solutions.

Let us consider the value \(\gamma := \sqrt{1 + \alpha}\) and define the set \(S_R := \{\phi \in C([-\tau, 0]; \mathbb{R}) \mid \|\phi\|_\infty \leq \gamma + R\}\) for any \(R \geq 0\). Here \(\|\cdot\|_\infty\) denotes the supremum norm in the space of continuous functions \(C([-\tau, 0]; \mathbb{R})\). At this point we can state some general dynamical properties of (2.1) as follows.
Figure 3. A self-excited attracting periodic orbit (cyan) coexists with two unstable periodic orbits (red) born after a homoclinic bifurcation. The unstable separatrices (blue and orange) tend to the attracting periodic orbit. The stable separatrices (pink) tend to the unstable periodic orbits in the negative direction of time.

(1) Equation \(2.1\) generates a dissipative semiflow \(\varphi^t\), where \(t \geq 0\), in the space \(E = C([-\tau, 0]; \mathbb{R})\). Moreover, the sets \(S_R\) are positively invariant, i.e. \(\varphi^t(S_R) \subset S_R\) for all \(R \geq 0\), and the \(\omega\)-limit set of any point \(\phi_0 \in E\) lies in \(S_0\).

(2) The \(\omega\)-limit set of any point \(\phi_0 \in E\) satisfies the Poincaré-Bendixson trichotomy, i.e. it can be either a stationary point, either a periodic orbit or a union of a set of stationary points and complete orbits connecting them.

Item (1) is shown in our paper [1]. Item (2) follows from the result of J. Mallet-Paret and G. R. Sell [17] since \(2.1\) is a monotone cyclic feedback system.

There are three stationary states: the origin \(\phi_0 \equiv 0\) and a pair of symmetric ones \(\phi^+ \equiv \sqrt{1 - \alpha}\) and \(\phi^- \equiv -\sqrt{1 - \alpha}\). Eigenvalues of the linearization at \(\phi^0\) are given by the roots \(p \in \mathbb{C}\) of the equation (see, for example, J. K. Hale [11])

\[
1 - \alpha e^{-\tau p} - p = 0. \tag{2.2}
\]

It is straightforward to show that \(2.2\) has exactly one positive root \(\lambda_1 > 0\), one negative root \(\lambda_2 < 0\) and the other roots are located to the left from the line \(\lambda_2 + i \mathbb{R}\). So there is always a one-dimensional unstable manifold at \(\phi^0\).

Linearization at the symmetric stationary states leads to the equation

\[
3\alpha - 2 - \alpha e^{-\tau p} - p = 0. \tag{2.3}
\]
Searching for purely imaginary roots \( p = i\theta \), we get that such roots appear for
\[
\theta = \pm \sqrt{\alpha^2 - (3\alpha - 2)^2} \quad \text{and} \quad \tau = \frac{\pm \arccos \frac{3\alpha - 2}{\alpha}}{\sqrt{\alpha^2 - (3\alpha - 2)^2}},
\]
where \( k = 0, 1, 2, \ldots \). In particular, when \( \alpha \) is fixed, the first (as \( \tau \) increases) pair of purely imaginary roots appear at
\[
\tau = \frac{\arccos \frac{3\alpha - 2}{\alpha}}{\sqrt{\alpha^2 - (3\alpha - 2)^2}}.
\]

Let \( \Omega_{st} \) denote the set of pairs \((\tau, \alpha)\), where \( \alpha \in (0, 1) \) and \( \tau > 0 \) is strictly smaller than the right-hand side of \((2.5)\), i.e. the point \((\tau, \alpha)\) lies below the curve \((2.5)\) which was called neutral curve in \([21]\). This curve is displayed at Fig. [4]. It can be verified that for any parameters from \( \Omega_{st} \) equation \((2.1)\) has only roots with negative real parts and, consequently, the symmetric stationary states \( \phi^+ \) and \( \phi^- \) are asymptotically stable. We call \( \Omega_{st} \) the linear stability region.

Numerical simulations described in \([21]\) (as well as in \([7]\)) did not detect periodic orbits for parameters from \( \Omega_{st} \). In fact, a more careful study allows to detect self-excited periodic orbits for parameters \((\tau, \alpha) \in \Omega_{st} \) close to the neutral curve \((2.5)\). For example, when \( \alpha = 0.75 \) the point \((\tau, \alpha)\) belongs to \( \Omega_{st} \) for any \( \tau < 1.74 \). Taking \( \tau = 1.65 \) and starting from a neighbourhood of \( \phi^0 \), one can observe a symmetric periodic orbit shown at Fig. [6]. Its period is \( \sigma \approx 12.3 \) (\( \approx 8.17 \) years).

**Remark 1.** To obtain the period in days, one should use the formula \( \sigma_{\text{days}} = (\sigma \cdot \Delta)/\tau \), where \( \Delta \) is the total delay depending on the Rossby and Kelvin waves propagation. This formula follows from the scaling used in \([21]\) to obtain the dimensionless form \((2.1)\). Usually one takes \( \Delta = 400 \) \([21]\) or \( \Delta = 359 \) \([7]\). We will use the value \( \Delta = 400 \). Below we will show the presence of hidden periodic orbits within a variety of periods such that any choice of \( \Delta \) within reasonable limits does not change our qualitative conclusions.

**Remark 2.** All presented numerical integrations are done via the Python “ddeint 0.2” package. To obtain Fig. [6] and Fig. [9] equation \((2.1)\) was integrated on the interval \([0, 200]\) with the step 0.001. To obtain Fig. [10] and Fig. [11] we used the interval \([0, 100]\) and the step 0.001. To obtain Fig. [12] we used the interval \([0, 100]\) and the step 0.0005.

Let us rewrite \((2.1)\) as
\[
\dot{x}(t) = (3\alpha - 2)x(t) - \alpha x(t - \tau) + f(x(t)), \quad (2.6)
\]
where \( f(y) = -3y^2 + 3(1 - \alpha)y \). For two numbers \( 0 < \mu_\infty < \mu \) we consider the nonlinearity \( g: \mathbb{R} \to \mathbb{R} \) defined as
\[
g(y) = \begin{cases} 
\mu_\infty \cdot (y - 1) + \mu & \text{for } y > 1, \\
\mu \cdot y & \text{for } |y| \leq 1,
\end{cases} \quad (2.7)
\]
Along with \((2.6)\) we consider the family of equations for \( \varepsilon \in [0, 1] \) defined as
\[
\dot{x}(t) = (3\alpha - 2)x(t) - \alpha x(t - \tau) + F_\varepsilon(x(t)), \quad (2.8)
\]
where \( F_\varepsilon(y) = \varepsilon f(y) + (1 - \varepsilon)g(y) \). The linearization of \((2.8)\) with \( \varepsilon = 0 \) at the zero stationary state leads to the characteristic equation
\[
3\alpha - 2 + \mu - \alpha e^{-\tau p} - p = 0. \quad (2.9)
\]
Assuming that $-4\alpha + 2 < \mu < -2\alpha + 2$, as above we get a curve (for a fixed $\alpha$)

$$
\tau = \frac{\arccos \frac{3\alpha - 2 + \mu}{\alpha}}{\sqrt{\alpha^2 - (3\alpha - 2 + \mu)^2}},
$$

(2.10)

whose points $(\tau, \mu)$ corresponds to the first (as $\tau$ increases) appearance of purely imaginary roots of (2.9). For $\alpha = 0.75$ this curve is displayed at Fig. 5. In particular, for $\tau = 1.58$ and $\alpha = 0.75$, putting $\mu = 0.45$ (the point $(\mu, \tau)$ is above the curve) and $\mu_\infty = 0.005$ (the point $(\mu_\infty, \tau)$ is below the curve), we get that (2.8) with $\varepsilon = 0$ has a unique stationary state with a two-dimensional unstable manifold and the system is dissipative (for the latter see [20]).
Figure 6. A self-excited symmetric periodic orbit with period $\sigma \approx 12.3$ ($\sigma_{\text{years}} \approx 8.16$ years) of system (2.1) for $\alpha = 0.75$ and $\tau = 1.65$ localized by two trajectories (red and green), which start following the unstable separatrices of the saddle equilibrium. Two curves (blue and orange) tend to the asymptotically stable symmetric equilibria. All the trajectories are projected onto the $(x(t-\tau), x(t))$-plane.

The transfer function (see [2]) of (2.8) is given by

$$W(p) = \frac{1}{3\alpha - 2 - \alpha e^{-\tau p} - p}. \quad (2.11)$$

Recall that for $\tau = 1.58$ and $\alpha = 0.75$ all the roots of (2.3) are pairs of complex-conjugate numbers with negative real parts. Moreover, numerical calculations show that the first (as the real part decreases) two pairs of roots can be estimated as $\lambda_{1,2} \approx -0.05 \pm i0.75$ and $\lambda_{3,4} = -1.2 \pm i4.78$. Thus, for $\nu := 0.88$ there are exactly two roots located to the right of the line $-\nu + i\mathbb{R}$. It is clear that the nonlinearity $F_0 = g$ is Lipschitz with the Lipschitz constant $\Lambda = \mu = 0.45$. The numerical estimation shown at Fig. 7 implies that

$$|W(-\nu + i\omega)| \leq 0.64 < \Lambda^{-1} = 2.2 \quad \text{for all } \omega \in \mathbb{R}. \quad (2.12)$$

Now we can state an auxiliary proposition.

**Proposition 1.** Consider (2.8) with $\varepsilon = 0$, $\alpha = 0.75$, $\tau = 1.58$, $\mu = 0.45$ and $\mu_{\infty} = 0.005$ and let (2.12) be satisfied. Then there exists a periodic orbit such that any point from a sufficiently small neighbourhood of the zero equilibrium (except the points from its stable manifold) tends to the periodic orbit.

**Proof.** Since the frequency-domain condition in (2.12) is satisfied, the semiflow generated by (2.8) (with the given parameters) satisfies all the assumptions of Theorem 6.2.2 from [3] with $\nu = 0.88$ and $j = 2$ that gives the existence of a two-dimensional inertial manifold $\mathfrak{M}$ (homeomorphic to the plane $\mathbb{R}^2$), which attracts all the trajectories by trajectories lying on the manifold $\mathfrak{M}$ at the uniform exponential rate with the exponent $\nu$. In particular, all the trajectories satisfy the Poincaré-Bendixson trichotomy. Note that the semiflow is
dissipative due to the choice of the nonlinearity \( g \). From this and since the equilibrium is unique, any point (except the equilibrium itself) from its unstable manifold (which lies in \( A \) by the construction; see [3]) must tend to a common periodic trajectory. Note that any point \( \phi_0 \) from the entire space, which is sufficiently close to the equilibrium, is exponentially attracted by a point \( \phi^*_0 \) from the unstable manifold (by Theorem 3.1.1. from [3]). Moreover, if \( \phi^*_0 \) coincides with the equilibrium then \( \phi_0 \) belongs to the stable manifold. This finishes the proof.

\[ \square \]

Remark 3. Note that Proposition [1] does not guarantee that the periodic orbit will be orbitally stable\(^1\). However, there is at least one orbitally stable periodic orbit and any orbitally stable periodic orbit is asymptotically orbitally stable provided that it is isolated from other periodic orbits (see [4]). In applications, we expect the periodic orbit from Proposition [1] to be the only periodic orbit and, consequently, to be asymptotically orbitally stable.

At Fig. 8 one may see some steps from the continuation by parameter procedure applied to (2.8) as \( \varepsilon \) varies from 0 to 1. This leads to the discovery of a hidden periodic orbit for the original system (2.1). Below in Section 3 we justify that it is indeed a hidden periodic orbit. Some trajectories near this orbit are shown at Fig. 9.

3. Unstable periodic orbits

A careful look at Fig. 9 suggests that the limit dynamics may take place on a two-dimensional manifold homeomorphic to a plane (in particular, this can be motivated by the symmetricity of the observed periodic orbits). If this is true, then there are not so many ways to describe the phase portrait, which follow from the Poincaré-Bendixson theorem.

\(^1\)Note that a similar statement in [4] needs a clarification.
Figure 8. Results of the continuation by parameter procedure. At the initial step $\varepsilon = 0$ we see the self-excited periodic orbit from Proposition 1. At $\varepsilon = 1$ this orbit evolves into a hidden periodic orbit (which is symmetric) for (2.1). All the trajectories are projected onto the $(x(t - \tau), x(t))$ plane.

Below we shall justify that Fig. 9 corresponds to the phase portrait displayed at Fig. 2. We will do this via combining analytical and numerical approaches again.

Let $\alpha = 0.75$ and $\tau = 1.58$ be fixed. Let us again consider the original equation (2.1) as a control system in the Lur’e form given by (2.6) and the transfer function $W(p)$ given by (2.11) for the chosen parameters. For the same as above $\nu = 0.88$ there are exactly two roots of (2.2) located to the right of the line $-\nu + i\mathbb{R}$.

We suppose that the global attractor of (2.1) lies in the ball of radius 0.8 as it is suggested by Fig. 9 and further numerical experiments. Clearly, the Lipschitz constant of $f(y) = -y^3 + 3(1 - \alpha)y$ on $[-0.8, 0.8]$ is $\Lambda = 3 \cdot 0.8^2 - 0.75 = 1.17$ and $\Lambda^{-1} = 0.854700 > 0.85$. Using the numerical estimate from (2.12) we get

$$|W(-\nu + i\omega)| \leq 0.64 < 0.85 < \Lambda^{-1}$$

for all $\omega \in \mathbb{R}$.  

$$(3.1)$$
A hidden symmetric periodic orbit with period $\sigma \approx 15.3$ (years $\approx 10.6$ years) of system (2.1) for $\alpha = 0.75$ and $\tau = 1.58$ localized by the green trajectory. Two trajectories (blue and orange), which start following the unstable separatrices of the saddle equilibrium, tend to the asymptotically stable equilibria. All the trajectories are projected onto the $(x(t-\tau), x(t))$ plane.

Let a nonlinear function $g: \mathbb{R} \to \mathbb{R}$ be such that it coincides with $f(x) = -x^3 + 3(1-\alpha)x$ on the interval $[-0.8, 0.8]$ and smoothly (at least $C^2$) prolonged outside of this interval with preserving the Lipschitz constant $\Lambda$. We consider the equation

$$\dot{x}(t) = (3\alpha - 2)x(t) - \alpha x(t - \tau) + g(t) \tag{3.2}$$

and the corresponding semiflow in $C([-\tau, 0]; \mathbb{R})$. Clearly, the dynamics of (2.1) and (3.2) coincide on the global attractor of (2.1).

**Proposition 2.** Suppose that for $\alpha = 0.75$ and $\tau = 1.58$ the global attractor of (2.1) is contained in the ball of radius 0.8 in $C([-\tau, 0]; \mathbb{R})$ and (3.1) is satisfied. Then the global attractor of (2.1) (with the above defined $g$) lies on a two-dimensional $C^1$-smooth normally hyperbolic invariant manifold for (3.2). In particular, if there is a periodic orbit of (2.1), surrounding the origin $\phi^0 \equiv 0$, then both stable separatrices of $\phi^0$ on the manifold are well-defined and must either tend to unstable periodic orbits (possibly different) in the negative direction of time or form a symmetric homoclinic “figure eight” (not necessarily containing $\phi^+$ and $\phi^-$).

**Proof.** The hypotheses allow us to apply to the semiflow generated by (3.2) Theorem 6.2.2 from [3] with $\nu = 0.88$ and $j = 2$, Theorem 6 from [1] (differentiability) and Theorem 3.6.1 from [3] (normal hyperbolicity). Thus, there exists a two-dimensional $C^1$-differentiable normally hyperbolic inertial manifold $\mathfrak{A}$, which is homeomorphic to the plane $\mathbb{R}^2$ and attracts exponentially fast (with the exponent $\nu = 0.88$) all trajectories of the semiflow generated by (3.2). From the normal hyperbolicity of $\mathfrak{A}$ (see [3]) and the inertial form obtained in [1] one can deduce that the tangent space to $\mathfrak{A}$ at the origin $\phi^0$ is given by the eigenvectors.
We have the Lipschitz constant $\Lambda$ of numerical experiments give the estimate $0$. The roots located to the right of the line $\lambda$ as parameters, the first (as the real part decreases) two pairs of roots of (2.2) can be estimated. This finishes the proof.

According to Fig. 3 the conclusion of Theorem 2 must hold. We have to justify that there must be a single symmetric unstable periodic orbit, which encloses all the equilibria and separate them from the hidden periodic orbit. Note that any periodic orbit, which encloses the origin $\phi^0$, must enclose the other symmetric equilibria $\phi^+, \phi^-$ due to the Poincaré index theorem and, consequently, it must be symmetric.

If the stable separatrices of the origin $\phi^0$ form a homoclinic “figure eight” (not necessarily containing $\phi^+$ and $\phi^-$), then there must exist trajectories, starting from a small neighborhood of $\phi^0$, which does not tend to $\phi^+$ or $\phi^-$. As it is shown on Fig. 10 this is not the case. So, the stable separatrices must tend to an unstable periodic orbit, which must enclose $\phi^0$ (in virtue of Fig. 10) and, consequently, $\phi^+$ and $\phi^-$ that leads to the symmetricality.

**Remark 4.** We are unable to localize this orbit numerically, so we use the first winding of the green trajectory (which corresponds to the initial condition $\phi_0(\theta) = 0.255 \cdot \cos^2(\theta)$ for $\theta \in [-\tau, 0]$) from Fig. 9 to approximate the unstable periodic orbit. This gives an estimate of its period as $\sigma \approx 23.9 \approx 16.58$ years. Note that according to Fig. 9 some parts of this orbit are located close to the unstable separatrices as well as to the hidden periodic orbit. If this is not an artifact of the projection, then the noise acting on the system in these states may lead to sudden transition to different basins of attraction. We conjecture that the corresponding phase portrait is the same as shown at Fig. 2 and a similar picture holds for the parameters from Section 5 where hidden orbits with periods agreeing with real ENSO events are observed.

Now lets take a more careful look at Fig. 4 and follow the same approach. Consider again the original equation (2.1) as a control system in the Lur’e form given by (2.6) with the transfer function $W(p)$ given by (2.11) for $\alpha = 0.75$ and $\tau = 1.65$. For the chosen parameters, the first (as the real part decreases) two pairs of roots of (2.2) can be estimated as $\lambda_{1,2} \approx -0.027 \pm i0.73$ and $\lambda_{3,4} = -1.12 \pm i4.58$. Thus for $\nu := 0.81$ there are exactly two roots located to the right of the line $-\nu + i\mathbb{R}$.

Again we suppose that the global attractor of (2.1) lies in the ball of radius 0.9 (direct numerical experiments give the estimate 0.85) as it is suggested by Fig. 6. Consider the Lipschitz constant $\Lambda$ of $f(y) = -y^3 + 3(1-\alpha)y$ on $[-0.9, 0.9]$ that is $\Lambda = 3 \cdot 0.9^2 - 0.75 = 1.68$. We have $\Lambda^{-1} = 0.5952380 > 0.59$. From numerical simulations we get that

$$W(-\nu + i\omega) \leq 0.56 < 0.59 < \Lambda^{-1} \text{ for all } \omega \in \mathbb{R}. \quad (3.3)$$

Let $g: \mathbb{R} \to \mathbb{R}$ be a smooth (at least $C^2$) continuation of $f$ from $[-0.9, 0.9]$ to $\mathbb{R}$, which preserves the Lipschitz constant $\Lambda$. We have the following analog of Proposition 2 which can be proved in the same way.

**Proposition 3.** Suppose that for $\alpha = 0.75$ and $\tau = 1.65$ the global attractor of (2.1) is contained in the ball of radius 0.9 in $C([-\tau, 0]; \mathbb{R})$ and (3.3) is satisfied. Then the conclusion of Proposition 2 holds.
Figure 10. Some trajectories near the origin $\phi^0$ of system (2.1) with $\alpha = 0.75$ and $\tau = 1.58$. Here trajectories from different saddle parts tend to the asymptotically stable equilibria. All the trajectories are projected onto the $(x(t - \tau), x(t))$ plane.

In virtue of Fig. 11, which shows trajectories starting from all the saddle parts, the stable separatrices cannot form a homoclinic “figure eight” orbit and cannot tend (in the negative direction of time) to a periodic orbit, which encloses $\phi^0$. Thus, each of the stable separatrices must tend to its personal unstable periodic orbit, which encloses only one of the symmetric stationary states. We expect that the global phase portrait corresponds to Fig. 3.

4. Homoclinic “figure eight” orbits

Observations made in Section 3 motivate the existence of a homoclinic “figure eight” corresponding to parameters $\alpha = 0.75$ and $1.58 < \tau < 1.65$. We have seen that a single unstable periodic orbit must appear when trajectories from all the saddle parts close to $\phi^0$ tend to the symmetric equilibria and a pair of unstable periodic orbits appear when trajectories from all the saddle parts close to $\phi^0$ tend to the attracting periodic orbit. If the dynamics can be different for distinct saddle parts, then there must exist homoclinic “figure eight” orbits. Since on the interval $[1.58, 1.65]$ there may be only one value, which corresponds to homoclinic orbits, such a value cannot be founded numerically. But near this value one should expect similar behaviour due to computation errors. We found such behaviour near $\tau = 1.596$. Some trajectories near $\phi^0$ are shown at Fig. 12.

Consider again the original equation (2.1) as a control system in the Lur’e form given by (2.6) with $\alpha = 0.75$ and $\tau = 1.596$ and the transfer function $W(p)$ given by (2.11). For the given parameters the first (as the real part decreases) two pairs of roots of (2.2) can be estimated as $\lambda_{1,2} \approx -0.044 \pm 0.75i$ and $\lambda_{3,4} = -1.18 \pm 4.74i$. Thus, for $\nu := 0.87$ there are exactly two roots located to the right of the line $-\nu + i \mathbb{R}$.

Fig. 12 suggests that the global attractor lies in a ball with radius 0.8. Thus, we use the above computed Lipschitz constant $\Lambda = 1.17$ with $\Lambda^{-1} > 0.85$. It can be verified numerically
Figure 11. Some trajectories near the origin \( \phi^0 \) of system (2.1) with \( \alpha = 0.75 \) and \( \tau = 1.65 \). Here trajectories from all the saddle regions tend to the attracting periodic orbit. All the trajectories are projected onto the \( (x(t-\tau), x(t)) \) plane.

that
\[
|W(-\nu + i\omega)| \leq 0.54 < 0.85 < \Lambda^{-1} \text{ for all } \omega \in \mathbb{R}. \tag{4.1}
\]

So, we have the following proposition.

**Proposition 4.** Suppose that for \( \alpha = 0.75 \) and \( \tau = 1.596 \) the global attractor of (2.1) is contained in the ball of radius 0.8 in \( C([-\tau, 0]; \mathbb{R}) \) and (4.1) is satisfied. Then the conclusion of Proposition 2 holds.

As we already noted Fig. 12 along with Proposition 4 indicate the presence of a homoclinic “figure eight” orbits for some parameter near \( \tau = 1.596 \). We expect that for any \( \alpha \) there is exactly one such \( \tau \), which corresponds to the point on the upper hidden curve introduced in the next section.

5. Dynamics in the linear stability region

From numerical experiments we made some observations. For each pair of parameters in the linear stability region there is at most one attracting symmetric periodic orbit, which “encloses” the global attractor, and this orbit depend continuously or disappears when the parameters change. Moreover, this attracting periodic orbit can be localized by a trajectory starting outside of the region \( S_0 \). For example, in the results below we use the initial condition \( \phi_0(\theta) = 3.5 \cdot \cos^2(\theta) - 1 \), where \( \theta \in [-\tau, 0] \), for any considered parameters. This along with the symmetricity indicate the existence of two-dimensional inertial manifolds.

Self-excited orbits are located close to the neutral curve. They become hidden and then disappear as we move towards the true stability region (by decreasing \( \tau \) or \( \alpha \)). This and the results from previous sections allow us to suggest a description of the dynamics in the linear stability region.
For any $\alpha \in [0.43, 1)$ we numerically estimated the minimal $\tau_{\text{min}}$ and the maximal $\tau_{\text{max}}$ parameter $\tau$ for which hidden periodic orbits can be observer. Thus, for $\tau < \tau_{\text{min}}$ we expect a gradient-like (convergent) behaviour, for $\tau > \tau_{\text{max}}$ there are self-exited oscillations and for $\tau_{\text{min}} < \tau < \tau_{\text{max}}$ there are hidden oscillations. From this we obtained two curves, which we call the lower hidden curve and the upper hidden curve. They are schematically presented at Fig. 13.

Remark 5. We conjecture the following description of the dynamics in the linear stability region. Fig. 2 describes a typical behaviour in the region $\Omega_{\text{hid}}$ between the lower and upper hidden curves and Fig. 3 describes a typical behaviour in the region $\Omega_{\text{sc}}$ between the upper hidden curve and the neutral curve. At points from the upper hidden curve we expect homoclinic “figure eight” orbits and phase portraits corresponding to Fig. 1 which bifurcate into the corresponding to $\Omega_{\text{hid}}$ or $\Omega_{\text{sc}}$ phase portraits. When moving from $\Omega_{\text{hid}}$ to the lower hidden curve, there occurs a collision of the single unstable periodic orbit and the hidden periodic orbit that leads to their collapse and a gradient-like behaviour in the region $\Omega_{\text{grad}}$ below the lower hidden curve. When moving from $\Omega_{\text{sc}}$ to the neutral curve, the pair of symmetric cycles collides with the symmetric equilibria and then disappear providing asymptotic stability of the equilibria (subcritical Andronov-Hopf bifurcation).

Note that periods of oscillations found in the linear stability region depend on parameters in the same way as oscillations in the unstable region [21, 7]. Namely, the dimensionless period $\sigma$ increases when $\tau$ increases, but the corresponding to it period in years $\sigma_{\text{years}}$ decreases, i.e. the ratio $\sigma/\tau$ decreases. For some points from the upper hidden curve estimates for $\sigma$ and $\sigma_{\text{years}}$ are collected in Table 1. From this we can see that periods $\sigma_{\text{years}}$ of hidden periodic orbits corresponding to parameters close to the upper hidden
As the delay value $\tau$ increases, the real parts of roots corresponding to (2.3) begin condensing near each other. This makes values of the transfer function in (2.11) on the line $-\nu + iR$ too large (since there are always roots close to this line) and, consequently, our slow manifolds theory is no longer applicable. However, we hope that this is only limitations of the analytical machinery and this does not affect the drawn in Remark 5 qualitative

Table 1. A table of points $(\alpha, \tau_{\text{min}})$ and $(\alpha, \tau_{\text{max}})$ approximating the lower and upper hidden curves respectively. For each point $(\alpha, \tau_{\text{max}})$ estimations of the dimensionless period $\sigma$ and the period in years $\sigma_{\text{years}}$ (see Remark 1) of the corresponding hidden periodic orbit are presented.

| $\alpha$ | $\tau_{\text{min}}$ | $\tau_{\text{max}}$ | $\sigma$ | $\sigma_{\text{years}}$ |
|---------|---------------------|---------------------|----------|--------------------------|
| 0.52    | 3.35                | 3.66                | 18.41    | 5.37                     |
| 0.51    | 3.53                | 3.92                | 18.41    | 5.14                     |
| 0.50    | 3.73                | 4.19                | 19.21    | 5.13                     |
| 0.49    | 3.95                | 4.50                | 19.81    | 4.8                      |
| 0.48    | 4.20                | 4.86                | 20.79    | 4.68                     |

| $\alpha$ | $\tau_{\text{min}}$ | $\tau_{\text{max}}$ | $\sigma$ | $\sigma_{\text{years}}$ |
|---------|---------------------|---------------------|----------|--------------------------|
| 0.47    | 4.49                | 5.28                | 22.03    | 4.57                     |
| 0.46    | 4.82                | 5.80                | 22.74    | 4.3                      |
| 0.45    | 5.19                | 6.43                | 24.75    | 4.22                     |
| 0.44    | 5.65                | 7.24                | 26.70    | 4.04                     |
| 0.43    | 6.18                | 8.3                 | 29.11    | 3.85                     |
Figure 14. A numerically obtained set (blue) of parameters \((\tau, \alpha)\) above the lower hidden curve (green), for which there exists a two-dimensional inertial manifold. Other curves are the same as at Fig. 13.

conclusions. On Fig. 14 it is displayed a numerically obtained set of parameters, for which the frequency-domain condition similar to conditions (3.1), (3.3) and (4.1) used in Sections 3 and 4 is satisfied.

6. AN ANALYTICAL NON-OscILLATORY REGION

Let \(\lambda_1 = \lambda(\alpha, \tau) > 0\) and \(\lambda_2 = \lambda_2(\alpha, \tau) < 0\) be the positive and negative roots of (2.2) respectively. From the dichotomy of linear autonomous systems (see J. K. Hale [11]) it follows that the inequality \(\lambda_1 + \lambda_2 < 0\) indicates the squeezing of two-dimensional volumes at the zero stationary state \(\phi^0\) of (2.1). In our work [1] it was stated a problem:

**Problem 1.** Is it true that there are no periodic orbits and homoclinics in (2.1) provided that \(\lambda_1 + \lambda_2 < 0\)?

The region in the space of parameters \((\tau, \alpha)\) determined by the inequality \(\lambda_1 + \lambda_2 < 0\), which we will denote as \(\Omega_{\text{dst}}\), is displayed at Fig. 15. It is included in the linear stability region \(\Omega_{\text{st}}\). As we have shown for parameters in \(\Omega_{\text{st}}\) there is a possibility of the existence of self-excited or hidden periodic orbits, but they are only observed above the lower hidden curve. Thus the above problem still remains open and our results only justify its nontriviality. Note that the description of dynamics proposed in Section 5 motivates an affirmative answer to Problem 1

Note that a positive answer to Problem 1 implies the convergent (non-oscillatory) behaviour in (2.1) for corresponding parameters from \(\Omega_{\text{dst}}\) due to the Poincaré-Bendixson trichotomy stated at the beginning of Section 2.

A motivation for Problem 1 comes from dimension estimates [1]. A more stronger conjecture may be stated in terms of the Lyapunov dimension (see N. V. Kuznetsov and V. Reitmann [13]) as

\[2\]

To speak about volumes one should consider the equation in a proper Hilbert space. See [1] for details.
Problem 2. Is it true that the local Lyapunov dimension at the zero stationary state $\phi^0$ coincides with the Lyapunov dimension of the global attractor of (2.1) for parameters from $\Omega_{dst}$?

A positive answer to Problem 2 will immediately lead to a positive answer to Problem 1 due to the criterion of non-existence of invariant curves proved by M. Y. Li and J. S. Muldowney [16]. We note that a difficulty in Problem 2 is linked with the problem of obtaining effective dimension estimates for delay equations, where the symmetrization approach (which has already proven to be relatively effective for ODEs [13]) does not work. We refer to our work [1] for examples and discussions in this direction. Note that our numerical experiments show that the answer to Problem 2 when $\Omega_{dst}$ is changed to $\Omega_{st}$ is negative.

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