A generalized Cole–Hopf transformation for nonlinear ODEs

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Abstract

We introduce a ‘hybrid’ Cole–Hopf–Darboux transformation to relate the solutions of nonlinear and linear second-order differential equations and derive classification and sufficient condition for this correspondence. We explore physical applications of this correspondence to nonlinear oscillations, the Duffing equation and a nonlinear form of the Schrödinger equation for the nonrelativistic hydrogen atom. In addition, we show that solutions of some nonlinear second-order equations are related to the special functions of mathematical physics through this transformation. These nonlinear equations can be viewed as the ‘class of special nonlinear equations’ which correspond to the linear differential equations which define the special functions of mathematical physics.

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1. Background

Many problems in modern theoretical physics (and other fields) lead to nonlinear differential equations whose exact solutions cannot be obtained by traditional methods. For some classes of these equations, the application of symmetry principles can yield exact solutions or at least the reduction of the original problem.

Another paradigm that has been used extensively in many applications was the derivation of an ‘ad hoc special transformation’ that linearizes the original problem. In this context, Hopf–Cole transformation [1, 2] and its generalizations [3, 4] have been used extensively to linearize some nonlinear partial differential equations such as the Burgers equation [5–7].

For ordinary differential equations, a ‘similar’ transformation has been used for some time. In fact, it is well known [8] that the Ricatti equation

\[ \psi(x)' + A(x)\psi(x)^2 + B(x)\psi(x) + C_1(x) = 0 \]  

(1.1)
where primes denote differentiation with respect to \(x\), can be linearized by the transformation

\[
\psi(x) = \frac{\phi(x)'}{A(x)\phi(x)}.
\]

(1.2)

The resulting linear equation for \(\phi(x)\) is

\[
A(x)\phi(x)'' + (B(x)A(x) - A(x)')\phi(x)'/A(x) + C_1(x)A(x)^2\phi(x) = 0.
\]

(1.3)

However, the operator

\[
T = \frac{d}{dx} + A(x)\psi(x) + B(x)
\]

(1.4)

can be used repeatedly to generate higher order differential equations that can be linearized by the transformation (1.2). That is, the differential equation

\[
T^n\psi(x) = Q(x),
\]

(1.5)

where \(T^n\psi(x) = T(T^{n-1})\psi(x)\) can be linearized by the transformation (1.2).

In particular, for \(n = 2\), we have

\[
T^2\psi = \left(\frac{d}{dx} + A(x)\psi(x) + B(x)\right)^2\psi(x).
\]

(1.6)

Thus for \(n = 2\), \(A(x) = 1\) and \(B(x) = 0\) the nonlinear differential equation (1.5) becomes

\[
\psi(x)''' + 3\psi(x)\psi(x)' + \psi(x)^3 = Q(x).
\]

(1.7)

Applying the transformation (1.2) to this equation yields

\[
\phi(x)''' = Q(x)\phi(x).
\]

(1.8)

This shows that the transformation (1.2) relates some nonlinear equations to a linear equation of higher order. However, there is another class of transformations that is used to relate the solutions of two linear differential equations of the same order. These are Darboux transformations [9–12] (which form the basis for the well-known factorization method [13, 15]). In this case, the operator that relates the two equations is of the form

\[
D = C(x) + \frac{d}{dx}.
\]

(1.9)

Our objective in this paper is to explore the possible use of some 'hybrid' form of (1.2) and (1.9) to relate the solutions of a nonlinear equation to those of a linear one of the same order. In part, we are motivated by the fact that the exact solutions of some nonlinear second-order equations can be related to the special functions of mathematical physics through a transformation similar to (1.2). In a certain sense, these equations then define a class of 'special nonlinear differential equations'. From a physical point of view, this correspondence demonstrates that ambiguity might exist about the nature of a physical system (whether it is linear or nonlinear) even if its states (or data) might be represented by functions which satisfy a familiar linear differential equation.

For direct physical applications, we apply this transformation to nonlinear oscillations [16–20], Duffing equation and a nonlinear form of the radial part of Schrödinger equation for the hydrogen atom [21]. In all cases, we present exact analytic solutions to these nonlinear equations through this transformation. Furthermore, we show that the Painleve II equation has a solution that can be expressed in terms of Airy functions for a special set of its parameters.

The plan of the paper is as follows. In section 2, we present the general technique which relates the solutions of the linear and nonlinear equations. In section 3, we specialize to a subset of this general method and provide an intrinsic test for the applicability of the method and general classification. Section 4 discusses physical applications. Section 5 explores the relationship between some classes of nonlinear equations and the special functions of mathematical physics. Section 6 provides some additional examples of nonlinear equations whose solutions are related to those of a linear equation. We end up in section 7 with some conclusions.
2. A generalized transformation

We shall say that the solutions of the equations

\[ \psi(x)'' = S(x) + V(x)\psi(x) + W(x)\psi(x)^2 + R(x)\psi(x)^3 + \lambda\psi(x) \]  
(2.1)

and

\[ \phi(x)'' = U(x)\phi(x) + K(x)\phi(x)' + \lambda\phi(x) \]  
(2.2)

are related if we can find the functions \(P(x)\) and \(Q(x)\), so that

\[ \psi(x) = P(x) + Q(x) \frac{\phi(x)'}{\phi(x)}. \]  
(2.3)

We observe that, in principle, the terms \(\lambda\psi(x)\) and \(\lambda\phi(x)\) in (2.1) and (2.2) can be absorbed by \(V(x)\) and \(U(x)\), respectively. Furthermore, (2.1) can take the more general form

\[ \psi(x)'' = S(x) + V(x)\psi(x) + V_1(x)\psi(x)' + W(x)\psi(x)^2 + R(x)\psi(x)^3. \]  
(2.4)

In this case, we can find \(p(x)\), so that \(V_1(x) = -2\frac{p(x)\psi'(x)}{p(x)\psi(x)}\). Introducing \(\xi(x) = p(x)\psi(x)\), (2.4) becomes

\[ \xi(x)'' = p(x)S(x) + \left(V(x) + \frac{p(x)\psi'}{p(x)}\right)\xi(x) + \frac{W(x)}{p(x)}\xi(x)^2 + \frac{R(x)}{p(x)^2}\xi(x)^3, \]  
(2.5)

which has the same form as (2.1).

To classify those nonlinear equations (2.1) which can be ‘paired’ with a linear equation of the form (2.2), we differentiate (2.3) twice and in each step replace the second-order derivative of \(\phi(x)\) by \(U(x)\phi(x) + K(x)\phi(x)' + \lambda\phi(x)\). We then use (2.1) to eliminate \(\psi(x)''\). As a result, we find that the following equation must hold:

\[ a_3(x)\phi(x)^3 + a_2(x)\phi(x)^2 + a_1(x)\phi(x) + a_0(x) = 0, \]  
(2.6)

where

\[ a_3(x) = -Q(x)^2R(x) + 2, \]
\[ a_2(x) = -2Q(x) - 3R(x)P(x)Q(x)^2 + W(x)Q(x)^2 - 3K(x)Q(x) \]  
(2.7)
\[ a_1(x) = Q(x)'' + 2K(x)Q(x)'\]  
(2.8)
\[ a_0(x) = 2(U(x) + \lambda)Q(x)' + Q(x)U(x)' + P(x)'' - W(x)P(x)^2 - V(x)P(x) \]
\[ -R(x)P(x)^3 - \lambda P(x) + K(x)Q(x)(U(x) + \lambda) - S(x). \]  
(2.9)

To satisfy (2.6), it is sufficient to let \(a_i(x) = 0, \) \(i = 0, 1, 2, 3\). We use these conditions to express \(S(x), V(x), W(x), R(x), K(x)\) and \(U(x)\) in terms of the parameters \(P(x)\) and \(Q(x)\). From (2.7), we obtain

\[ R(x) = \frac{2}{Q(x)^2}, \quad W(x) = -\frac{2(Q(x)'' + 3P(x)) + 3K(x)Q(x)}{Q(x)^2}. \]  
(2.10)

Substituting these results into (2.8), we obtain an equation which we can solve for \(V(x)\)

\[ V(x) = \frac{Q(x)Q(x)'' + 4P(x)Q(x)'}{Q(x)^2} + 6P(x)^2 - Q(x)^2(2U(x) + 3\lambda) \]
\[ + K(x)^2 + \frac{6P(x) + 2Q(x)'}{Q(x)} + K(x). \]  
(2.11)
Substituting these expressions into (2.9), we obtain the following equation for $S(x)$ (or if $S(x)$ is the known first-order linear equation for $U(x)$):

$$Q(x)U(x)' + (2Q(x)' + 2P(x) + K(x)Q(x))U(x) + P(x)''$$

$$+ 2 \left[ \lambda - \frac{P(x)}{Q(x)} \left(K(x) + \frac{P(x)}{Q(x)} \right) \right] Q(x)'$$

$$+ 2\lambda P(x) - \frac{P(x)^2}{Q(x)^2} (2P(x) + 3K(x)Q(x))$$

$$- \frac{P(x)Q(x)''}{Q(x)} + K(x) (\lambda Q(x) - K(x)P(x)) - P(x)K(x)' - S(x) = 0. \quad (2.12)$$

Equations (2.10)–(2.12) determine the coefficients of equation (2.1) in terms of (pre-selected) functions $U(t)$, $K(t)$, $P(t)$ and $Q(t)$. In this case, one starts with a linear differential equation and classifies those nonlinear equations that can be paired with it. However, the process can be reversed. Starting from a given nonlinear differential equation of the form (2.1), equation (2.10) determines $Q(x)$ in terms of $R(x)$ and $P(x)$ in terms of $K(x)$. Similarly, (2.11) can be used to determine $U(x)$ in terms of $K(x)$. Finally, (2.12) can be viewed as a differential equation for $K(x)$. (As an example of this process, we treat the Duffing equation in section 4.)

### 3. Solutions with $Q(x) = 1$

When $Q(x) = 1$, we have $R(x) = 2$ and

$$V(x) = 6P(x)(P(x) + K(x)) - 2U(x) - 3\lambda + K(x)^2 + K(x)',$$

$$W(x) = -6P(x) - 3K(x). \quad (3.1)$$

Equation (2.12) simplifies, and we have

$$U(x)' + 2P(x)(U(x) + \lambda - P(x)^2) + P(x)'' - S(x)$$

$$+ (U(x) + \lambda)K(x) - P(x)(K(x)^2 + 3K(x)P(x) + K(x)') = 0. \quad (3.2)$$

Since this equation contains two unknown functions $U(x)$ and $K(x)$, it is natural to split it into two equations

$$K(x)' + K(x)^2 + \left(3P(x) - \frac{\lambda}{P(x)} \right) K(x) + \frac{S(x)}{P(x)} = 0 \quad (3.3)$$

and

$$U(x)' + 2P(x)(U(x) + \lambda - P(x)^2) + P(x)'' + K(x)U(x) = 0. \quad (3.4)$$

Equation (3.3) is independent of $U(x)$ and can be solved for $K(x)$ once $P(x)$ and $S(x)$ have been specified. Actually, (3.3) is a Ricatti equation whose linear second-order form is

$$y(x)'' + \left(3P(x) - \frac{\lambda}{P(x)} \right) y(x)' + \frac{S(x)y(x)}{P(x)} = 0, \quad (3.5)$$

where $K(x) = \frac{\mu y(x)}{y(x)}$. We see that by a proper choice of $P(x)$ and $S(x)$, the solution $y(x)$ can be a special function of mathematical physics. In this case, $K(x)$ will be the logarithmic derivative of such functions.

Another possible decomposition of (3.2) is as follows:

$$K(x)' + K(x)^2 + \left(3P(x) - \frac{\lambda}{P(x)} \right) K(x) = 0 \quad (3.6)$$
and

\[ U(x)' + 2P(x)(U(x) + \lambda - P(x)^2) + P(x)'' + K(x)U(x) - S(x) = 0. \] (3.7)

If we let \( S(x) = -2P(x)^3 \), then (3.7) becomes linear in both \( U(x) \) and \( P(x) \). As a result, one may choose \( P(x) \) and solve this equation for \( U(x) \) or choose \( U(x) \) and solve for \( P(x) \).

Another option to find solutions to (3.2) is to cancel the nonlinear terms in \( P(x) \) by making the ansatz

\[ S(x) = -3P(x)^2K(x) - 2P(x)^3; \] (3.8)

then the equation becomes

\[ P(x)'' + (2U(x) + 2\lambda - K(x)' - K(x)^2)P(x) + K(x)(U(x) + \lambda) + U(x)' = 0, \] (3.9)

which is a linear equation for \( P(x) \) once \( U(x) \) and \( K(x) \) were chosen.

In the following, we use all these strategies to find nonlinear differential equations whose solutions are related to those of a linear differential equation by the transformation (2.3).

Finally, substituting these expressions for \( S(x) \) and \( U(x) \) into (3.2), condition (3.10) follows.

We now consider the following practical question. Suppose one considers a nonlinear differential equation of the form given by (2.1). Under what conditions can one find a linear differential equation (2.2) whose solutions are related to it by (2.3) with \( Q(x) = 1 \)?

**Theorem.** A sufficient condition for the solution of (2.1) to be related to an equation of the form (2.2) by (2.3) with \( Q(x) = 1 \) is that \( R(x) = 2 \) and

\[ S(x) = \frac{1}{2}[-V(x)' + \frac{1}{2}(-W(x)' + (V(x) + W(x)') + \lambda)W(x)] - \frac{1}{12}W(x)^3. \] (3.10)

When this condition is satisfied, one can choose \( K(x) = 0 \) and

\[ U(x) = -\frac{1}{2}V(x) + \frac{1}{12}W(x)^2 - \frac{1}{3}\lambda. \] (3.11)

We observe that condition (3.10) is an intrinsic condition on the coefficients of (2.1).

**Proof.** From (3.1), we have

\[ P(x) = -\frac{1}{6}W(x) - \frac{1}{2}K(x). \] (3.12)

Substituting this expression into the formula for \( V(x) \), it follows that

\[ V(x) = \frac{1}{6}K(x)^2 + \frac{1}{2}K(x)^2 + 2U(x) + 3\lambda - K(x)' = 0. \] (3.13)

Solving (3.13) for \( U(x) \), we find that

\[ U(x) = -\frac{1}{2}V(x) + \frac{1}{12}W(x)^2 - \frac{1}{2}K(x)^2 - \frac{3}{2}\lambda + \frac{1}{2}K(x)'. \] (3.14)

Finally, substituting these expressions for \( P(x) \) and \( U(x) \) into (3.2), condition (3.10) follows.

If we let \( K(x) = 0 \) in (3.12)–(3.14), the expression for \( U(x) \) reduces to the one given by (3.11).

To provide a general classification of the nonlinear equations of the form (2.1) whose solutions are related to those of the form (2.2) by equation (2.3), we first show that if \( K(x) \neq 0 \) in equation (2.2), then this equation can be transformed into an equation with \( K(x) = 0 \).

As a first step, we observe that equation (2.2) can be rewritten in the equivalent self-adjoint form

\[ \left[ \exp \left( \int -K(x) \, dx \right) \phi' \right]' = [U(x) + \lambda] \left[ \exp \left( \int -K(x) \, dx \right) \right] \phi(x). \] (3.15)

That is, equation (2.2) can be rewritten in the form

\[ [p(x)\phi']' + q(x)\phi(x) + \lambda p(x)\phi(x) = 0. \] (3.16)
Since the function $p(x)$ is non-negative, the transformation
\[ y(x) = p(x)^{1/2} \phi(x) \]
yields an equation of the form
\[ y''(x) + r(x)y(x) + \lambda y(x) = 0, \tag{3.17} \]
where
\[ r(x) = \frac{p(x)^2 - 2p(x)p(x)' + 4q(x)p(x)}{4p(x)^2}. \]
Equation (3.17) is of the same form as equation (2.2) with $K(x) = 0$. Thus, for classification purposes, we can, without loss of generality, assume that $K(x) = 0$ in equation (2.2).

For equations of the form (2.2) with $K(x) = 0$ and $Q(x) = 1$, we have $R(x) = 2$ and equations (3.1) and (3.2) become, respectively,
\[ V(x) = 6P(x)^2 - 2U(x) - 3\lambda, \quad W(x) = -6P(x) \tag{3.18} \]
and
\[ S(x) = U(x)' + 2P(x)(U(x) + \lambda - P(x)^2) + P(x)''. \tag{3.19} \]
Thus, all the coefficient functions that appear in equation (2.1) are determined by the parameter function $P(x)$ and the potential function $U(x)$.

We conclude then that equations (3.18) and (3.19) yield (in principle) a general classification of all equations of the form equation (2.1), whose solutions can be expressed in terms of the solutions of equation (2.2).

4. Physical applications

The Schrödinger equation for various forms of the potential function can be brought after the separation of variables into the form of equation (2.2) with $K(x) = 0$ (as we showed in section 2). It follows then that the transformation (2.3) relates each (linear) Schrödinger equation with a potential function $U(x)$ to a class of ‘nonlinear Schrödinger equations’. That is, the solutions of these nonlinear equations can be expressed in terms of the solutions of equation (2.2) by the transformation (2.3).

We consider two explicit examples.

Example 1: nonlinear oscillations. Nonlinear oscillations are of paramount importance in many branches of applied mathematics, theoretical physics, mathematical biology and others. Obtaining exact solutions for equations modeling nonlinear oscillators is an outstanding problem that has been addressed by many researchers, (We quote only few references [16–20] out of the many.) Here, we derive a class of nonlinear oscillators whose exact solutions are related to those of the harmonic oscillator.

The Schrödinger equation for the one-dimensional harmonic oscillator is
\[ \psi''(x) = x^2 \psi - \mu \psi. \tag{4.1} \]
That is, $U(x) = x^2$, $K(x) = 0$ and $\lambda = -\mu$ (in the notation of equation (2.2)). Using equations (3.18) and (3.19), we find that the class of nonlinear equations whose solutions are related to those of the harmonic oscillator are of the following form:
\[ \psi''(x) = 2x + 2P(x)(x^2 - \mu - P(x)^2) + P'(x) \\
+ (6P(x)^2 - 2x^2 + 2\mu)\psi(x) - 6P(x)\psi(x)^2 + 2\psi(x)^3. \tag{4.2} \]
In the particular case of $P(x) = \frac{4}{x^2}$, this becomes

$$
\psi''(x) = 2 \left\{ \psi(x)^3 - \frac{3a}{x} \psi(x)^2 + \left( \frac{3a^2}{x^2} - x^2 + \mu \right) \psi(x) + x(1 + a) - \frac{a}{x} \left( \mu + \frac{a^2 - 1}{x^2} \right) \right\}.
$$

In this equation, the linear part in $\psi$ represents the potential of the isotonic oscillator, but the equation also contains the nonlinear and forcing terms [19].

We point out that although there is no damping term in equation (4.2), it can be added using the procedure outlined by the transformation from equation (2.4) to (2.5) [20].

Example 2: the Duffing equation. The Duffing equation can written in the form

$$
\frac{d^2 \psi}{dt^2} + c \frac{d\psi}{dt} = \alpha \psi(t) + b \psi(t)^3 + f(t),
$$

(4.3)

where $\alpha$, $b$ and $c$ are constants and $f(t)$ is a forcing function. To bring this equation to the form of equation (2.1), we introduce

$$
y(t) = e^{\alpha t/2} \psi(t)
$$

to obtain

$$
\frac{d^2 y(t)}{dt^2} = ay(t) + b e^{-\alpha t} y(t)^3 + f(t) e^{\alpha t/2},
$$

(4.4)

where $a = \alpha + \frac{c^2}{4}$. The coefficients of equation (4.4) correspond to those of equation (2.1) by the following relations:

$$
S(t) = f(t) e^{\alpha t/2}, \quad V = a, \quad W = 0, \quad R = b e^{-\alpha t}, \quad \lambda = 0.
$$

(4.5)

From equation (2.10), we have

$$
Q(t) = \sqrt{\frac{7}{b}} e^{\alpha t/2} = C e^{\alpha t/2}
$$

(4.6)

and

$$
P(t) = -\frac{C}{6} e^{\alpha t/2} (c + 3K(t)).
$$

(4.7)

Similarly, from equation (2.11), we have

$$
U(t) = -\frac{a}{2} + \frac{c^2}{24} - \frac{1}{4} K(t)^2 + \frac{1}{2} \frac{dK(t)}{dt}.
$$

(4.8)

Finally, equation (2.12) yields

$$
S(t) = \frac{C}{108} e^{\alpha t/2} \left[ c(c^2 - 36a) + 54 \left( K(t) \frac{dK(t)}{dt} - \frac{d^2 K(t)}{dt^2} \right) \right].
$$

(4.9)

Using (4.5), this relationship can be expressed in terms of the original forcing function as

$$
f(t) = \frac{C}{108} \left[ c(c^2 - 36a) + 54 \left( K(t) \frac{dK(t)}{dt} - \frac{d^2 K(t)}{dt^2} \right) \right].
$$

(4.10)

From a practical point of view, if one desires to solve equation (4.3) with a forcing function $f(t)$, then using (4.9) (which is a linear differential equation with constant coefficients) one determines $K(t)$. Equations (4.6), (4.7) and (4.8) then determine $U(t)$, $P(t)$ and $Q(t)$, respectively. Then the solutions of equation (4.4) can be obtained by solving equation (2.2) and using (2.3).

As an example, consider the (simple) case where $f(t) = \frac{Cc}{108} (c^2 - 36a)$. Using (4.9), we have $K(t) = 0$ and the corresponding expressions for $Q$, $P$ and $U$ are

$$
Q(t) := C e^{\alpha t/2}, \quad P(t) = -\frac{1}{b} C c e^{\alpha t/2}, \quad U(t) = -\frac{1}{2} a + \frac{1}{24} c^2.
$$

(4.11)
The solution of equation (2.3) (assuming \(-\frac{1}{4}a + \frac{1}{12}c^2 < 0\)) is
\[
\phi(t) = D \sin(\omega t + \chi),
\] (4.12)
where \(\omega = \frac{1}{12}\sqrt{72a - 6c^2}\), and \(D\) and \(\chi\) are constants. A solution for \(y(t)\) is obtained using (2.2).

**Example 3: hydrogen atom.** The radial equation for the nonrelativistic hydrogen atom is
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \phi \right) + \left[ \frac{2}{r} - \frac{l(l + 1)}{r^2} + \mu \right] \phi = 0.
\]
Introducing \(\phi = r\tilde{\phi}\), this equation becomes
\[
\frac{d^2 \tilde{\phi}}{dr^2} + \left[ \frac{2}{r} - \frac{l(l + 1)}{r^2} + \mu \right] \tilde{\phi} = 0.
\] (4.13)
That is, in the notation of equation (2.2) \(U(r) = \frac{2}{r} - \frac{l(l+1)}{r^2}\), \(K(r) = 0\) and \(\lambda = -\mu\). Using equations (3.18) and (3.19), we find that the class of nonlinear equations whose solutions are related to those of equation (4.13) are of the form
\[
\psi'' = 2 \left\{ \psi^3 - 3P\psi^2 + \left[ 3P^2 - \frac{l(l+1)}{r^2} + \frac{2}{r} + \mu \right] \psi \right.
\]
\[
- \frac{l(l+1)}{r^3} + \frac{1}{r^2} + P \left[ \frac{l(l+1)}{r^2} - \frac{2}{r} - \mu - P^2 \right] \bigg\} + P''.
\] (4.14)
For \(P(r) = 1/r\), equation (4.14) becomes
\[
\psi'' = 2 \left\{ \psi^3 - \frac{3\psi^2}{r} + \left[ \frac{3 - l(l+1)}{r^2} + \frac{2}{r} + \mu \right] \psi - \frac{1}{r^2} - \frac{\mu}{r} \right\}.
\] (4.15)
The linear part of this equation resembles the original equation (4.13). However, the equation contains additional nonlinear terms. The solutions of equations (4.14) and (4.15) can be expressed in terms rational expressions of solutions for equation (4.13). Equations similar to equation (4.14) were considered in [21].

### 5. Relationships to the special functions

In this section, we explore the relationship between the solutions of some classes of nonlinear differential equations and the special functions of mathematical physics. We demonstrate that each class of these functions provides (exact) solutions to some nonlinear differential equations under the transformation (2.3). These differential equations should be considered then as ‘the nonlinear cousins’ of the linear differential equations whose solutions define the functions of mathematical physics. In all cases, we let \(Q(x) = 1\).

**Case 1: exponential and trigonometric functions.** We start by considering the harmonic oscillator equation,
\[
\phi(x)'' = -\omega^2 \phi(x), \quad \omega \neq 0.
\] (5.1)
For this equation, \(U(x) = -\omega^2\), \(K(x) = 0\) and \(\lambda = 0\). Hence from (3.9), we have
\[
P(x)'' - 2\omega^2 P(x) = 0.
\]
Therefore,
\[
P(x) = C_1 e^{\sqrt{2}ax} + C_2 e^{-\sqrt{2}ax}.
\]
Computing $V(x)$, $W(x)$ and $S(x)$ using (3.1) and (3.8), we finally obtain the following differential equation for $\psi(x)$:

$$\psi(x)'' - (6P(x)^2 + 2\omega^2)\psi + 6P(x)\psi(x)^2 - 2\psi(x)^3 = -2P(x)^3,$$

(5.2)

whose solutions are related to those of (5.2) by (2.3).

If we consider

$$\phi(x)'' = \omega^2 \phi, \quad \omega \neq 0,$$

(5.3)

then

$$P(x) = C_1 \sin(\sqrt{2\omega}x) + C_2 \cos(\sqrt{2\omega}x)$$

and the differential equation for $\psi(x)$ becomes

$$\psi(x)'' - (6P(x)^2 - 2\omega^2)\psi(x) + 6P(x)\psi(x)^2 - 2\psi(x)^3 = -2P(x)^3.$$

(5.4)

For the case where $\omega = 0$, i.e.

$$\phi(x)'' = 0;$$

(5.5)

then

$$P(x) = C_1 + C_2 x,$$

and the equation for $\psi(x)$ is

$$\psi(x)'' - 6P(x)^2 \psi(x) + 6P(x)\psi(x)^2 - 2\psi(x)^3 = -2P(x)^3.$$

(5.6)

A similar treatment can be made for the general second-order equation with constant coefficients. We omit the details.

**Case 2: Legendre polynomials.** The differential equation for the Legendre polynomials is

$$\phi(x)'' = \frac{2x}{1 - x^2} \phi(x)' - \frac{n(n + 1)}{1 - x^2} \phi(x).$$

(5.7)

That is, $U(x) = -\frac{n(n+1)}{1 - x^2}$, $K(x) = \frac{2x}{1 - x^2}$ and $\mu = 0$. Substituting this into (3.9), we find that a particular solution for $P(x)$ is

$$P(x) = -\frac{2n(n + 1)x}{(n^2 + n - 2)(1 - x^2)}, \quad n \neq 1.$$  

(5.8)

(The solution for $n = 1$ is available, but we shall not elaborate on it further.) Using (3.1), we have

$$V(x) = -\frac{48n(n + 1)x^2}{(n + 2)^2(n - 1)^2(1 - x^2)^2} + \frac{2n(n + 1)}{1 - x^2} + \frac{2(2x^2 + 1)}{(1 - x^2)^2}.$$  

(5.9)

From (3.8) and (3.1), we obtain

$$S(x) = \frac{8n^2(n + 1)^2(n + 3)(n - 2)x^3}{(n + 2)^3(n - 1)^3(x^2 - 1)^3},$$

(5.10)

$$W(x) = -6P(x) - 3K(x).$$  

(5.11)

For the functions $S(x)$, $V(x)$, $W(x)$ and $R(x) = 2$, the solutions of (2.1) are related to the solutions of the Legendre equation by the transformation (2.3).

**Case 3: Bessel functions.** The differential equation for Bessel functions is

$$\phi(x)'' = -\frac{\phi(x)'}{x} - \frac{x^2 - p^2}{x^2} \phi(x).$$

(5.12)
That is, \( U(x) = -\frac{z^2 - x^2}{2} \), \( K(x) = -\frac{1}{x} \) and \( \lambda = 0 \). Substituting these into (3.9), we find that the general solutions for \( P(x) \) with \( p = 0, 1 \), respectively, are

\[
P_0(x) = \frac{C_1 e^{x^2} (\sqrt{2} - 2x) + C_2 e^{-x^2} (2x + \sqrt{2})}{x} + \frac{1}{2x}, \quad z = \sqrt{2x}, \quad (5.13)
\]

\[
P_1(x) = C_1 e^{x^2} + C_2 e^{-x^2} - \frac{z^2 e^2 \Gamma(0, z) - z e^{-2} \Gamma(0, -z)}{2x} + \frac{3}{2x}, \quad (5.14)
\]

where

\[
\Gamma(0, z) = \int_z^\infty r^{-1} e^{-r} \, dr.
\]

If we choose for \( n = 0 \) the special solution \( P_0(x) = \frac{1}{2x} \), then we find that

\[
V(x) = 2 + \frac{1}{2x^2}, \quad W(x) = 0, \quad S(x) = \frac{1}{2x^3}.
\]

The differential equation for \( \psi(x) \) is

\[
\psi(x)^{\prime\prime} = \frac{1}{2x^2} + \left( 2 + \frac{1}{2x^2} \right) \psi + 2 \psi(x)^3. \quad (5.15)
\]

The solutions of this nonlinear equation are related to the Bessel functions of order zero by the transformation (2.3).

**Case 4: Hermite polynomials.** The differential equation for Hermite polynomials is

\[
\phi(x)^{\prime\prime} = 2x \phi(x)^{\prime} - 2n \phi(x). \quad (5.16)
\]

Hence \( U(x) = -2n, K(x) = 2x \). Substituting these into (3.9), we find that an explicit particular (and general) solution for \( P(x) \) in terms of elementary functions can be obtained for even \( n \). For \( n = 2, 4 \), these particular solutions are

\[
P_2(x) = -\frac{8\pi (1/4 + x^2) \exp(x^2) \text{erf}(x) + \sqrt{\pi} x}{\sqrt{\pi}}, \quad (5.17)
\]

\[
P_4(x) = -\frac{64\pi (3/16 + 3/2x^2 + x^4) \exp(x^2) \text{erf}(x) + \sqrt{\pi} x (1 + x^2)}{3\sqrt{\pi}}, \quad (5.18)
\]

where

\[
\text{erf}(x) = 2 \int_0^x \frac{\exp(-t^2)}{\sqrt{\pi}} \, dt.
\]

The computation of the functions \( S(x) \), \( V(x) \) and \( W(x) \) using (3.1) and (3.8) is straightforward.

**Case 5: Laguerre polynomials.** The differential equation for the Laguerre polynomials is

\[
\phi(x)^{\prime\prime} = -\frac{1-x}{x} \phi(x)^{\prime} - \frac{n}{x} \phi(x); \quad (5.19)
\]

That is, \( U(x) = -\frac{\pi}{x}, K(x) = -\frac{1-x}{x} \) and \( \lambda = 0 \). Substituting these into (3.9), we find that a particular solution for \( P(x) \) with \( n = 2 \) is

\[
P_2(x) = -\Gamma(0, -x) e^{-x} + e^x \Gamma(0, 0, x) \left( \frac{1}{x} + 2x - \frac{2}{x} \right) - 2 + \frac{2}{x}. \quad (5.20)
\]

We omit the computation of \( V(x) \), \( W(x) \) and \( S(x) \) using (3.1) and (3.8) which is straightforward.

**Case 6: the Painleve II equation.** The Painleve II equation is

\[
\psi(x)^{\prime\prime} = 2\psi(x)^3 + x\psi(x) + a. \quad (5.21)
\]
where $a$ is a constant. In the notation of the previous sections $V(x) = x, W(x) = 0, R(x) = 2, \ S(x) = a$ and $\lambda = 0$. With $Q(x) = 1$, (5.20) satisfies the constraint (3.10) when $a = -\frac{1}{2}$.

With $K(x) = 0$, (3.11) and (3.12) yield under the present settings $P(x) = 0$ and

$$ U(x) = -\frac{1}{2} V(x) = -\frac{x}{2}, $$
(5.22)

i.e. the differential equation for $\phi(x)$ is

$$ \phi(x)'' = -\frac{x}{2} \phi(x), $$
(5.23)

whose general solution is

$$ \phi(x) = C_1 Ai\left(-\frac{x}{2^{1/3}}\right) + C_2 Bi\left(-\frac{x}{2^{1/3}}\right), $$
(5.24)

where $Ai(x)$ and $Bi(x)$ are the Airy wavefunctions. This solution is related to the solution of (5.21) by the transformation (2.3) (with $P(x) = 0$). We observe that other solutions of this equation are possible if we let $K(x) \neq 0$ (However, it should be observed that the original nonlinear equation might have additional solutions.)

6. Some special cases

In this section, we present some explicit solutions to the equations which pair the solution of a nonlinear equation with a linear equation using the algorithm which was presented in the previous sections. In all cases, we let $Q(x) = 1$.

**Example 1.** In this example, we let $K(x) = 0$ and $S(x) = 0$ and use (3.2). Choosing $P(x) = \frac{b}{\pi}$, we obtain for $n = 1$,

$$ U(x) = C_1 x^{2b} + \frac{b(b + 1)}{x^2} - \lambda, $$
(6.1)

and the corresponding expression for $V(x)$ is

$$ V(x) = 2C_1 x^{2b} + \frac{2b(2b - 1)}{x^2} - \lambda. $$

For $n \neq 1$, we have

$$ U(x) = C_1 \exp\left(\frac{2b x^{1-n}}{n-1}\right) + \frac{nb}{x^{1+n}} + \frac{b^2}{x^{2n}} - \lambda $$
(6.2)

and

$$ V(x) = 6\left(\frac{b}{x}\right)^n - 2C_1 \exp\left(\frac{2b x^{1-n}}{n-1}\right) - \frac{2bn}{x^{1+n}} - \frac{2b^2}{x^{2n}} - \lambda. $$

For $n = 1$, the equation for $\phi(x)$ with $C_1 = 0$ is

$$ \phi(x)'' = \frac{b(b + 1)}{x^2} \phi(x) = 0, $$
(6.3)

whose general solution is

$$ \phi(x) = D_1 x^{b+1} + D_2 x^{-b}. $$
(6.4)

The corresponding nonlinear differential equation for $\psi(x)$ is

$$ \psi(x)'' - \frac{2b(2b + 1)}{x^2} \psi(x) + \frac{6b\psi(x)^2}{x} - 2\psi(x)^3 = 0. $$
(6.5)

In this case, the explicit relationship between $\phi(x)$ and $\psi(x)$ (as postulated in (2.3)) is

$$ \psi(x) = \frac{b}{x} + \phi(x). $$
(6.6)

It is straightforward to verify that this is actually a solution of (6.5).
Example 2. Let \( P(x) = -\frac{2x}{3}, S(x) = -\frac{4a}{x} \) and \( \lambda = 0 \). Equation (3.5) becomes
\[
y''(x) - 2xy'(x) + 2ny(x) = 0, \tag{6.7}
\]
which is the differential equation for the Hermite polynomials. Hence,
\[
K(x) = \frac{H_n(x)'}{H_n(x)}, \tag{6.8}
\]
where \( H_n(x) \) is a Hermite polynomial of order \( n \). Substituting these results into (3.4) leads to the following general solutions for \( U(x) \):
\[
U_n(x) = \frac{C_1 - \int \frac{16}{\pi x^3} H_n(x) \exp \left(-\frac{2x^2}{3}\right) \, dx \exp \left(\frac{2x^2}{3}\right)}{H_n(x)}. \tag{6.9}
\]
The integral in (6.9) can be computed explicitly for different values of \( n \). For \( H_0(x) = 1 \), we have
\[
U_0(x) = C_1 \exp \left(\frac{2x^2}{3}\right) - \frac{3}{8} (3 + 2x^2).
\]
For \( H_2(x) = 4x^2 - 2 \),
\[
U_2(x) = \frac{C_1 \exp \left(\frac{2x^2}{3}\right) - \frac{3}{4} (4x^4 + 10x^2 + 15)}{4x^2 - 2},
\]
etc.

From these expressions for \( K(x) \) and \( U(x) \), we find that
\[
V_n(x) = \frac{8x^2}{3} - 2U_n(x) + \frac{H_n(x)'}{H_n(x)}; \quad W_n(x) = 4x - \frac{3H_n(x)'}{H_n(x)}. \tag{6.10}
\]
We conclude that the solutions of the nonlinear equation (2.1) with \( R(x) = 2 \) and \( V_n(x) \), \( W_n(x) \) in (6.10) are related to those of the linear differential equation (2.2) with \( U(x) \) and \( K(x) \) given by (6.9) and (6.8) by the transformation (2.3).

Example 3. In this example, we let \( K(x) = 0 \), \( P(x) = a + bx \), \( S(x) = -2P(x)^3 \) and use (3.7) to compute \( U(x) \). This yields
\[
U(x) = C_1 \exp[-(2a + bx)x] - \lambda, \quad V(x) = 6(a + bx)^2 - \lambda - 2C_1 \exp[-(2a + bx)x]. \tag{6.11}
\]
and \( W(x) = -6P(x) \) (from (2.10)).

Example 4. Here as in the previous example we let \( K(x) = 0 \), \( S(x) = -2P(x)^3 \) and \( \lambda = 0 \). However, we now let \( U(x) = -\frac{a}{x} \) and use (3.7) to compute \( P(x) \). For \( a \neq 1 \), we obtain
\[
P(x) = \frac{6\ln x + 2}{x} + C_1 x^{1/2+\beta} + C_2 x^{1/2-\beta} \quad \beta = \sqrt{1 + 8a} \tag{6.12}
\]
For \( a = 1 \)
\[
P(x) = \frac{6\ln x + 2}{x} + C_1 x^{1/2} + C_2 x^{-1/2}. \tag{6.13}
\]
The corresponding \( V(x) \) in this case (viz \( a = 1 \)) is
\[
V(x) = \frac{2}{x^2} + P(x)^2. \tag{6.14}
\]

If we reverse the roles of \( P(x) \) and \( U(x) \), i.e. fix \( P(x) = -\frac{a}{x} \) and use (3.7) to compute \( U(x) \), we find
\[
U(x) = C_1 \exp \left(-\frac{2a}{x}\right) - 3 \left(\frac{1}{x^2} - \frac{1}{ax} + \frac{1}{2a^2}\right). \tag{6.15}
\]
Example 5. Under the same settings of example 4, we specify $U(x) = 2 \exp(2ax)$ and use (3.7) to compute $P(x)$. We find

$$P(x) = C_1 J_0(z) + C_2 Y_0(z), \quad z = \frac{2 \exp(ax)}{a},$$

(6.16)

where $J_0$ and $Y_0$ are the Bessel functions of the order zero of the first and second kinds. The corresponding values of $V(x)$ and $W(x)$ are

$$V(x) = 6P(x)^2 - 4 e^{2ax}, \quad W(x) = -6P(x).$$

7. Conclusions

We demonstrated in this paper that the hybrid ‘Cole–Hopf–Darboux’ operator (2.3) can be used to relate the solutions of some nonlinear and linear second-order differential equations. The algorithm is straightforward to apply and we presented several examples which demonstrated the scope of the method and its potential interest for problems in mathematical physics. In particular, we presented a class of (cubic) nonlinear oscillators whose exact analytic solutions are related to those of the linear harmonic oscillator.

From a perspective, those nonlinear differential equations whose solutions can be expressed in terms of the special functions of mathematical physics (through the transformation (2.3)) should obviously be viewed as a new ‘special class of nonlinear equations’.

Research into the possible application of the transformation (2.3) to partial differential equations is ongoing.

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