Fidelity plateaux from correlated noise in cold-atom quantum simulators

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We demonstrate that, in a quantum simulation protocol based on the Hubbard model, correlated noise in the Hubbard parameters leads to arbitrarily long plateaux in the state-preparation fidelity as a function of elapsed time. We argue that this correlated-noise scenario is the generic one in the cold-atom context, since all of the Hubbard-model parameters ultimately depend on the same set of lasers. We explain the formation of such a plateau using the Bloch-sphere representation, deriving analytical expressions for its start and end times and its height.

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Introduction. The study of disorder in quantum condensed matter systems goes back several decades. Key milestones include Anderson’s investigation of disorder-induced localization in one-dimensional lattices [1], the demonstration that all single-particle eigenstates are also localized in two dimensions [2], and the extension of these ideas to the case of interacting electrons, creating the burgeoning field of many-body localization [3].

These studies have generally concentrated on uncorrelated time-independent site disorder. This is a reasonable approach in solid-state systems with impurity-generated randomness, since the relaxation rate of the electrons is much faster than the dynamics of such impurities.

In the past twenty years great strides have also been made in using cold atom systems as ‘quantum simulators’ of models that have long been of interest in the condensed-matter context [4, 5]. This is often done by subjecting the atoms to a laser standing wave that mimics the periodic potential of the crystalline environment: an ‘optical lattice’. The almost total lack of disorder in such optical lattices is often an advantage. Nonetheless, there have also been experiments in which disorder is deliberately introduced, either via an additional incommensurate optical lattice [6] or by exposing the atoms in the set-up to a laser speckle pattern [7]. Both of these sources of disorder are also independent of time.

Another direction of research in cold-atom physics has been the deliberate preparation of the cold-atom system in a particular quantum state. To this end, one often wishes to alter the parameters of the Hubbard model describing the low-energy atomic motion (the hopping integral $J$ and the on-site repulsion $U$) in some deterministic way: $J(t)$ and $U(t)$. This is achieved by ‘ramping’ the intensities of the lasers that create the optical lattice according to some time-dependent function $V_0(t)$ [8–10].

This ideal ramp profile will in reality be perturbed by noise, whether due to ‘pixellation’ by the digital controller or the deliberate introduction of perturbations to the ramp. This motivates the consideration of time-dependent noise [11, 12] in the cold-atom context.

An important feature of such noise is that, because $J(t)$ and $U(t)$ are time-local functions of $V_0(t)$, any noise added to $V_0(t)$ will appear in both of them [4, 5]: the noise in the hopping integral and the noise in the on-site repulsion will be correlated. In this Letter, we show that such correlations can cause a plateau in the disorder-averaged fidelity as a function of total ramp time.

We begin by presenting a simple two-site Hubbard model, similar to one used in recent state-preparation experiments, in which this phenomenon can be easily illustrated. We then explain the physical origin of the plateau, and give an explicit expression for its height (i.e. the fidelity at which the system gets ‘stuck’). We present the results of numerical simulations in which the plateau can be clearly seen, and which show excellent agreement with our analytical theory. Finally, we give analytical estimates of the noise parameters for which the plateau is visible, as well as expressions for its start and end times.

The double-well optical lattice. The interference pattern between several laser fields can be engineered to generate a Hubbard model on a wide range of lattice geometries, including checkerboard, triangular, and brick-wall patterns [13]. Here we shall consider a simple case in which each lattice site is formed of a double well, as used in recent experiments by several groups [9, 10, 14].

One such double well is shown in Fig. 1(a). Two atoms occupying the same well incur a double-occupancy penalty $U$; the hopping between one well and the other is $J$. The system can be made such that the hopping between neighboring double wells $J'$ is very weak ($J' \ll J, U$). We therefore neglect such hopping, and restrict ourselves to an effective two-site Hubbard model.

We populate the double well with two fermions of opposite spin, for which there are four possible states. However, we shall choose initial conditions such that only two of them are ever populated: $|s\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ and $|d\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$. In this subspace, which always contains the ground state of the four-state problem, the Hamiltonian is

$$
\mathcal{H} = U|d\rangle\langle d| - 2J(|d\rangle\langle s| + |s\rangle\langle d|).
$$

We now consider a state-preparation process in which
the laser intensities are changed with time, from an initial configuration at time $t = 0$ to a final one at time $t = t_f$. It follows that the parameters in the Hubbard model also become time-dependent, i.e. $J \to J(t)$ and $U \to U(t)$.

We define the instantaneous eigenstates of the resulting time-dependent Hamiltonian $\mathcal{H}(t)$ as follows:

\begin{align}
\mathcal{H}(t)|g(t)\rangle &= E_g(t)|g(t)\rangle; \\
\mathcal{H}(t)|e(t)\rangle &= E_e(t)|e(t)\rangle,
\end{align}

where ‘g’ stands for ‘ground’, and ‘e’ for ‘excited’. The changing laser intensities induce transitions between the instantaneous eigenstates of the system, since $J$ and $U$ (shown by the thick lines in Fig. 1(b)) have different functional dependences on the lattice depth. We start the system at $t = 0$ in the instantaneous ground state of the Hamiltonian $\mathcal{H}(0)$, i.e. we set $|\psi(0)\rangle = |g(0)\rangle$. We define the fidelity, $\mathcal{F}(t)$, as the probability of finding the system in the instantaneous ground state of $\mathcal{H}(t)$ at a time $t$ into the ramp:

$$\mathcal{F}(t) = \left| \langle g(t)|\psi(t)\rangle \right|^2.$$  

Our measure of adiabaticity is the probability of finding the system in its ground state at the end of the ramp, $\mathcal{F}(t_f)$. In a noise-free system, provided the energy gap above the ground state remains finite during the ramp, $\mathcal{F}(t_f) \to 1$ as $t_f$ becomes large [15]. However, in a system with fast noise this is not the case, since the eigenbasis fluctuates rapidly even for large $t_f$.

We consider a ramp in which the laser intensity is

$$V(t) = V_0(t) \left[ 1 + \epsilon \eta(t) \right]$$

where $V_0(t)$ is the ‘target’ laser intensity, $\epsilon$ the noise strength, and $\eta(t)$ a Gaussian-distributed stochastic variable satisfying $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)f(t+\tau) \rangle = \delta(\tau)$, where bars denote averages over disorder realizations [16].

To first order in $\epsilon \eta$ this noise appears in the Hubbard parameters as

$$J(t) \approx J_0(t) + \epsilon \eta(t) J_n(t)$$

and

$$U(t) \approx U_0(t) + \epsilon \eta(t) U_n(t),$$

where $J_0(t)$ and $U_0(t)$ are the target Hubbard parameters (corresponding to laser intensity profile $V_0(t)$) and $J_n(t)$ and $U_n(t)$ are the coefficients of an expansion in $\epsilon \eta$. An example of the Hubbard parameters including noise is shown by thin lines in Fig. 1(b).

The resulting Hamiltonian is

$$\mathcal{H}(t) = \mathcal{H}_0(t) + \epsilon \eta(t) \mathcal{H}_n(t),$$

where $\mathcal{H}_0$ and $\mathcal{H}_n$ have the form (1), with the corresponding subscripts on the Hubbard parameters. Importantly, $J_0$, $U_0$, $J_n$, and $U_n$ evolve deterministically in time and depend only on $V_0$. Therefore the instantaneous eigenstates and eigenenergies of $\mathcal{H}_0$ and $\mathcal{H}_n$ also evolve deterministically, depending only on the target laser intensity profile. Despite the presence of the noise, we continue to initialize the system and measure the fidelity using the ground state of the target Hamiltonian $|g_0(t)\rangle$.

The physics of the fidelity plateau. When the noise is strong, the Hamiltonian (6) will be dominated by the contribution from the noise part, i.e.

$$\mathcal{H}(t) \approx \epsilon \eta(t) \mathcal{H}_n(t).$$

In this limit, we should therefore describe the system in the basis $\{ |g_n(t)\rangle, |e_n(t)\rangle \}$, the instantaneous eigenbasis of $\mathcal{H}_n(t)$. If the ramp is performed slowly enough that the system adiabatically follows the eigenstates of $\mathcal{H}_n(t)$, then the Hamiltonian (7) causes pure dephasing, i.e. the relative populations of the states $|g_n(t)\rangle$ and $|e_n(t)\rangle$ become independent of time. This, as we shall show, gives rise to a plateau in the disorder-averaged fidelity $\mathcal{F}(t_f)$.

This ‘pure dephasing’ time-evolution is given by

$$|\psi(t)\rangle = \sum_{m=g,e} c_m \exp \left( -\frac{i}{\hbar} \int_0^t dt' \epsilon_n(t') v_m(t') \right) |m_n(t)\rangle,$$

where $v_m(t)$ is the instantaneous eigenvalue of $\mathcal{H}_n(t)$ corresponding to the instantaneous eigenstate $|m_n(t)\rangle$, and $c_m = \langle m_n(0)|\psi(0)\rangle = \langle m_n(0)|g_0(0)\rangle$. When the relative phase of the two components of $|\psi(t)\rangle$ is fully randomized, the average over disorder realizations gives a fidelity of

$$\mathcal{F}_0(t_f) = \left| \langle g_n(0)|g_0(0)\rangle \langle g_0(t)|g_n(t)\rangle \right|^2 + \left| \langle e_n(0)|g_0(0)\rangle \langle g_0(t)|e_n(t)\rangle \right|^2.$$
which clearly depends only on the instantaneous properties of $H_0(0)$, $H_0(t)$, $H_n(0)$, and $H_n(t)$, but not on the ramp rate. Thus, counterintuitively, strong noise does not fully randomize the system’s state, instead spreading it evenly over a small fraction of the full state space.

We may visualize this in the Bloch sphere picture, by comparing the evolution on the Bloch spheres defined by the instantaneous eigenstates of $H_0(t)$ and $H_n(t)$. On the $H_n(t)$ Bloch sphere the initial state is located at a point on the sphere’s surface at some angle $\theta_0$ to the $|g_n(t)\rangle$ pole. Pure-dephasing behavior then corresponds to the state performing a random walk around the line of latitude $\theta = \theta_0$. Thus at $t = t_f$ the ensemble of final states is distributed on a ring at a constant angle $\theta_0$ from the $|g_n(t)\rangle$ pole, as shown on the left of Fig. 1(c).

The eigenstates of $H_0(t_f)$ are related to those of $H_n(t_f)$ by a unitary transformation, so the Bloch sphere of $H_0(t_f)$ is identical to that of $H_n(t_f)$ except for a rotation. This rotation preserves the form of the ring of final states, but changes the angular position of its center, as shown on the right of Fig. 1(c).

The fidelity is equal to the $z$-coordinate of the state on the Bloch sphere measured from the south pole. This results in a distribution of $\mathcal{F}(t_f)$ the average of which (equal to $\mathcal{F}_0(t_f)$) is the $z$-coordinate of the center of the ring. The distribution of fidelities corresponding to this ring is:

$$f(x) = \frac{1}{\pi \mathcal{F}_1} \left( 1 - \left( \frac{x - \mathcal{F}_0}{\mathcal{F}_1} \right)^2 \right)^{-1/2}$$

where $\mathcal{F}_1 = 2 |\langle g_0(t_f) | g_0(0) \rangle|^2$ with the superscripts $(i)$ and $(f)$ signifying the state evaluated at $t = 0$ and $t = t_f$ respectively.

Numerical demonstration of the fidelity plateau. We now demonstrate the predicted behavior by numerically time-evolving the noisy ramp process described by the Hamiltonian (6), using the Heun algorithm [16]. The results are shown in Figs. 2 and 3.

Fig. 2 shows the disorder-averaged fidelity as a function of ramp time, for various noise strengths. It decays from $\mathcal{F}(0) = |\langle g_0(t_f) | g_0(0) \rangle|^2$ to the value $\mathcal{F}_0(t_f)$, indicated by the dashed line. For strong noise, as predicted, it ‘sticks’ there, forming a long plateau; for weak noise, it almost immediately resumes decaying towards 1/2, the value naively expected for a two-state system.
Fig. 3 shows the disordered-averaged fidelity $\mathcal{F}(t_f)$ as a function of noise strength, for various ramp times. The predicted value of the fidelity plateau $\mathcal{F}_0$ is again shown by the dashed line. For all ramp times shown, the average fidelity tends to $\mathcal{F}_0$ as the noise becomes strong. The curves depart from the strong-noise prediction as the noise strength $\epsilon$ is reduced. This reflects the counterintuitive fact, established above, that weaker noise is better able to spread the ensemble of states across the entire Bloch sphere.

Fig. 3(b) shows $\mathcal{F}(t)$ for the ramp performed at $\epsilon = 0.07 s^{1/2}$ with $t_f = 0.3 \text{ ms}$. The solid black line shows $\mathcal{F}_0(t)$ (the center of the ring in the right-hand panel of Fig. 1(c)), and the dashed black lines show the upper and lower bounds on $\mathcal{F}(t)$ (the top and bottom of the ring in the right-hand panel of Fig. 1(c)). The solid colored line shows $\mathcal{F}(t)$, which falls exactly on the $\mathcal{F}_0(t)$ curve. An example of $\mathcal{F}(t)$ for an individual noise realization, shown by the colored dotted line, stays within the predicted bounds.

**When does the plateau occur?** There are three conditions that must be satisfied for the fidelity plateau to occur. They are most easily understood in terms of the equation of motion for $c_g(t)$ and $c_e(t)$, the decomposition of the system’s state in terms of the instantaneous eigenbasis of $\mathcal{H}_n$:

$$i\hbar \left[ \begin{array}{c} \dot{c}_g \\ \dot{c}_e \end{array} \right] = \left[ \begin{array}{cc} \langle g_n | \mathcal{H}_0 | g_n \rangle & \langle g_n | \mathcal{H}_e | e_n \rangle \\ \langle e_n | \mathcal{H}_0 | g_n \rangle & \langle e_n | \mathcal{H}_e | e_n \rangle \end{array} \right] \left[ \begin{array}{c} c_g \\ c_e \end{array} \right] + i\epsilon \left[ \begin{array}{c} v_g \\ 0 \\ 0 \\ v_e \end{array} \right].$$

The first condition is that transitions due to $[\mathcal{H}_0, \mathcal{H}_n] \neq 0$ must be negligible, i.e. the first term in the square brackets must be small compared to the third. The second condition is that non-adiabatic transitions between the instantaneous eigenstates of $\mathcal{H}_n$ must be negligible as well, i.e. the second term must be small compared to the third. The third condition is that the third term must be strong enough to completely dephase the components of the initial state by the end of the ramp.

To determine when the first condition is satisfied, we rescale the time variable: $t = st_f$ with $s \in [0, 1]$. For the first term in the square brackets in (11), this is equivalent to rescaling the matrix elements by a factor $t_f$. However, the third term must be treated more carefully. Examining the correlation function of the noise,

$$\langle \eta(t) \eta(t') \rangle = \delta(t - t') = \delta(t_f[s - s']) = t_f^{-1} \delta(s - s'),$$

we see that the appropriate rescaling is $\eta(t) \rightarrow t_f^{-1/2} \eta(s)$. Thus the Schrödinger equation becomes:

$$i\hbar \partial_s |\psi(s)\rangle = \left[ t_f \mathcal{H}_0(s) + \epsilon \sqrt{t_f} \eta(s) \mathcal{H}_e(s) \right] |\psi(s)\rangle.$$  

This shows that the ‘strength’ of the noise depends on how long the system is exposed to it. The influence of

The deterministic part becomes comparable to that of the noise when the two terms on the right-hand side of (13) are of similar magnitude: $t_f \Delta E_n \sim \epsilon \sqrt{t_f} \Delta E_n$, where $\Delta E_n$ is the typical energy difference between the eigenstates of $\mathcal{H}_n$. Thus transitions due to $\mathcal{H}_0$ may be neglected provided that

$$t_f \ll t_q \sim \left( \epsilon \frac{\Delta E_n}{\Delta E_0} \right)^2.$$  

The second condition requires that the third term on the right-hand side of (11) is much stronger than the second. Rescaling time by $t_f$ as before, we find that the second term gains no prefactor to balance the $\sqrt{t_f}$ enhancement of $\mathcal{H}_n$. The non-adiabatic effects of the ramp can therefore no longer be neglected when $\epsilon \sqrt{t_f} \Delta E_n \sim h D_n \Delta E_n^{-1}$, where $D_n$ is the typical magnitude of $\langle g_n(s) \partial_s \mathcal{H}_n(s) | e_n(s) \rangle \sim | \partial_s J_n(s) |$, $| \partial_s U_n(s) |$. Thus, as expected, non-adiabatic effects become weaker for ramps performed over a longer time, and become negligible provided that

$$t_f \gg t_a \sim \left( \frac{h D_n}{\epsilon \Delta E_n^2} \right)^2.$$  

Assuming that the first two conditions are satisfied, only the third term on the right-hand side of (11) need be retained. Thus, from equation (8), we see that the relative phase is given by $\Delta \phi(t) = \frac{1}{2} \int_0^t dt' \epsilon(t') \Delta E_n(t')$, where $\Delta E_n(t) = v_c(t) - v_g(t)$, the instantaneous gap between the eigenenergies of $\mathcal{H}_n$. It is straightforward to
show that $\Delta \phi(t) = 0$ and $(\Delta \phi)^2(t) = \epsilon^2 \hbar^{-2} \delta E_n^2 t$, where

$$
\delta E_n^2 = \epsilon t \int_0^t dt' \Delta E_n(t')^2.
$$

The distribution of $\Delta \phi$ becomes approximately uniform when the variance is of order one, and thus the phase is randomized provided that

$$
t_f \gg t_\phi \sim \left( \frac{\hbar}{\epsilon \delta E_n} \right)^2.
$$

(16)

Fig. 4 shows $|\mathcal{F}(t_f) - \mathcal{F}_0|$, the deviation of the numerically determined fidelity from the predicted plateau value, as a function of $\epsilon$ and $t_f$. The dashed white lines show the timescales (14) and (15), and we see that the fidelity attains its plateau value everywhere in the upper part of the graph between the two curves, exactly as predicted by the above analysis. The curve (16) is not marked on the graph as it follows the same functional form as $t_\alpha$, and for this system we find that $t_\phi < t_\alpha$, which can be seen from the rapid collapse of $\mathcal{F}(t)$ onto $\mathcal{F}_0(t)$ in Fig. 3(b).

**Outlook.** We have shown that the correlated nature of the noise in cold-atom experiments can lead to counterintuitive consequences for quantum state preparation, especially the occurrence of long plateaux at non-trivial values of the fidelity. While we have illustrated this phenomenon in the simplest case — an effective two-level system formed from a double-well Hubbard model — it seems likely that it will be of broader applicability. It is, in fact, a form of localization (i.e. failure of ergodicity) due to the separation of timescales. Links to other recently discovered instances of ergodicity-breaking, e.g. in Josephson junction chains [17], would be worthy of further investigation. It would also be worth investigating whether there are potential state-preparation protocols that make explicit use of this fidelity-plateau phenomenon.

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