Ergodic convergence rates for time-changed symmetric Lévy processes in dimension one

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Abstract
We obtain the lower bounds for ergodic convergence rates, including spectral gaps and convergence rates in strong ergodicity, for time-changed symmetric Lévy processes, by using harmonic function and reversible measure. As direct applications, explicit sufficient conditions for exponential and strong ergodicity are given. Some examples are also presented.

Keywords and phrases: Lévy process; spectral gap; convergence rate in strong ergodicity; harmonic function; time change.

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1 Main results and examples

Ergodicity for Lévy-type processes is an important topic in study of Markov processes. In general, criteria are obtained by using Lyapunov functions (cf. [16] for general one-dimensional Lévy-type operators, [17] for Lévy-driven SDEs, and [4] for time-changed symmetric stable processes), or coupling methods (see [10] for Lévy-driven SDEs).

Recently, [18] obtains the criteria for strong and exponential ergodicity of one-dimensional time-changed symmetric stable processes; the lower bounds for ergodic convergence rates, including spectral gaps and convergence rates in strong ergodicity are also estimated in [18]. Different from Lyapunov criteria and coupling methods, the main idea in [18] is to estimate the Green function for \( \mathbb{R} \setminus \{0\} \) by assuming that the process is pointwise-recurrent.

While [18] deals with the classical \( \alpha \)-stable processes for \( \alpha \in (1,2) \), this will exclude some significant pointwise-recurrent Lévy processes such as the diffusion operator with stable jump: \( L = a(x)(c_1\Delta + c_2\Delta^{\alpha/2}) \), where \( \alpha \in (1,2) \), \( a \) is a positive function such that \( a^{-1} \) is Lebesgue integrable, \( c_1 \) and \( c_2 \) are two constants, \( \Delta \) is the Laplacian operator and \( \Delta^{\alpha/2} \) is the fractional Laplacian.

The aim of this paper is to study the ergodic convergence rates for general pointwise-recurrent time-changed symmetric Lévy processes. To this end, we first recall some basic definitions.

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Let $X = (X_t)_{t \geq 0}$ be a one-dimensional symmetric Lévy process with Lévy measure $\nu$. The characteristic exponent of $X$ is defined as

$$
\psi(\xi) := -\log \mathbb{E} e^{i\xi X_1} = \int_{\mathbb{R}} (1 - \cos \xi x) \nu(dx) + \sigma^2 \xi^2, \quad \xi, \sigma \in \mathbb{R}.
$$

(1.1)

The generator $A$ is a one-dimensional Lévy operator given by

$$
Au(x) := -\int \psi(\xi) \hat{u}(\xi) e^{i\xi x} d\xi, \quad u \in C_0^\infty(\mathbb{R}),
$$

(1.2)

where $C_0^\infty(\mathbb{R})$ is the space of compactly supported smooth functions in $\mathbb{R}$ and $\hat{u}(\xi)$ is the Fourier transform of $u$. The corresponding regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ is given by

$$
\mathcal{E}(u, w) = \int_{\mathbb{R}} \frac{1}{2} \sigma^2 u'(x)w'(x)dx + \int_{\mathbb{R} \times \mathbb{R} \setminus \text{diag}} (u(x + h) - u(x))(w(x + h) - w(x))\nu(dh)dx,
$$

(1.3)

for $u, w \in \mathcal{F} := \{ f \in L^2(dx) : \mathcal{E}(f, f) < \infty \}$ (see [1, Example 3.13] for more details).

Let $p_t(y - x) := p_t(x, y)$ be the transition density of this process. Define

$$
H(x) = \int_0^\infty (p_s(0) - p_s(x))ds = \frac{1}{\pi} \int_0^\infty (1 - \cos xs) \frac{1}{\psi(s)} ds.
$$

(1.4)

Obviously, $H$ is an even function: $H(x) = H(-x)$. Let $P_t^0$ be the semigroup of the process killed upon hitting the origin, i.e.,

$$
P_t^0(x, A) = \mathbb{P}_x[X_t \in A, t < \tau_0], \quad \forall x \in \mathbb{R}, \forall A \in \mathcal{B}(\mathbb{R}),
$$

(1.5)

where $\tau_0 := \inf\{ t > 0 : X_t = 0 \}$. By [19, Theorem 1.1], $H(x)$ is the harmonic function for $P_t^0$, i.e., for any $x \neq 0$, $P_t^0 H(x) = H(x)$.

Now we consider the time-changed symmetric Lévy processes. Let $a$ be a positive and locally bounded measurable function on $\mathbb{R}$ with $a(x)^{-1}$ integrable. Define $\mu(dx) := a(x)^{-1}dx$, $A_t := \int_0^t 1/a(X_s)ds$, and the time-changed Lévy process $Y_t = X_{A_t}$, where

$$
\tau_t = \inf\{ s > 0 : A_s > t \}.
$$

(1.6)

Then, the generator of time-changed process $Y$ is just the operator $L = aA$ (for more details about this paragraph, see Section [2]).

Let $\mu(f) := \int f d\mu$ and $\| f \|_{L^2(\mu)} := \sqrt{\mu(f^2)}$. We say that $Y$ is $(L^2)$-exponentially ergodic, if there exist non-negative function $C(x) < \infty$ and $\lambda_1 > 0$, such that

$$
\| P_t f - \mu(f) \|_{L^2(\mu)} \leq e^{-\lambda_1 t}\| f - \mu(f) \|_{L^2(\mu)}.
$$

(1.7)

The optimal convergence rate $\lambda_1$ in (1.7) (i.e., the $L^2$-spectral gap) is defined by

$$
\lambda_1 = \inf\{ \mathcal{E}(f, f) : f \in \mathcal{F}, \mu(f^2) = 1, \mu(f) = 0 \}.
$$

(1.8)

Let $\| \eta \|_{\text{Var}} := \sup_{|f| \leq 1} |\eta(f)|$ be the total variation of a signed measure $\eta$. We say that $Y$ is strongly ergodic, if there exist constants $C < \infty$ and $\kappa > 0$, such that

$$
\sup_{x \in \mathbb{R}} \| P_t(x, \cdot) - \mu \|_{\text{Var}} \leq C e^{-\kappa t}.
$$
The optimal convergence rate (see \[11\] for more details)
\[
\kappa = -\lim_{t \to \infty} \frac{1}{t} \log \sup_{x \in E} \|P_t(x, \cdot) - \mu\|_{\text{var}} = -\lim_{t \to \infty} \frac{1}{t} \log \|P_t - \mu\|_{\infty \to \infty}.
\]

The following theorem gives the explicit sufficient conditions for exponential ergodicity and strong ergodicity. The explicit lower bounds for ergodic convergence rates are also obtained.

**Theorem 1.1.** Assume that \(\mu(\mathbb{R}) < \infty\). Consider the following conditions:

(A1) \(\int_0^\infty (q + \psi(x))^{-1} dx < \infty\), for any \(q > 0\);

(A2) \(\int_0^1 (\psi(x))^{-1} dx = \infty\);

(A3) \(\psi(t)/t \to \infty\), as \(t \to \infty\).

(1) If (A1)–(A3) hold, and
\[
\delta := \sup_x H(x) \mu((-|x|, |x|)^c) < \infty, \tag{1.9}
\]
then \(Y\) is exponentially ergodic and the \(L^2\)-spectral gap
\[\lambda_1 \geq \frac{1}{8\delta}.\]

(2) If (A1)–(A2) hold, and
\[
I := \int_{\mathbb{R}} a(x)^{-1} H(|x|) dx < \infty,
\]
then \(Y\) is strongly ergodic and
\[\kappa \geq \frac{1}{2I} > 0.\]

**Remark 1.2.** (1) For symmetric Lévy process, condition (A1) means that the process is not compound Poisson and the origin is regular for itself (see \[1\] Section 2). Condition (A2) means that the process is recurrent (see \[19\] Section 3.2). If (A1) and (A2) hold, then \(X\) is pointwise recurrent.

(2) (A3) indicates that \(H\) is differentiable (see the proof of Theorem \[17\]).

(3) Note that a time change does not change the recurrence (cf. \[13\] Corollary 4.3.7]). Therefore, under the conditions (A1)–(A2), \(Y\) is also pointwise recurrent, so \(Y\) is Lebesgue irreducible (see \[13\], Page 42] for the definition). Thus, by \[13\], Theorem 4.1.1 and Theorem 4.2.1, if \(\mu(\mathbb{R}) < \infty\), then \(Y\) is ergodic.

(4) Let \(\psi(x) = |x|^\alpha, \alpha \in (1, 2)\). Then \(Y\) is a time-changed symmetric \(\alpha\)-stable process. It is well known that the process is pointwise recurrent; by \[19\], Example 1.1], the harmonic function \(H(x) = \omega_\alpha |x|^{\alpha - 1}/2\), where \(\omega_\alpha = -(\cos(\pi\alpha/2)\Gamma(\alpha))^{-1} > 0\). Then we have \(Y\) is exponentially ergodic if
\[
\delta_1 := \sup_x |x|^{\alpha - 1}\mu((-|x|, |x|)^c) < \infty, \tag{1.10}
\]
and

\[ \lambda_1 \geq \frac{1}{4 \omega_1 \delta_1}, \]

\( Y \) is strongly ergodic if

\[ I_1 = \int_{\mathbb{R}} \sigma(x)^{-\alpha} |x|^{\alpha-1} dx < \infty, \tag{1.11} \]

and

\[ \kappa \geq \frac{1}{\omega_1 I_1} > 0. \]

This case is introduced in [18]. In fact, the conditions (1.10) and (1.11) are sufficient and necessary (see [18] for more details).

Next, we discuss a class of extended \( \alpha \)-stable processes, which is introduced in [9]. In general, the cases mean that there exists \( \alpha > 0 \) such that \( \psi(\theta)/\theta^\alpha \) is comparable to a non-decreasing function on \((0, \infty)\).

We say that \( \psi \) satisfies the global weak lower scaling condition, if there exist \( \delta > 0 \) and \( \beta \in (0, 1] \), such that for \( \lambda \geq 1 \) and \( \theta > 0 \),

\[ \psi(\lambda \theta) \geq \beta \lambda^\delta \psi(\theta). \]

In short, we write \( \psi \in \text{WLSC}(\delta, \beta) \) (see [9] for more details). Applying this condition, we have the following result which is a direct corollary by using Theorem 1.1 and [9, Lemma 2.14]:

**Corollary 1.3.** Let \( \psi^*(x) = \sup_{|u| \leq x} \psi(u) \), \( x \geq 0 \). Assume that there exists a constant \( c > 0 \), such that \( \psi \geq c \psi^* \), and \( \psi \in \text{WLSC}(\delta, \beta) \) (\( \delta > 1 \)). If

\[ \sup_x \frac{\mu((-|x|, |x|)^c)}{|x| \psi(1/x)} < \infty, \]

then \( Y \) is exponentially ergodic and the spectral gap

\[ \lambda_1 \geq \frac{\pi (\delta - 1)^2}{10} \inf_x \frac{|x| \psi^*(1/|x|)}{\mu((-|x|, |x|)^c)}, \]

If

\[ \int_{\mathbb{R}} \frac{1}{|x| a(x) \psi(1/x)} dx < \infty, \]

then \( Y \) is strongly ergodic and

\[ \kappa \geq \frac{\pi \beta^2 (\delta - 1)}{20 \int_{\mathbb{R}} (|x| a(x) \psi^*(1/|x|))^{-1} dx}. \]

Now we return to the diffusion operator with stable jump.

**Example 1.4.** Let \( \psi(x) = c_1 x^2 + c_2 |x|^\alpha \), where \( c_1, c_2 \) are two constants and \( \alpha \in (1, 2) \). Then \( L = a(x)(c_1 \Delta + c_2 \Delta^{\alpha/2}) \). Denote by \( Y \) the corresponding process with generator \( L \). By (1.4), the harmonic function for \( c_1 \Delta + c_2 \Delta^{\alpha/2} \) is

\[ H(x) = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos xs}{c_1 s^2 + c_2 s^{\alpha}} ds. \]
Obviously, (A1)–(A3) hold, $\psi^*(x) = \psi(x) = c_1 x^2 + c_2 |x|^\alpha$, and $\psi \in \text{WLSC}(\alpha, 1)$. According to [2, (12)], we have

$$H(x) \leq \frac{10}{\pi (\alpha - 1)} \frac{1}{c_1 |x|^{-1} + c_2 |x|^{1-\alpha}}.$$ 

Combining it with Corollary 1.3 if

$$\sup_x |x|^{\alpha-1} \mu((-|x|, |x|)^c) < \infty,$$

then $Y$ is exponentially ergodic, and the spectral gap

$$\lambda_1 \geq \frac{\pi (\alpha - 1)}{80} \inf_x \frac{c_1 |x|^{-1} + c_2 |x|^{1-\alpha}}{\mu((-|x|, |x|)^c)};$$

if

$$\int_{\mathbb{R}} \sigma(x)^{-\alpha} |x|^{\alpha-1} dx < \infty,$$

then $Y$ is strongly ergodic, and the convergence rate in strong ergodicity

$$\kappa \geq \frac{\pi (\alpha - 1)}{20 \int_{\mathbb{R}} (a(x))^{-1} (c_1 |x|^{-1} + c_2 |x|^{1-\alpha})^{-1} dx}.$$

**Remark 1.5.** Note that if $c_1 = 0$ (resp., $c_2 = 0$), then the result is reduced to the time-changed symmetric stable process (resp., time-changed Brownian motion).

**Example 1.6.** Let $\psi(x) = x^2 + |x|$. The process $X$ associated with $\Delta + \Delta^{1/2}$ is a sum of the Cauchy process and independent Brownian motion. Obviously, (A1)–(A3) hold. By [2, Lemma 2.14],

$$H(x) \leq \frac{10}{\pi} \int_{1/|x|}^{\infty} \frac{dr}{r + r^2} = \frac{10}{\pi} \log (1 + |x|).$$

Therefore, if

$$\delta_2 := \sup_x \log(1 + |x|) \mu((-|x|, |x|)^c) < \infty,$$

then $Y$ is exponentially ergodic, and the spectral gap

$$\lambda_1 \geq \frac{\pi}{80 \delta_2}.$$

If

$$I_2 := \int_{\mathbb{R}} a(x)^{-1} \log(1 + |x|) dx < \infty,$$

then $Y$ is strongly ergodic and

$$\kappa \geq \frac{\pi}{20 I_2}.$$
2 Time change and Green potential

Let $X = (X_t)_{t \geq 0}$ be a one-dimensional symmetric Lévy process with Lévy measure $\nu$, transition density $p_t(x, y) = p_t(y - x)$, and characteristic exponent $\psi$ given by (1.1).

Recalling that $a$ is a positive and locally bounded measurable function on $\mathbb{R}$ with $a(x)^{-1}$ is Lebesgue integrable, $A_t = \int_0^t 1/a(X_s)ds$ is the positive continuous additive functional and $Y$ is the time-changed Lévy process defined as (1.6). The Revuz measure $\mu$ of $A_t$ with respect to $dx$, is given by (cf. [7])

$$\mu(f) = \lim_{t \to \infty} \frac{1}{t} \int \mathbb{E}_x \left[ \int_0^t f(X_s) dA_s \right] dx.$$ 

Since $dx$ is the invariant measure of $X$, for nonnegative bounded function $f$, we have

$$\mu(f) = \lim_{t \to \infty} \frac{1}{t} \int \mathbb{E}_x \left[ \int_0^t f(X_s)a(X_s)^{-1}ds \right] dx = \lim_{t \to \infty} \frac{1}{t} \int \mathbb{E}_x \left[ \int_0^t (fa^{-1})(x) dx ds \right] = \int (fa^{-1})(x) dx,$$

thus the Revuz measure $\mu(dx) = a(x)^{-1}dx$. Combining this fact and [5] Theorem 5.2.2, Theorem 5.2.8 and Corollary 5.2.12, similar to [4] Page 2807, we know that $Y$ is $\mu$-symmetric and its Dirichlet form $(\mathcal{E}, \mathcal{F})$ is given by

$$\mathcal{E}(f, g) = \mathcal{E}(f, g), \quad f, g \in \mathcal{F} := \mathcal{F}_e \cap L^2(\mu),$$

where $\mathcal{F}_e$ is the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$, i.e., the family of functions $u$ satisfy that there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\} \subset \mathcal{F}$ such that for a.e. $x$, $\lim_{n \to \infty} u_n = u$ in $L^2(dx)$ and $\mathcal{E}(u, u) = \lim_{n \to \infty} \mathcal{E}(u_n, u_n)$. Therefore, the $L^2$ infinitesimal generator of $Y$ is $\mathcal{L} = a\mathcal{A}$.

By a similar argument to [1] Corollary 4.2, we can also prove that the extended infinitesimal generator $\tilde{\mathcal{L}}$ (see [15] Definition 2.1), is also $a\mathcal{A}$.

Recalling that $P^0_t$ is the killed semigroup of $X$ given by (1.5). Define the Green potential measure $G^0_X(x, A)$ for $P^0_t$ by

$$G^0_X(x, A) = \int_0^\infty P^0_t(x, A) dt, \quad \forall x \in \mathbb{R}, \forall A \in \mathcal{B}(\mathbb{R}).$$

Denote by $P^{0,Y}_t$ the semigroup of $Y$ killed upon hitting the origin, i.e.

$$P^{0,Y}_t(x, A) = \mathbb{P}_x[Y_t \in A, t < \tau^Y_0],$$

where $\tau^Y_0 = \inf\{t > 0 : Y_t \neq 0\}$. Let $G^0_Y(x, A)$ be the Green measure of $Y$ killed upon 0:

$$G^0_Y(x, A) = \int_0^\infty P^{0,Y}_t(x, A) dt, \quad \forall x \in \mathbb{R}, \forall A \in \mathcal{B}(\mathbb{R}).$$

Similar to [18] (14), for the time-changed process $Y$,

$$G^0_Y f(x) = \int_\mathbb{R} G^0_X(x, y)a(y)^{-1}dy.$$  

(2.3)
3 Proofs of main results

To finish the proof of Theorem 1.1, we need to consider the first Dirichlet eigenvalue

$$\lambda_0 = \inf \{ \mathcal{E}(f, f) : f \in \hat{F}, \mu(f^2) = 1 \text{ and } f(0) = 0 \},$$  \quad (3.1)

which will play a crucial role in the proof of Theorem 1.1.

First, we introduce the following dual variational inequality, which is mainly motivated by the dual variational formulas for one-dimensional diffusion processes (see [3, Theorem 6.1]).

**Lemma 3.1.** Let $G_Y^0$ be the Green operator of $Y$ killed upon $\{0\}$, $C(B)$ be the space of all continuous functions on a measurable set $B \subset \mathbb{R}$. Denote by

$$\mathcal{H} = \{ f : f(0) = 0, f \in C(\mathbb{R} \setminus \{0\}) \}$$

$$\tilde{\mathcal{H}} = \{ f : f(0) = 0, \text{ there exists } x_0 > 0 \text{ such that } f = f(\cdot \wedge x_0 \vee (-x_0)), f \in C(-x_0, x_0) \}.$$

Then

$$\inf_{f \in \tilde{\mathcal{H}}} \sup_{x \in \mathbb{R} \setminus \{0\}} \frac{f(x)}{G_Y^0 f(x)} \geq \lambda_0 \geq \sup_{f \in \mathcal{H}} \inf_{x \in \mathbb{R} \setminus \{0\}} \frac{f(x)}{G_Y^0 f(x)}.$$

**Proof.** First we proof the upper bound. Note that for any $g \in \tilde{\mathcal{H}}$, there exists $x_0 > 0$ such that $g(-x_0) \leq g \leq g(x_0)$. Therefore, $g \in L^2(\pi)$. By a similar argument to the proof of [18, Theorem 2] (see [18, Page 12]), we have $G_Y^0 g \in \hat{F}$, and

$$\mathcal{E}(G_Y^0 g, G_Y^0 g) = \int g G_Y^0 g d\pi.$$

By the definition (3.1),

$$\lambda_0 \leq \frac{\mathcal{E}(G_Y^0 g, G_Y^0 g)}{\pi((G_Y^0 g)^2)} \leq \sup_{x \in \mathbb{R} \setminus \{0\}} \frac{g(x)}{G_Y^0 g(x)}.$$

Now we get the upper bound by the arbitrariness of $g \in \tilde{\mathcal{H}}$.

Next, denote by

$$\lambda_0^{(n)} = \inf \{ \mathcal{E}(f, f) : \mu(f^2) = 1, f|_{(-\infty, -n) \cup (n, \infty)} = 0 \}.$$

According to [18, Lemma 11],

$$\lim_{n \to \infty} \lambda_0^{(n)} = \lambda_0.$$

By a similar argument to the proof of [18, Lemma 7], for regular set $B_n := (-n, 0) \cup (0, n)$ (see [6, Page 68] for the definition of regular set),

$$\lambda_0^{(n)} \geq \sup_{f \in C_b(B_n)} \inf_{x \in B_n} \frac{f(x)}{G_Y^{B_n} f(x)},$$

where $G_Y^{B_n}$ is the Green operator defined as $G_Y^{B_n}(x, A) := \int_0^\infty \mathbb{P}_x[Y_t \in A, t < \tau_{B_n}^Y] dt$, $\tau_{B_n}^Y$ is the exit time from $B_n$: $\tau_{B_n}^Y := \inf\{ t \geq 0 : Y_t \notin B_n \}$.
Next, since \( \phi_{B_{n}} \leq G_{Y}^{0} \phi \) and \( f \in C(\mathbb{R} \setminus \{0\}) \) is bounded on \( B_{n} \), we have that for any \( f \in \mathcal{H} \),

\[
\lambda_{0} = \lim_{n \to \infty} \lambda_{0}^{(n)} \geq \lim_{n \to \infty} \inf_{x \in B_{n}} \frac{f(x)}{G_{Y}^{0} f(x)} \geq \inf_{x \in \mathbb{R} \setminus \{0\}} \frac{f(x)}{G_{Y}^{0} f(x)},
\]

(3.2)

thus we obtain the lower bound. \( \square \)

The following explicit estimates for lower and upper bounds of \( \lambda_{0} \) is similar to Theorem \( 2 \), and can be obtained directly by Lemma \( 3.1 \).

**Theorem 3.2.** Let \( \delta \) be defined by (1.9) and

\[
\delta^{+} = \sup_{x > 0} H(x) \mu((x, \infty)), \quad \delta^{-} = \sup_{x < 0} H(x) \mu((\infty, -x)).
\]

Then

\[
\frac{1}{\delta^{+}} + \frac{1}{\delta^{-}} \geq \lambda_{0} \geq \frac{1}{8\delta}, \quad (3.3)
\]

**Proof.** First, we prove the upper bound. According to Lemma \( 3.1 \) for any \( g \in \mathcal{H} \),

\[
\lambda_{0} \leq \sup_{x \in \mathbb{R} \setminus \{0\}} \frac{g(x)}{G_{Y}^{0} g(x)} \leq \sup_{x > 0} \frac{g(x)}{G_{Y}^{0} g(x)} + \sup_{x < 0} \frac{g(x)}{G_{Y}^{0} g(x)}. \quad (3.4)
\]

Note that by [9, Proposition 2.4], for \( xy > 0 \),

\[
G_{Y}^{0}(x, y) \geq H(|x| \wedge |y|),
\]

so we get that for \( x > 0 \),

\[
G_{Y}^{0} g(x) \geq \int_{0}^{\infty} H(x \wedge y) g(y) a(y)^{-1} dy \geq H(x) \int_{x}^{\infty} g(y) a(y)^{-1} dy. \quad (3.5)
\]

Since \( g \in \mathcal{H} \), there exists \( x_{0} > 0 \) such that \( g(x) = g(x \wedge x_{0} \vee (-x_{0})) \). Now by choosing \( g(x) = H(x \wedge x_{0} \vee (-x_{0})) \), we obtain that for \( x > 0 \),

\[
\frac{G_{Y}^{0} g(x)}{g(x)} = \frac{G_{Y}^{0} g(x \wedge x_{0})}{g(x \wedge x_{0})} \geq H(x_{0}) \int_{x_{0}}^{\infty} a(y)^{-1} dy. \quad (3.6)
\]

Similarly, for \( x < 0 \),

\[
\frac{G_{Y}^{0} g(x)}{g(x)} = \frac{G_{Y}^{0} g(x \vee (-x_{0}))}{g(x \vee (-x_{0}))} \geq H(-x_{0}) \int_{-\infty}^{-x_{0}} a(y)^{-1} dy. \quad (3.7)
\]

Therefore, by combining (3.4), (3.6), and (3.7), we obtain the estimate for upper bound.

Next, we consider the lower bound. Let \( G_{Y}^{0} \) be the Green operator of \( Y \) killed upon \( \mathbb{R} \setminus \{0\} \). According to [9, Proposition 2.3 and 2.4],

\[
G_{Y}^{0}(x, y) = H(x) + H(y) - H(y - x) \leq 2(H(x) \wedge H(y)). \quad (3.8)
\]

By the property of time change and (3.8), for any \( f \) with \( G_{Y}^{0}|f| < \infty \),

\[
G_{Y}^{0} f(x) \leq \int_{\mathbb{R}} 2(H(y) \wedge H(x)) f(y) a(y)^{-1} dy. \quad (3.9)
\]
Since $\psi(t)/t \to \infty$ as $t \to \infty$, by Dirichlet criterion, $\int_0^\infty t \sin(xt)\psi(t)^{-1}dt$ is integrable. Therefore, $H$ is differentiable, and
\[
H'(x) = \frac{1}{\pi} \int_0^\infty \frac{t \sin xt}{\psi(t)} dt < \infty.
\]
Recalling that $p_t(x) = (2\pi)^{-1} \int_\mathbb{R} e^{-t\psi(\xi)} - i\xi x \, d\xi$ and $\psi(\xi) = \psi(-\xi)$. Since
\[
\int_\mathbb{R} e^{-t\psi(\xi)} d\xi = p_t(0) = p_t(x,x) < \infty,
\]
e$^{-t\psi} \in L^1(dx)$. By [8, Theorem 1.1], for $|\eta| = r$,
\[
-\frac{1}{2\pi r} \frac{d}{dr} p_t(r) = \int_\mathbb{R}^3 e^{-t\psi(|x|)} e^{-2\pi i x \cdot \eta} \, dx = \frac{1}{2\pi} q_t(\eta),
\]
where $q_t(\eta)$ is the transition density of a 3-dimensional Lévy process with characteristic exponent $\Psi(\eta) = \psi(|\eta|)$. Therefore, $q_t(\eta) \geq 0$, $p_t(x)$ is non-increasing for $x$, i.e. it is unimodal (see [9]). Thus by [9], $H(x)$ is non-increasing on $(0, \infty)$ and non-decreasing on $(-\infty, 0)$, so we have
\[
G_0^0 f(x) \leq 2 \left( \int_{\mathbb{R}\setminus(-|x|,|x|)} H(x)f(y) \mu(dy) + \int_{-|x|}^{x} H(y)f(y) \mu(dy) \right)
\]
\[
= 2 \int_{0}^{x} H'(z) \left( \int_{\mathbb{R}\setminus(-z,z)} f(y) \mu(dy) \right) dz.
\]
Choose $f(x) = \sqrt{H(x)}$. By using integration by parts, for any $y > 0$,
\[
\int_{\mathbb{R}\setminus(-y,y)} \sqrt{H(z)} \mu(dz) \leq \sqrt{H(y)} \mu((-y,y)^c)
\]
\[
+ \int_{y}^{\infty} \frac{H'(z) \mu((-z,z)^c)}{2\sqrt{H(z)}} \, dz.
\]
Since $\delta < \infty$, then
\[
\int_{\mathbb{R}\setminus(-y,y)} \sqrt{H(z)} \mu(dz) \leq \frac{\delta}{\sqrt{H(y)}} + \delta \int_{y}^{\infty} \frac{H(z)^{-3/2} dH(z)}{\sqrt{H(y)}} = \frac{2\delta}{\sqrt{H(y)}}.
\]
Thus
\[
\frac{G_0^0 \sqrt{H}}{\sqrt{H}}(x) \leq \frac{2}{\sqrt{H(x)}} \int_{0}^{x} H'(z) \frac{2\delta}{\sqrt{H(z)}} \, dz = 8\delta.
\]
Now by Lemma 3.1 and letting $f = \sqrt{H}$, we obtain that if $\delta < \infty$, then
\[
\lambda_0 \geq \inf_{x \neq 0} \frac{\sqrt{H(x)}}{G_0^0 \sqrt{H(x)}} \geq \frac{1}{8\delta}.
\]

\textbf{Proof of Theorem 1.1}
(1) By [2, Proposition 3.2], $\lambda_1 \geq \lambda_0$. Hence
\[ \lambda_1 \geq \lambda_0 \geq \frac{1}{8\delta}, \]
and $Y$ is exponential ergodicity.

(2) Specially, by choosing $f \equiv 1$ in (3.9), we have
\[ M_0 := \sup_x \mathbb{E}_x \tau_0^Y = \sup_x G_0^1(x) \leq \int_{\mathbb{R}} 2H(y)a(y)^{-1}dy < \infty, \]
thus by [12, Theorem 1.2(R2)], $\kappa \geq M_0^{-1} \geq (2I)^{-1} > 0$. \hfill \Box

Proof of Corollary 1.3

First, according to [9, Lemma 2.14], $H(x) \approx (|x|\psi(1/x))^{-1}$, thus by using Theorem 1.1, we obtain the exponential ergodicity and strong ergodicity.

By [9, Lemma 2.14 and (12)],
\[ H(x) \leq \frac{10}{\pi \beta} \int_{1/x}^{\infty} \frac{1}{\psi^*(s)} ds \leq \frac{10}{\pi \beta^2 (\alpha - 1) x \psi^*(1/x)}, \text{ for } x > 0. \]

Then the estimates for $\lambda_1$ and $\kappa$ follow from Theorem 1.1. \hfill \Box

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