THE K-RANK NUMERICAL RADII

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ABSTRACT. The $k$-rank numerical range $\Lambda_k(A)$ is expressed via an intersection of any countable family of numerical ranges $\{F(M^*_\nu AM_\nu)\}_{\nu \in \mathbb{N}}$ with respect to $n \times (n - k + 1)$ isometries $M_\nu$. This implication for $\Lambda_k(A)$ provides further elaboration of the $k$-rank numerical radii of $A$.

1. Introduction

Let $\mathcal{M}_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices and $k \geq 1$ be a positive integer. The $k$-rank numerical range $\Lambda_k(A)$ of a matrix $A \in \mathcal{M}_n$ is defined by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : X^*AX = \lambda I_k \text{ for some } X \in \mathcal{X}_k\} = \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some } P \in \mathcal{Y}_k\},$$

where $\mathcal{X}_k = \{X \in \mathcal{M}_{n,k} : X^*X = I_k\}$ and $\mathcal{Y}_k = \{P \in \mathcal{M}_n : P = XX^*, X \in \mathcal{X}_k\}$. Note that $\Lambda_k(A)$ has been introduced as a versatile tool to solving a fundamental error correction problem in quantum computing [3, 4, 6, 7, 9].

For $k = 1$, $\Lambda_1(A)$ reduces to the classical numerical range of a matrix $A$,$$
\Lambda_1(A) \equiv F(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\},$$which is known to be a compact and convex subset of $\mathbb{C}$ [5], as well as the same properties hold for the set $\Lambda_k(A)$, for $k > 1$ [7, 9]. Associated with $\Lambda_k(A)$ are the $k$-rank numerical radius $r_k(A)$ and the inner $k$-rank numerical radius $\tilde{r}_k(A)$, defined respectively, by

$$r_k(A) = \max \{|z| : z \in \partial \Lambda_k(A)\} \quad \text{and} \quad \tilde{r}_k(A) = \min \{|z| : z \in \partial \Lambda_k(A)\}.$$For $k = 1$, they yield the numerical radius and the inner numerical radius,$$r(A) = \max \{|z| : z \in \partial F(A)\} \quad \text{and} \quad \tilde{r}(A) = \min \{|z| : z \in \partial F(A)\},$$respectively.

In the first section of this paper, $\Lambda_k(A)$ is proved to coincide with an indefinite intersection of numerical ranges of all the compressions of $A \in \mathcal{M}_n$ to $(n - k + 1)$-dimensional subspaces, which has been also used in [3, 4]. Further elaboration led us to reformulate $\Lambda_k(A)$ in terms of an intersection of a countable family of numerical ranges. This result provides additional characterizations of $r_k(A)$ and $\tilde{r}_k(A)$, which are presented in section 3.

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2. ALTERNATIVE EXPRESSIONS OF $\Lambda_k(A)$

Initially, the higher rank numerical range $\Lambda_k(A)$ is proved to be equal to an infinite intersection of numerical ranges.

**Theorem 2.1.** Let $A \in \mathcal{M}_n(\mathbb{C})$. Then

$$\Lambda_k(A) = \bigcap_{M \in \mathcal{X}_{n-k+1}} F(M^*AM) = \bigcap_{P \in \mathcal{Y}_{n-k+1}} F(PAP).$$

**Proof.** Denoting by $\lambda_1(H) \geq \ldots \geq \lambda_n(H)$ the decreasingly ordered eigenvalues of a hermitian matrix $H \in \mathcal{M}_n(\mathbb{C})$, we have [7]

$$\Lambda_k(A) = \bigcap_{\theta \in [0, 2\pi)} e^{-i\theta} \{ z \in \mathbb{C} : \text{Re} z \leq \lambda_k(H(e^{i\theta}A)) \}$$

where $H(\cdot)$ is the hermitian part of a matrix. Moreover, by Courant-Fisher theorem, we have

$$\lambda_k(H(e^{i\theta}A)) = \min_M \max_{x \in \mathcal{S}} \frac{\mathbb{E}x^*H(e^{i\theta}A)x}{\|x\|^2}.$$

Denoting by $\mathcal{S} = \text{span}\{u_1, \ldots, u_{n-k+1}\}$, where $u_i \in \mathbb{C}^n$, $i = 1, \ldots, n - k + 1$ are orthonormal vectors, then any unit vector $x \in \mathcal{S}$ is written in the form $x = My$, where $M = [u_1 \cdots u_{n-k+1}] \in \mathcal{X}_{n-k+1}$ and $y \in \mathbb{C}^{n-k+1}$ is unit. Hence, we have

$$\lambda_k(H(e^{i\theta}A)) = \min_M \max_{y \in \mathbb{C}^{n-k+1}} \frac{y^*M^*H(e^{i\theta}A)My}{\|y\|^2} = \min_M \max_{y \in \mathbb{C}^{n-k+1}} y^*H(e^{i\theta}M^*AM)y \|y\|^2 = \min_M \lambda_1(H(e^{i\theta}M^*AM))$$

and consequently

$$\Lambda_k(A) = \bigcap_{\theta} e^{-i\theta} \{ z \in \mathbb{C} : \text{Re} z \leq \min_M \lambda_1(H(e^{i\theta}M^*AM)) \}$$

$$= \bigcap_{M} \bigcap_{\theta} e^{-i\theta} \{ z \in \mathbb{C} : \text{Re} z \leq \lambda_1(H(e^{i\theta}M^*AM)) \}$$

$$= \bigcap_{M \in \mathcal{X}_{n-k+1}} F(M^*AM).$$

Moreover, if we consider the $(n - k + 1)$-rank orthogonal projection $P = MM^*$ of $\mathbb{C}^n$ onto the aforementioned space $\mathcal{S}$, then $x = Px$, for $x \in \mathcal{S}$ and $P\hat{x} = 0$, for $\hat{x} \notin \mathcal{S}$. Hence, we have

$$\Lambda_k(A) = \bigcap_{P \in \mathcal{Y}_{n-k+1}} F(PAP).$$

$\square$
At this point, we should note that Theorem 2.1 provides a different and independent characterization of \( \Lambda_k(A) \) than the one given in [6, Cor. 4.9]. We focus on the expression of \( \Lambda_k(A) \) via the numerical ranges \( F(M^*AM) \) (or \( F(PAP) \)), since it represents a more useful and advantageous procedure to determine and approximate the boundary of \( \Lambda_k(A) \) numerically.

In addition, Theorem 2.1 verifies the “convexity of \( \Lambda_k(A) \)” through the convexity of the numerical ranges \( F(M^*AM) \) (or \( F(PAP) \)), which is ensured by the Toeplitz-Hausdorff theorem. A different way of indicating that \( \Lambda_k(A) \) is convex, is developed in [9]. For \( k = n \), clearly \( \Lambda_n(A) = \bigcap_{\nu \in \mathbb{C}^n, \|x\| = 1} F(x^*Ax) \) and should be \( \Lambda_n(A) \neq \emptyset \) precisely when \( A \) is scalar.

Motivated by the above, we present the main result of our paper, redescribing the higher rank numerical range as a countable intersection of numerical ranges.

**Theorem 2.2.** Let \( A \in M_n \). Then for any countable family of orthogonal projections \( \{P_\nu : \nu \in \mathbb{N}\} \subseteq \mathcal{Y}_{n-k+1} \) (or any family of isometries \( \{M_\nu : \nu \in \mathbb{N}\} \subseteq \mathcal{X}_{n-k+1} \)) we have

\[
\Lambda_k(A) = \bigcap_{\nu \in \mathbb{N}} F(P_\nu AP_\nu) = \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu). \tag{2.1}
\]

**Proof.** By Theorem 2.1, we have

\[
[\Lambda_k(A)]^c = \mathbb{C} \setminus \Lambda_k(A) = \bigcup_{P \in \mathcal{Y}_{n-k+1}} [F(PAP)]^c,
\]

whereupon the family \( \{F(PAP)^c : P \in \mathcal{Y}_{n-k+1}\} \) is an open cover of \( [\Lambda_k(A)]^c \).

Moreover, \( [\Lambda_k(A)]^c \) is separable, as an open subset of the separable space \( \mathbb{C} \) and then \( [\Lambda_k(A)]^c \) has a countable base [8], which obviously depends on the matrix \( A \).

This fact guarantees that any open cover of \( [\Lambda_k(A)]^c \) admits a countable subcover, leading to the relation

\[
[\Lambda_k(A)]^c = \bigcup_{\nu \in \mathbb{N}} [F(P_\nu AP_\nu)]^c,
\]

i.e. leading to the first equality in (2.1). Taking into consideration that there exists a countable dense subset \( \mathcal{J} \subseteq \mathcal{Y}_{n-k+1} \) with respect to the operator norm \( \|\cdot\| \) and \( P_\nu \in \mathcal{Y}_{n-k+1} \), for \( \nu \in \mathbb{N} \), clearly, \( \bigcap_{\nu \in \mathbb{N}} F(P_\nu AP_\nu) = \bigcap_{\nu \in \mathbb{N}, P_\nu \in \mathcal{J}} F(P_\nu AP_\nu). \)

That is in (2.1), the family of orthogonal projections \( \{P_\nu : \nu \in \mathbb{N}\} \) can be chosen independently of \( A \). Moreover, due to \( P_\nu = M_\nu M_\nu^* \), with \( M_\nu \in \mathcal{X}_{n-k+1} \), we derive the second equality in (2.1). \( \square \)

For a construction of a countable family of isometries \( \{M_\nu : \nu \in \mathbb{N}\} \subseteq \mathcal{X}_{n-k+1} \), see also in the Appendix.

Furthermore, using the dual “max-min” expression of the \( k \)-th eigenvalue,

\[
\lambda_k(H(e^{i\theta}A)) = \max_{\dim \mathcal{V} = k} \min_{x \in \mathcal{V}, \|x\| = 1} x^*H(e^{i\theta}A)x = \max_N \lambda_{\min}(H(e^{i\theta}N^*AN)),
\]

\( \lambda_{\min}(H(e^{i\theta}N^*AN)) \) represents the smallest eigenvalue of the Hermitian matrix \( H(e^{i\theta}N^*AN) \), where \( \mathcal{V} \) is a \( k \)-dimensional subspace of \( \mathbb{C}^n \).
where \( N \in \mathcal{X}_k \), we have
\[
\Lambda_k(A) = \bigcap_{\theta} e^{-i\theta} \{ z \in \mathbb{C} : \text{Re} z \leq \max_{N} \lambda_k(H(e^{i\theta} N^* A N)) \}
\]
\[
= \bigcup_{N} \bigcap_{\theta} e^{-i\theta} \{ z \in \mathbb{C} : \text{Re} z \leq \lambda_k(H(e^{i\theta} N^* A N)) \}
\]
\[
= \bigcup_{N \in \mathcal{X}_k} \Lambda_k(N^* A N), \quad (2.2)
\]
and due to the convexity of \( \Lambda_k(A) \), we establish
\[
\Lambda_k(A) = \co \bigcup_{N \in \mathcal{X}_k} \Lambda_k(N^* A N), \quad (2.3)
\]
where \( \co(\cdot) \) denotes the convex hull of a set. Apparently, \( \Lambda_k(N^* A N) \neq \emptyset \) if and only if \( N^* A N = \lambda I_k \) \([6]\) and then (2.3) is reduced to \( \bigcup_{N} \Lambda_k(N^* A N) = \bigcup_{N} \{ \lambda : N^* AN = \lambda I_k \} = \Lambda_k(A) \), where \( N \) runs all \( n \times k \) isometries.

In spite of Theorem 2.2, \( \Lambda_k(A) \) cannot be described as a countable union in (2.2), because if
\[
\Lambda_k(A) = \bigcup_{\nu \in \mathbb{N}} \{ \Lambda_k(N^*_\nu A N_\nu) : N_\nu \in \mathcal{X}_k \} = \bigcup_{\nu \in \mathbb{N}} \{ \lambda_\nu : N^*_\nu A N_\nu = \lambda_\nu I_k, N_\nu \in \mathcal{X}_k \},
\]
then \( \Lambda_k(A) \) should be a countable set, which is not true.

3. Properties of \( r_k(A) \) and \( \tilde{r}_k(A) \)

In this section, we characterize the \( k \)-rank numerical radius \( r_k(A) \) and the inner \( k \)-rank numerical radius \( \tilde{r}_k(A) \). Motivated by Theorem 2.2, we present the next two results.

**Theorem 3.1.** Let \( A \in \mathcal{M}_n \) and \( J_\nu(A) = \bigcap_{p=1}^{\nu} F(M^*_p A M_p) \), where \( M_p \in \mathcal{X}_{n-p+1} \). Then
\[
\begin{align*}
 r_k(A) &= \lim_{\nu \to \infty} \sup_{z \in \mathcal{J}_\nu(A)} |z| = \inf_{\nu \in \mathbb{N}} \sup_{z \in \mathcal{J}_\nu(A)} |z| = \inf_{\nu \in \mathbb{N}} \sup_{z \in \mathcal{J}_\nu(A)} |z| = \inf_{\nu \in \mathbb{N}} \sup_{z \in \mathcal{J}_\nu(A)} |z|.
\end{align*}
\]

**Proof.** By Theorem 2.2, we have
\[
\begin{align*}
\Lambda_k(A) &= \bigcap_{\nu=1}^{\infty} J_\nu(A) \subseteq J_\nu(A) \subseteq F(A) \subseteq \mathcal{D}(0, \| A \|_2), \quad (3.1)
\end{align*}
\]
for all \( \nu \in \mathbb{N} \), where the sequence \( \{ J_\nu(A) \}_{\nu \in \mathbb{N}} \) is nonincreasing and \( \mathcal{D}(0, \| A \|_2) \) is the circular disc centered at the origin with radius the spectral norm \( \| A \|_2 \) of \( A \in \mathcal{M}_n \). Clearly,
\[
r_k(A) = \max_{z \in \bigcap_{\nu=1}^{\infty} J_\nu(A)} |z| \leq \sup_{z \in J_\nu(A)} |z| \leq r(A) \leq \| A \|_2,
\]
then the nonincreasing and bounded sequence \( q_\nu = \sup \{ |z| : z \in J_\nu(A) \} \) converges. Therefore
\[
r_k(A) \leq \lim_{\nu \to \infty} q_\nu = q_0 \leq q_\nu.
\]
We shall prove that the above inequality is actually an equality. Assume that \( r_k(A) < q_0 \). In this case, there is \( \varepsilon > 0 \), where \( r_k(A) + \varepsilon < q_0 \leq q_\nu \) for all
\(\nu \in \mathbb{N}\). Then we may find a sequence \(\{\zeta_{\nu}\} \subseteq \mathcal{J}_{\nu}(A)\) such that \(q_0 \leq |\zeta_{\nu}|\) for all \(\nu \in \mathbb{N}\). Due to the boundedness of the set \(\mathcal{J}_{\nu}(A)\), the sequence \(\{\zeta_{\nu}\}\) contains a subsequence \(\{\zeta_{\nu_{\rho}}\}\) converging to \(\zeta_0 \in \mathbb{C}\) and clearly, we obtain \(q_0 \leq |\zeta_0|\). Because of the monotonicity of \(\mathcal{J}_{\nu}(A)\) (i.e. \(\mathcal{J}_{\nu+1}(A) \subseteq \mathcal{J}_{\nu}(A)\)), \(\zeta_{\nu}\) eventually belong to \(\mathcal{J}_{\nu}(A)\), \(\forall \nu \in \mathbb{N}\), meaning that \(\{\zeta_{\nu}\} \subseteq \bigcap_{\nu=1}^{\infty} \mathcal{J}_{\nu}(A) = \Lambda_k(A)\) and since \(\Lambda_k(A)\) is closed, \(\zeta_0 \in \Lambda_k(A)\). It implies \(|\zeta_0| \leq r_k(A)\) and then \(q_0 \leq r_k(A)\), a contradiction.

The second equality is apparent. \(\square\)

**Theorem 3.2.** Let \(A \in \mathcal{M}_n\) and \(\mathcal{J}_{\nu}(A) = \bigcap_{\nu=1}^{\infty} F(M_p^*AM_p)\), for some \(M_p \in \mathcal{X}_{n-k+1}\). If \(0 \notin \Lambda_k(A)\), then

\[
\tilde{r}_k(A) = \liminf_{\nu \to \infty} \{|z| : z \in \mathcal{J}_{\nu}(A)\} = \sup_{\nu \in \mathbb{N}} \inf_{\nu \in \mathbb{N}} \{|z| : z \in \mathcal{J}_{\nu}(A)\}.
\]

**Proof.** Obviously, \(0 \notin \Lambda_k(A)\) indicates \(\tilde{r}_k(A) = \min\{|z| : z \in \Lambda_k(A)\}\) and by the relation (3.1), it is clear that

\[
\|A\|_2 \geq r(A) \geq \tilde{r}_k(A) = \min_{z \in \bigcap_{\nu=1}^{\infty} \mathcal{J}_{\nu}(A)} |z| \geq \inf_{z \in \mathcal{J}_{\nu}(A)} |z|.
\]

Consequently, the sequence \(t_{\nu} = \inf\{|z| : z \in \mathcal{J}_{\nu}(A)\}, \nu \in \mathbb{N}\), is nondecreasing and bounded and we have

\[
\tilde{r}_k(A) \geq \lim_{\nu \to \infty} t_{\nu} = t_0.
\]

In a similar way as in Theorem 3.1, we will show that \(\tilde{r}_k(A) = \lim_{\nu \to \infty} t_{\nu}\). Suppose \(\tilde{r}_k(A) > t_0\), then \(t_{\nu} \leq t_0 < \tilde{r}_k(A) - \varepsilon, \forall \nu \in \mathbb{N}\) and \(\varepsilon > 0\). Considering the sequence \(\{\zeta_{\nu}\} \subseteq \mathcal{J}_{\nu}(A)\) such that \(|\zeta_{\nu}| \leq t_0\), let its subsequence \(\{\tilde{\zeta}_{s_{\nu}}\}\) converging to \(\tilde{\zeta}_0\), with \(|\tilde{\zeta}_0| = t_0\). Since \(\{\mathcal{J}_{\nu}(A)\}\) is nonincreasing, \(\tilde{\zeta}_{s_{\nu}}\) eventually belong to \(\mathcal{J}_{\nu}(A)\), \(\forall \nu \in \mathbb{N}\), establishing \(\{\tilde{\zeta}_{s_{\nu}}\} \subseteq \bigcap_{\nu \in \mathbb{N}} \mathcal{J}_{\nu}(A) = \Lambda_k(A)\). Hence, we conclude \(\tilde{\zeta}_0 \in \bigcap_{\nu=1}^{\infty} \mathcal{J}_{\nu}(A) = \Lambda_k(A)\), i.e. \(t_0 \geq |\tilde{\zeta}_0| \geq \tilde{r}_k(A)\), absurd.

The second equality is trivial. \(\square\)

The next proposition asserts a lower and an upper bound for \(r_k(A)\) and \(\tilde{r}_k(A)\), respectively.

**Proposition 3.3.** Let \(A \in \mathcal{M}_n\) and \(M_p \in \mathcal{X}_{n-k+1}, p \in \mathbb{N}\), then

\[
r_k(A) \leq \inf_{p \in \mathbb{N}} r(M_p^*A M_p).
\]

If \(0 \notin \Lambda_k(A)\), then

\[
\tilde{r}_k(A) \geq \inf_{p \in \mathbb{N}} \tilde{r}(M_p^*A M_p).
\]

**Proof.** By Theorem 2.2, we obtain \(\partial \Lambda_k(A) \subseteq \Lambda_k(A) \subseteq F(M_p^*A M_p)\) for all \(p \in \mathbb{N}\). Then

\[
r_k(A) = \max\{|z| : z \in \Lambda_k(A)\} \leq \max\{|z| : z \in F(M_p^*A M_p)\} = r(M_p^*A M_p).
\]

Denoting by \(c(M_p^*A M_p) = \min\{|z| : z \in F(M_p^*A M_p)\}\) for all \(p \in \mathbb{N}\), we have

\[
\tilde{r}_k(A) \geq \min\{|z| : z \in \Lambda_k(A)\} \geq c(M_p^*A M_p).
\]
Since $0 \leq c(M^*_p AM_p) \leq \tilde{r}(M^*_p AM_p) \leq r(M^*_p AM_p) \leq \|A\|_2$ for any $p \in \mathbb{N}$, immediately, we obtain

$$r_k(A) \leq \inf_{p \in \mathbb{N}} r(M^*_p AM_p) \quad \text{and} \quad \tilde{r}_k(A) \geq \sup_{p \in \mathbb{N}} c(M^*_p AM_p).$$

If $0 \notin \Lambda_k(A)$, then by Theorem 2.2, $0 \notin \mathcal{F}(M^*_l AM_l)$ for some $l \in \mathbb{N}$, $M_l \in \mathcal{X}_{n-k+1}$ and $c(M^*_l AM_l) = \tilde{r}(M^*_l AM_l)$. Hence

$$\tilde{r}_k(A) \geq \sup_{p \in \mathbb{N}} c(M^*_p AM_p) \geq \tilde{r}(M^*_l AM_l) \geq \inf_{p \in \mathbb{N}} \tilde{r}(M^*_p AM_p).$$

□

The numerical radius function $r(\cdot) : \mathcal{M}_n \to \mathbb{R}_+$ is not a matrix norm, nevertheless, it satisfies the power inequality $r(A^m) \leq [r(A)]^m$, for all positive integers $m$, which is utilized for stability issues of several iterative methods [2, 5]. On the other hand, the $k$-rank numerical radius fails to satisfy the power inequality, as the next counterexample reveals.

**Example 3.4.** Let the matrix $A = \begin{bmatrix} 1.8 & 2 & 3 & 4 \\ 0 & 0.8+i & 0 & i \\ -2 & 1 & -1.2 & 1 \\ 0 & 0 & 1 & 0.8 \end{bmatrix}$. Using Theorems 2.1 and 2.2, the set $\Lambda_2(A)$ is illustrated in the left part of Figure 1 by the uncovered area inside the figure. Clearly, it is included in the unit circular disc, which indicates that $r_2(A) < 1$. On the other hand, the set $\Lambda_2(A^2)$, illustrated in the right part of Figure 1 with the same manner, is not bounded by the unit circle and thus $r_2(A^2) > 1$. Obviously, $[r_2(A)]^2 < 1 < r_2(A^2)$.

![Figure 1](image-url)

**Figure 1.** The “white” bounded areas inside the figures depict the sets $\Lambda_2(A)$ (left) and $\Lambda_2(A^2)$ (right).

The results developed in this paper draw attention to the rank-$k$ numerical range $\Lambda_k(L(\lambda))$ of a matrix polynomial $L(\lambda) = \sum_{i=0}^m A_i \lambda^i$ ($A_i \in \mathcal{M}_n$), which has been extensively studied in [3, 4]. It is worth noting that Theorem 2.2 can be also generalized in the case of $L(\lambda)$, which follows readily from the proof. Hence, the rank-$k$ numerical radii of $\Lambda_k(L(\lambda))$ can be elaborated with the same spirit as here [1].
Following we provide another construction of a family of $n \times (n-k+1)$ isometries \( \{M_\nu : \nu \in \mathbb{N}\} \) presented in Theorem 2.2.

\textbf{Proof.} By Theorem 2.1, we have

\[
\Lambda_k(A) = \bigcap_{M \in \mathcal{X}_{n-k+1}} F(M^*AM), \tag{A.1}
\]

which is known to be a compact and convex subset of \( \mathbb{C} \). For any \( n \times (n-k+1) \) isometry \( M_\nu \ (\nu \in \mathbb{N}) \), we have

\[
\Lambda_k(A) \subseteq \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu). \tag{A.2}
\]

In order to prove equality in the relation (A.2), we distinguish two cases for the interior of \( \Lambda_k(A) \).

Suppose first that \( \text{int}\Lambda_k(A) \neq \emptyset \). Then by (A.2), we obtain

\[
\emptyset \neq \text{int}\Lambda_k(A) \subseteq \text{int} \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu)
\]

and since \( \bigcap_{\nu} F(M_\nu^*AM_\nu) \) is convex and closed, we establish

\[
\text{int} \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu) = \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu), \tag{A.3}
\]

where \( \overline{\cdot} \) denotes the closure of a set. Thus, combining the relations (A.2) and (A.3), we have

\[
\Lambda_k(A) \subseteq \text{int} \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu). \tag{A.4}
\]

Further, we claim that \( \text{int} \bigcap_{\nu} F(M_\nu^*AM_\nu) \subseteq \Lambda_k(A) \). Assume on the contrary that \( z_0 \in \text{int} \bigcap_{\nu} F(M_\nu^*AM_\nu) \) but \( z_0 \notin \Lambda_k(A) \), then there exists an open neighborhood \( \mathcal{B}(z_0, \varepsilon) \), with \( \varepsilon > 0 \), such that

\[
\mathcal{B}(z_0, \varepsilon) \subseteq \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu) \text{ and } \mathcal{B}(z_0, \varepsilon) \cap \Lambda_k(A) = \emptyset.
\]

Then, the set \( [\Lambda_k(A)]^c = \mathbb{C} \setminus \Lambda_k(A) \) is separable, as an open subset of the separable space \( \mathbb{C} \) and let \( \mathcal{Z} \) be a countable dense subset of \( [\Lambda_k(A)]^c \) [8]. Therefore, there exists a sequence \( \{z_p : p \in \mathbb{N}\} \) in \( \mathcal{Z} \) such that \( \lim_{p \to \infty} z_p = z_0 \) and \( z_p \in \mathcal{B}(z_0, \varepsilon) \). Moreover, \( z_p \in [\Lambda_k(A)]^c \) and by (A.1), it follows that for any \( p \) correspond indices \( j_p \in \mathbb{N} \) such that \( z_p \notin F(M_{j_p}^*AM_{j_p}) \). Thus \( z_p \notin \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu) \), which is absurd, since \( z_p \in \mathcal{B}(z_0, \varepsilon) \cap \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu) \). Hence \( z_0 \in \Lambda_k(A) \), verifying our claim and we obtain

\[
\text{int} \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu) \subseteq \Lambda_k(A) = \text{int}\Lambda_k(A). \tag{A.5}
\]

By (A.3), (A.4) and (A.5), the required equality is asserted.

Consider now that \( \Lambda_k(A) \) has no interior points, namely, it is a line segment or a singleton. Then there is a suitable affine subspace \( \mathcal{V} \) of \( \mathbb{C} \) such that \( \Lambda_k(A) \subseteq \mathcal{V} \) and with respect to the subspace topology, we have \( \text{int}\Lambda_k(A) \neq \emptyset \) and \( \mathcal{V} \setminus \Lambda_k(A) \)
be separable. Following the same arguments as above, let \( \tilde{Z} \) be a countable dense subset of \( \mathcal{V} \setminus \Lambda_k(A) \). Hence, there is a sequence \( \{ \tilde{z}_q : q \in \mathbb{N} \} \) in \( \tilde{Z} \) converging to \( z_0 \) and \( \tilde{z}_q \in \mathcal{B}(z_0, \varepsilon) \subset \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu) \). On the other hand, by (A.1), we have \( \tilde{z}_q \notin \bigcap_{q \in \mathbb{N}} F(M_{i_q}^*AM_{i_q}) \) for some indices \( i_q \in \mathbb{N} \). Clearly, we are led to a contradiction and we deduce \( \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu) \subseteq \Lambda_k(A) \). Hence, with (A.2), we conclude

\[ \Lambda_k(A) = \bigcap_{\nu \in \mathbb{N}} F(M_\nu^*AM_\nu). \]

\[\square\]

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