Shear coordinate description of the quantized versal unfolding of a $D_4$ singularity

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Abstract

In this communication, by using Teichmüller theory of a sphere with four holes/orbifold points, we obtain a system of flat coordinates on the general affine cubic surface having a $D_4$ singularity at the origin. We show that the Goldman bracket on the geodesic functions on the four-holed/orbifold sphere coincides with the Etingof–Ginzburg Poisson bracket on the affine $D_4$ cubic. We prove that this bracket is the image under the Riemann–Hilbert map of the Poisson–Lie bracket on $\mathfrak{sl}^\ast(2, \mathbb{C})$. We realize the action of the mapping class group by the action of the braid group on the geodesic functions. This action coincides with the procedure of analytic continuation of solutions of the sixth Painlevé equation. Finally, we produce the explicit quantization of the Goldman bracket on the geodesic functions on the four-holed/orbifold sphere and of the braid group action.

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1. Introduction

The main subject studied in this communication is the following irreducible affine cubic $\tilde{\phi} \in \mathbb{C}[u, v, w]$ having a simple $D_4$ singularity at the origin:

$$\tilde{\phi} = u^2 + v^2 + w^2 - uvw + r_1 u + r_2 v + r_3 w + r_4,$$

where $r_1, r_2, r_3, r_4$ are four complex parameters. It was proved in [7] that the following formulae define a Poisson bracket on $\mathbb{C}[u, v, w]$:

$$\{u, v\} = \frac{\partial \tilde{\phi}}{\partial w}, \quad \{v, w\} = \frac{\partial \tilde{\phi}}{\partial u}, \quad \{w, u\} = \frac{\partial \tilde{\phi}}{\partial v},$$

(1.1)
and \( \tilde{\phi} \) itself is a central element for this bracket, so that the quotient space

\[
\mathcal{M}_{\tilde{\phi}} := \mathbb{C}[u, v, w] / \langle \tilde{\phi} = 0 \rangle
\]

inherits the Poisson algebra structure. Note that \( \mathcal{M}_{\tilde{\phi}} \) is the manifold of the monodromy data of the sixth Painlevé equation [15].

In this communication we define an analytic surjective map

\[
\mu : \mathbb{C}^3 / \langle y_1 + y_2 + y_3 = \text{const.} \rangle \to \mathcal{M}_{\tilde{\phi}},
\]

giving rise to a system of flat coordinates \( Y_1, Y_2, Y_3 \) for the affine irreducible cubic surface \( \mathcal{M}_{\tilde{\phi}} \).

To achieve this, we represent a real slice \( \mathcal{M}_{\tilde{\phi}} \cap \{(u, v, w) \in \mathbb{R}^3, u, v, w > 2\} \) of \( \mathcal{M}_{\tilde{\phi}} \) in terms of geodesic functions (i.e. functions of the lengths of closed geodesic curves) on a four-holed/orbifold sphere in the Poincaré uniformization. These geodesic functions are merely finite Laurent polynomials of exponentials of the shear coordinates \( Y_1, Y_2, Y_3 \) introduced by Penner and Thurston, and they simultaneously satisfy skein relations and the Goldman–Poisson relations. In particular we prove that in the case of a four-holed/orbifold sphere the Goldman bracket coincides with (1.1). Despite the fact that this geometric interpretation is only valid for real-valued coordinates \( Y_1, Y_2, Y_3 \), resulting in real \( u, v, w > 2 \), the map \( \mu \) is analytic and can be extended to any \( Y_1, Y_2, Y_3 \in \mathbb{C} \).

We present the braid group action both on the level of geodesic functions and on the level of shear coordinates and provide its quantum version in terms of the quantum geodesic functions. The Poisson brackets are constants on the space of shear coordinates so their quantization is straightforward and gives rise to the quantum commutation relations between the quantum geodesic functions.

We stress that our quantization procedure can easily be extended to all of \( \mathbb{C}^3 / \langle y_1 + y_2 + y_3 = \text{const.} \rangle \), thus providing an explicit and natural quantization of the affine cubic surface \( \mathcal{M}_{\tilde{\phi}} \).

Finally we prove that the Poisson bracket (1.1) on the manifold of the monodromy data \( \mathcal{M}_{\tilde{\phi}} \) of the sixth Painlevé equation is the image under the Riemann–Hilbert map of the Poisson–Lie bracket on \( \oplus_{i=1}^{3} \mathfrak{sl}_2(2, \mathbb{C}) \).

2. Geodesic algebras for a sphere with four holes/orbifold points

In this section we compute the Poisson algebra of the geodesic length functions on a sphere with four holes or orbifold points.

We use the fat-graph description of the Teichmüller theory of surfaces developed in [8]. We are going to adapt this description to the case of a sphere \( \Sigma_{0,4-j,j} \) with \( 4-j \) holes and \( j \) orbifold points, \( j = 0-4 \). The holes have perimeters \( P_i, i = 1, \ldots, 4-j \), the orbifold points correspond to the case when the perimeters become imaginary numbers \( P_l = 2\pi i/k_l, l = 1, \ldots, j, k_l \) being the order of the orbifold point (the particular case of orbifold points of order 2 was treated in detail in [3, 4]).

Let us start with the case \( j = 0 \), i.e. a sphere \( \Sigma_{0,4} \) with four holes and no orbifold points. A fat graph associated to a Riemann surface with holes [8, 9] is a spine \( \Gamma_{g,s} \), which is a connected three-valent graph drawn without self-intersections on \( \Sigma_{g,s} \) with a prescribed cyclic ordering of labelled edges entering each vertex; it must be a maximal graph in the sense that its complement on the Riemann surface is a set of disjoint polygons (faces), each polygon containing exactly one hole (and becoming simply connected after gluing this hole).

By the Poincaré uniformization theorem, the Riemann surface \( \Sigma_{g,s} \) of genus \( g \) and with \( s \) holes can be obtained as

\[
\Sigma_{g,s} \sim \mathbb{H} / \Delta_{g,s},
\]
where
\[ \Delta_{g,s} = \langle \gamma_1, \ldots, \gamma_{2g+s-1} \rangle, \quad \gamma_1, \ldots, \gamma_{2g+s-1} \in PSL(2, \mathbb{R}) \]
is the fundamental group of the surface \( \Sigma_{g,s} \), in this case a Fuchsian group containing only hyperbolic elements.

In the Thurston shear coordinate description of the Teichmüller spaces of Riemann surfaces with holes [8], we decompose each hyperbolic matrix \( \gamma \in \Delta_{g,s} \) into a product of the form
\[ \gamma = (-1)^K R^{k_j} X_{Z_i} \ldots R^{k_i} X_{Z_1}, \quad i_j \in I, \quad k_{ij} = 1, 2, \quad K := \sum_{j=1}^p k_{ij}, \quad (2.1) \]
where \( I \) is a set of integer indices and the matrices \( R \) and \( X_{Z_i}, Z_i \in \mathbb{R} \), are defined as follows:
\[ R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_{Z_i} := \begin{pmatrix} 0 & -\exp\left(\frac{Z_i}{2}\right) \\ \exp\left(\frac{Z_i}{2}\right) & 0 \end{pmatrix}. \quad (2.2) \]

The set of closed geodesics on a Riemann surface \( \Sigma_{g,s} \) is in the one-to-one correspondence with conjugacy classes of elements of the Fuchsian group \( \Delta_{g,s} \) with the lengths \( \ell_\gamma \) of these geodesics to be determined as
\[ e^{\ell_\gamma/2} + e^{-\ell_\gamma/2} = \text{tr} \gamma, \]
where we take a trace of the matrix product (2.1). We call the combination \( e^{\ell_\gamma/2} + e^{-\ell_\gamma/2} \) the geodesic function \( G_\gamma \).

The fat graph for \( \Sigma_{0,4} \) has the form of the three-petal graph depicted in the figure shown in (2.3) where we also present the geodesic line corresponding to the element \( G_{1,2} \) (note that we label the six shear coordinates \( Z_\alpha \) by \( Y_1, Y_2, Y_3, P_1, P_2, P_3 \)):

(2.3)

The algebras of geodesic length functions were constructed in [2] by postulating the Poisson relations on the level of the shear coordinates \( Z_\alpha \) of the Teichmüller space:
\[ \{ f(Z), g(Z) \} = \sum_{\text{three-valent vertices } \alpha} \sum_{i=1}^{4g+2s+n-4} 3 \mod 3 \left( \frac{\partial f}{\partial Z_{\alpha_i}} \frac{\partial g}{\partial Z_{\alpha_{i+1}}} - \frac{\partial g}{\partial Z_{\alpha_i}} \frac{\partial f}{\partial Z_{\alpha_{i+1}}} \right), \quad (2.4) \]
where the sum ranges all the (three-valent) vertices of a graph and \( \alpha_i \) are the labels of the cyclically (counterclockwise) ordered \( (\alpha_4 \equiv \alpha_1) \) edges incident to the vertex with the label \( \alpha \).

This bracket gives rise to the Goldman bracket on the space of geodesic length functions [12].

In the case of \( \Sigma_{0,4} \), i.e. the sphere with four holes, we consider the following generators for the Fuchsian group \( \Delta_{0,4} \):

(2.3)
\[ \gamma_i = X_{Y_i} RX_{P_i} RX_{Y_i} = \begin{pmatrix} 0 & -e^{Y_i + \frac{P_i}{4}} \\ e^{-Y_i + \frac{P_i}{4}} & -e^{-P_i/2} - e^{P_i/2} \end{pmatrix} \]

\[ \gamma_2 = -RX_{Y_2} RX_{P_2} RX_{Y_2} L = -R \begin{pmatrix} 0 & -e^{Y_2 + \frac{P_2}{4}} \\ e^{-Y_2 + \frac{P_2}{4}} & -e^{-P_2/2} - e^{P_2/2} \end{pmatrix} L \]

\[ \gamma_3 = -LX_{Y_3} RX_{P_3} RX_{Y_3} R = -L \begin{pmatrix} 0 & -e^{Y_3 + \frac{P_3}{4}} \\ e^{-Y_3 + \frac{P_3}{4}} & -e^{-P_3/2} - e^{P_3/2} \end{pmatrix} R, \]

where \( L = -R^2 \).

**Theorem 2.1.** The Poisson algebra of geodesic functions on the sphere with four holes is generated by the three elements \( G_{1,2}, G_{2,3} \) and \( G_{1,3} \)

\[ G_{i,j} := -\text{Tr}(Y_i Y_j), \quad i < j, i, j = 1, 2, 3, \]

which correspond to closed paths encircling two holes without self-intersections as shown in figure (2.3). Their Poisson brackets are given by the formulae

\[ \{G_{1,2}, G_{2,3}\} = G_{1,2} G_{2,3} - 2G_{1,3} - \omega_{1,3}, \]

\[ \{G_{2,3}, G_{1,3}\} = G_{2,3} G_{1,3} - 2G_{1,2} - \omega_{1,2}, \]

\[ \{G_{1,3}, G_{1,2}\} = G_{1,2} G_{1,3} - 2G_{2,3} - \omega_{2,3}, \]

\[ \text{Formulae (2.6)–(2.8) define an abstract Poisson algebra satisfying the Jacobi relations for any choice of the constants } \omega_{i,j}. \]

\[ C = G_{1,2}^2 + G_{2,3}^2 + G_{1,3}^2 - G_{1,2} G_{2,3} G_{1,3} + G_{1,2} \omega_{1,2} + G_{2,3} \omega_{2,3} + G_{1,3} \omega_{1,3}. \]

**Proof.** For convenience, we perform the change of variable \( \tilde{Y}_i = Y_i - P_i/2, i = 1, 2, 3, \)

after which the matrix combination \( X_{\tilde{Y}_i} RX_{P_i} RX_{\tilde{Y}_i} \), which is the main building block in (2.5) becomes merely

\[ \left( \begin{array}{cc} 0 & -e^{\tilde{Y}_i} \\ e^{-\tilde{Y}_i} & -G_i \end{array} \right), \]

\[ \text{where } G_i = e^{P_i/2} + e^{-P_i/2} \text{ is the trace of the monodromy around the hole. Note that thanks to the shape of the fat graph,} \]

\[ \{Y_i, P_j\} = 0 \quad \text{for } i \neq j, \]

so that

\[ \{\tilde{Y}_i, \tilde{Y}_j\} = \{Y_i, Y_j\}. \]

The explicit forms of \( G_{i,j} \) are then

\[ G_{1,2} = e^{\tilde{Y}_1 + \tilde{Y}_2} + e^{-\tilde{Y}_1 - \tilde{Y}_2} + e^{-\tilde{Y}_1 + \tilde{Y}_2} + G_1 e^{\tilde{Y}_1} + G_2 e^{-\tilde{Y}_1}, \]

\[ G_{2,3} = e^{\tilde{Y}_2 + \tilde{Y}_3} + e^{-\tilde{Y}_2 - \tilde{Y}_3} + e^{-\tilde{Y}_2 + \tilde{Y}_3} + G_2 e^{\tilde{Y}_2} + G_3 e^{-\tilde{Y}_2}, \]

\[ G_{3,1} = e^{\tilde{Y}_3 + \tilde{Y}_1} + e^{-\tilde{Y}_3 - \tilde{Y}_1} + e^{-\tilde{Y}_3 + \tilde{Y}_1} + G_3 e^{\tilde{Y}_3} + G_1 e^{-\tilde{Y}_3}. \]

The Poisson brackets (2.6)–(2.8) can now be proved by brute force computation by applying the Poisson brackets (2.4).
Remark 2.2. As we mentioned in the above theorem, formulae (2.6)–(2.8) define an abstract Poisson algebra, i.e. we can think of $G_{i,j}$ as abstract quantities. If we impose the parametrization (2.11), then it is straightforward to prove that the central element $C$ satisfies the following relation originally from Fricke [19]:

$$C = 4 - G_1 G_2 G_3 G_\infty - G_1^2 - G_2^2 - G_3^2 - G_\infty^2.$$  \(2.12\)

This relation plays a fundamental role in the theory of the sixth Painlevé equation which will be discussed in section 5.

2.1. Sphere with orbifold points

Let us consider the case in which one or more holes in the sphere $\Sigma_{0,4}$ are substituted by orbifold points. As mentioned earlier, this corresponds to allowing the perimeters to become imaginary numbers $P_l = 2\pi i/k_l$, with $k_l$ being the order of the orbifold point. The Poincaré uniformization theorem still holds:

$$\Sigma_{0,4-i,j} \sim \mathbb{H}/\Delta_{0,4-i,j}^-,$$

where the Fuchsian group $\Delta_{0,4-i,j}^-$ is now generated by $4-j$ hyperbolic elements and by $j$ elliptic elements satisfying one relation. The hyperbolic elements are expressed in terms of the Thurston shear coordinate as in (2.1), while the elliptic elements are decomposed as follows: we take $G_i^j := 2 \cos(2\pi/k)$ and set the matrix $F(R)$ every time we go around the orbifold point counterclockwise and the matrix $F(L)$ every time we go around it clockwise. When all holes are replaced by orbifold points, the generators of the Fuchsian group become

$$\gamma_1 = XY_1 F_1^{(R)} X Y_1,$$
$$\gamma_2 = -RX Y_2 F_2^{(R)} X Y_2 L,$$
$$\gamma_3 = -LX Y_3 F_3^{(R)} X Y_3 R.$$ \(2.13\)

We see that the matrix combination $XY_i F_i^{(R)} X Y_i$ has exactly the form (2.10) in which now $\tilde{\gamma}_i = \gamma_i$ and the parameter $G_i := 2 \cos(2\pi/k_i)$, where $k_i$ is the order of the corresponding orbifold point. We can therefore treat in a uniform way both the case of a hole and of an orbifold point. In particular the quantities $G_{i,j} := -\text{Tr}(\gamma_i \gamma_j)$, $i < j$, $i, j = 1, 2, 3$, now have the same form as (2.11) with $\tilde{Y}_i = Y_i$ and the parameter $G_i = 2 \cos(2\pi/k_i)$, $2 > G_i \geq 0$:

$$G_{1,2} = e^{2\pi i/2} + e^{-2\pi i/2} + e^{2\pi i/2} + e^{-2\pi i/2} + G_1 e^{2\pi i/2} + G_2 e^{-2\pi i/2},$$
$$G_{2,3} = e^{2\pi i/3} + e^{-2\pi i/3} + e^{2\pi i/3} + e^{-2\pi i/3} + G_2 e^{2\pi i/3} + G_3 e^{-2\pi i/3},$$
$$G_{3,1} = e^{2\pi i/1} + e^{-2\pi i/1} + e^{2\pi i/1} + e^{-2\pi i/1} + G_3 e^{2\pi i/1} + G_1 e^{-2\pi i/1}. \quad (2.14)$$

As a result the following corollary of theorem 2.1 holds true:

Corollary 2.3. The Poisson algebra of geodesic length functions on the sphere with $4-j$ holes and $j$ orbifold points, $j = 1, 2, 3, 4$, is generated by the three elements $G_{1,2}$, $G_{1,3}$, and $G_{2,3}$:

$$G_{i,j} := -\text{Tr}(\gamma_i \gamma_j), \quad i < j, \quad i, j = 1, 2, 3.$$

Their Poisson brackets are given by formulae (2.6)–(2.8).
**Proof.** As explained above, this is a straightforward consequence of theorem 2.1. We present here an alternative proof that follows from evaluating the Poisson brackets between, say, $G_{1,2}$ and $G_{2,3}$ using the Goldman bracket [12] and the skein relation. For this, we introduce a new geodesic function
\[
\tilde{\mathcal{G}}_{1,3} := \text{Tr}(\gamma_1 \gamma_2 \gamma_3^{-1}).
\]
which corresponds to the geodesic that goes around the holes/orbifold points with the numbers 1 and 3 and goes twice around the hole/orbifold point with the number 2. It is then easy to see that
\[
\{G_{1,2}, G_{1,3}\} = \tilde{\mathcal{G}}_{1,3} - G_{1,3}
\]
and we can use the skein relation for the product of $G_{1,2}$ and $G_{1,3}$:
\[
G_{1,2}G_{1,3} = \tilde{\mathcal{G}}_{1,3} + G_{1,3} + G_1 G_3 + G_2 G_{\infty},
\]
where $G_{\infty} = -\text{Tr}(\gamma_1 \gamma_2 \gamma_3) = e^{\gamma_1 + \gamma_2 + \gamma_3} + e^{-\gamma_1 - \gamma_2 - \gamma_3}$ is the central element corresponding to the geodesic that goes around the last, fourth, hole. Expressing $\tilde{\mathcal{G}}_{1,3}$ from this relation, we immediately come to (2.6) in which we set $\omega_{1,3} := G_1 G_3 + G_2 G_{\infty}$. □

2.2. Braid group action on $\Sigma_{0,4-i,j}$

The action of the braid group element $\beta_{i,i+1}$, $i = 1, 2$, in terms of the geodesic functions corresponds to interchanging $i$th and $(i + 1)$st holes/orbifold points resulting in a deformation of the loops on the four-holed sphere. On the level of the Teichmüller space coordinates $Y_i$, we achieve this permutation by flipping edges.

Here we illustrate the action of $\beta_{1,2}$: first, the one with the label $Y_2$, and second, the one with the label $Y_2 + P_2$:

\[
Y_1 \quad P_1 \quad Y_2 \quad P_2
\]

In this picture, we also indicate the (continuous) transformation of $G_{1,3}$ that leaves it invariant (in the new variables $Y_i''$, $P_i''$).

The resulting transformation (in terms of shifted variables $Y_i$) reads
\[
Y_1'' = Y_1 + \log(1 + G_2 e^{Y_2} + e^{2Y_2}), \quad P_1'' = P_1,
Y_2'' = Y_3 - \log(1 + G_2 e^{-Y_2} + e^{-2Y_2}), \quad P_2'' = P_3,
Y_3'' = -Y_2, \quad P_3'' = P_2,
\]
and it produces the following formulae for the corresponding transformations of the geodesic functions:
\[
\beta_{1,2} G_{1,2} = G_{1,2}, \quad \beta_{1,2} \omega_{1,2} = \omega_{1,2},
\beta_{1,2} G_{2,3} = G_{1,3}, \quad \beta_{1,2} \omega_{2,3} = \omega_{1,3},
\beta_{1,2} G_{1,3} = G_{1,2} G_{1,3} - G_{2,3} - \omega_{2,3}, \quad \beta_{1,2} \omega_{1,3} = \omega_{2,3},
\]
and by the same procedure
\[ \beta_{2,3}G_{1,2} = G_{2,3}G_{1,2} - G_{1,3} - \omega_{1,3}, \quad \beta_{2,3}\omega_{1,2} = \omega_{1,3}, \]
\[ \beta_{2,3}G_{2,3} = G_{2,3}, \quad \beta_{2,3}\omega_{2,3} = \omega_{2,3}, \]
\[ \beta_{2,3}G_{1,3} = G_{1,2}, \quad \beta_{2,3}\omega_{1,3} = \omega_{1,2}. \] (2.18)

**Lemma 2.4.** The transformations (2.17), (2.18) satisfy the braid group relations and the element (2.9) is invariant w.r.t. the braid group transformations.

**Proof.** This result is proved by straightforward computations, and in the context of the Painlevé sixth equation was proved in [14]. \( \square \)

### 3. Quantised poisson algebra

In the quantum version, we introduce the Hermitian operators \( Y_i^h \) subject to the commutation inherited from the Poisson bracket of \( Y_i \):
\[ [Y_i^h, Y_{i+1}^h] = i\pi h [Y_i, Y_{i+1}] = i\pi h, \quad i = 1, 2, 3, \quad i + 3 \equiv i. \]

Observe that thanks to this fact, the commutators \( [Y_i^h, Y_j^h] \) are always numbers and therefore we have
\[ \exp(aY_i^h) \exp(bY_j^h) = \exp \left( aY_i^h + bY_j^h + \frac{ab}{2} \{Y_i^h, Y_j^h\} \right), \]
for any two constants \( a, b \). Therefore we have the Weyl ordering:
\[ e^{Y_i^h} e^{Y_j^h} = q^{\frac{1}{2}} e^{Y_i^h} e^{Y_j^h} = q^{-\frac{1}{2}} e^{Y_i^h} e^{Y_j^h}, \quad q \equiv e^{-i\pi h}. \]

After quantization, the central elements \( G_1, G_2, G_3 \) remain central and non-deformed, so we preserve the previous notation for them. We assume that the expression for \( G_{1,2}^h, G_{2,3}^h, G_{1,3}^h \), has precisely the form of (2.11) or (2.14) with \( Y_i^h \) substituted for the respective \( Y_i \), in order to ensure the Hermiticity of \( G_{1,2}^h, G_{2,3}^h, G_{1,3}^h \) as follows:
\[ G_{1,2}^h = G_{2,3}^h - G_{1,3}^h - \omega_{1,3}, \quad G_{1,2}^h = G_{2,3}^h, \quad G_{1,3}^h = G_{1,2}^h. \]

We have the corresponding deformations of the Poisson relations, which become commutation relations between \( G_{1,2}^h \):
\begin{align*}
q^{-1/2}G_{1,2}^h G_{2,3}^h - q^{1/2}G_{2,3}^h G_{1,2}^h &= (q^{-1} - q)G_{1,3}^h + (q^{-1/2} - q^{1/2})\omega_{1,3}, \\
q^{-1/2}G_{2,3}^h G_{1,3}^h - q^{1/2}G_{1,3}^h G_{2,3}^h &= (q^{-1} - q)G_{1,2}^h + (q^{-1/2} - q^{1/2})\omega_{1,2}, \\
q^{-1/2}G_{1,3}^h G_{1,2}^h - q^{1/2}G_{1,2}^h G_{1,3}^h &= (q^{-1} - q)G_{2,3}^h + (q^{-1/2} - q^{1/2})\omega_{2,3}. \end{align*} (3.1)

The action of the quantum braid group is given by
\begin{align*}
\beta_{1,2}G_{1,2}^h &= G_{1,2}^h, & \beta_{1,2}\omega_{1,2} &= \omega_{1,2}, \\
\beta_{1,2}G_{2,3}^h &= G_{2,3}^h, & \beta_{1,2}\omega_{2,3} &= \omega_{1,3}, \quad \beta_{1,2}\omega_{2,3} &= \omega_{2,3}, \quad \beta_{1,2}\omega_{2,3} &= \omega_{2,3}, \quad (3.2) \\
\beta_{1,2}G_{1,3}^h &= G_{1,2}^h, \quad G_{2,3}^h - G_{1,3}^h - \omega_{1,3} = q^{-1/2}G_{1,3}^h G_{1,2}^h - q^{-1/2}G_{2,3}^h G_{2,3}^h - \omega_{1,3}, \\
\beta_{2,3}G_{1,2}^h &= G_{2,3}^h, \quad G_{2,3}^h - G_{1,3}^h - \omega_{1,3} = q^{-1/2}G_{1,3}^h G_{1,2}^h - q^{-1/2}G_{2,3}^h G_{2,3}^h - \omega_{1,3}, \quad (3.3) \\
\beta_{2,3}G_{2,3}^h &= G_{2,3}^h, \quad \beta_{2,3}\omega_{2,3} &= \omega_{2,3}, \quad \beta_{2,3}\omega_{2,3} &= \omega_{2,3}, \quad \beta_{2,3}\omega_{2,3} &= \omega_{2,3}. \\
\beta_{2,3}G_{1,3}^h &= G_{1,2}^h, \quad \beta_{2,3}\omega_{1,2} &= \omega_{1,3}, \quad \beta_{2,3}\omega_{1,3} &= \omega_{1,3}.
\end{align*}
Finally the quantum central element

\[ M = q^{-1/2} G_{1,2}^h G_{2,3}^h G_{1,3}^h - q^{-1} (G_{1,2}^h)^2 - q (G_{2,3}^h)^2 - q^{-1} (G_{1,3}^h)^2 \]

\[ - q^{-1/2} \omega_{1,2} G_{1,2}^h - q^{1/2} \omega_{2,3} G_{2,3}^h - q^{-1/2} \omega_{1,3} G_{1,3}^h \]

is chosen to be Hermitian: \( (M^h)^\dagger = M^h \).

4. Versal unfolding of the \( D_4 \) singularity

Given any \( \phi \in \mathbb{C}[u, v, w] \), the following formulae define a Poisson bracket on \( \mathbb{C}[u, v, w] \):

\[ [u, v] = \frac{\partial \phi}{\partial w}, \quad [v, w] = \frac{\partial \phi}{\partial u}, \quad [w, u] = \frac{\partial \phi}{\partial v}, \]

and \( \phi \) itself is a central element for this bracket, so that the quotient space

\[ M_\phi := \mathbb{C}[u, v, w]/(\phi=0) \]

inherits the Poisson algebra structure [7].

For \( \phi \) given by

\[ \phi(u, v, w) = u^2 + v^2 + w^2 - uvw, \]

the quotient space \( M_\phi \) has a simple \( D_4 \) singularity at the origin. It was proved in [7] that all Poisson algebra deformations of \( (M_\phi, \{\cdot, \cdot\}) \) are obtained by deforming \( \phi \) to

\[ \tilde{\phi} = u^2 + v^2 + w^2 - uvw + r_1 u + r_2 v + r_3 w + r_4, \]

where \( r_1, r_2, r_3, r_4 \) are any four complex parameters. This means that on the deformed surface \( \tilde{M}_\phi := \mathbb{C}[u, v, w]/(\tilde{\phi}=0) \) the Poisson bracket is still given by formulae (4.1) with \( \phi \) substituted by \( \tilde{\phi} \).

The equation \( \tilde{\phi} = 0 \) defines an affine irreducible cubic surface \( \tilde{M}_\phi \) in \( \mathbb{C}^3 \) whose projective completion

\[ \tilde{M}_\phi := \{(u, v, w, t) \in \mathbb{P}^3 | u^2 t + v^2 t + w^2 t - uvw + r_1 ut^2 + r_2 vt^2 + r_3 wt^2 + r_4 t^3 = 0 \} \]

is a del Pezzo surface of degree 3 and differs from it by three smooth lines at infinity forming a triangle [21]:

\[ t = 0, \quad uvw = 0. \]

Observe that this Poisson algebra (4.1) on \( \tilde{M}_\phi \) coincides with our (2.6)–(2.8), while \( \tilde{\phi} \) coincides with the central element \( C \), after the appropriate identifications:

\[ G_{12} \rightarrow u, \quad G_{13} \rightarrow v, \quad G_{23} \rightarrow w, \quad \omega_{12} \rightarrow r_1, \quad \omega_{13} \rightarrow r_2, \quad \omega_{23} \rightarrow r_3, \]

and, thanks to (2.12),

\[ -4 + G_1 G_2 G_3 G_\infty + G_1^2 + G_2^2 + G_3^2 + G_\infty^2 \rightarrow r_4. \]

As a consequence our parametrization (2.11) \( G_{12}, G_{13}, G_{23} \) in terms of \( Y_1, Y_2, Y_3 \) defines an analytic surjective map

\[ \mu : \mathbb{C}^3/\{Y_1+Y_2+Y_3=\text{const.}\} \rightarrow \tilde{M}_\phi, \]

giving rise to a system of flat coordinates for the affine irreducible cubic surface \( \tilde{M}_\phi \).
Remark 4.1. It is straightforward to prove that for $G_1 = G_2 = G_3 = 0$, the map $\mu$ is always invertible apart from the symplectic leaves for which

$$Y_1 + Y_2 + Y_3 = i n \pi, \quad n \in \mathbb{Z}.$$  

In this case the Casimir element becomes

$$C = 4 - G_2^\infty = \begin{cases} 4 & \text{for } n \text{ even}, \\ 0 & \text{for } n \text{ odd}, \end{cases}$$

and the symplectic leaves degenerate. In particular there exist two points $(u, v, w) = (2, 2, 2)$ and $(u, v, w) = (0, 0, 0)$ for which each $u, v, w$ are Casimirs, so that the symplectic leaves reduce to a point.

We stress that our quantization procedure described in section 3 is not only valid in the geometric case (i.e. when we restrict this map to real non-negative $Y_1, Y_2, Y_3$) but can easily be extended to all of $\mathbb{C}^3/\langle Y_1 + Y_2 + Y_3 \rangle$, thus providing an explicit and natural quantization of the affine cubic surface $M_\tilde{\phi}$.

We note that Oblomkov [21] proved that the quantization of the affine cubic surface $M_\tilde{\phi}$ coincides with the spectrum of the centre of the generalized rank 1 double affine Hecke algebra studied in [22].

5. Poisson algebra structure on the monodromy data of the PVI equation

In this section, we show that the Poisson algebra (2.6)–(2.8) is the image under the Riemann–Hilbert map of the Lie–Poisson structure on $\oplus_1^3 sl_2(\mathbb{C})$. In order to do so, we need to recall some well-known facts about the Painlevé sixth equation and its relation to the monodromy preserving deformation equations [16, 17].

5.1. Isomonodromic deformations associated with the sixth Painlevé equation

The Painlevé sixth equation PVI [10, 11, 23],

$$y_{tt} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) y_t = \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) y_t,$$

$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left[ \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right],$$

is equivalent to the simplest non-trivial case of the Schlesinger equations [23]. These are Pfaffian differential equations

$$\frac{\partial}{\partial u_i} A_i = \frac{[A_i, A_j]}{u_i - u_j}, \quad i \neq j,$$

$$\frac{\partial}{\partial u_i} A_i = - \sum_{j \neq i} \frac{[A_i, A_j]}{u_i - u_j},$$

for $m \times m$ matrix-valued functions $A_1 = A_1(u), \ldots, A_n = A_n(u), u = (u_1, \ldots, u_n)$, where the independent variables $u_1, \ldots, u_n$ are pairwise distinct.

The case corresponding to the PVI equation is for $m = 2$ and $n = 3$ and describes the monodromy preserving deformations of a rank 2 meromorphic connection over $\mathbb{P}^1$ with four simple poles $u_1 = 0, u_2 = 1, u_3 = t$ and $\infty$:

$$\frac{d\Phi}{d\lambda} = \left( \frac{A_1(t)}{\lambda} + \frac{A_2(t)}{\lambda-t} + \frac{A_3(t)}{\lambda-1} \right) \Phi,$$
where
\[
eigen(A_i) = \pm \frac{\theta_i}{2}, \quad \text{for } i = 1, 2, 3, \quad A_\infty := -A_1 - A_2 - A_3 \quad (5.4)
\]
\[
A_\infty = \begin{cases} 
\left( \frac{\theta_\infty}{2} & \frac{\theta_\infty}{2} \\
0 & 1 \\
0 & 0
\end{cases}, & \text{for } \theta_\infty \neq 0 \\
\left( 1 & 0 \\
0 & 1 
\right), & \text{for } \theta_\infty = 0 
\end{cases} \quad (5.5)
\]
and the parameters \(\theta_i, i = 1, 2, 3, \infty\), are related to the PVI parameters by
\[
\alpha = (\theta_\infty - 1)^2, \quad \beta = -\frac{\theta_1^2}{2}, \quad \gamma = \frac{\theta_3^2}{2}, \quad \delta = \frac{1 - \theta_2^2}{2}.
\]
The precise dependence of the matrices \(A_1, A_2, A_3\) on the PVI solution \(y(t)\) and its first derivative \(y'(t)\) can be found in [17].

In this communication we take the monodromy matrices \(M_1, M_2, M_3, M_\infty\) of the Fuchsian system (5.3) defined w.r.t. the fundamental matrix \(\Phi_1\) normalized at \(\infty\):
\[
\Phi_\infty = (1 + \mathcal{O}(1/\lambda))\lambda^{-A_\infty}\lambda^{-R_\infty},
\]
where the term \(\lambda^{-R_\infty}\) only appears in the resonant case, i.e. when \(\theta_\infty \in \mathbb{Z}_+\) in which case all entries of \(R_\infty\) are zero apart from \(R_{\infty i}\), or when \(\theta_\infty \in \mathbb{Z}\) in which case all entries of \(R_\infty\) are zero apart from \(R_{\infty i}\), and w.r.t. the basis of loops \(l_1, l_2, l_3\) with base point at \(\infty\), where \(l_i\) encircles only once \(u_i, i = 1, 2, 3,\) and \(l_1, l_2, l_3\) are oriented in such a way that
\[
M_1M_2M_3M_\infty = 1,
\]
where \(M_\infty = \exp(2\pi i A_\infty)\exp(2\pi i R_\infty)\).

Denote by \(\mathcal{F}(\theta_1, \theta_2, \theta_3, \theta_\infty)\) the moduli space of rank 2 meromorphic connection over \(\mathbb{P}^1\) with four simple poles \(0, 1, t, \infty\) of the form (5.3) and by \(\mathcal{M}(\theta_1, \theta_2, \theta_3, \theta_\infty)\) the moduli space of monodromy representations
\[
\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, t, \infty\}) \to SL_2(\mathbb{C})
\]
with prescribed local monodromies:
\[
eigen(M_j) = \exp(\pm \pi i \theta_j), \quad j = 1, 2, 3, \infty.
\]
Then the Riemann–Hilbert correspondence
\[
\mathcal{F}(\theta_1, \theta_2, \theta_3, \theta_\infty) \backslash \mathcal{G} \rightarrow \mathcal{M}(\theta_1, \theta_2, \theta_3, \theta_\infty) \backslash GL_2(\mathbb{C}),
\]
where \(\mathcal{G}\) is the gauge group [1], is defined by associating with each Fuchsian system its monodromy representation class.

**Theorem 5.1.** [13] The Riemann–Hilbert correspondence is a Poisson map

Iwasaki [14] proved that \(\mathcal{M}(\theta_1, \theta_2, \theta_3, \theta_\infty) \backslash GL_2(\mathbb{C}) = M_\mathcal{G}\). We are now going to prove that the Poisson bracket (4.1) on \(M_\mathcal{G}\) is the image under the Riemann–Hilbert map of the Poisson–Lie bracket on \(\mathfrak{sl}_2^* \backslash \mathfrak{sl}_2^* \otimes (2, \mathbb{C})\).
5.2. Poisson bracket on the monodromy data of the general Painlevé sixth equation

The Schlesinger equations on \( g := \mathfrak{sl}(m, \mathbb{C}) \) admit Hamiltonian formulation with time-dependent quadratic Hamiltonians

\[
H_k = \sum_{i \neq k} \frac{\text{Tr}(A_k A_i)}{u_k - u_i},
\]

(5.6)

where \( \{ \cdot, \cdot \} \) is the standard Lie–Poisson bracket on \( g^* \), which can be represented in \( r \)-matrix formalism:

\[
\{ A(\lambda_1) \otimes A(\lambda_2) \} = \left[ \frac{1}{2} (A(\lambda_1) + A(\lambda_2), r(\lambda_1 - \lambda_2)) \right],
\]

where \( r(z) = \frac{\Omega}{z} \) is a classical \( r \)-matrix, i.e., a solution of the classical Yang–Baxter equation.

In the case of \( g := \bigoplus \mathfrak{sl}(m) \), \( \Omega \) is the exchange matrix \( \Omega = \sum_{i,j} \frac{1}{2} E_{ij} \otimes E_{ji} \) (we identify \( \mathfrak{sl}(m) \) with its dual by using the Killing form \( (A, B) = \text{Tr} AB, \ A, B \in \mathfrak{sl}(m) \)).

The standard Lie–Poisson bracket on \( \mathfrak{sl}(m, \mathbb{C}) \) is mapped by the Riemann–Hilbert correspondence to the Korotkin–Samtleben bracket [18]:

\[
\{ M_i \otimes M_j \} = \frac{1}{2} \left( M_i \Omega M_j - M_j \Omega M_i \right)
\]

(5.8)

\[
\{ M_i \otimes M_j \} = \frac{1}{2} \left( M_j \Omega M_i + M_i \Omega M_j - \Omega M_i M_j - M_j M_i \Omega \right), \quad \text{for} \ i < j.
\]

This bracket does not satisfy the Jacobi identity—however it restricts to a Poisson bracket on the adjoint invariant objects.

Theorem 5.2. In the PVI case the Korotkin–Samtleben bracket restricted to the adjoint invariant objects

\[
G_{i,j} := -\text{Tr}(M_i M_j)
\]

(5.9)

is given by formulae (2.6)–(2.8).

Proof. We show how to prove relation (2.6), all the others being equivalent. By definition of \( G_{i,j} \) we have

\[
\{ G_{1,2}, G_{2,3} \} = \{ \text{Tr}(M_1 M_2), \text{Tr}(M_2 M_3) \} = \text{Tr}\left( \{ M_1 \otimes M_2 \} M_2 M_3 \right)
\]

\[
+ M_2 \{ M_1 \otimes M_3 \} M_2 + M_1 \{ M_2 \otimes M_2 \} M_1
\]

\[
+ M_1 M_2 \{ M_2 \otimes M_3 \}.
\]

Applying the Korotkin–Samtleben bracket (5.8), one gets

\[
\{ G_{1,2}, G_{2,3} \} = \frac{1}{2} \sum_{i,j=1}^{12} \text{Tr}\left( \{ M_i \Omega M_j \} + M_2 \{ M_j \otimes M_1 \} M_2 M_3 \right)
\]

\[
+ M_2 \{ M_i \otimes M_2 \} M_2 + M_1 \{ M_2 \otimes M_2 \} M_1
\]

\[
+ M_1 M_2 \{ M_2 \otimes M_3 \}.
\]

(5.10)
In the subsequent calculations we use $\Omega$ as the exchange matrix, which implies that for every $i$, $j$
\[
\begin{align*}
\frac{1}{2} M_j \Omega M_i &= \Omega M_j M_i = \frac{1}{2} M_j M_i \Omega.
\end{align*}
\] (5.11)
We then obtain that the first two lines on the right-hand side of (5.10) cancel each other and
\[
\{G_{1,2}, G_{2,3}\} = \text{Tr}(M_1 M_2 M_3 M_2 - M_1 M_2 M_2 M_3).
\]
By repeated applications of the skein relation:
\[
\text{Tr}(AB) + \text{Tr}(AB^{-1}) = \text{Tr}(A)\text{Tr}(B),
\] (5.12)
which is valid for any $(2 \times 2)$-matrices $A$ and $B$ with unit determinants, we obtain the final result. In fact
\[
\begin{align*}
\text{Tr}(M_1 M_2 M_3 M_2) &= \text{Tr}(M_1 M_2)\text{Tr}(M_2 M_3) + \text{Tr}(M_1 M_3) - \text{Tr}(M_1)\text{Tr}(M_3) \\
\text{Tr}(M_1 M_2 M_2 M_3) &= \text{Tr}(M_1 M_2)\text{Tr}(M_2 M_3) - \text{Tr}(M_1)\text{Tr}(M_3) \\
&\quad + \text{Tr}(\omega)\text{Tr}(M_2) - \text{Tr}(M_1 M_2 M_3 M_2),
\end{align*}
\]
so that
\[
\{G_{1,2}, G_{2,3}\} = -2G_{1,3} + G_{1,2}G_{2,3} - \omega_{1,3}
\]
as we wanted to prove. The other relations can be obtained in a similar way. The Jacobi identity is a straightforward brute force computation.

**Remark 5.3.** Observe that Dubrovin produced the following Poisson bracket on the Stokes data associated with a three-dimensional Frobenius manifold [5]:
\[
\begin{align*}
\{S_{1,2}, S_{2,3}\} &= S_{1,2}S_{2,3} - 2S_{1,3}, \\
\{S_{2,3}, S_{1,3}\} &= S_{2,3}S_{1,3} - 2S_{1,2}, \\
\{S_{1,3}, S_{1,2}\} &= S_{1,2}S_{1,3} - 2S_{2,3}.
\end{align*}
\] (5.13) (5.14) (5.15)
It is a straightforward computation to show that this bracket coincides with our bracket for the case of PVI$_\mu$, i.e. the Painlevé sixth equation with parameters $\beta = \gamma = 0$, $\delta = \frac{1}{2}$ and $\alpha = \frac{2\mu - 1}{2}$ appearing in the Frobenius manifold theory, via the change of coordinates
\[
G_{i,j} = S_{i,j}^2 - 2.
\]
This change of coordinates actually corresponds to a quartic transformation on the sixth Painlevé equation [20].

### 5.3. Action of the braid group $B_3$

The procedure of the analytic continuation of the solutions of the PVI equation was described in [6] by the action of the braid group $B_3 = \langle \beta_{12}, \beta_{23} \rangle$ on the monodromy matrices $M_1$, $M_2$, $M_3$ given by
\[
\begin{align*}
\beta_{12}(M_1, M_2, M_3) &= (M_1 M_2 M_1^{-1}, M_1, M_3), \\
\beta_{23}(M_1, M_2, M_3) &= (M_1, M_2 M_3 M_2^{-1}, M_3).
\end{align*}
\] (5.16) (5.17)
By using the skein relation (5.12) it is a straightforward computation to prove that the action of the braid group on the Poisson algebra (2.6)–(2.8) is given by formulae (2.17) and (2.18).

**Remark 5.4.** In the Teichmüller space framework all coordinates $Y_1$, $Y_2$, $Y_3$ are assumed to be positive real numbers; therefore, the products $\gamma_i \gamma_j$ are always hyperbolic elements, i.e. $G_{ij} = -\text{Tr}(\gamma_i \gamma_j) > 2$. Therefore there is no Teichmüller space interpretation of the algebraic solutions of PVI [6].
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