RAMANUJAN-TYPE $1/\pi$-SERIES FROM BIMODULAR FORMS

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Abstract. We develop an approach to establish $1/\pi$-series from bimodular forms. Utilizing this approach, we obtain new families of 2-variable $1/\pi$-series associated to Zagier’s sporadic Apéry-like sequences.

1. Introduction

In 1914, Ramanujan [25] gave seventeen series representations for $1/\pi$ of the form

$$
\sum_{n=0}^{\infty} \frac{(1/2)_n(a)_n(1-a)_n}{(n!)^3} (An+B)C^n = \frac{D}{\pi}
$$

with $a \in \{1/2, 1/3, 1/4, 1/6\}$, $A, B, C \in \mathbb{Q}$, and $D \in \mathbb{Q}$, such as

$$
\sum_{n=0}^{\infty} \frac{(1/2)_n^3}{(6n+1)} \left(\frac{1}{4}\right)^n = \frac{4}{\pi},
$$

where $(a)_n$ is the Pochhammer symbol defined by $(a)_n = a(a+1)\ldots(a+n-1)$. The first proof of such identities was given by Borwein and Borwein [6]. Since then, mathematicians [1, 4, 7, 9, 10, 11, 12, 13, 14, 16, 18, 40] have generalized Ramanujan’s $1/\pi$-series and produced many series of similar nature with the coefficients $(1/2)_n(a)_n(1-a)_n/(n!)^3$ replaced by binomial sums, such as the Apéry numbers

$$
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.
$$

(See [3] for a more thorough survey.)

In [30], through some powerful insight, Z.-W. Sun discovered (conjectured) many two-variable analogues of Ramanujan’s $1/\pi$-series. For instance, if we let

$$
T_n(b, c) = \sum_{m=0}^{[n/2]} \binom{n}{2m} \binom{2m}{m} b^{n-2m} c^m
$$

denote the coefficient of $x^n$ in the expansion of $(x^2+bx+c)^n$, then one of Sun’s conjectural formulas states that

$$
\sum_{n=0}^{\infty} \frac{(1/2)_n^2}{(n!)^2} T_n(34, 1)(30n+7) \left(\frac{-1}{64}\right)^n = \frac{12}{\pi}
$$

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Among Sun’s conjectural formulas, three families are of the form
\[ \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(n!)^2} T_n(b,c)(An+B)C^n = \frac{D}{\pi}, \]
a \in \{1/2, 1/3, 1/4\}, b, c, A, B, C \in \mathbb{Q}, and D \in \mathbb{Q}. They were proved by Chan, Wan, and Zudilin [15]. They used cleverly a link between \( T_n(b,c) \) and Legendre polynomials, theory of hypergeometric functions, and a formula of Brafman [8] to convert standard Ramanujan-type \( 1/\pi \)-series to Sun’s formulas. Using similar techniques, Rogers and Straub [26] proved another family of Sun’s conjectural formulas.

It turns out that Sun’s series
\[ \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(n!)^2} T_n(b,c)z^n = \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(n!)^2} T_n(bz,cz^2) \]
actually appeared earlier in literature in a very different context. In [21], Lian and Yau explored relations among mirror maps and Picard-Fuchs differential equations for families of toric Calabi-Yau threefolds, modular forms, and hypergeometric functions. One family of toric Calabi-Yau threefolds they considered consists of hypersurfaces of degree 24 in the weighted projective space \( \mathbb{P}^4[1,1,2,8,12] \). In an earlier work [20] (A.36), it was shown that the Picard-Fuchs system for this family is given by
\[
\begin{align*}
\theta_x(\theta_x - 2\theta_y) & - x(\theta_x + 1/6)(\theta_x + 5/6), \\
\theta_y(\theta_y - 2\theta_x) & - y(2\theta_y - \theta_x + 1)(2\theta_y - \theta_x), \\
\theta_x^2 - z(2\theta_x - \theta_y + 1)(2\theta_x - \theta_y), 
\end{align*}
\]
where for a variable \( t = x, y, z \), we let \( \theta_t = t\partial/\partial t \). Letting \( z \to 0 \), one gets a new system
\[
\begin{align*}
\theta_x(\theta_x - 2\theta_y) & - x(\theta_x + 1/6)(\theta_x + 5/6), \\
\theta_y^2 & - y(2\theta_y - \theta_x + 1)(2\theta_y - \theta_x), 
\end{align*}
\]
which coincides with the Picard-Fuchs system for the family of toric \( K3 \) surfaces corresponding to hypersurfaces of degree 12 in \( \mathbb{P}^3[1,1,4,6] \). Up to scalars, this system has a unique solution holomorphic near \( (x,y) = (0,0) \). It is easy to see that this solution is given by
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{n/2} \frac{(1/6)_n(5/6)_n}{(n!)^2} \frac{n!}{2m!} \frac{(2m)!}{m!} x^ny^m = \sum_{n=0}^{\infty} \frac{(1/6)_n(5/6)_n}{(n!)^2} T_n(x,x^2y),
\]
which is Sun’s series \( (1) \) for the case \( a = 1/6 \). Likewise, Sun’s series for \( a = 1/4, 1/3, 1/2 \), when written in the same form as the left-hand side of \( (1) \), are solutions of the Picard-Fuchs systems
\[
\begin{align*}
\theta_x(\theta_x - 2\theta_z) & - x(\theta_x + a)(\theta_x + 1 - a), \\
\theta_y^2 & - y(2\theta_y - \theta_x + 1)(2\theta_y - \theta_x)
\end{align*}
\]
for some families of \( K3 \) surfaces.

One remarkable feature of these Picard-Fuchs systems \( (2) \) is that they admit parameterization by bimodular forms, first described by [21] Corollary 1.3] (for the case \( a = 1/6 \)) and later elaborated in more details in [38]. We will discuss the bimodular properties of Sun’s 2-variable \( 1/\pi \)-series in the next section.

Here the main goal of the paper is to present new families of 2-variable \( 1/\pi \)-series associated to Zagier’s sporadic Apéry-like sequences and discuss their bimodular properties.
Let $a$, $b$, and $c$ be real numbers and $u_n$ be the sequence of numbers defined recursively by $u_{-1} = 0$, $u_0 = 1$, and

$$
(n + 1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2u_{n-1}
$$

(3)

for $n \geq 0$. The series $f(t) = \sum_{n=0}^{\infty} u_n t^n$ is a solution of the differential equation

$$
(\theta^2 - t(a\theta^2 + a\theta + b) + c\theta^2(\theta + 1)^2)f = 0, \quad \theta = t\frac{d}{dt}.
$$

(4)

(Up to a scalar, this is the unique solution holomorphic near $t = 0$.) The definition of such a sequence is motivated by the Apéry numbers, corresponding to the case $(a, b, c) = (11, 3, -1)$, that first appeared in Apéry’s proof of irrationality of zeta-values [2][32]. A surprising feature of the Apéry numbers is that they are all integers. In literature, we say $\{u_n\}$ is an Apéry-like sequence if all numbers $u_n$ are integers. In [39], Zagier did an extensive search and found 36 Apéry-like sequences, up to scalings. Among them, some are terminating sequences, some are polynomials (meaning that $u_n = g(n)$ for some polynomial $g$), some are hypergeometric (the cases where $c = 0$), some are Legendrian (the cases where $a^2 = 4c$), where the differential equation in (4) has three singularities and can be reduced to a hypergeometric one, and there are six sporadic cases. All hypergeometric, Legendrian, and sporadic cases have modular-function origin. Here we are interested in the sporadic cases. Their modular properties and corresponding $u_n$ are given in Table 1. Here the notation $1^33^6/2^33^3$ etc. is a shorthand for $\eta(\tau)^3\eta(6\tau)^6/\eta(2\tau)^3\eta(3\tau)^9$ etc.

| $(a, b, c)$ | group | $t$ | $f$ | $u_n$ |
|-------------|-------|-----|-----|------|
| $(7, 2, -8)$ | $\Gamma_0(6)$ | $\frac{1^3 3^6}{2^3 3^3}$ | $\frac{2^2 3^6}{1^6 3^3}$ | $\sum_{k=0}^{n} \frac{n^2}{k^2}$ |
| $(10, 3, 9)$ | $\Gamma_0(6)$ | $\frac{1^4 3^8}{2^8 3^4}$ | $\frac{2^6 3^3}{1^6 6^2}$ | $\sum_{k=0}^{n} \frac{n!}{k!}$ |
| $(-17, -6, 72)$ | $\Gamma_0(6)$ | $\frac{1^2 3^5}{2^5 3^2}$ | $\frac{1^6 6^3}{2^3 3^2}$ | $\sum_{k=0}^{n} (-8)^{n-k} \frac{n^2}{k^2}$ |
| $(12, 4, 32)$ | $\Gamma_0(8)$ | $\frac{1^4 4^2 8^4}{2^4 3^2}$ | $\frac{2^{10}}{1^4 4^2}$ | $\sum_{k=0}^{\left\lfloor n/3 \right\rfloor} \frac{n^2}{k^2}$ |
| $(-9, -3, 27)$ | $\Gamma_0(9)$ | $\frac{9^3}{1^3}$ | $\frac{1^3}{3^1}$ | $\sum_{k=0}^{n} (-3)^{n-3k} \frac{n^2}{3k}$ |
| $(11, 3, -1)$ | $\Gamma_1(5)$ | see below | see below | $\sum_{k=0}^{n} \frac{n^2}{k^2}$ |

**Table 1.** Sporadic Apéry-like sequences

The modular function $t$ and the modular form $f$ in the case $(11, 3, -1)$ are

$$
t = q \prod_{n=1}^{\infty} (1 - q^n)^5(\frac{q}{n^3}),
$$

and

$$
f = \prod_{n=1}^{\infty} (1 - q^n)^2 \cdot \prod_{n \geq 0, n \equiv 1, 4 \text{ mod } 5} (1 - q^n)^{-3} \prod_{n \geq 0, n \equiv 2, 3 \text{ mod } 5} (1 - q^n)^2,
$$

etc.
respectively.

**Remark 1.** Note that the congruence subgroups in all six cases have genus 0, 4 cusps, and no elliptic points.

Note also that in [34] and [39], the case \((a, b, c) = (-9, -3, 27)\) is listed as \((9, 3, 27)\) instead. The reason is that Zagier normalized the parameter \(a\) to be positive. With \((9, 3, 27)\) instead of \((-9, -3, 27)\), the modular function \(t\) and the modular form \(f\) are

\[
\frac{\eta(\tau)^3 \eta(4\tau)^3 \eta(18\tau)^9}{\eta(2\tau)^9 \eta(9\tau)^3 \eta(36\tau)^3} = \frac{\eta(9(\tau + 1/2))^3}{\eta(\tau + 1/2)^3}
\]

and

\[
\frac{\eta(2\tau)^9 \eta(3\tau)^3 \eta(12\tau)^3}{\eta(\tau)^3 \eta(4\tau)^3 \eta(6\tau)^3} = \frac{\eta(\tau + 1/2)^3}{\eta(3(\tau + 1/2))^3},
\]

respectively. In other words, the congruence group with this choice of \((a, b, c)\) is

\[
\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \Gamma_0(9) \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}^{-1}.
\]

Likewise, the case \((-17, -6, 72)\) is listed as \((17, 6, 72)\) with the modular function \(t\) and \(f\) given by

\[
\frac{\eta(\tau)^5 \eta(3\tau)^3 \eta(4\tau)^3 \eta(6\tau)^2 \eta(12\tau)^3}{\eta(2\tau)^{14}} = \frac{-\eta(2(\tau + 1/2))^5 \eta(6(\tau + 1/2))^3}{\eta(\tau + 1/2)^3 \eta(3(\tau + 1/2))^3}
\]

and

\[
\frac{\eta(2\tau)^{15} \eta(3\tau)^2 \eta(12\tau)^2}{\eta(\tau)^6 \eta(4\tau)^3 \eta(6\tau)^3} = \frac{-\eta(\tau + 1/2)^6 \eta(6(\tau + 1/2))^2}{\eta(2(\tau + 1/2))^3 \eta(3(\tau + 1/2))^2},
\]

respectively, in [34][39].

**Theorem 1.** Let \(\{u_n\}\) be one of the six sporadic Apéry-like sequences with parameters \((a, b, c)\). Then we have

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor n/2 \rfloor} u_n \binom{2m}{m} \binom{n}{2m} (An + B)x^n y^m = \frac{C}{\pi}
\]

for \(A, B, C, x,\) and \(y\) given in Tables [2][7] in Appendix [A].

**Remark 2.** Using the same techniques, we also obtain many 2-variable \(1/\pi\)-series of the form \((5)\) for \(u_n = (a) n (1 - a) n / (n!)^2, a \in \{1/2, 1/3, 1/4\}\). They are given in Tables [8][10] in Appendix [B]. They include all series conjectured or proved in [15][50], along with many new series missed by [15][30].

Note that for each \(1/\pi\)-series in the tables, there is a companion series of the form

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor n/2 \rfloor} u_n \binom{2m}{m} \binom{n}{2m} (A^2 + B^2)x^n y^m = \frac{C'}{\pi}
\]

(see Theorem [2]). For example, for the case \((a, b, c) = (7, 2, -8)\), there are series of the form \((6)\) for

\[
(x, y, A', B', C') = (47/441, 1/47^2, 2835, 172, 402\sqrt{5}),
\]

\[
(-97/1176, 1/994^2, 1164240, 43269, 53627\sqrt{5}),
\]

etc.

Similarly, for \(u_n = (1/2) n / (n!)^2\), there are series of the form \((6)\) for

\[
(x, y, A', B', C') = (-1/16, 16, 105, 12, 44), \quad (-17/32, 1/34^2, 240, 11, 31),
\]

etc.
However, in general the constants $A'$, $B'$, and $C'$ are much more complicated (see the discussion following Theorem 2), so we will not list them in the paper.

**Remark 3.** After the paper was completed, we discovered that Z.-W. Sun [31] has also found (conjectured) most of the $1/\pi$-series in our tables in Appendix A independently. We have made a comparison of our series with those of Sun [30, 31] and Chan, Wan and Zudilin [15], and give the corresponding equation numbers in their papers when the series coincide.

2. **Bimodular properties of 2-variable $1/\pi$-series**

The notion of bimodular forms was first introduced by Stienstra and Zagier [29]. However, since bimodular forms are not frequently studied and there is no universally used definition, here we take the liberty to define bimodular forms as follows.

**Definition 4.** Let $\Gamma$ be a congruence subgroup of $SL(2, \mathbb{Z})$ and $N(\Gamma)$ be its normalizer in $SL(2, \mathbb{R})$. For a subgroup $G$ of $N(\Gamma)$ containing $\Gamma$, a character $\chi_1$ on $\Gamma$, and a character $\chi_2$ on $G$ such that $\chi_2|_{\Gamma} = \chi_1^2$, we say a function $F : \mathbb{H}^2 \to \mathbb{C}$ is a bimodular form of weight $k$ on $(\Gamma, G)$ with characters $(\chi_1, \chi_2)$ if

(i) $F$ is a modular form of weight $k$ with character $\chi_1$ in each of the two variables, i.e., for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$F(\gamma \tau_1, \gamma \tau_2) = \chi_1(\gamma)(c\tau_1 + d)^k F(\tau_1, \tau_2),$$

(7) $F(\tau_1, \gamma \tau_2) = \chi_1(\gamma)(c\tau_2 + d)^k F(\tau_1, \tau_2)$, and

(ii) for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$,

$$F(\gamma \tau_1, \gamma \tau_2) = \chi_2(\gamma)(c\tau_1 + d)^k (c\tau_2 + d)^k F(\tau_1, \tau_2).$$

(8)

Likewise, we say a meromorphic function $f : \mathbb{H}^2 \to \mathbb{C}$ is a bimodular function on $(\Gamma, G)$ if (7) and (8) hold with $k = 0$ and the characters $\chi_1$ and $\chi_2$ are trivial.

**Example 5.** Theorem 4.1 of [38] shows that the system (2) of partial differential equations admit bimodular parameterization for $a = 1/2, 1/3, 1/4, 1/6$. For example, for $a = 1/3$, if we set

$$t(\tau) = -2\frac{\eta(3\tau)^{12}}{\eta(\tau)^{12}}, \quad f(\tau) = \sum_{m, n \in \mathbb{Z}} q^{mn^2 + mn + n^2},$$

$$x(\tau_1, \tau_2) = \frac{t(\tau_1) + t(\tau_2)}{(t(\tau_1) - 1)(t(\tau_2) - 1)}, \quad y(\tau_1, \tau_2) = \frac{t(\tau_1)t(\tau_2)}{(t(\tau_1) + t(\tau_2))^2},$$

and

$$F(\tau_1, \tau_2) = f(\tau_1)f(\tau_2),$$

then $F$, as a function of $x$ and $y$, is a solution of (2). Here $x$ and $y$ are modular functions in each of the two variables on $\Gamma_0(3)$. Also, since the Atkin-Lehner involution $w_3$ maps $t$ to $1/t$, $x$ and $y$ are invariant under the substitution $(\tau_1, \tau_2) \mapsto (w_3 \tau_1, w_3 \tau_2)$. Hence $x$ and $y$ are bimodular functions on $(\Gamma_0(3), \Gamma_0(3) + w_3)$. Likewise, we can check that $F$ is a bimodular form of weight 1 on the same groups with $\chi_1$ being the character $(-)\chi_2$ of nebentype and $\chi_2(w_3) = -1$.

We now describe bimodular properties of series that are used to obtain 2-variable $1/\pi$-series. Since many functions are defined using the Dedekind eta function. Here let us recall its transformation formula.
Lemma 6 ([35, Pages 125–127]). For
\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}),
\]
the transformation formula for \( \eta(\tau) \) is given by, for \( c = 0 \),
\[
\eta(\tau + b) = e^{\pi ib/12} \eta(\tau),
\]
and, for \( c \neq 0 \),
\[
\eta(\gamma \tau) = \epsilon_1(a, b, c, d) \sqrt{\frac{c\tau + d}{i}} \eta(\tau)
\]
with
\[
\epsilon_1(a, b, c, d) = \begin{cases} \left( \frac{d}{c} \right) e^{(1-c)/2} e^{\pi i (bd(1-c^2) + c(a+d))/12}, & \text{if } c \text{ is odd,} \\ \left( \frac{c}{d} \right) e^{\pi i (ac(1-d^2) + d(b-c+3))/12}, & \text{if } d \text{ is odd}, \end{cases}
\]
where \( \left( \frac{d}{c} \right) \) is the Legendre-Jacobi symbol.

For \( \gamma \in \text{GL}(2, \mathbb{Q}) \) and a meromorphic modular form \( f \) with weight \( k \) on some congruence subgroup of \( \text{SL}(2, \mathbb{Z}) \), we define
\[
(f|\gamma)(\tau) := (\det \gamma)^{k/2} (c\tau + d)^{-k} f(\gamma \tau).
\]

2.1. Series involving sporadic Apéry-like sequences. Let \( \{u_n\} \) be the sequence defined by (3) for certain real parameters \( a, b, \) and \( c \). Let \( P_n(x) \) be the \( n \)th Legendre polynomial defined by
\[
P_n(x) = \sum_{m=0}^{n} \binom{n}{m} \left( \frac{x-1}{2} \right)^m \left( \frac{x+1}{2} \right)^{n-m}.
\]
In [34, Theorem 2], Wan and Zudilin proved that
\[
\sum_{n=0}^{\infty} u_n P_n \left( \frac{(X+Y)(1+cXY) - 2aXY}{(Y-X)(1-cXY)} \right) \left( \frac{Y-X}{1-cXY} \right)^n
\]
\[
= (1 - cXY) \left( \sum_{n=0}^{\infty} u_n X^n \right) \left( \sum_{n=0}^{\infty} u_n Y^n \right).
\]
Now it is straightforward to prove by induction that
\[
\sum_{m=0}^{n} \binom{n}{m}^2 r^m s^{n-m} = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n}{2m} \binom{n}{m} (rs)^m (r+s)^{n-2m}.
\]
Thus, setting
\[
x = \frac{(X+Y)(1+cXY) - 2aXY}{(1-cXY)^2}, \quad y = \frac{XY(1-aX+cX^2)(1-aY+cY^2)}{((X+Y)(1+cXY) - 2aXY)^2},
\]
we may write Wan and Zudilin’s formula as
\[
\sum_{n=0}^{\infty} u_n \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{2m}{m} \binom{n}{2m} x^n y^m = (1 - cXY) \left( \sum_{n=0}^{\infty} u_n X^n \right) \left( \sum_{n=0}^{\infty} u_n Y^n \right).
\]
Quite interestingly and remarkably, both $x$ and $y$ in (12) are invariant under the action of several linear fractional transformations. Namely, assuming that $c \neq 0$ and $a^2 - 4c \neq 0$, let $\alpha$ and $\beta$ be the two numbers such that

$$1 - aX + cX^2 = (1 - \alpha X)(1 - \beta X).$$

Then $x$ and $y$ are invariant under

$$(X, Y) \mapsto \left(\frac{1}{cX^*}, \frac{1}{cY}\right), \quad (X, Y) \mapsto \left(\frac{1 - \alpha X}{\alpha(1 - \beta X)}, \frac{1 - \alpha Y}{\alpha(1 - \beta Y)}\right)$$

and their composition

$$(X, Y) \mapsto \left(\frac{1 - \beta X}{\beta(1 - \alpha X)}, \frac{1 - \beta Y}{\beta(1 - \alpha Y)}\right),$$

in addition to the obvious symmetry $(X, Y) \mapsto (Y, X)$. When $(a, b, c)$ is one of the sporadic cases, these symmetries all come from normalizers of the corresponding congruence subgroups in $\text{SL}(2, \mathbb{R})$.

**Proposition 7.** Let $(a, b, c)$ be one of the six sporadic cases, and $\Gamma$, $t(\tau)$ and $f(\tau)$ be given as in Table[1]. Then the involutions $\sigma_1 : t \mapsto 1/ct$ and $\sigma_2 : t \mapsto (1 - \alpha t)/\alpha(1 - \beta t)$ correspond to the action of normalizers of $\Gamma$ in $\text{SL}(2, \mathbb{R})$ given by

| $(a, b, c)$ | $\alpha$ | $\beta$ | $\sigma_1$ | $\sigma_2$ |
|------------|----------|---------|------------|------------|
| $(7, 2, -8)$ | $-1$ | $8$ | $w_2$ | $w_3$ |
| $(10, 3, 9)$ | $1$ | $9$ | $w_3$ | $w_2$ |
| $(-17, -6, 72)$ | $-8$ | $-9$ | $w_6$ | $w_2$ |
| $(12, 4, 32)$ | $4$ | $8$ | $\left(\frac{4}{5} - \frac{1}{5}\right)$ | $\left(\frac{2}{5} - \frac{1}{5}\right)$ |
| $(-9, -3, 27)$ | $(-9 + 3\sqrt{-3})/2$ | $(-9 - 3\sqrt{-3})/2$ | $w_9$ | $\left(\frac{3}{5} - \frac{2}{5}\right)$ |
| $(11, 3, -1)$ | $(11 + 5\sqrt{5})/2$ | $(11 - 5\sqrt{5})/2$ | $\left(\frac{2}{5} - \frac{1}{5}\right)$ | $\left(\frac{3}{5} - \frac{2}{5}\right)$ |

(For convenience, we represent elements in $N(\Gamma)$ by matrices in $\text{GL}^+(2, \mathbb{Q})$, instead of matrices in $\text{SL}(2, \mathbb{R})$.) Moreover, set $t_1 = t(\tau_1)$, $t_2 = t(\tau_2)$,

$$x(\tau_1, \tau_2) = \frac{(t_1 + t_2)(1 + ct_1t_2) - 2at_1t_2}{(1 - ct_1t_2)^2},$$

$$y(\tau_1, \tau_2) = \frac{t_1t_2(1 - at_1 + ct_1^2)(1 - at_2 + ct_2^2)}{(t_1 + t_2)(1 + ct_1t_2) - 2at_1t_2^2},$$

and

$$F(\tau_1, \tau_2) = (1 - ct(\tau_1)t(\tau_2))f(\tau_1)f(\tau_2).$$

Then $x$ and $y$ are bimodular functions on $(\Gamma, G)$, and $F$ is a bimodular form of weight 1 on $(\Gamma, G)$ with characters $(\chi_1, \chi_2)$, where $\Gamma$, $G$, $\chi_1$, and $\chi_2$ are given by

| $(a, b, c)$ | $\Gamma$ | $\chi_1$ | $\chi_2$ |
|------------|--------|---------|---------|
| $(7, 2, -8)$ | $\Gamma_0(6)$ | $\chi_2(w_2) = 1, \chi_2(w_3) = -1$ |
| $(10, 3, 9)$ | $\Gamma_0(6)$ | $\chi_2(w_2) = -1, \chi_2(w_3) = 1$ |
| $(-17, -6, 72)$ | $\Gamma_0(6)$ | $\chi_2(w_2) = -1, \chi_2(w_3) = -1$ |
| $(12, 4, 32)$ | $\Gamma_0(8)$ | $\chi_2((\frac{3}{5} - \frac{2}{5})) = 1, \chi_2(w_9) = -1$ |
| $(-9, -3, 27)$ | $\Gamma_0(9)$ | $\chi_2((- \frac{3}{5} - \frac{2}{5})) = -1, \chi_2(w_9) = 1$ |
| $(11, 3, -1)$ | $\Gamma_0(5)$ | $\chi_2((\frac{2}{5} - \frac{1}{5})) = 1, \chi_2(w_5) = -1$ |
and \( w_m \) denotes the Atkin-Lehner involution. Here the character \( \chi \) for the case \((11,3,-1)\) is

\[
\chi(\gamma) = \begin{cases} 
1, & \text{if } \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod 5, \\
-1, & \text{if } \gamma \equiv \begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix} \mod 5.
\end{cases}
\]

(Note that since \( \chi_1 \) is a quadratic character in each case, the value of \( \chi_2 \) for \( w_m \) does not depend on the choice of representatives for \( w_m \).)

**Proof.** When \( t \) and \( f \) are eta-products and the normalizer comes from Atkin-Lehner involutions, it is straightforward to use the transformation law for the Dedekind eta function (Lemma 6) to verify the bimodular properties of \( t \) and \( F \). For instance, when the group is \( \Gamma_0(6) \) and the Atkin-Lehner involution is \( w_2 \), represented by the matrix \( \begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix} \), we have

\[
\eta \left( \begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix} \tau \right) = \eta \left( \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \right) = e^{-2\pi i/24} \sqrt{\frac{6\tau - 2}{i}} \eta(2\tau),
\]

\[
\eta \left( \begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix} \tau \right) = \eta \left( \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix} \right) = e^{2\pi i/24} \sqrt{\frac{3\tau - 1}{i}} \eta(\tau),
\]

\[
\eta \left( \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} \tau \right) = \eta \left( \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \right) = e^{-2\pi i/24} \sqrt{\frac{6\tau - 2}{i}} \eta(6\tau),
\]

\[
\eta \left( \begin{pmatrix} 6 & -1 \\ 2 & -2 \end{pmatrix} \tau \right) = \eta \left( \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix} \right) = e^{2\pi i/24} \sqrt{\frac{3\tau - 1}{i}} \eta(3\tau).
\]

Thus, for the case \((a, b, c) = (7, 2, -8)\), we have

\[
t(\tau)\big|_{w_2} = \frac{1}{8} \eta(2\tau)^3 \eta(3\tau)^9 = \frac{1}{8t},
\]

\[
f(\tau)\big|_{w_2} = \sqrt{2} e^{-2\pi i/4} \frac{4(3\tau - 1)}{6\tau - 2} \frac{\eta(\tau) \eta(6\tau)^6}{\eta(2\tau)^2 \eta(3\tau)^3} = -\sqrt{8t} f(\tau),
\]

and

\[
F(\tau_1, \tau_2)\big|_{w_2} = \left( 1 + \frac{1}{8t(\tau_1)t(\tau_2)} \right) 8t(\tau_1)t(\tau_2) f(\tau_1) f(\tau_2) = F(\tau_1, \tau_2),
\]

where, for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}^+(2, \mathbb{Q}) \), \( F(\tau_1, \tau_2)|\gamma \) is defined to be

\[
F(\tau_1, \tau_2)|\gamma := \frac{\det \gamma}{(c\tau_1 + d)(c\tau_2 + d)} F(\gamma \tau_1, \gamma \tau_2).
\]

Here we omit the proof of the remaining cases where \( t \) and \( f \) are eta-products and the normalizer comes from Atkin-Lehner involutions.
When the normalizer is not from Atkin-Lehner involutions, the computation is a little more complicated. In the case \( \Gamma = \Gamma_0(8) \) and the normalizer is \( \gamma = \left( \frac{3}{1} \frac{4}{1} \right) \), we have

\[
\eta \left( \left( \frac{4}{8} \frac{1}{4} \right) \tau \right) = \eta \left( \left( \frac{1}{2} \frac{0}{1} \right) (2\tau + 1/2) \right) = e^{2\pi i/24} \sqrt{\frac{4\tau + 2}{i}} \eta(2\tau + 1/2) = e^{2\pi i/16} \sqrt{\frac{4\tau + 2}{i}} \eta(4\tau)^3.
\]

\[
\eta \left( \left( \frac{2}{8} \frac{1}{4} \right) \tau \right) = \eta \left( \left( \frac{1}{1} \frac{1}{2} \right) (4\tau) \right) = e^{2\pi i/8} \sqrt{\frac{4\tau + 2}{i}} \eta(4\tau),
\]

\[
\eta \left( \left( \frac{4}{8} \frac{1}{4} \right) \tau \right) = \eta \left( \left( \frac{2}{1} \frac{1}{1} \right) (2\tau) \right) = e^{2\pi i/8} \sqrt{\frac{2\tau + 1}{i}} \eta(2\tau),
\]

\[
\eta \left( \left( \frac{8}{8} \frac{1}{4} \right) \tau \right) = \eta \left( \left( \frac{4}{1} \frac{1}{0} \right) (\tau + 1/2) \right) = e^{2\pi i/6} \sqrt{\frac{2\tau + 1}{2i}} \eta(\tau + 1/2).
\]

From these, we deduce that

\[
t(\tau) \left| \left( \begin{array}{cc} 4 & 1 \\ 8 & 4 \end{array} \right) \right| = \frac{1}{32t(\tau)},
\]

\[
f(\tau) \left| \left( \begin{array}{cc} 4 & 1 \\ 8 & 4 \end{array} \right) \right| = \frac{\sqrt{8} e^{2\pi i/2}}{8 \tau + 4} \frac{8(2\tau + 1) \eta(8\tau)^4}{i} \eta(4\tau)^2 = 4\sqrt{2} e^{2\pi i/4} t(\tau) f(\tau),
\]

and hence

\[
F(\tau_1, \tau_2) \left| \left( \begin{array}{cc} 4 & 1 \\ 8 & 4 \end{array} \right) \right| = F(\tau_1, \tau_2).
\]

For \( \Gamma = \Gamma_0(9) \) with normalizer \( \left( \frac{-3}{9} \frac{-2}{3} \right) \), we have

\[
\eta \left( \left( \frac{-3}{9} \frac{-2}{3} \right) \tau \right) = \eta \left( \left( \frac{-1}{3} \frac{0}{-1} \right) (\tau + 2/3) \right) = \sqrt{\frac{3\tau + 1}{i}} \eta(\tau + 2/3),
\]

\[
\eta \left( \left( \frac{3}{9} \frac{-2}{3} \right) \tau \right) = \eta \left( \left( \frac{-1}{1} \frac{-2}{1} \right) (3\tau) \right) = \sqrt{\frac{3\tau + 1}{i}} \eta(3\tau),
\]

\[
\eta \left( \left( \frac{9}{9} \frac{-2}{3} \right) \tau \right) = \eta \left( \left( \frac{-3}{1} \frac{-1}{0} \right) (\tau + 1/3) \right) = e^{-2\pi i/8} \sqrt{\frac{\tau + 1/3}{i}} \eta(\tau + 1/3).
\]

Hence,

\[
t(\tau) \left| \left( \begin{array}{cc} -3 & -2 \\ 9 & 3 \end{array} \right) \right| = e^{-6\pi i/8} \eta(\tau + 1/3)^3 = \frac{1 - \alpha t}{\alpha(1 - \beta t)},
\]

\[
f(\tau) \left| \left( \begin{array}{cc} -3 & -2 \\ 9 & 3 \end{array} \right) \right| = \frac{1}{i} \eta(\tau + 2/3)^3 \eta(3\tau) = e^{-2\pi i/6} (1 - \beta t(\tau)) f(\tau),
\]

and

\[
F(\tau_1, \tau_2) \left| \left( \begin{array}{cc} -3 & -2 \\ 9 & 3 \end{array} \right) \right| = -F(\tau_1, \tau_2),
\]

where \( \alpha = (-9 + 3\sqrt{-3})/2 \) and \( \beta = (-9 - 3\sqrt{-3})/2 \).
The case $(11, 3, -1)$ is more complicated. For integers $g$ and $h$ not congruent to $0$ modulo $5$ simultaneously, define the generalized Dedekind eta function $E_{g,h}(\tau)$ by

$$E_{g,h}(\tau) := q^{B(5/2)} \prod_{m=1}^{\infty} \left(1 - \zeta^h q^{m-1+g/5}\right) \left(1 - \zeta^{-h} q^{m-g/5}\right), \quad \zeta = e^{2\pi i/5},$$

where $B(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial. They are modular functions on some congruence subgroup of $SL(2, \mathbb{Z})$. Then the modular function $t(\tau)$ and the modular form $f(\tau)$ can be written as

$$t(\tau) = \frac{E_{1,0}(5\tau)^5}{E_{2,0}(5\tau)^5}, \quad f(\tau) = \frac{\eta(5\tau)^2 E_{2,0}(5\tau)^2}{E_{1,0}(5\tau)^3},$$

respectively. Using transformation laws for generalized Dedekind eta functions [36, Theorem 1], we find that

\begin{align*}
E_{1,0} \left( \frac{5}{2} \begin{array}{cc} 2 & -1 \\ 5 & -2 \end{array} \right) \tau &= E_{1,0} \left( \frac{2}{1} \begin{array}{cc} -5 & 0 \\ 1 & -2 \end{array} \right) (5\tau) = i e^{-2\pi i/5} E_{2,0}(5\tau), \\
E_{2,0} \left( \frac{5}{2} \begin{array}{cc} 2 & -1 \\ 5 & -2 \end{array} \right) \tau &= E_{2,0} \left( \frac{2}{1} \begin{array}{cc} -5 & 0 \\ 1 & -2 \end{array} \right) (5\tau) = -ie^{2\pi i/5} E_{1,0}(5\tau), \\
\eta \left( \frac{5}{2} \begin{array}{cc} 2 & -1 \\ 5 & -2 \end{array} \right) \tau &= \eta \left( \frac{2}{1} \begin{array}{cc} -5 & 0 \\ 1 & -2 \end{array} \right) (5\tau) = \sqrt{5\tau - 2} \eta(5\tau).
\end{align*}

It follows that

\begin{align*}
t(\tau) \left| \begin{array}{cc} 2 & -1 \\ 5 & -2 \end{array} \right| = -\frac{E_{2,0}(5\tau)^5}{E_{1,0}(5\tau)^5} = -\frac{1}{t(\tau)}, \\
\frac{\eta(5\tau)^2 E_{1,0}(5\tau)^2}{E_{2,0}(5\tau)^3} = -t(\tau) f(\tau),
\end{align*}

and

$$F(\tau_1, \tau_2) \left| \begin{array}{cc} 2 & -1 \\ 5 & -2 \end{array} \right| = F(\tau_1, \tau_2).$$

Also,

\begin{align*}
E_{1,0} \left( \frac{5}{0} \begin{array}{cc} 0 & -1 \\ 5 & 0 \end{array} \right) \tau &= E_{1,0} \left( \frac{0}{1} \begin{array}{cc} -1 & 0 \\ 1 & 0 \end{array} \right) (5\tau) = \frac{e^{2\pi i/10}}{i} E_{0,-1}(\tau), \\
E_{2,0} \left( \frac{5}{0} \begin{array}{cc} 0 & -1 \\ 5 & 0 \end{array} \right) \tau &= E_{2,0} \left( \frac{0}{1} \begin{array}{cc} -1 & 0 \\ 1 & 0 \end{array} \right) (5\tau) = \frac{e^{2\pi i/5}}{i} E_{0,-2}(\tau), \\
\eta \left( \frac{5}{0} \begin{array}{cc} 0 & -1 \\ 5 & 0 \end{array} \right) \tau &= \eta \left( \frac{0}{1} \begin{array}{cc} -1 & 0 \\ 1 & 0 \end{array} \right) (5\tau) = \sqrt{\frac{5}{\tau}} \eta(\tau).
\end{align*}

Hence,

\begin{align*}
t(\tau) \left| \begin{array}{cc} 0 & -1 \\ 5 & 0 \end{array} \right| = -\frac{E_{0,-1}(\tau)^5}{E_{0,-2}(\tau)^5} = \frac{1 - \alpha t(\tau)}{\alpha(1 - \beta t(\tau))}, \\
\frac{e^{2\pi i/10} \eta(\tau)^2 E_{0,-2}(\tau)^2}{\sqrt{5}} = -\frac{\alpha}{5\sqrt{5}} (1 - \beta t(\tau)) f(\tau),
\end{align*}

and

$$F(\tau_1, \tau_2) \left| \begin{array}{cc} 0 & -1 \\ 5 & 0 \end{array} \right| = -F(\tau_1, \tau_2),$$

where $\alpha = (11 + 5\sqrt{5})/2$ and $\beta = (11 - 5\sqrt{5})/2$. \hfill \Box
2.2. **Sun’s series.** In Sections 2 and 3 of [30], Sun listed several families of conjectural $1/\pi$-series of the form

$$
\sum_{n=0}^{\infty} (An + B) \sum_{m=0}^{n} a_{n,m} x^n y^m = \frac{C}{\pi}
$$

for some coefficients $a_{n,m}$ expressible in terms of binomial coefficients, some rational numbers $A, B, x, y$, and some algebraic number $C$ (Conjectures 2, 3(ii), 3(iii), 6(i), 6(ii), and I–VII). A close inspection suggests that $x$ and $y$ in several families, including those in Conjectures 2, 6(i), and 6(ii), are related by algebraic functions. In addition, Zudilin [31] found that $x$ and $y$ in Conjecture VII are parameterized by a single modular function on $\Gamma_0(7)$. Those series should be regarded as one-variable $1/\pi$-series and will not be discussed here. The bimodular properties of the remaining case are described below.

**Series in Conjectures I–III.** The series in Sun’s Conjectures I–III are of the form

$$
\sum_{n=0}^{\infty} \frac{(a)n(1-a)n}{(n!)^2} T_n(bz, cz) z^n = \sum_{n=0}^{\infty} \frac{(a)n(1-a)n}{(n!)^2} T_n(bz, cz) x^n y^m,
$$

with $a = 1/2, 1/3, \text{and} 1/4$, respectively. As mentioned in the introduction section, they can also be written as

$$
\sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} \frac{(a)n(1-a)n}{(n!)^2} \left( \frac{n}{2m} \right) \left( \frac{2m}{m} \right) x^n y^m, \quad x = bz, \quad y = \frac{c}{b^2}.
$$

The bimodular properties of these series were already studied in [38, Theorem 4.1].

**Proposition 8.** For $a = 1/2$, let

$$
t(\tau) = \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4}, \quad f(\tau) = \theta_4(\tau)^2,
$$

where

$$
\theta_2(\tau) = \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2}, \quad \theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^n, \quad \theta_4(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^n
$$

are the Jacobi theta functions. For $a = 1/3$, let

$$
t(\tau) = -2\frac{\eta(3\tau)^{12}}{\eta(\tau)^{12}}, \quad f(\tau) = \sum_{m, n \in \mathbb{Z}} q^{m^2 + mn + n^2}.
$$

For $a = 1/4$, let

$$
t(\tau) = -64 \frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}}, \quad f(\tau) = (2E_2(2\tau) - E_2(\tau))^{1/2},
$$

where $E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} nq^n / (1 - q^n)$ is the Eisenstein series of weight 2 on $\text{SL}(2, \mathbb{Z})$. Set $t_1 = t(\tau_1)$, $t_2 = t(\tau_2)$,

$$
x(\tau_1, \tau_2) = -\frac{t_1 + t_2}{(1 - t_1)(1 - t_2)}, \quad y(\tau_1, \tau_2) = \frac{t_1 t_2}{(t_1 + t_2)^2}, \quad F(\tau_1, \tau_2) = f(\tau_1) f(\tau_2).
$$

Then we have

$$
F(\tau_1, \tau_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} \frac{(a)n(1-a)n}{(n!)^2} \left( \frac{2m}{m} \right) \left( \frac{n}{2m} \right) x^n y^m.
$$
Moreover, for \( a = 1/2 \), \( x \) and \( y \) are bimodular functions and \( F \) is a bimodular form of weight 1 on \( (\Gamma_0(4), \Gamma_0(4) + w) \), \( w = (\frac{1}{2} \, \frac{0}{1}) \), with characters
\[
\chi_1 \left( \begin{array}{cc} * & * \\ c & d \end{array} \right) = (-1)^{c/2} \left( \frac{-4}{d} \right), \quad \chi_2(w) = -1.
\]
For \( a = 1/3 \), \( x \) and \( y \) are bimodular functions and \( F \) is a bimodular form of weight 1 on \( (\Gamma_0(3), \Gamma_0(3) + w_3) \) with characters
\[
\chi_1 \left( \begin{array}{cc} * & * \\ c & d \end{array} \right) = \left( \frac{-3}{d} \right), \quad \chi_2(w_3) = -1.
\]
For \( a = 1/4 \), \( x \) and \( y \) are bimodular functions and \( F^2 \) is a bimodular form of weight 2 on \( (\Gamma_0(2), \Gamma_0(2) + w_2) \) with trivial characters.

**Proof.** The identity \([13]\) was proved in \([38]\). It is clear from the criterion given in \([22, \text{ Proposition 3.2.8}]\) that the functions \( t(\tau) \) are modular on their respective groups. Moreover, for the cases \( a = 1/3 \) and \( a = 1/4 \), it is easy to see that their respective Atkin-Lehner involutions map \( t \) to \( 1/t \) and hence \( x \) and \( y \) are bimodular functions on the given groups.

For \( a = 1/2 \), we note that
\[
\theta_2(\tau) = \frac{\eta(4\tau)^2}{\eta(2\tau)}, \quad \theta_3(\tau) = \frac{\eta(2\tau)^5}{\eta(\tau)^2 \eta(4\tau)^2}, \quad \theta_4(\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)},
\]
and
\[
t(\tau) = 16 \frac{\eta(\tau)^8 \eta(4\tau)^{16}}{\eta(2\tau)^{24}}.
\]
We then compute that
\[
\eta \left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right) \tau = e^{2\pi i/24} \sqrt{\frac{2\tau + 1}{i \eta(\tau)}},
\]
\[
\eta \left( \begin{array}{cc} 2 & 1 \\ 2 & 1 \end{array} \right) \tau = \eta \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) (2\tau) = e^{2\pi i/12} \sqrt{\frac{2\tau + 1}{i \eta(2\tau)}},
\]
\[
\eta \left( \begin{array}{cc} 4 & 1 \\ 2 & 1 \end{array} \right) \tau = \eta \left( \begin{array}{cc} 2 & -1 \\ 1 & 0 \end{array} \right) (\tau + 1/2) = e^{2\pi i/12} \sqrt{\frac{2\tau + 1}{2i \eta(\tau + 1/2)}} \eta(2\tau)^3
\]
and hence
\[
t \left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \tau \right) = \frac{1}{t(\tau)}.
\]
It follows that \( x \) and \( y \) are bimodular functions on \( (\Gamma_0(4), \Gamma_0(4) + w) \), \( w = (\frac{1}{2} \, \frac{0}{1}) \) in the case \( a = 1/2 \).

The bimodular property of \( F^2 \) in the case \( a = 1/4 \) is obvious. That of \( F \) in the case \( a = 1/3 \) follows from general properties of the theta series \( \sum_{m,n} q^{m^2+mn+n^2} \). Finally, for the case \( a = 1/2 \), the computation above shows that
\[
f(\tau) \left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right) = \frac{1}{t} f(\tau), \quad F(\tau_1, \tau_2) \left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right) = -F(\tau_1, \tau_2),
\]
This completes the proof. \( \square \)
Series in Conjectures IV and 3(ii)(iii). In Conjecture IV of [30], Sun listed a family of \(1/\pi\)-series of the form
\[
\sum_{n=0}^{\infty} \left(\frac{2n}{n}\right)^2 T_{2n}(b,c)(An + B)C^n = \frac{D}{\pi}.
\]
These formulas were proved in [34]. The key ingredient in the proof is Wan and Zudilin’s identity ([34, Theorem 3])
\[
\sum_{n=0}^{\infty} \frac{(1/2)^n}{(n!)^2} P_{2n} \left(\frac{(X + Y)(1 - XY)}{X - Y}(1 + XY)\right) \left(\frac{X - Y}{X - Y}(1 + XY)\right)^{2n} = 1 + XY,
\]
valid for \((X, Y)\) close to \((1, 1)\), where \(P_n(x)\) is the \(n\)th Legendre polynomial defined by (10). Using (11), we see that
\[
P_n \left(\frac{(X + Y)(1 - XY)}{(X - Y)(1 + XY)}\right) = \frac{n}{(n-1)!} \left(\frac{Y(1-X)}{X-Y}(1+XY)\right)^{m} \left(\frac{X(1-X^2)}{(X-Y)(1+XY)}\right)^{n-m}.
\]
Thus, (15) can be written as
\[
\frac{1 + XY}{2} \sum_{n=0}^{\infty} \frac{(1/2)^n}{(n!)^2} T_{2n} \left(\frac{(X + Y)(1 - XY), XY(1 - X^2)(1 - Y^2)}{(X - Y)(1 + XY)}\right).
\]
Letting
\[
x = \frac{(X + Y)^2(1 - XY)^2}{(1 + XY)^4}, \quad y = \frac{XY(1 - X^2)(1 - Y^2)}{(X + Y)^2(1 - XY)^2},
\]
we then have
\[
\frac{1 + XY}{2} \sum_{n=0}^{\infty} \frac{(1/2)^n}{(n!)^2} \sum_{m=0}^{n} \left(\frac{2m}{2m}\right) \left(\frac{2n}{2m}\right) x^m y^n.
\]
Note that \(x\) and \(y\) are both invariant under
\[
(X, Y) \mapsto \left(\frac{1}{X}, \frac{1}{Y}\right), \quad (X, Y) \mapsto \left(\frac{X - 1}{X + 1}, \frac{Y - 1}{Y + 1}\right),
\]
in addition to the obvious symmetry \((X, Y) \mapsto (Y, X)\). The modular meanings of these symmetries are as follows.
Recall that
\[
\theta_3(\tau)^2 = \theta_2(\tau) \left(\frac{1}{2}; \frac{1}{2}; \frac{1}{\theta_3(\tau)^4}\right)
\]
and
\[
\theta_2(\tau)^4 + \theta_4(\tau)^4 = \theta_3(\tau)^4.
\]
Therefore, if we let
\[ t(\tau) = \frac{\theta_3(\tau)^2}{\theta_3(\tau)} = \frac{\eta(\tau)^8 \eta(4\tau)^4}{\eta(2\tau)^{12}}, \]
then \( t_1 = t(\tau_1), t_2 = t(\tau_2), \) and
\[
(18) \quad x = \frac{(t_1 + t_2)^2(1 - t_1) t_2^2}{(1 + t_1 t_2)^4}, \quad y = \frac{t_1 t_2 (1 - t_1^2)(1 - t_2^2)}{(1 + t_1 t_2)^2(1 - t_1 t_2)^2},
\]
then (16) just says that
\[
(19) \quad \frac{1 + t_1 t_2}{2} \theta_3(\tau_1)^2 \theta_3(\tau_2)^2 = \sum_{n=0}^{\infty} \frac{(1/2)_n^2}{(n!)^2} \sum_{m=0}^{n} \left( \frac{2m}{m} \right) \left( \frac{2n}{2m} \right) x^n y^m.
\]

The function \( t(\tau) \) is a modular function on \( \Gamma_0(8) \) holomorphic throughout the upper half-plane. Its values at the four cusps \( \infty, 1/4, 1/2, \) and 0, of \( X_0(8) \) are 1, \(-1, \) \( \infty, \) and 0, respectively. In particular, it generates the field of modular functions on \( X_0(8). \) Let \( N(\Gamma_0(8)) \) be the normalizer of \( \Gamma_0(8) \) in \( \text{SL}(2, \mathbb{R}). \) From the description of \( N(\Gamma_0(8)) \) given in [17], we know that \( N(\Gamma_0(8))/\Gamma_0(8) \) is isomorphic to the dihedral group \( D_8 \) of 8 elements and is generated by
\[
\sigma_1 = \left( \begin{array}{cc} 0 & -1 \\ 8 & -4 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & -1 \\ 8 & 0 \end{array} \right),
\]
with \( \sigma_1^4 = \sigma_2^2 = \text{id} \) and \( \sigma_2 \sigma_1 \sigma_2^{-1} = \sigma_1^{-1}. \) From the values of \( t \) at the four cusps, it is easy to see that
\[
t_i \sigma_1 = \frac{t - 1}{t + 1}, \quad t_i \sigma_2 = \frac{1 - t}{1 + t},
\]
Therefore, the invariances in [17] mean that \( x(\tau_1, \tau_2) \) and \( y(\tau_1, \tau_2) \) are bimodular functions on \( (\Gamma_0(8), N(\Gamma_0(8))). \)

Now let \( F(\tau_1, \tau_2) \) be the function on the left-hand side of (19). We have
\[
\eta \left( \begin{array}{cc} 0 & -1 \\ 8 & -4 \end{array} \right) = \sqrt{\frac{8\tau - 4}{i}} \cdot \eta(8\tau - 4) = e^{-2\pi i/6} \sqrt{\frac{8\tau - 4}{i}} \cdot \eta(8\tau),
\]
\[
\eta \left( \begin{array}{cc} 2 & 0 \\ 8 & -4 \end{array} \right) = \sqrt{\frac{4\tau - 2}{i}} \cdot \eta(4\tau - 2) = e^{-2\pi i/12} \sqrt{\frac{4\tau - 2}{i}} \cdot \eta(4\tau),
\]
\[
\eta \left( \begin{array}{cc} 4 & 0 \\ 8 & -4 \end{array} \right) = \sqrt{\frac{2\tau - 1}{i}} \cdot \eta(2\tau - 1) = e^{-2\pi i/24} \sqrt{\frac{2\tau - 1}{i}} \cdot \eta(2\tau).
\]
It follows that
\[
\theta_3(\tau)^2 \left| \frac{1}{2} \right. \left( \begin{array}{cc} 0 & -1 \\ 8 & -4 \end{array} \right) = \sqrt{\frac{7}{i}} \cdot \frac{\eta(4\tau)^{10}}{\eta(2\tau)^{4} \eta(8\tau)^4} \cdot \frac{1}{\sqrt{2t}(1 + t(\tau))} \theta_3(\tau)^2,
\]
and
\[
F(\tau_1, \tau_2) \left| \frac{1}{2} \right. \left( \begin{array}{cc} 0 & -1 \\ 8 & -4 \end{array} \right) = -\frac{1}{4} \left( 1 + \frac{t_1 - 1}{t_1 + 1} \right) \left( 1 + \frac{t_2 - 1}{t_2 + 1} \right) (1 + t_1)(1 + t_2) \theta_3(\tau_1)^2 \theta_3(\tau_2)^2
\]
\[
= -F(\tau_1, \tau_2).
\]
A similar computation shows that \( F(\tau_1, \tau_2) \left| \frac{1}{2} \right. \left( \begin{array}{cc} 0 & -1 \\ 0 & 0 \end{array} \right) = -F(\tau_1, \tau_2) \) as well. We summarize the bimodular properties of the series in Sun’s Conjecture IV in the following proposition.
Proposition 9. Let \( x(\tau_1, \tau_2) \) and \( y(\tau_1, \tau_2) \) be defined by (18). Let also \( F(\tau_1, \tau_2) \) be the function on the left-hand side of (19). Then \( x \) and \( y \) are bimodular functions and \( F \) is a bimodular form of weight 1 on \( \Gamma_0(8), N(\Gamma_0(8)) \) with characters \( (\chi_1, \chi_2) \) given by

\[
\chi_1 \left( \begin{array}{cc} * & * \\ * & d \end{array} \right) = \left( \frac{-1}{d} \right), \quad \chi_2 \left( \begin{array}{cc} 0 & -1 \\ 8 & -4 \end{array} \right) = \chi_2 \left( \begin{array}{cc} 0 & -1 \\ 8 & 0 \end{array} \right) = -1.
\]

We now consider Sun’s Conjectures (ii) and (iii), where the series are of the form

\[
\sum_{n=0}^{\infty} \left( An + B \right) \left( \frac{2n}{n} \right) \sum_{m=0}^{n} \left( \frac{n}{m} \right)^2 \left( \frac{2m}{n} \right) C^n D^{2m-n} = E
\]

and

\[
\sum_{n=0}^{\infty} \left( An + B \right) \left( \frac{2n}{n} \right) \sum_{m=0}^{n} \left( \frac{n}{m} \right)^2 \left( \frac{2m}{n} \right) (-1)^m C^n D^{2m-n} = E,
\]

respectively. The bimodular properties of the two types of series are clearly the same. In [26, Theorem 2.3], Rogers and Straub show that if \( s \) and \( t \) are related to \( X \) and \( Y \) by

\[
-st = \left( \frac{X - Y}{4(1 + XY)} \right)^2, \quad 1 + 4s t = \left( \frac{(X + Y)(1 - XY)}{(X - Y)(1 + XY)} \right)^2,
\]

then

\[
\sum_{n=0}^{\infty} \left( \frac{2n}{n} \right) \sum_{m=0}^{n} \left( \frac{n}{m} \right)^2 \left( \frac{2m}{n} \right) (-1)^m s^n t^{2m-n}
\]

\[
= \frac{1 + XY}{2} 2F_1(1/2, 1/2; 1; 1 - X^2) F_1(1/2, 1/2; 1 - X^2).
\]

Thus, setting

\[
x = \frac{s}{t} = \frac{XY(1 - X^2)/(1 - Y^2)}{(X - Y)^2(1 + XY)^2}, \quad y = -t^2 = \frac{(X - Y)^4}{16XY(1 - X^2)(1 - Y^2)},
\]

we have

\[
\sum_{n=0}^{\infty} \left( \frac{2n}{n} \right) \sum_{m=0}^{n} \left( \frac{n}{m} \right)^2 \left( \frac{2m}{n} \right) x^n y^m
\]

\[
= \frac{1 + XY}{2} 2F_1(1/2, 1/2; 1; 1 - X^2) F_1(1/2, 1/2; 1 - Y^2).
\]

As pointed out by Rogers and Straub [26], both expressions in (20) are invariant under (17). Therefore, the bimodular properties of this series is the same as those of the series in Conjecture IV.

Proposition 10. Let

\[
t(\tau) = \frac{\theta_4(\tau)^2}{\theta_3(\tau)^2} = \frac{\eta(\tau)^8 \eta(4\tau)^4}{\eta(2\tau)^{12}},
\]

\( t_1 = t(\tau_1), t_2 = t(\tau_2) \), and

\[
x = \frac{t_1 t_2 (1 - t_1^2)(1 - t_2^2)}{(t_1 - t_2)(1 - t_1 t_2)^2}, \quad y = \frac{(t_1 - t_2)^4}{16 t_1 t_2 (1 - t_1^2)(1 - t_2^2)}.
\]

Then

\[
\frac{1 + t_1 t_2}{2} \theta_3(\tau_1)^2 \theta_3(\tau_2)^2 = \sum_{n=0}^{\infty} \left( \frac{2n}{n} \right) \sum_{m=0}^{n} \left( \frac{2m}{n} \right)^2 \left( \frac{n}{m} \right)^2 x^n y^m.
\]

Their bimodular properties are the same as those in Proposition 9.
2.3. Series in Conjecture V. In [30] Conjecture V, Sun recorded one single $1/\pi$-series
\[
\sum_{n=0}^{\infty} \frac{(1/3)_n(2/3)_n}{(n!)^2} T_{3n}(61, 1) \frac{1638n + 277}{(-80)^{3n}} = \frac{44\sqrt{105}}{\pi}
\]
involving $T_{3n}$. In [34] (34), Wan and Zudilin gave another example of such series. Here we shall discuss the bimodular properties of these series.

According to [34] Theorem 3], one has
\[
\sum_{n=0}^{\infty} \frac{(1/3)_n(2/3)_n}{W(n!)^2} P_{3n} \left( X + Y - 2X^2Y^2 \right) \left( X - Y \right)^{3n} = \frac{W^3}{3} \sum_{n=0}^{\infty} \left( \frac{(1/3)_n(2/3)_n}{(n!)^2} \right) T_{3n}(X + Y - 2X^2Y^2, XY(1 - X^3)(1 - Y^3)).
\]

where $W = \sqrt{1 + 4XY(X + Y)}$, for $(X, Y)$ close to $(1, 1)$. Using (10) and (11), this can be written as
\[
\frac{W^3}{3} \sum_{n=0}^{\infty} \left( \frac{(1/3)_n(2/3)_n}{(n!)^2} \right) T_{3n}(X + Y - 2X^2Y^2, XY(1 - X^3)(1 - Y^3))
\]

Setting
\[
x = \frac{(X + Y - 2X^2Y^2)^3}{(1 + 4XY(X + Y))^3}, \quad y = \frac{XY(1 - X^3)(1 - Y^3)}{(X + Y - 2X^2Y^2)^3},
\]
we find that
\[
\frac{W^3}{3} \sum_{n=0}^{\infty} \left( \frac{(1/3)_n(2/3)_n}{(n!)^2} \right) T_{3n}(X + Y - 2X^2Y^2, XY(1 - X^3)(1 - Y^3))
\]

On the other hand, Borwein and Borwein [5] Theorem 2.3] showed if we set
\[
L(\tau) = \sum_{m,n \in \mathbb{Z}} q^{m^2+mn+n^2}
\]

and
\[
a(\tau) := L(\tau), \quad b(\tau) := \frac{1}{2} (3L(3\tau) - L(\tau)), \quad c(\tau) := \frac{1}{2} (L(\tau/3) - L(\tau)),
\]
then
\[
a(\tau) = 2F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; \frac{c(\tau)^3}{a(\tau)^3} \right)
\]

and
\[
a(\tau)^3 = b(\tau)^3 + c(\tau)^3.
\]

Therefore, letting
\[
t(\tau) = \frac{b(\tau)}{a(\tau)},
\]

$t_1 = t(\tau_1), t_2 = t(\tau_2)$, and
\[
x = \left( \frac{t_1 + t_2 - 2t_1^2 t_2^2}{1 + 4t_1 t_2(t_1 + t_2)} \right)^3, \quad y = \frac{t_1 t_2(1 - t_1^3)(1 - t_2^3)}{(t_1 + t_2 - 2t_1^2 t_2^2)^2},
\]

(21)
we have
\[
\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \frac{[3n/2]}{(n!)^2} \sum_{m=0}^{[3n/2]} \binom{2m}{m} \binom{3n}{2m} x^ny^m
\]
(22)
= \frac{1}{3} \sqrt{1 + 4t_1t_2(t_1 + t_2)} F_1(1/3, 2/3; 1; 1 - t_1^3) F_1(1/3, 2/3; 1; 1 - t_2^3)
= \frac{1}{3} \sqrt{1 + 4t_1t_2(t_1 + t_2)} a(\tau_1)a(\tau_2).

The function \( t(\tau) \) is modular on \( \Gamma_0(9) \). The relation between \( t \) and the Hauptmodul \( j_9(\tau) = \eta(\tau)^3/\eta(9\tau)^3 \) on \( \text{X}_0(9) \) is
\[
t = \frac{j_9}{j_9 + 9}.
\]

Note that
\[
\Gamma_0(9) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right) \Gamma(3) \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right)^{-1}.
\]

Since \( \Gamma(3) \) is a normal subgroup of \( \text{SL}(2, \mathbb{Z}) \), \( \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right) \) \( \text{SL}(2, \mathbb{Z}) \) \( \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right)^{-1} \) normalizes \( \Gamma_0(9) \).

From [17, Section 3], we see that this is precisely the normalizer \( N(\Gamma_0(9)) \) of \( \Gamma_0(9) \) in \( \text{PSL}(2, \mathbb{R}) \). The quotient group \( N(\Gamma_0(9))/\Gamma_0(9) \) is generated by
\[
\sigma_1 = \left( \begin{array}{cc} 0 & -1/3 \\ 3 & 0 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 1 & 0 \\ 3 & 1 \end{array} \right),
\]
of orders 2 and 3, respectively. Now, we have
\[
t|\sigma_1 = \frac{j_9}{j_9 + 9}|\sigma_1 = \frac{27/j_9}{27/j_9 + 9} = \frac{1 - t}{1 + 2t}.
\]

For \( \sigma_2 \), we compute that
\[
\eta\left( \left( \begin{array}{cc} 1 & 0 \\ 3 & 1 \end{array} \right) \tau \right) = \sqrt{\frac{3\tau + 1}{i}} \eta(\tau),
\]
\[
\eta\left( \left( \begin{array}{cc} 9 & 0 \\ 3 & 1 \end{array} \right) \tau \right) = \eta\left( \left( \begin{array}{cc} 3 & -1 \\ 1 & 0 \end{array} \right) (\tau + 1/3) \right) = e^{2\pi i/8} \sqrt{\frac{\tau + 1/3}{i}} \eta(\tau + 1/3),
\]
and
\[
j_9|\left( \begin{array}{cc} 1 & 0 \\ 3 & 1 \end{array} \right) = \sqrt{27} e^{-6\pi i/8} \frac{\eta(\tau)^3}{\eta(\tau + 1/3)^3} = \frac{\beta j_9}{j_9 - \alpha},
\]
where \( \alpha = (-9 + 3\sqrt{-3})/2 \) and \( \beta = (-9 - 3\sqrt{-3})/2 \). It follows that
\[
t|\sigma_2 = \rho^2 t, \quad \rho = e^{2\pi i/3}.
\]

We check that the functions \( x \) and \( y \) of \( t_1 \) and \( t_2 \) are invariant under
\[
(t_1, t_2) \mapsto \left( \frac{1 - t_1}{1 + 2t_1}, \frac{1 - t_2}{1 + 2t_2} \right), \quad (t_1, t_2) \mapsto (\rho t_1, \rho t_2).
\]

In other words, \( x \) and \( y \) are bimodular functions on \( (\Gamma_0(9), N(\Gamma_0(9))) \).

Let \( F(\tau_1, \tau_2) \) be the function in the last expression of (22). We now consider the bimodular property of \( F^2 \). We check that
\[
a(\tau)^2 = \frac{1}{2} (3E_2(3\tau) - E_2(\tau)), \quad E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n},
\]
Then
\[ a(\tau)^2 \sigma_1 = \frac{1}{2} (3E_2(3\tau) - 9E_2(9\tau)) = -a(\tau)^2 (1 + 2t(\tau))^2. \]

Also, since \( a(\tau)^2 \) is a modular form of weight 2 on \( \Gamma_0(3) \), \( a(\tau)^2 \sigma_2 = a(\tau)^2 \). It follows that \( F(\tau_1, \tau_2)^2 \) is a bimodular form of weight 2 on \( (\Gamma_0(9), \mathbb{N}(\Gamma_0(9))) \) with trivial characters. We summarize our findings in the following proposition.

**Proposition 11.** Let \( x(\tau_1, \tau_2) \) and \( y(\tau_1, \tau_2) \) be defined by (21) and \( F(\tau_1, \tau_2) \) be the function in the last expression of (22). Then \( x \) and \( y \) are bimodular functions and \( F^2 \) is a bimodular form of weight 2 with trivial characters on \( (\Gamma_0(9), \mathbb{N}(\Gamma_0(9))) \).

### 3. Ramanujan-type 1/π-series from bimodular forms

In this section we shall describe procedures to obtain Ramanujan-type 1/π-series from bimodular forms.

**3.1. General form of 1/π-series.** We first review the notion of CM-points on modular curves. Let \( \mathcal{O} \) be an order in \( M(2, \mathbb{Q}) \), that is, a \( \mathbb{Z} \)-module of rank 4 that is also a subring with unity of \( M(2, \mathbb{Q}) \). Let \( \mathcal{O}^1 \) be the group of elements of determinant 1 in \( \mathcal{O} \). We call the quotient space \( \mathcal{O}^1 / \mathbb{H} \), after compactification by adding cusps, the modular curve associated to \( \mathcal{O} \) and denote it by \( X(\mathcal{O}) \). Here we will only consider two types of orders

\[
\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{Z}) : N|c \right\}, \quad \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{Z}) : N|c, a \equiv d \mod N \right\},
\]

where \( N \) is a positive integer. The groups of elements of determinant 1 in the two cases are \( \Gamma_0(N) \) and \( \Gamma_1(N) \) and the modular curves are \( X_0(N) \) and \( X_1(N) \), respectively.

Let \( \mathcal{O} \) be an order in \( M(2, \mathbb{Q}) \). For an embedding \( \phi : \mathbb{Q}(\sqrt{d_0}) \hookrightarrow M(2, \mathbb{Q}) \) from a quadratic number field \( \mathbb{Q}(\sqrt{d_0}) \) to \( M(2, \mathbb{Q}) \), it can be shown that the intersection of \( \phi(\mathbb{Q}(\sqrt{d_0})) \) and \( \mathcal{O} \) is \( \phi(R) \) for some quadratic order \( R \) in \( \mathbb{Q}(\sqrt{d_0}) \). Let \( d \) be the discriminant of this quadratic order \( R \). We say \( \phi \) is an optimal embedding of discriminant \( d \) with respect to \( \mathcal{O} \). Now if \( d_0 < 0 \), then elements in \( \phi(R) \) share a (unique) common fixed point \( \tau_{\phi} \) in \( \mathbb{H} \). We call this point a CM-point of discriminant \( d \) on the modular curve \( X(\mathcal{O}) \). Notice that if \( \phi : \mathbb{Q}(\sqrt{d}) \hookrightarrow M(2, \mathbb{Q}) \) is an optimal embedding of discriminant \( d \) with respect to \( \mathcal{O} \), then so is \( -\phi \) and they share the same fixed point \( \tau_{\phi} \). We say \( \phi \) is normalized if

\[
\phi(\alpha) \left( \begin{array}{c} \tau_{\phi} \\ 1 \end{array} \right) = \alpha \left( \begin{array}{c} \tau_{\phi} \\ 1 \end{array} \right)
\]

for all \( \alpha \in \mathbb{Q}(\sqrt{d}) \). It is clear that this condition amounts to the assumption that the \((2, 1)\)-entry of \( \phi(\sqrt{d}) \) is positive.

**Lemma 12.** Let \( t \) be a nonconstant modular function on a modular curve \( X \) associated to some order \( \mathcal{O} \) in \( M(2, \mathbb{Q}) \). Let \( g = t'/2\pi i = \theta_{q,t} \). Let \( \tau_d \) be a CM-point of discriminant \( d \) on \( X \) with the corresponding normalized optimal embedding \( \phi \). Let \( \alpha = \phi(\sqrt{d}) \) and \( h = (g|2\alpha)/g \). We have

\[
\theta_{q,t}(\tau_d) - \frac{1}{2} (\theta_{q,t})'(\tau_d) = \frac{1}{2\pi \Im \tau_d}
\]

**Proof.** The proof is essentially the same as that of Equation (4) in [37]. For convenience of the reader, we reproduce the proof here.

Write

\[
\alpha = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.
\]
Since $\alpha$ is the image of $\sqrt{d}$ under the optimal embedding $\phi$, we have $\text{tr} \, \alpha = 0$ and $\det \alpha = |d|$. As $\tau_d$ is a fixed point of $\alpha$, it follows that

$$\tau_d = \frac{(b_1 - b_4) + \sqrt{(b_1 - b_4)^2 + 4b_2b_3}}{2b_3} = \frac{b_1 + \sqrt{d}}{b_3}$$

(note that $\phi$ is normalized, so $b_3 > 0$) and

$$\det \alpha = \frac{|d|}{d} = -1. \quad (24)$$

Consequently, by the definition of the slash operator,

$$h(\tau_d) = \frac{\det \alpha}{(b_3 \tau_d + b_4)^2} g(\alpha \tau_d) = -1. \quad (25)$$

Now recall that the Shimura-Maass operator $\partial_k$ of weight $k$ for a smooth function $f : \mathbb{H} \to \mathbb{C}$ and an integer $k$ is defined to be

$$(\partial_k f)(\tau) := \frac{1}{2\pi i} \left( f'(\tau) + \frac{k f(\tau)}{\tau - \tau} \right).$$

The Shimura-Maass operators satisfy

$$\partial_{k_1+k_2}(f_1 f_2) = (\partial_{k_1} f_1) f_2 + f_1 (\partial_{k_2} f_2) \quad (26)$$

and

$$\partial_k (f |_{k+\gamma}) = (\partial_k f) |_{k+\gamma} \quad (27)$$

for all smooth functions $f_1, f_2, f : \mathbb{H} \to \mathbb{C}$, any integers $k_1, k_2, k$, and any $\gamma \in \text{GL}^+ (2, \mathbb{R})$ (see Equations (1.5) and (1.8) of [28]). The second property in particular implies that if $f$ is modular of weight $k$ on a subgroup $\Gamma$ of $\text{GL}^+ (2, \mathbb{R})$, then $\partial_k f$ is modular of weight $k + 2$ on $\Gamma$.

Now consider $\partial_2(g |_{2\alpha})/g$. By (27), it is equal to $(\partial_2 g) |_{4\alpha}$. On the other hand, by (26),

$$\frac{(\partial_2 g) |_{4\alpha}}{g} = \frac{\partial_2(g |_{2\alpha})}{g} = \frac{\partial_2(gh)}{g} = h \frac{\partial_2 g}{g} + \partial_3 h = h \frac{\partial_2 g}{g} + \theta_q h. \quad (28)$$

Evaluating the two sides at $\tau_d$ and using (24) and (25), we find that

$$\frac{\partial_2 g}{g}(\tau_d) = -\frac{\partial_2 g}{g}(\tau_d) + (\theta_q h)(\tau_d). \quad (28)$$

Since

$$(\partial_2 g)(\tau) = (\theta_q g)(\tau) + \frac{2 g(\tau)}{2\pi i (\tau - \tau)} = (\theta_q g)(\tau) - \frac{g(\tau)}{2\pi \text{Im} \, \tau},$$

(28) can be written as

$$\theta_q g(\tau_d) - \frac{1}{2} (\theta_q h)(\tau_d) = \frac{1}{2\pi \text{Im} \, \tau_d}. \quad (28)$$

This completes the proof of the lemma. □

Then the general form of $1/\pi$-series can be stated as follows.

**Theorem 2.** Let $\{u_n\}$ be one of the six sporadic Apéry-like sequences with parameters $(a, b, c)$, congruence subgroup $\Gamma$, modular function $t$, and modular form $f$ given in Table 1. Let $x(X, Y)$ and $y(X, Y)$ be defined by (12). For two CM-points $\tau_1$ and $\tau_2$ in $\mathbb{H}$ of discriminant $d_1$ and $d_2$, respectively, such that $\mathbb{Q}(\sqrt{d_1}) = \mathbb{Q}(\sqrt{d_2})$, set

$$t_1 = t(\tau_1), \quad t_2 = t(\tau_2), \quad x_0 = x(t_1, t_2), \quad y_0 = y(t_1, t_2).$$
Assume that the series $\sum_n \sum_{m \leq \lfloor n/2 \rfloor} u_n \binom{2m}{m} \binom{n}{2m} x_0^n y_0^m$ converges absolutely. Then there exist algebraic numbers $B_j$ and $C_j$, $j = 1, 2$, such that

\begin{equation}
\sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor n/2 \rfloor} u_n \binom{2m}{m} \binom{n}{2m} (n + B_1) x_0^n y_0^m = \frac{C_1}{\pi}
\end{equation}

and

\begin{equation}
\sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor n/2 \rfloor} u_n \binom{2m}{m} \binom{n}{2m} (m + B_2) x_0^n y_0^m = \frac{C_2}{\pi}.
\end{equation}

To be more concrete, let $\phi_j$ be the optimal embeddings defining the CM-points $\tau_j$ and $\alpha_j = \phi_j(\sqrt{d_j})$, $j = 1, 2$. Set

\[ g = \theta_q t, \quad h_j = (g|2\alpha_j)/g, \]

and

\[ \delta_j = \frac{\theta_q h_j}{f_2}(\tau_j). \]

Set also $\epsilon = f(\tau_2)/f(\tau_1)$. Then

\[ B_1 = \frac{c(t_1(\theta_x Y)|_{(t_1,t_2)} + t_2(\theta_x X)|_{(t_1,t_2)})}{1 - ct_2 t_1} + \frac{(\theta_x X)|_{(t_1,t_2)} - \delta_1 + 2 - 4at_1 + 6ct_1^2}{4t_1(1 - at_1 + ct_1^2)} \]

\[ + \frac{(\theta_x Y)|_{(t_1,t_2)} - \delta_2 + 2 - 4at_2 + 6ct_2^2}{4t_2(1 - at_2 + ct_2^2)}. \]

and

\[ C_1 = \frac{\epsilon(1 - ct_2)(\theta_x Y)|_{(t_1,t_2)}}{4t_1(1 - at_1 + ct_1^2)\text{Im} \tau_1} + \frac{\epsilon^{-1}(1 - ct_2)(\theta_x X)|_{(t_1,t_2)}}{4t_2(1 - at_2 + ct_2^2)\text{Im} \tau_2}, \]

where $\theta_x = x\partial/\partial x$, and similar expressions with $\theta_x$ replaced by $\theta_y := y\partial/\partial y$ hold for $B_2$ and $C_2$.

We remark that

\[ (\theta_x X)(X, Y) = \frac{X(1 - cY^2)(1 - aX + cX^2)}{1 - a(X + Y) + c(X^2 + 4XY + Y^2) - acXY(X + Y) + c^2X^2Y^2} \]

and a similar expression with the roles of $X$ and $Y$ switched holds for $(\theta_y Y)(X, Y)$. Note that if we let $\alpha$ and $\beta$ be the two numbers such that $1 - aX + cX^2 = (1 - \alpha X)(1 - \beta X)$, then the denominator of the expression above can be factorized as

\[ (1 - \alpha(X + Y) + cXY)(1 - \beta(X + Y) + cXY). \]

The expressions for $\theta_x X$ and $\theta_y Y$ are more complicated. We have

\[ (\theta_x X)(X, Y) = \frac{XY(1 - aX + cX^2)(1 - aY + cY^2)}{(Y - X)(1 - cXY)^2} \]

\[ \times \frac{1 - 2aX + 3cX^2 + cXY(3 - 2aX + cX^2)}{1 - a(X + Y) + c(X^2 + 4XY + Y^2) - acXY(X + Y) + c^2X^2Y^2}. \]

Proof of Theorem 2 The proof consists mostly of straightforward calculus. Abusing notations, for variables $\tau_j \in \mathbb{H}$, $j = 1, 2$, we let $t_j = t(\tau_j)$, $f_j = \sum_{n=0}^{\infty} u_n t_0^n$, $g_j = g(\tau_j)$, $x = x(t_1, t_2)$, and $y = y(t_1, t_2)$. By (12),

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor n/2 \rfloor} u_n \binom{2m}{m} \binom{n}{2m} x^n y^m = (1 - ct_1 t_2)f_1 f_2. \]
Applying the differential operator \( \theta_x := x \partial / \partial x \) on the two sides and using the chain rule, we obtain
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} u_n \binom{2m}{m} \binom{n}{2m} n x^n y^m = -c(t_1 \theta_x t_2 + t_2 \theta_x t_1) f_1 f_2 \\
+ (1 - ct_1 t_2) \left( f_2 \frac{df_1}{dt_1} \theta_x t_1 + f_1 \frac{df_2}{dt_2} \theta_x t_2 \right).
\]

We check case by case that the modular forms \( f \) and \( g \) are related by
\[
g = f^2 t(1 - at + ct^2).
\]

Hence,
\[
f \frac{df}{dt} = \frac{1}{2t(1 - at + ct^2)} \frac{dg}{dt} - f^2 \frac{1 - 2at + 3t^2}{2t(1 - at + ct^2)}.
\]

Plugging this into (31), evaluating at two CM-points \( \tau_1 \) and \( \tau_2 \) of discriminants \( d_1 \) and \( d_2 \) with \( \mathbb{Q}(\sqrt{d_1}) = \mathbb{Q}(\sqrt{d_2}) \), and setting \( \epsilon = f(\tau_2) / f(\tau_1) \), we obtain
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} u_n \binom{2m}{m} \binom{n}{2m} n x^n y^m = -c(t_1 \theta_x t_2 + t_2 \theta_x t_1) f_1 f_2 \\
+ \epsilon(1 - ct_1 t_2) \theta_x t_1 \left( \frac{1}{2t_1(1 - at_1 + ct_1^2)} \frac{dg_1}{dt_1} - f_1^2 \frac{1 - 2at_1 + 3t_1^2}{2t_1(1 - at_1 + ct_1^2)} \right) \\
+ \frac{1}{\epsilon}(1 - ct_1 t_2) \theta_x t_2 \left( \frac{1}{2t_2(1 - at_2 + ct_2^2)} \frac{dg_2}{dt_2} - f_2^2 \frac{1 - 2at_2 + 3t_2^2}{2t_2(1 - at_2 + ct_2^2)} \right).
\]

Finally, substituting (23) into the last expression and simplifying, we obtain the claimed formula. \( \square \)

Remark 13. Following the same computation, we can also obtain the general form of \( 1/\pi \)-series for \( u_n = (a)_n(1 - a)_n/(n!)^2 \) with \( a \in \{1/2, 1/3, 1/4\} \).

Let \( t(\tau), x(\tau_1, \tau_2) \) and \( y(\tau_1, \tau_2) \) be given in Proposition 8 and assume other notations and conditions in Theorem 2. Then (29) and (30) hold with
\[
B_1 = \frac{(1 - t_1)(1 - t_2)}{4(1 - t_1 t_2)}(4 - \delta_1 - \delta_2), \\
C_1 = \frac{(1 - t_1)(1 - t_2)}{4(1 - t_1 t_2)} \left( \frac{\epsilon}{\text{Im} \tau_1} + \frac{\epsilon^{-1}}{\text{Im} \tau_2} \right),
\]
and
\[
B_2 = \frac{1}{4(1 - t_1 t_2)(t_2 - t_1)} \left( (2 - \delta_1)(1 - t_1^2) t_2 - (2 - \delta_2)(1 - t_2^2) t_1 \right),
\]
\[
C_2 = \frac{1}{4(1 - t_1 t_2)(t_2 - t_1)} \left( \frac{\epsilon}{\text{Im} \tau_1}(1 - t_1^2) t_2 - \frac{\epsilon^{-1}}{\text{Im} \tau_2}(1 - t_2^2) t_1 \right).
\]

3.2. \( 1/\pi \)-series with rational \( x \) and \( y \). In this section, we shall describe situations where the values of the bimodular functions \( x \) and \( y \) are rational numbers.

Let \( \{u_n\} \) be one of the sporadic Apéry-like sequences and let the notations \( (a, b, c), \alpha, \beta, \Gamma, G, t, x, \) and \( y \) be given as in Proposition 7.
Lemma 15. Consider the first three cases where $\Gamma = \Gamma_0(6)$ and $G = N(\Gamma_0(6))$. For a negative discriminant $d$, we let $CM(d)$ denote the set of CM-points of discriminant $d$ on $X_0(6)$. It is clear that the group of Atkin-Lehner involutions acts on $CM(d)$.

Lemma 14 ([23, Theorem 2]). Let $d$ be a negative discriminant. Write it as $d = r^2d_0$, where $d_0$ is a fundamental discriminant. Then the number $|CM(d)|$ of CM-points of discriminant $d$ on $X_0(6)$ is

$$|CM(d)| = h(d) \times \prod_{p \mid d} \nu_p(d),$$

where $h(d)$ is the class number of the quadratic order of discriminant $d$ and $\nu_p(d)$ is defined by

$$\nu_p(d) = \begin{cases} 1 + \left( \frac{d}{p} \right), & \text{if } p \nmid r, \\ 2, & \text{if } p \mid r. \end{cases}$$

Note that the function $\nu_p(d)$ is called the Eichler symbol (in the case $M(2, \mathbb{Q})$) in literature.

Lemma 15. Let $d$ be a negative discriminant such that the set $CM(d)$ of CM-points of discriminant $d$ on $X_0(6)$ is nonempty. Let $K = \mathbb{Q}^{\sqrt{d}}$, and $R$, $h(d)$, and $H$ be the quadratic order of discriminant $d$, its class number, and its ring class field, respectively.

(i) Assume that $6 \mid d$. Then $\mathbb{Q}(t(\tau)) = H \cap \mathbb{R}$ for any $\tau \in CM(d)$.

(ii) Assume that $6 \nmid d$. Then $\mathbb{Q}(t(\tau)) = H$ for any $\tau \in CM(d)$.

Moreover, assume that the class group of $R$ is an elementary 2-group, i.e., that each genus of the class group consists of one single class. Let $W$ be the full Atkin-Lehner group on $X_0(6)$. If $|CM(d)|$ is 2 or 4, then the values of $x(\tau_1, \tau_2)$ and $y(\tau_1, \tau_2)$ are both rational numbers for any $\tau_1, \tau_2 \in CM(d)$; if $|CM(d)| = 8$, then the same conclusion holds for any $\tau_1, \tau_2 \in CM(d)$ such that $\tau_2$ is not in the $W$-orbit of $\tau_1$.

Proof. The assertion about $|CM(d)|$ follows from the previous lemma, while that about $\mathbb{Q}(t(\tau))$ follows from Theorem 5.2 of [19]. For the assertion about rationality of $x$ and $y$, here we will only prove the case $|CM(d)| = 8$ since the proof of the case $|CM(d)| = 2$ or $|CM(d)| = 4$ is similar and simpler.

Assume that $|CM(d)| = 8$ and the class group of $R$ is an elementary 2-group. When $6 \nmid d$, by Lemma 5.9 of [19], the action of each element of $W$ coincides with that of some element of $\text{Gal}(H/K)$. Say, $G$ is the subgroup of $\text{Gal}(H/K)$ corresponding to $W$. Let $\tau_1$ and $\tau_2$ be elements in $CM(d)$ such that $\tau_2$ is not in the $W$-orbit of $\tau_1$. Consider $x(\tau_1, \tau_2)$.

Since $x(w\tau_1, w\tau_2) = x(\tau_1, \tau_2)$ for all $w \in W$, we have $x(\tau_1, \tau_2)^\sigma = x(\tau_1, \tau_2)$ for all $\sigma \in W$. Let $\sigma'$ be the element in $\text{Gal}(H/K)$ such that $t(\tau_1)^\sigma' = t(\tau_2)$. Since the class group is assumed to be an elementary 2-group, we also have $t(\tau_2)^\sigma' = t(\tau_1)$. It follows that $x(\tau_1, \tau_2)^\sigma' = x(\tau_2, \tau_1) = x(\tau_1, \tau_2)$ as well. Therefore, $x(\tau_1, \tau_2)$ is invariant under $\text{Gal}(H/K)$. By the first half of the lemma, we have $x(\tau_1, \tau_2) \in K \cap \mathbb{R} = \mathbb{Q}$. The same conclusion holds for $y(\tau_1, \tau_2)$. When $d = 3$, by Lemmas 5.9 and 5.10 of [19], the action of $w_3$ coincides with some element of $\text{Gal}(H/K)$ and that of $w_2$ coincides with complex conjugation. By a similar reasoning as in the case $6 \nmid d$, we see that $x(\tau_1, \tau_2)$ and $y(\tau_1, \tau_2)$ are rational numbers for all $\tau_1, \tau_2 \in CM(d)$ such that $\tau_2$ is not in the $W$-orbit of $\tau_1$. The proof of the remaining cases is omitted. \qed
Example 16. (i) The class group of the quadratic order of discriminant \(-420\) is an elementary 2-group of order 8. We represent the CM-points \(\tau\) by their corresponding quadratic forms \(A, B, C := Ax^2 + Bxy + Cy^2\) (i.e., \(\tau = (-B + \sqrt{B^2 - 4AC})/2A\)). The 8 CM-points of discriminant \(-420\) are represented by

\[
Q_1 = [6, 6, 19], \ Q_2 = [30, 30, 11], \ Q_3 = [42, 42, 13], \ Q_4 = [66, 30, 5], \\
Q_5 = [78, 42, 7], \ Q_6 = [114, 6, 1], \ Q_7 = [210, 210, 53], \ Q_8 = [318, 210, 35]
\]

The actions of \(w_2, w_3,\) and \(w_6\) map \(Q_1\) to \(Q_8, Q_7,\) and \(Q_6,\) respectively. Thus, we choose

\[
\tau_1 = \frac{-6 + \sqrt{-420}}{12}, \quad \tau_2 = \frac{-30 + \sqrt{-420}}{60},
\]

corresponding to \(Q_1\) and \(Q_2,\) respectively. Since \(-420 = -3 \times 140 = -7 \times 60 = -20 \times 21,\) we expect that the values of \(t\) at CM-points of discriminant \(-420\) belong to \(\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7}).\) Indeed, for the case \((a, b, c) = (-17, -6, 72)\) and \(t(\tau) = \eta(2\tau)\eta(6\tau^3)/\eta(\tau)^5\eta(3\tau),\) we find

\[
t(\tau_1) = \frac{3\sqrt{3} - 5}{12} \cdot \frac{5 - \sqrt{21}}{2} \cdot \frac{\sqrt{7} - \sqrt{3}}{2} \cdot (\sqrt{21} - 2\sqrt{5})(4 - \sqrt{15}),
\]

\[
t(\tau_2) = \frac{3\sqrt{3} + 5}{12} \cdot \frac{5 - \sqrt{21}}{2} \cdot \frac{\sqrt{7} - \sqrt{3}}{2} \cdot (\sqrt{21} - 2\sqrt{5})(4 + \sqrt{15}),
\]

and

\[
x(\tau_1, \tau_2) = -\frac{71}{1008}, \quad y(\tau_1, \tau_2) = \frac{1}{142^2}.
\]

They are indeed rational numbers.

(ii) Let \(d = -20.\) The 4 CM-points \(\tau_j, j = 1, \ldots, 4\) of discriminant \(-20\) are represented by

\[
Q_1 = [6, 2, 1], \ Q_2 = [6, -2, 1], \ Q_3 = [18, 14, 3], \ Q_4 = [18, -14, 3].
\]

The actions of \(w_2, w_3,\) and \(w_6\) map \(Q_1\) to \(Q_3, Q_2,\) and \(Q_4,\) respectively. In the case \((a, b, c) = (-17, -6, 72),\) we have

\[
t(\tau_1) = \frac{-2 + \sqrt{5} - i(2 + \sqrt{5})}{36}, \quad t(\tau_3) = \frac{-2 - \sqrt{5} + i(2 - \sqrt{5})}{36},
\]

\[
t(\tau_2) = t(\tau_1), \quad t(\tau_4) = t(\tau_3).
\]

We find

\[
x(\tau_1, \tau_2) = \infty, \quad x(\tau_1, \tau_3) = -\frac{9}{80}, \quad x(\tau_1, \tau_4) = -\frac{1}{8},
\]

and

\[
y(\tau_1, \tau_2) = \frac{1}{4}, \quad y(\tau_1, \tau_3) = y(\tau_1, \tau_4) = \frac{1}{324}.
\]

(iii) Occasionally, there are \(\tau_1 \in \text{CM}(d_1)\) and \(\tau_2 \in \text{CM}(d_2)\) with \(d_1 \neq d_2\) and \(\mathbb{Q}(\sqrt{d_1}) = \mathbb{Q}(\sqrt{d_2})\) such that \(x(\tau_1, \tau_2)\) and \(y(\tau_1, \tau_2)\) are rational numbers. For example, let \(d_1 = -12\) and \(d_2 = -48.\) By the lemma above, \(t(\tau) \in \mathbb{Q}\) for \(\tau \in \text{CM}(-12).\) Since \(h(-48) = 2\) and \(|\text{CM}(-48)| = 4,\) the Atkin-Lehner group \(W\) acts on \(\text{CM}(-48)\) transitively. It follows that \(x(\tau_1, \tau_2)\) and \(y(\tau_1, \tau_2)\) are rational for \(\tau_1 \in \text{CM}(-12)\) and \(\tau_2 \in \text{CM}(-48).\) For example, let

\[
\tau_1 = \frac{-3 + \sqrt{-3}}{6}, \quad \tau_2 = \frac{-3}{6},
\]
with corresponding quadratic forms \([6, 6, 2]\) and \([12, 0, 1]\). In the case \((a, b, c) = (-17, -6, 72)\), we have

\[
t(\tau_1) = -\frac{1}{12}, \quad t(\tau_2) = \frac{1 + \sqrt{3}}{6},
\]

and

\[
x(\tau_1, \tau_2) = -\frac{5}{36}, \quad y(\tau_1, \tau_2) = -\frac{1}{50}.
\]

**Case** \(\Gamma = \Gamma_0(8)\). We next consider the case \((a, b, c) = (12, 4, 32)\) with \(\Gamma = \Gamma_0(8)\) and \(G\) generated by \(\Gamma, w_8, \text{ and } (\frac{\sqrt{2}}{8})\). As before, we let \(\text{CM}(d)\) denote the set of CM-points of discriminant \(d\) on \(X_0(8)\). For our purpose, we need to know how \(w_8, (\frac{\sqrt{2}}{8})\), and the complex conjugation act on \(\text{CM}(d)\). In the subsequent discussion, we let \(d\) be a negative discriminant, \(K = \mathbb{Q}(\sqrt{d})\), and \(R, h(d)\), and \(H\) be the quadratic order of discriminant \(d\), its class number, and its ring class field, respectively. Write \(d = r^2d_0\), where \(d_0\) is a fundamental discriminant.

Recall that the Galois group \(\text{Gal}(H/K)\) acts on the set \(\text{CM}(d)\) through the Shimura reciprocity and each \(\text{Gal}(H/K)\)-orbit contains \(h(d)\) points. Moreover, two CM-points lie in the same \(\text{Gal}(H/K)\)-orbit if and only if their normalized optimal embeddings are locally equivalent at every finite place. Hence

\[
|\text{CM}(d)| = h(d) \prod_p \nu_p(d)
\]

where \(\nu_p(d)\) is the number of optimal embeddings of discriminant \(d\) from \(K\) into \(O_p := \left(\frac{\mathbb{Z}_p}{\mathbb{Z}_p^2}\right) \otimes \mathbb{Z}_p\) modulo conjugation by \(O_p^\times\) (see [33, Section 5, Chapter III]). Since \(O_p = (\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p)\) for odd primes \(p\), we have \(\nu_p(d) = 1\) for odd primes \(p\) and \(|\text{CM}(d)| = \nu_2(d)h(d)\).

We now determine \(\nu_2(d)\) and describe the \(\text{Gal}(H/K)\)-orbits of \(\text{CM}(d)\) and how \(w_8, (\frac{\sqrt{2}}{8})\), and the complex conjugation act on them. (Note that one can obtain a formula for \(\nu_2(d)\) from Theorem 2 of [23]).

**Lemma 17.** In the following, we represent a CM-point by the corresponding quadratic form \([A, B, C] := Ax^2 + Bxy + Cy^2\) with \(B^2 - 4AC = d\), \(8|A\), and \((A/8, B, C) = 1\) (i.e., \(\tau = (-B + \sqrt{d})/2A\)).

(i) If \(d_0 \equiv 5 \mod 8\) and \(4 \nmid r\) or if \(4|d_0\) and \(2 \nmid r\), then \(|\text{CM}(d)| = 0\).

(ii) Assume that \(d \equiv 1 \mod 8\). Let \(b\) be an integer such that \(b^2 \equiv d \mod 32\). Then \(\nu_2(d) = 2\) and the two \(\text{Gal}(H/K)\)-orbits of \(\text{CM}(d)\) are

\[
\{[A, B, C] : B \equiv b \mod 16\}, \quad \{[A, B, C] : B \equiv -b \mod 16\}.
\]

The actions of \(w_8\) and the complex conjugation swap the two orbits. The action of \((\frac{\sqrt{2}}{8})\) maps the two orbits to the first two orbits of \(\text{CM}(4d)\) in (iii).

(iii) Assume that \(d_0 \equiv 1 \mod 8\) and \(2| r\). Let \(b_1\) and \(b_2\) be integers such that \(b_1^2 - d_0 \equiv 8 \mod 16\) and \(b_2^2 - d_0 \equiv 0 \mod 16\). Then \(\nu_2(d) = 6\) and the six \(\text{Gal}(H/K)\)-orbits are

\[
C_1 : \{[A, B, C] : B \equiv 2b_1 \mod 16\},
C_2 : \{[A, B, C] : B \equiv -2b_1 \mod 16\},
C_3 : \{[A, B, C] : B \equiv 2b_2 \mod 16, C \text{ even}\},
C_4 : \{[A, B, C] : B \equiv 2b_2 \mod 16, C \text{ odd}\},
C_5 : \{[A, B, C] : B \equiv -2b_2 \mod 16, C \text{ even}\},
C_6 : \{[A, B, C] : B \equiv -2b_2 \mod 16, C \text{ odd}\}.
\]
The action of \( w_8 \) interchanges \( C_1 \) with \( C_2 \), \( C_3 \) with \( C_6 \), and \( C_4 \) with \( C_5 \). The action of the complex conjugation interchanges \( C_1 \) with \( C_2 \), \( C_3 \) with \( C_5 \), and \( C_4 \) with \( C_6 \). The action of \((\frac{4}{8})\) interchanges \( C_3 \) with \( C_5 \), \( C_4 \) with \( C_6 \), and maps \( C_1 \) and \( C_2 \) to the two orbits of \( \text{CM}(d/4) \) in (ii).

(iv) If \( d_0 \) is odd and \( 4 \| r \), then \( \nu_2(d) = 4 \) and the four \( \text{Gal}(H/K) \)-orbits are

\[
C_1 : \{(A, B, C) : B \equiv 4 \text{ mod } 16, \ C \text{ even}\},
C_2 : \{(A, B, C) : B \equiv 4 \text{ mod } 16, \ C \text{ odd}\},
C_3 : \{(A, B, C) : B \equiv -4 \text{ mod } 16, \ C \text{ even}\},
C_4 : \{(A, B, C) : B \equiv -4 \text{ mod } 16, \ C \text{ odd}\}.
\]

The action of \( w_8 \) interchanges \( C_1 \) with \( C_4 \) and \( C_2 \) with \( C_3 \). The action of the complex conjugation interchanges \( C_1 \) with \( C_3 \) and \( C_2 \) with \( C_4 \). The action of \((\frac{4}{8})\) interchanges \( C_1 \) with \( C_2 \) and \( C_3 \) with \( C_4 \).

(v) If \( 4 \| d_0 \) and \( 2 \| r \), then \( \nu_2(d) = 2 \) and the two \( \text{Gal}(H/K) \)-orbits are

\[
\{(A, B, C) : B \equiv 0 \text{ mod } 16\}, \quad \{(A, B, C) : B \equiv 8 \text{ mod } 16\}.
\]

All \( w_8 \), \((\frac{4}{8})\), and the complex conjugation switch the two orbits.

(vi) If \( 8 \| d_0 \) and \( 2 \| r \), then \( \nu_2(d) = 2 \) and the two \( \text{Gal}(H/K) \)-orbits are

\[
\{(A, B, C) : B \equiv 0 \text{ mod } 16\}, \quad \{(A, B, C) : B \equiv 8 \text{ mod } 16\}.
\]

All \( w_8 \), \((\frac{4}{8})\), and the complex conjugation fix every orbit.

(vii) If \( 64 \| d \), then \( \nu_2(d) = 4 \) and the four \( \text{Gal}(H/K) \)-orbits are

\[
C_1 : \{(A, B, C) : B \equiv 0 \text{ mod } 16, \ C \text{ even}\},
C_2 : \{(A, B, C) : B \equiv 0 \text{ mod } 16, \ C \text{ odd}\},
C_3 : \{(A, B, C) : B \equiv 8 \text{ mod } 16, \ C \text{ even}\},
C_4 : \{(A, B, C) : B \equiv 8 \text{ mod } 16, \ C \text{ odd}\}.
\]

The action of \( w_8 \) interchanges of \( C_1 \) with \( C_2 \) and \( C_3 \) with \( C_4 \). The action of \((\frac{4}{8})\) interchanges \( C_1 \) with \( C_4 \) and \( C_2 \) with \( C_3 \). The complex conjugation fixes every orbit.

In particular,

(i) if \( 32 \mid d \), then \( \mathbb{Q}(i(\tau)) = H \) for all \( \tau \in \text{CM}(d) \), and

(ii) if \( 32 \mid d \) and the class group of \( R \) is an elementary 2-group, then \( \mathbb{Q}(i(\tau)) = H \cap \mathbb{R} \) for all \( \tau \in \text{CM}(d) \).

Proof. For \( a \in \mathbb{Z}_2 \), let \( \nu_2(a) \) denote the 2-adic valuation of \( a \). Let \( \phi : R \to \mathcal{O}_2 \), \( \mathcal{O}_2 = \left( \mathbb{Z}_2^2 \mathbb{Z}_2^2 \right) \), be an optimal embedding of discriminant \( d \). We remark that \( \phi \) is completely determined by \( \phi(\sqrt{d}) \).

Consider the case \( d \) is odd first. We have

\[
\phi(\sqrt{d}) = \begin{pmatrix} a & b \\ 8c & -a \end{pmatrix}
\]

for some \( a, b, c \in \mathbb{Z}_2 \) with \( a^2 + 8bc = d \). It is clear that if \( d \) is not congruent to 1 modulo 8, then no such \( a, b, c \) exist and hence \( \nu_2(d) = 0 \). Now assume that \( d \equiv 1 \text{ mod } 8 \). Note that since \( \phi \) is an optimal embedding of discriminant \( d \), we must have \( \phi((1 + \sqrt{d})/2) \in \mathcal{O}_2 \). In other words, we have \( 2 \| b, c \) and \( a^2 \equiv d \text{ mod } 32 \). Now we check by direct computation the following properties.
(i) Among the residue classes modulo 16, there are precisely two residue classes \(a\) such that \(a^2 \equiv d \mod 32.

(ii) There exists \(k \in \mathbb{Z}_2\) such that the 2-adic valuation of the \((1, 2)\)-entry of

\[
\begin{pmatrix}
1 & k \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
a & b \\
8c & -a
\end{pmatrix}
\begin{pmatrix}
1 & -k \\
0 & 1
\end{pmatrix}
\]

is 1. In other words, \(\phi\) is \(O_2^\times\)-equivalent to the one given by \(\sqrt[d]{d} \mapsto \left(\frac{a}{8c - a}\right)\) with \(a^2 \equiv d \mod 32, 2\|b\) and \(2|c\), which we assume from now on.

(iii) Let \(\left(\frac{a}{8c - a}\right)\) be the matrix from (ii). We have

\[
\begin{pmatrix}
\frac{b}{2} & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
a & b \\
8c & -a
\end{pmatrix}
\begin{pmatrix}
\frac{b}{2} & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
a & 2 \\
8bc & -a
\end{pmatrix}.
\]

Hence \(\phi\) is \(O_2^\times\)-equivalent to \(\sqrt[d]{d} \mapsto \left(\frac{a}{d - a^2/2 - a}\right)\).

(iv) Two optimal embeddings \(\sqrt[d]{d} \mapsto \left(\frac{a_1}{(d - a^2)/2 - a_1}\right)\) and \(\sqrt[d]{d} \mapsto \left(\frac{a_2}{(d - a^2)/2 - a_2}\right)\) are \(O_2^\times\)-equivalent if and only if \(a_1 \equiv a_2 \mod 16\).

Based on these properties, we conclude that if \(d \equiv 1 \mod 8\), then there are two \(\text{Gal}(H/K)\)-orbits of CM-points of discriminant \(d\) and two points in \(\text{CM}(d)\) are in the same \(\text{Gal}(H/K)\)-orbit if and only if their corresponding quadratic forms \(A_1x^2 + B_1xy + C_1y^2\) and \(A_2x^2 + B_2xy + C_2y^2\) satisfy \(B_1 \equiv B_2 \mod 16\).

We next consider the case \(4|d\). We must have \(\phi(\sqrt[d]{d}/2) \in O_2\), say

\[
\phi(\sqrt[d]{d}/2) = \begin{pmatrix}
a & b \\
8c & -a
\end{pmatrix}
\]

for some \(a, b, c \in O_2\) satisfying \(a^2 + 8bc = d/4\). If \(d/4\) is odd, then \(a\) must be odd. If \(d/4 \equiv 1 \mod 8\), no such \(a, b, c\) can exist and hence \(\nu_2(d) = 0\). When \(d/4 \equiv 1 \mod 8\), it is clear that for each odd element \(a\) of \(O_2\), there are \(b\) and \(c\) in \(O_2\) such that \(a^2 + 8bc = d/4\). We now check the following properties.

(i) Among the 4 odd residue classes modulo 8, there are exactly 2 classes \(a\) such that \(8\|/(d/4 - a^2)\) and the other two classes satisfy \(16\|(d/4 - a^2)\).

(ii) In the case \(8\|/(d/4 - a^2)\), \(\phi\) is \(O_2^\times\)-equivalent to \(\sqrt[d]{d}/2 \mapsto \left(\frac{a}{d/4 - a^2 - a}\right)\). In the case \(16\|(d/4 - a^2)\), \(\phi\) is \(O_2^\times\)-equivalent to \(\sqrt[d]{d}/2 \mapsto \left(\frac{a}{d/4 - a^2 - a}\right)\) or \(\sqrt[d]{d}/2 \mapsto \left(\frac{a}{d/4 - a^2 - a}\right)\) and the two are inequivalent.

(iii) Two optimal embeddings \(\sqrt[d]{d}/2 \mapsto \left(\frac{a_1}{(d/4 - a_1^2) - a_1}\right)\) and \(\sqrt[d]{d}/2 \mapsto \left(\frac{a_2}{(d/4 - a_2^2) - a_2}\right)\) are \(O_2^\times\)-equivalent if and only if \(a_1 \equiv a_2 \mod 8\). The same property also holds for embeddings of the form \(\sqrt[d]{d}/2 \mapsto \left(\frac{a}{d/4 - a^2 - a}\right)\).

It follows that \(\nu_2(d) = 6\). Also, two CM-points of discriminant \(d\) are in the same \(\text{Gal}(H/K)\)-orbit if and only if their corresponding quadratic forms \(A_1x^2 + B_1xy + C_1y^2\) and \(A_2x^2 + B_2xy + C_2y^2\) satisfy \(B_1 \equiv B_2 \mod 8\) and \(C_1 \equiv C_2 \mod 2\).

The proof of the remaining cases is similar. We skip the details and just summarize our findings as follows.

(i) If \(8\|d\), no \(a, b, c \in \mathbb{Z}_2\) can satisfy \(a^2 + 8bc = d/4\). Thus, \(\nu_2(d) = 0\).
(ii) If $16|d$ and $d/16 \equiv 3 \mod 4$ (i.e., $2||r$), then any optimal embedding $\phi$ is $O_{d}^{+}$-equivalent to $\sqrt{d}/2 \mapsto \begin{pmatrix} 2 & 1 \\ d/4 - 4 & -2 \end{pmatrix}$ or $\sqrt{d}/2 \mapsto \begin{pmatrix} -2 & 1 \\ d/4 - 4 & 2 \end{pmatrix}$ and the two are not $O_{d}^{+}$-equivalent. (Note that $8|(d/4 - 4)$.) Hence $\nu_2(d) = 2$.

(iii) If $32|d$, then $2||r$ and any optimal embedding is $O_{d}^{+}$-equivalent to $\sqrt{d}/2 \mapsto \begin{pmatrix} 0 & 1 \\ d/4 & 0 \end{pmatrix}$ or $\sqrt{d}/2 \mapsto \begin{pmatrix} 4 & 1 \\ d/4 - 16 & -4 \end{pmatrix}$ and the two are not $O_{d}^{+}$-equivalent. Hence $\nu_2(d) = 2$.

(iv) If $64|d$, then $4||r$ and any optimal embedding is $O_{d}^{+}$-equivalent to one of $\sqrt{d}/2 \mapsto \begin{pmatrix} 0 & 1 \\ d/4 & 0 \end{pmatrix}$, $\sqrt{d}/2 \mapsto \begin{pmatrix} 0 & d/32 \\ 8 & 0 \end{pmatrix}$, $\sqrt{d}/2 \mapsto \begin{pmatrix} 4 & 1 \\ d/4 - 16 & -4 \end{pmatrix}$, and $\sqrt{d}/2 \mapsto \begin{pmatrix} 4 & 32 - 2 \\ 8 & -4 \end{pmatrix}$. Hence $\nu_2(d) = 4$.

Now the action $w_8$ maps the quadratic form $[A, B, C]$ to $[8C, -B, A/8]$ and the action of the complex conjugation maps $[A, -B, C]$ to $[A, B, C]$. Finally, we compute that

\[
\begin{pmatrix} 4 & 1 \\ 8 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ 8c & -a \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 8 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 3a - 4b + 4c & -a + 2b - c \\ 8a - 8b + 16c & -3a + 4b - 4c \end{pmatrix}.
\]

From these, we easily obtain our description of the actions in each case. Note that when $R$ is an elementary 2-group, the ring class field $H$ is of the form $\mathbb{Q}(\sqrt{d}, \sqrt{a_1}, \ldots, \sqrt{a_k})$ for some positive rational numbers $a_1, \ldots, a_k$. In the case $32|d$, since the complex conjugation fixes every $Gal(H/K)$-orbit and $Gal(H/K)$ acts transitively on each orbit, we must have $t(\tau) \in \mathbb{Q}(\sqrt{a_1}, \ldots, \sqrt{a_k})$ for all $\tau \in CM(d)$. This complete the proof of the lemma. \(\square\)

**Corollary 18.** Let $d$ be a negative discriminant such that the set $CM(d)$ of $CM$-points of discriminant $d$ on $X_0(8)$ is nonempty and write $d = r^2 d_0$, where $d_0$ is a fundamental discriminant. Let $R$, $h(d)$, and $H$ be the quadratic order of discriminant $d$, its class number, and its ring class field, respectively. Let $W$ be the subgroup of $N(\Gamma_0(8))/\Gamma_0(8)$ generated by the Atkin-Lehner involution $w_8$ and \(\left(\frac{4}{8} \right)\).

(i) Assume that $d \equiv 1 \mod 8$ and $h(d) = 1$ (i.e., $d = -7$). Let $\tau_1$ and $\tau_2$ be any two points in $CM(d)$ and the first two orbits of $CM(4d)$ described in the previous lemma, or let $\tau_1$ and $\tau_2$ be any two points in the last four orbits of $CM(4d)$, or let $\tau_1$ and $\tau_2$ be any two points in $CM(16d)$ such that $\tau_2$ is not in the $W$-orbit of $\tau_1$.

(ii) Assume that $d \equiv 1 \mod 8$ and $h(d) = 2$. Let $\tau_1$ and $\tau_2$ be any two points in $CM(d)$ and the first two orbits of $CM(4d)$ described in the previous lemma such that $\tau_2$ is not in the $W$-orbit of $\tau_1$, or let $\tau_1$ and $\tau_2$ be any two points in the last four orbits of $CM(4d)$ such that $\tau_2$ is not in the $W$-orbit of $\tau_1$.

(iii) Assume that $4|d_0$ and $2||r$. Assuming that $h(d) = 1$ or $h(d) = 2$, let $\tau_1$ and $\tau_2$ be any two points in $CM(d)$, or assuming that the class group of $R$ is the Klein 4-group, let $\tau_1$ and $\tau_2$ be any two points in $CM(d)$ such that $\tau_2$ is not in the $W$-orbit of $\tau_1$.

(iv) Assume that $8|d_0$ and $2||r$. Assuming that the class group of $R$ is an elementary 2-group of order 2 or 4, let $\tau_1$ and $\tau_2$ be any two points in the same $Gal(H/K)$-orbit in $CM(d)$, or assuming that the class group of $R$ is an elementary 2-group of order 8, let $\tau_1$ and $\tau_2$ be two points in the same $Gal(H/K)$-orbit in $CM(d)$ such that $\tau_2$ is not in the $W$-orbit of $\tau_1$.

(v) Assume that $d = -64$. Let $\tau_1$ and $\tau_2$ be any two points in $CM(-64)$ such that $\tau_2$ is not in the $W$-orbit of $\tau_1$.

Then the values of $x(\tau_1, \tau_2)$ and $y(\tau_1, \tau_2)$ are rational numbers.
Example 19. (i) Let $d = -112$. The four $\text{Gal}(H/K)$-orbits of $\text{CM}(-112)$ are
\[\{Q_1 = [8, 4, 4], Q_2 = [56, -28, 4]\}, \{Q_3 = [32, 4, 1], Q_4 = [32, -28, 7]\},\]
\[\{Q_5 = [8, -4, 4], Q_6 = [56, 28, 4]\}, \{Q_7 = [32, -4, 1], Q_8 = [32, 28, 7]\}\]
Let $\tau_j, j = 1, \ldots, 8$, be the corresponding CM-points. The actions of $w_8$, $(\frac{1}{2}, \frac{1}{2})$, and the complex conjugation map $Q_1$ to $Q_7$, $Q_4$, and $Q_5$, respectively. The $W$-orbit of $Q_1$ is $\{Q_1, Q_4, Q_6, Q_7\}$. We expect that $x(\tau_1, \tau_j)$ and $y(\tau_1, \tau_j)$ will be rational for $j = 2, 3, 5, 8$. Indeed, we find
\[x(\tau_1, \tau_2) = \frac{16}{63}, \quad x(\tau_1, \tau_3) = \frac{1}{8}, \quad x(\tau_1, \tau_5) = -\frac{1}{252}, \quad x(\tau_1, \tau_8) = \frac{1}{8},\]
and
\[y(\tau_1, \tau_2) = y(\tau_1, \tau_3) = \frac{1}{256}, \quad y(\tau_1, \tau_5) = y(\tau_1, \tau_8) = 16.\]
The case $j = 5$ will yield a $1/\pi$-series.

(ii) Let $d = -15$. The two $\text{Gal}(H/K)$-orbits of $\text{CM}(-15)$ are
\[\{Q_1 = [8, 7, 2], Q_2 = [16, 7, 1]\}, \{Q_3 = [8, -7, 2], Q_4 = [16, -7, 1]\}.\]
The first two $\text{Gal}(H/K)$-orbits of $\text{CM}(-60)$ are
\[\{Q_5 = [8, 6, 3], Q_6 = [24, 6, 1]\}, \{Q_7 = [8, -6, 3], Q_8 = [24, -6, 1]\}.\]
Let $\tau_j$ be the corresponding CM-points. The actions of $w_8$, $(\frac{1}{2}, \frac{1}{2})$, and the complex conjugation map $Q_1$ to $Q_4$, $Q_7$, and $Q_3$, respectively, and the $W$-orbit of $Q_1$ is $\{Q_1, Q_4, Q_6, Q_7\}$. Hence, $x(\tau_1, \tau_j)$ and $y(\tau_1, \tau_j)$ should be rational numbers for $j = 2, 3, 5, 8$. Indeed, we have
\[x(\tau_1, \tau_2) = \frac{7}{45}, \quad x(\tau_1, \tau_3) = -\frac{7}{40}, \quad x(\tau_1, \tau_5) = \frac{7}{12}, \quad x(\tau_1, \tau_8) = \frac{7}{32},\]
and
\[y(\tau_1, \tau_2) = y(\tau_1, \tau_3) = \frac{1}{49}, \quad y(\tau_1, \tau_5) = y(\tau_1, \tau_8) = \frac{16}{49}.\]

(iii) Let $d = -480$. There are two $\text{Gal}(H/K)$-orbits
\[\{Q_1 = [8, 0, 15], Q_2 = [24, 0, 5], Q_3 = [40, 0, 3], Q_4 = [120, 0, 1],\]
\[Q_5 = [88, 64, 13], Q_6 = [104, 64, 11], Q_7 = [136, 128, 31], Q_8 = [248, 128, 17]\]
and
\[\{Q'_1 = [8, 8, 17], Q'_2 = [136, 8, 1], Q'_3 = [24, 24, 11], Q'_4 = [88, 24, 3],\]
\[Q'_5 = [40, 40, 13], Q'_6 = [104, 40, 5], Q_7 = [120, 120, 31], Q'_8 = [248, 120, 15]\].
Let $\tau_j$ and $\tau'_j, j = 1, \ldots, 8$, be the corresponding CM-points. In the first orbit, the $W$-orbit of $Q_1$ is $\{Q_1, Q_4, Q_7, Q_8\}$. We find
\[x(\tau_1, \tau_2) = \frac{11}{240}, \quad x(\tau_1, \tau_3) = \frac{31}{320}, \quad x(\tau_1, \tau_5) = \frac{11}{16}, \quad x(\tau_1, \tau_6) = \frac{31}{96},\]
and
\[y(\tau_1, \tau_2) = y(\tau_1, \tau_3) = \frac{1}{22^2}, \quad y(\tau_1, \tau_5) = y(\tau_1, \tau_6) = \frac{1}{62^2}.\]
The cases of $\tau_2$ and $\tau_5$ will yield $1/\pi$-series. In the second orbit, the $W$-orbit of $Q'_1$ is $\{Q'_1, Q'_2, Q'_7, Q'_8\}$. We find that
\[x(\tau'_1, \tau'_2) = -\frac{7}{96}, \quad x(\tau'_1, \tau'_4) = \frac{49}{320}, \quad x(\tau'_1, \tau'_6) = -\frac{7}{16}, \quad x(\tau'_1, \tau'_6) = \frac{49}{240},\]
and
\[ y(\tau'_1, \tau'_3) = y(\tau'_1, \tau'_5) = \frac{1}{14^2}, \quad y(\tau'_1, \tau'_4) = y(\tau'_1, \tau'_6) = \frac{1}{98^2}. \]

The case of \( \tau'_3 \) will yield a \( 1/\pi \)-series.

**Case** \( \Gamma = \Gamma_0(9) \). As before, for a negative discriminant \( d \), let \( K = \mathbb{Q}(\sqrt{d}) \). Also, let \( R, h(d) \), and \( H \) be the quadratic order of discriminant \( d \), the class number of \( R \), and the ring class field of \( R \), respectively. Let \( \text{CM}(d) \) denote the set of CM-points of discriminant \( d \) on \( X_0(9) \).

**Lemma 20.** Write a negative discriminant \( d \) as \( d = r^2d_0 \), where \( d_0 \) is a fundamental discriminant. In the following, we represent a CM-point by the corresponding quadratic \( \tau \).

(i) If \( d_0 \not\equiv 1 \mod 3 \) and \( 3 \nmid r \), then \( |\text{CM}(d)| = 0 \).

(ii) Assume that \( d_0 \equiv 1 \mod 3 \) and \( 3 \nmid r \). Let \( b \) be an integer such that \( b^2 \equiv d \mod 9 \). Then \( \nu_3(d) = 2 \) and the two \( \text{Gal}(H/K) \)-orbits are
\[ \{ [A, B, C] : B \equiv b \mod 9 \}, \quad \{ [A, B, C] : B \equiv -b \mod 9 \}. \]

The action of \( w_9 \) and the complex conjugation both swap the two orbits. The action of \( \left( \frac{-3}{9} \right) \) maps all the points in \( \text{CM}(d) \) to the last orbit in (ii).

(iii) Assume that \( d_0 \equiv 1 \mod 3 \) and \( 3 | r \). Then \( \nu_3(d) = 5 \) and the five \( \text{Gal}(H/K) \)-orbits are
\[
\begin{align*}
C_1 &: \{ [A, B, C] : B \equiv 3 \mod 9, \ 3 \nmid C \}, \\
C_2 &: \{ [A, B, C] : B \equiv 3 \mod 9, \ 3 | C \}, \\
C_3 &: \{ [A, B, C] : B \equiv -3 \mod 9, \ 3 \nmid C \}, \\
C_4 &: \{ [A, B, C] : B \equiv -3 \mod 9, \ 3 | C \}, \\
C_5 &: \{ [A, B, C] : B \equiv 0 \mod 9 \}.
\end{align*}
\]

The action of \( w_9 \) interchanges \( C_1 \) with \( C_4 \), \( C_2 \) with \( C_3 \), and fixes \( C_5 \). The action of the complex conjugation interchanges \( C_1 \) with \( C_3 \), \( C_2 \) with \( C_4 \), and fixes \( C_5 \). The action of \( \left( \frac{-3}{9} \right) \) maps \( C_5 \) to \( \text{CM}(d/9) \) and permutes the other CM-points. (It may not map an orbit to an orbit.)

(iv) Assume that \( d_0 \equiv 2 \mod 3 \) and \( 3 | r \). Then \( \nu_3(d) = 3 \) and the three \( \text{Gal}(H/K) \)-orbits are
\[
\begin{align*}
C_1 &: \{ [A, B, C] : B \equiv 3 \mod 9 \}, \\
C_2 &: \{ [A, B, C] : B \equiv -3 \mod 9 \}, \\
C_3 &: \{ [A, B, C] : B \equiv 0 \mod 9 \}.
\end{align*}
\]

Both \( w_9 \) and the complex conjugation switch \( C_1 \) with \( C_2 \) and fix \( C_3 \). The action of \( \left( \frac{-3}{9} \right) \) fixes \( C_3 \) and permutes the other CM-points. (It may not map an orbit to an orbit.)

(v) Assume that \( 27 | d \). Then \( \nu_3(d) = 4 \) and the four \( \text{Gal}(H/K) \)-orbits are
\[
\begin{align*}
C_1 &: \{ [A, B, C] : B \equiv 3 \mod 9 \}, \\
C_2 &: \{ [A, B, C] : B \equiv -3 \mod 9 \}, \\
C_3 &: \{ [A, B, C] : B \equiv 0 \mod 9, \ 3 \nmid C \}, \\
C_4 &: \{ [A, B, C] : B \equiv 0 \mod 9, \ 3 | C \}.
\end{align*}
\]
The action of $w_3$ interchanges $C_1$ with $C_2$ and $C_3$ with $C_4$. The action of the complex conjugation interchanges $C_1$ with $C_2$ and fixes $C_3$ and $C_4$. The action of $\left(-\frac{3}{9},-\frac{2}{3}\right)$ may not map an orbit to an orbit.

Remark 21. The messiness of the action of $\left(-\frac{3}{9},-\frac{2}{3}\right)$ is due to the fact that $\left(-\frac{3}{9},-\frac{2}{3}\right)$ only normalizes $\Gamma_0(9)$, but not $\left(\frac{z}{9\mathbb{Z}},\frac{z}{9\mathbb{Z}}\right)$.

Proof. We omit most of the proof since it is very similar to that of Lemma 17. Here we merely indicate why the action of $\left(-\frac{3}{9},-\frac{2}{3}\right)$ fixes the orbit $C_3$ in (iv).

Recall that a $\text{Gal}(H/K)$-orbit is determined by its local optimal embeddings. In the case of $X_0(9)$, we only need to consider the prime 3. Let $O_3 = \left(\frac{z_3}{3\mathbb{Z}},\frac{z_3}{3\mathbb{Z}}\right)$. In the case $d_0 \equiv 2 \mod 3$ and $3\mid r$, we can check that two optimal embeddings of discriminant $d$ of $K$ into $O_3$ defined by

$$\sqrt{d} \mapsto \left(\begin{array}{cc}
a_1 & b_1 \\
9c_2 & -a_1
\end{array}\right), \quad \sqrt{d} \mapsto \left(\begin{array}{cc}
a_2 & b_2 \\
9c_2 & -a_2
\end{array}\right)$$

are $O_3^\mathbb{Z}$-equivalent if and only if $a_1 \equiv a_2 \mod 9$ and hence $\nu_3(d) = 3$. Now assume that $\phi : R \to O_3$ is an optimal embedding of discriminant $d$, say $\phi(\sqrt{d}) = \left(\frac{a}{9c},\frac{b}{9c}\right)$ with $a, b, c \in \mathbb{Z}$, $\gcd(a, b, c) = 1$, and $a^2 + 9bc = d$. We compute that

$$\left(\begin{array}{cc}
-3 & -2 \\
9 & 3
\end{array}\right) \left(\begin{array}{cc}
a & b \\
9c & -a
\end{array}\right) \left(\begin{array}{cc}
-3 & -2 \\
9 & 3
\end{array}\right)^{-1} = \left(\begin{array}{cc}
-3a + 3b - 6c & -4a/3 + b - 4c \\
6a - 9b + 9c & 3a - 3b + 6c
\end{array}\right),$$

When $a \equiv 0 \mod 9$, since $(a/3)^2 + bc = d/9 \equiv 2 \mod 3$, we have $b \equiv -c \mod 3$ and hence $-3a + 3b - 6c \equiv 0 \mod 9$. This shows that the action of $\left(-\frac{3}{9},-\frac{2}{3}\right)$ fixes the orbit $C_3$. □

When the class group of $R$ is an elementary 2-group, we can deduce rationality criteria for $x$ and $y$ analogous to those given in Corollary 13. However, there are situations where the class group is not an elementary 2-group, but the values of $x$ and $y$ are still rational. It is complicated to summarize the conditions into a formal statement, so we will just give one example.

Example 22. Let $d = -1008$. The class group is isomorphic to the direct product of a cyclic group of order 4 and a cyclic group of order 2. Consider the orbit $C_3$ in Part (iv) of Lemma 20. It contains

$$Q_1 = [9,0,28], \quad Q_2 = [36,0,7], \quad Q_3 = [63,0,4], \quad Q_4 = [252,0,1],$$
$$Q_5 = [99,90,23], \quad Q_6 = [99,-90,23], \quad Q_7 = [261,180,32], \quad Q_8 = [261,-180,32].$$

Let $\tau_j$, $j = 1,\ldots,8$, be the corresponding CM-points. Using the standard cycle notations, the actions of $w_3$, $\left(-\frac{3}{9},-\frac{2}{3}\right)$, and the complex conjugation are $(1,4)(2,3)(5,6)(7,8), (1,7)(2,5)(3,6)(4,8), \text{ and } (5,6)(7,8)$, respectively. Moreover, the form class group is generated by $Q_5$ of order 4 and $Q_2$ of order 2 with $Q_4$ being the identity element. In the cycle notation, the multiplications by $Q_5$ and by $Q_2$ are $(5,1,6,4)(2,8,3,7)$ and $(2,4)(1,3)(5,8)(6,7)$, respectively. Let $\sigma_5$ and $\sigma_2$ be their corresponding elements in $\text{Gal}(H/K)$.

Consider $x(\tau_1,\tau_2)$. According the the Shimura reciprocity law [27, Main Theorem II], if we let $\beta_1$ and $\beta_2$ be elements in $\mathcal{O} = \left(\frac{z}{9\mathbb{Z}},\frac{z}{9\mathbb{Z}}\right)$ with positive determinants such that

$$\left(\begin{array}{cc}
99\mathbb{Z} + \left(-\frac{45}{9},-\frac{28}{45}\right)\mathbb{Z} \\
\right)\mathcal{O} = \beta_1\mathcal{O}, \quad \left(99\mathbb{Z} + \left(-\frac{45}{36},-\frac{7}{45}\right)\mathbb{Z}\right)\mathcal{O} = \beta_2\mathcal{O}$$
then
\[ t(\tau_1)^{\sigma_5} = t(\beta_1^{-1} \tau_1), \quad t(\tau_2)^{\sigma_5} = t(\beta_2^{-1} \tau_2). \]

Here we may choose \( \beta_1 = \begin{pmatrix} 9 & -1 \\ 18 & 9 \end{pmatrix} \) and \( \beta_2 = \begin{pmatrix} 27 & 8 \\ 18 & 9 \end{pmatrix} \) and find that
\[ t(\tau_1)^{\sigma_5} = t(\tau_5), \quad t(\tau_2)^{\sigma_5} = t(\tau_7). \]

It follows that
\[ x(\tau_1, \tau_2)^{\sigma_5} = x(\tau_5, \tau_7) = x\left(\begin{pmatrix} -3 & -2 \\ 9 & 3 \end{pmatrix} \tau_2, \begin{pmatrix} -3 & -2 \\ 9 & 3 \end{pmatrix} \tau_1\right) = x(\tau_2, \tau_1) = x(\tau_1, \tau_2). \]

By a similar computation, we can also show that
\[ x(\tau_1, \tau_2)^{\sigma_2} = x(\tau_3, \tau_4) = x(w_9 \tau_2, w_9 \tau_1) = x(\tau_2, \tau_1) = x(\tau_1, \tau_2). \]

(Alternatively and slightly more abstractly, if we write \( t(\tau_j) \) as \( t([Q_j]) \) and let \( \sigma_j \) denote the element in \( Gal(H/K) \) corresponding to \( \tau_j \in CM(d) \), then the Shimura reciprocity law can be stated as \( t([Q_j])^{\sigma_j} = t([Q_j]^{-1}[Q_i]). \) From this, we may deduce the same conclusion.)

Since \( \sigma_2 \) and \( \sigma_5 \) generate \( Gal(H/K) \) and the value of \( x(\tau_1, \tau_2) \) is real, we find that \( x(\tau_1, \tau_2) \in \mathbb{Q} \) and so does \( y(\tau_1, \tau_2) \). Indeed, we have
\[ x(\tau_1, \tau_2) = \frac{52}{675}, \quad y(\tau_1, \tau_2) = \frac{1}{2704}. \]

In addition to \((\tau_1, \tau_2)\), we may also consider \((\tau_1, \tau_3)\) and find
\[ x(\tau_1, \tau_3) = \frac{13}{27}, \quad y(\tau_1, \tau_3) = \frac{1}{2704}. \]

**Case** \( \Gamma = \Gamma_1(5) \). When the parameters \((a, b, c)\) are \((11, 3, -1)\), the modular curve is \( X_1(5) \). In this case, the values of the modular function \( t \) at CM-points lie in the ray class fields of imaginary quadratic orders (see [24]).

As before, for a negative discriminant \( d \), let \( R \) and \( h(d) \) denote the quadratic order of discriminant \( d \) and its class number, respectively. Also, we write \( d \) as \( r^2 d_0 \), where \( d_0 \) is a fundamental discriminant. We let \( CM_0(d) \) and \( CM_1(d) \) denote the sets of CM-points of discriminant \( d \) on \( X_0(5) \) and \( X_1(5) \).

We first recall that the order of \( M(2, \mathbb{Q}) \) in the case of \( X_1(5) \) is
\[ \mathcal{O} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{Z}) : 5|c, a \equiv d \text{ mod } 5 \right\}. \]

Assume that \( \phi \) is an optimal embedding of discriminant \( d \) into \( \mathcal{O} \), say,
\[ \phi(\sqrt{d}) = \begin{pmatrix} a & b \\ 5c & -a \end{pmatrix} \]
with \( a^2 + 5bc = d \). In order for \( \phi(\sqrt{d}) \) to be in \( \mathcal{O} \), the integer \( a \) must be divisible by 5. It follows that the discriminant of an optimal embedding into \( \mathcal{O} \) must be a multiple of 5. Furthermore, if \( 5|r \) and \( \left\langle d/25 \right\rangle = -1 \) (respectively, \( \left\langle d/25 \right\rangle = 0 \)), then \( 6|h(d) \) (respectively, \( 5|h(d) \)) and the values of \( x \) and \( y \) at pairs of CM-points of discriminant \( d \) will not be rational numbers. Thus, we only need to consider the case \( 25|d \) and \( d/25 \equiv \pm 1 \text{ mod } 5 \) and the case \( 5|d \).

In the case \( 25|d \) and \( d/25 \equiv \pm 1 \text{ mod } 5 \), there are two types of points in \( CM_1(d) \). One is that the integers \( b \) and \( c \) in (35) are both divisible by 5 and the other is that one of \( b \) and \( c \) is not divisible by 5. Points of the first type are mapped to \( CM_0(d/25) \) under the covering
$X_1(5) \to X_0(5)$, while those of the second type are mapped to $CM_0(d)$. The number of points of the first type is
\[
\begin{cases}
2, & \text{if } d = -100, \\
4h(d/25), & \text{if } d \neq -100,
\end{cases}
\]
and the number of points of the second type is $4h(d) = 16h(d/25)$. We see that the values of $x$ and $y$ at pairs of points from $CM_1(d)$ can possibly be rational when $h(d/25) \leq 2$. However, in practice, we do not find any such discriminants that yield rational-valued $x$ and $y$.

In the case $5|d$, we have $|CM_1(d)| = 2h(d)$. The ray class groups in general are not elementary 2-groups even when the class groups without modulus are elementary 2-groups. Thus, it is complicated to state criteria for $x$ and $y$ to have rational values at points from $CM_1(d)$, so here we will simply give an example. (In fact, we find only two discriminants that yield $1/\pi$-series.)

**Example 23.** Let $d = -760$ with $h(d) = 4$ and $|CM_1(d)| = 8$. Let $K = \mathbb{Q}(\sqrt{d})$, $R$ be the ring of integers in $K$, and $p$ be the unique prime ideal of $R$ lying above $5$. Let $G$ and $H$ be the ray class group and the ray class field of $R$ of modulus $p$, respectively. In terms of the form class group, the $8$ elements of the class group are
\[
Q_1 = [5, 0, 38], \quad Q_2 = [10, 0, 19], \quad Q_3 = [95, 0, 2], \quad Q_4 = [190, 0, 1], \\
Q_5 = [970, 780, 157], \quad Q_6 = [515, 420, 86], \quad Q_7 = [430, 420, 103], \quad Q_8 = [785, 780, 194].
\]
Let $\tau_j$, $j = 1, \ldots, 8$, be the corresponding CM-points. The form class group is generated by $Q_1$ of order 4 and $Q_6$ of order 2. Using the cycle notations, the multiplication by $Q_1$ is given by $(1, 8, 5, 4)(2, 3, 6, 7)$ and that by $Q_6$ is $(1, 7)(2, 8)(3, 5)(4, 6)$. Also, the actions of $w_5$ and $\left(\frac{2}{5} - \frac{1}{2}\right)$ are $(1, 4)(2, 3)(5, 8)(6, 7)$ and $(1, 5)(2, 6)(3, 7)(4, 8)$, respectively. Let $\sigma_j$ denote the element in $\text{Gal}(H/K)$ corresponding to $\tau_j \in CM(d)$. Then following a similar computation as Example 22, we can show that
\[
x(\tau_1, \tau_3)^{\sigma_1} = x(\tau_4, \tau_2) = x(w_5\tau_1, w_5\tau_3) = x(\tau_1, \tau_3),
\]
\[
x(\tau_1, \tau_3)^{\sigma_6} = x(\tau_7, \tau_5) = x(\left(\begin{array}{cc}-2 & -1 \\ 5 & 2\end{array}\right)\tau_1, \left(\begin{array}{cc}-2 & -1 \\ 5 & 2\end{array}\right)\tau_3) = x(\tau_1, \tau_3)
\]
and hence $x(\tau_1, \tau_3) \in K$. Since $t(\tau_1)$ and $t(\tau_3)$ are clearly real, we find that $x(\tau_1, \tau_3) \in \mathbb{Q}$ and so does $y(\tau_1, \tau_3)$. Indeed, we find that
\[
x(\tau_1, \tau_3) = \frac{19601}{217800}, \quad y(\tau_1, \tau_3) = \frac{1}{392022}.
\]

### 3.3. Evaluations of the constants in Theorem 2
The constants $B_j$ and $C_j$ in Theorem 2 depend on the values of several modular functions at CM-points, including $t_j$, $\epsilon$, and $\delta_j$. Here we shall describe the strategies to determine their values.

**Determination of $t_j$.** The evaluation of $t_j$ is the easiest. The value of the modular function $1/t$ at a CM-point are algebraic integers since $1/t$ is integral over $\mathbb{Z}[j]$ and the value of the elliptic $j$-function at a CM-point is known to be an algebraic integer. Thus, one only needs to evaluate $1/t$ at all CM-points in the same Galois orbit to sufficient precision and recognize any symmetric sum of these values as a rational integer. This will give us the minimal polynomial of $t_j$ over $\mathbb{Q}$ and hence its exact value.
Determination of $\epsilon$. The determination of $\epsilon$ is a little more complicated. Here we first give an example and explain the general strategy later.

Example 24. Consider the case $(a, b, c) = (-17, -6, 72)$ with $\Gamma = \Gamma_0(6)$,

\begin{equation}
 t(\tau) = \frac{\eta(2\tau)\eta(6\tau)^5}{\eta(\tau)^5\eta(3\tau)}, \quad f(\tau) = \frac{\eta(\tau)^9\eta(6\tau)}{\eta(2\tau)^3\eta(3\tau)^2}.
\end{equation}

Let $\tau_1$ and $\tau_2$ be the two CM-points of discriminant $-420$ given by (34) in Example 16(i). To obtain a $1/\pi$-series using $\tau_1$ and $\tau_2$, we are required to determine the value of $\epsilon := f(\tau_2)/f(\tau_1)$.

Let $R$ be the quadratic order of discriminant $d = -420$. The class group of $R$ is generated by $p_2$, $p_3$, and $p_5$, the unique prime ideals lying above 2, 3, and 5, respectively. Since the quadratic forms associated to $\tau_1$ and $\tau_2$ are $[6, 6, 19]$ and $[30, 30, 11]$, respectively, i.e., the ideals of $R$ associated to $\tau_1$ and $\tau_2$ are $p_2p_3$ and $p_2p_3p_5$, respectively, there exists a matrix $\gamma$ of determinant 5 in $\left(\frac{\mathbb{Z}}{\mathbb{Z} \tau}\right)$ such that $\gamma \tau_1 = \gamma_2$. Indeed, we find that $\gamma = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 5 \end{smallmatrix}\right)$ has this property.

Now let $\gamma_0 = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\gamma_j = \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right)$, $j = 1, \ldots, 5$, be the six matrices defining the Hecke operator $T_5$ on spaces of modular forms on $\Gamma_0(6)$ and let $g_j := f^2j\gamma_j$. Then we have

\[ \prod_{j=0}^{5} \left( x - \frac{g_j}{f^2} \right) \in \mathbb{Q}(t)[x]. \]

Using Fourier expansions, we can determine this polynomial $P(x)$ (which is too complicated to be displayed here). We then specialize this polynomial $P(x) \in \mathbb{Q}(t)[x]$ at $t = t(\tau_1)$, whose value is given in Example 16(i), and obtain a polynomial $p(x)$ in $x$ of degree 6 over $\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7})$. Since $\tau_2 = (\frac{1}{0} - 2 \frac{1}{5}) \tau_1$, $f(\tau_2)^2/5f(\tau_1)^2 = g_3(\tau_1)/f(\tau_1)^2$ is a root of $p(x)$. Observe that $\gamma_j \tau_1$, $j \neq 3$, are CM-points of discriminant $-5^2 \cdot 420$. This means that among the six roots of $p(x)$, five are in the ring class field of the quadratic order of discriminant $-5^2 \cdot 420$ and the remaining root is in $\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7})$. This is the value of $f(\tau_2)^2/5f(\tau_1)^2$. After completing these tedious calculations, we find that

\[ \frac{f(\tau_2)}{f(\tau_1)} = \frac{\sqrt{5}}{2} (-5 + 3\sqrt{3})(3 + \sqrt{7})(4 - \sqrt{15})(6 + \sqrt{35}). \]

(Notice that $f(\tau_2)/\sqrt{5}f(\tau_1)$ is a unit in $\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{7})$. There should be a theoretical explanation for this phenomenon, but we will not pursue it here.)

From the example, it should be evident how to determine $\epsilon = f(\tau_2)/f(\tau_1)$. Namely, we find a matrix $\gamma$ in the relevant order $\left(\begin{smallmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{smallmatrix}\right)$ except for the case $(a, b, c) = (11, 3, -1)$ of $M(2, \mathbb{Q})$. Let $n = \det \gamma$ and $\gamma_j$ be the matrices defining the Hecke operators $T_n$, express the polynomial

\[ \prod \left( x - \frac{f^2j\gamma_j}{f^2} \right) \]

as a polynomial $P(x)$ in $\mathbb{Q}(t)[x]$, and specialize $P(x)$ at $t = t(\tau_1)$, say $p(x)$. Then $f(\tau_2)/f(\tau_1)$ will be the unique root of $p(x)$ that is in the same field as $t(\tau_1)$.

Determination of $\delta_j$. The determination of $\delta_j$ is the most complicated part. Traditionally, it is done by using modular equations and known values of modular functions at CM-points. Here we use a different approach.

Lemma 25. Let $t(\tau)$ and $f(\tau)$ be the modular function and the modular form of weight 1 associated to one of the sporadic Apéry-like sequences, as given in Table 1. Let $\tau_0$
be a CM-point of discriminant \( d \) with \( \phi \) being the corresponding optimal embedding. Let 
\[ \alpha = \phi(\sqrt{d}) \] 
and set 
\[ g = \theta_q t, \quad h = \frac{g^{\cdot 2} \alpha}{g}, \quad \delta = \frac{\theta_q h}{f^2}(\tau_0). \]

Then there exists an explicitly computable positive integer \( A \), depending only on the Galois orbit over \( \mathbb{Q} \) of the given CM-point, such that \( A\delta \) is an algebraic integer.

**Example 26.** Let us continue working on the case considered in Example 24, i.e., let \( t \) and \( f \) be given by (36). We check that 
\[ g = t(1 + 17t + 72t^2)f^2. \]
(In fact, \( \theta_q t = t(1 - at + ct^2)f^2 \) holds in all six cases.) Note that 
\[ \text{div} t = (\infty) - (0), \quad \text{div}(1 + 8t) = (1/3) - (0), \] 
\[ \text{div}(1 + 9t) = (1/2) - (0), \quad \text{div} f^2 = 2(0), \] 
and hence 
\[ \text{div} g = (\infty) + (1/2) + (1/3) - (0). \]

Consider \( \tau_1 = (-6 + \sqrt{-420})/12 \), we have \( \alpha = \left(\begin{smallmatrix} -3 & -2 \\ 6 & 3 \end{smallmatrix}\right) \). Observe that 
\[ \alpha = \left(\begin{smallmatrix} -3 & -2 \\ 6 & 3 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -17 \\ 0 & 35 \end{smallmatrix}\right) . \]

Let \( w_3 = \left(\begin{smallmatrix} -3 & -2 \\ 6 & 3 \end{smallmatrix}\right) \) and \( \gamma = \left(\begin{smallmatrix} 1 & -17 \\ 0 & 35 \end{smallmatrix}\right) \). We check that 
\[ t|w_3 = -\frac{1 + 9t}{9(1 + 8t)}, \quad f^2|w_3 = \frac{9(3\tau)^{12}\eta(2\tau)^2}{\eta(6\tau)^6\eta(\tau)^2} = 9(1 + 8t)^2f. \]

Thus, 
\[ g|\alpha = g|w_3|\gamma = \frac{g}{9(1 + 8t)^2}|\gamma. \]

Note that 
\[ \text{div} \frac{g}{(1 + 8t)^2} = (\infty) + (1/2) + (0) - (1/3). \]

Now let \( \gamma_{e,j} = \left(\begin{smallmatrix} e \\ j \end{smallmatrix}\right) \), \( e = 1, 5, 7, 35 \) and \( j = 0, \ldots, 35/e - 1 \), be the standard coset representatives defining the Hecke operator \( T_{35} \) on spaces of modular forms on \( X_0(6) \) and let 
\[ h_{e,j} = \frac{g/(1 + 8t)^2|\gamma_{e,j}}{9g} \]
(with \( h = h_{1,18} \) being one of them). Since 
\[ \frac{g}{(1 + 8t)^2}|\gamma_{e,j} = \frac{e}{35(1 + 8t((e^2 \tau + ej)/35))^2}, \]
we find that the Fourier coefficients of \( 105^2(\theta_q h_{e,j})/f^2 \) are all algebraic integers and that any symmetric sum of \( 105^2(\theta_q h_{e,j})/f^2 \) has integer Fourier coefficients. Futhermore, \( \gamma_{e,j}|\infty \) (respectively, \( \gamma_{e,j}|1/2 \), \( \gamma_{e,j}|1/3 \), \( \gamma_{e,j}|0 \)) are all equivalent to \( \infty \) (respectively, \( 1/2, 1/3, 0 \)) under \( \Gamma_0(6) \) for all \( e \) and \( j \). Hence, by (37), (38), and (39), we have 
\[ \text{div} \sum_{e,j} \left(\frac{\theta_q h_{e,j}}{f^2}\right)^k \geq -k((\infty) + (1/2) + 2(1/3) + 2(0)) \]
for any positive integer $k$. Thus, if we let $j_6 = 1/t$, which has values $\infty$, $-9$, $-8$, and $0$ at the cusps $\infty$, $1/2$, $1/3$, and $0$, respectively, then

$$\sum_{e,j} \left( 105^2 j_6^2 (j_6 + 9)(j_6 + 8)^2 \frac{\theta_q h_{e,j}}{\ell^2} \right)^k$$

is a modular function on $\Gamma_0(6)$ with integer Fourier coefficients and a unique pole at $\infty$ and therefore is equal to $P_h(j_6)$ for some $P_h(x) \in \mathbb{Z}[x]$. Now the value of $j_6$ at the CM-point $\tau_1$ is an algebraic integer, which implies that

$$105^2 j_6(\tau_1)^2 (j_6(\tau_1) + 9)(j_6(\tau_1) + 8)^2 \frac{\theta_q h_{e,j}}{\ell^2}(\tau_1)$$

are all algebraic integers for all $e$ and $j$. In particular, if we write the value of $j_6(\tau_1)^2 (j_6(\tau_1) + 9)(j_6(\tau_1) + 8)^2$ as $B/C$ for some rational integer $B$ and some algebraic integer $C$, then $A\delta$ is an algebraic integer for $A = 105^2B$. Here we find that

$$\frac{1}{j_6(\tau_1)^2 (j_6(\tau_1) + 9)(j_6(\tau_1) + 8)^2} = \frac{1}{1728} \left( 2 - \sqrt{3} \right)^6 \left( \frac{1 - \sqrt{5}}{2} \right)^{18}$$

$$\times (4 - \sqrt{15})^3 \left( \frac{5 - \sqrt{21}}{2} \right)^6 \left( 6 - \sqrt{35} \right)^3 (2\sqrt{5} - \sqrt{21})^3.$$

That is, we may choose $B = 1728$ and $A = 105^2 \cdot 1728$. From the argument, we see that the same integer $A$ works for all CM-points in the same Galois orbit of $\tau_1$. Thus, we can determine the value of $\delta$ by evaluating numerically $A\delta$ to sufficient precision for all CM-points in the Galois orbit and recognize every symmetric sum as a rational integer.

After a tedious computation, we arrive at

$$\delta = \frac{1}{3\sqrt{35}} \left( 2 - \sqrt{3} \right)^2 \left( \frac{1 + \sqrt{5}}{2} \right)^6 (8 + 3\sqrt{7})(4 - \sqrt{15}) \left( \frac{5 - \sqrt{21}}{2} \right)^2 (6 + \sqrt{35})$$

$$\left( \sqrt{21} - 2\sqrt{5} \right) \left( \frac{87 + 80\sqrt{3} + 84\sqrt{5} + 48\sqrt{7} + 14\sqrt{15} + 17\sqrt{21} - 34\sqrt{35} - 4\sqrt{105}}{2} \right).$$

The value of $\delta$ for $\tau_2 = (-30 + \sqrt{-120})/60$ is the Galois conjugate of the above number obtained by $\sqrt{3} \to -\sqrt{3}, \sqrt{5} \to \sqrt{5},$ and $\sqrt{7} \to -\sqrt{7}$. Combining this computation with the results from Examples [19](i) and [24](a), and applying Theorem [2], we obtain

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor n/2 \rfloor} u_n \frac{2m}{m} \left( \frac{n}{2m} \right) (5n - 2) \left( \frac{-71}{1008} \right)^n \left( \frac{1}{142^2} \right)^m = \frac{3\sqrt{35}}{\pi},$$

where $\{u_n\}$ is the sequence corresponding to $(a, b, c) = (-17, -6, 72)$ in Table [1].

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Appendix A. 2-variable $1/\pi$-series for sporadic Apéry sequences
\[ d \quad x \quad y \quad A \quad B \quad C \quad \text{Ref.} \]

-96  
-1/12  
-1/8  
9  
3  
4\sqrt{2}  
31 (6.1)

-192  
-1/36  
-1/32  
99  
23  
39\sqrt{2}  
31 (6.2)

-240  
1/60  
1/64  
315  
51  
92\sqrt{5}  
31 (6.3)

-660  
47/441  
1/47^2  
180  
-8  
483\sqrt{3}  
31 (6.4)

-840  
241/5080  
1/482^2  
630  
75  
374\sqrt{2}  
31 (6.5)

-1092  
-727/6776  
1/1454^2  
585  
172  
110\sqrt{7}  
31 (6.6)

-1320  
1057/50784  
1/2114^2  
630  
77  
92\sqrt{15}  
31 (6.7)

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| (a, b, c) = (7, 2, -8) |   |   |   |   |   |

| d  | x   | y   | A   | B   | C   | Ref. |
|----|-----|-----|-----|-----|-----|------|
| -96| -7/81| -8/7^2 | 32  | 12  | 9\sqrt{3} | 31 (7.1) |
| -192| -31/1089| -32/31^2 | 64  | 16  | 33  | 31 (7.2) |
| -240| 13/135| 1/52^2 | 7   | -1  | 30\sqrt{5} | 31 (7.3) |
| -660| -89/990| 1/178^2 | 280 | 93  | 20\sqrt{33} | 31 (7.4) |
| -840| 251/6300| 1/502^2 | 176 | 15  | 25\sqrt{42} | 31 (7.5) |
| -1320| 485/6534| 1/970^2 | 560 | -23 | 693\sqrt{3} | 31 (7.6) |
| -1380| 1079/52920| 1/2158^2 | 12880 | 1332 | 4410\sqrt{3} | 31 (7.7) |
| -1428| 5291/57132| 1/1058^2 | 6160 | -1824 | 15939\sqrt{3} | 31 (7.8) |
| -1848| 8749/255150| 1/1749^2 | 32032 | 2546 | 14175\sqrt{3} | 31 (7.9) |
| 21295/267696| 1/42590^2 | 560 | -67 | 286\sqrt{22} | 31 (7.10) |

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| (a, b, c) = (10, 3, 9) |   |   |   |   |   |   |

\[ d \quad x \quad y \quad A \quad B \quad C \quad \text{Ref.} \]

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| (a, b, c) = (7, 2, -8) |   |   |   |   |   |   |

\[ d \quad x \quad y \quad A \quad B \quad C \quad \text{Ref.} \]

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| (a, b, c) = (10, 3, 9) |   |   |   |   |   |   |
### Table 4. $1/\pi$-series for $(a, b, c) = (-17, -6, 72)$

| $d$   | $x$    | $y$    | $A$ | $B$ | $C$    |
|-------|--------|--------|-----|-----|--------|
| -96   | $-1/18$| $-1/32$| 1   | 0   | $\sqrt{6}$    |
| -180  | $-1/16$| $1/18^2$| 5   | -1  | $9\sqrt{3}$    |
|       | $-31/320$| $1/62^2$| 5   | -6  | $30\sqrt{3}$    |
| -192  | $-5/216$| $-1/50$| 11  | 1   | $6\sqrt{3}$    |
| -240  | $7/360$| $1/7^2$| 35  | 9   | $4\sqrt{15}$    |
| -420  | $-71/1008$| $1/142^2$| 5   | -2  | $3\sqrt{35}$    |
|       | $-161/1728$| $1/322^2$| 35  | -41 | $252\sqrt{3}$    |
|       | $-161/3240$| $1/322^2$| 385 | -57 | $360\sqrt{3}$    |
|       | $-1079/10584$| $1/2158^2$| 5   | -11 | $84\sqrt{3}$    |
|       | $161/2592$| $1/322^2$| 385 | 150 | $72\sqrt{3}$    |
| -1092 | $-1351/23400$| $1/2702^2$| 385 | -101| $210\sqrt{13}$   |
|       | $-6049/60984$| $1/12098^2$| 65  | -124| $132\sqrt{91}$   |
|       | $881/35280$| $1/1762^2$| 805 | 209 | $210\sqrt{3}$    |
|       | $-1351/73008$| $1/2702^2$| 1610| 76  | $585\sqrt{3}$    |
|       | $-51841/476928$| $1/103682^2$| 65  | -375| $4968\sqrt{3}$   |
|       | $-3401/75600$| $1/6802^2$| 13090| -1714| $9765\sqrt{3}$   |
|       | $-28799/278784$| $1/57598^2$| 595  | -1602| $2728\sqrt{51}$   |
| -1320 | $8049/121968$| $1/12098^2$| 30030| 10506| $4081\sqrt{6}$   |

### Table 5. $1/\pi$-series for $(a, b, c) = (-9, -3, 27)$

| $d$   | $x$    | $y$    | $A$ | $B$ | $C$    | Ref.   |
|-------|--------|--------|-----|-----|--------|--------|
| -180  | $-2/27$| $-1/4^2$| 26  | 5   | $27\sqrt{3}$    | [31] (9.2)|
| -288  | $-7/50$| $-1/14^2$| 13  | 3   | $30\sqrt{2}$    | [31] (9.3)|
| -315  | $-1/6$  | $1/18^2$| 14  | 5   | $27\sqrt{3}$    | [31] (9.3)|
| -576  | $-22/243$| $-1/968$| 76  | 8   | $81\sqrt{3}$    | [31] (9.4)|
| -819  | $-55/378$| $1/110^2$| 182 | 37  | $315\sqrt{3}$   | [31] (9.6)|
| -1008 | $52/675$| $1/52^2$| 182 | 64  | $45\sqrt{3}$    | [31] (9.5)|
| -3627 | $-12151/95256$| $1/24302^2$| 107198| 8989| $147420\sqrt{3}$| |
| -3843 | $-6049/151200$| $1/12098^2$| 410774| 33451| $182700\sqrt{3}$| |
\begin{table}
\centering
\begin{tabular}{l|cccccc}
\hline
$d$ & $x$ & $y$ & $A$ & $B$ & $C$ & Ref. \\
\hline
$-240$ & $13/225$ & $1/52^2$ & 145 & 9 & 285 & [31, (8.1)] \\
$-760$ & $19601/217800$ & $1/3920^2$ & 95 & $-1388$ & $9405\sqrt{19}$ & \\
\hline
\end{tabular}

\caption{1/$\pi$-series for $(a, b, c) = (11, 3, -1)$}
\end{table}

\begin{table}
\centering
\begin{tabular}{l|cccccc}
\hline
$d$ & $x$ & $y$ & $A$ & $B$ & $C$ & Ref. \\
\hline
$-60$ & $1/32$ & $1$ & 15 & 3 & $8(2 + \sqrt{5})$ & [31, (5.1)] \\
$-64$ & $1/36$ & $-2$ & 3 & 1 & 3 & [31, (5.3)] \\
$-32, -288$ & $7/96$ & $1/14^2$ & 3 & 0 & 8 & [31, (5.4)] \\
$-112$ & $-1/252$ & $16$ & 21 & 5 & $6\sqrt{7}$ & [31, (5.2)] \\
$-480$ & $-7/96$ & $1/14^2$ & 30 & 11 & 12 & [31, (5.5)] \\
$11/240$ & $1/22^2$ & 15 & 1 & $6\sqrt{10}$ & [31, (5.6)] \\
$31/320$ & $1/62^2$ & 30 & $-7$ & 160 & [31, (5.9)] \\
$-672$ & $-13/336$ & $1/26^2$ & 21 & 6 & $2\sqrt{21}$ & [31, (5.7)] \\
$17/576$ & $1/34^2$ & 21 & 2 & 18 & [31, (5.8)] \\
$97/896$ & $1/194^2$ & 6 & $-3$ & 56 & [31, (5.13)] \\
$-1120$ & $-71/720$ & $1/142^2$ & 210 & 85 & $33\sqrt{5}$ & [31, (5.12)] \\
$127/2304$ & $1/254^2$ & 210 & $-1$ & 288 & [31, (5.16)] \\
$251/2800$ & $1/502^2$ & 42 & $-10$ & $105\sqrt{2}$ & [31, (5.17)] \\
$-1248$ & $-49/4800$ & $1/98^2$ & 195 & 34 & 80 & [31, (5.10)] \\
$53/5616$ & $1/106^2$ & 195 & 22 & $27\sqrt{13}$ & [31, (5.11)] \\
$1249/10400$ & $1/2498^2$ & 78 & $-131$ & 2600 & [31, (5.20)] \\
$-1632$ & $-97/18816$ & $1/194^2$ & 1785 & 254 & 672 & [31, (5.14)] \\
$101/20400$ & $1/202^2$ & 210 & 23 & $15\sqrt{34}$ & [31, (5.15)] \\
$4801/39200$ & $1/960^2$ & 510 & $-1523$ & 33320 & [31, (5.23)] \\
$-2080$ & $-577/18496$ & $1/1154^2$ & 7410 & 1849 & 2992 & [31, (5.18)] \\
$721/28880$ & $1/1442^2$ & 6630 & 505 & $2014\sqrt{5}$ & [31, (5.19)] \\
$5201/46800$ & $1/1040^2$ & 570 & $-457$ & $1590\sqrt{13}$ & [31, (5.24)] \\
$-3040$ & $-2737/197136$ & $1/5475^2$ & 62985 & 11363 & $7659\sqrt{10}$ & [31, (5.21)] \\
$3041/243200$ & $1/608^2$ & 358530 & 33883 & 176280 & [31, (5.22)] \\
$52021/439280$ & $1/104042^2$ & 3705 & $-5918$ & $36499\sqrt{5}$ & \\
\hline
\end{tabular}

\caption{1/$\pi$-series for $(a, b, c) = (12, 4, 32)$}
\end{table}
## Appendix B. 2-Variable $1/\pi$-series for Hypergeometric Sequences

| $d$       | $x$    | $y$   | $A$ | $B$ | $C$ | Ref.   |
|-----------|--------|-------|-----|-----|-----|--------|
| $-7$      | $-1/16$ | 16    | 30  | 7   | 24  | [30] (I1) |
| $-192$    | $-17/32$ | $1/34^2$ | 30  | 7   | 12  | [30] (I2) |
| $-240$    | $31/128$ | $1/62^2$ | 42  | 5   | $16\sqrt{3}$ | [30] (I4) |
|           | $97/128$ | $1/194^2$ | 30  | $-1$ | 80  | [30] (I3) |

Table 8. $1/\pi$-series for $a = 1/2$

| $d$       | $x$    | $y$   | $A$ | $B$ | $C$ | Ref.   |
|-----------|--------|-------|-----|-----|-----|--------|
| $-12$, $-48$ | $1/2$  | $1/54$ | 60  | 8   | $45\sqrt{3}$ | [30] (I1) |
| $-20$     | $-7/20$ | $729/980$ | 54  | 12  | $15\sqrt{3} + \sqrt{15}$ | [30] (II11) |
| $-32$     | $-2/25$ | $729/800$ | 756 | 132 | $75\sqrt{3} + 100\sqrt{6}$ | [30] (II2) |
| $-35$     | $13/256$ | $729/676$ | 135 | 21  | $24\sqrt{3} + 8\sqrt{15}$ | [30] (II10) |
| $-72$     | $27/100$ | $1/100$ | 182 | 24  | $75\sqrt{3}$ | [30] (II2) |
| $-72$     | $73/100$ | $729/730^2$ | 18  | 1   | $25\sqrt{3}$ | [30] (II5) |
| $-84$     | $-1/4$  | $-1/108$ | 39  | 7   | $9\sqrt{3}$ | [30] (II1) |
| $-120$    | $1/12$  | $1/18^2$ | 210 | 25  | $54\sqrt{3}$ | [30] (II3) |
|           | $11/12$ | $1/198^2$ | 30  | $-8$ | $135\sqrt{3}$ | [30] (II3) |
| $-132$    | $-3/44$ | $-1/396$ | 45  | 6   | $5\sqrt{11}$ | [30] (II3) |
| $-168$    | $3/100$ | $1/30^2$ | 198 | 21  | $50\sqrt{2}$ | [30] (II3) |
|           | $97/100$ | $1/970^2$ | 42  | $-41$ | $525\sqrt{3}$ | [30] (II4) |
| $-228$    | $-1/100$ | $-1/2700$ | 17157 | 1654 | $2925\sqrt{3}$ | [30] (II3) |
| $-240$    | $488/1331$ | $1/188^2$ | 11310 | 976  | $4719\sqrt{3}$ | [30] (II3) |
| $-240$    | $843/1331$ | $1/843^2$ | 2520 | 48  | $1573\sqrt{3}$ | [30] (II3) |
| $-312$    | $3/1156$ | $1/102^2$ | 13860 | 1118 | $1445\sqrt{6}$ | [30] (II6) |
|           | $1153/1156$ | $1/3920^2$ | 390 | $-3967$ | $56355\sqrt{3}$ | [30] (II8) |
| $-372$    | $-1/900$ | $-1/24300$ | 105339 | 7843  | $14175\sqrt{3}$ | [30] (II3) |
| $-408$    | $1/1452$ | $1/198^2$ | 888420 | 62896 | $114345\sqrt{3}$ | [30] (II7) |
|           | $1451/1452$ | $1/287298^2$ | 210 | $-7157$ | $114345\sqrt{3}$ | [30] (II9) |
| $-435$    | $-107/256$ | $1/5778^2$ | 39585 | 7075  | $7344\sqrt{3}$ | [30] (II3) |
| $-555$    | $-9249/42592$ | $1/18498^2$ | 7245 | 1073  | $605\sqrt{15}$ | [30] (II3) |
| $-708$    | $-3/124844$ | $-1/1123596$ | 6367095 | 342786 | $140185\sqrt{59}$ | [30] (II3) |
| $-795$    | $-7361/96000$ | $1/132498^2$ | 62403 | 7049  | $10800\sqrt{3}$ | [30] (II3) |

Table 9. $1/\pi$-series for $a = 1/3$
| $d$   | $x$    | $y$    | $A$  | $B$   | $C$   | Ref. |
|-------|--------|--------|------|-------|-------|------|
| $-8, -72$ | $1/2$   | $1/98^2$ | 360  | 27    | $70\sqrt 21$ | $[30]$ (II(3)) |
| $-15$   | $-7/21^2$ | $64^2/7^2$ | 640  | 104   | $21\sqrt 42 + 14\sqrt 210$ | $[30]$ (II(5)) |
| $-48$   | $257/33^2$ | $16^2/257^2$ | 160  | 18    | $11\sqrt 66$ | $[30]$ (I(II)) |
|         | $832/33^2$ | $1/52^2$  | 85   | 2     | $33\sqrt 33$ | $[30]$ (II(4)) |
| $-64$   | $-511/63^2$ | $-512/511^2$ | 704  | 92    | $63\sqrt 14$ | $[30]$ (II(2)) |
| $-84$   | $-110/12^2$ | $1/110^2$  | 28   | 5     | $3\sqrt 6$ | $[30]$ (II(2)) |
| $-112$  | $4097/513^2$ | $64^2/4097^2$ | 19040 | 1682  | $513\sqrt 114$ | $[30]$ (II(2)) |
|         | $259072/513^2$ | $1/4048^2$ | 455  | $-784$ | $2052\sqrt 77$ | $[30]$ (II(6)) |
| $-120$  | $322/42^2$ | $1/322^2$  | 760  | 71    | $126\sqrt 7$ | $[30]$ (II(6)) |
|         | $1442/42^2$ | $1/1442^2$ | 40   | $-4$  | $7\sqrt 210$ | $[30]$ (II(7)) |
| $-132$  | $-398/48^2$ | $1/398^2$  | 260  | 33    | $32\sqrt 6$ | $[15]$ (II(6)) |
| $-168$  | $898/114^2$ | $1/898^2$  | 3080 | 276   | $95\sqrt 178$ | $[30]$ (II(8)) |
|         | $12098/114^2$ | $1/12098^2$ | 280  | $-139$ | $95\sqrt 399$ | $[30]$ (II(9)) |
| $-228$  | $-2702/336^2$ | $1/2702^2$ | 83980 | 7331  | $3360\sqrt 72$ | $[15]$ (II(7)) |
| $-280$  | $3920/378^2$ | $1/2702^2$ | 1840 | 136   | $135\sqrt 177$ | $[30]$ (II(10)) |
|         | $103682/378^2$ | $1/103682^2$ | 440  | $-25$ | $378\sqrt 3$ | $[30]$ (II(11)) |
| $-312$  | $10402/1302^2$ | $1/10402^2$ | 337480 | 24044 | $3689\sqrt 434$ | $[30]$ (II(10)) |
|         | $1684802/1302^2$ | $1/1684802^2$ | 8840 | $-50087$ | $7378\sqrt 8163$ | $[30]$ (II(11)) |
| $-340$  | $-103682/684^2$ | $1/103682^2$ | 97580 | 12197 | $2736\sqrt 95$ | $[30]$ (II(11)) |
| $-372$  | $-24302/3036^2$ | $1/24302^2$ | 2143960 | 142322 | $11385\sqrt 1518$ | $[15]$ (II(8)) |
| $-408$  | $3920/4902^2$ | $1/3920^2$ | 11657240 | 732103 | $80883\sqrt 817$ | $[30]$ (II(12)) |
|         | $23900402/4902^2$ | $1/23900402^2$ | 3080 | $-58871$ | $17974\sqrt 2481$ | $[30]$ (II(13)) |
| $-520$  | $1684802/5922^2$ | $1/1684802^2$ | 1382440 | 106756 | $14805\sqrt 658$ | $[30]$ (II(13)) |
|         | $33385282/5922^2$ | $1/33385282^2$ | 337480 | $-320300$ | $115479\sqrt 658$ | $[30]$ (II(13)) |
| $-532$  | $-5177198/3348^2$ | $1/5177198^2$ | 602140 | 88597 | $4185\sqrt 1302$ | $[15]$ (II(8)) |
| $-708$  | $1123598/140448^2$ | $1/1123598^2$ | 1898208780 | 90848259 | $4307072\sqrt 4389$ | $[15]$ (II(9)) |
| $-760$  | $33385282/55062^2$ | $1/33385282^2$ | 27724840 | 1877581 | $49266\sqrt 15295$ | $[15]$ (II(9)) |
|         | $2998438562/55062^2$ | $1/2998438562^2$ | 72760 | $-289964$ | $156009\sqrt 322$ | $[15]$ (II(9)) |

Table 10. 1/$\pi$-series for $a = 1/4$

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