ON THE NUMBER OF CLONOIDS

ATHENA SPARKS

Abstract. A clonoid is a set of finitary functions from a set $A$ to a set $B$ that is closed under taking minors. Hence clonoids are generalizations of clones. By a classical result of Post, there are only countably many clones on a 2-element set. In contrast to that, we present continuum many clonoids for $A = B = \{0, 1\}$. More generally, for any finite set $A$ and any 2-element algebra $B$, we give the cardinality of the set of clonoids from $A$ to $B$ that are closed under the operations of $B$.

1. Introduction

A clone on a set $D$ is a set of finitary operations on $D$ that contains all projections and is closed under composition of functions (see [5, page 97] for the definition). In particular, clones are closed under the usual manipulations of permuting variables, identifying variables, and introducing dummy variables in functions. For subsets $A, B$ of $D$, the restriction of a clone on $D$ to the functions from powers of $A$ into $B$ is not a clone anymore. However, this restriction is still closed under the variable manipulations mentioned above. More precisely, such a set of functions is closed under minors.

Definition 1.1. Let $A, B$ be sets and $f: A^k \to B$. Let $[k] := \{1, \ldots, k\}$. For $\ell \in \mathbb{N}$ and $\sigma : [k] \to [\ell]$, the function

$$f^\sigma : A^\ell \to B, (x_1, \ldots, x_\ell) \mapsto f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

is a minor of $f$.

Sets of functions that are closed under minors have been investigated by Pippenger in [4]. He developed a Galois theory for them and pairs of relations that generalizes the classical Galois theory for clones. These sets reappeared recently when Brakensiek and Guruswami classified Promise Constraint Satisfaction Problems (PCSP) via polymorphisms.
between relational structures $A$ and $B$ of the same type in $\mathbb{3}$ (see the definition of polymorphisms at the end of this section). Independently, they were used by Aichinger and Mayr to investigate equational theories of algebras in $\mathbb{1}$. Following the notion introduced in that last paper we define:

**Definition 1.2.** $\mathbb{1}$, Definition 4.1] Let $A$ be a set and $B = (B, \mathcal{F})$ an algebra. For a subset $C$ of $\bigcup_{n \in \mathbb{N}} B^A^n$ and $k \in \mathbb{N}$, we let $C_k := C \cap B^A_k$. We call $C$ a clonoid with source set $A$ and target algebra $B$ if

1. $C$ is closed under taking minors, and
2. for all $k \in \mathbb{N}$, $C_k$ is a subalgebra of $B^A_k$.

The set of all clonoids with source $A$ and target algebra $B$ is denoted $C_{A,B}$.

Note that every subset $C$ of $\bigcup_{n \in \mathbb{N}} B^A_n$ that is closed under taking minors is a clonoid with target algebra the set $(B, \emptyset)$. Further, every clone $C$ on a set $A$ is a clonoid with source set $A$ and target algebra $(A, C)$.

It is a well known result of Post that there are only countably many clones on a two element set $\mathbb{5}$, Theorem 3.1.1]. Janov and Mučnik showed that there are continuum many clones on any finite set with three or more elements $\mathbb{5}$, Theorem 8.1.3]. In light of these results, one may ask whether the number of clonoids for fixed source $A$ and target $B$ depends on the size of $A$ and $B$. We will show that there are already continuum many clonoids with source and target of size 2 (see Corollary $\mathbb{14}$).

The following main result of this paper gives more precise information about the cardinality of clonoids with target algebras of size 2.

**Theorem 1.3.** Let $C_{A,B}$ denote the set of all clonoids with finite source $A$ ($|A| > 1$) and target algebra $B$ of size 2. Then

1. $C_{A,B}$ is finite iff $B$ has an NU-term;
2. $C_{A,B}$ is countably infinite iff $B$ has a Mal’cev term but no majority term;
3. $C_{A,B}$ has size continuum iff $B$ has neither an NU-term nor a Mal’cev term.

Moreover, in cases (1) and (2) all clonoids in $C_{A,B}$ are finitely related.

See the end of this section for a definition of finitely related clonoids.

Following the case distinction of the theorem, we consider the size of $C_{A,B}$ for an arbitrary finite $B$ with an NU-term in Section $\mathbb{2}$, for $B$ with a cube term in Section $\mathbb{3}$ and for $B$ of size 2 without cube term in Section $\mathbb{4}$ (see $\mathbb{2}$, Definition 2.5] for the definition of a cube
term). In Section 5 we combine the results from these sections to prove Theorem 1.3.

Since each clonoid with a target \(\{0,1\}\) is also a clonoid with target \(\{0,\ldots,n\}\) for any \(n \geq 1\), Theorem 1.3 (3) immediately yields the following:

**Corollary 1.4.** For all \(m, n \geq 1\), there are continuum many clonoids with source \(\{0,\ldots,m\}\) and target \(\{0,\ldots,n\}\).

We introduce some more notation that will be needed in the following sections. Let \(A\) be a set and \(B = (B, \mathcal{F})\) an algebra. For a set \(F \subseteq \bigcup_{n \in \mathbb{N}} B^{A^n}\), the clonoid with source set \(A\) and target algebra \(B\) generated by the functions in \(F\) is denoted \(\langle F \rangle_B\). Let \(P\) and \(Q\) be a pair of \(k\)-ary relations on \(A\) and \(B\) respectively. A function \(f: A^k \to B\) is a polymorphism of \((P, Q)\) if \(f\) applied component-wise to any \(k\)-tuple of elements of \(P\) is an element of \(Q\). For a set of pairs of relations \(R := \{(P_i, Q_i) : i \in I\}\) on \(A\) and \(B\), the set of functions that are polymorphisms of all relations in \(R\) is denoted \(\text{Pol}(R)\). If \(R\) contains only a single pair of relations \((P, Q)\), we write \(\text{Pol}(P, Q)\) instead. Note that if \(f \in \text{Pol}(R)\), then any minor of \(f\) is in \(\text{Pol}(R)\). A clonoid \(C\) with source set \(A\) and target algebra \(B\) is finitely related if there exists a finite set of pairs of finitary relations \(R := \{(P, Q_i) : 1 \leq i \leq n\}\) on \(A\) and \(B\) such that \(C = \text{Pol}(R)\).

2. NU-terms

In this section, we will show that there are only finitely many clonoids with a finite source \(A\) and algebra \(B\) with an NU-term. In particular, we show that each such clonoid is the polymorphism clonoid of a single pair of relations on \(A\) and \(B\).

**Theorem 2.1.** Let \(A\) be a finite set of size greater than 1 and \(B\) a finite algebra with \(n\)-ary NU-term \((n \geq 3)\). Let \(C\) be a clonoid with source \(A\) and target \(B\). Then \(C = \text{Pol}(\Pi_{A}^{[A]^{n-1}}, C_{|A|^{n-1}})\) where \(\Pi_{A}^{[A]^{n-1}}\) is the set of all \(|A|^{n-1}\)-ary projections on \(A\). Hence there are only finitely many such clonoids with source \(A\) and target \(B\).

**Proof.** Let \(f: A^k \to B\). We claim that

\[(2.1) \quad f \in C \text{ iff all } |A|^{n-1}\text{-ary minors of } f \text{ are in } C.\]

This is equivalent to \(C = \text{Pol}(\Pi_{A}^{[A]^{n-1}}, C_{|A|^{n-1}})\).

The forward direction of (2.1) is immediate from the definition of clonoids. For the reverse direction, note that the \(k\)-ary functions in \(C\) form a subalgebra \(C_k\) of \(B^{A^k}\). By the Baker-Pixley Theorem, \(C_k\)
is uniquely determined by its projections onto the subsets of $A^k$ with $n - 1$ or fewer elements. More precisely,

$$(2.2) \quad f \in C_k \iff \forall I \subseteq A^k \text{ with } |I| \leq n - 1, \exists g \in C_k \text{ so that } f|_I = g|_I.$$ 

Let $Z$ be a matrix with $n - 1$ rows whose columns are the $|A|^{|A|^{n-1}}$ tuples of $A^{n-1}$ in some order. For fixed $x_1, \ldots, x_{n-1} \in A^k$, let $X$ denote the matrix with rows $x_1, \ldots, x_{n-1}$ and $k$ columns. Let $\sigma: [k] \to [|A|^{n-1}]$ such that the $i$-th column of $X$ is equal to the $\sigma(i)$-th column of $Z$. With functions acting on the rows of the corresponding matrices, we then have

$$(2.3) \quad f(X) = f^\sigma(Z).$$

With (2.3) and (2.2) it follows that $f \in C_k$. Thus (2.1) and the lemma are proved. \hfill \Box

3. Cube term

In this section, we will show that all clonoids with a finite source and a target algebra with a cube term, in particular, with a Mal'cev term, are finitely related. We will also construct infinitely many clonoids for a fixed algebra of size 2 with a Mal'cev term.

**Theorem 3.1.** Let $A$ be a finite set and $B$ a finite algebra with cube term. Then each clonoid $\mathcal{C}$ with source $A$ and target $B$ is finitely related. Hence there are at most countably many such clonoids.

**Proof.** Follows since $\mathcal{C}_{A,B}$ satisfies the DCC by [1, Theorem 5.3]. \hfill \Box

Next we show that there actually are infinitely many clonoids with target any Mal'cev algebra of size 2 without an NU-term. By Post's classification of Boolean clones, the clone of each such algebra is contained in the clone of $([0,1], +, 0, 1)$, where 0, 1 are the unary constant functions. For algebras $B$ and $B'$, if the clone of $B'$ is contained in the clone of $B$, then $\mathcal{C}_{A,B} \subseteq \mathcal{C}_{A,B'}$ for any set $A$. So it suffices to construct infinitely many clonoids with target algebra $B := ([0,1], +, 0, 1)$.

**Example 3.2.** Let $A := \{0, \ldots, d\}$ for $d \in \mathbb{N}$ and $B := ([0,1], +, 0, 1)$. For $k \in \mathbb{N}$ define

$$e_k: A^k \to \{0,1\}, x \mapsto \begin{cases} 1 & \text{if } x = (1, \ldots, 1), \\ 0 & \text{else.} \end{cases}$$

We will show that

$$\langle e_1 \rangle_B \subset \langle e_2 \rangle_B \subset \ldots$$
is an infinite ascending chain of clonoids with target \( B \). The idea for this example was used by Bulatov in [4] to construct countably many expansions of \((\mathbb{Z}_4, +)\).

It is enough to show that

\[
(3.1) \quad e_k \neq \sum_{i=1}^{k-1} a_i e_i^{\sigma_i} \text{ for any } a_i \in \{0, 1\} \text{ and } \sigma_i : [i] \to [k].
\]

For any \( i < k \) and \( \sigma_i : [i] \to [k] \), let the support of \( e_i^{\sigma_i} \) be \( \{ x \in [0,1]^k : e_i^{\sigma_i}(x) = 1 \} \). Note that the support of \( e_i^{\sigma_i} \) has even size for any \( i < k \) and \( \sigma_i : [i] \to [k] \). Hence \( \sum_{i=1}^{k-1} a_i e_i^{\sigma_i} \) has support of even size for all \( a_i, \sigma_i \). Since the support of \( e_k \) is odd, (3.1) follows immediately.

4. Without cube term

In this section, we will show that there are continuum many clonoids with finite sources and three specific target algebras all with size 2 and without NU-terms or Mal’cev terms.

By Post’s classification of Boolean clones, each clone on \{0, 1\} without an NU-term or Mal’cev term is contained in a clone generated by one of the following sets of operations:

1. \( \{\neg, 0\} \),
2. \( \{\to\} \) or \( \{\not\to\} \),
3. \( \{\land, 0, 1\} \) or \( \{\lor, 0, 1\} \).

Thus there are 3 cases up to duality. We will show that for each case there are continuum many clonoids with source \( A = \{0, 1, \ldots, d\} \) for \( d \in \mathbb{N} \) and corresponding target algebra \( B \). From this it follows that for algebras with smaller clone of term operations (e.g., the set \((\{0, 1\}, \emptyset)\)), there are continuum many clonoids as well.

We will use the following functions and relations for the proofs of the three cases. Define the following \( n \)-ary relations on \( A \) and \{0, 1\}, respectively, for all \( n \in \mathbb{N} \):

\[
P_n := \{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\} \subseteq A^n,
\]

\[
Q_n := \{0,1\}^n \setminus \{(1,\ldots,1)\} \subseteq B^n.
\]

For \( U \subseteq \mathbb{N} \), let \( R_U := \{(P_n, Q_n) : n \in U\} \). Note that 0 preserves \( R_U \) for any \( U \subseteq \mathbb{N} \).

Define the following \( k \)-ary functions for all \( k \in \mathbb{N} \):

\[
f_k : A^k \to \{0, 1\},
\]

\[
x \mapsto \begin{cases} 1 & \text{if } x \in P_k, \\ 0 & \text{otherwise}. \end{cases}
\]
For $U \subseteq \mathbb{N}$, let $F_U := \{ f_k : k \in U \}$.

We show some connections between these functions and relations that we need later.

**Lemma 4.1.**

1. Let $k, n \in \mathbb{N}$. Then $f_k$ preserves $(P_n, Q_n)$ iff $k \neq n$.
2. $\langle F_U \rangle \subseteq \text{Pol}(R_{\overline{U}})$ for each $U \subseteq \mathbb{N}$ where $\overline{U}$ is the complement of $U$.

**Proof.** For (1), we see that $f_k$ does not preserve $(P_k, Q_k)$ since

\[
\begin{array}{ccccccc}
1 & 0 & \cdots & 0 & \overset{f_k}{\rightarrow} & 1 \\
0 & 1 & \cdots & 0 & \overset{f_k}{\rightarrow} & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
0 & 0 & \cdots & 1 & \overset{f_k}{\rightarrow} & 1 \\
\cap & \cap & \cdots & \cap & \overset{\cap}{\cap} & \cap \\
P_k & P_k & \cdots & P_k & Q_k
\end{array}
\]

Next assume $n \neq k$ and $x_1, \ldots, x_k \in P_n$. Let $M$ be the $n \times k$ matrix where the $j$th column is $x_j$ for $1 \leq j \leq k$. If $n < k$, then at least one row of $M$ must have at least two entries equal to 1. Thus at least one entry of $f_k(M)$ is 0. Hence $f_k(M)$ is in $Q_n$. If $n > k$, then at least one row of $M$ is all zeros. So at least one entry of $f_k(M)$ is 0 and $f_k(M)$ is in $Q_n$. This concludes the proof of (1).

Item (2) is immediate from (1). \qed

We begin proving the 3 cases with the case where $B = (\{0,1\}, \neg, 0)$.

**Theorem 4.2.** The number of clonoids with finite source $A$ and target algebra $B = (\{0,1\}, \neg, 0)$ is continuum.

**Proof.** The statement is immediate from the following claim:

\[(4.1) \quad \langle F_U \rangle_B \cap F_n = F_U \text{ for each } U \subseteq \mathbb{N} \setminus \{1\}.
\]

The inclusion $\supseteq$ is clear. To prove the converse, let $U \subseteq \mathbb{N} \setminus \{1\}$ and $\ell \in \mathbb{N}$ such that $f_\ell \in \langle F_U \rangle_B$. Then $f_\ell = f_k^\sigma$ or $f_\ell = \neg(f_k^\sigma)$ for some $k \in U$ and map $\sigma : [k] \rightarrow [\ell]$. In the former case, Lemma 4.1 yields $\ell = k$ and further $\ell \in U$. To see that the latter cannot occur, let
m ∈ N, m ≠ ℓ, and let a = (1, 0, . . . , 0) ∈ P_m. We have

\[ f_\ell(a, \ldots, a) = \neg(f_k^\ell(a, \ldots, a)) \]

\[ = \neg f_k(a, \ldots, a) \]

\[ = (1, \ldots, 1) \text{ since } k > 1 \]

\[ \not\in Q_m. \]

Thus \( f_\ell \) does not preserve \((P_m, Q_m)\). This contradicts Lemma 4.1 and completes the proof of (4.1). □

Now we prove the case where \( B = (\{0, 1\}, \not\rightarrow) \).

**Theorem 4.3.** The number of clonoids with finite source \( A \) and target algebra \( B = (\{0, 1\}, \not\rightarrow) \) is continuum.

**Proof.** First we show

\[ (4.2) \quad \langle F_U \rangle_B \subseteq \text{Pol}(R_U) \text{ for each } U \subseteq N. \]

By Lemma 4.1, \( \langle F_U \rangle \subseteq \text{Pol}(R_U) \). Assume \( g, h \in \langle F_U \rangle_B \) of arity \( k \) preserve \((P_n, Q_n)\) and let \( a_1, \ldots, a_k \in P_n \). Let \( d := g \not\rightarrow h \). Then we have

\[ d(a_1, \ldots, a_k) = g(a_1, \ldots, a_k) ∧ (\neg h(a_1, \ldots, a_k)). \]

Since \( g \) preserves \((P_n, Q_n)\), there must be at least one zero entry in \( g(a_1, \ldots, a_k) \). Thus \( d(a_1, \ldots, a_k) \in Q_n \). Hence (4.2) is proved.

Let \( U, V \subseteq N \) such that \( U \neq V \). We claim that

\[ (4.3) \quad \langle F_U \rangle_B \neq \langle F_V \rangle_B. \]

Without loss of generality, assume there exists \( n \in U \setminus V \). From (4.2), we have that \( \langle F_V \rangle_B \) preserves \((P_n, Q_n)\). Since \( n \in U \), we have \( f_n \in F_U \) and thus \( \langle F_U \rangle_B \) does not preserve \((P_n, Q_n)\) by Lemma 4.1. Therefore \( \langle F_U \rangle_B \neq \langle F_V \rangle_B \). □

The final case, where \( B = (\{0, 1\}, \land, 0, 1) \), is given in the following theorem.

**Theorem 4.4.** The number of clonoids with finite source \( A \) and target algebra \( B = (\{0, 1\}, \land, 0, 1) \) is continuum.

**Proof.** Let \( B = (\{0, 1\}, \land, 0, 1) \) and \( B' = (\{0, 1\}, \land, 0) \). For any subset \( U \subseteq N \), we have \( \langle F_U \rangle_B \setminus \{1\} = \langle F_U \rangle_{B'}. \) Therefore it is enough to show there are continuum many Boolean clonoids of the form \( \langle F_U \rangle_{B'}. \) As in the proof of Theorem 4.3 this follows from showing that \( \langle F_U \rangle_{B'} \subseteq \text{Pol}(R_U) \). □
5. Proof of Main Theorem

In this section, we combine the results from the previous sections to give a proof of Theorem 1.3.

Proof of Theorem 1.3. The reverse direction of (1) follows immediately from Theorem 2.1.

To prove the reverse direction of (2), let $A$ be a finite set and $B$ an algebra with a Mal’cev term but no majority term. Then by Theorem 3.1, $C_{A,B}$ is at most countably infinite. Since $B$ has no majority term, by Post’s classification, the clone of $B$ is contained in the clone of $B' := (\{0,1\}, +, 1)$. In Example 3.2, we show that there are infinitely many clonoids in $C_{A,B'}$. Since $C_{A,B'} \subseteq C_{A,B}$, there are countably many clonoids in $C_{A,B}$.

Now let $A$ be a finite set and $B$ an algebra without an NU-term and without a Mal’cev term. As mentioned in the beginning of Section 4, it follows from Theorems 4.2, 4.3, and 4.4 and their duals that there are continuum many clonoids with target $B$. This proves the reverse direction of (3).

Now the forward directions of (1), (2), and (3) follow because the cases are mutually exclusive.

\[ \square \]

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Department of Mathematics, University of Colorado Boulder, USA

E-mail address: athena.sparks@colorado.edu