Brillouin lasing in whispering gallery micro-resonators

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Abstract

Thresholds of stimulated Brillouin scattering (SBS) in solid-state whispering gallery mode (WGM) microresonators are analyzed. It is shown that the SBS interaction is substantially different here from that known in the bulk case and in the case of water droplet resonators. The reason is the absence of pure longitudinal acoustic WGMs owing to strong coupling of the longitudinal (l) and transverse (t) acoustic displacements at the surface of the resonator. As a result, a considerable increase of the SBS thresholds takes place, and the lowest thresholds correspond to the hybrid l+t-modes with very large radial indices. Nevertheless, the thresholds lie in the μW range of the pump power. Dependence of the SBS power thresholds on the modal numbers and the possibility of self-tuning to the SBS resonance are analyzed.

1. Introduction

Stimulated Brillouin scattering (SBS) is one of the strongest nonlinear processes in optics [1, 2]. It is allowed in any solid-state material regardless of the properties of spatial symmetry. In particular, it is well investigated in transparent isotropic materials, such as glasses, as applied both to bulk samples and fibers. Here the lowest optical power thresholds occur for the excitation of counter-propagating light waves and longitudinal acoustic waves with wavevectors close to the doubled pump wavevector 2k_p.

During the last decade, there has been tremendous research interest in low-power nonlinear phenomena in optical whispering gallery mode (WGM) resonators [3–9]. In particular, the second and third harmonic generation, optical parametric oscillation, frequency comb generation, and temporal solitons were discovered with continuous-wave low-power light sources. The amazing strength of all these nonlinear WGM phenomena stems from a well-defined modal structure; extremely large quality factors, Q = 10^7–10^13; and very small modal volumes, V ∼ 10^3 μm^3, causing huge intensity enhancements inside the resonators [10–13]. The above-mentioned factors are relevant to all sizes of actual WGM resonators, from ~10^-3 to ~10^-7 cm.

Interest in interactions of the optical WGMs with low-frequency acoustic modes has also shown a strong upsurge. While Brillouin scattering (stimulated and spontaneous) in liquid droplets has been known for a long time [14, 15], controllable access to different opto-acoustic processes in micro resonators was achieved only recently. In particular, parametric excitation of high-Q radial mechanical vibrations by optical WGMs was discovered and used for laser cooling [16–18]. Lasing in the μW-to-mW pump range owing to backward SBS was detected [19, 20] and then further advanced (see [21–24] and references therein). Forward Brillouin scattering on the surface (Rayleigh) acoustic modes strongly localized near the rim of the resonator was also reported [25, 26] and used for cooling [27].

Our concern in this paper is lasing in optical WGM resonators owing to backward SBS on acoustic WGMs. This low-order nonlinear process involves three modes only. One can expect that it has the strongest overlap for the radial functions, the largest coupling coefficients and, consequently, the lowest thresholds. At first sight, the first detailed experiments [20] confirm these expectations: the excited optical mode was counter-propagating, and the difference in the optical frequencies was found to be close to the frequency of the longitudinal acoustic wave in the bulk case. However, the detected optical threshold power was much higher than expected from the model employed. Furthermore, it was found [28] that there are no longitudinal (or near-longitudinal) acoustic WGMs in solid-state resonators. This is due to a strong coupling of the longitudinal (l) and transverse (t) sound
waves at the surface of the resonator. As a result, the acoustic WGMs are of either the \( t \)- or the \( tl \)-type (with a strong \( t \)-component). The latter have no analogs among optical WGMs. Apparently, further studies are necessary to understand the nature of the SBS processes in optical WGM resonators.

Below we consider this problem theoretically for a spherical resonator of radius \( r_0 \) made of an isotropic material like glass. Three issues are crucial:

--- Phase matching: The SBS phase-matching conditions for the modal angular frequencies and azimuth numbers are \( \omega_p - \omega_s = \Omega \) and \( m_p - m_s = m \), where the subscripts ‘\( p \)’ and ‘\( s \)’ refer to the pump and Stokes optical modes. Owing to the discreteness of the frequency spectra of WGMs and very narrow optical line widths, the phase-matching conditions are not generally fulfilled. This means that additional tuning methods, such as temperature tuning, can be necessary to realize SBS processes.

--- Radial overlap: Since all WGM eigenfunctions are strongly localized near the rim of the resonator and the localization properties of the acoustical modes are highly specific [28], optimization of the radial overlap in terms of the radial mode numbers is crucial.

--- Vectorial structure: Even with a good radial overlap, coupling of optical and acoustic modes can be weak because of vectorial effects and the absence of pure longitudinal acoustic modes. This puts serious restrictions on the types of interacting modes.

While the WGM and SBS properties are generally complicated, there are important simplifying circumstances.

- All WGMs—optical and acoustic—can be viewed as quasi-1D waves propagating along the rim of the resonator. Here, the wavevectors acquire discrete values \( m/r_0 \). Thus, we have a rough estimate for the pump frequency \( \omega_p \approx m_p c/\pi r_0 \), where \( c \) is the speed of light and \( n \) is the modal refractive index, which is close to the bulk index. Similarly, for the acoustic frequencies we have \( \Omega \approx mv/r_0 \), where \( v \) is the relevant sound velocity. Expressing \( \omega_p \) by the vacuum pump wavelength \( \lambda_p \) as \( \omega_p = 2\pi c/\lambda_p \), for the azimuth pump number we obtain \( m_p \approx 2\pi m_0/\lambda_p \). For representative values \( \lambda_p = 1 \mu m, r_0 = 1 \ mm, \) and \( n = 1.5 \) we have an estimate \( m_p \approx 10^4 \). This is a major parameter of the WGM theory. By analogy with the bulk case, where the strongest SBS occurs for counter-propagating light waves, we have an estimate \( m \approx 2m_p \) for the acoustic azimuth number. Lastly, the quasi-1D character of WGMs strongly simplifies their polarization properties.

- The quality factors of acoustic WGMs are not very high in the GHz range owing to the so-called phonon viscosity [1]. One can expect that \( Q \lesssim 10^4 \) for these modes in accordance with the data of [20]. Correspondingly, the distances between neighboring discrete values of \( \Omega \) can be within the acoustic line width, i.e., the acoustic frequency spectrum can be considered as quasi-continuous.

Finally, we mention earlier theoretical studies of SBS in micro-droplets; see, e.g., [29–31] and references therein. They consider free-beam pumping, ignore the difference between \( l \)- and \( t \)-modes, and are not relevant to the actual experiments.

2. Coupling between optical and acoustical modes

Any acoustic and light modes can be characterized by spatio-temporal distributions of the displacement vector \( \mathbf{u} (\mathbf{r}, t) \) and of light electric field \( \mathbf{E} (\mathbf{r}, t) \). In transparent media, coupling of these modes is due to the elasto-optic effect [32, 33]: On one hand, light induces a driving force of density \( \mathbf{f} \) affecting \( \mathbf{u} \) and causing an elastic strain. On the other hand, the elastic strain changes the optical permittivity tensor \( \delta \varepsilon_{ij} \). Both \( \mathbf{f} \) and \( \delta \varepsilon_{ij} \) can be expressed by the same dimensionless fourth-rank elasto-optic tensor \( p_{ijmn} \). In isotropic media, this tensor has only two independent components, denoted conventionally as \( p_{12} \) and \( p_{44} \); typically \( p_{12} \gg p_{44} \). The basic relations for \( \delta \varepsilon_{ij} \) and \( \mathbf{f} \), here read:

\[
\delta \varepsilon_{ij} = - n^4 \left( p_{12} \delta_{ij} \varepsilon_{mm} + 2 p_{44} \varepsilon_{ij} \right),
\]

\[
f_j = \left( n^4 / 8\pi \right) \left( p_{12} \nabla_i E_j + 2 p_{44} E_i \nabla_j E_i \right), \tag{1}
\]

where \( \varepsilon_{ij} = (\nabla_i \mathbf{u}_j + \nabla_j \mathbf{u}_i)/2 \) is the strain tensor, \( \delta_{ij} \) is the Kronecker delta, the subscript \( \Omega \) indicates taking the relevant acoustic frequency component (see also below), and summation over the repeating indices is adopted. The trace of the strain tensor \( \varepsilon_{mm} = \nabla \cdot \mathbf{u} \), which gives the local relative volume change, is zero for the \( t \)-part of \( \mathbf{u}(\mathbf{r}) \). This means that the largest component \( p_{12} \) is responsible for coupling to the acoustic \( l \)-modes. According to equation (1), the function \( \mathbf{f}(\mathbf{r}) \) has only a non-zero \( l \)-part in the situation where two plane light waves possess parallel (or anti-parallel) wave vectors. Only the acoustic \( l \)-waves can be excited in this case during an SBS process. This known fact will have far-reaching consequences as applied to our quasi-1D WGM case.
In an isotropic medium, the elasticity theory equations [34] and Maxwell’s equations give straightforwardly
\[
\frac{\partial^2 u_i}{\partial t^2} - v_i^2 \nabla^2 u_i - \left( v_i^2 - v_j^2 \right) \nabla \nabla_j u_j = \frac{f_i}{\rho},
\]
\[
n^2 \frac{\partial^2 E_i}{\partial t^2} - \nabla^2 E_i = \frac{1}{n^2} \nabla \nabla_m \delta_{mj} E_j - \delta_{ij} \frac{\partial^2 E_i}{\partial t^2}.
\]
(2)

Here \( n \) is the bulk refractive index, \( c \) is the speed of light, \( \rho \) is the mass density, and \( v_i \) are the longitudinal (l) and transverse (t) sound velocities. The ratio \( s = v_t / v_l \), which is an important parameter of our theory, obeys the inequality \( s < 1 / \sqrt{2} \approx 0.71 \). The right-hand terms in equations (2) describe mutual coupling of the light and acoustic modes. Generally, this equation set has to be supplemented by proper boundary conditions for \( u \) and \( E \). These conditions are important for determination of the optical and acoustical eigenmodes within the linear approximation in the variables. For the acoustical modes this means that the boundary of the resonator can be treated as unperturbed, as was done in [28]. Knowledge of the eigenmodes is sufficient for further SBS studies.

Equation set (2) can be applied to the bulk case where the acoustic modes are plane waves. In particular, it enables one to calculate an important experimental characteristic—the 1D center-line gain factor \( g \) for the backward SBS [1]—as applied to solids. Employing the general calculation procedure of [1], we obtain
\[
g = \frac{2\pi^2 \rho_0^2 Q \omega_p^3}{\rho c^4 \Omega^2},
\]
(3)
where \( Q \) is the acoustic quality factor, such that \( Q/\Omega \) is the lifetime of the acoustic \( l \)-mode. This relation will be used in what follows.

Now we apply set (2) to the WGM case. Let the SBS phase-matching conditions for the WGM frequencies and azimuth numbers be fulfilled. Then we can introduce scalar slowly varying amplitudes of the modes using the relations
\[
E_i = a_p \hat{E}_{p,i} e^{-i\omega_{p,i}t} + a_s \hat{E}_{s,i} e^{-i\omega_{s,i}t} + c.c.
\]
\[
u_i = b \hat{u}_i e^{-i\omega_l t} + c.c., \quad \delta \varepsilon_{ij} = b \delta \varepsilon_{ij} e^{-i\omega_l t} + c.c.
\]
(4)

Here \( \hat{E}_{p,s}(r) \) and \( \hat{u}(r) \) are vectorial eigenfunctions expressing the spatial structure of the WGMs involved, \( \delta \varepsilon_{ij} \) is expressible by \( \hat{u} \) from equations (1), while \( a_{p,s}(t) \) and \( b(t) \) are slowly varying scalar modal amplitudes.

Normalization of the eigenfunctions is a matter of convenience. Combining equations (2) and (4) and introducing the modal acoustic and Stokes decay constants \( 1/2Q \) and \( \omega_d/2Q_{\ast} \), respectively, we obtain the set of coupled equations for \( b \) and \( a_{\ast}^* \):
\[
\dot{b} + \left( \Omega/2Q \right) b = i \kappa a_{\ast} a_{\ast}^* \\
\dot{a}_{\ast}^* + \left( \omega_d/2Q_{\ast} \right) a_{\ast}^* = - i \kappa^* a_{\ast}^* b,
\]
(5)
where the dot indicates differentiation in time, and the asterisk stands for complex conjugation. The scalar coupling coefficients \( \kappa \) and \( \kappa_{\ast} \) (generally complex) are given by
\[
\kappa = \frac{n^4}{8 \pi \rho \Omega \left| \hat{u} \right|^2} \left[ p_{12} \left( \hat{u}_{\ast} \nabla \hat{E}_{p,ij} \hat{E}_{s,ij} \right) + p_{44} \left( \hat{u}_{\ast} \left( \hat{E}_{p,ij} \nabla \hat{E}_{s,ij} + \hat{E}_{s,ij} \nabla \hat{E}_{p,ij} \right) \right) \right]
\]
\[
\kappa_{\ast} = \frac{\omega_d}{2 n^2 \left| \hat{E}_{s,ij} \right|^2} \left[ \left( \hat{E}_{s,ij} \hat{E}_{s,ij}^* \delta \varepsilon_{ij} \right) + \frac{c^2}{n^2 \omega_p} \left( \hat{E}_{s,ij} \nabla \delta \varepsilon_{mn} \hat{E}_{s,mn}^* \right) \right],
\]
(6)
(7)
where \( \langle \ldots \rangle \) indicates integration over the resonator volume. While the general expressions for \( \kappa \) and \( \kappa_{\ast} \) are cumbersome, they can be strongly simplified in important particular cases. It is not difficult to write an equation for \( a_{\ast}^* \); however, it is not necessary within the undepleted pump approximation, \( a_p = \text{const.} \).

Setting \( b, a_{\ast} \propto \exp(\gamma t) \) in equations (5), it is not difficult to find the threshold value \( |a_p|_{th}^2 \) for the SBS lasing in of the terms arbitrary complex coupling coefficients \( \kappa \) and \( \kappa_{\ast} \). Usually, the product \( \kappa \kappa_{\ast} \) is real or almost real. In this most important case we have immediately from equations (5):
\[
\left| a_{\ast} \right|_{th}^2 \approx \frac{\Omega \omega_s}{4 Q \kappa \kappa_{\ast}}.
\]
(8)

Further calculations require additional information about the WGM frequencies and eigenfunctions and also additional approximations.
3. WGM properties

We start from optical modes. They are transverse (solenoidal) so that the potential part of $\vec{E}(r)$ is zero. In the plane-wave case, this would mean orthogonality of $\vec{E}$ and $k$. There are two types of transverse WGMs, 1 and 2 (called also $H$- and $E$-modes). The corresponding expressions for $\vec{E}$ are $\vec{E}_1 = \nabla \times \left( r \vec{\Psi}_1 \right)$ and $\vec{E}_2 = \nabla \times \nabla \times \left( r \vec{\Psi}_2 \right)$ [12, 35]. In the spherical $r, \theta, \phi$ coordinate system, the potentials $\vec{\Psi}_{1,2}(r)$ are given by the expression $\vec{\Psi}_{1,2} = Y_{m\ell}(\theta, \varphi) f^\ell(kr)$. Here $\ell = 0, 1, …$ is the orbital number, $m$ is the azimuth number taking values from $-\ell$ to $\ell$, $Y_{m\ell}(\theta, \varphi)$ is the spherical function, $f^\ell(x) = \sqrt{\pi/2x} I_{\ell+1/2}(x)$ is expressed by the first-kind Bessel function of semi-integral index, and $k = m\omega/c$. The frequency $\omega$ is degenerate in $m$; it depends on $\ell$ and the radial modal number $q = 1, 2, …$. The spherical function can be represented as $Y_{m\ell}(\theta, \varphi) = \Theta_{m\ell}(\theta, \varphi) \exp \left( \text{i} m \varphi \right)$. It is normalized such that $\int \Theta_{m\ell}^* \sin \theta \text{d} \theta \text{d} \varphi \equiv 1$.

The WGM case corresponds to very high orbital numbers, $\ell = 10^3 \cdots 10^5$, and to the several lowest values of $q$ and $N \equiv \ell - |m|$. Here one can use the Airy asymptotic for $I_{\ell+1/2}(x)$ and the Hermite-Gaussian asymptotic for $\Theta_{m\ell}(\theta, \varphi)$, see equations (A.1) and (A.2). With good accuracy, the frequency is given by $\omega = (cE/\ell m)(1 + \xi_q/(2E^2)^{1/3})$, where $\xi_q$ is the $q$th zero of the Airy function $Ai(-\xi): \xi_1 \approx 2.34, \xi_2 \approx 4.09, \xi_3 \approx 5.55$, etc. Thus, the actual WGM frequencies lie slightly above $E/c \ell m$. The function $f^\ell(kr)$ is localized near $r$ on the scale of $\delta \approx r_0/\ell^{2/3}$, and the function $\Theta_{m\ell}(\theta, \varphi)$ is localized in $\theta$ near $\pi/2$ (near the equator) on the scale $\delta \theta \approx 1/\ell^{1/2}$.

For $\ell \gg 1$, the components of $\vec{E}_{1,2}$ in the spherical coordinate system are given by the expressions

$$
\vec{E}_1 \approx \text{im} \vec{\Psi}_1 \vec{e}_\theta \cdot \nabla \vec{\Psi}_1 = \frac{\partial \vec{\Psi}_1}{\partial \theta} \vec{e}_\varphi,
$$

$$
\vec{E}_2 \approx \frac{\ell^2 \vec{\Psi}_2}{\ell \theta} \vec{e}_\theta + \frac{\partial^2 \vec{\Psi}_2}{\partial \theta^2} \vec{e}_\theta + \text{im} \frac{\partial \vec{\Psi}_2}{\partial \theta} \vec{e}_\varphi,
$$

where $\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi$ are the unit basis vectors. Owing to the properties of localization in $r$ and $\theta$ and the proximity of $|m|$ to $\ell$, the dominating components of $\vec{E}_1$ and $\vec{E}_2$ are $\vec{E}_1 \propto \vec{\Psi}_1$ and $\vec{E}_2 \propto \vec{\Psi}_2$, respectively. This expresses the quasi-1D nature of the optical WGMs and simplifies the subsequent considerations.

Now we turn to the acoustic WGMs [28]. Here, there are not only similarities to but substantial distinctions from the optical modes. In addition to the transverse (solenoidal) part $\vec{u}(r)$ such that $\nabla \cdot \vec{u} = 0$, the displacement vector $\vec{u}$ generally has a longitudinal (potential) part $\vec{u}_p(r)$ with $\nabla \cdot \vec{u}_p = 0$ and $\nabla \cdot \vec{u}_r = 0$. It is worth mentioning that the $t$- and $l$-modes are controlled by the transverse and longitudinal sound velocities $v_t$ and $v_r$, respectively.

Similar to optical modes 1, there are acoustic $t$-modes of type 1 with the eigenfunctions $\vec{u}_t = \nabla \times \left( r \vec{\Phi}_t \right)$, $\vec{\Phi}_1 = Y_{m\ell}(\theta, \varphi)f^\ell(kr)$, $k_t = \Omega/v_t$, and the frequencies $\Omega(t, q) = (v_t r_0/\ell m)(1 + \xi_q/(2E^2)^{1/3})$. Here $\xi_q$ is the $q$th zero of the derivative $Ai'(-\xi)$: $\xi_2 \approx 1.02, \xi_3 \approx 2.35, \xi_4 \approx 4.82$, etc. The presence of $\xi_q$ instead of $\xi_q$ stems from the difference in the elastic and optical boundary conditions. The values of $\Omega$ lie slightly above $\ell v_t r_0$, and the localization and vectorial properties of the $t$-modes are similar to those of the optical modes. The components of $\vec{u}_t$ can be obtained from equation (9) for $\vec{E}_1$ if we replace $\vec{\Psi}_1$ by $\vec{\Phi}_t$.

Figure 1(a) shows the first two radial optical and acoustic functions, $f^\ell_1(kr)$ and $f^\ell_2(kr)$, for the pump orbital number $\ell_p = 10^4$ and the acoustic orbital number $\ell = 2\ell_p$. The vertical scale is of no importance. The difference at $r = r_0$ comes from the difference between the optical and acoustic boundary conditions. A notably stronger localization of the acoustic $t$-modes is due to the difference in the orbital numbers. The radial overlap between the optical and acoustic modes is, nevertheless, fairly good.

Acoustic WGMs of type 2—the hybrid $tl$-modes—differ strongly from the optical WGMs [28]. For each $tl$-mode, the eigenfunction $\vec{u}$ is the sum of the $t$- and $l$-parts: $\vec{u} \propto C_0 \nabla \Psi_0 + \nabla \times \left( r \vec{\Phi}_2 \right)$, where $C_0 = C_0(t, q)$ is a real modal constant, $\vec{\Psi}_0 = Y_{m\ell}(\theta, \varphi)f^\ell(k_0 r)$, $\vec{\Phi}_2 = Y_{m\ell}(\theta, \varphi)f^\ell(k_0 r), |m| \approx \ell$, and $k_{0l} = \Omega/v_{1l}$. The coupling between the $t$- and $l$-parts of $\vec{u}$, quantified by $C_0$, is due to the free-surface boundary condition of the elasticity theory [34]. While the angular properties of $\vec{\Psi}_0$ and $\vec{\Phi}_2$ are the same, the radial properties are strongly different because of the difference in the arguments of the Bessel functions $J_{\ell+1/2}(kr)$ and $J_{\ell+1/2}(kr)$ for $\ell \gg 1$, see appendix A and the subsequent considerations for details. As concerns the vectorial properties, the largest components of the $t$- and $l$-parts of $\vec{u}$ are $\vec{u}_t \propto \ell^2 \vec{\Phi}_2/r$ and $\vec{u}_l \propto \text{im} C_0 \vec{\Psi}_0/r$.

The frequencies $\Omega_{t,q}$ of the $tl$-modes and the corresponding values of $C_0(t, q)$ can be calculated from the relations given in appendix B. These relations include the Bessel functions of two arguments, $k_0 r_0$ and $k_0 r_0 \equiv s k_0 r_0$. Three asymptotic relations for $f^\ell(kr)$, given in appendix A, are relevant to the $tl$-modes.

It is important to compare the functions $f^\ell_1(kr)$ and $(C_0/\ell)f^\ell_2(kr)$ representing the radial dependence of the largest vectorial components of $\vec{u}_t$ and $\vec{u}_l$. This comparison has to be made in close conjunction with the dependence of $\Omega_{t,q}$ on the radial number $q$.
The lowest in frequency $\Omega_{eq}$ (we attribute it to $q = 0$) lies well below $\sqrt[6]{\nu r_0}$. It corresponds to the Rayleigh wave [34], which is localized near $r_0$ on the smallest possible scale of $r_0/\ell$ and only weakly disturbed by the curvature of the surface. The radial overlap of the corresponding acoustic eigenfunction with the optical eigenfunctions is rather weak, which excludes the lowest $tl$-mode from the candidates for the lowest SBS excitation threshold.

- For the next few modes ($q = 1, 2, \ldots$) the values of $\Omega_{eq}$ lie slightly above $\sqrt[6]{\nu r_0}$, and the function $f_k(kr)$ is localized near $r_0$ on the scale of $r_0/\ell^{2/3}$ typical for the optical modes. The function $(C_q/\ell)f_k(kr)$ is still localized on the scale of $r_0/\ell$, resulting in a bad overlap with the optical radial functions. This is illustrated by figures 1(b) and 1(c) for $s = \nu r_0/\ell^{2/3} = 2/3, \ell = 2 \times 10^4$, and $q = 1.2$. The localization scale for the $t$-part is $\approx 20$ times larger than it is for the $l$-part. While the absolute values of the functions at $r = r_0$ are comparable with each other, the maximum values are much larger for $f_k(kr)$. Note also that both the boundary value and the localization scale are slightly larger for $q = 2$ than for $q = 1$ in figure 1(c).

- When $q$ increases, the $l$-part becomes gradually stronger and less localized in $r$, while the $t$-part acquires an oscillating tail stretching up to $r_0 - r \approx q r_0/\ell^{2/3}$. For very large values of $q$, the frequency $\Omega_{eq}$ overpasses $\nu r_0/\ell^{2/3}$. Here the $l$-part is localized in $r$ on the scale of $r_0/\ell^{2/3}$, providing a good overlap with the optical functions, and one can use the Airy approximation of equation (A.1) for $f_k(kr)$. The oscillating tail of the $t$-part stretches up to $r \approx r_0/\ell^{2/3}$, and its amplitude is substantially smaller compared to the value of the $l$-part at $r = r_0$. The ratio of these quantities can be estimated as $\approx \ell^{1/6}$. This is illustrated by figure 1(d) for the $tl$-mode with $k_{r0} \approx 30043.8$. This value is calculated with equation (B.1); it corresponds to $q \approx 10^3$ and $\Omega_{eq} \approx 1.00146 \times \sqrt[6]{\nu r_0}$. The boundary value $(C_q/\ell)f_k(kr)_{r=r_0}$ exceeds the amplitude of the $t$-tail given by $f_k(kr)$ by a factor of $\approx 6$. The tail ends abruptly at $r \approx 2r_0/3$. For the next value of $q$ we have $k_{r0} \approx 30047.7$ and $\Omega_{eq} \approx 1.00159 \times \sqrt[6]{\nu r_0}$; the corresponding radial functions remain almost the same. While the relative tail is relatively weak in the amplitude, it carries the largest part of the modal power. The $tl$-modes in this range can be qualified as pseudo-longitudinal WGMs.

On the basis of this analysis, we can expect that the excitation of pseudo-longitudinal acoustic WGMs with frequencies $\Omega \approx 2\ell^{1/6}r_0/\ell^{2/3}$ is the main possibility for the SBS lasing. Here the negative influence of the $t$-tail on the
oscillation threshold has to be analyzed carefully. Other possibilities are related to the excitation of the transverse acoustic WGMs with the frequencies \( \Omega \approx 2 \ell v / r_0 \).

4. SBS thresholds

4.1. Excitation of pseudo-longitudinal acoustic modes

The lowest SBS thresholds correspond to the same type of optical modes—the polarization transformation is forbidden. The thresholds are almost the same for the pump modes of types 1 and 2. For definiteness, let the optical modes be of type 1.

With a high accuracy, we have for the eigenfunctions \( \hat{E}_p = i m_p \Psi_{1p} e_\theta, \hat{E}_\theta = i m_s \Psi_{1s} e_\phi \), and
\[ \hat{u} = C_0 (i m / n_0) \Phi_\theta e_\phi + \epsilon^2 (\Phi_\theta / r) e_\phi, \]
where \( m_s \approx -m_p, m \approx 2 m_\theta \), and \( \epsilon^2 \approx 4 m_\theta^2 \). Correspondingly, for the coupling coefficients \( \kappa \) and \( \kappa_\theta \) we obtain from equations (6) and (7):

\[
\kappa \approx - \frac{C_0 n^2 m_\theta^2 \epsilon^2}{8 \pi \rho \Omega} \left( \left| \Psi_{1p} \right|^2 \left\{ \Phi_{\theta}^2 / r^2 \right\} \right) \]

\[
\kappa_\theta \approx - \frac{\omega \epsilon^2 C_0 n^2 m_\theta^2 \left\{ \Psi_{1p}^* \Psi_{1s} \right\}}{2 \epsilon^2 \left\{ \left| \Psi_{1s} \right|^2 \right\}} .
\]

Only the first terms in equations (6) and (7) have contributed here. The potentials in the above expressions are complex only because of the azimuth exponential factors. Since \( m_p - m_s = m \), these factors can omitted. Then we have for the threshold value of \( |a_p|^2 \) from equation (8):

\[
|a_p|^2 = \frac{4 \pi \rho \Omega^2 r_0^2 \left\{ \Psi_{1s}^2 \right\}}{Q Q_{\theta} n^2 \epsilon^2 \left\{ \Psi_{1p}^* \Psi_{1s} \right\}} .
\]

The second term in the square bracket accounts for the negative influence of the \( t \)-tail of \( \hat{u}(r) \).

Next we express the squared pump amplitude inside the resonator, \( |a_p|^2 \), by the external pump power \( W \). In the first step we calculate the largest \( \varphi \)-component of the Pointing vector inside the cavity. In our notation, it is given by \( m_p^2 \left| \Psi_{1p} \right|^2 \) \( \left| a_p \right|^2 / n c / 2 \pi \), and the azimuth factor in \( \Psi_{1p} \) can be omitted. To get the internal circulating pump power \( W_{th} \), we integrate this quantity over a resonator cross-section \( \varphi = \text{const} \), linking \( W_{th} \approx m_p^2 \left| \Psi_{1p} \right|^2 \left| a_p \right|^2 / n c / 4 \pi r_0 \). In turn, the internal pump power in a WGM resonator is linked to the external power \( W \) by \( W_{th} \approx (\lambda, Q_{\theta} / 2 \pi r_0 r_0) W \). It is assumed that \( Q_{\theta} \), as well as \( Q_\theta \), is the loaded quality factor, which accounts not only for the internal losses, but also for the coupling losses [12]. It is implied also that the coupling to the WGM is optimum, i.e., critical. Finally, we get \( |a_p|^2 \approx Q \omega / \epsilon^2 n^2 m_p^2 \left| \Psi_{1p} \right|^2 \). Using this relation and equation (12), we represent the external power threshold in the following convenient form:

\[
W_{th} \approx \frac{W_{th}^0}{1 + \zeta / F} .
\]

Here \( W_{th}^0 \) is the characteristic power parameter,

\[
W_{th} = \frac{\rho c^2 \lambda_p V_p}{8 \pi n^2 \epsilon^2 Q Q_{\theta} Q_{\theta}} = \frac{2 \pi^3 n^2 V_p}{g Q \Omega_\lambda c_p^2} ,
\]

while \( F \) and \( \zeta \) are positive dimensionless quantities characterizing the modal overlap and the negative impact of the \( t \)-tail, respectively:

\[
F = \frac{V_p \left\{ \Psi_{1p}^* \Psi_{1s} \right\}}{Q \Omega_{\lambda} c_p^2 \left| \Psi_{1p} \right|^2} \left\{ \Phi_\theta^2 / r^2 \right\} , \quad \zeta = \frac{\epsilon^2 \left\{ \Psi_{1s}^* \Psi_{1s} \right\}}{C_0^2 \left| \Phi_\theta \right|^2} .
\]

In these relations \( g \) is the center-line SBS gain factor given by equation (3) and \( V_p \) is a characteristic pump-mode volume. The two right-hand-side expressions in equation (14) give two useful alternatives for evaluation of \( W_{th}^0 \), see also below. The choice of the modle volume \( V_\theta \) does not affect \( W_{th} \); it is arbitrary to a certain extent. We chose the pump mode volume as \( V_p = (2 \pi)^6 n^2 / \left| \epsilon^2 / n \right| \), which is in agreement with the above considerations of the localization scales and close to the estimate of [12] \( \left| m_p \right| = \epsilon \). One might expect that \( F \approx 1 \) in the case of a good overlap and \( F \ll 1 \) otherwise, see also below. In an imaginary case of a pure longitudinal acoustic mode we would get \( \zeta = 0 \). In reality \( \zeta = 0 \), and the question is how large is this positive quantity.

The characteristic power \( W_{th}^0 \) is a combination of well-defined material and experimental parameters. Setting \( \rho = 2.2 \text{ g cm}^{-3}, n = 1.5, \rho_{12} = 0.24 \) (typical of silica glass), \( \lambda_p = 1 \mu\text{m}, r_0 = 1 \text{ mm}, \Omega / 2 \pi = 10 \text{ GHz}, \)
Q = 10^3, and Q_p = Q_f = 10^8, we get the estimate \( W^0 \approx 10 \mu W \). This gives the scale of the expected threshold powers. Note that the chosen estimate of \( Q_p \), is conservative; experimentally available larger values of \( Q_p \), can substantially lower the threshold.

### 4.2. Analysis of \( F \) and \( \zeta \)

Expressions for \( F \) and \( \zeta \) have to be further analyzed and simplified. We start from equation (15) for \( F \). Evidently, the value of \( F \) does not depend on normalization of the potentials \( \Psi_{1p}, \Psi_{1f}, \) and \( \Phi_0 \). Recalling that the azimuth factors in the expressions for these potentials are omitted, we have \( \Psi_{1p} = \Theta_{q_0m}(\ell f_p(kr)), \nabla \Psi_{1f} = \Theta_{q_0m}(\ell f_p(kr)), \) and \( \Phi_0 = \Theta_{q_0m}(\ell f_p(kr)), \) where \( k_p = n_\omega/c, k_f = n_\omega/c, k_\Omega = v/kr, \) and each of the eigenfrequencies depends on the relevant orbital and radial numbers. With our definition of \( \Theta_{q_0m}(\ell) \), only the radial functions have to be taken into account in the denominator of the expression for \( F \). Furthermore, the quantity \((\ldots)^2\) in the numerator splits into the product of angular and radial factors. Employing the Airy approximation of equation (A.1) for the radial functions, the Hermite-Gaussian approximation (A.2) for \( \Theta_{q_0m}(\ell) \), and the equality \( \ell \approx 2x_p \), we represent \( F \) as the product \( F = F_p F_r \), where

\[
F_p = \left[ \pi^{-1/2} \int_{-\infty}^{\infty} H_N(x) H_{N_p}(x/\sqrt{2}) H_{N_q}(x/\sqrt{2}) \exp\left(-x^2\right) dx \right]^{1/2},
\]

(16)

\[
F_r = \int_{-\infty}^{\infty} \text{Ai}(2^{1/3}x - \xi_p) \text{Ai}(2^{1/3}x - \xi_s) \text{Ai}(2x - \xi) dx \left[ \int_{-\infty}^{\infty} \text{Ai}(2^{1/3}x - \xi_p) \text{Ai}(2x - \xi) dx \right]^{1/2},
\]

(17)

\( H_N(x) \) is the Hermite polynomial of degree \( N, \xi_p = \xi_q \), and \( \xi_s = \xi_q \) are again the first zeros of \( \text{Ai}(-x) \), and \( \xi = (kr_0 - \varepsilon)(2/\varepsilon)^{1/3} \) is a useful dispersion parameter. Generally speaking, \( \xi \) is discrete in \( \varepsilon \) and \( q \). However, the neighboring acoustic frequencies \( \Omega_{\zeta_\ell} \) are typically within the line width, so \( \xi \) can be treated as a continuous variable parameter, see also below.

The angular overlap \( F_r = F_r(N_p, N_s, N) \) depends on the integers \( N = \varepsilon - |m|, N_p = \varepsilon_p - |m_p|, \) and \( N_s = \zeta - |m| \). It equals 1 for \( N = N_p = N_s = 0 \). Nonzero values of the integers give considerably smaller values of \( F_r \) and, therefore, substantially higher thresholds. In particular, we have \( F_r(1, 0, 1) = F_r(0, 1, 1) = 1/2, F_r(1, 1, 0) = 1/4, F_r(2, 0, 0) = F_r(0, 2, 0) = 1/8, \) and \( F_r(2, 2, 2) \approx 2 \times 10^{-3} \). Odd values of the sum \( N + N_p + N_s \) give zero values of \( F_r \).

The function \( F_p = F_p(\zeta) \), which quantifies the radial overlap, depends on the optical radial numbers \( q_0 \) and \( q_s \). It is represented by figure 2 for three actual combinations of these numbers. For \( q_0 = q_s = 1 \) it reaches the maximum possible value 1/2 at \( \zeta = 2.8 \). The maximum is broad enough to harbor several discrete values of \( \zeta \). Increasing the optical radial numbers lowers the maximum and shifts it towards large values of \( \zeta \), i.e., to slightly larger values of \( \Omega \). Anyhow, \( \Omega \approx 2\varepsilon \nu_{\zeta}/r_0 \) within the actual range of \( \zeta \).

Now we switch to the quantity \( \zeta \) that accounts for the negative influence of the t-tail of \( \text{Ai}(r) \). The angular factors entering the expressions for \( \Psi_{0,2} \) make no contribution to \( \zeta \) according to equation (15), so only the radial functions \( f_p(kr) \) and \( f_s(kr) \) matter. These functions refer to the same radial acoustic number \( q \). First, for the tail characteristic we have \( \langle r^2 \text{Ai}^2/r^2 \rangle = (\pi r_0^2/2k_0) \int_0^{\infty} x^{-1/2} j^2_\ell(x) dx, \) where \( \nu = \nu_e + 1/2 \) and \( x_0 = k_0 r_0 \). Since \( \Omega \approx \nu \nu_e/r_0 \), we have \( x_0 \approx \nu/s > \sqrt{2} \nu_{\zeta}/r_0 \) for the upper limit with a good accuracy. The lower limit can be transferred to some point \( x_{\text{\ldots}} \approx \nu \) in view of a very sharp decrease of \( f_s(kx) \) for \( x < \nu_e \), see also appendix A. Our analysis shows that the trigonometric approximation of \( f_s(kx) \) for \( \nu < x_e \), given by equation (A.1), is valid with a high accuracy until the first maximum of the cosine function; for smaller values of \( x \) the Bessel function \( f_s(kx) \) drops sharply and does not contribute to the integral. The point \( x_{\ldots} \) can thus be placed just at this maximum. The dependence \( x_{\ldots} (\nu) \) is presented by figure 3(a). Note that \( x_{\ldots} \) is slightly above \( \nu \). Finally, we come to the estimate \( \langle r^2 \text{Ai}^2/r^2 \rangle \approx \pi r_0^2/\nu^2 \), where

\[
I = \int_{x_0}^{\sqrt{2} \nu_{\zeta}/r_0} \nu \cos^2\left[ \sqrt{x^2 - \nu^2} - \nu \arccos(\nu/x) - \pi/4 \right] dx / \left( x^2 - \nu^2 \right)^{1/2}.
\]

(18)

When \( \nu \) ranges from \( 10^3 \) to \( 10^5 \) and \( s \) ranges from 0.5 to 0.7, the integral \( I \) varies only between 0.38 and 0.5.

The dominant contribution to \( \langle \Phi_0^2 \rangle \) in equation (15) comes from the range \( |k| r - \varepsilon | \ll \varepsilon | \). Here we can use the Airy approximation of equation (A.1). It leads to the estimate \( \langle \Phi_0^2 \rangle \approx \pi r_0^2 g(\xi)/2^{1/3} \varepsilon^{1/3} \), where

\[ g(\xi) = \int_{-\infty}^{\xi} \text{Ai}(x) dx. \]

The function \( g(\xi) \) is represented by figure 3(b). It is very small for \( \xi < 0 \) (when \( \Omega_{\zeta_\ell} < \varepsilon \nu_{\zeta}/r_0 \), it grows rapidly for \( \Omega_{\zeta_\ell} > \varepsilon \nu_{\zeta}/r_0 \). Note that the growth occurs in steps—each step corresponds to formation of a new maximum of \( f_s(kr) \); compare to figure 1(d). Combining the above expressions, we obtain
The second factor on the right-hand side is not very large for $\ell \sim 10^4 \text{ and } 2$. The question now is how large is the ratio $\ell / C$. Our analysis, based on the formulas of appendix B and the asymptotic relations of appendix A, shows that the sign of $C_0(\ell, q)$ alternates with $q$, while the absolute value $C_q(\ell)$ changes around $\ell$ for $1 < x < 2$ showing both quasi-periodic and random features. The reason is the oscillatory non-periodic behaviors of both Bessel functions $J_\nu(k_0 r_0)$ and $J_{\nu+1}(k_0 r_0)$ entering equation (B.1) in the actual spectral range, as described by equation (A.1). An interplay between these functions gives not-quite-regular frequency distances between neighboring modes and not-quite-regular behavior of $C_0(\ell, q)$.

The discrete values of the dispersion parameter $\xi = (\Omega_{eq}/\nu_{1} - \ell)/(1/\ell)^{1/3}$ and the corresponding values of the ratio $|C_0|/|\ell|$ are presented in figure 4 for $\ell = 10^4 \text{ and } s = 2/3$. The range of $\xi$ shown includes more than 50 values, and it corresponds to $10030 < \Omega_{eq}/\nu_{1} < 10180$. The distance between neighboring values of $\xi$ ranges from $\approx 0.1$ to $0.16$, and the average relative distance between neighboring values of $\Omega_{eq}$ is about $2.5 \times 10^{-4}$. This means that two neighboring eigenfrequencies $\Omega_{eq}$ and $\Omega_{eq+1}$ are within the acoustic line width. The horizontal line at 1 separates large and small values of $|C_0|/\ell$. For about 25% of values we have $|C_0|/\ell > 1$; the corresponding modes can be regarded as SBS active. At the same time, we have $|C_0|/\ell < 1/2$ for about 50% of SBS passive modes. The data shown in figure 4 were obtained with the use of asymptotic relations (A.1). Similar data for $\ell = 10^3$ were obtained numerically without approximations.

Returning to equation (19), we can now make quantitative judgements about $\xi$. Setting $|C_0|/\ell = 1$, $I = 0.4$, $g = 0.7$, and $s = 2/3$, as representative values, we obtain that $\xi \approx 5$ for $\ell = 2 \times 10^4$. For the SBS active $tl$-modes, this estimate can drop to $\xi \approx 1$. For most of the modes in the range $\xi = 2$–$10$ we have $\xi \lesssim 10$.

Figure 2. The function $F_\ell(\xi)$ for $\ell = 10^4$ and three combinations of the optical radial numbers.

Figure 3. Attributes for calculation of $\zeta$: (a) the difference $x - \nu$ versus $\nu$; (b) the function $g(\xi)$.
Lastly, taking into account our estimates of the overlap $F$, we can expect that the minimum SPS power threshold exceeds our preliminary estimate ($W_{th}^0 \approx 10 \mu W$ for $Q_{th}^0 = 10^8$) by about one order of magnitude. This increase of $W_{th}$ is mostly due to the harmful influence of the $t$-tail. All excited acoustic modes must have frequencies very close to $\Omega = 2 \varepsilon_p v_l / r_0 \approx 4 \pi n v_l / \lambda_s$; this coincides with the estimate for the bulk case. Since not all $tl$-modes are equally active for SBS, the optical mode can experience additional difficulties, discussed in the next section.

4.3. Other SBS processes

One can show that employment of optical modes of type 2 (polarized predominantly along $e_\ell$) does not change the above estimates: it is sufficient to replace $\Psi_\ell$ by $\Psi_s$ in equation (12). However, the SBS process with a change in type of the optical mode has a much higher threshold.

Consider the excitation of the acoustic $t$-mode. Calculations using equations (6) and (7) show strong cancellations of the main contributions to the coupling coefficients $\kappa_\ell$ and $\kappa_0$. Each of them is of the order $\sim \varepsilon_p^{-3/2}$, which is small compared to the case of the $tl$-mode. Furthermore, both coefficients are proportional here to the elasto-optic constant $p_{t44}$, which is typically substantially smaller than $p_{t12}$. Therefore, the threshold $W_{th}$ becomes higher here by a factor of $(p_{t12} / p_{t44})^3 \varepsilon_p^3 \gg 1$, which makes the corresponding SBS process extremely weak.

A similar situation occurs for the excitation of the $tl$-modes with low radial numbers and frequencies close to $2\varepsilon_p v_l / r_0$. The threshold is higher here by at least a factor of $\varepsilon_p^6$.

5. Phase matching issues

Fulfillment of the phase-matching conditions $\omega(\varepsilon_p, q_p) = \omega(\varepsilon_s, q_s) = \Omega(\varepsilon, q), m_p - m_s = m$ is necessary for an SBS processes with WGMs to run. The discreteness of the optical WGM spectrum creates serious obstacles to the phase matching. Fortunately, the spectrum of pseudo-longitudinal acoustic WGMs is almost continuous. This gives a tool for self-adjustment. Importantly, the dispersion parameter $\xi$ can be used as a continuous adjustment parameter. The actual variation of this dispersion parameter is approximately from 2 to 10. Outside this range, the radial overlap of the optical and acoustic modes becomes bad, leading to high SBS thresholds. If the self-adjustment with the use of $\xi$ fails, it is necessary to employ an external tuning. For example, one can vary the refractive index $n$ by changing the temperature.

Since the WGM frequencies in spherical resonators are degenerate in the azimuth numbers, and nonzero values of the polar numbers increase the thresholds, we restrict ourselves to $m_p = \varepsilon_p, m_s = -\varepsilon_s$ and $m = \varepsilon, \ell = \ell_p + \ell_s$. Substituting it into the frequency condition, we arrive at a single equation for the optical orbital numbers $\varepsilon_p, \ell_p$ using the dispersion relations $\omega_{p,\ell} = (\varepsilon_p \ell_p / r_0) \sqrt{1 + \xi_p,\ell_p / (2\varepsilon_p^3 \ell_p^3)^{1/2}}$ and $\Omega = (\nu \ell / m) \sqrt{1 + \xi / (2\varepsilon_p^3 \ell_p^3)^{1/2}}$, we can analyze the phase matching with a sufficiently high precision. Only the lowest few optical radial numbers $q_p$ entering $\xi_p,\ell_p = (\ell_p, q_p)$ are important. The radius $r_0$ cancels out from the phase-matching equation, and cannot influence the values of $\varepsilon_p, \ell_p$.

Choosing certain discrete values of $\Delta = \varepsilon_p - \varepsilon_s$ and $q_p$, we can consider $\varepsilon_p$ formally as a function of $\xi$ and $n$. The question is whether discrete values of $\varepsilon_p$ can be achieved within the acceptable variation range of $\xi$ or whether an external tuning is necessary.

Figure 4. The ratio $|Q_\ell| / \varepsilon$ versus the dispersion parameter $\xi$ for $\varepsilon = 10^4$ and $s = 2/3$. The horizontal solid line separates the regions of large and small ratios.
We consider first the case of equal optical radial numbers, $q_p = q_s$, which corresponds to the lowest thresholds, in accordance with figure 2. A strong cancellation of the optical dispersion terms in the frequency condition occurs in this case. The phase matching is illustrated by figure 5(a) for $\Delta = 1$ and $v_l = 6 \times 10^5$ cm s$^{-1}$. The solid and dotted lines are plotted for the refractive indices $n$ and $n + \delta n$ with $n = 1.5$ and $\delta n = 10^{-4}$. The value $\delta n$ corresponds to the temperature change $\delta T \approx 5^\circ C$ for silica glass. One sees that many discrete values of $\ell_p$ lie within the acceptable range of $\xi$ for each curve. An additional tuning is not really necessary in this case, as it is easily available—temperature variation $\delta T \approx 5^\circ C$ is sufficient to obtain a discrete value of $\ell_p$ for any fixed $\xi$. The values of $\ell_p$ fit a rough estimate $\ell_p \approx c\Delta/2nv_l$ that follows from the frequency condition with neglected dispersion terms.

Now we consider the case $q_p = 1$, $q_s = 2$, when the frequency $\omega_p$ substantially exceeds $\omega_s$ for $\Delta \approx 1$ because of the dispersion terms. Here the value of $\Delta$ must be much larger than 1 to achieve the phase matching. This is illustrated by figure 5(b) for $\Delta = 36$ and two close values of the refractive index. The slope of the function $\xi(\ell_p)$ is approximately ten times larger than in case (a). Nevertheless, we have several discrete values of $\ell_p$ within the acceptable range of $\xi$. They are well below $\ell_p \approx 16630$ (case (a)). The temperature tuning is more demanding—it requires roughly ten times larger values of $\delta T$. Setting $\Delta = 35$ and 37, we achieve phase matching for $\ell_p \approx 14780$ and 17314, respectively. The expected thresholds are considerably higher, in accordance with figure 2. The case $q_p = 2$, $q_s = 1$ can be considered similarly. For $\Delta = -34$ and $-35$, discrete values of $\ell_p$ lie close to 15900 and 17440, respectively.

Increasing the difference in the optical radial numbers, we can reach even smaller and larger discrete values of $\ell_p$ compared to $c/2nv_l$. However, they correspond to much higher SBS thresholds. With the values $\ell_p$ determined, the pump wavelength can be linked to the radius of the resonator: $\lambda_p \approx 2\pi nr_0/\ell_p$.

6. Discussion

Starting from the basic relations for interaction of light and acoustic fields and using the results of [28] on acoustic whispering gallery modes (AWGMs), we have analyzed the power thresholds for SBS lasing in optical WGM resonators. The general feature of these resonators is the absence of longitudinal AWGMs—only $t$- and $tl$-modes are available. This feature makes a big difference for SBS in bulk materials and fibers, where pure longitudinal acoustic modes are present and provide the strongest coupling with the light waves. As a result, competition between the $t$- and $l$-parts the AWGMs controls the SBS thresholds, and this competition strongly depends on the radial number of the AWGM. The minimum thresholds can be achieved within a relatively narrow range of very large radial numbers, such that the acoustic frequency $\Omega \approx \nu_d/r_0$, and a partial analogy with the one-dimensional bulk case with regard to the SBS phase matching holds true. The minimum lasing thresholds are substantially higher than initially expected, but lie within the $\mu$W range of pump power.

Importantly, the AWGM spectrum is quasi-continuous in the actual range of radial acoustic numbers. Correspondingly, the phase matching conditions can be fulfilled automatically without an external (temperature) adjustment. This explains a surprising (for systems with discrete optical spectra) self-tuning of SBS lasing in experiments [28]. Another important feature of the lasing process is different SBS activity of close AWGMs—the coupling strength and the oscillation threshold vary substantially in the actual range of the acoustic radial numbers.
Let us compare our results for the external power threshold \( W_{th} \) with the estimates presented in [19, 20] and based on an analogy with the Raman lasing case. In both these papers, the \( il \)-hybridization effect is not taken into account, corresponding to \( \zeta = 0 \) in our equation (13). Next, our equation (14) for \( W_{th} \) is equivalent to equation (3) of [19] for \( W_{th} \), if we replace our \( 2 \pi N V_p \) by the volume \( V \). Most probably, the latter was regarded as a free parameter. Equation (1) of [20] for \( W_{th} \) includes the factor of \( V/B \), where \( V \) is the optical-mode volume and \( B \) is an undefined modal overlap. Furthermore, this equation includes an extra factor of \((1 + Q\lambda/2\pi\eta_0)^{-1}\). For sufficiently large values of \( Q \), it leads to the dependence \( W_{th} \propto 1/Q^2Q_pQ_s \), which is not found in the literature on three-wave processes. The physical meaning of our dependence \( W_{th} \propto 1/Q^2Q_sQ \) seems to be quite clear: there is neither input nor output for the acoustic WGM. This mode experiences only a driving force from the high-\( Q \)-optical modes, and this force is practically uniform along the circumference. The forced amplitude of the acoustic vibration is proportional to the acoustic decay constant and, consequently, to \( Q^{-1} \).

Now we compare our prediction for \( W_{th} \) with the results of experiment [20]. This experiment was conducted with a spherical silica-glass resonator of \( r_0 \sim 50 \mu m \) and \( Q_p \approx 3 \times 10^9 \) at \( \lambda_p \approx 1.5 \mu m \). In accordance with equation (13) and our considerations, the minimum threshold power is expected to be within the nW range for this small resonator. However, the observed threshold power was more than two orders of magnitude higher than expected, \( W_{th} \approx 26 \mu W \). Most probably, the reason for this was the unfavorable combination of \( r_0 \) and \( \lambda_p \); with the above mentioned experimental parameters, SBS processes with \( q_p = q_s \) were forbidden, and the allowed processes corresponded to \( F \ll 1 \) and much higher thresholds. On the other hand, the use of a wedge resonator with \( r_0 \approx 3 \text{ mm} \) at \( \lambda_p \approx 1.5 \mu m \), which allows the SBS processes with \( q_p = q_e \), resulted in \( W_{th} \approx 40 \mu W \) [21]. This value is close to the expected one. No additional adjustment tools were used in either experiment, in agreement with our theory.

Strictly speaking, the results obtained are valid for resonators made of isotropic materials, such as glass. Even in the case of optically isotropic cubic crystals, the azimuth symmetry for AWGMs is lost. The effects of elastic anisotropy on the AWGM and SBS properties in WGM resonators are practically unexplored.

### 7. Conclusions

In conclusion, we have presented a rigorous theory of SBS thresholds in WGM resonators made of isotropic materials. The theory predicts substantially higher than expected power thresholds owing to the absence of pure longitudinal acoustic WGMs. Owing to the quasi-continuity of the AWGM spectrum, the SBS phase matching can be fulfilled automatically despite the pronounced discreteness of the optical spectrum. The theory predicts the values and the wavelength dependencies of the SBS thresholds. It provides a basis for comparison with experiment and for new experiments.

#### Appendix A. Asymptotic relations for Bessel and spherical functions

For very large values of the index, \( \nu \gg 1 \), three asymptotic relations for the first-kind Bessel function \( J_{\nu}(x) \) are relevant to the WGM properties [36, 37]:

\[
J_\nu \approx \frac{\exp \left[ \sqrt{\nu^2 - x^2} - \nu \arccosh (\nu/x) \right]}{\sqrt{2\pi \nu^2 - x^2}}, \quad (x < \nu)
\]

\[
J_\nu \approx (2/\nu)^{1/3} A_{\nu} \left[ (2/\nu)^{1/3}(\nu - x) \right], \quad (|x - \nu| \ll \nu)
\]

\[
J_\nu \approx \frac{2 \cos \left[ \sqrt{x^2 - \nu^2} - \nu \arccos (\nu/x) - \pi/4 \right]}{\sqrt{2\pi \sqrt{x^2 - \nu^2}}}, \quad (x > \nu).
\]  

(A.1)

The first relation gives a very rapid growth of \( J_\nu(x) \), the second (the Airy approximation) describes several first oscillations, and the third (the trigonometric approximation) describes the oscillating tail. For \( \nu \gtrsim 10^3 \), the accuracy of these approximations is very high.

The second approximation is relevant to optical WGMs, whereas the treatment of acoustic WGMs requires all three approximations.

For the function \( \Theta_{\ell m}(\theta) \) with \( \ell \gg 1 \) and \( N = \ell - |m| = 0, 1, 2, \ldots \), one can use the relation

\[
\Theta_{\ell m}(\theta) \approx \left( \ell^2/4\pi^2 \right)^{1/4} \frac{H_N(\ell^{1/2} \hat{\theta})}{\sqrt{2N!}} e^{-i\ell\hat{\theta}/2},
\]

(A.2)

where \( H_N(x) \) is the Hermite polynomial of degree \( N \) and \( \hat{\theta} = \theta - \pi/2 \) is the polar angle measured from the equator.
Appendix B. Dispersion relations for acoustic \( tl \)-modes

The resonant frequencies of acoustic \( tl \)-modes obey the dispersion relation [28]

\[
\left[ \ell^2 - \ell - 0.5x^2 + 2yR_\ell(y) \right] \left[ \ell^2 - 1 - 0.5x^2 + xR_\ell(x) \right] - \ell (\ell + 1) \left[ \ell^2 - 1 - xR_\ell(x) \right] \left[ \ell^2 - 1 - yR_\ell(y) \right] = 0, \tag{B.1}
\]

where \( R_\ell(x) = I_{\ell+3/2}(x)/I_{\ell+1/2}(x), x = \Omega r_0/r_0 \) and \( y = \Omega r_0/v \equiv sx \). For any \( \ell \), it gives a discrete sequence of solutions \( x_{\ell,q} \) where \( q \) is the radial number. With \( x = x_{\ell,q} \) determined, one can calculate the modal constant \( C_0 = C_0(\ell, q) \) using the relation

\[
C_0 = -\sqrt{2}\ell (\ell + 1) \left[ (\ell - 1)I_{\ell+1/2}(x) - xI_{\ell+3/2}(x) \right] \left( \ell^2 - \ell - 0.5x^2 \right) I_{\ell+1/2}(y) + 2yI_{\ell+3/2}(y). \tag{B.2}
\]

For \( \ell \gtrsim 10^3 \), it is practical to employ the relevant asymptotic relations for \( J_\ell(x) \).

References

[1] Boyd R W 2008 Nonlinear Optics (London: Academic)
[2] Agrawal G P 2007 Nonlinear Fiber Optics (London: Academic)
[3] Ilchenko V S, Savchenkov A A, Matsko A B and Maleki L 2004 Phys. Rev. Lett. 92 043903
[4] Savchenkov A A, Matsko A B, Strekalov D, Mohageg M, Ilchenko V S and Maleki L 2004 Phys. Rev. Lett. 93 243905
[5] Carmon T and Vahala K J 2007 Nat. Phys. 3 430
[6] Fürst J U, Strekalov D V, Elser D, Lassen M, Andersen U L, Marquardt C and Leuchs G 2010 Phys. Rev. Lett. 104 153901
[7] Kippenberg T J, Holzwarth R and Diddams S A Science 332 555
[8] Beckmann T, Linnenbank H, Steigerwald H, Sturman B, Haertle D, Buse K and Breunig I 2011 Phys. Rev. Lett. 106 143903
[9] Herr T, Brash V, Jost J D, Wang C Y, Kondratiev N M, Gorodetsky M L and Kippenberg T J 2014 Nat. Photonics 8 145
[10] Braginsky V B, Gorodetsky M L and Ilchenko V S 1989 Phys. Lett. A 137 393
[11] Vahala K J 2003 Nature 424 839
[12] Matsko A B and Ilchenko V S 2006 IEEE J. Sel. Top. Quantum Electron. 12 3
[13] Schunk G, Fürst J U, Fortsch M, Strekalov D V, Vogl U, Sedlmeir F, Schwefel H G L, Leuchs G and Marquardt M 2014 Opt. Express 22 30795
[14] Zhang J-Z and Chang R K 1989 J. Opt. Soc. Am. B 6 151
[15] Ching S C, Leung P T and Young K 1990 Phys. Rev. A 41 5026
[16] Kippenberg T J and Vahala K J 2007 Opt. Express. 15 17172
[17] Schliesser A, Anetsberger G, Riviere R, Arcizet O and Kippenberg T J 2008 New J. Phys. 10 09515
[18] Kippenberg T J and Vahala K J 2008 Science 321 1172
[19] Gradim I S, Matsko A B and Maleki L 2009 Phys. Rev. Lett. 102 043902
[20] Tomes M and Carmon T 2009 Phys. Rev. Lett. 102 113601
[21] Li J, Lee H, Chen T and Vahala K J 2012 Opt. Express 20 20170
[22] Lee H, Chen T, Li J, Yang Y, Jeon S, Painter O and Vahala J 2012 Nat. Photon 6 369
[23] Li J, Lee H and Vahala K J 2014 Opt. Lett. 39 287
[24] Roh W et al 2015 Optica 2 225
[25] Matsko A B, Savchenkov A A, Ilchenko V S, Seidel D and Maleki L 2009 Phys. Rev. Lett. 103 257403
[26] Savchenkov A A, Matsko A B, Ilchenko V S, Seidel D and Maleki L 2011 Opt. Lett. 36 3338
[27] Bahl G, Tomes M, Marquardt F and Carmon T 2012 Nat. Phys. 8 203
[28] Sturman B and Breunig I 2015 J. Appl. Phys. 118 013102
[29] Chizhov S M and Cantrell C D 1989 J. Opt. Soc. Am. B 6 1326
[30] Cantrell C D 1991 J. Opt. Soc. Am. B 8 2158
[31] Belokopytov G V and Pushchekhin P N 1992 Radiophysics Quantum Electron. 35 324
[32] Nye J F 1985 Physical Properties of Crystals (Oxford: Oxford University Press)
[33] Kroll N M 1965 J. Appl. Phys. 36 34
[34] Landau L D and Lifshitz E M 1970 Theory of Elasticity vol 7 (Oxford: Pergamon)
[35] Oraevsky A N 2002 Quantum Electron. 35 377
[36] Abramowitz M and Stegun A 1972 Handbook of Mathematical Functions (New York: Dover) p 722
[37] Gradshteyn I S and Ryzhik I M 2007 Table of Integrals Series, and Products (London: Academic)