Covering b-Symbol Metric Codes and the Generalized Singleton Bound

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Abstract—Symbol-pair codes were proposed for the application in high density storage systems, where it is not possible to read individual symbols. Yaakobi, Bruck and Siegel proved that the minimum pair-distance \(d_2\) of binary linear cyclic codes satisfies \(d_2 \geq \lceil 3d_H/2 \rceil\) and introduced \(b\)-symbol metric codes in 2016. In this paper, covering codes in \(b\)-symbol metrics are considered. Some examples are given to show that the Delsarte bound and the Norse bound for covering codes in the Hamming metric do not hold true for covering codes in the pair metric. We give the redundancy bound on covering radius of linear codes in the \(b\)-symbol metric and give some optimal codes attaining this bound. Then we prove that there is no perfect linear symbol-pair code with the minimum pair-distance 7 and there is no perfect \(b\)-symbol metric code if \(b \geq \frac{n + d}{2}\). Moreover a lot of cyclic and algebraic-geometric codes are proved non-perfect in the \(b\)-symbol metric. The covering radius of the Reed-Solomon code in the \(b\)-symbol metric is determined. As an application, the generalized Singleton bound on the sizes of list-decodable \(b\)-symbol metric codes is also presented. Then an upper bound on lengths of general MDS symbol-pair codes is proved.

Index Terms—Covering code, Delsarte bound, Norse bound, redundancy bound, perfect \(b\)-symbol metric code, list-decodable \(b\)-symbol metric code, MDS symbol-pair code, generalized Singleton bound.

I. INTRODUCTION

The Hamming weight \(wt(a)\) of a vector \(a \in \mathbb{F}_q^n\) is the number of non-zero coordinate positions. The Hamming distance \(d_H(a, b)\) between two vectors \(a\) and \(b\) is the Hamming weight of \(a - b\). The minimum Hamming distance of a code \(C \subset \mathbb{F}_q^n\),

\[
d_H(C) = \min_{a \neq b} \{d_H(a, b) : a \in C, b \in C\},
\]

is the minimum of Hamming distances \(d_H(a, b)\) between any two different codewords \(a\) and \(b\) in \(C\). Theory of Hamming metric error-correcting codes has been extensively developed and numerous constructions have been proposed, we refer to [19], [21], [27], and [30]. The Singleton bound for a linear \([n, k, d]_q\) code is \(d_H \leq n - k + 1\). When the equality holds, this code is called a maximum distance separable (MDS) code. Let \(\mathbb{F}_q\) be an arbitrary finite field, \(P_1, \ldots, P_n\) be \(n \leq q\) distinct elements in \(\mathbb{F}_q\). The Reed-Solomon code is defined by

\[
RS(n, k) = \{ (f(P_1), \ldots, f(P_n)) : f \in \mathbb{F}_q[x], \deg(f) \leq k - 1 \}.
\]

This is an \([n, k, n - k + 1]_q\) linear MDS code attaining the Singleton bound, since a degree \(\deg(f) \leq k - 1\) nonzero polynomial has at most \(k - 1\) roots.

A code \(C \subset \mathbb{F}_q^n\) is called \((d, L)\) list-decodable in the Hamming metric, if each ball \(B_H(x, d) = \{ y : d_H(x, y) < d \} \subset \mathbb{F}_q^n\) contains at most \(L\) codewords of \(C\) for each \(x \in \mathbb{F}_q^n\), see [29]. The generalized Singleton bound

\[
|C| \leq Lq^n - \frac{(\frac{L+1}{2})^d}{x^n}
\]

for \((d, L)\) list-decodable Hamming metric codes was proved in [29]. When \(d = \frac{d_H(C) - 1}{2}\) and the list size \(L = 1\), this is the classical Singleton bound.

For a code \(C \subset \mathbb{F}_q^n\), we define its covering radius in the Hamming metric by

\[
R_H(C) = \max_{x \in \mathbb{F}_q^n} \min_{c \in C} \{wt(x - c)\}.
\]

Hence Hamming balls \(B(x, R_H(C))\) centered at all codewords \(x \in C\), with the radius \(R_H(C)\) cover the whole space \(\mathbb{F}_q^n\), and moreover this radius is the smallest possible such radius. For more details, we refer to the book [19, Chapter 11] and [7].

The following three bounds for covering codes in the Hamming metric are important. The Norse bound in [18] claims that for a binary linear code \(C\) with its dual distance at least 2, then its covering radius is at most \(R_H(C) \leq \frac{n}{2}\), see [19, Corollary 11.2.2]. The Delsarte bound in [8] asserts that the covering radius of a linear code is bounded from above by the number of nonzero weights of its dual, see [19, Theorem 11.3.3]. The redundancy bound asserts that the covering radius of a linear \([n, k, d]_q\) code is at most \(n - k\), see [19, Corollary 11.1.3].

A code in the Hamming metric is perfect if the covering radius of this code is \(R_H(C) = \lfloor(d_H(C) - 1)/2 \rfloor\). For a length \(n\) perfect code \(C\), the whole space \(\mathbb{F}_q^n\) is the disjoint union of the Hamming balls of the radius \(R_H(C) = \lfloor(d_H(C) - 1)/2 \rfloor\) centered at all codewords. This is an intersecting point of both packing and covering problems in the Hamming metric space \(\mathbb{F}_q^n\). Perfect codes in the Hamming metric have the same parameters as parameters of Hamming codes, or the binary [23, 12, 7/2] Golay code or the ternary [11, 6, 5/3] Golay code, see [7], [20], and [32]. We cite the comment “The classification of perfect codes as summarized in this theorem was a significant and difficult piece of mathematics” in [19, page 49]. Covering codes and in particular perfect codes in the rank-metric have been studied in [1], [6], and [17]. Another interesting intuition of the generalized covering radii of linear codes was introduced and discussed in a recent paper [13].
Symbol-pair codes were introduced for high density data storage, and we refer to [2] and [3]. Set \((F_q^2)^n = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n), (x_n, x_1)\} : x = (x_1, \ldots, x_n) \in F_q^n\). The space \((F_q^2)^n\) for \(b = 3, 4, \ldots, n\) can be defined similarly. The pair metric on \(F_q^n\) is defined as follows. For \(x = (x_1, \ldots, x_n) \in F_q^n\), the mapping \(\pi_2 : F_q^n \rightarrow (F_q^2)^n\) is defined by \(\pi_2(x) = ((x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n), (x_n, x_1)) \in (F_q^2)^n\). The pair weight of \(x \in F_q^n\) is \(w(t(x_2)) = |\{i : (x_i, x_{i+1}) \neq 0\}|\). The pair-distance \(d_2\) is defined by
\[
d_2(x, y) = w(t(\pi_2(x) - \pi_2(y)),
\]
see [2] and [3]. It follows from the definition that
\[
\min\{d_H(x, y) + 1, n\} \leq d_2(x, y) \leq \min\{2d_H(x, y), n\}.
\]
The ball in the pair metric is
\[
B_2(x, r) = \{y \in F_q^n : d_2(x, y) \leq r\}.
\]
For a code \(C \subset F_q^n\), the minimum pair-distance is
\[
d_2(C) = \min\{d_2(x, y) : x \neq y, x \in C, y \in C\}.
\]
Then we have
\[
\min\{d_H(C + 1, n)\} \leq d_2(C) \leq \min\{2d_H(C), n\}.
\]
The Singleton bound \(|C| \leq q^{n-d_2^2}\) for symbol-pair codes was proved in [4]. A symbol-pair code attaining this bound is called an MDS symbol-pair code. Many MDS symbol-pair codes have been constructed in [4], [5], [9], [11], [25], and [26]. In [33] the b-symbol metric was introduced and the following lower bound of the minimum pair distances for linear binary cyclic codes was proved,
\[
d_2 \geq \left\lceil \frac{3d_H}{2} \right\rceil.
\]
A general symbol-pair code \(C \subset F_q^n\) is called \((d_{list}, L)\) list-decodable if each ball \(B_2(x, d_{list}) \subset F_q^n\) contains at most \(L\) codewords of \(C\) for each \(x \in F_q^n\). The list-decodability and list-decoding of symbol-pair codes have been studied in [23]. Additional constructions and bounds for symbol-pair codes were given in [14]. Generalized pair weights of linear codes were introduced and studied in [22].

We recall the b-symbol metric on \(F_q^n\) for \(2 \leq b \leq n-1\). For \(x \in F_q^n\), set \(\pi_b(x) = ((x_1, \ldots, x_b), (x_2, \ldots, x_{b+1}), \ldots, (x_n, x_1, \ldots, x_b-1)) \in (F_q^b)^n\). The b-symbol metric on \(F_q^n\) was introduced in [33]. The b-symbol weight of \(x = (x_1, \ldots, x_n) \in F_q^n\) is \(w(t(\pi_b(x))) = |\{i : (x_i, x_{i+1}, \ldots, x_{i+b-1}) \neq 0\}|\). The b-symbol distance is defined by
\[
d_b(x, y) = w(t(\pi_b(x) - \pi_b(y)).
\]
The b-symbol ball in \(F_q^n\) is
\[
B_b(x, r) = \{y \in F_q^n : d_b(x, y) \leq r\}.
\]
It is clear that
\[
\min\{d_H(x, y) + b - 1, n\} \leq d_b(x, y) \leq \min\{bd_H(x, y), n\},
\]
see Proposition 9 in [33]. The minimum b-symbol distance of a code \(C\) is
\[
d_b(C) = \min\{d_b(x, y) : x \neq y, x \in C, y \in C\}.
\]
For recent results about minimum b-symbol weights of linear cyclic codes, we refer to [10] and [31].

For a general code \(C \subset F_q^n\), we define its covering radius in the b-symbol metric by
\[
R_b(C) = \max\{\min\{w_b(x - c)\} \mid c \in C\}.
\]
Hence the balls \(B_b(x, R_b(C))\) centered at all codewords \(x \in C\), with the radius \(R_b(C)\) cover the whole space \(F_q^n\), and moreover this radius is the smallest possible such radius. To the best of our knowledge, there has been no previous work on covering codes in the b-symbol metric. It is interesting to determine perfect codes in the b-symbol metric, that is these codes \(C \subset F_q^n\) satisfying
\[
R_b(C) = \left\lceil \frac{d_b(C) - 1}{2} \right\rceil.
\]
A code \(C \subset F_q^n\) is called \((d_{list}, L)\) list-decodable b-symbol metric code if the ball \(B_2(x, d_{list}) \subset F_q^n\) contains at most \(L\) codewords of \(C\) for each \(x \in F_q^n\). There is no previous work about list-decodable codes in the b-symbol metric.

The motivation of this paper is as follows. There have been extensive research on covering radii of codes in the Hamming metric. For example, in the paper [12], covering radii of more than six thousand binary cyclic codes were calculated and determined. There are many results about covering radius of codes in the Hamming metric, for example, the Delsarte bound in [8], the Norse bound in [18] and the redundancy bound, see [19, Chapter 11]. It is also interesting to study perfect codes in b-symbol metric. On the other hand many MDS symbol-pair codes have been constructed by cyclic codes or constacyclic codes, see [5], [9], [10], [25], and [26]. However, the is no general upper bound on lengths of MDS symbol-pair codes.

The main contributions of this paper are as follows. We give a similar redundancy upper bound on the covering radius of a linear code in b-symbol metric in Section II. Some codes attaining this redundancy bound are constructed. The covering radius of the Reed-Solomon code in b-symbol metric is determined. We also give some examples, which show that Delsarte and Norse bounds do not hold for covering codes in the pair metric in Section II. We prove that the perfect b-symbol metric code does not exist when \(b \geq \frac{n+3}{2}\) and the perfect linear pair-weight code of the minimum pair-distance 7 does not exist neither, in Section III. Moreover it is proved that a linear \([n, k, q]\) code can not be perfect in the \(2(k + 1)\)-symbol metric. Many well-known cyclic codes, constacyclic codes and algebraic-geometric codes are shown to be not perfect in the pair metric. Based on our previous results on covering codes in b-symbol metric, the generalized Singleton bound on list-decodable b-symbol metric codes is proved in Section IV. As an application of this bound, we also give an upper bound on lengths of MDS symbol-pair codes in Section IV.
II. COVERING CODES IN THE b-SYMBOL METRIC

The following two upper bound on the covering radius of a linear code in $\mathbb{F}_q^n$ in the $b$-symbol metric is clear from the property of the $b$-symbol weight.

Proposition 1: Let $C \subset \mathbb{F}_q^n$ be a general code. Then
$$\min\{R_H(C) + b - 1, n\} \leq R_0(C) \leq \min\{bR_H(C), n\}.$$

Proof: From the inequality $\min\{d_H(x, y) + b - 1, n\} \leq d_S(x, y)$, (see Proposition 9 in [33]), it follows that $R_0(C) \leq bR_H(C)$ follows from the inequality $d_S(x, y) \leq d_H(x, y)$.

In [24], the covering radii of many binary covering codes were determined. Then covering radii of these binary codes in the pair metric are ranging from $R_H + 1$ to $2R_H$. It is interesting to determine their covering radii in the pair metric exactly.

Let $S_n$ be the permutation group of the order $n!$ of all $n$ coordinate permutations. For any given code $C \subset \mathbb{F}_q^n$ and a permutation $s \in S_n$, the code after permutation is denoted by $s(C)$. If $C$ is a linear $[n, k, q]$ code, then $s(C)$ is also a linear $[n, k, q]$ code for any permutation $s \in S_n$. We recall that the covering radius in the Hamming metric of a linear $[n, k, q]$ code is at most $n - k$, this is the redundancy bound, see [7, page 217].

Theorem 1: Let $C \subset \mathbb{F}_q^n$ be a linear $[n, k, q]$ code, then there exists a permutation $s \in S_n$ such that the covering radius in $b$-symbol metric of $s(C)$ is at most $n - k + b - 1$, that is,
$$R_0(s(C)) \leq \min\{n - k + b - 1, n\}.$$

Proof: Suppose that the first $k$ columns of one generator matrix are linearly independent. Then for any given vector $y$ in $\mathbb{F}_q^k$ there is a codeword $c$ in $C$ such that the first $k$ coordinates of $c$ equal to $y$. Therefore in each coset $y + C$ we can find a vector with the first $k$ coordinates equal to zero. Then the smallest $b$-symbol weight in each coset is at most $n - k + b - 1$. If the first $k$ columns of this linear code are not linearly independent, suppose that the columns $i_1 < i_2 < \cdots < i_k$ are linearly independent. We take a permutation $s$ transforming $i_1, \ldots, i_k$ to $n - k + 1, \ldots, n$. Then the conclusion follows immediately.

The following is the redundancy bound on covering radius in $b$-symbol metric.

Corollary 1 (Redundancy Bound): Let $C$ be a linear $[n, k, q]$ code, if there are $k$ consecutive coordinate positions $i, i + 1, \ldots, i + k - 1$ such that columns in one generator matrix of $C$ at these positions are linear independent, then $R_0(C) \leq \min\{n - k + b - 1, n\}$.

For a linear $[n, k, q]$ code $C$, let $wt_{coseq}(v + C)$ be the minimum Hamming weight among all weights of vectors in this coset. Hence
$$R_H(C) = \max_{v \in \mathbb{F}_q^n}\{wt_{coseq}(v + C)\},$$
see [19, Theorem 11.1.2]. Then $R_H(C)$ is minimum positive integer $h$ such each nonzero vector in $\mathbb{F}_q^{n-k}$ can be represented as linear combinations of at most $h$ columns in its parity check matrix. Moreover such a linear combination with the smallest $h$ corresponds to a vector in one coset with the smallest Hamming weight. Then the redundancy bound on covering radii of linear codes follows immediately. This bound $R_H(C) \leq n - k$ for a linear $[n, k, q]$ code is attained when $C$ is a Reed-Solomon code, see [7]. Similarly for a linear $[n, k, q]$ code $C \subset \mathbb{F}_q^n$ let $wt_{b,coseq}(v + C)$ be the minimum $b$-symbol weight among all weights of vectors in this coset. Then
$$R_b(C) = \max_{v \in \mathbb{F}_q^n}\{wt_{b,coseq}(v + C)\}.$$

Example 1. Let $C$ be a binary linear self-dual $[n, k, 2]_2$ code with the generator matrix $(I_n, I_n)$, where $I_n$ is the $n \times n$ identity matrix. Then it is easy to verify that $R_H(C) = n + 1$. On the other hand the vectors in cosets can be of the form $(0, y)$, where $0$ and $y$ are vectors in $\mathbb{F}_2^n$. Hence the smallest possible pair weight is $n + 1$, and $R_2(C) = n + 1$. The above redundancy bound for symbol-pair code is attained. Example 2 shows that the condition in Corollary 1 is necessary.

Example 2. We use different coordinate ordering of binary linear $[4t, 2t, 2]_2$ code $C_1$ with the following generator matrix.
$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}$$
This code is a self-dual code and the above generator matrix is also a parity check matrix. Then the smallest possible pair weight vectors in cosets is of the form
$$(01101101010110).$$
Therefore the smallest possible pair weight of vectors in cosets is $3t$. We have $R_2(C_1) = 3t$.

The Delsarte bound in [8] asserts that the covering radius of a linear code in the Hamming metric is bounded from above by the number of nonzero weights of its dual, see [19, Theorem 11.3.3]. When $t = 2$ we have a self-dual binary $[4, 2, 2]_2$ code $C_2$ with four codewords, $(0000), (1100), (0011), (1111)$. Hence the nonzero pair weights of the dual code are 3 and 4. It is clear that the covering radius in the pair metric is $R_2(C_2) = 3$. This is a counterexample to the Delsarte bound in the pair metric.

Example 3. Let $C$ be a binary linear $[6, 3, 3]_2$ code with the following parity check matrix.
$$\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 \\
\end{pmatrix}$$
Since $(000111)$ is a codeword, $d_2(C) = 4$. Any vector in $F_2^3$ can be represented as the sum of at most two columns in the above parity check matrix. Hence the covering radius of the above binary linear $[6, 3, 3]_2$ code is 2. From Proposition 1, the covering radius of this linear $[6, 3, 3]_2$ code in the pair metric satisfies $4 \leq R_2(C) \leq 4$. If $R_2(C) = 3$, then in each coset $v + C \subset F_2^n$, the vector of Hamming weight 2 has to contain two consecutive support positions. This is not the case, since we can check $(1, 1, 1)$ has no representation as the sum of two consecutive columns in the above matrix. Therefore $R_2(C) = 4$ attains the above redundancy bound Corollary 1 for symbol-pair codes and the bound in Proposition 1.
The Norse bound in [18] claims that for a binary linear code $C$, if its dual distance is at least 2, that is no two columns in any generator matrix of $C$ are linear dependent, then the covering radius $R_H(C) \leq \lfloor \frac{n}{2} \rfloor$, see Corollary 11.2.2 in [19]. Example 3 is a counterexample to such claim in the pair metric. Hence the Norse bound on the covering radius in the $b$-symbol metric does not hold true.

Example 4. Let $C$ be the Hamming $[8,4,4]_2$ code with the following generator matrix. This is a binary linear code with $d_H(C) = 4$ with 16 codewords. Their weights are 4 and 8 and this code has two nonzero weights 4 and 8.

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

From the Delsarte bound it is easy to verify that the covering radius of $C$ in the Hamming metric is $R_H(C) = 2$. It is clear that all pair weights of codewords are in the set $\{5, 6, 7, 8\}$. We observe the vectors of weight 2 in each coset. It is easy to verify that there are many vectors with two support positions, one support position in $\{1, 2, 3, 4\}$ and one support position in $\{5, 6, 7, 8\}$, in some cosets. Then the smallest pair weight

\[
\max_{v \in \mathbb{F}_q^2} \{w_{t,\text{coset}}(v + C)\} = 4.
\]

We have $R_2(C) = 4$. This is an example with the covering radius in the pair metric smaller than the redundancy upper bound.

A symbol-pair code $C \subseteq \mathbb{F}_q^2$ satisfies the Singleton bound

\[
|C| \leq q^{n-d_2+2},
\]

see [4]. A code attaining this bound is called an MDS symbol-pair code. Many linear MDS codes have been constructed in [4], [5], [9], [10], [11], [25], and [26]. Similarly a linear $[n,k]_q$ code satisfying $d_b = n - k + b$ is called an MDS $b$-symbol metric code and some linear MDS $b$-symbol metric codes have been constructed in [10] and [11].

**Corollary 2:** Let $C$ be a linear MDS $b_1$-symbol metric code for $1 \leq b_1 \leq n - 1$, then its covering radius in the $b$-symbol metric satisfies $R_b(C) \leq \min\{n - k + b_1 - 1, n\}$.

**Proof:** Let $G$ be one generator matrix, then the last $k$ columns have to be linearly independent. Otherwise there is a nonzero codeword such that the last $k$ coordinates are zero. Then $d_{b_1}(C) \leq n - k + b_1 - 1$. This code $C$ is not an MDS $b_1$-symbol metric code. Then the conclusion follows from Corollary 1.

We determine the covering radius of Reed-Solomon codes in $b$-symbol metric.

**Theorem 2:** The covering radius of Reed-Solomon codes in $b$-symbol metric satisfies $R_b(RS(n,k)) = \min\{n - k + b - 1, n\}$.

**Proof:** Any $k$ consecutive coordinate positions satisfy the required property in Corollary 2 for Reed-Solomon code. On the other hand the covering radius of the code $RS(n,k)$ is $n - k$. The conclusion follows from the conclusion $R_H(C) + b - 1 \leq R_b(C)$ in Proposition 1, and the redundancy bound Corollary 1.

Since there are many MDS symbol-pair or MDS $b$-symbol linear cyclic codes constructed, see for example [10] and [25], it is an interesting open problem to determine the covering radii of these MDS codes in the pair metric or the $b$-symbol metric, as in Theorem 2. In Theorem 3 and Corollary 4 some lower bounds on their covering radii in the pair metric are given.

We recall some basic facts about algebraic geometry codes. We refer to Chapter 10 of [21], Chapter 13 of [19] and [30] for the basic notations and definitions in algebraic geometry.

Let $X$ be an absolutely irreducible projective smooth genus $g$ curve defined over $\mathbb{F}_q$. Let $P_1, \ldots, P_n$ be $n$ distinct rational points of $X$ over $\mathbb{F}_q$. Let $G$ be a rational divisor over $\mathbb{F}_q$ of degree $\deg(G)$ where $2g - 2 < \deg(G) < n$ and

\[
support(G) \cap P = \emptyset.
\]

Let $L(G)$ be the function space associated with the divisor $G$. The algebraic geometry function code associated with $G, P_1, \ldots, P_n$ is defined by

\[
C(P_1, \ldots, P_n, G, X) = \{(f(P_1), \ldots, f(P_n)) : f \in L(G)\}.
\]

The dimension of this code is

\[
k = \deg(G) - g + 1
\]

from the Riemann-Roch Theorem, see [30]. The minimum Hamming distance is

\[
d_H \geq n - \deg(G).
\]

The Reed-Solomon codes are just the algebraic-geometric codes over the genus 0 curve. Algebraic geometry residual code with the dimension $k = n - \deg(G) + g - 1$ and minimum Hamming distance $d_H \geq \deg(G) - 2g + 2$ were also defined, we refer to [19], [21], and [30] for the detail.

From the Riemann-Roch Theorem, any $k - g$ columns in one generator matrix of an algebraic-geometric $[n,k]_q$ code are linearly independent, see [30]. Hence we have the following upper bound on the covering radius of an algebraic-geometric code.

**Corollary 3:** Let $C$ be an algebraic-geometric $[n,k]_q$ code, then its covering radius in $b$-symbol metric is at most $n - k + g + b - 1$.

**Proof:** This conclusion follows from a similar argument as the proof of Theorem 1.

Some lower bounds on the covering radii in the $b$-symbol metric of linear codes are given in Theorem 3, 4 and Corollary 5.

**Theorem 3:** Let $C$ be a linear $[n,k]_q$ code, then its covering radius in $b$-symbol metric is at least

\[
(b + 1) \cdot \left\lfloor \frac{n}{2(k + 1)} \right\rfloor.
\]

**Proof:** We check the minimum $b$-symbol weight of vectors in the cosets of this linear code $C$. We divide $\{1, 2, \ldots, n\}$ to $t = \left\lfloor \frac{n}{2(k + 1)} \right\rfloor$ parts $A_1, \ldots, A_t$, each part has consecutive $k + 1$ coordinate positions. Since any $k + 1$ columns in the generator matrix $G$ of $C$ are linearly dependent, then we can find a vector $y$ such that, each vector in the coset $y + C$ has at least one nonzero coordinate in $A_1, \ldots, A_t$. In every
two consecutive parts, the vector in the coset has the smallest $b$-weight, only when it is of the form $(\ldots 011\ldots)$, where the first 1 is in the part $A_1$, and the second 1 is in the next part $A_{i+1}$. Then the conclusion follows immediately.

Covering radii of cyclic codes including BCH codes have been studied extensively, for example, see [7] and [12]. From the calculations of pair-distances and $b$-symbol distances of some cyclic codes and constacyclic codes in [9] and [10] we can give lower bounds on the covering radii of these codes in the pair or the $b$-symbol metric. The main point is the natural inclusion relation of these cyclic and constacyclic codes. It is obvious that similar lower bounds on the covering radius of some other cyclic or constacyclic codes can be obtained.

**Theorem 4:** Let $p$ be an odd prime. Let $C_{i,i}$ be the cyclic code over $F_{p^m}$ of the length $2p^s$ generated by $(x^2 - 1)^i$. Suppose that $p^s - p^{s-\epsilon} + \tau p^{s-\epsilon - 1} + 2 \leq i \leq p^s - p^{s-\epsilon} + (\tau + 1)p^{s-\epsilon - 1} + 1$, where $0 \leq \tau \leq p - 2$, $0 \leq \epsilon \leq s - 1$. Then

$$R_2(C_{i,i}) \geq 2(\tau + 1)p^\epsilon.$$

**Proof:** The cyclic codes $C_{i,i}$ and $C_{i-1,i-1}$ satisfy that $C_{i,i} \subset C_{i-1,i-1}$. Moreover these two codes are not the same. Then in the coset $v + C_{i,i}$ for some $v \in C_{i-1,i-1}\setminus C_{i,i}$, the pair weight of each vector is at least the minimum pair-distance $d_2(C_{i-1,i-1})$ of the code $C_{i-1,i-1}$. The conclusion follows from Theorem 9 in [9].

Let $p$ be a prime and $s, m$ be two positive integers, $s = r_1 m + r$, where $r_1$ and $r$ are two nonnegative integers satisfying $0 \leq r \leq m - 1$. Let $\lambda$ be a nonzero element in $F_{p^m}$. Set $\gamma = \lambda^{p^{r_1 + \lambda} + \gamma^2}$. Notice that $(r_1 + 1)m - s$ satisfies $1 \leq (r_1 + 1)m - s \leq m$. Then length $p^s - \lambda$-constacyclic code $C_{i}$ is generated by $(x - \gamma)^i$, $0 \leq i \leq p^s$, see [10] page 2. The following result follows from Theorem 10.

**Corollary 4:** Let $p$ be an odd prime. Suppose that $i$ satisfies $p^s - p^{s-\epsilon} + \tau p^{s-\epsilon - 1} + \beta + 1 \leq i \leq p^s - p^{s-\epsilon} + (\tau + 1)p^{s-\epsilon - 1} + 1$, where $0 \leq \epsilon \leq s - 2$, $0 \leq \tau \leq p - 2$ and $0 \leq \beta(\tau + 1)$. Then the covering radii of $C_{i}$ in $b$-symbol metric satisfies

$$R_b(C_{i}) \geq b(\tau + 2)p^\epsilon.$$

**Proof:** Let $C_{i}$ be the constacyclic code generated by $(x - \gamma)^i$, then $C_{i} \subset C_{i-1}$. Since $i - 1$ is in the range $p^s - p^{s-\epsilon} + \tau p^{s-\epsilon - 1} + \beta \leq i - 1 \leq p^s - p^{s-\epsilon} + (\tau + 1)p^{s-\epsilon - 1}$, $d_b(C_{i-1}) = b(\tau + 2)p^\epsilon$, from [10], Theorem 9]. Then for each vector $v \in C_{i-1} \subset F_{p^m}^\times$, the minimum $b$-weights of vectors in $v + C_{i-1}$ is at least $d_b(C_{i-1}) = b(\tau + 2)p^\epsilon$. The conclusion follows immediately.

**III. Perfect b-Symbol Metric Codes**

When $b = 2$, $B_2(x, 2)$ is exactly the $H_2(x, 1)$. We refer to [2], Example 2. Hence binary Hamming $[2^n - 1, 2^n - 1, 3]_2$ code with the pair distance 5 is a perfect pair metric code as showed in Theorem 19 in [2]. It is interesting to study perfect $b$-symbol metric codes in general.

**Proposition 2:** $B_b(x, r) \subset B_H(x, r - b + 1)$ and $B_H(x, r) \subset B_b(x, br)$. Moreover $B_b(x, r) = x$ for $r \leq b - 1$, $B_b(x, b) = B_H(x, 1)$.

**Proof:** The conclusions follows from a direct analysis about the shapes of the balls.

The following result is a direct generalization of Theorem 19 in [2].

**Proposition 3:** The minimum $b$-symbol distance of the Hamming $[2^n - 1, m, 3]_q$ code over $F_q$ is at most $2b + 1$. If the minimum $b$-symbol distance of this $q$-ary Hamming code is $2b + 1$, it is a perfect $b$-symbol metric code.

**Proof:** It is clear that there is one Hamming weight 3 codeword $c = (110\ldots 0x\ldots 0)$ in the $q$-ary Hamming code for some nonzero $x \in F_q$. Then from the definition of the $b$-symbol distance, we have $d_b(c, 0) \leq 2b + 1$. Since $B_b(x, b) = B_H(x, 1)$, then the whole space $F_q^{2^n - 1}$ is the disjoint union of $B_b(x, b)$, where $x$ takes over all codewords of this $q$-ary Hamming code with the $b$-symbol distance $2b + 1$.

It is interesting to observe that there is no perfect symbol-pair code of the minimum pair distance 7.

**Theorem 5:** For any given finite field $F_q$, there is no perfect symbol-pair code with the cardinality $q^2$ and the minimum pair-distance 7.

**Proof:** We analyze the shape of the ball

$$B_2(0, 3) = B_2(0, 2) \bigcup \{x : d_2(x, 0) = 3\}.$$

It is clear that $B_2(0, 2)$ and the sphere $\{x : d_2(x, 0) = 3\}$ are disjoint. For any nonzero $x$ satisfying $d_2(x, 0) \leq 2$, the vector $x$ has to be of the form $(0\ldots 0\ast \ldots 0\ast)$ with only one nonzero coordinate. When $x$ satisfies $d_2(x, 0) = 3$, it has to be of the form $(0\ldots 0\ast \ldots 0\ast \ast \ldots \ast)$, with two consecutive nonzero coordinates. Therefore for each $x \in B_2(0, 3)$, we have $d_2(x, 0) = 1$, or $d_2(x, 0) = 2$ and $x$ has two consecutive support positions. Hence the cardinality of the ball $B_2(0, 3)$ is $|B_2(0, 3)| = 1 + n(q - 1) + n(q - 1)^2$.

If there is a perfect symbol-pair code with the minimum pair-distance 7 and the cardinality $q^2$, $|B_2(0, 3)|q^2 = q^n$. Then $\frac{n + 1}{q^2} = \frac{n + 1}{q^n} = nq. Hence nq = q^{k+1+q^{k-1}+q^{k-2}+\cdots+q+1} = 1$ mod $q$. This is a contradiction.

**Theorem 6:** If $C$ is a perfect $b$-symbol metric code, then $d_b(C) \leq d_H(C) + 2b - 3$.

In particular if $b \geq \frac{n+4}{2}$, there is no perfect $b$-symbol metric code.

**Proof:** From Proposition 1, $R_b(C) \geq R_H(C) + b - 1 \geq \frac{[d_b(C) - 1]}{2} + b - 1$, we have

$$\frac{|d_b(C) - 1|}{2} \geq \frac{|d_H(C) - 1|}{2} + b - 1,$$

since $C$ is perfect in the $b$-symbol metric, and $R_b(C) = \frac{[d_b(C) - 1]}{2}$. Then we have $d_b(C) \geq d_H(C) + 2b - 3$. If $b \geq \frac{n+4}{2}$ and $C$ is perfect in the $b$-symbol metric, $d_b(C) \geq 2b - 3 \geq n + 1$. This is a contradiction, since $d_b(C) \leq n$ from the definition of the $b$-symbol weight. The conclusion follows immediately.

**Theorem 7:** Let $C$ be a linear $[n, k]_q$ code. Then $C$ is not perfect in the $(2k + 2)$-symbol metric.

**Proof:** From the proof of Theorem 3, there is a vector in some coset, every consecutive $(2k + 2)$ coordinate positions can not be zero. Therefore $R_{2(k+1)}(C) = n$ from...
Theorem 8: Let $C \subset C' \subset F_q^n$ be two linear codes satisfying $d_h(C') \geq \frac{b}{2} \cdot d_H(C)$ and $C \neq C'$. Then $C$ is not perfect in the $b$-symbol metric.

Proof: Let $v \in C' \setminus C$, then any vector in the coset $v + C$ has its $b$-symbol weight at least $d_b(C')$. Therefore $R_b(C) \geq d_b(C') \geq \frac{b}{2} \cdot d_H(C) > \frac{d_b(C)-1}{2}$, since $bd_H(C) \geq d_b(C)$. The conclusion follows directly.

The following result Corollary 5 follows from Theorem 8 for the pair metric case.

Corollary 5: Let $C'$ be an MDS symbol-pair $[n, k]_q$ linear code, then any linear subcode $C$ in $C'$ of dimension $k-1$ is not perfect in the pair metric.

Proof: From the Singleton bound in the Hamming metric $d_H(C) \leq n - (k-1) + 1 = n - k + 2 = d_2(C')$. The conclusion follows from Theorem 8 immediately.

Similarly we can prove the following result.

Corollary 6: Let $C$ be a linear subcode of a linear code $C_1$ and assume that these two codes are not the same. Assume $d_2(C_1) \geq d_H(C)$ or $d_H(C_1) \geq d_H(C) - 1$. Then $C$ is not perfect in the pair metric.

Proof: As in the argument in the proof of Theorem 4, $R_2(C) \geq d_2(C_1) \geq d_H(C) > \frac{d(C_1)-1}{2}$. Then $C$ is not perfect in the pair metric if $d_2(C_1) \geq d_H(C)$. On the other hand, $R_2(C) \geq d_2(C_1) \geq d_H(C_1) + 1 \geq d_H(C) > \frac{d(C_1)-1}{2}$, if $d_H(C_1) \geq d_H(C) - 1$. The second conclusion follows.

Theorem 3.1 and Corollary 6 indicate that many linear codes are not perfect in the pair metric. Theorem 3.4 and Corollary 6 can be used to exclude a lot of linear codes as perfect codes in the pair metric. For example from the computation of optimum distance profiles (ODPs) of self-dual binary codes in [15] and [16], many codimension 1 linear subcode of some self-dual binary codes are not perfect in the pair metric. Moreover a lot of cyclic codes and algebraic-geometric codes are not perfect in the $b$-symbol metric, as proved in the following two results. We refer to [19, Chapter 13] for the notations and basic results of algebraic geometry codes.

Corollary 7: Let $C$ be an algebraic geometry residue code with the dimension $n - m + g - 1$ and the minimum Hamming distance $m - 2g + 2$, defined by a degree $m$ divisor of the form $mQ$, where $m > 2g - 2$ and $Q$ is a rational point of the curve. Then $C$ is not perfect in the pair metric.

Proof: Let $C_1$ be the residual code defined by the divisor $(m - 1)Q$. Then $C$ is a real subcode of $C_1$. Since $d_H(C_1) \geq m - 1 - 2g + 2 \geq d_H(C) - 1$, the conclusion follows from Corollary 6.

The Hermite curve $x^n + x^3 + 1$ over $F_q^2$ is a genus $g = q^2 - q$ curve and it has $1 + q^3$ rational points. Let $m$ be a positive integer satisfying $2q - 2 < m \leq q^3 - 1$. The Hermite code $C_m$ is a linear $[q^3, q^3 - m + g - 1, \geq m - 2g + 2]_q$ code, see [28]. When $m \geq 2q^3 - 2q - 2$, the true minimum Hamming distance is $m - 2g + 2$. From Corollary 7, these Hermite codes are not perfect symbol-pair codes.

From [9, Theorem 8, 9], most cyclic codes of the length $2p^s$ over $F_{p^m}$ of the form $C_{i,j}$ as in Section II can not be perfect. We give the following result.

Corollary 8: Let $p$ be an odd prime. Let $C_{1,i}$ be the cyclic code over $F_{p^m}$ of the length $2p^s$ generated by $(x^2 - 1)^i$. Suppose that $p^s - p^{s-e} + r p^{s-1} + (r + 1) p^{s-e-1}$, where $0 \leq r \leq p - 2$, $0 \leq e \leq s - 1$. Then $C_{i,j}$ is not a perfect code in the pair metric.

Proof: It is clear that $C_{i,j}$ is a subcode of $C_{i-1,i-1}$.

From Theorem 8 and Theorem 9 in [9], the pair-distance of the code $C_{i-1,i-1}$ and the Hamming distance of the code $C_{i,j}$ are $d_2(C_{i-1,i-1}) = 2(\tau + 1) p^s$, and $d_H(C_{i,j}) = \tau (p + 1) p^s$, since both $i$ and $i - 1$ are in the range $[p^s - p^{s-e} + r p^{s-1} + 2, p^s - p^{s-e} + (r + 1) p^{s-e-1}]$. Then the condition in Corollary 5 is satisfied and the conclusion follows directly.

Similarly many constacyclic codes studied in [10] are not perfect in the $b$-symbol metric from Corollary 3.2. It is an interesting open problem to determine all perfect symbol-pair codes.

IV. THE GENERALIZED SINGLETON BOUND

When $b = n - 1$, $B_{n-1}(x, n - 1) = B_H(x, 1)$, hence any code is $(n-1, 1 + (n-q-1))$ list-decodable $(n-1)$-symbol metric code. It is not interesting to discuss list-decodable $b$-symbol metric codes when $b$ is big.

From the definitions above, for an $(d_{\text{list}}, L)$ list-decodable code $C \subset F_q^n$ in the $b$-symbol metric, and a given covering code $C_1 \subset F_q^n$ in the $b$-symbol metric with the covering radius at most $d_{\text{list}}$, we have the following upper bound on the size of the list-decodable code or the lower bound on the list size,

$$|C| \leq L / |C_1|.$$
Theorem 8: Let $q$ be a prime power, $m$ be a positive integer, $t$ be an even positive integer, and set $n = \frac{(q^m - 1)}{q - 1}$. Let $C \subset \mathbb{F}_q^v$ be a $(bt, L)$-list-decodable $b$-symbol code, where $b \leq \frac{q^m - 1}{q - 1}$. Then

$$|C| \leq L \cdot q^{n - tm}.$$  

Notice that $tm = \frac{m}{b} \cdot d_{list}$, where $d_{list} = bt$ is the list-decodable radius.

Proof: Let $H$ be the following $tm \times n$ matrix where $n = \frac{(q^m - 1)}{q - 1}$.

$$
\begin{pmatrix}
H_1 & 0 & 0 & \cdots & 0 \\
0 & H_2 & 0 & \cdots & 0 \\
0 & 0 & H_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & H_t
\end{pmatrix}
$$

Here $H_i$ are the parity check $m \times \frac{q^m - 1}{q - 1}$ matrix for the $[\frac{q^m - 1}{q - 1}, \frac{q^m - 1}{q - 1} - m]_q$ Hamming code. Let $C_1$ be the linear $[n = \frac{(q^m - 1)}{q - 1}, k = \frac{q^m - 1}{q - 1} - tm]_q$ code with the above parity check matrix $H$. This is the covering code in the $b$-symbol metric used in our above observation. From the equality

$$R_b(C_1) = \max_{v \in \mathbb{F}_q^v} \{wt_{b, coset}(v + C_1)\},$$

it holds $R_b(C_1) \leq bt = \frac{b(n-k)}{m}$, if $b \leq q^m - 1$. Actually, each vector in $\mathbb{F}_q^{n-k}$ can be represented as the sum of $t$ columns in the parity check matrix $H$.

The balls in the $b$-symbol metric centered at codewords of $C_1$ with the radius $bt$ cover the whole space, since $R_b(C_1) \leq bt$. Each such ball contains at most $L$ codewords of $C$. The conclusion follows directly.

Theorem 4.1 improves Proposition 3 significantly, since we take a better auxiliary covering code $C_1$. Generally if a covering code with the smaller size can be found and used in the above simple observation, a better generalized Singleton bound can be obtained.

When $\frac{m}{b}$ is larger than $2 + \epsilon$ and $b$ is small (for example $b = 2$), where $\epsilon$ is any small positive real number, the bound in Theorem 4.1 is better than the Singleton bound in Section I, see [4], when the size $L = 1$. Actually, we have $\frac{m}{b} d = \frac{m}{b} \left[ d_b(C) - 1 \right] > d_b(C)$. Theorem 4.1 is the generalized Singleton bound for list-decodable $b$-symbol codes, which can be argued simply from a covering $b$-symbol metric code.

Corollary 9: Let $q$ be a prime power and $n$ be a positive integer satisfying $n = \frac{q^m - 1}{q - 1} + v$, where $t$ is a positive integer and $0 \leq u \leq \frac{q^m - 1}{q - 1} - 1$. Let $C \subset \mathbb{F}_q^n$ be a length $n$ symbol-pair code with the minimum pair distance $D \geq 4t + 1$. Then

$$|C| \leq q^{n - 5t}.$$  

Proof: We take the auxiliary covering code $C_2 = C_1 \times \mathbb{F}_q^v$ where $C_1$ is the covering code in the proof of Theorem 4.1 for $m = 5$. Then $R_2(C_2) \leq R_2(C_1)$, since the last $v$ coordinates of codewords in $C_2$ are free. Hence $R_2(C_2) \leq R_2(C_1) \leq 2t$ from the proof of Theorem 4.1. Then balls in the pair metric centered at codewords of $C_2$ with the radius $2t$ cover the whole space. Moreover each such ball centered at one codeword of $C_2$ with the radius $2t \leq \frac{D - 1}{2}$ contains at most one codeword of $C$. Then $|C| \leq |C_2| = q^{n} \cdot |C_1| = q^{(q^m - 1)v + v - 5t} = q^{n - 5t}$. The conclusion is proved.

The above results assert that for a given minimum pair-distance when the code length is long, the above generalized Singleton bound $|C| \leq q^{n - 5t}$ is much stronger than the Singleton bound $|C| \leq q^{n - D + 2}$ in [4] on the sizes of the symbol-pair codes.

Since the construction of MDS symbol-pair codes in two papers [4], [25], there have been many constructions of the MDS symbol-pair codes in [5], [9], [10], [11], and [26] from cyclic or constacyclic codes. However no upper bound on the lengths of general MDS symbol-pair codes has been given. It was proved in [11] that the length of a linear MDS symbol-pair code over $\mathbb{F}_q$ of the minimum pair-distance 5 or 6 can not be larger than $q^2 + q + 1$ or $q^2$. From Corollary 4 we give the following upper bound on lengths of (even no linear) MDS symbol-pair codes.

Corollary 10: Let $C \subset \mathbb{F}_q$ be a symbol-pair code with the length $n$ and the minimum pair distance $D$. Suppose that $n$ and $D$ satisfy $D > 17$ and $n > \frac{(D-1)(q^2-1)}{4(q-1)}$. Then the symbol-pair code $C$ is not MDS. In particular there is no MDS symbol-pair binary code with the minimum pair distance $D > 17$ and the length $n \geq \frac{3(D-1)}{4}$.

Proof: Set $t = \frac{D - 1}{2}$. If $n \geq \frac{(D-1)(q^2-1)}{4(q-1)}$ and $D > 17$, then $|C| \leq q^{n - 5t} < q^{n - D + 2}$ from Corollary 8. The conclusion is proved.

If smaller covering codes in the $b$-symbol metric could be found, then Theorem 4.1 and Corollary 3 could be improved. To the best of our knowledge, Corollary 9 is the first general upper bound on lengths of MDS symbol-pair codes.

V. CONCLUSION

In this paper, covering codes and covering radii in the $b$-symbol metric of some famous linear codes are considered. First of all some results about covering codes in the Hamming metric, such as the Delosart bound and the Norse bound, do not hold in the pair metric. Some highly nontrivial upper and lower bounds on covering radii of some cyclic, constacyclic and algebraic geometry codes in the $b$-symbol metric are given. The covering radius of the Reed-Solomon code as a $b$-symbol metric code is determined. As an application, the generalized Singleton bound on list-decodable $b$-symbol metric codes is proved and an upper bound on lengths of general MDS symbol-pair codes is given. We give a simple sufficient condition for non-perfect code in the $b$-symbol metric and prove that many well-known codes are not perfect in the $b$-symbol metric. It is an interesting open problem to classify all perfect codes in $b$-symbol metric for small $b = 2$ or 3. In particular is there any perfect symbol-pair code of the minimum pair-distance bigger than or equal 9?

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