Maurer-Cartan Forms and Equations for Two-Dimensional Superdiffeomorphisms

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Abstract

We present explicit expressions for the Maurer-Cartan forms of the superdiffeomorphism group associated to a super Riemann surface. As an application to superconformal field theory, we use these forms to evaluate the effective action for the factorized superdiffeomorphism anomaly.

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1 Introduction

Maurer-Cartan (MC) forms and equations are familiar to mathematicians and physicists alike: they are used for instance to define connections on homogenous spaces \[1\] or to construct field theories on group manifolds \[2\]. While the case of finite-dimensional Lie groups is well known, the infinite-dimensional theory is less familiar. An example for the latter is provided by the group of diffeomorphisms of a smooth manifold \[3\]. The particular case of $C^\infty$-diffeomorphisms on a Riemann surface has recently been investigated \[4\]. The corresponding results have been applied to evaluate the Wess-Zumino (WZ) action associated to the chirally split diffeomorphism anomaly on the complex plane \[4\]; this anomaly occurs in conformal models and is equivalent to the Weyl anomaly \[3\]. It is quite remarkable that the WZ functional can be explicitly determined in this case despite the fact that the symmetry group is non-Abelian. (E.g., for non-Abelian gauge theories in four dimensions, a complete evaluation of the WZ action has not been achieved to date.) The computational success in the two-dimensional case ultimately relies on the low dimensionality of the underlying space-time manifold.

Some time ago, the MC-forms and equations have been generalized to finite-dimensional supergroups, e.g. to the graded Poincaré group \[6\] \[2\]. In the present paper, we consider the infinite-dimensional case and, more specifically, the superdiffeomorphism group associated with a $N = 1$ super Riemann surface (SRS). Along the lines of the bosonic theory \[4\], we determine the corresponding MC-forms and we use them to construct the WZ action associated to the chirally split superdiffeomorphism anomaly on the complex superplane. This anomaly has been known for some time \[7\] and the explicit expression for the associated WZ action has previously been postulated in analogy to the results of the non-supersymmetric theory \[8\].

As to the organization of the paper, we first discuss the case of generic SRS’s and subsequently consider the limitation of supercomplex structures which is frequently chosen for SRS’s (section 3).

2 General case

2.1 Supercomplex structures

Let us first recall the basic facts which are needed in the sequel (see \[7\] and references therein). A SRS is locally parametrized by coordinates $(Z, \bar{Z}, \Theta, \bar{\Theta})$. The super 1-forms $e^Z \equiv dZ + \Theta d\Theta$, $e^\Theta \equiv d\Theta$ (and c.c.) which span the cotangent space to the SRS can be expressed with respect to a reference system $e^z \equiv dz + \theta d\theta$, $e^\theta \equiv d\theta$ (and c.c.) by

$$e^Z = \left[ e^z + e^\theta H^z_{\bar{z}} + e^\theta H^{\bar{z}}_{\bar{z}} \right] \Lambda$$  \hspace{1cm} (1)
\( e^\Theta = \left[ e^z + e^\bar{z} H^z_{\bar{z}} + e^\theta H^\theta_{\bar{z}} + e^\bar{\theta} H^\theta_{\bar{z}} \right] \tau + \left[ e^\theta H^\theta_{\bar{z}} + e^\bar{\theta} H^\theta_{\bar{z}} + e^\bar{\theta} H^\theta_{\bar{z}} \right] \sqrt{\Lambda} \)

(and the complex conjugate expressions). The structure equations \( de^z + e^\Theta e^\Theta = 0 = de^\bar{z} \) (and c.c.) imply that the odd superfields \( H^z_{\bar{z}}, H^\theta_{\bar{z}} \) (and c.c.) are the only independent Beltrami coefficients and that all the others depend on them and their derivatives. Furthermore, these equations imply that the factor \( \tau \) depends on the integrating factor \( \Lambda \) and that \( \Lambda \) satisfies the linear differential equation

\[
\tilde{D} \ln \Lambda = \partial H^z_{\bar{z}} - \frac{H^\theta_{\bar{z}}}{H^\theta_{\bar{z}}} \partial H^\theta_{\bar{z}} .
\]  (2)

Here, \( \tilde{D} \) is a linear differential operator whose coefficients are functions of \( H^z_{\bar{z}}, H^\theta_{\bar{z}} \) and their derivatives; the canonical derivations in superspace are denoted by

\[
\partial \equiv \frac{\partial}{\partial z} , \quad D \equiv \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z} , \quad (D^2 = \partial) \quad \text{and c.c.} .
\]

The superfield \( H^z_{\bar{z}} \) admits a component field expansion

\[
H^z_{\bar{z}} = \sigma^z_{\bar{z}} + \theta v^z + \bar{\theta} \mu^z_{\bar{z}} + \theta \bar{\theta} [-i\alpha^\theta_{\bar{z}}] ,
\]  (3)

where \( \mu^z_{\bar{z}}(z, \bar{z}) \) and \( \alpha^\theta_{\bar{z}}(z, \bar{z}) \) denote the usual Beltrami coefficient and its fermionic partner, respectively.

Infinitesimal superdiffeomorphisms generated by the vector field

\[
\Xi \cdot \partial \equiv \Xi^z(z, \bar{z}, \theta, \bar{\theta}) \partial + \Xi^\theta(z, \bar{z}, \theta, \bar{\theta}) \tilde{D} + \Xi^\bar{\theta}(z, \bar{z}, \theta, \bar{\theta}) \bar{D}
\]

act on the basic variables according to

\[
s \Theta = i_{\Xi, \partial} d\Theta = i_{\Xi, \partial} e^\Theta = C^z \tau + C^\theta \sqrt{\Lambda}
\]

\[
s Z = i_{\Xi, \partial} dZ = i_{\Xi, \partial} \left[ e^z - \Theta e^\Theta \right] = C^z \Lambda - \Theta (s \Theta) .
\]  (4)

Here, \( i_{\Xi, \partial} \) denotes the interior product with the vector field \( \Xi \cdot \partial \) and the parameters

\[
C^z \equiv \Xi^z + \Xi^\theta H^\theta_{\bar{z}} + \Xi^\bar{\theta} H^\theta_{\bar{z}} + \Xi^\bar{\theta} H^\theta_{\bar{z}}
\]

\[
C^\theta \equiv \Xi^\theta H^\theta_{\bar{z}} + \Xi^\bar{\theta} H^\theta_{\bar{z}} + \Xi^\bar{\theta} H^\theta_{\bar{z}}
\]  (5)

are supposed to represent ghost superfields; the nilpotency requirement for the BRS-operation \( s \) then leads to the transformation laws

\[
s C^z = - \left[ C^z \partial C^z + C^\theta C^\theta \right]
\]

\[
s C^\theta = - \left[ C^z \partial C^\theta + \frac{1}{2} C^\theta (\partial C^z) \right] .
\]  (6)

The induced variations of the independent Beltrami fields and of the integrating factor read

\[
s H^z_{\bar{z}} = (\tilde{D} - H^z_{\bar{z}} \partial) C^z + (\partial H^z_{\bar{z}}) C^z - 2 H^\theta_{\bar{z}} C^\theta
\]

\[
s H^\theta_{\bar{z}} = (D - H^z_{\bar{z}} \partial) C^z + (\partial H^z_{\bar{z}}) C^z - 2 H^\theta_{\bar{z}} C^\theta
\]

\[
s \Lambda = \partial (C^z \Lambda) + 2 C^\theta \tau \sqrt{\Lambda} .
\]  (7)
2.2 Maurer-Cartan form for the superdiffeomorphism group

In this section, we determine the MC 1-form associated to the supergroup $\text{Diff}_0(\mathbf{S}\Sigma)$, i.e. the superdiffeomorphisms on a (compact) SRS $\mathbf{S}\Sigma$ which are homotopic to the identity.

By virtue of eqs.(4), the infinitesimal superdiffeomorphisms act on the local coordinates $(Z, \bar{Z}, \Theta, \bar{\Theta})$ according to

\[
sZ = C^z \Lambda - \Theta s\Theta \quad \text{and c.c.} \quad \quad (8)
\]

\[
s\Theta = C^\theta \tau + C^\theta \sqrt{\Lambda} \quad \text{and c.c.} \quad .
\]

These relations can be ‘solved’ for the ghost fields $C^z$ and $C^\theta$, respectively:

\[
C^z = \Lambda^{-1} \left[ sZ + \Theta s\Theta \right] \quad \text{and c.c.} \quad (9)
\]

\[
C^\theta = \Lambda^{-3/2} \left[ (\Lambda + \Theta \tau) s\Theta - \tau sZ \right] \quad \text{and c.c.} \quad .
\]

The integrating factors are locally given by $\Lambda = \partial Z + \Theta \partial \Theta$, $\tau = \partial \Theta$ and the induced variations of $C^z$, $C^\theta$ are those given by eqs.(6).

From the previous considerations, we can deduce the components $\Omega^z$, $\Omega^\theta$ (and c.c.) of the Maurer-Cartan 1-form associated to the group $\text{Diff}_0(\mathbf{S}\Sigma)$. To do so, we rely on the fact [9] [4] that there is a canonical morphism between the BRS algebra (with differential $s$ and generators $C^z, C^\theta, C^\bar{z}, C^\bar{\theta}$) and the algebra of differential forms on $\text{Diff}_0(\mathbf{S}\Sigma)$ (with differential $\delta$ and generators $\Omega^z, \Omega^\theta, \Omega^\bar{z}, \Omega^\bar{\theta}$). This morphism is realized by taking eqs.(9) and substituting $s$ by $\delta$ and $Z, \Theta$ by $Z^\varphi, \Theta^\varphi$ with $\varphi \in \text{Diff}_0(\mathbf{S}\Sigma)$:

\[
\Omega^z = (\partial Z^\varphi + \Theta^\varphi \partial \Theta^\varphi)^{-1} \left[ \delta Z^\varphi + \Theta^\varphi \delta \Theta^\varphi \right] \quad \text{and c(d0)}
\]

\[
\Omega^\theta = (\partial Z^\varphi + \Theta^\varphi \partial \Theta^\varphi)^{-3/2} \left[ (\delta Z^\varphi + 2\Theta^\varphi \partial \Theta^\varphi) \delta \Theta^\varphi - (\partial \Theta^\varphi) \delta Z^\varphi \right] \quad \text{and c.c.} \quad .
\]

By construction (cf.eqs.[3]), these 1-forms satisfy the Maurer-Cartan equations

\[
\delta \Omega^z = - \left[ \Omega^z \partial \Omega^z + \Omega^\theta \Omega^\theta \right] \quad \text{and c.c.} \quad (11)
\]

\[
\delta \Omega^\theta = - \left[ \Omega^z \partial \Omega^\theta + \frac{1}{2} \Omega^\theta \left( \partial \Omega^z \right) \right] \quad \text{and c.c.} \quad .
\]

For further mathematical details, we refer to [4].

2.3 Wess-Zumino action

Let $\int_{\mathbf{S}\Sigma} d^4z \ A(C^z, C^\theta; H_\bar{z}, H_{\bar{\theta}})$ + c.c. denote the chirally split superdiffeomorphism anomaly on $\mathbf{S}\Sigma$. According to the algebraic treatment of anomalies [9], the WZ action associated to this anomaly is given by

\[
\Gamma_{WZ}[\varphi; H_\bar{z}, H_{\bar{\theta}}] + \text{c.c.} = - \int_0^1 \int_{\mathbf{S}\Sigma} d^4z \ A \left( \Omega^z_t, \Omega^\theta_t; (H_\bar{z})^\varphi_t, (H_{\bar{\theta}})^{\bar{\varphi}}_t \right) + \text{c.c.} \quad ,
\]

(12)
where $\varphi \in \text{Diff}_0(\Sigma \Sigma)$ transforms like $s(H_\theta^z)^\varphi = 0$ or, equivalently, $s \varphi = \Xi \cdot \partial \varphi$ and where one has by construction

$$s \Gamma_{WZ}[\varphi; H_\theta^z, H_\theta^\bar{z}] = \int_0^1 \int_{\Sigma \Sigma} d^4z \, A(C^z, C^\theta; H_\theta^z, H_\theta^\bar{z}) .$$

In the expression (12), $\varphi_t$ denotes a smooth family of superdiffeomorphisms which interpolates between the identity and $\varphi$. The 1-form $\Omega_t$ is obtained from the MC 1-form $\Omega$ by replacing $\delta$ by $d_t \equiv dt \partial/\partial t$ and $\varphi$ by $\varphi_t$. The $'H^\varphi_t'$ follow from the $'H'$ by the action of the finite supercoordinate transformation $\varphi_t$.

The integral (12) is rather complicated, even if we consider the superplane, $\Sigma \Sigma = \Sigma C$, and if we restrict the geometry by $H_\theta^z = 0 = H_\theta^\bar{z}$ (see next section): in this case, the factorized superdiffeomorphism anomaly is explicitly given by

$$\int_{\Sigma C} d^4z \, A(C^z; H_\theta^z) + \text{c.c.} = \int_{\Sigma C} d^4z \, C^z \partial^2 DH_\theta^z + \text{c.c.} . \quad (13)$$

In section 3.2, we will work out the expression (12) for this case.

### 3 Restriction of the geometry ($H_\theta^z = 0$)

#### 3.1 Geometric framework

For $H_\theta^z = 0$ and

$$0 = sH_\theta^z = DC^z - 2C^\theta ,$$

the $s$-variations of the basic variables reduce to

$$sH_\theta^z = \left[ \bar{D} - H_\theta^z \partial + \frac{1}{2} (DH_\theta^z) D \right] C^z + (\partial H_\theta^z) C^z \quad (14)$$

$$sC^z = - \left[ C^z \partial C^z + \frac{1}{4} (DC^z)^2 \right]$$

$$s\Lambda = C^z \partial \Lambda + \frac{1}{2} (DC^z) D \Lambda + \Lambda \partial C^z .$$

Moreover, the integrating factor equation, eq.(3), takes the simple form

$$\left[ \bar{D} - H_\theta^z \partial + \frac{1}{2} (DH_\theta^z) D \right] \Lambda = (\partial H_\theta^z) \Lambda . \quad (15)$$

From the local form of $H_\theta^z$ [7], i.e. $H_\theta^z = (\partial Z + \Theta \partial \Theta)^{-1} (DZ - \Theta D \Theta)$, and the condition $H_\theta^z = 0$, we deduce

$$DZ = \Theta D \Theta$$

$$(D \Theta)^2 = \partial Z + \Theta \partial \Theta , \quad (16)$$

where the second equation follows from the first one by application of $D$. It should be emphasized that $Z$ and $\Theta$ still depend on both $z, \theta$ and $\bar{z}, \bar{\theta}$.
For consistency, we only consider those superdiffeomorphisms
\[(z, z', \theta, \bar{\theta}) \rightarrow (z'(z, \bar{z}, \theta, \bar{\theta}), z'(z, \bar{z}, \theta, \bar{\theta}), \theta'(z, \bar{z}, \theta, \bar{\theta}), \bar{\theta}'(z, \bar{z}, \theta, \bar{\theta}))\]
which respect the condition \(H_{\bar{\theta}z} = 0\), i.e. satisfy
\[
0 = \left[ Dz' - \theta' D\theta' \right] + \left[ D\bar{z}' - \bar{\theta}' D\bar{\theta}' \right] H_{\bar{\theta}z}' + \left[ D\bar{\theta}' \right] H_{\bar{\theta}z}' .
\tag{17}
\]

### 3.2 Construction of the WZ action on SC

For \(H_{\bar{\theta}z} = 0\), the MC equations, eqs.\([\color{blue}11]\), reduce to
\[
\delta \Omega^z = - \left[ \Omega^z \partial \Omega^z + \frac{1}{4} (D\Omega^z) (D\Omega^z) \right] \quad \text{and c.c.} ,
\tag{18}
\]
with
\[
\Omega^z = \frac{\delta Z^\varphi + \Theta^\varphi \delta \Theta^\varphi}{\partial Z^\varphi + \Theta^\varphi \partial \Theta^\varphi} = \frac{\delta Z^\varphi - (\delta \Theta^\varphi) \Theta^\varphi}{(D\Theta^\varphi)^2}.
\]

Furthermore, the WZ action, eq.\([\color{blue}12]\), becomes
\[
\Gamma_W^z[\varphi; H_{\bar{\theta}z}] + \text{c.c.} = - \int_0^1 \int_{\text{sc}} d^4z \ A(\Omega_i^z; (H_{\bar{\theta}z})^{\varphi t}) + \text{c.c.} .
\tag{19}
\]
Here, \(A\) is given by eq.\([\color{blue}13]\) and
\[
(H_{\bar{\theta}z})^{\varphi t} = \frac{\bar{D}Z^\varphi - \Theta^\varphi \bar{D}\Theta^\varphi}{\partial Z^\varphi + \Theta^\varphi \partial \Theta^\varphi} = \frac{\bar{D}Z^\varphi - (\bar{D}\Theta^\varphi) \Theta^\varphi}{(D\Theta^\varphi)^2} ,
\tag{20}
\]
where the superdiffeomorphism \(\varphi(z, \bar{z}, \theta, \bar{\theta}) \equiv (z', \bar{z}', \theta', \bar{\theta}')\) is subject to the condition \([\color{blue}17]\). The evaluation of the functional \([\color{blue}19]\) is fairly lengthy; roughly, it proceeds along the lines of the bosonic theory \([\color{blue}4]\), though some new arguments have to be invoked due to the anticommuting variables. Before discussing the calculation, we state the final result:
\[
-2 \Gamma_W^z[\varphi; H_{\bar{\theta}z}] = \int_{\text{sc}} d^4z \ (H_{\bar{\theta}z})^{\varphi t} \partial D \ln \Lambda^\varphi - \int_{\text{sc}} d^4z \ H_{\bar{\theta}z} \partial D \ln \Lambda
\tag{21}
\]
with \(\Lambda \equiv \partial Z + \Theta \partial \Theta = (D\Theta)^2\).

Let us now sketch the main steps of the derivation. By virtue of eq.\([\color{blue}13]\),
\[
\int_{\text{sc}} d^4z \ A(\Omega_i^z; (H_{\bar{\theta}z})^{\varphi t}) = \int_{\text{sc}} d^4z \ \Omega_i^z \ \partial^2 D (H_{\bar{\theta}z})^{\varphi t}
\tag{22}
\]
\[
= \frac{1}{2} \int_{\text{sc}} d^4z \ \left\{ \Omega_i^z \ \partial^2 D (H_{\bar{\theta}z})^{\varphi t} + (H_{\bar{\theta}z})^{\varphi t} \ \partial^2 D \Omega_i^z \right\} ,
\]
henceforth we consider the symmetrized integrand
\[
A(\Omega; H_{\bar{\theta}z}) = \frac{1}{2} \left[ \frac{\delta Z - (\delta \Theta) \Theta}{(D\Theta)^2} \partial^2 D \frac{DZ - (D\Theta) \Theta}{(D\Theta)^2} + (\delta \leftrightarrow \bar{D}) \right] .
\tag{23}
\]
Here and in the following, we have suppressed the suffixes \( t \) and \( \varphi \) in order to simplify the notation. Note that \( \delta \) and \( D \) cannot simply be exchanged, because the first is an even and the second an odd operator (e.g. \( \delta D = D\delta \), but \( \bar{D}D = -D\bar{D} \)); thus, the two contributions in eq. (23) have to be evaluated separately.

Our aim is to show that

\[
A(\Omega;H^{\bar{z}}) = \delta \left[ H^{\bar{z}} \partial D \ln D \Theta \right] + \partial \left[ ... \right] + \bar{D} \left[ ... \right] + D \left[ ... \right] + \bar{D} \left[ ... \right] , \tag{24}
\]

which relation immediately yields the result (21) by integration over the suppressed variable \( t \).

First, we evaluate \( \partial^2 D \left[ (D\Theta)^{-2}(\bar{D}Z - (D\Theta)\Theta) \right] \) and express all derivatives \( DZ \) and \( \partial Z \) in terms of \( D\Theta \) and \( \partial \Theta \) by virtue of eqs. (16). For the second term in eq. (23), we proceed in a similar way. Thus, one is led to

\[
- A(\Omega;H^{\bar{z}}) = \frac{\delta Z - (\delta \Theta)\Theta}{(D\Theta)^2} \frac{\bar{D}(\partial^2 \Theta)}{D\Theta} - \frac{\bar{D}Z - (D\Theta)\Theta}{(D\Theta)^2} \frac{\delta (\partial^2 \Theta)}{D\Theta} \tag{25}
\]

\[
+ \text{terms in } (D\Theta)^{-4}, ..., (D\Theta)^{-7} .
\]

According to the Leibniz rule, the first term on the r.h.s. can be rewritten as

\[
\bar{D} \left[ \frac{\delta Z - (\delta \Theta)\Theta}{(D\Theta)^2} \frac{\partial^2 \Theta}{D\Theta} \right] - \bar{D} \left[ \frac{\delta Z - (\delta \Theta)\Theta}{(D\Theta)^3} \right] \partial^2 \Theta .
\]

In this expression, we replace \( \partial^2 \Theta / D\Theta \) by the super Schwarzian derivative [10]:

\[
S(Z,\Theta ; z,\theta) \equiv \partial D \ln D \Theta - (D \ln D \Theta) (\partial \ln D \Theta) \tag{26}
\]

\[
= \frac{\partial^2 \Theta}{D\Theta} - 2 \frac{(\partial \Theta)(\partial D \Theta)}{(D\Theta)^2} .
\]

Proceeding in this way, one obtains a formula for \(-A\) which involves a lengthy expression; however the latter can be rewritten in a compact way by virtue of the following useful equations:

\[
-2 \frac{\delta Z - (\delta \Theta)\Theta}{(D\Theta)^2} \frac{\bar{D}Z - (D\Theta)\Theta}{(D\Theta)^2} \left[ \frac{\partial^2 \Theta}{D\Theta} - 3 \frac{(\partial^2 D \Theta)(\partial \Theta)}{(D\Theta)^2} - 6 \frac{(\partial D \Theta)(\partial^2 \Theta)}{(D\Theta)^2} + 12 \frac{(\partial D \Theta)^2(\partial \Theta)}{(D\Theta)^3} \right] \]

\[
= -2 \delta Z - (\delta \Theta)\Theta \left[ \bar{D}Z - (D\Theta)\Theta \right] D \left\{ \frac{1}{D\Theta} D \left[ \frac{S}{(D\Theta)^3} \right] \right\} \tag{27}
\]

\[
= 4 \frac{\delta Z - (\delta \Theta)\Theta}{(D\Theta)^2} \frac{DD \Theta}{D\Theta} S + 4 \frac{\bar{D}Z - (D\Theta)\Theta}{(D\Theta)^2} \frac{\delta D \Theta}{D\Theta} S + D \left[ ... \right] .
\]

Here, we introduced the notation \( S \equiv S(Z,\Theta ; z,\theta) \) and we repeatedly used the Leibniz rule and eqs. (10) to pass to the last line. In conclusion, one finds the
intermediate result

\[- \mathcal{A}(\Omega; H_\theta^z) = \bar{D} \left[ \frac{\delta Z - (\delta \Theta) \Theta}{(D \Theta)^2} S \right] - \delta \left[ \frac{\bar{D}Z - (\bar{D} \Theta) \Theta}{(D \Theta)^2} S \right] + 2 \frac{\delta Z - (\delta \Theta) \Theta}{(D \Theta)^2} S - 2 \frac{\delta Z - (\delta \Theta) \Theta}{(D \Theta)^2} \frac{D \Theta}{D \Theta} \delta D \Theta \delta D \Theta \left[ \bar{D} - H_\theta^z \partial \right] \delta D \Theta \delta D \Theta \right] S + D[...] . \] 

Next, we consider the contribution

\[2 \frac{\bar{D}Z - (\bar{D} \Theta) \Theta}{(D \Theta)^2} \frac{D \Theta}{D \Theta} S = 2 H_\theta^z (\delta \ln D \Theta) S . \] 

In order to evaluate the r.h.s., we apply \( \delta \) to the Beltrami equation (20), i.e. to

\[\bar{D}Z - (\bar{D} \Theta) \Theta - H_\theta^z (D \Theta)^2 = 0 , \]

and obtain the relation

\[- 2 H_\theta^z (\delta \ln D \Theta) = \delta H_\theta^z - \frac{\delta [\bar{D}Z - (\bar{D} \Theta) \Theta]}{(D \Theta)^2} . \] 

Substituting this result and eq. (28) into the expression (29), we get

\[2 H_\theta^z (\delta \ln D \Theta) S = -(\delta H_\theta^z) \partial D \ln D \Theta + (\delta H_\theta^z) (\partial \ln D \Theta) (D \ln D \Theta) + ... . \]

A convenient form for the second term on the r.h.s. follows by application of \( \delta \) to the integrating factor equation (15):

\[(\delta H_\theta^z) \partial \ln D \Theta = \frac{1}{2} \partial \delta H_\theta^z + \frac{1}{2} (D \delta H_\theta^z) (D \ln D \Theta) + \left[ \bar{D} - H_\theta^z \partial + \frac{1}{2} (D H_\theta^z) D \right] \delta \ln D \Theta . \]

By substituting the previous expressions into eq. (28), one is led to the advocated equation (24) and thereby to the final result (21).

4 Concluding remarks

As noted in reference [8], the superdiffeomorphism \( \varphi \) can be further restricted by the condition \((H_\theta^z)_{\varphi} = 0\). Then, the WZ action (21) reduces to the so-called Wess-Zumino-Polyakov functional,

\[\Gamma_{WZP}[H_\theta^z] = \frac{1}{2} \int_{\text{sc}} d^4 z H_\theta^z \partial D \ln \Lambda . \]

This expression represents a very compact notation for the superspace generalization of Polyakov’s chiral gauge action [11] [8].
We note that the case of \((p, q)\) supersymmetry can be treated along the lines of our previous discussion. As an application, we explicitly constructed the WZ action associated to the superdiffeomorphism anomaly \(\int_{\text{SC}} d^3 z \, C^z \, \partial^2 D H_{\bar{z}}^z\) \(\text{[12]}\) in the \(z\)-sector of the \((1, 0)\) supersymmetric theory. In this case, one has to evaluate the expression

\[
A^{(1,0)}(\Omega ; H_{\bar{z}}^z) = \frac{1}{2} \left[ \frac{\delta Z + \Theta (\delta \Theta)}{(D \Theta)^2} \partial^2 D \frac{\partial Z + \Theta (\bar{\partial} \Theta)}{(D \Theta)^2} - (\delta \leftrightarrow \bar{\partial}) \right]. \tag{33}
\]

Since both \(\delta\) and \(\bar{\partial}\) are even, the second contribution simply follows by antisymmetrizing the result of the first one. From the intermediate result

\[
A^{(1,0)}(\Omega ; H_{\bar{z}}^z) = \left\{ \bar{\partial} \left[ \frac{\delta Z + \Theta (\delta \Theta)}{(D \Theta)^2} S \right] + 2 \left( \frac{\bar{\partial} Z + \Theta (\bar{\partial} \Theta)}{(D \Theta)^2} \frac{\delta D \Theta}{D \Theta} \right) \right. \tag{34}

\begin{align*}
&\left. - \frac{(\delta \Theta)(\bar{\partial} \Theta)}{(D \Theta)^2} S \right\} - (\delta \leftrightarrow \bar{\partial}),
\end{align*}
\]

one finds that

\[
2 \Gamma_{\text{WZ}}^{(1,0)}[\phi ; H_{\bar{z}}^z] = \int_{\text{SC}} d^3 z \left\{ (H_{\bar{z}}^z) \phi \partial D \ln \Lambda^\phi - H_{\bar{z}}^z \partial D \ln \Lambda \right\}. \tag{35}
\]

For \((H_{\bar{z}}^z)^\phi = 0\), this expression reduces to the previously considered WZP functional \(\text{[8]}\).

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