LOGARITHMIC INTERTWINING OPERATORS AND $\mathcal{W}(2, 2p - 1)$-ALGEBRAS

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ABSTRACT. For every $p \geq 2$, we obtained an explicit construction of a family of $\mathcal{W}(2, 2p - 1)$-modules, which decompose as direct sum of simple Virasoro algebra modules. Furthermore, we classified all irreducible self-dual $\mathcal{W}(2, 2p - 1)$-modules, we described their internal structure, and computed their graded dimensions. In addition, we constructed certain hidden logarithmic intertwining operators among two ordinary and one logarithmic $\mathcal{W}(2, 2p - 1)$-modules. This work, in particular, gives a mathematically precise formulation and interpretation of what physicists have been referring to as “logarithmic conformal field theory” of central charge $c_{p,1} = 1 - \frac{6(p-1)^2}{p}$, $p \geq 2$. Our explicit construction can be easily applied for computations of correlation functions. Techniques from this paper can be used to study the triplet vertex operator algebra $\mathcal{W}(2, (2p - 1)^3)$ and other logarithmic models.

0. INTRODUCTION

The Virasoro algebra is one of the most fundamental structures in two-dimensional conformal field theory. The most important family of Virasoro algebra modules are certainly the minimal models, because these models give rise to rational conformal field theories. Interestingly, many non-rational models have recently appeared in studies of $\mathcal{W}$-algebras, which are certain extensions of Virasoro vertex algebras. Since there are several different types of $\mathcal{W}$-algebras (see for instance [16] for $\mathcal{W}$-algebras of positive integer central charge), in this paper we limit ourselves to $\mathcal{W}$-algebras closely related to representations of Virasoro algebra with central charge

$$c_{p,1} = 1 - \frac{6(p-1)^2}{p}, \quad p \in \mathbb{N}_{\geq 2}.$$ 

These central charges, belonging to the boundary of Kac’s table, are relevant in logarithmic conformal field theory [13], [14], [20], [24]. If we denote by $L(c_{p,1}, 0)$ the simple lowest weight Virasoro algebra module of central charge $c_{p,1}$ and conformal weight zero, then we have the following embedding of $\mathcal{W}$-algebras

$$L(c_{p,1}, 0) \hookrightarrow \mathcal{W}(2, 2p - 1) \hookrightarrow \mathcal{W}(2, (2p - 1)^3), \quad p \geq 2,$$

where $\mathcal{W}(2, 2p - 1)$ is also known as the singlet $\mathcal{W}$-algebra, and $\mathcal{W}(2, (2p - 1)^3)$ is the triplet $\mathcal{W}$-algebra. The theory of $\mathcal{W}$-algebras could be understood much better from vertex algebra point of view. In this setup, the singlet vertex algebra $\mathcal{W}(2, 2p - 1)$ is generated by the Virasoro element $\omega$ and another element $H$ of conformal weight $2p - 1$ (cf. [2], [8], [25], [30]). The singlet vertex algebra admits infinitely many nonisomorphic irreducible modules, so it fails to be rational. On the other hand, the triplet algebra $\mathcal{W}(2, (2p - 1)^3)$ (cf. [28], [29]) is of the right size and its rationality was discussed in [21], [22] (some further studies of the triplet algebra were pursued in [19], [23] etc.). The triplet algebra has only finitely many equivalence classes of irreducible
modules, but in addition it admits certain indecomposable logarithmic modules (i.e., modules that admit nontrivial Jordan blocks with respect to the action of the degree zero Virasoro generator). These indecomposable modules are needed to obtain the fusion closure and the modular invariance of characters. These logarithmic modules are also responsible for logarithmic behavior of matrix coefficients. All these properties make the triplet algebra CFT a rather odd looking "rational CFT"—so in order to distinguish it from ordinary rational CFTs—the triplet model and related models are usually dubbed as rational logarithmic CFTs.

Many important aspects of logarithmic CFT can be studied by using algebraic techniques. For instance, the appearance of non-diagonalizable representations can be easily explained with the use of Zhu’s associative algebra [20], [33], [34]. Similarly, logarithmic behavior of correlation functions can be explained via logarithmic intertwining operators [26], [27], [33], [34] (see also [11] for a related approach). The most interesting examples of logarithmic intertwining operators are those of type

\[
\begin{pmatrix}
\text{logarithmic module} \\
\text{ordinary module} \\
\text{ordinary module}
\end{pmatrix}.
\]

In [33] the second author provided a general construction of intertwining operators that arise from certain deformations of bosonic vertex operators. These operators are not present in the original non-logarithmic theory, which is the main reason why we called them hidden.

Several clues from the physics literature indicate that the triplet algebra and other related models should involve hidden logarithmic intertwiners. In this paper we make the first step in the direction of understanding these operators. Here we consider the intermediate singlet vertex algebra \(W(2, 2p - 1)\), which will be denoted by \(\overline{M(1)}_p\) throughout the paper (cf. [2]). We prove several general results about \(W(2, 2p - 1)\)-algebras. Firstly, we show that \(\overline{M(1)}_p\) is a simple vertex algebra (cf. Theorem 3.5). Secondly, we construct a distinguished irreducible \(\overline{M(1)}_p\)-module denoted by \(M(1, \beta)\), which decomposes further as a direct sum of simple Virasoro algebra modules (cf. Theorem 5.4). Then we classify all irreducible self-dual \(\overline{M(1)}_p\)-modules and we compute their graded dimensions (cf. Theorem 5.2). Furthermore, we construct a self-extension of \(M(1, \beta)\), which give rise to an indecomposable \(\overline{M(1)}_p\)-module (see Theorem 6.1). Finally, by using this self-extension and several results from [33] we obtain an explicit construction of a family of hidden intertwining operators of \(\overline{M(1)}_p\)-modules (cf. Corollary 11.1). Thus, we construct an algebraic counterpart of logarithmic conformal field theory of level \(c_{p,1}\). In a sequel we plan to study the triplet model and related models.

1. Feigin-Fuchs Modules

We shall introduce some notation first. We denote by \(h\) a one-dimensional abelian Lie algebra spanned by \(h\) with a bilinear form \(\langle \cdot, \cdot \rangle\), such that \(\langle h, h \rangle = 1\), and by

\[
\hat{h} = h \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c
\]

the affinization of \(h\) with bracket relations

\[
[a(m), b(n)] = m(a, b)\delta_{m+n,0}c, \quad a, b \in h,
\]

\[
[c, a(m)] = 0.
\]
Set $\hat{h}^+ = tC[t] \otimes \hat{h}; \quad \hat{h}^- = t^{-1}C[t^{-1}] \otimes \hat{h}$. Then $\hat{h}^+$ and $\hat{h}^-$ are abelian subalgebras of $\hat{h}$. Let $U(\hat{h}^-) = S(\hat{h}^-)$ be the universal enveloping algebra of $\hat{h}^-$. Let $\lambda \in \hat{h}$. Consider the induced $\hat{h}$-module

$$M(1, \lambda) = U(\hat{h}) \otimes_{U(\langle t \rangle) \otimes \hat{h} \otimes C_t)} C_\lambda \simeq S(\hat{h}^-) \text{ (linearly)},$$

where $tC[t] \otimes \hat{h}$ acts trivially on $C_\lambda = C$, $h$ acts as $\langle h, \lambda \rangle$, and $e$ acts as multiplication by $1$. For simplicity, we shall write $M(1)$ for $M(1, 0)$.

It is well-known that $M(1)$ has a vertex operator algebra structure and that each $M(1, \lambda)$ is a $M(1, 0)$-module [15], [31]. More precisely, there are infinitely many different (non-isomorphic) vertex operator algebra structures on $M(1)$, denoted by $M(1)_a, a \in \mathbb{C}$, where the conformal vector is chosen to be

$$(1.1) \quad \omega_a = \frac{h(-1)^21}{2} + ah(-2)1.$$ 

Similarly, each $M(1, \lambda)$, denoted now $M(1, \lambda)_a$, becomes an irreducible $M(1)_a$-module. Here the subscript $a$ indicates that Virasoro algebra acts differently; as vector spaces of course $M(1, \lambda) = M(1, \lambda)_a$.

It is a standard fact (which can be easily shown) that the vertex operator algebra $M(1)_a$ has central charge

$$c = 1 - 12a^2.$$ 

Also, $M(1, \lambda)_a$ is a Virasoro algebra module of lowest conformal weight

$$h_{\lambda} = \frac{1}{2}\lambda^2 - \lambda a.$$ 

The Virasoro algebra modules $M(1, \lambda)_a$ are usually called Feigin-Fuchs module [9]. Let us ignore for a moment the conformal structure and view $M(1, \lambda)$ only as a $\hat{h}$-module. If we denote by $M(1, \lambda)^\circ$ the contragradient $\hat{h}$-module of $M(1, \lambda)$ defined by using the anti-involution $\omega(h(n)) = -h(-n)$, then we have

$$M(1, \lambda)^\circ \cong M(1, -\lambda).$$ 

But if we use anti-involution $\omega(L(n)) = L(-n)$ and denote by $M(1, \lambda)^*_a$ the contragradient Virasoro module (or $M(1)_a$-module) the result is different as illustrated by the following lemma (cf. [9]). We will include the proof here for later purposes (see Section 6).

**Lemma 1.1.** We have the following isomorphism

$$M(1, \lambda)^*_a \cong M(1, 2a - \lambda)_a,$$

of Virasoro algebra modules (or $M(1)_a$-modules). In particular, if $a = \lambda$, then $M(1, \lambda)_a$ is self-dual.
Proof. From the formula for $L(-n)$ in terms of Heisenberg algebra generators $h(n)$, we have

$$\langle L(n) \cdot w', w \rangle = \langle w', L(-n)w \rangle$$

$$= \langle w', \left( \sum_{n \in \mathbb{Z}} : h(-m)h(-n + m) : - (n + 1)ah(-n) \right) w \rangle$$

$$= \langle \left( \sum_{n \in \mathbb{Z}} : h(m)h(n - m) : - (n - 1)ah(n) \right) w', w \rangle$$

$$= \langle \left( \sum_{n \in \mathbb{Z}} : h(m)h(n - m) : - (n + 1)ah(n) \right) w', w \rangle$$

$$= \langle \left( \sum_{n \in \mathbb{Z}} : h(m)h(n - m) : - (n - 1)ah(n) \right) w', w \rangle$$

$$= \langle \tilde{L}(n)w', w \rangle,$$

where $:\cdot:\cdot$ denotes the normal ordering,

$$\tilde{h}(n) = h(n) - 2a\delta_{n,0},$$

for all $n \in \mathbb{Z}$ and $\tilde{L}(-n)$ are Virasoro generators in terms of $\tilde{h}(n)$ generator. The map $h(n) \mapsto \tilde{h}(n)$ induces an automorphism of $\tilde{h}$, and with this new Heisenberg algebra generators the module is isomorphic to the dual of $M(1, \lambda - 2a)$, which is isomorphic to $M(1, 2a - \lambda)a$. □

The embedding structure of Feigin-Fuchs modules is well-known [9]. In the self-dual case it is particularly simple.

**Proposition 1.2.** As before, we let

$$\lambda_p = \frac{p - 1}{\sqrt{2p}}, \quad p \in \mathbb{N}_{\geq 2}.$$

The Feigin-Fuchs module $M(1, \lambda_p)$ is self-dual and completely reducible if and only if $a = \lambda_p$. Moreover, we have the following decomposition

$$M(1, \lambda_p) = \bigoplus_{n=0}^{\infty} L(c_{p,1}, h^p_n)$$

where $L(c_{p,1}, h^p_n)$ denotes irreducible lowest weight Virasoro module of central charge $c_{p,1}$ and lowest conformal weight

$$h^p_n = \frac{(2pn)^2 - (p - 1)^2}{4p}.$$

In the previous proposition $a = \lambda_p$, so the central charge is

$$c_{p,1} = 1 - \frac{6(p - 1)^2}{p}$$

and the lowest conformal weight of $M(1, \lambda_p)$ is

$$h^p_0 = -\frac{(p - 1)^2}{4p}.$$

Modules of central charge $c_{p,1}$ are also known as *logarithmic* minimal models in the physics literature (this should not to be confused with logarithmic modules that will appear later in the text).
2. **Virasoro Verma Modules of Central Charge $c_{p,1}$**

In the previous section we considered some special Feigin-Fuchs modules. Here we discuss closely related Verma modules. Their embedding structure is similar to those of Feigin-Fuchs modules (for a fixed $c$ and $h$ one has to “invert” one-half of embeddings in the Verma module to get the embeddings in the Feigin-Fuchs module with the same $c$ and $h$). As usual, we shall denote by $V(c, h)$ the Verma module of lowest conformal weight $h$ and central charge $c$, i.e.,

$$V(c, h) = \mathcal{U}(\text{Vir}) \otimes_{\mathcal{U}(\text{Vir})_{\geq 0}} \mathbb{C}v_{c, h},$$

where $L(n), n \geq 1$ acts trivially on the lowest weight vector $v_{c, h}$ and

$$L(0) \cdot v_{c, h} = hv_{c, h},$$
$$C \cdot v_{c, h} = cv_{c, h}.$$  

The Verma module is a cyclic $\mathbb{N}$-gradable Virasoro algebra module, where the grading is inherited from the action of $L(0)$. There exists a unique maximal submodule of $V(c, h)$, denoted by $V^1(c, h)$, not necessarily cyclic, such that $L(c, h) = V(c, h)/V^1(c, h)$ is irreducible. In fact, every irreducible lowest weight module of central charge $c$ and lowest conformal weight $h$ is isomorphic to $L(c, h)$.

Generically, Verma modules are irreducible and isomorphic to appropriate Feigin-Fuchs modules described in the previous section. For instance, for each $a, -a^2/2 \notin \mathbb{Q}$, we have an isomorphism $M(1, a, 1) \cong L(1 - 2a^2, -a^2/2)$. This class of representations will not be treated in our paper.

In addition to complete description of Feigin-Fuchs modules, Feigin and Fuchs classified all embeddings among Verma modules. Here is their result in the case of $c_{p,1}$ (cf. [7]):

**Proposition 2.1.** The Verma module $V(c_{p,1}, h)$ is reducible if and only if

$$h = h_{m,n} := \frac{(m - np)^2 - (p - 1)^2}{4p}, \quad n = 1, \quad m > 1.$$  

Moreover, for $m = kp$, $k \in \mathbb{N}$ and $n = 1$ we have the following chain of embeddings

$$V(c_{p,1}, h_{m,1}) \hookrightarrow V(c_{p,1}, h_{m+2p,1}) \hookrightarrow V(c_{p,1}, h_{m+4p,1}) \hookrightarrow V(c_{p,1}, h_{m+6p,1}) \hookrightarrow \cdots,$$

while for $1 \leq m \leq p - 1$ we have

$$V(c_{p,1}, h_{m,1}) \hookrightarrow V(c_{p,1}, h_{m+2p,1}) \hookrightarrow V(c_{p,1}, h_{m+4p,1}) \hookrightarrow V(c_{p,1}, h_{m+6p,1}) \hookrightarrow \cdots.$$  

From now until the end of this section we assume that $m$ is a multiple of $p$. From the previous proposition we clearly have

$$L(c_{p,1}, h_{m,1}) \cong V(c_{p,1}, h_{m,1})/V(c_{p,1}, h_{m+2p,1}).$$

Also, for every $r$ this yields a short exact sequence

$$(2.4) \quad 0 \rightarrow L(c_{p,1}, h_{m+(r+2)p,1}) \rightarrow V(c_{p,1}, h_{m+(r+2)p,1})/V(c_{p,1}, h_{m+(r+4)p,1}) \rightarrow L(c_{p,1}, h_{m+(r+4)p,1}) \rightarrow 0.$$  

Similarly, dual functor applied to $(2.4)$ yields

$$(2.5) \quad 0 \rightarrow L(c_{p,1}, h_{m+(r+2)p,1}) \rightarrow (V(c_{p,1}, h_{m+(r+2)p,1})/V(c_{p,1}, h_{m+(r+4)p,1}))^* \rightarrow L(c_{p,1}, h_{m+(r+4)p,1}) \rightarrow 0.$$  

$^1$Equivalently, we may assume that $n > 0$, $0 < m \leq p$
As in [34], [9] or [4] it is not hard to prove the following result, which we state without a proof.

Lemma 2.2. Let \( m \) be a multiple of \( p \). Extensions in (2.4) and (2.5) are the only nontrivial non-logarithmic extensions among two irreducible modules \( L(c, h) \) and \( L(c, h_{m+sp,1}) \).

In general there could be some logarithmic extensions between irreducible Virasoro algebra modules. For instance, the irreducible module \( L \) admits a nontrivial self-extension, which is logarithmic. It turns out that every non-logarithmic self-extension of \( L(c, h) \) is split exact. Indeed, suppose that \((W, \iota, \pi)\) is a self-extension of \( L(c, h) \) with the embedding map \( \iota \) and projection \( \pi \). Then we consider a preimage \( w = \pi^{-1}(v_2) \) in \( W \), where \( v_2 \) is the lowest weight vector in \( L(c, h) \).

It is easy to see that \( w \) is a lowest weight vector in \( W \) (this is not the case if \( W \) is logarithmic, because \( w \) can form a Jordan block with another vector). Thus, the submodule of \( W \) generated by \( w \) is a quotient of \( M(c, h) \). If \( v_1 \) is another lowest weight vector in \( L(c, h) \), then \( \iota(v_1) \) generates a copy of \( L(c, h) \) in \( W \). Now, \( \iota(L(c, h)) \cap U(Vir)w = 0 \), for if there is something nontrivial in the intersection this would contradict to either the irreducibility of \( L(c, h) \) or the exactness. Thus, \( \iota(L(c, h)) \oplus U(Vir_{\leq 0})w = W \) (\( W \) is generated by \( w \) and \( \iota(v_1) \)). Now, the kernel of \( \pi \) is precisely \( \iota(L(c, h)) \cong L(c, h) \), so \( U(Vir_{\leq 0})w \cong L(c, h) \), and the sequence is split exact.

3. Vertex operator algebra \( \overline{M(1)_p} \)

In what follows the vertex operator algebra \( M(1)_{\lambda_p} \) will be denoted by \( M(1)_p \), for simplicity. This vertex operator algebra has two important vertex subalgebras. First, \( \tau \)-invariant vertex subalgebra \( M(1)^+_p \), where \( \tau \) is the involution induced by \( \tau h(-n)1 = -h(-n)1 \) (see [7] for more about \( M(1)^+ \) and its modules). For \( \lambda_p = 0 \), \( M(1)^+ \) is in fact a vertex operator subalgebra of \( M(1) \) (the Virasoro element is fixed by \( \tau \)). Another subalgebra of interest is \( \overline{M(1)_p} \), in the physics literature usually denoted by \( \mathcal{W}(2, 2p−1) \) (the singlet \( \mathcal{W} \)-algebra). In this section we recall the definition of \( \overline{M(1)_p} \) and some structural results, following closely [2]. We will also present a result on simplicity of \( \overline{M(1)_p} \).

3.1. Definition of \( \overline{M(1)_p} \). In what follows we shall study a subalgebra of the vertex operator algebra \( V_L \) associated to the lattice \( L = \mathbb{Z}\alpha \), \( \langle \alpha, \alpha \rangle = 2p \), where \( p \in \mathbb{Z}_{\geq 2} \).

Let \( p \in \mathbb{Z}_{\geq 0} \), \( p \geq 2 \). Let \( L = \mathbb{Z}\alpha \) be a lattice of rank one with nondegenerate \( \mathbb{Z} \)-bilinear form \( \langle \cdot, \cdot \rangle \) given by

\[
\langle \alpha, \alpha \rangle = 2p.
\]

Let \( \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L \). Let \( \widehat{\mathfrak{h}} \) be as in Section 1. Extend the form \( \langle \cdot, \cdot \rangle \) on \( L \) to \( \mathfrak{h} \). As in Section 1 we shall denote by \( \widehat{\mathfrak{h}} \) an extended Heisenberg Lie algebra associated to \( \mathfrak{h} \), where

\[
\mathfrak{h} = \frac{\alpha}{\sqrt{2p}};
\]

and by \( M(1) \) the corresponding vertex algebra, which is also a level one \( \widehat{\mathfrak{h}} \)-module. Then \( M(1) \) is a vertex subalgebra of the generalized vertex algebra \( V_L \).

Consider the dual lattice \( \bar{L} \) of \( L \), so that \( \bar{L} = \mathbb{Z}\alpha/\mathbb{Z} \alpha \). Let \( Y \) be the vertex operator map that defines the generalized vertex algebra structure on \( V_L \). The vertex algebra \( V_L \) is then a vertex subalgebra of \( V_L \).
As in Section 1, we shall choose the following Virasoro element in $M(1) \subset V_L$:

$$\omega = \frac{1}{4p} \alpha(-1)^2 1 + \frac{p-1}{2p} \alpha(-2) 1.$$ 

In Section 1 the corresponding vertex algebra was denoted by $M(1)_{\lambda_p}$, but we shall write $M(1)_p$ for simplicity. The subalgebra of $M(1)_p$ generated by $\omega$ is isomorphic to the simple Virasoro vertex operator algebra $L(c_{p,1},0)$ where as before $c_{p,1} = 1 - 6\frac{(p-1)^2}{p}$. Let

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}.$$ 

The element $L(0)$ of the Virasoro algebra defines a $\mathbb{Z}_{\geq 0}$-gradation on $V_L$. In this article we shall consider $V_L$ as a $\mathbb{Z}_{\geq 0}$–graded vertex operator algebra of rank $c_{p,1}$. As in [2] define the following operators:

$$Q = e_0^a, \quad \tilde{Q} = e_{-1}^a,$$

where

$$e^a = 1 \otimes e^a \in V_L, \quad e^{-\tilde{\gamma}^{a}} = 1 \otimes e^{-\tilde{\gamma}^{a}} \in V_L,$$

$$Y(e^\gamma, x) = \sum_{n \in \mathbb{Z}} e^n x^{-n-1},$$

which denotes the Fourier expansion of $e^\gamma$. By using results from [2], we have

$$[Q, \tilde{Q}] = 0, \quad [L(n), Q] = [L(n), \tilde{Q}] = 0 \quad (n \in \mathbb{Z}).$$

Thus, the operators $Q$ and $\tilde{Q}$ are intertwining (or screening) operators among Virasoro algebra modules. In fact, the Virasoro vertex operator algebra $L(c_{p,1},0) \subset M(1)_p$ is the kernel of the screening operator $Q$ (cf. [2]). Define

$$\overline{M(1)_p} = \text{Ker}_{M(1)} \tilde{Q}.$$ 

Since $\tilde{Q}$ commutes with the action of the Virasoro algebra, we have

$$L(c_{p,1},0) \subset \overline{M(1)_p}.$$ 

This implies that $\overline{M(1)_p}$ is a vertex operator subalgebra of $M(1)_p$ in the sense of [17] (i.e., $\overline{M(1)_p}$ has the same Virasoro element as $M(1)_p$).

The following theorem describes the structure of the vertex operator algebra $\overline{M(1)_p}$ as a $L(c_{p,1},0)$–module.

**Theorem 3.1.** [2]

(i) The vertex operator algebra $\overline{M(1)_p}$ is a completely reducible Virasoro algebra module and the following decomposition holds:

$$\overline{M(1)_p} = \bigoplus_{n=0}^{\infty} U(Vir). u^{(n)} \cong \bigoplus_{n=0}^{\infty} L(c_{p,1}, n^2 p + np - n).$$

\footnote{This intertwining operator is not an intertwining operator among a triple of vertex algebra modules (the two are related though).}
where
\begin{equation}
(3.6) \quad u^{(n)} = Q^n e^{-na}.
\end{equation}

(ii) The vertex operator algebra $\overline{M(1)_p}$ is generated by $\omega$ and
\begin{equation}
(3.7) \quad H = Qe^{-\alpha}
\end{equation}
of conformal weight $2p - 1$.

### 3.2. Zhu’s algebra $A(\overline{M(1)_p})$.

We recall the definition of Zhu’s algebra for vertex operator algebras [39].

Let $(V,Y,1,\omega)$ be a vertex operator algebra. We shall always assume that
\begin{equation}
V = \prod_{n \in \mathbb{Z}_{\geq 0}} V_n, \quad \text{where } V_n = \{a \in V \mid L(0)a = nv\}.
\end{equation}

For $a \in V_n$, we shall write $\text{wt}(a) = n$.

For a homogeneous element $a \in V$ we define the bilinear maps $*: V \otimes V \to V$, $\circ: V \otimes V \to V$ as follows:
\begin{equation}
\begin{align*}
a * b & := \text{Res}_x Y(a,x) \frac{(1 + x)^{\text{wt}(a)}}{x} b = \sum_{i=0}^{\infty} \binom{\text{wt}(a)}{i} a_{i-1} b, \\
a \circ b & := \text{Res}_x Y(a,x) \frac{(1 + x)^{\text{wt}(a)}}{x^2} b = \sum_{i=0}^{\infty} \binom{\text{wt}(a)}{i} a_{i-2} b.
\end{align*}
\end{equation}

We extend $*$ and $\circ$ on $V \otimes V$ linearly, and denote by $O(V) \subset V$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$. The space $A(V)$ has an associative algebra structure, the Zhu’s algebra of $V$. For instance $A(M(1)_a)$ is isomorphic to a polynomial algebra in one variable. It is more difficult to prove

**Theorem 3.2.** [2] Zhu’s associative algebra $A(\overline{M(1)_p})$ is isomorphic to the commutative algebra $\mathbb{C}[x,y]/\langle P(x,y) \rangle$, where $\langle P(x,y) \rangle$ is the principal ideal generated by
\begin{equation}
(3.8) \quad P(x,y) = y^2 - \frac{(4p)^{2p-1}}{(2p-1)!^2} (x + \frac{(p-1)^2}{4p})^{p-2} \prod_{i=0}^{p-2} \left( x + \frac{i}{4p} (2p - 2 - i) \right)^2.
\end{equation}

The equation $P(x,y) = 0$ defines a genus zero algebraic curve, with a polynomial parametrization obtained in [2] (in a special case the same formula was previously obtained in [37, 38]).

Let
\begin{equation}
(3.9) \quad Y(H_{-1}1,x) = \sum_{n \in \mathbb{Z}} H_n x^{-n-1}.
\end{equation}

Since $\text{wt}(H_{-1}1) = 2p - 1$, it is sometimes more convenient to shift the index in $H_n$ and work with the generators $H(n) = H_{n+2p-2}$.

**Proposition 3.3.** Let $V$ be a finite-dimensional vector space and $P \in \mathbb{C}[x,y]$ as in (3.8). Assume that
\begin{equation}
M = M(1)_p \cdot V = \text{span}_\mathbb{C} \{ a_jv \mid a \in M(1)_p, v \in V, j \in \mathbb{Z} \}.
\end{equation}
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is a $\mathbb{Z}_{\geq 0}$-gradable $M(1)_p$-module generated by the vector space $V$ such that

$$L(m)v = \delta_{m,0} X \cdot v, \quad H(m)v = \delta_{m,0} Y \cdot v \quad v \in V, \quad m \in \mathbb{Z}_{\geq 0},$$

where $X, Y \in \text{End}(V)$. Then $P(X, Y) = 0$ as an operator on $V$.

**Proof.** Since $M$ is $\mathbb{N}$-gradable module, the top component of $M$ is an $A(M(1)_p)$-module. The rest follows from Theorem 3.2. □

If $V$ is one-dimensional, by slightly abusing language, we say that the module $M$ from Proposition 3.3 is a lowest weight $M(1)_p$-module with respect to the subalgebra $\text{Span}_\mathbb{C}\{L(0), H(0)\}$, and the lowest weight is $(x, y) \in \mathbb{C}^2$. Therefore the lowest weights have to satisfy the equation $P(x, y) = 0$.

3.3. **Simplicity of $\overline{M(1)_p}$**. For every $n \in \mathbb{Z}_{\geq 0}$ we define

$$Z_n = \text{Ker}_{M(1)_p} Q^{n+1}.$$

By using results from [2] we have:

$$Z_n = \bigoplus_{i=0}^{n} U(Vir). u^{(i)} \cong \bigoplus_{i=0}^{n} L(c_{p,1}, i^2p + ip - i).$$

**Lemma 3.4.** Assume that $n \geq 1$. There is $i \in \mathbb{Z}_{\geq 0}$ such that

$$0 \neq H_i u^{(n)} \in Z_{n-1}.$$

**Proof.** First we notice (see Lemma 4.3 below) that

$$Q^{2n+1} e^{-n\alpha} = 0 \quad (3.10)$$

Let $j \geq 0$. By using (3.6), (3.7), (3.10) and

$$Q(a_j b) = (Qa)_j b + a_j (Qb) \quad (a, b \in V_L),$$

we obtain

$$Q^n (H_j u^{(n)}) = \frac{1}{2n+1} Q^{2n+1} (e^{-\alpha} e^{-n\alpha}) = 0. \quad (3.11)$$

Therefore $H_j u^{(n)} \in Z_{n-1}$ for every $j \geq 0$. Assume now that $H_j u^{(n)} = 0$ for every $j \geq 0$. Since $u^{(n)}$ is a singular vector for the Virasoro algebra, we have that $\mathbb{C} u^{(n)}$ is the top level of the $\mathbb{Z}_{\geq 0}$-graded $\overline{M(1)_p}$-module $\overline{M(1)_p} \cdot u^{(n)}$. Therefore $\overline{M(1)_p} \cdot u^{(n)}$ is a lowest weight $\overline{M(1)_p}$-module with the lowest weight $(n^2p + np - n, 0)$. This is a contradiction since $P(n^2p + np - n, 0) \neq 0$ (see Corollary 3.3 and Section 6 of [2]). So there exists $j_1 \in \mathbb{Z}_{\geq 0}$ such that $H_{j_1} u^{(n)} \neq 0$. □

Let us recall (cf. [31]) that a vertex operator algebra $V$ is called simple if it has no proper left ideals. In fact, in the definition “left ideals” could be replaced by “right ideals” or “two-sided ideals” [31].

**Theorem 3.5.** The vertex operator algebra $\overline{M(1)_p}$ is simple.
Lemma 4.1. Following important lemma:

\[ u^{(n)} \in I \quad \text{for every } i < n_0. \]

Now Lemma 3.4 gives that there is \( I \notin I \cap Z_{n_0 - 1}. \) But this contradicts the relation (3.12). The proof follows.

\[ \square \]

4. SINGULAR VECTORS OF FEIGN-FUCHS MODULES

By using standard calculations in lattice vertex algebras (cf. [5], [6], [31], [36]) we obtain the following important lemma:

**Lemma 4.1.** In the generalized vertex algebra \( V_L \) the following formula holds:

\[
Y(e^\alpha, x_1) \cdots Y(e^\alpha, x_{2n}) = E^-(\alpha, x_1, \ldots, x_{2n})E^+(\alpha, x_1, \ldots, x_{2n})
\]

\[
\Delta_{2n}(x_1, \ldots, x_{2n})^{2p} u^{2n\alpha}(x_1 \cdots x_{2n})^2 \]

where

\[
E^\pm(\alpha, x_1, \ldots, x_{2n}) = \exp \left( \sum_{k=1}^{\infty} \frac{\alpha(\pm k)}{\pm k} (x_1^{\pm k} + \cdots + x_{2n}^{\pm k}) \right)
\]

and \( \Delta_{2n}(x_1, \ldots, x_{2n}) = \prod_{i < j}(x_i - x_j) \) is the Vandermonde determinant.

Let now \( i \in \{0, \ldots, p - 1\}. \) Define \( \gamma_i = \frac{i}{2p} \alpha \in V_L. \)

Define the operator \( A = e^{\alpha}_{p-1-i}. \) Note that in the case \( i = p - 1 \) we have that \( A = Q. \)

**Lemma 4.2.** We have

\[
A^{2n} e^{\gamma_i - n\alpha} = C_n e^{\gamma_i + n\alpha}
\]

where the nonzero constant \( C_n \) is \((-1)^{np(2np)}p^{n}.\)

**Proof.** First we notice that

\[
A^{2n} e^{\gamma_i - n\alpha} = \text{Res}_{x_1} \text{Res}_{x_2} \cdots \text{Res}_{x_{2n}} (x_1 \cdots x_{2n})^{p-i-1} (Y(e^\alpha, x_1) \cdots Y(e^\alpha, x_{2n}) e^{\gamma_i - n\alpha})
\]

By using Lemma 4.1 and (4.14) we have that

\[
A^{2n} e^{\gamma_i - n\alpha} = C_n e^{\gamma_i + n\alpha},
\]

where \( C_n \) is the constant term in the Laurent polynomial

\[
\Delta_{2n}(x_1, \ldots, x_{2n})^{2p}(x_1 \cdots x_{2n})^{-(2n-1)p}.
\]

Now Theorem 4.1 from [3] (famous Dyson’s conjecture) implies that \( C_n = (-1)^{np(2np)}p^{n}. \)

Recall that a vector in \( V_{L+\gamma_i} \) is called primary if it is a singular vector for the action of the Virasoro algebra.

Since \( e^{\gamma_i - n\alpha} \in V_{L+\gamma_i} \) is a primary vector for every \( n \in \mathbb{Z}_{\geq 0}, \) we have that

\[
Q^i e^{\gamma_i - n\alpha}
\]

is either zero or a primary vector.
Lemma 4.3. Assume that $n \geq 1$. We have:

1. $Q^{2n}e^{\gamma_i - n\alpha} \neq 0$.
2. $Q^{2n+j}e^{\gamma_i - n\alpha} = 0$ for $j > 0$.

Proof. The assertion (1) was proven [35] and [36]. The same assertion follows from Lemma 4.2 by using the fact that there exists $f \in \mathcal{U}(\mathfrak{h}^+)$, $f \neq 0$ such that

$$fQ^{2n}e^{\gamma_i - n\alpha} = A^{2n}e^{\gamma_i - n\alpha} \neq 0.$$

Note that for $j > 0$, $Q^{2n+j}e^{\gamma_i - n\alpha}$ is a singular vector of weight $h_i + 1 + 2n + 1$. But there are no (singular) vectors in the Feigin-Fuchs module $M(1) \otimes e^{\gamma_i + (n+j)\alpha}$ of this weight. \hfill \Box

Remark 1. The non-triviality of $Q^{2n}e^{\gamma_i - n\alpha}$ from Lemma 4.3 can be also proven by using methods developed in [9] and [10].

5. Irreducibility of Certain $M_p(1)$-Modules

Since $M(1)_p$ is a subalgebra of $M(1)_p$, we have that every $M(1)_p$-module $M(1, \lambda)_p$ carries a natural $M(1)_p$-module structure. In fact, in [2] it was proven that every $\mathbb{Z}_{\geq 0}$-graded irreducible $M(1)_p$-module is an irreducible subquotient of $M(1, \lambda)$ for certain $\lambda \in \mathfrak{h}$.

Define now

$$\beta = \frac{p-1}{2p}\alpha \in \bar{L}.$$

In this section we shall consider the $M(1)_p$-module $M(1, \beta) = M(1) \otimes e^\beta$. As we are about to see, this module is distinguished from several point of views. We will prove that $M(1, \beta)$ is an irreducible $M(1)_p$-module, where

$$M(1, \beta) = M(1, \lambda_p)_{\lambda_p}$$

is a self-dual Virasoro module studied in Section 2.

Firstly, in parallel with Section 3, we shall view $M(1, \beta)$ inside a module for the generalized lattice vertex algebra $V_L$. For these purposes let us consider $V_L$-module $V_{L+\beta}$ that contains $M(1, \beta)$. We shall now investigate the action of the operator $Q$ on $V_{L+\beta}$. Since operators $Q^j$, $j \in \mathbb{Z}_{>0}$, commute with the action of the Virasoro algebra, they are (again) intertwining (or screening) operators among Feigin-Fuchs modules inside the $V_L$-module $V_{L+\beta}$.

Next, we shall present a theorem describing the structure of the $M(1)$-module $M(1, \beta)$ as a module for the Virasoro vertex operator algebra $L(c_p, 1)$. The following theorem is just an improved version of Proposition 1.2 (see [9], [10]).

Theorem 5.1. For every $n \in \mathbb{Z}_{>0}$, the vector

$$v^{(n)} = Q^n e^{\beta - n\alpha}$$

is a non-trivial singular vector and

$$U(Vir)v^{(n)} \cong L(c_p, 1, h^n_p),$$

where $h^n_p = \frac{(2pn)^2 - (p-1)^2}{4p}$. 

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\( M(1, \beta) \) is a completely reducible \( L(c_{p,1}, 0) \)-module and we have the following decomposition

\[
M(1, \beta) = \bigoplus_{n=0}^{\infty} L(c_{p,1}, h_n^p).
\]

Theorem 5.1 immediately gives the following result.

**Proposition 5.2.** We have

\[
L(c_{p,1}, -\frac{(p-1)^2}{4p}) \cong \ker_{M(1,\beta)} Q.
\]

For every \( n \in \mathbb{Z}_{\geq 0} \) we define

\[
Z_n^\beta = \ker_{M(1,\beta)} Q^{n+1}.
\]

By using Lemma 4.3 and Theorem 5.1 we have:

\[
Z_n^\beta = \bigoplus_{i=0}^{n} U(Vir). Q^i e^{\beta - i \alpha} \cong \bigoplus_{i=0}^{n} L(c_{p,1}, h_i^p).
\]

**Lemma 5.3.**

(i) There is a nonzero constant \( C \in \mathbb{C} \) such that

\[
H_{j_0} v^{(n)} = C v^{(n+1)} + v', \quad v' \in Z_n^\beta
\]

where \( j_0 = -2np + p - 2 \).

(ii) Assume that \( n \geq 1 \). There is \( j_1 \in \mathbb{Z}_{\geq 0} \) such that

\[
H_{j_1} v^{(n)} \in Z_{n-1}^\beta, \quad H_{j_1} v^{(n)} \neq 0.
\]

**Proof.** By using Lemma 4.3 we have

\[
Q^{n+1}(H_{j_0} v^{(n)}) = \frac{1}{2n+1} Q^{2n+2} (e_{-\alpha}^{-1} e^{\beta - n \alpha}) = \frac{1}{2n+1} Q^{2n+2} (e^{\beta - (n+1) \alpha}) \neq 0.
\]

Therefore \( H_{j_0} v^{(n)} \in Z_{n+1}^\beta \setminus Z_n^\beta \). Next we notice that

\[
L(0) H_{j_0} v^{(n)} = ((2p-1) + h_n^p - j_0 - 1) H_{j_0} v^{(n)} = h_{n+1}^p H_{j_0} v^{(n)}.
\]

Since \( L(0) v^{(n+1)} = h_{n+1}^p v^{(n+1)} \), we conclude that there is a constant \( C, C \neq 0 \), such that \( H_{j_0} v^{(n)} = C v^{(n+1)} + v', v' \in Z_n^\beta \). This proves (i).

The proof of assertion (ii) is similar to that of Lemma 5.4. Let \( j > j_0 \). We have

\[
Q^n(H_j v^{(n)}) = \frac{1}{2n+1} Q^{2n+1} (e_{-\alpha}^{-1} e^{\beta - n \alpha}) \neq 0.
\]

Therefore \( H_j v^{(n)} \in Z_{n-1}^\beta \) for every \( j > j_0 \). Assume now that \( H_j v^{(n)} = 0 \) for every \( j \geq 0 \). Since \( v^{(n)} \) is a singular vector for the Virasoro algebra, we have that \( \mathbb{C} v^{(n)} \) is the top level of the \( \mathbb{Z}_{\geq 0} \)-graded \( M(1)_p \)-module

\[
\overline{M(1)_p} \cdot v^{(n)} = \text{span}_{\mathbb{C}} \{ a_i v^{(n)} | a \in \overline{M(1)_p}, i \in \mathbb{Z} \}.
\]

Therefore \( \overline{M(1)_p} \cdot v^{(n)} \) is a lowest weight \( \overline{M(1)_p} \)-module with the lowest weight \( (h_n^p, 0) \). This is a contradiction since \( P(h_n^p, 0) \neq 0 \) (see Proposition 5.3 and Section 6 of [2]). So there exists \( j_1 \in \mathbb{Z}_{\geq 0} \) such that \( H_{j_1} v^{(n)} \neq 0 \).

\( \square \)
**Theorem 5.4.** We have, 

\[ M(1, \beta) := M(1) \otimes e^\beta \]

is an irreducible \( \overline{M(1)}_p \)-module ( = \( \mathcal{W}(2, 2p - 1) \)-module).

**Proof.** By using Lemma 5.3(i) we see that \( v^{(n)} \in \overline{M(1)}_p \cdot e^\beta \) for every \( n \in \mathbb{Z}_{\geq 0} \). Now Theorem 5.1 implies that

\[ M(1, \beta) = \overline{M(1)}_p \cdot e^\beta, \; \text{i.e.,} \; e^\beta \text{ is a cyclic vector.} \]

Assume now that there is a \( \overline{M(1)}_p \)-submodule \( 0 \neq N \subset M(1, \beta) \). Then \( N \) is also \( L(0) \)-graded. One can easily show that \( v^{(n)} \in N \) for some \( n \in \mathbb{Z}_{\geq 0} \). Assume that \( N \neq M(1, \beta) \). Then there is \( n_0 \in \mathbb{Z}_{>0} \) such that 

\[ v^{(n_0)} \in N, \quad v^{(i)} \notin N \text{ for every } i < n_0. \]

Now Lemma 5.3(ii) gives that there is \( j_1 \in \mathbb{Z}_{\geq 0} \) such that \( 0 \neq H_{j_1} v^{(n_0)} \in N \cap \mathbb{Z}^\beta_{n_0-1} \). This contradicts the minimality of \( n_0 \). Therefore \( 1 \otimes e^\beta \in N \), which implies that \( N = M(1, \beta) \). \( \square \)

### 6. Indecomposable \( L(c_{p,1}, 0) \text{ and } \overline{M(1)}_p \)-modules

In the previous section we proved that \( M(1, \beta) \) is an irreducible \( \overline{M(1)}_p \)-module. In this part we construct a non-trivial self-extension of \( M(1, \beta) \) and describe this extension in terms of Virasoro algebra submodules. As in [34] we start from a two-dimensional vector space \( \Omega \) and define a \( \mathfrak{h} \)-module

\[ M(1)_p \otimes \Omega, \]

where \( h(0)|_\Omega \) act as

\[ \begin{bmatrix} \lambda_p & 1 \\ 0 & \lambda_p \end{bmatrix}, \]

in some basis \( \{w_1, w_2\} \) of \( \Omega \). Here, we identified \( 1 \otimes \Omega \) with \( \Omega \). Even though the action of \( h(0) \) admits a Jordan block of size two, the module \( M(1)_p \otimes \Omega \) is still an ordinary \( M(1)_p \)-module (i.e., it is diagonalizable with respect to the action of \( L(0) \)). Let us also recall that the lowest conformal weight of \( M(1)_p \otimes \Omega \) is

\[ -(p - 1)^2 \]

\[ \frac{4p}{4p} \]

The following result describes \( M(1)_p \otimes \Omega \) in more details.

**Theorem 6.1.** Let \( h(0) \) acts on \( \Omega \) via [6.13]. Then the space \( M(1)_p \otimes \Omega \) is an ordinary, self-dual (viewed as a Virasoro module, cyclic \( \overline{M(1)}_p \)-module. Moreover, we have a non-split exact sequence of \( \overline{M(1)}_p \)-modules

\[ 0 \rightarrow M(1, \beta) \rightarrow M(1)_p \otimes \Omega \rightarrow M(1, \beta) \rightarrow 0. \]

**Proof:** First we observe that \( M(1)_p \otimes \mathbb{C} w_1 \cong M(1, \beta) \). Thus we have an embedding (on the level of \( L(c_{p,1}, 0) \) and \( \overline{M(1)}_p \)-modules) \( M(1, \beta) \hookrightarrow M(1)_p \otimes \Omega \). Also, from \( h(0) w_2 = \lambda_p w_2 + w_1 \) it is clear that \( M(1)_p \otimes \Omega / M(1, \beta) \cong M(1, \beta) \). Thus we have an exact sequence [6.20].

To show that \( M(1)_p \otimes \Omega \) is a self-dual Virasoro algebra module, it suffices to show that the operator \( L(0) \) acting on \( \Omega^* \) is similar to the operator \( L(0) \) acting now on \( \Omega \). This is a consequence
of the following more general fact that applies for any pair of modules: Let \( \dim(\Omega_1) = \dim(\Omega_2) \). Then \( M(1)_p \otimes \Omega_1 \) and \( M(1)_p \otimes \Omega_2 \) are isomorphic as \( M(1)_p \)-modules if and only if \( h(0)|_{\Omega_1} \) is similar to \( h(0)|_{\Omega_2} \). Now, let us focus on the contragradient module \( (M(1)_p \otimes \Omega)^* \). By using the same argument as in Lemma 1.1 it follows that

\[
(M(1)_p \otimes \Omega)^* \cong M(1)_p \otimes \Omega^*,
\]

where \( h(0)|_{\Omega^*} \) is represented by

\[
\begin{bmatrix}
\lambda_p & 0 \\
-1 & \lambda_p
\end{bmatrix}.
\]

But this module is isomorphic to \( M(1)_p \otimes \Omega \), so our module is self-dual. Now, we prove that the extension in (6.20) is non-split. For these purposes we compute the action of \( H(0) \) on \( \Omega \). From [1], [2] we have

\[
H(0) \cdot w_1 = \frac{\alpha(\alpha - 1) \cdots (\alpha - 2p + 2)}{(2p - 1)!} \cdot w_1,
\]

where \( \alpha \) acts as

\[
\begin{bmatrix}
p - 1 & \sqrt{2p} \\
0 & p - 1
\end{bmatrix}.
\]

Now if we combine the previous two formulas we get that \( H(0) \) acts (in the same basis) via the nilpotent Jordan block

\[
\begin{bmatrix}
0 & \nu_p \\
0 & 0
\end{bmatrix}.
\]

Finally, since \( H(0)w_2 = \nu_p w_1 \), for some \( \nu_p \neq 0 \) and \( M(1, \beta) \) is irreducible (and hence cyclic) it is clear that \( M(1)_p \otimes \Omega \) is also cyclic. \( \square \)

**Remark 2.** The \( M(1)_p \)-module \( M(1)_p \otimes \Omega \) can be also constructed by using the Zhu’s theory (cf. [39]) starting from a two-dimensional \( A(M(1)_p) \)-module \( \Omega \).

Here is an interesting consequence of formula (3.8), which also indicates why \( M(1, \beta) \) is indeed a special \( M(1)_p \)-module.

**Proposition 6.2.** There are no logarithmic self-extensions of \( M(1, \beta) \).

**Proof.** Suppose that

\[
0 \rightarrow M(1, \beta) \rightarrow M \rightarrow M(1, \beta) \rightarrow 0,
\]

for some logarithmic module \( M \). Since \( M(1, \beta) \) is irreducible, \( M \) is generated by 2 vectors \( w_1 \) and \( w_2 \) of generalized conformal weight \( -(p - 1)^2/4p \). Thus, we may assume that \( w_1, w_2 \) form a Jordan block with respect to \( L(0) \), that is \( L(0)w_1 = \frac{-(p - 1)^2}{4p}w_1 \) and \( L(0)w_2 = \frac{-(p - 1)^2}{4p}w_2 + w_1 \).

Now, for every \( N \)-gradable \( M(1)_p \)-module \( M = \oplus_{n \in N} M_n \), the top component \( M_0 \) has a natural \( A(M(1)_p) \)-module structure. By Proposition 3.3, \( P(L(0), H(0)) = 0 \) as an operator on acting on \( M_0 \). The equation \( P(L(0), H(0)) = 0 \) implies that \( H^2(0) \cdot w_1 = 0 \) and \( H^2(0) \cdot w_2 = aw_1 \), for some \( a \neq 0 \), which depends on \( p \). But there is no linear operator with these properties. \( \square \)
7. Virasoro Algebra Structure of $M(1)_p \otimes \Omega$

In this part we describe $M(1)_p \otimes \Omega$ as a Virasoro algebra module. The embedding structure of $M(1)_p \otimes \Omega$ is similar to the case $c = 1$ studied in [34]. As in the previous section we shall denote by $\{w_1, w_2\}$ a basis of $\Omega$ with the action of $L(0)$ and $H(0)$ computed as above via (6.18).

In what follows we shall use the following graphical notation: a singular vector in $M(1)_p \otimes \Omega$ will be denoted by $\bullet$, and a cosingular vector (i.e., a vector that becomes singular in the quotient generated by all singular vectors) will be denoted by $\diamondsuit$. Then we have the following theorem

**Theorem 7.1.** As a Virasoro algebra module, $M(1)_p \otimes \Omega$ is generated by a sequence of singular vectors $v^{(n)}$, $n \geq 0$ and a sequence of cosingular vectors $v^{2,n}$, $n \geq 0$, of conformal weight $h^p_n$, $n \geq 0$, as on the following diagram:

(7.21)

where singular vectors are denoted by $\bullet$ and cosingular vectors with $\diamondsuit$. The arrows have usual meaning (i.e., an arrow pointing from $\diamondsuit$ to $\bullet$ indicates that $\bullet$ is in the submodule generated by $\diamondsuit$).

**Proof.** The proof is similar as in the $c = 1$ case examined in [34] so we omit some details. Let $w_1$ and $w_2$ be as in Section 6, so that

$$h(0) \cdot w_1 = \lambda_p w_1, \quad h(0) \cdot w_2 = \lambda_p w_2 + w_1.$$  

By using Theorem 6.1 it is clear that $M(1, \beta) \hookrightarrow M(1)_p \otimes \Omega$ and a sequence of lowest weight vectors $v^{(n)}$, described in the left column (7.21), corresponds to the sequence of lowest weight vectors in $M(1, \beta)$. For the cosingular vectors we choose any sequence $v^{2,n}$ satisfying $h(0) \cdot v^{2,n} = \lambda_p v^{2,n} + v^{(n)}$, so that $v^{2,n}$ become a (nonzero) singular vectors in the quotient $M(1)_p \otimes \Omega / M(1, \beta)$. These two sequences generate the whole module $M(1)_p \otimes \Omega$. As in [34], by an application of Lemma 2.2 we see that the set of arrows exiting from $v^{2,n}$ is a subset of arrows already displayed on the diagram (7.21). So it remains to show that from every $v^{2,n}$, $n \geq 1$ there are precisely two arrows, one pointing to $v^{(n-1)}$ and another pointing to $v^{(n+1)}$ (from $v^{2,0}$ there should be only one arrow pointing to $v^{(1)}$). Firstly, we show that from every subsingular vector $v^{2,n}$, $n \geq 1$ there is an arrow pointing up to the singular vector $v^{(n-1)}$. In order to see this consider

$$L(m)v^{2,n}, \quad m \geq 1.$$
We claim that for every \( n \geq 1 \) there exists \( m \geq 1 \) such that \( L(m)v^{2,n} \neq 0 \). Suppose that \( L(m)v^{2,n} = 0 \), for all \( m > 0 \). This amounts to \( h(m)v^{(n)} = 0 \), \( m > 0 \), which is impossible for \( n \geq 1 \). Since

\[
\text{wt}(L(m)v^{2,n}) < \text{wt}(v^{2,n}), \ m \geq 1,
\]

then \( L(m)v^{2,n} \in U(Vir)v^{(n-1)} \). It remains to show that from every subsingular vector \( v^{2,n} \) there is an arrow pointing down to the lowest weight vector \( v^{(n+1)} \). For these purposes let us consider \((M(1)_p \otimes \Omega)^*\), the dual of \( M(1)_p \otimes \Omega \). The duality functor reverses the roles of singular and cosingular vectors and reverses the orientation of arrows. More precisely, there is an isomorphism from \( M(1)_p \otimes \Omega \) to its dual that maps singular vector \( v^{(n)} \) to a cosingular vector \( w^{2,n} \) and the cosingular vector \( v^{2,n} \) to a singular vector \( w^{(n)} \), such that \( w^{(n)} \) and \( w^{2,n} \) form a Jordan block with respect to \( h(0) \) (i.e., \( h(0) \cdot w^{(n)} = \lambda_p w^{(n)} \), \( h(0) \cdot w^{2,n} = \lambda_p w^{2,n} + w^{(n)} \)). Thus, there will be a sequence of arrows pointing down from each cosingular vector \( w^{2,n} \) to the singular vector \( w^{(n+1)} \). By using the same argument as before, from each cosingular vector \( w^{2,n} \) in \((M(1)_p \otimes \Omega)^*\) there will be an arrow pointing up to \( w^{(n-1)} \). Finally, \( M(1)_p \otimes \Omega \) is self-dual, so the proof follows. \( \square \)

8. A REALIZATION OF SELF-DUAL \( M(1)_p \)-MODULES

From Theorem 3.2 follows that the irreducible self-dual \( M(1)_p \)-modules have lowest weight \((x,0)\) where \( x = \frac{i(2p-2-i)}{4p}, \ i = 0, \ldots, p-1 \). (Extremal) self-dual modules with lowest weights \((0,0)\) and \((-\frac{(p-1)^2}{4p},0)\) were constructed in Sections 3 and 5.

As before for \( i \in \{0, \ldots, p-2\} \), we define \( \gamma_i = \frac{i}{2p} \alpha \), and consider \( V_{L_{-\gamma_i}} \)-module \( V_{L_{+\gamma_i}} \).

Recall that \( h_{m,n} := \frac{(m-np)^2-(p-1)^2}{4p} \).

By using Lemma 4.3 and the structure theory of Feigin-Fuchs modules \[\] we get the following theorem.

**Theorem 8.1.** Assume that \( i \in \{0, \ldots, p-2\} \).

(i) The Feigin-Fuchs module \( M(1,\gamma_i) \), is generated by the family of singular and cosingular vectors \( \widetilde{\text{Sing}}_i \cup \widetilde{CSing}_i \), where

\[
\widetilde{\text{Sing}}_i = \{u^{(n)}_i \mid n \in \mathbb{Z}_{\geq 0}\}; \quad \widetilde{CSing}_i = \{w^{(n)}_i \mid n \in \mathbb{Z}_{>0}\}.
\]

These vectors satisfy the following relations:

\[
u^{(n)}_i = Q^n e^{\gamma_i-n\alpha}, \quad Q^n w^{(n)}_i = e^{\gamma_i+n\alpha},
\]

\[
U(Vir)u^{(n)}_i \cong L(c_{p,1}, h_{i+1,2n+1}).
\]

(ii) The submodule generated by vectors \( u^{(n)}_i \), \( n \in \mathbb{Z}_{\geq 0} \) is isomorphic to

\[
M(1,\gamma_i) \cong \bigoplus_{n=0}^{\infty} L(c_{p,1}, h_{i+1,2n+1}).
\]

(iii) The quotient module is isomorphic to

\[
M(1,\gamma_i)/M(1,\gamma_i) \cong \bigoplus_{n=0}^{\infty} L(c_{p,1}, h_{i+1,2n+1}).
\]
Theorem 8.2. Assume that $i \in \{0, \ldots, p-2\}$. Then $\overline{M(1, \gamma_i)}$ is an irreducible $\overline{M(1)}_p$-module. Moreover, $\overline{M(1, \gamma_i)}$ is an irreducible, self-dual $\overline{M(1)}_p$-module with the lowest weight $(\frac{-(2p-2i)}{4p}, 0)$.

Proof. By using Frenkel-Zhu’s formula [18] and the methods developed in [32] one can prove that the space of intertwining operators

$$I \left( \begin{array}{c} L(c_{p,1}, h) \\ L(c_{p,1,2p-1}) L(c_{p,1, h_{i+1,2n+1}}) \end{array} \right)$$

is non-trivial if and only if $h = h_{i+1,2n-1}$, $h = h_{i+1,2n+1}$ or $h = h_{i+1,2n+3}$, for $n \geq 1$. Since the multiplicities of these fusion rules are always one, we may write formally:

$$L(c_{p,1}, 2p-1) \times L(c_{p,1}, h_{i+1,2n+1}) =$$

$$L(c_{p,1}, h_{i+1,2n-1}) \oplus L(c_{p,1}, h_{i+1,2n+1}) \oplus L(c_{p,1}, h_{i+1,2n+3}) \quad (n \geq 1).$$

(8.22)

The same results has been known by physicists (cf. [12]). The fusion rules (8.22) implies that

$$H_j \overline{M(1, \gamma_i)} \subset \overline{M(1, \gamma_i)}, \quad \text{for every } j \in \mathbb{Z}.$$

Since $\overline{M(1)}_p$ is generated by $\omega$ and $H$ we have that $\overline{M(1, \gamma_i)}$ is an $\overline{M(1)}_p$-module. By using a completely analogous proof to those of Theorem 5.5 and Theorem 5.4 we get the $\overline{M(1, \gamma_i)}$ is an irreducible $\overline{M(1)}_p$-module. \hfill \Box

9. Some logarithmic $\overline{M(1)}_p$-modules

In this part we study certain logarithmic $\overline{M(1)}_p$-modules. First, we consider $\overline{M(1)}_p$-module $\overline{M(1)}_p \otimes \Omega_0$, where on $\Omega_0$ we have $h(0)^2 = 0$ and $h(0) \neq 0$. Then we have

$$0 \rightarrow \overline{M(1)}_p \rightarrow \overline{M(1)}_p \otimes \Omega_0 \rightarrow \overline{M(1)}_p \rightarrow 0.$$ 

Furthermore, $L(0)^2 = 0$ and $L(0) \neq 0$ on $\Omega_0$, so $L(0)|_{\Omega_0}$ is a nilpotent Jordan block of size two. The module $\overline{M(1)}_p \otimes \Omega_0$ is also an $\overline{M(1)}_p$-module, where on the two-dimensional top level $\Omega$, the generator $H(0)$ acts via a nonzero nilpotent Jordan block (this matrix can be easily computed by using Proposition 5.2 in [2] or relation (5.8)). In order to describe the structure of $\overline{M(1)}_p \otimes \Omega_0$ let us first examine $\overline{M(1)}_p$, viewed as an $\overline{M(1)}_p$-module. As before, we shall view $\overline{M(1)}_p$ embedded inside the lattice VOA $V_L$, which further sits inside the generalized vertex algebra $V_{\overline{L}}$. Now, the Virasoro algebra dual of $\overline{M(1)}_p$ is also contained in $V_{\overline{L}}$ and is isomorphic to $\overline{M(1)}_p \otimes e^{2\beta} = \overline{M(1, 2\beta)}$, by Lemma 1.1. Consider a two-dimensional space $\Omega_1$ such that $(h(0) - 2\lambda_p)^2 = 0$ and $h(0) \neq 2\lambda_p$, as operators on $\Omega_1$. Then we have an exact sequence

$$0 \rightarrow \overline{M(1, 2\beta)} \rightarrow \overline{M(1)}_p \otimes \Omega_1 \rightarrow \overline{M(1, 2\beta)} \rightarrow 0,$$

where $\overline{M(1)}_p \otimes \Omega_1$ is a logarithmic module $\overline{M(1)}_p$-module (keep in mind that $h(0) = \frac{\omega}{\sqrt{2p}}$ and $\beta = \frac{p-1}{\sqrt{2p}}$).

We observe an exact sequence of $\overline{M(1)}_p$-modules

$$0 \rightarrow \overline{M(1)}_p \rightarrow \overline{M(1)}_p \rightarrow W \rightarrow 0,$$
where $W \cong M(1)_p/M(1)_p$ is an $M(1)_p$-module. Recall that the screening $\tilde{Q} = e_0^{-\alpha/p}$ acts as a derivation operator on $V_L$, meaning that

$$\tilde{Q}(u_jv) = (\tilde{Q}u)_jv + u_j\tilde{Q}v, \quad u, v \in V_L.$$ 

Also, by definition (cf. [2])

$$\text{Ker}|_{M(1)_p} \tilde{Q} = \overline{M(1)_p}.$$ 

In view of that, the map

$$\tilde{Q} : M(1)_p \rightarrow M(1)_p \otimes e^{-\alpha/p}$$

is actually a homomorphism of $\overline{M(1)_p}$-modules, so we have an isomorphism

$$\text{Im}\tilde{Q} \cong W.$$ 

So $W$ is actually an $\overline{M(1)_p}$-submodule of $M(1)_p \otimes e^{-\alpha/p}$. This situation is depicted via the following diagram

![Diagram]

The middle column represents the vacuum module $M(1)_p$. The column to the right (resp. left) of $M(1)_p$ represents $M(1)_p \otimes e^\alpha$ (resp. $M(1)_p \otimes e^{-\alpha/p}$). Observe also that

$$W \cong \bigoplus_{n=1}^{\infty} L(c_{p,1}, n^2p - np + n).$$

In addition $W$ is a cyclic $\overline{M(1)_p}$-module of lowest weight $(L(0), H(0)) = (1, -2p)$. Now, $M(1, 2\beta)$ enjoys similar properties. Firstly, we observe a sequence

$$0 \rightarrow W^* \rightarrow M(1, 2\beta) \rightarrow \overline{M(1)_p} \rightarrow 0,$$

where we used self-duality of $\overline{M(1)_p}$. As a Virasoro algebra module

$$W^* \cong \bigoplus_{n=1}^{\infty} L(c_{p,1}, n^2p - np + n).$$

However, $W$ is not isomorphic to $W^*$ (the lowest weight of $W^*$ is $(1, 2p)$). On the other hand, both $\overline{M(1)_p}$ and $M(1, \beta)$ are self-dual as $\overline{M(1)_p}$-modules.

**Remark 3.** There are some obvious questions that could be pursued now regarding $W$ and other related modules (e.g., irreducibility). We shall address these issues in our future publications.
10. GRADED DIMENSIONS OF SELF-DUAL $M(1)_p$-MODULES

In this section we discuss graded dimensions of self-dual $M(1)_p$-modules appearing in the paper. As usual the graded dimensions are defined by using the formula

$$\text{ch}_M(\tau) = \text{tr}|_M q^{L(0)-c_p,1/24}, \quad q = e^{2\pi i \tau}.$$  

Firstly, the characters of $M(1, \beta)$ and $M(1)_p \otimes \Omega$ are easily computed:

(10.23) \hspace{1cm} \text{ch}_{M(1, \beta)}(\tau) = \frac{1}{\eta(\tau)},

(10.24) \hspace{1cm} \text{ch}_{M(1)_p \otimes \Omega}(\tau) = \frac{2}{\eta(\tau)}.

It is also not hard to see, by using Proposition 2.1 and Theorem 8.2, that

(10.25) \hspace{1cm} \text{ch}_{M(1, \gamma_i)}(\tau) = q^{(p-n-1)/4} \frac{\sum_{n=0}^{\infty} q^{p(n+1)-i-1} - \sum_{n=1}^{\infty} q^{n(pn+i+1)}}{\eta(\tau)},

where $i = 0, \ldots, p - 2$. The previous formula can be rewritten in a more compact way as

(10.26) \hspace{1cm} \text{ch}_{M(1, \gamma_i)}(\tau) = \sum_{n \in \mathbb{Z}} \text{sgn}(n) q^{(2n+1)p-1-i} \frac{1}{\eta(\tau)}, \quad i = 0, \ldots, p - 2,

where $\text{sgn}$ is the sign function (where $\text{sgn}(0) = 1$.) Just for the record notice also that

$$\text{ch}_{M(1)_p \otimes \Omega_0}(\tau) = \frac{q^{(p-1)/2} / 4(1 + 2\pi i \tau)}{\eta(\tau)}.$$

The numerators appearing in (10.26) are certain $\theta$-like constants. Their modular properties were studied in [12].

11. CONSTRUCTION OF HIDDEN LOGARITHMIC INTERTWINING OPERATORS FOR $W(2, 2p - 1)$-ALGEBRAS

In this part we apply results from [14] and the previous section in the case of $\lambda = \lambda_p$. Results from [34] provide us with a hidden logarithmic intertwining operator

$$\mathcal{Y} \in I \left( \begin{array}{c} M(1)_p \otimes \Omega_1 \\ M(1)_p \otimes \Omega M(1, \beta) \end{array} \right),$$

where $M(1)_p \otimes \Omega_1$ is the logarithmic $M(1)_p$-module described in the previous section. The explicit formulas for $\mathcal{Y}$ are given in Theorem 7.4, [34]. Here we give explicit formulas for $\mathcal{Y}(u, x)$ in two special cases: $u = w_i \in M(1)_p \otimes \Omega$, $i = 1, 2$. 
Corollary 11.1. Let $w_1$, $w_2$ and $\mathcal{Y}$ as above. Then we have
\[
\mathcal{Y}(w_1, x) = E^-(\lambda_p, x)E^+(\lambda_p, x)T_{\Omega, \beta}^{\Omega_1}(w_1)(1 + \lambda_p \log(x)h_n(0))x^{\lambda_p h_2(0)}
\]
\[
\mathcal{Y}(w_2, x) = \int^+ h(x)E^-(\lambda_p, x)E^+(\lambda_p, x)T_{\Omega, \beta}^{\Omega_1}(w_1)(1 + \lambda_p \log(x)h_n(0))x^{\lambda_p h_2(0)}
\]
\[
+ E^-(\lambda_p, x)E^+(\lambda_p, x)T_{\Omega, \beta}^{\Omega_2}(w_1)(1 + \lambda_p \log(x)h_n(0))x^{\lambda_p h_2(0)} \int^+ h(x)
\]
\[
(11.27)
\]
\[
+ E^-(\lambda_p, x)E^+(\lambda_p, x)T_{\Omega, \beta}^{\Omega_1}(w_2)(1 + \lambda_p \log(x)h_n(0))x^{\lambda_p h_2(0)}
\]

where $h_n(0)$ is the nilpotent part of $h(0)$, $T_{\Omega, \beta}^{\Omega_1} \in Hom(\Omega, Hom(\mathbb{C}, \Omega_1))$ is the operator that corresponds to the obvious $h$-isomorphism between $\Omega \otimes \mathbb{C}$ and $\Omega_1$, and
\[
\int^+ h(x) = h(0) \log(x) + \sum_{m>0} \frac{h(m)x^{-m}}{-m},
\]
\[
\int^- h(x) = \sum_{m<0} \frac{h(m)x^{-m}}{-m}.
\]

12. CONCLUSION AND FUTURE WORK

As we indicated in the introduction our future goal(s) are in the direction of understanding the triplet vertex operator algebra $\mathcal{W}(2, (2p - 1)^3)$ [14], [20], [21], [22] and its irreducible modules. This will require some modifications of the present paper since the triplet algebra $\mathcal{W}(2, (2p - 1)^3)$ is not contained inside $M(1)_p$, but rather inside $V_L$. Another interesting direction, which in our opinion does not have a satisfactory explanation, is to probe our method for certain logarithmic $A_1^{(1)}$-modules at admissible level studied by Gaberdiel [20].

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