LEARNING SPARSE GRAPH LAPLACIAN WITH $K$ EIGENVECTOR PRIOR VIA ITERATIVE GLASSO AND PROJECTION

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ABSTRACT

Learning a suitable graph is an important precursor to many graph signal processing (GSP) pipelines, such as graph spectral signal compression and denoising. Previous graph learning algorithms either i) make some assumptions on connectivity (e.g., graph sparsity), or ii) make simple graph edge assumptions such as positive edges only. In this paper, given an empirical covariance matrix $C$ computed from data as input, we consider a structural assumption on the graph Laplacian matrix $L$: the first $K$ eigenvectors of $L$ are pre-selected, e.g., based on domain-specific criteria, and the remaining eigenvectors are then learned from data. One example use case is image coding, where the first eigenvector is pre-chosen to be constant, regardless of available observed data. We first prove that the subspace of symmetric positive semi-definite (PSD) matrices $H^+_u$ with the first $K$ eigenvectors being $\{u_k\}$ in a defined Hilbert space is a convex cone. We then construct an operator to project a given positive definite (PD) matrix $L$ to $H^+_u$, inspired by the Gram-Schmidt procedure. Finally, we design an efficient hybrid graphical lasso / projection algorithm to compute the most suitable graph Laplacian matrix $L^* \in H^+_u$ given $C$. Experimental results show that the first $K$ eigenvectors as a prior, our algorithm outperforms competing graph learning schemes using a variety of graph comparison metrics.

Index Terms— Graph learning, Graph signal processing

1. INTRODUCTION

Graph signal processing (GSP) [1] is the study of signals residing on graphs. While GSP tools have been demonstrated to be effective in a wide range of applications from image compression and denoising to matrix completion [2–7], a fundamental first step in GSP is the selection of an appropriate graph $G$ (or graph Laplacian matrix $L$) that suitably describes pairwise (dis)similarities. Previous graph learning algorithms fall into two categories: i) statistical approaches like graphical lasso that, given an empirical covariance matrix $C$, estimate a sparse inverse matrix with no graph-structured assumptions [8, 9]; and ii) edge-based approaches that assume edge properties such as edge signs and absence of self-loops [10–12]. Neither of these two categories of methods make assumptions about the eigenstructure of the graph Laplacian matrix $L$.

In this paper, we introduce explicit eigen-structure assumptions on the graph Laplacian matrix $L$—that the first $K$ eigenvectors of $L$ are pre-selected or computed based on domain-specific criteria—into a graph learning framework. Consider the example of image compression, where one can deduce from domain knowledge that the most common pixel block is the constant signal, and thus should be the first eigenvector regardless of (possibly limited) training data. Consider also political voting records, where the most common pattern is voting along party affiliations in a two-party political system, and thus the first eigenvector should be piecewise constant (i.e., nodes of one political party are assigned 1, while nodes of the other party are -1). There are also practical cases where the first $K$ eigenvectors can be pre-chosen for fast computation. For example, one can use fast graph Fourier transform (FGFT) [13] to construct a set $K$ sparse eigenvectors based on Givens rotation matrices, so that the first $K$ transform coefficients can be computed speedily.

We propose an optimization strategy where a graph Laplacian matrix $L$ is learned optimally from data, while restricting the first $K$ eigenvectors to be those chosen ahead of time. We first prove that the subspace $H^+_u$ of symmetric positive semi-definite (PSD) matrices with the first $K$ eigenvectors taken from a given set $\{u_k\}_{k=1}^K$ is a convex cone. We next construct an operator to project a given positive definite (PD) matrix $L$ to $H^+_u$, inspired by the Gram-Schmidt procedure [14]. Finally, we design an efficient hybrid graphical lasso / projection algorithm to compute the most suitable graph Laplacian matrix $L^* \in H^+_u$ given input empirical covariance matrix $C$. Experimental results demonstrate that given the first $K$ eigenvectors as a prior, our algorithm outperformed competing graph learning methods in terms of different graph comparison metrics.

2. PRELIMINARIES

2.1. Graph Definitions

Suppose we are given a graph $G(\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = N$ nodes and edges $(i, j) \in \mathcal{E}$ connecting nodes $i$ and $j$ with weight $w_{ij} \in \mathbb{R}$. Denote by $W$ the adjacency matrix, where $W_{ij} = w_{ij}$. $W$ may include self-loops $W_{ii}$’s. Assuming that the edges are undirected, $W$ is symmetric. Define next the diagonal degree matrix $D$ where $D_{ii} = \sum_j W_{ij}$. The combinatorial graph Laplacian matrix $[1]$ is then defined as $L = D - W$. To properly account for self-loops, the generalized graph Laplacian matrix is defined as $\mathcal{L} = D - W + \text{diag}(W)$. In this paper, we assume that the target graph Laplacian to be learned from data may have self-loops, and edge weights can be positive or negative. This means that our graph Laplacian $\mathcal{L}$ can account for both pairwise correlation and anti-correlation.

2.2. Hilbert Space

Following terminologies in [15], we first define a vector space $\mathcal{X}$ of real, symmetric matrices in $\mathbb{R}^{N \times N}$. Note that $\mathcal{X}$ is closed under addition and scalar multiplication. We next define the standard inner product $\langle \cdot, \cdot \rangle$ for matrices $P, Q$ in $\mathcal{X}$ as:

$$\langle P, Q \rangle = \text{Tr}(Q^T P) = \sum_{i,j} P_{ij} Q_{ij} \quad \text{(1)}$$

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Further, we are interested in the subspace of given an empirical covariance matrix observations \( L \) computed using Hilbert space. We can now define a positive definite (PD) matrix to \( H_u^+ \). We describe our optimization algorithm using the projection operator in Section 4.

3. PROJECTION OPERATOR

We first overview our algorithm development. We prove that the subspace \( H_u^+ \) of symmetric PSD matrices that share a common ordered set of first \( K \) eigenvectors is a convex cone. This means that, given an empirical covariance matrix \( C \) estimated from \( M \) signal observations \( X = \{x_1, \ldots, x_M\} \), minimizing a convex objective computed using \( C \) and a graph Laplacian matrix \( L \) while restricting \( L \in H_u^+ \) is a convex optimization problem. We then develop a projection operator to project a positive definite (PD) matrix to \( H_u^+ \). We describe our optimization algorithm using the projection operator in Section 4.

3.1. Subspace of Matrices with Common First \( K \) Eigenvector

Denote by \( H_u^+ \subset H^+ \) the subspace of PSD matrices that share the first \( K \) eigenvectors \( \{u_k\}_{k=1}^K \), assumed to be orthonormal, i.e.,

\[
\begin{align*}
\forall j, k \in I_K, \quad u_j^T u_k = \delta_{j-k},
\end{align*}
\]

where \( I_K = \{1, \ldots, K\} \) and \( \delta_i \) is the discrete impulse function that evaluates to 1 if \( i = 0 \) and 0 otherwise. We can define \( H_u^+ \) using the Rayleigh quotient and the min-max [16] as

\[
H_u^+ = \left\{ L \in H^+ \mid u_k = \arg \min_{x \neq 0, y \neq 0} \frac{x^T L x}{x^T x}, \ k \in I_K \right\}
\]

(4)

Lemma 3.1. \( H_u^+ \) is a convex cone.

Proof. We prove by induction. For the base case when \( K = 1 \), let \( L_1, L_2 \in H^+ \) be two matrices that share the first eigenvector \( u_1 \). Consider a positive linear combination \( L = c_1 L_1 + c_2 L_2 \), where \( c_1, c_2 \geq 0 \). We examine the smallest eigenvalue \( \lambda_{\min}(L) \) of \( L \) using the Rayleigh quotient again, i.e.,

\[
\lambda_{\min}(L) = \min_x \frac{x^T L x}{x^T x} = \min_x \frac{x^T (c_1 L_1 + c_2 L_2) x}{x^T x}
\]

(5)

(6)

The inequality in (a) is true since right-hand side (RHS) has more degrees of freedom than left-hand side (LHS) to minimize the two non-negative terms \( L_1 \) and \( L_2 \) are PSD matrices in \( H^+ \). (b) is true by definition of Rayleigh quotient, with \( u_1 \) being the minimizing argument for both terms by assumption. Thus, the argument that minimizes the Rayleigh quotient for \( L \) is also \( u_1 \), and therefore \( u_1 \) is the first eigenvector of \( L \). Since this analysis is true for all \( c_1, c_2 \geq 0 \), \( H_u^+ \) is a convex cone for \( K = 1 \).

Consider next the inductive step, where we assume the \( K = K_o + 1 \) case is true, and we prove the \( K_o + 1 \) case. Let \( L_1, L_2 \in H^+ \) be two matrices that share the first \( K_o + 1 \) eigenvectors \( \{u_k\}_{k=1}^{K_o+1} \). Consider a positive linear combination \( L = c_1 L_1 + c_2 L_2 \), where \( c_1, c_2 \geq 0 \). By the inductive assumption, \( L \) shares the first \( K_o \) eigenvectors \( \{u_k\}_{k=1}^{K_o} \). For the \( (K_o + 1) \)-th eigenvector, consider the \( (K_o + 1) \)-th eigenvalue \( \lambda_{K_o+1}(L) \) computed as:

\[
\lambda_{K_o+1}(L) \geq c_1 \left( \min_{x \neq 0, y \neq 0} \frac{x^T L_1 x}{x^T x} \right) + c_2 \left( \min_{y \neq 0} \frac{y^T L_2 y}{y^T y} \right)
\]

(7)

Since the argument that minimizes the Rayleigh quotient for both \( L_1 \) and \( L_2 \), \( u_{K_o+1} \) is the argument that minimizes the Rayleigh quotient for \( L \). Thus the \( (K_o + 1) \)-th eigenvector for \( L \) is \( u_{K_o+1} \). Thus, we conclude that \( L \) has \( \{u_k\}_{K_o+1} \) as its first \( K_o + 1 \) eigenvectors. Since both the base case and the inductive step are true, \( H_u^+ \) is a convex cone.

3.2. Projection to Convex Cone \( H_u^+ \)

To project a PD matrix \( P \in H^+ \) to convex cone \( H_u^+ \), we focus on the inverse \( C = P^{-1} \) and ensure \( u_1 \) of the known eigenvector set \( \{u_k\}_{k=1}^{K_o} \) is the argument that maximizes the Rayleigh quotient of \( C \), then \( u_2 \) of \( \{u_k\}_{k=1}^{K_o} \) is the argument that maximizes the Rayleigh quotient while being orthogonal to \( u_1 \), and so on.

Specifically, we first compute \( C = P^{-1} \). Define \( \Omega = \{\alpha U_j \mid \alpha \in \mathbb{R}\} \) as the subspace spanned by rank-1 matrix \( U_j \) with scalar \( \alpha \). Define by \( \Omega^\perp \) the orthogonal subspace in \( H \) so that \( H = \Omega \oplus \Omega^\perp \). We first write \( C \) as its projection to \( \Omega \) plus its orthogonal component, i.e.,

\[
C = (C, U_1) U_1 + C \Omega^\perp
\]

(8)

where \( C \Omega^\perp \) is the orthogonal component of \( C \) in subspace \( \Omega^\perp \). We show that given \( C \) is PD, inner product \( (C, U_1) \) is non-negative.

Lemma 3.2. \( (C, U_1) \geq 0 \).

Proof. Since \( C \) is PD by assumption, without loss of generality we can write \( C = BB^T \). Then

\[
(C, U_1) = \text{Tr} \left( (BB^T)^T u_1^T u_1 \right) = \text{Tr} \left( BB^T u_1^T u_1 \right)
\]

\[
= \text{Tr} \left( u_1^T B B^T u_1 \right)
\]

The last term is a Frobenius norm \( \|u_1^T B\|^2 \geq 0 \).

Define \( \mu_1 = (C, U_1) \geq 0 \). Suppose \( \Omega^\perp \) can be expressed as the span of orthogonal \( U_{2j-1}, \ldots, U_K \) and \( V_{K+1}, \ldots, V_N \), where \( U_k = u_k u_k^T \), \( V_i = v_i v_i^T \) and \( \|v_i\| = 1 \). Suppose also that the inner products of \( C \) with the orthogonal components, \( (C, U_k)_k, (C, V_i)_i \), are within range \([0, \mu_1], \forall k,i \). This means that the respective Rayleigh quotients are no larger than \( \mu_1 \):

\[
\text{Tr}(u_k^T C u_k) = \text{Tr}(C u_k u_k^T) = (C, U_k) \leq \mu_1, \ \forall k
\]

(9)

Then \( u_1 \) is surely the last eigenvector of \( C \) corresponding to largest eigenvalue \( \mu_k \). If this is not the case, then we need to approximate
C’s projection $C_{\Omega^\perp}$ to $\Omega^\perp$ as $C_{\Omega^\perp}$ just to ensure $u_1$ is the last eigenvector:

$$
\min_{(V_1), C_{\Omega^\perp}} \|C_{\Omega^\perp} - \hat{C}_{\Omega^\perp}\|_2^2, \text{ s.t.}
\begin{array}{l}
(V_1, U_k) = 0, \forall i, k \\
(V_1, V_j) = 0, \forall i, j \\
V_1 = v_1v_1^\top, \forall i \\
\|v_1\|_2 = 1, \forall i \\
0 \leq \langle C_{\Omega^\perp}, U_k \rangle \leq \mu_1, \forall k \\
0 \leq \langle C_{\Omega^\perp}, V_i \rangle \leq \mu_1, \forall i
\end{array}
$$

(7)

Jointly optimizing $N - K$ rank-1 matrices $V_i$’s and $C_{\Omega^\perp}$ in (7) is difficult. We propose a greedy algorithm instead next.

### 3.3. Gram-Schmidt-inspired Algorithm

Our algorithm is inspired by the Gram-Schmidt procedure [14] that iteratively computes a set of orthonormal vectors given a set of linearly independent vectors. Similarly, our algorithm iteratively computes one orthogonal rank-1 matrix $V_t = v_t v_t^\top$ at each iteration $t$, for $t \geq K + 1$. Iteration $t \leq K$ uses $U_t = u_t u_t^\top$, which is known.

Consider iteration $t = 2$. From (5), we first define the residual signal as $E_1 = C - (C, U_1)U_1$. Consider first the case where $K \geq 2$, and thus rank-1 matrix $U_2 = u_2 u_2^\top$ orthogonal to $U_1$ is known. We need only that the inner product $\mu_2 = \langle C, U_2 \rangle$ of approximation $C$ and $U_2$ is no larger than $\mu_1$. Thus, we set

$$
\mu_2 = \begin{cases} 
\langle E_1, U_2 \rangle, & \text{if } \langle E_1, U_2 \rangle \leq \mu_1 \\
o.w. &
\end{cases}
$$

(8)

Consider next the case where $K = 1$, and we must identify a new rank-1 matrix $V_2 = v_2 v_2^\top$ orthogonal to $U_1$ to reconstruct $C$. In this case, to minimize error $\|C - \hat{C}\|_2^2$, we seek a $V_2$ “most aligned” with $E_1$, i.e.,

$$
\max_{v_2} \langle E_1, v_2 v_2^\top \rangle, \text{ s.t. } \begin{cases} 
\langle v_2 v_2^\top, U_1 \rangle = 0 \\
\|v_2\|_2 = 1
\end{cases}
$$

(9)

Essentially, (9) seeks a rank-1 approximation $\langle E_1, V_2 \rangle V_2$ of matrix $E_1$, while constraining $V_2$ to be orthogonal to $U_1$. The objective is equivalent to $v_2^\top E_1 v_2$, which is convex for a maximization problem.

To convexify the problem, we first approximate $E_1 \approx e e^\top$ where $e$ is the last eigenvector\(^1\) of $E_1$, and $e e^\top$ is the best rank-1 approximation of $E_1$. We then relax the norm equality constraint and rewrite (9) into the following optimization:

$$
\max_{v_2} e^\top v_2, \text{ s.t. } \begin{cases} 
v_2^\top u_1 = 0 \\
\|v_2\|_2^2 \leq 1
\end{cases}
$$

(10)

Optimization (10) is now convex and solvable in polynomial time, using algorithms such as proximal gradient (PG) [18].

Having computed $V_2 = v_2 v_2^\top$ in (10), we project $E_1$ onto $V_2$ and threshold its inner product to within $[0, \mu_1]$ as done in the $K \geq 2$ case, i.e.,

$$
\mu_2 = \begin{cases} 
\langle E_1, V_2 \rangle, & \text{if } \langle E_1, V_2 \rangle \leq \mu_1 \\
o.w. & \mu_1
\end{cases}
$$

(11)

Notice that $\langle E_1, V_2 \rangle \geq 0$. We omit the proof of the more general case $\langle E_k, V_k \rangle \geq 0$ for brevity. The projection of $E_1$ on $V_2$ is thus $\mu_2 V_2$. We then compute new residual signal $E_2$:

$$
E_2 = E_1 - \mu_2 V_2
$$

(12)

\(^1\)Extreme eigenvectors of sparse symmetric matrices can be computed efficiently using LOBPCG [17].

Given residual $E_2$, we again compute a rank-1 matrix $V_3 = v_3 v_3^\top$ orthogonal to $V_2$ and $U_1$—that is most aligned with $E_2$, and compute projection of $E_2$ to $V_3$, and so on.

More generally, at each iteration $t \leq K$, we threshold the inner product $\mu_{t+1} = \min(\langle E_t, U_{t+1} \rangle, \mu_t)$, and compute the next residual signal $E_{t+1} = E_t - \mu_{t+1} V_{t+1}$. On the other hand, at each iteration $t \geq K + 1$, we first approximate $E_t \approx e e^\top$ where $e$ is $E_t$’s last eigenvector. We then compute an optimal $v_{t+1}$:

$$
\max_{v_{t+1}} e^\top v_{t+1}, \text{ s.t. } \begin{cases} 
v_{t+1}^\top u_k = 0, i \in \mathcal{K} \\
v_{t+1}^\top v_\tau = 0, \tau \in \{K + 1, \ldots, t\} \\
\|v_{t+1}\|_2 \leq 1
\end{cases}
$$

(13)

We then threshold the inner product $\mu_{t+1} = \min(\langle E_t, V_{t+1} \rangle, \mu_t)$. We compute the next residual signal $E_{t+1} = E_t - \mu_{t+1} V_{t+1}$.

We finally note that our constructed operator $\text{Proj}(\cdot)$ is indeed a projection, since it is provably idempotent [15], i.e., $\text{Proj}(\text{Proj}(P)) = \text{Proj}(P)$ for any PD matrix $P \in \mathbb{H}^+$.\(^2\)

### 4. GRAPH LAPLACIAN MATRIX ESTIMATION

Given an empirical covariance matrix $C$ computed from data, we formulate the following optimization problem to estimate a graph Laplacian matrix $L$ starting from the GLASSO formulation in [19]:

$$
\min_{\text{LM} \in \mathbb{H}_+^N} \text{Tr}(LC) - \log \det L + \rho \|L\|_1
$$

(14)

where $\rho > 0$ is a shrinkage parameter for the $l_1$ norm. The only difference from GLASSO is that (14) has an additional constraint $\text{LM} \in \mathbb{H}_+^N$. Because $\mathbb{H}_+^N$ is a convex set, (14) is a convex problem.

We solve (14) iteratively using our developed projection operator in Section 3 and a variant of the block Coordinate descent (BCD) algorithm in [20]. Specifically, we first solve the dual of GLASSO as follows. We first note that the $l_1$ norm in (14) can be written as

$$
\|L\|_1 = \max_{\|U\|_\infty \leq 1} \text{Tr}(LU)
$$

(15)

where $\|U\|_\infty$ is the maximum absolute value element of the symmetric matrix $U$. Then the dual problem of GLASSO that solves for an estimated covariance matrix $C = L^{-1}$ is

$$
\min_{C \in \mathbb{H}_+} -\log \det C, \text{ s.t. } \|C - \hat{C}\|_\infty \leq \rho
$$

(16)

where $C = C + U$ implies that the primal and dual variables are related via $L = (C + U)^{-1}$ [21]. To solve (16), we update one row-column pair in $C$ in (16) in each iteration. Specifically, we first reorder the rows / columns of $C$ so that the optimizing row / column are swapped to the last. We then partition $C$ and $C$ into blocks:

$$
C = \begin{pmatrix} C_{11} & C_{12} \\
C_{21} & C_{22} \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} \hat{C}_{11} & \hat{C}_{12} \\
\hat{C}_{21} & \hat{C}_{22} \end{pmatrix}
$$

(17)

[21] showed that the optimal $c_{12}$ is the solution to the following linearly constrained quadratic programming problem:

$$
e_{12} = \arg \min_y \langle y, C_{11} y \rangle, \text{ s.t. } \|y - \hat{c}_{12}\|_\infty \leq \rho
$$

(18)

Our algorithm to solve (14) is thus as follows. We minimize the GLASSO terms in (14) by solving its dual (16)—iteratively updating one row / column of $C$ using (18). We then project $L = C^{-1}$ to convex cone $\mathbb{H}_+^N$ using our projection operator. We repeat these two steps till convergence. Note that both steps are computed using covariance $C$ directly, and thus inversion to graph Laplacian $L = C^{-1}$ is not necessary until convergence, when we output a solution.
5. EXPERIMENTAL RESULTS

5.1. Experiments Setups for Synthetic Datasets

We conducted experiments using synthetic datasets to evaluate the performance of our method (Proj-Lasso) and of competing schemes: Graphical Lasso (GLASSO) [8], graph Laplacian learning with Gaussian probabilistic prior (GL-SigRep) [12] and diagonally dominant generalized graph Laplacian estimation under structural and Laplacian constraints (DDGL) [10]. The convergence tolerance was set to $\epsilon = 10^{-4}$, and the regularization parameter was $\rho = e^{-6}$.

To simulate ground truth graphs, we first randomly located 20 nodes in 2D space and used the Erdos-Renyi model [22] to determine their connectivity with probability 0.6. We then computed edge weights using a Gaussian kernel, i.e., $w_{ij} = \exp(-d(i, j)^2/2\sigma^2)$, where $d(i, j)$ is the Euclidean distance between $i$ and $j$ and $\sigma$ is 0.5. Edge weights smaller than 0.75 were removed for sparsity. To introduce negative weights, we flipped the sign of each edge with probability 0.5 and then computed the generalized graph Laplacian $\mathcal{L} = D - W + \text{diag}(W)$. To generate data from $\mathcal{L}$, we first computed covariance $K = (\mathcal{L} + \epsilon I)^{-1}$ for $\epsilon = 0.5$. We then generated data set $\mathcal{X} = \{x_i\}_{i=1}^{20}$ from multivariate normal distribution $x \sim N(0, K)$. An empirical covariance matrix $C$ was then computed from $\mathcal{X}$ and used as input to different graph learning algorithms.

We employed three popular graph similarity metrics to evaluate graph Laplacian matrices learned: relative error (RE), DeltaCon and $\lambda$-distance [23–26]. Specifically, RE computes the relative Frobenius norm error between the ground truth Laplacian matrix $L$ and the learned matrix $\hat{L}$; DeltaCon compares the similarities between all node pairs in the two graphs, and $\lambda$-distance metric computes the eigenvectors’ distance between two matrices.

Table 1 shows the graph learning performance of different methods with 20 vertices and 20 signals on each node.

To show visual comparisons of learned Laplacian matrices in Fig. 1, demonstrating that Proj-Lasso had the best performance, i.e., Proj-Lasso is visually closer to the ground truth matrix than others.

5.3. Results using Eigenvectors from Givens Rotation Method

[13] developed a method based on Givens rotation matrices to approximately diagonalize Laplacian $L$, i.e.,

$$L \approx S_1 \cdots S_j \hat{A} S_j^\top \cdots S_1^\top$$

where $S_1, \ldots, S_j$ are Givens rotation matrices that are both sparse and orthogonal, and $\hat{A}$ is a near-diagonal matrix. $T = S_j^\top \cdots S_1^\top$ can be interpreted as a fast Graph Fourier Transform (FGFT) that approximates the original eigen-matrix of $L$. $T$ is sparse since each $S_i$ is sparse, and thus computation of transform coefficients $\alpha = T x$ can be efficient. $J$ is a parameter to trade off the complexity of the transform $T$ and the GFT approximation error.

In this experiment, we assumed that the first $K$ rows of $T$ are chosen as the first $K$ eigenvectors in our prior, which are the input to our algorithm Proj-Lasso. The algorithm then computed the remaining $N - K$ eigenvectors to compose Laplacian $L$. We set $J = 2000$. We compared the performance of Proj-Lasso against the scheme that uses all the $N$ rows of $T$. Table 2 shows learning performances using different numbers of rows from $T$ ($K = 1, \ldots, 3$ is Proj-Lasso, and $K = N$ is Givens method). We observe that Proj-Lasso has smaller error for both metrics RE and $\lambda$-distance.

6. CONCLUSION

Given observable graph signals, we propose a new graph learning algorithm, where we assume that the first $K$ eigenvectors of the graph Laplacian matrix are pre-chosen based on domain knowledge, or pre-computed using an alternative criterion. We construct an operator in Hilbert space to project a positive definite (PD) matrix into the convex cone of positive semi-definite (PSD) matrices that share the first $K$ eigenvectors. We design an efficient algorithm combining block coordinate descent (BCD) in GLASSO and our projection operator, both optimizing the covariance matrix directly. Experimental results show that our algorithm outperformed competing graph learning schemes when the first $K$ eigenvectors are known.
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