DECAY OF CORRELATIONS AND UNIQUENESS OF THE INFINITE-VOLUME GIBBS MEASURE OF THE CANONICAL ENSEMBLE OF 1D-LATTICE SYSTEMS

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Abstract. We consider a one-dimensional lattice system of unbounded, real-valued spins with arbitrary strong, quadratic, finite-range interaction. We show the equivalence of correlations of the grand canonical (gce) and the canonical ensemble (ce). As a corollary we obtain that the correlations of the ce decay exponentially plus a volume correction term. Then, we use the decay of correlation to verify a conjecture that the infinite-volume Gibbs measure of the ce is unique on a one-dimensional lattice. For the equivalence of correlations, we modify a method that was recently used by the authors to show the equivalence of the ce and the gce on the level of thermodynamic functions. In this article we also show that the equivalence of the ce and the gce holds on the level of observables. One should be able to extend the methods and results to graphs with bounded degree as long as the gce has a sufficient strong decay of correlations.

1. Introduction

The broader scope of this article is the study of phase transitions. A phase transition occurs if a microscopic change in a parameter leads to a fundamental change in one or more properties of the underlying physical system. The most well-known phase transition is when water becomes ice. Many physical systems, non-physical systems and mathematical models have phase transitions. For example, liquid to gas phase transitions are known as vaporization. Solid to liquid phase transitions are known as melting. Solid to gas phase transitions are known as sublimation. More examples are the phase transition in the 2-d Ising model (see for example [Sel16]), the Erdös-Renyi phase transition in random graphs (see for example [ER60], [ER61] or [KS13]) or phase transitions in social networks (see for example [FFH07]).

In this article, we study a one-dimensional lattice system of unbounded real-valued spins. The system consists of a finite number of sites $i \in \Lambda \subset \mathbb{Z}$ on the lattice $\mathbb{Z}$. For convenience, we assume that the set $\Lambda$ is given by $\{1, \ldots, N\}$. At each site $i \in \Lambda$ there is a spin $x_i$. In the Ising model the spins can take on the value 0 or 1. In this study, spins $x_i \in \mathbb{R}$ are real-valued and unbounded. A configuration of the lattice system is given by a vector $x \in \mathbb{R}^K$. The energy of a configuration $x$ is given by the Hamiltonian $H : \mathbb{R}^K \to \mathbb{R}$ of the system. For the detailed definition of the Hamiltonian $H$ we refer to Section 2. We consider arbitrary strong, quadratic, finite-range interaction.
We consider two different ensembles: The first ensemble is the grand canonical ensemble (gce) which is given by the finite-volume Gibbs measures

$$\mu^\sigma(dx) = \frac{1}{Z} \exp \left( \sigma \sum_{i=1}^N x_i - H(x) \right) dx.$$ 

Here, $Z$ is a generic normalization constant making the measure $\mu^\sigma$ a probability measure. The constant $\sigma \in \mathbb{R}$ is interpreted as an external field. The second ensemble is the canonical ensemble (ce). It emerges from the gce by conditioning on the mean spin

$$m = \frac{1}{N} \sum_{i=1}^N x_i.$$ 

The ce is given by the probability measure

$$\mu_m(dx) = \mu^\sigma \left( dx \mid \frac{1}{N} \sum_{i=1}^N x_i = m \right) = \frac{1}{Z} \mathbb{1}_{\left\{ \frac{1}{N} \sum_{i=1}^N x_i = m \right\}}(x) \exp(-H(x)) \mathcal{L}^{N-1}(dx),$$ 

where $\mathcal{L}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.

There are many different ways to characterize phase transitions. In this article we use the convention that an ensemble has no phase transition if the associated infinite-volume Gibbs measure is unique. On the two-dimensional lattice the gce has a phase transition (see for example [Pei36]). This is not the case if one considers the gce on the one-dimensional lattice. There, the gce does not have a phase transition if the interaction is finite-range or decays fast enough (see for example [Isi25, Dob68, Dob74, Rue68, MN14]). It is a natural question if those results extend from the gce to the ce. This is a non-trivial question since there are known examples where the gce has no phase transition but the ce has (see for example [SS96, BCR02, BCK03]).

If the spins are discrete, i.e. $\{0, 1\}$-valued, we guess that there is no phase transition for the ce on a one-dimensional lattice with nearest-neighbor interaction. This should follow from a result of Cancrini, Martinelli and Roberto [CMR02] (see also the introduction of [KM18a]). In this article we consider this question for real-valued and unbounded spins. Considering unbounded spins is harder because we lose compactness and one cannot transfer the arguments from the discrete case. It was conjectured in [KM18a] that the infinite-volume Gibbs measure of the canonical ensemble is unique on the one-dimensional lattice. A first step toward verifying this conjecture was taken in [KM18a]. There, it was shown that the gce and the ce are equivalent. This indicates that the gce and the ce should share similar properties. In this article, we show that the conjecture is indeed true: There is no phase transition for the ce on a one-dimensional lattice with finite-range interaction (see Theorem 2.16 from below).

For the proof of Theorem 2.16 we follow a standard argument (see e.g. [Yos03] or [Men14]) which is based on an ingredient of its own interest: decay of correlations. Decay of correlations is a classical ingredient when deducing the uniqueness of the infinite-volume Gibbs measure in lattice systems (see for example [DS87, Mar99, MN14] for the gce). In many lattice systems decay of correlations is one of many equivalent mixing conditions, including the Dobrushin-Slosshman mixing condition (see e.g. [DS85, DS87, DW90, Mar99, Yos03, HM16]). It is known that for one-dimensional lattice systems the correlations of the gce decay exponentially fast (see for example [Zeg96, MN14], references therein and Theorem 2.7 below). In this article, we extend results of [CM00] for discrete-spin systems to unbounded continuous-spins
(see also Section 3.2 in [CMR02]). More precisely, we show that the correlations of the ce and gce are equivalent (see Theorem 2.8 below). As a direct consequence we get that the correlations of the ce decay exponentially plus a volume correction term (see Theorem 2.9 below). We expect that this is a manifestation of a more general principle for systems on general lattices: If the gce has sufficient decay of correlations then the equivalence of correlations of the gce and ce should hold (see also [CT04]). In [KM18a], the decay of correlations of the ce (cf. Theorem 2.9) is an important ingredient in the proof that the ce with arbitrary strong, ferromagnetic, finite-range interaction satisfies a uniform log-Sobolev inequality on the one-dimensional lattice.

For the proof of Theorem 2.9 we apply a similar strategy as for the proof of the equivalence of ensembles in [KM18a]. We show that the decay of correlations can be transferred from the gce to the ce. The argument is robust and should apply to more general situations. The proof does not use that the lattice is one-dimensional. Instead, it only uses that the grand canonical ensemble on a one-dimensional lattice has an uniform exponential decay of correlations (see for example [MN14] and [Zeg96]). Under the assumption of fast enough decay of correlation, one should be able to use similar calculations to deduce decay of correlations of the ce for spin systems on arbitrary graphs, as long as the degree is uniformly bounded and the interaction has finite range. However, we only consider the one-dimensional lattice with finite-range interaction because less notational burden is better for explaining ideas and presenting the calculations.

There are many different aspects of equivalence of ensembles. For further background, we refer the reader to [SZ91, LPS94, Geo95, Ada06, Tou15]. In this article we follow the exposition of [CO17]. There, the equivalence of the ce and the micro-canonical ensembles was deduced for classical particle systems via a combination of an Edgeworth expansion, a local central limit theorem and a local large deviations principle. Equivalence of ensembles exists on the level of thermodynamic functions, on the level of observables and on the level of correlations. In [KM18a, Theorem 2], we showed a statement from which the equivalence of ensembles of the ce and gce on the level of thermodynamic functions follows. In this article, we will show that the equivalence of ensembles of the ce and gce also holds on the level of observables (see Theorem 2.4 below) and on the level of correlations (see Theorem 2.8). In order for correlations cov(f, g) to be equivalent, the involved functions f and g have to be local. If the functions are not local, the correlations and fluctuations, i.e. covariances and variances, depend on the ensemble. However, there is still a nice relation between the fluctuations expressed by the Lebowitz-Percus-Verlet formula [LPV67]. This formula was rigorously deduced for particle systems in [CO17]. It would be very interesting to deduce the Lebowitz-Percus-Verlet formula in our setting. This will be a lot harder than in [CO17]. The reason is that instead of deducing an Edgeworth expansion for independent random variables one would have to deduce an Edgeworth expansion for dependent random variables.

Let us mention some more open questions and problems:

- Instead of considering finite-range interaction, is it possible to deduce similar results for infinite-range, algebraically decaying interactions? More precisely, is it possible to extend the results of [MNT4] from the gce to the ce? Is the same algebraic order of decay sufficient, i.e. of the order $2 + \varepsilon$, or does one need a higher order of decay?
Is it possible to consider more general Hamiltonians? For example, our argument is based on the fact that the single-site potentials are perturbed quadratic, especially when we use the results of [KM18a]. One would like to have general super-quadratic potentials as was for example used in [MO13]. Also, it would be nice to consider general interactions than quadratic or pairwise interaction.

Is it possible to generalize the results to vector-valued spin systems?

We conclude the introduction by giving an overview over the article. In Section 2, we introduce the precise setting and present the main results. In Section 3, we provide several auxiliary results. In Section 4, we prove the equivalence of the the gce and the ce on the level of observables and correlations (cf. Theorem 2.6 and Theorem 2.8). We also prove the decay of correlations of the canonical ensemble (cf. Theorem 2.9) in Section 4. In Section 5, we show the uniqueness of the infinite-volume Gibbs measure (cf. Theorem 2.16).

Conventions and Notation

The symbol $T(k)$ denotes the term that is given by the line $(k)$.

We denote with $0 < C < \infty$ a generic uniform constant. This means that the actual value of $C$ might change from line to line or even within a line.

Uniform means that a statement holds uniformly in the system size $|\Lambda|$, the mean spin $m$, the boundary $x_{\Lambda}^A$ and the external field $s$.

$a \lesssim b$ denotes that there is a uniform constant $C$ such that $a \leq Cb$.

$a \sim b$ means that $a \lesssim b$ and $b \lesssim a$.

$\mathcal{L}^k$ denotes the $k$-dimensional Hausdorff measure. If there is no cause of confusion we write $\mathcal{L}$.

$Z$ is a generic normalization constant. It denotes the partition function of a measure.

For each $N \in \mathbb{N}$, $[N]$ denotes the set $\{1, \ldots, N\}$.

For a vector $x \in \mathbb{R}^Z$ and a set $A \subset \mathbb{Z}$, $x^A \in \mathbb{R}^A$ denotes the vector $(x^A)_i = x_i$ for all $i \in A$.

For a function $f : \mathbb{R}^Z \to \mathbb{C}$, denote $\text{supp} \ f$ by the minimal subset of $Z$ with $f(x) = f(x^{\text{supp} \ f})$.

A function $f : \mathbb{R}^Z \to \mathbb{C}$ is said to be local if $\text{supp} \ f$ is finite.

2. Setting and main results

We consider a system of unbounded continuous spins on the lattice $\mathbb{Z}$. The formal Hamiltonian $H : \mathbb{R}^Z \to \mathbb{R}$ of the system is defined as

$$H(x) = \sum_{i \in \mathbb{Z}} \left( \psi(x_i) + s_i x_i + \frac{1}{2} \sum_{j: 1 \leq |j-i| \leq R} M_{ij} x_i x_j \right)$$

$$= \sum_{i \in \mathbb{Z}} \left( \psi_b(x_i) + s_i x_i + \frac{1}{2} \sum_{j: |j-i| \leq R} M_{ij} x_i x_j \right),$$

where $\psi(z) := \frac{1}{2} z^2 + \psi_b(z)$ and $M_{ii} := 1$. We assume the following:

- The function $\psi_b : \mathbb{R} \to \mathbb{R}$ satisfies
  $$|\psi_b|_\infty + |\psi_b'|_\infty + |\psi_b''|_\infty < \infty.$$ 
  It is best to imagine $\psi(z) = \frac{1}{2} z^2 + \psi_b(z)$ as a double-well potential (see Figure 1).
The interaction is symmetric i.e.
\[ M_{ij} = M_{ji} \quad \text{for all distinct } i,j \in \mathbb{Z}. \]

The fixed, finite number \( R \in \mathbb{N} \) models the range of interactions between the particles in the system i.e. it holds that \( M_{ij} = 0 \) for all \( i,j \) such that \( |i - j| > R \).

The matrix \( M = (M_{ij}) \) is strictly diagonal dominant i.e. for some \( \delta > 0 \), it holds for any \( i \in \mathbb{Z} \) that
\[
\sum_{1 \leq |j-i| \leq R} |M_{ij}| + \delta \leq M_{ii} = 1.
\]

The vector \( s = (s_i) \in \mathbb{R}^\mathbb{Z} \) is arbitrary. It models the interaction with an inhomogeneous external field. Because the interaction is quadratic, this term also models the interaction of the system with the boundary.

Let us consider a finite sublattice \( \Lambda \subset \mathbb{Z} \). Given boundary values \( x^{\mathbb{Z} \setminus \Lambda} \in \mathbb{R}^{\mathbb{Z} \setminus \Lambda} \) we define the finite volume Hamiltonian \( H : \mathbb{R}^\Lambda \to \mathbb{R} \) as (using a small abuse of notation)
\[
H(x^\Lambda) := H(x^\Lambda, x^{\mathbb{Z} \setminus \Lambda})
= \sum_{i \in \Lambda} \left( \psi(x_i) + s_i x_i + \frac{1}{2} \sum_{j : 1 \leq |j-i| \leq R} M_{ij} x_i x_j \right)
= \sum_{i \in \Lambda} \left( \psi(x_i) + \left( s_i + \frac{1}{2} \sum_{j \in \mathbb{Z} \setminus \Lambda : 1 \leq |j-i| \leq R} M_{ij} x_j \right) x_i + \frac{1}{2} \sum_{j \in \Lambda : 1 \leq |j-i| \leq R} M_{ij} x_i x_j \right). \tag{1}
\]

We want to point out that in (1) the boundary values \( x^{\mathbb{Z} \setminus \Lambda} \) just modify the external field that is seen by a particular spin \( x_i \).

**Definition 2.1.** The gce \( \mu^{\Lambda,\sigma} \) associated to the Hamiltonian \( H \) with boundary values \( x^{\mathbb{Z} \setminus \Lambda} \) is the probability measure on \( \mathbb{R}^\Lambda \) given by the Lebesgue density
\[
\mu^{\Lambda,\sigma}(dx^\Lambda) := \frac{1}{Z} \exp \left( \sigma \sum_{k \in \Lambda} x_k - H(x^\Lambda, x^{\mathbb{Z} \setminus \Lambda}) \right) dx^\Lambda, \tag{2}
\]
where $\text{d}x^\Lambda$ denotes the Lebesgue measure on $\mathbb{R}^\Lambda$. The ce $\mu^\Lambda_m$ emerges from the gce by conditioning on the mean spin

$$\frac{1}{|\Lambda|} \sum_{k \in \Lambda} x_k = m.$$ More precisely, the ce $\mu^\Lambda_m$ is the probability measure on

$$X_{\Lambda,m} := \left\{ x^\Lambda \in \mathbb{R}^\Lambda : \frac{1}{|\Lambda|} \sum_{k \in \Lambda} x_k = m \right\} \subset \mathbb{R}^\Lambda$$

with density

$$\mu^\Lambda_m(dx^\Lambda) = \mu^\Lambda,\sigma \left( \frac{1}{|\Lambda|} \sum_{k \in \Lambda} x_k = m \right)$$

$$= \frac{1}{Z} \int_{\left\{ \frac{1}{|\Lambda|} \sum_{k \in \Lambda} x_k = m \right\}} \exp \left( -H(x^\Lambda, x^{\mathbb{Z}\setminus\Lambda}) \right) \mathcal{L}^{|\Lambda|-1}(dx^\Lambda),$$

where $\mathcal{L}^{|\Lambda|-1}(dx)$ denotes the $(|\Lambda|-1)$-dimensional Hausdorff measure supported on $X_{\Lambda,m}$.

**Definition 2.2** (The free energies of the gce and the ce). The free energy $A_{\text{gce}} : \mathbb{R} \to \mathbb{R}$ of the gce $\mu^\Lambda,\sigma$ is defined as

$$A_{\text{gce}}(\sigma) := \frac{1}{|\Lambda|} \ln \int_{\mathbb{R}^\Lambda} \exp \left( \sigma \sum_{k \in \Lambda} x_k - H(x^\Lambda, x^{\mathbb{Z}\setminus\Lambda}) \right) \text{d}x^\Lambda.$$ (4)

The free energy $A_{\text{ce}} : \mathbb{R} \to \mathbb{R}$ of the ce $\mu^\Lambda_m$ is

$$A_{\text{ce}}(\sigma) := \frac{1}{|\Lambda|} \ln \int_{X_{\Lambda,m}} \exp \left( \sigma \sum_{k \in \Lambda} x_k - H(x^\Lambda, x^{\mathbb{Z}\setminus\Lambda}) \right) \mathcal{L}^{|\Lambda|-1}(dx^\Lambda).$$

To relate the external field $\sigma$ of $\mu^\Lambda,\sigma$ and the mean spin $m$ of $\mu^\Lambda_m$ we make the following definition which will be justified in Section 3.

**Definition 2.3.** For each $m \in \mathbb{R}$, we choose $\sigma = \sigma(m) \in \mathbb{R}$ such that

$$\frac{d}{d\sigma} A_{\text{gce}}(\sigma) = m,$$

or vice versa. Setting $m_k := \int x_k \mu^\Lambda,\sigma (dx^\Lambda)$ we equivalently get

$$\frac{1}{|\Lambda|} \sum_{k \in \Lambda} m_k = m.$$ 

In this article we consider three different notions of equivalence of ensembles. The first one is the equivalence of the free energies in $C^2$ and was deduced in [KM18a].

**Theorem 2.4** (Theorem 2 in [KM18a]). It holds that

$$\lim_{\Lambda \to \mathbb{Z}} |A_{\text{gce}} - A_{\text{ce}}|_{C^2} = 0,$$

where the convergence is uniform in the external field $s$, the boundary $x^{\mathbb{Z}\setminus\Lambda}$ and the mean spin $m$. More precisely, for each $\varepsilon > 0$, there is an integer $N_0 \in \mathbb{N}$ such that for all $\Lambda \subset \mathbb{Z}$ with $|\Lambda| \geq N_0$,

$$\sup_{\sigma \in \mathbb{R}} |A_{\text{gce}}(\sigma) - A_{\text{ce}}(\sigma)| \leq \frac{1}{|\Lambda|},$$
Denote $C$ Theorem 2.8. Let

\[ \sup_{\sigma \in \mathbb{R}} \left| \frac{d}{d\sigma} A_{gce}(\sigma) - \frac{d}{d\sigma} A_{ce}(\sigma) \right| \lesssim \frac{1}{|\Lambda|^{1-\varepsilon}}, \]

\[ \sup_{\sigma \in \mathbb{R}} \left| \frac{d^2}{d\sigma^2} A_{gce}(\sigma) - \frac{d^2}{d\sigma^2} A_{ce}(\sigma) \right| \lesssim \frac{1}{|\Lambda|^{\frac{1}{2}-\varepsilon}}. \]

Remark 2.5. In [KM18a], we assumed that there is a uniform constant $C \in (0, \infty)$ with

\[ \frac{1}{|\Lambda|} \text{var}_{\mu^\Lambda,\sigma} \left( \sum_{k \in \Lambda} X_k \right) \geq C, \]

where $X = (X_k)_{k \in \Lambda}$ is a real-valued random variable distributed according to $\mu^{\Lambda,\sigma}$. However, this assumption can be removed in Theorem 2.4. Indeed, this lower bound of variance was only needed when deriving the uniform strict convexity of the free energy of the ce (cf. [KM18a, Corollary 1]) and the local Cramér theorem (cf. [KM18a, Theorem 3]).

The second one is the equivalence of observables.

**Theorem 2.6.** For a function $f : \mathbb{R}^Z \to \mathbb{C}$, denote $f$ by the minimal subset of $\mathbb{Z}$ with $f(x) = f(x^{\text{supp} f})$. Let $f : \mathbb{R}^Z \to \mathbb{R}$ be a local function, i.e. a function with finite support.

Then for each $\gamma > 2$ and $\varepsilon > 0$, there exist constants $N_0 \in \mathbb{N}$ and $\tilde{C} = \tilde{C}(\gamma, \varepsilon) \in (0, \infty)$ independent of the external field $s$, the boundary $x^{\mathbb{Z} \setminus \Lambda}$, and the mean spin $m$ such that for all $\Lambda \subset \mathbb{Z}$ with $|\Lambda| \geq N_0$ and $\Lambda \supset \text{supp} f$, it holds that

\[ \left| \mathbb{E}_{\mu^{\Lambda,\sigma}} [f] - \mathbb{E}_{\hat{\mu}^\Lambda} [f] \right| \leq \tilde{C} \frac{|\text{supp} f|^2}{|\Lambda|^{1-\varepsilon}} \|\nabla f\|_{L^2(\mu^{\Lambda,\sigma})}. \]

We outline the proof of Theorem 2.6 in Section 4.

In this article, we will study decay of correlations of the ce $\mu^\Lambda_m$. For that purpose let us, before we proceed, recall that for one-dimensional lattice systems the correlations of the gce decay exponentially fast ([KM18a, Lemma 6]. See also [MN14, Theorem 1.4]).

**Theorem 2.7** (Lemma 6 in [KM18a]). Let $f, g : \mathbb{R}^Z \to \mathbb{R}$ be local functions supported on $\Lambda$.

\[ \left| \text{cov}_{\mu^{\Lambda,\sigma}} (f, g) \right| \lesssim \| \nabla f \|_{L^2(\mu^{\Lambda,\sigma})} \| \nabla g \|_{L^2(\mu^{\Lambda,\sigma})} \exp (-C \text{dist} (\text{supp} f, \text{supp} g)). \] (5)

The last notion of equivalence of ensembles is the equivalence of correlations.

**Theorem 2.8.** Let $f, g : \mathbb{R}^Z \to \mathbb{R}$ be local functions and let $\gamma > 2$ be a fixed real number. Denote $C(f, g)$ by

\[ C(f, g) : = \| \nabla \left( (f(X) - \mathbb{E}_{\mu^{\Lambda,\sigma}} [f(X)]) \left( g(X) - \mathbb{E}_{\mu^{\Lambda,\sigma}} [g(X)] \right) \right) \|_{L^\gamma(\mu^{\Lambda,\sigma})} \]

\[ + \| \nabla f \|_{L^\gamma(\mu^{\Lambda,\sigma})} \| \nabla g \|_{L^\gamma(\mu^{\Lambda,\sigma})}. \] (6)

Then for each $\varepsilon > 0$, there exist constants $N_0$ and $\tilde{C} = \tilde{C}(\gamma, \varepsilon) \in (0, \infty)$ independent of the external field $s$, the boundary $x^{\mathbb{Z} \setminus \Lambda}$, and the mean spin $m$ such that for all $\Lambda \subset \mathbb{Z}$ with $|\Lambda| \geq N_0$ and $\Lambda \supset \text{supp} f$, it holds that

\[ \left| \text{cov}_{\mu^\Lambda_m} (f, g) - \text{cov}_{\mu^{\Lambda,\sigma}} (f, g) \right| \]

\[ \leq \tilde{C} C(f, g) \left( \frac{|\text{supp} f| + |\text{supp} g|^2}{|\Lambda|^{1-\varepsilon}} + \exp (-C \text{dist} (\text{supp} f, \text{supp} g)) \right). \] (7)
We give the proof of Theorem 2.8 in Section 4. A combination of Theorem 2.7 and Theorem 2.8 yields another main result of this article:

**Theorem 2.9** (Decay of correlations of the ce). Under the same assumptions as in Theorem 2.8, it holds that

\[
\left| \text{cov}_{\mu_m^A} (f,g) \right| \leq \tilde{C} \left( \left( \|f\| + \|g\| \right)^2 \frac{1}{|A|^{1-\varepsilon}} + \exp (-C \text{dist(supp } f, \text{ supp } g)) \right).
\]  

(8)

We give the proof of Theorem 2.9 in Section 4.

**Remark 2.10.** In [TP15] a similar result for classical particle systems was deduced. However, there are subtle and important differences between [TP15, Theorem 2.5] and Theorem 2.8. Obviously, one statement is about lattice systems and the other one is about particle systems. The next difference is that Theorem 2.8 yields the decay of correlations for arbitrary local functions \(f\) and \(g\) whereas [TP15, Theorem 2.5] only shows the decay of correlations for the two-point function, i.e. setting \(f(x) = x_i\) and \(g(x) = x_j\). Another subtle difference is that for [TP15, Theorem 2.5] one needs the validity of the cluster expansion for the ce. For deducing Theorem 2.8 on general lattices one would only need the decay of correlations for the gce. Cluster expansions and decay of correlations are closely related. For example, both hold in the high temperature regime. However, studying and deriving properties for the gce is a lot easier than for the ce.

**Remark 2.11.** The statement of Theorem 2.9 can be understood as an extension of a classical result of Cancrini and Martinelli from discrete spins to unbounded continuous spins (see Proposition 3.2 in [CMR02] or Section 7.3 in [CM00]).

In order to deduce decay of correlations via Theorem 2.9 one still needs to estimate the right hand side of (8). Fortunately, the right hand side only involves \(L^k\) norms with respect to the gce \(\mu_{m,\Lambda}^A\) and not the ce \(\mu_m^A\). Given the fact that under sufficient decay of correlations the gce \(\mu_{m,\Lambda}^A\) satisfies a uniform LSI and Poincaré inequality (cf. [HM16]), it is a lot easier to estimate \(L^k\) norms with respect to the gce \(\mu_{m,\Lambda}^A\) than with respect to the ce \(\mu_m^A\). For example, we have the following statement:

**Lemma 2.12.** For each \(i \in \Lambda\), define

\[
m_i := \int x_i \mu_{\Lambda,\sigma}^A (dx^\Lambda).
\]  

(9)

Then for any \(k \geq 1\), there is a constant \(C(k)\) independent of the system size \(|\Lambda|\), the external field \(s\), the boundary \(x^{\Lambda \setminus \Lambda}\) and the mean spin \(m\) such that

\[
E_{\mu_{\Lambda,\sigma}} \left[ |X_i - m_i|^k \right] \leq C(k) \quad \text{for all } i \in \Lambda.
\]

For the proof of Lemma 2.12 we refer to the proof of [KM18a, Lemma 5]. A combination of Theorem 2.9 and Lemma 2.12 yields the following statement.

**Corollary 2.13.** For given \(\varepsilon > 0\), there exist constants \(N_0 \in \mathbb{N}\) and \(\tilde{C} = \tilde{C}(\varepsilon)\) independent of the external field \(s\), the boundary \(x^{\Lambda \setminus \Lambda}\), and the mean spin \(m\) such that for any sublattice \(\Lambda \subset \mathbb{Z}\) with \(|\Lambda| \geq N_0\), it holds that for any \(i, j \in \Lambda\),

\[
\left| \text{cov}_{\mu_m^A} (X_i, X_j) \right| \leq \tilde{C} \left( \frac{1}{|\Lambda|^{1-\varepsilon}} + \exp (-C |i - j|) \right).
\]  

(10)
Remark 2.14. Comparing the estimate (10) of Corollary 2.13 with the estimate (5) of Theorem 2.7 one observes the additional term $\frac{1}{\Lambda^{1/2}}$ on the right hand side of (10). Eventually, this term could be improved to the order $\frac{1}{\Lambda}$ but not further. For example, assuming the random variables $X_1, \ldots, X_N$ are distributed according to a symmetric measure $\nu$ with fixed mean spin $m$, it holds that for all $i, j \in [N]$ with $i \neq j$

$$\text{cov}_\nu(X_1, X_2) = \text{cov}_\nu(X_i, X_j).$$

Therefore we have

$$\text{cov}_\nu(X_1, X_2) = \frac{1}{N-1} \text{cov}_\nu(X_1, X_2 + \cdots X_N)$$

$$= \frac{1}{N-1} \text{cov}_\nu(X_1, Nm - X_1) = -\frac{1}{N-1} \text{var}_\nu(X_1).$$

Let us turn to the next main result of this article, namely the uniqueness of the infinite-volume Gibbs measure of the ce.

Definition 2.15 (Infinite-volume Gibbs measure). Let $\mu$ be a probability measure on $\mathbb{R}^\mathbb{Z}$ with standard product Borel sigma-algebra. For any finite subset $\Lambda \subset \mathbb{Z}$ we decompose the measure $\mu$ into the conditional measure $\mu(dx^\Lambda|\mathbb{Z}\setminus\Lambda)$ and the marginal $\bar{\mu}(dx^{\mathbb{Z}\setminus\Lambda})$. This means that for any test function $f$ it holds

$$\int f(x)\mu(dx) = \int \int f(x^\Lambda, x^{\mathbb{Z}\setminus\Lambda}) \mu(dx^\Lambda|\mathbb{Z}\setminus\Lambda) \mu(dx^{\mathbb{Z}\setminus\Lambda}).$$

We say that $\mu$ is an infinite-volume Gibbs measure of the ce if the conditional measures $\mu(dx^\Lambda|\mathbb{Z}\setminus\Lambda)$ are given by the finite volume ce $\mu^{\Lambda}_m(dx^\Lambda)$ given by Definition 3 i.e.

$$\mu(dx^\Lambda|\mathbb{Z}\setminus\Lambda) = \mu^{\Lambda}_m(dx^\Lambda).$$

The equations of the last identity are called Dobrushin-Lanford-Ruelle (DLR) equations.

Theorem 2.16 (Uniqueness of the infinite-volume Gibbs measure of the ce). Let $H$ be a Hamiltonian that satisfies the assumptions described at the beginning of this section. Then there is only one infinite-volume Gibbs measure of the ce that satisfies the uniform bound

$$\sup_{i \in \mathbb{Z}} \text{var}_\mu(x_i) < \infty. \quad \text{(11)}$$

We deduce Theorem 2.16 in Section 5. The main ingredient in the proof is the decay of correlations (cf. Theorem 2.9).

Remark 2.17. In this article, we only show the uniqueness of infinite-volume Gibbs measure of the ce, not the existence. However, the authors of this article believe that with a cosmetic change, the existence should follow by a compactness argument (see for example [BHK82]).

3. Auxiliary Lemmas

In this section we provide several auxiliary results. All those results were proved in [KM18a] for lattice systems with nearest-neighbor interaction. However, it is not hard to see that the arguments in [KM18a] can be generalized in a straight-forward manner to lattice systems with
finite range interaction $R < \infty$, which is considered in this article. Recall the definition of gce

$$\mu^{\Lambda,\sigma}(dx^\Lambda) := \frac{1}{Z} \exp \left( \sigma \sum_{k \in \Lambda} x_k - H(x^\Lambda, x^{\mathbb{Z}\setminus\Lambda}) \right) dx^\Lambda.$$ 

The next lemma provides a variance estimate for the gce $\mu^{\Lambda,\sigma}$.

**Lemma 3.1** (Lemma 3 in [KM18a]). There is a constant $C \in (0, \infty)$ independent of the system size $|\Lambda|$, the external field $s$, the boundary $x^{\mathbb{Z}\setminus\Lambda}$ and $\sigma$ such that

$$0 < \frac{1}{|\Lambda|} \text{var}_{\mu^{\Lambda,\sigma}} \left( \sum_{k \in \Lambda} X_k \right) \leq C.$$ 

Moreover, if the interaction is attractive, i.e.

$$M_{ij} \leq 0 \quad \text{for all distinct} \ i, \ j,$$

it holds that

$$\frac{1}{C} \leq \frac{1}{|\Lambda|} \text{var}_{\mu^{\Lambda,\sigma}} \left( \sum_{k \in \Lambda} X_k \right) \leq C.$$ 

Recall the definition of the free energy $A_{gce} : \mathbb{R} \to \mathbb{R}$ of the gce $\mu^{\Lambda,\sigma}$ given by

$$A_{gce}(\sigma) := \frac{1}{|\Lambda|} \ln \int_{\mathbb{R}^\Lambda} \exp \left( \sigma \sum_{k \in \Lambda} x_k - H(x^\Lambda, x^{\mathbb{Z}\setminus\Lambda}) \right) dx^\Lambda.$$ 

A direct consequence of Lemma 3.1 is the strict convexity of the free energy $A_{gce}$. More precisely, it holds

**Lemma 3.2** (Lemma 1 in [KM18a]). There is a constant $C \in (0, \infty)$ independent of the system size $|\Lambda|$, the external field $s$, the boundary $x^{\mathbb{Z}\setminus\Lambda}$ such that for all $\sigma \in \mathbb{R}$,

$$0 < \frac{d^2}{d\sigma^2} A_{gce}(\sigma) \leq C,$$

Moreover, in the case of attractive interaction, the free energy $A_{gce}$ is uniformly strictly convex in the sense that

$$\frac{1}{C} \leq \frac{d^2}{d\sigma^2} A_{gce}(\sigma) \leq C,$$

Using Lemma 3.2 we are able to relate the external field $\sigma$ of $\mu^{\Lambda,\sigma}$ and the mean spin $m$ of $\mu^{\Lambda}_m$ as follows:

**Definition 3.3.** We choose $\sigma = \sigma(m) \in \mathbb{R}$ and $m = m(\sigma) \in \mathbb{R}$ such that

$$\frac{d}{d\sigma} A_{gce}(\sigma) = m.$$ 

Recalling the definition of $m_i$, we equivalently get

$$\frac{1}{|\Lambda|} \sum_{k \in \Lambda} m_k = \frac{1}{|\Lambda|} \int \left( \sum_{k \in \Lambda} x_k \right) \mu^{\Lambda,\sigma}(dx^\Lambda) = m.$$ 

By strict convexity of $A_{gce}(\sigma)$, for each $m \in \mathbb{R}$ there exists a unique $\sigma = \sigma(m)$ satisfying (12) or vice versa.
The following two statements are consequences of Lemma 3.1.

**Lemma 3.4** (Extension of (23) in [KM18a]). Recall the definition (9) of $m_i$. For each $k \geq 1$, there is a constant $C = C(k)$ such that for each $i \in \Lambda$

$$E_{\mu^\sigma} \left| X_i - m_i \right|^k \leq C(k).$$

**Lemma 3.5** (Lemma 5 in [KM18a]). For any finite set $B_i \subset \Lambda$ and $k \in \mathbb{N}$, it holds that

$$\left| E_{\mu^\Lambda, \sigma} \left[ \sum_{i_1 \in B_1} \cdots \sum_{i_k \in B_k} (X_{i_1} - m_{i_1}) \cdots (X_{i_k} - m_{i_k}) \right] \right| \lesssim |B_1| \cdots |B_k|.$$ 

Lastly, let $g$ be the density of the random variable

$$\frac{1}{\sqrt{|\Lambda|}} \sum_{k \in \Lambda} (X_k - m) \overset{(\ref{eq:13})}{=} \frac{1}{\sqrt{|\Lambda|}} \sum_{k \in \Lambda} (X_k - m_k),$$

where the random vector $X = (X_k)_{k \in \Lambda}$ is distributed according to $\mu^\Lambda, \sigma$. The following proposition provides estimates for $g(0)$.

**Proposition 3.6** (Proposition 1 in [KM18a]). For each $\alpha > 0$ and $\beta > \frac{1}{2}$, there exist constants $C \in (0, \infty)$ and $N_0 \in \mathbb{N}$ independent of the external field $s$, the boundary $x, Z \setminus \Lambda$, and the mean spin $m$ such that for all $\Lambda$ with $|\Lambda| \geq N_0$, it holds that

$$1 \overset{C}{\leq} g(0) \leq C, \quad \left| \frac{d}{d\sigma} g(0) \right| \lesssim |\Lambda|^{\alpha} \quad \text{and} \quad \left| \frac{d^2}{d\sigma^2} g(0) \right| \lesssim |\Lambda|^{\beta}.$$ 

4. **Proof of Theorem 2.6, Theorem 2.8 and Theorem 2.9**

Because proof of Theorem 2.6 shares a lot of similarities with proof of Theorem 2.8, we shall only present the proof of Theorem 2.8 and Theorem 2.9. The proof of Theorem 2.8 is very detailed and contains all the ideas that are needed for the proof of Theorem 2.6.

The main idea of the proof of Theorem 2.8 is to write the left hand side of (7) in terms of $L^k$ norms with respect to gce $\mu^\Lambda, \sigma$ via Cramér’s representation. Since the gce $\mu^\Lambda, \sigma$ on a one-dimensional lattice satisfies a uniform LSI and Poincaré inequality (cf. [HM16]), the $L^k$ norms with respect to the gce $\mu^\Lambda, \sigma$ can be estimated relatively easily.

One difficulty of this argument is that we consider general local functions $f$ and $g$. Indeed, the case when $f$ and $g$ are point functions, i.e. $f(x) = x_i$ and $g(x) = x_j$ for some $i, j \in \Lambda$ (cf. Corollary 2.13), one can easily prove the theorem with the help of moment estimates given in Lemma 3.4. However, we overcome this difficulty by combining Lemma 3.4 with Poincaré inequality and Hölder’s inequality. For more details, we refer to Section 4.1.

Let us begin with a convention which will reduce our notational burden.

**Convention.** We assume that $\Lambda = [N] = \{1, \cdots, N\}$. Moreover, if there is no source of confusion, we write (with some abuse of notations) $\mu^\sigma := \mu^{[N], \sigma}$, $\mu_m := \mu_m^{[N]}$, $x = x^{[N]}$ and $H(x) = H(x^{[N]}, x^{Z \setminus [N]}).$
Let us introduce auxiliary notations and settings. Fix local functions \( f, g \) and define modified gce \( \mu_{\sigma_i, \sigma_j}^{s_i, s_j} \) and ce \( \mu_{m, \sigma_j}^{s_i, s_j} \) depending on \( \sigma_i, \sigma_j \in \mathbb{R} \) by

\[
\mu_{\sigma_i, \sigma_j}^{s_i, s_j} (dx) := \frac{1}{Z} \exp \left( \sigma \sum_{k=1}^{N} x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) dx,
\]

\[
\mu_{m, \sigma_j}^{s_i, s_j} (dx) := \frac{1}{Z} \mathbb{1}_{\left\{ \frac{1}{N} \sum_{k=1}^{N} x_k = m \right\}} (x) \exp \left( \sigma \sum_{k=1}^{N} x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) \mathcal{L}^{N-1}(dx).
\]

Note in particular that we have

\[
\mu^{0,0} = \mu, \quad \mu_{m}^{0,0} = \mu_{m}.
\]

The associated free energies are

\[
A_{gce}^{f,g} (\sigma_i, \sigma_j) := \frac{1}{N} \ln \int \exp \left( \sigma \sum_{k=1}^{N} x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) dx,
\]

\[
A_{ce}^{f,g} (\sigma_i, \sigma_j) := \frac{1}{N} \ln \int_{\left\{ \frac{1}{N} \sum_{k=1}^{N} x_k = m \right\}} \exp \left( \sigma \sum_{k=1}^{N} x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) \mathcal{L}^{N-1}(dx).
\]

There are two ways to interpret free energies. First, the Cramér’s representation implies that the difference between two free energies is described by the density of a real-valued random variable distributed according to the modified gce \( \mu_{\sigma_i, \sigma_j}^{s_i, s_j} \). More precisely, we have

**Lemma 4.1.** Let \( Z = (Z_1, Z_2, \ldots, Z_N) \) be a real-valued random vector distributed according to \( \mu_{\sigma_i, \sigma_j}^{s_i, s_j} \). Then it holds that

\[
A_{ce}^{f,g} (\sigma_i, \sigma_j) - A_{gce}^{f,g} (\sigma_i, \sigma_j) = \frac{1}{N} \ln g_{\sigma_i, \sigma_j}(0),
\]

where \( g_{\sigma_i, \sigma_j} \) denotes the density of

\[
\frac{1}{\sqrt{N}} \sum_{k=1}^{N} (Z_k - m).
\]

**Proof of Lemma 4.1.** A direct calculation yields that

\[
A_{ce}^{f,g} (\sigma_i, \sigma_j) - A_{gce}^{f,g} (\sigma_i, \sigma_j) = \frac{1}{N} \ln \int_{\left\{ \frac{1}{N} \sum_{k=1}^{N} x_k = m \right\}} \exp \left( \sigma \sum_{k=1}^{N} x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) \mathcal{L}^{N-1}(dx)
\]

\[
- \frac{1}{N} \ln \int \exp \left( \sigma \sum_{k=1}^{N} x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) dx
\]

\[
= \frac{1}{N} \ln \int_{\left\{ \frac{1}{N} \sum_{k=1}^{N} x_k = m \right\}} \exp \left( \sigma \sum_{k=1}^{N} x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) \mathcal{L}^{N-1}(dx)
\]

\[
\int \exp \left( \sigma \sum_{k=1}^{N} x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) dx
\]

\[
= \frac{1}{N} \ln \int_{\left\{ \frac{1}{N} \sum_{k=1}^{N} (x_k - m) = 0 \right\}} \exp \left( \sigma \sum_{k=1}^{N} x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) \mathcal{L}^{N-1}(dx)
\]

\[
\int \exp \left( \sigma \sum_{k=1}^{N} x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) dx
\]
\[ = \frac{1}{N} \ln g_{\sigma_i, \sigma_j}(0). \]

Second, the following auxiliary lemma (Lemma 4.2) connects the free energies \( A_{f,g}^{\text{gce}} \) and \( A_{f,g}^{\text{ce}} \) of the measures \( \mu^{\sigma_i, \sigma_j} \) and \( \mu_{m, \sigma_i, \sigma_j} \) with covariances with respect to the gce \( \mu^\sigma \) and the ce \( \mu_m \).

**Lemma 4.2.** Let \( X \) and \( Y \) be real-valued random variables distributed according to the gce \( \mu^\sigma \) and the ce \( \mu_m \), respectively. Then it holds that

\[
\frac{d^2}{d\sigma_i d\sigma_j} A_{gce}^{f,g} \bigg|_{\sigma_i, \sigma_j = 0} = \frac{1}{N} \text{cov}_{\mu^\sigma} (f(X), g(X)),
\]

(15)

\[
\frac{d^2}{d\sigma_i d\sigma_j} A_{ce}^{f,g} \bigg|_{\sigma_i, \sigma_j = 0} = \frac{1}{N} \text{cov}_{\mu_m} (f(Y), g(Y)),
\]

(16)

**Proof of Lemma 4.2** We only provide the proof of (15). The formula (16) can be derived using the same type of argument. From the definition of \( A_{gce}^{f,g} \), it holds that

\[
\frac{d}{d\sigma_j} A_{gce}^{f,g} (\sigma_i, \sigma_j) = \frac{1}{N} \frac{\int g(x) \exp \left( \sigma \sum_{k=1}^N x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) dx}{\int \exp \left( \sigma \sum_{k=1}^N x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) dx}
\]

\[
= \frac{1}{N} \mathbb{E}_{\mu_{\sigma_i, \sigma_j}} [g(Z)],
\]

where \( Z = (Z_1, Z_2, \ldots, Z_N) \) is a random variable distributed according to \( \mu_{\sigma_i, \sigma_j} \). Taking partial derivative with respect to \( \sigma_i \) again yields

\[
\frac{d^2}{d\sigma_i d\sigma_j} A_{gce}^{f,g} (\sigma_i, \sigma_j) = \frac{1}{N} \mathbb{E}_{\mu_{\sigma_i, \sigma_j}} [f(Z)g(Z)] - \frac{1}{N} \mathbb{E}_{\mu_{\sigma_i, \sigma_j}} [f(Z)] \mathbb{E}_{\mu_{\sigma_i, \sigma_j}} [g(Z)]
\]

\[
= \frac{1}{N} \text{cov}_{\mu_{\sigma_i, \sigma_j}} (f(Z), g(Z)).
\]

Then (15) follows from the observation (14). \( \square \)

The next step towards the proof of Theorem 2.8 is the estimation of the density \( g_{0,0} \).

**Proposition 4.3.** Let \( g_{\sigma_i, \sigma_j} \) be defined as in Lemma 4.1. Given \( \gamma > 2 \) and \( \varepsilon > 0 \), there exist constants \( \tilde{C} = \tilde{C}(\gamma, \varepsilon) \in (0, \infty) \) and \( N_0 \in \mathbb{N} \) independent of the external field \( s \), the boundary \( x^\varepsilon \Lambda \), and the mean spin \( m \) such that for all \( N \geq N_0 \), the following estimation holds:

\[
\frac{1}{\tilde{C}} \leq g_{0,0}(0) \leq C,
\]

(17)

\[
\left| \frac{d}{d\sigma_i} g_{\sigma_i, \sigma_j}(0) \right|_{\sigma_i, \sigma_j = 0} \leq \tilde{C} \frac{\|f\|_{L^2(\mu^\sigma)} \|g\|_{L^2(\mu^\sigma)}}{N^{\frac{1}{2} - \varepsilon}},
\]

(18)

\[
\left| \frac{d}{d\sigma_j} g_{\sigma_i, \sigma_j}(0) \right|_{\sigma_i, \sigma_j = 0} \leq \tilde{C} \frac{\|g\|_{L^2(\mu^\sigma)} \|f\|_{L^2(\mu^\sigma)}}{N^{\frac{1}{2} - \varepsilon}},
\]

(19)

\[
\left| \frac{d^2}{d\sigma_i d\sigma_j} g_{\sigma_i, \sigma_j}(0) \right|_{\sigma_i, \sigma_j = 0} \leq \tilde{C} C(f, g) \left( \frac{(\|f\| + \|g\|)^2}{N^{1 - \varepsilon}} + \exp (-C \text{dist} (\text{supp } f, \text{supp } g)) \right),
\]

(20)
where \( C(f, g) \) is given by (6).

The statement of Proposition 4.3 should be compared to the Proposition 3.6. It is not surprising that the proof of Proposition 4.3 is similar to the proof of Proposition 3.6 (cf. proof of Proposition 1 in [KM18a]). For convenience, we do not give the proof of Proposition 4.3 in full detail. We only highlight the differences to the proof of Proposition 1 in [KM18a].

The proof of Proposition 4.3 is given in Section 4.1. Let us see how this proposition can be used to prove Theorem 2.8.

Proof of Theorem 2.8. A combination of Lemma 4.1 and Lemma 4.2 yields

\[
\text{cov} \mu_m(f(Y), g(Y)) - \text{cov} \mu^\sigma(f(X), g(X)) = \frac{d^2}{d\sigma_i d\sigma_j} \left( \ln g_{\sigma_i, \sigma_j}(0) \right) \bigg|_{\sigma_i, \sigma_j = 0} - \frac{1}{(g_{0, 0}(0))^2} \left( \frac{d}{d\sigma_i} g_{\sigma_i, \sigma_j}(0) \bigg|_{\sigma_i, \sigma_j = 0} \right) \left( \frac{d}{d\sigma_j} g_{\sigma_i, \sigma_j}(0) \bigg|_{\sigma_i, \sigma_j = 0} \right) + \frac{1}{g_{0, 0}(0)} \left( \frac{d^2}{d\sigma_i d\sigma_j} g_{\sigma_i, \sigma_j}(0) \bigg|_{\sigma_i, \sigma_j = 0} \right).
\]

Then the estimates (17), (18), (19) and (20) provide the desired bound (7). □

Let us now give the proof of Theorem 2.9.

Proof of Theorem 2.9. The desired statement is a direct consequence of Theorem 2.7 and Theorem 2.8. Indeed, it holds that

\[
\left| \text{cov} \mu^\Lambda_m(f, g) \right| \leq \left| \text{cov} \mu^\Lambda_m(f, g) \right| - \left| \text{cov} \mu^\Lambda \sigma_m(f, g) \right| + \left| \text{cov} \mu^\Lambda \sigma_m(f, g) \right| \leq C \left( \frac{|\text{supp } f| + |\text{supp } g|}{|\Lambda|^{1-\varepsilon}} \right) + \exp (-C \text{dist (supp } f, \text{supp } g))
\]

\[
+ C \| \nabla f \|_{L^2(\mu^\Lambda \sigma_m)} \| \nabla g \|_{L^2(\mu^\Lambda \sigma_m)} \exp (-C \text{dist (supp } f, \text{supp } g))
\]

\[
\lesssim C \left( \frac{|\text{supp } f| + |\text{supp } g|}{|\Lambda|^{1-\varepsilon}} \right) + \exp (-C \text{dist (supp } f, \text{supp } g))
\]

4.1. Proof of Proposition 4.3. Let us begin with an auxiliary computation. For any smooth function \( h : \mathbb{R}^N \rightarrow \mathbb{R} \), it holds that

\[
\frac{d}{d\sigma_i} \mathbb{E}_{\mu_{\sigma_i, \sigma_j}}[h(Z)] = \frac{d}{d\sigma_i} \int h(x) \frac{1}{Z} \exp \left( \sigma \sum_{k=1}^N x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) dx
\]

\[
= \int \frac{d}{d\sigma_i} h(x) \exp \left( \sigma \sum_{k=1}^N x_k + \sigma_i f(x) + \sigma_j g(x) - H(x) \right) dx
\]

\[
+ \int (f(x) - \mathbb{E}_{\mu_{\sigma_i, \sigma_j}}[f(Z)]) h(x) \exp \left( \sigma \sum_{k=1}^N x_k + \sigma_i x_i + \sigma_j x_j - H(x) \right) dx
\]
\[ = \mathbb{E}_{\mu}^{\sigma_i,\sigma_j} \left[ \frac{d}{d\sigma_i} h(Z) \right] + \mathbb{E}_{\mu}^{\sigma_i,\sigma_j} \left[ (f(Z) - \mathbb{E}_{\mu}^{\sigma_i,\sigma_j} [f(Z)]) h(Z) \right]. \] 

Using (21) we deduce the following formulas.

**Lemma 4.4.** Let \( m_k, k = 1, \ldots, N \) be given by (13). It holds that

\[ 2\pi g_{0,0}(0) = \int_{\mathbb{R}} \mathbb{E}_{\mu}^{\sigma} \left[ \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k) \xi \right) \right] d\xi, \] 

(22)

\[ 2\pi \frac{d}{d\sigma_i} g_{\sigma_i,\sigma_j}(0) \bigg|_{\sigma_i,\sigma_j=0} = \mathbb{E}_{\mu}^{\sigma} \left[ (f(X) - \mathbb{E}_{\mu}^{\sigma} [f(X)]) \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k) \xi \right) \right], \] 

(23)

\[ \frac{d^2}{d\sigma_i d\sigma_j} g_{\sigma_i,\sigma_j}(0) \bigg|_{\sigma_i,\sigma_j=0} = \mathbb{E}_{\mu}^{\sigma} \left[ (f(X) - \mathbb{E}_{\mu}^{\sigma} [f(X)]) (g(X) - \mathbb{E}_{\mu}^{\sigma} [g(X)]) \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k) \xi \right) \right]. \] 

(24)

The proof of Lemma 4.4 is a straightforward application of Fourier inversion and (21).

**Proof of Lemma 4.4.** We start with deriving (22). The inverse Fourier transform yields

\[ 2\pi g_{\sigma_i,\sigma_j}(0) = \int_{\mathbb{R}} \mathbb{E}_{\mu}^{\sigma_i,\sigma_j} \left[ \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (Z_k - m_k) \xi \right) \right] d\xi \]

(13)

\[ = \int_{\mathbb{R}} \mathbb{E}_{\mu}^{\sigma_i,\sigma_j} \left[ \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (Z_k - m_k) \xi \right) \right] d\xi. \] 

(25)

Setting now \( \sigma_i = \sigma_j = 0 \) in combination with (14) yields the desired formula (22).

Let us now turn to the verification of (23). Applying (21) to (25) yields

\[ 2\pi \frac{d}{d\sigma_i} g_{\sigma_i,\sigma_j}(0) = \int_{\mathbb{R}} \frac{d}{d\sigma_i} \mathbb{E}_{\mu}^{\sigma_i,\sigma_j} \left[ \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (Z_k - m_k) \xi \right) \right] d\xi \]

\[ = \int_{\mathbb{R}} \mathbb{E}_{\mu}^{\sigma_i,\sigma_j} \left[ (f(Z) - \mathbb{E}_{\mu}^{\sigma_i,\sigma_j} [f(Z)]) \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (Z_k - m_k) \xi \right) \right] d\xi. \]

Setting now \( \sigma_i = \sigma_j = 0 \) in combination with (14) yields the desired formula (23). Similar computations also yield formula (24). □

The statement of Proposition 4.3 follows directly from a combination of Lemma 4.4 and the estimates provided in the following statement.

**Lemma 4.5.** For each \( \gamma > 2 \) and \( \varepsilon > 0 \), there exist constant \( \tilde{C} = \tilde{C}(\gamma, \varepsilon) \) and \( N_0 \in \mathbb{N} \) independent of the external field \( s \), the boundary \( x_{Z\setminus \Lambda} \), and the mean spin \( m \) such that for all \( N \geq N_0 \),

\[ \frac{1}{\tilde{C}} \leq \int_{\mathbb{R}} \mathbb{E}_{\mu}^{\sigma} \left[ \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k) \xi \right) \right] d\xi \leq C, \] 

(26)
enough, there exists a positive constant \( C \).

**Lemma 4.6** (Extension of Lemma 7 in [KM18a]). For large enough \( N \) and \( \delta > 0 \) small enough, there exists a positive constant \( C > 0 \) such that the following inequalities hold for all \( \xi \in \mathbb{R} \) with \( \frac{|\xi|}{\sqrt{N}} \leq \delta \).

\[
\left| \mathbb{E}_{\mu^\sigma} \left[ f(X) - \mathbb{E}_{\mu^\sigma} [f(X)] \right] \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k) \xi \right) \right| \leq C \frac{\|\nabla f\|_{L^2(\mu^\sigma)} |\text{supp} f|}{N^{1/2 - \varepsilon}}
\]

where \( C(f,g) \) and \( C_2(f,g) \) is given by (6).

The rest of the section consists of the proof of Lemma 4.5 which is a lengthy calculation.

**Proof of Lemma 4.5.** We observe that the estimate (26) directly follows from Proposition 3.6. Here, we only provide a proof of (28). The estimate (27) can be derived by a similar computation.

Let us describe how we deduce the estimate (28). First of all, we may assume without loss of generality that

\[ \mathbb{E}_{\mu^\sigma} [f(X)] = \mathbb{E}_{\mu^\sigma} [g(X)] = 0. \]

We then make use of the idea presented the proof of [KM18a, Proposition 1]: We divide the integral into inner and outer parts and estimate them separately. Let us fix \( \delta > 0 \) small and decompose the integral as follows:

\[
\int_{\mathbb{R}} \mathbb{E}_{\mu^\sigma} \left[ f(X)g(X) \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k) \xi \right) \right] d\xi \\
= \int_{\{|(1/\sqrt{N})| \leq \delta\}} \mathbb{E}_{\mu^\sigma} \left[ f(X)g(X) \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k) \xi \right) \right] d\xi \quad (29) \\
+ \int_{\{|(1/\sqrt{N})| > \delta\}} \mathbb{E}_{\mu^\sigma} \left[ f(X)g(X) \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k) \xi \right) \right] d\xi. \quad (30)
\]

**Estimation of the inner integral (29):** We define the auxiliary sets \( F_1^{f,g} \) and \( F_2^{f,g} \) as (see Figure 2):

\[
F_1^{f,g} := \{1, 2, \ldots, N\} \cap \{k : \text{dist}(k, \text{supp}(f)) \leq L \text{ or } \text{dist}(k, \text{supp}(g)) \leq L\},
\]

\[
F_2^{f,g} := \{1, 2, \ldots, N\} \cap \{k : \text{dist}(k, \text{supp}(f)) > L \text{ and } \text{dist}(k, \text{supp}(g)) > L\},
\]

where \( L \ll N \) is a positive integer that will be chosen later. The main ingredients for this part are Theorem 2.7, Lemma 3.5, and an extension of [KM18a, Lemma 7].

**Lemma 4.6** (Extension of Lemma 7 in [KM18a]). For large enough \( N \) and \( \delta > 0 \) small enough, there exists a positive constant \( C > 0 \) such that the following inequalities hold for all \( \xi \in \mathbb{R} \) with \( \frac{|\xi|}{\sqrt{N}} \leq \delta \).

\[
\left| \mathbb{E}_{\mu^\sigma} \left[ \exp \left( i \sum_{k \in F_2^{f,g}} (X_k - m_k) \frac{\xi}{\sqrt{N}} \right) |\mathcal{F}_{f,g}\right] \right| \lesssim (1 + \xi^2) \exp (-C\xi^2).
\]
Figure 2. The sets $F_{1}^{f,g}$ and $F_{2}^{f,g}$ where $f(x) = x_i - m_i$ and $g(x) = x_j - m_j$.

where $\mathcal{F}_{f,g}$ denotes the sigma algebra defined by

$$\mathcal{F}_{f,g} := \sigma \left( X_k, \ k \in F_{1}^{f,g} \right).$$

**Remark 4.7.** The proof of Lemma 4.6 is almost similar to that of [KM18a, Lemma 7]. One should compare the sets $F_{1}^{f,g}, F_{2}^{f,g}$ with $F_{1}^{n,l}, F_{2}^{n,l}$ in [KM18a]. The main difference is that we assume finite range interaction with range $R$ instead of the nearest neighbor interaction. However, there is only a cosmetic difference between these two proofs. We leave the details to the reader.

Now, we have all ingredients to estimate (29). We define $e : \mathbb{R}^2 \to \mathbb{C}$ by

$$e(\xi_1, \xi_2) := \mathbb{E}_{\mu^\sigma} \left[ f(X)g(X) \exp \left( i \sum_{k \in F_{1}^{f,g}} (X_k - m_k) \xi_1 + i \sum_{l \in F_{2}^{f,g}} (X_l - m_l) \xi_2 \right) \right].$$

Then a Taylor expansion with respect to the first variable $\xi_1$ yields

$$e(\xi_1, \xi_2) = \mathbb{E}_{\mu^\sigma} \left[ f(X)g(X) \exp \left( i \sum_{l \in F_{2}^{f,g}} (X_l - m_l) \xi_2 \right) \right]$$

$$+ \mathbb{E}_{\mu^\sigma} \left[ f(X)g(X) \left( i \sum_{k \in F_{1}^{f,g}} (X_k - m_k) \right) \exp \left( i \sum_{l \in F_{2}^{f,g}} (X_l - m_l) \xi_2 \right) \right] \xi_1$$

$$+ \mathbb{E}_{\mu^\sigma} \left[ f(X)g(X) \left( i \sum_{k_1 \in F_{1}^{f,g}} (X_{k_1} - m_{k_1}) \right)^2 \right.$$

$$\times \exp \left( i \sum_{k_2 \in F_{1}^{f,g}} (X_{k_2} - m_{k_2}) \tilde{\xi}_1 + i \sum_{l \in F_{2}^{f,g}} (X_l - m_l) \xi_2 \right) \left. \right] \xi_1^2,$$

where $\tilde{\xi}_1$ is a real number between 0 and $\xi_1$. In particular for $(\xi_1, \xi_2) = \left( \frac{\xi}{\sqrt{N}}, \frac{\xi}{\sqrt{N}} \right)$, it holds that
It also follows from Theorem 2.7, Lemma 3.5 and Lemma 4.6 that

\[ \mathbb{E}_{\mu^*} \left[ f(X)g(X) \exp \left( i \sum_{k=1}^{N} (X_k - m_k) \frac{\xi}{\sqrt{N}} \right) \right] = \mathbb{E}_{\mu^*} \left[ f(X)g(X) \exp \left( i \sum_{l \in F_2^{f,g}} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right] \]

\[ + \mathbb{E}_{\mu^*} \left[ f(X)g(X) \left( i \sum_{k \in F_1^{f,g}} (X_k - m_k) \right)^2 \right. \]

\[ \times \exp \left( i \sum_{k_2 \in F_1^{f,g}} (X_{k_2} - m_{k_2}) \frac{\xi}{\sqrt{N}} + i \sum_{l \in F_2^{f,g}} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \left( \frac{\xi}{\sqrt{N}} \right)^2, \]

where \( \frac{\xi}{\sqrt{N}} \) is a real number between 0 and \( \frac{\xi}{\sqrt{N}} \).

Let us consider (32). By definition of covariances, it holds that

\[ T_{32} = \text{cov}_{\mu^*} \left( f(X)g(X), \exp \left( i \sum_{l \in F_2^{f,g}} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right) \]

\[ + \mathbb{E}_{\mu^*} \left[ f(X)g(X) \right] \mathbb{E}_{\mu^*} \left[ \exp \left( i \sum_{l \in F_2^{f,g}} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right]. \]

Then a combination of Theorem 2.7 and Lemma 3.5 yields

\[ |T_{32}| \lesssim \| \nabla (fg) \|_{L^2(\mu^*)} |\xi| \exp (-CL). \]

It also follows from Theorem 2.7, Lemma 3.5 and Lemma 4.6 that

\[ |T_{33}| \lesssim \| \text{cov}_{\mu^*} (f(X), g(X)) \|_{L^2(\mu^*)} (1 + \xi^2) \exp (-CL \xi^2) \]

\[ \lesssim \| \nabla f \|_{L^2(\mu^*)} \| \nabla g \|_{L^2(\mu^*)} \exp (-C \text{dist} (\text{supp}(f), \text{supp}(g))) (1 + \xi^2) \exp (-C \xi^2). \]

Therefore recalling the definition (30) of \( C(f, g) \), we obtain

\[ |T_{32}| \leq |T_{33}| + |T_{34}| \]

\[ \lesssim \| \nabla (fg) \|_{L^2(\mu^*)} (1 + \xi^2) \exp (-CL \xi^2) \]

\[ + \| \nabla f \|_{L^2(\mu^*)} \| \nabla g \|_{L^2(\mu^*)} \exp (-C \text{dist} (\text{supp}(f), \text{supp}(g))) (1 + \xi^2) \exp (-C \xi^2) \]

\[ \lesssim C(f, g) (1 + \xi^2) \exp (-CL \xi^2) \]

\[ + C(f, g) \exp (-C \text{dist} (\text{supp}(f), \text{supp}(g))) (1 + \xi^2) \exp (-C \xi^2). \]
Let us turn to the estimation of \((33)\). Denote
\[
\tilde{T}_{(33)} := \{ k \in F_1^{f,g} : \text{dist}(k, \text{supp}(f)) \leq \frac{L}{2} \text{ or } \text{dist}(k, \text{supp}(g)) \leq \frac{L}{2} \},
\]
and decompose \(T_{(33)}\) as
\[
T_{(33)} = i \frac{\xi}{\sqrt{N}} \sum_{k \in F_1^{f,g}} \mathbb{E}_{\mu^o} \left[ f(X)g(X) (X_k - m_k) \exp \left( i \sum_{l \in F_2^{f,g}} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right] \tag{37}
\]
\[+ i \frac{\xi}{\sqrt{N}} \sum_{k \in F_2^{f,g}} \mathbb{E}_{\mu^o} \left[ f(X)g(X) (X_k - m_k) \exp \left( i \sum_{l \in F_2^{f,g}} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right]. \tag{38}
\]

We estimate the terms \((37)\) and \((38)\) separately.

**Step 1.** Estimation of \((37)\).

For each \(k \in F_1^{f,g}\), we further decompose the summand in \(T_{(33)}\) as follows:
\[
\mathbb{E}_{\mu^o} \left[ f(X)g(X) (X_k - m_k) \exp \left( i \sum_{l \in F_2^{f,g}} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right] \tag{39}
\]
\[= \text{cov}_{\mu^o} \left( f(X)g(X) (X_k - m_k), \exp \left( i \sum_{l \in F_2^{f,g}} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right) \tag{40}
\]
\[+ \mathbb{E}_{\mu^o} [f(X)g(X) (X_k - m_k)] \mathbb{E}_{\mu^o} \left[ \exp \left( i \sum_{l \in F_2^{f,g}} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right]. \tag{41}
\]
Given \(\gamma > 2\), let \(\gamma^* \geq 1\) be a real number satisfying
\[
\frac{1}{2} = \frac{1}{\gamma} + \frac{1}{\gamma^*}.
\]
Then a combination of Theorem 2.7, Lemma 3.5 and Hölder’s inequality yields
\[
|T_{(33)}| \lesssim \| \nabla (f(X)g(X) (X_k - m_k)) \|_{L^2(\mu^o)} |\xi| \exp(-CL)
\]
\[\leq \left( \| \nabla (f(X)g(X)) (X_k - m_k) \|_{L^2(\mu^o)} + \| f(X)g(X) \|_{L^2(\mu^o)} \right) |\xi| \exp(-CL)
\]
\[\lesssim \left( \| \nabla (fg) \|_{L^2(\mu^o)} \| X_k - m_k \|_{L^2(\mu^o)} + \| f(X)g(X) \|_{L^2(\mu^o)} \right) |\xi| \exp(-CL)
\]
\[\text{Lemma 3.4} \lesssim \left( \| \nabla (fg) \|_{L^2(\mu^o)} + \| f(X)g(X) \|_{L^2(\mu^o)} \right) |\xi| \exp(-CL). \tag{42}
\]

Let us further estimate the term \(\| f(X)g(X) \|_{L^2(\mu^o)} \). Because \(\mu^o\) satisfies a uniform Poincaré inequality (cf. [HM16]), we have
\[
\| f(X)g(X) \|_{L^2(\mu^o)}^2 = \mathbb{E}_{\mu^o} [f(X)^2 g(X)^2]
\]
\[= \text{var}_{\mu^o} (f(X)g(X)) + \mathbb{E}_{\mu^o} [f(X)g(X)]^2
\]
A combination of (44) and (45) yields

\[ \| \nabla (fg) \|_{L^2(\mu^\sigma)}^2 + \text{cov}_{\mu^\sigma} (f(X), g(X))^2. \]

As a consequence, one gets by applying Theorem 2.7 that

\[ \| f(X)g(X) \|_{L^2(\mu^\sigma)} \lesssim \| \nabla (fg) \|_{L^2(\mu^\sigma)} + |\text{cov}_{\mu^\sigma} (f(X), g(X))| \]

\[ \lesssim \| \nabla (fg) \|_{L^2(\mu^\sigma)} + \| \nabla f \|_{L^2(\mu^\sigma)} \| \nabla g \|_{L^2(\mu^\sigma)} \exp (-C \text{dist (supp}(f), \text{supp}(g))) \]

\[ \lesssim C(f,g). \]

Plugging the estimate (43) into (42) yields

\[ |T_{11}| \lesssim C(f,g) |\xi| \exp (-CL) \lesssim C(f,g)(1 + \xi^2) \exp (-CL). \]

To estimate the term (41), we note by Pigeon hole principle that

\[ \text{max (dist (k, supp(f)), dist (k, supp(g)))} \geq \frac{\text{dist (supp}(f), \text{supp}(g))}{2}, \]

Assuming without loss of generality that \( \text{dist (k, supp}(f)) \geq \frac{\text{dist (supp}(f), \text{supp}(g))}{2} \), it holds

\[ |T_{11}| \lesssim |\text{cov}_{\mu^\sigma} (f(X), g(X) (X_k - m_k))| (1 + \xi^2) \exp (-C \xi^2) \]

\[ \leq \| \nabla f \|_{L^2(\mu^\sigma)} \| \nabla (g(X) (X_k - m_k)) \|_{L^2(\mu^\sigma)} \]

\[ \times \exp (-C \text{dist (supp}(f), \text{supp}(g))) (1 + \xi^2) \exp (-C \xi^2). \]

Note that we have

\[ \| \nabla (g(X) (X_k - m_k)) \|_{L^2(\mu^\sigma)} \leq \| \nabla g(X) (X_k - m_k) \|_{L^2(\mu^\sigma)} + \| g \|_{L^2(\mu^\sigma)} \]

\[ \leq \| \nabla g \|_{L^2(\mu^\sigma)} \| X_k - m_k \|_{L^2(\mu^\sigma)} + \| g \|_{L^2(\mu^\sigma)} \]

\[ \lesssim \| \nabla g \|_{L^2(\mu^\sigma)} \]

A combination of (44) and (45) yields

\[ |T_{11}| \lesssim C(f,g) \exp (-C \text{dist (supp}(f), \text{supp}(g))) (1 + \xi^2) \exp (-C \xi^2), \]

and as a consequence we obtain

\[ |T_{10}| \leq |T_{10}| + |T_{11}| \]

\[ \leq C(f,g) (1 + \xi^2) \exp (-CL) \]

\[ + C(f,g) \exp (-C \text{dist (supp}(f), \text{supp}(g))) (1 + \xi^2) \exp (-C \xi^2). \]

To conclude, we plug in this estimate into (37) and get

\[ |T_{17}| \lesssim C(f,g) |\xi| (1 + \xi^2) \exp (-CL) \frac{|\hat{f}_{1}\hat{g}|}{\sqrt{N}} \]

\[ + C(f,g) \exp (-C \text{dist (supp}(f), \text{supp}(g))) |\xi| (1 + \xi^2) \exp (-C \xi^2) \frac{|\hat{f}_{1}\hat{g}|}{\sqrt{N}}. \]
Step 2. Estimation of (38).
For $k \in \tilde{F}^{i,j}_2$, we decompose the summand in $T(38)$ by

$$\mathbb{E}_{\mu^o} \left[ f(X)g(X) (X_k - m_k) \exp \left( i \sum_{l \in F^{f,g}_2} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right] = \text{cov}_{\mu^o} \left( f(X)g(X), (X_k - m_k) \exp \left( i \sum_{l \in F^{f,g}_2} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right)$$

$$+ \mathbb{E}_{\mu^o} [f(X)g(X)] \mathbb{E}_{\mu^o} \left[ (X_k - m_k) \exp \left( i \sum_{l \in F^{f,g}_2} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right].$$

Then applying similar arguments as in Step 1 gives

$$|T(37)| \lesssim C(f,g) \left( 1 + \xi^2 \right) \exp (-CL),$$
$$|T(38)| \lesssim C(f,g) \exp (-C \text{dist} (\text{supp}(f), \text{supp}(g))) \left( 1 + \xi^2 \right) \exp (-C \xi^2).$$

Therefore we have

$$|T(39)| \leq |T(37)| + |T(38)| \lesssim C(f,g) \left( 1 + \xi^2 \right) \exp (-CL)$$

$$+ C(f,g) \exp (-C \text{dist} (\text{supp}(f), \text{supp}(g))) \left( 1 + \xi^2 \right) \exp (-C \xi^2),$$

and as a consequence

$$|T(38)| \lesssim C(f,g) \left| \xi \right| \left( 1 + \xi^2 \right) \exp (-CL) \frac{|\tilde{F}^{f,g}_2|}{\sqrt{N}}$$

$$+ C(f,g) \exp (-C \text{dist} (\text{supp}(f), \text{supp}(g))) \left| \xi \right| \left( 1 + \xi^2 \right) \exp (-C \xi^2) \frac{|\tilde{F}^{f,g}_2|}{\sqrt{N}}.$$

The two steps from above yields the desired estimate

$$|T(33)| \leq |T(37)| + |T(38)| \lesssim C(f,g) \left| \xi \right| \left( 1 + \xi^2 \right) \exp (-CL) \frac{|F^{f,g}_1|}{\sqrt{N}}$$

$$+ C(f,g) \exp (-C \text{dist} (\text{supp}(f), \text{supp}(g))) \left| \xi \right| \left( 1 + \xi^2 \right) \exp (-C \xi^2) \frac{|F^{f,g}_1|}{\sqrt{N}}.$$

Let us estimate (34). We consider the conditional expectation with respect to the sigma algebra $\mathcal{F}_{f,g} = \sigma (X_k, k \in F^{f,g}_1)$. Then it holds from Lemma 3.5 and Lemma 4.6 that
Then a combination of the estimates from above yields the desired estimate

Note also that

where

\( C \)

\[ T_{T\{31\}} = \left| \mathbb{E}_{\mu^e} \left[ f(X)g(X) \left( i \sum_{k_1 \in F_{1,g}^i} (X_{k_1} - m_{k_1}) \right)^2 \exp \left( i \sum_{k_2 \in F_{1,g}^i} (X_{k_2} - m_{k_2}) \frac{\xi}{\sqrt{N}} \right) \right] \times \mathbb{E}_{\mu^e} \left[ \exp \left( \sum_{l \in F_{2,g}^i} (X_l - m_l) \frac{\xi}{\sqrt{N}} \right) \right] \right| \frac{\xi^2}{N} \]

Lemma 4.0 \( \lesssim \) \( \mathbb{E}_{\mu^e} \left| f(X)g(X) \left( i \sum_{k_1 \in F_{1,g}^i} (X_{k_1} - m_{k_1}) \right)^2 \right| (1 + \xi^2) \exp (-C\xi^2) \]

Lemma 3.5 \( \lesssim \) \( \|f\|_{L^2(\mu^e)} \|g\|_{L^2(\mu^e)} \frac{|F_{1,g}^i|^2}{N} \xi^2 (1 + \xi^2) \exp (-C\xi^2) \)

where the last inequality follows from Poincaré inequality followed by Hölder’s inequality.

To conclude, we have

\[ |T_{T\{31\}}| \leq |T_{T\{32\}}| + |T_{T\{33\}}| + |T_{T\{34\}}| \]

\[ \lesssim C(f, g) \left( 1 + \xi^2 \right) \exp (-C\|L\|) + C(f, g) |\xi| \left( 1 + \xi^2 \right) \exp (-C\|L\|) \frac{|F_{1,g}^i|}{\sqrt{N}} \]

\[ + C(f, g) \exp (-C\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g))) \left( 1 + \xi^2 \right) \exp (-C\xi^2) \]

\[ + C(f, g) \exp (-C\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g))) |\xi| \left( 1 + \xi^2 \right) \exp (-C\xi^2) \frac{|F_{1,g}^i|}{\sqrt{N}} \]

\[ + C(f, g) \frac{|F_{1,g}^i|^2}{N} \xi^2 (1 + \xi^2) \exp (-C\xi^2) . \]

Note that for \( L = N^\varepsilon \ll N \) and \( N \) large enough, it holds that

\[ \int_{\{|\xi| \leq 1/\sqrt{N}\}} |\xi|^k (1 + \xi^2) \exp (-C\|L\|) \, d\xi \lesssim \frac{1}{N^2} \quad \text{for } k = 0, 1, \]

\[ \int_{\{|\xi| \leq 1/\sqrt{N}\}} |\xi|^k (1 + \xi^2) \exp (-C\xi^2) \, d\xi \lesssim 1 \quad \text{for } k = 0, 1, 2. \]

Note also that \( |F_{1,g}^i| \leq N \) and furthermore,

\[ |F_{1,g}^i| \leq (2L + |\operatorname{supp} f|) + (2L + |\operatorname{supp} g|) \]

\[ \leq 8L (|\operatorname{supp} f| + |\operatorname{supp} g|) . \]

Then a combination of the estimates from above yields the desired estimate

\[ |T_{T\{29\}}| \lesssim \tilde{C} C(f, g) \left( \frac{|\operatorname{supp} f| + |\operatorname{supp} g|}{N^{1-\varepsilon}} \right)^2 + \exp (-C\operatorname{dist}(\operatorname{supp} f, \operatorname{supp} g)) \]

where \( C(f, g) \) is defined by \[6\]
Estimation of the outer integral (30): This part is similar to the argument presented in [KM18a]. The main difference is again, we consider the finite range interaction with range $R$ instead of the nearest-neighbor interaction.

Consider the characteristic function $\varphi_W(\xi)$ of the random variable $W = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k)$

$$\varphi_W(\xi) = \mathbb{E}_{\mu^\sigma} \left[ \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k) \xi \right) \right].$$

Our strategy is to induce an artificial independence by conditioning. More precisely, let $[N]_R := \{R + 1, 2(R + 1), \ldots \} \cap [N]$ (see Figure 3). Because of the finite range interaction with range $R$ we have a product structure of conditional characteristic functions i.e.

$$\mathbb{E}_{\mu^\sigma} \left[ \exp \left( i \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (X_k - m_k) \xi \right) \right]$$

$$= \mathbb{E}_{\mu^\sigma} \left[ \exp \left( i \frac{1}{\sqrt{N}} \sum_{k \in [N] \setminus [N]_R} (X_k - m_k) \xi \right) \right] \times \mathbb{E}_{\mu^\sigma} \left[ \exp \left( i \frac{1}{\sqrt{N}} \sum_{l \in [N]_R} (X_l - m_l) \xi \right) \right] \mid X_j, j \in [N] \setminus [N]_R$$

$$= \mathbb{E}_{\mu^\sigma} \left[ \exp \left( i \frac{1}{\sqrt{N}} \sum_{k \in [N] \setminus [N]_R} (X_k - m_k) \xi \right) \right] \times \prod_{l \in [N]_R} \mathbb{E}_{\mu^\sigma} \left[ \exp \left( i \frac{1}{\sqrt{N}} (X_l - m_l) \xi \right) \right] \mid X_j, j \in [N] \setminus [N]_R.$$

Then the desired estimates easily follow from the same argument presented in [KM18a].

5. Proof of Theorem 2.16

Proof of Theorem 2.16: Suppose that there are two infinite-volume Gibbs measures $\mu$ and $\nu$ of the ce $\mu_m^\Lambda$. It suffices to prove that for any smooth function $f : \mathbb{R}^\mathbb{Z} \to \mathbb{R}$ with bounded support

$$\int f\mu = \int f\nu.$$

Let us fix a smooth function $f$ with bounded support. For each $r > 0$ define $B_r \subset \mathbb{Z}$ by

$$B_r := \{k \in \mathbb{Z} \mid -r < k < r\}$$
and choose $K > 0$ so that
\[ \text{supp } f \subset B_K. \] (49)

For each $r > K$ we decompose the measure $\mu$ into the conditional measure $\mu(dx^B_r|y \setminus B_r)$ and the marginal measure $\bar{\mu}(dy^{B_r})$, i.e. for any test function $g$
\[
\int g \mu = \int \int g(x^B_r, y^{B_r}) \mu(dx^B_r|y^{B_r}) \bar{\mu}(dy^{B_r}).
\]

Similarly, decompose the measure $\nu$ into $\nu(dx^B_r|y \setminus B_r)$ and $\bar{\nu}(dy^{B_r})$. Then it holds from (DLR) equations that
\[ \mu(dx^B_r|y \setminus B_r) = \nu(dx^B_r|y \setminus B_r) = \mu^B_r(dx^B_r|y \setminus B_r). \] (50)

For notational convenience we write $x = x^{B_r}$, $y = y^{B_r}$, and $z = z^{B_r}$. (51)

Note that (50) implies
\[
\left| \int f \mu - \int f \nu \right| = \left| \int \int f \mu(dx|y) \bar{\mu}(dy) - \int \int f \nu(dx|z) \bar{\nu}(dz) \right|
\leq \int \int \left| f \mu^B_r(dx|y) - f \mu^B_r(dx|z) \right| \bar{\mu}(dy) \bar{\nu}(dz).
\] (52)

We claim that the right hand side of $T_{(52)}$ becomes small when choosing $r > 0$ large enough. More precisely, we have the following estimate.

**Lemma 5.1.** Let $\varepsilon$ be a fixed positive number. Then it holds that
\[
\int \int \left| f \mu_m^B_r(dx|y) - f \mu_m^B_r(dx|z) \right| \bar{\mu}(dy) \bar{\nu}(dz)
\leq R^2 (|f|_{\infty} + |\nabla f|_{\infty}) \left( \frac{|\text{supp } f|^2}{r^{1-\varepsilon}} + \exp(-C(r - R - K)) \right).
\]

The statement from above finishes the proof of Theorem 2.16 by letting $r \to \infty$ and get
\[ \left| \int f \mu - \int f \nu \right| = 0. \]

□

Now let us turn to the proof of Lemma 5.1.

**Proof of Lemma 5.1.** By interpolation it holds that (recall the convention (51))
\[
\int f \mu^B_m(dx|y) - \int f \mu^B_m(dx|z) = \int_0^1 \left( \frac{d}{dt} \int f \mu^B_m(dx|ty + (1-t)z) \right) dt
= \int_0^1 \text{cov}_{\mu^B_m}(dx|ty + (1-t)z) f_{iB_r, j \notin B_r} \sum_{|i-j| \leq R} M_{ij} x_i(y_j - z_j) dt.
\] (53)
Let us consider the integrand in (53). To estimate the covariance with respect to the measure $\mu_{m}^{B_{r}}(dx|ty+(1-t)z)$, let us define the corresponding gce $\mu_{m}^{r,\tau}(dx|ty+(1-t)z)$ by

$$\mu_{m}^{r,\tau}(dx|ty+(1-t)z) = \frac{1}{Z} \exp \left( \tau \sum_{k \in B} x_{k} - H(x,ty+(1-t)z) \right) dx,$$

where we choose $\tau = \tau(m)$ such that (cf. Definition 3.3)

$$m = \frac{1}{|B_{r}|} \int \left( \sum_{k \in B_{r}} x_{k} \right) \mu_{m}^{r,\tau}(dx|ty+(1-t)z).$$

For a pair $(i,j)$ with $i \in B_{r}$, $j \notin B_{r}$ and $|i-j| \leq R$, the triangle inequality yields

$$|i| \geq |j| - |i-j| \geq r - R,$$

and in particular for $r > R + K$ (cf. (49) and Figure 4),

$$\text{dist}(\text{supp} f, \{i\}) \geq r - R - K.$$

Then a combination of Theorem 2.9 and Lemma 3.5 yields

$$\left| \text{cov}_{\mu_{m}^{r}}(dx|ty+(1-t)z) \left( f, \sum_{i \in B_{r}, \ j \notin B_{r} \atop |i-j| \leq R} M_{ij}x_{i}(y_{j} - z_{j}) \right) \right|$$

$$\leq \sum_{i \in B_{r}, \ j \notin B_{r} \atop |i-j| \leq R} |M_{ij}| |y_{j} - z_{j}| \left| \text{cov}_{\mu_{m}^{r}}(dx|ty+(1-t)z) (f, x_{i}) \right|$$

$$\lesssim (|f|_{\infty} + |\nabla f|_{\infty}) \left( \frac{|\text{supp} f|^{2}}{r^{1-\varepsilon}} + \exp(-C(r - R - K)) \right) \left( \sum_{i \in B_{r}, \ j \notin B_{r} \atop |i-j| \leq R} |y_{j} - z_{j}| \right). \quad (54)$$

Hence (53) and (54) imply

$$\int \int \left| \int f \mu_{m}^{B_{r}}(dx|y) - \int f \mu_{m}^{B_{r}}(dx|z) \right| \tilde{\mu}(dy) \tilde{\nu}(dz)$$

$$\leq (|f|_{\infty} + |\nabla f|_{\infty}) \left( \frac{|\text{supp} f|^{2}}{r^{1-\varepsilon}} + \exp(-C(r - R - K)) \right) \times \sum_{i \in B_{r}, \ j \notin B_{r} \atop |i-j| \leq R} \int |y_{j} - z_{j}| \tilde{\mu}(dy) \tilde{\nu}(dz).$$
Note that Cauchy’s inequality implies
\[
\int \int |y_j - z_j| \mu(dy) \mu(dz) \leq \left( \int \int (y_j - z_j)^2 \mu(dy) \mu(dz) \right)^{\frac{1}{2}}
\]
\[
= (2 \text{ var}_\mu (y_j))^{\frac{1}{2}} \lesssim 1.
\]
Because there are at most \(2R^2\) many pairs of \((i, j)\) with \(i \in B_r, j \notin B_r\) and \(|i - j| \leq R\), we have
\[
\int \int \left| \int f \mu_{m}^{B_r} (dx|y) - \int f \mu_{m}^{B_r} (dx|z) \right| \mu(dy) \nu(dz)
\]
\[
\lesssim R^2 (|f|_{\infty} + |\nabla f|_{\infty}) \left( \frac{|\text{supp} f|^2}{r^{1-\varepsilon}} + \exp \left( -C(r - R - K) \right) \right).
\]

\[\square\]

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