A Reduction Method for Semilinear Elliptic Equations and Solutions Concentrating on Spheres

by

Filomena Pacella
Dipartimento di Matematica, Università di Roma “Sapienza”
P.le A. Moro 2-00185, Roma -Italy
E-mail: pacella@mat.uniroma1.it

and

P.N. Srikanth
TIFR-CAM, Sharadanagar, Chikkabommasandra, Bangalore -560 065
E-mail:srikanth@math.tifrbng.res.in

Abstract

We show that any general semilinear elliptic problem with Dirichlet or Neumann boundary conditions in an annulus $A \subseteq \mathbb{R}^{2m}, m \geq 2$, invariant by the action of a certain symmetry group can be reduced to a nonhomogenous similar problem in an annulus $D \subseteq \mathbb{R}^{m+1}$, invariant by another related symmetry. We apply this result to prove the existence of positive and sign changing solutions of a singularly perturbed elliptic problem in $A$ which concentrate on one or two $(m-1)$ dimensional spheres. We also prove that the Morse indices of these solutions tend to infinity as the parameter of concentration tends to infinity.

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1 Introduction

In this paper we propose a method to reduce a semilinear elliptic problem of the type:

\[
\begin{cases}
-\Delta u = f(u) & \text{in} \quad A \subseteq \mathbb{R}^{2m} \\
u = 0 \quad \text{or} \quad \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial A
\end{cases}
\]  

(1.1)

where $A$ is an annulus in $\mathbb{R}^{2m}, m \geq 2$, $A = \{x \in \mathbb{R}^{2m} : a < |x| < b, \quad 0 < a < b\}$ and $f$ is a $C^{1,\alpha}$ nonlinearity, to the semilinear elliptic problem:

\[
\begin{cases}
-\Delta v = \frac{f(v)}{2|z|} & \text{in} \quad D \subseteq \mathbb{R}^{m+1} \\
v = 0 \quad \text{or} \quad \frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial D
\end{cases}
\]  

(1.2)

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where \( z \in \mathbb{R}^{m+1} \) and \( D \) is the annulus

\[
D = \left\{ z \in \mathbb{R}^{m+1} : \frac{a^2}{2} < |z| < \frac{b^2}{2} \right\}
\]

As will be clear from the construction, there will be a one to one correspondence between solutions of (1.1) invariant under the action of a symmetry group in \( H^1_0(A) \) or \( H^1(A) \) and solutions of (1.2) invariant under another symmetry in \( H^1_0(D) \) or \( H^1(D) \).

More precisely, writing \( x \in \mathbb{R}^{2m} \) as \( x = (y_1, y_2), y_i \in \mathbb{R}^m, i = 1, 2 \), we consider solutions \( u \) of (1.1) which are radially symmetric in \( y_1 \) and \( y_2 \) i.e. \( u(x) = w(|y_1|, |y_2|) \) and solutions \( v \) of (1.2) which are axially symmetric i.e. \( v(z) = h(|z|, \varphi) \) with \( \varphi = \arccos \left( \frac{\bar{z}}{|z|} \cdot p \right) \) for a unit vector \( p \in \mathbb{R}^{m+1} \).

Since the domains are annuli, by standard regularity theory all solutions we consider are classical \( C^{2,\alpha} \)-solutions. We set:

\[
X = \{ u \in C^{2,\alpha}(A) : u(x) = w(|y_1|, |y_2|) \}
\]
\[
Y = \{ v \in C^{2,\alpha}(D) : v \text{ axially symmetric } \}
\]

Our result is the following:

**Theorem 1.1.** There is a bijective correspondence between solutions of (1.1) in \( X \) and solutions of (1.2) in \( Y \).

The map which gives the bijection to prove Theorem 1.1 will be defined in Section 3 after choosing suitable coordinates in \( \mathbb{R}^{2m} \) and \( \mathbb{R}^{m+1} \).

The possibility of reducing a problem in dimension \( 2m \) to a problem in the lower dimension \( (m + 1) \) is of great importance in the study of semilinear elliptic equations. As example one can think of the case of power nonlinearities when critical or supercritical problems in \( \mathbb{R}^{2m} \) can become subcritical in \( \mathbb{R}^{m+1} \). Moreover solutions concentrating on sets of a certain dimension in \( \mathbb{R}^{m+1} \) (e.g. points) can give rise to solutions concentrating on higher dimensional manifolds on \( \mathbb{R}^{2m} \).

Indeed, the inspiration for our method came from the paper [14] where a reduction method was introduced to pass from a singularly perturbed problem in an annulus in \( \mathbb{R}^4 \) to a singularly perturbed problem in an annulus in \( \mathbb{R}^3 \) which allowed to prove the existence of solutions concentrating on \( S^1 \)-orbits in \( \mathbb{R}^4 \). Their construction is related to Hopf fibrations and an extension to other dimensions seems to be possible only in dimension 8 and 16 (see also [4]).

Our reduction works in all even dimensions, and in \( \mathbb{R}^4 \) allows to get the same result as in [14].

The new idea (compared to [14]) is to impose more symmetry on both problems and the key point is to identify symmetric solutions of (1.1) with axially symmetric solutions of (1.2), once the reduction has been made. In [14] the reduced problem in \( \mathbb{R}^3 \) did not have any particular symmetry.
Of course, this means that our method can be applied to “lift” solutions from $D \subset \mathbb{R}^{m+1}$ to $A \subset \mathbb{R}^{2m}$ when we know that the solutions in $D$ are axially symmetric. However, by the results of [12] and [13] (see also [8]) this is true for every solution of (1.2) with Morse index less than or equal to $(m + 1)$ if $f$ or $f'$ are convex. Actually in these papers only the Dirichlet problem is considered, but it is easy to see, arguing as in [9], that the symmetry results extend also to the Neumann problem. In particular the solutions in the annulus in $\mathbb{R}^3$ considered in [14] which are least energy positive solutions are axially symmetric. This is why our result applies to the case of [14]. As application of our reduction method we focus as in [14], on the following Dirichlet problem which was, indeed, the initial motivation for our result:

$$
\begin{cases}
-\Delta u + \lambda u = |u|^{p-1}u & \text{in } A \\
u = 0 & \text{on } \partial A
\end{cases}
$$

(1.3)

where $\lambda > 0$ and $p > 1$.

Note that (1.3) is equivalent to the singularly perturbed problem

$$
\begin{cases}
-\varepsilon^2 \Delta u + u = |u|^{p-1}u & \text{in } A \\
u = 0 & \text{on } \partial A
\end{cases}
$$

(1.4)

and to study (1.3) as $\lambda \to \infty$ is equivalent to studying (1.4) as $\varepsilon \to 0$. By applying Theorem 1.1, we reduce problem (1.3) to (1.2) with

$$
f(v) = |v|^{p-1}v - \lambda v
$$

(1.5)

If we take $p < \frac{(m+1)+2}{(m+1)-2}$, i.e. $p$ subcritical in dimension $(m+1)$, then the inhomogeneous problem (1.2), with $f$ given by (1.5), can be studied as in [14] adapting the methods of [10], [5] to produce least energy single peak positive solutions $v_\lambda$ which concentrated on the inner boundary of $D$ as $\lambda \to \infty$. Also, by adapting the method of [11] we show the existence of least energy two peaks nodal solutions $\tilde{v}_\lambda$ of (1.2) which again concentrate on the inner boundary as $\lambda \to +\infty$.

Since these solutions have Morse index 1 and 2 respectively, by the results of [12] and [13] (see also [2]) we deduce that they are foliated Schwarz symmetric, so, in particular, they are axially symmetric.

Then we can transform these solutions getting families of positive solutions of (1.3) concentrating on a $(m - 1)$ dimensional sphere and families of sign changing solutions of (1.3) concentrating on two $(m - 1)$- dimensional spheres. So, denoting by $(\partial A)_a = \{x \in \partial A, |x| = a\}$, we have

**THEOREM 1.2.** Let $1 < p < \frac{(m+1)+2}{(m+1)-2}$. Then there exists a family $\{u_\lambda\}$ of positive solutions of (1.3) concentrating on a $(m - 1)$- dimensional sphere $\Gamma \subset (\partial A)_a$ and a family $\{\tilde{u}_\lambda\}$ of sign changing solutions of (1.3) such that the positive part $\{\tilde{u}_\lambda^+\}$ and the negative part $\{\tilde{u}_\lambda^-\}$ concentrate on $(m - 1)$ dimensional spheres $\Gamma^+ \subset (\partial A)_a$ and $\Gamma^- \subset (\partial A)_a$, respectively, as $\lambda \to +\infty$.

Let us remark that the exponent $p \in \left(1, \frac{(m+1)+2}{(m+1)-2}\right)$ can be critical or supercritical for problem (1.3) in $\mathbb{R}^{2m}$. 

3
As it is clear from the construction these solutions have an $O(m) \times O(m)$ symmetry.

As far as we know the result of Theorem 1.2 is the first result for singularly perturbed Dirichlet problems about sign changing solutions concentrating on manifolds of dimension larger than or equal to 1. It is also a new result for positive solutions since the only previous ones concern concentration on $(2m-1)$ dimensional spheres ([1], [6]) or on 1-dimensional spheres in $\mathbb{R}^4$ ([14]).

Another interesting question connected with the concentration phenomena is the asymptotic behaviour of the Morse index of the solutions when $\lambda \to +\infty$. For the least energy positive or sign changing solutions concentrating in one or two points the Morse index is obviously independent of $\lambda$ and is 1 or 2. When the concentration takes place on a $(2m-1)$-dimensions sphere, which is the case of radial solutions, then it is easy to see that the Morse index tend to infinity, as $\lambda \to +\infty$. Indeed in this case the spectrum of the linearized operator can be split into a “radial” part and an “angular” part. For solutions concentrating on lower dimensional spheres, as in our case, a decomposition of the spectrum does not seem immediate. However we are able to show:

**THEOREM 1.3.** The Morse indices $m(u_\lambda)$ and $m(\tilde{u}_\lambda)$ of the solutions constructed in Theorems 1.2 tend to infinity as $\lambda \to +\infty$.

To prove Theorem 1.3. we test the quadratic form associated to the linearized operator at $u_\lambda$ and $\tilde{u}_\lambda$ by some functions obtained using eigenfunctions of the Laplace-Beltrami operator on $S^{m-1}$. Moreover we exploit that the first eigenvalue of the linearised operator at these solutions tends to $-\infty$ as $\lambda \to \infty$.

The result of Theorem 1.3 shows that the concentrating solutions we find become more and more unstable as $\lambda \to \infty$. This indicates that many local bifurcations should occur. However, since we do not know if that the families $\{u_\lambda\}$ or $\{\tilde{u}_\lambda\}$ give a curve in $X$, we cannot prove it rigorously. Note that if we knew that the least energy solutions (positive or nodal) in $D$ were unique (up to symmetry) then this would be true.

Finally we believe that this reduction method can be used to get different kind of results for other semilinear elliptic problems (as for example in [4]). Moreover it should be possible to generalize this approach to reduce problems in $\mathbb{R}^k$ to problem in $\mathbb{R}^h$, for suitable $h < k$ exploiting other symmetries.

The paper is organized as follows. In Section 2 we recall some results on symmetry of solutions of general semilinear elliptic equations. In Section 3 we introduce suitable coordinates and symmetries and prove Theorem 1.1. In Section 4 we show the concentration of the least energy solutions of problem (1.2). Finally in Section 5 we prove Theorem 1.2 and Theorem 1.3.
2 Axial Symmetry of solutions of semilinear elliptic equations.

Let us consider a general semilinear elliptic problem of the type:

\[
\begin{aligned}
-\Delta v &= f(|z|,v) \quad \text{in } B \\
v &= 0 \text{ or } \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B
\end{aligned}
\]  

(2.1)

where \(B\) is either an annulus or a ball centered at the origin of \(\mathbb{R}^N, N \geq 2, z \in \mathbb{R}^N\), and \(f: \overline{B} \times \mathbb{R} \to \mathbb{R}\) is (locally) a \(C^{1,\alpha}\)-function. We give the following definitions:

**DEFINITION 2.1.** We say that a function \(v \in C(B)\) is axially symmetric if there is a unit vector \(p \in \mathbb{R}^N, |p| = 1\) such that \(v(x)\) only depends on \(\rho = |z|\) and \(\varphi = \arccos \left( \frac{z}{|z|} \cdot p \right)\).

**DEFINITION 2.2.** If an axially symmetric function is also non increasing in the polar angle then it is called foliated Schwarz symmetric.

**REMARK 2.1.** Let us write \(z \in \mathbb{R}^N\) as \(z = (z_1, \ldots, z_n)\) and consider the spherical coordinates \((\rho, \varphi_1, \varphi_2, \ldots, \varphi_{N-1}), \varphi_i \in [0, \pi], i = 1, \ldots, N-2, \varphi_{N-1} \in [0, 2\pi], \rho = |z|, \) then

\[
\begin{aligned}
z_1 &= \rho \sin \varphi_1 \ldots \sin \varphi_{N-1} \\
z_2 &= \rho \sin \varphi_1 \ldots \sin \varphi_{N-2} \cos \varphi_{N-1} \\
\vdots \\
z_{N-1} &= \rho \sin \varphi_1 \cos \varphi_{N-2} \\
z_N &= \rho \cos \varphi_1
\end{aligned}
\]

(2.2)

Then if \(v\) is a axially symmetric function, without loss of generality, we can think that the vector \(p\) is \(p = (0,0,\ldots,1)\) i.e. the symmetry axis is the \(z_n\)-axis and therefore \(v\) depends only on \(\rho\) and \(\varphi_1\).

**REMARK 2.2.** If \(v\) is an axially symmetric function belonging to \(C^2(B)\) then the Laplace operator, using the above coordinates and the fact that \(v(x)\) only depends on \(\rho\) and \(\varphi_1\), reduces to

\[
\Delta_{R^N} v = v_{\rho\rho} + \frac{N-1}{\rho} v_\rho + \frac{N-2 \cos \varphi_1}{\rho^2} \sin \varphi_1 v_{\varphi_1} + \frac{1}{\rho^2} v_{\varphi_1\varphi_1}
\]

(2.3)

Some sufficient conditions on the nonlinearity and on a solution \(v\) of (2.1) for the foliated Schwarz symmetry have been obtained in [12], [13] (see also [8]) for the Dirichlet problem and they extend easily to the Neumann problem (see [9]).

We recall them here:

**THEOREM 2.1.** Let \(f(|z|, s)\) be either convex in the \(s\)-variable or with a convex first derivative \(f'(|z|, s) = \frac{\partial f}{\partial s}(|z|, s),\) for every \(z \in B\). Then any classical solution of (2.1) with Morse index \(j \leq N\) is foliated Schwarz symmetric.
We recall that the Morse index of a solution \( v \) of (2.1) is the number of the negative eigenvalues of the linearized operator at the solution \( L_v = -\Delta - f'(|z|, v) \) with the same boundary condition on \( \partial B \).

For least energy nodal solutions of (2.1) it is also useful to recall a similar result obtained in [2].

**Theorem 2.2.** If \( f(|z|, s) \) is subcritical i.e \( \exists p \in \left( 2, \frac{2N}{N-2} \right) \), if \( N \geq 3, p \in (2, \infty) \), if \( N \leq 2 \) and \( C > 0 \) such that

\[
f(|z|, s) \leq C(|s| + |s|^{p-1})
\]

and if the function \( s \to \frac{f(|z|, s)}{|s|} \) is strictly increasing on \( \mathbb{R}^- \) and \( \mathbb{R}^+ \) for all \( z \in B \), then the least energy nodal solution of (2.1) is foliated Schwarz symmetric.

### 3 Reduction and proof of Theorem 1.1.

Let us consider \( \mathbb{R}^{2m}, m \geq 2, \) as the product of two copies of \( \mathbb{R}^m \), i.e \( \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m \) and denote a point \( x \in \mathbb{R}^{2m} \) by \( x = (y_1, y_2), y_i \in \mathbb{R}^m, i = 1, 2. \)

Taking in each \( \mathbb{R}^m \) the spherical coordinates

\[
(\rho_1, \theta_1^1, \ldots, \theta_1^{m-1}), (\rho_2, \theta_2^1, \ldots, \theta_2^{m-1})
\]

\[
\rho_1 = |y_1|, \quad \rho_2 = |y_2|, \quad \theta_1^i \in [0, 2\pi],
\]

\[
\theta_2^j \in [0, \pi] \text{ for } i = 1, 2, \quad j = 2, \ldots, m - 1.
\]

(see Remark 2.1) and observing that

\[
\rho_1 = r \cos \theta, \quad \rho_2 = r \sin \theta, \quad r = |x|, \quad \theta \in [0, \frac{\pi}{2}]
\]

we have that a point \( x \in \mathbb{R}^{2m} \) can be represented by the coordinates

\[
x = (r, \theta_1^1, \ldots, \theta_1^{m-1}, \theta_2^1, \ldots, \theta_2^{m-1}, \theta)
\]

(3.1)

Then if we consider the annulus \( A \subset \mathbb{R}^{2m}, \)

\[
A = \{x \in \mathbb{R}^{2m}, a < |x| < b\}, \quad 0 < a < b
\]

a function \( u \in C^{2,\alpha}(\overline{A}) \) which is invariant under rotations in \( y_1 \) and \( y_2 \), i.e.

\[
u \in X = \{u \in C^{2,\alpha}(\overline{A}) : u(x = w(|y_1|, |y_2|)) \}
\]

(3.2)

in the above coordinates will depend only on \( r \) and \( \theta \), i.e. \( u = u(r, \theta) \)

Therefore for such functions the Laplace operator in \( \mathbb{R}^{2m} \), written in the above coordinates, reduces to

\[
\Delta_{\mathbb{R}^{2m}}u = u_{rr} + \frac{(2m-1)}{r}u_r + \frac{(m-1)}{r^2}u_\theta \left[ \frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} \right] + \frac{u_{\theta\theta}}{r^2}
\]
Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1** Let \( u \) be a solution of (1.1) in \( X \) and define the new variables:

\[
\rho = \frac{1}{2} r^2, \quad \varphi = 2\theta
\]  

and the function

\[
v(\rho, \varphi) = u(r(\rho), \theta(\varphi)) = u(\sqrt{2\rho}, \frac{\varphi}{2})
\]  

By easy computations we have

\[
u_r = v_\rho \sqrt{2\rho}, \quad u_{rr} = 2\rho v_{\rho\rho} + v_\rho \]

\[
u_\theta = 2v_\varphi, \quad u_{\theta\theta} = 4v_{\varphi\varphi}
\]

Therefore, by (3.2) and (1.1) we get that \( v \) satisfies

\[-2\rho \left[v_{\rho\rho} + \frac{mv_\rho}{\rho} + \frac{(m - 1)}{\rho^2} v_\rho \cos \varphi + \frac{v_{\varphi\varphi}}{\rho^2}\right] = f(v), \quad \rho \in \left(\frac{a^2}{2}, \frac{b^2}{2}\right), \varphi \in [0, \pi]
\]

Thus, by Remark 2.2 (with \( N = m + 1, \varphi_1 = \varphi, B = D \)) the function \( v(\rho, \varphi) \) is an axially symmetric solution of (1.2) in \( D \subset \mathbb{R}^{m+1} \), i.e. belongs to \( Y \). On the other hand starting, with an axially symmetric solution \( v = v(\rho, \varphi) \) of (1.2) and defining \( u(r, \theta) = v(\rho(r), \theta(\varphi)) \) with the same change of variables we get that \( u \in X \) and is a solution of (1.1) invariant by rotation in \( y_1 \) and \( y_2 \). Hence the theorem holds. 

\[\square\]

**REMARK 3.1.** To understand better the transformation (3.3) let us consider in \( \mathbb{R}^{2m} \) the group action:

\[T(x) = T(y_1, y_2) = (T_1(y_1), T_2(y_2)), \quad T_i \in O(m), \ i = 2.\]

It is easy to see that this action does not have fixed points in the annulus \( A \). Moreover the orbit of a point \( Q = (q_1, q_2) \in A, q_1 \in \mathbb{R}^m, q_2 \in \mathbb{R}^m \) is either \( S^{m-1} \times S^{m-1} \) if \( |q_i| \neq 0, i = 1, 2 \), or just \( S^{m-1} \) if one among \( |q_1| \) or \( |q_2| \) is zero.

Analogously we could consider in \( \mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R} \) the group action:

\[\tau(z) = \tau(z_1, \ldots, z_m, z_{m+1}) = (\tau_1(z_1, \ldots, z_m), z_{m+1}), \quad \tau_1 \in (O(m)).\]

In this case all points of the \( z_{m+1} \) axis are fixed by this action. Therefore by the change of variables (3.3) used in the proof of Theorem 1.1 it is easy to understand that any point \( P \) on the \( z_{m+1} \) axis in \( D \subset \mathbb{R}^{m+1} \), i.e. any fixed point under rotations about the \( z_{m+1} \) axis in the annulus \( D \subset \mathbb{R}^{m+1} \) is mapped into an \( S^{m-1} \) orbit in \( A \subset \mathbb{R}^{2m} \). Indeed \( P \) has spherical coordinates in \( \mathbb{R}^{m+1} \) equal to

\[P \equiv (\rho, 0, 0, \varphi), \ \text{with} \ \rho = |z_{m+1}|, \ \varphi = 0 \ or \ \varphi = \pi.
\]

and hence corresponds in \( \mathbb{R}^{2m} \) to points \( Q \) with \( \theta \) coordinate equal to 0 or \( \frac{\pi}{2} \). Therefore, taking in each \( \mathbb{R}^m \) the spherical coordinates (see the beginning of this section)

\[(\rho_1, \theta_1^1, \ldots, \theta_1^{m-1}), \ (\rho_2, \theta_2^1, \ldots, \theta_2^{m-1})\]

either \( \rho_1 \) or \( \rho_2 \) (but never both!) will be zero.
4 Concentrating Solutions

In this section we consider the problem

\[ \begin{align*}
-\Delta v &= \frac{1}{2|z|} \left[ |v|^{p-1} v - \lambda v \right] \quad \text{in } D \\
v &= 0 \quad \text{on } \partial D
\end{align*} \]  

(4.1)

where \( D = \{ z \in \mathbb{R}^N, R_1 < |z| < R_2 \} \) \( 0 < R_1 < R_2 \), \( 1 < p < \frac{N+2}{N-2} \), \( N \geq 3, \lambda > 0 \). By known results we have

**Proposition 4.1.** For every \( \lambda > 0 \) problem (4.1) has a positive solution \( v_\lambda \) and a sign changing solution \( \tilde{v}_\lambda \) such that

(i) \( v_\lambda \) minimize the functional

\[ J_\lambda(u) = \int_D \left[ \frac{1}{2} |\nabla v|^2 + \frac{\lambda}{4|z|} v^2 - \frac{1}{2(p+1)|z|} |v|^{p+1} \right] \]  

(4.2)

on the Nehari manifold, in \( H^1_0(D) \),

\[ N_\lambda = \{ v \in H^1_0(D) : v \neq 0, \langle J'_\lambda(v), v \rangle = 0 \} \]  

(4.3)

(ii) \( \tilde{v} \) minimize \( J_\lambda(v) \) in \( H^1_0(D) \) on the nodal Nehari set

\[ N^+_\lambda = \{ v \in H^1_0(D) : v^\pm \neq 0, \langle J'_\lambda(v^\pm), v^\pm \rangle = 0 \} \]  

(4.4)

where \( v^\pm \) denotes either the positive or the negative part of \( v \).

(iii) \( v_\lambda \) has Morse index 1 while \( \tilde{v}_\lambda \) has Morse index 2 and only two nodal regions.

(iv) \( v_\lambda \) and \( \tilde{v}_\lambda \) are foliated Schwarz symmetric.

**Proof.** Since \( p < \frac{N+2}{N-2} \) (i) is a standard result in critical point theory. The existence of \( \tilde{v}_\lambda \) satisfying (ii) is proved in [3] and [2]. The Morse index claim (iii) is again classical for \( v_\lambda \) and proved in [2] for \( \tilde{v}_\lambda \) where it is also proved that \( \tilde{v}_\lambda \) has only two nodal regions. Finally the foliated Schwarz symmetry of \( v_\lambda \) and \( \tilde{v}_\lambda \) is a consequence of (iii) and Theorem 2.1 and Theorem 2.2.

We are interested in the asymptotic behaviour of \( v_\lambda \) and \( \tilde{v}_\lambda \) as \( \lambda \to +\infty \). For the positive solution \( v_\lambda \) we have

**Theorem 4.1.** For \( \lambda \) sufficiently large:

(i) \( v_\lambda \) has only one local maximum point \( P_\lambda \in D \) and \( \sqrt{\lambda}d(P_\lambda, \partial D) \to +\infty \) as \( \lambda \to \infty \), where \( d(\cdot, \rho, \partial D) \) denotes the distance from \( \partial D \).

(ii) \( P_\lambda \) belongs to the symmetry axis of \( v_\lambda \),

\[ P_\lambda \to P \in \{ z \in \partial D, |z| = R_1 \} \text{ and } v_\lambda \to 0 \text{ in } C^1_{\text{loc}}(D \setminus \{ P \}) \text{, as } \lambda \to +\infty. \]
Proof. The result (i) is proved in [14], adapting a theorem of [10]. The location of \( P_\lambda \) on the symmetry axis is a consequence of the foliated Schwarz symmetry since it implies that all critical points of \( v_\lambda \) are on the symmetry axis. Finally the convergence of \( P_\lambda \) to a point on the inner boundary and the concentration of \( v_\lambda \) in \( P \) have been proved in [14] again following the proof of [10] and [5].

To study the asymptotic behaviour of the least energy nodal solution \( \tilde{v}_\lambda \) of (4.1), as \( \lambda \to \infty \) we adapt the proofs of [11] where the asymptotic behaviour of the least energy nodal solution is studied for the autonomous singularly perturbed problem

\[
\begin{cases}
  -\varepsilon^2 \Delta v + v = |v|^{p-1}v & \text{in } \Omega \\
  v = 0 & \text{on } \partial \Omega.
\end{cases}
\]

in a smooth bounded domain \( \Omega \). We also use the modifications of the proofs of [10] and [5] made in [14].

To start with, let us assume, without loss of generality, that the symmetry axis of \( \tilde{v}_\lambda \) is the \( x_N \)-axis. Then, as a consequence of the foliated Schwarz symmetry of \( \tilde{v}_\lambda \) we have that all critical points belong to the \( x_N \)-axis and, by the monotonicity with respect to the polar angle (see Definition 2.2) all local maximum points \( P_\lambda^+ \) are on the set \( \{ x = (x_1, \ldots, x_N) \in D, x_N > 0 \} \) while all local minimum points belong to the set \( \{ x = (x_1, \ldots, x_N) \in D, x_N < 0 \} \). This implies that \( |P_\lambda^+ - P_\lambda^-| > 2R_1 \) for all such points so that they cannot converge to the same point as \( \lambda \to +\infty \). We can prove

**THEOREM 4.2.** (i) For \( \lambda \) sufficiently large \( \tilde{v}_\lambda \) has only one positive local maximum point \( P_\lambda^+ \) and only one negative local minimum point \( P_\lambda^- \). Moreover \( \sqrt{\lambda}d(P_\lambda^+, \partial D) \to \infty \) as \( \lambda \to \infty \), as before \( d(\cdot, \partial D) \) denotes the distance from \( \partial D \).

(ii) Let \( P^+ \) and \( P^- \) be the limit points of \( P_\lambda^+ \) and \( P_\lambda^- \) respectively as \( \lambda \to \infty \). Then \( \tilde{v}_\lambda \to 0 \) in \( C^1_{\text{loc}}(D \setminus \{P^+, P^-\}) \)

(iii) \( P^+ \) and \( P^- \) belong to set \( \{ z \in \partial D, |z| = R_1 \} \)

**Proof.** The first part of assertion i) is similar to the proof of Lemma 3.3 in [11] which in turn, uses the same argument of [10]. The fact that \( \sqrt{\lambda}d(P_\lambda^+, \partial D) \to +\infty \) as \( \lambda \to +\infty \) can be deduced as in [14] (Proposition 4 there) by using a “boundary straightening”, a rescaling argument and the fact that limit problem

\[
\begin{cases}
  -\Delta u + \frac{u}{2|P^+|} - \frac{u^p}{2|P^-|} = 0 & \text{in } \Omega \\
  u > 0 \\
  u = 0 & \text{on } \partial \Omega
\end{cases}
\]

does not have solutions if \( \Omega \) is the half-space, by Theorem 1.1 of [7].

To prove ii) we use the modification of Proposition 3.4 of [10] derived in [14] and apply it to the positive part \( \tilde{v}_\lambda^+ \) and to the negative part \( \tilde{v}_\lambda^- \). This is based on the comparison with
the known radial solution $w_d$ of the problem.

\[
\begin{align*}
-\Delta w + \frac{1}{2d}w - \frac{1}{2d}w^p &= 0 \quad \text{in } \mathbb{R}^N \\
w(x) &\to 0 \quad \text{as } |x| \to \infty
\end{align*}
\]  \tag{4.6}

where $d$ is either $|P^+|$ or $|P^-|$. Thus, defining $\tilde{w}_\lambda^\pm(y) = \tilde{v}_\lambda \left( P^\lambda_\pm \frac{y}{\sqrt{\lambda}} \right)$ we get, as in Proposition 3.4 of \cite{14}:

For any $\delta \in (0,1)$ \exists $C > 0$ such that

\[
\tilde{w}_\lambda^\pm(y) \leq Ce^{\frac{\sqrt{\lambda}}{R_2}|y|}
\]

for $y \in \tilde{D}_\lambda^+ = \left\{ y \in \mathbb{R}^N : P^\lambda_\pm + \frac{y}{\sqrt{\lambda}} \in D \right\}$

Then, as in \cite{10} we get the assertion ii) - To conclude we prove that the limit points $P^+$ and $P^-$ belong to the inner boundary of $D = \{ z \in \partial D, |z| = R_1 \}$ i.e claim iii).

To do this, we simplify also the proof of \cite{14} for the least energy positive solution exploiting the fact that our domain is an annulus centred at the origin. Let us observe that the energy functional (4.2) on the solution $\tilde{v}_\lambda$ can be written as

\[
J_\lambda(\tilde{v}_\lambda) = J_\lambda(\tilde{v}_\lambda^+) + J_\lambda(\tilde{v}_\lambda^-)
\]

Then by rescaling $\tilde{v}_\lambda^\pm$ about $P^\lambda_\pm$ in the usual way we obtain, as in \cite{11} and arguing as in \cite{10}, that

\[
J_\lambda(u^\pm_\lambda) = I_{d^\pm}(w_{d^\pm}^\pm) + o(1) \quad \text{as } \lambda \to \infty
\]

where $w_{d^+}$ and $w_{d^-}$ are the positive solutions of

\[
\begin{align*}
-\Delta w + \frac{1}{2d}w - \frac{1}{2d}w^p &= 0 \quad \text{in } \mathbb{R}^N \\
w(x) &\to 0 \quad \text{as } |x| \to \infty
\end{align*}
\]  \tag{4.7}

with $d = d^+ = |P^+|$ or $d = d^- = |P^-|$ and $P^\lambda_\pm \to P^+, P^-_\lambda \to P^-$ respectively, and

\[
I_{d^\pm}(w_{d^\pm}) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w_{d^\pm}|^2 dx + \frac{1}{2d^\pm} \int_{\mathbb{R}^N} \frac{1}{2} |w_{d^\pm}|^2 dx - \frac{1}{d^\pm} \int_{\mathbb{R}^N} \frac{1}{p+1} |w_{d^\pm}|^{p+1} dx
\]

As observed in \cite{14} we have

\[
I_{d^\pm}(w_{d^\pm}) = \sqrt{2d^\pm} I(z) \quad \tag{4.8}
\]

where $z$ is the solution of the equation

\[
-\Delta z + z - z^p = 0 \quad \text{on } \mathbb{R}^N
\]

since $w_{d^\pm}(|x|) = z \left( \frac{|x|}{\sqrt{2d^\pm}} \right)$. 

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From (4.8) it is easy to understand that in order to reduce the energy the points \( P^\pm \) should converge to points \( P^\pm \) in the annulus which have the smallest distance from the origin. These are in fact the points on the inner boundary.

To prove it rigorously assume that one of the two points \( \{ P^\pm \} \), say \( P^+ \), converge to a point \( P^+ \in \overline{D} \) with \( |P^+| = R_1 + \alpha \) for some \( \alpha > 0 \). Then we could consider the ball \( B(Q, \alpha/3) \) with center in \( Q = (0, \ldots, 0, R_1 + \frac{\alpha}{3}) \) and define the function

\[
h^+(x) = \varphi(x)w_{d_{\alpha}}((x - Q)\sqrt{\lambda}) \quad \text{for } x \in D.
\]

where \( w_{d_{\alpha}} \) is the solution of (4.6) with \( d = d_{\alpha} = R_1 + \frac{\alpha}{3} \) and \( \varphi \) is a suitable cut off function such that \( h^+ \in C_0^2(B(Q, \alpha/3)) \). Hence \( h^+(x) \) is just the solution of (4.6) (for \( d_{\alpha} = R_1 + \frac{\alpha}{3} \)) suitably translated rescaled and cut to be defined in \( D \). Then

\[
J_\lambda(h^+) \to I_{d_{\alpha}}(w_{d_{\alpha}}) = \sqrt{2d_{\alpha}}I(z) < \sqrt{2d^+}I(z)
\]

as \( \lambda \to \infty \), by the choice of \( d_{\alpha} \). Then we consider the function

\[
h^-(x) = \psi(x)\tilde{v}^+(x)
\]

where \( \psi(x) \) is a suitable cut off function such that \( h^- \) and \( h^+ \) have disjoint supports. Note that this can be always done since \( |P^+ - P^-| > 2R_1 \) as pointed out before.

Finally, we can have that the function

\[
h(x) = h^+(x) - h^-(x)
\]

belong to the nodal Nehari set (4.3). Then we get

\[
J_\lambda(h) \to I_{d_{\alpha}}(w_{d_{\alpha}}) + I_{d^-}(w_{d^-}) < I^+(w_{d^+}) + I^-((w_{d^-}) \quad \text{as } \lambda \to \infty
\]

by comparison with (4.8) and (4.9). For \( \lambda \) large this contradicts the fact that \( \tilde{v}_\lambda \) is the least energy nodal solution as stated in Proposition 4.1. Hence (iii) is proved.

5 Proofs of Theorem 1.2 and Theorem 1.3

We start with the proof of Theorem 1.2 which, at this stage, is a direct consequence of the results of the previous sections.

**Proof of Theorem 1.2** Let \( 1 < p < \frac{(m+1)+2}{(m+1)-2} \) and consider problem (4.1) with such exponents \( p \) in the annulus \( D \subset R^{m+1}, (i.e \ N = m + 1, \text{ in Section 4}) \) with radii \( R_1 = \frac{a^2}{2}, R_2 = \frac{b^2}{2} \). By Proposition 4.1 there exist two families of solutions \( \{ v_\lambda \} \) and \( \{ \tilde{v}_\lambda \} \) which are foliated Schwarz symmetric, hence, in particular, they are axially symmetric and so belong to the space \( Y \).

Thus Theorem 1.1 applies and we get families of solutions \( \{ u_\lambda \} \) and \( \{ \tilde{u}_\lambda \} \) in \( X \) for problem (1.3). Finally, by Theorem 4.2 and Theorem 4.3, we have that \( v_\lambda \) concentrates in a point \( P_\lambda \) while \( \tilde{v}_\lambda \) concentrate in two points \( P^+_\lambda \) and \( P^-_\lambda \). All these points belong to the
symmetry axis, which is the $z_{m+1}$ axis and converge to points $P$, $P^+$ and $P^-$ lying on the inner boundary of $D$. Therefore by the transportation map (3.3) and Remark 3.1 we get that $u_\lambda$ and $\tilde{u}_\lambda$ have the claimed concentration properties.

Next we prove that the Morse indices $m(u_\lambda)$ of $u_\lambda$ and $m(\tilde{u}_\lambda)$ of $\tilde{u}_\lambda$ tend to infinity as $\lambda \to +\infty$. To this aim let us set

$$L_{u_\lambda} = -\Delta + \lambda I - pu_\lambda^{p-1}I$$

and

$$L_{\tilde{u}_\lambda} = -\Delta + \lambda I - p|\tilde{u}_\lambda|^{p-1}I$$

the linearised operators at $u_\lambda$ and $\tilde{u}_\lambda$ and define the associated quadratic forms:

$$Q_{u_\lambda}(\psi) = \int_A |\nabla \psi|^2 dx + \lambda \int_A |\psi|^2 dx - p \int_A u_\lambda^{p-1} \psi^2 dx$$

for $\psi \in H^1_0(A)$ and $Q_{\tilde{u}_\lambda}(\psi)$ defined analogously.

Let us denote by $\mu_j = \mu_j(\lambda)$ (respectively, $\tilde{\mu}_j = \tilde{\mu}_j(\lambda)$) the eigenvalues of $L_{u_\lambda}$ (resp. $L_{\tilde{u}_\lambda}$) in $H^1_0(A), j \in \mathbb{N}$. We have

**Lemma 5.1.** The eigenvalues $\mu_1, \tilde{\mu}_1, \tilde{\mu}_2$ tend to $-\infty$ as $\lambda \to \infty$.

**Proof.** Let us show it for $\mu_1$. We evaluate the quadratic form (5.3) on $u_\lambda$ itself. By the equation (1.3). We have

$$Q_{u_\lambda}(u_\lambda) = (1 - p) \int_A (|\nabla u_\lambda|^2 + \lambda |u_\lambda|^2) dx$$

Hence

$$\mu_1 \leq \frac{Q_{u_\lambda}(u_\lambda)}{\int_A |u_\lambda|^2 dx} = (1 - p) \left[ \frac{\int_A |\nabla u_\lambda|^2 dx}{\int_A |u_\lambda|^2 dx} + \lambda \right] \leq (1 - p)\lambda \to -\infty \text{ as } \lambda \to +\infty.$$ 

The same holds for $\tilde{\mu}_i, i = 1, 2$, using $\tilde{u}_\lambda^+$ and $\tilde{u}_\lambda^-$ as test functions to evaluate the quadratic form.

To show the asymptotic behavior of the Morse index of our solutions we construct a sequence $\{\Phi_k\}$ of $L^2$-orthogonal functions, on which the quadratic form (5.3) is negative for $\lambda$ large.

We need some preliminary notations and remarks. As in Section 3 a point $x \in \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$ is represented by $x = (y_1, y_2), y_i \in \mathbb{R}^m, i = 1, 2$. Then $y_1 = (\rho_1, \sigma_1), y_2 = (\rho_2, \sigma_2)$
with $p_i = |y_i|$, $\sigma_i \in S^{m-1} \subset \mathbb{R}^m(y_i)$, $i = 1, 2$ and $\rho_1 = r \cos \theta$, $\rho_2 = r \sin \theta$, $r = |x|$, $\theta \in [0, \pi/2]$. Thus we can represent $x \in \mathbb{R}^{2m}$ by
\[ x = (r, \sigma_1, \sigma_2, \theta). \]

Then the Laplace operator in $\mathbb{R}^{2m}$ can be expanded as:
\[ \Delta_{\mathbb{R}^{2m}} u = u_{rr} + \frac{(2m - 1)}{r} u_r + \frac{(m - 1)}{r^2} u_{\theta\theta} + \frac{1}{r^2 \cos^2 \theta} \Delta_{S^{m-1}}^1 u + \frac{1}{r^2 \sin^2 \theta} \Delta_{S^{m-1}}^2 u \]
where $\Delta_{S^{m-1}}^i$, $i = 1, 2$ is the Laplace-Beltrami operator on $S^{m-1}$ in the $\sigma_i$ -variable. Since the solutions $u_\lambda$ and $\tilde{u}_\lambda$ are radially symmetric in $y_1$ and $y_2$, i.e. belong to the space $X$ (see 3.2) the linearized operators are invariant under the same symmetry. Denoting by $g_1 = g_1(\lambda)$ (resp. $\tilde{g}_1 = \tilde{g}_1(\lambda)$) the first $L^2$- normalized eigenfunction of $L_{u_\lambda}$ (resp $L_{\tilde{u}_\lambda}$) in $H_0^1(A)$ easily have

**Lemma 5.2.** The eigenfunctions $g_1$ and $\tilde{g}_1$ belong to $H_0^1(A) \cap X$ i.e depend only on $(r, \theta)$.

**Proof.** Since $u_\lambda$ and $\tilde{u}_\lambda$ belong to $X$ then $L_{u_\lambda}$ and $L_{\tilde{u}_\lambda}$ are invariant by the same symmetry. Therefore the symmetry of $g_1$ and $\tilde{g}_1$ derives by the uniqueness of the first eigenfunction (up to normalization). Indeed the first eigenfunction of $L_{u_\lambda}$ and $L_{\tilde{u}_\lambda}$ in $H_0^1(A)$ and $H_0^1(A) \cap X$ must be the same. \hfill $\square$

Let us observe that, since $g_1$ depends only on $(r, \theta)$ it satisfies the problem
\[
\begin{aligned}
- (g_1)_{rr} &- \frac{2(m-1)}{r} (g_1)_r - \frac{m-1}{r^2} (g_1)_\theta - \frac{2 \cos 2\theta}{\sin 2\theta} \, \frac{(g_1)_\theta}{r^2} + \lambda g_1 - \mu p_\lambda^{p-1} g_1 = \mu_1 g_1 \\
\end{aligned}
\]
(5.5)

The analogous statement holds for $\tilde{g}_1$. Then let $\psi_k$ be the $k$-th eigenfunction of $-\Delta_{S^{m-1}}$ corresponding to the eigenvalue $\nu_k = k(k + m - 2)$, $m \geq 2, k \geq 1$. We have

**Lemma 5.3.** Define for any $k \geq 1$
\[ \Phi^k = g_1(r, \theta) \left[ \cos^2 \theta \psi_k(\sigma_1) + \sin^2 \theta \psi_k(\sigma_2) \right] \] (5.6)

Then, for any $k \geq 1$,
\[ Q_{u_\lambda}(\Phi^k) < 0, \quad \text{for } \lambda \text{ sufficiently large.} \]

**Proof.** Note that, by Lemma 5.2, only $g_1$ depends on $r$ while only $\psi_k$ depends on $\sigma_1$ or $\sigma_2$. Hence, the relevant terms in evaluating the quadratic form come only from the $\theta$ derivatives. Then by (5.4), (5.5), since $\psi_k(\sigma_1), \psi_k(\sigma_2)$ are eigenfunctions of $-\Delta_{S^{m-1}}$ corresponding to the same eigenvalue $\nu_k$ we obtain
\[
L_{u_\lambda} \Phi^k = \mu_1 \Phi^k + \frac{\nu_k}{r^2} g_1 \psi_k(\sigma_1) + \frac{\nu_k}{r^2} g_1 \psi_k(\sigma_2) - \frac{(m-1)}{r^2} g_1 2 \cos 2\theta [\psi_k(\sigma_2) - \psi_k(\sigma_1)] 
\]
\[
+ \frac{2}{r^2} [(g_1)_\theta \sin 2\theta + g_1 \cos 2\theta][\psi_k(\sigma_2) - \psi_k(\sigma_1)]
\]

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Using that $g_1$, $\psi_k$ can be taken as $L^2$-normalized and that $\psi_k$ has mean value zero on $S^{m-1}$, i.e
\[
\int_A g_1^2 = 1 \quad \int_{S^{m-1}} |\psi_k|^2 = 1 \quad \text{and} \quad \int_{S^{m-1}} \psi_k = 0
\]
and multiplying by $\Phi^k$ and integrating we get:
\[
Q_{u_\lambda}(\Phi^k) = \langle L_{u_\lambda} \Phi^k, \Phi^k \rangle = \mu_1 \int_A |\Phi^k|^2 dx + \nu_k \int_A \frac{1}{r^2} g_1^2 [\psi_k^2(\sigma_1) \cos^2 \theta + \psi_k^2(\sigma_2) \sin^2 \theta] dx +
\]
\[-2(m-1) \int_A \frac{1}{r^2} g_1^2 \cos 2\theta [\psi_k(\sigma_2) - \psi_k(\sigma_1)][\cos^2 \theta \psi_k(\sigma_1) + \sin^2 \theta \psi_k(\sigma_2)] dx +
\]
\[+ \int_A \frac{2}{r^2} (g_1)_\theta \cdot g_1 \sin 2\theta [\psi_k(\sigma_2) - \psi_k(\sigma_1)][\cos^2 \theta \psi_k(\sigma_1) + \sin^2 \theta \psi_k(\sigma_2)] dx +
\]
\[+ \int_A \frac{2}{r^2} (g_1)^2 \cos 2\theta [\psi_k(\sigma_2) - \psi_k(\sigma_1)][\cos^2 \theta \psi_k(\sigma_1) + \sin^2 \theta \psi_k(\sigma_2)] dx
\]
For the first term we have, by the previous remarks,
\[
\int_A |\Phi^k|^2 = \int_A g_1^2 [\cos^4 \theta + \sin^4 \theta] \geq \delta > 0
\]
where $\delta = \min_{[0, \pi/2]} (\sin^4 \theta + \cos^4 \theta) > 0$ Then, taking into account that $2(g_1)_\theta \cdot g_1 = (g_1^2)_\theta$, we finally get
\[
Q_{u_\lambda}(\Phi^k) \leq \mu_1 \delta + C_k \int_A |g_1|^2 = \mu_1 \delta + C_k
\]
for some constant $C_k$ independent of $\lambda$. Since $\mu_1 = \mu_1(\lambda) \to -\infty$ as $\lambda \to \infty$ we get the assertion.

Of course the statement of Lemma 5.3 holds also if we substitute $g_1$ with $\tilde{g}_1$. Thus

**Proof of Theorem 1.3.** We consider the sequences $\Phi^k$ defined by (5.8) and $\tilde{\Phi}^k = \tilde{g}_1(r, \theta)[\cos^2 \theta \psi_k(\sigma_1) + \sin^2 \theta \psi_k(\sigma_2)]$, $k \geq 1$ and observe that
\[
\int_A \Phi^k \Phi^j dx = \int_A \tilde{\Phi}^k \tilde{\Phi}^j = 0 \quad \text{for} \quad j \neq k
\]

By Lemma 5.3, for any $k \geq 1$ there exists $\lambda(k)$ such that $Q_{u_\lambda}(\Phi^k) < 0$ for $\lambda > \lambda(k)$. Thus for $\lambda \to -\infty$, the Morse index $m(u_\lambda)$ tends to infinity. The same applies to $m(\tilde{u}_\lambda)$.
References

[1] Ambrosetti, Antonio; Malchiodi, Andrea; Ni, Wei-Ming; Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. II. Indiana Univ. Math. J. 53 (2004), no. 2, 297–329.

[2] Bartsch, Thomas; Weth, Tobias; Willem, Michel; Partial symmetry of least energy nodal solutions to some variational problems. J. Anal. Math. 96 (2005), 1–18.

[3] Castro, Alfonso; Cossio, Jorge; Neuberger, John M.; A sign-changing solution for a superlinear Dirichlet problem. Rocky Mountain J. Math. 27 (1997), no. 4, 1041–1053.

[4] M. Clapp, J. Faya and A. Pistoia; Nonexistence and multiplicity of solutions to elliptic problems with supercritical exponents (preprint).

[5] Del Pino, Manuel; Felmer, Patricio L.; Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting. Indiana Univ. Math. J. 48 (1999), no. 3, 883–898.

[6] Esposito, P.; Mancini, G.; Santra, Sanjiban; Srikanth, P. N.; Asymptotic behavior of radial solutions for a semilinear elliptic problem on an annulus through Morse index. J. Differential Equations 239 (2007), no. 1, 1–15.

[7] Esteban, Maria J.; Lions, Pierre Louis; Existence and nonexistence results for semilinear elliptic problems in unbounded domains. Proc. Roy. Soc. Edinburgh Sect. A. 93 (1982), 1–14.

[8] Gladiali, Francesca; Pacella, Filomena; Weth, Tobias; Symmetry and nonexistence of low Morse index solutions in unbounded domains. J. Math. Pures Appl. (9)93 (2010), no. 5, 536–558.

[9] Montefusco, Eugenio; Axial symmetry of solutions to semilinear elliptic equations in unbounded domains. Proc. Roy. Soc. Edinburgh. A133 (2003), no. 5, 1175–1192.

[10] Ni, Wei-Ming; Wei, Juncheng; On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems. Comm. Pure Appl. Math. 48 (1995), no. 7, 731–768.

[11] Noussair, Ezzat S.; Wei, Juncheng; On the effect of domain geometry on the existence of nodal solutions in singular perturbations problems. Indiana Univ. Math. J. 46 (1997), no. 4, 1255–1271.

[12] Pacella, Filomena; Symmetry results for solutions of semilinear elliptic equations with convex nonlinearities. J. Funct. Anal. 192 (2002), no. 1, 271–282.

[13] Pacella, Filomena; Weth, Tobias; Symmetry of solutions to semilinear elliptic equations via Morse index. Proc. Amer. Math. Soc.135 (2007), no. 6, 1753–1762 (electronic).

[14] Ruf, Bernhard; Srikanth, P. N; Singularly perturbed elliptic equations with solutions concentrating on a 1-dimensional orbit. J. Eur. Math. Soc. 12 (2010), no. 2, 413 – 427.