Lie and conditional symmetries of the three-component diffusive Lotka–Volterra system

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Abstract
Lie and $Q$-conditional symmetries of the classical three-component diffusive Lotka–Volterra system in the case of one space variable are studied. The group-classification problems for finding Lie symmetries and $Q$-conditional symmetries of the first type are completely solved. Notably, non-Lie symmetries ($Q$-conditional symmetry operators) for a multi-component nonlinear reaction–diffusion system are constructed for the first time. The results are compared with those derived for the two-component diffusive Lotka–Volterra system. The conditional symmetry obtained for the non-Lie reduction of the three-component system used for modeling competition between three species in population dynamics is applied and the relevant exact solutions are found. Particularly, the exact solution describing different scenarios of competition between three species is constructed.

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1. Introduction

In 1952, Turing wrote a remarkable paper [1], in which a revolutionary idea about the mechanism of morphogenesis (the development of structures in an organism during the life) was proposed. From the mathematical point of view, Turing’s idea immediately led to the construction of reaction–diffusion (RD) systems (not single equations!) exhibiting the so-called Turing instability (see, e.g., chapter 14.3 in [2]). It should be stressed that nonlinear RD systems govern equations for many well-known nonlinear second-order models used to describe various processes in physics [3], biology [2, 4] and ecology [5]. Since 1952, nonlinear RD systems have been extensively studied by means of different mathematical methods, including group-theoretical methods. Nevertheless, finding Lie symmetries of two-component RD systems was initiated about 30 years ago [6], while a complete solution of this group-classification problem was obtained only a few years ago in [7, 8] (for constant diffusivities),
[9–11] (for non-constant diffusivities) and [12] (for constant cross-diffusion). It should be stressed that there are only a few papers devoted to the search for non-Lie (conditional, non-classical) symmetries of such systems [13–15]. Because finding non-Lie symmetries for RD systems is a very difficult problem [14, 16], only special cases of the general two-component RD system were examined in these papers.

All the papers cited above, except [11], deal with the two-component RD systems only. In section 4 of [11], Lie symmetries of a class of multi-component RD systems are completely described. To the best of our knowledge, there are no papers devoted to the construction of conditional symmetries for the multi-component RD systems. On the other hand, nonlinear multi-component RD systems are an important tool for mathematical modeling of a wide range of processes involving several kinds of species (cells, chemicals etc). Moreover, such systems possess some properties, which are not common for relevant two-component systems. Thus, it is time to extend the results obtained for two-component RD systems to the multi-component systems. It turns out that this is a highly non-trivial problem and this paper is devoted to solving this problem for the well-known three-component RD system.

In this paper, we shall consider the diffusive Lotka–Volterra (DLV) system

\[
\begin{align*}
\lambda_1 u_t &= u_{xx} + u(a_1 + b_1 u + c_1 v + d_1 w), \\
\lambda_2 v_t &= v_{xx} + v(a_2 + b_2 u + c_2 v + d_2 w), \\
\lambda_3 w_t &= w_{xx} + w(a_3 + b_3 u + c_3 v + d_3 w),
\end{align*}
\]

where \( u(t, x) \), \( v(t, x) \) and \( w(t, x) \) are unknown concentrations, and \( a_k, b_k, c_k, d_k \) and \( \lambda_k > 0 \) are arbitrary constants (hereafter \( k = 1, 2, 3 \) and the subscripts \( t \) and \( x \) denote differentiation with respect to these variables). These constants have a relevant biological (chemical) interpretation depending on the type of interaction between populations (cells, chemicals), which the system (1) describes [2, 4, 5]. Setting formally \( w = 0 \), we obtain the two-component DLV system

\[
\begin{align*}
\lambda_1 u_t &= u_{xx} + u(a_1 + b_1 u + c_1 v), \\
\lambda_2 v_t &= v_{xx} + v(a_2 + b_2 u + c_2 v),
\end{align*}
\]

which is the most common RD system. Lie symmetries of such a system have been completely described in [17], while our recent paper [15] was devoted to the search for its \( Q \)-conditional symmetries.

The system (1) is the standard generalization of the classical Lotka–Volterra system that takes into account the diffusion process for interacting species (see terms \( u_{xx}, v_{xx} \) and \( w_{xx} \)). Nevertheless, the classical Lotka–Volterra system was independently introduced by Lotka and Volterra about 90 years ago; its different generalizations are widely studied at the present time because of their importance for mathematical modeling of many processes in physics, biology, ecology etc. Notably, there are several recent papers devoted to the rigorous study of a multi-component DLV system (see [18–20] and references therein).

2. Main definitions

Because the DLV system (1) belongs to the general class of three-component RD systems

\[
\begin{align*}
\lambda_1 u_t &= u_{xx} + C^1(u, v, w), \\
\lambda_2 v_t &= v_{xx} + C^2(u, v, w), \\
\lambda_3 w_t &= w_{xx} + C^3(u, v, w),
\end{align*}
\]

where \( C^k(u, v, w) \) are arbitrary smooth functions, we formulate the main definitions for any system of the form (3).
Definition 1. Operator (4) is called the $Q$-conditional symmetry of the first type for the RD system (3) if the following invariance conditions are satisfied [16]:

\[
\begin{align*}
\frac{Q}{2} (S_1) &= \frac{Q}{2} (\lambda_1 u_t - u_{tx} - C^1(u, v, w)) \big|_{\mathcal{M}} = 0, \\
\frac{Q}{2} (S_2) &= \frac{Q}{2} (\lambda_2 v_t - v_{tx} - C^2(u, v, w)) \big|_{\mathcal{M}} = 0, \\
\frac{Q}{2} (S_3) &= \frac{Q}{2} (\lambda_3 w_t - w_{tx} - C^3(u, v, w)) \big|_{\mathcal{M}} = 0,
\end{align*}
\]

where the manifold $\mathcal{M}$ is either $\{S_1 = 0, S_2 = 0, S_3 = 0, Q(u) = 0\}$ or $\{S_1 = 0, S_2 = 0, S_3 = 0, Q(w) = 0\}$. Hereafter, the notations $Q(u) = \xi^0 u_t + \xi^1 u_x - \eta^1 v_t - \eta^2$, $Q(v) = \xi^0 u_t + \xi^1 v_x - \eta^1 v_t - \eta^2$, and $Q(w) = \xi^0 w_t + \xi^1 w_x - \eta^3$ are used.

Definition 2. Operator (4) is called the $Q$-conditional symmetry (non-classical symmetry in terminology used, e.g., in [24]) for the RD system (3) if the following invariance conditions are satisfied [16]:

\[
\begin{align*}
\frac{Q}{2} (S_1) &= \frac{Q}{2} (\lambda_1 u_t - u_{tx} - C^1(u, v, w)) \big|_{\mathcal{M}_3} = 0, \\
\frac{Q}{2} (S_2) &= \frac{Q}{2} (\lambda_2 v_t - v_{tx} - C^2(u, v, w)) \big|_{\mathcal{M}_3} = 0, \\
\frac{Q}{2} (S_3) &= \frac{Q}{2} (\lambda_3 w_t - w_{tx} - C^3(u, v, w)) \big|_{\mathcal{M}_3} = 0,
\end{align*}
\]

where the manifold $\mathcal{M}_3 = \{S_1 = 0, S_2 = 0, S_3 = 0, Q(u) = 0, Q(v) = 0, Q(w) = 0\}$.

It is easily seen that $\mathcal{M}_3 \subset \mathcal{M}_1 \subset \mathcal{M}$; hence, each Lie symmetry is automatically a $Q$-conditional symmetry of the first type, while each $Q$-conditional symmetry of the first type is also $Q$-conditional (non-classical) symmetry (see also [16] for an extensive discussion about a hierarchy of conditional symmetry operators).

Proposition 1. Let us assume that $X_1 = h^1_t (t, x, u, v, w) \partial_u + h^2_t (t, x, u, v, w) \partial_v$ (hereafter $h^1$ and $h^2$ are the given functions) is a Lie symmetry operator of the RD system (3), while $Q_1$ is a $Q$-conditional symmetry of the first type, which was found using the manifold $\mathcal{M}_1 = \{S_1 = 0, S_2 = 0, S_3 = 0, Q(u) = 0\}$. Then, any linear combination $C_1 X_1 + C_2 Q_1$ (hereafter $C_1$ and $C_2$ are arbitrary constants) produces another $Q$-conditional symmetry of the first type.

It should be stressed that this statement is not valid for arbitrary given $Q$-conditional symmetry, but only for that of the first type. Proposition 1 will be applied in the following section to prove theorem 2.
3. Main results

Obviously, the RD system (1) for arbitrary functions $C^k(u, v, w)$ admits the two-dimensional Lie algebra, called principal (or trivial) algebra, with the basic operators

$$P_t = \partial_t, \quad P_x = \partial_x.$$  

(7)

It can be easily shown that (7) is the principal algebra also for the DLV system (1). Note that we want to exclude the semi-coupled systems, i.e. those containing an autonomous equation; hence, hereafter the restrictions

$$c_1^2 + d_1^2 \neq 0, \quad b_2^2 + d_2^2 \neq 0, \quad b_3^2 + c_3^2 \neq 0.$$  

(8)

are assumed.

To find all possible extensions of principal algebra in the case of the DLV system (1), one needs to apply the invariance criteria (5) and to solve the obtained system of determining equations (DEs). Omitting rather standard calculations we present the system of DEs in question:

$$\xi_0^0 = \xi_0^1 = \xi_0^2 = \xi_0^3 = 0,$$  

(9)

$$\eta^1_{uv} = \eta^1_{vw} = \eta^1_{ww} = \eta^1_{uu} = \eta^1_{uv} = 0, \quad k = 1, 2.$$  

(10)

$$\eta^1_{ux} = \eta^1_{ux} = \eta^2_{ux} = 0,$$  

(11)

$$\eta^1_{ux} = (\lambda_1 - \lambda_2) \eta^2_{ux} = 0,$$  

(12)

$$\eta^1_{ux} = (\lambda_2 - \lambda_3) \eta^3_{ux} = 0,$$  

(13)

$$2\xi_1^0 - \xi_0^0 = 0,$$  

(15)

$$2\xi_0^2 + (\lambda_2 - \lambda_1) \xi_1^2 = 0,$$  

(16)

$$2\xi_0^3 + (\lambda_3 - \lambda_1) \xi_1^3 = 0,$$  

(17)

$$2\eta^1_{ux} + \lambda_1 \xi_1^1 = 0,$$  

(18)

$$2\eta^2_{ux} + \lambda_2 \xi_1^1 = 0,$$  

(19)

$$2\eta^3_{ux} + \lambda_3 \xi_1^1 = 0,$$  

(20)

$$\eta^0_{uw} + \eta^0_{vw} + \eta^0_{uw} + \eta^0_{uw} - \lambda_1 \eta_{1 w} + (2\xi_1^0 - \eta_{1 u}) C^1 - \eta_{1 v} C^2 - \eta_{1 w} C^3 = 0,$$  

(21)

$$\eta^0_{uw} + \eta^0_{vw} + \eta^0_{uw} + \eta^0_{uw} - \lambda_2 \eta_{2 w} + (2\xi_1^0 - \eta_{1 u}) C^2 - \eta_{1 v} C^1 - \eta_{1 w} C^3 = 0,$$  

(22)

$$\eta^0_{uw} + \eta^0_{vw} + \eta^0_{uw} + \eta^0_{uw} - \lambda_3 \eta_{3 w} + (2\xi_1^0 - \eta_{1 u}) C^3 - \eta_{1 v} C^1 + \eta_{1 w} C^2 = 0.$$  

(23)

Because the functions $C^k(u, v, w)$ have the known structure defined in the system (1) (otherwise the problem is very difficult even in the case of a two-component RD system [7, 8]), this system of DEs can be solved in a straightforward way. In fact, solving the subsystem (9)–(11), one obtains

$$\xi_0^0 = \xi_0^1 (t, x), \quad \xi_1^1 = \xi_1^1 (t, x),$$  

$$\eta^1 = r^1(t, x) u + q^1(t) v + h^1(t) w + p^1(t, x),$$  

$$\eta^2 = r^2(t, x) v + q^2(t, x) u + h^2(t) w + p^2(t, x),$$  

$$\eta^3 = r^3(t, x) w + q^3(t, x) u + h^3(t) v + p^3(t, x),$$  

(24)

where $\xi_0^0, \xi_1^1, \eta^1, \eta^2, \eta^3, r^1, h^1, h^2$ and $p^3$ are unknown functions at the moment. They can be found by the substitution of (24) into (12)–(23), and thus the integration of the linear system obtained. The result will depend on the coefficients $a_1, b_1, c_1, d_1$ and $\lambda_1 > 0$, and can be formulated as theorem 1.
Theorem 1. The DLV system (1) admits a non-trivial Lie algebra of symmetries if and only if one and the corresponding symmetry operator(s) have the forms listed in table 1. Any other DLV system admitting three-order and higher order Lie algebra is reduced to one of those from table 2 by a local transformation from the set (25). Simultaneously this Q-conditional operator is transformed to the corresponding operator listed in table 2.

Table 1. Lie symmetry operators of the DLV system (1).

| Reaction terms | Restrictions | Additional Lie symmetries |
|----------------|--------------|--------------------------|
| $u(b_1u + c_1v + d_1w)$ | $D = 2\partial_\lambda + x\partial_\lambda - 2(a\partial_\lambda + v\partial_\lambda + w\partial_\lambda)$ |
| $v(b_2u + c_2v + d_2w)$ |
| $w(b_3u + c_3v + d_3w)$ |
| $u(c_1v + d_1w)$ | $u\partial_\lambda$ |
| $v(c_2v + w)$ |
| $w(c_3v + w)$ |
| $u(a_1 + av + v)$ | $\lambda_2 = \lambda_3 = 1$ | $\exp(-a_2)u\partial_\lambda, \ w\partial_\lambda$ |
| $v(a_2 + u + cv)$ |
| $w(u + cv)$ |
| $u(a_1 + av + v)$ | $\lambda_1 = \lambda_2 = \lambda_3 = 1$ | $\exp(-a_2)u\partial_\lambda, \ w\partial_\lambda, \ (a_2u + a_1v)\partial_\lambda$ |
| $v(a_2 + u + cv)$ |
| $w(u + av)$ |
| $u(a + u + v)$ | $\lambda_1 = \lambda_2 = \lambda_3 = 1$ | $\exp(-a)u\partial_\lambda, \ w\partial_\lambda, \ (a + av)\partial_\lambda$ |
| $v(u + v)$ |
| $w(u + v)$ |
| $u(bu + v)$ | $\lambda_1 = \lambda_2 = \lambda_3 = 1$ | $D, \ w\partial_\lambda, \ ((b - 1)u + (1 - c)v)\partial_\lambda$ |
| $v(u + cv)$ |
| $w(bu + cv)$ |

where $c_{ij}$ are some correctly specified constants in each case ($i = 1, \ldots, 5, j = 0, \ldots, 3$).

Now we present the main result of this section.

Theorem 2. The DLV system (1) is invariant under Q-conditional operator(s) of the first type (4) (with $\xi^0 \neq 0$) if and only if one and the corresponding operator(s) have the forms listed in table 2. Any other DLV system admitting a Q-conditional operator of the first type is reduced to one of those from table 2 by a local transformation from the set (25). Simultaneously this Q-conditional operator is transformed to the corresponding operator listed in table 2 (up to equivalent representations generated by adding a Lie symmetry operator of the form $h^2(t, x, u, v, w)\partial_\lambda + h(t, x, u, v, w)\partial_\lambda$).

In table 2, the following designations are introduced:

$Q^2_i = Q^4_i$ with $\alpha = 0$, $i = 1, \ldots, 6$;

$Q^4_1 = \partial_t + \frac{a_1 - a_2}{\lambda_1 - \lambda_2}u(\partial_u - \partial_v) + \alpha u(\partial_v - \partial_w)$, $Q^4_2 = \partial_t + \frac{a_1 - a_2}{\lambda_1 - \lambda_2}v(\partial_v - \partial_w) + \alpha v(\partial_u - \partial_w)$,
\[
Q^3 = a + \alpha v(\partial_x - a) - \beta v - 1 - a(v\partial_v - a);
\]
\[
Q^4 = a + \beta(\partial_x + b) + \alpha v(\partial_x - a) + \beta v - \alpha v(\partial_v - a) - \beta v - 1 - a(v\partial_v - a);
\]
\[
Q^5 = a + \beta(\partial_x + b) + \alpha v(\partial_x - a) - \beta v + \beta v(\partial_v - a) - 1 - a(v\partial_v - a);
\]
\[
Q^6 = a + \beta(\partial_x + b) + \alpha v(\partial_x - a) + \beta v - \beta v(\partial_v - a) - 1 - a(v\partial_v - a).
\]

**Table 2.** 6 conditional symmetries of the first type of the MA-system (1).

| Reaction terms | Restrictions | Symmetry operators |
|----------------|--------------|--------------------|
| 1 $u(\partial_x + b) + \beta(\partial_x + b)(a - \alpha v) + \beta v(\partial_v - a)$ | $\alpha - \alpha v(\partial_v - a) \neq 0$ | $\beta(\partial_x + b)$ |
| 2 $u(\partial_x + b) + \beta(\partial_x + b)(a - \alpha v) + \beta v(\partial_v - a)$ | $\beta \neq 0$ | $\beta(\partial_x + b)$ |
| 3 $u(\partial_x + b) + \beta(\partial_x + b)(a - \alpha v) + \beta v(\partial_v - a)$ | $\beta \neq 0$ | $\beta(\partial_x + b)$ |
| 4 $u(\partial_x + b) + \beta(\partial_x + b)(a - \alpha v) + \beta v(\partial_v - a)$ | $\beta \neq 0$ | $\beta(\partial_x + b)$ |
| 5 $u(\partial_x + b) + \beta(\partial_x + b)(a - \alpha v) + \beta v(\partial_v - a)$ | $\beta \neq 0$ | $\beta(\partial_x + b)$ |
| 6 $u(\partial_x + b) + \beta(\partial_x + b)(a - \alpha v) + \beta v(\partial_v - a)$ | $\beta \neq 0$ | $\beta(\partial_x + b)$ |
where the functions $\varphi_i(t)$ ($i = 1, \ldots, 4$):

$$
\varphi_1(t) = \begin{cases} 
\beta_1 t + \beta_2, & \text{if } \alpha_2 = 0, \\
\beta_2 \exp(-\alpha_2 t) + \frac{\beta_1}{\alpha_2}, & \text{if } \alpha_2 \neq 0;
\end{cases}
\varphi_2(t) = \begin{cases} 
\beta_1 t, & \text{if } \alpha_2 = 0, \\
\frac{\beta_1}{\alpha_2}, & \text{if } \alpha_2 \neq 0;
\end{cases}
\varphi_3(t) = \begin{cases} 
\beta_1 t + \beta_2, & \text{if } \alpha_1 = 0, \\
\beta_2 \exp(-\alpha_1 t) + \frac{\beta_1}{\alpha_1}, & \text{if } \alpha_1 \neq 0;
\end{cases}
\varphi_4(t) = \begin{cases} 
\lambda + \beta, & \text{if } \alpha = 0, \\
\beta \exp(-\alpha t) + \frac{1}{\alpha}, & \text{if } \alpha \neq 0
\end{cases}
$$

while $\alpha$ and $\beta$ (with and without the subscripts 1 and 2) are the arbitrary constants.

**Sketch of the proof.** Let us apply definition 1 to find $Q$-conditional symmetries of the first type of the form (4) with $\xi^0 \neq 0$. In contrast to the standard $Q$-conditional (non-classical) symmetry, here we cannot simply set $\xi^0 = 1$ because some operators will be missed, as was shown in [16]. From the formal point of view, one needs to use three different manifolds in the criteria (6). However, we take into account the fact that the DLV system (1) has a symmetric structure and admits two discrete transformations of the form straightforwardly by inserting the functions symmetry operators of the first type.

Let us apply definition 1 to find the operators (4) with $\xi^0 \neq 0$. Moreover, all the solutions obtained should be excluded (it means that the solutions obtained should not satisfy equations (12)). Moreover, all the $Q$-conditional symmetries obtained were reduced to the simplest forms using proposition 1. Having this done, we arrived at exactly nine DLV systems with correctly specified coefficients, which admit $Q$-conditional symmetries of the first type. It turns out that three different cases should be examined depending on diffusion coefficients, namely (i) $\lambda_1$ are arbitrary positive constants, (ii) either $\lambda_1 = \lambda_2$ or $\lambda_1 = \lambda_3$ (both conditions are equivalent up to transformations (25)) and (iii) $\lambda_2 = \lambda_3$.

The second step of the algorithm consists in solving the obtained system of DEs. This step can be realized in a similar way as for the system of DEs (9)–(23); however, all solutions leading to the Lie operators should be excluded (it means that the solutions obtained should not satisfy equations (12)). Moreover, all the $Q$-conditional symmetries obtained were reduced to the simplest forms using proposition 1. Having this done, we arrived at exactly nine DLV systems with correctly specified coefficients, which admit $Q$-conditional symmetries of the first type.

Let us examine case (i) in detail. The system of DEs in explicit form can be obtained straightforwardly by inserting the functions

$$
C_1 = u(a_1 + b_1 u + c_1 v + d_1 w),
C_2 = u(b_2 + b_2 u + c_2 v + d_2 w),
C_3 = w(a_3 + b_3 u + c_3 v + d_3 w),
$$

where the functions $\varphi_i(t)$ ($i = 1, \ldots, 4$):

$$
\varphi_1(t) = \begin{cases} 
\beta_1 t + \beta_2, & \text{if } \alpha_2 = 0, \\
\beta_2 \exp(-\alpha_2 t) + \frac{\beta_1}{\alpha_2}, & \text{if } \alpha_2 \neq 0;
\end{cases}
\varphi_2(t) = \begin{cases} 
\beta_1 t, & \text{if } \alpha_2 = 0, \\
\frac{\beta_1}{\alpha_2}, & \text{if } \alpha_2 \neq 0;
\end{cases}
\varphi_3(t) = \begin{cases} 
\beta_1 t + \beta_2, & \text{if } \alpha_1 = 0, \\
\beta_2 \exp(-\alpha_1 t) + \frac{\beta_1}{\alpha_1}, & \text{if } \alpha_1 \neq 0;
\end{cases}
\varphi_4(t) = \begin{cases} 
\lambda + \beta, & \text{if } \alpha = 0, \\
\beta \exp(-\alpha t) + \frac{1}{\alpha}, & \text{if } \alpha \neq 0
\end{cases}
$$

while $\alpha$ and $\beta$ (with and without the subscripts 1 and 2) are the arbitrary constants.

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The second step of the algorithm consists in solving the obtained system of DEs. This step can be realized in a similar way as for the system of DEs (9)–(23); however, all solutions leading to the Lie operators should be excluded (it means that the solutions obtained should not satisfy equations (12)). Moreover, all the $Q$-conditional symmetries obtained were reduced to the simplest forms using proposition 1. Having this done, we arrived at exactly nine DLV systems with correctly specified coefficients, which admit $Q$-conditional symmetries of the first type. It turns out that three different cases should be examined depending on diffusion coefficients, namely (i) $\lambda_1$ are arbitrary positive constants, (ii) either $\lambda_1 = \lambda_2$ or $\lambda_1 = \lambda_3$ (both conditions are equivalent up to transformations (25)) and (iii) $\lambda_2 = \lambda_3$.

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into (9)–(21) (excluding (12) as was noted above) and (26) and (27)). Because $\lambda_k$ ($k = 1, 2, 3$) are the arbitrary constants and formulae (24) take place, equations (13) and (14) immediately produce $q^j = h^k = 0$, $k = 1, 2, 3$. Thus, the system of DEs has the form

\begin{align*}
    c_1 p^1 &= d_1 p^1 = d_2 p^2 = c_3 p^3 = 0, \\
    (b_1 - b_2) q^2 &= 0, \quad (d_1 - d_2) q^2 = 0, \\
    (b_1 - b_2) q^3 &= 0, \quad (c_1 - c_3) q^3 = 0, \\
    \xi^0_1 - 2 \xi^1_1 &= 0, \\
    2r^k_1 + \lambda_4 \xi^1_1 &= 0, \\
    c_1 (r^2 + 2 \xi^1_1) &= 0, \quad d_4 (r^3 + 2 \xi^1_1) = 0, \\
    c_1 q^2 + d_1 q^3 + b_1 (r^1 + 2 \xi^1_1) &= 0, \\
    (2c_2 - c_1) q^2 + d_2 q^3 + b_2 (r^1 + 2 \xi^1_1) &= 0, \\
    c_3 q^2 + (2d_3 - d_1) q^3 + b_3 (r^1 + 2 \xi^1_1) &= 0, \\
    (\lambda_j - \lambda_i) q^2 + 2 \xi^0 q^j &= 0, \\
    r^1_{xx} - \lambda_1 r^1_1 + 2a_1 \xi^1_1 + 2b_1 p^1 + c_1 p^2 + d_1 p^3 = 0, \\
    r^2_{xx} - \lambda_2 r^2_1 + 2a_2 \xi^1_1 + b_2 p^1 + c_2 p^2 + d_2 p^3 = 0, \\
    r^3_{xx} - \lambda_3 r^3_1 + 2a_3 \xi^1_1 + b_3 p^1 + c_3 p^2 + 2d_3 p^3 = 0, \\
    q^j_{xx} - \lambda_j q^j_1 + (a_j - a_1) q^j + b_1 p^1 + \frac{\lambda_1 - \lambda_j}{\xi^0} q^j r^1 = 0, \\
    p^j_{xx} - \lambda_k p^j_1 + a_k p^j + \frac{\lambda_1 - \lambda_k}{\xi^0} q^j p^1 = 0,
\end{align*}

where $k = 1, 2, 3, j = 2, 3$. Now one should solve this system with respect to the functions $\xi^0, \xi^1, r^k, q^j, h^k$ and $p^j$ taking into account the fact that the form of these functions depends essentially on the system parameters. First of all, one observes that

\begin{equation}
    (q^2)^2 + (q^3)^2 \neq 0;
\end{equation}

otherwise the system coincides with that for searching Lie symmetry operators. Moreover, equations (30) and (31) show us that two different subcases should be examined: (i) $q^2 \neq 0, q^3 = 0$ and (ii) $q^2 = 0, q^3 \neq 0$. Formally speaking, there is the third subcase (iii) $q^2 = 0, q^3 \neq 0$; however, one is equivalent to (i) up to the discrete transformations

\begin{equation}
    v \leftrightarrow w, \quad a_2 \leftrightarrow a_3, \quad b_2 \leftrightarrow b_3, \quad c_2 \leftrightarrow d_3, \quad d_2 \leftrightarrow c_3.
\end{equation}

Let us consider subcase (i). Since $c_1^2 + d_1^2 \neq 0$ (see (8)), equation (29) gives $p^1 = 0$, while equations (32)–(34) lead to the equations

\begin{equation}
    \xi^1_{xx} = r^k_1 = \xi^1_1 = 0 \quad (k = 1, 2, 3).
\end{equation}

Hence, equations (30) and (35)–(36) produce the coefficient restrictions $b_1 = b_2 = b, c_1 = c_2 = c, d_1 = d_2 = d$. Moreover, (42) with $j = 3$ is reduced to the algebraic condition $b_3 p^3 = 0 \Rightarrow p^3 = 0, b_3 \neq 0$ (otherwise $b_3 = 0$ and simultaneously (37) leads to $c_3 = 0$, but this contradicts to restriction (8)).

Having $b_3 \neq 0$ and assuming $c_3 \neq 0$ (the special subcase $c_3 = 0$ immediately leads to case 5 of table 2 only), we obtain $q^2_1 = 0$ differentiating equation (37) with respect to $x$ and
taking into account (46). Because $q^2 = 0$, equation (38) with $j = 2$ immediately gives $\xi^1 = 0$ and equation (34) leads to $r^2 = 0$. Now (40) is reduced to the algebraic condition $c p^2 = 0$; hence, equations (29) produce $p^1 = p^2 = 0$. Particularly it means that equations (43) vanish.

Finally, we find $q^2 = -\frac{b_2}{c^2} r^1$, $b = \frac{b_2}{c^2}$ from (35) and (37), $r^1 = \frac{a_1-a_2}{\lambda_1-\lambda_2}$ from (39), (46) and (42) with $j = 2$, and $r^3 = \beta = \text{const}$ from (41). Thus, the system of DEs is completely solved in subcase (i). Substituting the functions and parameter values obtained into (1), (24) and (4), we conclude that the DLV system

$$\begin{align*}
\lambda_1 u_t &= u_{xx} + u \left( a_1 + \frac{b_3}{c_3} cu + cv + dw \right), \\
\lambda_2 v_t &= v_{xx} + v \left( a_2 + \frac{b_3}{c_3} cu + cv + dw \right), \\
\lambda_3 w_t &= w_{xx} + w(a_3 + b_3 u + c_3 v + d_3 w)
\end{align*}$$

admits a $Q$-conditional symmetry operator of the first type

$$Q = \partial_t + \frac{a_1 - a_2}{\lambda_1 - \lambda_2} u \left( \partial_u - \frac{b_3}{c_3} \partial_v \right) + \beta w \partial_w. \tag{48}$$

Because of equations (34) there are the coefficient restrictions: either $d = d_3 = 0$ or $\beta = 0$. However, in the case $d = d_3 = 0$, the operator $Q^* = \beta w \partial_w$ is the Lie symmetry of (47). So, taking into account proposition 1 we may set $\beta = 0$ into (48) without losing generality, i.e.

$$Q = \partial_t + \frac{a_1 - a_2}{\lambda_1 - \lambda_2} u \left( \partial_u - \frac{b_3}{c_3} \partial_v \right). \tag{49}$$

Now one realizes that the system and the first operator listed in case 1 of table 2 are obtained from (47) and (49) by the transformation $b_3 u \rightarrow u$, $c_3 v \rightarrow v$ and denotation $c \rightarrow c_3 b$. The second operator listed in case 1 of table 2 and the operators from the case 2 of table 2 were obtained in the third step of the proof algorithm, i.e., definition 1 with $M^1_2$ and $M^1_3$ was subsequently applied to (47).

Thus, subcase (i) is completely examined and cases 1, 2 and 5 of table 2 were obtained. Subcase (ii) has been investigated in the quite similar way so that cases 3, 4 and 6 of table 2 were derived. The examination of case (I) is now completed. It turns out that case (II) does not produce any new system and symmetry, while the last three cases of table 2 were obtained by the investigation of case (III).

The proof is now completed. $\square$

**Remark 1.** We point out that (i) the inequalities listed in the second column of table 2 guarantee that the relevant operators from the third column are not equivalent to any Lie symmetry operators listed in table 1; (ii) if the given DLV system from table 2 contains as a particular case (up to local transformations of the form (25)) a simpler system listed in another case of table 2 then, in order to obtain the complete list of symmetries, one should go the case involving the simpler system.

### 4. Exact solutions for the system describing competition of three species

Let us apply the results obtained to a biologically motivated system of the form (1). One notes that the DLV system in case 4 of table 2 is equivalent (the relevant substitution is $u \rightarrow -bu$, $v \rightarrow -cv$, $w \rightarrow -dw$) to the system

$$\begin{align*}
\lambda_1 u_t &= u_{xx} + u(a_1 - bu - cv - dw), \\
\lambda_2 v_t &= v_{xx} + v(a_2 - bu - cv - dw), \\
\lambda_3 w_t &= w_{xx} + w(a_3 - bu - cv - dw), \tag{50}
\end{align*}$$
where the coefficients $a_1$, $b$, $c$ and $d$ are known positive constants. It is well known that the system (50) is used for modeling competition between three species in population dynamics [2, 4]. Substituting $u \to -bu$, $v \to -cv$ and $w \to -dw$ into the $Q$-conditional symmetry operator $Q^1_1$ (see case 4 of table 2), one obtains

$$Q^1_1 \to Q = \partial_t + \frac{a_1 - a_2}{\lambda_1 - \lambda_2} \left( \partial_u - \frac{b}{c} \partial_v \right) + a b u \left( \frac{1}{c} \partial_v - \frac{1}{d} \partial_w \right).$$

(51)

Applying the standard procedure for reducing the given PDE system to an ODE system via the known symmetry operator (51), we easily find the ansatz

$$bu = \psi_1(x) e^{\delta t},$$
$$cv = \psi_2(x) + \left( \frac{\alpha}{\delta} - 1 \right) \psi_1(x) e^{\delta t},$$
$$dw = \psi_3(x) - \frac{\alpha}{\delta} \psi_1(x) e^{\delta t},$$

(52)

where $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$ are new unknown functions. Substituting ansatz (52) into (50) and taking into account the restriction $(\lambda_2 - \lambda_3) a_1 - (\lambda_1 - \lambda_3) a_2 + (\lambda_1 - \lambda_2) a_3$ (see case 4 of table 2), one obtains the reduced system of ODEs

$$\psi''_1 + \psi_1 \left( \frac{\lambda_1 a_2 - \lambda_2 a_1}{\lambda_1 - \lambda_2} - \psi_2 - \psi_3 \right) = 0,$$
$$\psi''_2 + \psi_2 \left( a_2 - \psi_2 - \psi_3 \right) = 0,$$
$$\psi''_3 + \psi_3 \left( a_3 - \psi_2 - \psi_3 \right) = 0.$$  

(53)

Now exact solutions of the 3D competition system (50) can be easily derived by inserting solutions of this system into ansatz (52).

Because the system (53) is a three-component system of nonlinear second-order ODE, its general solution is unknown in an explicit form. Let us assume that the triplet $(\psi_1^0(x), \psi_2^0(x), \psi_3^0(x))$ is a particular solution of (53) and the functions $\psi_i^0$ are non-negative and bounded on the space interval $I$. Then, one observes that the exact solution (52) with $\psi_k = \psi_k^0$ tends to the steady-state solution $\left(0, \frac{1}{\lambda_2 \psi_2^0}, \frac{1}{\lambda_3 \psi_3^0}\right)$ of the DLV system (50) with $\delta < 0$ provided $t \to +\infty$. In the general case, the solution $\left(0, \frac{1}{\lambda_2 \psi_2^0}, \frac{1}{\lambda_3 \psi_3^0}\right)$ produces a curve in the phase space $(u, v, w)$, which lies in the plane $(0, v, w)$. Assuming that competition between three populations take place at the space interval $I$, we may conclude that solution (52) with $\psi_k = \psi_k^0$ describes such competition between them that the species $u$ die, while the species $v$ and $w$ coexist. Particularly, the interesting phenomena, a limit cycle, may occur if the concentrations $v = \frac{1}{\lambda_2 \psi_2^0}$ and $w = \frac{1}{\lambda_3 \psi_3^0}$ create a closed curve.

Let us consider example in the simplest case when $\psi_2^0(x)$ and $\psi_3^0(x)$ are constants and all the functions can be written in explicit form. In fact, the constant solution $\psi_2 = v_0$ and $\psi_3 = a_2 - v_0$ of the second and third equations of (53) with $a_2 = a_3$ produces the following exact solution of the 3D competition system (50) with $a_1 \neq a_2 = a_3$:

$$u = \frac{1}{b} \psi_1(x) e^{\delta t},$$
$$v = \frac{v_0}{c} + \frac{1}{c} \left( \frac{\alpha}{\delta} - 1 \right) \psi_1(x) e^{\delta t},$$
$$w = \frac{a_2 - v_0}{d} - \frac{\alpha}{d \delta} \psi_1(x) e^{\delta t},$$

(54)

where $\psi_1(x)$ is a solution of the linear ODE

$$\psi_1'' - \lambda_2 \delta \psi_1 = 0.$$  

(55)
It should be stressed that solution (54) is not obtainable by any Lie symmetry because the system (50) is invariant only under the principal algebra (7), so that only traveling wave solutions can be constructed.

Obviously, equation (55) with \( \delta < 0 \) possesses the general solution
\[
\varphi(x) = C_1 \cos(\sqrt{-\delta \lambda^2} x) + C_2 \sin(\sqrt{-\delta \lambda^2} x),
\]
where \( C_1 \) and \( C_2 \) are arbitrary constants. Setting \( C_1 = 0 \) and \( C_2 = 1 \) in (56) and substituting \( \varphi(x) \) into (54), we arrive at the exact solution
\[
\begin{align*}
\varphi(x) &= \frac{1}{b} \sin(\sqrt{-\delta \lambda^2} x) e^{\delta t}, \\
v &= \frac{v_0}{c} + \frac{1}{c} \left( \frac{\alpha}{\delta} - 1 \right) \sin(\sqrt{-\delta \lambda^2} x) e^{\delta t}, \\
w &= \frac{a_2 - v_0}{d} - \frac{\alpha}{d \delta} \sin(\sqrt{-\delta \lambda^2} x) e^{\delta t},
\end{align*}
\]
of the system (50) with \( a_1 \neq a_2 = a_3 \).

Let us provide a biological interpretation of the solution obtained. Assuming that competition between three populations take place at the space interval \( [0, \frac{\pi}{\sqrt{-\delta \lambda^2}}] \), we observe that all the components of the solution (57) are bounded and non-negative for arbitrary given \( t > 0 \) and \( x \in [0, \frac{\pi}{\sqrt{-\delta \lambda^2}}] \) provided the coefficient restrictions,
\[
\begin{align*}
0 &\leq v_0 \leq a_2 - \frac{\alpha}{\delta}, \quad \text{if } \alpha \leq \delta, \\
1 - \frac{\alpha}{\delta} &\leq v_0 \leq a_2 - \frac{\alpha}{\delta}, \quad \text{if } \delta < \alpha \leq 0, \\
1 &\leq v_0 \leq a_2, \quad \text{if } \alpha > 0,
\end{align*}
\]
take place. In this case, this exact solution tends to the steady-state point \((0, \frac{v_0}{c}, \frac{a_2 - v_0}{d})\) if \( t \to +\infty \). Thus, the solution (57) can describe the following types of competition between three species: (i) the species \( v \) and \( w \) eventually coexist while the species \( u \) die if \( 0 < v_0 < a_2 \); (ii) the species \( v \) eventually dominate while the species \( u \) and \( w \) die if \( v_0 = a_2 \); (iii) the species \( w \) eventually dominate while the species \( u \) and \( v \) die if \( v_0 = 0 \).

5. Conclusions

It is well known that a new approach for finding symmetries was proposed in 1969 by Bluman and Cole [23, 24]. Nevertheless, the approach is based on the classical Lie scheme; generally speaking, the resulting symmetries cannot be any Lie symmetries of the equation in question; therefore they were called non-classical symmetries (\( Q \)-conditional symmetries). Since 1987 when the Bluman–Cole approach was rediscovered in [25, 26] and successfully applied to a wide range of nonlinear PDEs (see, e.g., the recent book [24] for references), several other definitions of non-Lie symmetries have been introduced and applied for solving nonlinear PDEs (weak symmetry [25, 27, 28], conditional symmetry [29, 30], generalized conditional symmetry [31, 32] etc). A common property that underlies all these symmetries can be described as follows: to find the relevant non-Lie symmetry, one needs to consider the given equation together with differential constraint(s), i.e. a system of equations. The main problem is how to define suitable constraint(s) in such a way that the over-determined system obtained will produce new symmetries leading to new solutions of the given equation. Moreover, solving the over-determined system is another non-trivial problem because the system is usually nonlinear. When the given equation is a multi-component nonlinear system of PDEs the problem becomes much more complicated.
In this paper, Lie and $Q$-conditional symmetries for the 3D diffusive Lotka–Volterra (DLV) system (1) are studied. The main results are presented in theorem 1 and theorem 2 giving an exhaustive list of the systems admitting Lie symmetry and $Q$-conditional symmetry of the first type, respectively. To the best of our knowledge, there is only paper [33] (see section 4.1) where the authors attempted to construct $Q$-conditional symmetry operators for a three-component PDE system, the classical Prandtl system, using the so-called iterating non-classical method. However, all the results obtained therein are obtainable by the Lie method (this is clearly indicated in [33]). Here, the $Q$-conditional symmetry operators listed in table 2 are not reducible to Lie operators in table 1. Thus, we have constructed $Q$-conditional symmetries for a multi-component ($n > 2$) nonlinear system of PDEs for the first time.

It should be stressed that theorem 2 is not a straightforward generalization of the relevant theorem for 2D DLV system (2). In fact, according to theorem 2 [15] there is one case only when the system (2) with $c_1^2 + b_2^2 \neq 0$ (otherwise the system will contain an autonomous equation) admits $Q$-conditional symmetry of the first type, while there are nine inequivalent cases for the 3D DLV system (1). It is quite interesting that the main result of theorem 2 [15] can be obtained from case 1 (see table 2) by formal setting $w = 0$ in the reaction terms but not vice versa. Thus, the results of conditional symmetry search for DLV systems with different numbers of components can be summarized as follows.

1. In the case of a 1D system, i.e., the famous Fisher equation $u_t = u_{xx} + u(a_1 + b_1 u)$, there is no $Q$-conditional symmetry of the first type (in this case this symmetry coincides with non-classical symmetry). This follows from the known results derived independently by different authors [29, 34] for the single reaction–diffusion equation.

2. In the case of a 2D system, there is a unique system admitting two $Q$-conditional symmetries of the first type. Any other system admitting a conditional symmetry operator is either equivalent to this unique system or contains the autonomous Fisher equation [15].

3. In the case of a 3D system, there are nine inequivalent systems admitting $Q$-conditional symmetries of the first type. Any other system admitting a conditional symmetry operator is either equivalent (up to transformations of the form (25)) to one of these systems or contains the autonomous 2D system, or the autonomous Fisher equation.

4. In the case of an $n$-component system ($n > 3$), the problem is still open and cannot be solved by a simple generalization of the results obtained for an $(n-1)$-component system. Of course, some particular results can be derived in such way. For example, we have checked that the straightforward generalization of the system and the operators listed in case 2 of table 2 can be done as follows: an $n$-component system

\[ \lambda_i u'_i = u_{xx} + u'(a_i + u^1 + \cdots + u^n), \quad i = 1, \ldots, n \]

admits $n(n-1)$ operators of the form

\[ Q_{ij} = \partial_t + \frac{a_i - a_j}{\lambda_i - \lambda_j} u'(\partial_{u^i} - \partial_{u^j}), \quad i \neq j = 1, \ldots, n. \]

It should be noted that $Q$-conditional symmetry operators were found under the restriction $\xi^0 \neq 0$ (see formula (4)); hence, the case $\xi^0 = 0$ should be analyzed separately. It is well known that the problem of finding $Q$-conditional symmetry operators with $\xi^0 = 0$ for each single evolution PDE is equivalent to one of the constructions of the general solution for the given PDE (see [35, 36] for details). Thus, this problem cannot be completely solved for any non-integrable evolution equation. Of course, particular solutions of the equation in question may help us to solve this problem partly; for example, an interesting approach was presented in [37]. We expect that the situation can be different for the systems of evolution of...
the PDEs if one seeks the $Q$-conditional symmetry operators of the first type and we are going to demonstrate this in a forthcoming paper.

Finally, we point out that the structure of the operator coefficients in (4) can be generalized by involving the derivatives of the vector-function $(u, v, w)$ and the relevant definitions and algorithm for finding generalized conditional symmetries of three-component systems can be worked out. However, the system of DEs obtained for finding generalized conditional symmetries will be much more complicated than (9)–(23). Of course, some particular solutions of that system may be obtained, but we do not expect that any complete result (like one presented in theorem 2) will be derived.

In conclusion, we have applied the $Q$-conditional symmetry operator (51) to reduce the DLV system (50) used for modeling competition between three species in population dynamics. An analysis of the ODE system obtained with aim to provide a biological interpretation has been done. Particularly, this was shown that even the simplest particular solution of the ODE system leads to the exact solution (57), describing different scenarios of competition between three species.

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