Backward Touchard congruence

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Abstract

The celebrated Touchard congruence states that \( B_{n+p} \equiv B_n + B_{n+1} \pmod{p} \), where \( p \) is a prime number and \( B_n \) denotes the Bell number. In this paper we study divisibility properties of \( B_{n-p} \) and their generalizations involving higher powers of \( p \) as well as the \( r \)-Bell numbers. In particular, we show a closely relation of the considered problem to the Sun-Zagier congruence, which is additionally improved by deriving a new relation between \( r \)-Bell and derangement numbers. Finally, we conclude some results on the period of the Bell numbers modulo \( p \).

Keywords: \( r \)-Bell numbers, Touchard’s congruence, periodicity, Derangement numbers

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1 Introduction

The Bell numbers \( B_n \) are one of the most classical sequences in Combinatorics and describe the number of partitions of a given set of \( n \) elements into non-empty subsets. Their various aspects have been studied for more than a hundred years. In particular, the divisibility properties are of special interest (see, among others, [7, 8, 10, 12, 14, 18, 20, 21, 22]).

The first remarkable result in this direction comes from 1933, when Jackues Touchard [21] obtained the congruence

\[
B_{n+p^m} \equiv mB_n + B_{n+1} \pmod{p},
\]

valid for any natural \( n, m \) and prime \( p \). Nevertheless, the term ‘Touchard congruence’ refers usually to the case \( m = 1 \) only. Another interesting relation was discovered by Sun and Zagier [20]

\[
\sum_{k=1}^{p-1} \frac{B_k}{(-m)^k} \equiv (-1)^{m-1}D_{m-1}, \quad (\text{mod } p\mathbb{Z}_p),
\]

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where \( m \geq 1 \), \( p \nmid m \) is a prime number, \( D_n \) stands for the \( n \)-th derangement number and \( \mathbb{Z}_p \) denotes the ring of \( p \)-adic integers.

The \( r \)-Bell numbers \( B_{n,r} \), \( r \geq 0 \), are a natural generalization of the Bell numbers and count partitions of a set of \( n+r \) elements such that \( r \) chosen elements are separated [13]. The case \( r = 0 \) clearly corresponds to the standard Bell numbers \( B_{n,0} = B_n \). It turns out that the \( r \)-Bell numbers satisfy (1.1) as well [14]. Furthermore, the Sun-Zagier congruence (1.2) has been improved [1, 14, 19, 18] reaching the following form involving the \( r \)-Bell numbers [16]

\[
\sum_{i=1}^{p^a-1} \frac{B_{n+i,r}}{(-m)^i} \equiv a \sum_{k=0}^{n} \binom{n}{k} \left(-1\right)^{k+m+r-1} D_{k+m+r-1}, \quad (\text{mod } p\mathbb{Z}_p), \quad (1.3)
\]

where \( a \geq 1 \) and \( \binom{n}{k} \) are the \( r \)-Stirling numbers of the second kind - see Section 2.1 for more details. In fact, these results have been obtained in the polynomial version as well, where \( r \)-Bell and derangement numbers are replaced by \( r \)-Bell and derangement polynomials, respectively. Note that the \( r \)-Bell numbers appear naturally when studying divisibility properties of the classical Bell numbers, which has been shown in [16] and will be confirmed in Section 4.

In this article, we study divisibility properties of \( B_{n-p^m} \), which might be described as backward analog of (1.1), as well as their generalizations for the \( r \)-Bell numbers. In particular, we show their relation to the generalized Sun-Zagier congruence (1.3), which is additionally improved by providing equivalent forms where number of terms in the sum does not depend on \( n \). This is achieved by deriving some new identities bonding \( r \)-Bell and derangement numbers.

One of the simplest and, at the same time, most elegant results of the paper (see Corollary 4.2) states that

\[
B_{n-p} \equiv V_n, \quad (\text{mod } p),
\]

where \( V_n \) is the number of partitions of the set \( \{1, 2, ..., n\} \) without singletons. Apparently, \( V_n \) may be considered as \( B_{n,-1} \). It turns out to be a part of a more general rule, which motivated us to introduce the \( r \)-Bell numbers for negative values of the index \( r \). In fact, this could be avoided due to the relation \( B_{n,r} \equiv B_{n,r+p^m} (\text{mod } p) \), \( r \in \mathbb{Z} \), however, it is definitely clearer that \( B_{n,-l} \) is independent of \( p \) than \( B_{n,p-l} \) is, for some \( l \geq 0 \). Furthermore, we apply the established equivalences to conclude some result on periodicity of the sequences \( B_{n,r} \mod p \). In particular, we address the hypothesis that \( \frac{p^{l+1}}{p-1} \) is their minimal period by excluding a class of some other potential periods.
2 Preliminaries

2.1 The $r$-Stirling numbers

We denote by $\left[ \begin{array}{c} n \\ k \end{array} \right]_r$ and $\left\{ \begin{array}{c} n \\ k \end{array} \right\}_r$, $r \geq 0$, the $r$-Stirling numbers of the first and second kind, respectively. The number $\left[ \begin{array}{c} n \\ k \end{array} \right]_r$ counts permutations of the set $\{1, 2, ..., n+r\}$ having $k+r$ cycles such that the numbers $1, 2, ..., r$ are in distinct cycles, while $\left\{ \begin{array}{c} n \\ k \end{array} \right\}_r$ represents the number of partitions of the set $\{1, 2, ..., n+r\}$ into $k+r$ non-empty disjoint subsets, such that the numbers $1, 2, ..., r$ are in distinct subsets. For $r = 0$ we obtain the classical Stirling numbers.

The $r$-Stirling numbers were introduced and described in details by A. Z. Broder in [2] (see also [4, 5]), where slightly different notation was used ($\left[ \begin{array}{c} n+r \\ k+r \end{array} \right]_r$ instead of $\left[ \begin{array}{c} n \\ k \end{array} \right]_r$ and similarly $\left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r$ instead of $\left\{ \begin{array}{c} n \\ k \end{array} \right\}_r$). Nevertheless, the convention used in this paper seems more natural and makes expressions less complicated.

The $r$-Stirling numbers may be also characterized by the following expansions

\begin{equation}
(x+r)^n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_r x^k,
\end{equation}

\begin{equation}
(x+r)^n_\pi = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_r x^k;
\end{equation}

where $x^\uparrow = x(x-1) \cdots (x-k+1)$ and $x^\downarrow = x(x+1) \cdots (x+k-1)$ are the falling and rising factorials, respectively. They admit the orthogonality relation

\[ \sum_{k=m}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_r \left\{ \begin{array}{c} k \\ m \end{array} \right\}_r (-1)^k = \sum_{k=m}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_r \left[ \begin{array}{c} k \\ m \end{array} \right]_r (-1)^k = (-1)^n \delta_{mn}. \]

The exponential generating functions are given by

\[ \sum_{n=k}^{\infty} \left[ \begin{array}{c} n \\ k \end{array} \right]_r \frac{x^n}{n!} = \frac{1}{k!} \left( \frac{1}{1-x} \right)^r \left( \ln \left( \frac{1}{1-z} \right) \right)^k, \]

\[ \sum_{n=k}^{\infty} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_r \frac{x^n}{n!} = \frac{1}{k!} e^{rx} (e^x - 1)^k. \]

Consequently, defining the $r$-Stirling transform (involving the $r$-Stirling numbers of the second kind only) of a sequence $(a_n)_{n \geq 1}$ by

\[ b_n = \sum_{k=1}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_r a_k, \]
there holds the following relation between the exponential generating functions $A(x)$, $B(x)$ of $(a_n)$ and $(b_n)$, respectively,

$$B(x) := \sum_{n=1}^{\infty} \frac{b_n}{n!} x^n = e^{rx} A(e^x - 1).$$

### 2.2 The $r$-Bell numbers

As mentioned in Introduction, the $r$-Bell numbers $B_{n,r}$, $r \geq 0$, count partitions of the set $\{1, ..., n + r\}$ into non-empty disjoint subsets such that the numbers $1, ..., r$ are separated. In particular, we have

$$B_{n,r} = \sum_{i=0}^{n} \left\{\begin{array}{c} n \\ k \end{array}\right\}_r.$$

The exponential generating function takes the form

$$B_r(t) = \sum_{n=0}^{\infty} B_{n,r} \frac{t^n}{n!} = e^{e^t - 1 + rt}. \quad (2.3)$$

Treating it as the definition, we can clearly extend the range of $r$ onto all integer numbers $\mathbb{Z}$. The combinatorial interpretation of the Bell numbers with a negative index $r$ is only known, up to the authors knowledge, in the case $r = -1$. Namely, $B_{n,-1}$, denoted usually by $V_n$, represents the number of partitions of the set of $n$ elements containing no singletons.

Applying the general Leibniz rule to (2.3), we obtain

$$\sum_{k=0}^{n} \binom{n}{k} B_{k,r} m^{n-k} = B_{n,r+m} \quad (2.4)$$

for all $n \geq 0$, $r \in \mathbb{Z}$. As a consequence, we get the following periodicity property

$$B_{n,r+p} \equiv B_{n,r} \pmod{p}, \quad (2.5)$$

valid for any prime $p$ and $n \geq 0$, $r \in \mathbb{Z}$. In particular, applying this to Theorem 5 in [14], which generalized (1.1) onto $r$-Bell numbers for $r \geq 0$, we arrive at

$$B_{n+p^a,r} \equiv B_{n+1,r} + aB_{n,r} \pmod{p}, \quad (2.6)$$

where $a, n \in \mathbb{N}$ and $r \in \mathbb{Z}$ is any integer number. We close this section with another useful recurrence ([4], eq. (3.22-3.23))

$$B_{n+m,r} = \sum_{k=0}^{m} \left\{\begin{array}{c} m \\ k \end{array}\right\}_r x^k B_{n,k+r}, \quad (2.7)$$

$$B_{n,r+m} = \sum_{k=0}^{m} (-1)^{m-k} \left[\begin{array}{c} m \\ k \end{array}\right]_r x^k B_{n+k,r}, \quad (2.8)$$

where $m, n, r \geq 0$. 

4
3 Generalized Sun-Zagier congruence

The main feature of the generalized Sun-Zagier congruence (1.3) is that the right-hand side does not depend on $p$. Nevertheless, since $r$ and $m$ are typically supposed to be fixed and $n$ may vary, the number of terms in the sum could be arbitrarily large. In the next theorem we present two identities removing this inaccuracy.

**Theorem 3.1** For $n \geq 0$ and $r + m \geq 1$ we have

$$\sum_{k=0}^{n} \binom{n}{k} r (-1)^{k+m+r-1} D_{k+m+r-1} = \sum_{k=0}^{m} \sum_{i=0}^{r} \left[ \binom{m}{k} \binom{r}{i} \right] (-1)^{r-i} B_{n+k+i-1,-m}$$

$$= \sum_{k=0}^{m+r-1} (-1)^k k! B_{n,k-1} \pmod{p}.$$  

**Proof.** In order to derive the first identity from the assertion we will employ the umbral calculus. It allows us to represent $B_n$ as $\mathbb{B}^n$, where $\mathbb{B}$ is a symbol called umbra. Such an approach has been already used in e.g. in [1, 14, 17, 18] in the context of Bell numbers.

Taking $r = 0$ in (2.4) and by the binomial theorem we obtain

$$B_{n,m} = (\mathbb{B} + m)^n, \quad m \in \mathbb{Z}. \quad (3.1)$$

Furthermore, from Lemma 2.2 in [18] we know that for any $k \geq 0$ it holds

$$(\mathbb{B} - 1) \mathbb{B}^k = (\mathbb{B} - 1)(\mathbb{B} - 2)\ldots(\mathbb{B} - k) = (-1)^k D_k. \quad (3.2)$$

Thus, we get

$$\sum_{k=0}^{n} \binom{n}{k} r (-1)^{k+r+m-1} D_{k+r+m-1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (\mathbb{B} - 1)(\mathbb{B} - 2)\ldots(\mathbb{B} - k - r - m + 1)$$

$$= (\mathbb{B} - 1)^{r+m-1} \sum_{k=0}^{n} \binom{n}{k} (\mathbb{B} - r - m)^k$$

$$= (\mathbb{B} - 1)^{r+m-1} (\mathbb{B} - m)^n,$$

where we used (2.1). For $n \geq 1$, we rewrite it as follows

$$(\mathbb{B} - 1)^{r+m-1} (\mathbb{B} - m)^n = ((\mathbb{B} - m) + (m - 1))^{r+m-1} (\mathbb{B} - m)^n$$

$$= ((\mathbb{B} - m) + (m - 1))\ldots((\mathbb{B} - m) - r + 1) (\mathbb{B} - m)^n.$$
\[(
\begin{align*}
B - m
\end{align*}
\right)^{r}
(m - B)
\right)^{r}
(B - m)^{n-1}.
\]

Hence, the identity (2.2) gives us
\[
L = \sum_{k=0}^{m} \binom{m}{k} (B - m)^{k} (-1)^{r} \sum_{i=0}^{r} \binom{r}{i} (m - B)^{i} (B - m)^{n+k+i-1}
\]
\[
= \sum_{k=0}^{m} \sum_{i=0}^{r} \binom{m}{k} \binom{r}{i} (-1)^{r-i} (B - m)^{n+k+i-1}
\]
\[
= \sum_{k=0}^{m} \sum_{i=0}^{r} \binom{m}{k} \binom{r}{i} (-1)^{r-i} B_{n+k+i-1,-m},
\]
as required. Eventually, let us observe that due to the assumption \(r + m \geq 0\) the above calculations hold true for \(n = 0\). Indeed, assuming \(m \geq 1\), we have \(\binom{m}{0} = 0\) and \(x^{m} = xP(x)\) for some polynomial \(P\), so from (2.2) we have
\[
x^{m} x^{-1} = \sum_{k=0}^{m} \binom{m}{k} x^{-1},
\]
which may be applied to \((B - m)^{m} (B - m)^{-1}\). We proceed similarly in the case \(r \geq 1, m = 0\).

To obtain the other identity from the assertion, let us write
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^{k+m+r-1} D_{k+m+r-1} = \sum_{k=0}^{n} \binom{n}{k} a_{k}, \tag{3.3}
\]
where
\[
a_{k} = (-1)^{k+m+r-1} D_{k+m+r-1}.
\]
Since \(e^{-t}/(1 - t)\) is the exponential generating function of the sequence \(D_{k}\), the exponential generating function of \(a_{k}\) is given by
\[
A(x) = \frac{d^{m+r-1}}{dx^{m+r-1}} \frac{e^{t}}{1 + t} = \sum_{i=0}^{m+r-1} \frac{e^{t}}{(1 + t)^{1+i}} i!(-1)^{i},
\]
where we used the general Leibniz rule. Hence, since (3.3) is the \(r\)-Stirling transform of \(a_{k}\), its exponential generating function takes the form
\[
e^{rt} A(e^{t} - 1) = \sum_{k=0}^{r+m-1} \frac{e^{t}}{(e^{t})^{1+k}} k!(-1)^{k} = \sum_{k=0}^{r+m-1} (-1)^{k} k! e^{t-1-(k+1)t}.
\]
We can identify the exponents in the sum as exponential generating functions of the \(r\)-Bell numbers (2.3), which ends the proof. \(\square\)
Remark 3.2 Repeating the arguments from the proof, one can easily obtain a ‘polynomial’ version of the theorem, i.e. with $B_{n,r}(x) = \sum_{k=0}^{n}{n \choose k}x^k$ and $D_n(x) = \sum_{k=0}^{n}{n \choose k}k!(x-1)^{n-k}$ instead of $B_{n,r}$ and $D_n$, respectively. It is enough to know that the corresponding exponential generating functions are $e^{x(e^t-1)+rx}$ and $e^{(x-1)t}/(1-t)$.

The first identity from Theorem 3.1 takes especially simple form for $m = 0$. Namely, by virtue of (2.8), we get

Corollary 3.3 For $n \geq 1$ and $r \geq 1$ it holds

$$\sum_{k=0}^{n}{n \choose k}(-1)^{k+r-1}D_{k+r-1} = B_{n-1,r}.$$  

Remark 3.4 The above result is valid for $r = 0$ as well. This very special case, in the ‘polynomial’ version, is covered by Lemma 2.2 in [18].

Finally, let us formulate the new version of the Sun-Zagier congruence. Applying Theorem 3.1 to (1.3), we obtain

Corollary 3.5 For any $a, m \geq 1, n \geq 0$ and any prime number $p \nmid m$, we have

$$\sum_{i=1}^{p^n-1} \frac{B_{n+i,r}}{(-m)^i} \equiv a \sum_{k=0}^{m} \sum_{i=0}^{r} \left[ \begin{array}{c} m \\ k \end{array} \right] \left[ \begin{array}{c} r \\ i \end{array} \right] (-1)^{r-i}B_{n+k+i-1,-m}$$

$$\equiv a \sum_{k=0}^{r+m-1} (-1)^kk!B_{n,-k-1}, \quad (\text{mod } p\mathbb{Z}_p).$$

4 Backward Touchard congruence

We start this section with an equivalence, which simplifies (2.6) in some cases and was one of the motivations of research presented in the article.

Proposition 4.1 For $n, r \geq 0$ we have

$$B_{n+p^r,-r} \equiv B_{n,-r+1}, \quad (\text{mod } p).$$

Proof. From (2.7) with $m = 1$ we get for any $l \in \mathbb{Z}$

$$B_{n,l+1} \equiv B_{n+1,l} - lB_{n,l}, \quad (\text{mod } p).$$

Hence, substituting $l = -r \leq 0$ and using (2.6), we get

$$B_{n,-r+1} \equiv B_{n+1,-r} + rB_{n,-r} \equiv B_{n+p^r,-r}, \quad (\text{mod } p),$$

which ends the proof. \qed
In particular, for \( r = 1 \) we obtain the below-given elegant congruence, which may be called the backward Touchard congruence.

**Corollary 4.2** For a prime \( p \) and natural \( n \geq p \) we have

\[
B_{n-p} \equiv V_n, \quad (\text{mod } p).
\]

Next, using simple induction argument, one can generalize it as follows.

**Corollary 4.3** We have

\[
B_{n-\sum_{k=1}^r p^k} \equiv B_{n-r}, \quad (\text{mod } p).
\]

This equivalence has further consequences on the period of the \( r \)-Bell numbers modulo \( p \) - see the next section for details. Nevertheless, the most natural direction of research is to investigate divisibility properties of \( B_{n-p^m,r} \), which is executed in the next theorem.

**Theorem 4.4** For a prime \( p \) and integers \( n, m \geq 1, r \geq 0 \) such that \( n \geq p^m \) and \( p \nmid m \) we have

\[
B_{n-p^m,r} \equiv \sum_{k=0}^n \binom{n}{k} (-1)^{k+m+r-1} D_{k+m+r-1}^k B_n \equiv \sum_{k=0}^m \sum_{i=0}^r \left[ \begin{array}{c} m \\ k \end{array} \right] [r] \binom{r}{i} (-1)^{r-i} B_{n+k+i-1,m-m}, \quad (\text{mod } p).
\]

**Proof.** We will show only the first equivalence. The other ones follow from Theorem 3.1.

From (2.6) we have

\[
mB_{n,r} \equiv B_{n+p^m,r} - B_{n+1,r}, \quad (\text{mod } p). \tag{4.1}
\]

More generally, for any \( N \geq 1 \) it holds

\[
m^N B_{n,r} \equiv \left( \sum_{k=0}^{N-1} (-1)^k m^{N-1-k} B_{n+p^m+k,r} \right) + (-1)^N B_{n+N,r}, \quad (\text{mod } p), \tag{4.2}
\]

which may be shown by induction. Indeed, multiplying it by \( m \) and using (4.1), we get

\[
m^{N+1} B_{n,r} \equiv \left( \sum_{k=0}^{N-1} (-1)^k m^{N-k} B_{n+p^m+k,r} \right) + (-1)^N m B_{n+N,r} \equiv (\text{mod } p)
\]
\[
\begin{align*}
= & \left( \sum_{k=0}^{N-1} (-1)^k m^{(N+1)-1-k} B_{n+p^m+k,r} \right) + (-1)^N (B_{n+p^m+N,r} - B_{n+N+1,r}) \\
= & \left( \sum_{k=0}^{(N+1)-1} (-1)^k m^{(N+1)-1-k} B_{n+p^m+k,r} \right) + (-1)^{N+1} B_{n+(N+1),r},
\end{align*}
\]
as required. Next, substituting \( N = p^m \) in (4.2) and exploiting the congruence \( m^{p^m} \equiv m \), valid by virtue of Fermat’s little theorem, we arrive at

\[
mB_{n,r} \equiv m^{p^m} B_{n,r} \equiv \left( \sum_{k=0}^{p^m-1} (-1)^k m^{p^m-1-k} B_{n+p^m+k,r} \right) + (-1)^{p^m} B_{n+p^m}
\]
\[
\equiv \sum_{k=1}^{p^m-1} \frac{B_{n+p^m+k,r}}{(-m)^k} + \left( 1 + (-1)^{p^m} \right) B_{n+p^m+k,r}, \quad \text{(mod } pZ_p).\]

The equivalence \( 1 + (-1)^{p^m} \equiv 0 \mod p \), which holds for any prime \( p \), and application of the generalized Sun-Zagier congruence (1.3) end the proof. \( \square \)

**Remark 4.5** For \( m = 0 \) we the first equivalence from Theorem 4.4 becomes an identity - see Corollary 3.3 and Remark 3.4. However, it is not true anymore for \( m \geq 1 \). For example, one can verify that for \( m = 1, r = 0, p = 2, n = 4 \).

## 5 The \( r \)-Bell numbers modulo a prime number

In this section we focus on the periodicity of the sequence \( B_{n,r} \mod p \), for a prime \( p \), which is simply the sequence of reminders of a division \( B_{n,r} \) by \( p \). Hall [9] discovered that \( B_n \) has the period

\[
N_p = \frac{p^p - 1}{p - 1} = 1 + p + p^2 + ... + p^{p-1}.
\]

We can easily recover it combining Corollary 4.3 with (2.5) and the equality \( B_{n,1} = B_{n+1} \):

\[
B_{n-N_p} = B_{n-1-\sum_{k=1}^{p-1} p^k} \equiv B_{n-1-p+1} \equiv B_{n-1,1} = B_n, \quad \text{(mod } p).\]

Mezó and Ramírez [14] extended the Hall’s result for \( r \)-Bell numbers by showing that \( N_p \) is a period of \( B_{n,r} \), for a fixed \( r \geq 0 \). Apparently, the relation between the sequences for different \( r \)’s is much stronger, which is presented below.

**Corollary 5.1** For any \( r \in \mathbb{Z} \) the sequence \( B_{n,r} \mod p \) is equal to the sequence \( B_n \mod p \) shifted in the following manner

\[
B_{n,r} \equiv B_{n-K}, \quad \text{(mod } p), \quad n \geq K,
\]
where \( K = \sum_{k=1}^{r \mod p} p^k \).

**Proof.** The case \( r \in \{0, \ldots, p - 1\} \) is cover by Corollary 4.3. One can extend it onto all \( r \in \mathbb{Z} \) by virtue of (2.5).

\( N_p \) was proven [11, 15, 23] to be the minimal period for \( p < 126 \) as well as for \( p = 137, 149, 157, 163, 167 \) and 173. For other primes the problem is open, however, there exist in the literature some partial results. They are usually related to the divisibility properties of \( N_p \). A quantitative bound [12, 3] states that the minimal period is greater than

\[
\frac{1}{2} \left( \frac{2p}{p} \right) + p.
\]

In the next theorem, we provide a result of new type, referring to the representation of the period in the base \( p \) numerical system.

**Theorem 5.2** Let

\[
P_p = \sum_{k=0}^{p-2} a_k p^k < p^{p-1}, \quad a_k \in \{0, 1, \ldots, p - 1\}, \hspace{1cm} (5.1)
\]

be a period (not necessarily the minimal one) of \( B_n \) modulo a prime \( p \). Then

\[
\sum_{k=0}^{p-2} a_k \geq p + 1.
\]

**Proof.** First, we will justify that we can assume \( a_0 > 0 \). Namely, if \( a_0 = a_1 = \ldots = a_{k_0} = 0 \) and \( a_{k_0+1} > 0 \), then \( P_p/p^{k_0} \) is a period as well with the same sum of digits (in the base \( p \) numerical system). This follows from the fact that both: \( N_p \) and \( P_p \) are multiplicities of the minimal period, while all the dividers of \( N_p \) are of the form \( 2kp + 1, k \in \mathbb{N} \) (see [6], p. 381), and hence the minimal period is not divisible by \( p \), so \( P_p/p^{k_0} \) has to a multiplicity of the minimal period.

Denote \( M = \sum_{k=1}^{p-2} a_k \). Next, starting from \( B_{n+P_p} \), we exploit the congruence (1.1)

\[
B_{n+p^m} \equiv B_{n+1} + mB_n, \hspace{1cm} (\text{mod } p),
\]

in \( M - a_0 \) steps. In each step we apply it to every term that already appeared and with the same \( m \geq 1 \). Eventually, this procedure leads to

\[
B_{n+P_p} \equiv \sum_{k=a_0}^{M} b_k B_{n+k}, \hspace{1cm} (\text{mod } p), \hspace{1cm} (5.2)
\]
for some $b_k \geq 0$, $k \in \{0, ..., M\}$, such that $b_M = 1$. Since the sum in (5.1) is up to $p - 2$, in each step the sum of coefficients of appearing Bell numbers increases $(m + 1)$ times for some $2 \leq m \leq p - 1$. Thus, the sum $S := \sum_{k=a_0}^{M} b_k$ is a product of $M - a_0$ positive numbers smaller than $p$, and consequently $S \not\equiv 0 \pmod{p}$. Furthermore, since $P_p$ is a period, we get

$$\sum_{k=a_0}^{M} b_k B_{n+k} \equiv B_n \pmod{p}.$$ 

Let as suppose that $M \leq p$. Subtracting $B_n$ from both sides and applying the Touchard congruence to $B_{n+M}$, we obtain for $n \geq p - M$

$$\sum_{k=M-p}^{M-1} c_k B_{n+k} \equiv 0 \pmod{p},$$

where

$$\sum_{k=M-p}^{M-1} c_k = \sum_{k=a_0}^{M-1} b_k + 2b_M - 1 = \sum_{k=a_0}^{M-1} b_k + 1 = S \not\equiv 0 \pmod{p}.$$ 

This contradicts Theorem (3.3) in [12], which implies that if $\sum_{k=0}^{A} \alpha_k B_{n+k} \equiv 0$ for all $n \geq 0$, where $A < p$, then $\alpha_k \equiv 0$ for all $k \in \{0, ..., A\}$. Putting $\alpha_k = c_{k+M-p}$, $k \in \{0, ..., p-1\}$, we have $A = p - 1 < p$, while $\sum_{k=0}^{A} \alpha_k \not\equiv 0$ and hence there exists $k \in \{0, ..., p-1\}$ such that $\alpha_k \not\equiv 0$.  

\hspace{1cm} \square

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