Brownian motion at the speed of light

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Abstract

In this paper we construct a class of processes which transform one in the other by Lorentz boost. The particle only moves at light speed and the velocity performs a non uniform Wiener process on the surface of a sphere of radius $c$. Therefore, the trajectories of the velocity are almost everywhere continuous but they are not differentiable, on the contrary the trajectories of the positions are continuous and differentiable. For large times the behavior of the position is diffusive while for short times is ballistic.

This class of processes generalizes to 3+1 dimensions the 1956 idea of the Polish physicist and mathematician Marek Kac which considered a (1+1)-dimensional class of processes where the particle travels at light speed (left or right) and randomly inverts its velocity. Although the generalization to 3+1 dimensions could be obtained in a different way, for example considering a velocity which performs jumps, it seems to us that our proposal is the most natural having as a constitutive ingredient the Wiener process.

Keywords: Brownian motion, Wiener process, relativity, Lorentz boost, Ito calculus.

1 Introduction

Brownian motion is a physical phenomenon which can be mathematically modeled by the Wiener process, both directly identifying the particle trajectories with the realizations of the process and indirectly via the Langevin equation (Ornstein-Uhlenbeck process). Due to its historical connection with the physical phenomenon, the Wiener process is often simply called Brownian motion. Although in the last century it has been used to describe a variety of phenomenon in finance, biology, engineering, electronics and so on, its name remains strictly tied to the description of the motion of random particles.

If one tries to extend its use to the description of the random motion of relativistic particles he clashes against one of its more characteristic properties: trajectories are not differentiable, which means infinite speed while relativity only allows luminal or subluminal velocities. This fact doesn’t imply that it is useless, on the contrary the relativistic Brownian motion can be still modeled via a variety of modified Langevin equations which produce trajectories with a speed which is never superluminal. We just quote a few of the studies which followed this strategy in the last fifty years: [1-18], more bibliography can be found therein.

One more reason for searching a Wiener description of relativistic random particles is the possibility to extend the analogy between the Feynman integral and the Wiener integral (Feynman-Kac formula) to the relativistic quantum domain. The Schrödinger equation is solved by the Feynman integral while the heat equation, which is connected to the first by analytic continuation, is solved by the Wiener integral. The relativistic versions of the Schrödinger equation are the Klein-Gordon (zero spin particles) and Dirac (spin one half) equations. Both are hyperbolic equations (Dirac equation in its second order formulation). Analytic continuation gives rise to elliptic equations, the point is: which process is associated to the elliptic equations?

Indeed, the answer is again the Wiener process, but a four dimensional one where both position and time follow trajectories which are the realizations of a Wiener process with the proper time as index [19, 20, 21]. The proper time is then eliminated by a procedure based on hitting times. Nevertheless, the resulting process is unphysical since the speed is not bounded and the whole construction only results in a tool for obtaining
a probabilistic solution of some elliptic equations. If one forces the approach to the realm of physics has to abandon Markov property and the single particle picture [22].

There is a third way to approach the relativistic problem with a process which is physical and allows to construct the solution of the quantum hyperbolic equations. In 1956 the Polish physicist and mathematician Marek Kac considered a (1+1)-dimensional process where the particle travels at light speed (left or right) and randomly inverts its velocity and he proved that the associated probability density satisfies the telegrapher equation [23].

Later Gaveau et al. noticed that the telegrapher equation can be easily associated both to the Dirac equation in 1+1 dimensions (first order formulation) and to the Klein-Gordon equation also in 1+1 dimensions (second order formulation). Using this equivalence they were able to give a probabilistic solution (by the backward Kolmogorov equation) to these fundamental quantum equations [24]. The weak point was that both Kac and Gaveau constructions only worked for particles in 1+1 dimensions (one space dimension + time).

Indeed, the process considered by Gaveau et al. is part of a larger class, in fact by Lorentz boosts new processes can be obtained with particles moving at the speed of light (a simple consequence of the fact that a light speed particle in an inertial frame is also light speed in any other inertial frame). The processes of this generalized class have in general an unbalanced probability rate of velocity inversion i.e., the inversions from right to left occur with a different probability rate of those from left to right, as a consequence, the particle may have a non-vanishing average velocity.

The class of these one-dimensional light speed process was further extended by considering inversion rates which not only depend on the sign of the velocity but also on the position. This extension gave the possibility to reformulate the quantum mechanics of a relativistic particle in terms of stochastic processes [25] in the spirit of Nelson’s Stochastic Mechanics [26]. Again, this construction was limited to 1+1 dimensions.

In this paper we propose a process which generalize the Kac approach to the 3+1 dimensional case. The goal is to construct a Brownian motion which is the most similar to the Wiener process among all those processes which do not conflict with relativity. Our particle moves constantly at the speed of light c, but velocity performs a uniform Brownian motion on the surface of a sphere of radius c. In this way the speed is always c which is the largest among those compatible with relativity, but velocity direction changes. As a consequence, the position trajectories are always differentiable.

While the particle is ballistic at short times (position changes proportionally to time), the behavior of the position, at large times, is ordinary diffusive ($E[x^2] \sim t$) and the average velocity vanishes. This is our process 'at rest' i.e, a process with a velocity which performs a uniform Brownian motion on the surface of the sphere so that the average velocity vanishes at large times. Then we consider the class all those process which are generated by Lorentz boosts. The instantaneous velocity of these process must be also luminal, because a luminal particle is luminal in any inertial frame. Therefore, the velocity still remains on the surface of the sphere, nevertheless, the Brownian motion is not homogeneous and average velocity is different from zero at large times. In order to characterize this class of processes which transform one in the other by Lorentz boost we have to use Ito calculus.

The paper is simply organized: in section 2 we introduce the process 'at rest'. The velocity process on the sphere is formulated in a new and more economic way which allow a simpler use of Ito calculus. In section 3 we characterize the entire class of processes generated by Lorentz boosts. Nevertheless, the very long application of Ito calculus to reach this goal is postponed in an Appendix that eventually the reader can skip. Averages are computed in section 4 where we also highlight some relevant variables which can be isolated from the others. Summary and outlook can be finally found in section 5.

2 Stochastic equations for the process 'at rest'

We assume that the particle velocity performs a uniform Brownian motion on the surface of a sphere of radius c. In this way, while the velocity direction changes, the speed always equals c which is the largest among those compatible with relativity.
The equations governing this process are:

\[ dx(t) = c(t)dt, \]

\[ dc(t) = -\omega^2 c(t) dt + \omega c \, dw(t) \]  \hspace{1cm} (1)

where, according to Ito, \( dc(t) = c(t + dt) - c(t) \) and \( dw(t) = w(t + dt) - w(t) \) is a two component Wiener increment on the plane perpendicular to \( c(t) \) such that \( E[|dw(t)|^2] = 2dt \).

The velocity \( dc(t) \) remains on the surface of a sphere i.e., \( |c(t)| = c \) that we assume to be the speed of light. In fact, it is straightforward to verify that according to Ito, \( dc^2(t) = 0 \) (in next section this equality is explicitly proven for the general class of processes generated by Lorentz boosts).

Indeed, the second equation describes the uniform Wiener process on a sphere which was studied for the first time at least 70 years ago \[27\,28\]. We already mentioned that \( dw(t) = w(t + dt) - w(t) \) is a two component Wiener increment on the plane perpendicular to \( c(t) \), nevertheless, the recipe for the construction of the underlying process is not univocal.

In the following pages we will omit the time as an explicit argument when it is not strictly necessary. For example, we will simply write \( c, dc, w \) and \( dw \) for \( c(t), dc(t), w(t) \) and \( dw(t) \).

In the early seventies Strook and then Ito \[29\,30\] constructed the increment \( dw \) in the second of the equations \[1\] as

\[ dw = (I - nn^T) dB \]  \hspace{1cm} (2)

where \( n(t) = c(t)/c \) is a time dependent unitary vector, \( B \) is a standard three dimensional Brownian motion, \( I \) is the \( 3 \times 3 \) identity matrix and the row vector \( n^T \) is the transposed of the column vector \( n \). One gets

\[ dc = -\omega^2 c \, dt + \sigma \, dB \]  \hspace{1cm} (3)

where \( \sigma = \omega c (I - nn^T) \) is a \( 3 \times 3 \) matrix.

About ten years later a simpler choice was considered \[31\,32\]:

\[ dw = n \times dB, \]  \hspace{1cm} (4)

which leads to \( dc = -\omega^2 c \, dt + \hat{\sigma} dB \) where \( \hat{\sigma} = \omega c[n] \) with \( n \) being the \( 3 \times 3 \) skew matrix representation of the vector \( n \).

It is easy to check that \( \hat{\sigma} \hat{\sigma}^T = \hat{\sigma}' \hat{\sigma}'^T = \omega^2 c^2 (I - nn^T) \) which implies that the Fokker-Planck equation is the same for choices \[2\] and \[3\]. See \[33\,34\] for properties and applications.

Both implementations of the bi-dimensional increment \( dw \) are made by a three dimensional Wiener process \( B(t) \), which is somehow redundant for the construction of a bi-dimensional Brownian motion. We propose here to use in place of the standard three dimensional Wiener process \( B(t) = B_1(t), B_2(t), B_3(t) \) a standard bi-dimensional Wiener process \( w_2(t), w_3(t) \) (we write \( w_2(t), w_3(t) \) in place of \( B_2(t), B_3(t) \) to avoid confusion). Our choice is

\[ dw = n_2 \, dw_2 + n_3 \, dw_3 \]  \hspace{1cm} (4)

where \( n_2(t) \) and \( n_3(t) \) are two unitary vectors perpendicular each other and also perpendicular to \( n(t) \). The Wiener increments are independent which implies \( E[dw_2(t) \, dw_3(t)] = 0 \) and they are standard which means \( E[(dw(t))^2] = 2dt \). The orientation of \( n_2(t) \) and \( n_3(t) \) can be arbitrarily chosen on the plane perpendicular to \( n(t) \). For example, given a constant vector \( v \), one can chose

\[ n_2 = \frac{v \times n}{|v \times n|}, \quad n_3 = n \times n_2, \]  \hspace{1cm} (5)

so that \( v, n = c/c \) and \( n_3 \) are on the same plane and \( n_2 \) is perpendicular to it. To fix the ideas one can put the north pole in the \( v \) direction, so that \( n_3 \) is tangent to a meridian and points to north, while \( n_2 \) is tangent to a parallel and points to est. In this way it is simple to pass to spherical coordinates. At the poles (where \( n \) equals \( \pm v/|v| \)) the unitary vectors \( n_2 \) and \( n_3 \) can be arbitrarily chosen perpendicularly to \( v \).

According to our representation the second equation in \[1\] rewrites as:

\[ dc = -\omega^2 c \, dt + \omega c (n_2 \, dw_2 + n_3 \, dw_3), \]  \hspace{1cm} (6)

the advantage being that we use only a two component Wiener process in place of a three component one, moreover this stochastic equation is straightforwardly associated to the velocity spherical laplacian in the Fokker-Planck equation when it is expressed in terms of longitude and latitude.
3 Lorentz boosts and stochastic equations in a generic inertial frame

In the frame where the process is ’at rest’ the velocity \( \mathbf{c}(t) \) of the particle evolves according to equation (6) where \( \mathbf{n}_2 \) and \( \mathbf{n}_3 \) are defined by (8). Then assume that this ’rest frame’ moves at constant velocity \( \mathbf{u} \) (without rotating) with respect to a second inertial frame. Since the choice of \( \mathbf{v} \) in (5) is arbitrary, we can leave it to coincide with \( \mathbf{u} \). In the next we will only use \( \mathbf{v} \) to indicate both the velocity in (5) and the velocity of the ’rest frame’.

According to special relativity, the velocity \( \mathbf{c}'(t) \) of the particle in the second frame is

\[
\mathbf{c}'(t) = \frac{1}{1 + \frac{\mathbf{v} \cdot \mathbf{c}(t)}{c^2}} \left[ \alpha \mathbf{c}(t) + \mathbf{v} + (1 - \alpha) \frac{\mathbf{v} \cdot \mathbf{c}(t)}{v^2} \mathbf{v} \right]
\]

where \( \mathbf{v}, c, \) and \( \alpha = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \) are constant. By special relativity the velocity in this second inertial frame is also luminal (\(|\mathbf{c}'| = c\)) (at the end of this section we will show that indeed \((c')^2 = c^2 = c^2\)). If one also takes into account that the time increment \( dt' \) in the second frame satisfies

\[
\frac{dt}{dt'} = \frac{1}{\alpha} \left(1 - \frac{\mathbf{v} \cdot \mathbf{c}'}{c^2}\right),
\]

he should be able to write from (7) and (8) a stochastic equation for \( \mathbf{c}'(t') \) analogous to (6) and (5). Notice that the argument of \( \mathbf{c}' \) is now \( t' \) and that the new equation should express the increment \( d\mathbf{c}' = \mathbf{c}'(t' + dt') - \mathbf{c}'(t') \) in terms of \( \mathbf{c}'(t'), dt' \) and of the increments \( dw_2' = w_2(t' + dt') - w_2(t') \) and \( dw_3' = w_3(t' + dt') - w_3(t') \).

In the second frame the particle will still instantaneously move at light speed but, contrarily to the case of the process in the ’rest frame’, its average velocity will not vanish at large times but it will equal \( \mathbf{v} \).

Define \( \delta \mathbf{c}' = \mathbf{c}'(t + dt) - \mathbf{c}'(t) \) (notice the difference with \( d\mathbf{c}' \)), then a long and tedious application of Ito calculus (see the Appendix) leads to

\[
\delta \mathbf{c}' = -\frac{\omega^2}{\alpha^2} \left[ 1 - \frac{\mathbf{v} \cdot \mathbf{c}'}{c^2} \right]^2 \mathbf{c}' dt' + \frac{\omega c}{\alpha} \left(1 - \frac{\mathbf{v} \cdot \mathbf{c}'}{c^2}\right) (n'_2 dw_2 + n'_3 dw_3)
\]

where \( n'_2 \) and \( n'_3 \) are the two unitary vectors perpendicular to \( \mathbf{c}' \) defined as in (5) (with \( \mathbf{n}, \mathbf{n}_2 \) and \( \mathbf{n}_3 \) replaced by \( \mathbf{n}' = \mathbf{c}'/c, n'_2 \) and \( n'_3 \)). Then, taking into account (5) and remembering that \( dw_3/dw_3' = dw_2/dw_2' = (dt/dt') \), one gets

\[
d\mathbf{c}' = -\frac{\omega^2}{\alpha^3} \left[ 1 - \frac{\mathbf{v} \cdot \mathbf{c}'}{c^2} \right]^3 \mathbf{c}' dt' + \omega c \left(1 - \frac{\mathbf{v} \cdot \mathbf{c}'}{c^2}\right) \left[\alpha \left(1 - \frac{\mathbf{v} \cdot \mathbf{c}'}{c^2}\right)\right]^\frac{3}{2} d\mathbf{w}'
\]

where \( d\mathbf{w}' = n'_2 dw_2' + n'_3 dw_3' \) is a two component increment perpendicular to \( \mathbf{c}' \) in complete analogy with the process in the ’rest frame’. Notice that by (7) the three vectors \( \mathbf{v}, \mathbf{c}, \) and \( \mathbf{c}' \) are on the same plane so that \( \mathbf{n}_3, n'_3 \) also are on the same plane. As a consequence \( n'_2 \) and \( n'_2 \) are both perpendicular to that plane so that \( n'_2 = \mathbf{n}_2 \).

One can easily prove from (8) that the speed of the particle remains constantly luminal i.e, \( d|\mathbf{c}'(t)| = 0 \), in fact by Ito calculus

\[
d|\mathbf{c}'|^2 = -2\frac{\omega^2}{\alpha^3} \left[ 1 - \frac{\mathbf{v} \cdot \mathbf{c}'}{c^2} \right] |\mathbf{c}'|^2 dt' + 2c^2 \omega^2 \frac{1}{\alpha^3} \left[ 1 - \frac{\mathbf{v} \cdot \mathbf{c}'}{c^2} \right] |\mathbf{c}'|^2 dt' + 2\frac{c^3 \omega^3}{\alpha^3} \left[ 1 - \frac{\mathbf{v} \cdot \mathbf{c}'}{c^2} \right] \mathbf{c}' \cdot d\mathbf{w}'
\]

where the second term at the right comes from the second order contribution to Ito differential. Since \( \mathbf{c}' \) and \( d\mathbf{w}' \) are perpendicular the equation reduces to

\[
d|\mathbf{c}'|^2 = -2\frac{\omega^2}{\alpha^3} \left[ 1 - \frac{\mathbf{v} \cdot \mathbf{c}'}{c^2} \right] \left(|\mathbf{c}'|^2 - c^2\right) dt' = 0
\]

where he last equality holds if the initial velocity is luminal i.e, \(|\mathbf{c}(0)| = c\).
The fact that the process remains luminal is not astonishing since a particle moving at the speed of light also moves at the speed of light in any other inertial frame. Therefore, the equation (10) defines a class of light speed processes which transform one in the other by Lorentz boost.

Notice that in a generic inertial frame, although the particle still only moves at the speed of light, it has an average velocity \( \mathbf{v} \) for large times. This is a consequence of the fact that, according to (10), the diffusion of \( \mathbf{c}' \) slows down the more \( \mathbf{v} \cdot \mathbf{c}' \) is large. This means that the particle spends more time with values of \( \mathbf{c}' \) aligned with \( \mathbf{v} \) and less time when it is anti-aligned. This is exactly the same one has with the \( (1+1) \)-dimensional Kac process associated to the telegrapher equation. In this simpler case a non vanishing average velocity is determined by the fact that unbalanced rates of inversions lead to a longer permanence of the velocity in one of the two directions.

We also would like to stress again that the Kac process could be generalized to \( 3+1 \) dimensions in a different way considering a velocity which performs jumps in place of having continuous trajectories, but we think the process presented here, having as a constitutive ingredient the Wiener process, is the most natural choice for this generalization.

4 Averages and special variables

In this section we only consider the process in the 'rest frame', all results concerning averages can be eventually Lorentz transformed for the processes in a generic inertial frame.

The stochastic equation (11) can be recast in an integral equation:

\[
x(t) = x(0) + \int_0^t c(s)ds,
\]

\[
c(t) = e^{-\omega^2t} \left[ c(0) + \omega c \int_0^t e^{\omega^2s} dw(s) \right].
\]

This is not a solution because \( dw(s) \), according to (11) and (3), depends on \( c(t) \). Let us mention that the proof that the particle velocity remains constantly luminal i.e, \( |c(t)| = c \) can be eventually also obtained by the second integral equation in (13).

Starting from second integral equation in (13) one can easily find out that the following averages hold for \( t \geq s \geq 0 \):

\[
E[c(t)] = e^{-\omega^2t}c(0),
\]

\[
E[c(t) \cdot c(s)] = c^2 e^{-\omega^2(t+s)} + 2\omega^2 c^2 e^{-\omega^2(t+s)} \int_0^s e^{2\omega^2u} du = c^2 e^{-\omega^2(t-s)}.
\]

Notices that the first of the equalities above says that the average velocity \( E[c(t)] \) vanishes for large \( t \) however, for a generic inertial frame, \( E[c(t')] \to \mathbf{v} \) for large \( t' \). Then, using these averages and the first of the integral equations in (13), one also obtains

\[
E[x(t)] = x(0) + \frac{1-e^{-\omega^2t}}{\omega^2} c(0),
\]

\[
E[(x(t) - x(0))^2] = 2c^2 \int_0^t \int_0^s e^{-\omega^2(s-u)} du ds = \frac{2c^2}{\omega^2} t - \frac{2c^2}{\omega^4} (1-e^{-\omega^2t}).
\]

The above averages imply, for large times, a diffusive behaviour with coefficient \( \frac{c^2}{\omega^2} \), in this limit one has in fact \( E[(x(t) - x(0))^2] \sim \frac{2c^2}{\omega^2} t \). On the contrary, for short times \( E[(x(t) - x(0))^2] \sim c^2 t^2 \) which means ballistic behavior at the speed of light.

It is interesting to note that it is possible to find out a pair of autonomous variables, i.e, two variables whose stochastic equations can be written and eventually solved independently from the equations of the other variables. Consider first the equation for the variable \( \mathbf{x} \cdot \mathbf{n} \):

\[
d(\mathbf{x} \cdot \mathbf{n}) = \mathbf{x} \cdot d\mathbf{n} + \mathbf{n} \cdot d\mathbf{x} = cdt - \omega^2 (\mathbf{x} \cdot \mathbf{n}) dt + \omega \mathbf{x} \cdot dw
\]

(16)
where the second equality is derived from (1). On the other side: $x \cdot dw(t) = x \cdot n_2 dw_2 + x \cdot n_3 dw_3 = [(x \cdot n_2)^2 + (x \cdot n_3)^2]^{1/2} dw = [\|x\|^2 - (x \cdot n)^2]^{1/2} dw$ where the second equality only use the composition rules of independent Gaussian increments ($dw$ also is a unitary Gaussian increment) and the last equality only use the identity $(x \cdot n_2)^2 + (x \cdot n_3)^2 = \|x\|^2 - (x \cdot n)^2$. Moreover

$$d\|x\|^2 = 2x \cdot dx = 2c(x \cdot n) dt$$

where the second equality also simply derives from (1). In conclusion the pair of variables $x \cdot n, \|x\|^2$ satisfy the autonomous system

$$d(x \cdot n) = c dt - \omega^2 (x \cdot n) dt + \omega \left[ \|x\|^2 - (x \cdot n)^2 \right]^{1/2} dw,$$

$$d\|x\|^2 = 2c(x \cdot n) dt.$$  

(17)

Taking an average ad solving the resulting system one finds out

$$E[x(t) \cdot n(t)] = \frac{c}{\omega^2} - \frac{c}{\omega^2} e^{-\omega^2 t} + e^{-\omega^2 t} x(0) \cdot n(0),$$

$$E[\|x(t)\|^2] = \frac{2c^2}{\omega^2} t - \frac{2c^2}{\omega^2} (1-e^{-\omega^2 t}) + 2 \frac{1-e^{-\omega^2 t}}{\omega^2} x(0) \cdot c(0) + \|x(0)\|^2.$$  

(19)

While the first equality is new, the second can be obtained combining the two equalities in (18).

The fact that these two variables can be isolated obviously does not mean that the probability density depends only on these two. Nevertheless, in case the process starts from the origin position $(x(0) = 0)$ and the initial velocity distribution is uniform on the surface of the sphere, it is easy to realize that the density only depends on these two variables at any time.

The condition $x(0)=0$ can be eventually replaced by the more generic $x(0)=x_0$, to return to the previous case it is in fact sufficient to consider the variables $(x(t) - x_0) \cdot c(t)$ and $\|x(t) - x_0\|^2$ which also satisfy (18).

5 Summary and outlook

In conclusion we have found that equation (10) describes a class of light speed processes which transform one in the other by Lorentz boost. Their main characteristics can be resumed as follows:

- the particle only moves at light speed and the velocity performs a Brownian motion on a sphere of radius $c =$ light speed. For large times it has an average velocity $v$, this is is a consequence of the fact that according to (10) the diffusion of the velocity slows down the more $v \cdot c'$ is large. In turn, this means that the particle spends more time with values of $c'$ aligned with $v$ and less time when it is anti-aligned. This is exactly the same situation one has with the $(1+1)$-dimensional Kac process associated to the telegrapher equation since the probability rate of inversion of velocity can be different for left/right and right/left inversions;

- the class of processes that we propose generalizes to 3+1 dimensions the 1956 idea of the Polish physicist and mathematician Marek Kac which considered a (1+1)-dimensional process where the particle travels at light speed (left or right) and randomly inverts its velocity. Although the Kac process could be generalized to 3+1 dimensions in a different way, for example considering a velocity which performs jumps in place of having continuous trajectories, we think the process presented here, having as a constitutive ingredient the Wiener process, is the most natural choice for this generalization. Moreover, since the speed is always the maximum possible it is posses the trajectories which better mimics the (infinite speed) Wiener trajectories given the relativistic constraint;

- for large times the behavior of the position is diffusive with coefficient $\frac{c^2}{\omega^2}$, one has in fact $E[(x(t) - x(0))^2] \sim \frac{2c^2}{\omega^2} t$. On the contrary, for short times $E[(x(t) - x(0))^2] \sim c^2 t^2$ which means ballistic behavior at the speed of light.
This process is the natural candidate for modeling the diffusion of mass-less particles, nevertheless, its use should not limited to this case. The situation is similar in the non-relativistic realm; thought that a particle with infinite speed is unphysical, the Wiener process is largely used to model its erratic movement.

Another point that deserves investigation and which contributed to prompt this work is the possible connection of the Fokker-Plank backward equation associated to this process with relativistic equations as Klein-Gordon and Dirac. The goal would be to find a generalization of the Gaveau et al. approach to the (3+1)-dimensional case. This topic is presently under study.

Appendix: Ito calculus

In this appendix we apply Ito calculus, in order to obtain equation (9) from equation (7). Since we defined \( \delta c' = c'(t + dt) - c'(t) \), then from (7) we immediately get

\[
\delta c' = \frac{1}{1 + \frac{v \cdot c}{c^2}} \left[ \alpha(c + dc) + v + (1 - \alpha) \frac{v \cdot (c + dc)}{v^2} \right] - \frac{1}{1 + \frac{v \cdot c}{c^2}} \left[ \alpha c + v + (1 - \alpha) \frac{v \cdot c}{v^2} \right]
\]

(20)

where \( c + dc = c(t + dt) \) and \( c = c(t) \). This is still not a Ito increment, but it is the trivial application of the definition \( \delta c' = c'(t + dt) - c'(t) \). This equation can be exactly rewritten as

\[
\delta c' = \frac{dc}{1 + dc} \left[ (1 - \alpha) \frac{v^2}{c^2} v \right] - \frac{1}{1 + \frac{v \cdot c}{c^2}} \left( \alpha c + v + (1 - \alpha) \frac{v \cdot c}{v^2} \right) + \frac{1}{1 + \frac{v \cdot c}{c^2}} \alpha dc
\]

(21)

where \( dc \) is given by (6) and where

\[
dc = \frac{1}{1 + \frac{v \cdot c}{c^2}} \frac{v \cdot dc}{c^2} = -\frac{\omega^2}{c^2} \frac{v \cdot c}{1 + \frac{v \cdot c}{c^2}} dt + \frac{\omega}{c} \frac{v \cdot n_3}{1 + \frac{v \cdot c}{c^2}} dw_3.
\]

(22)

The next step is to rewrite the right side of the equation (21) in terms of the new variables \( c', n_2', n_3' \). First of all, using (7) we immediately rewrite the equation (21) as

\[
\delta c' = \frac{1}{1 + dc} \left[ dc \left( (1 - \alpha) \frac{v^2}{c^2} v - c' \right) + \frac{1 - \frac{v \cdot c}{c^2}}{\alpha} dc \right],
\]

(23)

but we also need to rewrite \( dc \) and \( dc \) in terms of the new coordinates. In order to reach this goal we need to recall that

\[
c = \frac{1}{1 - \frac{v \cdot c}{c^2}} \left[ \alpha c' - v + (1 - \alpha) \frac{v \cdot c'}{v^2} v \right], \quad \rightarrow \quad v \cdot c = \frac{1}{1 - \frac{v \cdot c}{c^2}} \left[ v \cdot c' - v^2 \right],
\]

(24)

moreover

\[
c n_3 = \frac{1}{1 - \frac{v \cdot c}{c^2}} \left[ \alpha cn_3' - v \times n_2' + (1 - \alpha) \frac{v \cdot c'}{v^2} v \times n_2' \right], \quad \rightarrow \quad v \cdot n_3 = \alpha \frac{v \cdot n_3'}{1 - \frac{v \cdot c}{c^2}}.
\]

(25)

From these two last equations one easily realize that the three vectors \( c, v, c' \) are co-planar. As well, \( n_3 \) and \( n_3' \) lie on the same plane. Moreover, \( n_2' = n_2 \) is perpendicular to that plane. We get

\[
dc = -\frac{\omega^2}{c^2} \frac{v \cdot c - v^2}{\alpha^2} dt + \frac{\omega}{c} \frac{v \cdot n_3'}{\alpha} dw_3,
\]

(26)

where \( dc \) is given by (6) with \( n_2 = n_2' \) and \( c \) and \( n_3 \) given respectively by the first equation in (24) and the first equation in (25).

We are now ready compute the Ito differential \( i.e., we are ready to rewrite the differential \( \delta c' \) keeping only terms of order \( dt \). To obtain this result we have first of all to expand (25) to the second order with respect to the differentials:
\[ \delta c' \simeq \left[ dc \left( (1 - \alpha) \frac{c^2}{v^2} v - c' \right) + \frac{1 - \frac{v \cdot c'}{c}}{\alpha} dc \right] - \left[ (dc)^2 \left( (1 - \alpha) \frac{c^2}{v^2} v - c' \right) + d\delta c \right] \frac{1 - \frac{v \cdot c'}{c}}{\alpha}, \quad (27) \]

then, we have replace the second order differentials \((dc)^2\) and \(d\delta c\) by the terms proportional to \(dt\) of their averages:

\[ (dc)^2 \simeq \left( \frac{\omega}{c} \right)^2 \left( \frac{v \cdot n_3'}{c^2} \right)^2 dt, \quad d\delta c \simeq \frac{\omega}{\alpha} (v \cdot n_3)n_3 dt \tag{28} \]

where \(n_3\) must be expressed in terms of the new variables by the first equation in \(\text{[25]}\).

Let us rewrite equation \(\text{[27]}\) as

\[ \delta c' = \delta A + \delta B + \delta C \tag{29} \]

where \(\delta A\) is the term proportional to \(dt\) which comes from the first order differentials (the deterministic part of the first term between square parenthesis in \(\text{[27]}\)), \(\delta B\) is the term proportional to \(dt\) which comes from the second order differential (the second term between square parenthesis in \(\text{[27]}\)) and \(\delta C\) is the random term (the random part of the first term between square parenthesis in \(\text{[27]}\)). After having expressed all the old variables in terms of the new ones (except \(n_3\)), we have

\[ \delta A = - \left[ \frac{\omega^2}{c^2} \frac{v \cdot c' - v^2}{\alpha^2} \left( (1 - \alpha) \frac{c^2}{v^2} v - c' \right) + \frac{\omega}{\alpha} \left( \frac{\omega^2}{c^2} \frac{v \cdot c' - v^2}{\alpha^2} \right) \right] dt, \]

\[ \delta B = - \frac{\omega^2}{\alpha^2} (v \cdot n_3') \left[ \frac{v \cdot n_3}{c^2} \left( (1 - \alpha) \frac{c^2}{v^2} v - c' \right) + 1 - \frac{v \cdot c'}{c^2} \right] n_3 dt, \]

\[ \delta C = \left[ \frac{\omega}{c} \frac{v \cdot n_3'}{\alpha} \left( (1 - \alpha) \frac{c^2}{v^2} v - c' \right) \right] dw_3 + \frac{1 - \frac{v \cdot c'}{c^2}}{\alpha} \omega c (n_2' dw_2 + n_3 dw_3) \tag{30} \]

with \(n_3\) given by \(\text{[25]}\) in terms of the new variables. After some rearrangement of terms we get:

\[ \delta A = - \left[ \frac{\omega^2}{c^2} \frac{v \cdot c' - v^2}{\alpha^2} \right] v dt + \left[ \frac{\omega^2}{c^2} \frac{v \cdot c' - v^2}{\alpha^2} \right] c' dt = \frac{\omega^2}{\alpha^2} \left[ 1 - \frac{v \cdot c'}{c^2} \right] (v - c') dt. \tag{31} \]

This differential lies in the plane of \(c'\) and \(n_3'\) and can be decomposed along these two vectors:

\[ \delta A = \frac{\omega^2}{\alpha^2} \left[ 1 - \frac{v \cdot c'}{c^2} \right] c' dt + \frac{\omega^2}{\alpha^2} \left[ 1 - \frac{v \cdot c'}{c^2} \right] (v \cdot n_3') n_3' dt. \tag{32} \]

Analogously, the term \(\delta B\), after decomposition along \(c'\) and \(n_3'\), can be rewritten as

\[ \delta B = - \frac{\omega^2}{\alpha^2} (v \cdot n_3') \left( 1 - \frac{v \cdot c'}{c^2} \right) \left[ \frac{v \cdot n_3}{c^2} \left( (1 - \alpha) \frac{c^2}{v^2} v - c' \right) + n_3 \cdot c' \right] c' dt \]

\[ \left\{ - \frac{\omega^2}{\alpha^2} (v \cdot n_3') \left( 1 - \frac{v \cdot c'}{c^2} \right) \left[ \frac{v \cdot n_3}{c^2} \left( (1 - \alpha) \frac{c^2}{v^2} v - c' \right) + n_3 \cdot c' \right] \right\} n_3' dt. \tag{33} \]

Since \(c' \cdot n_3 = -c \cdot n_3'\), from \(\text{[24]}\) by scalar product with \(n_3'\), one gets

\[ c' \cdot n_3 = \frac{1}{1 - \frac{v \cdot c'}{c^2}} \left[ 1 - (1 - \alpha) \frac{v \cdot c'}{v^2} \right] v \cdot n_3', \tag{34} \]

therefore the first term at the right side of equality \(\text{[33]}\) vanishes, moreover from \(\text{[24]}\) and by the definitions of \(n_3\) and \(n_3'\) one has that

\[ n_3 \cdot n_3' = \frac{c \cdot c'}{c^2} = \frac{1}{1 - \frac{v \cdot c'}{c^2}} \left[ \alpha c^2 - v \cdot c' + (1 - \alpha) \left( \frac{v \cdot c'}{v^2} \right)^2 \right]. \tag{35} \]
which, can be substituted in the second term of (33) in order to obtain
\[ \delta B = -\frac{\omega^2}{\alpha^2}(v \cdot n_3')(1 - \frac{v \cdot c'}{c^2}) n_3' dt. \] (36)

A scalar product of the third of (30) with \( c' \) gives
\[ c' \cdot \delta C = \frac{\omega c}{\alpha} \left[ v \cdot n_3' \left( 1 - \alpha \frac{v \cdot c'}{v^2} - 1 \right) + \left( 1 - \frac{v \cdot c'}{c^2} \right) (c' \cdot n_3) \right] dw_3 = 0 \] (37)
where the equality is obtained using (34). Construction is coherent since the Wiener increment of \( c' \) has no component parallel to \( c' \) itself. Therefore, by decomposition along \( n_2' = n_2' \) and \( n_3' \) we have
\[ \delta C = \frac{\omega c}{\alpha} \left[ 1 - \frac{v \cdot c'}{c^2} \right] n_2' dw_2 + \frac{\omega c}{\alpha} \left[ (1 - \alpha) \frac{(v \cdot n_3')^2}{v^2} + \left( 1 - \frac{v \cdot c'}{c^2} \right) \frac{c' \cdot c}{c^2} \right] n_3' dw_3, \] (38)
which given (35) can be rewritten as
\[ \delta C = \frac{\omega c}{\alpha} \left( 1 - \frac{v \cdot c'}{c^2} \right) (n_2' dw_2 + n_3' dw_3). \] (39)

Finally, by \( \delta c' = \delta A + \delta B + \delta C \) we finally obtain
\[ \delta c' = -\frac{\omega^2}{\alpha^2} \left[ 1 - \frac{v \cdot c'}{c^2} \right]^2 c' dt + \frac{\omega c}{\alpha} \left( 1 - \frac{v \cdot c'}{c^2} \right) (n_2' dw_2 + n_3' dw_3), \] (40)
which is the equation (39) that we use in section 3 and which allows to find out the equation (40) which characterizes the general class of the light speed processes.

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