Gauge Theory at Large $N$ and New $G_2$ Holonomy Metrics

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We find a one-parameter family of new $G_2$ holonomy metrics and demonstrate that it can be extended to a two-parameter family. These metrics play an important role as the supergravity dual of the large $N$ limit of four dimensional supersymmetric Yang-Mills. We show that these $G_2$ holonomy metrics describe the M theory lift of the supergravity solution describing a collection of D6-branes wrapping the supersymmetric three-cycle of the deformed conifold geometry for any value of the string coupling constant.

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1. Introduction

There are at least two important reasons to study M theory on manifolds admitting a metric with $G_2$ holonomy. The first one is that the condition of four-dimensional $\mathcal{N} = 1$ supersymmetry that follows from the low energy approximation to M theory – eleven-dimensional supergravity – is precisely that the internal seven-dimensional manifold admit a $G_2$ holonomy metric. The massless four-dimensional fields that arise from such compactifications and the classical four-dimensional effective supergravity description can be computed from the topology of the corresponding $G_2$ manifold \cite{1}. The familiar obstruction to obtaining a chiral four-dimensional spectrum still holds at the level of the supergravity approximation but non-perturbative effects – arising for instance from singularities – can lead to interesting models with chiral matter and non-abelian gauge fields from compactification of M theory on spaces with $G_2$ holonomy. A more recent motivation for the study of $G_2$ holonomy manifolds is the rôle they play as geometric dual descriptions of the large $N$ limit of $\mathcal{N} = 1$ four-dimensional gauge theories \cite{2,3,4}. In \cite{3} a duality conjectured by Vafa \cite{5} between Type IIA on the deformed conifold with D-branes and Type IIA on the resolved conifold with Ramond-Ramond flux was derived by lifting the two Type IIA backgrounds to M theory, where they take a purely geometrical form in terms of a compactification \cite{6} on two different $G_2$ holonomy manifolds admitting a smooth interpolation in M theory. Since the Type IIA background with D-branes naturally contains gauge fields this duality allows one to study the infrared dynamics of gauge theories by analyzing M theory on spaces with $G_2$ holonomy. This new type of duality has been further developed and generalized in \cite{3,4,5}.

Only three examples of complete metrics with $G_2$ holonomy are known in the literature \cite{16,17}. However, supersymmetry together with the familiar Type IIA duality with M theory indicate \cite{2,3,4,10,11} that there must exist a large class of $G_2$ holonomy manifolds describing the M theory lift of Type IIA D6-branes wrapped on a special Lagrangian three-cycle of a Calabi-Yau three-fold. Constructing new metrics with $G_2$ holonomy is therefore an important enterprise which might eventually lead to interesting four-dimensional supersymmetric chiral models. Moreover, new $G_2$ holonomy metrics can lead to an improved understanding of the strongly coupled infrared dynamics of gauge theories. The search for new complete metrics of exceptional holonomy was revived recently and the list of

\footnote{1 The $G_2$ holonomy manifolds that appear in the M theory lift are noncompact, so strictly speaking we are not compactifying.}
examples with Spin(7) holonomy [16,17] was extended [18,19]. See also [20] in which a somewhat different approach is pursued.

In this paper we construct a new metric with $G_2$ holonomy. The metric we find describes the M theory lift of a configuration of D6-branes wrapping the $S^3$ of the deformed conifold geometry for a finite asymptotic value of the string coupling. The metric considered in [3] describes the uplift in the limit where the string coupling is infinite far from the D6-branes. Our solution, as opposed to the previously known metric which is asymptotically conical, has at infinity a circle of finite radius which we identify with the M theory circle. In the interior, our new metric just reduces to one of the previously known ones.

The plan of the rest of the paper is as follows. In section 2 we use symmetries and general properties of $G_2$ holonomy manifolds to write down the metric ansatz and find a set of first order equations whose solutions give rise to a $G_2$ holonomy metric. We analyze our system of equations in two different interesting limits and in this way we recover the known $G_2$ holonomy metric on the spin bundle over $S^3$ and the $SU(3)$ holonomy metric on the deformed conifold. In section 3 we find explicitly a one-parameter family of new metrics with $G_2$ holonomy. We analyze its geometry and asymptotic behavior. Furthermore, we provide convincing evidence that there exists a two-parameter family of metrics using perturbative arguments. In section 4 we follow a different route based on Ricci flatness to rederive the system of first order equations found in section 2 as the conditions for $G_2$ holonomy. Section 5 considers the reduction of our one-parameter family of metrics down to a Type IIA solution. By analyzing the solution we show that it describes the supergravity background corresponding to wrapped D6-branes on the deformed conifold. In sections 6 and 7 we discuss dynamical aspects of $\mathcal{N} = 1$ gauge theory on D6-branes. In particular, in section 6 we present a puzzle where a mismatch of the massless spectrum between Type IIA and M theory is exhibited. In section 7 we compute the non-perturbative superpotential directly in M theory by counting membrane instantons, including the contributions of multiple covers. In section 8 we find a system of first order equations whose solutions give rise to $G_2$ holonomy metrics of reduced symmetry. We were not able to find metrics of this type in closed form but by analogy with the well known Taub-NUT and Atiyah-Hitchin metrics it is plausible that solutions to these equations describe M theory lifts of Type IIA orientifold six-planes on the deformed conifold. Finally, section 9 contains a discussion of our results and further remarks on the duality studied by [3].
2. $G_2$ Holonomy Metric and Special Solutions

2.1. Symmetries and Ansatz

We want to find a metric with $G_2$ holonomy on a seven-dimensional manifold which describes the M theory lift of $N$ D6-branes wrapping the $S^3$ in the deformed conifold geometry. The starting point to accomplish this is to write down the most general metric ansatz with a prescribed symmetry. One way to determine the appropriate symmetry of our ansatz is to notice that the symmetry of the Type IIA configuration that we want to describe in eleven dimensions is $SU(2) \times SU(2) \times Z_2$. This can be easily understood as follows. The deformed conifold is described by the following equation in $C^4$

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = r.$$  \hspace{1cm} (2.1)

This equation has an obvious $SO(4) \sim SU(2) \times SU(2)$ symmetry which rotates the $z_i$’s. Moreover when $r = 0$, where the space develops a conical singularity, (2.1) has a $U(1)$ symmetry which acts by a common phase rotation $z_i \to e^{i\alpha}z_i$. When $r \neq 0$, then the $U(1)$ symmetry is broken to $Z_2$, which acts by $z_i \to -z_i$. Therefore, the deformed conifold geometry with wrapped D6-branes has an $SU(2) \times SU(2) \times Z_2$ symmetry. Furthermore, once this background is lifted to M theory there is an extra $U(1)$ symmetry which acts by shifts on the M theory circle. Therefore, the symmetry that we are going to impose on our ansatz for the new $G_2$ holonomy metric is going to be $SU(2) \times SU(2) \times U(1) \times Z_2$. Another way to understand why this is the appropriate symmetry to impose on the metric ansatz is to notice that the geometry found in [3] describing the infinite string coupling limit of the wrapped D6-branes on the deformed conifold has an $SU(2) \times SU(2) \times SU(2)$ symmetry. If one wants a circle of finite radius at infinity corresponding to a finite value of the string coupling and not an asymptotically conical geometry in which the M theory circle decompactifies then one of the $SU(2)$’s in the metric of [3] must be broken to $U(1)$.

It is convenient to realize our ansatz with the required symmetry by using two sets of left-invariant $SU(2)$ forms. In this basis the precise implementation of the symmetry will be as $SU(2)_L \times \widetilde{SU(2)}_L \times U(1)^{\text{diag}}_R \times Z_2$, where $SU(2)_L$ and $\widetilde{SU(2)}_L$ are associated to the two sets of left invariant one-forms

$$\sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi, \quad \Sigma_1 = \cos \tilde{\psi} d\tilde{\theta} + \sin \tilde{\psi} \sin \tilde{\theta} d\tilde{\phi},$$

$$\sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi, \quad \Sigma_2 = -\sin \tilde{\psi} d\tilde{\theta} + \cos \tilde{\psi} \sin \tilde{\theta} d\tilde{\phi},$$

$$\sigma_3 = d\psi + \cos \theta d\phi \quad \text{and} \quad \Sigma_3 = d\tilde{\psi} + \cos \tilde{\theta} d\tilde{\phi}.$$  \hspace{1cm} (2.2)
which satisfy the $SU(2)$ algebra

$$
d\sigma_a = -\frac{1}{2} \epsilon_{abc} \sigma_b \wedge \sigma_c \quad d\Sigma_a = -\frac{1}{2} \epsilon_{abc} \Sigma_b \wedge \Sigma_c. \tag{2.3}
$$

Then $U(1)_{R}^{\text{diag}} = (U(1)_R \times \widetilde{U(1)}_R)^{\text{diag}}$ acts by a diagonal rotation on the two sets of left-invariant $SU(2)$ forms. If we embed the $U(1)_{R}^{\text{diag}}$ along the Cartan generator of $SU(2)$ and $SU(2)$ then $U(1)_{R}^{\text{diag}}$ acts by a rotation

$$
\begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix} \rightarrow
\begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix}
\tag{2.4}
$$

and likewise on $(\Sigma_1, \Sigma_2)$. Lastly, the $Z_2$ symmetry acts by exchanging the two sets of left-invariant one-forms

$$
Z_2: \quad \sigma_a \leftrightarrow \Sigma_a. \tag{2.5}
$$

Then, the most general metric ansatz compatible with these symmetries is given by

$$
ds^2 = \sum_{a=1}^{7} e^a \otimes e^a, \tag{2.6}
$$

with the following vielbeins

$$
e^1 = A(r)(\sigma_1 - \Sigma_1), \quad e^2 = A(r)(\sigma_2 - \Sigma_2), \\
e^3 = D(r)(\sigma_3 - \Sigma_3), \quad e^4 = B(r)(\sigma_1 + \Sigma_1), \\
e^5 = B(r)(\sigma_2 + \Sigma_2), \quad e^6 = C(r)(\sigma_3 + \Sigma_3), \\
e^7 = dr/C(r), \tag{2.7}
$$

where we have made a particular choice for the radial coordinate. The ansatz depends on four functions.

### 2.2. $G_2$ Holonomy and First Order Equations

Local reduction of the holonomy group of a seven-manifold from $SO(7)$ to $G_2$ is determined by the $G_2$-structure. This is a globally defined three-form $\Phi$, usually called the associative three-form, which is covariantly constant with respect to the Levi-Civita connection determined by the metric. An equivalent statement is that it is closed and co-closed:

$$
d\Phi = 0 \quad d^* \Phi = 0. \tag{2.8}
$$

\footnote{This is actually a stronger condition than that $\Phi$ be harmonic, because on a non-compact manifold, closed and co-closed implies harmonic, but not the converse.}
The choice of a $G_2$-structure on the seven-dimensional manifold breaks the $GL(7, \mathbb{R})$ tangent space symmetry to precisely $G_2$. This follows because locally, given a vielbein basis $e^a$, where $a = 1, \ldots, 7$, the associative three-form is given by

$$
\Phi = \frac{1}{3!} \psi_{abc} e^a e^b e^c
\tag{2.9}
$$

where $\psi_{abc}$ – which are totally antisymmetric – are the structure constants of the imaginary octonions,

$$
i_{a}i_{b} = -\delta_{ab} + \psi_{abc} i_{c}, \quad a, b, c = 1, \ldots, 7
\tag{2.10}
$$

and $G_2$ is the automorphism group of the imaginary octonions. In a choice of basis the non-zero structure constants are given by

$$
\psi_{abc} = +1, \quad (abc) = \{(123), (147), (165), (246), (257), (354), (367)\}. \tag{2.11}
$$

Likewise, the coassociative four-form $\ast \Phi$ is locally given by

$$
\ast \Phi = \frac{1}{4!} \psi_{abcd} e^a e^b e^c e^d
\tag{2.12}
$$

where the totally antisymmetric structure constants $\psi_{abcd}$ in the basis (2.11) are given by

$$
\psi_{abcd} = +1, \quad (abcd) = \{(4567), (2356), (2374), (1357), (1346), (1276), (1245)\} \tag{2.13}
$$

and also have $G_2$ symmetry. We now use the vielbeins (2.7) of our ansatz to construct by means of (2.9) and (2.12) $\Phi$ and $\ast \Phi$. This construction gives a candidate $G_2$-structure $\Phi$. To prove that our metric has $G_2$ holonomy we must impose that the associative three-form $\Phi$ is closed and co-closed. These conditions imposed on the associative three-form $\Phi$ constructed from our vielbeins leads to the following system of first order differential equations:

$$
\begin{align*}
\frac{dA}{dr} &= \frac{1}{4} \left[ \frac{B^2 - A^2 + D^2}{BCD} + \frac{1}{A} \right] \\
\frac{dB}{dr} &= \frac{1}{4} \left[ \frac{A^2 - B^2 + D^2}{ACD} - \frac{1}{B} \right] \\
\frac{dC}{dr} &= \frac{1}{4} \left[ \frac{C}{B^2} - \frac{C}{A^2} \right] \\
\frac{dD}{dr} &= \frac{1}{2} \left[ \frac{A^2 + B^2 - D^2}{ABC} \right].
\end{align*}
\tag{2.14}
$$

\footnote{For completeness, we have included in the Appendix the exterior calculus of the vielbeins of the ansatz (2.7) needed to verify the calculations in this paper.}
The general solution to these equations gives rise to a metric with $G_2$ holonomy. In section 3 we find a solution of these equations in terms of elementary functions. We conclude this subsection with two clarifying remarks. First, the conditions (2.8) actually only guarantees that the holonomy group of the seven manifold is contained in $G_2$. Therefore, we need a further criterion which determines when the holonomy group is precisely $G_2$. Such a criterion is known [16], and it requires that there exist no non-zero covariantly constant one-forms on the seven manifold or equivalently that the fundamental group of the manifold be pure torsion. Informally, this condition is the statement that the seven-manifold cannot be written as the direct product of two spaces. As will be clear in the next section when we analyze the geometrical properties of our ansatz the holonomy of our metric is precisely $G_2$. Another important point to keep in mind is as follows. As explained above, the choice of a $G_2$-structure on the manifold breaks the tangent space symmetry to $G_2$. On the other hand, the choice of a metric $g$ on the manifold breaks the tangent space symmetry to $SO(7)$. Therefore, a given $G_2$-structure $\Phi$ uniquely determines a $G_2$ holonomy metric $g$ since $G_2 \subset SO(7)$. The converse is not true, given a metric $g$ there is no canonical choice of a $G_2$-structure. In the case under study, we have succeeded in constructing the associative three-form from the metric because we have used a very suitable choice of vielbeins for the metric. Generically it is difficult to choose the right set of vielbeins of the metric to construct the associative three-form. In section 4 we follow a more traditional approach where we rederive the first order differential equations (2.14) from the second order equations derived by imposing Ricci flatness on the metric ansatz (2.6)(2.7). In this approach one has to prove that the metric has $G_2$ holonomy either by constructing the $G_2$-structure or by showing that there is only one component of the $SO(7)$ spinor which is covariantly constant.\footnote{The covariantly constant spinor is the singlet in the decomposition of the spinor of $SO(7)$ 8 $\rightarrow$ 1 $\oplus$ 7 under $G_2$.} We have presented the non-canonical approach first because it is the fastest way to getting the first order equations for the metric and because by construction it guarantees that the metric has holonomy contained in $G_2$.

2.3. Special Solutions

In this section we specialize our ansatz (2.6)(2.7) to have an enhanced $SU(2) \times SU(2) \times SU(2) \times Z_2$ symmetry and we show that the metric we get is the previously known $G_2$ holonomy metric on the spin bundle of $S^3$. We also analyze our system of first order
equations when one of the functions vanishes. In this way we recover the known metric of $SU(3)$ holonomy on the deformed conifold geometry \cite{21}. By setting $A = D$ and $B = C$ in our metric ansatz \eqref{2.6} \eqref{2.7} the metric acquires an enhanced $SU(2) \times SU(2) \times SU(2) \times Z_2$ symmetry. Then the metric takes the following form

$$ds^2 = A^2 \sum_{a=1}^{3} (\sigma_a - \Sigma_a)^2 + B^2 \sum_{a=1}^{3} (\sigma_a + \Sigma_a)^2 + dr^2 / B^2.$$ \hfill (2.15)

In order to understand the geometry it is useful to make the following coordinate transformation. Let $U$ and $V$ be the $SU(2)$ group elements from which one constructs the left invariant $SU(2)$ one-forms $\sigma_a$ and $\Sigma_a$

$$\sigma = \sigma_a T^a = U^{-1} dU$$
$$\Sigma = \Sigma_a T^a = V^{-1} dV,$$ \hfill (2.16)

where $T^a$ are the $SU(2)$ group generators. Then one can construct the following new set of left-invariant $SU(2)$ forms

$$\tilde{w} = V(\sigma - \Sigma)V^{-1}.$$ \hfill (2.17)

where $\tilde{w} = \tilde{W}^{-1} d\tilde{W}$ for $\tilde{W} = UV^{-1}$. Moreover it follows from \eqref{2.17} that the following relation holds

$$V(\sigma + \Sigma)V^{-1} = \tilde{w} - 2w,$$ \hfill (2.18)

where $w$ is a left-invariant $SU(2)$ one-form such that $w = W^{-1} dW$ with $W = V^{-1}$. With this new set of left invariant one forms $w$ and $\tilde{w}$ the metric \eqref{2.15} takes the simple form

$$ds^2 = A^2 \sum_{a=1}^{3} \tilde{w}_a^2 + 4B^2 \sum_{a=1}^{3} (w_a - \frac{1}{2} \tilde{w}_a )^2 + dr^2 / B^2.$$ \hfill (2.19)

For this more symmetric ansatz, when $A = D$ and $B = C$, our first order equations \eqref{2.14} reduce to the following simple ones

$$\frac{dA}{dr} = \frac{1}{2A}$$
$$\frac{dB}{dr} = \frac{1}{4B} \left( 1 - \frac{B^2}{A^2} \right)$$ \hfill (2.20)

\footnote{This transformation has been independently studied in a forthcoming paper \cite{22}.}
which agree with the first order equations for the metric on the spin bundle over $S^3$ [17,18]. Solution to these equations yields the following $G_2$ holonomy metric [16,17]

$$ds^2 = \frac{\rho^2}{12} \sum_{a=1}^{3} \tilde{w}_a^2 + \frac{\rho^2}{9} (1 - \frac{\rho^3}{\rho_0^3}) \sum_{a=1}^{3} (w_a - \frac{1}{2} \tilde{w}_a)^2 + \frac{d\rho^2}{(1 - \frac{\rho^3}{\rho_0^3})} \tag{2.21}$$

where $\rho \geq \rho_0$ and $\rho_0$ determines the size of the $S^3$ generated by the $\tilde{w}_a$'s at $\rho = \rho_0$. This space is topologically $R^4 \times S^3$ so the space is simply connected which together with the explicit construction of the $G_2$-structure (2.9), which is given by

$$\sqrt{3} \Phi = \frac{\rho_0^3}{144} \epsilon_{abc} \tilde{w}_a \wedge \tilde{w}_b \wedge \tilde{w}_c + \frac{\rho^2}{6} d\rho \wedge \tilde{w}_a \wedge w_a + \frac{(\rho^3 - \rho_0^3)}{36} \epsilon_{abc} (\tilde{w}_a \wedge \tilde{w}_b \wedge w_c - \tilde{w}_a \wedge w_b \wedge w_c) \tag{2.22}$$

guarantees that (2.21) has $G_2$ holonomy. We note for future reference that this metric is asymptotically conical and the base of the cone is topologically $S^3 \times S^3$.

It is also interesting to consider our ansatz (2.6)(2.7) when $C = 0$. Then the system of equations (2.14) reduces to

$$\begin{align*}
\frac{dA}{dt} &= \frac{1}{4} \left[ \frac{B^2 - A^2 + D^2}{BD} \right] \\
\frac{dB}{dt} &= \frac{1}{4} \left[ \frac{A^2 - B^2 + D^2}{AD} \right] \\
\frac{dD}{dt} &= \frac{1}{2} \left[ \frac{A^2 + B^2 - D^2}{AB} \right],
\end{align*} \tag{2.23}$$

where the new radial coordinate $t$ is related to the old one by $dr = Cdt$ and the metric of the resulting six-dimensional manifold looks like

$$ds^2 = A^2 \left( (\sigma_1 - \Sigma_1)^2 + (\sigma_2 - \Sigma_2)^2 \right) + B^2 \left( (\sigma_1 + \Sigma_1)^2 + (\sigma_2 + \Sigma_2)^2 \right) + D^2 (\sigma_3 - \Sigma_3)^2 + dt^2. \tag{2.24}$$

We should note that setting $C = 0$ reduces the symmetry of the ansatz to just $SU(2) \times SU(2) \times Z_2$ which is precisely the symmetry of the deformed conifold geometry [21]. In order to identify the geometry we rewrite the metric in a way that is conducive to comparison with the known deformed conifold metric. It is straightforward to change coordinates to rewrite the metric (2.24) in the following way:

$$ds^2 = \frac{B^2}{2} ((g^1)^2 + (g^2)^2) + \frac{A^2}{2} ((g^3)^2 + (g^4)^2) + D^2 (g^5)^2 + dt^2, \tag{2.25}$$
where
\[ g^1 = E^1 - E^3, \quad g^2 = E^2 - E^4, \quad g^3 = E^1 + E^3, \quad g^4 = E^2 + E^4, \quad g^5 = E^5 \] (2.26)
with
\begin{align*}
E^1 &= -\sin \theta_1 d\phi_1 \\
E^2 &= d\theta_1 \\
E^3 &= \cos \psi_1 \sin \theta_2 d\phi_2 - \sin \psi_1 d\theta_2 \\
E^4 &= \sin \psi_1 \sin \theta_2 d\phi_2 + \cos \psi_1 d\theta_2 \\
E^5 &= d\psi_1 + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2.
\end{align*} \tag{2.27}

This is a popular set of vielbeins in which to write the deformed conifold metric. The metric as written in (2.25) is the familiar ansatz for the deformed conifold metric. In order to show that the \( C = 0 \) truncation of our first order equations indeed describes the deformed conifold we must show that the left-over functions \( A, B \) and \( D \) equations (2.23) coincide with the first order equations of the deformed conifold. A straightforward calculation shows that the \( C = 0 \) truncation of our metric indeed yields the deformed conifold first order equations \[23\] whose solution gives the \( SU(3) \) holonomy metric \[21\] which in a convenient choice of the radial coordinate is given by
\[
ds^2 = K(\tau) \left[ \frac{1}{3K^3(\tau)} (d\tau^2 + (g^5)^2) + \frac{1}{4}\sinh^2 \left( \frac{\tau}{2} \right) [(g^1)^2 + (g^2)^2] + \frac{1}{4}\cosh^2 \left( \frac{\tau}{2} \right) [(g^3)^2 + (g^4)^2] \right] \tag{2.28}\]
with
\[
K(\tau) = \frac{(\sinh(2\tau)/2 - \tau)^{1/3}}{\sinh(\tau)}. \tag{2.29}
\]
Asymptotically this metric is also conical and the base of the cone is topologically \( S^2 \times S^3 \). This is quite satisfactory since setting \( C = 0 \), roughly speaking, removes the twisting on the M theory circle due to the wrapped D6-branes on the \( S^3 \) of the deformed conifold. Once the twisting is removed one just gets the unperturbed deformed conifold metric as we have just shown. When \( C \neq 0 \) one gets a non-trivial \( U(1) \) fibration which we analyze next.
3. A New Complete $G_2$ Holonomy Metric and Its Geometry

As shown in section 2 the most general metric on a seven-dimensional manifold having $SU(2) \times SU(2) \times U(1) \times Z_2$ symmetry depends on four functions and the requirement of having a $G_2$-structure resulted in the following system of coupled first order equations

\[
\begin{align*}
\frac{dA}{dr} &= \frac{1}{4} \left[ \frac{B^2 - A^2 + D^2}{BCD} + \frac{1}{A} \right] \\
\frac{dB}{dr} &= \frac{1}{4} \left[ \frac{A^2 - B^2 + D^2}{ACD} - \frac{1}{B} \right] \\
\frac{dC}{dr} &= \frac{1}{4} \left[ C - C \right] \\
\frac{dD}{dr} &= \frac{1}{2} \left[ \frac{A^2 + B^2 - D^2}{ABC} \right].
\end{align*}
\]

This complicated system of equations has the following discrete $Z_2$ symmetry

\[
Z_2: \begin{cases} 
  r \rightarrow -r \\
  A \leftrightarrow B \\
  D \rightarrow -D.
\end{cases}
\]

Therefore, the most general solution to (3.1) invariant under this symmetry depends on three parameters. However, one of them is trivial since it just corresponds to shifting by a constant the radial coordinate. Therefore, the general solution to (3.1) depends on two nontrivial parameters. As we shall see later, the two parameters can be interpreted from the Type IIA perspective as determining the string coupling $g_s$ and the size of the $S^3$ in the deformed conifold geometry on which the D6-branes are wrapped. In the eleven-dimensional description these two parameters correspond respectively to the size of the $U(1)$ fiber at infinity and to the volume of an $S^3$ inside the seven-dimensional manifold of $G_2$ holonomy.

3.1. A family of solutions

We have been able to find the following solution to the equations (3.1)

\[
\begin{align*}
A &= \frac{1}{\sqrt{12}} \sqrt{(r - 3/2)(r + 9/2)} \\
B &= \frac{1}{\sqrt{12}} \sqrt{(r + 3/2)(r - 9/2)} \\
C &= \sqrt{\frac{(r - 9/2)(r + 9/2)}{(r - 3/2)(r + 3/2)}} \\
D &= r/3.
\end{align*}
\]

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One can then use (3.3) to find a one-parameter family of metrics (2.6) (2.7) with $G_2$ holonomy. Since (3.3) solve the first order equations (2.8) which follow from the existence of a $G_2$-structure and if we let $g$ be the corresponding metric then the rescaled metric $r_0^2 g$ also admits a $G_2$-structure. After rescaling the radial coordinate $r \to r/r_0$ one gets the following one-parameter family of metrics

$$ds^2 = \sum_{a=1}^{7} e^a \otimes e^a,$$

with the following vielbeins

$$
\begin{align*}
e^1 &= A(r)(\sigma_1 - \Sigma_1), & e^2 &= A(r)(\sigma_2 - \Sigma_2), \\
e^3 &= D(r)(\sigma_3 - \Sigma_3), & e^4 &= B(r)(\sigma_1 + \Sigma_1), \\
e^5 &= B(r)(\sigma_2 + \Sigma_2), & e^6 &= r_0 C(r)(\sigma_3 + \Sigma_3), \\
e^7 &= dr/C(r)
\end{align*}
(3.5)
$$

where now

$$
\begin{align*}
A &= \frac{1}{\sqrt{12}} \sqrt{(r - 3r_0/2)(r + 9r_0/2)}, \\
B &= \frac{1}{\sqrt{12}} \sqrt{(r + 3r_0/2)(r - 9r_0/2)}, \\
C &= \sqrt{(r - 9r_0/2)(r + 9r_0/2)/(r - 3r_0/2)(r + 3r_0/2)}, \\
D &= r/3.
\end{align*}
(3.6)
$$

The resulting metric is Ricci flat and complete for either $r \geq 9r_0/2$ or $r \leq -9r_0/2$. These two solutions are related to each other by the action of the non-trivial $\mathbb{Z}_2$ automorphism (3.2) of the first order differential equations (3.1). For concreteness, we will consider from now on the solution whose radial coordinate is constrained to be $r \geq 9r_0/2$. The metric is Ricci flat and has a $G_2$-structure that we construct in terms of the vielbeins (3.5) together with (3.6). The $G_2$-structure can be conveniently written as

$$
\Phi = \frac{9r_0^3}{16} \epsilon_{abc} \left( \sigma_a \wedge \sigma_b \wedge \sigma_c - \Sigma_a \wedge \Sigma_b \wedge \Sigma_c \right) + d \left( \frac{r}{18} \left( r^2 - \frac{27r_0^2}{4} \right) \sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2 \right) + \frac{r_0}{3} \left( r^2 - \frac{81r_0^2}{8} \right) \sigma_3 \wedge \Sigma_3).
(3.7)
$$

The existence of this covariantly constant three-form guarantees that our metric has holonomy contained in $G_2$. 

11
3.2. The Geometry of the Solution and Asymptotics

The metric we found in the previous subsection is a $U(1)$ bundle over a six-dimensional manifold. The circle, which is parameterized by the vielbein $e^6$, has its size at infinity set by $r_0$ since $C \to 1$ as $r \to \infty$. In the interior, when $r \to 9r_0/2$, then $C \to 0$ so that the circle shrinks to zero size. This behavior is very similar to that of the Taub-NUT metric which is not surprising since our solution describes the M theory lift of a wrapped D6-brane. In particular, the size of the circle at infinity – given by $r_0$ – determines the Type IIA string coupling constant. One can generalize the coordinate transformation presented in section 2.3 and rewrite the metric as follows

$$ds^2 = A^2((g_1)^2 + (g_2)^2) + B^2((g_3)^2 + (g_4)^2) + D^2(g_5)^2 + r_0 C^2(g_6)^2 + dr^2/C^2, \quad (3.8)$$

where $g^1, \ldots, g^5$ are defined in (2.26)-(2.27) and

$$g^6 = d\psi_2 + \cos \theta_1 d\phi_1 - \cos \theta_2 d\phi_2. \quad (3.9)$$

Then the asymptotic behavior of the metric at infinity is given by

$$ds^2 = dr^2 + r^2 \left( \frac{1}{9} \left( d\psi_1 + \sum_{i=1}^{2} \cos \theta_i d\phi_i \right)^2 + \frac{1}{6} \sum_{i=1}^{2} (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) \right) + r_0 (g^6)^2. \quad (3.10)$$

This geometry is that of a $U(1)$ bundle over the singular conifold metric with $SU(3)$ holonomy. The base of the cone is described by the Einstein metric on the homogeneous space $T^{1,1} = (SU(2) \times SU(2))/U(1)$ where the $U(1)$ is diagonally embedded along the Cartan generator of the $SU(2)$'s. Therefore, at infinity our metric is topologically $\mathbb{R}_+ \times S^1 \times S^2 \times S^3$.

We now analyze the geometry in the interior. The metric is non-singular everywhere and near $r = 9r_0/2$ it behaves like

$$ds^2 \sim d\rho^2 + \frac{9}{4} r_0^2 ((g_1)^2 + (g_2)^2 + (g_5)^2) + \frac{\rho^2}{16} ((g_3)^2 + (g_4)^2 + (g_6)^2), \quad (3.11)$$

where $\rho^2 = 8r_0(r - 9r_0/2)$. Therefore, there is an $S^3$ of finite size and topologically the space becomes $\mathbb{R}^4 \times S^3$. Since $A = D$ and $C = D$ as $r \to 9r_0/2$, in the interior our solution has enhanced $SU(2) \times SU(2) \times SU(2) \times Z_2$ symmetry. This is exactly the situation discussed in detail in section 2.3. Hence, in the interior our new metric approaches the behavior of
the previously known asymptotically conical metric on the spin bundle over $S^3$. In fact, both $G_2$ manifolds have the same Betti numbers.

The space on which we put the $SU(2) \times SU(2) \times U(1) \times Z_2$ metric (3.4) – (3.5) is homotopic to $\mathbb{R}^4 \times S^3$, so it is simply connected. As explained in section 2.2 this implies that the holonomy of our metric is exactly $G_2$. Therefore, the $G_2$-structure (3.7) provides a local reduction of the holonomy group from $SO(7)$ to precisely $G_2$ and guarantees the existence of a unique covariantly constant spinor.

3.3. Evidence for a two-parameter family of solutions

A peculiarity of our solution (3.6) is that the size of the topologically non-trivial $S^3$ in the interior and the size of the circle at infinity are in fixed ratio: both are determined by $r_0$. In section 5, when we reduce our solution to Type IIA this translates into the statement that the string coupling constant and the size of the $S^3$ are not independent. On the Type IIA side, one should be free to adjust the dilaton at infinity independently of the size of the $S^3$. Thus we expect that our one-parameter family of solutions can be generalized to a two-parameter family. We have not been able to find two-parameter solutions in closed form, but numerical integration of the BPS equations (3.1) indicates that they exist. Rather than mapping out the parameter space with extensive numerics, we will be satisfied to give here a perturbative argument which uses numerics only to check one important point.

Suppose we start with the solution (3.6) and wish to make a uniformly small perturbation. We write

$$
A = A_0 + \phi_A, \quad B = B_0 + \phi_B, \quad C = C_0 + \phi_C, \quad D = D_0 + \phi_D,
$$

(3.12)

where $A_0$, $B_0$, $C_0$, and $D_0$ are the solutions given explicitly in (3.6). Plugging these expressions into (3.1) and expanding to linear order in the $\phi_i$’s, we obtain an equation of the form

$$
\frac{d\vec{\phi}}{dr} = \mathbf{M}(r)\vec{\phi}
$$

(3.13)

where $\mathbf{M}(r)$ is some $4 \times 4$ matrix whose entries can be simply expressed in terms of $A_0$, $B_0$, $C_0$, and $D_0$. The asymptotic forms of $\mathbf{M}(r)$ near $r = 9/2$ (the rounded tip of the cone) and $r = \infty$ will be helpful:

$$
\mathbf{M}(r) = \frac{m_0}{r - 9/2} + \frac{m_1}{\sqrt{r - 9/2}} + O(1)
$$

(3.14)
for $r$ close to $9/2$, where

$$\mathbf{m}_0 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1/2 & -1 & 0 \\ 0 & -1 & 1/2 & 0 \\ 2 & 0 & 0 & -2 \end{pmatrix} \quad \mathbf{m}_1 = \begin{pmatrix} 0 & 1/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{2}/3 & -\sqrt{2}/3 & 0 \end{pmatrix}$$

and

$$\mathbf{M}(r) = \mathbf{M}_0 + \frac{\mathbf{M}_1}{r} + O\left(\frac{1}{r^2}\right)$$

for large $r$, where

$$\mathbf{M}_0 = \begin{pmatrix} 0 & 0 & -1/2\sqrt{3} & 0 \\ 0 & 0 & -1/2\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \end{pmatrix} \quad \mathbf{M}_1 = \begin{pmatrix} -3/2 & 1/2 & \sqrt{3}/2 & \sqrt{3}/2 \\ 1/2 & -3/2 & -\sqrt{3}/2 & \sqrt{3}/2 \\ 0 & 0 & 0 & 0 \\ 4/\sqrt{3} & 4/\sqrt{3} & 0 & -4 \end{pmatrix}$$

Because of the $1/(r - 9/2)$ term in (3.14), a perturbation $\phi(r)$ will blow up logarithmically near $r = 9/2$ unless $\lim_{r \to 9/2} \phi(r)$ is annihilated by $\mathbf{m}_0$. In fact, $\mathbf{m}_0$ has only one null eigenvector, namely $(1, 0, 0, 1)$, which corresponds to expanding the unshrunk $\mathbf{S}^3$. So there is only one perturbation regular at the origin. Because the third row of both $\mathbf{M}_0$ and $\mathbf{M}_1$ vanish completely, and the additional contributions are $O(1/r^2)$, we see that any perturbation will lead to a finite change in the radius of the circle at infinity.

We have now shown that the perturbation is a well-defined deformation of the solution (3.6). The only remaining detail is to ensure that it is not the deformation that we have already studied—that is, a rigid rescaling. A completely analytic way to check this would be to solve (3.13) in a series around $r = 9/2$ and check that $\phi_A/A_0 \neq \phi_B/B_0$ at some order in $r - 9/2$. However, it is perhaps more to the point to demonstrate that the circle at infinity changes its radius by a different amount from the unshrunk $\mathbf{S}^3$. It is straightforward to obtain

$$\lim_{r \to 9/2} \frac{\phi_A}{A_0} = \lim_{r \to 9/2} \frac{\phi_D}{D_0} = \frac{2}{3} \lambda \neq \lim_{r \to \infty} \frac{\phi_C}{C_0} \approx 0.11 \lambda,$$

for the unique perturbation regular at $r = 9/2$. Here $\lambda$ is a parameter measuring the strength of the perturbation, and to get the crucial 0.11 we numerically integrated (3.13) with initial conditions $\phi_A = \phi_D = 1$, $\phi_B = \phi_C = 0$ imposed very close to $r = 9/2$. Thus this perturbation does correspond to changing the unshrunk $\mathbf{S}^3$ by a different scale factor from the circle at infinity—this is to be compared with our explicitly known one-parameter

\[\text{14}\]
family, all of which were obtained by rigidly scaling the solution \((3.0)\). Standard implicit function arguments suffice to show that the perturbative result implies the existence of a two-parameter family of non-singular solutions in some finite neighborhood of the one-parameter slice which we have exhibited in closed form.

The perturbation problem was particularly benign in this case because \(\phi_i\) could be taken uniformly small as compared to the unperturbed solution. By way of comparison, let us investigate another corner of the parameter space: if the circle at infinity is very small but the unshrunk \(S^3\) is finite, then we should be able to probe a little ways into the full two-parameter family of solutions by perturbing around the deformed conifold solution, \((2.28)\). Let us use a radial variable \(\tau\) such that \(dr = C\nu d\tau\), where \(\nu\) is some function of radius to be set at our convenience. The first order equations \((3.1)\) become

\[
\begin{align*}
\frac{dA}{d\tau} &= \frac{\nu}{4} \left[ \frac{B^2 - A^2 + D^2}{BD} \right] + C^2 \frac{1}{A}, \\
\frac{dB}{d\tau} &= \frac{\nu}{4} \left[ \frac{A^2 - B^2 + D^2}{AD} \right] - C^2 \frac{1}{B}, \\
\frac{dC}{d\tau} &= \frac{\nu}{4} \left[ \frac{C^2}{B^2 - A^2} \right], \\
\frac{dD}{d\tau} &= \frac{\nu}{2} \left[ \frac{A^2 + B^2 - D^2}{AB} \right],
\end{align*}
\]

and we make the gauge choice \(\nu = 1/\sqrt{3}K(\tau)\) so that the unperturbed solution can be written as

\[
A = \sqrt{\frac{K}{2}} \cosh \frac{\tau}{2}, \quad B = \sqrt{\frac{K}{2}} \sinh \frac{\tau}{2}, \quad C = 0, \quad D = \frac{1}{\sqrt{3}K}.
\]

See \((2.29)\) for a definition of \(K(\tau)\). We might expect to be able to do a straightforward perturbation expansion in small \(C\). This almost works: plugging the unperturbed solution \((3.20)\) into the third equation in \((3.19)\), one obtains

\[
\frac{d}{d\tau} \left( \frac{1}{C} \right) = -\frac{2}{\sqrt{3}} (\sinh(2\tau)/2 - \tau)^{-2/3}.
\]

While there is no elementary expression for \(C(\tau)\), it is clear that the limit \(C_\infty = \lim_{\tau \to \infty} C(\tau)\) is finite and represents the one integration constant of \((3.21)\). Furthermore,

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\(^6\) We thank E. Witten for a discussion which motivated us to carry out this analysis.
because the right hand side of (3.21) is everywhere negative, $1/C(\tau)$ is monotonically decreasing in $\tau$. Assuming $C_\infty > 0$ we learn that $C(\tau)$ is uniformly bounded by $C_\infty$.

Solving (3.21) in a series around $\tau = 0$ results in

$$C = \frac{\tau}{\sqrt{12}} \left(1 + O(\tau^2)\right).$$

(3.22)

The problem is that near $\tau = 0$, the term $-C/B$ in the second equation of (3.19) is no longer small compared to the other term in square brackets. This “back-reaction” of the $U(1)$ fiber on the deformed conifold geometry invalidates the straightforward perturbation expansion in a small neighborhood of $\tau = 0$. This is analogous to the phenomenon of boundary layers. One can see that there is a problem by observing that if we use (3.22), then $\lim_{\tau \to 0} C/B = 2$, which means that the shrinking $S^3$ is not round. This would actually mean that there is a singularity at $\tau = 0$. The resolution is to give a different perturbation analysis near $\tau = 0$ by assuming $A \sim O(1)$, $B \sim O(\tau)$, $C \sim O(\tau)$, and $D \sim O(1)$. At leading order in $\tau$, we obtain

$$\frac{dA}{d\tau} = \frac{1}{4\sqrt{12}} \left[\frac{-A^2 + D^2}{BD}\right]$$
$$\frac{dB}{d\tau} = \frac{1}{4\sqrt{12}} \left[\frac{A^2 + D^2}{AD} - \frac{C}{B}\right]$$
$$\frac{dC}{d\tau} = \frac{1}{4\sqrt{12}} \left[\frac{C^2}{B^2}\right]$$
$$\frac{dD}{d\tau} = \frac{1}{2\sqrt{12}} \left[\frac{A^2 - D^2}{AB}\right].$$

(3.23)

The only regular solution is $A = D = (\text{const})$ and $B = C = \frac{1}{4\sqrt{12}} \tau$. We do not have a proof, but we expect it is possible to match this “boundary layer” solution smoothly onto the straightforward perturbation theory at $\tau \sim C_\infty$, leading to a uniform approximation to a two-parameter family of solutions. One parameter is $C_\infty$, and we can get the other by rigidly rescaling the whole metric.

To summarize, we have developed perturbation theory around the solutions (3.6) and (2.28). If we parametrize our solutions with the radius $R_1$ of the circle at infinity and the radius $R_2$ of the unshrunk $S^3$, then we can describe slivers of the parameter space with $R_1/R_2$ small or with $R_1/R_2 \approx 3/\sqrt{2}$. 

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4. Effective Lagrangian approach

In this section we rederive using a different method the first order equations (3.1) that we obtained previously by imposing the existence of a $G_2$-structure. In this other method we first find the equations of motion which follow from imposing that the metric ansatz be Ricci flat. If we express the metric ansatz (2.6)(2.7) in terms of the new functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $\delta(t)$ as

$$ds^2 = e^{4\alpha+4\beta+2\gamma+2\delta}dt^2 + e^{2\alpha}[(\sigma_1 - \Sigma_1)^2 + (\sigma_2 - \Sigma_2)^2] + e^{2\delta}(\sigma_3 - \Sigma_3)^2 + e^{2\beta}[(\sigma_1 + \Sigma_1)^2 + (\sigma_2 + \Sigma_2)^2] + e^{2\gamma}(\sigma_3 + \Sigma_3)^2,$$

then a convenient way of obtaining the Ricci flatness equations is to realize that these follow from the equations of motion which are derived from the Einstein-Hilbert action $S = \int dx^7 \sqrt{|g|} R$. Therefore, by computing the Einstein-Hilbert action on the ansatz one can reduce the problem to an effective quantum mechanics problem. Evaluating the action on the ansatz (4.1) leads to the following $0+1$-dimensional effective Lagrangian:

$$L_{\text{eff}} = T - V = 16(\alpha')^2 + 64\alpha'\beta' + 16(\beta')^2 + 32\alpha'\gamma' + 32\beta'\gamma' + 32\alpha'\delta' + 32\beta'\delta' + 16\gamma'\delta'$$

$$- 2e^{2(3\alpha+\beta+\gamma)} - 2e^{2(\alpha+3\beta+\gamma)} + 4e^{4\alpha+4\beta+2\gamma} + 8e^{2(2\alpha+\beta+\gamma+\delta)}$$

$$+ 8e^{2(\alpha+2\beta+\gamma+\delta)} - 2e^{2(\alpha+\beta+\gamma+2\delta)} - e^{4\alpha+4\gamma+2\delta} - e^{4\beta+4\gamma+2\delta}$$

where a prime stands for a derivative with respect to the "time" coordinate $t$. The first line in this expression should be understood as the kinetic term for the scalar fields $\alpha_i = (\alpha, \beta, \gamma, \delta)$. It can be written as

$$T = \frac{1}{2}G_{ij} \frac{\partial \alpha_i}{\partial t} \frac{\partial \alpha_j}{\partial t}$$

with the following constant scalar-manifold metric

$$G_{ij} = \begin{pmatrix} 32 & 64 & 32 & 32 \\ 64 & 32 & 32 & 32 \\ 32 & 32 & 0 & 16 \\ 32 & 32 & 16 & 0 \end{pmatrix}.$$

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7 As usual in this approach we have replaced the second order terms by first order ones by integrating by parts.
The last two lines in the effective Lagrangian (4.2) represent the scalar potential. One can derive first order equations from (4.2) if one manages to write the scalar potential of the auxiliary Lagrangian (4.2) in terms of a superpotential

\[ V = -\frac{1}{2} G^{ij} \frac{\partial W}{\partial \alpha^i} \frac{\partial W}{\partial \alpha^j}. \] (4.5)

This can be accomplished and the corresponding superpotential reads

\[ W = 2\sqrt{2} \left[ 2e^{3\alpha+\beta+\gamma} + 2e^{\alpha+3\beta+\gamma} + e^{2\alpha+2\gamma+\delta} - e^{2\beta+2\gamma+\delta} + 2e^{\alpha+\beta+\gamma+2\delta} \right]. \] (4.6)

Therefore, the following first order BPS-like equation

\[ \frac{\partial \alpha^i}{\partial t} = G^{ij} \frac{\partial W}{\partial \alpha^j}. \] (4.7)

along with the constraint \( T + V = 0 \) guarantees that any solution of (4.7) is a solution of the Ricci flatness equations. By rewriting these first order equations in terms of the original variables

\[ A = e^\alpha, \quad B = e^\beta, \quad C = e^\gamma, \quad D = e^\delta, \quad dr = \sqrt{2} e^{2\alpha+2\beta+2\gamma+\delta} dt \] (4.8)

we obtain the first order equations (2.14) which we obtained before by a more direct method. In the new variables \( A, B, C, \) and \( D, \) the superpotential (4.6) takes the form of a homogeneous polynomial of degree five:

\[ W = 2\sqrt{2} \left[ 2A^3 BC + 2AB^3 C + A^2 C^2 D - B^2 C^2 D + 2ABC D^2 \right]. \] (4.9)

Every solution to the BPS-like equation (4.7) should interpolate between critical points of the superpotential. Since in our case \( W \) is a homogeneous polynomial, the only critical points are \( W = 0 \) and \( W = \infty. \) Therefore, for any given solution the image of \( W(r) \) should be given by a semi-infinite line going from \( W = 0 \) to \( W = \infty. \) The space of such solutions is parametrized by three real numbers (not including the trivial shift of the radial variable \( r), which can be identified, for example, with the value of \( C \) at \( W = \infty \) and with values of \( A \) and \( B \) at \( W = 0. \) Roughly speaking, these parameters represent the size of the \( S^1 \) fiber and the volumes of two 3-spheres, respectively. We are interested in solutions which have \( B = 0 \) in the interior.\(^8\) The solutions satisfying this extra condition are parametrized in general by two real parameters, in agreement with our analysis in section 3. In the effective Lagrangian approach one has to prove that the metric has reduced holonomy. Luckily, we have already constructed the corresponding \( G_2 \)-structure for this metric and have in fact shown that the one-parameter solution (3.6) has \( G_2 \) holonomy.

\(^8\) It follows from the extremality conditions that for non-zero \( A, \) the value of \( C \) has to vanish at \( W = 0 \) as well.
5. Type IIA Reduction and Wrapped D6-Branes

The $G_2$ holonomy metric is a solution of eleven-dimensional supergravity. It can be used to describe a four-dimensional vacuum with four-dimensional $\mathcal{N} = 1$ supersymmetry of the Type $\mathbf{R}^{1,3} \times X_7$ where $X_7$ is the seven manifold under consideration. The metric we found has a $U(1)$ isometry which act by shifts on an angular coordinate. Therefore, we can reduce the solution along this $U(1)$ isometry to obtain a Type IIA solution by using

$$ds_{11}^2 = e^{-2\phi/3}ds_{10}^2 + e^{4\phi/3}(dx_{11} + C_{\mu}dx^\mu)^2,$$

where $\phi$ and $C$ are respectively the Type IIA dilaton and Ramond-Ramond one-form gauge field. Thus, reducing the solution we found in (3.8) and identifying $x_{11} = \psi_2$ one obtains the following Type IIA solution

$$ds_{10}^2 = r_0^{1/2}C (dx_{1,3}^2 + A^2 ((g^1)^2 + (g^2)^2) + B^2 ((g^3)^2 + (g^4)^2) + D^2(g^5)^2) + r_0^{1/2}dr^2/C ,
 e^\phi = r_0^{3/4}(C)^{3/2} ,
 F_2 = (\sin \theta_1 d\phi_1 \wedge d\theta_1 - \sin \theta_2 d\phi_2 \wedge d\theta_2) ,$$

where the $g^i$'s are given by (2.26). This solution describes a D6-brane wrapping the $\mathbf{S}^3$ in the deformed conifold geometry. At infinity the Type IIA metric becomes that of the singular conifold and the flux is through the $\mathbf{S}^2$ surrounding the wrapped D6-brane. Moreover, the dilaton is constant at infinity. One can also analyze the solution in the interior. For $r - 9r_0/2 = \epsilon \to 0$ the string coupling goes to zero $e^\phi \sim \epsilon^{3/4}$ whereas the curvature blows up as $R \sim \epsilon^{-3/2}$ just like in the near horizon region of a flat D6-brane. This means that classical supergravity is valid for sufficiently large radius. However, the singularity in the interior is the same as the one of flat D6 branes, as expected. On the other hand the dilaton continuously decreases from a finite value at infinity, set by the radius $r_0$, to zero, so that for small $r_0$ classical string theory is valid everywhere. The global geometry is that of a warped product of flat Minkowski space and a non-compact space, $Y_6$, which for large radius is simply the conifold since the backreaction of the wrapped D6 brane becomes less and less important. In the interior however the backreaction induces changes of $Y_6$ away from the conifold geometry. At $r \to 9r_0/2$ the $\mathbf{S}^2$ has shrunk to zero size whereas an $\mathbf{S}^3$ of finite size remains. Note, that this behavior is similar to that of

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9 Since the space is non-compact gravity lives in eleven dimensions.
the deformed conifold but the two metrics are different. One can mod out the original eleven-dimensional metric by the following $Z_N$ action:

$$Z_N: \psi_2 \to \psi_2 + \pi \frac{1}{N}$$

(5.3)

The fixed points of this $Z_N$ action are located on the $S^3$, where the size of the circle parameterized by $\psi_2$ goes to zero. Thus, the local geometry at $r \approx 9r_0/2$ is singular, with $A_{N-1}$ singularity fibered over $S^3$, the so-called singular quotient \[3\]. After compactification to Type IIA theory it describes $N$ coincident D6-branes wrapped on the supersymmetric $S^3$ of the deformed conifold.

6. A $U(1)$ Puzzle

Having established in section 5 that the $G_2$ manifold given by (3.4)(3.6) reduces in the Type IIA language to a collection of wrapped D6-branes we analyze in this section the spectrum of massless fields both in Type IIA and in M theory and we point out that there is an apparent discrepancy in the spectrum. In the Type IIA side we expect to have a $U(N)$ gauge theory living on the brane arising from the usual massless open strings of the N D6-branes. The only subtlety here is that the D6-brane normal bundle is non-trivial and the gauge theory on the branes is topologically twisted. Since the supersymmetric $S^3$ is rigid the gauge theory one obtains at low energies is a four-dimensional $\mathcal{N} = 1$ $U(N)$ gauge theory. The M theory origin of the $SU(N)$ part of the gauge group is familiar. The M theory geometry is that of an $A_{N-1}$ singularity fibered over $S^3$ and the massless gauge fields come from membranes wrapping the shrunken cycles of the $A_{N-1}$ singularity. So we now search for the origin of the overall $U(1)$ in M theory.

In flat space, the $U(1)$ gauge field on a D6-brane is realized in M theory by performing a Kaluza-Klein reduction of the eleven-dimensional supergravity three-form gauge field $C$ on the harmonic two-form $\eta$ of the Taub-NUT geometry. This gives rise to a $U(1)$ gauge field since we take $C = A_\mu \wedge \eta$ and $A$ becomes a fluctuating dynamical field since $\eta$ is a $L^2$-normalizable harmonic two-form. Likewise, the $U(1)$ mode on a D6-brane which wraps the deformed conifold geometry should come from reducing $C$ along the $L^2$-normalizable harmonic two-form $\eta$ of the geometry which describes the wrapped D6-brane in M theory and which is given by (3.4)(3.6). We therefore must investigate whether our $G_2$ holonomy manifold admits an $L^2$-normalizable harmonic two-form $\eta$. Our space is homotopic to $\mathbb{R}^4 \times S^3$ which implies that our two-form, if it exists, must be topologically trivial, i.e.
exact. Moreover, this two-form must have the full symmetry of the metric, namely $SU(2) \times SU(2) \times U(1) \times Z_2$. Such a form can be constructed from the one-form which integrates to a constant along the finite size $S^1$ at infinity. Even though this one-form cannot be extended as a closed one-form in the interior it must be harmonic. Therefore, the candidate two-form is given by

$$\eta = d(F(r)(\sigma_3 + \Sigma_3)). \quad (6.1)$$

It is a straightforward exercise to show that the non-singular harmonic two-form corresponds to $F = C^2$ and that in fact the one-form $F(r)(\sigma_3 + \Sigma_3)$ is itself harmonic. It turns out that this one-form has an interesting geometrical origin. As mentioned earlier the metric (2.6)(2.7) has a $U(1)$ isometry whose Killing vector is given by $\partial_\psi + \partial_\tilde{\psi}$. It is a well known fact that given a $U(1)$ Killing vector $K = K^M \partial_M$ on a Ricci flat manifold, the dual one-form $K = K_M dx^M$ with

$$K_M = g_{MN}K^N, \quad (6.2)$$

where $g_{MN}$ is the Ricci flat metric is harmonic. Therefore, the candidate two-form is given by

$$\eta = d(C^2(\sigma_3 + \Sigma_3)). \quad (6.3)$$

In order for this form to give rise to a $U(1)$ gauge field in four dimensions it has to be $L^2$-normalizable. Unfortunately, its norm is badly divergent

$$||\eta|| = \int_{X_7} \eta \wedge *\eta \sim \Lambda^2, \quad (6.4)$$

where $\Lambda$ is an IR regulator. This suggests that the $U(1)$ mode obtained by Kaluza-Klein reduction is a parameter and not a fluctuating dynamical field. It would be nice to reconcile this calculation with the Type IIA expectation.

One can likewise compute the norm of the associative three-form. Reducing the supergravity three-form $C$ along it gives rise to a scalar mode. However, since $\Phi$ is not $L^2$-normalizable

$$||\Phi|| = \int_{X_7} \Phi \wedge *\Phi \sim \Lambda^6, \quad (6.5)$$

the scalar mode is a real parameter. This real parameter combines with a real scalar parameter from the metric into a complex scalar which is part of a four-dimensional $N = 1$ chiral multiplet.
It is possible to find other harmonic two-forms $\eta$ preserving the $SU(2) \times SU(2) \times U(1) \times Z_2$ symmetry by projecting onto irreps of $G_2$. That is, we can demand $\eta \wedge \Phi = 2\eta$ or $\eta \wedge \Phi = -\eta$ to restrict to the 7 or 14 of $G_2$. Then it turns out that $\eta$ is harmonic if it is closed. The result is that there is one more harmonic form in the 7, but it is singular at $r = 9r_0/2$; and there is one regular harmonic form in the 14, but it is highly non-normalizable at infinity. Neither of these two-forms seems to resolve the $U(1)$ puzzle.

7. Summing up Membrane Instantons

Another interesting aspect of the IR dynamics of the four-dimensional effective $\mathcal{N} = 1$ gauge theory is the superpotential generated by instantons. In this section we explain the origin of the effective superpotential directly in M theory on the $G_2$ manifold $X$. Since there are no background fluxes or branes in the M theory compactification on $X$, the effective superpotential $W$ in M theory is generated only by instantons corresponding to Euclidean membranes wrapped on supersymmetric 3-cycles in $X$. Note, there are no five-brane instantons since $G_2$-holonomy manifolds in general do not have supersymmetric 6-cycles. In fact, in our model\footnote{The 7-manifold $X$ of $G_2$ holonomy constructed in this paper has $b_3 = 1$. Therefore, it has one modulus, which could be interpreted as a scalar component of a chiral multiplet in the four-dimensional $\mathcal{N} = 1$ effective field theory \cite{1}. However, as we explained in the previous section, this field is non-dynamical since the corresponding harmonic form is not $L^2$-normalizable. In this sense, in our model $W$ is a function of the coupling constant. However, following the notations of \cite{5,24}, here we refer to the function $W$ as a superpotential, bearing in mind applications to more general models.} we have $H_6(X) = 0$.

The problem of counting the contributions of multiple covers of membrane instantons usually prevents one from doing the calculation beyond the one-instanton approximation \cite{25}. Here, summing up the entire instanton series for our model, we demonstrate how geometric dualities conjectured by Vafa \cite{3} open an avenue for such calculations, reducing the problem to counting world-sheet instantons in type IIA string theory. This calculation provides a further evidence for the membrane multiple cover formula proposed by Ooguri and Vafa \cite{26}, at least in the case of rational homology spheres:

$$c_n = \frac{1}{n^2} \quad (7.1)$$
Depending on the orientation of the supersymmetric 3-cycle with respect to the $U(1)$ fiber ("M theory circle") each membrane instanton can become in type IIA:

\( i \) an open string world-sheet instanton;

\( ii \) a D2-brane instanton;

\( iii \) a closed string world-sheet instanton.

The first option is realized when the 3-cycle can be represented as a $U(1)$-bundle over a disk, so that the size of the fiber goes to zero at the boundary of the disk. This possibility was explained in a recent paper [24]. For the effects of open string instantons see [27, 28, 24]. The second option, when a membrane instanton reduces to a D2-brane instanton is trivial, and occurs when the $U(1)$ is "orthogonal" to the 3-cycle. The last option occurs when the 3-cycle can be represented as a non-trivial Hopf fibration of $U(1)$ over $S^2$. In this case, a membrane instanton is reduced to the genus zero closed string world-sheet instanton. Notice, that in all cases, a particular membrane instanton is reduced to the corresponding instanton in type IIA theory. This property will allow us to make an identification of their contributions to the superpotential instanton-by-instanton.

The contribution of a single membrane instanton to the non-perturbative superpotential was analyzed by Harvey and Moore [25]:

\[
\Delta W \sim |H_1(\Sigma, Z)| \exp \left( - \int_{\Sigma} (\Phi + iC) \right),
\]

where, $\Sigma$ denotes a supersymmetric 3-cycle. Our space $X$ has only one compact supersymmetric 3-cycle $\Sigma \equiv S^3$ calibrated with respect to the three-form (2.4), so the only possible membrane instantons are those, which wrap $\Sigma \subset X$. Following [3], we may consider a non-singular quotient of our metric by the group $Z_N$ which acts freely on $S^3$. Hence, in this case $\Sigma = S^3/Z_N$ and $H_1(\Sigma, Z) = Z_N$. If we denote the complexified volume of $\Sigma$ by $z = \int_{\Sigma} (\Phi + iC)$, we can write the sum over multiple covers of $\Sigma$ in the form:

\[
W(z) = N \sum_{n=1}^{\infty} c_n \exp(-nz)
\]

where $c_n$ are numerical coefficients that can be obtained using the dual type IIA descriptions. One of the dual descriptions involves type IIA string theory on the resolved conifold geometry $Y_6$ given by an $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{P}^1$ with $N$ units of Ramond-Ramond two-form flux $F$ through the basic 2-cycle [33]. In this picture the effective $\mathcal{N} = 1$
four-dimensional theory has a tree-level superpotential generated by the Ramond-Ramond two-form flux \[29,30\]:

\[
W = \int_{Y_6} F \wedge \mathcal{K} \wedge \mathcal{K}
\]

where the two-form \( \mathcal{K} \) is obtained by integrating the associative three-form \( \Phi \) over the \( S^1 \) fiber as outlined in \[3\]. The two-form \( \mathcal{K} \) agrees with the Kähler form on the resolved conifold before we turn on Ramond-Ramond flux. This expression for the superpotential, which is valid only in the large volume limit, is corrected by world-sheet instantons. The resulting superpotential can be written in terms of genus zero closed topological string amplitude \( F_0(t) \) \[5\]:

\[
W(t) = \int_{Y_6} F \wedge \mathcal{K} \wedge \mathcal{K} + \text{instantons} = N \frac{\partial F_0(t)}{\partial t} = N \sum_{n=1}^{\infty} \frac{e^{-nt}}{n^2} ,
\]

where \( t = \int_{P_1} \mathcal{K} \) denotes the size of the basic two-cycle.

The same expression for the superpotential (modulo an ambiguous polynomial piece \[1,21\]) can be obtained from the disk instanton calculation in another type IIA dual description including D6-branes \[27,28,24\]. The parameters in the Type IIA compactification on \( Y_6 \) and in the M theory compactification on \( X \), in the large volume limit (i.e. when \( z \) is large) should be identified as \( z \approx t \) \[3\]. If we now compare the contribution of every single instanton in \( (7.5) \) and in \( (7.3) \), we find that the two expressions agree, provided the coefficients \( c_n \) are given by the formula \( (7.1) \) for multiple membrane wrapping in M theory, as proposed in \[26\]. Hence, when \( \Sigma \) is a rational homology 3-sphere it is natural to conjecture the following general formula for the effective superpotential that includes the contributions of multiple covers:

\[
W = \sum \frac{|H_1(\Sigma, Z)|}{n^2} \exp \left( -n \int_{\Sigma} (\Phi + iC) \right)
\]

This should be compared with the coefficients \( c_n = n^{-3} \) in the genus zero topological string partition function \( F_0(t) \) that account for multiple covers by fundamental strings \[32,33\]. It is amusing to check formula \( (7.6) \) in more general compactifications of M theory on \( G_2 \) manifolds. For example, a similar analysis\[4\] for quotients by dihedral groups suggests

\[11\] We thank C. Vafa for extensive discussions on this point.
that one has to sum over all membrane topologies in (7.6) in order to reproduce the factor 
\( N \pm 4 \) in the corresponding Type IIA calculation (7.5). In fact, the 4 in this expression 
comes as before from the formula (7.2) since \( H_1(\Sigma, \mathbb{Z}) \) is an abelian group of order 4 for 
\( \Sigma \cong S^3/D_N \). The leading contribution \( N \) comes from bound states of two basic membrane 
instantons, which transform into each other under the action of one of the generators of the 
dihedral group \( D_N \). These membrane instantons become genus zero world-sheet instantons 
after reduction to Type IIA string theory [6]. It would be very interesting to extend this 
analysis to more general models.

8. A More General Ansatz

In view of the numerous applications \( G_2 \) holonomy metrics and the scarcity of ex-

cplicitly known metrics it is important to search for more examples. Our ansatz (2.6)(2.7) 
allows for a straightforward generalization by introducing six independent functions which 
breaks the symmetry from \( SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2 \) down to \( SU(2) \times SU(2) \times \mathbb{Z}_2 \). The 
metric is given by

\[
    ds^2 = \sum_{a=1}^{7} e^a \otimes e^a, 
\]

in terms of the following vielbeins

\[
    e^1 = A_1(r)(\sigma_1 - \Sigma_1), \quad e^2 = A_2(r)(\sigma_2 - \Sigma_2), \\
    e^3 = A_3(r)(\sigma_3 - \Sigma_3), \quad e^4 = B_1(r)(\sigma_1 + \Sigma_1), \\
    e^5 = B_2(r)(\sigma_2 + \Sigma_2), \quad e^6 = B_3(r)(\sigma_3 + \Sigma_3), \\
    e^7 = dr. 
\]

One can construct the associative three-form \( \Phi \) as in section 2. Imposing closure and co-
closure of the $G_2$-structure yields the following system of first order differential equations\(^\text{(8.3)}\):

\[
\begin{align*}
\frac{dA_1}{dr} &= -\frac{1}{4} \left[ \frac{A_1^2 - A_3^2 - B_2^2}{A_3 B_2} + \frac{A_1^2 - A_2^2 - B_3^2}{A_2 B_3} \right] \\
\frac{dA_2}{dr} &= -\frac{1}{4} \left[ \frac{A_2^2 - A_3^2 - B_1^2}{A_3 B_1} + \frac{A_2^2 - A_1^2 - B_3^2}{A_1 B_3} \right] \\
\frac{dA_3}{dr} &= -\frac{1}{4} \left[ \frac{A_3^2 - A_2^2 - B_1^2}{A_2 B_1} + \frac{A_3^2 - A_1^2 - B_2^2}{A_1 B_2} \right] \\
\frac{dB_1}{dr} &= \frac{1}{4} \left[ \frac{A_2^2 + A_3^2 - B_1^2}{A_2 A_3} + \frac{B_1^2 - B_2^2 - B_3^2}{B_2 B_3} \right] \\
\frac{dB_2}{dr} &= \frac{1}{4} \left[ \frac{A_1^2 + A_3^2 - B_2^2}{A_1 A_3} + \frac{B_2^2 - B_1^2 - B_3^2}{B_1 B_3} \right] \\
\frac{dB_3}{dr} &= \frac{1}{4} \left[ \frac{A_1^2 + A_2^2 - B_3^2}{A_1 A_2} + \frac{B_3^2 - B_1^2 - B_2^2}{B_1 B_2} \right]
\end{align*}
\]

We note that these equations can also be found by the effective Lagrangian method we used in section 4.

Unfortunately, we were not able to find closed solutions to this complicated set of differential equations. It is worthwhile however to end this section with a speculation on the solutions of (8.3). At several places we mentioned the similarity of our solution with the Taub-NUT metric which described the M theory lift of D6 branes. Taub-NUT itself has an $SU(2) \times U(1)$ symmetry, but there exists another four-dimensional self-dual metric without the $U(1)$ symmetry which has an interesting interpretation in M theory. It is the Atiyah-Hitchin metric \(^{[34]}\) which describes the uplift of an $O6^-$ orientifold plane in Type IIA \(^{[35],[36]}\). Hence, it is tempting to conjecture that the solutions of (8.3), for certain values of the parameters, describe an $O6^-$ plane — possibly in the presence of additional D6 branes — wrapped on $S^3$ of the deformed conifold. Such backgrounds would be the supergravity duals of $\mathcal{N} = 1$ supersymmetric Yang-Mills with gauge group $SO(2N)$.

9. Discussion

In this paper we found explicitly a new one-parameter family of Ricci flat metrics, which have $G_2$ holonomy group and also the structure of $U(1)$ bundle over the conifold

\footnote{After completing this work, we were informed about \(^{[22]}\), where these equations and some aspects of their solutions have been discussed. We would like to thank M. Cvetić, G.W. Gibbons, H. Lü and C.N. Pope for sharing their results prior to publication.}
geometry $T^*S^3$. The topology of these metrics is that of the spin bundle over $S^3$. These metrics, unlike the previously known asymptotically conic $G_2$ holonomy metrics, have a circle of finite radius at infinity. The size of the circle has an interpretation in string theory as the size of the Type IIA string coupling constant. This new metric, if used as a vacuum solution of M theory, has a nice interpretation as a Type IIA string theory solution. Once the $G_2$ holonomy metric is reduced to Type IIA along it provides the supergravity description of a collection of D6-branes wrapping the supersymmetric $S^3$ in the deformed conifold geometry. As such it plays a rôle as a supergravity dual to supersymmetric Yang-Mills theory.

The metric we have constructed pertains to the duality conjectured by Vafa [3] and recently geometrized in [2,4,4]. This duality states that Type IIA string theory with wrapped D6-branes on the deformed conifold is dual to Type IIA string theory on the resolved conifold with flux. In fact there are two sides of the duality with flux related by the familiar flop transition. Since these Type IIA backgrounds involve only the metric, dilaton and Ramond-Ramond one-form gauge field they have a purely geometric description in M theory. Supersymmetry dictates that the M theory geometry is that of a manifold with $G_2$ holonomy. As shown in [3,4] these dual theories are indeed connected in M theory since the three $G_2$ holonomy metrics reside on the same moduli space. Moreover, for arbitrary value of the $\theta$-angle of the supergravity three-form one can in M theory smoothly interpolate between the three geometries without a phase transition. According to [24,21], each of these three phases has a further discrete choice of behavior of the $G_2$ metric at large distances related to the so-called framing ambiguity. It implies the existence of infinitely many $G_2$ metrics of the same topology, which have the same behavior in the interior, but differ in the choice of the $U(1)$ fiber at large distance. In this context, the new metric constructed here should correspond to the canonical framing, $p = 0$ in the notations of [31].

In this paper we have found one of these metrics, namely, the one describing the deformed conifold geometry with branes. Needles to say, it would be very interesting to find the $G_2$ holonomy metrics which upon reduction to Type IIA describe the resolved conifold with flux and/or corresponding to the different choice of framing. Finding these and analogous metrics will hopefully teach us – among other things – about compactifications

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13 We thank C. Vafa for pointing this out.
on $G_2$ holonomy manifolds and improve our understanding of the type of dualities suggested by Vafa [5] between branes and fluxes.

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Appendix I. The Vielbein Algebra

In this appendix we present the basic vielbein algebra required to perform the computations in this paper. The metric ansatz we want to consider is

\[ ds^2 = \sum_{a=1}^{7} e^a \otimes e^a, \tag{I.1} \]

with the following vielbeins

\[ e^1 = A(r)(\sigma_1 - \Sigma_1), \quad e^2 = A(r)(\sigma_2 - \Sigma_2), \]
\[ e^3 = D(r)(\sigma_3 - \Sigma_3), \quad e^4 = B(r)(\sigma_1 + \Sigma_1), \]
\[ e^5 = B(r)(\sigma_2 + \Sigma_2), \quad e^6 = C(r)(\sigma_3 + \Sigma_3), \]
\[ e^7 = dr/C(r). \tag{I.2} \]

The exterior calculus of these vielbeins is

\[ de^1 = C'A' \frac{e^7}{A} \wedge e^1 + A \left( \frac{e^3 \wedge e^5}{BD} + \frac{e^6 \wedge e^2}{AC} \right), \]
\[ de^2 = C'A' \frac{e^7}{A} \wedge e^2 + A \left( \frac{e^4 \wedge e^3}{BD} + \frac{e^1 \wedge e^6}{AC} \right), \]
\[ de^3 = CD' \frac{e^7}{D} \wedge e^3 + \frac{D}{2AB} \left( e^2 \wedge e^4 + e^5 \wedge e^1 \right), \]
\[ de^4 = CB' \frac{e^7}{B} \wedge e^4 - \frac{B}{2} \left( \frac{e^2 \wedge e^3}{AD} + \frac{e^5 \wedge e^6}{BC} \right), \]
\[ de^5 = CB' \frac{e^7}{B} \wedge e^5 - \frac{B}{2} \left( \frac{e^3 \wedge e^1}{AD} + \frac{e^6 \wedge e^4}{BC} \right), \]
\[ de^6 = C' e^7 \wedge e^6 - \frac{C}{2} \left( \frac{e^1 \wedge e^2}{A^2} + \frac{e^4 \wedge e^5}{B^2} \right), \]
\[ de^7 = 0 \tag{I.3} \]

where \( F' \equiv \frac{dF(r)}{dr} \).
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