New interpretation for energy gap $\Delta$ of the cut-off approximation in the BCS theory of superconductivity

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This paper concerns the solution of the self-consistency equation for energy gap parameter $\Delta_k$ in the BCS theory of superconductivity. We show that there exists a well-defined relation between the solution for energy gap parameter amplitude $|\Delta_k|$ for a general interaction $V_{k,k'}$ and energy gap $\Delta$ obtained by using the cut-off approximation. The relation between $|\Delta_k|$ and $\Delta$ indicates that $\Delta$ is a weighted average over $|\Delta_k|$ of electronic states within cut-off energy $\xi_c$ around the Fermi surface. In this interpretation for $\Delta$, $\xi_c$ is not a property of $V_{k,k'}$, but a parameter specifying the energy range within which the weighted average over $|\Delta_k|$ is taken. We show that the proper choice for the value of $\xi_c$ is only a few $k_BT_c$ (i.e., $\xi_c/k_BT_c$ is about 3 or 4). We also show that the cut-off approximation, even with $\xi_c/k_BT_c = \infty$, is a good approximation when it is used to calculate quantities such as the condensation energy and the specific heat, but it leads to significant overestimation for the Josephson critical current density of a Josephson junction if $\xi_c/k_BT_c \gg 1$ is assumed.

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In the BCS theory of superconductivity, the superconducting state is characterized by the existence of energy gap parameter $\Delta_k$ in quasi-particle excitation energy $E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$ (where $\xi_k$ is the normal state electronic energy, measured relative to the Fermi level). Energy gap parameter $\Delta_k$ is determined self-consistently via the equation

$$\Delta_k = - \sum_{k'} V_{k,k'} \frac{\tanh(E_{k'}/2k_BT)}{2E_{k'}} \Delta_{k'},$$

(1)

where $V_{k,k'}$ is the pairing interaction matrix element.

In principle, once $\Delta_k$ is determined, thermodynamic quantities in the superconducting state can be quantitatively calculated as functions of temperature $T$ by starting with the diagonalized Hamiltonian

$$\hat{H} = \sum_k \left[ U_k + E_k \left( \gamma_{k+}^\dagger \gamma_{k+} + \gamma_{k-}^\dagger \gamma_{k-} \right) \right],$$

(2)

where $U_k = \xi_k - E_k + |\Delta_k|^2 \tanh(E_{k'}/2k_BT)/2E_k$, and $\gamma_{k+}$ and $\gamma_{k-}$ are the Fermi operators for quasi-particles in the superconducting state.

In general, $\Delta_k$ is a complex quantity, i.e., $\Delta_k = |\Delta_k|e^{i\theta_k}$, and both amplitude $|\Delta_k|$ and phase $\theta_k$ can be wave vector $k$ dependent. The cut-off approximation in which $V_{k,k'}$ is approximated by

$$V_{k,k'} = \begin{cases} -V & \text{if } |\xi_k|, |\xi_{k'}| < \xi_c \\ 0 & \text{otherwise} \end{cases}$$

(3)

and $\theta_k = \theta$ is assumed to be a constant (which can be arbitrary), suppresses the $k$-dependence of $\Delta_k$ so that

$$|\Delta_k| = \begin{cases} \Delta & \text{if } |\xi_k| < \xi_c \\ 0 & \text{otherwise} \end{cases}$$

(4)

and Eq. (1) becomes

$$1 = V \sum_{|\xi_k|<\xi_c} \frac{\tanh \left( \sqrt{\xi_k^2 + \Delta^2}/2k_BT \right)}{2\sqrt{\xi_k^2 + \Delta^2}}.$$  

(5)

Cut-off energy $\xi_c$ was thought to be of the same order as Debye energy $\hbar\omega_D$, i.e., $\xi_c/k_BT_c \simeq \hbar\omega_D/k_BT_c \gg 1$ and $\xi_c/k_BT_c = \infty$ was often assumed in practical calculation of various quantities.

Major quantitative results of the BCS theory were first derived by using the cut-off approximation. Despite the fact that the approximation is oversimplified, the quantitative results have shown, in general, good agreement with experiments on a variety of (conventional) superconductors. However, there are also noteworthy discrepancies. An example is that the predicted value for the magnitude of the Josephson critical current density of a Josephson junction is much too large compared to what experimentally observed even though the prediction for the temperature dependence of the normalized Josephson critical current density has been found to be in excellent agreement with experiments.

We have derived in Ref. 11 a solution for energy gap parameter amplitude $|\Delta_k|$ for a general interaction $V_{k,k'}$. The solution for $|\Delta_k|$ shows that reduced energy gap parameter amplitude $|\Delta_k|/k_BT_c$ is a function only of reduced variables $|\xi_k|/k_BT_c$ and $T/T_c$, which contains no explicit $V_{k,k'}$-dependence. The solution also shows that $|\Delta_k|$ is appreciable only for energies within a few $k_BT_c$ around the Fermi level. This latter feature of $|\Delta_k|$ is very different from what one would expect from the cut-off approximation if $\xi_c/k_BT_c \gg 1$ is assumed. Despite this difference, as we have shown in Ref. 12, the results for thermodynamic critical magnetic field $H_c(T)$, specific heat $C(T)$ and normalized Josephson critical current density $I_c(T)/I_c(0)$, obtained by using the solution of Ref. 11 for $|\Delta_k|$, are not much different from those obtained by using the cut-off approximation (with $\xi_c/k_BT_c = \infty$). However, there is one significant difference: the value of $I_c(0)$, obtained by using the solution of Ref. 11 for $|\Delta_k|$, is only a third of that obtained by using the cut-off approximation (with $\xi_c/k_BT_c = \infty$). The reason behind these is further analyzed and made clear in this paper.
In the following, we show that there exists a well-defined relation between the solution for $|\Delta_k|$ obtained in Ref. 11 for a general interaction $V_{k,k'}$ and energy gap $\Delta$ of the cut-off approximation. The relation between $|\Delta_k|$ and $\Delta$ indicates that $\Delta$ is a weighted average over $|\Delta_k|$ of electronic states within $\xi_c$ around the Fermi surface. In this interpretation for $\Delta$, cut-off energy $\xi_c$ is not a property of the interaction, but a parameter specifying the energy range within which the weighted average over $|\Delta_k|$ is taken. We show that the proper choice for the value of $\xi_c$ is only a few $k_BT_c$ (i.e., $\xi_c/k_BT_c$ is about 3 or 4). We also show that the cut-off approximation, even with $\xi_c/k_BT_c = \infty$, is a good approximation when it is used to calculate quantities such as condensation energy $H_c^2(T)/8\pi$, specific heat $C(T)$ and normalized Josephson critical current density $I_c(T)/I_c(0)$, but it leads to significant overestimation for the magnitude of the Josephson critical current density if $\xi_c/k_BT_c \gg 1$ is assumed.

As we have shown in Refs. 11 and 12, the following critical temperature constraint, which, as we will see in the following, allows unique determination of $|\Delta(T)|$.

Note that interaction $V_{k,k'}$ and phase $\theta_k$ do not appear explicitly in Eq. (6). Instead, critical temperature $T_c$ is involved through the condition that $|\Delta_k| = 0$ at $T = T_c$. This indicates that $|\Delta_k|$ depends on $V_{k,k'}$ only implicitly via $T_c$.

To see how critical temperature $T_c$ depends on interaction $V_{k,k'}$, we turn to Eq. (1). In the limit of $T \rightarrow T_c$, we have $|\Delta_k| \rightarrow 0$ so that Eq. (1) can be linearized, and we have an eigenvalue problem:

$$\Delta_k = -\sum_{k'} V_{k,k'} \frac{\tanh(|\xi_k|/2k_BT_c)}{2|\xi_k|} \Delta_{k'}.$$  \hspace{1cm} (7)

In principle, critical temperature $T_c$ and phase $\theta_k$ are determined by solving this eigenvalue problem for given interaction $V_{k,k'}$ and electronic energy spectrum $\xi_k$.

We turn back to Eq. (6) to consider a solution for $|\Delta_k|$ can be obtained. Clearly,

$$\frac{\tanh\left(\sqrt{\xi_k^2 + |\Delta_k|^2}/2k_BT\right)}{\sqrt{\xi_k^2 + |\Delta_k|^2}} = \frac{\tanh(|\xi_k|/2k_BT_c)}{|\xi_k|}$$  \hspace{1cm} (8)

is a solution of Eq. (6). This equation, which was previously obtained in Ref. 11, is an implicit solution for $|\Delta_k|$ as a function of $|\xi_k|$ and $T$ for given $T_c$, and satisfies the condition that $|\Delta_k| = 0$ at $T = T_c$.

However, Eq. (8) is not the only possible solution of Eq. (6). Actually, as one can see, Eq. (8) can have infinite number of solutions. For example, the solution of the form of Eq. (1) in the case of the cut-off approximation is also a solution of Eq. (6). This can be seen by substituting Eq. (4) into Eq. (6) to obtain

$$\sum_{|\xi_k| < \xi_c} \frac{d}{dT} \left[ \frac{\tanh\left(\sqrt{\xi_k^2 + |\Delta|^2}/2k_BT\right)}{\sqrt{\xi_k^2 + |\Delta|^2}} \right] = 0,$$  \hspace{1cm} (9)

and noticing that this equation can also be obtained from Eq. (5) by operating $d/dT$ on it. Similarly, a solution of the form $\Delta_k = \Delta\omega_k$, as in the case of a separable interaction $V_{k,k'} = -V\omega_k\omega_{k'}$, is also a solution of Eq. (6). We therefore need an additional constraint so that $|\Delta_k(T)|$ can be uniquely determined.

Note that the solution for $\xi_k$ and $S_k$ derived from diagonalized Hamiltonian $\hat{H}$ of Eq. (2) is $T$-dependent because of its dependence on $|\Delta_k(T)|$. This implies the existence of an additional self-consistency constraint, which, as we will see in the following, allows unique determination of $|\Delta_k(T)|$.

As we discussed in Ref. 11, since diagonalized Hamiltonian $\hat{H}$ of Eq. (2) is $T$-dependent, there should be no coupling (except pair correlation) between the quasi-particle excitations. Therefore, we expect the thermal energy and the entropy associated with each pair of $(k\uparrow, -k\downarrow)$ excitations to be

$$\varepsilon_k = U_k + 2f_kE_k$$  \hspace{1cm} (10)

and

$$S_k = -2k_B \left[ f_k \ln f_k + (1 - f_k) \ln(1 - f_k) \right],$$  \hspace{1cm} (11)

respectively where $f_k = (e^{E_k/k_BT} + 1)^{-1}$ is the Fermi function. However, as compared to these standard expressions for $\varepsilon_k$ and $S_k$, those derived from diagonalized Hamiltonian $\hat{H}$ of Eq. (2) contain additional terms involving $dU_k/dT$, $dE_k/dT$ and $df_k/dT$. Letting the sum of the additional terms in each expression to be zero, one gets a first order differential equation for $|\Delta_k(T)|$, of which the solution satisfying the condition $|\Delta_k(T)| = 0$ is Eq. (8). Namely, the solution for $|\Delta_k(T)|$ given by Eq. (8) is the only solution that both satisfies the original self-consistency equation [Eq. (11)] and ensures that Eqs. (10) and (11) hold. [A complete solution for $\Delta_k$ is therefore a combination of the solutions of Eq. (7) for $T_c$ and $\theta_k$ and the solution of Eq. (8) for $|\Delta_k|$].


\[
\sum_{|\xi_k|<\xi_c} \left[ \frac{\tanh\left( \sqrt{\xi_k^2 + \Delta^2}/2k_BT \right)}{\sqrt{\xi_k^2 + \Delta^2}} - \frac{\tanh(|\xi_k|/2k_BT_c)}{|\xi_k|} \right] = 0. \tag{12}
\]

By substituting Eq. (8) into the above equation, we obtain

\[
\sum_{|\xi_k|<\xi_c} \left[ \frac{\tanh\left( \sqrt{\xi_k^2 + \Delta^2}/2k_BT \right)}{\sqrt{\xi_k^2 + \Delta^2}} - \frac{\tanh(|\xi_k|/\Delta_{k,T_c})}{|\xi_k|} \right] = 0. \tag{13}
\]

This equation defines \( \Delta \) as a weighted average over \( |\Delta_k| \) of electronic states with \( |\xi_k|<\xi_c \); we therefore can so interpret \( \Delta \). In this interpretation for \( \Delta, \xi_c \) is not a property of the interaction, but a parameter specifying the energy range within which the weighted average over \( |\Delta_k| \) is taken.

We next examine how \( \Delta \) depends on \( \xi_c \). From Eq. (9), by making the usual substitution \( \sum_{|\xi_k|<\xi_c} \rightarrow N(0) \int_0^{\xi_c} d\xi \) [where \( N(0) \) is the density of states at the Fermi level] and a rearrangement, we obtain the following expression:

\[
\frac{d(\Delta^2)}{dT} = \frac{\int_0^{\xi_c} d\xi \left[ \frac{\tanh\left( \sqrt{\xi^2 + \Delta^2}/2k_BT \right)}{\sqrt{\xi^2 + \Delta^2}} \right]}{\int_0^{\xi_c} d\xi \left[ \frac{\tanh\left( \sqrt{\xi^2 + \Delta^2}/2k_BT \right)}{(\sqrt{\xi^2 + \Delta^2})^{3/2} - \tanh^2\left( \sqrt{\xi^2 + \Delta^2}/2k_BT \right)} \right]} / T^2, \tag{14}
\]

where energies are measured in units of \( k_BT_c \). With this expression for \( d(\Delta^2)/dT \) and the initial value \( \Delta^2 = 0 \) at \( T = 1 \) (temperature \( T \) is measured in unit of \( T_c \)), we can numerically calculate \( \Delta(T) \) for arbitrary \( \xi_c \) by using the Runge-Kutta method. The integrals involved in the expression for \( d(\Delta^2)/dT \) are calculated by using the Simpson method.

We show in Fig. 1 the energy dependence of the gap parameter amplitude at \( T = 0 \). The results for \( \Delta(0) \) obtained for \( \xi_c/k_BT_c = 1, 2, 3.37, 5, 10 \) and \( \infty \) are shown as plots (a), (b), (c), (d), (e) and (f), respectively. (Here the number 3.37 is special, as we will see later.) Plot (g) shows \( |\Delta_k(0)| \) versus \( |\xi_k| \).

Also note that if a different cut-off range is assumed so that \( |\Delta_k| = \Delta \) for \( \xi_c<|\xi_k|<\xi_c2 \) and zero otherwise, then \( \Delta \) is simply a weighted average over \( |\Delta_k| \) of electronic states with \( \xi_k \) in the range \( \xi_c < |\xi_k| < \xi_c2 \). As examples, the results for \( (\xi_c1/k_BT_c, \xi_c2/k_BT_c) = (6, 7), \) (7, 8) and (8, 9) are shown as plots (h), (i) and (j), respectively, in Fig. 1.

Comparing plots (a)-(f) with plot (g), it is evident that \( \Delta \) is indeed a weighted average over \( |\Delta_k| \) of energies within a specific range. This relation between \( \Delta \) and \( |\Delta_k| \) is mathematically expressed by Eq. (13). We can also see from Fig. 1 that electronic states with lower \( |\xi_k| \) contribute with larger weights to the average. The fact that the difference between the values of \( \Delta(0) \) for \( \xi_c/k_BT_c = 10 \) [plot (e)] and \( \xi_c/k_BT_c = \infty \) [plot (f)] is less than 1\% indicates that contributions from electronic states with \( |\xi_k|/k_BT_c \gg 1 \) are negligibly small.

The minimum single quasi-particle excitation energy is \( E_{k_{\text{min}}} = |\Delta_{k_F}| \) (where \( k_F \) is a Fermi wave vector). We have \( |\Delta_{k_F}(0)|/k_BT_c = 2 \). The result \( \Delta(0)/k_BT_c = 1.764 \) for \( \xi_c/k_BT_c = \infty \) [plot (f) in Fig. 1] is about 88\% of \( |\Delta_{k_F}(0)|/k_BT_c \).

In Fig. 2, we compare the \( T \)-dependence of normalized energy gap \( |\Delta_{k_F}(T)|/|\Delta_{k_F}(0)| \) with that of \( \Delta(T)/\Delta(0) \). Four practically indistinguishable curves are shown in Fig. 2: \( |\Delta_{k_F}(T)|/|\Delta_{k_F}(0)| \) and \( \Delta(T)/\Delta(0) \) for \( \xi_c/k_BT_c = 1, 100 \) and \( \infty \) of the cut-off approximation \( \Delta(T)/\Delta(0) \) for \( \xi_c/k_BT_c = \infty \) was previously calculated by MÜHLSCHLEGEL. As shown in Fig. 2, \( \Delta(T)/\Delta(0) \) is practically \( \xi_c \)-independent, and practically the same as \( |\Delta_{k_F}(T)|/|\Delta_{k_F}(0)| \).

We next examine how other quantities such as thermodynamic critical magnetic field \( H_c(T) \), specific heat \( C(T) \) and Josephson critical current density \( I_c(T) \) depend on

\[
FIG. 1: Energy dependence of the energy gap parameter amplitude at \( T = 0 \). Plots (a)-(f): \( \Delta(0)/k_BT_c = 1.975, 1.922, 1.859, 1.786 \) and 1.764 for \( \xi_c/k_BT_c = 1, 2, 3.37, 5, 10 \) and \( \infty \), respectively. Plot (g): \( |\Delta_k(0)| \) versus \( |\xi_k| \).
\]
FIG. 2: Temperature dependence of the normalized energy gap. Four practically indistinguishable curves are plotted in this figure: $|\Delta_{kF}(t)|/|\Delta_{kF}(0)|$; and $\Delta(t)/\Delta(0)$ for $\xi_c/k_B T_c = 1, 100$ and $\infty$.

FIG. 3: Zero-temperature condensation energy $H^2_e(0)/8\pi$ versus cut-off energy $\xi_c$. The maximum $H^2_e(0)/8\pi = 1.613$ [in units of $N(0)(k_B T_c)^2$] is located at $\xi_c/k_B T_c = 3.37$; $H^2_e(0)/8\pi = 1.556$ for $\xi_c/k_B T_c = \infty$. For comparison, the results calculated by using $|\Delta_k|$ is $H^2_e(0)/8\pi = 1.645$.

FIG. 4: Deviation function $D_H = H_e(T)/H_e(0) - [1 - (T/T_c)^2]$ versus $(T/T_c)^2$, calculated by using $|\Delta_k|$ and by using the cut-off approximation with different values of $\xi_c/k_B T_c$ as indicated on the curves.

FIG. 5: Electronic specific heat $C$ versus temperature $T$, calculated by using $|\Delta_k|$ and by using the cut-off approximation with different values of $\xi_c/k_B T_c$ as indicated on the curves. The $C(T_c)/\gamma T_c$ values for the four curves shown in the figure are 2.597, 2.525, 2.482 and 2.318, respectively.

cut-off energy $\xi_c$ when they are calculated by using the cut-off approximation.

Once $\Delta(T)$ is obtained, quantities such as $H_e(T), C(T)$ and $I_c(T)$ can be calculated straightforwardly. In Fig. 3, we show the $\xi_c$-dependence of zero-temperature condensation energy $H^2_e(0)/8\pi$. The $H^2_e(0)/8\pi$-versus-$\xi_c$ curve shows a maximum $H^2_e(0)/8\pi = 1.613$ [in units of $N(0)(k_B T_c)^2$] at $\xi_c/k_B T_c = 3.37$. In the limit of $\xi_c/k_B T_c = \infty$, $H^2_e(0)/8\pi = 1.556$. Note that these two values of $H^2_e(0)/8\pi$ are only about 2% and 5%, respectively, smaller than the results $H^2_e(0)/8\pi = 1.645$ calculated by using $|\Delta_k|$. In Fig. 4, we show the $T$-dependence of thermodynamic critical magnetic field $H_c$. The results are plotted as deviations from the $1 - (T/T_c)^2$ law. For comparison, the result calculated by using $|\Delta_k|$ and the results calculated by using the cut-off approximation for several different values of $\xi_c$ are plotted in the figure as $C/\gamma T_c$ versus $T/T_c$ [where $\gamma = (2\pi^2/3)k_B N(0)$]. At $T = T_c$, we have $C/\gamma T_c = 2.597$ by using $|\Delta_k|$, and $C/\gamma T_c = 2.525, 2.482$ and 2.318 by using the cut-off approximation for $\xi_c/k_B T_c = 3.37, 100$ and $2$, respectively. The result $C(T_c)/\gamma T_c = 2.426$ for $\xi_c/k_B T_c = \infty$ was previously obtained by Mühlschlegel.

From Figs. 3-5, we can see that, for calculating $H_e$ and $C$, the cut-off approximation gives results that are not much different from those obtained by using $|\Delta_k|$. The approximation is optimized when $\xi_c/k_B T_c = 3.37$, suggesting the proper choice for $\xi_c$ is about a few $k_B T_c$. We also note that the results for $H_c$ and $C$ show only weak $\xi_c$-dependence when $\xi_c$ is a few $k_B T_c$ or larger, so that the cut-off approximation remains a good approximation even with $\xi_c/k_B T_c \gg 1$. The reason for the weak $\xi_c$-dependence of $H_c$ and $C$ is that, as one can see from the expressions for $H_c$ and $C$, the relevant quantity for $H_c$ and $C$ is quasi-particle excitation en-
FIG. 6: The $\xi_c$-dependence of zero-temperature Josephson critical current density $I_c(0)$ of a symmetric Superconductor-Insulator-Superconductor junction. As indicated in the figure, $I_c(0) = 0.834$ and $2.771$ (in units of $k_B T_c/R_n e$) for $\xi_c/k_B T_c = 3.37$ and $\infty$, respectively. The result obtained by using $|\Delta_k|$ is $I_c(0) = 1.103$.

FIG. 7: Comparison between the $T$-dependence of normalized Josephson critical current density $I_c(T)/I_c(0)$ of a symmetric Superconductor-Insulator-Superconductor junction calculated by using $|\Delta_k|$ and those by using the cut-off approximation with different values of $\xi_c/k_B T_c$ as indicated in the figure.

energy $E_k = \sqrt{\xi_k^2 + \Delta^2}$ (or $E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$), which becomes $E_k \simeq |\xi_k| = \text{independent of } \Delta$ (or $|\Delta_k|$) for $|\xi_k| \gg k_B T_c$, so that only electronic states within a few $k_B T_c$ around the Fermi surface contribute significantly to the difference between the superconducting and normal states. The situation is different in the case of Josephson critical current density $I_c$ of a Josephson junction, because, as one can see from the expression for $I_c$, the relevant quantity for $I_c$ is energy gap parameter $\Delta_k$ itself. This leads to a strong $\xi_c$-dependence for $I_c$ when it is calculated by using the cut-off approximation, as we will see next.

In Fig. 6, we show the $\xi_c$-dependence of zero-temperature Josephson critical current density $I_c(0)$ of a symmetric Superconductor-Insulator-Superconductor junction. As shown in the figure, $I_c(0)$ is a monotonically increasing function of $\xi_c$ that does not saturate until $\xi_c/k_B T_c \approx 10^3$. As indicated in the figure, we have $I_c(0) = 0.834$ and $2.771$ (in units of $k_B T_c/R_n e$) for $\xi_c/k_B T_c = 3.37$ and $\infty$, respectively (the result of $I_c$ for $\xi_c/k_B T_c = \infty$ was previously obtained by Ambegaokar and Baratoff [2]). For comparison, the result $I_c(0) = 1.103$ calculated by using $|\Delta_k|$ is also indicated in the figure, which is only about 40% of the result calculated by using the cut-off approximation with $\xi_c/k_B T_c = \infty$. It is evident from the results shown in Fig. 6 that the cut-off approximation significantly overestimates $I_c$ if $\xi_c/k_B T_c \gg 1$ is assumed.

In Fig. 7, we show the $T$-dependence of normalized Josephson critical current density $I_c/I_c(0)$ of a symmetric Superconductor-Insulator-Superconductor junction. For comparison, results calculated by using the cut-off approximation for several different values of $\xi_c$ and that by using $|\Delta_k|$ are plotted in the figure. As we can see from the figure, for lower values of $\xi_c/k_B T_c$, the difference between the result obtained by using the cut-off approximation and that by using $|\Delta_k|$ is appreciable. However, the difference becomes much less significant for $\xi_c/k_B T_c \gg 1$.

Experimentally [7,8,9,10,16,17] the $T$-dependence of $I_c/I_c(0)$ obtained by using the cut-off approximation with $\xi_c/k_B T_c = \infty$, which is practically indistinguishable from or very close to that obtained by using $|\Delta_k|$, has been well confirmed, but the magnitude of $I_c$ has always been found to be much too small compared to that predicted by using the cut-off approximation with $\xi_c/k_B T_c = \infty$. Clearly, as we can see from Figs. 6 and 7, the results obtained by using $|\Delta_k|$ can help improve agreement between theory and experiments.

In summary, we have shown in this paper that energy gap $\Delta$ that was originally obtained by using the cut-off approximation can be interpreted as a weighted average over $|\Delta_k|$ of electronic states within $\xi_c$ around the Fermi surface [here $|\Delta_k|$ is the energy gap parameter amplitude for a general interaction $V_{kk'}$, and satisfies Eq. (3)]. In this interpretation for $\Delta$, cut-off energy $\xi_c$ is not a property of the interaction, but a parameter specifying the energy range within which the weighted average over $|\Delta_k|$ is taken. We have shown that the proper choice for the value of $\xi_c$ is only a few $k_B T_c$ (i.e., $\xi_c/k_B T_c$ is about 3 or 4). We have also shown that the cut-off approximation, even with $\xi_c/k_B T_c = \infty$, is a good approximation when it is used to calculate quantities such as condensation energy $H^c_2(T)/8\pi$, specific heat $C(T)$ and normalized Josephson critical current density $I_c(T)/I_c(0)$, but it leads to significant overestimation for the magnitude of the Josephson critical current density if $\xi_c/k_B T_c \gg 1$ is assumed.

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14 See Ref. 12 or references cited therein for details about expressions for $H_c$, $C$ and $I_c$. In calculating these quantities, as usual, the substitution $\sum_k \rightarrow N(0) \int \! d\xi$ is made, and resulting integrals are numerically calculated by using the Simpson method.
15 The value $\xi_c/k_BT_c = 3.37$ is obtained by maximizing condensation energy $H_c^2/8\pi$ at $T = 0$. The optimize value for $\xi_c$ may show some degree of $T$-dependence, but it is not important, as our point here is to show that the proper choice for $\xi_c$ is only a few $k_BT_c$.
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