ON PROPERLY CONVEX REAL-PROJECTIVE MANIFOLDS WITH
GENERALIZED CUSPS

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ABSTRACT. Suppose $E$ is an end of an irreducible, properly convex, real-projective $n$-manifold $M$. If $\pi_1 E$ contains a subgroup of finite index isomorphic to $\mathbb{Z}^{n-1}$, and $E \hookrightarrow M$ is $\pi_1$-injective, then $E$ is a generalized cusp. We list some consequences when all ends are of this type. Under certain hypotheses we prove the holonomy of a properly convex manifold is irreducible.

A generalized cusp is a properly convex, real-projective manifold $C$ that is diffeomorphic to $[0, \infty) \times \partial C$ such that $\partial C$ contains no line segment, and $\pi_1 C$ is virtually nilpotent.

Generalized cusps were introduced in [13] and their theory developed in [4, 5]. It was shown in [4] that if $C$ has compact boundary then the fundamental group is virtually abelian, but see [12] and [14] (5.9) for counter-examples when $\partial C$ is not compact. In the rest of this paper the term generalized cusp will be only be used in the narrow sense that the boundary is compact.

There is a growing literature concerning properly convex manifolds with ends of this type [1, 3, 4, 5, 6, 2, 8, 9, 12, 14, 13, 17, 18].

In applications it is desirable to replace the geometric hypothesis on the boundary of a generalized cusp by an algebraic one. This is done in Theorem (0.1), and is needed for forthcoming work by the authors [13]. Theorem (3.5) lists some consequences when all the ends are generalized cusps, and the fundamental group is relatively hyperbolic.

A properly convex set $\Omega \subset \mathbb{RP}^n$ is reducible if there are disjoint proper subspaces $\mathbb{RP}^a, \mathbb{RP}^b \subsetneq \mathbb{RP}^n$ and every point in $\Omega$ is contained in a line segment in $\Omega$ with one endpoint in $\mathbb{RP}^a \cap \text{cl} \Omega$ and the other in $\mathbb{RP}^b \cap \text{cl} \Omega$. Otherwise $\Omega$ is irreducible. A properly convex manifold $M = \Omega / \Gamma$ is irreducible if $\Omega$ is irreducible. Given $\Omega \subset \mathbb{RP}^n$ recall that $\text{Fr} \Omega = \text{cl}(\Omega) \setminus \text{int}(\Omega)$ and $\partial \Omega = \Omega \cap \text{Fr} \Omega$.

A non-compact submanifold $E \subset M$ is called an end of $M$ if $E$ is the closure of a component of $M \setminus \partial E$, and for all $i \in \mathbb{N}$ there are compact submanifolds $K_i \subset K_{i+1}$ with $\partial E \subset K_i \subset E$, and $E \setminus K_i$ is connected, and $E = \cup K_i$. This is a special case of a more general definition of end that suffices for our applications, and makes various statements simpler. An end, $E$, of a properly convex manifold is called a generalized cusp of $M$ if $E$ deformation retracts to a generalized cusp $C$. A subspace $A \subset B$ is $\pi_1$-injective if, whenever a loop in $A$ is contractible in $B$, then the loop is contractible in $A$.

Theorem 0.1 (irreducible implies generalized cusps). Suppose $M$ is an irreducible properly convex $n$-manifold, and $C$ is an end of $M$. If $C$ is $\pi_1$-injective, and $\pi_1 C$ contains a subgroup of finite index isomorphic to $\mathbb{Z}^{n-1}$, then $C$ is a generalized cusp of $M$.

Recall that a radial flow ([13] p. 1384) is a one-parameter subgroup of $\text{PGL}(n+1, \mathbb{R})$ such that the orbit of every point is a proper subset of a projective line, and there is a point $c \in \mathbb{RP}^n$ called the center of the flow that is in the closure of every flowline. The stationary set is $H \cup \{c\}$, where $H \cong \mathbb{RP}^{n-1}$. A displacing hyperplane ([13] p. 1385) for a radial flow $\Phi$ is a hyperplane $P \neq H$ such that $c \notin P$.

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Sketch proof of (0.1). We may assume \( \pi_1 C \cong \mathbb{Z}^{n-1} \), and the holonomy of \( C \) is a lattice \( \Gamma \) in an abelian upper-triangular subgroup \( T \subset \text{PGL}(n+1, \mathbb{R}) \). Moreover there is a radial flow \( \Phi \) that centralizes \( T \) and fixes a hyperplane \( H \), that is disjoint from \( \Omega \). Furthermore the orbit of a generic point under \( T \oplus \Phi \) is open. The \( T \)-orbit of some point deep inside \( \Omega \) is a convex hypersurface \( S \subset \Omega \). Then \( C \) is a generalized cusp if and only if \( S \) contains no line segment. If \( S \) is strictly convex we are done, otherwise \( S \) contains a maximal flat \( F \). The subgroup \( \text{stab}(F) \subset T \) that preserves \( F \) acts simply transitively on \( F \). Hence \( F \) is an open simplex, and there is a one-parameter subgroup \( L \subset \text{stab}(F) \) whose orbits in \( F \) are line segments. Since \( L \) commutes with \( T \oplus \Phi \) there is an open set of points whose \( L \)-orbits are line segments. This implies \( L \) preserves \( \Omega \), and hence \( \Omega \) is reducible. ∎

A complete proof of Theorem (0.1) is given in Section 1. Section 2 provides the following:

**Theorem 0.2** (irreducible holonomy). Suppose \( \Omega \) is properly convex and \( \Gamma \subset \text{PGL}(\Omega) \) is finitely generated, discrete and torsion free, and contains no non-trivial normal abelian subgroup. Suppose either \( \Omega/\Gamma \) is closed, or there is a subgroup \( G \cong \mathbb{Z}^{n-1} \) and \( |\Gamma : G| = \infty \). Then \( \Gamma \) does not preserve any proper projective subspace.

For example, this applies if \( M \) is the interior of a compact \( n \)-manifold that contains an embedded \( \pi_1 \)-injective torus. Theorems (0.1) and (0.2) imply:

**Theorem 0.3** (Generalized cusps are completely general). Suppose \( M \) is a properly convex \( n \)-manifold, and \( C \) is an end of \( M \) and

- \( \pi_1 C \) contains a subgroup of finite index isomorphic to \( \mathbb{Z}^{n-1} \),
- \( C \) is \( \pi_1 \)-injective,
- \( \pi_1 M \) does not contain a non-trivial normal abelian subgroup,
- \( |\pi_1 M : \pi_1 C| = \infty \),

then \( C \) is a generalized cusp.

In particular this applies if \( M \) is homeomorphic to a complete hyperbolic manifold of finite volume. Theorem (3.5) in Section 3 gives some useful properties of manifolds whose ends are generalized cusps.

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1. **Generalized cusps are completely general**

Let \( W = \mathbb{R}^{n+1} \). If \( \Omega \subset \mathbb{P} W \), then \( \text{PGL}(\Omega) \subset \text{PGL} W \) is the subgroup that preserves \( \Omega \). We write \( \text{SL} W \) for the subgroup of \( \text{GL} W \) with determinant \( \pm 1 \), and \( \text{SL} \Omega \) is the preimage in \( \text{SL} W \) of \( \text{PGL} \Omega \). An element \( A \in \text{SL} \Omega \) is hyperbolic if \( A \) has an eigenvalue \( \lambda \) with \( |\lambda| \neq 1 \), and is strongly hyperbolic if, in addition, \( A \) has unique eigenvalues of largest and smallest modulus, and they are real, and have algebraic multiplicity one. It is elliptic if it is conjugate into \( \text{O}(n+1) \) and is parabolic if it is not hyperbolic and not elliptic. The same terms are applied to \( [A] \in \text{PGL} \Omega \). A subgroup is parabolic if every element is parabolic or trivial.

Suppose \( M^n = \Omega / \Gamma \) is properly convex and \( \Gamma \cong \mathbb{Z}^{n-1} \). We will show that \( \Gamma \) is a lattice in a subgroup \( T \cong \mathbb{R}^{n-1} \) of \( \text{PGL} W \). Moreover the orbit of a generic point under \( T \) is a convex hypersurface \( S \), and \( M \) is a generalized cusp if and only if \( S \) is strictly convex. In the remaining case \( T \) contains a one-parameter subgroup called a linear flow whose orbits are contained in lines. Moreover \( \Omega \) is foliated by orbits and is reducible.

A linear flow is an injective homomorphism \( \Phi : \mathbb{R} \to \text{PGL}(n+1, \mathbb{R}) \) such that the orbit of every point in \( \mathbb{R} \mathbb{P}^n \) is a proper subset of a projective line. If \( \phi : \mathbb{R} \to \mathbb{R} \) is an isomorphism then \( \Phi \circ \phi \) is
called a reparameterization of $\Phi$. There is a homomorphism $\Psi : \mathbb{R} \to \text{GL}(n+1, \mathbb{R})$ with $\Phi = [\Psi]$, and $\Psi_t = \exp(tM)$ for some matrix $M$. The eigenvalues of $M$ correspond to weights for $\Psi$. If $\Psi'$ is another homomorphism and $\Phi = [\Psi']$, then $\Psi'_t = \exp(\lambda t)\psi_t = \exp(t(M + \lambda I))$. This operation is called rescaling. We will abuse notation by also referring to $\Psi$ as a linear flow.

The stationary subset of $\Phi$ is the subset of $\mathbb{R}P^n$ consisting of all points that are fixed by the flow. A linear flow is parabolic if, after reparamerization, there is $\pi \in \text{End}(\mathbb{R}^{n+1})$ with $\pi^2 = 0$ and $\Phi_t[x] = [x + t\pi(x)]$, in which case the stationary subset is $\mathbb{P}(\ker \pi)$. It is hyperbolic if, after reparamerization, there is a direct sum decomposition $\mathbb{R}^n = A \oplus B$ and $\Phi_t[a + b] = [a + \exp(t)b]$, where $a \in A$ and $b \in B$; in which case the stationary subset is $\mathbb{P}(A) \cup \mathbb{P}(B)$.

**Lemma 1.1.** Every linear flow is parabolic or hyperbolic.

**Proof.** Let $\Psi$ be a linear flow. The hypothesis implies every point $0 \neq x \in \mathbb{R}^{n+1}$ is contained in some 2-dimensional subspace $V$ that is preserved by the flow. Moreover $\Psi[V]$ has real weights, otherwise the orbit of $[x]$ is $\mathbb{R}P^1$. Hence all the eigenvalues of $M$ are real. Suppose there are 3 distinct eigenvalues $\alpha$, corresponding to three points $[x_i] \in \mathbb{R}P^n$ each fixed by the flow. Then there is an orbit $\sum \exp(\alpha t)x_i$ that is not contained in any $\mathbb{R}P^1$. If there is only one eigenvalue, by rescaling $\Psi$, we may assume it is 0. If $M^2 \neq 0$ consideration of Jordan normal form contradicts that $\Psi$ is a linear flow. In this case $\Psi$ is parabolic.

This leaves the case there are exactly two eigenvalues $\alpha \neq \beta$. If there is a Jordan block for $\alpha$ of size bigger than 1, then we may assume $\alpha = 0$ by rescaling, and there is a 2-dimensional subspace $V = \langle a, b \rangle$ with $\Psi_t(a + b) = a + t \cdot b$. There is also $c \neq 0$ with $\Psi_t(c) = \exp(\beta t)c$. The orbit of $a + b + c$ is $[(a + tb) + \exp(\beta t)c]$ which is not contained in any $\mathbb{R}P^1$. Thus $M$ is diagonalizable. By rescaling we may assume one eigenvalue $\alpha = 0$ and let $A$ be the eigenspace for $\exp \alpha$ and $B$ the other eigenspace. Then $\mathbb{R}^n = A \oplus B$ and $\Psi_t$ is hyperbolic.

**Lemma 1.2.** Suppose $\Phi \subset \text{PGL}(n, \mathbb{R})$ is a one parameter subgroup and $U \subset \mathbb{R}P^{n-1}$ is a non-empty open set such that the orbit under $\Phi$ of each point in $U$ is a proper subset of a projective line. Then $\Phi$ is a linear flow.

**Proof.** Let $\Phi = [\Psi]$. The set $V$ consists of all triples $(a, b, c)$ with $a, b, c \in \mathbb{R}^n$ that are contained in some 2-dimensional linear subspace of $\mathbb{R}^n$. Then $V$ is defined by the polynomial equations given by setting the determinants of all $3 \times 3$ sub-matrices of $(a : b : c)$ equal to zero. Let $W \subset \mathbb{R}^n$ consist of all $a \in \mathbb{R}^n$ such that the flow line containing $[a]$ is contained in a projective line. Then $W$ equals the set of all $a$ such that $(a, \Psi(a), \Psi^2(a)) \in V$ for all $s, t \in \mathbb{R}$, and is therefore also a real algebraic variety. Since $W$ contains the non-empty open set $U$ it follows that $W = \mathbb{R}^n$ and therefore every orbit of $\Phi$ is a subset of a projective line. It only remains to show that no orbit is an entire projective line. If there is such an orbit then $\Phi$ has a weight that is not real.

If $\Phi$ has a weight that is not real then there are $b_1, b_2 \in \mathbb{R}^n$ and $\gamma, \delta \in \mathbb{R}$ with $\gamma \neq 0$ such that the orbit of $b_1$ is $\Phi_t(b_1) = \exp(\gamma t)(\cos(\delta t)b_1 + \sin(\delta t)b_2)$. Choose $[a] \in U$. For all small $\epsilon$, then $c = a + \epsilon b \in U$. First assume the orbit of $[a]$ limits on two distinct points $[a_1], [a_2] \in \mathbb{R}P^{n-1}$. Then $a_1, a_2$ and $\Phi$ can be chosen so that $\Phi_t(a) = a_1 + \exp(\alpha t)a_2$ with $\alpha \neq 0$. Moreover $\langle a_1, a_2 \rangle \cap \langle b_1, b_2 \rangle = 0$. The four functions $1$, $\exp(\alpha t)$, $\exp(\gamma t)\cos(\delta t)$, $\exp(\gamma t)\sin(\delta t)$ are linearly independent. It follows that the orbit of $c$ contains four linearly independent vectors, which contradicts that the orbit of $[c]$ is contained in a line.

The remaining case is that $\Phi_t(a) = a + td$. The four functions $1$, $t$, $\exp(\gamma t)\cos(\delta t)$, $\exp(\gamma t)\sin(\delta t)$ are linearly independent, which is again a contradiction.

Suppose $\Phi \subset \text{PGL}(n, \mathbb{R})$ is a 1-parameter group. A subgroup $\Gamma \subset \text{PGL}(n, \mathbb{R})$ approaches $\Phi$ at infinity if for every neighborhood $U \subset \text{PGL}(n, \mathbb{R})$ of the identity, and every $s \in \mathbb{R}$ there are $\gamma, \gamma' \in \Gamma$ and $t > s$ and $t' < -s$ such that

$$\gamma \in U \cdot \Phi(t), \quad \gamma' \in U \cdot \Phi(t')$$
In particular, if $\Gamma$ is a lattice in $T \cong \mathbb{R}^n$, and $\Phi \subset T$ then $\Gamma$ approaches $\Phi$ at infinity.

**Lemma 1.3.** Suppose $\Omega$ is properly convex, and $\Gamma \subset \text{PGL}(\Omega)$ approaches a linear flow $\Phi$ at infinity. Then $\Omega$ is preserved by $\Phi$.

**Proof.** Suppose $p \in \Omega$ is not fixed by $\Phi$. Let $\ell = [a, b]$ be the closure of the flowline containing $p$. There is sequence $\gamma_n \in \Gamma$ with $\gamma_n = e_n \circ \Phi(t_n)$ with $e_n \rightarrow 1$ and $t_n \rightarrow \infty$. Then $\Phi(t_n)(p) \rightarrow b$ so $\gamma_n(p) \rightarrow b$. Thus $b \in \text{cl} \Omega$. Similarly $a \in \text{cl} \Omega$ so $(a, b) \subset \Omega$.

**Lemma 1.4.** If a closed properly convex domain $\Omega$ is preserved by a linear flow $\Phi$, then $\Omega$ is reducible and $\Phi$ is hyperbolic.

**Proof.** Given $p \in \text{int} \Omega$ the closure, $\ell$, of the flowline containing $p$ is contained in $\Omega$. If $\Phi$ is parabolic, then $\ell \cong \mathbb{R}P^1$. Thus $\Phi$ is hyperbolic and $\ell = [a, b]$ has endpoints $a \in A \cap \Omega$ and $b \in B \cap \Omega$, where $A$ and $B$ are the stationary subsets of $\Phi$. It follows that $\Omega$ is the convex hull of $A \cap \Omega$ and $B \cap \Omega$, and it is therefore reducible.

The following is central to our approach.

**Corollary 1.5.** Suppose $M = \Omega/\Gamma$ is properly convex and $\Gamma$ approaches a linear flow at infinity. Then $M$ is reducible.

**Proof.** By (1.3) $\Omega$ is preserved by a linear flow, and by (1.4) $\Omega$ is reducible.

**Lemma 1.6.** Suppose $\Omega$ is open, properly convex, and there is an abelian group $T \subset \text{PGL}(\Omega)$ that acts simply transitively on $\Omega$. Then $\Omega$ is the interior of a simplex.

**Proof.** Since $\dim T > \dim \text{Fr} \Omega$, for every $p \in \text{Fr} \Omega$ there is $1 \neq A_p \subset T$ with $A_p(p) = p$. Let $C_p$ be the component of $\text{Fr} \Omega \cap \text{Fix}(A_p)$ that contains $p$. Then $C_p$ is a non-empty compact convex subset of $\text{Fr} \Omega$. Since $T$ is abelian and connected it follows that $C_p$ is preserved by $T$. Hence $\text{Fr} \Omega$ is the union of $T$-invariant convex sets. Let $q \in \text{cl} \Omega$ be an extreme point. Then $q$ equals the intersection of those $C_p$ that contain it. Hence $q$ is fixed by all of $T$. Every point in $\Omega$ is in the convex hull of an $n$-simplex, $\Delta$, with vertices that are extreme points of $\text{cl} \Omega$. The vertices of $\Delta$ are fixed by $T$, therefore $T$ preserves the interior of $\Delta$. Since $T$ acts transitively on $\Omega$, and $\text{int} \Delta$ contains a point in $\Omega$, it follows that $\Omega = \text{int} (\Delta)$.

**Definition 1.7.** A quasi-cusp is a properly convex $n$-manifold $Q = \Omega/\Gamma$ such that $\Gamma$ contains a finite index subgroup isomorphic to $\mathbb{Z}^{n-1}$.

Every cusp is a quasi-cusp. Observe that there is no requirement on $\partial \Omega$. If $Q$ is a quasi-cusp with boundary, then $\text{int} Q$ is also a quasi-cusp. The proof of Theorem (0.1) amounts to showing quasi-cusps are generalized cusps under some extra hypotheses. The definitions imply:

**Lemma 1.8.** Suppose $Q$ is a quasi-cusp of dimension $n$. Then $H^{n-1}(Q; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Suppose $C_i = \Omega_i/\Gamma_i$ are generalised cusps for $i = 1, 2$ and $\dim C_i = n_i$. Then there is a quasi-cusp $\Omega/\Gamma$, where $\Omega = \Omega_1 * \Omega_2 \subset \mathbb{RP}^{n_1+n_2+1}$ and $\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus K$ and $K \cong \mathbb{Z}^2$ is a discrete subgroup of $\Phi_1 \otimes \Phi_2 \otimes \Phi \cong \mathbb{R}^3$, where $\Phi_i$ is a radial flow for $C_i$ and $\Phi$ is the linear flow that fixes each point in $\Omega_1$ and $\Omega_2$.

If $Q \cong \partial Q \times [0, 1) = \Omega/\Gamma$ is a quasi-cusp with compact boundary, there is a decomposition of $\text{Fr} \Omega$ into three parts

$$\text{Fr} \Omega = \partial \Omega \cup \text{Fr}_v \Omega \cup \partial_{\infty} \Omega$$

that is described below. Moreover $\partial_{\infty} \Omega = \mathbb{RP}^{n-1}_{\infty} \cap \text{cl} \Omega$ and $\text{Fr}_v \Omega$ is empty for generalized cusps. 

**Example.** In the following example the quasi-cusp is $Q = \Omega/\Gamma$, with $\Omega \subset \mathbb{R}^3$, and $\Gamma \subset \text{Aff}(3)$. As an affine manifold $Q = S^1 \times R$ where $S^1$ is the quotient of $(0, \infty)$ by a homothety, and $R$ is the quotient of the parabolic model $\{(y, z) : 2y \geq z^2\}$ of $\mathbb{H}^2$ by a cyclic group of parabolics.
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Figure 1. $\Omega_M = \{(x, y, z) : 2y \geq z^2, x > 0\}$

This can be referred to during the proof of Theorem (1.9). Define $\Omega = \text{int}(\Omega_M)$ where $\Omega_M \subset \mathbb{R}^3$ and $\tau : \mathbb{R}^2 \to \text{Aff}(\Omega)$ are given by

$$\Omega_M = \{(x, y, z) : 2y \geq z^2, x > 0\}, \quad \tau(a, b) = \begin{pmatrix} e^a & 0 & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then $T = \tau(\mathbb{Z}^2)$ is a lattice in $T = \text{Im}\tau$. Observe that $\Omega$ is properly convex and preserved by $T$, so $Q = \Omega/\Gamma$ is a quasi-cusp. Moreover, $T$ acts simply transitively on $\partial \Omega_M$. The radial flow $\Psi_t(x, y, z) = (x, y - t, z)$ is centralized by $T$ and $\Psi \oplus T$ acts simply transitively on $\Omega^+ = \cup_t \Psi_t(\Omega) = \{(x, y, z) : x > 0\}$. The plane $H$ where $y = -1$ is a displacing hyperplane for $\Psi$ that is disjoint from $\Omega$.

Also $\Omega$ is backwards invariant, i.e. $\Psi_t(\Omega) \subset \Omega$ for all $t \leq 0$. The surface

$$\partial \Omega_M = \{(x, y, z) : 2y = z^2, x > 0\}$$

is convex and every point in $\partial \Omega_M$ is contained in a straight line segment. Moreover, $\partial \Omega_M$ is transverse to the flow lines of $\Psi$ and is called the flow boundary. The vertical frontier

$$\text{Fr}_v \Omega := (\text{Fr} \Omega \setminus \partial \Omega) \cap \mathbb{R}^3 = \{(x, y, z) : 2y \geq z^2, x = 0\}$$

is backwards invariant under $\Psi$. The ideal boundary is a 1-simplex

$$\partial_\infty \Omega = \text{Fr} \Omega \cap \mathbb{R}P^2_\infty = \{[x : 1 - x : 0 : 0] : x \in [0, 1]\}$$

The quasi-cusp $Q$ is foliated by convex ruled tori that are covered by level sets of $2y - z^2$. These levels sets are permuted by $\Psi$. Also $Q$ is a convex submanifold of $Q^+ = \Omega^+/\Gamma \cong T^2 \times \mathbb{R}$ where $\Omega^+ = \{(x, y, z) : x > 0\} \supset Q$. There is another radial flow $L_t(x, y, z) = (\exp(t)x, y, z)$ that commutes with $T$. This flow preserves $\Omega$ and shows it is reducible: $\Omega_M$ is a cone from $[1 : 0 : 0 : 0]$ to $\text{Fr}_v \Omega$, which is a closed disk with one point deleted from the boundary. Moreover $L_t$ is a subgroup of $T \oplus \Psi$.

The subgroup $\text{UT}(n) < \text{GL}(n, \mathbb{R})$ consists of upper-triangular matrices with positive diagonal entries. Let $\text{D}(n) < \text{UT}(n)$ be the subgroup of diagonal matrices. Suppose $\Omega$ is properly convex and $K \subset \Omega$. The convex hull $\text{CH}(K)$ is the intersection of all closed convex subsets of $\Omega$ that contains $K$.

**Theorem 1.9.** Suppose $Q = \Omega/\Gamma$ is a quasi-cusp of dimension $n$. Then either $Q$ contains a generalized cusp or $\Omega$ is reducible.
Proof. We may assume $\Omega$ is open in $\mathbb{RP}^n$. By [13](6.18) there is a finite index subgroup $\Gamma_0 \subset \Gamma$ such that (after conjugacy) $\Gamma_0$ is a lattice in an upper-triangular connected nilpotent Lie subgroup $T \subset \text{UT}(n+1)$. It follows from the definition that if $Q$ has a finite cover that is a generalized cusp then $Q$ is a generalized cusp. Thus we may assume $\Gamma_0 = \Gamma$. By [13](6.19) there is a radial flow $\Phi$ that centralizes $T$.

Here is a sketch of the proof when the radial flow is parabolic. We want to show that the $T$-orbit of a point $x$ deep inside $\Omega$ is a convex hypersurface inside $\Omega$. Since $\Gamma$ is a lattice in $T$ and $\Gamma \cdot x \subset \Omega$ this seems reasonable. Let $\Omega_\ell = \Phi_t(\Omega)$. To prove it, we enlarge $\Omega$ to be the set $\Omega^+ = \cup_\ell \Omega_\ell$ that is the union of flow lines thorough $\Omega$. This is preserved by $\Gamma$ because $\Phi$ centralizes $\Gamma$. Then $Q^+ = \Omega^+/\Gamma$ is $Q$ plus an open collar. A bit of work shows that $\Omega^+$ is preserved by all of $T$. If $x \in \Omega$ then $\Omega' = T \cdot x$ is a hypersurface in $\Omega^+$. We do not yet know that $\Omega'$ is convex. We use the fundamental-domains trick from [13](6.24) to produce from $\Omega'$ a convex codimension-1 submanifold $S \subset \Omega_\ell$ (for some $t$) that is preserved by $T$. The action of $T$ on $S$ is simply transitive. If $S$ is strictly convex we are done. Otherwise there is a maximal flat $F \subset S$ and the stabilizer $H \subset T$ of $F$ acts simply transitively on $F$. There is a subgroup $L$ of $H$ such that the $H$-orbit of every point in $F$ is a line segment. Since $L$ commutes with $T \oplus \Phi$ it follows that the orbit under $L$ of every point in $\Omega^+$ is a line segment, thus $L$ is a linear flow, and $\Omega$ is reducible. Now for the details.

If $T \cap \Phi \neq 1$, then $\Phi \subset T$ so $\Gamma$ approaches $\Phi$ at infinity, and this implies $\Omega$ is reducible. Thus we may assume $T \cap \Phi = 1$. Let $H$ be the stationary hyperplane and $c$ the center for $\Phi$. We may assume $H$ is disjoint from $\Omega$, otherwise replace $\Omega$ by one of the components, $\Omega_\ell$, of $\Omega$. Since $\Omega^+ = \cup_\ell \Omega_\ell$ is a hypersurface in $\Omega^+$, we are done. If $\Omega^+ / \Gamma$ does not contain a generalized cusp, then below we show that $\Gamma$ approaches a linear flow at infinity, and then it follows from (1.5) that $\Omega$ is reducible.

The first case is that $\Phi$ is parabolic, so $c \in H$. By [13](6.19) we may choose $\Phi$ to be parabolic whenever $T$ is not diagonal. The affine patch $\mathbb{R}^n = \mathbb{RP}^n \setminus H$ contains $\Omega$. Since $\Omega$ is properly convex there is a displacing hyperplane $P \subset \mathbb{RP}^n$ that is disjoint from $\text{cl}(\Omega)$.

After reversing the direction of the radial flow if needed, we may assume $P_\ell := \Phi_t(P)$ moves away from $\Omega$ as $t$ increases. After an affine change of coordinates we may assume that $P$ is the hyperplane $x_1 = 0$ and $x_1 > 0$ on $\Omega$ and $\Phi_t(x) = x - t \cdot e_1$. Since $\Phi$ centralizes $T$, it centralizes $\Gamma$, so $\Gamma$ preserves $\Omega_\ell = \Phi_t(\Omega)$.

It follows from Claims 2, 3 and 4 of [13](6.23) that $\Omega_\ell \subset \Omega$, whenever $t < 0$, i.e. $\Omega$ is backwards invariant. Moreover it follows that there is a properly convex set $\Omega_M = \partial \Omega_M \times [0, \infty)$ with $\Omega \subset \Omega_M \subset \text{cl}(\Omega)$, and that $Q = \text{int}(M)$, where $M = \Omega_M / \Gamma \cong \partial M \times [0, \infty)$. Then $\Omega^+ = \cup_\ell \Omega_\ell$ is the union of flowlines that contain a point of $\partial M$. Moreover $\Omega^+ \subset \mathbb{R}^n$ is convex and $F : \partial \Omega_M \times \mathbb{R} \to \Omega^+$ given by $F(x, t) = \Phi_t(x)$ is a homeomorphism. Also $\Gamma$ acts freely and properly discontinuously on $\Omega^+ / \Gamma \cong \partial M \times \mathbb{R}$.

We claim that $T$ preserves $\Omega^+$. Given $g \in T$, then $V(g) := \Omega^+ \cap g(\Omega)^+ = \Omega^+$ is convex, open and $\Gamma$–invariant, but it might be empty. Let $W \subset T$ be the set of all $g \in T$ such that $V(g)$ is not empty. Then $W$ is open. Moreover if $V(g) \neq \emptyset$ then $V(g) = \Omega^+$ because $V(g) / \Gamma$ is a convex submanifold of $\Omega^+ / \Gamma$ and it is a union of flowlines. Thus $V(g) = N \times \mathbb{R}$ for some submanifold $N \subset \partial M$. But $N$ and $M$ are $K(\Gamma, 1)$’s so $V(g)$ is a closed submanifold and therefore $N = \partial M$. It follows that a neighborhood of the identity in $T$ preserves $\Omega^+$, and therefore $T$ preserves $\Omega^+$.

Following the proof of the first claim of [13](6.24) there is a compact $X \subset T$ and compact $D \subset \partial M$ so that $\Gamma \cdot D = \partial M$ and $\Gamma \cdot X = T$. Let $\pi : \Omega^+ \to \Omega^+ / \Gamma$ be the projection and define

$$F : T \times \partial M \to \Omega^+ / \Gamma$$

by $F(g, x) = \pi(g \cdot x)$. Then $K = \text{Im}(F) = F(X \times D)$ is compact. Thus there is $t$ such that $K \subset \Omega_\ell / \Gamma$. Hence $\partial M = \Gamma \cdot K \subset \Omega_\ell$. Now $\Omega_\ell$ is properly convex, so

$$Y = \text{cl}(\text{CH}(\pi^{-1}(K))) \subset \Omega_{t+1}$$
is properly convex, and $T$-invariant, and a closed subset of $\Omega_{t+1}$. Hence $S = \partial Y$ is a convex hypersurface that is $T$-invariant.

Since $\dim T = \dim S$, and $T$ contains no elliptics, $T$ acts freely on $S$. This action is transitive since otherwise $(T \cdot x)/\Gamma$ is a $K(\Gamma, 1)$ that is a proper submanifold of $\Sigma$, which is impossible. If $S$ is strictly convex then $Y$ is a generalized cusp.

Otherwise there is a line segment in $S$. Hence $S$ contains a maximal flat $F$. Then the subgroup $H \subset T$ that preserves $F$ acts simply transitively on $F$. By (1.6) $F \cong \text{int}(\Delta^k)$ is the interior of a simplex and $H/F$ is the projective diagonal group. There is a 1-parameter subgroup $L \subset H$ given by $L_t = \text{Diag}(t, 1, \cdots, 1)$ such that every orbit of $L$ in $F$ is a segment of a line. Now $T \oplus \Phi$ acts simply transitively on $\Omega^+$. Since $L$ commutes with $T$ and $\Phi$, it follows that the orbit under $L$ of every point in $\Omega^+$ is contained in a line. Then by (1.2) $L$ is a linear flow. But $\Gamma$ approaches $L$ at infinity so by (1.3) $\Omega$ is preserved by $L$. Then by (1.4) $\Omega$ is reducible. This completes the proof when $\Phi$ is parabolic.

If $\Phi$ is hyperbolic, then $G = T \times \Phi$ is the diagonal group in $\text{UT}(n + 1)$, and it follows that $\Omega^+$ is the interior of the $n$-simplex $\Delta$ with vertices $[e_1], \cdots, [e_{n+1}]$. Let $x \in \text{int}(\Delta)$ and consider the hypersurface $S = T \cdot x \subset \Delta$. If $S$ is strictly convex then $C$ is a generalized cusp. If $S$ is not convex then $\Omega = \text{int}(\Delta)$ is reducible. If $S$ is convex, but contains a flat, the argument above implies that $\Omega$ is reducible.

2. Discreteness and Irreducibility

This section shows that if $\pi_1 M$ satisfies certain algebraic conditions then the holonomy of a properly convex structure on $M$ is irreducible (0.2), and that a limit of such holonomies is always discrete and faithful (2.2).

**Theorem 2.1** (Cluckrow’s theorem [11],[16](8.4)). Suppose $\Gamma$ is a finitely generated group that does not contain a normal infinite nilpotent subgroup $N$. Then the subset of $\text{Hom}(\Gamma, \text{GL}(n, \mathbb{R}))$ consisting of discrete faithful representations is closed in the usual (Euclidean) topology.

If $\Omega \subset \mathbb{RP}^n$, then $\text{PGL}(\Omega) \subset \text{PGL}(n + 1, \mathbb{R})$ is the subgroup that preserves $\Omega$. We will make frequent use of the following implication.

**Corollary 2.2.** Suppose $\Gamma$ is finitely generated and does not contain a non-trivial normal abelian subgroup. Then the subset of $\text{Hom}(\Gamma, \text{GL}(n + 1, \mathbb{R}))$ consisting of discrete faithful representations is closed.

**Proof.** Suppose $\Gamma$ contains an infinite normal nilpotent subgroup $G$. Let $Z$ be the center of $G$. Then $Z$ is non-trivial and abelian. Since $Z$ is characteristic in $G$, and $G$ is normal in $\Gamma$, it follows that $Z$ is normal in $\Gamma$. This contradicts a hypothesis. The result now follows from (2.1). □

The next result, is due to Benoist [7], see also [10].

**Lemma 2.3.** If $M$ is closed and properly convex, then $\pi_1 M$ contains a non-trivial normal abelian subgroup if and only if $\pi_1 M$ has non-trivial virtual center.

**Proof.** Set $G = \pi_1 M$. Suppose $Z = Z(H) \neq 1$ is the center of a finite index subgroup $H \subset G$. Since the universal cover of $M$ is contractible, $G$ is torsion-free, so $Z$ is infinite. There is a subgroup $H' \subset H$ of finite index with $H' \lhd G$. Then $H' \cap Z$ has finite index in $Z$, and is central in $H'$, therefore $Z' = Z(H')$ is non-trivial. Thus, after replacing $H$ by $H'$, we may assume $H \lhd G$. Now $Z$ is characteristic in $H$ and thus normal in $G$. Hence $Z \neq 1$ is an infinite normal abelian subgroup of $G$.

For the converse, suppose $1 \neq A \lhd G$ and $A$ is abelian. Let $\Gamma$ be the image of the holonomy $\rho : \pi_1 M \to \text{SL}_k^+(n + 1, \mathbb{R})$ of $M$. Every element of $\Gamma$ is hyperbolic because $M$ is closed. Let $d_\Omega$ be
the Hilbert metric on $\Omega$. The displacement function $\tau : \Gamma \to \mathbb{R}$ is given by
\[
\tau(g) = \inf\{d_\Omega(x, gx) : x \in \Omega\}
\]
and $\mu = \min \tau(\pi_1 M) > 0$ because $M$ is compact. Since $A$ is abelian, the moduli of the weights of $\rho|A$ give a homomorphism
\[
\lambda : A \to \mathbb{R}_+^{n+1}
\]
By (2.1) in [14] for $a \in A$ we have $\tau(a) = \log(\lambda_+ / \lambda_-)$, where $\lambda_+, \lambda_-$ are the maximum and minimum moduli of eigenvalues of $\rho(a)$. Since $\mu > 0$ it follows that $\lambda$ is discrete and faithful. Hence the subset $B \subset A$ of elements that minimize $\tau|A$ is finite. Now $\tau(ghg^{-1}) = \tau(h)$ so the action of $G$ on $A$ by conjugation permutes $B$. Since $B$ is finite, the kernel of this action is a finite index subgroup $H \subset \pi_1 M$ that fixes each element of $B$. Thus $B \subset Z(H)$. 

However when one deals with manifolds that are not closed, these statements are not equivalent, as the following example illustrates.

Example. Let $M$ be a $3$-manifold that is a torus bundle over $S^1$ with monodromy $A \in \text{SL}(2, \mathbb{Z})$. Then $M$ is Euclidean if $A$ is periodic; NIL if $\text{tr}(A) = \pm 2$ and $A \neq \pm \text{Id}$; and otherwise $M$ has a SOLV geometry. For each $s > 0$ there is an affine realization of this structure as $\mathbb{R}^3 / \Gamma_s$, where $\Gamma_s \cong \pi_1 M \cong \mathbb{Z} \times A \mathbb{Z}^2$ is the image of
\[
\rho_s(m, n, p) = \begin{pmatrix} A^p & 0 & s \begin{pmatrix} m \\ n \end{pmatrix} \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix} \quad m, n, p \in \mathbb{Z}
\]
This gives a path of discrete faithful representations $\rho_s : \pi_1 M \to \text{PGL}(4, \mathbb{R})$ whose images converge to a cyclic group as $s \to 0$.

Now $\text{PGL}(4, \mathbb{R})$ acts on $\text{SL}(4, \mathbb{R}) / \text{SO}(4)$ realized as the properly convex domain $\Omega \subset \mathbb{R}P^5$ obtained by projectivizing the space of positive definite quadratic forms on $\mathbb{R}^3$. This gives a sequence of properly convex projective 5-manifolds, homeomorphic to an $\mathbb{R}^3$-bundle over $M$, and the holonomies converge to a non-faithful representation. In the SOLV case $\pi_1 M$ has trivial virtual center, but contains $\mathbb{Z}^2$ as a normal subgroup. 

Let $W = \mathbb{R}^{n+1}$ and suppose $\Omega \subset \mathbb{P}W$ is open and $M = \Omega / \Gamma$ is a properly convex manifold. We say $\Gamma$ is reducible if there is a proper projective subspace $P = \mathbb{P}U \subset \mathbb{P}W$ that is preserved by $\Gamma$. Let $W^* = \text{Hom}(W, \mathbb{R})$ denote the dual vector space. The dual manifold $M^* = \Omega^* / \Gamma^*$ is properly convex and diffeomorphic to $M$. Now $W = U \oplus V$ and $W^* = U^* \oplus V^*$ where $V^* = \{ \phi \in W^* : \phi(U) = 0 \}$ and similarly for $U^*$. Moreover $\Gamma$ preserves $\mathbb{P}(U)$ if and only if $\Gamma^*$ preserves $\mathbb{P}(V^*)$.

Lemma 2.4. With the notation above, if $\Omega = P \cap \Omega \neq \emptyset$ then $L = \Omega' / \Gamma$ is a convex submanifold of $M$ and $\dim L = \dim P$, and the inclusion $L \hookrightarrow M$ is a homotopy equivalence.

If $P \cap \text{cl} \Omega = \emptyset$ then $M$ contains a closed submanifold $L$ of codimension $\dim P$ and $L \hookrightarrow M$ is a homotopy equivalence.

Proof. The first conclusion is immediate. Now suppose that $P \cap \text{cl} \Omega = \emptyset$. There is a projective hyperplane $H$ that contains $P$ and is disjoint from $\text{cl} \Omega$. Then $H = \ker \phi$ for some $\phi \in V^*$ with $\phi(U) = 0$. It follows that $[\phi] \in W = \mathbb{P}(V^*) \cap \Omega^* \neq \emptyset$. The result now follows from the first part, using the diffeomorphism $M \cong M^*$.

Lemma 2.5. With the hypotheses of (2.4) suppose either that $M$ is closed or else that $\Gamma$ contains a subgroup $\Gamma'$ of infinite index and $H_{n-1}(\Omega / \Gamma'; \mathbb{Z}_2) \neq 0$, then $\emptyset \neq P \cap \text{cl} \Omega \subset \text{Fr} \Omega$

Proof. Otherwise by (2.4) there is a submanifold $L$ of $M$ such that the inclusion $L \hookrightarrow M$ is a homotopy equivalence and $\dim L < \dim M$. If $M$ is closed then $H_n(M; \mathbb{Z}_2) \neq 0$ but $\dim L < \dim M$ so $H_n(L; \mathbb{Z}_2) = 0$ which is a contradiction.
If $M$ is not closed, let $M'$ and $L'$ be the covers of $M$ and $L$ corresponding to $\Gamma'$. Then $H_{n-1}(L'; \mathbb{Z}) \cong H_{n-1}(\Omega/\Gamma'; \mathbb{Z}) \neq 0$ and it follows that $L'$ is a closed manifold of dimension $(n - 1)$, and therefore a finite cover of $L$. Hence $|\Gamma : \Gamma'| < \infty$, which contradicts $|\Gamma : \Gamma'| = \infty$.  

We will apply this when $M$ contains a convex, closed submanifold $N = \Omega'/\Gamma'$ of codimension one with $\Omega' \subset \Omega$ and $|\Gamma : \Gamma'| = \infty$. The following applies to a properly convex manifold that contains a generalized cusp.

**Lemma 2.6.** Suppose $M = \Omega/\Gamma$ is a properly convex manifold, and $\Omega' \subset \Omega$ is a convex closed subset bounded by a smooth, connected hypersurface $\partial \Omega' \neq \emptyset$, that contains no line segment. Let $\Gamma' \subset \Gamma$ be the subgroup that preserves $\Omega'$ and suppose that $|\Gamma : \Gamma'| = \infty$, and $N = \partial \Omega'/\Gamma'$ is a compact manifold. Suppose there is $\gamma \in \Gamma$ such that $\gamma \Omega' \cap \Omega' = \emptyset$. Then $\Gamma$ does not preserve a proper projective subspace.

**Proof.** Suppose $\Gamma$ preserves $P = \mathbb{P}(U)$. By (2.5) we may assume that $W^+ = \text{cl} \Omega \cap P$ is not empty, convex, and is contained in $\text{Fr} \Omega$. Let $W = W^+ \setminus \partial W^+$, so that $W$ is an open properly convex set and $\dim W < \dim M$.

First suppose there is $x \in W \cap \partial \Omega'$. Then $\gamma x \in W \cap \Omega'$ and $d_{W^+}(x, \gamma x) < \infty$. Given $p \in \partial \Omega'$ then, since $\Omega'$ is convex, the segment $\ell = [p, x]$ is contained in $\Omega'$ and $\gamma \ell \subset \Omega'$. If $y$ is on $\ell$ and close enough to $x$ then $d_{\Omega'}(y, \gamma y) \leq d_W(x, \gamma x) + 1$. This implies $d_{\Omega'}(y, \partial \Omega') \leq d_W(x, \gamma x) + 1$. But $d_{\Omega'}(x, \partial \Omega') \to \infty$ as $y \to x$. It follows that $W \cap \partial \Omega' = \emptyset$.

Since $\Omega'$ is closed in $\Omega$, for each $x \in \Omega$ there is a point $y \in \Omega'$ that minimizes $d_{\Omega'}(x, y)$. Since $\Omega'$ is convex, and $\partial \Omega'$ smooth, and contains no line segment, $y$ is unique and the map $\pi : \Omega \to \Omega'$ given by $\pi x = y$ is distance non-increasing. Moreover, if $x \in \partial \Omega'$ then $\pi^{-1} x$ has closure a segment $[x, y]$ with $y \in \partial \Omega$. Thus there is a continuous extension $\pi : \text{cl} \Omega \to \text{cl} \Omega'$ where the closures are in projective space. Let $W' = \text{Fr} \Omega \setminus \partial \Omega'$, then $\pi | W \to \partial \Omega'$ is a homeomorphism.

Clearly $\pi$ is $\Gamma'$ equivariant. Restricting gives an injective map $\pi : W \to \partial \Omega'$. The action of $\Gamma$ on $\Omega'$ is free and properly discontinuous. Thus the same is true for the action on $W$, and $\pi$ covers a map $f : W/\Gamma' \to \Omega'/\Gamma' = N$ that is a homotopy equivalence. Since $N$ is closed it follows that $f$ is surjective and thus $\pi$ is a surjective. Hence $W = W' = \text{Fr} \Omega \setminus \partial \Omega'$. But the same is true when $\Omega$ is replaced by $\gamma \Omega$. Thus there is $x \in \text{Fr} \Omega' \cap \text{Fr} \Omega = \text{Fr} \Omega' \cap \text{Fr} \gamma \Omega$. As before there is a line $\ell = [p, x] \in \Omega$ and another line $\ell' = [p', x] \in \Omega'$ and this is a contradiction.  

The following restricts the fundamental group of a reducible manifold.

**Lemma 2.7.** Suppose $\Omega$ is reducible and $M = \Omega/\Gamma$ is properly convex. If $\Gamma$ contains no non-trivial normal abelian subgroup then there is a properly convex manifold $\Omega/\Gamma^+$ with $\Gamma^+ \cong \Gamma \times \mathbb{Z}$.

**Proof.** We have $\Omega = \Omega_U \ast \Omega_V$ with $\Omega_U \subset \mathbb{P} U$ and $\Omega_V \subset \mathbb{P} V$. Let $k = \dim U$ and $l = \dim V$. Let $\rho : \pi_1 M \to \text{GL} U \oplus \text{GL} V$ be the holonomy of $M$, so that $\Gamma = \rho(\pi_1 M)$. Given $s \in \mathbb{R}$ there is a homomorphism $\theta_s : \text{GL} U \oplus \text{GL} V \to \text{GL} U \oplus \text{GL} V$ given by

$$
\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto \begin{pmatrix} |\det A|^{-s/k} A & 0 \\ 0 & |\det B|^{-s/l} B \end{pmatrix}
$$

If $s \neq 1$ then $\theta_s$ is injective and has inverse $\theta_{1/(1-s)}$. For $s \in [0, 1]$ define $\rho_s = \theta_s \circ \rho$.

Observe that $\alpha I_U + \beta I_V \in \text{GL}(U) \oplus \text{GL}(V)$ preserves $\Omega$. Hence $\Gamma_s = \rho_s(\pi_1 M)$ preserves $\Omega$. When for $s \neq 1$ then $\theta_s : \Gamma \to \Gamma_s$ has an inverse, so $\rho_s$ is discrete and faithful. It follows from (2.2) that $\rho_1 : \pi_1 M \to \text{SL} U \oplus \text{SL} V$ is discrete faithful. Moreover $\rho_1$ preserves $\Omega$, and $\Omega$ is properly convex, so this action is free and properly discontinuous.

Observe that $\Omega$ is preserved by the hyperbolic linear flow for $(U, V)$ given by $\Phi : \mathbb{R} \to \text{GL} U \oplus \text{GL} V$ where $\Phi(t) = I_U + \exp(t) I_V$. Let

$$
\tau : \pi_1 M \oplus \mathbb{Z} \to \text{GL} U \oplus \text{GL} V \quad \text{given by} \quad \tau(\alpha, n) = \sigma(\alpha) \circ \Phi(n)
$$
Then $\Gamma^{+} = \tau(\pi_{1} M \oplus \mathbb{Z})$ preserves $\Omega$ and $\tau$ is discrete and faithful, because $\det \tau(\alpha, n) = \exp(n)$. Thus $\Gamma^{+}$ acts freely and properly discontinuously on $\Omega$, so $R = \Omega / \Gamma^{+}$ is a properly convex manifold. \hfill $\square$

**Theorem 2.8** (Benoist). Suppose $M$ is a closed properly convex manifold and $\pi_{1} M$ has trivial virtual center. Then the holonomy of $M$ is irreducible.

**Proof.** Let $M = \Omega / \Gamma$. By (2.5) we may assume that $\emptyset \neq X = \mathbb{P}(U) \cap \overline{\Omega} \subset \text{Fr} \; \Omega$. Let $\Omega_{U} = X \setminus \partial X$ be the relative interior of $X$. Thus $\Omega_{U}$ is properly convex, and $\dim \Omega_{U} \leq \dim \mathbb{P} U$. Now $W = U \oplus V$ and $\Gamma$ preserves $U$. We may assume $U$ is chosen to minimize $\dim V$. The representation $\rho : \pi_{1} M \to \Gamma$ is given in matrix form by

$$
\rho = \begin{pmatrix}
A & 0 \\
0 & C
\end{pmatrix}
$$

where $A : \pi_{1} M \to \text{GL}(U)$ and $C : \pi_{1} M \to \text{GL}(V)$.

We claim that replacing $B$ by 0 gives a discrete faithful representation, $\rho_{0} = A \oplus C : \pi_{1} M \to \text{GL} U \oplus \text{GL} V$, that preserves a reducible properly convex set $\Omega_{0} = \Omega_{U} \ast \Omega_{V}$. Then by (2.7) there is a properly convex manifold $N = \Omega_{0} / \Gamma^{+}$. But $N$ is an $n$-manifold that is homotopy equivalent to $M \times S^{1}$. The latter is a closed manifold of dimension $(n+1)$, and this is a contradiction.

It only remains to prove the claim. Observe that $A$ preserves the properly convex set $\Omega_{U} = \mathbb{P}(U) \cap \overline{\Omega} \subset \mathbb{P}(U)$. The holonomy of the dual manifold $M^{*}$ is the dual representation $\rho_{0}^{*} : \pi_{1} M \to \text{GL}(\mathbb{P}^{*} V)$ which preserves $\mathbb{P}^{*} V^{*}$. By (2.5) $\Omega' = \mathbb{P}(V^{*}) \cap \overline{\Omega}$ is a non-empty subset of $\text{Fr} \; \Omega^{*}$ and therefore properly convex. Also $\dim \Omega' = \dim \mathbb{P}^{*} V^{*}$, otherwise $\Omega'$ lies in a proper projective subspace of $\mathbb{P}^{*} V^{*}$ that is preserved by $\rho'(\pi_{1} M)$, and this contradicts minimality of $\dim V$. Hence $\Omega' = (\Omega')^{*} \subset \mathbb{P}(V)$ is a non-empty properly convex open set that is preserved by $C$.

For $0 < t \leq 1$ define

$$
P_{t} = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{and} \quad \rho_{t} = \begin{pmatrix} A & tB \\ 0 & C \end{pmatrix}.
$$

Observe that $\rho_{1} = \rho$ and $\rho_{0}$ is defined for $t = 0$ and $\rho_{0} = A \oplus C$. For $t > 0$ notice that $\rho_{t} = P_{t} \rho_{1} P_{t}^{-1}$, so $\rho_{t}$ is discrete and faithful. Since $\rho_{t} \to \rho_{0}$ as $t \to 0$ and $\pi_{1} M$ has trivial virtual center, it follows that $\rho_{0}$ is discrete and faithful by (2.2). The action of $\rho$ and $\rho_{0}$ on $U$ are both equal to $A$ and $\rho_{0}$ preserves $\Omega_{U}$.

Now $\rho_{0}$ preserves $\Omega'$ and the action of $\rho_{0}$ and $\rho$ on $\mathbb{P} V$ are both given by $C$ and are thus equal. Hence $\rho_{0}$ preserves $\Omega_{V}$.

Thus $\Omega' = \Omega_{U} \ast \Omega_{V}$ is properly convex and preserved by $\rho_{0}$. We claim that $\dim \Omega_{0} = \dim \Omega$. Let $W' \subset W$ be the vector subspace of minimal dimension such that $\Omega' \subset \mathbb{P} W'$. Then $\dim \Omega' = \dim \mathbb{P} W'$ and $\rho_{0}$ preserves $\Omega_{0}$ and thus preserves $W_{0}$. Let $\rho_{0} : \pi_{1} M \to \text{GL}(W')$ be the restricted action. Replace $\Omega'$ be the interior of $\Omega'$ in $W'$.

We claim $\rho_{0}$ is discrete and faithful. Suppose $g \in \pi_{1} M$ and $\rho(g) = [L]$ is hyperbolic. Then there are $w_{\pm} \in W$ such that $[w_{\pm}] \in \text{Fr} \; \Omega$ and $w_{+}$ is an attracting, and $w_{-}$ a repelling, fixedpoint of $[L]$. This means $Lw = \lambda_{\pm} v$ with $\lambda_{+} > 0$ real and $\lambda_{-}$ is the spectral radius of $L^{\pm 1}$. Moreover the displacement distance of $\rho(g)$ for the Hilbert metric $d_{h}$ is $\log \lambda_{+} / \lambda_{-}$.

Write $w_{\pm} = u_{\pm} \pm v_{\pm}$ with $u_{\pm} \in U$ and $v_{\pm} \in V$. Now $\rho_{0}$ preserves the properly convex domain $\Omega_{t} = P_{t} \Omega$ and $P_{t}(w_{\pm}) = u_{\pm} + t^{-1} v_{\pm}$. If $v_{\pm} = 0$ then $P_{t}(w_{\pm}) = u_{\pm}$ gives a point $\Omega_{0}$. If $v_{\pm} \neq 0$ then $\lim_{t \to 0} P_{t}[w_{\pm}] = [v_{\pm}]$ is in $\text{Fr} \; \Omega_{V}$. Thus in both cases $\lim_{t \to 0} P_{t}[w_{\pm}]$ is in $\text{Fr} \; \Omega'$. It follows that $\rho$ and $\rho_{0}$ have the same the displacement distance. Since $M$ is compact there is an element of $\pi_{1} M$ of shortest length. Hence $\rho_{0}$ is discrete faithful. \hfill $\square$

**Proof of irreducible holonomy (0.2).** If $M$ is closed this follows from (2.8). Otherwise there is a subgroup $G \cong \mathbb{Z}^{n-1}$ of $\Gamma$. If $\Omega / G$ is a generalized cusp the result follows from (2.6). Otherwise $\Omega$ is reducible by (1.9). Then by (2.7) there is a properly convex manifold $P = \Omega / \Gamma^{+}$. Now $\Gamma^{+}$ contains the subgroup $G^{+} = G \times Z \cong \mathbb{Z}^{n}$. Then $N = \Omega / G^{+}$ is an $n$-manifold with $\pi_{1} N \cong \mathbb{Z}^{n}$, so
Lemma 3.2. Suppose that $\pi$ is a subgroup of $H$ such that $p \in H$ and $H \cap \text{int} \Omega = \emptyset$. The point $p$ is a smooth point if $H$ is unique, and is a strictly convex point if there is $H$ with $p = H \cap \text{cl} \Omega$, and $p$ is a round point if it is both conditions hold.

If $\Omega$ is properly convex then a properly embedded triangle or PET in $\Omega$ is a flat triangle $\Delta$ with $\text{int} \Delta \subset \text{int} \Omega$ and $\partial \Delta \subset \text{Fr} \Omega$. We say that the generalized cusp $C \cong \partial C \times [0,1)$ is minimal size if the only convex submanifold of $\text{cl} C$ that contains $\partial C$ is $\text{cl} C$. This is always the case unless the holonomy is diagonalizable, see [4](1.4) It follows from [4](1.24) that $\text{cl}(\Omega)$ is a properly convex and $\Gamma \subset \text{Aff}(\mathbb{R}^n)$ such that $\Gamma = \Omega$. Moreover $\partial \Omega \subset \mathbb{R}^n$ is a properly embedded, strictly-convex hypersurface and $\text{Fr} \Omega = \partial \Omega \cup \Delta^r$, where $\Delta^r \subset \mathbb{RP}^{n-1}$ is a flat simplex called the end flat, and $0 \leq r \leq n-1$ is the rank of $C$. In particular $\Omega$ does not contain a PET.

If $\Omega$ is properly convex, then a flat in $\text{Fr} \Omega := \text{cl} \Omega \setminus \text{int} \Omega$ is a convex set that contains more than one point. A flat is maximal if it is not a proper subset of another flat in $\text{Fr} \Omega$. Every flat is contained in at least one maximal flat. Suppose $M$ is properly convex and $C \subset M$ is a generalized cusp. Let $\pi : \Omega \to M$ be the projection and let $U \subset \Omega$ be a component of $\pi^{-1} C$. Then $V = \text{Fr} \Omega \cap \text{cl} U = \partial_U U$ is called the end flat of $\Omega$ corresponding to $U$. By (3.1) it is a flat simplex of dimension $r$.

If $M$ is properly convex we say all the ends of $M$ are generalized cusps if there are pairwise disjoint $\pi_1$-injective generalized cusps $E_1, \cdots, E_n \subset M$ such that $\text{cl}(M \setminus \cup E_i)$ is compact. In this case we say $\pi_1 M$ is hyperbolic rel ends if $\pi_1 M$ is hyperbolic rel the subgroups $\{\pi_1 E_i : 1 \leq i \leq n\}$ in the sense of Drutu [15]. Note that it follows from the definitions that $\text{cl}(M \setminus \cup E_i)$ is connected. A subgroup of $\pi_1 M$ is an end group if it is conjugate to some $\pi_1 E_i$.

Lemma 3.2. Suppose $M = \Omega/\Gamma$ is properly convex and $V, V'$ are end flats of $M$ corresponding to $U, U' \subset \Omega$. If $U$ and $U'$ are disjoint, then $V$ and $V'$ are disjoint.

Proof. Suppose $x \in \text{cl}(U) \cap \text{cl}(U') \cap \text{Fr} \Omega$. Generalized cusps are convex, therefore there are line segments $\ell : [0, \infty) \to \text{cl} U$ and $\ell' : [0, \infty) \to \text{cl} U'$ both parameterized by arc length that both limit to $x$. Then $d(\ell(t), \ell'(t'))$ is not increasing, and so is bounded above. Let $C = \pi U \subset M$ be the end covered by $U$. Since $U$ and $U'$ are disjoint $d(\ell(t), \ell'(t'')) \geq d(\ell(t), \partial U)$. But $d(\ell(t), \partial U) = d(\pi(\ell(t)), \partial C) \to \infty$ as $t$ increases, a contradiction.

Theorem 3.3. Suppose $M = \Omega/\Gamma$ is properly convex with ends that are generalized cusps, and $\pi_1 M$ is hyperbolic rel the ends. Then the end flats of $\Omega$ are pairwise disjoint, and every flat in $\text{Fr} \Omega$ is contained in an end flat. Moreover $\Omega$ does not contain a PET.

Proof. Suppose $T$ is a PET in $\Omega$. Then by [15], there is $R > 0$, and an end $E$ of $M$, and a component $U$ of $\pi^{-1} E$ so that $T$ is contained in the $R$-neighborhood, $W$, of $U$. However $W$ is the universal cover of a generalized cusp. By (3.1) $W$ does not contain a PET, hence $\Omega$ does not contain a PET.

The pairwise disjoint property follows from (3.2). Let $B \subset M$ be compact such that the closure of each component of $M \setminus B$ is a generalized cusp. Suppose $\ell \subset \text{Fr} \Omega$ is a closed non-trivial line segment. Choose $p \in \text{int} \Omega$ and let $T = \text{cl} \Omega$ be the triangle that is the convex hull of $p$ and $\ell$. Given $s > 0$ let $T(s) = \{x \in \text{int} T : d_{\Omega}(x, \partial T) \geq s\}$.

Let $\pi : \Omega \to M$ be the projection. The first case is that for some $s > 0$ the set $\pi(T(s))$ is disjoint from $B$. Then $\pi(T(s))$ is contained in some end $C$ of $M$. This implies $\ell$ is contained in an end flat corresponding to a component of $\pi^{-1}(C)$. 

$H_n(N) \cong \mathbb{Z}$ and $N$ is closed. But $N$ covers the manifold $P = \Omega/G^+$ so this covering is finite. This implies $|\Gamma : G^+| < \infty$. This contradicts $|\Gamma : G| = |\Gamma^+ : G^+| = \infty$. $\square$
Suppose Lemma 3.4. Then the dual manifold $M^* = \Omega^*/\Gamma^*$ has the same structure.

Proof. If $C$ is a generalized cusp then it follows from the definition that $C^*$ is also a generalized cusp. Suppose $C \subset M$ is a generalized cusp. Then $C^* \supset M^*$ and $C^*$ is a generalized cusp. By the classification, [4](0.2), $C^*$ contains a smaller generalized cusp that is contained in an end of $M^*$. □

Theorem 3.5 (properties of generalized cusps). Suppose $M = \Omega/\Gamma$ is a properly convex manifold without boundary, and all the ends of $M$ are generalized cusps with compact boundary. Also suppose $\pi_1M$ is hyperbolic rel the ends, and $\pi_1M$ is not the union of the end groups. Let $\mathcal{F} \subset \text{Fr } \Omega$ be the union of the flats. Then

1) Maximal flats are pairwise disjoint.
2) Every maximal flat is an end flat.
3) Every end flat is a maximal flat.
4) The stabilizer in $\Gamma$ of a maximal flat is an end group.
5) Every parabolic subgroup is contained in an end group.
6) Every parabolic subgroup is conjugate in $\text{PGL}(n+1,\mathbb{R})$ into $\text{PO}(n,1)$.
7) Every element of $\pi_1M$ is strongly hyperbolic or contained in an end group.
8) The set $X = \{x \in \text{Fr }\Omega : \gamma(x) = x \text{ and } \gamma \text{ is strongly hyperbolic}\}$ consists of round points.
9) $\Omega' = \text{int}(\text{CH}X)$ is the unique minimal, non-empty, properly convex set preserved by $\Gamma$.
10) $X$ is dense in $\text{Fr } (\Omega) \setminus \mathcal{F}$.
11) $\Omega'$ does not contain a PET.
12) The dual manifold $M^*$ has the same properties.

Proof. Theorem (3.3) implies (1), (2), (3) and (11). Let $V = \text{cl}(\tilde{C}) \cap \text{Fr }\Omega$ be the end flat corresponding to the component $\tilde{C} \subset \pi^{-1}(C)$ for a generalized cusp $C \subset M$. If $\gamma \in \Gamma$ stabilizes $V$, then $\gamma(\tilde{C}) = \tilde{C}$, so $\gamma$ is an endgroup of $C$. This proves (4).

If $\Gamma_0 \subset \Gamma$ is a parabolic subgroup, then by [14](4.7) $\Gamma_0$ preserves a hyperplane $H$ and a point $p \in H \cap \text{Fr }\Omega$. The only points in $\text{Fr }\Omega$ that are fixed by a parabolic are points in the end flat. Hence the end flat for every element of $\Gamma_0$ is the same one. Thus $\Gamma_0$ is conjugate into an endgroup, which proves (5). The classification of generalized cusps in [4], with (5) implies (6).

Suppose $\gamma \in \Gamma$ is hyperbolic. The attracting and repelling sets $S_{\pm} \subset \text{Fr }\Omega$ of $\gamma$ are flat. If one of them is not a single point, then it is contained in a flat, and thus an end flat. But this implies $\gamma$ preserves the end flat and is therefore in an endgroup. Thus, if $\gamma$ is not in an end group, then the proof of [14](2.8) now shows that $\gamma$ is strongly hyperbolic, which proves (7).

The set $X$ is not empty because there is $\gamma \in \pi_1M$ that is not conjugate into an end group, so by (7) $\gamma$ is strictly hyperbolic. Thus if $\gamma(x) = x$ then $x$ is an attracting or repelling fixed point of $\gamma$ and is a strictly convex point. Moreover $x$ is a smooth point because the fixed points of the action of $\gamma$ on the dual domain are strictly convex points. This proves (8).

Since $X$ is preserved by $\Gamma$ it follows that $\Omega'$ is $\Gamma$-invariant, so $\dim \Omega' = n$ because $\rho$ is irreducible. Clearly $\Omega' \subset \Omega$, so $\Omega'$ is properly convex, which proves (9).

Every point $p \in \text{Fr }\Omega'$ is in the limit of a sequence $n$-simplices with vertices in $X$. If this limit is not a single point then it is a flat that contains $p$. Hence $p$ is in a maximal flat. Otherwise $p$ is a limit of points in $X$, which proves (10). Finally, (3.4) implies (12). □
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