We discuss the ten dimensional black holes made of D0-branes in the regime where the effective coupling is large, and yet the 11D geometry is unimportant. We suggest that these black holes can be interpreted as excitations over the threshold bound state. Thus, the entropy formula for the former is used to predict a scaling region of the wave function of the latter. The horizon radius and the mass gap predicted in this picture agree with the formulas derived from the classical geometry.
The 10 dimensional black hole consisting of D0-branes in the regime where the classical 10 dimensional geometry defined by the string metric is reliable has been hard to understand. This difficulty is caused partly by the large effective coupling constant in the D0-brane quantum mechanics, and partly by the fact that unlike other higher dimensional D-branes, D0-branes are highly dynamical due to the Heisenberg uncertainty relation.

Our basic observation in this paper is that the black hole in the regime of interest can be regarded as a small excitation over the large N threshold bound state. Thus, it is hopeless to make a first principle calculation concerning the black hole without understanding the bound state first. What we shall do here is to postulate that there is a scaling regime where the one-particle wave function obeys a power law, matching the entropy formulas fixes the exponent. As a consistency check, we show that the correct formula for the horizon radius emerges naturally in this picture. We also estimate the mass gap in the black hole background.

The near horizon geometry of a D0-brane black hole is given by \[ ds^2 = -f dt^2 + f^{-1} dU^2 + \sqrt{\lambda} U^{-3/2} d\Omega_8^2, \]
\[ e^\phi = g_s \left( \frac{\lambda}{U} \right)^{3/4}, \]
where \[ f = \frac{U^{7/2}}{\sqrt{\lambda}} \left( 1 - \frac{U_0}{U} \right), \quad \lambda = 60 \pi^3 g_s N. \]

We have set \( \alpha' = 1 \). It was argued by Polchinski that there is a duality between M theory defined on the background geometry of the threshold bound state and quantum mechanics of D0-branes in the large N limit. For related discussions, see [3].

This duality reduces to that between the IIA theory and D0-brane quantum mechanics when the curvature is smaller than the string scale, and the dilaton is sufficiently small. The conditions are just \[ 1 \ll \sqrt{\lambda} U_0^{-3/2} \ll N^{2/7}. \]

The Hawking temperature is \[ T = \frac{1}{4\pi} \partial_U g_{00}(U = U_0) = \frac{7}{4\pi} \lambda^{-1/2} U_0^{5/2}. \]

In the M theory unit, we have \[ r_0 = U_0 \alpha' = N^{1/5} l_p^{9/5} (\beta R)^{-2/5}, \]

1
where we ignored a numerical factor. Define the effective coupling by \( \lambda_{\text{eff}} = N R^3 / l_p^6 \), the entropy of the black hole is given by

\[
S = N^2 \lambda_{\text{eff}}^{-3/5},
\]

again a numerical factor is dropped. Conditions (3) are rewritten in terms of the effective coupling

\[
1 \ll \lambda_{\text{eff}} \ll N^{10/7}.
\]

Since the effective coupling is large, it is hopeless to describe this black hole using the perturbative quantum mechanics of D0-branes. Also notice that there is an upper bound on the effective coupling, this is to avoid the M theory region where the effective 11th dimension becomes larger than the Planck length. Thus, the 10 dimensional black hole we are aiming to describe in this paper is different from the matrix black holes discussed in [4,5]. This upper bound also indicates that for a given temperature, we do not have to hold the 't Hooft coupling fixed. Indeed, the natural limit in matrix theory is to hold the Yang-Mills coupling fixed. It is unclear whether a different phase should occur when one crosses the upper bound for \( \lambda_{\text{eff}} \). The entropy of the black hole is of order \( N^{8/7} \) at this bound, this is still larger than \( N \) where a localization transition occurs, for a systematically discussion on the phase diagram, see [6]. We shall here adopt a conservative viewpoint, and obey the upper bound.

As in the background of any black hole in any dimension, there is a mass gap. The mass gap is determined by the Hawking temperature. A WKB solution of the dilaton equation of motion clearly indicates that indeed this mass gap is order of the temperature [7]. We shall not quote the precise formula here, since we will not try to calculate the exact coefficient. Again, this behavior is hard to understand in terms of the perturbative nonabelian quantum mechanics, which is valid only in the high temperature regime. A rather straightforward calculation shows that the one-loop mass gap \( m^2 \) is actually negative, and is proportional to \( \lambda_{\text{eff}} T^2 \), see the appendix.

Recently it has been argued that the size of the large N threshold bound state grows with N as \( N^{1/3} \) [2,8]. The bound state is a highly nonabelian object. It is impossible to separate the D0-brane degrees of freedom from the off-diagonal elements. It is still reasonable to talk about the density of D0-branes. The situation is similar to the matrix membrane [9], where the object is also highly nonabelian, nevertheless so long if N is
sufficiently large, it is meaningful to define D0-brane or the longitudinal momentum density along the membrane.

We shall not repeat the argument which leads to the estimate of the size of the bound state $L \sim N^{1/3} l_p$. For our purpose, it is sufficient to know the order of magnitude of a few relevant quantities. The size can be measured by the quantity $1/N \langle \text{tr} X^2 \rangle = L^2$, $X$ is one of the nine bosonic matrices. The typical frequency of an off-diagonal element is of order

$$\omega = N^{1/3} R l_p^{-2},$$

thus the “zero-point energy” of an off-diagonal state is of this order. This huge amount of energy is supposed to be canceled by contribution from the fermionic fluctuations. If we assume that the black hole is a “small excitation” over the threshold bound state, the surplus energy must come from excitations of the off-diagonal elements. This is because the energy is of order $N^2$ when $\lambda_{\text{eff}}$ is held fixed. The typical energy carried by such elementary quanta is just $T$. This is much smaller than $\omega$ since $\omega/T = \lambda_{\text{eff}}^{1/3} \gg 1$. Indeed this is a small fluctuation over the bound state configuration as long as the bosonic degrees are concerned. So the wave function will not be significantly changed by the fluctuation. This is our basic observation.

It is easy to see that the condition $\lambda_{\text{eff}} \gg 1$ is also equivalent to the condition $r_0 \ll L$. Physically, one would imagine that it is sufficient to know the shape of the wave function in this region in order to calculate the thermodynamic quantities of the black hole. This will turn out to be true. For our purpose, it it enough to know the density of D0-branes in this region, $\rho(r) \sim |\psi(r)|^2$. We shall adopt the following scaling ansatz

$$\rho(r) = C N L^{-9} \left( \frac{r}{L} \right)^\alpha,$$

where $\alpha$ is to be determined, $C$ is a numeric constant which we will drop henceforth. We used rotational invariance in this ansatz. Obviously, $L$ must be the only relevant scale in this problem. Beyond $L$, there may be another scaling region which can not be probed by black hole. This is the perturbative region in quantum mechanics, however.

The above ansatz is valid only when we smear over scales much larger than the average spacing between two neighboring D0-branes. As we have emphasized, it is difficult to define such spacing with a highly nonabelian configuration. When we make an observation only in one direction, say $X$, it makes sense to define such a spacing. It is $L/N \sim N^{-2/3} l_p$, a rather small quantity in the large $N$ limit. The high density in one dimension is caused by
the projection. A naive definition of the true spacing is $a \sim (L^9/N)^{1/9} \sim N^{2/9}l_p$. This is a rather large quantity. Lacking a better definition, we shall use this naive one here.

The frequency for a diagonal element stretched between a D0-brane at point $X$ and another D0-brane at point $Y$ is $R|X - Y|l_p^{-3}$, thus the Boltzmann weight is a function

$$f(\beta R|X - Y|l_p^{-3}).$$

(10)

We shall not specify this function. For a bosonic degree of freedom, $f(x) = (e^x - 1)^{-1}$, and for a fermionic degree of freedom, it is $f(x) = (e^x + 1)^{-1}$. To include all possible states, $f$ can be more complicated. (There is no chemical potential, since individual elementary quanta associated to an off-diagonal element are not conserved.) The contribution of these states to the free energy is another function $g(x)$. For a boson, it is $\ln(1 - e^{-x})$ and for a fermion, it is $\ln(1 + e^{-x})$. The total free energy is

$$\beta F = \int \rho(X) \rho(Y) d^9X d^9Y g(\beta R|X - Y|l_p^{-3}).$$

(11)

Rescaling $X \rightarrow L X$ and using the ansatz (9) we obtain

$$\beta F = N^2 \int d^9X d^9Y |X|^\alpha |Y|^\alpha g(\lambda_{eff}^{1/3}|X - Y|).$$

(12)

The entropy can be obtained using the standard formula $S = \beta E - \beta F$. It is obvious that the entropy is a function of $N^2$ and $\lambda_{eff}$ only, just like $\beta F$. This result justifies our observation that indeed the black hole is a thermal excitation over the threshold bound state.

To obtain the right dependence on $\lambda_{eff}$, let us do a further rescaling $X \rightarrow \lambda_{eff}^{-1/3} X$. The free energy reads

$$\beta F = N^2 \lambda_{eff}^{-2(\alpha+18)/3} \int d^9X d^9Y |X|^\alpha |Y|^\alpha g(|X - Y|).$$

(13)

Now the integral is a pure number. Matching onto the scaling in (9) determines the exponent

$$\alpha = -\frac{81}{10}.$$

(14)

This is our main result in this paper. With this exponent, it is obvious that the integral $\int d^9X \rho(X)$ is convergent at the origin, and divergent at large $X$. This is just fine, since our ansatz (9) is supposed to be valid only for $|X| < L$. We shall see shortly that this restriction agrees with the black hole physics as governed by formula (13). Beyond the size
L, we enter into the perturbative region of quantum mechanics. For a threshold bound state, the decay of the wave function at sufficiently large distance also obeys a power law. The exponent in the asymptotic region must be smaller than $-9$.

We still have to worry about the convergence of the integral in (13). The function $g(x)$ damps fast for large $x$, peaks at $x = 0$. Integrating over $Y$ first, we shall obtain another factor $|X|^\alpha$ for large $|X|$, thus integration over $X$ is convergent for large $|X|$. For small $X$, integrating over $Y$ first will give rise to a order $O(1)$ number, so integration over $X$ near the origin is also convergent. We conclude that the integral in (13) is a finite pure number.

The contribution to the integral in (13) mainly comes from the region $|X| \sim L^{\lambda - 1/3}_{\text{eff}}$, in terms of the coordinates before rescaling. This size is smaller than $L$, but independent of $N$. Thus it can not be identified with the horizon radius $\frac{\alpha}{N}$. There may be a few different definitions of the horizon radius. One simple definition is the subtracted quantity $1/N \langle \text{tr} X^2 \rangle$. By subtraction we mean that the physical contribution to the average comes only from the thermal fluctuations. This is reasonable, since by scattering an object against a threshold bound state, the wave function size should be not probed, as a basic assumption in a holographic theory. Another more precise definition is proposed in [10]. The proposal is that the horizon radius is where a tachyonic mode develops between a probing D0-brane and the black hole. The authors of [10] argue that the horizon radius is determined by the first negative eigen-value of the following matrix

$$\langle M^2_{ij} \rangle = \langle \delta_{ij} X^2 + 2[X_i, X_j] \rangle.$$  

(15)

It is expected that the negative eigen-value will be of order $1/N \langle \text{tr} X^2 \rangle$. Since we are not trying to compute the exact coefficient, we will not endeavor to use the above definition.

The amplitude $X^2$ of an off-diagonal element stretched between point $X$ and point $Y$ is $n l^3_p |X - Y|^{-1}$, when it is in its $n$-th harmonic state. The thermal average is than

$$\frac{l^3_p}{|X - Y|} f(\beta R |X - Y| l^3_p^{-3}),$$  

(16)

where $f$ is the Boltzmann factor. We have the following thermal average

$$\frac{1}{N} \langle \text{tr} X^2 \rangle_\beta = \frac{l^3_p}{N} \int d^9 X d^9 Y \rho(X) \rho(Y) |X - Y|^{-1} f(\beta R |X - Y| l^3_p^{-3}).$$  

(17)

Performing the same rescaling as before, we obtain

$$\frac{1}{N} \langle \text{tr} X^2 \rangle_\beta = \frac{N l^3_p}{L} \lambda^{-4/15}_{\text{eff}} \int d^9 X d^9 Y |X|^{-81/10} |Y|^{-81/10} |X - Y|^{-1} f(|X - Y|).$$  

(18)
Now, the integral is also convergent, since the superficial pole in $|X - Y|^{-1}$ as well as in $f(|X - Y|)$ is not severe in 9 dimensions. Thus this integral is a pure number. Substituting what we know about $L$ and $\lambda_{eff}$, the above quantity is of order

$$N^{2/5} l_p^{18/5} (\beta R)^{-4/5}.$$  \hfill (19)

This is precisely $r_0^2$, as in \((\mathbb{E})\). $r_0$ can be also expressed as $L\lambda_{eff}^{-2/15}$. There are three scales appearing in our problem

$$L \gg L\lambda_{eff}^{-2/15} \gg L\lambda_{eff}^{-1/3}.$$  \hfill (20)

We are fortunate that it is not the last scale being identified with the horizon radius. The correct horizon radius we obtained is a strong consistency check of our picture.

The next thing we wish to do is to identify the mass gap. Our definition of the mass gap is the quantity $m^2$ appearing in the following effective action

$$S = \frac{1}{R} \int dt \text{tr} \left( (\dot{X})^2 + m^2 X^2 + \ldots \right),$$  \hfill (21)

where we use the Euclidean time, the dots denote high order terms. The quadratic term $\text{tr}X^2$ comes from thermal fluctuations. We expect the answer be $m^2 \sim T^2$. As explained in the appendix, this is going to be a nonperturbative result.

It is rather easy to see how to compute $m^2$ in our picture. It is just the average mass square of stretched strings. From the quadratic term $Rl_p^{-6}\text{tr}[X_i, X_j]^4$ in the matrix action, we find the right quantity to be averaged over is $R^2 l_p^{-6}|X - Y|^2$. An additional factor $R$ comes from our definition in \((21)\): Obviously, up to a numerical factor, one can replace $|X - Y|^2$ by $(L\lambda_{eff}^{-1/3})^2$. Therefore the mass gap is

$$m^2 \sim R^2 l_p^{-6}(L\lambda_{eff}^{-1/3})^2 = T^2.$$  \hfill (22)

We see that all factors depending on $R$, $l_p$ and $N$ cancel. In a way, we should not be surprised by this result.

It is possible to follow the strategy of \([11]\) to compute the Hawking radiation rate. It may also be possible to use the exact coefficient in the entropy formula to fix the coefficient in the formula of D0-brane density, then coefficients in other physical quantities may be computed within our picture. We are only beginning to investigate the fascinating subject of the large $N$ D0-brane threshold bound state, and its role in the duality between the large $N$ D0-brane quantum mechanics and M theory. We have only performed some qualitative analysis. Hard work remains to be done.
I understand that D. Kabat and G. Lifschytz have been investigating the ten-dimensional black holes along a different line.

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Appendix

There are a few papers discussing D0-brane statistical mechanics in the perturbative regime [12]. To our knowledge, the one-loop mass gap has never been computed. We will give a formula at the one-loop level here.

The Euclidean action of D0-brane quantum mechanics is

\[
S_E = \int_0^\beta d\tau \text{tr} \left( \frac{1}{2R} (D_\tau X)^2 - \frac{R}{16\pi^2 l_p^6} [X_i, X_j]^2 + \theta D_\tau \theta + \frac{R}{2\pi l_p^3} \theta \gamma_i [X_i, \theta] \right). \tag{23}
\]

Performing rescaling \( \tau \rightarrow \beta \tau, \ X_i \rightarrow \sqrt{\beta R} X_i \), we reach the following action

\[
S_E = \int_0^1 d\tau \text{tr} \left( \frac{1}{2} (D_\tau X)^2 - \frac{g^2}{4} [X_i, X_j]^2 + \theta D_\tau \theta + g \theta \gamma_i [X_i, \theta] \right), \tag{24}
\]

where the effective coupling is \( g^2 = (\beta R)^3 l_p^{-6} (4\pi^2)^{-1} \).

The mass gap is the coefficient appearing in the effective quadratic potential. One way to compute the static potential is to use the background field method. The first step is to fix the gauge. In general, the gauge field \( A \) can be put into a diagonal form, each diagonal element represents a U(1) holonomy. We are only interested in the mass gap, so we set \( A = 0 \). For fermions, since the action is quadratic, one can even compute the exact static potential coming from this part. The simplest way to compute it is to start with the Hamiltonian formalism. For a given static background \( X_i \), the Hamiltonian is simply

\[
H_f = \frac{g}{4} \text{tr} (\theta \gamma_i [X_i, \theta]) = \frac{1}{2} \theta^a_{\alpha} N_{a\alpha, b\beta} \theta^b_{\beta}, \tag{25}
\]

where \( a, b \) are indices of the su(N) Lie algebra, and \( \alpha, \beta \) are spinor indices. We have recaled \( \theta \)’s such that \( \{\theta^a_{\alpha}, \theta^b_{\beta}\} = 2\delta_{ab}\delta_{\alpha\beta} \). The \( 16(N^2 - 1) \times 16(N^2 - 1) \) matrix \( N \) is

\[
N_{a\alpha, b\beta} = \frac{1}{2} i g \gamma^i_{\alpha} X^c_i f_{abc}. \tag{26}
\]
It is easy to check that $\mathcal{N}$ is an antisymmetric matrix.

Now $\theta^a_x$ form a $16(N^2 - 1)$ dimensional Clifford algebra. The fermionic Hilbert space is a spinor representation of this algebra. The static potential is encoded in $\text{tr} e^{-H_f}$. Since $\mathcal{N}$ is an even rank antisymmetric matrix, by an orthogonal rotation, it can be put into the Jordan form. Once this done, we see that

$$\text{tr} e^{-H_f} = (\det(e^N + e^{-N}))^{1/2}.$$  \hspace{1cm} (27)

The determinant is invariant under the orthogonal rotation, thus the above formula is valid for a general $\mathcal{N}$. For a small $\mathcal{N}$, one can perform a perturbative expansion. The first term in the effective action is quadratic in $\mathcal{N}$. Indeed we have

$$\text{tr} e^{-H_f} = e^{1/4\text{tr}N^2+...},$$

where the trace is taken over $(a, \alpha)$. In the end, we have

$$\text{tr} e^{-H_f} = e^N g^2 \text{tr} X^2+....$$  \hspace{1cm} (28)

Apparently, this expansion is valid when the coupling $\lambda_{eff} = Ng^2$ is small. We see that the contribution of fermionic fluctuations to the mass gap $m^2$ is negative.

It is a little more complicated to compute the contributions from bosonic fluctuations. We separate bosonic matrices into a sum of the background and the fluctuation $X_i + Y_i$. The one-loop contribution is determined by the quadratic part in the fluctuation $Y_i$ in the action

$$S_b = \frac{1}{2}(\partial_\tau Y_i^a)^2 + \frac{1}{2}Y_i^a \mathcal{M}_{ai,bj}Y_j^b,$$  \hspace{1cm} (29)

with

$$\mathcal{M}_{ai,bj} = g^2[-i[X_i, X_j]^c f_{abc} + X_k^c X_k^d f_{ace} f_{bde} \delta_{ij} - X_j^c X_i^d f_{ace} f_{bde}].$$  \hspace{1cm} (30)

The path integral can be performed in two steps. First one converts it into the form $\text{tr} e^{-H_b}$. This gives the result

$$\text{tr} e^{-H_b} = \left(\det[\sinh(\sqrt{M}/2)]\right)^{-1}.$$  \hspace{1cm} (31)

This result is singular when $M = 0$. The problem is caused by zero modes. We have already included the zero modes in the background fields, the contribution of this part must be subtracted. The subtracted result is then

$$\frac{\det[\sqrt{M}/2]}{\det[\sinh(\sqrt{M}/2)]}.$$  \hspace{1cm} (32)
For small $\mathcal{M}$, the contribution to the effective action from bosonic fluctuations is then

$$\frac{-1}{24} \text{tr}\mathcal{M} = -\frac{1}{3}Ng^2\text{tr}X^2 + \ldots$$

(33)

Although this contribution to the mass square is positive, its magnitude is smaller than that from fermionic fluctuations. The overall mass gap $m^2$ is negative. If we rescale back to the original time and fields $X$, we see that $m^2 \sim Ng^2T^2 \sim \lambda_{eff}T^2$.

It is not surprising to have a negative mass square at a finite temperature. The presence of temperature breaks supersymmetry, thus the cancellation between bosonic fluctuations and fermionic fluctuations is spoiled. The phase space of bosons is overwhelmed by that of fermions, that at short distance there is a repulsive force. This agrees with the result for two D0-branes in the last paper of [12], where integration over the holonomy of the gauge field is also taken into account. We expect that high order terms in the static potential takes over for large $X$, thus the net force will be attractive.

As we have seen in the main body of this paper, the mass gap $m^2$ is positive in the strong coupling regime, and is independent of $\lambda_{eff}$. When one tunes down $\lambda_{eff}$, there must be a point where the mass gap vanishes. One might take this point as a phase transition point. This phase transition is generally predicted in [13].
References

[1] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, hep-th/9802042.
[2] J. Polchinski, unpublished.
[3] V. Balasubramanian, R. Gopakumar and F. Larsen, hep-th/9712077; A. Jevicki and T. Yoneya, hep-th/9805069; S. Hyun, hep-th/9802026; S. Hyun and Y. Kiem, hep-th/9805136.
[4] T. Banks, W. Fischler, I. R. Klebanov and L. Susskind, hep-th/9709091; hep-th/9711005; I. R. Klebanov and L. Susskind, hep-th/9709108; G. Horowitz and E. Martinec, hep-th/9710217.
[5] M. Li, hep-th/9710226; M. Li and E. Martinec, hep-th/9801070.
[6] M. Li, E. Martinec and V. Sahakian, hep-th/9809061; E. Martinec and V. Sahakian, hep-th/9810224; hep-th/9901135.
[7] J. A. Minahan, hep-th/9811156.
[8] L. Susskind, hep-th/9901070.
[9] T. Banks, W. Fischler, S. Shenker and L. Susskind, hep-th/9610043.
[10] D. Kabat and G. Lifschytz, hep-th/9806214.
[11] T. Banks, W. Fischler and I. Klebanov, hep-th/9712236.
[12] N. Ohta, J-G. Zhou, hep-th/9801023; M. L. Meana, M. A. R. Osorio, J. P. Penalba, hep-th/9803058; J. Ambjorn, Y. M. Makeenko and G. W. Semenoff, hep-th/9810170.
[13] Y.-H. Gao and M. Li, hep-th/9810053.