\textbf{\(\eta\)-invariant and flat vector bundles}

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\textit{Dedicated to the memory of Professor Shiing-shen Chern}

\begin{abstract}
We present an alternate definition of the mod \(Z\) component of the Atiyah-Patodi-Singer \(\eta\) invariant associated to (not necessary unitary) flat vector bundles, which identifies explicitly its real and imaginary parts. This is done by combining a deformation of flat connections introduced in a previous paper with the analytic continuation procedure appearing in the original article of Atiyah, Patodi and Singer.
\end{abstract}

\textbf{Keywords} flat vector bundle, \(\eta\)-invariant, \(\rho\)-invariant

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\section{Introduction}

Let \(M\) be an odd dimensional oriented closed spin manifold carrying a Riemannian metric \(g^TM\). Let \(S(TM)\) be the associated Hermitian bundle of spinors. Let \(E\) be a Hermitian vector bundle over \(M\) carrying a unitary connection \(\nabla^E\). Moreover, let \(F\) be a Hermitian vector bundle over \(M\) carrying a unitary flat connection \(\nabla^F\). Let

\begin{equation}
D^{E\otimes F} : \Gamma(S(TM) \otimes E \otimes F) \longrightarrow \Gamma(S(TM) \otimes E \otimes F)
\end{equation}

denote the corresponding (twisted) Dirac operator, which is formally self-adjoint (cf. [BGV]).

For any \(s \in \mathbb{C}\) with \(\text{Re}(s) \gg 0\), following [APS1], set

\begin{equation}
\eta(D^{E\otimes F}, s) = \sum_{\lambda \in \text{Spec}(D^{E\otimes F}) \setminus \{0\}} \frac{\text{Sgn}(\lambda)}{|\lambda|^s}.
\end{equation}

Then by [APS1], one knows that \(\eta(D^{E\otimes F}, s)\) is a holomorphic function in \(s\) when \(\text{Re}(s) > \frac{\dim M}{2}\). Moreover, it extends to a meromorphic function over \(\mathbb{C}\), which is holomorphic at \(s = 0\). The \(\eta\) invariant of \(D^{E\otimes F}\), in the sense of Atiyah-Patodi-Singer [APS1], is defined by

\begin{equation}
\eta(D^{E\otimes F}) = \eta(D^{E\otimes F}, 0),
\end{equation}

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while the corresponding reduced $\eta$ invariant is defined and denoted by

$$\tilde{\eta}(D\otimes F) = \frac{\dim(\ker D\otimes F) + \eta(D\otimes F)}{2}. \quad (1.4)$$

The $\eta$ and reduced $\eta$ invariants play an important role in the Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary $[APS1]$.

In $[APS2]$ and $[APS3]$, it is shown that the following quantity

$$\rho(D\otimes F) := \tilde{\eta}(D\otimes F) - \mathrm{rk}(F)\tilde{\eta}(D) \mod \mathbb{Z} \quad (1.5)$$

does not depend on the choice of $g^{TM}$ as well as the metrics and (Hermitian) connections on $E$. Also, a Riemann-Roch theorem is proved in $[APS3]$ (5.3)], which gives a $K$-theoretic interpretation of the analytically defined invariant $\rho(D\otimes F) \in \mathbb{R}/\mathbb{Z}$. Moreover, it is pointed out in $[APS3]$ Page 89, Remark (1)] that the above mentioned $K$-theoretic interpretation applies also to the case where $F$ is a non-unitary flat vector bundle, while on $[APS3]$ Page 93 it shows how one can define the reduced $\eta$-invariant in case $F$ is non-unitary, by working on non-self-adjoint elliptic operators, and then extend the Riemann-Roch result $[APS3]$ (5.3)] to an identity in $\mathbb{C}/\mathbb{Z}$ (instead of $\mathbb{R}/\mathbb{Z}$). The idea of analytic continuation plays a key role in obtaining this Riemann-Roch result, as well as its non-unitary extension.

In this paper, we show that by using the idea of analytic continuation, one can construct the $\mathbb{C}/\mathbb{Z}$ component of $\tilde{\eta}(D\otimes F)$ directly, without passing to analysis of non-self-adjoint operators, in case where $F$ is a non-unitary flat vector bundle. Consequently, this leads to a direct construction of $\rho(D\otimes F)$ in this case. We will use a deformation introduced in $[MZ]$ for flat connections in our construction.

In the next section, we will first recall the above mentioned deformation from $[MZ]$ and then give our construction of $\tilde{\eta}(D\otimes F) \mod \mathbb{Z}$ and $\rho(D\otimes F) \in \mathbb{C}/\mathbb{Z}$ in the case where $F$ is a non-unitary flat vector bundle.

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2 The $\eta$ and $\rho$ invariants associated to non-unitary flat vector bundles

This section is organized as follows. In Section 2.1 we construct certain secondary characteristic forms and classes associated to non-unitary flat vector bundles. In Section 2.2 we present our construction of the $\mod \mathbb{Z}$ component of the reduced $\eta$-invariant, as well as the $\rho$-invariant, associated to non-unitary flat vector bundles. Finally, we include some further remarks in Section 2.3.

2.1 Chern-Simons classes and flat vector bundles

We fix a square root of $\sqrt{-1}$ and let $\varphi : \Lambda(T^*M) \to \Lambda(T^*M)$ be the homomorphism defined by $\varphi : \omega \in \Lambda^i(T^*M) \to (2\pi\sqrt{-1})^{-i/2}\omega$. The formulas in what
follows will not depend on the choice of the square root of $\sqrt{-1}$.

If $W$ is a complex vector bundles over $M$ and $\nabla_0^W$, $\nabla_1^W$ are two connections on $W$. Let $W_t$, $0 \leq t \leq 1$, be a smooth path of connections on $W$ connecting $\nabla_0^W$ and $\nabla_1^W$. We define Chern-Simons form $CS(\nabla_0^W, \nabla_1^W)$ to be the differential form given by

\[ CS(\nabla_0^W, \nabla_1^W) = -\left(\frac{1}{2\pi \sqrt{-1}}\right)^{\frac{1}{2}} \varphi \int_0^1 \text{Tr} \left[ \frac{\partial \nabla_t^W}{\partial t} \exp\left(-((\nabla_t^W)^2)\right) \right] dt. \]

Then (cf. [Z1, Chapter 1])

\[ dCS(\nabla_0^W, \nabla_1^W) = \text{ch}(W, \nabla_1^W) - \text{ch}(W, \nabla_0^W). \]

Moreover, it is well-known that up to exact forms, $CS(\nabla_0^W, \nabla_1^W)$ does not depend on the path of connections on $W$ connecting $\nabla_0^W$ and $\nabla_1^W$.

Let $(F, \nabla^F)$ be a flat vector bundle carrying the flat connection $\nabla^F$. Let $g^F$ be a Hermitian metric on $F$. We do not assume that $\nabla^F$ preserves $g^F$. Let $(\nabla^F)^*\omega$ be the adjoint connection of $\nabla^F$ with respect to $g^F$.

From [BZ, (4.1), (4.2)] and [BL, §1(g)], one has

\[ (\nabla^F)^*\omega = \nabla^F + \frac{1}{2} \omega(F, g^F) \]

with

\[ \omega(F, g^F) = (g^F)^{-1} (\nabla^F g^F). \]

Then

\[ \nabla^{F,e} = \nabla^F + \frac{1}{2} \omega(F, g^F) \]

is a Hermitian connection on $(F, g^F)$ (cf. [BL, (1.33)]) and [BZ (4.3)]).

Following [MZ (2.47)], for any $r \in \mathbb{C}$, set

\[ \nabla^{F,e, (r)} = \nabla^{F,e} + \frac{\sqrt{-1}}{2} \omega(F, g^F). \]

Then for any $r \in \mathbb{R}$, $\nabla^{F,e, (r)}$ is a Hermitian connection on $(F, g^F)$.

On the other hand, following [BL (0.2)], for any integer $j \geq 0$, let $c_{2j+1}(F, g^F)$ be the Chern form defined by

\[ c_{2j+1}(F, g^F) = (2\pi \sqrt{-1})^{-j} \frac{2^{-(2j+1)}}{2} \text{Tr} \left[ \omega^{2j+1}(F, g^F) \right]. \]

Then $c_{2j+1}(F, g^F)$ is a closed form on $M$. Let $c_{2j+1}(F)$ be the associated cohomology class in $H^{2j+1}(M, \mathbb{R})$, which does not depend on the choice of $g^F$.

For any $j \geq 0$ and $r \in \mathbb{R}$, let $a_j(r) \in \mathbb{R}$ be defined as

\[ a_j(r) = \int_0^1 (1 + u^2 r^2)^j du. \]

With these notation we can now state the following result first proved in [MZ, Lemma 2.12].

**Proposition 2.1.** The following identity in $H^{\text{odd}}(M, \mathbb{R})$ holds for any $r \in \mathbb{R}$,

\[ CS\left(\nabla^{F,e}, \nabla^{F,e, (r)}\right) = -\frac{r}{2\pi} \sum_{j=0}^{+\infty} \frac{a_j(r)}{j!} c_{2j+1}(F). \]
2.2 $\eta$ and $\rho$ invariants associated to flat vector bundles

We now make the same assumptions as in the beginning of Section 1, except that we no longer assume $\nabla^F$ there is unitary.

For any $r \in \mathbb{C}$, let

$$D^{E \otimes F}(r) : \Gamma(S(TM) \otimes E \otimes F) \longrightarrow \Gamma(S(TM) \otimes E \otimes F)$$

(2.10)

denote the Dirac operator associated to the connection $\nabla^{F,e,(r)}$ on $F$. Since when $r \in \mathbb{R}$, $\nabla^{F,e,(r)}$ is Hermitian on $(F, g^F)$, $D^{E \otimes F}(r)$ is formally self-adjoint and one can define the associated reduced $\eta$-invariant as in (1.4).

By the variation formula for the reduced $\eta$-invariant (cf. [APS1] and [BF]), one gets that for any $r \in \mathbb{R}$,

$$\eta(D^{E \otimes F}(r)) - \eta(D^{E \otimes F}(0)) \equiv \int_M \hat{A}(TM) \text{ch}(E) CS(\nabla^{F,e}, \nabla^{F,e,(r)}) \mod \mathbb{Z},$$

where $\hat{A}$ and ch are standard notations for the Hirzebruch $\hat{A}$-class and Chern character respectively (cf. [Z1, Chapter 1]).

Let $D^{E \otimes F,e}$ denote the Dirac operator $D^{E \otimes F}(0)$.

From (2.9) and (2.11), one gets that for any $r \in \mathbb{R}$,

$$\eta(D^{E \otimes F}(r)) \equiv \eta(D^{E \otimes F,e}) - \frac{r}{2\pi} \int_M \hat{A}(TM) \text{ch}(E) \sum_{j=0}^{+\infty} \frac{a_j(r)}{j!} c_{2j+1}(F) \mod \mathbb{Z}.$$  

(2.12)

Recall that even though when $\text{Im}(r) \neq 0$, $D^{E \otimes F}(r)$ might not be formally self-adjoint, the $\eta$-invariant can still be defined, as outlined in [APS3, page 93].

On the other hand, from (2.5) and (2.6), one sees that

$$\nabla^F = \nabla^{F,e,(\sqrt{-1})}.$$  

(2.13)

We denote the associated Dirac operator $D^{E \otimes F}(\sqrt{-1})$ by $D^{E \otimes F}$.

We also recall that

$$\int_0^1 (1 - u^2)^j du = \frac{2^{2j}(j!)^2}{(2j + 1)!}.$$  

(2.14)

We can now state the main result of this paper as follows.

**Theorem 2.2.** Formula (2.12) holds indeed for any $r \in \mathbb{C}$. In particular, one has

$$\eta(D^{E \otimes F}) \equiv \eta(D^{E \otimes F,e}) - \frac{\sqrt{-1}}{2\pi} \int_M \hat{A}(TM) \text{ch}(E) \sum_{j=0}^{+\infty} \frac{2^{2j} j!}{(2j + 1)!} c_{2j+1}(F) \mod \mathbb{Z}.$$  

(2.15)

Equivalently,

$$\text{Re} (\eta(D^{E \otimes F})) \equiv \eta(D^{E \otimes F,e}) \mod \mathbb{Z},$$

$$\text{Im} (\eta(D^{E \otimes F})) = -\frac{1}{2\pi} \int_M \hat{A}(TM) \text{ch}(E) \sum_{j=0}^{+\infty} \frac{2^{2j} j!}{(2j + 1)!} c_{2j+1}(F).$$  

(2.16)
Proof. Clearly, the right hand side of (2.12) is a holomorphic function in \( r \in \mathbb{C} \). On the other hand, by [APS3, page 93], \( \eta(D \otimes F(r)) \mod \mathbb{Z} \) is also holomorphic in \( r \in \mathbb{C} \). By (2.12) and the uniqueness of the analytic continuation, one sees that (2.12) holds indeed for any \( r \in \mathbb{C} \). In particular, by putting together (2.12) and (2.13), one gets (2.15). Q.E.D.

Recall that when \( \nabla^F \) preserves \( g^F \), the \( \rho \)-invariant has been defined in (1.5).

Now if we no longer assume that \( \nabla^F \) preserves \( g^F \), then by Theorem 2.2, one sees that one gets the following formula of the associated (extended) \( \rho \)-invariant.

**Corollary 2.3.** The following identity holds,

\[
\rho \left( D \otimes F \right) = \eta \left( D \otimes F, e \right) - \operatorname{rk}(F) \eta \left( D \right) - \frac{\sqrt{-1}}{2\pi} \int_M \hat{A}(TM) \operatorname{ch}(E) \sum_{j=0}^{\infty} \frac{2^{2j} j!}{(2j+1)!} c_{2j+1}(F) \mod \mathbb{Z}.
\]

Equivalently,

\[
\operatorname{Re} \left( \rho \left( D \otimes F \right) \right) = \eta \left( D \otimes F, e \right) - \operatorname{rk}(F) \eta \left( D \right) \mod \mathbb{Z},
\]

\[
\operatorname{Im} \left( \rho \left( D \otimes F \right) \right) = -\frac{1}{2\pi} \int_M \hat{A}(TM) \operatorname{ch}(E) \sum_{j=0}^{\infty} \frac{2^{2j} j!}{(2j+1)!} c_{2j+1}(F).
\]

It is pointed out in [APS3] that the Riemann-Roch formula proved in [APS3, (5.3)] still holds for \( \rho(D \otimes F) \) in the case where \( \nabla^F \) does not preserve \( g^F \). One way to understand this is that the argument in the proof of [APS3 (5.3)] given in [APS3] works line by line to give a K-theoretic interpretation of \( \eta(D \otimes F, e) - \operatorname{rk}(F) \eta(D) \). By (2.17) it then gives such an interpretation for \( \rho(D \otimes F) \).

### 2.3 Further remarks

**Remark 2.4.** The argument in proving Theorem 2.2 works indeed for any twisted vector bundles \( F \), not necessarily a flat vector bundle. This gives a direct formula for the mod \( \mathbb{Z} \) part of the \( \eta \)-invariant for non-self-adjoint Dirac operators.

**Remark 2.5.** In [Z2, Theorem 2.2], a K-theoretic formula for \( D \otimes F(r) \mod \mathbb{Z} \) has been given in the \( r \in \mathbb{R} \) case. As a consequence, one gets an alternate K-theoretic formula for \( \rho(D \otimes F) \) in [Z2 (4.6)] which holds in the case where \( \nabla^F \) preserves \( g^F \). By combining the arguments in [Z2] with Theorem 2.2 proved above, one can indeed extend [Z2 Theorem 2.2] and [Z2 (4.6)] to the case where \( \nabla^F \) might not preserve \( g^F \). We leave this to the interested reader. Here we only mention that this will provide an alternate K-theoretic interpretation of \( \rho \)-invariants in the case where \( \nabla^F \) does not preserve \( g^F \).

**Remark 2.6.** We refer to [MZ] where we have employed deformation (2.6) to study and generalize certain Riemann-Roch-Grothendieck formulas due to Bismut-Lott ([BL]) and Bismut ([B]), for flat vector bundles over fibred spaces.
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