Fermion Mass Hierarchy in the Grand Unified Theory on $S_1/(Z_2 \times Z'_2)$ Orbifold

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We suggest a simple grand unified theory where the fifth dimensional coordinate is compactified on an $S_1/(Z_2 \times Z'_2)$ orbifold. This model is based on the supersymmetric flipped $SU(5) \times U(1)$ grand unified theory, which can realize not only the triplet-doublet splitting but also the natural fermion mass hierarchies. The triplet-doublet splitting is realized by $S_1/(Z_2 \times Z'_2)$ orbifolding, which also reduces the gauge group as $SU(5) \times U(1) \rightarrow SU(3)_c \times SU(2)_L \times U(1)_Z \times U(1)_X$. The suitable fermion mass hierarchies are generated by integrating out extra three sets of vector-like heavy fields which can propagate in five dimensions. The radiative corrections to can reduce the gauge group to $SU(3)_c \times SU(2)_L \times U(1)_Y$ by a simple extension of the model.

§1. Introduction

In the grand unified theory (GUT), one of the most serious problems is how to realize the mass splitting between the triplet and the doublet Higgs particles in the Higgs sector. This problem is so-called triplet-doublet (TD) splitting problem. For solving this serious problem, people have suggested various solutions, for example, the missing partner mechanism, the idea of Higgs doublets as pseudo Nambu-Goldstone bosons, the Dimopoulos-Wilczek mechanism, sliding singlet mechanism, and so on. Recently, the new idea for solving the TD splitting problem has been suggested in five dimensional $SU(5)$ GUT where the fifth dimensional coordinate is compactified on an $S_1/(Z_2 \times Z'_2)$ orbifold. In this model, only Higgs and gauge fields can propagate in five dimensions, and the TD splitting is realized by the same origin as the gauge group reduction.

In this paper, we consider the supersymmetric flipped $SU(5) \times U(1)$ GUT in five dimensions. The fifth dimensional coordinate is compactified on an $S_1/(Z_2 \times Z'_2)$ orbifold. The model we propose can realize not only TD splitting but also the natural fermion mass hierarchies. The TD splitting is realized by $S_1/(Z_2 \times Z'_2)$ orbifolding, which also reduces the gauge group as $SU(5) \times U(1) \rightarrow SU(3)_c \times SU(2)_L \times U(1)_Z \times U(1)_X$. The $U(1)_R$ symmetry protects the Higgs doublets from gaining heavy masses. The higher order operators do not destroy the TD splitting in this model. In addition to three generation chiral matter fields, we introduce extra three sets of vector-like matter fields which can propagate in five dimensions. The suitable fermion mass hierarchies are generated by integrating out extra three sets of vector-like heavy fields which can propagate in five dimensions. The radiative corrections to can reduce the gauge group to $SU(3)_c \times SU(2)_L \times U(1)_Y$ by a simple extension of the model.
hierarchies are generated by integrating out these extra vector-like heavy fields. The large (small) flavor mixings in the lepton (quark) sector are naturally explained. The proton-decay process through the dimension five operator is strongly suppressed by $U(1)_R$ symmetry, and the dominant proton-decay mode is $p \rightarrow e^+ \pi^0$ via the exchange of $X, Y$ gauge bosons with Kaluza-Klein masses. In the model we propose, the radiative corrections of the large Yukawa coupling of right-handed neutrinos can reduce the gauge group as $SU(3)_c \times SU(2)_L \times U(1)_X \rightarrow SU(3)_c \times SU(2)_L \times U(1)_Y$ by a simple extension of the model.

In section 2, we show the field contents of this model and the gauge group reduction. In section 3, we will see the mechanism of generating the fermion mass hierarchies. Section 4 gives summary and discussions.

§2. Flipped $SU(5) \times U(1)$ GUT on $S_1/(Z_2 \times Z'_2)$

We denote the five dimensional coordinate as $y$, which is compactified on an $S_1/(Z_2 \times Z'_2)$ orbifold. Under the parity transformation of $Z_2$ and $Z'_2$, which transforms $y \rightarrow -y$ and $y' \rightarrow -y'$ ($y' = y + \pi R/2$), respectively, a field $\phi(x^\mu, y)$ which can propagate in five dimensions transforms as
\[
\phi(x^\mu, y) \rightarrow \phi(x^\mu, -y) = P\phi(x^\mu, y),
\]
\[
\phi(x^\mu, y') \rightarrow \phi(x^\mu, -y') = P'\phi(x^\mu, y'),
\]
where $P$ and $P'$ are operators of $Z_2$ and $Z'_2$ transformations, respectively. Two walls at $y = 0$ ($\pi R)$ and $\pi R/2$ ($-\pi R/2$) are fixed points under $Z_2$ and $Z'_2$ transformations, respectively. The physical space can be taken to be $0 \leq y \leq \pi R/2$, since the walls at $y = \pi R$ and $-\pi R/2$ are identified with those at $y = 0$ and $\pi R/2$, respectively. On this orbifold, the field $\phi(x^\mu, y)$ is divided into
\[
\phi_{++}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2n+1\pi R}} \phi_{++}^{(2n+1)}(x^\mu) \cos \frac{(2n+1)y}{R},
\]
\[
\phi_{+-}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{+-}^{(2n+1)}(x^\mu) \cos \frac{(2n+1)y}{R},
\]
\[
\phi_{-+}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{-+}^{(2n+1)}(x^\mu) \sin \frac{(2n+1)y}{R},
\]
\[
\phi_{--}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{--}^{(2n+2)}(x^\mu) \sin \frac{(2n+2)y}{R},
\]
according to the eigenvalues $(\pm, \pm)$ of the parity $(Z_2, Z'_2)$.

Now let us see the supersymmetric flipped $SU(5) \times U(1)$ GUT, which produces the natural fermion mass hierarchies. The fifth dimensional coordinate is compactified on the $S_1/(Z_2 \times Z'_2)$ orbifold. We introduce three sets of extra vector-like

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* The five dimensional SUSY standard model compactified on an $S_1/(Z_2 \times Z'_2)$ orbifold has been constructed in Refs. [1]. There had been several works of the SUSY reduction by the compactification, for example, in Refs. [2, 3, 4]. The extensions of the discrete symmetry and the gauge symmetry are also discussed in Refs. [5, 6, 7, 8], respectively.
matter fields which can propagate in the bulk. We take the \( Z_2 \) parity operator as \( P = \text{diag}.(1,1,1,1,1) \) and the \( Z'_2 \) parity operator as \( P' = \text{diag}.(-1,-1,-1,1,1) \) acting on a \( 5 \) representation in \( SU(5) \).

At first let us show matter multiplets. The ordinal chiral matter fields are given by

\[
(10)_i = (Q_L, D_R^c, \bar{N}_R^c) \, , \\
(5 - \bar{3})_i = (U_R^c, L_L)_i \, , \\
(15)_i = (T_R^c)_i \, ,
\]

where the index number shows the charge of \( U(1) \), and the generation is denoted by \( i = 1, 2, 3 \). We assume that these chiral matter fields can not propagate in the bulk and are localized on the four dimensional wall at \( y = 0 \) (\( \pi R \)). The gauge quantum numbers after the compactification and \( U(1)_R \) charges of these fields are shown in Table I. The superpotential of the Yukawa sector on the wall \( y = 0 \) (\( \pi R \)) is given by

\[
W_Y = y_{ij}^d H_5^d 10_i 10_j + y_{ij}^u H_5^u 5_i 10_j + y_{ij}^e H_5^e 5_i 1_j + y_{ij}^\nu H_\nu 10_i \phi_j + M_{ij}^\phi \phi_i \phi_j. \tag{2.8}
\]

The field \( \phi_3 \) with the symmetric mass matrix \( M_{ij}^\phi \), being the origin of lepton number violation, is the gauge singlet matter field localized on the four dimensional wall at \( y = 0 \) (\( \pi R \)), which plays a crucial role for making neutrino masses be light as will be shown in the next section. We assume the eigenvalues of \( M_{ij}^\phi \) is much larger than the electroweak scale. All Yukawa couplings in Eq.(2.8) are assumed to be of \( O(1) \) independently of the generation index. \( Hs \) represent Higgs fields which can propagate in the five dimensions.

In addition to above three generation chiral matter fields, we introduce extra three sets of vector-like matter fields which can propagate in the bulk. As we will show later, the suitable fermion mass hierarchies are generated by integrating out these extra vector-like heavy fields. The gauge quantum numbers after the compactification, the charges of \( U(1)_R \) symmetry, parity eigenvalues of \( Z_2 \times Z'_2 \), and mass spectra at the tree level are shown in Table II, where the index \( I = 4, 5, 6 \) denotes the label of three sets of vector-like matter fields. The Yukawa interactions which mix the ordinal chiral matter fields and extra vector-like matter fields on the wall \( y = 0 \) (\( \pi R \)) are given by

\[
W_Y' = y_{ij}^d H_5^d 10_i 10_j + y_{ij}^u H_5^u 5_i 10_j + y_{ij}^e H_5^e 5_i 1_j + y_{ij}^\nu H_\nu 10_i \phi_j + y_{ij}^\nu H_\nu 10_i \phi_j + M_{ij}^\phi \phi_i \phi_j + y_{ij}^\nu H_\nu 10_i \phi_j + M_{ij}^\phi \phi_i \phi_j. \tag{2.9}
\]

The volume suppression of extra dimension suggests \( y^{\text{brane}} \ll 1 \). The extra matters can have gauge invariant vector-like mass terms,

\[
W_M = M_{11}^d 10_i 10_j + M_{11}^u 5_i 10_j + M_{11}^e 5_i 1_J + m_i 10_i 10_J + m_i 5_i 10_J + m_i 5_i 1_J + m_i 5_i 1_J \tag{2.10}
\]

on \( y = 0 \) (\( \pi R \)). For simplicity, we take \( M_{11}^d = M_{11}^u = M_{11}^e = M_{11}^\phi \) and \( m_i = m_i' = m_i'' = m_i \delta_{i(J-3)} \). \( M_I \) and \( m_i \) include the volume suppression factors of extra
dimension, and their values are assumed to be smaller than the compactification scale of $M_c(\sim 1/R)$ but much larger than the SUSY breaking scale to avoid the blow-up of the gauge coupling constants. The Kaluza-Klein zero modes of vector-like matter fields obtain supersymmetric mass terms of $M_I$ and $m_i$. The ratios of $M_I$ and $m_i$ play crucial roles for generating fermion mass hierarchies as will be seen in the next section.

Table III shows the gauge quantum numbers after the compactification, the charges of $U(1)_R$ symmetry, parity eigenvalues of $Z_2 \times Z_2'$, and mass spectra at the tree level of Higgs super-multiplets. From this table it is found that the TD splitting is realized automatically by the compactification, since the doublet (triplet) Higgs fields $H_W$ and $H_{\overline{\pi}}$ ($H_C$ and $H_{\overline{\tau}}$) are (not) containing the Kaluza-Klein zero mode. This is the great benefit of the orbifold compactification of $S_1/(Z_2 \times Z_2')$.

The superpotential of the Higgs sector on the four dimensional wall at $y = 0$ ($\pi R$) is $W_H = 0$ due to the $U(1)_R$ symmetry. We should notice that the conventional flipped $SU(5) \times U(1)$ GUT in four dimensions have the Higgs superpotential as $W_H^{4d} \simeq H_{10} H_{10} H_{5} + H_{\overline{\pi}} H_{\overline{\pi}} H_{\overline{\tau}}$, which are needed for TD splitting. However, these Higgs interactions are not required in our five-dimensional theory, since the TD splitting is already realized by the orbifolding. As for Higgs mass parameters, $\mu$ term, $\mu H_{5} H_{\overline{\tau}}$, and soft SUSY breaking mass term, $B \mu h_{5} h_{\overline{\tau}}$, where $h_{iS}$ denote the scalar components of Higgs superfields, are assumed to be generated after the supersymmetry is broken. On the other hand, $\mu H_{10} H_{\overline{\pi}}$ and $B' \mu h_{10} h_{\overline{\pi}}$ terms must be forbidden by introducing $Z_3$ $R$ symmetry unless $H_{10}$ and $H_{\overline{\pi}}$ do not take large vacuum expectation values (VEVs) which are needed for inducing the light neutrino masses.

Let us consider the supersymmetry breaking mechanism. We introduce a gauge and $U(1)_R$ singlet field $S = F_S \theta^2$ on the $y = \pm \pi R/2$ branes. The bulk gauge fields can couple to the SUSY breaking field $S$, and generate gaugino masses and the fermion soft breaking masses are induced through radiative corrections on the four dimensional wall at $y = 0(\pi R)$. This SUSY breaking scenario is so-called gaugino mediation mechanism. $\mu H_{5} H_{\overline{\tau}}$ and $B \mu h_{5} h_{\overline{\tau}}$ terms are also induced from the interactions between bulk Higgs fields and $S$.

Next let us study the gauge group reduction. The gauge group is reduced by the $Z_2$ parity as $SU(5) \times U(1) \rightarrow SU(3)_c \times SU(2)_L \times U(1)_Z \times U(1)_X$ at the compactification scale $M_c$. The VEVs of $\langle h_N \rangle$ and/or $\langle h_{\overline{\tau}} \rangle$ reduce the gauge group as $SU(3)_c \times SU(2)_L \times U(1)_Z \times U(1)_X \rightarrow SU(3)_c \times SU(2)_L \times U(1)_Y$. It is worth noting that our model do not demand the VEVs of $h_N$ and $h_{\overline{\tau}}$ to be the GUT scale contrary to the ordinary four-dimensional flipped $SU(5) \times U(1)$ GUT. Are the components of $h_N$ and $h_{\overline{\tau}}$ really take the VEVs in $h_{10}$ and $h_{\overline{\pi}}$ in our model??

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* The higher order superpotential interaction $(H_{10} H_{\overline{\pi}})^n$ $(n \geq 2)$ is also forbidden by this symmetry.

** In the ordinary four-dimensional flipped $SU(5) \times U(1)$ theory, the VEVs of $h_{10}$ and $h_{\overline{\pi}}$ can be always identified with those of $h_N$ and $h_{\overline{\tau}}$, respectively, by the field rotation. This field redefinition is not available in our five-dimensional theory.

*** The most simple way to obtain the VEVs of $h_N$ and $h_{\overline{\tau}}$ is setting the four-dimensional
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Apparently, $h_Q$ and $h_{\overline{Q}}$ components are too heavy to take VEVs, since they have Kaluza-Klein masses. Thus, in order to realize the suitable gauge reduction, that means $h_N$ and/or $h_{\overline{N}}$ take VEVs whereas $h_D$ and/or $h_{\overline{D}}$ do not, the field contents in this model need a little extension. We show a simple example here. We introduce vector-like matter fields $5'_2 = (D', L'), \overline{5}'_2 = (\overline{D}', L'), 10''_1 = (Q'', \overline{D}'', N'')$, and $\overline{10''}_1 = (\overline{Q}'', D'', N''')$, which are localized on the four dimensional wall at $y = 0$ ($\pi R$).

Introducing an additional discrete symmetry, $P_2$, we assume only these fields possess $P_2$ odd parity. Then, they have Yukawa couplings,

$$W' = fH_{10}10''_1 + \tilde{f}H_{\overline{10}}\overline{10''}_1 + M_n5'_2 + M_m10''_1\overline{10''}_1,$$

(2.11)

where there are no mixings with other matter fields due to $P_2$ symmetry. The values of $M_n$ and $M_m$ are smaller than the compactification scale but much larger than the SUSY breaking scale to avoid the blow-up of the gauge coupling constants. We can evaluate the relevant soft SUSY breaking masses using the following renormalization group equations (RGEs):

\[
\frac{dm_{h_D}^2}{dt} = \frac{16}{3} g_3^2 M_3^2 + \frac{16}{9} g_Z^2 M_Z^2 + 4 g_X^2 M_X^2 - \text{tr}(y'^\nu y'^\nu^\dagger) m_{h_D}^2 - \text{tr}(y'^\nu y'^\dagger m_{h_D}^2 + m_{N''}^2 + m_{\overline{D}''}^2),
\]

(2.12)

\[
\frac{dm_{h_N}^2}{dt} = 4g_2^2 M_Z^2 + 4g_X^2 M_X^2 - \text{tr}(y'' y'^\dagger m_{h_N}^2) - \text{tr}(y'' m_{\overline{N}''}^2 + y'^\dagger m_{\overline{N}''}^2)
- 3 |f|^2 (m_{h_N}^2 + m_{D''}^2 + m_{\overline{D}''}^2),
\]

(2.13)

\[
\frac{dm_{h_{\overline{D}}}^2}{dt} = 4g_2^2 M_Z^2 + 4g_X^2 M_X^2 - 3 |f|^2 (m_{h_{\overline{D}}}^2 + m_{\overline{D}''}^2 + m_{\overline{D}''}^2),
\]

(2.14)

\[
\frac{d(m_{ij}^2)}{dt} = -2m_{\overline{N}''}^2 y'^\nu y'' - m_{\overline{D}''}^2 y'^\nu y'' m_{\overline{N}''}^2 - (y'^\nu y'')_{ij} m_{h_N}^2 - (y'^\nu m_{\overline{N}''}^2 y'')_{ij}
- 3(y'^\nu y'')_{ij} m_{h_{\overline{D}}}^2 - 3(y'^\nu m_{\overline{D}''}^2 y'')_{ij}.
\]

(2.15)

Here $t = -1/(4\pi^2)\ln(\mu^2)$. The above RGEs are available in the energy scale of $M_{ij} < \mu < M_c$. Here we neglect the small Yukawa couplings between matter fields and extra generations in Eq.(2.9)$^{[3]}$. $g_3$, $g_Z$, and $g_X$ are gauge coupling constants for $SU(3)_c$, $U(1)_Z$, and $U(1)_X$, respectively. The RGEs show that gauge couplings give the positive contributions whereas the Yukawa couplings give the negative contributions to the soft breaking masses toward the low energy scale. Then, in the case that the Yukawa coupling $y''$ is sufficiently large, scalar squared masses can become negative through the radiative corrections$^{[4]}$. $h_N$ and $h_{\overline{N}}$ have the $D$-flat direction, $|(h_N)| = |(h_{\overline{N}})| (\equiv v_N)$, in the SUSY limit. The soft SUSY breaking terms superpotential $W_N \sim X(H_N h_{\overline{N}} - v_N^2)$ at $y = \pm \pi R/2$ branes where there are no $SU(5) \times U(1)$ gauge transformations. Here $X$ is the gauge singlet field localized on the $y = \pm \pi R/2$ branes.

$^{[3]}$ The small couplings of $y'^\nu$ evade the large flavor changing neutral current (FCNC) induced from the destruction of sfermion mass degeneracies.
$m_{h_N}^2$ and $m_{h'_{\tilde N}}^2$ induce a small deviation from flatness, and the Higgs “effective potential” \[ V \simeq [m_{h_N}^2(v_N) + m_{h'_{\tilde N}}^2(v_N)]v_N^2 \] is given by $m^2(\mu)$ shows the value of RGE running parameter $m^2$ at energy scale $\mu$. Starting from high energy with positive soft masses squared for $m_{h_N}^2$ and $m_{h'_{\tilde N}}^2$, the RGE effects make a reversal of sign for $m_{h_N}^2 + m_{h'_{\tilde N}}^2$. This reversal of sign shows the development of the symmetry breaking minimum along the flat direction with VEV of order $v_N$. We can check the large magnitudes of $f$ and $\bar{f}$ really generate the large value of $v_N$. The value of $m_{h_N}^2 + m_{h'_{\tilde N}}^2$ becomes negative faster than that of $m_{h_D}^2 + m_{h'_{\tilde D}}^2$, since $m_{h_D}^2$ and $m_{h'_{\tilde D}}^2$ receive positive corrections from the QCD effects (see Eqs.(2.12)\~(2.14)). Therefore we can conclude that the suitable gauge reduction is really realized in this model.

As for the physical squared mass of gauge singlet field $\phi_i$, it is positive due to the large supersymmetric masses of $M_{ij}$ in Eq.(2.15), even if the large value of $y'$ causes the negative contributions to $m_{\phi}^2$ in Eq.(2.15).

Here we should estimate the corrections of higher order operators induced from the VEVs of $(h_N) \simeq (h'_{\tilde N})$. The higher order operators $\{(h_N)\langle h'_{\tilde N}\rangle/M_c\}$ induce the corrections to the mass parameters in Eq.(2.10). In order not to destroy the fermion mass hierarchies which will be shown in the next section, the relation of $(h_N)\langle h'_{\tilde N}\rangle/M_c \ll M_I, m_i$ should be satisfied. This constraint is satisfied when the magnitudes of $(h_N)$ and $(h'_{\tilde N})$ are smaller than $M_I$ and $m_i$. Under this condition, the corrections of $\mu$ parameters, which is $(\langle h_N\rangle\langle h'_{\tilde N}\rangle/M_c^2)\mu H_W H_{\tilde W}$, is negligibly small. Thus the higher order operator does not destroy the TD splitting in this model.

As for the proton stability, the $p \to e^+\pi^0$ process is dominant via $X,Y$ gauge boson exchange, which has mass of order $M_c(\sim 1/R)$ as shown in Table IV. The proton-decay process through the dimension five operator is strongly suppressed by $U(1)_R$ symmetry. It is because the colored Higgs $H_C (H_{\tilde C})$ has Kaluza-Klein mass of order $M_c$ with $H_C^2 (H_{\tilde C}^2)$, and the conjugate fields $H_C$ and $H_{\tilde C}$ do not couple directly to the quarks and leptons.

§3. Fermion Mass Hierarchy

Let us see the mechanism which produces the fermion mass hierarchies in this model. The fermion mass hierarchies in the chiral matter fields are generated by integrating out the heavy extra vector-like generations. Let us see, for example, the quark doublet $(Q_i)$ sector. The mass terms of the quark doublet sector in Eq.(2.10)

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\(^{\dagger}\) Equation (2.13) shows that the large magnitude of $y'$ might induce negative contributions to the RGEs of soft squared masses. However, the running parameter $m_{h_N}^2 + m_{h'_{\tilde N}}^2$ does not become negative since the value of $m_{h'_{\tilde N}}^2$, existing in the 3rd and 4th terms in the R.H.S. of Eq.(2.13), also become small by the RGE effects. That is why we introduce $5', 5, 10', \text{ and } 10^-$.

\(^{**}\) When we consider the gravity mediated SUSY breaking scenario, the large value of soft SUSY breaking $A$ parameters might realize radiative symmetry breaking of $SU(5) \times U(1)$ as shown in Ref. by using the $A$ term contribution appearing in Eqs.(2.13) and (2.14). However, our model cannot find this solution under the color and charge conserving sufficient condition, $A < 3m_{soft}$. 

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are given by

\[ W_M = \sum_{i=1}^{3} (m_i Q_i \overline{Q}_{i+3} + M_{i+3} Q_{i+3} \overline{Q}_i). \]  

(3.1)

All these fields represent Kaluza-Klein zero mode. Then the light eigenstate \( Q_i^l \), which is just the quark doublet at the low energy, and the heavy eigenstate \( Q_i^H \) are given by

\[ Q_i^l = \frac{M_{i+3}}{\sqrt{M_{i+3}^2 + m_i^2}} Q_i - \frac{m_i}{\sqrt{M_{i+3}^2 + m_i^2}} Q_{i+3}, \]  

(3.2)

\[ Q_i^H = \frac{M_{i+3}}{\sqrt{M_{i+3}^2 + m_i^2}} Q_i + \frac{m_i}{\sqrt{M_{i+3}^2 + m_i^2}} Q_{i+3}. \]  

(3.3)

We consider the case of \( \epsilon_1 \simeq M_4/m_1 \ll 1 \), \( \epsilon_2 \simeq M_5/m_2 < 1 \), and \( M_6/m_3 \sim 1 \), where \( \epsilon_i \equiv M_{i+3}/\sqrt{M_{i+3}^2 + m_i^2} \). Then, the mass hierarchy is generated in the mass matrix of the light eigenstate \( Q_i^l \). The fields \( \overline{U}_i \) and \( \overline{E}_i \) also receive the same effects as Eq.(3.2) in the light eigenstates, but \( \overline{D}_i, L_i, \) and \( \overline{N}_i \) do not receive these effects since their extra vector-like generations do not have zero modes as shown in Table II.

Bellow the electroweak scale, the light eigenstates mass matrices of up quark sector, down quark sector, and charged lepton sector are given by

\[
m_u^l \simeq \begin{pmatrix} \epsilon_1^2 & \epsilon_2 \epsilon_1 & \epsilon_1 \\ \epsilon_1 \epsilon_2 & \epsilon_2^2 & \epsilon_2 \\ \epsilon_1 & \epsilon_2 & 1 \end{pmatrix} v, \quad m_d^l \simeq \begin{pmatrix} \epsilon_1 & \epsilon_1 & \epsilon_1 \\ \epsilon_1 & \epsilon_2 & \epsilon_2 \\ 1 & 1 & 1 \end{pmatrix} v, \quad m_e^l \simeq \begin{pmatrix} \epsilon_1 & \epsilon_2 & 1 \\ \epsilon_1 & \epsilon_2 & 1 \\ 1 & 1 & 1 \end{pmatrix} v,
\]  

(3.4)

respectively, where \( v \equiv \langle h_W \rangle, \overline{v} \equiv \langle h_{h_W} \rangle \). Each element is understood to be multiplied by \( O(1) \) coefficient. We write the mass matrices that the left-handed fermions are to the left and the right-handed fermions are to the right. Setting the values of \( M_{i+3} \) and \( m_i \) as \( \epsilon_1 \simeq \lambda^4 \) and \( \epsilon_2 \simeq \lambda^2 \), where \( \lambda \) is the Cabbibo angle estimated as 0.2, we can obtain the suitable mass hierarchies. Moreover, the small (large) flavor mixings in the quark (lepton) sector are naturally obtained.

The neutrino mass matrix is given by

\[
m_{\nu} = \begin{pmatrix}
L_i & \overline{N}_i & \phi_i & N^c_{1(2n+1)} & N^c_{2(2n+1)} \\
0 & m^D_{\nu} & 0 & y^C \overline{h} & 0 \\
m^D_{\nu} & 0 & y^C \langle h_N \rangle & m_i & 0 \\
0 & y^C \langle h_N \rangle & M^\phi_i & y^F \langle h_{h_N} \rangle & 0 \\
y^C \overline{h} & m_i & y^F \langle h_{h_N} \rangle & (2n+1)/R & 0 \\
0 & 0 & 0 & (2n+1)/R & 0
\end{pmatrix}
\]  

(3.5)

where the neutrino Dirac mass matrix is given by

\[
m^D_{\nu} \simeq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \overline{v},
\]  

(3.6)
because the fields $L_I$, $\overline{L}_I$, $N_I$, and $\overline{N}_I$ do not have the Kaluza-Klein zero mode. Each element of $m_\nu^D$ has $O(1)$ coefficient. Neglecting the contributions from the super-heavy Kaluza-Klein masses $(2n + 1)/R$, Eq.(3.5) induces the mass matrix of three light neutrinos as

$$m_\nu^{(l)} \simeq \frac{m_\nu^D m_\nu^{DT}}{y^\nu_2 v_N^2 / M^\phi} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi^2 M^\phi \\ \pi^2 v^2 \\ y^\nu_2 v^2_N \end{pmatrix}. \quad (3.7)$$

$M^\phi$ in Eq.(3.7) means the typical scale of $M^\phi_{ij}$. We can obtain the suitable mass scale for the neutrino oscillation experiments by choosing the values of $M^\phi$ and $y^\nu_2 v_N$. The suitable choice of $O(1)$ coefficients in the mass matrix can derive the suitable flavor mixings consistent with the neutrino oscillation experiments.

Above fermion mass matrices can give the suitable mass hierarchies of quarks and leptons [19, 20, 21]. They also give us the natural explanation why the flavor mixing in the quark sector is small while the flavor mixing in the lepton sector is large. Since all components have undetermined $O(1)$ coefficients, we have to determine the explicit values of $O(1)$ coefficients in order to predict the experimentally observable quantities.

§4. Summary and Discussion

In this paper, we have proposed a supersymmetric flipped $SU(5) \times U(1)$ GUT in five dimensions where the fifth dimensional coordinate is compactified on an $S_1/(Z_2 \times Z'_2)$ orbifold. We have shown that the model can realize not only the TD splitting but also the natural fermion mass hierarchies. The TD splitting is realized by the $S_1/(Z_2 \times Z'_2)$ orbifolding, which also reduces the gauge group as $SU(5) \times U(1) \rightarrow SU(3)_c \times SU(2)_L \times U(1)_Z \times U(1)_X$. The triplet Higgs fields have the Kaluza-Klein masses of order of $1/R$ whereas the $U(1)_R$ symmetry protects the Higgs doublets from gaining heavy masses. The higher order operators do not destroy the triplet-doublet splitting in this model. The proton-decay process through the dimension five operator is strongly suppressed by the $U(1)_R$ symmetry, and the dominant proton-decay mode is $p \rightarrow e^+ \pi^0$ via the exchange of the $X, Y$ gauge bosons which have Kaluza-Klein masses.

A simple extension of the model can make the SUSY breaking squared mass of $h_N$ and $h_{\overline{N}}$ be negative, which reduces the gauge group as $SU(3)_c \times SU(2)_L \times U(1)_Z \times U(1)_X \rightarrow SU(3)_c \times SU(2)_L \times U(1)_Y$.

In addition to three generation chiral matter fields, we have introduced extra three sets of vector-like matter fields which can propagate in the bulk. The suitable fermion mass hierarchies are generated by integrating out these extra vector-like heavy fields. Moreover, the large (small) flavor mixings in the lepton (quark) sector are naturally explained.
Acknowledgment

We would like to thank Y. Kawamura, Y. Nomura, and T. Kondo for useful discussions. Research of KU is supported in part by the Japan Society for Promotion of Science under the Predoctoral Research Program. This work is supported in part by the Grant-in-Aid for Science Research, Ministry of Education, Science and Culture, Japan (No.12004276, No.12740146, No.13001292).

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Fermion Mass Hierarchy in the Grand Unified Theory on $S_1/(Z_2 \times Z'_2)$ Orbifold

Table I. The gauge quantum numbers after the compactification and $U(1)_R$ charges of the chiral matter fields confined on the wall at $y = 0$ ($\pi R$) are shown. The index $i = 1, 2, 3$ denotes the generation.

| 4d matter fields | (SU(3)$_C$, SU(2)$_L$, U(1)$_Z$, U(1)$_X$) | $U(1)_R$ |
|------------------|---------------------------------|-----------|
| $Q_i$            | $(3, 2, 1/6, 1)$                | 1         |
| $D_i$            | $(3, 1, -2/3, 1)$               | 1         |
| $\nu_i$          | $(1, 1, 1, 1)$                  | 1         |
| $U_i$            | $(3, 1, 1/3, -3)$               | 1         |
| $L_i$            | $(1, 2, -1/2, -3)$              | 1         |
| $E_i$            | $(1, 1, 0, 5)$                  | 1         |
| $\phi_i$         | $(1, 1, 0, 0)$                  | 1         |

Table II. The gauge quantum numbers after the compactification, the charges of $U(1)_R$, and mass spectra at the tree level of the three sets of vector-like extra matter fields are shown. The index $I = 4, 5, 6$ denotes the label of extra matter fields. The first column represents the corresponding extra matter fields before the compactification.
Table III. The gauge quantum numbers after the compactification, the charges of $U(1)_R$ symmetry, parity eigenvalues of $Z_2 \times Z_2'$, and mass spectra at the tree level of Higgs supermultiplets are shown. The first column represents the corresponding Higgs fields before the compactification.

| Higgs fields $(SU(5), U(1)_X)$ | Higgs fields $(SU(3)_C, SU(2)_L, U(1)_Z, U(1)_X)$ | $U(1)_R$ | $(Z_2, Z_2')$ | mass |
|--------------------------------|-----------------------------------------------|----------|----------------|------|
| $H_{10}^{c}(10, 1)$          | $H_Q^{(2n+1)}(3, 2, 1/6, 1)$                    | 0        | (+, -)         | $2n+1$ |
|                               | $H_Q^{(2n)}(\Sigma, 1, -2/3, 1), H_U^{(2n)}(1, 1, 1, 1)$ | 0        | (+, +)         | $2n+1$ |
| $H_{10}^{c}(10, -1)$         | $H_Q^{(2n+1)}(\Sigma, 2, -1/6, -1)$             | 2        | (-, +)         | $2n+1$ |
|                               | $H_Q^{(2n)}(3, 1, 2/3, -1), H_U^{(2n)}(1, 1, -1, -1)$ | 2        | (-, -)         | $2n+1$ |
| $H_{10}^{c}(10, 1)$          | $H_Q^{(2n+1)}(\Sigma, 3, 1, 2/3, -1), H_U^{(2n)}(1, 1, -1, -1)$ | 0        | (+, -)         | $2n+1$ |
|                               | $H_Q^{(2n)}(\Sigma, 3, 2, 1/6, 1), H_U^{(2n)}(1, 1, 1, 1)$ | 2        | (-, +)         | $2n+1$ |
|                               | $H_Q^{(2n+2)}(\Sigma, 1, -2/3, 1), H_U^{(2n+2)}(1, 1, 1, 1)$ | 2        | (-, -)         | $2n+1$ |
| $H_5(5, -2)$                 | $H_Q^{(2n+1)}(\Sigma, 3, 1, -1/3, -2)$            | 0        | (+, -)         | $2n+1$ |
|                               | $H_Q^{(2n)}(1, 2, 1/2, -2)$                      | 0        | (+, +)         | $2n+1$ |
| $H_5(5, 2)$                  | $H_Q^{(2n+1)}(\Sigma, 3, 1, 1/3, 2)$              | 2        | (-, +)         | $2n+1$ |
|                               | $H_Q^{(2n+2)}(1, 2, -1/2, 2)$                    | 2        | (-, -)         | $2n+1$ |
| $H_5(5, 2)$                  | $H_Q^{(2n+1)}(\Sigma, 3, 1, 1/3, 2)$              | 0        | (+, -)         | $2n+1$ |
|                               | $H_Q^{(2n)}(1, 2, -1/2, 2)$                      | 0        | (+, +)         | $2n+1$ |
| $H_5^{c}(5, -2)$             | $H_Q^{(2n+1)}(\Sigma, 3, 1, -1/3, -2)$            | 2        | (-, +)         | $2n+1$ |
|                               | $H_Q^{(2n+2)}(1, 2, 1/2, -2)$                    | 2        | (-, -)         | $2n+1$ |

Table IV. The gauge quantum numbers after the compactification, the charges of $U(1)_R$ symmetry, parity eigenvalues of $Z_2 \times Z_2'$, and mass spectra at the tree level of gauge supermultiplets are shown.

| gauge fields $(SU(3)_C, SU(2)_L, U(1)_Z, U(1)_X)$ | $U(1)_R$ | $(Z_2, Z_2')$ | mass |
|-------------------------------------------------|----------|----------------|------|
| $V^a$                                           | $(8, 1, 0, 0), (1, 3, 0, 0), (1, 1, 0, 0), (1, 1, 0, 0)$ | 0      | (+, +)         | $2n+1$ |
| $V^b$                                           | $(3, 2, -5/6, 0) (\overline{3}, 2, 5/6, 0)$ | 0      | (+, -)         | $2n+1$ |
| $\Sigma^a$                                      | $(3, 2, -5/6, 0) (\overline{3}, 2, 5/6, 0)$ | 0      | (+, -)         | $2n+1$ |
| $\Sigma^b$                                      | $(8, 1, 0, 0), (1, 3, 0, 0), (1, 1, 0, 0), (1, 1, 0, 0)$ | 0      | (+, +)         | $2n+1$ |