Normalized Hierarchical SVM

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Abstract

We present improved methods of using structured SVMs in a large-scale hierarchical classification problem, that is when labels are leaves, or sets of leaves, in a tree or a DAG. We examine the need to normalize both the regularization and the margin and show how doing so significantly improves performance, including allowing achieving state-of-the-art results where unnormalized structured SVMs do not perform better than flat models. We also describe a further extension of hierarchical SVMs that highlight the connection between hierarchical SVMs and matrix factorization models.

1. Introduction

We consider the problem of hierarchical classification. That is, a classification problem when the labels are leaves in a large hierarchy or taxonomy specifying the relationship between labels. Such hierarchies have been extensively used to improve accuracy (Mccallum et al., 1998; Silla and Freitas, 2011; Vural and Dy, 2004) in domains such as document categorization (Cai and Hofmann, 2004), web content classification (Dumais and Chen, 2000), and image annotation (Huang et al., 1998). In some problems, taking advantage of the hierarchy is essential since each individual labels (leaves in the hierarchy) might have only a few training examples associated with it.

We focus on hierarchical SVM (Cai and Hofmann, 2004), which is a structured SVM problem with the structure specified by the given hierarchy. Structured SVMs are simple compared to other hierarchical classification methods, and yield convex optimization problems with straight-forward gradients. However, as we shall see, adapting structured SVMs to large-scale hierarchical problems can be problematic and requires care. We will demonstrate that “standard” hierarchical SVM suffers from several deficiencies, mostly related to lack of normalization with respect to different path-length and different label sizes in multi-label problems, which might result in poor performance, possibly not providing any improvement over a “flat” method which ignores the hierarchy. To amend these problems, we present the Normalized Hierarchical SVM (NHSVM). The NHSVM is based on normalization weights which we set according to the hierarchy, but not based on the data. We then go one step further and learn these normalization weights discriminatively. Beyond improved performance, this results in a model that can be viewed as a constrained matrix factorization for multi-class classification, and allows us to understand the relationship between hierarchical SVMs and matrix-factorization based multi-class learning (Amit et al., 2007).

We also extend hierarchical SVMs to issues frequently encountered in practice, such as multi-label problems (each document might be labeled with several leaves) and taxonomies that are DAGs rather then trees.

We present a scalable training approach and apply our methods to large scale problems, with up to hundreds of thousands of labels and tens of millions of instances, obtaining significant improvements over standard hierarchical SVMs and state-of-the-art results on a hierarchical classification benchmark.

2. Related Work

Much research was conducted regarding hierarchical multi-class or multi-label classification. The differences with other methods lies in normalization of structure, scalability of the optimization, and utilization of the existing label
structure.

Our work is based upon hierarchical classification using SVM which is introduced in Cai and Hofmann (2004). The model extends the multi-class SVM to hierarchical structure. An extension to the multi-label case was presented by Cai and Hofmann (2007). In Rouu et al. (2006), an efficient dual optimization method for a kernel-based structural SVM and weighted decomposable losses are presented for a tree structured multi-label problem. These methods focus on dual optimization which does not scale up to our focused datasets with large instances and large number of labels. Also, the previous methods do not consider the normalization of the structures, which is important for such large structures.

For instance, we focus on the Wikipedia dataset in Large Scale Hierarchical Text Classification Competition (LSHTC)
[1]. It has 400K instances with a bag of words representation of wikipedia pages which are multi-labeled to its categories. The labels are the leaves from a DAG structure with 65K nodes. Notice that the scale of dataset is very large compared to dataset considered in previous mentioned methods. For instance, in Rouu et al. (2006) the largest dataset has 7K instances and 233 nodes. Extensions of KNN, meta-learning, and ensemble methods were popular methods in the competition.

Gopal and Yang (2013) presented a model with a multi-task objective and an efficient parallelizable optimization method for dataset with a large structure and number of instances. However, its regularization suffers the same normalization issue, and relies on the other meta learning method (Gopal and Yang, 2010) in the post-processing for high accuracy in multi-label problems.

There are alternatives to SVMs approaches (Weinberger and Chapelle, 2009; Vural and Dy, 2004; Cesa-Bianchi et al., 2006), however, the approaches are not scalable to large scale dataset with large structures.

Another direction is to learn the structure rather than utilizing given structure. Bravo et al. (2009); Blaschko et al. (2013) focus on learning a small structure from the data, which is is very different from using a known structure. A fast ranking method (Prabhu and Varma, 2014) is proposed for a large dataset. It builds a tree structure for ranking of labels. However, it does not utilize given hierarchy, and is not directly a multi-label classifier.

3. Preliminaries

Let $G$ be a tree or a directed acyclic graph (DAG) representing a label structure with $M$ nodes. Denote the set of leaves nodes in $G$ as $L$. For each $n \in [M]$, define the sets of parent, children, ancestor, and descendent nodes of $n$ as $\mathcal{P}(n)$, $\mathcal{C}(n)$, $\mathcal{A}(n)$, and $\mathcal{D}(n)$, respectively. Additionally, denote the ancestor nodes of $n$ including node $n$ as $\mathcal{A}(n) = \{n\} \cup \mathcal{A}(n)$, and similarly, denote $\mathcal{D}(n) = \{n\} \cup \mathcal{D}(n)$. We also extend the notation above for sets of nodes to indicate the union of the corresponding sets, i.e., $\mathcal{P}(A) = \cup_{n \in A} \mathcal{P}(n)$.

Let $\{(x_i, y_i)\}_{i=1}^N$ be the training data of $N$ instances. Each $x_i \in \mathbb{R}^d$ is a feature vector and it is labeled with either a leaf (in single-label problems) or a set of leaves (in multi-label problems) of $G$. We will represent the labels $y_i$ as subsets of the nodes of the graph, where we include the indicated leaves and all their ancestors. That is, the label space (set of possible labels) is $\mathcal{Y} = \{\mathcal{A}(l) | l \in L\}$ for single-label problems, and $\mathcal{Y}_m = \{\mathcal{A}(L) | L \subseteq L\}$ for multi-label problems.

4. Hierarchical Structured SVM

We review the hierarchical structured SVM introduced in Cai and Hofmann (2004) and extended to the multi-label case in Cai and Hofmann (2007). Consider $W \in \mathbb{R}^{M \times d}$, and let the $n$-th row vector $W_n$ be be weights of the node $n \in [M]$. Define $\gamma(x, y)$ to be the potential of label $y$ given feature $x$, which is the sum of the inner products of $x$ with the weights of node $n \in y$, $\gamma(x, y) = \sum_{n \in y} W_n \cdot x$. If we vectorize $W$, $w = vec(W) = [W_1^T \ W_2^T \ldots W_M^T]^T \in \mathbb{R}^{d \cdot M}$, and define the class-attribute $\land(y) \in \mathbb{R}^M$, $[\land(y)]_n = 1$ if $n \in y$ or 0 otherwise, then

$$\gamma(x, y) = \sum_{n \in y} W_n \cdot x = w \cdot (\land(y) \otimes x) \quad (1)$$

where $\otimes$ is the Kronecker product. With weights $W_n$, prediction of an instance $x$ amounts to finding the maximum response label

$$\hat{y}(x) = \arg \max_{y \in \mathcal{Y}} \gamma(x, y) = \arg \max_{y \in \mathcal{Y}} \sum_{n \in y} W_n x$$

Given a structural error $\triangle(y', y)$, for instance a hamming distance $\triangle_H(y', y) = |y' - y| = \sum_{n \in [M]} |1_{n \in y'} - 1_{n \in y}|$, a training a hierarchical structured SVM is optimizing:

$$\min_W \lambda \sum_n \|W_n\|^2_2 + \sum_i \max_{y \in \mathcal{Y}} \left\{ \sum_{n \in y} W_n x_i - \sum_{n \in y_i} W_n x_i + \triangle(y, y_i) \right\} \quad (2)$$

[1]http://lshtc.iit.demokritos.gr/
Equivalently, in terms of $w$ and class-attribute $\land(y)$,
\[
\min_{w} \lambda ||w||_{2}^{2} + \sum_{i} \max_{y \in Y} \{ w \cdot ((\land(y) - \land(y_{i})) \otimes x_{i}) + \triangle(y_{i}) \} \geq 0 \quad \text{s.t.} \quad \Lambda w = y_{i} \quad \forall y \in Y, \quad \forall y \in Y
\]

(3)

5. Normalized Hierarchical SVM

A major issue we highlight is that unbalanced structures (which are frequently encountered in practice) lead to non-uniform regularization with the standard hierarchical SVM. To illustrate this issue, consider the two binary tree structures with two leaves shown in Figure 1. Implicitly both structures describes the same structure. Recall that the regularization penalty is $||W||_{2}^{2} = \sum_{n} ||W_{n}||_{2}^{2}$ where each row of $W$ is a weight vector for each node. In the left structure, the class attributes are $\land(y_{1}) = [1 \ 0 \ 0]^{T}$, and $\land(y_{2}) = [0 \ 1 \ 0]^{T}$, assume $||x||_{2} = 1$, and let the optimal weights of node 1 and node 2 in the left structure be $W_{1}$ and $W_{2}$. Now add a new node 3 as a child of node 1, so that $M = 3$, $\land(y_{3}) = [1 \ 0 \ 1]^{T}$, and $\land(y_{2}) = [0 \ 1 \ 0]^{T}$. Let $W_{1}$ and $W_{2}$ be the new weights for the nodes 1 and 3. If we assume $W_{1} = W_{2} = \frac{1}{2} W^{*}$, the potential function, and thus the decision boundary remain the same, but the regularization penalty for $y_{1}$ is halved so that $||W_{1}||_{2}^{2} = \frac{1}{2} ||W^{*}||_{2}^{2}$ and $||W_{2}||_{2}^{2} > ||W^{*}||_{2}^{2}$. This can be generalized to any depth, and the regularization penalty can differ arbitrarily for the model with the same decision boundary for different structures. In the given example, the structure on the right imposes half the penalty for the predictor of $y_{1}$ than that of $y_{2}$.

The issue can also be understood in terms of the difference between the norms of $\land(y)$ for $y \in Y$. Let $\phi(x, y) \in \mathbb{R}^{d}$. From (1),
\[
w \cdot (\land(y) \otimes x) = w \cdot \phi(x, y)
\]

$\land(y) \otimes x$ behaves as a feature map in hierarchical structured SVM. While the model regularizes $w$, the norm of $\phi(x, y)$ is different for $y$ and scales as $||\land(y)||_{2}$.

\[
||\gamma(x, y)||_{2} = ||\land(y) \otimes x||_{2} = ||\land(y)||_{2} \cdot ||x||_{2}
\]

Note that $||\land(y)||_{2} = \sqrt{\Lambda(y)}$ and the differences in regularization can grow linearly with the depth of the structure.

To remedy this effect, for each node $n$ we introduce a weight $\alpha_{n} \geq 0$ such that the sum of the weights along each path to a leaf is one, i.e.,
\[
\sum_{n \in \Lambda(l)} \alpha_{n} = 1, \quad \forall l \in L.
\]

(4)

Given such weights, we define the normalized class-attribute $\tilde{\land}(y) \in \mathbb{R}^{M}$ and the normalized feature map $\tilde{\phi}(x, y) \in \mathbb{R}^{d} M$,
\[
\tilde{\phi}(x, y) = \phi(x, y) / \sqrt{\alpha_{n}}
\]

(5)

The norm of these vectors are normalized to 1, independent of $y$, i.e., $||\tilde{\land}(y)||_{2} = 1, ||\tilde{\phi}(x, y)||_{2} = ||x||_{2}$ for $y \in Y$, and the class attribute for each node $n$ is fixed to 0 or $\sqrt{\alpha_{n}}$ for all labels. The choice of $\alpha$ is crucial and we present several alternatives (in our experiments, we choose between them using a hold-out set). For instance, using $\alpha_{n} = 1$ on the leaves $n \in L$ and 0 otherwise will recover the flat model and lose all the information in the hierarchy. To refrain from having a large number zero weight and preserve the information in the hierarchy, we consider setting $\alpha$ optimizing:

\[
\min_{\alpha} \sum_{n \in \Lambda(l)} \alpha_{n} \quad \text{s.t.} \quad \sum_{n \in \Lambda(l)} \alpha_{n} = 1, \quad \forall l \in L
\]

\[
\alpha_{n} \geq 0, \quad \forall n \in [M]
\]

(6)

where $\alpha > 1$. In Section 5.2, we will show that as $\alpha \to 1$, we obtain weights that remedy the effect of the redundant nodes shown in Figure 1.

We use (6) with $\alpha = 2$ as a possible way of setting the weights. However, when $\alpha = 1$, the optimization problem (6) is no longer strongly convex and it is possible to recover weights of zeroes for most nodes. Instead, for $\alpha = 1$, we consider the alternative optimization for selecting weights:

\[
\max_{\alpha} \min_{n} \alpha_{n} \quad \text{s.t.} \quad \sum_{n \in \Lambda(l)} \alpha_{n} = 1, \quad \forall l \in L
\]

\[
\alpha_{n} \geq 0, \quad \forall n \in [M]
\]

(7)

\[
\alpha_{n} \geq \alpha_{p}, \quad \forall n \in [M], \forall p \in P(n)
\]

We refer to the last constraint as a “directional constraint”, as it encourage more of the information to be carried by the leaves and results more even distribution of $\alpha$. 

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Figure 1. Regularization penalty for label $y_{1}$ (left branch) is halved to without changing decision boundary due to difference in the label structure.
For some DAG structures, constraining the sum \( \sum_{n \in \mathcal{X}(i)} \alpha_n \) to be exactly one can result in very flat solution. For DAG structures we therefore relax the constraint to
\[
1 \leq \sum_{n \in \mathcal{X}(i)} \alpha_n \leq T, \quad \forall l \in \mathcal{L}.
\]
for some parameter \( T \) (\( T = 1.5 \) in our experiments).

Another source of the imbalance is the non-uniformity of the required margin, which results from the norm of the differences of class-attributes, \( \| \wedge(y) - \wedge(y') \|_2 \). The loss term of each instance in (3) is, \( \max_{y \in Y} w \cdot (\wedge(y) - \wedge(y_i)) \otimes x + \Delta(y, y_i) \). And to have a zero loss \( \forall y \in \mathcal{Y}, \)
\[
\Delta(y, y_i) \leq w \cdot (\wedge(y) - \wedge(y_i)) \otimes x
\]
\( \Delta(y, y_i) \) works as the margin requirement to have a zero loss for \( y \). The RHS of the bound scales as norm of \( \wedge(y) - \wedge(y_i) \) scales.

This calls for the use of structural error that scales with the bound. Define normalized structural error \( \hat{\Delta}(y, y_i) \)
\[
\hat{\Delta}(y, y_i) = \| \wedge(y) - \wedge(y_i) \| = \sqrt{\sum_{n \in \mathcal{Y} \cup \mathcal{Y}_i} \alpha_n}
\]
and \( y \Delta y' = (y_i - y) \cup (y - y_i) \), and \( \wedge(y) \) and \( \alpha \) are defined in (5)(6). Without the normalization, this is the square root of the hamming distance, and is similar to a tree induced distance in (Dekel et al., 2004). This view of nonuniform margin gives a justification that the square root of hamming distance or tree induced distance is preferable to hamming distance.

5.1. Normalized Hierarchical SVM model

Summarizing the above discussion, we propose the Normalized Hierarchical SVM (NHSVM), which is given in terms of the following objective:
\[
\min_W \lambda \sum_n \|W_n\|^2_2 + \sum_i \max_{y \in Y} \sum_{n \in y} \sqrt{\alpha_n} W_n x_i - \sum_{n \in y_i} \sqrt{\alpha_n} W_n x_i + \hat{\Delta}(y, y_i)
\]
(10)

Instead of imposing a weight for each node, with change of variables \( U_n = \sqrt{\alpha_n} W_n \), we can write optimization (10) as changing regularization,
\[
\min_U \lambda \sum_n \|U_n\|^2_2 \alpha_n + \sum_i \max_{y \in Y} \sum_{n \in y} U_n x_i - \sum_{n \in y_i} U_n x_i + \hat{\Delta}(y, y_i)
\]
(11)

Also optimization (10) is equivalently written as
\[
\min_W \lambda \|w\|^2_2 + \sum_i \max_{y \in Y} w \cdot (\wedge(y) - \wedge(y_i)) \otimes x_i + \hat{\Delta}(y, y_i)
\]
(12)

Note that for the single-label problem, normalized hierarchical SVM can be viewed as a multi-class SVM changing the feature map function to (5) and the loss term to (9). Therefore, it can be easily applied to problems where flat SVM is used, and also popular optimization method for SVM, such as Shalev-Shwartz et al. (2007); Lacoste-Julien et al. (2013), can be used.

Another possible variant of optimization (11) which we experiment with is obtained by dividing inside the max with \( \|\wedge(y) - \wedge(y_i)\|_2 \):
\[
\min_W \lambda \|w\|^2_2 + \sum_i \max_{y \in Y} w \cdot \left( \frac{\wedge(y) - \wedge(y_i)}{\|\wedge(y) - \wedge(y_i)\|_2} \otimes x_i \right) + 1
\]
(13)

There are two interesting properties of the optimization (13). The norm of the vector right side of \( w \) is normalized, i.e.,
\[
\frac{\|\wedge(y) - \wedge(y_i)\|_2 \otimes x_i}{\|\wedge(y) - \wedge(y_i)\|_2} = \|x_i\|_2.
\]

Also the loss term per instance at the decision boundary, which is also the required margin, is normalized to 1. However, because normalized class attribute in (13) does not decompose w.r.t nodes as in (10), loss augmented inference in (13) is not efficient for multi-label problems.

5.2. Invariance property of the normalized hierarchical SVM

As we saw in Figure 1, different hierarchical structures can be used to describe the same data, and this causes undesired regularization problems. However, this is a common problem in real-world datasets. For instance, an action movie label can be further categorized into a cop-action movie and a hero-action movie in one dataset whereas the other dataset uses a action movie as a label. Therefore, it is desired for the learning method of hierarchical model to adapt to this difference and learn a similar model if given dataset describes similar data. Proposed normalization can be viewed as an adaptation to this kind of distortions. In particular, we show that NHSVM is invariant to node duplication.

Define duplicated nodes as follows. Assume that there are no unseen nodes in the dataset, i.e., \( \forall n \in [\mathcal{M}], \exists i, n \in \mathcal{A}(y_i) \). Define two nodes \( n_1 \) and \( n_2 \) in \( [\mathcal{M}] \) to be duplicated if \( y_i, n_1 \in y_i \iff n_2 \in y_i \). Define the minimal graph \( \mathcal{M}(G) \) to be the graph having a representative node per each
duplicated node set by merging each duplicated node set to a node. For the proof, see Appendix A.

Theorem 1 (Invariance property of NHSVM). Decision boundary of NHSVM with $G$ is arbitrarily close to that of NHSVM with the minimum graph $M(G)$ as $\rho$ in (6) approaches $1, \rho > 1$.

6. Shared SVM: Learning with Shared Frobenius norm

In the NHSVM, we set the weights $\alpha$ based on the graphically structured tree hierarchy, but disregard the data itself. We presented several options for setting the weights, and it is not clear what the best setting would be, or whether a different setting altogether would be preferable. Instead, here we consider discriminative learning the weights from the data by optimizing a joint objective over the weights and the predictors. The resulting optimization is equivalent to regularization with a new norm which we consider as a new norm which we call Structured Shared Frobenius norm or Structured Shared norm. It explicitly incorporates the information of the label structure $G$. Regularization with the structured shared Frobenius norm promotes the model to utilize shared information, thus it is a complexity measure suitable for structured learning. Notice that we only consider multi-class problem in this section. An efficient algorithm for tree structure is discussed in section 7.

Consider the formulation (11) as a joint optimization over both $\alpha$ and $U = [U_1^T \; U_2^T \; \ldots \; U_M^T]^T$ with fixed $\hat{\Delta}(y_i) = \Delta(l, l_i)$ (i.e. we no longer normalize the margins, only the regularization):

$$
\min_{U, \alpha} \lambda \sum_n \alpha_n \sum_{i \in [Y]} ||U_n||_F^2 + \sum_{i \in [Y]} \max_{l_i \in [\mathcal{A}(l)]} \sum_{n \in \mathcal{A}(l)} U_n x_i - \sum_{n \in \mathcal{A}(l)} U_n x_i + \Delta(l, l_i)
$$

subject to

$$
\sum_{n \in \mathcal{A}(l)} \alpha_n \leq 1, \quad \forall l \in [Y]
$$

$$
\alpha_n \geq 0, \quad \forall n \in [M]
$$

We can think of the first term as a regularization norm $\| \cdot \|_{s,G}$ and write

$$
\min_{U} \lambda ||U||_{s,G}^2 + \sum_{i \in [Y]} \max_{l_i \in [\mathcal{A}(l_i)]} U_{l_i} \cdot x_i - U_{l_i} \cdot x_i + \Delta(l, l_i)
$$

where the the structured shared Frobenius norm $\| \cdot \|_{s,G}$ is defined as:

$$
\|U\|_{s,G} = \min_{A V = U} \|A\|_{2 \rightarrow \infty} \|V\|_F
$$

subject to

$$
A_{l,n} = \begin{cases} 0 & \text{otherwise} \\ a_n & n \in \mathcal{A}(l) \end{cases}, \quad \forall l, \forall n
$$

$$
a_n \geq 0, \quad \forall n \in [M]
$$

where $\|A\|_{2 \rightarrow \infty}$ is the maximum of the $\ell_2$ norm of row vectors of $A$. Row vectors of $A$ can be viewed as coefficient vectors, and row vectors of $V$ as factor vectors which decompose the matrix $U$. The factorization is constrained, though, and must represent the prescribed hierarchy. We will refer (15) to Shared SVM or SSVM.

To better understand the SSVM, we can also define the Shared Frobenius norm without the structural constraint as

$$
\|U\|_s = \min_{A V = U} \|A\|_{2 \rightarrow \infty} \|V\|_F
$$

The Shared Frobenius norm is a norm between the trace-norm (aka nuclear norm) and the max-norm (aka $\gamma_2 : 1 \rightarrow \infty$ norm), and an upper bounded by Frobenius norm:

Theorem 2. For $\forall U \in \mathbb{R}^{r \times c}$

$$
\frac{1}{\sqrt{r c}} \|U\|_s \leq \frac{1}{\sqrt{c}} \|U\|_s \leq \|U\|_{\max}
$$

$$
\|U\|_s \leq \|U\|_{s,G} \leq \|U\|_F
$$

where $\|U\|_s = \min_{AW = U} \|A\|_F \|W\|_F$ is then the trace norm, and $\|U\|_{\max} = \min_{AW = U} \|A\|_{2 \rightarrow \infty} \|W\|_{2 \rightarrow \infty}$ is so-called the max norm (Srebro and Shraibman, 2005).

Proof. The first inequality follows from the fact that $\frac{1}{\sqrt{r c}} \|U\|_F \leq \|U\|_{2 \rightarrow \infty}$, and the second inequality is from taking $A = I$, or $A_{l,n} = 1$ when $n$ is an unique node for $l$ or 0 for all other nodes in (16) respectively. \[\square\]

We compare the Shared norm to the other norms to illustrate the behavior of the Shared norm, and summarize in Table 1. Shared norm is upper bounded by Frobenius norm, and reduce from it only if sharing the factors $V$ is beneficial. If there is no reduction from sharing as in disjoint feature case in Table 1, it equals to Frobenius norm, which is the norm used for multi-class SVM. Therefore, this justifies the view of SSVM that it extends multi-class SVM to shared structure, i.e., SSVM is equivalent to multi-class SVM if no sharing of weights is beneficial. This differs from the trace norm, which we can see specifically in disjoint feature case.
Normalized Hierarchical SVM

|                  | \[\|U\|_s\] | \[\|U\|_{s,G}\] | \[\|U\|_F\] | \[\|U\|_s\] |
|------------------|------------|----------------|------------|------------|
| Full sharing     | \[\|u\|_2\] | \[\|u\|_2\] | \[\sqrt{Y}\|u\|_2\] | \[\sqrt{Y}\|u\|_2\] |
| No sharing       | .          | \[\|U\|_F\] | \[\|U\|_F\] | .           |
| Disjoint feature | \[\sqrt{\sum_i |u_i|^2}\] | \[\sqrt{\sum_i |u_i|^2}\] | \[\sum_i |u_i|^2\] | \[\sum_i |u_i|^2\] |
| Factor scaling   | \[max_i |a_{i,u}| \|u\|_2\] | . | \[\sqrt{\sum_i a_{i,u}^2 \|u\|_2}\] | \[\sqrt{\sum_i a_{i,u}^2 \|u\|_2}\] |

Table 1. Comparing \[\|U\|_s\], \[\|U\|_{s,G}\], \[\|U\|_F\] and \[\|U\|_s\] in different situations. (1) Full sharing, \[U = [u \ u ... u]^T, \forall n', \forall l, n' \in \hat{A}(l)\]. (2) No sharing, \[\forall l \neq l', \hat{A}(l) \cap \hat{A}(l') = \emptyset\]. (3) Disjoint feature, \[U = [u_1 \ u_2 ... \ u_T]^T, \forall l_1 \neq l_2, \text{Supp}(u_1) \cap \text{Supp}(u_2) = \emptyset\]. (4) Factor scaling, \[U = [a_{1,u} a_{2,u} ... a_{T,u}]\]. Unlike trace norm, Shared Frobenius norm reduces to Frobenius norm if no sharing is beneficial as in case of (3) disjoint feature. See the text and Appendix B for details.

7. Optimization

In this section, we discuss the details of optimizing objectives (10) and (14). Specifically, we show how to obtain the most violating label for multi-labels problems for objective (10) and an efficient algorithm to optimize objective (14).

7.1. Calculating the most violating label for multi-label problems

We optimize our training objective (10) using SGD (Shalev-Shwartz et al., 2007). In order to do so, the most challenging part is calculating

\[
\hat{y}_i = \arg\max_{y \in \mathcal{Y}} \sum_{n \in y} \sqrt{\alpha_n} W_n x_i - \sum_{n \in y_i} \sqrt{\alpha_n} W_n x_i + \tilde{\Delta}(y, y_i)
\]

\[
= \arg\max_{y \in \mathcal{Y}} L_i(y)
\]  

(18)

at each iteration. For single label problems, we can calculate \(\hat{y}_i\) by enumerating all the labels. However, for a multi-label problem, this is intractable because of the exponential size of the label set. Therefore, in this subsection, we describe how to calculate \(\hat{y}_i\) for multi-label problems.

If \(L_i(y)\) decomposes as a sum of functions with respect to its nodes, i.e., \(L_i(y) = \sum_n L_{i,n}(\mathbf{1}\{n \in y\})\), then \(\hat{y}_i\) can be found efficiently. Unfortunately, \(\tilde{\Delta}(y, y_i)\) does not decompose with respect to the nodes. In order to allow efficient computation for multi-label problems, we actually replace \(\tilde{\Delta}(y, y_i)\) with a decomposing approximation \(\Delta_L(y, y_i) = ||y - \mathcal{L}|| - ||y \cap \mathcal{L}|| = \sum_{l \in \mathcal{L}} (\mathbf{1}\{l \in y - y_i\} - \mathbf{1}\{l \in y \cap y_i\}) + ||y \cap \mathcal{L}||\) instead. When \(\Delta_L(y, y_i)\) is used and the graph \(\mathcal{G}\) is a tree, \(\hat{y}_i\) can be computed in time \(O(M)\) using dynamic programming.

When the graph \(\mathcal{G}\) is a DAG, dynamic programming is not applicable. However, finding (18) in a DAG structure can be formulated into the following integer programming.

\[
\hat{z} = \arg\min_{z \in \{0,1\}^M} \sum_{n=1}^M z_n \cdot r_n \\
\text{s.t.} \quad \sum_{c \in C(n)} z_c \geq z_n, \quad \forall n \\
\quad \sum_{l \in \mathcal{L}} z_l \geq 1
\]  

(19)

where \(r_n = \sqrt{\alpha_n} W_n x_i + \mathbf{1}\{n \notin y, n \in \mathcal{L}\} - \mathbf{1}\{n \in y, n \in \mathcal{L}\}\). The feasible label from (19) is the set of labels where if a node \(n\) is in the label \(y\), at least one of its child node is in \(y\), i.e., \(\forall n \notin \mathcal{L}, n \in y \implies \exists c \in C(n), c \in y, \text{and all the parents of} n \text{are in the label}, \forall n \in y \implies \forall p \in \mathcal{P}(n), p \in y\). The feasible set is equivalent to \(\mathcal{Y}_n\). The search problem (19) can be shown to be NP-hard by reduction from the set cover problem. We relax the integer program into a linear program for training. Last constraint of \(\sum_{l \in \mathcal{L}} z_l \geq 1\) is not needed for an integer program, but yields a tighter LP relaxation. In testing, we rely on the binary integer programming only if the solution to LP is not integral. In practice, integer programming solver is effective for this problem, only 3 to 7 times slower than linear relaxed program using gurobi solver (Gurobi Optimization, 2015).

7.2. Optimizing with the shared norm

Optimization (14) is a convex optimization jointly in \(U\) and \(\alpha\), and thus, has a global optimum. For the proof, see Appendix C.

Lemma 1. Optimization (14) is a convex optimization jointly in \(U\) and \(\alpha\).

Since it is not clear how to jointly optimize efficiently with respect to \(U\) and \(\alpha\), we present an efficient method to optimize (14) alternating between \(\alpha\) and \(V_n = \frac{U_n}{\sqrt{\alpha_n}}\). Specifically, we show how to calculate the optimal \(\alpha\) for a fixed \(U\) in closed form in time \(O(M)\) when \(\mathcal{G}\) is a tree where \(M\) is the number of nodes in the graph, and for fixed \(\alpha\), we optimize the objective using SGD (Shalev-Shwartz et al., 2007).
Normalized Hierarchical SVM

Algorithm 1 Calculate the optimal $\alpha$ in (20) for the tree structure $G$ in $O(M)$. We assume nodes are sorted in increasing order of depth, i.e., $\forall n, p \in \mathcal{P}(n), n > p$.

1: Input: $U \in \mathbb{R}^{M \times d}$, a tree graph $G$
2: Output: $\alpha \in \mathbb{R}^M$.
3: Initialize: $\alpha = N = E = [0 \ldots 0] \in \mathbb{R}^M, L = [1 \ldots 1] \in \mathbb{R}^M$
4: for $n = M \rightarrow 1$ do
5:   $N_n \leftarrow \|U_n\|^2 \|2\|, E_n \leftarrow 0$
6:   if $|C(n)| \neq 0$ then
7:      $S \leftarrow \sum_{c \in C(n)} N_c$
8:      if $\sqrt{S} + \sqrt{N_n} \neq 0$ then
9:         $N_n \leftarrow (\sqrt{N_n} + \sqrt{S})^2, E_n \leftarrow \frac{\sqrt{N_n}}{\sqrt{S} + \sqrt{N_n}}$
10:   end if
11: end if
12: end for
13: $L_1 \leftarrow 1 - E_1$
14: for $n = 2 \rightarrow M$ do
15:   $p \leftarrow \mathcal{P}(n)$
16:   $\alpha_n = \sqrt{L_p \cdot E_n}$
17: $L_n = L_p \cdot (1 - E_n)$
18: end for
19: return $\alpha$

2007) with change of variables $V_n = \frac{U_n}{\sqrt{\alpha_n}}$.

$$\min_{\alpha_n} \sum_{n \in [M]} \frac{\|U_n\|^2}{\alpha_n} \tag{20}$$

s.t. $\sum_{n \in A(y)} \alpha_n \leq 1, \quad \forall y \in \mathcal{Y}$

$\alpha_n \geq 0, \quad \forall n \in \mathcal{N}$.

Algorithm 1 shows how to calculate optimum $\alpha$ in (20) in time $O(M)$ for a tree structure. See Appendix D.

**Lemma 2.** For a tree structure $G$, algorithm 1 finds optimal $\alpha$ in (20) in $O(M)$ in a closed form.

In the experiments, we optimize $\alpha$ using algorithm 1 with $U_n = \sqrt{\alpha_n} V_n$ after a fixed number of epochs of SGD with respect to $V$, and repeat this until the objective function converges. We find that the algorithm is efficient enough to scale up to large datasets.

**8. Experiments**

We present experiments on both synthetic and real data sets. In Section 8.1, we consider synthetic data sets with both balanced structures and unbalanced structures (i.e. when some leaves in the class hierarchy are much deeper then others). We use this to demonstrate empirically the vulnerability of un-normalized Hierarchical SVM to structure imbalance, and how normalization solves this problem. In particular, we will see how un-normalized HSVM does not achieve any performance gains over "flat" learning (completely ignoring the structure), but our NHSVM model does leverage the structure and achieves much higher accuracy. Then, in Section 8.2, we compare our method to competing methods on mid-sized benchmark data sets, including ones with multiple labels per instance and with DAG structured hierarchies. Finally, in Section 8.3 we demonstrate performance on the large-scale LSHTC competition data, showing significant gains over the previously best published results and over other recently suggested methods. Data statistics is summarized in table 2.

| Model            | $M$ | $d$ | $N$ | $|\mathcal{L}|$ | $\bar{y}$ |
|------------------|-----|-----|-----|-----------------|---------|
| Synthetic(B)     | 15  | 1K  | 8K  | 8               | 1       |
| Synthetic(U)     | 19  | 1K  | 10K | 11              | 1       |
| IPC              | 553 | 228K| 75K | 451             | 1       |
| WIKI d5          | 1512| 1000| 41K | 1218            | 1.1     |
| ImageNet         | 1676| 51K | 100K| 1000            | 1       |
| DMOZ10           | 17221| 165K| 163K| 12294            | 1       |
| WIKI             | 50312| 346K| 456K| 36504           | 1.8     |

**Table 2.** Data statistics: $M$ is the number of nodes in the graph. $d$ is the dimension of the features. $N$ is the number of the instances. $|\mathcal{L}|$ is the number of labels. $\bar{y}$ is the average labels per instance. $\mathcal{L} = 1$ denotes a single-label dataset.

**8.1. Synthetic Dataset**

In this subsection, we empirically demonstrate the benefit of the normalization with the intuitive hierarchical synthetic datasets. While even for a perfectly balanced structure, we gain from the normalization, we show that the regularization of the HSVM can suffer significantly from imbalance of the structure (i.e. when the depths of the leaves are very different). Notice that for a large structured dataset such as wikipedia dataset, the structure is very unbalanced.

Balanced synthetic data is created as follows. A weight vector $W_n \in \mathbb{R}^d$ for each node $n \in [2^d - 1]$ in the complete balanced tree with depth 4 and an instance vector $x_i \in \mathbb{R}^d, i \in [N], N = 15000$, for each instances are sampled from the standard multivariate normal distribution. Instances are assigned to labels which have maximum potential.

To create the unbalanced synthetic data, we sample $x_i \in \mathbb{R}^d$ from the multivariate normal distribution with $d = 1,000, i \in [N], N = 10,000,$ and normalize its norm to 1. We divide the space $\mathbb{R}^d$ with a random hyperplane recursively so that the divided spaces form an unbalanced binary tree structure, a binary tree growing only in one direction. Specifically, we divide the space into two spaces with a random hyperplane, which form two child spaces, and recursively divide only one of the child space with a ran-
Table 3. Accuracy on synthetic datasets. † shows that the the improvements over FlatSVM and HSV is statistically significant. ‡ shows that the improvement over NHSVM is statistically significant.

Table 4. Accuracy on benchmark datasets. * denotes that the algorithm was not be applied due to the graphical structure of the data.

Table 5. Results on full WIKI. *The inference of HR-SVM relies on the other meta learning method (Gopal and Yang, 2010) for high accuracy. ** NHSVM is used to predict the number of labels in the inference.

ImageNet have DAG structures, and the others have tree structures.

8.3. Result on LSHTC Competition

We also compared our methods with the competition dataset, Large Scale Hierarchical Text Classification Challenge 2. We compared with the winner of the competition as well as the the best published method we acknowledge so far, HR-SVM (Gopal and Yang, 2013). We also added comparisons with FastXML (Prabhu and Varma, 2014) in the competition dataset. FastXML is a very fast ranking method suitable for a large dataset. Since FastXML predicts rankings of full labels rather than list of labels, we predicted with the same number of labels as NHSVM, and compared the result. In table 5, we show the result on full competition dataset, WIKI-2011, and compare with results currently reported. NHSVM was able to adapt to the large scale of WIKI-2011 dataset with the state-of-the-art results. Only 98,519 features that appear in the test set are used weather we managed to successfully predict all the full labels in WIKI-2011 with current best published method. With a computer with Intel Xeon CPU E5-2620 processor, optimization took around 1.5 weeks in matlab without a warm start.

9. Summary

In this paper we considered the problem of large-scale hierarchical classification, with a given known hierarchy. Our starting point was hierarchical structured SVM of Cai and Hofmann (2004), and we also considered extensions for
handling multi-label problems (where each instance could be tagged with multiple labels from the hierarchy) and of label hierarchies given by DAGs, rather than rooted trees, over the labels. Our main contribution was pointing out a normalization problem with this framework, both in the effective regularization for labels of different depths, and in the loss associated with different length paths. We suggested a practical correction and showed how it yields to significant improvement in prediction accuracy. In fact, we demonstrate how on a variety of large-scale hierarchical classification tasks, including the Large-scale Hierarchical Text Classification Competition data, our Normalized Hierarchical SVMs outperform all other relevant methods we are aware of (that work using the same data and can be scaled to the data set sizes). We also briefly discussed connections with matrix factorization approaches to multi-label classification and plan on investigating this direction further in future research.

References

Amit, Y., Fink, M., Srebro, N., and Ullman, S. (2007). Uncovering shared structures in multiclass classification. In Proceedings of the 24th international conference on Machine learning, pages 17–24. ACM.

Blaschko, M. B., Zaremba, W., and Gretton, A. (2013). Taxonomic prediction with tree-structured covariances. In Machine Learning and Knowledge Discovery in Databases, pages 304–319. Springer.

Bravo, H. C., Wright, S., Eng, K. H., Keles, S., and Wahba, G. (2009). Estimating tree-structured covariance matrices via mixed-integer programming. Journal of machine learning research: JMLR, 5:41.

Cai, L., and Hofmann, T. (2004). Hierarchical document categorization with support vector machines. In Proceedings of the thirteenth ACM international conference on Information and knowledge management, pages 78–87, New York, NY, USA. ACM.

Cai, L., and Hofmann, T. (2007). 2007, exploiting known taxonomies in learning overlapping concepts. In In: Proceedings of the 20th International Joint Conference on Artificial Intelligence (IJCAI ’07), pages 714–719.

Cesa-Bianchi, N., Gentile, C., and Zaniboni, L. (2006). Hierarchical classification: combining bayes with svm. In Proceedings of the 23rd international conference on Machine learning, ICML ’06, pages 177–184, New York, NY, USA. ACM.

Dekel, O., Keshet, J., and Singer, Y. (2004). Large margin hierarchical classification. In Proceedings of the twenty-first international conference on Machine learning, page 27. ACM.

Dumais, S., and Chen, H. (2000). Hierarchical classification of web content. In Proceedings of the 23rd annual international ACM SIGIR conference on Research and development in information retrieval, SIGIR ’00, pages 256–263, New York, NY, USA. ACM.

Gopal, S., and Yang, Y. (2010). Multilabel classification with meta-level features. In Proceedings of the 33rd international ACM SIGIR conference on Research and development in information retrieval, pages 315–322. ACM.

Gopal, S., and Yang, Y. (2013). Recursive regularization for large-scale classification with hierarchical and graphical dependencies. In Proceedings of the 19th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 257–265. ACM.

Gurobi Optimization, I. (2015). Gurobi optimizer reference manual.

Huang, J., Kumar, S. R., and Zabih, R. (1998). An automatic hierarchical image classification scheme. In Proceedings of the sixth ACM international conference on Multimedia, MULTIMEDIA ’98, pages 219–228, New York, NY, USA. ACM.

Lacoste-Julien, S., Jaggi, M., Schmidt, M., and Pletscher, P. (2013). Block-coordinate frank-wolfe optimization for structural svms. In ICML 2013 International Conference on Machine Learning, pages 53–61.

Mccallum, A., Rosenfeld, R., Mitchell, T., and Ng, A. (1998). Improving text classification by shrinkage in a hierarchy of classes.

Prabhu, Y., and Varma, M. (2014). Fastxml: A fast, accurate and stable tree-classifier for extreme multi-label learning. In Proceedings of the 20th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 263–272. ACM.

Robertson, S., and Zaragoza, H. (2009). The probabilistic relevance framework: BM25 and beyond. Now Publishers Inc.

Rousu, J., Saunders, C., Szendmak, S., and Shawe-Taylor, J. (2006). Kernel-based learning of hierarchical multilabel classification models. JOURNAL OF MACHINE LEARNING RESEARCH, 7:1601–1626.

Russakovsky, O., Deng, J., Su, H., Krause, J., Satheesh, S., Ma, S., Huang, Z., Karpathy, A., Khosla, A., Bernstein, M., Berg, A. C., and Fei-Fei, L. (2014). ImageNet Large Scale Visual Recognition Challenge.

Shalev-Shwartz, S., Singer, Y., and Srebro, N. (2007). Pegasos: Primal estimated sub-gradient solver for svm. In Proceedings of the 24th international conference on Machine learning, ICML ’07, pages 807–814, New York, NY, USA. ACM.

Silla, Jr., C. N. and Freitas, A. A. (2011). A survey of hierarchical classification across different application domains. Data Min. Knowl. Discov., 22(1-2):31–72.

Srebro, N. and Shraibman, A. (2005). Rank, trace-norm and max-norm. In Learning Theory, pages 545–560. Springer.

Vural, V. and Dy, J. G. (2004). A hierarchical method for multi-class support vector machines. In Proceedings of the twenty-first international conference on Machine learning, ICML ’04, pages 105–, New York, NY, USA. ACM.

Weinberger, K. and Chapelle, O. (2009). Large margin taxonomy embedding with an application to document categorization.
**Appendix A. Invariance property of NHSVM**

**Theorem 1** (Invariance property of NHSVM). Decision boundary of NHSVM with $G$ is arbitrarily close to that of NHSVM with the minimum graph $M(G)$ as $\rho$ in (6) approaches $1$, $\rho > 1$.

**Proof.** We prove by showing that for any $G$, variable $\alpha$ in (6) can be reduced to one variable per each set of duplicated nodes in $G$ using the optimality conditions, and optimizations (6)(10) are equivalent to the corresponding optimizations of $M(G)$ by change of the variables.

Assume there are no duplicated leaves, however, the proof can be easily generalized for the duplicated leaves by introducing an additional constraint on $\gamma$.

Let $F(n')$ be a mapping from node $n'$ in graph $M(G)$ to a corresponding set of duplicated nodes in $G$. Denote the set of nodes in $G$ as $N_G$, and the set of nodes in $M(G)$ as $N'$, and the set of leaves in $M(G)$ as $L'$.

Consider (6) for $G$. Note that (6) has a constraint on sum of $\alpha_n$ to be $1$ for $n \in \{n \in A(l) | l \in L\}$. By the definition of the duplicity, if two nodes $n_1$ and $n_2$ are duplicated nodes, they are the ancestors of the same set of the leaves, and term $\alpha_{n_1}$ appears in the first constraints of (6) if and only if term $\alpha_{n_2}$ appears, thus we conclude that all the duplicated nodes will appear altogether. Consider a change of variable for each $n' \in N'$

$$K_{n'} = \sum_{n \in F(n')} \alpha_n$$

(21)

Then, (6) are functions of $K_{n'}$, and (6) decompose w.r.t $K_{n'}$. From the convexity of function $x^\rho$ with $\rho > 1$, $x > 0$, and Jensen’s inequality, $(\frac{1}{|F(n')|} \sum_{n \in F(n')} \alpha_n)^\rho \leq \frac{1}{|F(n')|} \sum_{n \in F(n')} \alpha_n$, minimum of (6) is attained when $\alpha_n = \frac{1}{|F(n')|} K_{n'}$ for $\forall n \in F(n')$. As $\epsilon$ approaches $0$, where $\epsilon = \rho - 1 > 0$,

$$\sum_{n \in N} \alpha_n = \sum_{n' \in N'} |F(n')| (\frac{K_{n'}}{|F(n')|})^\rho = |F(n')|^\rho K_{n'}$$

(22)

Plugging (22) (21) into (6),

$$\min_{n' \in N'} \sum_{n' \in N'} K_{n'}$$

s.t.

$$\sum_{n' \in N'} K_{n'} = 1, \quad \forall y' \in Y'$$

These formulations are same as (6) for $M(G)$.

Thus given $n'$, $\alpha_n = \frac{K_{n'}}{|F(n')|}$ is fixed for $\forall n \in F(n')$, and with the same argument for $W_n$ in (10), change of variables gives , $W_{n'} = \sum_{n \in F(n')} W_n$. Then (10) is a minimization w.r.t $W_{n'}$, and the minimum is when $W_{n'} = \frac{W_n}{|F(n')|}$ for $\forall n \in F(n')$, plugging this in (10),

$$\lambda \sum_{n \in N} |F(n')| \frac{|W_n|^2}{|F(n')|^2} + \sum_{i} \max_{y \in Y} (\sum_{n \in F(n')} |F(n')| - \sqrt{K_n/|F(n')|})$$

$$W_{n'} - \sum_{n \in y} |F(n')| \sqrt{K_n/|F(n')|} \cdot x_i + \tilde{\Delta}(y, y_i)$$

(23)

By substituting $W_n'' = \frac{1}{\sqrt{|F(n')|}} W_{n'}$,

$$\lambda \sum_{n} |F(n')| \frac{|W_n''|^2}{|F(n')|^2} + \sum_{i} \max_{y \in Y} \left( \sum_{n \in y} \sqrt{K_n} W_n'' - \sum_{n \in y_i} \sqrt{K_n} W_n'' \right) \cdot x_i + \tilde{\Delta}(y, y_i)$$

(10),(6) for $G$ are equivalent to those of $M(G)$, thus two solutions are equivalent with a change of variables and the decision boundaries are the same.

**Appendix B. Behavior of Shared Frobenius Norm**

We first show a lower bound for $\|\cdot\|_s, \|\cdot\|_{s,G}$ which will be useful for the later proofs.

**Lemma 3.** For $U \in \mathbb{R}^{y \times d}$,

$$\|U\|_{s,G} \geq \|U\| \geq \max_{y \in Y} \|U_y\|_2^2$$

where $U_y$ is $y$-th row vector of $U$.

**Proof.** Let $A \in \mathbb{R}^{y \times M}, V \in \mathbb{R}^{M \times d}$ be the matrices which attain minimum in $\|U\|_s = \min_{A,V} \|AV = U\|_F$. Since $A_{r,c} = U_{r,c}$ and from the cauchy-schwarz, $\|U_{r,c}\| \leq \|A_r\| \cdot \|V_{c,2}\| = \|V_{c,2}\|$, and if we square both sides and sum over $c$, $\|U_r\|_2^2 \leq \|V\|_F^2 = \|U\|_s^2$ which holds for all $r$.

Following are the detailed descriptions for table 1 and the sketch of the proofs.

**Full sharing** If all weights are same for all classes, i.e., $U = [u, u, \ldots, u] \in \mathbb{R}^{y \times d}$ for $u \in \mathbb{R}^d$, and there exists a node $n$ that it is shared among all $Y_i$, i.e., $\exists n, \forall l \in A(l)$, then $\|U\|_s^2 = \|u\|_2^2$ whereas $\|U\|_F^2 = \|U\|_s^2 = \|Y\|_2^2$.

$\|U\|_F^2 = \|u\|_2^2$ can be shown with matrix $A = [1_{Y \cdot 1} 0_{Y \times M - 1}] \in \mathbb{R}^{y \times M}$ and $V = \begin{bmatrix} u \\ 0_{M - 1, d} \end{bmatrix}$ where $n_{r,c} \in \mathbb{R}^{y \times d}$ is a matrix with all elements set to $n$, $U = AV$ and the factorization attains the minimum of (17) since it attains the lower bound from lemma 3, $\|U\|_2^2 = \|Y\|_2^2$ is easily shown from the fact that $U$ is a rank one matrix with a singular value of $\sqrt{\|Y\|_2}$.

**No sharing** If there is no shared node, i.e., $\forall l, l' \in Y, l \neq l', A(l) \cap A(l') = \emptyset$, then $\|U\|_s^2 = \|U\|_F^2$.

To show this, let $A$ and $V$ be the matrices which attain the minimum of (16) $m$-th element of $A_g$ is zero for all $y$ except one and $V_{m,d}$ is nonzero only for one $y$ such that $m \in y$. Therefore, (16) decomposes w.r.t $A_g$ and $V_{g,d}$, where $V_{g,d}$ is the $d$-th column vector of $V$ taking only for row $y$. 

$$\min_{A_{V} = U} \|V\|_F^2 = \sum_{d} \sum_{A_g V_{d,y} = U_{y,d}} \|V_{g,d}\|_2^2$$
Given \( \|A\|_2 = 1 \),
\[
\|V_{y,d}\|_2 = \|A\|_2 \|V_{y,d}\|_2 \geq \|A \cdot V_{y,d}\| = \|V_{y,d}\|
\]

And let \( A_y = V_{y,d} / \|V_{y,d}\|_2 \) which attains the lower bound.
\[
\therefore \min_{A = U} \|V\|_F^2 = \sum_y \sum_d |U_{y,d}|^2 = \|U\|_F^2
\]

**Disjoint feature** If \( U = [u_1 u_2 \ldots u_Y]^T \in \mathbb{R}^{Y \times d} \) for \( l \in [Y] \),
\( u_l \in \mathbb{R}^d \), and the support of \( u_y \) are all disjoint, i.e., \( \forall y_1 \neq y_2, \text{Supp}(u_{y_1}) \cap \text{Supp}(u_{y_2}) = \emptyset \), then \( \|U\|_2 = \|U\|_F^2 = \sum_y \|u_y\|_2^2 \) and \( \|U\|_F^2 = (\sum_y \|u_y\|_2)^2 \).

For \( \| \cdot \|_n \), it is similar to no sharing. The factorization decomposes w.r.t. each column \( u \). For the trace norm, since the singular values are invariant to permutations of rows and columns, \( U \) can be transformed to a block diagonal matrix by permutations of rows and columns, and the singular values decompose w.r.t. block matrices with corresponding singular values of \( \|u_y\|_2 \).

**Factor scaling** If \( U = [a_1 u_2 \ldots a_Y u] \in \mathbb{R}^{Y \times d} \) for \( l \in [L] \),
\( u \in \mathbb{R}^d \), then \( \|U\|_2 = \max a_i \|u\|_2 \) and \( \|U\|_F = \|U\|_2 = (\sum_i \|a_i\|^2 \cdot \|u\|_2^2) \).

Proof is similar to full sharing. For \( \| \cdot \|_n \), \( A = \frac{1}{\max_i a_i} \)
\([a_1 a_2 \ldots a_Y]^T \text{0}_Y \text{M} - 1 \) and \( V = \max_i a_i \begin{bmatrix} u^T \text{0}_Y \text{M} - 1.d \end{bmatrix} \)
is a feasible solution which attains the minimum in lemma 3. For the trace norm, singular values can be easily computed with knowing \( U \) is a rank 1 matrix.

**Appendix C. Convexity of Shared Frobenius Norm optimization**

**Lemma 1.** Optimization (14) is a convex optimization jointly in \( U \) and \( \alpha \).

Proof. Let \( f(U, \alpha) = \sum_n \sum_d f_{n,d}(U_n, \alpha_n) \) where \( f_{n,d} = (U_n^2 + \alpha_n) \). The Hessian of each \( f_{n,d} \) can be calculated easily by differentiating twice. Then, the Hessian is a positive-semidefinite matrix for \( \forall \alpha_n \geq 0 \), since
\[
\text{if } \alpha_n > 0, \nabla^2 f_{n,d} = \begin{bmatrix} \frac{\partial^2 f_{n,d}}{\partial u_{n,d}^2} & \frac{\partial^2 f_{n,d}}{\partial u_{n,d} \partial \alpha_n} \\ \frac{\partial^2 f_{n,d}}{\partial u_{n,d} \partial \alpha_n} & \frac{\partial^2 f_{n,d}}{\partial \alpha_n^2} \end{bmatrix}
\]
\[
= \frac{2}{\alpha_n} \begin{bmatrix} U_{n,d} & U_{n,d}^2 \\ \alpha_n & \alpha_n^2 \end{bmatrix}, \text{ and if } \alpha_n = 0 \text{ we can assume } \|U_n\|_2 = 0 \text{ by restricting the domain and the hessian to be a zero matrix. Thus, } \sum_n \|U_n\|_2^2 \text{ is a convex function jointly in } U_n \text{ and } \alpha_n, \text{ and the lemma follows from the fact that the rest of the objective function in (14) is convex in } U_n. \]

**Proof.** Let \( f(n, l) = \min_{\alpha_n \leq 1; \sum_n \|U_n\|_2^2} \frac{\|U_n\|_2^2}{\alpha_n} \)
where \( \bar{D} \) denotes the union set of \( \{n\} \) and descendents nodes of \( n \), the following recursive relationship holds, since \( G \) has a tree structure.
\[
\begin{aligned}
f(n, l) &= \begin{cases} 
\frac{\|U_n\|_2^2}{l} & \text{if } n \text{ is a leaf node} \\
\min_{0 < k < l} \left( \frac{\|U_n\|_2^2}{l} \cdot k \right) + \sum_{c \in C(n)} f(c, l(1 - k)) & \text{otherwise}
\end{cases} \\
\end{aligned}
\]
(24)

If \( n \) is a parent node of leaf nodes,
\[
f(n, l) = \min_{0 < k < l} \left( \frac{B_1}{l} - \frac{B_2}{l(1 - k)} \right)
\]
(25)

where \( C(n) \) denotes the set of children nodes of \( n \), \( B_1 = \|U_n\|_2^2 \) and \( B_2 = \sum_{c \in C(n)} \|U_c\|_2^2 \). This has a closed form solution,
\[
f(n, l) = \frac{1}{\sqrt{l}} \left( \sqrt{B_1} + \sqrt{B_2} \right)^2
\]
(26)

and the minimum is attained at \( k = \frac{\sqrt{B_2}}{\sqrt{B_1} + \sqrt{B_2}} \). For nodes \( p \) of \( n \), \( f(p, l) \) will also have a form of (26), since the equation (26) has a form of leaf node, and the recursive relationship (24) holds. We continue this process until the root node \( r \) is reached, and \( f(r, 1) \) is the optimum. The optimal \( \alpha \) can be calculate backward.

**Appendix D. Closed form optimization of \( \alpha \)**

**Lemma 2.** For a tree structure \( G \), algorithm 1 finds optimal \( \alpha \) in (20) in \( O(M) \) in a closed form.