Noncommutative Network Models

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Outline

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- Network models
- Eckmann–Hilton for network models
- Kneser graphs
- Graph products of monoids
- Free undirected network models
Species

**Definition**

Let $S$ denote the **symmetric groupoid**, the category consisting of sets $n = \{1, \ldots, n\}$ for objects ($0 = \emptyset$), and bijections for the morphisms.

Notice, all morphisms are automorphisms. This is a skeleton of FinBij, the maximal subgroupoid in FinSet.

**Definition ([Joy81])**

A **combinatorial species** is a functor $F : S \to \text{Set}$.

**Example**

$SG : S \to \text{Set}$ by $SG(n) = \text{the set of simple graphs with } n \text{ nodes}$.
Network models

Definition ([BFMP20])

A network model is a symmetric lax monoidal functor of the form $(F, \sqcup) : (S, +) \to (\text{Mon}, \times)$.

“overlay” $\sqcup_n : F(n) \times F(n) \to F(n)$

and “disjoint union” $\sqcup_{m,n} : F(m) \times F(n) \to F(m + n)$

Example

- simple graphs
- directed edges
- multiple edges
- edge colors
- hypergraphs
- Petri nets

Nonexample

acyclic graphs
Operad from a network model

The original motivation for network models is to construct operads modeling network design:

\[
\text{NetMod} \xrightarrow{\int} \text{SMC} \xrightarrow{U} \text{Oprd}
\]
Constructing network models

**Construction**

You can get a network model from any monoid. There's a functor $\text{Mon} \to \text{NetMod}$ given by $M \mapsto (n \mapsto M(n/2))$.

**Example**

- $M = (\mathbb{B}, +)$ recovers the simple graphs network model.
- $M = (\mathbb{N}, +)$ gives graphs with multiple (indistinguishable) edges.
- $M = (2^X, \cup)$ gives graphs with edges labeled in $X$.

But notice different edge components automatically commute with each other:

$\quad 1 \overset{m_1}{\rightarrow} 2 \quad \cup \quad 4 \overset{m_2}{\rightarrow} 3$

$= \quad 1 \overset{m_1}{\rightarrow} 2 \quad \cup \quad 4 \overset{m_2}{\rightarrow} 3$

$\quad (m_1, 0, 0, 0, 0, 0) \cup (0, m_2, 0, 0, 0, 0)$

$= \quad (0, m_2, 0, 0, 0, 0) \cup (m_1, 0, 0, 0, 0, 0)$

Can we define $\Gamma_M : S \to \text{Set}$ by $\Gamma_M(n) = \bigsqcup^{(n)}_2 M$?
Eckmann–Hilton for network models

Disjoint components must commute with each other: Let $a \in F(m)$ and $b \in F(n)$. Then

$$(a \sqcup \emptyset) \sqcup (\emptyset \sqcup b) = (a \sqcup \emptyset) \sqcup (\emptyset \sqcup b)$$

$$= (\emptyset \cup a) \sqcup (b \cup \emptyset)$$

$$= (\emptyset \cup b) \sqcup (a \cup \emptyset)$$
So what this means is that we want to define $\Gamma_M$ to be $n \mapsto \bigsqcup_{(n)} M/\sim$ where $\sim$ tells us to impose commutativity for edge components which are disjoint. Let's take a look at the first few levels to see how this relation looks.

\begin{align*}
0 & \mapsto 1, \quad 1 \mapsto 1, \quad 2 \mapsto M, \quad 3 \mapsto \bigsqcup M, \\
4 & \mapsto \bigsqcup M/\langle a_{1,2}b_{3,4} = b_{3,4}a_{1,2}, a_{1,3}b_{2,4} = b_{2,4}a_{1,3}, a_{1,4}b_{2,3} = b_{2,3}a_{1,4} \rangle \\
5 & \mapsto \bigsqcup M/\langle a_{1,2}b_{3,4} = b_{3,4}a_{1,2}, a_{1,2}b_{3,4} = b_{3,4}a_{1,2}, a_{1,2}b_{3,5} = b_{3,5}a_{1,2}, \quad a_{1,3}b_{2,4} = b_{2,4}a_{1,3}, a_{1,3}b_{2,5} = b_{2,5}a_{1,3}, a_{1,3}b_{4,5} = b_{4,5}a_{1,3}, \quad a_{1,4}b_{2,3} = b_{2,3}a_{1,4}, a_{1,4}b_{2,5} = b_{2,5}a_{1,4}, a_{1,4}b_{3,5} = b_{3,5}a_{1,4}, \quad a_{1,5}b_{2,3} = b_{2,3}a_{1,5}, a_{1,5}b_{2,4} = b_{2,4}a_{1,5}, a_{1,5}b_{3,4} = b_{3,4}a_{1,5}, \quad a_{2,3}b_{4,5} = b_{4,5}a_{2,3}, a_{2,4}b_{3,5} = b_{3,5}a_{2,4}, a_{2,5}b_{3,4} = b_{3,4}a_{2,5} \rangle \end{align*}
So what this means is that we want to define $\Gamma_M$ to be $n \mapsto \coprod_{(n)} M/\sim$ where $\sim$ tells us to impose commutativity for edge components which are disjoint. Let’s take a look at the first few levels to see how this relation looks.

$3 \mapsto \coprod \quad M$

$4 \mapsto \coprod M/\langle \quad \rangle$

$5 \mapsto \coprod M/\langle \quad \rangle$
Graph products of monoids

**Definition ([Gre90, Vel01])**

Let $G$ be a graph with $N$ nodes, and $M_i$ a family of $N$ monoids. The **graph product** is the monoid

$$G(M_i) = \coprod M_i / \langle a_k b_\ell = b_\ell a_k \text{ if } (k, \ell) \in G \rangle.$$ 

Define $IC : \text{SimpleGrph} \to \text{Cat}$ by

Let $D : IC(G) \to \text{Mon}$ be the diagram

$$\begin{array}{cccc}
M_1 & \to & M_1 \times M_2 & \leftarrow & M_2 \\
\downarrow & & \downarrow & & \downarrow \\
M_1 \times M_4 & \to & M_1 \times M_3 & \leftarrow & M_2 \times M_3 \\
\uparrow & & \uparrow & & \uparrow \\
M_4 & \leftarrow & M_3 & \leftarrow & \\
\end{array}$$

**Proposition (M.)**

$$G(M_i) \cong \text{colim}D.$$
Now for a given monoid $M$ we define a network model $\Gamma_M: (S, +, 0) \to (\text{Mon}, \times, 1)$ by $n \mapsto KG_{n,2}(M)$.

**Theorem (M.)**

$\Gamma_M$ defined above is a network model. Moreover, we have an adjunction

\[
\begin{array}{ccc}
\Gamma & \cong & \text{ev}_2 \\
\downarrow & & \downarrow \\
\text{Mon} & & \text{NetMod}^+ \\
\end{array}
\]

where $\text{NetMod}^+$ is the subcategory of $\text{NetMod}$ of network models with trivial involution (thanks to Mike Shulman for pointing out a mistake in the original).

Network models with trivial involution are essentially “undirected network models” (thanks Mike Shulman).
J. C. Baez, J. Foley, J. Moeller, and B. S. Pollard.
Network models.
*Theory and Applications of Categories, 35(20):700–744, 2020.*
Available at http://www.tac.mta.ca/tac/volumes/35/20/35-20abs.html.

Elisabeth R. Green.
*Graph Products of Groups.*
PhD thesis, University of Leeds, 1990.

André Joyal.
Une théorie combinatoire des séries formelles.
*Advances in Mathematics, 42:1–82, 1981.*

Joe Moeller.
Noncommutative network models.
*Mathematical Structures in Computer Science, 30(1):14–32, 2020.*

Joe Moeller and Christina Vasilakopoulou.
Monoidal Grothendieck construction.
*Theory and Applications of Categories, 35(31):1159–1207, 2020.*
Antonio Veloso da Costa.
Graph products of monoids.
*Semigroup Forum, 63:247–277, 2001.*