**Abstract**

\( \mathbb{Z}_2 \) monopoles are important for the understanding of the Goddard-Nuyts-Olive duality when the scalar field is not in the adjoint representation. We analyze the \( \mathbb{Z}_2 \) monopole solutions in \( SU(n) \) Yang-Mills-Higgs theories spontaneously broken to \( Spin(n)/\mathbb{Z}_2 \) by a scalar in the \( n \times n \) representation. We construct explicitly a \( \mathbb{Z}_2 \) monopole asymptotic form for each of the weights of the defining representation of the dual algebra \( so(n)^\vee \).
1 Introduction

Electromagnetic duality in Yang-Mills-Higgs theories was proposed in the work of Goddard, Nuyts, and Olive (GNO) [1]. In this work they consider a theory with gauge group $G$ and a scalar field $\phi$ in an arbitrary representation which spontaneously breaks $G$ to $G_0$ in such a way that $\pi_2(G/G_0)$ is nontrivial, which is the necessary condition for the existence of topological monopoles. Since then, the monopole’s solutions [2, 3, 4, 5] and the electromagnetic duality conjecture [6] were studied intensely in the particular case where the scalar field was in the adjoint representation. In this case, the unbroken gauge group $G_0$ necessarily has a $U(1)$ factor which is generated by the scalar field vacuum solution. This $U(1)$ factor guarantees that $\pi_2(G/G_0) = \mathbb{Z}$, in which case the theory can have monopole solutions which are a generalization of 't Hooft-Polyakov solution. We shall call them $\mathbb{Z}$ monopoles. The Bogomol’nyi-Prasad Sommerfield (BPS) $\mathbb{Z}$ monopoles are conjectured to be dual to particles in the adjoint representation in the dual theory for suitable supersymmetric theories [6]. On the other hand, much less is known in the cases where $\phi$ is not in the adjoint representation and $G_0$ is semisimple. In these cases, since $\pi_2(G/G_0)$ is a cyclic group $\mathbb{Z}_n$ or a product of cyclic groups, the monopoles are called $\mathbb{Z}_n$ monopoles. Some general properties of these $\mathbb{Z}_n$ monopoles were analyzed in [1, 7, 8], and in [9] it was proposed that the $\mathbb{Z}_2$ monopole in $\mathcal{N} = 1$ $SU(n)$ super Yang-Mills should satisfy a duality transformation which is alternative to the GNO conjecture.

One of the main motivations for the study of monopoles and electromagnetic dualities is their possible application to the problem of confinement in QCD. Following the ideas of 't Hooft and Mandelstam, the confinement of particles in QCD must be a phenomenon dual to confinement of monopoles in a superconductor. In this model, the formation of chromoelectric flux tubes in QCD must be due to a monopole condensate. However, it is not clear yet if this condensate is made of $\mathbb{Z}$ monopoles, $\mathbb{Z}_n$ monopoles, or Dirac monopoles. There are some lattice results which indicate that confinement could be related to $\mathbb{Z}_n$ monopoles’ condensates [10]. Another more recent application of GNO duality is in the geometric Langlands program [11].

In the last years, the ideas of 't Hooft and Mandelstam were applied to supersymmetric non-Abelian theories which satisfy electromagnetic duality with $\mathbb{Z}$ monopoles. In particular in [12] the confinement of $\mathbb{Z}$ monopoles by the formation of magnetic flux tubes or $\mathbb{Z}_n$ strings in soft broken $\mathcal{N} = 4$ super Yang-Mills theories with an arbitrary simple gauge group was analyzed. It was shown that the tensions of these $\mathbb{Z}_n$ strings satisfy the Casimir scaling law in the BPS limit, which is believed to be the behavior that the chromoelectric flux tubes in QCD must satisfy. This result indicates that these $\mathbb{Z}_n$ strings can be dual to QCD chromoelectric strings.

In order to understand better the properties of the $\mathbb{Z}_n$ monopoles, in the present work we obtain explicitly the asymptotic form of the $\mathbb{Z}_2$ monopoles in $SU(n)$ Yang-Mills-Higgs theories with the gauge group broken to $Spin(n)/\mathbb{Z}_2$ by a scalar in the $n \times n$ representation of $SU(n)$ or its symmetric part. In this case, one could in principle embed the theory in a (deformed) $\mathcal{N} = 2$ $SU(n)$ super Yang-Mills with a hypermultiplet in the $n \times n$ representation which has vanishing $\beta$ function, similarly to $\mathcal{N} = 4$ super Yang-Mills theory.
In the work of GNO, they obtained that the possible magnetic weights of the \( \mathbb{Z}_n \) monopoles must belong to the weight lattice \( \Lambda_\omega(\tilde{G}_0^\vee) \) of the dual unbroken gauge group \( \tilde{G}_0 \), where \( \tilde{G}_0 \) means the covering group of \( G_0 \). The magnetic weights satisfy a further constraint to belong to particular cosets in \( \Lambda_\omega(\tilde{G}_0^\vee) \). In order to obtain that constraint we used the fact that for a theory with an unbroken gauge group \( G_0 = \tilde{G}_0/K \), the \( \mathbb{Z}_n \) monopole’s topological charge sectors are associated to the elements of the group \( K \) which is a subgroup of \( Z(\tilde{G}_0) \), the center of the group \( \tilde{G}_0 \). Then, we used the result that the elements of \( Z(\tilde{G}_0) \) are associated to cosets which are related to nodes of the extended Dynkin diagram of \( \tilde{G}_0^\vee \) related to the node 0 by a symmetry transformation. Therefore, the \( \mathbb{Z}_n \) monopoles must be associated to weights of a subset of these cosets. This form of writing elements of the center of a group was also used to obtain \( \mathbb{Z}_n \) strings solutions\(^{12,14,15}\). Then, using the elegant general construction of Weinberg et al.\(^7\) for the monopoles in theories where the gauge group \( SU(n) \) is broken to \( Spin(n)/\mathbb{Z}_2 \), we associated to each weight of the defining representation of the dual algebra \( so(n)^\vee \) a \( su(2) \) subalgebra and constructed explicitly a \( \mathbb{Z}_2 \) monopole asymptotic solution which we called fundamental, generalizing the \( \mathbb{Z}_2 \) monopole solution for the \( SU(3) \) gauge group \(^7\). This is consistent with the result in \(^8\) where the authors concluded that the \( \mathbb{Z}_2 \) monopoles in these theories should be associated to weights in the coset with the highest weight of the defining representation of \( so(n)^\vee \). From these fundamental \( \mathbb{Z}_2 \) monopoles we constructed other \( \mathbb{Z}_2 \) monopole asymptotic solutions. Differently from Weinberg’s general construction where the \( \mathbb{Z}_n \) monopole’s topological charge sectors were associated to integers modulo \( n \), in our construction they are associated to cosets in the weight lattice \( \Lambda_\omega(\tilde{G}_0^\vee) \), which gives in principle a larger number of possible monopoles. We construct the monopole solutions considering two symmetry breakings of \( su(n) \) to \( so(n) \): one in which \( so(n) \) is invariant under outer automorphism and another in which it is invariant under Cartan automorphism. In the first case, the monopole’s magnetic flux is in the Cartan subalgebra of \( su(n) \) but \( n \) must be odd, and in the second case the magnetic flux is not in the Cartan subalgebra of \( su(n) \). This general procedure can be generalized to other gauge groups. We expect that this explicit construction of the \( \mathbb{Z}_2 \) monopole asymptotic solutions can be useful in order to understand better the electromagnetic duality in the theories where the scalar field is not in the adjoint representation.

This paper is organized as follows: we start in Sec. 2 giving some mathematical conventions. Then, we give a brief review of the results of GNO in Sec. 3 and explain in Sec. 4 how the topological charge sectors of the \( \mathbb{Z}_n \) monopoles are associated to particular cosets in the weight lattice \( \Lambda_\omega(\tilde{G}_0^\vee) \). In Sec. 5, we obtain two scalar field configurations which break \( su(n) \) to \( so(n) \) where for the first configuration \( so(n) \) is invariant under Cartan automorphism and for the second configuration it is invariant under outer automorphism. Finally, in Sec. 6 we construct the fundamental \( \mathbb{Z}_2 \) monopoles for both symmetry breaks. We also include an appendix where we analyze the center elements of \( Spin(3) \), \( Spin(5) \), \( Spin(6) \).
2 Mathematical conventions

Let us start by giving some conventions which will be used later on. Let $g$ be the Lie algebra of rank $r$ associated to the group $G$. Let us adopt the Cartan-Weyl basis. In this basis, the commutation relations read

\begin{align}
[H_i, E_\alpha] &= (\alpha)^i E_\alpha, \\
[E_\alpha, E_{-\alpha}] &= \frac{2\alpha}{\alpha^2} H,
\end{align}

where generators $H_i, i = 1, 2, ..., r$, form a basis for the Cartan subalgebra (CSA) $h$, $\alpha$ are roots and the upper index in $(\alpha)^i$ means the component $i$ of $\alpha$.

Given a representation of $g$, we can take a basis $\{|\mu\rangle\}$ in which the elements of the CSA, $H_i, i = 1, 2, ..., r$, are diagonal,

$$H_i |\mu\rangle = (\mu)^i |\mu\rangle, \quad i = 1, 2, ..., r.$$ 

The vector $\mu$ with the $r$ eigenvalues $(\mu)^i$ as components is called weight and $|\mu\rangle$ is called weight state.

We shall denote by $\alpha_i, i = 1, 2, ..., r$, the simple roots of $g$ which is a basis of the root space and by $\lambda_i, i = 1, 2, ..., r$, the fundamental weights of $g$. Moreover we shall call

$$\alpha_i^\vee = \frac{2\alpha_i}{\alpha^2}, \quad \lambda_i^\vee = \frac{2\lambda_i}{\alpha^2}$$

the simple coroots and fundamental coweights respectively. They satisfy the relations

$$\alpha_i \cdot \lambda_j^\vee = \alpha_i^\vee \cdot \lambda_j = \delta_{ij}.$$  \hspace{1cm} (3)

$\alpha_i^\vee$ and $\lambda_i^\vee$ are respectively simple roots and fundamental weights of the dual algebra $g^\vee$.

Let us denote by $\tilde{G}$ the covering group of $G$. Then, the fundamental weights form a basis for the weight lattice of $\tilde{G}$,

$$\Lambda_w(\tilde{G}) = \left\{ \mu = \sum_{i=1}^{r} n_i \lambda_i, \quad n_i \in \mathbb{Z} \right\}. \hspace{1cm} (4)$$

This lattice includes as a subset, the root lattice of $G$,

$$\Lambda_r(G) = \left\{ \beta = \sum_{i=1}^{r} n_i \alpha_i, \quad n_i \in \mathbb{Z} \right\}, \hspace{1cm} (5)$$

which has the simple roots $\alpha_i$ as the basis. Similarly, the fundamental coweights $\lambda_i^\vee$ are the basis of the weight lattice of the dual group $\tilde{G}^\vee$

$$\Lambda_w(\tilde{G}^\vee) = \left\{ \mu = \sum_{i=1}^{r} n_i \lambda_i^\vee, \quad n_i \in \mathbb{Z} \right\} \hspace{1cm} (6)$$

\[^{3}\text{We shall adopt the convention of using capital letters to denote Lie groups and lower letters for Lie algebras.}\]
which is also called the coweight lattice of $\tilde{G}$ and which has the root lattice of the dual group $G^\vee$ (or coroot lattice of $G$)

$$\Lambda_r(G^\vee) = \left\{ \beta = \sum_{i=1}^{r} n_i \alpha_i^\vee, \quad n_i \in \mathbb{Z} \right\}$$  \hspace{1cm} (7)

as a subset.

3 Magnetic monopoles in non-Abelian theories

In a theory with gauge group $G$ spontaneously broken to $G_0$, the monopole’s solutions are associated to elements of the second homotopy group

$$\pi_2(G/G_0) = \text{Ker} \left( \pi_1(G_0) \to \pi_1(G) \right).$$  \hspace{1cm} (8)

This result implies that monopoles are associated with nontrivial elements of $\pi_1(G_0)$ which correspond to trivial elements of $\pi_1(G)$. Therefore, the relation (8) is equivalent to

$$\pi_2(\tilde{G}/G_0') = \pi_1(G_0')$$

where $G_0'$ is the unbroken subgroup of $\tilde{G}$. Therefore for simplicity, without loss of generality, we shall consider that $G$ is simply connected.

Let us therefore consider a Yang-Mills-Higgs theory with gauge group $G$ which we shall consider to be simple and simply connected. Let us also consider that in the theory there is a scalar $\phi$ in a representation $R$ of $G$ and $\phi_0$ is a vacuum configuration which spontaneously breaks $G$ to a subgroup $G_0$, in such a way that $\pi_2(G/G_0)$ is nontrivial, which allows the existence of magnetic monopoles. The generators of $G_0$ are those which annihilate $\phi_0$, that is,

$$T_a \phi_0 = 0.$$

We shall denote by $g_0$ the algebra formed by these generators.

Let us briefly review some general properties of these monopoles which will also be useful to fix our notation. Following GNO [1], we shall consider static finite energy monopoles with the asymptotic form of the magnetic field of the form

$$B_i(\theta, \varphi) = \frac{r_i}{4\pi r^3} X(\theta, \varphi), \quad \text{with} \quad D_i X(\theta, \varphi) = 0,$$  \hspace{1cm} (9)

where $D_i \equiv \partial_i + ieW_i$ and $\theta, \varphi$ are the angular spherical coordinates. The finite energy asymptotic condition $D_i \phi = 0$, implies that asymptotically we can write [1]

$$\phi(\theta, \varphi) = g(\theta, \varphi)\phi_0,$$  \hspace{1cm} (10)

where $g(\theta, \varphi) \in G$. Then, the condition that $D_i X(\theta, \varphi) = 0$ implies that

$$X(\theta, \varphi) = g(\theta, \varphi)X_0 g(\theta, \varphi)^{-1}$$  \hspace{1cm} (11)
with \( X_0 \equiv X(\theta = 0, \varphi = 0) \). The asymptotic condition \( D_i \phi = 0 \) and the definition of the field strength as the commutator of covariant derivatives, implies that asymptotically \( B_i \phi = 0 \). Then, using (9), (10), and (11) results that \( X_0 \phi_0 = 0 \) and therefore, \( X_0 \in g_0 \). Moreover, one can write [1]

\[
X_0 = \omega \cdot h \tag{12}
\]

where \( \omega \cdot h = \sum \omega_i h_i \), with \( h_i \) being the elements of the Cartan subalgebra (CSA) of \( g_0 \) and \( \omega \) is a constant vector. Note that in general, the elements of the CSA of \( g_0 \) do not necessarily belong to the CSA of \( g \), the Lie algebra of \( G \). Therefore, we shall denote by \( h_i, f_\alpha \) the generators of \( g_0 \) and \( H_i \) and \( E_\alpha \) the generators of \( g \).

One can show the quantization condition [1]

\[
\exp [ieX_0] = \exp [ie\omega \cdot h] = 1. \tag{13}
\]

Let us consider that \( G_0 \) is semisimple[4]. In this case \( G_0 \) can be written as

\[
G_0 = \tilde{G}_0/K(G_0)
\]

where \( \tilde{G}_0 \) is the universal covering group of \( G_0 \) and the factor \( K(G_0) \) is the kernel of the homomorphism \( \tilde{G}_0 \to G_0 \). The factor \( K(G_0) \) is a discrete subgroup of the center of \( \tilde{G}_0 \) which we will denote by \( Z(\tilde{G}_0) \).

Therefore, the topological charge sectors of the theory are associated to

\[
\pi_2 (G/G_0) = \pi_1 (G_0) = K(G_0) \subset Z(\tilde{G}_0). \tag{14}
\]

Since \( K(G_0) \) is a cyclic group \( \mathbb{Z}_n \) or a product of cyclic groups, then these monopoles are called \( \mathbb{Z}_n \) monopoles.

Considering the condition (13) in \( \tilde{G}_0 \) rather than in \( G_0 \) implies that [1]

\[
\exp [i e \omega \cdot h] \in K(G_0) \subset Z(\tilde{G}_0), \tag{15}
\]

where \( \exp \) denotes the exponential mapping in \( \tilde{G}_0 \). Using the fact that the elements of the center \( Z(\tilde{G}_0) \) of a group \( G_0 \), have the form

\[
\exp [2\pi i v \cdot h] \tag{16}
\]

where \( v \) is a vector of the coweight lattice \( \Lambda_w(\tilde{G}_0^\vee) \), Goddard, Nuyts, and Olive concluded that the so-called magnetic weights must satisfy

\[
e\omega/2\pi \in \Lambda_w(\tilde{G}_0^\vee)\]

together with condition (15). From this result they conjectured that the monopoles should be dual to particles in a theory with unbroken gauge group \( \tilde{G}_0^\vee \).

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[4] Note that if the scalar field \( \phi \) is in the adjoint representation, then \( G_0 \) is not semisimple since it has a \( U(1) \) factor generated by the vacuum configuration. This case is considered in detail in [3].
4 Topological charge sectors of $\mathbb{Z}_n$ monopoles

Let us now analyze how the different values of $e\omega/2\pi$ are associated to the different elements of $K(G_0) \subset Z(\tilde{G}_0)$ or topological charge sectors \([14]\) of the theory. We shall also restrict the possible values $e\omega/2\pi$ can take. In order to do this we must remember that since $\Lambda_r(G'_0)$ is a sublattice of $\Lambda_w(G'_0)$, we can define the quotient $\Lambda_w(G'_0)/\Lambda_r(G'_0)$. This quotient can be represented by the cosets \([17]\)

$$
\Lambda_r(G'_0), \quad \lambda^\vee_{\tau(0)} + \Lambda_r(G'_0), \quad \lambda^\vee_{\tau(2)(0)} + \Lambda_r(G'_0), \ldots, \quad \lambda^\vee_{\tau(n)(0)} + \Lambda_r(G'_0)
$$

where the weights $\lambda^\vee_{\tau(0)}$ are associated to nodes of an extended Dynkin diagram of $G'_0$ related to the node 0 by a symmetry transformation. In Table 1 we used black nodes to denote these nodes in the extended Dynkin diagrams. One can then show that the center of $G_0$ is a discrete group isomorphic to the classes\([13]\)

$$
Z(\tilde{G}_0) = \{ \exp[2\pi i \Lambda_r(G'_0) \cdot h], \exp[2\pi i (\lambda^\vee_{\tau(0)} + \Lambda_r(G'_0)) \cdot h], \ldots, \exp[2\pi i (\lambda^\vee_{\tau(n)(0)} + \Lambda_r(G'_0)) \cdot h], \ldots \}.
$$

In other words, all the group elements \([16]\) with the vector $v$ in the same coset of \([17]\) correspond to the same element of the center $Z(\tilde{G}_0)$. In particular when $v$ belongs to $\Lambda_r(G'_0)$, the group elements \([16]\) correspond to the identity of $Z(\tilde{G}_0)$. Such a way of writing the center elements is also quite useful to analyze the $\mathbb{Z}_N$ string solutions which appear when the gauge group $G$ is broken to its center group $Z(G)$\([12]\).\([14]\).\([15]\). In \([14]\) the center group of some groups is analyzed in some more detail.

Since $K(G_0) \subset Z(\tilde{G}_0)$, it will be formed by a subset of elements of \([18]\). Therefore, the magnetic weights $e\omega/2\pi$ cannot belong to an arbitrary coset of \([17]\), but only to those cosets associated to elements of $K(G_0)$, the kernel of the homomorphism $\tilde{G}_0 \rightarrow G_0$. Moreover, $\mathbb{Z}_n$ monopoles will be in the same topological sector if their associated magnetic weights are in the same coset. In the next sections we will analyze the values the magnetic weights can take when the gauge group $SU(n)$ is broken to $Spin(n)/\mathbb{Z}_2$.

5 Gauge symmetry breaking

Let us now consider a Yang-Mills theory with gauge group $SU(n)$ and with a scalar field $\phi$ in the representation which is the direct product $n \times n$ of $SU(n)$. In this case the theory can in principle be embedded in a (deformed) $\mathcal{N} = 2$ $SU(n)$ Super Yang-Mills with an hypermultiplet in the $n \times n$ representation which has a vanishing $\beta$ function. Such a theory has already been considered in \([18]\) but with a different gauge symmetry breaking, which gave rise to $\mathbb{Z}_k$ strings and $\mathbb{Z}$ monopole confinement. We can also consider that $\phi$ is in the symmetric part of $n \times n$, since the vacuum solutions $\phi_0$ which we will consider are nontrivial only in the symmetric part of $n \times n$. The specific form of the potential is not important to determine the asymptotic form of the $\mathbb{Z}_2$ monopoles.
Table 1: Extended Dynkin diagrams, nodes symmetrically related to the node 0 and center groups $Z(G)$.
In order for \( \mathbb{Z}_2 \) monopoles to exist, we want to find configurations of the scalar field which break \( SU(n) \) to \( Spin(n)/\mathbb{Z}_2 \), \( n \geq 3 \). We first analyze some \( so(n) \) subalgebras of \( su(n) \). We shall consider that \( n \neq 4 \), since \( so(4) = su(2) \oplus su(2) \) is not simple. We shall consider \( so(n) \) invariant subalgebras under Cartan or outer automorphisms which are order two automorphisms or involutions. Remembering that, under an order two automorphism \( \sigma \) of a Lie algebra \( g \), \( \sigma \) has eigenvalues \( \exp(\pi ip) \), \( p = 0,1 \) and the Lie algebra split into \( g(0) \) and \( g(1) \), where \( g(p) \) is formed by the generators \( T_a \), such that \( \sigma(T_a) = e^{\pi ip}T_a \), \( p = 0,1 \). Moreover, \( g(0) \) forms a subalgebra of \( g \). The quotient of the group generated by \( g \) modulo the group generated by \( g(0) \) is a symmetric space and \( g(1) \) is associated to a representation of \( g(0) \).\(^{[19]}\)

5.1 Breaking of \( su(n) \) to \( so(n) \) invariant under Cartan automorphism

The Cartan automorphism for a general Lie algebra \( g \) is defined by

\[
\sigma(H_i) = -H_i, \\
\sigma(E_\alpha) = -E_{-\alpha}.
\]

For this automorphism, \( g(0) \) and \( g(1) \) are formed by the generators

\[
g^{(0)} = \{E_\alpha - E_{-\alpha}, \alpha > 0\}, \\
g^{(1)} = \{H_a, a = 1,2,...,\text{rank}(g); E_{\alpha} + E_{-\alpha}, \alpha > 0\}.
\]  

Let us consider \( g = su(n) \). The generators of \( su(n) \) in the \( n \)-dimensional representation can be written in terms of the \( n \times n \) matrices \( E_{ij} \) with components \( (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \) or

\[
E_{ij}|e_j\rangle = |e_i\rangle
\]  

where \( |e_i\rangle \) are the weight states of the \( n \)-dimensional representation. Then, the basis elements of the CSA of \( su(n) \) correspond to the traceless combinations \( E_{ij} - E_{j+1,i+1,j} = 1,2,...,n-1 \). On the other hand, the generator \( E_{ij}, i \neq j \), is the step operator associated to the root \( e_i - e_j \), where \( e_i \) are orthonormal vectors in the \( n \)-dimensional vector space with \( (e_i)_k = \delta_{ik} \). The root \( e_i - e_j \) is positive (negative) if \( i < j \) (\( i > j \)) and is a simple root if \( j = i + 1 \). The weight associated to \( |e_i\rangle \) can be written as

\[
e_i = \frac{1}{n} \sum_{j=1}^{n} e_j.
\]

From \(^{[19]}\), we can conclude that the generators of \( g^{(0)} \), in the \( n \)-dimensional representation are

\[
M_{ij} = -i (E_{ij} - E_{ji}), \ i < j,
\]  

which are \( n(n-1) \) antisymmetric \( n \times n \) matrices which form a \( so(n) \) subalgebra of \( su(n) \). Therefore, for \( g = su(n) \), \( g^{(0)} = so(n) \) for \( n \geq 3 \)\(^{[19]}\).
For example, in the $su(3)$ case, the Gell-Mann matrices

\begin{align*}
\lambda_2 &= -i (E_{12} - E_{21}) = -i (E_{\alpha_1} - E_{-\alpha_1}), \\
\lambda_5 &= -i (E_{13} - E_{31}) = -i (E_{\alpha_1+\alpha_2} - E_{-\alpha_1-\alpha_2}), \\
\lambda_7 &= -i (E_{23} - E_{32}) = -i (E_{\alpha_2} - E_{-\alpha_2})
\end{align*}

form a $so(3)$ invariant subalgebra under Cartan automorphism.

Let us consider the scalar field configuration

\[ \phi_0 = v \sum_{i=1}^{n} |e_i \rangle \otimes |e_i \rangle \quad (23) \]

where $v$ is a constant. Using the fact that in a tensor product representation a generator $T$ acts as $T \otimes 1 + 1 \otimes T$, it is straightforward to conclude that $\phi_0$ is annihilated only by the generators (22) and hence breaks $su(n)$ to $so(n)$ subalgebra generated by (22). Therefore, we shall consider that $\phi_0$ is the vacuum configuration responsible for the symmetry breaking.

Let us determine an orthogonal basis of the Cartan subalgebra (CSA) of the above $so(n)$ invariant subalgebra (22). For $so(3)$, we can consider the Gell-Mann matrix $\lambda_2$ as the generator of the Cartan subalgebra. Recalling that $so(2m)$ and $so(2m+1)$ have same rank equal to $m$, one can check that the generators

\[ h_k = -i (E_{\alpha_{2k-1}} - E_{-\alpha_{2k-1}}), \quad k = 1, 2, ..., m, \quad (24) \]

form an orthogonal basis of the Cartan subalgebras of $so(n)$ for $n = 2m, 2m + 1$, and where $E_{\alpha_k}$ are generators of $su(n)$. It is important to note that for this $so(n)$ subalgebra invariant under Cartan automorphism, the CSA of $so(n)$ is not in the CSA of $su(n)$. We shall denote by $h_i, f_{\alpha}$ the generators of the subalgebra $so(n)$ in order to distinguish from generators $H_i, E_{\alpha}$ of $su(n)$.

### 5.2 Breaking of $su(2m+1)$ to $so(2m+1)$ invariant under outer automorphism

A Dynkin diagram which is invariant under a transformation of the nodes, $i \rightarrow \tau(i)$, implies that the corresponding Cartan matrix satisfies

\[ K_{\tau(i)\tau(j)} = K_{ij}. \]

As a consequence the associated Lie algebra $g$ has an outer automorphism $^\tau$:

\begin{align*}
\tau(\alpha \cdot H) &= \tau(\alpha) \cdot H, \\
\tau(E_{\alpha}) &= \chi_{\alpha} E_{\tau(\alpha)},
\end{align*}

\footnote{For a review see for example [19, 17].}
where $\chi_\alpha = \pm 1$.

In particular, for $su(2m+1)$, the Dynkin diagram is invariant under the transformation of the nodes $j \to 2m+1-j$. Let $\langle j \rangle$ denote the orbit of nodes $j$ and $2m+1-j$ connected under this transformation. Using the so-called folding procedure one can show that the invariant subalgebra of $su(2m+1)$ under this automorphism is $so(2m+1)$ \cite{17,19}. Let $H_\alpha$ and $E_\alpha$, be generators of $su(2m+1)$. Then, the invariant $so(2m+1)$ subalgebra has the following generators

$$
H_{\langle l \rangle} = \sum_{i \in \langle l \rangle} \alpha_i \cdot H = H_{\alpha_l} + H_{\alpha_{2m+1-l}}, \quad \text{for } l = 1, 2, ..., m-1,
$$

$$
H_{\langle m \rangle} = 2 \sum_{i \in \langle m \rangle} \alpha_i \cdot H = 2 \left( H_{\alpha_m} + H_{\alpha_{m+1}} \right),
$$

$$
E_{\pm \langle l \rangle} = \sum_{i \in \langle l \rangle} E_{\pm \alpha_i} = E_{\pm \alpha_l} + E_{\pm \alpha_{2m+1-l}}, \quad \text{for } l = 1, 2, ..., m-1,
$$

$$
E_{\pm \langle m \rangle} = \sqrt{2} \sum_{i \in \langle m \rangle} E_{\pm \alpha_m} = \sqrt{2} \left( E_{\pm \alpha_m} + E_{\pm \alpha_{m+1}} \right),
$$

where $H_\alpha = 2\alpha \cdot H/\alpha^2$.

Let $\alpha_i$ and $\lambda_i$ be, respectively, simple roots and fundamental weights of $su(2m+1)$. Then, the simple roots, simple coroots, fundamental weights and coweights of the invariant subalgebra $so(2m+1)$ are

$$
\alpha_{\langle l \rangle} = \frac{1}{2} \left( \alpha_l + \alpha_{2m+1-l} \right), \quad \text{for } l = 1, 2, ..., m
$$

$$
\alpha_{\langle m \rangle}^\vee = \frac{2\alpha_{\langle m \rangle}}{\alpha_{\langle l \rangle}^2} = \alpha_l + \alpha_{2m+1-l}, \quad \text{for } l = 1, 2, ..., m-1,
$$

$$
\alpha_{\langle m \rangle}^\vee = \frac{2\alpha_{\langle m \rangle}}{\alpha_{\langle l \rangle}^2} = 2 \left( \alpha_m + \alpha_{m+1} \right),
$$

$$
\lambda_{\langle l \rangle} = \frac{1}{2} \left( \lambda_l + \lambda_{2m+1-l} \right), \quad \text{for } l = 1, 2, ..., m-1,
$$

$$
\lambda_{\langle m \rangle} = \frac{1}{4} \left( \lambda_m + \lambda_{m+1} \right),
$$

$$
\lambda_{\langle l \rangle}^\vee = \frac{2\lambda_{\langle l \rangle}}{\alpha_{\langle l \rangle}^2} = \lambda_l + \lambda_{2m+1-l}, \quad \text{for } l = 1, 2, ..., m.
$$

One can check easily that the scalar products between the simple roots give the Cartan matrix of $so(m+1)$ and that simple roots and fundamental weights satisfy the right orthonormality conditions.

As in the Cartan automorphism, we are looking for a vacuum configuration $\phi_0$ which is annihilated by the generators given by (25), that is, which breaks $su(2m+1)$ to the $so(2m+1)$ subalgebra invariant by outer automorphism. Let us consider the scalar field configuration.
\[ \phi_0 = \nu \sum_{l=1}^{2m+1} (-1)^{l+1} |e_l\rangle \otimes |e_{2m+2-l}\rangle , \]  

(27)

where \( \nu \) is a constant. Since \( \alpha_l + \alpha_{2m+1-l} = e_l - e_{l+1} + e_{2m+1-l} - e_{2m+2-l} \), we can obtain directly that

\[ H_{(l)} \phi_0 = 0, \quad \text{for } l = 1, 2, \ldots, m. \]

With respect to the folded step operators \( E_{(l)} \) we can use the fact that

\[ E_{\alpha_j} = E_{\gamma_j+1} \]

which implies that

\[ E_{(l)} \phi_0 = 0, \quad \text{for } l = 1, 2, \ldots, m. \]

Therefore we can consider that \( \phi_0 \) given by Eq. (27) is a configuration which breaks \( su(2m+1) \) to the \( so(2m+1) \) subalgebra invariant by outer automorphism.

### 5.3 Unbroken gauge group

The above \( so(n) \) subalgebras of \( su(n) \) generates subgroups \( G_0 = Spin(n)/K(G_0) \) of \( SU(n) \), where \( K(G_0) \) is a subgroup of the center of \( Spin(n) \) which we want to determine.

Following [20], in order to determine the factor \( K(G_0) \) of the subgroup \( G_0 = \tilde{G}_0/K(G_0) \) of \( G = \tilde{G}/K(G) \), we must first choose a representation \( R_{\lambda}(\tilde{G}) \) of \( \tilde{G} \) with highest weight \( \lambda \) such that the Ker\( (R_{\lambda}(\tilde{G})) = K(G) \). If \( R_{\lambda}(\tilde{G}) \) branches to the representation \( R_{\tilde{\lambda}}(\tilde{G}_0) \) of \( \tilde{G}_0 \) with highest weight \( \tilde{\lambda} \), then \( K(G_0) = \text{Ker}(R_{\tilde{\lambda}}(\tilde{G}_0)) \).

Therefore, in order to determine the discrete group \( K(G_0) \) of the unbroken gauge subgroup \( Spin(n)/K(G_0) \) of \( SU(n) \) we shall choose the \( n \)-dimensional representation \( R_{\lambda_1}(SU(n)) \) of \( SU(n) \) with highest weight \( \lambda_1 \) since in this representation Ker\( (R_{\lambda_1}(SU(n))) = 1 \) is well known. Then, for the above two different embeddings of \( so(n) \) in \( su(n) \), the \( n \)-dimensional irrep. of \( su(n) \) branches to the \( n \)-dimensional irreducible representation (irrep) of \( so(n) \), which has \( \lambda_1 \) as highest weight. The weight states of this representation are of the form \( |\lambda_1 - \gamma\rangle \), where \( \gamma \) are positive roots of \( so(n) \). The kernel Ker\( (R_{\lambda_1}(Spin(n))) \) of this representation is made by the elements \( g \in Spin(n) \) such that

\[ g|\lambda_1 - \gamma\rangle = |\lambda_1 - \gamma\rangle , \]

for all weight states \( |\lambda_1 - \gamma\rangle \) of the representation. Since Ker\( (R_{\lambda_1}(Spin(n))) \) is a subgroup of the center \( Z(Spin(n)) \), we just need to act the elements of \( Z(Spin(n)) \) on the weight states of the representation. Let us consider \( Spin(n) \) with \( n \geq 7 \). In the Appendix we analyze the particular cases of the groups \( Spin(n) \) for \( n = 3, 5, 6 \). From the symmetry of the extended Dynkin diagram in Table 1 we can conclude that the weight lattice \( \Lambda_w(Spin(2n+1)^\vee) \), \( n \geq 3 \) split in

\[ \Lambda_r(Spin(2n+1)^\vee), \quad \Lambda_r^\vee + \Lambda_r(Spin(2n+1)^\vee) \]  

(28)
and the center of $\text{Spin}(2n+1)$, $n \geq 3$ is

$$Z(\text{Spin}(2n+1)) = \mathbb{Z}_2 \cong \{\exp(2\pi i \alpha^\vee \cdot h), \exp[2\pi i (\lambda_1^\vee + \alpha^\vee) \cdot h]\},$$

where $\alpha^\vee \in \Lambda_r(\text{Spin}(2n+1)^\vee)$. Acting these elements on the weight states of the $(2n+1)$-dimensional representation of $\text{so}(2n+1)$ we obtain

$$ \exp(2\pi i \alpha^\vee \cdot h) |\lambda_1 - \gamma\rangle = |\lambda_1 - \gamma\rangle, $$

$$ \exp[2\pi i (\lambda_1^\vee + \alpha^\vee) \cdot h] |\lambda_1 - \gamma\rangle = |\lambda_1 - \gamma\rangle. $$

(29)

For $\text{Spin}(2n)$, we have

$$Z(\text{Spin}(2n)) = \{ \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ if } 2n = 4k, \mathbb{Z}_4 \text{ if } 2n = 4k + 2, \}$$

where $k \in \mathbb{N}$. In both cases the weight lattice split in the four cosets,

$$\Lambda_r(\text{Spin}(2n)^\vee), \; \lambda_1^\vee + \Lambda_r(\text{Spin}(2n)^\vee), \; \lambda_{n-1}^\vee + \Lambda_r(\text{Spin}(2n)^\vee), \; \lambda_n^\vee + \Lambda_r(\text{Spin}(2n)^\vee), \; \lambda_r^\vee,$$

(30)

and

$$Z(\text{Spin}(2n)) \cong \{ \exp(2\pi i \alpha^\vee \cdot h), \exp[2\pi i (\lambda_1^\vee + \alpha^\vee) \cdot h],$$

$$ \exp[2\pi i (\lambda_{n-1}^\vee + \alpha^\vee) \cdot h] \exp[2\pi i (\lambda_n^\vee + \alpha^\vee) \cdot h] \} ,$$

where $\alpha^\vee \in \Lambda_r(\text{Spin}(2n)^\vee)$. Acting these elements on the weight states of the $2n$-dimensional representation of $\text{so}(2n)$ we obtain

$$ \exp(2\pi i \alpha^\vee \cdot h) |\lambda_1 - \gamma\rangle = |\lambda_1 - \gamma\rangle, $$

$$ \exp[2\pi i (\lambda_1^\vee + \alpha^\vee) \cdot h] |\lambda_1 - \gamma\rangle = |\lambda_1 - \gamma\rangle, $$

$$ \exp[2\pi i (\lambda_{n-1}^\vee + \alpha^\vee) \cdot h] |\lambda_1 - \gamma\rangle = -|\lambda_1 - \gamma\rangle, $$

$$ \exp[2\pi i (\lambda_n^\vee + \alpha^\vee) \cdot h] |\lambda_1 - \gamma\rangle = -|\lambda_1 - \gamma\rangle. $$

(31)

Therefore, from (29) and (31), we can conclude that for $n$ odd or even,

$$K(G_0) = \text{Ker}(R_{\lambda_1}(\text{Spin}(n))) $$

$$ = \mathbb{Z}_2 \cong \{ \exp[2\pi i \lambda_1(\text{Spin}(n)^\vee) \cdot h], \exp[2i\pi (\lambda_1^\vee + \Lambda_r(\text{Spin}(n)^\vee)) \cdot h] \} $$

and the symmetry breaking is of the form

$$\text{SU}(n) \rightarrow \text{Spin}(n)/\mathbb{Z}_2$$

Hence we can conclude that for this symmetry breaking, the magnetic weights $e\omega/2\pi$ must belong to the cosets

$$\Lambda_r(\text{Spin}(n)^\vee), \; \lambda_1^\vee + \Lambda_r(\text{Spin}(n)^\vee),$$

(32)

where $\lambda_1^\vee$ is the highest weight of the defining representation of $\text{so}(n)^\vee$ which has dimension $2m$ for $\text{so}(2m)^\vee = \text{so}(2m)$ and for $\text{so}(2m+1)^\vee = \text{sp}(2m)$. This result holds also for the special cases $\text{Spin}(3)$, $\text{Spin}(5)$ and $\text{Spin}(6)$, as is analyzed in the Appendix. A similar result was obtained in [8] using a different approach.
6 $\mathbb{Z}_2$ monopole’s asymptotic configuration

We want to construct explicitly the asymptotic form for static spherically symmetric $\mathbb{Z}_2$ monopole’s solutions. In order to do that let us define

$$T_3^\beta = \frac{\beta \cdot h}{2}, \quad (33)$$

where

$$\beta = \frac{\epsilon \omega}{2\pi} \in \Lambda_\omega(\tilde{G}_0^\vee)$$

and from Eq. (15) implies that

$$\exp\left[4\pi i T_3^\beta\right] \in K(G_0).$$

Let us also consider other two generators $T_1^\beta$, $T_2^\beta$ of $g$, but not of $g_0$, such that

$$\left[T_i^\beta, T_j^\beta\right] = i\epsilon_{ijk} T_k^\beta.$$

The choice of these two generators $T_1^\beta$ and $T_2^\beta$ will be discussed in detail in sections 6.1 and 6.2. Following E. Weinberg et al. [7], we can construct a spherically symmetric monopole, consistent with GNO results, with the asymptotic form of the scalar field given by (10) with the group element $g(\theta, \varphi)$ of the form

$$g(\theta, \varphi) = \exp[-i\varphi T_3^\beta] \exp[-i\theta T_2^\beta] \exp[i\varphi T_3^\beta]$$

and the asymptotic form for the gauge field given by [7]

$$W_i(\theta, \varphi) = g(\theta, \varphi) W_i^0 g(\theta, \varphi)^{-1} + \frac{i}{e} \left(\partial_i g(\theta, \varphi)\right) g(\theta, \varphi)^{-1} \quad (34)$$

where

$$W_r^0 = W_\theta^0 = 0, \quad W_\varphi^0 = \frac{T_3^\beta}{e} (1 - \cos \theta).$$

This gauge field produces the magnetic field

$$B_i(\theta, \varphi) = \frac{r_i}{4\pi r^3} g(\theta, \varphi) T_3^\beta g(\theta, \varphi)^{-1}$$

$$= \frac{r_i}{4\pi r^3} g(\theta, \varphi) \omega \cdot h g(\theta, \varphi)^{-1}$$

$$= \frac{r_i}{4\pi r^3} X(\theta, \varphi),$$

consistent with Eqs. (9), (11).
Note that in our construction, we have a difference from Weinberg’s construction. In his construction, the $\mathbb{Z}_k$ monopoles in a given topological charge class were associated to the same integer modulo $k$. On the other hand, in our construction, $\mathbb{Z}_k$ monopoles are in the same topological sector when they are associated to magnetic weights $\beta = \omega/2\pi$ in the same coset in (17).

Using the identity
\[ \exp (iaT_j) T_i \exp (-iaT_j) = (\cos a) T_i + (\sin a) \epsilon_{ijk} T_k, \quad i \neq j \] (36)
where $a$ is a constant and $T_i, i = 1, 2, 3$ are generators of a $su(2)$ algebra, we can write the asymptotic form of the gauge fields (34) as
\[ W_\theta(\theta, \phi) = -\frac{1}{e} \left[ (\cos \varphi) T_2^\beta - (\sin \varphi) T_1^\beta \right], \]
\[ W_\phi(\theta, \phi) = \frac{\sin \theta}{e} \left[ - (\sin \theta) T_3^\beta + \cos \theta \left( (\cos \varphi) T_1^\beta + (\sin \varphi) T_2^\beta \right) \right], \]
\[ W_r(\theta, \phi) = 0. \]

Using Eq. (36) we can also rewrite the $\mathbb{Z}_2$ monopole asymptotic magnetic field (35) as
\[ B_i(\theta, \phi) = \frac{r_i}{er^3} \left[ (\sin \theta \cos \varphi) T_1^\beta + (\sin \theta \sin \varphi) T_2^\beta + (\cos \theta) T_3^\beta \right] \]
\[ = \frac{r_i}{er^3} \sum_{j=1}^{3} r_j T_j^\beta, \] (37)
which is the standard hedgehog form for the magnetic field.

Let us now analyze the possible monopole solutions for both symmetry breakings discussed in the previous sections.

### 6.1 so(n) invariant under Cartan automorphism

Let us determine the possible $su(2)$ subalgebra’s generators $T_i^\beta$ for the symmetry breaking of $SU(n) \to Spin(n)/\mathbb{Z}_2$ where $Spin(n)$ is the subgroup invariant under Cartan automorphism. We have that $T_3^\beta = \beta \cdot \hbar/2$ where the Cartan elements $\hbar_i$ are given by (24). Then, from Eq. (32), we conclude that for the $\mathbb{Z}_2$ monopoles associated to the nontrivial sector, the vector $\beta$ must belong to the coset
\[ \lambda_i^\vee + \Lambda_r(Spin(n)^\vee). \]

This coset has in particular the weights of the defining representation of the dual algebra $so(n)^\vee$ which has $\lambda_i^\vee$ as the highest weight. We know that $so(2m)^\vee = so(2m)$ and $so(2m + 1)^\vee = sp(2m)$, and that the weights of the defining representation of $so(2m)$ and $sp(2m)$ have dimension $2m$. In terms of the orthonormal vectors these weights can be written in both cases as
\[ \pm e_i, \quad i = 1, 2, ..., m. \]
For each weight \( e_k \), we can construct a \( su(2) \) subalgebra

\[
T_3^{\pm e_k} = \pm \frac{1}{2} e_k \cdot h = \pm \frac{1}{2} h_k = \pm \frac{E_{\alpha_{2k-1}} - E_{-\alpha_{2k-1}}}{2i},
\]

\[
T_1^{\pm e_k} = \frac{\alpha_{2k-1} \cdot H}{\alpha_{2k-1}},
\]

\[
T_2^{\pm e_k} = \pm \frac{E_{\alpha_{2k-1}} + E_{-\alpha_{2k-1}}}{2},
\]

for \( k = 1, 2, ..., m \). From Eq. (19) we can conclude that \( T_3^{\pm e_k} \in so(n) \) and \( T_1^{\pm e_k}, T_2^{\pm e_k} \notin so(n) \). Therefore, for each weight of the defining representation of \( so(n)^\vee \) we have a \( \mathbb{Z}_2 \) monopole solution (10), (34), (37).

We can construct monopole asymptotic forms with magnetic charge associated to others elements of the cosets (32). However, the \( su(2) \) generators associated to these new monopoles seem to be always combination of the generators (38) and therefore these monopoles can be interpreted as superpositions of the above monopoles which we call fundamental. Some examples of these \( su(2) \) subalgebras are

\[
T_3^{\pm n_k e_k} = \pm \sum_{k=1}^{m} \frac{1}{2} n_k e_k \cdot h = \pm \sum_{k=1}^{m} \frac{1}{2} n_k h_k = \pm \sum_{k=1}^{m} n_k \frac{E_{\alpha_{2k-1}} - E_{-\alpha_{2k-1}}}{2i},
\]

\[
T_1^{\pm n_k e_k} = \sum_{k=1}^{m} n_k \frac{\alpha_{2k-1} \cdot H}{\alpha_{2k-1}},
\]

\[
T_2^{\pm n_k e_k} = \pm \sum_{k=1}^{m} n_k \frac{E_{\alpha_{2k-1}} + E_{-\alpha_{2k-1}}}{2},
\]

for \( n_k = 0, 1 \).

We can understand easily the \( \mathbb{Z}_2 \) nature of these monopoles: writing the fundamental weight \( \lambda_1 \) in the basis of simple roots, we have that for \( sp(2m) = so(2m + 1)^\vee \)

\[
\lambda_1 = \alpha_1 + \alpha_2 + \ldots + \alpha_{m-1} + \frac{1}{2} \alpha_m
\]

and for \( so(2m) = so(2m)^\vee \)

\[
\lambda_1 = \alpha_1 + \alpha_2 + \ldots + \alpha_{m-2} + \frac{1}{2} (\alpha_{m-1} + \alpha_m).
\]

In both cases we see that \( 2\lambda_1 \in \Lambda_r (Spin(n)^\vee) \), and therefore a combination of an even number of fundamental \( \mathbb{Z}_2 \) monopoles will result in a configuration associated to \( \Lambda_r (Spin(n)^\vee) \) which corresponds to the trivial element \( 1 \) of the group \( \mathbb{Z}_2 \), and an odd combination will result on a configuration associated to the nontrivial coset, which corresponds to element \( -1 \) of the group \( \mathbb{Z}_2 \). We must also note that two configurations belonging to the same topological sector does not have necessarily the same magnetic charge or magnetic weight.
From the vacuum (23), we obtain that the asymptotic form of the scalar field for the monopole associated to the magnetic weight \( \beta = \pm e_k \) is

\[
\phi(\theta, \varphi) = g(\theta, \varphi)\phi_0 = \phi_0 + v \{ (\cos \theta - 1) \mp i \sin \theta \cos \varphi \} \{ |2k - 1, 2k - 1\rangle + |2k, 2k\rangle \}
\]

\[
\mp iv \sin \theta \cos \varphi \{ |2k, 2k - 1\rangle + |2k - 1, 2k\rangle \}
\]

where we defined \(|i, j\rangle = |e_i\rangle \otimes |e_j\rangle\).

From the kinetic term for the scalar field, expanding \( \phi \) around the vacuum \( \phi_0 \), we obtain the term

\[
D_\mu \phi_0^\dagger D^\mu \phi_0 = e^2 \phi_0^\dagger \{ T_a, T_b \} \phi_0 W_{ab} W^\mu,
\]

which implies that the mass squared matrix for the gauge particles is

\[
(M^2)_{ab} = e^2 \phi_0^\dagger \{ T_a, T_b \} \phi_0.
\]

For \( su(n) \) broken to \( so(n) \), there are \( (n + 2)(n - 1)/2 \) massive gauge particles which can be associated to the generators

\[
T_{1}^{ij} = \frac{E_{ij} + E_{ji}}{2}, \quad i < j, \quad i = 1, 2, \ldots, n - 1,
\]

\[
T_{3}^{i,i+1} = \frac{E_{ii} - E_{i+1,i+1}}{2}, \quad i = 1, 2, \ldots, n - 1.
\]

Then, using the definition of \( E_{ij} \) and adopting the normalization \( \langle i, j | k, l \rangle = \delta_{ik} \delta_{jl} \) one can obtain directly that all massive gauge particles have same mass equal to

\[
m = 2ev.
\]

This result coincides with the one obtained in [21] for the \( SU(3) \) case, up to a global factor due to a different normalization.

### 6.2 \( so(2m+1) \) invariant under outer automorphism

Let us determine the possible \( su(2) \) subalgebra generators \( T_i^\beta \) we can have for the symmetry breaking of \( SU(2m+1) \rightarrow Spin(2m+1)/Z_2 \) where \( Spin(2m+1) \) is the subgroup invariant under outer automorphism. From Eq. (32), we can conclude that for the \( Z_2 \) monopoles in the nontrivial sector, the vector \( \beta \) must belong to coset

\[
\beta = \lambda_{(1)}^\vee + \sum_{i=1}^{m} c_{(i)} \alpha_{(i)}^\vee,
\]

where \( c_{(i)} \) are integer numbers and \( \lambda_{(i)}^\vee \) and \( \alpha_{(i)}^\vee \) are, respectively, coweights and coroots of \( so(2m+1) \) given by the set of Eqs. (26). From these equations and the fact that

\[
\lambda_{(1)}^\vee = \lambda_1 + \lambda_{2m} = \psi = \alpha_1 + \alpha_2 + \ldots + \alpha_{2m}
\]
where \( \psi \) is the highest root of \( su(2m + 1) \), we can conclude that

\[
\beta = \sum_{i=1}^{m} (1 + c_{(i)}) \left( \alpha_i + \alpha_{2m+1-i} \right).
\] (39)

Therefore, \( \beta \) must also belong to the subspace of \( \Lambda_{\tau}(SU(2m + 1)) \) invariant under the outer automorphism transformation \( \tau(\alpha_i) = \alpha_{2m+1-i} \) of the \( su(2m + 1) \) algebra. The fact that at the same time \( \beta \in \Lambda_{\nu}(Spin(2m + 1)^{\gamma}) \) and \( \beta \in \Lambda_{\tau}(SU(2m + 1)) \) is consistent with Eq. (8) which means that the \( \mathbb{Z}_2 \) monopoles must be associated to elements of \( \pi_1(Spin(2m + 1)/\mathbb{Z}_2) = \mathbb{Z}_2 \) which correspond to the identity of \( \pi_1(SU(2m + 1)) = 1 \). In order to construct \( su(2) \) subalgebras we consider that \( \beta \) must satisfy not only condition (39) but also that it is a root of \( su(2m + 1) \). Then we can define

\[
T_3^\beta = \frac{\beta \cdot H}{2} = \frac{\beta \cdot H}{\beta^2},
\]
\[
T_1^\beta = \frac{E_{\beta} + E_{-\beta}}{2},
\]
\[
T_2^\beta = \frac{E_{\beta} - E_{-\beta}}{2i},
\] (40)

where we used the fact that \( \beta^2 = 2 \) since it is a root of \( su(2m + 1) \). Since the roots of \( su(2m + 1) \) are of the form

\[
\alpha_p + \alpha_{p+1} + \alpha_{p+2} + \ldots + \alpha_{p+q}
\]

where \( 0 \leq q \leq 2m - p \), and \( \beta \) must satisfy (39), we arrive to the conclusion that \( \beta \) can be the following \( 2m \) roots,

\[
\begin{align*}
\alpha_1 + \alpha_2 + \ldots + \alpha_{2m} &= \lambda_1^\vee, \\
\alpha_2 + \alpha_3 + \ldots + \alpha_{2m-1} &= \lambda_1^\vee - \alpha_1^\vee, \\
\vdots & \quad \vdots \\
\alpha_m + \alpha_{m+1} &= \lambda_1^\vee - \alpha_1^\vee - \ldots - \alpha_{(m-1)}^\vee, \\
-(\alpha_m + \alpha_{m+1}) &= \lambda_1^\vee - \alpha_1^\vee - \ldots - \alpha_{(m-1)}^\vee - \alpha_m^\vee, \\
-(\alpha_{m-1} + \alpha_m + \alpha_{m+1} + \alpha_{m+2}) &= \lambda_1^\vee - \alpha_1^\vee - \ldots - 2\alpha_{(m-1)}^\vee - \alpha_m^\vee, \\
\vdots & \quad \vdots \\
-(\alpha_1 + \alpha_2 + \ldots + \alpha_{2m}) &= \lambda_1^\vee - 2\alpha_1^\vee - \ldots - 2\alpha_{(m-1)}^\vee - \alpha_m^\vee,
\end{align*}
\] (41)

where we wrote the roots of \( su(2m + 1) \) as coweights of \( so(2m + 1) \), using Eq. (26). From the fact that fundamental coweights and simple coroots of \( so(2m + 1) \) are, respectively, fundamental weights and simple roots of \( sp(2m) \), we can recognize this set as the weights of the \( 2m \)-dimensional defining representation of \( sp(2m) \).
It remains to show that \( T_1^\beta, T_2^\beta \not\in so(2m + 1) \). In order to do that we can write the set of roots \( \Pi \) in terms of the orthonormal basis vectors \( e_i, i = 1, 2, \ldots, 2m + 1 \). Since \( \alpha_i = e_i - e_{i+1} \), the roots in \( \Pi \) are of the form
\[
e_p - e_{2m+2-p}, \quad p = 1, 2, \ldots, m,
\]
consistent with the fact that under the outer automorphism of \( su(2m+1) \), \( \tau(e_p) = -e_{2m+2-p} \). The step operator associated to the root \( (42) \) in the \( n \)-dimensional representation is proportional to the matrix \( E_{p,2m+2-p} \). Therefore, denoting
\[
E_{ij} = E_{ij} \otimes 1 + 1 \otimes E_{ij},
\]
we can write \( T_1^\beta \) and \( T_2^\beta \) in terms of these matrices and acting on the vacuum \( (27) \) we obtain
\[
2T_1^{p,2m+2-p} \phi_0 = (E_{p,2m+2-p} - E_{2m+2-p,p}) \cdot 2 \sum_{l=1}^{2m+1} (-1)^{l+1} |e_l\rangle \otimes |e_{2m+2-l}\rangle
\]
\[
= 2(-1)^{p+1}v (|e_p\rangle \otimes |e_p\rangle + |e_{2m+2-p}\rangle \otimes |e_{2m+2-p}\rangle)
\]
\[
\ne 0,
\]
\[
2iT_2^{p,2m+2-p} \phi_0 = (E_{p,2m+2-p} + E_{2m+2-p,p}) \cdot 2 \sum_{l=1}^{2m+1} (-1)^{l+1} |e_l\rangle \otimes |e_{2m+2-l}\rangle
\]
\[
= 2(-1)^{p+1}v (|e_p\rangle \otimes |e_p\rangle - |e_{2m+2-p}\rangle \otimes |e_{2m+2-p}\rangle)
\]
\[
\ne 0.
\]
Hence, \( T_1^\beta, T_2^\beta \not\in so(2m + 1) \). Therefore, we can conclude that to each weight \( \Pi \) of the defining representation of \( sp(2m) = so(2m + 1)^\vee \) we can associate a \( su(2) \) subalgebra \( (40) \) and \( \mathbb{Z}_2 \) monopole.

Similarly to the previous case, we can construct monopole asymptotic forms with magnetic charge associated to others elements of the cosets \( (32) \). However, the \( su(2) \) generators associated to these new monopoles seem to always be combination of the generators \( (40) \) and therefore these monopoles can be interpreted as superpositions of the above monopoles. Some examples of these \( su(2) \) subalgebras are generated by
\[
T_3^{n_p(e_p - e_{2m+2-p})} = \sum_{p=1}^{m} n_p \frac{(e_p - e_{2m+2-p}) \cdot H}{2},
\]
\[
T_1^{n_p(e_p - e_{2m+2-p})} = \sum_{p=1}^{m} n_p \frac{E(e_p - e_{2m+2-p}) + E(i(e_p - e_{2m+2-p}))}{2},
\]
\[
T_2^{n_p(e_p - e_{2m+2-p})} = \sum_{p=1}^{m} n_p \frac{E(e_p - e_{2m+2-p}) - E(i(e_p - e_{2m+2-p}))}{2i},
\]
\[
18
\]
where $n_p = 0, 1$.

For the vacuum \[^{27}\text{27}\] we obtain that the asymptotic form of the scalar field for the monopole associated to $\beta = e_p - e_{2m+2-p}$ is

$$
\phi(\theta, \varphi) = g(\theta, \varphi)\phi_0 = \phi_0 + (-1)^{p+1} \left\{ -\sin \theta \left[ e^{-i\varphi} |p, p\rangle + e^{i\varphi} |2m + 2 - p, 2m + 2 - p\rangle \right] + (\cos \theta - 1) \left[ |p, 2m + 2 - p\rangle + |2m + 2 - p, p\rangle \right] \right\}.
$$

**Appendix**

Let us analyze the elements of the center group and the kernel $K(G_0)$ for the special cases of $Spin(n)$ for $n = 3, 5, 6$:

For $Spin(3) \cong SU(2)$, the weight lattice splits in two cosets,

$$
\Lambda_r(SU(2)), \, \lambda^\vee_1 + \Lambda_r(SU(2)),
$$

and

$$
Z(SU(2)) = Z_2 \cong \{ \exp (2\pi i\alpha^\vee \cdot h), \exp [2\pi i (\lambda^\vee_1 + \alpha^\vee) \cdot h] \}.
$$

The branching of the irrep. with highest weight $\lambda_1$ of $su(3)$ in $su(2)$ is $(10) \simeq (2)$. Acting the center elements on the weight states $|2\lambda_1 - \gamma\rangle$ of $su(2)$, we obtain that

$$
Ker(R_{\lambda_1}(Spin(2))) = Z_2 \cong \{ \exp [(2\pi i\Lambda_r(SU(2))) \cdot h], \exp [2\pi i (\lambda^\vee_1 + \Lambda_r(SU(2))) \cdot h] \},
$$

where $\lambda^\vee_1$ is the highest weight of the $2$-dimensional irrep. of $su(2)$.

For $Spin(5) \cong Sp(4)$, the weight lattice splits in two cosets,

$$
\Lambda_r(Sp(4)^\vee), \, \lambda^\vee_2 + \Lambda_r(Sp(4)^\vee),
$$

and

$$
Z(Sp(4)) = Z_2 \cong \{ \exp (2\pi i\alpha^\vee \cdot h), \exp [2\pi i (\lambda^\vee_2 + \alpha^\vee) \cdot h] \}.
$$

The branching of the irrep. with highest weight $\lambda_1$ of $su(5)$ in $sp(4)$ is $(1000) \simeq (01)$. Acting the center elements on the weight states $|\lambda_2 - \gamma\rangle$ of $sp(4)$, we obtain that

$$
Ker(R_{\lambda_2}(Spin(5))) = Z_2 \cong \{ \exp [(2\pi i\Lambda_r(Sp(4)^\vee)) \cdot h], \exp [2\pi i (\lambda^\vee_2 + \Lambda_r(Sp(4)^\vee)) \cdot h] \}
$$

where $\lambda^\vee_2$ is the highest weight of the $4$-dimensional irrep. of $sp(4)$.

For $Spin(6) \cong SU(4)$, the weight lattice splits in four cosets,

$$
\Lambda_r(SU(4)), \, \lambda_1 + \Lambda_r(SU(4)), \, \lambda_2 + \Lambda_r(SU(4)), \, \lambda_3 + \Lambda_r(SU(4)),
$$

and

$$
Z(SU(4)) \cong \{ \exp (2\pi i\alpha^\vee \cdot h), \exp [2\pi i (\lambda^\vee_1 + \alpha^\vee) \cdot h], \exp [2\pi i (\lambda^\vee_2 + \alpha^\vee) \cdot h], \exp [2\pi i (\lambda^\vee_3 + \alpha^\vee) \cdot h] \}.
$$
The branching of the irrep. with highest weight $\lambda_1$ of $su(6)$ in $su(4)$ is $(1 0 0 0) \simeq (0 1 0)$. Acting the center elements on the weight states $|\lambda_2 - \gamma\rangle$ of $su(4)$, we obtain that

$$\text{Ker}(R_{\lambda_1}(\text{Spin}(6)) = \mathbb{Z}_2 \cong \{ \exp [(2\pi i \Lambda_r(SU(4))) \cdot h], \exp [2\pi i (\lambda^\vee_2 + \Lambda_r(SU(4))) \cdot h]\}$$

where $\lambda^\vee_2$ is the highest weight of the 6-dimensional irrep. of $su(4) \cong so(6)$. Therefore for all the three cases, similarly to the general case of $\text{Spin}(n)$, $n \geq 7$,

$$\text{Ker}(R_{\lambda_i}(\text{Spin}(n)) = \mathbb{Z}_2 \cong \{ \exp [(2\pi i \Lambda_r(Spin(n)^\vee)) \cdot h], \exp [2\pi i (\lambda^\vee_i + \Lambda_r(Spin(n)^\vee)) \cdot h]\},$$

where $\lambda^\vee_i$ is the highest weight of the defining representation of $so(n)^\vee$ which has dimension $2m$ for $so(2m)^\vee = so(2m)$ and for $so(2m + 1)^\vee = sp(2m)$.

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