Energy Dissipation and Regularity for a Coupled Navier-Stokes and Q-Tensor System

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Abstract

We study a complex non-newtonian fluid that models the flow of nematic liquid crystals. The fluid is described by a system that couples a forced Navier-Stokes system with a parabolic-type system. We prove the existence of global weak solutions in dimensions two and three. We show the existence of a Lyapunov functional for the smooth solutions of the coupled system and use the cancellations that allow its existence to prove higher global regularity, in dimension two. We also show the weak-strong uniqueness in dimension two.

1 Introduction

In this paper we study the global existence of solutions for a system describing the evolution of a nematic liquid crystal flow. The system couples a forced Navier-Stokes system, describing the flow, with a parabolic-type system describing the evolution of the nematic crystal director fields (Q-tensors). The coupled system has a Lyapunov functional made of two parts: the free energy due to the director fields and the kinetic energy of the fluid. This functional describes, from a physical point of view, the dissipation of the energy of the complex fluid.

In the first part of the paper we use, in a classical manner, the apriori bounds provided by the energy dissipation to prove the existence of global weak solutions in the natural energy space. In the second part, we study the case where the fluid evolves in the two dimensional space and prove the existence of a global regular solution issued from an appropriately regular initial data. In the two dimensional space we also show that for an appropriately regular initial data the weak and the strong solutions coincide. The main contribution of this paper is to show how to use the specific coupling of the system (the coupling structure that allows the system to dissipate energy) not only at the level of regularity of weak solutions but also to transport arbitrarily large (enough) regularity of the initial data. Thus we show that for this type of complex fluids the existence of an energy dissipation is intrinsically related to the high regularity of the solutions.

There exist several competing theories that attempt to capture the complexity of nematic liquid crystals, and a comparative discussion and further references are available for instance in [16], [21]. In the present paper we use one of the most comprehensive description of nematics, the Q-tensor description, proposed by P.G. de Gennes [14]. There exist various specific models that all use the Q-tensor description and a comparative discussion of the main models is available for instance in [23].

In this paper we use a model proposed by Beris and Edwards [3], that one can find in the physics literature for instance in [11], [24]. An important feature of this model is that if one assumes smooth solutions and one formally takes \( Q(x) = s_+(n(x) \otimes n(x) - \frac{1}{3} I_d) \), with \( s_+ \) a constant (depending on the parameters of the system, see for instance [21]) and \( n : \mathbb{R}^d \to S^{d-1} \) smooth, then the equations reduce (see [11]) to the generally accepted

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equations of Ericksen, Leslie and Parodi \[15\]. The system we study is related structurally to other models of complex fluids coupling a transport equation with a forced Navier-Stokes system \[1, 6, 7, 8, 10, 17, 18, 22, 26\]. In our case the Navier-Stokes equations are coupled with a parabolic type system, but we also have two more derivatives (than in the previously mentioned models) in the forcing term of the Navier-Stokes equations. The Ericksen-Leslie-Parodi system describing nematic liquid crystals, whose structure is closer to our system (but that has one less derivative in the forcing term of the Navier-Stokes equations) was studied in \[12, 13, 19\].

In the following we use a partial Einstein summation convention, that is we assume summation over repeated greek indices, but not over the repeated latin indices. We consider the equations as described in \[11, 24\] but assume that the fluid has constant density in time.

We denote

\[
S(\nabla u, Q) \overset{def}{=} (\xi D + \Omega)(Q + \frac{1}{3}Id) + (Q + \frac{1}{3}Id)(\xi D - \Omega) - 2\xi(Q + \frac{1}{3}Id)\text{tr}(Q\nabla u) \tag{1}
\]

where \(D \overset{def}{=} \frac{1}{2}(\nabla u + (\nabla u)^T)\) and \(\Omega \overset{def}{=} \frac{1}{2}(\nabla u - (\nabla u)^T)\) are the symmetric part and the antisymmetric part, respectively, of the velocity gradient tensor \(\nabla u\). The term \(S(\nabla, Q)\) appears in the equation of motion of the order-parameter, \(Q\), and describes how the flow gradient rotates and stretches the order-parameter. The constant \(\xi\) depends on the molecular details of a given liquid crystal and measures the ratio between the tumbling and the aligning effect that a shear flow would exert over the liquid crystal directors.

We also denote:

\[
H \overset{def}{=} -aQ + b|Q|^2 - \frac{\text{tr}(Q^2)}{3}Id - cQ\text{tr}(Q^2) + L\Delta Q \tag{2}
\]

where \(L > 0\).

With the notations above we have the coupled system:

\[
\begin{cases}
(\partial_t + u \cdot \nabla)Q - S(\nabla, Q) = \Gamma H \\
\partial_\tau u_\alpha + u_\beta \partial_\beta u_\alpha = \nu \partial_\beta \partial_\beta u_\alpha + \partial_\alpha p + \partial_\beta \tau_{\alpha\beta} + \partial_\beta \sigma_{\alpha\beta} \\
\partial_\gamma u_\gamma = 0
\end{cases} \tag{3}
\]

where \(\Gamma > 0, \nu > 0\) and we have the symmetric part of the additional stress tensor:

\[
\tau_{\alpha\beta} = -\xi \left( Q_{\alpha\gamma} + \frac{\delta_{\alpha\gamma}}{3} \right) H_{\gamma\beta} - \xi H_{\alpha\gamma} \left( Q_{\gamma\beta} + \frac{\delta_{\gamma\beta}}{3} \right) + 2\xi(Q_{\alpha\beta} + \frac{\delta_{\alpha\beta}}{3})Q_{\gamma\delta}H_{\gamma\delta} - L \left( \partial_\beta Q_{\gamma\delta} \partial_\alpha Q_{\gamma\delta} + \frac{\delta_{\alpha\beta}}{3} Q_{\nu\varepsilon}Q_{\nu\varepsilon} \right) \tag{4}
\]

and an antisymmetric part:

\[
\sigma_{\alpha\beta} = Q_{\alpha\gamma} H_{\gamma\beta} - H_{\alpha\gamma} Q_{\gamma\beta} \tag{5}
\]

In the rest of the paper we restrict ourselves to the case \(\xi = 0\). This means that the molecules are such that they only tumble in a shear flow, but are not aligned by such a flow. In this case the system \[3\] reduces to:

\[
\begin{cases}
(\partial_t + u_\gamma \cdot \partial_\gamma)Q_{\alpha\beta} - \Omega_{\alpha\gamma}Q_{\gamma\beta} + Q_{\alpha\gamma}\Omega_{\gamma\beta} = \Gamma \left( L\Delta Q_{\alpha\beta} - aQ_{\alpha\beta} + b[Q_{\alpha\gamma}Q_{\gamma\beta} - \frac{\delta_{\alpha\beta}}{3} \text{tr}(Q^2)] - cQ_{\alpha\beta}\text{tr}(Q^2) \right) \\
\partial_\tau u_\alpha + u_\beta \partial_\beta u_\alpha = \nu \Delta u_\alpha + \partial_\alpha p - L\partial_\beta \left( \partial_\alpha Q_{\gamma\delta} \partial_\beta Q_{\gamma\delta} - \frac{\delta_{\alpha\beta}}{3} \partial_\lambda Q_{\xi\delta} \partial_\lambda Q_{\xi\delta} \right) + L\partial_\beta (Q_{\alpha\gamma} \Delta Q_{\gamma\beta} - \Delta Q_{\alpha\gamma} Q_{\gamma\beta}) \\
\partial_\gamma u_\gamma = 0
\end{cases} \tag{6}
\]

in \(\mathbb{R}^d\), \(d = 2, 3\).
We also need to assume from now on that
\[ c > 0 \]
This assumption is necessary from a modelling point of view (see \[20], \[21\]) so that the energy \( F \) (see next section, relation \( \ref{eq:energy} \)) is bounded from below, and it is also necessary for having global solutions (see Proposition \( \ref{prop:global} \) and its proof).

We restrict ourselves to the case \( \xi = 0 \) for technical simplicity. However, we think that our method can also be used in the general case \( \xi \neq 0 \) and we will study this in a forthcoming paper \[25\].

The paper is organised as follows: in the second section we show that the equation admits a Lyapunov functional, whose existence is based on a certain cancellation that will prove to be crucial in the proof of higher regularity and the weak-strong uniqueness of solutions in dimension two. The appendix contains a technical calculation and that the strong norms increase in time at most triply exponentially. Finally in the last section we show the weak-strong uniqueness of solutions in dimension two. The appendix contains a technical calculation necessary in the fourth section.

**Notations and conventions** Let \( S_0 \subset M^{3 \times 3} \) denote the space of Q-tensors, i.e.
\[ S_0 \overset{\text{def}}{=} \{ Q \in M^{3 \times 3}, Q_{ij} = Q_{ji}, \text{tr}(Q) = 0, i,j = 1,2,3 \} \]

We use the Frobenius norm of a matrix \( |Q| \overset{\text{def}}{=} \sqrt{\text{tr}(Q^2)} = \sqrt{Q_{\alpha \beta}Q_{\alpha \beta}} \) and define Sobolev spaces of Q-tensors in terms of this norm. For instance \( H^1(\mathbb{R}^d, S_0) \overset{\text{def}}{=} \{ Q : \mathbb{R}^d \to S_0, \int_{\mathbb{R}^d} |\nabla Q(x)|^2 + |Q(x)|^2 dx < \infty. \}

For \( A, B \in S_0 \) we denote \( A \cdot B = \text{tr}(AB) \) and \( |A| = \sqrt{\text{tr}(A^2)} \). We also denote \( |\nabla Q|^2(x) \overset{\text{def}}{=} Q_{\alpha \beta, \gamma}(x)Q_{\alpha \beta, \gamma}(x) \) and \( |\Delta Q|^2(x) \overset{\text{def}}{=} \Delta Q_{\alpha \beta}(x)\Delta Q_{\alpha \beta}(x) \). We recall also that \( \Omega_{\alpha \beta} \overset{\text{def}}{=} \frac{1}{2} (\partial_{\beta} u_\alpha - \partial_{\alpha} u_\beta) \) and \( u_{\alpha \beta} \overset{\text{def}}{=} \partial_\beta u_\alpha, \)
\[ Q_{i,j,k} \overset{\text{def}}{=} \partial_{\beta} Q_{i,j}. \]

## 2 The dissipation principle and apriori estimates

Let us denote the free energy of the director fields:
\[ F(Q) = \int_{\mathbb{R}^d} \frac{L}{2} |\nabla Q|^2 + \frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}^2(Q^2) \, dx \]

In the absence of the flow, when \( u = 0 \) in the equations \( \ref{eq:flow} \), the free energy is a Lyapunov functional of the system. If \( u \neq 0 \) we still have a Lyapunov functional for \( \ref{eq:flow} \) but this time one that includes the kinetic energy of the system. More precisely we have:

**Proposition 1.** The system \( \ref{eq:flow} \) has a Lyapunov functional:
\[ E(t) \overset{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} |u|^2(t,x) \, dx + \int_{\mathbb{R}^d} \frac{L}{2} |\nabla Q|^2(t,x) + \frac{a}{2} \text{tr}(Q^2(t,x)) - \frac{b}{3} \text{tr}(Q^3(t,x)) + \frac{c}{4} \text{tr}^2(Q^2(t,x)) \, dx \]

If \( d = 2,3 \) and \((Q,u)\) is a smooth solution of \( \ref{eq:flow} \) such that \( Q \in L^\infty(0,T;H^1(\mathbb{R}^d)) \cap L^2(0,T;H^2(\mathbb{R}^d)) \) and \( u \in L^\infty(0,T;L^2(\mathbb{R}^d)) \cap L^2(0,T;H^1(\mathbb{R}^d)) \) then, for all \( t < T \), we have:
\[ \frac{d}{dt} E(t) = -\nu \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d} \text{tr} \left( L\Delta Q - aQ + b|Q^2 - \frac{\text{tr}(Q^2)}{3}I|d| - cQ\text{tr}(Q^2)^2 \right) \, dx \leq 0 \]
Proof. We multiply the first equation in (6) to the right by \(-L\Delta Q - aQ + b|Q^2| - \frac{1}{3}tr(Q^2)Id - c\)tr(Q^2), take the trace, integrate over \(\mathbb{R}^d\) and by parts and sum with the second equation multiplied by \(u\) and integrated over \(\mathbb{R}^d\) and by parts (let us observe that because of our assumptions on \(Q\) and \(u\) we do not have boundary terms, when integrating by parts). We obtain:

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2}|u|^2 + \frac{L}{2}|\nabla Q|^2 + \frac{a}{2}\tr(Q^2) - \frac{b}{3}\tr(Q^3) + \frac{c}{4}u^2(Q^2) \, dx + \nu \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \Gamma \int_{\mathbb{R}^d} |L\Delta Q - aQ + b|Q^2| - \frac{1}{3}\tr(Q^2)Id - c\tr(Q^2)|^2 \, dx
\]

\[
= \int_{\mathbb{R}^d} u \cdot \nabla Q_{\alpha\beta} \left(-a\delta_{\alpha\beta} + b[Q_{\alpha\gamma}Q_{\gamma\beta} - \frac{\delta_{\alpha\beta}}{3}\tr(Q^2)] - c\tr(Q^2)\right) \, dx
\]

\[
+ \int_{\mathbb{R}^d} (\Omega\alpha\gamma Q_{\gamma\beta} + Q_{\alpha\gamma}\Omega_{\gamma\beta}) \left(-a\delta_{\alpha\beta} + b[Q_{\alpha\delta}Q_{\delta\beta} - \frac{\delta_{\alpha\beta}}{3}\tr(Q^2)] - c\tr(Q^2)\right) \, dx
\]

\[
+ \frac{L}{2} \int_{\mathbb{R}^d} u_{\gamma,\alpha}Q_{\gamma\beta}\Delta Q_{\alpha\beta} \, dx - \frac{L}{2} \int_{\mathbb{R}^d} u_{\alpha,\gamma}Q_{\gamma\beta}\Delta Q_{\alpha\beta} \, dx
\]

\[
+ \frac{L}{2} \int_{\mathbb{R}^d} Q_{\alpha\gamma}u_{\gamma,\beta}\Delta Q_{\alpha\beta} \, dx - \frac{L}{2} \int_{\mathbb{R}^d} Q_{\alpha\gamma}u_{\alpha,\beta}\Delta Q_{\alpha\beta} \, dx
\]

\[
= -L \int_{\mathbb{R}^d} u_{\alpha,\gamma}Q_{\gamma\beta}\Delta Q_{\alpha\beta} \, dx + L \int_{\mathbb{R}^d} u_{\gamma,\alpha}Q_{\gamma\beta}\Delta Q_{\alpha\beta} \, dx
\]

\[
- \int_{\mathbb{R}^d} Q_{\alpha\gamma}\Delta Q_{\gamma\beta}u_{\alpha,\beta} \, dx + L \int_{\mathbb{R}^d} Q_{\alpha\gamma}\Delta Q_{\gamma\beta}u_{\alpha,\beta} \, dx = 0
\]

(11)

where \(\mathcal{I} = 0\) (since \(\nabla \cdot u = 0\), \(\mathcal{II} = 0\) (since \(Q_{\alpha\beta} = Q_{\beta\alpha}\)) and for the second equality we used

\[
\int_{\mathbb{R}^d} u_{\gamma,\alpha}Q_{\gamma\beta}\Delta Q_{\alpha\beta} \, dx + \int_{\mathbb{R}^d} Q_{\delta,\alpha}Q_{\gamma,\delta}u_{\alpha,\beta} \, dx = \int_{\mathbb{R}^d} u_{\gamma,\alpha}Q_{\gamma\beta}\Delta Q_{\alpha\beta} \, dx
\]

\[
- \int_{\mathbb{R}^d} Q_{\gamma,\delta}Q_{\gamma,\delta}u_{\delta,\alpha} \, dx - \int_{\mathbb{R}^d} Q_{\gamma,\delta}Q_{\gamma,\delta}u_{\alpha,\delta} \, dx = \int_{\mathbb{R}^d} \frac{1}{2}Q_{\gamma,\delta}Q_{\gamma,\delta}u_{\delta,\alpha} \, dx = 0
\]

(12)

while for the last equality in (11) we used \(2B + BB = 2C + CC = 0\). □

In the following we assume that there exists a smooth solution of (6) and obtain estimates on the behaviour of various norms:

**Proposition 2.** Let \((Q, u)\) be a smooth solution of (6), with restriction (7), and smooth initial data \((Q(x), u(x))\), that decays fast enough at infinity so that we can integrate by parts in space (for any \(t \geq 0\)) without boundary terms.
(i) If $Q \in L^p$ for some $p \geq 2$ we have

$$
\|Q(t, \cdot)\|_{L^p} \leq e^{Ct}\|\dot{Q}\|_{L^p}, \forall t \geq 0
$$

with $C = C(a, b, c, p, \Gamma)$.

(ii) For $d = 2, 3$ (and $(\ddot{Q}, \dddot{u})$ so that the right hand side of the expression below is finite) we have:

$$
\|u(t, \cdot)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(s, \cdot)\|_{L^2}^2 ds + L\|\nabla Q(t, \cdot)\|_{L^2}^2 + \Gamma L^2 \int_0^t \|\Delta Q(s, \cdot)\|_{L^2}^2 ds \leq \|u(0, \cdot)\|_{L^2}^2 + \|\nabla Q(0, \cdot)\|_{L^2}^2 + Ce^{Ct} \left(\|Q(0, \cdot)\|_{L^2}^2 + \|Q(0, \cdot)\|_{L^6}^p\right)
$$

with the constant $C = C(a, b, c, d, L, \Gamma)$.

Proof.

(i) Multiplying the first equation in (9) by $2pQtr^{p-1}(Q^2)$ and taking the trace we obtain:

$$
(\partial_t + u \cdot \nabla)tr^p(Q^2) = \Gamma \left(2pL \Delta Q_{\alpha\beta}Q_{\alpha\beta}tr^{p-1}(Q^2) - 2p\tau tr^p(Q^2) + 2p\tau tr(Q^3)tr^{p-1}(Q^2) - 2p\tau tr^{p+1}(Q^2)\right)
$$

Let us observe that for $Q$ a traceless, symmetric, $3 \times 3$ matrix we have:

$$
\text{tr}(Q^3) \leq \frac{3c}{8} \text{tr}^2(Q^2) + \frac{1}{\varepsilon} \text{tr}(Q^2), \forall \varepsilon > 0
$$

Indeed, if $Q$ has the eigenvalues $x, y, -x - y$ then $\text{tr}(Q^3) = -3xy(x + y)$, $\text{tr}(Q^2) = 2(x^2 + y^2 + xy)$ and the inequality (16) follows.

Integrating over $\mathbb{R}^d$, integrating by parts (we have no boundary terms because of our assumption), as well as using that $\nabla \cdot u = 0$, together with (16) (where $\varepsilon = \frac{1}{4\nu}$) and the assumption $c > 0$ we obtain:

$$
\begin{align*}
\partial_t \int_{\mathbb{R}^d} tr^p(Q^2) dx &\leq -2p\Gamma L \int_{\mathbb{R}^d} \nabla Q_{\alpha\beta} \nabla Q_{\alpha\beta} tr^{p-1}(Q^2) dx \\
-4p(p - 1)\Gamma L \int_{\mathbb{R}^d} Q_{\alpha\beta\gamma} Q_{\alpha\beta\delta} Q_{\beta\gamma\delta} tr^{p-2}(Q^2) dx + C \int_{\mathbb{R}^d} tr^p(Q^2) dx
\end{align*}
$$

where the constant $C$ depends on $a, b, c, p$ and $\Gamma$. Thus we have

$$
\int_{\mathbb{R}^d} tr^p(Q^2(t, x)) dx \leq e^{Ct} \int_{\mathbb{R}^d} tr^p(Q^2(0, x)) dx
$$

with $C = C(a, b, c, p, \Gamma)$.

(ii) Relation (10) implies

$$
\begin{align*}
\frac{L}{2} \|\nabla Q(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|u(t, \cdot)\|_{L^2}^2 + \nu \int_0^t \|\nabla u(s, \cdot)\|_{L^2}^2 ds + \Gamma L^2 \int_0^t \|\Delta Q(s, \cdot)\|_{L^2}^2 ds \\
\leq C \int_{\mathbb{R}^d} tr(Q^2(t, x)) dx + C \int_{\mathbb{R}^d} tr(Q^2(0, x)) dx + C \int_{\mathbb{R}^d} tr(Q^3(0, x)) dx + \frac{L}{2} \|\nabla Q(0, \cdot)\|_{L^2}^2 + \frac{1}{2} \|u(0, \cdot)\|_{L^2}^2 \\
+ \Gamma \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} tr\left((aQ - bQ^2 + cQ tr^2(Q^2))\right) dx ds + \Gamma \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} tr\left((aQ - bQ^2 + cQ tr^2(Q^2))L\Delta Q\right) dx ds
\end{align*}
$$

In the last inequality we use Holder inequality to estimate $\Delta Q$ in $L^2$ and absorb it in the left hand side while the terms without gradients are estimated using (13) and interpolation between the $L^2$ and $L^6$ norms. □
3 Weak solutions

A pair \((Q, u)\) is called a weak solution of the system \([5]\), subject to initial data

\[
Q(0, x) = Q(x) \in L^2(\mathbb{R}^d), \quad u(0, x) = \bar{u}(x) \in L^2(\mathbb{R}^d), \quad \nabla \cdot \bar{u} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d)
\]  

for \(Q \in L^\infty_{\text{loc}}(\mathbb{R}^d; H^1) \cap L^2_{\text{loc}}(\mathbb{R}^d; H^2), \quad u \in L^\infty_{\text{loc}}(\mathbb{R}^d; L^2) \cap L^2_{\text{loc}}(\mathbb{R}^d; H^1)\) and for every compactly supported \(\varphi \in C^\infty([0, \infty) \times \mathbb{R}^d; S_0), \quad \psi \in C^\infty([0, \infty) \times \mathbb{R}^d)\) with \(\nabla \cdot \psi = 0\) we have

\[
\int_0^\infty \int_{\mathbb{R}^d} (-Q \cdot \partial_t \varphi - \Gamma L \Delta Q \cdot \varphi) - Q \cdot u \nabla_x \varphi - \Omega Q \cdot \varphi + Q \Omega \cdot \varphi \, dx \, dt
\]

subject to initial conditions \((19)\). The solution uniqueness we have \((\subseteq)\)

\[
\int_0^\infty \int_{\mathbb{R}^d} \frac{d}{dt}(Q^2) \cdot \varphi \, dx \, dt - \int_{\mathbb{R}^d} \bar{u}(x) \varphi(0, x) \, dx = L \int_0^\infty \int_{\mathbb{R}^d} Q_{\gamma \delta, \alpha} Q_{\gamma \delta, \beta} \psi_{\alpha, \beta} - Q_{\alpha \gamma} \Delta Q_{\gamma \beta} \psi_{\alpha, \beta} + \Delta Q_{\alpha \gamma} Q_{\gamma \beta} \psi_{\alpha, \beta} \, dx \, dt
\]

\[
\int_0^\infty \int_{\mathbb{R}^d} \frac{d}{dt}(Q^2) \cdot \varphi \, dx \, dt - \int_{\mathbb{R}^d} \bar{u}(x) \varphi(0, x) \, dx = L \int_0^\infty \int_{\mathbb{R}^d} Q_{\gamma \delta, \alpha} Q_{\gamma \delta, \beta} \psi_{\alpha, \beta} - Q_{\alpha \gamma} \Delta Q_{\gamma \beta} \psi_{\alpha, \beta} + \Delta Q_{\alpha \gamma} Q_{\gamma \beta} \psi_{\alpha, \beta} \, dx \, dt
\]

Proposition 3. For \(d = 2, 3\) there exists a weak solution \((Q, u)\) of the system \([9]\), with restriction \([7]\), subject to initial conditions \([17]\). The solution \((Q, u)\) is such that \(Q \in L^\infty_{\text{loc}}(\mathbb{R}^d; H^1) \cap L^2_{\text{loc}}(\mathbb{R}^d; H^2)\) and \(u \in L^\infty_{\text{loc}}(\mathbb{R}^d; L^2) \cap L^2_{\text{loc}}(\mathbb{R}^d; H^1)\).

Proof. We define the mollifying operator

\[
\hat{J}_n f(\xi) = 1_{[\frac{1}{n}, n)}(\|\xi\|) \hat{f}(\xi)
\]

and consider the system:

\[
\begin{cases}
\partial_t Q^{(n)} + J_n \left( P J_n u^n \nabla J_n Q^{(n)} \right) - J_n \left( P J_n Q^{(n)} J_n Q^{(n)} \right) + J_n \left( J_n Q^{(n)} P J_n Q^{(n)} \right) = \Gamma L \Delta J_n Q^{(n)} \\
\quad + \left( -a J_n Q^{(n)} + b J_n (J_n Q^{(n)} J_n Q^{(n)}) - \frac{\text{tr}(J_n (J_n Q^{(n)} J_n Q^{(n)}))}{d} \right) I_d - c J_n Q^{(n)} \text{tr}(J_n (J_n Q^{(n)} J_n Q^{(n)})) \right)
\end{cases}
\]

where \(P\) denotes the Leray projector onto divergence-free vector fields.

The system above can be regarded as an ordinary differential equation in \(L^2\) verifying the conditions of the Cauchy-Lipschitz theorem. Thus it admits a unique maximal solution \((Q^{(n)}, u^{(n)}) \in C^1([0, T_n]; L^2(\mathbb{R}^d; \mathbb{R}^d) \times L^2(\mathbb{R}^d, \mathbb{R}^d))\). As we have \((P J_n)^2 = P J_n\) and \(J_n^2 = J_n\) the pair \((J_n Q^{(n)}, P J_n u^{(n)})\) is also a solution of \((22)\). By uniqueness we have \((J_n Q^{(n)}, P J_n u^{(n)}) = (Q^{(n)}, u^{(n)})\) hence \((Q^{(n)}, u^{(n)}) \in C^1([0, T_n], H^\infty)\) and \((Q^{(n)}, u^{(n)})\) satisfy the system:

\[
\begin{cases}
\partial_t Q^{(n)} + J_n \left( u^n \nabla Q^{(n)} \right) - J_n \left( \nabla Q^{(n)} Q^{(n)} \right) = \Gamma L \Delta Q^{(n)} \\
\quad + \left( -a Q^{(n)} + b J_n (Q^{(n)} Q^{(n)}) - \frac{\text{tr}(J_n (Q^{(n)} Q^{(n)}))}{d} \right) I_d - c Q^{(n)} \text{tr}(J_n (Q^{(n)} Q^{(n)})) \right)
\end{cases}
\]

\[
\begin{cases}
\partial_t Q^{(n)} + J_n \left( u^n \nabla Q^{(n)} \right) - J_n \left( \nabla Q^{(n)} Q^{(n)} \right) = \Gamma L \Delta Q^{(n)} \\
\quad + \left( -a Q^{(n)} + b J_n (Q^{(n)} Q^{(n)}) - \frac{\text{tr}(J_n (Q^{(n)} Q^{(n)}))}{d} \right) I_d - c Q^{(n)} \text{tr}(J_n (Q^{(n)} Q^{(n)})) \right)
\end{cases}
\]

We can argue as in the proof of the a priori estimates and the same estimates hold for the approximating system \(\tilde{(22)}\). These estimates allow us to conclude that \(T_n = \infty\) and we also get the following a priori bounds:
\[
\sup_n \|Q^{(n)}\|_{L^2(0,T;H^2)} + \|Q^{(n)}\|_{L^\infty(0,T;H^1)} < \infty
\]
\[
\sup_n \|u^n\|_{L^\infty(0,T;L^2)} + \|u^n\|_{L^2(0,T;H^2)} < \infty
\]

for any \( T < \infty \).

The pair \((Q^{(n)}, u^n)\) is also a weak solution of the approximating system \((22)\) hence for every compactly supported \( \varphi \in C^\infty((0,\infty) \times \mathbb{R}^d; S_0) \), \( \psi \in C^\infty((0,\infty) + \times \mathbb{R}^d; \mathbb{R}^d) \) with \( \nabla \cdot \psi = 0 \) we have:

\[
\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^d} (-Q^{(n)} \cdot \partial_t \varphi - \Gamma L \Delta Q^{(n)} \cdot \varphi) - J_n(Q^{(n)} \cdot u^n) \nabla_x \varphi - J_n(\Omega^n Q^{(n)}) \cdot \varphi + J_n(Q^{(n)}\Omega^n) \cdot \varphi \, dx \, dt \\
= \int_{\mathbb{R}^d} Q(x) \cdot \varphi(0, x) \, dx + \Gamma \int_0^\infty \int_{\mathbb{R}^d} \left\{ -aQ^{(n)} + b[J_n\left(\left( Q^{(n)} \right)^2 \right) - \frac{\text{tr}(J_n((Q^{(n)})^2))}{d} I] - cQ^{(n)} \text{tr}(J_n(Q^{(n)})^2) \right\} \cdot \varphi \, dx \, dt
\end{aligned}
\]

and weak convergence arguments, that there exists a \( Q \) for any \( \psi \) and the first term, \( Q \) and the last term in \((25)\), namely

\[
\int_{\mathbb{R}^d} L \int_0^\infty \int_{\mathbb{R}^d} Q^{(n)} dx dt
\]

We consider the solutions of \((22)\) and taking into account the bounds \((23)\) we get, by classical compactness and weak convergence arguments, that there exists a \( Q \in L^\infty(\mathbb{R}_+; H^2) \cap L^2(\mathbb{R}_+; H^1) \) and a \( u \in L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; H^1) \) so that, on a subsequence, we have:

\[
Q^{(n)} \rightharpoonup Q \text{ in } L^2(0,T; H^2) \text{ and } Q^{(n)} \rightharpoonup Q \text{ in } L^2(0,T; H^1_{\text{loc}}) \text{, } \forall \epsilon > 0
\]

\[
Q^{(n)}(t) \rightharpoonup Q(t) \text{ in } H^1 \text{ for all } t \in \mathbb{R}_+
\]

\[
u^n \rightharpoonup u \text{ in } L^2(0,T; H^1) \text{ and } u^n \rightharpoonup u \text{ in } L^2(0,T; H^1_{\text{loc}}) \text{, } \forall \epsilon > 0
\]

\[
u^n(t) \rightharpoonup u(t) \text{ in } L^2 \text{ for all } t \in \mathbb{R}_+
\]

These convergences allow us to pass to the limit in the weak solutions \((24), (25)\) to obtain a weak solution of \((20)\), namely \((26), (21)\). The term that is the most difficult to treat in passing to the limit is the last term in \((25)\), namely

\[
L \int_0^\infty \int_{\mathbb{R}^2} J_n\left( Q_{\alpha\gamma}^{(n)} \Delta Q^{(n)}_{\alpha\gamma} - \Delta Q_{\alpha\gamma}^{(n)} Q_{\alpha\gamma}^{(n)} \right) \psi_{\alpha\beta} \, dx \, dt = L \int_0^\infty \int_{\mathbb{R}^d} \left( Q_{\alpha\gamma}^{(n)} \Delta Q_{\alpha\gamma}^{(n)} - \Delta Q_{\alpha\gamma}^{(n)} Q_{\alpha\gamma}^{(n)} \right) \cdot J_n \psi_{\alpha\beta} \, dx \, dt
\]

Recalling that \( \psi \) is compactly supported we have that there exists a time \( T > 0 \) so that \( \psi(t, x) = J_n \psi(t, x) = 0, \forall t > T, x \in \mathbb{R}^d, n \in \mathbb{N} \). Taking into account that \( \psi \) is compactly supported and the convergences \((26)\) one can easily pass to the limit the terms \( \partial_\beta J_n \psi_{\alpha} Q_{\alpha\gamma}^{(n)} \) and \( \partial_\beta J_n \psi_{\alpha} Q_{\alpha\gamma}^{(n)} \) strongly in \( L^2(0,T; L^2) \). Indeed we have:

\[
\partial_\beta J_n \psi_{\alpha} Q_{\alpha\gamma}^{(n)} - \partial_\beta \psi_{\alpha} Q_{\alpha\gamma} = \left( \partial_\beta J_n \psi_{\alpha} - \partial_\beta \psi_{\alpha} \right) Q_{\alpha\gamma}^{(n)} + \partial_\beta \psi_{\alpha} \left( Q_{\alpha\gamma}^{(n)} - Q_{\alpha\gamma} \right)
\]

and the first term, \( I \), converges to 0, strongly in \( L^2(0,T; L^2) \) because \( \psi \) is smooth and compactly supported, hence \( \partial_\beta J_n \psi - \partial_\beta \psi \) converges to zero in any \( L^p(0,T; L^p) \) and \( Q^{(n)} \) is bounded in \( L^\infty \) in time and \( L^p \) in space.
(1 < p < ∞ if d = 2 and 2 ≤ p ≤ 6 if d = 3, due to the bounds (23)). On the other hand the second term \( T \) converges strongly to zero in \( L^2(0; T; L^2) \) because of (26) and the fact that \( \psi \) is compactly supported.

Relations (26) give that \( Q^{(n)}(\alpha) \), \( Q^{(n)}(\beta) \) converges weakly in \( L^2(0; T; L^2) \). Thus we get convergence to the limit

\[
L \int_0^\infty \int_{\mathbb{R}^d} (\Delta Q_{\gamma\beta})(\partial_\beta \psi_0 Q_{\alpha\gamma}) dx dt = L \int_0^\infty \int_{\mathbb{R}^d} (\Delta Q_{\alpha\gamma})(\partial_\beta \psi_0 Q_{\gamma\beta}) dx dt
\]

\[
= L \int_0^T \int_{\mathbb{R}^d} (\Delta Q_{\gamma\beta})(\partial_\beta \psi_0 Q_{\alpha\gamma}) dx dt - L \int_0^\infty \int_{\mathbb{R}^d} (\Delta Q_{\alpha\gamma})(\partial_\beta \psi_0 Q_{\gamma\beta}) dx dt.
\]

\( \square \)

## 4 Higher regularity in 2D, using the dissipation principle

In this section we restrict ourselves to dimension two and show that starting from an initial data with some higher regularity, we can obtain more regular solutions. More precisely, we have:

**Theorem 1.** Let \( s > 1 \) and \((\hat{Q}, \hat{u}) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)\). There exists a global a solution \((Q(t, x), u(t, x))\) of the system (4), with restriction (5), subject to initial conditions

\[ Q(0, x) = \hat{Q}(x), u(0, x) = \hat{u}(x) \]

and \( Q \in L^2_{\text{loc}}(\mathbb{R}_+; H^{s+2}(\mathbb{R}^2)) \cap L^\infty_{\text{loc}}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)), u \in L^2_{\text{loc}}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)) \cap L^\infty_{\text{loc}}(\mathbb{R}_+; H^s). \)

Moreover, we have:

\[
L \|\nabla Q(t, \cdot)\|_{H^s(\mathbb{R}^2)} + \|u(t, \cdot)\|_{H^s(\mathbb{R}^2)} \leq \left( e + \|Q\|_{H^{s+1}(\mathbb{R}^2)} + \|\hat{u}\|_{H^s(\mathbb{R}^2)} \right) e^{Ct}
\]

where the constant \( C \) depends only on \( \hat{Q}, \hat{u}, a, b, c, \Gamma \) and \( L \).

The proof of the theorem is mainly based on \( H^s \) energy estimates and the following cancelation(that is also used implicitly in showing the dissipation of the energy in Proposition 1):

**Lemma 1.** For any symmetric matrices \( Q', Q \in \mathbb{R}^{d \times d} \) and \( \Omega_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} - u_{\beta,\alpha}) \in \mathbb{R}^{d \times d} \) we have

\[
\int_{\mathbb{R}^d} tr((\Omega Q' - Q' \Omega) \Delta Q) dx = \int_{\mathbb{R}^d} \partial_\beta Q_{\alpha\gamma} Q_{\gamma\beta} \Delta Q_{\alpha\gamma} - \Delta Q_{\alpha\gamma} Q_{\gamma\beta} u_\alpha dx = 0
\]

**Proof.** We note that

\[
\int_{\mathbb{R}^d} tr((\Omega Q' - Q' \Omega) \Delta Q) dx = \int_{\mathbb{R}^d} \Omega_{\alpha\gamma} Q'_{\gamma\beta} \Delta Q_{\alpha\beta} - Q'_{\alpha\gamma} \Omega_{\gamma\beta} \Delta Q_{\beta\alpha} = \int_{\mathbb{R}^d} \Omega_{\alpha\gamma} Q'_{\gamma\beta} \Delta Q_{\alpha\beta} + \Omega_{\beta\gamma} Q_{\gamma\alpha} \Delta Q_{\alpha\beta}
\]

\[
= 2 \int_{\mathbb{R}^d} tr(Q' \Delta Q) dx = \int_{\mathbb{R}^d} u_{\alpha,\beta} Q'_{\beta\gamma} \Delta Q_{\gamma\alpha} dx - \int_{\mathbb{R}^d} u_{\beta,\alpha} Q'_{\beta\gamma} \Delta Q_{\gamma\alpha} dx
\]

and on the other hand

\[
- \int_{\mathbb{R}^d} \partial_\beta (Q_{\alpha\gamma} \Delta Q_{\gamma\beta}) u_\alpha = \int_{\mathbb{R}^d} Q_{\alpha\gamma} \Delta Q_{\gamma\beta} \partial_\beta u_\alpha = \int_{\mathbb{R}^d} Q'_{\alpha\gamma} \Delta Q_{\gamma\alpha} \partial_\beta u_\alpha = I_2
\]

and also

\[
\int_{\mathbb{R}^d} \partial_\beta (\Delta Q_{\alpha\gamma} Q_{\gamma\beta}) u_\alpha = - \int_{\mathbb{R}^d} Q'_{\beta\gamma} \Delta Q_{\gamma\alpha} \partial_\beta u_\alpha = - I_1
\]

which finishes the proof. \( \square \)
Remark 1. The main point in the proof of the theorem is to use the previous lemma to eliminate the highest derivatives in \(u\) in the first equation of the system \([\mathcal{Q}]\) and the highest derivatives in \(Q\) in the second equation of the system. The proof could have been done, alternatively, by differentiating the equations \(k \geq 1\) times and using the previous lemma. However, that would have required estimating some delicate commutators and would have restricted the initial data to \((\mathcal{Q}, \bar{u}) \in H^3 \times H^2\). The Littlewood-Paley approach that we use allows for \((\mathcal{Q}, \bar{u}) \in H^{s+1} \times H^s\) with \(s > 1\).

In order to prove the theorem we need to introduce some technical preliminaries:

### 4.1 Littlewood-Paley theory

We define \(\mathcal{C}\) to be the ring of center 0, of small radius \(1/2\) and great radius 2. There exist two nonnegative radial functions \(\chi\) and \(\varphi\) belonging respectively to \(\mathcal{D}(B(0, 1))\) and to \(\mathcal{D}(\mathcal{C})\) so that

\[
\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \forall \xi \in \mathbb{R}^d \tag{31}
\]

\[
|p - q| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-q}\cdot) \cap \text{Supp } \varphi(2^{-p}\cdot) = \emptyset. \tag{32}
\]

For instance, one can take \(\chi \in \mathcal{D}(B(0, 1))\) such that \(\chi \equiv 1\) on \(B(0, 1/2)\) and take

\[
\varphi(\xi) = \chi(\xi/2) - \chi(\xi).
\]

Then, we are able to define the Littlewood-Paley decomposition. Let us denote by \(\mathcal{F}\) the Fourier transform on \(\mathbb{R}^d\). Let \(h, \tilde{h}, \Delta_q, S_q (q \in \mathbb{Z})\) be defined as follows:

\[
h = \mathcal{F}^{-1}\varphi \quad \text{and} \quad \tilde{h} = \mathcal{F}^{-1}\chi,
\]

\[
\Delta_q u = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u) = 2^qd \int h(2^qy)u(x - y)dy,
\]

\[
S_q u = \mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}u) = 2^qd \int \tilde{h}(2^qy)u(x - y)dy.
\]

We recall that for two appropriately smooth functions \(a\) and \(b\) we have Bony’s paraproduct decomposition \([2]\):

\[
ab = T_a b + T_b a + R(a, b) \tag{33}
\]

where

\[
T_a b = \sum_{q'} S_{q' - 1} a \Delta_q b, \quad T_b a = \sum_{q'} S_{q' - 1} b \Delta_q a \quad \text{and} \quad R(a, b) = \sum_{q \in \{0, \pm 1\}} \Delta_q a \Delta_{q+1} b.
\]

Then we have

\[
\Delta_q(ab) = \Delta_q T_a b + \Delta_q T_b a + \Delta_q R(a, b) = \Delta_q T_a b + \Delta_q \tilde{R}(a, b) \tag{34}
\]

where \(\tilde{R}(a, b) = T_b a + R(a, b) = \Sigma_q S_{q' + 2} b \Delta_{q'} a\). Moreover:

\[
\Delta_q(ab) = \Sigma_{|q' - q| \leq 5} \Delta_q(S_{q' - 1} a \Delta_q b) + \Sigma_{q' > q - 5} \Delta_q(S_{q' + 2} b \Delta_{q'} a)
\]

\[
= \Sigma_{|q' - q| \leq 5} [\Delta_q, S_{q' - 1} a] \Delta_q b + \Sigma_{|q' - q| \leq 5} S_{q' - 1} a \Delta_q b + \Sigma_{q' > q - 5} (S_{q'} + 2 b) \Delta_{q'} a
\]

\[
= \Sigma_{|q' - q| \leq 5} [\Delta_q, S_{q' - 1} a] \Delta_q b + \Sigma_{|q' - q| \leq 5} S_{q' - 1} a \Delta_q b + \Sigma_{q' > q - 5} \Delta_q(S_{q' + 2} b \Delta_{q'} a) + \Sigma_{|q' - q| \leq 5} S_{q' - 1} a \Delta_q \Delta_{q'} b
\]

\[
= S_{q + 1} a \Delta_{q + 1} b \tag{35}
\]
In terms of this decomposition we can express the Sobolev norm of an element \( u \) in the space \( H^s \) as:

\[
\|u\|_{H^s} = \left( \|S_0u\|_{L^2}^2 + \sum_{q \in \mathbb{N}} 2^{2qs} \|\Delta_q u\|_{L^2}^2 \right)^{1/2}
\]

We will use the following well-known estimates:

**Lemma 2.** ([8], [10])

(i) *(Bernstein inequalities)*

\[
2^{-q}\|\nabla S_q u\|_{L^p} \leq C\|u\|_{L^p}, \forall 1 \leq p \leq \infty
\]

\[
\|\Delta_q u\|_{L^p} \leq C2^{-q}\|\nabla u\|_{L^p}, \forall 1 \leq p \leq \infty
\]

(ii) *(commutator estimate)*

\[
\|\{S_{q'-1}a, \Delta_q\}b\|_{L^2} \leq C2^{-q}\|\nabla S_{q'-1}a\|_{L^\infty}\|b\|_{L^2}
\]

4.2 Proof of theorem [1]

**Step 1. Estimates of the high frequencies**

Applying \( \Delta_q \) to the first equation in (60) we get:

\[
\partial_t \Delta_q Q_{\alpha\beta} - \Gamma L \Delta_q \Omega_{\alpha\gamma} S_{q'-1}Q_{\gamma\beta} + S_{q'-1}Q_{\alpha\gamma} \Delta_q \Omega_{\gamma\beta} = -\Delta_q (u, Q_{\alpha\beta,\gamma})
\]

\[
+ \Gamma \Delta_q [-aQ_{\alpha\beta} + b \left( Q_{\alpha\gamma} \Omega_{\gamma\beta} - \frac{\delta_{\alpha\beta}}{2} \text{tr}(Q^2) \right) - cQ_{\alpha\beta} \text{tr}(Q^2)]
\]

\[
+ \sum_{|q'|-q \leq 5} [\Delta_q; S_{q'-1}Q_{\gamma\beta}] \Delta_q \Omega_{\alpha\gamma} + \sum_{|q'|-q \leq 5} (S_{q'-1}Q_{\gamma\beta} - S_{q-1}Q_{\gamma\beta}) \Delta_q \Omega_{\alpha\gamma} + \sum_{|q'|-q \leq 5} (S_{q'-2} \Omega_{\alpha\gamma} \Delta_q Q_{\gamma\beta})
\]

\[
- \sum_{|q'|-q \leq 5} (S_{q'-1}Q_{\alpha\gamma} - S_{q-1}Q_{\alpha\gamma}) \Delta_q \Omega_{\gamma\beta} - \sum_{|q'|-q \leq 5} (S_{q'-2} \Omega_{\gamma\beta} \Delta_q Q_{\alpha\gamma})
\]

Multiplying the previous equation by \(-L \Delta \Delta_q Q_{\alpha\beta}\) and integrating over \( \mathbb{R}^2 \) and by parts we obtain:

\[
\frac{L}{2} \partial_t \|\nabla \Delta_q Q\|_{L^2}^2 + \Gamma L^2 \|\Delta \Delta_q Q\|_{L^2}^2 + L \int \Delta_q \Omega_{\alpha\gamma} S_{q'-1}Q_{\gamma\beta} \Delta_q Q_{\alpha\beta} - L \int S_{q'-1}Q_{\alpha\gamma} \Delta_q \Omega_{\gamma\beta} \Delta_q Q_{\alpha\beta}
\]

\[
= L \left( \Delta_q (u \nabla Q_{\alpha\beta}), \Delta_q Q_{\alpha\beta} \right) - \sum_{|q'|-q \leq 5} ((S_{q'-1}Q_{\gamma\beta} - S_{q-1}Q_{\gamma\beta}) \Delta_q \Omega_{\alpha\gamma}, \Delta_q Q_{\alpha\beta})
\]

\[
- \sum_{|q'|-q \leq 5} ((S_{q'-1}Q_{\alpha\gamma} - S_{q-1}Q_{\alpha\gamma}) \Delta_q \Omega_{\gamma\beta}, \Delta_q Q_{\alpha\beta})
\]

\[
+ L \sum_{|q'|-q \leq 5} (S_{q'-1}Q_{\alpha\gamma} - S_{q-1}Q_{\alpha\gamma}) \Delta_q \Omega_{\gamma\beta}, \Delta_q Q_{\gamma\beta}) + L \sum_{|q'|-q \leq 5} (S_{q'-2} \Omega_{\alpha\gamma} \Delta_q Q_{\gamma\beta})
\]

\[
- L \sum_{|q'|-q \leq 5} (S_{q'-2} \Omega_{\gamma\beta} \Delta_q Q_{\alpha\gamma})\Delta_q Q_{\alpha\beta}
\]

\[
- L \Gamma \left( -aQ_{\alpha\beta} + bQ_{\alpha\gamma} \Omega_{\gamma\beta} - cQ_{\alpha\beta} \text{tr}(Q^2), \Delta_q Q_{\alpha\beta} \right)
\]

Applying \( \Delta_q \) to the second equation in (60) we get:
\[\partial_t \Delta_q u_{\alpha} - \nu \Delta \Delta_q u_{\alpha} = \partial_q \Lambda_q p + L \partial_q' \left( S_{q-1} Q_{\alpha \gamma} \Delta_q \Delta Q_{\gamma} - \Delta_q \Delta Q_{\alpha \gamma} S_{q-1} Q_{\gamma} \right) \]

\[= - L \partial_q \Delta_q \left( \partial_c Q_{\alpha \gamma} \partial_q Q_{\gamma} - \frac{\delta_{\alpha \beta}}{3} \partial_q Q_{\beta} \partial_q Q_{\gamma} \right) \Delta_q (u_{\beta} q^\beta u_{\alpha}) + L \partial_q \left( \Sigma_{|q' - q| \leq 5} |\Delta_q S_{q' - 1} Q_{\alpha \gamma} \Delta_q' Q_{\gamma} - \Sigma_{|q' - q| \leq 5} (S_{q' - 1} Q_{\alpha \gamma} - S_{q' - 1} Q_{\alpha \gamma}) \Delta_q \Delta_q' Q_{\gamma} \right) \]

\[+ L \partial_q \left( \Sigma_{|q' > q - 5} (S_{q' + 2} Q_{\alpha \gamma} \Delta_q Q_{\gamma} - \Sigma_{|q' > q - 5} (S_{q' - 1} Q_{\alpha \gamma} - S_{q' - 1} Q_{\alpha \gamma}) \Delta_q \Delta_q') \right) \]

\[= - L \partial_q \left( \Sigma_{|q' > q - 5} (S_{q' + 2} Q_{\alpha \gamma} \Delta_q Q_{\gamma} - \Sigma_{|q' > q - 5} (S_{q' - 1} Q_{\alpha \gamma} - S_{q' - 1} Q_{\alpha \gamma}) \Delta_q \Delta_q') \right) \]

\[+ L \Sigma_{|q' > q - 5} (S_{q' + 2} Q_{\alpha \gamma} \Delta_q Q_{\gamma} - \Sigma_{|q' > q - 5} (S_{q' - 1} Q_{\alpha \gamma} - S_{q' - 1} Q_{\alpha \gamma}) \Delta_q \Delta_q') \Delta_q u_{\alpha} \beta \]

\[= - L \partial_q \left( \Sigma_{|q' > q - 5} (S_{q' + 2} Q_{\alpha \gamma} \Delta_q Q_{\gamma} - \Sigma_{|q' > q - 5} (S_{q' - 1} Q_{\alpha \gamma} - S_{q' - 1} Q_{\alpha \gamma}) \Delta_q \Delta_q') \right) \Delta_q u_{\alpha} \beta \]

\[+ L \Sigma_{|q' > q - 5} (S_{q' + 2} Q_{\alpha \gamma} \Delta_q Q_{\gamma} - \Sigma_{|q' > q - 5} (S_{q' - 1} Q_{\alpha \gamma} - S_{q' - 1} Q_{\alpha \gamma}) \Delta_q \Delta_q') \Delta_q u_{\alpha} \beta \]

We multiply the last equation by \(\Delta_q u_{\alpha} \), integrate over \(\mathbb{R}^2\) and by parts to obtain:

\[\frac{1}{2} \partial_t \| \Delta_q u \|_{L^2}^2 + \nu \| \Delta_q \nabla u \|_{L^2}^2 + L \int S_{q-1} Q_{\alpha \gamma} \Delta_q \Delta Q_{\alpha \gamma} \Delta_q u_{\alpha} \beta - L \int \Delta_q \Delta Q_{\alpha \gamma} S_{q-1} Q_{\gamma} \Delta_q u_{\alpha} \beta \]

\[= - \left( \Delta_q (u_{\beta} \partial_q u_{\alpha}) \right) + L \int \Delta_q \left( \partial_c Q_{\alpha \gamma} \partial_q Q_{\gamma} - \frac{\delta_{\alpha \beta}}{3} \partial_q Q_{\beta} \partial_q Q_{\gamma} \right) \Delta_q u_{\alpha} \beta \]

\[+ L \Sigma_{|q' > q - 5} \left( S_{q' + 2} Q_{\alpha \gamma} \Delta_q Q_{\gamma} - \Sigma_{|q' > q - 5} (S_{q' - 1} Q_{\alpha \gamma} - S_{q' - 1} Q_{\alpha \gamma}) \Delta_q \Delta_q' Q_{\gamma} \right) \Delta_q u_{\alpha} \beta \]

\[+ L \Sigma_{|q' > q - 5} (S_{q' + 2} Q_{\alpha \gamma} \Delta_q Q_{\gamma} - \Sigma_{|q' > q - 5} (S_{q' - 1} Q_{\alpha \gamma} - S_{q' - 1} Q_{\alpha \gamma}) \Delta_q \Delta_q') \Delta_q u_{\alpha} \beta \]

\[= - L \partial_q \left( \Sigma_{|q' > q - 5} (S_{q' + 2} Q_{\alpha \gamma} \Delta_q Q_{\gamma} - \Sigma_{|q' > q - 5} (S_{q' - 1} Q_{\alpha \gamma} - S_{q' - 1} Q_{\alpha \gamma}) \Delta_q \Delta_q') \right) \Delta_q u_{\alpha} \beta \]

\[+ L \Sigma_{|q' > q - 5} (S_{q' + 2} Q_{\alpha \gamma} \Delta_q Q_{\gamma} - \Sigma_{|q' > q - 5} (S_{q' - 1} Q_{\alpha \gamma} - S_{q' - 1} Q_{\alpha \gamma}) \Delta_q \Delta_q') \Delta_q u_{\alpha} \beta \]

Summing (37) and (39) and using Lemma 11 we get:

\[\partial_t \left( \frac{L}{2} \| \nabla \Delta_q Q \|_{L^2}^2 + \frac{1}{2} \| \Delta_q u \|_{L^2}^2 \right) + \nu \| \Delta_q \nabla u \|_{L^2}^2 + \frac{\Gamma L^2}{2} \| \Delta \Delta_q Q \|_{L^2}^2 = \sum_{i=1}^{8} I_i + \sum_{j=1}^{8} J_j \]

\[\frac{d}{dt} \varphi_2 + \sum_{\nu \in \mathbb{N}} 2^{2\nu} \left( \frac{\Gamma L^2}{2} \| \Delta \Delta_q Q \|_{L^2}^2 + \nu \frac{L^2}{2} \| \nabla \Delta_q u \|_{L^2}^2 \right) \leq C(1 + \| \nabla Q \|_{L^\infty} + \| u \|_{L^\infty}^2 + \| Q \|_{L^\infty}^2) \left( \| \nabla Q \|_{H^\nu} + \| u \|_{H^\nu} \right) + \frac{\Gamma L^2}{50} \| \Delta Q \|_{H^\nu}^2 + \frac{\nu}{50} \| \nabla u \|_{H^\nu}^2 \]

Step 2. Estimates of the low frequencies
This is much easier than the previous step. We apply \( S_0 \) to the first equation in (3), multiply by \(-LS_0\Delta Q_{\alpha\beta}\), take the trace, integrate over \( \mathbb{R}^2 \) and by parts and we get:

\[
\frac{L}{2} \partial_t \| S_0 \nabla Q \|^2_{L^2} + \Gamma \| \Delta S_0 Q \|^2_{L^2} \leq C \| u \|_{L^\infty} \| S_0 \nabla Q \|_{L^2} \| \Delta S_0 Q \|_{L^2} + C \| Q \|_{L^\infty} \| \nabla u \|_{L^2} \| \Delta S_0 Q \|_{L^2} \\
+ C \| \nabla S_0 Q \|^2_{L^2} (1 + \| Q \|_{L^\infty} + \| Q \|^2_{L^\infty})
\]

hence

\[
\frac{L}{2} \partial_t \| S_0 \nabla Q \|^2_{L^2} + \Gamma \frac{L^2}{2} \| \Delta S_0 Q \|^2_{L^2} \leq C \| S_0 \nabla Q \|_{L^2}^2 (\| u \|^2_{L^\infty} + 1 + \| Q \|_{L^\infty} + \| Q \|^2_{L^\infty}) + C \| u \|^2_{H^1} \| Q \|^2_{L^\infty} \quad (42)
\]

We apply \( S_0 \) to the second equation in (3), multiply by \( S_0 u \) and integrate over \( \mathbb{R}^2 \) and by parts to obtain:

\[
\frac{1}{2} \partial_t \| S_0 u \|^2_{L^2} + \nu \| \nabla S_0 u \|^2_{L^2} \leq C \| u \|_{L^\infty} \| \nabla u \|_{L^2} \| S_0 u \|_{L^2} + C \| \nabla S_0 u \|_{L^2} \left( \| \nabla Q \|_{L^\infty} \| \nabla Q \|_{L^2} + \| Q \|_{L^\infty} \| \Delta Q \|_{L^2} \right)
\]

hence

\[
\frac{1}{2} \partial_t \| S_0 u \|^2_{L^2} + \nu \frac{L^2}{2} \| \nabla S_0 u \|^2_{L^2} \leq C \| u \|^2_{L^\infty} (1 + \| u \|_{L^\infty}) + C \| \nabla Q \|^2_{H^1} \left( \| \nabla Q \|^2_{L^2} + \| Q \|^2_{L^\infty} \right) \quad (43)
\]

Summing (42) and (43) we obtain:

\[
\partial_t \varphi_1 + \frac{\nu}{2} \| \nabla S_0 u \|^2_{L^2} + \Gamma \frac{L^2}{2} \| \Delta S_0 Q \|^2_{L^2} \leq \| u \|^2_{L^\infty} \varphi + C \left( 1 + \| Q \|^2_{L^\infty} + \| \nabla Q \|^2_{L^2} \right) \varphi \quad (44)
\]

**Step 3. The estimates of the high norms**

Summing (41) and (44) we obtain:

\[
\frac{1}{2} \varphi'(t) \leq C(\| \nabla Q \|^2_{L^\infty} + \| u \|^2_{L^\infty}) \varphi + C \left( 1 + \| Q \|^2_{L^\infty} + \| \nabla Q \|^2_{L^2} \right) \varphi
\]

Now we use a fundamental ingredient in the global existence, namely the logarithmic estimate (see [4]), for \( s > 1 \),

\[
\| \nabla Q \|_{L^\infty} + \| u \|_{L^s} \leq C(\| Q \|^2_{H^2} + \| u \|_{H^1}) \sqrt{\ln(e + \frac{\| \nabla Q \|^2_{H^2} + \| u \|^2_{H^1}}{\| Q \|^2_{H^2} + \| u \|^2_{H^1}})},
\]

and be denoting \( f(t) \overset{def}{=} \| Q \|^2_{H^2} + \| u \|^2_{H^1} \) and \( g(t) \overset{def}{=} 1 + \| Q \|^2_{L^\infty} + \| \nabla Q \|^2_{L^2} \) we obtain

\[
\varphi'(t) \leq C f(t) \left( \ln(e + \frac{\varphi(t)}{f(t)}) + g(t) \right) \varphi(t).
\]

Observing that the function \( h(x) \overset{def}{=} x \ln(e + \frac{x}{x}) \) is increasing the last relation implies:

\[
\varphi'(t) \leq C (1 + f(t)) \left( \ln(e + \varphi(t)) + g(t) \right) \varphi(t)
\]

By integrating this differential inequality, we obtain:

\[
\varphi(t) \leq (e + \| Q_0 \|^2_{H^1} + \| u_0 \|^2_{H^1}) \left( e^{C f(t) g(t) dt} \right) \leq C(t, Q, \bar{u}, s),
\]

and this uniform bound, imply the global existence of a regular solution for regular enough initial data. Taking into account Proposition [2] we have that \( \int f(s) ds \) increases exponentially and this gives the rate in [29]. \( \square \)
5 Weak-Strong uniqueness in 2D

In this section we consider a global weak solution and a strong one, starting from the same initial data \( (\bar{Q}, \bar{u}) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \) with \( s > 1 \) and we show that they are the same. More precisely:

**Proposition 4.** Let \( (\bar{Q}, \bar{u}) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \) with \( s > 1 \). By Proposition 3 there exists a weak solution \((Q_1, u_1)\) of the system \(\mathcal{B}\), subject to restriction \(\mathcal{A}\) and starting from initial data \((Q, \bar{u})\), such that

\[
Q_1 \in L^\infty_{lo}(\mathbb{R}_+; H^1(\mathbb{R}^2)) \cap L^2_{lo}(\mathbb{R}_+; H^2(\mathbb{R}^2)) \quad \text{and} \quad u_1 \in L^\infty_{lo}(\mathbb{R}_+; L^2(\mathbb{R}^2)) \cap L^2_{lo}(\mathbb{R}_+; H^1(\mathbb{R}^2))
\]  

(45)

**Theorem 7** gives the existence of a strong solution \((Q_2, u_2)\) such that

\[
Q_2 \in L^\infty_{lo}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)) \cap L^2_{lo}(\mathbb{R}_+; H^{s+2}(\mathbb{R}^2)) \quad \text{and} \quad u_2 \in L^\infty_{lo}(\mathbb{R}_+; H^s(\mathbb{R}^2)) \cap L^2_{lo}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2))
\]

(46)

with \( s > 1 \) and the same initial data \((\bar{Q}, \bar{u}) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)\). Then \((Q_1, u_1) = (Q_2, u_2)\).

**Proof.** We denote by \( \delta Q = Q_1 - Q_2 \) and \( \delta u = u_1 - u_2 \) which verify the following system

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
(\partial_t + \delta u \nabla) \delta Q - \delta Q \delta \Omega + \delta Q \delta \Omega + \delta u \nabla Q_2 + u_2 \nabla \delta Q + Q_2 \delta \Omega + \delta Q \Omega_2 - \delta \Omega Q_2 - \Omega_2 \delta Q \\
= \Gamma \left( L \delta \Omega - a \delta Q + b [\delta Q Q_1 + Q_2 \delta Q - \frac{\text{tr}(\delta Q Q_1 + Q_2 \delta Q)}{2} \text{Id}] - c \delta Q \text{tr}(\delta Q_1) - c_2 \text{tr}(Q_1 \delta Q + \delta Q Q_2) \right)
\end{array}
\right.
\end{aligned}
\]

(47)

\[
\begin{aligned}
\partial_t \delta u + \nabla (\delta u \nabla \delta u) + \nu \Delta \delta u - \nabla \left( \nabla \delta Q \nabla \delta Q - \frac{1}{2} |\nabla \delta Q|^2 \right) + L \left( \nabla \cdot (\delta Q \delta \Omega - \Delta \delta \Omega Q) \right)
\end{aligned}
\]

We proceed similarly as in the proof of Proposition 3 namely we multiply the first equation in (47) to the right by \(-L \Delta \delta Q + \delta \Omega\), integrate over \(\mathbb{R}^2\) and by parts, take the trace and sum with the second equation in (47) multiplied by \( \delta u \) and integrated over \(\mathbb{R}^2\) and by parts. Taking into account the cancellations analogous to the ones in (41) we obtain:

\[
\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^2} \frac{L}{2} |\nabla \delta Q(x)|^2 + \frac{1}{2} |\delta Q(x)|^2 + \frac{1}{2} |\delta u(x)|^2 \, dx + \int_{\mathbb{R}^2} \nu |\nabla \delta u(x)|^2 + \Gamma \Delta |\delta Q(x)|^2 \, dx
\end{aligned}
\]

(48)
\[-c\Gamma \int_{\mathbb{R}^2} \text{tr}(Q_1)\delta Q \, dx - c\Gamma \int_{\mathbb{R}^2} \text{tr}(Q_2\delta Q)\text{tr}(Q_1\delta Q + \delta Q Q_2) \, dx\]

\[-\int_{\mathbb{R}^2} (u_2\nabla u + \delta u \nabla u_2) \delta u \, dx + L\int_{\mathbb{R}^2} (\nabla \delta Q \nabla Q_2 + \nabla Q_2 \nabla \delta Q) \cdot \nabla \delta u \, dx\]

\[-L\int_{\mathbb{R}^2} (\delta Q \nabla Q_2 - \Delta Q_2 \delta Q) \cdot \nabla \delta u \, dx - \int_{\mathbb{R}^2} (u_2\nabla u + \delta u \nabla u_2) \delta u \, dx + L\int_{\mathbb{R}^2} (Q_2 \delta \Delta Q_2 - \Delta Q_2 \delta Q) \cdot \nabla \delta u \, dx\]  

\((49)\)

Let us observe that Lemma \(\textbf{A}\) implies \(\mathcal{A} - \mathcal{A} = 0\). Also \(\mathcal{I} + \mathcal{II} = 0\) and then we easily obtain

\[
\frac{1}{2} \frac{d}{dt} \left( L\|\nabla \delta Q\|_{L^2}^2 + \|\delta Q\|_{L^2}^2 + \|\delta u\|_{L^2}^2 \right) + \nabla \delta Q \|_{L^2}^2 + L\|\nabla \delta Q\|_{L^2}^2 + 2L\|\delta Q\|_{L^2} \|\Omega_2\|_{L^\infty} \|\Delta \delta Q\|_{L^2} + t \|\delta \mathcal{Q}\|_{L^2}^2 + |b| \|\delta \mathcal{Q}\|_{L^2} \|\Omega_1\|_{L^4} + |b| \|\mathcal{Q}_2\|_{L^4} \|\delta \mathcal{Q}\|_{L^2} \|\Omega_2\|_{L^4} + c \|\nabla \mathcal{Q}_2\|_{L^4} \|\delta \mathcal{Q}\|_{L^2} \|\Omega_2\|_{L^4} \|\Delta \delta \mathcal{Q}\|_{L^4} + C \left( 1 + \|\nabla \mathcal{Q}_2\|_{L^4}^2 + \|\delta \mathcal{Q}\|_{L^2}^2 \right) \|\Delta \mathcal{Q}_2\|_{L^4}^2 + \|\nabla \mathcal{Q}_2\|_{L^4} \|\delta \mathcal{Q}\|_{L^2} \|\Omega_1\|_{L^4} + \|\nabla \mathcal{Q}_2\|_{L^4} \|\delta \mathcal{Q}\|_{L^2} \|\Omega_2\|_{L^4} \|\Delta \delta \mathcal{Q}\|_{L^4} \|\Omega_1\|_{L^4} + \|\nabla \mathcal{Q}_2\|_{L^4} \|\delta \mathcal{Q}\|_{L^2} \|\Omega_2\|_{L^4} \|\Delta \delta \mathcal{Q}\|_{L^4} \|\Omega_2\|_{L^4} \]  

\[
\leq \frac{\nu}{2} \|\nabla \delta u\|_{L^2}^2 + \|\mathcal{T}_f\|_{L^4}^2 \|\Delta \delta Q\|_{L^2}^2 + \|\mathcal{T}_f\|_{L^4}^2 \|\mathcal{T}_f\|_{L^4}^2 \|\Delta \mathcal{Q}_2\|_{L^4}^2 + \left( 1 + \|\nabla \mathcal{Q}_2\|_{L^4}^2 + \|\delta \mathcal{Q}\|_{L^2}^2 \right) \|\Delta \mathcal{Q}_2\|_{L^4}^2 + \left( 1 + \|\nabla \mathcal{Q}_2\|_{L^4}^2 + \|\delta \mathcal{Q}\|_{L^2}^2 \right) \|\Delta \mathcal{Q}_2\|_{L^4}^2 \]  

\[(50)\]

We are in 2D so \(\|\delta Q\|_{L^2}^2\) is controlled by \(\|\delta Q\|_{L^2}^2 + \|\nabla \delta Q\|_{L^2}^2\). The hypothesis, namely relations \((48)\) and \((49)\), ensure that the terms \(\mathcal{T}_i, i = 1, 2, 3, 4\) are integrable in time thus using the last inequality and Gronwall Lemma we obtain the uniqueness of the solution. \(\square\)

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## A  Proof of estimate \((41)\)

In the following, \(a_q(t)\) denotes a sequence in \(l^2_{\eta} \) for all \( t > 0 \) and \( b_q(t) \) is a sequence in \( l^1_{\eta} \), \( \forall t \geq 0 \), sequences that can change from one line to the next. Moreover \( ||(a_q(t))_{q \in \mathbb{N}}||_{l^2_{\eta}}, ||(b_q(t))_{q \in \mathbb{N}}||_{l^1_{\eta}} \leq C \) where the constant \( C \) is independent of \( t \geq 0 \).

\[|\mathcal{I}_1| = |(\Delta \mathcal{Q}, \Delta \mathcal{Q})| = \left| \int_{\mathbb{R}^2} S_{q-1} u \Delta \mathcal{Q} \nabla \mathcal{Q}_{\alpha \beta} \Delta \mathcal{Q} \Delta \mathcal{Q}_{\alpha \beta} + \sum_{|q'-q| \leq 5} \left( |\Delta \mathcal{Q}|, S_{q'-1} u \Delta \mathcal{Q} \nabla \mathcal{Q}_{\alpha \beta} \Delta \mathcal{Q} \Delta \mathcal{Q}_{\alpha \beta} \right) \right| \]  

\[\sum_{|q'-q| \leq 5} \left( S_{q'-1} u - S_{q-1} u \right) \Delta \mathcal{Q} \Delta \mathcal{Q} \nabla \mathcal{Q}_{\alpha \beta} \Delta \mathcal{Q} \Delta \mathcal{Q}_{\alpha \beta} + \sum_{q' \geq q-5} \left( \Delta \mathcal{Q}(S_{q'+2} \nabla \mathcal{Q}_{\alpha \beta} \Delta \mathcal{Q} u), \Delta \mathcal{Q} \Delta \mathcal{Q}_{\alpha \beta} \right) \]  

\[(51)\]

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\[ |I_{1a}| \leq C \|u\|_{L^\infty} \|\Delta_q \nabla Q\|_{L^2} \|\Delta \Delta_q Q\|_{L^2} \leq C 2^{-2q^*} b_q(t) \|u\|_{L^\infty} \|\nabla Q\|_{H^s} \|\Delta Q\|_{H^s} \quad (52) \]

\[ |I_{1b}| \leq \sum_{|q' - q| \leq 5} \|\Delta_q : S_{q' - 1} u \Delta_q \nabla Q_{\alpha \beta}\|_{L^2} \|\Delta_q \Delta \Delta_q Q_{\alpha \beta}\|_{L^2} \leq \sum_{|q' - q| \leq 5} 2^{-q} \|\nabla S_{q' - 1} u\|_{L^\infty} \|\nabla \Delta_q Q_{\alpha \beta}\|_{L^2} \|\Delta_q \Delta \Delta_q Q_{\alpha \beta}\|_{L^2} \leq C \|u\|_{L^\infty} 2^{-2q^*} b_q(t) \|\nabla Q\|_{H^s} \|\Delta Q\|_{H^s} \quad (53) \]

\[ |I_{1c}| \leq C \|u\|_{L^\infty} \|\Delta_q \nabla Q\|_{L^2} \|\Delta \Delta_q Q\|_{L^2} \leq C 2^{-2q^*} b_q(t) \|u\|_{L^\infty} \|\nabla Q\|_{H^s} \|\Delta Q\|_{H^s} \quad (54) \]

\[ |I_{1d}| \leq \sum_{q' > q - 5} \|\Delta_q : (S_{q' + 2} \nabla Q_{\alpha \beta} \Delta_q u, \Delta_q \Delta \Delta_q Q_{\alpha \beta})\|_{L^\infty} \leq \|\nabla Q\|_{L^\infty} \sum_{q' > q - 5} 2^{-q} \|\nabla S_{q' - 1} u\|_{L^\infty} \|\Delta_q u\|_{L^2} \|2^{q^*} \Delta_q \Delta Q\|_{L^2} \leq \|\nabla Q\|_{L^\infty} \sum_{q' > q - 5} 2^{-q} \|\nabla S_{q' - 1} u\|_{H^s} \|\Delta Q\|_{H^s} \leq \|\nabla Q\|_{L^\infty} 2^{-2q^*} \tilde{b}_q(t) \|u\|_{H^s} \|\Delta Q\|_{H^s} \quad (55) \]

where \( \tilde{b}_q(t) = \sum_{q' \geq q - 5} 2^{-(q - q') \cdot q^*} b_q(t) \).

\[ |I_2| = \sum_{|q' - q| \leq 5} \|\Delta_q : S_{q' - 1} Q_{\gamma \beta} \|_{L^\infty} \|\nabla \Delta_q Q_{\alpha \beta}\|_{L^2} \leq \sum_{|q' - q| \leq 5} 2^{-q} \|S_{q' - 1} \nabla Q_{\gamma \beta}\|_{L^\infty} \|\Delta_q \Omega_{\alpha \gamma}\|_{L^2} \|\Delta \Delta_q Q_{\alpha \beta}\|_{L^2} \leq C \sum_{|q' - q| \leq 5} \|\nabla Q\|_{L^\infty} \|\Delta_q u\|_{L^2} \|\Delta \Delta_q Q\|_{L^2} \leq C 2^{-2q^*} b_q(t) \|\nabla Q\|_{L^\infty} \|u\|_{H^s} \|\Delta Q\|_{H^s} \quad (56) \]

\[ |I_3| = \sum_{|q' - q| \leq 5} \|\Delta_q : (S_{q' - 1} Q_{\gamma \beta} - S_{q - 1} Q_{\gamma \beta}) \|_{L^\infty} \|\Delta_q : \Delta_q \Omega_{\alpha \gamma}, \Delta \Delta_q Q_{\alpha \beta}\|_{L^2} \leq \sum_{|q' - q| \leq 5} \|S_{q' - 1} Q_{\gamma \beta} - S_{q - 1} Q_{\gamma \beta}\|_{L^\infty} \|\Delta_q \Omega_{\alpha \gamma}\|_{L^2} \|\Delta \Delta_q Q_{\alpha \beta}\|_{L^2} \leq C \sum_{|q' - q| \leq 5} \|\nabla Q_{\gamma \beta}\|_{L^\infty} \|\Delta_q \Omega_{\alpha \gamma}\|_{L^2} \|\Delta \Delta_q Q_{\alpha \beta}\|_{L^2} \leq C 2^{-q} \|\nabla Q\|_{L^\infty} \|u\|_{H^s} \|\Delta Q\|_{H^s} \quad (57) \]

where \( \tilde{\Delta}_q = \sum_{|j| \leq 5} \Delta_j \).

\[ |I_4| = \sum_{q' > q - 5} \|\Delta_q : (S_{q' + 2} \Omega_{\alpha \gamma}, \Delta_q Q_{\gamma \beta}) \|_{L^\infty} \|\Delta_q : \Delta \Delta_q Q_{\alpha \beta}\|_{L^2} \leq \sum_{q' > q - 5} \|S_{q' + 2} \Omega_{\alpha \gamma}\|_{L^\infty} \|\Delta_q Q_{\gamma \beta}\|_{L^2} \|\Delta \Delta_q Q_{\alpha \beta}\|_{L^2} \leq C \sum_{q' > q - 5} 2^{q^*} \|S_{q' + 2} u\|_{L^\infty} \|\Delta_q \Omega_{\alpha \gamma}\|_{L^2} \|\Delta \Delta_q Q_{\alpha \beta}\|_{L^2} \leq \|u\|_{L^\infty} C \sum_{q' > q - 5} 2^{-2q^*} 2^{q^*} 2^{q} \|\Delta_q \Omega_{\alpha \gamma}\|_{L^2} \|\Delta \Delta_q Q_{\alpha \beta}\|_{L^2} \leq C \|u\|_{L^\infty} \|\nabla Q\|_{H^s} \|\Delta Q\|_{H^s} \leq C 2^{-2q^*} b_q(t) \|u\|_{L^\infty} \|\nabla Q\|_{H^s} \|\Delta Q\|_{H^s} \]
where \( b_q(t) = \sum_{q' > q-5} 2^{-2(q'-q)^s} a_{q'}(t) \bar{a}_{q'}(t) \).

The term \( I_k, k = 5, 6, 7 \) is estimated exactly as the term \( I_{k-3} \) that we have already studied above.

\[
|I_k| = |(\Delta_y(u \nabla u), \Delta_y u)| = | \int_{S_{q-1}} \Delta_y u \nabla u \Delta_y u | + \sum_{q' > q-5} | \int_{S_{q'} \setminus S_{q-1}} \Delta_y \nabla u \Delta_y u | + \sum_{q'' \leq q-5} | \int_{S_{q''} \setminus S_{q-1}} \Delta_y \nabla u \Delta_y u | + \sum_{q'' \leq q-5} | \int_{S_{q''} \setminus S_{q-1}} \Delta_y \nabla u \Delta_y u | + \sum_{q'' \leq q-5} | \int_{S_{q''} \setminus S_{q-1}} \Delta_y \nabla u \Delta_y u | \tag{59}
\]

with

\[
|J_{1a}| \leq \|S_{q-1} u\|_{L^\infty} \|\Delta_y \nabla u\|_{L^2} \|\Delta_y u\|_{L^2} \leq \|u\|_{L^\infty} 2^{-2q^s} b_q(t) \|\nabla u\|_{H^s} \|u\|_{H^s} \tag{60}
\]

\[
|J_{1b}| = \sum_{q' > q-5} | \int \Delta_y (S_q \setminus S_{q-1}) u \Delta_y \nabla u \Delta_y u | \leq C 2^{-q} \|S_{q-1} u\|_{L^\infty} \|\Delta_y \nabla u\|_{L^\infty} \|\Delta_y u\|_{L^2} \leq C \|u\|_{L^\infty} 2^{-2q^s} b_q(t) \|\nabla u\|_{H^s} \|u\|_{H^s} \tag{61}
\]

\[
|J_{1c}| \leq \sum_{q' > q-5} \|S_{q-1} - S_{q'} u\|_{L^\infty} \|\Delta_y \nabla u\|_{L^2} \|\Delta_y u\|_{L^2} \leq C \|u\|_{L^\infty} 2^{-2q^s} b_q(t) \|\nabla u\|_{H^s} \|u\|_{H^s} \tag{62}
\]

\[
|J_{1d}| = | \sum_{q' > q-5} (\Delta_y (S_q \setminus S_{q-1}) u \Delta_y u) | \leq \sum_{q' > q-5} \|\Delta_y (S_q \setminus S_{q-1}) u \Delta_y u\|_{L^2} \|\Delta_y u\|_{L^2} \leq \sum_{q'' \leq q-5} \|S_{q''} \setminus S_{q-1} u\|_{L^\infty} \|\Delta_y u\|_{L^2} \|\Delta_y u\|_{L^2} \leq C \|u\|_{L^\infty} \sum_{q'' \leq q-5} c 2^{-2q^s} \|\Delta_y \nabla u\|_{L^2} \|\Delta_y u\|_{L^2} \leq C \|u\|_{L^\infty} \|\nabla u\|_{H^s} \|u\|_{H^s} \sum_{q'' \leq q-5} c 2^{-2q^s} b_q(t) \|\nabla u\|_{H^s} \|u\|_{H^s} \leq C 2^{-2q^s} b_q(t) \|\nabla u\|_{H^s} \|u\|_{H^s} \tag{63}
\]

where \( b_q(t) = \sum_{q' > q-5} 2^{-2(q'-q)^s} a_{q'}(t) \bar{a}_{q'}(t) \in l_q^1, \forall t \geq 0. \)

\[
|J_2| = \int |\Delta_y (\partial_\alpha Q_{\gamma,\delta} \partial_\beta Q_{\gamma,\delta} \partial_\eta u_{\alpha,\beta})| \leq \|\Delta_y (\partial_\alpha Q_{\gamma,\delta} \partial_\beta Q_{\gamma,\delta} \partial_\eta u_{\alpha,\beta})\|_{L^2} \|\Delta_y \nabla u\|_{L^2} \leq C 2^{-2q^s} b_q(t) \|\partial_\alpha Q_{\gamma,\delta} \partial_\beta Q_{\gamma,\delta} \partial_\eta u_{\alpha,\beta}\|_{H^s} \|\nabla u\|_{H^s} \leq C 2^{-2q^s} b_q(t) \|\nabla Q\|_{L^\infty} \|\nabla u\|_{H^s} \|\nabla u\|_{H^s} \tag{64}
\]
\[ |J_3| = \left| \sum_{|q'| < q} \int \Delta_q \left( S_{q' - 1} Q_{\alpha} \right) \Delta_q \Delta Q_{\gamma} \Delta_q u_{\alpha, \beta} \right| \leq \sum_{|q'| \leq 5} \left| \sum_{|q'| < q} \right| \int \Delta_q \left( S_{q' - 1} Q_{\alpha} \right) \Delta_q \Delta Q_{\gamma} \Delta_q u_{\alpha, \beta} \right| \leq C 2^{-q} \|S_{q' - 1} \nabla Q_{\alpha}\|_{L^\infty} \|\Delta_q \Delta Q_{\gamma}\|_{L^2} \|\Delta_q u\|_{L^2} \leq C 2^{-q} \|\nabla Q\|_{L^\infty} \|\Delta_q \nabla Q\|_{L^2} \|\Delta_q u\|_{L^2} \leq C 2^{-2q} b_q(t) \|\nabla Q\|_{L^\infty} \|\Delta_q \nabla Q\|_{L^2} \|\Delta_q u\|_{L^2} \] (65)

Concerning the term \( J_4 \) we use that \( (S_{q' - 1} Q_{\alpha'}) - S_{q - 1} Q_{\alpha'} \) is localized in a dyadic ring, so we have

\[ \|S_{q' - 1} Q_{\alpha'} - S_{q - 1} Q_{\alpha'}\|_{L^\infty} \leq C 2^{-q} \|\nabla Q\|_{L^\infty}, \]

and we obtain

\[ |J_4| = \int \sum_{|q'| \leq 5} (S_{q' - 1} Q_{\alpha'}) - S_{q - 1} Q_{\alpha'} \Delta_q \Delta Q_{\gamma} \Delta_q u_{\alpha, \beta} \leq C 2^{-q} \|\nabla Q\|_{L^\infty} 2^q \|\Delta_q \nabla Q\|_{L^2} \|\Delta_q u_{\alpha, \beta}\|_{L^2}. \]

Using the fact that \( \|\Delta_q u_{\alpha, \beta}\|_{L^2} \leq C 2^{-q} a_q^1(t) \|\nabla u\|_{H^s} \) and \( \|\Delta_q \nabla Q\|_{L^2} \leq C 2^{-q} a_q^2(t) \|\nabla Q\|_{H^s} \) and denoting \( b_q(t) \) we find

\[ |J_4| \leq C 2^{-2q} b_q(t) \|\nabla Q\|_{L^\infty} \|\nabla Q\|_{H^s} \|\nabla u\|_{H^s}. \]

The following term to estimate is \( J_5 \). Using Bernstein inequalities \( \|S_{q' + 2} \Delta Q\|_{L^\infty} \leq C 2^q \|\nabla Q\|_{L^\infty} \) and \( \|\Delta_q Q_{\gamma}\|_{L^2} \leq C 2^{-q} \|\nabla \Delta_q Q_{\gamma}\|_{L^2} \), we obtain

\[ |J_5| = \left| \sum_{q' > q - 5} \int \Delta_q \left( S_{q' + 2} \Delta Q_{\gamma} \Delta_q Q_{\alpha'} \right) \Delta_q u_{\alpha, \beta} \right| \leq \left| \sum_{q' > q - 5} \int \Delta_q \left( S_{q' + 2} \Delta Q_{\gamma} \Delta_q Q_{\alpha'} \right) \Delta_q u_{\alpha, \beta} \right| \leq 2^q \|\nabla Q\|_{L^\infty} 2^{-q} \|\Delta_q \nabla Q\|_{L^2} \|\Delta_q u\|_{L^2} \leq C 2^{-q} b_q(t) \|\nabla Q\|_{L^\infty} \|\nabla \Delta_q Q_{\gamma}\|_{L^2} \|\nabla u\|_{H^s} \leq C 2^{-2q} b_q(t) \|\nabla \Delta_q Q_{\gamma}\|_{L^2} \|\nabla u\|_{H^s} \]

where \( b_q(t) = \sum_{q' > q - 5} 2^{-(q' - q)s} a_q(t) \).

The term \( J_k, k = 6, 7, 8 \) is estimated exactly as the term \( J_{k-3} \) that we have already studied above.

Putting together all this estimates, multiplying by \( 2^{2q} \) and taking the sum in \( q \), observing that we can write any sequence \( b_q \) as \( b_q = a_q \cdot \bar{a}_q \) with \( a_q, \bar{a}_q \in l^2_q \), using \( ab \leq C^{-1} a^2 + \epsilon b^2 \) with appropriately chosen \( \epsilon \), we obtain the claimed estimate (11)

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