CRAMÉR-TYPE MODERATE DEVIATION OF NORMAL APPROXIMATION FOR EXCHANGEABLE PAIRS

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Abstract. In Stein’s method, an exchangeable pair approach is commonly used to estimate the convergence rate of normal and nonnormal approximation. Using the exchangeable pair approach, we establish a Cramér-type moderate deviation theorem of normal approximation for an arbitrary random variable without a bound on the difference of the exchangeable pair. A Berry–Esseen-type inequality is also obtained. The result is applied to the subgraph counts in the Erdős–Rényi random graph, local dependence, and graph dependency.

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1. Introduction

Let $W$ be a random variable, and we say $(W, W')$ is an exchangeable pair if $(W, W')$ has the same joint distribution as $(W', W)$. The exchangeable pair approach of Stein’s method is commonly used in the normal and nonnormal approximation to estimate the convergence rates. Using exchangeable pair approach, Chatterjee and Shao [9] and Shao and Zhang [25] provided a concrete tool to identify the limiting distribution of the target random variable as well as the $L_1$ bound of the approximation. Recently, Shao and Zhang [26] obtained a Berry–Esseen-type bound of normal and nonnormal approximation for unbounded exchangeable pairs. In this paper, we focus only on the normal approximation. Let $(W, W')$ be an exchangeable pair and $\Delta = W - W'$. Assume that there exists a constant $\lambda \in (0, 1)$ and a random variable $R$ such that

$$E(\Delta | W) = \lambda(W + R),$$

and Shao and Zhang [26] proved that

$$\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq E \left| 1 - \frac{1}{2\lambda} E(\Delta^2 | W) \right| + \frac{1}{\lambda} E|E(\Delta^* \Delta | W)| + E|R|,$$

where $\Delta^* := \Delta^*(W, W')$ is any random variable satisfying that $\Delta^*(W, W') = \Delta^*(W', W)$ and $\Delta^* \geq |\Delta|$. We refer to Stein [29], Rinott and Rotar [22], Chatterjee, Diaconis and Meckes [7], Chatterjee and Meckes [8] and Meckes [18] for other related results for the exchangeable pair approach. Chatterjee [5], Chatterjee and Dey [6] and Mackey, Jordan, Chen, Farrell and Tropp [17] also proved the concentration inequality results for exchangeable pairs.

The Berry–Esseen bound (1.2) provides an optimal convergence rate for many applications. However, in practice, it may not be easy to check (1.1) in general. Alternatively, in Section 2, we prove a new version of (1.2) under the condition (D1), which is a natural generalization of (1.1).

While the Berry–Esseen-type bound describes the absolute error for the distribution approximation, the Cramér-type moderate deviation reflects the relative error. More precisely, let $\{Y_n, n \geq 1\}$ be a sequence of random variables that converge to $Y$ in distribution, the Cramér-type moderate deviation is

$$\frac{P(Y_n > x)}{P(Y > x)} = 1 + \text{error term} \to 1$$

for $0 \leq x \leq a_n$, where $a_n \to \infty$ as $n \to \infty$. Specially, for independent and identically distributed (i.i.d.) random variables $X_1, \cdots, X_n$ with $E X_i = 0$, $E X_i^2 = 1$ and
\[ E e^{t_0 \sqrt{|X_1|}} < \infty, \] where \( t_0 > 0 \) is a constant, put \( W_n = n^{-1/2}(X_1 + \cdots + X_n) \),

\[
\frac{P(W_n > x)}{1 - \Phi(x)} = 1 + O(1)n^{-1/2}(1 + x^3),
\]

for \( 0 \leq x \leq n^{1/6} \), where \( \Phi(x) \) is the standard normal distribution function. We refer to Linnik [16] and Petrov [19] for details. The condition \( E e^{t_0 \sqrt{|X_1|}} < \infty \) is necessary, and the range \( 0 \leq x \leq n^{1/6} \) and the order of the error term \( n^{-1/2}(1 + x^3) \) are optimal.

Since introduced by Stein [28] in 1972, Stein’s method has been widely used in recent years, and shows its importance and power for estimating the approximation errors of normal and nonnormal approximation. Moderate deviation results were also obtained by Stein’s method in the literature. For instance, using Stein’s method, Račić [20] considered the moderate deviation under certain local dependence structures. In the context of Poisson approximation, Barbour, Holst and Janson [4], Chen and Choi [10] and Barbour, Chen and Choi [2] applied Stein’s method to prove moderate deviation results for sums of independent indicators, whereas Chen, Fang and Shao [11] studied sums of dependent indicators. Moreover, Chen, Fang and Shao [12] and Shao, Zhang and Zhang [24] obtained the general Cramér-type moderate deviation results of normal and nonnormal approximation for dependent random variables whose dependence structure is defined in terms of a Stein identity.

For normal approximation, assume that there exists a constant \( \delta > 0 \), a random function \( \hat{K}(u) \geq 0 \) and a random variable \( \hat{R} \) such that for any absolutely continuous function \( f \),

\[
E(Wf(W)) = \mathbb{E} \int_{|t| \leq \delta} f'(W + u)\hat{K}(u) \, du + \mathbb{E}(\hat{R}f(W)).
\]

Let \( K := \int_{|t| \leq \delta} \hat{K}(u) \, du \), and assume that there exists constants \( \theta_0, \theta_1 \) and \( \theta_2 \) such that

\[
|K| \leq \theta_0,
\]

\[
|\mathbb{E}(K_1 | W) - 1| \leq \theta_1(1 + |W|),
\]

\[
|\mathbb{E}(R | W)| \leq \theta_2(1 + |W|).
\]

By Chen, Fang and Shao [12, Theorem 3.1], the random variable \( W \) has the following moderate deviation result:

\[
\frac{P(W > z)}{1 - \Phi(z)} = 1 + O(1)\theta_0^3(1 + z^3)(\delta + \theta_1 + \theta_2),
\]

for \( 0 \leq z \leq \theta_0^{-1}(\delta^{-1/3} + \theta_1^{-1/3} + \theta_2^{-1/3}) \) where \( O(1) \) is bounded by a universal constant. However, in their results, a boundedness assumption on \( |\Delta| \) is required and the conditions in (1.4) may be difficult to check in general. This motivates us to...
prove a more general Cramér-type moderate deviation result for the exchangeable pair approach.

This paper is organized as follows. We present our main results in Section 2. In Section 3, we give some applications of our main result. The proofs of Theorems 2.1 and 2.2 are put in Section 4. The proofs of theorems in Section 3 are postponed to Section 5.

2. Main results

Let $X$ be a field of random variables and $W = \varphi(X)$ be the random variable of interest. We consider the following condition:

(D1) Let $(X, X')$ be an exchangeable pair. Let $D = F(X, X')$ be an anti-symmetric function with respect to $X$ and $X'$ satisfying $\mathbb{E}(D | X) = \lambda(W + R)$ where $0 < \lambda < 1$ is a constant and $R$ is a random variable.

The condition (D1) is a natural generalization of (1.1). Specially, if (1.1) is satisfied, we can simply choose $D = \Delta$.

The following theorem provides a uniform Berry–Esseen bound in the normal approximation.

**Theorem 2.1.** Let $(W, W')$ be an exchangeable pair satisfying the condition (D1), and $\Delta = W - W'$. Let $D^* := D^*(W, W')$ be any random variable such that $D^*(W, W') = D^*(W', W)$ and $D^* \geq |D|$. Then,

\[
\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq \mathbb{E} \left| 1 - \frac{1}{2\lambda} \mathbb{E}(D\Delta | W) \right| + \frac{1}{\lambda} \mathbb{E} \left| \mathbb{E}(D^* \Delta | W) \right| + \mathbb{E} |R|.
\]

If (1.1) is satisfied, then (2.1) covers (1.2) by taking $D = \Delta$.

The following theorem provides a a Cramér-type moderate deviation result under the condition (D1) without the assumption that $|\Delta|$ is bounded.

**Theorem 2.2.** Let $(W, W')$ be an exchangeable pair satisfying the condition (D1), and $\Delta = W - W'$. Let $D^* := D^*(W, W')$ be any random variable such that $D^*(W, W') = D^*(W', W)$ and $D^* \geq |D|$. Assume that there exists a constant $A > 0$ and increasing functions $\delta_1(t), \delta_2(t)$ and $\delta_3(t)$ such that for all $0 \leq t \leq A$,

(A1) $\mathbb{E} e^{teW} < \infty$,
(A2) $\mathbb{E} \left| 1 - \frac{1}{\lambda} \mathbb{E}(D\Delta | X) \right| e^{teW} \leq \delta_1(t) \mathbb{E} e^{teW},$
(A3) $\mathbb{E} \left| \frac{1}{\lambda} \mathbb{E}(D^* \Delta | X) \right| e^{teW} \leq \delta_2(t) \mathbb{E} e^{teW}$, and
(A4) $\mathbb{E} |R| e^{teW} \leq \delta_3(t) \mathbb{E} e^{teW}$.
For $d_0 > 0$, let

$$A_0(d_0) := \max \left\{ 0 \leq t \leq A : \frac{t^2}{2}(\delta_1(t) + \delta_2(t)) + \delta_3(t)t \leq d_0 \right\}.$$  

Then, for any $d_0 > 0$,

$$\left| \frac{P(W > z)}{1 - \Phi(z)} - 1 \right| \leq 20 e^{d_0}((1 + z^2)(\delta_1(z) + \delta_2(z)) + (1 + z)\delta_3(z)),$$

provided that $0 \leq z \leq A_0(d_0)$.

Remark 2.1. Under the condition (D1) and assume that $|\Delta| \leq \delta$, then (1.3) is satisfied with

$$\hat{K}(u) = \frac{1}{2\lambda}D\left(1_{-\Delta \leq t \leq 0} - 1_{0 < t \leq -\Delta}\right),$$

$\hat{R} = -R$ and $K_1 = (D\Delta)/(2\lambda)$. Under the condition (A1), it can be shown that (see, e.g. Chen, Fang and Shao [12, Lemma 5.1] and Shao, Zhang and Zhang [24, Section 4]) the condition (1.4) implies conditions (A2) and (A4) with $\delta_1(t) = \theta_1(1 + t)$ and $\delta_3(t) = \theta_2(1 + t)$.

3. Applications

3.1. Subgraph counts in the Erdős–Rényi random graph. Let $G$ be a graph with $N$ vertices and $V := \{v_i, 1 \leq i \leq N\}$ be the vertex set. For any integer $k \geq 1$, let

$$[N]_k := \{(i_1, \ldots, i_k) : 1 \leq i_1 < i_2 < \cdots < i_k \leq N\}.$$  

For $(i, j) \in [N]_2$, define

$$\xi_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are connected}, \\ 0, & \text{otherwise.} \end{cases}$$

We say $G(N, p)$ is an Erdős–Rényi random graph if $\{\xi_{ij}, (i, j) \in [N]_2\}$ are independent and for each $(i, j) \in [N]_2$, $P(\xi_{ij} = 1) = 1 - P(\xi_{ij} = 0) = p$. For any graph $H$, let $v(H)$ and $e(H)$ denote the number of its vertices and edges, respectively.

Let $G$ be a given fixed graph and denote $v := v(G)$ and $e := e(G)$. Let $S_N$ be the number of copies (not necessarily induced) of $G$ in $G(N, p)$. Let $\mu_N = \mathbb{E}(S_N)$, $\sigma_N = \sqrt{\text{Var}(S_N)}$ and $W_N = (S_N - \mu_N)/\sigma_N$.

Theorem 3.1. Let

$$\psi = \min_{H \subset G, e(H) > 0} \left\{ N^{v(H)}p^e(H) \right\}.$$
We have

\[
(3.1) \quad \sup_{z \in \mathbb{R}} |P(W_N \leq z) - \Phi(z)| \leq \begin{cases} 
C\psi^{-1/2}, & 0 < p \leq 1/2, \\
CN^{-1}(1-p)^{-1/2}, & 1/2 < p < 1,
\end{cases}
\]

where $C > 0$ is a constant depending only on $G$. Moreover,

\[
(3.2) \quad \frac{P(W_N > z)}{1 - \Phi(z)} = 1 + O(1)(1 + z^2)b_N(p, z),
\]

for $0 \leq z \leq (1-p)N^2p^c\psi^{-1/2}$ such that $(1+z^2)b_N(p, z) \leq 1$, where $O(1)$ is bounded by a constant depending only on $G$ and

\[
(3.3) \quad b_N(p, z) = \begin{cases} 
\psi^{-1/2}(1 + z), & 0 < p < 1/2, \\
N^{-1}(1-p)^{-1/2}(1 + (1-p)^{-1/2}z), & 1/2 < p < 1.
\end{cases}
\]

The proof of Theorem 3.1 is put in Section 5.

**Remark 3.1.** Barbour, Karoski and Ruciski [3] and Fang [15] proved the same bound as in (3.1) for the Wasserstein-1 distance and for the Wasserstein-2 distance, respectively.

**Remark 3.2.** For fixed $p$ which is bounded away from 0 and 1, and independent of $N$, then for sufficiently large $N$, we have $\psi = O(N^2)$ and $p = O(1)$. In this case, (3.2) yields

\[
\frac{P(W_N > z)}{1 - \Phi(z)} = 1 + O(1)N^{-1}(1 + z^3),
\]

for $z \in (0, N^{1/3})$.

**Remark 3.3.** Specially, when $G$ is a triangle, the bound (3.1) reduces to

\[
\sup_{z \in \mathbb{R}} |P(W_N \leq z) - \Phi(z)| \leq \begin{cases} 
CN^{-3/2}p^{-3/2}, & 0 < p \leq N^{-1/2}, \\
CN^{-1}p^{-1/2}, & N^{-1/2} < p \leq 1/2, \\
CN^{-1}(1-p)^{-1/2}, & 1/2 < p < 1,
\end{cases}
\]

where $C$ is an absolute constant. This is as same as the result in Röllin [23]. For the Cramér-type moderate deviation, (3.2) reduces to the following four cases. Here, $O(1)$ is bounded by an absolute constant.

(1) If $0 < p \leq N^{-1/2}$,

\[
\frac{P(W_N > z)}{1 - \Phi(z)} = 1 + O(1)N^{-3/2}p^{-3/2}(1 + z^3),
\]

for $0 \leq z \leq N^{1/2}p^{3/2}$. 

(2) If $N^{-1/2} < p \leq N^{-2/7}$, 
\[
\frac{P(W_N > z)}{1 - \Phi(z)} = 1 + O(1)N^{-1/2}(1 + z^3),
\]
for $0 \leq z \leq N^{5/2}$.
(3) If $N^{-2/7} < p \leq 1/2$, 
\[
\frac{P(W_N > z)}{1 - \Phi(z)} = 1 + O(1)N^{-1/2}(1 + z^3),
\]
for $0 \leq z \leq N^{1/3}p^{1/6}$.
(4) If $1/2 < p < 1$, 
\[
\frac{P(W_N > z)}{1 - \Phi(z)} = 1 + O(1)N^{-1}(1 - p)^{-1/2}(1 + z^2)(1 + (1 - p)^{-1/2}z),
\]
for $0 \leq z \leq N^{1/3}(1 - p)^{1/3}$.

3.2. Local dependence. Let $\mathcal{J}$ be an index set and $\{X_i, i \in \mathcal{J}\}$ be a local dependent random field with zero mean and finite variances. Put $W = \sum_{i \in \mathcal{J}} X_i$ and assume that $\text{Var}(W) = 1$. For $A \subset \mathcal{J}$, let $X_A = \{X_i, i \in A\}$, $A^c = \{j \in \mathcal{J} : j \notin A\}$ and let $|A|$ be the cardinality of $A$.

Assume that $\{X_i, i \in \mathcal{J}\}$ satisfies the following conditions.

(LD1) For any $i \in \mathcal{J}$, there exists $A_i \subset \mathcal{J}$ such that $X_i$ is independent of $X_{A_i}$.
(LD2) For any $i \in \mathcal{J}, j \in A_i$, there exists $A_{ij}$ such that $A_i \subset A_{ij} \subset \mathcal{J}$ and $X_i, X_j$ is independent of $X_{A_{ij}}$.

We have the following Berry–Esseen-type bound.

**Theorem 3.2.** Assume that conditions (LD1) and (LD2) are are satisfied. Then,

\[
\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq 12r^{1/2},
\]

where

\[
r = \sum_{i \in \mathcal{J}} \sum_{j \in A_i} \sum_{\substack{i' \in A_{ij} \setminus A_i \setminus A_{ij} \setminus A_i \setminus A_{ij}}} \left\{E|X_i|^4 + E|X_j|^4 + E|X_{i'}|^4 + E|X_{j'}|^4 \right\}.
\]

Remark 3.4. In particular, if $E|X_i|^4 \leq \delta^4$ for some $\delta > 0$ and for each $i \in \mathcal{J}$, then

\[
\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq 12n^{1/2} \kappa_1 \kappa_2 \delta^2,
\]

where $\kappa_1$ and $\kappa_2$ are positive constants such that

\[
\max_{i \in \mathcal{J}} |A_i| \leq \kappa_1, \quad \text{and} \quad \max_{\substack{i \in \mathcal{J} \setminus \mathcal{J} \setminus A_i \setminus A_{ij}}} |A_{ij}| \leq \kappa_2.
\]
Remark 3.5. The conditions (LD1) and (LD2) is another version of Barbour, Karoski and Ruciski [3, Eqs. (2.3)–(2.5)], which is also studied in Fang [15]. Also, Chen and Shao [14] considered the following condition:

\[(LD2') \text{ There exist } A_i \subset B_i \subset J \text{ such that } X_i \text{ is independent of } X_{A_i^c} \text{ and } \{X_j, j \in A_i\} \text{ is independent of } \{X_j, j \in B_i^c\}.\]

The size of $A_{ij}$ is smaller than that of $B_i$. Let $N(B_i) = \{j : J : B_i \cap B_j = \emptyset\}$ and $\kappa' = \max_{i \in J} |N(B_i)|$. Chen and Shao [14] proved that, under the condition (LD2'), for $2 < p \leq 4$,

\[
\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq (13 + 11\kappa') \sum_{i \in J} \left( \mathbb{E}|X_i|^{3\lambda_p} + \mathbb{E}\sum_{j \in A_i} X_j^{3\lambda_p} \right)
+ 2.5 \left( \kappa' \sum_{i \in J} \left( \mathbb{E}|X_i|^{p} + \mathbb{E}\sum_{j \in A_i} |X_j|^p \right) \right)^{1/2}.
\]

It is well known that

\[
\mathbb{E}\left| \sum_{j \in A_i} X_j \right|^p \leq |A_i|^{p-1} \sum_{j \in A_i} \mathbb{E}|X_j|^p, \quad p \geq 1.
\]

In this point of view, the result (3.4) covers (3.5) with $p = 4$.

We refer to Shergin [27], Baldi, Rinott and Stein [1] and Rinott [21] for more existing results of Berry–Esseen bound.

For the Cramér-type moderate deviation, we require two additional conditions:

(LD3) For any $i \in J$, $j \in A_i$ and $k \in A_{ij}$, there exists $A_{ijk}$ such that $A_{ij} \subset A_{ijk} \subset J$ and $\{X_i, X_j, X_k\}$ is independent of $X_{A_{ijk}^c}$.

(LD4) Assume that for each $i \in J$, there exist two positive constants $\alpha > 0$ and $\beta \geq 1$, a random variable $U_i \geq 0$ that is independent of $\{X_j, j \in A_i^c\}$, such that

\[
\sum_{j \in A_i \setminus \{i\}} |X_j| \leq U_i,
\]

and for each $i \in J$,

\[
\mathbb{E} e^{\alpha(|X_i| + U_i)} \leq \beta,
\]

and $\mathbb{E}|X_i|^6 e^{\alpha(|X_i| + U_i)} < \infty$.

Remark 3.6. The condition (LD3) is an extension of conditions (LD1) and (LD2) and condition (LD4) is on the moment generating function for the neighborhood of $X_i$. Raić [20] proposed some different conditions, but those conditions depend on not only $\{X_j, j \in A_i\}$ but also $\{X_k, k \in A_{ij}\}$.
For $i \in J$, $j \in A_i$ and $k \in A_{ij}$, let $\kappa_{ij} = |A_{ij} \setminus (A_i \cup A_j)|$ and $\kappa_{ijk} = |A_{ijk} \setminus (A_i \cup A_j \cup A_k)|$. Let $\gamma_{p,i}(t) = \mathbb{E}(|X_i|^p e^{t(U_i+|X_i|)})$ and 

\begin{equation}
(3.6) \quad \Gamma_3(t) = \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in A_{ij}} \beta^{2\kappa_{ij} + 2\kappa_{ik}} \left\{ \gamma_{3,i}(t) + \gamma_{3,j}(t) + \gamma_{3,k}(t) \right\},
\end{equation}

\begin{equation}
\Gamma_4(t) = \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in A_{ij}} \sum_{l \in A_{ijk}} |A_{ij}|^{-1} \beta^{2\kappa_{ij}}
\end{equation}

\begin{equation}
\times \left\{ \gamma_{4,i}(t) + \gamma_{4,j}(t) + \gamma_{4,l}(t) + \gamma_{4,k}(t) \right\},
\end{equation}

\begin{equation}
\Gamma_6(t) = \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in A_{ij}} \sum_{l \in A_{ijk}} \sum_{m \in A_{ijkl}} \beta^{2\kappa_{ij} + 2\kappa_{ik}} \left\{ \sum_{m \in \{i,j,i',j',k,l\}} \gamma_{6,m}(t) \right\}.
\end{equation}

We have the following moderate deviation result.

**Theorem 3.3.** Assume that conditions (LD1)–(LD4) are are satisfied. Then, for any $d_0 > 0$,

\[
\left| \frac{P(W > z)}{1 - \Phi(z)} - 1 \right| \leq 240 e^{d_0}(1 + z^2) \left\{ \beta^{5/2} \Gamma_4^{1/2}(z) + \beta^6 \Gamma_3(z)z + \beta^3 \Gamma_6^{1/2}(z)z \right\},
\]

provided that $0 \leq z \leq \alpha$ such that \(\beta^{5/2} \Gamma_4^{1/2}(z) + \beta^6 \Gamma_3(z)z + \beta^3 \Gamma_6^{1/2}(z)z \) $z^2 \leq 2d_0$.

For bounded random variables, condition (LD4) can be replaced by the following condition:

(K1) Assume that there exist positive constants $\delta > 0$, $\kappa_1 \geq 1$, $\kappa_2 \geq 1$, $\kappa_3 \geq 1$ and $\kappa_4 \geq 0$ such that $|X_i| \leq \delta$ and

\begin{equation}
(3.7) \quad \max_{i \in J} |A_i| \leq \kappa_1, \quad \max_{ij \in A_{ij}} |A_{ij}| \leq \kappa_2, \quad \max_{ij \in A_{ij}} \kappa_{ij} \leq \kappa_3, \quad \max_{ijk \in A_{ijk}} \kappa_{ijk} \leq \kappa_4.
\end{equation}

Taking $U_i = (\kappa_2 - 1)\delta$, $\alpha = \delta^{-1}\kappa_1^{-1}(1 + \kappa_4)^{-1}$ and $\beta = e^{1/(1+\kappa_4)}$, Theorem 3.3 implies the following corollary:

**Corollary 3.1.** Let $n = |J|$. Assume that conditions (LD1)–(LD3) and (K1) are satisfied. Then,

\[
\left| \frac{P(W > z)}{1 - \Phi(z)} - 1 \right| \leq C \kappa_1^{1/2} \left\{ n^{1/2}\delta^2 + n\delta^3(\kappa_2^{1/2} + \kappa_3^{1/2})z \right\}(1 + z^2),
\]

provided that $0 \leq z \leq \delta^{-1}\kappa_1^{-1}(1 + \kappa_4)^{-1}$ and $\kappa_1 \kappa_2^{1/2} \left\{ n^{1/2}\delta^2 + n\delta^3(\kappa_2^{1/2} + \kappa_3^{1/2})z \right\}(1 + z^3) \leq 1$, where $C > 0$ is an absolute constant.
3.3. **Graph dependency.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges. Consider a set of random variables $\{X_i, i \in \mathcal{V}\}$. The graph $\mathcal{G}$ is said to be a dependency graph if for any pair of disjoint sets $\Gamma_1$ and $\Gamma_2$ in $\mathcal{V}$ such that no edge in $\mathcal{E}$ has one endpoint in $\Gamma_1$ and the other in $\Gamma_2$, the sets of random variables $\{X_i, i \in \Gamma_1\}$ and $\{X_i, i \in \Gamma_2\}$ are independent. The degree $d(v)$ of a vertex $v$ in $\mathcal{G}$ is the number of edges connected to this vertex. The maximal degree of a graph is $D(\mathcal{V}) = \max_{v \in \mathcal{V}} d(v)$.

Assume that $E(X_i) = 0$ for each $i \in \mathcal{V}$. Put

$$W = \sum_{i \in \mathcal{V}} X_i / \sigma,$$

where $\sigma = \sqrt{\text{Var}(\sum_{i \in \mathcal{V}} X_i)}$. The uniform Berry–Esseen bound was obtained by Baldi, Rinott and Stein [1] and Rinott [21] and the nonuniform Berry–Esseen bound by Chen and Shao [14].

For any $i, j, k \in \mathcal{V}$, let $A_i = \{j \in \mathcal{V} : \text{there is an edge connecting } j \text{ and } i\}$, $A_{ij} = A_i \cup A_j$ and $A_{ijk} = A_i \cup A_j \cup A_k$. Then $\kappa_{ijk} = \kappa_{ij} = 0$ and

$$\max \{|A_i|, |A_{ij}|, |A_{ijk}|\} \leq 3D(\mathcal{V}).$$

If there exists a constant $B > 0$ such that $|X_i| \leq B$. Then (K1) is satisfied with $\delta = B\sigma^{-1}$. Applying Corollary 3.1 yields the following theorem:

**Theorem 3.4.** Let $W = \sum_{i \in \mathcal{V}} X_i / \sigma$, where $\sigma^2 = \text{Var}(\sum_{i \in \mathcal{V}} X_i)$. Assume that for each $i \in \mathcal{V}$, $E(X_i) = 0$ and $|X_i| \leq B$. Then

$$(3.8) \quad \sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq CB^2 \sigma^{-2} n^{1/2} D(\mathcal{G})^{3/2},$$

and

$$(3.9) \quad \frac{P(W > z)}{1 - \Phi(z)} = 1 + O(1) \left\{ B^2 \sigma^{-2} n^{1/2} D(\mathcal{G})^{3/2} + B^3 \sigma^{-3} n D(\mathcal{G})^2 z \right\} (1 + z^2),$$

for $z \geq 0$ such that $\left\{ B^2 \sigma^{-2} n^{1/2} D(\mathcal{G})^{3/2} + B^3 \sigma^{-3} n D(\mathcal{G})^2 z \right\} (1 + z^2) \leq 1$. Here, $C$ is an absolute constant and $O(1)$ is also bounded by an absolute constant.

**Remark 3.7.** Under the same assumptions as in Theorem 3.4, Rinott [21] obtained the following Berry–Esseen bound:

$$\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq \sqrt{\frac{1}{2\pi} DB \sigma^{-1} + 16B^2 \sigma^{-2} n^{1/2} D(\mathcal{G})^{3/2}} + 10B^3 \sigma^{-3} n D(\mathcal{G})^2,$$

which is covered by (3.8). Moreover, the moderate deviation (3.9) is new.
Remark 3.8. Note that when $D(G)$ and $B$ are bounded, and $\sigma^2 \geq cn$ for some constant $c > 0$, and (3.9) yields

$$\frac{P(W > z)}{1 - \Phi(z)} = 1 + O(1)n^{-1/2}(1 + z^3), \text{ for } 0 \leq z \leq n^{1/6},$$

where both the range $[0, n^{1/6}]$ and the convergence rate $O(1)n^{-1/2}(1 + z^3)$ are optimal.

4. Proofs of Theorems 2.1 and 2.2

In this section, we give the proofs of our main results in Section 2. Before proving Theorems 2.1 and 2.2, we first present some preliminary lemmas. In the proofs, we apply the ideas in Chen, Fang and Shao [12, Lemmas 5.1–5.2] and Shao and Zhang [26, pp. 71–73].

Lemma 4.1. Let $\phi$ be a nondecreasing function. Then,

$$\frac{1}{2\lambda} \left| \mathbb{E} \left( D \int_{-\Delta}^{0} \{ \phi(W + u) - \phi(W) \} \, du \right) \right| \leq \frac{1}{2\lambda} \mathbb{E} (D^* \Delta \phi(W)),$$

where $D^*$ is as defined in Theorem 2.2.

Proof of Lemma 4.1. Since $\phi(\cdot)$ is increasing, it follows that

$$0 \geq \int_{-\Delta}^{0} (\phi(W + u) - \phi(W)) \, du$$

$$\geq -\Delta (\phi(W) - \phi(W')).$$

Therefore, as $(W, W')$ is exchangeable,

$$\frac{1}{2\lambda} \left| \mathbb{E} \left( D \int_{-\Delta}^{0} \{ \phi(W + u) - \phi(W) \} \, du \right) \right|$$

$$\leq \frac{1}{2\lambda} \mathbb{E} D^* 1_{\{D > 0\}} \Delta (\phi(W) - \phi(W'))$$

$$= \frac{1}{2\lambda} \mathbb{E} D^* \Delta (1_{\{D > 0\}} + 1_{\{D < 0\}}) \phi(W)$$

$$= \frac{1}{2\lambda} \mathbb{E} D^* \Delta \phi(W). \quad \square$$

The following lemma provides a bound for the moment generating function of $W$.

Lemma 4.2. For $0 \leq t \leq A$, we have

$$\mathbb{E} e^{tW} \leq \exp \left\{ \frac{t^2}{2} (1 + \delta_1(t) + \delta_2(t)) + \delta_3(t)t \right\}. \quad (4.1)$$

For $d_0 > 0$, let

$$A_0(d_0) := \max \left\{ 0 \leq t \leq A : \frac{t^2}{2} (\delta_1(t) + \delta_2(t)) + \delta_3(t)t \leq d_0 \right\}.$$
Then, for $0 \leq t \leq A_0(d_0)$,

$$
(4.2) \quad \mathbb{E} e^{tW} \leq e^{d_0 t^{2/2}}.
$$

**Proof of Lemma 4.2.** Let $h(t) = \mathbb{E} e^{tW}$. Since $\mathbb{E} e^{tW} < \infty$, and by the continuity of the exponential function, we have $h'(t) = \mathbb{E}(W e^{tW})$. Since $\mathbb{E}(D \mid W) = \lambda(W + R)$, it follows that

$$
(4.3) \quad \mathbb{E}(W e^{tW}) = t \frac{1}{2\lambda} \mathbb{E}\left\{D \int_{-\Delta}^{0} e^{(W+u)} du\right\} - \mathbb{E}(R e^{tW})
$$

$$
\leq t \mathbb{E}(e^{tW}) + t \frac{1}{2\lambda} \mathbb{E}\left\{D \int_{-\Delta}^{0} (e^{(W+u)} - e^{tW}) du\right\}
$$

$$
+ t \mathbb{E}\left|1 - \frac{1}{2\lambda} \mathbb{E}(D \Delta \mid W)\right| e^{tW} + \mathbb{E}(|R| e^{tW}).
$$

By condition (A3) and Lemma 4.1, we have for $0 \leq t \leq A$,

$$
(4.4) \quad \frac{t}{2\lambda} \mathbb{E}\left\{D \int_{-\Delta}^{0} (e^{(W+u)} - e^{tW}) du\right\} \leq t \delta_2(t) \mathbb{E} e^{tW}.
$$

By conditions (A2) and (A4), for $0 \leq t \leq A$,

$$
(4.5) \quad t \mathbb{E}\left|1 - \frac{1}{2\lambda} \mathbb{E}(D \Delta \mid W)\right| e^{tW} \leq t \delta_1(t) \mathbb{E} e^{tW},
$$

$$
(4.6) \quad \mathbb{E}|R| e^{tW} \leq \delta_3(t) \mathbb{E} e^{tW}.
$$

Combining (4.3)–(4.6), we have for $0 \leq t \leq A$,

$$
h'(t) = \mathbb{E}(W e^{tW})
$$

$$
\leq th(t) + \left\{t(\delta_1(t) + \delta_2(t)) + \delta_3(t)\right\} h(t).
$$

Noting that $h(0) = 1$, and $\delta_1, \delta_2$ and $\delta_3$ are increasing, we complete the proof by solving the foregoing differential inequality. \[\square\]

**Lemma 4.3.** Let $A_0(d_0)$ be as defined in Lemma 4.2. We have for $0 \leq z \leq A_0(d_0)$,

$$
(4.7) \quad \mathbb{E}\left\{1 - \frac{1}{2\lambda} \mathbb{E}(D \Delta \mid W)\right\} |W e^{W^{2/2}} 1_{\{0 \leq W \leq z\}}| \leq 6 e^{d_0} (1 + z^2) \delta_1(z),
$$

$$
(4.8) \quad \frac{1}{2\lambda} \mathbb{E}\left\{\mathbb{E}(D \Delta \mid W)\right\} |W e^{W^{2/2}} 1_{\{0 \leq W \leq z\}}| \leq 6 e^{d_0} (1 + z^2) \delta_2(z),
$$

and

$$
(4.9) \quad \mathbb{E}\left\{|R| e^{W^{2/2}} 1_{\{0 \leq W \leq z\}}\right\} \leq 3 e^{d_0} (1 + z) \delta_3(z).
$$
Proof of Lemma 4.3. We apply the idea in Chen, Fang and Shao [12, Lemma 5.2] in this proof. For \( a \in \mathbb{R} \), denote \( [a] = \max\{n \in \mathbb{N} : n \leq a\} \). It follows that

\[
\mathbb{E} \left\{ \left| 1 - \frac{1}{2\lambda} \mathbb{E} \left( D\Delta \mid W \right) \right| W e^{W^2/2} 1_{\{0 \leq W \leq z\}} \right\}
\]

\[
= \sum_{j=1}^{[z]} \mathbb{E} \left\{ \left| 1 - \frac{1}{2\lambda} \mathbb{E} \left( D\Delta \mid W \right) \right| W e^{W^2/2} 1_{\{j-1 \leq W < j\}} \right\}
\]

\[
+ \mathbb{E} \left\{ \left| 1 - \frac{1}{2\lambda} \mathbb{E} \left( D\Delta \mid W \right) \right| W e^{W^2/2} 1_{\{j \leq W \leq z\}} \right\}
\]

\[
\leq \sum_{j=1}^{[z]} e^{(j-1)^2/2 - j(j-1)} \mathbb{E} \left\{ \left| 1 - \frac{1}{2\lambda} \mathbb{E} \left( D\Delta \mid W \right) \right| e^{W} 1_{\{j-1 \leq W < j\}} \right\}
\]

\[
+ z e^{[z]^2/2 - [z]z} \mathbb{E} \left\{ \left| 1 - \frac{1}{2\lambda} \mathbb{E} \left( D\Delta \mid W \right) \right| e^{W} 1_{\{z \leq W \leq z\}} \right\}
\]

\[
\leq 3 \sum_{j=1}^{[z]} e^{-j^2/2} \mathbb{E} \left\{ \left| 1 - \frac{1}{2\lambda} \mathbb{E} \left( D\Delta \mid W \right) \right| e^{W} 1_{\{j-1 \leq W < j\}} \right\}
\]

\[
+ 3z e^{-z^2/2} \mathbb{E} \left\{ \left| 1 - \frac{1}{2\lambda} \mathbb{E} \left( D\Delta \mid W \right) \right| e^{W} 1_{\{z \leq W \leq z\}} \right\}
\]

By condition (A2) and (4.2), and recalling that \( \delta_1 \) is increasing, for any \( 0 \leq x \leq z \leq A_0(d_0) \),

\[
e^{-x^2/2} \mathbb{E} \left\{ \left| 1 - \frac{1}{2\lambda} \mathbb{E} \left( D\Delta \mid W \right) \right| e^{xW} \right\} \leq \delta_1(x) \mathbb{E} e^{xW - x^2/2} \leq e^{d_0} \delta_1(x) \leq e^{d_0} \delta_1(z).
\]

By the foregoing inequalities,

\[
\mathbb{E} \left\{ \left| 1 - \frac{1}{2\lambda} \mathbb{E} \left( D\Delta \mid W \right) \right| W e^{W^2/2} 1_{\{0 \leq W \leq z\}} \right\}
\]

\[
\leq 3 e^{d_0} \delta_1(z) \sum_{j=1}^{[z]} j + z \leq 6 e^{d_0} (1 + z^2) \delta_1(z).
\]

This proves (4.7). The inequalities (4.8) and (4.9) can be obtained similarly. \( \square \)

Now we are ready to give the proof of Theorems 2.1 and 2.2.

Proofs of Theorems 2.1 and 2.2. Let \( z \geq 0 \) be a real number. Let \( f_z \) be the solution to the Stein equation:

\[
f'(w) - wf(w) = 1_{\{w \leq z\}} - \Phi(z),
\]

\[ (4.10) \]
where \( \Phi(\cdot) \) is the distribution function of the standard normal distribution. It is well known that (see, e.g., Chen, Goldstein and Shao [13])

\[
(4.11) \quad f_z(w) = \begin{cases} 
\frac{\Phi(w)(1 - \Phi(z))}{p(w)}, & w \leq z, \\
\frac{\Phi(z)(1 - \Phi(w))}{p(w)}, & w > z,
\end{cases}
\]

where \( p(w) = (2\pi)^{-1/2} e^{-w^2/2} \) is the density function of the standard normal distribution.

Since \((W, W')\) is exchangeable satisfying that \( \mathbb{E}(D \mid W) = \lambda(W + R) \), it follows that

\[
\mathbb{E}(W f_z(W)) = \frac{1}{2\lambda} \mathbb{E} \left( D \int_{-\Delta}^{0} f_z'(W + t) \, dt \right) - \mathbb{E} \left( R f_z(W) \right),
\]

and thus,

\[
(4.12) \quad P(W > z) - (1 - \Phi(z)) = \mathbb{E} \left( f_z'(W) - W f_z(W) \right) - \mathbb{E}(R f_z(W))
\]

where

\[
J_1 = \mathbb{E} \left( f_z'(W) \left\{ 1 - \frac{1}{2\lambda} \mathbb{E}(D \Delta \mid W) \right\} \right),
\]

\[
J_2 = \frac{1}{2\lambda} \mathbb{E} \left( D \int_{-\Delta}^{0} \{ f_z'(W + u) - f_z'(W) \} \, du \right),
\]

\[
J_3 = \mathbb{E}(R f_z(W)).
\]

We first prove (2.1) by bounding \( J_1 \), \( J_2 \) and \( J_3 \), separately. By Chen, Goldstein and Shao [13, Lemma 2.3], we have

\[
\|\tilde{f}_z\| \leq 1, \quad \|f_z\| \leq 1.
\]

Then,

\[
(4.13) \quad |J_1| \leq \mathbb{E} \left| 1 - \frac{1}{2\lambda} \mathbb{E}(D \Delta \mid W) \right|,
\]

\[
|J_3| \leq \mathbb{E} |R|.
\]

For \( J_2 \), observe that \( f_z'(w) = w f_z(w) - 1_{\{w > z\}} + (1 - \Phi(z)) \), and both \( w f_z(w) \) and \( 1_{\{w > z\}} \) are increasing functions (see, e.g. Chen, Goldstein and Shao [13, Lemma...
by Lemma 4.1,

\[ |J_2| \leq \frac{1}{2\lambda} \left| E\left( D \int_{-\Delta}^{0} \{(W + u) f_z(W + u) - W f'_z(W)\} \, du \right) \right| + \frac{1}{2\lambda} \left| E\left( D \int_{-\Delta}^{0} \{1_{\{w+u>z\}} - 1_{\{w>z\}}\} \, du \right) \right| \leq \frac{1}{2\lambda} E\left| (D^* \Delta \mid W) \right| \left( |W f_z(W)| + 1_{\{w>z\}} \right) \leq J_{21} + J_{22}, \]

where

\[ J_{21} = \frac{1}{2\lambda} E\left\{ |E(D^* \Delta \mid W)| \cdot |W f_z(W)| \right\}, \]
\[ J_{22} = \frac{1}{2\lambda} E\left\{ |E(D^* \Delta \mid W)|1_{\{w>z\}} \right\}. \]

Then, \( |J_2| \leq \frac{1}{\lambda} E\left| (D^* \Delta \mid W) \right| \). This proves Theorem 2.1 together with (4.13).

Now we move to prove Theorem 2.2. Without loss of generality, we only consider \( J_2 \), because \( J_1 \) and \( J_3 \) can be bounded similarly.

For any \( w > 0 \), it is well known that

\[ \frac{1 - \Phi(w)}{p(w)} \leq \max \left\{ \frac{1}{w}, \frac{\sqrt{2\pi}}{2} \right\}. \]

Then, for \( w > z \),

\[ |f_z(w)| \leq \frac{\sqrt{2\pi}}{2} \Phi(z), \quad |w f_z(w)| \leq 1, \]

and by symmetry, for \( w < 0 \),

\[ |f_z(w)| \leq \frac{\sqrt{2\pi}}{2} (1 - \Phi(z)), \quad |w f_z(w)| \leq 1 - \Phi(z). \]

For \( J_{21} \), by (4.11), (4.14) and (4.15), we have

\[ J_{21} \leq \frac{1}{2\lambda} (1 - \Phi(z)) E\left\{ |E(D^* \Delta \mid W)|1_{\{w < 0\}} \right\} + \frac{\sqrt{2\pi}}{2\lambda} (1 - \Phi(z)) E\left\{ |E(D^* \Delta \mid W)|W e^{W^2/2} 1_{\{0 \leq W \leq z\}} \right\} + \frac{1}{2\lambda} E\left\{ |E(D^* \Delta \mid W)|1_{\{w > z\}} \right\}. \]

Thus,

\[ |J_2| \leq \frac{1}{2\lambda} (1 - \Phi(z)) E\left\{ |E(D^* \Delta \mid W)|1_{\{w < 0\}} \right\} + \frac{\sqrt{2\pi}}{2\lambda} (1 - \Phi(z)) E\left\{ |E(D^* \Delta \mid W)|W e^{W^2/2} 1_{\{0 \leq W \leq z\}} \right\} + \frac{1}{\lambda} E\left\{ |E(D^* \Delta \mid W)|1_{\{w > z\}} \right\}. \]
For the first term of (4.16), by condition (A3) with \( t = 0 \), and noting that \( \delta_2 \) is increasing,

\[
\frac{1}{2\lambda} \mathbb{E}\left\{ \left| \mathbb{E}(D^* \Delta | W)\right| 1_{\{W < 0\}} \right\} \leq \frac{1}{2\lambda} \mathbb{E}\left\{ \left| \mathbb{E}(D^* \Delta | W)\right| \right\} \leq \delta_2(0) \leq e^{d_0} \delta_2(z).
\]

For the second term of (4.16), by Lemma 4.3, we have

\[
\frac{1}{2\lambda} \mathbb{E}\left\{ \left| \mathbb{E}(D^* \Delta | W)\right| W e^{W^2/2} 1_{\{0 \leq W \leq z\}} \right\} \leq 6 e^{d_0} (1 + z^2) \delta_2(z).
\]

For the third term of (4.16), by condition (A3) and (4.2), for \( 0 \leq z \leq A_0(d_0) \),

\[
\frac{1}{2\lambda} \mathbb{E}\left\{ \left| \mathbb{E}(D^* \Delta | W)\right| 1_{\{W > z\}} \right\} \leq \delta_2(z) e^{-z^2} \mathbb{E}e^W \leq e^{d_0} \delta_2(z) e^{-z^2/2}. 
\]

It is well known that for \( z > 0 \),

\[
e^{-z^2/2} \leq \sqrt{2\pi}(1 + z)(1 - \Phi(z)) \leq \frac{3\sqrt{2\pi}}{2} (1 + z^2)(1 - \Phi(z)).
\]

Then, for \( 0 \leq z \leq A_0(d_0) \),

\[
\frac{1}{2\lambda} \mathbb{E}\left\{ \left| \mathbb{E}(D^* \Delta | W)\right| 1_{\{W > z\}} \right\} \leq \frac{3 e^{d_0} \sqrt{2\pi}}{2} (1 + z^2) \delta_2(z)(1 - \Phi(z)).
\]

Therefore, combining (4.16)–(4.19), for \( 0 \leq z \leq A_0(d_0) \), we have

\[
|J_2| \leq 20 e^{d_0} (1 + z^2) \delta_2(z)(1 - \Phi(z)).
\]

Similarly,

\[
|J_1| \leq 20 e^{d_0} (1 + z^2) \delta_1(z)(1 - \Phi(z)),
\]

and

\[
|J_3| \leq 20 e^{d_0} (1 + z) \delta_3(z)(1 - \Phi(z)).
\]

This completes the proof together with (4.12). \( \square \)

5. Proofs of other results

5.1. Proof of Theorem 3.1. In this subsection, the constants \( C \) depend only on the fixed graph \( G \), which may take different values in different places. Recall that \( v = v(G) \), \( e = e(G) \), and let \( \{e_j\}_{1 \leq j \leq \binom{N}{2}} \) denote the edges of the complete graph on \( N \) vertices. Define

\[
\mathcal{I}_N = \left\{ i = \{i_1, \cdots, i_e\} : 1 \leq i_1 < i_2 < \cdots < i_e \leq \binom{N}{2} \right\},
\]

\[
G_i := \{e_{i_1}, \cdots, e_{i_e}\} \text{ is a copy of } G.
\]
Let $X_i = \left( \prod_{l=1}^{e} \varepsilon_{i_l} - p^e \right) / \sigma_N$. We have

$$W_N = \sum_{i \in \mathcal{I}_N} X_i = \frac{1}{\sigma_N} \sum_{i \in \mathcal{I}_N} \left( \prod_{l=1}^{e} \varepsilon_{i_l} - p^e \right),$$

where $\sigma^2_N = \text{Var}(S_N)$ and $\varepsilon_{i_l}$ is the indicator of the event that the edge $e_{i_l}$ is connected in $G(N, p)$. It is known that (see, e.g., Barbour, Karoski and Ruciski [3, p. 132])

$$\sigma^2_N \geq C(1 - p)N(1 - 2p)^{-1}. \quad (5.1)$$

Now we construct $A_i, A_{ij}$ and $A_{ijk}$. By Fang [15, pp. 11–12], for each $i \in \mathcal{I}_N$,

$$A_i = \{ j \in \mathcal{I}_N : |i \cap j| > 0 \} ;$$

for $i \in \mathcal{I}_N$ and $j \in A_i$,

$$A_{ij} = \{ k \in \mathcal{I}_N : |k \cap (i \cup j)| > 0 \} ;$$

and for $i \in \mathcal{I}_N, j \in A_i$ and $k \in A_{ij}$,

$$A_{ijk} = \{ l \in \mathcal{I}_N : |l \cap (i \cup j \cup k)| > 0 \} .$$

Then, $A_{ij} = A_i \cup A_j$ and $A_{ijk} = A_i \cup A_j \cup A_k$, and thus $\kappa_{ijk} = \kappa_{ij} = 0$. Also, $|A_i| \leq CNv^{-2}$. We now construct the exchangeable pair for $\{ X_i, i \in \mathcal{I}_N \}$. For each $l \in e(G)$, let $\varepsilon'_l$ be an independent copy of $\varepsilon_l$, which is also independent of $\{ \varepsilon_k, k \neq l \}$ and $\{ \varepsilon'_k, k \neq l \}$. For each $i = (i_1, \cdots, i_e) \in \mathcal{I}_N$, define

$$X_i^{(i)} = \frac{1}{\sigma_N} \left( \prod_{l=1}^{e} \varepsilon'_i - p^e \right),$$

and for $j = (j_1, \cdots, j_e) \in A_i$,

$$X_j^{(i)} = \frac{1}{\sigma_N} \left( \prod_{k \in i \cap j} \varepsilon'_k \prod_{l \in j \cap i} \varepsilon_l - p^e \right).$$

Let $I$ be random index uniformly distributed in $\mathcal{I}_N$ which is independent of all others. Let $D = X_I - X_I^{(I)}$, and

$$W^{(I)} = \sum_{j \in A_I} X_j + \sum_{j \in A'_I} X_j^{(I)} .$$

Then, $(W, W^{(I)})$ is an exchangeable pair and $\Delta = \sum_j (X_j - X_j^{(I)})$. Let $\mathcal{F} = \sigma\{ \varepsilon_i, 1 \leq i \leq \binom{N}{2} \}$. It follows that

$$\mathbb{E}(D \mid \mathcal{F}) = \frac{1}{|\mathcal{I}_N|} W.$$

This implies that condition (D1) is satisfied with \( \lambda = 1/|I_N| \). Note that by exchangeability and recall that \( \mathbb{E} W^2 = 1 \),

\[
\mathbb{E}(D\Delta) = 2\mathbb{E}(DW) = 2\lambda \mathbb{E}(W^2) = 2\lambda.
\]

Moreover,

\[
\mathbb{E}\left((X_i - X_i^{(i)})(X_j - X_j^{(i)}) \middle| \mathcal{F}\right) = \frac{1}{\sigma^2_N}\left(1 - p^{[i,j]}\right) \prod_{k \in i, j} \varepsilon_k - \frac{1}{\sigma^2_N} p^c \prod_{k \in j} \varepsilon_k + \frac{1}{\sigma^2_N} p^{e+[i,j]} \prod_{k \in [i,j]} \varepsilon_k := \nu_{ij},
\]

and

\[
\mathbb{E}(X_i - X_i^{(i)})(X_j - X_j^{(i)}) = \frac{1}{\sigma^2_N} p^{[i,j]}(1 - p^{[i,j]}) := \tilde{\nu}_{ij}.
\]

Also, with \( \mu_{ij} := \mathbb{E}\left(|X_i - X_i^{(i)}|(X_j - X_j^{(i)}) \middle| \mathcal{F}\right) \), we have \( \mathbb{E}\mu_{ij} = 0 \) by exchangeability. Then,

\[
\frac{1}{2\lambda} \mathbb{E}(D\Delta \mid \mathcal{F}) - 1 = \frac{1}{2} \sum_{i \in I_N} \sum_{j \in A_i} (\nu_{ij} - \tilde{\nu}_{ij}),
\]

\[
\frac{1}{\lambda} \mathbb{E}(|D\Delta \mid \mathcal{F}) = \sum_{i \in I_N} \sum_{j \in A_i} \mu_{ij}.
\]

Similar to (5.2),

\[
\mathbb{E}\left(|X_i - X_i^{(i)}||X_j - X_j^{(i)}| \mid \mathcal{F}\right) \leq \frac{1}{\sigma^2_N}\left(1 + p^{[i,j]}\right) \prod_{k \in i, j} \varepsilon_k + p^c \prod_{k \in j} \varepsilon_k + p^{e+[i,j]} \prod_{k \in [i,j]} \varepsilon_k \right).}
\]

By Barbour, Karoski and Ruciski [3, p. 132], we have \( \sigma_N |X_i| \leq 1 \) and

\[
\sigma_N |X_i| \leq \left|1 - \prod_{k \in i} \varepsilon_k\right| + \mathbb{E}\left|1 - \prod_{k \in i} \varepsilon_k\right|.
\]

By (5.4),

\[
\mathbb{E}\left(|X_i - X_i^{(i)}||X_j - X_j^{(i)}| \mid \mathcal{F}\right) \leq \frac{C}{\sigma^2_N}\left(1 - \prod_{k \in i} \varepsilon_k\right) + \mathbb{E}\left|1 - \prod_{k \in i} \varepsilon_k\right|.
\]

It follows from (5.3) and (5.5) that

\[
\max\{|\nu_{ij}|, |\mu_{ij}|\}
\]

\[
\leq \mathbb{E}\left(|X_i - X_i^{(i)}||X_j - X_j^{(i)}| \mid \mathcal{F}\right)
\]

\[
\leq \frac{C}{\sigma^2_N}\min\left\{\left(1 + p^{[i,j]}\right) \prod_{k \in i, j} \varepsilon_k + p^c \prod_{k \in j} \varepsilon_k + p^{e+[i,j]} \prod_{k \in [i,j]} \varepsilon_k, \left|1 - \prod_{k \in i} \varepsilon_k\right| + \mathbb{E}\left|1 - \prod_{k \in i} \varepsilon_k\right|\right\}.
\]

The following lemma provides some other properties for \( \nu_{ij} \) and \( \mu_{ij} \).
Lemma 5.1. For $0 \leq t \leq (1 - p)N^{2p}\psi^{-1/2}$,

$$\mathbb{E}\left(\sum_{i \in I_N} \sum_{j \in A_i} (\nu_{ij} - \bar{\nu}_{ij})^2 e^{tW}\right)$$

(5.7)

$$\leq \begin{cases} 
C\psi^{-1}(1 + t^2)\mathbb{E}e^{tW}, & 0 < p \leq 1/2, \\
CN^{-2}(1 - p)^{-1}(1 + (1 - p)^{-1}t^2)\mathbb{E}e^{tW}, & 1/2 < p < 1,
\end{cases}$$

and

$$\mathbb{E}\left(\sum_{i \in I_N} \sum_{j \in A_i} \mu_{ij}^2 e^{tW}\right)$$

(5.8)

$$\leq \begin{cases} 
C\psi^{-1}(1 + t^2)\mathbb{E}e^{tW}, & 0 < p \leq 1/2, \\
CN^{-2}(1 - p)^{-1}(1 + (1 - p)^{-1}t^2)\mathbb{E}e^{tW}, & 1/2 < p < 1,
\end{cases}$$

where $C$ is a constant depending only on the fixed graph $G$.

The proof of Lemma 5.1 is put at the end of this subsection. By (2.1) and Lemma 5.1 with $t = 0$, we prove the Berry–Esseen bound (3.1).

Note that for fixed $N$, we have $|W_N| \leq N^\nu/\sigma_N$, and then $\mathbb{E}e^{tW_N} < \infty$. By Lemma 5.1, conditions (A1)–(A4) are satisfied with

$$\delta_1(t) = \delta_2(t) = \begin{cases} 
C\psi^{-1/2}(1 + t), & 0 < p \leq 1/2, \\
CN^{-1}(1 - p)^{-1/2}(1 + (1 - p)^{-1/2}t), & 1/2 < p < 1.
\end{cases}$$

Applying Theorem 2.2 yields the moderate deviation (3.2), as desired.

Proof of Lemma 5.1. Without loss of generality, we only prove (5.7), because (5.8) can be shown similarly.

For any $i, j, i', j', k, q \in I_N$, denote

$$W_{ij'} = \sum_{l \in I_N} X_l I(l \in A_i \cup A_{i'} \cup A_j \cup A_{j'}),$$

$$W_{ij} = \sum_{l \in I_N} X_l I(l \not\in A_i \cup A_{i'} \cup A_j \cup A_{j'}),$$

$$W_{ij'k} = \sum_{l \in I_N} X_l I(l \in A_i \cup A_{i'} \cup A_j \cup A_{j'} \cup A_k),$$

$$W_{ij'k} = \sum_{l \in I_N} X_l I(l \not\in A_i \cup A_{i'} \cup A_j \cup A_{j'} \cup A_k),$$

$$W_{ijq} = \sum_{l \in I_N} X_l I(l \in A_i \cup A_j \cup A_{i'} \cup A_{j'} \cup A_k \cap A_q),$$

$$W_{ijq} = \sum_{l \in I_N} X_l I(l \not\in A_i \cup A_j \cup A_{i'} \cup A_{j'} \cup A_k \cap A_q).$$
For any $\mathcal{T} \subset \mathcal{I}_N$, define

$$W_\mathcal{T} = \sum_{j \in \mathcal{T}} X_j,$$

Note that $|X_j| \leq \sigma_N^{-1}$ for each $j \in \mathcal{I}_N$, and by the Jensen inequality, it follows that

$$\mathbb{E} e^{tW} \geq e^{-|\mathcal{T}| - t\sigma_N^{-1}} \mathbb{E} e^{t(W_\mathcal{T})}.$$

Then, for $0 \leq t \leq (1 - p)N^2\psi^{-1/2}$, we have

$$\max \left\{ e^{tW_{ij,j'}}, e^{tW_{ij',j'}}, e^{tW_{ij'j'}} \right\} \leq C e^{tW}.$$

It is well known that

$$|e^{x} - 1 - x| \leq \frac{1}{2} x^2 (1 + e^{x}).$$

Expanding the squared term and by (5.10), we have

$$\mathbb{E} \left( \sum_{i \in \mathcal{I}_N} \sum_{j \in A_{ij}} (\nu_{ij} - \bar{\nu}_{ij}) \right)^2 e^{tW}$$

$$= \sum_{i \in \mathcal{I}_N} \sum_{j \in A_{ij}} \sum_{i' \in \mathcal{I}_N} \sum_{j' \in A_{ij'}} \mathbb{E} (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW}$$

$$\leq \sum_{i \in \mathcal{I}_N} \sum_{j \in A_{ij}} \sum_{i' \in \mathcal{I}_N} \sum_{j' \in A_{ij'}} |\mathbb{E} (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ij,j'}}|$$

$$+ \sum_{i \in \mathcal{I}_N} \sum_{j \in A_{ij}} \sum_{i' \in \mathcal{I}_N} \sum_{j' \in A_{ij'}} t |\mathbb{E} W_{ij,j'} (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ij,j'}}|$$

$$+ \frac{1}{2} \sum_{i \in \mathcal{I}_N} \sum_{j \in A_{ij}} \sum_{i' \in \mathcal{I}_N} \sum_{j' \in A_{ij'}} t^2 \mathbb{E} W_{ij,j'}^2 |\nu_{ij} - \bar{\nu}_{ij}| |\nu_{ij'} - \bar{\nu}_{ij'}| e^{tW_{ij,j'}}$$

$$+ \frac{1}{2} \sum_{i \in \mathcal{I}_N} \sum_{j \in A_{ij}} \sum_{i' \in \mathcal{I}_N} \sum_{j' \in A_{ij'}} t^2 \mathbb{E} W_{ij,j'}^2 |\nu_{ij} - \bar{\nu}_{ij}| |\nu_{ij'} - \bar{\nu}_{ij'}| e^{tW}$$

$$:= Q_1 + Q_2 + Q_3 + Q_4.$$

For $Q_1$, observe that $(\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'})$ and $W_{ij,j'}^c$ are independent, then

$$\mathbb{E} (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ij,j'}} = \mathbb{E} (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) \mathbb{E} e^{tW_{ij,j'}}.$$

If $i', j' \in A_{ij}$, then $\nu_{ij}$ and $\nu_{ij'}$ are independent, and thus,

$$\mathbb{E} (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) = 0.$$

If $|i \cap j| = m_1$, $|i' \cap j'| = m_2$ and $|(i \cup j) \cap (i' \cup j')| = m_3$, where $1 \leq m_1, m_2 \leq e$, and $1 \leq m_3 \leq 2e - 1$, then, by (5.6) and Barbour, Karoski and Ruciski [3, Eq. (3.8)], it follows that

$$|\mathbb{E} (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'})| \leq \begin{cases} C\sigma_N^{-4} p^{4e-m_1-m_2-m_3}, & 0 < p \leq 1/2, \\ C\sigma_N^{-4}(1-p), & 1/2 < p < 1. \end{cases}$$
For $0 < p < 1/2$, noting that $|(i \cup j) \cap (i' \cup j')| \geq \max \{|i' \cap (i \cap j)|, |j' \cap (i \cap j)|\}$, we have

$$
\sum_{i \in \mathcal{I}_N} \sum_{j \in A_i} \sum_{i' \in \mathcal{I}_N} \sum_{j' \in A_{i'}} \left| \mathbb{E}(\nu_{ij} - \bar{\nu}_{ij})(\nu_{i'j'} - \bar{\nu}_{i'j'}) \right|
\leq C \sigma_N^{-4} \sum_{i \in \mathcal{I}_N} \sum_{i' \in \mathcal{I}_N} \sum_{j \in A_i} \sum_{j' \in A_{i'}} p^{4e - |i' \cap j| - |i' \cap (i \cap j)|}
+ C \sigma_N^{-4} \sum_{i \in \mathcal{I}_N} \sum_{i' \in \mathcal{I}_N} \sum_{j \in A_i} \sum_{j' \in A_{i'}} p^{4e - |i' \cap j'| - |j' \cap (i \cap j)|}
= C \sigma_N^{-4} \sum_{i \in \mathcal{I}_N} \sum_{i' \in \mathcal{I}_N} \sum_{j \in A_i} \sum_{j' \in A_{i'}} \sum_{e(K) \geq 1} p^{4e - |i' \cap j| - e(K) - |i' \cap (i \cap j)|}
\times \sum_{i'' \in \mathcal{I}_N} \sum_{e(K) = K} p^{4e - |i'' \cap j| - e(K) - |j'' \cap (i \cap j)|}
\leq C \sigma_N^{-4} \psi^{-1} n^p p^{e} \sum_{i \in \mathcal{I}_N} \sum_{i' \in \mathcal{I}_N} \sum_{j \in A_i} \sum_{j' \in A_{i'}} p^{3e - |i' \cap j| - |i' \cap (i \cap j)|}
+ C \sigma_N^{-4} \psi^{-1} n^p p^{e} \sum_{i \in \mathcal{I}_N} \sum_{i' \in \mathcal{I}_N} \sum_{j \in A_i} \sum_{j' \in A_{i'}} p^{3e - |i' \cap j'| - |j' \cap (i \cap j)|}
\leq C \sigma_N^{-2} (\psi^{-1} n^p p^e)^2 \leq C \psi^{-1},
\tag{5.11}
$$

where we used (5.1) and Barbour, Karoski and Ruciski [3, Eq. (3.10)] in the last line. For $1/2 < p < 1$, by (5.1) again, we have

$$
\sum_{i \in \mathcal{I}_N} \sum_{j \in A_i} \sum_{i' \in \mathcal{I}_N} \sum_{j' \in A_{i'}} \left| \mathbb{E}(\nu_{ij} - \bar{\nu}_{ij})(\nu_{i'j'} - \bar{\nu}_{i'j'}) \right|
\leq C \sigma_N^{-4} n^{4e - 6} (1 - p) \leq CN^{-2}(1 - p)^{-1}.
$$

Then, combining (5.1) and (5.9), for $0 \leq t \leq (1 - p)N^2p^e\psi^{-1/2}$, we have

$$
|Q_t| \leq \begin{cases} 
C \psi^{-1} \mathbb{E}e^{tW}, & 0 < p \leq 1/2, \\
CN^{-2}(1 - p)^{-1} \mathbb{E}e^{tW}, & 1/2 < p < 1.
\end{cases}
\tag{5.12}
$$
For $Q_2$, we have

$$Q_2 \leq \sum_{i \in I_N} \sum_{j \in A_i} \sum_{i' \in I_N} \sum_{j' \in A_{i'}} \sum_{k \in A_i \cup A_{j'}} t |\mathbb{E} X_k (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ijj'}^c}| \tag{5.13}$$

By symmetry, we only consider $Q_{21}$. By the Taylor expansion,

$$|\mathbb{E} X_k (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ijj'}^c}| \leq |\mathbb{E} X_k (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ijj'}^c}| + t |\mathbb{E} X_k (W_{ijj'}^c - W_{ijj'})(\nu_{ij} - \bar{\nu}_{ij})(\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ijj'}^c}| + t |\mathbb{E} X_k (W_{ijj'}^c - W_{ijj'})(\nu_{ij} - \bar{\nu}_{ij})(\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ijj'}^c}|. \tag{5.14}$$

Note that $X_k (\nu_{ij} - \bar{\nu}_{ij})(\nu_{ij'} - \bar{\nu}_{ij'})$ is independent of $W_{ijj'}^c$. Then,

$$\mathbb{E} X_k (\nu_{ij} - \bar{\nu}_{ij})(\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ijj'}^c} = \mathbb{E} X_k (\nu_{ij} - \bar{\nu}_{ij})(\nu_{ij'} - \bar{\nu}_{ij'}) \mathbb{E} e^{tW_{ijj'}^c}.$$

If $|\{i' \cup j'\} \cap (A_i \cup A_j \cup A_k)| = 0$, then $X_k (\nu_{ij} - \bar{\nu}_{ij})$ is independent of $(\nu_{ij'} - \bar{\nu}_{ij'})$, and thus

$$\mathbb{E} X_k (\nu_{ij} - \bar{\nu}_{ij})(\nu_{ij'} - \bar{\nu}_{ij'}) = 0.$$

Denote $|i \cap j| = m_1$, $|i' \cap j'| = m_2$, $k \cap (i \cup j) = m_4$, and $|\{i \cup j\} \cap (i' \cup j')| = m_3$, where $1 \leq m_1, m_2, m_4 \leq c$, and $1 \leq m_3 \leq 2c - 1$, then, by (5.6),

$$|\mathbb{E} X_k (\nu_{ij} - \bar{\nu}_{ij})(\nu_{ij'} - \bar{\nu}_{ij'})| \leq \begin{cases} C \sigma_N^{-5} p^{5e - m_1 - m_2 - m_3 - m_4}, & 0 < p \leq 1/2, \\ C \sigma_N^{-5} (1 - p), & 1/2 < p < 1. \end{cases}$$
For $0 < p \leq 1/2$,

\[
\sum_{i \in I_N} \sum_{j \in A_i} \sum_{i' \in I_N} \sum_{j' \in A_{i'}} \sum_{k \in A_{i \cup A_j}} \left| \mathbb{E} X_k (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) \right| \\
\leq C \sigma_N^{-5} \sum_{i \in I_N} \sum_{j \in A_i} \sum_{i' \in I_N} \sum_{j' \in A_{i'}} \sum_{k \in A_{(i \cup j) \cap (i' \cup j')}} \frac{1}{p^{5e-|i| - |i'| - |(i \cup j) \cap (i' \cup j')| - |k|}} \\
\times \sum_{k \in A_{i \cup A_j}} p^{5e-|i| - |i'| - |(i \cup j) \cap (i' \cup j')| - m_4} \\
\leq C \sigma_N^{-5} \psi^{-1} n^v p^e \sum_{i \in I_N} \sum_{j \in A_i} \sum_{i' \in I_N} \sum_{j' \in A_{i'}} \sum_{k \in A_{(i \cup j) \cap (i' \cup j')}} \frac{1}{p^{4e-|i| - |i'| - |(i \cup j) \cap (i' \cup j')|}} \\
\leq C \sigma_N^{-3} (\psi^{-1} n^v p^e)^3 \leq C \psi^{-3/2},
\]

where we used (5.11) in the last line. For $1/2 < p < 1$,

\[
\sum_{i \in I_N} \sum_{j \in A_i} \sum_{i' \in I_N} \sum_{j' \in A_{i'}} \sum_{k \in A_{i \cup A_j}} \left| \mathbb{E} X_k (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) \right| \\
\leq C \sigma_N^{-5} N^{5e-8} (1 - p) \leq C N^{-3} (1 - p)^{-3/2}.
\]

Note that $|X_k| \leq \sigma_N^{-1}$ and $|A_k| \leq C N^{v-2}$, $e^{W_{ij'} j' k} \leq C e^{W_{ij'} j' k}$. For $|i \cap j| = m_1$, $|i' \cap j'| = m_2$, $|i \cup j| \cap (i' \cup j') = m_3$, $|k \cap (i \cup j \cup i' \cup j')| = m_4$, and $|q \cap (i \cup j \cup i' \cup j' \cup k)| = m_5$, where $1 \leq m_1, m_2 \leq e$, $1 \leq m_4, m_5 \leq e$, and $0 \leq m_3 \leq 2e - 1$, then for $0 \leq t \leq (1 - p) N^2 p^e \psi^{-1/2}$, by (5.6), we have

\[
\mathbb{E} X_k (W_{ij'} j' k - W_{ij'} j' k) (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) e^{W_{ij'} j' k} \\
\leq C \sum_{q \in A_k} \mathbb{E} X_q X_k (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) e^{W_{ij'} j' k} \\
\leq C \sum_{q \in A_k} \mathbb{E} X_q X_k (\nu_{ij} - \bar{\nu}_{ij}) (\nu_{ij'} - \bar{\nu}_{ij'}) \mathbb{E} e^{W_{ij'} j' k} \\
\leq \begin{cases} 
C \sum_{q \in A_k} \sigma_N^{-6} P^{p_0 - m_1 - m_2 - m_3 - m_4 - m_5} e^{W}, & 0 < p \leq 1/2, \\
C N^{v-2} \sigma_N^{-6} (1 - p) e^{W}, & 1/2 < p < 1,
\end{cases}
\]
where we used (5.9) in the last line. Similar to (5.11) and (5.15), for \(0 < p \leq 1/2\) and for \(0 \leq t \leq (1 - p)N^2p^\varepsilon\psi^{-1/2}\), it follows that

\[
\sum_{i \in I_N} \sum_{j \in A_i} \sum_{i' \in I_N} \sum_{j' \in A_i} \sum_{k \in A_i \cup A_j} \sum_{\mathcal{A}_k} \|X_k(W_{ij}W_{ij'} - W_{ij'}) \times (\nu_{ij} - \bar{\nu}_{ij})(\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ij}W_{ij'}}| \\
\leq C \mathbb{E} e^{tW} \sum_{i \in I_N} \sum_{j \in A_i} \sum_{i' \in I_N} \sum_{j' \in A_i} \sum_{k \in A_i \cup A_j} \sum_{\mathcal{A}_k} \sigma_N^6 p^6e^{-m_1 - m_2 - m_3 - m_4 - m_5} \\
\leq C \mathbb{E} e^{tW} \sigma_N^{-6} (\psi^{-1}N^p\epsilon)^2 \left(\sum_{i \in I_N} \sum_{j \in A_i} p^{2e - |\nu_{ij}|}\right)^2 \\
\leq C \mathbb{E} e^{tW} \sigma_N^{-2} (\psi^{-1}N^p\epsilon)^2 \leq C \psi^{-1} \mathbb{E} e^{tW},
\]

where \(m_1 = |i \cap j|, \ m_2 = |i' \cap j'|, \ m_3 = |(i \cup j) \cap (i' \cup j')|, \ m_4 = |k \cap (i \cup j \cup i' \cup j')|\) and \(m_5 = |q \cap (i \cup j \cup i' \cup j' \cup k)|\) and we used Barbour, Karoski and Ruciski [3, Eq. (3.7)] in the last line.

For \(1/2 < p < 1\) and for \(0 \leq t \leq (1 - p)N^2p^\varepsilon\psi^{-1/2}\), we have

\[
\sum_{i \in I_N} \sum_{j \in A_i} \sum_{i' \in I_N} \sum_{j' \in A_i} \sum_{k \in A_i \cup A_j} \sum_{\mathcal{A}_k} \|X_k(W_{ij}W_{ij'} - W_{ij'}) \times (\nu_{ij} - \bar{\nu}_{ij})(\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ij}W_{ij'}}| \\
\leq CN^{-2}(1 - p)^{-2} \mathbb{E} e^{tW}.
\]

Similar to (5.17) and (5.18), for \(0 \leq t \leq (1 - p)N^2p^\varepsilon\psi^{-1/2}\), it follows that

\[
\sum_{i \in I_N} \sum_{j \in A_i} \sum_{i' \in I_N} \sum_{j' \in A_i} \sum_{k \in A_i \cup A_j} \sum_{\mathcal{A}_k} \|X_k(W_{ij}W_{ij'} - W_{ij'}) \times (\nu_{ij} - \bar{\nu}_{ij})(\nu_{ij'} - \bar{\nu}_{ij'}) e^{tW_{ij}W_{ij'}}| \\
\leq \begin{cases} C\psi^{-1} \mathbb{E} e^{tW}, & 0 < p \leq 1/2, \\
CN^{-2}(1 - p)^{-2} \mathbb{E} e^{tW}, & 1/2 < p < 1. \end{cases}
\]

Substituting (5.9) and (5.15)–(5.19) to (5.13) and (5.14), for \(0 \leq t \leq (1 - p)N^2p^\varepsilon\psi^{-1/2}\), we have

\[
Q_2 \leq \begin{cases} C(t\psi^{-3/2} + t^2\psi^{-1}) \mathbb{E} e^{tW}, & 0 \leq p \leq 1/2, \\
C(tN^{-3}(1 - p)^{-3/2} + t^2N^{-2}(1 - p)^{-2}) \mathbb{E} e^{tW}, & 1/2 < p < 1. \end{cases}
\]

For any \(H \subset G\) such that \(e(H) > 0\), we have \(v(H) \geq 2\) and \(e(H) \leq \epsilon\), and it follows that

\[N^{\psi(H)}p^\epsilon(H) \geq N^2p^\epsilon.\]

Thus, for \(0 < p \leq 1/2\) and \(0 \leq t \leq (1 - p)N^2p^\varepsilon\psi^{-1/2}\),

\[0 \leq t\psi^{-1/2} \leq N^2p^\epsilon \psi^{-1} \leq 1.\]
Hence, (5.20) becomes

\[ Q_2 \leq \begin{cases} C\psi^{-1}(1 + t^2)Ee^{tW}, & 0 < p \leq 1/2, \\ C(tN^{-3}(1 - p)^{-3/2} + t^2N^{-2}(1 - p)^{-2})Ee^{tW}, & 1/2 < p < 1. \end{cases} \]  

(5.21)

Similar to (5.21), we have for 0 ≤ t ≤ (1 - p)N^2p\psi^{-1/2},

\[ Q_3 + Q_4 \leq \begin{cases} Ct^2\psi^{-1}Ee^{tW}, & 0 < p \leq 1/2, \\ Ct^2N^{-2}(1 - p)^{-2}Ee^{tW}, & 1/2 < p < 1. \end{cases} \]

This proves (5.7) together with (5.12) and (5.21). \( \square \)

5.2. Proofs of Theorems 3.2 and 3.3. We use Theorems 2.1 and 2.2 to prove Theorems 3.2 and 3.3. The proof is organized as follows. We first construct the exchangeable pair, and check the conditions conditions (A1)–(A4). The properties for the exchangeable pair are given in Lemmas 5.2 and 5.3, whose proofs are put in subsection 5.4.

Let \( X = \{X_i, i \in J\} \). We now construct \( \{(X_k^{(i)})_{k \in A_i}\} \) as follows. For each \( i \in J \), let \( X_i^{(i)} \) be an independent copy of \( X_i \), which is also independent of \( \{X_j, j \neq i\} \) and \( \{X_j^{(j)}, j \neq i\} \). Given \( X_i^{(i)} = x \), define the vector \( \{X_j^{(i)}, j \in A_i \setminus \{i\}\} \) to have the same distribution of \( \{X_j, j \in A_i \setminus \{i\}\} \) conditional on \( X_i = x \) and \( \{X_j, j \in A_i^{c}\} \). For each \( i \in J \), define \( X^{(i)} = \{(X_i)_{i \in A_i}, (X_i^{(i)})_{i \not\in A_i}\} \). Let \( I \) be a random index independent of all other random variables and uniformly distributed among \( J \). Then, \( (X, X^{(i)}) \) is an exchangeable pair.

Define \( D = X_I - X^{(i)}_I \). We have

\[ E(D | X) = \frac{1}{n} \sum_{i \in J} E \left( X_i - X_i^{(i)} \mid X \right) \]

(5.22)

\[ = \frac{1}{n} W, \]

since \( X_i^{(i)} \) is independent of \( X \). Thus, the condition (D1) is satisfied with \( \lambda = 1/n \) and \( R = 0 \).

Let \( W' = \sum_{j \not\in A_I} X_j + \sum_{j \in A_I} X^{(i)}_j \). Thus, \( (W, W') \) is an exchangeable pair, and \( \Delta := W - W' = \sum_{j \in A_I} (X_j - X^{(i)}_j) \). Let \( \mathcal{F} = \sigma(X, \{X^{(i)}, i \in J\}) \).

For any \( i \in J \), define

\[ \eta_{A_i} = \frac{1}{2} \sum_{j \in A_i} \left\{ (X_i - X_i^{(i)})(X_j - X_j^{(i)}) - E(X_i - X_i^{(i)})(X_j - X_j^{(i)}) \right\}, \]

\[ \zeta_{A_i} = \frac{1}{2} \sum_{j \in A_i} \left\{ |X_i - X_i^{(i)}|(X_j - X_j^{(i)}) \right\}. \]
By the assumption that \( E(W^2) = 1 \) and by (5.22) and the exchangeability,
\[
E(D\Delta) = 2E(DW) = 2\lambda E W^2 = 2\lambda.
\]

For any \( i \in J \) and \( j \in A_i \), we have \( \{X_i, X_i^{(i)}, X_j, X_j^{(i)}\} \in \mathcal{F} \), and it follows that
\[
\frac{1}{2\lambda} E(D\Delta | \mathcal{F}) - 1 = \frac{1}{2} \sum_{i=1}^{n} \left( (X_i - X_i^{(i)})(X_j - X_j^{(i)}) \right) \bigg| \mathcal{F} - 1 = \frac{1}{2} \sum_{i=1}^{n} \left\{ (X_i - X_i^{(i)})(X_j - X_j^{(i)}) - E(X_i - X_i^{(i)})(X_j - X_j^{(i)}) \right\} = \sum_{i \in J} \eta_{A_i}.
\]
Moreover,
\[
\frac{1}{2\lambda} E(D\Delta | \mathcal{F}) = \sum_{i \in J} \zeta_{A_i}.
\]

The following two lemmas provide some properties for \( \sum_{i \in J} \eta_{A_i} \) and \( \sum_{i \in J} \zeta_{A_i} \).

**Lemma 5.2.** Assume that conditions (LD1) and (LD2) are satisfied, then
\[
E \left( \sum_{i \in J} \eta_{A_i} \right)^2 \leq 10 \sum_{i, j \in A, i \neq j} \sum_{i' \in A_i, j' \in A_j} \left\{ E\left|X_i\right|^4 + E\left|X_j\right|^4 + E\left|X_i'\right|^4 + E\left|X_j'\right|^4 \right\},
\]
\[
E \left( \sum_{i \in J} \zeta_{A_i} \right)^2 \leq 4 \sum_{i \in J, j \in A_i} \sum_{i' \in A_i, j' \in A_j} \left\{ E\left|X_i\right|^4 + E\left|X_j\right|^4 + E\left|X_i'\right|^4 + E\left|X_j'\right|^4 \right\}.
\]

**Lemma 5.3.** Assume that conditions (LD1)–(LD4) are satisfied. For \( i \in J, j \in A_i \) and \( k \in A_j \), let \( \kappa_{ij} = |A_{ij} \setminus (A_i \cup A_j)| \) and \( \kappa_{ijk} = |A_{ijk} \setminus (A_i \cup A_j \cup A_k)| \). For \( i \in J \) and \( k \geq 1 \), \( \gamma_{k,i}(t) = E|X_i|^k e^{t(U_i + |X_i|)} \). For \( 0 \leq t \leq \alpha \), we have
\[
\max \left\{ E \left( \sum_{i \in J} \eta_{A_i} \right)^2 e^{tW}, E \left( \sum_{i \in J} \zeta_{A_i} \right)^2 e^{tW} \right\} \leq 132 E e^{tW} \left\{ \beta^5 \Gamma_4(t) + t^2 \left( \beta^3 \Gamma_3(t) + \beta^6 \Gamma_6(t) \right) \right\},
\]
where \( \Gamma_3, \Gamma_4 \) and \( \Gamma_6 \) are as defined in (3.6).

By the Cauchy inequality and Lemma 5.2, Theorem 2.2 implies Theorem 3.2. Again, by the Cauchy inequality and Lemma 5.3, it follows that
\[
E \left| \sum_{i \in J} \eta_{A_i} \right| e^{tW} \leq 12 \left\{ \beta^{5/2} \Gamma_4^{1/2}(t) + \beta^6 \Gamma_3(t)t + \beta^3 \Gamma_6^{1/2}(t)t \right\} \times E e^{tW},
\]
\[
E \left| \sum_{i \in J} \zeta_{A_i} \right| e^{tW} \leq 12 \left\{ \beta^{5/2} \Gamma_4^{1/2}(t) + \beta^6 \Gamma_3(t)t + \beta^3 \Gamma_6^{1/2}(t)t \right\} \times E e^{tW}.
\]
By condition (LD4) (see also Lemma 5.4), we have \( E e^{tW} \leq \beta^2 |\mathcal{J}| < \infty \) for \( 0 \leq t \leq \alpha \). This completes the proof of Theorem 3.3 by Theorem 2.2.

5.3. Some preliminary lemmas. In this subsection, we present some preliminary lemmas, which are important in the proofs of Lemmas 5.2 and 5.3. For the sake of brevity, define

\[
\chi_{ij} = (X_i - X_i^{(i)}) (X_j - X_j^{(i)}) - E(X_i - X_i^{(i)}) (X_j - X_j^{(i)}),
\]

and

\[
\rho_{ij} = |X_i - X_i^{(i)}|(X_j - X_j^{(i)}).
\]

For each \( i \in \mathcal{J} \) and \( T \subset \mathcal{J} \), define \( W_T = \sum_{j \in T} X_j, T_i = T \cap A_i, T_i^c = T \cap A_i^c \).

Lemma 5.4. Under the conditions as in Lemma 5.3. For any \( T \subset \mathcal{J} \) and for \( 0 \leq t \leq \alpha \), we have

\[
\frac{1}{\beta^2} E(e^{tW_T}) \leq E(e^{tW_{T \setminus \{i\}}}) \leq \beta^2 E(e^{tW_T}).
\]

Also,

\[
\frac{1}{\beta} E(e^{tW_T}) \leq E(e^{tW_{T_i^c}}) \leq \beta E(e^{tW_T}).
\]

Proof of Lemma 5.4. Let \( \mathcal{F}_i^c = \sigma \{ X_j, j \in \mathcal{J} \setminus A_i \} \). By the total expectation formula,

\[
E(e^{tW_T}) = E \left\{ e^{tW_{T_i^c}} E(e^{tW_{T_i^c}} \mid \mathcal{F}_i^c) \right\}.
\]

By condition (LD4), we have \( |W_{T_i^c}| \leq U_i + |X_i| \). For \( 0 \leq t \leq \alpha \),

\[
\beta \geq E(e^{t(U_i + |X_i|)}) \geq E(e^{tW_{T_i^c}} \mid \mathcal{F}_i^c) \geq E(e^{-t(U_i + |X_i|)}) \geq \left\{ E(e^{t(U_i + |X_i|)}) \right\}^{-1} \geq 1/\beta.
\]

Thus, the inequality (5.25) follows from (5.26) and (5.27). Similarly,

\[
\frac{1}{\beta} E(e^{tW_{T_i^c}}) \leq E(e^{tW_{T \setminus \{i\}}}) \leq \beta E(e^{tW_{T_i^c}}).
\]

This proves (5.24) together with (5.25). \( \square \)
Lemma 5.5. Under the conditions in Lemma 5.3. Let \( T \) be a subset of \( J \), and let \( W_T = \sum_{j \in T} X_j \). For any \( i \in J \) and \( j \in A_i \), we have

\[
E|\chi_{ij}|^q e^{tW_T} \leq 2^{3q-1} \beta^3 \left\{ E(|X_i|^{2q} e^{t(U_j+\chi_{ij})}) \right\},
\]

(5.28)

\[
\begin{align*}
&\quad + E\left\{ |X_j|^{2q} e^{t(U_j+\chi_{ij})} \right\} \cdot E e^{tW_T}, \quad \text{for } q \geq 1, \\
&\quad \quad + E\left\{ |X_j|^{2q} e^{t(U_j+\chi_{ij})} \right\} \cdot E e^{tW_T}, \quad \text{for } q \geq 1,
\end{align*}
\]

(5.29)

and

\[
E|X_i|^q e^{tW_T} \leq \beta E\left\{ |X_i|^q e^{t(U_j+|X_j|)} \right\} \cdot E(e^{tW_T}), \quad \text{for } q \geq 0.
\]

(5.30)

Proof of Lemma 5.5. We first prove (5.30). Recall that for each \( i \in J \), we have \( T_i = T \cap A_i \), \( T_i^c = T \cap A_i^c \), \( W_{T_i} = \sum_{j \in T_i} X_j \) and \( W_{T_i^c} = \sum_{j \in T_i^c} X_j \). Thus, \( T = T_i \cup T_i^c \) and \( W_T = W_{T_i} + W_{T_i^c} \). Let \( F_i^c = \sigma(X_j, j \in A_i^c) \). By condition (LD4), we have \( |W_{T_i}| \leq U_i + |X_i| \), where \( \{U_i, X_i\} \) is independent of \( F_i^c \). For \( 0 \leq t \leq \alpha \), it follows that

\[
E(|X_i|^p e^{tW_{T_i^c}}) = E\left\{ E\left( |X_i|^p e^{tW_{T_i^c}} \mid F_i^c \right) \right\}
\]

(5.31)

\[
\begin{align*}
&\quad = E\left\{ e^{tW_{T_i}} E\left( |X_i|^p e^{tW_{T_i}} \mid F_i^c \right) \right\} \\
&\quad \leq E\left( |X_i|^p e^{t(U_j+|X_j|)} \right) \cdot E(e^{tW_{T_i^c}}),
\end{align*}
\]

This proves (5.30) together with (5.25).

We now move to prove (5.28). Observe that

\[
E|\chi_{ij}|^q e^{tW_T} \leq 2^{q-1} \left\{ E\left( (X_i - X_i^{(i)})(X_j - X_j^{(i)}) \right)^q e^{tW_T} \right\}
\]

(5.32)

\[
\begin{align*}
&\quad + E\left( (X_i - X_i^{(i)})(X_j - X_j^{(i)}) \right)^q E e^{tW_T} \right\}.
\end{align*}
\]

By the Cauchy inequality,

\[
E\left( (X_i - X_i^{(i)})(X_j - X_j^{(i)}) \right)^q e^{tW_T}
\]

\[
\leq \frac{1}{2} E|X_i - X_i^{(i)}|^{2q} e^{tW_T} + \frac{1}{2} E|X_j - X_j^{(i)}|^{2q} e^{tW_T}
\]

(5.33)

\[
\leq 2^{2q-2} \left( E|X_i|^{2q} e^{tW_T} + E|X_i^{(i)}|^{2q} e^{tW_T} \right)
\]

\[\quad + E|X_j|^{2q} e^{tW_T} + E|X_j^{(i)}|^{2q} e^{tW_T}.\]

Since \( X_i^{(i)} \) is independent of \( W_T \), it follows that

\[
E|X_i^{(i)}|^{2q} e^{tW_T} = E|X_i|^{2q} E e^{tW_T}.
\]

(5.34)
By the construction of $X^{(i)}_j$, conditional on $\mathcal{F}^*_i$, $X^{(i)}_j$ is conditionally independent of $\{X_k, k \in \mathcal{A}_i\}$ and has the same distribution as $X_j$. Thus,

\[
E|X^{(i)}_j|^2 e^{tW_T} = E \left\{ e^{tW_T} E \left( |X_j|^{2q} \mid \mathcal{F}^*_i \right) E \left( e^{tW_T} \mid \mathcal{F}^*_i \right) \right\}
\]
(5.35)

\[
\leq E e^{t(U_i+|X_i|)} E(|X_j|^{2q} e^{tW_T})
\]
\[
\leq \beta^3 E(|X_j|^{2q} e^{t(U_i+|X_i|)}) E e^{tW_T},
\]

where we used condition (LD4) and (5.25) and (5.30) in the last line. By (5.33)–(5.35), and recalling that $\beta \geq 1$, we have

\[
E|X_i - X^{(i)}_i(X_j - X^{(i)}_j)|^q e^{tW_T}
\]
(5.36)

\[
\leq 2^{2q-1} \beta^3 \left\{ E(|X_i|^{2q} e^{t(U_i+|X_i|)}) + E(|X_j|^{2q} e^{t(U_i+|X_j|)}) \right\} E e^{tW_T}.
\]

Taking $t = 0$,

\[
E\left| (X_i - X^{(i)}_i)(X_j - X^{(i)}_j) \right|^q
\]
(5.37)

\[
\leq 2^{2q-1} \beta^3 \{ E |X_i|^q + E |X_j|^q \}.
\]

By (5.32), (5.36) and (5.37), we have

\[
E |\chi_{ij}|^q e^{tW_T} \leq 2^{2q-1} \beta^3 \left\{ E(|X_i|^{2q} e^{t(U_i+|X_i|)}) + E(|X_j|^{2q} e^{t(U_i+|X_j|)}) \right\} E e^{tW_T}.
\]

This proves (5.28). The inequality (5.29) follows from (5.36), and this completes the proof.

**Lemma 5.6.** We have

\[
E \left| \chi_{ij} \chi^{(i')}_{ij'} e^{tW_{ij'}} \right| \leq 20 \beta^3 E e^{tW_{ij'}} \left\{ E(|X_i|^4 e^{t(U_i+|X_i|)})
\right.\]
(5.38)

\[
+ \left. E(|X_j|^4 e^{t(U_j+|X_j|)}) \right\}.
\]

and

\[
E |\rho_{ij} \rho_{ij'} e^{tW_{ij'}}| \leq 8 \beta^3 E e^{tW_{ij'}} \left\{ E(|X_i|^4 e^{t(U_i+|X_i|)})
\right.\]
(5.39)

\[
+ \left. E(|X_j|^4 e^{t(U_j+|X_j|)}) \right\}.
\]

**Proof of Lemma 5.6.** Without loss of generality, we only prove (5.38), because (5.40) can be shown similarly. By the Cauchy inequality, we have

\[
E \left| \chi_{ij} \chi^{(i')}_{ij'} e^{tW_{ij'}} \right| \leq \frac{1}{2} E \left( \chi^2_{ij} e^{tW_{ij'}} \right) + \frac{1}{2} E \left( \chi^2_{ij'} e^{tW_{ij'}} \right)
\]

\[
= \frac{1}{2} E \left( \chi^2_{ij} \right) E \left( e^{tW_{ij'}} \right) + \frac{1}{2} E \left( \chi^2_{ij'} \right) e^{tW_{ij'}}.
\]

For the first term, by (5.37) with $q = 2$, it follows that

\[
E \left( \chi^2_{ij} \right) \leq 8 \beta^3 (E |X_i|^4 + E |X_j|^4).
\]
For the second term, by (5.28), we have
\[ \mathbb{E}\left( \chi_{i,j}^{2} e^{iW_{ij}} \right) \leq 32 \beta^{3} \left( \mathbb{E}\left( |X_{i}|^{4} e^{i(U_{i} + |X_{i}|)} \right) + \mathbb{E}\left( |X_{j}|^{4} e^{i(U_{j} + |X_{j}|)} \right) \right) \mathbb{E} e^{iW_{ij}}. \]

This completes the proof of (5.38). \( \Box \)

5.4. **Proof of Lemmas 5.2 and 5.3.** We first prove Lemma 5.3, and the proof of Lemma 5.2 is put at the end of this subsection. Without loss of generality, we only prove the bound for \( \mathbb{E}\left( \sum_{i \in J} \eta_{A_i} \right)^2 e^{iW} \). Again, let
\[ \chi_{ij} = (X_i - X_i^{(i)})(X_j - X_j^{(i)}) - \mathbb{E}(X_i - X_i^{(i)})(X_j - X_j^{(i)}). \]

For \( i \in J \) and \( j \in J_i \), define \( W_{ij} = \sum_{i \in J_i} X_i, \quad W_{ij}^{c} = W - W_{ij}. \)

It follows from (5.10) that
\[
\mathbb{E}\left( \sum_{i \in J} \eta_{A_i} \right)^2 e^{iW} = \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \mathbb{E}\left( \chi_{ij} \chi_{i'j'} e^{iW} \right) \\
= \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \mathbb{E}\left( \chi_{ij} \chi_{i'j'} e^{iW_{ij} + iW_{ij}^{c}} \right) \\
\leq \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \mathbb{E}\left( \chi_{ij} \chi_{i'j'} e^{iW_{ij}} \right) \\
\quad + \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij}} t \mathbb{E}\left( X_k \chi_{ij} \chi_{i'j'} e^{iW_{ij}^{c}} \right) \\
\quad + \frac{1}{8} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij}} t^2 \mathbb{E}\left| W_{ij}^{c} \chi_{ij} \chi_{i'j'} e^{iW_{ij}^{c}} \right| \\
\quad + \frac{1}{8} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij}} t^2 \mathbb{E}\left| W_{ij}^{c} \chi_{ij} \chi_{i'j'} e^{iW} \right| \\
=: H_1 + H_2 + H_3 + H_4,
\]

where
\[
H_1 = \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \mathbb{E}\left( \chi_{ij} \chi_{i'j'} e^{iW_{ij}} \right), \\
H_2 = \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij}} t \mathbb{E}\left( X_k \chi_{ij} \chi_{i'j'} e^{iW_{ij}} \right), \\
H_3 = \frac{1}{8} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij}} t^2 \mathbb{E}\left| W_{ij}^{c} \chi_{ij} \chi_{i'j'} e^{iW_{ij}} \right|, \\
H_4 = \frac{1}{8} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij}} t^2 \mathbb{E}\left| W_{ij}^{c} \chi_{ij} \chi_{i'j'} e^{iW} \right|.
\]

In what follows, we will give the bounds of \( H_1, H_2, H_3 \) and \( H_4 \), separately.
For \( i' \in \mathcal{A}_{ij} \) and \( j' \in \mathcal{A}_{ij}' \), by condition (LD2), the random variables \( \chi_{ij} \) and \( \chi_{ij'} e^{iW_{ij}} \) are independent, and then \( E(\chi_{ij} \chi_{ij'} e^{iW_{ij}}) = 0 \). Let

\[
\mathring{A}_j = \{ i : j \in \mathcal{A}_i \}.
\]

If \( i \in \mathring{A}_j \), then \( \{ j \} \cap \mathcal{A}_i = \emptyset \), which means \( X_i \) and \( X_j \) are independent and thus \( i \in \mathcal{A}_j^c \). This shows that \( \mathcal{A}_j \subset \mathring{A}_j \). Similarly, \( \mathring{A}_j \subset \mathcal{A}_j \). Thus, \( \mathcal{A}_j = \mathring{A}_j \).

For \( H_1 \), we have

\[
(5.42) \quad H_1 \leq H_{11} + H_{12},
\]

where

\[
H_{11} = \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \left| E \left( \chi_{ij} \chi_{ij'} e^{iW_{ij}} \right) \right| I(i' \in \mathcal{A}_{ij}, j' \in \mathcal{A}_{i'}),
\]

\[
H_{12} = \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \left| E \left( \chi_{ij} \chi_{ij'} e^{iW_{ij}} \right) \right| I(j' \in \mathcal{A}_{ij}, i' \in \mathring{A}_{i'}). \]

Also, note that \( \mathcal{A}_{ij} \subset \mathcal{A}_i^c \cap \mathcal{A}_j^c \). By (5.24) and (5.25),

\[
(5.43) \quad E(e^{iW_{A_i^c \cap A_j^c}}) \leq \beta^{2\kappa_{ij}} E(e^{iW_{A_i \cup A_j}}) \leq \beta^{2\kappa_{ij} + 2} E(e^{iW}),
\]

where \( \kappa_{ij} = |\mathcal{A}_{ij} \setminus (\mathcal{A}_i \cup \mathcal{A}_j)| \). By (5.38) and (5.43) and the Cauchy inequality,

\[
E(\chi_{ij} \chi_{ij'} e^{iW_{ij}}) \leq 20\beta^3 E e^{iW_{ij}} \left\{ E \left( |X_i|^4 e^{i(U_i + |X_i|)} \right) + E \left( |X_j|^4 e^{i(U_j + |X_j|)} \right) \right. \]

\[
+ \left. E \left( |X_{i'}|^4 e^{i(U_{i'} + |X_{i'}|)} \right) + E \left( |X_{j'}|^4 e^{i(U_{j'} + |X_{j'}|)} \right) \right\}
\]

\[
\leq 20\beta^{2\kappa_{ij} + 5} E e^{iW} \left\{ E \left( |X_i|^4 e^{i(U_i + |X_i|)} \right) + E \left( |X_j|^4 e^{i(U_j + |X_j|)} \right) \right. \]

\[
+ \left. E \left( |X_{i'}|^4 e^{i(U_{i'} + |X_{i'}|)} \right) + E \left( |X_{j'}|^4 e^{i(U_{j'} + |X_{j'}|)} \right) \right\}.
\]

By (5.42) and (5.44), and recalling that \( \gamma_{p,j}(t) = E |X_j|^p e^{i(U_j + |X_j|)} \),

\[
(5.45) \quad H_1 \leq 10\beta^5 \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in A_{i'} \setminus \mathcal{A}_i} \sum_{j' \in A_{i'}} \beta^{2\kappa_{ij}} \left\{ \gamma_{4,i} + \gamma_{4,j} \right\} E e^{iW}.
\]

\[
= 10\beta^5 \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in A_{ij}} \sum_{i' \in A_{i'} \setminus \mathcal{A}_i} \sum_{j' \in A_{i'}} \frac{1}{|\mathcal{A}_{ij}|} \beta^{2\kappa_{ij}} \times \left\{ \gamma_{4,i} + \gamma_{4,j} + \gamma_{4,i'} + \gamma_{4,j'} \right\} E e^{iW}.
\]

\[
\leq 10\beta^5 E e^{iW} \Gamma_{4}(t),
\]
where

\[ \Gamma_4(t) = \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in A_{ij}} \sum_{i' \in A_{ijk}} \sum_{j' \in A_{ijk}} |A_{ij}|^{-1} \beta^{2 \kappa_{ij}} (\gamma_{4,i}(t) + \gamma_{4,j}(t) + \gamma_{4,j'}(t)), \]

as defined in (3.6).

Now we move to give the bound of \( H_2 \). Let \( W_{ijk} = \sum_{t \in A_{ijk}} X_t \), \( \tilde{W}_{ijk} = \sum_{t \in A_{ijk} \setminus A_{ij}} X_t \) and \( W^c_{ijk} = W_{ij} - \tilde{W}_{ijk} \). It follows that

\[ W^c_{ijk} = W^c_{ij} - \tilde{W}_{ijk}. \]

Observe that

\[ E \left( X_k \chi_{ij} \chi_{i'j'} e^{tW^c_{ij}} \right) = E \left( X_k \chi_{ij} \chi_{i'j'} e^{tW^c_{ij}} \right) + E \left\{ X_k \chi_{ij} \chi_{i'j'} e^{tW^c_{ij}} \left( e^{t\tilde{W}_{ijk}} - 1 \right) \right\}, \]

and

\[ \left| e^{t\tilde{W}_{ijk}} - 1 \right| \leq \sum_{t \in A_{ijk} \setminus A_{ij}} |X_t| \left( 1 + e^{t\tilde{W}_{ijk}} \right). \]

Thus,

\[ H_2 \leq \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} k \in A_{ij} \sum_{i \in A_{ijk} \setminus A_{ij}} t \left| E \left( X_k \chi_{ij} \chi_{i'j'} e^{tW^c_{ij}} \right) \right| \]

\[ + \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} k \in A_{ij} \sum_{i \in A_{ijk} \setminus A_{ij}} t^2 E \left| X_k X_i \chi_{ij} \chi_{i'j'} e^{tW^c_{ij}} \right| \]

\[ + \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} k \in A_{ij} \sum_{i \in A_{ijk} \setminus A_{ij}} t^2 E \left| X_k X_i \chi_{ij} \chi_{i'j'} e^{tW^c_{ij}} \right| \]

\[ := H_{21} + H_{22} + H_{23}, \]

where

\[ H_{21} = \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} k \in A_{ij} \sum_{i \in A_{ijk} \setminus A_{ij}} t \left| E \left( X_k \chi_{ij} \chi_{i'j'} e^{tW^c_{ij}} \right) \right|, \]

\[ H_{22} = \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} k \in A_{ij} \sum_{i \in A_{ijk} \setminus A_{ij}} t^2 E \left| X_k X_i \chi_{ij} \chi_{i'j'} e^{tW^c_{ij}} \right|, \]

\[ H_{23} = \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} k \in A_{ij} \sum_{i \in A_{ijk} \setminus A_{ij}} t^2 E \left| X_k X_i \chi_{ij} \chi_{i'j'} e^{tW^c_{ij}} \right|. \]

We first consider \( H_{22} \) and \( H_{23} \). Recall that

\[ \kappa_{ijk} = |A_{ijk} \setminus (A_i \cup A_j \cup A_k)|. \]

By Lemma 5.4 and similar to (5.43), we have

\[ E e^{t \sum_{i \in A_{ijk} \setminus A_{ij}} X_i} \leq \beta^{2 \kappa_{ij} + 2} E e^{t \sum_{i \in A_{ijk} \setminus A_{ij}} X_i} \leq \beta^{2 \kappa_{ijk} + 2 \kappa_{ij} + 5} E e^{tW^c_{ij}}. \]
For any \(i, j, i', j', k\) and \(l\), by (5.28), (5.30) and (5.46), we have
\[
\mathbb{E}[X_k X_l \chi_{ij} \chi_{i'j'} e^{tW_{ijl}}] \\
\leq 88 \beta^3 \left\{ \sum_{m \in \{i, j, i', j', k, l\}} \gamma_{6,m}(t) \right\} \mathbb{E}[e^{tW_{ijl}}] \\
\leq 88 \beta^{2\kappa_{i,j,k}+6} \left\{ \sum_{m \in \{i, j, i', j', k, l\}} \gamma_{6,m}(t) \right\} \mathbb{E}[e^{tW}].
\] (5.47)
Therefore,
\[
|H_{22}| \leq 22 \beta^{2\kappa_{i,j,k}+6} \left\{ \sum_{m \in \{i, j, i', j', k, l\}} \gamma_{6,m}(t) \right\} \mathbb{E}[e^{tW}].
\] (5.48)
Similarly,
\[
|H_{23}| \leq 22 \beta^{2\kappa_{i,j,k}+5} \left\{ \sum_{m \in \{i, j, i', j', k, l\}} \gamma_{6,m}(t) \right\} \mathbb{E}[e^{tW}].
\] (5.49)
Now we move back to \(H_{21}\). Note that for \(i' \in A_{ijk}^c\) and \(j' \in A_{ijk}^c\), we have
\[
\mathbb{E}[\chi_{i'j'} e^{tW_{ijl}}] = \mathbb{E}[\chi_{i'j'} e^{t\sum_{l \in A_{ijk} \setminus A_{i'j'}} X_l}] \\
+ \mathbb{E}\left( \chi_{i'j'} \left( e^{t\sum_{l \in A_{ijk} \cap A_{i'j'}} X_l} - 1 \right) e^{t\sum_{l \in A_{ijk} \setminus A_{i'j'}} X_l} \right).
\] (5.50)
Note that \(\chi_{i'j'}\) and \(\{X_l, l \in A_{ijk}^c \setminus A_{i'j'}\}\) are independent, and \(\mathbb{E}[\chi_{i'j'}] = 0\), then
\[
\mathbb{E}[\chi_{i'j'} e^{t\sum_{l \in A_{ijk} \setminus A_{i'j'}} X_l}] = 0.
\] (5.51)
For the second term of (5.50),
\[
\left| \mathbb{E}\left( \chi_{i'j'} \left( e^{t\sum_{l \in A_{ijk} \cap A_{i'j'}} X_l} - 1 \right) e^{t\sum_{l \in A_{ijk} \setminus A_{i'j'}} X_l} \right) \right| \\
\leq t \sum_{m \in A_{ijk} \cap A_{i'j'}} \mathbb{E}|\chi_{i'j'} X_m| e^{t\sum_{l \in A_{ijk} \setminus A_{i'j'}} X_l} \\
+ t \sum_{m \in A_{ijk} \cap A_{i'j'}} \mathbb{E}|\chi_{i'j'} X_m| e^{t\sum_{l \in A_{ijk} \setminus A_{i'j'}} X_l}.
\]
By (5.28), (5.30) and (5.46) and the Cauchy inequality, we have
\[
\mathbb{E}[|\chi_{i'j'}|^{3/2} e^{t\sum_{l \in A_{ijk} \setminus A_{i'j'}} X_l}] \leq 16\beta^{2\kappa_{i,j,k}+6} (\gamma_{3,i'j'}(t) + \gamma_{3,j'}(t)) \mathbb{E}[e^{tW}],
\]
and
\[
\mathbb{E}[|X_m|^{3/2} e^{t\sum_{l \in A_{ijk} \setminus A_{i'j'}} X_l}] \leq \beta^{2\kappa_{i,j,k}+6} \gamma_{3,m}(t) \mathbb{E}[e^{tW}].
\]
Then, by (5.46) and the inequality \(|xy| \leq (2/3)|x|^{3/2} + (1/3)|y|^{3/2}\),
\[
\mathbb{E}[\chi_{i'j'} X_m| e^{t\sum_{l \in A_{ijk} \setminus A_{i'j'}} X_l}] \\
\leq 11 \beta^{2\kappa_{i,j,k}+6} \left( \gamma_{3,i'j'}(t) + \gamma_{3,j'}(t) + \gamma_{3,m}(t) \right) \mathbb{E}[e^{tW}],
\] (5.52)
and similarly,

$$
E \left| \chi_{i'j'} X_m e^{t \sum_{i \in A_{ijk}^c} X_i} \right| \leq 11 \beta^3 \{ \gamma_{3,i'}(t) + \gamma_{3,j'}(t) + \gamma_{3,m}(t) \} E e^{t \sum_{i \in A_{ijk}^c} X_i} 
$$

$$
\leq 11 \beta^{2 \kappa_{ijk} + 6} \{ \gamma_{3,i'}(t) + \gamma_{3,j'}(t) + \gamma_{3,m}(t) \} E e^{tW}.
$$

By (5.50)–(5.53),

$$
\left| E \chi_{i'j'} e^{tW_{ijk}} \right| \leq 22 t \sum_{m \in A_{i'j'}} \beta^{2 \kappa_{ijk} + 2 \kappa_{i'j'} + 8} \times \{ \gamma_{3,i'}(t) + \gamma_{3,j'}(t) + \gamma_{3,m}(t) \} E e^{tW}.
$$

By Lemma 5.5 and (5.30) and the Cauchy inequality, we have for $t \geq 0$,

$$
E |X_k \chi_{ij}| \leq 11 \beta^3 (\gamma_{3,i}(0) + \gamma_{3,j}(0) + \gamma_{3,k}(0)) 
$$

$$
\leq 11 \beta^3 \{ \gamma_{3,i}(t) + \gamma_{3,j}(t) + \gamma_{3,k}(t) \}.
$$

If $i' \in A_{ijk}^c$ and $j' \in A_{ijk}^c$, then $X_k \chi_{ij}$ is independent of $\chi_{i'j'}$ and $W_{ijk}$, and thus by (5.54) and (5.55),

$$
\begin{align*}
&|t| E \left( X_k \chi_{ij} \chi_{i'j'} e^{tW_{ijk}} \right) \\
&= t E \left( X_k \chi_{ij} E \left( \chi_{i'j'} e^{tW_{ijk}} \right) \right) \\
&\leq 242 t^2 E e^{tW} \sum_{m \in A_{i'j'}} \beta^{2 \kappa_{ijk} + 2 \kappa_{i'j'} + 12} \times \{ \gamma_{3,i'}(t) + \gamma_{3,j'}(t) + \gamma_{3,m}(t) \}
\end{align*}
$$

For any $i, j, i', j'$ and $k$ such that $k \in A_{ij}$ and $\{i', j'\} \in A_{ijk} \neq \emptyset$, by (5.24), (5.28) and (5.30) and the Cauchy inequality, we have

$$
\begin{align*}
&|E \left( tX_k \chi_{ij} \chi_{i'j'} e^{tW_{ijk}} \right) | \\
&\leq \left| E \left( tX_k \chi_{ij} \chi_{i'j'} e^{tW_{ijk}} \right) \right| + \left| E \left( tX_k \chi_{ij} \chi_{i'j'} (e^{tW_{ijk}} - e^{tW_{ijk}}) \right) \right| \\
&\leq \frac{1}{2 |A_{ij}|} \left| E \left[ \chi_{ij} \chi_{i'j'} e^{tW_{ijk}} \right] \right| + \frac{1}{2} |A_{ij}| t^2 E \left| X^2_k \chi_{ij} \chi_{i'j'} e^{tW_{ijk}} \right| \\
&+ \sum_{t \in A_{ijk} \setminus A_{ij}} t^2 E \left| X_k \chi_{ij} \chi_{i'j'} e^{tW_{ijk}} \right| \\
&+ \sum_{t \in A_{ijk} \setminus A_{ij}} t^2 E \left| X_k \chi_{ij} \chi_{i'j'} e^{tW_{ijk}} \right| \\
&\leq \frac{1}{2 |A_{ij}|} \left| E \left[ \chi_{ij} \chi_{i'j'} e^{tW_{ijk}} \right] \right| + \frac{1}{2} |A_{ij}| t^2 E \left| X^2_k \chi_{ij} \chi_{i'j'} e^{tW_{ijk}} \right| \\
&+ \sum_{t \in A_{ijk}} t^2 E \left| X_k \chi_{ij} \chi_{i'j'} e^{tW_{ijk}} \right| \\
&+ \sum_{t \in A_{ijk}} t^2 E \left| X_k \chi_{ij} \chi_{i'j'} e^{tW_{ijk}} \right|.
\end{align*}
$$
By (5.44) and (5.47), (5.57) can be bounded by
\[
\begin{align*}
| E(tX_k\chi_{ij}\chi_{i'j'}e^{iW_{ij}^e}) | & \leq \frac{10}{|A_{ij}|} \beta^{2\kappa_{ij}} + 5 Ee^{iW} (\gamma_{4,i}(t) + \gamma_{4,j}(t) + \gamma_{4,i'}(t) + \gamma_{4,j'}(t)) \\
& + 8t^2 |A_{ij}| \beta^{2\kappa_{ijk} + 6} \left\{ \sum_{m \in \{i,j,i',j',k,l\}} \gamma_{6,m}(t) \right\} Ee^{iW} \\
& + 8t^2 \sum_{l \in \mathcal{A}_{ijk}} \beta^{2\kappa_{ijk} + 6} \left\{ \sum_{m \in \{i,j,i',j',k,l\}} \gamma_{6,m}(t) \right\} Ee^{iW} \\
& + 8t^2 \sum_{l \in \mathcal{A}_{ijk}} \beta^{2\kappa_{ij} + 5} \left\{ \sum_{m \in \{i,j,i',j',k,l\}} \gamma_{6,m}(t) \right\} Ee^{iW}.
\end{align*}
\]  
(5.58)
Observe that
\[
H_{21} \leq \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij}} \sum_{l \in A_{ij}} t \left| E \left( X_k\chi_{ij}\chi_{i'j'}e^{iW_{ij}^e} \right) \right| I(i' \in \mathcal{A}_{ijk}, j' \in \mathcal{A}_{ijk}) \\
+ \frac{1}{4} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij}} \sum_{l \in A_{ij}} t \left| E \left( X_k\chi_{ij}\chi_{i'j'}e^{iW_{ij}^e} \right) \right| I(i \cap j \cap \mathcal{A}_{ijk} \neq \emptyset).
\]

By (5.56),
\[
H_{211} \leq 61\beta^{12}t^2 Ee^{iW} \left\{ \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in A_{ij}} \beta^{2\kappa_{ijk} + 2\kappa_{ij}} (\gamma_{3,i}(t) + \gamma_{3,j}(t) + \gamma_{3,k}(t)) \right\}^2.
\]

By (5.58),
\[
H_{212} \leq 3\beta^5 Ee^{iW} \sum_{i \in J} \sum_{j \in A_i} \sum_{k \in A_{ij}} \sum_{i' \in J} \sum_{j' \in A_{i'}} \sum_{k' \in A_{i'j'}} |A_{ij}|^{-1} \beta^{2\kappa_{ij}} (\gamma_{4,i}(t) + \gamma_{4,j}(t) + \gamma_{4,i'}(t) + \gamma_{4,j'}(t)) \\
+ 66\beta^6 Ee^{iW} \sum_{i \in J} \sum_{j \in A_i} \sum_{i' \in J} \sum_{j' \in A_{i'}} \sum_{k \in A_{ij}} \sum_{k' \in A_{i'j}} \beta^{2\kappa_{ijk} + 2\kappa_{ij}} \left\{ \sum_{m \in \{i,j,i',j',k,l\}} \gamma_{6,m}(t) \right\}.
\]

By the foregoing inequalities and (5.48) and (5.49), we have
\[
|H_2| \leq 61\beta^{12}t^2 Ee^{iW} \Gamma_3(t)^2 + 3\beta^5 Ee^{iW} \Gamma_4(t) + 110\beta^6 Ee^{iW} \Gamma_6(t),
\]  
(5.59)
where \(\Gamma_3, \Gamma_4\) and \(\Gamma_6\) are as defined in (3.6).

By (5.47),
\[
|H_3| \leq 11t^2\beta^6 Ee^{iW} \Gamma_6(t),
\]  
(5.60)
and
\[
|H_4| \leq 11t^2\beta^6 Ee^{iW} \Gamma_6(t).
\]  
(5.61)
From (5.45) and (5.59)–(5.61), it follows that
\[
\mathbb{E} \left( \sum_{i \in \mathcal{J}} \eta A_i \right)^2 e^{tW} \leq 61 \beta^{12} \mathbb{E} e^{tW} \Gamma_3(t)^2 + 13 \beta^5 \mathbb{E} e^{tW} \Gamma_4(t) + 132 \beta^6 \mathbb{E} e^{tW} \Gamma_6(t).
\]
This completes the proof of (5.23) and hence Lemma 5.3 holds.

Now we move to prove Lemma 5.2. Note that with \( t = 0 \),
\[
\mathbb{E} e^{t(U_i + |X_i|)} = 1.
\]
Hence, Lemma 5.2 follows from (5.41) and Lemma 5.6 with \( t = 0 \) and \( \beta = 1 \).

REFERENCES

[1] Baldi, P., Rinott, Y. and Stein, C. (1989). A normal approximation for the number of local maxima of a random function on a graph. In Probability, Statistics, and Mathematics 59–81. Elsevier.

[2] Barbour, A., Chen, L. H. and Choi, K. (1995). Poisson approximation for unbounded functions, I: Independent summands. Stat. Sin. 749–766.

[3] Barbour, A., Karoski, M. and Ruciski, A. (1989). A central limit theorem for decomposable random variables with applications to random graphs. J. Comb. Theory B. 47 125–145.

[4] Barbour, A. D., Holst, L. and Janson, S. (1992). Poisson approximation. 2. The Clarendon Press, Oxford University Press, New York. Oxford Science Publications.

[5] Chatterjee, S. (2006). Stein’s method for concentration inequalities. Probab. Theory Relat. Fields. 138 305–321.

[6] Chatterjee, S. and Dey, P. S. (2010). Applications of Stein’s method for concentration inequalities. Ann. Probab. 38 2443–2485.

[7] Chatterjee, S., Diaconis, P. and Meckes, E. (2005). Exchangeable pairs and Poisson approximation. Probab. Surveys. 2 64–106.

[8] Chatterjee, S. and Meckes, E. (2008). Multivariate normal approximation using exchangeable pairs. ALEA Lat. Am. J. Probab. Math. Stat. 4 257–283.

[9] Chatterjee, S. and Shao, Q.-M. (2011). Nonnormal approximation by Stein’s method of exchangeable pairs with application to the Curie–Weiss model. Ann. Appl. Probab. 21 464–483.

[10] Chen, L. H. Y. and Choi, K. P. (1992). Some asymptotic and large deviation results in poisson approximation. Ann. Probab. 1867–1876.

[11] Chen, L. H. Y., Fang, X. and Shao, Q.-M. (2013). Moderate deviations in Poisson approximation: a first attempt. Stat. Sin. 1523–1540.
[12] Chen, L. H. Y., Fang, X. and Shao, Q.-M. (2013). From Stein identities to moderate deviations. Ann. Probab. 41 262–293.

[13] Chen, L. H. Y., Goldstein, L. and Shao, Q.-M. (2011). Normal Approximation by Stein’s Method. Probability and its Applications. Springer Berlin Heidelberg, New York.

[14] Chen, L. H. Y. and Shao, Q.-M. (2004). Normal approximation under local dependence. Ann. Probab. 32 1985–2028.

[15] Fang, X. (2018). Wasserstein-2 bounds in normal approximation under local dependence. Available at arXiv:1807.05741.

[16] Linnik, Y. V. (1961). On the probability of large deviations for the sums of independent variables. In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability 2 289–306. Univ of California Press.

[17] Mackey, L., Jordan, M. I., Chen, R. Y., Farrell, B. and Tropp, J. A. (2014). Matrix concentration inequalities via the method of exchangeable pairs. Ann. Probab. 42 906–945.

[18] Meckes, E. (2009). On Stein’s Method for Multivariate Normal Approximation. Institute of Mathematical Statistics.

[19] Petrov, V. V. (1975). Sums of Independent Random Variables. Springer Berlin Heidelberg.

[20] Raič, M. (2007). Clt-related large deviation bounds based on stein’s method. Adv. Appl. Probab. 39 731–752.

[21] Rinott, Y. (1994). On normal approximation rates for certain sums of dependent random variables. J. Comput. Appl. Math. 55 135–143.

[22] Rinott, Y. and Rotar, V. (1997). On coupling constructions and rates in the clt for dependent summands with applications to the antivoter model and weighted u-statistics. Ann. Appl. Probab. 1080–1105.

[23] Röllin, A. (2017). Kolmogorov bounds for the normal approximation of the number of triangles in the Erdős-Rényi random graph. Available at arXiv:1704.00410.

[24] Shao, Q.-M., Zhang, M. and Zhang, Z.-S. (2018). Cramér-type moderate deviation theorems for nonnormal approximation. Available at arXiv:1809.07966.

[25] Shao, Q.-M. and Zhang, Z.-S. (2016). Identifying the limiting distribution by a general approach of Stein’s method. Sci. China Math. 59 2379–2392.

[26] Shao, Q.-M. and Zhang, Z.-S. (2019). Berry–Esseen bounds of normal and nonnormal approximation for unbounded exchangeable pairs. Ann. Probab. 47 61–108.
[27] Shergin, V. V. (1980). On the convergence rate in the central limit theorem for m-dependent random variables. *Theory Probab. Appl.* **24** 782–796.

[28] Stein, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* **2** 583–602. University of California Press, Berkeley.

[29] Stein, C. (1986). *Approximate Computation of Expectations*. 7. IMS, Hayward, CA.

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