Pebble Exchange Group of Graphs

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Abstract
A graph puzzle Puz(G) of a graph G is defined as follows. A configuration of Puz(G) is a bijection from the set of vertices of a board graph G to the set of vertices of a pebble graph G. A move of pebbles is defined as exchanging two pebbles which are adjacent on both a board graph and a pebble graph. For a pair of configurations f and g, we say that f is equivalent to g if f can be transformed into g by a sequence of finite moves.

Let Aut(G) be the automorphism group of G, and let 1_G be the unit element of Aut(G). The pebble exchange group of G, denoted by Peb(G), is defined as the set of all automorphisms f of G such that 1_G and f are equivalent to each other.

In this paper, some basic properties of Peb(G) are studied. Among other results, it is shown that for any connected graph G, all automorphisms of G are contained in Peb(G^2), where G^2 is a square graph of G.

keywords: pebble motion, motion planning, graph puzzle, automorphism

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1 Introduction

Let $G$ be a finite and undirected graph with no multiple edge or loop. The vertex set of $G$ and the edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. Let $P = \{1, \ldots, k\}$ be a set of pebbles with $k < |V(G)|$. An arrangement of $P$ on $G$ is defined as a function $f$ from $V(G)$ to $\{0, 1, \ldots, k\}$ with $|f^{-1}(i)| = 1$ for $1 \leq i \leq k$, where $f^{-1}(i)$ is a vertex occupied with the $i$th pebble for $1 \leq i \leq k$ and $f^{-1}(0)$ is a set of unoccupied vertices. A move is defined as shifting a pebble from a vertex to its unoccupied neighbour.

The pebble motion problem on the pair $(G, P)$ is to decide whether a given arrangement of pebbles reachable from another by executing a sequence of moves. The well-known puzzle named “15-puzzle” due to Loyd [10] is a typical example of this problem where the graph $G$ is a $4 \times 4$-grid. The pebble motion problem is studied intensively [1–5, 8, 9, 11–14], because of its considerable theoretical interest as well as its wide range of applications for computer science and robotics, such as management of indivisible packets of data moving on wide-area communication network and motion planning of independent robots. In 1974, Wilson [14] solved completely the feasibility problem (i.e. the problem of determining whether all the configurations of the puzzle are rearrangeable from one another or not) for the case of $|f^{-1}(0)| = 1$ on general graphs, and it followed by the result of Kornhauser, Miller and Spirakis (FOCS ’84) [9] for the case of $|f^{-1}(0)| \geq 2$. In 2012, Fujita, Nakamigawa and Sakuma [5] generalized the problem to the case of “colored pebbles”, where each pebble of $P$ is distinguished by its color. They also completely solved the feasibility problem for their model. Note that Papadimitriou, Raghavan, Sudan and Tamaki (FOCS ’94) [11] also treat a special case of this model in [5]. They consider the case that there exist two colors (“blue:robot” and “red:obstacle”) of pebbles and that the number of blue colored pebbles is one (i.e. with single robot), and they focus on the time complexity problems for optimal number of moves from a given arrangement to a goal arrangement on a tree.

In 2015, Fujita, Nakamigawa and Sakuma [6] generalized the pebble motion problem as follows: For two graphs $G$ and $H$ with a common number of vertices, let us consider a puzzle $\text{Puz}(G, H)$, where $G$ is a board graph and $H$ is a pebble graph. We call a bijection $f$ from $V(G)$ to $V(H)$ a configuration of $\text{Puz}(G, H)$, and we denote the set of all configurations of $\text{Puz}(G, H)$ by $\mathcal{C}(G, H)$. Given a configuration $f$, if $f(x) = y$, we consider that the vertex $x$ of the board is occupied by the pebble $y$. In $\text{Puz}(G, H)$, two pebbles $y_1 = f(x_1)$ and $y_2 = f(x_2)$ can be exchanged if $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. Then the resultant configuration $g$ satisfies that $g(x_1) = y_2$, $g(x_2) = y_1$ and $g(x) = f(x)$ for any $x \in V(G) \setminus \{x_1, x_2\}$. We call the opera-
Let $E$ and $E$ not bipartite, and (3) $G, K$ are bipartite graphs with at least three vertices. It is not difficult to see that $E$ is not a cycle, and (2) every vertex of $P$, the join $G, H$ is considered as generalizations of Example 1. Suppose that both $G$ and $H$ are considered as equivalent to each other. For two graphs $G$ and $H$, let $G \times H$ denote a Cartesian product of $G$ and $H$, where $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(u_1, v_1)(u_2, v_2) \in V(G \times H)^2 : u_1u_2 \in E(G) \text{ and } v_1 = v_2, \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E(H)\}$. Let $P_k$ be the path with $k$ vertices, and let $K_{1,\ell}$ be the star with $\ell$ pendant vertices.

**Example 1** $Puz(P_4 \times P_4, K_{1,15})$ corresponds to the 15-puzzle, by considering that the center $z$ of $K_{1,15}$ corresponds to a vacant position. For two configurations $f, g \in C(P_4 \times P_4, K_{1,15})$ with $f^{-1}(z) = g^{-1}(z)$, $f$ is equivalent to $g$ if and only if $g^{-1} \circ f$ is an even permutation on $V(G)$ (cf. [8],[3]).

We will show some more examples, from Example 2 to Example 5, which are considered as generalizations of Example 1. Suppose that both $G$ and $H$ are not bipartite graphs with at least three vertices. It is not difficult to see that $Puz(G, H)$ is not feasible because of the parity of configurations (cf. [6]).

Let $\theta(1,2,2)$ be a graph such that $V(\theta(1,2,2)) = \{v_i : 1 \leq i \leq 7\}$ and $E(\theta(1,2,2)) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_1v_7, v_4v_7\}$.

**Example 2** Let $G$ be a 2-connected non-bipartite graph with $n$ vertices. If $G$ is not a cycle or $\theta(1,2,2)$, then $Puz(G, K_{1,n-1})$ is feasible (cf. [7],[4]).

For a positive integer $k$, a path $P = v_1v_2\cdots v_k$ of a graph $G$ is called a $k$-isthmus if (1) every edge of $P$ is a bridge of $G$, (2) every vertex of $P$ is a cut-vertex of $G$, and (3) $\deg(v_i) = 2$ for $1 < i < k$. For two graphs $G$ and $H$, the join $G + H$ is defined as $V(G + H) = V(G) \cup V(H)$, $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. For a graph $G$, let $\overline{G}$ be the complement of $G$.

**Example 3** Let $2 \leq k \leq n$. A pebble graph $K_k + \overline{K_{n-k}}$ is considered as a set of $n - k$ labeled pebbles and $k$ unlabeled pebbles, in which two labeled pebbles cannot directly exchange their positions with each other. Let $G$ be a connected graph with $n$ vertices except a cycle. Then $Puz(G, K_k + \overline{K_{n-k}})$ is feasible if and only if $G$ has no $k$-isthmus (cf. [9]).

**Example 4** Let $2 \leq k \leq n/2$. Let $G$ be a graph with $n$ vertices. Then $Puz(G, K_{k,n-k})$ is feasible if and only if (1) $G$ is not a cycle, and (2) $G$ is not bipartite, and (3) $G$ has no $k$-isthmus (cf. [3]).
Example 5  Let $r \geq 3$ and let $2 \leq n_1 \leq \ldots \leq n_r$. Let $G$ be a graph with $n = n_1 + n_2 + \cdots + n_r$ vertices. Then Puz($G, K_{n_1,n_2,\ldots,n_r}$) is feasible if and only if (1) $G$ is not a cycle, and (2) $G$ has no $(n - n_r)$-isthmus (cf. [5]).

In [6], the above mentioned graph puzzle was formally introduced and some more necessary/sufficient conditions of the feasibility of the puzzle was studied.

This model has again a wide range of real world applications, especially for robot motion planning problems and facility relocation problems. Here we will quote two examples from the paper [6].

**Application 1** Let $G$ be a simple graph whose vertex set $V(G)$ is the set of workplaces for robots. Each of the workplaces has a unique electrical outlet and a single robot is working there. Two workplaces $u$ and $v$ in $V(G)$ are adjacent with an edge $e$ of $G$ if there exists a unique passageway from $u$ to $v$. Each of the passageway is so narrow that at most two robots can pass at the same time. Moreover, there exists a pair of robots such that they have no common method of taking mutual communication and hence the two robots may collide with each other on such a narrow passageway. Let $H$ be a simple graph whose vertex set $V(H)$ is the set of robots working in the workplaces and two robots in $V(H)$ are adjacent if the robots can take mutual communication to avoid their collision. In this case, the rearrangements of the robots $V(H)$ on the workplaces $V(G)$ can be described only by the pebble exchange model Puz($G, H$).

**Application 2** Let $G$ be a simple graph whose vertex set is the set of chemical storerooms. In each chemical storeroom, we can store only one type of chemical. Two chemical storerooms $u$ and $v$ in $V(G)$ are adjacent with an edge $e$ of $G$ if there exists a unique passageway from $u$ to $v$. Again, each of the passageway is so narrow that at most two trucks of chemicals can pass at the same time. There exist several dangerous pairs of chemicals such that, for each of the pairs, a near miss of the two chemicals can cause serious chemical reaction with the possibility of explosion. Let $H$ be a simple graph whose vertex set $V(H)$ is the set of chemicals stored in the chemical storerooms $V(G)$ and two chemicals of $V(H)$ are adjacent if the pair is safe (i.e. no chemical reaction occurs). Now the investigation of rearrangements of the chemicals $V(H)$ in the chemical storerooms $V(G)$ leads us again to treat the pebble exchange model Puz($G, H$).
In this paper, we will shed light on some algebraic property of the puzzle, which is of not only theoretical interest, but also practical importance, as will be discussed later.

In the following, we only consider the case where a board graph and a pebble graph are the same, and we denote $C(G, G)$ and $\text{Puz}(G, G)$ simply by $C(G)$ and $\text{Puz}(G)$, respectively.

The automorphism group of a graph $G$, denoted by $\text{Aut}(G)$, is the group which consists of all bijections $f$ from $V(G)$ to $V(G)$ such that $f(x_1)f(x_2) \in E(G)$ if and only if $x_1x_2 \in E(G)$. Let $1_G$, or simply 1, denote the identity element of $\text{Aut}(G)$. Let us introduce the pebble exchange group of $G$, denoted by $\text{Peb}(G)$, as the group which consists of all automorphisms $f$ of $G$ such that $1_G$ and $f$ are equivalent in $\text{Puz}(G)$. If $f, g \in \text{Peb}(G)$, there exists a finite sequence of configurations $f = f_0, f_1, \ldots, f_s = g$, where $f_i$ is generated from $f_{i-1}$ by a move of $\text{Puz}(G)$ for all $1 \leq i \leq s$. We remark that for $1 \leq i \leq s - 1$, $f_i$ is not necessarily an automorphism of $G$. It is not difficult to see that $\text{Peb}(G)$ turns out a normal subgroup of $\text{Aut}(G)$.

For example, please recall Application 2 in the above. From a given stable disposition of chemicals with no more information about the store system, one of the most moderate assumptions would be that two chemical store-rooms are adjacent with a passageway only if the pair of corresponding two chemicals is safe. This situation leads us to treat the pebble exchange model $\text{Puz}(G)$. Moreover, in such case, any rearrangement from a stable disposition to another stable disposition should be corresponding to an element of $\text{Aut}(G)$. Hence if we can show the equation $\text{Peb}(G) = \text{Aut}(G)$ here, it means that practically all the necessary and sufficient dispositions are rearrangeable from one another.

However, it seems to be highly nontrivial and difficult problem to characterize completely the graphs whose pebble exchange groups are equal to their automorphism groups. Hence, before to attack this problem directly, in this paper we will show that the class of graphs $G$ satisfying $\text{Peb}(G) = \text{Aut}(G)$ is considerably large. Especially, we prove (Theorem 5) that, for any connected graph $G$, the pebble exchange group of the square of $G$ contains a subgroup isomorphic to the automorphism group of $G$. This result is somewhat surprising since the square of a sparse graph $G$ is apt to sparse and the puzzle $\text{Puz}(G^2)$ is also far from feasible in general. By using this result, for example, we can show that, for any connected graph $G$, if we 2-subdivide all the edges of $G$, and if we take its square, the resulting graph $H$ satisfies the equation $\text{Peb}(H) = \text{Aut}(H)$. 

5
2 Main Results

It is known that for any finite group $\Gamma$, there exists a graph $G$ such that $\text{Aut}(G) \simeq \Gamma$ (cf. [7]). By using this fact, we have the following result.

**Proposition 1** For any finite group $\Gamma$, there exists a graph $G$ such that $\text{Peb}(G) \simeq \Gamma$.

**Proof.** Let us take a graph $H$ such that $\text{Aut}(H) \simeq \Gamma$. Since at least one of $H$ and $\overline{H}$ is connected, and $\text{Aut}(H) \simeq \text{Aut}(\overline{H})$, by replacing $H$ with $\overline{H}$, if necessary, we may assume $H$ is connected.

Let $T_{uv}$ be a tree such that $V(T_{uv}) = \{u, v, x_1, x_2, x_3\}$ and $E(T_{uv}) = \{ux_1, vx_1, x_1x_2, x_2x_3\}$. Let us build $H'$ from $H$ by replacing all edges $uv \in E(H)$ with $T_{uv}$. Then we have $\text{Aut}(H) \simeq \text{Aut}(H')$. Now, let us consider $G = H' + z$ with $n$ vertices, which is the join of $H'$ and an additional vertex $z$. Then $G$ is 2-connected and it contains $K_{1,n-1}$ as a spanning subgraph. Hence, by Wilson’s theorem, Example 2, $\text{Puz}(G)$ is feasible. Therefore, we have $\text{Peb}(G) = \text{Aut}(G)$. Since $\text{Aut}(G) \simeq \text{Aut}(H') \simeq \Gamma$, we have $\text{Peb}(G) \simeq \Gamma$, as required.

Second, we note a simple observation about $\text{Peb}(G)$, where $G$ contains no small cycle. For a graph $G$, the *girth* of $G$, denoted by $\text{girth}(G)$, is the order of the smallest cycle contained in $G$. If $G$ contains no cycle, $\text{girth}(G)$ is defined as $\infty$. A *matching* of a graph $G$ is a set of independent edges of $G$. For a matching $M$ of a graph $G$, let $f(M)$ be a configuration of $\text{Puz}(G)$ such that for all $x \in V(M)$, $f(x) = y$, where $xy \in E(M)$, and $f(x) = x$ for all $x \notin V(M)$. Let $\mathcal{M}(G) = \{f(M) : M$ is a matching of $G\}$.

**Proposition 2** Let $G$ be a connected graph with at least three vertices. If $\text{girth}(G) \geq 5$, then $\text{Peb}(G) \simeq \{1_G\}$.

**Proof.** It is sufficient to show that if $f \in \mathcal{C}(G)$ satisfies $1_G \sim f$, then $f$ is not an automorphism of $G$.

**Claim.** Let $f$ be a configuration of $G$. Then $f \sim 1_G$ if and only if $f \in \mathcal{M}(G)$.

First, suppose that $f = f(M) \in \mathcal{M}(G)$, where $M$ is a matching of $G$. Starting from $1_G$, by exchanging all pairs of pebbles $u$ and $v$ satisfying $uv \in M$, we have $f(M) \sim 1_G$.

Second, suppose that $f \sim 1_G$. Let $f_0 = 1_G, f_1, f_2, \ldots, f_{s-1}, f_s = f$ be a sequence of configurations, where $f_i$ is generated from $f_{i-1}$ by a move for all $1 \leq i \leq s$. By induction, we may assume $f_{s-1} = f(M) \in \mathcal{M}(G)$. Let us assume we have $f$ from $f_{s-1}$ by a move, in which two pebbles $u$ and $v$ are
exchanged. What we need to show is that \( f \in \mathcal{M}(G) \).

**Case 1.** Both \( u \) and \( v \) are contained in \( V(M) \) and \( uv \in V(M) \).

In this case, we have \( f = f(M') \), where \( M' = M \setminus \{uv\} \).

**Case 2.** Both \( u \) and \( v \) are contained in \( V(M) \) and \( uv \notin V(M) \).

Suppose that \( ux \in E(M) \) and \( vy \in E(M) \). In order to exchange \( u = f_{s-1}(x) \) and \( v = f_{s-1}(y) \), we have \( uv \in E(G) \) and \( xy \in E(M) \). Hence, \( uxyv \) forms a cycle of length 4, a contradiction.

**Case 3.** Exactly one of \( u \) and \( v \) is contained in \( V(M) \).

We may assume \( u \in V(M) \) and \( v \notin V(M) \). Suppose that \( ux \in E(M) \). In order to exchange \( u = f_{s-1}(x) \) and \( v = f_{s-1}(v) \), we have \( uv \in E(G) \) and \( vx \in E(G) \). Hence, \( uvx \) forms a cycle of length 3, a contradiction.

**Case 4.** None of \( u \) and \( v \) is contained in \( V(M) \).

In this case, we have \( f = f(M') \), where \( M' = M \cup \{uv\} \).

Suppose to a contradiction that there exists an automorphism \( f \) of \( G \) with \( f \sim 1_G \) and \( f \neq 1_G \). By the above claim, we have a matching \( M \) of \( G \) such that \( f = f(M) \). Since \( f \neq 1_G \), we have \( E(M) \neq \emptyset \). Let \( uv \in E(M) \). Because \( |V(G)| \) is at least 3 and \( G \) is connected, we may assume there exists a vertex \( x \in V(G) \setminus \{u, v\} \) such that \( ux \in E(G) \). If \( x \in V(M) \), there exists an edge \( xy \in E(M) \). Since \( f \) is an automorphism of \( G \), we have \( f(u)f(x) = vy \in E(G) \). Hence, \( uvyx \) forms a cycle of length 4, a contradiction. If \( x \notin V(M) \), since \( f \) is an automorphism of \( G \), we have \( f(u)f(x) = vx \in E(G) \). Hence, \( uvx \) forms a cycle of length 3, a contradiction.

The next result is about pebble exchange group of a product of graphs.

**Theorem 3** For any two connected graphs \( G_1 \) and \( G_2 \), \( \text{Peb}(G_1 \times G_2) \simeq \text{Peb}(G_1) \times \text{Peb}(G_2) \).

The proof of Theorem 3 will be given in Section 3.

Let \( Q_n \) be the \( n \)-dimensional hypercubic graph. Since \( Q_n = P_2^n \) and \( \text{Peb}(P_2) \simeq \mathbb{Z}/2\mathbb{Z} \), we have the following corollary as an immediate consequence of Theorem 3.

**Corollary 4** For \( n \geq 1 \), \( \text{Peb}(Q_n) \simeq (\mathbb{Z}/2\mathbb{Z})^n \).

As a graph \( G \) becomes sparse, the number of possible moves on \( G \) decrease. Hence, it is interesting to show the existence of graphs \( G \) such that \( \text{Peb}(G) \) has a rich structure and \( |E(G)| = O(|V(G)|) \).
For a graph $G$, the square graph $G^2$ of $G$ is defined as $V(G^2) = V(G)$ and $E(G^2) = \{uv \in V(G)^2 : d_G(u, v) = 1 \text{ or } 2\}$, where $d_G(u, v)$ is the distance between $u$ and $v$ in $G$.

The main result of the paper is the following theorem.

**Theorem 5** For any connected graph $G$, $\text{Peb}(G^2) \supset \text{Aut}(G)$.

In order to prove Theorem 5, we first deal with the simplest but the most important case, where $G$ is a path.

**Lemma 6** For $n \geq 2$, $\text{Peb}(P^2_n) \supset \text{Aut}(P_n)$.

The proof of Proposition 6 will be given in Section 4.

Second, let us introduce a new operation, path flip, for a configuration $f \in C(G)$. Let $P = v_0v_1\ldots v_n$ be a path of $G$. If $f(v_0)f(v_1)\ldots f(v_n)$ is also a path of $G$, by a path flip, $f$ can be replaced with $g \in C(G)$ such that $g(v_i) = f(v_{n-i})$ for $0 \leq i \leq n$, and $g(x) = f(x)$ for all $x \in V(G) \setminus V(P)$.

The following lemma may arouse an independent interest apart from pebble-exchange puzzles.

**Lemma 7** For a connected graph $G$, and for any two configurations $f, g \in \text{Aut}(G)$, $f$ can be transformed into $g$ by a finite sequence of path flips.

The proof of Lemma 7 will be given in Section 5.

By Lemma 6 and 7, Theorem 5 follows.

## 3 Proof of Theorem 3

First, we will show that $\text{Peb}(G_1) \times \text{Peb}(G_2) \subset \text{Peb}(G_1 \times G_2)$. For $\sigma \in \text{Peb}(G_1)$ and $\tau \in \text{Peb}(G_2)$, it suffices to show that $\sigma \times \tau \in \text{Peb}(G_1 \times G_2)$. In the first part of moves, we proceed a sequence of moves corresponding to $\sigma$ on all copies of $G_1$ in parallel. In the second part of moves, we proceed a sequence of moves corresponding to $\tau$ on all copies of $G_2$ in parallel. The sequence of total moves yields $\sigma \times \tau$.

Second, we will show that $\text{Peb}(G_1 \times G_2) \subset \text{Peb}(G_1) \times \text{Peb}(G_2)$.

**Claim 1.** Let $f \in C(G_1 \times G_2)$ such that $1_{G_1 \times G_2} \sim f$. For two pebbles $x$ and $y$, if $f^{-1}(x)$ and $f^{-1}(y)$ are in a common copy of $G_i$ for some $i = 1, 2$, then $x$ and $y$ are not in a common copy of $G_{3-i}$.

Suppose to a contradiction that there exists a pair of pebbles $x$ and $y$ and a configuration $f$ with $1 \sim f$ such that $f^{-1}(x)$ and $f^{-1}(y)$ are in a common
copy of $G$, and $x$ and $y$ are in a common copy of $G_{3-i}$. We may assume that $f$ can be made from 1 with the minimum number $s$ of moves violating the condition of the claim. We may assume that $y$ is exchanged with a pebble $z$ in the $s$-th move.

**Case 1.** $f^{-1}(y)$ and $f^{-1}(z)$ are in a common copy of $G_{3-i}$.

In this case, by the minimality of $s$, $y$ and $z$ are in a common copy of $G_{3-i}$. Since $x$ and $y$ are in a common copy of $G_{3-i}$, $x$ and $z$ are in a common copy of $G_{3-i}$. Then $x$ and $z$ violate the condition of the claim just after the $(s-1)$-th steps. This contradicts the minimality of $s$.

**Case 2.** $f^{-1}(y)$ and $f^{-1}(z)$ are in a common copy of $G_i$.

In this case, $x$ and $y$ already violate the condition of the claim after the $(s-1)$-th steps. This contradicts the minimality of $s$.

**Claim 2.** Let $f$ be an automorphism of $G_1 \times G_2$ such that $1_{G_1 \times G_2} \sim f$. For two pebbles $x$ and $y$, if $f^{-1}(x)$ and $f^{-1}(y)$ are in a common copy of $G_i$ for some $i = 1, 2$, then $x$ and $y$ are in a common copy of $G_i$.

Let us assume that $f^{-1}(x)$ and $f^{-1}(y)$ are in a common copy of $G_i$. Since $f$ is an automorphism of $G_1 \times G_2$, there exists a path $P$ from $x$ to $y$ such that a path $f^{-1}(V(P))$ is in a common copy of $G_i$. By Claim 1, all pairs of vertices in $V(P)$ are in a mutually different copy of $G_{3-i}$. Hence, any pair of adjacent vertices in $V(P)$ are in a common copy of $G_i$. Therefore, $x$ and $y$ are in a common copy of $G_i$.

By Claim 2, $f$ induces a permutation $\sigma_i$ on the set of all copies of $G_{3-i}$ for $i = 1, 2$, where $\sigma_i$ naturally corresponds to $\sigma_i \in \text{Aut}(G_i)$. Then, we have $f = \sigma_1 \times \sigma_2 \in \text{Aut}(G_1) \times \text{Aut}(G_2)$. Furthermore, by Claim 1, if two pebbles $x$ and $y$ are in a common copy of $G_i$, $x$ and $y$ can be exchanged only if they occupy a common copy of $G_i$. Hence, we have $\sigma_i \in \text{Peb}(G_i)$ for $i = 1, 2$. Therefore, we have $f \in \text{Peb}(G_1) \times \text{Peb}(G_2)$. 

4 Proof of Lemma 6

It is not difficult to see that $\text{Puz}(P_n^2)$ is feasible for $n \leq 5$. Hence, we have $\text{Peb}(P_n^2) = \text{Aut}(P_n^2) \supset \text{Aut}(P_n)$. Suppose that $n \geq 6$. In this case, since $\text{Aut}(P_n^2) = \text{Aut}(P_n) \cong \mathbb{Z}/2\mathbb{Z}$, it suffices to prove $\text{Peb}(P_n^2) = \mathbb{Z}/2\mathbb{Z}$. Let us label the vertices of $P_n$ as $V(P_n) = \{1, 2, \ldots, n\}$ and $E(P_n) = \{ij : j - i = 1\}$. Note that $\text{Aut}(P_n^2) = \{1_n, \alpha_n\}$, where $1_n(i) = i$ for $1 \leq i \leq n$ and $\alpha_n(i) = n - i + 1$ for $1 \leq i \leq n$. It suffices to show that $1_n \sim \alpha_n$ in $\text{Puz}(P_n^2)$.

In the following, besides $\text{Puz}(P_n^2)$, we consider two additional puzzles $\text{Puz}(P_{n+1}^2 \setminus \{n\}, P_n^2)$ and $\text{Puz}(P_n^2, P_{n+1}^2 \setminus \{n\})$. For configurations $f \in C(P_n^2, P_n^2)$,
\[ g \in \mathbb{C}(P_{n+1}^2 \setminus \{n\}, P_n^2) \text{ and } h \in \mathbb{C}(P_n^2, P_{n+1}^2 \setminus \{n\}) \], we will use notations as
\[
\begin{align*}
  f &= (f(1), f(2), \ldots, f(n-1), f(n)), \\
  g &= (g(1), g(2), \ldots, g(n-1), *, g(n+1)), \\
  h &= (h(1), h(2), \ldots, h(n-1), h(n)).
\end{align*}
\]
By using this notation, \(1_n, \alpha_n\) is expressed as
\[
1_n = (1, 2, \ldots, n-1, n), \quad \alpha_n = (n, n-1, \ldots, 2, 1).
\]
Let us define \(1'_n, \beta_n \in \mathbb{C}(P_{n+1}^2 \setminus \{n\}, P_n^2)\) as
\[
1'_n = (1, 2, \ldots, n-1, *, n), \quad \beta_n = (n, n-1, \ldots, 2, *, 1),
\]
and let us define \(1''_n, \gamma_n \in \mathbb{C}(P_n^2, P_{n+1}^2 \setminus \{n\})\) as
\[
1''_n = (1, 2, \ldots, n-1, n+1), \quad \gamma_n = (n+1, n-1, \ldots, 2, 1).
\]
What we want to show is that \(1_n \sim \alpha_n, 1'_n \sim \beta_n, 1''_n \sim \gamma_n\) for all \(n \geq 1\).

Note that \(P_{n+1}^2 \setminus \{n\}\) is naturally considered as a subgraph of \(P_n^2\). Hence, \(1'_n \sim \beta_n\) implies that \(1_n \sim \alpha_n\). Furthermore, \(Puz(P_{n+1}^2 \setminus \{n\}, P_n^2)\) and \(Puz(P_n^2, P_{n+1}^2 \setminus \{n\})\) are isomorphic as puzzles, since these puzzles can be switched to each other by interchanging the roles of a board graph and a pebble graph. Hence, \(1'_n \sim \beta_n\) holds if and only if \(1''_n \sim \gamma_n\) holds.

We proceed by induction on \(n\). For \(n \leq 2\), it is not difficult to see that the conclusion holds. Let \(n \geq 3\). It suffices to show that \(1'_n \sim \beta_n\) by using the inductive assumptions \(1_k \sim \alpha_k, 1'_k \sim \beta_k, 1''_k \sim \gamma_k\) for \(2 \leq k \leq n-1\). We have
\[
\begin{align*}
  1'_n &= (1, 2, \ldots, n-2, n-1, *, n) \\
       &\sim (1, n, \ldots, 4, 3, *, 2) \quad \text{by } 1'_{n-1} \sim \beta_{n-1} \\
       &\sim (1, 3, \ldots, n-1, n, *, 2) \quad \text{by } 1_{n-2} \sim \alpha_{n-2} \\
       &\sim (n, n-1, \ldots, 3, 1, *, 2) \quad \text{by } 1''_{n-1} \sim \gamma_{n-1} \\
       &\sim (n, n-1, \ldots, 3, 2, *, 1) \quad \text{by the exchange of 1 and 2} \\
       &= \beta_n,
\end{align*}
\]
as required.

\section{Proof of Lemma \ref{lemma7}}

In the following, for a configuration \(f \in \mathbb{C}(G)\), we say that \(f\) is realizable by path flips, if \(f\) can be transformed from \(1_G\) by a finite sequence of path flips.
Suppose to a contradiction that there exists a pair \((G, \sigma)\) of a graph \(G\) and an automorphism \(\sigma \in \text{Aut}(G)\) such that \(\sigma\) is not realizable by path flips. Note that the order of an automorphism \(\sigma \in \text{Aut}(G)\) is the smallest integer \(k\) such that \(\sigma^k = 1_G\). Let us choose a counter example \((G, \sigma)\) such that (1) \(|V(G)|\) is minimum, and (2) the order of \(\sigma\) is minimum subject to (1). Let \(n\) be the order of \(\sigma\). First, we claim that \(n\) is a prime power. Indeed, if \(n\) is not a prime power, there exist relatively prime two integers \(r \geq 2\) and \(s \geq 2\) with \(n = rs\). Since the order of \(\sigma^r\) is \(s < n\) and the order of \(\sigma^s\) is \(r < n\), by the choice of \(n\), both \(\sigma^r\) and \(\sigma^s\) are realizable by path flips. Since \(r\) and \(s\) are relatively prime, there exist two integers \(x\) and \(y\) such that \(rx + sy = 1\). Hence, we have \(\sigma = (\sigma^r)^x(\sigma^s)^y\) and so \(\sigma\) is also realizable by path flips.

Let \(n = p^\alpha\), where \(p\) is a prime and \(\alpha\) is a positive integer. Let \(C(\sigma)\) denote a cyclic group generated by \(\sigma\). If \(\sigma'\) is another generator of \(C(\sigma)\), \(\sigma'\) is realizable by path flips if and only if \(\sigma\) is realizable by path flips. Let us denote the orbit of \(x\) in \(C(\sigma)\) by \(C(\sigma) \cdot x\). Let us choose a pair \((\sigma', x)\), where \(\sigma'\) is a generator of \(C(\sigma)\) and \(x\) is a vertex of \(G\) such that (1) \(d_G(x, \sigma'(x))\) is minimum, and (2) \(|C(\sigma) \cdot x|\) is minimum subject to (1).

We redefine \(\sigma\) as a chosen element \(\sigma'\), and put \(d = d_G(x, \sigma(x))\) and \(m = |C(\sigma) \cdot x|\). Note that \(m\) is a power of \(p\), since \(m\) divides \(n = p^\alpha\). First, we deal with the case, where \(m = 1\).

**Case 1.** \(m = 1\).

In this case, we have \(C(\sigma) \cdot x = \{x\}\) and \(d = 0\). Let \(G' = G - x\). Since \(\sigma(V(G')) = V(G')\), it follows that \(\sigma|_{G'}\), the restriction of \(\sigma\) on \(G'\), is an element of \(\text{Aut}(G')\). If \(G'\) is connected, by the inductive hypothesis, \(\sigma|_{G'}\) is realizable by path flips in \(G'\). Since there is no need to move \(x\) for \(\sigma\), \(\sigma\) is also realizable by path flips. Hence, in the following, we assume that \(G'\) is disconnected. Then there exists an integer \(s \geq 2\), and a vertex partition \(V(G') = V_1 \cup V_2 \cup \cdots \cup V_s\), where \(G[V_i]\) is a connected component of \(G'\) for all \(1 \leq i \leq s\).

Then \(\sigma\) induces a permutation \(\tilde{\sigma}\) on \(\{1, \ldots, s\}\) such that \(\sigma(V_i) = V_{\tilde{\sigma}(i)}\) for all \(1 \leq i \leq s\). By the inductive hypothesis, all automorphisms of \(G[V_i]\) are realizable by path flips for all \(1 \leq i \leq s\). Since any permutation is written as a product of transpositions, it suffices to prove the assertion under the condition where \(V(G) = V_1 \cup V_2\) and \(\tilde{\sigma}(1) = 2\), \(\tilde{\sigma}(2) = 1\). In this case, let us take \(v_1 \in V_1\) such that \(d_G(x, v_1)\) is maximum, and let \(v_2 = \sigma(v_1)\). Then we have \(v_i \in V_i\) for \(i = 1, 2\), \(\sigma(v_1) = v_2\) and \(\sigma(v_2) = v_1\).

Let \(P\) be a path of \(G\) from \(v_1\) to \(v_2\), and set a path \(P' = P \setminus \{v_1, v_2\}\). Now, let us flip \(P\), and let us flip \(P'\) subsequently. Set \(H = G \setminus \{v_1, v_2\}\). Since \(\sigma(V(H)) = V(H)\), we have \(\sigma|_H \in \text{Aut}(H)\). Since \(H\) is connected, by the inductive hypothesis, \(\sigma|_H\) is realizable by path flips.
Hence, $\sigma$ is also realizable by path flips, as required.

**Case 2.** $m \geq 2$.

Let us take a shortest path $P = y_0 y_1 \ldots y_{d-1} \sigma(x)$ from $x$ to $\sigma(x)$, where we set $x = y_0$. Let $Y = V(P) \setminus \{ \sigma(x) \}$.

**Claim 1.** If $0 \leq i < j \leq d - 1$, then $\sigma^s(y_i) \neq \sigma^t(y_j)$ for all integers $s$ and $t$.

Suppose to a contradiction that $\sigma^s(y_i) = \sigma^t(y_j)$ for some $s$ and $t$. We have $y_i = \sigma^k(y_j)$, where $k = t - s$. Since $d(y_i, \sigma^k(y_j)) = d(\sigma^k(y_j), \sigma^k(y_i)) = d(y_j, y_i) < d$, by the choice of $d$, $\sigma^k$ is not a generator of $C(\sigma)$. Since $C(\sigma)$ is a cyclic group of the order $n = p^s$, we have $k \equiv 0 \pmod{p}$. Furthermore, we have $d(y_i, \sigma^{k+1}(y_j)) = d(\sigma^k(y_j), \sigma^{k+1}(y_j)) \leq d - j + i < d$. Therefore, we have $k + 1 \equiv 0 \pmod{p}$, a contradiction.

**Claim 2.** For all $0 \leq i \leq d - 1$, if $s \not\equiv t \pmod{m}$, then $\sigma^s(y_i) \neq \sigma^t(y_i)$.

Suppose to a contradiction that $\sigma^s(y_i) = \sigma^t(y_j)$ for some $s$ and $t$ with $s \not\equiv t \pmod{m}$. We have $y_i = \sigma^k(y_j)$ with some $k \not\equiv 0 \pmod{m}$. Since $y_i = \sigma^s(y_i)$ also holds, we have $|C(\sigma) : y_i| \leq \gcd(k, n) < m$, because $n$ is a power of $p$ and $k \not\equiv 0 \pmod{m}$. With the fact $d(y_i, \sigma(y_i)) \leq d$, this contradicts to the choice of $m$.

For $0 \leq k \leq m - 1$, let us define $X_k = \bigcup_{0 \leq i \leq m - 1} \{ \sigma^i(V(P)) \}$, and $X'_k = X_k \setminus \{ \sigma(x) \}$. Then we have $\sigma(X_k) = X_{k+1}$ for $0 \leq k \leq m - 2$, and $\sigma(X_{m-1}) = X_0$. Furthermore, by Claim 1 and Claim 2, $X'_k \cap X'_{k+1} = \emptyset$ for $k \neq \ell$.

Let us define a subgraph $H$ of $G$ such that $V(H) = \bigcup_{0 \leq k \leq m - 1} X_k$ and $E(H) = \bigcup_{0 \leq i \leq n - 1} \{ \sigma^i(E(P)) \}$. Since $\sigma(V(H)) = V(H)$ and $\sigma(E(H)) = E(H)$, $\sigma|_H$ is an automorphism of $H$.

**Claim 3.** $\sigma|_H$ is realizable by path flips on $H$.

For $0 \leq k \leq m - 1$, let $H_k = H[X_k]$. If $m = n$, $H_k$ is simply a path $\sigma^k(P)$ for $0 \leq k \leq m - 1$, and $H$ is a cycle. Hence, $\text{Aut}(H)$ is isomorphic to a dihedral group, which is generated by a pair of reflections of cycles. Since a reflection is realizable by a path flip, the claim is proved. In the following, we assume that $m < n$. Let us define a configuration $\tau \in \mathcal{C}(H)$ such that $\tau(v) = \sigma(v)$ for $v \in V(H) \setminus X_{m-1}$ and $\tau(v) = \sigma^{1-m}(v)$ for $v \in X_{m-1}$.

We claim that $\tau$ is an automorphism of $H$, because for $0 \leq k \leq n - 1$, $\sigma|_{H_k}$ is an automorphism from $H_k$ to $H_{k+1}$ and $\sigma^{1-m}|_{H_{m-1}}$ is an automorphism from $H_{m-1}$ to $H_0$. Furthermore, by definition, the order of $\tau$ is $m$. Since $m < n$, by the minimality of $n$, $\tau$ is realizable by path flips. On the other hand, $\sigma^m|_{H_0}$ is an automorphism of $H_0$. Since the order of $\sigma^m|_{H_0} = n/m < n$, by the minimality of $n$, $\sigma^m|_{H_0}$ is realizable by path flips. Since $\sigma$ is a composition of $\tau$ and $\sigma^m|_{H_0}$, $\sigma$ is also realizable by path flips.
We may assume $V(G) \setminus V(H) \neq \emptyset$. Choose a vertex $z \in V(G) \setminus V(H)$ such that $d_G(z, V(H))$ is maximum. Let $Q$ be a shortest path from $z$ to $V(H)$, and let $y \in V(Q) \cap V(H)$ be the end vertex of $Q$. Then $y$ is contained in $\sigma^i(Y)$ with some $i$ for $0 \leq i \leq n - 1$. By replacing $V(Q)$ with $\sigma^{-i}(V(Q))$ if necessary, we may assume $y$ is contained in $Y$ from the beginning. Let us define a subgraph $F$ of $G$ such that $V(F) = V(H) \cup \cup_{0 \leq i \leq n-1} \sigma^i(V(Q))$ and $E(F) = E(H) \cup \cup_{0 \leq i \leq n-1} \sigma^i(E(Q))$. Since $\sigma(V(F)) = V(F)$ and $\sigma(E(F)) = E(F)$, $\sigma|_F$ is an automorphism of $F$.

Case 2.1. $V(F) \neq V(G)$.

In this case, by the minimality of $|V(G)|$, $\sigma|_F$ is realizable by path flips. Let us define two more subgraphs $F' = F - C(\sigma) \cdot z$ and $G = G - C(\sigma) \cdot z$. Then both $F'$ and $G'$ are connected, and $\sigma(V(F')) = V(F')$, $\sigma(V(G')) = V(G')$. Hence, $\sigma|_{F'}$ and $\sigma|_{G'}$ are automorphisms of $F'$ and $G'$, respectively. Again by the minimality of $|V(G)|$, $\sigma|_{F'}$ and $\sigma|_{G'}$ are realizable by path flips. Since $\sigma$ is a composition of $\sigma|_{F}$, $\sigma^{-1}|_{F'}$ and $\sigma|_{G'}$, $\sigma$ is realizable by path flips.

Case 2.2. $V(F) = V(G)$.

For $0 \leq k \leq m - 1$, let us define $W_k = \cup_{0 \leq i \leq n-1} \cup_{i \equiv k \ (\text{mod } m)} \sigma^i(V(Q))$. Note that $W_k$'s are not necessarily disjoint to each other. Let us define a configuration $\tau \in C(F)$ such that $\tau(v) = \sigma(v)$ for $v \in V(H) \setminus (X'_{m-1} \cup W_{m-1})$ and $\tau(v) = \sigma^{1-m}(v)$ for $v \in X'_{m-1} \cup W_{m-1}$. We need to check that this definition is well-defined. Suppose that there exists a vertex $v \in V(F)$ such that $v \in (X'_{m-1} \cup W_{m-1}) \cap (X'_{k} \cup W_{k})$ for some $k$ with $0 \leq k \leq m - 2$. Then there exists an integer $i$ with $i \not\equiv 0 \ (\text{mod } m)$ such that $\sigma^i(v) = v$. Since $m$ is a prime power, we have $|C(\sigma) \cdot v| \leq \gcd(i, n) < m$. Hence, we have $\sigma^m(v) = v$, which implies $\sigma(v) = \sigma^{1-m}(v)$.

For $0 \leq k \leq m - 1$, let $F_k = F[X_k \cup W_k]$. We claim that $\tau$ is an automorphism of $F$, because $\sigma|_{F_k}$ is an isomorphism from $F_k$ to $F_{k+1}$ and $\sigma^{1-m}|_{F_{m-1}}$ is an isomorphism from $F_{m-1}$ to $F_0$. Furthermore, by definition, the order of $\tau$ is $m$.

Case 2.2.1. $m < n$.

In this case, by the minimality of $n$, $\tau$ is realizable by path flips. Furthermore, since $\sigma^m|_{F_0}$ is an automorphism of $F_0$ and the order of $\sigma^m|_{F_0} = n/m < n$, by the minimality of $n$, $\sigma^m|_{F_0}$ is realizable by path flips. Since $\sigma$ is a composition of $\tau$ and $\sigma^m|_{F_0}$, $\sigma$ is realizable by path flips.

Case 2.2.2. $m = n$.

In this case, $H$ is a cycle of order $dn$. Put $r = dn$. We relabel the vertices of $F$ as follows: let us label $V(Q)$ as $Q = w_0w_1 \ldots w_s$, where $w_0 = z$ and $w_s = y$. For $0 \leq i \leq n - 1$ and $0 \leq j \leq s$, let $w_{i,j} = \sigma^i(w_j)$. Note that $w_{i,j}$
may coincide with \( w_{k,j} \) for some \( i \neq k \). We also write \( Q_i = w_{i,0}w_{i,1} \ldots w_{i,s} \) for \( 0 \leq i \leq n-1 \). Let us label the vertices of \( H \), which is a cycle of length \( r \), as \( H = z_0z_1 \ldots z_r \), where \( z_r = z_0 \) and \( z_{id} = w_{i,s} \) for \( 0 \leq i \leq n-1 \).

For a positive integer \( N \) and for an integer \( t \), let us define permutations \( \pi(N, t) \) on \( \{0, 1, \ldots, N-1\} \) such that \( \pi(N, t)(i) \equiv t - i \pmod{N} \) for \( 0 \leq i \leq N-1 \). For an integer \( t \), let us define a bijection \( \rho_t \) on \( V(F) \) satisfying \( \rho_t^2 = 1 \), as follows:

\[ \rho_t(w_{i,j}) = w_{\pi(n,t)(i),j} \quad \text{for} \quad 0 \leq i \leq n-1 \quad \text{and} \quad 0 \leq j \leq s, \quad \text{and} \quad \rho_t(z_i) = z_{\pi(r,dt)(i)} \quad \text{for} \quad 0 \leq i \leq r-1. \]

We need to check that this definition is well-defined. If \( w_{i,j} = w_{k,j} \) with some \( i, k, j \) with \( i \neq k \), we have \( \sigma^k(w_j) = \sigma^j(w_j) \). For any integer \( t \), Since \( \pi(n, t)(k) \equiv i - k \pmod{n} \), we have \( w_{\pi(n, t)(i),j} = \sigma^\pi(n, t)(i)(w_j) = \sigma^\pi(n, t)(k)(w_j) = w_{\pi(n, t)(k),j} \), as claimed.

Let \( H' = F' \mid \{0 \leq i \leq n-1 \}Q_i \). Since \( \rho_t \mid V(H') \) is a permutation of \( Q_i \) for \( 0 \leq i \leq n-1 \) and \( \rho_t \mid V(H) \) is a reflection of \( H \), \( \rho_t \) is an automorphism of \( F \).

**Claim 4.** \( \rho_t \) is realizable by path flips on \( F \).

For all \( 0 \leq i \leq n-1 \) with \( i \leq \pi(n, t)(i) \), let us choose a shortest path \( R_i \) from \( w_{i,0} \) to \( \rho_t(w_{i,0}) = w_{\pi(n, t)(i),0} \). By consecutive path flips of \( R_i \) and \( R_i - \{w_{i,0}, \rho_t(w_{i,0})\} \), we can exchange \( w_{i,0} \) and \( \rho_t(w_{i,0}) \) for \( 0 \leq i \leq n-1 \). In the remaining graph \( F' = F - C(\sigma) \cdot w_0, \rho_t \mid F' \) is realizable by path flips by the minimality of \( |V(G)| \), as claimed.

Since \( \sigma \) is a composition of \( \rho_1 \) and \( \rho_0 \), by Claim 4, it is realizable by path flips.

\[ \square \]

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