Observables in terms of connection and curvature variables for Einstein’s equations with two commuting Killing vectors

P Kordas
35 Square Marie-Louise
1000 Bruxelles
Belgium
e-mail: panayiotis.kordas@physics.org

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Abstract
Einstein’s equations with two commuting Killing vectors and the associated Lax pair are considered. The equations for the connection $A(\zeta, \eta, \gamma) = \Psi, \gamma \Psi^{-1}$, where $\gamma$ the variable spectral parameter are considered. A transition matrix $T = A(\zeta, \eta, \gamma)A^{-1}(\xi, \eta, \gamma)$ for $A$ is defined relating $A$ at ingoing and outgoing light cones. It is shown that it satisfies equations familiar from integrable pde’s theory. A transition matrix on $\zeta = constant$ is defined in an analogous manner.

These transition matrices allow us to obtain a hierarchy of integrals of motion with respect to time, purely in terms of the trace of a function of the connections $g, g^{-1}$ and $g, g^{-1}$. Furthermore a hierarchy of integrals of motion in terms of the curvature variable $B = A, A^{-1}$, involving the commutator $[A(1), A(-1)]$, is obtained.

We interpret the inhomogeneous wave equation that governs $\sigma = \ln N$, $N$ the lapse, as a Klein-Gordon equation, a dispersion relation relating energy and momentum density, based on the first connection observable this first observable corresponds to mass. The corresponding quantum operators are $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial z}$ and this means that the full Poincare group is at our disposal.

Keywords: General Relativity, Integrable Systems, Gravitation, Quantum Field Theory, Klein- Gordon equation

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1 Introduction

The general relativistic equations in the presence of two commuting Killing vectors have received a great amount of interest over the years. The first important contribution, relevant to the considerations here, was the proof in [1] that there exists an infinite hierarchy of solutions which can be mapped to one another via transformations of what we now call the Geroch group. Later it was shown [2, 3, 4, 5, 6, 7] by different authors and in differing approaches, that the field equations (2) are integrable (in the sense understood in the inverse scattering field) and a variety of solutions was obtained and analyzed. These results are presented and reviewed in [8]. Relatively more recently techniques of quantum inverse scattering were used [9] to quantize appropriate functions of the gravitational field appearing in (2) with the emphasis on the metric. The main motivation of the present paper is to present an approach for future work in terms of connections which are appropriate variables for Gravity.

The organization of the present work is as follows. In section (2) we introduce the equations and briefly present the solitonic technique. Further in section (3) $A(\gamma)$ is defined. Transition matrices for $A$ relating it at points on ingoing and outgoing null coordinates is defined, and it is proved that it obeys equations similar to the ones of other integrable pde’s, a result absent from the literature thus far, for connection variables. This presents the possibility of obtaining integrals of motion, from the trace of the derivative of appropriate combinations of the two transition matrices, in terms of classical connections of General Relativity. These transition matrices first appeared in [10], where it is also noted that the possibility of observables in terms of connections can be attained. These may be appropriate for quantization as they are the fundamental variables of the classical theory and connections are fundamental also in Quantum Canonical General Relativity [11, 12]. It is shown that $Tr(A^n(1)A^{-n}(-1))$ is an integral of motion. In section (4) we consider the curvature $B(\gamma) = A, A^{-1}$ and obtain observables in a similar fashion as for the connection, but where now $B(\pm 1)$ involve the commutator $[A(1), A(-1)]$. We discuss these observables in Section (5), interpreting the first one of the connection observables as mass due to the fact that it appears, as the inhomogeneous term, in the wave equation governing the lapse $N$ which is interpreted as a Klein-Gordon equation. This puts the associated Quantum Field Theory in our hands for quantization.

2 The Einstein equations with two commuting Killing vectors

The metric in the presence of two commuting Killing vectors, and assuming the existence of 2-surfaces orthogonal to the group orbits, is given by [13, 2, 8]:

$$ds^2 = f(t, z)(dz^2 - dt^2) + g_{ab}(t, z)dx^a dx^b,$$

where $g_{ab}(t, z)$ real and symmetric tensor, $a, b = 1, 2$. 


Einstein’s equations corresponding to this metric are, (in null coordinates $\varsigma = \frac{1}{2}(z + t), \eta = \frac{1}{2}(z - t)$)

\[
(\alpha g, g^{-1})_{,\varsigma} + (\alpha g, g^{-1})_{,\eta} = 0,
\]

(2)

\[
(ln f)_{,\varsigma}(ln \alpha)_{,\varsigma} = (ln \alpha)_{,\varsigma^2} + \frac{1}{4\alpha^2} Tr A^2(1),
\]

(3)

\[
(ln f)_{,\eta}(ln \alpha)_{,\eta} = (ln \alpha)_{,\eta^2} + \frac{1}{4\alpha^2} Tr A^2(-1),
\]

(4)

where

\[
A(1) = \alpha g, g^{-1}, \quad A(-1) = \alpha g, g^{-1},
\]

(5)

det $g = \alpha^2$ and the reason for the labelling $A(\pm 1)$ will become apparent in the following. Further from the above equations (taking the trace of (2)) it follows that $\alpha$ satisfies

\[
\alpha_{,\varsigma \eta} = 0.
\]

(6)

Belinsky-Zakharov have shown that equation (2) is integrable [2, 8, 14]. This means the existence of a linear system of differential equations (which have as consistency relation the equation (2))

\[
\frac{d\Psi}{d\varsigma} = \frac{A(1)}{\alpha(1 - \gamma)} \Psi, \quad \frac{d\Psi}{d\eta} = \frac{A(-1)}{\alpha(1 + \gamma)} \Psi,
\]

(7)

where we use the notation of [9] because the symmetry $\varsigma \leftrightarrow \eta$ of (2) becomes more evident in the system (7, 8, 9). Reality is ensured via $\Psi^*(\gamma^*) = \Psi(\gamma)$. The system (7) has as compatibility conditions equation (2) and the zero-curvature condition [15] associated with this integrable system [8, p. 15, eq. 1.48]. The differentials are given by

\[
\frac{d}{d\varsigma} = \frac{\partial}{\partial \varsigma} + \gamma_{,\varsigma} \frac{\partial}{\partial \gamma}, \quad \frac{d}{d\eta} = \frac{\partial}{\partial \eta} + \gamma_{,\eta} \frac{\partial}{\partial \gamma},
\]

(8)

where $\alpha = a(\varsigma) - b(\eta)$ is a solution of (3) and $\gamma$ is a 'variable' spectral parameter satisfying

\[
\gamma_{,\varsigma} = \frac{\alpha_{,\varsigma}}{\alpha} \frac{\gamma(1 + \gamma)}{(1 - \gamma)}, \quad \gamma_{,\eta} = \frac{\alpha_{,\eta}}{\alpha} \frac{\gamma(1 - \gamma)}{(1 + \gamma)},
\]

(9)

which are equivalent to

\[
\gamma + \frac{1}{\gamma} = 2 \left( \frac{w - \beta}{\alpha} \right)
\]

(10)

and are solved by

\[
\gamma_{\pm}(w; \varsigma, \eta) = \frac{1}{\alpha} \left\{ w - \beta \pm \sqrt{(w - \beta)^2 - \alpha^2} \right\} = 1/\gamma_{\mp},
\]

(11)

where $w$ complex constant and $\beta = a(\varsigma) + b(\eta)$ is a second solution of (6). The choice $\alpha = t$, $\beta = z$ corresponds to the cosmological case while $\alpha = \rho$, $\beta = t$ corresponds to cylindrically symmetric gravitational waves.
Solutions of equations (7) can be reproduced from a known background solution according to the B-Z dressing procedure [2, 14, 8]. One starts with $\Psi_0$, a solution of (7) (corresponding to a background metric $g_0$) and form

$$\Psi = \chi(\varsigma, \eta, \gamma) \Psi_0,$$

where

$$\chi = I + \sum_{k=1}^{N} \frac{R_k(\varsigma, \eta)}{\gamma - \gamma_k}.$$  \hspace{1cm} (13)

where $I$ unit matrix, and the poles $\gamma_k$, which have to be solutions of (10), i.e. are given by (11) for $w = w_k$, correspond to solitons in the sense understood in the inverse scattering literature. Reality is ensured via $\chi^*(\gamma^*) = \chi(\gamma)$.

It is important that the variables $R_k$ have no $\gamma$ dependence and are only functions of $\varsigma$ and $\eta$. The poles $\gamma_k$ can be interpreted as the null trajectories of perturbations propagating on the background solution and can be thought of as gravitational solitons [8] although they are not fully analogous to solitons as they are known in other integrable pde’s [16, 17, 18]. The poles come in pairs either real $(\gamma_k^+, \gamma_k^-)$ or $(\gamma_k, \gamma_k^*)$ in order to ensure reality [8, 9].

It is ensured that $g$ is symmetric via

$$g^{-1}(\gamma) = (\chi^{-1}(1/\gamma))^T g_0^{-1}$$  \hspace{1cm} (14)

3 The connection $A$, the Transition Matrix and observables

Following [9] we consider the Lie algebra-valued connection or logarithmic derivative of $\Psi A(\gamma)$ given by

$$A(\gamma) = \Psi(\gamma)\Psi^{-1}.$$  \hspace{1cm} (15)

From (8, 15, 7) at $\gamma = \pm 1$ we see how the definitions (5) arise. Now we consider the pde’s for $A(\gamma)$. Differentiating the r.h.s. of (15) w.r.t. $\eta$, $\varsigma$ and using the Lax pair (7) we obtain

$$A_{,\varsigma}(\gamma) = [A_+, A(\gamma)] + A_{+,\gamma},$$ \hspace{1cm} (16)

$$A_{,\eta}(\gamma) = [A_-, A(\gamma)] + A_{-,\gamma},$$ \hspace{1cm} (17)

where

$$A_+ = \frac{A(1)}{\alpha(1 - \gamma)} - \gamma_\varsigma A(\gamma) (= \Psi_{,\varsigma}\Psi^{-1}), \hspace{0.5cm} A_- = \frac{A(-1)}{\alpha(1 + \gamma)} - \gamma_\eta A(\gamma) (= \Psi_{,\eta}\Psi^{-1})$$  \hspace{1cm} (18)

where we see that the equations 'decouple' for the two variables $\varsigma$ and $\eta$ a fact first observed in [19]. These equations have appeared in Chapter 6 of [20]. $A_\pm$ satisfy the zero curvature condition

$$A_{+,\eta}(\gamma) - A_{-,\varsigma}(\gamma) = [A_-(\gamma), A_+(\gamma)]$$  \hspace{1cm} (19)
Also it can be shown \[2\] that \( A(\gamma) \) satisfies

\[
\Psi^T \left( \frac{1}{\gamma} \right) \varsigma, \eta \right) g^{-1}(\varsigma, \eta) \Psi(\gamma, \varsigma, \eta) = g^{-1}(\varsigma, \eta),
\]

which upon differentiation gives \[9, 20\],

\[
\gamma^2 A(\gamma) g = g A^T(1/\gamma).
\]

Also it can be seen from \(5\), \( A(\pm 1) \) satisfies

\[
A(\varsigma, \eta, \gamma = 1) = -A(\eta, \varsigma, \gamma = -1),
\]

since \( \alpha = \varsigma - \eta \). This relation appears in \(9\) as a reality condition for \( A(\pm 1) \) since for stationary axisymmetric systems \( \alpha = ip \). Now observing \(16\), \(17\) and \(19\) we see that the transformation \( \tau : (\varsigma, \eta, \gamma) \rightarrow (\eta, \varsigma, -\gamma) \) is an ‘involution’ of the differential equations sending essentially the one to the other ie

\[
\tau : \left( \frac{d}{d\varsigma}, \frac{d}{d\eta}, \gamma \right) \rightarrow \left( \frac{d}{d\eta}, \frac{d}{d\varsigma}, \gamma' = -\gamma \right)
\]

It may be further noticed that the transformation

\[
\tau : (\varsigma, \eta, \gamma) \rightarrow (\eta, \varsigma, -\gamma)
\]

which is equivalent, for the particular choice of \( \alpha = t \), to \( (\alpha, \beta, \gamma) \rightarrow (-\alpha, \beta, -\gamma) \) is an involution of \( A(\gamma) \) (from \(15\) and \(23\)), that is it satisfies

\[
\tau : A(\gamma) \rightarrow -A(-\gamma)
\]

Having noticed \(22\) we define the transition matrix \( \mathcal{T}(\xi, \varsigma, \gamma) \)

\[
\mathcal{T}(\varsigma, \eta, \xi, \gamma) \equiv -A(\varsigma, \eta, \gamma)A^{-1}(\eta, \xi, -\gamma) = A(\varsigma, \eta, \gamma)A^{-1}(\xi, \eta, \gamma)
\]

Considering \( \mathcal{T}_\varsigma \) we obtain

\[
\mathcal{T}(\xi, \varsigma, \gamma) = \mathcal{P} e^{\int_{\xi'} U(\varsigma', \eta, \gamma)d\varsigma'}
\]

\[
U = A_{\varsigma'}(\varsigma', \eta, \gamma)A^{-1}(\varsigma', \eta, \gamma)
\]

\[
\eta = w_2 = \text{constant}
\]

where \( \mathcal{P} \) denotes path ordered exponential. Further considering \( \mathcal{T}_{\eta} \) we obtain

\[
\mathcal{T}(\xi, \varsigma, \gamma) = V(\xi, \gamma)\mathcal{T}(\xi, \varsigma, \gamma) - \mathcal{T}(\xi, \varsigma, \gamma)V(\varsigma, \gamma)
\]

\[
V(\varsigma, \eta, \gamma) = A_{\gamma}(\varsigma, \eta, \gamma)A^{-1}(\varsigma, \eta, \gamma)
\]

that is we have obtained a transition matrix analogous to the one that is very common in integrable pde’s \[15, 21\] with the null coordinate \( \eta \) playing the role of time. It should be mentioned that such a matrix was lacking for the equations of gravity in the presence of two commuting Killing vectors in the connection \( A \)
formulation. Of course the roles of \( \eta \) and \( \varsigma \) can be reversed with the definition

\[
\mathcal{T} = A(\varsigma, \eta, \gamma)A^{-1}(\zeta, \vartheta, \gamma)
\]

which gives,

\[
\mathcal{T}'(\eta, \vartheta, \gamma) = \mathcal{P} e^{\int_\eta^\vartheta V(\varsigma, \eta, \gamma) d\eta'}
\]

\[
V = A_{\eta}^{-1}
\]

\[
\zeta = w_2 = constant
\]

The transition matrix is extensively used in integrable PDE’s and in their quantization \[15\] \[21\] \[22\]. It should be stressed that the poles of \( \chi \) and hence \( A(\gamma) \) correspond essentially to the null trajectories of the solitons and are the light cones \( w_k - \beta = \pm \alpha \) \[9\].

It is clear that for \( \eta = w_2, \gamma = -1 \) from \[11\] with \( w = w_2 \) and \( \gamma = 1 \) for \( \varsigma = w_2 \). This holds in general for appropriate expression of \( \gamma \) for \( \eta, \varsigma \) constant.

In Fig. 1 we see the path, on which the transition matrix carries \( A \). We define \( O \)

\[
O = \mathcal{T}'(\eta, \vartheta, \varsigma = w_2, \gamma = 1) \mathcal{T}_B(\varsigma, \eta, -1, 1) \mathcal{T}(\xi, \varsigma, \eta = w_2, \gamma = -1)
\]

where

\[
\mathcal{T}_B(\gamma, \gamma') = A(\varsigma, \eta, \gamma)A^{-1}(\varsigma, \eta, \gamma') = \mathcal{P} e^{\int_\gamma^\gamma' A_{\gamma} A^{-1} d\gamma'}
\]

is included to ensure continuity of \( \gamma \) in \( A \) along the path from \( (\xi, \eta) \rightarrow (\varsigma, \eta) \rightarrow (\zeta, \vartheta) \) in Fig. 1. In \( (36) \) the path can be \( \gamma = e^{i\phi} \in [-\pi, 0] \) which corresponds, from \( (11) \) to the branch cut \( w \in [\beta - \alpha, \beta + \alpha] \). (It should be stressed that at no point do we attain the singularity \( \alpha = 0 \) which corresponds to \( \eta = \varsigma \) and from \( (11) \) to either \( \gamma = 0 \) or \( \gamma = \infty \). With \( w = w_2, \alpha \cos \phi = w_2 - \beta \). Now we start from \( O^2 \)

\[
O^2 = \mathcal{T}' \mathcal{T}_B \mathcal{T}' \mathcal{T}_B \mathcal{T}
\]

Now

\[
\mathcal{T}' \mathcal{T} = \mathcal{P} e^{\int_\eta^{\gamma'} A_{\eta'}^{-1}(\varsigma, \eta', 1) A^{-1}(\varsigma, \eta, 1) d\eta'} \mathcal{P} e^{\int_{\gamma'}^\zeta A_{\gamma'}^{-1}(\varsigma, \eta, 1) A^{-1}(\varsigma', \eta, -1) d\varsigma'}
\]

\[
= \mathcal{P} e^{\int_\eta^{\gamma'} A_{\eta'}^{-1}(\varsigma, \eta, 1) A^{-1}(\varsigma, \eta, 1) d\eta'} \mathcal{P} e^{\int_{\gamma'}^\zeta A_{\gamma'}^{-1}(\varsigma, \eta, 1) A^{-1}(\varsigma', \eta, -1) d\varsigma'} = I
\]

where we have used \[22\], the particular path of Fig. 1 which implies \( \xi = \vartheta \) and \( \eta = \varsigma \) in the boundaries of the above path integrals and the basic feature of path integrals that \( (\mathcal{P} e^{\int y ...})^{-1} = \mathcal{P} e^{\int y ...} \).

Also, in a similar way

\[
\mathcal{T}_B \mathcal{T}_B = \mathcal{P} e^{\int_{\gamma = -1}^{\gamma'} A_{\gamma} A^{-1}(\varsigma, \eta, \gamma) d\gamma} \mathcal{P} e^{\int_{\gamma = -1}^{\gamma'} A_{\gamma} A^{-1}(\varsigma, \eta, -1) d\gamma} = A(\varsigma, \eta, 1) A^{-1}(\varsigma, \eta, 1) A(\varsigma, \eta, -1) A^{-1}(\varsigma, \eta, -1) = I
\]

where we have used \( (22), (23) \) and \( \gamma(\varsigma, \eta, w) = -\gamma(\eta, \varsigma, w) \) for the case \( \alpha = t \).

Since \( O^2 = I = \mathcal{P} e^{\int y ...} \), implies taking the square root of both sides \( O = \pm I \).
Now we consider $Tr\mathcal{O}$, using (38)

$$\frac{\partial}{\partial t} Tr T_B = \left( \frac{\partial}{\partial \varsigma} - \frac{\partial}{\partial \eta} \right) Tr T_B = 0 \tag{40}$$

which implies by actually performing the integration

$$\frac{\partial}{\partial t} Tr (A(\varsigma, \eta, 1)A^{-1}(\varsigma, \eta, -1)) = 0. \tag{41}$$

Further it is clear that $\frac{\partial}{\partial t} Tr (\mathcal{O}^n) = 0$ since $\mathcal{O}^n = I = \mathcal{P}e_n \mathcal{P}$ which implies $\frac{\partial}{\partial t} Tr (T_B^n) = 0$ and hence

$$\frac{\partial}{\partial t} Tr \left[ (A(\varsigma, \eta, 1))^n (A^{-1}(\varsigma, \eta, -1))^n \right] = 0 \tag{42}$$

Fig.1 the path $(\xi, \eta) \rightarrow (\varsigma, \eta) \rightarrow (\varsigma, \vartheta)$

In the case $\alpha = \rho, \beta = t$ which corresponds to cylindrically symmetric gravitational waves, i.e. $\alpha$ spacelike, we have

$$\gamma_\pm = \frac{1}{\rho} \left( w - t \pm \sqrt{(w-t)^2 - \rho^2} \right) \tag{43}$$

with $\varsigma = \frac{1}{2}(\rho + t), \eta = \frac{1}{2}(\rho - t)$ In this case the necessary involution is $(\varsigma \leftrightarrow \eta)$ (which corresponds to $t \rightarrow -t$) along with $w \rightarrow -w$ which corresponds to $\gamma_+(\rho, t, w) \rightarrow \gamma_+(\rho, -t, -w) = -\gamma_+(\rho, t, w) = -1/\gamma_+(\rho, t, w)$. It is again evident from (8) that the involution $\tau'$ defined by

$$\tau' : (\varsigma, \eta, w) \rightarrow (\eta, \varsigma, -w) \tag{44}$$
has the effect $\tau' : A(\varsigma, \eta, \gamma) \rightarrow \gamma^2 A(\eta, \varsigma, -\gamma_-)$ and $A(\gamma_-)$ satisfies the same differential equation as $A(\gamma_+)$ (because $\gamma_{+}^{\pm}$ are the two solutions of $(10)$). Hence $\tau'$ is an involution of the linear system $(7)$

$$\tau' : \left( \frac{d}{d\varsigma}, \frac{d}{d\eta}, \frac{d}{d\gamma} \right) \rightarrow \left( \frac{d}{d\eta}, \frac{d}{d\varsigma}, \gamma' = -\gamma_- \right)$$

Hence transition matrices can be defined for the case $\alpha = \rho$ (i.e. $\alpha$ spacelike) as

$$\begin{align*}
T &= A(\xi, \eta, \gamma_+) A^{-1}(\eta, \varsigma, -\gamma_-) = A(\xi, \eta, \gamma_+) A^{-1}(\varsigma, \eta, \gamma_+) \quad (46) \\
T' &= A(\varsigma, \eta, \gamma_+) A^{-1}(\vartheta, \varsigma, -\gamma_-) = A(\varsigma, \eta, \gamma_+) A^{-1}(\varsigma, \vartheta, \gamma_+) \quad (47)
\end{align*}$$

Hence we have obtained transition matrices for $\alpha$ spacelike in a similar way to the timelike case above. (In this and the next chapter all commutators are matrix commutators.)

The metric $(1)$ corresponds to a wide variety of solutions including cosmological, cylindrically symmetric gravitational waves, and stationary axisymmetric space-times (with appropriate transcription of the coordinates). The solitonic ansatz, among other methods, may be employed to obtain solutions (from a diagonal background usually but not exclusively) from a seed solution including Schwarzschild and Kerr among many others. Although the considerations here involve mainly space-times with one of the two significant coordinates timelike, the observables may be relevant for the axistationary case, e.g. inside a Black Hole horizon where one of the two significant coordinates becomes timelike.

A generic diagonal seed spacetime for the solitonic technique [8] is

$$\begin{align*}
(g_0)_{11} &= \alpha e^\beta \\
(g_0)_{22} &= \alpha e^{-\beta}
\end{align*}$$

The corresponding solution of $(11)$ is given by

$$\begin{align*}
(\Psi_0)_{11} &= \alpha (\gamma^2 + 2 \frac{\beta}{\alpha} \gamma + 1)^{1/2} e^{\frac{\gamma}{2} \alpha \gamma + \beta} \\
(\Psi_0)_{22} &= \alpha (\gamma^2 + 2 \frac{\beta}{\alpha} \gamma + 1)^{1/2} e^{-\frac{\gamma}{2} \alpha \gamma - \beta}
\end{align*}$$

It is straightforward to check that (e.g. with $\alpha = \varsigma - \eta$, $\beta = \varsigma + \eta$), $(\frac{\partial}{\partial \gamma} - \frac{\partial}{\partial \eta} \text{Tr}(g_0, g_0^{-1} g_0, g_0^{-1})) = 0$. It is well-known [23] that the equations $(2)$ are also valid for $d$-dimensional general relativity in the presence of $d - 2$ Killing vectors. The considerations here are valid in that case also and the observables are available to use in string-theoretic considerations.

4 The curvature $B$ and observables

We now consider

$$B = A, \gamma A^{-1}$$

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8
appearing in (36). Taking the derivative of $B$ with respect to $\varsigma$, $\eta$ turning the partial derivatives on the rhs of (52) and using (16, 17) one obtains

$$B_\varsigma = [B_+,B] + B_{+,\gamma}$$
$$B_{\eta} = [B_-,B] + B_{-,\gamma}$$

where

$$B_+ = ([A_+,A] + A_{+,\gamma}) A^{-1}$$
$$B_- = ([A_-,A] + A_{-,\gamma}) A^{-1}$$

From (16, 17) we see that

$$B_+(\varsigma, \eta, -1 + \delta \gamma) = A_\varsigma(\varsigma, \eta, -1 + \delta \gamma)A^{-1}(\varsigma, \eta, -1 + \delta \gamma)$$

$$= \left( \begin{array}{c} A(1) \\ \alpha(1-\gamma) \end{array} \right) A^{-1}(1 + \delta \gamma) + \frac{A(1)A^{-1}(1 + \delta \gamma)}{\alpha(1-\gamma)^2} - (\gamma, \varsigma, \gamma I - A_{,\gamma}A^{-1}(1 + \delta \gamma)A^{-1}(1 + \delta \gamma))$$

Also

$$B_-(\varsigma, \eta, 1 + \delta \gamma) = A_{,\eta}(1 + \delta \gamma)A^{-1}(1 + \delta \gamma) =$$

$$\left[ \begin{array}{c} A(-1) \\ 2\alpha \end{array} \right] A^+(1 + \delta \gamma) A^{-1}(1 + \delta \gamma) + \frac{A(-1)A^{-1}(1 + \delta \gamma)}{4\alpha} - \frac{I}{2\alpha}$$

$$+ \frac{A^{-1}A_{,\gamma}(1 + \delta \gamma)A^{-1}(1 + \delta \gamma)}{2\alpha^3}$$

We form,

$$\frac{A_{,\eta}(1 + \delta \gamma)A^{-1}(1 + \delta \gamma)}{2\alpha} - \frac{A_{,\eta}(1)A^{-1}(1)}{2\alpha} - 2\alpha \left( \frac{A_{,\eta}(\gamma)A^{-1}(\gamma)}{\gamma} \right)_{\gamma=1} = 0$$

and hence, using (17, 16, 58, 57)

$$B(1) = \left[ \begin{array}{c} A(1), A(-1) \end{array} \right] A^{-1}(1) + \frac{1}{2} A(-1)A^{-1}(1) - I$$

and,

$$B(-1) = \left[ \begin{array}{c} A(-1), A(1) \end{array} \right] A^{-1}(-1) - \frac{1}{2} A(1)A^{-1}(-1) + I$$

Further we want to calculate the limit $B_+(\gamma \to 1)$ and $B_-(\gamma \to -1)$. From (55, 56) in the limit $\gamma \to 1$ we get

$$A_\varsigma(\varsigma, \eta, 1)A^{-1}(1) = B_+(\varsigma, \eta, \gamma \to 1) =$$
\[-A(1), \frac{B(1)}{\alpha} + \frac{\langle B, \gamma \rangle_{\gamma=1}}{\alpha} \quad (62)\]
\[A_n(\zeta, \eta, -1)A^{-1}(-1) = B_-(\zeta, \eta, \gamma \rightarrow -1) = \]
\[\left[ A(-1), \frac{B(-1)}{\alpha} \right] + \frac{\langle B, \gamma \rangle_{\gamma=-1}}{\alpha} \quad (64)\]

where we have used
\[\gamma, \varsigma \mid_{(1+\delta \gamma)} \approx -\frac{2}{\alpha \delta \gamma} \approx -\gamma, \eta \mid_{(-1+\delta \gamma)} \quad (65)\]
\[(\gamma, \varsigma) \mid_{(1+\delta \gamma)} \approx \frac{2}{\alpha \delta \gamma^2} \approx -(\gamma, \eta) \mid_{(-1+\delta \gamma)} \quad (66)\]
\[\frac{\delta A^{-1}}{\delta \gamma} \frac{\delta (A^{-1})^{-1}}{\delta \gamma} A^{-1} = -I \quad (67)\]

We see from (25, 52) (in the case \(\alpha = \tau\) i.e. \(\alpha\) timelike)
\[\tau : B_+(\varsigma, \eta, \gamma) \rightarrow B_-(\eta, \varsigma, -\gamma) \quad (68)\]
\[\tau : B(\varsigma, \eta, \gamma) \rightarrow -B(\eta, \varsigma, -\gamma) \quad (69)\]

The transformation \(\tau\) is again an involution of the equations (53, 54) so like in section 3 we can define transition matrices for \(B\). The process of obtaining observables this way must involve the derivatives of \(A_+, B_\pm\) and commutators thereof, always modulo the integrable systems zero curvature condition, the field equations (2) and Bianchi identity.

So we have (for \(\alpha = t\) timelike)
\[T = -B(\varsigma, \eta, \gamma)B^{-1}(\eta, \varsigma, -\gamma) = \mathcal{P} \mathcal{E} \int d\varsigma \int d\eta B(\varsigma, \eta, \gamma)B^{-1}(\varsigma, \eta, -\gamma) \quad (70)\]
\[T' = -B(\varsigma, \theta, \gamma)B^{-1}(\eta, \varsigma, -\gamma) = \mathcal{P} \mathcal{E} \int d\varsigma \int d\eta B(\varsigma, \eta, \gamma)B^{-1}(\varsigma, \eta, -\gamma) \quad (71)\]
\[T_C(\gamma, \gamma') = B(\varsigma, \eta, \gamma)B^{-1}(\varsigma, \eta, -\gamma) = \mathcal{P} \mathcal{E} \int d\gamma B(\varsigma, \eta, \gamma)B^{-1}(\varsigma, \eta, -\gamma) \quad (72)\]

and the observable
\[\text{Tr} OB = \text{Tr} \left( T' T_C T \right) \quad (73)\]

which gives
\[\left( \frac{\partial}{\partial \varsigma} - \frac{\partial}{\partial \eta} \right) \text{Tr} \left( B^n(1)B^n(-1) \right) \quad (74)\]

since
\[O_B B(\xi, \eta, \gamma) = B(\xi, \eta, -\gamma) \quad (75)\]

and with the boundary condition (69) on the path of Fig. 1 we get (74).

We may consider obtaining observables directly from \(\text{Tr} \mathcal{O}\). Indeed
\[\left( \frac{\partial}{\partial \varsigma} - \frac{\partial}{\partial \eta} \right) \text{Tr} \mathcal{O} \]
\[
\begin{align*}
\frac{\partial}{\partial \kappa} - \frac{\partial}{\partial \eta} \text{ Tr} \left( T' T \right) \\
= \left( \frac{\partial}{\partial \kappa} - \frac{\partial}{\partial \eta} \right) \left( P e^{-\int_{\eta}^\phi B_{-}(\xi, \eta', 1) d\eta'} P e^{-\int_{\phi}^\xi B_{+}(\xi', \eta, -1) d\xi'} \right)
\end{align*}
\]
Taking the derivative in (76), and using (38), we have
\[
\text{Tr}\mathcal{O} = \\
\text{Tr} \left( \int_0^\phi B_{-}(\kappa, \eta', 1) d\eta' \right) T_B - T_B \left( \int_\xi^\kappa B_{+}(\kappa', \eta, -1) d\kappa' \right) + \\
T_B B_{+}(\kappa, \eta, -1) - B_{-}(\kappa, \eta, +1) T_B + T_B - T_B = 0
\]
So we have obtained in (77) and a relation giving observables in terms of the curvature variables \( B_{\pm}(\pm 1) B(\gamma) \) in what is a form of generalised zero curvature condition. It is also clear that there exists a countable infinite hierarchy of hierarchies of constants of motion built from \( A(\gamma), B(\gamma), ... \)

5 Field equations and interpretation of the connection observable

First consider (41). Using the fact that the inverse of a \( 2 \times 2 \) matrix \( M \) can be written as
\[
M^{-1} = \frac{\text{Tr} M I - M}{\det M}
\]
(80) may be written as
\[
\text{Tr}(A(1)A^{-1}(-1)) = \frac{\text{Tr}A(1)\text{Tr}A(-1) - \text{Tr}(A(1)A(-1))}{\det A(-1)}
\]
Notice immediately that from (80) the alpha dependence cancels. So the observables obtained here are well behaved, in the first instance, on the axis \( \alpha = 0 \), which for \( \alpha \) timelike is the cosmological singularity and in the cylindrically symmetric case is the axis of symmetry. Further we may traverse, in Fig. 3, the path in the opposite direction \( (\kappa, \theta) \to (\kappa, \eta) \to (\xi, \eta) \), in which case everything in Sec. 3 may be repeated to obtain the observable \( \text{Tr} (A(-1)A^{-1}(1)) \) This implies that
\[
\det (A(1)A^{-1}(-1))
\]
is a constant of motion. This suggests that the first essential constant of motion is \( \text{Tr}(A(1)A(-1)) \). Indeed if we consider
\[
[\partial_\kappa - \partial_\eta] \text{ Tr} \left( g_{\kappa \kappa} g^{-1} g_{\eta \eta} g^{-1} \right) = F(\kappa, \eta) - F(\eta, \kappa) = E(\kappa, \eta) \text{ where,}
\]
\[
F(\kappa, \eta) = \text{Tr} \left( g_{\kappa \kappa} g^{-1} g_{\eta \eta} g^{-1} + g_{\eta \eta} g^{-1} g_{\eta \eta} g^{-1} - 2g_{\kappa \kappa} g^{-1} g_{\kappa \kappa} g^{-1} g_{\eta \eta} g^{-1} \right)
\]
and we have used (22). Upon \( \varsigma \leftrightarrow \eta \) (which corresponds to \( t \to -t \)) \( E(\varsigma, \eta) = -E(\varsigma, \eta) \) hence \( E(\varsigma, \eta) = 0 \) and so \( Tr(A(1)A(-1)) \) is a constant of motion. There has been indication [24] that \( Tr(A(1)A(-1)) \) is a constant of motion. Of course upon quantization if \( A, B \in \text{su}(2) \) or \( \text{so}(2) \) we have \( Tr(AB^{-1}) = Tr(AB) \).

The Einstein-Hilbert action is [25]

\[
S = \int d^4x \sqrt{|\det g_{\mu\nu}|} R^{(4)}
\]

Using [25],

\[
R^{(D+1)} = R^{(D)} - s \left[ K_{\mu\nu}K^{\mu\nu} - (K^{\mu})^2 \right] + 2s \nabla_\mu (n^\nu K_{\nu\mu} - n^\mu K_{\nu\nu})
\]

twice, where \( K_{\mu\nu} \) is the extrinsic curvature, \( s \) is the signature of the metric and \( n^\mu = \frac{1}{N} \frac{\partial}{\partial t} \) is the normal of the \( D \)-surface for \( D = 3 \), \( n^\mu = \frac{1}{N} \frac{\partial}{\partial z} \) for \( D = 2 \). We view the two hypersurface orthogonal Killing vectors metric (1) as \( 1 - 1 - 2 \) metric and apply (85) twice we obtain

\[
\mathcal{L} = \frac{\alpha}{4} g^{ac} g^{cd} g_{de} g^{ea} - \frac{\alpha}{4} g^{ac} g^{cd} g_{de} g^{ea} - 2\alpha \sigma, t + 2\alpha \sigma, z - \frac{\alpha t^2}{\alpha} + \frac{\alpha z^2}{\alpha} - 2\alpha \sigma, z - 2\alpha N^2 \partial_t \left( \frac{\sigma, t}{\alpha N^2} + \frac{\alpha t}{\alpha N^2} \right)
\]

where the last term is a boundary term. The Euler-Lagrange equations for the metric functions \( g_{ab}(z, t) \) give

\[
(g, z g^{-1}), z - (g, t g^{-1}), t = 0
\]

which is exactly (2) in the variables \((z, t)\). The trace of (87) gives

\[
\alpha, zz = \alpha, tt
\]

ie (11). If \( \alpha = a(\varsigma) - b(\eta) \) is a timelike solution of (11), then \( \beta = a(\varsigma) + b(\eta) \) is an a second independent spacelike solution of (11) [14]. The Euler-Lagrange equations give the equation for \( \sigma \)

\[
\sigma, zz - \sigma, tt = \frac{1}{8} Tr \left( (g, t g^{-1})^2 - (g, z g^{-1})^2 \right)
\]

It is clear from the above discussion that the right hand side of (89) is a constant. Further we see that \( \alpha, z = a' - b' = \beta, t \), \( \beta \) spacelike. Since \( N \) is the lapse that is it measures proper time and also proper distance in the \( z \) direction we interpret (89) as a dispersion relation that is an equation of the form \( E^2 - p^2 = m^2 \). That means that our connection observable is interpreted as a constant that corresponds to mass. This makes equation (89) a typical Klein-Gordon equation and makes available all the tools of the corresponding field theory for
quantization. With the definite choice of variables $\alpha = t$, $\beta = z$ equation (39) is solved by (40) (transcribed to the variables used here)

$$\sigma_{t} = \alpha T r\left(\left(g, t g^{-1}\right)^{2} + \left(g, z g^{-1}\right)^{2}\right) + C_{t}, \quad \sigma_{z} = \frac{\alpha}{4} T r\left(g, t g^{-1} g_{z} g^{-1}\right) + C_{z}$$

(90)

to essentially a solution of the wave equation (88) $C(z, t)$. To stress the interpretation of (39), with variables and conjugate momenta to be $(\alpha, p_{\alpha})$, $(\beta, p_{\beta})$, $(g, p_{g})$ where

$$p_{\alpha} = 2(\sigma_{t} + \frac{\alpha t}{\alpha}) , \quad p_{\beta} = 2(\sigma_{z} + \frac{\beta z}{\alpha}) , \quad p_{g}^{ab} = \frac{\alpha}{2} g^{ac} g_{cd} g^{db}$$

(91)

we have for the Hamiltonian density $H$

$$H = p_{\mu} \dot{q}^{\mu} - L$$

(92)

which gives

$$H = \alpha T r\left(\left(g, t g^{-1}\right)^{2} + \left(g, z g^{-1}\right)^{2}\right)$$

(93)

after fixing the freedom in (38) to $C_{t} = -\frac{1}{\alpha}$. So the energy of the system is $p_{t}$ and is positive definite. Further it is clear that upon quantization $(q, p) \rightarrow (q, \frac{\partial}{\partial q})$ so the generators of $\sigma_{t}, \sigma_{z}$ are $i \frac{\partial}{\partial q}$, $i \frac{\partial}{\partial p}$ which clearly commute by construction here. The other translation generators are trivial. Rotation generators are

$$J_{1} = -i (x_{2} \partial_{z} - z \partial_{2}) , \quad J_{2} = -i (z \partial_{1} - x_{1} \partial_{z}) , \quad J_{z} = -i (x_{1} \partial_{2} - x_{2} \partial_{1})$$

(94)

Boost generators are

$$K_{1} = i (t \partial_{1} + x_{1} \partial_{t}) , \quad K_{2} = i (t \partial_{2} + x_{2} \partial_{t}) , \quad K_{z} = i (t \partial_{z} + z \partial_{t})$$

(95)

We form (26)

$$L_{1} = K_{1} - J_{2} = i ((t + z) \partial_{1} + x_{1} \partial_{t - z}) , \quad L_{2} = K_{2} + J_{1} = i ((t + z) \partial_{2} + x_{2} \partial_{t - z})$$

(96)

(97)

This gives

$$[K_{1}, J_{2}] = K_{z}, \quad [K_{1}, J_{z}] = K_{2}, \quad [K_{2}, J_{1}] = -K_{z}, \quad [K_{1}, K_{2}] = -i J_{z}, \quad [K_{2}, J_{z}] = 0 (\text{nosummation}),$$

(98)

(99)

(100)

(101)

(102)

$$[L_{1}, L_{2}] = 0, \quad [J_{3}, L_{1}] = i L_{2}, \quad [L_{2}, J_{3}] = i L_{1}$$

(103)

(104)

(105)

The last three relations represent the Euclidean group in two dimensions ISO(2).
6 Conclusion

We have obtained transition matrices \( \mathbf{T} \) in terms of connection variables satisfying equations similar to the ones satisfied by the transition matrices for other integrable pde’s. Using these, hierarchies of observables in terms of connection and curvature variables have been obtained. The first in the hierarchy of connection observables has been shown to correspond to the mass of a Klein-Gordon equation that governs the lapse variable \( N \).

7 Data accessibility

Not applicable

8 Competing interests

We have no competing interests

9 Authors’ contributions

Not applicable

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