A stochastic thermalization of the Discrete Nonlinear Schrödinger Equation

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Abstract
We introduce a mass conserving stochastic perturbation of the discrete nonlinear Schrödinger equation that models the action of a heat bath at a given temperature. We prove that the corresponding canonical Gibbs distribution is the unique invariant measure. In the one-dimensional cubic focusing case on the torus, we prove that in the limit for large time, continuous approximation, and low temperature, the solution converges to the steady wave of the continuous equation that minimizes the energy for a given mass.

Keywords Discrete Nonlinear Schrödinger Equation · Thermalization · Large Deviations · Hypoelliptic diffusions · Solitary waves

1 Introduction

Consider the Nonlinear Schrödinger Equation in $d$ space dimension:

$$i \partial_t \phi(y, t) = -\tilde{\Delta} \phi(y, t) + \kappa |\phi|^{p-1} \phi(y, t); \quad p > 1,$$

$$\phi : \Omega^d \times \mathbb{R}_+ \to \mathbb{C}; \quad \phi(y, 0) := \phi_0(y),$$

(1.1)

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where $\Omega = \mathbb{R}$, or $\Omega = \mathbb{T}_L^1$, the circle of length $L$, for the periodic boundary conditions case, 
$\kappa = -1$ corresponds to the focusing case, and $\kappa = 1$ to the defocusing, 
and $\tilde{\Delta}$ denotes the usual Laplacian. This equation has many conserved quantities, 
in particular the most important are the energy and the mass:

$$
\mathcal{H}(\phi) = \frac{1}{2} \int |\nabla \phi|^2 dy + \frac{\kappa}{p+1} \int |\phi|^{p+1} dy, \quad \mathcal{M}(\phi) = \int |\phi|^2 dy. \quad (1.2)
$$

In some particular cases (like for $d = 1$ and $p = 3$), the dynamics is completely integrable.

We are particularly interested in the focusing case $\kappa = -1$, where the non-linearity 
contrast the dispersive effect of the Laplacian. Notice that, thanks to the Gagliardo-Nirenberg inequality (cf. (C.1)), $\mathcal{H}(\phi)$ is still bounded below if $\mathcal{M}(\phi)$ is fixed, and $p < 1 + \frac{1}{d}$, also known as mass sub-critical case. In the one-dimensional mass sub-critical NLS with periodic boundary conditions ($d = 1$, $p < 5$, and $\Omega = \mathbb{T}_L^1$), it has been proven that the canonical Gibbs measure at temperature $\beta^{-1}$, formally defined as

$$
\mathcal{Z}^{-1} \exp(-\beta \mathcal{H}(\phi)) \delta (\mathcal{M}(\phi) = m) \prod_y d\phi(y) \quad (1.3)
$$

is invariant for the dynamics defined by (1.1). Rigorous definition of (1.3) can be found in [31], while its invariance for the dynamics is proven in [5], see also [6, 33–36]. For $p = 3$, $d = 1$, (1.1) is completely integrable; hence it is obvious that (1.3) cannot be ergodic, not even conditioned to a value of the energy $\mathcal{H}$ (i.e., the microcanonical Gibbs measure) as there are other conserved quantities beyond energy and mass. A natural question is then how to define a stochastic perturbation of (1.1) such that it acts as a heat bath at temperature $\beta^{-1}$, and such that the resulting stochastic dynamics has (1.3) as the unique stationary measure. This implies that the only conserved quantity of the dynamics should be the mass $\mathcal{M}$.

Formally, one way to define such stochastic dynamics is to consider the stochastic partial differential equation

$$
i \partial_t \phi(y, t) = -\tilde{\Delta} \phi(y, t) + \kappa |\phi|^{p-1} \phi(y, t) - \gamma \phi(y, t) \left(i \beta^{-1} - \frac{\delta \mathcal{H}(\phi)}{\delta \theta(y)} \right) + \sqrt{2\gamma \beta^{-1}} \phi(y, t) W(y, t), \quad (1.4)
$$

where $\theta(y)$ is the phase of $\phi(y)$ ($\phi(y) = |\phi(y)| e^{i\theta(y)}$), $W(y, t)$ is the standard space-time white noise, and $\gamma > 0$ is a parameter that regulates the intensity of the contact with the heat bath. Notice that $\frac{\delta \mathcal{H}(\phi)}{\delta \theta(y)} = \Im[\phi(y)^* \Delta \phi(y)]$, and that (1.4) should be intended in the Ito’s sense. Consequently, the mass $\mathcal{M}(\phi)$ is still formally conserved by this dynamics. The heat bath acts with random but continuous rotations of the phase of $\phi(y)$ at each point $y$. Because of the singularity in space of the multiplicative white noise $W$ and the non-linearities present in (1.4), it is very hard to give sense to
the solution of this equation. There is an extensive literature on the NLSE with space correlated multiplicative noise (cf. [12, 13]), but it does not include non linearities like $\phi(y) \Delta \phi^*(y)$. Additive noises have also been studied (cf. [7, 8, 14, 30]) but usually do not conserve the mass, and the corresponding dynamics have the Grand Canonical Gibbs measure as stationary.

We introduce instead a space discretization of (1.4), see (2.8), whose solution can be defined globally. The infinite temperature version of this stochastic evolution was introduced in [32]). This is a $n^{nd}$ (complex) dimensional stochastic evolution that conserves the mass, and for any given initial mass, the Gibbs measure on the corresponding complex sphere defined by (2.10), discrete analogous of (1.3), is well defined and invariant. We prove in Sect. 2 that this Gibbs measure is the unique invariant measure, and that the distribution of the process starting from an arbitrary initial condition converges exponentially in total variation to this stationary measure (cf. Theorem 2.1 and Proposition 2.3). These results on the ergodicity of the stochastic dynamics contained in Sect. 2 are general and are valid for any $d$, $p > 1$, $\kappa = \pm 1$, $n$ and more general non-linearity. Let us emphasize that, to the best of our knowledge, the novelty of this dynamics is that it is the first mass conserving perturbation of the DNLS (Discrete Non-Linear Schrödinger), such that the canonical Gibbs measure is the unique invariant measure, determining the dynamics long-time behavior. We should mention that in Section 6 of [7], a mass conserving noise is proposed such that the Canonical Gibbs measure remains invariant by the dynamics. However, this dynamics is not studied, and [7] mainly concerns another dynamics, which does not conserve the mass and converges to the Grand Canonical measure. Moreover, the above-mentioned mass conserving dynamics is different from ours; in fact, a straightforward analysis suggests that our dynamics is more degenerate.

From Sect. 3 and after, we concentrate on the one-dimensional focusing cubic case with periodic boundary conditions ($d = 1$, $p = 3$, $\kappa = -1$). For the continuous model, the minimizers of the energy $\mathcal{H}(\phi)$ under the mass constrain $\mathcal{M}(\phi) = m$ are known explicitly [20]. These minimizers, that we denote by $Q_{m,L}(y)$, are unique up to translations and multiplication by a constant phase. To these minimizers correspond a class of standing waves $\phi(y, t) = e^{i \omega t} Q_{m,L}(y)$, which are solutions of (1.1), where the frequency $\omega$ is determined by $m$ and $L$. We call solitons these ground state standing waves, in analogy to the traveling solitary waves of the dynamics in $\mathbb{R}$. If $m \leq \frac{\pi^2}{L}$ these solitons are constant in space, while for $m > \frac{\pi^2}{L}$ are given by the dnoidal elliptic Jacobi functions (cf. Appendix D for the definition, and Chapter 2,3 of [29] for properties of these functions) properly rescaled. These non-trivial solitons catch the 0-temperature behavior of the dynamics. The purpose of our work is to show that the solution $\psi_n(x, t)$ of the stochastic discrete dynamics (3.14), for large time $t$, large $n$, and small temperature $\beta^{-1}$, is close, in an opportune norm, to the continuous soliton. The result is contained in Theorem 3.2, where it is first taken the limit $t \rightarrow \infty$ then $n \rightarrow \infty$, rescaling the temperature with $n$, i.e. $\beta_n \sim \infty$ faster than $n$. This is a way to interpret the soliton resolution conjecture (SRC) in the periodic case, where there is no possibility for the energy to escape to infinity. Intuitively, in the periodic case, our dynamics in the zero temperature limit dissipate the excess of the energy without losing any mass, forcing the system to approach the ground state as $t \rightarrow \infty$. At a
physical level, this mechanism is somehow mimicking the dynamics of DNLS in [10], where energy disperse to infinity via a "radiating" part of the field carrying arbitrarily small mass. In fact, our dynamics is partially motivated by [10], where Chatterjee proves a "probabilistic" version of the SRC. In particular, in Theorem 3.1 in [10] it is proven that almost every ergodic invariant measure satisfies the SRC in the time average sense.

Our stochastic dynamics provides the uniqueness of the invariant ergodic measure and the time mixing property.

In Theorem 3.2 the limit for \( t \to \infty \) follows from the ergodic and time mixing properties of the dynamics proven in Sect. 2. Then we have to prove that the discrete Gibbs measure (finite \( n \)) concentrate fast enough in a small neighborhood of the corresponding lowest energy configurations, that we call discrete solitons, who converge to the continuous one as \( n \to \infty \). This relies on large deviation properties of the discrete Gibbs measure, proven in Sect. 5. These large deviations estimates are based on some precise large deviations of the uniform probability measure \( \mu_{\beta_m}^{n} \) on the complex \( 2n \)-dimensional sphere \( S_m^n \), that we prove in Appendix A, and the discrete version of the Gagliardo-Nirenberg inequality, stated in Appendix C. The Gibbs measure has a density \( \exp(-\beta_n \mathcal{H}_n) \) with respect to the uniform measure \( \mu_{\beta_m}^{n} \). Splitting the energy \( \mathcal{H}_n = G_n - V_n \), where \( G_n \) is the kinetic part, and \( V_n \) the potential part (3.8), one can observe that a "typical" configuration w.r.t \( \mu_{\beta_m}^{n} \) has kinetic energy \( G_n \sim n^2 \).

The large deviations estimates in Section A, in particular Lemma A.1, combined with Gagliardo-Nirenberg inequality (C.9) yields: for \( 0 \leq a < 2 \), the "entropy factor" behaves as \( \mu_{\beta_m}^{n} (G_n \sim n^a) \sim \mu_{\beta_m}^{n} (\mathcal{H}_n \sim n^a) \sim e^{-(2-a)n \ln n} \).

Therefore, taking into account the Boltzmann factor \( \exp(-\beta_n \mathcal{H}_n) \), we have for \( 0 \leq a < 2 \): \( \mu_{\beta_{n,m}}^{n} (\mathcal{H}_n \sim n^a) \sim e^{-\beta_n n^a} e^{-(2-a)n \ln n} \). Optimizing this estimate on \( a \in [0,2) \), if \( \beta_n \sim O(1) \), then \( a = 1 \) is the optimal value and the Gibbs measure concentrates on rather rough configurations with \( |\psi(j) - \psi(j-1)| \sim \frac{1}{\sqrt{n}} \), so that \( G_n \sim n \). This corresponds to the fact that Wiener measure is concentrated on configurations of Hölder regularity less than \( \frac{1}{2} \). Instead, if \( \beta_n \sim O(n) \) we have that \( a = 0 \) is the optimal value and this suggests that \( \mu_{\beta_{n,m}}^{n} \) to concentrates on smooth configurations (i.e., with \( |\psi(j) - \psi(j-1)| \sim \frac{1}{n} \)) with \( \mathcal{H}_n \sim O(1) \). Notice that minimal energy configurations (the discrete solitons), have energy of order one as well. However, this scaling is not enough for this measure to concentrate on a small neighborhood of discrete solitons, and we need to go further. Finally, thanks to large deviation estimate (A.11), we deduce in Theorem 5.1 that scaling \( \beta_n \gg n \) is sufficient.

In the last step of the proof, we show in Proposition 4.1 that if \( \psi_n \) is a configuration with energy close to \( E_0(m) \), then its linear interpolation \( \tilde{\psi}_n \) (see (3.17)) is close to the continuous soliton \( Q_{m,L} \) in \( H^1 \) norm (up to a translation and multiplication by a phase, see (3.18)), for \( n \) sufficiently large. In that regard, first we observe that having energy close to \( E_0(m) \) means the configuration is smooth \( G_n \sim O(1) \), thanks to the discrete Gagliardo-Nirenberg inequality. Subsequently, since for smooth configurations \( \mathcal{H}_n(\psi_n) \) is close to \( \mathcal{H}(\tilde{\psi}_n) \) (See Corollary (4.3.1)), one can conclude by compactness of the minimizing sequence corresponding to the continuous minimization problem characterizing solitons (3.6).
Appendix B contains the proof of the hypoellipticity of the discrete stochastic dynamics, necessary for the proof of the ergodicity of Sect. 2. Since the real and complex part of our field are somehow symmetric in the noise, this makes the proof of the hypoellipticity more complicated than usual, and computing three nested commutators is necessary (see (B.4)).

The Gibbs measure of DNLS have been studied both in Mathematics (cf. [10, 11]) and Physics community (cf. [26] and references therein: in particular: [25, 40, 41]; See also [19]). In the physics community, one usually takes the Kinetic energy with a negative sign and study the measure corresponding to Hamiltonian (2.2), by taking $h = 1$. Although this regime is substantially different from ours, and does not correspond to discretization of a continuous profile anymore, interesting phenomena such as discrete breathers is observed (cf. [16, 43]).

In mathematics community, we can mention most notably [11], and [10] (cf. [27], for a review). In [11], the Hamiltonian (2.2) is considered such that $Nh^2 \to 0$, as $h \to 0$, and $N \to \infty$, where $N$ denotes the number of particles, and $h$ is the interparticle distance. These assumptions only seems natural in $d \geq 3$. In this regime, certain phase transition happens: When $\beta m^2 < \theta_c$ the Gibbs measure concentrates on configurations such that $\psi_n(j) \sim o(n)$, whereas for $\beta m^2 > \theta_c$ breather-like structures appears, where a single site has macroscopic mass.

In [10], the model is defined on the box $[0, nh]^d$, such that $h \to 0$, $n \to \infty$, with $nh \to \infty$. In this regime, the microcanonical measure corresponding to energy $E$ concentrates on soliton-like configurations in $\mathbb{R}^d$.

Comparing our result with [11], and [10], we highlight the fact that different scaling among the parameters $h$, $n$ leads to substantially different phenomena: In [11], $Nh^2 \to 0$ makes the Gradient term negligible and phase transition is a consequence of competition among potential energy and mass constraint. In [10], $nh \to \infty$, kinetic and potential energy become comparable; however, the mass per particle goes to zero in the limit, demonstrating the macroscopic infinite volume, facilitating escape of the energy to infinity and resulting in soliton like behavior. In contrast, in our case we take $n \to \infty$, and $nh = 1$, representing the finite macroscopic volume, and positive mass per particle in the macroscopic limit. This scaling yields a dominant kinetic energy for typical configurations on the sphere of constant mass, and rescaling $\beta_n$ makes the kinetic and potential energy comparable.

In particular, these different scaling change our large deviation estimates (A.1), and (A.11) comparing to estimates in [10] (See Section 10 of [10]).

2 Stochastic dynamics

Fix $n \in \mathbb{N}$, let $\chi = \mathbb{C}^n$ be the configuration space, and denote a typical element of $\chi$ by $\{\psi(x)\}_{x \in \tilde{T}_n^d}$, where $\tilde{T}_n = \mathbb{Z}/n\mathbb{Z}$ is the discrete Torus of size $n$.

Equivalently, one can see a function on $\tilde{T}_n^d$, $\psi : \tilde{T}_n^d \to \mathbb{C}$, as the discretization of a function $u$ on the $d$-dimensional torus of length size $nh$, $u : \mathbb{T}^d_{nh} \to \mathbb{C}$, with mesh size $h > 0$, i.e., $\psi(x) = u(hx)$, for $x \in \tilde{T}_n^d$. Then the discrete nonlinear Schrödinger
equation (DNLS) is the following system of ODEs:

$$i \frac{d\psi(x, t)}{dt} = -\Delta_d \psi(x, t) + \kappa |\psi(x, t)|^{p-1} \psi(x, t), \quad x \in \mathbb{T}^d_n \quad (2.1)$$

where $\Delta_d$ is the $d$-dimensional discrete Laplacian:

$$\Delta_d \psi(x) = h^{-2} \sum_{|y-x|=1} (\psi(y) - \psi(x)),$$

where $|.|$ should be understood modulus $n$.

These equations conserve the energy, given by the Hamiltonian

$$\mathcal{H}_n(\psi) = s \sum_{x, y \in \mathbb{T}^d_n, |x-y|=1} \frac{h^{-2}}{2} |\psi(x) - \psi(y)|^2 + \frac{s\kappa}{p+1} \sum_{x \in \mathbb{T}^d_n} |\psi(x)|^{p+1}, \quad (2.2)$$

and the mass, given by the $\ell^2$ norm:

$$\mathcal{M}_n(\psi) = s \sum_{x \in \mathbb{T}^d_n} |\psi(x)|^2. \quad (2.3)$$

Here $s > 0$ is a scaling parameter that we will choose opportunistically later.

Denote $\psi(x) = \psi_r(x) + i \psi_i(x) = |\psi(x)| e^{i \theta(x)}$, the deterministic evolution equation (2.1) can be regarded as a Hamiltonian dynamics with the following generator:

$$\mathcal{A} = s^{-1} \sum_{x \in \mathbb{T}^d_n} \left( \partial_{\psi_i(x)} \mathcal{H}_n(x) \partial_{\psi_r(x)} - \partial_{\psi_r(x)} \mathcal{H}_n(x) \partial_{\psi_i(x)} \right). \quad (2.4)$$

Moreover, define the operator $\partial_{\theta(x)}$ acting on a suitable function $F : \chi \to \mathbb{C}$ as

$$\partial_{\theta(x)} F(\psi) = (\psi_r(x) \partial_{\psi_i(x)} - \psi_i(x) \partial_{\psi_r(x)}) F(\psi). \quad (2.5)$$

Corresponding to a positive temperature $\beta^{-1} > 0$, define:

$$S = \beta^{-1} \sum_{x \in \mathbb{T}^d_n} e^{\beta \mathcal{H}_n(\psi)} \partial_{\theta(x)} e^{-\beta \mathcal{H}_n(\psi)} \partial_{\theta(x)}. \quad (2.6)$$

Fix $\beta > 0, \gamma > 0$, and consider the Markov process with values in $\chi$, generated by

$$L = \mathcal{A} + \gamma S, \quad (2.7)$$
where $S$ and $A$ are defined in (2.6) and (2.4). Since $\partial_{\theta(x)} \psi(x) = i \psi(x)$, we have

$$S \psi(x) = -\psi(x) \left( \beta^{-1} + i \partial_{\theta(x)} \mathcal{H}_n(\psi) \right) = -\psi(x) \left( \beta^{-1} + i s \Im[\psi^*(x) \Delta_d \psi(x)] \right),$$

and system of stochastic differential equations generated by (2.7) reads:

$$d \psi(x, t) = i [\Delta_d \psi(x, t) - \kappa |\psi(x, t)|^{p-1} \psi(x, t)] dt - \gamma \psi(x, t) (\beta^{-1} + i \partial_{\theta(x)} \mathcal{H}_n(\psi)) dt$$

$$- i \sqrt{2 \gamma \beta^{-1}} \psi(x(t)) d w(x(t)), \quad x \in \mathbb{R}^n,$$

where $\{w(x, t), x \in \mathbb{R}^n\}$ are real independent Wiener processes.

We observed that $\mathcal{M}_n(\psi) = 0$, one can check that $S \mathcal{M}_n(\psi) = 0$. Therefore, mass is a conserved quantity for the dynamics (2.7). Hence, if we assume the initial condition $\psi(0) = \psi_0 \in \mathbb{C}^n$, such that $\mathcal{M}_n(\psi_0) = m$, then our dynamics is confined in the compact manifold with $\mathcal{M}_n(\psi) = m$, which is a $(2n^d - 1)$-sphere. We denote this sphere by $S_{m,s}^n$:

$$S_{m,s}^n = \{ \psi \in \mathbb{C}^{n^d} | \mathcal{M}_n(\psi) = m \}. \quad (2.9)$$

**Proposition 2.1** The generator $L$ is hypoelliptic on $S_{m,s}^n$.

The proof follows from Hörmander characterization, i.e., that the Lie algebra generated by $[A, \partial_{\theta(x)}, x \in \mathbb{T}_n]$ generates the tangent space of $S_{m,s}^n$. This is proven in Appendix B.

Let $d \mu_{\beta,m,s}^n$ be the uniform probability measure on $S_{m,s}^n$, one can define this measure as the projection of the Lebesgue measure on $S_{m,s}^n$, properly normalized.

Define the canonical Gibbs measure with inverse temperature $\beta$ on $S_{m,s}^n$ as

$$d \mu_{\beta,m,s}^n = \frac{1}{Z_n(\beta, m, s)} e^{-\beta \mathcal{H}_n(\psi)} d \mu_{m,s}^n, \quad (2.10)$$

Here $Z_n(\beta, m, s)$ is the partition function:

$$Z_n(\beta, m, s) = \int_{S_{m,s}^n} e^{-\beta \mathcal{H}_n(\psi)} d \mu_{m,s}^n. \quad (2.11)$$

Note that, since $\mathcal{H}_n$ is a smooth function on a compact set and therefore, bounded from below, $Z_n(\beta, m, s)$ is finite, and consequently, the existence of $d \mu_{\beta,m,s}^n$ is evident.

Denote by $C_b(S_{m,s}^n)$, the set of continuous bounded functions on $S_{m,s}^n$. The following observation $\forall f \in C_b(S_{m,s}^n), \int_{\mathbb{C}^n} L f d \mu_{\beta,m,s}^n = 0$, implies that $d \mu_{\beta,m,s}^n$ is an invariant measure for the dynamics (2.7)(2.8)). In fact, if we fix $m$, $\gamma$, $\beta > 0$, this measure is the unique invariant probability measure:

**Theorem 2.1** Fix real positive parameters $h, s, \gamma > 0$, the mass of the field $m > 0$, inverse temperature $\beta > 0$, and $\kappa \in \mathbb{R}$, the measure $d \mu_{\beta,m,s}^n$ is the unique invariant measure for the dynamics generated by (2.7).
Proof Without losing generality we can fix $h = s = 1$. Since the generator $L$ is hypoelliptic, the stationary measure must have density w.r.t $d\mu^n_{\beta,m,s}$, and then also w.r.t $d\mu^n_{\beta,m,s}$. Denoting $f(\psi)$ the density w.r.t $d\mu^n_{\beta,m,s}$, it must satisfy the equation

$$0 = L^* f = (-A + \gamma S) f,$$  

(2.12)

where $L^*$ denotes the adjoint of $L$ in $L^2(d\mu^n_{\beta,m,s})$. Since $L$ is hypoelliptic, $f$ is smooth and (2.12) is valid pointwise. Multiplying by $A$ and (2.12) is valid pointwise. Multiplying by $f$ and integrating w.r.t $d\mu^n_{\beta,m,s}$, we have

$$0 = \gamma < f(-S) f > = \gamma \sum_{x \in \tilde{T}_n^d} < (\partial_{\theta(x)} f)^2 >,$$  

(2.13)

where $\langle \cdot, \cdot \rangle$ denotes integration w.r.t $d\mu^n_{\beta,m,s}$. This means that $\partial_{\theta(x)} f = 0$ $d\mu^n_{\beta,m,s}$-a.e., and $Af = 0$. We want to conclude that $f = 1$, $d\mu^n_{\beta,m,s}$-a.e.. Since $\partial_{\theta(x)} f = 0$ for any $x$, then $f = \tilde{f}(\langle \psi, x \rangle^2, x \in \tilde{T}_n^d)$. The operator $A$ can be written as $A^0 + A^p$, with

$$A^0 = \sum_{x \in \tilde{T}_n^d} \left\{ (\Delta_d \psi_f(x)) \partial_{\psi_f} \right\} - (\Delta_d \psi_f(x)) \partial_{\psi_f},$$  

(2.14)

and

$$A^p = \kappa \sum_{x \in \tilde{T}_n^d} |\psi(x)|^{-1} \left\{ \psi_f(x) \partial_{\psi_f} - \psi_f(x) \partial_{\psi_f} \right\} = \kappa \sum_{x \in \tilde{T}_n^d} |\psi(x)|^{-1} \partial_{\theta(x)}.$$  

(2.15)

It is immediate that $A^p f = 0$, hence, $A^0 f = 0$, pointwise. Let us denote $\alpha(x) := |\psi(x)|^2$, and the canonical basis of $\mathbb{R}^d$ by $\{e_j\}_{j=1}^d$, then we have:

$$0 = A^0 f = 2 \sum_{x \in \tilde{T}_n^d} \{ (\Delta_d \psi_f(x)) \psi_f(x) - (\Delta_d \psi_f(x)) \psi_f(x) \} \left[ \partial_{u(x)} \tilde{f} \right] (|\psi|^2, y \in \tilde{T}_n^d)$$

$$= 2 \sum_{x \in \tilde{T}_n^d} \sum_{j=1}^d \nabla\tilde{f}(\psi_f(x) \psi_f(x - e_j) - \psi_f(x) \psi_f(x - e_j)) \left[ \partial_{u(x)} \tilde{f} \right] (|\psi|^2, y \in \tilde{T}_n^d)$$

$$= 2 \sum_{x \in \tilde{T}_n^d} \sum_{j=1}^d \left[ \psi_f(x) \psi_f(x - e_j) - \psi_f(x) \psi_f(x - e_j) \right] \left[ \partial_{u(x)} - \partial_{u(x - e_j)} \right] \tilde{f} (|\psi|^2, y \in \tilde{T}_n^d)$$

$$= 2 \sum_{x \in \tilde{T}_n^d} \sum_{j=1}^d \sin(\theta_{x - e_j} - \theta_{x})|\psi(x)| \left| \psi(x - e_j) \right| \left[ \partial_{u(x)} - \partial_{u(x - e_j)} \right] \tilde{f} (|\psi|^2, y \in \tilde{T}_n^d),$$

(2.16)

where $\nabla\tilde{f}$ denotes the discrete gradient in the $e_j$ direction $(\nabla\tilde{f} g)(x) = g(x + e_j) - g(x)$. Since this relation is true pointwise for any $\psi \in S_{m,s}^n$, by choosing a proper

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ψ (for example one can take \( \theta_y \) equal to zero, for \( y \in \tilde{T}_n^d \), except \( \theta(x) \), and take \(|\psi(y)| = 0\), for all \(|y - x|_d = 1\), except \( x - e_j \), we have that

\[
(\partial_{a(x)} - \partial_{a(x-e_j)}) \tilde{f}(a(y), y \in \tilde{T}_n^d) = 0,
\]

pointwise for every \( x \in \tilde{T}_n^d \), and any \( 1 \leq j \leq d \).

From (2.17), we conclude that \( (\partial_{a(x)} - \partial_{a(z)}) \tilde{f}(a(y), y \in \tilde{T}_n^d) = 0 \) for any \( x, z \in \tilde{T}_n^d \). This implies

\[
\tilde{f}(a(y), y \in \tilde{T}_n^d) = F \left( \sum_{y \in \tilde{T}_n^d} a(y) \right) = F(m).
\]

which yields the result.

Remark 2.2 Similar to Remark B.1, the proof of Theorem 2.1, can be adapted to the case where the non-linearity appearing in the Hamiltonian is given by \( N_n(\psi(x)) \sim f(|\psi(x)|^2) \), with \( f \) sufficiently smooth, instead of \( N_n(\psi(x)) \sim |\psi(x)|^4 \).

By classical theorems in control theory, given the Hörmander condition, and the existence of a unique invariant measure with full support on \( S_{m,s}^n \), it follows the strict positivity of the probability transition (cf. [21], proof of Theorem 2.1) and the following proposition. For any proper signed measure \( \mu \), denote its total variation norm by \( \|\mu\|_{TV} := \sup_A \mu(A) - \inf_A \mu(A) \) (cf. Sect. 3 of [4]). Then we have:

Proposition 2.3 Consider the dynamics generated by (2.7), denote the law of this process by \( \mu_{t}^{\beta,n,m} \) with initial condition \( \mu_0^{\beta,n,m} = \delta_{\psi_0} \), where \( \psi_0 \in S_{m,s}^n \). There exists \( \gamma_0 = \gamma_0(n, m, \beta, s) > 0 \), and \( C = C(n, m, \beta, s) > 0 \), such that for any \( \psi_0 \in S_{m,s}^n \), we have:

\[
\|\mu_t^{\beta,n,m} - \mu_{\beta,m,s}^n\|_{TV} \leq C e^{-\gamma_0 t}.
\]

In particular, we have the weak convergence:

\[
\mu_t^{\beta,n,m} \underset{t \to \infty}{\longrightarrow} \mu_{\beta,m,s}^n.
\]

Proof Since \( \mu_{\beta,m,s}^n \) is the unique invariant measure (ergodicity), with full support (for any open set \( A \subset S_{m,s}^n \), \( \mu_{\beta,m,s}^n(A) > 0 \)), given the Hörmander condition we can use the result of [21], (proof of Theorem 2.1 in [21]) and deduce the strict positivity of the probability transition. Furthermore, having the strict positivity of the probability transition, compactness of the phase space, as well as the generator’s hypoellipticity, we can conclude by Theorem 8.9 of [4]. Notice that the fact that \( C \) is independent of \( \psi_0 \) is deduced from compactness of the phase space.

1 Notice that with our definition, \( \mu_t^{\beta,n,m} \) actually depends on \( \psi_0 \). However, we omit \( \psi_0 \) to lighten the notation.
The novelty of the stochastic perturbation (2.6) can be described as follows: it’s a mass-conserving white noise, such that the Gibbs measure is the unique invariant measure for the dynamics, and it provides good ergodic properties as in Proposition 2.3. This perturbation is quite "powerful" in the sense that its ergodic properties do not depend on the non-linearity, and we can consider either focusing or defocusing non-linearity. In either of these cases the long time behavior is given by the corresponding Gibbs measure. However, depending on the choice of parameters $d, s, h, \kappa$ many interesting phenomena can be observed in the large scale limit. In the rest of this note, we focus on one particular case: one-dimensional focusing nonlinear Schrödinger Equation on the torus. As an example of a mass conserving stochastic perturbation of dispersive evolution equations, we can mention [15], falling into the general framework of [37]. However, noise of [15] does not thermalize the system in the above sense i.e., the long time behavior of the system is not given by the Gibbs measure.

3 Large scale limit and main result

3.1 Preliminaries about periodic cubic nonlinear Schrödinger equation

In this section, we recall rather basic results about the focusing nonlinear Schrödinger equation (NLS) with periodic boundary conditions. Consider the following nonlinear cubic Schrödinger equation:

$$i \frac{\partial_t \phi(x,t)}{= - \partial_{xx} \phi(x,t) - |\phi(x,t)|^2 \phi(x,t), \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \ \phi(x,0) = \phi_0(x), \ \phi_0 \in H^1(\mathbb{T}_L), (3.1)$$

where we assume the periodic boundary conditions by the definition of $H^1(\mathbb{T}_L)$ as:

$$H^1(\mathbb{T}_L) = \{ u \in H^1_{loc}(\mathbb{R}, \mathbb{C}) | \forall x \in \mathbb{R}, \ u(x+L) = u(x) \},$$

with the following norms and inner product ($\bar{v}$ indicates the complex conjugate):

$$\|u\|_{L^p} = \left( \int_{\mathbb{T}_L} |u|^p \, dx \right)^{\frac{1}{p}}, \quad (u,v) = \int_{\mathbb{T}_L} u \bar{v} \, dx, \quad \|u\|_{H^1} = \left( \int_{\mathbb{T}_L} (|\partial_x u|^2 + |u|^2) \, dx \right)^{\frac{1}{2}}. (3.2)$$

Global wellposedness of this problem is established in [6, 9]; in particular, $\forall t > 0, \ \phi(x,t) \in H^1(\mathbb{T}_L)$. Note that this equation has two important conserved quantities $^2$:

- the energy or Hamiltonian $\mathcal{H}$, and $L^2$ norm or mass $\mathcal{M}$, defined by

$$\mathcal{H}(\phi) = \frac{1}{2} \int_{\mathbb{T}_L} |\partial_x \phi|^2 \, dx - \frac{1}{4} \int_{\mathbb{T}_L} |\phi|^4 \, dx, \quad \mathcal{M}(\phi) = \int_{\mathbb{T}_L} |\phi|^2 \, dx. (3.3)$$

$^2$ In fact, since this equation is completely integrable, we have infinite conserved quantities. However, most of the results in this note can be generalized to the sub-critical non-linearities that are not integrable, i.e., we can change the nonlinearity term in (3.1) into $|\phi|^{p-1} \phi$ with $1 \leq p < 5$. Notice that if $p \neq 3$, we do not have the explicit characterization of the Solitons.
One of the main features of this equation is the existence of a special class of solutions called the "standing waves" or "periodic waves". These are time periodic solutions having the following form:

$$\phi(x, t) = e^{i\omega t} u(x).$$  \hfill (3.4)

If $\phi(x, t) = e^{i\omega t} u(x)$ be a solution of (3.1), then $u(x)$ should satisfy the following ODE, with periodic boundary condition:

$$u''(x) - \omega u(x) + |u(x)|^2 u(x) = 0.$$  \hfill (3.5)

Notice that the solution of (3.5) characterizes the minimum of the energy $\mathcal{H}(u)$, under the constrain $\mathcal{M}(u) = m$, where the frequency $\omega$ plays the role of Lagrange multiplier.

In general, we should consider complex valued solutions of (3.4). On the other hand, writing this solution as $u(x) = \rho(x) e^{i\theta(x)}$, the corresponding energy is given by

$$\mathcal{H}(u) = \frac{1}{2} \int_{\mathbb{T}_L} \left( |\rho'(x)|^2 + \rho(x)^2 |\theta'(x)|^2 \right) dx - \frac{1}{4} \int_{\mathbb{T}_L} |\rho(x)|^4 dx.$$  

This shows that the minimum of the energy $\mathcal{H}(u)$, under the constrain $\mathcal{M}(u) = m$ is attained for $\theta(x) = \text{constant}$.

Consequently, this minimum are defined up to a constant phase and we can choose positive real solutions. Also notice that translations $u_y(x) = u(x + y)$ do not change energy and mass.

Here, if we fix the $L$, and assume $u$ to be real-valued, and positive, and fix the mass of $u$ to be $\mathcal{M}(u) = m$, then under these assumptions, (3.5) has a unique (up to a translation) smooth solution, this solution can be written in terms of Jacobi elliptic functions as $u(x) = \alpha \text{dn}(\lambda x, k)$, where $k \in (0, 1)$, $\alpha$, and $\lambda > 0$, $\omega > 0$ are uniquely determined by $m$, and $L$ (cf. [17, 18, 20, 38], cf. Appendix D for the definition of $\text{dn}$).

We recall the following crucial result from [20], Proposition 3.2, which characterizes this solution as the minimizer of $\mathcal{H}(\phi)$ under the constraint that $\mathcal{M}(\phi) = m$.

\textbf{Theorem 3.1} Fix $m, L \in \mathbb{R}_+$, and consider the following minimization problem:

$$E_0(m, L) := \inf\{\mathcal{H}(u)|\mathcal{M}(u) = m, \ u \in H^1(\mathbb{T}_L)\},$$  \hfill (3.6)

then we have: $-\infty < E_0(m, L) < 0$, and

1. If $0 < m \leq \frac{\pi^2}{L}$, then the constant function $Q_{m,L}(x) = \left(\frac{m}{L}\right)^\frac{1}{2}$ is the unique minimizer of (3.6). This uniqueness is up to a multiplication by a constant phase.
2. If $\frac{\pi^2}{L} < m$, then $Q_{m,L}(x) := \alpha \text{dn}(\lambda x, k)$ is the unique minimizer of (3.6), up to a translation and multiplication by a constant phase. Moreover, $\alpha, \lambda > 0, k \in (0, 1)$ are determined uniquely by $m, L$. 

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Furthermore, we have compactness of the minimizing sequence up to a phase shift and translation in $H^1(T_L)$, i.e., for any sequence $u_n$ in $H^1(T_L)$, such that $\mathcal{H}(u_n) \to E_0(m, L)$, as $n \to \infty$, there is a subsequence $u_{n_k}$, and sequences $\gamma_k \in [0, 2\pi)$, and $x_k \in \mathbb{T}_L$, where $e^{i\gamma_k}u_{n_k}(\cdot + x_k) \to Q_{m,L}$, in $H^1(T_L)$.

Since each solution of (3.5) (and consequently a solution to (3.1)) corresponds to the minimization problem (3.6), by abusing the terminology, we use the term "standing wave" or Soliton for $Q_{m,L}$.

Notice that multiplying (3.5) by $\bar{u}$ and integrating, we obtain the following relation

$$E_0(m, L) = \frac{1}{4} \int_{T_L} u^4(x) dx - \frac{\omega m}{2}.$$  

that implies $\omega \geq \frac{1}{2m} \int_{T_L} u^4(x) dx + \frac{m}{2L^2}$.

### 3.2 Stochastic perturbation of discrete focusing NLS

In this section, we are going to perturb the NLS (3.1), with the stochastic heat bath, which we defined in Sect. 2, namely (2.6). Without losing generality, in order to simplify notation, we fix the macroscopic length $L = 1$. This means that we fix the following parameters $h = \frac{1}{n}$, $s = \frac{1}{n}$, $d = 1$, $p = 3$, $\kappa = -1$.

Here, we briefly recall the dynamics of Sect. 2 in this particular setup, in order to set the notations.

Fix $n \in \mathbb{N}$, the configuration space is $\chi = \mathbb{C}^n$ and denote a typical element of $\chi$ by $\{\psi(x)\}_{x \in \mathbb{T}_n}$, with $\mathbb{T}_n = \mathbb{Z}/n\mathbb{Z}$ is the discrete torus of size $n$. Equivalently, a function $\psi$ on $\mathbb{T}_n$ can be seen as discretization of a function $u$ on a unit torus, $u : \mathbb{T} \to \mathbb{C}$, with mesh size $\frac{1}{n}$, i.e., $\psi(x) = u(\frac{x}{n})$, for $x \in \mathbb{T}_n$. Then the discrete cubic focusing nonlinear Schrödinger equation (DNLS) is the following system of ODEs:

$$i \frac{d\psi(x, t)}{dt} = -\Delta \psi(x, t) - |\psi(x, t)|^2 \psi(x, t), \quad (3.7)$$

where $\Delta \psi(x, t) = n^2 (\psi(x + 1) - 2\psi(x) + \psi(x - 1))$, and we imposed periodic boundary condition $\psi(0) \equiv \psi(n)$. Notice that we define $\Delta$ such that formally in the limit $n \to \infty$, this definition coincides with the continuous Laplacian on a unit torus.

Similar to the continuous case, we have the energy or Hamiltonian $\mathcal{H}_n : \mathbb{C}^n \to \mathbb{R}$ as a conserved quantity, that is defined by:

$$\mathcal{H}_n(\psi) = \frac{1}{n} \sum_{x \in \mathbb{T}_n} \frac{n^2}{2} |\psi(x) - \psi(x - 1)|^2 - \frac{1}{4n} \sum_{x \in \mathbb{T}_n} |\psi(x)|^4 = G_n(\psi) - V_n(\psi), \quad (3.8)$$

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where we have denoted the kinetic energy $G_n(\psi)$ and the potential energy $V_n(\psi)$ as:

$$G_n(\psi) = \frac{1}{n} \sum_{x \in \tilde{T}_n} \frac{n^2}{2} |\psi(x) - \psi(x - 1)|^2, \quad V_n(\psi) = \frac{1}{4n} \sum_{x \in \tilde{T}_n} |\psi(x)|^4.$$  \hspace{1cm} (3.9)

The other conserved quantity is given by the mass $M_n(\psi)$:

$$M_n(\psi) = \frac{1}{n} \sum_{x \in \tilde{T}_n} |\psi(x)|^2.$$  \hspace{1cm} (3.10)

Notice that we scaled (3.8) and (3.10), such that in the limit as $n \to \infty$, we recover $H$ and $M$ formally.

The stochastic perturbation we consider will only conserve the mass. Recall $\psi(x) = \psi_r(x) + i \psi_i(x) = |\psi(x)| e^{i \theta(x)}$, the generators of the Hamiltonian and stochastic noise at temperature $\beta^{-1}$ read

$$A_n = n \sum_{x \in \tilde{T}_n} (\partial_{\psi_i(x)} H_n) \partial_{\psi_r(x)} - (\partial_{\psi_r(x)} H_n) \partial_{\psi_i(x)},$$

$$S_n = \beta^{-1} \sum_{x \in \tilde{T}_n} e^{\beta H_n} \partial_{\theta(x)} e^{-\beta H_n} \partial_{\theta(x)}.$$  \hspace{1cm} (3.11)

Fix $\beta > 0, \gamma > 0$, then the generator of the dynamics and corresponding system of stochastic partial differential equations with values in $\chi$, are as follows:

$$L_n = A_n + \gamma S_n,$$  \hspace{1cm} (3.13)

$$d\psi(x, t) = i[\Delta \psi(x, t) + |\psi(x, t)|^2 \psi(x, t)]dt - \gamma \psi(x, t)(\beta^{-1} + i \partial_{\theta(x)} H_n(\psi))dt$$

$$- i \sqrt{2\gamma \beta^{-1}} \psi(x, t) dw(x, t),$$  \hspace{1cm} (3.14)

where $\{w(x, t), x \in \tilde{T}_n\}$ are real independent Wiener processes.

Due to the mass conservation, having an initial condition $\psi(0, t) = \psi_0 \in \mathbb{C}^n$ such that $\mathcal{M}_n(\psi_0) = m$, our dynamic will be confined in the sphere $S^n_m = \{\psi \in \mathbb{C}^n | \mathcal{M}_n(\psi) = m\}$. Denote the uniform probability measure on $S^n_m$ by $d\mu^n_m$, and define the canonical Gibbs measure with inverse temperature $\beta$ on $S^n_m$ as

$$d\mu^n_{\beta, m} = \frac{1}{Z_n(\beta, m)} e^{-\beta H_n(\psi)} d\mu^n_m.$$  \hspace{1cm} (3.15)

Here $Z_n(\beta, m) = \int_{S^n_m} e^{-\beta H_n(\psi)} d\mu^n_m$. As we observed, $Z_n(\beta, m)$ is finite, and consequently, the existence of $d\mu^n_{\beta, m}$ is evident, since $H_n(\psi)$ is bounded from below in $S^n_m$. However, one can find a lower bound for $H_n(\psi)$, which is uniform in $n$, using a version of Gagliardo-Nirenberg inequality in the discrete periodic setup. This will be discussed broadly in the Sect. 4 and Appendix C.
Applying results of Sect. 2, we have the following: By Theorem 2.1 we know that $d\mu_{n,\beta,m}$ is the unique invariant measure for the dynamics (3.13)((3.14)). Moreover, let $\mu_t$ denotes the law of the process at time $t \geq 0$, generated by (3.13), with initial condition $\mu_0 = \delta_{\psi_0}$, where $\psi_0$ is an arbitrary element of $S^n_m$. Then, thanks to Proposition 2.3, there exists $C = C(n,m) > 0$ and $\gamma_0 = \gamma_0(n,m) > 0$, such that for any $\psi_0 \in S^n_m$,

$$\|\mu_t - \mu_{n,\beta,m}\|_{TV} \leq Ce^{-\gamma_0 t}.$$ (3.16)

If we run our dynamics for a long time, then take the limit of large $n$ and small temperature $\beta^{-1}$ properly, we end-up near Solitons or standing waves ($Q_{1,m}$ from Theorem 3.1), with probability one. Notice that here we can take the limit in $\beta$ and $n$ simultaneously, where we scale $\beta$ by a factor of $\vartheta(n)$. In order to make these words rigorous, and connect the discrete setup to the continuous one, we need to introduce some notations. For any $\psi_n \in C^n$, we define its linear interpolation $\bar{\psi}_n : \mathbb{T} \to \mathbb{C}$, on a unit torus by

$$\bar{\psi}_n(y) = \psi_n([ny])([ny] + 1 - ny) + \psi_n([ny] + 1)(ny - [ny]), \quad \forall y \in \mathbb{T}, \quad (3.17)$$

where $[ny]$ denotes the greatest integer less than $ny$. Denote $H^1_{\text{per}}([0,1]) = H^1_{\text{per}}$. For $x \in \mathbb{T}$, let $\tau_x$ denotes the translation operator on $H^1_{\text{per}}$, i.e., $\tau_x f(x + y)$. In order to deal with the phase multiplication and translation, define the following seminorm as in [10]:

$$\forall f,g \in H^1_{\text{per}}, \quad \|f - g\|_{\tilde{H}^1_{\text{per}}} := \inf_{\gamma \in [0,2\pi], x \in \mathbb{T}} \|e^{i\gamma \tau_x f} - g\|_{H^1_{\text{per}}}. \quad (3.18)$$

In the following we set $Q_{1,m} =: Q_m$. Now we can state the main theorem of this section:

**Theorem 3.2** Fix $m > 0$, $\gamma > 0$, and $\beta > 0$, let $\beta_n = \vartheta(n)\beta$, where $\vartheta(n) > 0$ is a scaling parameter, such that

$$\lim_{n \to \infty} \frac{\vartheta(n)}{n} \to \infty. \quad (3.19)$$

Let $\mu^{n,m}_{t,\beta_n,n}$ be the law of the process given by its generator (3.13), with the initial condition $\mu^{n,m}_{0,\beta_n,n} = \delta_{\psi^{n,m}_0}$, where $\psi^{n,m}_0$ is a sequence of proper initial conditions, i.e., for all $n$, $\psi^{n,m}_0 \in S^n_m$.

Then $\forall \epsilon > 0$, we have:

$$\lim_{n \to \infty} \lim_{t \to \infty} \mu^{n,m}_{t,\beta_n,n,m}(\|\bar{\psi}_n - Q_m\|_{\tilde{H}^1_{\text{per}}} < \epsilon) \to 1. \quad (3.20)$$

\footnote{Notice that as before, with our definition, $\mu_t$ actually depends on $\psi_0$. However, we omit $\psi_0$ to lighten the notation.}
We briefly sketch the proof: we have already proved that $\mu_{n,m}^{\beta_n}$ is the limit in $t$ of $\mu_{t,n,m}^{\beta_n}$. Consequently, all we have to prove is that

$$\lim_{n \to \infty} \mu_{n,m}^{\beta_n} \left( \| \psi_n - Q_m \|_{H^1_{\text{per}}} < \epsilon \right) \to 1. \quad (3.21)$$

We can prove that the measure $\mu_{n,m}^{\beta_n}$ concentrates all its mass on the (discrete) configurations having close to the minimal energy, when we send temperature to zero with a proper speed. It turns out that the proper speed here is to scale $\beta$ by $\vartheta(n)$, satisfying (3.19). Finally, we show that if a configuration has energy close to the minimal, it will be close to $Q_m$ in the sense of (3.20). This can be done by adapting certain form of concentration compactness argument to the discrete setup.

**Remark 3.1** About the exchange of limits in (3.20): In the evolution equation (3.14) the drift term $\partial_{\theta(x)} \mathcal{H}_n(\psi) = \frac{1}{n} \text{Im} [\psi(x) \Delta \psi^*(x)]$ would became very singular when $n \to \infty$ keeping the temperature positive. But with $\beta_n \to \infty$ fast enough the solution should became enough regular in space so that the corresponding limit as $n \to \infty$ should be given by the continuous deterministic NLS. This will be investigated in a future work [23]. The later suggests that one could study the joint limit $n, t \to \infty$, with $t_n = n^{\alpha} t$. We address the case $t_n \ll \beta_n$ in [23]. However, the case $t_n \gg \beta_n$ seems more challenging.

### 4 Discrete "Solitons"

As we already observed in Theorem 3.1, the function $Q_m$ (Solitons) can be characterized as the minimizer of a certain variational problem, where we have the compactness of the minimizing sequence. Therefore, one can observe that for a function $u \in H^1_{\text{per}}([0, 1])$, with $\mathcal{M}(u) = m$, having "close to minimal" energies, means the function itself is close to $Q_m$ in the following sense:

**Lemma 4.1** Assume $u \in H^1_{\text{per}}$ and $\mathcal{M}(u) = m$, then, $\forall \epsilon > 0, \exists \delta > 0$, such that if $\mathcal{H}(u) \leq E_0^0(m) + \delta$, then there exists $\gamma \in [0, 2\pi], x \in [0, 1], such that \| e^{i\gamma} u(\cdot + x) - Q_m \|_{H^1_{\text{per}}} < \epsilon$, equivalently $\| u(x) - Q_m \|_{H^1_{\text{per}}} < \epsilon$.

**Proof** This is straightforward by the compactness of the minimizing sequence in Theorem 3.1. $\square$

In this section, we establish a similar result in the discrete setup (cf. Proposition 4.1). There are several works in the literature regarding relation among continuous and "discrete" solitons also known as discrete breathers (cf. [2, 3] and references therein for unbounded domain). Since our proposition is slightly different from these existing works, to be self-contained and for the sake of completeness, we bring a different proof for the above-mentioned proposition. This proof is adapted to our case with bounded domain.

Similar to (3.6), fix $n > 1, m > 0$ and define $E_0^n(m)$ as follows:

$$E_0^n(m) := \inf \{ \mathcal{H}_n(\psi_n) | \psi_n \in S^n_m \}. \quad (4.1)$$
Since $\mathcal{H}_n(\psi_n)$ is a continuous function from the compact set $S^n_m$ to $\mathbb{R}$, the image of this function is compact, hence, $-\infty < E^n_0(\psi_n)$, and this infimum is achieved in a compact set, which will be called the set of "discrete Solitons" and denoted by $\emptyset \neq Q^n_m \subset S^n_m$. By the same argument as in the continuous case, discrete solitons are real-valued and positive up to a constant phase.

For $\psi_n \in S^n_m$, we define $\|\psi_n\|_{L^p(\mathbb{T}_n^m)}^p = \frac{1}{n} \sum_j |\psi_n(j)|^p$. Then we can write $\mathcal{H}_n(\psi_n) = G_n(\psi_n) - \frac{1}{2} \|\psi_n\|_{L^4(\mathbb{T}_n^m)}^4$, and by using the discrete Gagliardo-Nirenberg inequality (C.11), we have:

$$-\theta(m) \leq E^n_0(m) < 0,$$

(4.2)

where $\theta(m) = \frac{C^2}{6|\theta|} m^3 + \frac{C}{4} m^2$. First inequality is a direct consequence of (C.11), and the second one can be deduced by considering the constant function $\psi_n(x) = \sqrt{m}$, for all $x \in \mathbb{T}_n^m$.

From (4.2) we establish a simple but useful lemma:

**Lemma 4.2** For every $\epsilon > 0$, there exists $C(m, \epsilon)$, such that for all $n \in \mathbb{N}$ and $\psi_n \in S^n_m$, with $\mathcal{H}_n(\psi_n) \leq E^n_0(m) + \epsilon$, we have $G_n(\psi_n) \leq C(m, \epsilon)$.

**Proof** Consider the inequality (4.2), and (C.11); denote $x = G_n(\psi_n)^{\frac{1}{2}}$ so $x \geq 0$. If $\mathcal{H}_n(\psi_n) \leq E^n_0 + \epsilon$ then, thanks to (C.11) we have

$$E^n_0(m) + \epsilon \geq x^2 - c'm^2 - c'm^2 \implies x \leq C^{\frac{1}{2}}(m, \epsilon).$$

(4.3)

where $C^{\frac{1}{2}}(m, \epsilon)$ is given by

$$C^{\frac{1}{2}}(m, \epsilon) = \frac{c'm^3 + \sqrt{c'm^3 + 4(c'm^2 + E^n_0(m) + \epsilon)}}{2},$$

where the expression under the square root is clearly positive, thanks to the expression of $\theta(m)$.

Lemma 4.2 states that if the energy is "small" ($O(1)$), then the configuration should be "smooth" i.e., $G_n \sim O(1)$.

In the rest of this section, we prove that $\tilde{\psi}_n$, the linear interpolation of a configuration $\psi_n$, is arbitrarily close to $Q_m$ in $H^1_{per}$, if we take $n$ sufficiently large, and the energy of $\psi_n$, $\mathcal{H}_n(\psi_n)$, sufficiently close to $E^n_0(m)$. The proof relies on the fact that configurations with close to minimal energies are smooth in the sense that their linear interpolation’s norm $(L^p, H^1$ or even the energy) is close to the corresponding discrete norms. This result heavily depends on the inequality of Appendix C. We begin by stating this result:

**Proposition 4.1** Fix $m > 0$, for any $\epsilon > 0$, there exists $\eta(\epsilon)$ and $N_0(\epsilon)$, such that for $n > N_0(\epsilon)$, if $\mathcal{H}_n(\psi_n) \leq E^n_0(m) + \eta$, then we have: $\|\psi_n - Q_m\|_{H^1_{per}} < \epsilon$.

We divide the proof of (4.1), into a couple of simple lemmas. The advantage of the linear interpolation (3.17) is that it conserves the kinetic energy, i.e., $G_n(\psi_n) = \frac{1}{2} \int_0^1 |\partial_t \tilde{\psi}_n|^2$. But unfortunately, in general, we have $\|\tilde{\psi}_n\|_{L^p(\mathbb{T}_n^m)} \geq \|\tilde{\psi}_n\|_{L^p}$ for $p \geq 1$, thanks to the Jensen inequality. Consequently, in general we have $\mathcal{H}_n(\psi_n) \leq \mathcal{H}(\tilde{\psi}_n)$ and $\mathcal{M}_n(\psi_n) \geq \mathcal{M}(\tilde{\psi}_n)$. However, the following lemma helps to establish the fact that these quantities are "close", for configurations with near minimal energies.
Lemma 4.3 For all \( n \in \mathbb{N} \), if \( \psi_n \in S^n_m \) we have:

\[
\left| \| \tilde{\psi}_n \|_{L^p(T)}^p - \| \psi_n \|_{L^p(\tilde{T}_n)}^p \right| \leq \frac{p(2G_n(\psi_n))^{1/2}(m^{1/2} + G_n(\psi_n)^{1/2})^{p-1}}{n},
\]

Proof We have \( \psi_n \in S^n_m \), and define \( \ell_n = \min\{|\psi(x)| \mid x \in \tilde{T}_n\} \), clearly \( \ell_n \leq \sqrt{m} \). Moreover, for any \( x \in \tilde{T}_n \), we have:

\[
|\ell_n - \psi_n(x)| \leq \sum_{j=1}^{n} |\psi_n(j) - \psi_n(j - 1)| \leq \sqrt{n} \left( \sum_{j=1}^{n} |\psi_n(j) - \psi_n(j - 1)|^2 \right)^{1/2} = \sqrt{2G_n(\psi_n)},
\]

where we used a Cauchy-Schwartz inequality. Therefore, we can deduce that

\[
\sup_x |\psi(x)| \leq c_1 = m^{1/2} + (2G_n(\psi_n))^{1/2}.
\]

Moreover, thanks to the definition of \( \tilde{\psi}_n(y) \), we have \( |\tilde{\psi}_n(y)| \leq c_1 \), for all \( y \in \mathbb{T} \). Then we can simply compute:

\[
\left| \| \tilde{\psi}_n \|_{L^p(\mathbb{T})}^p - \| \psi_n \|_{L^p(\tilde{T}_n)}^p \right| \leq \sum_{x=0}^{n-1} \int_{x/n}^{(x+1)/n} \left| |\psi_n(x)|^p - |\tilde{\psi}_n(y)|^p \right| dy
\]

\[
\leq \frac{pc_1^{p-1}}{n} \sum_{x=1}^{n} \int_{x-1/n}^{x} \left| |\psi_n(x) - \psi_n(x-1)| \right| dy
\]

\[
\leq \frac{pc_1^{p-1}}{n} \sum_{x=1}^{n} |\psi_n(x) - \psi_n(x-1)|
\]

\[
\leq \frac{pc_1^{p-1}(2G_n(\psi_n))^{1/2}}{n}, \tag{4.4}
\]

where the first inequality comes from the definition, in the second inequality we used the fact that \( \psi_n(x) \) and \( \tilde{\psi}_n(y) \) are bounded uniformly in \( x \) and \( y \), and in the third inequality we used the definition of \( \tilde{\psi}_n(y) \):

\[
|\psi_n(x) - \tilde{\psi}_n(y)| \leq |\psi_n(x) - \psi_n(x-1)|.
\]

Notice that the last inequality in (4.4) is obtained as above.

As a straightforward consequence of Lemma 4.3, we can deduce the following corollaries:

Corollary 4.3.1 For any \( c > 0 \), there exist \( C_1(c, m) \), such that for every \( n \in \mathbb{N} \), and \( \psi_n \in S^n_m \), such that \( G_n(\psi_n) < c \), then \( |\mathcal{H}(\tilde{\psi}_n) - \mathcal{H}_n(\psi_n)| \leq \frac{C_1(c, m)}{n} \).

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Proof Thanks to the definition of $\bar{\psi}_n$ (3.17), the weak derivative of $\bar{\psi}_n$ is given as follows: for any $y \in [0, 1]$, if $\frac{1}{n} \leq y < \frac{1 + 1}{n}$ with $x \in \mathbb{Z}_n$, then $\partial_y \bar{\psi}_n(y) = n(\psi_n(x + 1) - \psi_n(x))$. Therefore, we have: $\frac{1}{2} \int_0^1 |\partial_y \bar{\psi}_n|^2 = \frac{n}{2} \sum_{x=1}^{\infty} |\psi(x) - \psi(x - 1)|^2 = G_n(\psi_n)$. Hence, we have:

$$|\mathcal{H}_n(\psi_n) - \mathcal{H}(\bar{\psi}_n)| = \frac{1}{4} \left| \bar{\psi}_n \|_{L^4(T)}^4 - \|\psi_n\|_{L^4(\mathbb{T}_n)}^4 \right|,$$

and we can conclude thanks to Lemma 4.3. □

Corollary 4.3.2 For any $\delta > 0$, there exist $\eta > 0$ and $N_0(\delta)$, such that for $n > N_0(\delta)$ if $\mathcal{H}_n(\psi_n) \leq E_0^n(m) + \eta$ then $\mathcal{H}(\bar{\psi}_n) \leq E_0^n(m) + \delta$.

Proof It follows immediately from Corollary 4.3.1 and Lemma 4.2. □

Proposition 4.2

$$\lim_{n \to \infty} E_0^n(m) \to E_0(m). \quad (4.5)$$

Proof Before proceeding, we emphasize the fact that all the constants $c, c_1, c_2, c', \ldots$ are independent of $n$ in this proof.

Recall the definition of $Q_m$ as the minimizer of (3.6). Moreover, recall the definition of the set of discrete Solutions $Q_m^n$, as the set of minimizer of (4.1). Take $q_n \in Q_m^n$, notice that thanks to the inequality $|\psi_n(x) - \psi_n(x - 1)| \geq ||\psi_n(x) - \psi_n(x - 1)||$, we can take $q_n$ to be real-valued and positive. Then we have: $\mathcal{H}(Q_m) = E_0(m)$, and for all $n$, $\mathcal{H}_n(q_n) = E_0^n(m)$. thanks to Lemma 4.2 there exists $c > 0$ uniform in $n$, such that $G_n(q_n) \leq c$. Therefore, we can use the result of Corollary 4.3.1, and deduce that there exists $C_1$ independent of $n$, such that:

$$|\mathcal{H}(\bar{q}_n) - \mathcal{H}(q_n)| \leq \frac{C_1}{n}. \quad (4.6)$$

For any $\psi \in H_{per}^1([0, 1])$, define $\lambda_n(\psi)$ as follows:

$$\lambda_n(\psi) = \left( \frac{m}{\mathcal{M}(\psi)} \right)^{\frac{1}{2}}. \quad (4.7)$$

In particular, let $\lambda_n = \lambda(\bar{q}_n)$ and observe that for $n$ sufficiently large, $|\lambda_n^2 - 1| \leq \frac{c_0}{n}$, with $c_0$ independent of $n$, thanks to Lemma 4.3.

More precisely, we can take $c_0 = \frac{c}{m}$, for $n$ sufficiently large, where $\tilde{c}$ is given by Lemma 4.3. Now, if we use the definition of $\mathcal{H}$, for $n$ sufficiently large we obtain:

$$|\mathcal{H}(\lambda_n \bar{q}_n) - \mathcal{H}(\bar{q}_n)| \leq |\lambda_n^2 - 1| \int_0^1 \frac{1}{2} |\partial_y \bar{q}_n(y)|^2 \, dy + \frac{|\lambda_n^4 - 1|}{4} \int_0^1 |\bar{q}_n(y)|^4 \, dy \leq \frac{c_1}{n}, \quad (4.8)$$
where $c_1$ is independent of $n$, and we used the estimate $|\lambda_n^2 - 1| \leq \frac{c_0}{n}$; moreover, in order to treat the first term, we take advantage of the fact that $G_n(q_n) = \int_0^1 \frac{1}{2} |\partial_y \tilde{q}_n(y)|^2 dy \leq c$.

Lastly, the second term is bounded as follows: we used the bound $\|q_n\|_{C^4(\bar{T}_n)} \leq c'$ (thanks to Lemma 4.2 and (4.2)), then we conclude by using the fact $\|q_n\|_{C^4(\bar{T}_n)} - \|\tilde{q}_n\|_{L^4(\Omega)} \leq \frac{c'}{n}$ which is a direct consequence of Lemma 4.3.

Notice that $M(\lambda_n \bar{q}_n) = m$; therefore, $E_0(m) \leq H(\lambda_n \bar{q}_n)$. Combining this fact with (4.6) and (4.8), for $n$ large enough we have:

$$E_0(m) \leq E_0^0(m) + \frac{c''}{n},$$

where $c''$ is a constant independent of $n$, and we used the fact that $H_n(q_n) = E_0^0(m)$.

On the other hand, recall that $Q_m$ is smooth, real-valued and non-negative thanks to Theorem 3.1. Define $Q_n : \tilde{T}_n \rightarrow \mathbb{C}$ as $Q_n(x) = Q_m(\frac{x}{n})$, for $x \in \tilde{T}_n$.

Let $\tilde{\lambda}_n := \left( \frac{m}{\mathcal{M}_n(Q_n)} \right)^{\frac{1}{2}}$.

Thank to the properties of $Q_m$ (in particular the fact that $Q_m$ is smooth with bounded $H^1$ and $L^4$ norm), for $n$ large enough we have:

$$|\tilde{\lambda}_n^2 - 1| \leq \frac{c_2}{n},$$

where one can take $c_2 = 4 \frac{\|Q_m\|_{L^\infty} \|Q'_m\|_{L^\infty}}{m}$ ($Q'$ denotes the derivative of $Q$). Moreover, since $Q_m$ is smooth, $G_n(Q_n)$ and $V_n(Q_n)$ are bounded uniformly in $n$ by $\frac{\|Q_m\|_{L^\infty}}{2}$, and $\frac{\|Q_m\|_{L^\infty}^3}{4}$, respectively. Hence, thanks to (4.10) for $n$ sufficiently large we have:

$$|H_n(\tilde{\lambda}_n Q_n) - H_n(Q_m)| \leq \frac{c_3}{n}.$$  

Again, since $Q_m$ is at least $C^3$, by a simple computation we get for $n$ sufficiently large:

$$|H_n(Q_n) - H(Q_m)| \leq \frac{1}{2} \sum_{x=1}^n \int_{x-\frac{1}{n}}^{x} n^2 \left| Q_m(\frac{x}{n}) - Q_m(\frac{x-1}{n}) \right|^2 |\partial_y Q_m(y)|^2 dy$$

$$+ \frac{1}{4} \sum_{x=1}^n \int_{x-\frac{1}{n}}^{x} \left| Q_m(\frac{x}{n}) - Q_m(y) \right|^4 dy$$

$$\leq \frac{\|Q'_m\|_{L^\infty} \|Q'_m\|_{L^\infty}}{n} + \frac{\|Q'_m\|_{L^\infty} \|Q_m\|_{L^3}}{n}$$

$$\leq \frac{c_4}{n}.$$
Therefore, combining the estimates (4.11) and (4.12), and recalling the fact that $\mathcal{H}(Q_m) = E_0(m)$, we have for $n$ large enough:

$$|\mathcal{H}_n(\tilde{\lambda} Q_m^n) - \mathcal{H}(Q_m)| \leq \frac{c}{n} \implies E_0^n(m) \leq E_0(m) + \frac{c}{n}, \quad (4.13)$$

where we used the fact that $\mathcal{M}(\tilde{\lambda}_n Q_m^n) = m$, hence $E_0^n(m) \leq \mathcal{H}_n(\tilde{\lambda}_n Q_m^n)$. Finally, taking the limit of $n \to \infty$ in (4.13) and (4.9), properly (lim sup and lim inf, respectively), we deduce the result (4.5).

We finish this section by proving the Proposition 4.1:

**Proof of Proposition 4.1** In consequence of corollary 4.3.2 and Proposition 4.5, we have that for any $\delta > 0$, there exist $\eta > 0$ and $N_0(\delta)$, such that for $n > N_0(\delta)$ if $\mathcal{H}_n(\psi_n) \leq E_0^n(m) + \eta$ then $\mathcal{H}(\tilde{\psi}_n) \leq E_0(m) + 2\delta$. Define

$$\lambda_{\tilde{\psi}_n} = \left(\frac{m}{\mathcal{M}(\psi_n)}\right)^{\frac{1}{2}} \geq 1,$$

so that $\mathcal{M}(\lambda_{\tilde{\psi}_n} \tilde{\psi}_n) = m$. Furthermore by Lemma 4.3 $\lambda_{\tilde{\psi}_n} \to 1$. We also have that

$$\mathcal{H}(\lambda_{\tilde{\psi}_n} \tilde{\psi}_n) = \mathcal{H}(\tilde{\psi}_n) + (\lambda_{\tilde{\psi}_n}^2 - 1)G_n(\psi_n) - (\lambda_{\tilde{\psi}_n}^4 - 1)V(\tilde{\psi}_n)$$

$$\leq E_0(m) + 2\delta + (\lambda_{\tilde{\psi}_n}^2 - 1)C, \quad (4.14)$$

where we bounded $G_n$ thanks to Lemma 4.2. By lemma 4.1, we have $\|\lambda_{\tilde{\psi}_n} \psi_n - Q_m\|_{\tilde{H}_1} < \epsilon / 2$, and since

$$\|\lambda_{\tilde{\psi}_n} \psi_n - \tilde{\psi}_n\|_{\tilde{H}_1} \leq |\lambda_{\tilde{\psi}_n} - 1|^{1/2}\|\psi_n\|_{\tilde{H}_1} < \epsilon / 2$$

for $n$ large enough, we conclude the proof.

**5 Large deviation estimates**

In Proposition 4.1, we proved that if the energy $\mathcal{H}_n(\psi_n)$ is sufficiently close to the minimal energy $E_0^n(m)$ for $n$ sufficiently large, then the linear interpolation of a configuration $\psi_n$ is close to $Q_m$ in $\tilde{H}_1$-norm. In this section, we prove that the measure $\mu_{\beta_n,m}^n$ (3.15) concentrate on configurations with minimal energy as $n \to \infty$, if we set $\beta_n = \vartheta(n)\beta$, where $\vartheta(n)$ satisfies (3.19).

In fact, thanks to Proposition 4.1, and (2.19) Theorem 3.2 can be reduced to the following theorem. As $m$ is fixed in this section, we will drop it from the notations.

**Theorem 5.1** For any $\epsilon > 0$, we have:

$$\lim_{n \to \infty} \mu_{\beta_n,m}^n(\mathcal{H}_n(\psi_n) \leq E_0^n + \epsilon) = 1, \quad (5.1)$$

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where we recall $\beta_n = \vartheta(n)\beta$, where $\vartheta(n)$ satisfies (3.19), i.e., $\lim_{n \to \infty} \vartheta(n)/n \to \infty$.

The proof of Theorem 5.1 depends on two large deviation estimates for the uniform probability measure $d\mu^n_m$ that are proven in appendix A:

1. For any $n$ and any $0 < g$, we have:

$$
\mu^n_m(G_n(\psi_n) < g) \leq \exp(-2n \ln n) \left(\frac{2g}{m}\right)^{n-1}2^n,
$$

(5.2)

This bound is proven in Lemma A.1, following the same spirit as in [10], Section 10. However, because of our special scaling in $G_n$, one should follow the dependence of the rate function on $n$ carefully, in contrast to the estimate in [10]. This lemma provides the aforementioned upper bounds. Combining (5.2) with Gagliardo-Nirenberg inequality, we can deduce a suitable upper bound for $H_n$.

2. For any $\epsilon > 0$, there exists $d = d(\epsilon)$ such that for $n \geq 2$:

$$
\mu^n_m(\mathcal{H}_n(\psi_n) < E^n_0 + \epsilon) \geq d^n e^{-2n \ln n}.
$$

(5.3)

This is proven in Lemma A.2.

We will proceed as follows: first, we state a proof of (5.1), when $\beta_n = \beta n \ln n$. This proof is quite simple and illustrates how does the above estimates are involved.

Finally, we prove the general case $\beta_n = \vartheta(n)\beta$.

Proof of Theorem 5.1 with $\beta_n = \beta n \ln n$ Assume $0 < \epsilon < 1$, in order to prove (5.1), it is sufficient to prove that:

$$
\left(\int_{S^n_m} \mathbb{1}_{\mathcal{H}_n(\psi_n) - E^n_0 \geq \epsilon} e^{-\beta_n \mathcal{H}_n(\psi_n)} d\mu^n_m \right)^{-1} \leq e^{\beta_n(E^n_0 + \frac{\epsilon}{2})} \left[\mu^n_m \left( \mathcal{H}_n(\psi_n) - E^n_0 \leq \frac{\epsilon}{2} \right) \right]^{-1} \leq e^{n \ln n \left(\beta(E^n_0 + \frac{\epsilon}{2}) + 2\right) - n \ln d}.
$$

(5.5)
Let \( c > \epsilon + \frac{2}{\beta} \); recall Lemma 4.2 and let \( c' = C(m, c) > 0 \), which is given by this lemma. By using (5.2) there exists \( N_1 \), such that for \( n > N_1 \):

\[
\int_{S^m_n} \mathbb{1}_{\{ n \mathcal{H}_n - E_n^0(m) \geq \epsilon \}} e^{-\beta_n \mathcal{H}_n} d\mu^n_m \\
= \int_{S^m_n} \mathbb{1}_{\{ n \mathcal{H}_n - E_n^0(m) \geq \epsilon \}} e^{-\beta_n \mathcal{H}_n} d\mu^n_m + \int_{S^m_n} \mathbb{1}_{\{ n \mathcal{H}_n - E_n^0(m) \geq 0 \}} e^{-\beta_n \mathcal{H}_n} d\mu^n_m \\
\leq e^{-\beta_n \ln n (E_n^0(m) + \epsilon)} \mu_n^m (G_n < c') + e^{-\beta_n \ln n (E_n^0(m) + \epsilon)} \\
\leq e^{-n \ln n (\beta E_n^0(m) + \epsilon) + 2n \ln (4c'/m)} + e^{-\beta_n \ln n (E_n^0(m) + \epsilon)},
\]

(5.6)

where in the second line we used the fact that \( \{ c > \mathcal{H}_n - E_n^0(m) \geq \epsilon \} \subset \{ G_n \leq c' \} \), thanks to the choice of \( c' \), see Lemma 4.2. Finally, taking \( N > N_1 \), and combining (5.5) and (5.6), gives us the following:

\[
0 \leq p_n \leq e^{-n \left( \ln n \left( \frac{c \beta}{\epsilon} - 2 \right) - \ln d \right)} + e^{-n \left( \ln n \left( \frac{c \beta}{\epsilon} - 2 \right) - \ln d \right)} \xrightarrow{n \to \infty} 0,
\]

(5.7)

thanks to the choice of \( c \).

Now we prove Theorem 5.1 in the general situation with \( \beta_n = \beta \vartheta(n) \) satisfying (3.19); Proof of Theorem 5.1 with \( \beta_n = \vartheta(n) \beta \) Fix \( 0 < \epsilon < 1 \) (the other cases will be straightforward). In order to prove (5.1), it is sufficient to prove (5.4).

As for (5.5), there exist \( d > 0 \) such that:

\[
\left( \int_{S^m_n} \mathbb{1}_{\{ \mathcal{H}_n - E_n^0 < \frac{\epsilon}{\beta} \}} e^{-\beta_n \mathcal{H}_n} d\mu^n_m \right)^{-1} \leq e^{\beta \vartheta(n)(E_n^0 + \frac{\epsilon}{\beta}) (d) - n \ln n}.
\]

(5.8)

Let us decompose the numerator of (5.4) into two parts and denote them by \( q_n \) and \( q'_n \):

\[
\int_{S^m_n} \mathbb{1}_{\{ \mathcal{H}_n - E_n^0 \geq \epsilon \}} e^{-\beta_n \mathcal{H}_n} d\mu^n_m = \int_{S^m_n} \mathbb{1}_{\{ \ln n \mathcal{H}_n - E_n^0 \geq \epsilon \}} e^{-\beta_n \mathcal{H}_n} d\mu^n_m \\
+ \int_{S^m_n} \mathbb{1}_{\{ \mathcal{H}_n - E_n^0 \geq \ln n \}} e^{-\beta_n \mathcal{H}_n} d\mu^n_m.
\]

(5.9)

We simply bound \( q'_n \leq e^{-\beta_n (E_n^0 + \ln n)} \) and observe that:

\[
\left( \int_{S^m_n} \mathbb{1}_{\{ \mathcal{H}_n - E_n^0 < \frac{\epsilon}{\beta} \}} e^{-\beta_n \mathcal{H}_n} d\mu^n_m \right)^{-1} q'_n \leq e^{-n \ln n (\beta \vartheta(n) - 2n) \ln n} e^{\beta \vartheta(n) \frac{\epsilon}{\beta} - n \ln d} \xrightarrow{n \to \infty} 0.
\]

(5.10)
as \( n \to \infty \), where we used the fact that \( d \) is a constant independent of \( n \), as well as the condition \( \lim_{n \to \infty} \frac{\vartheta(n)}{n} = \infty \).

Now we treat the term corresponding to \( q_n \), thanks to (5.2). First, observe that for any \( E_0^n(m) < a \leq n \), thanks to the inequality (C.11), if we have \( \mathcal{H}_n(\psi_n) \leq a \), we can deduce \( G_n(\psi_n) \leq c_1(m) + 2a \), where \( c_1(m) \) is a constant independent of \( n \) (in fact, \( c_1(m) = (\tilde{c}^2m^3 + 2\tilde{c}m^2) \), with \( \tilde{c} = \frac{C}{4} \) and \( C \) is the constant in (C.11)). Consequently, we have for any \( E_0^n(m) < a \leq n \):

\[
\mu^n_m(\mathcal{H}_n \leq a) \leq \mu^n_m(G_n \leq 2a + c_1(m)). \tag{5.11}
\]

Recall (5.2): for any \( 0 < \alpha < 2n \) denote \( \alpha_o = \frac{2\alpha}{m} \), then

\[
\mu^n_m(G_n \leq \alpha) \leq 2^n \alpha_o^{n-1} e^{-2n \ln n} \tag{5.12}
\]

holds.

Therefore, for large \( n \) thanks to (5.11), (5.12), we have for any \( E_0^n(m) < a \leq n \):

\[
\mu^n_m(\mathcal{H}_n \leq a) \leq 2^n e^{-2n \ln n} \left( \frac{4a + 2c_1(m)}{m} \right)^{n-1}. \tag{5.13}
\]

Take \( h > 0 \) independent of \( n \), let \( N = \frac{\ln n}{h} \). Then we have for \( n \) sufficiently large:

\[
q_n = \sum_{j=0}^{N-1} \int_{S_m^n} 1_{\{E_0^n + \epsilon + jh \leq \mathcal{H}_n < E_0^n + \epsilon + (j+1)h\}} e^{-\beta_n \mathcal{H}_n} d\mu^n_m
\]

\[
\leq \sum_{j=0}^{N-1} e^{-\beta_n(E_0^n + \epsilon + jh)} \mu^n_m \left( E_0^n + \epsilon + jh \leq \mathcal{H}_n < E_0^n + \epsilon + (j+1)h \right)
\]

\[
\leq 2^n e^{-2n \ln n} \sum_{j=0}^{N-1} e^{-\beta_n(E_0^n + \epsilon + jh)} \left( \frac{4}{m} \right)^{n-1} e^{(n-1) \ln(E_0^n + \epsilon + (j+1)h + \frac{c_1(m)}{2})}
\]

\[
= 2^n \left( \frac{4}{m} \right)^{n-1} e^{-2n \ln n} e^{\beta_n(h + \frac{c_1(m)}{2})} \sum_{j=1}^{N} \exp \left( -\beta_n \left( E_0^n + \epsilon + jh + \frac{c_1(m)}{2} \right) \right)
\]

\[
+ (n-1) \ln \left( E_0^n + \epsilon + jh + \frac{c_1(m)}{2} \right), \tag{5.14}
\]

where we take advantage of the estimate (5.13) in the second line. Notice that \( E_0^n + \frac{c_1(m)}{2} > 0 \) thanks to the lower bound (4.2). Recall that \( \beta_n = \frac{\vartheta(n)}{n} \), with \( \lim_{n \to \infty} \frac{\vartheta(n)}{n} \to \infty \). Therefore, for \( n \) sufficiently large \( -\beta_n + \frac{n-1}{x} < 0 \), for any \( x \in [h, 2\ln n] \). However, the latter expression is the derivative of \( -\beta_n x \) \((n-1) \ln(x)\), hence, this function is decreasing on the interval \([h + E_0^n + \epsilon + \frac{c_1(m)}{2}, 2 \ln n] \) for any \( n \) sufficiently large, and \( -\beta_n x \) \((n-1) \ln(x)\) achieves its minimum at \( x = h + E_0^n + \epsilon + \frac{c_1(m)}{2} \) in the aforementioned interval. Combining this fact with
(5.14), we get:

\[
q_n \leq 2^n \left( \frac{4}{m} \right)^{n-1} e^{-2n \ln n} e^{\beta_n (h + \frac{c_1(m)}{2})} \\
\times N \exp(-\beta_n (E_0^n + \epsilon + h + \frac{c_1(m)}{2}) + \ln(E_0^n + \epsilon + h + \frac{c_1(m)}{2})) \\
= 2^n \left( \frac{4}{m} \right)^{n-1} e^{-2n \ln n} N \exp(-\beta_n (E_0^n + \epsilon) + (n-1) \ln(E_0^n + \epsilon + h + \frac{c_1(m)}{2})).
\]

Notice that \(0 < (E_0^n + \epsilon + h + \frac{c_1(m)}{2}) < (\epsilon + h + \frac{c_1(m)}{2}) =: c',\) and \(c'\) is a constant independent of \(n.\) Combining the latter estimate (5.15), with (5.8) we get for \(n\) sufficiently large:

\[
\left( \int_{\mathcal{H}_m - E_0^n(m) < \frac{x}{\sqrt{2}}} e^{-\beta_n \mathcal{H}_m} d\mu_m \right)^{-1} q_n \leq e^{-\beta_n \frac{c'}{2}} \left( d^{-n/2n} \left( \frac{4}{m} \right)^{n-1} (c')^{n-1} \right) \frac{\ln n}{h} \to 0,
\]

(5.16)
as \(n \to \infty.\) Notice that (5.16) is evident, since the first term \(e^{-\beta_n n}\) is super-exponentially small thanks to the assumption \(\beta_n = \beta \vartheta (n)\) with \(\lim_{n \to \infty} \frac{\vartheta (n)}{n} = \infty\) and the second term is bounded by \(e^{n \tilde{c}},\) where \(\tilde{c}\) is a constant independent of \(n.\) Finally, recalling the decomposition (5.9) and combining (5.16) with (5.10) gives us (5.4) and finishes the proof.

Finally, the proof of Theorem 3.2 is a direct consequence of Proposition 4.1, and Theorem 5.1:

**Proof of Theorem 3.2** Fix \(\epsilon > 0,\) thanks to the Proposition 2.3, in particular (2.19), we have:

\[
\lim_{n \to \infty} \lim_{t \to \infty} \mu_{t,n,m}^{\beta_n} (\| \tilde{\psi}_n - Q^n \| \tilde{\mu}_{per} < \epsilon) = \lim_{n \to \infty} \mu_{\beta_n,m}^{n} (\| \tilde{\psi}_n - Q^n \| \tilde{\mu}_{per} < \epsilon).
\]

On the other hand let us take \(\delta = \delta (\epsilon),\) which is given by Proposition 4.1, then for all \(n > N_0 (\epsilon)\) thank to this proposition we have:

\[
1 \geq \mu_{\beta_n,m}^{n} (\| \tilde{\psi}_n - Q^n \| \tilde{\mu}_{per} < \epsilon) \geq \mu_{\beta_n,m}^{n} (|\mathcal{H}_n (\psi_n) - E_0^n(m)| < \delta).
\]

(5.17)

However, notice that \(\lim_{n \to \infty} \mu_{\beta_n,m}^{n} (|\mathcal{H}_n (\psi_n) - E_0^n(m)| < \delta) = 1,\) thanks to Theorem 5.1, in particular (5.1), and this finishes the proof of Theorem 3.2, i.e.,(3.20). □

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Appendix A some large deviations for the uniform probability on the sphere

We collect here some large deviation estimates concerning $\mu^n_m$, the uniform probability on the complex $n$-dimensional sphere $S^n_m$, and in particular the estimates (5.2) and (5.3). Note that in this appendix we slightly change our notations and denote the elements of $\mathbb{C}^n$ by $z$ or $\bar{z}$ instead of $\psi$.

**Lemma A.1** For any $n \in \mathbb{N}$, and $0 < g$, we have:

$$\mu^n_m(G_n(\psi_n) < g) \leq \frac{1}{\delta(1-\delta)^n-1} \left(\frac{2g}{m}\right)^{n-1} \exp(-2n \ln n), \quad (A.1)$$

for any $0 < \delta < 1$. Therefore, for any $n \in \mathbb{N}$, and $0 < g$:

$$\mu^n_m(G_n(\psi_n) < g) \leq 4n \left(\frac{2g}{m}\right)^{n-1} \exp(-2n \ln n). \quad (A.2)$$

**Proof** Let $\{Z_j\}_{j=1}^{\infty}$ be a sequence of i.i.d standard complex normal random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, i.e, for any $n > 0$, the probability density function of $(Z_1, \ldots, Z_n)$ is given by:

$$f(z) = \prod_{j=1}^{n} \frac{e^{-|z_j|^2}}{\pi}, \quad z := (z_1, \ldots, z_n) \in \mathbb{C}^n. \quad (A.3)$$

Consequently, the random vector $\{\Psi_n(j) = \sqrt{mn}Z_j / \left(\sum_{l} |Z_l|^2\right)^{1/2}, \quad j = 1, \ldots, n\}$ is distributed uniformly on $S^n_m$. For $k \in \mathbb{T}_n$, let the random variable $\hat{Z}_k$ be defined as the Fourier transform of $Z_1, \ldots, Z_n$:

$$\hat{Z}_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{-2\pi j k/n} Z_j. \quad (A.4)$$

Notice that $(\hat{Z}_1, \ldots, \hat{Z}_n)$ has the same distribution as $(Z_1, \ldots, Z_n)$. Moreover, we have the following identities thanks to the properties of discrete Fourier transform:

$$\sum_{j=1}^{n} |Z_j|^2 = \sum_{k=1}^{n} |\hat{Z}_k|^2. \quad (A.5)$$
\[
\sum_{j=1}^{n} |Z_j - Z_{j-1}|^2 = \sum_{k=1}^{n} \omega_k^2 |\hat{Z}_k|^2, \quad \text{(A.6)}
\]

where \( \omega_k = 2|\sin(\pi \frac{k}{n})| \).

Denote
\[
g_o := \frac{2g}{n^2m}, \quad \text{(A.7)}
\]

and take \( 0 < \lambda \) such that, \( 0 < 1 - g_o \lambda \). By using Chebyshev’s inequality, as well as (A.5) and (A.6), we have:

\[
\begin{align*}
\mu_m^n(G_n(\psi_n) \leq g) &= \mathbb{P}\left( \frac{n^2m \sum_{j=1}^{n} |Z_j - Z_{j-1}|^2}{\sum_{j=1}^{n} |Z_j|^2} \leq 2g \right) \\
&= \mathbb{P}\left( \sum_{k=1}^{n} \omega_k^2 |\hat{Z}_k|^2 \leq \sum_{k=1}^{n} |\hat{Z}_k|^2 g_o \right) \\
&= \mathbb{P}\left( \exp \left( -\sum_{k=1}^{n} \lambda (\omega_k^2 - g_o) |\hat{Z}_k|^2 \right) \geq 1 \right) \\
&\leq \mathbb{E}\left( \exp \left( -\sum_{k=1}^{n} \lambda (\omega_k^2 - g_o) |\hat{Z}_k|^2 \right) \right) \\
&= \prod_{k=1}^{n} \mathbb{E}\left( \exp \left( -\lambda (\omega_k^2 - g_o) |\hat{Z}_1|^2 \right) \right) \\
&= \prod_{k=1}^{n} \frac{1}{\lambda (\omega_k^2 - g_o) + 1}.
\end{align*}
\]

Notice that in the first line, we used the fact that \( \Psi_n \) is uniformly distributed on \( S^n_m \), and in the last line we used the fact that \( \hat{Z}_k \) are independent complex Gaussian variable with the same distribution as \( Z_i \), as well as the choice of \( \lambda \), which permits us to compute the last expectation. We emphasize the fact that the last bound holds for any \( 0 < \lambda < g_o^{-1} = \frac{n^2m}{2g} \), which can depend on \( n \). In fact, our choice of \( \lambda \) depends on \( n \).

Before proceeding, let us recall the following trigonometric identity:

\[
\prod_{k=1}^{n-1} \sin \left( \frac{\pi k}{n} \right) = \frac{n}{2^{n-1}}, \quad \Rightarrow \quad \frac{1}{\prod_{k=1}^{n-1} \omega_k^2} = \frac{1}{n^2}. \quad \text{(A.9)}
\]

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For any $0 < \delta < 1$, let us take $\lambda = \frac{(1-\delta)}{\omega_o}$. Notice that we have $1 - \lambda \omega_o = \delta$. Thanks to the choice of $\lambda$, by using (A.8) and (A.9), we obtain

$$
\mu_n^m(G_n(\psi_n) < \varrho) \leq \prod_{k=1}^{n} \frac{1}{\lambda(\omega_k^2 - \omega_o)} + 1 \leq \frac{1}{\delta} \prod_{k=1}^{n-1} \frac{1}{\lambda \omega_k} \leq \frac{1}{\delta} \frac{1}{(1-\delta)^{n-1}} \varrho^{n-1} \frac{1}{n^2} = \frac{1}{\delta(1-\delta)^{n-1}} \exp(-2n \ln n) \left(\frac{2\varrho}{m}\right)^{n-1}.
$$

(A.10)

Notice that the bound (5.2) corresponds to the choice $\delta = \frac{1}{2}$. Moreover, optimizing (A.1) over $\delta$ yields: for any $n \in \mathbb{N}$, and $\varrho > 0$ we get (A.2).

We obtain now the lower bound (5.3), indicating that set of configurations with close to minimal energy is "large enough".

**Lemma A.2** For any $\epsilon > 0$, a constant $c = c(\epsilon)$, independent of $n$, such that for $n > 2$, we have:

$$
\mu_n^m(\mathcal{H}_n < E_0^n + \epsilon) \geq c^n e^{-2n \ln n}.
$$

(A.11)

**Proof** Denote by $Q$ a discrete Soliton, we have that $E_0^n = \mathcal{H}_n(Q) = G_n(Q) - V_n(Q)$. We know from the results of Sect. 4 that $Q$ is uniformly bounded in $n$ as well as $G_n(Q)$ and $V_n(Q)$. Observe that

$$
\psi \in S_m^n : \mathcal{H}_n(\psi) < E_0^n + \epsilon \\supset \{ \psi \in S_m^n : |G_n(\psi) - G_n(Q)| \leq \epsilon/2, |V_n(\psi) - V_n(Q)| \leq \epsilon/2 \}.
$$

(A.12)

Consequently, we need to construct a neighborhood $\tilde{A} \subset S_m^n$ of $Q$ that is contained in the set on the RHS of (A.12), and such that $\mu_n^m(\tilde{A}) \geq c^n e^{-2n \ln n}$ for some constant depending on $\epsilon$.

Let us identify $\mathbb{C}^n \sim \mathbb{R}^{2n}$, and denote the corresponding real components of $Q$ by $(q_1, \ldots, q_{2n})$, and the components of a generic $\psi \in S_m^n \sim \mathbb{R}^{2n}$ as $(x_1, \ldots, x_{2n})$. We can choose the discrete Soliton $Q$, such that $q_{2n} \geq q_j \geq 0$.

For any small $\delta > 0$, define the set $\tilde{A}_\delta \subset \mathbb{R}^{2n-1}$ as follows:

$$
\tilde{A}_\delta = \left\{ \xi \in \left[ -\frac{\delta}{2n}, \frac{\delta}{2n} \right]^{2n-1} : \left| \sum_{j=1}^{2n-1} \xi_j q_j \right| \leq \frac{q_{2n} \delta}{2\sqrt{n}} \right\}.
$$

(A.13)

The volume of this set can be easily estimated by

$$
\text{vol}(\tilde{A}_\delta) \geq \frac{1}{3} \left( \frac{\delta}{n} \right)^{2n-1}.
$$

(A.14)
We postpone the proof of (A.14) later.

We now define our neighborhood of $Q$ as

$$\tilde{A}_\delta = \left\{ x \in S^{2n-1}_{\sqrt{nm}} : x_j = q_j + \xi_j, j = 1, \ldots, 2n-1; \xi \in \tilde{A}_\delta \right\}. \tag{A.15}$$

Notice that if $x \in \tilde{A}_\delta$, we have automatically that $x_{2n} = (nm - \sum_{j=1}^{2n-1} (q_j + \xi_j)^2)^{1/2}$. Furthermore, we have that

$$|x_{2n} - q_{2n}| \leq \frac{2\delta}{\sqrt{n}}. \tag{A.16}$$

It is easy to check that if $x \in \tilde{A}_\delta$, then $|V_n(x) - V_n(Q)| \leq 2C_v \delta/n$, where $C_v$ is a constant independent of $n$. About the gradients term, denoting $\xi_{2n} = x_{2n} - q_{2n}$, we have

$$G_n(\xi) = \frac{n}{2} \sum_{i=1}^{n-1} (\xi_{i+1} - \xi_i)^2 + \frac{n}{2} (\xi_n - \xi_1)^2$$

$$+ \frac{n}{2} \sum_{i=1}^{n-2} (\xi_{n+i+1} - \xi_{n+i})^2 + \frac{n}{2} (\xi_{2n} - \xi_{2n-1})^2 + \frac{n}{2} (\xi_{2n} - \xi_{n+1})^2 \leq 6\delta^2, \tag{A.17}$$

and

$$|G_n(x) - G_n(Q)| = \left| \sum_j (q_j - q_{j-1})(\xi_j - \xi_{j-1}) + G_n(\xi) \right|$$

$$\leq (2G_n(Q))^{1/2} \left( 2G_n(\xi) \right)^{1/2} + G_n(\xi)$$

$$\leq C \left( 2G_n(\xi) \right)^{1/2} + G_n(\xi) \leq C_g \delta, \tag{A.18}$$

where $C_g$ is a constant independent of $n$. It follows that, choosing $\delta < \min\{\epsilon/2C_g, \epsilon/2C_v\}$, the set $\tilde{A}_\delta$ is contained in the set defined in (A.12).

In order to compute $\mu^n_m(\tilde{A}_\delta)$ we use the following change of variable formula for any measurable $f : S^2_{2n} \rightarrow \mathbb{R}$: (cf. Appendix A of [1])

$$\int_{S^2_{2n}} f(x) d\sigma_r^{2n}(x) = \frac{r^2}{2r^{2n}n V(\mathbb{B}^{2n}_1)} \int_{\mathbb{B}^{2n-1}} \frac{f(y, \sqrt{r^2 - \|y\|^2}) + f(y, -\sqrt{r^2 - \|y\|^2})}{\sqrt{r^2 - \|y\|^2}} dy_1 \ldots dy_{2n-1}, \tag{A.19}$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^{2n-1}$, and $V(\mathbb{B}^{2n}_1) = \frac{\pi^n}{n!}$ denotes the volume of the unit ball. Applying the above formula and noticing that $nm \geq \sqrt{nm - \|y\|^2}$,
we have

\[ \mu_m^n(\tilde{A}_k) = \frac{mn!}{2(nm)^n \pi^n} \int_{\mathbb{S}^{2n-1}_{\tilde{A} \delta}} 1_{\tilde{A}_k} (y, \sqrt{nm - \|y\|^2}) + 1_{\tilde{A}_k} (y, -\sqrt{nm - \|y\|^2}) \frac{1}{\sqrt{nm - \|y\|^2}} dy_1 \ldots dy_{2n-1} \]

\[ \geq \frac{n!}{2n(nm)^n \pi^n} \int_{\mathbb{S}^{2n-1}_{\tilde{A} \delta}} 1_{\tilde{A}_k} (y, \sqrt{nm - \|y\|^2}) + 1_{\tilde{A}_k} (y, -\sqrt{nm - \|y\|^2}) \frac{1}{\sqrt{nm - \|y\|^2}} dy_1 \ldots dy_{2n-1} \]

\[ = \frac{n!}{n(nm)^n \pi^n} \int_{\frac{\delta}{n}} \frac{\delta}{n} \ldots \int_{\frac{\delta}{n}} \frac{\delta}{n} 1_{\tilde{A}_k} (\xi) d\xi_1 \ldots d\xi_{2n-1} \geq \frac{n!}{n(nm)^n \pi^n} \frac{1}{3} \left( \frac{\delta}{n} \right)^{2n-1} \] (A.20)

and by Stirling approximation we have the desired lower bound. \( \Box \)

**Proof of (A.14)** Let \( \{\xi_j\}_{j=1}^\infty \) be a sequence of i.i.d random variables uniformly distributed on \( [-\frac{\delta}{2n}, \frac{\delta}{2n}] \). Thanks to Chebyshev’s inequality we get:

\[ \mathbb{P} \left( \left| \sum_{j=1}^{2n-1} \xi_j q_j \right| \geq \frac{q_{2n} \delta}{2 \sqrt{n}} \right) = 1 - \mathbb{P} \left( \left| \sum_{j=1}^{2n-1} \xi_j q_j \right|^2 > \left( \frac{q_{2n} \delta}{2 \sqrt{n}} \right)^2 \right) \]

\[ \geq 1 - \frac{4n}{q_{2n}^2 \delta^2} \mathbb{E} \left( \left| \sum_{j=1}^{2n-1} \xi_j q_j \right|^2 \right) \]

\[ \geq 1 - \frac{4n}{\delta^2} \mathbb{E} (\xi_1^2) \sum_{j=1}^{2n-1} \left( \frac{q_j}{q_{2n}} \right)^2 = 1 - \frac{1}{3n} \sum_{j=1}^{2n-1} \left( \frac{q_j}{q_{2n}} \right)^2 \geq 1 - \frac{2}{3}, \] (A.21)

where we used our choice of the discrete Soliton \( 0 \leq q_j \leq q_{2n} \). \( \Box \)

We conclude this section mentioning some more precise limits on the large deviations for the uniform measure on the sphere, with a matching lower bound for Large deviation estimate (A.1). *These results are not used for proving theorem 5.1 and theorem 3.2, so their proof would be published in a future work* [23]. For \( 0 \leq a < 2 \) we have:

\[ \limsup_{n \to \infty} \frac{1}{n} \ln(e^{(2-a)n \ln n} \mu_m^n(G_n < cn^a)) \leq \ln \left( \frac{2c}{m} \right). \] (A.22)

\[ \liminf_{n \to \infty} \frac{1}{n} \ln(e^{(2-a)n \ln n} \mu_m^n(G_n \leq cn^a)) \geq \ln \left( \frac{2c}{m} \right). \] (A.23)

**Appendix B Hypoellipticity**

In this section, we prove that the generator (2.7), is hypoelliptic, and therefore the invariant measure has a smooth density. Notice that we add the subscript \( n \), to emphasize the dependence on \( n \).
Lemma B.1 Recall the operator \( L_n = A_n + S_n \), where \( A_n = A \) and \( S_n = S \) are given by (2.4), (2.6), respectively. Then \( L_n \) is hypoelliptic. Consequently, the invariant measure has smooth density with respect to \( d\mu_n^m \).

Proof Let us the fix the parameters \( h = s = \gamma = 1 \), the proof for other cases is similar. In order to prove this lemma, it is sufficient to show that \( L_n \) satisfies the so-called Hörmander condition. Then the hypoellipticity, and smoothness of the invariant measure follow by Hörmander’s Theorem (hypoellipticity follows from Theorem 22.2.1 of [24], for a general review one can also see [4], and [22]). We prove this condition in the case \( d = 1 \) in details, the generalization to higher dimensions is a matter of messier algebra (we comment on this at the end of the proof). Let us denote \( Y_x \) of [24], for a general review one can also see [4], and [22]). We prove this condition called Hörmander condition. Then the hypoellipticity, and smoothness of the invariant measure follow by Hörmander’s Theorem (hypoellipticity follows from Thorem 22.2.1 of [24], for a general review one can also see [4], and [22]). We prove this condition in the case \( d = 1 \) in details, the generalization to higher dimensions is a matter of messier algebra (we comment on this at the end of the proof). Let us denote \( Y_0 = A_n \) and \( Y_x = \partial_{\theta(x)} \) for \( x \in \widetilde{T}_n \). \( L_n \) satisfies Hörmander’s condition if the Lie algebra generated by the family

\[
\{Y_x\}_{x=1}^n, \quad \{[Y_x, Y_y]\}_{x,y=0}, \quad \{[[Y_x, Y_y], Y_z]\}_{x,y,z=0, \ldots}
\]

has full rank (here \( 2n - 1 \)) at every point \( \psi \in S_n^m \).

Let us define the following notation: for \( x, y \in \widetilde{T}_n \) and symbols \( i, r \), we define \( R_{x^i, y^r}, R_{x^i, y^r}, R_{x^i, y^r}, \) and \( R_{x^r, y^i} \) as the following rotations:

\[
R_{x^r, y^r} = \psi_r(x)\partial_{\psi_r(y)} - \psi_r(y)\partial_{\psi_r(x)}, \quad R_{x^r, y^i} = \psi_r(x)\partial_{\psi_i(y)} - \psi_i(y)\partial_{\psi_r(x)},
\]

\[
R_{x^i, y^r} = \psi_i(x)\partial_{\psi_r(y)} - \psi_r(y)\partial_{\psi_i(x)}, \quad R_{x^i, y^i} = \psi_i(x)\partial_{\psi_i(y)} - \psi_i(y)\partial_{\psi_i(x)}.
\]

We can rewrite \( \partial_{\theta(x)} \), and the Hamiltonian operator \( A_n \) in terms of these rotations:

\[
\partial_{\theta(x)} = R_{x^r, x^i},
\]

\[
A_n = \sum_{x \in \widetilde{T}_n} R_{(x+1)^r, x^i} + R_{x^r, (x+1)^i} - 2R_{x^r, x^i} + \kappa|\psi(x)|^{p-1}R_{x^r, x^i}.
\]

Observe that for any \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \{x^\mu | x \in \widetilde{T}_n, \mu \in \{r, i\} \} \), (these indices are of the form \( x^i, x^r \)), we have (recall \( [a, b] = ab - ba \)):

\[
[R_{\alpha_1, \alpha_2}, R_{\alpha_3, \alpha_4}] = \delta_{\alpha_1, \alpha_4}R_{\alpha_2, \alpha_3} + \delta_{\alpha_2, \alpha_3}R_{\alpha_1, \alpha_4} - \delta_{\alpha_1, \alpha_3}R_{\alpha_2, \alpha_4} - \delta_{\alpha_2, \alpha_4}R_{\alpha_1, \alpha_3}
\]

\[
= \sum_{i,j=1}^4 \delta_{[i+j-5]}R_{\alpha_i, \alpha_j} \delta_{\alpha_k, \alpha_l} (-1)^{i+j+1},
\]

where \( \{k, l\} := \{1, 2, 3, 4\} \setminus \{i, j\} \).

We rewrite the following commutators in terms of these rotations for every \( x \in \widetilde{T}_n \):

\[
\partial_{\theta(x)} = R_{x^r, x^i},
\]

\[
A_x := [A_n, \partial_{\theta(x)}] = R_{x^r, (x+1)^r} + R_{x^i, (x+1)^i} - R_{(x-1)^r, x^r} - R_{(x-1)^i, x^i},
\]

\[
A_{x, x+1} := [[A_n, \partial_{\theta(x)}], \partial_{\theta(x+1)}] = R_{x^r, (x+1)^i} - R_{x^i, (x+1)^r},
\]

\[
A_{x, x+1, x} := [[[A_n, \partial_{\theta(x)}], \partial_{\theta(x+1)}], \partial_{\theta(x)}] = R_{x^r, (x+1)^r} + R_{x^i, (x+1)^i}.
\]
We can compute the following commutators: \( A_{x,x+1}^{(2)} := [A_x, x+1, A_{x+1,x+2,x+1}] \) and \( [A_{x,x+1}, \partial \Phi(x+2)] \) thanks to (B.3), and observe that \( R_{x'}(x+2)^i - R_{x'}(x+2)^{i'} \) and \( R_{x',y'}(x+2)^{i'} + R_{x',y'}(x+2)^i \) belong to our Lie algebra. Repeating this process, following an induction, we observe that for \( x, y \in \mathbb{T}_n \), the following terms are in the Lie algebra generated by \( \{Y_x\}^{n}_{x=1} \), \( \{[Y_x, Y_y]\}^{n}_{x,y=0} \), \( \{[[Y_x, Y_y], Y_z]\}^{n}_{x,y,z=0} \) : 

\[
G_n^o := \{R_{x',x}, R_{x',y} - R_{x',y'}, R_{x',y'} + R_{x',y'}|x, y \in \mathbb{T}_n\}. \tag{B.5}
\]

Notice that in the linear case (absence of non-linearity i.e., \( p = 2 \)), the terms appeared in (B.5) represent a basis for the Lie algebra (all elements are linear combinations of these terms).

In the following, we observe that \( G_n^o \) has rank \( 2n - 1 \) for any \( \psi \in S^n = \{\psi \in \mathbb{C}^n | \sum_{x=1}^n |\psi(x)|^2 = 1\} \), notice that we consider \( S^n \) as a \( 2n - 1 \) real sphere \( \mathbb{S}^{2n-1} \) (the case where we replace \( S^n \) by \( S_m^n \) can be treated similarly). Let us prove by an induction. The case \( n = 1 \) is trivial, since \( \partial \Phi = \psi_r(1)\psi_r(1) - \psi_r(1)\psi_r(1) \) has rank one for any \( \psi_r(1) \in \mathbb{S}^1 \) (|\( \psi_r(1)|^2 + |\tilde{\psi}_r(1)|^2 = 1\)).

Assume \( G_n^o \) has rank \( 2n - 1 \) at every point of \( \mathbb{S}^{2n-1} \), we prove that \( G_{n+1}^o \) has rank \( 2n + 1 \) at every point of \( \mathbb{S}^{2n+1} \). We split the proof into two cases:

Case 1. Take \( \psi \in S^{n+1} \), and assume that there exists at least one point \( x \), such that \( |\psi(x)| = 0 \), we can take \( x = n + 1 \), since we are in the periodic setup. We have \( \psi_r(n+1) = \psi_r(n+1) = 0 \); therefore, \( \hat{\psi} = (\psi_1, \ldots, \psi_n) \in S^n \), and \( G_n^o \) has rank \( 2n - 1 \) by induction hypothesis. On the other hand, since \( \hat{\psi} \in S^n \), there exists \( y \in \mathbb{T}_n \), such that \( |\psi(y)| \neq 0 \). First, observe that

\[
B := \{\psi_r(y)\partial \psi_r(n+1) - \psi_i(y)\partial \psi_r(n+1), \psi_r(y)\partial \psi_r(n+1) + \psi_i(y)\partial \psi_r(n+1)\}
\]

\[
= \{R_{y',(n+1)^i} - R_{y',(n+1)^{i'}}, R_{y',(n+1)^{i'}} - R_{y',(n+1)^i}\} \subset G_{n+1}^o,
\]

has rank two (this is straightforward, since \( (\psi_r(y), \psi_i(y)) \neq 0 \), and one can see a linear combination of elements of \( B \) is zero iff \( |\psi(y)| = 0 \)). Then the result follows from the induction hypothesis, as well as the fact that \( B \) is orthogonal to \( G_n^o \).

Case 2. Take \( \psi \in S^{n+1} \) and assume \( |\psi(x)| \neq 0 \) for all \( x \in \mathbb{T}_{n+1} \). In this case, we claim the set

\[
G_{n+1}^i := \{R_{(n+1)^i,(n+1)^i}, R_{(n+1)^i,(n+1)^{i'}}, R_{(n+1)^{i'}}, R_{(n+1)^{i'}}, x^r \}
\]

\[
+ R_{(n+1)^{i'},x^i}|x \in \mathbb{T}_n\} \subset G_n^o,
\]

has rank \( 2n + 1 \). In fact, this set has \( 2n + 1 \) elements, where we observe that they are linearly independent. Take real 4 coefficients \( \{a_x, b_x, c_x\}^n_{x=1} \) such that

\[
cR_{(n+1)^{i'},(n+1)^i} + \sum_{x=1}^n a_x \left( R_{(n+1)^{i'},x^i} - R_{(n+1)^{i'},x^r} \right)
\]

\[
+ b_x \left( R_{(n+1)^i},x^r + R_{(n+1)^i},x^r \right) = 0.
\]

\(^4\) Notice that we are considering the Field \( \mathbb{R} \) here, by decomposing \( \psi \) into real and imaginary parts.
Computing the coefficients of $\partial_{\psi_r(x)}$ and $\partial_{\psi_l(x)}$, for any $x \in T_n$ we get:

\[
\begin{align*}
(b_x \psi_r(n + 1) - a_x \psi_l(n + 1))\partial_{\psi_r(x)} &= 0, \\
(a_x \psi_r(n + 1) + b_x \psi_l(n + 1))\partial_{\psi_l(x)} &= 0.
\end{align*}
\] (B.8)

Notice that if $(a_x, b_x) \neq 0$, then det \( \left( \frac{b_x - a_x}{a_x} \right) \) \( \neq 0 \). However, in order to (B.8) holds, the later cannot happen, since $(\psi_r(n + 1), \psi_l(n + 1)) \neq 0$; therefore $a_x = b_x = 0$ for all $x \in T_n$, and we can deduce $c = 0$, which yields the result in the case $d = 1$.

In order to prove the result for $d > 1$, for any $x, y \in T_n^d$, and any $\mu, \nu \in \{r, i\}$, we define $\mathcal{R}_{x^\mu, y^\nu}$, similar to (B.1). Recall $\{e_j\}_{j=1}^d$ as the canonical basis of $\mathbb{R}^d$, then (B.2) will be modified as:

\[
\partial_{\theta(x)} = \mathcal{R}_{x^\mu, x^\mu},
\]

\[
A_n = \sum_{x \in T_n} \sum_{j=1}^d \left( \mathcal{R}_{(x+e_j)^\nu, x^\mu} + \mathcal{R}_{x^\mu, (x+e_j)^\nu} - 2 \mathcal{R}_{x^\mu, x^\mu} \right) + \sum_{x \in T_n} \kappa |\psi(x)|^{p-1} \mathcal{R}_{x^\mu, x^\mu}.
\] (B.9)

The identity (B.3) remains true by taking $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \{x^\mu | x \in T_n^d, \mu \in \{r, i\}\}$. This leads to the following modification of (B.4), for any $x \in T_n^d$, and $1 \leq k \leq d$:

\[
A_x := [A_n, \partial_{\theta(x)}] = \sum_{j=1}^d \mathcal{R}_{x^\mu, (x+e_j)^\nu} + \mathcal{R}_{x^\mu, (x+e_j)^\nu} - \mathcal{R}_{(x-e_j)^\mu, x^\mu} - \mathcal{R}_{(x-e_j)^\mu, x^\mu}.
\]

\[
A_{x,x+e_k} := [[A_n, \partial_{\theta(x)}], \partial_{\theta(x+e_k)}] = \mathcal{R}_{x^\mu, (x+e_k)^\nu} - \mathcal{R}_{x^\mu, (x+e_k)^\nu},
\]

\[
A_{x,x+e_k,x} := [[[A_n, \partial_{\theta(x)}], \partial_{\theta(x+e_k)}], \partial_{\theta(x)}] = \mathcal{R}_{x^\mu, (x+e_k)^\nu} + \mathcal{R}_{x^\mu, (x+e_k)^\nu}.
\] (B.10)

Following the exact same strategy as in the previous case, by an induction we observe that all the terms of the form $\mathcal{R}_{x^\mu, (x+l_k e_k)^\nu} - \mathcal{R}_{x^\mu, (x+l_k e_k)^\nu}$ and $\mathcal{R}_{x^\mu, (x+l_k e_k)^\nu} + \mathcal{R}_{x^\mu, (x+l_k e_k)^\nu}$, for any $x \in T_n^d$, any $1 \leq k \leq d$, and any $l_k \in T_n$, belong to our Lie algebra. Notice that thanks to (B.3), we have:

\[
\begin{align*}
[\mathcal{R}_{x^\mu, (x+l_k e_k)^\nu} - \mathcal{R}_{x^\mu, (x+l_k e_k)^\nu}, \mathcal{R}_{x^\mu, (x+l_k e_k)^\nu} + \mathcal{R}_{x^\mu, (x+l_k e_k)^\nu}] &= - \mathcal{R}_{x^\mu, (x+l_k e_k + l_k e_k)^\nu} + \mathcal{R}_{x^\mu, (x+l_k e_k + l_k e_k)^\nu},
\end{align*}
\]

\[
\begin{align*}
[\mathcal{R}_{x^\mu, (x+l_k e_k + l_k e_k)^\nu} + \mathcal{R}_{x^\mu, (x+l_k e_k + l_k e_k)^\nu}] &= \mathcal{R}_{x^\mu, (x+l_k e_k + l_k e_k)^\nu} - \mathcal{R}_{x^\mu, (x+l_k e_k + l_k e_k)^\nu}.
\end{align*}
\] (B.11)

Repeating the above procedure for $d - 1$ times, we can deduce the following set is included in our Lie algebra:

\[
\mathcal{G}_{n}^{\mu,\nu} := \{\mathcal{R}_{x^\mu, x^\mu}, \mathcal{R}_{x^\mu, y^\mu} - \mathcal{R}_{x^\mu, y^\nu}, \mathcal{R}_{x^\mu, y^\nu} + \mathcal{R}_{x^\mu, y^\nu} | x, y \in T_n^d \}.
\] (B.12)
Recall that we observed that the rank of \( G_0^n \) is \( 2n - 1 \). However, due to symmetry one can observe that \( G_{0,d}^n \) and \( G_{0,d}^{n-1} \) has the same rank and this finishes the proof. \( \square \)

**Remark B.1** In the above proof, \( A_n \) is the Hamiltonian generator corresponding to the Hamiltonian (2.2), where the non-linearity is given by \( N_n(\psi(x)) \sim |\psi(x)|^4 \). The proof of Lemma B.1 can be adapted to more general cases where the non-linearity is given by \( N_n(\psi(x)) \sim f(|\psi(x)|^2) \), where \( f \) is a sufficiently smooth function.

## Appendix C Discrete Gagliardo-Nirenberg inequality

We present different versions of the Gagliardo-Nirenberg inequality. This inequality is crucial in the study of the sub-critical nonlinear focusing Schrödinger equation, for proving the well-posedness and characterization of the Solitons (cf. [9, 20, 39, 42]). In particular, this inequality has been used in the the proof of Theorem 3.1 in [20]. We take advantage of the discrete version of this inequality, so we can establish properties of configurations with minimal or close to minimal energy.

**Gagliardo-Nirenberg inequality** states that for every \( u \in H^1(\mathbb{R}^d) \), and \( 1 < p < 1 + \frac{4}{d} \), there exists a constant \( C(p, d) \), such that (cf. [9, 39, 42]):

\[
\|u\|_{L^p}^{p+1} = \int |u|^{p+1} \leq C(d, p) \left( \int |\nabla u|^2 \right)^{d(p-1)/4} \left( \int |u|^2 \right)^{(p+1)/2 - d(p-1)/4}. \tag{C.1}
\]

While we are focusing on the case where \( d = 1 \), and \( p = 3 < 1 + \frac{4}{d} = 5 \) and the domain is periodic, we state the following version from [20] Section 3.2, [31] Lemma 4.1. For all \( u \in H^1_{\text{per}} = H^1(\mathbb{T}) \), there exists a constant \( C > 0 \):

\[
\|u\|_{L^4}^4 = \int_0^1 |u|^4 \leq C(\|\partial_x u\|_{L^2} \|u\|_{L^2}^3 + \|u\|_{L^2}^2) = C \left( \left( \int_0^1 |u|^2 \right)^{3/2} \left( \int_0^1 |\partial_x u|^2 \right)^{1/2} + \left( \int_0^1 |u|^2 \right) \right). \tag{C.2}
\]

We bring the counterparts of these inequalities in the discrete setting, from [28], and [10] Section 17. First, let us define a handful of notations. Fix \( n > 0 \), and consider a function \( f : \mathbb{T}_n \to \mathbb{C} \), define the discrete \( \ell^p(\mathbb{T}_n) \) norm of \( f \), for \( p \geq 1 \), as:

\[
\|f\|_{\ell^p(\mathbb{T}_n)} = \left( \frac{1}{n} \sum_{x \in \mathbb{T}_n} |f(x)|^p \right)^{1/p}. \tag{C.3}
\]

Notice that our definition differs from the conventional one by a factor \( n^{-1/p} \). This difference is motivated by the fact that in the limit as \( n \to \infty \), we can recover the
continuous $L^p$ norm, formally. Define the $H^1_{\text{per}}(\mathbb{T}_n)$ norm of $f$ as follows:

$$
\|f\|_{H^1(\mathbb{T}_n)} := \left( \frac{1}{n} \sum_{x \in \mathbb{T}_n} n^2 |f(x) - f(x-1)|^2 + \frac{1}{n} \sum_{x \in \mathbb{T}_n} |f(x)|^2 \right)^{\frac{1}{2}}. \quad (C.4)
$$

We can also define the space $\ell^p(\mathbb{Z})$, with the following norm: For $f : \mathbb{Z} \to \mathbb{C}$ and $p \geq 1$ define:

$$
\|f\|_{\ell^p(\mathbb{Z})} = \left( \sum_{x \in \mathbb{Z}} |f(x)|^p \right)^{\frac{1}{p}}. \quad (C.5)
$$

As usual, we have: $\ell^p(\mathbb{Z}) = \{ f : \mathbb{Z} \to \mathbb{C} \| f \|_{\ell^p(\mathbb{Z})} < \infty \}$. We denote the discrete gradient energy of $f : \mathbb{Z} \to \mathbb{C}$ by $G(f)$, and define it as:

$$
G(f) := \frac{1}{2} \sum_{x \in \mathbb{Z}} |f(x) - f(x-1)|^2. \quad (C.6)
$$

Note the difference between $G(f)$ and $G_n(f)$ in (3.9), where we scale the second definition by $n^2$ in order to get the continuous counterpart, formally.

The first version of the discrete Gagliardo-Nirenberg inequality can be recalled from Proposition 17.6 of [10] with a small modification: For every $1 < p \leq \infty$, let $\theta = \frac{1}{2} - \frac{1}{p+1}$, obviously $\theta \in (0, 1)$, we have: there exists a constant $C(p) > 0$, such that $\forall f \in \ell^p(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$:

$$
\|f\|_{\ell^{p+1}(\mathbb{Z})} \leq C(p) \left( \|f\|_{\ell^2(\mathbb{Z})} \right)^{1-\theta} \left( G(f) \right)^{\theta/2}. \quad (C.7)
$$

In particular, for $p = 3$, we have: there exists a constant $C > 0$, such that $\forall f \in \ell^4(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$:

$$
\|f\|_{\ell^4(\mathbb{Z})}^4 \leq C(\|f\|_{\ell^2(\mathbb{Z})})^3 (G(f))^{\frac{1}{2}}. \quad (C.8)
$$

We express following lemma from [28], which is crucial for our purposes.

**Lemma C.1** Recall the definition of $G_n$ (3.9), and $\|\cdot\|_{\ell^p(\mathbb{T}_n)}$ (C.3), there exist a constant $C > 0$ independent of $n$, such that for every $f : \mathbb{T}_n \to \mathbb{C}$:

$$
\|f\|_{\ell^4(\mathbb{T}_n)}^4 \leq C(\|f\|_{\ell^2(\mathbb{T}_n)})^3 (G_n(f))^{\frac{1}{2}}. \quad (C.9)
$$
we write this inequality in this open form:

\[
\frac{1}{n} \sum_{x \in \tilde{T}_n} |f(x)|^4 \leq C \left( \left( \frac{1}{n} \sum_{x \in \tilde{T}_n} n^2 |f(x) - f(x - 1)|^2 \right)^{\frac{1}{2}} \left( \frac{1}{n} \sum_{x \in \tilde{T}_n} |f(x)|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{n} \sum_{x \in \tilde{T}_n} |f(x)|^2 \right)^{\frac{1}{2}} \right).
\]

(C.10)

Usually we have \( \frac{1}{n} \sum_{x \in \tilde{T}_n} |f(x)|^2 = m \), hence, we have:

\[
\frac{1}{n} \sum_{x \in \tilde{T}_n} |f(x)|^4 \leq C \left( m^\frac{3}{2} G_n(f)^{\frac{1}{2}} + m^2 \right).
\]

(C.11)

**Proof** This lemma is a special case of Theorem 3.2., and Theorem 3.3. of [28] (see also (1.6) of [28]).

\[\square\]

### Appendix D Jacobi elliptic functions

Given \( k \in (0, 1) \), the incomplete elliptic integral of the first kind, for any \( \phi \in \mathbb{R} \) is defined as:

\[
x = F(\phi; k) := \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}.
\]

Consequently, one can define \( \text{cn}(-) \), \( \text{sn}(-) \), \( \text{dn}(-) \) via the inverse of \( F(\cdot, k) \):

\[
\text{sn}(x, k) := \sin(\phi), \quad \text{cn}(x, k) := \cos(\phi), \quad \text{dn}(x, k) := \sqrt{1 - k^2 \sin^2(\phi)}.
\]

(D.1)

From (D.1), it is straightforward to see for all \( x \)

\[
\text{sn}^2(x, k) + \text{cn}^2(x, k) = k^2 \text{sn}^2(x, k) + \text{dn}^2(x, k) = 1.
\]

(D.2)

Moreover, the derivative (w.r.t \( x \)) of these functions can be obtained directly from the definition:

\[
\partial_x \text{sn}(x, k) = \text{cn}(x, k) \partial_x \text{dn}(x, k), \quad \partial_x \text{cn}(x, k) = -\text{sn}(x, k) \partial_x \text{dn}(x, k),
\]

\[
\partial_x \text{dn}(x, k) = -k^2 \text{cn}(x, k) \text{sn}(x, k).
\]

(D.3)

Moreover, the period of these functions is given via the following complete elliptic integral:

\[
K(k) := F(\frac{\pi}{2}; k),
\]

(D.4)
where $\partial n$ is $2K$ periodic and even, $s_n$ and $c_n$ are $4K$ periodic, where $s_n$ is $2K$ anti periodic and odd, and $c_n$ is $2K$ anti periodic and even.

Notice the limiting cases: $K(k) \to \frac{\pi}{2}$ as $k \to 0$, and $K(k) \to \infty$ as $k \to 1$. Moreover, as for $k = 0$, $s_n(x, 0) = \sin(x)$, $c_n(x, 0) = \cos(x)$, $\partial n(x, 0) = 1$. Furthermore, $s_n(x, 1) = \tanh(x)$, $c_n(x, 1) = \partial n(x, 1) = \text{sech}(x)$.

Finally, notice that from (D.3) one can deduce that $\frac{1}{\alpha} \partial n(\frac{x}{\beta}, k)$, $\frac{1}{\alpha} c_n(\frac{x}{\beta}, k)$, and $\frac{1}{\alpha} s_n(\frac{x}{\beta}, k)$ are solutions to (3.5), where $\alpha, \beta, k$ are determined by $\omega, L$ in each case, respectively.

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