Higher torsion in $p$-groups, Casimir operators and the classifying spectral sequence of a Lie algebra

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Abstract We study exceptional torsion in the integral cohomology of a family of $p$-groups associated to $p$-adic Lie algebras. A spectral sequence $E_r^{*,*}[[g]]$ is defined for any Lie algebra $g$ which models the Bockstein spectral sequence of the corresponding group in characteristic $p$. This spectral sequence is then studied for complex semisimple Lie algebras like $\mathfrak{sl}_n(\mathbb{C})$, and the results there are transferred to the corresponding $p$-group via the intermediary arithmetic Lie algebra defined over $\mathbb{Z}$.

Over $\mathbb{C}$, it is shown that $E_1^{*,*}[[g]] = H^*(g, U(g)^*) = H^*(ABG)$ where $U(g)^*$ is the dual of the universal enveloping algebra of $g$ and $ABG$ is the free loop space of the classifying space of a Lie group $G$ associated to $g$. In characteristic $p$, a phase transition is observed. For example, it is shown that the algebra $E_1^{*,*}[[\mathfrak{sl}_2[\mathbb{F}_p]]]$ requires at least 17 generators unlike its characteristic zero counterpart which only requires two.

Keywords Lie algebra · cohomology · $p$-group · free loop space · Bockstein spectral sequence

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1 Introduction

Let $k$ be a PID throughout this paper (in a lot of cases we’ll be using a field but sometimes we need $k = \mathbb{Z}$ or other decent rings and still use the term algebra through abuse of notation). By a $k$-Lie algebra $\mathfrak{g}$ we mean a free $k$-module equipped with a bilinear bracket $[-,-] : \mathfrak{g} \otimes_k \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity:

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

for all $x, y, z \in \mathfrak{g}$. We refer to the dimension of $\mathfrak{g}$ as the rank of the underlying free $k$-module, and will usually assume this is finite unless otherwise specified.

Lie algebras arise in a variety of contexts: over $\mathbb{R}$ as the tangent space at the identity of a Lie group or as the collection of smooth vector fields on a smooth manifold, over $\mathbb{Q}$ as the rational homotopy Lie algebra of a space, over $\mathbb{Z}$ as the Lie algebra associated to the descending central series of a residually nilpotent group and over the $p$-adic integers $\mathbb{Z}_p$ or finite fields in the theory of pro-$p$ groups (see [DS]) and $p$-groups respectively. For basic facts on Lie algebras and their cohomology used in this paper see [Bo] or [GG]. For the basic facts on the cohomology of groups used in this paper see [Be] or [B].

Due to our primary motivation let us give a few more details on the correspondence between $p$-groups and Lie algebras. Consider the formal power series for $e^x$ in $\mathbb{Q}[[x]]$, i.e., $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, it is a trivial formality to check $e^0 = 1$ and $e^{-x} e^x = 1$ and that if $x$ and $y$ commute then $e^{x+y} = e^x e^y$. However working in the (completed) free associative algebra over $\mathbb{Q}$ on two variables one finds that if $x$ and $y$ don’t commute, then one has the fundamental Baker-Campbell-Hausdorff identity:

$$e^x e^y = e^{x+y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \frac{1}{120}[y,[x,y]] + \cdots} = e^{x+y + I}$$

where $I$ is an infinite sum of iterated brackets of $x$ and $y$ with rational coefficients and where $[a, b] = ab - ba$. Due to this, whenever $\mathbb{Q} \subseteq k$, we can associate a formal group $e^g$ to every $k$-Lie algebra $\mathfrak{g}$. When $k = \mathbb{C}$ and the usual Euclidean metric is used, the defining series converge and one recovers...
the classical “Exponential-Log” correspondence between Lie algebras and Lie groups.

Over the $p$-adic integers $\mathbb{Z}_p$ with the $p$-adic metric, and $p$ an odd prime, $e^{p\pi}$ can be shown to converge for every $x \in \mathbb{Z}_p$. Thus if we let $g = gl_n(\mathbb{Z}_p)$ be the Lie algebra of $n \times n$, $p$-adic matrices with the bracket $[A, B] = AB - BA$, one has that $e^{p\pi}$ is a group. In fact since $e^{p\pi} \equiv 1 \mod p$, it is not hard to show that $e^{p\pi}$ is the group of $p$-adic matrices which are congruent to the identity matrix mod $p$, which is called the $p$-congruence subgroup $\Gamma_{gl_n}$ of $GL_n(\mathbb{Z}_p)$. (See exponential-log correspondence in [Ro]). From this formal group, one can form a tower of finite $p$-groups $\Gamma_{g,k} = e^{p\pi} \mod p^{k+1}$ for $k = 1, 2, 3, \ldots$ for which $e^{p\pi}$ is the inverse limit. It is not hard to see that $\Gamma_{g,1}$ is elementary abelian and can be identified naturally with the residue Lie algebra of the $p$-adic Lie algebra $g$ i.e., $g/p^2g = g \otimes \mathbb{F}_p$. In fact $e^{p\pi} = 1 + pA \mod p^2$ so $e^{p\pi} \mod p^2$ corresponds to $A \mod p$. A little more analysis shows that we have a central short exact sequence

$$0 \to g \otimes \mathbb{F}_p \to \Gamma_{g,2} \to g \otimes \mathbb{F}_p \to 0,$$

and so $\Gamma_{g,2} = \Gamma_0 \mod p^3$ is a $p$-group given by a central extension of an elementary abelian $p$-group by itself.

More generally, for $p$ odd, one can show given any $\mathbb{F}_p$-Lie algebra $L$ (whether it lifts to the $p$-adics or not) there exists a unique $p$-power exact sequence

$$0 \to L \to G(L) \to \mathbb{F}_p \to 0,$$

where $G(L)$ is a $p$-group of order $p^{2 \dim(L)}$. In fact the construction $L \to G(L)$ is part of a covariant functor from the category of $\mathbb{F}_p$-Lie algebras to the category of $p$-groups. (See [BP]). For $p = 2$ certain phenomena involving quadratic forms arise and the correspondence needs to be modified (see [PY1] and [PY2]). Due to this, throughout this paper, $p$ denotes an odd prime.

In [BP] it was shown that if $n$ is the dimension of $L$, then

$$H^*(G(L), \mathbb{F}_p) \cong A^*(L^*) \otimes Poly(L^*) \cong A(x_1, \ldots, x_n) \otimes \mathbb{F}_p[y_1, \ldots, y_n]$$

where $L^*$ is the dual of $L$. Here $A^*(V)$ denotes the exterior algebra on the vector space $V$ where $V$ is given grading 1 while $Poly(V) = \mathbb{F}_p[V]$ denotes the polynomial algebra on $V$ where $V$ is given grading 2.

Furthermore the differential provided by the Bockstein $\beta$ was shown to be the same as the differential that makes this algebra into the Koszul resolution computing $H^*(L, Poly(adj^*))$ where $adj^*$ is the dual adjoint representation of $L$ on $L^*$ and the action of $L$ is extended to $Poly(adj^*)$ by declaring it to act via derivations.

Thus the $\beta$-cohomology of these $p$-groups was shown to be $H^*(L, Poly(adj^*))$ and this is interesting as this cohomology gives the 2nd page of the Bockstein spectral sequence used to analyze the integral cohomology $H^*(G(L); \mathbb{Z})$. Explicitly if we decompose the polynomial algebra $Poly(adj^*)$ into its homogeneous components (Hodge decomposition) $Poly(adj^*) = \bigoplus_{n=0}^{\infty} S^n$ we have

$$B_2^* = \bigoplus_{n=0}^{\infty} H^*(L, S^n)$$
and so the higher torsion in the integral cohomology of the $p$-group $G(\mathfrak{L})$ is reflected in these Lie algebra cohomology groups.

In fact an analysis of the integral cohomology of $G(\mathfrak{sl}_2(\mathbb{F}_p))$ using these techniques was used to provide a counterexample in [PR] to a conjecture of Adem (see [A]) at odd primes.

In this paper, we study a fundamental differential graded algebra (dga) associated to any Lie algebra $\mathfrak{L}$ given by the Koszul resolution whose underlying algebra is $\Lambda^*(\mathfrak{L}^*) \otimes \text{Poly}(\mathfrak{L}^*)$ with a differential such that the cohomology calculates $H^*(\mathfrak{L}, \text{Poly}(\mathfrak{ad}^*))$. While our motivation is to calculate higher torsion in the integral cohomology of $p$-groups and hence primarily concerns Lie algebras over $\mathbb{F}_p$, our basic technique is to relate these Lie algebras to Lie algebras defined over $\mathbb{Z}$ and to then compare these to the corresponding Lie algebras defined over $\mathbb{Q}$ or $\mathbb{C}$ where classical results can be appealed to. This translates then to results which hold for “all but finitely many primes” when studying the corresponding $p$-groups. This philosophy was motivated by previous work with A. Adem in [AP], though the implementation is quite different here.

The dga we construct $E[\mathfrak{L}] = \Lambda^*(\mathfrak{L}^*) \otimes \text{Poly}(\mathfrak{L}^*)$ comes equipped with a spectral sequence that is constructed in the appendix and functions in some sense as an algebraic classifying complex for the co-Lie algebra $\mathfrak{L}^*$ much like $EG$ does for a Lie group $G$.

Interestingly enough, this seemingly specialized dga carries a lot of structure. Over fields of characteristic zero and for nondegenerate Lie algebras (Killing form nondegenerate), one has an isomorphism of $\mathfrak{L}$-modules between $\text{ad}$ and $\text{ad}^*$. Furthermore using the Poincare-Birkoff-Witt theorem one can identify $\text{Poly}(\mathfrak{ad})$ with $U(\mathfrak{L})$ as filtered modules and $\text{Poly}(\mathfrak{ad}^*)$ with $U(\mathfrak{L})^*$ as graded modules where $U(\mathfrak{L})$ is the universal enveloping algebra of $\mathfrak{L}$ equipped with the adjoint action. (Throughout this paper $U(\mathfrak{L})$ and its dual will always be equipped with the adjoint action and not the left or right translation action!) Thus $E[\mathfrak{L}]$ as a dga is nothing other than the canonical Koszul complex computing $H^*(\mathfrak{L}, U(\mathfrak{L})^*)$. In this context it is important to point out that over fields of characteristic zero, $H^*(\mathfrak{L}, U(\mathfrak{L})^*)$ can be identified with the cohomology of the free loop space of $BG$ in the case that $G$ is a compact, connected Lie group with Lie algebra a real form for $\mathfrak{L}$ as we will see in this paper in the case that $G$ is semisimple as a byproduct of our analysis of this spectral sequence.

The cohomology of the free loop space $\Lambda M$ on a manifold $M$ has been an object of intense study in the last decade due to the existence of a “string multiplication” introduced by Chas and Sullivan (See [CS]) and the structure of a Batalin-Vilkovisky algebra with string theoretic interpretations, (see [GW], [Ma]).

Furthermore $H^0(\mathfrak{L}, U(\mathfrak{L}))$ consists of the central elements of $U(\mathfrak{L})$, the so-called “Casimir ring” which is very important in the representation theory of $\mathfrak{L}$. We will see that their dual elements, the “dual Casimirs” in $H^0(\mathfrak{L}, U(\mathfrak{L})^*)$ play an important role in the higher torsion of $p$-groups.

As one of the main tools of this paper, we construct a spectral sequence whose $E_0$ term is this dga and whose $E_\infty$ term is the cohomology of a point.
which gives a lot of structure to the underlying dga. Though this spectral sequence is defined over any coefficient \( k \), over \( \mathbb{F}_p \) it models the associated Bockstein Spectral sequence of the corresponding \( p \)-group mentioned above. More precisely we show that the \((E_0,d_0)\) term of this spectral sequence is identical to the \((B_1,\beta)\) term of the Bockstein spectral sequence of the \( p \)-group \( G(\mathfrak{L}) \). Though both spectral sequences converge to the cohomology of a point, it is unknown if the higher pages coincide. Nevertheless, using the structure theorems available for both spectral sequences, and that the first one can be used to compare the situation for the \( \mathbb{F}_p \)-Lie algebra with that of corresponding integral and complex Lie algebras, one can obtain results. Throughout this paper \((E_r,d_r)\) will always refer to the algebraic spectral sequence constructed in the appendix and \((B_r,\beta_r)\) will denote the Bockstein spectral sequence of the group \( G(\mathfrak{L}) \) in the case that \( \mathfrak{L} \) is a \( \mathbb{F}_p \)-Lie algebra.

The spectral sequence \( E_r \) carries a lot of information, indeed even over \( \mathbb{C} \) it can be used to recover Harish-Chandra’s calculation of the Casimir ring of \( \mathfrak{sl}_2(\mathbb{C}) \) among many other results. (We discuss the physical relevance of this within the paper.) In this introduction we quote just this classical corollary for simplicity to show the main frame of the arguments. In some of these statements the “Hodge decomposition” \( \text{Poly}(\text{ad}^*) = \bigoplus_{n=0}^{\infty} S^n \) of the polynomial algebra into its homogeneous components is used and \( S^0 \) always denotes the base ring \( k \) with trivial \( \mathfrak{L} \) action.

**Theorem 1 (Harish-Chandra calculation for \( \mathfrak{sl}_2(\mathbb{C}) \))**

\[
H^*(\mathfrak{sl}_2(\mathbb{C}), \text{Poly}(\text{ad}^*)) \cong H^*(\mathfrak{sl}_2(\mathbb{C}), U(\mathfrak{sl}_2(\mathbb{C}))^*) \cong A^*(u) \otimes \mathbb{C}[\kappa]
\]

where \( \kappa = H^2 + EF \in H^0(\mathfrak{sl}_2(\mathbb{C}), S^2) \) is (a nonzero scalar multiple) of the Killing form (an example of a “dual Casimir” element) and \( u \in H^3(\mathfrak{sl}_2(\mathbb{C}), S^0) \) is the volume form. \( \kappa \) is dual to the central Casimir element \( H^2 + 2EF + 2FE \) in \( H^0(\mathfrak{sl}_2, U(\mathfrak{sl}_2)) \) which corresponds to “total angular momentum squared” in spin systems.

As mentioned before, this calculation when used in conjunction with the spectral sequence we construct can be used to derive results for the corresponding Lie algebras \( \mathfrak{sl}_2(\mathbb{F}_p) \) after passing through \( \mathfrak{sl}_2(\mathbb{Z}) \). To state these results let us note that for \( \mathbb{F}_p \)-Lie algebras \( \mathfrak{L} \) we have \((E_0,d_0) = (B_1,\beta) = A^*(\mathfrak{L}^*) \otimes \text{Poly}(\mathfrak{L}^*) \) where \((B_r,\beta_r)\) is the Bockstein spectral sequence of the \( p \)-group \( G(\mathfrak{L}) \). We will call the polynomial degree of a homogeneous element of this algebra its “Hodge degree”.

This leads to the following:

**Theorem 2 (Higher torsion in \( G(\mathfrak{sl}_2(\mathbb{F}_p)) \))** Let \( G(\mathfrak{sl}_2(\mathbb{F}_p)) \) be the kernel of the reduction homomorphism \( SL_2(\mathbb{Z}/p^2\mathbb{Z}) \to SL_2(\mathbb{F}_p) \). Then if \( B^* \) denotes the Bockstein spectral sequence for \( G(\mathfrak{sl}_2(\mathbb{F}_p)) \) we have:

\[
E^*_1 \cong = B^*_2 = H^*(\mathfrak{sl}_2(\mathbb{F}_p), \text{Poly}(\text{ad}^*)) \sim A^*(u) \otimes \mathbb{F}_p[\kappa]
\]

where \( \sim \) denotes isomorphism in the range of Hodge degree \( \leq N \) for all but finitely many primes \( p \) (depending on \( N \)).
Furthermore $B_3^*$ is finite dimensional and we have $B_3^* \sim \Lambda^*(u) \otimes \mathbb{F}_p[\kappa]$. In particular though $B_3^*$ is always finite dimensional, there is no fixed bound on its dimension that holds for all primes $p$.

Let us try to explain this in a less technical manner. For a finite $p$-group $G$ let $\text{exp}(G)$ be the exponent of $G$, i.e., the smallest positive integer $n$ such that $g^n = e$ for all $g \in G$. Let $\tilde{H}^*(G, \mathbb{Z})$ denote the reduced integral cohomology of $G$ and $e(G)$ denote its exponent. Let $e_\infty(G)$ denote the asymptotic exponent of $G$, i.e., the smallest positive integer such that $e_\infty(G)\tilde{H}^*(G, \mathbb{Z})$ is finite. It is known that $\text{exp}(G) \mid e_\infty(G) \mid e(G) \mid |G|$ and there exist $G$ such that $e_\infty(G) \neq e(G)$. (See [PK]). When $e_\infty(G) \neq e(G)$, we can define the highest dimension of an element in the finite graded group $e_\infty(G)\tilde{H}^*(G, \mathbb{Z})$ as the “exceptional dimension” of $G$. Thus all torsion elements of exceptionally high order lie at or below the exceptional dimension of $G$.

**Corollary 1** For all odd primes $p$, $e_\infty(G(\mathfrak{sl}_2(\mathbb{F}_p))) = p^2$ while $e(G(\mathfrak{sl}_2(\mathbb{F}_p))) = p^3$. Moreover by the calculations above, for any $N$, for all but a finite number of primes $p$, the exceptional dimension of $G(\mathfrak{sl}_2(\mathbb{F}_p))$ is bigger than $N$.

Thus for every odd prime there are exceptional elements of order $p^3$ in the reduced integral cohomology of $G(\mathfrak{sl}_2(\mathbb{F}_p))$, but there is no bound on the exceptional dimension that holds for all primes. Thus by suitable choice of primes $p$, one can find elements of order $p^3$ in as high a dimension as one likes. However, for any fixed prime $p$, asymptotically the exponent of the integral cohomology of $G(\mathfrak{sl}_2(\mathbb{F}_p))$ is always $p^2$.

For details on these calculations and their implications a more thorough and leisurely development can be found in the paper itself. The basic idea is as follows:

1. For every simple Lie algebra over $\mathbb{C}$, a Cartan-Serre basis can be taken to extract a corresponding integral Lie algebra. The algebraic spectral sequences for these can be compared and, through characteristic zero techniques, computations of the required dual Casimirs can be done.
2. Since each piece in the Hodge decomposition of the dga of the corresponding integral Lie algebra is of finite type, when looking at a chunk corresponding to terms with Hodge degree $N$ or less, one can use the universal coefficient theorems to say that for all but a finite number of primes, the corresponding dga over $\mathbb{F}_p$ will look similar to the one over $\mathbb{C}$.
3. In every case though there is a breakdown in the dga over $\mathbb{F}_p$ when the Hodge degree becomes close to $p - 1$ and there is generally a phase transition between characteristic zero behaviour and characteristic $p$ behaviour.
4. In many cases we find that the “exceptional torsion” is created by the part which corresponds to the characteristic zero case and the transition to characteristic $p$ kills this exceptional torsion and is signaled by a divided power issue in the dga for the integral Lie algebra. For example for $\mathfrak{sl}_2$ we prove the key identity $\kappa^{p-1}u = 0$ in the $\mathbb{F}_p$-dga as the left hand side of the identity.
is $p$ times a generator in the corresponding $\mathbb{Z}$-dga. Detailed pictures of the algebraic SS are available in the paper to show how this identity helps cause a "phase-transition" between the char 0 and char $p$ behaviour in the case of $\mathfrak{sl}_2$.

In the paper, all semisimple Lie algebras are studied, including $\mathfrak{sl}_n$ for $n > 2$, and the exceptional Lie algebra $\mathfrak{g}_2$. These are studied over $\mathbb{C}$, $\mathbb{Z}$ and finite fields. Over $\mathbb{C}$ one finds the behaviour of $E^*_0(\mathfrak{g})$ is like that of $EG$ which motivates us calling it the "classifying spectral sequence for the Lie algebra" in general.

**Theorem 3** (Complex semisimple Lie algebras) For any complex simple Lie algebra $\mathfrak{g}$ with corresponding compact form $\mathfrak{g}_R$ and compact connected Lie group $G$ with Lie algebra $\mathfrak{g}_R$ we have:

$$E^*_1(\mathfrak{g}) = H^*(\mathfrak{g}, U(\mathfrak{g}^*)) \cong H^*(G, \mathbb{C}) \otimes H^*(\mathbb{C}G, \mathbb{C}) \cong H^*(\Lambda \mathbb{C}G, \mathbb{C}).$$

where $\Lambda \mathbb{C}G$ denotes the free loop space of the classifying space $BG$. For example, for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ one has

$$E^*_1 = H^*(\mathfrak{sl}_n, U(\mathfrak{sl}_n^*)) \cong H^*(\mathbb{SU}(n), \mathbb{C}) \otimes H^*(\mathbb{BSU}(n), \mathbb{C})$$

$$\cong A^*(u_1, u_2, \ldots, u_{2n-1}) \otimes \mathbb{C}[c_2, \ldots, c_n]$$

$$\cong A^*(u_1, u_2, \ldots, u_{2n-1}) \otimes \mathbb{C}[\sigma_2, \ldots, \sigma_n]$$

where $c_i$ are the universal Chern classes and $\sigma_i$ are the adjoint invariant polynomial functions on $\mathfrak{sl}_n$ given by the elementary symmetric functions on the eigenvalues expressed in terms of the coefficients of the matrix. This invariant theory picture is used to obtain these results and is explained completely in the relevant sections of the paper. In general, there are 3 core pictures for $E^*_1 = H^*(\mathfrak{g}, U(\mathfrak{g}^*))$ discussed in this picture, (a) as elements dual to central elements in $U(\mathfrak{g})$, i.e., as dual Casimirs, (b) as adjoint invariant polynomial functions on $\mathfrak{g}$ and (c) as the cohomology algebra $H^*(BG, \mathbb{C})$.

As a final example, for $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ one has

$$E^*_1 = H^*(\mathfrak{sp}_{2n}(\mathbb{C}), U(\mathfrak{sp}_{2n}(\mathbb{C})^*)) \cong H^*(\mathbb{ASp}(n), \mathbb{C})$$

$$\cong A^*(u_1, u_7, \ldots, u_{4n-1}) \otimes H^*(P_1, P_2, \ldots, P_n)$$

where $P_i$ are the universal Pontryagin classes.

As mentioned above, each of these results over $\mathbb{C}$ yield a picture for the spectral sequence of the associated Lie algebra over $\mathbb{F}_p$ and hence the Bockstein spectral sequence of the corresponding $p$-group, at least for low Hodge degree and for all but finitely many primes. This is discussed in detail in the paper, as is the behaviour of higher pages of the spectral sequence.

Over $\mathbb{C}$, these results are probably, by and large, repackaging of classical results in the cohomology of Lie groups and their classifying spaces, invariant
theory and Casimir theory into a spectral sequence, but we go through the process in detail in the paper as we need the spectral sequence for our work in $p$-groups.

When working in prime characteristic, computations become much more difficult and a “weight stratification” is required. If $R$ is a system of roots for the semisimple Lie algebra $\mathfrak{g}$ with root lattice $\Lambda(R)$, we show that there is a decomposition of the spectral sequence by weight which is very helpful for computations of associated Lie algebras in prime characteristic:

**Theorem 4 (Weight Stratification)** Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Let $\mathfrak{g}_\mathbb{Z}$ be a corresponding integral Lie algebra (always exists by Cartan-Serre basis). We can then get a corresponding Lie algebra $\mathfrak{g}_k$ over any ring of definition $k$.

The root decomposition of $\mathfrak{g}$ induces a weight decomposition of spectral sequences:

$$E_r^{\ast,\ast}[\mathfrak{g}_k] = \bigoplus_{\alpha \in \Lambda(R)} E_r^{\ast,\ast}[\alpha]$$

where $\Lambda(R)$ is the root lattice of $\mathfrak{g}$ and is a free abelian group of rank equal to the rank of $\mathfrak{g}$, i.e., the dimension of a Cartan subalgebra of $\mathfrak{g}$.

In addition we show that if $k$ is a field of characteristic zero then the $E_1$-page and beyond only has contributions from the weight 0 term while for a field of characteristic $p$, the “polynomial line” $E_1^{\ast,0}$ has contributions only from weights which are zero modulo $p$.

In sections 8 and 9 using a DeRham complex and D-module language together with the weight stratification mentioned above, explicit mod $p$ calculations are performed for the spectral sequence $E_r^{\ast,\ast}[\mathfrak{sl}_2(\mathbb{F}_p)]$ and the “almost all” caveat is removed in a range of Hodge degrees:

**Theorem 5 (\(\mathfrak{sl}_2(\mathbb{F}_p))\)-computation)**

$$E_1^{\ast,\ast}[\mathfrak{sl}_2(\mathbb{F}_p)] = H^\ast(\mathfrak{sl}_2(\mathbb{F}_p), Poly(ad^\ast)) \sim \Lambda^\ast(u) \otimes \mathbb{F}_p[k]$$

where $\sim$ indicates isomorphism for all odd primes in the range of Hodge degrees strictly less than $p - 1$. Thus for all odd primes $p$, the characteristic 0 to $p$ phase transition does not occur before Hodge degree $p - 1$ for the Lie algebra scheme $\mathfrak{sl}_2(-)$. (For $p = 2$ the breakdown occurs immediately at Hodge degree 0.)

However a char 0 to char $p$ phase transition occurs at Hodge degree $p - 1$ and it is shown that one needs minimally 17 generators to generate $H^\ast(\mathfrak{sl}_2(\mathbb{F}_p), Poly(ad^\ast))$ through Hodge degree $p$. (see section 9 for details on these generators and the algebra).

It follows trivially that $H^\ast(\mathfrak{sl}_2(\mathbb{Z}), Poly(ad^\ast))$ has $p$-torsion for every prime $p$ and hence cannot be a finitely generated ring.

From this it follows that the exceptional dimension (maximal dimension for which exceptional high order torsion exists) for the $p$ groups $G(\mathfrak{sl}_2(\mathbb{F}_p))$ is greater than or equal to $2p - 2$ for all odd primes $p$. 
As a byproduct of the analysis of the spectral sequence, the following exact sequence is obtained for $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{F}_p)$ and $i \geq p - 2$:

$$0 \to H^3(\mathfrak{sl}_2, S^i) \to H^2(\mathfrak{sl}_2, S^{i+1}) \to H^1(\mathfrak{sl}_2, S^{i+2}) \to H^0(\mathfrak{sl}_2, S^{i+3}) \to 0,$$

where $S^i$ is the module of homogeneous, degree $i$ polynomials on $\mathfrak{sl}_2$ equipped with the dual adjoint action.

Finally while many of the techniques apply to solvable (or nilpotent) Lie algebras also, we concentrate on semisimple Lie algebras in this paper, postponing the discussion for nilpotent Lie algebras such as those which would arise from torsion-free, pro-$p$, finite index, subgroups of the Morava Stabilizer group until another time.

2 The Classifying Spectral Sequence $E[\mathfrak{g}]$ and Universal Coefficient Arguments

Associated to any $\mathbf{k}$-Lie algebra $\mathfrak{g}$ is a spectral sequence $(E^s[\mathfrak{g}]^{s,t}, d_r)$, constructed in the appendix. We call $t$ the exterior degree and $s$ the Hodge or polynomial degree and $2s + t$ the total degree. This is a first quadrant spectral sequence and often we will plot the Hodge degree along the $x$-axis and either the exterior degree $t$ or the total degree $2s + t$ along the $y$-axis. The reader is encouraged to draw such a diagram when following the arguments. Often we will suppress $\mathfrak{g}$ explicitly from the notation when it is understood. This spectral sequence has the following properties:

1. As a $\mathbf{k}$-algebra $E^0_{s,t} = \Lambda^s(\mathfrak{g}^*) \otimes \mathbf{k}[\mathfrak{g}^*]$; i.e., it is the tensor product of the exterior algebra on the dual of $\mathfrak{g}$ with the polynomial algebra on the dual of $\mathfrak{g}$. If $s$ denotes the natural isomorphism $\Lambda^1(\mathfrak{g}^*) \to \text{Poly}^1[\mathfrak{g}^*] = \mathfrak{g}^*$, then we can write $E^0_{s,t} = \Lambda^s(\mathfrak{g}^*) \otimes \mathbf{k}[s(\mathfrak{g}^*)]$ as a bigraded algebra where $\mathfrak{g}^*$ is given bigrading $(s,t) = (0,1)$ while $s(\mathfrak{g}^*)$ is given bigrading $(s,t) = (1,0)$. Note however that graded commutativity of the dga is determined by total degree and that the elements $s(\mathfrak{g})$ have even total degree as required for them to generate a polynomial algebra. Explicitly, if $\{e_1, \ldots, e_n\}$ is a $\mathbf{k}$-basis for $\mathfrak{g}$ and $\{x_1, \ldots, x_n\}$ the canonical dual basis for $\mathfrak{g}^*$ determined by $x_i(e_j) = \delta_{i,j}$, and we set $y_i = s(x_i)$, then

$$E^0_{s,t} = \Lambda^s(x_1, \ldots, x_n) \otimes \mathbf{k}[y_1, \ldots, y_n].$$

2. Each page of the spectral sequence is a dga over $\mathbf{k}$ such that

$$d_r : E^{s,t}_r \to E^{s+r,t-(2r-1)}_r$$

i.e., the $r$th differential raises the Hodge degree by $r$ and reduces the exterior degree by $(2r - 1)$, and hence raises total degree by $1$. The differential $d_r$ is a derivation with respect to the induced algebra structure; i.e.,

$$d_r(\alpha \beta) = d_r(\alpha) \beta + (-1)^{|\alpha|} \alpha d_r(\beta),$$
where \( \alpha, \beta \) are homogeneous elements of \( E_r^{*,*} \) and \( |\alpha| \) denotes the total degree of \( \alpha \).

Of course as usual \( E_r^{*,*} = H^*(E_r^{*,*}, d_r) \).

(3) The differential \( d_0 \) is induced naturally as follows. It is the unique derivation on \( E_0^{*,*} \) such that

\[
d_0 : A^1(g^*) \to A^2(g^*)
\]

is minus the dual of the Lie-bracket \([-,-] : A^2(g) \to g \) and

\[
d_0 : s(g^*) \to s(g^*) \otimes A^1(g^*)
\]

is the dual of the Lie-bracket followed by \( \text{Identity} \otimes s^{-1} \). More explicitly if \( c^k_{ij} \) are the structure constants of \( g \) with respect to the \( k \)-basis \( \{e_1, \ldots, e_n\} \), \( \{x_i\} \) is the dual basis to \( \{e_i\} \), and \( y_i = s(x_i) \), then

\[
d_0(x_i) = -\sum_{j<k} c^i_{jk} x_j x_k \quad \text{and} \quad \quad d_0(y_i) = \sum_{1 \leq j, k \leq n} c^i_{jk} y_j x_k.
\]

(4) Thus the dga \( (E_0^{*,*}, d_0) \) breaks up as a direct sum of finite dimensional dga’s \( E_0^{s,*} = \bigoplus_{s=0}^{\infty} E_0^{s,*} \) where the decomposition is based on polynomial degree and will be refered to as the “Hodge decomposition”. As explained in the appendix, each of these turns out to be the Koszul resolution for the Lie algebra \( g \) for the module \( Poly^s(g^*) \) of homogeneous degree \( s \) polynomials on the dual of \( g \). (For infinite fields, \( Poly^s(g^*) \) can be identified as homogeneous degree \( s \) polynomial functions \( f : g \to k \) on \( g \).) Thus one has

\[
E_1^{*,*} = \bigoplus_{s=0}^{\infty} H^*(g, Poly^s(g^*)) = H^*(g, Poly(g^*)�)
\]

When \( k \) is a field of characteristic zero, one can identify the dual of the Universal Enveloping algebra of \( g \) i.e., \( U(g)^* \) with \( Poly(g^*) \) as an algebra where \( U(g)^* \) is given the dual algebra structure coming from the comultiplication in the primitively generated Hopf algebra \( U(g) \). This follows essentially from the Poincare-Birkoff-Witt theorem (we need a field of characteristic zero as in general the dual of a primitively generated Hopf algebra is a divided power algebra which only is a polynomial algebra over fields of characteristic zero). This isomorphism is also one of \( g \)-modules as long as we equip \( U(g)^* \) with the dual adjoint action coming from the adjoint action of \( g \) on \( U(g) \). Thus over fields of characteristic zero,

\[
E_1^{*,*} = H^*(g, U(g)^*)�.
\]

(5) The spectral sequence comes from two anticommuting differentials on \( E_0^{*,*} \), \( d_0 \) and \( d_1 \). \( d_1 \) is defined as follows: \( d_1 \) is the unique derivation which is \( s \) on \( A^1(g^*) \) and zero on \( Poly^1(s(g^*)) \) i.e., \( d_1(x_i) = y_i \) and \( d_1(y_i) = 0 \) for all \( 1 \leq i \leq n \).
Once define $K$ as a zig-zag construction (see [BT]), all higher differentials can be determined from these two as follows: If $\omega$ represents a class in $E_r$ then one inductively can define $d_1(\omega) = d_0(\alpha_1)$, $d_1(\alpha_k) = d_0(\alpha_{k+1})$ for $1 \leq k \leq r-2$ and finally $d_1(\alpha_{r-1})$ represents $d_r(\omega)$ and, as is typical with these sorts of constructions, the final class can be shown to be independent of the choices made along the way.

(6) Once $2r - 1 > \dim(g)$ we have $d_r = 0$ and so $E_{m+1}^{*,*} = E_{\infty}^{*,*}$ once $m > \frac{\dim(g)+1}{2}$. Furthermore as shown in the appendix,

$$E_{\infty}^{s,t} = \begin{cases} k & \text{if } s = t = 0; \\ 0 & \text{otherwise}. \end{cases}$$

(7) $E[-]^{*,*}$ defines a contravariant functor from the category of $k$-Lie algebras to the category of $k$-differential graded algebra spectral sequences. In other words, if $\theta : h \to g$ is a map of $k$-Lie algebras then it is not hard to show that $\theta^* : h^* \to g^*$ commutes with the dual Adjoint action in the sense that if we make $h^*$ a $g$-module using $\theta$ then $\theta^*$ is a $g$-module map. It is then a routine exercise to check that the algebra map induced from $\theta$,

$$\Lambda^*(h^*) \otimes k[s(h^*)] \to \Lambda^*(g^*) \otimes k[s(g^*)],$$

commutes with $d_0$ and $d_1$ and hence induces a morphism of spectral sequences $(E[h]^{*,*}, d_r) \to (E[g]^{*,*}, d_r)$.

(8) The construction $E[-]^{*,*}$ is natural with respect to extension of scalars. In other words if $k \to K$ is a homomorphism of rings between two PIDs and $g$ is a $k$-Lie algebra then $g \otimes_k K$ is naturally a $K$-Lie algebra and

$$(E[g \otimes_k K]^{*,*}, d_0) = (E[g]^{*,*} \otimes_k K, d_0 \otimes_k Id)$$

as $K$-dga’s. Thus if $K$ is flat over $k$ (this is needed to avoid tor terms in the universal coefficient theorem) one has

$$(E[g \otimes_k K]^{*,*}, d_r) = (E[g]^{*,*} \otimes_k K, d_r \otimes_k Id)$$

as $K$-spectral sequences.

(9) For $k$ a field with $p$ elements, $p$ an odd prime and $g$ a $k$-Lie algebra which lifts over $\mathbb{Z}/p^2\mathbb{Z}$, one has the following result proven in [BP]:

If $(B^*_r, \beta_r)$ is the Bockstein spectral sequence for the $p$-group $G(g)$, then $B_1 = (E_0, d_0)$ and so $B^0_1 = \oplus_{s,t}^{2s+t=r} E_1^{s,t}$. Furthermore both $B_\infty$ and $E_\infty$ give the cohomology of a point. Thus $E_\infty^{*,*}$ can be viewed as an algebraic model for the Bockstein spectral sequence and is computationally useful as we will see. It is known that $E_1 = B_2$, and the properties of $E_1$ will allow us to compute it more readily. (It is unknown whether $E_r = B_{1+r}$ for $r > 1$ but it is true in all computed examples.) In the following we concentrate on computations of $B_2^* = E_1^{*,*} = H^*(g, Poly(g^*))$. The flexibility to change coefficient rings in $E_1[-]^{*,*}$ will play a very fundamental role, as will the fact that we know the differentials $d_r$ in $E_r$ raise polynomial degree by $r$. 

$n$. This $d_1$ induces the differential on $E_1^{*,*}$. Then via a completely standard zig-zag construction (see [BT]), all higher differentials can be determined from these two as follows: If $\omega$ represents a class in $E_r$ then one inductively can define $d_1(\omega) = d_0(\alpha_1)$, $d_1(\alpha_k) = d_0(\alpha_{k+1})$ for $1 \leq k \leq r-2$ and finally $d_1(\alpha_{r-1})$ represents $d_r(\omega)$ and, as is typical with these sorts of constructions, the final class can be shown to be independent of the choices made along the way.
The fundamental idea is encoded in the following application of the universal coefficient theorem. This might be a bit abstract here but in the next few sections where explicit examples are worked out it will become clearer. To avoid needless generality we state this theorem for a special case of complex simple Lie algebras though results can easily be extended.

**Theorem 6 (Fundamental Comparison Theorem)** Let $\mathfrak{g}$ be a $\mathbb{Z}$-Lie algebra such that $\mathfrak{L} = \mathfrak{g} \otimes \mathbb{C}$ is a complex simple Lie algebra. (Cartan-Serre basis shows all complex simple Lie algebras arise this way.) Let $H^0(\mathfrak{L}, \text{Poly}(\mathfrak{L}^*)) = H^0(\mathfrak{L}, U(\mathfrak{L}^*)) \subseteq \text{Poly}(\mathfrak{L}^*)$ be denoted by $\Gamma$. Then we have

$$E_1[\mathfrak{L}]^* = H^*(\mathfrak{L}, U(\mathfrak{L}^*)) = H^*(\mathfrak{L}, \mathbb{C}) \otimes c \Gamma$$

where $H^*(\mathfrak{L}, \mathbb{C})$ can be identified with the cohomology of a certain (compact form) Lie group $G$ and is an exterior algebra on odd degree generators. (We will see in future sections, that $\Gamma$ will correspond to $H^*(BG, \mathbb{C})$ which is a polynomial algebra.)

Now since the part of $E_0[\mathfrak{g}]$ corresponding to elements of Hodge degree $\leq N$ is finitely generated, its cohomology $E_1[\mathfrak{g}]$ can have torsion only for finitely many primes.

Thus for all but a finite number of primes $p$, we have

$$\dim(E_{s,t}^1[\mathfrak{g} \otimes \mathbb{F}_p]) = \dim(E_{s,t}^1[\mathfrak{g} \otimes \mathbb{C}])$$

for all $s, t$ with $s \leq N$. On the other hand we will see that if a restriction on Hodge degree is not imposed, this will always break down. For example for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{Z})$ we will see that $E_1^*[\mathfrak{g}] = H^*([\mathfrak{g}, \text{Poly}(\mathfrak{g}^*)])$ has $p$-torsion for every prime $p$ and hence is not a finitely generated ring.

**Proof** The proof of the form of $E_1[\mathfrak{L}]^* \otimes \mathbb{C}$ hinges on two facts.

1. As $\mathfrak{L}$ is a simple Complex Lie algebra, every finite dimensional complex representation $V$ decomposes as a sum of irreducible representations.
2. If $V$ is an irreducible representation other than the trivial one-dimensional representation, then $H^*(\mathfrak{L}, V) = 0$. (This is called Whitehead’s Lemma, see [GG].) With this we can decompose $\text{Poly}(\mathfrak{L}^*) = \Gamma \oplus W$ as $U(\mathfrak{L})$-modules where $\Gamma$ are the adjoint invariant polynomials; i.e., $\Gamma = H^0(\mathfrak{L}, \text{Poly}(\mathfrak{L}^*))$, as mentioned in the statement of the theorem. Note $W$ decomposes as a sum of irreducibles, none of which is the trivial one-dimensional representation. Thus

$$H^*(\mathfrak{L}, \text{Poly}(\mathfrak{L}^*)) = H^*(\mathfrak{L}, \Gamma) \oplus H^*(\mathfrak{L}, W)$$

$= H^*(\mathfrak{L}, \Gamma)$ by Whitehead’s Lemma

$= H^*(\mathfrak{L}, \mathbb{C}) \otimes c \Gamma$ as the action of $\mathfrak{L}$ on $\Gamma$ is trivial.

For every simple complex Lie algebra $\mathfrak{L}$, there exists a real Lie algebra $\mathfrak{L}_R$ whose complexification is $\mathfrak{L}$ and which is the Lie algebra of a compact connected Lie group $G$. It is known classically that $H^*(\mathfrak{L}_R, \mathbb{R}) = H^*(G, \mathbb{R})$ is an exterior algebra on odd generators (as it is a Hopf algebra). Thus its complexification $H^*(\mathfrak{L}, \mathbb{C})$ is also an exterior algebra on odd generators.
The final parts are a direct application of the universal coefficient theorem to the complex \((E^*_{0r}(g),d_0)\) and will be left to the reader. The example for \(\mathfrak{sl}_2(\mathbb{Z})\) will be discussed later in the paper.

In the next section we compute the spectral sequence \(E^*_{r,\ast}(\mathfrak{sl}_2(\mathbb{C}))\) completely using two separate methods, one using just properties of the spectral sequence itself and the second using classical invariant theory for Lie groups. Both recover the classical calculation of Harish-Chandra of the Casimir algebra for \(\mathfrak{sl}_2(\mathbb{C})\). The invariant theory viewpoint provides a concrete meaning to the answer. Then in section 4 we compute the behaviour of the spectral sequence \(E^*_{r,\ast}(\mathfrak{L})\) for any complex simple Lie algebra and relate the answer using invariant theory.

Using the Fundamental Comparison Theorem above we then will get results for the corresponding Lie algebras of corresponding \(p\)-groups which we will use to understand higher torsion in their integral cohomology.

### 3 Computation of Dual Casimirs and Invariant Theory

For a good background discussion of the theory of complex simple Lie algebras and corresponding Lie groups that we use in this paper, see [FH].

Recall \(\mathfrak{sl}_2(\mathbb{C})\) is a Lie algebra of dimension 3 with \(\mathbb{C}\)-basis\[
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix}, 
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
\end{bmatrix}, 
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
\end{bmatrix}
\]
and commutation relations
\([x_h, x_e] = 2x_e, [x_h, x_f] = -2x_f, [x_e, x_f] = x_h\).

Let \(h, e, f\) denote the corresponding dual basis in \(A^1(\mathfrak{L}^*)\) and let \(H, E, F\) denote the “suspended” dual basis in \(Poly^1(\mathfrak{L}^*)\).

Thus
\[E^*_{\ast,\ast}(\mathfrak{sl}_2(\mathbb{C})) = \Lambda^\ast(h, e, f) \otimes \mathbb{C}[H, E, F]\]
and using Theorem 6 we find
\[E_1^*\ast(\mathfrak{sl}_2(\mathbb{C})) = \Lambda^\ast(u) \otimes \Gamma\]

where \(\Lambda^\ast(u) = H^\ast(\mathfrak{sl}_2(\mathbb{C}), \mathbb{C})\) with \(u = hef\) (a simple computation) and \(\Gamma = H^0(\mathfrak{sl}_2(\mathbb{C}), U(\mathfrak{sl}_2(\mathbb{C}))^\ast)\) is the dual Casimir algebra that we seek to find.

It is not hard to see that \(d_r = 0\) on \(\Gamma\) for \(r \geq 1\) as these differentials lower exterior degree and so \(\Gamma\) consists of permanent cycles in the spectral sequence \(E^*_{\ast,\ast}(\mathfrak{sl}_2(\mathbb{C}))\). However, we also know that \(E^*_{\infty,\ast}(\mathfrak{sl}_2(\mathbb{C}))\) is the cohomology of a point, so everything needs to be killed off, thus \(u\) must support some differential. Note that \(H^\ast(\mathfrak{L}, Poly^1(ad^\ast)) = 0\) by Whitehead’s lemma as \(Poly^1(ad^\ast) = ad^\ast\) is a nontrivial irreducible representation of \(\mathfrak{L}\). Thus \(d_1(u) = 0\) and so \(E_1 = E_2\). Now since the dimension of the Lie algebra is
three, and \( d_2 \) lowers exterior degree by three, this is the final possible nonzero differential and so \( E_3 = E_\infty \) and so we know \((E_2, d_2)\) is an acyclic complex.

Let \( \kappa_2 = d_2(u) \) then \( \kappa_2 \in \Gamma \) is an invariant quadratic polynomial. The \( E_2 \) page is concentrated on two horizontal lines: a line with exterior degree 3 with term \( u \otimes \Gamma \) and a line with exterior degree 0 with term \( 1 \otimes \Gamma \). The differential \( d_2 \) between these two lines is given by multiplication by \( \kappa_2 : \Gamma \rightarrow \Gamma \). Since the complex is acyclic we conclude that \( \kappa_2 : \Gamma \rightarrow \Gamma^+ \) is an isomorphism where \( \Gamma^+ \) are the elements of \( \Gamma \) of positive degree. A simple induction then shows \( \Gamma \cong \mathbb{C}[\kappa_2] \). This yields the following theorem:

**Theorem 7** \( E_1[\mathfrak{sl}_2(\mathbb{C})]\)-computation:

\[
E_1^{*,*}[\mathfrak{sl}_2(\mathbb{C})] \cong H^*(\mathfrak{sl}_2(\mathbb{C}), U(\mathfrak{sl}_2(\mathbb{C}))^*) \cong \Lambda^*(u) \otimes \mathbb{C}[\kappa_2]
\]
where \( u \in E_1^{0,3} \) and \( \kappa_2 \in E_1^{2,0} \). The element \( \kappa_2 \) is a permanent cycle in the spectral sequence and \( d_2(u) = \kappa_2 \). \( \kappa_2 \) is a nonzero multiple of the Killing form of \( \mathfrak{sl}_2(\mathbb{C}) \) and hence is a nonzero multiple of \( H^2 + EF \).

**Proof** Everything besides the last comment has been already proven. Note for any semisimple complex Lie algebra \( \mathfrak{g} \), the Killing form

\[
\kappa(A, B) = \text{Trace}(\text{ad}(A) \circ \text{ad}(B) : \mathfrak{g} \rightarrow \mathfrak{g})
\]
is a symmetric 2-form on \( \mathfrak{g} \) which is nondegenerate and hence non-zero. Furthermore it has the property that \( \kappa([A, B], C) + \kappa(B, [A, C]) = 0 \) for all \( A, B, C \in \mathfrak{g} \) which implies that it represents a nonzero adjoint-invariant element in

\[
H^0(\mathfrak{g}, \text{Poly}^2(\mathfrak{g}^*)) = E_1^{2,0}[\mathfrak{g}].
\]
Since in our case \( H^0(\mathfrak{g}, \text{Poly}^2(\mathfrak{g}^*)) \) is one-dimensional and spanned by \( \kappa_2 \), the final comment follows. A simple computation shows that the Killing form is a nonzero multiple of \( H^2 + EF \) when the standard identification of quadratic forms with symmetric inner products is made.

A few things to note about the last calculation:

1. The number of exterior/polynomial generators in \( E_1 = H^*(\mathfrak{sl}_2, U(\mathfrak{sl}_2)^*) \) is one each which is equal to the rank of \( \mathfrak{sl}_2(\mathbb{C}) \), i.e., the dimension of a Cartan subalgebra in this case the span of \( x_h \). This will hold in general for any classical complex simple Lie algebra.

2. The computation of the dual Casimirs can be used to find the Casimirs, i.e., the central elements in the non-commutative algebra \( U(\mathfrak{sl}_2) \). This is because the Killing form sets up an isomorphism between the representations \( \text{ad} \) and \( \text{ad}^* \) and hence between \( H^0(\mathfrak{sl}_2, U(\mathfrak{sl}_2(\mathbb{C}))^*) \) and \( H^0(\mathfrak{sl}_2, U(\mathfrak{sl}_2(\mathbb{C})))^* \) i.e., between the Casimir algebra and the dual Casimir algebra. One has to be a bit careful though as \( U(\mathfrak{sl}_2) \) is only a filtered module and it is noncommutative; the Killing form isomorphism is only between the associated graded of \( U(\mathfrak{g}) \) and \( U(\mathfrak{g})^* \) in general.

What this means is that when finding the corresponding central elements in \( U(\mathfrak{sl}_2) \), one has to do a few things:
(a) Find the duals under the Killing form and view them in the associated graded of \( U(\mathfrak{sl}_2) \). Simple computations show \( \kappa(x_e, x_f) = 4 = \kappa(x_f, x_e) \), \( \kappa(x_h, x_h) = 8 \) and all other inner products of the three basis elements with respect to the Killing form are zero. Thus with respect to the Killing form, the dual of \( H \) is \( \frac{1}{4}x_h \), the dual of \( E \) is \( \frac{1}{4}x_f \) and the dual of \( F \) is \( \frac{1}{4}x_e \). Thus the dual of \( H^2 + EF \in H^0(\mathfrak{sl}_2, U(\mathfrak{sl}_2)^+) \) is \( \frac{1}{4\pi}x_h^2 + \frac{1}{16}x_fx_e \) up to scaling, \( x_h^2 + 4x_fx_e \in H^0(\mathfrak{sl}_2, A) \) where \( A \) is the associated graded of the noncommutative filtered algebra \( U(\mathfrak{sl}_2) \).

(b) Find the correct invariant lift from the symmetric algebra \( A \) to the noncommutative universal enveloping algebra \( U(\mathfrak{sl}_2) \). An invariant lift, i.e., a lift to a central element of \( U(\mathfrak{sl}_2) \) is always possible by semisimplicity and this procedure works in general for any complex semisimple Lie algebra. In this case, the correct lift can be obtained by symmetry considerations as \( x_h^2 + 2x_fx_e + 2x_exf \).

The physical importance of this computation is as follows. In angular momentum or spin systems, \( x_h, x_e, x_f \) represent \( L_z, L_+ = L_x + iL_y, L_- = L_x - iL_y \) respectively, angular momentum operators in 3 directions in \( \mathbb{R}^3 \). Thus their span \( \mathfrak{sl}_2(\mathbb{C}) \) represents a space of angular momentum operators taken with respect to all the possible axis directions. In quantum mechanics, a simultaneous measurement of two quantities can be taken if and only if their operators commute. Thus in this case since \( \mathfrak{sl}_2(\mathbb{C}) \) has rank one, one can only measure one of these quantities at a time and cannot simultaneously measure say \( L_z \) and \( L_x \). Since it is beneficial to be able to simultaneously measure as many observables as possible to nail down properties of the system, one then is asked if there are any operators in the operator algebra generated by \( L_z, L_x, L_y \) which commute with all these axis-specific angular momentum operators. Since \( U(\mathfrak{sl}_2) \) is exactly this operator algebra, we find that mathematically we are asking for exactly the center of \( U(\mathfrak{sl}_2) \) i.e., the Casimir algebra.

By this calculation, we see the only such operators are polynomials in \( x_h^2 + 2x_fx_e + 2x_exf \), which after identifications comes out to be a scalar multiple of the total angular momentum squared \( L^2 = L_x^2 + L_y^2 + L_z^2 \). Thus in spin systems one can measure \( L^2 \) and, say, \( L_z \) simultaneously and the corresponding values turn out to be crucial in the physics and chemistry of spin systems. In general we will not comment anymore about finding Casimirs from dual Casimirs as it is not the primary aim of this paper.

While it is possible to perform a similar calculation of the spectral sequence for \( \mathfrak{sl}_3(\mathbb{C}) \) purely using properties of the spectral sequence \( E^*_a \), for \( \mathfrak{sl}_n(\mathbb{C}), n \geq 4 \) ambiguities arise in differentials. Thus to perform the analogous calculation for an arbitrary complex simple Lie algebra, a supplementary approach has to be taken involving invariant theory. We will repeat the calculation for \( \mathfrak{sl}_2(\mathbb{C}) \) using this invariant theory approach now before doing the general calculations over \( \mathbb{C} \) in the next section. This recalculation will give us additional insight into what we are calculating.

Here are the basic observations needed for the invariant theory viewpoint:

1. Fix a semisimple Lie algebra \( \mathfrak{L} \) over \( \mathbb{C} \), then \( Poly^*(\mathfrak{L}^*) \) can be naturally identified as the algebra of polynomial functions on \( \mathfrak{L} \). (This identification is
natural and injective over infinite fields in general, though over finite fields, different polynomials can induce the same function on the underlying vector space.)

(2) If $Poly^*(\mathfrak{L}^*)$ is given the dual adjoint action, $H^0(\mathfrak{L}, Poly(\mathfrak{L}^*)) = \Gamma$ can be identified with the ad-invariant polynomial functions on $\mathfrak{L}$. (Indeed a connected Lie group is generated by a small neighborhood of the identity and the image of the exponential map contains such. The exponential map itself need not be onto, indeed the exponential map $exp: \mathfrak{sl}_2(\mathbb{C}) \to SL_2(\mathbb{C})$ is not onto.) It can then be shown that the Adjoint action of the Lie group $G$ on $\mathfrak{L}$ has exactly the same invariant polynomial functions on it. Thus

$$H^0(G, Poly(\mathfrak{L}^*)) = H^0(\mathfrak{L}, Poly(\mathfrak{L}^*)) = \Gamma.$$  

(4) It follows from the remarks above that in the case of $G = SL_n(\mathbb{C})$ and $\mathfrak{L} = \mathfrak{sl}_n(\mathbb{C})$, $\Gamma$ can be identified with the polynomial functions $f: \mathfrak{sl}_n(\mathbb{C}) \to \mathbb{C}$ that are invariant under $SL_n(\mathbb{C})$-conjugation.

(5) Consider the characteristic polynomial of a matrix,

$$P_\lambda(x) = det(xI - \lambda) = x^n - \sigma_1(\lambda)x^{n-1} + \sigma_2(\lambda)x^{n-2} - \cdots + (-1)^n \sigma_n(\lambda).$$

Note that $\sigma_j: \mathfrak{gl}_n \to \mathbb{C}$ is a homogeneous polynomial of degree $j$ in the entries of $\lambda$ with value equal to the $j$th elementary symmetric function of the eigenvalues of $\lambda$. Thus for example $\sigma_1 = trace$ is a linear polynomial in the entries of $\lambda$, while $\sigma_n = det$ is a degree $n$ polynomial in the entries of $\lambda$. Since the characteristic polynomial of a matrix is unchanged by conjugation/similarity, we conclude that $\sigma_1, \sigma_2, \ldots, \sigma_n$ represent elements in $H^0(GL_n(\mathbb{C}), Poly(\mathfrak{gl}_n^*)) = H^0(\mathfrak{gl}_n, Poly(\mathfrak{gl}_n^*))$. It is also clear that these elements are algebraically independent polynomials as they restrict to the elementary symmetric functions when restricted as functions over the diagonal matrices, and these are well-known to be algebraically independent.

(6) It is not hard to show that when restricted to functions over $\mathfrak{sl}_n(\mathbb{C})$, $\sigma_1 = 0$ but $\sigma_2, \ldots, \sigma_n$ remain algebraically independent. (Just restrict to the diagonals of $\mathfrak{sl}_n$ to check this). Thus we have found a polynomial subalgebra $\mathbb{C}[\sigma_2, \ldots, \sigma_n]$ of $\Gamma = H^0(\mathfrak{sl}_n, Poly(\mathfrak{sl}_n^*))$. Comparing with our previous calculation we see that

$$H^0(\mathfrak{sl}_2(\mathbb{C}), Poly(\mathfrak{sl}_2(\mathbb{C}))^*) = \mathbb{C}[\kappa_2] = \mathbb{C}[\sigma_2],$$

and thus the only conjugation-invariant polynomial functions on $\mathfrak{sl}_2(\mathbb{C})$ are polynomials in the determinant function, which is itself a quadratic function. (In the next section, we will show the analogous result holds for $\mathfrak{sl}_n(\mathbb{C})$, i.e., that the conjugation invariant polynomials will be a polynomial algebra on $\sigma_2, \ldots, \sigma_n$.)

(7) From these theoretical considerations, it follows that the Killing form and determinant are both conjugation-invariant homogeneous quadratic functions
\[ sl_2(\mathbb{C}) \to \mathbb{C} \text{ and, from the computations, must be a nonzero scalar multiple of each other. Of course this can be explicitly checked:} \]
\[
\det \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = -a^2 - bc = -(a^2 + bc).
\]

Using the same dual basis stated in the beginning of this section, we then see \( \det = -(H^2 + EF) \) and so indeed we have checked that \( \det \) and the Killing quadratic form \( \kappa_2 \) are scalar multiples of each other as functions on \( sl_2(\mathbb{C}) \).

4 Computation of \( E_r^* \) where \( g \) is a complex simple Lie algebra

In this section all Lie algebras will be over \( \mathbb{C} \) and so we will suppress that from the notation. Let us start first with the family \( sl_n \). Many of the arguments are similar for other complex simple Lie algebras.

Recall a Cartan subalgebra \( h \) (maximal abelian Lie subalgebra consisting of semisimple/diagonalizable elements) is given by the diagonal matrices inside \( sl_n \). For a semisimple complex Lie algebra \( g \), all Cartan subalgebras are Adjoint-conjugate and the dimension of a Cartan subalgebra is called the rank of the Lie algebra. Thus \( sl_n \) has rank \( n - 1 \). The adjoint action of \( h \) on \( g \) is diagonalizable and a simultaneous nonzero eigenvalue function \( \alpha : h \to \mathbb{C} \) is called a root with corresponding eigenvector \( v \) called a root vector. Thus

\[ [h, v] = \alpha(h)v \]

for all \( h \in h \). A root \( \alpha \) gives a complex linear functional on \( h \), i.e., \( \alpha \in h^* \).

The vector space \( g_\alpha = \{ v \in g | [h, v] = \alpha(h)v \text{ for all } h \in h \} \) is called the root space corresponding to the root \( \alpha \). It is standard (see section 14.1 of [FH]) that the root spaces are 1-dimensional and \( g = h \oplus (\oplus_{\alpha \in R} g_\alpha) \) where \( R \) is the set of roots. The Weyl group \( W(g) \) is a group generated by reflections which acts on \( h \) via automorphisms of the ambient Lie algebra \( g \) which map the subspace \( h \) back into itself. In the case of \( sl_n \), the Weyl group is the symmetric group on \( n \) letters \( \Sigma_n \) which acts on the diagonal matrices \( h \) by permuting the diagonal entries.

The following lemma is basic to the invariant theory approach to computing the dual Casimirs of \( g \):

**Lemma 1 (Invariants Lemma)** Let \( g \) be a complex simple Lie algebra, \( h \) a Cartan subalgebra, and \( W(g) \) the Weyl group of \( g \). Let

\[ \Gamma = H^0(g, Poly^*(g^*)) = H^0(U(g)^*) \]

be the algebra of adjoint-invariant polynomial functions on \( g \) i.e., the dual Casimir algebra. Let \( H^0(W(g), Poly^*(h^*)) \) denote the Weyl group invariant polynomial functions on \( h \). Then there is an algebra monomorphism

\[ \theta : \Gamma \to H^0(W(g), Poly^*(h^*)) , \]
and furthermore $H^0(W(\mathfrak{g}), \text{Poly}^*(\mathfrak{h}^*))$ can be identified as $H^* (BG, \mathbb{C})$, where $G$ is the classifying space of a simply connected, compact Lie group $G$ corresponding to a compact form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$.

**Proof** In this proof, note that a polynomial function on $\mathfrak{g}$ being invariant under the adjoint action of $\mathfrak{g}$ is the same as it being invariant under the Adjoint action of $G$ where $G$ is a connected Lie group having Lie algebra $\mathfrak{g}$ (though the meaning of invariant is slightly different for Lie algebras versus Lie groups).

We will capitalize Adjoint when we are specifically referring to the Lie group Adjoint action.

Restriction of adjoint-invariant polynomial functions on $\mathfrak{g}$ to polynomial functions on $\mathfrak{h}$ gives an algebra homomorphism $\theta : \Gamma \to H^0(W(\mathfrak{g}), \text{Poly}^*(\mathfrak{h}^*))$ as the restriction of an adjoint-invariant polynomial to $\mathfrak{h}$ will yield a polynomial function on $\mathfrak{h}$ which is invariant under the Weyl group (this is because each element of the Weyl group is induced by an Adjoint automorphism of $g$). Thus it remains to show injectivity. If $f$ is a nonzero element of $\Gamma$ then $f(v) \neq 0$ for some $v \in \mathfrak{g}$. Since $f : \mathfrak{g} \to \mathbb{C}$ is given by a polynomial, it is continuous. Thus using Jordan decompositions, we can find a semisimple (diagonalizable) element $w$ close to $v$ such that $f(w) \neq 0$. (Matrices with distinct eigenvalues form an open dense subset of $\mathfrak{sl}_n$ for example.) Then $w$ lies in a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$ and since any two Cartan subalgebras are conjugate, there is an Adjoint Lie algebra automorphism taking $\mathfrak{h}' \to \mathfrak{h}$ which takes $w$ to some element $z \in \mathfrak{h}$. By invariance under Adjoint automorphisms, $f(w) = f(z) \neq 0$ and so $f|_\mathfrak{h} \neq 0$ and so $\theta$ is a monomorphism.

The final statement is not needed for any of our $p$-group results but provides a picture for the complex results so we will only sketch it. If $G$ is a compact, simply connected Lie group $G$ corresponding to a compact form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$, then a maximal torus $T$ corresponds to a compact form for the Cartan subalgebra $\mathfrak{h}$ and $N_G(T)/T$ corresponds to the Weyl group $W$.

Thus

$$H^0(W, \text{Poly}^*(\mathfrak{h}^*)) = H^0(W, H^*(BT, \mathbb{C})) = H^*(BG, \mathbb{C}).$$

The final equality is a well-known result of Borel and will not be reproven here. In the middle we used that $H^*(BT, \mathbb{C})$ can be identified with a polynomial algebra on $H^2(BT, \mathbb{C}) = H^1(T, \mathbb{C}) = H^1(\mathfrak{h}, \mathbb{C}) = \mathfrak{h}^*$.

As there are a lot of different important equivalences going on, let us look at the explicit case $\mathfrak{sl}_n$ first. In this case note that a compact form is given by the real Lie algebra $\mathfrak{su}_n$ (see section 26.1 in [FH]) with corresponding compact, simply connected Lie group $G = SU(n)$. One has $H^*(BSU(n), \mathbb{C}) = \mathbb{C}[c_2, c_3, \ldots, c_n]$ a polynomial algebra on the universal Chern classes with $\deg(c_i) = 2i$ and $\theta(\mathcal{I}) \subseteq \mathbb{C}[c_2, c_3, \ldots, c_n]$.

In the last section we saw that $\mathbb{C}[\sigma_2, \sigma_3, \ldots, \sigma_n] \subseteq \mathcal{I}$ with $\deg(\sigma_i) = 2i$. ($\sigma_i$ is a degree $i$ polynomial on $\mathfrak{sl}_n$ but in our setup as an element in
$H^0(\mathfrak{sl}_n, Poly^*(\mathfrak{sl}_n)^*)$ is an element of total degree 2: due to the suspension of the polynomial variables.) Comparing these set inclusions of graded algebras we conclude:

**Theorem 8 (\(\mathfrak{sl}_n\) invariant calculation)** Let $\Gamma = H^0(\mathfrak{sl}_n, Poly^*(ad^*)) = H^0(\mathfrak{sl}_n, U(\mathfrak{sl}_n)^*)$ be the algebra of adjoint-invariant polynomial functions on $\mathfrak{sl}_n$, or equivalently the dual Casimir algebra. Then $\Gamma = \mathbb{C}[[\sigma_2, \sigma_3, \ldots, \sigma_n]]$ where the $\sigma_i$ have degree $2i$ and express the $i$th elementary symmetric polynomial of the eigenvalues of a matrix in terms of the entries of the matrix.

Restriction defines an isomorphism

$$\theta : H^0(\mathfrak{sl}_n, U(\mathfrak{sl}_n)^*) \to H^0(\Sigma_n, Poly^*(\mathfrak{h}^*)) = H^*(BSU(n), \mathbb{C}) = \mathbb{C}[c_2, c_3, \ldots, c_n].$$

Thus in this case the dual Casimirs of $\mathfrak{sl}_n(\mathbb{C})$ can be identified with the polynomial algebra on the universal Chern classes $c_2, c_3, \ldots, c_n$. Finally

$E_1^{*,*}[\mathfrak{sl}_n] = H^*(SU(n) \times BSU(n)) = A^*(u_3, u_5, \ldots, u_{2n-1}) \otimes C[c_2, c_3, \ldots, c_n]$ where $u_i \in E_1^{0,i}$ and $c_i \in E_1^{1,0}$.

It is now a simple matter to work out the full behaviour of the spectral sequence $E_1^{*,*}$. We will not really use this for any $p$-group results so we will be brief. We recall the definition of indecomposables in a connected graded algebra:

**Definition 1** Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded algebra and $A^+$ be the ideal of positive degree elements. (We’ll assume $A_0 = k$ is a copy of the base field.)

$$A^+ \cdot A^+ = \left\{ \sum_{j=1}^{m} a_j b_j | a_j, b_j \in A^+ \right\}$$

is called the ideal of decomposables.

$$Q = A/(A^+ \cdot A^+)$$

is called the space of indecomposables.

We now make a series of observations:

1. Let $A^*$ be $E_1^{*,*}[\mathfrak{sl}_n]$ graded by total degree. Then the space of indecomposables $Q$ is spanned by $\{u_3, u_5, \ldots, u_{2n-1}, c_2, c_3, \ldots, c_n\}$, and furthermore the $c_i$ are permanent cycles and algebraically independent.

2. Since the $d_r$ are derivations the image of a decomposable under $d_r$ is decomposable.

3. The spectral sequence converges to the cohomology of a point and for dimensional reasons the only differentials that can be supported between indecomposables are $d_k(u_{2k-1}) = \mu_k c_k$ modulo decomposables, where $\mu_k \in \mathbb{C} - \{0\}$ for $2 \leq k \leq n$. As the original Chern classes were algebraically independent, redefining $c_k$’s as necessary we can assume $\mu_k = 1$ for $2 \leq k \leq n$.

Thus we conclude:
Theorem 9 The spectral sequence $E^{s,r}_1[\mathfrak{sl}_n]$ is given by

$$E^{s,r}_1[\mathfrak{sl}_n] = \Lambda^s(u_3, u_5, \ldots, u_{2n-1}) \otimes \mathbb{C}[c_2, c_3, \ldots, c_n]$$

where $u_i \in E^{0,i}_1$ and $c_i \in E^{i,0}_1$. The higher differentials are determined by the fact that the $c_i$ are permanent cycles and $d_r(u_{2r-1}) = c_r$ for $2 \leq r \leq n$. Thus in particular $E^{*,*}_1 = E^{*,*}_\infty$ is the cohomology of a point.

If $B^*_\ell$ denotes the Bockstein spectral sequence of the $p$-group $G(\mathfrak{sl}_n(F_p))$, which is the congruence kernel of the reduction $SL_n(\mathbb{Z}/p^k\mathbb{Z}) \to SL_n(F_p)$, then for any Hodge degree $N$,

$$B^*_\ell = E^{*,*}_1[\mathfrak{sl}_2(F_p)] \sim \Lambda^*(u_3, u_5, \ldots, u_{2n-1}) \otimes F_p[c_2, c_3, \ldots, c_n]$$

where $\sim$ indicates isomorphism of complexes up to Hodge degree $N$ for all but finitely many primes (depending on $N$).

Proof All but the last paragraph follows from the observations before the statement of the theorem. The last paragraph follows from the fundamental comparison theorem discussed in previous sections. In this instance we compare $B^*_\ell = E^{*,*}_1[\mathfrak{sl}_2(F_p)]$ to $E^{*,*,*}_1[\mathfrak{sl}_n(\mathbb{C})]$ through $E^{*,*,*}_1[\mathfrak{sl}_n(\mathbb{Z})]$. Excluding the finite set of primes where torsion occurs in $E^{*,*,*}_1[\mathfrak{sl}_n(\mathbb{Z})]$ in Hodge degrees $\leq N$, we have by Theorem 8 that

$$\dim_{\mathbb{F}_p}(E^{*,t}_1[\mathfrak{sl}_n(F_p)]) = \dim_{\mathbb{C}}(E^{*,t}_1[\mathfrak{sl}_n(\mathbb{C})])$$

for arbitrary $t$ and $s \leq N$.

There is a subtlety regarding the ring structure, as the fundamental comparison theorem just shows that the dimensions match up; there is a divided power issue that needs to be addressed when excluding the finite set of primes. To clarify, we know that $A^{*,*} = E^{*,*,*}_1[\mathfrak{sl}_n(\mathbb{Z})] = \Lambda^*(E^{0,*}_1[\mathfrak{sl}_n(\mathbb{Z})], d_0)$ modulo torsion is a bigraded abelian group where $A^{0,i}$ is free abelian with rank equal to the dimension (as a complex vector space) of $E^{*,*}_1[\mathfrak{sl}_n(\mathbb{C})]$. Thus one can choose generators in $A^{0,i}$ corresponding to the $u_i$ (and call them the same thing) and in $A^{i,0}$ corresponding to the $c_i$ (and call them the same thing). However in $A^{*,*}$ it is not true in general that a combination $u_3^{\alpha_3} \cdots u_{2n-1}^{\alpha_{2n-1}} c_2^{\beta_2} \cdots c_n^{\beta_n}$ is a generator in its corresponding group; all that is known is that it will be a nonzero integer times a generator. Thus while this element is part of a basis in $E^{*,*,*}_1[\mathfrak{sl}_n(\mathbb{C})]$ it is not necessarily a generator in $E^{*,*,*}_1[\mathfrak{sl}_n(\mathbb{Z})]$ but only a nonzero integral multiple $v$ of one. (In the case where more than one of these elements lie in the same location $A^{i,j}$, $v$ should be replaced by the determinant of the nonsingular matrix which expresses the collection of these elements in $A^{i,j}$ in terms of a $\mathbb{Z}$-basis of $A^{i,j}$.)

Now when we reduce to $E^{*,*,*}_1[\mathfrak{sl}_n(F_p)]$, if the nonzero multiple $v$ mentioned above is a multiple of $p$, the corresponding product becomes zero (or linearly dependent in the case of more than one element in the same position in the $E_1$-page), while if $v$ is not a multiple of $p$, then the mod $p$ reduction will be a basis element in the corresponding position in the $E_1$-page. Since we know the complex and mod $p$ dimensions match up thru Hodge degree $N$, to ensure
the correct algebra structure, we just have to avoid all primes $p$ dividing the aforementioned multiples $v$ in the finite number of locations up through Hodge degree $N$ in addition to all the primes where $p$-torsion occurs in $E_1^{*,*}[\mathfrak{sl}_n(\mathbb{Z})]$ up through the same Hodge degree. Since this collection of primes is a finite set the theorem follows.

We will see a bit more detail on this issue when working out the ring structure of $E_1^{*,*}[\mathfrak{sl}_2(\mathbb{F}_p)]$ in later sections.

Note we would like to emphasize that it is only known that $(E_0^{*,*}, d_0) = (B_1^*, \beta)$ and so $E_1^{*,*} = B_2^*$ by the work in [BP]. It is not known that the higher differentials agree though often this is forced for dimensional reasons.

As a consequence of the last theorem we see for a generic odd prime $p$, in the Bockstein spectral sequence for $G(\mathfrak{sl}_n(\mathbb{F}_p))$, $B_2^* = E_1^{*,*}[\mathfrak{sl}_2(\mathbb{F}_p)]$ looks like $\Lambda^*(u_3, u_5, \ldots, u_{2n-1}) \otimes \mathbb{F}_p[c_2, c_3, \ldots, c_n]$ for low Hodge degrees. However, when the Hodge degree approaches $p$, this will break down. One way to see that this has to happen is to note that for low Hodge degree, $E_1^{*,*}[\mathfrak{sl}_n(\mathbb{F}_p)] = B_2^*$ will look like it has Krull dimension $n - 1$, the rank of $\mathfrak{sl}_n$. However once the Hodge degree passes $p$, the existence of an invariant polynomial algebra on the $p$-powers of the original dual basis of $\mathfrak{sl}_n[\mathbb{F}_p]$ will exhibit a Krull dimension of $n^2 - 1$, the dimension of $\mathfrak{sl}_n$.

Thus as we will see explicitly in future sections, in the low dimensions the higher torsion $B_2^*$ is governed by a “characteristic zero” contribution which has Krull dimension given by the rank of the underlying Lie algebra. However when the Hodge degree approaches $p$, a phase transition occurs and $B_2^*$ explodes behaving now like an algebra which has Krull dimension given by the dimension of the underlying Lie algebra. This phenomenon occurs for all the simple Lie algebras and we will work it out explicitly in further sections for $\mathfrak{sl}_2$.

In the remainder of this section, we run quickly through the other families of simple complex Lie algebras as well as the exceptional ones. We will not put in as many details as for the family $\mathfrak{sl}_n$, as that would make the paper unwieldy, and our primary concern is the characteristic $p$ considerations. All of the basic calculations needed can be found in Mimura and Toda’s treatise, [MT] together with Borel’s result that for a compact Lie group $G$ with maximal torus $T$ we have

$$H^*(BG, \mathbb{Q}) \cong H^*(BT, \mathbb{Q})^W,$$

where $W$ is the Weyl group. Letting $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebra and Cartan subalgebra one then can show as done above for $\mathfrak{sl}_n$ that

$$H^*(BG, \mathbb{C}) \cong H^*(BT, \mathbb{C})^W \cong Pol(y(\mathfrak{h}^*))^W \cong H^0(\mathfrak{g}, Pol(y(\mathfrak{ad}^*))) = H^0(\mathfrak{g}, U(\mathfrak{g}))^*).$$

Thus in all cases for a complex simple Lie algebra $\mathfrak{g}$ with corresponding compact form $\mathfrak{g}_\mathbb{R}$ and compact, connected Lie group $G$, we have

$$E_1^{*,*}[\mathfrak{g}] \cong H^*(G, \mathbb{C}) \otimes H^*(BG, \mathbb{C}).$$

However, it is important to note that everything in $E_1^{*,*}$ is expressed in terms of the Lie algebra $\mathfrak{g}$ and invariant polynomial functions and forms on it. In
each case the differentials transgress from the “fiber” \( H^*(G, \mathbb{C}) \) to the “base” \( H^*(BG, \mathbb{C}) \) for pretty much the same reasons as in the \( \mathfrak{sl}_n \) case. Thus over the complex numbers, \( E_{r, *}[\mathfrak{g}] \) functions very much like the spectral sequence for the fibration \( G \to E_G \to BG \).

This is why we will refer to \( E_{r, *}[\mathfrak{g}] \) as the **classifying spectral sequence** of the Lie algebra \( \mathfrak{g} \). It provides an algebraic model which can be used even when the Lie algebras are defined over fields of characteristic \( p \).

Now without further ado, we summarize the calculation of this spectral sequence for the rest of the complex simple Lie algebras, in each case this gives a picture of the higher torsion in the Bockstein spectral sequence of corresponding \( p \)-groups, before the “char 0 to char p” phase transition.

The family \( \mathfrak{sp}_{2n}(\mathbb{C}) \) can be described as follows after suitable choice of basis (see section 16.1 of [FH]):

\[
\mathfrak{sp}_{2n}(\mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid B^T = B, C^T = C, D = -A^T, A, B, C, D \in \mathfrak{gl}_n(\mathbb{C}) \right\}.
\]

It has dimension \( 2n^2 + n \) and rank \( n \) with a Cartan subalgebra given by diagonal matrices subject to the constraint \( D = -A^T \). The Weyl group in this picture is generated by permutations of the entries of \( A \) and \( D \) simultaneously and consistently with \( D = -A^T \) and also by pairwise swaps between diagonal elements in \( A \) with their negatives in \( D \). With this, it is not hard to show that Weyl group invariant polynomial functions on \( \mathfrak{h} \) restrict to polynomials in the elementary symmetric functions on the \( n \) diagonal entries in \( A \) together with the condition that they be invariant under negation of any variable, and thus polynomials in \( S_2, S_4, \ldots \) where \( S_{2i} \) is the \( i \)th elementary symmetric polynomial in the squares of the variables. Note \( S_{2i} \in H^0(\mathfrak{g}, \mathrm{Poly}(ad^*)) \) is polynomial of degree \( 2i \) and hence in \( E_{2i,0}^{0} \), and of total degree \( 4i \).

It turns out \( \mathfrak{so}_{2n+1}(\mathbb{C}) \) has the same rank \( n \) and Weyl group as \( \mathfrak{sp}_{2n}(\mathbb{C}) \) even though they are not isomorphic as Lie algebras in general. This however means that the two corresponding compact forms, the compact symplectic group \( \text{Sp}(n) \) for \( \mathfrak{sp}_{2n}(\mathbb{C}) \) and \( \text{SO}(2n+1) \) for \( \mathfrak{so}_{2n+1}(\mathbb{C}) \) have the same cohomology with rational (and hence complex) coefficients. The same thing goes for the cohomology of their classifying spaces by Borel’s theorem. Thus we get:

**Theorem 10 (Symplectic and odd orthogonal Lie algebras)**

Let \( \mathfrak{g} \) be either \( \mathfrak{so}_{2n+1}(\mathbb{C}) \) or \( \mathfrak{sp}_{2n}(\mathbb{C}) \). Then

\[
E_{r, *}[\mathfrak{g}] \cong H^*(\text{SO}(2n+1), \mathbb{C}) \otimes H^*(\text{BSO}(2n+1), \mathbb{C})
\]

\[
\cong H^*(\text{Sp}(n), \mathbb{C}) \otimes H^*(\text{BSp}(n), \mathbb{C})
\]

\[
\cong \Lambda^*(u_3, u_7, \ldots, u_{4n-1}) \otimes \mathbb{C}[P_1, P_2, P_3, \ldots, P_n],
\]

where \( u_i \in E_{1,1} \), and \( P_i \in E_{2i,0} \) correspond to the universal Pontryagin classes and \( \text{Sp}(n) \) is the compact symplectic group. In the spectral sequence, the \( P_i \)'s are permanent cycles and we have \( d_{2r}(u_{4r-1}) = P_r \) for \( 1 \leq r \leq n \). Thus
\( E^{*,*}_{2n+1} = E^{*,*}_n \) is the cohomology of a point. Note the rank of \( \mathfrak{so}_{2n+1}(\mathbb{C}) \) and \( \mathfrak{sp}_{2n}(\mathbb{C}) \) is \( n \) while the dimension is \( 2n^2 + n \).

As usual this means that the 2nd term of the Bockstein spectral sequence for the \( p \)-groups \( G(\mathfrak{so}_{2n+1}(\mathbb{F}_p)) \) and \( G(\mathfrak{sp}_{2n}(\mathbb{F}_p)) \) satisfy:

\[
B^*_2 = E^{*,*}_1[\mathfrak{sp}_{2n}(\mathbb{F}_p)] \sim \Lambda^*(u_3, u_7, \ldots, u_{4n-1}) \otimes \mathbb{F}_p[P_1, P_2, \ldots, P_n],
\]

where \( \sim \) means that the two complexes are isomorphic in the range of Hodge degrees \( \leq N \) for all but finitely many primes (that depend on \( N \)).

We now will just list the results for the other complex simple Lie algebras, the implications for higher torsion in the corresponding \( p \)-groups being understood:

**Theorem 11 (Even orthogonal Lie algebras)** For \( n \geq 4 \),

\[
E_1^{*,*}[\mathfrak{so}_{2n}] \cong H^*(SO(2n), \mathbb{C}) \otimes H^*(B SO(2n), \mathbb{C}) \\
\cong \Lambda^*(u_3, u_7, \ldots, u_{4n-5}, v) \otimes \mathbb{C}[P_1, P_2, P_3, \ldots, P_{n-1}, e],
\]

where \( u_i \in E^{0,1}_1, \, v \in E^{0,2n-1}_1 \), \( P_i \in E^{2i,0}_1 \) correspond to the universal Pontryagin classes and \( e \in E^{1,0}_1 \) corresponds to the universal Euler class and both are permanent cycles in this spectral sequence. Furthermore we have \( d_2(u_{4i-1}) = P_r \) for \( 1 \leq r \leq n-1 \) and \( d_n(v) = e \). Note \( \mathfrak{so}_{2n} \) has rank \( n \) and dimension \( 2n^2 - n \).

**Theorem 12 (Exceptional simple Lie Algebras)**

\[
E_1^{*,*}[\mathfrak{g}_2] \cong \Lambda^*(u_3, u_{11}) \otimes \mathbb{C}[K_2, K_6] \\
E_1^{*,*}[\mathfrak{i}_4] \cong \Lambda^*(u_3, u_{11}, u_{15}, u_{23}) \otimes \mathbb{C}[K_2, K_6, K_8, K_{12}] \\
E_1^{*,*}[\mathfrak{e}_6] \cong \Lambda^*(u_3, u_9, u_{11}, u_{15}, u_{17}, u_{23}) \otimes \mathbb{C}[K_2, K_5, K_6, K_8, K_9, K_{12}] \\
E_1^{*,*}[\mathfrak{e}_7] \cong \Lambda^*(u_3, u_{11}, u_{15}, u_{19}, u_{23}, u_{27}, u_{35}) \otimes \mathbb{C}[K_2, K_6, K_8, K_{10}, K_{12}, K_{14}, K_{18}] \\
E_1^{*,*}[\mathfrak{e}_8] \cong \Lambda^*(u_3, u_{15}, u_{23}, u_{27}, u_{35}, u_{39}, u_{47}, u_{59}) \otimes \mathbb{C}[K_2, K_8, K_{12}, K_{14}, K_{18}, K_{20}, K_{24}, K_{30}].
\]

In all cases, the dimension of the Lie algebra is the sum of the subscripts of the \( u_i \) in \( E^{0,1}_1 \) in its cohomology. As usual, the dual Casimirs \( K_i \) in \( E^{1,0}_1 \) represent homogeneous degree 1 adjoint invariant polynomial functions on the Lie algebra and \( K_2 \) always is the quadratic form corresponding to the Killing form.

The results above cover all the complex simple Lie algebras. More generally a semisimple Lie algebra is a direct sum of simple Lie algebras and is covered by the fact that if \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) then \( E^{*,*}_0[\mathfrak{g}] = E^{*,*}_0[\mathfrak{g}_1] \oplus E^{*,*}_0[\mathfrak{g}_2] \) as differential graded algebras. Furthermore the complex also decomposes as a tensor product.
with respect to the commuting differential used to define the spectral sequence for the double complex. Thus one has that

\[ E_r^{*,*}[g] = E_r^{*,*}[g_1] \otimes E_r^{*,*}[g_2] \]

as dga’s for any \( r \geq 0 \), i.e., the spectral sequence of the sum is the tensor product of the spectral sequences of the factors. (These comments in fact work over any base field \( k \).) Thus one can work out the spectral sequence for any complex semisimple Lie algebra from the results in this section. A general complex Lie algebra \( \mathfrak{L} \) fits in a short exact sequence

\[ 0 \to \text{rad}(\mathfrak{L}) \to \mathfrak{L} \to \mathfrak{g} \to 0 \]

where \( \text{rad}(\mathfrak{L}) \) is the largest solvable ideal of \( \mathfrak{L} \) and \( \mathfrak{g} \) is semisimple, so it would remain to treat solvable (and in particular nilpotent) Lie algebras. Generally the cohomology of these is much more complicated, though definitely interesting but we will not consider these much in this paper nor the corresponding \( p \)-groups as the treatment would require different methods.

We have shown that for any complex semisimple Lie algebra \( \mathfrak{g} \), with compact form \( \mathfrak{g}_R \) corresponding to a compact connected Lie group \( G \), that

\[ E_1^{*,*}[g] = H^*(\mathfrak{g}, U(\mathfrak{g})^*) = H^*(G \times BG, \mathbb{C}). \]

We note finally that it is well known that

\[ H^*(\mathcal{L}BG, \mathbb{C}) \cong H^*(G \times BG, \mathbb{C}) \]

by considering the fibration \( G \simeq \Omega BG \to \mathcal{L}BG \to BG \).

A quick comment regarding related physics: For \( \mathfrak{g} \) a complex semisimple Lie algebra we have seen that the Casimir algebra will be a polynomial algebra on \( h \) generators where \( h \) is the rank of the Lie algebra. In general this means the \( h \) linearly independent, commuting operators represented by the Cartan algebra \( \mathfrak{h} \) will commute with an additional \( h \) Casimir operators which generate the Casimir algebra. This gives a total of \( 2h \) commuting and simultaneously measurable operators coming from the Cartan algebra and the Casimir generators, which form (part of) a good system of variables in the underlying physical system.

Thus for example \( \mathfrak{sl}_3(\mathbb{C}) \) which plays a role in quark/anti-quark theory as well as meson/baryon classification will have two variables coming from the Cartan algebra, the corresponding measurements of which will correspond to the root weights described in later sections of this paper and an additional two Casimir generators dual to the Chern classes \( c_2, c_3 \), to give a total of 4 commuting operators from this discussion. For a nice mathematical article on the related physics, see [HH].
5 Exponent theory for finite p-groups

In this section we recall the exponent theory of finite groups. Many of these results were introduced in work of A. Adem and Ian Leary (see \[A\] and \[Le\]) while the fact that $e_\infty(P) \neq e(P)$ was first shown in \[Pk\]. We summarize the main results regarding these exponents in this section and include a few more for the category of p-groups we will be dealing with.

In this section let $G$ be a finite group. It is well known that the integral cohomology groups $H^n(G, \mathbb{Z}), n > 0$ are finite abelian groups and $|G| \cdot H^n(G, \mathbb{Z}) = 0$ for all $n > 0$. Let $\bar{H}(G) = \oplus_{n=1}^{\infty} H^n(G, \mathbb{Z})$ denote the total reduced integral cohomology of $G$. We define the following exponents:

**Definition 2** (Exponent Definitions)

$\exp(G) = \min\{n \geq 1 | g^n = e$ for all $g \in G\}$.

$e_\infty(G) = \min\{n \geq 1 | n\bar{H}(G)$ is a finite set $\}$.

$e(G) = \min\{n \geq 1 | n\bar{H}(G) = 0\}$.

The following propositions are discussed in detail in \[Pk\] so we will not reproduce proofs here. As mentioned before many of them occur in previous independent work of Adem and Leary. Failure of various propositions for exactly one of $e_\infty$ or $e$ were shown in \[Pk\]. We write $n|m$ for “the integer $n$ divides the integer $m$”.

**Proposition 1** (Basic Relationship between Exponents) If $P$ is a finite p-group then

$$\exp(P) \mid e_\infty(P) \mid e(P) \mid |P|$$

and furthermore there are examples of p-groups that show that these four quantities are different in general.

**Proposition 2** (Subgroups) Let $P_1 \leq P_2$ be finite p-groups. Then $e_\infty(P_1) | e_\infty(P_2), \exp(P_1)|\exp(P_2)$ and $|P_1| | |P_2|$. However $e(P_1) \nmid e(P_2)$ in general.

**Proposition 3** (Cyclic groups and Symmetric Sylow subgroups) Let $C$ be a finite cyclic group and $S(m)$ be the Sylow p-subgroup of the symmetric group on $m$ letters.

1. $\exp(C) = e_\infty(C) = e(C) = |C|$
2. $p^n = \exp(S(p^n)) = e_\infty(S(p^n)) = e(S(p^n))$.

**Proposition 4** (Products) If $G = G_1 \times \cdots \times G_n$, then

$$\exp(G) = \text{lcm}_{i=1, \ldots, n}\{\exp(G_i)\},$$

$$e_\infty(G) = \text{lcm}_{i=1, \ldots, n}\{e_\infty(G_i)\},$$

$$e(G) = \text{lcm}_{i=1, \ldots, n}\{e(G_i)\}$$
where lcm stands for least common multiple and can be replaced with max in the case of $p$-groups. Thus if $p^n$ is the largest power of $p$ less than or equal to $m$, then

$$p^n = \exp(S(m)) = e_\infty(S(m)) = e(S(m)).$$

**Proof** The first statement follows from an application of K"unneth’s Theorem. The second follows from the structure of Sylow $p$-subgroups of symmetric groups. More precisely, if the $p$-adic expansion of $m$ is given by $m = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n$ then $S(m)$ is the direct product of $a_0$ copies of $S_1$, $a_1$ copies of $S(p)$, up to $a_n$ copies of $S(p^n)$.

**Theorem 13** (Exponent Characterizations) Let $P$ be a $p$-group.

1. [Nakayama-Rim] If $e_\infty(P) = 1$ then $P = \{e\}$ and $e(P) = 1$ also.
2. [Adem] If $e_\infty(P) = p$ then $P$ is elementary abelian and $e(P) = \exp(P) = p$ also.
3. [Pakianathan] For odd primes $p$, the group $G(sl_2(F_p))$ has $\exp(P) = e_\infty(P) = p^2$ and $e(P) = p^3$ while $|P| = p^6$. Thus $e_\infty(P) \neq e(P)$ in general.

**Proposition 5** (Faithful actions on finite sets) Let $P$ be a finite $p$-group.

1. If $P$ has a faithful action on a set of size $m$ and $p^n$ is the largest power of $p$ less than or equal to $m$ then $e_\infty(P) | p^n$. Thus $e_\infty(P)$ is a lower bound on the size of a set on which $P$ can act faithfully.
2. If the intersection of all subgroups of $P$ of index $p^n$ is trivial then $e_\infty(P) | p^n$. This statement does not hold with $e(P)$ replacing $e_\infty(P)$.

**Theorem 14** (Browder’s exponent theorem and consequences) Let $G$ be a finite group and $X$ a finite dimensional free $G$–$CW$-complex, with homologically trivial $G$-action. Then

$$|G| \mid \prod_{n=2}^{\infty} \exp(H^n(G, H_{n-1}(X))) \mid e(G)^s(X)$$

where $s(X)$ is the number of positive dimensions in which the integral homology of $X$ is nontrivial.

A corollary of this is if $E$ is an elementary abelian group acting freely and homologically trivially on a product of $N$ equal dimensional spheres then $\text{rank}(E) \leq N$.

**Proof** The short and elegant proof of this theorem and corollary can be found in [Br]. The second division comes from the definition of $e(G)$. It is unknown to the authors at this time whether the theorem would hold for $e_\infty(G)$ but we see no reason why it should.

The two fundamental exponents can be recharacterized in terms of the behaviour of the Bockstein spectral sequence of the $p$-group $P$. A discussion of this spectral sequence can be found in many places in the literature and is described for example in [BP]. Recall that when applied to the classifying space of a finite $p$-group $P$ one has that $B^*_1 = H^*(P, \mathbb{F}_p)$ and $\beta_1$ is the Bockstein.
Nonzero permanent cycles in $B_r$ represent elements of order $p^r$ or greater in the integral cohomology of $P$ and $B_\infty$ is the cohomology of a point. Thus one has the following:

**Lemma 2 (Bockstein Spectral Sequence Characterization)** If $P$ is a finite $p$-group, then:

1. $e_\infty(P) = p^k$ if and only if $B^k_{k+1}$ is the first page of the Bockstein spectral sequence whose total complex is finite.
2. $e(P) = p^k$ if and only if $B^k_{k+1}$ is the first page of the Bockstein spectral sequence whose total complex is concentrated in degree 0. (equivalently is 1-dimensional).

Finally for $p$-groups $G(\mathfrak{L})$ corresponding to $\mathbb{F}_p$-Lie algebras $\mathfrak{L}$ as in [BP], where $0 \to \mathfrak{L} \to G(\mathfrak{L}) \to \mathfrak{L} \to 0$, we have the following:

**Proposition 6 (Exponent bounds for $G(\mathfrak{L})$-p-groups.)** Let $\mathfrak{L}$ be a $\mathbb{F}_p$ Lie algebra such that the intersection of index $p^3$ sub Lie algebras is zero. Then $e_\infty(G(\mathfrak{L}))|p^{3\beta}$.

**Proof** In the construction of the $p$-groups $G(\mathfrak{L})$ one has in general that $|G(\mathfrak{L})| = |\mathfrak{L}|^2$ and so an index $p^3$ sub Lie algebra of $\mathfrak{L}$ gives a corresponding index $p^{3\beta}$ subgroup of $G(\mathfrak{L})$. Since the zero Lie subalgebra corresponds to the trivial group, the rest follows from the previous theorems on faithful group actions.

As the reader can see, these exponents play an interesting role in the cohomology of groups limited only by the difficulty in carrying out complete calculations of the integral cohomology of groups. In some results the integral cohomology exponent $e(G)$ arises while in others the asymptotic exponent $e_\infty(G)$ arises. Since these are not equal in general it is useful to define the extent to which they differ, and the following concept helps do that.

**Definition 3 (Exceptional Torsion elements and Exceptional Dimension)** Let $P$ be a finite $p$-group.

By definition $e_\infty(P) \cdot H^*(P, \mathbb{Z})$ is a finite graded ring. Let the **exceptional dimension** of $P$, denoted by $ED(P)$, be defined as the largest dimension in which $e_\infty(P) \cdot H^*(P, \mathbb{Z})$ is nonzero. Note that $ED(P)$ is always a finite nonnegative integer and is zero if and only if $e_\infty(P) = e(P)$.

Any element $x \in H^*(P, \mathbb{Z})$ with $0 < \beta \leq ED(P)$ such that $e_\infty(P) \cdot x \neq 0$ is called an **exceptional torsion element**. Note if $ED(P) > 0$ then an exceptional torsion element always exists in $H^{ED(P)}(P, \mathbb{Z})$ by definition and no exceptional torsion elements exist in dimensions higher than $ED(P)$, i.e., $e_\infty(P) \cdot \oplus_{n=ED(P)+1}^{\infty} H^n(P, \mathbb{Z}) = 0$.

There is no bound on exceptional dimension in general, i.e., it can be arbitrarily high as the next example shows:

**Theorem 15** Let $P$ be the $N$-fold direct product of $G(\mathfrak{sl}_2)$'s. Then $e_\infty(P) = p^2$, $e(P) = p^3$, $|P| = p^{6N}$ and $ED(P) = N \cdot ED(G(\mathfrak{sl}_2)) > 0$.

Thus $\lim_{N \to \infty} ED(P) = \infty$. 


Proof Follows from the results in this section together with the fact that \( e_\infty(G(sl_2)) = p^2 \) and \( e(G(sl_2)) = p^3 \) proven in [P]. Note that the Bockstein spectral sequence of \( P \) is the \( N \)-fold tensor product of the Bockstein spectral sequence of \( G(sl_2) \) (this means at each page it’s the tensor product as dga’s.). Also note that \( ED(G(sl_2)) \) is the maximum nonzero dimension of \( B^3_2(G(sl_2)) \) while \( ED(P) \) is the maximum nonzero dimension of \( B^3_j(P) = \otimes_{j=1}^{N} B^3_j(G(sl_2)) \) from which the results for essential dimension follow.

The last example is not so strong in the sense that \( \log_p(|P|) \) has to increase in order to get the increase in \( ED(P) \). In the next sections we will study the Bockstein spectral sequence for \( G(sl_2) \) again more carefully using the machinery of Lie algebras we have mentioned in previous sections. More analysis shows in fact the following stronger facts:

1. \( ED(G(sl_2(F_p))) \geq 2p - 2 \) for odd primes \( p \). Thus the essential dimension grows with the prime of definition of \( G(sl_2(F_p)) \) and is therefore, in some sense, not even bounded for the fixed “group scheme” \( G(sl_2(-)) \).

2. Furthermore we will see that the “exceptional torsion elements correspond to the characteristic zero contribution of the Lie algebra scheme \( sl_2(-) \) and has Krull dimension given by the rank of the Lie algebra (scheme). Then in the vicinity of the essential dimension a “char 0 to char p” phase transition occurs and the asymptotic torsion has a larger Krull dimension equal to the dimension of the underlying Lie algebra.

3. In this sense, the word “exceptional torsion” transition to “asymptotic torsion” can be seen in some situations at least, to correspond to a char 0 to char p phase transition in corresponding Lie algebra schemes.

This should become clearer after a few examples.

6 Non-abelian Lie algebra of dimension 2

We will work out an easy example which will introduce the useful concept of weight stratification of the spectral sequence.

Let \( g \) be the non-abelian Lie algebra (scheme) of dimension 2 with basis \( x_h, x_e \) and commutator given by \([x_e, x_h] = x_e\). Note \( g \) can be defined over any base ring \( k \) and we will write \( g(k) \) when we want to make the base ring clear. Let \( h, e \) denote the dual basis in \( g^* \) and \( H, E \) suspended copies in \( Poly^1(g^*) \). Then

\[ E_0^{*,*}[g] = A^*(h, e) \otimes k[H, E] \]

with differential given by \( d_0(h) = d_0(H) = 0 \) and \( d_0(e) = he, d_0(E) = Eh - He \).

The (anti)commuting differential \( d_1 \) giving rise to the spectral sequence is given by \( d_1(h) = H, d_1(e) = E, d_1(E) = d_1(H) = 0 \). Furthermore by \([BP]\), \((E_0, d_0) = (H^*(G(g), F_p)), \beta) \) where \( \beta \) is the Bockstein operator when \( k = F_p \).

Now if we assign \( h, H \) to have weight 0 regarded as an integer and \( e, E \) to have weight 1, and extend weight to \( E_0^{*,*}[g] \) so that the weight of a product is the sum of the weights of the individual factors, then it is easy to see that \( d_0 \) and \( d_1 \) preserve weight and hence so do all differentials of the spectral
sequence. Thus the spectral sequence decomposes into a direct sum of spectral sequences given by isolating terms of specific weight:

\[ E^*_r = \bigoplus_{w \in \mathbb{N}} E^*_r[w] \]

We will see this occurs in general in the next section, and in the case of semisimple Lie algebras, the sum will be indexed by the root lattice of the corresponding complex semisimple Lie algebra which will be a free abelian group of rank equal to the rank of the Lie algebra. The weight stratification is overkill in this example, but becomes necessary in more complicated computations.

Let us first look at the weight 0 contribution; since the weights in this example are all nonnegative integers, it is easy to see that

\[ E^*_0[0] = \Lambda^*(h) \otimes k[H], \]

with \( d_0 = 0 \) identically and \( d_1(h) = H \). Thus \( E^*_0[0] = E^*_1[0] \) and \( E^*_2[0] = E^*_\infty[0] \) is the cohomology of a point.

Now consider a nonzero weight \( n > 0 \). Then \( E^*_0[n] \) is a right \( E^*_0[0] \)-module and since \( d_0 \) is a derivation which vanishes on \( E^*_0[0] \), we have that \( d_0 : E^*_0[n] \to E^*_0[n] \) is a right \( E^*_0[0] \)-module map. Furthermore \( E^*_0[n] \) is a free \( E^*_0[0] \)-module of rank 2 with (ordered) basis \( \{ eE^{n-1}, E^n \} \). Thus we can represent \( d_0 \) on this weight \( n \)-piece as a \( 2 \times 2 \) matrix with entries in \( E^*_0[0] = \Lambda^*(h) \otimes k[H] \).

Computing we have:

\[ d_0(eE^{n-1}) = (he)E^{n-1} - e(n-1)E^{n-2}(Eh - He) = eE^{n-1}(-nh) \]

and

\[ d_0(E^n) = nE^{n-1}(Eh - He) = eE^{n-1}(-nH) + E^n(nh) \]

and hence

\[ d_0(eE^{n-1} \gamma + E^n \delta) = eE^{n-1} \gamma' + E^n \delta' \]

for any \( \gamma, \delta \in E^*_0[0] \) with

\[
\begin{bmatrix}
\gamma' \\
\delta'
\end{bmatrix} =
\begin{bmatrix}
-nh & -nH \\
0 & +nh
\end{bmatrix}
\begin{bmatrix}
\gamma \\
\delta
\end{bmatrix},
\]

and thus \( d_0 : E^*_0[n] \to E^*_0[n] \) is represented by the matrix \( \begin{bmatrix}
-nh & -nH \\
0 & +nh
\end{bmatrix} \) with respect to the ordered basis mentioned above.

Restricting to the case where \( k \) is a field, there are now two cases to consider:

Case 1: \( n = 0 \) in \( k \).

In this case \( d_0 = 0 \) on \( E^*_0[n] \) and \( E^*_0[n] = E^*_1[n] \). Direct computation shows \( d_1(eE^{n-1}) = E^n \) and we conclude that \( E^*_2[n] = E^*_\infty[n] = 0. \)
Case 2: $n \neq 0$ in the field $k$.

Note that a typical element in $E^*_0[0] = A^*(h) \otimes k[H]$ can be written in the form $\gamma_1 + h\gamma_2$ where $\gamma_i$ are $k$-polynomials in $H$. If $\begin{bmatrix} \gamma_1 + h\gamma_2 \\ \delta_1 + h\delta_2 \end{bmatrix}$ is in the kernel of the matrix then

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -nh - nhH \\ 0 \end{bmatrix} \begin{bmatrix} \gamma_1 + h\gamma_2 \\ \delta_1 + h\delta_2 \end{bmatrix} = \begin{bmatrix} -nh(\gamma_1 + H\delta_2) - nh\delta_1 \\ + nh\delta_1 \end{bmatrix}.$$ 

Since $n$ is a unit in $k$ in this case, it follows quickly that $\delta_1 = 0$ and $\gamma_1 + H\delta_2 = 0$. Thus the typical element in the kernel of $d_0$ looks like $\begin{bmatrix} -H\delta_2 + h\gamma_2 \\ h\delta_2 \end{bmatrix}$. However such an element is in the image of $d_0$ as

$$\begin{bmatrix} -nh - nhH \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{n}(\gamma_2) \\ \frac{1}{n}(\delta_2) \end{bmatrix} = \begin{bmatrix} -H\delta_2 + h\gamma_2 \\ h\delta_2 \end{bmatrix}.$$ 

Thus the cohomology of the complex $(E^*_0[0], d_0)$ is identically zero, i.e., $E^*_1 = 0$ and $E^*_2 = 0$.

Summarizing to the main fields of interest $\mathbb{C}$ and $\mathbb{F}_p$, we get:

**Theorem 16 (Non-abelian Lie algebra of dimension 2)** Let $g$ be the $k$-Lie algebra with basis $x_h, x_e$ and bracket given by $[x_e, x_h] = x_e$. Then $E^*_0 = A^*(h, e) \otimes k[H, E]$. For $k$ a field of characteristic zero zero we have

$$E^*_1 = H^*(g, U(g^*)) = E^*_1[0] = A^*(h) \otimes k[H]$$

and $E^*_2 = E^*_\infty$ is the cohomology of a point. Note the nonzero contribution to $E^*_1$ is only from weight 0 and that the Krull dimension of $E_1$ is the rank of the Lie algebra which is one.

For $k$ a field of characteristic $p$, we have

$$E^*_1 = H^*(g, Poly(g^*)) = A^*(h, e^{E^{p-1}}) \otimes k[H, E^p]$$

with $d_1(h) = H$ and $d_1(e^{E^{p-1}}) = E^p$. Note the nonzero contribution to $E_1^{*\ast}$ is only from weights that are congruent to 0 modulo $p$. Also the Krull dimension is now two, which is the dimension of the Lie algebra. The phase transition from the characteristic 0 answer first occurs in dimension where $e^{E^{p-1}}$ lies i.e., in total dimension $2p - 1$ (Hodge degree $p - 1$).

Following [BP], this yields the following corollary:

**Corollary 2** Let $g$ be the nonabelian Lie algebra above defined over $\mathbb{F}_p$, $p$ an odd prime. Let $G(g)$ be the corresponding $p$-group of order $p^4$. Then $E^*_1 = B^*_{p+1}$ for all $r$ where $B^*_r$ is the Bockstein spectral sequence of the group. Thus

$$E^*_1 = B^*_2 = A^*(h, e^{E^{p-1}}) \otimes \mathbb{F}_p[H, E^p]$$

and $E^*_2 = B^*_1$ is the cohomology of a point. Thus $e_\infty(G(g)) = e(G(g)) = p^2 = exp(G(g))$. 


Proof As mentioned before it was shown in [BP] that \((E_1^{*}, d_0) = (B_1^*, \beta)\) in general and hence that \(E_1^{*} = B_2^*\). Using comparisons to the cyclic \(p^2\)-subgroups generated by the kernel of \(e\) and \(h\) respectively (note \(A^1(h, e) = H^1(G(g), \mathbb{F}_p) = \text{Hom}(G(g), \mathbb{F}_p)\)), one sees that one must have \(\beta_2(h) = H\) and \(\beta_2(e^{p-1}) = E^p\) at least up to nonzero scalars. This shows that \(B_3^*\) has the cohomology of a point and that the two spectral sequences coincide on all pages. The rest follows immediately.

Even though \(e = e_\infty\) in this example, note that there is still a transition in the higher torsion \(B_2^*\) from behaving like the characteristic zero contribution (Krull dimension one) before Hodge degree \(p - 1\) and then changing to a Krull dimension two behaviour after Hodge degree \(p\). Thus for \(E_1^{*}\) the contribution \(\Lambda^*(h, H)\) is universal while the contribution \(\Lambda^*(e^{p-1}) \otimes \mathbb{F}[E^p]\) only occurs in characteristic \(p\).

We will see that this behaviour is relatively generic and occurs also in higher dimensional examples.

7 Weight Stratification of the Spectral Sequence \(E_r^{*,*}\)

Let \(g\) be any complex semisimple Lie algebra and let \(h\) be a Cartan subalgebra. The adjoint action of \(h\) on \(g\) is (simultaneously) diagonalizable and decomposes \(g\) as

\[ g = h \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \]

where \(R\) is the set of roots of the Lie algebra. Recall a root \(\alpha\) with its root vector \(v\) satisfies

\[ [h, v] = \alpha(h)v \]

for all \(h \in h\). It turns out the values of \(\alpha\) are always integral, and so each root \(\alpha\) can be viewed in \(\mathbb{Z}^n\) where \(n\) is the rank of the Lie algebra, i.e., the dimension of its Cartan subalgebra. The correspondence comes by taking a basis \(\{h_1, \ldots, h_n\}\) of \(h\) and sending \(\alpha\) to \((\alpha(h_1), \ldots, \alpha(h_n))\).

The Cartan algebra itself corresponds to \(v\) such that \([h, v] = 0 = 0v\) for all \(h \in h\), so we will say it has weight 0 (though 0 is usually not considered a root for various reasons).

For a semisimple Lie algebra the set of roots is always finite, and if \(\alpha\) is a root, then so is \(-\alpha\). Furthermore, the sublattice of \(\mathbb{Z}^n\) spanned by the roots is of rank \(n\) also and is called the root lattice.

We now extend this root decomposition to a weight decomposition of \(E_0^{*,*} = A^*(g^*) \otimes \text{Poly}(g^*)\) as follows. Decompose \(A^1(g^*) = g^* = \text{Poly}^1(g^*)\) by declaring something to have weight \(\alpha\) if it is dual to a (nonzero) element of weight \(-\alpha\) in \(g\). Then extend this weighing to all of \(E_0^{*,*}\) by declaring the weight of a product to be the sum of the weights. It is not hard to see that this decomposes \(E_0^{*,*}\) into a direct sum of “weight subspaces” where the weights are any nonnegative integer combination of the root weights, i.e., are the elements of the root lattice.
Though the minus sign is not really important, the reason the dual of a weight $\alpha$ element is said to have weight $-\alpha$ is that if $v \in g$ is a root vector, $[h, v] = \alpha(h) v$ for all $h \in h$, and $\phi \in g^*$ is a dual functional to $v$ (which vanishes on any root vector of different weight), then in the dual adjoint action,

$$h \cdot \phi(s) = \phi([s, h]) = -\phi([h, s])$$

and so we have $h \cdot \phi = -\alpha(h) \phi$ for all $h \in h$. Thus it is reasonable to define $\phi$ to have weight $-\alpha$. (However, the weight stratification and all the important properties we need would still hold if we dropped the minus sign.)

Finally, it is well known that $[g_\alpha, g_\beta] \subseteq g_{\alpha + \beta}$. From this it is clear from the definition of $d_0$ that it preserves weight as it does on the generating set. Note $d_0(x_k) = -\sum_{i<j} c_{ij}^k x_i x_j$ and it is clear that $c_{ij}^k = 0$ unless the weight of $x_k$ is the sum of the weights of $x_i$ and $x_j$. Since $d_1$ also preserves weight trivially, we see since all differentials are derived from $d_0$ and $d_1$ that the spectral sequence respects the weight decomposition.

Thus letting $A(R)$ denote the root lattice we have a direct sum decomposition of spectral sequences

$$E_r^{*,*}[g] = \bigoplus_{\alpha \in A(R)} E_r^{*,*}[\alpha].$$

Note it is not necessary for a Lie algebra to be semisimple to have a weight decomposition. Indeed the example in the last section is not semisimple. All that is needed is a decomposition of $g$ into $g_\alpha$ over some abelian semigroup index $R \subseteq \mathbb{Z}^n$ with $[g_\alpha, g_\beta] \subseteq g_{\alpha + \beta}$. Of course the corresponding weight decomposition of $E_1^{*,*}$ will use as index set the set of nonnegative integer combinations of $R$ which might not be a lattice if $-R \neq R$. This was the case in the example in the last section as we only got a ray and not a lattice in $\mathbb{Z}$ as our weights.

Over a field of characteristic zero, $E_1^{*,*} = H^*(g, U(g^*)) = H^*(g) \otimes \Gamma$ where $\Gamma$ are the adjoint invariant polynomial functions on $g$. Note that if $f$ is an adjoint invariant polynomial function, then by definition $[g, f] = 0 = 0 f$ for all $g \in g$ and so in particular for all $g \in h$. Thus an adjoint invariant polynomial is by necessity of weight zero. So $\Gamma \subseteq E_1^{*,*}[0]$. However we have previously seen that the generators in $H^*(g)$ transgress to elements in $\Gamma$ in this spectral sequence and so they also must have weight 0. Thus we conclude for a complex semisimple Lie algebra, $H^*(g, U(g^*)) = E_1^{*,*} = E_1^{*,*}[0]$ and so only the weight zero contributions survive in the $E_1$-page and beyond.

Over a field of characteristic $p$, it is still true that $E_1^{*,0}$ consists of invariant polynomials and hence must have eigenvalue 0 under the adjoint action of $g$ on itself. However since our original weight decomposition used integral weights, we can only conclude that the weight of an invariant polynomial is congruent to 0 modulo $p$ as all those weights will be zero in the field. Furthermore not every element of the spectral sequence transgresses to the line of invariant polynomials in general but those which do will have weight congruent to 0 mod $p$ of course.

We summarize this section’s discussion in:
Theorem 17 (Weight stratification) Let \( g \) be a complex semisimple Lie algebra. Let \( g_{\mathbb{Z}} \) be a corresponding integral Lie algebra (always exists by Cartan-Serre basis). We can then get a corresponding Lie algebra \( g_k \) over any ring of definition \( k \).

The root decomposition of \( g \) induces a weight decomposition of spectral sequences:

\[
E_\ast^r[g_k] = \bigoplus_{\alpha \in \Lambda(R)} E_r^{\ast,\ast}[\alpha]
\]

where \( \Lambda(R) \) is the root lattice of \( g \) and is a free abelian group of rank equal to the rank of \( g \).

If \( k \) is a field of characteristic zero then

\[
E_1^{\ast,\ast}[g_k] = E_1^{\ast,\ast}[0] = H^\ast(g, U(g)^\ast);
\]

i.e., only weight zero terms contribute anything to the \( E_1 \)-page or beyond.

If \( k \) is a field of characteristic \( p \) then \( E_1^{\ast,0} \) consists of polynomials of weight zero modulo \( p \). Furthermore any elements that transgress to the polynomial line also have to have weight zero modulo \( p \).

Though this decomposition might seem technical, as the example in the last section shows, it helps organize computations very effectively.

8 Witt Lie algebras and Algebraic de Rham Cohomology

Since the computations in prime characteristic will be more difficult by far than in characteristic zero, we will have to bring in the additional important computational viewpoint of algebraic D-modules. We use only the most basic algebraic settings of this theory and recall the essential definitions below, but for an excellent introduction in the basic theory the reader is referred to \([C3]\).

A basic discussion about a large family of infinite semisimple Lie algebras related to the classical Witt Lie algebra can be found in \([NP]\).

Let \( k \) be a field and \( P = k[x_1, \ldots, x_n] \) be a polynomial algebra in \( n \) variables. Formal differentiation \( \frac{\partial}{\partial x_i} \) and multiplication by \( x_i \) define operators i.e., elements of \( \text{End}_k(P, P) \). The operators \( \frac{\partial}{\partial x_i} \) and multiplication operators generate under composition a formal operator algebra on the polynomial algebra called the Weyl algebra in \( n \) variables. A typical element is a formal differential operator with polynomial coefficients, e.g., \( x_1 x_2 \frac{\partial}{\partial x_1} x_2 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_2} \).

If we define the usual commutator Lie bracket of operators \( [\psi, \tau] = \psi \circ \tau - \tau \circ \psi \) then the multiplication operators and formal differentiation operators \( \frac{\partial}{\partial x_i}, 1 \leq i \leq n \), can be used to define a Lie subalgebra of the Weyl algebra called the Witt algebra in \( n \) variables which will be denoted by \( \mathbb{W}_n(k) \).

The typical element of the Witt Lie algebra looks like \( \sum_{i=1}^n P_i \frac{\partial}{\partial x_i} \) where \( P_i \) are polynomials, i.e., is a first order formal differential operator. Equivalently
the Witt Lie algebra can be thought of as the Lie algebra of polynomial vector fields on affine space $k^n$. The Lie bracket is given by

$$\left[ P \frac{\partial}{\partial x_i}, Q \frac{\partial}{\partial x_j} \right] = \left( P \frac{\partial Q}{\partial x_i} \right) \frac{\partial}{\partial x_j} - \left( Q \frac{\partial P}{\partial x_j} \right) \frac{\partial}{\partial x_i}. $$

Before we see how the Witt Lie algebra plays a role in our computations, we have to look at the dga $(E^*_0, d_0)$ as an algebraic De Rham complex.

Let $A^k_{DR}(g) = E^*_0 \otimes k[g]$, where we have let $P$ denote the polynomial algebra $k[g] = Poly(g)$. The differential $d_0$ then defines a formal algebraic De Rham complex

$$0 \rightarrow A^0_{DR} \rightarrow A^1_{DR} \rightarrow \cdots \rightarrow A^n_{DR} \rightarrow 0,$$

where $n$ is the dimension of $g$. Recall elements in $Poly(g)$ can be thought of as formal polynomial functions on $g$ (formal in the case of finite fields) and so the elements in $A^m_{DR}$ can be thought of as polynomial $m$-forms on $g$.

Let us call the cohomology of this complex the De Rham cohomology of $g$, and denote it $H^*_DR(g)$. Note the (De Rham) cohomology is then

$$H^m_{DR}(g) = E^*_1 \otimes [g] = H^m(g, Poly(g^*))$$

Although, this is essentially just rephrasing, the viewpoint will be useful when we consider $g = sl_2(F_p)$, which we do next as an explicit example. Recall in this case $E^*_0[sl_2(F_p)] = A^*(h, e, f) \otimes F_p[H, E, F]$ is the cohomology of the $p$-group $G(sl_2(F_p))$ and $d_0$ is the Bockstein $\beta$.

Here we are using the same generators as were used in the previous discussion of $sl_2(C)$. Since we will be using the weight stratification a lot in the computations, we will rename the generators now.

The generators $h$ and $H$ are dual to the Cartan algebra element $v_h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and so have weight 0. We will denote them by $x_0$ and $y_0$ respectively now. The generators $e$ and $E$ are dual to the weight 2 element $v_e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and so have weight $-2$. We will denote them by $x_-$ and $y_-$ respectively now. The generators $f$ and $F$ are dual to the weight $-2$ element $v_f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and so have weight $+2$. We will denote them by $x_+$ and $y_+$. respectively now.

Thus explicitly $H^*(G(sl_2(F_p)), F_p) = E^*_0 = A^*(x_0, x_-, x_+) \otimes F_p[y_0, y_-, y_+]$. Since $[v_h, v_e] = 2 v_e, [v_h, v_f] = -2 v_f$, the (Bockstein) differential $d_0 = \beta$ can be obtained from equation 1 in Section 2 and is given by
\[ \beta(x_0) = x_+ x_- \]
\[ \beta(x_+) = 2x_0 x_+ \]
\[ \beta(x_-) = -2x_0 x_- \]
\[ \beta(y_0) = y_- x_+ - y_+ x_- \]
\[ \beta(y_+) = -2y_0 x_+ + 2y_+ x_0 \]
\[ \beta(y_-) = 2y_0 x_- - 2y_- x_0 \]

Here anything with a subscript of zero has weight zero, a negative subscript has weight \(-2\) and a positive subscript has weight \(+2\). Note the root lattice of \(\mathfrak{sl}_2\) is \(2\mathbb{Z}\).

Since \(\mathfrak{sl}_2(\mathbb{F}_p)\) is 3-dimensional, we can make the following identifications.

We can identify \(A_{DR}^0(\mathfrak{sl}_2(\mathbb{F}_p))\) with the polynomial algebra \(P = \mathbb{F}_p[y_0, y_- , y_+]\). We can identify a typical element \(f_0 x_0 + f_- x_- + f_+ x_+\) of \(A_{DR}^1(\mathfrak{sl}_2(\mathbb{F}_p))\) with \(\hat{f} = (f_0, f_-, f_+) \in P^3\). We will view \(\hat{f}\) is a vector-valued polynomial function.

We can identify a typical element \(g_0 x_- x_+ + g_- x_- x_0 + g_+ x_0 x_-\) of \(A_{DR}^2(\mathfrak{sl}_2(\mathbb{F}_p))\) with \(\hat{g} = (g_0, g_-, g_+) \in P^3\). Finally we can identify a typical element \(\theta x_0 x_- x_+ \in A_{DR}^3(\mathfrak{sl}_2(\mathbb{F}_p))\) with \(\theta \in P\).

We then define the \(\mathfrak{sl}_2\) gradient, curl and divergence, \(\nabla_{\mathfrak{sl}_2}, \text{Curl}_{\mathfrak{sl}_2}, \nabla_{\mathfrak{sl}_2}\), by saying the following diagram commutes:

\[
\begin{array}{ccc}
A_{DR}^3 & \longrightarrow & P \\
\uparrow \beta & & \uparrow \nabla_{\mathfrak{sl}_2} \\
A_{DR}^2 & \longrightarrow & P^3 \\
\uparrow \beta & & \uparrow \text{Curl}_{\mathfrak{sl}_2} \\
A_{DR}^1 & \longrightarrow & P^3 \\
\uparrow \beta & & \uparrow \nabla_{\mathfrak{sl}_2} \\
A_{DR}^0 & \longrightarrow & P \\
\end{array}
\] (2)

where the horizontal maps are the identifications mentioned in the preceding paragraph.

For \(P \in A_{DR}^0\) we can write \(\beta(P) = x_0 \beta_0(P) + x_- \beta_-(P) + x_+ \beta_+(P)\) where \(\beta_i : P \rightarrow P\) are easily seen to be derivations. Since the derivations \(P \rightarrow P\) can in general be identified with the Witt Lie algebra, let us find such identifications for \(\beta_0, \beta_-, \beta_+\). Since these derivations are uniquely determined by their action on the generators \(y_0, y_- , y_+\) of \(P\), simple computations using the
Bockstein formulas listed above show that we have the following identifications:

\[ \beta_0 = 2y_+ \frac{\partial}{\partial y_+} - 2y_- \frac{\partial}{\partial y_-} \]
\[ \beta_- = -y_+ \frac{\partial}{\partial y_0} + 2y_0 \frac{\partial}{\partial y_-} \]
\[ \beta_+ = y_- \frac{\partial}{\partial y_0} - 2y_0 \frac{\partial}{\partial y_+} \]

It is a simple computation to find the operator commutators \([\beta_0, \beta_-] = 2\beta_-\), \([\beta_0, \beta_+] = -2\beta_+\) and \([\beta_-, \beta_+] = \beta_0\) and hence these operators give a faithful representation of the Lie algebra \(sl_2\) in the Witt Lie algebra on 3 generators.

Straightforward computations then express the \(sl_2\)-gradient, curl and divergence in terms of these linear differential operators.

\[ \nabla_{sl_2}(\theta) = (\beta_0(\theta), \beta_-(\theta), \beta_+(\theta)); \]
\[ \text{Curl}_{sl_2}(\hat{f}) = (\beta_- f_+ - \beta_+ f_- - f_0, \beta_+ f_0 - \beta_0 f_+ - 2f_+, \beta_0 f_- - \beta_- f_0 - 2f_-); \]
\[ \nabla_{sl_2} \cdot \hat{f} = \beta_0 f_0 + \beta_- f_- + \beta_+ f_+; \]

for all vector polynomials \(\hat{f} = (f_0, f_-, f_+) \in P^3\) and scalar polynomials \(\theta \in P\).

Since \(\beta \circ \beta = 0\), one has the usual calculus identities:

\[ \nabla_{sl_2} \cdot \text{Curl}_{sl_2}(\hat{f}) = 0 \]
\[ \text{Curl}_{sl_2}(\nabla_{sl_2}(\theta)) = 0 \]

for all \(\hat{f} \in P^3\) and \(\theta \in P\). Thus \(H^3_{DR} = H^3(sl_2, Poly(sl_2)^*)\) measures the obstructions to writing a polynomial as a divergence of a vector polynomial, \(H^2_{DR} = H^2(sl_2, Poly(sl_2)^*)\) measures the obstructions to writing a divergence-free polynomial vector as a curl of another polynomial vector, \(H^1_{DR} = H^1(sl_2, Poly(sl_2)^*)\) measures the obstructions to writing a curl-free polynomial vector as the gradient of a polynomial, and \(H^0_{DR} = H^0(sl_2, Poly(sl_2)^*)\) are the polynomials with gradient identically equal to zero. Thus the DeRham cohomology plays exactly the same role as it does in calculus, except that the basic notions of gradient, curl and divergence have been altered in accordance with the Lie algebra \(sl_2\).

The goal now is to compute \(E^*_1 \Rightarrow [sl_2(F_p)] = H^*(sl_2(F_p), Poly(sl_2(F_p))^*)\) as it gives \(B^2_2\) in the Bockstein spectral sequence of the \(p\)-group \(G(sl_2(F_p))\). This is equivalent to computing \(H^*_{DR}(sl_2(F_p))\) as mentioned above. We have seen general comparison theorems that give the answer in low Hodge degrees relative to the prime \(p\) but we seek now to compute the answer more explicitly in order to understand the “char 0 to char \(p\)” phase transition.

We record some basic facts about \(A^*_2(sl_2)\) that will be used in our computations in the next section. As mentioned before, we weigh the polynomials in \(P = F_p[y_0, y_-, y_+]\) by giving \(y_0\) weight 0, \(y_-\) weight \(-2\) and \(y_+\) weight \(+2\) and declaring the weight of a product to be the sum of the weights.
Proposition 7 (Eigenfunctions of $\beta_0$) If $f \in P$ is a polynomial of weight $w$, then $\beta_0(f) = w f$. Furthermore the monomial basis of $P$ is a basis of $\beta_0$-eigenfunctions.

Proof A simple computation shows that the identity holds for the generators $y_0, y_-, y_+$. To finish the proof we note that if $\phi_1$ has weight $a_1$ and $\phi_2$ has weight $a_2$ then as $\beta_0$ is a derivation, we have $\beta_0(\phi_1 \phi_2) = \beta_0(\phi_1) \phi_2 + \phi_1 \beta_0(\phi_2) = (a_1 + a_2) \phi_1 \phi_2$ and the proposition then follows from simple induction and the fact that $P$ has a basis of monomials which are products of generators.

Proposition 8 (Weight of exterior 0 and 3 lines.) $H^0_{DR} = H^3(\mathfrak{sl}_2(\mathbb{F}_p), \text{Poly}(\mathfrak{sl}_2(\mathbb{F}_p))^*)$ and $H^0_{DR} = H^0(\mathfrak{sl}_2(\mathbb{F}_p), \text{Poly}(\mathfrak{sl}_2(\mathbb{F}_p))^*)$ consist completely of elements of weights congruent to 0 mod $p$.

The polynomials $\kappa = y_0^3 + y_+ y_- y_0^p, y_- y_0^p$ lie in $H^0_{DR}$.

Proof If $\theta \in H^0_{DR} - \{0\}$ is of weight $w$ then $\beta(\theta) = 0$ so in particular $\beta_0(\theta) = w \theta = 0$. Thus $w = 0$ in $\mathbb{F}_p$. Since $w$ is regarded as an integer, this is the same as saying $w$ must be congruent to 0 mod $p$.

Let $u = x_0 x_--x_+$, and let $\psi u$ with $\psi \in P$ be a typical element in $A^3_{DR}$. Suppose the weight $w$ of $\psi$ is not zero modulo $p$; then $\beta_0 \psi = w \psi \neq 0$ and $\beta_0(\frac{\psi}{w}) = \psi$. It is then easy to check that $\nabla_{\mathfrak{sl}_2} \cdot (\frac{\psi}{w}, 0, 0) = \psi$ and so $\psi u$ is a coboundary and represents zero in $H^3_{DR}$. Thus $H^3_{DR}$ consists exclusively of elements of weight congruent to 0 mod $p$ as $u$ itself has weight 0.

The final line of the proposition is a trivial computation.

If $f_0, f_-, f_+ \in P$ have weights $m, n, k$ respectively, we will refer to the weight of the vector polynomial $f = (f_0, f_-, f_+)$ as $(m, n, k) \in \mathbb{Z}^3$.

Proposition 9 (Weight of exterior 1 line.) Let $\theta = f_0 x_0 + f_- x_- + f_+ x_+ \in A^1_{DR}$ have weight $w \in \mathbb{Z}$. The corresponding vector $\hat{f} = (f_0, f_-, f_+)$ has weight $(w, w+2, w-2)$. Then we have $\beta(\theta) = 0 \iff \text{Curl}_{\mathfrak{sl}_2}(\hat{f}) = 0$ if and only if the following equations hold:

$$f_0 = \beta_-(f_+ + f_-)$$
$$\beta_+(f_0) = w f_+$$
$$\beta_-(f_0) = w f_-$$

$H^1_{DR} = H^1(\mathfrak{sl}_2(\mathbb{F}_p), \text{Poly}^*(\mathfrak{sl}_2(\mathbb{F}_p))^*)$ consists completely of elements of weight congruent to 0 mod $p$.

Proof The equations follow from $\text{Curl}_{\mathfrak{sl}_2}(\hat{f}) = 0$ using the formula for $\text{Curl}_{\mathfrak{sl}_2}$ listed above together with simplifications which come from Proposition 7.

Thus it remains to show that $H^1_{DR}$ consists only of elements of weight congruent to 0 mod $p$. Let $\theta \in A^1_{DR}$ be an element with $\beta(\theta) = 0$ and weight $w$ not congruent to 0 mod $p$. Since $w \neq 0 \in \mathbb{F}_p$, we have

$$\nabla_{\mathfrak{sl}_2} \left( \frac{f_0}{w} \right) = \frac{1}{w} (\beta_0(f_0), \beta_-(f_0), \beta_+(f_0)) = \frac{1}{w} (w f_0, w f_-, w f_+) = \hat{f}$$
Thus in particular, \( \sum_{i=0}^{\dim(\mathfrak{g})} (-1)^i \dim(E_{1}^{i} \mathfrak{g}) = 0 \) for all nonnegative integers \( m \).
Proof. $H^*(g, M)$ can be computed as the cohomology of a Koszul complex of the form $\Lambda^*(g^*) \otimes M$. Letting $C^i = \Lambda^i(g^*) \otimes M$, one has from the Euler-Poincare lemma:

$$\sum_{i=0}^{n} (-1)^i \dim(C^i) = \sum_{i=0}^{n} (-1)^i \dim(H^i(g, M)).$$

Since $\dim(C^i)$ is easily seen to be $\binom{n}{i} \dim(M)$, the alternating sum on the left comes out to

$$\dim(M) \sum_{i=0}^{n} (-1)^i \binom{n}{i} = \dim(M)(1 - 1)^n = 0.$$

The last statement follows as $E_1^{\text{diff}, i}[g] = H^i(g, Poly^m(\text{ad}^*))$.

**Proposition 11.** ($H_{DR}^0$-computation) Let $\kappa = y_0^2 + y_0 y_+, s_0 = y_0^p$, $s_- = y_-$, and $s_+ = y_+^p$. Then $H_{DR}^0 = H^0(\mathfrak{sl}_2(F_p), Poly(\text{ad}^*))$ is generated as an $F_p$-algebra by $\kappa, s_0, s_-, s_+$. Furthermore, $s_0, s_-, s_+$ generate a polynomial algebra and $\kappa$ is integral over this polynomial algebra satisfying minimal relation $\kappa^p = s_0^2 + s_- s_+.

Proof. Let $A \subseteq F_p[y_0, y_-, y_+]$ be the $F_p$-algebra generated by $\kappa, s_0, s_-, s_+$. $A \subseteq H_{DR}^0$ by proposition 5. Also we have seen that $H_{DR}^0$ is a subset of the weight 0 mod $p$ polynomials which lie in the $F_p$-algebra generated by $y_0$ and $A$. Let $f \in H_{DR}^0$ have degree $N$. Then we can write

$$f = \sum_{i=0}^{N} y_0^i P_i = \sum_{i=0}^{p-1} y_0^i Q_i,$$

where $P_i, Q_i \in A$ for all $i$. (We used that $y_0^p = s_0 \in A$ in the second equality.) Thus $f = Q_0 + \sum_{i=1}^{p-1} y_0^i Q_i$. Since $\beta(f) = \beta(Q_i) = 0$, we have

$$0 = \sum_{i=1}^{p-1} \beta_+(y_0^i) Q_i = \sum_{i=1}^{p-1} iy_0^{i-1} y_- Q_i,$$

and so $\sum_{i=1}^{p-1} iy_0^i Q_i = 0$, where in the last step we scaled by $y_0$ and used that $y_-$ is not a zero divisor in $F_p[y_0, y_-, y_+]$ to cancel it. If $G = \sum_{i=1}^{p-1} y_0^i Q_i = f - Q_0$, then we have seen that since $\beta(G) = 0$, $\sum_{i=1}^{p-1} iy_0^i Q_i = 0$. Since the numbers $0 < i < p$ are nonzero in $F_p$, we can use this relation to solve for $y_0^{p-1} Q_{p-1}$ as a linear combination of $y_0^i Q_i$ for $0 < i < p - 1$. Thus $G = \sum_{i=1}^{p-2} y_0^i T_i$ for new elements $T_i \in A$. We can now repeat the argument until we see that $G = y_0 T$ for some $T \in A$. However $\beta_+(G) = 0 = y_- T$ then gives $T = 0$ and hence $G = 0$. Thus $f = Q_0 + G = Q_0 \in A$ and hence we have shown that $H_{DR}^0 \subseteq A$. Thus $A = H_{DR}^0$.

Finally $s_0, s_-, s_+$ are algebraically independent as they are the image of the algebraically independent elements $y_0, y_-, y_+$ under the injective Frobenius
algebra endomorphism of $\mathbb{F}_p[y_0, y_-]$ given by $\theta(\alpha) = \alpha^p$. Since $\kappa = y_0^j + y_- y_+$, the Frobenius endomorphism also shows that $\kappa^p = s_0^j + s_- s_+$ and hence $\kappa$ is integral over $\mathbb{F}_p[s_0, s_-, s_+]$ as it satisfies the monic polynomial $t^p - (s_0^j + s_- s_+)$. It is clear that this is the minimal polynomial as $\kappa$ has degree 2 and the $s_i$ have degree $p$, and $p$ is an odd prime.

**Proposition 12** \((H^3_{DR}\text{-computation})\) Let $u \in E^{0,3}_{1} \mathbb{K}$ and $\tau \in E^{p-1,3}_{1} \mathbb{K}$ be $\beta$-cohomology classes representing $x_0 x_- x_+ + y_0^{p-1} x_0 x_- x_+$ respectively. Then $u, \tau \neq 0$ and $E^{p-1,3}_{1} \mathbb{K}$ is generated as an $E^{0,0}_{1} \mathbb{K}$-module by $u$ and $\tau$. Furthermore we have the relations $u x_+ = u s_+ = 0$ and the fundamental relation $\kappa \frac{\partial}{\partial \kappa} u = 0$.

This implies $\tau H^0_{DR} \cap u H^0_{DR} = 0$ and so $H^3_{DR} = \tau H^0_{DR} \oplus u H^0_{DR}$.

**Proof** By proposition 5, any element of $H^3_{DR}$ has weight congruent to 0 mod $p$. Since $x_0 x_- x_+$ has weight 0, such an element has to be represented by $x_0 x_- x_+ + y_0^{p-1} x_0 x_- x_+$ where $F$ is a linear combination of things of the form $\kappa^j y_0^a s_0^b s_-^c s_+^d$ for $\ell, j, a, b, c$ nonnegative integers and $0 \leq j \leq p - 1$. We can write $F$ as $\sum_{i=0}^{p-1} f_i y_0^i$ where $f_i \in H^0_{DR}$. (Note $\beta$ is a $H^0_{DR}$-module-map so the elements $f_i$ of gradient identically zero function as “constants”.) Thus we see $H^3_{DR}$ is generated as an $H^0_{DR}$-module by $[x_0 x_- x_+ y_0^{p-1}]$ with $\kappa \frac{\partial}{\partial \kappa} \bigg|_{\kappa = \tau} u = u x_+ = u s_+ = 0$ and the fundamental relation $\kappa \frac{\partial}{\partial \kappa} u = 0$.

Let us write $F_1 \sim F_2$ if they differ by $\nabla_{x_+} : G$ for some $G \in \mathbb{K}^3$. In this case $x_0 x_- x_+ F_1$ and $x_0 x_- x_+ F_2$ are $\beta$-cohomologous and so $u F_1 = u F_2 \in H^3_{DR}$.

Let us fix a weight $w$ which is 0 mod $p$ for $F$ and ask about the image of $\nabla_{x_+}$ in this weight $w$ component. To hit this component, the input $G = (g_0, g_-, g_+)$ has to have weight $(w, w - 2, w + 2)$ and the corresponding output will be

$$\beta_0 g_0 + \beta_- (g_-) + \beta_+ (g_+) = \beta_- (g_-) + \beta_+ (g_+),$$

as $\beta_0 g_0 = 0$. Since $g_-$ has weight $w - 2$ congruent to $-2$ mod $p$, it is a combination of things of the form $y_- y_0^a$ or $y_-^{p-1} y_0^a$ with $\alpha \in H^0_{DR}$ and $0 \leq i \leq p - 1$. Since $g_+$ has weight $w + 2$ congruent to $+2$ mod $p$, it is a combination of things of the form $y_+ y_0^a$ or $y_+^{p-1} y_0^a$, $\alpha \in H^0_{DR}$, $0 \leq i \leq p - 1$. Thus, as $g_-$ and $g_+$ can be chosen independently, we see that the weight $w$ component of the image of the divergence $\nabla_{x_+}$ is the $H^0_{DR}$-module spanned by the elements $\beta_-(g_- y_0^i), \beta_+(g_+ y_0^i), \beta_- (y_- y_0^i), \beta_+ (y_+ y_0^i)$ with $0 \leq i \leq p - 1$ (or equivalently for all nonnegative integers $i$ as $y_0^i = s_0 \in H^0_{DR}$).

After simple computations we find that the image of the divergence is generated as an $H^0_{DR}$-module by the elements $i \kappa y_0^{i-1} - (i + 2) y_0^{i+1}, s_- i y_0^{i-1}, s_+ i y_0^{i-1}$ for $0 \leq i \leq p - 1$ (or equivalently all nonnegative $i$).

Setting $\theta_j = j y_0^{j-1}$ we find that $\theta_j = 0$ if $j$ is congruent to 0 mod $p$ and $\frac{\partial}{\partial \theta_j} y_0^{-1}$ if $j$ is not congruent to 0 mod $p$. In this new language the image of the divergence can be computed as the $H^0_{DR}$-module spanned by $\kappa \theta_j - \theta_{j+2}, s_+ \theta_j, s_- \theta_j$ for all nonnegative integers $j$. Thus $\kappa \theta_j \sim \theta_{j+2}$ for all
nonnegative $j$, and by a simple induction we have that all $\theta_{2k} \sim \kappa^k \theta_0 = 0$ and $\theta_{2k+1} \sim \kappa^k \theta_1 = \kappa^k$.

Thus $u[\theta_{2k}] = 0$ and $u[\theta_{2k+1}] = u \kappa^k$ in $H_{DR}^3$. Since $\theta_{i+1}$ are nonzero multiples of $y_0^i$ when $1 \leq i < p - 1$ it follows that $u[y_0^i] = 0$ when $i$ is odd in this range and $u[y_0^{2k}] = \kappa^k$ when $i = 2k$ in this range. Since we had previously seen that $H_{DR}^3$ is generated by $u[y_0^i]$ for $0 \leq i \leq p - 1$ as $H_{DR}^0$-module, we now see that all these generators except $u[y_0^0] = u$ and $u[y_0^{p-1}] = \tau$ are redundant.

Hence we have shown that $H_{DR}^3$ is generated by $u$ and $\tau$ as a $H_{DR}^0$-module. Since 1 is not in the image of the divergence, $u \neq 0$. Since $y_0^{p-1}$ is not in the image of the divergence, $\tau \neq 0$. Since $s_+ \theta_1 = s_+, s_- \theta_1 = s_-$ are in the image of the divergence, $s_+ u = s_- u = 0$. Since $y_0^p = \theta_{p+1} \sim \kappa^{\frac{p-1}{2}} \theta_0 = 0$, we also have $s_0 u = 0$. Finally since $0 = \theta_p \sim \kappa^{\frac{p-1}{2}} \theta_1 = \kappa^{\frac{p-1}{2}}$ we have $\kappa^{\frac{p-1}{2}} u = 0$.

Note the relations that we just found imply that $u H_{DR}^0$ is concentrated in Hodge degrees less than or equal to $p - 2$ as $u, u\kappa, \ldots, u \kappa^{\frac{p-1}{2}}$ are. On the other hand $\tau H_{DR}^0$ is concentrated in Hodge degrees greater than or equal to $p - 1$, the Hodge degree of $\tau$. Thus $\tau H_{DR}^0 \cap u H_{DR}^0 = 0$ and the proof of the proposition is complete.

Since it is important to know that $\kappa^i u \neq 0$ for $0 \leq i < \frac{p-1}{2}$ in order to get lower bounds on essential dimension, we show that with a dedicated proof for clarity:

**Proposition 13 (Fundamental nonvanishing)** In $H_{DR}^3$ we have

$$\kappa^{\frac{p-1}{2}} u \neq 0.$$

**Proof** To ease notation, we set $i = \frac{p-1}{2}$. Assume to the contrary that there is an element $x_0 x_+ F + x_0 x_- G + x_+ x_- H$ with

$$\beta(x_0 x_+ F + x_0 x_- G + x_+ x_- H) = \kappa^i u,$$

where $F$, $G$, and $H$ are polynomials of degree $p - 3$. Then $F$ must have weight $-2$, and hence is a linear combination of terms of the form $y_0^{2j+1} y_+^{i-j} y_-^{i-j}$ for $0 \leq j \leq i$. Similarly, $G$ is a linear combination of terms of the form $y_0^{i+1} y_+^{j-i} y_-^{i-j}$, and $H$ is a linear combination of terms of the form $y_0^i y_+^i y_-^i$. A straightforward induction shows that

$$\beta(x_0 x_+ y_0^k y_+^{i-1} y_-^j) = x_0 x_+ x_- (2 y_0^k y_+^{i-1} y_-^j - k y_0^{k-1} y_+^j y_-^j) = \beta(x_0 x_- y_0^k y_+^i y_-^j);$$

$$\beta(x_+ x_- y_0^k y_+^i y_-^j) = 0.$$

Thus, without loss of generality, we may assume $G = H = 0$. 

Now we compute
\[
\beta \left( \sum_{j=0}^{i-1} a_j x_0 x_+ y_0^{2j+1} y_+^{-j-1} y_-^{-j} \right)
\]
\[
= \sum_{j=0}^{i-1} a_j x_0 x_- x_+ \left( (2j - 2i)y_0^{2j+2} y_+^{-j-1} y_-^{-j-1} + (2j + 1)y_0^{2j} y_+^{-j-1} y_-^{-j} \right)
\]
\[
= u \left( a_0 y_+ y_-^2 - 2a_{i-1} y_0^{2i} + \sum_{j=1}^{i-1} \left( a_j (2j + 1) + a_{j-1} (2j - 2i - 2) \right) y_0^{2j} y_+^{-j-1} y_-^{-j} \right)
\]
Setting this equal to
\[
u \kappa^i = u \sum_{j=0}^{i} \binom{i}{j} y_0^j y_+^{-j} y_-^{-j},
\]
and comparing the $y_0$-degree 0 terms, we obtain $a_0 = 1$. We claim that $a_j = \binom{i}{j}$ for $j \leq i$, which we show by induction.

Assume that $a_{j-1} = \binom{i-1}{j-1} = \binom{i}{j-1}$, then the coefficient of the $y_0$-degree 2$j$ term for $1 \leq j \leq i - 1$ is
\[
a_j (2j + 1) + a_{j-1} (2j - 2i - 2) = a_j (2j + 1) + \binom{i}{j-1} (2j - 2i - 2).
\]
Now, $2j + 1 \not\equiv 0 \pmod{p}$, since $j \leq i = \frac{p-3}{2}$, so upon setting this equal to $\binom{i}{j}$, we obtain
\[
a_j = \frac{1}{2j + 1} \left( \binom{i}{j} + \binom{i}{j-1} \frac{j}{i-j+1} (2i - 2j + 2) \right)
\]
\[
= \frac{1}{2j + 1} \left( \binom{i}{j} + 2j \binom{i}{j-1} \right) = \binom{i}{j}.
\]
Finally, we compare the $y_0$-degree 2$i$ terms, and find that $-2a_{i-1} = \binom{i}{i} = 1$.
However since we also have $a_j = \binom{i}{j}$, we get $-2a_{i-1} = -2\binom{i}{i-1} = -2i = 3 - p \equiv 3 \not\equiv 1 \pmod{p}$, and this contradiction completes the proof.

We now show that for every odd prime $p$, the behaviour of $E_1^{\ast,\ast}[\mathfrak{s}l_2(F_p)]$ is the same as that of $E_1^{\ast,\ast}[\mathfrak{s}l_2(C)]$ for Hodge degrees less than $p - 1$.

**Proposition 14 (Low Hodge degree computation)** For every odd prime $p$, $E_1^{\ast,\ast}[\mathfrak{s}l_2(F_p)] \cong \Lambda^\ast(u) \otimes F_p[\kappa]$ for Hodge degrees less than $p - 1$. (The isomorphism breaks down at Hodge degree $p - 1$, for example we have seen $\kappa \otimes u = 0$).

**Proof** We have seen that the $H^0_{DR} = E_1^{0,0}$-line is generated by $\kappa, s_0, s_+,$ and so for Hodge degree less than $p$, the only elements are $\kappa^i, 0 \leq i \leq \frac{p-3}{2}$.
We have seen that the $H^1_{DR}$-line is generated over $H^0_{DR}$ by $\tau$ and $u$ and so for Hodge degree less than $p - 1$, the only elements are $u \kappa^i, 0 \leq i < \frac{p-1}{2}$.
Let $a_i = \dim(E_i^{0,0}), b_i = \dim(E_i^{1,0}), c_i = \dim(E_i^{1,1}), d_i = \dim(E_i^{1,3})$ then we have for $0 \leq i \leq \frac{p-3}{2}, a_i = d_i = 1$ if $i$ is even and $a_i = d_i = 0$ for $i$ odd. Thus in this same range, the Euler-Poincare identity gives $0 = a_i - b_i + c_i - d_i = -b_i + c_i$ and so $b_i = c_i$ for $0 \leq i \leq \frac{p-3}{2}$.

Note since $E_1^{1,0} = 0$, we know that any nonzero element of $E_1^{0,1}$ would survive to $E_\infty$ contradicting that $E_\infty$ is the cohomology of a point. Thus $E_1^{0,1} = 0$ and so $b_0 = c_0 = 0$.

Now $d_1(u) = d_1(x_0 x_- x_+ + y_0 x_- x_+ + x_0 x_- y_+) = 0$ as $y_0 x_- x_+ - x_0 y_- x_+ + x_0 x_- y_+ = \beta(x_0 y + \frac{1}{2}(x_- y_+ + x_+ y_-))$. Note as $d_1(x_0 y + \frac{1}{2}(x_- y_+ + x_+ y_-)) = y_0^2 + y_- y_+,$ we see directly that $d_2(u) = \kappa$.

This establishes then that $d_1(uc^i) = 0$ and $d_2(uc^i) = [\kappa i+1]$ in $E_2$. Since by proposition $\exists \text{ } uc^i \neq 0$ in $E_2^{2,3}$ for $0 \leq i < \frac{p-1}{2}$, this forces $d_2$ to be nontrivial in this range, as these elements must support some differential by $E_\infty$ and $d_2$ is the last possible nonzero differential. Note this means also that $\kappa i+1$ is not hit by any $d_1$-differential for this range of $i$, as it must survive until $E_2$ to be hit by $d_2(uc^i)$.

Since this accounts for the behaviour of all of the exterior 0 and 3 line in the range of Hodge degree less than $p - 1$, we conclude that we must have $d_1 : E_1^{i,2} \rightarrow E_1^{i+1,1}$ is an isomorphism in the range $0 \leq i < p - 2$. Thus $c_i = b_{i+1}$ in this range.

Thus $0 = b_0 = c_0 = b_1 = c_1 = b_2 = c_2 = \cdots = b_{p-2} = c_{p-2}$ which completes the proof of the proposition.

This completes the bulk of the computations. In the next section we lay out a picture of generators of $E_1^{*,*} = H^*(\mathfrak{sl}_2, Poly(ad^*))$ with tables and spectral sequence diagrams to help summarize the picture. Results about the corresponding $p$-groups are then found.

9 Computation of $E_1^{*,*}[\mathfrak{sl}_2(\mathbb{F}_p)]$

The table below gives a list of important elements in $E_1^{*,*}[\mathfrak{sl}_2(\mathbb{F}_p)]$. All elements are $d_0 = \beta$-cycles and so represent elements in $E_1$. We have seen that the elements $\kappa, \upsilon, \tau, s_0, s_-, s_+$ generate the exterior 0 and 3 lines as well as all elements of Hodge degree less than $p - 1$. In the table, $s$ is the Hodge degree, $t$ is the exterior degree and $w$ is the weight.
Note that multiplication by $y_+^{-1}$ is defined where it appears, since every term it is multiplied by contains a factor of $y_+$. From what we have so far, the following short exact sequence is for $c_1$:

\[ 0 \rightarrow E_1^{p-1,0} \rightarrow E_1^{p,0} \rightarrow E_1^{p+1,0} \rightarrow 0. \]

\begin{tabular}{|c|c|c|c|}
\hline
$s$ & $t$ & $w$ & \\
\hline
$u$ & 0 & 3 & 0 \hspace{0.5cm} x_0 x_-- x_+ \\
$\kappa$ & 2 & 0 & 0 \hspace{0.5cm} y_0^2 + y_+ y_- \\
$\lambda_0$ & $p-1$ & 1 & 0 \hspace{0.5cm} x_0 \kappa^{(p-1)/2} - x_+ y_0 y_+^{-1} \left( \kappa^{(p-1)/2} - y_0^{p-1} \right) \\
$\lambda_+$ & $p-1$ & 1 & 2p \hspace{0.5cm} x_+ y_+^{p-1} \\
$\lambda_-$ & $p-1$ & 1 & -2p \hspace{0.5cm} x_- y_+^{p-1} \\
$\mu_0$ & $p-1$ & 2 & 0 \hspace{0.5cm} x_0 y_0^{p-2} \beta(y_0) = x_0 y_0^{p-2} (x_+ y_- - x_- y_+) \\
$\mu_+$ & $p-1$ & 2 & 2p \hspace{0.5cm} x_+ y_+^{p-2} \beta(y_+) = -2 x_0 x_+ y_0^{p-1} \\
$\mu_-$ & $p-1$ & 2 & -2p \hspace{0.5cm} x_- y_+^{p-2} \beta(y_-) = 2 x_0 x_- y_0^{p-1} \\
$\tau$ & $p-1$ & 3 & 0 \hspace{0.5cm} x_0 x_- x_+ y_0^{p-1} \\
$s_0$ & $p$ & 0 & 0 \hspace{0.5cm} y_0^2 = d_1(\lambda_0) \\
$s_+$ & $p$ & 0 & 2p \hspace{0.5cm} y_+^2 = d_1(\lambda_+) \\
$s_-$ & $p$ & 0 & -2p \hspace{0.5cm} y_-^2 = d_1(\lambda_-) \\
f_0$ & $p$ & 1 & 0 \hspace{0.5cm} y_0^{p-1} \beta(y_0) = y_0^{p-1} (x_+ y_- - x_- y_+) = d_1(\mu_0) \\
f_+$ & $p$ & 1 & 2p \hspace{0.5cm} y_+^{p-1} \beta(y_+) = -2 y_0^{p-1} (x_+ y_0 - x_0 y_-) = d_1(\mu_+) \\
f_-$ & $p$ & 1 & -2p \hspace{0.5cm} y_-^{p-1} \beta(y_-) = 2 y_0^{p-1} (x_- y_0 - x_0 y_+) = d_1(\mu_-) \\
$\gamma$ & $p$ & 1 & 0 \hspace{0.5cm} x_+ y_+^{p-1} \left( \kappa^{(p+1)/2} - y_0^{p+1} \right) + x_0 y_0^{p-1} \\
$\epsilon$ & $p$ & 2 & 0 \hspace{0.5cm} y_0^{p-1} (x_+ x_0 y_0 - x_0 x_+ y_- + x_0 x_- y_+) = d_4(\tau) \\
\hline
\end{tabular}

Let $\mu_0, \mu_-, \mu_+$ be a basis of $E_1^{p-1,2}$. (Candidates are listed in the table above but we still need to show these candidates are linearly independent in $E_1$. This will be done in Lemma 4 below.)

Note the explicit element $\gamma$ in the table has $d_1(\gamma) = \kappa^{p+1}$ and so we see explicitly that $[\kappa^{\frac{p+1}{2}}] = 0$ in $E_2$. This now leaves $\tau$ in a dilemma: it can support no nonzero $d_2$ differential as the corresponding target location is zero (spanned by $[\kappa^{\frac{p+1}{2}}] = 0$). Thus $d_1(\tau) = \epsilon \neq 0$ in $E_1$. From what we have so far, the following short exact sequence is forced:

\[ 0 \rightarrow E_1^{p-1,2} \rightarrow E_1^{p,1} \rightarrow E_1^{p+1,0} \rightarrow 0. \]
We know that the left hand group has basis the $\mu_i$ and dimension 3. We also know the right hand group has basis $\kappa^{\pm 1}$ and dimension 1. Thus $E^{p,1}_1$ has dimension exactly four generated by the element $\gamma$ and the $d_i$-images of the $\mu_i$. In the table above we have noted these $d_i$-images as $f_i$ i.e., $f_i = d_i(\mu_i)$.

Now note that $E^{p,3}_1 = 0$ as the exterior 3 line is generated by $\tau$ and $u$ over $E^{0,0}_1$ and $u$’s contribution lies entirely before Hodge degree $p - 1$, while nothing nonzero times $\tau$ lies in that location. Thus we know for certain that the dimensions of $E^{0,1}_1$, $E^{1,1}_1$ and $E^{p,3}_1$ are 3, 4 and 0 respectively. The Euler-Poincare count now forces $\dim(E^{p,2}_1) = 1$. Since we have already argued for the existence of the nonzero element $e$ in that location, it must be a basis for $E^{p,2}_1$.

We now prove that the $\mu_i$ and hence the $f_i$ listed in the table are indeed the basis for $E^{p,1}_1$ and $E^{1,1}_1$ respectively:

**Lemma 5** The elements $\{\mu_0, \mu_+, \mu_-\}$ in the table above are a basis for $E^{p,1}_1$ and the elements $\{f_0, f_+, f_-\}$ are a basis for $E^{1,1}_1$.

**Proof** Since the elements have distinct integral weight, to show they are linearly independent, it is sufficient to show they are nonzero. Since we also have argued that $\dim E^{p,1}_1 = \dim E^{1,1}_1 = 3$ this is enough to show they are a basis and complete the lemma.

Furthermore since $d_i(\mu_i) = f_i$ for $i = 0, -, +$, we can show either $\mu_i$ or $f_i$ is nonzero, and it will follow that the other in the pair is also nonzero.

Recall the root lattice of $\mathfrak{s}\mathfrak{l}_2$ is $2\mathbb{Z}$, it is easy to check that $E^{p-1,1}_0[2\rho]$, the weight $2p$ component of $E^{p-1,1}_0$, is one dimensional, spanned by $\lambda_+ = x_+ y_+^{p-1}$. A direct check shows that $d_0(x_+ y_+^{p-1}) = 0$ and so we find that $d_0 : E^{p-1,1}_0[2\rho] \to E^{p-1,2}_0[2\rho]$ is identically zero. Since $\mu_+ \in E^{p-1,2}_0[2\rho]$ and the spectral sequence preserves weight, this shows $\mu_+$ is not a $d_0$-coboundary and so $\mu_+ \not\in E^{p-1,2}_1$ as desired. A similar proof works to show $\mu_- \not\in E^{p-1,2}_1$.

It remains to show $\mu_0 \not\in E^{p-1,2}_1$. We will do this by showing $d_1(\mu_0) = f_0 \not\in E^{1,1}_1$. Since $f_0$ is represented by $y_0^{-1} \beta(y_0)$ and has integral weight 0, it is enough to show that $y_0^{-1} \beta(y_0)$ is not in the image of $d_0 : E^{0,0}_1[0] \to E^{1,1}_1[0]$. The typical element $\alpha$ of $E^{0,0}_1[0]$ is of the form $\alpha = \sum_{\ell=0}^{\infty} c_\ell \kappa^\ell y_0^{-2\ell}$.

Computing we find:

$$d_0(\alpha) = (\sum_{\ell=0}^{\infty} c_\ell \kappa^\ell (p - 2\ell) y_0^{-1 - 2\ell}) \beta(y_0).$$

Now $\beta(y_0) = y_+ x_+ - y_+ x_-$ has the easily verified property that for any nonzero polynomial $f \in \mathbb{F}_p[r, y_0]$, $\beta(y_0) f \not\in E^{0,0}_1$. Thus if $d_0(\alpha) = y_0^{-1} \beta(y_0)$ we would have

$$\left(\sum_{\ell=0}^{\infty} c_\ell (p - 2\ell) \kappa^\ell y_0^{-1 - 2\ell} - y_0^{-1}\right) = 0.$$

Since $y_0$ and $\kappa$ are algebraically independent, equating coefficients of $y_0^{-1}$ we get $c_0 p - 1 = 0$, i.e., $-1 \equiv 0 \mod p$ which is ridiculous. Thus $f_0 = y_0^{-1} \beta(y_0)$ is not in the image of $d_0$ and hence $f_0 \not\in E^{1,1}_1$ and we are done.
Thus after lots of work we have found:

**Theorem 18** *(H*(\(\mathfrak{s}_2, \text{Poly(ad^*)}\)) description).* The 17 elements listed in the table above generate all parts of \(E_i^{*,*}[\mathfrak{s}_2(\mathbb{F}_p)] = H^*(\mathfrak{s}_2, \text{Poly(ad^*)})\) in Hodge degrees \(p\) or less, and on the exterior 0 and 3 lines. They are a minimal set of generators that do so. At most a finite number of additional generators lying on exterior 1, 2 lines of Hodge degree \(> p\) are required to generate the whole algebra.

Furthermore,

\[
E_i^{*,*}[\mathfrak{s}_2(\mathbb{F}_p)] = \Lambda^*(u) \otimes \mathbb{F}_p[k]/<u \kappa^p, \kappa^{p+1}>.
\]

**Proof** Most of the theorem has been proved already; we need only make the following additional observations. As \(E_0\) is finitely generated as a module over the Noetherian polynomial algebra \(R = \mathbb{F}_p[s_0, s_-, s_+]\) and \(d_0\) is a \(R\)-module homomorphism, we conclude \(\ker(d_0)\) and hence \(E_1\) are finitely generated \(R\)-modules. It follows that \(E_1\) is finitely generated as an algebra and that at most a finite number of additional generators might be required.

Regardless, since we know \([u \kappa^p, \kappa^{p+1}] = [s_0] = [s_] = [s_+] = 0\) in \(E_2\), we know that no \(d_2\) differential is supported on the exterior 3 line, in Hodge degree \(p - 2\) or more as \(E_2^{*,0} = 0\) for \(s \geq p\). From this it is an easy argument to conclude that the only things that survive to the \(E_2\) page are as described in the theorem. \((E_\infty\) is known to be the cohomology of a point and \(d_2\) is the last possible nonzero differential.)

As a byproduct of all this analysis we have also shown:

**Corollary 4** Let \(\mathfrak{s}_2\) denote \(\mathfrak{s}_2(\mathbb{F}_p)\) for \(p\) an odd prime and let \(S^i\) denote the module of homogeneous, degree \(i\) polynomials equipped with the dual adjoint action. Then for \(i \geq p - 2\), the following sequence is exact:

\[
0 \to H^3(\mathfrak{s}_2, S^i) \to H^2(\mathfrak{s}_2, S^{i+1}) \to H^1(\mathfrak{s}_2, S^{i+2}) \to H^0(\mathfrak{s}_2, S^{i+3}) \to 0.
\]

**Proof** The above sequence is just

\[
0 \to E_1^{i,3} \xrightarrow{d_1} E_1^{i+1,2} \xrightarrow{d_1} E_1^{i+2,1} \xrightarrow{d_1} E_1^{i+3,0} \to 0
\]

in the spectral sequence \(E_1^{*,*}[\mathfrak{s}_2(\mathbb{F}_p)]\).

It is exact as \(E_\infty^{s,t} = E_2^{s,t} = 0\) for \(s + t > p\).

Note that \(d_1(H_{DR}^3) = d_1(H_{DR}^3 + H_{DR}^3 + H_{DR}^0) = H_{DR}^0\) so any missing “ghost generator” on the exterior 2 line has to inject under \(d_1\) to a missing “ghost generator” on the exterior 1 line as the image of \(d_1\) from the exterior 3 line is accounted for.

The following diagram illustrates \(E_1^{*,*}[\mathfrak{s}_2(\mathbb{F}_p)]\) through Hodge degree \(p + 1\) for a typical odd prime \(p\). The given elements are cohomology generators in Hodge degree \(s\) and exterior degree \(t\). The dashed arrows represent the action of the differential \(d_2\), and the solid arrows represent the action of the differential \(d_1\).
The next diagram illustrates $E^*_1[E^*_1(\mathfrak{sl}_2(\mathbb{F}_p))]$ through Hodge degree 20. The numbers indicate cohomology dimensions in Hodge degree $s$ and exterior degree $t$. The dashed arrows represent the differential $d_2$, which is an isomorphism in each case. The solid arrows represent the differential $d_1$, which gives a short exact sequence along each diagonal line.

Computer calculations indicate that the 17 generators we have listed are sufficient to generate the whole algebra through high Hodge degree so we conjecture:

1

Conjecture 1 The 17 elements $u, \kappa, \tau, \epsilon, \gamma, \lambda_i, \mu_i, f_i, s_i$ generate the algebra $H^*(\mathfrak{sl}_2(\mathbb{F}_p), Poly(ad^*))$. 
10 Bounds on the exceptional torsion in congruence subgroups

The $p$-group $G(sl_2(F_p))$ is the kernel of the mod $p$ reduction $SL_2(Z/p^3Z) \to SL_2(F_p)$. Since $G(g) \times Z/pZ$ (where $g$ is the nonabelian Lie algebra of dimension 2) is an index $p$ subgroup of $G(sl_2(F_p))$, it follows that $e(G(sl_2(F_p^2)) \leq pe(G(g)) = p^3$. In [PR], it was shown that $e_{\infty}(G(sl_2(F_p))) = p^2$.

By the results in [BP], it is known that $B_2^2(G(sl_2(F_p))) = E_{2,2}^0*[sl_2(F_p)]$ which was computed (partially) in the last section. Using comparisons to the cyclic group of order $p^2$ defined by $E = F = 0$, it was shown in [BP] that $\beta_2(u) = \beta_2(\kappa) = 0$ and $\beta_3(u)$ is a nonzero multiple of $\kappa$. Thus in particular, as has been the case in all other computations, we have $(E_{1,*}, d_r) = (B_{r+1}, \beta_{r+1})$ for $sl_2(F_p)$ at least in the range between Hodge degree 0 and $p - 2$.

Since we have shown that $ue_{\infty} = 0 \in E_1 = B_2^2$, it follows that we have argued for a nonzero differential $\beta_3(u \kappa e_{\infty}) = ce_{\infty}^2$ with $c \neq 0$. Since it is known from [PR] that $B_2^2$ is finite dimensional, it follows that $\kappa e_{\infty}$ lifts to an exceptional integral cohomology class of order $p^3$ in $H^{2p-2}(G(sl_2(F_p)), Z)$.

Thus $ED(G(sl_2(F_p))) \geq 2p - 2$.

If Conjecture [H] holds and furthermore $E_{1,*} = B_{r+1}$ in all Hodge degrees, then the inequality above would be an equality. (We feel this is almost certainly true but are unable to prove it at this time.)

11 Comments on Modular Forms and Orthogonal Steenrod Structures

11.1 Modular Forms

Following [FTY], $H^*(SL_2(Z), Poly_C(V))$ where $V$ is the canonical complex representation of $SL_2(Z)$ can be identified directly via an “Eichler-Shimura” correspondence with modular forms of certain flavors.

It is well known that the Hodge decomposition $Poly_C(V) = \oplus_{i=0}^{\infty} Sym^i(V)$ yields all the finite dimensional complex irreducible representations of $SL_2(C)$ and equivalently $sl_2(C)$. In this picture $V = Sym^1(V)$ and $ad = Sym^2(V)$. Whitehead’s lemma shows that $H^0(SL_2(C), Poly(V)) = H^0(sl_2(C), Poly(V))$ is the cohomology of a point and hence uninteresting. Thus it is crucial in the Eichler-Shimura correspondence that one has passed to the arithmetic subgroup $SL_2(Z)$.

The closest analog in our work would be $E_{1,*}*[sl_2(Z)] = H^*(sl_2(Z), Poly(ad^*))$, which we have shown has $p$-torsion for all primes $p$. Here there is no longer a nice exponential correspondence between $sl_2(Z)$ and $SL_2(Z)$ and the adjoint representation has been used, rather than the canonical representation, so there is no direct relation to modular forms, though it does seem reasonable to suspect an indirect one.

However, this arithmetic object shares properties with $H^*(SL_2(Z), Poly(W))$, where $W$ is the canonical integral representation of $SL_2(Z)$, which has also
been shown to have p-torsion for all primes p in unpublished work of F. Cohen, M. Salvetti and F. Callegaro.

11.2 Orthogonal Steenrod Structures.

The spectral sequence $E_r^{*,*}$ used in this paper was motivated by the Bockstein spectral sequence of p-groups associated to p-adic Lie algebras.

For a Lie algebra $\mathfrak{g}$, the spectral sequence $E_r^{*,*}[\mathfrak{g}]$ arises as that of a double complex with differentials $d_0$ and $d_1$.

Over $\mathbb{F}_p$, $E_0^{*,*} = A^*(V) \otimes \text{Poly}(V)$ and $d_0$ is the Bockstein arising from the p-group $G(\mathfrak{sl}_2(\mathbb{F}_p))$ while $d_1$ is the Bockstein arising from the elementary abelian p-group. The higher Steenrod P-power operations are axiomatically determined and agree for the two p-groups.

Thus while the two p-groups have the same $\mathbb{F}_p$-cohomology and Steenrod p-power operations, their Bocksteins act “orthogonally” in the following sense:

**Definition 4** (Orthogonal Steenrod Structures) Let $\mathfrak{U}$ be the category of unstable modules over the mod p Steenrod algebra $A_p$. A bigraded module $F^{m,n}$ is an orthogonal Steenrod bimodule if

1. There are two actions of $A_p$ on $F^{m,n}$ such that the $P$-power operations agree under the two actions but the Bockstein operators act differently say via $\beta_0$ and $\beta_1$ respectively.
2. $\beta_0$ raises m-degree by one and preserves n-degree while $\beta_1$ raises n-degree by one and preserves m-degree.
3. $\beta_0 \circ \beta_1 = -\beta_1 \circ \beta_0$.
4. The total complex determined by the pair of commuting differentials $\beta_0$ and $\beta_1$ is acyclic.

(Note the bigrading above does not coincide with the one we have been using in the paper, a regrading was made for convenience and is explained in the appendix.)

In our case the resulting spectral sequence derived from these anti-commuting Bocksteins proved very useful. Considering that every indecomposable injective $\mathfrak{U}$-module is of the form $L \otimes J[n]$ where $L$ is a summand of the cohomology of an elementary abelian p-group and $J[n]$ is a Brown-Gitler module (see [S]), it seems that considering which equivalence classes of extensions in $\text{Ext}_\mathfrak{U}(k,k)$ can be equipped with such “orthogonal structures” might be interesting.

12 Acknowledgments

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A Construction of the Algebraic Spectral Sequence $E[g]$

In this section, we will derive a spectral sequence for Lie algebra cohomology which is based on the Bockstein spectral sequence of the groups mentioned in the introduction but is defined for Lie algebras over an arbitrary PID $k$. We give what are probably too many details so the reader is referred to the short summary of the final results given in an earlier section.

Let $k$ be an arbitrary PID. Let $L$ be a $k$-Lie algebra and $V$ be a finite dimensional $L$-module. Set $V^{(s)} = V \otimes \cdots \otimes V$ ($s$ times) for $s \geq 1$ and $V^{(0)} = k$. $V^{(s)}$ is given the usual $L$-module structure, i.e.

$$u \cdot (v_1 \otimes \cdots \otimes v_s) = (u \cdot v_1 \otimes \cdots \otimes v_s) + (v_1 \otimes u \cdot v_2 \otimes \cdots \otimes v_s) + \cdots + (v_1 \otimes \cdots \otimes u \cdot v_s)$$

for all $u \in L$. ($V^{(0)}$ is given the trivial action). The $L$-action on $V$ is extended to $V^{(s)}$ by saying it acts via derivations.

Let $T^*(V) = k \oplus V \oplus V^{(2)} \oplus \cdots$ be the usual tensor algebra. We will make it a graded algebra by setting $V^n$ to have grading 2n. Thus $T^*(V)$ becomes a $L$-module by the actions defined above. Let $I$ be the ideal generated by elements $v \otimes w - w \otimes v$ for all $v, w \in V$. Then it is easy to check that the action of $L$ preserves $I$. Thus $S^*(V) = T^*(V)/I$ is a graded algebra which inherits an $L$ action from $T^*(V)$. Note that the $L$ action preserves the grading of $S^*(V)$ so that $S^i(V)$ is a $L$-module for all $i \geq 0$. It is well known that $S^*(V)$ is a polynomial algebra on $k$ variables where $k = \dim(V)$. Fix $u \in L$; then for $v_1, \ldots, v_s \in V$ we have

$$u \cdot (v_1 \ldots v_s) = \sum_i v_1 \ldots v_{i-1} \cdot (u \cdot v_i) \cdot v_{i+1} \ldots v_s,$$

so $u$ acts as a derivation on $S^*(V)$. Given a $L$-module $M$ let $\wedge^*(L, M)$ be the usual Koszul resolution for $H^*(L, M)$. Thus $\wedge^*(L, M)$ consists of $M$-valued alternating $s$-forms on $L$ and

$$(d\omega)(x_0, \ldots, x_s) = \sum_{i<j} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, x_i, \ldots, x_j, \ldots, x_s) + \sum_{i=0}^s (-1)^i x_i \cdot \omega(x_0, \ldots, x_i, \ldots, x_s)$$

for all $\omega \in \wedge^s(L, M)$ and all $x_0, x_s \in L$.

Consider

$$H = \bigoplus_{i=0}^{\infty} \wedge^i(L, S^i(V)),$$

the direct sum of Koszul complexes equipped with differential $d$ the direct sum of the differentials of each of the complexes involved. Using the algebra structure of $S^*(V)$ one can give $H$ an algebra structure with grading where the grading of $\wedge^i(L, S^i(V))$ is 1 and that of $\wedge^0(L, S^i(V))$ is 2. With these conventions $H$ is isomorphic to $\wedge^i(x_1, \ldots, x_n) \otimes k[s_1, \ldots, s_k]$ where $n = \dim(L, x_1, \ldots, x_n)$ a dual basis to a basis of $L$, $k = \dim(V)$ and $s_1, \ldots, s_k$ a basis of $V$. We will show that the differential $d$ on $H$ is a derivation with respect to this algebra structure. Thus

$$d(uv) = (du)v + (-1)^{\deg(u)}u(dv)$$

for $u, v$ homogeneous elements in $H$. First note that $d$ is a derivation on $\wedge^i(L, S^0(V))$. This is because this is the same as $d$ for $\wedge^i(L, k)$ which is well known to be a derivation. Also on $\wedge^0(L, S^i(V)) = S^i(V)$ we have

$$d(s_{i_1} \ldots s_{i_m})(u) = u \cdot (s_{i_1} \ldots s_{i_m})$$

$$= \sum_{j=1}^{m} s_{j_1} \ldots u \cdot s_{j_m}$$

$$= \sum_{j=1}^{m} s_{j_1} \ldots (ds_{i_j}) \ldots s_{i_m}$$

$$(u)$$
for all $u \in \mathcal{L}$. Thus $d$ is a derivation on $\wedge^0(\mathcal{L}, S^*(V))$. Now the elements

$$x_{\mu_1} \ldots x_{\mu_i} \cdot s_{\lambda_1} \ldots s_{\lambda_m}$$

for $\mu_1 < \cdots < \mu_i$ and $\lambda_i$ arbitrary form a basis for $H$. Let $x_{\mu} = x_{\mu_1} \ldots x_{\mu_i}$ and $s_{\lambda} = s_{\lambda_1} \ldots s_{\lambda_m}$. We also say $l(\lambda) = m$ etc. So using this sort of notation let $x_{\mu} s_{\lambda}$ and $x_{\eta} s_{\kappa}$ be two arbitrary basis elements. Then

$$d(x_{\mu} s_{\lambda} x_{\eta} s_{\kappa}) = d(x_{\mu} x_{\eta} s_{\lambda} s_{\kappa}).$$

Suppose we have the identity

$$d(x_{\mu} s_{\lambda}) = d(x_{\mu}) s_{\lambda} + (-1)^{l(\mu)} x_{\mu} d(s_{\lambda}), \tag{4}$$

then we have

$$d(x_{\mu} s_{\lambda} x_{\eta} s_{\kappa}) = d(x_{\mu} x_{\eta}) s_{\lambda} s_{\kappa} + (-1)^{l(\mu)+l(\eta)} x_{\mu} x_{\eta} d(s_{\lambda} s_{\kappa})$$

$$= d(x_{\mu} x_{\eta}) s_{\lambda} s_{\kappa} + (-1)^{l(\mu)} x_{\mu} d(x_{\eta}) s_{\lambda} s_{\kappa}$$

$$+ (-1)^{l(\mu)+l(\eta)} x_{\mu} x_{\eta} d(s_{\lambda} s_{\kappa})].$$

On the other hand,

$$d(x_{\mu} s_{\lambda}) x_{\eta} s_{\kappa} + (-1)^{l(\mu)} x_{\mu} s_{\lambda} d(x_{\eta} s_{\kappa})$$

$$= d(x_{\mu}) x_{\eta} s_{\lambda} s_{\kappa} + (-1)^{l(\mu)+l(\eta)} x_{\mu} x_{\eta} d(s_{\lambda} s_{\kappa}) + (-1)^{l(\mu)} x_{\mu} d(x_{\eta}) s_{\lambda} s_{\kappa}$$

$$+ (-1)^{l(\mu)+l(\eta)} x_{\mu} x_{\eta} d(s_{\lambda} s_{\kappa}).$$

Comparing the two expressions we get

$$d(x_{\mu} s_{\lambda} x_{\eta} s_{\kappa}) = d(x_{\mu} s_{\lambda}) x_{\eta} s_{\kappa} + (-1)^{l(\mu)} x_{\mu} s_{\lambda} d(x_{\eta} s_{\kappa}).$$

So using an easy linearity argument we see $d$ is a derivation on $H$. So it remains to prove equation (4). To do this let $\{e_1, \ldots, e_n\}$ be a basis of $\mathcal{L}$ with $x_i(e_j) = \delta_{i,j}$ the Kronecker delta function. Then for $\alpha_0 < \cdots < \alpha_t$ we have

$$d(x_{\mu} s_{\lambda})(e_{\alpha_0}, \ldots, e_{\alpha_t}) = \sum_{i<j} (-1)^{i+j} [x_{\mu} s_{\lambda}](e_{\alpha_i}, e_{\alpha_j}) (e_{\alpha_0}, \ldots, e_{\alpha_i}, \ldots, e_{\alpha_j}, \ldots, e_{\alpha_t})$$

$$+ \sum_{i=0}^t (-1)^i [x_{\mu} e_{\alpha_i} \cdot s_{\lambda}](e_{\alpha_0}, \ldots, e_{\alpha_i}, \ldots, e_{\alpha_t})$$

$$= d(x_{\mu} s_{\lambda})(e_{\alpha_0}, \ldots, e_{\alpha_t})$$

$$+ \sum_{i=0}^t (-1)^i [x_{\mu} e_{\alpha_i} \cdot s_{\lambda}](e_{\alpha_0}, \ldots, e_{\alpha_i}, \ldots, e_{\alpha_t}).$$

So to show equation (4) it is enough to show

$$\sum_{i=0}^t (-1)^i [x_{\mu} e_{\alpha_i} \cdot s_{\lambda}](e_{\alpha_0}, \ldots, e_{\alpha_i}, \ldots, e_{\alpha_t}) = (-1)^t x_{\mu} d(s_{\lambda})(e_{\alpha_0}, \ldots, e_{\alpha_t}).$$

This is done by an easy case-by-case analysis and is left to the reader.
A.1 Differentials

In the last section we showed that given a finite dimensional \( \mathfrak{L} \)-module \( V \), the differential \( d \) on \( H = \bigoplus_{i=0}^{\infty} \wedge^i(\mathfrak{L}, S^i(V)) \) given by the direct sum of the Koszul differentials is a derivation. We now discuss a method of putting differentials \( D : H \to H \) on \( H \). This means \( D \) is linear, \( D \circ D = 0 \) and \( D \) is a derivation with respect to the graded algebra structure of \( H \). We choose a basis \( \{e_1, \ldots, e_n\} \) of \( \mathfrak{L} \) and let \( \{x_1, \ldots, x_n\} \) be a dual basis. Let \( \{s_1, \ldots, s_k\} \) be a basis of \( V \). Then, as discussed before,

\[
H = \wedge^\ast(x_1, \ldots, x_n) \otimes k[s_1, \ldots, s_k]
\]
as graded algebras where \( \deg(x_i) = 1, \deg(s_i) = 2 \). We will use this notation freely in the proof of the next lemma.

**Lemma 6** Let \( H \) be as in the paragraph above. Then if

\[
\psi : \wedge^1(\mathfrak{L}, S^0(V)) \to \wedge^0(\mathfrak{L}, S^1(V))
\]
is a linear map then it extends to a unique derivation

\[
D : H \to H
\]
which vanishes on

\[
\wedge^0(\mathfrak{L}, S^*(V)) = S^*(V).
\]
Furthermore the derivation is a differential i.e. \( D \circ D = 0 \).

**Proof** First let us define \( D \) as a linear map \( H \to H \). For \( \alpha_0 < \cdots < \alpha_s \) define

\[
D(x_{\alpha_0} \cdots x_{\alpha_s}) = \sum_{l=0}^s (-1)^l x_{\alpha_0} \cdots \psi(x_{\alpha_l}) \cdots x_{\alpha_s}
\]
and

\[
D(1) = 0.
\]
As \( \{x_{\alpha_0} \cdots x_{\alpha_s} : -1 \leq s < n, \alpha_0 < \cdots < \alpha_s \} \) is a basis for \( \wedge^\ast(\mathfrak{L}, S^0(V)) \) we see we have defined a unique linear map

\[
D : \wedge^\ast(\mathfrak{L}, S^0(V)) \to H
\]
extending \( \psi \). It is routine to show (and is left to the reader) that \( D \) defined in this way is a derivation \( \wedge^\ast(\mathfrak{L}, S^0(V)) \to H \). Now we extend \( D \) to \( \wedge^0(\mathfrak{L}, S^*(V)) = S^*(V) \) by defining it to be identically 0 on this part. Finally we define \( D \) on an arbitrary basis element \( x_{\alpha} s_{\bar{\alpha}} \) of \( H \) by

\[
D(x_{\alpha} s_{\bar{\alpha}}) = D(x_{\alpha}) s_{\bar{\alpha}}.
\]
Note this agrees with the parts already defined and hence defines a linear map \( D : H \to H \). It is now routine (and left to the reader) to verify that \( D \) is a derivation with respect to the graded algebra structure on \( H \). Now we argue why this is the unique extension of \( \psi \) to such a derivation which vanishes on \( \wedge^0(\mathfrak{L}, S^*(V)) \). Clearly the definition of \( D \) on \( \wedge^\ast(\mathfrak{L}, S^0(V)) \) was forced if we want a derivation. Then as we require \( D \) to vanish on \( \wedge^0(\mathfrak{L}, S^*(V)) \) we see that the definition on an arbitrary basis element is also forced, and uniqueness follows. Note that \( D \) raises grading by 1. Now we are left only to show \( D \circ D = 0 \) on \( H \). As \( D \) is linear it is enough to check on basis elements \( x_{\alpha} s_{\bar{\alpha}} \) of \( H \). We have

\[
D \circ D(x_{\alpha} s_{\bar{\alpha}}) = D(D(x_{\alpha}) s_{\bar{\alpha}}) = D(D(x_{\alpha})) s_{\bar{\alpha}} + (-1)^{1(\bar{\alpha})+1} D(x_{\alpha}) D(s_{\bar{\alpha}}) = D(D(x_{\alpha})) s_{\bar{\alpha}}.
\]
So we will be done once we show \( D(D(x_{\alpha})) = 0 \). For this let \( \bar{\alpha} \) be \( \alpha_1 < \cdots < \alpha_s \) for some \( 1 \leq s \leq n \). Note for \( s = 1 \) we have \( D(x_{\alpha_1}) \in \wedge^0(\mathfrak{L}, S^1(V)) \) so we have \( D(D(x_{\alpha_1})) = 0 \). Also
D(D(1)) = 0. So we will prove D(D(x_\bar{a})) = 0 by induction on s = l(\bar{a}). Assume we have shown it for s < l where l > 1; then
\[
D(D(x_{a_1} \ldots x_{a_l})) = D(D(x_{a_1})x_{a_2} \ldots x_{a_l} - x_{a_1}D(x_{a_2} \ldots x_{a_l}))
\]
\[
= D(D(x_{a_1})x_{a_2} \ldots x_{a_l} + D(x_{a_1})D(x_{a_2} \ldots x_{a_l}))
\]
\[
- D(x_{a_1})D(x_{a_2} \ldots x_{a_l}) + x_{a_1}D(D(x_{a_2} \ldots x_{a_l}))
\]
\[
= 0,
\]
where in the last step we used cancellation and the inductive hypothesis. Thus by induction we are done and we have shown D \circ D = 0 on H, as desired.

Now we study when the differential D constructed in Lemma 8 has the additional property d \circ D = -D \circ d on H, where d is the Koszul differential of H as in section 1. Let \{e_1, \ldots, e_n\} be a basis for \mathcal{L} as before, \{x_1, \ldots, x_n\} the dual basis, and \epsilon^j_k = [e_j, e_k]_k the structure constants of the Lie algebra \mathcal{L}. Let \{s_1, \ldots, s_k\} be a basis of V as before.

**Lemma 7** Assume the notation of the preceding paragraph. Let
\[
\psi : \Lambda^1(\mathcal{L}, S^0(V)) \to \Lambda^0(\mathcal{L}, S^1(V))
\]
be a linear map. Let D be the differential extending \psi as given by Lemma 8. Then d \circ D = -D \circ d if and only if
\[
e_j \circ \psi(x_i) = \sum_{l=1}^{n} \epsilon^j_l \psi(x_i)
\]
and
\[
\sum_{l=1}^{n} (e_l \cdot s_l) \psi(x_l) = 0
\]
for all 1 \leq i, j \leq n and 1 \leq l \leq k.

**Proof** Since D and d are linear it is enough to check d \circ D = -D \circ d on basis elements x_\bar{a} s_\lambda. So we have
\[
(d \circ D)(x_\bar{a} s_\lambda) = d(D(x_\bar{a}) s_\lambda) = d(D(x_\bar{a})) s_\lambda + (-1)^{(l(\bar{a})+1)} D(x_\bar{a}) d(s_\lambda).
\]
On the other hand we have
\[
(-D \circ d)(x_\bar{a} s_\lambda) = -D(d(x_\bar{a}) s_\lambda) =- D(d(x_\bar{a})) s_\lambda + (-1)^{(l(\bar{a})+1)} D(x_\bar{a}) d(s_\lambda).
\]
So we see by comparing the two expressions that we have d \circ D = -D \circ d if we can show d(D(x_\bar{a})) = -D(d(x_\bar{a})) and D(d(s_\lambda)) = 0. Let us work with the second condition. Let s_\lambda = s_{\lambda_1} \ldots s_{\lambda_m} for some m \geq 1 then
\[
D(d(s_\lambda)) = D \left( \sum_{i=1}^{m} s_{\lambda_1} \ldots d(s_{\lambda_i}) \ldots s_{\lambda_m} \right)
\]
\[
= D \left( \sum_{i=1}^{m} d(s_{\lambda_i}) s_{\lambda_1} \ldots s_{\lambda_i} \ldots s_{\lambda_m} \right)
\]
\[
= \sum_{i=1}^{m} D(d(s_{\lambda_i})) s_{\lambda_1} \ldots s_{\lambda_i} \ldots s_{\lambda_m}.
\]
So we see that to show D(d(s_\lambda)) = 0 one needs only show D(d(s_i)) = 0 for all 1 \leq i \leq k. Now we work with the first condition above. Suppose one has shown d(D(x_i)) = -D(d(x_i)) for 1 \leq i \leq n. Then let us use induction on t = l(\bar{a}) to show d(D(x_\bar{a})) = -D(d(x_\bar{a})). So by
hypothesis we have this for $t = 1$ so assume we have shown it for $t < l$ for some $l > 1$. Then one has:

$$-D(d(x_1 \ldots x_{n-1})) = -D(d(x_1)x_{n-1} - x_1d(x_2 \ldots x_{n-1})) = -D(d(x_1)x_2 \ldots x_{n-1} - d(x_1)D(x_2 \ldots x_{n-1})) + D(x_1)d(x_2 \ldots x_{n-1}) - x_1d(D(x_2 \ldots x_{n-1})) = d(D(x_1)x_2 \ldots x_{n-1} - d(x_1)D(x_2 \ldots x_{n-1})) + D(x_1)d(x_2 \ldots x_{n-1}) - x_1D(D(x_2 \ldots x_{n-1})) = d(D(x_1)x_2 \ldots x_{n-1} - x_1D(x_2 \ldots x_{n-1})) = d(D(x_1 \ldots x_{n-1})).$$

Thus by induction we have the second condition. We conclude from all the above that if we have $d(D(x_1)) = -D(d(x_1))$ for all $1 \leq i \leq n$ and $D(d(s_i)) = 0$ for all $1 \leq i \leq k$ then we have $d \circ D = -D \circ d$ on $H$ as desired. Let us translate the condition $d(D(x_1)) = -D(d(x_1))$ noting that $D(x_1) = \psi(x_1) \in \wedge^0(\mathfrak{L}, S^1(V)) = S^1(V)$. Thus $d(D(x_1))(e_j) = e_j \cdot \psi(x_1)$. On the other hand,

$$-D(d(x_1))(e_j) = D \left( \sum_{l=1}^{n} c^l_m x_l x_m \right) (e_j) = \sum_{l=1}^{n} c^l_m \left[D(x_l)x_m - x_l D(x_m)\right] (e_j) = \sum_{l=1}^{n} c^l_m \psi(x_l) - \sum_{j=1}^{m} c^j_m \psi(x_m) = \sum_{l=1}^{n} c^l_m \psi(x_l) + \sum_{j=1}^{m} c^j_m \psi(x_m) = \sum_{l=1}^{n} c^l_m \psi(x_l).$$

So $d(D(x_1)) = -D(d(x_1))$ for all $1 \leq i \leq n$ if and only if

$$e_j \cdot \psi(x_1) = \sum_{l=1}^{n} c^l_m \psi(x_l)$$

for all $1 \leq i, j \leq n$. Now let us translate the condition $D(d(s_i)) = 0$. Note $d(s_i) \in \wedge^1(\mathfrak{L}, S^1(V))$ is a 1-form on $\mathfrak{L}$ with values in $S^1(V) = V$ so as $d(s_i)(u) = u \cdot s_i$ for all $u \in \mathfrak{L}$ we have $d(s_i) = \sum_{j=1}^{n} d(s_i)(e_j)x_j = \sum_{j=1}^{n} (e_j \cdot s_i)x_j$. Thus

$$D(d(s_i)) = D \left( \sum_{j=1}^{n} (e_j \cdot s_i)x_j \right) = \sum_{j=1}^{n} (e_j \cdot s_i)\psi(x_j)$$

where we have used that $D(e_j \cdot s_i) = 0$ as $e_j \cdot s_i \in \wedge^0(\mathfrak{L}, S^1(V))$. Thus $D(d(s_i)) = 0$ for all $1 \leq i \leq k$ if and only if $\sum_{j=1}^{n} (e_j \cdot s_i)\psi(x_j) = 0$ for all $1 \leq i \leq k$. This ends the proof of the lemma.

### A.2 The spectral sequence

Now we apply the results of the last section to obtain a spectral sequence for Lie algebra cohomology. Let $\mathfrak{L}$ be a Lie algebra over the PID $\mathbb{k}$, and $V = ad^*$ be the $\mathfrak{L}$-module which is the dual space $\mathfrak{L}^*$ of $\mathfrak{L}$ with action

$$(u \cdot \alpha)(v) = \alpha([v, u])$$
for \(u,v \in \mathcal{L}\) and \(\alpha \in \mathcal{L}^*\). It is easy to check using the Jacobi identity that this is indeed a \(\mathcal{L}\)-module. Fix a basis \(\{e_1, \ldots, e_n\}\) of \(\mathcal{L}\). Let \(\{x_1, \ldots, x_n\}\) be the dual basis viewed in \(\wedge^j(\mathcal{L}, S^0(V)) = \wedge^j(\mathcal{L}, k)\) and \(\{s_1, \ldots, s_n\}\) the dual basis viewed in \(\wedge^0(\mathcal{L}, S^j(V)) = V\). Then let \(\psi : \wedge^j(\mathcal{L}, S^0(V)) \to \wedge^0(\mathcal{L}, S^j(V))\) be the linear map given by taking \(x_i\) to \(s_i\) for all \(1 \leq i \leq n\). Recall \(H\) and \(d\) from the last sections. By Lemma \ref{lemma:psi}, one has \(\psi\) extends to a differential \(d\) on \(H\). Note we have

\[
e_j \cdot s_i = \sum_{l=1}^n (e_j, e_l) s_i = \sum_{l=1}^n c_{ij} s_l,
\]

where the first equality is verified by noting both sides evaluate to the same thing when evaluated on \(e_k\) for \(1 \leq k \leq n\) and the second equality uses the definition of the structure constants \(c_{ij}\). Thus as \(\psi(x_i) = s_i\) we have

\[
e_j \cdot \psi(x_i) = \sum_{l=1}^n c_{ij} \psi(x_l).
\]

We also have

\[
\sum_{j=1}^n (e_j \cdot s_i) \psi(x_j) = \sum_{j=1}^n (e_j \cdot s_j) s_i = \sum_{j,l=1}^n c_{ij} s_l s_j = 0.
\]

Thus we have verified the conditions of Lemma \ref{lemma:psi} so we conclude \(d \circ d = -D \circ d\). Thus we have two commuting differentials \(D, d\) on \(H = \oplus_{i=0}^\infty \wedge^i(\mathcal{L}, S^*(ad^*))\) which we note is bigraded. Note \(S^*(ad^*)\) is isomorphic to the vector space of symmetric \(s\)-forms on \(\mathcal{L}\). We will use \(S^1 = S^1(ad^*)\) for short. Recall we can view

\[
H = \wedge^*(\{x_1, \ldots, x_n\}) \otimes k[\{s_1, \ldots, s_n\}]
\]

as graded algebras where \(\deg(x_i) = 1, \deg(s_i) = 2\) and when we do this \(D\) is a derivation. But \(D(x_i) = s_i\) for all \(1 \leq i \leq n\) by construction so we see that the cohomology of the complex \((H, D)\) is concentrated in the zero grading where it is \(k\). (This follows from Künneth’s theorem for example). Now let

\[
F_s^q = \wedge^{q-s}(\mathcal{L}, S^s)
\]

for \(s,q \geq 0\). Then we see \(d\) gives a vertical differential \(F_0^s \to F_0^{s+1}\) and \(D\) gives a horizontal differential \(F_0^s \to F_0^{s+1}\) and \(D, d\) commute (up to sign). Such a complex is called a double complex. If we let \(M\) be the graded complex with \(M^r = \oplus_{s+q=r} F_0^{s,q}\) and differential \(T = D + d\) then one gets by a standard construction two first quadrant \(F_0\) spectral sequences with \(F_0^{s,q}\) given by the formula above converging to the cohomology of \((M^*, T)\). (See \cite{B}). One of these spectral sequences has \(E_1^{r,*}\) equal to the cohomology of \(F_0\) with respect to \(D\). However we have argued before that this is acyclic so \(E_1^{r,*} = 0\) for \(s+q > 0\) and \(E_1^{0,0} = k\). Thus \(E_\infty^{s,q} = 0\) for \(s+q > 0\) and \(F_\infty^{0,0} = k\) for this spectral sequence.

Since this abuts to the cohomology of \(M\) we conclude that \((M^*, T)\) is an acyclic complex. (This means that the only nonzero cohomology is in the zero grading and \(H^0(M^*; k) = k\)). The other spectral sequence has \(E_1^{r,*}\) equal to the cohomology of \(F_0\) with respect to \(d\). Thus it is easy to see \(E_1^{s,q} = H^{q-s}(\mathcal{L}, S^q)\) for \(s,q \geq 0\). However this spectral sequence also abuts to the cohomology of \(M^*\) so we conclude it also has \(E_\infty^{s,q} = 0\) for \(s+q > 0\) and \(F_\infty^{0,0} = k\). Also as \(D\) and \(d\) were derivations one concludes as usual that all the differentials \(d_r\) in this spectral sequence are derivations with respect to the induced graded algebra structures on the \(F_r\). We summarize in the following theorem:

**Theorem 19** Fix an arbitrary PID \(k\). Let \(\mathcal{L}\) be a \(k\)-Lie algebra. Let \(S^*\) be the \(\mathcal{L}\)-module of symmetric \(s\)-forms on \(\mathcal{L}\) with the dual adjoint action discussed before. Then there is a first quadrant \(F_0\)-spectral sequence with \(F_0^{s,q} = M^{q-s}(\mathcal{L}, S^s)\) and \(E_1^{s,q} = H^{q-s}(\mathcal{L}, S^s)\) for \(s,q \geq 0\) which has \(E_\infty^{s,q} = 0\) for \(s+q > 0\) and \(F_\infty^{0,0} = k\). Furthermore all the differentials \(d_r : F_r^{s,q} \to F_r^{s+r,q-r+1}\) are derivations with respect to the induced graded algebra structure on \(F_r\).
In the above spectral sequence, \( s \) denotes the polynomial degree, \( q - s \) represents the exterior degree and \( (q - s) + 2s = q + s \) represents the total degree. This unusual grading is used so that in the derivation, the initial differentials would be horizontal and vertical. However, this bigrading is not so easy to work with in applications so finally we will tweak the definition of the bigrading of the above spectral to:

\[
E_{0}^{s,t} = E_{0}^{s,s+t}.
\]

This yields the final form of the main spectral sequence that we use in this paper:

**Theorem 20** Fix an arbitrary PID \( k \) and \( k \)-Lie algebra \( \mathfrak{g} \). Let \( S^s \) be the \( \mathfrak{g} \)-module of symmetric degree \( s \)-polynomials on \( \mathfrak{g} \), equipped with the dual adjoint action.

Then there is a first quadrant spectral sequence

\[
E_{0}^{s,t} = A^t(\mathfrak{g}, S^s)
\]

with differentials \( d_r : E_r^{s,t} \to E_{r+1}^{s+r,t-(2r-1)} \) which are derivations with respect to the algebra structure induced from

\[
E_{0}^{s,*} = A^*(\mathfrak{g}^*) \otimes S^*(\mathfrak{g}^*).
\]

The number \( s \) is called the polynomial (or Hodge) degree, \( t \) the exterior degree and \( 2s + t \) the total degree. Thus note the differential \( d_r \) raises polynomial degree by \( r \) while raising total degree by one (and hence lowers exterior degree by \( (2r - 1) \)).

The spectral sequence converges to the cohomology of a point. Explicit formulas for the differentials are stated within the paper itself and are easily derived from the construction in the appendix.

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