On the Spectral and Wave Propagation Properties of the Surface Maryland Model

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Abstract

We study the discrete Schrödinger operator $H$ in $\mathbb{Z}^d$ with the surface potential of the form $V(x) = g\delta(x_1) \tan(\pi (\alpha \cdot x_2 + \omega))$, where for $x \in \mathbb{Z}^d$ we write $x = (x_1, x_2)$, $x_1 \in \mathbb{Z}^{d_1}$, $x_2 \in \mathbb{Z}^{d_2}$, $\alpha \in \mathbb{R}^{d_2}$, $\omega \in [0, 1)$. We first consider the case where the components of the vector $\alpha$ are rationally independent, i.e. the case of the quasi periodic potential. We prove that the spectrum of $H$ on the interval $[-d, d]$ (coinciding with the spectrum of the discrete Laplacian) is absolutely continuous. Then we show that generalized eigenfunctions corresponding to this interval have the form of volume (bulk) waves, which are oscillating and non decreasing (or slow decreasing) in all variables. They are the sum of the incident plane wave and of an infinite number of reflected or transmitted plane waves scattered by the "plane" $\mathbb{Z}^{d_2}$. These eigenfunctions are orthogonal, complete and verify a natural analogue of the Lippmann-Schwinger equation. We also discuss the case of rational vectors $\alpha$ for $d_1 = d_2 = 1$, i.e. a periodic surface potential. In this case we show that the spectrum is absolutely continuous and besides volume (Bloch) waves there are also surface waves, whose amplitude decays exponentially as $|x_1| \to \infty$. The part of the spectrum corresponding to the surface states consists of a finite number of bands. For large $q$ the bands outside of $[-d, d]$ are exponentially small in $q$, and converge in a natural sense to the pure point spectrum, that was found in [19] in the case of the Diophantine $\alpha$'s.
1 Introduction

The progress of the last decades in spectral theory of differential and finite difference operators with random ergodic and almost periodic coefficients in the whole space makes natural the study of operators with same type of coefficients supported on a subspace only. Being of evident interest from the point of view of wave physics, they provide a class of operators "intermediate" between operators whose coefficients decay in all coordinates (scattering theory) and operators, having coefficients of the same order of magnitude in all coordinates. We mention recent papers [2], [6], [19], [8], [14], [15], [19], devoted to the study of the spectral and related properties for operators of such a kind. These operators are either defined on the half-space by random, almost periodic or periodic boundary conditions or have the same type of coefficients supported on certain subspaces of \( \mathbb{R}^d \) or \( \mathbb{Z}^d \).

As in [19] we consider here the discrete Schrödinger operator

\[
H = H_0 + V
\]

acting on \( l^2(\mathbb{Z}^d) \), where

\[
(H_0 \Psi)(x) = -\frac{1}{2} \sum_{|x-y|=1} \Psi(y),
\]

is the discrete Laplacian,

\[
V(x) = \delta(x_1) v(x_2), \quad x = (x_1, x_2), \quad x_1 \in \mathbb{Z}^{d_1}, \quad x_2 \in \mathbb{Z}^{d_2}, \quad d_1 + d_2 = d,
\]

with

\[
v(x_2) = g \tan \pi (\alpha \cdot x_2 + \omega)
\]

is the multiplication operator ("surface" potential), whose support is the subspace \( \mathbb{Z}^{d_2} \) of the space \( \mathbb{Z}^d \), and

\[
d_1, d_2 \in \mathbb{N}, \quad g > 0, \quad \alpha \in \mathbb{R}^{d_2}, \quad \omega \in [0, 1)
\]

are the parameters determining the potential.

It was shown in [19] that for any \( g \neq 0, \omega \in [0, 1) \), and for \( \alpha \in \mathbb{R}^{d_2} \), satisfying the Diophantine condition i.e. there exists \( \varepsilon > 0 \) such that

\[
|\alpha \cdot x_2 - m| \geq \text{const} / |x_2|^{d_2+\varepsilon}, \quad \forall x_2 \in \mathbb{Z}^{d_2} \setminus \{0\}, \quad \forall m \in \mathbb{Z},
\]

the spectrum of \( H = H_0 + V \), lying outside the spectrum \([−d, d]\) of the discrete Laplacian (1.2), is pure point, dense, of multiplicity one, and the respective eigenfunctions decay exponentially at infinity.

The "volume" version of this operator, corresponding to the case \( d_1 = 0 \), has been studied earlier in [1, 26]. The operator has a complete system of exponentially decaying eigenfunctions, corresponding to the pure point dense spectrum of multiplicity one occupying the whole real axis. This spectral structure is caused by strong and irregular fluctuations of the quasi periodic
potential (1.3). It is the extreme case of the strong localization regime, which in general appears either if, for a fixed energy, the amplitude of the potential (random or almost periodic) is large enough or, if for a fixed potential, the energy is close enough to the spectrum edges (see [22] for related results and references).

In the case $d_1 = 1$ the support of the potential is the hyperplane $\mathbb{Z}^{d-1}$ of the space $\mathbb{Z}^d$. This is why it is natural to call the respective operator (1.1) - (1.3) the surface Maryland model. This operator is closely related to the boundary value problem (3.3), considered in [15, 11, 14]. We may also call the operator (1.1) - (1.5), for $d_1 \geq 2$, the subspace Maryland model.

These models can be analyzed in great detail, thereby providing examples of spectral types which are only partly known for general random or almost periodic function $v$ in (1.3). All these versions of the Maryland model have an absolutely continuous component of the spectrum. This component was first indicated in [15], and then was studied in [11] in the context of the boundary value problem defined by (1.4) and by formula (3.5) below. It was proven that if the components of the vector $\alpha \in \mathbb{R}^{d_2}$ are rationally independent, this part of the spectrum of $H$ is purely absolutely continuous and also that the properly defined wave operators corresponding to this part exist and are complete. Besides, it was proven in [14] that the surface states (see [15, 14] for definitions) are absent.

In this paper we develop several general ideas and results of the theory by considering the explicitly soluble model, defined by formulas (1.1) - (1.3). We begin by showing that the Green function of the model can be written in a rather convenient form (Section 2). By using this form we study first the quasi-periodic case of rationally independent components of the vector $\alpha$ in (1.3) (Section 3). We prove that the spectrum of the operator is purely absolutely continuous on the interval $[-d, d]$ (on the spectrum of discrete Laplacian) and that the wave operators, corresponding to this part of the spectrum exist (these facts were proved in [11, 14] by other methods). Then we find an explicit form of the generalized eigenfunctions (polynomially bounded solutions of the respective equation), corresponding to this part of the spectrum. These eigenfunctions possess properties, similar to those of the Sommerfeld solutions of scattering theory. Along the $x_2$ direction, they behave like Bloch-Floquet solutions. They are orthogonal and complete on the interval $[-d, d]$ of the spectrum. As they do not decay in the longitudinal coordinates $x_1$ we call them volume states. We consider also the case of rationally dependent components of the vector $\alpha$ in (1.4), where the respective surface potential is periodic in $x_2$, restricting ourselves to the technically simplest case of $d_1 = d_2 = 1$ (Section 4). In this case the whole spectrum is absolutely continuous. It consists of the interval $[-d, d]$ as in the quasiperiodic case, and of a certain number of intervals, some of them possibly intersecting $[-d, d]$. To the interval $[-d, d]$ correspond generalized eigenfunctions which do not decay in the longitudinal coordinates $x_1$. Instead the generalized eigenfunctions corresponding to the other intervals decay exponentially in $x_1$, being of the Bloch-Floquet form in the longitudinal coordinate $x_2$. Such a type of surface states (see Definition 3.1 below) were first found by Rayleigh in the problem of oscillation of an homogeneous elastic half-space (see e.g. [21]), and since then were found and studied in a number of problems, described by differential and finite difference equations whose coefficients are strongly varying in coordinates $x_1$ (see e.g. [19] for a list of references on respective physics results and applications). All these results concerned the problems where coefficients were constant in the $x_2$ coordinates. We analyze also the case where $\alpha_n = p_n/q_n$ approaches an irrational $\alpha$ as $n \to \infty$, and we show that there exists a certain
continuity of the spectrum in this asymptotic regime. In particular, the width of surface bands, lying outside of $[-d, d]$ is exponentially small in $q_n$ as $n \to \infty$, and the bands approach the dense set of eigenvalues, found in [19].

2 Generalities

Recall that we are studying the self-adjoint operator $H$, acting in $l^2(\mathbb{Z}^d)$ and defined in (1.1) - (1.3). The operator is self-adjoint as the sum of the multiplication self-adjoint operator $V$ of (1.3), and of the bounded self-adjoint operator $H_0$ of (1.2). We will use an analogue of the Cayley transform introduced in [5] for the ”volume” potential ($d_1 = 0$) and in [19] for the ”surface” case ($d_1 = 1$), in both cases to study the pure point spectrum for the Diophantine $\alpha'$s (see (1.6)).

To put the subsequent simple argument in a more general context, we rewrite the potential (1.3) as

$$V(x) = v(x_2)\chi_S(x), \quad (2.1)$$

where $\chi_S$ is the indicator of the subspace $S = \mathbb{Z}^{d_2}$ and we assume that $g > 0$ (the case $g < 0$ can be treated analogously). We define the orthogonal projection $P$ of $l^2(\mathbb{Z}^d)$ : 

$$(P\Phi)(x) = \chi_S(x)\Phi((0, x_2)), \quad (2.2)$$

and we write the potential (2.1) in the form

$$V = PvP, \quad (2.3)$$

Here and in the following we use lower cases to denote operators acting on $l^2(S)$ defined by the restriction on $Pl^2(\mathbb{Z}^d)$ of the corresponding operator.

We use as a starting point the well known formulas for the resolvent $G(z) = (H - z)^{-1}$ of a selfadjoint operator $H = H_0 + V$:

$$G(z) = G_0(z) - G_0(z)T(z)G_0(z), \Im z \neq 0 \quad (2.4)$$

with

$$G_0(z) = (H_0 - z)^{-1}, \quad T(z) = V - T(z)G_0(z)V. \quad (2.5)$$

It follows from (2.3) and from (2.4) that the operator $T(z)$ has the form:

$$T(z) = Pt(z)P, \quad (2.6)$$

where the operator $t(z)$, acting on $l^2(S)$, satisfies the equation

$$t(z) = v - t(z)\gamma_0(z)v, \quad (2.7)$$
in which $\gamma_0(z)$ is defined from the restriction of $G_0(z)$ to the subspace $l^2(\mathbb{Z}^d)$. The formal solution of the equation is

$$t(z) = v(1 + \gamma_0(z)v)^{-1} = (v^{-1} + \gamma_0(z))^{-1}. \quad (2.8)$$

Let $u$ be the unitary operator in $l^2(S)$ defined by the relation:

$$(uv)(x_2) = e^{-2i\pi \alpha \cdot x_2} \psi(x_2), \quad x_2 \in S. \quad (2.9)$$

Then, by using the Euler formula for the function $x \mapsto \tan x$ and the notations above, we can write the potential (1.4) as

$$v = g^i \cdot \frac{1 - \sigma u}{1 + \sigma u}, \quad (2.10)$$

where

$$\sigma = e^{-2i\pi \omega}. \quad (2.11)$$

Formulas (1.3) - (2.11) motivate the following abstract statement.

**Lemma 2.1.** Let $H$ be a selfadjoint operator, acting on $l^2(\mathbb{Z}^d)$, and having the form $H = H_0 + V$, where $H_0$ is a selfadjoint operator and $V$ is given by formulas (2.3) and (2.10) in which $S$ is any subset of $\mathbb{Z}^d$ and $|\sigma| \leq 1$. Define the following operators in $l^2(S)$

$$b(z) = (g\gamma_0(z) - i)(g\gamma_0(z) + i)^{-1}, \quad (2.12)$$

assuming that $b(z)$ is bounded. If the operator $g\gamma_0(z) + i$ is invertible and if

$$||b(z)|| < 1, \quad (2.13)$$

then the operator $t(z)$, defined in (2.6) and in (2.8), can be represented in the form:

$$t(z) = g(1 - \sigma u)(1 - \sigma b(z)u)^{-1}(g\gamma_0(z) + i)^{-1}, \quad (2.14)$$

or in the form

$$t(z) = g(g\gamma_0(z) + i)^{-1}[1 - 2i\sigma u \sum_{l=0}^{q-1}(\sigma b(z)u)^l(1 - (\sigma b(z)u)^q)^{-1}(g\gamma_0(z) + i)^{-1}], \quad (2.15)$$

where $\sigma$ is defined in (2.11), and $q \geq 1$ is an integer.

**Proof.** Note that the conditions $||b(z)|| < 1$ and $|\sigma| \leq 1$ allow us to define the operator $(1 - \sigma b(z)u)^{-1}$ by the Neumann-Liouville series. Consider first the case, where the modulus of the complex number $\sigma$ in (2.10) is strictly less than 1. In this case the operator $(1 + \sigma u)^{-1}$ is well defined and we obtain from (2.10), and from (2.12):

$$1 + \gamma_0 v = \left[i(1 + \sigma u) + g\gamma_0(1 - \sigma u)\right](i(1 + \sigma u))^{-1}
= (g\gamma_0 + i) \left(1 - (g\gamma_0 - i)(g\gamma_0 + i)^{-1}\sigma u\right)(i(1 + \sigma u))^{-1},$$

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or $1 + \gamma_0 v = (g\gamma_0 + i)(1 - b(z)\sigma u)(i(1 + \sigma u))^{-1}$, where the operators $\gamma_0(z)$, and $b(z)$ are defined in (2.12). Formulas (2.7), (2.10), and the hypotheses of the lemma lead to (2.14) for $|\sigma| < 1$. According to inequality (2.13) the Neumann-Liouville series for $(1 - b(z)\sigma u)^{-1}$ converges for $|\sigma| = 1$, and since the operator $(1 + \sigma u)^{-1}$ is not present in formula (2.14), we can make the limit $|\sigma| \to 1$ in the formula, proved for $|\sigma| < 1$, and obtain representation (2.14) in the case $|\sigma| = 1$.

**Proposition 2.1.** Let $H$ be the selfadjoint operator defined in Lemma 2.7 and $G(z) = (H - z)^{-1}, \exists z > 0$ be its resolvent. Assume that $z$ is such that the conditions of the Lemma 2.7 hold. Then $G(z)$ can be represented as follows:

$$G(z) = G_0(z) - gG_0(z)P(g\gamma_0(z) + i)^{-1}PG_0(z) + 2igG_0(z)P(g\gamma_0(z) + i)^{-1}$$

$\times \sigma u \sum_{l=0}^{q-1}(\sigma b(z)u)^l(1 - (\sigma b(z)u)^q)^{-1}(g\gamma_0(z) + i)^{-1}PG_0(z),$ (2.16)

where $q \geq 1$ is an integer, $u$ is defined in (2.3) and the operators $\gamma_0(z), b(z)$ are defined in (2.12).

**Proof.** The proposition follows easily from (2.4), and from Lemma 2.1.

**Remarks.** 1. In formula (2.10) the unitary operator $\sigma u$ can be viewed as the Cayley transform of $v$ (see [4] for the definition of the Cayley transform). Likewise, the contraction operator $b(z)$ can be viewed as the Cayley transform of the dissipative operator $i\gamma_0(z)$ ($\Re i\gamma > 0$). Hence, we can say that the passage from the operators $v^{-1}$ and $\gamma_0(z)$ in (2.8) to their Cayley transforms $\sigma u$ and $b(z)$ in the case of the potential (1.3) - (1.4) leads to formulas (2.14) - (2.16). This will allow us to study the absolutely continuous spectrum of the operator $H$ for any $d_1 \geq 0$, as it was done in papers [22] and [19] for the pure point spectrum, despite that the subsequent techniques to study the resolvent (2.11) are different in these two cases. 2. Integrate formula (2.16) with respect to $\omega \in [0, 1)$ and denote this operation by $\langle \cdots \rangle$. We obtain:

$$\langle G(z) \rangle = G_0(z) - gG_0(z)P(g\gamma_0(z) + i)^{-1}PG_0(z).$$

In view of the general formula (2.8), valid for any surface potential $v$, we can interpret the equality $\langle t(z) \rangle = g(g\gamma_0(z) + i)^{-1} = (-ig)^{-1} + \gamma_0(z))^{-1}$ as the fact that $\langle G(z) \rangle$ is the resolvent of the Schrödinger operator whose surface potential is the complex constant $V(x) = -ig\chi_S(x).$ This fact plays an important role in the interpretation of results of analysis of the point spectrum of $H$ outside $[-d, d]$ in [18]. Similar fact is known also in the case of the volume potential (2.3), i.e. for the case $S = \mathbb{Z}^d$ [4].

Now we are going to show that the above proposition is applicable to the operator defined by (1.1) - (1.3) where $S$ is chosen as $\mathbb{Z}^{d_2}$. To check the conditions of the lemma and the proposition we will use the Fourier transformation which we define as follows:

$$\hat{\Phi}(k) = \sum_{x \in \mathbb{Z}^\nu} e^{-2i\pi x \cdot k} \Phi(x), \quad k \in \mathbb{T}^\nu, \quad \Phi(x) = \int_{\mathbb{T}^\nu} dk e^{2i\pi x \cdot k} \hat{\Phi}(k), \quad x \in \mathbb{Z}^\nu,$$ (2.17)

where $\mathbb{T}^\nu = [0, 1)^\nu$ is the $\nu$-dimensional unit torus.
By using the Fourier transformation we can write the following representation of the Green function \( G_0^{(\nu)}(x - y; z) \) of the \( \nu \)-dimensional Laplacian (operator (1.2) for \( d = \nu \)):

\[
G_0^{(\nu)}(x - y; z) = \int_{\mathbb{T}^\nu} dk \frac{e^{2i\pi k \cdot (x - y)}}{E_\nu(k) - z}, \quad \Im z \neq 0, \quad (2.18)
\]

where

\[
E_\nu(k) = -\sum_{i=1}^\nu \cos 2\pi k_i, \quad (k_1, \ldots, k_\nu) = k \in \mathbb{T}^\nu. \quad (2.19)
\]

These formulas imply that the operator \( \gamma_0(z) \) of (2.12) has the following matrix in \( l^2(\mathbb{Z}^{d_2}) \):

\[
\gamma_0(x_2; z) = G_0^{(d_1)}((0, x_2) - (0, y_2); z), \quad (2.20)
\]

i.e. \( \gamma_0(z) \) is a convolution operator in \( l^2(\mathbb{Z}^{d_2}) \). In view of (2.18) we have:

\[
\gamma_0(x_2; z) = \int_{\mathbb{T}^{d_2}} dk_2 e^{2i\pi k_2 \cdot x_2} \int_{\mathbb{T}^{d_1}} \frac{dk_1}{E_d(k) - z}, \quad (2.21)
\]

or

\[
\gamma_0(x_2; z) = \int_{\mathbb{T}^{d_2}} dk_2 e^{2i\pi k_2 \cdot x_2} G_0^{(d_1)}(0, z - E_d(k_2)). \quad (2.22)
\]

We shall denote

\[
\hat{\gamma}_0(k_2; z) := G_0^{(d_1)}(0, z - E_{d_2}(k_2)), \quad (2.23)
\]

i.e. \( \hat{\gamma}_0(k_2; z) \) is the symbol, representing the operator \( \gamma_0(z) \) in \( L^2(\mathbb{T}^{d_2}) \) as a multiplication operator. These formulas allow us to show that the hypotheses of Lemma 2.1 and Proposition 2.1 are valid for any \( z, \Im z > 0 \) (see Lemma 5.2). Besides, we have

**Lemma 2.2.** Let \( b(z) \) and \( u \) be the operators, defined by (2.12) and (2.9). Then for any integer \( m \geq 1 \),

\[
((b(z)u)^m \varphi)(k_2) = \sigma^m (\prod_{l=0}^{m-1} \hat{b}(k_2 + l\alpha; z)) \hat{\varphi}(k_2 + m\alpha), \quad k_2 \in \mathbb{T}^{d_2}, \quad (2.24)
\]

where \( \hat{\varphi} \) denotes the Fourier transform of \( \varphi \in l^2(\mathbb{Z}^{d_2}) \) and

\[
\hat{b}(k_2; z) = \frac{g\hat{\gamma}_0(k_2; z) - i}{g\hat{\gamma}_0(k_2; z) + i} \quad (2.25)
\]

where \( \hat{\gamma}_0(k_2, z) \) is defined in (2.23).

**Proof.** It follows from (2.9) that the operator \( u \) is the shift by \( \alpha \) in the space \( L^2(\mathbb{T}^{d_2}) \):

\[
\widehat{(u\varphi)}(k_2) = \hat{\varphi}(k_2 + \alpha). \quad (2.26)
\]

From this and the fact that \( b(z) \) of (2.12) is the multiplication by the function \( \hat{b}(k_2; z) \) of (2.25) in the space \( L^2(\mathbb{T}^{d_2}) \) prove the lemma.

We will obtain now a representation of the Green function of \( H \) which will be central in the subsequent spectral analysis of the absolutely continuous spectrum of the operator.
Theorem 2.1. Let $H$ be the operator, defined by (1.1)-(1.5). Then the Green function of $H$ (the matrix in $l^2(Z^d)$ of its resolvent $G(z) = (H - z)^{-1}$) can be written in the form:

$$G(x, y; z) = G^{(d)}_0(x - y; z) + \sum_{m=0}^{\infty} \int_{\mathbb{T}^d} dk_2 e^{2\pi i k_2 \cdot (x_1 - y_2)} t_m(k_2; z) \times G^{(d)}_0(x_1; z - E_{d_2}(k_2)) G^{(d)}_0(y_1; z - E_{d_2}(k_2 + m\alpha)) e^{-2\pi i m\alpha \cdot y_2},$$  \hspace{1cm} (2.27)

where

$$t_m(k_2; z) = \frac{g}{g^{(d)}_0(k_2; z) + i} \left\{ \begin{array}{ll} -1, & m = 0 \\ 2i\sigma(g^{(d)}_0(k_2 + \alpha; z) + i)^{-1}, & m = 1 \\ 2i\sigma^{m+1}(g^{(d)}_0(k_2 + m\alpha; z) + i)^{-1} \frac{d}{dz} \hat{t}(k_2 + l\alpha; z), & m \geq 2 \end{array} \right. \hspace{1cm} (2.28)$$

$G^{(d)}_0(x_1; z)$ is the Green function (2.18) of the $d_1$-dimensional Laplacian, $E_{d_2}(k_2)$ is defined in (2.13) for $\nu = d_2$, and $g^{(d)}_0(k_2; z)$, $\hat{t}(k_2; z)$ are defined respectively in (2.23) and (2.23).

Besides, the (generalized) kernel of the operator $T(z)$ of (2.4) and of Lemma 2.1 has the following form in $L^2(\mathbb{T}^d)$:

$$T(k, p; z) = \sum_{m=0}^{\infty} t_m(k_2; z) \delta(k_2 + m\alpha - p_2),$$  \hspace{1cm} (2.29)

where $t_m(k_2; z)$ is defined in (2.28). In particular, the kernel is independent of the components $k_1, p_1 \in \mathbb{T}^{d_1}$ of its arguments $k, p \in \mathbb{T}^d$.

Remark. Formulas (2.27) and (2.29) have to be compared with the formulas for respective quantities for point potential: $V(x) = \nu \delta(x)$, $(d_2 = 0)$ and for the constant surface potential: $V(x) = \nu \delta(x_1)$, $\nu = \text{const}$. In the first case we have:

$$G(x, y; z) = G^{(d)}_0(x - y; z) - \frac{\nu}{1 + v G^{(d)}_0(0; z)} G^{(d)}_0(x; z) G^{(d)}_0(y; z),$$  \hspace{1cm} (2.30)

and

$$T(k, p; z) = \frac{\nu}{1 + v G^{(d)}_0(0; z)},$$  \hspace{1cm} (2.31)

while in the second case:

$$G(x, y; z) = G^{(d)}_0(x - y; z) - \nu \int_{\mathbb{T}^d} dk_2 \frac{e^{2\pi i k_2 \cdot (x_2 - y_2)}}{1 + v G^{(d_1)}_0(0; z - E_{d_2}(k_2))} \times G^{(d_1)}_0(x_1; z - E_{d_2}(k_2)) G^{(d_1)}_0(y_1; z - E_{d_2}(k_2)), $$  \hspace{1cm} (2.32)

and

$$T(k, p; z) = \frac{\nu \delta(k_2 - p_2)}{1 + v G^{(d_1)}_0(0; z - E_{d_2}(k_2))}. $$  \hspace{1cm} (2.33)
In particular the term, corresponding to \( m = 0 \) in (2.27), coincides with the second of (2.32) in which \( v \) is replaced by \( ig \).

**Proof of Theorem (2.4).** According to (2.13), \(|b(z)| < 1\) if \( \Im z \neq 0 \). Hence we can write the operator \((1 - \sigma bu)^{-1}\) in (2.14) for \( q = 1 \) as the Neumann-Liouville series in powers of \( \sigma bu \). Applying lemma 2.1 to each term of the series, we get (2.27) after simple algebra. Formula (2.29) follows from (2.4) and (2.27). Theorem 2.1 is then proved.

**Remark.** Formulas (2.27) and (2.29) are the basic tools of spectral and scattering analysis of the operator (1.1) presented in this paper. An advantage of these formulas is that they are valid for all values of the spectral parameter \( z = E + i\varepsilon \), up to the real values \( z = E \pm i0 \), for \( |E| < 2 \), in the case of \( \alpha \)'s with rationally independent components (quasi-periodic in \( x_2 \) potential \( V(x) \)) and they are valid for all \( E \in \mathbb{R} \) in the case of \( \alpha \)'s with rational components (periodic in \( x_2 \) potential \( V(x) \)).

One more general fact, concerning the operator \( H \) and necessary in the sequel, is given by

**Theorem 2.2.** Let \( H = H_0 + V \) be the operator defined by (1.1), (1.2) and (2.1). Then its spectrum \( \sigma(H) \) contains the interval \([-d, d] = \sigma(H_0)\) for all \( g \in \mathbb{R}, \alpha \in \mathbb{R}^{d_2} \) and \( \omega \in [0, 1] \).

**Proof.** We will apply the H. Weyl criterion, according to which \( E \in \mathbb{R} \) belongs to the spectrum of a self-adjoint operator \( H \) if and only if there exists a sequence \( \{\Psi_n\}_{n \in \mathbb{N}} \) of vectors of respective Hilbert space such that \( \|\Psi_n\| = 1 \), and that \( \lim_{n \to \infty} \|(H - E)\Psi_n\| = 0 \).

Denote by \( 1_r \) the indicator of the ball \( \{x \in \mathbb{Z}^d : |x| \leq r \} \) and set for all \( k \in \mathbb{T}^d \)

\[
\Psi_n(x) = 1_n(x)(1 - \delta(x_1))e^{2i\pi k \cdot x}/N_n; \quad N_n^2 = \sum_{x \in \mathbb{Z}^d} |1_n(x)(1 - \delta(x_1))|^2 = O(n^d), \quad n \to \infty.
\]

It is easy to find that

\[
(H\Psi_n)(x) = \begin{cases} E_d(k)\Psi_n(x), & |x| \leq n - 2, |x_1| \geq 2; \\ A_n(x), & n - 2 \leq |x| \leq n + 2; \\ b_n(x), & |x_1| \leq 1; \\ 0, & |x| \geq n + 3,
\end{cases}
\]

where \( \|A_n\| = O(n^{-1/2}), \|b_n\| = O(n^{-d_1/2}) \) as \( n \to \infty \). This proves the theorem.

### 3 Absolute Continuous Spectrum in the Almost Periodic Case

In this section we assume that the vector \( \alpha \in \mathbb{R}^{d_2} \) from (1.3) has rationally independent components, i.e. that the relation \( \alpha_1 r_1 + \ldots + \alpha_{d_2} r_{d_2} = 0 \) with rational coefficients \( r_1, \ldots, r_{d_2} \) implies that all these coefficients are equal to zero.

**Theorem 3.1.** Let \( H = H_0 + V \) be the self-adjoint operator defined by (1.1) - (1.3) in which the vector \( \alpha \in \mathbb{R}^{d_2} \) has rationally independent components. Then \( H \) has purely absolutely continuous spectrum on the interval \((-d, d)\).
Proof. According to the general principles (see e.g. \cite{24}), it suffices to prove that for any vector \( \Phi \in l^2(\mathbb{Z}^d) \) of a dense set the limit \( \Im(G(E + i0)\Phi, \Phi) \) exists and is bounded for all \( E \in (-d, d) \). Restricting ourselves to the vectors concentrated at a point \( x \in \mathbb{Z}^d \), i.e. to the vectors \( \delta_x = \{ \delta(x - y) \}_{y \in \mathbb{Z}^d} \), we have to prove that for any \( \Phi \in l^2(\mathbb{Z}^d) \) of a dense set the limit \( \Im(G(x, x; E + i0)) \) exists and is bounded for all \( E \in (-d, d) \). We shall prove more, namely that \( G(x, y; E + i0) \) exists and is bounded for all \( E \in (-d, d) \) and all \( x, y \in \mathbb{Z}^d \). In view of Theorem 2.1, we have to prove that the series of (2.27) converges not only for \( \Im(z) > 0 \) but also for \( \Im(z) = 0 \).

Since the vector \( \alpha \) has rationally independent components, we have uniformly in \( k_2 \in \mathbb{T}^d \) and for any \( \gamma > 0 \) (see e.g. \cite{3}):

\[
\lim_{m \to \infty} \sharp \{ l \in \mathbb{Z} : k_2 + l\alpha \in K_\gamma(E), \ 1 \leq l \leq m \} m^{-1} = |K_\gamma(E)|, \tag{3.1}
\]

where

\[
K_\gamma(E) = \{ k_2 \in \mathbb{T}^d : E - E_{d_2}(k_2) \in [-d_1 + \gamma, d_1 - \gamma] \}, \tag{3.2}
\]

and \( |K_\gamma(E)| \) denotes the Lebesgue measure of the set \( K_\gamma(E) \subset \mathbb{T}^d \). It is easy to check that for any \( |E| < d \) there exists \( \gamma > 0 \) such that \( K_\gamma(E) \) is an open set of \( \mathbb{T}^d \). According to Lemma 5.3, in this case there exists \( \delta > 0 \) such that \( |\hat{b}(k_2, E + i0)| \leq 1 - \delta, \forall k_2 \in K_\gamma(E) \), and according to (3.1), there exists \( m_0 > 0 \) such that

\[
\sharp \{ l \in \mathbb{Z} : k_2 + l\alpha \in K_\gamma(E), \ 1 \leq l \leq m \} \geq \frac{m}{2} |K_\gamma(E)|
\]

for all \( m \geq m_0 \). Hence we have the following bound for the product in the r.h.s. of (2.27):

\[
\prod_{l=0}^{m-1} |\hat{b}(k_2 + l\alpha; E + i0)| \leq (1 - \delta)^{m|K_\gamma(E)|/2}, \quad m \geq m_0, \tag{3.3}
\]

and the series in the r.h.s. of (2.27) converges uniformly in \( k_2 \in \mathbb{T}^d \). Besides, by using bound (3.3) and Lemma 5.3, it can be shown that for \( |E| \leq d - \gamma, \gamma > 0 \), the series is bounded in \( k_2 \) and \( E \), hence we can integrate the series with respect to \( k_2 \). Theorem is proved.

Remarks. 1). Another form to express (3.1) - (3.3) is to write the relation:

\[
\lim_{m \to \infty} \left| \prod_{l=0}^{m-1} \hat{b}(k_2 + l\alpha; E + i0) \right|^{1/m} = \exp \left\{ \int_{\mathbb{T}^d} dq_2 \log |\hat{b}(q_2; E + i0)| \right\}, \tag{3.4}
\]

valid uniformly in \( k_2 \in \mathbb{T}^d \) (see \cite{3}) and showing that if \( |E| \leq d - \gamma, \gamma > 0 \), then the integral in the r.h.s. is negative, thus the product in the l.h.s. is exponentially decaying in \( m \) as \( m \to \infty \).

2). Theorem 5.1 reveals a fairly simple mathematical mechanism responsible for the absolutely continuous spectrum for the "subspace" potential (1.3) - (1.4) with \( d_1 \geq 1 \) (recall that in the "volume" case \( d_1 = 0, d_2 = d \), the absolutely continuous spectrum is absent, moreover if \( \alpha \) is Diophantine then the spectrum is pure point \( \mathbb{E} \)). The mechanism is the positiveness of the imaginary part of \( \hat{\gamma}_0(k_2; E + i0) = G_0^{(d_1)}(0, E + i0 - E(k_2)) \) in a certain domain of \( (E, k_2) \). This is most transparent in the "genuine surface" case \( d_1 = 1 \), where \( G_0^{(1)}(0, E + i0) \) is pure
imaginary if $|E| < 1$ and is pure real if $|E| \geq 1$, (see formula (3.37) below). In the latter case $|\hat{b}(k_2; E + i0))| = 1$ and the series (2.27) diverges for a dense set of energies (see [13]). This leads to the pure point spectrum everywhere outside of the spectrum $\sigma(H_0)$ of the Laplacian (similarly to the volume case [5], where the analogue of $\gamma_0(k_2; E)$ in (2.25) is real for all $E \in \mathbb{R}$). In the former case $|\hat{b}(k_2; E + i0))|$ is strictly less than 1 for any $E \in (-d, d)$ on an open set of $k_2 \in \mathbb{T}^{d_2}$, the series is convergent and the spectrum inside of $\sigma(H_0) = [-d, d]$ is pure absolutely continuous.

As usual in scattering theory, a fact of primary interest is the existence and completeness of wave operators $\Omega_{\pm} = s \cdot \lim_{t \to \pm \infty} e^{itH} e^{-it H_0} \mathcal{E}_0(\Delta)$, where $\mathcal{E}_0$ is the resolution of identity of $H_0$, and $\Delta$ is an interval of the spectral axis. In the next theorem we prove these properties in our case.

We mention first that in papers [8, 9, 11, 14], the scattering theory was developed for the operator $H_1$, acting in $l^2(\mathbb{Z}^d_+), \mathbb{Z}^d_+ = \{(x_1, x_2) \in \mathbb{Z}^d; x_1 \geq 0, x_2 \in \mathbb{Z}^{d-1}\}$, and defined as:

$$
(H_1 \Psi)(x) = \left\{ \begin{array}{ll}
\sum_{|x-y|=1} \Psi(y), & x_1 \geq 1; \\
\Psi(1, x_2) + \sum_{|x_2-y_2|=1} \Psi(0, y_2) + v(x_2)\Psi(0, x_2), & x_1 = 0
\end{array} \right. \quad (3.5)
$$

for certain random and almost periodic surface potentials $v$. The operator can be viewed as a boundary value problem for the discrete Laplacian in $l^2(\mathbb{Z}^d_+)$ with the boundary condition $\Psi(-1, x_2) = v(x_2)\Psi(0, x_2), x_2 \in \mathbb{Z}^{d-1}$. The “unperturbed” operator $H_0$ here is the discrete Dirichlet Laplacian, corresponding to $v \equiv 0$ in (3.5). The operator $H_1$ is closely related to our operator $H$ of (1.1) for the surface case $d_1 = 1, d_2 = d - 1$ via standard Green’s formulas.

**Theorem 3.2.** Under the conditions of the Theorem 3.1, the wave operators $\Omega_{\pm}$ for the pair $(H, H_0)$, defined by (1.1)- (1.3), exist and are complete for any closed interval $\Delta = [a, b] \subset (-d, d)$.

**Proof.** Existence of wave operators is a rather general fact. It was proved in [8] for a general surface perturbation $v$ in (3.3). In our case the proof is practically the same. Thus we have to prove the completeness. Mimicking the argument of [11, 14], developed for the boundary value problem (3.4), it is easy to reduce the proof of completeness to the proof of the relation:

$$
\sup_{E > 0, \varepsilon \in [a, b]} \sum_{x_2 \in \mathbb{Z}^{d_2}} |G((x_1, x_2), y; E \pm i\varepsilon)|^2 < \infty \quad (3.6)
$$

for any fixed $x_1 \in \mathbb{Z}^{d_1}, y \in \mathbb{Z}^d$ and $[a, b] \subset (-d, d)$. Our formulas (2.27) - (2.28) for the Green function of $H$ can be written in the form:

$$
G((x_1, x_2), y; z) = \int_{\mathbb{T}^{d_2}} dk_2 e^{2\pi i k_2 \cdot x_2} G((x_1, k_2), y; z),
$$

where

$$
G((x_1, k_2), y; z) = G^{(d_1)}_0(x_1 - y_1, z - E_{d_2}(k_2)) - \sum_{m=0}^{\infty} t_m(k_2, z) G^{(d_1)}_0(x_1; z - E_{d_2}(k_2))
\times G^{(d_1)}_0(y_1; z - E_{d_2}(k_2 + m\alpha)) e^{2\pi i y_2 \cdot (k_2 + m\alpha)}. \quad (3.7)
$$
Thus, applying the Parseval equality for the Fourier transform with respect to the variable $x_2$, we can present the sum in (3.6) as:

$$
\int_{\mathbb{T}^d_2} dk_2 |G((x_1, k_2), y; E + i\varepsilon)|^2.
$$

We have shown in the proof of Theorem 3.1 that the series (3.7) converges uniformly in $k_2 \in \mathbb{T}^d_2$ for $z = E + i\varepsilon, E \in [a, b] \subset (-d, d); \varepsilon > 0$. Hence the integral in (3.8) is finite for these values of $E$ and $\varepsilon$. This proves (3.6).

In the next theorem we construct a family of generalized eigenfunctions of $H$, relating them to the Green function of the operator, as in the conventional scattering theory [23, 25].

**Theorem 3.3.** Let $G(x, y; z)$ be the Green function of the operator $H = H_0 + V$, defined by (1.1) - (1.3), in which the vector $\alpha$ is rationally independent. Set

$$
G(x, k; z) = \sum_{y \in \mathbb{Z}^d} G(x, y; z) e^{2i\pi k \cdot y}, \quad k \in \mathbb{T}^d,
$$

(3.9)

and

$$
\hat{\Psi}_{\pm}(x, k) = (E_d(k) - z) G(x, k; z),
$$

(3.10)

where

$$
\hat{\Psi}_{\pm}(x, k) = \lim_{\varepsilon \to +0} \Psi_{\pm}(x, k) |_{z = E_d(k) \mp i\varepsilon} = \lim_{\varepsilon \to +0} \pm i\varepsilon G(x, k; E_d(k) \mp i\varepsilon),
$$

(3.12)

exist for all $k \in \mathbb{T}^d$, are bounded in $x \in \mathbb{Z}^d$ for any $k \in \mathbb{T}^d$, are continuous in $k$ varying in any compact set of $\mathbb{T}^d$, and have the form

$$
\Psi_{\pm}(x, k) = e^{2i\pi k \cdot x} + \sum_{m=0}^{\infty} t_m(k_2 - m\alpha; z) G_0^{(d_1)}(x_1; z - E_d(k_2 - m\alpha)) \bigg|_{z = E_d(k) \mp i0} e^{2i\pi(k_2 - m\alpha) \cdot x_2},
$$

(3.13)

where the coefficients $t_m(k_2, z)$ are defined in (2.28).

Moreover:

(i) the functions $\Psi_{\pm}(x, k)$ satisfy the Schrödinger equation in $x$ for any $k \in \mathbb{T}^d$:

$$
((H_0 + V)\Psi_{\pm})(x, k) = E_d(k)\Psi_{\pm}(x, k);
$$

(3.14)

(ii) the functions $\Psi_{\pm}(x, k)$ are the unique solutions of the equation:

$$
\Psi_{\pm}(x, k) = e^{2i\pi k \cdot x} - \sum_{y \in \mathbb{Z}^d} G_0^{(d)}(x - y; E_d(k) \mp i0) V(y)\Psi_{\pm}(y, k).
$$

(3.15)
for any $k \in \mathbb{T}^d$ in the class of sequences $\Psi = \{\Psi(x)\}_{x \in \mathbb{T}^d}$ whose restrictions $\psi = \{\Psi(0, x_2)\}_{x_2 \in \mathbb{T}^{d_2}}$ and the sequences $\{(1 + \sigma - 2i\pi \alpha \cdot x_2)\psi(x_2)\}_{x_2 \in \mathbb{T}^{d_2}}$ are representable as the Fourier transforms of measures of bounded variation on $\mathbb{T}^{d_2}$, and the sum of the r.h.s. of (3.14) is understood as the generalized convolution of respective functions and measures;

(iii) the families $\{\Psi_{\pm}(\cdot, k)\}_{k \in \mathbb{T}^d}$ are orthonormalized, i.e. if for any continuous function $\hat{\Phi}$ of compact support in $\mathbb{T}^d$ we set:

$$\Phi_{\pm}(x) = \int_{\mathbb{T}^d} \Psi_{\pm}(x, k) \hat{\Phi}(k) dk,$$

then for any two such functions $\hat{\Phi}^{(1)}$ and $\hat{\Phi}^{(2)}$ we have:

$$\sum_{x \in \mathbb{Z}^d} \Phi_{\pm}^{(1)}(x) \Phi_{\pm}^{(2)}(x) = \int_{\mathbb{T}^d} dk \Phi_{\pm}^{(1)}(k) \Phi_{\pm}^{(2)}(k);$$

(iv) the functions $\Psi: \mathbb{Z}^d \times \mathbb{T}^d \to \mathbb{C}$ are the kernels of the wave operators $\Omega_{\pm}$, whose existence and completeness are proved in Theorem 3.2, i.e. for any $\Phi \in l^2(\mathbb{Z}^d)$ such that the support of its Fourier transform $\hat{\Phi}$ is a compact set in $\mathbb{T}^d$ we have:

$$(\Omega_{\pm}\Phi)(x) = \int_{\mathbb{T}^d} \Psi_{\pm}(x, k) \hat{\Phi}(k) dk.$$  

Proof. We use again our basic formulas (2.27) - (2.28) for the resolvent of $H$. Making the Fourier transform of (2.27) with respect to $y$ and multiplying the result by $E_d(k) - z$, we present (3.10) in the form:

$$\Psi_z(x, k) = e^{2i\pi k \cdot x} + \sum_{m=0}^{\infty} t_m(k_2 - m\alpha; z) G_0^{(d_1)}(x_1; z - E_d(k_2 - m\alpha)) e^{2i\pi (k_2 - m\alpha) \cdot x_2}. \quad (3.19)$$

Each term in this series is continuous in $k_2$ and $z$, $\Re z > 0$ and can be extended to real $z = E + i0, E \in [a, b]$, if the closed interval $[a, b]$ lies strictly inside $(-d, d)$. According to bound (3.3), the series converges uniformly in $k_2 \in \mathbb{T}^{d_2}$ and $a \leq \Re z \leq b, \Im z \geq 0$, hence it defines a continuous function in this domain. This allows us to perform the limits (3.12) for $k_2 \in \mathbb{T}^{d_2}$ and to obtain formula (3.13).

Our limitation $k \in \mathbb{T}^d$, where $\mathbb{T}^d$ is defined in (3.11) is necessary because for $k \in \mathbb{T}^d \setminus \mathbb{T}^d$ and for the respective two values of $E = \pm d$ we cannot guarantee the validity of bound (3.3), thus the convergence of the series in formula (3.13).

Let us prove now property (i) of $\Psi_{\pm}(x, k)$. We have obviously:

$$\sum_{|t-x|=1} H_0(x-t) G(t, y; z) - EG(x, y; z) + V(x) G(x, y; z) = i\varepsilon G(x, y; z) + \delta(x - y).$$

The definition (3.10) of $\Psi_z(x, k)$ and an easy justification of the interchange of the multiplication by $V(x)$ and of the Fourier transformation in $y$ in the third term of l.h.s. lead to the equality:

$$\sum_{|t-x|=1} H_0(x-t) \Psi_z(t, k) - E \Psi_z(x, k) + V(x) \Psi_z(x, k) = i\varepsilon (\Psi_z(x, k) + e^{2i\pi k \cdot x}).$$
Now, in view of relation (3.12), the limit of the r.h.s. of the last equality is zero as \( \varepsilon \to 0 \), and we get (3.14).

Let us prove now assertion (ii) of the theorem, i.e. that \( \Psi_\pm(x,k) \) satisfy (the Lippmann-Schwinger) equation (3.15). We remark first that any solution \( \Psi \) of (3.15) is uniquely determined by its restriction \( \psi(x_2) = \Psi((0, x_2)) \) to the subspace \( \mathbb{Z}^d \), and that \( \psi \) verifies the equation, that can be symbolically written as:

\[
\psi(x_2) = e^{2i\pi k_2 \cdot x_2} - (\gamma_0 v \psi)(x_2), \quad x_2 \in \mathbb{Z}^d. \tag{3.20}
\]

Hence we have to verify that the restriction \( \psi(x_2, k) \) of (3.13) to \( \mathbb{Z}^d \) satisfies (3.20). By using (3.19) and (3.13), we can write the restriction symbolically in the form:

\[
\psi = \left\{ 1 - (g\gamma_0 + i)^{-1}g\gamma_0[1 - 2i\sigma u(1 - \sigma bu)^{-1}(g\gamma_0 + i)^{-1}] \right\} e_2 \bigg|_{z = E_d(k) \mp i0}, \tag{3.21}
\]

where \( e_2(x_2) = e^{2i\pi k_2 \cdot x_2} \) and we used definition (2.12) of \( \gamma_0 \). The symbols \( \gamma_0 \), \( b \) and \( u \) in the formula denote now not operators on \( l^2(\mathbb{Z}^d_\pm) \) or in \( L^2(\mathbb{T}^d) \), defined in (2.12) and in (2.9), but just operations acting on sequences (functions of \( x_2 \in \mathbb{Z}^d \)) and representable as Fourier transforms of measures of bounded variation depending on the parameter \( z = E(k) \mp i0, \quad k \in \mathbb{T}^d \). In order words, they belong to the linear manifold:

\[
\mathcal{L}_k = \{ f(x_2), x_2 \in \mathbb{Z}^d : f(x_2) = \int_{\mathbb{T}^d} e^{2i\pi p_2 \cdot x_2} \mathcal{M}_k(dp_2); \quad \text{Var}\mathcal{M}_k < \infty \}. \tag{3.22}
\]

The operations \( b \) and \( \gamma_0 \) are multiplications of \( \mathcal{M}_k \) by \( \hat{b}(p_2, z) \) and by \( \hat{\gamma}_0(p_2, z) \) with \( z = E_d(k) \mp i0 \), and \( u \) is the shift by \( \alpha \) of the measure. The operation \( (1 - bu)^{-1} \) is defined by the series \( \sum_{m=0}^{\infty}(bu)^m \), whose terms are given by (2.24), and which converges for all \( k \in \mathbb{T}^d \). By using these facts and a simple algebra, we can rewrite (3.21) as:

\[
\psi = i(1 + \sigma u)(1 - \sigma bu)^{-1}(g\gamma_0 + i)^{-1}e_2 \bigg|_{z = E_d(k) \mp i0}. \tag{3.23}
\]

Hence we have for the r.h.s. of (3.20):

\[
e_2 - \gamma_0 v \psi = e_2 - g\gamma_0(1 - \sigma u)(1 + \sigma u)^{-1}(1 + \sigma u)(1 - \sigma bu)^{-1}(g\gamma_0 + i)^{-1} e_2
\]

or

\[
e_2 - \gamma_0 v \psi = \{ 1 - g\gamma_0(1 - \sigma u)(1 - \sigma bu)^{-1}(g\gamma_0 + i)^{-1} \} e_2,
\]

meaning that the complex spectral parameter \( z \) is replaced by \( E(k) \mp i0 \). The r.h.s. of the relation coincides with \( \psi \). To prove this fact we have to repeat the arguments leading to (3.13) and (3.21), but starting from formula (2.14) for the operator \( T(z) \) instead formula (2.13). Thus we have proved that (3.13) solves (3.15).

Let us prove that (3.13) is the unique solution of (3.15) in \( \mathcal{L}_k \) and such that their multiplication by \( (1 + \sigma e^{2\pi i k_2 \cdot x_2}) \) belongs also to \( \mathcal{L}_k \). Consider the homogeneous equation, corresponding to (3.13):

\[
\chi = \gamma_0 v \chi \tag{3.24}
\]
on the same manifold, and write the equality $\chi = (1 + u)\varphi$, where $\varphi$ also belongs to (3.22). Then we obtain the following equation for $\varphi$:

$$(1 + g\gamma_0)(1 - \sigma bu)\varphi = 0,$$

where the symbols $\gamma_0$, $b$, and $u$ are again understood as operations in the class $\mathcal{L}_k$. Applying to this relation the operation $(1 - \sigma bu)^{-1} (g\gamma_0 + i)^{-1}$, which is well defined in $\mathcal{L}_k$, we obtain: $\varphi = 0$.

According to the above considerations the second term in the r.h.s. of (3.15) is the Fourier transform of the product of $G_0^{(d_1)}(x_1; E_d(k) - E_{d_2}(k_2) + i0)$ (the Fourier transform of $G_0^{(d)}(x;z)|_{z = E_d(k + i0}$ in $x_2$) and of the measure $M_k$, corresponding to $v\psi$:

$$\int_{T_{d_2}} G_0^{(d_1)}(x_1; E_d(k) - E_{d_2}(p_2) + i0) M_k(d p_2).$$

(3.25)

In view of (2.10), and (3.23) we have:

$$v\psi = g(1 - \sigma u)(1 - \sigma bu)^{-1} (g\gamma_0 + i)^{-1}e_2 |_{z = E_d(k) + i0}$$

$$= g(g\gamma_0 + i)^{-1} (1 - 2i \sum_{m=0}^{\infty} \sigma u(\sigma bu)^m (g\gamma_0 + i)^{-1}e_2) |_{z = E_d(k) + i0}. \tag{3.26}$$

By using this relation and the notations introduced in Lemma 2.1 and in Theorem 2.1, we obtain that the measure corresponding to the second term of the r.h.s. of (3.15) is

$$\sum_{m=0}^{\infty} t_m(k_2 - m\alpha; E_d(k) \pm i0)\delta(k_2 - m\alpha - p_2).$$

Combining these formulas we obtain (3.15).

Let us prove now the orthogonality of $\Psi_{\pm}(x,k)$, corresponding to different $k$’s, i.e. relation (3.17). It is clear that it is sufficient to prove (3.17) for $\Phi^{(1)} = \Phi^{(2)}$. The proof is rather technical and we outline only its scheme, considering, say $\Phi_-$.

The first step is the proof of the relation:

$$\lim_{\varepsilon \to +0} \sum_{x \in \mathbb{Z}^d} |\Phi_-(x) - \Phi_\varepsilon(x)|^2 = 0, \tag{3.27}$$

where (cf (3.16))

$$\Phi_\varepsilon(x) = \int_{T_{d}} \Psi_{E_d(k)+i\varepsilon}(x,k)\hat{\Phi}(k)dk, \tag{3.28}$$

and $\Psi_z(x,k)$ is defined in (3.10), i.e. $\Psi_{E_d(k)+i\varepsilon}(x,k) = -i\varepsilon G(x,k; E_d(k) + i\varepsilon)$. The proof is based on formulas (3.13), and (3.13), and on the continuity of $G_0^{(d)}(x,E + i\varepsilon)$ with respect to $\varepsilon > 0$. It is given in Lemma 3.1 below.
The second step is the proof of the relation:

$$\lim_{\varepsilon \to +0} \sum_{x \in \mathbb{Z}^d} |\hat{\Phi}_x(x)|^2 = \int_{T^d} |\hat{\Phi}(k)|^2 dk,$$

(3.29)

which implies (3.17). We will just sketch a proof of this relation.

Write the resolvent identity for the pair $G(z')$ and $G(z'')$:

$$\sum_{t \in \mathbb{Z}^d} G(t, y; z'')G(t, x; z') = (z' - z'')^{-1}(G(t, x; z') - G(t, y; z'')).$$

(3.30)

Replace in the r.h.s. of the identity $G$ by $G_0 - G_0 T G_0$ (see (2.4)). We obtain after a simple algebra:

$$G_0' G_0'' + (z' - z'')^{-1}(G_0' T' G_0' - G_0'' T'' G_0''),$$

(3.31)

where $G_0' = (H_0 - z')^{-1}$, $G_0'' = (H_0 - z'')^{-1}$ and $T'$ and $T''$ are the $T$-operators for the spectral parameters $z'$ and $z''$ respectively. Now we make the Fourier transformation with respect to $x$ and $y$, multiplying (3.30) and (3.31) by $e^{2\pi i p y - 2\pi i x}$ and summing the result over $x, y \in \mathbb{Z}^d$. The l.h.s. of the obtained relation is $(\Psi_{z''}, \Psi_{z'})$. As for the r.h.s., it can be written symbolically as:

$$\delta(k - p) - T(k, p; z') \left( \frac{1}{E_d(p) - z'} - \frac{1}{z'' - z'} \right) - T(k, p; z'') \left( \frac{1}{E_d(k) - z''} - \frac{1}{z' - z''} \right),$$

(3.32)

where $T(k, p; z)$ is the kernel in $L^2(\mathbb{T}^d)$ of the $T$-operator, whose expression is given in Theorem 2.1. Setting in (3.32), $z' = E_d(p) + i\varepsilon$, and $z'' = E_d(k) + i\varepsilon$, we obtain:

$$\delta(k - p) - \left( T(k, p; E_d(k) + i\varepsilon) + T(k, p; E_d(p) + i\varepsilon) \right) \times \left( \frac{1}{E_d(p) - E_d(k) + i\varepsilon} - \frac{1}{E_d(p) - E_d(k) + 2i\varepsilon} \right).$$

After multiplication by $\hat{\Phi}(k)\overline{\hat{\Phi}(p)}$, where $\hat{\Phi}(k)$ is a smooth function whose compact support is strictly inside $\mathbb{T}^d$, and after the subsequent integration with respect to $k, p \in \mathbb{T}^d$, the second term of the last expression tends (weakly) to zero as $\varepsilon \to 0$. We use the explicit form (2.24) of the kernel $T(k, p; z)$ to prove that $T(k, p; E + i\varepsilon)$ is weakly bounded in $\varepsilon \geq 0$, if $k, p$ are strictly inside of $\mathbb{T}^d$ and $|E| < d$. After that we are left to prove that the expression in the parentheses tends weakly to zero as $\varepsilon \to 0$. This proves assertion (iii) of the theorem.

Let us prove assertion (iv), according to which the solutions $\Psi_{\pm}(x, k)$ are the kernels of the wave operators $\Omega_{\pm}$, whose existence and completeness are proved in Theorem 3.2 (see also [14] for similar results). We will base the proof on the formula (see [23]):

$$\Omega_{\pm} \Phi = s - \lim_{\varepsilon \to \pm 0} \int_{-\infty}^{\infty} G(E + i\varepsilon)\mathcal{E}_0(dE)\Phi,$$
where $E_0$ is the resolution of identity of the Laplacian $H_0$ of $(\mathbb{Z}^2)$, and $G(z) = (H - z)^{-1}$. In the $(x, k)$ representation, usual in the scattering theory, this formula can be written as follows:

$$(\Omega_{\pm} \Phi)(x) = s - \lim_{\varepsilon \to \mp 0} (\Omega_{\pm} \Phi)(x),$$

where

$$(\Omega_{\pm} \Phi)(x) = \int_{\mathbb{T}^d} \Psi_{E_d(k) + i\varepsilon}(x, k) \hat{\Phi}(k) dk,$$

and $\Psi_{\pm}(x, k)$ is defined in (3.10).

According to general principles, it suffices to prove (3.33) for a dense set of vectors of $L^2(\mathbb{T}^d)$. We choose a set of functions of the form $\hat{\Phi}((k_1, k_2)) = \hat{\Phi}_1(k_1)\hat{\Phi}_2(k_2)$, where $\hat{\Phi}_1, \hat{\Phi}_2$ are smooth and the support of $\hat{\Phi}_1$ does not contain the critical points of $E_{d_1}$. Denoting the r.h.s. of (3.32) by $\Phi_{\varepsilon}(x)$, we have to prove the relations:

(a) $\lim_{\varepsilon \to \mp 0} \Phi_{\varepsilon}(x) = \Phi_{\pm}(x)$;

(b) $\lim_{\varepsilon \to \mp 0} \sum_{x \in \mathbb{Z}^d} |\Phi_{\varepsilon}(x) - \Phi_{\pm}(x)|^2 = 0$.

where $\Phi_{\pm}(x)$ are defined in (3.16). Both facts are proved in the Lemma 3.1 below. Theorem 3.3 is proved.

Remarks. 1). Functions $\Psi_{\pm}(x, k)$ are analogs of the Sommerfeld solutions, which appear in the scattering theory for potentials decaying in all directions and which provide a complete set of generalized eigenfunctions for the part of the spectrum that coincides with the spectrum of the Laplacian \cite{23, 25}. Likewise, (3.15) is an analogue of the Lippmann-Schwinger equation of scattering theory.

2). According to formula (3.13), $\Psi_{\pm}(x, k)$ depends on the component $x_2$ of $x = (x_1, x_2)$, $x_1 \in \mathbb{Z}^{d_1}, x_2 \in \mathbb{Z}^{d_2}$ via the product of $e^{ik_2 x_2}$ and of a 1-periodic function of the argument $\alpha \cdot x_2$, i.e. of a quasi periodic function of $x_2 \in \mathbb{Z}^{d_2}$ (recall that we assume in this section that the components of the vector $\alpha$ in (1.4) are rationally independent). This fact is in agreement with the widely accepted idea, according to which generalized eigenfunctions of absolutely continuous spectrum of differential and finite difference operators with almost periodic coefficients have the "almost Bloch" form, i.e. the form of the product of a plane wave and an almost periodic function with the same frequencies as the coefficients (see e.g \cite{22}).

3). According to formula (2.18), if $|E| > \nu$, the Green function $G^{(\nu)}_0(x; E + i0)$ of the $\nu$-dimensional Laplacian decays exponentially and if $|E| < \nu$ it decays as $1/|x|^{\nu-1}$ for $\nu \ge 2$ (in the one dimensional case for $|E| < 1$, $G^{(1)}_0(x; E + i0)$ behaves as $e^{\eta(E)|x|}$, where $\eta(E)$ is a real valued function, see formula (3.37) and (3.38) below). As $m$ varies the expression $E_{d_1}(k) - E_{d_2}(k - m\alpha)$ has values inside $(-d_1, d_1)$ as well as outside this interval, then the Green function

$$G^{(d_1)}_0(x_1; E_{d_1}(k) - E_{d_2}(k - m\alpha)),$$

entering the expression (3.13), may be exponentially decaying or slowly decaying (i.e. as $1/|x|^{\nu-1}$). In other words we can write, say for $\Psi_{-}$:

$$\Psi(x, k) = e^{i\pi k \cdot x} + \Psi_{vol}(x, k) + \Psi_{surf}(x, k),$$

(3.35)
where $\Psi_{\text{vol}}$ is the part of the sum in (3.13), containing only slow decaying terms, and $\Psi_{\text{surf}}$ is the part, containing the exponentially decaying terms.

Recall now the definition of the surface states according to [13] (for other definitions see [4],[10],[14]).

**Definition 3.1.** Let $\Psi_E$ be a generalized eigenfunction $\Psi_E$, corresponding to a point $E$ of the spectrum of the operator $H$ of (1.1) - (1.3). We say that $\Psi_E$ is a surface state, if for any $\varepsilon > 0$ we have

$$\sup_{x_2 \in \mathbb{Z}^{d_2}} (1 + |x_2|^{d_2/2 + \varepsilon})^{-1} \sum_{x_1 \in \mathbb{Z}^{d_1}} |\Psi_E((x_1, x_2))|^2 < \infty.$$  \hspace{1cm} (3.36)

Since the part $e^{i\pi k \cdot x} + \Psi_{\text{vol}}(x, k)$ of the solution (3.35) $\Psi(x, k)$ is not decaying in the $x_1$-direction, the solution is not a ”surface” state but a ”volume” state. Hence, we can say that Theorem 3.3 above implies the existence of the volume states for all $E \in (-d, d)$. Theorem 3.4 below implies that these generalized eigenfunctions are complete in the interval $(-d, d)$. We conclude that there is no surface states in the interval $(-d, d)$ of the spectrum of the operator $H$ in the considered case of quasi periodic surface potential (1.3) - (1.4). However, despite that surface states are absent, the volume states (3.35) contain both a term, $e^{i\pi k \cdot x} + \Psi_{\text{vol}}(x, k)$ which slowly decays or even only oscillates in $|x_1|$, and a term, $\Psi_{\text{surf}}(x, k)$, which exponentially decays in $|x_1|$. They are respectively the superposition of reflected or transmitted waves which propagate inside the bulk and of waves which propagate only along the subspace $\mathbb{Z}^{d_2}$.  

4). The scattering interpretation (3.35) of generalized eigenfunction (3.13) allows us to introduce transmission and reflection amplitudes and coefficients (the latter as square of the modulus of the former). Consider the simplest case of $d_1 = 1$ and recall that:

$$G_0^{(1)}(x; z) = \frac{ie^{i\eta(|z|)x}}{\sin \eta(z)} = -\frac{e^{i\eta(|z|)x}}{\sqrt{z^2 - 1}},$$  \hspace{1cm} (3.37)

where $-\cos \eta = z$, or

$$\eta(z) = -i \log(-z + \sqrt{z^2 - 1}),$$  \hspace{1cm} (3.38)

and we use the branch of the logarithm that has the cut along the negative semi-axis and the branch of $\sqrt{z^2 - 1}$ fixed by the condition $\sqrt{z^2 - 1} = z(1 + O(z^{-1})), z \to \infty$. In particular $\Im \eta(z) \geq 0$ for $\Im z \geq 0$ and

$$\eta(E + i0) \in \left\{ \begin{array}{ll} (0, \pi), & |E| < 1, \\
\pi + i\mathbb{R}_+, & E > 1, \\
i\mathbb{R}_+, & E < -1. \end{array} \right.$$  \hspace{1cm} (3.39)

Combining these formulas and (3.13), we can present $\Psi_{\text{vol}}(x, k)$ in (3.33) for $d_1 = 1$ as

$$\Psi_{\text{vol}}(x, k) = \sum_{m} \Psi_m(k)e^{i\eta_m(k)|x_1|+2i\pi(k_2-m\alpha)x_2},$$  \hspace{1cm} (3.40)

where $\sum_m$ denotes the sum of those terms in (3.13) for which $\eta_m(k) := \eta(\lambda_m(k) + i0)$ is real, and $\lambda_m(k)$ is defined by the equation: $\lambda_m(k) = E_d(k) - E_{d-1}(k_2 - m\alpha)$. Recall that in the
one-dimensional scattering problem for the potential \( v \delta(x), x \in \mathbb{Z} \), the Sommerfeld solutions are (cf \((2.30), \text{ and } (2.31)\)):

\[
\Psi_-(x, k) = e^{2i\pi kx} - \frac{iv}{iv + \sin 2\pi k} e^{2i\pi \eta_-(k)|x|},
\]

where \( \eta_-(k) = \eta(\cos 2\pi k + i0), k \in \mathbb{T} \). Hence in this case

\[
t(k) = \frac{\sin 2\pi k}{iv + \sin 2\pi k}, \quad r_-(k) = -\frac{iv}{iv + \sin 2\pi k}
\]

are the transmission and the reflection amplitudes. This makes natural to view

\[
\eta_0 \text{ propagating in the directions (3.40) as the transmission and the reflection amplitudes of the scattered plane waves.}
\]

By using formula \((2.29)\) for the kernel of the \( T \)-operator, it can be shown that the generalized kernel \( T(k, p) \) of the \( T \)-matrix of \((3.41)\) is:

\[
\mathcal{S} = 1 - \mathcal{T}, \quad \mathcal{T} = (-2i\pi) s - \lim_{\varepsilon_1 \to 0} \int_{\varepsilon_2 \to 0} \delta_{\varepsilon_2}(H_0 - \lambda) T(\lambda + i\varepsilon_1) \mathcal{E}_0(dE), \quad (3.41)
\]

where \( \delta_{\varepsilon}(A) = (2i\pi)^{-1}[(A + i\varepsilon)^{-1} - (A - i\varepsilon)^{-1}], T(z) \) is defined in \((2.4)\), \( \mathcal{E}_0 \) is the resolution of identity of \( H_0 \), and the limits have to be carried out in the following order: first \( \varepsilon_1 \to 0 \), second \( \varepsilon_2 \to 0 \). Formula \((3.41)\) implies that for any sufficiently smooth function \( \hat{f} \) on \( \mathbb{T}^d \) we have:

\[
(\mathcal{T} \hat{f})(k) = (-2i\pi) s - \lim_{\varepsilon_1 \to 0} \int_{\varepsilon_2 \to 0} \delta_{\varepsilon_2}(E_d(k) - E_d(p)) T(k, p; E_d(p) + i\varepsilon_1) \hat{f}(p) dp.
\]

By using formula \((2.29)\) for the kernel of the \( T \)-operator, it can be shown that the generalized kernel \( T(k, p) \) of the \( T \)-matrix of \((3.41)\) is:

\[
\mathcal{T}(k, p) = -2i\pi \delta(E_d(k) - E_d(p)) T(k, p; E_d(p) + i0)
\]

\[
= -2i\pi \delta(E_d(k) - E_d(p)) \sum_{m=0}^{\infty} t_m(k_2; E_d(k) + i0) \delta(k_2 + m\alpha - p_2).
\]

Now we formulate and prove the lemma that was used in the proofs of assertions (iii) and (iv) of Theorem 3.3.

**Lemma 3.1.** Let \( \hat{\Phi}_{1,2} : \mathbb{T}^{d_1,2} \to \mathbb{C} \) be smooth functions. Assume that the support of \( \hat{\Phi}_1 \) does not contain the critical points of \( E_{d_1} \):

\[
\text{supp } \hat{\Phi}_1 \cap \{k_1 \in \mathbb{T}^{d_1} : \nabla_1 E_{d_1}(k_1) = 0\} = \emptyset. \quad (3.43)
\]
Set for $\varepsilon \neq 0$:

$$\Phi_\varepsilon(x) = \int_{\mathbb{T}^d} \Psi_E(x, k) \hat{\Phi}(k) dk,$$

where $\Psi_\varepsilon(x, k)$ is defined in (3.9), (3.10), and in (3.19), and $\hat{\Phi}(k)$ is of the form $\hat{\Phi}(k_1, k_2) = \hat{\Phi}(k_1)\hat{\Phi}(k_2)$. Then:

$$\lim_{\varepsilon \to \mp 0} \sum_{x \in \mathbb{Z}^d} |\Phi_\varepsilon(x) - \Phi_\pm(x)|^2 = 0,$$

(3.44)

where $\Phi_\pm(x, k)$ are defined in (3.12) and in (3.16).

**Proof.** By using (3.19), we find that for any $\varepsilon \neq 0$:

$$\Phi_\varepsilon(x) = \Phi(x) + \sum_{m=0}^\infty \int_{\mathbb{T}^d} dp_1 \hat{\Phi}_1(p_1) \int_{\mathbb{T}^d} dp_2 e^{2i\pi(k_2-m\alpha)x_2} \hat{\Phi}_2(p_2)$$

$$\times t_m(k_2, E_d(k) + i\varepsilon) G_0^{(d_1)}(x_1, E_d(k) - E_d(k_2 - m\alpha) + i\varepsilon),$$

where $\hat{\Phi}$ is the Fourier transform of $\Phi$. The integrals and the series in this formula converge and can be written in any order because of the bound (3.3) applicable in view of (3.43). The integral representation (2.18) for $G_0^{(d_1)}$ allows us to rewrite the last formula as follows:

$$\Phi_\varepsilon(x) = \Phi(x) + \int_{\mathbb{T}^d} e^{2i\pi k \cdot x} \hat{\Psi}_\varepsilon(k) dk,$$

(3.46)

where

$$\hat{\Psi}_\varepsilon(k) = \sum_{m=0}^\infty \int_{\mathbb{T}^d} dp_1 \frac{\hat{\Phi}_1(p_1) \hat{\Phi}_2(k_2 + m\alpha)}{E_d(k) - E_d(p_1) - E_d(k_2 + m\alpha) + i\varepsilon}$$

$$\times t_m(k_2, E_d(p_1) + E_d(k_2 + m\alpha) + i\varepsilon).$$

(3.47)

This series and the integral are convergent because the modulus of the denominator is bounded from below for $\varepsilon \neq 0$, and because of bound (3.3).

Now we will prove that for any $k \in \mathbb{T}^d$, the limits $\lim_{\varepsilon \to \mp 0} \hat{\Psi}_\varepsilon(k) \equiv \hat{\Psi}_\mp(k)$ exist and that the convergence is bounded. Consider the case $\hat{\Psi}_-$ for the sake of definiteness. The building block of the coefficient $t_m(k_2, E + i\varepsilon)$ in (3.47) is the function $\hat{\gamma}_0(k_2, E + i\varepsilon) = G_0^{(d_1)}(0, E - E_d(k_2 + i\varepsilon))$. This function is real analytic in $k_2 \in \mathbb{T}^{d_2}$ (see (3.11) for the definition of $\mathbb{T}^{d_2}$), and in $E \in (-d + \gamma, d - \gamma)$ for any fixed (small) $\gamma > 0$, (see (2.18) and (2.13)). By using identity (5.8) for $G_0^{(d_1)}$, we can write the mth term of formula (3.47) as:

$$\int_0^\infty dt e^{-\varepsilon t - it(E_d(k) - E_d(k_2 + m\alpha))} \hat{\Phi}_2(k_2 + m\alpha)$$

$$\times \int_{\mathbb{T}^{d_1}} dp_1 \hat{\Phi}_1(p_1) e^{-itE_d(p_1)} t_m(k_2, E_d(p_1) + E_d(k_2 + m\alpha) + i\varepsilon)).$$

(3.48)

Since the support of $\hat{\Phi}_1$ does not contain critical points of $E_d$, and since $G_m(k_2, E + i\varepsilon)$ is real analytic in $k_2 \in \mathbb{T}^{d_2}$ and in $E \in (-d + \gamma, d - \gamma)$, $\gamma > 0$ for all $\varepsilon \geq 0$, we can integrate by parts
twice in components of \( p_1 \in \mathbb{T}^d_1 \), and obtain an expression of the form \( t^{-2}\Phi_m(k, p_1, \varepsilon) \), where \( \Phi_m \) is bounded in \( k \in \mathbb{T}^d \), \( p_1 \in \mathbb{T}^d_1 \) and \( \varepsilon > 0 \). This allows us to make the limit \( \varepsilon \to +0 \) in (3.48) and obtain a bounded in \( k \) expression.

Besides, \( \Phi_m \) is a linear combination of the first and second partial derivatives in components of \( p_1 \in \mathbb{T}^d_1 \) of the integrand in (3.48). The derivatives are linear combination of products of bounded (and smooth) in \( k \in \mathbb{T}^d \), \( p_1 \in \mathbb{T}^d_1 \) and \( \varepsilon > 0 \). This allows us to make the limit \( \varepsilon \to +0 \) in (3.47) and obtain a bounded in \( k \) expression.

Subtracting this relation from (3.46) and applying to the result the Parseval equality, we obtain that:

\[
\sum_{x \in \mathbb{Z}^d} |\Psi_\varepsilon(x) - \Psi_-(x)|^2 = \int_{\mathbb{T}^d} |\hat{\Psi}_\varepsilon(k) - \hat{\Psi}_-(k)|^2 dk.
\]

(3.50)

Thus (3.49 and the Lebesgue theorem imply (3.44). Lemma is proved.

**Theorem 3.4.** Let \( H = H_0 + V \) be the self-adjoint operator on \( l^2(\mathbb{Z}^d) \), defined by (1.1) - (1.3) in which the vector \( \alpha \in \mathbb{R}^d_2 \) has rationally independent components. Then the family \( \{\Psi_\varepsilon(x, k) : x \in \mathbb{Z}^d \}_{k \in \mathbb{T}^d_1} \), defined in Theorem 3.3 (see (3.9), (3.12), and (3.13)), is the complete system of generalized eigenfunctions of \( H \) in the part \((-d, d)\) of the spectrum of \( H \), i.e.:

(i) for any \( f \in l^2(\mathbb{Z}^d) \), the series:

\[
F_\pm(k) = \sum_{x \in \mathbb{Z}^d} \overline{\Psi_\pm(x, k)} f(x)
\]

(3.51)

converges in \( l^2(\mathbb{Z}^d) \);

(ii) if \( \mathcal{E}_H(\Delta) \) is the spectral projection of \( H \), corresponding to the closed interval \( \Delta = [a, b] \subset (-d, d) \), then

\[
\|\mathcal{E}_H(\Delta)f\|^2 = \int_{k \in \mathbb{T}^d : E_d(k) \in \Delta} |F_\pm(k)|^2 dk;
\]

(3.52)

where \( E_d(k) \) is defined in (2.13));
(iii) the following relation is valid
\[
\|H\mathcal{E}_H(\Delta)f\|^2 = \int_{k \in \mathbb{T}^d : E_d(k) \in \Delta} |E_d(k)F_\pm(k)|^2 dk.
\] (3.53)

Proof. We write the Hilbert identity for the Green function \(G(x, y; z_{1,2})\), \(\Im z_{1,2} \neq 0\):
\[
G(x, y, z_1) - G(x, y, z_2) = (z_1 - z_2) \sum_{s \in \mathbb{Z}^d} G(x, s; z_1) \overline{G(y, s; z_2)}.
\] (3.54)

By using the Parseval equality for the Fourier transform with respect to the variable \(s\) in the r.h.s. of this identity, we rewrite it as follows:
\[
\int_{k \in \mathbb{T}^d} dk G(x, k, z_1) \overline{G(y, k, z_2)},
\]
where \(G(x, k, z)\) is the Fourier transform of \(G(x, y, z)\) in the second variable \(y\), defined in (3.3). Multiply now resulting relation by \(f(x)f(y)\), where \(f\) has compact support in \(\mathbb{Z}^d\) and sum over \(x, y \in \mathbb{Z}^d\). This yields:
\[
((G(z_1) - G(z_2))f, f) = \int_{\mathbb{T}^d} dk \frac{z_1 - z_2}{(E_d(k) - z_1)(E_d(k) - z_2)} \overline{F_{z_1}(k)}F_{z_2}(k),
\]
where
\[
F_z(k) = \sum_{x \in \mathbb{Z}^d} \Psi_z(x, k)f(x),
\] (3.55)
and \(\Psi_z(x, k)\) is defined in (3.10). Setting \(z_1 = \tilde{z}_2 = E + i\varepsilon, \varepsilon > 0\), we get:
\[
\frac{1}{\pi} \Im (G(E + i\varepsilon)f, f) = \frac{1}{\pi} \int_{\mathbb{T}^d} dk \frac{\varepsilon}{(E_d(k) - E)^2 + \varepsilon^2} |F_{E+i\varepsilon}(k)|^2.
\] (3.56)

\(\Delta = [a, b] \in (-d, d)\), we obtain in the l.h.s. of the resulting relation the expression \(\|\mathcal{E}_H(\Delta)f\|^2\).

\(\Delta = [a, b] \in (-d, d)\), we obtain in the l.h.s. of the resulting relation the expression \(\|\mathcal{E}_H(\Delta)f\|^2\). Can be continued in \(z\) to the real \(z = E_d(k) + i0 \in \Delta\), and that the continued function is uniformly continuous in \(k \in \{k \in \mathbb{T}^d : E_d(k) \in \Delta\}\), where \(\mathbb{T}^d\) is defined in (3.11). Since \(f\) is of compact support in \(\mathbb{Z}^d\), it suffices to show that \(\Psi_z(x, k)\) possess this property for any fixed \(x \in \mathbb{Z}^d\). But this fact is proved Theorem 3.3. Thus we have established (ii) for the case where \(f\)'s of finite support. The extension to \(f\)'s belonging to \(l^2(\mathbb{Z}^d)\) is based on the standard arguments of spectral theory (see e.g. [23, 24]). This proves assertions (i) and (ii). As for assertion (iii), it follows from (ii) and from the spectral theorem.

4 The Periodic Case

In this section we consider the operator \(H = H_0 + V\) of (1.1) - (1.3) in which \(d_1 = d_2 = 1\) and \(\alpha\) is a rational number: \(\alpha = p/q, p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}\), i.e. for periodic potentials \(v\) of (1.2). We show that in this case the whole spectrum of \(H\) is absolutely continuous and we construct corresponding generalized eigenfunctions. It turns out that there are two types of generalized
eigenfunctions. Both types have the Bloch-Floquet form in the longitudinal coordinates \( x_2 \) but behave differently in the transverse coordinate \( x_1 \).

We will follow the same strategy as in the preceding section namely the construction of generalized eigenfunctions based on the formulas for the Green function of Section 2 and on formulas (3.9), (3.10) and (3.12) of Theorem 3.3. Thus we have to analyze the behavior of the Green function as the spectral parameter tends to the real axis. Our first goal is to find the set of energies for which the limit \( G(x, y; E + i0) \) exists and is bounded, i.e. the purely absolutely continuous part of the spectrum. We shall see that unlike the quasiperiodic case, where this set is \([-d, d]\), in the periodic case the whole spectrum is pure absolutely continuous. The spectrum which lies outside \([-d, d]\) consists of surface states only. As for the part in the interior of \([-d, d]\), it consists of the volume states whose energies occupy the whole interval \([-d, d]\), and of the surface states that may exist under certain conditions.

For any \( z \in \mathbb{C} \), \( \Im z \neq 0 \), and \( m = 1, \ldots, q \) define the function:

\[
P_m(k_2; z) = \sigma^m \prod_{l=1}^{m} \hat{b}(k_2 + l\alpha; z), \forall k_2 \in \mathbb{T},
\]

where \( \sigma \) and \( \hat{b} \) are defined by (2.11), (2.12), (2.25) and \( \mathbb{T} = (0, 1] \). Then, by using Lemma 2.2, we obtain for \( \alpha = p/q \):

\[
((bu)^q \varphi)(k_2) = \sigma^q \prod_{l=1}^{q} \hat{b}(k_2 + l\alpha; z) \hat{\varphi}(k_2) = P_q(k_2; z) \hat{\varphi}(k_2),
\]

where the operator \( u \) is defined in (2.9). We conclude that \( (bu)^q \) is a multiplication operator by the function \( P_q \) in the space \( L^2(\mathbb{T}) \).

**Theorem 4.1.** Let \( H = H_0 + V \) be the operator defined by (1.4) - (1.5) in which \( d_1 = d_2 = 1 \) and \( \alpha = p/q, p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\} \) is a rational parameter. Then the Green function \( G(x, y; z) = (H - z)^{-1}(x, y) \), \( x, y \in \mathbb{Z}^2 \) of \( H \) can be written in the form:

\[
G(x, y; z) = G^{(2)}_0(x - y; z) + \sum_{m=0}^{q} \int_{\mathbb{T}} dk_2 e^{2i\pi k_2(x_2 - y_2)} t_m(k_2; z) \]

(4.3)

\[
\times G^{(1)}_0(x_1; z + \cos 2\pi k_2) G^{(1)}_0(y_1; z + \cos 2\pi(k_2 + m\alpha)) e^{-2i\pi m\gamma y_2},
\]

where

\[
t_m(k_2; z) = \frac{g}{g_0(k_2; z) + \tilde{\gamma} + i}
\]

(4.4)

\[
\times \begin{cases} -1, & m = 0; \\
1/2 - \sigma, & m = 1; \\
1 - P_q(k_2; z) g_0(k_2 + \alpha; z) + i, & m = 2, \\
1 - P_q(k_2; z) g_0(k_2 + m\alpha; z) + i, & m \geq 2,
\end{cases}
\]

\( G^{(\nu)}_0(\cdot; z), \nu = 1, 2 \) is the Green function (3.37) of the \( \nu \)-dimensional discrete Laplacian, \( \tilde{\gamma}_0(\cdot; z) \), and \( b(\cdot; z) \) are defined in (2.22) and in (2.23).
The proof of the theorem is based on the same argument as that used in the proof of Theorem 2.1.

Formulas (4.3) and (4.4) suggest that the spectrum $\sigma(H)$ of $H$ contains the set $S = \{E \in \mathbb{R} : \exists k_2 \in \mathbb{T}; P_q(k_2, E) = 1\}$. We prove below that indeed, the limit $G(x, y, E + i0)$ exists and is bounded for all $E \in S \setminus D$ where $D$ is a discrete set.

For any $1 > \gamma > 0$, $E \in \mathbb{R}$ and $l = 1, \ldots, q$, define the sets:

$$K_\gamma^l(E) = \{k_2 \in \mathbb{T} : E + \cos 2\pi(k_2 + l\alpha) \in [-1 + \gamma, 1 - \gamma]\},$$

and

$$K_\gamma(E) = \bigcup_{l=1}^{q} K_\gamma^l(E), \quad K_\gamma^c(E) = \mathbb{T} \setminus K_\gamma(E).$$

It follows from formula (4.2), Lemma 5.3, and from the argument of the proof of Theorem 3.1, that for any $0 < \gamma < 1$ there exists $\delta(\gamma) > 0$ such that the inequality $\sup_{\epsilon>0}|P_q(k_2, E+i\epsilon)| < 1-\delta$ is valid uniformly in $k_2 \in K_\gamma(E)$. This means that the function $(1 - P_q(k_2, E + i0))^{-1}$ is well defined and bounded on the sets,

$$K(E) = \bigcup_{0<\gamma<1} K_\gamma(E)$$

and that possible singularities of this function which are given by the “band-equation”:

$$P_q(k_2, E) = 1,$$

where $P_q(k_2, E) = P_q(k_2, E + i0)$, are localized on $K_\gamma^c(E)$. It is natural to think that energies, satisfying the band equation (4.7) for some $k_2 \in \mathbb{T}$ belong to the spectrum of $H$. The following proposition describes properties of solutions of the band equation.

**Proposition 4.1.** For any $2 \leq q < \infty$ the band equation (4.7) admits a finite number $N_1^q$ of positive solutions $0 \leq E_1(k_2) < \ldots < E_{N_1^q}(k_2) < \infty$ (the positive energy band functions), and a finite number $N_2^q$ of negative solutions $-\infty < E_{-N_2^q}(k_2) < \ldots < E_{-1}(k_2) \leq 0$ (the negative energy band functions).

The functions $E_j$, $j = -N_2^q, \ldots, -1, 1, \ldots N_1^q$ are $1/q$-periodic in $k_2$, and are real analytic in the interior of their respective domains $\mathcal{D}_j \subset \mathbb{T}$ (each domain $\mathcal{D}_j$ is a closed subset of $\mathbb{T}$).

Moreover, the band functions are separated in the sense that:

(i) for any $j = -N_2^q, \ldots, -1, 1, \ldots N_1^q$ there exists a finite subset $\mathcal{D}_j'$ of $\mathcal{D}_j$, such that for all $k \in \mathbb{T} \times (\mathcal{D}_j \setminus \mathcal{D}_j')$ we have:

$$|E_j(k_2) - E_2(k)| > 0,$$

where $E_2(k) = -\cos 2\pi k_1 - \cos 2\pi k_2$;

(ii) there exists a positive constant $\eta_q > 0$ such that for any $j, j' = -N_2^q, \ldots, -1, 1, \ldots N_1^q$, $j \neq j'$ we have:

$$\inf_{k_2 \in \mathcal{D}_j \cap \mathcal{D}_j'} |E_j(k_2) - E_{j'}(k_2)| \geq \eta_q > 0.$$
The proof of the proposition will be given after the proof of Theorem 4.6.

The band function $E_j$, $j = -N''_q, \ldots, -1, 1, \ldots N''_q$ defined in Proposition (4.1) determine the band-gap structure of the spectrum of the periodic in $x_2$ operator $H$ in the following sense,

**Theorem 4.2.** Let $H = H_0 + V$ be the operator defined in Theorem 4.1. Then for all rational parameter $\alpha$, $g \neq 0$, and $\omega \in [0, 1]$ the spectrum $\sigma(H)$ of $H$ is a finite union of closed intervals (energy bands):

$$\sigma(H) = \bigcup_{j=-N''_q}^{N''_q} \text{Ran} E_j \cup [-2, 2]. \quad (4.10)$$

The assertion that $(-d, d)$ is in the spectrum of $H$ is a consequence of Theorem 4.2, the rest of the theorem will be proved after the proof of Theorem 4.3.

Let us now define the set $\mathcal{E}_c$ of critical energies as

$$\mathcal{E}_c = \{E \in \mathbb{R} : \exists j \in \{-N''_q, \ldots, N'_q\}, \exists k_2 \in \mathbb{T}, E_j(k_2) = E \text{ and either } \frac{dE_j}{dk_2}(k_2) = 0 \text{ or } k_2 \in \partial\mathcal{D}_j\}. \quad (4.11)$$

Denote

$$D = \mathcal{E}_c \cup \{-d, d\}, \quad (4.11)$$

and notice that because of Proposition 4.1, $D$ is a discrete subset of $\mathbb{R}$.

**Theorem 4.3.** Let $H = H_0 + V$ be the operator defined in Theorem 4.1, and let $G(x, y; z)$ be its Green function. Then for any rational $\alpha$, $g \in \mathbb{R}$, and $\omega \in [0, 1]$ the limit $G(x, y; E + i0)$ exists and is bounded for any $E \in \sigma(H) \setminus D$ and $x, y \in \mathbb{Z}^2$, where $D$ is defined in (4.14). In particular the spectrum of $H$ is absolutely continuous.

**Proof.** For any $E \in \sigma(H) \setminus D$ set $z = E + i\varepsilon, \varepsilon > 0$ and fix $0 < \gamma < 1$. By using formula (4.3) we can write that

$$G(x, y; z) = G_{1, \gamma}(x, y; z) + G_{2, \gamma}(x, y; z), \quad x, y \in \mathbb{Z}^2, \quad (4.12)$$

where

$$G_{1, \gamma}(x, y; z) = G_0^{(2)}(x - y; z) + \int_{K_\gamma(E)} dk_2 e^{2\pi ik_2(x_2 - y_2)} \sum_{m=0}^{q} t_m(k_2; z) \quad (4.13)$$

$$\times G_0^{(1)}(x_1; z + \cos 2\pi k_2)G_0^{(1)}(y_1; z + \cos 2\pi (k_2 + m\alpha))e^{-2\pi m\gamma y_2},$$

$K_\gamma(E)$ and $t_m$ are defined in (4.6) and in (4.4), and

$$G_{2, \gamma}(x, y; z) = G(x, y; z) - G_{1, \gamma}(x, y; z). \quad (4.14)$$

Since the inequality $\sup_{\varepsilon > 0} |P_q(k_2, E + i\varepsilon)| < 1 - \delta$ is valid uniformly on $K_\gamma(E)$, the same arguments as in the proof of Theorem 4.1 imply that the limit $G_{1, \gamma}(x, y; E + i0)$ exists and is bounded.
Hence, to prove the theorem we have to show the same property for the term \( G_{2,\gamma}(x, y; z) \) of (1.12). We first note that by Proposition 4.1 this term can be rewritten as

\[
G_{2,\gamma}(x, y; z) = \int_{K^c(E)} dk_2 \frac{g_{2,\gamma}(x, y, k_2; z)}{1 - P_q(k_2; z)},
\]

where for any \( 0 < \gamma < 1, \varepsilon \geq 0 \) and \( (x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \), \( g_{2,\gamma}(x, y; z) \) are smooth functions on \( K^c(E) \). Now in order to compute the integral in the r.h.s. of (4.15), consider the level sets:

\[
S_j = S_j(E, \gamma) = \{ k_2 \in K^c(E) : E_j(k_2) = E \}, \quad j = -N'_q, \ldots, N'_q,
\]

and the following neighborhoods \( \nu_j \) of \( S_j \):

\[
\nu_j = \nu_j(E, \gamma, \eta) = \{ k_2 \in K^c(E) : |E_j(k_2) - E| \leq \eta \}.
\]

If \( \eta \) is small enough, then Proposition 4.1 implies the relation: \( \nu_j \cap \nu_{j'} = \emptyset \) if \( j \neq j' \). Thus to prove that \( G_{2,\gamma}(x, y; E + i\varepsilon) \) exists and is bounded as \( \varepsilon \to 0 \), it suffices to show that this holds for

\[
G_{2,\gamma,j}(x, y; E + i\varepsilon) = \int_{\nu_j} dk_2 \frac{g_{2,\gamma}(x, y, k_2; E + i\varepsilon)}{1 - P_q(k_2; z)}, \quad j = -N'_q, \ldots, N'_q.
\]

Since \( \eta \) is small enough and \( E \not\in D \), we can parameterize \( \nu_j \) by the local coordinate \( \tilde{E} \) defined by the relation \( \tilde{E} = E_j(k_2) \). Denoting \( \varphi_j \) the respective change of variables and \( J_{\varphi_j} \) its Jacobian, we have

\[
G_{2,\gamma,j}(x, y; E + i\varepsilon) = \int_{-\eta}^{\eta} d\tilde{E} \frac{g_{2,\gamma}(x, y, \varphi_j(\tilde{E}); E + i\varepsilon)}{1 - P_q(\varphi_j(\tilde{E}); E + i\varepsilon)} J_{\varphi_j}, \quad j = -N'_q, \ldots, N'_q.
\]

Suppose now that \( \eta \) and \( \varepsilon \) are so small that we can write:

\[
1 - P_q(\varphi_j(\tilde{E}), E + i\varepsilon) = (\tilde{E} - E - i\varepsilon)p_j(\tilde{E}; E + i\varepsilon), \quad \tilde{E} \in [-\eta, \eta],
\]

where \( p_j, \quad j = -N'_q, \ldots, N'_q \) are smooth and non vanishing functions on the interval \( [-\eta, \eta] \) such that

\[
|p_j(., E)| \geq C|\partial E P_q(., E)| + O(\eta) + O(\varepsilon)
\]

for some strictly positive constant \( C \). Moreover it follows from the proof of Proposition 4.1 (see formula (4.15)) that

\[
|\partial E P_q(\varphi_j(\tilde{E}), E)| \neq 0, \quad \tilde{E} \in [-\eta, \eta].
\]

Then standard arguments imply the existence and the boundedness of \( G_{2,\gamma,j}(x, y; E + i0), \quad j = -N'_q, \ldots, N'_q \) hence the existence and the boundedness of \( G_{2,\gamma}(x, y; E + i0) \). The theorem is proved.

The last theorem together with the arguments of the proof of Theorem 3.2 lead to:

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Theorem 4.4. Under the conditions of the Theorem 4.1, the wave operators $\Omega_{\pm}$ for the pair $(H, H_0)$ defined in (1.1) - (1.3) with a rational $\alpha$ exist and are complete for any closed interval $\Delta = [a, b] \subset (-d, d) \setminus \bigcup_{j=-N_q'}^{N_q'} \text{Ran} E_j$.

Our next theorem shows that surface states (see definition 3.1) exist and are bounded. They can be labelled by the ”quasi-momentum” $k_2 \in \mathbb{T}/q$, such that respective eigenvalues are given by the band functions: $E = E_j(k_2)$. The ”volume” states that do not belong to $l^2(\mathbb{Z})$ in $x_1$ are labelled by the ”momentum” $k \in \mathbb{T}^2$, such that the corresponding eigenvalues are given by the dispersion law of the Laplacian: $E = E_2(k) = -\left(\cos 2\pi k_1 + \cos 2\pi k_2\right)$. We consider here only the non-degenerate case, i.e. the case where chosen pairs $(k_2, E = E_j(k_2))$, and $(k, E = E_2(k))$ are such that $E_j(k_2) \neq E_2(k)$. By Proposition 4.1 this property is valid for all energies except a finite set.

Consider the set:

$$T^2_j = \{k = (k_1, k_2) \in \mathbb{T}^2, \ k_2 \in T_j\}, \ j = -N''_q, ..., N'_q,$$

where $\mathbb{T}^2$ is defined in (3.11), $T_j = D_j \setminus D'_j$, and $D_j, D'_j$ are defined in Proposition (4.1), and the set

$$\mathbb{T}^2 = \bigcup_{j=-N''_q}^{N'_q} \{k = (k_1, k_2) \in \mathbb{T}^2, \ k_2 \in \mathbb{T} \setminus D'_j\}.$$

Hence the set of degenerate energies is

$$\mathcal{E}_c' = \{E \in \mathbb{R}, \exists k = (k_1, k_2) \in \mathbb{T}^2, \exists j = -N''_q, ..., N'_q, \ E = E_2(k) = E_j(k_2)\},$$

By Proposition (4.1) $\mathcal{E}_c'$ is a discrete set as well as the set

$$D' = D \cup \mathcal{E}_c',$$

where $D$ is defined in (4.11).

Theorem 4.5. Let $H = H_0 + V$ be the operator defined in Theorem 4.1, $G(x, y; z)$ be its Green function, and $G(x, k; z)$ be defined in (3.9). Then:

(i) for $z = E_2(k) \mp i\varepsilon$ the limits

$$\Psi_{v, \pm}(x, k) = \lim_{\varepsilon \to +0} \Psi_z(x, k) \bigg|_{z = E_2(k) \mp i\varepsilon} = \lim_{\varepsilon \to +0} \pm i\varepsilon G(x, k; (E_2(k) \mp i\varepsilon)), \quad (4.18)$$

exist for all $k \in \mathbb{Z}^2$, are bounded in $x \in \mathbb{Z}^2$ for any $k \in \mathbb{Z}^2$, are continuous in $k$ on any compact subset of $\mathbb{T}^2$ for any $x \in \mathbb{Z}^2$, and satisfy the Schrödinger equation:

$$(H_0 + V)\Psi_{v, \pm}(x, k) = E_2(k)\Psi_{v, \pm}(x, k); \quad (4.19)$$
(ii) For $z = E_j(k_2) \mp i\varepsilon$, $j = -N''_q, \ldots, N'_q$ the limits

$$\Psi_{s,j,\pm}(x, k_2) = \lim_{\varepsilon \to +0} \mp i\varepsilon I_j(k; z)G(x, k; (E_j(k_2) \mp i\varepsilon)),$$

in which

$$I_j(k; z) = (E_2(k) - z) \left[ \int_{\mathbb{T}} \frac{1}{|E_2(k) - z|^2} dk \right]^{1/2}$$

exist for any $k = (k_1, k_2) \in \mathbb{T}^2$, are bounded in $x \in \mathbb{Z}^2$ for any $k_2 \in \mathbb{T}_j$, are continuous in $k_2$ on any compact subset of $\mathbb{T}_j$ and satisfy the Schrödinger equation:

$$((H_0 + V)\Psi_{s,j,\pm})(x, k_2) = E_j(k_2)\Psi_{s,j,\pm}(x, k_2). \quad (4.20)$$

(iii) $\Psi_{s,j,\pm}(\cdot, k_2)$, $k_2 \in \mathbb{T}_j$ are surface states in the sense of Definition 3.1.

Remarks. 1). It can be shown that for all $k \in \mathbb{T}^2$ such that $E_2(k) \in (-d, d) \setminus \bigcup_{j=1}^{N''_q} \text{Ran} E_j$ the function $\Psi_{v, \pm}$, defined by (1.18), is the unique solution of the integral equation:

$$\Psi(x, k) = e^{2i\pi k \cdot x} - \sum_{y \in \mathbb{Z}^d} G^{(2)}_0(x - y; E_2(k) \mp i0)V(y)\Psi(y, k), \quad (4.21)$$

that has to be understood in the same way as in Theorem 3.3 (ii). On the other hand, it is easy to check that for any $j = -N''_q, \ldots, N'_q$, $k = (k_1, k_2) \in \mathbb{T}_j^2$ and $E_j(k_2) \notin [-d, d]$, $\Psi_{s,j,\pm}(x, k_2)$ is a solution of the homogeneous integral equation:

$$\Psi(x, k_2) = -\sum_{y \in \mathbb{Z}^d} G^{(2)}_0(x - y; E_j(k_2) \mp i0)V(y)\Psi(y, k_2). \quad (4.22)$$

2). One can view the above results from the point of view of the direct integral decomposition technique for finite difference operators with periodic coefficients [1]. Namely by using the periodicity in $x_2$ of the operator $H$ with $\alpha = p/q$, we can write the direct integral decomposition

$$H = \int_{|k_2 - 1/2| \leq 1/2q}^\oplus H_q(k_2)dk_2. \quad (4.23)$$

Here $H_q(k_2)$ is the selfadjoint operator defined by the restriction of $H$ to the linear manifold of functions $\Psi_{k_2}(x) = e^{2i\pi k_2x_2}\Phi_{k_2}(x)$, where $\Phi_{k_2}$ is $q$-periodic in $x_2$. Thus $H_q(k_2)$ acts in the strip $\{x_1 \in \mathbb{Z}, 1 \leq x_2 \leq q\}$, and is the perturbation of the respective Laplacian by the $q$ rank potential (1.3) with $1 \leq x_2 \leq q$. This implies that the spectrum of $H_q(k_2)$ consists of two parts. The first is the absolutely continuous component: the union of values of the functions $-\cos 2\pi k_1 - \cos 2\pi (k_2 + l/q)$, $k_1 \in \mathbb{T}$, $l = 1, \ldots, q$ and $k_2 \in [1/2 - 1/2q, 1/2 + 1/2q)$ is fixed, the corresponding eigenfunctions are deformed plane waves in $x_1$. The second part is discrete spectrum, consisting of $N_q \leq q$ eigenvalues $E_j(k_2)$, lying outside of the above absolutely continuous spectrum, and having exponentially decaying in $x_1$ eigenfunctions. As $k_2$ varies in the direct integral the absolutely continuous spectrum of $H_q(k_2)$ gives rise to the volume states of the operator $H$, while the discrete spectrum of $H_q(k_2)$ gives rise to the surface states.
Proof of Theorem 4.3. Take \((E, \mathbf{k}) \in \sigma(H) \times \mathbb{T}^2\), \(z = E \pm i\varepsilon, \varepsilon > 0\) and denote \(\Psi_z(x, k) = (E - z)G(x, k; z)\), where \(G(x, k; z)\) is defined in (3.3). We know from the proof of Theorem 3.3 that if for any \(x \in \mathbb{Z}^2\) the limit \(\Psi_E(x, k) = \lim_{\varepsilon \to 0} \Psi_z(x, k)\) exists, then \(\Psi_E\) is a solution of the Schrödinger equation \(H\Psi_E = E\Psi_E\). By Theorem 4.1 we can write the representation:

\[
G(x, k, z) = \frac{1}{E_2(k) - z} \left[ e^{2i\pi k \cdot x} + \sum_{m=0}^q t_m(k_2 - m\alpha; z)G_0^{(1)}(x_1; z + \cos 2\pi(k_2 - m\alpha))e^{2i\pi(k_2 - m\alpha)x_2} \right].
\]

Choose first a pair \((k \in \mathbb{T}^2, E = E_2(k))\), as it was done in the proof of Theorem 3.3 for the quasiperiodic case. By Proposition 4.1 the denominator \(1 - P_q(E, k_2)\) in \(t_m\) of (4.4) is nonzero and we obtain from (4.24):

\[
\Psi_{\nu, \pm}(x, k) = e^{2i\pi k \cdot x}\left(1 - \sum_{m=0}^q t_m(k_2 - m\alpha; z)G_0^{(1)}(x_1; z + \cos 2\pi(k_2 - m\alpha))\right)_{\bigg|_{z=E_2(k)\mp i0}}e^{2i\pi(k_2 - m\alpha)x_2}.
\]

This proves the first assertion of the theorem.

Consider now the case where \(k = (k_1, k_2) \in \mathbb{T}^2_j\), and \(E = E_j(k_2)\) for some \(j = -N''_j, ..., N'_j\). We know that the pair \((k, E)\) is such that \(E + \cos 2\pi(k_2 + l\alpha) \notin (-1, 1)\) for any \(l = 1, ..., q\). Hence, by using the separability property (4.9) and the periodicity of the \(E_j\)'s, given by Proposition 4.1, we find that \(E - \cos 2\pi(k_2 + l\alpha) \notin [-1, 1]\) for any \(l = 1, ..., q\), i.e. all that these energies belong to the resolvent set of the 1-dimensional Laplacian.

This observation implies the existence of the limit

\[
\Psi_{s,j, \pm}(x, k_2) = \lim_{\varepsilon \to 0+} \left[ I(k, z)\Psi_z(x, k) \right]_{\bigg|_{z=E_j(k_2)\mp i\varepsilon}},
\]

provided that the limit

\[
\lim_{\varepsilon \to 0+} \frac{\varepsilon}{1 - \sum_{j=-N''_j}^{N'_j} P_q(E_j(k_2)\mp i\varepsilon)}.
\]

exists. This can be proved by using the relations

\[
1 - P_q(E_j(k_2)\mp i\varepsilon) = \pm i\varepsilon \partial_E P_q(E_j(k_2)\mp i\varepsilon) + O(\varepsilon^2),
\]

valid for sufficiently small \(\varepsilon\), and the relation \(\partial_E P_q \neq 0\). Now, it easy to verify that

\[
\Psi_{s,j, \pm}(x, k_2) = I_j(k_2)\sum_{m=1}^q \bar{t}_m(k_2 - m\alpha)G_0^{(1)}(x_1; E_j(k_2) + \cos 2\pi(k_2 - m\alpha))e^{2i\pi(k_2 - m\alpha)x_2},
\]

where

\[
I_j(k_2) = \left[ \int_{\mathbb{T}} dk_1 \frac{1}{|(E_2(k) - z)|^2} \right]_{\bigg|_{z=E_j(k_2)\mp i0}}^{1/2}.
\]
\[ \tilde{t}_1(k_2) = 2i\sigma\left( (g\gamma_0(k_2; z) + i)\partial_E P_q(k_2, z)(g\gamma_0(k_2 + \alpha; z) + i) \right)^{-1} \left|_{z=E_j(k_2)} \right. , \]

and for \( m \geq 2 \)

\[ \tilde{t}_m(k_2; E) = 2ig\sigma\left( (g\gamma_0(k_2; z) + i)\partial_E P_q(k_2, z)(g\gamma_0(k_2 + m\alpha; z) + i) \right)^{-1} \times P_{m-1}(k_2; z) \left|_{z=E_j(k_2)} \right. . \]

(4.29)

By using the same argument as that in the proof of (4.19), we find that \( \Psi_{s,j,\pm} \) satisfies (4.20). Let us prove now that \( \Psi_{s,j,\pm} \) is a surface state. We know that \( E + \cos 2\pi(k_2 + l\alpha) \notin [-1, 1] \) for any \( l = 1, \cdots, q \). Since all these energies are in the resolvent set of the 1-dimensional Laplacian, each term of the sum of the r.h.s. of (4.27) decays exponentially with respect to the transverse coordinate \( x_1 \in \mathbb{Z} \). Since the number of these terms is finite, we conclude that for any \( x_2 \in \mathbb{Z}, \Psi_{s,j,\pm}(., x_2) \in l^2(\mathbb{Z}) \). The proof of the theorem is complete.

We now use the last theorem, where we have constructed the generalized eigenfunctions (4.25) and (4.27), to prove Theorem 4.2.

**Proof of Theorem 4.2.** It follows from the proof of the Theorem 4.3 that \( \sigma(H) \subset [-d, d] \cup (\bigcup_{j=-N_q}^{N_q} \text{Ran} E_j) \). Hence we have to prove the opposite inclusion. For the part \([-d, d]\) of the spectrum the inclusion was proved in Theorem 2.2. So assume that \( E \in \bigcup_{j=-N_q}^{N_q} \text{Ran} E_j \setminus [-d, d] \) is such that there exists a surface state \( \Psi_s(x) \) satisfying the Schrödinger equation: \( (\mathcal{H}\Psi_s)(x) = E\Psi_s(x) \). We apply again the H. Weyl criterion, setting

\[ \Psi_n(x) = 1_n(x_2)\Psi_s(x)/N_n; \quad N_n = \|1_n\Psi_s\|_{l^2(\mathbb{Z}^2)}, \]

where \( 1_n \) is the indicator of the ball \( \{x_2 \in \mathbb{Z} : |x_2| \leq n\} \). A straightforward calculation shows that \( C_1n^{1/2} \leq N_n \leq C_2n^{1/2} \) as \( n \to \infty \) for some strictly positive constants \( C_{1,2} \), and that

\[ (\mathcal{H}\Psi_n)(x) = \begin{cases} E\Psi_n(x), & |x_2| < n; \\ A_n(x), & n \leq |x_2| \leq n + 1; \\ E\Psi_n(x) = 0, & |x_2| \geq n + 2, \end{cases} \]

where \( \|A_n\|_{l^2(\mathbb{Z}^2)} = O(n^{-1/2}), n \to \infty \). It is easy to check that \( \Psi_n \) is a Weyl sequence for \( \mathcal{H} \) at the energy \( E \). This proves the theorem.

Our next result concerns the completeness of the system of generalized eigenvectors (4.25) and (4.27), defined in Theorem 4.3.

**Theorem 4.6.** Let \( H = H_0 + V \) be the selfadjoint operator in \( l^2(\mathbb{Z}^2) \) defined in Theorem 4.4. Consider the family \( \mathcal{F} = \{\Psi_s(x, k), x \in \mathbb{Z}^2\}_{k \in \hat{T}_2} \bigcup_{j=-N_q}^{N_q} \{\Psi_{s,j}(x, k_2); x \in \mathbb{Z}^2\}_{k_2 \in T_j} \), defined by (4.28) and by (4.27). Then \( \mathcal{F} \) is a complete system of generalized eigenfunctions of \( H \) in any sufficiently small interval \( \Delta \) of \( \sigma(H) \) such that \( \Delta \cap D' = \emptyset \), i.e.

(i) for any \( f \in l^2(\mathbb{Z}^2) \) the series

\[ F_v(k) = \sum_{x \in \mathbb{Z}^d} \overline{\Psi_v(x, k)}f(x), \quad k \in \hat{T}_2, \]

\[ \psi(x, k) = \sum_{\mathbb{Z}^d} \overline{\psi(x, k)} \delta(x - \mathbb{Z}^d) \delta(k - \hat{k}) \]

is complete in \( l^2(\mathbb{Z}^2) \).
and

\[ F_{s,j}(k_2) = \sum_{x \in \mathbb{Z}^d} \overline{\Psi}_{s,j}(x, k_2) f(x), \quad k_2 \in \mathbb{T}_j, \quad j = -N'_q, ..., N'_q \]

converge in \( l^2(\mathbb{Z}^d) \);

(ii) if \( \mathcal{E}_H(\Delta) \) is the spectral projection of \( H \) corresponding to the interval \( \Delta \in \sigma(H) \), then

\[
\| \mathcal{E}_H(\Delta) f \|^2 = \int_{\{ k \in \mathbb{T}^2 : E_2(k) \in \Delta \}} |F_v(k)|^2 d\mathbf{K} + \sum_{j = -N'_q}^{N'_q} \int_{\{ k \in \mathbb{T}^2 : E_j(k) \in \Delta \}} |F_{j,s}(k_2)|^2 d\mathbf{K}_2.
\]

(iii) for the same interval we have

\[
\| H \mathcal{E}_H(\Delta) f \|^2 = \int_{\{ k \in \mathbb{T}^2 : E_2(k) \in \Delta \}} |E_2(k)|^2 |F_v(k)|^2 d\mathbf{K} + \sum_{j = -N'_q}^{N'_q} \int_{\{ k \in \mathbb{T}^2 : E_j(k) \in \Delta \}} |E_j(k_2)|^2 |F_{s,j}(k_2)|^2 d\mathbf{K}_2.
\]

Proproof. For any compact interval \( \Delta \subset \sigma(H) \setminus D' \) consider the sets:

\[ \nu = \{ k \in \mathbb{T}^2 : E_2(k) \in \Delta \}, \quad \nu_j = \{ k = (k_1, k_2) \in \mathbb{T}^2 : E_j(k_2) \in \Delta \}, \quad j = -N''_q, ..., N'_q. \] (4.30)

Proposition 4.1 implies that there exists a constant \( \eta > 0 \) such that:

\[
\min_j \inf_{k \in \nu \cap \mathbb{T}^2_j} |E_2(k) - E_j(k_2)|, \min_j \inf_{k \in \nu_j \cap \mathbb{T}^2} |E_2(k) - E_j(k_2)| \geq \eta.
\]

Notice that \( \eta \) depends only on the \( \text{dist}(\Delta, D') \). Moreover, if \( \Delta \) is sufficiently small, then the sets \( \nu, \nu_j, j = -N''_q, ..., N'_q \) are disjoint. The subsequent argument uses this property of \( \nu, \) and \( \nu_j, j = -N''_q, ..., N'_q \).

We will follow now the proof of Theorem 3.4 Hence we have to prove assertion (ii) first for a function \( f \) with compact support. We have for \( z = E + \epsilon \), where \( E \in \Delta \) and \( \epsilon > 0 \):

\[
\frac{1}{\pi} \Im(G(z) f, f) = \frac{1}{\pi} \int_{\nu} d\mathbf{K} \frac{\epsilon}{|(E_2(k) - z)|^2} |F_v(k)|^2 + \frac{1}{\pi} \sum_{j = -N'_q}^{N'_q} \int_{\nu_j} d\mathbf{K} \frac{\epsilon}{|(E_j(k_2) - z)|^2} |F_{s,j}(k_2)|^2 + O(\epsilon), \quad (4.31)
\]

where

\[ F_z(k) = \sum_{x \in \mathbb{Z}^d} \overline{\Psi}_z(x, k) f(x), \quad \Psi_z(x, k) = (\tilde{E} - z) G(x, k; z), \] (4.32)

\( \tilde{E} = E_2(k) \) or \( \tilde{E} = E_j(k_2) \) and \( G(x, k; z) \) is defined by (3.3) and (4.3)-(4.4). Since for every \( k = (k_1, k_2) \in \nu \) and \( E \in \Delta \), \( P_q(k_2, E) - 1 \) is not zero, the limit \( \lim_{\epsilon \to 0} \Psi_{E_2 + i\epsilon}(x, k) \) exists for
any $x \in \mathbb{Z}^2$ uniformly in $k = (k_1, k_2) \in \nu$ and in $E \in \Delta$. Applying to the first term of the r.h.s of (4.31) the operation $\lim_{\varepsilon \to 0} \int_{\Delta} \ldots \, dE$, we get:

$$\lim_{\varepsilon \to 0} \int_{\Delta} \frac{1}{\pi} \int_{\nu} \frac{\varepsilon}{|E_j(k) - z|^2} |F_z(k)|^2 \, dk = \int_{\{k \in \mathbb{T}^2 : E_j(k) \in \Delta\}} |F_{\nu}(k)|^2 \, dk. \tag{4.33}$$

So we are left with the second term of the r.h.s of (4.31). For every $j \in \{-N_q', \ldots, N_q'\}$, $k = (k_1, k_2) \in \nu_j$, and $E \in \Delta$, we have:

$$\lim_{\varepsilon \to 0} F_{E + i\varepsilon}(k) = \sum_{x \in \mathbb{Z}^2} \Psi(x, k; E) f(x), \tag{4.34}$$

where

$$\Psi(x, k, E) = \frac{(E_j(k) - E)}{E_j(k) - E} \left[ e^{2i\pi k \cdot x} \right. \tag{4.35}$$

$$\quad + \sum_{m=0}^{q} t_m(k_2 - m\alpha, E + i0) G_0^{(1)}(x_1; E + i0 + \cos 2\pi(k_2 - m\alpha)) e^{2i\pi(k_2 - m\alpha)x_2}],$$

which in particular corresponds to $[(E_j(k) - E_j(k_2))I_j(k_2)]^{-1} \Psi_{s,j}(x, k_2)$ for $E = E_j(k_2)$. The limit (4.33) is also uniform in $k = (k_1, k_2) \in \nu_j$ and in $E \in \Delta$. Applying again the same operation: $\lim_{\varepsilon \to 0} \int_{\Delta} \ldots \, dE$ to the $j$th term of the sum in r.h.s of (4.31) we get

$$\lim_{\varepsilon \to 0} \int_{\Delta} \frac{1}{\pi} \int_{\nu_j} \frac{\varepsilon}{|E_j(k) - z|^2} |F_{z}(k)|^2 \, dk = \int_{\{k \in \mathbb{T}^2 : E_j(k) \in \Delta\}} |F_{j,s}(k_2)|^2 \, dk_2. \tag{4.36}$$

Relations (4.33), and (4.36) imply assertions (i) and (ii) of the theorem for the case of a function $f$ with compact support. The proofs of these assertions for an arbitrary function $f \in l^2(\mathbb{Z}^2)$, and the proof of assertion (ii) require standard means of spectral theory (see the proof of Theorem 3.4).

**Proof of Proposition 4.1.** According to (4.1), we can write equation (4.7) for $\alpha = p/q$ as

$$P_{q}(k_2, E) = \sigma q \prod_{l=1}^{q} \hat{b}(k_2 + l/q, E + i0) = 1, \tag{4.37}$$

where

$$\hat{b}(k_2, z) = \frac{gG_0^{(1)}(0, z + \cos 2\pi k_2) - i}{gG_0^{(1)}(0, z + \cos 2\pi k_2) + i}. \tag{4.38}$$

Since the product in the l.h.s. of the equation (4.37) is periodic in $k_2$ with period $1/q$, its solutions are also periodic in $k_2$ with period $1/q$, and we can restrict ourselves to the interval $[1/2 - 1/(2q), 1/2 + 1/(2q)]$. By Lemmas 5.1 - 5.2 we have $|\hat{b}(k_2, z)| \leq 1$, $k_2 \in \mathbb{T}$, $z \in \mathbb{C}$, thus the band equation (4.37) admits a solution if and only if the modulus of each factor $\hat{b}(k_2 + l/q, E + i0)$, $l = 1, \ldots, q$ in its l.h.s is 1. Hence, we can write the representation

$$\hat{b}(k_2, E) = \exp\{2\pi i\phi(k_2, E)\}.$$
In what follows we will consider the case where \( E \) is positive (the arguments for negative \( E \) are similar and will be omitted). In this case we have from (3.37):

\[
G_0^{(1)}(0, E + i0) = -\frac{1}{\sqrt{E^2 - 1}}, \quad E > 1,
\]

and we can choose the phase \( \phi(k_2, E) \) as

\[
\phi(k_2, E) = \frac{1}{\pi} \arctan \left( \frac{1}{g} \sqrt{(E + \cos 2\pi k_2)^2 - 1} \right). \tag{4.39}
\]

For any \( k_2 \in [1/2 - 1/(2q), 1/2 + 1/(2q)] \) \( \phi \) is a non-negative and an increasing function of \( E \geq 1 - \cos 2\pi k_2 \), satisfying the inequalities:

\[
0 \leq \phi(k_2, E) < 1/2.
\]

The above formulas show that the l.h.s of equation (4.37) is real analytic in the domain

\[
\{(k_2, E) : k_2 \in [1/2 - 1/(2q), 1/2 + 1/(2q)), \quad E > 1 - \cos 2\pi k_2 \},
\]

hence solutions of the equation, if they exist, are real analytic in \( k_2 \) (notice that here the condition \( E > 1 - \cos 2\pi k_2 \) is equivalent to the conditions \( E > 1 - \cos 2\pi(k_2 + l\alpha), \forall l = 1, ..., q \)).

We will use (4.37) in the form

\[
\Phi_q(k_2, E) - q\omega = 0 \pmod{1}, \tag{4.40}
\]

where

\[
\Phi_q(k_2, E) = \sum_{l=1}^{q} \phi(k_2 + l\alpha, E) = \sum_{l=0}^{q-1} \phi(k_2 + l\alpha, E). \tag{4.41}
\]

For any fixed \( k_2 \in [1/2 - 1/(2q), 1/2 + 1/(2q)] \), \( \Phi_q(k_2, E) \) is a positive and an increasing function of \( E \geq 1 - \cos 2\pi k_2 \), bounded by \( q/2 \).

Fix now \( q \) and \( \omega \) and denote by \( \alpha_\omega \) the integer part of the minimum

\[
\min_{k_2 \in [1/2 - 1/(2q), 1/2 + 1/(2q)]} \Phi_q(k_2, 1 - \cos 2\pi k_2) - q\omega.
\]

For a fixed integer \( j \) denote by \( E_j(k_2) \) the energy such that

\[
\Phi_q(k_2, E_j(k_2)) - q\omega = \alpha_\omega + j \tag{4.42}
\]

and denote by \( \mathcal{D}_j \) the set of \( k_2 \in \mathbb{T} \) such that (4.42) is satisfied. The sets \( \mathcal{D}_j, j = -N'_q, ..., N'_q \) form an increasing family of the closed subset of \( \mathbb{T} \). For all \( j \) larger than some \( j_0 \), \( \mathcal{D}_j \) coincides with \( \mathbb{T} \).

Hence \( E_j \) is the \( j \)-th energy band function and \( \text{Ran} E_j \) is the \( j \)-th surface energy band. It is clear that the maximum value \( N'_q \) of \( j \) for which such a solution exists, is such that \( N'_q \leq q/2 \).

Since \( \Phi_q(k_2, 1 - \cos 2\pi k_2) - q\omega \) is analytic in \( k_2 \in \mathbb{T} \), it may exist a discrete set \( \mathcal{D}'_j \) of \( k_2 \in \mathbb{T} \), for which \( \Phi_q(k_2, 1 - \cos 2\pi k_2) - q\omega \) is equal to the integer \( \alpha_\omega + j \). Numerical experiments show
that for small \( q \) there are at most two values of \( k_2 \) in the interval \([1/2 - 1/(2q), 1/2 + 1/(2q)]\) for which this event occurs, so the number of points in \( \mathcal{D}_j' \) is 2\( q \) and the other \( \mathcal{D}_j' \) are empty. We have proved that if \( k_2 \in \mathbb{T}_j = \mathcal{D}_j \setminus \mathcal{D}_j' \), \( E_j(k_2) \) exceeds \( 1 - \cos 2\pi k_2 \), then we have for all \( k \in \mathbb{T}_j^2 \):

\[
E_j(k_2) > 1 - \cos 2\pi k_2 \geq -\cos 2\pi k_1 - \cos 2\pi k_2 = E_2(k),
\]

i.e. the separation property (4.8) between the band of the volume states and the surface bands.

Let us now discuss separation between the surface bands \( E_{-N_k'}, ... E_{N_k'} \). We will use the relation

\[
1 = \Phi_q(k_2, E_{j+1}(k_2)) - \Phi_q(k_2, E_j(k_2)),
\]

implied by (4.40).

Consider first the energy range \( E \geq \epsilon_m \) for some \( \epsilon_m > 2 \). It follows from (4.40) that the maximum energy \( E_q \) for which the equation is soluble is finite (this is the upper edge of the spectrum of the operator \( H \) for a given \( q \)). Hence the partial derivative

\[
\frac{\partial \Phi_q}{\partial E} = \frac{1/\pi}{\pi} \sum_{l=0}^{q-1} \frac{g}{g^2 + [E + \cos 2\pi(k_2 + l/q)]^2 - 1} \frac{E + \cos 2\pi(k_2 + l/q)}{\sqrt{(E + \cos 2\pi(k_2 + l/q))^2 - 1}}
\]

satisfies the inequalities:

\[
0 < \frac{\partial \Phi_q}{\partial E} \leq \frac{q}{\pi g} \frac{E_q + 1}{\sqrt{(\epsilon_m - 2)\epsilon_m}} := (\eta'_q)^{-1} < \infty.
\]

This bound and (4.44) lead to the relations

\[
1 = \int_{E(k_2)}^{E_{j+1}(k_2)} \frac{\partial \Phi_q(k_2, E)}{\partial E} dE \leq (E_{j+1}(k_2) - E_j(k_2))(\eta'_q)^{-1},
\]

implying the separation property (4.9) in the case where \( E_j(k_2) > 2 \).

In the case, where

\[
0 \leq 1 - \cos 2\pi k_2 \leq E_j(k_2) \leq 2, \ k_2 \in [1/2 - 1/(2q), 1/2 + 1/(2q)],
\]

the r.h.s of (4.43) can be infinite because of the contribution of the first term (for \( E = 1 - \cos 2\pi k_2 \), and of the second term (for \( E = 1 - \cos 2\pi k_2 \), and \( k_2 = 1/2 - 1/(2q) \)) or of the \((q-1)\)th term (for \( E = 1 - \cos 2\pi k_2 \), and \( k_2 = 1/2 + 1/(2q) \)). Since, however, each term in the phase (4.41) is non-negative and

\[
\phi_q := \max_{k_2 \in [1/2 - 1/(2q), 1/2 + 1/(2q)]} \phi(k_2, 2)
\]

is strictly less than 1/2, the contribution of these terms in the difference (4.44) is bounded from above by \( 2\phi_q < 1 \), and we obtain from (4.44) the inequality

\[
0 < 1 - 2\phi_q < \int_{E_j(k_2)}^{E_{j+1}(k_2)} \frac{\partial \Phi_q(k_2, E)}{\partial E} dE,
\]

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where $\tilde{\Phi}_q(k_2, E)$ is the sum in (4.41), in which the terms corresponding to $l = 0$ and to $l = 1$
if $k_2 \in [1/2 - 1/(2q), 1/2)$, and to $l = q - 1$ if $k_2 \in [1/2, 1/2 + 1/(2q))$ are omitted. It is easy

to check that the partial derivative of $\tilde{\Phi}_q(k_2, E)$ with respect to $E$ is bounded from above by a
constant $(\eta'_q)^{-1} < \infty$. This leads to the bound (4.46) in which $(\eta'_q)^{-1}$ is replaced by $(\eta''_q)^{-1}$
and $\Phi_q$ by $\tilde{\Phi}_q$. The obtained bounds imply the separation property (4.9) with $\eta_q = \min\{\eta'_q, \eta''_q\}$. Proposition 4.4 is proved.

Remark. It can be seen from the proof above that the distance between the bands increases as $|j|$ increases. Besides, the distance between the two first bands is of order $O(1/q)$ when $q$ is large.

Denote from now on the operator of (1.1) - (1.4) as $H_\alpha$. We conclude this section by discussing

correspondence between the spectrums of the operators $H_\alpha$ with an irrational number $\alpha$ and with its rational approximations $p_n/q_n$:

$$
\lim_{n \to \infty} \frac{p_n}{q_n} = \alpha. \quad (4.47)
$$

It is easy to prove, by using the basic formula (2.19) for the resolvent, that under condition (4.47)
$H_{p_n/q_n}$ converges to $H_\alpha$ in the strong resolvent sense. Hence, according to general principles [13],
the spectrum $\sigma(H_{p_n/q_n})$ is upper semi-continuous in $n$ in the limit (4.47). Here is a statement,
that gives a more detailed behavior of $\sigma(H_{p/q})$ for large $q$. Recall, that $\sigma(H_{p/q})$ is the union
of the interval $[-2, 2]$ and of $N'_q + N''_q$ surface bands, part of which can belong or intersect the
interval $[-2, 2]$.

**Theorem 4.7.** Assume that $q$ sufficiently large. Then there exists at most one negative surface
energy band above $E = -2$ and at most one positive surface energy band below $E = 2$. These
bands, if they exist, have the width of order $O(1/q^2)$ as $q \to \infty$. The width of the surface energy
bands lying in $(-\infty, -2)$ and in $(2, +\infty)$, are of order $O(\exp\{-\text{const} \cdot q\})$ as $q \to \infty$.

**Proof.** We start from the dispersion equation (4.41) - (4.40) for the surface energy bands. Since
the function $\Phi_q(k_2, E)$ has period $1/q$ in $k_2$, its Fourier series is:

$$
\Phi_q(k_2, E) = \sum_{n \in \mathbb{Z}} \hat{\Phi}_{q,n}(E)e^{-2\pi i k_2 q n},
$$

where

$$
\hat{\Phi}_{q,n}(E) = q \int_0^{1/q} dk_2 e^{2\pi i k_2 q n} \phi(k_2 + l/q, E) = q \int_0^1 dk_2 e^{2\pi i k_2 q n} \phi(E, k_2) := q \hat{\phi}_{q,n}(E),
$$

and $\hat{\phi}_{q,n}$ is the $qn$-th Fourier coefficient of the function $\phi(., E)$. Hence

$$
\Phi_q(k_2, E) = q \left( \hat{\phi}_0(E) + \hat{\phi}_q(E)e^{-2\pi i k_2 q} + ... \right). \quad (4.48)
$$

According to (4.33), the function $\phi(., E)$ is analytic for $|E| > 2$, thus its Fourier coefficient
$\hat{\phi}_q(E)$ is of order $\exp\{-\text{const} \cdot q\}$ as $q \to \infty$. In addition, formula (4.40) implies the relation

$$
dE_j dk_2 = \frac{\partial \Phi_q}{\partial k_2} \cdot \left( \frac{\partial \Phi_q}{\partial E} \right)^{-1}.
$$
It follows now from (4.48) and from the exponential decay of the Fourier coefficient \( \hat{\Phi}_{q,n}(E) \) that the upper bound for the derivative \( \frac{\partial \Phi_q}{\partial k_2} \) is of order \( O(q^2e^{-\text{const} \cdot q}) \), while the lower bound for \( \frac{\partial \Phi_q}{\partial E} \), which is reached for the highest energy band \( E_{N_q}(k_2) \), is of order \( O(q) \). Thus the derivative \( \frac{dE_j}{dk_2} \) is of order \( O(qe^{-\text{const} \cdot q}) \). Since \( E_j(k_2) \) is periodic in \( k_2 \) with period \( 1/q \), then denoting respectively by \( E_j^{\max} \) and \( E_j^{\min} \) the maximum and the minimum of the \( j \)-th band function \( E_j \), we see that \( |E_j^{\max} - E_j^{\min}| \) is of the order \( \exp\{-\text{const} \cdot q\} \) if \( E_j^{\min} > 2 \).

Let us fix \( k_2 \in [1/2 - 1/(2q), 1/2 + 1/(2q)] \). To see how many bands are in between the lowest possible energy \( E = 1 - \cos 2\pi k_2 \) and the energy, \( E = 2 \), let us calculate \( \delta \Phi = \Phi_q(k_2, 2) - \Phi_q(k_2, 1 - \cos 2\pi k_2) \). We have:

\[
\delta \Phi = 1/\pi \sum_{l=0}^{q-1} \int_{1 - \cos 2\pi k_2}^{2} \frac{g}{g^2 + (E + \cos 2\pi(k_2 + l/q))^2 - 1} \frac{E + \cos 2\pi(k_2 + l/q)}{\sqrt{(E + \cos 2\pi(k_2 + l/q))^2 - 1}}.
\]

Performing the integration for the different values of \( l \) and summing respective contributions we obtain that \( \delta \Phi \) is of the order \( (1/q) \log q \) as \( q \to \infty \), thus \( \delta \Phi \to 0 \) as \( q \to \infty \). Remembering that for each \( k_2 \) the energy of a band corresponds to an entire value of \( \Phi_q + q\omega \), we deduce that for large \( q \) there is at most one band in the interval to the left of 2. Since the minimum of \( E_1 \) for \( k_2 \in [1/2 - 1/(2q), 1/2 + 1/(2q)] \) is larger than \( 1 + \cos \pi/q \), the width of any band, lying inside the interval \( [1 + \cos \pi/q, 2] \), is bounded by \( \pi^2/2q^2 \).

It can also occur that \( E_1^{\max} > 2 \). In this case, the same argument as above show that the part of the energy band in \((2, \infty)\) is exponentially small in \( q \). Thus the total width in that case is at most of the order \( 1/q^2 \).

**Remark.** The assertion of the theorem can be interpreted as a kind of continuity of the spectrum with respect to the limiting transition (4.47). Indeed, according to the theorem, the width of the surface bands of \( H_{p_n/q_0} \), lying outside the interval \([-d, d]\), is exponentially small in \( q_n \to \infty \). It can also be shown that the gaps between these bands are of the order \( 1/q_n \). This is in agreement with the “limiting” form of this part of the spectrum of \( H_\alpha \) for irrational \( \alpha \)'s, satisfying the Diophantine condition (1.4). Indeed, according to [19], the spectrum of \( H_\alpha \) in this case is pure point and dense on \( \mathbb{R} \setminus [-d, d] \). Here is one more manifestation of this continuity.

Recall that according to [19] the eigenvalues of \( H_\alpha \) outside \([-d, d]\) are indexed by \( x_2 \in \mathbb{Z}^{d_2} \), and for each \( x_2 \in \mathbb{Z}^{d_2} \) the eigenvalue \( E_{x_2} \) is the unique solution of the equation

\[
f(E_{x_2}) \equiv \alpha \cdot x_2 + \omega \pmod{1},
\]

where \( f: \mathbb{R} \setminus [-d, d] \to \mathbb{R} \) is the monotone increasing function, defined for \( E > d \) as

\[
f(E) = -\frac{1}{\pi} \int_{\mathbb{T}^{d_2}} dk_2 \arctan\left(g_{\gamma_0}(k_2, E)\right)^{-1},
\]

or, in view of (2.33) and (3.31), and for \( d_2 = 1 \)

\[
f(E) = -\frac{1}{\pi} \int_{\mathbb{T}^1} dk_2 \arctan g^{-1} \sqrt{(E + \cos 2\pi k_2)^2 - 1}.
\]

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On the other hand, we can write the band equation (4.40) as
\[
\frac{1}{q_n} \Phi_q(k_2, E) = \frac{l}{q_n} + \omega
\]  
(4.52)
for some integer \(l\). Choosing \(l\) in the form \(l = p_n x_2 + m q_n\) for some integer \(m\), we can write the last equation as
\[
\frac{1}{q_n} \Phi_{q_n}(k_2, E) = \frac{p_n}{q_n} x_2 + \omega.
\]  
(4.53)
Recalling now the expression (4.41) for the function \(\Phi_q(k_2, E)\), we conclude that for the limiting transition (4.47) and \(E > 2\) the equation (4.40), defining the surface bands of \(H_{p_n/q_n}\) outside \([-d, d]\), converges to the equation (4.49), defining the all eigenvalues of \(H_\alpha\) for a Diophantine \(\alpha\) outside \([-d, d]\).

5 Auxiliary Facts

We present here useful facts on the Green function (2.18) of the \(\nu\)-dimensional Laplacian and on related quantities.

**Lemma 5.1.** Let \(G_0^{(\nu)}(x - y; z)\), \(x, y \in \mathbb{Z}^\nu, \Im z \neq 0\), be the Green function (2.18) of the \(\nu\)-dimensional Laplacian (1.2). Write
\[
G_0^{(\nu)}(0; z) = R_\nu(z) + i I_\nu(z), \quad R_\nu, I_\nu \in \mathbb{R}.
\]  
(5.1)

Then
\(i\) for any \(\varepsilon > 0\), and \(E \in \mathbb{R}\)
\[
|R_\nu(E + i\varepsilon)| < \infty, \quad 0 < I_\nu(E + i\varepsilon) < \infty;
\]  
(5.2)

\(ii\) the limits \(R_\nu(E + i0)\) and \(I_\nu(E + i0)\) exist for \(|E| \neq \nu\), satisfy inequality (5.3) for \(|E| < \nu\), and \(I_\nu(E + i0) = 0\) if and only if \(|E| > \nu\).

**Proof.** The part \((i)\) of the lemma follows from the integral representation (2.3). It is also easy to prove that the limits \(R_\nu(E + i0)\) and \(I_\nu(E + i0)\) exist and are finite for \(|E| \neq \nu\) (in fact, for \(\nu \geq 3\) they are finite even for \(|E| = \nu\), see Lemma 5.4 below). Thus we have to prove that \(I_\nu(E + i0)\) is strictly positive for \(|E| < \nu\). By using (2.18), it easy to show that for \(\nu = 1\)
\[
\pi^{-1} I_1(E + i0) := \rho_1(E) = \begin{cases} 
(1 - E^2)^{-1/2}, & |E| < 1, \\
0, & |E| > 1,
\end{cases}
\]
and that \(\pi^{-1} I_\nu(E + i0)\) is the \(\nu\)th convolution of \(\rho_1\). These two observations imply that \(\pi^{-1} I_\nu(E + i0)\) is strictly positive if \(|E| < \nu\), and is zero for \(|E| > \nu\). Lemma is proved.
Lemma 5.2. Let $\gamma_0(z)$ be the operator in $l(\mathbb{Z}^{d_2})$, defined as

$$
\gamma_0(z) = P_{\mathbb{Z}^{d_2}} G_0^{(d)}(z) P_{\mathbb{Z}^{d_2}}, \quad d_2 < d,
$$

and

$$
b(z) = \frac{g\gamma_0(z) - i}{g\gamma_0(z) + i}. \quad (5.3)
$$

Then the operator $\gamma_0(z) + i$ is invertible for $\Re z \geq 0$, and the operator $b(z)$ is a contraction for $\Re z \neq 0$:

$$
||b(z)|| < 1.
$$

Proof. According to (2.18) and (2.22) $\gamma_0(z)$ is the a convolution operator in $l^2(\mathbb{Z}^{d_2})$ and its symbol $\hat{\gamma}_0(k_2; z)$ satisfies the inequality: $\Re \hat{\gamma}_0(k_2; z) \geq 0$, $\Re z \geq 0$. Since the symbol of $\gamma_0(z) + i$ is $\hat{\gamma}_0(k_2; z) + i$, we have that $|\hat{\gamma}_0(k_2; z) + i| \geq \Re (\hat{\gamma}_0(k_2; z) + i) \geq 1$. Hence $\gamma_0(z) + i$ is invertible and $||\hat{\gamma}_0(z) + i||^{-1} \leq 1$.

The operator $b(z)$ is a rational function of $\gamma_0(z)$, thus its norm can be found as

$$
||b(z)|| = \sup_{k_2 \in \mathbb{T}^{d_2}} |\hat{b}(k_2; z)|.
$$

By using (5.4), we obtain that

$$
|\hat{b}(k_2, z)| = \left| \frac{R_{d_2}^2 + (I_{d_1} - 1)^2}{R_{d_2}^2 + (I_{d_1} + 1)^2} \right|_{z \rightarrow z - E_{d_2}(k_2)}, \quad (5.4)
$$

where $R_{\nu}$ and $I_{\nu}$ are defined in (2.11). This formula and Lemma 5.1 lead to (5.3).

Lemma 5.3. Let $\hat{b}(k_2; z)$ be defined by (2.22). Then

(i) $|\hat{b}(k_2; E + i0)| \leq 1$, $\forall E \in \mathbb{R}$;

(ii) for any $\gamma > 0$ and $|E| \leq d - \gamma$ there exists an open set $K_{\gamma}(E) \subset \mathbb{T}^{d_2}$, such that

$$
\hat{b}(k_2; E + i0) < 1, \quad k_2 \in K_{\gamma}(E). \quad (5.5)
$$

Proof. The part (i) of the lemma follows from Lemma 5.1 and from (5.4). To prove assertion (ii) we have to find that for any $\gamma > 0$ and $|E| < d - \gamma$ there exists an open set $K_{\gamma}(E)$ such that for $k_2 \in K_{\gamma}(E)$, $|E - E_{d_2}(k_2)| < d_1$. Then $I_{d_1}(E + i0)$ will be strictly positive and $\hat{b}(k_2; E + i0)$ will be strictly less than 1 in view of (5.4). Since $E_{d_2}$ is a continuous function in $k_2$ on $\mathbb{T}^{d_2}$, varying between $-d_2$ and $d_2$, respective open set $K_{\gamma}(E)$ always exists if $|E| < d$. Lemma is proved.

Lemma 5.4. Let $G_0^{(\nu)}(x; z)$ be the Green function of the $\nu$-dimensional Laplacian and $g > 0$. Then the expression

$$
\frac{G_0^{(\nu)}(x; E + i0)}{gG_0^{(\nu)}(0; E + i0) + i}
$$

is bounded in $x \in \mathbb{Z}^\nu$ and in $E \in \mathbb{R}$.
Proof. Consider first the one-dimensional case \( \nu = 1 \). Then it follows from (1.14) that the expression (5.6) is
\[
e^{i\eta(E+i0|x|)} \\
g + \sin \eta(E+i0),
\]
and, according to (3.38) - (3.39), the modulus of the last expression is bounded by \( g^{-1} \).

For \( \nu \geq 2 \) we will use the integral representation of \( G_0^{(\nu)}(x; z) \) of (2.18), valid for \( \Im z > 0 \):
\[
G_0^{(\nu)}(x; z) = i \int_0^\infty dt e^{izt} \prod_{l=1}^\nu J_{x_l}(t)e^{i\pi x_l/2},
\]
where \( x = \{x_i\}_{i=1}^\nu \), and \( J_n(t) \) is the Bessel function of the order \( n \):
\[
J_n(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{in\vartheta + it\sin \vartheta} d\vartheta.
\]
The representation follows easily from (2.5), and from the identity
\[
(\lambda - z)^{-1} = i \int_0^\infty dt e^{-it(\lambda - z)}, \quad \lambda \in \mathbb{R}, \Im z > 0.
\]
By using the asymptotic formula
\[
J_n(t) = \left( \frac{2}{\pi t} \right)^{1/2} \cos \left( t - \frac{(n + 1/2)\pi}{2} \right) + O\left( \frac{1}{t} \right), \quad t \to \infty,
\]
we find that \( \nu \geq 3 \) \( G_0^{(\nu)}(x; E + i0) \) is bounded in \( x \) and in \( E \). Since, in addition, \( |gG_0^{(\nu)}(0; E + i0) + i| \geq g\Im G_0^{(\nu)}(0; E + i0) + 1 \geq 1 \) (recall that in view of (2.18) \( \Im G_0^{(\nu)}(0; z) \) is nonnegative), we obtain the assertion of the lemma for \( \nu \geq 3 \).

Thus, we are left with the case \( \nu = 2 \). By using again (5.7) and (5.9), we find that \( G_0^{(\nu)}(x; E + i0) \) is bounded in \( x \) and in \( E \) everywhere except \( |E| = 2 \), and that in a sufficiently small neighborhood of \( E = 2 \)
\[
G_0^{(\nu)}(x; E + i0) = A(x) \log |E - 2| + B_{\pm}(x) + O(|E - z|), \quad E - z \to 0,
\]
where \( A(x) \) and \( B(x) \) are bounded in \( x \), \( A(0) \neq 0 \), and \( B_{\pm}(x) \) correspond to \( \operatorname{sign}(E - 2) \). The same asymptotic representation is valid in a neighborhood of \( E = -2 \). This shows that the ratio \( G_0^{(\nu)}(x; E + i0)/G_0^{(\nu)}(0; E + i0) \) is bounded and continuous in \( E \in \mathbb{R} \) for any \( x \in \mathbb{Z}^\nu \). In addition we have:
\[
\left| \frac{G_0^{(\nu)}(0; E + i0)}{gG_0^{(\nu)}(0; E + i0) + i} \right| = \frac{1}{g + i \left[ G_0^{(\nu)}(0; E + i0) \right]^{-1}} \leq \frac{1}{g - \Im \left[ G_0^{(\nu)}(0; E + i0) \right]^{-1}} \leq g^{-1},
\]
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because

\[-\Im \left[ G_0^{(\nu)}(0; E + i0) \right]^{-1} = \Im G_0^{(\nu)}(0; E + i0)/|G_0^{(\nu)}(0; E + i0)|^2 \geq 0.\]

Lemma is proved.

**Lemma 5.5.** *The expression*

\[G_0^{(d_1)}(x; E - E_{d_2}(k_2) + i0)\]

\[g\gamma_0(k_2, E + i0) + i\]

*is bounded in* \(E \in \mathbb{R}, k_2 \in \mathbb{T}^{d_2}, \) *and* \(x \in \mathbb{Z}^{d_1}.\)

**Proof.** According to (2.18), \(\hat{\gamma}(k_2, z) = G_0^{(d_1)}(0; z - E_{d_2}(k_2)).\) Hence, we can apply Lemma 5.4.

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