The vertex coloring of local antimagic total labeling on corona product graphs

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Abstract. Let $G$ be a graph with the vertex set $V(G)$ and edge set $E(G)$. A function $h$ is a bijective function of domain the union of vertex set and edge set of $G$ and range the natural number $\{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$. We called the function $h$ as a vertex local antimagic total labeling if for any two adjacent vertices $x$ and $x'$, $w(x) \neq w(x')$, where $w(x) = \sum_{e \in E(x)} h(e) + h(x)$, and $E(x)$ is the set of edges which are incident to $x$. It is considered to be a proper coloring on vertices of graph $G$ if we assign colour to all vertices with $w(x)$. The minimum number of colors by the vertex local antimagic total labeling of $G$ is called the vertex local antimagic chromatic number, denoted by $\chi_{lat}(G)$. We study on the vertex local antimagic total labeling of graphs and determined chromatic number on some corona product graphs, namely corona $(G \odot C_m)$ where $G$ is isomorph with path, star, broom, cycle, and sunlet graph.

1. Introduction
Graph is a mathematical object that involves all the vertex and edge. Graph consists of two sets, namely the set of vertices $(V(G))$ and the set of edges $(E(G))$. All graphs that used in this study are finite, simple, nontrivial, and connected graphs. The simple graph in this article is a graph that has a set of vertices and also has a set of edges. The graph $G$ is connected if every two vertices of $G$ are connected [2, 3].

A graph labeling is an assignment of positive integers to the vertices or edges, or both of them, subject to certain conditions. Many kind of labeling were introduced by previous researcher since 1960. Garceful labelings and harmonius labelings are kind of labelings. Magic labelings is on of the kind of labelings which was introduced by Sedlacek in 1963. The magic type of labelings are edge magic total labelings, super edge magic total labelings, vertex magic total labelings, $H$ magic total labelings, magic labelings of type $(a, b, c)$, sigma labelings, $l$-vertex magic labelings, distance magic, and other types of magic labelings [1].

Antimagic labeling is one of the type of labelings. Hartsfield and Ringel [4] introduced antimagic graph in 1990. A graph with $|E(G)|$ edges is called antimagic if its edges can be labeled with $1, 2, 3, \ldots, |E(G)|$ without repetition such that the sums of the labels of the edges incident to each vertex are distinct. Let $G$ be a graph with the vertex set $V(G)$ and edge set $E(G)$. The antimagic total labeling of graph $G$ is a bijective function $h : \{V(G) \cup E(G)\} \rightarrow$
\{1, 2, 3, \ldots, |V(G)| + |E(G)|\} such that the sums of the label vertex and the labels of all edges incident to each vertex are distinct \cite{9}. The graph \(G\) is called antimagic if \(G\) has an antimagic labeling. Since Hartshfield and Ringel \cite{4} have implemented antimagic labeling, the topic has become a very common topic in graph theory, see Gallian \cite{1} for more information.

In the further research, antimagic labeling has a new type, namely local antimagic coloring on graph that introduced by Arumugam et.al \cite{5}. Let \(G\) be a graph with the vertex set \(V(G)\) and edge set \(E(G)\). A local antimagic edge labeling of graph \(G\) is a bijective function \(h : \{E(G)\} \rightarrow \{1, 2, 3, \ldots, |E(G)|\}\) such that every two adjacent vertices have different weight. The vertex weight, denoted by \(w(x)\) is the sum of the label vertex and the labels of all edges incident to its vertex. The graph \(G\) is called local antimagic if \(G\) has an local antimagic edge labeling. It is considered to be a proper coloring on vertices of graph \(G\) if \(w(x)\) is assigned as the colour of vertices. The local antimagic chromatic number of graph \(G\), denoted by \(\chi_{lat}(G)\) is the minimum number of colors taken over all coloring induced by local antimagic edge labeling. The labeling is not only at the vertices, but also on the edges of the graph. Agustin et.al \cite{6} studied the local edge antimagic coloring of graphs. They determined the local edge antimagic chromatic number of some graphs. The previous results on local antimagic of graphs, vertex local antimagic or edge local antimagic can be seen in \cite{6, 7, 8, 10, 11, 12}.

Furthermore, Putri, D.F et al. extended the concept of local antimagic to a vertex local antimagic total labeling. Let \(G\) be a graph with the vertex set \(V(G)\) and edge set \(E(G)\). A vertex local antimagic total labeling of graph \(G\) is a bijective function \(h' : \{V(G) \cup E(G)\} \rightarrow \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}\) such that every two adjacent vertices have different weight. The vertex weight, denoted by \(w(x')\) is the sum of the label vertex and the labels of all edges incident to its vertex. The graph \(G\) is called local antimagic total if \(G\) has an local antimagic total labeling. It is considered to be a proper coloring on vertices of graph \(G\) if \(w(x')\) is assigned as the colour of vertices. The vertex local antimagic total chromatic number of graph \(G\), denoted by \(\chi_{lat}(G)\) is the minimum number of colors taken over all coloring induced by vertex local antimagic total labeling \cite{12}. They determined the vertex local antimagic total chromatic number on some graphs.

In this paper, we continued the studying about the vertex local antimagic total labeling on some corona product graphs. The definition of corona product graphs can be seen in \cite{13, 14}.

2. Local Antimagic Total Labeling on Corona Graph

In the following theorems, we show the results on vertex coloring of local antimagic total labeling on corona product graphs, \(G \odot C_m\) where \(G\) isomorphis with path, star, broom, cycle, and sunlet graph.

**Theorem 1** For even \(m \geq 4\), \(\chi_{lat}(G \odot C_m) = \chi_{lat}(G) + 3\), where \(G\) is isomorphis with path, star, broom, cycle, and sunlet graph.

**Proof.** Given that a graph \(G\) of order \(n\). The graph \(G \odot C_m\) is formed from one copy of \(G\) and \(n\) copies of \(C_m\), followed by connecting the \(i^{th}\) vertex of \(G\) to each vertex of \(i^{th}\) copies of graph \(C_m\). There are \(m\) vertices in \(C_m\), but they are not adjacent with the others \(C_m\), thus it is possible to have the same colors. Based on the previous results, the vertex local antimagic total chromatic number of cycle is 3, such that the vertex local antimagic chromatic number \(n\) copies of \(C_m\) is also 3. The graph graph \(G\) has a local antimagic vertex coloring as \(G\) itself has. It concludes that, \(\chi_{lat}(G \odot C_m) \geq \chi_{lat}(G) + 3\). Furthermore, we will determine the upper bound of local antimagic vertex coloring by defining the function of vertex and edge labeling in the following. To prove it, We will divide into five cases based on the graph given graph \(G\). Define the bijective function \(f\) and \(g\) in the Table 1 and Table 2. For \(1 \leq i \leq n\) and \(1 \leq j \leq m\), we have the following functions.
Table 1. The Bijective Function $g(i,j)$

| $j/i$ | 1   | 2   | 3   | 4   | $n-2$ | $n-1$ | $n$        |
|-------|-----|-----|-----|-----|-------|-------|-----------|
| 1     | 1   | 2   | 3   | 4   | $n-2$ | $n-1$ | $n$       |
| 2     | $2n$| $2n-1$| $2n-2$| $2n-3$| ... | $n+3$| $n+2$| $n+1$  |
| 3     | $2n+1$| $2n+2$| $2n+3$| $2n+4$| ... | $3n-2$| $3n-1$| $3n$   |
| 4     | $4n$| $4n-1$| $4n-2$| $4n-3$| ... | $3n-2$| $3n+2$| $3n+1$ |
| ...   | ... | ... | ... | ... | ... | ... | ...       |
| $m$   | $mn$| $mn-1$| $mn-2$| $mn-3$| ... | $(m-1)n+3$| $(m-1)n+2$| $(m-1)n+1$ |

Table 2. The Bijective Function $f(i,j)$

| $j/i$ | 1 | 2 | 3 | $n-1$ | $n$ | $m$ | 1 | 2 | 3 | $n-1$ | $n$ |
|-------|---|---|---|-------|-----|----|---|---|---|-------|-----|
| 1     | $mn$| $mn-1$| $mn-2$| ... | $(m-1)n+2$| $(m-1)n+1$|
| 2     | $(m-2)n+1$| $(m-2)n+2$| $(m-2)n+3$| ... | $(m-1)n-1$| $(m-1)n$|
| 3     | $(m-2)n$| $(m-2)n-1$| $(m-2)n-2$| ... | $(m-3)n$| $(m-3)n+1$|
| 4     | $(m-4)n+1$| $(m-4)n+2$| $(m-4)n+3$| ... | $(m-3)n-1$| $(m-3)n$|
| ...   | ... | ... | ... | ... | ... | ... |

Case 1. For $G \cong P_n$

Let $P_n \odot C_m$ with vertex set $V(P_n \odot C_m) = \{a_i, v_{i,j} : i \in [1, n], j \in [1, m]\}$ and edge set $E(P_n \odot C_m) = \{e_{i,i+1} : i \in [1, n-1]\} \cup \{e_{(i,j)}, e'_{(i,j)} : i \in [1, n], j \in [1, m]\}$. We have the bijective function of vertices and edges in $P_n \odot C_m$ as follows.

For even $n$, the labeling on $a_i a_{i+1}$ and $a_i$ are

$$g_1(a_i a_{i+1}) = \begin{cases} 
\frac{i}{2}, & \text{if } 2 \leq i \leq n-2; i \text{ even} \\
\frac{n+1-i}{2}, & \text{if } 1 \leq i \leq n-1; i \text{ odd}
\end{cases}$$

$$f_1(a_i) = \begin{cases} 
2n-1-i, & \text{if } 1 \leq i \leq n-1; i \text{ odd} \\
2n+1-i, & \text{if } 2 \leq i \leq n; i \text{ even}
\end{cases}$$

For odd $n$, the labeling on $a_i a_{i+1}$ and $a_i$ are

$$g_1(a_i a_{i+1}) = \begin{cases} 
\frac{i}{2}, & \text{if } 2 \leq i \leq n-1; i \text{ even} \\
\frac{n+1}{2}, & \text{if } 1 \leq i \leq n; i \text{ odd}
\end{cases}$$

$$f_1(a_i) = \begin{cases} 
2n-2-i, & \text{if } 1 \leq i \leq n; i \text{ odd }, i \neq n \\
2n-i, & \text{if } 2 \leq i \leq n-1; i \text{ even} \\
2n-1, & \text{if } i = n
\end{cases}$$

$$g_1(e_{i,j}) = g(i, j) \oplus (2n-1)$$

$$f_1(v_{i,j}) = f(i, j) \oplus (mn+2n-1)$$
\[ g_1(e_{ij}) = \begin{cases} \frac{1}{2}(j + 1) + \frac{1}{2}(i - 1) + (2mn + 2n - 1), & \text{if } 1 \leq i \leq n; 1 \leq j \leq m - 1, j \text{ odd} \\ 3mn + 2n - \frac{i}{2} - \frac{n}{2}(i - 1), & \text{if } 1 \leq i \leq n; 2 \leq j \leq m, j \text{ even} \end{cases} \]

Then we have the vertex weights as follows,

\[ W_1(v_{ij}) = \begin{cases} 7mn + 8n - 2, & \text{if } 1 \leq j \leq m - 1; j \text{ odd and } 1 \leq i \leq n \\ 7mn + 8n - 1, & \text{if } 2 \leq j \leq m - 2; j \text{ even and } 1 \leq i \leq n \\ 7mn + 8n - 1 - \frac{m}{2}, & \text{if } j = m \text{ and } 1 \leq i \leq n \end{cases} \]

For odd \( n \), the vertex weight on \( a_i \) are

\[ W_1(a_i) = \begin{cases} \frac{5n-1}{2} + \frac{m}{4}(2nm + 2) + m(2n - 1), & \text{if } 2 \leq i \leq n - 1; i \text{ even and } n \text{ odd} \\ \frac{5n-1}{2} - 2 + \frac{m}{4}(2nm + 2) + m(2n - 1), & \text{if } 1 \leq i \leq n - 2; i \text{ odd and } n \text{ odd} \\ \frac{5n-1}{2} - 1 + \frac{m}{4}(2nm + 2) + m(2n - 1), & \text{if } i = n; n \text{ odd} \end{cases} \]

For even \( n \), the vertex weight on \( a_i \) are

\[ W_1(a_i) = \begin{cases} \frac{5n}{2} + \frac{m}{4}(2nm + 2) + m(2n - 1), & \text{if } 2 \leq i \leq n - 1; i \text{ even and } n \text{ even} \\ \frac{5n}{2} - 2 + \frac{m}{4}(2nm + 2) + m(2n - 1), & \text{if } 1 \leq i \leq n - 2; i \text{ odd and } n \text{ even} \\ \frac{5n}{2} - 1 + \frac{m}{4}(2nm + 2) + m(2n - 1), & \text{if } i = n; n \text{ even} \end{cases} \]

From the all vertex weight in \( P_n \circ C_m \), we have 3 different weight on \( P_n \) and 3 different weight on \( v_{i,j} \), such that \( \chi_{\text{lat}}(P_n \circ C_m) = \chi_{\text{lat}}(P_n) + 3 = 3 + 3 = 6 \).

**Case 2. For \( G \cong S_t \)**

Let \( S_t \circ C_m \) with vertex set \( V(S_t \circ C_m) = \{A\} \cup \{x_i; i \in [1, t]\} \cup \{v_{i,j}; i \in [1, n], j \in [1, m]\} \) and edge set \( E(S_t \circ C_m) = \{Ax_i; i \in [1, t]\} \cup \{e_{(i,j)}'; e_{(i,j)}; i \in [1, n], j \in [1, m]\} \), where \( n = t + 1 \).

\[ f_2(A) = 2t + 1 \]
\[ f_2(x_i) = 2t + 1 - i; 1 \leq i \leq t \]
\[ g_2(Ax_i) = i; 1 \leq i \leq t \]
\[ g_2(e_{(i,j)}') = g(i, j) \oplus (2t + 1) \]
\[ f_2(v_{i,j}) = f(i, j) \oplus (mn + 2t + 1) \]

\[ g_2(e_{ij}) = \begin{cases} \frac{1}{2}(j + 1) + \frac{1}{2}(i - 1) + (2mn + 2t + 1), & \text{if } 1 \leq i \leq n; 1 \leq j \leq m - 1, j \text{ odd} \\ 3mn + 2t + 2 - \frac{i}{2} - \frac{n}{2}(i - 1), & \text{if } 1 \leq i \leq n; 2 \leq j \leq m, j \text{ even} \end{cases} \]
Then we have the vertex weights as follows,

\[
W_2(v_{ij}) = \begin{cases} 
7mn + 8t + 6, & \text{if } 1 \leq j \leq m - 1; j \text{ odd and } 1 \leq i \leq n \\
7mn + 8t + 7, & \text{if } 2 \leq j \leq m - 2; j \text{ even and } 1 \leq i \leq n \\
7mn + 8t + 7 - \frac{m}{2}, & \text{if } j = m \text{ and } 1 \leq i \leq n 
\end{cases}
\]

\[
W_2(A) = \frac{t^2 + 5t + 2}{2}
\]

\[
W_2(x_i) = 2t + 1 + \frac{m}{4}(2nm + 2) + m(2t + 1); i \in [1, t]
\]

From the all vertex weight in \(S_t \circ C_m\), we have 2 different weight on \(S_t\) and 3 different weight on \(v_{i,j}\), such that \(\chi_{lat}(S_t \circ C_m) = \chi_{lat}(S_t) + 3 = 2 + 3 = 5\).

**Case 3. For** \(G \cong Br_{t,s}\)

Let \(Br_{t,s} \circ C_m\) with vertex set \(V(\text{Br}_{t,s} \circ C_m) = \{x_i; i \in [1, t]\} \cup \{y_j; j \in [1, s]\} \cup \{v_{ij}; i \in [1, n], j \in [1, m]\}\) and edge set \(E(\text{Br}_{t,s} \circ C_m) = \{x_ix_{i+1}; i \in [1, t - 1]\} \cup \{x_iy_j; j \in [1, s]\} \cup \{e_{i,j}, e'_{i,j}; i \in [1, n], j \in [1, m]\}\), where \(t + s = n\).

**For odd** \(t\), the labeling on \(x_i, y_j, x_ix_{i+1}, \) and \(x_iy_j\) are

\[
f_3(x_i) = \begin{cases} 
1, & \text{if } i = 1 \\
2t - i + 1, & \text{if } 3 \leq i \leq t; i \text{ odd} \\
2n - i + 2, & \text{if } 2 \leq i \leq t - 1; i \text{ even}
\end{cases}
\]

\[
f_3(y_j) = j + 1; j \in [1, s]
\]

\[
g_3(x_ix_{i+1}) = \begin{cases} 
3t + i, & \text{if } 1 \leq i \leq t - 2; i \text{ odd} \\
3t + \frac{i}{2} + 3, & \text{if } 2 \leq i \leq t - 1; i \text{ even}
\end{cases}
\]

\[
g_3(x_ix_{i+1}) = 2t + j; j \in [1, s]
\]

**For even** \(t\), the labeling on \(x_i, y_j, x_ix_{i+1}, \) and \(x_iy_j\) are

\[
f_3(x_i) = \begin{cases} 
1, & \text{if } i = 1 \\
n, & \text{if } i = t \\
2t - i - 1, & \text{if } 2 \leq i \leq t - 2; i \text{ even} \\
2t - i + 1, & \text{if } 3 \leq i \leq t - 1; i \text{ odd}
\end{cases}
\]

\[
f_3(y_j) = j + 1; j \in [1, s]
\]

\[
g_3(x_ix_{i+1}) = \begin{cases} 
3t + \frac{i + 1}{2} - 2, & \text{if } 1 \leq i \leq t - 1; i \text{ odd} \\
3t + \frac{i}{2} + 2, & \text{if } 2 \leq i \leq t - 2; i \text{ even}
\end{cases}
\]
\[ g_3(x_t y_j) = 2t + j - 1; j \in [1, s] \]

\[ g_3(e_{i,j}) = g(i, j) \oplus (2t + 2s - 1) \]

\[ f_3(v_{i,j}) = f(i, j) \oplus (mn + 2t + 2s - 1) \]

\[ g_3(e'_{i,j}) = \begin{cases} 
\frac{1}{2}(j + 1) + \frac{1}{2}(i - 1) + (2mn + 2t + 2s - 1), & \text{if } 1 \leq i \leq n; 1 \leq j \leq m - 1, j \text{ odd} \\
3mn + 2t + 2s - \frac{i}{2} - \frac{n}{2}(i - 1), & \text{if } 1 \leq i \leq n; 2 \leq j \leq m, j \text{ even}
\end{cases} \]

Then we have the vertex weights as follows,

\[ W_3(v_{i,j}) = \begin{cases} 
7mn + 8t + 8s - 2, & \text{if } 1 \leq j \leq m - 1; j \text{ odd and } 1 \leq i \leq n \\
7mn + 8t + 8s - 1, & \text{if } 2 \leq j \leq m - 2; j \text{ even and } 1 \leq i \leq n \\
7mn + 8t + 8s - 1 - \frac{m}{2}, & \text{if } j = m \text{ and } 1 \leq i \leq n
\end{cases} \]

For even \( t \), the vertex weight on \( x_i \) are

\[ W_3(x_i) = \begin{cases} 
3t + \frac{m}{4}(2mn + 2) + m(2t + 2s - 1), & \text{if } i = 1 \\
8t - 1 + \frac{m}{4}(2mn + 2) + m(2t + 2s - 1), & \text{if } 2 \leq i \leq t - 2; i \text{ even} \\
8t + 1 + \frac{m}{4}(2mn + 2) + m(2t + 2s - 1), & \text{if } 3 \leq i \leq t - 1; i \text{ odd} \\
12n - t - 2 + \frac{m}{4}(2mn + 2) + m(2t + 2s - 1), & \text{if } i = t
\end{cases} \]

\[ W_3(y_j) = 3t + \frac{m}{4}(2mn + 2) + m(2t + 2s - 1); j \in [1, s] \]

For even \( t \), the vertex weight on \( x_i \) are

\[ W_3(x_i) = \begin{cases} 
3t + i + 1 + \frac{m}{4}(2mn + 2) + m(2t + 2s - 1), & \text{if } i = 1 \\
9t - 3 + \frac{m}{4}(2mn + 2) + m(2t + 2s - 1), & \text{if } 2 \leq i \leq t - 1; i \text{ even} \\
9t - 1 + \frac{m}{4}(2mn + 2) + m(2t + 2s - 1), & \text{if } 3 \leq i \leq t - 2; i \text{ odd} \\
12n - t - 2 + \frac{m}{4}(2mn + 2) + m(2t + 2s - 1), & \text{if } i = t
\end{cases} \]

\[ W_3(y_j) = 3t + 2 + \frac{m}{4}(2mn + 2) + m(2t + 2s - 1); j \in [1, s] \]

From the all vertex weight in \( Br_{t,s} \odot C_m \), we have 4 different weight on \( Br_{t,s} \) and 3 different weight on \( v_{i,j} \), such that \( \chi_{lat}(Br_{t,s} \odot C_m) = \chi_{lat}(Br_{t,s}) + 3 + 4 = 7 \).

**Case 4.** For \( G \cong C_m \)

Let \( C_n \odot C_m \) with vertex set \( V(C_n \odot C_m) = \{x_i; i \in [1, n]\} \cup \{v_{i,j}; i \in [1, n]; j \in [1, m]\} \) and edge set \( E(C_n \odot C_m) = \{x_i x_{i+1}; i \in [1, n-1]\} \cup \{x_1 x_n\} \cup \{e_{i,j}, e'_{i,j}; i \in [1, n], j \in [1, m]\} \). We have the bijective function of vertices and edges in \( C_n \odot C_m \) as follows.
For even $n$, the labeling on $x_i, x_{i+1}, x_1, x_n$ and $x_i$ are

$$g_4(x_i, x_{i+1}) = \begin{cases} \frac{i+1}{2}, & \text{if } 1 \leq i \leq n-1; i \text{ odd} \\ \frac{n+i}{2}, & \text{if } 2 \leq i \leq n-2; i \text{ even} \end{cases}$$

$$g_4(x_1, x_n) = n$$

$$f_4(x_i) = \begin{cases} n+2, & \text{if } i = 1 \\ 2n-i+1, & \text{if } 2 \leq i \leq n; i \text{ even} \\ 2n-i+3, & \text{if } 3 \leq i \leq n-1; i \text{ odd} \end{cases}$$

For odd $n$, the labeling on $x_i, x_{i+1}, x_1, x_n$ and $x_i$ are

$$g_4(x_i, x_{i+1}) = \begin{cases} \frac{i+1}{2}, & \text{if } 1 \leq i \leq n-2; i \text{ odd} \\ \frac{n+i+1}{2}, & \text{if } 2 \leq i \leq n-1; i \text{ even} \end{cases}$$

$$g_4(x_1, x_n) = \frac{n+1}{2}$$

$$f_4(x_i) = \begin{cases} 2n, & \text{if } i = 1 \\ 2n-i, & \text{if } 2 \leq i \leq n-1; i \text{ even} \\ 2n-i+2, & \text{if } 3 \leq i \leq n; i \text{ odd} \end{cases}$$

$$g_4(e_{i,j}) = g(i, j) \oplus (2n-1)$$

$$f_4(v_{i,j}) = f(i, j) \oplus (2mn+2n-2)$$

$$g_4(e'_{i,j}) = \begin{cases} \frac{1}{2}(j+1) + \frac{1}{2}(i-1) + (2mn+2n), & \text{if } 1 \leq i \leq n; 1 \leq j \leq m-1, j \text{ odd} \\ 3mn+2n - \frac{j}{2} - \frac{n}{2}(i-1), & \text{if } 1 \leq i \leq n; 2 \leq j \leq m, j \text{ even} \end{cases}$$

Then we have the vertex weights as follows,

$$W_4(v_{ij}) = \begin{cases} 7mn+8n-2, & \text{if } 1 \leq j \leq m-1; j \text{ odd and } 1 \leq i \leq n \\ 7mn+8n-1, & \text{if } 2 \leq j \leq m-2; j \text{ even and } 1 \leq i \leq n \\ 7mn+8n-\frac{m}{2}, & \text{if } j = m \text{ and } 1 \leq i \leq n \end{cases}$$
For even \( n \), the vertex weight on \( x_i \) are

\[
W_4(x_i) = \begin{cases} 
3n - 3, & \text{if } i = 1 \\
3n - 2, & \text{if } 2 \leq i \leq n; \text{ even} \\
3n, & \text{if } 3 \leq i \leq n - 1; \text{ odd}
\end{cases}
\]

For odd \( n \), the vertex weight on \( x_i \) are

\[
W_4(x_i) = \begin{cases} 
3n - 2, & \text{if } i = 1 \\
\frac{5n+1}{2}, & \text{if } 2 \leq i \leq n - 1; \text{ even} \\
3n - 1, & \text{if } 3 \leq i \leq n; \text{ odd}
\end{cases}
\]

From the all vertex weight in \( C_n \odot C_m \), we have 3 different weight on \( C_n \) and 3 different weight on \( v_{i,j} \), such that \( \chi_{lat}(C_n \odot C_m) = \chi_{lat}(C_n) + 3 = 3 + 3 = 6 \). \( \square \)

Case 5. For \( G \cong S_r \)

Let \( S_r \odot C_m \) with vertex set \( V(S_r \odot C_m) = \{x_i, y_i : i \in [1, r] \cup \{v_{i,j} : i \in [1, n], j \in [1, m]\} \} \) and edge set \( E(S_r \odot C_m) = \{x_i x_{i+1} : i \in [1, r - 1]\} \cup \{x_1 x_r\} \cup \{x_i y_i : i \in [1, n]\} \cup \{e_{i,j}, e'_{i,j} : i \in [1, n], j \in [1, m]\} \) where \( 2r = n \).

For even \( r \), the labeling on \( x_i, y_i, x_i x_{i+1}, \text{ and } x_i y_i \) are

\[
f_5(x_i) = i; i \in [1, r] \\
f_5(y_i) = r + i; i \in [1, r]
\]

\[
g_5(x_i x_{i+1}) = \begin{cases} 
3r + \frac{i+1}{2}, & \text{if } 1 \leq i \leq r - 1; \text{ odd} \\
4r - \frac{i}{2} + 1, & \text{if } 2 \leq i \leq r - 2; \text{ even}
\end{cases}
\]

\[
g_5(x_1 x_r) = \frac{7}{2}r + 1
\]

\[
g_5(x_i y_i) = 3r + 1 - i; i \in [1, r]
\]

For odd \( r \), the labeling on \( x_i, y_i, x_i x_{i+1}, \text{ and } x_i y_i \) are

\[
f_5(x_i) = i; i \in [1, r] \\
f_5(y_i) = r + i; i \in [1, r]
\]

\[
g_5(x_i x_{i+1}) = \begin{cases} 
3r + \frac{i+1}{2}, & \text{if } 1 \leq i \leq r - 1; \text{ odd} \\
4r - \frac{i}{2} + 1, & \text{if } 2 \leq i \leq r - 2; \text{ even}
\end{cases}
\]
\[ g_5(x_1x_r) = \frac{7r - 1}{2} - r + 1 \]
\[ g_5(x_iy_i) = 3r + 1 - i; i \in [1, r] \]
\[ g_5(e_{i,j}) = g(i, j) \oplus (4r) \]
\[ f_5(v_{i,j}) = f(i, j) \oplus (mn + 4r) \]
\[ g_5(e'_{i,j}) = \begin{cases} 
\frac{1}{2}(j + 1) + \frac{1}{2}(i - 1) + (2mn + 4r), & \text{if } 1 \leq i \leq n; 1 \leq j \leq m - 1, j \text{ odd} \\
3mn + 4r + 1 - \frac{1}{2} - \frac{n}{2}(i - 1), & \text{if } 1 \leq i \leq n; 2 \leq j \leq m, j \text{ even} 
\end{cases} \]

Then we have the vertex weights as follows,
\[ W_5(v_{ij}) = \begin{cases} 
7mn + 16r + 2, & \text{if } 1 \leq j \leq m - 1; j \text{ odd and } 1 \leq i \leq n \\
7mn + 16r + 3, & \text{if } 2 \leq j \leq m - 2; j \text{ even and } 1 \leq i \leq n \\
7mn + 16r + 3 - \frac{m}{2}, & \text{if } j = m \text{ and } 1 \leq i \leq n 
\end{cases} \]

For even \( r \), the vertex weight on \( x_i \) are
\[ W_5(x_i) = \begin{cases} 
\frac{19}{2}r + 3 + \frac{m}{4}(2nm + 2) + 4mr, & \text{if } i = 1 \\
10r + 2 + \frac{m}{4}(2nm + 2) + 4mr, & \text{if } 2 \leq i \leq r; i \text{ even} \\
10r + 3 + \frac{m}{4}(2nm + 2) + 4mr, & \text{if } 3 \leq i \leq r - 1; i \text{ odd} 
\end{cases} \]
\[ W_5(y_i) = 4r + 1; i \in [1, r] \]

For odd \( r \), the vertex weight on \( x_i \) are
\[ W_5(x_i) = \begin{cases} 
\frac{10r - 1}{2}r + 3 + \frac{m}{4}(2nm + 2) + 4mr, & \text{if } i = 1 \\
10r + 2 + \frac{m}{4}(2nm + 2) + 4mr, & \text{if } 2 \leq i \leq r; i \text{ even} \\
10r + 3 + \frac{m}{4}(2nm + 2) + 4mr, & \text{if } 3 \leq i \leq r - 1; i \text{ odd} 
\end{cases} \]
\[ W_5(y_i) = 4r + 1 + \frac{m}{4}(2nm + 2) + 4mr; i \in [1, r] \]

From the all vertex weight in \( S_r \odot C_m \), we have 4 different weight on \( S_r \) and 3 different weight on \( v_{i,j} \), such that \( \chi_{lat}(S_r \odot C_m) = \chi_{lat}(S_r) + 3 = 4 + 3 = 7 \).

The illustration of labeling on \( S_5 \odot C_4 \) can be seen in Figure 1. From the labeling on \( S_5 \odot C_4 \), there are 3 different weight in \( C_4 \) and two different weight in \( S_5 \), such that the local antimagic total chromatic number of \( S_5 \odot C_4 \) is 5.

3. Concluding Remark

We have obtained the chromatic number on vertices of some corona product graphs by local antimagic total labeling. Those graphs are \( G \odot C_m \) where \( G \) isomorphs with path, star, broom, cycle, and sunlet graph. Since finding an exact values of local antimagic total chromatic number \( \chi_{lat}(G) \) is considered to be NP hard problem, it remains giving some open problems.
4. Open Problem

The open problems are given in the following:

(i) Determine the local antimagic total chromatic number of any operation graphs apart from those families.

(ii) Determine the sharpest bounds, lower and upper bound of the local antimagic total chromatic number of any graphs, and continue to determine the gap between lower and upper bound, and finally solve the existence of the gap.

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