CONVEXITY OF CONSTANT MEAN CURVATURE GRAPHS IN $\mathbb{R}^{n+1}$ WITH PLANAR BOUNDARY

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ABSTRACT. We prove that the optimal solvability condition of constant mean curvature graphs with planar boundary also suffices to prove the strict convexity.

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1. INTRODUCTION

The study of convexity properties of elliptic pde’s in convex domains is a very classical subject with a rich history. The bulk of the literature has focused on the question of the convexity of level sets and there are fewer results on convexity, especially in geometric problems. In this paper, we are interested in the geometric problem of when constant mean curvature graphs in $\mathbb{R}^{n+1}$ with planar boundary and (normalized) constant mean curvature $H > 0$ defined over a strictly convex domain $\Omega$ are convex. That is, we consider the Dirichlet problem,

$$\frac{1}{\sqrt{1 + |Du|^2}} \left( \delta_{ij} - \frac{u_i u_j}{1 + |Du|^2} \right) u_{ij} = nH \quad \text{in} \quad \Omega \subset \mathbb{R}^n$$

$$u = 0 \quad \text{on} \quad \Gamma = \partial \Omega$$

and we ask for conditions on $\Gamma$ such that there is a unique strictly convex solution $u$.

Recently [14], the authors proved such a theorem for the translator equation for the mean curvature flow via a continuity method using the constant rank theorem of Bian-Guan [1]. A constant rank theorem states that the hessian $(u_{ij})$ of a convex solution $u$ of an elliptic partial differential equation must have constant rank. Thus a natural approach to this problem is to use a continuity method to connect a given strictly convex domain $\Omega$ to a canonical typically small nearly spherical domain $U$, where the solution is almost radial and strictly convex. This leaves us with the problem of showing that the rank of the hessian of a smooth family of strictly convex solutions $u^t$ cannot drop at the boundary (or possibly everywhere simultaneously) during the homotopy of the initial domain $U$ to $\Omega$. It is well known [8], [18] that even the convexity of level sets fails in general and thus appropriate
conditions on $\Gamma := \partial \Omega$ are needed. Here we show that the optimal solvability condition for the normalized mean curvature $h_\Gamma$ also suffices to prove the strict convexity. In particular,

**Theorem 1.1.** Let $\Omega$ be a strictly convex domain with $C^{2+\alpha}$ boundary $\Gamma$ having (normalized) mean curvature $h = h_\Gamma$ satisfying $h \geq (1 + \varepsilon)H$ for some $\varepsilon > 0$ and $H > 0$ constant. Then there is a unique solution $u \in C^2(\Omega)$ of (1.1) which is strictly convex in $\Omega$.

The limiting case $\varepsilon = 0$ can be handled by an approximation argument to obtain

**Theorem 1.2.** Let $\Omega$ be a strictly convex domain with $C^2$ boundary $\Gamma$ having (normalized) mean curvature $h = h_\Gamma$ satisfying $h \geq H$ with $H > 0$ constant. Then there is a unique solution $u \in C^2(\Omega) \cap C^0(\Omega)$ of (1.1) which is strictly convex in $\Omega$.

An interesting feature of the proof is that we use the Simons’ identity and a fully non-linear elliptic equation satisfied by the smallest principal curvature of the graph $\Sigma^t = \text{graph}(u^t) = \{(x, u^t(x)) : x \in \Omega^t\}$ of solutions where $\Gamma^t = \partial \Omega^t$ is a special analytic foliation of $\Omega$ constructed using the mean curvature flow of $\Gamma$, to show that the rank of the hessian of $u^t$ cannot drop.

2. **Existence with zero boundary data**

The Dirichlet problem for the (normalized) constant mean curvature $H > 0$ equation in a domain $\Omega$ with zero boundary data on $\Gamma := \partial \Omega$ consists of solving:

\[
\frac{1}{\sqrt{1 + |Du|^2}} \left( \delta_{ij} - \frac{u_i u_j}{1 + |Du|^2} \right) u_{ij} = nH \quad \text{in} \quad \Omega
\]

\[
u = 0 \quad \text{on} \quad \Gamma.
\]

The solvability condition of Serrin [12] is too restrictive because he treats general boundary data. For the benefit of the reader we sketch the well known proof [15] of existence under relaxed boundary conditions. By the maximum principle, we always have the uniqueness of solutions. We first prove the existence statement of Theorem [1.1].

**Theorem 2.1.** Assume that the (normalized) mean curvature $h = h_\Gamma$ of $\Gamma \in C^{2+\alpha}$ satisfies $h \geq (1 + \varepsilon)H$ for some $\varepsilon > 0$. Then there is a unique solution $u \in C^{2+\beta}$, $\beta \in (0, \alpha)$ of (1.1).

**Proof.** A standard method of proof [5] is the continuity method, starting from the solution $u^0 \equiv 0$ for $H = 0$ and continuously deforming to the solution $u^t$ for $tH$ until we reach the desired solution $u$ at $t = 1$. For this method to succeed we need only derive $C^1$ estimates.
for the solution $u^t$. Since the argument is the same for all values of $t$, we do them for $t = 1$.

We use the well known identities \[ \Delta \Sigma u = \frac{nH}{W} \]
\[ \Delta \Sigma \frac{1}{W} + |A|^2 \frac{1}{W} = 0, \]
where $\Delta \Sigma$ is the Laplace-Beltrami operator on $\Sigma = \text{graph } u$, $A$ is the second fundamental form of $\Sigma$ and $\frac{1}{W}$ is the last component of the upward pointing unit normal

\[
N = \left( -\frac{Du}{\sqrt{1 + |Du|^2}}, \frac{1}{\sqrt{1 + |Du|^2}} \right).
\]

Using that $|A|^2 \geq nH^2$, we see from (2.2), (2.3) that $Hu + \frac{1}{W}$ and $\frac{1}{W}$ are superharmonic in $\Omega$. In particular,

\[ -\frac{1}{H} \leq u \leq 0 \text{ in } \Omega \]
and both functions $Hu + \frac{1}{W}$ and $\frac{1}{W}$ achieve their minimum on $\Gamma$ at a point $P$ where $|Du|$ achieves its maximum.

Introduce a local orthonormal frame $e_1, \ldots, e_n$ at $P$ with $e_n$ the exterior unit normal and $e_k$, $k < n$ the principal curvature directions of $\Gamma$ at $P$. Then $u_n(P) > 0$, $u_{nn}(P) > 0$ and we have at $P$:

\[ D_n(Hu + \frac{1}{W})(P) = Hu_n(P) - \frac{u_n(P)u_{nn}(P)}{W^3(P)} \leq 0, \]
that is,

\[ \frac{u_{nn}}{W^3}(P) \geq H. \]
We also have at $P$,

\[ \sum_{k<n} u_{kk}(P) = (n - 1)u_n(P)h(P) \geq (n - 1)(1 + \varepsilon)Hu_n(P), \]
by our assumption on $\Gamma$. Using (2.5), (2.6) in our equation (1.1) gives at $P$:

\[ nH \geq (n - 1)(1 + \varepsilon)H \frac{u_n}{W}(P) + H, \]
or

\[ (1 + \varepsilon)\frac{u_n}{W}(P) \leq 1. \]
Thus,
\begin{equation}
\max_{\Omega} |\nabla u| \leq \frac{1}{\sqrt{2\varepsilon + \varepsilon^2}}.
\end{equation}

Therefore the estimates (2.4), (2.7) give the required apriori \(C^1\) estimate for \(u\), which completes the proof. \(\Box\)

3. CONVEXITY OF CONSTANT MEAN CURVATURE GRAPHS WITH PLANAR BOUNDARY

In this section we prove the convexity of graphical solutions of the constant mean curvature graph equation (1.1) in a \(C^{2+\alpha}\) strictly convex domain with zero boundary values satisfying the solvability condition of Theorem 1.1.

For the graph of \(u\), the induced metric \(g_{ij}\), its inverse matrix \(g^{ij}\), and its second fundamental form \(h_{ij}\) are given by, respectively,
\[
g_{ij} = \delta_{ij} + u_i u_j, \quad g^{ij} = \delta_{ij} - \frac{u_i u_j}{W^2},
\]
and
\[
h_{ij} = \frac{u_{ij}}{W}, \quad W = \sqrt{1 + |Du|^2}.
\]

Moreover, the principal curvatures of the graph of \(u\) are the eigenvalues of the symmetric matrix \(A[u] = (A_{ij})\):
\begin{equation}
A_{ij} = \gamma^{ik} \frac{u_{kl}}{W} \gamma_{lj},
\end{equation}
where \((\gamma^{ik})\) and its inverse matrix \((\gamma_{ik})\) are given respectively, by
\[
\gamma^{ik} = \delta_{ik} - \frac{u_i u_k}{W(1 + W)}; \quad \gamma_{ik} = \delta_{ik} + \frac{u_i u_k}{1 + W}.
\]

Geometrically, \((\gamma^{ik})\) is the square root of the metric, i.e., \(\gamma_{ik} \gamma_{kj} = g_{ij}\).

Let \(\lambda_1 \leq \ldots \leq \lambda_n\) and \(\kappa_1 \leq \ldots \leq \kappa_n\) be the ordered eigenvalues of \(D^2 u\) and \(A[u]\), respectively. For any \(\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n\), we have [6]
\begin{equation}
u_{ij} \xi_i \xi_j = W A_{kl} \gamma_{ik} \gamma_{lj} \xi_i \xi_j = W A_{kl} \xi_i' \xi_i',
\end{equation}
where \(\xi_i' = \gamma_{ik} \xi_k = \xi_i + \frac{(\xi \cdot Du) u_i}{1 + W}\). Note that
\[
|\xi|^2 \leq |\xi'|^2 = |\xi|^2 + |\xi \cdot Du|^2 \leq W^2 |\xi|^2,
\]
where \(\xi' = (\xi_1', \ldots, \xi_n')\). If both \(D^2 u\) and \(A[u]\) are positive semidefinite, it follows from (3.2) and the minimax characterization of eigenvalues that
\begin{equation}
W \kappa_k \leq \lambda_k \leq W^3 \kappa_k, \quad 1 \leq k \leq n.
\end{equation}

In particular \(\kappa_1\) vanishes if and only if \(\lambda_1\) vanishes.
**Theorem 3.1.** Let $\Omega$ be a strictly convex domain with $\Gamma := \partial \Omega \subseteq C^{2+\alpha}$ satisfying $h_{\Gamma} \geq (1 + \varepsilon)H$ for some $\varepsilon > 0$ and $H > 0$ constant. Then there exists a unique strictly convex solution $u$ of (1.1) in $\Omega$.

**Proof.** Let $\Gamma^t$ be a foliation of $\Omega$ by strictly convex domains with $\Gamma^t := \partial \Omega^t$ analytic, $\Gamma^0 = \Gamma$ so that as $t$ tends to one, $\Gamma^t$ shrinks to a point $O$ and become asymptotically spherical. Moreover assume $h_{\Gamma^t} \geq (1 + \varepsilon)H$. For example, we can let $\Gamma^t$ be the mean curvature flow of $\Gamma$. We reindex so that $\Gamma^1 = \Gamma$ and $\Gamma^0 = \{O\}$. Then $h^t := h_{\Gamma^t}$ satisfies (see [7]):

$$\frac{\partial h^t}{\partial t} = \Delta_t h + |A|^2 h^t.$$  

It follows that $\min_{\Gamma^t} h^t$ is strictly increasing in $t$, $0 \leq t < 1$ and so for $t < 1$, $h_{\Gamma^t} \geq (1 + \varepsilon)H$, that is, $\Gamma^t$ satisfies the hypothesis of Theorem 2.1. Note that here we have claimed the analyticity in space for smooth solutions of the mean curvature flow. One can see this clearly in the stationary level set formulation [4] for this flow:

$$\left(\delta_{ij} - \frac{w_i w_j}{|\nabla w|^2}\right) w_{ij} = -1 \quad \text{in } \Omega$$

$$w = 0 \text{ on } \Gamma$$

$$\Gamma^t = \{x \in \Omega : w(x) = t\}.$$  

Since $|\nabla w| > 0$ before $\Gamma^t$ disappears at the unique point $O$ where $w$ achieves its maximum, it follows that $\Gamma^t$ is analytic for $0 < t < w(O)$ by the implicit function theorem, since $w$ is analytic and $|\nabla w| > 0$.

Hence by Theorem 2.1 there is a unique solution $u^t$ of (1.1) in $\Omega^t$, $u^t = 0$ on $\Gamma^t$. Moreover since for $t$ very small, $\Gamma^t$ becomes spherical exponentially fast, $u^t$ is asymptotically a small piece of a sphere of radius $\frac{1}{H}$ centered at the origin, and therefore strictly convex. Let $T = \sup \{t : u^t \text{ is strictly convex in } \Omega^t\}$ and suppose for contradiction that $T < 1$.

**Claim 1.** $\max_{\Omega^T} |\nabla u^T|$ is achieved at a point $Q$ where $\det (u^T_{ij}(Q)) > 0$.

By (2.3), $\frac{1}{W_T}$ is superharmonic on the graph of $u^T$ so the minimum of $\frac{1}{W_T}$ is achieved on $\Gamma^T$, say at $Q$. Choose an orthonormal curvature frame $\{e_1, \ldots, e_n\}$ at $Q$ with $e_n$ the outer unit normal. By the Hopf boundary point lemma, $u^T_n(Q), u^T_{nn}(Q) > 0$. Then $u^T_{nk}(Q) = 0$ and $u_{kk}(Q) = \lambda_k(Q) u_n(Q) > 0$, $k < n$. Hence $\det (u^T_{ij}(Q)) = u_{nn}(Q) \Pi_{k=2}^n u_{kk}(Q) > 0$. Hence Claim [1] is proved.

By the constant rank theorem of Bian and Guan (see Corollary 1.3 of [1]), the rank of $(u^T_{ij})$ is $n$ in $\Omega^T$ by Claim [1]. Therefore by the definition of $T$, we must have $\det u^T_{ij}(P) = 0$.
for some \( P \in \Gamma^T \). Again choose an orthonormal curvature frame \( \{e_1, \ldots, e_n\} \) at \( P \) with \( e_n \) the outer unit normal. As above, we have
\[
 u^T_n(P) > 0, \quad \text{and} \quad u^T_{kk}(P) > 0, \ k < n.
\]
Then,
\[
 (u^T_{ij}(P)) = \begin{pmatrix}
  u^T_{11}(P) & 0 & u^T_{1n}(P) \\
  \vdots & \ddots & \vdots \\
  0 & u^T_{n-1n-1}(P) & u^T_{n-1n}(P) \\
  u^T_{n1}(P) & \cdots & u^T_{n(n-1)}(P) & u^T_{nn}(P)
\end{pmatrix}
\]
and so
\[
 0 = \det u^T_{ij}(P) = u^T_{nn}(P) \prod_{k<n} u^T_{kk}(P) - \sum_{k<n} (u^T_{kn})^2 \prod_{l \neq k} u^T_{ll}(P).
\]
It follows that
\[
u^T_{nn}(P) = \sum_{k<n} \frac{(u^T_{kn})^2}{u^T_{kk}(P)}.
\]

**Claim 2.** The multiplicity of \( \lambda_1(u^T_{ij}(P)) \) is one.

**Case 1:** \( u^T_{nn}(P) = u^T_{kn}(P) = 0, \ k < n. \)
Then \( u^T_{ij}(P)\xi_i\xi_j = \sum_{k<n} u^T_{kk}(P)\xi_k^2 \) is minimized (\(|\xi| = 1\)) when \( \xi_k = 0, \ k < n \) and \( \xi_n = 1 \). Hence the dimension of the eigenspace for \( \lambda^T_1(P) = 0 \) is one.

**Case 2:** \( u^T_{nn}(P) > 0. \)
Then
\[
 u^T_{ij}(P)\xi_i\xi_j = \sum_{k<n} \left( u^T_{kk}(P)\xi_k^2 + 2u^T_{kn}(P)\xi_k\xi_n + \frac{(u^T_{kn})^2}{u^T_{kk}(P)}\xi_n^2 \right)
\]
\[
 = \sum_{k<n} \left( \sqrt{u^T_{kk}(P)}\xi_k + \frac{u^T_{nk}(P)}{\sqrt{u^T_{kk}(P)}}\xi_n \right)^2,
\]
is minimized (\(|\xi| = 1\)) when \( \xi_k = \frac{-u^T_{kn}(P)}{u^T_{kk}(P)}\xi_n \) and \( \xi_n = (1 + \sum_{k<n} (\frac{(u^T_{kn})^2}{u^T_{kk}(P)}))^{-1}. \) Thus the claim is proven.

Since \( \Gamma^T \) is analytic, \( u^T \) extends to a solution of (1.1) in a small neighborhood \( B \) of \( P \) by the Cauchy-Kowalewski theorem. Moreover since \( \kappa^T_i(P) \geq \theta > 0 \) for \( i \geq 2, \kappa_1(P) = 0, \)
we may choose $B$ so small that the smallest principal curvature $\kappa_1^T(x)$ of $\Sigma^T = \text{graph}(u^T)$ : $x \in B$ is smooth.

**Lemma 3.2.** $\Delta^{\Sigma^T} \kappa_1^T \leq 0$ in $\Omega^T \cap B$.

**Proof.** We use the Simons’ identity [10] for the Laplacian of the second fundamental $A^T = (h^T_{ij})$ form of $\Sigma^T$:

\begin{equation}
\Delta^{\Sigma^T} A^T + |A^T|^2 A^T = (\text{trace} A^T)(A^T)^2.
\end{equation}

For a dense open set in $\Omega^T \cap B$, we can introduce a smooth orthonormal frame $\{e_i\}$ of eigenvectors of $A^T$ corresponding to the ordered principal curvatures $0 < \kappa_1^T < \kappa_2^T \leq \ldots \leq \kappa_n^T$ (see Singley [13]). Then in this special frame, we can rewrite (3.7) as

\begin{equation}
\Delta^{\Sigma^T} h^T_{ij} + |A^T|^2 h^T_{ij} = n H (h^T_{ij})^2 \delta_{ij}.
\end{equation}

Let $\mu := f(\kappa_1^T, \ldots, \kappa_n^T) = F(h_{ij}^T)$ be any smooth symmetric homogeneous degree one concave approximation of $\kappa^T_{\min} = \min_i \kappa_i^T$.

For example, such an approximation is constructed in our paper [14]. Then using (3.7) and a well known computation,

\begin{equation}
\Delta^{\Sigma^T} \mu = F^{ij} \Delta^{\Sigma^T} h^T_{ij} + F^{ijkl} h^T_{ij} h^T_{kl} h^T_{rs}
\end{equation}

\begin{equation}
\leq F^{ij} (-|A^T|^2 h^T_{ij} + n H (h^T_{ij})^2 \delta_{ij})
\end{equation}

\begin{equation}
= -|A^T|^2 \mu + n H \sum f_i (\kappa_i^T)^2.
\end{equation}

Since $\kappa_1^T$ is smooth in $\Omega \cap B$ and separated from the other principal curvatures, $\mu$ converges smoothly to $\kappa_1^T$ and $f_1 \to 1$, $f_i \to 0$, $i \geq 2$ uniformly. Hence we have

\begin{equation}
\Delta^{\Sigma^T} \kappa_1^T + (|A^T|^2 - n H \kappa_1^T) \kappa_1^T \leq 0 \text{ in } \Omega^T \cap B.
\end{equation}

But

\begin{equation}
|A^T|^2 - n H \kappa_1^T = \sum_i \{(\kappa_i^T)^2 - \kappa_1^T \kappa_i^T\} = \sum_{i \geq 2} \kappa_1^T (\kappa_i^T - \kappa_1^T) > 0,
\end{equation}

and so from (3.11) we conclude $\Delta^{\Sigma^T} \kappa_1^T \leq 0$, completing the proof. \hfill \Box

An alternative but trickier proof of Lemma 3.2, without using approximation, may be given along the following lines [2].
Lemma 3.3. For a dense open subset of $\Omega$, we have

$$\Delta^T \kappa_1^T + |A|^2 \kappa_1^T = nH(\kappa_1^T)^2 - 2 \sum_{p=1}^{n} \sum_{l>1} \frac{(h_{1l,p}^T)^2}{\kappa_l^T - \kappa_1^T} \leq nH(\kappa_1^T)^2.$$  

Proof. Taking $i = j = 1$ in (3.8) gives

$$\sum_{p=1}^{n} h_{11,pp}^T + |A|^2 \kappa_1^T = nH(\kappa_1^T)^2.$$  

Since $\nabla e_1^p \perp e_1$ for any $p > 1$, we have

$$h_{11,p}^T = (\nabla e_1^p h^T)(e_1, e_1) = e_p(\kappa_1^T) - 2h^T(\nabla e_1^p e_1, e_1) = e_p(\kappa_1^T).$$

Then

$$h_{11,pp}^T = (\nabla^2 e_1^p h^T)(e_1, e_1)$$

$$= e_p(\nabla e_1^p h^T)(e_1, e_1) - 2(\nabla e_1^p h^T)(\nabla e_1^p e_1, e_1) - (\nabla^2 e_1^p h^T)(e_1, e_1)$$

$$= e_p(\kappa_1^T) - 2 \sum_{l>1} (\nabla e_1^p e_1, e_1) h_{11,lp}^T - (\nabla^2 e_1^p)(\kappa_1^T)$$

$$= e_p(\kappa_1^T) - (\nabla e_1^p)(\kappa_1^T) + 2 \sum_{l>1} \frac{(h_{11,lp}^T)^2}{\kappa_l^T - \kappa_1^T}.$$  

Summing over $p$ from 1 to $n$ leads to

$$\sum_{p=1}^{n} h_{11,pp}^T = \Delta^T \kappa_1^T + 2 \sum_{p=1}^{n} \sum_{l>1} \frac{(h_{11,lp}^T)^2}{\kappa_l^T - \kappa_1^T}.$$  

Inserting this back into (3.13) gives (3.12).

Note that in graph coordinates (see [15]),

$$\Delta^T = (g^T)^{ij} \partial_i \partial_j + nH \frac{u_k^T}{W^T} \partial_k.$$  

Therefore $\kappa_1^T > 0$ satisfies

$$\left( (g^T)^{ij} \partial_i \partial_j + nH \frac{u_k^T}{W^T} \partial_k \right) \kappa_1^T \leq 0 \text{ in } \Omega^T \cap B.$$  

Since $\kappa_1^T(P) = 0$, the Hopf boundary point lemma implies that $|\nabla \kappa_1^T(P)| \neq 0$. Recall that $\kappa_1^T$ is analytic in a small ball centered at $P$. By the implicit function theorem, $\{\kappa_1^T = 0\}$ is a smooth hypersurface $\Lambda$ passing through $P$. Moreover, $\Lambda$ is transversal to $\Gamma^T$ at $P$ since $u_{kk}^T(P) > 0$, contradicting that the rank($u_{ij}^T$) = $n$ in $\Omega^T$. Thus $T = 1$. Repeating the
arguments we made above shows that \( u = u^1 \) is a strictly convex solution in \( \overline{\Omega} \). This completes the proof of Theorem 1.1.

\[ \square \]

**Corollary 3.4.** Let \( \Omega \) be a strictly convex domain with \( \Gamma := \partial \Omega \in C^2 \) satisfying \( h_\Gamma \geq H \) and \( H > 0 \) constant. Then there exists a unique solution \( u \in C^2(\Omega) \cap C^0(\Omega) \) of \((1.1)\) with \( u \) strictly convex in \( \Omega \).

**Proof.** The inner parallel surface \( \Gamma^\varepsilon \) to \( \Gamma \) at distance \( \varepsilon \) has normalized mean curvature

\[
(3.15) \quad h^\varepsilon = \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{\kappa_i}{1 - \varepsilon \kappa_i} = \frac{1}{n-1} \left\{ \sum_{i=1}^{n-1} \kappa_i + \sum_{i=1}^{n-1} \frac{\varepsilon \kappa_i^2}{1 - \varepsilon \kappa_i} \right\} \\
= h + \frac{\varepsilon}{n-1} \sum_{i=1}^{n-1} \frac{\kappa_i^2}{1 - \varepsilon \kappa_i} \geq h + \frac{\varepsilon}{(n-1)^2} h^2 \geq (1 + \frac{\varepsilon H}{(n-1)^2}) H.
\]

By approximation of \( \Gamma^\varepsilon \) from the outside, we may assume \( \Gamma^\varepsilon \) is smooth. Thus we can apply Theorem 3.1 to \( \Gamma^\varepsilon = \partial \Omega^\varepsilon \) and find a strictly convex solution \( u^\varepsilon \) of \((1.1)\) in \( \Omega^\varepsilon \). By interior gradient estimates [5], we can pass to the limit as \( \varepsilon \to 0 \) and obtain a convex solution \( u \) of \((1.1)\) in \( \Omega \). Let \( u(0) = \min_{\Omega} u < 0 \) and suppose that \( u_{11}(0) = 0 \) for some direction \( e_1 \).

Then \( u_{11} = 0 \) along a line \( L \) joining \( 0 \) to a point \( R \in \partial \Omega \). But then \( u_1 = 0 \) on \( L \) and so \( u \) is constant on \( L \), a contradiction. Hence \( u \) is strictly convex in \( \Omega \).

\[ \square \]

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