ON SOME CLASSES OF OPEN TWO-SPECIES EXCLUSION PROCESSES

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ABSTRACT. We investigate some properties of the nonequilibrium stationary state (NESS) of a one dimensional open system consisting of first and second class (type 1 and type 2) particles. The dynamics are totally asymmetric but the rates for the different permitted exchanges (1 0 → 0 1, 1 2 → 2 1, and 2 0 → 0 2) need not be equal. The entrance and exit rates of the different species can also be different. We show that for certain classes of rates one can compute the currents and phase diagram, or at least obtain some monotonicity properties. For other classes one can obtain a matrix representation of the NESS; this generalizes previous work in which second class particles can neither enter nor leave the system. We analyze a simple example of this type and establish the existence of a randomly located shock at which the typical density profiles of all three species are discontinuous.

1. Introduction

The properties of open systems kept in a nonequilibrium stationary state (NESS) through contact with very large (formally infinite) reservoirs at different temperatures and chemical potentials is a central issue in statistical mechanics. Of particular interest in these systems are the fluxes of locally conserved quantities, such as particle numbers, transported by the system from one reservoir to another. The only nontrivial systems for which the NESS is known (more or less) explicitly are one dimensional lattice systems with stochastic dynamics corresponding to simple exclusion processes or zero range processes [1, 2]. The latter case is rather exceptional in that the NESS is described fully by a product measure and thus does not exhibit the long range correlations expected to be generic for NESS of realistic systems [3, 4].

In this note we investigate the particle fluxes and densities in the NESS of the open two-species totally asymmetric simple exclusion process (2-TASEP). This is a system on a finite one-dimensional lattice of \( L \) sites, in which each site is either empty (empty sites are also referred to as holes) or is occupied by a type 1 or a type 2 particle. The internal (bulk) dynamics are those in which a type 1 particle can exchange with a hole or type 2 particle to its right and a type 2 particle with a hole to its right or a type 1 particle to its left; that is, a type 1 particle at site \( j \) can...
jump to site $j + 1$ unless that site is occupied by another type 1 particle, and a hole at site $j$ can jump to site $j - 1$ if that site is nonempty. For the moment we allow different rates for each type of exchange, setting the time scale by normalizing the particle-hole exchange rate to be 1, but in many of the special cases to be considered below we will specialize to have equal rates for all bulk exchanges. We will follow the convention of [5] and refer to type 1 and type 2 particles as \textit{first class} and \textit{second class}, respectively, when the bulk exchange rates are all equal, as in the original definition of second class particles [6]. Particles enter the system from the left, i.e., at site 1; type 1 particles can do so either by filling a hole or by displacing a type 2 particle, and type 2 particles by filling a hole. Similarly, particles exit at the right with the replacement of type 1 particles by either holes or type 2 particles, and of type 2 particles by holes.

The one-dimensional 2-TASEP dynamics have been studied earlier, primarily using the so-called \textit{matrix ansatz}; see [5] for an extensive review of this method and its applications. The case in which all exchange rates are equal was considered both for the closed system on a ring [7] and for the system on the infinite lattice [8, 9]. The case of unequal rates for the system on the ring has also been discussed [10, 11], with a concentration on the situation in which there is only one type 2 or “defect” particle. The equal-rates dynamics were also studied, and the NESS obtained, for a semipermeable open system, i.e., one in which second class particles can neither enter nor leave [12, 13, 14]; this restriction of course forces the current of second class particles to be zero. In [14] it was shown that the projection of the NESS on the second class parties (whose total number was fixed) was given by a Gibbs measure in which the interaction was long range (logarithmic) but restricted to pairs of neighboring second class particles (that is, pairs separated only by holes and first class particles).

In the present work the restriction to semipermeable systems is dropped. This permits us to investigate the dependence of the type 2 particle current on the different bulk and boundary rates. Unfortunately, we obtain explicit results only under various restrictions on the rates. In Section 2 we discuss results obtained from projecting the two-species TASEP onto a one-species TASEP in two different ways, a process we call \textit{coloring}. In Section 3 we use coupling arguments to obtain some information on how the currents of the different species depend on the boundary rates. In Section 4 we discuss cases in which the system is solvable by a matrix ansatz, and in Section 4.1 we describe one such system in which, using the matrix ansatz, we establish the occurrence of a shock at which all three density profiles are discontinuous.

Table 1 gives our general notation for the rates of the bulk and boundary processes; in some sections below we will adopt simpler notation more suited to the restricted rates considered there. We will let $J_0$, $J_1$, and $J_2$ denote the signed currents of the various particle species in the NESS; we always have of course $J_0 + J_1 + J_2 = 0$. Note that always $J_1 \geq 0$ and $J_0 \leq 0$, while $J_2$ can have either sign. Finally, we remark that, given a specification of the rates, an equivalent system is obtained by interchanging type 1 particles with holes, left with right, $u$ with $w$, and $\alpha_i$ with $\beta_i$, $i = 1, \ldots, 3$. 


In this section we assume that the rates for particle exchanges in the bulk are independent of the types of particle involved, that is, we take \( w = v = 1 \) in the notation of Table 1. It is well known that for a system with these bulk rates and no boundaries, that is, a system on a ring or on the (doubly) infinite lattice, one may project the system onto a one-species TASEP system, say consisting of black and white particles, in two distinct ways: we always color the first class particles black and the holes white, but may color the second class particles either (all) black or (all) white; we will refer to the systems arising from these distinct colorings as the \((1, 2 + 0)\) system and the \((1 + 2, 0)\) system, respectively. Such coloring arguments have been used in the past to understand the two-species TASEP on the ring \([7]\) as well as an open semipermeable system (in which the second class particles are confined) \([14]\); in the latter case, however, they provided only an intuitive guide, since the boundary rates for the semipermeable system do not satisfy the requirements for consistency with the coloring discussed in the next paragraph.

To extend fully these colorings to the open system we must assume appropriate boundary rates. For the \((1, 2 + 0)\) system we require that first class particles enter the system at a rate which does not depend on whether site 1 is occupied by a hole or a second class particle, that is, we must take \( \alpha_1 = \alpha_3 \). Similarly for the \((1 + 2, 0)\) system we must take \( \beta_1 = \beta_3 \). In this note we consider primarily the case in which both colorings are possible (but see Remark 2.2), and for notational simplicity we will then modify the notation of Table 1 and write the transition rates in the following form:

\[
\begin{align*}
0 \rightarrow 1 \text{ with rate } \alpha_1 & \quad 10 \rightarrow 01 \text{ with rate } \alpha_1 \\
0 \rightarrow 2 \text{ with rate } \alpha_2 & \quad 20 \rightarrow 02 \text{ with rate } \beta_2 \\
2 \rightarrow 1 \text{ with rate } \alpha_3 & \quad 12 \rightarrow 21 \text{ with rate } v
\end{align*}
\]

**Table 2. Rates for the case in which both colorings are possible**
Similarly, the asymptotic density of particles in the bulk is given by

\[ \tilde{\rho}(\tilde{\alpha}, \tilde{\beta}) = \begin{cases} \frac{1}{2} & \text{if } \tilde{\alpha}, \tilde{\beta} \geq \frac{1}{2} \text{ (maximal current)}, \\ \tilde{\alpha} & \text{if } \tilde{\alpha} < \frac{1}{2} \text{ and } \tilde{\beta} \leq \frac{1}{2} \text{ (low density)}, \\ \text{linear}(\tilde{\alpha}, 1 - \tilde{\alpha}) & \text{if } \tilde{\alpha} = \tilde{\beta} < \frac{1}{2} \text{ (shock line)}, \\ 1 - \tilde{\beta} & \text{if } \tilde{\beta} < \frac{1}{2} \text{ and } \tilde{\beta} < \tilde{\alpha} \text{ (high density)}, \end{cases} \]

where “linear(\alpha, 1 - \alpha)” means the profile is linear with densities \( \alpha \) on the left and \( 1 - \alpha \) on the right. As is well known, this linear profile occurs due to the formation of a shock, which performs an unbiased random walk within the system. Note that because the current \( \tilde{J} \) is continuous across the shock line \( \tilde{\alpha} = \tilde{\beta} < 1/2 \) there is no need for a separate entry for this region in (2.1).

Now if the two-species TASEP has rates given by Table 2 then the \((1, 2 + 0)\) system is a one-species TASEP with entry rate \( \alpha \) and exit rate is \( \beta + \delta \), so that the current \( J_1 \) and density \( \rho_1 \) in the two-species system are given by \( J_1 = \tilde{J}(\alpha, \beta + \delta) \) and \( \rho_1 = \tilde{\rho}(\alpha, \beta + \delta) \), where \( \tilde{J} \) and \( \tilde{\rho} \) are obtained from (2.1) and (2.2). Similarly, the \((1 + 2, 0)\) system has entry rate \( \alpha + \gamma \) and exit rate \( \beta \), so that in the two-species TASEP we have \( J_0 = -\tilde{J}(\alpha + \gamma, \beta) \) and \( \rho_0 = 1 - \tilde{\rho}(\alpha + \gamma, \beta) \). Finally, \( J_2 = -J_1 - J_0 \) and \( \rho_2 = 1 - \rho_1 - \rho_0 \). Thus the phase diagram of the two-species system is completely determined by the phase diagrams of the two colored systems.

We summarize in Table 3 the currents and in Table 4 the bulk densities in the two-species system; the second of these tables has one more row and column than the first for the reason mentioned above in regard to (2.1)–(2.2). Detailed information about the density profile near the boundary of the system is obtainable similarly from the results of [15].

**Remark 2.1.** Examination of Tables 3 and 4 shows several interesting features of these systems.

(a) In regions of parameter space in which neither colored subsystem is on its shock line the densities are uniform in the bulk. In fact we see from Table 4 that, given any \( \rho = (\rho_0, \rho_1, \rho_2) \) with \( \sum_i \rho_i = 1 \), there exist parameters \( \alpha, \beta, \gamma, \delta \) which yield the \( \rho_i \) as the (constant) bulk densities. Recall [8] that for any such \( \rho \) there exists a unique steady state \( \mu^\rho \) for the 2-TASEP dynamics on an infinite lattice having these densities. It is then easy to see that in the open system with bulk densities \( \rho \) the local state \([15, 14]\) at any point a fraction \( x, 0 < x < 1 \), of the way through the system will be \( \mu^\rho \); that is, in the \( L \to \infty \) limit the NESS near the site \( i = [xL] \) (where \([\xi]\) denotes the integer part of \( \xi \)) will look like \( \mu^\rho \).

(b) When both colored subsystems are in the maximal current region, i.e., when \( \alpha, \beta \geq 1/2 \), second class particles are excluded from the bulk and their current vanishes. It follows from the results of [14], however, that second class particles are present in the system, with a density decreasing as the inverse square root of the distance from either boundary.

(c) When both colored systems are on their respective shock lines, necessarily \( \alpha = \beta \) and \( \gamma = \delta = 0 \); the system is then a one-species TASEP, again on the shock line.

(d) When the \((1, 2 + 0)\) system is on its shock line and the \((1 + 2, 0)\) system is not, the latter must be in its high density phase. In that case there is a uniform density \( \beta \) of holes in the system, together with a shock, with random location uniformly distributed over the system, separating a region on the left with densities \( \rho''_0 = \beta, \rho'_1 = \alpha, \) and \( \rho'_2 = 1 - \alpha - \beta, \) from a region on the right with \( \rho''_0 = \beta, \rho''_1 = 1 - \alpha, \)
Table 3. Currents for each class of particle. Note that the order of regions in the rows and columns is different. Always $J_2 = -(J_0 + J_1)$; the table gives the sign of $J_2$, when this is determined. Regions marked $X$ cannot be realized, and the region in the lower right corner can be realized only with $\alpha = \beta$ and $\gamma = \delta = 0$.

| Region | $\alpha + \gamma, \beta \geq \frac{1}{2}$ (Maximal current) | $\alpha < \frac{1}{2}, \alpha < \beta + \delta$ (Low density) | $\beta + \delta < \frac{1}{2}, \beta + \delta \leq \alpha$ (High density) |
|--------|------------------------------------------------------------|-------------------------------------------------------------|---------------------------------------------------------------|
| $\alpha + \gamma, \beta \geq \frac{1}{2}$ | $J_1 = \frac{1}{4}$ | $J_1 = \alpha(1 - \alpha)$, $J_0 = -\frac{1}{4}$, $J_2 > 0$ | $X$ |
| $\beta < \frac{1}{2}, \beta < \alpha + \gamma$ (High density) | $J_1 = \frac{1}{4}$, $J_0 = -\beta(1 - \beta)$, $J_2 < 0$ | $J_1 = \alpha(1 - \alpha)$, $J_0 = -\beta(1 - \beta)$, $J_2$ sign undetermined | $J_1 = (\beta + \delta)(1 - \beta - \delta)$, $J_0 = -\beta(1 - \beta)$, $J_2 < 0$ |
| $\alpha + \gamma < \frac{1}{2}, \alpha + \gamma \leq \beta$ (Low density) | $X$ | $J_1 = \alpha(1 - \alpha)$, $J_0 = -(\alpha + \gamma)(1 - \alpha - \gamma)$, $J_2 > 0$ | $J_1 = \alpha(1 - \alpha)$, $J_0 = -\beta(1 - \beta) = -J_1$,$J_2 = 0$ |
\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{(1, 2 + 0)} & \alpha, \beta + \delta \geq \frac{1}{2} & \alpha < \frac{1}{2}, \alpha < \beta + \delta & \alpha = \beta + \delta < \frac{1}{2} & \beta + \delta < \frac{1}{2}, \beta + \delta < \alpha \\
\hline
\text{(1 + 2, 0)} & \text{(Maximal current)} & \text{(Low density)} & \text{(Shock line)} & \text{(High density)} \\
\hline
\alpha + \gamma, \beta \geq \frac{1}{2} & \rho_1 = \frac{1}{2}, & \rho_1 = \alpha, & \rho_1 = \text{linear}(\alpha, 1 - \alpha), & \rho_1 = 1 - \beta - \delta, \\
& \rho_0 = \frac{1}{2}, & \rho_0 = \frac{1}{2}, & \rho_0 = \beta, & \rho_0 = \beta, \\
& \rho_2 = 0 & \rho_2 = \frac{1}{2} - \alpha & \rho_2 = \text{linear}(1 - \alpha - \beta, \alpha - \beta) & \rho_2 = \delta \\
\hline
\beta < \frac{1}{2}, \beta < \alpha + \gamma & \rho_1 = \frac{1}{2}, & \rho_1 = \alpha, & \rho_1 = \text{linear}(\alpha, 1 - \alpha), & \rho_1 = 1 - \beta - \delta, \\
& \rho_0 = \beta, & \rho_0 = \beta, & \rho_0 = \beta, & \rho_0 = \beta, \\
& \rho_2 = \frac{1}{2} - \beta & \rho_2 = 1 - \alpha - \beta & \rho_2 = \text{linear}(1 - \alpha - \beta, \alpha - \beta) & \rho_2 = \delta \\
\hline
\alpha + \gamma = \beta < \frac{1}{2} & X & \rho_1 = \alpha, & \rho_1 = \text{linear}(\alpha, 1 - \alpha), & \rho_1 = 1 - \beta - \delta, \\
& \text{(Shock line)} & \rho_0 = \text{linear}(1 - \beta, \beta), & \rho_0 = \beta, & \rho_0 = \beta, \\
& & \rho_2 = \text{linear}(\beta - \alpha, 1 - \alpha - \beta) & \rho_2 = \text{linear}(1 - \alpha - \beta, \alpha - \beta) & \rho_2 = \delta \\
\hline
\alpha + \gamma < \frac{1}{2}, \alpha + \gamma < \beta & X & \rho_1 = \alpha, & \rho_1 = \text{linear}(\alpha, 1 - \alpha), & \rho_1 = 1 - \beta - \delta, \\
& \text{(Low density)} & \rho_0 = 1 - \alpha - \gamma, & \rho_0 = \beta, & \rho_0 = \beta, \\
& & \rho_2 = \gamma & \rho_2 = \text{linear}(1 - \alpha - \beta, \alpha - \beta) & \rho_2 = \delta \\
\hline
\end{array}
\]

Table 4: Currents for each class of particle. Note that the order of regions in the rows and columns is different. Regions marked X cannot be realized, and the region in which both subsystems are on their respective shock lines can be realized only if \(\alpha = \beta\) and \(\gamma = \delta = 0\).
and $\rho_2^i = \alpha - \beta$. The local state a fraction $x$ of the way through the system is then the superposition $x\mu_1^o + (1-x)\mu_3^o$. Similar conclusions hold when the $(1+2,0)$ system is on its shock line and the $(1,2+0)$ system is in its low density phase.

**Remark 2.2.** One can, of course, also discuss values of the boundary rates which respect one but not both of the colorings. Much less can then be said, but there is one simple case in which the currents can be completely determined. Suppose that $\alpha_1 = \alpha_3 = \alpha$, so that the $(1,2+0)$ coloring is possible; then since the exit rate of type 1 particles is $\beta_1 + \beta_2$ the current $J_1$ is related to the density $\rho_1(L)$ of type 1 particles at site $L$ by

$$J_1 = (\beta_1 + \beta_2)\rho_1(L).$$

Now suppose further that $\beta_3 = 0$, so that type 2 particles cannot exit at the right boundary and holes can enter there only when a type 1 particle exits. The currents across the right boundary (and thus across any bond) must then satisfy

$$J_0 = -\beta_1\rho_1(L) = -\frac{\beta_1}{\beta_1 + \beta_2}J_1, \quad J_2 = -\beta_2\rho_1(L) = -\frac{\beta_2}{\beta_1 + \beta_2}J_1.$$

Note that these results apply for any bulk rate $w$, although we still require that $v = 1$ in order that the $(1,2+0)$ coloring be possible.

### 3. Coupling and monotonicity of currents

We do not know how to calculate the currents of the various species in our model for general rates, but in some cases it is possible to show by means of coupling arguments that the currents depend monotonically on the rates. The next two theorems give results of this nature.

**Theorem 3.1.** Suppose that $v = w = 1$. Then in the region of parameter space in which $\alpha_3 \geq \alpha_1$ and $\beta_3 \geq \beta_1$:

(a) The current $J_1$ is monotonically increasing in $\alpha_1$, $\alpha_2$, and $\alpha_3$; it is monotonically decreasing in $\beta_3$ and under increase of $\beta_1$ when $\beta_1 + \beta_2$ is constant.

(b) The (negative) current $J_0$ is monotonically increasing in magnitude as a function of $\beta_1$, $\beta_2$, and $\beta_3$; it is monotonically decreasing in magnitude in $\alpha_3$ and under increase of $\alpha_1$ when $\alpha_1 + \alpha_2$ is constant.

**Theorem 3.2.** Suppose that $v = w = 1$. Then:

(a) If $\alpha_1 = \alpha_3$ then the current $J_2$ is monotonically increasing in $\alpha_2$, in $\beta_3$, and under increase of $\beta_1$ when $\beta_1 + \beta_2$ is constant.

(b) If $\beta_1 = \beta_3$ then the current $J_2$ is monotonically decreasing in $\beta_2$, in $\beta_3$, and under increase of $\alpha_1$ when $\alpha_1 + \alpha_2$ is constant.

To develop the proofs of these theorems, given later in this section, we introduce the natural order $0 \prec 2 \prec 1$ on the three species of particles, and if $\tau$ and $\tau'$ are two configurations of our system we write $\tau \preceq \tau'$ if $\tau_i \preceq \tau'_i$ for $i = 1, \ldots, L$. To *couple* two evolving systems, denoted as primed and unprimed, is to define a joint dynamics for the pair $(\tau'(t), \tau(t))$ in such a way that the induced dynamics on each separate system is that of Table I with rates $v', w', \alpha'_i, \beta'_i$ and $v, w, \alpha_i, \beta_i$, respectively. We are interested only in order preserving couplings, for which if $\tau'(0) \succeq \tau(0)$ then $\tau'(t) \succeq \tau(t)$ for all $t \geq 0$.

We now define the coupling in the bulk and on the boundaries.
Bulk: For this coupling it is necessary that all the bulk rates for the two systems be the same:

\[ v = v' = w = w' = 1. \]

Under this assumption we define the coupled dynamics by supposing that a Poisson alarm clock which rings with rate 1 is associated with each bond, and that when the alarm rings any possible exchange across that bond, in either the primed or unprimed system, takes place. Equivalently one may describe the dynamics by giving rates for the coupled process: if \((\tau_i', \tau_i) \neq (\tau_i'_{t+1}, \tau_{t+1})\) then at rate 1,

\[
(\tau_i', \tau_i)(\tau_{i+1}', \tau_{i+1}) \rightarrow (\tau_{i+1}', \tau_{i+1})(\tau_i', \tau_i) \quad \text{if} \quad \tau_i' \geq \tau_i'_{t+1} \quad \text{and} \quad \tau_i \geq \tau_i_{t+1},
\]

\[
(\tau_i', \tau_i)(\tau_{i+1}', \tau_{i+1}) \rightarrow (\tau_i', \tau_{i+1})(\tau_{i+1}', \tau_i) \quad \text{if} \quad \tau_i' < \tau_i'_{t+1} \quad \text{and} \quad \tau_i \geq \tau_i_{t+1},
\]

\[
(\tau_i', \tau_i)(\tau_{i+1}', \tau_{i+1}) \rightarrow (\tau_{i+1}', \tau_i)(\tau_i', \tau_{i+1}) \quad \text{if} \quad \tau_i' \geq \tau_i'_{t+1} \quad \text{and} \quad \tau_i < \tau_i_{t+1},
\]

It is easy to see that no coupling of this general sort will succeed if (3.4) does not hold.

Boundaries: As an example we begin by considering the particular case in which the state on site \(L\) is \((\tau_L', \tau_L) = (1', 1)\). We must assign rates to the five allowed transitions on site \(L\) arising from the exit of the first class particle in one or both of the coupled systems, that is, transitions from \((1', 1)\) to \((0', 0)\), \((2', 0)\), \((2', 2)\), \((1', 0)\), or \((1', 2)\); on the other hand, transitions to \((0', 1)\), \((0', 2)\), or \((2', 1)\), which would violate the ordering of the two configurations, must not occur. The total rate for each transition in each of the two coupled systems must be given by Table 1 so that for example the rate for \(1' \rightarrow 2'\), which is the sum of the rates for \((1', 1) \rightarrow (2', 0)\) and \((1', 1) \rightarrow (2', 2)\) must be \(\beta_2'\), the sum of the rates for \((1', 1) \rightarrow (0', 0)\), \((1', 1) \rightarrow (2', 0)\) and \((1', 1) \rightarrow (1', 0)\), must be \(\beta_1\), etc. There are thus five equations to be satisfied by five rates; it is easy to see that a solution exists with nonnegative rates if and only if \(\beta_1 \geq \beta_1'\) and \(\beta_1 + \beta_2 \geq \beta_1' + \beta_2'\), and that there is then in general one free parameter.

All other boundary transitions may be analyzed similarly; the resulting rates are given in Table 5. In this table \(x, y, s, \) and \(t\) are parameters which may be chosen freely subject to the condition that all the resulting rates be nonnegative, that is, to the inequalities

\[
\max\{0, \alpha_2 - \alpha_2'\} \leq x \leq \min\{\alpha_2, \alpha_1' - \alpha_1\},
\]

\[
0 \leq y \leq \min\{\alpha_2, \alpha_3' - \alpha_1\},
\]

\[
\max\{0, \beta_2 - \beta_2'\} \leq s \leq \min\{\beta_2, \beta_1 - \beta_1'\},
\]

\[
0 \leq t \leq \min\{\beta_2, \beta_3 - \beta_1'\}.
\]

The inequalities (3.3) will have a solution, and all rates of Table 5 will be nonnegative, if and only if the parameters of the two systems satisfy

\[
\alpha_1' \geq \alpha_1, \quad \alpha_1' + \alpha_2' \geq \alpha_1 + \alpha_2, \quad \alpha_3' \geq \alpha_3, \quad \alpha_3' \geq \alpha_1,
\]

\[
\beta_1' \leq \beta_1, \quad \beta_1' + \beta_2' \leq \beta_1 + \beta_2, \quad \beta_3' \leq \beta_3, \quad \beta_3' \leq \beta_3.
\]

In summary: an order preserving coupling of the two systems is possible if (3.1), (3.4), and (3.5) are satisfied, and is then given by Table 5.

Remark 3.3. (a) Note in particular that the marginal rates for boundary transitions in either the primed or unprimed system are those of Table 1 separately in each boundary configuration. For example, the rate for \(1' \rightarrow 2'\) at the right
The left-right, particle-hole symmetry described in Section 1 can be extended (1)
boundary is $\beta$

Theorem 3.4. Proof of Theorem 3.1:

We have no evidence that the restrictions in Theorem 3.1 that

Remark 3.5. □

Then the currents of first-class particles satisfy

\[ \alpha_1 (1', 2) \rightarrow (0', 0) \beta_1' \]

\[ \alpha_2 (1', 2) \rightarrow (0', 0) \beta_2' \]

\[ \alpha_3 (1', 0) \rightarrow (0', 0) \beta_3' \]

\[ \alpha_2 (1', 2) \rightarrow (0', 0) \beta_2' \]

\[ \alpha_3 (1', 0) \rightarrow (0', 0) \beta_3' \]

\[ \alpha_2 (1', 2) \rightarrow (0', 0) \beta_2' \]

\[ \alpha_3 (1', 0) \rightarrow (0', 0) \beta_3' \]

\[ \alpha_2 (1', 2) \rightarrow (0', 0) \beta_2' \]

\[ \alpha_3 (1', 0) \rightarrow (0', 0) \beta_3' \]

Table 5. Rates for the coupled process

| Left       | Rate | Right       | Rate |
|------------|------|-------------|------|
| $(0', 0)$  | $\alpha_1$ | $(1', 1)$  | $\beta_1$ |
| $(0', 0)$  | $\alpha_2'$ | $(1', 1)$  | $\beta_2$ |
| $(0', 0)$  | $\alpha_2' - \alpha_1 - x$ | $(1', 1)$  | $\beta_2' - \beta_1 - s$ |
| $(0', 0)$  | $\alpha_2 - x$ | $(1', 1)$  | $\beta_2' - s$ |
| $(0', 0)$  | $\alpha_2' - \alpha_2 + x$ | $(1', 1)$  | $\beta_2' + s$ |
| $(2', 0)$  | $\alpha_1$ | $(1', 2)$  | $\beta_1' |
| $(2', 0)$  | $\alpha_1$ | $(1', 2)$  | $\beta_1' |
| $(2', 0)$  | $\alpha_2 - y$ | $(1', 2)$  | $\beta_1' |
| $(2', 0)$  | $\alpha_2 - y$ | $(1', 2)$  | $\beta_1' |
| $(2', 0)$  | $\alpha_3$ | $(1', 0)$  | $\beta_3$ |
| $(2', 0)$  | $\alpha_3$ | $(1', 0)$  | $\beta_3$ |
| $(2', 0)$  | $\alpha_3 - \alpha_3$ | $(1', 0)$  | $\beta_3$ |
| $(2', 0)$  | $\alpha_3$ | $(1', 0)$  | $\beta_3$ |
| $(2', 0)$  | $\alpha_3$ | $(1', 0)$  | $\beta_3$ |

boundary is $\beta_2'$ if $(\tau_{L}', \tau_L) = (1', 1)$, as discussed in the example above, and also if

$(\tau_{L}', \tau_L) = (1', 2)$ or $(\tau_{L}', \tau_L) = (1', 0)$.

(b) The left-right, particle-hole symmetry described in Section 1 can be extended to the coupled system by also interchanging the primed and unprimed systems.

From now on we will write $J_1'$ and $J_i$ for the currents in the primed and unprimed systems.

**Theorem 3.4.** (a) Suppose that the coupling is possible, i.e., that (3.1), (3.5) hold, and that in addition

\[ \beta_1 + \beta_2 = \beta_1' + \beta_2'. \]

Then the currents of first-class particles satisfy $J_1' \geq J_1$.

(b) Similarly, $J_0' \leq J_0$ when $\alpha_1 + \alpha_2 = \alpha_1' + \alpha_2$.

**Proof.** (a) By (3.3) and (3.6) we must have $s = \beta_1 - \beta_1' = \beta_2' - \beta_2$ in the coupling, so that the rates are zero for the processes $(1', 1) \rightarrow (1', 0)$ and $(1', 1) \rightarrow (1', 2)$ at the right end of the system. Thus a type 1 particle cannot exit the system unless a type 1' does also, and this implies the result. For (b) one may give a similar argument or appeal to the symmetry of Remark 3.3(b).

We can now prove the first monotonicity result given above.

**Proof of Theorem 3.4.** We show monotonicity of $J_1$ in $\alpha_1$; all other cases are similar. If $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ satisfy $\alpha_3 \geq \alpha_1$ and $\beta_3 \geq \beta_1$, and if $\alpha_i'$ satisfies $\alpha_3 \geq \alpha_i' \geq \alpha_1$, then by defining $\alpha_2' = \alpha_2, \alpha_2' = \alpha_3$, and $\beta_i' = \beta_i$, $i = 1, 2, 3$, we obtain two systems to which Theorem 3.4 may be applied.

**Remark 3.5.** We have no evidence that the restrictions in Theorem 3.4 that $\alpha_3 \geq \alpha_1$ and $\beta_3 \geq \beta_1$ are in fact necessary for the monotonicity; they may well be an artifact of the method of proof.
We now turn to the proof of Theorem 3.2 for which we introduce a more restrictive coupling with a larger set of excluded pairs. Specifically, the coupling has the property that if the initial configuration of the coupled systems contains none of the pairs \((\tau_i', \tau_i) = (0', 2), (0', 1), (2', 1), (1', 0)\) or \((1', 2)\) then these pairs are also absent from configurations \((\tau'(t), \tau(t))\) for all \(t \geq 0\). For this coupling the parameters must satisfy

\[
\begin{align*}
(3.7) & \quad u = u' = w = w' = 1, \\
(3.8) & \quad \alpha'_1 = \alpha_1 = \alpha_3 = \alpha'_3, \quad \alpha'_2 \geq \alpha_2, \\
(3.9) & \quad \beta'_1 \leq \beta_1, \quad \beta'_1 + \beta'_2 = \beta_1 + \beta_2, \quad \beta'_3 \leq \beta_3.
\end{align*}
\]

Rates for the coupled process are then given in Table 6.

Note that in \((3.7) - (3.9)\) we have added conditions to those required for the previous coupling (that is, to \((3.1), (3.4),\) and \((3.5)\)) but also omitted the conditions \(\alpha'_3 \geq \alpha_1\) and \(\beta'_1 \leq \beta_3\). Note also that \((3.8)\) implies that both the primed and unprimed systems can be given the \((1, 2 + 0)\) coloring discussed in Section 2 and hence that \(J'_1 = J_1\); this also follows from the fact that particles 1' and 1 appear in the coupled system only in the pair \((1', 1)\). Thus any inequality between \(J'_2\) and \(J_2\) implies the opposite inequality between \(J'_0\) and \(J_0\).

| Left          | Rate | Right          | Rate |
|---------------|------|----------------|------|
| \((0', 0)\) → \((1', 1)\) | \(\alpha_1\) | \((1', 1)\) → \((0', 0)\) | \(\beta'_1\) |
| \((0', 0)\) → \((2', 2)\) | \(\alpha_2\) | \((1', 1)\) → \((2', 0)\) | \(\beta'_2 - \beta_2\) |
| \((0', 0)\) → \((2', 0)\) | \(\alpha'_2 - \alpha_2\) | \((1', 1)\) → \((2', 2)\) | \(\beta'_2\) |
| \((2', 0)\) → \((1', 1)\) | \(\alpha_1\) | \((2', 2)\) → \((0', 0)\) | \(\beta'_3\) |
| \((2', 0)\) → \((2', 2)\) | \(\alpha_2\) | \((2', 2)\) → \((2', 0)\) | \(\beta_3 - \beta'_3\) |
| \((2', 2)\) → \((1', 1)\) | \(\alpha_3\) | \((2', 0)\) → \((0', 0)\) | \(\beta'_3\) |

Table 6. Rates for the coupled process giving monotonicity of \(J_2\)

**Theorem 3.6.** Suppose that \((3.7) - (3.9)\) hold. Then:

(a) If \(\beta_i = \beta'_i\) for \(i = 1, 2, 3\) then \(J'_2 \geq J_2\).

(b) If \(\alpha_2 = \alpha'_2\) then \(J'_2 \leq J_2\).

**Proof.** Under the hypothesis of (a) no \((2', 0)\) pair can enter the system at the right boundary, while under (b) no \((2', 0)\) pair can enter the system at the left boundary. \(\square\)

**Proof of Theorem 3.5.** (a) is obtained from Theorem 3.6 as in the proof of Theorem 3.1 and then (b) follows from the symmetry of Remark 3.3(b). \(\square\)

One can check easily that these results are confirmed by those of Table 3 when both colorings are possible; in that case they imply that \(J_2\) should be increasing in \(\beta\) and \(\gamma\) and decreasing in \(\alpha\) and \(\delta\).

4. **Matrix solutions**

In this section we observe that for various special choices of the boundary rates of Table 4 the probabilities of configurations of the open two-species TASEP can be computed using a variant of the usual matrix product ansatz. Specifically, this
means that for these rates there exist matrices $X_0$, $X_1$, and $X_2$ and vectors $|V\rangle$ and $\langle W|$ such that the probability of the configuration $\tau_1 \ldots \tau_L$, where $\tau_i$ is 0, 1, or 2, is given by

\begin{equation}
\mu_L(\tau_1, \ldots, \tau_L) = Z_L^{-1} w_L(\tau_1, \ldots, \tau_L);
\end{equation}

the weight $w_L$ is given by

\begin{equation}
w_L(\tau_1, \ldots, \tau_L) = \langle W|X_{\tau_1}X_{\tau_2}\cdots X_{\tau_L}|V\rangle,
\end{equation}

and the normalizing factor $Z_L$ by

\begin{equation}
Z_L = \sum_{\tau_1, \ldots, \tau_L} w_L(\tau_1, \tau_2, \ldots, \tau_L) = \langle W|(X_0 + X_1 + X_2)^L|V\rangle.
\end{equation}

Such matrix solutions exist for two different families of boundary rates, which we call permeable and semipermeable. We can impose either type of boundary condition at either end of the system, leading to four distinct families of open systems.

**Bulk:** The exchange rates in the bulk are those of Table 1, although we shall on occasion specialize to take $v$ and/or $w$ equal to 1. For the weights (4.1) to be stationary the matrices $X_0$, $X_1$, $X_2$ should satisfy

\begin{equation}
X_1X_0 = X_1 + X_0,
\end{equation}

\begin{equation}
X_1X_2 = \frac{1}{v}X_2,
\end{equation}

\begin{equation}
X_2X_0 = \frac{1}{w}X_2,
\end{equation}

the usual algebraic rules for the 2-TASEP \[10\] \[11\].

**Semipermeable boundaries:** A semipermeable boundary is one through which type 2 particles cannot pass, that is, a semipermeable left boundary has $\alpha_2 = \alpha_3 = 0$ and a semipermeable right boundary $\beta_2 = \beta_3 = 0$. A semipermeable boundary is thus characterized by a single rate, which we will write as $\alpha = \alpha_1$ for a left boundary and $\beta = \beta_1$ for a right boundary. The vectors $\langle W|$ for the left boundary and $|V\rangle$ for the right boundary should satisfy

\begin{equation}
\text{Semipermeable left boundary: } \langle W|X_0 = \frac{1}{\alpha}\langle W|,
\end{equation}

\begin{equation}
\text{Semipermeable right boundary: } X_1|V\rangle = \frac{1}{\beta}|V\rangle.
\end{equation}

The open system with two semipermeable boundaries and with $w = v = 1$ was studied in \[13\], \[12\], and \[14\].

**Permeable boundaries:** At a permeable boundary type 2 particles can enter and leave the system, but the rates are constrained by

\begin{equation}
\text{Permeable left boundary: } \alpha_1 + \alpha_2 = w,
\end{equation}

\begin{equation}
\text{Permeable right boundary: } \beta_1 + \beta_2 = v.
\end{equation}

We will choose a convenient parametrization of such rates which will be discussed further below. At the left boundary we write $\alpha_3 = a$ for the rate of exit of type 2 particles; there is a second free parameter, $z$, which plays the role of a fugacity for type 2 particles, and $\alpha_1 = wa z$, $\alpha_2 = w(1 - az)$. At the right boundary we write similarly $\beta_3 = b$, $\beta_2 = vbz$, and $\beta_1 = v(1 - bz)$. If both boundaries are permeable then the parameter $z$ must be the same on the left and the right, and to insure nonnegative rates these parameters must satisfy

\begin{equation}
a z \leq 1, \quad b z \leq 1.
\end{equation}
The vectors vectors $|V\rangle$ and $\langle W|$ used to construct the steady state should satisfy

\begin{align}
\langle W|X_0 &= \frac{1}{w} |W\rangle, \quad \langle W|X_2 = z |W\rangle, \\
\end{align}

Permeable left boundary:

\begin{align}
\langle W|X_0 &= 1, \\
\langle W|X_2 &= \langle W|, \\
\langle W|X_2 &= z \langle W|.
\end{align}

(4.10)

Permeable right boundary:

\begin{align}
X_1|V\rangle &= 1, \\
X_2|V\rangle &= \langle V|.
\end{align}

(4.11)

| Left | Bulk | Right |
|------|------|-------|
| $|W\rangle$ | $\langle W_\alpha|$ | $\langle W_w|$ | $|V\rangle$ | $|V_\beta\rangle$ | $|V_v\rangle$ |
| $0 \to 1$ | $\alpha$ | $w(1-az)$ | $0 \to 1$ | $1$ | $1 \to 0$ | $\beta$ | $v(1-bz)$ |
| $0 \to 2$ | $0$ | $waz$ | $2 \to 0$ | $2$ | $1 \to 2$ | $0$ | $v\beta z$ |
| $2 \to 1$ | $0$ | $a$ | $12 \to 21$ | $v$ | $2 \to 0$ | $0$ | $b$ |

Table 7. Rates for permeable (P) and semipermeable (SP) boundary conditions. The last line shows the boundary vectors $\langle W|$ and $|V\rangle$ to be used in (4.1).

It is a straightforward exercise, using standard arguments, to verify that if there exist matrices and vectors $X_0$, $X_1$, $X_2$, $|V\rangle$, and $\langle W|$ satisfying (4.4), either (4.5) or (4.10), and either (4.6) or (4.11), then (4.1) defines an invariant measure for the corresponding dynamics. We establish this existence below. One striking consequence is that if the system has one or more permeable boundaries then the parameters $a$ and/or $b$, which control the entry and exit of type 2 particles, do not appear in the algebraic conditions (4.4) and (4.10) and/or (4.11), so that the steady state of the system is independent of these parameters.

To establish the existence, recall [15] that there exist matrices $X_0$ and $X_1$, and vectors $\langle W_\alpha|$ and $|V_\beta\rangle$ satisfying

\begin{align}
X_1X_0 &= X_1 + X_0, \\
\langle W_\alpha|X_0 &= \frac{1}{\alpha} \langle W_\alpha|, \quad X_1|V_\beta\rangle = \frac{1}{\beta} |V_\beta\rangle \\
\langle W_\alpha|V_\beta\rangle &= \frac{\alpha \beta}{\alpha + \beta - 1}, \quad \alpha + \beta > 1.
\end{align}

(4.12) (4.13) (4.14)

We now define

\begin{align}
X_2 &= \frac{w + v - 1}{wv} |V_v\rangle \langle W_w|;
\end{align}

(4.15)

then the matrices $X_0$, $X_1$, $X_2$ satisfy (4.4) (in fact, this requires only $X_2 = \lambda |V_v\rangle \langle W_w|$ for some non-zero $\lambda$ [10] [11]). Note that for a system with two semipermeable boundaries we are now introducing an additional parameter $z$ (which is again a fugacity, as discussed below). For semipermeable boundary conditions on the left (respectively right) we take $\langle W| = \langle W_\alpha|$ (respectively $|V| = |V_\beta\rangle$), thus satisfying (4.5) (respectively (4.6)). For permeable boundary conditions on the left (right) we take $\langle W| = \langle W_w|$ ($|V| = |V_v\rangle$); (4.10) (respectively (4.11)) will then be satisfied due to the choice of overall constant in the definition (4.15) of $X_2$. This
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To extend the discussion to all parameter values we must distinguish two cases.

**Case 1: One or both boundaries permeable.** In this case all weights (1.2) have the form $QP(1/\alpha, 1/\beta, 1/w, 1/v)$, with $P$ a polynomial having positive coefficients and $Q$ a common factor depending only on the type of system under consideration: $Q = (w + v - 1)^{-1}$ in the case $P/P$ (that is, two permeable boundaries), $Q = (w + \beta - 1)^{-1}$ in the case $SP/P$, and $Q = (\alpha + v - 1)^{-1}$ in the case $P/SP$. The factor $Q$ cancels in the probabilities (1.1) and the resulting probabilities are well defined and positive for all positive values of the parameters; the algebraic identities among them which express stationarity hold by analytic continuation from the region (1.10).

**Case 2: Both boundaries semipermeable.** In this case the number $N_2$ of type 2 particles in the system is conserved by the dynamics, and the NESS decomposes into ergodic components corresponding to the $L + 1$ possible values of $N_2$. Within each component an argument as in Case 1 allows one to extend the NESS to all parameter values; however, the factor $Q$ referred to there depends on $N_2$, with $Q = Q_0 \equiv (\alpha + \beta - 1)^{-1}$ if $N_2 = 0$ and $Q = Q_* \equiv (w + v - 1)(\alpha + v - 1)^{-1}(w + \beta - 1)^{-1}$ if $N_2 > 0$. One may also consider grand canonical ensembles in which $N_2$ fluctuates. The algebraic conditions (4.4)–(4.6) do not determine the relative weights of configurations with different values of $N_2$, but the particular choice of representation of the algebra discussed above does, and thus defines a grand canonical ensemble in the region (1.10) via (4.1)–(4.3). In general the inequality of $Q_0$ and $Q_*$ is an obstruction to extending this ensemble to all values of the parameters—negative probabilities can arise. This difficulty can be avoided, and a grand canonical NESS defined, by excluding the component with $N_2 = 0$ from the superposition. Alternatively, a grand canonical NESS including all components can be defined when $Q_0 = Q_*$, that is, when $w = \alpha$ or $v = \beta$.

**Remark 4.1.** (a) The cases of semipermeable and permeable boundary conditions are not mutually exclusive; on the left, the case $a = 0$ of permeable conditions corresponds to the case $\alpha = w$ of semipermeable, with a similar common case on the right.

(b) The system with permeable boundaries on both ends was studied, using other methods, in [17]. The authors took (in our language) $a = b = vw/(1 + vwz)$ and did not observe that the NESS is in fact independent of $a$ and $b$.

As noted above, the case of two semipermeable boundaries with $w = v = 1$ was studied, in the canonical ensemble with fixed $N_2$, in [13] and [14], and it is easy to see that several features discussed there are still present for arbitrary $v$ and $w$. The current $J_2$ is always zero. The operator $X_2$ has rank one (see (4.15)) which implies that a type 2 particle at site $i$ decouples sites to the left of $i$ from those to the right, except that the total number of type 2 particles in these two subsystems must be $N_2 - 1$. As a consequence [14] the marginal measure on configurations of type 2 particles is an equilibrium measure for a certain Hamiltonian with nearest-particle interactions. In any of the grand canonical ensembles discussed under Case 2 above a second class particle completely decouples the system; here we use the fact that...
the grand canonical ensembles for the left and right subsystems are well defined (see the last remark under Case 2).

Let us consider then the case in which there are permeable boundary conditions at one or both ends of the system. As noted above the NESS is then independent of the parameters \( a \) and/or \( b \), and so is determined by \( v, w, z, \) and \( \alpha \) (respectively \( \beta \)) if semipermeable boundary conditions are used at the left (respectively right) boundary. In particular, we may achieve this steady state with \( a = 0 \) or \( b = 0 \), so that certainly we still have \( J_2 = 0 \). That is, all models with the rates of Table 7 have zero current of type 2 particles.

In all of the grand canonical ensembles discussed above the parameter \( z \) plays the role of a fugacity for the type 2 particles. To see this, suppose that \( \tau \) is a configuration with \( N_2 \) type 2 particles and note from (4.5) and (4.6) that the weight assigned to \( \tau \) by the algebra above will contain a factor \( z^{N_2} \). Now let \( w_{L,N_2}(\tau) \) denote the weight assigned to \( \tau \) by the algebra (4.4)–(4.6) of two semipermeable boundaries with \( \alpha = w \) and \( \beta = v \) (for \( v = w = 1 \) this is the weight assigned to \( \tau \) in the algebra of ([14])). Then we see that

\[
\mu_L(\tau) = Z_L^{-1} \sum_{N_2=0}^{L} z^{N_2} w_{L,N_2}(\tau).
\]

That is, \( \mu_L \) is just the “grand canonical” ensemble, with fugacity \( z \) for the type 2 particles, for the semipermeable system with entry and exit rates \( w \) and \( v \), respectively, in the sense that this term is often used when discussing matrix ansatz models. It then follows, for example, that the type 2 particles are in a grand-canonical version of the equilibrium ensemble mentioned above.

4.1. A simple special case. We close this section with the consideration of a special case in which various quantities can be computed rather easily: we take permeable boundary conditions on the left and semipermeable on the right, and specialize to \( v = 1 \) (there is a corresponding analysis with left and right reversed and \( w = 1 \)). The algebra then reduces to

\[
\begin{align*}
X_1X_0 &= X_1 + X_0, & X_1X_2 &= X_2, & X_2X_0 &= \frac{1}{w}X_2, \\
X_1|V\rangle &= \frac{1}{\beta}|V\rangle, & \langle W|X_0 &= \frac{1}{w}\langle W|, & \langle W|X_2 &= z\langle W|,
\end{align*}
\]

(4.17)

and is realized by taking \( |V\rangle = |V_\beta\rangle \), \( \langle W| = \langle W_w| \), and \( X_2 = z|V_1\rangle \langle W_w| \). If we now define new matrices \( X'_1 \) and \( X'_2 \) by \( X'_1 = X_1 \) and \( X'_0 = X_0 + X_2 \) then these satisfy

\[
\begin{align*}
X'_1X'_0 &= X'_1 + X'_0, & X'_1|V\rangle &= \frac{1}{\beta}|V\rangle, & \langle W|X'_0 &= \frac{1 + wz}{w}\langle W|.
\end{align*}
\]

(4.18)

This is the algebra of an open one-species TASEP system [15] with entry rate \( \alpha = w/(1 + wz) \) and exit rate \( \beta \), which implies that if we color the type 1 particles black and the holes and type 2 particles white, as for the \((1, 2 + 0)\) system discussed in Section 2 then the marginal of the NESS on the black/white system is precisely the NESS of this 1-TASEP system. (The situation here is somewhat different from that of Section 2 however; the correspondence here is on the level of measures rather than dynamics, since the rate at which a type 1 particle enters the system depends on whether there is a hole or a type 2 particle on site 1.)
Many properties of the 2-TASEP may now be obtained. The current and density profile of type 1 particles are given by (4.19) and (4.20):

\[ J_1 = \tilde{J}(\alpha, \beta), \quad \rho_1 = \tilde{\rho}(\alpha, \beta). \]

The current of type 2 particles is zero since the right boundary conditions are semipermeable, and the density profile of type 2 particles,

\[ \rho_2(x) = \lim_{L \to \infty, i \to \pm L} \text{Prob}(\tau_i = 2), \]

may be obtained from the fact that \( X_2 \) is a rank one matrix:

\[
\text{Prob}(\tau_i = 2) = \frac{\langle W|(X_0 + X_1 + X_2)^{i-1}X_2(X_0 + X_1 + X_2)^{L-i}|V \rangle}{\langle W|(X_0 + X_1 + X_2)^{L}|V \rangle}
\]

\[ = \frac{Z^\alpha_{i-1}Z^{\alpha,\beta}_{L-i}}{Z^\alpha_{L}}, \]

and the known asymptotics [15] of \( Z^\gamma_{L} = \langle W|X_0 + X_1 |V \rangle \).

As an example we discuss the case \( \alpha = \beta < 1/2 \); for convenience we will eliminate \( z \) and \( \beta \) and use \( \alpha \) and \( w \) as basic variables to describe the system in this case. We will need the asymptotic formulas [15]

\[ Z^\gamma_{L} \approx \begin{cases} 
\frac{L(1-2\gamma)^2}{\gamma L(1-\gamma)L+2}, & \text{if } \gamma = \delta < 1/2, \\
\frac{\delta(1-2\gamma)}{(\delta-\gamma)(1-\alpha)^{L+1}}, & \text{if } \gamma < \delta \text{ and } \gamma < 1/2.
\end{cases} \]

In this case all density profiles are linear:

\[ \rho_i = \text{linear}(r_i, r'_{i}), \quad i = 0, 1, 2, \]

where

\[ r_1 = \alpha, \quad r_2 = \frac{(1-2\alpha)(w-\alpha)}{w(1-\alpha)}, \quad r_0 = \frac{\alpha(1-2\alpha+\alpha w)}{w(1-\alpha)} \]

\[ r'_1 = 1-\alpha, \quad r'_2 = 0, \quad r'_0 = \alpha. \]

For \( \rho_1 \) this follows from (4.19), for \( \rho_2 \) from (4.20) and (4.21), using \( z = (w-\alpha)/(\alpha w) \) and (4.22), and then for \( \rho_0 \) from \( \sum_i \rho_i = 1 \).

We now show that these linear profiles arise, as usual, from the occurrence of a shock in the typical (not averaged) density profiles: a randomly located point at which (in general) all three typical profiles are discontinuous. In this case the shock location is marked by the position of the rightmost type 2 particle in the system; the probability that this particle is located at site \( i \) is, from (4.22),

\[
\frac{\langle W|(X_0 + X_1 + X_2)^{i-1}X_2(X_0 + X_1 + X_2)^{L-i}|V \rangle}{\langle W|(X_0 + X_1 + X_2)^{L}|V \rangle} = \frac{Z^{\alpha,\beta}_{i-1}Z^{\gamma}_{L-i}}{Z^\alpha_{L}} \approx \frac{1}{L},
\]

so that this particle (and hence, as we will see below, the shock) is uniformly located throughout the system. When conditioned on the location \( i \) of this particle the NESS decomposes into the product of the NESS’s of two independent subsystems, and by finding the densities in these subsystems we exhibit the discontinuity in the overall (conditioned) density profile that is the characteristic of a shock:

**Left subsystem:** On sites \( 1, \ldots, i-1 \) the NESS is that of a semipermeable/permeable 2-TASEP like the full system, but with exit rate \( \beta = 1 \). One may again introduce a
(1, 2+0) coloring for this system for which the type 1 particles form an open system with entrance rate $\alpha$ and exit rate 1; the colored system is in its low density phase and thus has (up to boundary effects) a uniform density of type 1 particles: $\rho_1 \equiv \alpha = r_1$. The density profile $\rho_2$ can then be calculated from (4.20) and (4.21) with the replacement $\beta \to 1$; the result is a constant profile with value $r_2$. The profile $\rho_0$ is then also constant with $\rho_0 \equiv r_0$.

**Right subsystem:** On sites $i+1, \ldots, L$ the NESS is that of an open one-component TASEP with entry rate $w$ and exit rate $\beta = \alpha$. Since $\alpha < w$ this system is in its high density phase; density profiles are constant with $\rho_1 \equiv 1 - \alpha$ and $\rho_0 \equiv \alpha$.

Note that if $w = 1$ then $r_0 = r_0'$, so that the density of holes is constant through out the system and the densities of first and second class particles have compensating discontinuities at the shock position. Situations of this kind were encountered in Section 2 (see Remark 2.1(d)) and in the study of the “fat shock” in the system with $v = w = 1$ and two semipermeable boundaries [14]. For general $w$ all three densities are discontinuous at the shock positions, which so far as we know is a new phenomenon.

The shock is stationary in the sense that it has no net drift velocity, although its position does of course fluctuate. Let us write $J_i(\vec{\rho}; v, w)$ for the current of species $i$ in a uniform steady state of an infinite system with densities $\vec{\rho} = (\rho_1, \rho_2, \rho_3)$, where the exchange rates in the bulk are 1, $v$, and $w$, as in Table 1. The condition for a stationary shock as described above is then that

$$J_i(\vec{r}; 1, w) = J_i(\vec{r}'; 1, w) \quad \text{for } i = 0, 1, 2.$$  

It would be useful to have analytic expressions for the currents $J_i(\vec{\rho}; v, w)$ so that these relations could be checked; such expressions were obtained for the case $v = w = 1$ in [2], essentially by a coloring argument. However, for any given $\vec{\rho}, v$, and $w$ one may compute $J_i(\vec{\rho}; v, w)$ with good accuracy by Monte Carlo simulation of the uniform system on a large ring; we have done so and thus verified that (4.27) holds for a variety of choices of $\alpha$ and $w$.

**Remark 4.2.** A simple mean field calculation of the currents in a uniform two species system yields $J_1(\vec{\rho}; v, w) = \rho_1(\rho_0 + v\rho_2)$, $J_2(\vec{\rho}; v, w) = \rho_2(\rho_0 - v\rho_1)$, and $J_0(\vec{\rho}; v, w) = \rho_0(-\rho_1 - w\rho_2)$. These mean field values are in fact correct when $v = w = 1$, as is well known, but the Monte Carlo simulations referred to above show that they are not correct in general.

**References**

[1] M. R. Evans and T. Hanney. Nonequilibrium statistical mechanics of the zero-range process and related models. *J. Phys. A: Math Gen.* 38 (2005), R195–R240.

[2] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Stochastic interacting particle systems out of equilibrium. *J. Stat. Mech.* (2007), P07014, 35pp.

[3] H. Spohn, Long range correlations for stochastic lattice gases in a non-equilibrium steady state, *J. Phys A.* 16, 4275–4291 (1983).

[4] R. Schmitz, Fluctuations in nonequilibrium fluids, *Phys. Reports* 171, 1–58 (1988).

[5] R. A. Blythe and M. R. Evans, Nonequilibrium steady states of matrix-product form: a solver’s guide, *J. Phys. A: Math. Theor.* 40 (2007), R333–R441.

[6] E. D. Andjel, M. Bramson, and T. M. Liggett, Shocks in the asymmetric exclusion process, *Prob. Th. Rel. Fields* 78 (1988), 231–247.

[7] B. Derrida, S. A. Janowsky, J. L. Lebowitz, and E. R. Speer, Exact solution of the totally asymmetric simple exclusion process: shock profiles, *J. Stat. Phys.* 73 (1993), 813–842.
[8] E. R. Speer, The two species totally asymmetric exclusion process, in *On Three Levels: The Micro-, Meso-, and Macroscopic Approaches in Physics*, edited by M. Fannes, C. Maes and A. Verbeure, NATO ASI Series B: Physics 324, pp. 91–112, Plenum, New York, 1994.

[9] P. A. Ferrari, L. R. G. Fontes, and Y. Kohayakawa, Invariant measures for a two-species asymmetric process, *J. Stat. Phys.* 76 (1994), 1153–1177.

[10] B. Derrida, Systems out of equilibrium: some exactly solvable models, in *Statphys 19, the 19th IUPAP International Conference on Statistical Physics, Xiamen, China, July 31-August 4, 1995*, ed. B. Hao, Singapore, World Scientific, 1996.

[11] K. Mallick, Shocks in the asymmetric exclusion model with an impurity, *J. Phys. A.* 29 (1996), 5375–5386.

[12] C. Arita, Phase transitions in the two-species totally asymmetric exclusion process with open boundaries, *J. Stat. Mech.* P12008 (2006), 18pp.

[13] C. Arita, Exact analysis of two-species totally asymmetric exclusion process with open boundary conditions, *J. Phys. Soc. Jpn.* 75 (2006), 065003, 2pp.

[14] A. Ayyer, J. L. Lebowitz, and E. R. Speer, On the Two Species Asymmetric Exclusion Process with Semi-Permeable Boundaries, *J. Stat. Phys.* 135 (2009), 1009–1037.

[15] B. Derrida, M. R. Evans, V. Hakim, and V. Pasquier, Exact solution of a 1D asymmetric exclusion model using a matrix formulation, *J. Phys. A* 26 (1993), 1493–1517.

[16] T. M. Liggett, *Stochastic interacting systems: contact, voter and exclusion processes*, Springer-Verlag, Berlin, 1999.

[17] E. Duchi and G. Schaeffer, A combinatorial approach to jumping particles, *J. Combin. Theory A* 110 (2005), 1–29.

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