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On Problem of Efficient Determination of Elastic Critical Moment of Beams with Selected Types of Cross-Sections

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Abstract. Assessment of the lateral-torsional buckling resistance of slender metal beams with no intermediate restraints requires the determination of the critical moment. Nowadays, its magnitude can be found using numerical analysis e.g. by means of widely used finite element method but also available derived formulae for the calculation of the critical moment based on the mathematical solution of the eigenvalue problem of differential equations of bending are still of considerable importance. For some common cases of support and load conditions and some specific types of cross-sections of metal beams, they allow to calculate the desired magnitude of the critical moment required for the buckling resistance check practically and reliably. The paper focuses on problem of derivation of the elastic critical moment of beams of double symmetrical cross-sections and channels loaded perpendicularly to the axis of symmetry. Starting with the Vlasov’s theory of stability of thin-walled members and variational methods, the process of derivation of the critical moment is briefly described. Whereas in case of beams of prismatic cross-sections the application of this method can subsequently result in general formula for calculation of the critical moment for various support and load conditions, the solution for members with variable cross-sections is much more complex and requires application of specific methods. The paper deals with application of selected methods of numerical mathematics on problem of determination of the elastic critical moment of metal beams and, when possible, compares the obtained values with analytical solution. Special attention is paid to members with variable cross-sections where primarily numerical methods can be used. Based on comparison of results, suitability of the utilized methods applied on problem of lateral-torsional buckling of metal beams is evaluated with significant emphasis on members with variable cross-sections.

1. Introduction
Calculation of the critical moment is a crucial step of the assessment of the buckling resistance of slender metal beams. In case of beams with no intermediate restraints along the span out-of-plane, buckling (lateral-torsional buckling) might occur having two components of deformation: lateral displacement $v$ and rotation $\phi$ (figure 1). For an ideal beam (with no initial imperfections), this stability problem occurs when critical moment $M_c$ is reached. It plays an essential role for the process of the lateral-torsional buckling check and the magnitude of the critical moment is one of the important input values necessary for the assessment of the actual beam.
The stability problem of beams was dealt with by Prandtl [1], Timoshenko [2], Winter [3] and others. A significant contribution to the theory of thin-walled members was developed by Vlasov [4] who derived differential equations of stability of thin-walled members in bending and compression. The theory of stability of thin-walled members was later developed by Czech and Slovak specialists as well [5–8]. The stability of a channel with expression of a formula for calculation of the elastic critical moment was dealt with in detail in [9] and [10]. An energetic approach applied on the solution of the buckling problem of tapered members was focused on e.g. in [11] and [12].

In terms of members in bending (beams), the general problem is defined by three homogenous differential equations of fourth order (1), (2) and (3) with boundary conditions depending on the type of the supports [4]:

$$EI_y \cdot u'''' + (M_z \cdot \varphi)'' = 0$$  \hspace{1cm} (1)

$$EI_z \cdot v'''' + (M_y \cdot \varphi)'' = 0$$  \hspace{1cm} (2)

$$EI_\omega \cdot \varphi'''' + \left[2 \cdot b_y \cdot M_z - 2 \cdot b_z \cdot M_y - GI_t\right] \cdot \varphi'' + \left[q_z \cdot (e_z - a_z) + q_y \cdot (e_y - a_y)\right] \cdot \varphi + M_y \cdot v'' + M_z \cdot u'' = 0$$  \hspace{1cm} (3)

where $E$ is the Young modulus, $G$ is the shear modulus, $I_\ell$ and $I_e$ are the second moments of area, $I_\omega$ is the warping constant, $I_t$ is the torsion constant, $q_z$ and $q_y$ are the loads applied in the XZ and XY plane, respectively, $M_y$ and $M_z$ are the appropriate bending moments, $u$, $v$ and $\varphi$ are the components of deformation (lateral displacements and rotation), $a_y$ and $a_z$ are the coordinates of the shear centre related to the gravity centre, $e_y$ and $e_z$ are the coordinates of the point of load application related to the centre of gravity and $b_y$ and $b_z$ are the Wagner's coefficients.

The critical moment $M_{cr}$ can be found as a solution of the eigenvalue problem of the fourth order differential equations of bending with appropriate boundary conditions. For selected types of cross-sections, it can result in expression for calculation of the critical moment with certain factors depending on the type of load and support conditions. Practically applicable formulas for some usual cases of cross-sections and loads can be found e.g. in [5] and [7]. The Czech national annex to the standard [13] states a general formula (based on the eigenvalue problem solution) for calculation of the critical moment of beams with double symmetrical and monosymmetrical cross-sections for some common cases of loads with factors taking into account the types of the supports and variation of the bending moment along the span.
2. Derivation of the critical moment

A simply supported beam of a span of \( L \) with double symmetrical or monosymmetrical cross-section (channel) is considered (figure 2). As the cross-sections are symmetrical about \( y \) axis, the parameters \( a_z \) and \( b_z \) are both equal to zero. The load is applied in the XZ plane, which results in the bending moment \( M_y \) only provided the load passes the shear centre \( C_s \) of the cross-section. Considering these assumptions, the differential equations of the problem (4) and (5) with boundary conditions (6) are obtained. All derivatives are with variable \( x \) spanning from 0 (left support) to \( L \) (right support).

\[
EI_z \cdot v''' + (M_y \cdot \varphi)'' = 0
\]  
(4)

\[
EI_\omega \cdot \varphi''' - GL_y \cdot \varphi'' + q_z \cdot e_z \cdot \varphi + M_y \cdot v'' = 0
\]  
(5)

\[v(0) = v(L) = 0, v''(0) = v''(L) = 0, \varphi(0) = \varphi(L) = 0, \varphi''(0) = \varphi''(L) = 0\]

(6)

Figure 2. Cross-sections

In terms of the Vlasov’s solution [4], parameters \( a, b \) and \( c \) and the function \( \kappa \) are defined using following expressions:

\[a = \frac{EI_\omega}{L^2 \cdot GL_i}\]  
(7)

\[b = \frac{e_z \cdot EI_z}{L \cdot GI_i}\]  
(8)

\[c = \frac{EI_z}{GI_i}\]  
(9)

\[\kappa = \frac{L}{EI_z} \cdot M_y\]  
(10)

To simplify subsequent operation, the problem is transformed to new interval spanning from 0 to 1 instead of 0 to \( L \) using new variable \( \zeta \). The boundary conditions change accordingly. To approach to the eigenvalue problem solution, the Bubnov-Galerkin variational method [4] is applied. The function \( \varphi \) is substituted by new function (12) where \( C \) is an arbitrary constant.

\[x = \zeta \cdot L\]  
(11)

\[\varphi = C \cdot \sin \pi \zeta\]  
(12)
After application of this method on equations (4) and (5), modifications of the expressions, multiplying by \( \sin(\pi \zeta) \) and integration, one differential equation (13) is obtained. It is a general equation valid for beams with double symmetrical and monosymmetrical cross-sections. The procedure of modifications is in more detail described in [4].

\[
\pi^4 \cdot \int_0^1 a \cdot \sin^2 \pi \zeta \, d\zeta + \pi^2 \cdot \int_0^1 \sin^2 \pi \zeta \, d\zeta + \int_0^1 b \cdot \kappa'' \cdot \sin^2 \pi \zeta \, d\zeta - \int_0^1 c \cdot \kappa' \cdot \sin^2 \pi \zeta \, d\zeta = 0 \quad (13)
\]

2.1. Prismatic members

In case of prismatic members, the parameters \( a, b \) and \( c \) are constant due to consistent cross-section parameters of the beam along the span. Equation (13) can be simplified resulting in equation (14) [4].

\[
a \cdot \pi^2 + 1 + \int_0^1 \left(4 \cdot b \cdot \kappa + \frac{c}{\pi^2} \cdot \kappa^2\right) \cdot \cos 2\pi \zeta \, d\zeta - \int_0^1 \frac{c}{\pi^2} \cdot \kappa^2 \, d\zeta = 0 \quad (14)
\]

The function \( \kappa \) (10) is a function of \( \zeta \) and involves the variation of the bending moment along the span. The general procedure is to express the function of the bending moment for given type of load, calculation of the definite integrals and their substitution to equation (14) which results in quadratic equation with two roots (eigenvalues) being the critical loads. The critical moment is calculated using the positive one. The examples of the calculations for common cases of loads can be found in [14].

The formula for calculation of the critical moment \( M_{cr} \) can be generalized. One of the possible forms is given by the Czech national annex (NA) to the standard [13]. For beams with double symmetrical and monosymmetrical cross-sections (provided the load passes the shear centre) it is expressed by equation (15) where \( C_1 \) and \( C_2 \) are factors depending on support conditions and load types. Their values for selected common cases of loads can be found in tables in the national annex. The application of this formula is restricted to prismatic members.

\[
M_{cr} = \frac{\pi \cdot \sqrt{E \cdot I_z} \cdot G \cdot I_t}{L} \cdot C_1 \left(-C_2 \cdot \frac{\pi \cdot e_z \cdot \sqrt{E \cdot I_z}}{L \cdot \sqrt{G \cdot I_t}} + \frac{C_2^2 \cdot e_z^2 \cdot E \cdot I_z}{L^2 \cdot G \cdot I_t} + \frac{\pi^2 \cdot E \cdot I_{z}}{L^2 \cdot G \cdot I_t} + 1\right) \quad (15)
\]

2.2. Members with variable cross-sections

Members with variable cross-sections are characterized by variable cross-section parameters along the span, i.e. the parameters \( I_y, I_z, I_t, I_{z0} \) and \( e_z \) as well as the parameters \( a, b \) and \( c \) given by expressions (7), (8) and (9) are functions of \( x \) or \( \zeta \), respectively. It results in complexity of integrals in equation (13), which can therefore be efficiently solved e.g. using selected numerical methods.

3. Analysis

3.1. Numerical integration

For solution of the definite integrals in equation (13), composite Simpson method for numerical solution of definite integrals can be used. The method allows numerical calculation of a definite integral \( I \) on given interval using equation (16). The interval of solution is divided into \( n \) subintervals using equidistant nodes with length of one subinterval being \( H \) given by equation (17).

\[
I = \int_a^b f(x) \, dx = \frac{H}{3} \left[f(x_a) + 4 \cdot f(x_1) + 2 \cdot f(x_2) + 4 \cdot f(x_3) + ... + f(x_n)\right] \quad (16)
\]

\[
H = \frac{b - a}{n} \quad (17)
\]
The suitability of the numerical integration applied on problem of solution of the critical moment of beams with double symmetrical and monosymmetrical cross-sections defined by equation (13) is studied. Three types of beams with variable cross-sections along the span as shown in figure 3 are considered: two types of tapered beams and a beam with parabolic variation of the bottom flange. The interval of solution spanning from 0 to 1 is divided into 100 subintervals.

![Figure 3. Members with variable cross-sections: a) tapered beam (A), b) parabolic beam, c) tapered beam (B)](image)

Figure 4. Cross-section

The process of solution is explained on an example of constant variation of bending moment along the span of the simply supported beam (the ratio between the bending moments at both supports \( \psi \) is therefore equal to 1) with cross-section according to figure 4. First, prismatic member is studied, i.e. parameters \( a, b, \) and \( c \) are constant. The dimensions are \( b_f = 200 \) mm, \( t_f = 20 \) mm, \( t_w = 10 \) mm and \( h = 200 \) mm. The span is 10 m. Considering the constant bending moment along the span of the magnitude of \( M \), its second derivative is equal to zero and equation (13) is modified to equation (18). After numerical calculation of the definite integrals using composite Simpson method with cross-section parameters of the considered cross-section, quadratic equation (19) is obtained. Solution of this equation results in two roots – critical moments: \( M_c = \pm 231.36 \) kNm.

For comparison, solution according to the national annex [13], i.e. according to formula (15) with factors taken from tables in the standard (national annex) for appropriate load and support conditions, gives the identical magnitude of the critical moment 231.36 kNm.

As the comparison shows good agreement between results of both methods, the composite Simpson method is used for calculation of critical moment of the three above mentioned types of members with variable cross-sections. The ratio \( \delta \) between minimum and maximum depth of the cross-section \( h_{min} \)
and $h_{\text{max}} (\delta = h_{\text{min}} / h_{\text{max}})$ is set to 0.25, 0.50, 0.75 and 1.00 (prismatic member – for comparison with the solution according to [13]). Six types of loads are investigated: uniformly distribute load (UDL) and five types of moment load with different ratios $\psi$ between end moments (1.0, 0.5, 0.0, -0.5, -1.0) as shown in figure 5. The variations of selected cross-section parameters are shown in figure 6.

Figure 5. Types of loads and variations of bending moments

Figure 6. Variation of selected cross-section parameters along the span (second moment of area $I_z$, warping constant $I_\omega$, torsion constant $I_t$)

The variations of the bending moments along the span are considered according to figure 5. The moments are functions of $\zeta$. For each bending moment variation, the function $\kappa$ is expressed and applied to the equation (13). The integrals are calculated using the composite Simpson method to obtain the quadratic equations with critical loads being its roots. The positive one is used for calculation of the critical moment $M_{cr}$. The results are summarized in figure 7. The charts show the relationship between the ratio $\delta = h_{\text{min}} / h_{\text{max}}$ and the critical moment $M_{cr}$. 

3.2. Finite element method

For comparison and verification of results, selected cases were investigated using finite element method (FEM) system ANSYS 15.0 [15]. A simply supported steel beam of a span of 10 m with tapered cross-section \((h_{\text{min}} = 200 \text{ mm}, h_{\text{max}} = 800 \text{ mm}, \text{ratio } \delta = 0.25)\) and with fork support conditions (lateral displacement at the supports prevented, warping not prevented) was considered. The model was created using SHELL181 finite elements with edge size 20 mm. The material model was considered as ideal elastic (Young’s modulus 210 GPa, Poisson’s ratio 0.3). Two cases were investigated: uniformly distributed load of a magnitude of 1 kN/m and constant moment load along the span of the beam of a magnitude of 1 kNm. The process of the numerical analysis consisted of linear static analysis (LA) and linear buckling analysis (LBA) which provided eigenvalues and buckling modes. The first positive eigenvalue was considered as critical load and was used to calculate the critical moment by means of principles of structural mechanics. Selected results of the finite element analysis of the considered tapered member are shown in figure 8 (eigenvalues for uniformly distributed load and constant moment load along the span).

![Figure 8. Eigenvalues (constant moment load, uniformly distributed load)](image)

4. Results and discussions

The results of selected investigated cases are compared in table 1. In case of the tapered member (type A in figure 3) with ratio \(\delta = 0.25\), results of numerical integration and finite element method are summarized. For prismatic members (\(\delta = 1.00\)), comparison of three methods is performed: numerical
integration, finite element method and solution according to [13] (which use is restricted to prismatic members). The table contains the results for uniformly distributed load and constant moment load along the span of the beam.

**Table 1.** Comparison of results – critical moments

| Tapered member \((\delta = 0.25)\) | Prismatic member \((\delta = 1.00)\) |
|----------------------------------|----------------------------------|
| **Numerical integration** | **Finite elem. method** | **Numerical integration** | **Finite elem. method** |
| **Unif. distr. load** | 246.35 | 229.64 | 234.83 | 214.96 | 234.73 |
| **Const. mom. load** | 273.42 | 254.65 | 231.36 | 213.35 | 231.36 |

The results for prismatic members show very good agreement between the calculations using numerical integration and calculation of the critical moment according to [13]. The finite element method provided slightly lower magnitudes of the critical moments. This difference might be caused by the fact that the numerical integration as well as the calculation according to [13] are based on differential equations of bending neglecting possible small effects of local instabilities. Stability problems of metal members are always mixed usually with certain prevailing buckling mode [5]. As the referenced differential equations of bending were developed for global stability problems of metal members (provided no distortion of the cross-section itself occurs) they do not take into account possible effects of local buckling. Nevertheless, their application of usual (especially hot-rolled) members is relevant. The process of calculation of the critical moment according to [11] has been widely used. The finite element method takes into consideration the behaviour of the investigated member as a whole and therefore the possible local instabilities are included which (in the investigated cases) might result in slightly lower critical moments. This effect was also observed e.g. in [16]. The comparison of results obtained from the analysis of tapered members indicates similar behaviour with certain difference between results of numerical integration (based on differential equations of bending) and results obtained from the analysis performed using finite element method. The difference is approximately 7%.

Performed calculations indicate that for uniformly distributed load the difference between critical moments of prismatic beams and beams with variable cross-sections are relatively low. The variations of the critical moments and the ratio \(\delta\) are shown in figure 7. In case of tapered member (A), the critical moment of a beam with the ratio \(\delta = 0.25\) is approximately 105% of the critical moment of a prismatic beam \((\delta = 1.00)\), i.e. the difference is approximately 5%. In cases of parabolic beams and tapered beams (B) this difference is 14% or 15%, respectively. Greater difference was observed in cases of beams with constant moment load. For tapered beams (A), this difference varies from 16% to 18%, for parabolic beams and tapered beams (B), the difference is between 34% and 37%. The increase of the critical moment is therefore more significant.

5. **Conclusions**

The paper focuses on lateral-torsional buckling problem of metal beams with double symmetrical and monosymmetrical cross-sections, especially on problem of determination of the elastic critical moment, which is necessary for buckling resistance check. In the frame of the paper, attention was paid to critical moment of members with variable cross-sections and their comparison with critical moments of prismatic beams. The influence of the variable cross-section of the beam on the magnitude of the elastic critical moment was quantified using selected methods – numerical integration and finite element method based analysis applied on selected cases of loads and cross-section variations. Both methods provided comparable results. The application of the numerical integration allows to efficiently solve various cases of cross-section variations and load types by change of selected parameters e.g. in commonly available table processors. The quantification and evaluation of the differences between the magnitudes of the critical moments of prismatic beams and beams with variable cross-sections were performed for selected examples of cross-sections.
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References

[1] L. Prandtl, “Kipperscheinungen (Lateral-torsional buckling)”, 1899. Cit. in: V. Březina, “Vzpěrná únosnost kovových prutů a nosníků (Buckling Resistance of Metal Members)”, Nakladatelství Československé akademie věd (Czechoslovak Academy of Sciences Publishing), Prague, 1962.

[2] S. Timoshenko, “Einige Stabilitätsprobleme der Elastizitätstheorie (Some Stability Problems of Theory of Elasticity)”, Zeitschrift für Mathematik und Physik, 1910. Cit. in: V. Březina, “Vzpěrná únosnost kovových prutů a nosníků (Buckling Resistance of Metal Members)”, Nakladatelství Československé akademie věd (Czechoslovak Academy of Sciences Publishing), Prague, 1962.

[3] G. Winter, “Lateral Stability of Unsymmetrical I-beams and Trusses in Bending”, Trans. ASCE, 1943. Cit. in: V. Březina, “Vzpěrná únosnost kovových prutů a nosníků (Buckling Resistance of Metal Members)”, Nakladatelství Československé akademie věd (Czechoslovak Academy of Sciences Publishing), Prague, 1962.

[4] V. Z. Vlasov, “Tenkostěnné pružné pruty (Thin-Walled Elastic Members)”, Státní nakladatelství technické literatury (State Publishing of Technical Literature), Prague, 1962.

[5] V. Březina, “Vzpěrná únosnost kovových prutů a nosníků (Buckling Resistance of Metal Members)”, Nakladatelství Československé akademie věd (Czechoslovak Academy of Sciences Publishing), Prague, 1962.

[6] A. Mrázík, J. Gruska, “Výpočet tenkostenných prútov (Calculation of Thin-Walled Members)”, Vydavateľstvo Slovenskej akadémie vied (Slovak Academy of Science Publishing), Bratislava, 1965.

[7] J. Melcher, “Ohyb, kroucení a stabilita ocelových nosníků (Bending, Torsion and Stability of Steel Members)”, Vysoké učení technické v Brně (Brno Univ. of Technology), Brno, 1975.

[8] I. Baláž, Y. Koleková, “Critical moments”, Stability and Ductility of Steel Structures, pp. 31–38, 2002, ISBN: 963-05-7950-2.

[9] J. Melcher, “Stabilita ohybu tenkostěnného nosníku průřezu U (Stability of Bending of a Thin-Walled Channel)”, Inženýrské stavby, vol. 18 (2), pp. 98–107, 1970.

[10] J. Melcher, “Kippen von Trägern als Stabilitätsproblem zweier Gruppen von Querschnitttypen (Lateral beam buckling as a stability problem of two groups of cross-section types)”, Stahlbau, vol. 68 (1), pp. 24–29, 1999, ISSN: 0038-9145, DOI: 10.1002/stab.199900060.

[11] N. Trahair, “Lateral buckling of tapered members”, Engineering Structures, vol. 151, pp. 518–526, 2017, ISSN: 0141-0296, DOI: 10.1016/j.engstruct.2017.08.038.

[12] L. Zhang, G. S. Tong, “Lateral buckling of web-tapered I-beams: A new theory”, Journal of Constructional Steel Research, vol. 64 (12), pp. 1379–1393, 2008, ISSN: 0143-974X, DOI: 10.1016/j.jcsr.2008.01.014.

[13] ČSN EN 1993-1-1, “Eurocode 3: Design of steel structures – Part 1-1: General rules and rules for buildings”, Český normalizační institut (Czech Standard Institute), Prague, 2006.

[14] I. Baláž, J. Melcher, “Lateral-torsional buckling of beams of monosymmetrical cross-sections loaded perpendicularly to the axis of symmetry: Theoretical analysis”, ce/papers, vol. 1 (2-3), pp. 1086–1095, 2017, ISSN: 2509-7075, DOI: 10.1002/cropa.149.

[15] ANSYS® Academic Research, Release 15.0.

[16] I. Baláž, J. Melcher, “Stability of Thin-Walled Beams with Lateral Continuous Restraint”, Transactions of the VŠB – Technical University of Ostrava, Civil Engineering Series, vol. 15 (1), pp. 1–10, 2015, ISSN: 1804-4824, DOI: 10.1515/tvsb-2015-0001.