ON TAME, PET, DOMESTIC, AND MISERABLE IMPARTIAL GAMES

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Abstract. Playing impartial games under the normal and misère conventions may differ a lot. However, there are also many “exceptions” for which the normal and misère plays are very similar. As early as in 1901 Bouton noticed that this is the case with the game of Nim. In 1976 Conway introduced a large class of such games that he called tame games. Here we introduce a proper subclass, pet games, and a proper superclass, domestic games. For each of these three classes we provide an efficiently verifiable characterization based on the following property. These games are closely related to another important subclass of the tame games introduced in 2007 by the first author and called miserable games. We show that tame, pet, and domestic games turn into miserable games by “slight modifications” of their definitions. We also show that the sum of miserable games is miserable and find several other classes that respect summation. The developed techniques allow us to prove that very many well-known impartial games fall into classes mentioned above. Such examples include all subtraction games, which are pet; game Euclid, which is miserable (and, hence, tame), as well as many versions of the Wythoff game and Nim, which may be miserable, pet, or domestic.

1. Sprague-Grundy theory of impartial games

Combinatorial games were analyzed in the comprehensive books [3] and earlier in [14]; an introductory theory can be found in [1, 35]. Readers familiar with the subject can skip this section. We restrict ourselves to a special case. A game is called

- impartial if both players have the same possible moves in each position;
- acyclic if each position can be visited at most once;
- finite if the set of positions is finite;

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- locally finite if the subgame defined by any fixed initial position is finite.

In this paper we consider only the locally finite acyclic impartial games of two players, calling them simply games, for brevity. We say that a game is played under the normal (resp., misère) convention if the player who makes the last move wins (resp., loses). We will consider both.

Games are modeled by finite acyclic digraphs whose vertices are interpreted as positions and arcs as moves. In case there exists a move \((x, y)\) from a position \(x\) to \(y\), we write \(x \to y\) and say that \(x\) is movable to \(y\), or \(y\) is reachable from \(x\), or \(y\) is an option of \(x\).

Similarly, given two sets of positions \(U, W \subseteq V\), we say that \(U\) is movable to \(W\) or that \(W\) is reachable from \(U\), if from every position \(x \in U\) there is a move to some position \(y \in W\).

A position without available moves is called terminal. The set of terminal positions is denoted by \(V_T\). A position is called an \(N\)-position (resp., a \(P\)-position) if the next (resp., previous) player wins when both players play optimally starting from that position.

Given a set \(S\) of non-negative integers, the minimum excludant of \(S\), denoted by \(\text{mex}(S)\), defined as the least non-negative integer that is not in \(S\). In particular, \(\text{mex}(S) = 0\) whenever \(0 \notin S\), for example, if \(S = \emptyset\).

The Sprague-Grundy (SG) function of a game \(G\), denoted by \(G\), is defined recursively as follows
\[
G(x) = \text{mex}\{G(y) \mid y \text{ is an option of } x\}.
\]

The value \(G(x)\) is called the SG value, or alternatively the nim-value, of position \(x\). By the above definition, \(G(x) = 0\) whenever \(x\) is a terminal position. It is both obvious and well-known \([36, 37, 22]\) that the SG values are characterized as follow.

**Lemma 1.** We have \(G(x) = n\) if and only if the next two conditions hold:

(i) \(G(y) \neq G(x)\) whenever there is a move from \(x\) to \(y\), in particular, \(G(y) \neq n\) if \(G(x) = n\);

(ii) for each integer \(k\) such that \(0 \leq k < n\) there exists a move \(x \to y\) such that \(G(y) = k\).

In particular, the \(P\)-positions are exactly the zeros of the SG function. □

Given two games \(G\) and \(H\), their disjunctive sum \(G + H\) is defined as a game in which every move consists of choosing one game and making
a move in it. The SG function of the sum is characterized by the following well-known statement. Let $\oplus$ be the bitwise addition in the binary number system without carrying, or in other words, the bitwise mod2 addition.

**Theorem 1.** [22, 36, 37] The SG value of the position $(x, y)$ in the sum $G + H$ is $G(x) \oplus G(y)$. □

This result can be obviously extended to the sums of $k$ games for any integer $k \geq 2$, since $\oplus$ is an associative and commutative operation.

The mis`ere SG value $G^-(x)$ of a position $x$ in a game $G$ is defined by the same recursion (1), but the initialization is different: $G^-(x) = 1$ (rather than $G^-(x) = 0$) for all terminal positions $x \in V_T$.

**Remark 1.** For an individual game, the mis`ere version can be easily reduced to the normal one by the following simple transformation of the graph $G = (V, E)$. Add to $V$ one new position $x_T$ and an arc $(x, x_T)$ from each former terminal position $x \in V_T$ to $x_T$. Thus, $x_T$ becomes a unique terminal in the obtained graph $G^-$. It is easy to verify that for every position $x \in V$ its mis`ere SG value $G^-(x)$, in the original digraph $G$, equals the normal SG value $G(x)$ the extended digraph $G^-$. However, then the mis`ere version of a sum and the sum of the mis`ere versions of the summands are not the same. In the first case we add only one new terminal position for the whole sum, while in the second case we have to add one for each game summand.

As early as in 1956 Grundy and Smith [23] noticed that playing a game under the mis`ere convention may be difficult in general. In this paper we focus on the exceptions, that is, on the games for which functions $G$ and $G^-$ are closely related.

### 2. Main concepts and results

A position $x$ will be called an $i$-position (resp., an $(i, j)$-position) if $G(x) = i$ (resp., if $G(x) = i$ and $G^-(x) = j$). We will denote by $V_i$ (resp., $V_{i,j}$) the set of $i$-positions (resp., $(i, j)$-positions). A position $x \in V_{0,1} \cup V_{1,0}$ will be called a swap position.

**Definition 1.** An impartial game will be called

(i) *domestic* if it has neither $(0, k)$-positions nor $(k, 0)$-positions with $k \geq 2$;

(ii) *tame* if it has only $(0, 1)$-positions, $(1, 0)$-positions, and $(k, k)$-positions with $k \geq 0$;

(iii) *pet* if it has only $(0, 1)$-positions, $(1, 0)$-positions, and $(k, k)$-positions with $k \geq 2$. 
Tame games were introduced in [14, Chapter 12] (see page 178), pet games were introduced recently in the preprints [25, 26], while domestic games are introduced in this paper. According to the above definitions, domestic, tame, and pet games form nested classes: a pet game is tame and a tame game is domestic. Furthermore, both containments are strict. Figures 1, 2, 3, and 4 distinguish these three classes.

**Figure 1.** This game is not domestic since it contains a (2,0)-position.

**Figure 2.** This game is domestic but not tame since it contains a (1,2)-position.

**Figure 3.** This game is tame but not pet since it contains a (0,0)-position; also it is miserable but not strongly miserable.

**Figure 4.** This game is pet.

The following two “technical” properties appear to be closely related to the above three classes of games.
Definition 2. A game $G$ is said to be
(i) *forced* if each move from a $(0, 1)$-position results in a $(1, 0)$-position and vice versa;
(ii) *returnable* if the following, weaker, implications hold: let $x$ be a $(0, 1)$-position (resp., a $(1, 0)$-position) movable to a non-terminal position $y$, then $y$ is movable to a $(0, 1)$-position (resp., to a $(1, 0)$-position).

Obviously, the forced games are returnable. Figures 5 and 6 give examples of a non-returnable game and a returnable game that is not forced, respectively. Note that the games in Figures 1, 2, 3, and 4 are all forced.

![Figure 5](image5.png)

**Figure 5.** This game is not returnable.

![Figure 6](image6.png)

**Figure 6.** This game is returnable but not forced.

Definition 3. For each position $x$ let us consider the following properties:
(a) $x$ is a swap position, $x \in V_{0,1} \cup V_{1,0}$;
(b) $x$ is not movable to $V_{0,1} \cup V_{1,0}$;
(c) $x$ is movable to $V_{0,1}$ and to $V_{1,0}$ simultaneously;
(d) $x$ is movable to $V_{0,1}$ and to $V_{0,0}$ simultaneously;
(e) $x$ is movable to $V_{1,0}$ and to $V_{0,0}$ simultaneously;
(f) $x$ is movable to $V_{0,0}$ and to $V_{1,1}$ simultaneously.

A game is called
(i) *strongly miserable* if either (a) or (c) hold for every position;
(ii) miserable if (a), or (b), or (c) hold for every position;
(iii) \( t \)-miserable if (a\( _0 \)), or (c), or (c) hold for every position;
(iv) weakly miserable if (a), or (b), or (c), or (c\( _0 \)), or (c\( _1 \)) hold for every position.

The classes of miserable and strongly miserable games were introduced in [24] and [25, 26], respectively. It is not difficult to verify that four classes of Definition 3 are nested. Furthermore, three examples in Figures 2, 3, and 8 show that the containments are strict.

The paper is organized as follows. In Section 3 we show that a game is domestic, tame, or pet if and only if it is weakly miserable, \( t \)-miserable, or strongly miserable, respectively.

Let us note, however, that these effective characterizations in terms of “miserability” still do not provide efficient membership tests, because verifying properties of Definition 3 requires knowledge of sets \( V_{0,1} \), \( V_{1,0} \), \( V_{0,0} \), and \( V_{1,1} \) that are defined recursively. In Section 4 we reformulate “slightly” these properties to obtain an efficient way to verify the membership in the classes of domestic, tame, and pet games.

We say that a class of games is preserved under summation if the sum of games from this class also belongs to it. In Section 5 we prove that the classes of tame games, miserable games, forced and miserable games, returnable and miserable games are preserved under summation, while pet and domestic games are not. For tame games the result was stated in [14] and proved in [35]; we provide a simpler proof.

In Section 6 we apply the results of Section 4 for several well-known classes of games, including SUBTRACTION GAMES, EUCLID, NIM, WYTHOFF, as well as for several modifications and generalizations of these games.

3. Containment and equalities

3.1. Summary. The following classes of games are shown to be identical:

- domestic games and weakly miserable games;
- tame games and \( t \)-miserable games;
- pet games and strongly miserable games.

Furthermore, the following strict containments hold:

- the pet (strongly miserable) games are miserable and the latter are tame.

We illustrate relations between the six considered classes by the diagram in Figure 7
strongly miserable games = pet games
miserable games
tame games = \(t\)-miserable games

Figure 7. The diagram of containments.

The following concept will be instrumental. Given a position \(x\) of a game \(G\), we denote by \(d(x)\) the greatest number of successive moves from \(x\) to the terminal position. Let us denote by \(G_{[x]}\) the subgame of \(G\) defined by the initial position \(x\). Obviously, \(G_{[x]}\) contains \(x\) and all positions that can be reached from \(x\) (by one or several moves; recall that this set is finite) and all arcs between these positions.

3.2. Domestic games and weakly miserable games coincide.

Lemma 2. In a domestic game, from each \((1, 0)\)-position there is a move to a \((0, 1)\)-position and from each non-terminal \((0, 1)\)-position there is a move to a \((1, 0)\)-position.

Proof. From each \((1, 0)\)-position (resp., non-terminal \((0, 1)\)-position) there is a move to a \((0, k)\)-position (resp., \((k, 0)\)-position); obviously, \(k \neq 0\) (resp., \(k \neq 1\)). Furthermore, \(k \leq 1\) since the game is domestic.

Theorem 2. A game is weakly miserable if and only if it is domestic.

Proof. 
\(\Rightarrow\) Assume that \(G\) is weakly miserable but not domestic. Let \(x\) be a \((0, k)\)-position with \(k \geq 2\) for which \(d(x)\) takes the smallest possible value. Then, there is a move from \(x\) to a \((k', 0)\)-position \(x'\). Since \(d(x') < d(x)\), from our assumption we conclude that \(G_{[x']}\) is domestic and, hence, \(k' \leq 1\). Furthermore, \(k' = \mathcal{G}(x') \neq 0\) since \(\mathcal{G}(x) = 0\) and \(x\) is movable to \(x'\); hence, \(k' = 1\). Thus, \(x'\) is a \((1, 0)\)-position and (b) fails for \(x\). Note that (a) does not hold for \(x\) either.

Similarly, \(x\) is movable to no position \(y\) with \(\mathcal{G}(y) = 0\), because \(\mathcal{G}(x) = 0\). Therefore, (c), \((c_0)\), and \((c_1)\) fail for \(x\), resulting in a contradiction. Thus, \(G\) is domestic.
The case when $x$ is a $(k,0)$-position, rather than $(0,k)$-position, is similar.

($\Leftarrow$) Assume that $G$ is domestic. If $x$ is a swap position, then (a) holds for $x$. If $x$ is a $(0,0)$-position or a $(1,1)$-position, then (b) holds for $x$. If $x$ is an $(a,b)$-position such that $\max(a,b) \geq 2$, then $\min(a,b) \geq 1$, because $G$ is domestic.

Without loss of generality, assume that $a \leq b$. Since $a \geq 1$ and $b \geq 1$, there is a move from $x$ to a $(0, i)$-position $y$ and to a $(j,0)$-position $z$. Then, $i \leq 1$ and $j \leq 1$, because $G$ is domestic. If $i = 1$ and $j = 1$, then (c) holds for $x$. Otherwise, $x$ is movable to a $(0,0)$-position ($\star$).

If (b) fails for $x$, then $x$ is movable to either a $(0,1)$-position or a $(1,0)$-position ($\star \star$). By ($\star$) and ($\star \star$), either (c$_0$) or (c$_1$) holds for $x$. Hence, the game is weakly miserable. □

3.3. Tame games and $t$-miserable games coincide.

**Theorem 3.** A game $G$ is tame if and only if it is $t$-miserable.

**Proof.**

($\Rightarrow$) Let us assume that $G$ is tame and prove that for every position $x$ at least one of three properties (a$_0$), (c), (e) holds.

Furthermore, (a$_0$) holds for $x$ if $x$ is either a swap, or a $(0,0)$-position or a $(1,1)$-position. Assume that $x$ is a $(k,k)$-position for some $k \geq 2$. By Lemma 1 and its misère version, there are moves from $x$: to a $(0,i')$-position $x'$, to a $(1,i''')$-position $x''$, to a $(i'''',0)$-position $x'''$, and to a $(i''''',1)$-position $x''''$. Furthermore, $\max(i', i'', i'''', i''''') \leq 1$, since the game is tame.

If $i' = 1$ and $i'' = 0$, (c) holds. If $i' = 0$ and $i'' = 1$, (e) holds. If $i' = 1$ and $i'' = 1$, we consider $x'''$. If $i''' = 1$, (e) holds; otherwise, (e) holds. If $i' = 0$ and $i'' = 0$, consider $x''''$. If $i'''' = 1$, (e) holds; otherwise, (e) holds.

($\Leftarrow$) Let us assume that (a$_0$), or (c), or (e) holds for every position and prove by induction on $d(x)$ that each $x$ is either a $(k,k)$-position for some $k \geq 0$ or a swap position. Note that the claim holds when $d(x) \leq 1$. Indeed, $d(x) = 0$ if and only if position $x$ is terminal; in this case $x$ is a $(0,1)$-position. Furthermore, $d(x) = 1$ if and only if every move from $x$ results in a terminal position; in this case $x$ is a $(1,0)$-position.

Let us proceed by induction. Assume that the claim holds for every position $x$ with $d(x) \leq n$, for some $n \geq 1$, and prove it for $x$ with $d(x) = n + 1$.

Assume that (a$_0$) fails for an $(a,b)$-position $x$. Then, obviously, $a \geq 2$ or $b \geq 2$. Without loss of generality, assume that $a \geq 2$ and consider
two sets

\[ M = \{G(y) \mid y \text{ is a option of } x\} \] and \[ M^- = \{G^-(y) \mid y \text{ is a option of } x\}. \]

If (e) or (e) holds for \( x \), both \( M \) and \( M^- \) contain both 0 and 1. Furthermore, if \( y \) is a option of \( x \) and \( y \notin V_{0,1} \cup V_{1,0} \cup V_{0,0} \cup V_{1,1} \), then \( y \) is a \((k, k)\)-position for some \( k \geq 2 \) by the inductive hypothesis. Therefore, \( M = M^- \), implying that \( G(x) = \text{mex}(M) = \text{mex}(M^-) = G^-(x) \) and, hence, \( x \) is a \((k, k)\)-position for some \( k \geq 0 \).

3.4. Miserable games are tame.

Theorem 4. A miserable game is tame.

This statement was announced in [24] and shown in [25, 26]. Here we provide simpler arguments.

Proof. Assume that \( G \) is miserable and prove by induction on \( d(x) \) that every position \( x \) is either a swap position or a \((k, k)\)-position for some \( k \geq 0 \).

The case \( d(x) \leq 1 \) was already considered in the proof of Theorem 3 above (if \( d(x) = 0 \) then \( x \) is a \((0, 1)\)-position; if \( d(x) = 1 \) then \( x \) is a \((1, 0)\)-position).

Let us assume that the claim holds for every position \( x \) with \( d(x) \leq n \), for some \( n \geq 1 \), and prove that it holds for every position \( x \) with \( d(x) = n + 1 \).

Since \( G \) is miserable, (a), or (b), or (c) holds for \( x \).

(i) If (a) holds, \( x \) is a swap position and we are done.

(ii) If (b) holds, by the inductive hypothesis, each option \( y \) of \( x \) is a \((k_y, k_y)\)-position for some \( k_y \geq 0 \). Therefore, \( x \) is a \((k, k)\)-position in which

\[ k = \text{mex}\{k_y \mid y \text{ is a option of } x\}. \]

(iii) If (c) holds, by the inductive hypothesis, each option \( y \) of \( x \) is either a swap position or a \((k_y, k_y)\)-position for some \( k_y \geq 0 \). Therefore, \( x \) is a \((k, k)\)-position in which

\[ k = \text{mex}\{0, 1, k_y \mid y \text{ is a option of } x \text{ and } y \text{ is a } \((k_y, k_y)\)-position\}. \]

Note that in this case, \( k \geq 2 \).

\[ \square \]

Figure 8 provides a tame game that is not miserable showing that the containment of Theorem 4 is strict.
3.5. Pet games and strongly miserable games coincide. These pet games can be characterized in many equivalent ways; the following list was suggested in [26].

Theorem 5. The following properties of a game $G$ are equivalent.

(i) $G$ is strongly miserable.

(ii) $G$ is pet.

(iii) $G$ has no $(0,0)$-position.

(iv) $G$ has neither $(0,0)$-position nor $(1,1)$-position.

(v) If $G(x) = 0$ and $x$ is not terminal then $x$ is movable to some $x'$ with $G(x') = 1$.

(vi) If $G^-(x) = 0$ then $x$ is movable to some $x'$ with $G^-(x') = 1$.

Interestingly, property (v), claiming that any non-terminal 0-position is movable to a 1-position, was introduced (for some other purposes) already in 1974 by Ferguson [16] who proved that it holds for all subtraction games; see Section 6.

Some proofs were given in [26]. Here we give the complete analysis.

Proof of Theorem 5

(i) $\Rightarrow$ (ii). Every strongly miserable game is miserable and hence tame, by Theorem 4. It remains to show that $G$ has neither $(0,0)$-position nor $(1,1)$-position. Indeed, assume that $x$ is such a position. Then, properties (a) and (c) of Definition 3 fail for $x$, which is contradiction.

(ii) $\Rightarrow$ (i). Let $x$ be a non-swap position of $G$. Since $G$ is pet, $x$ is a $(k,k)$-position for some $k \geq 2$. By Lemma 1 and its misère version, there are moves from $x$ to a $(0,i)$-position and to a $(j,0)$-position. Since $G$ is pet, $i = j = 1$. Thus, (c) holds for $x$.

(ii) $\Rightarrow$ (iii). This implication is straightforward.

(iii) $\Rightarrow$ (ii). Assume that $G$ has no $(0,0)$-position and prove by induction on $d(x)$ that every position $x$ is either a swap position or a $(k,k)$-position for some $k \geq 2$. Standardly, the claim can be verified for the case $d(x) \leq 1$. 
Suppose that position $x$ is a counterexample with the smallest value of $d(x)$. The following case analysis results in a contradiction:

(a) Case 1: $x$ is a $(1,1)$-position. Then $x$ is movable to a $(0,e)$-position $x_1$ with $e \neq 1$. Since $d(x_1) < d(x)$, our choice of $x$ implies $e = 1$, which is impossible.

(b) Case 2: $x$ is a $(0,a)$-position (the case where $x$ is a $(a,0)$-position is treated similarly) with $a \geq 2$. Then $x$ is movable to some $(e,1)$-position $x_2$ with $e \neq 0$. Since $d(x_2) < d(x)$, our choice of $x$ implies $e = 0$, which is impossible.

(c) Case 3: $x$ is a $(b,c)$-position with $1 \leq b < c$. Then, there must be three options $x_3, x_4, x_5$ of $x$ such that

- $x_3$ is a $(0,i)$-position for some $i \geq 1$,
- $x_4$ is a $(j,0)$-position for some $j \geq 1$, and
- $x_5$ is a $(k,b)$-position for some $k \geq 0$

By the choice of $x$, we have $j = 1$, and hence, $b \geq 2$. Furthermore, since $b \geq 2$ and $d(x_5) < d(x)$, we have $k = b$ or equivalently $G(x_5) = G(x)$, which is impossible.

(iii) $\Leftrightarrow$ (iv). We already proved that (iii) $\Rightarrow$ (ii). Furthermore, (ii) $\Rightarrow$ (iv) results immediately from the definition of pet games. Thus, (iii) $\Rightarrow$ (iv) holds.

(ii) $\Rightarrow$ (v) (resp., (vi)). Assume that $G$ is pet. Let $x$ be a position with $G(x) = 0$ (resp., $G^-(x) = 0$). Since $G$ is pet, $x$ must be a $(0,1)$-position (resp., $(1,0)$-position). Since $x$ is not a terminal position, it is movable to a $(l,0)$-position (resp., to a $(0,l)$-position) for some $l$. Since $G$ is pet, we have $l = 1$, as required.

(v) $\Rightarrow$ (ii). Assume that (v) holds for a game $G$ that is not pet. Then, $G$ contains a position $x$ that is neither swap nor a $(k,k)$-position for any $k \geq 2$. Due to symmetry, we can assume that $x$ is either

- (1) a $(0,0)$-position, or
- (2) a $(1,1)$-position, or
- (3) an $(m,n)$-position with $0 \leq m < n$ and $n \geq 2$.

As usual, let us choose such an $x$ with the smallest $d(x)$. Then,

\begin{itemize}
  \item [(\star)] every position $x'$ with $d(x') < d(x)$ is a swap or a $(k,k)$-position for some $k \geq 2$
\end{itemize}

In case (1) (resp., (2)), $x$ is movable to a position $x'$ with $G(x') = 1$, by (v) (resp., $G(x') = 0$, by the SG Theorem). Then, $x'$ is a $(1,0)$-position (resp., a $(0,1)$-position), by (\star) and the assumption $d(x') < d(x)$. Hence, $G^-(x') = 0 = G^-(x)$ (resp., $G^-(x') = 1 = G^-(x)$), resulting in a contradiction.
Since $n \geq 2$, in case (3) there are moves $x \to x'$ and $x \to x''$ such that $G^-(x') = 0$ and $G^-(x'') = 1$, by Lemma 1 and its misère version. Since $d(x') < d(x)$ and $d(x'') < d(x)$, by ($\star$) we conclude that $x'$ and $x''$ are a $(1,0)$-position and $(0,1)$-position, respectively. Hence, $m \geq 2$. Since $G^-(x') = n > m$, there exists a move $x \to x'''$ such that $G^-(x''') = m$, that is, $x'''$ is a $(r,m)$-position for some $r$. Since $d(x''') < d(x)$ and $m \geq 2$, by ($\star$) we have $r = m$. Thus, that $G(x) = m = G^-(x''')$, resulting in a contradiction.

(vi) \implies (ii). This case is similar to the case (v) \implies (ii).

\section*{4. Constructive characterizations of domestic, tame, miserable, and strongly miserable games}

\subsection*{4.1. A general plan.}

We could make use of Definitions 1 and 3 to verify whether a game is miserable or strongly miserable, but to do so we have to know its swap positions. It may be even more difficult to verify membership in the other considered classes, because the sets $V_{0,0}$ and/or $V_{1,1}$ become also involved. Since the SG values are defined recursively, it looks difficult to guarantee in advance that a given subset contains all, for example, $(0,1)$-positions; see Definition 3.

To avoid this problem and obtain constructive characterizations, we will modify Definitions 1, 3 and obtain Theorems 6, 7, 8, 9 characterizing strongly miserable (pet), miserable, $t$-miserable (tame), weakly miserable (domestic) games, respectively. In these theorems, sets $V_{0,0}$, $V_{1,0}$, $V_{0,1}$ of Definition 3 are replaced by some “abstract” sets $V'_{0,0}$, $V'_{1,0}$, $V'_{0,1}$, $V'_{1,1}$. Requiring (almost) the same properties from these sets, we characterize all above classes and show that the old and new sets are equal, that is, $V'_{i,j} = V_{i,j}$ for all $i, j \in \{0,1\}$.

We will prove only Theorem 6; the remaining three theorems can be proven in a similar way and we leave them to the reader.

\subsection*{4.2. Strongly miserable games.}

Let us begin with the strongly miserable (pet) games.

**Theorem 6.** A game $G$ is strongly miserable if and only if there exist two disjoint sets $V'_{0,1}$ and $V'_{1,0}$ satisfying the following conditions:
Let us assume that it holds for every position \( \leq x \) specifically, it means that there is no move from \( x \) to terminal position. Moreover, there are no other moves from \( x \) and so \( V \) contains \( V'_1 \). By Theorem 5.

Proof. The “only if” part is straightforward, by setting \( V'_0 = V_0 \) and \( V'_1 = V_1 \). Let us prove the “if” part. Actually, it is enough to prove that \( V'_0 = V_0 \) and \( V'_1 = V_1 \). It then follows from condition \( SM(v) \) that the game does not have \((0, 0)\)-position, and so it is strongly miserable by Theorem 5.

As usual, we proceed by induction on \( d(x) \) to show the following claims:

1. If \( x \) is a \((0, 1)\)-position, then \( x \in V'_0 \);
2. If \( x \) is a \((1, 0)\)-position, then \( x \in V'_1 \);
3. If \( x \) is not a swap position then \( x \) is a \((k, k)\)-position for some \( k \geq 2 \) and, moreover, \((c') \) holds for \( x \).

If \( d(x) = 0 \) then \( x \) is a terminal position and (1) holds, since \( V'_0 \) contains \( V_0 \). If \( d(x) = 1 \) then \( x \) is a \((1, 0)\)-position that is movable to terminal position. Moreover, there are no other moves from \( x \). In particular, it means that there is no move from \( x \) that terminates in \( V'_1 \). The condition \( SM(v) \) implies that \( x \in V'_0 \). By (i), \( x \notin V'_0 \) and so \( x \in V'_1 \). Thus (2) holds for \( x \).

The claims (1) – (3) are standardly verified for \( d(x) = 0 \) and \( d(x) = 1 \). Let us assume that it holds for every position \( x \) with \( d(x) \leq n \) for some \( n \geq 1 \) and prove it for \( x \) such that \( d(x) = n + 1 \).

1. Let \( x \) be a \((0, 1)\)-position. Then, \( x \) is not movable to \( V'_1 \cap G_{[x]} \), because each position of this set is a \((0, 1)\)-position, by the inductive hypothesis on (1), meaning \( x \) is not movable to \( V'_0 \). From this fact and \( SM(v) \) it follows that \( x \in V'_0 \cup V'_1 \). We show that \( x \notin V'_1 \).

Assume for contradiction that \( x \in V'_0 \). It follows from (iv) that \( x \) is movable to a position \( y \in V'_0 \). By induction, if \( y \in (G_{[x]} \cap V'_1) \setminus \{x\} \) then \( y \) is a \((0, 1)\)-position. But \( x \) is a \((0, 1)\)-position too and, hence, it cannot be movable to such \( y \). This give a contradiction.
Thus, \( x \notin V'_{1,0} \), implying that \( x \in V'_{0,1} \) or, equivalently, that (1) holds.

(2) Similarly, assuming that \( x \) is a \((1,0)\)-position. We can show that \( x \in V'_{1,0} \).

(3) Assume that \( x \) is not a swap position. We show that (3) holds. First, note that \( x \) is neither a \((0,0)\)-position nor a \((1,1)\)-position as well, because \((a')\) or \((c')\) holds for \( x \).

Let \( x \) be a \((k,l)\)-position such that either \( k \geq 2 \) or \( l \geq 2 \). Without loss of generality, assume that \( k \geq 2 \). Then, \((a')\) fails for \( x \) and, hence, \((c')\) holds. It follows that \( x \) is movable to a position \( x' \) in \( V'_{0,1} \) and to a position \( x'' \) in \( V'_{1,0} \). It remains to show that \( l = k \).

Let us consider two sets

\[
M = \{ G(y) \mid y \text{ is a option of } x \} \quad \text{and} \quad M^- = \{ G^-(y) \mid y \text{ is a option of } x \}.
\]

We have \( \{0, 1\} \subseteq M \) and \( \{0, 1\} \subseteq M^- \), since both \( x' \) and \( x'' \) are options of \( x \). Moreover, by the inductive hypothesis, if an option \( y \) of \( x \) is not a swap position then \( y \) is a \((m,m)\)-position. Therefore, \( M = M^- \) and, hence,

\[
k = G(x) = \text{mex}(M) = \text{mex}(M^-) = G^-(x) = l.
\]

\( \square \)

4.3. Miserable games. Miserable games can be characterized in a similar way; only property SM(v) of Theorem 3 is slightly changed.

**Theorem 7.** A game \( G \) is miserable if and only if there exist two disjoint sets \( V'_{0,1} \) and \( V'_{1,0} \) satisfying (i) – (iv) of Theorem 3 and every position \( x \) satisfies at least one of the following three conditions:

(a') \( x \in V'_{0,1} \cup V'_{1,0} \);

(b') \( x \) is not movable to \( V'_{0,1} \cup V'_{1,0} \);

(c') \( x \) is movable to \( V'_{0,1} \) and to \( V'_{1,0} \).

Moreover, if all above conditions hold then \( V'_{0,1} = V_{0,1} \) and \( V'_{1,0} = V_{1,0} \).

4.4. Tame games. Similarly, we characterize tame games as follows.

**Theorem 8.** A game is tame if and only if there exist four disjoint sets \( V'_{0,1}, V'_{1,0}, V'_{0,0}, V'_{1,1} \) satisfying the following conditions:

(i) all four sets are independent;

(ii) \( V'_{0,1} \) contains the terminal position, \( V_T \subseteq V'_{0,1} \);

(iii) if \( x \in V'_{0,1} \) then \( x \) is movable to \( V'_{1,0} \) but not to \( V'_{0,0} \cup V'_{1,1} \);

(iv) if \( x \in V'_{1,0} \) then \( x \) is movable to \( V'_{0,1} \) but not to \( V'_{0,0} \cup V'_{1,1} \);

(v) if \( x \in V'_{0,0} \) then \( x \) is not movable to \( V'_{0,1} \cup V'_{1,0} \).
(vi) if \( x \in V_{0,1}' \) then \( x \) is movable to \( V_{0,0}' \) but not to \( V_{0,1}' \cup V_{1,0}' \);
(vii) if \( x \not\in V_{0,1}' \cup V_{1,0}' \cup V_{0,0}' \) then \( x \) is movable to \( V_{0,1}' \cup V_{1,0}' \cup V_{0,0}' \).

\( T(viii) \) Every position \( x \) satisfies at least one of the following three conditions:
\[
\begin{align*}
(a'_x) & \ x \in V_{0,1}' \cup V_{1,0}' \cup V_{0,0}' \cup V_{1,1}' ; \\
(c'_x) & \ x \text{ is movable to } V_{0,1}' \text{ and to } V_{1,0}' ; \\
(c''_x) & \ x \text{ is movable to } V_{0,0}' \text{ and to } V_{1,1}' .
\end{align*}
\]
Moreover, \( V_{0,1}' = V_{0,1}, V_{1,0}' = V_{0,0}, \) and \( V_{1,1}' = V_{1,1} \) whenever all above conditions hold.

\[
\begin{array}{ccc}
A & - & B \\
& - & \\
& - & C
\end{array}
\]

Figure 9. \( V_{0,1} \neq V_{0,1}' \), although conditions (i) - (viii) of Theorem \( T \) hold.

One may ask, whether conditions (i) - (vii) of Theorem \( T \) themselves result in equalities \( V_{0,1}' = V_{0,1} \) and \( V_{1,0}' = V_{1,0} \). This is not the case. The game in Figure 9 provides a counterexample with setting \( V_{0,1}' = A, V_{1,0}' = B, \) and \( C \in V_{0,1} \setminus V_{0,1}' \neq \emptyset \).

4.5. Domestic games. Finally, a similar characterization holds for the domestic games.

**Theorem 9.** A game is domestic if and only if there exist three disjoint sets \( V_{0,1}', V_{1,0}', \) and \( V_{0,0}' \) such that the following conditions hold:

(i) all three sets are independent;
(ii) \( V_{0,1}' \) contains all terminal positions;
(iii) if \( x \in V_{0,1}' \) is non-terminal, \( x \) is movable to \( V_{1,0}' \) but not to \( V_{0,0}' \);
(iv) If \( x \in V_{1,0}' \), \( x \) is movable to \( V_{0,1}' \) but not to \( V_{0,0}' \);
(v) If \( x \in V_{0,0}' \), \( x \) is not movable to \( V_{0,1}' \cup V_{1,0}' \);
(vi) if \( x \not\in V_{0,1}' \cup V_{1,0}' \cup V_{0,0}' \), \( x \) is movable to \( V_{0,1}' \cup V_{1,0}' \cup V_{0,0}' \);

\( D(vii) \) every position \( x \) satisfies at least one of conditions
\[
\begin{align*}
(a'_x) & \ x \in V_{0,1}' \cup V_{1,0}' ; \\
(b'_x) & \ x \text{ is not movable to } V_{0,1}' \cup V_{1,0}' ; \\
(c'_x) & \ x \text{ is movable to } V_{0,1}' \text{ and to } V_{1,0}' ; \\
(c''_x) & \ x \text{ is movable to } V_{0,1}' \text{ and to } V_{0,0}' ; \\
(c''_x) & \ x \text{ is movable to } V_{1,0} \text{ and to } V_{0,0}' .
\end{align*}
\]
Moreover, if all above conditions hold then \( V_{0,1}' = V_{0,1}, V_{1,0}' = V_{0,1}, \) and \( V_{0,0}' = V_{0,0} \). \( \square \)
5. Sums of games

We say that a class of games is preserved under summation if the sum of games from this class belongs to it too. In this section, we show the classes of tame, miserable, miserable and forced, miserable and returnable games are preserved under summation. For the tame games, this property was claimed by Conway in [14] and proven in [35]; we suggest a simpler proof.

In contrast, the classes of domestic (weakly miserable) and of pet (strongly miserable) games are not preserved under summation. Already the classic \( n \)-pile Nim is a counterexample for the second case. Indeed, one-pile Nim is pet but the \( n \)-pile Nim, which is the sum of \( n \) one-pile Nim games, is not whenever \( n > 1 \); see Subsection 6.1 for more details.

The sum of domestic games may be not domestic; Figure 10 gives an example.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{Games \( G_1 \) and \( G_2 \) are domestic but their sum \( G_1 + G_2 \) is not. Notation \( P(i, j) \) means that \( P \) is an \((i, j)\)-position in a summand, while \( PQ(i, j) \) means that the sum \( PQ \) of \( P \) and \( Q \) is an \((i, j)\)-position.}
\end{figure}

5.1. The sum of tame games is tame. Recall that a swap position is either a \((0,1)\)-position or a \((1,0)\)-position. We will call two swap
positions opposite if one of them is a \((0,1)\)-position while the other is a \((1,0)\)-position, and we will call them parallel otherwise.

**Theorem 10.** If games \(G_1\) and \(G_2\) are tame then their sum \(G_1 + G_2\) is tame too. Moreover, \(x = (x_1, x_2)\) is a swap position of \(G_1 + G_2\) if and only if \(x_i\) is a swap position of \(G_i\) for \(i = 1, 2\). Furthermore, \(x\) is a \((1,0)\)-position of \(G\) if and only if either \(x_1\) is a \((1,0)\)-position in \(G_1\) and \(x_2\) is a \((0,1)\)-position in \(G_2\) or vise versa.

The first claim was stated (without a proof) in 1976 by Conway; see [14] page 178. A proof based on the genus theory appeared in [35]. Here we give an alternative proof based on the characterization of tame games by Theorem \(\text{8}\).

**Proof.** For non-negative integers \(i, j, k\) and \(l\), denote by \([(i,j),(k,l)]\) the set of positions \(x = (x_1, x_2)\) in the sum \(G_1 + G_2\) such that \(x_1\) is an \((i,j)\)-position in \(G_1\) and \(x_2\) is a \((k,l)\)-position in \(G_2\). Let us set

\[
\begin{align*}
V'_{0,1} &= \{[(0,1),(0,1)], [(1,0),(1,0)]\}, \\
V'_{1,0} &= \{[(0,1),(1,0)], [(1,0),(0,1)]\}, \\
V'_{0,0} &= \{[(0,1),(0,0)], [(0,0),(0,1)], [(n,n),(n,n)] | n \in \mathbb{Z}_{\geq 0}\}, \\
V'_{1,1} &= \{[(0,0),(1,0)], [(1,0),(0,0)], [(0,1),(1,1)], [(1,1),(0,1)], \\
&\quad [(n,n),(n+1,n+1)], [(n+1,n+1),(n,n)] | n = 2k, k \in \mathbb{Z}_{\geq 0}\}
\end{align*}
\]

Recall that \(\mathbb{Z}_{\geq 0}\) denotes the set of non-negative integers.

It can be verified that the above four sets satisfy conditions (i) - (vii) of Theorem \(\text{8}\). We now prove by induction on \(d(x)\) that every position \(x\) of the sum \(G = G_1 + G_2\) satisfies (at least) one of the conditions \((\mathbf{a}_5'), (\mathbf{c}'), (\mathbf{e}')\) of Theorem \(\text{8}\) and so the sum is tame.

Note that in this proof, when we recall conditions \((\mathbf{a}_5), (\mathbf{c}), (\mathbf{e})\) (resp., \((\mathbf{a}_5'), (\mathbf{c}'), (\mathbf{e}')\)), we refer them in Definition \(\text{3}\) (resp., Theorem \(\text{8}\)).

By definition, \(x = (x_1, x_2)\) is a terminal position of the sum \(G = G_1 + G_2\) if and only if each \(x_i\) is a terminal position of the summand \(G_i\), for \(i = 1, 2\). Hence, \((\mathbf{a}_5')\) holds for \((x_1, x_2)\). If \(d(x_1, x_2) = 1\) then either \(d(x_1) = 0\) \((x_1\) is terminal\) and \(d(x_2) = 1\) or vise versa and so \((x_1, x_2) \in V'_{1,0}\), meaning \((\mathbf{a}_5')\) holds for \((x_1, x_2)\).

We assume that at least one of the conditions \((\mathbf{a}_5'), (\mathbf{c}'), (\mathbf{e}')\) holds for every position \((x_1, x_2)\) in \(G\) such that \(d(x_1, x_2) \leq n\) for some \(n \geq 1\) and will show that at least one of these conditions holds for each position \((x_1, x_2)\) in \(G\) such that \(d(x_1, x_2) = n + 1\).
Suppose that \((a'_n)\) fails for \(x = (x_1, x_2)\). Then there exists a move from \(x\) to a position \(x' \in V'_{0,1} \cup V'_{1,0} \cup V'_{0,0}\), by (vii) of Theorem 8. Assume such move \(x_1 \rightarrow x'_1\) is made in \(G_1\).

(1) Case \(x' = (x'_1, x_2) \in V'_{0,1}\). In this case \(x'_1\) and \(x_2\) are two parallel swap positions. Since \(x_1\) is movable to the swap position \(x'_1\), condition \((a)\) fails for \(x_1\) and, hence, \((c)\) or \((e)\) holds for \(x\), since \(G_1\) is tame.

(a) If \((e)\) holds for \(x_1\) then \(x_1\) is movable to a position \(x''_1\) such that \(x'_1\) and \(x''_1\) are two opposite swap positions, then \(x''_1\) and \(x_2\) are two opposite swap positions and, hence, \(x'' = (x''_1, x_2) \in V'_{1,0}\), by definition. Recall that \(x\) can also be moved to \(x' \in V'_{0,1}\). Then \((e')\) holds for \(x\).

(b) If \((e)\) holds for \(x_1\) then \(x_1\) is movable to some \((0,0)\)-position \(x'''_1\) and to some \((1,1)\)-position \(x''''_1\). It is not difficult to verify that one of these two positions belongs to \(V'_{0,0}\), while the other to \(V'_{1,1}\) and, hence, \((e')\) holds for \(x\).

(2) Case \(x' = (x'_1, x_2) \in V'_{1,0}\) is similar to the case (1): just swapping “opposite” and “parallel”, as well as “0,1” and “1,0”.

(3) Case \(x' = (x'_1, x_2) \in V'_{0,0}\). Consider the following three options for \(x'\):

(a) If \((x'_1, x_2) \in [(0,0), (0,1)]\) then either \(x_1\) is a \((1,1)\)-position or \((a)\) fails for \(x_1\). Yet, the former case cannot occur as otherwise, \(x = (x_1, x_2) \in V'_{1,1}\), giving a contradiction. In the latter case, either \((c)\) or \((e)\) holds for \(x_1\), since \(G_1\) is tame. It is easily seen that if \((c)\) (resp., \((e)\)) holds for \(x_1\) then \((c')\) (resp., \((e')\)) holds for \(x\).

(b) Case \(x' = (x'_1, x_2) \in [(0,1), (0,0)]\) is similar to the case \((x'_1, x_2) \in [(0,0), (0,1)]\) treated in (a).

(c) If both \(x'_1\) and \(x_2\) are \((n,n)\)-positions, we consider two possibilities for \(n\): \(n\) is odd and \(n\) is even. By checking carefully possible cases for \(n\), one can verify that \((a'_n)\), or \((e')\), or \((e')\) holds for \(x\). We leave the checking task to the reader.

By induction, we conclude that each position satisfies \((a'_n)\), or \((e')\), or \((e')\) and, by Theorem 8 sum \(G_1 + G_2\) is tame. Moreover, \(V_{0,1} = V'_{0,1}\) and \(V_{1,0} = V'_{1,0}\), implying that \(x = (x_1, x_2)\) is a swap position of the sum \(G_1 + G_2\) if and only if \(x_i\) is a swap position of the summand \(G_i\) for \(i = 1, 2\).

The following obvious generalization results from Theorems 10 and 11.
Corollary 1. If games $G_1, \ldots, G_n$ are tame then their sum $G = G_1 + \ldots + G_n$ is tame too. Moreover, a position $x = (x^1, \ldots, x^n)$ of $G$ is a swap position of $G$ if and only if $x^i$ is a swap position of $G_i$ for $i \in \{1, \ldots, n\}$. Furthermore, $x$ is a $(1,0)$-position if and only if the number of $(1,0)$-positions in the set $\{x^1, \ldots, x^n\}$ is odd. \hfill $\square$

5.2. Sums of miserable, returnable, and forced games.

Theorem 11. If games $G_1$ and $G_2$ are miserable then their sum $G_1 + G_2$ is miserable too. Moreover, $x = (x_1, x_2)$ is a swap position of $G_1 + G_2$ if and only if each $x_i$ is a swap position of $G_i$ for $i = 1, 2$. Furthermore, $x$ is a $(1,0)$-position of $G$ if and only if either $x_1$ is a $(1,0)$-position in $G_1$ and $x_2$ is a $(0,1)$-position in $G_2$ or vice versa.

Proof. We proceed by induction on $d(x)$ and prove that every position $x$ in $G = G_1 + G_2$ satisfies condition (a), or (b), or (c) of Definition 3. Note that $G_1$ and $G_2$ are tame, by Theorem 10 and, hence, $G$ is tame, by Theorem 10.

Let $x = (x_1, x_2)$ be a position of $G$. Clearly, (a) holds when $d(x) = 0$, since in this case both $x_1$ and $x_2$ are terminal positions.

Assume that (a), or (b), or (c) holds for every position $x$ with $d(x) \leq n$ for some $n \geq 1$.

Then, by induction, $x$ is either a swap position or a $(k,k)$-position. We prove that every position $x$ with $d(x) = n + 1$ satisfies (a), or (b), or (c). Assume that (a) and (b) fail for $x$ and show that then (c) holds.

Indeed, $x \notin V_{0,1} \cup V_{1,0}$, since (a) fails for $x$, and $x \notin V_{0,0} \cup V_{1,1}$ since (b) fails for $x$. Therefore, $x$ is a $(m,m)$-position for some $m \geq 2$ since $G$ is tame. Furthermore, $x$ is movable to a swap position $x'$, because (b) fails for $x$.

Assume that $x'$ is a $(0,1)$-position.

Furthermore, without loss of generality, we can assume that move $x \to x'$ in $G$ is realized by a move $x_1 \to x'_1$ in $G_1$. Since $G$ is tame and $x' = (x'_1, x_2)$ is a $(0,1)$-position, both $x'_1$ and $x_2$ are swap positions, by Theorem 10. Moreover, $G(x'_1) \oplus G(x_2) = 0$ implies that $G(x'_1) = G(x_2)$ and that $x'_1$ and $x_2$ are parallel.

In the case when $x'$, $x'_1$, and $x_2$ are $(1,0)$-positions rather than $(0,1)$-positions, similar arguments are applicable.

Since $G_1$ is miserable, $x_1$ satisfies (a), or (b), or (c).

Since $x_1$ is movable to $x'_1$, which is a $(0,1)$-position, (b) fails for $x_1$. We claim that (a) fails for $x_1$. Indeed, otherwise $x_1$ is a swap position. Note that $x_2$ is also a swap position and so $x$ is a swap position by Theorem 10. But this contradicts our assumption that $x$ is a $(m,m)$-position. Therefore (a) fails and (c) holds for $x_1$. 

Then, there is also a move from $x_1$ to a $(1,0)$-position $x''_1$. Note that $x'_1$ and $x''_1$ are opposite while $x'_1$ and $x_2$ are parallel. Hence, $x''_1$ and $x_2$ are opposite. By Theorem 10, $x'' = (x''_1, x_2)$ is a swap position. Moreover, it is a $(1,0)$-position and an option of $x$. Thus, (c) holds for $x$.

Then, by induction, (a), or (b), or (c) holds for every position. Therefore, $G$ is miserable. □

The following generalization results directly from Theorems 11 and 1.

**Corollary 2.** If games $G_1, \ldots, G_n$ are miserable then their sum $G = G_1 + \ldots + G_n$ is miserable too. Moreover, a position $x = (x_1, \ldots, x_n)$ of $G$ is a swap position of $G$ if and only if $x_i$ is a swap position of $G_i$ for $i \in \{1, \ldots, n\}$. Furthermore, $x$ is a $(1,0)$-position if and only if the number of $(1,0)$-positions in set $\{x_1, \ldots, x_n\}$ is odd. □

The subclasses of forced or returnable miserable games are preserved under summation, as well.

**Proposition 2.** The sum of miserable games is returnable whenever all summands are returnable.

*Proof.* It is sufficient to prove that $G_1 + G_2$ is returnable whenever $G_1$ and $G_2$ are miserable and returnable. Let $x = (x_1, x_2)$ be a swap position in $G$. By Theorem 11, both $x_1$ and $x_2$ are swap positions. Assume that $x$ is movable to some $x'$ in $G_1 + G_2$. Without loss of generality, assume that this move is realized by the move $x_1 \rightarrow x'_1$ in $G_1$. Since $G_1$ is returnable, there exists a move $x'_1 \rightarrow x''_1$ in $G_1$ such that $x_1$ and $x''_1$ are either both $(0,1)$-positions or both $(1,0)$-positions. Set $x' = (x'_1, x_2)$ and $x'' = (x''_1, x_2)$ and consider moves $x \rightarrow x'$ and $x' \rightarrow x''$ in $G_1 + G_2$. By Theorems 1 and 11, $x$ and $x''$ are either both $(0,1)$-positions or both $(1,0)$-positions in $G$. □

**Proposition 3.** The sum of miserable games is forced whenever all summands are forced.

*Proof.* It is sufficient to prove that $G = G_1 + G_2$ is forced whenever $G_1$ and $G_2$ are miserable and forced. Let $x = (x_1, x_2)$ be a swap position in $G$. By Theorem 11, both $x_1$ and $x_2$ are swap positions. If $x'_1$ is an option of $x_1$ in $G_1$ then $x'_1$ and $x' = (x'_1, x_2)$ are swap positions, by Theorem 11. Moreover, Theorems 1 and 11 imply that if $x$ is a $(0,1)$-position (resp., $(1,0)$-position) then $x'$ is a $(1,0)$-position (resp., $(0,1)$-position). These arguments are applicable to any option of $x$ in $G$. □
6. Applications

In this section, we show that many classical games fall into classes considered above.

6.1. The game of Nim. This game is played with \( k \) piles of tokens. By each move a player chooses one pile and removes an arbitrary (positive) number of tokens from it. The complete analysis of Nim is was given by Charles Bouton in \([8]\), who solved both the normal and misère versions.

Let us start with the trivial case \( k = 1 \). The next statement is obvious.

**Lemma 3.** One-pile Nim is a strongly miserable game with exactly one \((0, 1)\)-position, which is the terminal position, and exactly one \((1, 0)\)-position, which is the single pile of size 1, while the pile of size \( n \) is an \((n, n)\)-position for all \( n \geq 2 \).

Already the two-pile Nim is not strongly miserable. For example, \( \text{Nim}(2, 2) \) is a \((0, 0)\)-position.

**Proposition 4.** The game of Nim is miserable and forced.

**Proof.** By Lemma 3 and Theorem 11, Nim is miserable. Let us show that it is forced. Let \( x = (x_1, \ldots, x_k) \) be a swap position. By Theorem 11, each \( \text{Nim}(x_i) \) is a swap position, implying either \( x_i = 0 \) or \( x_i = 1 \) for every \( i \). Obviously, every move from a swap position ends in another swap position and changes the parity of the number of ones.

The above arguments also prove that the \((0, 1)\)-positions and \((1, 0)\)-positions alternate. This immediately results in the following characterization of the sets \( V_{0,1} \) and \( V_{1,0} \).

**Proposition 5.** \( V_{0,1} = \{(1, \ldots, 1) \mid k \geq 0\} \) and \( V_{1,0} = \{(1, \ldots, 1) \mid 2k \text{ entries } 1, 2k+1 \text{ entries } 1 \} \).

6.2. Subtraction games. Subtraction game, denoted by \( S(X) \), is played with a finite pile of tokens and a set \( X \) of positive integers, which may be finite or infinite. A move is to choose an element of \( X \) and remove this number of tokens from the pile. Various aspects of this game are exposed in \([1, 2, 3, 10, 16]\).

In \([16]\), Ferguson shows that in any subtraction game each non-terminal 0-position is movable to a 1-position. This and Theorem 5 imply the following statement.

**Proposition 6.** Subtraction games are strongly miserable.
Since the proof by Ferguson [16] is very short and elegant, we copy it here for the reader’s convenience.

**Proposition 7 ([16]).** Every subtraction game satisfies property (v) of Theorem 5.

Proposition 7 is based on the following lemma.

**Lemma 4 ([16]).** Set \( k = \min(X) \). Then \( G(x) = 0 \) if and only if \( G(x + k) = 1 \).

**Proof.** Since \( k \in X \), \( G(x) = 0 \) implies \( G(x + k) \neq 0 \) for all \( x \).

For the necessary condition, assume for contradiction that there exists the smallest \( x \) such that \( G(x) = 0 \) and \( G(x + k) > 1 \). By the definition of SG values, there exists \( s \in X \) such that \( G(x + k - s) = 1 \). Since \( k = \min(X) \), \( k - s \leq 0 \). Moreover, \( x + k - s \geq k \) or \( x - s \geq 0 \) (otherwise, there is no move from \( x + k - s \) while \( G(x + k - s) = 1 \)). Furthermore, \( G(x) = 0 \) implies \( G(x - s) > 0 \). Thus there exists \( s' \in X \) such that \( G(x - s - s') = 0 \) by the definition of SG values.

Let \( y = x - s - s' \). Then \( y < x \) and \( G(y) = 0 \), implying that \( G(y + k) = 1 \), by the choice of the smallest \( x \). However, the last equation implies that \( G(y + k + s') \neq 1 \) or, equivalently, \( G(x - s + k) \neq 1 \), contradicting \( G(x + k - s) = 1 \) as above.

Conversely, if \( G(x) = 1 \) and \( G(x - k) \neq 0 \), there exists \( s \in X \) such that \( G(x - k - s) = 0 \). By the necessary condition, \( G(x - s) = 1 \), which contradicts \( G(x) = 1 \).

**Proof of Proposition 7.** Given any non-terminal \( x \) such that \( G(x) = 0 \), one has \( G(x - k) \neq 0 \), where \( k \) is the smallest element of \( X \). This implies that there is an \( s \in X \) such that \( G(x - k - s) = 0 \). From Lemma 4, \( G(x - s) = 1 \).

**6.3. Game MARK.** A game played with a single pile is called a single-pile Nim-like game if two players take turns removing tokens from that pile. After Subsections 6.1 and 6.2, one may ask whether each single-pile Nim-like game is strongly miserable. The is not the case. Moreover, such a game may be not even domestic. For example, let us consider the following single-pile Nim-like game suggested by Fraenkel [20] and called MARK. By one move a pile of size \( n \) should be reduced to either \( n - 1 \) or \( \lfloor \frac{n}{2} \rfloor \).

**Proposition 8.** Game MARK is not domestic.

**Proof.** It is not difficult to verify that 8 is a (0,2)-position.
6.4. Game Euclid. In 1969 Cole and Davie [12] introduced game Euclid. It is played with two piles of tokens. By one move a player has to remove from the greater pile any number of tokens that is an integer multiple of the size of the smaller pile. The game ends when one of the piles is empty. A position of two piles of sizes \(x\) and \(y\) is denoted by \((x, y)\). It was shown in [12] that \((x, y)\) is a \(P\)-position if and only if \(x < y < \phi x\), where \(\phi = (1 + \sqrt{5})/2\) is the golden ratio [12].

In 1997, Grossman [21] proposed a modification of this game in which the entries must stay positive. In particular, move \((x, y) \rightarrow (x, 0)\) is not allowed even if \(y\) is a multiple of \(x\). Thus, the terminal positions of this game are \((x, x)\) for some positive \(x\).

Note that Grossman’s variant is not the misère version of Euclid by Cole and Davie. Also note that in the literature the examples [24, 31, 32, 34] referred to as Euclid are Grossman’s version, not Cole and Davie’s version.

The SG function of Grossman’s variant was solved in [34] and that of the original game Euclid was solved later in [11], where it was shown that these two SG functions are very similar. Some other variants were also studied in [9, 13, 30].

We now analyze miserability of these two games. Miserability of Grossman’s variant was analyzed in [24].

Proposition 9. Both Cole and Davie’s game and Grossman’s game of Euclid are miserable and forced.

Proof. We first prove that Cole and Davie’s game miserable. Set \(V'_{0,1} = \{(0, x), (x, 0) \mid x \in \mathbb{Z}_{>0}\}\) and \(V'_{1,0} = \{(x, x) \mid x \in \mathbb{Z}_{>0}\}\) in which \(\mathbb{Z}_{>0}\) is the set of positive integers. Note that if \(v \in V'_{0,1}\), then \(v\) is a terminal and, hence, a \((0, 1)\)-position. If \(v \in V'_{1,0}\) then \(v\) is movable to a terminal position and, moreover, this is the only move available from \(v\); hence, \(v\) is a \((1, 0)\)-position.

It is easily seen that if \(v \notin V'_{0,1} \cup V'_{1,0}\) then either \(v\) is not movable to \(V'_{0,1} \cup V'_{1,0}\) or \(v\) is movable to \(V'_{0,1}\) and to \(V'_{1,0}\). Then, by Theorem 7, the game is miserable and, moreover, \(V_{0,1} = V'_{0,1}\) and \(V_{1,0} = V'_{1,0}\). It follows also that this game is forced.

For Grossman’s game, we set \(V'_{0,1} = \{(x, x) \mid x \in \mathbb{Z}_{>0}\}\) and \(V'_{1,0} = \{(x, 2x), (2x, x) \mid x \in \mathbb{Z}_{>0}\}\) and the same arguments work. \(\square\)

6.5. Game Wythoff. The Wythoff game [39] is a modification of the two-pile Nim in which a player by one move is allowed to remove either

(i) an arbitrary number of tokens from one pile, or
(ii) the same number of tokens from both.

Two piles of sizes $x$ and $y$ define a position $(x, y)$. By symmetry, $(x, y)$ and $(y, x)$ are equivalent; we will assume that $x \leq y$ unless the converse is explicitly said.

Let $(x_n, y_n)_{n \geq 0}$, where $x_i < x_j$ if $i < j$, be the sequence of $\mathcal{P}$-positions of the game. Wythoff [39] proved that $(x_n, y_n)$ is a $\mathcal{P}$-position if and only if $x_n = \lfloor \phi n \rfloor$ and $y_n = \lfloor \phi^2 n \rfloor$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. Note that $\lfloor \phi^2 n \rfloor = \lfloor \phi n \rfloor + n$.

The game Wythoff and numerous modifications of it were studied intensively in the literature: [4, 7, 15, 17, 19, 18, 27, 29, 38]. However, no explicit formula is known for the SG function of this game.

In [18], Fraenkel analyzed the misère version of Wythoff and characterized its $\mathcal{P}$-positions. Interestingly, the $\mathcal{P}$-positions of the normal and misère versions of Wythoff differ only by six positions: \{(0, 0), (1, 2), (2, 1)\} is a (0, 0)-position, \{(0, 1), (1, 0), (2, 2)\} is a (1, 0)-position, \{(0, 1), (1, 0), (2, 2)\} is a (2, 2)-position, \{(0, 0), (1, 2), (2, 1)\} is a (0, 0)-position, \{(0, 1), (1, 0), (2, 2)\} is a (1, 0)-position, \{(0, 1), (1, 0), (2, 2)\} is a (2, 2)-position. One can check this claim by comparing [15, Proposition 2] and [18, Theorem 2.1]. Using these results, one can verify directly that the game Wythoff is miserable. Here, we provide an alternative proof using Theorem 7.

**Proposition 10.** Game Wythoff is miserable.

**Proof.** Let us set $V'_{0,1} = \{(0, 0), (1, 2), (2, 1)\}$ and $V'_{1,0} = \{(0, 1), (1, 0), (2, 2)\}$. One can easily verify the containments $V'_{0,1} \subseteq V_{0,1}$ and $V'_{1,0} \subseteq V_{1,0}$. Let $(x, y)$ be a position that does not belong to $V'_{0,1} \cup V'_{1,0}$. It is easily seen that either $(x, y)$ is not movable to $V'_{0,1} \cup V'_{1,0}$ or $(x, y)$ is movable to both $V'_{0,1}$ and $V'_{1,0}$. Thus, by Theorem 7, the game Wythoff is miserable and, moreover, $V_{0,1} = V'_{0,1}$ and $V_{1,0} = V'_{1,0}$. \[\square\]

Note that (3, 5) is a (0, 0)-position and, thus, Wythoff is not strongly miserable.

**Proposition 11.** The game Wythoff is returnable but not forced.

**Proof.** There is a move from (2, 2), which is a (1, 0)-position, to (1, 1), which is a (2, 2)-position; hence, the game is not forced. It is easily seen that the game is returnable. \[\square\]

6.6. **Game Wyt(a).** In [17] Fraenkel, for any positive integer $a$, introduced the following generalization Wyt(a) of the game Wythoff. This game is also played with two piles of tokens and by one move a player is allowed

(i) to remove an arbitrary number of tokens from one pile, or
(ii) to remove $k$ tokens from one pile and $l$ tokens from the other pile such that $|k - l| < a$.

The game $\text{Wyt}(a)$ was studied by Fraenkel [17, 18]. Note that $\text{Wyt}(1)$ is Wythoff and, hence, it is miserable.

**Proposition 12.** Game $\text{Wyt}(a)$ is strongly miserable whenever $a \geq 2$.

We first recall results on $\mathcal{P}$-positions of the normal and misère versions.

**Proposition 13** ([17]). For $a \geq 2$, the sequence $(x_n, y_n)_{n \geq 0}$ of $\mathcal{P}$-positions of $\text{Wyt}(a)$ satisfies the following conditions:

(i) $(x_0, y_0) = (0, 0)$;

(ii) for $n \geq 1$, $x_n = \text{mex}\{x_i, y_i \mid 0 \leq i < n\}$ and $y_n = x_n + an$.

**Proposition 14** ([18]). For $a \geq 2$, the sequence $(x'_n, y'_n)_{n \geq 0}$ of $\mathcal{P}$-positions of misère $\text{Wyt}(a)$ satisfies the following conditions:

(i) $(x'_0, y'_0) = (0, 1)$;

(ii) for $n \geq 1$, $x'_n = \text{mex}\{x'_i, y'_i \mid 0 \leq i < n\}$ and $y'_n = x'_n + an + 1$.

**Corollary 3.** For $a \geq 2$, two sets of $\mathcal{P}$-positions of $\text{Wyt}(a)$ and its misère version are disjoint.

**Proof.** Let $(x_m, y_m)$ be a $\mathcal{P}$-position of $\text{Wyt}(a)$ and let $(x'_n, y'_n)$ be a $\mathcal{P}$-position of misère $\text{Wyt}(a)$. If these two positions are coincident then $x_m = x'_n$ and $x_m + am = x'_n + an + 1$. One can then simplify to obtain the equation $a(m - n) = 1$, giving a contradiction as 1 cannot be multiple of $a$. □

**Proof of Proposition 12.** It follows immediately from Corollary 3 and Theorem 5 (iii). □

### 6.7. Game $\text{Wyt}(a, b)$

Game $\text{Wyt}(a, b)$ was introduced in [27], for any two non-negative integers $a$ and $b$, as follows. Like Wythoff, it is played with two piles of tokens. By one move a player is allowed to delete $x \geq 0$ tokens from one pile and $y \geq 0$ tokens from the other such that $x + y > 0$ and $(\min(x, y) < b$ or $|x - y| < a)$. Note that $\text{Wyt}(0, 1)$ is the two-pile Nim, $\text{Wyt}(1, 1)$ is Wythoff, and $\text{Wyt}(a, 1)$ is $\text{Wyt}(a)$.

The following recursive solution of the normal and misère versions of the game was given in [27].

Given an integer $b \geq 1$ and a finite set $S$ of $m$ non-negative integers $s_1, \ldots, s_m$ such that $s_1 < \cdots < s_m$, let us set $s_0 = -b$ and $s_{m+1} = +\infty$. Then, there exists the smallest index $i$ such that $s_{i+1} - s_i > b$. Let us define a function $\text{mex}_b$ of $S$ as follows:

$$\text{mex}_b(S) = s_i + b$$
It is easily seen that \( \mex_b(\emptyset) = 0 \) and that \( \mex_b(S) \) equals \( \mex(S) \) when \( b = 1 \), that is, \( \mex_1 = \mex \).

The \( \mathcal{P} \)-positions of the normal and its misère versions of game \( \text{WYT}(a, b) \) are characterized in [27] as follows.

**Proposition 15 ([27])**. The sequence \((x_n, y_n)_{n \geq 0}\) of the \( \mathcal{P} \)-positions of the normal version of game \( \text{WYT}(a, b) \) satisfies the following recursion:

\[
x_n = \mex_b\{x_i, y_i \mid 0 \leq i < n\}, \quad y_n = x_n + an.
\]

**Proposition 16 ([27])**. The sequence \((x'_n, y'_n)_{n \geq 0}\) of the \( \mathcal{P} \)-positions of misère version of game \( \text{WYT}(a, b) \) satisfies the following recursion:

(i) if \( a = 1 \), then \((x'_0, y'_0) = (b + 1, b + 1)\) and \( x'_n = \mex_b\{x'_i, y'_i \mid 0 \leq i < n\}, \quad y'_n = x'_n + an; \)

(ii) if \( a \geq 2 \), then \( x'_n = \mex_b\{x'_i, y'_i \mid 0 \leq i < n\}, \quad y'_n = x'_n + an + 1. \]

**Proposition 17 ([27])**. Game \( \text{WYT}(a, b) \) is strongly miserable whenever \( a \geq 2 \).

**Proof.** We only need to show that the normal and misère versions of \( \text{WYT}(a, b) \) do not share \( \mathcal{P} \)-positions, or in other words, that there is no \((0, 0)\)-position. Then, game \( \text{WYT}(a, b) \) is strongly miserable, by Theorem 5.

Let \((x_n, y_n)\) and and \((x'_m, y'_m)\) be \( \mathcal{P} \)-positions of the normal and misère versions of \( \text{WYT}(a, b) \), respectively. Suppose these two positions coincide, \( x'_m = x_m \) and \( y'_m = y_n \). By Propositions 13 and 16 for case \( a \geq 2 \), one obtains equality \( a(n - m) = 1 \), which is a contradiction since 1 cannot be a multiple of \( a \).

The case \( a \leq 1 \) was studied in [25, 27]. Combining these results with Proposition 17 we obtain the following criterion.

**Proposition 18.** Game \( \text{WYT}(a, b) \) is miserable and returnable if \((a = 1 \text{ and } b \geq 1)\) or \((b = 1 \text{ and } a \leq 1)\). Otherwise, the game is strongly miserable.

6.8. Moore’s \( \text{NIM}_{n, \leq k} \) and its variants.

6.8.1. Moore’s \( \text{NIM}_{n, \leq k} \). The following game was introduced in 1910 by Moore [33]. Let \( k \) and \( n \) be two positive integers such that \( k \leq n \). By one move a player has to reduce (strictly) at least 1 and at most \( k \) from given \( n \) piles of \((x_1, \ldots, x_n)\) tokens. Moore denoted this game \( \text{NIM}_k \), but we use notation \( \text{NIM}_{n, \leq k} \) to include \( n \).

We will show that game of \( \text{NIM}_{n, \leq k} \) is miserable. For \( k = 1 \), it is known.
Proposition 19. The game of $\text{Nim}_{n,k}$ is miserable for $2 \leq k < n$. Moreover, let $x = (x_1, \ldots, x_n)$ be a position in $\text{Nim}_{n,k}$ and $l$ be the number of non-empty piles in $x$. Then

(a) $x$ is a $(0,1)$-position if and only if $x_i \leq 1$ for all $i$ and $l \equiv 0 \mod (k + 1)$;
(b) $x$ is a $(1,0)$-position if and only if $x_i \leq 1$ for all $i$ and $l \equiv 1 \mod (k + 1)$.

Proof. Let us set

\[ V_{0,1}' = \{(x_1, \ldots, x_n) \mid x_i \leq 1 \text{ for all } i \text{ and } l \equiv 0 \mod (k + 1)\}; \]
\[ V_{1,0}' = \{(x_1, \ldots, x_n) \mid x_i \leq 1 \text{ for all } i \text{ and } l \equiv 1 \mod (k + 1)\}. \]

We verify the conditions (i) - $M(v)$ of Theorem 4. Condition (i) holds since there is no move between two arbitrary positions in each set since such a move must reduce $k + 1$ piles. Condition (ii) holds since $V_{0,1}'$ contains the terminal position $(0,0, \ldots, 0)$. Condition (iii) holds since from every non-terminal position in $V_{0,1}'$, the move removing exactly $k$ tokens terminates in $V_{1,0}'$. Condition (iv) holds since from every position in $V_{1,0}'$, the move removing exactly one token terminates in $V_{0,1}'$. It remains to verify condition $M(v)$.

Let $x$ be a position not in the set $V_{0,1}' \cup V_{1,0}'$. If there is no move from $x$ that terminates in $V_{0,1}' \cup V_{1,0}'$ then the condition $M(v)$ holds and we are done. Assume that this is not the case. Then there exists one move $M_1$ from $x$ that terminates in either $V_{0,1}'$ or $V_{1,0}'$. We need to prove that $x$ is movable to both $V_{0,1}'$ and $V_{1,0}'$.

Note that a move from $x$ reduces at most $k$ piles $x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}$ for a permutation $\pi$, meaning

\[(M_1) \colon (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) \rightarrow (x_{\pi(1)} - y_1, x_{\pi(2)} - y_2, \ldots, x_{\pi(k)} - y_k) \]

with at least some $y_j \geq 1$.

1. If the move $(M_1)$ terminates in $V_{0,1}'$, then it leaves $m(k + 1)$ entries of size 1.
   (a) If $x_{\pi(i)} - y_i = 1$ for all $i$, then the corresponding move
   \[(M_2) : (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) \rightarrow (x_{\pi(1)} - y_1 - 1, x_{\pi(2)} - y_2 - 1, \ldots, x_{\pi(k)} - y_k - 1) \]
   terminates in $V_{1,0}'$, leaving $(m - 1)(k + 1) + 1$ entries of size 1.
   (b) If $x_{\pi(i)} - y_i = 0$ for some $i$, then either there exists $i_0$ such that $y_{i_0} \geq 2$ or there exist $i_0$ and $j_0$ such that $y_{i_0} \geq 1$ and $y_{j_0} \geq 1$. In fact, if otherwise, $x \in V_{1,0}'$, giving a contradiction. In either of cases, we can choose $(y_1', y_2', \ldots, y_k')$ such
that $0 \leq y_i' \leq y_i$ and $y_1' + y_2' + \cdots + y_k' = y_1 + y_2 + \cdots + y_k - 1$.

Then the corresponding move

$$(M_3) \quad (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) \rightarrow (x_{\pi(1)} - y_1', x_{\pi(2)} - y_2', \ldots, x_{\pi(k)} - y_k')$$

terminates in $V_{1,0}'$, leaving $m(k + 1) + 1$ entries of size 1.

(2) If the move $(M_1)$ terminates in $V_{1,0}'$, then it leaves $m(k + 1) + 1$ entries of size 1.

(a) If $x_{\pi(i_0)} > y_{i_0}$ for some $i_0$, then we define $y_i' = y_i$ for all $i$, except for $y_{i_0}' = x_{\pi(i_0)}$. The move

$$(M_4) \quad (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) \rightarrow (x_{\pi(1)} - y_1', x_{\pi(2)} - y_2', \ldots, x_{\pi(k)} - y_k')$$

terminates in $V_{0,1}'$, leaves $m(k + 1)$ entries of size 1. Here $(M_4)$ imitates $(M_1)$ before removing the whole pile $x_{\pi(i_0)}$.

(b) If $x_{\pi(i)} = y_i$ for all $i$, we consider two cases.

(i) If $y_{i_0} = 0$ for some $i_0$, we can choose some pile $x_{j_0} \notin \{x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}\}$ of size 1 which is not touched in the move $(M_1)$. Then the move

$$(M_5) : \quad (x_{j_0}, x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) \rightarrow (0, x_{\pi(1)} - y_1, x_{\pi(2)} - y_2, \ldots, x_{\pi(k)} - y_k)$$

terminates in $V_{0,1}'$. Note that $(M_5)$ imitates $(M_1)$ before removing the pile $x_{j_0}$, resulting in $m(k + 1)$ entries of size 1.

(ii) If $x_{\pi(i)} = y_i > 0$ for all $i$, then there exists $i_0$ such that $x_{i_0} \geq 2$. Otherwise, $x \in V_{0,1}'$. Now, we have $y_{i_0} - 1 = x_{i_0} - 1 \geq 1$. Then the move

$$(M_6) : \quad (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)})$$

$$\rightarrow (x_{\pi(1)} - (y_1 - 1), x_{\pi(2)} - (y_2 - 1), \ldots, x_{\pi(k)} - (y_k - 1))$$

terminates in $V_{0,1}'$, leaving $(m + 1)(k + 1)$ entries of size 1.

\[ \square \]

6.8.2. An extension of $\text{Nim}_{n, \leq k}$. We extend $\text{Nim}_{n, \leq k}$ to a game called \textit{Extended Nim$_{n, \leq k}$} that has an extra pile with $x_0$ tokens. By one move, it is allowed to reduce $x_0$ and at most $k$ of the remaining $n$ piles. Note that at least one pile must be reduced strictly; reducing $x_0$ is not compulsory and reducing only $x_0$ is legal. When $k = n - 1$, the game \textit{Extended Nim$_{n, \leq n-1}$} is called \textit{Extended Complementary Nim}, or \textit{Exco-Nim}, for short, [5].
Proposition 20. Let $n \geq 3$ and $1 \leq k < n$. Game Extended Nim$_{n,\leq k}$ is miserable. Moreover, $x = (x_0, x_1, \ldots, x_n)$ is a $(0, 1)$- (resp. $(1, 0)$)-position if and only if $x_0 = 0$ and $(x_1, \ldots, x_n)$ is a $(0, 1)$- (resp. $(1, 0)$)-position of Nim$_{n,\leq k}$.

The proof is essentially similar to that of Proposition 19 and we leave it to the reader.

6.8.3. Exact $k$-Nim. Let us consider a modification of Nim$_{n,\leq k}$ in which by one move a player must (strictly) reduce exactly $k$ piles. We denote this game by Nim$^{'}_{n,\leq k}$. A closed formula for its SG function was obtained in [6] for the case $n \leq 2k$.

We prove that the game is miserable when $n \leq 2k$. We start with the case $n = 2k$.

Proposition 21. Game Nim$^{'}_{2k,\leq k}$ is miserable. Moreover, $x = (x_1, \ldots, x_n)$ is a

(i) $(0, 1)$-position if and only if $x_1 = \cdots = x_{k+1} \leq 1$;
(ii) $(1, 0)$-position if and only if $d(x) = 1$.

Recall that $d(x)$ denotes the greatest number of successive moves from $x$ to the terminal position.

Proof. We leave to the reader to check that two sets $V_{0,1}^{'} = \{x = (x_1, \ldots, x_n) \mid x_1 = \cdots = x_{k+1} \leq 1\}$ and $V_{1,0}^{'} = \{x \mid d(x) = 1\}$ satisfy conditions in Theorem 7; hence the game is miserable with $V_{0,1} = V_{0,1}^{'}$ and $V_{1,0} = V_{1,0}^{'}$.

Recall that $d(x)$ is the largest number of moves from $x$ to the terminal position.

Proposition 22. Game Nim$^{'}_{n,\leq k}$ with $n < 2k$ is strongly miserable.

Proof. Note that if $x = (x_1, \ldots, x_n)$ is a $P$-position then $x$ is terminal. Indeed, every non-terminal position is movable to the terminal position by eliminating the piles $x_1, \ldots, x_k$ and, thus, leaving at most $n - k = k - 1$ nonempty piles. By definition, a positions with at most $k - 1$ nonempty piles is terminal.

In other words, $G(x) = 0$ if and only if $x$ is the terminal position, which is also a $(0, 1)$-position. In particular, there are no $(0, 0)$-position and, by Theorem 5, the game is strongly miserable.

Many games Nim$^{'}_{n,\leq k}$ with $k < n/2$ are not even domestic. For example, our computations show that $(1, 2, 3, 3, 3)$ is a $(0, 2)$-position of Nim$^{'}_{5,\leq 2}$. 
6.8.4. **Slow $k$-Nim.** Let us now consider a modification of $\text{Nim}_{n,k}$ in which a move consists of choosing at least one and at most $k$ from $n$ piles and removing exactly one token from each of them. The obtained game is denoted by $\text{Nim}_{n,k}^1$; it was analyzed in [28].

Relations between the normal and misère versions are summarized by the following statement.

**Proposition 23.** For $k \geq n-1$, the game of Slow $k$-Nim is miserable. Moreover,

(i) if $k = n$, $V_{0,1} = \{(0,0,\ldots,0,2j) \mid j \in \mathbb{Z}_2\}$ and $V_{1,0} = \{(0,0,\ldots,0,2j+1) \mid j \in \mathbb{Z}_2\}$;
(ii) if $k = n-1$, $V_{0,1} = \{(i,i,\ldots,i,i+2j) \mid i,j \in \mathbb{Z}_2\}$ and $V_{1,0} = \{(i,i,\ldots,i,i+2j+1) \mid i,j \in \mathbb{Z}_2\}$.

**Proof.** For $k = n$ and For $k = n-1$ let us respectively set

$V_{0,1} = \{(0,0,\ldots,0,2j) \mid j \in \mathbb{Z}_2\}$ and $V_{1,0} = \{(0,0,\ldots,0,2j+1) \mid j \in \mathbb{Z}_2\}$.

$V_{0,1}' = \{(i,i,\ldots,i,i+2j) \mid i,j \in \mathbb{Z}_2\}$ and $V_{1,0}' = \{(i,i,\ldots,i,i+2j+1) \mid i,j \in \mathbb{Z}_2\}$.

We leave to the reader to verify that these two sets $V_{0,1}'$ and $V_{1,0}'$ satisfy all conditions of Theorem 7 and, hence, the game is miserable with $V_{0,1} = V_{0,1}'$ and $V_{1,0} = V_{1,0}'$. \hfill $\Box$

Our computations show that game $\text{Nim}_{4}^1$ is not domestic; for example, $(1,1,2,3)$ is a $(4,0)$-positions. Thus, case $k = n-2$ differs a lot from the case $k = n-1$ corresponding to the Complementary Nim.

6.9. **Heap overlapping Nim.** The following generalization of Nim was introduced in [2] and called HO-Nim, where HO stands for “Heap Overlapping”. Given a ground set $V$, a position of this game involves a family of its subsets $\mathcal{H} = \{H_1,\ldots,H_n\}$. Furthermore, a move from this position consists of choosing a non-empty subset $S$ of some set $H_i$, deleting $S \cap H_j$ from each $H_j$, and getting thus a new position $\{H_1 \setminus S,\ldots,H_n \setminus S\}$. Note that HO-Nim ($\mathcal{H}$) is the classic Nim whenever the subsets $H_i$ are pairwise disjoint.

In this subsection we construct examples of domestic but not tame HO-Nim games.

**Definition 4.** Given a ground set $V$ partitioned by $n \geq 3$ pairwise disjoint subsets $V_1,\ldots,V_n$, let us set $H_1 = V_1 \cup V_2, H_2 = V_2 \cup V_3,\ldots,H_{n-1} = V_{n-1} \cup V_n, H_n = V_n \cup V_1$, and $\mathcal{H} = \{H_1,H_2,\ldots,H_n\}$. We denote the corresponding position by $\langle |V_1|,|V_2|,\ldots,|V_n| \rangle$ and game by $\mathcal{H}(C_n)$. \hfill $\Box$
Proposition 24. HO-Nim $\mathcal{H}(C_4)$ is miserable and forced. HO-Nim $\mathcal{H}(C_5)$ is domestic but not tame. HO-Nim $\mathcal{H}(C_6)$ is not domestic.

Proof. By symmetry, the positions $(x_1, x_2, \ldots, x_n)$ and $(x_2, x_3, \ldots, x_n, x_1)$ are equivalent. We denote by $[(x_1, x_2, \ldots, x_n)]$ the set of positions equivalent with $(x_1, x_2, \ldots, x_n)$.

For $\mathcal{H}(C_4)$, set $V'_{0,1} = \{(0,0,0,0)\} \cup [(0,1,0,1)]$ and $V'_{1,0} = [(0,0,0,1)]$. By Theorem 7, the game is miserable; moreover, $V'_{0,1} = V_{0,1}$ and $V'_{1,0} = V_{1,0}$. Furthermore, every move from a position in $V_{1,0}$ ends in $(0,0,0,0)$, which is the (unique) terminal, while every move from a position of $V_{0,1}$ terminates in $V_{1,0}$. Hence, the game is forced. Note that the $(0,0)$-positions of this game are $\{(a,b,a,b)\mid a,b \in \mathbb{Z}_{\geq 0}, a+b \geq 2\}$.

For $\mathcal{H}(C_5)$, direct computation shows that $x = (2,0,1,1,1)$ is a $(5,1)$-position. Therefore, HO-Nim $\mathcal{H}(C_5)$ is not tame. Let us show that $\mathcal{H}(C_5)$ is domestic.

It can be easily verified that the set of $(0,0)$-positions is

$$V_{0,0} = \{a+c+a+b+a, a+c+b+a\} \cup \{a+c+a+b+a\}$$
$$\cup \{a+c+a+b+a, c+b+a\} \quad \text{with} \quad c,b,a \in \mathbb{Z}_{\geq 0}.$$

Let us set

$$V'_{0,0} = V_{0,0},$$
$$V'_{0,1} = \{(0,0,0,0), (1,1,1,1)\} \cup [(0,0,1,0,1)],$$
$$V'_{1,0} = [(0,0,0,0,1)] \cup [(0,1,1,1,1)].$$

It is easily seen that three sets $V'_{0,0}, V'_{0,1},$ and $V'_{1,0}$ satisfy conditions of Theorem 9 and, thus, the game is domestic.

Game $\mathcal{H}(C_6)$ is not domestic, since $(1,1,1,1,1,1)$ is a $(0,2)$-position in it. \qed

Definition 5. Given a ground set $V$ partitioned by $n \geq 3$ pairwise disjoint subsets $V_1, \ldots, V_n$, let us set $H_i = V_i \cup V_{i+1}$ for $1 \leq i \leq n-1$ and $\mathcal{H} = \{H_i \mid 1 \leq i \leq n-1\}$. We denote the corresponding position by $|V_1|, |V_2|, \ldots, |V_n|)$ and the game by HO-Nim $\mathcal{H}(P_n)$. \qed

Proposition 25. HO-Nim $\mathcal{H}(P_3)$ is miserable. HO-Nim $\mathcal{H}(P_4)$ and $\mathcal{H}(P_5)$ are domestic but not tame. HO-Nim $\mathcal{H}(P_6)$ is not domestic.

Proof. By Theorem 7 it can be checked that $\mathcal{H}(P_3)$ is miserable with

$$V_{0,1} = \{(0,0,0),(1,0,1)\},$$
$$V_{1,0} = \{(0,0,1),(0,1,0),(1,0,1)\}.$$ 

Moreover, 0,0-positions form the set $\{(a,0,a)\mid a \in \mathbb{Z}^+, a \geq 2\}$. 


By Theorem 6 it can be checked that $\mathcal{H}(P_4)$ is domestic with
\[
V_{0,0} = \{(a, b, 0, a + b) \mid a, b \in \mathbb{Z}^+, a + b \geq 2\},
V_{0,1} = \{(0, 0, 0, 0), (0, 1, 0, 1), (1, 0, 1, 0), (1, 0, 0, 1)\},
V_{1,0} = \{(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0), (1, 1, 1, 1)\}.
\]

Yet, $\mathcal{H}(P_4)$ is not tame, since $(1, 1, 1, 2)$ is a $(5, 1)$-position.

Similarly, $\mathcal{H}(P_5)$ is domestic with
\[
V_{0,0} = \{(a, b, c, 0, a + b + c) \mid (a, b, 0, d, e) \cup [(f, 0, h, f, h)\right]
\text{with } a + b = d + e, f < h,
V_{0,1} = \{(0, 0, 0, 0, 0), [(1, 0, 1, 0, 0)]\},
V_{1,0} = \{(0, 0, 0, 0, 1), (0, 1, 1, 1, 1)\}.
\]

Yet, $\mathcal{H}(P_5)$ is not tame, since $(1, 1, 1, 2, 0)$ is a $(5, 1)$-position.

Finally, $\mathcal{H}(P_6)$ is not domestic, since $(1, 0, 1, 1, 1, 2)$ is a $(4, 0)$-position.

Based on our calculations, we conjecture that the family of domestic but not tame games is large; for example, it contains the next two subfamilies.

**Definition 6.** Given a ground set $V$ partitioned by four pairwise disjoint subsets $V_1, V_2, V_3, V_4$, let us set $H_1 = V_1 \cup V_4, H_2 = V_2 \cup V_4, H_3 = V_3 \cup V_4, H_4 = V_1 \cup V_2 \cup V_3$, and $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$. We denote the corresponding position by $(|V_1|, |V_2|, |V_3|, |V_4|)$.

The game in Definition 6 is not tame: $(1, 2, 2, 2)$ is a $(7, 1)$-position.

**Definition 7.** Given a ground set $V$ partitioned by five pairwise disjoint subsets $V_1, V_2, V_3, V_4$, let us set $H_1 = V_1 \cup V_2 \cup V_3, H_2 = V_3 \cup V_4, H_3 = V_1 \cup V_3 \cup V_5, H_4 = V_2 \cup V_4, H_5 = V_5$, and $\mathcal{H} = \{H_1, H_2, H_3, H_4, H_5\}$. We denote the corresponding position by $(|V_1|, |V_2|, |V_3|, |V_4|, |V_5|)$.

The game in Definition 6 is not tame: $(1, 1, 1, 1, 1)$ is a $(1, 5)$-position.

7. Closing Remarks

After misère play was considered by Grundy and Smith [23] in 1956, it is a commonplace that the SG theory for the misère play is much more difficult than for the normal play. The reason is as follows. Although, by Remark 1 a simple transformation of the digraph of a game allows to convert the misère play in $G$ to the normal play in $G^-$ yet, a problem appears for the sums. The misère play of a sum $G^- = (G_1 + \cdots + G_n)^-$ differs from the sum of the corresponding misère games $G'' = G_1^- + \cdots + G_n^-$. Indeed, by Remark 1 in the first case we add one new terminal,
and an extra move, to the whole sum, while in the second case we add them to each game-summand. Thus, in general, the SG functions $G$ and $G^-$ may differ a lot.

The main goal of this paper is to outline cases when the above two functions are similar. Although the SG theory is not directly applicable to the misère playing sums, in general, but it is applicable, in case when each summand is pet, or miserable and forced, or (a weaker requirement) tame and returnable.

This idea should be attributed to Bouton, who applied it to Nim as early as in 1901, long before the SG theory was developed. The classical Nim is the sum of $n$ games, each of which (the one-pile Nim) is trivial. It is pet and forced. For a pile of $k$ tokens the normal SG function $G(k) = k$, while the misère one $G^-(k) = k$ for $k \geq 2$, but $G^-(0) = 1$ and $G^-(1) = 0$. Thus, there are only two swap positions: $k = 0$ is the $(0,1)$-position, and $k = 1$ is the $(1,0)$-position. Each of them can be reached by one move from any non-swap, $(k,k)$, position with $k \geq 2$.

Nim is the sum of $n$ such games and it has similar properties. Namely, $x = (x_1, \ldots, x_n)$ is a swap swap position of Nim if and only if $x_i$ is 0 or 1 for every $i \in [n] = \{1, \ldots, n\}$. Furthermore, $x$ is a $(0,1)$-position when the number of ones in $x$ is even, and $x$ is a $(1,0)$-position when this number is odd.

Given a non-swap position $x = (x_1, \ldots, x_n)$, obviously, a swap position can be reached from $x$ by one move if and only if $x_i > 1$ for exactly one $i \in [n]$. But in this case, obviously, there is a move from $x$ to a $(0,1)$- as well as another move to a $(1,0)$-position. Thus, Nim is miserable (and hence, tame) but it is not pet. In a pet game a $(0,1)$- as well as a $(1,0)$-position is reachable in one move from every non-swap position.

Moreover, Nim is forced, since after a swap position is reached, the $(0,1)$- and $(1,0)$-positions alternate in any play, since the number of piles containing one token will decrease one by one. From these observations Bouton concluded that the normal and misère plays of Nim are similar: the winning moves, if any, coincide in each position, unless a swap position can be reached by one move. Only in such (critical) position the player should inquire which version, normal or misère, is actually played, and then make a move to the swap position of the corresponding parity.

In fact, the same properties hold whenever each game-summand is tame (not necessarily pet or miserable) and returnable (not necessarily forced). Surprisingly many games have these properties. Let us recall, for example, the game Euclid. Its swap positions are the Fibonacci
pairs \((F_j, F_{j+1})\), which are \((0, 1)\)- or \((1, 0)\)-positions if and only if \(j\) is even or odd, respectively. There is only one move from \((F_j, F_{j+1})\) and it leads to \((F_j, F_{j-1})\). Moreover, for every non-swap position either there is no move to a swap one, or there is a move to an even Fibonacci pair, as well as some other move to an odd one \([24]\). Thus, the game Euclid is miserable and forced.

Every subtraction game is pet, as it was shown by Ferguson \([16]\) in 1974; all considered versions of Wythoff’s games are miserable; both are returnable but not forced; see Section \([6]\).

Thus, the misère play of any (possibly, mixed) sum of the games mentioned above, Nim, Euclid, or Wythoff, is not more difficult than the normal play.

Let us note however that both may be difficult. For example, no closed formula is known for the \(SG\)-function of the standard Wythoff game or any of its versions considered in Section \([6]\), but if such a formula, for the normal play, would be discovered, it will immediately allow us to solve both the normal and misère play of a sum that may include Wythoff-summands among others.

The sum is tame (resp., miserable, miserable and forced, miserable and returnable) whenever every summand is, in which case \(G^-\) is simply equal to \(G\) in all positions but swap ones. Thus, the winning player makes a move to a \((0, 0)\)-position from every positions, except a critical one, in which case (s)he makes a move to a \((0, 1)\) position of the sum.

At the end of 19th century students usually played the misère version of Nim, which was considered standard. So, this game was the goal of Bouton. Yet, a nicer formula, so called Nim-sum, describes the \(SG\) function of the normal version. For this reason, Bouton solved it first and then noticed that solution of the standard (that is, misère) version can be easily obtained from it, since the game of Nim is miserable and forced. Thus, in \([8]\) Bouton introduced, for the special case of Nim, five fundamental concepts of game and graph theories that appears in general only much later: (i) the \(P\)-positions, or in other words, the \textit{kernel} of an acyclic directed graph, (ii) the \(SG\) function, (iii) the misère play, (iv, v) miserable and forced games.

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