Competition and evolution in restricted space

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Abstract. We study the competition and the evolution of nodes embedded in Euclidean restricted spaces. The population evolves by a branching process in which new nodes are generated when up to two new nodes are attached to the previous ones at each time unit. The competition in the population is introduced by considering the effect of overcrowding of nodes in the embedding space. The branching process is suppressed if the newborn node is closer than a distance $\xi$ to the previous nodes. This rule may be relevant to describe a competition for resources, limiting the density of individuals and therefore the total population. This results in an exponential growth in the initial period, and, after some crossover time, approaching some limiting value. Our results show that the competition among the nodes associated with geometric restrictions can even, for certain conditions, lead the entire population to extinction.

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1. Introduction

A growing tree-like network can model different processes, such as technological or biological systems, represented by a set of nodes, in which each element of the network can create new elements. Innovation and discovery [1], artistic expression and culture [2], language structures [3, 4] and the evolution of life [5, 6] can naturally be represented by a branching tree [7], along with a wide range of real-life processes and phenomena [8–14].

The general branching process is defined mathematically as a set of objects (nodes) that do not interact: at each time step each object can give rise to new objects. By contrast, interacting branching processes are much more interesting and difficult to analyse [1]. A generalized tree, with one (or more) ancestor(s), has been used to depict evolutionary relationships between interacting nodes, such as genes, species, cultures. Besides the interaction among nodes, one can consider spatially embedded nodes. The evolution of networks embedded in metric spaces has attracted much attention [15–20].

In this work we study the evolution of a population: i.e., the number of nodes in the network influenced by the interaction among existing nodes and confined to a limited area, representing a competition of individuals for resources. We assume that the growing tree is embedded in a metric space and we consider that spatially close nodes, previously placed in the network, will suppress their ability to give birth to new nodes. In other words, overcrowding of nodes will drain the resources and suppress the offspring. In our model each node lives for three generations.

The evolution of the population of nodes is actually determined by two parameters: the minimum distance between any pair of nodes $\xi$, and the area in which the network is embedded, namely the linear size of the area, $L$. For simplicity, we assume that this area does not change in time. The population evolves in two different regimes. At the initial generations (time steps), one can see an exponential evolution, followed by a crossover time.

In the saturation regime, the size of the network will finally approach some limiting value. The network can even extinguish itself, if at some moment all its nodes occur in a small area. We investigated this possibility of complete extinction. The term

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`extinction`, for our model, implies the end of evolution and the absence of new generations. Interaction among the nodes inside the radius is defined by a parameter $\xi$ and the value of $L$ regulates the population’s dynamics. Our results show that, under certain conditions, the entire population can be led to extinction.

This paper is organized as follows. In section 2 we present our model details and obtain simple estimates for its growth. In section 3 we describe the populational evolution. The possibility of extinction for the model embedded in a bounded space is discussed in section 4, and, finally, in section 5, we summarize the results and present our conclusions.

2. The model

In our model, the population consists of interacting nodes spatially separated by some distance. We start our process from a single root node at time $t = 0$, as one can see in figure 1. The single root node (black circle in figure 1), can branch to produce up two new daughter nodes (dark gray circles) in a future generation, i.e., at the next time step. The position of each new node is randomly chosen inside a circle with a given radius ($0 \leq \text{radius} \leq 1$) centered in the parents’ positions. The attempt to add a newborn node is refused if the chosen position is closer than a distance $\xi$ from other nodes.

The attempt to generate offspring takes place at the time step after introduction of a new node: each node can produce daughter nodes only at this time. At the next time step—after three generations—the node is removed from the network.

At each time step, each of the nodes previously introduced attempts to branch; so, at each time step a new generation of nodes is born. The nodes are chosen uniformly at random one by one and, during a unit of time, we update the entire network. The total area of the system is limited: it has a natural spatial restriction. The first node is settled as the origin of the space; from this origin we set a maximum length for each spatial coordinate in a two-dimensional space. In other words, the geometric position of each node in the network, for our model, is restricted in the range $-L/2 \leq x \leq L/2$, $-L/2 \leq y \leq L/2$. The linear size of the area, $L$, is introduced as a parameter of the model and we assume that this area does not change in time. In our simulations we used open boundary conditions.
3. Population size evolution

If one allows the population dynamics to evolve embedded in an infinitely large system \((L \rightarrow \infty)\), the population always increases in size. The number of new nodes grows very fast as \(N = 2^t\) initially; after a certain crossover time \(t_x\), the growth is slower than exponential, as one can see in the figure 2.

At this regime the total population as a function of the time is \(P(t) - (t/\xi)^2\), for \(t\) greater than \(t_x\). We can estimate, very roughly, \(t_x\) from \(N(t < t_x) = 2^t\) and \(N(t > t_x) - (t/\xi)^2\), we have

\[2^t - \left(\frac{t_x}{\xi}\right)^2,\]

which leads to the estimate

\[t_x \approx \frac{2}{\ln 2} \ln \left(\frac{1}{\xi}\right),\]

at small \(\xi\). Our numerical results are considering that \(t \gg t_x\), for the estimates of the total population in the saturation regime. We should emphasize that, in our model, the population is confined to a limited area and it is not possible to grow indefinitely.

The general result of our simulations for this model is exhibited in figure 3, where we consider a two-dimensional space and a sufficiently small value of \(\xi\), in comparison with the linear size of the system \(L\). Initially, the population grows exponentially and, after a certain crossover time, one can see that the population reaches a steady state. After the crossover time, \(P(t)\) is nearly constant. The maximum value of the population is \(P_{\text{max}} = (L/\xi)^2\), since we are considering a two-dimensional space for the simulations. The growth of the population is \(f(t) \sim 2^t\), is also exhibited. Data are averaged over 50 samples.

Figure 2. Time evolution of the number of new nodes for different values of \(\xi\). Behavior for the initial time steps, \(f(t) \sim 2^t\), is also exhibited. Data are averaged over 50 samples.
One can see that, in the saturation regime, the total population is smaller than $P_{\text{max}}$. This is due to the fact that the interaction among the nodes does not allow that each possible offspring may be created at some generation, keeping the total population below this limit.

4. Extinction

For a small population, the possibility of extinction is higher. There is even a chance that the network will collapse if the offspring created are too few and all its nodes occur in a small area at the same time. It is a well-known characteristic that the smaller a population, the more susceptible it is to extinction by various causes [22].

Figure 3. Time evolution of a single realization of the population dynamics, for $L = 10$ and $\xi = 0.2$. The inset shows that, even for one sample, the population reaches a constant value after some time.

Figure 4. Evolution of the population for a single realization of our model. The population grows for the initial time steps, but it can also decrease and, after some time, extinguish. This behavior is maintained for a long time, as one can see in the inset. The parameters are $\xi = 0.9$ and $L = 1$. 

One can see that, in the saturation regime, the total population is smaller than $P_{\text{max}}$. This is due to the fact that the interaction among the nodes does not allow that each possible offspring may be created at some generation, keeping the total population below this limit.
Figure 4 demonstrates an example of the evolution of the population, which in this case has $\xi = 0.9$ and $L = 1$. The population rapidly increases and the system enters the fluctuation regime, in which the population fluctuates around a mean value for a few generations. After a while, the population decreases and is extinguished.

The picture which we observe agrees with traditional views of extinction processes, which show ‘relatively long periods of stability alternating with short-lived extinction events’ (Raup) [23]. The competition for resources, combined with the restricted space, limits the total population and, for some values of the parameters in the model, the population will become extinct. In real situations, extinction may require external factors, such as environmental stress [24], or an internal mechanism, such as mutation [25]. One can see an example in the case of extinction of reindeer population on St. Matthew Island [26]. The US Coast Guard released 29 reindeer on the island during World War II, in 1944. The reindeer population grew exponentially and, in 1963, was about 6000 animals—47 reindeer per square mile. Overpopulation, limited food supply and the exceptionally severe winter of 1963–4 significantly affected subsequent offspring: the reindeer population of St. Matthew Island dropped to 42 animals in 1966 and died out by the 1980s.

This kind of extinction may also occur in branching annihilating random walks and other related processes, studied in [27, 28], in which random processes play the role of an external factor, internal mechanism or environmental stress that may lead to extinction.

In our model, if we choose a large enough value of the parameter $\xi$, the population will be small and the number of new nodes, after some generations, is likely to decrease and, sometimes, vanish. When the nodes’ competition increases, the population may decay or even vanish, as we can see by the rapid decrease of the new nodes in figure 4.

We investigated the state of the branching process after a long period of time, $t_{\text{observation}} = 10^5$ generations (i.e., time steps). For the case when $\xi \to 1$ the population always dies off, since no offspring is allowed, for any value of $L$. We simulated our model for 100 different samples and for various values of $L$ and $\xi$. From this data we obtain the probability of extinction $\Pi_{\text{ext}}$, i.e., the fraction of samples in which the population dies off before the $10^6$th generation, for different $L$, as one can see in figure 5.

In figure 6 one can see a diagram where extinction regions (below the curve) and non-extinction regions (above the curve) are shown. Each point is figure 6 is defined as
follows. For a given value of $L$, and considering $10^6$ generations, the value of $\xi$ for which the probability of extinction goes to one defines one point in the graphic of figure 6. Our results show that high values of populational density, represented in our model for small $L$ and large $\xi$, can lead to extinction. This picture is a different representation of the probability of extinction in which we are considering the values of the parameters $L$ and $\xi$ corresponding to $\Pi_{\text{ext}} \to 1$.

5. Conclusions

We studied the evolution of a population embedded into a restricted space in which the interaction among the population is determined by the relative position of nodes in space. Our model generates competition between species or individuals (represented by the nodes). Starting from a single root node and, at each time step, each existent node in the network can branch to produce up to two new daughter nodes at the next generation. The new nodes are not allowed to emerge closer than a certain distance of a pre-existent node, defined by a parameter $\xi$, i.e., overcrowding suppresses the ‘fertility’ of population. Evolutionary processes are usually considered in low dimensions and, for this case, our results do not depend qualitatively on the system’s dimension for $D \leq 3$.

We have demonstrated that the embedding of the network into a restricted area, which is natural for general populational evolution, set limits to growth and, for some values of the model’s parameters, can result in complete extinction. The simple model we studied can schematically describe a real process in nature.

Acknowledgments

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References

[1] Sood V, Mathieu M, Shreim A, Grassberger P and Paczuski M 2010 Phys. Rev. Lett. 105 178701
[2] Cowlishaw G and Mace R 1996 Ethol. Sociobiol. 17 87
[3] Schwammenthe V and de Oliveira P M C 2009 Physica A 388 2874
[4] Dorogovtsev S N and Mendes J F F 2001 Proc. R. Soc. Lond. B 268 2603
[5] Koonin E V 2007 Biol. Direct. 2 21
[6] Moret B M E, Nakhleh L, Warnow T, Linder C R, Tholse A, Padolina A, Sun J and Timme R 2004 IEEE ACM Trans. Comput. Biol. Bioinform. 1 13
[7] Huson D H and Bryant D 2006 Mol. Biol. Evol. 23 254
[8] Dorogovtsev S N and Mendes J F F 2002 Evolution of Networks: from Biological Nets to the Internet and WWW (Oxford: Clarendon)
Dorogovtsev S N 2010 Lectures on Complex Networks (Oxford: Oxford University Press)
[9] Alava M J and Lauritsen K B 2009 Encyclopedia of Complexity and Systems Science ed R Meyers (Heidelberg: Springer) p 644
[10] Gray R D and Atkinson Q D 2003 Nature 426 435
[11] Dorogovtsev S, Krapivsky P L and Mendes J F F 2008 Eurphys. Lett. 81 30004
[12] Dorogovtsev S N and Mendes J F F 2002 Adv. Phys. 51 1079
[13] Albert R and Barabási A-L 2002 Rev. Mod. Phys. 74 47
[14] Newman M E J 2003 SIAM Review 45 167
[15] Dall J and Christensen M 2002 Phys. Rev. E 66 016121
[16] Kleinberg J 2000 Nature 406 845
[17] Carmi S, Carter S, Sun J and ben-Avraham D 2009 Phys. Rev. Lett. 102 238702
[18] Cartozo C C and De Los Rios P 2009 Phys. Rev. Lett. 102 238703
[19] Boguñá M and Krioukov D 2009 Phys. Rev. Lett. 102 058701
[20] Krioukov D, Papadopoulos F, Kitsak M, Vahdat A and Boguñá M 2010 Phys. Rev. E 82 036106
[21] Vandermeers J 2010 Nature Educ. Knowledge 3 15
[22] Sznajd-Weron K and Weron R 2001 Physica A 293 559
[23] Raup M D 1986 Science 231 4745
[24] Roberts B W and Newman M E J 1996 J. Theor. Biol. 180 39
[25] Sibani P, Schmidt M R and Alstrøm P 1995 Phys. Rev. Lett. 75 2055
[26] Klein D R 1968 Wildl. Manag. 32 350
[27] Takayasu H and Tretyakov A Y 1992 Phys. Rev. Lett. 68 3060
[28] Jensen I 1994 Phys. Rev. E 50 3623

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