Explicit Boij-Söderberg theory of ideals from a graph isomorphism reduction

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Abstract
In the origins of complexity theory Booth and Lueker showed that the question of whether two graphs are isomorphic or not can be reduced to the special case of chordal graphs. To prove that, they defined a transformation from graphs $G$ to chordal graphs $BL(G)$. The projective resolutions of the associated edge ideals $I_{BL(G)}$ is manageable and we investigate to what extent their Betti tables also tell non-isomorphic graphs apart. It turns out that the coefficients describing the decompositions of Betti tables into pure diagrams in Boij-Söderberg theory are much more explicit than the Betti tables themselves, and they are expressed in terms of classical statistics of the graph $G$.

1 Introduction

According to the main theorem of Boij-Söderberg theory every Betti table can be expressed as a weighted non-negative sum of particularly elementary tables, called pure Betti tables [3][8][11]. These weight coefficients are usually even more cumbersome to express than the Betti numbers themselves, and from explicit calculations they tend to involve many binomial coefficients and alternating signs. In our setting it turns out that they are more straightforward to state than the Betti numbers, and we can provide elementary formulas for them.

In one of the first results of complexity theory, Booth and Lueker [5] proved that two graphs are isomorphic if and only if their corresponding Booth-Lueker graphs are isomorphic. This reduced the problem of graph isomorphism to the special class of Booth-Lueker graphs, and they have several attractive properties. The main result of this paper, Theorem 3.5, regards the ideal of the Booth-Lueker graph of a graph on $n$ vertices and $d_i$ vertices of degree $i$, for $i = 0, \ldots, n−1$. If $c_j$ is the weight of the pure 2-linear table with $j$ non-zero entries

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on the second row, then
\[ c_j = \frac{d_{j-n+1}}{j} + \frac{1}{j(j+1)} \sum_{i=n}^{n-1} d_i \quad \text{for} \quad n \leq j \leq 2n - 2, \quad c_{n-1} = \frac{d_0}{n}, \]
and the other Boij-Söderberg coefficients vanish. Further on, we also describe in Section 3 the explicit Betti numbers and anti-lecture hall compositions of these ideals.

In Theorem 4.2 we study the dual graph situation, constructing the ideal of the complement of the Booth-Lueker graph. We show that if the original graph has \( n \) vertices and \( m \) edges, then
\[ c_j = \frac{m}{j(j+1)} \quad \text{for} \quad m \leq j \leq m + n - 4, \quad c_{m+n-3} = \frac{m}{m + n - 3} \]
and the other coefficients vanish. We also give explicit results on Betti numbers and anti-lecture hall compositions for these ideals in Section 4.

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2 Tools and background results

In this section we list some useful tools from graph theory, commutative algebra and combinatorics, without introducing anything new.

2.1 Basic definitions and standard tools

2.1.1 Graph Theory

All graphs are assumed to be simple. The degree of a vertex is the number of edges incident to it. It is common to discuss the degree sequence of a graph, that is, the degrees sorted downwards. We will not do that.

Definition 2.1. The degree vector or degree statistics of a graph \( G \) on \( n \) vertices is the column vector 
\[ d_G := (d_0, d_1, \ldots, d_{n-1})^T \]
where \( d_i \) is the number of vertices of degree \( i \) in \( G \).

Remark 2.2. Actually the word degree was brought from algebra to graph theory by Petersen [13] in 1891, addressing a problem in invariant theory by Hilbert.

The complement of a graph \( G \) is denoted by \( \overline{G} \) and the induced subgraph of \( G \) on the set of vertices \( W \) by \( G[W] \).

Notation 2.3. Given a graph \( G \), we denote by \( I_G \) its edge ideal, in a polynomial ring \( S \) with as many variables as the vertices of \( G \).
2.1.2 Combinatorics

The following well-known result is used several times later on. It is known as Vandermonde’s identity or convolution, and proved in for example Section 1.2.6.I of [13].

Lemma 2.4. For all integers \( r, s, \) and \( n \),

\[
\sum_k \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n} \quad \text{and} \quad \sum_k (-1)^{r-k} \binom{r}{k} \binom{s+k}{n} = \binom{s}{n-r}.
\]

We will also make use several times of the following combinatorial result.

Lemma 2.5. If \( P \in \mathbb{Q}[x] \) is a polynomial of degree less than \( N \) and such that \( P(n) \in \mathbb{Z} \) for all \( n \in \mathbb{Z} \), then

\[
\sum_{i=0}^{N} (-1)^i \binom{N}{i} P(i) = 0.
\]

Proof. By Lemma 4.1.4 in [4], there are \( a_0, \ldots, a_{N-1} \) in \( \mathbb{Z} \) such that

\[
P(x) = \sum_{j=0}^{N-1} a_j \binom{x+j}{j}.
\]

Therefore, it is enough to prove that the statement holds for any polynomial of the form \( (1 + x)^N \), with \( j < N \). This follows from differentiating \( j \) times the binomial formula \( (1 + x)^N = \sum_{i=0}^{N} \binom{N}{i} x^i \), and setting \( x = -1 \).

2.2 Booth-Lueker graphs and the graph isomorphism problem

In this section we introduce our object of study, namely that of the Booth-Lueker graph \( BL(G) \) associated to a given graph \( G \), but before doing so we explain the motivation behind the construction.

A fundamental question in graph theory (with applications primarily in computer science) that goes by the name of “graph isomorphism problem” is the following: are two given graphs \( G \) and \( G' \) isomorphic? To the best of our knowledge, it is still an open question whether the isomorphism for general graphs can be determined in polynomial time or not. It is also unknown whether the graph isomorphism problem is NP-complete. There are some upper bounds: in 1983 Babai and Luks [2] showed that the problem can be solved in moderately exponential \( O(\sqrt{n \log n}) \) time, where \( n \) is the number of vertices, and in 2015 Babai claimed that the problem can actually be solved in quasipolynomial time. A mistake in the proof was found by Helfgott, then Babai fixed it, and in 2017 he re-claimed the quasipolynomial time (see [1]). As far as we know, the proof has not been fully checked yet, though.
However, there are many results known about special classes of graphs. The case of interest to us is that of chordal graphs, because of the following polynomial-time reduction, an old complexity theory result proved by Booth and Lueker [5].

**Theorem 2.6.** Arbitrary graph isomorphism is polynomially reducible to chordal graph isomorphism.

To prove this, they made use of the following construction, which we therefore name after them.

**Definition 2.7.** For any graph $G$ let $BL(G)$ be the graph with vertex set $V(G) \cup E(G)$ and edges $uv$ for every pair of vertices in $G$ and $ue$ for every vertex $u$ incident to an edge $e$ in $G$. We call $BL(G)$ the Booth-Lueker graph of $G$.

Notice that both $BL(G)$ and its complement are chordal, for every $G$. They are split graphs. Thus, we have two interesting ideals to define actually. In Section 3 we study algebraic invariants of the edge ideals of Booth-Lueker graphs, and in Section 4 we do the same for the complements.

**Example 2.8.** Consider the path on four vertices. It is depicted in Figure 1 with its Booth-Lueker graph.

We can think of the Booth-Lueker constructions in the following terms: we take the original graph $G$ on the left, add on the right new vertices corresponding to the edges of $G$ and connect these vertices with the respective ends of the associated edge of $G$. Then we complete the left part of the graph. Later on we will use again this terminology of a “left” and “right” part of $BL(G)$.

### 2.3 Boij-Söderberg theory

Denote as usual $S = k[x_1, \ldots, x_n]$. Boij-Söderberg theory deals with writing the Betti table of a finitely generated graded $S$-module as a sum of simpler pieces, coming from the so-called “pure Betti tables”; to each sequence $n = (n_0, \ldots, n_k)$ of strictly increasing non-negative integers, we associate the table $\pi(n)$ with entries

$$
\pi(n)_{i,j} := \begin{cases} 
\prod_{k \neq 0,j} \frac{n_k - n_0}{n_k - n_i} & \text{if } i \geq 0, j = n_i, \\
0 & \text{otherwise},
\end{cases}
$$
and this is called the pure Betti table associated to $n$. We can moreover give a partial order to such sequences by setting
\[(n_0, \ldots, n_s) \geq (m_0, \ldots, m_t)\]
whenever $s \leq t$ and $n_i \geq m_i$ for all $i \in \{0, \ldots, s\}$.

**Remark 2.9.** It is customary to refer to a sequence of strictly increasing integers as above as a *degree sequence*. Since the same name also corresponds to a concept in graph theory, we just do not refer to them, in order to avoid confusion.

**Theorem 2.10.** For every finitely generated graded $S$-module $M$, there is a strictly increasing chain $n_1 < \cdots < n_p$ of strictly increasing sequences of $n + 1$ non-negative integers and there are numbers $c_1, \ldots, c_p \in \mathbb{Q}_{\geq 0}$ such that
\[
\beta(M) = c_1 \pi(n_1) + \cdots + c_p \pi(n_p).
\]

**Definition 2.11.** We refer to the non-negative rational numbers $c_1, \ldots, c_p$ as in the theorem above as Boij-Söderberg coefficients of $M$.

**Example 2.12.** Take $I = (x^2, xy, y^3) \subset S = K[x, y]$. Then one can compute that
\[
\beta(S/I) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 1 \\
0 & 1 & 1
\end{pmatrix}
= \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 3 & 2
\end{pmatrix}
+ \frac{1}{4} \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{pmatrix}
+ \frac{1}{4} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 4 & 3
\end{pmatrix}
= \frac{1}{2} \pi(0, 2, 3) + \frac{1}{4} \pi(0, 2, 4) + \frac{1}{4} \pi(0, 3, 4),
\]
so that $n_1 = (0, 2, 3)$, $n_2 = (0, 2, 4)$, $n_3 = (0, 3, 4)$, and the Boij-Söderberg coefficients are $c_1 = 1/2$, $c_2 = 1/4$, and $c_3 = 1/4$.

**Remark 2.13.** In the same way as for the graded Betti numbers, there is a very straightforward algorithm to compute Boij-Söderberg coefficients, but (as far as we know) there are no general explicit formulas, so we don’t know *a priori* what to expect in general.

For a more detailed account of Boij-Söderberg theory, see for instance the survey [11] by Floystad. There the notation for a degree sequence is $d$ instead of $n$, but we already use $d_G$ to denote the degree vector of a graph $G$.

As motivated in Section 2.4 we will mostly be interested in ideals $I$ with 2-linear resolutions, i.e., such that the Betti table of $S/I$ looks like
\[
\beta(S/I) = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \beta_{1, 2} & \beta_{2, 3} & \cdots & \beta_{p, p+1}
\end{pmatrix}.
\]
By Boij-Söderberg theory, such a Betti table will be the weighted average of certain pure tables of the form $\pi(0, 2, 3, \ldots, s, s + 1)$ like for instance
\[
\pi(0, 2) = \begin{pmatrix}
1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots
\end{pmatrix}.
\]
or
\[
\pi(0, 2, 3) = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & 3 & 2 & 0 & \cdots
\end{pmatrix}.
\]

Following the discussion around converting Betti diagrams and Boij-Söderberg coefficients into each other from the paper [10] we record these pure diagrams as the two vectors
\[
\pi_1 = (1, 0, \ldots) \quad \text{and} \quad \pi_2 = (3, 2, 0, \ldots),
\]
i.e., we denote by \(\pi_s\) the second row of \(\pi(0, 2, 3, \ldots, s, s+1)\), except for the first entry, which is always zero. Slightly larger examples are
\[
\pi_7 = (28, 112, 210, 224, 140, 48, 7, 0, 0, \ldots),
\]
\[
\pi_8 = (36, 168, 378, 504, 420, 216, 63, 8, 0, 0, \ldots),
\]
\[
\pi_9 = (45, 240, 630, 1008, 1050, 720, 315, 80, 9, 0, \ldots).
\]

The zeros in the end extend indefinitely, and it will be convenient to us to write a specific amount of them, as it will become clear in the next sections.

### 2.4 On 2-linear resolutions

The following was proved by Fröberg [12] and refined by Dochtermann and Engström [7].

**Theorem 2.14.** For a simple graph \(G\), the edge ideal \(I_G\) has 2-linear resolution if and only if the complement of \(G\) is chordal. Moreover, given a graph \(G\) whose complement is chordal, the Betti numbers of \(S/I_G\) can be computed as follows:
\[
\beta_{i,i+1}(S/I_G) = \sum_{W \subseteq V(G), \#W = i+1} (-1 + \text{the number of connected components of } \overline{G}[W]).
\]

Engström and Stamps showed in Lemma 3.1 and Theorem 3.2 of [10] how Betti tables and Boij-Söderberg coefficients are related for 2-linear resolutions. First of all, let us introduce the following notation: if a graded \(S\)-module \(M\) has Betti table
\[
\beta(M) = \begin{pmatrix}
m & 0 & 0 & \cdots & 0 \\
0 & \beta_{1,2} & \beta_{2,3} & \cdots & \beta_{p,p+1}
\end{pmatrix},
\]
we denote by \(\omega(M) = (\beta_{1,2}, \ldots, \beta_{p,p+1})\) the Betti vector of \(M\).

**Lemma 2.15.** Let \(\Omega\) be the square matrix of size \(n + m - 1\) whose \(ij\)-entry is \(j^{(i+1)}\). Then \(\Omega\) is invertible and the inverse \(\Omega^{-1}\) has \(ij\)-entry \((-1)^{i-j} \binom{i+j}{i+1}\).

Furthermore, if \(M\) has a Betti table as above and if \(c = (c_1, \ldots, c_{n+m-1})\) is the vector with the Boij-Söderberg coefficients of \(M\), then we have
\[
c = \omega(M)\Omega^{-1}.
\]
### 2.4.1 Anti-lecture hall compositions

Recall that a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) of integers such that

\[
t \geq \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \cdots \geq \frac{\lambda_n}{n} \geq 0
\]

is called **anti-lecture hall composition** of length \( n \) bounded above by \( t \). For further information about anti-lecture hall compositions, see [6], where they were introduced, and also [15].

To a 2-linear ideal (equivalently, a graph with chordal complement) we can associate a unique anti-lecture hall composition with \( t = 1 \) and \( \lambda_1 = 1 \), see Section 4 of [10]. The following result is Proposition 4.11 from that paper.

**Lemma 2.16.** Let \( I \) be a 2-linear ideal, denote by \( \lambda = (\lambda_1, \ldots, \lambda_{m-1}) \) the anti-lecture hall composition associated to \( I \), \( \omega(S/I) \) the Betti vector of \( S/I \) and \( \Psi \) the invertible \( m \times m \) matrix with \( ij \)-entry \( \Psi_{ij} = \binom{j-1}{i-1} \). Then we have

\[
\lambda = \omega(S/I)\Psi^{-1}.
\]

### 3 The Boij-Söderberg theory of ideals of Booth-Lueker graphs

In this section we determine the Betti tables, Boij-Söderberg coefficients, and anti-lecture-hall compositions of the edge ideals of Booth-Lueker graphs. It turns out that the information carried over to the algebraic setting from the graphs are all compiled in its degree vector. We will employ several results by Engström and Stamps [10] on 2-linear resolutions and Boij-Söderberg theory to reach these results.

#### 3.1 Degree vector and Betti numbers

For an \( S \)-module \( S/I \) with 2-linear resolution of length \( n \), we denote the non-trivial part of its Betti table as

\[
\omega(S/I) := (\beta_{1,2}, \beta_{2,3}, \ldots, \beta_{n,n+1})
\]

and call it the **reduced Betti vector** of \( S/I \). If the ideal \( I \) is the edge ideal of a graph \( G \), we just write \( \omega(G) \).

**Proposition 3.1** (From the degree vector to the Betti numbers). If \( G \) is a graph on \( n \) vertices and \( m \) edges, \( A \) is the matrix of size \((n+m-1) \times n \) defined by \( A_{ij} = \binom{i+n-2}{i} \), and \( v \) is the column \((n+m-1)\)-vector defined by \( v_i = \binom{n}{i+1} \), then

\[
\omega(BL(G)) = Ad_G - v,
\]

where \( d_G = (d_0, d_1, \ldots, d_{n-1})^T \) is the degree vector of \( G \).
Proof. On the right-hand side we have

\[(Ad_G - v)_i = \sum_{j=1}^{n} \binom{j + n - 2}{i} d_{j-1} - \binom{n}{i+1}.\]

We want to use the formula in Theorem 2.14. Let \(W\) be a set of \(i+1\) vertices. If all of \(W\) is in the left (independent) part of \(BL(G)\), then the induced subgraph has \(i+1\) connected components. If one of the vertices of \(W\) is in the right part of \(BL(G)\), then this vertex is connected in \(BL(G)\) to \(n-2\) of the \(n\) vertices on the left. So actually if \(W\) has some vertices in the right part there are not many possibilities for the number of connected components of the induced subgraph: they can either be one, two or three. By applying the formula we find that

\[\beta_{i,i+1} = i \binom{n}{i+1} + 2 \binom{n-2}{i-2} m + 2 \binom{n-2}{i-1} m + \sum_{k=2}^{i} \sum_{j=1}^{n} d_{j-1} \binom{j-1}{i-k} \binom{n-1}{i-k},\]

where we used that \(\sum_{j=1}^{n} (j-1)d_{j-1} = 2m\). So we want to prove that

\[(i+1) \binom{n}{i+1} + \sum_{j=1}^{n} d_{j-1} \sum_{k=1}^{i} \binom{j-1}{i-k} \binom{n-1}{i-k} = \sum_{j=1}^{n} \binom{j+n-2}{i} d_{j-1},\]

which can be rewritten as

\[\sum_{j=1}^{n} d_{j-1} \sum_{k=0}^{i} \binom{j-1}{k} \binom{n-1}{i-k} = \sum_{j=1}^{n} \binom{j+n-2}{i} d_{j-1}\]

since \((i+1)^{n} \binom{n}{i+1} = n^{n-1}\). We can now conclude thanks to the special case of Vandermonde’s identity

\[\sum_{k=0}^{i} \binom{j-1}{k} \binom{n-1}{i-k} = \binom{j+n-2}{i},\]

(see Lemma 2.4).

Example 3.2. Consider the graph \(G\) on 7 vertices and 8 edges depicted in Figure 2. Thanks to Proposition 3.1 we compute the Betti numbers of \(BL(G)\). We
consider the matrix

$$A = \begin{pmatrix}
6 & 7 & 8 & 9 & 10 & 11 & 12 \\
15 & 21 & 28 & 36 & 45 & 55 & 66 \\
20 & 35 & 56 & 84 & 120 & 165 & 220 \\
15 & 35 & 70 & 126 & 210 & 330 & 495 \\
6 & 21 & 56 & 126 & 252 & 462 & 792 \\
1 & 7 & 28 & 84 & 210 & 462 & 924 \\
0 & 1 & 8 & 36 & 120 & 330 & 792 \\
0 & 0 & 1 & 9 & 45 & 165 & 495 \\
0 & 0 & 0 & 0 & 1 & 11 & 66 \\
0 & 0 & 0 & 0 & 0 & 1 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

constructed by \(A_{ij} = \binom{j+5}{i}\). The degree vector of \(G\) is the column vector

$$d_G = (0, 0, 0, 5, 2, 0, 0, 0)^T.$$  

Then

$$Ad_G = (58, 212, 448, 602, 532, 308, 112, 23, 2, 0, 0, 0, 0, 0)^T.$$ 

Now let

$$v = (21, 35, 35, 21, 7, 1, 0, 0, 0, 0, 0, 0, 0, 0)^T,$$

where \(v_i = \binom{7}{i+1}\) for \(i = 1, 2, \ldots, 14\). Then the Betti vector is

$$\omega(BL(G) = Ad_G - v = (37, 177, 413, 581, 525, 307, 112, 23, 2, 0, 0, 0, 0, 0)^T$$

and the Betti table is

$$\beta(BL(G)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 37 & 177 & 413 & 581 & 525 & 307 & 112 & 23 & 2 & 0 & 0 & 0 & 0 \end{pmatrix}. $$
Remark 3.3. The reason for the extra zeros comes from Hilbert’s Syzygy Theorem, giving a bound on the projective dimension: we simply write all the possibly non-zero entries.

Proposition 3.4 (From the Betti numbers to the degree vector). Denoting $\Delta(G)$ the largest vertex degree in $G$, and denoting by $B$ the square submatrix of $A$ obtained by taking the first $\Delta(G) + 1$ columns and the rows from $n - 1$ to $n + \Delta(G) - 1$, we have $(B^{-1})_{ij} = (-1)^{i+j}B_{ij}$ and

$$d_G = B^{-1}(\beta_{n-1,n+1}, \beta_{n,n+1}, \ldots, \beta_{n+\Delta(G)-1,n+\Delta(G)}).$$

That is, we can compute the degree vector in terms of the (last non-zero) Betti numbers.

Proof. We notice that the “effective length” of the vector $v$ is $n - 1$, while that of $Ad_G$ is $n + \Delta(G) - 1$. Therefore the Betti number $\beta_{i,i+1}$ is equal to the entry $(Ad_G)_{i,j}$, for $i = n, \ldots, n + \Delta(G) - 1$, while $\beta_{n-1,n} = (Ad_{G-v})_{n-1,n} = (Ad_G)_{n-1,n} - 1$. The entries of the matrix $A$ that we use in these computations are just

$$A_{ij}, \quad \text{for} \quad i = n - 1, \ldots, n + \Delta(G) - 1, \quad j = 1, \ldots, \Delta(G) + 1,$$

and therefore we define the square matrix $B$ as in the statement above. More explicitly, this submatrix of $A$ has the form

$$B = \begin{pmatrix}
1 & B_{12} & B_{13} & \cdots & B_{1,\Delta(G)+1} \\
0 & 1 & B_{23} & \cdots & B_{2,\Delta(G)+1} \\
0 & 0 & 1 & \cdots & B_{3,\Delta(G)+1} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}.$$

This matrix is invertible and so we clearly have the stated formula. It only remains to prove that the matrix $C$ with entries $C_{ij} := (-1)^{i+j}B_{ij}$ is the inverse of $B$. Let’s see that $BC$ is the identity matrix: it is clear that $(BC)_{ij} = 0$ if $j < i$, and also in the sum

$$(BC)_{ii} = (-1)^{i} \sum_{k=1}^{\Delta(G)+1} (-1)^{k} \binom{k+n-2}{i+n-2} \binom{i+n-2}{k+n-2}$$

the only non-zero summand corresponds to $k = i$ and it is 1. So let $i < j$ and consider

$$(BC)_{ij} = (-1)^{j} \sum_{k=i}^{j} (-1)^{k} \binom{k+n-2}{i+n-2} \binom{j+n-2}{k+n-2}$$

$$= (-1)^{i+j} \frac{(j+n-2)!}{(i+n-2)!} \sum_{k=0}^{j-i} (-1)^{k} \frac{1}{k!(j-i-k)!}.$$

This is zero because $\sum_{k=0}^{N}(-1)^{k}\frac{1}{k!(N-k)!} = \frac{1}{N!} \sum_{k=0}^{N}(-1)^{k}\binom{N}{k}$ is zero for all $N > 0$ (see Lemma 2.5). □
3.2 From degree vector to Boij-Söderberg coefficients

Recall that, for a graph $G$, by Boij-Söderberg coefficients of $G$ we mean the Boij-Söderberg coefficients of $S/I_G$.

Theorem 3.5. Let $G$ be a graph with $n$ vertices and $m \geq n$ edges, and let $d_G = (d_0, d_1, \ldots, d_{n-1})$ be its degree vector. Then the $j$th Boij-Söderberg coefficient of $BL(G)$ is

$$c_j = \begin{cases} 
0 & \text{if } j \leq n - 2, \\
\frac{d_0}{j} + \frac{\sum_{i=1}^{n-1} d_i}{j(j+1)} - \frac{n}{j(j+1)} & \text{if } j = n - 1, \\
\frac{d_{j-n+1}}{j} + \frac{\sum_{i=1}^{n-j} d_i}{j(j+1)} & \text{if } n - 1 < j \leq 2n - 2, \\
0 & \text{if } j > 2n - 2.
\end{cases}$$

Proof. Putting Lemma 2.15 and (2) together, we find that the $j$th Boij-Söderberg coefficient of $BL(G)$ is

$$c_j = \frac{(-1)^j \sum_{k=1}^n d_{k-1} \sum_{i=1}^{n+m-1} (-1)^i \frac{1}{i} \binom{i+1}{j+1} \binom{k+n-2}{i}}{j(j+1)} + \frac{(-1)^j \sum_{i=1}^{n+m-1} (-1)^i \frac{1}{i} \binom{i+1}{j+1} \binom{n}{i+1}}{j(j+1)} \frac{\sum_{i=1}^{n-j} d_i}{j(j+1)}$$

If $j \leq n - 1$,

$$c_j = \frac{(-1)^j \sum_{k=1}^n d_{k-1} \sum_{i=1}^{n+m-1} (-1)^i \frac{1}{i} \binom{i+1}{j+1} \binom{k+n-2}{i} - \frac{n}{j(j+1)}}{j(j+1)}$$

If $j > n - 1$,

$$c_j = \frac{(-1)^j \sum_{k=1}^n d_{k-1} \sum_{i=1}^{n+m-1} (-1)^i \frac{1}{i} \binom{i+1}{j+1} \binom{k+n-2}{i}}{j(j+1)}$$

For convenience of notation, write $b_{j,k} := \sum_{i=1}^{n+m-1} (-1)^i \frac{1}{i} \binom{i+1}{j+1} \binom{k+n-2}{i}$. Clearly for $j > 2n - 2$ we have that $b_{j,k} = 0$ as every summand is zero. Now fix $k$ and suppose $j \leq 2n - 2$. In order for $b_{j,k}$ to be non-zero, $j \leq k + n - 2$ must be satisfied. In the boundary condition $j = k + n - 2$ we can easily compute the sum as there is only the value $i = j$ for which both the binomial coefficients are non-zero, so that the sum is

$$b_{j,j-n+2} = \frac{(-1)^j}{j}.$$

Finally, if $j < k + n - 2$, then

$$b_{j,k} = \sum_{i=1}^{n+m-1} (-1)^i \frac{1}{i} \binom{i+1}{j+1} \binom{k+n-2}{i}$$

$$= \frac{1}{j(j+1)} \sum_{i=1}^{n+m-1} (-1)^i \binom{i+1}{j+1} \binom{k+n-2}{i}.$$
Consider now the polynomial $P(x) := (x + 1)(x - 1)$, which has degree $j$. If we set $N := k + n - 2$, by Lemma 2.5 we have that

$$b_{j,k} = \frac{1}{j(j+1)} \sum_{i=1}^{n+m-1} (-1)^i (i+1) \binom{k+n-2}{i}$$

$$= \frac{1}{j(j+1)} \left[ \sum_{i=0}^{n+m-1} (-1)^i (i+1) \binom{k+n-2}{i} - (-1)^{j+1} \right]$$

$$= \frac{(-1)^j}{j(j+1)}.$$

Putting these things together, we find the stated expressions for the Boij-Söderberg coefficients.

Example 3.6. Consider again the graph $G$ in Example 3.2. Recall that

$$\pi_7 = (28, 112, 210, 224, 140, 48, 7, 0, 0, 0, 0, 0, 0, 0),$$
$$\pi_8 = (36, 168, 378, 504, 420, 216, 63, 8, 0, 0, 0, 0, 0, 0),$$
$$\pi_9 = (45, 240, 630, 1008, 1050, 720, 315, 80, 9, 0, 0, 0, 0, 0, 0).$$

With Theorem 3.5, we can easily find that the Betti vector for $BL(G)$, computed in Example 3.2, can be encoded as

$$(37, 177, 413, 581, 525, 307, 112, 23, 2, 0, 0, 0, 0, 0) = \frac{1}{8} \pi_7 + \frac{47}{72} \pi_8 + \frac{2}{9} \pi_9.$$

Or to be more concise, the only non-zero Boij-Söderberg coefficients are

$$c_7 = \frac{d_1}{7} + \frac{d_2 + d_3 + d_4 + d_5 + d_6}{7 \cdot 8} = \frac{1}{8},$$
$$c_8 = \frac{d_2}{8} + \frac{d_3 + d_4 + d_5 + d_6}{8 \cdot 9} = \frac{47}{72},$$
$$c_9 = \frac{d_3}{9} + \frac{d_4 + d_5 + d_6}{9 \cdot 10} = \frac{2}{9},$$

where $d_i$ is the number of vertices of degree $i$ in $G$.

### 3.3 From degree vector to anti-lecture hall compositions

For a brief introduction to anti-lecture hall compositions, see Section 2.4.1. For an illustration of the next result, see Example 3.8

**Proposition 3.7.** Take a graph $G$ with $n$ vertices and $m$ edges, and assume that $m \geq n-1$. Denote by $d_k$ the number of vertices of degree $k$ in $G$ and denote by $\lambda$ the anti-lecture hall composition associated to $BL(G)$. Then we have

$$\lambda_j = \begin{cases} 
 j & \text{for } j = 1, \ldots, n, \\
 d_{n-1} + d_{n-2} + \cdots + d_{j-n+1} & \text{for } j = n, \ldots, 2n-2, \\
 0 & \text{for } j > 2n-2.
\end{cases}$$

Notice in particular that for $j = n$ we get $\lambda_n = d_{n-1} + d_{n-2} + \cdots + d_0 = n$. 

12
Proof. By applying Lemma 2.16 to $BL(G)$, we have $\lambda = \omega(BL(G))\Psi^{-1}$, where $\Psi$ is the invertible square matrix of size $n + m - 1$ with entries $\Psi_{ij} = \binom{i-1}{j-1}$ and $\lambda = (\lambda_1, \ldots, \lambda_{n+m-1})$. Using (2) and denoting $\Xi := (\Psi^{-1})^T$, we simply find
\[
\lambda^T = (\Psi^{-1})^T A_{dG} - (\Psi^{-1})^T v = \Xi A_{dG} - \Xi v. \tag{3}
\]
Thus we only need to see explicitly what $\Xi A$ and $\Xi v$ are. We claim that $\Xi A$ has the following shape (see also Example 3.8): the entry $(\Xi A)_{jk}$ is 1 for $j \leq k + n - 2$ and it is 0 for $j > k + n - 2$. To prove this, let $j \leq k + n - 2$ and note that
\[
(\Xi A)_{jk} = \sum_{i=1}^{n+m-1} \Xi_{ji} A_{ik} = \sum_{i=1}^{n+m-1} (-1)^{i+j} \binom{i-1}{j-1} \binom{k+n-2}{i}
\]
\[
= (-1)^j \sum_{i=j}^{k+n-2} (-1)^i \binom{i-1}{j-1} \binom{k+n-2}{i}
\]
\[
= (-1)^j \left[ \sum_{i=0}^{k+n-2} (-1)^i P(i) \binom{k+n-2}{i} - P(0) \right]
\]
\[
= (-1)^{j-1} P(0) = 1,
\]
where we apply Lemma 2.5 by considering the polynomial
\[
P(x) := \binom{x-1}{j-1} = \frac{(x-1)(x-2)\cdots(x-j+1)}{(j-1)!}.
\]
Moreover, for $j > k + n - 2$ every term in the sum expressing the entry $(\Xi A)_{jk}$ vanishes, and then $(\Xi A)_{jk} = 0$.

For what concerns $\Xi v$, in order to obtain the formulas for the $\lambda_j$’s starting from (3), we only need to prove that
\[
(\Xi v)_j = \begin{cases} n-j & \text{if } j < n \\ 0 & \text{if } j \geq n. \end{cases}
\]
This is true since for $j < n$ we have
\[
(\Xi v)_j = \sum_{i=1}^{n+m-1} \Xi_{ji} v_i = \sum_{i=1}^{n+m-1} (-1)^{i+j} \binom{i-1}{j-1} \binom{n}{i+1}
\]
\[
= (-1)^j \sum_{i=j}^{n-1} (-1)^i \binom{i-1}{j-1} \binom{n}{i+1}
\]
\[
= (-1)^j+1 \sum_{i=j+1}^{n} (-1)^i \binom{i-2}{j-1} \binom{n}{i}
\]
\[
= (-1)^{j+1} \left[ \sum_{i=0}^{n} (-1)^i P(i) \binom{n}{i} - P(0) + P(1)n \right]
\]
\[
= (-1)^{j+1} [ - (-1)^{j+1} j + (-1)^{j+1} n]
\]
\[
= n - j.
\]
where the polynomial we choose this time, in order to apply Lemma 2.5, is
\[ P(x) := \binom{x - 2}{j - 1} = \frac{(x - 2)(x - 3) \ldots (x - j)}{(j - 1)!}, \]
and for \( j \geq n \) we have \((\Xi w)_j = 0\) as every term in the big sum has as a multiplying factor a vanishing binomial coefficient. \(\square\)

**Example 3.8.** Consider the graph \( G \) in Example 3.2 and Example 3.6. By Proposition 3.7, we find that
\[
\begin{align*}
\lambda_1 &= 1 \\
\lambda_2 &= 2 \\
\lambda_3 &= 3 \\
\lambda_4 &= 4 \\
\lambda_5 &= 5 \\
\lambda_6 &= 6 \\
\lambda_7 &= 7 \\
\lambda_8 &= d_3 + d_2 = 7 \\
\lambda_9 &= d_3 = 2 \\
\lambda_{10} &= \lambda_{11} = \lambda_{12} = 0.
\end{align*}
\]

Moreover, the matrix and vector seen in the proof of Proposition 3.7 have the following forms:
\[
\Xi A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \Xi v = \begin{pmatrix}
6 \\
5 \\
4 \\
3 \\
2 \\
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

4 About the complement \( BL(G) \)

**Proposition 4.1.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Then, for every integer \( j \geq 1 \),
\[
\beta_{j+1}(BL(G)) = m \binom{m + n - 3}{j} - \binom{m}{j + 1}.
\]
Proof. The proof is in two steps:

(1) We generalize the Booth-Lueker construction to multi-graphs and prove the following: let $G$ be a multi-graph with, in particular, vertices $u, v, w$ and an edge $uv$, and let $G'$ be the same multi-graph except for the fact that we remove that edge $uv$ and add an edge $uw$; then the Betti numbers of $BL(G)$ and $BL(G')$ are equal.

(2) We prove the stated formula for a particular multi-graph $H$: the one with all the $m$ edges between two fixed vertices, and $n - 2$ other isolated vertices. Thanks to the fact that one can reach $H$ starting from any $G$ with $n$ vertices and $m$ edges iterating the transformation in the first step, and since the Betti numbers stay constant at each iteration, we will have the formula for any $G$.

The generalization of the Booth-Lueker construction to multi-graphs is done in the natural intuitive way: $BL(G)$ will be a simple graph with a “left part” with the original vertices of $G$, all connected to each other with one edge, and a “right part” with as many vertices as the edges of $G$, connected to the respective ends in the left part.

Let us prove part (1), by using the formula in Theorem 2.14. Let us denote by $e$ the edge between $u$ and $v$ that we “move” between $u$ and $w$. The vertex sets of $BL(G)$ and $BL(G')$ can be written in the same way, the only difference is that the vertex $e$ in the right part of $BL(G)$ is connected to $u$ and $v$ on the left, whereas $e$ in the right part of $BL(G')$ is connected to $u$ and $v$.

Let $S$ be a subset of the vertices of $BL(G)$ and let’s see how the number of connected components of $BL(G)[S]$ and $BL(G')[S]$ differ.

(i) If $e \notin S$, then $BL(G)[S] = BL(G')[S]$.

(ii) If $u \in S$, then $e$ is path-connected to $v$ and $w$ if they are in the graph, whether the “moving edge” is there or not, so the numbers of connected component of $BL(G)[S]$ and $BL(G')[S]$ are the same.

(iii) Let $e \in S$ and $u \notin S$. Then we have three possibilities:

(a) If both $v$ and $w$ are in $S$, then the number of connected components of $BL(G)[S]$ and $BL(G')[S]$ are the same.

(b) If $v \in S$ and $w \notin S$, then the number of connected components of $BL(G)[S]$ increases by one after the move to $BL(G')[S]$, as $e$ becomes a new isolated vertex.

(c) If $v \notin S$ and $w \in S$, then the number of connected components of $BL(G)[S]$ decreases by one after the move to $BL(G')[S]$, as $e$ is no longer a new isolated vertex.

So, to sum up, we have found the following: only if $e \in S$, $u \notin S$ and exactly one of $v$ and $w$ is in $S$ we have a change in the number of connected components.
Figure 3: The Booth-Lueker graph looking like a pineapple.

Consider the map
\[
\varphi \{ S \subseteq V(BL(G)) \mid v \in S, w \notin S, e \in S, u \notin S \} \to \\
\{ S \subseteq V(BL(G)) \mid v \notin S, w \in S, e \in S, u \notin S \}
\]
defined by \( S \mapsto (S \cup \{ w \}) \setminus \{ v \} \). Then \( \#S = \#\varphi(S) \), and \( \varphi \) is a bijection between those subsets there the number of connected components increases by one and those where it decreases by one. So in total the graded Betti numbers stay constant.

Let us now prove the second part. First, let us denote \( H \) the multi-graph on \( n \) vertices and \( m \) edges, where all the edges are between two fixed vertices \( u \) and \( v \). For instance, if \( n = 8 \) and \( m = 4 \), then the Booth-Lueker graph of \( H \) is depicted in Figure 3 and therefore we will refer to \( BL(H) \) as the pineapple.

For each integer \( j \geq 1 \), let us compute \( \beta_{j,j+1}(BL(H)) \) by using the formula in Theorem 2.14. For any subset \( W \) of \( j + 1 \) vertices in \( BL(H) \), the number of connected components of \( BL(H)[S] \) can vary from 1 to \( j + 1 \). We are not interested in the cases with only one connected component, as they give zero in the sum. If \( i + 1 \) is the number of connected component, with \( 1 \leq i \leq j - 1 \), then we have \( \binom{n - 2 - i}{j + 1 - i} \binom{m}{i} \) choices for such a \( W \), by selecting \( i \) vertices in the right part of the pineapple and \( j + 1 - i \) in the left part, except for \( u \) and \( v \). For \( j + 1 \) connected components, we can either choose \( j + 1 \) vertices in the right part of the pineapple, or one vertex among \( n - 2 \) vertices on the left and \( j \) on
the right. Therefore, we find

\[ \beta_{j,j+1}(BL(H)) = \sum_{i=1}^{j-1} i \left( \binom{n-2}{j+1-i} \binom{m}{i} + j \left( \binom{m}{j+1} + (n-2) \binom{m}{j} \right) \right) \]

\[ = m \sum_{i=1}^{j-1} \left( \binom{n-2}{j+1-i} \binom{m-1}{i-1} + j \left( \binom{m}{j+1} + (n-2) \binom{m}{j} \right) \right) \]

\[ = m \left[ \sum_{i=0}^{j} \left( \binom{n-2}{j-1} \binom{m-1}{i} - (n-2) \binom{m-1}{j-1} - \binom{m-1}{j} \right) + j \left( \binom{m}{j+1} + (n-2) \binom{m}{j} \right) \right] \]

\[ \equiv m \left( \binom{m+n-3}{j} \right) - m \left( \binom{m-1}{j} \right) + j \left( \binom{m}{j+1} \right) \]

\[ = m \left( \binom{m+n-3}{j} \right) - \left( \binom{m}{j+1} \right) , \]

where for (\(\ast\)) we used that \(\sum_{i=0}^{j} \binom{n-2}{j-1} \binom{m-1}{i} = (\binom{m+n-3}{j}) \) (see Lemma 2.4) and that \(m \binom{m-1}{j-1} = j \binom{m}{j} \).

\[ \Box \]

**Theorem 4.2.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Then the \( i \)-th Boij-Söderberg coefficient of \( BL(G) \) is

\[ c_i = \begin{cases} 
0 & \text{if } i < m, \\
\frac{m}{(i+1)!} & \text{if } m \leq i \leq m + n - 4, \\
\frac{m}{i} & \text{if } i = m + n - 3, \\
0 & \text{if } i > m + n - 3.
\end{cases} \]
Proof. We use again the formula in Lemma 2.15. We find
\[
c_i = \sum_{j=1}^{n+m-1} \beta_{j,i+1}(BL(G))(-1)^{j-i} \binom{j+1}{i+1}
\]
\[
= m \sum_{j=1}^{n+m-1} (-1)^{j-i} \binom{m+n-3}{j} \binom{j+1}{i+1} - \sum_{j=1}^{n+m-1} (-1)^{j-i} \binom{m}{j+1} \binom{j+1}{i+1}
\]
\[
= -m(m+n-3) \binom{m+n-3}{i+1} \sum_{j=0}^{n+m-4} (-1)^{m+n-4-j} \binom{m+n-4}{j} \binom{j+1}{i-1}
\]
\[
+ \frac{m}{(i+1)!} (-1)^{i+m+n+1} \sum_{j=0}^{n+m-4} (-1)^{m+n-3-j} \binom{m+n-3}{j} \binom{j-1}{i-1}
\]
\[
- \frac{m}{(i+1)!} (-1)^{i+m+n+1} \sum_{j=0}^{n+m-4} (-1)^{m-1-j} \binom{m-1}{j} \binom{j-1}{i-1} - \frac{m}{(i+1)!},
\]
from which we conclude, thanks to Lemma 2.4.

Proposition 4.3. For $G$ a simple graph with $n$ vertices and $m$ edges, the associated anti-lecture hall composition $\lambda = (\lambda_1, \ldots, \lambda_{n+m-1})$ is such that
\[
\lambda_j = \begin{cases} 
  j & \text{if } j \leq m, \\
  m & \text{if } m < j \leq m+n-3, \\
  0 & \text{if } j > m+n-3.
\end{cases}
\]

Proof. If we apply Lemma 2.16 to $BL(G)$, since the corresponding matrix $\Psi^{-1}$ has $ij$-entry equal to $(-1)^{i+j} \binom{j+1}{i-1}$, we find that
\[
\lambda_j = \sum_{i=1}^{n+m-1} \beta_{i,j+1}(BL(G)) \Psi^{-1}_{ij}
\]
\[
= \sum_{i=1}^{n+m-1} (-1)^{i+j} \binom{m+n-3}{i} \binom{i-1}{j-1} - \sum_{i=1}^{n+m-1} (-1)^{i+j} \binom{m}{i+1} \binom{i-1}{j-1}
\]
\[
= (-1)^{j+n+m+1}m \sum_{i=0}^{n+m-1} (-1)^{m+n-3-i} \binom{m+n-3}{i} \binom{i-1}{j-1}
\]
\[
+ (-1)^{j+m} \sum_{i=0}^{n+m} (-1)^{m-i} \binom{i-2}{j-1} - (-1)^j \binom{-2}{j-1}
\]
\[
= (-1)^{j+m+n+1}m \binom{j-m-n+2}{-1} + (-1)^{j+m} \binom{j-m-1}{-2} - (-1)^j \binom{-2}{j-1},
\]
where we applied Lemma 2.4. Now, depending on $j$, we find the stated expressions for $\lambda_j$. \qed

18
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