A directional coupler attack against the Kish key distribution system

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The Kish key distribution system has been proposed as a classical alternative to quantum key distribution. The idealized Kish scheme elegantly promises secure key distribution by exploiting thermal noise in a transmission line. However, it is vulnerable to nonidealities in its components, such as the finite resistance of the transmission line connecting its endpoints. We introduce a novel attack against this nonideality using a directional coupler, and experimentally demonstrate its efficacy.

The Kish key distribution (KKD) system, based on Kirchhoff’s laws and Johnson noise (KLJN) [1] has been proposed as a classical alternative to quantum key distribution (QKD) [2]. Eschewing expensive and environmentally-sensitive optics, it can be implemented economically in a wider variety of systems than QKD.

![Figure 1](image)

**FIG. 1.** We determine the forward- and reverse-travelling waves in this idealized KKD system. Practical systems would include low-pass filters and instrumentation that do not affect the steady-state signal. The mean-squared voltages \(\langle V_A^2(t)\rangle\) and \(\langle V_B^2(t)\rangle\) are proportional to the resistances \(R_A\) and \(R_B\) respectively. We perform our analysis in terms of the reflection coefficients \(\Gamma_A\) and \(\Gamma_B\).

The KKD system is claimed [1] to derive unconditional security from the second law of thermodynamics—the idea being that net power cannot flow from one resistor to the other under equilibrium.

An idealised KKD system is shown in Figure 1. Alice and Bob each apply a noise signal to a line through a series resistor. The voltage on the line is unchanged if the terminals of Alice and Bob are swapped; if the mean-square voltages applied by Alice and Bob are proportional to \(R_A\) and \(R_B\) respectively then no average power flows through the line, and in the ideal case an eavesdropper, Eve, cannot determine which end has which resistance [1]. If Alice and Bob randomly choose their resistances—resulting in corresponding noise amplitudes—to be either \(R_h\) or \(R_l\), three possibilities avail themselves: both choose \(R_h\), both choose \(R_l\), or one chooses \(R_h\) and the other chooses \(R_l\). In this third case, Alice knows the value of her own resistor, and so can deduce Bob’s resistor via noise spectral analysis, and vice-versa. However, an eavesdropper lacks this knowledge, and so in the ideal case Alice and Bob have secretly shared one bit of information.

It has been claimed [3] that transmission line theory does not apply to the the KKD system when operated at frequencies below \(f_c = \nu/(2L)\), where \(L\) is the transmission line length and \(\nu\) the signal propagation velocity, because wave modes do not propagate below this cutoff. We demonstrate that this is not the case by constructing a directional coupler that is then used for a successful finite-resistance attack against the system. The position that frequencies below \(f_c\) do actually propagate is also supported by the fact that, at low frequencies, a coaxial cable is known to only support TEM modes—these modes are known to have no low frequency cutoff [4, p. 358]. An exception occurs when the two ends of the line are held at equal potential; only standing waves possessing a frequency that is an integer multiple of \(\nu/(2L)\) can fulfill these boundary conditions [5, p. 31]. However, the the KKD system differs in allowing arbitrary potentials to appear at the ends of the line, and so need not support standing waves at the frequency of operation.

Several attacks against the KKD system exist, however none thus far have been shown experimentally to substantially reduce the security of the system [6].

The first attacks, proposed by Schaefer and Yariv [7], rely upon imperfections in the line connecting the two terminals; the first exploits transients generated by the resistor-switching operation, while the second exploits the line’s finite resistance. The former is foiled by the addition of low-pass filters to the terminals [8], while the latter was shown to leak less than 1% of bits [6, 8] in a practical system.

An attack by Hao [9, 10] instead focuses upon imperfections of the terminals; inaccuracies in the noise temperatures of Alice and Bob create an information leak. However, it was demonstrated [6, 10] that noise can be digitally generated with a sufficiently accurate effective noise temperature to prevent this attack from being useful in practice.

A theoretical argument has been made by Bennett and Riedel [11] that no purely classical electromagnetic system can be unconditionally secure due to the structure of

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Maxwell’s equations. It is argued that the upper bound on secrecy rate by Maurer [12] must be zero because of the locally-causal nature of classical electromagnetics, and so an eavesdropper can perfectly reconstruct the key with the aid of a directional coupler. Kish, et al. [13] respond that a nonzero secrecy rate is unnecessary in practice, provided it can be achieved in the ideal limit.

We begin our attack by analyzing the system in Figure 1 to determine the forward- and reverse-travelling waves through the transmission line. Let us denote the equivalent noise voltages of Alice and Bob \( V_a(t) \) and \( V_b(t) \) respectively, and the waves injected onto the line \( V_a'(t) \) and \( V_b'(t) \). These are related by

\[
\begin{align*}
V_a'(t) &= \frac{1}{2}(1 - \Gamma_A) V_a(t) \\
V_b'(t) &= \frac{1}{2}(1 - \Gamma_B) V_b(t).
\end{align*}
\]  

(1)

(2)

Noting that the mean-squared thermal noise voltage \( (V^2) = 4kTB_R \), we find that

\[
\begin{align*}
\langle V_a'^2 \rangle &= kTBZ_0(1 - \Gamma_A^2) \\
\langle V_b'^2 \rangle &= kTBZ_0(1 - \Gamma_B^2).
\end{align*}
\]  

(3)

(4)

As the transmission line in the KKD system is short—and so the forward- and reverse-travelling waves are equal throughout the line except for a loss factor \( \alpha \)—we may write the left- and right-travelling waves at Bob’s and Alice’s ends of the line respectively as

\[
\begin{align*}
V_+(t) &= V_0'(t) + \alpha \Gamma_A V_-(t) \\
V_-(t) &= V_0'(t) + \alpha \Gamma_B V_+(t)
\end{align*}
\]  

(5)

(6)

and so

\[
\begin{align*}
V_+(t) &= -\frac{V_0'(t) + \alpha \Gamma_A \Gamma_B V_0'(t)}{1 - \alpha^2 \Gamma_A \Gamma_B} \\
V_-(t) &= -\frac{V_0'(t) + \alpha \Gamma_A \Gamma_B V_0'(t)}{1 - \alpha^2 \Gamma_A \Gamma_B}.
\end{align*}
\]  

(7)

(8)

We may write this in matrix form \( \mathbf{v}'(t) = \mathbf{A} \mathbf{v}(t) \) and so find the covariance matrix \( \mathbf{C} = \mathbf{A} \mathbf{C} \mathbf{A}^t \) of the directional components:

\[
\mathbf{C} = \frac{kTBZ_0}{1 - \alpha^2 \Gamma_A \Gamma_B} \begin{bmatrix}
1 - \alpha^2 \Gamma_A^2 \Gamma_B^2 + (\alpha^2 - 1) \Gamma_A^2 \\
\alpha \Gamma_A (1 - \Gamma_B^2) + \alpha \Gamma_B (1 - \Gamma_A^2) \\
\alpha \Gamma_A (1 - \Gamma_B^2) + \alpha \Gamma_B (1 - \Gamma_A^2) \\
1 - \alpha^2 \Gamma_A^2 \Gamma_B^2 + (\alpha^2 - 1) \Gamma_B^2
\end{bmatrix}.
\]  

(9)

When the line is lossless and so \( \alpha = 1 \), Eqn. 9 is invariant under permutation of \( \Gamma_A \) and \( \Gamma_B \), and so the covariance matrix provides no information on the choice of resistors. However, when \( \alpha < 1 \) this property fails to hold, allowing the choices of \( \Gamma_A \) and \( \Gamma_B \) to be determined from the distribution of \( (v_+, v_-) \).

A directional coupler separates forward- and reverse-travelling waves on a transmission line [14]. We have constructed such a device using differential measurements across a delay line, shown in Figure 2.

Consider the d’Alembert solution [4, Eqn. 7.7] to the wave equation in a medium with propagation velocity \( \nu \),

\[
v(t, x) = v_+(t - \frac{x}{\nu}) + v_-(t + \frac{x}{\nu}).
\]  

(10)

The forward-travelling component \( v_+(\tau) \) differs from the reverse-travelling component \( v_- (\tau) \) in the sign of its spatial argument. We use this to our advantage by computing the linear combinations

\[
\begin{align*}
\frac{\partial v}{\partial t} + \nu \frac{\partial v}{\partial x} &= 2 \frac{dv_+}{dt} \\
\frac{\partial v}{\partial t} - \nu \frac{\partial v}{\partial x} &= 2 \frac{dv_-}{dt}.
\end{align*}
\]  

(11)

(12)

yielding the forward- and reverse-travelling waves as we desire. All that remains, then, is to determine \( \partial v/\partial t \) and \( \partial v/\partial x \).

The time derivative \( \partial v/\partial t \) may be determined digitally from sampled values of \( v(t) \). The spatial derivative is approximated as being proportional to the voltage across a short delay line, shown in Figure 2.

FIG. 2. The analog frontend of the directional coupler. Buffering, offset, gain control, and clamping are not shown. An instrumentation amplifier is used to measure the voltage across a 1.5 m length of coaxial cable, providing an estimate of \( \partial v/\partial x \). After offset and gain adjustments, the signals are simultaneously sampled by the 12-bit ADCs of an STM32F407 microcontroller.

After digitisation, the signals \( V \) and \( V_x \) are combined to produce the left- and right-travelling waves. The
time-derivative $\partial v/\partial t$ can be approximated by a difference operator, however in order to accommodate for the unknown propagation velocity and delay line length, common-mode leakage into $V_x$, and losses in the delay line, we instead use a first-order least-mean-squares (LMS) adaptive filter [15] for initial calibration. A signal source is applied to one port and the other is terminated; this produces a right-travelling wave on the line, but none travelling to the left. The left-travelling output $V_-$ is used as an error signal for the LMS filter, suppressing any contribution from the right-travelling wave.

![Diagram](image)

FIG. 3. The digital signal processing of the directional coupler, implemented on an STM32F407 microcontroller. Offset removal is not shown. A least-mean-squares filter is used at startup to determine the necessary filter coefficients; a signal is applied to one port while the other is connected to a terminator, and the filter coefficients adjusted to force $V_- = 0$. Filter updates are disabled once the apparent reflection coefficient becomes sufficiently small.

The real part of the reflection coefficient, seen looking out of the right port, is computed by a cross-correlation between left- and right-travelling waves. When this falls below 0.01, calibration is declared complete and filter updates cease. After calibration, we validate the system by configuring it as a reflectometer. Open and shorted measurements are made, yielding reflection coefficients of $+1$ and $-1$ respectively. The reflection coefficients of several resistors were also measured, again yielding the expected values.

We have described the implementation of a directional coupler using differential measurements across a delay line. While we might measure the power travelling in each direction in order to determine the resistor configuration, the distributions to be distinguished are very similar, resulting in a relatively large bit-error rate (BER) as was shown in [8]. However, comparison of the variances of $v_+$ and $v_-$ is suboptimal. We derive an improved test using Bayesian methods and demonstrate that the two cases can be far more easily distinguished.

Knowing the covariance matrices of $v_+(t)$ and $v_-(t)$ for each hypothesis, we may use Bayes’ theorem [16] to determine the probability of each configuration. Let $C = 0$ and $C = 1$ refer to the events that $(R_a, R_b) = (R_h, R_l)$ and vice-versa, respectively. Then,

$$P[C = 0 | v_+ \cap v_-] = \frac{P[v_+ \cap v_- | C = 0]P[C = 0]}{P[v_+ \cap v_-]}$$

(13)

$$= \frac{1}{2}p_0(v_+, v_-)$$

(14)

$$= \frac{1}{1 + \frac{p_1(v_+, v_-)}{p_0(v_+, v_-)}}$$

(15)

where $p_0(\cdot, \cdot)$ and $p_1(\cdot, \cdot)$ are the multivariate Gaussian PDFs for the measurements from each respective configuration.

The most probable state, then, is given by the maximum-likelihood estimator [16]

$$\hat{C} = \begin{cases} 0 & \text{if } p_0(v_+, v_-) > p_1(v_+, v_-) \\ 1 & \text{if } p_0(v_+, v_-) < p_1(v_+, v_-) \end{cases}$$

(16)

The results of simulation for various values of loss are shown in Figure 4.

Having demonstrated our attack in simulation, we proceed to experimental validation of the model. The estimation of $\partial v/\partial x$ is key to the operation of the directional coupler, however the synthesis provided above is dependent upon a wave-based analysis of the system. We therefore measure the frequency response of the electronically-estimated $\partial v/\partial x$, shown in Figure 5, with a wave travelling in a single direction in order to verify that this analysis is appropriate.

We expect to see a magnitude response linear in frequency and a constant $+90^\circ$ phase response. This agrees with the experimental results shown in Figure 5, validating our analysis, and demonstrating that the signal through a short transmission line indeed propagates as a wave, in contradiction to the theoretical claims of [3].

![Figure 4](image)

FIG. 4. Simulated eavesdropper bit-error-rate as a function of averaging time, for line attenuations of 0.01, 0.02, 0.05, 0.1, 0.5, 1.0, and 3.0 decibels respectively from top to bottom. Note that the averaging time is expressed in multiples of 200 μs. This is the correlation time (i.e. reciprocal of the system bandwidth) so that the results are bandwidth independent. Transmission lines with greater loss are more susceptible to attack, with substantial attenuations providing little protection.
FIG. 5. Measured frequency response of the $\partial v/\partial x$ estimation circuit in Figure 2. The derivative increases linearly with frequency, as would be expected from the d’Alembert solution to the wave equation. The response $H(0)$ at DC is subtracted in order to remove the effect of wire resistance, yielding the ‘compensated’ curves above. After this correction we see $\angle H(f)$ approximating the expected $+90^\circ$ constant phase response.

We have implemented the attack described above, using resistances $R_l = 1\, \text{k}\Omega$, $R_h = 10\, \text{k}\Omega$, and a coaxial transmission line of characteristic impedance $Z_0 = 50\, \Omega$. The voltage sources are produced by an arbitrary waveform generator, producing independent normally-distributed voltages over a frequency range of 500 Hz–5500 Hz. The bandwidth $B = 5\, \text{kHz}$ results in an approximate correlation time of $B^{-1} = 200\, \mu\text{s}$ [17]. Each configuration was set and the covariance matrices from Eqn. 9 measured during the setup phase. Resistor configurations are randomly selected as would be the case in an operational system, and the log-likelihood ratios are computed for the measured values of $v_+$ and $v_-$. Their differences are thresholded to compute (16), providing the bit-error rates in Figure 6. Even modest losses allowed almost all bits to be determined correctly.

By applying a threshold to the likelihood ratios, we may estimate the agreed bits and so determine the error rate of Eve. We see that even with the minuscule losses of the test system, Eve can acquire a substantial proportion of the agreed bits.

The technique above exploits imperfections in the KKD implementation; while it might be theoretically possible to counter this attack by reduction of losses as proposed in [8], the reduction of losses substantially below 0.1 dB ensures that this will be infeasible for all but the shortest or slowest of links.

This raises the question of why this attack should succeed where existing finite-resistance attacks have failed. The attack of Scheuer and Yariv [7] considered only the variances of the measured variables. Our attack exploits the large correlation between waves in each direction; the estimator used above partially removes this common signal, increasing the ability to distinguish between the two cases statistically.

We have demonstrated an attack against the KKD key distribution system that exploits losses within the connecting transmission line. The attack has been shown experimentally to correctly determine more than 99.9% of bits transmitted over a 2 m transmission line within 20 correlation times. As this attack requires that losses be reduced to a fraction of a decibel in order to maintain a meaningful level of security, modifications to the system will be necessary in order to produce a secure link of any significant length and bitrate.

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