The Earliest Phase Transition?

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Abstract

The question of a phase transition in exiting the Planck epoch of the early universe is addressed. An order parameter is proposed to help decide the issue, and estimates are made concerning its behavior. Our analysis is suggestive that a phase transition occurred.

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At about $\sim 10^{-12}$ seconds, the universe underwent a phase transition corresponding to the spontaneous breaking of the $SU(2) \times U(1)$ electroweak theory. The breaking provided elementary particles with mass. This event was followed by another phase transition – quark confinement – at around $10^{-6}$ seconds. The evolution that happened earlier than $10^{-12}$ seconds involves some speculation.\footnote{see for example, \cite{1}} When the universe was about $10^{-35}$ seconds old, it is believed that inflation caused space to stretch by an enormous factor.\footnote{Gravity was classical until times earlier than $10^{-42}$ seconds and definitely quantum mechanical during the Planck epoch (earlier than $10^{-43}$ seconds).} The purpose of this article is to discuss the transition from the Planck epoch to the period when classical gravity prevailed. Some qualitative statements can be made but we will mostly be concerned with defining an order parameter to probe this region and with making estimates about its behavior to determine whether a phase transition took place. We shall also briefly discuss our ideas in string theory.

It is difficult to rigorously determine whether there was a phase transition in exiting the Planck epoch because the quantum version of gravity realized in nature is not known. However, even lacking such a theory, one has a fairly good idea as to what qualitatively transpired because two things are known: (1) classical gravity given by Einstein’s general theory of relativity, and (2) the general features of quantum mechanics. By combining these, one obtains a qualitative picture of quantum gravity.

Quantum mechanics leads to uncertainty, fluctuations in degrees of freedom, and the incorporation of all possible histories. The degrees of freedom in general relativity are the components $g_{\mu \nu}$ of the metric, which describe the space-time geometry. It follows that any quantum theory of gravity should involve variations in $g_{\mu \nu}$ and in the space-time manifold.\footnote{Despite the lack of a consistent mathematical version of quantum gravity, it is clear that, during the Planck epoch, the geometry of spacetime was varying greatly.} The universe in the Planck epoch was extremely hot with a temperature around $E_{\text{Planck}}/k$, where $E_{\text{Planck}}$ is the Planck energy of about $10^{19}$ GeV and $k$ is Boltzmann’s constant. Quantum gravitational effects led to tunneling among universes,
strongly interacting gravitons, and significant black hole production and destruction. There was no single background manifold in which all things moved since, according to the rules of quantum mechanics, the space-time manifolds were probabilistically determined. Space and time were dynamic and significantly fluctuating.

When the universe cooled sufficiently, decoherence set in, and essentially a single space-time manifold dominated. The metric eventually became well described by a Friedmann-Robertson-Walker solution to the classical equations of Einstein’s general theory of relativity.

So did a phase transition take place as gravity evolved from being quantum mechanical to classical? To decide this issue, observables need to be examined and computed. In particular, an order parameter to probe the physics is helpful. The purpose of the first part of this article is to propose such an order parameter. We make use of the affine connection given by

\[
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} \partial^\sigma \left( \frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right).
\]

Under a general coordinate transformation, the one-forms \( \Gamma^\lambda_{\mu\nu} dx^\nu \) transform as

\[
\Gamma'^\lambda_{\mu\nu} dx'^\nu = U^\lambda_\rho \Gamma^\rho_\tau d\sigma^\sigma \left( U^{-1} \right)_\mu^\rho + U^\lambda_\rho d\sigma d\rho \left( U^{-1} \right)_\mu^\rho ,
\]

where \( U^\lambda_\rho = \partial \sigma^\lambda / \partial x^\rho \). This is similar to the change that a non-abelian potential undergoes in a gauge transformation. Treat \( \Gamma^\lambda_{\mu\nu} dx^\nu \) as a \( D \times D \) matrix in the indices \( \lambda \) and \( \mu \), where \( D \) is the number of space-time dimensions. Then the path ordered line integral

\[
T^\lambda_\rho (C) = \left[ \prod_{\tau \leq \sigma \leq 0} P \exp \left( -\int_0^\tau \Gamma \right) \right] _\rho^\lambda
\]

associated with the curve \( C \) transforms as \( T'^\lambda_\rho (x_f) = U^\lambda_\mu (x_f) T^\mu_\rho (x_0) \), where \( x_0 = X(0) \) and \( x_f = X(\tau) \) are the initial and final points of \( C \). Here, \( C \) is generated by \( X(\sigma) \) as \( \sigma \) varies from \( 0 \) to \( \tau \). It is well known that this path factor parallel transports a vector along \( C \): If \( v^\lambda (\tau) = T^\lambda_\mu v^\mu (0) \) then \( v \) satisfies

\[
\frac{dv^\lambda (\tau)}{d\tau} + \Gamma^\lambda_{\mu\nu} \frac{dx^\nu}{d\tau} v^\mu (\tau) = 0.
\]

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It follows from the above that the trace of the transport factor for a closed curve transforms as a scalar.

A candidate order parameter, which is the gravitational version of a Wilson loop, is thus

$$O'(C) = \left\langle \frac{1}{D} \sum_{\lambda} T^{\lambda}_{\lambda}(C) \right\rangle,$$  \hspace{1cm} (5)

where $C$ is a space-like curve, meaning that its tangent vectors are space-like $dX/d\tau \cdot dX/d\tau > 0$. Here, $\langle \rangle$ is the quantum expectation value, that is, the value of the trace of the closed parallel transporter averaged over the various geometries.

However, Eq.(5) is not quite well-defined because it is not possible in general to relate a closed curve $C$ in one manifold to a corresponding curve in another manifold. Furthermore, one would like to probe the properties of a manifold in all locations and not simply in the vicinity of a particular curve. This leads one to average over $C$. However, all curves should not be included: One wants to avoid highly irregularly shaped $C$ since one is interested in the non-smoothness property of the manifold and not of the curve. In the Appendix, a set of curves $C(P)$ is defined that have fixed length $P$ but are locally of maximal area. Roughly speaking, these curves can be thought of as generalized circles.

Our proposed order parameter $O(P)$ is

$$O(P) = \left\langle \left\langle \frac{1}{D} \sum_{\lambda} T^{\lambda}_{\lambda}(C) \right\rangle_{C \in C(P)} \right\rangle,$$  \hspace{1cm} (6)

where the averaging is done first over the class of $C$ with fixed perimeter length $P$ for a fixed geometry and then over geometries. When done in this order, the computational procedure in Eq.(6) is well-defined even if spacetime is fluctuating greatly. The order parameter $O$ is a function of the length $P$ of curves and is general coordinate invariant.

In the classical phase, $O$ can be reliably computed. For asymptotically late times, the universe is well described by a Friedmann-Robertson-Walker metric $-dt^2 + R^2(t)\{(1 - kr^2)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2\}$ with three possibilities $k = -1$ (hyperbolic), $k = 0$ (flat), and $k = 1$ (spherical). The dimensionful parameter $R(t)$ is the expansion factor.

\footnote{Our convention follow those of Weinberg\cite{3}. In particular, $(-1,1,1,1,...)$ is the signature for the Minkowski metric.}
For a “circle” of circumference $2\pi r R$, a lengthy but straightforward computation of Eq.(6) gives

$$O(P) = \frac{1}{2} \left(1 + \cos(2\pi\sqrt{1 - kr^2 - \dot{R}^2r^2})\right).$$

(7)

Even though space is flat for $k = 0$, $O(p)$ is not identically equal to one because spacetime is curved.

Equation (7) can be simplified using the standard solution to Einstein’s equations for the evolution of the universe in the presence of matter given by $\dot{R}^2 + k = \frac{8\pi G N}{3} \rho R^2$ where $\rho$ is the density of matter:

$$O(P) = \frac{1}{2} \left(1 + \cos(2\pi\sqrt{1 - \frac{P^2}{L_{\text{cp}}^2}})\right),$$

(8)

where

$$L_{\text{cp}} = \sqrt{\frac{3\pi}{2G N \rho}}.$$  

(9)

The present-day value of $L_{\text{cp}}$ is about 90 billion light-years, roughly the circumference of a circle with a radius of the size of the visible universe. Thus, $O(P)$ as a function $P$ is almost one until $P$ is enormous, and even then $O(P)$ is greater than $1/2$. This behavior of $O(P)$ in the classical phase is displayed as curve (a) in Figure 1.

Let us qualitatively determine the behavior of the order parameter during the Planck epoch. For a collapsed curved with $P = 0$, $O(P)$ is still 1. Consider what happens as $P$ increases. The trace of any matrix $M_{ij}$ is the sum $\sum_k e_{(k)}^i M_{ij} e_{(k)}^j$ over a complete orthonormal set of vectors $e_{(k)}$. Use this method to evaluate the trace in Eq.(4). If the curve $C$ passes through a region of a manifold that is highly curved then the final direction of $e_{(k)}$ as determined from Eq.(4) will be significantly different from its initial direction. The trace thus generates a value much less than one. During the Planck epoch, this trace will decrease rapidly with the size of $C$ because randomizing effects will be bigger. Thus, $O(P)$ quickly drops off as a function of $P$. Curve (b) in Figure 1 displays the result.

Given the similarity because non-abelian gauge theories and gravity, one reasonable guess is that $O(P)$ has area law behavior during the Planck epoch:

$$O(P) \approx \exp[-P^2/L_{\text{qp}}^2],$$

(10)
where $L_{\text{qp}}$ is some length scale. Possible values for $L_{\text{qp}}$ are the Planck scale $L_{\text{Planck}} = \sqrt{G_N}$, the Hubble length $L_{\text{Hubble}} = H^{-1} = R/\dot{R}$, the energy density scale $L_{\rho_c} = \rho_c^{-1/4}$, and the thermal length $L_{\text{thermal}} = (kt)^{-1}$ in units for which $\hbar = c = 1$. Even if area law behavior is not achieved, one expects some type of exponential falloff for $O(P)$ governed by one of the above length scales. A precise computation of $O$ in the quantum regime is currently premature and is not attempted in the present work.

Evaluated today, the above mentioned length scales take on widely different values: $L_{\text{Hubble}}$ is about 15 billion light-years, $L_{\text{thermal}}$ and $L_{\rho_c}$ are about a millimeter, and $L_{\text{Planck}}$ is of order $10^{-35}$ meters. As one goes back in time, these different scales approach each other and become comparable at the Planck time. In an ordinary field theory, one would expect $L_{\text{thermal}}$ to be the relevant length scale appearing in Eq.(10) during the Planck epoch. However, string theory, which is discussed below, suggests a different result.

While it is possible to find a function that interpolates between the behavior in Eq.(8) and that of Eq.(10), it seems unlikely that a quantum theory of gravity would yield such a function. For large $P$, it is difficult to go from an exponentially value small (in the quantum phase) value to a value above $1/2$ (in the classical phase) without jumping. Our non-rigorously qualitative analysis suggests that the early Universe underwent a first order phase transition.

One might worry that some of the above concepts do not make sense in the quantum theory of gravity realized in nature. For example, perhaps quantum gravity is not formulated in terms of a connection, in which case the formula for $O$ given above does not exist. In such a situation, however, there should exist an order parameter $\tilde{O}$ that generalizes $O$. This is expected because of the correspondence principle, which historically has maintained continuity between old, slightly incorrect theories and new, more accurate theories.

Suppose, for example, that string theory turns out to be realized in nature. In this case, the graviton is one of the vibrational modes of the string. In second quantization, the metric emerges as a component of the string field wave function:
For the bosonic string, the string field $\Psi$ has an expansion beginning as

$$\Psi = g_{\mu\nu} \partial X^\mu(0) \overline{\partial X^\nu(0)}|0\rangle + \ldots,$$

where the $X^\mu(z)$ are the first-quantized string variables promoted to operators. (For the superstring, similar expansions exist.) One could then construct the connection $\Gamma$ in Eq.(11) from the metric in Eq.(11) and obtain $\hat{O}$ using Eqs.(3) and (6). However, this would not be correct. String field theory has an infinite set of gauge invariances that must be respected. To construct gauge-covariant quantities and gauge-invariant observables, additional terms involving the other vibrational modes of the string must be added to Eqs.(1), (3) and (6). Such a construction would lead to the operator $\tilde{O}$ mentioned above.

Although it is still not completely clear how spacetime emerges from string theory, the correspondence principle virtually guarantees that spacetime, or an analogous concept, will appear. In the Planck epoch, fluctuations in geometry or its generalization in string theory should likewise occur, although the fluctuations might be less severe than in the naive quantization of Einstein’s general theory of relativity because string theory is renormalizable (probably even finite) with degrees of freedom spread over the Planck length.

Perturbative string theory leads to a limiting maximum temperature. If such a constraint survives quantum corrections, then the universe cannot become hotter than the Hagedorn temperature. This should not prevent extrapolating back in time to before a certain point; rather it should indicate that the dynamics of the early universe are changed. Likewise, it is sometimes said that T-duality implies a minimum size for the universe. Actually, T-duality relates a manifold of size smaller than the Planck length to a manifold of size larger than the Planck length. It does not limit a manifold’s size, but it does imply a change in dynamics if the universe were to become smaller than the Planck length. The above suggests that the length scales $L_{\text{Hubble}}$, $L_{\text{thermal}}$ and $L_{\text{pe}}$ are are effectively cutoff at the Planck length during the Planck epoch. String theory supports the idea that the relevant length scale $L_{\text{p}}$ in the quantum phase is $L_{\text{Planck}}$ or $L_{\text{String}}$.

The analysis of this article is different from that of ref. where a possible phase
transition was argued to arise because of the infinite number of degrees of freedom in string theory. It is possible, however, that there is a connection. For another work on the early universe from the viewpoint of string theory, see ref. [9].

Finally, string theory gives rise to additional spatial dimensions. If they are associated with an internal microscopic compactified manifold, then, at around the Planck time, one would expect ordinary and internal dimensions to have been approximately the same size. As the universe evolved, the compactified manifold remained small or became smaller, while familiar three-space expanded. The possibility of additional dimensions should not affect the above discussion. However, it does raise further dynamical questions. It is possible to orient \( O \) in the internal manifold as a way of probing the behavior of extra dimensions in the early universe.

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Appendix

This Appendix defines the class \( \mathcal{C}(P) \) of curves over which the average in Eq. (6) is to be performed. Let \( M \) be a fixed manifold endowed with a metric. Let \( C \) be a closed curve of fixed length \( P \). The area of \( C \) shall be defined as \( \inf S A(S) \), where \( S \) is a spanning surface of \( C \) and \( A(S) \) is the area of \( S \). If not even a single spanning surface of \( C \) exists, then \( C \) shall not be included in \( \mathcal{C}(P) \). For situations in which there is a spanning surface of minimal area of \( C \), the area of \( C \) is the area of that surface. The class \( \mathcal{C}(P) \) shall consist of space-like curves subject to the constraint that they have length \( P \) and are locally of maximal area. A curve \( C \) is defined to be locally of maximal area if deforming any small arc of \( C \) reduces its area. In Minkowski space, \( \mathcal{C}(P) \) consists of all space-like circles of perimeter \( P \). It remains to determine a measure for the curves. A possibility is to use the one associated with the Feynman path integral. [11]

Figure Captions

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3 The possibility of large internal dimensions has been considered in refs. [10].
Figure 1. The Behavior of the Order Parameter $O(P)$ as a Function of Perimeter Size $P$ in the Classical Regime (curve (a)) and during the Planck epoch (curve (b)).

References

[1] E. W. Kolb and M. S. Turner, *The Early Universe*, (Addison Wesley, 1990).

[2] A. H. Guth, Phys. Rev. **D23** (1981) 347; A. D. Linde, Phys. Lett. **108B** (1982) 389.

[3] *Quantum Cosmology and Baby Universe*, Eds. S. Coleman, J. B. Hartle, T. Piran and S. Weinberg, (World Scientific, 1991); *Euclidean Quantum Gravity*, Eds. G. W. Gibbins and S. W. Hawking, (World Scientific, 1993).

[4] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, New York, 1972)

[5] E. Witten, Nucl. Phys. **B268** (1986) 253.

[6] S. Samuel, in *Strings and Superstrings* (XVIIIth Int. GIFT, El Escorial, Spain, June, 1987) ed. J. P. Mittelbrunn, M. Ramón-Medrano and G. S. Rodero, (World Scientific, Singapore, 1988).

[7] R. Hagedron, Nuov Cim. Suppl. **3** (1965) 147; Nuovo Cim. **56A** (1968) 1027; Astron. Astrophys. **5** (1970) 184.

[8] J. J. Atick and W. Witten, Nucl. Phys. **B310** (1988) 291.

[9] G. Veneziano, Phys. Lett. **265B** (1991) 287; M. Gasperinie and G. Veneziano, Astropart. Phys. **1** (1993) 317; Mod. Phys. Lett. **1** (1993) 3701; Phys. Rev. **D50** (1994) 2519; G. Veneziano, in the proceedings of *International School of Subnuclear Physics, 35th Course: Highlights: 50 Years Later*, hep-th/9802057 and references therein.
[10] I. Antoniadis, Phys. Lett. **246B** (1990) 377;
   V. A. Kostelecký and S. Samuel, Phys. Lett. **270B** (1991) 21;
   I. Antoniadis, K. Benakli and M. Quirós, Phys. Lett. **331B** (1994) 313;
   J. D. Lykken, Phys. Rev. **D54** (1996) 3693;
   E. Cáceres, V. S. Kaplunovsky and M. Mandelberg, Nucl. Phys. **B493** (1997) 73;
   N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **429B** (1998) 263;
   K. R. Dienes, E. Dudas and T. Gherghetta, Phys. Lett. **436B** (1998) 55.

[11] R. Feynmann and A. Hibbs, *Quantum Mechanics and Path Integrals*, (McGraw-Hill, 1965).
Figure 1