SHARP AND PRINCIPAL ELEMENTS
IN EFFECT ALGEBRAS

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Abstract. In this paper we characterize the effect algebras whose sharp and principal elements coincide. We also give examples of two non-isomorphic effect algebras having the same universum, partial order and orthosupplementation.

1. Introduction

Effect algebras have been introduced by Foulis and Bennett in 1994 (see [5]) for the study of foundations of quantum mechanics (see [4]). Independently, Chovanec and Kôpka introduced an essentially equivalent structure called D-poset (see [9]). Another equivalent structure was introduced by Giuntini and Greuling in [6].

The most important example of an effect algebra is \((E(H), 0, I, \oplus)\), where \(H\) is a Hilbert space and \(E(H)\) consists of all self-adjoint operators \(A\) on \(H\) such that \(0 \leq A \leq I\). For \(A, B \in E(H)\), \(A \oplus B\) is defined if and only if \(A + B \leq I\) and then \(A \oplus B = A + B\). Elements of \(E(H)\) are called effects and they play an important role in the theory of quantum measurements ([2], [3]).

A quantum effect may be treated as two-valued (it means 0 or 1) quantum measurement that may be unsharp (fuzzy). If there exist some pairs of effects \(a, b\) which possess an orthosum \(a \oplus b\) then this orthosum correspond to a parallel measurement of two effects.

In this paper we solved the following Open Problem: Characterize the effect algebras whose sharp and principal elements coincide (see [8]). So far it was known (see Theorem 3.16 in [1]) that if effect algebra \(E\) is lattice-ordered then \(e \in E\) is principal iff \(e \wedge e' = 0\). It also was known that in every effect algebra any principal element is sharp (see Lemma 3.3 in [7]).

Definition 1.1. In [5] an effect algebra is defined to be an algebraic system \((E, 0, 1, \oplus)\) consisting of a set \(E\), two special elements \(0, 1 \in E\)

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called the zero and the unit, and a partially defined binary operation \( \oplus \) on \( E \) that satisfies the following conditions for all \( p, q, r \in E \):

1. [Commutative Law] If \( p \oplus q \) is defined, then \( q \oplus p \) is defined and \( p \oplus q = q \oplus p \).
2. [Associative Law] If \( q \oplus r \) is defined and \( p \oplus (q \oplus r) \) is defined, then \( p \oplus q \) is defined, \( (p \oplus q) \oplus r \) is defined, and \( p \oplus (q \oplus r) = (p \oplus q) \oplus r \).
3. [Orthosupplementation Law] For every \( p \in E \) there exists a unique \( q \in E \) such that \( p \oplus q \) is defined and \( p \oplus q = 1 \).
4. [Zero-unit Law] If \( 1 \oplus p \) is defined, then \( p = 0 \).

For simplicity, we often refer to \( E \), rather than to \((E, 0, 1, \oplus)\), as being an effect algebra.

If \( p, q \in E \), we say that \( p \) and \( q \) are orthogonal and write \( p \perp q \) iff \( p \oplus q \) is defined in \( E \). If \( p, q \in E \) and \( p \oplus q = 1 \), we call \( q \) the orthosupplement of \( p \) and write \( p' = q \).

It is shown in [5] that the relation \( \leq \) defined for \( p, q \in E \) by \( p \leq q \) iff \( \exists r \in E \) with \( p \oplus r = q \) holds for all \( p \in E \). It is also shown that the mapping \( p \mapsto p' \) is an order-reversing involution and that \( q \perp p \) iff \( q \leq p' \). Furthermore, \( E \) satisfies the following cancellation law: If \( p \oplus q \leq r \oplus q \), then \( p \leq r \).

An element \( a \in E \) is sharp if the greatest lower bound of the set \( \{a, a'\} \) equals \( 0 \) (i.e. \( a \land a' = 0 \)). We denote the set of sharp elements of \( E \) by \( S_E \).

An element \( a \in E \) is said to be principal iff for all \( p, q \in E \), \( p \perp q \) and \( p, q \leq a \Rightarrow p \oplus q \leq a \). We denote the set of principal elements of \( E \) by \( P_E \).

**Definition 1.2.** For effect algebras \( E_1, E_2 \) a mapping \( \phi: E_1 \to E_2 \) is said to be an isomorphism if \( \phi \) is a bijection, \( a \perp b \iff \phi(a) \perp \phi(b) \), \( \phi(1) = 1 \) and \( \phi(a \oplus b) = \phi(a) \oplus \phi(b) \).

Let us observe that if \( \phi: E_1 \to E_2 \) is an isomorphism then \( \phi(0) = 0 \), because \( \phi(0) \oplus 0 = \phi(0) = \phi(0 \oplus 0) = \phi(0) \oplus \phi(0) \) so by cancellation law \( 0 = \phi(0) \).

**Definition 1.3.** A quasigroup \((Q, \cdot)\) consists of a non-empty set \( Q \) equipped with a one binary operation \( \cdot \) such that if any two of \( a, b, c \) are given elements of a quasigroup, \( ab = c \) determines the third uniquely as an element of the quasigroup.

Moreover if \( a \cdot b = c \iff c \cdot a = b \) then \( Q \) is called semisymmetric (see [10]). Commutative semisymmetric quasigroups are called totally symmetric (see [11]).
2. Main Theorem

**Theorem 2.1.** [7, Theorem 3.5] If $p, q \in E$, $p \perp q$, and $p \lor q$ exists in $E$, then $p \land q$ exists in $E$, $p \land q \leq (p \lor q)' \leq (p \land q)'$ and $p \lor q = (p \land q) \oplus (p \lor q)$.

**Theorem 2.2.** Let $(E, 0, 1, \oplus)$ be an effect algebra. Then

$$P_E = \{x \in E: x \in S_E \text{ and } \forall t \in E t \leq x \Rightarrow t \lor x' \text{ exists in } E\}$$

**Proof.** Suppose that $x \in P_E$ then $x \in S_E$ (see Lemma 3.3 in [7]).

Let $t \in E$ and $t \leq x$ hence $t \perp x'$. We show that $t \lor x'$ is the join of $t$ and $x'$.

Obviously $t \leq t \lor x'$ and $x' \leq t \lor x'$. Suppose that $u \in E$, $t \leq u$ and $x' \leq u$ then

$$t \perp u'$$

(3)

and

$$u' \leq x \Rightarrow t \leq x.$$  

(4)

Now (3) and (4) implies $t \lor u' \leq x$ since $x \in P_E$. Hence $x' \perp (t \lor u')$ and by associativity $x' \perp t$ and $(x' \lor t) \perp t$ thus $t \lor x' \leq u$ so $t \lor x'$ is the smallest upper bound of the set $\{t, x'\}$ thus $t \lor x' = t \lor x'$.

Suppose that $x \in S_E$ and

$$\forall t \in E t \leq x \Rightarrow t \lor x' \text{ exists in } E.$$  

(8)

We show that $x \in P_E$.

If $u, s \in E$, $u \leq x$, $s \leq x$ and $u \perp s$ then

$$u \land x' = 0$$

(5)

because: if $y \leq x'$ and $y \leq u \leq x$ then $y = 0$ since $x \land x' = 0$.

Moreover $u \leq x$ so $u \lor x'$ exists by (8). By **Theorem 2.1**

$$u \lor x' = (u \land x') \lor (u \lor x')$$

(6)

We show that

$$u' \land x = (u \lor x')'$$

(7)

We show that $(u \lor x')'$ is a lower bound of the set $\{u', x\}$: $u \leq u \lor x' \Rightarrow u' \geq (u \lor x')'$ and $x' \leq u \lor x' \Rightarrow x \geq (u \lor x')'$.

If $v$ is a lower bound of $\{u', x\}$ then $u' \geq v$, $x \geq v$ then $u \leq v'$ and $x' \leq v'$ then $u \lor x' \leq v'$ and $(u \lor x')' \geq v$ and it implies that $(u \lor x')'$ is the greatest lower bound of the set $\{u', x\}$ so (7) is satisfied.

Moreover $s \leq u'$ (since $u \perp s$) and $s \leq x$ so $s \leq u' \land x$. Hence by (6) and (7) we have

$$s \leq u' \land x = (u \lor x')' = (u \lor x')'$$
so \( s \perp (u \oplus x') \) and by associativity \( s \oplus u \perp x' \) hence \( s \oplus u \leq x \) and \( x \in P_E \). \hfill \Box

In the following theorem we prove that in every effect algebra \( E \) sharp and principal elements coincide if and only if there exists in \( E \) join of every two orthogonal elements such that one of them is sharp.

**Theorem 2.3.** Let \((E, 0, 1, \oplus)\) be an effect algebra. Then \( S_E = P_E \) if and only if
\[
\forall t, x \in E \quad (t \perp x' \text{ and } x \wedge x' = 0) \Rightarrow t \lor x' \text{ exists in } E \quad (1)
\]

**Proof.** Suppose that \( S_E = P_E \). We show that (1) is satisfied.

Let \( x, t \in E, t \perp x' \) and \( x \wedge x' = 0 \). Then \( t \leq x, x \in P_E \) and by Theorem 2.2 we know that \( t \lor x' \) exists in \( E \).

Suppose that condition (1) is fullfilled. Obviously \( P_E \subseteq S_E \) (see Lemma 3.3 in [7]).

Now our task is to show that \( S_E \subseteq P_E \). Let \( x \in S_E \). If \( t \in E, t \leq x \) then \( t \perp x' \) and by condition (1) \( t \lor x' \) exists in \( E \) hence \( x \in P_E \) by Theorem 2.2. Thus \( S_E \subseteq P_E \). \hfill \Box

Let us observe that by Theorem 2.2 principal elements in an effect algebra are determined by partial order \( \leq \) and orthosupplementation \( ' \). We will see that there exist effect algebras \( E_1 = (E, 0, 1, \oplus_1) \) and \( E_2 = (E, 0, 1, \oplus_2) \) such that orthosupplementation \( ' \) in \( E_1 \) and orthosupplementation \( ' \) in \( E_2 \) are equal and also the same is true for partial order \( \leq \), but \( E_1 \) and \( E_2 \) are not isomorphic.

**Definition 2.4.** Let \((Q, \cdot)\) be a totally symmetric quasigroup.

We define \( E(Q, \cdot) := ((Q \times \{0\}) \cup (Q \times \{1\}) \cup \{0\} \cup \{1\}, 0, 1, \oplus) \) where
- \((q_1, 0) \oplus (q_2, 0) = (q_1 \cdot q_2, 1)\) for all \( q_1, q_2 \in Q \),
- \((q, 0) \oplus (q, 1) = (q, 1) \oplus (q, 0) = 1\) for all \( q \in Q \),
- \(0 \oplus x = x \ominus 0 = x\) for all \( x \in (Q \times \{0\}) \cup (Q \times \{1\}) \cup \{0\} \cup \{1\} \).

In remaining cases orthosum \( x \oplus y \) is not defined.

**Theorem 2.5.** If \((Q, \cdot)\) is a totally symmetric quasigroup then \( E(Q, \cdot) \) is an effect algebra.

**Proof.** The Commutative Law and Zero-unit Law are obvious. If \( q \in Q \) then there exists a unique element \( x = (q, 1) \) such that \((q, 0) \oplus x = 1\) so \((q, 0)' = (q, 1)\). Similarly \((q, 1)' = (q, 0)\) so the Orthosupplementation Law is satisfied.
It remains to show that the Associative Law is also fulfilled. Let $x, y, z \in (Q \times \{0\}) \cup (Q \times \{1\}) \cup \{0\} \cup \{1\}$. If $x = 0$ or $y = 0$, or $z = 0$ then the Associative Law is true. If $y \oplus z$ is defined and $x \oplus (y \oplus z)$ is defined and $x, y, z \neq 0$ then $x, y, z \in Q \times \{0\}$, so there exist $p, q, r \in Q$ such that $x = (p, 0)$, $y = (q, 0)$, $z = (r, 0)$, so $(q, 0) \oplus (r, 0)$ is defined and $(p, 0) \oplus ((q, 0) \oplus (r, 0))$ is defined, then $(p, 0) \oplus (q \cdot r, 1)$ is defined so $q \cdot r = p$ hence $p \cdot q = r$ thus $(p \cdot q, 1) \oplus (r, 0)$ is defined so $((p, 0) \oplus (q, 0)) \oplus (r, 0)$ is defined and $(p, 0) \oplus ((q, 0) \oplus (r, 0)) = ((p, 0) \oplus (q, 0)) \oplus (r, 0) = 1$. Therefore $(x \oplus y) \oplus z$ is defined and $x \oplus (y \oplus z) = (x \oplus y) \oplus z = 1$. □

Example 2.6. Let $Q = \{1, 2, 3\}$ and

| $\oplus_1$ | 1 | 2 | 3 | $\oplus_2$ | 1 | 2 | 3 |
|------------|---|---|---|------------|---|---|---|
| 1          | 1 | 3 | 2 | 1          | 1 | 2 | 1 |
| 2          | 3 | 2 | 1 | 2          | 1 | 3 | 2 |
| 3          | 2 | 1 | 3 | 3          | 3 | 2 | 1 |

then $E(Q, \cdot_1)$ and $(Q, \cdot_2)$ are totally symmetric quasigroups (see Example 2 and 3 in [11]). Then by Theorem 2.5 $E(Q, \cdot_1)$ and $E(Q, \cdot_2)$ are effect algebras with the following $\oplus$ tables. In this tables we do not include 0 and 1, since they have trivial sums and a dash means that the corresponding $\oplus$ is not defined:

| $\oplus_1$ | $a_1$ | $a_2$ | $a_3$ | $a'_1$ | $a'_2$ | $a'_3$ |
|------------|-------|-------|-------|--------|--------|--------|
| $a_1$      | $a_1$ | $a_3$ | $a_2$ | 1      | 1      | 1      |
| $a_2$      | $a_3$ | $a_2$ | $a_1$ | 1      | 1      | 1      |
| $a_3$      | $a_2$ | $a_3$ | $a_1$ | 1      | 1      | 1      |
| $a'_1$     | 1     | 1     | 1     | 1      | 1      | 1      |
| $a'_2$     | 1     | 1     | 1     | 1      | 1      | 1      |
| $a'_3$     | 1     | 1     | 1     | 1      | 1      | 1      |

| $\oplus_2$ | $a_1$ | $a_2$ | $a_3$ | $a'_1$ | $a'_2$ | $a'_3$ |
|------------|-------|-------|-------|--------|--------|--------|
| $a_1$      | $a_2$ | $a_3$ | 1     | 1      | 1      | 1      |
| $a_2$      | $a_1$ | $a_3$ | 1     | 1      | 1      | 1      |
| $a_3$      | $a_2$ | $a_1$ | 1     | 1      | 1      | 1      |
| $a'_1$     | 1     | 1     | 1     | 1      | 1      | 1      |
| $a'_2$     | 1     | 1     | 1     | 1      | 1      | 1      |
| $a'_3$     | 1     | 1     | 1     | 1      | 1      | 1      |

where $a_i = (i, 0)$ and $a'_i = (i, 1)$ for $i = 1, 2, 3$. In effect algebras $E(Q, \cdot_1)$ and $E(Q, \cdot_2)$ partial order $\le$ is the same: $a_1, a_2, a_3$ are minimal nonzero elements, $a'_1, a'_2, a'_3$ are maximal elements not equal to 1,
moreover $a_i \leq a'_j$ for all $i, j \in \{1, 2, 3\}$. Obviously orthosupplementation $'$ is the same in both effect algebras mentioned above. But $E(Q, \cdot_1)$ and $E(Q, \cdot_2)$ are not isomorphic:

Suppose that a mapping $\phi: (Q \times \{0\}) \cup (Q \times \{1\}) \cup \{0\} \cup \{1\} \rightarrow (Q \times \{0\}) \cup (Q \times \{1\}) \cup \{0\} \cup \{1\}$ is an isomorphism of $E(Q, \cdot_1)$ onto $E(Q, \cdot_2)$. Then

$$\phi(a_1) \oplus_2 \phi(a_1) = \phi(a_1 \oplus_1 a_1) = \phi(a_1) = \phi(a_1) \oplus_2 0$$

so $\phi(a_1) = 0$, but $\phi(0) = 0$ hence $a_1 = 0$ and we obtain a contradiction.
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