EXTENDED HIGHER HERGLOTZ FUNCTION II

RAJAT GUPTA AND RAHUL KUMAR

Abstract. Very recently, Radchenko and Zagier revived the theory of Herglotz functions. The main goal of the article is to show that one of the formulas on page 220 of Ramanujan’s Lost Notebook actually lives in the realms of this theory. As a consequence of our general theorem, we derive an interesting identity analogous to Ramanujan’s formula for \( \zeta(2m+1) \). We also introduce a character analogue of the Herglotz function and initiate its theory by obtaining an elegant functional equation governed by it.

1. Introduction

In his seminal work on the Kronecker limit formula for a real quadratic field, Zagier [19] found an interesting function which is now known as the Herglotz function. It is defined by

\[
F(x) := \sum_{n=1}^{\infty} \frac{\psi(nx) - \log(nx)}{n}, \quad x \in \mathbb{C} \setminus (-\infty, 0],
\]

where \( \psi(x) := \Gamma'(x)/\Gamma(x) \) is the digamma function. Prior to Zagier, a function quite similar to \( F(x) \) is also appeared in the work of Herglotz [10]. This is why Radchenko and Zagier call it the Herglotz function in [14]. Zagier establishes beautiful functional equations for \( F(x) \), namely, [19, Equations (7.4), (7.8)], for \( x \in \mathbb{C} \setminus (-\infty, 0] \),

\[
F(x) - F(x + 1) - F\left(\frac{x}{x + 1}\right) = -F(1) + \text{Li}_2\left(\frac{1}{1 + x}\right),
\]

\[
F(x) + F\left(\frac{1}{x}\right) = 2F(1) + \frac{1}{2} \log^2(x) - \frac{\pi^2}{6x}(x - 1)^2,
\]

where \( \text{Li}_2(x) := \sum_{n=1}^{\infty} \frac{1}{n^2} \) is dilogarithm function and [19, Equation (7.12)]

\[
F(1) = -\frac{1}{2} \gamma^2 - \frac{\pi^2}{12} - \gamma_1.
\]

Here \( \gamma \) and \( \gamma_1 \) denote the Euler and first Stieltjes constants respectively. In the same paper [19, Section 8], Zagier used these functional equations to prove Meyer’s theorem [13].

Recently, Radchenko and Zagier [14] studied \( F(x) \) extensively and unearthed a myriad of connections that \( F(x) \) has with other areas of number theory. They provided many exciting properties of \( F(x) \), including functional equations, special values at rational or quadratic irrational arguments, connections with Stark’s conjecture and with 1-cocycles for the modular group \( \text{PSL}(2, \mathbb{Z}) \).

Many generalizations of \( F(x) \) or its variants have been studied in the literature. Vlasenko and Zagier [18], in their work on higher Kronecker limit formula, considered the study of the...
higher Herglotz function:

$$F_k(x) := \sum_{n=1}^{\infty} \frac{\psi(nx)}{nk}, \quad k \in \mathbb{N}, \ k > 1, \ x \in \mathbb{C}\setminus(-\infty,0].$$  \hspace{1cm} (1.3)

They found the following functional equations for $F_k(x)$ [18, Equations (11), (12)]:

$$F_k(x) + (-x)^{k-1}F_k\left(\frac{1}{x}\right) = -\gamma \zeta(k) \left(1 + (-x)^{k-1}\right) - \sum_{r=2}^{k-1} \zeta(r)\zeta(k+1-r)(-x)^{r-1}

+ \zeta(k+1) \left((-x)^k - \frac{1}{x}\right),$$  \hspace{1cm} (1.4)

and

$$F_k(x) - F_k(x+1) + (-x)^{k-1}F_k\left(\frac{x+1}{x}\right)

= (-x)^{k-1} \left(\zeta(k,1) + \zeta(k+1) - \gamma \zeta(k)\right) - \sum_{r=1}^{k-1} \zeta(k+1-r,r)(-x)^{r-1}

+ \zeta(k+1) \left((-x)^k - \frac{1}{x}\right),$$

where, $\zeta(k)$ is the Riemann zeta function and

$$\zeta(m,n) := \sum_{p>q>0}^{\infty} \frac{1}{p^m q^n} \quad (m \geq 2, n \geq 1).$$

Vlasenko and Zagier [18, p. 54, Section 3] also dealt with another twisted extension of (1.3) to treat the zeta functions of ray classes. Ishibashi studied $j$th order Herglotz function to find an explicit representation of the Laurent series coefficients of the zeta function associated to indefinite quadratic forms [11, Theorem 3].

Let $\alpha : \mathbb{Z} \to \mathbb{C}$ be a periodic function with period $M$. Masri [12] defined the following $L$-series generalization of $F(x)$:

$$F(\alpha, s, x) := \sum_{n=1}^{\infty} \frac{\alpha(n) \left(\psi(nx) - \log(nx)\right)}{n^s}, \quad \text{Re}(s) > 0, \ x > 0,$$

and provided its analytic continuation and evaluated it at integer points.

Very recently, the authors along with Dixit [7] studied the extended higher Herglotz function

$$\mathcal{F}_{k,N}(x) := \sum_{n=1}^{\infty} \frac{\psi(n^N x) - \log(n^N x)}{n^k}, \quad x \in \mathbb{C}\setminus(-\infty,0],$$

where $k$ and $N$ are positive real numbers such that $k + N > 1$. Note that the function $\mathcal{F}_{k,N}(x)$ subsumes Herglotz function $F(x)$ as well as Vlasenko and Zagier’s higher Herglotz function $F_k(x)$. One of the nice functional equations which $\mathcal{F}_{k,N}(x)$ satisfies is

$$\frac{1-k}{x^N} \mathcal{F}_{k,N}(x) - \frac{(-1)^k}{N} \sum_{j=-(N-1)}^{(N-1)} e^{i\pi j(k-1)/N} \mathcal{F}_{N+k-1,\infty} \left(\frac{e^{-i\pi j}}{x^1/N}\right).$$

\[1\]The interpretation of $\zeta(1, k-1)$ given in [18, p. 28] has a minus sign missing in front of the whole term, that is, the correct form is $-(\zeta(k-1,1) + \zeta(k) - \gamma \zeta(k-1)).$
\[
\frac{x^{k+1}}{N} \left( \frac{1}{k} \right) + x^{k+1/N} \left( - (\gamma + \log x) \zeta(k) + N\zeta'(k) \right) - \frac{1}{x^{N/k}} \zeta(k + N) + \mathcal{B}(k, N, x),
\]

(1.5)

where \( k, N \in \mathbb{N} \) such that \( 1 < k \leq N \) and \( x \in \mathbb{C} \setminus (-\infty, 0] \) and

\[
\mathcal{B}(k, N, x) := \begin{cases} 
(-1)^{k+N+1}x^{1/N} \zeta(1 + \frac{k}{N}) & \text{if } k = N \\
0 & \text{if } k \neq N.
\end{cases}
\]

Here and throughout this paper, the notation \( \sum_{j=-(N-1)}^{(N-1)} \) denotes a sum over \( j = -(N-1), -(N-3), \ldots, N-3, N-1 \). For many other nice properties of \( F_{k,N}(x) \) and non-trivial applications of its functional equations, we refer the reader to [7].

In the last section of their paper, the present authors with Dixit [7] posed several questions. Here in this article, we answer some of those questions affirmatively. Before we do this though, we record an elegant formula from Ramanujan’s Lost Notebook on page 220. If

\[
\phi(x) := \psi(x) + \frac{1}{2x} - \log x,
\]

then for \( \alpha, \beta \) positive such that \( \alpha \beta = 1 \), we have [16, p. 220]

\[
\sqrt{\alpha} \left\{ \frac{\gamma - \log(2\pi \alpha)}{2\alpha} + \sum_{n=1}^{\infty} \varphi(n\alpha) \right\} = \sqrt{\beta} \left\{ \frac{\gamma - \log(2\pi \beta)}{2\beta} + \sum_{n=1}^{\infty} \varphi(n\beta) \right\}.
\]

(1.6)

He also provided the following surprising integral representation which is equal to either sides of (1.6):

\[
-\frac{1}{\pi^{3/2}} \int_{0}^{\infty} \left| \Xi \left( \frac{t}{2} \right) \Gamma \left( \frac{1 + it}{4} \right) \right|^2 \cos \left( \frac{t}{4} \log \alpha \right) \frac{dt}{1 + t^2},
\]

(1.7)

where \( \Xi(t) \) denotes Riemann’s function defined by [17, p. 16, Equations (2.1.14), (2.1.12)]

\[
\Xi(t) := \xi \left( \frac{1}{2} + it \right),
\]

and

\[
\xi(s) := \frac{s}{2} (s - 1) \pi^{-\frac{s}{2}} \Gamma \left( \frac{1}{2} \right) \zeta(s).
\]

The proof of the above formula can be found in [3]. The above transformation and various other transformations of its type have generated interesting mathematics. We refer the reader to recent papers [6], [9], and references therein for the extensive literature related to this formula.

It was commented in [7, p. 27] that “it seems that (1.6) lives in the realms of the theory of Herglotz function” by observing the similarity between the definition of \( F(x) \) in (1.1) and series appearing on both sides of (1.6). One of the main objectives of this paper is to show that this is in-fact the case, that is, Ramanujan’s formula (1.6) is actually a special case of two of our more general results on the Herglotz functions (Theorems 2.1, and 2.4). Note that the functional equation (1.5) is valid only when \( 1 < k \leq N \). It was also asked in [7, Section 5, p. 28] if there exists a functional equation for \( \mathcal{F}_{k,N}(x) \) where \( k \) is greater than \( N \)? We answer this question affirmatively in Theorem 2.5.

Although \( \mathcal{F}_{k,N}(x) \) is a generalization of Valsenko-Zagier’s higher Herglotz function \( \mathcal{F}_{k}(x) \), the functional equation of \( \mathcal{F}_{k}(x) \) in (1.4) could not be obtained from (1.5) as a special case due to the restriction \( 1 < k \leq N \). This problem is now circumvented through our Theorem 2.5.
To achieve our goals, we work with the following version of the extended higher Herglotz function:

\[ \mathfrak{H}_{k,N}(x) := \sum_{n=1}^{\infty} \frac{1}{n^k} \left( \psi(n^N x) - \log(n^N x) + \frac{1}{2n^N x} \right), \quad (1.8) \]

for \( x \in \mathbb{C} \setminus (-\infty, 0] \), and \( k + 2N > 1 \). The series is absolutely and uniformly convergent for \( k + 2N > 1 \) since [p. 259, formula 6.3.18]

\[ \psi(x) \sim \log(x) - \frac{1}{2x} - \frac{1}{12x^2} + \cdots, \quad \text{as} \ x \to \infty, \ |\arg(x)| < \pi. \]

Note that for \( k = N = 1 \), \( \mathfrak{H}_{k,N}(x) \) reduces to the function \( \mathfrak{H}^*(x) \):

\[ \mathfrak{H}^*(x) := \sum_{n=1}^{\infty} \frac{1}{n} \left( \psi(nx) - \log(nx) + \frac{1}{2nx} \right). \]

The function \( \mathfrak{H}^*(x) \) was used by Radchenko and Zagier [14, p. 17, Section 7.3] to reveal the cocycle nature of the Herglotz function \( \mathfrak{H}(x) \) by finding an Eichler-type integral for it.

The Herglotz function \( \mathfrak{H}(x) \) is a special case of \( \mathfrak{H}_{k,N}(x) \):

\[ \mathfrak{H}_{1,1}(x) = \mathfrak{H}(x) + \frac{\pi^2}{12x}. \]

Also,

\[ \mathfrak{H}_{k,N}(x) = \mathfrak{H}_{k,N}(x) + \frac{1}{2x}\zeta(k + N). \]

**Remark 1.1.** Ramanujan’s formula (1.6) can be rephrased in terms of \( \mathfrak{H}_{k,1}(x) \) by

\[ \mathfrak{H}_{k,1}(x) - \frac{1}{x} \mathfrak{H}_{k,1}(\frac{1}{x}) = \frac{1}{2} \left( \gamma - \log\left(\frac{2\pi}{x}\right) \right) - \frac{1}{2x} (\gamma - \log(2\pi x)). \quad (1.9) \]

Thus Ramanujan understood the importance of such a function. Therefore, it would not be unfair to say that Ramanujan studied Herglotz-type functions even before Herglotz and Zagier. Further, he went beyond the functional equation to find a surprising integral representation for this Herglotz-type function which involves the Riemann \( \Xi \)-function.

The next objective of this paper is to initiate the theory of a character analogue of the Herglotz function. We define it by

\[ F_k(x, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n) \psi(n^k x, \chi)}{n^k}, \quad x \in \mathbb{C} \setminus (-\infty, 0] \quad (1.10) \]

here and throughout the article \( \chi(n) \) is a primitive, nonprincipal character modulo \( d \), \( k \) is a non-negative integer, and

\[ \psi(x, \chi) := -\sum_{k=1}^{\infty} \frac{\chi(k)}{k + x}. \]

For a real character \( \chi \), the function \( \psi(n, \chi) \) is equivalent to the character analogue of the digamma function obtained by the logarithmic differentiation of the Weierstrass product form of the character analogue of the gamma function for real characters introduced by Berndt [2]:

\[ \Gamma(s, \chi) := e^{-sL(1, \chi)} \prod_{n=1}^{\infty} \left( 1 + \frac{s}{n} \right)^{-\chi(n)} e^{s\chi(n)/n}, \quad s \in \mathbb{C} \setminus \mathbb{Z} < 0. \]
Note that the series in (1.10) is convergent for any \( k \geq 0 \) since [5, p. 334, Corollary 4.4]

\[
\psi(x, \chi) \sim -\frac{L(0, \chi)}{x} - \chi(-1) \sum_{j=2}^{\infty} \frac{B_j(\chi)}{jx^j}, \quad |\text{arg}(x)| < \pi,
\]
as \( x \to \infty \). Here \( B_j(\chi) \) is the generalized Bernoulli numbers [2, p. 426].

2. Main Results

We divide this section into two parts. The first part is devoted to extending the modular relation of Ramanujan (1.6), and the other is dedicated to initiating the theory of \( F_k(x, \chi) \) defined in (1.10).

2.1. Extension of the formula on page 220 of Ramanujan’s Lost Notebook. In this subsection, we state three of our main theorems. The first is Theorem 2.1, which is valid for \( k \in \mathbb{N} \cup \{0\} \) and \( N = 1 \). The second is Theorem 2.4 which holds for \( k = 1 \) and \( N \in \mathbb{N} \). We conclude this subsection by providing Theorem 2.5 which holds for any natural numbers \( k \) and \( N \) greater than 1.

Theorem 2.1. Let \( \tilde{F}_{k,N}(x) \) be defined in (1.8). For any \( k \in \mathbb{N} \cup \{0\} \) and \( \text{Re}(x) > 0 \), the following identity holds:

\[
\tilde{F}_{k,1}(x) = (-1)^k x^{k-1} \left\{ \tilde{F}_{k,1} \left( \frac{1}{x} \right) - B_k(x) \right\},
\]

where, for \( k = 0 \),

\[
B_k(x) := \frac{x}{2} \left( \log \left( \frac{2\pi}{x} \right) - \gamma \right) + \frac{1}{2} (\gamma - \log(2\pi x)),
\]

for \( k = 1 \),

\[
B_k(x) := \frac{1}{6} \left( 3 \log^2(x) + \pi^2 - 6\gamma^2 - 12\gamma_1 \right) - \frac{\pi^2}{12} \left( x + \frac{1}{x} \right),
\]

and for \( k \geq 2 \),

\[
B_k(x) := \zeta(k) (\log(x) - \gamma) + \zeta'(k) + (-1)^k x^{1-k} \left( (\log(x) + \gamma)\zeta(k) - \zeta'(k) \right) + \frac{(-1)^k}{2x^k} \zeta(k+1)
\]
\[ - \frac{x}{2} \zeta(1+k) + \sum_{j=1}^{k-2} (-1)^{j-1} \zeta(j+1) \zeta(k-j)x^{-j}.
\]

Remark 2.2. It is easy to see that Ramanujan’s formula (1.6) (or (1.7)), Zagier’s functional equation (1.2), and Vlasenko-Zagier’s functional equation (1.4) are special cases of the above theorem for \( k = 0, \ k = -1, \) and \( k \geq 2 \), respectively.

An immediate consequence of the above theorem is the following beautiful symmetric formula.
Theorem 2.3. Let $k$ be any natural number greater than 1. Let $\text{Re}(\alpha) > 0$ and $\text{Re}(\beta) > 0$ such that $\alpha \beta = 1$, then the following relation holds:

$$
\alpha^{\frac{1-k}{2}} \left\{ (\gamma + \log(\alpha))\zeta(k) - \zeta'(k) + \frac{1}{2\alpha} \zeta(k+1) + \mathfrak{F}_{k,1}(\alpha) \right\} = (-1)^k \beta^{\frac{1-k}{2}} \left\{ (\gamma + \log(\beta))\zeta(k) - \zeta'(k) + \frac{1}{2\beta} \zeta(k+1) + \mathfrak{F}_{k,1}(\beta) \right\} + \sum_{j=1}^{k-2} (-1)^j \zeta(j+1)\zeta(k-j) \alpha^{\frac{1-k+j}{2}} \beta^{-\frac{j}{2}}.
$$

(2.1)

This formula is analogous to the Ramanujan’s formula for $\zeta(2m+1)$ [5] p. 173, Ch. 14, Entry 21(i), namely, for $\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \alpha \beta = \pi^2$ and $m \in \mathbb{Z}\setminus\{0\}$, by

$$
\alpha^{-m} \left\{ \frac{1}{2} \zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\pi mn} - 1} \right\} = (-\beta)^{-m} \left\{ \frac{1}{2} \zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\beta mn} - 1} \right\} - 2^{2m} \sum_{j=0}^{m-1} (-1)^j B_{2j} B_{2m+2-2j} \alpha^{m+1-j} \beta^j,
$$

where $B_n$ is the $n^\text{th}$ Bernoulli number.

The next theorem is another extension of (1.6) in the second parameter $N$.

Theorem 2.4. Let $N$ be any natural number and $\text{Re}(x) > 0$. Then

$$
\tilde{\mathfrak{F}}_{0,N}(x) = \frac{x^{-1/N}}{N} \sum_{j=-(N-1)}^{N-1} e^{-\frac{j \pi x}{N}} \tilde{\mathfrak{F}}_{N-1,N} \left( \frac{e^{i \pi j x}}{x^{1/N}} \right) - \frac{1}{2} \left( \log \left( \frac{(2\pi)^N}{x} \right) - \gamma \right) + R(N, x),
$$

(2.2)

where

$$
R(N, x) := \begin{cases} 
\frac{\pi x}{2N} (\log(2\pi x) - \gamma), & \text{if } N = 1 \\
-\frac{\pi x^{-1/N}}{N \sin(\pi/N)} \zeta(1-1/N) - \frac{1}{2} \zeta(N), & \text{if } N > 1.
\end{cases}
$$

(2.3)

Now we state a general theorem which is valid for any $k, N \in \mathbb{N}\setminus\{1\}$.

Theorem 2.5. Let $k, N$ be any natural numbers greater than 1. Then for $\text{Re}(x) > 0$, we have

$$
\tilde{\mathfrak{F}}_{k,N}(x) = \frac{(-1)^k x^{-k-1/N}}{N} \sum_{j=-(N-1)}^{N-1} e^{i \pi j x^{1/N}} \tilde{\mathfrak{F}}_{N+k-1,N} \left( \frac{e^{i \pi j x}}{x^{1/N}} \right) - \frac{x^{\frac{1-k}{N}}}{N} \mathfrak{N}_{N,k}(x),
$$

where, for $N \nmid (k - 1)$ and $N \nmid k$,

$$
\mathfrak{N}_{N,k}(x) = \frac{N}{2} x^{\frac{1-k-N}{N}} \zeta(k + N) - \frac{\pi \zeta \left( \frac{N+k-1}{N} \right)}{\sin \left( \frac{x}{N} (k - 1) \right)} - N x^{\frac{1-k}{N}} (N \zeta'(k) - (\gamma + \log x) \zeta(k)) + N \sum_{j=1}^{\left\{ \frac{k-1}{N} \right\}} (-1)^j \zeta(1+j) \zeta(k-N) x^{\frac{k-1}{N}+j},
$$

+ N \sum_{j=1}^{\left\{ \frac{k-1}{N} \right\}} (-1)^j (k-1) \zeta(k-N) x^{\frac{k-1}{N}+j},
$$

(2.4)

where $\mathfrak{F}_{N,k}(x)$ is the Ramanujan’s formula for $\zeta(2m+1)$.
and, for $N|k$,

$$
R_{N,k}(x) = \frac{N}{2} x^{\frac{1-k-N}{N}} \zeta(k+N) - x^{\frac{1-k}{N}} \sin \left( \frac{k+1}{N} \right) - N x^{\frac{1-k}{N}} \left( N \zeta'(k) - (\gamma + \log x) \zeta(k) \right)
$$

and, for $N/(k-1)$,

$$
R_{N,k}(x) = \frac{N}{2} x^{\frac{1-k-N}{N}} \zeta(k+N) - x^{\frac{k-1}{N}} \left( (N \gamma - \log x) \zeta \left( \frac{N+k-1}{N} \right) - \zeta' \left( \frac{N+k-1}{N} \right) \right)
$$

When we restrict $1 < k \leq N$ in the above theorem, it gives (1.5) as a special case:

**Corollary 2.6.** The functional equation (1.5) holds true.

### 2.2. Character analogue of the Herglotz function.

For a non-principal Dirichlet character $\chi(n)$, let $L(s, \chi)$ denote the Dirichlet $L$-function defined by

$$
L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}
$$

for $\Re(s) > 1$. This series converges conditionally for $0 < \Re(s) < 1$. Also, it can be analytically continued to an entire function of $s$.

The two-term functional equation governed by $F_k(x, \chi)$ is given in the following theorem.

**Theorem 2.7.** Let $F_k(x, \chi)$ be defined in (1.10). Let $\chi(n)$ be a primitive, nonprincipal character modulo $d$. Then $k \in \mathbb{N} \cup \{0\}$ and $\Re(x) > 0$, we have

$$
F_k(x, \chi) - (-1)^k x^{k-1} F_k \left( x, \frac{1}{\chi} \right) = \sum_{j=0}^{k-1} (-x)^{k-j-1} L(1 + j, \chi) L(-j + k, \chi).
$$

Now to state our next theorem, we define

$$
b := \begin{cases} 
0 & \text{if } \chi(-1) = 1 \\
1 & \text{if } \chi(-1) = -1.
\end{cases}
$$

Then the $\Xi(s, \chi)$ can be defined as

$$
\Xi(t, \chi) := \xi \left( \frac{1}{2} + it, \chi \right), \quad (2.4)
$$

where

$$
\xi(s, \chi) := \left( \frac{\pi}{d} \right)^{-s+b}/2 \Gamma \left( \frac{s+b}{2} \right) L(s, \chi). \quad (2.5)
$$

Our next theorem gives an elegant modular relation along with the integral comprising $\Xi(t, \chi)$-function.
Theorem 2.8. Let $\chi(n)$ be a primitive, nonprincipal character modulo $d$. Further assume
$\Re(\alpha) > 0$, $\Re(\beta) > 0$ such that $\alpha\beta = 1$, then

$$\sqrt{\alpha} F_0(\alpha, \chi) = \sqrt{\beta} F_0(\beta, \chi)$$

$$= -\frac{1}{2\pi} \left( \frac{\pi}{d} \right)^{b+\frac{1}{2}} \int_0^\infty \left| \frac{\Gamma \left( \frac{1}{4} + \frac{b}{2} \right)}{\Gamma \left( \frac{1}{4} + \frac{b}{2} + \frac{1}{2} \right)} \right|^2 \Xi \left( -\frac{t}{2}, \chi \right) \Xi \left( \frac{t}{2}, \chi \right) \cos \left( \frac{t}{2} \log \alpha \right) dt.$$

The above theorem reduces to [5, Corollaries 5.1 and 5.2] depending on the parity of the character.

3. Proofs

We first present the proof of our general Theorem 2.5. Throughout the proofs, the notation $\int_{(c)} ds$ will denote the line integral $\int_{c-i\infty}^{c+i\infty} ds$ with $c = \Re(s)$.

**Proof of Theorem 2.5** Kloosterman’s formula is given by [17, p. 25, Equation (2.9.1)]

$$\psi(x + 1) - \log(x) = \frac{1}{2\pi i} \int_{(c)} -\frac{\pi \zeta(1 - z)}{\sin(\pi z)} x^{-z} dz,$$ (3.1)

which is valid for $0 < c = \Re(z) < 1$. An application of the functional equation

$$\psi(x + 1) = \psi(x) + 1/x$$

in (3.1) implies that

$$\psi(x) - \log(x) + \frac{1}{x} = \frac{1}{2\pi i} \int_{(c)} -\frac{\pi \zeta(1 - z)}{\sin(\pi z)} x^{-z} dz.$$ (3.2)

For our purpose we need the integral representation for the function $\psi(x) - \log(x) + \frac{1}{2x}$. For that, we shift the line of the integration to $1 < d = \Re(z) < 2$. Let $C$ be a positively oriented rectangular contour formed by the points $c - iT, d - iT, d + iT, c + iT$. Observe that the integrand has only one simple pole at $s = 1$ due to $\sin(\pi z)$ and its residue is $-1/(2x)$ as $\zeta(0) = -1/2$. Therefore, residue theorem immediately yields

$$\frac{1}{2\pi i} \left( \int_{d-iT}^{d+iT} - \int_{c-iT}^{c+iT} \right) -\frac{\pi \zeta(1 - z)}{\sin(\pi z)} x^{-z} dz = -\frac{1}{2x}. $$ (3.3)

Note that the integrals along the horizontal lines goes to zero by using the Stirling formula in the vertical strip $p \leq \sigma \leq q$ [4, p. 224]:

$$|\Gamma(s)| = \sqrt{2\pi} |t|^p \frac{1}{2} e^{-\frac{1}{2} \pi |t|} \left( 1 + O \left( \frac{1}{|t|} \right) \right)$$ (3.4)

as $|t| \to \infty$. Thus, from (3.3), we have

$$\frac{1}{2\pi i} \int_{(d)} -\frac{\pi \zeta(1 - z)}{\sin(\pi z)} x^{-z} dz = \frac{1}{2\pi i} \int_{(c)} -\frac{\pi \zeta(1 - z)}{\sin(\pi z)} x^{-z} dz - \frac{1}{2x}. $$ (3.5)

Combining (3.2) and (3.5) together, we obtain

$$\psi(x) - \log(x) + \frac{1}{2x} = \frac{1}{2\pi i} \int_{(d)} -\frac{\pi \zeta(1 - z)}{\sin(\pi z)} x^{-z} dz.$$ (3.6)
Replacing \( x \) by \( n^N x \) in the above equation and invoking (1.3), we obtain
\[
\mathfrak{I}_{k,N}(x) = \sum_{n=1}^{\infty} \frac{1}{n^k 2\pi i} \int_{(d)} -\frac{\pi \zeta(1-z)}{\sin(\pi z)} (n^N x)^{-z} \, dz
\]
\[
= \frac{1}{2\pi i} \int_{(d)} -\frac{\pi \zeta(1-z)\zeta(k+Nz)}{\sin(\pi z)} x^{-z} \, dz,
\]
(3.7)
where in the last step we interchanged the order of the summation and integration and used the series definition of \( \zeta(s) \). We now make change of variable \( s = 1 - k - Nz \) in (3.7) to deduce, for \( 1 - k - 2N < c < 1 - k - N \),
\[
\mathfrak{I}_{k,N}(x) = \frac{x^{k-1}}{N} \left( \frac{1}{2\pi i} \int_{(\lambda)} \frac{\pi \zeta \left(1 + \frac{s+k-1}{N}\right) \zeta(1-s)}{\sin \left(\frac{\pi}{N} (s+k-1)\right)} \, ds - \mathcal{R}_{k,N}(x) \right).
\]
(3.9)
We next shift the line of integration to \( 1 < \lambda < 2 \). Consider the contour formed by the line segments \([\lambda - iT, \lambda + iT], [\lambda + iT, c + iT], [c + iT, c - iT]\) and \([c - iT, \lambda - iT]\). In order to do so, we encounter several poles of different orders depending upon \( k \) and \( N \). We denote the sum of the residues by \( \mathcal{R}_{k,N}(x) \). By invoking Stirling’s formula (3.4), we see that the integrals along the horizontal lines vanish as \( T \to \infty \). Thus, by residue theorem and (3.8), we have
\[
\mathfrak{I}_{k,N}(x) = \frac{x^{k-1}}{N} \left( \int_{(\lambda)} \frac{\pi \zeta \left(1 + \frac{s+k-1}{N}\right) \zeta(1-s)}{\sin \left(\frac{\pi}{N} (s+k-1)\right)} \, ds \right) - \mathcal{R}_{k,N}(x).
\]
Now, the main task is to evaluate the line integral present on the right-hand side of the above equation. Let us denote it by \( \mathcal{I}_{k,N}(x) \). To that end, we invoke following result [8, Lemma 4.1]
\[
\frac{1}{\sin(z)} = \sum_{j=-(N-1)}^{N-1} e^{ijz}
\]
with \( z = \frac{\pi}{N} (s+k-1) \) in \( \mathcal{I}_{k,N}(x) \), so that
\[
\mathcal{I}_{k,N}(x) = \sum_{j=-(N-1)}^{(N-1)} e^{i\frac{j(k-1)}{N}} \int_{(\lambda)} \frac{\pi \zeta(1-s)}{\sin \left(\frac{\pi}{N} (s+k-1)\right)} \left( e^{-\frac{\pi iz}{N}} \right)^{-s} \, ds.
\]
We can use the series definition of \( \zeta \left(1 + \frac{s+k-1}{N}\right) \) since \( 1 < \lambda = \text{Re}(s) < 2 \). Thus, we arrive at
\[
\mathcal{I}_{k,N}(x) = (-1)^k \sum_{j=-(N-1)}^{(N-1)} e^{i\frac{j(k-1)}{N}} \int_{(\lambda)} \frac{\pi \zeta(1-s)}{\sin \left(\frac{\pi}{N} (s+k-1)\right)} \left( e^{-\frac{\pi iz}{N}} \right)^{-s} \, ds.
\]
Invoking (3.6) in the above equation so as to obtain
\[
\mathcal{I}_{k,N}(x) = (-1)^k \sum_{j=-(N-1)}^{(N-1)} e^{i\frac{j(k-1)}{N}} \int_{(\lambda)} \left\{ \psi \left( \frac{n}{x} \frac{1}{N} e^{-\frac{\pi iz}{N}} \right) - \log \left( \frac{n}{x} \frac{1}{N} e^{-\frac{\pi iz}{N}} \right) \right\} \, ds
\]
\[
\hspace{1cm} + \frac{1}{2} \left( \frac{n}{x} \right)^{-\frac{1}{N} e^{-\frac{\pi iz}{N}}} \left( e^{-\frac{\pi iz}{N}} \right) \right),
\]
(3.10)
where in the last step we employed the definition of $\mathfrak{H}_{k,N}(x)$ from (1.8).

We next calculate the residues. We break this calculation into different cases depending on whether $N$ divides $k$ and $k-1$ or not. It boils down to only three cases: $N \nmid k$ and $N \nmid (k-1)$, $N|k$ and $N|(k-1)$ as if $N$ divides $k$ then it cannot divide $k-1$ and vice-versa.

**Case I:** Let $N \nmid k$ and $N \nmid (k-1)$. Observe that the integrand has simple poles at $s = 0$ due to $\zeta(1-s)$, and at $s = 1-k-N, 1-k+Nj$, where $1 \leq j \leq \lfloor \frac{k-1}{N} \rfloor$, due to $\sin \left( \frac{\pi}{N} (1-s-k) \right)$. It also has double pole at $s = 1 - k$ due to $\sin \left( \frac{\pi}{N} (1-s-k) \right)$ and $\zeta \left( 1 + \frac{s+k-1}{N} \right)$. These can be evaluated to as:

\[
R_0 = -\frac{\pi \zeta \left( \frac{N+k-1}{N} \right)}{\sin \left( \frac{\pi}{N} (k-1) \right)},
\]

\[
R_{1-k-N} = \frac{N}{2} \zeta(k + N)x^{\frac{k-1}{N}} \zeta(k-Nj),
\]

\[
R_{1-k} = N \{N \zeta'(k) - (\gamma + \log(x))\zeta(k)\} x^{\frac{k-1}{N}},
\]

\[
R_{1-k+Nj} = N \sum_{j=1}^{\lfloor \frac{k-1}{N} \rfloor} (-1)^j \zeta(1+j)\zeta(k-Nj)x^{\frac{k-1}{N}+j}.
\] (3.11)

**Case II:** Let $N|k$. Therefore, $k = mN$ for some $m \in \mathbb{N}$. This implies $\sin \left( \frac{\pi}{N} (1-s-k) \right) = (-1)^{m+1} \sin \left( \frac{\pi}{N} (1-s) \right)$. Hence, the integrand has an extra simple pole at $s = 1$ besides all other poles in the previous case. The residue at the pole $s = 1$ is:

\[
R_1 = -\frac{N}{2} (-1)^{k/N} x^{1/N} \zeta \left( \frac{k + N}{N} \right).
\] (3.12)

**Case III:** Let $N|(k-1)$. In this case, $\sin \left( \frac{\pi}{N} (1-s-k) \right)$ also has pole at $s = 0$ which was not the case before (this was $j = \lfloor \frac{k-1}{N} \rfloor$ case in Case I). This implies that the integrand has now double pole at $s = 0$ along with other poles in the Case I, but now $j$ runs between $1 \leq j \leq \lfloor \frac{k-1}{N} \rfloor - 1$. Therefore, we only need to calculate the residue at $s = 0$, that is:

\[
R_0 = (-1)^{\frac{k-1}{N}} \left\{ (N \gamma - \log(x))\zeta \left( \frac{N+k-1}{N} \right) - \zeta' \left( \frac{N+k-1}{N} \right) \right\}.
\] (3.13)

We now combine all of the residues from (3.11), (3.12), and (3.13) together, then it is nothing but the term $\mathfrak{H}_{k,N}(x)$ defined in the statement of the theorem according to the different cases.

Theorem now follows upon using the above facts and (3.10) in (3.9).

**Proof of Theorem** [22]. Note that up to (3.8), the calculation holds for any $N \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$. Thus, letting $N = 1$ in (3.8) gives for $-1 - k < c < -k$,

\[
\mathfrak{H}_{k,1}(x) = (-1)^k x^{k-1} \frac{1}{2\pi i} \int_{(c)} \frac{-\pi \zeta(s+k)\zeta(1-s)}{\sin(\pi s)} x^s ds.
\]

We shift the line of integration to $1 < \lambda < 2$ in the similar manner as we did in after (3.8). Thus, we have

\[
\mathfrak{H}_{k,1}(x) = (-1)^k x^{k-1} \left( \frac{1}{2\pi i} \int_{(\lambda)} \frac{-\pi \zeta(s+k)\zeta(1-s)}{\sin(\pi s)} x^s ds - \mathcal{R}_{k,1}(x) \right).
\] (3.14)
The calculation of the residues depends on the values of \( k \) is 0, 1 and \( k \geq 2 \).

**Case I:** When \( k = 0 \), the integrand has double pole at \( s = 0 \) due to \( \zeta(1-s) \) and \( \sin(\pi s) \) and \( s = 1 \) due to \( \zeta(s) \) and \( \sin(\pi s) \). The residues at these points are:

\[
R_0 = \frac{1}{2} \left( \gamma - \log(2\pi x) \right) \\
R_1 = \frac{x}{2} \left( \log\left( \frac{2\pi}{x} \right) - \gamma \right).
\]  
(3.15)

**Case II:** Let \( k = 1 \). Now integrand has simple pole at \( s = 1, -1 \) due to \( \sin(\pi s) \) and pole of order three at \( s = 0 \) due to \( \sin(\pi s), \zeta(s+1) \) and \( \zeta(1-s) \) with residues:

\[
R_1 = -\frac{\pi^2}{12}x \\
R_{-1} = \frac{\pi^2}{12}x \\
R_0 = \frac{1}{6} \left( 3\log^2(x) + \pi^2 - 6\gamma^2 - 12\gamma \right).
\]  
(3.16)

**Case III:** For \( k \geq 2 \), the integrand has pole of order 2 at \( s = 0 \) and \( s = 1-k \) because of \( \sin(\pi s), \zeta(1-s) \) and \( \zeta(s+k) \), respectively. It has also simple poles at \( s = 1, -k \) and \( s = -j, 1 \leq j \leq k-2 \) due to \( \sin(\pi s) \). The residues at these poles can be evaluated to

\[
R_0 = \zeta(k)(\log(x) - \gamma) + \zeta'(k) \\
R_{1-k} = (-1)^k x^{1-k} \left\{ \zeta(k)(\log(x) + \gamma) - \zeta'(k) \right\} \\
R_1 = -\frac{x}{2} \zeta(1+k) \\
R_{-k} = \frac{(-1)^k}{2x^k} \zeta(k+1) \\
R_j = \sum_{j=1}^{k-2} (-1)^{j-1} \zeta(j+1) \zeta(k-j)x^{-j}.
\]  
(3.17)

Upon combining the residues from (3.15), (3.16) and (3.17) together in (3.14), we see that \( R_{k,1}(x) \) is nothing but \( B_k(x) \) defined in the statement of the theorem.

The same argument, we used to obtain (3.10), can be adapted to show that

\[
\frac{1}{2\pi i} \int_{(\lambda)} \frac{-\pi \zeta(s+k)\zeta(1-s)}{\sin(\pi s)} x^s ds = \mathfrak{F}_{k,1}(\frac{1}{x}) \cdot \mathfrak{F}_{k,1}(\frac{1}{x})
\]  
Substituting the residues and the above integral in (3.14), we complete the proof. \( \square \)

**Proof of Theorem 2.4** The argument is similar to the proof of Theorem 2.3. One can proceed along the same lines with letting \( k = 0 \) and then only difference is while calculating the residual terms. When \( N > 1 \), the integrand has simple poles at \( s = 0, s = 1-N \) and a pole of order two at \( s = 1 \). Whereas, for \( N = 1 \), it has poles of order two at \( s = 0 \) and \( s = 1 \). Therefore, we have

\[
\mathfrak{F}_{0,N}(x) = \frac{x^{-\frac{1}{N}}}{N} \left( \frac{1}{2\pi i} \int_{(\lambda)} \frac{\pi \zeta(1+s/\sqrt[N]{N})\zeta(1-s)}{\sin(\frac{s}{\sqrt[N]{N}})} x^{\frac{s}{\sqrt[N]{N}}} ds - \mathfrak{H}_{0,N}(x) \right).
\]  
(3.18)
It can be easily seen that the residual term \( R_{0,N} \) present in above equation is
\[
- \frac{N}{2} x^{1/N} \left( \log \left( \frac{(2\pi)^N}{x} \right) - \gamma \right) + R(N, x),
\]
where \( R(N, x) \) is defined in (2.3).

The line integral in (3.18) can be evaluated by letting \( k = 0 \) in (3.10). Hence, now (2.2) follows straightforward upon using the above facts. \( \square \)

**Proof of Theorem 2.7.** For \( k \geq 2 \), Theorem 2.1 gives
\[
x^{-\frac{1}{2}} \tilde{F}_{k,1}(x) = (-1)^k x^{-\frac{k-1}{2}} \tilde{F}_{k,1} \left( \frac{1}{x} \right) - x^{-\frac{1}{2}} \left( (\log x + \gamma) \zeta(k) - \zeta'(k) \right) + \frac{1}{2} x^{-\frac{1}{4} - j} \zeta(k + 1) + \sum_{j=1}^{k-2} (-1)^{k+j-1} \zeta(j + 1) \zeta(k-j) x^{-\frac{j-1}{2}.}
\]

Let \( x = \alpha \) and \( \alpha \beta = 1 \). Then the above equation can be rephrased as
\[
\alpha^{-\frac{1}{2}} \left\{ (\log \alpha + \gamma) \zeta(k) - \zeta'(k) + \frac{1}{2\alpha} \zeta(k + 1) + \tilde{F}_{k,1}(\alpha) \right\}
\]
\[
= (-1)^k \beta^{-\frac{1}{2}} \left\{ (\log \beta + \gamma) \zeta(k) - \zeta'(k) + \frac{1}{2\beta} \zeta(k + 1) + \tilde{F}_{k,1}(\beta) \right\}
\]
\[
+ \sum_{j=1}^{k-2} (-1)^{k+j-1} \zeta(j + 1) \zeta(k-j) \alpha^{-\frac{k-1}{2} - j}. (3.19)
\]
Replacing \( j \) by \( k - 1 - j \) in the finite sum of (3.19) and using the fact \( \alpha \beta = 1 \), we arrive at (2.7). \( \square \)

We now present the proofs of our results on the character analogue of the Herglotz function \( F_k(x, \chi) \). We first prove the two-term functional equation.

**Proof of Theorem 2.7.** We first evaluate the sum
\[
\sum_{n=1}^{\infty} \frac{\chi(n) \psi(nx, \chi)}{n^k},
\]
by employing [5] Corollary 4.2\(^2\) hence we have for \( 0 < c < 1 \),
\[
\sum_{n=1}^{\infty} \frac{\chi(n) \psi(nx, \chi)}{n^k} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^k} \frac{1}{2\pi i} \int_{(c)} -\pi L(1 - s, \chi) (nx)^{-s} ds
\]
\[
= \frac{1}{2\pi i} \int_{(c)} -\pi L(1 - s, \chi) L(k + s, \chi) x^{-s} ds.
\]

Upon performing the change of variable from \( s \) to \( 1 - s - k \), we obtain
\[
\sum_{n=1}^{\infty} \frac{\chi(n) \psi(nx, \chi)}{n^k} = \frac{(-1)^k}{2\pi i} \int_{(d)} -\pi L(s + k, \chi) L(1 - s, \chi) x^{s+k-1} ds, \tag{3.20}
\]
where \( -k < d < 1 - k \). To evaluate it further, we shift the line of integration to \( 0 < c = \text{Re}(s) < 1 \). Let \( \mathcal{C} \) be a positively oriented rectangular contour formed by the points

\(^2\)It is easy to see that this result is valid for \( \text{Re}(x) > 0 \).
Observe that the integrand has simple poles at \( s = 0, 1, \ldots, 1 - k \) due to \( \sin(\pi s) \). Therefore, residue theorem immediately yields

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left( \int_{c-iT}^{d+iT} - \int_{d+iT}^{d-iT} \right) \frac{(-1)^{k+1} x^{k-1}}{\sin(\pi s)} x^s \, ds = -\sum_{j=0}^{k-1} R_{-j}.
\]

Note that, as \( T \to \infty \) the contribution from integrals along the vertical lines is zero. Hence,

\[
(\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{-\pi L(1-s, \chi)L(s, \chi)}{\sin(\pi s)} x^s \, ds)
= \sum_{j=0}^{k-1} R_{-j},
\]

where,

\[
R_{-j} = (-1)^{1+j-k} x^{-1-j+k} L(1+j, \chi)L(-j+k, \chi).
\]

Finally, from (3.20) and invoking the definition (1.10) we complete our proof. □

**Proof of Theorem 2.8.** The first-equality follows by letting \( k = 0 \) in Theorem 2.7. Now, for the second equality we evaluate

\[
\sum_{n=1}^{\infty} \chi(n) \psi(nx, \chi).
\]

Hence, for \( 0 < c < 1 \), from (3.20) and by employing the reflection formula for the gamma function \( \Gamma(s) \Gamma(1-s) = \pi / \sin(\pi s) \) in the second step, we see that

\[
\sum_{n=1}^{\infty} \chi(n) \psi(nx, \chi) = \sum_{n=1}^{\infty} \chi(n) \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{-\pi L(1-s, \chi)}{\sin(\pi s)} (nx)^{-s} \, ds
= -\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \Gamma(s) \Gamma(1-s) L(1-s, \chi)L(s, \chi)x^{-s} \, ds.
\]

Applying the definition (2.5) in the above equation to obtain

\[
\sum_{n=1}^{\infty} \chi(n) \psi(nx, \chi) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \Gamma(s) \Gamma(1-s) \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1-s+b}{2})} \left( \frac{\pi}{d} \right)^{\frac{1+b}{2}} \frac{\xi(1-s, \chi)\xi(s, \chi)}{\Gamma(\frac{s+b}{2})} \left( \frac{\pi}{q} \right)^{\frac{1+b}{2}} x^{-s} \, ds.
\]

Letting \( c = \frac{1}{2} \), we see that

\[
\sum_{n=1}^{\infty} \chi(n) \psi(nx, \chi) = -\left( \frac{\pi}{d} \right)^{\frac{1+b}{2}} \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(s) \Gamma(1-s)}{\Gamma(\frac{1-s+b}{2}) \Gamma(\frac{s+b}{2})} \xi(1-s, \chi)\xi(s, \chi)x^{-s} \, ds.
\]

Now we perform the change of variable \( s = \frac{1}{2} + it \) and employing (2.4),

\[
\sum_{n=1}^{\infty} \chi(n) \psi(nx, \chi) = -\frac{1}{\sqrt{x}} \left( \frac{\pi}{d} \right)^{\frac{1+b}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\frac{1}{2} + it)}{\Gamma(\frac{1}{4} - \frac{1}{2} + \frac{it}{2})} \frac{\Gamma(\frac{1}{2} - it)}{\Gamma(\frac{1}{4} + \frac{it}{2} + \frac{b}{2})} \Xi(-t, \chi)\Xi(t, \chi)x^{-it} \, dt.
\]

We perform the change of variable by \( t \to t/2 \), and after some considerable simplification, we arrive at

\[
\sum_{n=1}^{\infty} \chi(n) \psi(nx, \chi)
\]
\[ = - \frac{1}{2\pi \sqrt{x}} \left( \frac{\pi}{2} \right)^{b+\frac{1}{2}} \int_0^\infty \frac{\Gamma \left( \frac{1}{2} + \frac{it}{2} \right) \Gamma \left( \frac{1}{2} - \frac{it}{2} \right)}{\Gamma \left( \frac{1}{4} - \frac{it}{4} + \frac{b}{2} \right) \Gamma \left( \frac{1}{4} + \frac{it}{4} + \frac{b}{2} \right)} \Xi \left( -\frac{t}{2}, \chi \right) \Xi \left( \frac{t}{2}, \chi \right) \cos \left( \frac{t \log x}{2} \right) dt. \]

This completes the proof of the theorem. \( \square \)

4. Acknowledgements

The authors would like to show their sincere gratitude to Prof. Atul Dixit for fruitful discussions and suggestions on the manuscript. The first author’s research was supported by the SERB-DST CRG grant CRG/2020/002367 of Prof. Atul Dixit and partly by his institute IIT Gandhinagar. The second author’s research was supported by the grant IBS-R003-D1 of the IBS-CGP, POSTECH, South Korea. Both authors sincerely thank their respective funders for the support.

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables, 9th edition, Dover publications, New York, 1970.
[2] B. C. Berndt, Character analogues of the Poisson and Euler–MacLaurin summation formulas with applications, J. Number Theory 7:4 (1975), 413–445.
[3] B. C. Berndt and A. Dixit, A transformation formula involving the Gamma and Riemann zeta functions in Ramanujan’s Lost Notebook, The legacy of Alladi Ramakrishnan in the mathematical sciences, K. Alladi, J. Klauder, C. R. Rao, Eds, Springer, New York, 2010, pp. 199–210.
[4] E. T. Copson, Theory of Functions of a Complex Variable, Oxford University Press, Oxford, 1935.
[5] A. Dixit, Character analogues of Ramanujan type integrals involving the Riemann \( \Xi \)-function, Pacific J. Math., 255, No. 2 (2012), 317–348.
[6] A. Dixit and R. Gupta, Koshliakov zeta functions I. Modular relations, Adv. Math. 393 (2021), Paper No. 108093.
[7] A. Dixit, R. Gupta and R. Kumar, Extended higher Herglotz functions I. Functional equations, submitted for publication, 2021.
[8] A. Dixit, R. Gupta, R. Kumar and B. Maji, Generalized Lambert series, Raabe’s cosine transform and a two-parameter generalization of Ramanujan’s formula for \( \zeta (2m + 1) \), Nagoya Math. J. 239 (2020), 232–293.
[9] A. Dixit and R. Kumar, Superimposing theta structure on a generalized modular relation, Res. Math. Sci. 8 (2021), no. 3, Paper No. 41, 83 pp.
[10] G. Herglotz, "Uber die Kroneckersche Grenzformel für reelle, quadratische Körper I", Ber. Verhandl. Sächsischen Akad. Wiss. Leipzig 75 (1923), 3–14.
[11] M. Ishibashi, Laurent coefficients of the zeta function of an indefinite quadratic form, Acta Arith. 106 No. 1 (2003), 59–71.
[12] R. Masri, The Herglotz-Zagier function, double zeta functions, and values of L-series, J. Number Theory 106 No. 2(2004), 219–237.
[13] C. Meyer, Die Berechnung der Klassenzahl abelscher Körper über quadratischen Zahlkörpern, Berlin, 1957.
[14] D. Radchenko and D. Zagier, Arithmetic properties of the Herglotz Function, submitted for publication, arXiv:2012.15805, https://arxiv.org/abs/2012.15805.
[15] S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957; second ed., 2012.
[16] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.
[17] E. C. Titchmarsh, The Theory of the Riemann Zeta Function, 2nd ed., Revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.
[18] M. Vlasenko and D. Zagier, Higher Kronecker “limit” formulas for real quadratic fields, J. reine angew. Math. 679, pp. 23–64 (2013).
[19] D. Zagier, A Kronecker limit formula for real quadratic fields, Math. Ann. 213 (1975), 153–184.

Discipline of Mathematics, Indian Institute of Technology, Gandhinagar, Palaj, Gandhinagar 382355, Gujarat, India

Email address: rajat_gupta@iitgn.ac.in
