Semilinear response for the heating rate of cold atoms in vibrating traps

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Abstract – The calculation of the heating rate of cold atoms in vibrating traps requires a theory that goes beyond the Kubo linear response formulation. If a strong “quantum chaos” assumption does not hold, the analysis of transitions shows similarities with a percolation problem in energy space. We show how the texture and the sparsity of the perturbation matrix, as determined by the geometry of the system, dictate the result. An improved sparse random matrix model is introduced: it captures the essential ingredients of the problem and leads to a generalized variable range hopping picture.

The rate of energy absorption by particles that are confined by vibrating walls was of interest in past studies of nuclear friction\,[1–3], where it leads to the damping of the wall motion. More recently, it has become of interest in the context of cold atoms physics. In a series of experiments\,[4–6] with “atom-optics billiards” some surprising predictions\,[7] based on linear response theory (LRT) have been verified.

In this study, we consider the case where the billiard is fully chaotic\textsuperscript{1}, but with nearly integrable shape (fig. 1). We explain that in such circumstances LRT is not applicable (unless the driving is extremely weak such that relaxation dominates). Rather, the analysis that is relevant to the typical experimental conditions should go beyond LRT and involve a “resistor network” picture of transitions in energy space, somewhat similar to a percolation problem. Consequently, we predict that the rate of energy absorption would be suppressed by orders of magnitude and provide some analytical estimates that are supported by a numerical calculation.

We assume that an experimentalist has control over the position ($R$) of a wall element that confines the motion of cold atoms in an optical trap. We consider below the effect of low-frequency, noisy (non-periodic) driving. This means that $R$ is not strictly constant in time, either because of drifts\,[8] that cannot be eliminated in realistic circumstances, or else deliberately as a way to probe the dynamics of the atoms inside the trap [9]. We assume the usual Markovian picture of FGR transitions between energy levels, which is applicable in typical circumstances (see, e.g., [10]). These transitions lead to diffusion in the energy space. If the atomic cloud is characterized by a temperature $T$, then the diffusion in energy would lead to heating with the rate $\dot{E} = D/T$ (see footnote \textsuperscript{2}) and hence to an increase in the temperature of the cloud.

Naively one expects to observe an LRT behavior. That means to have $D \propto [\text{RMS}(\dot{R})]^2$, and more specifically to have a linear relation between the diffusion coefficient and the power spectrum of the driving,

$$D \equiv G \times \text{RMS}(\dot{R})^2 = \int_0^{\infty} \tilde{C}(\omega) \tilde{S}(\omega) \, d\omega,$$

Here $\tilde{S}(\omega)$ is the power spectrum of $\dot{R}$, and $\tilde{C}(\omega)$ is related to the susceptibility of the system. From the experimentalist’s point of view the second equality in eq. (1) can be regarded as providing a practical definition for $\tilde{C}(\omega)$, if the response is indeed linear.

We shall explain in this paper that the applicability of LRT in our problem is very limited, namely LRT would lead to wrong predictions in typical experimental circumstances. Rather we are going to use a more refined theory, which we call semilinear response theory (SLRT) \[11,12\], in order to determine $D$. The theory is called SLRT because on the one hand the power spectrum $\tilde{S}(\omega) \mapsto \lambda \tilde{S}(\omega)$ leads to $D \mapsto \lambda D$, but on the other hand

\textsuperscript{1}Our interest is in systems that are classically chaotic. This means exponential sensitivity to change in initial conditions, without having a mixed phase space.

\textsuperscript{2}For a more general version of $E = D/T$ that does not assume a Boltzmann-like distribution with a well-defined temperature, see sect. 4 of ref. \[3\].
\(\tilde{S}(\omega) = \tilde{S}_1(\omega) + \tilde{S}_2(\omega)\) does not lead to \(D \rightarrow D_1 + D_2\). This semilinearity can be tested in an experiment in order to distinguish it from linear response. Accordingly, in SLRT, the spectral function \(\tilde{G}(\omega)\) of eq. (1) becomes ill defined, while the coefficient \(G\) is still physically meaningful, and can be measured in an actual experiment.

If we assume a small driving amplitude the Hamiltonian matrix can be written as \(H = \{E_n\} + f(t)\{V_{nm}\}\), where

\[
V_{nm} = \left\langle n \left| \frac{dH}{dR} \right| m \right\rangle
\]

is the perturbation matrix. More than 50 years ago, Wigner had proposed to regard the perturbation matrix of a complex system as a random matrix (RMT) whose elements are taken from a Gaussian distribution. Later, Bohigas had conjectured that the same philosophy applies to quantized chaotic systems. For such matrices the validity of LRT can be established on the basis of the FGR picture, and the expression for \(G\) is the Kubo formula

\[
G_{\text{LRT}} = \pi g_E \left\langle \left| V_{nm} \right|^2 \right\rangle,
\]

where the “average” \(\langle x \rangle\) is defined as in refs. [11,12] via a resistor-network calculation [13]. (For mathematical details see “the SLRT calculation” paragraph below.)

Within the RMT framework an element \(x\) of \(\left| V_{nm} \right|^2\) is regarded as a random variable, and the histogram of all \(x\) values is used in order to define an appropriate ensemble. For the sake of later discussion we define, besides the algebraic average \(\langle x \rangle_a\), also the harmonic average as \(\langle x \rangle_h = \left| \langle 1/x \rangle \right|^{-1}\) and the geometric average as \(\langle x \rangle_g = \exp(\ln \langle x \rangle)\). The result of the resistor network calculation is labeled as \(\langle x \rangle\) (without subscript).

Our interest is in the circumstances where the strong “quantum chaos” assumption of Wigner fails. This would be the case if the distribution of \(x\) is wide in the log scale. If \(x\) has (say) a log-normal distribution, then it means that the typical value of \(x\) is much smaller compared with the algebraic average. This means that the perturbation matrix \(V_{nm}\) is effectively sparse (\textit{a lot of vanishingly small elements}). We can characterize the sparsity by the parameter \(q = \langle x \rangle_g / \langle x \rangle_a\). We are going to explain that for typical experimental conditions we might encounter sparse matrices for which \(q \ll 1\). Then the energy spreading process is similar to a percolation in energy space, and the SLRT formula, eq. (3), replaces the Kubo formula.

\footnote{The average is taken over all the elements within the energy window of interest as determined by the preparation temperature. The weight of \(\left| V_{nm} \right|^2\) in this average is determined by the spectral function as \(S(E_n - E_m)\).}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{model_systems}
\caption{Model systems: the atoms are held by a potential that may consist of static walls (solid lines), a vibrating wall (shaded lines), and bumps (thick points). The numerics has been done for (b) with a Gaussian bump. We work with two different aspect ratios. For the aspect ratio \(AS = 20\), we take \(L_x = 200\) and \(L_y = 10\). For the aspect ratio \(AS = 1\), we take \(L_x = 40\) and \(L_y = 40\). The position of the Gaussian bump was randomly chosen within the region \([0.4, 0.6]L_x \times [0.4, 0.6]L_y\). The width of the Gaussian is \(\sigma_x = \sigma_y = \sigma\). We have assumed noisy driving with \(\omega_c = 7\Delta\), where \(\Delta = 1/\rho_E\) is the mean level spacing, and the units were such that \(M = 1\).}
\end{figure}

Outline. – In what follows we present our model system, analyze it within the framework of SLRT, and then introduce an RMT model with log-normal distributed elements, that captures the essential ingredients of the problem. We show that a generalized resistor network analysis for the transitions in energy space leads to a generalized variable range hopping (VRH) picture (the standard VRH picture has been introduced by Mott in [14] and later refined by ref. [15] using the resistor network perspective of ref. [13]). Our RMT-based analytical estimates are verified against numerical calculation. Finally, we discuss the experimental aspect, and in particular define the physical circumstances in which SLRT rather than LRT applies. These two theories give results that can differ by orders of magnitude.

Modeling. – Consider a strictly rectangular billiard whose eigenstates are labeled by \(n = (n_x, n_y)\). The perturbation due to the movement of the “vertical” wall does not couple states that have different mode index \(n_y\). Due to this selection rule the perturbation matrix is sparse. If we deform slightly the potential (fig. 1(a)), or introduce a bump (fig. 1(b)), then states with different mode index are mixed. Consequently, the numerous zero elements become finite but still very tiny in magnitude, which means a very wide size distribution featuring a small fraction of large elements. Similar considerations apply for the circular cavity of fig. 1(c), where an off-center scatterer couples radial and angular motion, and which is more suitable for a real experiment (but less convenient for numerical analysis).

Typically, the perturbation matrix is not only sparse but also textured. This means (see fig. 2) that there are stripes where the matrix elements are larger, and bottlenecks where they are all small. The emergence of texture (\textit{i.e.}, non-random arrangement of the sparse large elements along the diagonals) is most obvious if we
Contrary to the naive expectation the theory does not lead to the Kubo formula. This is because the rate of absorption depends crucially on the possibility to make connected sequences of transitions. It is implied that both the texture and the sparsity of the $|V_{nm}|^2$ matrix play a major role in the calculation of $G$. Consequently, SLRT leads to eq. (3), where $\langle\langle x\rangle\rangle$ is defined using a resistor network calculation. Namely, the energy levels are regarded as the nodes of a resistor network, and the FGR transition rates as the bonds that connect different nodes. Following [12] the inverse resistance of a bond is defined as
\begin{equation}
 g_{nm} \equiv 2\rho E^{-3} \frac{|V_{nm}|^2}{(E_n - E_m)^2} \tilde{F}(E_m - E_n) \tag{5}
\end{equation}
and $\langle\langle|V_{nm}|^2\rangle\rangle$ is defined as the inverse resistivity of the network. It is a simple exercise to verify that if all the matrix elements are the same, say $|V_{nm}|^2 = c$, then $\langle\langle|V_{nm}|^2\rangle\rangle = c$ too. But if the matrix is sparse or textured then typically
\begin{equation}
 \langle\langle|V_{nm}|^2\rangle\rangle_h \ll \langle\langle|V_{nm}|^2\rangle\rangle \ll \langle\langle|V_{nm}|^2\rangle\rangle_a. \tag{6}
\end{equation}
In the case of sparse matrices this is a mathematically strict inequality, and we can use a generalized VRH scheme that we describe below in order to get an estimate for $\langle\langle x\rangle\rangle$. If the element-size distribution of $\log(x)$ is not too stretched, then a reasonable approximation is $\langle\langle x\rangle\rangle \approx \langle\langle x\rangle\rangle_g$, simply because the geometric mean is the typical (median) value for the size of the elements. However, if $|V_{nm}|^2$ has either a very stretched element-size distribution, or if it has texture, then our VRH analysis below show that the geometric average becomes merely an improved lower bound for the actual result.

Analysis. – We consider a particle of mass $M$ in a two-dimensional box of length $L_x$ and width $L_y$, such that $0 < x < L_x$ and $0 < y < L_y$ (see fig. 1(b)). With the driving the length of the box becomes $R = L_x + f(t)$. The Hamiltonian is
\begin{equation}
 \mathcal{H} = \text{diag}\{E_n\} + u\{U_{nm}\} + f(t)\{V_{nm}\}, \tag{7}
\end{equation}
where $n = (n_x, n_y)$ is a composite index that labels the energy levels $E_n$ of a particle in a rectangular box of size $L_x \times L_y$. The deformation is described by a normalized Gaussian potential $U(x, y)$ of width $(\sigma_x, \sigma_y)$ positioned at the central region of the box. Its matrix elements are $U_{nm}$, and it is multiplied in the Hamiltonian by a parameter $u$ which signifies the strength of the deformation. Note that the limit $\sigma \to 0$ is well defined and corresponds to an “s-scatterer”. The perturbation matrix due to the $f(t)$ displacement of the wall is
\begin{equation}
 V_{nm} = -\delta_{n_x, n_y} \frac{x^2}{ML_x^2} n_x m_x. \quad \tag{8}
\end{equation}
The power spectrum of $f$ is assumed to be constant within the frequency range $|\omega| < \omega_c$ and zero otherwise. This means that $\tilde{F}(\omega) = 1$ up to this cutoff frequency.
We have also considered (not presented) an exponential line shape $\tilde{F}(\omega) = \exp(-|\omega/\omega_c|)$, leading to qualitatively similar results. After diagonalization of $\{E_n\} + u\{U_{nm}\}$, the Hamiltonian takes the form

$$\mathcal{H} = \text{diag}\{E_n\} + f(t)\{V_{nm}\},$$

where $n$ (not bold) is a running index that counts the energies in ascending order. The DOS remains essentially the same as for $u = 0$, namely

$$\varrho(E) = \frac{1}{2\pi} ML_x L_y.$$  

The perturbation matrix $|V_{nm}|^2$ is sparse and textured (see fig. 2). First we discuss the sparsity, and the effect of the texture will be addressed later on.

Considering first zero deformation ($u = 0$) it follows from eq. (8) that the non-zero elements of the perturbation matrix are $|V_{nm}|^2 \approx |M_{qE}^2|/L_x^2$, where $c_E = \sqrt{2E}/M$. The algebraic average of the near-diagonal elements equals this value (of the large-size elements) multiplied by their percentage $p_0$. To evaluate $p_0$ let us consider an energy window $dE$. The number of near-diagonal elements $V_{nm}$ within the stripe $|E_{n-mv} - E_{n,mv}| < dx$ is $\varrho(E)dx$. It is a straightforward exercise to find out that the number of non-zero elements (i.e., with $n = m_y$) is the same number multiplied by $p_0 = [2\pi M v E dL_y]^{-1}$. Consequently,

$$\langle(|V_{nm}|^2)\rangle \approx \frac{1}{2\pi M v E dL_y} \left[ \frac{M_{qE}^2}{L_x} \right]^2 = \frac{M_{qE}^2}{2\pi L_y L_x^2}.$$  

Somewhat surprisingly, this result turns out to be the same (disregarding an order unity numerical prefactor) as for a strongly chaotic cavity (see eq. (13) of ref. [3]), as if there is no sparsity issue. This implies that irrespective of the deformation $u$, the LRT Kubo result is identical to the 2D version of the wall formula (see sect. 7 of ref. [3]):

$$G_{\text{LRT}} = \frac{4}{3\pi} \frac{M_{qE}^2}{L_x}.$$  

Our interest below is not in $G_{\text{LRT}}$ but in $G_{\text{SLRT}}$, which can differ by many orders of magnitudes. For sufficiently small $u$ the large-size matrix elements are not affected, and therefore the algebraic average stays the same. But in the SLRT calculation we care about the small-size matrix elements, that are zero if $u = 0$. Due to the first-order mixing of the levels, the typical overlap $|\langle m|n\rangle|$ between perturbed and unperturbed states is $|u U_{nm}/(E_m - E_n)|$. The typical size of a small $V_{nm}$ element is the multiplication of this overlap (evaluated for nearby levels) by the size of a non-zero $V_{nm}$ element. Consequently, the small-size matrix elements are proportional to $u^2$. The geometric average simply equals their typical size, leading to

$$\langle(|V_{nm}|^2) \rangle \approx \langle M_{qE}^2 \rangle^2 e^{-2M_{qE}^2(\sigma_x^2 + \sigma_y^2)} u^2.$$  

Motivated by the discussion below eq. (6) a crude estimate for the SLRT result is $G_{\text{SLRT}} \approx q \times G_{\text{LRT}}$, where for small deformation

$$q = \frac{\langle(|V_{nm}|^2) \rangle}{\langle(|V_{nm}|^2) \rangle} \propto u^2,$$  

see eqs. (11) and (13). It follows from the above (and see fig. 3) that for small deformations $q \ll 1$, and consequently we expect $G_{\text{SLRT}} \ll G_{\text{LRT}}$. This should be contrasted with the case of strongly deformed box for which all the elements are of the same order of magnitude and $q$ becomes of order unity. Our next task is to further improve the SLRT estimate using a proper resistor network calculation.

### RMT modeling

The $|V_{nm}|^2$ matrix looks like a random matrix with some distribution for the size of the elements (see fig. 4). It might also possess some non-trivial texture that we ignore within the RMT framework. The RMT perspective allows us to derive a quantitative theory for $G$ using a generalized VRH estimate. Let us demonstrate the procedure in the case of a homogeneous (neither banded nor textured) random matrix with log-normal distributed elements. The mean and the variance of $\ln(x)$ are trivially related to geometric and the algebraic averages, namely $\langle\ln(x)\rangle = \ln\langle x\rangle$, and $\text{Var}(x) = -2\ln(q)$. Given a hopping range $|E_m - E_n| \lesssim \omega$, we can look for the typical matrix element $x_\omega$ for connected sequences of transitions, which we find by solving the equation $g_E\omega F(x_\omega) \sim 1$, where $F(x)$ is the probability to find a matrix element larger than $x$. This gives

$$x_\omega \approx \langle x \rangle \exp \left[2\sqrt{-\ln q}\right],$$

where $\alpha = \ln(g_E\omega_c)$. From this equation we deduce the following: For $q \ll 1$, meaning that the distribution is not too wide, $x_\omega \approx \langle x \rangle$ as anticipated. But as the matrix gets more sparse ($q \ll 1$), the result deviates from the

\footnote{For a very small $u$, an optional route that bypass the resistor network calculation is to analyze the slow ($\propto u^2$) transitions between noise-broadened energy levels.}
generalized VRH estimate is based on optimization of the integral $\int \omega \tilde{F}(\omega) d\omega$. For the rectangular $\tilde{F}(\omega)$, which has been assumed below eq. (8), this optimization is trivial and gives $\approx \omega c$, leading to

$$G_{\text{SLRT}} = q \exp \left[ 2\sqrt{-\ln q^2} \right] \times G_{\text{LRT}},$$

where $G_{\text{LRT}}$ is given by eq. (12) and $q$ is given by eq. (14). We have also tested the standard VRH that assumes an exponential $\tilde{F}(\omega)$ (not presented).

**Numerical results.** – The analytical estimates in eqs. (11) and (13) are supported by the histograms of fig. 4. For each choice of the parameters $(AS, \sigma, u)$, we calculate the algebraic, geometric and the SLRT resistor network averages of $\langle |V_{nm}|^2 \rangle$ (see figs. 5 and 6). We also compare the actual results for $G_{\text{SLRT}}$ with those that were obtained from a log-normal RMT ensembles with the same algebraic and geometric averages as that of the physical matrix\(^5\). As further discussed in the next paragraph one concludes that the agreement of the physical results with the associated VRH estimate eq. (16) is very good whenever the perturbation matrix is not textured, which is in fact the typical case for non-extreme aspect ratios.

In order to figure out whether the result is fully determined by the distribution of the elements or else texture is important we repeat the calculation for untextured versions of the same matrices. The untextured version of a matrix is obtained by performing a random permutation of its elements along the diagonals. This procedure affects neither the bandprofile nor the $\langle |V_{nm}|^2 \rangle$ distribution, but merely removes the texture. In fig. 5 we see that the physical results cannot be distinguished from the untextured results, and hence are in agreement with the RMT and with the associated VRH estimate. On the other hand, in fig. 6, which is for large aspect ratio, we see that the physical results deviate significantly from the untextured result. As the width of the Gaussian potential becomes larger (smoother deformation), the texture becomes more important. These observation are in complete agreement.
with the expectations that were discussed in the modeling section.

**Experiment.** – As in [4–6], a collection of \( N \sim 10^6 \)
atoms, say \(^{85}\text{Rb} \) atoms (\( M = 1.4 \times 10^{-25} \text{kg} \)), are laser
cooled to low temperature of \( T \sim 10 \mu \text{K} \), such that the
the typical thermal velocity is \( v_T \sim 0.05 \text{m/s} \). The atoms are
trapped in an optical billiard whose blue-detuned light walls confine the atoms by repulsive optical dipole
potential. The motion of the atoms is limited to the billiard plane by a strong perpendicular optical standing
wave. The thickness of the billiard walls (~10 \( \mu \text{m} \)) is much smaller than its linear size (\( L \sim 200 \mu \text{m} \)). The
2D mean level spacing is \( \Delta = \varphi_E^{-1} \sim 2.5 \times 10^{-34} \text{J} \), which is
2.4 Hz. One or more of the billiard walls can be vibrated
with several kilohertz frequency by modulating the laser
intensity. The dimensionless spectral bandwidth of this
driving can be set as say \( \omega_c/\Delta \sim 1000 \), with an amplitude
\( \sim 10 \mu \text{m} \), such that \( \dot{R} \sim 0.015 \text{m/s} \). The temperature
of the trapped atoms can then be measured as a function
of the time by the time-of-flight method. The LRT estimate
\( G_{\text{LRT}} \sim 1.3 \times 10^{-51} \text{J/s}/\text{m}^2 \) would lead to heating rate \( \dot{E} \sim 2 \times 10^{-27} \text{J/s} \), which is \( \sim 0.15 \text{mK/s} \). Considering (say) the
global figure (1(c)), the deformation \( u \) is achieved
by introducing an off-center optical “spot”, or by
deforming slightly the optical walls (such precise control
on the geometry has been demonstrated in previous exper-
iments). Having control over \( u \) we can have \( q \sim 10^{-5} \)
that would imply factor 100 suppression, i.e., an estimated
heating rate of few \( \mu \text{K/s} \). Such heating rate can be accu-
rate measured, yielding high sensitivity to the energy
diffusion process studied here.

**SLRT vs. LRT.** – Typically the environment intro-
duces in the dynamics an incoherent relaxation effect.
If the relaxation rate is strong compared with the rate of
the externally driven transitions, then the issue of
having “connected sequences of transitions” becomes
irrelevant, and the SLRT slowdown of the absorption Is
not expected. In the latter case, LRT rather than SLRT is
applicable. It follows that for finite relaxation rate
there is a crossover from LRT to SLRT behavior as a
function of the intensity of the driving. In cold atom
experiments, the relaxation effect can be controlled, and
typically it is negligible. Hence, SLRT rather than LRT behavior
should be expected. This implies, as discussed above, a much smaller absorption rate. Furthermore, as
discussed at the beginning of this paper, one can verify
experimentally the signature of SLRT: namely, the effect
of adding independent driving sources is expected to be
non-linear with respect to their spectral content.

**Conclusions.** – In this work, we have introduced a
tool for the calculation of the heating rate of cold atoms in
vibrating traps. This theory, which treats the diffusion in
energy space as a resistor network problem, is required
if the cavity is not strongly chaotic and if the relaxation
effect is small. The SLRT result, unlike the LRT (Kubo)
result, is extremely sensitive to the sparsity and the
textures that characterize the perturbation matrix of the
driving source. For typical geometries the ratio between
them is determined by the sparsity parameter \( q \) as in
eq. (16), and hence is roughly proportional to the deformation
\( (u^2) \) of the confining potential. If the cavity has a large
aspect ratio, and the deformation of the confining poten-
tial is smooth, then the emerging textures in the perturba-
tion matrix of the driving source become important, and
then the actual SLRT result becomes even smaller.

By controlling the density of the trapped atoms, or their
collisional cross-section (e.g., via the Feshback resonance),
the atomic collision rate can be tuned by many orders
of magnitude. Their effect on the dynamics can thus be
made either negligible (as assumed above) or significant,
thereby serving as an alternative (but formally similar)
mechanism for weak breakdown of integrability. It follows
that heating rate experiments can be used not only to
probe the deformation \( u \) of the confining potential, but
also to probe the interactions between the atoms.

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