Representations of relativity, quantum gravity and cosmology

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We review an attempt to set a suitable foundational principle for consistent quantization of gravity, based on the canonical formulation. It requires extending the spacetime description of the relativistic postulates to also encompass an alternative formulation in momentum-energy continuum where the inertial physical laws can be equivalently described. The extension to noninertial frames breaks such an equivalence, leaving a new dynamical field which, together with gravity, allows to construct a canonical scenario where the Dirac’s quantization method leads to consistent definitions of hermitian ordering for the operators of the canonical quantum theory.

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I. INTRODUCTION

It is a well-known fact that relativity relegates space and time to the subjective role of the elements of a language that any observer may use to describe the laws that govern the behaviour of the objective reality when this is interpreted as a set of matter bits. Actually, it is the relation between the only two kinds of the objetivized entities allowed by the theory -namely the observer and the bits of matter, that one must take as the fundamental building blocks of relativity, and it makes no difference to the predictions of the theory which part of the objective physical system is identified with the observer and which part is called an observed bit of matter.

Objectivizing bits of matter is nevertheless just a particular way to analyse physical reality along the task of dividing it into its minutest pieces. There still exists another approach to look at the physical reality that manner. It consists of objectivizing bits of space and time rather than matter, and taking them as the fundamental constituent parts of reality. Adopting such an approach would be allowed from two fundamental developments in gravitational physics. One of them is the realization that there exist very small, but still nonzero fundamental length and time, possibly at the Planck scale, which determine the finite maximum resolution for all experiments. The other development is relativistic cosmology itself. Here, physical reality can be looked at as being described by the set of relations between distances and times that characterize the large scale structure of the universe. In its canonical formulation, moreover, the dynamical content of cosmology is given by the Hamiltonian constraint which, being zero, is prepared to be treated both as an energy-momentum object, or as a space-time object, depending on whether we divide or multiply by the Planck-length squared. In the latter case, it would be momentum and energy which should be relegated to the subjective role of the elements of a mere language that the observers would use to describe a physical reality made up of objectivized bits of space and time, with the building blocks of the theory now being the relation between observers and such objectivized bits. One would expect that it again makes no difference to the predictions of the theory in this representation which part of the physical system is identified with the observer and which part is taken to play the role of an observed bit of space or time.

Clearly, the kinematics of Einstein relativity for inertial systems gives, through Lorentz transformations, the relativistic changes of time durations and space distances that an observer may measure with clocks and meters. It appears then that for momentum and energy to be relegated to elements of a subjective language, they should enter a relativistic formalism formally identical to that for spacetime relativity and reproduce the same transformations for momentum and energy, and space and time as well, though the latter two quantities of such a formalism should be taken as objective elements of the physical reality.

On the other hand, a traditional debate about wave mechanics refers to whether it requires relativity theory to be consistently formulated [1]. Of course, de Broglie derived his known relation based on a relativistic foundation [2]. In order to explain x-ray diffraction in crystals by means of the corpuscular theory of light though, Duane postulated [3] the momentum rule before de Broglie without relativistic foundation. But, as formulated by Duane, the rule becomes obscure without de Broglie’s idea of the correspondence of particle and wave, and without his relativistic proof that the group velocity of a wave packet coincides with the velocity of the corresponding particle [4]. Thus, if we adhere to the currently most accepted view that wave mechanics is only derivable from relativistic concepts and has an in principle well-defined nonrelativistic limit, then one could formulate a wave mechanics which would follow from the alternate relativistic approach relegating momentum and energy to the role of subjective quantities. Since the above two relativistic representations should be expected to be
II. REPRESENTATIONS OF RELATIVITY

There is one sense in which quantum-mechanical position and momentum representations are not formally equivalent if wave-particle duality is, as usual, meant to imply equal contributions from the two pictures (wave and particle) to that duality. When one presupposes a system to be an elementary particle with mass \( m \), the particle is being assumed to be point-like and its wave function in \( p \)-representation \( \Psi(p) \) can also be written as a function of the wavelength, namely \( \Psi(\lambda) \), by using the de Broglie relation \( p = \frac{\hbar}{\lambda} \), which also holds in the non-relativistic limit. Therefore, \( \Psi(p) = \Psi(\lambda) \) can always be interpreted as the probability amplitude for the presupposed particle-like system to behave as a wave with wavelength \( \lambda \). At the same time, in \( x \)-representation, \( \Psi(x) \) is regarded as the probability amplitude for the presupposed point-like particle to be localized in space at \( x \). However, if one would presuppose the system to be a wave with wavelength \( \lambda \), whereas \( \Psi(p) \) could equivalently be regarded as the probability amplitude for the wave to propagate with a momentum “localized” at the single value \( p \), there is no known fundamental quantum relation allowing the spatial distance \( x = R \) in \( \Psi(x) \) (with \( R \) being the objectivized bit of spatial distance characterizing the system in wave representation) to be discretized so that this wave function can be re-written as a function of a corresponding particle property (which we take to be the mass), namely \( \Psi(m) \), interpretable as the probability amplitude for the wave-like system to behave like a particle with mass \( m \).

In the relativistic formalism, \( x \) could still be discretized in terms of a relativistic Compton wavelength of the system, \( R \equiv \lambda_c = \frac{h}{mc} \), i.e. in terms of the spatial scale at which the system undergoes purely relativistic interactions with effects such as the fine-structure originating from its spin. However, this relation would be lost in the limit \( c \to \infty \), where \( R \to 0 \), and in any case, cannot be considered as a fundamental quantum relation that could be regarded to be at the same footing as the de Broglie formula, in this case relating a measurable bit of objectivized space distance to mass. Moreover, even at the relativistic level, there exists no known quantum relation whatsoever which would link a discretized bit of objectivized time, \( T \) (characterizing the system), to the mass of that system, leading to a transformation \( \Psi(T) \to \Psi(m) \), analogous to as the Einstein-de Broglie relation \( E = h\nu \) does with energy and frequency to allow the transformation \( \Psi(E) \to \Psi(\nu) \).

Although, given the mass of the electron, it is our choice whether to measure its position or momentum, and this is still enough to describe objective reality in the inertial approximation, the alluded inequivalence appears to be detrimental to the beauty of the underlying theory and leads, in fact, to the known difficulties encountered in any attempt to quantize gravity (see Subsec. II-D and Secs. III and IV). The electron has a mass, but e.g. in experiments where its interaction with the Coulomb field of the hydrogen nucleus is measured, it also shows another element of its objective reality which, like mass, only depends on relative velocity (through the relativistic factor): the spatial domain given by the Compton wavelength where the Darwin interaction takes place, or equivalently, the time interval that a train a light waves would take in traversing that spatial domain.

The lack of a fundamental quantum relation between \( R \) and \( m \) and between \( T \) and \( m \) leading to the above for-
mal inequivalence appears to be related to the fact that wave mechanics originated from a relativistic mechanics where one just objetivizes bits of matter relative to an observer, but leaves spacetime to always play the role of coordinates labeling events that occur through the emergence in spacetime of such bits of matter. It is the author’s contention that, relative to an observer, one would also need a relativistic theory of momentum-energy itself in order to objetivize bits of the spacetime -i.e. bits of space distances and time durations, and hence derive the missing relations between \( R \) and \( m \) and between \( T \) and \( m \), following steps parallel to de Broglie’s. On the other hand, a priori presupposing that a microscopic system is a particle or a wave would require some appropriate physical conditions to be satisfied by the system.

In Einstein relativity space and time are relegated to play the subjective role of elements of a language that is used by the observer to describe his environment, and it is the relation of the bit of matter (with its own spacetime trajectory) with the observer what makes the objective reality out of which the world is constructed [5]. What would be new in a relativistic formalism described in terms of a momentum-energy continuum is the explicit renouncement to presuppose the Einsteinian objective relation between the observer and bits of matter as a necessarily established and unique element of the possible physical reality. Instead, we take all three notions, space, time and matter -when considered independent of the observer- as a priori being merely the elements of a subjective language. The observer can then get related to either bits of matter or bits of space and time by some introspective process that leads to either a distinct, purely theoretical world picture, or to the design of related experiments and observations, so that, depending on the very nature of the system and the predisposition of the observer toward it, either the bits of matter or the bits of space and time become objetivized relative to the observer, while space-time or momentum-energy remains respectively relegated to play the subjective role of coordinates.

On the other hand, in order to presuppose “nothing” about the system an abstract relativistic formalism should be established in which the coordinate labeling events do not imply any objetivization either of matter or of spacetime. Consistently imposing then the appropriate physical conditions on this formalism would finally result in usual spacetime relativity or the alternate description in terms of momentum-energy relativity for objetivized bits of, respectively, matter or space and time. Quantities that one may take to play the role of the coordinates labeling events in the generalized, abstract formalism are the components of some unobjetivized action \( q^\alpha \), \( \alpha = 0, 1, ..., 3 \). Note that one can make these coordinates simple dimensionless numbers by using the Planck constant, thus showing the abstract character of them. The usual line element of Einstein relativity would then generalize to an action element

\[
ds^{(q)} = \left[ (dq^0)^2 - \sum_{i=1}^{3} (dq^i)^2 \right]^{1\over 2}. \tag{2.1}\]

An inertial reference system for action coordinates \( q^\alpha \) will then be an orthonormal frame, \( q^0, q^1, q^2, q^3 \), characterized by a constant value of the dimensionless quantity \( {dq^0 \over dq^3} \). We assume (2.1) to be relativistically invariant in any of such action reference frames. Note however that since they do not correspond to visualizable objetivized elements of the physical reality, the values of these intervals cannot be measured by any experimental devices. This abstract action interval should follow an action line of the universe which at every point has a tangent whose direction in action space is defined by a vector with unit length given by

\[
u^{\alpha}_{(q)} = {dq^\alpha \over ds^{(q)}}, \tag{2.2}\]

with \( u^{\alpha}_{(q)} u^{\alpha}_{(q)} = 1 \).

We regard the appropriate physical conditions that allow an abstract wave-particle entity to be objetivized so that it contains a bit of either space and time (wave picture) or matter (particle picture) as being described by a mapping of the action coordinates onto coordinates of, respectively, 4-momentum, \( dq^\alpha \rightarrow T_0 c dp^\alpha \), and 4-position, \( dq^\alpha \rightarrow m_0 c dx^\alpha \), where \( T_0 \) and \( m_0 \) are objetivized bits of time and matter, and \( c \) is the velocity of light. In the first case, we allow the system to accommodate null rays (null geodesics) along which repetitive, reliable measurements of its “objetive” spacetime characteristics are enabled, while the resulting unobjetivized 4-momentum components \( dp^\alpha \) are kept as coordinates that label events with the above objetivized spacetime characteristics. In the second case, the mapping allows the system to evolve along lines with constant values of \( {dq^0 \over dq^3} \) (which we call null cosmodesics) and this permits repetitive, reliable measurements of “objetive” particle-like characteristics of the system, while the resulting spacetime components \( dx^\alpha \) are kept as coordinates that are used to label events with the above objetivized particle-like characteristics.

The allowance of null cosmodesics to probe the evolution of the system makes then the action line element (2.1) and the action velocity vector (2.2) to transform as

\[
ds^{(q)} \rightarrow m_0 c ds^{(x)} \tag{2.3}\]

\[
u^{\alpha}_{(q)} \rightarrow {dx^\alpha \over ds^{(x)}} = u^{(x)\alpha}, \tag{2.4}\]

where \( ds^{(x)} \) is the usual line element of spacetime Einstein relativity and \( u^{(x)\alpha} \) the corresponding velocity of the universe. If we allow the system to accommodate null geodesics in order to probe its evolution, then it is instead obtained

\[
ds^{(q)} \rightarrow T_0 c ds^{(p)} \tag{2.5}\]
with $ds^{(p)}$ the line element in momentum-energy coordinates and $u^{(p)\alpha}$ the velocity of the universe defined on them. The invariance of the interval $ds^{(p)}$ would give rise to a formulation of relativity which is formally equivalent to that of Einstein spacetime relativity for inertial frames.

A. Special relativity in momentum-energy

In what follows I will formulate a momentum-energy representation for relativity. In order for the resulting theory to be self-consistent, such a formulation should satisfy the following requirements.

(i) The kinematics of special relativity (i.e. the relations between coordinate labels) in the momentum-energy representation must satisfy all mechanical Einstein four-momentum transformations, and its associated mechanics (i.e. the quantities derived from an action principle) must in turn obey the usual Lorentz transformations.

(ii) Whereas description of a given system in space-time implies that such a system occupies just a space-time part (often just a point) from a necessarily larger system where at least an external observer is also included, its description in momentum-energy continuum requires considering the system and the observer as located at distinct particular values of momentum and energy intervals on the same frame, so that no evolution of a system independent of the observer is possible.

(iii) The nonrelativistic limit $c \to \infty$ of the resulting mechanical relations between time durations and space distances should produce either known or rather trivial results, or not exist at all. The nonrelativistic limit of the kinematical transformations of momentum and energy must predict values of the energy which depend on the chosen reference system, and values of the momentum such that this behaved as an absolute quantity. The latter requirement needs some further explanation. Consider a system $S$ which evolves uniformly (i.e. at a constant rate $\frac{dp}{ds}$) in the vacuum momentum-energy continuum. Since, after requirement (i), its evolution rate is $\frac{dp}{ds} = \frac{\gamma}{c^2}$, we can see why the components of momentum must become absolute quantities in the nonrelativistic limit, where energy will still depend on the bare velocity $v$. In such a limit, one would not expect the system $S$ with energy $e_1$ to interact with itself with a different energy $e_2$ because, then, the maximum rate of signal propagation in momentum-energy, $\frac{1}{c^2}$, becomes zero.

Passing to the domain where $c$ is finite, we see that the maximum rate of signal propagation in momentum-energy is no longer zero and, therefore, the momentum components become no longer absolute quantities. This will give rise to the emergence of a purely relativistic interaction of the system $S$ with itself when it evolves along different values of the energy. We can then introduce momentum-energy reference systems evolving uniformly relative to each other with relative constant rates $\frac{\Delta p}{\Delta c}$, so as an extended principle of relativity according to which all the laws of nature are identical in all “inertial” momentum-energy reference systems, if the equations expressing the laws and the events that take place in such reference systems are all described in terms of momenta and energies. Such laws must then be invariant with respect to transformations of momenta and energies from one momentum-energy reference system to another.

A differential interval defined in one of such reference systems can be given by

$$ds^{(p)} = \frac{de^2}{c^2} - dp_x^2 - dp_y^2 - dp_z^2.$$  \hspace{1cm} (2.7)

The principle of relativity for momentum-energy continuum implies that $ds^{(p)}$ will be the same in all inertial momentum-energy systems, and leads to the definition of a proper energy given by

$$de = \frac{de'}{\gamma}, \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}.$$  \hspace{1cm} (2.8)

Let us consider two inertial momentum-energy reference systems independently evolving with a relative rate $\frac{\Delta x}{c}$. From the above discussion it follows that if the energy origin is chosen at the point where both systems coincide, and such systems evolve so that their $p_x$-axes always coincide, then we will have in the limit $c \to \infty$

$$p_x = p'_x, \quad p_y = p'_y, \quad p_z = p'_z, \quad e = e' + p_xv.$$  \hspace{1cm} (2.9)

On the other hand, if $c$ is kept finite, it is easy to see that the transformations that leave invariant the interval are

$$p_x = \frac{p'_x + \frac{v}{\gamma}e'}{\gamma}, \quad p_y = p'_y,$$

$$p_z = p'_z, \quad e = \frac{p'_zv + e'}{\gamma},$$  \hspace{1cm} (2.10)

which, in turn, coincide with the transformation formulas for momentum-energy 4-vector of Einstein relativistic mechanics. Equations (2.10) lead to expressions for the transformations of velocities, general 4-vectors, and unit 4-velocities, which exactly coincide with those of Einstein relativistic kinematics, and reduce to (2.9) as $c \to \infty$. Thus, the transformations (2.10) do satisfy the kinematical parts of the requirements in (i) and (iii).

In order to formulate the relativistic mechanics in momentum-energy representation, let us consider a free system evolving in the momentum-energy continuum. For such a system there should exist a certain integral (the counterpart to action of Einstein relativity in momentum-energy continuum) which has the minimum value for actual evolution of the system in the
momentum-energy continuum. This integral must have the form
\[
P = -\beta \int_a^b ds^{(p)} = -\frac{\beta}{c} \int_{e_1}^{e_2} d\gamma = \int_{e_1}^{e_2} \tilde{L} de, \quad (2.11)
\]
where \( f^b_a \) is an integral along a momentum-energy world line of the system between two particular events characterizing the momentum of the system when it has energies \( e_1 \) and \( e_2 \), and \( \beta \) is some constant that characterizes the system. The coefficient \( \tilde{L} \) of \( de \) plays the role of a Lagrangian and has the physical dimensions of a time. For \( P \) to have the dimensions of an action, unlike Einstein relativity where each system is characterized by its rest energy \( mc^2 \), here each system should be characterized by the complementary quantity to its rest energy, that is its rest time \( T_0 \). We take therefore \( \beta = eT_0 \), and hence the integral \( P \) for a free temporal system becomes
\[
P = -T_0 \int_{e_1}^{e_2} d\gamma, \quad (2.12)
\]
with \( \tilde{L} = -T_0 \gamma \).

Instead of a momentum and an energy, the mechanical system will now be described by a space distance \( R \) and a time duration \( T \). Assuming the momentum-energy coordinate space to be homogeneous, so that the properties of the system remain invariant under infinitesimal parallel displacements of rate \( \frac{dx}{dT} \) and energy \( e \), the quantities \( R \) and \( T \) would be conserved and can be obtained using the same Lagrangian principles as in classical mechanics, but in our complementary representation, i.e.
\[
R = \frac{\partial \tilde{L}}{\partial (\frac{\partial x}{\partial T})} = \frac{T_0 v}{c}, \quad T = \frac{R v}{c^2} - \tilde{L} = \frac{T_0}{\gamma}. \quad (2.13)
\]

We have to check that the relativistic mechanics expressed by (2.11)-(2.13) is consistent with the full relativistic picture, i.e. we have to check that by substituting space distance and time duration given in (2.13), expressed as a 4-vector in terms of the corresponding 4-velocity, in the transformation formulas for a general 4-vector, one obtains usual Lorentz transformations. That this is indeed the case can be readily seen by using the principle of least action, \( \delta P = 0 \), and \( ds^{(p)} = (dp_\alpha dp^\alpha)^* \), with \( p^0 = \frac{\tilde{L}}{\gamma}, \quad p^1 = p_x, \quad p^2 = p_y, \quad p^3 = p_z \). We then obtain
\[
\delta P = -T_0 u^{(p)}_\alpha \delta p^\alpha, \quad \text{where} \quad u^{(p)}_\alpha = \frac{dp^\alpha}{ds^{(p)}} = u^{(x)}_\alpha = u_\alpha, \quad u^{(x)}_\alpha \quad \text{being the Einstein unit 4-velocity (see the next subsection). It follows that}
\]
\[
x_\alpha = -\frac{\partial P}{\partial p^\alpha} = (cT, R) = T_0 u_\alpha \quad (2.14)
\]
is the distance 4-vector. It turns out that the square of the length of momentum 4-vector, \( \beta^2 - \sum_{i=1}^{3} (p^i)^2 \), is invariant under transformations (2.10). Generalizing to any 4-vector \( A^\alpha \) which transforms like the components of the momentum 4-vector under (2.10), we recover the usual transformation formulas for 4-vectors of Einstein relativity. It is now immediately seen that by substituting (2.14) into such formulas, one obtains usual Lorentz transformations. This completes fulfilment of requirement (i).

We also note that the formula for \( T \) in (2.13) has no nonrelativistic counterpart. In fact, in the limit \( c \to \infty \), we obtain from (2.13)
\[
T = T_0 + \frac{T_0 v^2}{2c^2} \approx T_0, \quad R \approx T_0 v, \quad (2.15)
\]
i.e. the nonrelativistic limit of \( T \) and \( R \) reduces, respectively, to the rest time and a distance-velocity law which may be trivially interpreted as the customary definition of velocity.

On the other hand, it also follows from (2.13)
\[
R = Tv \quad (2.16)
\]
Expression (2.16) should now correspond to the relativistic expression for the definition of velocity of the object. We finally note that (2.17) must correspond to the analogue of the usual relativistic Hamiltonian in our complementary momentum-energy formalism for relativity. If we express time \( T \) in terms of the distance \( R \), then we have a complementary relativistic "Hamiltonian"
\[
T = H_T = \frac{1}{c} (R^2 + T_0^2 c^2)^{\frac{1}{2}}, \quad (2.18)
\]
which has the physical dimension of a time. Law (2.18) must correspond to the Minkowskian function \( F \) which is the conjugate counterpart to Hamiltonian and whose existence has been recently suggested [6]. It describes the way in which objectivized bits of space distance, \( R \), and time interval, \( T \), are related to each other.

We still have to check that our mechanical relation (2.17) satisfies requirement (iii). Unlike the conventional Hamiltonian of Einstein relativity, which in the limit \( c \to \infty \) produces the known nonrelativistic Hamiltonian \( \frac{p^2}{2m} \) plus the rest energy, the relation (2.18) gives only the rest time \( T_0 \) in that limit where, therefore, it induces no mechanical effects. Of course, for high-velocity experiments one would expect time \( T \) to increase with velocity \( v \) and \( T_0 \), such as it is also predicted by Einstein relativity and verified many times in laboratory experiments. When suitably generalized to noninertial frames so that it becomes applicable to the whole universe, this law will describe the cosmological evolution in the vacuum momentum-energy continuum (see Sec. IIIIB).

### B. The R-m and T-m relations

We note now that the velocities of the universe \( u^{(x)}_\alpha \), \( u^{(p)}_\alpha \) and \( u^{(q)}_\alpha \) are all the same; i.e.
\[ u_\alpha^{(x)} = u_\alpha^{(p)} = u_\alpha^{(q)} = u_\alpha = \frac{\nu_\alpha}{c\gamma} \] (2.19)

This invariance would be a particular example of an invariance notion which refers to quantities that preserve their values in all the above three types of coordinate systems. Of course, all dimensionless quantities that can be formed in the theory should respect this kind of invariance which we hereafter refer to as representation invariance. Thus, the de Broglie theorem of phase harmony [2] can be regarded to be a consequence from this invariance. We can in fact visualize any microscopic entity as evolving along lines of the universe on three distinct sheets. Evolution on the action sheet would describe an unobjetivized wave-particle entity propagating with rate \( \frac{da}{dq} \) and having a pure action phase \( \varphi^q \). On the action line of the universe the entity would carry no definite observable energy or characteristic time. The action sheet can be unfolded by the above-mentioned mappings into the usual spacetime sheet and a momentum-energy sheet, each with the corresponding line of the universe projected on it. Along the spatial line of the universe on the spacetime sheet, the entity would manifest as a bit of energy propagating on that sheet with given velocity \( v \), and along the momentum line of the universe on the momentum-energy sheet, it manifested like a bit of time “propagating” with corresponding rate \( \frac{dv}{dx} = \varphi^x \) in momentum-energy, or like the phase wave with phase \( \varphi^{(x)} \) and velocity \( \frac{c^2}{\varphi^{(x)}} \), relative to the spacetime sheet. Likewise, projected on the spacetime sheet, the entity would manifest like the phase wave with phase \( \varphi^{(p)} \) and propagation rate \( \frac{1}{\nu_\alpha} \), relative to the momentum-energy sheet. Since the phase is dimensionless, we must then have

\[ \varphi^{(q)} = \varphi^{(x)} = \varphi^{(p)} = \varphi. \] (2.20)

These equalities would in fact represent a generalization of the de Broglie theorem of phase harmony.

Let us consider any two points \( P \) and \( Q \) along the action line of the universe on the action sheet. We can then form the integral

\[ -\int_P^Q ds^{(q)} = -\int_P^Q u_\alpha^{(q)} dq^\alpha, \] (2.21)

which should have a stationary value. It is then possible to introduce a general vector of the universe

\[ J_\alpha = u_\alpha, \] (2.22)

and a principle of least action such that

\[ \delta \int_P^Q J_\alpha dq^\alpha = m_0 c \delta \int_P^{P'} J_\alpha dx^\alpha = T_0 c \delta \int_P^{P''} J_\alpha dp^\alpha = 0, \] (2.23)

where \( P' = \frac{P}{T_0 c} \), \( P'' = \frac{P}{m_0 c} \), and similarly for \( Q' \) and \( Q'' \).

On the other hand, rays of the universe will be determined by the Fermat principle [2], i.e.

\[ \delta \int_P^Q d\varphi = 0, \] (2.24)

for the representation-invariant phase \( d\varphi \)

\[ d\varphi = 2\pi O_\alpha^{(q)} dq^\alpha = 2\pi O_\alpha^{(x)} dx^\alpha = 2\pi O_\alpha^{(p)} dp^\alpha, \] (2.25)

where the \( O_\alpha \)'s are the wave vector of the universe [2] on the respective representation.

If null cosmodesics are allowed to occur and be used as probes to follow the evolution of the system, then

\[ O_\alpha^{(q)} = \frac{O_\alpha^{(x)}}{m_0 c}, \] (2.26)

and if, alternatively, null geodesics are permitted to probe the evolution of the system, we obtain

\[ O_\alpha^{(p)} = \frac{O_\alpha^{(q)}}{T_0 c}. \] (2.27)

The de Broglie’s extension of the quantum relation [2] generalizes then to read

\[ hO_\alpha^{(q)} = u_\alpha, \] (2.28)

with \( h \) the Planck constant. Thus, whereas (2.28) yields the known Einstein-de Broglie relations between momentum and energy and, respectively, wavelength and frequency whenever null cosmodesics are allowed to occur and be used to follow the evolution of the system, as far as null geodesics are used to do that, (2.28) gives rise to the new fundamental quantum relations

\[ \mu = mc = \frac{h}{R}, \quad T = h\Omega, \] (2.29)

where Eqn. (2.14) has been used, and \( \Omega \) is the energy frequency in momentum-energy continuum.

The first of relations (2.29) provides us with the wanted relation between a discretized \( R \) and mass \( m \). It allows the interpretation of the wave function \( \Psi(x) \) in \( x \)-representation as the probability amplitude for a microscopic system to be a particle with mass \( m \) when one presupposes the system to be a wave. One could say that a particle is not but just a wave propagating in momentum-energy continuum with characteristic “wavelength” \( \mu \). The relation \( R \mu = h \) promotes the definition of the relativistic Compton wavelength to the same fundamental status as that played by the de Broglie relation \( p\lambda = h \). This new fundamental relation already has been therefore tested in all those atomic-physics experiments aiming at e.g. measuring the relativistic interaction between the electron and the Coulomb field produced by the hydrogen nucleus, corresponding to the fine-structure Darwin term. In particular, the first of equations (2.29) would
predict that an electron undergoing Darwin interaction would be sensible to the ensemble of values taken by the Coulomb field within a spatial domain which would decrease as the electron is excited to upper energy levels.

Equivalently, the fourth-component relation \( T = \hbar \Omega \) provides us with the missing relation between time and a particle-like property, and discretizes an objectivized time \( T \) for the system which corresponds to the time scale that light waves (null geodesics) would last in traversing the spatial domain \( R \). According to it, the objectivized time appears to be quantized in discrete portions, each carrying the total energy of the system. This entails no violation of energy conservation as the time portions are independent of each other. Both relations in (2.29) have no counterpart in the nonrelativistic limit \( c \to \infty \).

C. Wave mechanics in momentum-energy

A quantum-mechanical wave equation can also be derived from (2.18) by introducing the operators \( \hat{T} = i\hbar \frac{\partial}{\partial e} \) and \( \hat{R} = i\hbar \frac{\partial}{\partial p} \). Using a wave function \( \Psi \equiv \Psi(p,e) \), we obtain

\[
-\hbar^2 \frac{\partial^2 \Psi}{\partial e^2} = \frac{1}{c^2} \left( -\hbar^2 \frac{\partial^2 \Psi}{\partial p^2} + T_0^2 c^2 + V(p) \right) \Psi, \tag{2.30}
\]

where we have introduced a generic potential \( V(p) \). This is the counterpart in momentum-energy to the Klein-Gordon equation. If, as it is the case for the whole universe, the system is closed, then one would expect a discrete \( T \)-spectrum which would associate with an infinite set of universes "frozen" at the given eigenvalues of \( T \). This spectrum would only tend to become continuous in the classical region that corresponded to very large values of \( T \). We finally note that the quantum description of systems that show time asymmetry could only be accounted for whenever we assume a half-integer intrinsic angular momentum for the whole system, so that, instead of (2.30), one would have a Dirac-like wave equation

\[
\left( \gamma^\alpha \frac{\partial}{\partial p^\alpha} + cT_0 + V(p) \right) \Psi(p) = 0, \tag{2.31}
\]

with \( \gamma^\alpha \) the \( 4 \times 4 \) Dirac matrices, which is invariant under \( e \to -e \), but not under \( T \to -T \). Indeed, just as for antimatter in momentum representation, the negative time states could not be physically ignored, since there is nothing to prevent a system from making a transition from a state of positive time to a state of negative time. Equivalence between the two relativistic quantum-mechanical representations manifests here in the sense that states with negative time in momentum representation should be equivalent to states with negative energy in position representation as far as an antiparticle moving forward in time is equivalent to the corresponding particle moving backward in time.

Actually, in Einstein relativity the Minkowskian coordinates \( x^0 = t \) and \( x^i \) have a double function: they serve as labels for the events but at the same time they also inform us through the Lorentz transformations about actual time durations and space distances, measurable with clocks and meters. Moreover, although in Einstein relativity momentum components and energy can never be taken to label real events, they can be nevertheless obtained as actual quantities from the associated relativistic mechanics where mass is introduced as an objectivized bit of matter. Likewise, in the momentum-energy representation of special relativity one would expect the coordinates energy \( p^0 = e \) and momentum \( p^i \) to have also a double function: serving as labels of events characterized by objectivized bits of spatial sizes and time durations, and informing us about the actual values of the energy and momentum of the system. Such values should be the same as those predicted by Einstein relativistic mechanics. In momentum-energy representation of relativity, one would also obtain the same transformation formulas for time durations and space distances as in spacetime relativity, though in this case these quantities are given as mechanic rather than kinematic quantities.

Since the Klein-Gordon relativistic wave equation gives the eigenenergies of the system in terms of mass eigenvalues, \( e_n = m_n c^2 \), and time periods are related to the corresponding wavelengths by an explicit relation, one should expect the quantum theory derived from relativity in momentum-energy coordinates to be formulated in terms of wave functions which admits a completely equivalent interpretation to that of the wave functions of the quantum theory derived from Einstein relativity, when both are applied to inertial systems. Therefore, the non-gravitational quantum theory formulated above must be completely equivalent to that derived from Einstein special relativity.

D. The cosmological field

The conclusion obtained in the precedent subsection is no longer valid for noninertial frames. In general relativity we must distinguish between coordinate labels from proper intervals, entering at the two totally different levels that correspond, respectively, to differential topology and metric geometry. In spacetime general relativity showings of a physical clock are predicted not only by the labels that distinguish events, but also by the metric, and the change of the metric respect to spacetime coordinates describes at the same time a dynamical quantity: the gravitational field. In general theory of relativity formulated in terms of momentum-energy coordinates, besides mechanical time durations and space distances, one would likewise expect the emergence of a new quantity: the metric of the momentum-energy continuum, \( f_{\alpha\beta} \equiv f_{\alpha\beta}(p^i) \), which would help, together with the \( p^i \)-coordinate labels, to construct actual momentum and energy intervals.

Although as far as it describes the geometry of the
momentum-energy continuum, the dimensionless metric tensor \( f_{\alpha\beta} \) would be the same as the usual tensor \( g_{\alpha\beta} \) by representation invariance, its variations with respect to momentum-energy coordinates should, at the same time, describe an independent quantity with “dynamical” content by itself; i.e.: a new field which would generally differ from gravity and only coincides with this under particular, limiting conditions. These two fields would in general induce different behaviours in systems acted upon by them.

It is in this sense that curved spacetime and curved momentum-energy are not equivalent representations of a unique general-relativity theory. I will give now some arguments in support of the interpretation that the variations of the momentum-energy metric \( f_{\alpha\beta}(p') \) with respect to the momentum-energy coordinates must describe cosmological interactions.

(i) Because the dimensionless quantities \( f_{\alpha\beta} \), as regarded as the components of a metrical tensor, are representation invariant, we should have \( f_{\alpha\beta} = g_{\alpha\beta} \) and therefore, if \( f_{\alpha\beta} \) is taken to describe a cosmological field, appropriate solutions of the usual Einstein equations satisfying Weyl and cosmological principles are also cosmological solutions.

(ii) For the reasons discussed in the precedent subsections, one would not expect a cosmological field to have nonrelativistic counterpart. This must actually be the case for the interactions described by the field equations derived from \( f_{\alpha\beta} \). Nevertheless, one can consider the limit of very small but yet nonzero values of \( \frac{v}{c} \), where a very weak but still nonzero cosmological field with potential \( \rho \) is present. Assuming this field to be described from the metric \( f_{\alpha\beta} \), \( \rho \equiv \rho(p') \), the time-Lagrangian in momentum space could be written as

\[
\tilde{L} = -T + \frac{T v^2}{2c^2} - T c^2 \rho,
\]

(2.32)

where, similarly to as the nonrelativistic gravitational potential goes like \( \left( \frac{dp}{dt} \right)^2 \), i.e. like a squared velocity, the potential \( \rho \) should go like \( \left( \frac{dp}{dt} \right)^2 = \left( \frac{d\tau}{dt} \right)^2 \), i.e. like the inverse of a squared velocity. Hence, in the nonrelativistic limit \( \frac{\tau}{c} \to 0 \), \( \rho \) will in fact strictly vanish. The situation we shall nevertheless consider is one where

\[
0 < c^2 \rho << 1.
\]

(2.33)

Comparing in this case the action derived from (2.32),

\[
P = \int \tilde{L} \, de,
\]

with the general expression for action in momentum-energy relativity, \( P = -T c \int ds(p) \), we get

\[
ds(p)^2 = \left( \frac{1}{c^2} + \frac{v^4}{4c^6} + c^2 \rho^2 - \frac{v^2}{c^3} + 2 \rho - \frac{v^2 \rho}{c^2} \right) \, de^2.
\]

Taking into account that \( dp = \frac{v \, de}{c} \) and \( dp^0 = \frac{de}{c} \), we obtain then for very small \( \frac{v}{c} \)

\[
f_{00} \simeq 1 + 2c^2 \rho, \quad f_{0i} = 0, \quad f_{ii} \simeq -(1 + c^2 \rho),
\]

(2.34)

where only the second and third terms in the r.h.s. of the expression for \( ds(p)^2 \) above have been disregarded.

On the other hand, by analogy with the Einstein equations, the field equations in momentum-energy representation can be written

\[
R(f, p)_{\alpha\beta} = \frac{4\pi K}{c^4} \left( S_{\alpha\beta} - \frac{1}{2} f_{\alpha\beta} S \right),
\]

(2.35)

where \( R(f, p)_{\alpha\beta} \) is the Ricci tensor expressed in terms of tensor \( f_{\alpha\beta} \) and coordinates \( p' \), and \( K \) is the coupling constant for the new field. Since this constant has the dimension of a conventional force, one can regard it as the universal force exerted upon the system by a universal constant field other than \( \rho \), which is defined in spacetime.

The sole field which appears to be able to generate such a force is gravity and since this is attractive, \( K \) must be negative: \( K = -|K| \). Finally, the tensor \( S_{\alpha\beta} \) is a spacetime 4-tensor, the counterpart of the momentum-energy 4-tensor of Einstein equations in momentum-energy relativity. When the involved velocities are small compared to the velocity of light, we have

\[
S_{\alpha\beta}^0 \simeq u_{\alpha}u^{\beta}\tau_0,
\]

(2.36)

so that the dominating term in this tensor becomes \( S_{\alpha\beta}^0 \simeq \tau_0 \). The parameter \( \tau_0 = \frac{V_p}{c} \), where \( V_p \) is the 3-volume in momentum space, accounts for the time that characterizes the system in momentum-energy continuum in a unit momentum volume. Then, Eqn. (2.35) reduces to

\[
R(f, p)^0_{0} \simeq -4\pi |K| \tau_0.
\]

(2.37)

We note that the terms in (2.37) which contain derivatives of the affine connections in momentum-energy, \( \Gamma(f, p)_{\beta\gamma}^\alpha \), with respect to \( \frac{\tau}{c} \) involve extra power \( c \) and therefore are large as compared to the derivative with respect to the momenta \( p' \), \( i = 1, 2, 3 \). Hence, we can approximate

\[
\partial_e \left( \frac{1}{2} f^{ik} \partial_\epsilon f_{ik} \right) \simeq 4\pi |K| \tau_0.
\]

(2.38)

From (2.34) and (2.38) we obtain

\[
\dot{\rho} \simeq 4\pi |K| \tau_0,
\]

(2.39)

in which an overhead dot means derivative with respect to the energy-coordinate \( e \). Direct integration of (2.39) yields

\[
\rho(e) \simeq 4\pi |K| \tau_0 e^2 + K_1 e + K_2,
\]

(2.40)

where the integration constants \( K_1 \) and \( K_2 \) should be both zero since \( \rho \) must vanish at the sourceless limit \( \tau_0 \to 0 \).

Let us now consider the more important case of a constant \( \rho \)-field, meaning by that a field \( \rho \) which does not
Thus, \( \rho = \rho(p') \simeq |K| \int \frac{\tau_0 V_\rho}{c^2 p} \sim |K| |T| \frac{T'}{p^2}. \) 

We define the p-force between two sources \( T \) and \( T' \) as

\[
F_p = -c^2 T' \frac{\partial \rho}{\partial p} \simeq \frac{|K| |T'|}{p^2}. \tag{2.43}
\]

It is worth noticing that the p-force \( F_p \) has the dimension of the inverse of a conventional force, that is the dimension of the Newton constant \( \frac{G}{\hbar c} \). Therefore, we can regard \( G_N \) as the quantity that characterizes the constant \( \rho \)-field in our universe. Since \( F_p \) is repulsive \( G_N \) is then positive. On the other hand, since \( \rho \) does not depend on the position \( x' \), it must take on the same value (\( \rho_0 \) say) at any two distinct spatial points in a system, provided these points are characterized by the same momenta and times. Assuming then that the total mass of the system is \( M_0 \) and taking \( l = cT, V = \frac{p}{M_0} \), we obtain from (2.42)

\[
V = \left( \frac{|K|}{\rho_0 M_0 c^3} \right) l = Hl. \tag{2.44}
\]

Thus, \( H \) can be interpreted as a Hubble constant and (2.44) as a cosmological law.

(iii) For a constant \( \rho \)-field, \( p^0 \) should be related to the proper energy \( e \) by \( e = \sqrt{\mathcal{M}p^0} \). In the case of weak \( f_{\alpha \beta} \)-fields, \( e \simeq cp^0(1 + c^2 \rho) \). Let us consider then the propagation of a light ray in momentum-energy continuum when a weak constant \( \rho \)-field is present. The light ray will be characterized by an energy frequency \( \Omega \) which would be given by the derivative with respect to the energy coordinate of the phase-eikonal in momentum-energy, \( \eta = -\tau_\alpha p^\alpha + \phi \) (with \( \phi \) an arbitrary constant), for a “plane wave” \( e^{i\eta} \) in momentum-energy. As expressed in terms of the energy \( p^0 \), the energy frequency becomes \( c\Omega_0 = -\frac{\partial \eta}{\partial c} \), and if we express it in terms of the proper energy \( e \), we have

\[
c\Omega = -c^2 \frac{\partial \eta}{\partial c} = \frac{1}{\sqrt{\mathcal{M}p^0}} \frac{\partial \eta}{\partial p^0} \simeq \frac{\Omega_0 c}{1 + c^2 \rho}. \tag{2.45}
\]

We lift then the above restriction that spatial points of a system have all the same local momenta and hence the same values of field \( \rho \). Thus, if a ray of light is emitted at a point where the potential is \( \rho_1 \) and the energy frequency is \( \Omega \), then upon arriving at a point where the potential is \( \rho_2 \) it will have an energy frequency \( \Omega \frac{1 + \rho_2}{1 + \rho_1} \). For an observer at the arrival point the energy frequency would then be shifted by an amount \( \Delta \Omega = \Omega \frac{1 + \rho_2}{1 + \rho_1} - \Omega \) that corresponds to a proper-energy shift given by

\[
\Delta E \simeq E c^2 (\rho_2 - \rho_1), \tag{2.46}
\]

where \( E \) is the proper energy at the emission point where the potential is \( \rho_1 \).

If we assume that the system is our universe and that every point considered represents a galaxy of approximately the same size and luminosity, then the light coming to our galaxy from the inner regions of any other galaxy would be produced in a physical environment similar to our own. In this case, \( \rho_1 \approx \rho_2 \) and hence \( \Delta E \simeq 0 \). However, as the light source separates from the core and enters outer regions of the emitting galaxy where the momenta become smaller, \( \rho_1 > \rho_2 \) and from (2.46) \( \Delta E < 0 \). Although the approximation used in (2.46) breaks down as \( \rho_2 \) increases, the above discussion appears to point out that, rather than attributing this actually observed effect [7] to the presence of some sort of dark matter, it would instead be attributed to the noninvariance of proper energy under propagation in curved momentum-energy.

In what follows, the above results will be taken to imply that the field derived from variations of the metric tensor \( f_{\alpha \beta} \) with respect to coordinates \( p^i \) essentially describes cosmological interactions. We shall therefore refer to this field as the cosmological field.

### III. EXTENDED GEOMETRODYNAMICS

Let us introduce an arbitrary system of coordinates \( X^i \) in a Riemannian spacetime, and an arbitrary system of coordinates \( P^\mu \) in a Riemannian momentum-energy. Describe then a hypersurface in spacetime and a hypersurface in momentum-energy by giving four functions \( X^i(q^j) \) of three action coordinates \( q^j \) and four functions \( P^\mu(q^j) \) of the same action coordinates \( q^j \), respectively; i.e.:

\[
X^i = X^i(q^j), \quad P^\mu = P^\mu(q^j), \tag{3.1}
\]

with \( i = 0, 1, 2, 3 \) and \( \mu = 1, 2, 3 \); the 0-component in momentum-energy corresponds to energy. These two hypersurfaces are thus labeled hypersurfaces [8], i.e.: in this case two hypersurfaces together with a common intrinsic action coordinate system \( q^j \) for them. Expressions (3.1) tell us that the point of the \( X^i(P^\mu) \)-hypersurface carrying the intrinsic label \( q^j \) is located in spacetime (momentum-energy) at the point carrying the spacetime (momentum-energy) label \( X^i(P^\mu) \). This implements the unfolding discussed in Sec. II in the geometrodynamical formalism.
A. Deformations and relabelings

Changes in a labeled hypersurface on a given projected sheet (spacetime or momentum-energy) will generally induce changes in the labeled hypersurface on the other projected sheet. These labeled hypersurfaces are changed either by leaving both fixed in the respective embedding spaces (spacetime and momentum-energy) but relabeling uniquely their points, or by deforming both hypersurfaces into other pair of hypersurfaces, while leaving their labeling fixed. Any arbitrary change of a pair of such hypersurfaces may be decomposed into these two changes [8].

The first kind of changes represents a pure deformation of the hypersurfaces in the Riemannian space without changing of labeling. It can be carried out as follows. Start from hypersurfaces \( X^i(q^i) \) and \( P^i(q^i) \). Draw geodesics perpendicular to \( X^i(q^i) \) and cosmodesics perpendicular to \( P^i(q^i) \). Move then along the geodesic and cosmodesic that start from the point \( q^i \), eventually meeting a point of the deformed hypersurface \( \bar{X}^i \) and a point on the deformed hypersurface \( \bar{P}^i \), respectively. Attach to these points the same label \( q^i \) as that of the starting points, and describe the displacement of \( \bar{X}^i \) with respect to \( X^i \) by giving the proper time \( \tau(q^i) \) measured along the geodesic, and that of \( \bar{P}^i \) with respect to \( P^i \) by the proper energy \( \varepsilon(q^i) \) measured along the cosmodesic. Repeating this operation at each of the two original hypersurfaces will give rise to two single functions \( \tau(q^i) \) and \( \varepsilon(q^i) \) that describe the operations of pure deformation of the two surfaces; i.e.: \( \varrho[\tau(q^i)] \) and \( \varrho[\varepsilon(q^i)] \) (see Fig. 2).

Since in the curvilinear formalism the cosmodesic does not match the respective geodesic, the label of the end point on \( \bar{X}^i(q^i) \) will not coincide with the corresponding label of the end point on \( \bar{P}^i(q^i) \), and therefore the sheet \( \Pi_E \) (or the sheet \( \Pi_T \) of Fig. 2) should be deformed in an amount that allows these two final labels to exactly coincide. But deforming e.g. the sheet \( \Pi_E \) induces an additional deformation of hypersurface \( \bar{X}^i(q^i) \) itself. Hence, the action of an infinitesimal deformation of hypersurface \( \bar{X}^i(q^i) \) will be given by

\[
\varrho_T[\tau(q)]X^i(q^i) = \varrho_E[\delta N_T(q^i), \delta N_E(q^i)]X^i(q^i)
\]

\[
= X^i(q^i) + \nu^i(q^i) \left( \delta N_T(q^i) + \left( \frac{\partial N_T}{\partial N_E} \right)(q^i) \delta N_E(q^i) \right)
\]

\[
= X^i(q^i) + \nu^i(q^i) \delta N_X(q^i),
\]

(3.2)

where \( \delta N_T \) and \( \delta N_E \) account for the proper time and the proper energy, respectively, \( \nu^i(q^i) \) is the unit normal to the hypersurface \( X^i \), and \( N_X(q^i) \) is a generalized lapse function having the dimension of a time and is given by

\[
N_X(q^i) = N_T(q^i) + \frac{\Omega}{\nu} N_E(q^i),
\]

(3.3)
in which we have used \( \frac{\partial N_T}{\partial N_E} = \frac{\Omega}{\nu} \), with \( \Omega \) as given by the second of expressions (2.29) and \( \nu \) is the usual frequency defined by the Einstein-de Broglie relation \( E = h\nu \). Had we deformed sheet \( \Pi_T \), instead of \( \Pi_E \), then we had obtained the infinitesimal deformation

\[
\varrho_E[\varepsilon(q)]P^i(q^i) = P^i(q^i) + m^i(q^i) \delta N_P(q^i),
\]

(3.4)

where \( m^i(q^i) \) is the unit normal to the hypersurface \( P^i(q^i) \), and

\[
N_P(q^i) = N_E(q^i) + \frac{\nu}{\Omega} N_T(q^i).
\]

Similarly, relabeling is the operation (which we denote by \( \varrho[q(q^k)] \)) that takes the label \( q^k \) from fixed spacetime and momentum-energy points \( X^i \) and \( P^i \) and reattaches it to the points \( \bar{X}^i \) and \( \bar{P}^i \) which originally had the label \( \bar{q}^i(q^k) \). Here deformations of the spacetime (or momentum-energy) sheet are again necessary. We have

\[
\varrho_E[q(q^k)]X^i(q^i) = X^i(q^i) + X^i(q^k) \frac{\delta N^i(q^i)}{\delta N^k(q^k)} + \frac{\lambda}{\mu} N^i_X(q^i),
\]

(3.5)

with the subscript \( , i \) meaning the derivative with respect to \( q^i \), and we have used a generalized shift function which is given by

\[
N^i_X(q^k) = N^i_{T}(q^k) + \frac{\lambda}{\mu} N^i_{E}(q^k),
\]

(3.6)

where use of the de Broglie relation and the first of expressions (2.29) has been made.

It is also obtained

\[
\varrho_E[q(q^k)]P^i(q^i) = P^i + P^i_{\varepsilon}(q^k) \delta N_P(q^i),
\]

(3.7)
in which

\[
N^i_P(q^k) = N^i_{E}(q^k) + \frac{\mu}{\lambda} N^i_{T}(q^k).
\]

Note that \( \delta N_X \), as defined from (3.3), will give the actual proper time separation, \( T(q^i) \), say, between any two hypersurfaces and is generally different from \( \tau(q^i) \). The set of deformations of hypersurfaces turns out to be an infinitely dimensional set [8] whose elements are characterized by functions \( T(q^i), \bar{q}^i(q^k) \). One can define generators for the relabeling. Let us use the notation such
that e.g. $H_i(q^k) \equiv H_{iq}$, $N^i_X(q^k) \equiv N^i_X$, etc. Then if the action of group on the function space is expressed as

$$
\vartheta_E[N^i_X]X'^q = \tilde{X}'^q[X'^{\kappa}, N^i_X],
$$

(3.8) then the generators can be identified through the infinitesimal transformation

$$
\vartheta_E(\delta N^i_X)X'^q = X'^q + \frac{\delta \tilde{X}'^q[X'^{\kappa}, N^i_X]}{\delta N^i_X} \bigg|_{N^i_X=0} \delta N^i_X,
$$

(3.9)
in the neighborhood of the identity $N^i_X = 0$.

If we denote the coefficient for $\delta N^i_X$ in (3.9) by $\xi^i_{iq}$, the operators

$$
X_{iq} = \xi^i_{iq} \frac{\delta}{\delta X'^i(q)},
$$

(3.10)
will be the generators of the relabelings. The vectors $\xi^i_{iq}$ are obtained by comparing (3.9) with (3.5). It follows

$$
\xi^i_{iq} = X'_i(q') \delta(q,q'),
$$

(3.11)
so that the generators of relabeling are

$$
X_{iq} = X'_i(q) \frac{\delta}{\delta X'^i(q)}.
$$

(3.12)
Proceeding similarly, we can also identify the generators of pure deformations. They are:

$$
X_q = n^i(q) \frac{\delta}{\delta X'^i(q)}
$$

(3.13)
The structure constants of the infinitely dimensional group corresponding to relabelings and deformations are determined from the commutation relations of their generators (3.12) and (3.13). Of most interest is the commutator between two generators (3.13)

$$
[X_q, X_{q'}] = -n^i(q') \frac{\delta n^i(q)}{\delta X'^i(q')} \frac{\delta}{\delta X'^i(q)} + (q,q'),
$$

(3.14)
where $(q,q')$ means the same expression with $q$ and $q'$ interchanged. This antisymmetrization kills all terms in which $\frac{\delta n^i(q)}{\delta X'^i(q')}$ are proportional to the delta function $\delta(q,q')$ and, therefore, only the “tilting” term of $\frac{\delta n^i(q)}{\delta X'^i(q)}$ remains to contribute [8]. Such a tilting term has in this case the form

$$
-X'^i n_i \left( X'^j - \frac{\lambda}{\mu} \delta P^i_j \right),
$$

(3.15)
where the first term gives the change of $n^i$ when the hypersurface $X^\kappa$ is displaced directly by pure $X$-deforming by an amount $\delta X^\kappa(q)$, and the second term accounts for the change of $n^i$ produced by the displacement of hypersurface $X^\kappa$ induced by displacing $P^\kappa$ by an amount $\delta P^\kappa(q)$. In (3.15) the Greek indices are raised and lowered by $g_{\alpha \beta}$ (first term) and $f_{\alpha \beta}$ (second term), and the Latin indices by, respectively, the metric tensors

$$
g_{ik} = g_{\alpha \beta} X^i_\alpha X^k_\beta, \quad f_{ik} = f_{\alpha \beta} P^i_\alpha P^k_\beta,
$$

(3.16)
where $X^i_\alpha \equiv X^i_\lambda$ and $P^i_\alpha \equiv P^i_\lambda$. The terms (3.15) then contribute by an amount

$$
\frac{\delta_i n^i(q)}{\delta X^\kappa(q')} = -X'^i \eta{\kappa}(q) \left( \frac{\delta X^\kappa_\lambda(q)}{\delta X^\kappa(q')} + \frac{\lambda}{\mu} \frac{\delta P^\kappa_\mu(q)}{\delta X^\kappa(q')} \right)
$$

$$
\equiv -X'^i \eta{\kappa}(q) \delta_i(q,q') - (q,q'),
$$

(3.17)
in which the indices in the first term of (3.17) are raised and lowered by $g_{\alpha \beta}$, $g_{ik}$ and those of the second term in the same equation and in (3.18) by $f_{\alpha \beta}$, $f_{ik}$ and also $g_{\alpha \beta}$, $g_{ik}$. Substituting (3.18) in (3.14), we obtain

$$
[X_q, X_{q'}] = n^\kappa(q') n_i(q) X^i(q) \delta_i(q,q') \frac{\delta}{\delta X^\kappa(q')} - (q,q').
$$

(3.19)
By employing then the usual procedure [8], we finally get commutators with exactly the same formal structure as in conventional geometrodynamics, but with the indices in the r.h.s. being raised and lowered by $g_{ik}$ and also by $f_{ik}$, which are defined in (3.16).

**B. Hamiltonian formalism**

A minimal representation of this extended formulation of geometrodynamics should use as canonical variables both the metric tensor $g_{ik}$ and the metric tensor $f_{ik}$ as well as their respective conjugate momenta $\pi^{ik}$ and $\omega^{ik}$. Our task now is to find the superhamiltonian $H$ and the supermomentum $H_i$, which should be constructed out of the above metric tensors and momenta, while respecting the commutation relations of geometrodynamics, with an action functional

$$
S = \int d^4 q (\pi^{ik} g_{qik} + \omega^{ik} f_{qik})
$$

$$
- N_{Xq} H_q - N_{Xq} H_{iq},
$$

(3.20)
where $* \equiv \frac{\delta}{\delta q}$ and $N_X$ and $N_{Xq}$ are given by (3.3) and (3.6), respectively. The Hamiltonian $H$ that corresponds
to this action functional determines then the change, 
\[ \delta F, \]\ of any arbitrary function \( F \) of the geometrodynamic variables \( (\eta_{ik}, f_{ik}, \pi^{ik}, \omega^{ik}) \) induced by the deformation \( \delta N \). Therefore, the covariant derivatives in (3.23) can be written by

\[ \delta F = [F, \mathcal{H}_{q'} \delta N^q |^q_X + \mathcal{H}_{q'} \delta N^q |^q_X], \]  
(3.21)

where \( N^q |^q_X \) and \( N^q |^q_X \) are given by (3.3) and (3.6). Specializing to pure relabeling \( (\delta N = 0) \),

\[ \delta F = [F, \mathcal{H}_{q'} \delta N^q |^q_X], \]  
(3.22)

and taking into account that both \( g_{ik} \) and \( f_{ik} \) transform like tensors and both \( \pi^{ik} \) and \( \omega^{ik} \) do like tensor densities of weight 1 under relabeling, so that the respective changes are given by the Lie derivatives of a tensor and a tensor density, we obtain a set of equalities, i.e.

\[ [g_{ikq}, \mathcal{H}_{q'} \delta N^q |^q_X] = \frac{\delta \mathcal{H}_{q'}}{\delta \pi_{ikq}} \delta N^q |^q_X \]

\[ = g_{ik,l} \delta N^l |^l_X + g_{il} \delta N^l |^l_X, \]

\[ [f_{ikq}, \mathcal{H}_{q'} \delta N^q |^q_X] = \frac{\delta \mathcal{H}_{q'}}{\delta \pi_{ikq}} \delta N^q |^q_X \]

\[ = f_{ik,l} \delta N^l |^l_X + f_{il} \delta N^l |^l_X, \]

\[ [\pi^{ikq}, \mathcal{H}_{q'} \delta N^q |^q_X] = \frac{\delta \mathcal{H}_{q'}}{\delta \pi_{ikq}} \delta N^q |^q_X \]

\[ = (\pi^{ik} \delta N^k |^k_X)_l - \pi^{il} \delta N^l |^l_X - \pi^{lk} \delta N^k |^k_X, \]

\[ [\omega^{ikq}, \mathcal{H}_{q'} \delta N^q |^q_X] = \frac{\delta \mathcal{H}_{q'}}{\delta \pi_{ikq}} \delta N^q |^q_X \]

\[ = (\omega^{ik} \delta N^k |^k_X)_l - \omega^{il} \delta N^l |^l_X - \omega^{lk} \delta N^k |^k_X, \]

whose unique solution reads:

\[ \mathcal{H}_{iq} = -2 \left( g_{ik} \pi_{i|j}^{kl} + f_{ik} \omega_{j|l}^{kl} \right), \]  
(3.23)

where the subscript \( i \) means the corresponding covariant derivative. All derivatives in (3.23) are taken with respect to the action-like coordinates \( q^i \). These coordinates were however defined such that hypersurface \( X^t(q^i) \) would correspond to a constant value of \( q^i \). Due to the mutual complementary character of \( X \) and \( F \), \( q^i \) may either be given by \( q^i = \mu x^i \), when it is projected onto spacetime, or by \( q^i = \lambda p^i \) if it is projected onto momentum-energy. Therefore, the covariant derivatives in (3.23) can be written

\[ \pi^{kl |i} = \frac{\pi^{kl |i}}{\mu}, \ \omega^{kl |i} = \frac{\omega^{kl |i}}{\lambda}. \]  
(3.24)

It then follows

\[ \mathcal{H}_{iq} = -2 \left( q_{ik} \pi^{kl |i} + \mu \omega^{kl |i} \right) \]

\[ = \mathcal{H}_i^T (x) + \mu \mathcal{H}_E (p) \equiv \mathcal{H}_{ix} \]  
(3.25)

Using the same ansatz as in usual geometrodynamics [8], we can similarly obtain the superHamiltonian

\[ \mathcal{H}_q = G_{qiklm} \pi_{ik} \omega_{lm} - (\sqrt{g} R)_q \]

\[ + F_{qiklm} \omega_{ik} \omega_{lm} - \left( \sqrt{T} C \right)_q, \]  
(3.26)

where

\[ G_{qiklm} \]

\[ = \frac{1}{2 \sqrt{g}} \left( g_{ik} g_{lm} + g_{im} g_{kl} - g_{ik} g_{lm} \right) \]  
(3.27)

is the metric on usual superspace and \( F_{qiklm} \) is the metric on the equivalent superspace constructed from momentum-energy coordinates. Finally, \( R \) and \( C \) are the scalar curvatures in the respective 3-space.

The superHamiltonian \( \mathcal{H}_q \) will correspond to the operation \( \mathcal{H}_q \equiv \frac{\delta}{\delta q} \). Depending on which of the two subspaces it is projected onto, \( \mathcal{H}_q \) can be written either as

\[ \mathcal{H}_q = \Omega \frac{\delta}{\delta x} = \Omega \mathcal{H}_X \]  
(3.28)

or as

\[ \mathcal{H}_q = \nu \frac{\delta}{\delta \tau} = \nu \mathcal{H}_P. \]  
(3.29)

Therefore,

\[ \mathcal{H}_q = \Omega \left( G_{qiklm} \pi_{ik} \pi_{lm} - (\sqrt{g} R)_X \right) \]

\[ + \nu \left( F_{qiklm} \pi_{ik} \pi_{lm} - \left( \sqrt{T} C \right)_P \right) \]

\[ \equiv \Omega \left( \mathcal{H}_i^T (x) + \nu \mathcal{H}_E (p) \right) = \Omega \mathcal{H}_X. \]  
(3.30)

Using (3.2), (3.5), (3.24) and (3.29) in action (3.20) we obtain

\[ S_X \propto \int d^4 x (\pi^{ik} \omega_{Xik}) \]
where \( \omega^i \) when it is over \( g \) and \( \dot{\omega} = \frac{\delta}{\delta \tau} \) when it is over \( f \). From \( \frac{S}{\delta X} \) we obtain the new Hamiltonian constraint

\[
\mathcal{H}^T + \frac{\nu}{\Omega} \mathcal{H}^E = 0, \quad (3.32)
\]

with \( \mathcal{H}^T \) and \( \mathcal{H}^E \) the superHamiltonians of geometrodynamics and cosmodynamics which, separately, are no longer zero in the present formalism.

Of course, one could re-formulate the above canonical formalism in terms of the cosmological field rather than the gravitational field. We would then derive an action functional

\[
S_P \propto \int d^4p(\pi^{ik}_{\nu} g_{Xik}
+ \omega^i_f \hat{p}_{rik} - N_X \mathcal{H}_X - N_X^i \mathcal{H}_X^i), \quad (3.33)
\]

where \( N_P \) and \( N_P^i \) are as given in Sec. IIIA, and

\[
H_P = \mathcal{H}^E(p) + \frac{\Omega}{\nu} \mathcal{H}^T(x) \quad (3.34)
\]

\[
H_{iP} = \mathcal{H}^E_i(p) + \frac{\lambda}{\nu} \mathcal{H}^T_i(x). \quad (3.35)
\]

These are the basic equations for the canonical formulation of the cosmological field which we may call cosmodynamics. From \( \frac{S}{\delta N_P^i} \), we would then obtain again (after multiplying by \( \nu \) and dividing by \( \Omega \) the resulting expression) the constraint (3.32).

Clearly, physical systems that show observable gravitational effects are usually of large size (even astrophysical black holes are remarkably large). Such systems will then be characterized by small values of \( \nu \) and rather huge values of \( \Omega \). Hence, using the constraint \( \mathcal{H}^T = 0 \) for them becomes an excellent approximation. However, for primordial black holes or in the very early universe, one would expect the quantum characteristics of the systems to be exactly the opposite — i.e. such systems would have large \( \nu \) and small \( \Omega \). In this case, it would be the cosmological Hamiltonian which became approximately constrained so that \( \mathcal{H}^E \simeq 0 \). Therefore, one would also expect this constraint rather than the usual one to contain almost all the relevant dynamical information required to describe the latest stages of black-hole evaporation or the earliest stages of the evolution of the universe.

Finally, we note that by independently varying any of the two above action functionals with respect to either metric \( g_{ik} \) or metric \( f_{ik} \), we would respectively obtain [9] Einstein equations and the cosmological field equations (2.35).

### IV. QUANTIZATION

All of the essential steps that we shall adopt in what follows are not but hints and guesses as they concern the quantization of the canonical formalism developed in Sec. III. To my knowledge, there is no other way to proceed with the quantization of any field, not even for inertial systems. We start with the action functional obtained in the previous section for a gravitating system, i.e.

\[
S_X \propto \int d^4x(\pi^{ik}_{\nu} g_{Xik}
+ \omega^i_f \hat{p}_{rik} - N_X \mathcal{H}_X - N_X^i \mathcal{H}_X^i), \quad (4.1)
\]

which has the same form as that of parametrized field theories, but contains the additional (second) term, and differs in the specific form of the superHamiltonian and supermomentum which, in (4.1), read

\[
\mathcal{H}_X = \mathcal{H}^T(x) + \frac{\nu}{\Omega} \mathcal{H}^E(p),
\]

\[
\mathcal{H}_{iX} = \mathcal{H}^E_i(x) + \frac{\lambda}{\nu} \mathcal{H}^T_i(p), \quad (4.2)
\]

both being equal to zero.

Instead of (4.1), one could use the action functional relative to the momentum-energy sheet, \( S_P \), which is given in terms of the superquantities \( \mathcal{H}_P \) and \( \mathcal{H}_{iP} \). In the form given by (4.1) and (4.2), our action is prepared to be quantized just on spacetime. Spacetime quantization would proceed by turning into operators the metric \( g_{ik}(x) \), the momentum \( \pi^{ik}(x) \) and, as a consequence from the fact that they are given in terms of \( g_{ik}(x) \)'s and \( \pi^{ik}(x) \)'s, the quantities \( \mathcal{H}^T \) and \( \mathcal{H}^E \), as well as the quantities \( \mathcal{H}^E_i \) and \( \mathcal{H}^T_i \) by themselves. The resulting operators are assumed to satisfy the commutation relations (in what follows we set \( \hbar = c = G = K = 1 \))

\[
[g_{ik}(x), \pi^{lm}(x')] = \frac{1}{2} \left( \delta^l_i \delta^m_k + \delta^m_i \delta^l_k \right) \delta(x, x') \quad (4.3)
\]

\[
[g_{ik}(x), g_{lm}(x')] = [\pi^{ik}(x), \pi^{lm}(x')] = 0 \quad (4.4)
\]

\[
[\mathcal{H}^T(x), \mathcal{H}^E(p')] = -i \frac{\Omega}{\nu} g^{br}(x) \delta(x) \delta(x, x'), \quad (4.5)
\]

\[
[\mathcal{H}^T_i(x), \mathcal{H}^{Ei}(p')] = -\frac{\lambda}{\nu} g^{br}(x) \delta(x) \delta(x, x'), \quad (4.6)
\]

where the subscript \( i, l \) means derivation with respect to \( x_i \).
In order to proceed with the quantization of the complementary momentum-energy canonical formalism, we would start with (3.33)-(3.35) and similarly turn into operators $f_{ik}(p)$, $\omega_{ik}(p)$ and hence the quantities $\mathcal{H}^E$, $\mathcal{H}^T$, as well as the quantities $\mathcal{H}^T$ and $\mathcal{H}^T_i$ by themselves. The resulting operators would then satisfy the commutation relations

$$
[f_{ik}(p), \omega_{lm}(p')] = \frac{1}{2} i (\delta^i_j \delta^m_k + \delta^m_i \delta^j_k) \delta(p, p')
$$

(4.7)

$$
[f_{ik}(p), f_{lm}(p')] = [\omega^{ik}(p), \omega^{lm}(p')] = 0
$$

(4.8)

$$
[\mathcal{H}^E(p), \mathcal{H}^T(x')] = -i \frac{\nu}{\Omega} f^{br}(p) \delta(p) \delta(p, p'), x
$$

(4.9)

$$
[\mathcal{H}_i^E(p), \mathcal{H}^T_j(x')]
$$



= \left( \frac{i}{\lambda} f^{br}(p) \right) \delta(p) \delta(p, p'), x
$$

where the subscript $_j$ denotes now derivation with respect to $p_i$, instead of $x_i$.

A. Spacetime quantization

Here, we shall restrict ourselves to explicitly deal with quantization in spacetime. We shall adopt the metric representation in which the state functional $\Psi$ will become a functional of a $3$-metric $g_{ik}(x)$ and the quantities $\mathcal{H}^E(p)$ and $\mathcal{H}^T_i(p)$, in such a way that

$$
\Psi \equiv \Psi \left[ g_{ik}, \mathcal{H}^E, \mathcal{H}^T_i \right]
$$

(4.11)

can be interpreted as containing the information about the showings of clocks and meters among its arguments. Then [10]:

(i) the $3$-momentum $\pi^{ik}(x)$ is replaced by the variational derivative with respect to the metric $g_{ik}(x)$

$$
\hat{\pi}^{ik}(x) = -i \frac{\delta}{\delta g_{ik}(x)},
$$

(4.12)

and (ii) the quantities $\mathcal{H}^T(x)$ and $\mathcal{H}^T_i(x)$ are replaced by the functional derivatives with respect to $\mathcal{H}^E(p)$ and $\mathcal{H}_i^T(p)$, respectively, i.e.:

$$
\hat{\mathcal{H}}^T(x) = -i \frac{\delta}{\delta \mathcal{H}^E(p)}, \quad \hat{\mathcal{H}}^T_i(x) = -i \frac{\delta}{\delta \mathcal{H}_i^T(p)}.
$$

(4.13)

Following this procedure, we substitute these operators into the superHamiltonian and supermomentum in spacetime representation, and impose the general constraints (4.2) as restrictions on the state functional, that is

$$
\hat{\mathcal{H}}^T \Psi = -G_{iklm}(x) \frac{\delta^2 \Psi}{\delta g_{ik}(x) \delta g_{lm}(x)}
$$

$$
\hat{\mathcal{H}}^T_i \Psi = 2i \frac{\delta \Psi}{\delta g_{ik}(x)} \bigg|_k - i \frac{\delta \Psi}{\delta \mathcal{H}_i^T(p)}.
$$

(4.14)

(4.15)

These equations should always be different of zero, unless for systems of infinite size. We have therefore deconstrained our wave equations, leaving them in a manifest Schrödinger-like form. As in the parametrized field theories, equation (4.15) implies that the state functional is unchanged under relabeling of the hypersurfaces. Indeed, by a relabeling of the hypersurface the metric must change into

$$
g_{ik} \rightarrow g_{ik} = g_{ik} - \delta N_{X_{i|k}} - \delta N_{X_{k|i}}
$$

while, since $\mathcal{H}_i^E$ has the dimension of a spacetime distance, it undergoes the transformation

$$
\mathcal{H}_i^E \rightarrow \mathcal{H}_i^E = \mathcal{H}_i^E + \delta N_{X_i}.
$$

For the state functional to be kept unchanged, one should then have

$$
\int d^3x \left( 2 \frac{\delta \Psi}{\delta g_{ik}} \delta N_{X_{i|k}} - \frac{\delta \Psi}{\delta \mathcal{H}_i^E} \delta N_{X_i} \right) = 0.
$$

By integrating by parts the first of these integrals and taking into account the arbitrariness of $\delta N_{X_i}$, we recover in fact the supermomentum wave equation (4.15). The state functional thus depends on the spatial geometry $\mathcal{G}^S$ and physical distances $\mathcal{D}$, but not on the particular metric and position chosen to represent it. Likewise, one can show [8] the invariance of (4.14) under pure deformations of the hypersurfaces, so that now the wave functional will also depend on a generic time $\mathcal{T}$, but not on any of the particular moments that may be chosen to represent it. Thus,

$$
\Psi \equiv \Psi \left[ \mathcal{G}^S, \mathcal{D}, \mathcal{T} \right].
$$

(4.16)

It follows that the proper domain of the state functional is an extended superspace which, besides on the $3$-geometry, depends also on suitable distance and time concepts. The specific mathematical characteristics of such an extended superspace will be considered in a future publication. We have in this way succeeded in separating suitably defined space and time concepts from the dynamical variables.

B. Momentum-energy quantization

By following a completely parallel procedure, we finally obtain in the case of the cosmological field
\[ \dot{\omega}^{ik}(p) = -i \frac{\delta}{\delta f_{ik}(p)}. \]  \hspace{1cm} (4.17)

\[ \hat{H}^E(p) = -i \frac{\delta}{\delta \hat{H}^T(x)} \]  \hspace{1cm} (4.18)

\[ \hat{H}^E \Phi = -F_{iklm}(p) \frac{\delta^2 \Phi}{\delta f_{ik}(x) \delta f_{lm}(x)} \]
\[ + \sqrt{f(p)}C(p) \Phi = -i \frac{\delta \Phi}{\delta \hat{H}^T(x)} \]  \hspace{1cm} (4.19)

\[ \hat{H}^E \Phi \equiv 2i \left( \frac{\delta \Phi}{\delta f_{ik}(p)} \right)_{|k} = -i \frac{\delta \Phi}{\delta \hat{H}^T(x)} \]  \hspace{1cm} (4.20)

with
\[ \Phi \equiv \Phi \left[ g^M, \mathcal{M}, \mathcal{E} \right], \]  \hspace{1cm} (4.21)

where the subscript \(_{|k}\) now means covariant derivative with respect to the metric of momentum-energy, \( g^M \) denotes the geometry of a 3-momentum superspace, and \( \mathcal{M} \) and \( \mathcal{E} \) some concepts of generic momentum and energy, defined parallelly to as for generic space and time concepts in the case of spacetime quantization. Eqns. (4.17)-(4.21) form up the essentials of the formulation of what we may call quantum cosmodynamics, with the first two ones being different of zero always unless for systems of zero size.

C. Consistent operator-ordering

Let us now see how the operator-ordering problem which appears in conventional geometrodynamics can be worked out in our extended formalism. The problem can be expressed by using the hermitian ordering that corresponds to the quantum operators proposed by Anderson [11]. In our extended formalism of geometrodynamics, Anderson's ordering translates into
\[ \mathcal{H}_{iX} = \frac{1}{2} \left[ g_{ik} \pi^{kl} + \pi^{kl} g_{ik} + \frac{\mu}{\lambda} \left( f_{kl} \omega^{il} + \omega^{kl} f_{il} \right) \right] \]
\[ \mathcal{H}_X = \frac{\pi^{ik}}{\sqrt{g}} \left( g_{im} g_{kl} - g_{ik} g_{lm} \right) \pi^{lm} - \sqrt{g} R \]
\[ + \frac{\nu}{\Omega} \left( g, f; \pi, \omega; R, C \right), \]

where \((g, f; \pi, \omega; R, C)\) denotes the same expression as in all the explicated terms but with the \(g's, \pi's\) and \(R\) replaced for, respectively, the \(f's, \omega's\) and \(C\). The ordering problem is manifested through the commutator between Hamiltonian constraints. For the ordering chosen, in the present case one can find
\[ 2i [\mathcal{H}_X(x), \mathcal{H}_X(x')] = \delta(x, x') \left[ g^{rs}(x) \hat{H}_s^T(x) + \right] \]
\[ \frac{+ \delta \hat{H}_s^T(x) g^{rs}(x) + g^{rs}(x') \hat{H}_s^T(x')} \]
\[ + \frac{\nu}{\Omega} \left[ \left( \mathcal{H}^T(x), \mathcal{H}^E(y) \right) + \left[ \mathcal{H}^E(p), \mathcal{H}^T(x') \right] \right]. \]  \hspace{1cm} (4.22)

We can readily check that the troublesome terms (those that have factors \(g^{rs}\) or \(r^{rs}\) occurring to the right of the \(\hat{H}_s^T \) or \(\hat{H}_s^E \) [12] in the second and third lines of (4.22)) are all canceled by the commutators mixing Hamiltonian in \(x\) with that in \(p\) in the last line of (4.22). Using then \(\delta_r(p, p') = \frac{1}{2} \delta_r(x, x')\) and (4.5) and (4.9), we finally obtain
\[ [\mathcal{H}_X(x), \mathcal{H}_X(x')] = - \frac{1}{2} i \left( g^{rs}(x) \hat{H}_s^T(x) \right) \]
\[ + g^{rs}(x') \hat{H}_s^T(x') + \frac{\nu}{\Omega} \left( x, p; g, f; \hat{H}_s^T, \hat{H}_s^E \right) \delta_r(x, x') \]
\[ = - \frac{1}{2} i \delta_r(x, x') \left( \mathcal{H}_X(x) + \mathcal{H}_X(x') \right). \]  \hspace{1cm} (4.23)

Thus, (4.23) must vanish weakly. Since in the covariant form \(\mathcal{H}_i^T\) or \(\mathcal{H}_i^E\): (1) the interchange of momenta and coordinates only leads to terms with \(\delta_r(x, x')\) or \(\delta_r(p, p')\) which can be put equal to zero, and (2) the commutators
\[ [\mathcal{H}_i^T(x), \mathcal{H}_j^E(p')] = -i \frac{\lambda}{\mu} \delta_r(x, x'), \]  \hspace{1cm} (4.24)
\[ [\mathcal{H}_i^E(p), \mathcal{H}_j^T(x')] = -i \frac{\mu}{\lambda} \delta_r(p, p'), \]  \hspace{1cm} (4.25)

will also give terms with derivatives of \(\delta\)-functions, and no factor-ordering problem does not lead to any factor-ordering problem. Hence, one can have a closed algebra of the generalized constraints also in the quantized theory, and therefore \(\mathcal{H}_{iX} \Psi = 0\) and \(\mathcal{H}_X \Psi = 0\) can be satisfied simultaneously [10]. The same conclusion can also be obtained in the quantum-mechanical description of cosmodynamics. Thus, the quantization of the extended formalism of both geometrodynamics and cosmodynamics leads to no problem with a hermitian order of operators. The issue of quantizing the gravitational field may then be persued without restricting to domains where the factor-ordering problem is circumvented or replacing the dynamical content of Eqns. (4.19) and (4.20) for a cosmological constant.
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Legends for figures

Fig. 1: Relation of points of the two hypersurfaces which carry the same intrinsic label and their location in space-time and momentum-energy sheets.

Fig. 2: Related changes of the normals to the two hypersurfaces when each of these hypersurfaces is displaced an infinitesimal amount.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9712099v1
This figure "fig1-2.png" is available in "png" format from:

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