STABILIZERS OF \( \mathbb{R} \)-TREES WITH FREE ISOMETRIC ACTIONS
OF \( F_N \)

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ABSTRACT. We prove that if \( T \) is an \( \mathbb{R} \)-tree with a minimal free isometric action of \( F_N \), then the \( \text{Out}(F_N) \)-stabilizer of the projective class \([T]\) is virtually cyclic.

For the special case where \( T = T_+(\varphi) \) is the forward limit tree of an atoroidal iwip element \( \varphi \in \text{Out}(F_N) \), this is a consequence of the results of Bestvina, Feighn and Handel [6], via very different methods.

We also derive a new proof of the Tits alternative for subgroups of \( \text{Out}(F_N) \) containing an iwip (not necessarily atoroidal): we prove that every such subgroup \( G \leq \text{Out}(F_N) \) is either virtually cyclic or contains a free subgroup of rank two. The general case of the Tits alternative for subgroups of \( \text{Out}(F_N) \) is due to Bestvina, Feighn and Handel.

1. INTRODUCTION

The action of the mapping class group of a (closed) surface on its Teichmüller space has been a central theme in geometry, topology and ergodic theory, and it has served as model case for many related subjects. One of those is the outer automorphism group \( \text{Out}(F_N) \) of a free group \( F_N \) of finite rank \( N \geq 2 \), and its action on \textit{Outer space} \( CV_N \): This is the projectivized space of metric simplicial trees \( T \), provided with an action of \( F_N \) by isometries which is free and minimal. It is compactified (just as is Teichmüller space) by adding a Thurston boundary \( \partial CV_N \), and the points of this compactification \( CV_N = CV_N \cup \partial CV_N \) are homothety classes \([T]\) of \( \mathbb{R} \)-trees \( T \) provided with isometric \( F_N \)-actions that are minimal and very small. These terms are defined and discussed below in section 2.

A boundary point \([T] \in \partial CV_N \) may well be given by an \( \mathbb{R} \)-tree \( T \) where the \( F_N \)-action is free; however, contrary to trees in the “interior” \( CV_N \) such a free action will not be discrete.

Our main result is:

\textbf{Theorem 1.1.} Let \( N \geq 2 \), let \( T \) be an \( \mathbb{R} \)-tree with a minimal free isometric action of \( F_N \), and let \([T] \in CV_N \) be the projective class of \( T \). Then the stabilizer \( \text{Stab}_{\text{Out}(F_N)}([T]) \) is virtually cyclic.

If, in addition, the \( F_N \)-orbits of branch points are not dense if \( T \), then \( \text{Stab}_{\text{Out}(F_N)}([T]) \) is finite.

Theorem 1.1 is established in Theorem 4.4 below, which actually provides a more detailed description of the stabilizer \( \text{Stab}_{\text{Out}(F_N)}([T]) \). We show in Theorem 4.4 that in \( \text{Stab}_{\text{Out}(F_N)}([T]) \) there is always a canonically defined finite normal subgroup \( P_T \triangleleft \text{Stab}_{\text{Out}(F_N)}([T]) \) such that either \( \text{Stab}_{\text{Out}(F_N)}([T]) = P_T \) or else \( \text{Stab}_{\text{Out}(F_N)}([T]) \) is a semi-direct product \( \text{Stab}_{\text{Out}(F_N)}([T]) = P_T \rtimes \mathbb{Z} \).

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We also show that, if the tree $T$ from Theorem 1.1 does not have dense $F_N$-orbits of branch points, then $\text{Stab}_{\text{Out}(F_N)}([T])$ is finite and thus equal to $P_T$. A result of Wang and Zimmermann [46] shows that every finite subgroup of $\text{Out}(F_N)$ has order $\leq N! \cdot 2^N$. Therefore Theorem 4.4 implies that, for $T \in \overline{\text{CV}}_N$ as in Theorem 1.1, the stabilizer $\text{Stab}_{\text{Out}(F_N)}([T])$ has a cyclic subgroup (either trivial or infinite cyclic) of index at most $N! \cdot 2^N$.

The study in [6] of $\text{Out}(F_N)$-stabilizers of particular points in compactified Outer space (corresponding to forward limiting $\mathbb{R}$-trees, explained below) was a starting point in the solution by Bestvina, Feighn and Handel of the Tits Alternative for $\text{Out}(F_N)$ [7, 8]. Their paper [6] was a substantial source of motivation for the present paper.

Theorem 1.1 can be viewed as a part of a general theme, which originates from Kleinian groups and from Teichmüller theory, that aims to investigate the dynamics of the action of elements or of subgroups of $\text{Out}(F_N)$ on the space $\overline{\text{CV}}_N$ (or on related spaces), and then to deduce algebraic information from the geometric data obtained. In this spirit, a very useful dynamic information about a group acting on a (compact) space is the fact that the action of certain group elements has North-South dynamics: By this we mean that there are precisely two fixed points (the two “poles”), and that every other point has the “north pole” as forward limit point and the “south pole” as backwards limit point. Even stronger implications are possible if the convergence is uniform on compact subsets (for a precise definition see Proposition 5.4 below). For example, it is known that a pseudo-Anosov mapping class has such a strong form of North-South dynamics on the Thurston compactification of the Teichmüller space of a hyperbolic surface.

For $\text{Out}(F_N)$ the analogous elements to pseudo-Anosov mapping classes are atoroidal iwip automorphisms, see Definition 5.1. In [40] it has been shown that their induced dynamics on compactified Outer space $\overline{\text{CV}}_N$ is precisely of this North-South type with uniform convergence on compact subsets.

The two “poles”, i.e. the fixed points $[T_+(\varphi)], [T_-(\varphi)] \in \overline{\text{CV}}_N$, of such an atoroidal iwip automorphism $\varphi \in \text{Out}(F_N)$ are given by $\mathbb{R}$-trees $T_+(\varphi)$ and $T_-(\varphi)$ for which the isometric action of $F_N$ is free, so that our Theorem 1.1 applies. There is also a fairly explicit way of understanding the forward limiting tree $T_+(\varphi)$ of $\varphi$ via train-track representatives of $\varphi$ (see [6, 9, 19]). As noted above, in [6] Bestvina, Feighn and Handel proved that if $\varphi \in \text{Out}(F_N)$ is an iwip (which they do not require to be atoroidal) then $\text{Stab}_{\text{Out}(F_N)}([T_+(\varphi)])$ is virtually cyclic.

Theorem 1.1 generalizes this result for atoroidal iwips. As noted below, we also recover the conclusion that $\text{Stab}_{\text{Out}(F_N)}([T_+(\varphi)])$ is virtually cyclic for iwips that are not atoroidal, via a direct reduction of that case to known facts in surface theory.

Note that Theorem 1.1 applies to a greater class of trees than the forward limit trees of atoroidal iwips. In particular, it is possible for a non-iwip to fix the projective class of a free $F_N$-tree. For example, if $\varphi \in \text{Out}(F_N)$ is an atoroidal iwip, then the “double” $\overline{\varphi} \in \text{Out}(F_{2N})$ of $\varphi$ is not an iwip but the forward limit tree of $\overline{\varphi}$ (obtained from doubling a train-track representative of $\varphi$ and then applying the same construction as in [19]) has a free $F_{2N}$-action and is projectively fixed by $\overline{\varphi}$.

In [6] Bestvina, Feighn and Handel also introduced the notion of a “stable lamination” $\Lambda^+_\varphi$ of an iwip $\varphi$, defined explicitly in terms of a train-track representative of $\varphi$. The main technical result (Theorem 2.14) of [6] states that for an iwip
\( \varphi \in \text{Out}(F_N) \) the \( \text{Out}(F_N) \)-stabilizer of \( \Lambda^+_\varphi \) is virtually cyclic. In \cite{35} we use Theorem \ref{1.1} to recover this result for atoroidal iwips via geodesic currents on \( F_N \) and the intersection form between \( \mathbb{R} \)-trees and currents (see \cite{32, 33, 34, 35, 13, 29, 30} for background information regarding geodesic currents on free groups).

In the case where \( \varphi \in \text{Out}(F_N) \) is an iwip which is not atoroidal, the action of \( F_N \) on \( T_+(\varphi) \) is not free. However, it is known \cite{9} that such \( \varphi \) must come from a pseudo-Anosov homeomorphism of a compact surface with a single boundary component. It turns out that in this case one can reduce the proof that \( \text{Stab}_{\text{Out}}(F_N)([T_+(\varphi)]) \) is virtually cyclic to known facts about mapping class groups (see Proposition \ref{5.7} below). Thus, Theorem \ref{1.1} and Proposition \ref{5.7} imply the following result originally established in \cite{6}.

**Corollary 1.2.** \cite{6} Let \( N \geq 2 \) and let \( \varphi \in \text{Out}(F_N) \) be an iwip. Then \( \text{Stab}_{\text{Out}}(F_N)([T_+(\varphi)]) \) is virtually cyclic.

The proofs in \cite{6} rely on exploiting the train-track machinery for elements of \( \text{Out}(F_N) \). Our proof of Theorem \ref{1.1} uses an alternative approach and uses the machinery of “homotheties” and “eigenrays” for trees projectively fixed by some elements of \( \text{Out}(F_N) \). We will now give a brief overview of our argument:

If \( \varphi \in \text{Stab}_{\text{Out}}(F_N)([T]) \) then \( T \) is \( F_N \)-equivariantly isometric to the tree \( \lambda(\varphi)T \Phi \) where \( \lambda(\varphi) > 0 \) is the “stretching factor” of \( \varphi \) and where \( \Phi \in \text{Aut}(F_N) \) is a lift of \( \varphi \) to \( \text{Aut}(F_N) \). This means that there exists a homothety \( H : T \to T \) with stretching factor \( \lambda(\varphi) \) such that for every \( g \in F_N \) and \( x \in T \) we have

\[
H(gx) = \Phi(g)H(x).
\]

Such homotheties \( H \) represent elements of the stabilizer \( \text{Stab}_{\text{Out}}(F_N)([T]) \), and they turn out to have a number of nice properties (compare \cite{19, 40, 22}), which are recalled below in section \ref{3}. In particular, if \( H \) fixes a branch point of \( T \) and a “direction” \( d \) at that branch point, then \( H \) possesses a well-defined “eigenray” \( \rho \) starting at \( x \) in direction \( d \) such that \( H(\rho) = \rho \), so that \( H \) acts on the ray \( \rho \) as multiplication by \( \lambda(\varphi) \) homothetic. The stretching-factor map \( \lambda : \text{Stab}_{\text{Out}}(F_N)([T]) \to \mathbb{R}_{>0} \) is a group homomorphism. To prove Theorem \ref{1.1} we show that the image of \( \lambda \) is cyclic and the kernel of \( \lambda \) is finite. The finite normal subgroup \( P_T \triangleleft \text{Stab}_{\text{Out}}(F_N)([T]) \) mentioned above is precisely the kernel of the homomorphism \( \lambda \):

\[
P_T = \text{Ker}(\lambda) = \{ \varphi \in \text{Stab}_{\text{Out}}(F_N)([T]) : \lambda(\varphi) = 1 \}.
\]

Thus \( P_T \) consists precisely of all those elements of \( \text{Stab}_{\text{Out}}(F_N)([T]) \) which are represented by isometries of \( T \).

\footnote{As far as we were able to understand, there seems to be a gap in the proof of the main technical result, Theorem 2.14, in \cite{6}. Namely, the arguments presented there seem insufficient for proving Proposition 2.6 (1) in \cite{6} for the case where, for example, \( \psi \in \text{Stab}(\Lambda) \) is a reducible polynomially growing automorphism. Specifically, the statement “Notice that all \( H_0 \)-segments are Nielsen (periodic) or else \( h \)-iteration will produce arbitrarily long leaf segments contained in \( H_0 \) contradicting quasiperiodicity” in the proof Proposition 2.6 (1) in \cite{6} is incorrect and a more involved substitute argument is required to complete the proof of Proposition 2.6 (1). The gap is fillable and a subsequent paper \cite{7} of Bestvina, Feighn and Handel gives an independent and more detailed proof of generalizations of the main results from \cite{6}, via more elaborate train track arguments. Also, Arnaud Hilion suggested to us a different direct way of patching the proof of Proposition 2.6 (1) in \cite{6}, via the improved relative train track methods developed in \cite{7}. Our paper presents an alternative argument for stabilizers of forward limiting trees of iwips, avoiding the train track machinery altogether.}
Our proof that the image of $\lambda$ is cyclic can be pushed through to work for the case of arbitrary very small tree $T \in \overline{\Gamma F N}$. However, the argument that $\text{Ker}(\lambda)$ is finite relies crucially on the fact that $T$ is a free $F N$-tree.

In particular, the conclusion of Theorem 1.1 fails if we allow very small trees with non-trivial point stabilizers and trivial arc stabilizers. For example, if $T$ is the Bass-Serre tree corresponding to a proper free product decomposition $F N = A \ast B$ (where both $A$ and $B$ are nontrivial and at least one of $A$ or $B$ is non-cyclic), then there are many automorphisms of $F N$ that preserve this free product decomposition and hence fix $T$ (and thus $\overline{T}$). Nevertheless, the proof of Theorem 1.1 provides an approach for understanding stabilizers $\text{Stab}_{\text{Out}}(F N)(\overline{T})$ for arbitrary $T \in \overline{\Gamma F N}$. Since the image of $\lambda$ is also cyclic here, the main task becomes to understand the structure of the kernel $\text{Ker}(\lambda)$.

As a consequence of Theorem 1.1 and of the “North-South” dynamics of atoroidal iwips on $\overline{\Gamma F N}$ we derive (see §5) without much effort a new proof of the Tits Alternative for subgroups of $\text{Out}(F N)$ that contain an arbitrary iwip (not necessarily atoroidal):

**Corollary 1.3.** Let $G \leq \text{Out}(F N)$ be a subgroup such that $G$ contains an iwip element $\varphi$. Then exactly only of the following occurs:

1. The group $G$ is virtually cyclic.
2. There exists $g \in G$ and $M \geq 1$ such that $\langle \varphi^M, g^{-1}\varphi^M g \rangle \leq G$ is free of rank two with free basis $\varphi^M, g^{-1}\varphi^M g$.

This result has been proved in [6]. The general case of the Tits alternative for subgroups of $\text{Out}(F N)$ has been established by Bestvina, Feighn and Handel in a series of deep papers [7, 8] using the improved relative train-track technology.

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2. **Preliminaries**

An $\mathbb{R}$-tree $T$ is a path-connected non-empty metric space, such that for any two points $x, y \in T$ there is a unique embedded arc $[x, y] \subseteq T$ which joins $x$ to $y$, and this arc is isometric to the interval $[0, d(x, y)] \subseteq \mathbb{R}$. All $\mathbb{R}$-trees in this paper are equipped with a (left) isometric action of the free group $F N$ of finite rank $N \geq 2$. Such an $\mathbb{R}$-tree $T$ is called **minimal** if there is no non-empty $F N$-invariant subtree $T' \subseteq T$ different from $T$.

For any element $w \in F N$ the **translation length** on $T$ is defined by

$$||w||_T = \inf_{x \in T} \{d(x, wx)\}.$$ 

This infimum is always attained, and in case where $||w||_T > 0$ the set of points $x \in T$ which realize $d(x, wx) = ||w||_T$ is isometric to $\mathbb{R}$ and is called the **axis** of $w$, denoted by $\text{Ax}(w)$. En element $w \in F N$ (or more precisely: the isometry $T \to T, x \mapsto wx$) is called **hyperbolic** if $||w||_T > 0$ and **elliptic** if $||w||_T = 0$.

An $\mathbb{R}$-tree $T$ with an isometric $F N$-action is called **small** if for any $x \neq y$ in $T$ the stabilizer $\text{Stab}_{F N}([x, y]) \subseteq F N$ is cyclic. The tree $T$ is **very small** if in addition no $w \in F N$ inverts a non-degenerate segment or fixes a non-degenerate tripod in $T$.

--2Gilbert Levitt has shown us that the fact, that the image of the map $\lambda$ is cyclic, can alternatively be derived from Theorem 4.3 of [41].
The following is well known (see, for example, \cite{14}).

**Lemma 2.1.** Let $T$ be an $\mathbb{R}$-tree equipped with a minimal nontrivial (i.e. without a global fixed point) isometric action of $F_N$. Then $T$ is equal to the union of the axes $Ax(w)$ for all hyperbolic $w \in F_N$. \hfill \Box

The unprojectivized Outer space $cv_N$ consists of all $\mathbb{R}$-trees equipped with a minimal free discrete isometric actions of $F_N$. Two such trees are considered as equal if there exists an $F_N$-equivariant isometry between them. The closure $\overline{cv}_N$ of $cv_N$ in the equivariant Gromov-Hausdorff convergence topology is known \cite{13,2} to consists of all very small minimal isometric actions of $F_N$ on $\mathbb{R}$-trees, where again two trees are considered to be equal if there exists an $F_N$-equivariant isometry between them. Although the trivial action of $F_N$ on a tree consisting of a single point can be realized as the limit of free and discrete $F_N$-trees, this action by convention is excluded from $\overline{cv}_N$, so that all points of $\overline{cv}_N$ represent non-trivial minimal actions of $F_N$.

There is a natural (right) action of $\text{Aut}(F_N)$ on $\overline{cv}_N$ that leaves $cv_N$ invariant. Namely, for $\Phi \in \text{Aut}(F_N)$ and $T \in \overline{cv}_N$, the point $T\Phi \in \overline{cv}_N$ is defined as follows. The tree $T\Phi$ is equal to $T$ as a metric space, but the action of $F_N$ is twisted via $\Phi$:

$$w \cdot x := \Phi(w) \cdot x$$

for any $w \in F_N, x \in T$.

It is easy to see that $\text{Inn}(F_N)$ is contained in the kernel of this action of $\text{Aut}(F_N)$ on $\overline{cv}_N$ and therefore the action factors through to the action of $\text{Out}(F_N)$ on $\overline{cv}_N$ as follows: for $T \in \overline{cv}_N$ and $\varphi \in \text{Out}(F_N)$ we have $T\varphi := T\Phi$ where $\Phi \in \text{Aut}(F_N)$ is any representative of $\varphi$.

The projectivization $\overline{CV}_N$ of $\overline{cv}_N$ is defined as $\overline{CV}_N = \overline{cv}_N / \sim$, where for $T_1, T_2 \in \overline{cv}_N$ we have $T_1 \sim T_2$ if there exists a constant $c > 0$ such that $T_1 = cT_2$. The latter condition means that there exists an $F_N$-equivariant isometry between $T_1$ and the tree $cT_2$, which is obtained from $T_2$ by multiplying the metric on $T_2$ by $c$, while using the same $F_N$-action as given on $T_2$. The $\sim$-equivalence class of $T \in \overline{cv}_N$ is denoted by $[T]$. The image of $cv_N$ in $\overline{CV}_N$ under the canonical projection map is denoted by $CV_N$. Thus $CV_N$ is the projectivization of $cv_N$ and $CV_N = \{[T] | T \in cv_N\}$.

The actions of $\text{Aut}(F_N)$ and $\text{Out}(F_N)$ respect the $\sim$-equivalence relation and hence they quotient through to actions on $\overline{CV}_N$: for $\varphi \in \text{Out}(F_N)$ and $[T] \in \overline{CV}_N$ we have $[T]\varphi := [T\varphi]$.

For $T \in \overline{cv}_N$ and $x \in T$ we define the valence $val(x)$ of $x$ to be the number of connected components of $T \setminus \{x\}$. These connected components themselves are called directions at $x$. We can also think of a direction at $x$ as an equivalence class of nondegenerate geodesic segments starting at $x$, where two such segments are equivalent if they have an overlap of positive length.

The following is well known and follows directly from the definition of an $\mathbb{R}$-tree:

**Lemma 2.2.** Let $T \in \overline{cv}_N$. Then for any two points $y, y'$ contained in the same direction $d$ at some point $x \in T$, the segments $[x, y]$ and $[x, y']$ intersect in a non-degenerate segment

$$[x, y] \cap [x, y'] = [x, z]$$

with $x \neq z$. \hfill \Box

Note that for every $x \in T$ we have $val(x) \geq 2$. Indeed, if $val(x) = 1$, then we can remove the $F_N$-orbit of $x$ from $T$ to get a proper $F_N$-invariant subtree,
contradicting the minimality of the action of \(F_N\) on \(T\). We say that \(x \in T\) is a branch point if \(\text{val}(x) \geq 3\).

Note that for \(T \in \overline{\mathbb{N}}\), the group \(F_N\) acts on the set of branch points and on the set of directions at branch points in \(T\). We will need the following useful fact:

**Theorem 2.3** (Gaboriau-Levitt [20]). Let \(T \in \overline{\mathbb{N}}\) be arbitrary. Then there are finitely many \(F_N\)-orbits of branch points, and only finitely many \(\text{Stab}_{F_N}(Q)\)-orbits of directions at any branch point \(Q\). In particular, there are only finitely many \(F_N\)-orbits of directions at branch points.

The result of Gaboriau-Levitt is actually much more specific, in that it gives an upper bound formula in terms of what they call the index of \(T\). For the case where the \(F_N\)-action on \(T\) is free, this formula reduces to the following:

**Corollary 2.4** (Gaboriau-Levitt [20]). Let \(T \in \overline{\mathbb{N}}\) be with free \(F_N\)-action. Then the following holds:

1. The number of \(F_N\)-orbits of branch points is bounded above by \(2N - 2\).
2. For each branch point \(Q \in T\), the number of directions at \(Q\) is bounded above by \(2N\).
3. The total number of \(F_N\)-orbits of directions at branch points in \(T\) is bounded above by \(6N - 6\).

### 3. Stretching factors, homotheties and eigenrays

In this section we present some of the basics of \(\mathbb{R}\)-trees with isometric action of a free group \(F_N\) of finite rank \(N \geq 2\). The material of this section is known to the experts, but it is a little scattered in the literature (see e.g. [19], [42]).

Recall here that a homothety with stretching factor \(\lambda > 0\) is a bijection between \(H : T \to T'\) metric spaces \(T\) and \(T'\) which satisfies \(d(Hx, Hy) = \lambda d(x, y)\) for any \(x, y \in T\).

**Definition 3.1** (Stretching factors and homotheties). Let \(T \in \overline{\mathbb{N}}\) and let \(\Phi \in \text{Aut}(F_N)\) be such that \([T]\Phi = [T]\), or equivalently, \(\Phi \in \text{Stab}_{\text{Aut}(F_N)}([T])\). Thus for some \(\lambda = \lambda(\Phi) > 0\), called the stretching factor of \(\Phi\), there exists an \(F_N\)-equivariant isometry between the trees \(\lambda T\) and \(T^\varphi\).

By definition of \(T\Phi\) this means that there is a homothety \(H = H_\varphi : T \to T\) with stretching factor \(\lambda\), such that

\[
H(wx) = \Phi(w)H(x) \quad \text{for any } x \in T, w \in F_N
\]

In this case we say that \(H\) is a homothety of \(T\) representing \(\Phi\).

If \(\varphi \in \text{Out}(F_N)\) is such that \([T]\varphi = [T]\), if \(\Phi \in \text{Aut}(F_N)\) is a lift of \(\varphi\) to \(\text{Aut}(F_N)\) and if \(H : T \to T\) is a homothety of \(T\) representing \(\Phi\), we will also say that \(H\) is a homothety of \(T\) representing \(\varphi\). We also put \(\lambda(\varphi) = \lambda(\Phi)\) in this case and call \(\lambda(\varphi)\) the stretching factor of \(\varphi\).

If \(T\Phi = \lambda_1 T = \lambda_2 T\) then \(\lambda_1 = \lambda_2\). The reason is that in this case, at the level of translation length functions, one has \(\| \cdot \|_{T\Phi} = \lambda_1 \| \cdot \|_{T} = \lambda_2 \| \cdot \|_{T}\) on \(F_N\). Thus for \(\Phi \in \text{Stab}_{\text{Aut}(F_N)}([T])\) the stretching factor \(\lambda(\Phi)\) is well defined. It is also easy to see that for \(\varphi \in \text{Stab}_{\text{Out}(F_N)}([T])\) the stretching factor \(\lambda(\varphi)\) is well-defined.

Note that a homothety of \(T\) with stretching factor \(\lambda = 1\) is an isometry of \(T\).

The following is well known:
Lemma 3.2. Let $T \in \mathcal{N}$ and let $H : T \to T$ be a homothety with stretching factor $\lambda \neq 1$ which represents some automorphism of $F_N$. Then:

1. The branch points are dense in $T$ (in fact, for every branch point its $F_N$-orbit is dense in $T$).
2. The stabilizer in $F_N$ of any non-degenerate segment is trivial.

Corollary 3.3. Let $T \in \mathcal{N}$ and $H : T \to T$ be as in Lemma 3.2. Then for any $x \neq y$ in $T$ the branch points of $T$ are dense in $[x, y]$.

Proof. Let $x \neq y$ be two points in $T$. Let $z \in [x, y]$ such that $z \neq x, z \neq y$ be arbitrary. We claim that there exist branch points of $T$ in $[x, y]$ that are arbitrary close to $z$.

Let $\varepsilon > 0$ be any such that $10\varepsilon < \min\{d(z, x), d(z, y)\}$. By Lemma 3.2 there exists a branch point $q$ of $T$ such that $d(z, q) \leq \varepsilon$. If $q \in [x, y]$, we are done. If $q \notin [x, y]$, let $q' \in [x, y]$ be the nearest point projection of $q$ to $[x, y]$. By the choice of $\varepsilon$ and of $q$ we see that $q' \neq x, q' \neq y$, $d(q', z) \leq \varepsilon$ and that $q'$ is a branch point of $T$. This establishes the claim and completes the proof. $\square$

The following is essentially an immediate corollary of the definitions:

Lemma 3.4. For any $T \in \mathcal{N}$ we have:

1. Let $\Phi_1, \Phi_2 \in \text{Stab}_{\text{Aut}(F_N)}([T])$ and let $H_1, H_2$ be homotheties of $T$ representing $\Phi_1$ and $\Phi_2$ accordingly. Then $H_1 H_2$ is a homothety representing $\Phi_1 \Phi_2$ and thus $\lambda(\Phi_1 \Phi_2) = \lambda(\Phi_1) \lambda(\Phi_2)$.
2. Suppose $H$ is a homothety representing $\Phi \in \text{Stab}_{\text{Aut}(F_N)}([T])$. Then for any $u \in F_N$ the homothety $uH$ represents $\Phi_1 = I_u \circ \Phi \in \text{Stab}_{\text{Aut}(F_N)}([T])$, where $I_u \in \text{Inn}(F_N)$ is the inner automorphism defined by $I_u(w) = uwu^{-1}$ for every $w \in F_N$.
3. Let $\Phi \in \text{Stab}_{\text{Aut}(F_N)}([T])$ and let $H$ be a homothety representing $\Phi$. Then $H^{-1}$ is a homothety representing $\Phi^{-1}$. $\square$

Lemma 3.4 implies that

$$\lambda : \text{Stab}_{\text{Aut}(F_N)}([T]) \to \mathbb{R}_{>0}$$

is a homomorphism to the multiplicative group $\mathbb{R}_{>0}$, and that for every $\Phi \in \text{Inn}(F_N)$ we have $\lambda(\Phi) = 1$.

Thus considered as a function on $\text{Stab}_{\text{Out}(F_N)}([T])$, the stretching factor map

$$\lambda : \text{Stab}_{\text{Out}(F_N)}([T]) \to \mathbb{R}_{>0}$$

is also a homomorphism.

It is easy to see that any homothety $H$ representing some $\Phi \in \text{Stab}_{\text{Aut}(F_N)}([T])$ takes geodesic segments to geodesic segments, preserves valences of points of $T$, takes branch points to branch points and directions at branch points to directions at branch points. Moreover, equation (1) of Definition 3.1 implies that $H$ acts by permutations on $F_N$-orbits of branch points and on $F_N$-orbits of directions at branch points.

Lemma 3.5. Let $T \in \mathcal{N}$ and let $H$ be a homothety representing some $\Phi \in \text{Stab}_{\text{Aut}(F_N)}([T])$ such that $H$ preserves every $F_N$-orbit of branch points and every $F_N$-orbit of directions at branch points. Suppose $H$ fixes a branch point $x$ of $T$.

If the $F_N$-action on $T$ is free, then $H$ fixes every direction at $x$. 
Proof. Suppose $d$ is a direction at $x$. Since $H$ fixes the $F_N$-orbit of every direction in $T$, we have $Hd = ud$ for some $u \in F_N$. Hence $ux = x$ and therefore $u = 1$, since the action of $F_N$ on $T$ is free. Thus $Hd = d$, as required. \hfill $\square$

**Lemma 3.6.** Let $T \in \mathcal{CN}$, and let $H$ be a homothety with stretching factor 1 (that is an isometry of $T$) that represents $\Phi \in \text{Stab}_{\text{Aut}(F_N)}([T])$. Then the following hold:

1. If $\Phi = \text{Id}_{F_N} \in \text{Aut}(F_N)$, then $H = \text{Id}_T$.
2. If $H$ represents $I_u \in \text{Inn}(F_N)$, then $H(x) = ux$ for every $x \in T$.

Proof. (1) Let $w \in F_N$ be a hyperbolic element. Then, since $\Phi = \text{Id}_{F_N} \in \text{Aut}(F_N)$, by (i) of Definition 3.4 we have $H\text{Ax}(w) = Hw\text{Ax}(w) = wH\text{Ax}(w)$. Thus $w$ preserves the line $H\text{Ax}(w)$, and therefore $H\text{Ax}(w) = \text{Ax}(w)$ since $\text{Ax}(w)$ is the only $w$-invariant line in $T$.

Since $H$ is an isometry of $T$, it is either hyperbolic or elliptic. If $H$ is a hyperbolic isometry, then it preserves a unique line in $T$, namely $\text{Ax}(H)$, which contradicts the fact that $H$ leaves invariant the axis of every hyperbolic element of $F_N$. Thus $H$ is elliptic. Hence for every hyperbolic element $w$ of $F_N$ the isometry $H$ either fixes pointwise or acts as a reflection on the axis of $w$. We claim that in fact $H$ fixes every axis of a hyperbolic element pointwise. If not, then there exists a hyperbolic element $w \in F_N$ such that $H$ acts as a reflection on the axis of $w$. Let $x_0 \in \text{Ax}(w)$ be the unique point of $\text{Ax}(w)$ fixed by $H$. Since $H$ represents $\text{Id}_{F_N}$, we have $H(wx_0) = wHx_0 = wx_0$, so that $H$ fixes $wx_0$. However, $wx_0 \in \text{Ax}(w)$ and $d(wx_0, x_0) = ||u||_T > 0$, so that $wx_0 \neq x_0$. This contradicts the fact that $H$ acts a reflection centered at $x_0$ on $\text{Ax}(w)$. Thus the claim is established and $H$ fixes every axis pointwise. Then it follows from Lemma 3.1 that $H = \text{Id}_T$.

(2) Suppose now that $H$ is an isometry of $T$ that represents the inner automorphism $I_u \in \text{Aut}(F_N)$. Then, by Lemma 3.4 $u^{-1}H$ is an isometry that represents $\text{Id}_{F_N} \in \text{Aut}(F_N)$. Therefore, by part (1), $u^{-1}H = \text{Id}_T$, so that $H(x) = ux$ for every $x \in T$, as required. \hfill $\square$

**Corollary 3.7.** For any $T \in \mathcal{CN}$ we have:

1. Let $\Phi \in \text{Stab}_{\text{Aut}(F_N)}([T])$ and let $H_1, H_2$ be two homotheties of $T$ representing $\Phi$. Then $H_1 = H_2$.
2. Let $\varphi \in \text{Stab}_{\text{Out}(F_N)}([T])$ and let $H_1, H_2$ be two homotheties of $T$ representing $\varphi$. Then there is $u \in F_N$ such that $uH_1 = H_2$.

Proof. (1) Both $H_1$ and $H_2$ are $F_N$-equivariant isometries between the trees $T$ and $\chi(\Phi)^{-1}T\Phi$.

Hence, by Lemma 3.4 (2), the map $H := H_2^{-1}H_1 : T \to T$ is an $F_N$-equivariant isometry of $T$, that is $H(wx) = wH(x)$ for every $w \in F_N$ and every $x \in T$.

In particular $H$ is an isometry of $T$ representing the identity $\text{Id}_{F_N} \in \text{Aut}(F_N)$. Part (1) of Lemma 3.6 implies that $H = \text{Id}_T$, so that $H_1 = H_2$, as required.

(2) There are representatives $\Phi_1, \Phi_2 \in \text{Aut}(F_N)$ of $\varphi$ such that $H_1$ represents $\Phi_1$ and $H_2$ represents $\Phi_2$. Since $\Phi_1, \Phi_2$ both represent $\varphi$, there is $u \in F_N$ such that $\Phi_2(g) = u\Phi_1(g)u^{-1}$ for every $w \in F_N$.

Therefore by Lemma 3.4 the isometry $uH_1$ represents $\Phi_2$. Hence by part (1) we have $uH_1 = H_2$, as required. \hfill $\square$
Remark 3.8. Part (2) of Corollary 3.7 and Lemma 3.4 imply that if $\varphi \in \text{Stab}_{\text{Out}(F_N)}([T])$ and $H$ is a homothety representing $\varphi$ then a homothety $H'$ represents $\varphi$ if and only if $H'$ has the form $H' = uH$ where $u \in F_N$.

Convention 3.9 (Subgroup $K_T$). Let $T \in \overline{\text{wN}}$. As we have already observed, any homothety $H$ representing some $\varphi \in \text{Stab}_{\text{Out}(F_N)}([T])$ acts by permutations on $F_N$-orbits of branch points and on $F_N$-orbits of directions at branch points. Let $D$ be the set of $F_N$-orbits of directions at branch points of $T$. Thus there is a natural homomorphism from $\text{Stab}_{\text{Out}(F_N)}([T])$ to the group $\text{Sym}(D)$ of permutations of $D$. We denote by $K_T$ the kernel of this homomorphism.

By Theorem 2.3 the set $D$ is finite, with upper bound to its cardinality given by Corollary 2.4 (c), so that we we obtain:

Corollary 3.10. Let $T \in \overline{\text{wN}}$. Then the subgroup $K_T \subseteq \text{Stab}_{\text{Out}(F_N)}([T])$ is of finite index.

From now on we will restrict most of our attention to automorphisms in $K_T$.

Remark 3.11. If $\varphi \in K_T$, $x \in T$ is a branch point and $H'$ is a homothety representing $\varphi$, we can always choose another homothety $H = wH'$ representing $\varphi$, for some $w \in F_N$, such that $H$ fixes $x$. Namely, because $H'$ preserves the $F_N$-orbit of $x$, there is $u \in F_N$ such that $H'(x) = ux$. Then $H = u^{-1}H'$ fixes $x$, as required.

Definition 3.12 (Eigenray). Let $T \in \overline{\text{wN}}$ and let $H$ be a homothety representing some automorphism $\varphi \in \text{Stab}_{\text{Out}(F_N)}([T])$ with $\lambda(\varphi) > 1$. A (closed) geodesic ray $\rho \subseteq T$, which starts at some point $x \in T$ such that $\rho - \{x\}$ is contained in a direction $d$ at $x$, is called an eigenray of $H$ at $x$ in the direction $d$, if one has:

$$H(\rho) = \rho.$$ 

In this case it follows that $H(x) = x$ and $H(d) = d$. Furthermore, note that $H$ acts on $\rho$ as multiplication by $\lambda$ acts on $\mathbb{R}_{\geq 0}$.

Proposition 3.13. Let $T \in \overline{\text{wN}}$ and let $H$ be a homothety representing some $\varphi \in \text{Stab}_{\text{Out}(F_N)}([T])$ with $\lambda(\varphi) > 1$. Let $d$ be a direction at $x \in T$, and assume that $H(x) = x$ and $H(d) = d$.

Then there exists a unique eigenray $\rho = \rho_d$ in $T$ which starts at $x$ in direction $d$.

Proof. Since $H(d) = d$, it follows for any point $y \in d$ that the segments $[x,y]$ and $H[x,y] = [x,H(y)]$ overlap non-trivially, by Lemma 2.2. Consider a point $z \neq x$ in $[x,y] \cap H[x,y]$. Then $[x,z]$ is a subsegment of $H[x,z] = [x,H(z)] \subseteq [x,H(y)] \cap [x,H^2(y)]$, and the infinite nested union $\bigcup \{[x,H^n(z)] \mid n \in \mathbb{N}\}$ forms a ray $\rho$ which by construction satisfies $\rho = H(\rho)$, i.e. it is an eigenray at $x$ in the direction $d$.

The uniqueness of $\rho$ follows from the fact that another such eigenray $\rho'$ must have (by Lemma 2.2) a non-degenerate initial segment in common with $\rho$, but the bifurcation point (that is, the endpoint of the maximal common initial segment) $y$ must have $H$-image contained in both, $\rho$ and $\rho'$ (by the $H$-invariance of either). Since $d(x,H(y)) = \lambda(\varphi)d(x,y) > d(x,y)$, this yields a contradiction to the above definition of the point $y$. □

The following proposition shows how eigenrays transform under the action of elements of $F_N$ and under the action by other homotheties. The statement of this proposition is known (see [19]) but we present a proof for completeness:
Proposition 3.14. Let $T \in \mathcal{T}_N$ and let $H$ be a homothety that represents some $\Phi \in \text{Aut}(F_N)$ such that the outer automorphism $\varphi \in K_T$ class $\varphi$ of $\Phi$ belongs to $K_T$, and such that $\lambda = \lambda(\varphi) > 1$. Suppose that $H$ fixes a branch point $x \in T$. Let $\rho = \rho_d$ be the eigenvary of $H$ in a direction $d$ at $x$. Then the following hold:

1. Assume that the $F_N$-action on $T$ is free. Suppose that $H' = wH$, for some $w \in F_N$, is such that $H'$ fixes a branch point $x'$, with $v \in F_N$. Then $H'$ fixes the direction $d' = vd$ at $x'$, and $v\rho_d$ is the eigenvary of $H'$ in the direction $d'$. Furthermore, in this case $w = v \Phi(v)^{-1}$.

2. Let $H'$ be a homothety of $T$ which represents some element $\psi$ of $\text{Stab}_{\text{Out}}(F_N)([T])$. Then $H_1 = H'H^{-1}$ is another homothety that represents $\psi \varphi \psi^{-1} \in K_T$, the stretching factors of $H$ and $H_1$ are equal, $H_1$ fixes the point $H'(x)$ and the direction $H'(d)$ at $H'(x)$ and, moreover, $H'(\rho)$ is the eigenvary of $H_1$ at $H'(x)$ in the direction $H'(d)$.

3. Let $H'$ be another homothety of $T$ which represents some element $\psi$ of $K_T$ with $\lambda = \lambda(\psi) > 1$ such that $H'$ fixes $x$ (and hence $H'$ fixes every direction at $x$). Let $\rho' = \rho'_d$ be the eigenvary of $H'$ in direction $d$ and let $C > 0$ be the length of the maximal common initial segment of $\rho_d$ and $\rho'_d$.

Then the homothety $H'' = HH'H^{-1}$ fixes $x$ and represents $\varphi \psi \varphi^{-1}$. Let $\rho'' = \rho''_d$ be the eigenvary of $H''$ in direction $d$. Then $\rho$ and $\rho''$ have a common initial segment of length $\lambda(\varphi)C$.

Proof. 1. Since $H' = wH$, it follows that $H'$ also represents $\varphi \in K_T$, so that $H'$ fixes every $F_N$-orbit of directions of $T$. Thus, since by assumption the action of $F_N$ on $T$ is free, Lemma 3.3 implies that $H'$ fixes any direction at its fixed point $x'$, so that we have $H'(d') = d'$.

We deduce:

$$d' = H'(d') = H'(vd) = wHv(d) = w\Phi(v)H(d) = w\Phi(v)d = w\Phi(v)v^{-1}d'.$$

Thus the isometry of $T$ given by the action of the element $w\Phi(v)v^{-1}$ fixes the direction $d'$ and therefore it fixes the initial point $x' = vx$ of $d'$. Since the action of $F_N$ on $T$ is free, it follows that $w\Phi(v)v^{-1} = 1$ in $F_N$, so that $w = \psi(v)^{-1}$.

Then:

$$H'(v\rho_d) = wHv(\rho_d) = w\Phi(v)H(\rho_d) = vH(\rho_d) = v\rho_d,$$

which shows that $v\rho_d$ is the eigenvary of $H'$ at $x'$ in the direction of $d'$.

2. We note that

$$H_1(H'(\rho)) = H'H'H^{-1}H'(\rho) = H'H(\rho) = H'(\rho).$$

Thus $H_1(H'(\rho)) = H'(\rho)$ which, since $H_1$ is a homothety and $H'(\rho)$ is a ray at $H'(x)$ in the direction $H'(d)$, implies the statement of part (2) of the proposition.

3. This is a direct consequence of part (2). The length estimate for the common initial segment of $\rho$ and $\rho''$ follows from that for $\rho$ and $\rho'$ and from the stretching property of the homothety $H$.

4. Proof that stabilizers of projectivized free $F_N$-trees are virtually cyclic

Our strategy for the proof that $\text{Stab}_{\text{Out}}(F_N)([T])$ is virtually cyclic if $T \in \mathcal{T}_N$ is a free $F_N$-tree will be to show that the map

$$\lambda : \text{Stab}_{\text{Out}}(F_N)([T]) \to \mathbb{R}_{>0}$$
has image $Im(\lambda) \subseteq \mathbb{R}_{>0}$ which is cyclic, and that the kernel $Ker(\lambda) \subseteq Stab_{Out(F_N)}([T])$ is finite. This will imply that $Stab_{Out(F_N)}([T])$ is virtually cyclic.

Recall from Convention 3.9 that an outer automorphism $\varphi \in Out(F_N)$ belongs to the finite index normal subgroup $K_T \leq Stab_{Out(F_N)}([T])$ if and only if any homothety $H$ representing $\varphi$ preserves every $F_N$-orbit of branch points in $T$, as well as every $F_N$-orbit of directions at a branch point in $T$.

**Proposition 4.1.** Let $T \in \mathbb{TN}$, and suppose that the $F_N$-action on $T$ is free. Let $H$ be an isometry of $T$ that represents some $\Phi \in K_T \subseteq Stab_{Aut(F_N)}([T])$. Suppose that $H$ fixes a branch point $x_0$ of $T$. Then $H = Id_T$ and $\Phi = Id_{F_N}$.

**Proof.** We define $T_0 := Fix(H)$ to be the set of fixed points of $H$. Thus $T_0 \subseteq T$ is a closed subtree of $T$ which contains $x_0$. We claim that $T = T_0$. Indeed, by way of contradiction let us suppose $T_0 \neq T$. Then there exists a point $y \in T_0 \cap T - T_0$.

If $val(y) = 2$, there are precisely two distinct directions at $y$, one of which, denoted by $d_1$, can be represented by a non-degenerate segment $[y, z_1]$ contained in $T_0$, while the other, denoted by $d_2$, is represented by a non-degenerate segment $[y, z_2]$, and $y$ can be approximated by points in $[y, z_2]$ different from $y$, i.e. points of $T - T_0$. Since $H$ fixes $[y, z_1]$ pointwise and thus $H(d_1) = d_1$, it follows that we must have $H(d_2) = d_2$. Since $H$ is an isometry, it follows from Lemma 2.2 that $H$ fixes pointwise a non-degenerate initial segment of $[y, z_2]$, which contradicts the choice of $[y, z_2]$.

Suppose now that $val(y) \geq 3$, so that $y$ is a branch point of $T$. By Lemma 3.3 the homothety $H$ fixes every direction at $y$. Therefore, since $H$ is an isometry, $H$ fixes a non-degenerate initial segment of every direction at $y$. This contradicts the fact that $y \notin T - T_0$.

Thus indeed, $T = T_0$, so that $H = Id_T$. Using the fact that $H$ represents $\Phi$, we get

$$wx = H(wx) = \Phi(w)H(x) = \Phi(w)x$$

for every $x \in T, w \in F_N$.

Since the action of $F_N$ on $T$ is free, this implies that $w = \Phi(w)$ for every $w \in F_N$. Thus $\Phi = Id_{F_N}$ as claimed. \(
\)

Recall from Lemma 3.4 and the subsequent discussion that

$$\lambda : Stab_{Out(F_N)}([T]) \to \mathbb{R}_{>0}$$

is the stretching factor homomorphism.

**Proposition 4.2.** If the $F_N$-action on $T \in \mathbb{TN}$ is free, then

$$Ker(\lambda|_{K_T}) = \{1_{Out(F_N)}\}.$$ 

**Proof.** Suppose $\varphi \in Ker(\lambda|_{K_T})$, that is $\varphi \in K_T$ and $\lambda(\varphi) = 1$. This means that every homothety $H$ representing $\varphi$ is actually an isometry of $T$.

Recall that, since $\varphi \in K_T$, every homothety which represents $\varphi$ acts as an identity permutation on the set of $F_N$-orbits of branch points and directions at branch points in $T$. By Remark 3.11 we can find a lift $\Phi$ of $\varphi$ to $Aut(F_N)$ and a homothety $H$ representing $\Phi$ such that $H$ fixes some branch point of $T$.

Thus we can apply Proposition 4.1 to obtain that $H = Id_T$ and $\Phi = Id_{F_N}$.

Therefore $\varphi$ is the trivial outer automorphism, and $Ker(\lambda|_{K_T}) = \{1_{Out(F_N)}\}$ as required. \(\square\)

**Proposition 4.3.** Let $T \in \mathbb{TN}$, and assume that the $F_N$-action on $T$ is free.

Then the set $\lambda(K_T) \subseteq \mathbb{R}_{>0}$ is a cyclic subgroup of the multiplicative group $\mathbb{R}_{>0}$. 

Proof. Suppose, on the contrary, that the subgroup $\lambda(K_T) \subseteq \mathbb{R}_{>0}$ is not cyclic. Since $(\mathbb{R}_{>0}, \cdot)$ is isomorphic to $(\mathbb{R}, +)$, it follows that a subgroup of $(\mathbb{R}_{>0}, \cdot)$ is either discrete and cyclic or else it is dense in $\mathbb{R}_{>0}$. Thus the subgroup $\lambda(K_T) \subseteq \mathbb{R}_{>0}$ is dense.

Therefore we can find a sequence $\psi_i \in K_T$ such that $\lambda(\psi_i) \in [2, 2.001]$ are all distinct and $\lim_{i \to \infty} \lambda(\psi_i) = 2$. Since there are only finitely many $F_N$-orbits of branch points and directions at branch points, by part (3) of Proposition 3.14 we can find $\varphi_1 = \psi_1$ and $\varphi_i = \psi_i^{n_i} \psi_1 {n_i}^{-1}$ for $i \geq 2$, with the following property:

Whenever $x$ is a branch point of $T$, $d$ is a direction at $x$, $\hat{H}$ are homotheties fixing $x$ and representing $\varphi_i$ and $\hat{\rho}_i$ are their eigenrays in the direction $d$ then for every $i \geq 2$ the rays $\hat{\rho}_i$ and $\hat{\rho}_i$ have an overlap of length at least 100.

Note that $\lambda(\varphi_i) = \lambda(\psi_i)$ for every $i \geq 1$.

Now choose a branch point $Q$ of $T$ and a direction $d$ at $Q$. For every index $i \geq 1$ let $H_i$ be the homothety representing $\varphi_i$ and fixing $Q$ and let $\rho_i$ be its eigenray in the direction $d$. Let $Q'$ be the point at distance 100 from $Q$ on $\rho_i$. By assumption on $\psi_i$ we have $[Q, Q'] \subseteq \rho_i$ for all $i \geq 1$.

Since $T$ is a free $F_N$-tree with dense orbits, branch points are dense in every non-degenerate segment in $T$, particularly in $[Q, Q']$. Let $P \in [Q, Q']$ be a branch point such that $0 < d(Q, P) < 10$.

By Corollary 3.3 such a point $P$ always exists since we assumed that $\lambda(K_T) \neq \{1\}$ and thus branch points of $T$ are dense in every nondegenerate segment of $T$.

Let $S \in [Q, Q']$ be such that $d(Q, S) = 11$.

We have $H_i([P, S]) \subseteq [Q, Q']$ for every $i \geq 1$, since the stretching factors $\lambda(\varphi_i)$ belong to the interval $[2, 2.01]$. Let $d_1$ be the direction at $P$ determined by $[P, Q']$.

Let $H'_i$ be the homotheties representing $\varphi_i$ and fixing $P$ and let $\rho'_i$ be their eigenrays in the direction $d_1$. Note that $\rho'_i$ has an overlap of positive length $\geq c > 0$ for some $0 < c \leq 1$ with $[P, Q']$. Since each $\rho'_i$ has overlap of length $\geq 100$ with $\rho_i$, it follows that each $\rho'_i$ has overlap of length $\geq c$ with $[P, Q']$. Put $P_i = H_iP \in [P, Q']$, and let $d_2$ be the direction at $P_i$ determined by $[P_i, Q']$.

Since $S \in d_1$ and $H_i([P, S]) \subseteq [Q, Q']$, it follows that $H_i(d_1) = d_2$. Since $H_i$ preserves $F_N$-orbits of branch points and of directions at branch points, it follows that for every $i \geq 1$ there is $w_i \in F_N$ such that $P_i = w_iP$ and $d_2 = w_i d_1$.

For every $i \geq 1$ let $H''_i$ be the homothety representing $\varphi_i$ and fixing $P_i$. Let $\rho''_i$ be the eigenray of $H''_i$ in the direction $d_2$.

Then part (1) of Proposition 3.14 implies that $\rho''_i = w_i \rho'_i$. On the other hand, recall that $H'_i$ has the form $H'_i = v_i H_i$ for some $v_i \in F_N$.

Consider the ray $H_i(\rho'_i)$. Part (2) of Proposition 3.14 implies that $H_i(\rho'_i)$ is the eigenray of the homothety $H_iH'_iH_i^{-1}$ that fixes $H_i(P) = P_i$ and in the direction $H_i(d_1) = d_2$.

Also, $H_iH'_iH_i^{-1} = H_i\Phi_i v_i H_i^{-1} = H_i v_i = \Phi_i(v_i) H_i$ so that this homothety represents $\varphi_i$. Since furthermore it fixes $P_i$, it must be equal to $H''_i$. It follows that $H_i(\rho'_i) = \rho''_i = w_i \rho'_i$.

Note that by construction the ray $H_i(\rho'_i)$ has overlap of length $\geq 2c$ with $[P_i, Q']$. Let $Z \in [P, Q']$ be such that $d(P, Z) = c$. Then we have $w_i[P, Z] \subseteq [P, Q]$ for every $i \geq 1$.

It follows that $w_j^{-1} w_i$, as $i, j \to \infty$ almost fixes $[P, Z]$ and acts on a fixed non-degenerate subsegment $J$ of the interior of $[P, Z]$ with positive translation length.
which tends to zero. This easily leads to a contradiction with the fact that the action of $F_N$ on $T$ is free. Indeed, the points $P_i = H_i P$ converge to the point $P_\infty \in [Q, Q]$ where $d(Q, P_\infty) = 2d(Q, P)$. Hence the commutators $[w_j^{-1}w_i, w_k^{-1}w_l]$ fix $J$ pointwise as $i, j, k, l \to \infty$ and therefore $[w_j^{-1}w_i, w_k^{-1}w_l] = 1$. Since $F_N$ is free, it follows that $w_j^{-1}w_i$ belong to a common maximal cyclic subgroup $\langle g \rangle \leq F_N$. However, this contradicts the fact that $w_j^{-1}w_i$ can be made to have arbitrarily small positive translation length.

We can now prove the main result of this section. We define:

$$P_T = \{ \varphi \in \text{Stab}_{\text{Out}(F_N)}([T]) \mid \lambda(\varphi) = 1 \}$$

**Theorem 4.4.** Let $T \in \overline{\text{w}_N}$ be a tree with free $F_N$-action. Then we have:

(a) The group $P_T$ is finite, and the stabilizer $\text{Stab}_{\text{Out}(F_N)}([T])$ is virtually cyclic. Moreover:

$$\text{Stab}_{\text{Out}(F_N)}([T]) = \begin{cases} P_T & \text{if } \text{Stab}_{\text{Out}(F_N)}([T]) \text{ is finite,} \\ P_T \rtimes \mathbb{Z} & \text{if } \text{Stab}_{\text{Out}(F_N)}([T]) \text{ is infinite} \end{cases}$$

(b) If $T$ does not have dense $F_N$-orbits of branch-points, then $\text{Stab}_{\text{Out}(F_N)}([T])$ is finite.

**Proof.** (a) Let $K_T \subseteq \text{Stab}_{\text{Out}(F_N)}([T])$ be a normal subgroup of finite index chosen as in Convention 4.4 and recall that $\lambda : \text{Stab}_{\text{Out}(F_N)}([T]) \to \mathbb{R}_{>0}$ is the stretching factor homomorphism. Then we know from Proposition 4.2 that $\text{Ker}(\lambda|_{K_T}) = \{1\}$. Moreover, Proposition 4.3 implies that $\lambda(K_T) \subseteq \mathbb{R}_{>0}$ is cyclic. Therefore $\text{Stab}_{\text{Out}(F_N)}([T])$ is virtually cyclic.

Since by Corollary 4.10 the subgroup $K_T$ has finite index in $\text{Stab}_{\text{Out}(F_N)}([T])$, we see that $P_T$ contains $\text{Ker}(\lambda|_{K_T}) = \{1\}$ as a subgroup of finite index and thus $P_T$ is finite.

Also, since every $\varphi \in \text{Stab}_{\text{Out}(F_N)}([T])$ with $\lambda(\varphi) \neq 1$ has infinite order, we conclude that $P_T = \text{Stab}_{\text{Out}(F_N)}([T])$ if and only if $\text{Stab}_{\text{Out}(F_N)}([T])$ is finite, that is, if and only if $\lambda(\text{Stab}_{\text{Out}(F_N)}([T])) = \{1\}$.

Additionally, $P_T \triangleleft \text{Stab}_{\text{Out}(F_N)}([T])$ is normal in $\text{Stab}_{\text{Out}(F_N)}([T])$. Thus if $\lambda(\text{Stab}_{\text{Out}(F_N)}([T])) \neq \{1\}$, then $\lambda(\text{Stab}_{\text{Out}(F_N)}([T]))$ is infinite cyclic and for any $\varphi \in \text{Stab}_{\text{Out}(F_N)}([T])$ mapped by $\lambda$ to the generator of $\lambda(\text{Stab}_{\text{Out}(F_N)}([T]))$ we have $P_T \cap \langle \varphi \rangle = \{1\}$. Therefore $\text{Stab}_{\text{Out}(F_N)}([T]) = P_T \rtimes \langle \varphi \rangle$ in this case.

(b) Suppose now that $T$ is a minimal free $F_N$-tree which does not have dense orbits of branch-points. We claim that $\lambda(K_T) = \{1\}$ in this case. Indeed, suppose not. Then there exists $\varphi \in K_T$ with $0 < \lambda(\varphi) < 1$. Let $\Phi \in \text{Aut}(F_N)$ be a lift of $\varphi$ to $\text{Aut}(F_N)$ and let $H$ be a $\lambda(\varphi)$-homothety representing $\Phi$.

Since $T$ does not have dense $F_N$-orbits of branch-points, the canonical simplicial metric quotient tree $T/\overline{\text{T}_i}$, obtained from $T$ by contracting every maximal subtree $\overline{\text{T}_{i_{\text{dense}}}}$, where an $F_N$-orbit is dense, is a non-trivial $\mathbb{R}$-tree with isometric $F_N$-action that has trivial stabilizers (see [39] for more details). Hence it has only finitely many orbits of edges. On the other hand, the union of maximal open nondegenerate segments in $T$, whose interiors do not contain any branch point, is mapped by the $F_N$-equivariant quotient map $T \to T/\overline{\text{T}_{i_{\text{dense}}}}$, injectively to the union of open edges in $T/\overline{\text{T}_{i_{\text{dense}}}}$. Hence in $T$ there exist only finitely many $F_N$-orbits of maximal closed nondegenerate segments whose interiors do not contain
any branch points, and moreover, there is at least one such segment. In particular, the lengths of maximal closed nondegenerate segments in $T$, whose interiors do not contain any branch point, are bounded below by some constant $c > 0$. Choose a maximal nondegenerate segment $[a, b] \subseteq T$ whose interior does not contain any branch points of $T$. Since $H$ is a homothety of $T$, for every $n \geq 1$ the segment $H^n[a, b]$ has the same property: it is a maximal closed nondegenerate segments in $T$ whose interiors do not contain any branch points. However, the length of $H^n[a, b]$ is $\lambda^n(\varphi)d(a, b)$ which converges to 0 as $n \to \infty$, yielding a contradiction.

Thus indeed $\lambda(K_T) = \{1\}$. Since we already know that $\ker(\lambda|_{K_T}) = \{1\}$, it follows that $K_T = \{1\}$. Since $K_T$ has finite index in $\text{Stab}_{\text{Out}(F_N)}([T])$, we conclude that $\text{Stab}_{\text{Out}(F_N)}([T])$ is finite, as required. This completes the proof of the theorem.

As pointed out in the Introduction, since by a result of [46] every finite subgroup of $\text{Out}(F_N)$ has order at most $N!2^N$, it follows that in Theorem 4.4 we have $|P_T| \leq N!2^N$, so that $\text{Stab}_{\text{Out}(F_N)}([T])$ always has a cyclic subgroup (possibly trivial) of index at most $N!2^N$.

Remark 4.5. (a) The statement of Proposition 4.3 holds also in the case where the $F_N$-action on $T$ is only very small and not necessarily free. The proof follows the same lines, but gets technically a little more involved.

(b) The conclusion of Theorem 4.4 however, becomes wrong, if one omits the hypothesis that the $F_N$-action on $T$ is free. Easy counterexamples are provided for example by simplical trees $T$ with trivial edge stabilizers and large vertex stabilizers: As those are automatically free factors, one has many automorphisms which act non-trivially on some of the vertex groups, but leave invariant the free product structure that is realized by $T$.

5. TITS ALTERNATIVE FOR DYNAMICALLY LARGE SUBGROUPS OF $\text{Out}(F_N)$

In this section we apply Theorem 4.4 to give a new proof of the Tits alternative for subgroups of $\text{Out}(F_N)$ which contain an iwip automorphism.

Definition 5.1. (a) An outer automorphism $\varphi \in \text{Out}(F_N)$ is called irreducible with irreducible powers (iwip) if no positive power of $\varphi$ preserves the conjugacy class of a proper free factor of $F_N$.

(b) An outer automorphisms $\varphi \in \text{Out}(F_N)$ is called atoroidal if it has no non-trivial periodic conjugacy classes, i.e. if there do not exist $t \geq 1$ and $w \in F_N - \{1\}$ such that $\varphi^t$ fixes the conjugacy class $[w]$ of $w$ in $F_N$.

It was proved in [9] that for $N \geq 2$ an iwip automorphisms $\varphi \in \text{Out}(F_N)$ is non-atoroidal if and only if $\varphi$ is induced, via an identification of $F_N$ with the fundamental group of a compact surface $S$ with a single boundary component, by a pseudo-Anosov homeomorphisms $h : S \to S$.

Remark 5.2. The terminology “iwip” derives from the groundbreaking paper [9]: Bestvina-Handel call an element $\varphi \in \text{Out}(F_N)$ is reducible if there exists a free product decomposition $F_N = C_1 \ast \ldots \ast C_k \ast F'$, where $k \geq 1$ and $C_i \neq \{1\}$, such that $\varphi$ permutes the conjugacy classes of subgroups $C_1, \ldots, C_k$ in $F_N$. An element $\varphi \in \text{Out}(F_N)$ is called irreducible if it is not reducible.
It is not hard to see that an element \( \varphi \in \text{Out}(F_N) \) is an iwip if and only if for every \( n \geq 1 \) the power \( \varphi^n \) is irreducible (sometimes such automorphisms are also called \textit{fully irreducible}).

It is known by a result of Levitt and Lustig [40] that iwips have a simple “North-South” dynamics on the compactified Outer space \( \overline{CV}_N \):

\textbf{Proposition 5.3.} [40] Let \( \varphi \in \text{Out}(F_N) \) be an iwip. Then there exist unique \( [T_+] = [T_+(\varphi)], [T_-] = [T_-(\varphi)] \in \overline{CV}_N \) with the following properties:

1. The elements \( [T_+], [T_-] \in \overline{CV}_N \) are the only fixed points of \( \varphi \) in \( \overline{CV}_N \).
2. For any \( [T] \in \overline{CV}_N \), \( [T] \neq [T_-] \) we have \( \lim_{n \to \infty} [T \varphi^n] = [T_+] \) and for any \( [T] \in \overline{CV}_N \), \( [T] \neq [T_+] \) we have \( \lim_{n \to \infty} [T \varphi^{-n}] = [T_-] \).
3. We have \( T_+ \varphi = \lambda_+ T \) and \( T_- \varphi^{-1} = \lambda_- T_- \) where \( \lambda_+ > 1 \) and \( \lambda_- > 1 \). Moreover \( \lambda_+ \) is the Perron-Frobenius eigenvalue of any train-track representative of \( \varphi \) and \( \lambda_- \) is the Perron-Frobenius eigenvalue of any train-track representative of \( \varphi^{-1} \).

In [40] it is also proved that convergence in (2) is locally uniform and hence uniform on compact subsets. Alternatively, we proved in [35], using geodesic currents and the intersection form, that pointwise North-South dynamics for the action of an atoroidal iwip \( \varphi \) on \( \overline{CV}_N \) and on \( \text{Curr}(F_N) \) already implies that the convergence in part (2) of Proposition 5.3 is uniform on compact subsets.

We give a precise statement:

\textbf{Proposition 5.4.} [40] [35] Let \( \varphi \in \text{Out}(F_N) \) be an iwip, and let \( [T_+] = [T_+(\varphi)] \) be as in Proposition 5.3.

Then for any compact subset \( K \subseteq \overline{CV}_N - [T_-] \) and any neighborhood \( U \) of \( [T_+] \) there exists \( M \geq 1 \) such that for every \( n \geq M \) we have \( K \varphi^n \subseteq U \).

We can now show:

\textbf{Corollary 5.5.} Let \( \varphi, \psi \in \text{Out}(F_N) \) be atoroidal iwips such that \( [T_+(\varphi)], [T_-(\psi)] \) are four distinct points in \( \overline{CV}_N \). Then there exists \( M \geq 1 \) such that for any \( m,n \geq M \) the subgroup \( \langle \varphi^m, \psi^n \rangle \leq \text{Out}(F_N) \) is free of rank two with free basis \( \varphi^m, \psi^n \).

\textbf{Proof.} In \( \overline{CV}_N \) choose disjoint open neighborhoods \( U_+, U_-, V_+, V_- \) of \( [T_+(\varphi)], [T_-(\varphi)], [T_+(\psi)], [T_-(\psi)] \) respectively. By Proposition 5.4 there exists \( M \geq 1 \) such that for every \( n \geq M \) we have \( (\overline{CV}_N - U_-) \varphi^n \subseteq U_+, (\overline{CV}_N - V_-) \psi^n \subseteq V_+, (\overline{CV}_N - U_+ ) \varphi^{-n} \subseteq U_-, (\overline{CV}_N - V_+) \psi^{-n} \subseteq V_- \). Then the standard ping-pong argument implies that for every \( m,n \geq M \) the subgroup \( \langle \varphi^m, \psi^n \rangle \leq \text{Out}(F_N) \) is free with free basis \( \varphi^m, \psi^n \). \( \Box \)

A subgroup of \( \text{Out}(F_N) \) will be called \textit{dynamically large} if it contains an atoroidal iwip automorphism. Such subgroups have many nice properties, and their “negative”, \textit{dynamically small} subgroups (i.e. subgroups without atoroidal iwips) seem to be rather special, compare [26].

\textbf{Theorem 5.6} (Tits alternative for dynamically large subgroups). Let \( G \leq \text{Out}(F_N) \) be a subgroup and such that there exists an atoroidal iwip \( \varphi \in G \). Let \( [T_+(\varphi)], [T_-(\varphi)] \in \overline{CV}_N \) be the attracting and repelling fixed points of \( \varphi \) in \( \overline{CV}_N \). Then exactly one of the following occurs:
(1) The group $G$ is virtually cyclic and preserves the set $\{[T_+ (\varphi)], [T_- (\varphi)]\} \subseteq CV_N$.

(2) The group $G$ contains an iwip $\psi = g\varphi g^{-1}$ for some $g \in G$ such that $\{[T_+ (\varphi)], [T_- (\varphi)]\} \cap \{[T_+ (\psi)], [T_- (\psi)]\} = \emptyset$. Moreover, in this case there exists an exponent $M \geq 1$ such that the subgroup $\langle \varphi^M, \psi^M \rangle \leq G$ is free of rank two.

Proof. It is well-known (see, for example, [19]) that if $\varphi$ is an atoroidal iwip, then $T_+ (\varphi)$ and $T_- (\varphi)$ are free $F_N$-trees.

Therefore by Theorem 4.4 we have $\text{Stab}_{\text{Out}(F_N)}[T_+ (\varphi)]$ and $\text{Stab}_{\text{Out}(F_N)}[T_- (\varphi)]$ are virtually cyclic and contain $\langle \varphi \rangle$ as a subgroup of finite index.

If $G$ preserves the set $\{[T_+ (\varphi)], [T_- (\varphi)]\}$, then $G$ has a subgroup of index at most 2 that fixes each of $[T_\pm (\varphi)]$ and hence $G$ is virtually cyclic. Thus we may assume that $G$ does not preserve the set $\{[T_+ (\varphi)], [T_- (\varphi)]\}$. So there exists $g \in G$ such that $[T_+ (\varphi)]g \notin \{[T_+ (\varphi)], [T_- (\varphi)]\}$ or $[T_- (\varphi)]g \notin \{[T_+ (\varphi)], [T_- (\varphi)]\}$. We assume the former as the other case is symmetric. Thus $[T_+ (\varphi)]g \neq [T_{\pm} (\varphi)]$. Note that $\psi = g^{-1}\varphi g \in G$ is also an atoroidal iwip and that $[T_+ (\psi)] = [T_+ (\varphi)]g$. We claim that $[T_- (\varphi)] \neq [T_\pm (\varphi)]g$. Indeed, otherwise we have $[T_- (g^{-1}\varphi g)] = [T_\pm (\varphi)]$ or $[T_- (g^{-1}\varphi g)] = [T_- (\varphi)]$ and hence $g^{-1}\varphi g \in \text{Stab}_{\text{Out}(F_N)}[T_+ (\varphi)]$ or $g^{-1}\varphi g \in \text{Stab}_{\text{Out}(F_N)}[T_- (\varphi)]$. In either case (since both stabilizers contain $\langle \varphi \rangle$ as subgroup of finite index) $g^{-1}\varphi^k g = \varphi^l$ for some $k \neq 0, l \neq 0$ and therefore $g^{-1}\varphi^k g$ has the same fixed points in $CV_N$ as does $\varphi^l$, namely, $[T_\pm (\varphi)]$. This contradicts the fact that $g^{-1}\varphi g$ fixes the point $[T_+ (\varphi)]g \neq [T_{\pm} (\varphi)]$. Thus $[T_{\pm} (\varphi)], [T_\pm (\psi)]$ are four distinct points. Therefore, by Corollary 5.5 sufficiently high powers $\varphi^M, \psi^M$ freely generate a free subgroup of rank two in $G$, as required. 

It is possible to prove Theorem 5.6 also for subgroups $G \leq \text{Out}(F_N)$ which contain an iwip $\varphi \in G$ that is not atoroidal. Indeed, precisely the same proof applies, except that in this case the limit trees $T_\pm (\varphi)$ and $T_- (\varphi)$ do not have a free $F_N$-action, so that we can not apply Theorem 4.4 to show that for such $\varphi$ the group $\text{Stab}_{\text{Out}(F_N)}([T_+ (\varphi)])$ is virtually cyclic. However, we can provide an alternative argument:

**Proposition 5.7.** Let $N \geq 2$ and let $\varphi \in \text{Out}(F_N)$ be an iwip which is not atoroidal. Then:

1. $\text{Stab}_{\text{Out}(F_N)}([T_+ (\varphi)])$ is virtually cyclic.

2. The same conclusion as in Theorem 5.6 holds for subgroups of $\text{Out}(F_N)$ which contain an iwip $\varphi$ which is not atoroidal.

Proof. As pointed out above, it suffices to prove statement (1). Part (2) then follows by exactly the same argument as in the proof of Theorem 5.6. The only difference is that in the case where $G \leq \text{Out}(F_N)$ contains an iwip $\varphi$ that is not atoroidal, in order to show that $\text{Stab}_{\text{Out}(F_N)}([T_+ (\varphi)])$ is virtually cyclic we invoke part (1) of this proposition rather than Theorem 4.4. Thus it is enough to establish (1):

A result of Bestvina and Handel (see Theorem 4.1 in [9]) shows that if $\varphi \in \text{Out}(F_N)$ is an iwip which is not atoroidal then there exists a compact connected surface $S$ with a single boundary component and an identification $F_N = \pi_1(S, x_0)$ (where we assume that $x_0$ belongs to the boundary circle of $S$) such that $\varphi$ is induced by a pseudo-Anosov homeomorphism $f$ of $S$. Let $c \in \pi_1(S)$ correspond to the boundary circle of $S$. Thus, if $S$ is orientable, then $N = 2k$ is even and $F_N$ has
a free basis $a_1, b_1, \ldots, a_k, b_k$ such that $c = [a_1, b_1] \ldots [a_k, b_k]$. If $S$ is non-orientable, there is a free basis $a_1, \ldots, a_N$ of $F_N$ such that $c = a_1^2 a_2^2 \ldots a_N^2$.

In this case $T_+ := T_+(\varphi)$ is, up to rescaling, exactly the $\mathbb{R}$-tree $T_G$ which is dual to the stable measured lamination $\mathcal{L} \in \mathcal{ML}(S)$ of the pseudo-Anosov $f$ (see Ch. 11.12 in [3] for details related to the construction of a dual $\mathbb{R}$-tree defined by a measured lamination on $S$). Moreover, by construction of $T_G$, a nontrivial element $g \in F_N$ acts with a fixed point on $T_G = T_+$ if and only if $g$ is conjugate in $F_N$ to a nonzero power of $c$.

Let $Mod(S)^\pm$ be the full mapping class group of $S$ (including isotopy classes of orientation-reversing homeomorphisms of $S$ if $S$ is orientable). It is well known that in this case $Mod(S)^\pm \leq Out(F_N)$ and in fact

$$Mod(S)^\pm = \{ \varphi \in Out(F_N) : \varphi([c]) = [c^{\pm 1}] \} \leq Out(F_N).$$

Let $\psi \in Stab_{Out(F_N)}([T_+])$, that is $T_+ \psi = \lambda T_+$ for some $\lambda > 0$. Since $||c||_{T_+} = 0$, we have

$$||\psi(c)||_{T_+} = ||c||_{T_+} \psi = ||c||_{\lambda T_+} = \lambda ||c||_{T_+} = 0.$$

Thus $||\psi(c)||_{T_+} = 0$ and hence $\psi(c)$ is conjugate to a power of $c$ in $F_N$. Moreover, since $\psi$ is an (outer) automorphism, we get $\psi([c]) = [c]^{\pm 1}$. Hence $\psi \in Mod(S)^\pm \leq Out(F_N)$. This shows that $Stab_{Out(F_N)}([T_+]) \leq Mod(S)^\pm$ and in fact $Stab_{Out(F_N)}([T_+]) \leq Stab_{Mod(S)^\pm}([T_G])$.

It is well known (see, for example, [1]) that the map from the space of projective measured laminations $\mathcal{PML}(S)$ to the space of projectivized $\mathbb{R}$-trees, that takes an element of $\mathcal{PML}(S)$ and sends it to the projective class of its dual $\mathbb{R}$-tree is $Mod(S)^\pm$-equivariant and injective. Hence $Stab_{Mod(S)^\pm}([T_G]) = Stab_{Mod(S)^\pm}([\mathcal{L}])$. Since $\mathcal{L}$ is the stable measured lamination associated to a pseudo-Anosov $f$, the group $Stab_{Mod(S)^\pm}([\mathcal{L}])$ is virtually cyclic (see Lemma 5.10 in [27]). Therefore $Stab_{Out(F_N)}([T_+])$ is virtually cyclic, as claimed. \qed

Recall that if $G$ is a group and $H \leq G$ is a subgroup, the commensurator or virtual normalizer of $H$ in $G$ is the subgroup

$$Comm_G(H) = \{ g \in G \mid [H : H \cap g^{-1}Hg] < \infty, \text{ and } [g^{-1}Hg : H \cap g^{-1}Hg] < \infty \}.$$

The commensurator $Comm_G(H)$ always contains the normalizer of $H$ in $G$. As a consequence of Theorem 5.4 and part (1) of Proposition 5.7, we obtain:

**Corollary 5.8.** Let $N \geq 2$ and let $\varphi \in Out(F_N)$ be an iwip. Then the commensurator $Comm_{Out(F_N)}(\langle \varphi \rangle)$ is virtually cyclic.

**Proof.** Let $g \in Comm_{Out(F_N)}(\langle \varphi \rangle)$. For some non-zero integers $n, m$ we have $g^{-1} \varphi^m g = \varphi^n$.

Thus $\{ [T_{\pm}(g^{-1} \varphi^m g)] \} = \{ [T_{\pm}(g^{-1} \varphi g)] \}$ are the only two fixed points of the iwip $g^{-1} \varphi g$ in $CV_N$. On the other hand $\{ [T_{\pm}(\varphi)] \}$ are the only two fixed points of the iwip $\varphi^n$ in $CV_N$. Therefore $\{ [T_{\pm}(\varphi)] \} = \{ [T_{\pm}(\varphi)] \}$, that is $g \in Stab_{Out(F_N)}(\{ [T_{\pm}(\varphi)] \})$, which gives $Comm_{Out(F_N)}(\langle \varphi \rangle) \leq Stab_{Out(F_N)}(\{ [T_{\pm}(\varphi)] \}).$

By Theorem 5.4 and part (1) of Proposition 5.7, the group $Stab_{Out(F_N)}(\{ [T_{\pm}(\varphi)] \})$ is virtually cyclic. Hence $Comm_{Out(F_N)}(\langle \varphi \rangle)$ is also virtually cyclic. \qed

Corollary 5.8 can also be derived, by a similar argument to the one given above, directly from Theorem 2.4 in [6].
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