Dynamical phase and quantum heat transport at fractional frequencies

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We demonstrate a genuine quantum feature of heat: the power emitted by a qubit (quantum two-level system) into a reservoir under continuous driving shows well-defined peaks as a function of frequency $f$. These resonant features appear due to the accumulation of the dynamical phase during the driving. The position of the $n$th maximum is given by $f = f_M/n$, where $f_M$ is the mean frequency of the qubit in the cycle, and their appearance is independent of the form of the drive and the number of heat baths attached, and even the presence or absence of spectral filtering. We propose that this non-trivial quantum heat can be detected by observing the steady-state power absorbed by a resistor acting as a bolometer attached to a driven superconducting qubit. This quantum heat is expected to play a crucial role in the performance of driven thermal devices such as quantum heat engines and refrigerators. We also show that by optimizing the cycle protocol, we recover the favorable classical limit in fast driven systems without the use of counter-diabatic drive protocols.

**Introduction:** Quantum heat transport [1-4] and quantum heat engines and refrigerators [5-8] attract currently attention because of their role in thermodynamics in quantum domain, and their applicability in such areas as heat management in quantum circuits, qubit resetting, and also of the form of the drive. For simplicity without spectral filtering, and of the number of baths attached, and even the presence or absence of spectral filtering. We propose that this non-trivial quantum heat can be detected by observing the steady-state power absorbed by a resistor acting as a bolometer attached to a driven superconducting qubit. This quantum heat is expected to play a crucial role in the performance of driven thermal devices such as quantum heat engines and refrigerators. We also show that by optimizing the cycle protocol, we recover the favorable classical limit in fast driven systems without the use of counter-diabatic drive protocols.

**Model:** First we consider a driven qubit coupled to a heat bath at temperature $T$. The Hamiltonian of the qubit is given as

$$H(t) = \frac{\hbar g f(t)}{2} \sigma_z + \frac{\hbar \omega_0}{2} \sigma_x,$$  \hspace{1cm} (1)

where $\sigma_z$ and $\sigma_x$ are the Pauli matrices and $\omega_0$ is the minimum transition frequency of the qubit. Here we have the driving protocol

$$f(t) = 1 + \frac{\tanh \left[ a \cos \left( \omega_L t \right) \right]}{\tanh a},$$  \hspace{1cm} (2)

where $\omega_L$ is the frequency, $2g$ is the amplitude of the drive and $a$ is a real parameter. When $a \rightarrow 0$, $f(t) = 1 + \cos (\omega_L t)$ and when $a \rightarrow \infty$, $f(t)$ is a square wave. From Eq. (1), we get the energy gap $\Delta E(t) = \hbar \sqrt{g^2 f(t)^2 + \omega_0^2}$. For $f(t) = 2$, and $f(t) = 0$, we get maximum and minimum energy level spacings $\Delta E_1$ and $\Delta E_2$, respectively. Corresponding transition frequencies are denoted as $\omega_1 = \Delta E_1/\hbar$ and $\omega_2 = \Delta E_2/\hbar$. We consider a weak coupling between the system and the bath. The density matrix $\rho$ of the system undergoes a non-unitary evolution as [28]

$$\frac{d\rho}{dt} = -i \frac{\hbar}{\hbar} [H(t), \rho] + \mathcal{L}\rho + \mathcal{L}^\phi \rho,$$  \hspace{1cm} (4)

where the dissipator is given by

$$\mathcal{L}\rho = \Gamma^\dagger (\sigma^- \rho \sigma^+ - 1/2 \{\sigma^- \sigma^+, \rho\}) + \Gamma^\dagger (\sigma^+ \rho \sigma^- - 1/2 \{\sigma^+ \sigma^-, \rho\}),$$  \hspace{1cm} (5)

and pure dephasing by

$$\mathcal{L}^\phi \rho = \Gamma^\phi (\sigma_z \rho \sigma_z - \rho),$$  \hspace{1cm} (6)

where $\sigma^+$ and $\sigma^-$ are raising and lowering operators, respectively. For the case of coupling to the bath in $\sigma_z$ direction, the transition rates are [6, 8]

$$\Gamma^\dagger = \kappa \frac{\omega_0^2}{\omega_0^2 + g^2 f(t)^2} \frac{\Delta E}{\hbar} (N(\Delta E) + 1).$$  \hspace{1cm} (7)
Here $\kappa$ is dimensionless coupling parameter, $N(\Delta E) = 1/(\exp(\Delta E/k_B T) - 1)$, $\Gamma^\prime = \exp(\Delta E/k_B T)\Gamma^3$ due to detailed balance, and from zero frequency noise spectrum, we get the pure dephasing rate as

$$\Gamma^\phi = \frac{\kappa g^2 f(t)^2 k_B T}{\omega_0^2 + g^2 f(t)^2}.$$

(8)

**Origin of peaks in a fast driven system:** Here we analyse a case where $\omega_0 > g$. In this limit, we can ignore the effects due to pure dephasing as the prefactor $g^2 f(t)^2/(\omega_0^2 + g^2 f(t)^2) \ll 1$. For sufficiently large $a$, the drive is close to a square-wave as shown in Fig. 1(a).

Then for half of the period $\delta t = \pi/\omega_L$, the energy of the system is $\Delta E = \Delta E_2$ and for another half of the period, it is $\Delta E = \Delta E_1$ and the transitions between these legs during the drive can be approximated as sudden processes. Thus for $a \to \infty$, baths are acting on the system only in the branches $\Delta E = \Delta E_2$ and $\Delta E = \Delta E_1$, and the relaxation rates (see Eq. 7) at these branches are denoted as $\Gamma^s_2$ and $\Gamma^r_1$, respectively. Thus we can identify four steps during the drive: $p \to q$, $r \to s$ are the thermalization steps and $q \to r$, $s \to p$ represent the sudden changes as shown in Fig. 1(a). In the fast driven system, since the Hamiltonians at two different instances are not commuting, the coherences are effectively destroyed by the thermal bath. This condition can also be achieved by considering the time-evolution of the density matrix in a fast driven system as we will now demonstrate.

Let us next find the analytical expression for the density matrix and the dissipated power in the limit $a \to \infty$, i.e. for the abrupt changes of the energy of the qubit. We define $R_q$ and $I_q$ as the real and imaginary parts of the off-diagonal terms of the density matrix of the system, respectively. These are defined in the instantaneous eigenstates of the Hamiltonian. Their evolution can be obtained by applying the sudden approximation of quantum mechanics in legs $q \to r$ and $s \to p$ and relaxation in legs $p \to q$ and $r \to s$. They evolve as

$$D_q = D_p + [\Gamma^s_1 - \Gamma^r_2(D_p + 1/2)]\delta t_2$$

$$R_q = [R_p \cos(\omega_2 \delta t_2) - I_p \sin(\omega_2 \delta t_2)](1 - 1/\Gamma^s_2(\delta t_2))$$

$$I_q = [I_p \cos(\omega_2 \delta t_2) + R_p \sin(\omega_2 \delta t_2)](1 - 1/\Gamma^s_2(\delta t_2))$$

$$D_r = \sqrt{1 - \eta^2} D_q - \eta R_q$$

$$R_r = \sqrt{1 - \eta^2} R_q + \eta D_q, I_r = I_q$$

$$D_s = D_r + [\Gamma^s_1 - \Gamma^r_2(D_r + 1/2)]\delta t_1$$

$$R_s = [\cos(\omega_1 \delta t_1)R_r - I_q \sin(\omega_1 \delta t_1)](1 - 1/\Gamma^s_1(\delta t_1))$$

$$I_s = [\cos(\omega_1 \delta t_1)I_r + R_r \sin(\omega_1 \delta t_1)](1 - 1/\Gamma^s_1(\delta t_1))$$

$$D_p = \sqrt{1 - \eta^2} D_s + \eta R_s$$

$$R_p = \sqrt{1 - \eta^2} R_s - \eta D_s, I_p = I_s.$$  

(13)

Here we consider $\Gamma^s_1(\delta t_1) \ll 1$ and $\Gamma^r_2(\delta t_2) \ll 1$, where $\Gamma^s_{1(2)} = \Gamma^r_{1(2)}$ and $\delta t_1$ and $\delta t_2$ are the durations of the legs $p \to q$ and $r \to s$, respectively such that $\omega_L = 2\pi/(\delta t_1 + \delta t_2)$, and $\eta = \sqrt{1 - \omega^2/\omega^2}$. For the symmetric case $\delta t_1 = \delta t_2$, Eq. (13) represents the evolution of the system corresponding to the driving protocol in Eqs. (1) and (2) with $a \to \infty$. As seen from Fig. 1(c), the condition for the peaks in the power corresponds to the maximum occupation probability of the excited state before the system is coupled to the bath. Then the excess energy tends to relax to this bath. From Eq. (13), and considering $\Gamma^s_1 = \Gamma^r_2 = 0$, $\delta t_2 = \delta t_1 = \delta t$ and $\Gamma^r_2(\delta t) \ll 1$, we get

$$P = \frac{\Delta E_2(2\Gamma^r_2 - \Gamma^s_2)(1 - \cos(\omega_1 \delta t))(\omega^2 - \omega^2_1)}{2(4\omega^2_1 - (\omega_1 + \omega_2)^2 \cos(\omega_2 + \omega_1)\delta t - K)}.$$  

(14)
and

$$\rho_{ee,p} = \frac{1}{2} \left[ 1 \mp \frac{2 (\Gamma_1^+ - \Gamma_1^-)}{\Gamma_1^2} \right]$$

$$\times 4 \left[ \omega_1 \cos \frac{\omega_1 \delta t}{2} \sin \frac{\omega_1 \delta t}{2} + \omega_2 \cos \frac{\omega_2 \delta t}{2} \sin \frac{\omega_2 \delta t}{2} \right]$$

$$\left( \omega_1^2 - (\omega_1 + \omega_2)^2 \cos(\omega_2 + \omega_1) \delta t - K \right), \quad (15)$$

with \( K = 2 (\omega_1^2 - \omega_2^2) \cos(\omega_2 \delta t) + (\omega_1 - \omega_2)^2 \cos(\omega_1 - \omega_2) \delta t \).

When \( \omega_1 \rightarrow \omega_2 \), we get from Eqs. (14), (15), maximum value for \( P \) and \( \rho_{ee,p} \) for \( \delta t = 2n\pi/\omega_1 + \omega_1 \), which corresponds to Eq. (12) and to the peaks obtained in the power. The condition used in Eqs. (14), (15), \( \Gamma_1^+ - \Gamma_1^- = 0 \), can be easily achieved by using spectral filters [3]. Interestingly, when \( \delta t = 2n\pi/\omega_1 \), \( P = 0 \), and \( \rho_{ee,p} = \Gamma_2^+ / \Gamma_2^- \), which is the classical limit [6]. In this case, \( \mathcal{R}_r = \mathcal{R}_s \), \( \mathcal{I}_r = \mathcal{I}_s \), and \( \mathcal{D}_r = \mathcal{D}_s \), due to which the coherence created during the ramp \( q \rightarrow r \) is annihilated in the ramp \( s \rightarrow q \).

Generally, transitionless quantum driving is achieved by counter-diabatic driving [30], [31]. Here, we achieve the classical limit with minimal power without such counter-diabatic driving but with suitable choice of the driving protocol.

Another implication of \( \Gamma_1^+ = \Gamma_1^- = 0 \) is that the non-unitary step where the system is in contact with the heat bath should preserve the purity of the qubit. Consider \( U_{ij} \), as the unitary process representing the ramp \( i \rightarrow j \) where \( i, j = \{p, q, r, s\} \). From the cyclic process described in Fig. 1(a), we have

$$\mathcal{V}_{pq} \{ U_{sp} U_{qr} \rho_q U_{rq} U_{qs} \} \rho_p \mathcal{V}_{qp} = \rho_q, \quad (16)$$

where \( \rho_q \) is the density matrix at the beginning of the ramp \( q \rightarrow r \) and \( \mathcal{V}_{pq} \) represents the map corresponding to non-unitary process in the branch \( p \rightarrow q \). Unitary processes preserve the von-Neumann entropy (purity) of the system. To achieve cyclicity as shown in Eq. (16), purity of the system in branch \( p \rightarrow q \) should also be preserved. This implies,

$$\mathcal{D}_q^2 + \mathcal{R}_q^2 + \mathcal{I}_q^2 = \mathcal{D}_p^2 + \mathcal{R}_p^2 + \mathcal{I}_p^2. \quad (17)$$

Under Lindblad evolution, due to decoherence, we have

$$\sqrt{\mathcal{R}_q^2 + \mathcal{I}_q^2} < \sqrt{\mathcal{R}_p^2 + \mathcal{I}_p^2},$$

which implies \( \mathcal{D}_q > \mathcal{D}_p \) and thereby \( P > 0 \) as expected.

**Bloch sphere dynamics:** The driven system shows interesting trajectories in the Bloch sphere representation. The co-ordinates of the Bloch vector at a given instant of time are \( \langle \sigma_x(t) \rangle, \langle \sigma_y(t) \rangle, \langle \sigma_z(t) \rangle \) where \( \langle \sigma_i(t) \rangle = \text{Tr} [\sigma_i \rho(t)] \) and \( \rho(t) \) is obtained from Eq. (4) for \( t \rightarrow \infty \). Since we are depicting steady state cyclic processes, all the trajectories in the Bloch sphere for a complete cycle should be closed. Depending on the driving frequency or in other words, the position of the peak, the number of turns in the Bloch sphere trajectory is also fixed. For the peak corresponding to \( n = 1 \), there is only one turn, and for \( n = 2 \), there are two turns in the Bloch sphere (See Fig. 1(b)). For driving frequencies away from \( f_M/n \), the trajectories are close to the surface (not shown in Fig. 1(b)). They move towards the centre of the Bloch sphere when the driving frequencies are close to \( f_M/n \) (peaks).

This is due to the fact that at driving frequencies near \( f_{L,n} \), the off diagonal coherence terms are created which then dissipate to the bath. Or in other words, the driving increases the entropy which in turn increases the mixedness of the system. Thus trajectories reflects the amount of dissipation. The trajectory on the Bloch sphere can also be constructed from Eqs. (13). At a given instant, the co-ordinates are \( 2(\mathcal{R}, \mathcal{I}, \mathcal{D}) \). As an example, we represent the state of the system obtained from Eqs. (13) at a few instances as dots in Fig. 1(b). This shows an excellent agreement between analytical solution from Eq. (13) and numerical simulations.

**Experimental setup:** The predicted quantum heat might be observable in a basic qubit based setup. For a transmon qubit [32] with \( \Delta E = \hbar \omega \), coupled to a resistor as bath [33] through a capacitor with capacitance \( C_c \).
for $\delta t$ achieved at high frequency by suitably choosing $\delta t$ regime is defined as $P$ in Refs. [3, 8]. Now we can define power dissipated allowing almost unitary evolution in between as discussed in Section 3. The resonators (spectral filters) will help to couple the qubit to the bath 1 with temperature $T_1$ when $\Delta E = \Delta E_1$ and to the bath 2 at temperature $T_2$ when $\Delta E = \Delta E_2$, and allow almost unitary evolution in between as discussed in Refs. [3, 8]. Now we can define power dissipated to the baths 1 and 2 as $P_1 = [\Delta E_1(D_s - D_r)]\omega_L/2\pi$ and $P_2 = [\Delta E_2(D_s - D_r)]\omega_L/2\pi$, respectively. The cooling regime is defined as $P_2 < 0$ and $P_1 > 0$ and can be achieved at high frequency by suitably choosing $\delta t_2$ and $\delta t_1$. If we consider the case $\delta t_2 = \pi/\omega_2$, we get cooling for $\delta t_1 = 2n\pi/\omega_1$ as shown in Fig. 3. The transition rate due to the bath in the presence of resonators is given as

$$\Gamma_1 = \frac{\omega_0^2}{\omega_0^2 + g^2 f(t)^2} \frac{C_L^2}{C_C^2} \omega_r (N(\Delta E) + 1),$$

where $C_J$ is the capacitance of each junction, $C_C = C_J + 2C_J$, quality factor of the junction $Q = \sqrt{L_J/C_J}/R = 1/\omega C_J R$, and $L_J$ is the Josephson inductance which can be modulated with flux $\Phi$. For $gf(t)/\omega_0 \ll 1$ and for low temperature $\hbar \omega > k_B T$, we get

$$\Gamma_1 \approx \frac{C_L^2}{(C_J + C_r)^2} \omega_r^2 RC_J.$$  

For typical values of a transmon, $C_J = 30 fF$, $C_r = 8 fF$, $\omega/2\pi = 6 GHz$, and $R = 200 \Omega$, we have $\Gamma_1/\omega \approx 0.01$ which represents weak coupling and the proposed model is applicable.

**Cooling regime:** We have seen that the classical limit or in other words suppression of the coherence induced dissipation, can be achieved by considering $\Gamma_1 = \Gamma_1^{\uparrow} = 0$ and $\delta t = 2n\pi/\omega_1$. We can extend this approach for two baths coupled to the qubit via resonators to approach this limit. Such a setup will be useful in constructing quantum heat engines and refrigerators [27]. The resonators (spectral filters) will help to couple the qubit to the bath 1 with temperature $T_1$ when $\Delta E = \Delta E_1$ and to the bath 2 at temperature $T_2$ when $\Delta E = \Delta E_2$, and allow almost unitary evolution in between as discussed in Refs. [3, 8]. Now we can define power dissipated to the baths 1 and 2 as $P_1 = [\Delta E_1(D_s - D_r)]\omega_L/2\pi$ and $P_2 = [\Delta E_2(D_s - D_r)]\omega_L/2\pi$, respectively. The cooling regime is defined as $P_2 < 0$ and $P_1 > 0$ and can be achieved at high frequency by suitably choosing $\delta t_2$ and $\delta t_1$. If we consider the case $\delta t_2 = \pi/\omega_2$, we get cooling for $\delta t_1 = 2n\pi/\omega_1$ as shown in Fig. 3. The transition rate due to the bath in the presence of resonators is given as

$$\Gamma_1^\uparrow = \frac{\omega_0^2}{\omega_0^2 + g^2 f(t)^2} \frac{\Delta E(N(\Delta E) + 1)}{1 + Q_r^2 \left(\frac{\omega_r}{\omega} - \frac{\omega_r}{\omega_r}\right)^2},$$

where $r = 1, 2$, $Q_r$ is the quality factor of the $r$th resonator and $\omega = \Delta E/\hbar$. Moreover, the dynamical phase

is $\varphi = \frac{1}{\pi} \int_0^{2\pi/\omega_1} D(t) dt = \pi + \omega_1 (2\pi/\omega_1 - \pi/\omega_1)$. Invoking the condition $\varphi = 2n\pi$ as in Eq. (12), we get the power maxima (see Fig. 3) at frequencies

$$f_{\text{asy}} = \frac{1}{2\pi} \frac{2\omega_2 \omega_1}{(2n-1)\omega_2 + \omega_1}.$$  

The validity of Eq. (13) is in the regime $\Gamma_1^{\uparrow} \delta t_1 \ll 1$ and $\Gamma_2^{\downarrow} \delta t_2 \ll 1$, because we consider only the linear terms in $\delta t_1$ and $\delta t_2$ in the dissipators (see Eqs. (5) and (13)) in branches $p \rightarrow q$ and $r \rightarrow s$. Therefore we can see a slight mismatch between simulation and analytical solution in Fig. 3.

Irrespective of the initial state of the system, after sufficiently many periods of drive, the system reaches a steady state cycle with the same trajectory in the Bloch sphere for all the subsequent cycles. At this point, an interesting question in the periodically driven open quantum system is: what would be the minimum relaxation rate required to make the evolution of the system cyclic since for fully unitary (closed system) such steady state is not reached. This can be understood from Eq. (13) and Eq. (15). For $\Gamma_1^{\uparrow} = 0$, the cyclic condition is satisfied for any

FIG. 2. A possible experimental setup consists of a superconducting qubit capacitively coupled to a normal metal resistor with resistance $R$. (see Fig. 2), the transition rate is given as

$$\Gamma_1 = \frac{\omega_0^2}{\omega_0^2 + g^2 f(t)^2} \frac{C_L^2}{C_C^2} \omega_r (N(\Delta E) + 1),$$

FIG. 3. (a), Power $P_2$ versus frequency. Bottom figure (b) shows the enlarged cooling regime. Inset shows the asymmetric square-wave driving protocol ($\delta t_2 \neq \delta t_1$) used in (a) and (b). In (a) and (b), the brown curve depicts the theoretical model from Eq. (13) and the pink curve shows the simulation. The dashed curve indicates the classical limit with same energy level spacings and transition rates as in the quantum system. Vertical dashed lines in (a) are obtained from Eq. (21). Here $\delta t_1$ is varied and the vertical red-dashed lines in (b) correspond to $\delta t_2 = 2n\pi/\omega_2$, where cooling is achieved.

We take $\delta t_1 = 200 \Omega$, we have $\Gamma_1^{\downarrow} = 0.01$ when $\Delta E = \Delta E_1$, and $\delta t = 2n\pi/\omega_1$. If we consider the case $\delta t_1 = \pi/\omega_1$, we get cooling for $\delta t_1 = 2n\pi/\omega_1$ as shown in Fig. 3. The transition rate due to the bath in the presence of resonators is given as

$$\Gamma_1^{\downarrow} = \frac{\omega_0^2}{\omega_0^2 + g^2 f(t)^2} \frac{\Delta E(N(\Delta E) + 1)}{1 + Q_r^2 \left(\frac{\omega_r}{\omega} - \frac{\omega_r}{\omega_r}\right)^2},$$

where $r = 1, 2$, $Q_r$ is the quality factor of the $r$th resonator and $\omega = \Delta E/\hbar$. Moreover, the dynamical phase
\[ \Gamma_2^+ = 0. \text{ As } \Gamma_2^- = 0, \text{ the cyclic trajectory of the system approaches the center of the Bloch sphere. So in Eq. (15), we get } \rho_{\text{ee,q}} \rightarrow 1/2 \text{ and thereby } D_\gamma \rightarrow 0. \text{ Similar analysis can be done for } \mathcal{R} \text{ and } \mathcal{I}. \text{ As } \Gamma_2^- \rightarrow 0, \text{ the Bloch vector } 2(\mathcal{R}, \mathcal{I}, D) \rightarrow (0, 0, 0). \text{ Thus for an arbitrary initial state, away from the steady state trajectory, the system takes infinitely many cycles for } \Gamma_2^+ \rightarrow 0 \text{ to reach the steady state cycle. But for any non-vanishing } \Gamma_2^+, \text{ a steady state cycle is eventually reached.} 

To conclude, we have established the relation between the quantum heat in driven systems and the dynamical phase acquired during the drive. By manipulating the cycle protocol, one can approach the favorable classical limit without counter-diabatic drive. We discussed the trajectories traversed by the qubit on the Bloch sphere and the impact of dissipation on cyclicity. Our work can be extended to many interesting directions such as experimental verification of the proposed model, analysing whether the system can outperform classical limit, and the role of quantum measurement in stochastic thermodynamics, Contemp. Phys. 22, 012117 (2011).

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