Algebro-geometric solitonic solutions
and
Differential Galois Theory

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Abstract
The main goal of this work is to give the solutions in closed form of the spectral problem
$-\Psi'' + u\Psi = \lambda \Psi$, when $u$ is a stationary potential of the KdV hierarchy. We describe the
centralizer of the Schrödinger operator $L = -\partial^2 + u$, and we use differential resultants to ob-
tain the corresponding spectral curves $\Gamma$, in general of arbitrary genus. We define the spectral
Picard-Vessiot extension of the operator $L - \lambda$ over the spectral curve $\Gamma$, proving its existence.
In this work, we find closed form formulas for the solutions of $L - \lambda$ by means of differential subresultant operators, which allow effective computations. A key ingredient of our contribution
is to consider a global parametrization of the spectral curve to provide a one-parameter form of
the solutions.

Keywords: Schrödinger operator, differential resultant, Burchnall-Chaundy polynomial,
Picard-Vessiot extension
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1. Introduction

In 1967, Gardner Green Kruskal and Miura published a paper with a new approach, the
inverse spectral method, to solve the initial value problem for the Korteweg-de Vries (KdV)
equation

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under boundary conditions decaying rapidly to zero at infinity \[29\]. In 1968, Gardner, Kruskal and Miura obtained an infinite number of local conserved densities, expressed as differential polynomials in \(u, v_n = v_n[u]\), for the KdV equation \[41\]. The same year \[37\], motivated by these papers, Lax wrote the KdV equation as a “Lax pair”

\[
\partial_t L = [A, L],
\]

being \(L\) the stationary Schrödinger operator with potential \(u\) and \(A\) a suitable third order operator. Associated to the conserved densities are the higher KdV equations, that can also be written as Lax pairs

\[
\partial_t [A_{2n+1}, L] = 0,
\]

which are non-linear ordinary differential equations of order \(2n + 1\), called the Novikov equations. The solutions of the Novikov equations are called the \textit{algebro-geometric KdV potentials} and the motivation to study them was given by the KdV solutions with periodic (or more general, quasi-periodic) boundary conditions. In the seventies of the last century several mathematicians as Novikov, Lax, Matveev, Marchenko, Dubrovin, Its, Krichever, McKean, van Moerbeke, etc., become interested for these problems. A nice account of the history of the problem until 2008 is given in the reference \[39\]. For periodic boundary conditions the algebro-geometric potentials are finite-gap potentials, thus the instability region (gaps) of the spectral parameter \(\lambda\) is given by a finite union of intervals, while an infinite number of gaps appears in the generic situation. Of course, the algebro-geometric solutions were extended to other families of “solitonic” partial differential equations, like Sine-Gordon, Non-linear Schrödinger and Kadomtsev-Petviashvili equations. In particular, the Baker-Akhiezer function introduced by Krichever plays a fundamental role in all the field. A geometrical interpretation of the Krichever work in the context of fiber bundles is given by Mumford \[44\]. Two standard monographs about the algebro-geometric solutions of the solitonic partial differential equations are \[5\] and \[31\].

In the eighties of the last century, people become aware of the connection between algebro-geometric solutions of the solitonic partial differential equations and several old fundamental papers by Burchnall and Chaundy on one side and by Drach on the other side.

In 1928, Burchnall and Chaundy addressed the problem of the commutativity of general ordinary differential operators \[12, 13\],

\[
[Q, P] = 0.
\]

In particular, they proved that associated to two commuting operators there exist an algebraic curve \(f(\mu, \lambda) = 0\), the \textit{spectral curve} with the property that the differential operators satisfy the
curve, i.e., \( f(P, Q) = 0 \). The connection with the stationary solutions of the KdV hierarchy is clear. The spectral curve is in this case of hyperelliptic type \( \mu^2 = R_{2n+1}(\lambda) \), where \( R_{2n+1} \) is a polynomial of degree \( 2n + 1 \). From the commutativity of the operators it is deduced that they have a common non-trivial space of eigenfunctions. The dimension of this space is the rank of the problem. For the KdV Schrödinger operators the rank is one. The study of commuting operators with rank bigger than one is a much more difficult problem, see for instance the recent paper [40]. We call an operator like \( P \), or \( Q \), above an algebro-geometric operator, i.e., such that there exist another non-trivial operator that commute with it, see [15].

One of our guiding ideas is the strong connection between the integrability (i.e., solvability in closed form) of the direct and the inverse problems for the Schrödinger equation. By “direct problem” we understand given the potential to obtain the eigenfunctions \( \psi \) and the eigenvalues. The “inverse problem” would be to obtain the potential from some suitable spectral data. In fact, this was also the motivation for Drach in his 1919 papers [20, 21], about the integrability in closed form of the equation

\[
\frac{d^2 \psi}{dx^2} = (\varphi(x) + h) \psi
\]

where \( h \) is the spectral parameter. So he wrote in [20]:

“\( Il \text{ est donc très important de connaître les cas où une simplification se présente dans l’intégration de } (I), \text{ en laissant le paramètre } h \text{ arbitraire. Nous avons réussi à déterminer la fonction } \varphi \text{ dans tous les cas où l’intégrale } \psi \text{ peut s’obtenir par quadratures... “},

Thus, Drach’s motivation was to study the cases in which the direct problem is simplified for any value of the spectral parameter and, in fact, he was able to obtain all the potentials \( \varphi \) such that the eigenfunction \( \psi \) was obtained by quadratures. Moreover he said that to study this problem it is possible to use the classical theory of Picard about linear equations (Picard-Vessiot theory). In other words, he considered the integrability of the direct problem in the sense of the Picard-Vessiot theory. Along the paper Drach indicates that the constants, for instance the parameters in the “rationality group” (i.e., the Galois group), are functions of the spectral parameter \( h \). According to Ehlers and Knörrer [24], Drach’s work was rescued from oblivion by the Chudnovsky brothers. We point out that the connection between the integrability of the direct problem and the inverse problem is not constrained to the algebro-geometric situation. This is the case for the reflectionless potentials (see [43]).

In this paper we present a new framework to study and compute closed form solutions of the KdV Schrödinger equation

\[
L\psi = \lambda \psi
\]

for an algebro-geometric KdV (stationary) potential \( u \). We consider this work the first part of a project devoted to develop the Differential Galois Theory of Schrödinger equations for the general case of algebro-geometric potentials.

Our idea, continuing with Drach’s philosophy, is to consider the integrability of the direct problem in the context of Picard-Vessiot (PV) theory. This theory is a Galois theory of linear differential equations

\[
\frac{d\psi}{dx} = A(x)\psi,
\]

being \( A(x) \) a square matrix in some differential field. From our point of view PV theory is an integrability theory. It gives rigorous criteria for the solvability in closed form (integrability) of equation (3). Even when the equation is not integrable, it is possible to characterize the
reducibility to simpler equations in terms of the structure of the Galois group. Here we will assume some knowledge of PV theory. Two standard monographs on this theory are [19, 56], for a complex analytic introduction to PV see [42].

We mention some previous papers using PV theory in connection with algebro-geometric operators. Papers [6, 16] are devoted to multidimensional algebraically integrable differential operators, which in some cases coincide with algebro-geometric differential operators. Algebro-geometric (ordinary) operators with elliptic functions as coefficients are explicitly studied in [25, 33, 10]. We remark that in two other papers Brezhnev [8, 9], like us, is also partially motivated by the works of Drach. But in our opinion in all of these previous papers the role of the spectral parameter is not completely clarified (as signaled by Drach). In fact, we show that the spectral curve determines the field of constants of the PV extension of the KdV Schrödinger equation (2). Moreover, the aim our paper is to clarify this new spectral Picard-Vessiot extension. The problem is not trivial at all, because in the standard PV theory it is usually assumed that the constant field is algebraically closed, but this is not the case here. Thus the objective of our paper is to fill this gap. Furthermore, our approach has an effective algebraic computational flavor as a natural consequence of our theoretical results. We illustrate all these facts using some well known families of examples. All computations were done using Maple 18.

The paper is structured as follows. In Section 2 we set notation and we review the construction of the KdV hierarchy, of differential polynomials in a differential variable $u$. In Section 3, we identify the first KdV equation of the KdV hierarchy satisfied by a given potential and we describe the centralizer of a KdV Schrödinger operator $L$ with algebro-geometric KdV potential. In [49], E. Previato used differential resultants to compute spectral curves, opening the door to symbolic techniques for its effective computation. In this work, we explore the benefits of using differential resultants to compute the spectral curve $\Gamma$ of the Lax pair $(L, A_{2n+1})$, see sections 4 and 5. The central results of the paper are contained in sections 6 and 7. First we define appropriate coefficient fields $K(\Gamma)$ to give factorization algorithms for $L - \lambda$, for a generic spectral parameter $\lambda$. We prove the existence of a spectral Picard-Vessiot field for $L - \lambda$ and give a method to obtain the one-parameter form of its solutions. As an application of our results, in Section 8, we establish the foundations to compute Picard-Vessiot extensions of $L - \lambda_0$ for almost all points $P_0 = (\lambda_0, \mu_0)$ of the spectral curve $\Gamma$.

2. Preliminaries

Let $\mathbb{N}$ be the set of positive integers including 0. For concepts in differential algebra and differential Galois theory we refer the reader to [19], [56] or [42]. Let $K$ be a differential field of characteristic zero with derivation $\partial$ whose field of constants $C$ is algebraically closed. Let us consider algebraic variables $\lambda$ and $\mu$ with respect to $\partial$. Thus $\partial \lambda = 0$ and $\partial \mu = 0$ and we can extend the derivation $\partial$ of $K$ to the polynomial ring $K[\lambda, \mu]$ by

$$
\partial \left( \sum a_{i,j} \lambda^i \mu^j \right) = \sum \partial(a_{i,j}) \lambda^i \mu^j, \quad a_{i,j} \in K.
$$

Hence $(K[\lambda, \mu], \partial)$ is a differential ring whose ring of constants is $C[\lambda, \mu]$.

Let us consider a differential indeterminate $u$ over $C$. We will call formal Schrödinger operator to the operator $L(u) = -\partial^2 + u$ with coefficients in the ring of differential polynomials $B = C[u] = C[u, u', u'', \ldots]$, where $u'$ stands for $\partial(u)$ and $u^{(n)} = \partial^n(u)$, $n \in \mathbb{N}$. 4
Given a differential commutative ring $R$ with derivation $\partial$, let us denote by $R[\partial]$ the ring of differential operators with coefficients in $R$ and commutation rule

$$[\partial, a] = \partial a - a \partial = \partial(a), a \in R,$$

where $\partial a$ denotes the product in the noncommutative ring $R[\partial]$ and $\partial(a)$ is the image of $a$ by the derivation map. The ring of pseudo-differential operators in $\partial$ will be denoted by $R[\partial^{-1}]$ (see [32])

$$R[\partial^{-1}] = \left\{ \sum_{i=0}^{n} a_i \partial^i \mid a_i \in R, n \in \mathbb{Z} \right\},$$

where $\partial^{-1}$ is the inverse of $\partial$ in $R[\partial^{-1}]$, $\partial^{-1} \partial = \partial \partial^{-1} = 1$.

2.1. KdV polynomials and their Lax pair representations

In this section we will work with the formal Schrödinger operator $L = L(u)$ as defined in Section 2. In a convenient way to be used in this paper, we review well known algorithms to compute the differential polynomials in $u$ of the KdV hierarchy and the family of differential operators of its Lax representation. This was studied for the first time in the paper [30]. We follow the normalization in [31], see also [46] for other presentations.

Let us consider the pseudo differential operator

$$R = -\frac{1}{4} \partial^2 + u + \frac{1}{2} u' \partial^{-1}$$

and its adjoint $R^* = -\frac{1}{4} \partial^2 + u - \frac{1}{2} \partial^{-1} u'$.

Observe that $R^* = \partial^{-1} R \partial$. The operator $R^*$ is a recursion operator of the KdV equation (see [46], p. 319). Applying the recursion operator $R$, we define:

$$kdv_0 := u', \quad kdv_n := R(kdv_{n-1}), \text{ for } n \geq 1.$$  \hspace{1cm} (6)

Applying $R^*$ we define:

$$v_0 := 1, \quad v_n := R^*(v_{n-1}), \text{ for } n \geq 1.$$  \hspace{1cm} (7)

Hence for $n \in \mathbb{N}$ it holds

$$2\partial(v_{n+1}) = kdv_n.$$  \hspace{1cm} (8)

**Example 2.1.** Observe that the first two iterations of (6) give the differential polynomials in $u$

$$kdv_1 = -\frac{1}{4} u'' + \frac{3}{2} uu',$$

$$kdv_2 = \frac{1}{16} u^{(5)} - \frac{5}{8} uu'' - \frac{5}{4} u' u'' + \frac{15}{8} u^2 u'.$$

The first two iterations of (7) give differential polynomials that are shown in Example 2.4.

We will prove next that for all $n$, $kdv_n$ and $v_n$ are differential polynomials in $u$, elements of $\mathcal{B} = C[u]$. The proof is similar to the one of [46], Theorem 5.31 but we include details for completion. We will call the differential polynomials $kdv_n$ the KdV differential polynomials.

**Lemma 2.2.** The formulas for $kdv_n$ and $v_n$ give differential polynomials in $\mathcal{B} = C[u]$. 5
Proof. Observe that $\mathcal{R}(\text{kd}v_{n-1})$ is well defined if and only if $\text{kd}v_{n-1}$ is a total derivative. We will prove this by induction on $n$. It is trivial for $n = 1$ since $\text{kd}v_0 = \partial(u)$. Let us assume that $\text{kd}v_{n-1} = \partial(g_{n-1})$, $g_{n-1} \in \mathcal{B}$.

Since $\mathcal{R}$ and $\mathcal{R}'$ are adjoint operators we have $p\mathcal{R}(q) = q\mathcal{R}'(p) + \partial(a)$, $p, q, a \in \mathcal{B}$. Thus for $p = u$ and $q = u'$ we get

$$u\mathcal{R}'(u') = u'(\mathcal{R}')^+(u) + \partial(a_k), \text{ for } a_k \in \mathcal{B}.$$ 

Then

$$\text{kd}v_n' = (\partial u - u\partial)(\mathcal{R}')^{n-1}(u) + \partial(a_{n-1}) = -u\partial(\mathcal{R}')^{n-1}(u) + \partial(b), \quad b \in \mathcal{B},$$

which implies that $\text{kd}v_{n-1} = \partial(b/2)$ is the total derivative of a differential polynomial in $\mathcal{B}$.

Since

$$\mathcal{R} = \partial \left( -\frac{1}{3} \partial + \frac{1}{2} \partial^{-1} u + \frac{1}{2} u \partial^{-1} \right)$$

we obtain that $\text{kd}v_n = \mathcal{R}(\text{kd}v_{n-1})$ is a total derivative.

The fact that $\text{kd}v_n$ is a total derivative and (8) imply that $v_{n+1}$ are also elements of $\mathcal{B}$.

As in [31], we define a family of differential operators in $\mathcal{B}[\partial]$ of odd order (see also [22], [45])

$$P_1 := \partial, \quad P_{2n+1} := v_n \partial - \frac{1}{2} \partial(v_n) + P_{2n-1}L, \quad \text{for } n \geq 1. \quad (9)$$

Observe that

$$P_{2n+1} = \sum_{l=0}^{n} \left( \frac{n-l}{l!} \partial - \frac{1}{2} \partial(v_{n-l}) \right) L^l.$$ 

The operators $P_{2n+1}$ have the important property that the commutator $[P_{2n+1}, L]$ is a differential operator in $\mathcal{B}[\partial]$ but Lemma 2.3 shows that it has order zero, it is the multiplication operator by the $\text{kd}v_{n}$ differential polynomial. This is the famous Lax representation of $\text{kd}v_{n}$, see [31], [45]. We will call the differential operators $P_{2n+1}(u)$ the KdV differential operators.

**Lemma 2.3.** For $n \in \mathbb{N}$ it holds $[P_{2n+1}, L] = \text{kd}v_n$.

Proof. One can easily check that $v_1 = \mathcal{R}'(1) = u/2$ and $[P_1, L] = u' = 2\partial(v_1)$. We prove the result by induction on $n$. Since $P_{2n+3} = O + P_{2n-1}L$ where $O = v_{n+1} \partial - \frac{1}{2} \partial(v_{n+1})$, we have

$$[L, P_{2n+3}] = [L, O] + [L, P_{2n+1}]L = [L, O] - 2\partial(v_{n+1})L,$$

with

$$[L, O] = 2\partial(v_{n+1})\partial^3 + (1/2)\partial^3(v_{n+1}) - v_{n+1}u',
\quad 2\partial(v_{n+1})L = -2\partial(v_{n+1})\partial^2 + 2\partial(v_{n+1})u.$$ 

Observe that $\mathcal{R}' = -\frac{1}{2} \partial^{-1} S$ where $S = \partial^3 - 4u\partial - 2u'$. Thus

$$[L, P_{2n+3}] = (1/2)\partial^3(v_{n+1}) - v_{n+1}u' - 2\partial(v_{n+1})u = (1/2)S(v_{n+1}) = -2\partial\mathcal{R}'(v_{n+1}) = -2\partial(v_{n+2}).$$

By (8) the result is proved.
Now let us consider algebraic indeterminates \( c_n, n \geq 1 \) over \( C \). We define an extended family of KdV differential polynomials \( \text{KdV}_n(u, c^n) \), \( n \in \mathbb{N} \) in the differential indeterminate \( u \) and the list of algebraic indeterminates \( c^n = (c_1, \ldots, c_n) \).

\[
\text{KdV}_0 := u', \quad \text{KdV}_n := \text{kdv}_n + \sum_{l=0}^{n-1} c_{n-l} \text{kdv}_l, \text{ for } n \geq 1
\]

and an extended family of KdV differential operators whose coefficients are differential polynomials in \( u \) and \( c^n \),

\[
\hat{P}_1 := \partial \text{ and } \hat{P}_{2n+1} := P_{2n+1} + \sum_{l=0}^{n-1} c_{n-l} P_{2l+1}.
\]

One can easily check that

\[
[\hat{P}_{2n+1}, L] = \text{KdV}_n = 2\partial(f_{n+1}),
\]

for

\[
f_0 := v_0 = 1 \text{ and } f_n := v_n + \sum_{l=0}^{n-1} c_{n-l} v_l.
\]

**Example 2.4.** We illustrate (11) for \( n = 2 \):

1. \( v_1 = \frac{u}{3}, v_2 = \frac{3}{4}u^2 - \frac{1}{8}u'' \).
2. \( P_3 = -4\partial^3 + \frac{3}{2}u\partial + \frac{3}{2}u', P_5 = \partial^5 - \frac{5}{2}u\partial^3 - \frac{15}{4}u'\partial^2 + \frac{15}{8}u^2\partial - \frac{25}{16}u''\partial - \frac{15}{16}u''' + \frac{15}{16}uu' \).
3. \( \hat{P}_5 = P_5 + c_1 P_3 + c_2 P_1 \).

**3. Centralizers of KdV Schrödinger operators**

In this section, we specialize \( u \) to a potential \( \bar{u} \) in the differential field \( K \), with field of constants \( C \) as in Section 2. We are going to study the centralizer in \( K[\partial] \) of the Schrödinger operator \(-\partial^2 + \bar{u} \), with the potential \( \bar{u} \) subject to constrains given by the KdV hierarchy.

For a KdV Schrödinger equation

\[
(-\partial^2 + \bar{u})\Psi = \lambda \Psi,
\]

\( \lambda \) is not a free constant but it is governed by the equation of the spectral curve \( \Gamma \) associated to (14), by means of a differential operator that determines the centralizer of \(-\partial^2 + \bar{u} \).

We will clarify first what it means for a potential to be subject to the KdV hierarchy. This means that it satisfies a \( \text{KdV}_n = 0 \) equation given by (10), for some \( n \). More precisely, the solutions \( \bar{u} \) of \( \text{KdV}_n(u, c^n) = 0 \) depend on the existence of a set of constants \( \bar{c}^n \in C^n \) such that \( \text{KdV}_n(\bar{u}, \bar{c}^n) = 0 \). We will work with three families of examples for which this situation holds (see Examples 3.4) to illustrate this phenomenon.

**3.1. Flag of constants for KdV potentials**

Given a potential \( \bar{u} \) in \( K \), we will study next the determination of a set of constants \( \bar{c}^n \) satisfying the equation \( \text{KdV}_n(\bar{u}, c^n) = 0, n \in \mathbb{N} \), in the set of algebraic variables \( c^n = (c_1, \ldots, c_n) \). We will explore the structure of sets of constants verifying the KdV equations for a given potential \( \bar{u} \). Our method was motivated by [31].
Recall that $kdv_n$ is the differential polynomial in $C[u]$ given by (6). After replacing $u = \tilde{u}$ in $kdv_n$ we obtain an element of $K$ denoted by $k_n = kdv_n(\tilde{u})$. Observe that the linear equation in $c_1, \ldots, c_n$

$$kdv_n(\tilde{u}, c^n) = kdv_0(\tilde{u}) + \sum_{i=0}^{n-1} kdv_1(\tilde{u})c_{n-1} = k_n + k_{n-1}c_1 + \ldots + k_1c_{n-1} + k_0c_n = 0 \quad (15)$$

determines an affine hyperplane in $K^n$. Let $\mathcal{H}_n$ be its intersection with $C^n$

$$\mathcal{H}_n := \{\xi \in C^n \mid kdv_n(\tilde{u}, \xi) = 0\}.$$

**Definition 3.1.** We call a potential $\tilde{u}$ in a differential field $K$, a KdV potential if there exists $n \geq 1$ such that $\mathcal{H}_n \neq \emptyset$. Let $s$ be the smallest positive integer such that $\mathcal{H}_s \neq \emptyset$, we call $s$ the KdV level of $\tilde{u}$. We will write $u_s$ for a KdV potential $\tilde{u}$ of KdV level $s$.

Thus the level $s$ of a potential indicates the first equation $kdv_n = 0$ that is satisfied by $u_s$ for a given set of constants. Furthermore, the next proposition explains that $u_s$ satisfies $kdv_n = 0$ for all $n > s$. In addition the choice of constants is unique in the first level but not in the remaining ones.

**Example 3.2.** As a first example, let us consider $\tilde{u} = 6/x^2$ in $K = \mathbb{C}(x)$. One can easily check that $\tilde{u}$ is a KdV potential of level 2. It does not verify $kdv_1(u, c_1) = kdv_1(u) + c_1kdv_0(u) = 0$ for any $c_1 \in \mathbb{C}$ but $kdv_2(\tilde{u}, (0, 0)) = kdv_2(\tilde{u}) = 0$ and $kdv_3(u, c^n) = 0$ is satisfied by $u = \tilde{u}$ for infinitely many choices for $c^n \in C^n$.

**Proposition 3.3.** Given a potential $u_s$, the following statements are satisfied:

1. $\mathcal{H}_s = \{\tilde{c}^s\}$ with $\tilde{c}^s = (c_1^s, \ldots, c_n^s) \in C^n$.
2. For all $n > s$, the $C$ vector space $\mathcal{V}_n := \{\xi \in C^n \mid kdv_n(u_s, \xi) - k_n = 0\}$ has dimension $n - s$, namely

$$\mathcal{V}_{s+1} = \langle (1, \tilde{c}^s) \rangle, \quad \mathcal{V}_{n+1} = \mathcal{V}_n \oplus \mathcal{W}_n, \text{ with } \mathcal{W}_n = \langle (1, \tilde{c}^s, 0, \ldots, 0) \rangle, \quad (16)$$

identifying $\mathcal{V}_n$ with its natural embedding in $\mathcal{V}_{n+1}$ defined by $x \mapsto (0, x)$. Furthermore, there is an infinite flag

$$\mathcal{V}_s \subset \cdots \subset \mathcal{V}_n \subset \cdots, \quad (17)$$

that we call the flag of constants for $u_s$.
3. For all $n > s$, we have $\mathcal{H}_n = \tilde{c}_n + \mathcal{V}_n$, with $\tilde{c}_n = (c_1^s, \ldots, c_n^s, 0, \ldots, 0) \in C^n$. Furthermore, there is an infinite flag of affine spaces

$$\mathcal{H}_s \subset \cdots \subset \mathcal{H}_n \subset \cdots, \quad (18)$$

identifying $\mathcal{H}_n$ with its natural embedding in $\mathcal{H}_{n+1}$ defined by $x \mapsto (x, 0)$.

**Proof.** 1. If there exists $\xi = (\xi_1, \ldots, \xi_s) \neq \tilde{c}^s$ in $\mathcal{H}_s$ then for some $1 \leq i \leq s$, $\xi_i - c_i^s \neq 0$ and

$$k_{s-i}^{s-i+1} \frac{\xi_{s-i+1} - c_{s-i+1}}{\xi_i - c_i^s} + \ldots + k_0^{s-i+1} \frac{\xi_s - c_s^s}{\xi_i - c_i^s} = 0,$$

contradicting that $\mathcal{H}_{s-i} = \emptyset$. 

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2. By (6) and (10) we have

$$\mathcal{R}^{n-r}(\text{KdV}_{\xi}(u, c^\xi)) = \mathcal{R}^{n-r}(\text{KdV}_{\xi}(u)) + \sum_{i=0}^{n-1} c_i \mathcal{R}^{i}(\text{KdV}_{\xi}(u)) = k \text{dV}(u) + \sum_{i=0}^{n-1} c_i k \text{dV}_{n-i}(u).$$

(19)

Let us consider the recursion operator (5) for $u = u_s$, that is $\mathcal{R}_s = -\frac{1}{4} \partial^2 + u_s + \frac{1}{2} u'_s \partial^{-1}$. Replacing $u$ by $u_s$ and $c^\xi$ by $\bar{c}^\xi$ in (19) we obtain

$$k_n = -k_{n-1} c_1^\xi - \cdots - k_{n-s} c_s^\xi,$$

(20)

since $\mathcal{R}_s$ is a linear operator acting on $\mathcal{C}(u_s)$ and $\text{KdV}_{\xi}(u_s, \bar{c}^\xi) = 0$.

We prove (16) by induction on $n$. An element $\xi = (\xi_1, \ldots, \xi_{s+1})$ of $\mathcal{V}_{s+1}$ verifies $k_0 \xi_1 + \cdots + k_0 \xi_{s+1} = 0$, and taking $n = s$ in (20) we get

$$k_{s-1}(\xi_2 - \xi_1 c_1^\xi) + \cdots + k_0 (\xi_{s+1} - \xi_1 c_s^\xi) = 0.$$

Then $\xi = \xi_1(1, \bar{c}^\xi)$, because 1. implies that $\mathcal{V}_s$ is the null space. Let us assume that $\mathcal{V}_n$ has basis $\{w_1, \ldots, w_{n-s}\}$. Observe that

$$\mathcal{V}_{n+1} \cap \{\xi \in C^{n+1} \mid \xi_1 = 0\} = \{0\} \times \mathcal{V}_n$$

has basis $\mathcal{B} = \{(0, w_1), \ldots, (0, w_{n-s})\}$. Using (20) we can prove that $\xi \in \mathcal{V}_{n+1}$ verifies

$$k_{n-1}(\xi_2 - \xi_1 c_1^\xi) + \cdots + k_{n-s}(\xi_{s+1} - \xi_1 c_s^\xi) + k_{n-1} \xi_{s+2} + \cdots + k_0 \xi_{s+1} = 0.$$

Let $w = (1, \bar{c}^\xi, 0, \ldots, 0) \in C^{n+1}$, then $\xi - \xi_1 w \in \{0\} \times \mathcal{V}_n$, which proves that $\{w\} \cup \mathcal{B}$ is a basis of $\mathcal{V}_{n+1}$ of size $n + 1 - s$.

3. Substituting (20) in $\text{KdV}_{\xi}(u_s, c^\xi)$ gives

$$\text{KdV}_{\xi}(u_s, c^\xi) = k_0 (c_1 - c_1^\xi) + \cdots + k_{n-1} (c_s - c_s^\xi) + k_{n-1} x + \cdots + k_0 c_n,$$

proving that $\mathcal{H}_n = \bar{c}^\xi + \mathcal{V}_n$. Similarly we can prove that given $\xi \in \mathcal{H}_i$, $s \leq i \leq n-1$ then

$$\text{KdV}_{\xi+1}(u_s, c^\xi+1) = k_0 (c_1 - \xi_1) + \cdots + k_1 (c_i - \xi_i) + k_0 c_{i+1}.$$

Therefore $\mathcal{H}_i \times \{0\} \subset \mathcal{H}_{i+1}$ and (18) follows.

The previous proposition shows that the flag of constants of a KdV potential $u_s$ of KdV level $s$ is determined by $\mathcal{H}_s = \{c^\xi\}$. We call $c^\xi$ the basic constants vector of $u_s$.

**Examples 3.4.** We will illustrate all the results of the paper by means of some well known families of potentials in [57]. To start, we check that they are families of KdV potentials and we compute their basic constants vector.

1. Let us consider the family of rational potentials $u_s = s(s+1)/x^2$, $s \geq 1$, in $K = \mathbb{C}(x)$ with $\partial = d/dx$. One can easily check that the KdV level of $u_s$ is $s$ and its basic constants vector $\bar{c}^\xi = (0, \ldots, 0)$.
2. Now consider a family of Rosen-Morse potentials \( u_s = \frac{-s(x+1)}{\cos(x)}, s \geq 1 \), in the differential field \( K = \mathbb{C}(\phi) = \mathbb{C}(\cosh(x)) \) with \( \phi = d/dx \). To compute the KdV level of \( u_s \), we collect the coefficients of \( \eta = \cosh(x) \) and \( \eta' \) in \( \text{KdV}_0(u_s, c^2) \) taking into account that \( (\eta')^2 = \eta^2 - 1 \).

(a) **Level** \( s = 1 \). For \( s = 1 \) we have

\[
\text{KdV}_0(u_1) = -4 \frac{\eta'}{\eta}, \quad \text{KdV}_1(u_1, c^1) = -4 \frac{\eta'}{\eta} (1 - c_1).
\]

Thus \( \text{KdV}_0(u_1) \neq 0 \) and \( \text{KdV}_1(u_1, c^1) = 0 \) for \( c^1 = (1) \).

(b) **Level** \( s = 2 \). For \( s = 2 \) we have

\[
\begin{align*}
\text{KdV}_0(u_2) &= -12 \frac{\eta'}{\eta}, \quad \text{KdV}_1(u_2, c^1) = -12 \frac{\eta'}{\eta} \left( -6 + (-1 + c_1)\eta^2 \right), \\
\text{KdV}_2(u_2, c^2) &= 12 \frac{\eta'}{\eta^2} \left( 6c_1 - 30 + (c_1 - 1 - c_2)\eta^2 \right).
\end{align*}
\]

Thus \( \text{KdV}_0(u_2) \neq 0 \) and \( \text{KdV}_1(u_2, c^1) \neq 0 \) for all \( c^1 \in \mathbb{C} \). Finally \( \text{KdV}_2(u_2, c^2) = 0 \) for \( c^2 = (3, 4) \).

(c) **Level** \( s = 3 \). For \( s = 3 \) we have

\[
\begin{align*}
\text{KdV}_0(u_3) &= -24 \frac{\eta'}{\eta}, \quad \text{KdV}_1(u_3, c^1) = -24 \frac{\eta'}{\eta} \left( -15 + (-1 + c_1)\eta^2 \right), \\
\text{KdV}_2(u_3, c^2) &= 12 \frac{\eta'}{\eta^2} \left( -225 + (30c_1 - 150)\eta^2 + (-2c_2 + 2c_1 - 2)\eta^3 \right), \\
\text{KdV}_3(u_3, c^3) &= -12 \frac{\eta'}{\eta^3} \left( -3150 + 225c_1 + (150c_1 - 630 - 30c_2)\eta^2 + (-2c_2 + 2c_1 + 2c_3 - 3)\eta^3 \right).
\end{align*}
\]

Thus \( \text{KdV}_0(u_3) \neq 0 \) and \( \text{KdV}_1(u_3, c^1) \neq 0 \) for all \( c^1 \in \mathbb{C} \). Finally \( \text{KdV}_3(u_3, c^3) = 0 \) for \( c^3 = (14, 49, 36) \).

3. Next we consider a family of elliptic potentials \( u_s = s(s+1)\phi(x; g_2, g_3), s \geq 1 \), where \( \phi \) is the Weierstrass \( \wp \)-function for \( g_2, g_3 \), satisfying \( (\wp')^2 = 4\wp^3 - g_2\wp - g_3 \). In this case \( K = \mathbb{C}(\wp) = \mathbb{C}(\phi, \wp') \) with \( \phi = d/dx \). We take \( \eta = \wp \) and use that \( (\eta')^2 = 4\eta^3 - g_2\eta - g_3 \in C(\eta) \) to compute \( \text{KdV}_n(u_s, c^2) \), \( n = 0, \ldots, s \), for some values of \( s \).

(a) **Level** \( s = 1 \). For \( s = 1 \) we have

\[
\text{KdV}_0(u_1) = -2\eta', \quad \text{KdV}_1(u_1, c^1) = -2\eta' c_1.
\]

Thus \( \text{KdV}_0(u_1) \neq 0 \) and \( \text{KdV}_1(u_1, c^1) = 0 \) for \( c^1 = (0) \).

(b) **Level** \( s = 2 \). For \( s = 2 \) we have

\[
\begin{align*}
\text{KdV}_0(u_2) &= -6\eta', \quad \text{KdV}_1(u_2, c^1) = -6\eta' (6 + c_1\eta), \\
\text{KdV}_2(u_2, c^2) &= \eta' (-144c_1\eta - 63g_2 - 24c_2).
\end{align*}
\]

Thus \( \text{KdV}_0(u_2) \neq 0 \) and \( \text{KdV}_1(u_2, c^1) \neq 0 \) for all \( c^1 \in \mathbb{C} \). Finally \( \text{KdV}_2(u_2, c^2) = 0 \) for \( c^2 = (0, 21g_2/8) \).

(c) **Level** \( s = 3 \). For \( s = 3 \) we have

\[
\begin{align*}
\text{KdV}_0(u_3) &= -12\eta', \quad \text{KdV}_1(u_3, c^1) = -\eta' (180\eta + 12c_1), \\
\text{KdV}_2(u_3, c^2) &= \eta' (-360c_1\eta^2 - 2700\eta + 153g_2 - 24c_2), \\
\text{KdV}_3(u_3, c^3) &= \eta' \left( -5670g_2 - 360c_2\eta^2 - 2700c_1\eta - 153g_2c_1 - 1782g_3 - 24c_3 \right).
\end{align*}
\]
Thus \( \text{KdV}_0(u_3) \neq 0 \) and \( \text{KdV}_0(u_3, \bar{e}^n) \neq 0 \) for all \( \bar{e}^n \in \mathbb{C}^n \), \( n = 1,2 \). Finally \( \text{KdV}_3(u_3, \bar{e}^3) = 0 \) for \( \bar{e}^3 = (0, -63g_3/4, -297g_3/4) \).

**Remark 3.5.** The families 1 and 2 in Example 3.4 are degenerate cases of 3.

### 3.2. Centralizers and Burchnall-Chaundy polynomials

In this section we will study the centralizers of Schrödinger operators \( L_s = L(u_s) = -\partial^2 + u_s \), where \( u_s \) is a KdV potential of KdV level \( s \) and basics constant vector \( \bar{e}^s \), as defined in Section 3.1. We will call \( L_s \) a KdV Schrödinger operator or KdV for short.

To start we summarize some results from [32] about centralizers of differential operators. Let \( P \in R[\partial] \) be an operator \( P = a_n \partial^n + \cdots + a_1 \partial + a_0 \). Let us denote by \( C_R(P) \) the centralizer of \( P \) in \( R[\partial] \), that is

\[
C_R(P) = \{ Q \in R[\partial] | PQ = QP \}.
\]

By [32], Theorem 4.1, if \( n \) and \( a_n \) are non zero divisors in \( R \) then \( C_R(P) \) is commutative. Let \( C^\infty \) be the ring of infinitely-many times differentiable complex-valued functions on the real line. By [32], Corollary 4.4, \( C^\infty(P) \) is commutative if and only if there is no nonempty open interval on the real line on which the functions \( \partial(a_0), a_1, \ldots, a_0 \) all vanish.

Details of the evolution of these results from various previous works are given in [32]. We chose this reference because it simplifies the existing methods and applies them in as wide a context as reasonable. Precursors of the commutativity results are Schur [51], Flanders [26], Krichever [36], Amitsur [2], Carlson and Goodearl [14]. Results describing centralizers \( C_R(P) \) as a free module of finite rank appear in [26], [2], [14] and in Ore’s well known paper [47].

Recall that a commutative ring \( R \) is called reduced if it has no nonzero nilpotent element. Observe that \( C^\infty \) is not a field, but it is a reduced ring whose ring of constants is the field \( \mathbb{C} \).

**Theorem 3.6.** Let \( R \) be a reduced differential ring whose subring \( F \) of constants is a field. Let us assume that \( n \) is invertible in \( F \) and \( a_n \) is invertible in \( R \).

1. ([32], Theorem 4.2) \( C_R(P) \) is a commutative integral domain.

2. ([32], Theorem 1.2) Let \( X \) be the set of those \( i \) in \( \{0, 1, 2, \ldots, n-1\} \) for which \( C_R(P) \) contains an operator of order congruent to \( i \) module \( n \). For each \( i \in X \) choose \( Q_i \) such that \( \text{ord}(Q_i) \equiv i(\text{mod} \ n) \) and \( Q_i \) has minimal order for this property (in particular \( 0 \in X \) and \( Q_0 = 1 \)). Then \( C_R(P) \) is a free \( F[P] \)-module with basis \( \{Q_i | i \in X \} \). Moreover, the rank \( t \) of \( C_R(P) \) as a free \( F[P] \)-module is a divisor of \( n \).

We are ready now to describe the centralizer \( C_K(L_s) \) in \( K[\partial] \) of the KdV Schrödinger operator \( L_s \). We do so by generalizing an example in [32], Section 1.2. In addition, by [14], Theorem 1.6 we know that \( C_K(L_s) \) has rank 2.

Replacing \( u \) by \( u_s \) and \( e^n \) by \( \bar{e}^s_n = (\bar{e}^s, 0, \ldots, 0) \) in the family of KdV differential operators \( \hat{P}_{2n+1}(u_s, \bar{e}^s_n) \) defined in (11), we obtain a family of differential operators in \( K[\partial] \)

\[
A_{2n+1} := \hat{P}_{2n+1}(u_s, \bar{e}^s_n), \text{ for all } n \geq s. \quad (21)
\]

As a consequence of (12) and Proposition 3.3 we have

\[
[A_{2n+1}, L_s] = \text{KdV}_n(u_s, \bar{e}^s_n) = 0, \text{ for all } n \geq s. \quad (22)
\]
Thus $A_{2s+1} \in C_k(L_s)$, for all $n \geq s$. The next result shows that $A_{2s+1}$ has an important role in the description of the centralizer of $L_s$, it is the differential operator that determines the centralizer of $L_s$.

**Theorem 3.7.** Let $L_s$ be a KdV Schrödinger operator. The centralizer of $L_s$ in $K[\partial]$ equals the free $C[L_s]$-module of rank 2 with basis $\{1, A_{2s+1}\}$, that is

$$C_k(L_s) = \{p_0(L_s) + p_1(L_s)A_{2s+1} \mid p_0, p_1 \in C[L_s]\} = C[L_s](1, A_{2s+1}).$$

**Proof.** We will prove that there does not exist an operator of odd order smaller that 2$s + 1$ in $C_k(L_s)$. By Theorem 3.6, 2, this implies that $C_k(L_s) = C[L_s](1, A_{2s+1})$.

Let us consider a monic differential operator $Q \in K[\partial]$ of order 2$n + 1$ with $n < s$. Let $P_{2n+1}(u)$ be the family of KdV differential operators defined in (9) and denote by $P_{2n+1}^i := P_{2n+1}(u_i)$. Since $P_{2(n+1)}^{[s]}_2$ and $L_{2s+1}$ are families of operators in $K[\partial]$ of odd and even orders less than 2$n + 1$ respectively, we divide $Q$ by those families and write

$$Q = \sum_{i=0}^{n} q_{2i+1}P_{2i+1}^i + \sum_{i=0}^{n} q_{2i}L_{s}^i$$

with $q_{2n+1} = 1$ and $q_{2i+1}, q_{2i} \in K$. To compute $[Q, L_s]$, observe that $[a, L_s] = \partial^2(a) + 2\partial(a)\partial$, for $a \in K$ and

$$[q_{2i+1}P_{2i+1}^i, L_s] = -\partial^2(q_{2i+1})P_{2i+1}^i - 2\partial(q_{2i+1})\partial P_{2i+1}^i + q_{2i+1}kdv_i(u_s)$$

and

$$[q_{2i}L_{s}^i, L_s] = [q_{2i}, L_s]L_{s}^i = (\partial^2(q_{2i}) + 2\partial(q_{2i})\partial)L_{s}^i.$$ 

Thus in $[Q, L_s]$ the only term of order $2i + 2$ is the leading term of $\partial P_{2i+1}^i$ and the only term of order $2i + 1$ is the leading term of $\partial L_{s}^i$. If $[L_s, Q] = 0$ then $\partial(q_{2i}) = 0$ and $\partial(q_{2i+1}) = 0$. Therefore $[q_{2i}L_{s}^i, L_s] = 0$ and $q_{2i+1} \in C$, $i = 0, \ldots, n$ implies that

$$0 = [Q, L_s] = \sum_{i=0}^{n} q_{2i+1}kdv_i(u_s)$$

contradicting that $u_s$ has KdV level $s$. We conclude that $Q \notin C_k(L_s)$, which proves the result. 

A polynomial $f(x, y)$ with constant coefficients satisfied by a commuting pair of differential operators $P$ and $Q$ is called a Burchnall-Chanundy (BC) polynomial of $P$ and $Q$, since the first result of this sort appeared is the 1923 paper [12] by Burchnall and Chaundy. Generalizations (more general rings $R$) were later studied in [36], [14] and [50]. The next result shows that associated to the centralizer of a differential operator $P$ there are as many BC polynomials as operators in the centralizer. We will compute these polynomials using differential resultants, as it will be explained in Section 4.

**Theorem 3.8.** ([32], Theorem 1.13) Let $R$ be a reduced differential ring whose subring $F$ of constants is a field. Given any operator $Q \in C_k(P)$ there exist polynomials $p_0(P), \ldots, p_{r-1}(P) \in F[P]$ such that

$$p_0(P) + p_1(P)Q + \cdots + p_{r-1}(P)Q^{r-1} + Q' = 0.$$ 

That is, there exists a nonzero polynomial $f_0(x, y) \in F[x, y]$ such that $f_0(P, Q) = 0$. 

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4. Differential resultant and subresultants

Let $K$ be a differential field as in Section 2. Let us consider differential operators $P$ and $Q$ in $K[\partial]$ of orders $n$ and $m$ respectively and leading coefficients $a_n$ and $b_m$. We are interested in the common solutions of the system of linear differential equations

\[
\begin{align*}
P & = 0 \\
Q & = 0.
\end{align*}
\]

The tools we have chosen to study this problem are differential resultant and subresultants. Differential resultants of pairs of ordinary differential operators were defined by Berkovich and Tsirulik [4] and studied by Chardin [17], who also defined the subresultant sequence. They are clearly an adaptation of the algebraic resultant of two algebraic polynomials in one variable [18] to a noncommutative situation. We summarize next the definition and some important properties of differential resultants to be used in this note.

4.1. Differential resultant for ODO’s and main properties

The Sylvester matrix $S_0(P, Q)$ is the coefficient matrix of the extended system of differential operators

\[
S_0(P, Q) = \left[ \begin{array}{cccc}
\partial^{n-1} P & \ldots & \partial P & \partial^{m-1} Q \\
\partial^n P & \partial^{n-1} Q & \ldots & \partial Q \\
\end{array} \right].
\]

Observe that $S_0(P, Q)$ is a squared matrix of size $n + m$ and entries in $K$. We define the differential resultant of $P$ and $Q$ to be

\[
\partial \text{Res}(P, Q) := \det(S_0(P, Q)).
\]

Example 4.1. Given $P = a_2 \partial^2 + a_1 \partial + a_0$ and $Q = b_3 \partial^3 + b_2 \partial^2 + b_1 \partial + b_0$ in $K[\partial]$,

\[
S_0(P, Q) = \left[ \begin{array}{cccc}
a_2 & a_1 + 2 \partial(a_2) & a_0 + 2 \partial(a_1) + \partial^2(a_2) & 2 \partial(a_0) + \partial^2(a_1) \\
0 & a_2 & a_1 + \partial(a_2) & a_0 + \partial(a_1) \\
0 & 0 & a_2 & a_1 \\
b_3 & b_2 + \partial(b_3) & b_1 + \partial(b_2) & b_0 + \partial(b_1) \\
0 & b_3 & b_2 & b_1 \\
\end{array} \right].
\]

The next propositions state the most relevant properties of the differential resultant.

Proposition 4.2 ([17]). Let $(P, Q)$ be the left ideal generated by $P, Q$ in $K[\partial]$.

1. $\partial \text{Res}(P, Q) = AP + BQ$ with $A, B \in K[\partial]$, ord($A$) < $m$, ord($B$) < $n$, that is $\partial \text{Res}(P, Q)$ belongs to the elimination ideal $(P, Q) \cap K$.

2. $\partial \text{Res}(P, Q) = 0$ if and only if $P = \bar{P}R, Q = \bar{Q}R$, with ord($R$) > 0 and $\bar{P}, \bar{Q}, R \in K[\partial]$.

Observe that Proposition 4.2, 1, indicates that $AP + BQ$ is an operator of order zero, that is the terms in $\partial$ than one have been eliminated. Furthermore, Proposition 4.2, 2 states that $\partial \text{Res}(P, Q) = 0$ is a condition on the coefficients of the operators that guarantee a right common factor.

Given a fundamental system of solutions $y_1, \ldots, y_n$ of $P = 0$, let us denote by $W(y_1, \ldots, y_n)$ the Wronskian matrix

\[
W(y_1, \ldots, y_n) = \left[ \begin{array}{cccc}
y_1 & \cdots & y_n \\
\partial y_1 & \cdots & \partial y_n \\
\vdots & \vdots & \vdots \\
\partial^{n-1} y_1 & \cdots & \partial^{n-1} y_n \\
\end{array} \right].
\]
and by \( w(y_1, \ldots, y_n) \) its determinant. As in the case of the classical algebraic resultant there is a Poisson formula for \( \partial \text{Res}(P, Q) \).

**Proposition 4.3** ([17], Theorem 5, see also [49]). Given \( P, Q \in K[\partial] \) with respective orders \( n \) and \( m \), leading coefficients \( a_m \) and \( b_m \) and fundamental systems of solutions \( y_1, \ldots, y_n \) and \( z_1, \ldots, z_m \) respectively of \( P = 0 \) and \( Q = 0 \). It holds,

\[
\partial \text{Res}(P, Q) = (-1)^{nm} a_m^{\text{w}} \frac{w(Q(y_1), \ldots, Q(y_n))}{w(y_1, \ldots, y_n)} = b_m^{\text{w}} \frac{w(P(z_1), \ldots, P(z_m))}{w(z_1, \ldots, z_m)}.
\]

**Example 4.4.** Let us consider the formal Schrödinger operator \( L = -\partial^2 + u \) and \( \hat{P}_3(u, c^1) = P_3 + c_1 P_1 \) (as in Example 2.4). They are differential operators in \( \mathcal{B}[\partial] \) with \( \mathcal{B} = C[u] \). Let \( \lambda \) and \( \mu \) be algebraic indeterminates as in Section 2. The next differential resultant will be of interest

\[
\partial \text{Res}(L - \lambda, \hat{P}_3 - \mu) = -\mu^2 - \lambda^3 - 2c_1 \lambda^2 + p_1(u, c_1) \lambda + p_0(u, c_1)
\]

where

\[
p_1(u, c_1) = \frac{1}{2}u'' + \frac{3}{2}u^2 + c_1 u - c_1^2
\]

\[
p_0(u, c_1) = \frac{1}{4}u^4 + \frac{3}{2}u^3 - \frac{1}{2}u'u - \frac{1}{4}u''' + u^2 c_1 + uc_1^2.
\]

with \( \partial(p_1(u, c_1)) = KdV_1(u, c^1) \),

\[
\partial(p_0(u, c_1)) = \left( \frac{u}{2} + c_1 \right) KdV_1(u, c^1).
\]

### 4.2. Subresultant sequence

We introduce next the subresultant sequence for \( P \) and \( Q \), which was defined in [17], see also [38]. For \( k = 0, 1, \ldots, N := \min(n, m) - 1 \) we define the matrix \( S_k(P, Q) \) to be the coefficient matrix of the extended system of differential operator

\[
\Xi_k(P, Q) = (\partial^{n-k} P, \ldots, \partial P, \partial^{m-k} Q, \ldots, \partial Q, Q).
\]

Observe that \( S_k(P, Q) \) is a matrix with \( n + m - 2k \) rows, \( n + m - k \) columns and entries in \( K \). For \( i = 0, \ldots, k \) let \( S^i_k(P, Q) \) be the squared matrix of size \( n + m - 2k \) obtained by removing the columns of \( S_k(P, Q) \) indexed by \( \partial^i, \ldots, \partial^i \), except for the column indexed by \( \partial^i \). Whenever there is no room for confusion we denote \( S_k(P, Q) \) and \( S^i_k(P, Q) \) simply by \( S_k \) and \( S^i_k \) respectively. The **subresultant sequence** of \( P \) and \( Q \) is the next sequence of differential operators in \( K[\partial] \):

\[
\mathcal{L}_k = \sum_{i=0}^k \det(S^i_k) \partial^i, \ k = 0, \ldots, N.
\]

In this paper we will need \( \mathcal{L}_1 = \det(S^1_0) + \det(S^1_1) \partial \) where

\[
S^0_1 := \text{submatrix}(S_1, \partial)
\]

and

\[
S^1_1 := \text{submatrix}(S_1, 1)
\]

are the submatrices of \( S_1 \) obtained by removing columns indexed by \( \partial \) and 1 respectively.

Recall that \( K[\partial] \) is a left Euclidean domain. If \( \text{ord}(P) \geq \text{ord}(Q) \) then \( P = qQ + r \) with \( \text{ord}(r) < \text{ord}(Q) \), \( q, r \in K[\partial] \). Let us denote by \( \text{gcd}(P, Q) \) the greatest common (left) divisor of \( P \) and \( Q \).
Theorem 4.5 ([17], Theorem 4. Differential Subresultant Theorem). Given differential operators 
P and Q in K[\partial], \gcd(P, Q) is a differential operator of order r if and only if:

1. \( L_k \) is the zero operator for \( k = 0, 1, \ldots, r - 1 \) and.
2. \( L_r \) is nonzero.

Then \( \gcd(P, Q) = L_r. \)

Remark 4.6. 1. Given \( L_r = \gcd(P, Q) \) then \( P = \hat{P} L_r \) and \( Q = \hat{Q} L_r, \hat{P}, \hat{Q} \in K[\partial]. \)
2. The \( \gcd(P, Q) \) is nontrivial (it is not in K) if and only if \( L_0 = \partial \text{Res}(P, Q) = 0. \)

Example 4.7. The differential subresultant of \( L - \lambda \) and \( \hat{P}_3 - \mu \) (see Example 4.4) is

\[ L_1 = \det(S^0_1) + \det(S^1_1) \partial = \left( -\mu - \frac{u'}{4} \right) + \left( \frac{u}{2} + c_1 + \lambda \right) \partial \]

with

\[ S^0_1 = \begin{bmatrix} -1 & 0 & u' \\ 0 & -1 & u - \lambda \\ -1 & 0 & \frac{1}{2} u' - \mu \end{bmatrix} \quad \text{and} \quad S^1_1 = \begin{bmatrix} -1 & 0 & u - \lambda \\ 0 & -1 & 0 \\ -1 & 0 & \frac{1}{2} u + c_1 \end{bmatrix} \]

The next technical result will be necessary later in this paper.

Proposition 4.8. Let us consider the formal Schrödinger operator \( L = -\partial^2 + u \) and the differential operator \( \hat{P}_{2s+1}(u, c') \) defined in (11). The following statements hold:

1. \( \partial \text{Res}(L - \lambda, \hat{P}_{2s+1} - \mu) = -\mu^2 + R_{2s+1}(u, c', \lambda) \) where \( R_{2s+1}(u, c', \lambda) \) is a polynomial in \( C[u][\lambda] \) of degree \( 2s + 1 \) of the form

\[ R_{2s+1}(u, c', \lambda) := -\lambda^{2s+1} - 2c_1\lambda^{2s} + \cdots. \]

2. The determinant of \( S^1_1(L - \lambda, \hat{P}_{2s+1} - \mu) \) is a polynomial \( \varphi_2 \) in \( C[u][\lambda] \) of degree \( s \) of the form

\[ \varphi_2 := \lambda^s + \left( \frac{u}{2} + c_1 \right) \lambda^{s-1} + \cdots. \]

3. The determinant of \( S^0_1(L - \lambda, \hat{P}_{2s+1} - \mu) \) equals \( -\mu - \alpha \), where \( \alpha \in C[u][\lambda] \) has degree at most \( s - 1 \) in \( \lambda. \)

Proof. By induction on \( s \), we prove first that

\[ \hat{P}_{2s+1} = (-1)^s \partial^{2s+1} + \left( -1 \right)^{s-1} \left( \frac{2s + 1}{2} u + c_1 \right) \partial^{2s} + \cdots. \]  \hspace{1cm} (25)

By Example 4.7, the claim holds for \( s = 1. \) Assume the claim is true for \( \hat{P}_{2s-1} \)

\[ \hat{P}_{2s-1} = (-1)^{s-1} \partial^{2s-1} + \left( -1 \right)^{s-1} \left( \frac{2s - 1}{2} u + c_1 \right) \partial^{2s-2} + \cdots (-1)^{s-1} u \partial^{2s-1} \]

to obtain (25) from (11).
1. The Sylvester matrix $S_0(L - \lambda, \hat{P}_2 s + 1 - \mu)$ equals the following matrix of size $2s + 3$:

$$
M = \begin{pmatrix}
-1 & 0 & u - \lambda & \ast & \cdots & \ast & \ast & \ast \\
0 & -1 & 0 & u - \lambda & \ast & \ast & \ast & \ast \\
0 & 0 & -1 & 0 & \cdots & \ast & \ast & \ast \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -1 & 0 & u - \lambda \\
(-1)^s & 0 & (-1)^{s-1} \begin{pmatrix} 2s + 1 \\ 2 \end{pmatrix} u + c_1 & \ast & \cdots & \ast & -\mu & \ast \\
0 & (-1)^s & 0 & (-1)^{s-1} \begin{pmatrix} 2s + 1 \\ 2 \end{pmatrix} u + c_1 & \ast & \cdots & \ast & + \mu \\
\end{pmatrix}
$$

To compute $\partial \text{Res}(L - \lambda, \hat{P}_2 s + 1 - \mu)$ we develop the determinant of $M$ by its first column (being $M_{i,j}$ the minor determinant obtained by removing column $i$ and row $j$)

$$|M| = -M_{1,1} - (-1)^s M_{2s+2,1}.$$  

We show next the diagonal of $M_{1,1}$ with highest order in $\lambda$

$$M_{1,1} = \begin{pmatrix}
0 & 2s + 1 \\ -1 & 0 \\
\end{pmatrix}(u - \lambda)^{2s} + \cdots = \begin{pmatrix} 2s + 1 \\ 2 \end{pmatrix} u + c_1 \lambda^{2s} + \text{lower degree terms in } \lambda.
$$

Developing $M = M_{2s+2,1}$ by its first column we obtain $A_{2,1} = (-1)^s A_{2s+2,1}$ where

$$A_{2,1} = (-1)^s \left( \begin{pmatrix} 2s + 1 \\ 2 \end{pmatrix} u + c_1 \right) (u - \lambda)^{2s} + \cdots = (-1)^s \left( \begin{pmatrix} 2s + 1 \\ 2 \end{pmatrix} u + c_1 \right) \lambda^{2s} + \text{lower degree terms in } \lambda$$

$$A_{2s+2,1} = (u - \lambda)^{2s+1} + \cdots = -\lambda^{2s+1} + (2s + 1) u \lambda^{2s} + \text{lower degree terms in } \lambda.$$

Thus from $|M| = -M_{1,1} - (-1)^s A_{2,1} + A_{2s+2,1}$ the claim easily follows.

2. The matrix $C = S_1^L$ defined in (24) equals the following matrix of size $2s + 1$:

$$S_1^L = \begin{pmatrix}
-1 & 0 & u - \lambda & \ast & \cdots & \ast \\
0 & -1 & 0 & u - \lambda & \ast & \ast & \ast \\
0 & 0 & -1 & 0 & \cdots & \ast \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ast \\
0 & 0 & 0 & 0 & \cdots & N \\
(-1)^s & 0 & (-1)^{s-1} \begin{pmatrix} 2s + 1 \\ 2 \end{pmatrix} u + c_1 & \ast & \cdots & \ast \\
\end{pmatrix}$$

with $N = \begin{pmatrix} 0 & u - \lambda \\ -1 & 0 \end{pmatrix}$

Observe that $|C| = -C_{2,2}$ and that $N$ appears $s - 1$ times. We can reorganize the rows of $C_{2,2}$ to compute its determinant by the main box diagonal and prove the claim.

$$|C| = (-1)^{2s-1} \begin{pmatrix}
-1 & 0 & (u - \lambda) & \ast & \ast \\
0 & -1 & 0 & (u - \lambda) & \ast \\
0 & 0 & -1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & \cdots \\
(-1)^s & 0 & (-1)^{s-1} \begin{pmatrix} 2s + 1 \\ 2 \end{pmatrix} u + c_1 & \ast & \ast \\
\end{pmatrix} N^{-1} = (-1)^s \begin{pmatrix} 2s + 1 \\ 2 \end{pmatrix} u + c_1 |N|^{s-1} + |N|^{s} =$$

$$= (-1)^s \begin{pmatrix} 2s + 1 \\ 2 \end{pmatrix} u + c_1 (-1)^{s-1} \lambda^{s-1} + (-1)^s \lambda + (-1)^{s-1} u \lambda^{s-1} + \text{lower degree terms in } \lambda.$$
3. The matrix $S_1^0$ defined in (23) equals the following matrix of size $2s + 1$:

$$
S_1^0 = \begin{pmatrix}
-1 & * & N & * & \cdots & * & * \\
0 & * & * & N & \cdots & * & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & * & \cdots & * & * \\
0 & 0 & 0 & 0 & 0 & u' & * \\
0 & 0 & 0 & 0 & -1 & u - \lambda & * \\
(-1)^s & 0 & (-1)^{s-1} \left(\frac{2s+1}{\lambda} u + c_1\right) & * & \cdots & * & -\mu
\end{pmatrix}
$$

Observe first that $|S_1^0| = (-1)^{2s+1} (-\mu) - \alpha$. We can reorganize the columns of $S_1^0$ to develop its determinant by the main box diagonal

$$
\det(S_1^0) = \begin{vmatrix}
0 & 0 & u' \\
0 & -1 & u - \lambda \\
(-1)^s & * & -\mu
\end{vmatrix}
|N|^{s-1} + \text{lower degree terms in } \lambda.
$$

Which shows that its degree in $\lambda$ is less than or equal to $s - 1$. 

\[ \square \]

5. Spectral curves of KdV Schrödinger operators

The relation between Burchnell and Chaundy polynomials (see Section 3.2) and the differential resultant was given by E. Previato in [49]. We state Previato’s theorem in the general case of differential operators in $K[\partial]$ and give an alternative proof using the Poisson formula for the differential resultant in Proposition 4.3. Then we compute BC polynomials of KdV Schrödinger operators. We apply Previato’s Theorem 5.2 to the computation of the spectral curve of the Lax pair $[L_s, A_{2s+1}]$, showing the irreducible polynomials $f_s(\lambda, \mu)$ defining the spectral curve $\Gamma_s$.

5.1. Previato’s Theorem

The operators $P - \lambda$ and $Q - \mu$ have coefficients in the differential ring $(K[\lambda, \mu], \partial)$, see Section 2. By means of the differential resultant, Proposition 4.2, 1 ensures that we compute a nonzero polynomial,

$$
\partial \text{Res}(P - \lambda, Q - \mu) = a_0^0 \mu^n - b_0^n \lambda^m + \cdots
$$

in the elimination ideal $(P - \lambda, Q - \mu) \cap K[\lambda, \mu]$. The next result implies that if $P$ and $Q$ commute then

$$
\partial \text{Res}(P - \lambda, Q - \mu) \in (P - \lambda, Q - \mu) \cap C[\lambda, \mu].
$$

Remark 5.1. Note that to guarantee the existence of a fundamental system of solutions of $P - \lambda$ we may see this operator in the algebraic closure of $K(\lambda, \mu)$, whose field of constants is the algebraic closure $\mathbb{C}$ of $C(\lambda, \mu)$.

Theorem 5.2 (E. Previato, [49]). Given $P, Q \in K[\partial]$ such that $[P, Q] = 0$ then $g(\lambda, \mu) = \partial \text{Res}(P - \lambda, Q - \mu) \in C[\lambda, \mu]$ and $g(P, Q) = 0$.
Proof. Let \( y_1, \ldots, y_n \) be a fundamental system of solutions of \( (P - \lambda)(Y) = 0 \). Since \( 0 = [P, Q] = [P - \lambda, Q - \mu] \) we have \( (P - \lambda)(Q - \mu)(y_i) = (Q - \mu)(P - \lambda)(y_i) = 0 \) then \((Q - \mu)(y_i), i = 1, \ldots, n\) are solutions of \((P - \lambda)(Y) = 0\). By Remark 5.1, there exists a matrix \( M \) with entries in \( \mathbb{C} \) such that

\[
W((Q - \mu)(y_1), \ldots, (Q - \mu)(y_n)) = W(y_1, \ldots, y_n)M.
\]

By Proposition 4.3,

\[
\frac{\partial \text{Res}(P - \lambda, Q - \mu)}{\partial \lambda} = \frac{w((Q - \mu)(y_1), \ldots, (Q - \mu)(y_n))}{w(y_1, \ldots, y_n)} \quad \text{det}(M) = \text{det}(M),
\]

which belongs to \( K[\lambda, \mu] \cap \mathbb{C} = C[\lambda, \mu] \).

The last statement of this theorem follows from the fact that \( g(\lambda, \mu) = \frac{\partial \text{Res}(P - \lambda, Q - \mu)}{\partial \lambda} \) belongs to the differential ideal generated by \( P - \lambda \) and \( Q - \mu \) in \( K[\lambda, \mu][\partial] \). Therefore

\[
g(\lambda, \mu) = \lambda(P - \lambda) + B(Q - \mu), \text{ with } A, B \in K[\lambda, \mu][\partial].
\]

Since \( P \) and \( Q \) commute then \( g(P, Q) = 0 \).

The previous theorem shows that BC polynomials (defined in Section 3.2) can be computed using differential resultants. Let us suppose that \( [P, Q] = 0 \) and let \( f(\lambda, \mu) \) be the square free part of \( \frac{\partial \text{Res}(P - \lambda, Q - \mu)}{\partial \lambda} \) \( C[\lambda, \mu] \) (i.e. the product of the different irreducible components of \( g \)).

The affine algebraic curve defined by \( f \)

\[
\Gamma := \{ (\lambda, \mu) \in C^2 | f(\lambda, \mu) = 0 \}
\]

is known as the spectral curve of the pair \( [P, Q] \).

Let us suppose that \( f(\lambda, \mu) \) is an irreducible polynomial in \( K[\lambda, \mu] \) and denote by \( (f) \) the prime ideal generated by \( f \) in \( K[\lambda, \mu] \). As a polynomial in \( C[\lambda, \mu] \) is also irreducible and the ideal generated by \( f \) in \( C[\lambda, \mu] \) is also prime, abusing the notation we will also denote it by \( (f) \) and distinguish it by the context. Let us denote by \( C(\Gamma) \) and \( K(\Gamma) \) the fraction fields of the domains \( C[\lambda, \mu]/(f) \) and \( K[\lambda, \mu]/(f) \) respectively. Observe that \( C(\Gamma) \) and \( K(\Gamma) \) are usually interpreted as rational function on \( \Gamma \).

Remark 5.3. As differential operators in \( K[\lambda, \mu][\partial] \), the operators \( P - \lambda \) and \( Q - \mu \) have no common nontrivial solution, see (26), but as elements of \( K(\Gamma)[\partial] \) they have a common non constant factor. By Theorem 4.5 the first nonzero subresultant \( L_r = \text{gcd}(P - \lambda, Q - \mu) \) is the greatest common divisor of \( P - \lambda \) and \( Q - \mu \) in \( K(\Gamma)[\partial] \). We will use subresultants in Section 6 to compute factorizations of KdV Schrödinger operators.

5.2. Spectral curves

Let us consider the KdV Schrödinger operators \( L_s = L(u_s) = -\partial^2 + u_s \), where \( u_s \) is a KdV potential of KdV level \( s \) and basic constants vector \( \bar{c}^s \), as defined in Section 3.1. Let \( A_{2s+1} \) be the differential operator that determines the centralizer of \( L_s \), see (21).

Corollary 5.4. The spectral curve \( \Gamma_s \) of the pair \( [L_s, A_{2s+1}] \) is defined by the polynomial in \( C[\lambda, \mu] \),

\[
f_s(\lambda, \mu) := \frac{\partial \text{Res}(L_s - \lambda, A_{2s+1} - \mu)}{\partial \lambda} = -\mu^2 - R_{2s+1}(\lambda),
\]

where \( R_{2s+1}(\lambda) \) is a polynomial of degree \( 2s + 1 \) in \( C[\lambda] \). The polynomial \( f_s(\lambda, \mu) \) is irreducible in \( K[\lambda, \mu] \). In addition, the coefficients of \( R_{2s+1}(\lambda)(u, \bar{c}^s, \lambda) \) in Proposition 4.8 are first integrals of KdV\(_{\lambda}(u, \bar{c}^s)\).
Proof. By (22), \([A_{2n+1}, L_n] = \text{KdV}_n(u, \bar{e}_n) = 0\). Thus by Theorem 5.2, \(f_s \in C[\lambda, \mu]\). In addition, by Proposition 4.8, \(f_s = -\mu^2 - R_{2s+1}(\lambda)\), which can be easily proved to be irreducible in \(K[\lambda, \mu]\) because it has odd degree in \(\lambda\).

Example 5.5. Let us continue working with the families of potentials of Examples 3.4, where the basic constants vectors \(\bar{e}^s\) were computed. The next tables show the computation of \(f_s\) for some values of \(s\), using the differential resultant, which we implemented in Maple 18.

1. Family of rational potentials \(u_s = (s + 1)/x^2\), \(s \geq 1\):

| \(s\) | \(\bar{e}^s\) | \(f_s\) |
|---|---|---|
| 1 | (0) | \(-\mu^2 - \lambda^3\) |
| 2 | (0, 0) | \(-\mu^2 - \lambda^5\) |
| 3 | (0, 0, 0) | \(-\mu^2 - \lambda^7\) |

2. Family of Rosen-Morse potentials \(u_s = \frac{-s(s+1)}{\cosh(x)}\), \(s \geq 1\):

| \(s\) | \(\bar{e}^s\) | \(f_s\) |
|---|---|---|
| 1 | (1) | \(-\mu^2 - \lambda(\lambda + 1)^2\) |
| 2 | (5, 4) | \(-\mu^2 - \lambda(\lambda + 1)^2(\lambda + 4)^2\) |
| 3 | (14, 49, 36) | \(-\mu^2 - \lambda(\lambda + 1)^2(\lambda + 4)^2(\lambda + 9)^2\) |

3. Family of elliptic potentials \(u_s = (s + 1)\varphi(x; g_2, g_3)\), \(s \geq 1\):

| \(s\) | \(\bar{e}^s\) | \(f_s\) |
|---|---|---|
| 1 | (0) | \(-\mu^2 - \lambda^3 + (1/4)g_2\lambda - (1/4)g_3\) |
| 2 | (0, \(\frac{21}{2}g_2, \frac{297}{4}g_3\)) | \(-\mu^2 - \frac{1}{4}(-\lambda + 3g_2)(-\lambda + 9g_2 + 27g_3)\) |
| 3 | (0, \(\frac{61}{4}g_2, -\frac{297}{4}g_3\)) | \(-\mu^2 - R_7(\lambda)\) |

with

\[
R_7 = \frac{1}{16}(\lambda(-16\lambda^6 + 504g_2\lambda^4 + 2376g_3\lambda^3 - 4185g_2^2\lambda^2 + 3375g_2^3\lambda - 36450g_2g_3\lambda - 91125g_3^2)).
\]

For example, if

\[
u_1 = -\frac{2}{\cosh^2(x)}\text{ then } [L_1, A_3] = \text{KdV}_1(u_1, (1)) = 0,
\]

the spectral curve \(\Gamma_1\) of the Lax pair \([L_1, A_3]\) is defined by the polynomial

\[
f_1(\lambda, \mu) = -\mu^2 - \lambda(\lambda + 1)^2 =
\]

\[
\begin{vmatrix}
-1 & 0 & -2 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
-1 & 0 & \frac{3}{\cosh(x)} + 1 \\
0 & -1 & 0
\end{vmatrix} =
\begin{vmatrix}
-\lambda & 8 \frac{\sinh(x)}{(\cosh(x))^2} & 4 \frac{\sinh(x)}{(\cosh(x))^2} - 12 \frac{(\sinh(x))^2}{(\cosh(x))^4} \\
0 & -\lambda & 4 \frac{\sinh(x)}{(\cosh(x))^2} \\
0 & 0 & -\lambda \\
-1 & 0 & \frac{3}{\cosh(x)} + 1 \\
0 & -1 & \frac{3}{\cosh(x)} + 1
\end{vmatrix}
\]
6. KdV factors of KdV Schrödinger operators \( L_s - \lambda \)

Let \( u_s \) be a KdV potential of KdV level \( s \) and basic constants vector \( \bar{c}^s \). Let \( A_{2s+1} \) be the differential operator that determines the centralizer of the KdV Schrödinger \( L_s = -\partial^2 + u_s \) as in Theorem 3.7. By Corollary 5.4, the spectral curve \( \Gamma_s \) of the pair \( \{ L_s, A_{2s+1} \} \) is defined by the irreducible polynomial

\[
\phi_s(\lambda, \mu) = \partial \text{Res}(L_s - \lambda, A_{2s+1} - \mu) = \mu^2 - R_{2s+1}(\lambda) \in \mathbb{C}[\lambda, \mu].
\]

Let \( C(\Gamma_s) \) and \( K(\Gamma_s) \) be the fraction fields of the domains \( \mathbb{C}[\lambda, \mu]/(f_s) \) and \( K[\lambda, \mu]/(f_s) \). In this section, we explain how to factor \( L_s - \lambda \) as an operator in \( K(\Gamma_s)[\partial] \). For this purpose, we compute first the field of constants of \( K(\Gamma_s) \).

Afterwards, by means of the differential subresultant \( L_1 = \phi_1 + \phi_2 \partial \), we give a symbolic algorithm to compute the right factor \( \partial - \phi_s \) of \( L_s - \lambda \) over \( K(\Gamma_s) \), based on the differential subresultant theorem. It is well known that the factorization of \( L_s - \lambda \) is equivalent to the computation of a solution \( \phi_s \) of the Riccati equation \( \phi' + \phi^2 = u_s - \lambda \) associated to \( L_s - \lambda \). Thus we are computing a solution \( \phi_s \) in closed form of the Riccati equation as the quotient of two determinants \(-\phi_1/\phi_2\), which is a well defined function over the spectral curve. As far as we know, there are no "general" algorithms to obtain these factors explicitly in terms of the potential \( u_s \). There are some differential recursive expressions but no final closed formulas are exhibited [31] and the well known Kovacic algorithm for rational potential [35]. This symbolic factorization structure allows an specialization process to points \( \lambda_0 = (\lambda_0, \mu_0) \) of \( \Gamma_s \) (see Proposition 8.1).

An important remark should be made. There are no symbolic algorithms (as far as we know) to compute a solution \( \Psi \), of \((\partial - \phi_s)\Psi = 0 \) over the curve \( \Gamma_s \). Let us assume for a moment that a global parametrization of the algebraic affine curve \( \Gamma_s \) in \( \mathbb{C}^2 \), in the sense of [3], \( \text{N}(\tau) = (\chi_1(\tau), \chi_2(\tau)) \) has been computed. Then we can give a one-parameter form of the factorization algorithm that returns a factor \( \partial - \rho(\phi_s) \) of \( L_s - \chi_1(\tau) \), by means of a differential isomorphism

\[
\rho : K(\Gamma_s) \longrightarrow \mathcal{F} := K(\chi_1(\tau), \chi_2(\tau)).
\]

The symbolic integration techniques of M. Bronstein [11] can be used then to obtain a one-parameter closed form solution \( \Psi(\tau) \) of our problem (2), integrating the Risch differential equation \((\partial - \rho(\phi_s))\Psi = 0 \). Hence we solve (2) for KdV potentials as an ODE with coefficient field \( \mathcal{F} \) given by the spectral curve, following Drach’s ideology.

6.1. Structure of the coefficient field

In this subsection we obtain the canonical form of the elements of the coefficient field \( K(\Gamma_s) \), that is, a reduced form of the rational functions over the curve. We then prove that \( C(\Gamma_s) \) is the field of constants of \( K(\Gamma_s) \).

We extend the derivation \( \partial \) of \( K \) to the polynomial ring \( K[\lambda, \mu] \) by (4), with ring of constants \( C[\lambda, \mu] \). Let \( (f_s) \) be the ideal generated by \( f_s \) in \( K[\lambda, \mu] \). Observe that for any \( p \in K[\lambda, \mu] \) it holds that

\[
\partial(p f_s) = \partial(p) f_s + p \partial(f_s) = \partial(p) f_s,
\]

since \( f_s \in C[\lambda, \mu] \). This implies that \( (f_s) \) is a differential ideal in \( (K[\lambda, \mu], \partial) \). Let us consider the rings

\[
C[\Gamma_s] = \frac{C[\lambda, \mu]}{(f_s)} \quad \text{and} \quad K[\Gamma_s] = \frac{K[\lambda, \mu]}{(f_s)}
\]
Proposition 6.2. Let $q$ be a polynomial in $K$. Secondly we consider the standard differential structure of the quotient ring $K$ given by the derivation $	ilde{\partial}$ as follows:

$$
\tilde{\partial}(q + (f_3)) = \partial(q) + (f_3), \quad q \in K.[\lambda, \mu].
$$

(29)

Observe that $\tilde{\partial}$ is a derivation in $K$ since $(f_3)$ is a differential ideal. By abuse of notation we also denote by $\tilde{\partial}$ its extension to the fraction field $K$.

To work in the ring $K$ we will consider special representatives of its elements as follows.

Remark 6.1. Observations will be very important in what follows.

Theorem 6.3. Let us consider the monomial lexicographical order with $\lambda > \mu$ in $K$. Given $p + (f_3) \in K$, with representative $p \in K[\lambda, \mu]$, let us denote by $p_N$ the normal form of $p$ with respect to $(f_3)$ (see [18]), that is, $p_N$ can be obtained by replacing $\mu^2$ by $R_{\lambda+1}(\lambda)$ in $p$. We will call $p_N$ the normal form of $p$ on $\Gamma$. Observe that $p_N$ is a polynomial in $K[\lambda, \mu]$ of degree one in $\mu$. The following observations will be very important in what follows.

Proposition 6.2. Let $q$ be a polynomial in $K[\lambda, \mu]$ of degree greater or equal than one in $\mu$.

1. If $h$ is a factor of $q$ then $q = \beta h$ for some nonzero $\beta$ in $K[\lambda]$.
2. $q = a + b\gamma$ is irreducible in $K[\lambda, \mu]$ if and only $\gcd(a, b) = 1$.
3. Given $q$ of degree one in $\mu$, we can factor it as $q = \gamma b$ where $\Lambda \in K[\lambda]$ and $\gamma$ irreducible of degree one in $\mu$.

Proof. The polynomials $q$ and $f_3$ can be seen as polynomials in $\mu$ with coefficients in the field $K$. By hypothesis, $f_3$ does not divide $q$ in $K[\lambda, \mu]$, therefore they are coprime. There exist $A, B \in K[\lambda, \mu]$ such that $Ah + Bf_3 = 1$, see [18]. Chapter 1, § 5, Proposition 6. We can write $A(\lambda, \mu) = g(\lambda, \mu)/a(\lambda)$ and $B(\lambda, \mu) = h(\lambda, \mu)/b(\lambda)$, with $g, h \in K[\lambda, \mu]$ and nonzero $a, b \in K$. Thus $bhq + ahf_3 = ab$ in $K[\lambda, \mu]$. In $K$ we have

$$(T + (f_3))(q + (f_3)) = \Lambda(\lambda) + (f_3), \quad T = bg$$

with $T = \partial q$. Hence $T = (\partial q)$ in $K[\lambda, \mu]$ and for $r = \partial q$ statement 2 follows.

Theorem 6.3. 1. The ring of constants of $(K[\lambda, \mu], \partial)$ is $C[\lambda, \mu]$.

2. The field of constants of $(K[\lambda, \mu], \partial)$ is $C(\lambda)$.

Proof. Let us consider $p + (f_3)$ in $K[\lambda, \mu]$ such that $\partial(p + (f_3)) = 0$. Let $p_N = a + b\gamma$ be its normal form on $\Gamma$, then $\partial(a) + \partial(b)\mu = (f_3)$ in $K[\lambda, \mu]$. Then $\partial(a) = \partial(b) = 0$. Hence $p + (f_3)$ belongs to $C[\lambda, \mu]$, which proves statement 1.

Let us also consider $\nu \in K[\lambda, \mu]$ such that $\partial(\nu) = 0$. By Proposition 6.2, we have $\nu = \frac{\partial(\nu)}{\partial(\lambda)}$ in $K[\lambda, \mu]$, with $p \in K[\lambda, \mu]$ and $\Lambda_1 \in K[\lambda, \mu]$.

If $\nu = \Lambda_2 \in K[\lambda, \mu]$ then $\partial(\nu) = \frac{\partial(\nu)}{\partial(\lambda)}$ in $K[\lambda, \mu]$ with $H = \partial(\Lambda_2)\Lambda_1 - \partial(\Lambda_1)\Lambda_2$. Thus $0 = \Lambda_2^2 \partial(\nu) = H + (f_3)$. Since $H$ is a polynomial in $K[\lambda, \mu]$ then $H = 0$. Hence $\Lambda_2 = \gamma \Lambda_1$, with $\gamma$ in the field of constants $C(\lambda)$ of $K[\lambda, \mu]$. Consequently $\nu \in C(\lambda)$.
If $p \notin K[\lambda]$, by Remark 6.1 we can write $v = \frac{p_N}{\lambda}$ in $K(\Gamma_s)$, where the normal form on $\Gamma_s$ of $p$ equals $p_N = a + b\mu$. Now $0 = \Lambda^2 \tilde{\partial}(v) = H + (f_i)$, with

$$H = \Lambda_1 \partial(p_N) - \partial(\Lambda_1)p_N.$$  

Since the degree in $\mu$ of $H$ equals one then $H = 0$ in $K[\lambda, \mu]$. This implies that $p_N$ divides $\partial(p_N)$ and by Remark 6.1, $\partial(p_N) = \beta p_N$, with $\beta \in K[\lambda]$. Thus $a$ and $b$ are solutions of the linear differential equation $\tilde{\partial}(\Psi) = \beta(\lambda)\Psi$ then $a = cb$, $c = c_1/c_2$ with $c_1, c_2 \in C[\lambda]$. Therefore $v = \frac{b(c_2 + \mu)}{c_1 \lambda}$ in $K(\Gamma_s)$ and

$$0 = \tilde{\partial}(v) = (c_1 + \mu)\tilde{\partial}(w) \iff \tilde{\partial}(w) = 0,$$

with $w = \frac{b}{c_1 \lambda}$ in $K(\Gamma_s)$. Thus $w \in C(\Gamma_s)$, which proves that $v \in C(\Gamma_s)$, and statement 2 is proved. \qed

The next commutative diagram summarizes the situation:

$$\begin{array}{ccc}
K[\Gamma_s] & \rightarrow & K(\Gamma_s) \\
\downarrow & & \downarrow \\
C[\Gamma_s] & \rightarrow & C(\Gamma_s)
\end{array}$$

6.2. KdV factors on $\Gamma_s$

In this section, we consider the operators $L_s - \lambda$ and $A_{2s+1} - \mu$ as elements of $K(\Gamma_s)[\partial]$. Let $L_1 = \varphi_2 \partial + \varphi_1$ be the subresultant of $L_s - \lambda$ and $A_{2s+1} - \mu$ as in Section 4.2.

**Theorem 6.4.** The greatest common factor of the differential operators $L_s - \lambda$ and $A_{2s+1} - \mu$ in $K(\Gamma_s)[\partial]$ is the order one operator $L_1$.

**Proof.** Since $L_0 = \partial \text{Res}(L_s - \lambda, A_{2s+1} - \mu)$ is zero in $K(\Gamma_s)$ by Theorem 4.5 the result follows. \qed

We can take the monic greatest common factor of $L_s - \lambda$ and $A_{2s+1} - \mu$ to be

$$\partial - \phi_s,$$

where $\phi_s = -\frac{\varphi_1}{\varphi_2}$.

The fact that $\partial - \phi_s$ is a right factor implies that

$$L_s - \lambda = (-\partial - \phi_s)(\partial - \phi_s), \text{ in } K(\Gamma_s)[\partial]$$

and moreover $\phi_s$ is a solution of the Riccati equation associated to the Schödinger operator $L_s - \lambda$

$$\partial(\phi) + \phi^2 = u_s - \lambda \tag{31}$$

on the spectral curve $\Gamma_s$. Therefore, we compute a solution of (31) by means of the differential subresultant $L_1$. We will give next some details about $\phi_s$.

**Lemma 6.5.** The following formula holds

$$\phi_s = \frac{\mu + \alpha(\lambda)}{\varphi(\lambda)}, \tag{32}$$

where $\alpha$ and $\varphi$ are nonzero polynomials in $K[\lambda]$. Moreover $\phi_s$ is nonzero in $K(\Gamma_s)$. 22
Proof. By (23),(24), we have that \( \varphi_1 = \det(S(L_2 - \lambda, A_{2s+1} - \mu)) \) and \( \varphi_2 = \det(S(L_2 - \lambda, A_{2s+1} - \mu)) \). Now by Proposition 4.8, \( \varphi_1 = -\mu - \alpha \) and \( \alpha, \varphi = \varphi_2 \) are non-zero polynomials in \( K[\lambda] \). Observe that \( \varphi_2 = 0 \) in \( K(\Gamma_s) \) if and only if \( \mu + \alpha + (f_s) = 0 \) in \( K[\Gamma_s] \). But this is not possible since \( f_s \), which has degree 2 in \( \mu \), is not a factor of \( \mu + \alpha \) in \( K[\lambda, \mu] \). This proves the last claim. \( \square \)

To keep notation as simple as possible, we will also write \( \phi_s \) to denote the element \( \phi_s \) in \( K(\Gamma_s) \). The next algorithm takes as an input a KdV potential to obtain the factor \( \partial - \phi_s \) of \( L_s - \lambda \) in \( K(\Gamma_s)[\partial] \).

**Algorithm 6.6. (Factorization)**

- **Given** \( u_s \) a KdV potential of KdV level \( s \) and given \( \bar{c}^s \) the basic constants vector of \( u_s \).
- **Return** the defining polynomial \( f_s \) of the spectral curve \( \Gamma_s \) and the monic greatest common divisor \( \partial - \phi_s \) of \( L_s - \lambda \) and \( A_{2s+1} - \mu \) in \( K(\Gamma_s)[\partial] \).

1. Define \( L_s := -\partial^2 + u_s \) and \( A_{2s+1} := \hat{P}_{2s+1}(u_s, \bar{c}^s) \) as in (21).
2. Compute \( f_s = \partial \text{Res}(L_s - \lambda, A_{2s+1} - \mu) \), the defining polynomial of the spectral curve \( \Gamma_s \).
3. Compute \( L_1 = \varphi_1 + \varphi_2 \partial \), the subresultant of \( L_s - \lambda \) and \( A_{2s+1} - \mu \) as in Section 4.2.
4. Define \( \phi_s := -\frac{\partial}{\varphi} \).
5. Return \( f_s \) and \( \partial - \phi_s \).

**Remark 6.7.** Observe that We are computing \( \phi_s \) in closed form as the quotient of two determinants \( -\varphi_1 \varphi_2 \), which is a well defined function over the spectral curve. As far as we know, there were no algorithms to obtain the factors \( \partial - \phi_s \) of \( L_s - \lambda \) over the spectral curve. In [31], there are some differential recursive expressions but no final closed formulas.

**Examples 6.8.** We implemented the Factorization Algorithm 6.6 in Maple 18. Let us continue working with the families of potentials of Examples 5.5, where the values of \( f_s \) were computed. The next tables show the results of the computation of \( \phi_s \).

1. **Family of rational potentials** \( u_s = s(s + 1)/x^2 \), \( s \geq 1 \):

   \[
   s & \quad \bar{c}^s & \quad \phi_s \\
   1 & (0) & \mu x^3 - 1 \\
   2 & (0, 0) & -\mu x^3 + 3 \lambda x^2 + 18 \\
   3 & (0, 0, 0) & -\mu x^3 + 6 \lambda x^2 + 9 + 90 \lambda x^2 + 675 \\
   4 & (0, 0, 0, 0) & -\mu x^3 + 10 \lambda x^2 + 270 \lambda x^2 + 4725 \lambda x^2 + 44100 \\
   & & x (x^4 + 3 \lambda x^3 + 135 \lambda x^2 + 1575 \lambda x^2 + 44100)
   \]

2. **Family of Rosen-Morse potentials** \( u_s = \frac{-\mu(x^{s+1})}{\cosh(x)} \), \( s \geq 1 \):

   \[
   s & \quad \bar{c}^s & \quad \phi_s \\
   1 & (1) & \mu \cosh(x)^3 + \sinh(x) \\
   2 & (5, 4) & \frac{\mu \cosh(x)^2 + 3 \cosh(x)^2 \cosh(x) l + 12 \sinh(x) \cosh(x)^2 - 18 \sinh(x)}{\cosh(x)^2 l^2 + 5 \cosh(x)^4 l + 4 \cosh(x)^4 l^2 + 3 \cosh(x)^2 l^2 + 12 \cosh(x)^2 l^2 + 9 \cosh(x)} \\
   3 & (14, 49, 36) & \frac{\mu + \alpha l}{\varphi(d)}
   \]
with
\[ \alpha = \frac{6 \cosh(x)^4 \sinh(x) \lambda^2 + 78 \cosh(x)^4 \sinh(x) \lambda - 90 \cosh(x)^2 \sinh(x) \lambda + a}{\cosh(x)^7}, \]
\[ a = 27 \sinh(x)(8 \cosh(x)^4 - 30 \cosh(x)^2 + 25), \]
\[ \varphi = \frac{\cosh(x)^6 \lambda^3 + 14 \cosh(x)^6 \lambda^2 + 49 \cosh(x)^6 \lambda - 6 \cosh(x)^4 \lambda^2 - 78 \cosh(x)^2 \lambda + 45 \cosh(x)^2 \lambda + b}{\cosh(x)^6}, \]
\[ b = 9 \sinh(x)^2(4 \cosh(x)^4 - 20 \cosh(x)^2 + 25) \]

3. Family of elliptic potentials \( u_s = s(s+1)\varphi(x; g_2, g_3), s \geq 1 \):

\[
\begin{array}{c|c|c}
\lambda & \varphi & \mu + \frac{1}{2}\varphi' \\
-1 & 0 & -\mu - \frac{g_2}{2} \\
0 & \left(0, \frac{-21}{8} g_2\right) & -\mu - 9\varphi + \frac{3g_2}{2} \\
1 & \left(0, \frac{-63}{8} g_2 - \frac{225}{8} s_1\right) & \frac{\mu + 9\varphi - \frac{225}{8} g_2}{\lambda^3 + 6\varphi \lambda^2 + (45\varphi^2 - 15g_2)\lambda - 225\varphi^2} \\
\end{array}
\]

where \( \varphi \) and \( \varphi' \) denote \( \varphi(x; g_2, g_3) \) and \( \varphi'(x, g_2, g_3) \) respectively.

We would like to obtain a univariate expression of \( \phi_t \) using a parametric representation of \( \Gamma_x \), whenever it is possible. This will allow us to give a functional representation of \( \phi_t \) and as a byproduct, we will obtain a domain of definition of the solutions of \( L_x - \lambda \), see Section 9.5.

### 6.3. One-parameter form of KdV factors

Once we have solved the factorization problem in Algorithm 6.6, what remains is to replace \((\lambda, \mu)\) by a parametric representation \((\chi_1(\tau), \chi_2(\tau))\) of \( \Gamma_x \). This procedure strongly depends on the genus of the algebraic curve \( \Gamma_x \). We summarize next what are the parametrization possibilities (as far as we know) and emphasize on the algorithmic aspects of the process. Furthermore, we give a new algorithm that provides a one-parameter form of the factor \( \partial \phi_t \) of \( \Gamma_x \).

In this subsection we take \( C = \mathbb{C} \). Let \( \tau \) be an algebraic indeterminate over \( K \), that is, \( \tau \) is not the root of a polynomial with coefficients in \( K \) and \( \partial(\tau) = 0 \). Based on the [3] we give the following definition.

**Definition 6.9.** Let \( \mathbb{P}^1 \) denote the projective space of dimension one over \( \mathbb{C} \) and let us consider a simply connected subdomain \( D \) of \( \mathbb{P}^1 \). An affine algebraic curve \( \Gamma \) in \( \mathbb{C}^2 \) defined by a square free polynomial \( f(\lambda, \mu) \) is parametrizable if there are meromorphic functions \( \chi_1 \) and \( \chi_2 \) in \( D \) such that

1. for all \( \tau_0 \in D \), but a finite number of exceptions, the point \((\chi_1(\tau_0), \chi_2(\tau_0))\) is on \( \Gamma \), and
2. for all \((\lambda_0, \mu_0) \in \Gamma \), but a finite number of exceptions, there exists \( \tau_0 \in D \) such that \((\lambda_0, \mu_0) = (\chi_1(\tau_0), \chi_2(\tau_0)) \).

In this case \( N(\tau) = (\chi_1(\tau), \chi_2(\tau)) \) is called a (global or meromorphic) parametrization of \( \Gamma \).

**Remark 6.10.** 1. An affine algebraic curve \( \Gamma \) admits at any point \( P \in \Gamma \) a local parametrization in the field of Puiseux series, see for instance [48], Section 2.5. In this paper, as far as possible, we would like to maintain a global treatment of the curve, and therefore we will use global parametrizations of \( \Gamma_x \).
2. In the case of rational curves there are algorithms to obtain a global parametrization [48]. For elliptic curves we can define a meromorphic parametrization by means of the Weierstrass $\wp$-function. For all other cases, as far as we know, there are no algorithms to obtain a global parametrization.

A key point to have a one-parameter form factorization algorithm is to obtain a global parametrization of the spectral curve. How complicated is to obtain a global parametrization

**Proof.** $P$ but a finite number of points $\aleph$ for standard notation). Since $C$ restriction to the field of constants $\aleph$ and they determine a

**1. Genus** $g = 0$. If the curve $\Gamma$ is rational then $\Gamma$ can be parametrized by rational functions. In fact, the global parametrization can be defined on $D = C$, for any algebraically closed field $C$ of characteristic zero. A rational parametrization $\mathcal{N}(\tau)$ of $\Gamma$ gives an isomorphism from $C(\Gamma)$ to the field of rational functions $C(\tau)$, see [48], Section 4.1.

**2. Genus** $g = 1$. If $\Gamma$ is an elliptic curve, it can be parametrized by elliptic functions in $D = C$. A meromorphic parametrization is given by $(\chi_1(\tau), \chi_2(\tau)) = (\wp(\tau), \wp'(\tau))$, where $\wp(\tau) = \wp(\tau, g_2, g_3)$ is the Weierstrass $\wp$-function in $\tau$ and $\wp'(\tau) = \frac{d\wp}{d\tau}(\tau)$. In this case, the field $C(\Gamma)$ is isomorphic to $C(\wp(\tau), \wp'(\tau))$, [52].

**3. Genus** $g \geq 2$. We distinguish two cases. If the polynomial $R_{2g+1}(\lambda) = 0$ has simple roots then the affine curve $\Gamma_\lambda$ is nonsingular; $\Gamma_\lambda$ is a hyperelliptic curve. In this case, the work of Brezhnev ([10] and references there in) expresses $\phi_\lambda$ in some sense in terms of $\Theta$-functions associated to the spectral curve. When the spectral curve $\Gamma_\lambda$ is singular, one could use the desingularization of the curve, as in [3], to construct a global parametrization $\chi(\tau)$ of $\Gamma_\lambda$ defined on $D$. The domain $D$ corresponds to a Zariski open set of $\Gamma_\lambda \setminus E$, where $\Gamma_\lambda$ is the Riemann surface associated to $\Gamma_\lambda$ and $E$ is the exceptional divisor in the desingularization of $\Gamma_\lambda$. Hence the global parametrization $\mathcal{N}(\tau)$ can be expressed in terms of automorphic functions, see [27], [53].

Let $\mathcal{N}(\tau) = (\chi_1(\tau), \chi_2(\tau))$ be a global parametrization of the affine algebraic curve $\Gamma$ in $\mathbb{C}^2$, whose existence is guarantied in [3]. Let us consider the fraction field $\mathcal{F} = K(\chi_1(\tau), \chi_2(\tau))$ of the polynomial ring $K[\chi_1(\tau), \chi_2(\tau)]$. Since $\tau$ is an algebraic indeterminate over $K$, by condition 2 in Definition 6.9, it is natural to assume that $\partial(\chi_1(\tau)) = 0$ and $\partial(\chi_2(\tau)) = 0$, which allows to extend the derivation $\partial$ of $K$ to have a differential field $(\mathcal{F}, \partial)$.

**Theorem 6.11.** There exists a differential field isomorphism $\rho$ from $(K(\Gamma_\lambda), \tilde{\partial})$ to $(\mathcal{F}, \partial)$, whose restriction to the field of constants $\mathbb{C}(\Gamma_\lambda)$ of $K(\Gamma_\lambda)$ is an isomorphism to the field of constants $\mathbb{C}(\chi_1(\tau), \chi_2(\tau))$ of $\mathcal{F}$.

**Proof.** Let $\rho : K[\lambda, \mu] \rightarrow K[\chi_1(\tau), \chi_2(\tau)]$ be a ring homomorphism defined by

$$\rho(\lambda) = \chi_1(\tau), \quad \rho(\mu) = \chi_2(\tau), \quad \rho(a) = a, \quad \forall a \in K.$$ 

Observe that the ideal $I = (f_\lambda)$ generated by $f_\lambda$ in $K[\lambda, \mu]$ is contained in $J = \text{Ker}(\rho)$. Let us prove that $I = J$. Recall that $f_\lambda$ is an irreducible polynomial in $K[\lambda, \mu]$. Thus $I$ and $J$ are prime ideals and they determine affine algebraic irreducible varieties in $K^2$ such that $V(J) \subset V(I)$ (see [18] for standard notation). Since $N$ is a global parametrization of $\Gamma_\lambda$ in $D$ (see definition 6.9), for all but a finite number of points $P \in \Gamma_\lambda$, $P = (\chi_1(\tau_0), \chi_2(\tau_0))$, $\tau_0 \in D$. Thus $h(P) = 0$, for all $h \in J$, which implies $P \in V(J)$. This proves that $V(J)$ is an infinite variety. By [28], p. 9 since $V(I)$ is irreducible we conclude that $V(I) = V(J)$. Finally, $J = I$ because $V(J)$ is also irreducible.
We therefore have a ring isomorphism \( \rho : K(\Gamma_s) \to \mathcal{F} \), which induces the field isomorphism \( \rho : K(\Gamma_s) \to \mathcal{F} \). One can easily check that \( \partial \circ \rho = \rho \circ \tilde{\partial} \). Thus \( \rho \) is a differential field isomorphism and, by Theorem 6.3, the field of constants of \( \mathcal{F} \) equals \( \rho(C(\Gamma_s)) = C(\chi_1(\tau), \chi_2(\tau)) \). \( \square \)

We illustrate Theorem 6.11 with the following commutative diagram:

\[
\begin{array}{ccc}
K(\Gamma_s) & \xrightarrow{\rho} & \mathcal{F} \\
\uparrow \rho & & \uparrow \\
C(\Gamma_s) & \xrightarrow{\rho} & C(\chi_1(\tau), \chi_2(\tau))
\end{array}
\] (33)

We define \( \tilde{\phi}_s := \rho(\phi_s) \). Observe that \( \tilde{\phi}_s \) is a nonzero element of \( \mathcal{F} \) since by Lemma 6.5 \( \phi_s \) is nonzero in \( K(\Gamma_s) \). In particular, in the genus \( g = 0 \) case \( \tilde{\phi}_s \) belongs to \( \mathcal{F} = K(\tau) \). If the genus \( g = 1 \) then \( \tilde{\phi}_s \) belongs to \( \mathcal{F} = K(\phi(\tau), \phi'(\tau)) \), and if \( g \geq 2 \) then \( \tilde{\phi}_s \) belongs to \( \mathcal{F} \).

We extend naturally \( \rho \) to an isomorphism between the rings of differential operators \( \mathcal{O} : K(\Gamma_s)[\partial] \to \mathcal{F}[\partial] \) as follows:

\[ \mathcal{O}\left(\sum_j a_j \partial^j\right) = \sum_j \rho(a_j) \partial^j. \]

For instance \( \mathcal{O}(L_s - \lambda) = L_s - \chi_1(\tau) \) and \( \mathcal{O}(\partial - \phi_s) = \partial - \tilde{\phi}_s \). Furthermore, since the isomorphism respects the ring structure, we have

\[ L_s - \chi_1(\tau) = (-\partial - \tilde{\phi}_s)(\partial - \tilde{\phi}_s) \]

where \( \tilde{\phi}_s \) is a solution of the Ricatti equation \( \partial(\phi) + \phi^2 = u_s - \chi_1(\tau) \), since \( \rho \) respects the differential field structure.

Observe that diagram (33) extends to the following commutative diagram of rings of differential operators:

\[
\begin{array}{ccc}
K(\Gamma_s)[\partial] & \xrightarrow{\mathcal{O}} & \mathcal{F}[\partial] \\
\uparrow \rho & & \uparrow \\
C(\Gamma_s)[\partial] & \xrightarrow{\rho} & C(\chi_1(\tau), \chi_2(\tau))[\partial]
\end{array}
\]

As a consequence of the previous discussion we obtain the following algorithm.

**Algorithm 6.12. (One-parameter form factorization)**

- **Given** a KdV potential \( u_s \) of KdV level \( s \) and given \( \tilde{\phi}_s \) the basic constants vector of \( u_s \).
- **Return** a factor \( \partial - \tilde{\phi}_s(\tau) \) in \( \mathcal{F}[\partial] \) of \( L_s - \chi_1(\tau) \), for a global parametrization \( \mathcal{S}_s(\tau) = (\chi_1(\tau), \chi_2(\tau)) \) of \( \Gamma_s \), and a simply connected domain \( D \) for \( \tau \).

1. Run the Factorization Algorithm to obtain \( f_s \) and \( \phi_s \).
2. Compute the genus \( g \) of \( f_s \).
3. If \( g = 0 \), \( D := \mathbb{C} \) and compute a rational parametrization \( (\chi_1(\tau), \chi_2(\tau)) \) of \( \Gamma_s \).
4. If \( g = 1 \), \( D := \mathbb{C} \) and compute the Weierstrass normal form \( \mu^2 = 4\lambda^3 - g_2\lambda - g_3 \) of the curve \( \Gamma \) and define \( (\chi_1(\tau), \chi_2(\tau)) := (\wp(\tau, g_2, g_3), \wp'(\tau, g_2, g_3)) \).

5. If \( g \geq 2 \) return No algorithm is known to compute a global parametrization of \( \Gamma \), defined by \( f_1 \). If known please insert \( (\chi_1(\tau), \chi_2(\tau)) \) and \( D \), else return \( f_2 \) and \( \phi_1 \) in \( K(\Gamma) \).

6. Substitute \( \lambda = \chi_1(\tau), \mu = \chi_2(\tau) \) in \( \phi_1 \) to obtain \( \phi_2(\tau) \).

7. Return \( \partial - \phi_2(\tau) \) and \( D \).

Regarding Step 4, algorithms to compute the Weierstrass normal form of an algebraic curve are well known, see for instance [55] used to compute it in Maple 18. In Step 5 of the algorithm, for case \( g \geq 2 \), as far as we know there are no effective algorithms to compute a global parametrization \( (\chi_1(\tau), \chi_2(\tau)) \) of \( \Gamma \). This is a difficult open problem. Some contributions have been made in this direction, for instance by Y.V. Brezhnev in [7] where the connection with the uniformization problem was considered. A natural question arises, what is the behavior of the factorization when different parameterizations of the curve \( \Gamma \) are considered. We leave this interesting topic for future research.

**Examples 6.13.** We implemented the Parameterized Factorization Algorithm 6.12 in Maple 18. Let us continue working with the families of potentials of Examples 6.8, where the values of \( \phi_1 \) were computed. The next tables show the value of \( \phi_2(\tau) \) for a chosen global parametrization \( N_\gamma(\tau) \) of \( \tau \in \mathbb{C} \) of \( \Gamma \).

1. Family of rational potentials \( u_s = s(s+1)/x^2 \), \( s \geq 1 \):

| \( s \) | \( N_\gamma(\tau) \) | \( \phi_2(\tau) \) |
| --- | --- | --- |
| 1 | \((-\tau^2, -\tau)\) | \(-\frac{x(-\tau^2x^2 + 1)}{\tau^3x^3 + 1} \) |
| 2 | \((-\tau^2, -\tau^2)\) | \(-\frac{x(\tau^2x^2 - 3\tau^2x^2 + 9)}{\tau^3x^3 + 6\tau^4x^4 - 90\tau^2x^2 + 675} \) |
| 3 | \((-\tau^3, -\tau^2)\) | \(-\frac{x(\tau^6x^6 + 6\tau^4x^4 - 45\tau^2x^2 + 225)}{\tau^3x^3 - 10\tau^5x^8 + 270\tau^3x^4 - 4725\tau^2x^2 + 44100} \) |
| 4 | \((-\tau^2, -\tau^3)\) | \(-\frac{x^3x^9}{\tau^3x^3 - 10\tau^5x^8 + 135\tau^3x^4 - 1575\tau^2x^2 + 11025} \) |

2. Family of Rosen-Morse potentials \( u_s = \frac{x(s+1)}{\cosh(\tau)} \), \( s \geq 1 \):

| \( s \) | \( N_\gamma(\tau) \) | \( \phi_3(\tau) \) |
| --- | --- | --- |
| 1 | \((-\tau^3, -\tau(\tau^2 - 1))\) | \(\frac{(\tau - \tau)w^3 + \left(2\tau^2 - 4\right)w + \tau^2 + \tau}{((\tau - 1)w + \tau + 1)(w + 1)}\) |
| 2 | \((-\tau^3, -\tau(\tau^2 - 1)(\tau^2 - 4))\) | \(\frac{a_3(\tau)w^3 + b_2(\tau)w^2 + a_1(\tau)w + a_0(\tau)}{(b_2(\tau)w^2 + b_1(\tau)w + a_0(\tau))(w + 1)}\) |
| 3 | \((-\tau^2, -\tau(\tau^2 - 1)(\tau^2 - 4)(\tau^2 - 9))\) | \(\frac{c_3(\tau)w^3 + c_2(\tau)w^2 + c_1(\tau)w + c_0(\tau)}{(d_3(\tau)w^3 + d_2(\tau)w^2 + d_1(\tau)w + d_0(\tau))(w + 1)}\) |

where \( w = e^x \),

\[
\begin{align*}
a_3 = -\tau^3 - 3\tau^2 - 2\tau, & \quad a_2 = -3\tau^3 - 3\tau^2 + 18\tau + 24, & \quad a_1 = -3\tau^3 + 3\tau^2 + 18\tau + 24, \\
a_0 = -\tau^3 + 3\tau^2 - 2\tau, & \quad b_2 = \tau^2 + 3\tau + 2, & \quad b_1 = 2\tau^2 - 8, & \quad b_0 = \tau^2 - 3\tau + 2.
\end{align*}
\]
and

\[ c_4 = r^4 - 6 r^3 + 11 r^2 - 6 r, \quad c_3 = 4 r^4 - 12 r^3 - 40 r^2 + 168 r - 144, \quad c_2 = 6 r^4 - 102 r^2 + 432, \]
\[ c_1 = 4 r^4 + 12 r^3 - 40 r^2 - 168 r - 144, \quad c_0 = r^4 + 6 r^3 + 11 r^2 + 6 r, \quad d_3 = r^3 - 6 r^2 + 11 r - 6, \]
\[ d_2 = 3 r^3 - 6 r^2 - 27 r + 54, \quad d_1 = 3 r^3 + 6 r^2 - 27 r - 54, \quad d_0 = r^3 + 6 r^2 + 11 r + 6. \]

3. Let us consider the elliptic potential \( u_1 = 2\wp(x; g_2, g_3) \). In this case, one can easily prove that \( N_1(\tau) = \left(-\wp(\tau), \frac{1}{2}\wp'(\tau)\right) \) is a global parametrization of the spectral curve \( \Gamma \) whose defining polynomial is the irreducible polynomial \( f \) as in Examples 6.13, 3. In this case

\[ \tilde{\phi}_1 = \frac{\phi'(x) - \wp'(\tau)}{\wp(x) - \wp(\tau)}. \]

7. Spectral Picard-Vessiot fields for KdV Schrödinger operators

We are ready now to introduce the main concept of this paper, the spectral Picard-Vessiot field of the equation

\[ (L_s - \lambda)\Psi = 0. \tag{34} \]

We will first prove the existence of the spectral PV field highlighting the importance of the field of constants \( C(\Gamma_s) \) in the determination of closed formulas for the solutions. Afterwards we obtain a one-parameter version of the spectral PV field of (34). For algebro-geometric potentials, Brezhnev gave in [9] integral formulas, using \( \Theta \)-functions, for the solutions in terms of the spectral parameter. Whereas, for KdV potentials, our approach gives one-parameter closed form solutions \( \Psi \) of (34), using the essential transcendental functions provided by the parametrization of the spectral curve. This approach brings us back to Drach’s ideology: to solve problem (34) finding the minimum set of transcendental functions necessary to parametrize the spectral curve and compute its Liouvillian extensions. Obviously the development of parametrization algorithms for the algebraic curve \( \Gamma_s \) depends on its genus, as discussed in Section 6.3.

7.1. Definition and existence

As before \( u_s \) is a KdV potential of KdV level \( s \) and basic constants vector \( \vec{c}' \) and we consider \( (L_s - \lambda)(\Psi) = 0 \) as homogeneous linear differential equation of order two with coefficients in \( (K(\Gamma_s), \tilde{\partial}) \). Since \( \tilde{\partial} \) extends the derivation \( \partial \) of \( K \), when there is not room for confusion we write \( \tilde{\partial} \) instead of \( \partial \).

**Definition 7.1.** A differential field extension \( \mathcal{L} \) of \( K(\Gamma_s) \) is called a spectral Picard-Vessiot field over the curve \( \Gamma_s \) of the equation \( (L_s - \lambda)(\Psi) = 0 \) if the following conditions are satisfied:

1. \( \mathcal{L} = K(\Gamma_s)(\Psi_1, \Psi_2) \), the differential field extension of \( K(\Gamma_s) \) generated by \( \Psi_1, \Psi_2 \), where \( \{\Psi_1, \Psi_2\} \) is a fundamental set of solutions of \( (L_s - \lambda)(\Psi) = 0 \).
2. \( \mathcal{L} \) and \( K(\Gamma_s) \) have the same field of constants \( C(\Gamma_s) \).

Our next goal is to prove the existence of a spectral Picard-Vessiot field of \( (L_s - \lambda)(\Psi) = 0 \). By Section 6.2 we have

\[ L_s - \lambda = (-\tilde{\partial} - \phi_+)(\tilde{\partial} - \phi_+), \quad \text{where} \quad \phi_+ := \phi_s. \]
with $\phi$, as in Lemma 6.5. Let us consider now the operator $A_{2r+1} + \mu$ and observe that

$$f_5(\lambda, \mu) = \partial \text{Res}(L_s - \lambda, A_{2r+1} + \mu) = \mu^2 - R_{2r+1}(\lambda).$$

Applying Theorem 6.4 changing $\mu$ by $-\mu$ we obtain another factorization of $L - \mu$ in $K(\Gamma_\alpha)[\partial]$, namely

$$L_s - \lambda = (-\partial - \phi_+)(\partial - \phi_-), \quad \text{in } K(\Gamma_\alpha)[\partial].$$

The analogous result of Lemma 6.5 in this case gives

$$\phi_- = \frac{-\mu + \alpha(\lambda)}{\varphi(\lambda)}.$$

Let us consider nonzero solutions $\Psi_+$ and $\Psi_-$ respectively of the differential equations

$$\partial(\Psi) = \phi_+\Psi \text{ and } \partial(\Psi) = \phi_-\Psi.$$ 

Therefore

$$\frac{\partial(\Psi_+)}{\Psi_+} = \phi_+ \text{ and } \frac{\partial(\Psi_-)}{\Psi_-} = \phi_-$$

belong to $K(\Gamma_\alpha)$ and $K(\Gamma_\alpha)(\Psi_+)$ and $K(\Gamma_\alpha)(\Psi_-)$ are Liouvillian extensions of $K(\Gamma_\alpha)$.

**Lemma 7.2.** Let $\Psi_+$ and $\Psi_-$ as in (47), it holds that:

1. $\{\Psi_+, \Psi_-\}$ is a fundamental set of solutions of $(L_s - \lambda)(\Psi) = 0$.
2. $\Psi_+, \Psi_- \in K(\Gamma_\alpha)$.

**Proof.** Trivially $\Psi_+$ and $\Psi_-$ are nonzero solutions of $L_s - \lambda$. We will prove that their wronskian is nonzero. Observe that $\mu$ is a constant in $K(\Gamma_\alpha)$, that is $\mu \in C(\Gamma_\alpha)$ and furthermore it is nonzero. Since $\partial(w(\Psi_+, \Psi_-)) = 0$, $w(\Psi_+, \Psi_-)$ belongs to $C(\Gamma_\alpha)$. The following computation

$$\frac{w(\Psi_+, \Psi_-)}{\Psi_+ \Psi_-} = \frac{\partial(\Psi_+)}{\Psi_+} - \frac{\partial(\Psi_-)}{\Psi_-} = \phi_+ - \phi_- = \frac{2\mu}{\varphi}$$

implies that $w(\Psi_+, \Psi_-) \neq 0$ in $C(\Gamma_\alpha)$. This formula implies that

$$\Psi_+ \Psi_- = \frac{w(\Psi_+, \Psi_-)}{2\mu} \in K(\Gamma_\alpha),$$

which completes the proof.

**Remark 7.3.** Given a KdV Schrödinger operator $L_\alpha$, the second symmetric power of $(L_s - \lambda)\Psi = 0$ is the differential equation $S^2(\Psi) = 0$, whose solution space is generated by $\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$, (see [54]), with

$$S^2 := \partial^3 - 4(u_s - \lambda)\partial - 2u_s'.$$

By (37), observe that there exists a constant $\beta = 1/(2\mu w(\Psi_+, \Psi_-))$ in $C(\Gamma_\alpha)$ such that

$$\Psi_+ \Psi_- = \beta \varphi.$$ 

Therefore $\varphi$ is a solution of $S^2(\Psi) = 0$, i.e. it verifies the fundamental linear differential equation given in [31], equation (1.12) and [9], equation (11)

$$\partial^3(\varphi) - 4(u_s - \lambda)\partial(\varphi) - 2u_s'\varphi = 0.$$
By Lemma 7.2, it holds that
\[ K(\Gamma_s) (\Psi_+, \Psi_-) = K(\Gamma_\gamma) (\Psi_+, \Psi_-) = K(\Gamma_\gamma). \] (39)
Let us denote \( \Psi'_s := \Psi_+ \). Now we will proceed to determine the subfield of constants of the field \( K(\Gamma_\gamma) (\Psi'_s) \). For this purpose we will use results of M. Bronstein in [11].

**Definition 7.4** ([11], Definition 3.4.3). Let \((F, \delta)\) be a differential field. We say that \( \phi \in F \) is a logarithmic derivative of a \( F \)-radical if there exist a nonzero \( v \in F \) and an integer \( n \neq 0 \) such that \( n\phi = \delta v / v \).

Observe that if there exist a nonzero \( v \in K(\Gamma_\gamma) \) and a nonzero integer \( n \) such that
\[ \frac{\tilde{\delta}(v)}{nv} = \phi_s = \frac{\tilde{\delta}(\Psi'_s)}{\Psi'_s} \]
then for \( c := \Psi'_s / v \) we have \( \tilde{\delta}(c) = 0 \) and also \( \Psi'_s - cv = 0 \). This means that \( \Psi'_s \) is algebraic over a differential field that is generated by \( K(\Gamma_\gamma) \) and a possibly new constant \( c \). We will prove that this is not the case for \( \Psi'_s \).

**Proposition 7.5.** There does not exists a nonzero \( v \in K(\Gamma_\gamma) \) such that \( \phi_s = \frac{\tilde{\delta}(v)}{nv} \) for a nonzero integer \( n \). That is, \( \phi_s \) is not a \( K(\Gamma_\gamma) \)-radical.

**Proof.** Let us assume that there exists \( v \in K(\Gamma_\gamma), v \neq 0, \tilde{\delta}(v) \neq 0 \) such that \( \phi_s = \frac{\tilde{\delta}(v)}{nv} \) for a nonzero integer \( n \). By Proposition 6.2, 2, we can write \( v = \frac{\tilde{\delta}(\psi)}{\psi} \) in \( K(\Gamma_\gamma) \) for some nonzero \( p \in K[\lambda, \mu] \) and \( \Lambda_1 \in K[\lambda]. \) Let \( p_N \) be the normal form of \( p \) on \( \Gamma_\gamma \).

Let us consider the polynomial in \( K[\lambda, \mu] \)
\[ H = [\tilde{\delta}(p_N)\Lambda_1 - p_N\tilde{\delta}(\Lambda_1)] \varphi - np_N\Lambda_1(\mu + \alpha). \] (40)
Recall that \( \phi_s = \frac{\mu + \alpha}{\varphi} \) as in Lemma 6.5, hence \( \alpha, \varphi \in K[\lambda] \) and
\[ \frac{\tilde{\delta}(v)}{nv} - \phi_s = \frac{H}{np_N\Lambda_1 \varphi} = 0 \quad \text{in} \ K(\Gamma_\gamma). \]

Now we apply Proposition 6.2, 1, for \( q = np_N\Lambda_1 \varphi \). Then there exists \( T \in K[\lambda, \mu] \) and a nonzero \( \Lambda_2 \in K[\lambda] \) such that
\[ 0 = \frac{\Lambda_2 H}{np_N\Lambda_1 \varphi} = \frac{\Lambda_2 H}{q} = \frac{T H}{1} \quad \text{in} \ K(\Gamma_\gamma). \]

Therefore in \( K(\lambda, \mu) \)
\[ \frac{\Lambda_2 H}{np_N\Lambda_1 \varphi} = N f_s, \quad \text{for some} \ N \in K[\lambda, \mu]. \]
Finally we obtain
\[ \Lambda_2 \left[ [\tilde{\delta}(p_N)\Lambda_1 - p_N\tilde{\delta}(\Lambda_1)] \varphi - np_N\Lambda_1(\mu + \alpha) \right] = np_N\Lambda_1 \varphi N f_s. \] (41)

If \( p_N \in K[\lambda] \) then the degree in \( \mu \) of the LHS of (41) is 1 and of RHS of (41) is at least 2. Thus this is not possible. We have proved that \( v \) cannot be equal to \( \gamma \) in \( K(\Gamma_\gamma) \), with \( \gamma \in K(\lambda) \).

Hence it remains to check the case where \( p_N \) is not in \( K[\lambda] \).
Let us assume that \( p_N \notin K[\lambda] \). By (41), \( p_N \) is a factor of \( \Lambda_1 \Lambda_2 \partial(p_N) \varphi \). Then by Remark 6.1, \( \partial(p_N) = \Lambda_3 p_N \), with \( \Lambda_3 \in K[\lambda] \). Hence equality (41) becomes
\[
\Lambda_2 \left[ \left[ \Lambda_3 \Lambda_1 - \partial(\Lambda_1) \right] \varphi - n \Lambda_1 (\mu + \alpha) \right] = n \lambda_1 \varphi N f_s. 
\] (42)
Observe that the degree in \( \mu \) of the LHS of (42) is 1 and of RHS of (42) is at least 2. But this is a contradiction. Therefore we conclude that such \( \nu \) does not exist, which proves the result.

Now we can apply [11], Theorem 5.1.2 to the hyperexponential \( t = \Psi_s \) and the differential field \( (K(\Gamma_s), \partial) \).

**Theorem 7.6.** The following statements are equivalent:

1. \( \phi_s \) is not a logarithmic derivative of a \( K(\Gamma_s) \)-radical.
2. \( \Psi_s \) is transcendental over \( K(\Gamma_s) \) and the field of constants of \( K(\Gamma_s)(\Psi_s) \) equals the field of constants of \( K(\Gamma_s) \).

The following theorem is a direct consequence of Proposition 7.2, Theorem 7.6 and Definition 7.1. In particular, it proves the existence of the spectral Picard-Vessiot field over the curve \( \Gamma_s \) of the equation \( (L_s - \lambda)(\Psi) = 0 \).

**Theorem 7.7.** The following statements hold:

1. The field \( K(\Gamma_s)(\Psi_s) \) is a Liouvillian extension of \( K(\Gamma_s) \).
2. The field of constants of \( K(\Gamma_s)(\Psi_s) \) is \( C(\Gamma_s) \).
3. \( K(\Gamma_s)(\Psi_s) \) is a spectral Picard-Vessiot field over the curve \( \Gamma_s \) of the equation \( (L_s - \lambda)(\Psi) = 0 \).

We illustrate Theorem 7.7 with the following commutative diagram, whose second row shows the fields of constants:

\[
\begin{array}{ccc}
K(\Gamma_s) & \longrightarrow & K(\Gamma_s)(\Psi_s) \\
\downarrow & & \downarrow \\
C(\Gamma_s) & \longrightarrow & C(\Gamma_s)
\end{array}
\]

Since \( \phi_s + (f_s) \) belong to \( K(\Gamma_s) \), which is not a polynomial ring, there is no effective method, as far as we know, to compute \( \Psi_s \), i.e. to solve \( (\partial - \phi_s)\Psi = 0 \) in an effective manner. For this reason, we use the one-parameter form \( \tilde{\phi}_s \) of \( \phi_s \) to compute the solutions of \( (L_s - \lambda)(\Psi) = 0 \) in the following section. We will illustrate these results with the examples in Section 7.3.

### 7.2. One-parameter spectral Picard-Vessiot fields

Our next goal is to give an effective method to describe the spectral Picard-Vessiot field of the equation \( (L_s - \lambda)(\Psi) = 0 \). For this purpose we need to use a parametric representation of the spectral curve \( \Gamma_s \). Let \( N_\alpha(\tau) = (\chi_1(\tau), \chi_2(\tau)) \) be a global parametrization of \( \Gamma_s \), as defined in 6.9.

Recall that an isomorphism \( \rho : K(\Gamma_s)[\partial] \to \mathcal{T}[\partial] \) was defined in Theorem 6.11. Furthermore, we have factorizations
\[
L_s - \chi_1(\tau) = (-\partial - \tilde{\phi}_1)(\partial - \tilde{\phi}_1) = (-\partial - \bar{\phi}_2)(\partial - \bar{\phi}_2)
\]
where \( \tilde{\phi}_1 := \tilde{\phi}_s \) and \( \bar{\phi}_2 := \rho(\phi_2) \) are solutions of the Ricatti equation \( \partial(\phi) + \phi^2 = u_s - \chi_1(\tau) \).

Let us consider nonzero solutions \( Y_+ \) and \( Y_- \) respectively of the differential equations
\[
\partial(\bar{Y}) = \tilde{\phi}_1 Y \quad \text{and} \quad \partial(Y) = \bar{\phi}_2 Y.
\] (43)
Lemma 7.8. Let $\Upsilon_+$ and $\Upsilon_-$ as in (43), it holds that:

1. $\{\Upsilon_+, \Upsilon_\pm\}$ is a fundamental set of solutions of $(L_s - \chi_1(\tau))(\Upsilon) = 0$.
2. $\Upsilon_+, \Upsilon_- \in \mathcal{F}$.

Proof. The proof is analogous to the proof of Lemma 7.2, noting that

$$\frac{w(\Upsilon_+, \Upsilon_-)}{\Upsilon_+ \Upsilon_-} = \phi_+ - \phi_- = \rho(\phi_+ - \phi_-) = \rho\left(\frac{2\mu}{\nu}\right) \neq 0$$

since $\mu \neq 0$ and $\rho$ is an isomorphism. \hfill $\Box$

Therefore $\mathcal{F}(\Upsilon_+, \Upsilon_-) = \mathcal{F}(\Upsilon_+) = \mathcal{F}(\Upsilon_-)$. We denote $\Upsilon_+ := \Upsilon_+$. We extend naturally $\rho$ to an isomorphism between differential fields $\bar{\rho} : K(\Gamma_s)(\Psi_s) \rightarrow \mathcal{F}(\Upsilon_s)$ by sending $\Psi_s$ to $\Upsilon_s$. Hence diagram 33 extends to the following commutative diagram of differential fields, whose second row shows the fields of constants:

$$
\begin{array}{c}
K(\Gamma_s)(\Psi_s) & \xrightarrow{\rho} & \mathcal{F}(\Upsilon_s) \\
C(\Gamma_s) & \xrightarrow{\rho} & C(\chi_1(\tau), \chi_2(\tau))
\end{array}
$$

Lemma 7.9. The element $\phi_+$ of $\mathcal{F}$ is not a logarithmic derivative of a $\mathcal{F}$-radical.

Proof. Let us assume that there exists $w \in \mathcal{F}$, $w \neq 0$, $\delta(w) \neq 0$ such that $\phi_+ = \frac{\delta(w)}{nw}$ for a nonzero integer $n$. Since $\rho$ is an isomorphism, $w = \rho(v)$, with $v \in K(\Gamma_s)$, $v \neq 0$, $\delta(v) \neq 0$. Then

$$\phi_+ = \rho(\phi_+) = \rho\left(\frac{\delta(v)}{nv}\right) = \frac{\delta(w)}{nw}$$

implies $\phi_+ = \frac{\delta(w)}{nw}$ contradicting Lemma 6.5. \hfill $\Box$

Now we can apply [11], Theorem 5.1.2 to the hyperexponential $t = \Upsilon_+$ and the differential field $(\mathcal{F}, \delta)$.

Theorem 7.10. The following statements are equivalent:

1. The element $\phi_+$ of $\mathcal{F}$ is not a logarithmic derivative of a $\mathcal{F}$-radical.
2. $\Upsilon_+$ is transcendental over $\mathcal{F}$ and the field of constants of $\mathcal{F}(\Upsilon_+)$ equals the field of constants of $\mathcal{F}$.

Summarizing we have proved the next result.

Theorem 7.11. The following statements hold:

1. The field $\mathcal{F}(\Upsilon_+)$ is a Liouvillian extension of $\mathcal{F}$.
2. The field of constants of $\mathcal{F}(\Upsilon_+)$ is $C(\chi_1(\tau), \chi_2(\tau))$.
3. $\mathcal{F}(\Upsilon_+)$ is isomorphic to a spectral Picard-Vessiot field over the curve $\Gamma_s$ of the equation $(L_s - \lambda)(\Psi) = 0$.  

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Remark 7.12. Whenever the spectral curve $\Gamma_s$ is hyperelliptic, the solution $\Upsilon_s$ is expressed in [31], Theorem 1.20 in terms of $\Theta$-functions. In this format it is the well known Baker-Akhiezer function (see [31], (1.41)). However, $\Gamma_s$ in some cases is a singular curve but whenever a global parametrization is available a closed form formula is provided by the previous method.

We compute next several examples of the one-parameter closed form solution $\Upsilon_s$ that determines the spectral Picard-Vessiot field of the equation $(L_s - \chi_1(\tau))(\zeta') = 0$ for several KdV potentials $u_s$.

7.3. Examples

Let us continue working with the families of potentials of Examples 6.13, where the values of $\phi_s$ were computed. We use the symbolic integration package of Maple 18 to compute $\Upsilon_s$.

1. Family of rational potentials $u_s = s(s + 1)/x^2$, $s \geq 1$:

$$
\begin{align*}
  s & \quad \mathcal{N}_s(\tau) & \quad \Upsilon_s \\
  1 & \quad \left(-\tau^2, -\tau^3\right) & \quad \frac{\Upsilon_s}{x^2 - 1} e^{x\tau} \\
  2 & \quad \left(-\tau^2, -\tau^5\right) & \quad \frac{\tau^2 x^2 + 3 x\tau + 3}{x^2} e^{-\tau} \\
  3 & \quad \left(-\tau^2, -\tau^7\right) & \quad \frac{\tau^3 x^3 - 6 \tau^2 x^2 + 15 x\tau - 15}{x^2} e^{-\tau} \\
  4 & \quad \left(-\tau^2, -\tau^9\right) & \quad \frac{\tau^4 x^4 + 10 \tau^3 x^3 + 45 \tau^2 x^2 + 105 x\tau + 105}{x^2} e^{-\tau}
\end{align*}
$$

2. Family of Rosen-Morse potentials $u_s = \frac{\phi(s+1)}{\cosh(\tau)}$, $s \geq 1$:

$$
\begin{align*}
  s & \quad \mathcal{N}_s(\tau) & \quad \Upsilon_s \\
  1 & \quad \left(-\tau^2, -\tau(\tau^2 - 1)\right) & \quad \frac{\Upsilon_s}{(\tau - 1) w + \tau + 1} e^{\tau} \\
  2 & \quad \left(-\tau^2, -\tau(\tau^2 - 1)(\tau^2 - 4)\right) & \quad \frac{\left(\tau^2 + 3 \tau + 2\right) w^2 + \left(2 \tau^2 - 8\right) w + \tau^2 - 3 \tau + 2}{(w + 1)^2} e^{-\tau} \\
  3 & \quad \left(-\tau^2, -\tau(\tau^2 - 1)(\tau^2 - 4)(\tau^2 - 9)\right) & \quad \frac{p_3(\tau) w^3 + p_2(\tau) w^2 + p_1(\tau) w + p_0(\tau)}{(w + 1)^3} e^{\tau}
\end{align*}
$$

where $w = e^{\tau}$ and

$$
\begin{align*}
  p_3 & = \tau^3 - 6 \tau^2 + 11 \tau - 6, \quad p_2 = 3 \tau^3 - 6 \tau^2 - 27 \tau + 54, \\
  p_1 & = 3 \tau^3 + 6 \tau^2 - 27 \tau - 54, \quad p_0 = \tau^3 + 6 \tau^2 + 11 \tau + 6.
\end{align*}
$$

3. Let us consider the elliptic potential $u_1 = 2\phi(x; g_2, g_3)$. We take the global parametrization $\mathcal{N}_1(\tau) = \left(-\phi(\tau), \frac{1}{2}\phi'(\tau)\right)$ of $\Gamma_1$. In this case

$$
\Upsilon_1 = \frac{\sigma(x + \tau)}{\sigma(x)\sigma(\tau)} e^{-\zeta(\tau)},
$$

where $\zeta(\tau)$ denotes the Weierstrass Zeta function $\zeta(\tau, g_2, g_3)$ and $\sigma(\tau)$ denotes the Weierstrass Sigma function $\sigma(\tau, g_2, g_3)$.
8. Factorization at a point \( P_0 \) in \( \Gamma_s \) and Picard-Vessiot fields

So far in this paper \( \lambda \) and \( \mu \) were variables over \( K \), furthermore \( \partial \lambda = 0 \) and \( \partial \mu = 0 \). In this section we will give some examples of the specialization process of \((\lambda, \mu)\) to a point \( P_0 = (\lambda_0, \mu_0) \) of the spectral curve \( \Gamma_s \). The spectral PV fields admit a specialization process, at a point \( P_0 = (\lambda_0, \mu_0) \) in \( \Gamma_s \), that allows us to give examples of standard Picard-Vessiot extensions \( \Sigma / K \) of \( L_s - \lambda_0 \). The nature of the point \( P_0 \) (which could be singular) determines the type of PV extension \( \Sigma / K \) as the given examples reveal.

**Proposition 8.1.** Given \( P_0 = (\lambda_0, \mu_0) \) in \( \Gamma_s \), the differential operators \( L_s - \lambda_0 \) and \( A_{2s+1} - \mu_0 \) have a common factor over \( K \). Furthermore

\[
L_s - \lambda_0 = (-\partial - \phi_0)(\partial - \phi_0),
\]

where \( \phi_0 = \phi_s(P_0) \) with \( \phi_s \) as in (32) and

\[
\phi_0 = \frac{\varphi_1(P_0)}{\varphi_2(P_0)} = \frac{\mu_0 + \alpha(\lambda_0)}{\varphi_2(\lambda_0)} \tag{45}
\]

with \( \varphi_2(\lambda_0) \neq 0 \).

**Proof.** By Proposition 4.2, 2, the differential operators \( L_s - \lambda_0 \) and \( A_{2s+1} - \mu_0 \) in \( K[\partial] \) have a common factor since

\[
\partial \text{Res}(L_s - \lambda_0, A_{2s+1} - \mu_0) = f_s(\lambda_0, \mu_0) = 0.
\]

With the notation of Lemma 6.5, observe that \( \varphi_1(P_0) + \varphi_2(P_0)\partial \) is the subresultant of \( L_s - \lambda_0 \) and \( A_{2s+1} - \mu_0 \) as in Section 4.2. We will prove next that \( L_1 \) is an operator of order one and then by Theorem 4.5, we have the factorization

\[
L_s - \lambda_0 = (-\partial - \phi_0)(\partial - \phi_0),
\]

where \( \phi_0 = \phi_s(P_0) \) and the given formula follows by Lemma 6.5.

Let us suppose that \( L_1 \) is the zero operator. Then the second subresultant \( L_2 \) equals to \( L_s - \lambda_0 \). Hence \( L_s - \lambda_0 \) is a factor of \( A_{2s+1} - \mu_0 \). That is

\[
A_{2s+1} - \mu_0 = Q(L_s - \lambda_0)
\]

for some monic differential operator \( Q \) of order \( 2s+1 \) in \( K[\partial] \). Computing the commutator with \( L_s \), we obtain

\[
0 = [A_{2s+1} - \mu_0, L_s] = [QL_s, L_s] - [\lambda_0, L_s] = [Q, L_s]L_s.
\]

Since \( K[\partial] \) is a domain \( \{Q, L_s\} = 0 \) and \( Q \) belong to the centralizer of \( L_s \) in \( K[\partial] \), which contradicts Theorem 3.7 since \( Q \) has even order less than \( 2s+1 \). We have proved that \( L_1 \) is an operator of order one, in other words \( \varphi_2(\lambda_0) \neq 0 \). \( \square \)

**Remark 8.2.** Observe that if \( \phi_0 = 0 \) then, due to the Riccati equation, \( u_s \) is the constant potential \( \lambda_0 \), and conversely. From now on we will assume that \( u_s \) is not a constant potential.

We must distinguish two different types of points in the curve, the ones with \( \mu_0 \neq 0 \) and those with \( \mu_0 = 0 \), that is the finite set

\[
Z_s = \Gamma_s \cap (C \times \{0\}) = \{(\lambda, 0) \mid R_{2s+1}(\lambda) = 0\}.
\]
Observe that $Z_s$ contains all the affine singular points of $\Gamma_s$. We are going to show that the behavior of the Picard-Vessiot extension of the equation $(L_s - \lambda_0)(y) = 0$ associated to a point $P_0 = (\lambda_0, \mu_0)$ of $\Gamma_s \subset C^2$, with $\mu_0 \neq 0$, typically resembles the generic case, see Theorem 7.7. For the convenience of the reader we include next the definition of Picard-Vessiot extension, see [19] and also [56].

**Definition 8.3.** Let us consider $\lambda_0 \in C$. A differential field extension $\Sigma$ of $K$ is called a Picard-Vessiot extension of the equation $(L_s - \lambda_0)(y) = 0$ if the following conditions are satisfied:

1. $\Sigma = K(y_1, y_2)$, the differential field extension of $K$ generated by $y_1, y_2$, where $\{y_1, y_2\}$ is a fundamental set of solutions of $(L_s - \lambda_0)(y) = 0$.
2. $\Sigma$ and $K$ have the same field of constants $C$.

For a given point $P_0 = (\lambda_0, \mu_0) \in C^2$ of the curve $\Gamma_s$, we will assume $\mu_0 \neq 0$ from now on. Let us consider $\phi_0$ as in (45), in this section we will use the following notation

$$\phi_{0+} = \phi_0 = \frac{\mu_0 + \alpha(\lambda_0)}{\varphi_2(\lambda_0)} \quad \text{and} \quad \phi_{0-} = \frac{-\mu_0 + \alpha(\lambda_0)}{\varphi_2(\lambda_0)}, \quad (46)$$

pointing out that $\phi_{0+} \neq \phi_{0-}$ since $\mu_0 \neq 0$. Applying Proposition 8.1 to the point $(\lambda_0, -\mu_0)$ we obtain the following factorization of $L_s - \lambda_0$

$$L_s - \lambda_0 = (-\partial - \phi_{0-})(\partial - \phi_{0+}).$$

Let us consider nonzero solutions $\Psi_{0+}$ and $\Psi_{0-}$ respectively of the differential equations

$$\partial(\Psi) = \phi_{0+}\Psi \quad \text{and} \quad \partial(\Psi) = \phi_{0-}\Psi.$$ \hspace{1cm} (47)

Then the equality

$$\frac{W(\Psi_{0+}, \Psi_{0-})}{\Psi_{0+}, \Psi_{0-}} = \phi_{0+} - \phi_{0-} = \frac{2}{\varphi_2(\lambda_0)} \mu_0 \neq 0.$$

implies that $W_0 = w(\Psi_{0+}, \Psi_{0-}) \neq 0$ in $C$. Therefore $\{\Psi_{0+}, \Psi_{0-}\}$ is a fundamental set of solutions of $(L_s - \lambda_0)(\Psi) = 0$. Moreover

$$\Psi_{0+}, \Psi_{0-} = \frac{\varphi_2(\lambda_0)W_0}{2\mu_0} \in K,$$

hence

$$K(\Psi_{0+}, \Psi_{0-}) = K(\Psi_{0+}) \quad \text{and} \quad K(\Psi_{0+}, \Psi_{0-}) = K(\Psi_{0+}). \quad (48)$$

What remains is to prove that the field of constants of the differential extension $K(\Psi_{0+})$ is again $C$. We check that this is the case for some examples and leave this question for future research.

**8.1. Some examples of Picard-Vessiot extensions for KdV$_1$ Schrödinger operators**

Let us continue working with the families of potentials of Section 7.3. More precisely we treat the following three cases for $s = 1$ and $C = \mathbb{C}$. 35
1. **Rational potential** $u_1 = 2/x^2$. As shown in Section 7.3, a point $P_0 = (\lambda_0, \mu_0)$, with $\mu_0 \neq 0$ of the spectral curve $\Gamma_1$ can be written as $P_0 = \mathcal{N}_1(\tau_0) = (-\tau_0^2, -\tau_0^3)$, with $\tau_0 \neq 0$. As in (47), we consider the differential equation

$$\partial(\Psi) = \phi_{0_1} \Psi \text{ with } \phi_{0_1} = \frac{-\tau_0^3 x^3 - 1}{x (\tau_0 x^2 + 1)}.$$  

A solution is

$$\Psi_{0_1} = \frac{x\tau_0 - 1}{x} e^{\tau_0 x}.$$  

Observe that $\Psi_{0_1}$ is the specialization of $T_1$ as in Section 7.3, 1. Recall that in this case $K = \mathbb{C}(x)$ with $\partial = \partial/\partial x$ and field of constants $\mathbb{C}$. Thus equality (48) gives $K(\Psi_{0_1}) = K(e^{\tau_0 x})$. Now we can apply [11], Theorem 5.1.2 to the hyperexponential $t = e^{\tau_0 x}$, and the differential field $(K, \partial)$. We conclude that $t$ is transcendental over $K$ and the field of constants of $K(e^{\tau_0 x})$ is $\mathbb{C}$. By Definition 8.3, the field $K(e^{\tau_0 x})$ is the Picard-Vessiot extension of the Schrödinger operator $-\partial^2 - \frac{2}{\cosh^2(x)} + \tau_0^2$.

2. **Rosen-Morse potential** $u_1 = \frac{2}{\cos^2(x)}$. As shown in Section 7.3, a point $P_0 = (\lambda_0, \mu_0)$, with $\mu_0 \neq 0$ of the spectral curve $\Gamma_1$ can be written as $P_0 = \mathcal{N}_1(\tau_0) = (-\tau_0^2, -\tau_0^3 - 1)$, with $\tau_0 \neq 0, \pm 1$. As in (47), we consider the differential equation

$$\partial(\Psi) = \phi_{0_1} \Psi \text{ with } \phi_{0_1} = \frac{(\tau_0^2 - \tau_0) e^{x^2} + (2 \tau_0^2 - 4) e^{x^2} + \tau_0^2 + \tau_0}{((\tau_0 - 1) e^{2x} + \tau_0 + 1) (e^{2x} + 1)}.$$  

A solution is

$$\Psi_{0_1} = \frac{(\tau_0 - 1) e^{x^2} + \tau_0 + 1}{e^{x^2} + 1} e^{\tau_0 x}.$$  

Observe that $\Psi_{0_1}$ is the specialization of $T_1$ as in Section 7.3, 2. Recall that in this case $K = \mathbb{C}(e^x)$ with $\partial = \partial/\partial x$ and field of constants $\mathbb{C}$. Thus equality (48) gives $K(\Psi_{0_1}) = K(e^{\tau_0 x})$. This example shows that after specialization the field extension $K \subset K(\Psi_{0_1})$ is not transcendental for all $\tau_0$. For details:

- If $\tau_0 \in \mathbb{Z} \setminus \{0, \pm 1\}$ then $K(\Psi_{0_1}) = K$. Hence the Picard-Vessiot extension is trivial.
- If $\tau_0 \in \mathbb{Q} \setminus \mathbb{Z}$ then $K(\Psi_{0_1})$ is a finite algebraic extension of $K$. Therefore the field of constants of $K(\Psi_{0_1})$ is the algebraically closed field $\mathbb{C}$ (see [11], Corollary 3.3.1). Thus the field $K(e^{\tau_0 x})$ is the Picard-Vessiot extension of the Schrödinger operator $-\partial^2 - \frac{2}{\cos^2(x)} + \tau_0^2$.
- If $\tau_0 \in \mathbb{C} \setminus \mathbb{Q}$. Since $\{1, \tau_0\}$ are $\mathbb{Q}$-linearly independent, then by classical Lindemann’s Theorem $e^{\tau_0 x}$ is transcendental over $K$. Now we can apply [11], Theorem 5.1.2 to the hyperexponential $t = e^{\tau_0 x}$ to guarantee that the field of constants of $K(e^{\tau_0 x})$ is $\mathbb{C}$. By Definition 8.3, the field $K(e^{\tau_0 x})$ is the Picard-Vessiot extension of the Schrödinger operator $-\partial^2 - \frac{2}{\cos^2(x)} + \tau_0^2$.

3. **Elliptic potential** $u_1 = 2\varphi(x; g_2, g_3)$. As shown in Section 7.3, 3, a point $P_0 = (\lambda_0, \mu_0)$, with $\mu_0 \neq 0$ of the spectral curve $\Gamma_1$ can be written as $P_0 = \mathcal{N}_1(\tau_0) = \left(-\varphi(\tau_0), \frac{\varphi'(\tau_0)}{\varphi(\tau_0)}\right)$, with $\varphi'(\tau_0) \neq 0$. As in (47), we consider the differential equation

$$\partial(\Psi) = \phi_{0_1} \Psi \text{ with } \phi_{0_1} = \frac{-1}{\varphi(x) - \varphi(\tau_0)}.$$  

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A solution is

$$\Psi_{0^*} = \frac{\sigma'(x + \tau_0)}{\sigma(x) \sigma'(\tau_0)} e^{-\chi(\tau_0)}.$$ 

Observe that $\Psi_{0^*}$ is the specialization of $\mathcal{T}_1$ as in Section 7.3. Recall that in this case $K = \mathbb{C}(\varphi(x), \varphi'(x))$ with $\vartheta = \partial / \partial x$ and field of constants $\mathbb{C}$. If $t = \frac{\sigma'(x + \tau_0)}{\sigma(x)} e^{-\chi(\tau_0)}$ is transcendental over $K$ then we can apply [11], Theorem 5.1.2 to guarantee that the field of constants of $K(\Psi_{0^*}) = K(t)$ is $\mathbb{C}$. In this case, by Definition 8.3, the field $K(\Psi_{0^*})$ is the Picard-Vessiot extension of the Schrödinger operator $-\partial^2 + 2\varphi(x) + \varphi'(\tau_0)$. It would be very interesting to investigate for which points $P_0$ of the spectral curve is $t$ transcendental over $K$. We leave this question for future research.

The previous examples illustrate a general specialization framework. In fact a commutative diagram can be established as follows:

$$\begin{array}{ccc}
C(\Gamma_x) & \longrightarrow & K(\Gamma_x) \\
\downarrow & & \downarrow \text{Liouvillian} \\
C & \longrightarrow & K
\end{array}$$

for $K = C(u_i)$. If $P_0 \in \Gamma_x$ then we can apply Theorem 5.1.2 to guarantee that the field of constants of $K(\Psi_{0^*}) = K(t)$ is $\mathbb{C}$. In this case, by Definition 8.3, the field $K(\Psi_{0^*})$ is the Picard-Vessiot extension of the Schrödinger operator $-\partial^2 + 2\varphi(x) + \varphi'(\tau_0)$. It would be very interesting to investigate for which points $P_0$ of the spectral curve is $t$ transcendental over $K$. We leave this question for future research.

Finally, the integrability of the Schrödinger equation of the Rosen-Morse family $u_i = \frac{\sigma'(x+1)}{\cosh(x)}$ was studied in [43] where their spectral problem was also analyzed.

9. Final remarks

We summarize next some of the main achievements of this paper together with a wider scope of application of the results presented, that we plan to study in a near future. Optimally, we would like to extend the methods of this paper to the broader scope of algebro-geometric problems.

We have generalized, in Definition 7.1, the concept of Picard-Vessiot extension of $L_x - \lambda_0$ over $K$, with algebraically closed constant field $C$, to define a spectral Picard-Vessiot field for $L_x - \lambda$ over $K(\Gamma_x)$ with constant field $C(\Gamma_x)$, which is not necessarily algebraically closed. These spectral Picard-Vessiot fields admit a specialization process that allowed us to give examples of Picard-Vessiot extensions $\Sigma/K$ of $L_x - \lambda_0$ in Section 8. We plan to further develop a general specialization method in a future work, in particular for elliptic potentials. In addition, we will analyze the Differential Galois group $Gal(\Sigma/K)$ and the Picard-Vessiot extensions comparing them with the approaches given in [1], or in [10], [54].

The parametric equations of the spectral curve allowed us to exhibit a one-parameter form of the factorization of the KdV Schrödinger operator. Thus, standard theorems, on the dependence of a parameter $\tau$ of a differential equation, will allow us to guarantee the existence of a complex domain where the solutions of $(L_x - \lambda)\Psi = 0$ can be represented as graphs of function of $(\tau, x)$. We could exhibit a fundamental set of solutions of $(L_x - \chi(\tau))\Gamma = 0$ that varies continuously with the parameter $\tau$, for $\tau$ in an open Zariski set of the parametrization domain. In principle, different parameterizations give different factors, and hence different solutions of the given operator. We are working on establishing their relations and the consequences thereof.

Overall, we presented a new view on the integration of problem (2), establishing the foundations to approach KdV stationary Schrödinger equations by means of symbolic-effective com-
putation. These methods provide a new framework for calculating closed formulas for wave functions in the case of stationary KdV potentials. They also allow the study of the classical Picard-Vessiot extensions by means of a specialization process at a point of the spectral curve. This provides a deeper understanding of the fundamental role played by the field of constants.

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