Auxiliary Space Preconditioners for a $C^0$ Finite Element Approximation of Hamilton–Jacobi–Bellman Equations with Cordes Coefficients

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Received: 31 December 2021 / Revised: 2 June 2022 / Accepted: 4 June 2022 / Published online: 3 August 2022 © The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
In the past decade, there are many works on the finite element methods for the fully nonlinear Hamilton–Jacobi–Bellman (HJB) equations with Cordes condition. The linearised systems have large condition numbers, which depend not only on the mesh size but also on the parameters in the Cordes condition. This paper is concerned with the design and analysis of auxiliary space preconditioners for the linearised systems of a $C^0$ finite element discretization of HJB equations [Calcolo, 58, 2021]. Based on the stable decomposition on the auxiliary spaces, we propose both the additive and multiplicative preconditioners which converge uniformly in the sense that the resulting condition number is independent of both the number of degrees of freedom and the parameter $\lambda$ in Cordes condition. Numerical experiments are carried out to illustrate the efficiency of the proposed preconditioners.

Keywords Non-divergence form · Hamilton–Jacobi–Bellman · Cordes condition · $C^0$ finite element methods · Auxiliary space precondition

1 Introduction
Let $\Omega$ be a bounded, open, convex polytopal domain in $\mathbb{R}^d$, where $d = 2, 3$ represents the dimension. In this paper, we are interested in the Hamilton–Jacobi–Bellman (HJB) equations of the following type:

$$\sup_{\alpha \in \Lambda} (L^\alpha u - f^\alpha) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \quad (1)$$

where $\Lambda$ is a compact metric space, and

$$L^\alpha v := A^\alpha : D^2 v + b^\alpha \cdot \nabla v - c^\alpha v.$$
Here, $A : B := \sum_{i,j=1}^d A_{i,j} B_{i,j}$ denotes the Frobenius inner product of two matrices. $D^2 u$ and $\nabla u$ denote the Hessian and gradient of a real-valued function $u$, respectively. The coefficient $A^\alpha \in C(\overline{\Omega} \times \Lambda; \mathbb{R}^{d \times d})$ is assumed to be uniformly elliptic, i.e., there exist constants $\overline{\nu}, \underline{\nu} > 0$ such that

$$\overline{\nu} |\xi|^2 \leq \xi^t A^\alpha(x) \xi \leq \underline{\nu} |\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \text{ a.e. in } \Omega, \forall \alpha \in \Lambda. \quad (2)$$

Further, $b^\alpha \in C(\overline{\Omega} \times \Lambda; \mathbb{R}^d)$ and $c^\alpha \geq 0$, $f^\alpha \in C(\overline{\Omega} \times \Lambda; \mathbb{R})$.

The HJB equations arise in many applications including stochastic optimal control, game theory, and mathematical finance [1]. In [2, 3], the HJB equations are shown to admit $H^2$ strong solutions under the following Cordes condition.

**Definition 1.1** (Cordes condition for (1)) The coefficients satisfy

1. Whenever $b^\alpha \neq 0$ or $c^\alpha \neq 0$ for some $\alpha \in \Lambda$, there exist $\lambda > 0$ and $\varepsilon \in (0, 1]$ such that

$$\frac{|A^\alpha| + |b^\alpha|^2/(2\lambda) + (c^\alpha/\lambda)^2}{(\text{tr } A^\alpha + c^\alpha/\lambda)^2} \leq \frac{1}{d - \varepsilon} \quad \text{a.e. in } \Omega, \forall \alpha \in \Lambda. \quad (3a)$$

2. Whenever $b^\alpha \equiv 0$ and $c^\alpha \equiv 0$ for all $\alpha \in \Lambda$, there is an $\varepsilon \in (0, 1]$ such that

$$\frac{|A^\alpha|^2}{(\text{tr } A^\alpha)^2} \leq \frac{1}{d - 1 + \varepsilon} \quad \text{a.e. in } \Omega, \forall \alpha \in \Lambda. \quad (3b)$$

In this special case, the parameter $\lambda$, which reflects the strength of the lower-order terms, is set to be $0$.

In the past decade, several studies have been taken on the finite element approximation of $H^2$ strong solutions of the HJB equations with Cordes coefficients (3). The first discontinuous Galerkin (DG) method was proposed in [3], which has been extended to the parabolic HJB equations in [4]. The $C^0$-interior penalty DG methods were developed by [5]. A mixed method based on the stable finite element Stokes spaces was proposed in [6]. Recently, the $C^0$ (non-Lagrange) finite element method with no stabilization parameter was proposed in [7], where the element is required to be $C^1$-continuous at $(d - 2)$-dimensional subsimplex, e.g., $P_k$-Hermite family ($k \geq 3$) in 2D and $P_k$-Argyris family ($k \geq 5$) [8, 9] in 3D. The above discretizations can be naturally applied to the linear elliptic equations in non-divergence form [5–7, 10, 11]. Other related topics include the unified analysis of DGFEM and $C^0$-IPDG [12], and the adaptivity of $C^0$-IPDG [13, 14].

For all these discretizations, the discrete well-posedness is analysed under the broken $H^2$-norm with possible jump terms across the boundary. This, after linearization, leads to the ill-conditioned systems with condition number $O(h^{-4})$ on quasi-uniform meshes, where $h$ represents the mesh size. Due to the similar performance to the discrete system for fourth-order problems, it is conceivable that the linearised system from HJB equations can be effectively solved by the solvers for fourth-order problems, e.g., geometric multigrid [15–19] or domain decomposition [20, 21]. In [22], the nonoverlapping domain decomposition preconditioner was studied for the DGFEM discretization of HJB equations.

Traditional geometric multigrid methods depend crucially on the multilevel structures of underlying grids. On unstructured grids, the more user-friendly option is the algebraic multigrid method (AMG) that has been extensively studied for second-order equations. In [23], the first biharmonic equation was converted to a Poisson system based on the boundary operator proposed in [24]. Under the framework of auxiliary space preconditioning [25], [26] proposed a class of optimal solvers based on the auxiliary discretization of mixed form for the fourth-order problems. As a generalization of [15, 23, 27], it works for a variety of
conforming and nonconforming finite element discretizations on both convex and nonconvex domains with unstructured triangulation.

The purpose of this work is to study the auxiliary space preconditioner to a \( C^0 \) finite element discretization of HJB equations. More specifically, the numerical scheme for fully nonlinear HJB equations leads to a discrete nonlinear problem that can be solved iteratively by a semi-smooth Newton method [3, 5, 7]. The linear systems obtained from the semi-smooth Newton linearization are generally non-symmetric but coercive. To handle the non-symmetry, the existing GMRES theory [28] will lead to a guaranteed minimum convergence rate with a symmetric FOV-equivalent preconditioner \( P_{\lambda,h} \) that satisfies (20). The construction of \( P_{\lambda,h} \) under the auxiliary preconditioning framework follows two steps:

1. Construct appropriate auxiliary spaces and corresponding transfer operators mapping functions from original space to the auxiliary spaces;
2. Devise solvers on auxiliary spaces so that the bounds in (20) are uniform with respect to both \( h \) and the parameter \( \lambda \) in the Cordes condition.

Based on the stable decomposition for auxiliary spaces, both additive and multiplicative preconditioners are shown to be efficient and \( \lambda \)-uniform for the linearised system. Further, the preconditioners only involve the Poisson-like solver which can be efficiently solved by AMG with nearly optimal complexity.

In general cases, the auxiliary space preconditioner is additive [25, 26, 29], which usually leads to a stable but relatively large condition number in practical applications. The first contribution of this work is the construction and analysis of a multiplicative preconditioner based on the specific structure of auxiliary spaces. Having a coarse subspace, the symmetrized two-level multiplicative precondition was shown to be positive definite provided that the smoother on the fine level has a contraction property [30]. The condition number estimate of multiplicative precondition at the matrix level can be found in [31]. In this work, we show that the contracted smoother together with the stable decomposition for auxiliary spaces leads to a robust multiplicative preconditioner, which is also numerical verified with better performance than the additive version.

The parameter \( \lambda \) in the Cordes condition balances the diffusion and the constant term. We emphasize that this parameter is not involved in the monotonicity constant (2.1), which makes it possible to consider the preconditioner with uniformity on \( \lambda \). In this work, we carefully define the norm on the auxiliary space so that the induced preconditioner is uniform with respect to \( \lambda \). Although the preconditioner is designed for the \( C^0 \) finite element approximation, a similar idea can be applied to other discretizations.

The rest of the paper is organized as follows. In Sect. 2, we establish the notation and state some preliminary results. In Sect. 3, we apply the FOV-equivalence preconditioner for the linear system, which can be used to solve non-symmetric systems appearing in applications to the HJB equations. In Sect. 4, we construct both the additive and multiplicative auxiliary space preconditioners. We also show that the condition numbers of the preconditioned systems are uniformly bounded with the stable decomposition assumption, which is verified in Sect. 5. Several numerical experiments are presented in Sect. 6 to illustrate the theoretical results.

For convenience, we use \( C \) to denote a generic positive constant which may depend on \( \Omega \), shape regularity of mesh and polynomial degree, but is independent of the mesh size \( h \) and the parameter \( \lambda \). The notation \( X \lesssim Y \) means \( X \leq CY \). \( X \simeq Y \) means \( X \lesssim Y \) and \( Y \lesssim X \).
2 Preliminaries

In this section, we first review the $H^2$ strong solutions to the HJB equations (1) under the Cordes condition (3). Then we give a brief statement about the $C^0$ finite element scheme in [7].

Given an integer $k \geq 0$, let $H^k(\Omega)$ and $H^k_0(\Omega)$ be the usual Sobolev spaces, $\| \cdot \|_{H^k(\Omega)}$ and $| \cdot |_{H^k(\Omega)}$ denote the Sobolev norm and semi-norm. We also denote $V = H^2(\Omega) \cap H^1_0(\Omega)$. For any Hilbert space $X$, we denote $X'$ for the dual space of $X$, and $\langle \cdot, \cdot \rangle$ for the corresponding dual pair. Let $M : X \to X'$ be an SPD operator, we denote $(\cdot, \cdot)_M := \langle M \cdot, \cdot \rangle$ as an inner product on $X$ (similar for the norm $\| \cdot \|_M$). We also denote $| \cdot |$ as the Euclidian norm for vectors and the Frobenius norm for matrices.

2.1 $H^2$ Strong Solutions to the HJB Equations

We now invoke the theory of $H^2$ strong solutions of the HJB equations. In view of the Cordes condition (3), for each $\alpha \in \Lambda_1$, define $\gamma_\alpha := \begin{cases} \text{tr } A^\alpha + c^\alpha / \lambda, & b^\alpha \neq 0 \text{ or } c^\alpha \neq 0 \text{ for some } \alpha \in \Lambda_1, \\ \frac{|A^\alpha|^2 + |b^\alpha|^2/(2\lambda) + (c^\alpha/\lambda)^2}{\text{tr } A^\alpha}, & b^\alpha = 0 \text{ and } c^\alpha \equiv 0 \text{ for all } \alpha \in \Lambda. \end{cases}$ (4)

And for $\lambda$ as in (3), define a linear operator $L_\lambda : H^2(\Omega) \to L^2(\Omega)$ by $L_\lambda u := \Delta u - \lambda u, \quad u \in H^2(\Omega).$ (5)

Next, we define the operator $F_\gamma : H^2(\Omega) \to L^2(\Omega)$ by $F_\gamma[u] := \sup_{\alpha \in \Lambda} \{ \gamma_\alpha L^\alpha u - \gamma_\alpha f^\alpha \}.$ (6)

Note that the continuity of data implies $\gamma_\alpha \in C(\overline{\Omega} \times \Lambda; \mathbb{R})$. As a consequence, it is readily seen that the HJB equation (1) is equivalent to the problem $F_\gamma[u] = 0$ in $\Omega$, and $u = 0$ on $\partial \Omega$. The Cordes condition leads to the following lemma; See [3, Lemma 1] for proof.

**Lemma 2.1** (property of Cordes condition) Under the Cordes condition (3), for any open set $U \subset \Omega$ and $u, v \in V = H^2(\Omega) \cap H^1_0(\Omega)$, $|\gamma_\alpha|^2 \leq \|v\|_{H^2(\Omega)}$, (7)

Another key ingredient for the well-posedness of (1) is the Miranda-Talenti estimate stated as follows.

**Lemma 2.2** (Miranda-Talenti estimate, [2, 32]) Suppose $\Omega$ is a bounded convex domain in $\mathbb{R}^d$. Then, for any $v \in V = H^2(\Omega) \cap H^1_0(\Omega)$, $|v|_{H^2(\Omega)} \leq \|\Delta v\|_{L^2(\Omega)} \leq C |v|_{H^2(\Omega)},$ (8)

where the constant $C$ depends only on the dimension.

Let the operator $M : V \to V'$ be $\langle M[w], v \rangle := \int_\Omega F_\gamma[w] L_\lambda v \, dx.$ (9)
By using the Miranda-Talenti estimate (8) and Cordes condition (3), one can show the strong monotonicity of $M$,
\[
(M[v] - M[w], z) \geq (1 - \sqrt{1 - \epsilon})\|z\|_2^2 \quad \forall v, w \in V,
\]
where $z = v - w$ and $\|z\|_2^2 := \|D^2 z\|_{L^2(\Omega)}^2 + 2\lambda \|\nabla z\|_{L^2(\Omega)}^2 + \lambda^2 \|z\|_{L^2(\Omega)}^2$. Together with the Lipschitz continuity of $M$, the compactness of $\Lambda$ and the Browder-Minty Theorem [33, Theorem 10.49], one can show the existence and uniqueness of the following problem: Find $u \in V$ such that
\[
\langle M[u], v \rangle = 0 \quad \forall v \in V. \tag{10}
\]
We refer to [3, Theorem 3] for detailed proof.

### 2.2 A $C^0$ Finite Element Approximation of the HJB Equations

Let $T_h$ be a conforming shape regular simplicial triangulation of polytope $\Omega$ and $\mathcal{F}_h$ be the set of all faces of $T_h$. $\mathcal{F}_h^d := \mathcal{F}_h \setminus \partial \Omega$ and $\mathcal{F}_h^d := \mathcal{F}_h \cap \partial \Omega$. Let $\mathcal{N}_h$ be the set of all the vertexes of $T_h$ and $\mathcal{N}_h^d$ be the vertexes on the boundary $\partial \Omega$. Here $h_T$ is the diameter of $T \in T_h$. We also denote $h_T$ as the diameter of $F \in \mathcal{F}_h$. For $F \in \mathcal{F}_h$ and $T \in T_h$, we use $(\cdot, \cdot)_T$, respectively $(\cdot, \cdot)_F$, to denote the $L^2$-inner product over $T$, respectively $F$. Similarly, $(\cdot, \cdot)_{\Omega}$ is used to denote the $L^2$-inner product over $\Omega$.

For each $F \in \mathcal{F}^d_h$, we define the tangential Laplace operator $\Delta_T : H^s(F) \rightarrow H^{s-2}(F)$ as follows, where $s \geq 2$. Let $\{t_i\}_{i=1}^{d-1}$ be an orthogonal coordinate system on $F$. Then, for $w \in H^s(F)$ define
\[
\Delta_T w = \sum_{i=1}^{d-1} \frac{\partial^2}{\partial t_i^2} w.
\]
Next, we define the jump of a vector function $v$ on an interior face $F = \partial T^+ \cap \partial T^-$ as follows:
\[
[v]_F := v^+ \cdot n^+|_F + v^- \cdot n^-|_F,
\]
where $v^\pm = v|_{T^\pm}$ and $n^\pm$ is the unit outward normal vector of $T^\pm$, respectively. For scaler function $w$ we define
\[
[w] := w|_{T^+} - w|_{T^-}.
\]

Following [7], we adopt the $P_k$-Hermite finite elements ($k \geq 3$) in 2D and $P_k$-Argyris finite elements in 3D to discretize the HJB equations (1), both of which have the $C^0$-continuity on the face (namely, the $(d-1)$-dimensional subsimplex) and $C^1$-continuity on the $(d-2)$-dimensional subsimplex.

### 2D Hermite Elements

Following the description of [34, 35], the geometric shape of Hermite elements is triangle $T$. The shape function space is given as $\mathcal{P}_k(T)$ ($k \geq 3$), where $\mathcal{P}_k(T)$ denotes the set of polynomials with total degree not exceeding $k$ on $T$. In $T$ the degrees of freedom are defined as follows (cf. Fig. 1):

- Function value $v(a)$ and first order derivatives $\partial_i v(a)$, $i = 1, 2$ at each vertex;
- Moments $\int_v v q \, dx$, $\forall q \in \mathcal{P}_{k-4}(e)$ on each edge $e$;
- Moments $\int_T v q \, dx$, $\forall q \in \mathcal{P}_{k-3}(T)$ on element $T$. 

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Here, $\bar{f}$ represents the averaged integral.

### 3D Argyris Elements

In 3D case, the finite elements are required to be $C^0$ on face and $C^1$ on edge. The typical elements that meet the requirement are the 3D Argyris elements [9], which coincide with each component of the velocity finite elements in the 3D smooth de Rham complex [8]. Given a tetrahedron $T$, the shape function space is given as $P_k(T)$, for $k \geq 5$. In $T$ the degrees of freedom are defined as follows (cf. Fig. 2):

- One function value and (nine) derivatives up to second order at each vertex;
- Moments $\int_e f_e q \, ds$, $\forall q \in P_{k-6}(e)$ on each edge $e$;
- Moments $\int_e \frac{\partial f_e}{\partial n_{e,i}} q \, ds$, $\forall q \in P_{k-5}(e), i = 1, 2$ on each edge $e$;
- Moments $\int_F f_F q \, ds$, $\forall q \in P_{k-6}(F)$ on each face $F$;
- Moments $\int_T f_T q \, dx$, $\forall q \in P_{k-4}(T)$ on element $T$.

Here, $n_{e,i}$ ($i = 1, 2$) are two unit orthogonal normal vectors that are orthogonal to the edge $e$.

For every triangulation $T_h$ of the polytope $\Omega_1$, we define the global finite element spaces $V_h$ as follow:

1. For $d = 2$, with $k \geq 3$ (cf. Fig. 1),
   \[ V_h := \{ v \in H^1_0(\Omega) : \exists \bar{v} \in P_k(T), \forall T \in T_h, v \text{ is } C^1 \text{ at all vertices} \} \]
2. For $d = 3$, with $k \geq 5$ (cf. Fig. 2),
   \[ V_h := \{ v \in H^1_0(\Omega) : \exists \bar{v} \in P_k(T), \forall T \in T_h, v \text{ is } C^1 \text{ on all edges, } v \text{ is } C^2 \text{ at all vertices} \} \]

The $C^0$-continuity on the face and the $C^1$-continuity on the $(d-1)$-dimensional sub-simplex of the finite element space $V_h$ leads to the following lemma, which is critical in the design and analysis of finite element approximation of HJB equations (1).

**Lemma 2.3** (discrete Miranda-Talenti identity, [7]) Let $\Omega \subset \mathbb{R}^d$ be a convex polytopal domain and $T_h$ be a conforming triangulation. For each $v_h \in V_h$, it holds that

\[ \sum_{T \in T_h} \| \Delta v_h \|_{L^2(T)}^2 = \sum_{T \in T_h} \| D^2 v_h \|_{L^2(T)}^2 + 2 \sum_{F \in \mathcal{F}_h} \langle [\nabla v_h], \Delta_T v_h \rangle_F. \]

![Fig. 1](image-url) Degrees of freedom of 2D $P_k$ Hermite elements, in the case of $k = 3$ and $k = 4$
In light of (9), we define the operator $M_h : V + V_h \rightarrow V_h'$ by
\[
\langle M_h[w], v_h \rangle := \sum_{T \in T_h} (F_T[w], L_\lambda v_h)_T \\
- (2 - \sqrt{1 - \varepsilon}) \sum_{F \in F_h} \langle [\nabla w], \Delta_T v_h - \lambda v_h \rangle_F.
\]

The following finite element scheme is proposed to approximate the solutions to the HJB equations (1): Find $u_h \in V_h$ such that
\[
\langle M_h u_h, v_h \rangle = 0 \quad \forall v_h \in V_h.
\]
We define the inner product on $V_h$ as
\[
(w_h, v_h)_{\lambda,h} := \sum_{T \in T_h} (D^2 w_h, D^2 v_h)_T + 2\lambda(\nabla w_h, \nabla v_h)_\Omega + \lambda^2(w_h, v_h)_\Omega,
\]
and the norm $\|v_h\|_{\lambda,h}^2 := (v_h, v_h)_{\lambda,h}$. For the convergence of this scheme, we have the following theorem (see [7] for more details):

**Lemma 2.4** (quasi-optimal error estimate, [7]) Let $\Omega$ be a bounded, convex polytope in $\mathbb{R}^d$, and let $T_h$ be a simplicial, conforming, shape-regular mesh. Let $\Lambda$ be a compact metric space. Suppose that the coefficients satisfy the Cordes condition (3). Then, there exists a unique solution $u_h \in V_h$ satisfying (11). Moreover, there holds that
\[
\|u - u_h\|_{\lambda,h}^2 \leq C \sum_{T \in T_h} h_T^{2s-4} \|u\|_{H^s(T)}^2,
\]
where $s = \min\{s, k + 1\}$ provided that $u \in H^s(\Omega) \cap H_0^1(\Omega)$ for some $s \geq 2$.

### 2.3 Semi-smooth Newton Method

It is shown in [3] that the discretized nonlinear system (11) can be solved by a semi-smooth Newton method, which leads to a sequence of discretized linear systems. We summarized the main ideas on semi-smooth Newton here and refer to [3] for more details.
Following the discussion in [3], we define the admissible maximizers set for any \( v \in V_h + V \),

\[
\Lambda[v] := \left\{ \alpha(\cdot) : \Omega \to \Lambda \ \, | \, \alpha(x) \in \arg \max_{\alpha \in \Lambda} (A^\alpha : D_h^2 v + b^\alpha \cdot \nabla v - c^\alpha v - f^\alpha) \right\},
\]

where \( D_h^2 v \) denotes the broken Hessian of \( v \). As shown in [3, Lemma 9 & Theorem 10], the set \( \Lambda[v] \) is not empty for any \( v \in V_h + V \).

The semi-smooth Newton method is now stated as follows. Start by choosing an initial iterate \( u_0^h \in V_h \). Then, for each nonnegative integer \( j \), given the previous iterate \( u_j^h \in V_h \), choose an \( \alpha_j \in \Lambda[u_j^h] \). Next the function \( f_{\alpha_j} : \Omega \to \mathbb{R} \) is defined by \( f_{\alpha_j}(x) \); the functions \( A_{\alpha_j}, b_{\alpha_j}, c_{\alpha_j}, \text{and } \gamma_{\alpha_j} \) are defined in a similar way. Then find the solution \( u_{j+1}^h \in V_h \) of the linearised system

\[
b_{\lambda, h}^j (u_{j+1}^h, v_h) = \sum_{T \in T_h} (\gamma_{\alpha_j} f_{\alpha_j}, \Delta v_h)_T \quad \forall v_h \in V_h,
\]

where the bilinear form \( b_{\lambda, h}^j : V_h \times V_h \to \mathbb{R} \) is defined by

\[
b_{\lambda, h}^j (w_h, v_h) := \sum_{T \in T_h} (\gamma_{\alpha_j} L_{\alpha_j} w_h, L_{\lambda} v_h)_T - (2 - \sqrt{1 - \varepsilon}) \sum_{F \in F_h} (\lVert \nabla w_h \rVert, \Delta_T v_h - \lambda v_h)_F.
\]

It is also shown in [7] that the bilinear forms \( b_{\lambda, h}^j \) are uniformly coercive and bounded on \( V_h \) with norm \( \lVert \cdot \rVert_{\lambda, h} \), with constants independent of iterates. Since the preconditioners in this work take advantage of the coercivity and boundedness of \( b_{\lambda, h}^j \), we summarize the relevant results in the following lemma, see [7, Lemmas 4.1 & 4.2] for detailed proof.

**Lemma 2.5** (coercivity and boundedness of bilinear form) *For every \( w_h, v_h \in V_h \), we have*

\[
b_{\lambda, h}^j (v_h, w_h) \leq C \lVert v_h \rVert_{\lambda, h} \lVert w_h \rVert_{\lambda, h},
\]

\[
b_{\lambda, h}^j (v_h, v_h) \geq (1 - \sqrt{1 - \varepsilon}) \lVert v_h \rVert_{\lambda, h}^2.
\]

*Here, the constant \( C \) depends only on \( \Omega, \text{shape regularity of the grid and polynomial degree } k. \)*

### 3 FOV-equivalent Preconditioners for GMRES Methods

The preconditioned GMRES (PGMRES) methods are among the most effective iterative methods for non-symmetric linear systems arising from discretizations of PDEs. Our study will start by discussing PGMRES methods in an operator form. Let \( G : \mathcal{X} \to \mathcal{X} \) be a linear operator which may be non-symmetric or indefinite, defined on a finite dimensional space \( \mathcal{X} \), and \( g \) be a given functional in its dual space \( \mathcal{X}' \). The linear equation considered here is of the following form

\[
Gx = g.
\]
Let $(\cdot, \cdot)_M$ be an inner product on $\mathcal{X}$, and $P : \mathcal{X}' \to \mathcal{X}'$ be the preconditioner. The PGMRES method for solving (14) is stated as follows: Begin with an initial guess $x_0 \in \mathcal{X}$ and denote $r_0 = g - Gx_0$ the initial residual, the $k$-th steps of PGMRES method seeks $x_k$ such that

$$x_k = \arg\min_{\tilde{x}_k \in K_k(PG, Pr_0) + x_0} \|PG(x - \tilde{x}_k)\|_M,$$

where $K_k(PG, Pr_0)$ is the Krylov subspace of dimension $k$ generated by $PG$ and $Pr_0$.

In the semi-smooth Newton steps, the discrete linear equations (13) have a common form: Find $u_h \in V_h$ such that

$$b_{\lambda,h}(u_h, v_h) = f_h(v_h) \quad \forall v_h \in V_h,$$

where we shall omit to denote the dependence of the bilinear form $b_{\lambda,h}$ and the right-hand side $f_h$ on the iteration number of the semi-smooth Newton method. Define the operator $B_{\lambda,h} : V_h \to V'_h$ by

$$\langle B_{\lambda,h} u_h, v_h \rangle := b_{\lambda,h}(u_h, v_h) \quad \forall u_h, v_h \in V_h,$$

and the right-hand side $f_h$ on the iteration number of the semi-smooth Newton method. Define the operator $B_{\lambda,h} : V_h \to V'_h$ by

$$\langle B_{\lambda,h} u_h, v_h \rangle := b_{\lambda,h}(u_h, v_h) \quad \forall u_h, v_h \in V_h,$$

then the discrete system (15) can be written in an operator form, namely

$$B_{\lambda,h} u_h = f_h.$$

Moreover, a general operator $P_{\lambda,h} : V'_h \to V_h$ is used to denote the preconditioner. Given an inner product $(\cdot, \cdot)_{M_{\lambda,h}} := (M_{\lambda,h}(\cdot, \cdot))$ with an SPD operator $M_{\lambda,h} : V_h \to V'_h$, we can estimate the convergence rate of the PGMRES method. It is proved in [28, 36] that if $u_h^m$ is the $m$-iteration of PGMRES method and $u_h$ is the exact solution of (17), then

$$\frac{\|P_{\lambda,h} B_{\lambda,h}(u_h - u_h^m)\|_{M_{\lambda,h}}}{\|P_{\lambda,h} B_{\lambda,h}(u_h - u_h^0)\|_{M_{\lambda,h}}} \leq \left( 1 - \frac{\gamma^2}{\Gamma^2} \right)^{m/2},$$

where

$$\gamma \leq \frac{(v_h, P_{\lambda,h} B_{\lambda,h} v_h)_{M_{\lambda,h}}}{(v_h, v_h)_{M_{\lambda,h}}}, \quad \frac{\|P_{\lambda,h} B_{\lambda,h} v_h\|_{M_{\lambda,h}}}{\|v_h\|_{M_{\lambda,h}}} \leq \Gamma \quad \forall v_h \in V_h.$$ (18)

Therefore, we conclude that as long as we find an operator $P_{\lambda,h}$ and a proper inner product $(\cdot, \cdot)_{M_{\lambda,h}}$ such that condition (18) is satisfied with constants $\gamma$ and $\Gamma$ independent of the discretization parameter $h$ and the Cordes condition parameter $\lambda$, then $P_{\lambda,h}$ is a uniform preconditioner for the GMRES method. Such preconditioners are usually referred to as FOV-equivalent preconditioners. In what follows, we always take $P_{\lambda,h}$ to be an SPD operator, and $M_{\lambda,h} = P_{\lambda,h}^{-1}$.

Next, we give a general principle for constructing $P_{\lambda,h}$. Define an SPD operator $A_{\lambda,h} : V_h \to V'_h$ by

$$\langle A_{\lambda,h} w_h, v_h \rangle := (w_h, v_h)_{\lambda,h} \quad \forall w_h, v_h \in V_h.$$ (19)

Recalling Lemma 2.5 (coercivity and boundedness of bilinear form), $b_{\lambda,h}(\cdot, \cdot)$ is coercive and bounded on $V_h$ with the inner product $(\cdot, \cdot)_{\lambda,h}$. It is therefore that an efficient preconditioner for $A_{\lambda,h}$ can also be used as an FOV-preconditioner for the GMRES algorithm applied to $B_{\lambda,h}$, which is shown in the following lemma.
Lemma 3.1 (FOV-equivalent preconditioner) Let $A_{\lambda,h}$ and $B_{\lambda,h}$ be the operators defined in (19) and (16), respectively. If an SPD operator $P_{\lambda,h} : V_h' \rightarrow V_h$ satisfies

$$\alpha \langle P_{\lambda,h}^{-1}v_h, v_h \rangle \leq \langle A_{\lambda,h}v_h, v_h \rangle \leq \beta \langle P_{\lambda,h}^{-1}v_h, v_h \rangle \quad \forall v_h \in V_h,$$

(20)

with constants $\alpha, \beta$ independent of both $\lambda$ and $h$, then $P_{\lambda,h}$ is a uniform FOV-equivalent preconditioner of $B_{\lambda,h}$. The constants in (18) $\gamma = (1 - \sqrt{1 - \epsilon}) \alpha$ and $\Gamma = C \beta$, where $C$ is independent of both $\epsilon$ and $\lambda$.

Proof From (16), (19), and Lemma 2.5 (coercivity and boundedness of bilinear form), we see that for any $u_h, v_h \in V_h$

$$(1 - \sqrt{1 - \epsilon}) \langle A_{\lambda,h}u_h, u_h \rangle \leq \langle B_{\lambda,h}u_h, u_h \rangle$$

(21a)

$$\langle B_{\lambda,h}u_h, v_h \rangle \leq C \langle A_{\lambda,h}u_h, u_h \rangle^{1/2} (A_{\lambda,h}v_h, v_h)^{1/2},$$

(21b)

where $C$ is independent of both $\epsilon$ and $\lambda$.

Recalling $M_{\lambda,h} := P_{\lambda,h}^{-1}$, then for any $u_h \in V_h$, we have

$$\| P_{\lambda,h} B_{\lambda,h} u_h \|_{P_{\lambda,h}^{-1}} = \sup_{v_h \in V_h, v_h \neq 0} \frac{\langle P_{\lambda,h} B_{\lambda,h} u_h, v_h \rangle_{P_{\lambda,h}^{-1}}}{\langle v_h, v_h \rangle^{1/2}_{P_{\lambda,h}^{-1}}}$$

$$\leq \beta^{1/2} \sup_{v_h \in V_h, v_h \neq 0} \frac{\langle B_{\lambda,h} u_h, v_h \rangle}{\langle A_{\lambda,h} v_h, v_h \rangle^{1/2}} \quad \text{(by (20))}$$

$$\leq C \beta^{1/2} \langle A_{\lambda,h} u_h, u_h \rangle^{1/2} \quad \text{(by (21b))}$$

$$\leq C \beta \| u_h \|_{P_{\lambda,h}^{-1}}, \quad \text{(by (20))}$$

which yields the second inequality of (18) with $\Gamma = C \beta$. On the other side, we have

$$(u_h, P_{\lambda,h} B_{\lambda,h} u_h)_{P_{\lambda,h}^{-1}} = \langle B_{\lambda,h} u_h, u_h \rangle$$

$$\geq (1 - \sqrt{1 - \epsilon}) \langle A_{\lambda,h} u_h, u_h \rangle \quad \text{(by (21a))}$$

$$\geq (1 - \sqrt{1 - \epsilon}) \alpha \langle u_h, u_h \rangle_{P_{\lambda,h}^{-1}}, \quad \text{(by (20))}$$

which yields the first inequality of (18) with $\gamma = (1 - \sqrt{1 - \epsilon}) \alpha$. \hfill \Box

4 Fast Auxiliary Space Preconditioners

In this section, we construct both additive and multiplicative auxiliary space preconditioners for SPD operator $A_{\lambda,h}$. From Lemma 3.1 (FOV-equivalent preconditioner), those preconditioners can be applied to the discrete linearised systems (13) arising from each semi-smooth Newton step of solving the HJB equations.

4.1 The Auxiliary Space

Let $V_0 \subset H^1_0(\Omega)$ denotes the continuous piecewise linear element space on $T_h$ with homogeneous Dirichlet boundary condition. We use $V_0$ as the auxiliary space, i.e.,

$$V_h = V_h + \Pi_0 V_0,$$

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where \( \Pi_0 : V_0 \rightarrow V_h \) is a linear injective map that will be defined later. As a result, the induced operator \( A_0 := \Pi_0^* A_{\lambda,h} \Pi_0 : V_0 \rightarrow V_0' \) is also SPD, and hence we define \( \| \cdot \|_{A_0}^2 := \langle A_0 \cdot, \cdot \rangle \) on \( V_0 \). We also introduce a projection \( P_0 : V_h \rightarrow V_0 \) by
\[
\langle A_0 P_0 v_h, w_h \rangle := \langle v_h, \Pi_0 w_h \rangle_{\lambda,h} \quad \forall v_h \in V_h, w_h \in V_0.
\]
A direct calculation shows the following identity
\[
A_0 P_0 = \Pi_0^* A_{\lambda,h}.
\] (22)

**Smoother and Norm on \( V_0 \)**

Define the discrete Laplacian operator \(-\Delta_h : V_0 \rightarrow V_0 \) by
\[
(-\Delta_h w_h, v_h)_{\Omega} := \langle \nabla w_h, \nabla v_h \rangle_{\Omega} \quad \forall w_h, v_h \in V_0.
\] (23)
Then, the smoother on \( V_0 \), denoted by \( R_0 : V_0' \rightarrow V_0 \), is defined by
\[
(R_0^{-1} w_h, v_h) := \langle \lambda w_h - \Delta_h w_h, \lambda v_h - \Delta_h v_h \rangle_{\Omega} \quad \forall w_h, v_h \in V_0.
\] (24)

Note that for any given \( f_h \in V_0', u_h = R_0 f_h \in V_0 \) can be obtained by solving the following two discrete Poisson-like equations
\[
\langle f_h, v_h \rangle = \langle \lambda (z_h, v_h)_{\Omega} + (\nabla z_h, \nabla v_h)_{\Omega} \rangle \quad \forall v_h \in V_0,
\]
(25a)
\[
\langle z_h, w_h \rangle_{\Omega} = \langle \lambda (u_h, w_h)_{\Omega} + (\nabla u_h, \nabla w_h)_{\Omega} \rangle \quad \forall w_h \in V_0.
\] (25b)

It can be shown that the above two equations can be solved within \( O(N \log N) \) operations, where \( N \) denotes the number of degrees of freedom. We will give a detailed explanation in Remark 4.7 (computational complexity). The smoother \( R_0 \) induces a norm on \( V_0 \), i.e., \( \| \cdot \|_{R_0^{-1}}^2 := \langle R_0^{-1} \cdot, \cdot \rangle \). By using (23) and (24), it is straightforward to show that
\[
\| v_h \|^2_{R_0^{-1}} = \| \Delta_h v_h \|^2_{L^2(\Omega)} + 2\lambda \| \nabla v_h \|^2_{L^2(\Omega)} + \lambda^2 \| v_h \|^2_{L^2(\Omega)}.
\]

Here, we assume that the norm induced by the smooth \( R_0 \) on \( V_0 \) has the following property, which will be verified in Sect. 5.

**Property 4.1** *(spectral equivalence of \( R_0 \))* It holds that
\[
\| v_0 \|_{R_0^{-1}} \simeq \| v_0 \|_{A_0} \quad \forall v_0 \in V_0,
\]
with hidden constants independent of both \( \lambda \) and \( h \).

**Smoother on \( V_h \)**

Let \( R_h \) denote the Gauss-Seidel smoother for \( A_{\lambda,h} \) and \( \tilde{R}_h \) be the symmetric Gauss-Seidel smoother, i.e.,
\[
I - \tilde{R}_h A_{\lambda,h} = (I - R_h A_{\lambda,h})(I - R_h A_{\lambda,h}).
\]
We also define \( \| \cdot \|_{R_h^{-1}}^2 := \langle \tilde{R}_h^{-1} \cdot, \cdot \rangle \) as a norm on \( V_h \). Note that \( A_{\lambda,h} \) is an SPD operator, thus \( R_h \) has the following contraction property
\[
\| I - R_h A_{\lambda,h} \|_{\lambda,h} < 1,
\] (26)
here the norm for the operator is the induced norm by \( \| \cdot \|_{\lambda,h} \) on \( V_h \).
Transfer Operator
We first specify a set of global degrees of freedom for $V_h$ and denote it by $\Sigma_h = \{N_\alpha(\cdot), \alpha = 1, 2, \ldots, N\}$, where $N = \dim V_h$. Let $\varphi_\alpha \in V_h$ be the nodal basis function corresponding to $N_\alpha$, denote $\omega_\alpha := \bigcup \{T \cap \text{supp}(\varphi_\alpha) \neq \emptyset, T \in T_h\}$ and $\#\omega_\alpha := \#\{T \cap \text{supp}(\varphi_\alpha) \neq \emptyset, T \in T_h\}$. Let $k_\alpha$ represent the order of the derivative value involved in $N_\alpha$. Now we are ready to give the definition of the transfer operator (spectral equivalence of additive preconditioner, [25]).

Theorem 4.1

The following theorem plays a fundamental role in the theory of auxiliary space preconditioners to the verification of the following two key assumptions.

Assumption 4.1 (stable decomposition, [25]) There exists a uniform constant $c_0$ independent of both $\lambda$ and $h$, such that for any $v \in V_h$, there exist $v_h \in V_h$ and $v_0 \in V_0$ satisfy

\[
\|v\|_{\hat{R}^{-1}_h}^2 + \|v_0\|_{R_0^{-1}}^2 \leq c_0^2 \|v\|_{\lambda,h}^2.
\]

Assumption 4.2 (boundedness, [25]) There exist uniform constants $c_1$ and $c_2$ independent of both $\lambda$ and $h$ such that

\[
\|v_0\|_{A_h} \leq c_1 \|v_0\|_{R_0^{-1}} \quad \forall v_0 \in V_0,
\]

\[
\|v_h\|_{\lambda,h} \leq c_2 \|v_h\|_{\hat{R}^{-1}_h} \quad \forall v_h \in V_h.
\]

4.2 Additive Preconditioner

Firstly, we introduce the additive preconditioner $P_\lambda : V_h' \to V_h$ as

\[
P_\lambda := \tilde{R}_h + \Pi_0 R_0 \Pi_0'.
\]

The following theorem plays a fundamental role in the theory of auxiliary space preconditioning.

Theorem 4.1 (spectral equivalence of additive preconditioner, [25]) Let $P_\lambda : V_h' \to V_h$ be the preconditioner defined in (30). If Assumption 4.1 (stable decomposition) and Assumption 4.2 (boundedness) hold, then we have

\[
c_0^{-2}(v_h, v_h)_{\lambda,h} \leq (P_\lambda A_{\lambda,h} v_h, v_h)_{\lambda,h} \leq (c_1^2 + c_2^2)(v_h, v_h)_{\lambda,h} \quad \forall v_h \in V_h.
\]

That is, $P_\lambda$ is a uniform spectral equivalence preconditioner of $A_{\lambda,h}$. 

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In light of the above theorem and Lemma 3.1 (FOV-equivalent preconditioner), one can see that \( P_a \) is a uniform FOV-equivalent preconditioner of \( B_{\lambda,h} \) as long as Assumption 4.1 (stable decomposition) and Assumption 4.2 (boundedness) are verified. We postpone those verifications to Sect. 5.

**Remark 4.2** (additive preconditioner with Jacobi smoother) Let \( D_h^{-1} \) be the Jacobi smoother of \( A_{\lambda,h} \). From the norm equivalence between \( \tilde{R}_h \) and \( D_h^{-1} \) [37, Lemma 4.6], the additive preconditioner

\[
\tilde{P}_a := D_h^{-1} + \Pi_0 R_0 \Pi_0'
\]

is also a uniform spectral equivalence preconditioner of \( A_{\lambda,h} \).

**Remark 4.3** (additive preconditioner with scaled parameter) When implementing the additive preconditioners, a positive parameter \( \omega \) is usually introduced to balance the two components, namely,

\[
P_a := \tilde{R}_h + \omega \Pi_0 R_0 \Pi_0'.
\]

A proper choice of \( \omega \) may lead to a better preconditioning performance of \( P_a \) in practice.

### 4.3 Multiplicative Preconditioner

We introduce the multiplicative preconditioner \( P_m : V_h' \rightarrow V_h \) by

\[
I - P_m A_{\lambda,h} := (I - R_h A_{\lambda,h})(I - \Pi_0 R_0 \Pi_0' A_{\lambda,h})(I - R_h' A_{\lambda,h}).
\]  

(32)

Let \( \tilde{R}_0 = A_0^{-1} \) be the exact solver on \( V_0 \). To analyse the multiplicative preconditioner \( P_m \), we introduce an auxiliary multiplicative preconditioner \( \hat{P}_m : V_h' \rightarrow V_h \) by

\[
I - \hat{P}_m A_{\lambda,h} := (I - R_h A_{\lambda,h})(I - \Pi_0 \hat{R}_0 \Pi_0' A_{\lambda,h})(I - R_h' A_{\lambda,h}).
\]

(33)

We emphasize that \( \hat{P}_m \) is never computed but is useful for theoretical purposes in Theorem 4.5 (spectral equivalence of multiplicative preconditioner), which can be divided into two steps: (i) The spectral equivalence between \( P_m A_{\lambda,h} \) and \( \hat{P}_m A_{\lambda,h} \) by using Property 4.1 (spectral equivalence of \( R_0 \)); (ii) Estimate of \( \hat{P}_m A_{\lambda,h} \) by two-level convergence results. For the second step, let \( E := (I - R_h A_{\lambda,h})(I - \Pi_0 \hat{R}_0 \Pi_0' A_{\lambda,h}) \) be the error propagation operator of the two-level method corresponding to \( \hat{P}_m \). Since the solver on \( V_0 \) is exact, the convergence rate can be obtained in the following theorem. We refer to [37, Theorem 5.3] for more details.

**Theorem 4.4** (two-level convergence rate, [37]) The following identity holds

\[
\| E \|_{\lambda,h}^2 = 1 - \frac{1}{K(V_0)},
\]

where

\[
K(V_0) := \max_{v_h \in V_h} \min_{v_0 \in V_0} \frac{\| v_h - \Pi_0 v_0 \|_{\tilde{R}_h^{-1}}^2}{\| v_h \|_{\lambda,h}^2}.
\]

Note that the identity (22) implies that \( \hat{R}_0 \Pi_0' A_{\lambda,h} = P_0 \). For any \( v_h, w_h \in V_h \), we have

\[
(\Pi_0 P_0 v_h, w_h)_{\lambda,h} = (P_0 v_h, \Pi_0' A_{\lambda,h} w_h) = (P_0 v_h, A_0 P_0 w_h) = (v_h, \Pi_0 P_0 w_h)_{\lambda,h}.
\]

\[
(R_h' A_{\lambda,h} v_h, w_h)_{\lambda,h} = (A_{\lambda,h} v_h, R_h A_{\lambda,h} w_h) = (v_h, R_h A_{\lambda,h} w_h)_{\lambda,h}.
\]
which means that \( I - \Pi_0 P_0 \) and \( I - R_h' A_{\lambda,h} \) are respectively the dual operators of \( I - \Pi_0 P_0 \) and \( I - R_h A_{\lambda,h} \) under the inner product \((\cdot, \cdot)_{\lambda,h}\). As a consequence,
\[
\| E \|_{\lambda,h}^2 = \|(I - R_h A_{\lambda,h})(I - \Pi_0 P_0)\|_{\lambda,h}^2 = \|(I - \Pi_0 P_0)(I - R_h' A_{\lambda,h})\|_{\lambda,h}^2. \tag{34}
\]
Moreover, under Assumption 4.1 (stable decomposition), we have \( K(V_0) \leq c_0^2 \) and hence 
\[
\| E \|_{\lambda,h}^2 \leq 1 - \frac{1}{c_0^2},
\]
which leads to the following theorem.

**Theorem 4.5** (spectral equivalence of multiplicative preconditioner) Let \( R_h \) be the Gauss-Seidel smoother for \( A_{\lambda,h} \). Under Assumption 4.1 (stable decomposition), the multiplicative preconditioner \( P_m \) defined in (32) satisfies
\[
(P_m A_{\lambda,h} v, v)_{\lambda,h} \simeq (v, v)_{\lambda,h} \quad \forall v \in V_h,
\]
with hidden constants independent of both \( \lambda \) and \( h \). That is, \( P_m \) is a uniform spectral equivalence preconditioner of \( A_{\lambda,h} \).

**Proof** Step (i): For any \( v \in V_h \), denote \( w = (I - R_h' A_{\lambda,h}) v \). By the definition of \( P_m \) (32), we have
\[
(P_m A_{\lambda,h} v, v)_{\lambda,h} = (v, v)_{\lambda,h} - ((I - \Pi_0 R_0 \Pi_0 A_{\lambda,h}) w, w)_{\lambda,h}
\]
\[
= \|v\|_{\lambda,h}^2 - \|w\|_{\lambda,h}^2 + (\Pi_0 R_0 \Pi_0 A_{\lambda,h} w, w)_{\lambda,h}
\]
\[
= \|v\|_{\lambda,h}^2 - \|w\|_{\lambda,h}^2 + (R_0 A_0 P_0 w, P_0 w)_{A_0},
\]
where we use the identity (22) in the last step. Similarly, for \( \hat{P}_m \) defined in (33), we have
\[
(\hat{P}_m A_{\lambda,h} v, v)_{\lambda,h} = \|v\|_{\lambda,h}^2 - \|w\|_{\lambda,h}^2 + (P_0 w, P_0 w)_{A_0},
\]
since \( \hat{R}_0 = A_0^{-1} \). Invoking Property 4.1 (spectral equivalence of \( R_0 \)), we have
\[
C_1(P_0 w, P_0 w)_{A_0} \leq (R_0 A_0 P_0 w, P_0 w)_{A_0} \leq C_2(P_0 w, P_0 w)_{A_0}, \tag{35}
\]
where \( C_1, C_2 \) are constants independent of \( \lambda \) and \( h \). Then,
\[
\min\{1, C_1\} \{\hat{P}_m A_{\lambda,h} v, v\}_{\lambda,h} \leq (P_m A_{\lambda,h} v, v)_{\lambda,h} \leq \max\{1, C_2\} \{\hat{P}_m A_{\lambda,h} v, v\}_{\lambda,h}. \tag{36}
\]
Here, we use the contraction property of \( R_h \), namely \( \|v\|_{\lambda,h}^2 - \|w\|_{\lambda,h}^2 \geq (1 - \|I - R_h A_{\lambda,h}\|_{\lambda,h}^2) \|v\|_{\lambda,h}^2 = (1 - \|I - R_h A_{\lambda,h}\|_{\lambda,h}^2) \|v\|_{\lambda,h}^2 > 0 \).

Step (ii): In light of (36), we only need to show
\[
(\hat{P}_m A_{\lambda,h} v, v)_{\lambda,h} \simeq (v, v)_{\lambda,h} \quad \forall v \in V_h,
\]
with hidden constants independent of \( \lambda \) and \( h \). From the identity (22), we see that \( (I - \Pi_0 P_0) = (I - \Pi_0 P_0)^2 \) since \( P_0 \Pi_0 = A_0^{-1} \Pi_0 A_{\lambda,h} \Pi_0 = I \). Then, we obtain the upper bound of \( \hat{P}_m A_{\lambda,h} \):
Next, we estimate the lower bound of $\hat{P}_m A_{\lambda,h}$ by Theorem 4.4 (two-level convergence rate) and Assumption 4.1 (stable decomposition),

$$
(\hat{P}_m A_{\lambda,h} v, v)_{\lambda,h} = \|v\|_{\lambda,h}^2 - \|(I - \Pi_0 P_0)(I - R_h A_{\lambda,h})v\|_{\lambda,h}^2 \\
\geq (1 - \|(I - \Pi_0 P_0) (I - R_h A_{\lambda,h})\|_{\lambda,h}^2) \|v\|_{\lambda,h}^2 \\
= (1 - \|(I - R_h A_{\lambda,h}) (I - \Pi_0 P_0)\|_{\lambda,h}^2) \|v\|_{\lambda,h}^2 \\
= (1 - \|E\|_{\lambda,h}^2) \|v\|_{\lambda,h}^2 \\
= \frac{1}{K(V_0)} \|v\|_{\lambda,h}^2 \geq c_0^{-2} \|v\|_{\lambda,h}^2.
$$

Combining Step (i) and Step (ii), we obtain

$$
\min\{1, C_1\} c_0^{-2} \|v\|_{\lambda,h}^2 \leq (P_m A_{\lambda,h} v, v)_{\lambda,h} \leq \max\{1, C_2\} \|v\|_{\lambda,h}^2.
$$

The proof is thus complete. \(\square\)

**Remark 4.6** In the Proof of Theorem 4.5 (spectral equivalence of multiplicative preconditioner), Assumption 4.2 (boundedness) is not directly used. That is because Property 4.1 (spectral equivalence of $R_0$) leads to the boundedness on coarse space (29a), and the boundedness on fine space (29b) is a direct consequence of the contraction property (26) of Gauss–Seidel smoother $R_h$.

**Remark 4.7** (computational complexity) We now discuss the computational complexity of the action of preconditioner $P_\alpha$ (30) and $P_m$ (32). Let $N_p$ be the number of degrees of freedom, $N_p$ be the number of interior points of the grids. Since the transfer operator $\Pi_0$ is local, the action except $R_0$ can be done within $O(N_h)$ operations.

Invoking the definition of $R_0$ in (24), we can see for any given $f_h \in V'_0$, $u_h = R_0 f_h$ can be obtained by solving two discrete Poisson-like equations (25). The computational complexity of $(\lambda I - \Delta_h)^{-1}$ with classic geometric multigrid methods was shown to be optimal for $\lambda = O(1)$ [38] and for arbitrary $\lambda > 0$ [39]. For unstructured shape-regular grids with $\lambda = O(1)$, the computational complexity turns to be $O(\log N_p)$ by constructing an auxiliary coarse grid hierarchy where the geometric multigrid can be applied [29]. Therefore, the nearly optimal computational complexity for arbitrary $\lambda > 0$ on unstructured grids is to be expected by combining the techniques from [39] and [29].

**Remark 4.8** (Implement of action $R_0$) Let $\{|\psi_i\rangle\}_{i=1}^{N_p}$ be the nodal basis functions of $V_0$. Denote $A = ((\nabla \psi_j, \nabla \psi_i)_\Omega) \in \mathbb{R}^{N_p \times N_p}$, $M = ((\psi_j, \psi_i)_\Omega) \in \mathbb{R}^{N_p \times N_p}$ as the stiffness and mass matrix, respectively. For any $f_h \in V'_0$, let $f \in \mathbb{R}^{N_p}$ be its vector representation, i.e., $(f)_i = \langle f_h, \psi_i \rangle$. Then $u_h = R_0 f_h$ can be obtained by solving the following two linear systems successively.

$$
f = (\lambda M + A) z, \\
M z = (\lambda M + A) u,
$$

where $u \in \mathbb{R}^{N_p}$ is the vector representation of $u_h$, i.e., $u_h = \sum_{i=1}^{N_p} (u)_i \psi_i$.

## 5 Analysis of the Auxiliary Space Preconditioners

In this section, we first show some properties about the space $V_0$ and the transfer operator $\Pi_0$ introduced in Sect. 4. We refer to [26, 35] for details on those results. Next we will verify
Assumption 4.1 (stable decomposition) and Assumption 4.2 (boundedness) by combing those lemmas.

To begin with, we state the following lemma corresponding to the scaling of basis functions and $L^2$ norm. The proof follows from the scaling argument, see e.g. [40, Sect. 2.2.7].

**Lemma 5.1** (scaling) For any global nodal basis function $\varphi_\alpha$ of $V_h$, let $T \in T_h$ with $T \subset \omega_\alpha$, it holds that

$$
\int_T |D^j \varphi_\alpha|^2 \, dx \simeq h_T^{2k_\alpha + d - 2j}, \quad j = 0, 1, 2.
$$

Moreover, for any $v_h|_T \in \mathcal{P}_k(T)$, $T \in T_h$, it holds that

$$
\|v_h\|_{L^2(T)}^2 \simeq \sum_{\omega_\alpha \cap \partial T} h_T^{2k_\alpha + d} (N_\alpha(v_h))^2.
$$

Here $d = 2, 3$ represents the dimension and $k_\alpha$ represents the order of differentiation involved in $N_\alpha$.

The next lemma shows the stability and approximation property of the transfer operator defined in (27).

**Lemma 5.2** (see [26], Lemma 3.6) Let $\Pi_0 : V_0 \to V_h$ be the transfer operator defined in (27). For any $v_0 \in V_0$, it holds that

$$
\sum_{T \in T_h} h_T^{-4} \|\Pi_0 v_0 - v_0\|_{L^2(T)}^2 \lesssim \sum_{T \in T_h} h_T^{-1} \int_T \left| \frac{\partial v_0}{\partial \mathbf{n}_T} \right|^2 ds \quad \forall v_0 \in V_0.
$$

Moreover, we have

$$
\|\Pi_0 v_0\|_{L^2(\Omega)} \lesssim \|v_0\|_{L^2(\Omega)} \quad \text{and} \quad |\Pi_0 v_0|_{H^1(\Omega)} \lesssim |v_0|_{H^1(\Omega)}.
$$

**Proof** In [26, Lemma 3.6], the proof of (39) is given for the case $d = 2$. We give below the proof for the general case $d = 2, 3$ based on the techniques in [26, Lemma 3.6]. For any $v_0 \in V_0$, by (38) in Lemma 5.1 (scaling), we have

$$
\sum_{T \in T_h} h_T^{-4} \|\Pi_0 v_0 - v_0\|_{L^2(T)}^2 \lesssim \sum_{\omega_\alpha} \sum_{k_\alpha = 1}^{\#\omega_\alpha} (N_\alpha(\Pi_0 v_0) - N_\alpha(v_0|_{T_i}))^2 h_T^{d-2}.
$$

Here, the patch $\omega_\alpha$ is written as $\omega_\alpha = \bigcup_{l=1}^{\#\omega_\alpha} T_l$, where $T_l$ and $T_{l+1}$ share a common $d - 1$ dimensional face $F_l$. By the definition of $\Pi_0$ (27) and the fact that $v_0|_T \in \mathcal{P}_1(T)$, we find $(N_\alpha(\Pi_0 v_0) - N_\alpha(v_0|_{T_l}))$ is non-zero only if $k_\alpha = 1$. In the following, we divide $N_\alpha$ satisfying $k_\alpha = 1$ into two cases.

First, if $N_\alpha$ is an interior degree of freedom or corresponding to one of the edges on the boundary, by (27b) and the triangle inequality, we have

$$
\sum_{l=1}^{\#\omega_\alpha} (N_\alpha(\Pi_0 v_0) - N_\alpha(v_0|_{T_l}))^2 h_T^{d-2} \lesssim \sum_{l=1}^{\#\omega_\alpha - 1} (N_\alpha(v_0|_{T_{l+1}}) - N_\alpha(v_0|_{T_l}))^2 h_{F_l}^{d-2}
$$

$$
\lesssim \sum_{l=1}^{\#\omega_\alpha - 1} h_{F_l}^{-1} \int_{F_l} \left| \frac{\partial v_0}{\partial \mathbf{n}_{F_l}} \right|^2 ds.
$$

Here we use the fact that the derivative of $v_0$ is a piecewise constant.
Second, if $N_\alpha$ corresponds to the first derivative value at some boundary node $x_\alpha \in N_h^\partial$, i.e., $N_\alpha(\phi) = e_i \cdot \nabla \phi(x_\alpha)$, where $i = 1, 2, \ldots, d$. We divide $N_\alpha(v_0|T)$ in (41) into two parts, i.e.,

$$\sum_{l=1}^{\#_\omega_\alpha} (N_\alpha(\Pi_0 v_0) - N_\alpha(v_0|T))h_{T_l}^{d-2}$$

$$\lesssim \sum_{l=1}^{\#_\omega_\alpha} \left( \sum_{j=1}^{d-1} (t_j \cdot e_i) \frac{\partial(v_0|T)}{\partial t_j}(x_\alpha) \right)^2 h_{T_l}^{d-2}$$

(43)

$$+ \sum_{l=1}^{\#_\omega_\alpha} \left( \sum_{j=1}^{d-1} (t_j \cdot e_i) \frac{\partial(v_0|T)}{\partial t_j}(x_\alpha) \right)^2 h_{T_l}^{d-2}.$$

Here, $t_j$ is the tangential direction on $\partial\Omega$, and $n$ is the outer normal direction. Similar to the first case discussed above, the normal part $I_1$ can be controlled by $\sum_{l=1}^{\#_\omega_\alpha} h_{T_l}^{-1} \int_{F_l} \| \frac{\partial v_0}{\partial n_{F_l}} \|^2 ds$.

For the tangential part $I_2$, there exists $T^* \subset \omega_\alpha$ with face $F^*$ such that $x_\alpha \in F^*$ and $\overline{F^*} \subset \partial\Omega$. Note that $\frac{\partial(v_0|T)}{\partial t_j}(x_\alpha) = 0$ for $j = 1, 2, \ldots, d - 1$ due to the boundary condition of $v_0$. Adding these terms to $I_2$ and using the triangle inequality, we find that $I_2$ can also be controlled by $\sum_{l=1}^{\#_\omega_\alpha} h_{T_l}^{-1} \int_{F_l} \| \frac{\partial v_0}{\partial n_{F_l}} \|^2 ds$. Hence, (39) is obtained by combining (41), (42), and (43).

As for (40), it can be obtained by combining (39) with the triangular inequality, the standard inverse estimate and the trace estimate [35].

For any $v_h \in V_h$, define nodal interpolation $I_h : V_h \to V_0$ as

$$I_h v_h(x) = v_h(x) \quad \forall x \in N_h.$$  

(44)

According to standard polynomial approximation theory [35], we have the following lemma.

**Lemma 5.3** (see [35], Chapter 4) For any $v_h \in V_h$ and $T \in T_h$, it holds that

$$\| v_h - I_h v_h \|_{L^2(T)} \lesssim h_T^2 \| v_h \|_{H^2(T)} \quad \text{and} \quad \| \frac{\partial(v_h - I_h v_h)}{\partial n} \|_{L^2(\partial T)} \lesssim h_T^{1/2} \| v_h \|_{H^2(T)}.$$  

(45)

Moreover, we have

$$\| I_h v_h \|_{L^2(\Omega)} \lesssim \| v_h \|_{L^2(\Omega)} \quad \text{and} \quad \| I_h v_h \|_{H^1(\Omega)} \lesssim \| v_h \|_{H^1(\Omega)}.$$  

(46)

Note that, by the definition of $I_h$ and $\Pi_0$, $I_h \Pi_0$ is indeed the identity operator on $V_0$. The following lemma is an estimate of $\Pi_0 I_h : V_h \to V_h$.

**Lemma 5.4** For any $v_h \in V_h$, it holds that

$$\sum_{T \in T_h} \left( h_T^{-2} + \lambda^2 \right) \| v_h - \Pi_0 I_h v_h \|_{L^2(T)}^2 \lesssim \sum_{T \in T_h} \left( 1 + \lambda h_T^2 \right)^2 \| D^2 v_h \|_{L^2(T)}^2.$$  

(47)

**Proof** It suffices to show:

$$\sum_{T \in T_h} h_T^{-2\epsilon} \| v_h - \Pi_0 I_h v_h \|_{L^2(T)}^2 \lesssim \sum_{T \in T_h} h_T^{-2\epsilon+4} \| D^2 v_h \|_{L^2(T)}^2,$$
for $\ell = 0, 1, 2$. The case when $\ell = 2$ is proved in [26, Lemma 3.7]. When $\ell = 0, 1$, it can be proved by the similar arguments in [26, Lemma 3.7].

The following lemma is the estimate of the discrete Laplace operator $\Delta_h$, which is critical in our analysis. We refer [26] for more details.

**Lemma 5.5** Suppose $\Omega \subset \mathbb{R}^d$ is a convex polytopal domain. Let $\Delta_h$ be the discrete Laplacian operator defined in (23). Then it holds that

$$\|\Delta_h v_0\|^2_{L^2(\Omega)} \simeq \sum_{F \in \mathcal{F}_h^i} h_F^{-1} \int_F \left[ \frac{\partial v_0}{\partial n_F} \right]^2 ds \quad \forall v_0 \in V_0. \quad (48)$$

**Proof** (48) can be obtained by combining a similar technique in [26, Lemma 3.1] and the elliptic regularity in the convex polytopal domain in $\mathbb{R}^d$ [32, Chapter 3].

Applying these lemmas above, we are now ready to give the Proof of Property 4.1 (spectral equivalence of $R_0$) introduced in Sect. 4.

**Proof of Property 4.1** (spectral equivalence of $R_0$). *Step (i):* For any $v_0 \in V_0$, by combining Lemma 5.2 and Lemma 5.5 we have

$$\sum_{T \in T_h} \|D^2 \Pi_h v_0\|^2_{L^2(T)} = \sum_{T \in T_h} \|D^2 (\Pi_h v_0 - v_0)\|^2_{L^2(T)} \leq \sum_{T \in T_h} h_T^{-4} \|\Pi_h v_0 - v_0\|^2_{L^2(T)} \quad \text{(by inverse estimate)} \leq \sum_{F \in \mathcal{F}_h^i} h_F^{-1} \int_F \left[ \frac{\partial v_0}{\partial n_F} \right]^2 ds \quad \text{(by Lemma 5.2)} \leq \|\Delta_h v_0\|^2_{L^2(\Omega)}. \quad \text{(by Lemma 5.5)}$$

For the lower term, the boundedness of $\Pi_0$ (40) in Lemma 5.2 leads to:

$$2\lambda \|\nabla \Pi_0 v_0\|^2_{L^2(\Omega)} + \lambda^2 \|\Pi_0 v_0\|^2_{L^2(\Omega)} \lesssim 2\lambda \|\nabla v_0\|^2_{L^2(\Omega)} + \lambda^2 \|v_0\|^2_{L^2(\Omega)}.$$

By combining the above two estimates, we have:

$$\|v_0\|_{A_0}^2 = \sum_{T \in T_h} \|D^2 \Pi_h v_0\|^2_{L^2(T)} + 2\lambda \|\nabla \Pi_0 v_0\|^2_{L^2(\Omega)} + \lambda^2 \|\Pi_0 v_0\|^2_{L^2(\Omega)} \lesssim \|\Delta_h v_0\|^2_{L^2(\Omega)} + 2\lambda \|\nabla v_0\|^2_{L^2(\Omega)} + \lambda^2 \|v_0\|^2_{L^2(\Omega)} \lesssim \|v_0\|^2_{R_0^{-1}}.$$

*Step (ii):* Note that $v_0 = I_h \Pi_0 v_0$ for any $v_0 \in V_0$, using Lemma 5.5 and the approximation property of $I_h$ in Lemma 5.3, we have

$$\|\Delta_h v_0\|^2_{L^2(\Omega)} = \|\Delta_h I_h \Pi_0 v_0\|^2_{L^2(\Omega)} \lesssim \sum_{F \in \mathcal{F}_h^i} h_F^{-1} \int_F \left[ \frac{\partial I_h \Pi_0 v_0}{\partial n_F} \right]^2 ds \lesssim \sum_{F \in \mathcal{F}_h^i} h_F^{-1} \int_F \left[ \frac{\partial (I_h \Pi_0 v_0 - \Pi_0 v_0)}{\partial n_F} \right]^2 ds + \sum_{F \in \mathcal{F}_h^i} h_F^{-1} \int_F \left[ \frac{\partial \Pi_0 v_0}{\partial n_F} \right]^2 ds \lesssim \sum_{T \in T_h} \|D^2 \Pi_0 v_0\|^2_{L^2(T)}.$$
Here in the last step, note $\Pi_0 v_0 \in V_h$, the standard scaling argument \cite{35} gives that
\[ h_F^{-1} \int_F \left[ \frac{\partial \Pi_0 v_0}{\partial n_F} \right]^2 \, ds \lesssim \sum_{T \in \{ T^+, T^\cdot \}} \| D^2 \Pi_0 v_0 \|_{L^2(T)}^2, \]
holds for any interior face $F = \partial T^+ \cap \partial T^\cdot$, where the $C^0$-continuity at face and $C^1$-continuity at $(d - 2)$-dimensional subsimplex guarantee that the piecewise linear function on $\omega_F = T^+ \cup T^\cdot$ has to be a linear function on the $\omega_F$.

Similarly, the lower term can be estimate by applying the boundedness of $I_h$ \cite{46} in Lemma 5.3:
\[ 2\lambda \| \nabla I_h \Pi_0 v_0 \|_{L^2(\Omega)}^2 + \lambda^2 \| I_h \Pi_0 v_0 \|_{L^2(\Omega)}^2 \lesssim 2\lambda \| \nabla \Pi_0 v_0 \|_{L^2(\Omega)}^2 + \lambda^2 \| \Pi_0 v_0 \|_{L^2(\Omega)}^2. \]

By combining the above two estimates, we have:
\[ \| v_0 \|_{R_0^{-1}}^2 = \| I_h \Pi_0 v_0 \|_{R_0^{-1}}^2 \]
\[ \lesssim \sum_{T \in \mathcal{T}_h} \| D^2 \Pi_0 v_0 \|_{L^2(T)}^2 + \| \nabla \Pi_0 v_0 \|_{L^2(\Omega)}^2 + \| \Pi_0 v_0 \|_{L^2(\Omega)}^2 \]
\[ = \| \Pi_0 v_0 \|_{\hat{\mathcal{A}}_h}^2 = \| v_0 \|_{A_0}. \]
\[ \square \]

The next lemma gives an equivalence form of $\| \cdot \|_{\hat{R}_h^{-1}}$ which will be used in the verification of Assumption 4.1 (stable decomposition) and Assumption 4.2 (boundedness).

**Lemma 5.6** (norm equivalence of $\hat{R}_h$) Let $\mathcal{T}_h$ be a conforming shape regular triangulation of $\Omega$. Let $D_h^{-1}$ and $R_h$ be the Jacobi and Gauss-Seidel smoother for $A_{\lambda, h}$, respectively. Then, we have
\[ \| v_h \|_{\hat{R}_h^{-1}}^2 \simeq \| v_h \|_{D_h}^2 \simeq \sum_{T \in \mathcal{T}_h} (h_T^{-2} + \lambda)^2 \| v_h \|_{L^2(T)}^2 \quad \forall v_h \in V_h. \]

**Proof** By the classical theory of iterative method \cite{37, 46}, the symmetric Gauss-Seidel smoother and the Jacobi smoother are spectral equivalent for sparse SPD operator, namely $\| v_h \|_{\hat{R}_h^{-1}} \simeq \| v_h \|_{D_h}$. Combining (37) and (38) in Lemma 5.1 (scaling), we have
\[ \sum_{T \in \mathcal{T}_h} h_T^{-4} \| v_h \|_{L^2(T)}^2 \simeq \sum_{T \in \mathcal{T}_h} h_T^{-4} \sum_{\omega_\alpha \supset T} h_T^{2k_\alpha + d} \left( N_\alpha(v_h) \right)^2 \]
\[ = \sum_{\alpha} \left( \sum_{T \subset \omega_\alpha} h_T^{2k_\alpha + d - 4} \right) \left( N_\alpha(v_h) \right)^2. \]
\[ \simeq \sum_{\alpha} \left( \sum_{T \subset \omega_\alpha} \int_T |D^2 \varphi_\alpha|^2 \, dx \right) \left( N_\alpha(v_h) \right)^2. \]

A similar argument leads to
\[ \sum_{T \in \mathcal{T}_h} h_T^{-2} \| v_h \|_{L^2(T)}^2 \simeq \sum_{\alpha} \left( \| \nabla \varphi_\alpha \|_{L^2(\Omega)}^2 \right) \left( N_\alpha(v_h) \right)^2, \]
\[ \square \]
and
\[ \sum_{T \in T_h} \| v_h \|_{L^2(T)}^2 \simeq \sum_{\alpha} \left( \| \varphi_{\alpha} \|_{L^2(\Omega)}^2 \right) (N_{\alpha}(v_h))^2. \tag{52} \]

Multiplying (50), (51) and (52) respectively by 1, 2 and \( \lambda \), then summing these equations, we obtain
\[ \sum_{T \in T_h} \left( h^{-2} T + \lambda^2 \right) \| v_h \|_{L^2(T)}^2 \simeq \sum_{\alpha} \left( \sum_{T \in T_h} \int_T |D^2 \varphi_{\alpha}|^2 + 2\lambda |\nabla \varphi_{\alpha}|^2 + \lambda^2 \varphi_{\alpha}^2 \, dx \right) (N_{\alpha}(v_h))^2 = \| v_h \|_{D_h}^2, \]

which yields the norm equivalence (49). \( \square \)

**Theorem 5.7** (verification of boundedness) There exist constants \( c_1, c_2 \) independent of \( \lambda \) and \( h \), such that
\[ \| v_0 \|_{A_0} \leq c_1 \| v_0 \|_{R_0^{-1}} \quad \forall v_0 \in V_0, \tag{53a} \]
\[ \| v_h \|_{\lambda, h} \leq c_2 \| v_h \|_{R_h^{-1}} \quad \forall v_h \in V_h. \tag{53b} \]

**Proof** (53a) is a direct consequence of the Property 4.1 (spectral equivalence of \( R_0 \)). (53b) follows from the standard inverse estimate and Lemma 5.6 (norm equivalence of \( \tilde{R}_h \)).

\[ \| v_h \|_{\lambda, h}^2 = \sum_{T \in T_h} \| D^2 v_h \|_{L^2(T)}^2 + 2\lambda \| \nabla v_h \|_{L^2(\Omega)}^2 + \lambda^2 \| v_h \|_{L^2(\Omega)}^2 \]
\[ \leq \sum_{T \in T_h} (h^{-4} + 2\lambda h^{-2} + \lambda^2) \| v_h \|_{L^2(T)}^2 \]
\[ = \sum_{T \in T_h} (h^{-2} + \lambda) \| v_h \|_{L^2(T)}^2 \simeq \| v_h \|_{R_h^{-1}}^2. \]

This completes the proof. \( \square \)

**Theorem 5.8** (verification of stable decomposition) For any \( v \in V_h \), there exist \( v_h \in V_h \) and \( v_0 \in V_0 \) such that
\[ v = v_h + \Pi_0 v_0, \tag{54a} \]
\[ \| v_h \|_{R_h^{-1}}^2 + \| v_0 \|_{R_0^{-1}}^2 \leq c_0^2 \| v \|_{\lambda, h}^2, \tag{54b} \]

where \( c_0 \) is a constant independent with \( \lambda \) and \( h \).

**Proof** Recall the nodal interpolation \( I_h : V_h \rightarrow V_0 \) defined in (44). For any \( v \in V_h \), take \( v_0 = I_h v \) and \( v_h = v - \Pi_0 I_h v \) so that (54a) is satisfied. It follows from Lemmas 5.4 and 5.6.
(norm equivalence of $\bar{R}_h$) and inverse estimate that
\[
\|v_h\|^2_{\bar{R}_h^{-1}} \simeq \sum_{T \in T_h} (h_T^{-2} + \lambda)^2 \|v_h\|^2_{L^2(T)} \\
= \sum_{T \in T_h} (h_T^{-2} + \lambda)^2 \|v - \Pi_0 I_h v\|^2_{L^2(T)} \\
\lesssim \sum_{T \in T_h} (1 + \lambda h_T^2)^2 \|D^2 v\|^2_{L^2(T)} \quad \text{(by Lemma 5.4)} \\
\lesssim \sum_{T \in T_h} \|D^2 v\|^2_{L^2(T)} + 2\lambda \|\nabla v\|^2_{L^2(\Omega)} + \lambda^2 \|v\|^2_{L^2(\Omega)} \quad \text{(by inverse estimate)} \\
= \|v\|^2_{\mathcal{L}_h}.
\]

On the other hand, combining Lemma 5.5 and the approximation property of $I_h$ in Lemma 5.3, we have
\[
\|\Delta_h I_h v\|^2_{L^2(\Omega)} \lesssim \sum_{F \in F_h^i} h_F^{-1} \int_F \left\| \frac{\partial I_h v}{\partial n_F} \right\|^2 ds \\
\lesssim \sum_{F \in F_h^i} h_F^{-1} \int_F \left\| \frac{\partial (I_h v - v)}{\partial n_F} \right\|^2 ds + \sum_{F \in F_h^i} h_F^{-1} \int_F \left\| \frac{\partial v}{\partial n_F} \right\|^2 ds \quad (55) \\
\lesssim \sum_{T \in T_h} \|D^2 v\|^2_{L^2(T)}.
\]

As in the case of the Proof of Property 4.1 (spectral equivalence of $R_0$), here in the last step, the standard scaling argument [35] gives that
\[
h_F^{-1} \int_F \left\| \frac{\partial v}{\partial n_F} \right\|^2 ds \lesssim \sum_{T \in \{T^+, T^-\}} \|D^2 v\|^2_{L^2(T)},
\]
holds for any interior face $F = \partial T^+ \cap \partial T^-$. Combining the boundedness of $I_h$ in Lemma 5.5 and (55), we get
\[
\|v_0\|^2_{R_0^{-1}} = \|\Delta_h I_h v\|^2_{L^2(\Omega)} + 2\lambda \|\nabla I_h v\|^2_{L^2(\Omega)} + \lambda^2 \|I_h v\|^2_{L^2(\Omega)} \\
\lesssim \sum_{T \in T_h} \|D^2 v\|^2_{L^2(T)} \quad \text{(by (55))} \\
+ 2\lambda \|\nabla v\|^2_{L^2(\Omega)} + \lambda^2 \|v\|^2_{L^2(\Omega)} \quad \text{(by (46))} \\
= \|v\|^2_{\mathcal{L}_h}.
\]
The proof is thus complete.

\section{6 Numerical Experiments}

In this section, we present numerical experiments to illustrate the performance of PGMRES preconditioners for solving both linear and nonlinear problems.
Denote $\kappa(\cdot)$ for the condition number of operator or matrix, and DOF for the number of degrees of freedom. On the fine level, we use the Gauss-Seidel method for $A_{\lambda,h}$ with three iterations as the smoother on $V_h$, which shares similar properties as the Gauss-Seidel smoother $R_0$. For the actor of subspace smoother $R_0$, we apply the algebraic multigrid method (AMG) with stop criterion $\|b - Ax\|_2 / \|b\|_2 \leq 10^{-8}$ for a linear system $Ax = b$. We run the code on the laptop with Intel Core i5-5350U CPU (1.80 GHz).

6.1 Preconditioning Effect of $P_a$ and $P_m$ for $A_{\lambda,h}$

We test the theoretical results in Theorem 4.1 (spectral equivalence of additive preconditioner) and Theorem 4.5 (spectral equivalence of multiplicative preconditioner) by examining the condition number of $P_m A_{\lambda,h}$ and $P_a A_{\lambda,h}$. To showcase the flexibility of the preconditioner on non-uniform grids, we illustrate the performance of preconditioners on a sequence of graded bisection grids $\{T_\ell\}_{\ell \in N_0}$ with grading factor $1/2$ on $\Omega = (-1, 1)^2$, see Fig. 3. More specifically, we mark the elements which satisfy $|T| > C(\|x_T\|_2 - 1/2)^2/#T_\ell$, where $x_T$ is the barycenter of the element $T$, $#T_\ell$ is the number of elements in $T_\ell$. In the experiment, we set $C = 1000$. Further, in the case of additive preconditioners, we apply the scaled form (31) in Remark 4.3 (additive preconditioner with scaled parameter) with $\omega = 1/10$.

The resulting condition numbers for additive and multiplicative preconditioners at different bisection levels are listed respectively in Tables 1 and 2. We observe that both $P_m$ and $P_a$ are uniform preconditioners with respect to both $\lambda$ and DOF, which is in agreement with the theoretical results in Theorem 4.5 (spectral equivalence of multiplicative preconditioner) and Theorem 4.1 (spectral equivalence of additive preconditioner). We also observe that the multiplicative preconditioners perform better than additive ones.

### Table 1 Condition number of additive preconditioning for Experiment 6.1

| DOF         | $\lambda = 10^{-3}$ | $\lambda = 10^{-2}$ | $\lambda = 10^{-1}$ | $\lambda = 1$ | $\lambda = 10$ | $\lambda = 10^2$ | $\lambda = 10^3$ |
|-------------|---------------------|---------------------|---------------------|----------------|----------------|----------------|----------------|
| 3,147       | 1.64e2              | 1.64e2              | 1.60e2              | 1.40e2         | 9.41e1         | 5.43e1         | 2.48e1         |
| 4,467       | 1.65e2              | 1.64e2              | 1.60e2              | 1.40e2         | 9.49e1         | 6.43e1         | 3.42e1         |
| 6,587       | 1.61e2              | 1.60e2              | 1.57e2              | 1.35e2         | 9.22e1         | 7.09e1         | 4.52e1         |
| 10,027      | 1.61e2              | 1.61e2              | 1.57e2              | 1.36e2         | 9.62e1         | 8.04e1         | 5.49e1         |
| 15,927      | 1.62e2              | 1.62e2              | 1.58e2              | 1.36e2         | 9.95e1         | 8.78e1         | 6.46e1         |
Table 2  Condition number of multiplicative preconditioning for Experiment 6.1

| DOF    | $\kappa(P_m A_{\lambda, \lambda})$ | $\lambda = 10^{-3}$ | $\lambda = 10^{-2}$ | $\lambda = 10^{-1}$ | $\lambda = 1$ | $\lambda = 10$ | $\lambda = 10^2$ | $\lambda = 10^3$ |
|--------|----------------------------------|----------------------|----------------------|----------------------|----------------|----------------|----------------|----------------|
| 3,147  | 5.76                             | 5.76                 | 5.76                 | 5.75                 | 5.65           | 5.21           | 4.16           |                |
| 4,467  | 5.76                             | 5.76                 | 5.75                 | 5.75                 | 5.68           | 5.42           | 4.77           |                |
| 6,587  | 5.63                             | 5.63                 | 5.63                 | 5.62                 | 5.56           | 5.39           | 4.99           |                |
| 10,027 | 6.03                             | 6.03                 | 6.03                 | 6.02                 | 5.94           | 5.59           |                |                |
| 15,927 | 6.07                             | 6.07                 | 6.07                 | 6.06                 | 6.00           | 5.75           |                |                |

6.2 Uniform Preconditioning for the Linearised Problems

In the second experiment, we consider the linearised problems in the semi-smooth Newton steps, i.e., elliptic equations in non-divergence form:

$$A : D^2u + b^{\theta} \cdot \nabla u - c^{\theta}u = f^{\theta},$$

on the domain $\Omega = (-1, 1)^2$. The coefficients are set to be

$$A = \begin{pmatrix} 2 & x_1 x_2 \\ x_1 x_2 & |x_1 x_2| \end{pmatrix} / 2$$

$$b^{\theta} = \sqrt{\theta}(x_1, x_2)^T$$

$$c^{\theta} = 3\theta,$$

where $\theta$ is a parameter so that the $\lambda$ in Cordes condition differs with varying $\theta$. Let the exact solution be

$$u(x_1, x_2) = (x_1 e^{1-|x_1|} - x_1)(x_2 e^{1-|x_2|} - x_2),$$

the right-hand side $f^{\theta}$ is directly calculated from the Eq. (56). For any given $\theta > 0$, we may set $\lambda = \theta$, which yields

$$\frac{|A|^2 + |b^{\theta}|^2 / 2\lambda + (c^{\theta} / \lambda)^2}{(\text{Tr} A + c^{\theta} / \lambda)^2} = \frac{10 + 1/2(x_1^2 + x_2^2) + 9}{(4 + 3)^2} \leq \frac{20}{49},$$

which means that the Cordes condition is satisfied for $\varepsilon = 9/20$. The PGMRES method is applied to solve the linear system arising from the discretization of (56) with varying parameters $\theta > 0$. We stop the iteration when the relative residual is smaller than $10^{-6}$. The iteration steps and CPU time (in seconds) of both additive and multiplicative preconditioning are given in Tables 3 and 4 below. We observe that CPU time is nearly linear with respect to the number of DOF, which confirms Remark 4.7 (computational complexity). Similarly, better performance of multiplicative preconditioner is observed.

6.3 Application to the HJB Equations

In this experiment, we solve the nonlinear HJB equations (1) on the domain $\Omega = (0, 1)^2$. Following [3], we take $\Lambda = [0, \pi/3] \times \text{SO}(2)$, where $\text{SO}(2)$ is the set of 2 rotation matrices. The coefficients are given by $b^{\alpha} = 0$, $c^{\alpha} = \pi^2$, and

$$A^{\alpha} = \frac{1}{2} \sigma^{\alpha} (\sigma^{\alpha})^T, \quad \sigma^{\alpha} = R^T \begin{pmatrix} 1 \sin \theta \\ 0 \cos \theta \end{pmatrix}, \quad \alpha = (\theta, R) \in \Lambda.$$
Table 3 The iteration steps and CPU time (in seconds) of additive preconditioning for Experiment 6.2

| DOF     | $\lambda = 10^{-3}$ steps | $\lambda = 10^{-2}$ steps | $\lambda = 10^{-1}$ steps | $\lambda = 1$ steps | $\lambda = 10^1$ steps | $\lambda = 10^2$ steps | $\lambda = 10^3$ steps |
|---------|---------------------------|---------------------------|---------------------------|---------------------|---------------------|---------------------|---------------------|
| 5,055   | 120(18)                  | 119(13)                  | 116(15)                  | 117(14)             | 99(15)              | 71(6)              | 36(2)              |
| 20,351  | 133(72)                  | 132(77)                  | 132(58)                  | 130(72)             | 114(44)             | 95(38)             | 59(14)             |
| 81,663  | 137(241)                 | 137(223)                 | 136(205)                 | 134(172)            | 118(146)            | 109(120)           | 72(57)             |
| 327,167 | 138(500)                 | 137(509)                 | 135(494)                 | 135(498)            | 118(418)            | 117(370)           | 80(228)            |
| 1309,695| 132(1,480)               | 132(1,920)               | 131(1,930)               | 139(2,180)          | 127(1,890)          | 119(1,560)         | 86(1,040)          |

Table 4 The iteration steps and CPU time (in seconds) of multiplicative preconditioning for Experiment 6.2

| DOF     | $\lambda = 10^{-3}$ steps | $\lambda = 10^{-2}$ steps | $\lambda = 10^{-1}$ steps | $\lambda = 1$ steps | $\lambda = 10^1$ steps | $\lambda = 10^2$ steps | $\lambda = 10^3$ steps |
|---------|---------------------------|---------------------------|---------------------------|---------------------|---------------------|---------------------|---------------------|
| 5,055   | 28(6)                    | 28(4)                    | 28(4)                    | 28(4)               | 28(7)               | 27(5)               | 24(2)               |
| 20,351  | 26(25)                   | 26(22)                   | 26(23)                   | 26(25)              | 25(25)              | 25(25)              | 29(19)              |
| 81,663  | 25(68)                   | 24(69)                   | 24(69)                   | 24(65)              | 23(65)              | 23(63)              | 28(46)              |
| 327,167 | 23(190)                  | 23(175)                  | 23(174)                  | 22(163)             | 21(165)             | 21(134)             | 26(134)             |
| 1309,695| 22(426)                  | 22(420)                  | 22(412)                  | 22(415)             | 21(379)             | 20(330)             | 23(337)             |

Table 5 Average PGMRES iterations (Newton steps)

| DOF     | $h$  | Average PGMRES iterations (Newton steps) |
|---------|------|------------------------------------------|
| 71      | $2^{-2}$ | 14 (6)                                   |
| 303     | $2^{-3}$ | 18 (6)                                   |
| 1,247   | $2^{-4}$ | 18 (6)                                   |
| 5,055   | $2^{-5}$ | 18 (7)                                   |
| 20,351  | $2^{-6}$ | 18 (8)                                   |
| 81,663  | $2^{-7}$ | 17 (9)                                   |
| 327,167 | $2^{-8}$ | 17 (10)                                  |
| 1309,695| $2^{-9}$ | 17 (11)                                  |

We choose $f^\alpha = \sqrt{3} \sin^2 \theta / \pi^2 + g$, $g$ independent of $\alpha$ such that the exact solution of the HJB equations (1) is $u(x_1, x_2) = \exp(x_1 x_2) \sin(\pi x_1) \sin(\pi x_2)$. That is

$$g = \sup_{\alpha \in \Lambda} \{ A^\alpha : D^2 u - c^\alpha u - \sqrt{3} \sin^2 \theta / \pi^2 \}. \quad (61)$$

In the semi-smooth Newton algorithm, the initial guess is $u_h^0 = 0$. For the discrete linearised systems arising from each Newton step, the multiplicative preconditioners are applied. We computed the average number of PGMRES iterations per Newton step required to reduce the residual norm below a relative tolerance of $10^{-4}$. Convergence of the Newton method was determined by requiring a step-increment $L^2$-norm below $10^{-6}$. These tolerances are chosen to balance the different sources of error originating from discretization.

The numbers of semi-smooth Newton iterations and average PGMRES iterations are listed in Table 5. As can be observed from [3, 7], the semi-smooth Newton algorithm convergences...
fast (within eight iterations in the numerical experiment). In each Newton step, we apply the PGMRES with the multiplicative preconditioner due to its better performance than the additive one. Based on the results shown in Table 5, we can conclude that our multiplicative preconditioner is also effective and robust in the application to the HJB equations.

Acknowledgements The authors would like to express their gratitude to Prof. Jun Hu in Peking University for his helpful discussions.

Funding The work of Shuonan Wu is supported in part by the National Natural Science Foundation of China grant No. 11901016 and the startup grant from Peking University grant No. 7100601681.

Data Availability Enquiries about data availability should be directed to the authors.

Declarations

Competing interests The authors have not disclosed any competing interests.

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