ℵ₀-categorical Banach spaces contain ℓₚ or c₀

Karim Khanaki¹,²,∗

¹ Department of Science, Arak University of Technology, P.O. Box 38135-1177, Arak, Iran
² School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran

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This paper has three parts. First, we establish some of the basic model theoretic facts about MT, the Tsirelson space of Figiel and Johnson [20]. Second, using the results of the first part, we give some facts about general Banach spaces. Third, we study model-theoretic dividing lines in some Banach spaces and their theories. In particular, we show: (1) MT has the non independence property (NIP); (2) every Banach space that is ℵ₀-categorical up to small perturbations embeds c₀ or ℓₚ (1 ≤ p < ∞) almost isometrically; consequently the (continuous) first-order theory of MT does not characterize MT, up to almost isometric isomorphism.

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1 Introduction

A famous conjecture in Banach space theory had predicted that every Banach space contains at least one of the classical spaces c₀ or ℓₚ for some 1 ≤ p < ∞. Tsirelson’s example [55] was the first space not containing isomorphic copies of any of the classical sequence spaces, and the first space whose norm was defined implicitly rather than explicitly. This phenomenon, i.e., implicit definability, later played an important role in Banach space theory when new spaces with implicitly defined norms yielded solutions to many of the most long standing problems in the theory (cf. [24]). Given the fact that all spaces whose norms are implicitly defined do not contain any of the classical sequences, some Banach space theorists, such as Gowers [24, 25] and Odell [47], asked the following question:

(Q1) Must an ‘explicitly defined’ Banach space contain ℓₚ or c₀?

To provide a positive answer to this question one must first provide a precise definition of ‘explicitly defined space’. Second, he/she must show that explicitly defined spaces contain ℓₚ for some 1 ≤ p < ∞, or c₀. Third, all classical Banach spaces such as Lₚ spaces, Lorentz spaces, Orlicz spaces, Schreier spaces, etc., are explicitly defined but the Tsirelson example and similar spaces are not (cf. [25] for more discussions).

On the other hand, the notion of definability plays a basic and key role in model theory and its applications in algebraic structures. Recall that a subset A of a first-order structure M is definable if A is the set of solutions in M of some formula ϕ(x). However, the problem there is that the usual first-order logic does not work very well for structures in Analysis.

Chang and Keisler [16] and Henson [28] produced the continuous logics and the logic of positive bounded formulas, respectively, to study structures in analysis using model theoretic techniques. Since then, numerous attempts to provide a suitable logic have been going on (e.g., cf. [7, 29, 31]), and eventually led to the creation of the first order continuous logic (cf. [9, 12] for more details). In many ways the latter logic is the best and ultimate.

One of the main purposes of this article is to take a step towards answering Gowers’s question using logical tools presented in [9, 12] and to prove that Banach spaces whose norms are determined by first order statements (i.e., their first order theories have exactly one separable model) contain ℓₚ for some 1 ≤ p < ∞ or c₀.¹ Since the dichotomy stable/unstable (in both logics, classic and continuous) has been widely studied in literature, and some

¹ E-mail: khanaki@arakut.ac.ir

¹ In this paper, when we say a Banach space is determined by first order statements (or its theory) we mean its continuous theory is ℵ₀-categorical in the sense of Definitions 5.12 or A.1 below. That is, the first order axioms determine a unique separable space up to isometry.
model theoretic properties, such as IP and SOP, in continuous logic (or Banach space theory) have received less attention, the second goal of this paper is to provide examples of these concepts. The third and final purpose is to get a better understanding of communication between both fields (model theory and Banach space theory) so that techniques from one field might become useful in the other.

It is worth pointing out that some of the results in the present paper, such as Corollary 5.13, seem to be folklore to experts in the field, including Henson and Usvyatsov. Of course, we are not sure that this observation itself appears somewhere in the literature. On the other hand, some results, such as Corollary 4.19, Theorem 5.9 and others, seem to be new even to experts. In fact our main result is stronger than the name of the paper.

To summarize the results of this paper, in the first part (§ 4), we define the notion of ‘NIP on a model’ and study the example of Tsirelson as a guidance. Since, by the Krivine–Maurey theorem which asserts that every stable space embeds some $\ell_p$ almost isometrically, Tsirelson’s space is not stable, so we consider a weaker property and ask the following question: Does Tsirelson’s space have NIP?\(^2\) By a result due to Odell [48] which asserts that the type space of Tsirelson’s space is strongly separable, we show that the answer to the latter question is positive.\(^3\) On the other hand, it is easy to show that an $\aleph_0$-categorical space is stable if and only if its type space is strongly separable.\(^4\) Using these observations, we conclude that the theory of Tsirelson’s space (in any countable language) is not $\aleph_0$-categorical; equivalently, this space cannot be determined by first order axioms in continuous logic. This fact leads us to the following question:

(Q2) Does an $\aleph_0$-categorical Banach space contain $\ell_p$ or $c_0$?

In the second part of the paper (§ 5), we answer this question. In fact, the answer is “yes”, and this is a step towards answering the question (Q1) above; furthermore we explain why $\aleph_0$-categoricity (up to small perturbations) is in some ways the appropriate notion for our purpose and why this notion is not a complete answer to the question (Q1). We prove that separable structures which are $\aleph_0$-categorical up to small perturbations contain $\ell_p$ or $c_0$. We will discuss this subject in detail and offer other observations that could be interesting in themselves.

In the third part of the paper (§ 6), we first remark some results of [38] about correspondences between dividing lines in model theory and Banach space theory which will be used in the paper. Then, we study model theoretic dividing lines in some Banach spaces and their theories.

In Appendix A, we revisit $\aleph_0$-categoricity and in Appendix B, we remark an observation about Rosenthal’s dichotomy communicated to us by Megrelishvili which asserts that Fact B.4 below is not a dichotomy in non-compact Polish case. To our knowledge this observation itself does not appear somewhere in literature in a clear form.

To simplify to read through the paper, we list some more important observations and results: Theorem 5.9, Observations 4.6 & 4.17, Corollaries 4.19, 5.13 & A.5, Examples 4.5, 6.6 & B.8. Corollary 4.19 and Theorem 5.9 are new.

It is worth recalling another line of research. In [32], Iovino studied the connection between the definability of types and the existence of $\ell_p$ and $c_0$ in the framework of Henson’s logic, i.e., the logic of positive bounded formulas. [32, Theorem 1.1] asserts that the existence of enough definable types guarantees the existence of $\ell_p$ or $c_0$ as a subspace. After preparing of the present paper we came to know that, independently from us, Usvyatsov [56] also observed that $\aleph_0$-categorical spaces contain some classical sequences.

This paper is organized as follows: In the § 2, we briefly review the notions of types in model theory and Banach space theory. In § 3, we recall the definition of Tsirelson’s space and some facts which are used later. Also, we present the notion NIP on a model and show that every Banach space with strongly separable types, as well as Tsirelson’s space, has NIP. In the § 4, we prove that every perturbed $\aleph_0$-categorical space embeds $c_0$ or $\ell_p$ almost isometrically (Theorem 5.9). In § 5, we briefly review some model theoretic dividing lines and their connections to some functional analysis notions, and study some examples and investigate their model theoretic dividing lines. In Appendix A, we revisit $\aleph_0$-categoricity and in Appendix B, we give some remarks on Rosenthal’s dichotomy.

\(^2\) Cf. Definition 4.1 for the definition of NIP on a model.
\(^3\) Cf. Definition 4.15 for the definition of strongly separable space.
\(^4\) Cf. Definitions 5.12 and A.1 for the definition of $\aleph_0$-categorical structures.
2 Preliminaries from functional analysis

In this section we review some basic notions from functional analysis. Further details can be found in [1].

A Banach space $M = (M, \| \cdot \|)$ is a complete normed linear space. The dual (resp. bidual) space of $M$ is denoted by $M^*$ (resp. $M^{**}$). Recall that a Banach space $X$ is reflexive if the mapping $\hat{\cdot} : X \to X^{**}$, given by $\hat{x}(x^*) = x^*(x)$, is an isometry of $X$ onto $X^{**}$.

Let $1 \leq p < \infty$, $\ell_p$ is the Banach space of all sequences $x = (x_n)$ of reals so that $\|x\|_p = (\sum_1^\infty |x_n|^p)^{1/p} < \infty$. $c_0$ is the Banach space of all null sequences $(x_n)$ under $\|(x_n)\|_\infty = \sup_n |x_n|$. The (nonseparable) Banach space $\ell_\infty$ is the space of all bounded sequences $(x_n)$ under $\|(x_n)\|_\infty = \sup_n |x_n|$.

Let $M$ be a Banach space. The weak topology of $M$ is the weakest topology on $M$ such that each $x \in M^*$ is continuous. That is, any net $(x_n)$ converges weakly to $x_0 \in M$ if and only if for each $x^* \in M^*$, $x^*(x_n) \to x^*(x_0)$.

Let $X$ be a compact Hausdorff space. The space of continuous real-valued functions on $X$ is denoted by $C(X)$. Since $X$ is a compact space, every $f \in C(X)$ is bounded and $C(X)$ is a complete normed linear space with the supremum norm. A net $(f_\alpha) \subseteq C(X)$ converges pointwise to a function $f \in \mathbb{R}^X$ if and only if for each $x \in X$, $|f_\alpha(x) - f(x)| \to 0$. It is a well-known fact that for each compact topological space $X$, the weak topology and the pointwise convergence topology on norm-bounded subsets of $C(X)$ are the same (cf. [22, 462F]).

**Definition 2.1** Let $M$ be a Banach space and $A \subseteq M$.

(i) $A$ is relatively weakly compact if it has compact closure in the weak topology on $M$.

(ii) $A$ is relatively weakly sequentially compact if every sequence $(x_n)$ of it has a subsequence $(y_n)$ such that it weakly converges to an element of $M$.

Let $X$ be compact and $A \subseteq C(X)$ norm-bounded. Then, as the weak topology and the pointwise convergence topology on $A$ are the same, $A$ is relatively weakly compact if it has compact closure in the pointwise convergence topology on $C(X)$. $A \subseteq C(X)$ is relatively weakly sequentially compact if every sequence of it has a pointwise convergent subsequence such that its limit is a continuous function.

**Definition 2.2** Let $M$ be Banach space and $A$ a norm-bounded subset of it.

(i) $A$ is Rosenthal if every sequence in $A$ has a weak Cauchy subsequence. $M$ is called Rosenthal if its unit ball is Rosenthal. In particular, if $M = C(X)$ (where $X$ is compact) and $A \subseteq C(X)$ is Rosenthal, we say that $A$ is relatively (weakly) sequentially compact in $\mathbb{R}^X$ (short RSC). That is, every bounded sequence in $A$ has a pointwise convergent subsequence in $\mathbb{R}^X$.

(ii) $A$ is said to be weakly sequentially complete (short WS-complete) if every weak Cauchy sequence of $A$ has a weak limit (in $M$). $M$ is called weakly sequentially complete if its unit ball is weakly sequentially complete. In particular, if $M = C(X)$ (where $X$ is compact) and $A \subseteq C(X)$, we say that $A$ has the weak sequential completeness property (or short SCP) if the limit of each pointwise convergent sequence $(f_\alpha) \subseteq A$ is continuous.

**Fact 2.3** (Eberlein–Šmulian Theorem) Let $M$ be Banach space and $A \subseteq M$ norm-bounded. Then the following are equivalent:

1. $A$ is relatively weakly compact in $M$.
2. The following two properties hold:
   (i) $A$ is Rosenthal, and
   (ii) $A$ is weakly sequentially complete.

**Remark 2.4** Cf. [57] for a short proof of the Eberlein–Šmulian Theorem. (Cf. also [1, Theorem 1.6.3].) Notice that (2) is precisely the condition B in the main theorem of [57]. Indeed, every sequence $(x_n)$ has a subsequence $(y_n)$ such that it weakly converges to an element of $M$ (i.e., relatively weakly sequentially compact) if and only if (i) each sequence has a weakly Cauchy subsequence (i.e., Rosenthal), and (ii) every weak Cauchy sequence has a weak limit in $M$ (i.e., weakly sequentially complete). Clearly, (1) is the condition A in [57].
3 Types

We assume that the reader is familiar with continuous logic (cf. [9] or [12]). We will study the notion of type in continuous logic and its connection with the notion of type in Banach space theory. In this paper, local types are more important for us.

3.1 Types in model theory

Although in the current paper we study unbounded metric structures (i.e., Banach spaces) but without loss of generality we can assume that all structures are bounded and so the usual framework of continuous logic is sufficient for our purposes.

Convention: In this paper, we study bounded metric structures. More precisely, we study the unit balls of Banach spaces. Furthermore, each formula can have an arbitrary bound, although our focus is on the formula \( \varphi(x, y) = \|x + y\| \). So, we can assume that the atomic formulas are \([0,2]\)-valued.

As mentioned, model theory notation is standard [9]. We fix an \( L \)-formula \( \varphi(x, y) \) and a complete \( L \)-theory \( T \). We let \( \tilde{\varphi}(x, y) = \varphi(x, y) \). Let \( M \) be an \( L \)-structure, \( A \subseteq M \) and \( T_A = \text{Th}(M, a)_{a \in A} \). Let \( p(x) \) be a set of \( (L_A) \)-statements in free variable \( x \). We shall say that \( p(x) \) is a type over \( A \) if \( p(x) \cup T_A \) is satisfiable. A complete type over \( A \) is a maximal type over \( A \). The collection of all such types over \( A \) is denoted by \( S^M(A) \), or simply by \( S(A) \) if the context makes the theory \( T_A \) clear. The type of \( a \) in \( M \) over \( A \), denoted by \( \text{tp}^M(a/A) \), is the set of all \( (L_A) \)-statements satisfied in \( M \) by \( a \). If \( \varphi(x, y) \) is a formula, a \( \varphi \)-type over \( A \) is a maximal consistent set of statements of the form \( \varphi(x, a) \geq r \) and \( \varphi(x, a) \leq s \), for \( a \in A \) and \( r, s \in \mathbb{R} \). The set of \( \varphi \)-types over \( A \) is denoted by \( S_{\varphi}(A) \). Similarly, we define \( S_{\varphi}(A) \).

3.1.1 The logic topology and \( \varphi \)-metric

We now give a characterization of complete types in terms of functional analysis. Let \( \mathcal{L}_{A} \) be the family of all interpretations \( \varphi^M \) in \( M \) where \( \varphi \) is an \( (L_A) \)-formula with a free variable \( x \). Then \( \mathcal{L}_A \) is an Archimedean Riesz space of measurable functions on \( M \) (cf. [21]). Let \( \sigma_A(M) \) be the set of Riesz homomorphisms \( I : \mathcal{L}_A \to \mathbb{R} \) such that \( I(1) = 1 \). The set \( \sigma_A(M) \) is called the spectrum of \( T_A \). Note that \( \sigma_A(M) \) is a weak\(^*\) compact subset of \( \mathcal{L}^*_A \). The next proposition shows that a complete type can be coded by a Riesz homomorphism and gives a characterization of complete types. In fact, by the compactness theorem, the map \( S^M(A) \to \sigma_A(M) \), defined by \( p \mapsto I_p \), where \( I_p(\varphi^M) = r \) if \( \varphi(x) = r \) is in \( p \), is a bijection.

Remark 3.1 For an Archimedean Riesz space \( U \) with order unit \( 1 \), write \( X \) for the set of all normalized Riesz homomorphisms from \( U \) to \( \mathbb{R} \), or equivalently, the positive extreme points of unit ball of the dual space \( U^* \), with its weak\(^*\) topology. Then \( X \) is compact by Alaoglu’s theorem and the natural map \( u \mapsto \hat{u} : U \to \mathbb{R}^X \) defined by setting \( \hat{u}(x) = x(u) \) for \( x \in X \) and \( u \in U \), is an embedding from \( U \) to an order-dense and norm-dense embedding subspace of \( C(X) \).

Now, by the above remark, \( \mathcal{L}_A \) is dense in \( C(S^M(A)) \). It is easy to show that:

Fact 3.2 Suppose that \( M, A \) and \( T_A \) are as above.

(i) The map \( S^M(A) \to \sigma_A(M) \) defined by \( p \mapsto I_p \) is bijective.

(ii) \( p \in S^M(A) \) if and only if there is an elementary extension \( N \) of \( M \) and \( a \in N \) such that \( p = \text{tp}^N(a/A) \).

We equip \( S^M(A) = \sigma_A(M) \) with the related topology induced from \( \mathcal{L}^*_A \). Therefore, \( S^M(A) \) is a compact and Hausdorff space. For any complete type \( p \) and formula \( \varphi \), we let \( \varphi(p) = I_p(\varphi^M) \). It is easy to verify that the topology on \( S^M(A) \) is the weakest topology in which all the functions \( p \mapsto \varphi(p) \) are continuous. This topology is sometimes called the logic topology. The same things are true for \( S_{\varphi}(A) \).

Definition 3.3 (The \( \varphi \)-metric) The space \( S_{\varphi}(M) \) has another topology. Indeed, define

\[
\varphi(p, q) = \sup_{a \in M, |a| \leq 1} |\varphi(p, a) - \varphi(q, a)|.
\]

Clearly, \( \varphi \) is a metric on \( S_{\varphi}(M) \) and it is called the \( \varphi \)-metric on the type space \( S_{\varphi}(M) \).
Remark 3.4 For unbounded logics, we define $\varrho_a(p, q) = \sup_{a \in M} |\varphi(p, a) - \varphi(q, a)|$. The $\varrho_a$-metric sometimes is called the uniform topology. In this case, there is another topology on the type space, namely the strong topology. Indeed, define $\varrho_a(p, q) = \sum_k \sup_{|a| \leq k} |\varphi(p, a) - \varphi(q, a)|$. For the formula $\varphi(x, y) = \|x + y\|$, Hajdon has informed us that he has constructed a Banach space $M$ for which the space of $\varphi$-types over $M$ is strongly separable but not uniformly separable. Despite this, for bounded continuous logic, and hence in this paper, the $\varrho$-metric and the strong and uniform topologies are the same.

Definition 3.5 A (compact) topometric space is a triplet $\langle X, \tau, \varrho \rangle$, where $\tau$ is a (compact) Hausdorff topology and $\varrho$ a metric on $X$, satisfying:

(i) The metric topology refines the topology.

(ii) The metric $\varrho : X^2 \to [0, \infty]$ is lower semi-continuous, that is, for all $r \in \mathbb{R}^+$ the set $\{(a, b) \in X^2 : \varrho(a, b) \leq r\}$ is closed in $(X^2, \tau \times \tau)$, where $\tau \times \tau$ is the product topology.

Fact 3.6 Let $M$ be a structure and $\varphi(x, y)$ a formula. The triplet $\langle S_\varphi(M), \tau, \varrho \rangle$ is a compact topometric space where $\tau$ and $\varrho$ are the logic topology and the $\varrho$-metric on the type space $S_\varphi(M)$, respectively.

Proof. Since $(S_\varphi(M), \tau)$ is compact, it suffices to show that $\varrho$ is lower semi-continuous (cf. [8, Lemma 1.4]). Indeed, suppose that $\kappa$ is a cardinal number and $(p_\alpha, q_\alpha) \in (S_\varphi(M))^2, \alpha < \kappa$, such that $\varrho(p_\alpha, q_\alpha) \leq r$ for all $\alpha < \kappa$. Let $U$ be an ultrafilter on $\kappa$. It is a well-known fact that, since $S_\varphi(M)$ is $\tau$-compact, there are $p, q \in S_\varphi(M)$ such that the $U$-limit of $(p_\alpha, q_\alpha), \alpha < \kappa$ is $(p, q)$. (Recall that the $U$-limit of $(x_\alpha)_\alpha \in X$ is $x$ if for every neighborhood $V$ of $x$, the set $\{\alpha : x_\alpha \in V\}$ is in $U$. In this case, we write $\lim_U x_\alpha = x$.) Therefore,

$$\varrho(p, q) = \sup_{|\alpha| \leq 1} |\varphi(p, \alpha) - \varphi(q, \alpha)|$$

$$= \sup_{|\alpha| \leq 1} \lim_U |\varphi(p_\alpha, b) - \varphi(q_\alpha, b)|$$

$$\leq \lim_U \sup_{|\alpha| \leq 1} |\varphi(p_\alpha, b) - \varphi(q_\alpha, b)| \leq r.$$

As $\kappa$ and $U$ are arbitrary, $\varrho$ is lower semi-continuous. □

Note that $(S_\varphi(M), \tau)$ is the weakest topology in which all functions $p \mapsto \varphi(p, a), a \in M$, are continuous, and all functions $p \mapsto \varphi(p, a), a \in M$, are uniformly continuous relative to $\varrho$ with a modulus of uniform continuity $\Delta_\varrho$ (cf. [9]).

3.2 Types in Banach space theory

Let $M$ be a Banach space and let $a$ be in the unit ball $B_M = \{x \in M : \|x\| \leq 1\}$. In Banach space theory, a function $f_a : B_M \to [0, 2]$ defined by $x \mapsto \|x + a\|$ is called a trivial type over the unit ball of $M$. A type on the unit ball of $M$ is a pointwise limit of a family of trivial types. The set $S(M)$ of all types over the unit ball of $M$ is called the space of types. In fact $S(M) \subseteq [0, 2]^M$ is a compact topological space with respect to the product topology. This topology is called the weak topology on types, and denoted by $\tau'$. $S(M)$ has another topology. Define $\varrho'(f, g) = \sup_{a \in M, |a| \leq 1} |f(a) - g(a)|$ for $f, g \in S(M)$. Clearly, $\varrho'$ is a metric on $S(M)$.

Remark 3.7 Notice that, in the original definition of types in the Banach space theory, types are on the entire space (not just the unit ball), and to be realized by anything in the space not just in the unit ball. However, this does not affect the results of this paper, because we use positive results for the original definition. As an example, strong separability in the sense of [48] implies strong separability in Definition 4.15 below.

Fact 3.8 The triplet $\langle S(M), \tau', \varrho' \rangle$ is a compact topometric space.

Proof. The proof is a straightforward adaptation of the proof of Fact 3.6. □

Let $\varphi(x, y) = \|x + y\|$. Then there is a correspondence between $S_\varphi(M)$ and $S(M)$. Indeed, let $a \in B_M$, and consider the quantifier-free type $a$ over $B_M$, denoted by $\text{tp}_{\varphi}(a/B_M)$. It is easy to verify that $\text{tp}_{\varphi}(a/B_M)$ corresponds
to the function \( f_a : B_M \to [0, 2] \) defined by \( x \mapsto \| x + a \| \). Also, there is a correspondence between \( \text{tp}_{\text{qf}}(a / B_M) \) and the Riesz homomorphism \( J_a : \mathcal{L}_M \to [0, 2] \) defined by \( I_a(\varphi(x, y)) = \varphi^M(x, a) \) for all \( x \in B_M \). To summarize, the following maps are bijective:

\[
\begin{align*}
tp_{\text{qf}}(a / B_M) & \sim f_a, \\
tp_{\text{qf}}(a / B_M) & \sim I_a.
\end{align*}
\]

Now, it is easy to check that their closures are the same:

**Fact 3.9** Assume that \( M \) is a Banach space and \( \varphi(x, y) = \| x + y \| \). The compact topometric spaces \((S_\varphi(M), \tau, \varrho)\) and \((S(M), \tau', \varrho')\) are the same. More exactly, \((S_\varphi(M), \tau)\) and \((S(M), \tau')\) are homeomorphic, and \((S_\varphi(M), \varrho)\) and \((S(M), \varrho')\) are isometric.

## 4 Tsirelson’s space and NIP

In this section we show that the Tsirelson space has a property weaker than stability, namely the non independence property (NIP). First we remind the notion NIP on a model from [42].

### 4.1 NIP on a model

**Definition 4.1** (i) Let \( M \) be a structure, and \( \varphi(x, y) \) a formula. We say that \( \varphi(x, y) \) has the independence property (short IP) on \( M \) if there are a sequence \( (a_n) \subseteq M \), and \( r > s \), such that for each finite disjoint subsets \( E, F \) of \( \mathbb{N} \):

\[
\left\{ b \in M : \left( \bigwedge_{n \in E} \varphi^M(a_n, b) \leq s \right) \land \left( \bigwedge_{n \in F} \varphi^M(a_n, b) \geq r \right) \right\} \neq \emptyset.
\]

In this case, we say that \( \left( \varphi^M(a_n, y) : i < \omega \right) \) is an independent sequence. We say that \( \varphi(x, y) \) has NIP on \( M \) if it does not have IP on \( M \).

(ii) A complete theory \( T \) has IP if there are a formula \( \varphi \) and a model \( M \) of it such that \( \varphi \) has IP on \( M \); otherwise it is said that \( T \) has NIP.

**Lemma 4.2** Let \( M \) be a structure, and \( \varphi(x, y) \) a formula. Then the following are equivalent.

(i) \( \varphi(x, y) \) has NIP on \( M \).

(ii) For each sequence \( (a_n) \subseteq M \), each elementary extension \( N \supseteq M \), and \( r > s \), there are some possibly infinite disjoint subsets \( E, F \) of \( \mathbb{N} \) such that

\[
\left\{ b \in N : \left( \bigwedge_{n \in E} \varphi^N(a_n, b) \leq s \right) \land \left( \bigwedge_{n \in F} \varphi^N(a_n, b) \geq r \right) \right\} = \emptyset.
\]

(iii) For each sequence \( \varphi(a_n, y) \) in the set \( A = \{ \varphi(a, y) : S_\varphi(M) \to \mathbb{R} \mid a \in M \} \), where \( S_\varphi(M) \) is the space of all complete \( \varphi \)-types on \( M \), and \( r > s \) there are some finite disjoint subsets \( E, F \) of \( \mathbb{N} \) such that

\[
\left\{ q \in S_\varphi(M) : \left( \bigwedge_{n \in E} \varphi(a_n, q) \leq s \right) \land \left( \bigwedge_{n \in F} \varphi(a_n, q) \geq r \right) \right\} = \emptyset.
\]

(iv) The condition (iii) holds for arbitrary disjoint subsets \( E, F \) of \( \mathbb{N} \).

(v) Every sequence \( \varphi(a_n, y) \) in \( A \) has a pointwise convergent subsequence.

**Proof.** Cf. [42, Lemma 3.12] for a proof. \( \square \)

**Remark 4.3** (i) A formula \( \varphi \) has NIP for a theory \( T \) (in the obvious sense) if and only if it has NIP on every model \( M \) of \( T \) if it has NIP on some approximately \( \omega \)-saturated model of \( T \). (Cf. [12, Definition 1.3] for the definition of approximate \( \omega \)-saturation.)

(ii) Note that for each separable Banach space \( M \), every sequence \( \varphi^M(a_n, y) : M \to \mathbb{R} \) has a convergent subsequence (cf. Lemma 4.16 below), but this does not imply that every sequence \( \varphi(a_n, y) : S_\varphi(M) \to \mathbb{R} \) in \( A \) has a convergent subsequence (cf. Examples 4.5 and B.8 below).
(iii) The notion of NIP on a model was introduced in [42]. Recently, in [43] some equivalences of this notion are given. For example, it is shown that a formula $\varphi$ has NIP on $M$ iff every coheir of a $\varphi$-type over $M$ is Borel (Baire 1) definable.

**Definition 4.4** A Banach space (or Banach structure) $M$ has NIP (resp. IP) if the formula $\varphi(x, y) = \|x + y\|$ has NIP (resp. IP) on $M$.

The following example shows that $c_0$ has IP. Also, we will see later that this example shows that Fact B.4 below is not a dichotomy in noncompact spaces (cf. Example B.8 below). We thank Megrelishvili for communicating to us the example.

**Example 4.5** Let $c_0 = \{x = (x_n)_n \in \ell_\infty : \lim_n x_n = 0\}$ with $\|x\| = \sup_{n \in \mathbb{N}} |x_n|$. Let $B_{c_0}$ be the unit ball of $c_0$, i.e., $B_{c_0} = \{x \in c_0 : \|x\| \leq 1\}$. For each $a \in B_{c_0}$, define $f_a : B_{c_0} \to [0, 2]$ by $x \mapsto \|x + a\|$. Let $\mathcal{F}_{c_0} = \{f_a : a \in B_{c_0}\}$. Then the family $\mathcal{F}_{c_0}$ contains an independent sequence. Indeed, for each $n$, define $f_n(x) = \|x + e_n\|$ for all $x$ where $(e_n)_n$ is the standard basis of $c_0$. We show that $(f_n)_n$ is an independent sequence: For every finite disjoint subsets $I$ and $J$ of $\mathbb{N}$ (the naturals) define a binary vector $x \in c_0$ as the characteristic function of $J$: $x_j = 1$ for every $j \in J$ and $x_k = 0$ for every $k \notin J$ (in particular, for every $i \in I$). Then $x \in c_0$, $\|x\| = 1$ and $f_i(x) = 1$ for every $i \in I$, $f_j(x) = 2$ for every $j \in J$. Therefore, $(f_n)_n$ is an independent sequence.

**Observation 4.6** Suppose that $X$ is a Banach space structure containing a subspace isomorphic to $c_0$. Then $X$ has the independence property (IP).

**Proof.** Recall that every space isomorphic to $c_0$ has subspaces that embed $c_0$ almost isometrically (cf. [45, Proposition 2.e.3]). So, the sequence $(f_n)_n$ in Example 4.5 works well.

The converse does not hold, and we will give a counterexample in a future work. We now recall the notion of non order property (NOP) or stability on a model.

**Definition 4.7** Let $M$ be a structure, and $\varphi(x, y)$ a formula. The following are equivalent and in any of the cases we say that $\varphi(x, y)$ is stable on $M$ (or has NOP on $M$).

(i) Whenever $a_n, b_m \in M$ form two sequences we have

$$\lim_{n} \lim_{m} \varphi(a_n, b_m) = \lim_{m} \lim_{n} \varphi(a_n, b_m),$$

when the limits on both sides exist.

(ii) The set $A = \{\varphi(x, b) : S_x(M) \to \mathbb{R} : b \in M\}$ is relatively weakly compact in $C(S_x(M))$. (Cf. Definition 2.1 above.)

**Remark 4.8** (i) The equivalence (i) $\iff$ (ii) of Definition 4.7 is called the Eberlein–Grothendieck criterion (cf. [42]). In [37], stability is only defined for the formula $\|x + y\|$, and it is called stability for the whole space (and not NOP for the formula $\|x + y\|$). Definition 4.7(i) was based on Shelah’s general notion of NOP or stability of a formula. The general analogue of NOP in continuous setting appeared for the first time in [12].

(ii) A formula $\varphi$ has NOP for a theory $T$ (in the obvious sense) iff it has NOP on every model $M$ of $T$. A theory has NOP (or is stable) if every formula has NOP for the theory.

**Definition 4.9** A Banach space (or Banach structure) $M$ is stable (or has NOP) if the formula $\varphi(x, y) = \|x + y\|$ is stable on $M$.

The following is a consequence of the well-known compactness theorem of Eberlein and Šmulian (Fact 2.3). Indeed, if the set $A$ above is relatively weakly compact in $C(S_x(M))$, then every sequence in $A$ has a pointwise convergent subsequence in $\mathbb{R}^{S_x(M)}$, namely $A$ is relatively sequentially compact in $\mathbb{R}^{S_x(M)}$. (Cf. Definition 2.2(i).)

**Fact 4.10** ([42, Fact 3.4]) Let $M$ be a structure, and $\varphi(x, y)$ a formula. If $\varphi$ is stable on $M$ then $\varphi$ has NIP on $M$. In particular, every stable Banach space has NIP.

As $L_p[0, 1]$ $(1 \leq p < \infty)$ embeds $\ell_p$, and the former is stable (cf. [9, Theorem 17.11]), so $\ell_p$ is stable. Therefore, by the above fact, $\ell_p$ $(1 \leq p < \infty)$ has NIP.
4.2 Tsirelson’s space

Although we only use some known properties of Tsirelson’s example [55] or actually the dual space of the original example as described by Figiel and Johnson [20], for the sake of completeness, we present Tsirelson’s space and remind some its properties which are used in this paper. The key property for our purpose will be presented in Fact 4.18.

A collection \( \{E_i\}_i \) of finite subsets of natural numbers is called admissible if \( n \leq E_1 < E_2 < \cdots < E_n \). By \( E < F \) we mean \( \max E < \min F \) and \( n \leq F \) means \( n < \min F \). For \( x \in c_00 \) and \( E \subseteq \mathbb{N} \), by \( Ex \) we mean the restriction of \( x \) to \( E \), i.e., \( Ex(n) = x(n) \) if \( n \in E \) and 0 otherwise.

Now we recall the definition \( M_T \) (the Tsirelson of [20]). For all \( x \in c_00 \) set

\[
\|x\| = \max \left( \|x\|_{\infty}, \sup \left\{ \frac{1}{2} \sum_{i=1}^{n} \|E_i x\| : (E_i)_1^a \text{ is admissible} \right\} \right).
\]

\( M_T \) is then defined to be the completion of \((c_00, \| \cdot \|)\). Note that this norm is given implicitly rather than explicitly.

One can define this norm on \( c_00 \) by induction. Indeed, for \( x \in c_00 \), set \( \|x\|_0 = \|x\|_{\infty} \) and inductively

\[
\|x\|_{n+1} = \max \left( \|x\|_n, \sup \left\{ \frac{1}{2} \sum_{i=1}^{n} \|E_i x\|_n : (E_i)_1^a \text{ is admissible} \right\} \right).
\]

Then \( \|x\| = \lim_n \|x\|_n \) is the desired norm.

**Fact 4.11** ([1, Theorem 11.3.2]) Tsirelson’s space has the following properties.

(i) \( M_T \) is reflexive.

(ii) \( M_T \) does not contain a subspace isomorphic to \( c_0 \) or \( \ell_p \) \((1 \leq p < \infty)\).

In addition, it is easy to verify that \( (e_p) \) is a normalized unconditional basis for \( M_T \) and any spreading model of \( M_T \) is isomorphic to \( \ell_1 \). Cf. [1] for definitions and proofs.

The following deep result, due to Krivine and Maurey [37], gives a partial answer to the main question of the present paper. First, recall that:

**Definition 4.12** Let \( X, Y \) be two Banach spaces.

(i) \( X \) and \( Y \) are \( \lambda \)-isomorphic (for some \( \lambda \geq 1 \)), if there is a linear isomorphism \( T \) from \( X \) onto \( Y \) such that for all \( x \in X \), \( \lambda^{-1} \|x\| \leq \|T(x)\| \leq \lambda \|x\| \).

(ii) \( X \) and \( Y \) are called almost isometric if for each \( \lambda > 1 \), they are \( \lambda \)-isomorphic.

(iii) \( X \) embeds \( Y \) almost isometrically, if for each \( \lambda > 1 \) there is a subspace \( X_\lambda \) of \( X \) such that \( X_\lambda \) and \( Y \) are \( \lambda \)-isomorphic.

**Fact 4.13** (Krivine–Maurey theorem) Every (separable) stable Banach space embeds \( \ell_p \) almost isometrically, for some \( 1 \leq p < \infty \).

**Corollary 4.14** \( M_T \) is not stable.

**Proof.** Immediate by Facts 4.11 and 4.13.

A Banach space is weakly stable if the condition (i) in Definition 4.7 holds whenever \( (a_n) \) and \( (b_n) \) are both weakly convergent. It is proved that every weakly stable space embeds \( c_0 \) or \( \ell_p \), for some \( 1 \leq p < \infty \), almost isometrically (cf. [3]). To summarize, \( M_T \) is not stable, even worse, it is not weakly stable.

4.3 Strong separability

We will show that Tsirelson’s space has NIP. Indeed, we show that every (separable) Banach structure with strongly separable space of types has NIP. Then the desired result will be achieved from a deep result of Odell (Fact 4.18 below).

**Definition 4.15** We say a (separable) Banach space is strongly separable, or equivalently its type space is strongly separable, if \( (S_\varphi(M), \varrho) \) is separable where \( \varphi(x, y) = \|x + y\| \) and \( \varrho \) is the \( \varrho \)-metric on \( \varphi \)-types.
Recall that, by the Krivine–Maurey theorem, Tsirelson’s space is not stable (cf. Corollary 4.14 above). By Odell’s result below, it is easy to show that this space has a weaker property, namely NIP. For this, we recall some notion and fact.

Recall that a function \( f \) form a metric space \( X \) to a metric space \( Y \) is \( k \)-Lipschitz \((k \geq 0)\) if for all \( x, y \) in \( X \), 
\[
d(f(x), f(y)) \leq k \cdot d(x, y).
\]
Now we give the following easy lemma.

**Lemma 4.16** Assume that \((X, d)\) is a metric space, and \( F \) a bounded family of \( 1 \)-Lipschitz functions from \( X \) to \( \mathbb{R} \). If \( X \) is separable then every sequence in \( F \) has a (pointwise) convergent subsequence.

**Proof.** The proof is an easy diagonal argument.

**Observation 4.17** Assume that \( M \) is a separable Banach structure and \( \varphi(x, y) = \|x + y\| \). If the type space \( S_\varphi(M) \) is strongly separable, then \( \varphi \) has NIP on \( M \).

**Proof.** Recall that \((S_\varphi(M), \tau, \varphi)\) is a compact topometric space where \((S_\varphi(M), \tau)\) is compact and Polish, and \((S_\varphi(M), \varphi)\) is a complete metric space. (Here, \( \tau \) is the logic topology and \( \varphi \) is the \( \varphi \)-metric.) Note that \( \varphi \) is separable since the type space is strongly separable. Also, the functions of the form \( \varphi(\cdot, a), a \in M \), are \( 1 \)-Lipschitz with respect to \( \varphi \). Now, use Lemma 4.16 and the equivalence \((i) \iff (v)\) of Lemma 4.2.

In [48, Proposition 1], it is shown that the type space of a separable stable Banach space is separable with respect to the metric \( \varphi_1 \) (cf. Remark 3.4). For the unit ball of a separable and stable space \( M \), this is actually a consequence of the separability of \( M \) and the definability of types in stable models (cf. [4] and [38]). Odell showed that the converse does not hold:

**Fact 4.18** (Odell [48]) Tsirelson’s space is strongly separable. Furthermore, its type space in unbounded continuous logic is uniformly separable (cf. Remark 3.4 above).

Let \( M \) be a Banach space, \( \varphi(x, y) = \|x + y\|, X = S_\varphi(M) \) the space of complete \( \varphi \)-types on \( M \). For \( a \in M \), define \( \varphi(a, y) : X \to [0, 2] \) by \( q \mapsto \varphi(a, q) \). (Notice that \( \varphi(a, y) : X \to [0, 2], a \in M \), is continuous.) We say that the space \( \varphi \) of definable predicates on \( M \) is weakly sequentially complete if the set \( F = \{ \varphi(a, y) : X \to [0, 2] | a \in M \} \) has the weak sequential completeness property (cf. Definition 2.2(ii)).

The following observation is new, though it is really just a corollary of Odell’s deep result above.

**Corollary 4.19** (i) Tsirelson’s space has NIP.

(ii) The space of definable predicates on Tsirelson’s space is not weakly sequentially complete.

**Proof.** (i) Immediate, since the type space of Tsirelson’s space is strongly separable (cf. Fact 4.18 above).

(ii) Note that, by the Eberlein–Šmulian theorem (Fact 2.3), NIP and the weak sequential completeness property is equivalent to stability. (Cf. also [42, Theorem 4.3].)

It seems to be open whether the theory of Tsirelson’s space has NIP.

**Remark 4.20** Note that one can not expect a converse to Observation 4.17. There are some indications that the tree Banach spaces can be counterexamples. With the previous observations, we divide separable Banach spaces to some classes such as stable, containing some \( \ell_p \) \((1 \leq p < \infty)\), with strongly separable type space, with NIP, not containing \( c_0 \), and these classes form a hierarchy

\[
\text{stable} \subset \text{strongly separable} \subset \text{NIP} \subset \text{not containing } c_0.
\]

Two questions arise. Must a space not containing \( c_0 \) have NIP? Must a space with NIP have strongly separable types? These questions seem to be open, although we know that the answer to one of these questions is certainly negative. Indeed, in the 90’s Gowers [23] has constructed a space \( X_G \) not containing \( \ell_p \) and no reflexive subspace. By the result of Haydon and Maurey [27] which asserts that the spaces with strongly separable types contain \( \ell_1 \) or have a reflexive subspace, the type space of \( X_G \) is not strongly separable and this space does not contain \( c_0 \), so \{strongly separable\} \( \not\subset \) \{not containing \( c_0 \). Also, we strongly believe that the answers to both questions are negative. We will give sharper answers in a future work.
5 \( c_0 \)- and \( \ell_p \)-subspaces of perturbed \( \mathbb{N}_0 \)-categorical spaces

As we saw earlier the norm of Tsirelson’s space is given implicitly rather than explicitly. An easy observation (cf. Corollary A.5 below) shows that the norm of Tsirelson’s space can not be characterized by axioms in continuous logic. This and the fact that Tsirelson’s space does not contain \( c_0 \) or \( \ell_p \) lead us to a natural question as Gowers and Odell asked: Must an “explicitly defined” Banach space contain \( c_0 \) or \( \ell_p \)? (Cf. [24, 25, 47].) Now, we give an explicit definition of a similar alternative of an “explicitly defined” Banach space, and give a positive answer to the above question.

**Convention:** In this section the language is the usual language of Banach spaces (in continuous logic). Otherwise, we explicitly state what is our desired language.

5.1 \( c_0 \)- and \( \ell_p \)-types

**Definition 5.1** If \( p \in [0, \infty) \) and \( \varepsilon > 0 \), the following set of statements with countable variables \( \bar{x} = (x_0, x_1, \ldots) \) will be called the \( \varepsilon - \ell_p \)-type: For every natural number \( n \), and scalars \( r_0, \ldots, r_n \),

\[
(1 + \varepsilon)^{-1} \left\| \sum_0^n r_i x_i \right\| \leq \left( \sum_0^n |r_i|^p \right)^{\frac{1}{p}} \left\| x_0 \right\| (1 + \varepsilon) \left\| \sum_0^n r_i x_i \right\|.
\]

The following set of statements with countable variables \( \bar{x} = (x_0, x_1, \ldots) \) will be called the \( \varepsilon - c_0 \)-type: For every natural number \( n \), real number \( \varepsilon > 0 \) and scalars \( r_0, \ldots, r_n \),

\[
(1 + \varepsilon)^{-1} \left\| \sum_0^n r_i x_i \right\| \leq \left( \max_{0 \leq i \leq n} |r_i| \right) \left\| x_0 \right\| (1 + \varepsilon) \left\| \sum_0^n r_i x_i \right\|.
\]

**Remark 5.2** The notion of \( \varepsilon - \ell_p \)-type (\( \varepsilon - c_0 \)-type) is a formal definition, and is different from the notion of types in model theory or Banach space theory. Recall that if \( T \) is a complete theory and there exists a model \( M \) of \( T \) such that the \( \varepsilon - \ell_p \)-type (\( \varepsilon - c_0 \)-type) is realized in \( M \), then it is a type in the sense of model theory, i.e., it belongs to \( S_c(\mathbb{O}) \). However, the following deep classical theorem, due to Krivine, guarantees that for every Banach space there exists some \( p \) such that the notion of \( \varepsilon - \ell_p \)-type is really a type in the sense of model theory.

5.2 Krivine’s theorem

We recall the well-known theorem of Krivine.

**Definition 5.3** Let \( p \in [1, \infty] \) and \( (x_i) \) a sequence of elements of some Banach space. We say that \( \ell_p \) (resp. \( c_0 \)) is block finitely represented in \( (x_i) \) if \( p \in [1, \infty) \) (resp. \( p = \infty \)) and for every \( \varepsilon > 0 \) and positive integer \( n \), there are \( n + 1 \) finite subsets \( F_0, \ldots, F_n \) of the positive integers \( \mathbb{N} \) with \( \max F_j < \min F_{j+1} \), for all \( 0 \leq j \leq n - 1 \) and elements \( y_0, \ldots, y_n \) with \( y_j \) in the linear span of \( \{x_i : i \in F_j\} \) for all \( i \) so that for all scalars \( r_1, \ldots, r_n \),

\[
(1 - \varepsilon) \left\| \sum_0^n |r_i|^p \right\|^\frac{1}{p} \left\| y_0 \right\| \leq \left\| \sum_0^n r_i y_i \right\| \leq (1 + \varepsilon) \left\| \sum_0^n |r_i|^p \right\|^\frac{1}{p} \left\| y_0 \right\|
\]

(\( \text{where} \ (\sum_0^n |r_i|^p) = \sup |r_i|, \text{if} \ p = \infty \)).

In the year in which Tsirelson’s example appeared in print [55], Krivine [36] published his celebrated theorem.

**Fact 5.4** (Krivine’s theorem) Let \( (x_n) \) be a sequence in a Banach pace with infinite-dimensional linear span. Then either there exists a \( 1 \leq p < \infty \) so that \( \ell_p \) is block finitely represented in \( (x_n) \) or \( c_0 \) is block finitely represented in \( (x_n) \).

5.3 Main theorem

In this section, we prove the main theorem of the paper; that is, any (suitable) perturbed \( \mathbb{N}_0 \)-categorical space contains some classical sequences \( c_0 \) or \( \ell_p \).
Let $M, N$ be two Banach spaces and $a \in M$ and $b \in N$. Recall from Definition 4.12 that, for $\varepsilon > 0$, a $(1 + \varepsilon)$-isomorphic between $(M, a)$ and $(N, b)$ is a linear map $f : M \to N$ such that $f(a) = b$ and $\|f\|, \|f^{-1}\| \leq 1 + \varepsilon$. Let us fix a complete theory $T$ (in a countable language of Banach spaces) and a monster model $\mathcal{U}$ of it. In the following $S_n(T)$ denotes the set of $n$-types over $\emptyset$.

**Definition 5.5 (Banach–Mazur perturbation)** The Banach–Mazur perturbation system $p_{BM}$ (for theory $T$) is the perturbation system defined by the Banach–Mazur distance $d_{BM}$ in the sense of [6, Definition 1.23], that is, for each $\varepsilon > 0$, $p_{BM}(\varepsilon) = \{(p, q) \in S_n(T)^2 : n < \omega, d_{BM}(p, q) < \varepsilon\}$ where

$$d_{BM}(p, q) = \inf \{\log(1 + \varepsilon) : \text{there is a $(1 + \varepsilon)$-isomorphism between $(\mathcal{U}, a)$ and $(\ell, b)$, and } \mathcal{U} \models p(a), q(b)\}.$$  

Cf. also Iovino [33].

**Definition 5.6 ($p_{BM}$-isomorphism)** (i) Two separable models $M, N$ are called $p_{BM}$-isomorphic if for each $\varepsilon > 0$ there is a bijective map $f : M \to N$ such that $(tp^M(a), tp^N(f(a))) \in p_{BM}(\varepsilon)$ for all $a \in M^n, n < \omega$.

(ii) A theory $T$ is called $p_{BM}$-$\aleph_0$-categorical if every two separable models $M, N \models T$ are $p_{BM}$-isomorphic.

A separable Banach space is called $p_{BM}$-$\aleph_0$-categorical if its continuous first order theory (in the usual language) is $p_{BM}$-$\aleph_0$-categorical.

Clearly, every two models of a $p_{BM}$-$\aleph_0$-categorical theory are almost isometric (cf. Definition 4.12 above).

We want to give a general result to any perturbation system, so we generalize the above notions. Let $p$ be an arbitrary perturbation system in the sense of [6, Definition 1.23]. The notions $p$-isomorphism and $p$-$\aleph_0$-categoricity are defined similar to Definition 5.6, by replacing $p_{BM}$ with $p$. Similarly, a separable Banach space is called $p$-$\aleph_0$-categorical if its continuous first order theory (in the usual language) is $p$-$\aleph_0$-categorical.

The following terminology is not standard, so we single it out. Before that, we recall some notations. If $p(x)$ is any partial type and $\varepsilon > 0$ then $p(x')$ denotes the partial type $3\exists x'(p(x') \land d(x, x') \leq \varepsilon)$. If $\bar{x}$ is a countable tuple of variables then $p(\bar{x}')$ means $p(\bar{x}', x_1, \ldots)$, where the metric is the supremum metric on countable tuples. For a perturbation system $p$, the $p(\varepsilon)$-neighbourhood around $p(\bar{x})$ is defined as follows: $p(\varepsilon)(\bar{x}) = \{\bar{q}(\bar{x}) : (\bar{p}(\bar{x}), \bar{q}(\bar{x})) \in p(\varepsilon)\}$. (Cf. [6, Definition 1.1].)

**Definition 5.7 ($p$-Saturation)** Let $p$ be a perturbation system. We will say that a structure $M$ is $p$-saturated if for every type $p(\bar{x}) \in S_\omega(\emptyset)$ (in countable variable $\bar{x}$) and $\varepsilon > 0$, the partial type $p(\varepsilon)(\bar{x})$ is realised in $M$.

Notice that the notion $p$-saturation above is weaker than $p$-approximately $\aleph_0$-saturated in the sense of [6, Definition 2.2]. Indeed, recall from [6, Remark 2.4] that the latter notion can be defined for $\omega$-types in $S_\omega(\emptyset)$ (i.e., types with $\omega$-tuples).

**Definition 5.8 (Perturbed embedding)** Let $p$ be a perturbation system, $M$ a Banach space and $p(\bar{x})$ the $0 - \ell_p$-type (or $0 - c_0$-type). We say that $M$ embeds $\ell_p$ (or $c_0$) $p$-approximately if for every $\varepsilon > 0$, the partial type $p(\varepsilon)(\bar{x})$ is realised in $M$.

Now we are ready to give the main result of the paper.

**Theorem 5.9 (Main theorem)** Let $p$ be a perturbation system and $M$ a $p$-$\aleph_0$-categorical space. Then, $M$ embeds $c_0$ or $\ell_p (1 \leq p < \infty)$ $p$-approximately. Moreover, $M$ embeds $\ell_2$ $p$-approximately. In particular, if $p$ is the Banach–Mazur perturbation system $p_{BM}$, then $M$ embeds $\ell_2$ almost isometrically.

**Proof.** Let $M$ be a separable model of $T$. By Krivine’s theorem, some $\ell_p$ (or $c_0$) is block finitely representable in $M$. So, for every $\varepsilon > 0$, the $\varepsilon - \ell_p$-type is a type in $S_\omega(T)$ (i.e., $S_\omega(\emptyset)$). (Note that in this case the $\varepsilon - \ell_p$-type is a partial type, but we can expand it to a complete type in $S_\omega(T)$.) As $\varepsilon$ is arbitrary, the $0 - \ell_p$-type is a complete type. Let $p(\bar{x})$ be this complete type. By [6, Lemma 3.4], $M$ is $p$-saturated. (Indeed, recall that the notion $p$-approximately $\aleph_0$-saturation in [6, Definition 2.2] is stronger than $p$-saturation.) By the $p$-saturation, for every $\varepsilon > 0$, $p(\varepsilon)(\bar{x})$ is realised in $M$, and so $M$ embeds $\ell_p$ $p$-approximately. Moreover, it is known that $\ell_2$ is finitely representable in $c_0$ and $\ell_p$. (Indeed, in the case of $c_0$, every Banach space is finitely representable in $c_0$ (cf. [1, Example 12.1.2]). If $\ell_p$ is finitely representable, then so is $L_p$ (cf. [1, Proposition 12.1.8]), and since $\ell_2$ is isometric to a subspace of...
$L_p$ (cf. [1, Theorem 6.4.12]), $\ell_2$ is finitely representable.) So $\varepsilon - \ell_2$-type is a partial type in the sense of model theory, and so $M$ embeds $\ell_2$, $p$-approximately.

In particular, if $p$ is the Banach–Mazur perturbation system $p_{BM}$, $M$ embeds $\ell_p$ (or $c_0$) $p_{BM}$-approximately. That is, for each $\varepsilon > 0$ there is a sequence $\tilde{a} \in M$ such that it realises the partial type $p_{BM}^{(\varepsilon)}(\tilde{a})$. This means that there is a sequence $\tilde{b}$ in an elementary extension of $M$ such that $d(\tilde{a}, \tilde{b}) \leq \varepsilon$ and $p_{BM}^{(\varepsilon)}(\tilde{b})$. So, by the definition, $d_{BM}(tp(\tilde{a}), \ell_p) \leq \varepsilon$. As $\varepsilon$ is arbitrary, there is a sequence $\tilde{a} \in M$ (for $\varepsilon > 0$) such that $d_{BM}(tp(\tilde{a}), \ell_p) \leq \varepsilon$. This means that there is a sequence in $M$ which is $(1 + \varepsilon)$-isomorphic to $\ell_p$ (or $c_0$). Again, as $\ell_2$ is finitely representable in $c_0$ and $\ell_p$, $M$ embeds $\ell_2$ almost isometrically. 

**Remark 5.10**

Recall that for two perturbation systems $p$, $p'$, the perturbation system $p$ is stricter than $p'$, denoted by $p < p'$, if for each $\varepsilon > 0$ there is $\delta > 0$ such that $p(\delta) \subset p'(\varepsilon)$. Clearly, if $p < p'$ and $N$, $p$ are $p$-isomorphic then they are $p'$-isomorphic too. Therefore, if $p$ is any perturbation system stricter than $p_{BM}$, then the “almost isometrically” version of the above theorem holds for $p$.

**Example 5.11**

(i) Let $T = Th(AN \subseteq \omega)$ be the theory of Nakano spaces presented in [5]. The theory $T$ is not $\aleph_0$-categorical, but it is $\lambda$-stable for $\lambda \geq \varepsilon$ (cf. [50, Theorem 3.10.9]). Yaacov proved that $T$ is $\aleph_0$-categorical and $\aleph_0$-stable up to small perturbations of the exponent function. This perturbation system is stricter than $p_{BM}$ in the language of Banach spaces (cf. [5, Proposition 4.6]), and so the theory of Nakano spaces is $p_{BM}$-$\aleph_0$-categorical.

(ii) (Non-example) The theory of the $2$-convexification of Tsirelson space, is denoted by $M_{T,2}$, as presented by Johnson (in the language of Banach spaces) is not $p_{BM}$-$\aleph_0$-categorical because this space does not contain any $\ell_p$ or $c_0$. It is known that every ultrapower of this space is linearly homeomorphic to a canonical direct sum of the space and a Hilbert space of suitable dimension. Indeed, cf. [35, Proposition 2.4.a] for the proof; of course this depends on the argument of Pisier that is pointed to in [35] in the paragraph just before Proposition 2.4.a (from the book [49]). Henson suggested the following argument that there are separable models $N$ of the theory of $M_{T,2}$ such that $N$ and $M_{T,2} \oplus \ell_2$ are isomorphic. Indeed, for an ultrafilter $U$, let $H$ be a subspace of $(M_{T,2} \oplus \ell_2)_U$ isomorphic to a Hilbert space for which $(M_{T,2} \oplus \ell_2)_U$ is the direct sum internally of $M_{T,2}$ and $H$ (which is possible by the result pointed above). Let $H'$ be any separable, infinite dimensional subspace of $H$ and let $N$ be a separable infinite dimensional elementary subspace of $(M_{T,2} \oplus \ell_2)_U$ that contains $M_{T,2}$ and $H'$. Letting $H_0 = Y \cap H$ it is easy to show that $N = M_{T,2} \oplus H_0$ (again, the sum $\oplus$ is the one internal to $(M_{T,2} \oplus \ell_2)_U$). Of course this says essentially nothing about the precise way in which the norm on $N = M_{T,2} \oplus H_0$ is defined—and it seems likely that there will be a large number of different (non-isometric) possibilities. To summarize, $M_{T,2}$ and $M_{T,2} \oplus H_0$ are two non $p_{BM}$-isomorphic separable models of this theory.

**Definition 5.12**

A separable Banach space is called $\aleph_0$-categorical if it is $p_{id}$-$\aleph_0$-categorical, where $p_{id}$ is the strictest perturbation system. (Cf. also Definition A.1 below.)

**Corollary 5.13**

Every $\aleph_0$-categorical Banach space $M$ contains isometric copies of $c_0$ or $\ell_p$ ($1 \leq p < \infty$). Moreover, $M$ contains an isometric copy of $\ell_2$.

**Proof.** This is a consequence of [6, Lemma 2.5] and the main theorem (Theorem 5.9) assuming the identity perturbation system $p_{id}$. Indeed, by Theorem 5.9, for every $\varepsilon > 0$, $p_{id}^{(\varepsilon)}(\tilde{x})$ is realised in $M$, where $p(\tilde{x})$ is the $0 - \ell_p$-type (or $0 - c_0$-type). By [6, Lemma 2.5], for every $\varepsilon > 0$, $p_{id}^{(\varepsilon)}(\tilde{x})$ is realised in $M$. As $p_{id}$ is the identity system, i.e., $p_{id}(p) = \{p\}$ for all $\varepsilon > 0$, there is a sequence in $M$ which is isometric to $\ell_p$ (or $c_0$). Similarly, $M$ contains an isometric copy of $\ell_2$. □

**Remark 5.14**

(i) The spaces $L_p[0, 1]$ ($1 \leq p < \infty$) are $\aleph_0$-categorical (cf. [9]). The Gurarij space (a universal, ultrahomogeneous Banach space) is also $\aleph_0$-categorical (cf. [10]).

(ii) Note that since the nature of our question, i.e., the existence of good subspaces, is local and $\aleph_0$-categoricity is a global condition on norm, one cannot expect a converse to Theorem 5.9. For example, the space $M_T \oplus \ell_p$ has a complex norm (since it contains the Tsirelson space $M_T$), although it contains a good subspace, i.e., $\ell_p$.

(iii) As we mentioned earlier, Corollary 5.13 is folklore to experts in the field. Henson informed us that one can prove that every $\aleph_0$-categorical space contains $\ell_2$ using Dvoretzky’s Theorem.

**Remark 5.15**

Note that sequence spaces $\ell_p$ ($p \neq 2$) and $c_0$ are not $\aleph_0$-categorical. Despite this, $\ell_p$ ($p \neq 2$) is almost categorical, i.e., its theory has exactly two separable models $\ell_p$ and $L_p[0, 1] \oplus \ell_p$. This fact is folklore to experts in field, including Berenstein, Henson and Iovino. We thank Henson for communicating to us the following
argument and references. The main point behind this fact is the classification of \( L_p(\mu) \) spaces under elementary equivalence in [28, Theorem 2.2]. The proof has four steps. First, it was shown in [30, Corollary 2.5] that for a Banach space \( M \) with nonstandard hull \( \hat{M} \):

\[
M \text{ is an } L_p\text{-space if and only if } \hat{M} \text{ is an } L_p\text{-space.}
\]

Therefore, if two Banach spaces \( \ell_p \) and \( M \) have isometric nonstandard hulls, then \( M \) is an \( L_p\)-space. (Recall that, by the Kakutani representation theorem, every \( L_p\)-space \( M \) is of the form \( L_p(\mu) \) for some measure \( \mu \).) Second, by the Keisler–Shelah Theorem for metric structures (cf. [28, Theorem 1.13]), two metric structures are elementarily equivalent iff they have linearly isometric ultrapowers iff they have linearly isometric nonstandard hulls. (We believe this theorem should be credited to Henson [28] and to Stern [53] equally and independently.) Using this theorem, if \( \ell_p \) and \( M \) are elementary equivalent, then \( M \) is of the form \( L_p(\mu) \) for some measure \( \mu \). Third, by [28, Theorem 2.2], if \( \ell_p \) and \( L_p(\mu) \) are elementary equivalent, then the measure space of \( \mu \) has infinitely many atoms. Fourth, if \( M = L_p(\mu) \) is separable and the measure space of \( \mu \) has infinitely many atoms, then either (1) the atomless part of the measure space is \([0] \) (so \( M \) is linearly isometric to \( \ell_p \)), or (2) the atomless part of the measure space is isomorphic to the measure space of \([0,1] \) with Lebesgue measure (so \( M \) is linearly isometric to \( \ell_p \oplus \mu L_0(0,1) \)). (The latter statement is a well-known fact due to Carathéodory [44, § 14, Theorem 5].) So putting everything together we see that the complete theory of \( \ell_p \) ( \( p \neq 2 \) ) has exactly two separable models \( \ell_p \) and \( L_p(0,1) \oplus \ell_p \) up to isometry. This fact suggests the following conjecture: Do all models of a theory with finitely many separable models contain some \( \ell_p \) or \( c_0 \)?

Still, one can say more:

**Remark 5.16** (i) Let \( M \) be an \( \aleph_0\)-categorical Banach space. Then \( M \) embeds (isometrically) every Banach space with a basis which is finitely representable in \( M \). For this, repeat the argument of the proof of Theorem 5.9.

(ii) The *spectrum* \( \Sigma(X) \) of a Banach space \( X \) consist of all \( p \in [1, \infty] \) for which \( X \) contains \( \ell_p^n \)'s uniformly, that is, there exists \( K > 0 \) such that for all \( n \), there is a subspace of \( X \) which is \( K\)-isomorphic to \( \ell_p^n \). (Cf. [52, p. 57]) By Krivine’s theorem, the spectrum \( \Sigma(X) \) consist of all \( p \in [1, \infty] \) for which \( \ell_p \) is finitely representable in \( X \) (cf. [26, Theorem II.5.13]). Dvoretzky’s theorem says that \( 2 \in \Sigma(X) \). It is known that \( \Sigma(X) \) is a closed subset of \([1, \infty], \) and \( \Sigma(X) \cap [1, 2] \) is always an interval; for example, \( \Sigma(L_p) = [p, 2] \) if \( 1 \leq p < 2 \), \( \Sigma(L_p) = [2, p] \) if \( 2 \leq p < \infty \), and \( \Sigma(L_\infty) = [1, \infty] \). Therefore: Every \( \aleph_0\)-categorical Banach space \( M \) contains isometric copies of \( \ell_p \) for all \( p \in \Sigma(M) \).

### 5.4 Concluding remarks

(1) As previously mentioned, Krivine and Maurey [37] proved that every stable Banach space contains a copy of \( \ell_p \) for some \( 1 \leq p < \infty \). Also, it was noticed that the type space of every separable stable Banach space is strongly separable. Later, Haydon and Maurey [27] showed that a Banach space with strongly separable types contains either a reflexive subspace or a subspace isomorphic to \( \ell_1 \).

(2) Caicedo [15] showed that for any suitable extension \( L \) of continuous logic with the compactness and the separable downward Löwenheim-Skolem theorems, each sentence of \( L \) is equivalent to a sentence of continuous logic. On the other hand, using these two theorems (in continuous logic), one can give a proof of Corollary 5.13. Indeed, suppose that \( M \) is \( \aleph_0\)-categorical. By Krivine’s theorem, some \( \ell_p \) (or \( c_0 \)) is finitely representable in \( M \), and so, using the compactness theorem, there is an elementary extension \( N \) of \( M \) such that it contains \( \ell_p \) (or \( c_0 \)). By the separable downward Löwenheim-Skolem theorem, there is a separable elementary substructure \( M' \) of \( N \) such that it contains \( \ell_p \) (or \( c_0 \)). By the \( \aleph_0\)-categoricity, \( M \) and \( M' \) are isometric, and so \( M \) contains some classical sequence space. As \( \ell_2 \) is finitely representable in \( \ell_p \) and \( c_0 \), it is easy to see that \( M \) contains \( \ell_2 \). Therefore, by Caicedo’s result, the above argument does not work in a logic stronger than continuous logic.

(3) There is still another stronger property: A separable Banach space \( X \) is said to be finitely determined if for each separable space \( Y \) such that \( X \) is finitely representable (f.r.) in \( Y \) and \( Y \) is f.r. in \( X \) then \( Y \) is isometric to \( X \). Clearly, every finitely determined space is \( \aleph_0\)-categorical, but not converse. In [39], a more direct proof of existence of classical sequence spaces in finitely determined spaces is given.

(4) One might expect that \( \aleph_0\)-categoricity implies that a large number of \( \ell_p, p \in [1, \infty) \), or \( c_0 \), are involved in the space. This is not true; for example, the theory \( \ell_2 \cong L_2 = L_2([0, 1], \lambda) \) is \( \aleph_0\)-categorical (cf. [9]), but \( \ell_2 \) does
not have any subspace isomorphic to $c_0$ or $\ell_q$ for all $q \neq 2$ (cf. [1], Corollary 2.1.6). In other words, a direct proof of Theorem 5.9, without using Krivine’s Theorem, does not imply a stronger result.

(5) As previously mentioned, [32, Theorem 1.1] asserts that the existence of enough definable types guarantees the existence of $\ell_p$ subspaces. Note that our approach in this paper is different. In fact we counted separable models and the main result (Theorem 5.9) is about complete theories.

(6) Although $\aleph_0$-categoricity and $\aleph_0$-stability do not have any connection in general, but to our knowledge the most examples of $\aleph_0$-stable theories are studied in continuous model theory are $\aleph_0$-categorical (cf. [9]).

(7) (A Krivine–Maurey type theorem). We believe that the following statement holds: For any separable NIP space $X$ there exists a spreading model of $X$ containing $c_0$ or $\ell_p$ for some $1 \leq p < \infty$. The proof used Borel definability of types and Krivine’s Theorem. This result provides answers to the questions in Remark 4.20. Full proof will be presented elsewhere.

6 Dividing lines in Banach spaces and model theory

This section aims to provide a classification of Banach spaces similar to Shelah’s classification in classical logic. For the sake of completeness, we recall some facts which were observed in [38], and then we use them to give some examples which are (in our view) very illuminating.

6.1 Banach space for a formula

Let $M$ be an $L$-structure, $\varphi(x, y) : M \times M \to \mathbb{R}$ a formula (we identify formulas with real-valued functions defined on models).

Let $S_\varphi(M)$ be the space of complete $\varphi$-types over $M$ and set $A = \{ \varphi(x, a), -\varphi(x, a) \in C(S_\varphi(M)) : a \in M \}$. The (closed) convex hull of $A$, denoted by $\overline{\text{conv}}(A)$, is the intersection of all (closed) convex sets that contain $A$. $\overline{\text{conv}}(A)$ is convex and closed, and $\|f\| \leq \|\varphi\|$ for all $f \in \overline{\text{conv}}(A)$. So, by normalizing we can assume that $\|f\| \leq 1$ for all $f \in \overline{\text{conv}}(A)$. Set $B = \overline{\text{conv}}(A)$ and $V = \bigcup_{\lambda \geq 0} \lambda B$. It is easy to verify that $V$ is a Banach space with the normalized norm and $B$ is its unit ball. This space will be called the space of \emph{linearly-definable relations on $M$}.

\textbf{Fact 6.1} ([38]) Assume that $\varphi(x, y)$, $M$, $B$ and $V$ are as above. Then the following are equivalent:

(i) $\varphi$ is stable on $M$.

(ii) $B$ is weakly compact.

(iii) The Banach space $V$ is reflexive.

Recall that for an infinite compact Hausdorff space $X$, the space $C(X)$ is not reflexive.

\textbf{Fact 6.2} ([38]) Assume that $\varphi(x, y)$, $M$, and $V$ are as above. Then the following are equivalent:

(i) $\varphi$ has NIP on $M$.

(ii) $V$ is Rosenthal Banach space.

(iii) $V$ does not contain an isomorphic copy of $\ell_1$.

\textbf{Definition 6.3} Let $M$ be a model (of theory $T$) and $\varphi(x, y)$ a formula. We say $\varphi(x, y)$ has the \emph{strict order property on $M$} (short SOP on $M$) if there exists a sequence $(a_i, b_i : i < \omega)$ in $M$ and $\varepsilon > 0$ such that for all $i < j$,

$$
\varphi(M, a_i) \leq \varphi(M, a_{i+1}) \quad \text{and} \quad \varphi(b_j, a_i) + \varepsilon < \varphi(b_j, a_j).
$$

The acronym SOP stands for the strict order property and NSOP is its negation. A complete theory $T$ has SOP if there are a formula $\varphi$ and a model $M$ of it such that $\varphi$ has SOP on $M$, and otherwise it is said that $T$ has NSOP.

\textbf{Remark 6.4} (i) Let $U$ be a monster model (of theory $T$), $\varphi(x, y)$ a formula and $M$ a small model. Suppose that $\varphi$ has SOP on $M$. Then it is easy to verify that $\varphi$ has SOP on the monster model $U$. (Indeed, suppose that $p$ is a $\varphi$-type over $M$, $(b_j) \in M$ and $t(p(b_j/M)) \rightarrow p$. By SOP on $M$, $\varphi(b_j, a_i) \leq \varphi(b_j, a_{i+1})$, and since $\varphi(x, a_i)$’s are continuous, so $\varphi(b, a_i) \leq \varphi(b, a_{i+1})$ for all $b \models p$. This means that $\varphi(b, a_i) \leq \varphi(b, a_{i+1})$ for all $b \in U$.)

(ii) SOP strictly implies instability. In classical ($\langle 0, 1 \rangle$-valued) logic, this is a known fact.
(iii) Suppose that $M = \mathcal{U}$ and $V$ is as above. If $V$ is weakly sequentially complete, i.e., every weak Cauchy sequence has a weak limit (in $V$), then $\varphi$ has NSOP. This is a consequence of the Eberlein–Šmulian theorem (Fact 2.3). (Cf. [40, Remark 3.4(ii)].)

(iv) Recently, in [40], it was shown that the converse of (iii) above does not hold (for classical logic). That is, if $\varphi$ has NSOP, we cannot conclude that $V$ is weakly sequentially complete (cf. [40, Example 3.5(ii)]). On the other hand, we strongly believe that there is no perfect analog of Shelah’s theorem (i.e., a complete theory is stable if and only if it has NIP and NSOP) in continuous logic. This is discussed in detail in [41]. For the sake of completeness we give the diagram of these observations for classical logic [40]:

\[
\begin{array}{ccc}
\text{Stable} & \overset{\text{Shelah}}{\Longrightarrow} & \text{NIP} \quad \& \quad \text{NSOP} \\
\text{Eberlein–Grothendieck} & \downarrow & \\
\text{Reflexive} & \overset{\text{Eberlein–Šmulian}}{\iff} & \text{Rosenthal} \quad \& \quad \text{WS-complete}
\end{array}
\]

6.2 Dividing lines in classical spaces

In this subsection, we list some classical Banach spaces and some theories and point out their model theoretic dividing lines. Many of the following examples have been studied previously (mostly by Henson and co-authors in the 70’s), and much of what is said about them is well-known (cf. [29] and references therein). Nevertheless, there is a definite merit to gathering all of these facts together in one place.

It should be noted that since the notions NIP in a model (in both classic and continuous logics) and SOP for continuous logic are new, the following observations on these notions do not appear somewhere in literature. In fact, in our view, the examples form the most interesting and illuminating part of the article.

Example 6.5

(i) $L_p$ Banach lattices ($1 \leq p < \infty$) are reflexive and so Rosenthal and weakly sequentially complete. By the Kakutani representation theorem of $L$-spaces (cf. [2, 4.27]), it was proved that the classes of $L_p$ Banach lattices are axiomatizable (in a suitable language) and their theories, denoted by $\text{AL}L_p$, are stable (cf. [9, § 17]).

(ii) $L_1$ Banach lattices are neither reflexive nor Rosenthal in general (e.g., $\ell_1$). Although, they are weakly sequentially complete.

Nevertheless, their theory, $\text{AL}L_1$, is stable (cf. (i) above). However, it is not established in some extensions of the language. For example, Alexander Berenstein showed that $L_1(\mathbb{R})$ with convolution is unstable (cf. [13]), later did it appear that $\ell_1(\mathbb{Z}, +)$ with convolution is unstable (cf. [18], Proposition 6.2). In fact it has SOP (cf. Example 6.6 below). On the other hand, since the theory of atomless probability measure algebra, denoted by $\text{APA}$, is interpretable in the theory $L_1$ of Banach lattices, the prior also is stable (cf. [9]).

(iii) Hilbert spaces are reflexive. The theory of infinite dimensional Hilbert spaces, as described in [9], is stable.

(iv) $c_0$ is neither reflexive nor weakly sequentially complete, but it is Rosenthal. Let $(e_n)^\infty_1$ be a standard basis for $c_0$ and $s_n = e_1 + \cdots + e_n$. $(s_n)^\infty_1$ is called summing basis for $c_0$. Then $\|e_m + s_n\| = 2$ if $m \leq n$ and $= 1$ if $m > n$. So, the formula $\varphi(x, y) = \|x + y\|$ has the order property. Now let $x = (a_n)^\infty_1 \in \ell_1 = (c_0)^*$ then $x^*(s_n) \rightarrow \sum a_i$. So $(s_n)^\infty_1$ is a weak Cauchy sequence with no weak limit. It is trivial that $c_0$ is Rosenthal, because $(c_0)^* = \ell_1$ is separable but $(\ell_1)^* = \ell_\infty$ is nonseparable. Let $\psi(x, s_m) = \max(\|x + s_n\|, \|x - s_n\|)$. Then $\psi(x, s_m) \leq \psi(x, s_n)$ and $\psi(e_n, s_n) + 1 \leq \psi(e_m, s_n)$ for all $m < n$. So $\psi(x, y)$ has SOP on $c_0$.

By Example 4.5 above, $\varphi(x, y)$ has IP on $c_0$. Therefore, if $c_0$ is a model of a theory $T$, then $T$ has IP. (Also, see Example B.8 for more information.)

(v) $\ell_\infty$ is neither stable nor Rosenthal or weakly sequentially complete because $c_0$ and $\ell_1$ live inside $\ell_\infty = C(\beta\mathbb{N})$ (cf. below). The formula $\psi$ as described above has SOP in $\ell_\infty$. This fact explains why the case $p = \infty$ is excluded from (i) above. Since $c_0$ has IP, $\ell_\infty$ has IP.

(vi) $C(X, \mathbb{R})$ space for an (infinite) compact Hausdorff space is neither reflexive nor Rosenthal nor weakly sequentially complete (cf. (vii) below). By the Kakutani representation theorem of $M$-spaces, the class of
C(X)-spaces is axiomatizable in the language of Banach lattices. Indeed, replace the axioms of abstract L_p-spaces in [9] by the M-property \(|x^+ + y^+| = \max(|x^+|, |y^+|)\). Now we add a constant symbol 1 to our language as an order unite, i.e., for all x, \(|x| = 1\) implies that \(x^+ \leq 1\); equivalently, \(\sup_x (|x| - 1) = 1 + \sup_x (|x^+| \vee 1 - 1)\). Note that \(|\cdot|\) is an order unite norm if for all x and \(\alpha, x^+ \leq \alpha 1\) implies that \(|x| \leq \alpha\); equivalently \(\sup_{x} (|x^+| \vee \alpha 1 - 1) = \sup_{x} (|x| - \alpha)\). Since \(\ell_\infty\) has SOP, the theory of C(X)-spaces has SOP. Since \(\ell_\infty\) has IP, then the theory of C(X)-spaces has the independence property.

(vii) \(C_0(X, \mathbb{C})\) for an (infinite) locally compact space X is neither reflexive nor Rosenthal nor weakly sequentially complete. Note that, by the Gelfand–Naimark representation theorem of Abelian C*-algebras, the class \(C_0(X, \mathbb{C})\)-spaces is axiomatizable in a suitable language (cf. [18] for the noncommutative theory). First, it is known that all Banach spaces, in particular \(\ell_1\) and \(c_0\), live inside C(K)-spaces where K’s are compact. So C(K) is neither reflexive nor Rosenthal or weakly sequentially complete in general. Second, by an example of [18], we will show that the theory \(C_0(X)\)-spaces has SOP and IP. Indeed, in X find a sequence of distinct \(x_n\) that converges to some x (with x possibly in the compactification of X). For each n find a positive \(a_n \in C_0(X)\) such that \(\|a_n\| = 1\) and \(a_n(x_i) = 1\) if \(i \leq n\) and \(= 0\) if \(i > n\). By replacing \(a_n\) with maximum of \(a_j\) for \(j \leq n\), we may assume that this sequence is increasing. Also, we can assume that the support \(K_n\) of each \(a_n\) is compact and \(a_n(K_n) = 1\) for \(n < m\). So \(a_n a_m = a_n\) if \(n < m\). Now let \(\varphi(x, y) = \|(x - 1)y\|\). Then \(\varphi(x, a_i) \leq \varphi(x, a_j)\) and \(\varphi(a_i, a_j) + 1 \leq \varphi(a_i, a_j)\) for \(i < j\). So \(\varphi\) has SOP. Since \(c_0\) has IP, this theory has IP.

(viii) C*-algebras are neither reflexive nor Rosenthal nor weakly sequentially complete because some infinite-dimensional Abelian *-subalgebra lives inside a C*-algebra and so by the Gelfand–Naimark theorem it has a \(C_0(X)\)-subalgebra (cf. [18, Lemma 5.3]). Note that the Gelfand–Naimark–Segal theorem ensure that this class is axiomatizable (cf. [18]). So, the theory of C*-algebras has SOP. Since \(c_0\) has IP, this theory has IP.

(ix) The class of (Abelian) tracial von Neumman algebra is axiomatizable (cf. [19]). A tracial von Neumann algebra is stable if and only if it is of type I. The theory of Abelian tracial von Neumann algebra is stable because it is interpretable in the theory of probability measure algebras (cf. above and also [18, Lemma 4.5]).

To our knowledge the following observation itself does not appear somewhere in literature.

**Example 6.6** (\(\ell_1(\mathbb{Z}, +)\) with convolution has SOP) In [19] it is shown that the formula \(\varphi(x, y) = \inf_{|\xi| \leq 1} \|x \ast z - y\|\) has order property in \(\ell_1(\mathbb{Z}, +)\). We show that \(\varphi\) also has SOP. Indeed, we need to show that: (i) \(\varphi(x, y, z) \leq \varphi(x, y, z)\) for \(i \leq j\), and (ii) \(\varphi(x, y, z) < \varphi(x, y, z)\) for \(i < j\), where \(x\)'s are elements of \(\ell_1\) as described in [19, Proposition 6.2]. It is easy to verify that (ii) holds. We check that (i) also is true. Indeed, for any z, \(|z|^n \leq 1\) and any y, we have \(|y| - |z| | \leq \sup |z| (t - y(t)) = \|y\| + 1\) where \(|\cdot|\) is uniform norm on C(T).

So, if we assume that \(|y| \geq 1\), then \(\inf_{|\xi| \leq 1} |y - z| = 1\) where \(|\cdot|\) is uniform norm on C(T).

In fact we define \(\varphi(x, y) = \inf_{|\xi| \leq 1} \|x \ast z - y\|\). Therefore if \(i < j\), then \(\inf_{|\xi| \leq 1} |(\cos t)^i z - y| = |(\cos t)^j z - y|\). Equivalently, \(\varphi(x, y) \leq \varphi(x, y)\). To summarize, \(\varphi\) has SOP. A question arises: *Does \(\ell_1(\mathbb{Z}, +)\) with convolution have NIP?*

The following interesting fact affirms some of our observations.

**Fact 6.7** ([26, Theorem III.5.1]) *Every stable (separable) Banach space is weakly sequentially complete.*

So, every Banach space containing \(c_0\) is unstable. This directly implies that the theories of C(X)-spaces, \(C_0(X, \mathbb{C})\)-spaces, and C*-algebras are not stable. (cf. (vi)–(viii) above). (There is even something stronger (cf. Proposition 4.6).) Of course, the converse does not hold, e.g., Tsirelson’s space and its dual are unstable but they are reflexive and so weakly sequentially complete.

**Remark 6.8** In [46] the author proved that for \(1 \leq p < \infty, p \neq 2\), the non-commutative \(L_p\) space \(L_p(M)\) is stable if \(M\) is of type I. (Cf. also [34, p. 1479].) Later, in [18] the authors showed that a tracial von Neumann algebra is stable if and only if it is of type I. Two questions arise: *Which tracial von Neumann algebra (or non-commutative \(L_p\) spaces) have NIP? Which of them have NSOP?*

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Appendix A: \( \aleph_0 \)-categoricity, continued

In this appendix, we revisit the notion of \( \aleph_0 \)-categoricity. Some observations are not new for model theorists, but one reason for recalling them is to make the paper more accessible to other interested readers.

Recall that the density character of a topological space \( X \) is the least infinite cardinal number of a dense subset of \( X \). When measuring the size of a structure we will use its density character (as a metric space), denoted \( \| M \| \), rather than its cardinality. Similarly, for a separable structure \( M \), since \( S_\infty(M) \) is a metric space, we measure the size \( S_\infty(M) \) by its density character \( \| S_\infty(M) \| \).

Definition A.1 A complete theory \( T \) in a countable language is \( \aleph_0 \)-categorical (or \( \omega \)-categorical) if it has an infinite model and any two models of size \( \aleph_0 \) are isomorphic. Equivalently, \( T \) has an infinite model and separable models are isometric. An \( \aleph_0 \)-categorical (or \( \omega \)-categorical) structure is a separable structure \( M \) whose theory is \( \aleph_0 \)-categorical. (Compare with Definition 5.12.)

Let \( L \) be a language and \( T \) a complete \( L \)-theory. Suppose that \( M \) is a model of \( T \) and \( A \subseteq M \). Denote the \( L(A) \)-structure \(( M, a )_{a \in A} \) by \( M_A \), and set \( T_A \) to be the \( L(A) \)-theory of \( M_A \).

Remark A.2 It is easy to check that for a separable model \( M \), and countable subsets \( A \subseteq B \) of \( M \), if \( T_B \) is \( \aleph_0 \)-categorical then \( T_A \) is also \( \aleph_0 \)-categorical (cf. [9, Corollary 12.13]); however, the converse does not hold in a strong form (cf. [9, Remark 12.14]).

The following fact is folklore, and we give it without a proof. Recall that a formula \( \varphi \) is stable for a theory \( T \) if for every model \( M \) of \( T \), \( \varphi \) is stable on \( M \); equivalently, \( \varphi \) has not the order property on every model \( M \) of \( T \).

Fact A.3 ([4, Corollary 4]) Let \( T \) be a countable theory and \( \varphi(x, y) \) a formula. Then the following are equivalent.
(i) For every separable model \( M \) of \( T \), \( S_\varphi(M) \) is strongly separable.
(ii) \( \varphi \) is stable on every separable model of \( T \).
(iii) \( \varphi \) is stable for \( T \).
(iv) For every model \( M \) of \( T \), \( \| S_\varphi(M) \| \leq \| M \| \).

**Corollary A.4** Let \( M \) be an \( \aleph_0 \)-categorical structure. The complete theory \( T \) of \( M \) is stable if and only if for every formula \( \varphi \), the space \( S_\varphi(M) \) is strongly separable.

**Proof.** If \( T \) is stable, every \( \varphi \)-type is definable (cf. [4, Corollary 4]), and so \( S_\varphi(M) \) is strongly separable. The converse follows from the \( \aleph_0 \)-categoricity and the equivalence (i) \( \iff \) (iii) of Fact A.3 above. \( \square \)

The fact that Tsirelson’s space can not be \( \aleph_0 \)-categorical is a consequence of Corollary 5.13. Alternatively, it is clear by instability and the above fact:

**Corollary A.5** Suppose that \( T \) is the complete theory of Tsirelson’s space \( M_T \) (in a countable language). Then the following holds:

(i) \( T \) is not \( \aleph_0 \)-categorical.
(ii) There exists a separable model \( M \) of \( T \) which its type space is not strongly separable.

**Proof.**

(i) Suppose, if possible, that \( T \) is \( \aleph_0 \)-categorical. Then \( \varphi(x, y) \) is stable on \( M_T \), since \( S_\varphi(M) \) is \( \varphi \)-separable where \( \varphi(x, y) = \| x + y \| \). By Corollary 4.14, this is a contradiction.

(ii) Immediate from Fact A.3 and Corollary 4.14. (Clearly, this model is different from Tsirelson’s space by Fact 4.18.) \( \square \)

**Remark A.6** Note that the \( \aleph_0 \)-categorical assumption in Corollary A.4 is too strong and a weaker assumption is sufficient. Indeed, suppose that \( M \) is a \( p_{BM} \)-saturated structure, where \( p_{BM} \) is the Banach–Mazur perturbation system. (Cf. Definitions 5.5, 5.7 above.) Then, it is easy to show that, \( M \) is stable if and only if for every formula \( \varphi \) the space \( S_\varphi(M) \) is strongly separable.

**Appendix B: A remark on Rosenthal’s dichotomy**

Note that a well-known result of Rosenthal (Lemma B.3 below) is used in the proof of the direction (i) \( \implies \) (v) of Lemma 4.2 (cf. [42, Lemma 3.12]). So it is worthwhile to study it more carefully.

**Definition B.1** (i) A sequence \( \{ f_n \} \) of real valued functions on a set \( X \) is said to be independent if there exist real numbers \( s < r \) such that

\[
\bigcap_{n \in P} f_n^{-1}(\infty, s) \cap \bigcap_{n \in M} f_n^{-1}(r, \infty) \neq \emptyset \tag{\( \square \)}
\]

for all finite disjoint subsets \( P, M \) of \( \mathbb{N} \). A family \( F \) of real valued functions on \( X \) is called independent if it contains an independent sequence; otherwise it is called strongly (or completely) dependent.

(ii) We say that the sequence \( \{ f_n \} \) is strongly (or completely) independent if \( \square \) holds for all infinite disjoint subsets \( P, M \) of \( \mathbb{N} \). A family \( F \) of real valued functions is called strongly (or completely) independent if it contains a strongly independent sequence; otherwise it is called dependent.

It is an easy exercise to check that when \( X \) is a compact space and functions are continuous the above two notions are the same, but this does not hold in general (cf. Example B.8 below).

**Lemma B.2** Let \( X \) be a compact space and \( F \subseteq C(X) \) a bounded subset. Then the following conditions are equivalent:

(i) \( F \) is dependent.
(ii) \( F \) is strongly dependent.
Lemma B.3 (Rosenthal’s lemma) Let X be a compact space and \( F \subseteq C(X) \) a bounded subset. Then the following conditions are equivalent:

(i) \( F \) does not contain an independent subsequence.

(ii) Each sequence in \( F \) has a convergent subsequence in \( \mathbb{R}^X \).

Rosenthal [51] used the above lemma for proving his famous \( \ell_1 \) theorem: A sequence in a Banach space is either ‘good’ (it has a subsequence which is weakly Cauchy) or ‘bad’ (it contains an isomorphic copy of \( \ell_1 \)). We will shortly discuss this topic (cf. below).

**Fact B.4** ([51]) If \( X \) is a compact space and \( \{f_n\} \) a pointwise bounded sequence in \( C(X) \), then

1. either \( \{f_n\} \) has a (pointwise) convergent subsequence, or
2. \( \{f_n\} \) has a \( \ell_1 \)-subsequence, equivalently, \( \{f_n\} \) has an independent subsequence.

Note that for non-compact spaces, Fact B.4 is not a dichotomy. Megrelishvili informed us with details about this observation (cf. Example 4.5 above and Example B.8 below) and also pointed to an example which is in [17]. Of course, we are not sure that the observation itself (that Fact B.4 is not always a dichotomy in noncompact Polish case) appears somewhere in the literature in a clear form. For compact space, independence and strong independence are the same.

**Fact B.5** (Rosenthal’s \( \ell_1 \)-theorem) If \( (x_n) \) is a bounded sequence in a Banach space \( V \) then

1. \( (1)' \) either \( (x_n) \) has a weakly Cauchy subsequence, or
2. \( (2)' \) \( (x_n) \) has a \( \ell_1 \)-subsequence.

Fact B.5 is a consequence of Fact B.4. Indeed, let \( B^* \) be the unit ball of \( V^* \), and define \( f_n : B^* \rightarrow \mathbb{R} \) by \( x^* \mapsto x^*(x_n) \) for all \( x^* \in B^* \).

**Fact B.6** Rosenthal’s dichotomy, [54] If \( X \) is a Polish space and \( \{f_n\} \) a pointwise bounded sequence in \( C(X) \), then

1. \( (1)'' \) either \( \{f_n\} \) has a (pointwise) convergent subsequence, or
2. \( (2)'' \) \( \{f_n\} \) has a subsequence whose closure in \( \mathbb{R}^X \) is homeomorphic to \( \beta\mathbb{N} \), equivalently, it has a strong independent subsequence.

**Remark B.7** Note that \((1)' \implies (1) = (1)'', \text{ but } (1)'' \not\implies (1)' \text{ in general. Also, } (2)' \implies (2)'', \text{ but } (2)'' \not\implies (2)' \text{ (cf. below). In fact } (2)' \text{ is equivalent to independence property. For Polish space } X, (2)' \text{ holds if and only if there is a compact subset } K \subseteq X \text{ such that } \{f_n|_K\} \text{ contains a strong independent subsequence, equivalently, } \{f_n\} \text{ has a strong independent subsequence (cf. [14, Lemma 2B, Theorem 2F & Corollary 4G]).}

**Example B.8** We revisit Example 4.5. Recall that the family \( \mathcal{F}_{c_0} \) (in Example 4.5) contains an independent sequence. (i) The family \( \mathcal{F}_{c_0} \) is weakly precompact in \( \mathbb{R}^{c_0} \). Indeed, since \( c_0 \) is separable, and the functions \( f_n \) are 1-Lipschitz, so by a diagonal argument (cf. Lemma 4.16), every sequence has a pointwise convergent subsequence. This shows that Fact B.4 is not a dichotomy for non-compact spaces. (ii) By Fact B.5, the sequence \( \{f_n\} \) has not a weakly Cauchy subsequence. (iii) By Fact B.6, \( \{f_n\} \) has not a strong independent subsequence.

**Remark B.9** (i) Define \( f_n : B_{c_0} \rightarrow [0,2] \) by \( f_n(x) = \|x + s_n\| \). Then \( f_n \) converges to the continuous function \( f(x) = 1 + \|x\| \). Let \( c_0 \) be the \( \hat{\text{C}} \)-Stone compactification of \( c_0 \). Then \( \hat{f}_n \Rightarrow \hat{f} \). Indeed, note that \( \hat{f}_n|_{c_0} = f_n \) and \( \hat{f}|_{c_0} = f \), and \( f(e) = \lim_{m} \hat{f}_m(e_m) = \lim_{m} f(e_m) = \lim_{m} f(e_m) = 2 \neq 1 = \lim_{m} \lim_{n} f_n(e_m) = \lim_{n} \lim_{m} \hat{f}_n(e_m) = \lim_{n} \hat{f}(e) \text{ where } (e_m) \text{ is the standard basis of } c_0 \text{ and } e \text{ is a cluster point of } (e_m) \text{ in } \hat{c_0}. \text{ Note that } c_0 \text{ is dense in } \hat{c_0} \text{ and } f_n \Rightarrow f, \text{ but } \hat{f}_n \not\Rightarrow \hat{f}.

(ii) Define \( g_n(x) = \|x + e_n\| \). Then \( g_n \rightarrow g \) where \( g(x) = \max(\|x\|,1) \). But \( g_n \) does not contain a convergent subsequence; because \( (g_n) \) has independence property (cf. Example B.8 above).

(iii) \( g_n \) is a pointwise convergent sequence but it has not a weakly Cauchy subsequence. Because, by the Rosenthal \( \ell_1 \)-theorem, either a sequence has a weakly Cauchy subsequence, or it has a \( \ell_1 \)-subsequence (equivalently, it contains an independent subsequence). Note that weakly Cauchy is stronger than pointwise convergence, in general. But, for locally compact spaces, they are equivalent.