LEADER-FOLLOWER DYNAMICS

Hsin-Lun Li
1National Sun Yat-sen University, Kaohsiung 804, Taiwan

Abstract. The original Leader-Follower model, proposed in [5], categorizes agents with opinions in $[-1, 1]$ into a follower group, a leader group with a positive target opinion in $[0, 1]$, and a leader group with a negative target opinion in $[-1, 0]$. Leaders maintain a constant attraction to their target, blending it with the average opinion of their group neighbors at each update. Followers, on the other hand, have a constant attraction to the average opinion of their leader group’s opinion neighbors, also integrating it with their group neighbors’ average opinion. This model was numerically studied in [5].

This paper extends the Leader-Follower model to include a social relationship, variable degrees over time, high-dimensional opinions, and a flexible number of leader groups. We theoretically investigate conditions for asymptotic stability or consensus, particularly in scenarios where a few leaders can dominate the entire population.

1. Introduction

The Leader-Follower model contains the Hegselmann-Krause model in [2, 3] which involves two types of individuals: leaders and followers. The authors in [5] proposed a leader-follower model that partitions agents whose opinion is in $[-1, 1]$ to a follower group, a leader group with a positive target opinion in $[0, 1]$ and a leader group with a negative target opinion in $[-1, 0]$. Individual $j$ is an opinion neighbor of individual $i$ if their opinion distance does not exceed the confidence threshold of individual $i$. If all individuals share the same confidence threshold, two are opinion neighbors if their distance does not exceed that threshold. A leader’s opinion depends on the opinion neighbors in its group and its group target, while a follower’s opinion depends on all opinion neighbors. Define $[n] = \{1, 2, \ldots, n\}$. Say $N$ agents including $N_1$ followers, $N_2$ positive target agents and $N_3$ negative target agents, set as $[N_1]$, $[N_1 + N_2] - [N_1]$ and $[N] - [N_1 + N_2]$. The mechanism is as follows:

$$x_i(t+1) = \frac{1 - \alpha_i - \beta_i}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t) + \frac{\alpha_i}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t) + \frac{\beta_i}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t), \quad i = 1, \ldots, N_1,$$

$$x_i(t+1) = \frac{(1 - w_i)}{|N_i^P(t)|} \sum_{j \in N_i^P(t)} x_j(t) + w_i d, \quad i = N_1 + 1, \ldots, N_1 + N_2,$$

$$x_i(t+1) = \frac{1 - z_i}{|N_i^N(t)|} \sum_{j \in N_i^N(t)} x_j(t) + z_i g, \quad i = N_1 + N_2 + 1, \ldots, N,$$  

2020 Mathematics Subject Classification. 91C20, 91D25, 91D30, 93D50, 94C15.

Key words and phrases. Social network, leader-follower dynamics, consensus, Hegselmann-Krause dynamics, averaging dynamics.
where
\[ x_i(t) = \text{opinion of agent } i \text{ at time } t, \]
\[ d \in [0, 1] \text{ is the positive target opinion,} \]
\[ g \in [-1, 0] \text{ is the negative target opinion,} \]
\[ \epsilon_i = \text{confidence threshold of agent } i, \]
\[ N_i^F(t) = \{ j \in [N_1] : \| x_i(t) - x_j(t) \| \leq \epsilon_i \}, \]
\[ N_i^S(t) = \{ j \in [N_1 + N_2] - [N_1] : \| x_i(t) - x_j(t) \| \leq \epsilon_i \}, \]
\[ N_i^D(t) = \{ j \in [N] - [N_1 + N_2] : \| x_i(t) - x_j(t) \| \leq \epsilon_i \}, \]
\[ \alpha_i = \text{degree to the average opinion of agent } i's \]
\[ \text{positive target neighbors,} \]
\[ \beta_i = \text{degree to the average opinion of agent } i's \]
\[ \text{negative target neighbors,} \]
\[ w_i = \text{degree to the positive target of agent } i, \]
\[ z_i = \text{degree to the negative target of agent } i, \]
\[ \alpha_i, \beta_i, \quad w_i, \quad z_i \in [0, 1]. \]

The authors in [4] pointed out that it can be an application in e-commerce. The Leader-Follower model we investigate now includes the following:

- There is a social relationship.
- The degree toward the average opinion of a group can vary over time.
- The number of leader groups is decidable.
- Opinions can be high dimensional.

Let \( \mathcal{N}_i^S(t) \) be the collection of all social and opinion neighbors of individual \( i \) in set \( S \) at time \( t \) and \( x_i(t) \in \mathbb{R}^d \) be the opinion of individual \( i \) at time \( t \) where \( x_i(0) \) is a random variable. The leader group \( L \) model with target \( g \in \mathbb{R}^d \) is given by:
\[
x_i(t + 1) = \frac{\alpha_i(t)}{|\mathcal{N}_i^L(t)|} \sum_{j \in \mathcal{N}_i^L(t)} x_j(t) + (1 - \alpha_i(t))g, \quad i \in L
\]  
where \( \alpha_i(t) \in [0, 1] \) is a random variable indicating the degree of individual \( i \) toward the average opinion of its group neighbors at time \( t \) and \( L \) is the collection of all leader group members. Say the leader groups are \( L_1, \ldots, L_m \) with targets \( g_1, \ldots, g_m \). The follower group \( F \) model is given by:
\[
x_i(t + 1) = \left(1 - \frac{\sum_{k=1}^{m} \beta_i^k(t)}{\left| \mathcal{N}_i^F(t) \right|} \right) \sum_{k \in \mathcal{N}_i^F(t)} x_k(t) + \sum_{k=1}^{m} \frac{\beta_i^k(t)}{|\mathcal{N}_i^{L_k}(t)|} \sum_{k \in \mathcal{N}_i^{L_k}(t)} x_k(t), \quad i \in F
\]  
where \( \beta_i^k(t) \in [0, 1] \) is a random variable indicating the degree toward the average opinion of its social and opinion neighbors in leader group \( L_k \) at time \( t \) and \( F \) consists of all followers. \( \beta_i^k(t) = 0 \) if \( \mathcal{N}_i^{L_k}(t) = \emptyset \). Observe that (2) and (3) reduce to (1) when there are two leader groups and a follower group. (2) reduces to the synchronous Hegselmann-Krause model when \( \alpha_i(t) = 1 \) for all \( i \in L \) at all times. Similarly, (3) reduces to the synchronous Hegselmann-Krause model when \( \beta_i^k(t) = 0 \) for all \( k \in [m] \) and \( i \in F \) all the time. In (2) and (3), a leader's opinion depends on the social and opinion neighbors in its group and its group target. In contrast, a follower's opinion depends on all social and opinion neighbors.

Interpreting in a graph, a vertex represents an individual and an edge symbolizes a relationship between two individuals. Saying

- \( G(t) = (V, E(t)) \) is the social graph at time \( t \) with vertex set and edge set \( V \) and \( E(t) \) and
- \( \mathcal{G}(t) = (V, \mathcal{E}(t)) \) is the opinion graph at time \( t \) with vertex set and edge set \( V \) and \( \mathcal{E}(t) \).
Leader-Follower dynamics

Edge \((i, j) \in E(t)\) if individual \(i\) is socially connected with individual \(j\), or if individual \(j\) is a social neighbor of individual \(i\). Similarly, edge \((i, j) \in E(t)\) if individual \(i\) is opinion connected with individual \(j\), or if individual \(j\) is an opinion neighbor of individual \(i\). We can interpret a social relationship with an undirected social graph if edge \((i, j) \in E(t)\) implies \((j, i) \in E(t)\). For instance, if individual \(i\) is a relative of individual \(j\), then the reverse is also true. However, not all social relationships are reciprocal. For example, if individual \(i\) knows individual \(j\), it does not necessarily imply that individual \(j\) knows individual \(i\). In such cases, we use a directed social graph to represent the relationship. On the other hand, if all individuals share the same confidence threshold in an opinion relationship, we can interpret this opinion relationship with an undirected graph. A graph is \(\delta\)-trivial if the opinion distance between any two vertices does not exceed \(\delta\). Denote \(B(a, r)\) as the open ball centered at \(a\) with radius \(r\), i.e., \(B(a, r) = \{x : \|x - a\| < r\}\). A profile \(G \cap \mathcal{G}\) is the intersection of the social and opinion graphs.

2. Main results

Since leader groups are independent, and similarly for the follower groups, we respectively investigate sufficient conditions for asymptotic stability or a consensus in the leader group \(L\) and the follower group \(F\) with \(m\) leader groups. The sufficient condition in Theorem 1 is independent of social and opinion relationships. In fact, even a slight tendency toward the target by all leaders in \(L\) guarantees a consensus equal to the target.

**Theorem 1.** There is a consensus equal to the target in (2) when

\[
\lim \inf_{t \to \infty} \max_{i \in L} \alpha_i(t) < 1.
\]

The sufficient condition in Theorem 2 assumes an undirected social graph and an undirected opinion graph on \(L\). Specifically, the synchronous Hegselmann-Krause model meets this condition, thus ensuring asymptotic stability. Asymptotic stability of the synchronous Hegselmann-Krause model illustrates finite time convergence.

**Theorem 2.** Assume that the social graph and opinion graph are undirected on \(L\), the social graph becomes constant after some time, and

\[
\sum_{t \geq 0} (1/ \min_{i \in L} \alpha_i(t) - 1) < \infty. \text{ Then, asymptotic stability holds in (2).}
\]

The sufficient condition in Theorem 3 specifies that the social graph and opinion graph can be directed, provided that all followers are socially connected with a leader in each leader group. This condition also identifies circumstances under which a few leaders can dominate the entire population.

**Theorem 3.** Assume that all followers have one social neighbor in each leader group, that

\[
\{x_i(t), g_k\}_{i \in (\bigcup_{k=1}^m L_k) \cup F, k \in [m]} \subset B(g_j, \min_{i \in F} \epsilon_i)
\]

for some \(j \in [m]\) and \(t \geq 0\), that \(\beta^k_i(t) = \beta^k_i\) for all \(k \in [m]\) and \(i \in F\) all the time, and that

\[
\max_{i \in F} (1 - \sum_{k=1}^m \beta^k_i) < 1 \quad \text{and} \quad \sup_{s \geq t} \max_{i \in L_k, k \in [m]} \alpha^F_i(s) < 1.
\]

Then,

\[
\lim_{t \to \infty} \max_{i \in L_k, k \in [m]} \|x_i(t) - g_k\| = 0 \quad \text{and} \quad \lim_{t \to \infty} \max_{i \in F} \|x_i(t) - \sum_{k=1}^m \beta^k_i g_k\| = 0.
\]
The sufficient condition in Theorem 4 assumes an undirected social graph and an undirected opinion graph on $F$, allowing the social graph and opinion graph on leader groups to be directed. Specifically, the synchronous Hegselmann-Krause model satisfies this condition.

**Theorem 4.** Assume that the social graph and opinion graph are undirected on $F$, the social graph becomes constant after some time, and

$$\sum_{t \geq 0} \max_{i \in F;k \in [m]} \beta_{i}^k(t) < \infty.$$  

Then, asymptotic stability holds in (3).

### 3. The Leader Group Model

All leader groups are independent. Therefore, we investigate the behavior of a leader group. Let $y_i(t) = x_i(t) - g$, (2) becomes

$$y_i(t + 1) = \alpha_i(t) \sum_{j \in X_i^L(t)} y_j(t)$$  

(4)

It is clear that (4) is the synchronous Hegselmann-Krause model when $\alpha_i(t) = 1$ for all $i \in L$ all the time. Asymptotic stability holding in (2) is equivalent to holding in (4).

**Lemma 5.** We derive $x_i(t) \to g$ if $\lim_{t \to \infty} \alpha_i(t) = 0$ for all $i \in L$.

**Proof.** It follows from the triangle inequality that

$$\|y_i(t + 1)\| \leq \alpha_i(t) \max_{j \in X_i^L(t)} \|y_j(t)\|.$$ 

Taking $\limsup$ on both sides, we get

$$\lim_{t \to \infty} \|y_i(t + 1)\| \leq \max_{i \in L} \|y_i(0)\| \lim_{t \to \infty} \alpha_i(t) = 0.$$ 

This indicates that $y_i(t) \to 0$ as $t \to \infty$. Hence, $x_i(t) \to g$ as $t \to \infty$.  

**Lemma 6.** Let $Z_t = \max_{i \in L} \|x_i(t) - g\|$. Then, $(Z_t)_{t \geq 0}$ is nonincreasing and

$$Z_t - Z_{t+1} \geq (1 - \max_{i \in L} \alpha_i(t)) Z_t.$$  

(5)

**Proof.** By the triangle inequality, we get

$$Z_{t+1} = \max_{i \in L} \|y_i(t + 1)\| \leq \max_{i \in L} \alpha_i(t) \max_{i \in L} \|y_i(t)\| = \max_{i \in L} \alpha_i(t) Z_t.$$ 

It turns out that $(Z_t)_{t \geq 0}$ is nonincreasing and

$$Z_t - Z_{t+1} \geq (1 - \max_{i \in L} \alpha_i(t)) Z_t.$$  

\[\square\]

Next, we show circumstances in which social relationships and opinion relationships do not influence the achievement of consensus.

**Proof of Theorem 1.** It follows from Lemma 6 that $(Z_t)_{t \geq 0}$ is a nonnegative supermartingale. Via the martingale convergence theorem, $Z_t$ converges to some random variable $Z_\infty$ with finite expectation as $t \to \infty$. Letting $\alpha_t = \max_{i \in L} \alpha_i(t)$ and taking $\limsup$ on (5), we derive

$$0 = \lim_{t \to \infty} \sup_{t \geq 0} (Z_t - Z_{t+1}) \geq Z_\infty \lim_{t \to \infty} \sup_{t \geq 0} (1 - \alpha_t) = (1 - \liminf_{t \to \infty} \alpha_t) Z_\infty.$$ 

This implies $Z_\infty = 0$.  

\[\square\]
Lemma 7. Assume that the social graph is undirected on $L$, the opinion graph is undirected on $L$ with confidence threshold $\epsilon$, and $E(t) \subset E(t+1)$. Let $W_t = \sum_{i,j \in L} (\|x_i(t) - x_j(t)\|^2 - \|x_i(t+1) - x_j(t+1)\|^2) \vee \epsilon^2 \mathbb{1}\{(i, j) \notin E(t)\}$. Then, we derive

$$W_t - W_{t+1} \geq 4 \sum_{i \in L} \|x_i(t) - x_i(t+1)\|^2 - 4|L|^2(1/\min \alpha_i(t) - 1)$$

\[
\times \max_{i \in L} |x_i(0) - g| \left( \max_{i \in L} |x_i(0) - g| \vee \max_{i,j \in L} |x_i(0) - x_j(0)| \right). \tag{6}
\]

Proof. Let $\mathcal{M}_i^L = \mathcal{M}_i^L(t)$, $\alpha_i = \alpha_i(t)$, $x_i = x_i(t)$, $x_i^* = x_i(t+1)$, $y_i = y_i(t)$, $y_i^* = y_i(t+1)$, $E = E(t)$ and $E^* = E(t+1)$. It turns out that

$$W_t - W_{t+1} = \sum_{i \in L} \{ \sum_{j \in \mathcal{M}_i^L} (\|x_i - x_j\|^2 - \|x_i^* - x_j^*\|^2 + \epsilon^2)$$

\[
+ \sum_{j \in \mathcal{M}_i^L \cup \{i\}} \epsilon^2 - (\|x_i^* - x_j^*\|^2 + \epsilon^2) \vee \epsilon^2 \mathbb{1}\{(i, j) \notin E^*\} \}
\]

\[
\geq \sum_{i \in L} \sum_{j \in \mathcal{M}_i^L} (\|x_i - x_j\|^2 - \|x_i^* - x_j^*\|^2) = \sum_{i \in L} \sum_{j \in \mathcal{M}_i^L} (\|y_i - y_j\|^2 - \|y_i^* - y_j^*\|^2)
\]

\[
= \sum_{i \in L} \sum_{j \in \mathcal{M}_i^L} (2 < y_i - y_j, y_i^* - y_j > - 2 < y_i^* - y_j, y_i^* - y_j^* >)
\]

\[
= 2 \sum_{i \in L} (|\mathcal{M}_i^L| < y_i - y_j, y_i^* > (1 - 1/\alpha_i) - 2 \sum_{j \in \mathcal{M}_i^L} < y_i^* - y_j, y_i^* - y_j^* >
\]

\[
- 2 \sum_{j \in \mathcal{M}_i^L \cup \{i\}} < y_i - y_j, y_i^* - y_j^* >
\]

\[
= 2 \sum_{i \in L} (|\mathcal{M}_i^L| - 1) < y_i^* - y_i, y_i^* > (1 - 1/\alpha_i) + 2 \sum_{i \in L} \|y_i - y_i^*\|^2
\]

\[
- 2 \sum_{i \in L} \sum_{j \in \mathcal{M}_i^L \cup \{i\}} < y_i^* - y_i, y_i^* - y_j^* > - 2 \sum_{j \in \mathcal{M}_i^L} |\mathcal{M}_j^L| < y_j^*/\alpha_j - y_j, y_j - y_j^* >
\]

\[
\geq 2 \sum_{i \in L} (|\mathcal{M}_i^L| - 1) < y_i^* - y_i, y_i^* > (1 - 1/\alpha_i) + 2 \sum_{i \in L} \|y_i - y_i^*\|^2
\]

\[
- 2 \sum_{i \in L} \sum_{j \in \mathcal{M}_i^L \cup \{i\}} \|y_i^* - y_i\| \|y_j^* - y_j\|
\]

\[
- 2 \sum_{j \in \mathcal{M}_i^L \cup \{i\}} |\mathcal{M}_j^L| (1/\alpha_j - 1) < y_j^*, y_j - y_j^* > + 2 \sum_{j \in \mathcal{M}_i^L} |\mathcal{M}_j^L| \|y_j - y_j^*\|^2
\]

\[
= -4 \sum_{i \in L} (|\mathcal{M}_i^L| - 1) < y_i^*, y_i - y_i^* > + 2 \sum_{i \in L} \|y_i - y_i^*\|^2
\]

\[
+ \sum_{i \in L} \sum_{j \in \mathcal{M}_i^L \cup \{i\}} [\|y_j^* - y_i - \|y_j^* - y_i\|^2 - \|y_i^* - y_i\|^2 - \|y_j^* - y_j\|^2]
\]

\[
+ 2 \sum_{j \in \mathcal{M}_i^L} |\mathcal{M}_j^L| \|y_j - y_j^*\|^2
\]

\[
\geq -2 \sum_{i \in L} (|\mathcal{M}_i^L| - 1) \|y_i^* - y_i\|^2 - 4 \sum_{i \in L} |\mathcal{M}_i^L| (1/\alpha_i - 1) < y_i^*, y_i - y_i^* >
\]

\[
+ 2 \sum_{j \in \mathcal{M}_i^L} |\mathcal{M}_j^L| \|y_j - y_j^*\|^2 + 2 \sum_{i \in L} \|y_i - y_i^*\|^2
\]

\[
= 4 \sum_{i \in L} \|y_i - y_i^*\|^2 - 4 \sum_{i \in L} |\mathcal{M}_i^L| (1/\alpha_i - 1) < y_i^*, y_i - y_i^* >
\]
Hsin-Lun Li

\[ \geq 4 \sum_{i \in L} \|y_i - y_i^*\|^2 - 4|L|^2 \left( \frac{1}{\min_{i \in L} \alpha_i} - 1 \right) \max_{i \in L} \|x_i(0) - g\| \]
\[ \times \left( \max_{i \in L} \|x_i(0) - g\| \vee \max_{i,j \in L} \|x_i(0) - x_j(0)\| \right). \]

By finiteness of the social graph, the social graph monotone after some time is equivalent to the social graph constant after some time.

**Lemma 8** (Cheeger’s Inequality [1]). Assume that \( G = (V, E) \) is an undirected graph with the Laplacian \( \mathcal{L} \). Define \( \iota(G) = \min \left\{ \frac{\partial S}{|S|} : S \subset V, 0 < |S| \leq \frac{|G|}{2} \right\} \) where \( \partial S = \{(u, v) \in E : u \in S, v \in S^c\} \). Then,

\[ 2\iota(G) \geq \lambda_2(\mathcal{L}) \geq \frac{i^2(G)}{2\Delta(G)} \text{ where } \Delta(G) = \text{ maximum degree of } G. \]

**Lemma 9** ([2]). Assume that \( Q \) is a real square matrix and that \( V \) is invertible such that the matrix \( VQ = \mathcal{L} \) is the Laplacian of some connected graph. Then, \( 0 \) is a simple eigenvalue of \( Q'Q \) corresponding to the eigenvector \( \mathbb{1} = (1, 1, \ldots, 1)' \).

In particular, we have

\[ \lambda_2(Q'Q) = \min\{x'|Q'Qx : \|x\| = 1 \text{ and } x \perp \mathbb{1}\}. \]

**Lemma 10.** Assume that the social graph and opinion graph are undirected on \( L \). If some component \( H \) of the profile \( G(t) \cap \mathcal{G}(t) \) on \( L \) is \( \delta \)-nontrivial, then

\[ \sqrt{\sum_{i \in L} \|x_i(t) - x_i(t + 1)\|^2} \]
\[ \geq \sqrt{2}\delta \min_{i \in L} \alpha_i(t)/|L|^4 - (1 - \min_{i \in L} \alpha_i(t))\sqrt{|L| \max_{i \in L} \|x_i(0) - g\|}. \]

**Proof.** Letting \( V(H) \), the vertex set of \( H \), be \( [h] \) and \( \mathbb{1} = (1, 1, \ldots, 1)' \in \mathbb{R}^h \), express \( \mathbb{R}^h = W \oplus W^{\perp} \) for \( W = \text{Span}(\{\mathbb{1}\}) \). For \( y(t) = (y_1(t), \ldots, y_n(t))' \), write

\[ y(t) = [c_1 \mathbb{1} | c_2 \mathbb{1} | \cdots | c_d \mathbb{1}] + \left[ \hat{c}_1 u^{(1)} | \hat{c}_2 u^{(2)} | \cdots | \hat{c}_d u^{(d)} \right] \]

where \( c_i \) and \( \hat{c}_i \) are constants and \( u^{(i)} \in \mathbb{1}^{\perp} \) is a unit vector for all \( i \in [d] \). Observe that

\[ \|y_i(t) - y_j(t)\|^2 = \sum_{k \in [d]} c_k^2 (u^{(k)}_i - u^{(k)}_j)^2 \leq 2 \sum_{k \in [d]} c_k^2 ((u^{(k)}_i)^2 + (u^{(k)}_j)^2) \leq 2 \sum_{k \in [d]} c_k^2 \]

for all \( i, j \in [h] \). Since \( x_i - x_j = y_i - y_j, \)

component \( H \) \( \delta \)-nontrivial implies \( \sum_{k \in [d]} c_k^2 > \delta^2/2. \)

Letting \( \alpha(t) = (\alpha_1(t), \ldots, \alpha_h(t))' \) and \( B(t) = \text{diag}(\alpha(t))A(t) \) for \( A(t) \in \mathbb{R}^{h \times h} \) with \( A_{i,j}(t) = \mathbb{1}_{\{j \in \mathcal{M}^L(t) / \mathcal{M}^L(t) \}} \), we get

\[ y(t) - y(t + 1) = (I - B(t))y(t) = [C(t) + F(t)\mathcal{L}(t)]y(t) \]

where \( C(t) = I - \text{diag}(\alpha(t)), F(t) = \text{diag}(\alpha(t)) (\text{diag}(d_i)^{\alpha_{i-1}} + I)^{-1} \) with \( d_i \) the degree of vertex \( i \) in component \( H \), and \( \mathcal{L}(t) \) is the Laplacian of component \( H \). It follows from Lemmas 8 and 9 that

\[ \lambda_2(\mathcal{L}) > \frac{(2/h)^2}{2h} = 2/h^3, \]
Proof of Theorem 2. We claim the following:

1. All components of profile $G \cap \mathcal{G}$ on $L$ are $\delta$-trivial after some time for all $\delta > 0$.

2. No components of profile $G \cap \mathcal{G}$ on $L$ interact with each other after some time.

Without loss of generality, we assume the social graph on $L$ remains constant over time, say $G(t)|_{L} = G|_{L} = (L, E)$ for all $t \geq 0$. Observe that

$$\sum_{t \geq 0} (1/ \min_{i \in L} \alpha_i(t) - 1) < \infty \implies \lim_{t \to \infty} \min_{i \in L} \alpha_i(t) = 1 \iff \lim_{t \to \infty} \alpha_i(t) = 1 \text{ for all } i \in L.$$ 

Hence, we derive

$$\sqrt{2} \min_{i \in L} \alpha_i(t)/|L|^4 \to \sqrt{2} \delta/|L|^4 \text{ and } (1 - \min_{i \in L} \alpha_i(t)) \sqrt{|L|} \max_{i \in L} \|x_i(0) - g\| \to 0$$
after $t \to \infty$. There is $t_0 \geq 0$ such that

$$\sqrt{2} \min_{i \in L} \alpha_i(t)/|L|^4 - (1 - \min_{i \in L} \alpha_i(t)) \sqrt{|L|} \max_{i \in L} \|x_i(0) - g\| \geq \delta/|L|^4$$
for all $t \geq t_0$. Assume that asymptotic stability does not hold in (2). Then, there are $\delta > 0$ and $(s_k)_{k \geq 0}$ increasing with $s_0 \geq t_0$ and some component in profile $G(t_k) \cap \mathcal{G}(t_k)$ on $L$ $\delta$-nontrivial for all $k \geq 0$. Letting

$$M_0 = 4|L|^2 \max_{i \in L} \|x_i(0) - g\| \left( \max_{i \in L} \|x_i(0) - g\| \lor \max_{i, j \in L} \|x_i(0) - x_j(0)\| \right),$$
it turns out from Lemma 7 that

$$W_0 + M_0 \sum_{t=0}^{m} (1/ \min_{i \in L} \alpha_i(t) - 1) \geq \sum_{t=0}^{m} (W_i - W_{i+1}) + M_0 \sum_{t=0}^{m} (1/ \min_{i \in L} \alpha_i(t) - 1)$$
$$\geq 4 \sum_{t=0}^{m} \sum_{i \in L} \|x_i(t) - x_i(t + 1)\|^2 \text{ for all } m \geq 0.$$ 

As $m \to \infty$, we derive

$$\infty > W_0 + M_0 \sum_{t \geq 0} (1/ \min_{i \in L} \alpha_i(t) - 1) \geq 4 \sum_{t \geq 0} \sum_{i \in L} \|x_i(t) - x_i(t + 1)\|^2$$
$$\geq 4 \sum_{k \geq 0} \sum_{i \in L} \|x_i(s_k) - x_i(s_k + 1)\|^2 \geq 4 \sum_{k \geq 0} \delta^2/|L|^8 = \infty,$$ a contradiction.
Lemma 11. Assume that all followers have one social neighbor in leader group $F$ that vertices $i$ and $j$ belong to distinct components of profile $G \cap \mathcal{F}(t_k)$ on $L$,

$$(i,j) \in E \cap \mathcal{E}(t_k) \cap (i,j) \in E \cap \mathcal{E}(t_k + 1).$$

Letting $y_i = y_{i}(t_k)$, $y^*_i = y_{i}(t_k + 1)$ and $\alpha_i = \alpha_i(t_k)$ for all $i \in L$ and $k \geq 0$, it turns out from the triangle inequality that

$$\epsilon < ||y_i - y_j|| \leq ||y_i - y^*_i/\alpha_i|| + ||y^*_i/\alpha_i - y^*_j/\alpha_j|| + ||y^*_j/\alpha_j - y^*_j||$$

for the last inequality following from $||y_i - y^*_i/\alpha_i|| \leq \epsilon/4$ and $||y^*_j/\alpha_j - y^*_j|| \leq \epsilon/4$. Since $\limsup_{k \to \infty} ||y^*_i/\alpha_i - y^*_j|| = 0 = \limsup_{k \to \infty} ||y^*_j/\alpha_j - y^*_j||$, we derive

$$\epsilon/2 \leq \liminf_{k \to \infty} ||y^*_i/\alpha_i - y^*_j|| = \liminf_{k \to \infty} ||x^*_i - x^*_j||,$$ a contradiction.

□

(7) reduces to the synchronous Hegselmann-Krause model when $\alpha_i(t) = 1$ for all $i \in L$ at all times. Therefore, $\sum_{t \geq 0} (1/\min_{i \in F} \alpha_i(t) - 1) = 0 < \infty$. From Theorem 2, it follows that all components of a profile on $L$ become $\epsilon$-trivial, and no components interact with each other after some time under undirected social and opinion graphs. This indicates that all components achieve their consensus at the next time step, substantiating the finite time convergence property of the synchronous Hegselmann-Krause model under undirected social and opinion graphs.

4. The Follower Group Model

Follower groups are independent. We first consider a leader group $L$ and a follower group $F$. (3) becomes

$$x_i(t + 1) = \frac{(1 - \beta_i(t))}{|\mathcal{N}^F_i(t)|} \sum_{j \in \mathcal{N}^F_i(t)} x_j(t) + \frac{\beta_i(t)}{|\mathcal{N}^F_i(t)|} \sum_{j \in \mathcal{N}^L_i(t)} x_j(t), \quad i \in F,$$

which is equivalent to

$$y_i(t + 1) = \frac{(1 - \beta_i(t))}{|\mathcal{N}^F_i(t)|} \sum_{j \in \mathcal{N}^F_i(t)} y_j(t) + \frac{\beta_i(t)}{|\mathcal{N}^F_i(t)|} \sum_{j \in \mathcal{N}^L_i(t)} y_j(t), \quad i \in F.$$ (7)

Lemma 11. Assume that all followers have one social neighbor in leader group $L$ with target $g$, that $\{x_i(t)\}_{i \in L \cup F} \subset \mathbf{B}(g, \min_{i \in F} \epsilon_i)$ for some $t \geq 0$ and that

$$\sup_{s \geq t} \max_{i \in L} (1 - \beta_i(s)), \max_{i \in L} \alpha_i(s) < 1.$$

Then,

$$\lim_{t \to \infty} \max_{i \in L \cup F} ||x_i(t) - g|| = 0.$$

Proof. $\{x_i(t)\}_{i \in L \cup F} \subset \mathbf{B}(g, \min_{i \in F} \epsilon_i)$ for some $t \geq 0$ implies $\{x_i(s)\}_{i \in L \cup F} \subset \mathbf{B}(g, \min_{i \in F} \epsilon_i)$ for all $s \geq t$. Via Theorem 4 the $\lim_{t \to \infty} \max_{k \in L} \|y_k(t)\| = 0$ therefore $\max_{k \in L} \|y_k(s)\| < \delta$ for all $\delta > 0$, for some $p \geq t$

and for all $s \geq p$. For all $s \geq p$, $\delta > 0$, $i \in F$ and $j \in L$, we have

$$||y_i(s) - y_j(s)|| \leq ||y_i(s)|| + ||y_j(s)|| < \min_{i \in F} \epsilon_i + \delta$$

8
Leader-Follower dynamics

therefore \(\|x_i(s) - x_j(s)\| \leq \|g_i(s) - g_j(s)\| \leq \min_{i \in F} \epsilon_i \leq \epsilon_i\) and \(\mathcal{M}^L(s) \neq \emptyset\).

Let \(\alpha_t = \max_{k \in L} \alpha_k(t)\), \(\bar{\beta}_i = \max_{k \in F} (1 - \beta_k(t))\), \(\gamma = \sup_{s \geq t} \{\bar{\beta}_i, \alpha_k\}\), \(A_t = \max_{k \in F} \|y_k(t)\|\) and \(Z_t = \max_{k \in L} \|y_k(t)\|\). Applying the triangle inequality on (7), for all \(i \in F\) and \(t > p\),

\[
A_{t+1} \leq \bar{\beta}_i A_t + Z_t
\]

\[
\leq \bar{\beta}_i \bar{\beta}_i \ldots \bar{\beta}_p A_p + \bar{\beta}_i \ldots \bar{\beta}_{p+1} Z_p + \ldots + \bar{\beta}_i Z_{t-1} + Z_t
\]

\[
\leq \gamma^{t-p+1} A_p + (t - p + 1) \gamma^{t-p} Z_p
\]

therefore

\[
\limsup_{t \to \infty} A_{t+1} \leq 0.
\]

This completes the proof. \(\square\)

We move on to the follower group model with \(m\) leader groups.

**Proof of Theorem** \(3\) \(\{x_i(t), g_k\}_{i \in (\bigcup_{k=1}^m L_k) \cup F, k \in [m]} \subset B(g_j, \min_{i \in F} \epsilon_i)\) for some \(t \geq 0\) implies

\[
\{x_i(s)\}_{i \in (\bigcup_{k=1}^m L_k) \cup F} \subset B(g_j, \min_{i \in F} \epsilon_i) \text{ for all } s \geq t.
\]

It follows from Theorem \(1\) that \(\lim_{t \to \infty} \max_{k \in [m]} \max_{i \in L_k} \|x_i(t) - g_k\| = 0\) therefore

\[
\max_{k \in [m]} \max_{i \in L_k} \|x_i(t) - g_k\| < \delta \text{ for all } \delta > 0, \text{ for some } p \geq t \text{ and for all } s \geq p.
\]

For all \(s \geq p, \delta > 0, i \in F\) and \(j \in L_k\), we have

\[
\|x_i(s) - x_j(s)\| \leq \|x_i(s) - g_k\| + \|g_k - x_j(s)\| < \min_{i \in F} \epsilon_i + \delta
\]

therefore \(\|x_i(s) - x_j(s)\| \leq \min_{i \in F} \epsilon_i \leq \epsilon_i\) and \(\mathcal{M}^L(s) \neq \emptyset\). Letting

\[
\bar{\beta} = \max_{i \in F} (1 - \sum_{k=1}^m \beta^k_i), \quad \gamma = \sup_{s \geq t} \{\max_{k \in [m]} \max_{i \in L_k} \alpha_k(s), \bar{\beta}\}, \quad g = \sum_{k=1}^m \beta^k_i g_k / \sum_{k=1}^m \beta^k_i,
\]

\[
A_t = \max_{i \in F} \|x_i(t) - g\| / m, \quad C_t = \max_{k \in [m]} \|x_i(t) - g_k\| / m.
\]

Letting

\[
\bar{x}^F(t) = \frac{1}{|\mathcal{M}^F(t)|} \sum_{j \in \mathcal{M}^F(t)} x_j(t) \quad \text{and} \quad \bar{x}^L_k(t) = \frac{1}{|\mathcal{M}^L_k(t)|} \sum_{j \in \mathcal{M}^L_k(t)} x_j(t),
\]

write (3) as

\[
x_i(t+1) - g = (1 - \sum_{k \in [m]} \beta^k_i)(\bar{x}^F(t) - g) + \sum_{k \in [m]} \beta^k_i (\bar{x}^L_k(t) - g_k)
\]

and apply the triangle inequality, for all \(i \in F\) and \(t > p\),

\[
A_{t+1} \leq \bar{\beta}_i A_t + C_t
\]

\[
\leq \bar{\beta}_i \bar{\beta}_i \ldots \bar{\beta}_p A_p + \bar{\beta}_i \ldots \bar{\beta}_{p+1} C_p + \ldots + \bar{\beta}_i C_{t-1} + C_t
\]

\[
\leq \gamma^{t-p+1} A_p + (t - p + 1) \gamma^{t-p} C_p
\]

therefore

\[
\limsup_{t \to \infty} A_{t+1} \leq 0.
\]

This completes the proof. \(\square\)
Lemma 12. Assume that the social graph is undirected on $F$ and the opinion graph is undirected on $F$ with confidence threshold $\epsilon$. Let $X_t = \sum_{i,j \in F}(\|x_i(t) - x_j(t)\|^2 \wedge \epsilon^2) \vee \epsilon^2 \{i,j \notin E(t)\}$ and $E(t) \subseteq E(t + 1)$. Then,

$$X_t - X_{t+1} \geq 4 \sum_{i \in F} \|x_i(t) - x_i(t+1)\|^2 - 4m|F|^2 \max_{i \in F, k \in [m]} \beta_i^k(t)$$

$$\times \left( \max_{i,j \in \bigcup_{k \in [m]} L_k \cup F} \|x_i(0) - x_j(0)\| \vee \max_{i \in \bigcup_{k \in [m]} L_k \cup F} \|x_i(0) - g\| \right)^2.$$ 

Proof. Let $x_i = x_i(t), \ x_i^* = x_i(t+1), \ \beta_i(t) = \beta_i^k(t), \ \mathcal{M}_i^F = \mathcal{M}_i^{LF}(t), \ \mathcal{M}_i^L = \mathcal{M}_i^L(t), \ \bar{x}_i^F = \sum_{j \in \mathcal{M}_i^F} x_j/|\mathcal{M}_i^F|$ and $\bar{x}_i^L = \sum_{j \in \mathcal{M}_i^L} x_j/|\mathcal{M}_i^L|$. Observe that

$$X_t - X_{t+1} \geq \sum_{i \in F} \sum_{j \in \mathcal{M}_i^F} (\|x_i - x_\bar{x}_i^F\|^2 - \|x_i - x_j\|^2)$$

$$= 2 \sum_{i \in F} \sum_{j \in \mathcal{M}_i^F} \left( <x_i - x_j, x_i^* - x_j> - <x_i^* - x_j, x_j - x_j^*> \right)$$

$$= 2 \sum_{i \in F} |\mathcal{M}_i^F| <x_i - x_i^*, x_i^* - x_\bar{x}_i^F> + 2 \sum_{i \in F} \sum_{j \in \mathcal{M}_i^F} <x_i^* - x_i, x_j - x_j^*>$$

$$- 2 \sum_{i \in F} \sum_{j \in \mathcal{M}_i^F} <x_i - x_j, x_j - x_j^*>$$

$$\geq 2 \sum_{i \in F} |\mathcal{M}_i^F| <x_i - x_i^*, x_i^* - x_\bar{x}_i^F> + 2 \sum_{i \in F} \|x_i^* - x_i\|^2$$

$$- \sum_{i \in F} \sum_{j \in \mathcal{M}_i^F - \{i\}} (\|x_i^* - x_i\|^2 + \|x_j^* - x_j\|^2) - 2 \sum_{j \in F} \sum_{i \in \mathcal{M}_j^F} |\mathcal{M}_j^F| <x_j^F - x_j, x_j - x_j^*>$$

$$= 4 \sum_{i \in F} |\mathcal{M}_i^F| <x_i - x_i^*, x_i^* - x_\bar{x}_i^F> + 2 \sum_{i \in F} \|x_i^* - x_i\|^2$$

$$- 2 \sum_{i \in F} (|\mathcal{M}_i^F| - 1) \|x_i - x_i^*\|^2 + 2 \sum_{i \in F} |\mathcal{M}_i^F| \|x_i - x_i^*\|^2$$

$$= 4 \sum_{i \in F} \beta_i |\mathcal{M}_i^F| <x_i - x_i^*, x_i^L - x_i^F> + 4 \sum_{i \in F} \|x_i - x_i^*\|^2$$

$$\geq 4 \sum_{i \in F} \|x_i - x_i^*\|^2 - 4m|F|^2 \max_{i \in F, k \in [m]} \beta_i^k(t)$$

$$\times \left( \max_{i,j \in \bigcup_{k \in [m]} L_k \cup F} \|x_i(0) - x_j(0)\| \vee \max_{i \in \bigcup_{k \in [m]} L_k \cup F} \|x_i(0) - g\| \right)^2.$$ 

□

Lemma 13. Assume that the social graph and opinion graph are undirected on $F$. If some component $H$ of the profile $G(t) \cap \mathbb{F}(t)$ on $F$ is $\delta$-nontrivial, then

$$\sqrt{\sum_{i \in F} \|x_i(t) - x_i(t+1)\|^2} \geq \sqrt{2\delta}(1 - \max_{i \in F} \sum_{k \in [m]} \beta_i^k(t))/|F|^4 - 2m|F|^2 \max_{k \in [m] \setminus i \in F} \beta_i^k(t)$$

$$\times \left( \max_{i \in \bigcup_{k \in [m]} L_k \cup F} \|x_i(0)\| \vee \max_{k \in [m]} \|g_k\| \right).$$ 

Proof. Letting $V(H)$, the vertex set of $H$, be $[h]$ and $\mathbb{1} = (1, \ldots, 1) \in \mathbb{R}^h$, express $\mathbb{R}^h = W \oplus W^\perp$ for $W = \text{Span}\{\mathbb{1}\}$. For $x(t) = (x_1(t), \ldots, x_h(t))^\top$, write

$$x(t) = [c_1 \mathbb{1} | c_2 \mathbb{1} | \cdots | c_d \mathbb{1}] + [\hat{c}_1 u^{(1)} | \hat{c}_2 u^{(2)} | \cdots | \hat{c}_d u^{(d)}]$$

$$10$$
where \( c_i \) and \( \hat{c}_i \) are constants and \( u^{(i)} \in \mathbb{1}^n \) is a unit vector for all \( i \in [d] \). Observe that

\[
\|x_i(t) - x_j(t)\|^2 = \sum_{k \in [d]} \hat{c}_k^2 (u_i^{(k)} - u_j^{(k)})^2 \leq 2 \sum_{k \in [d]} \hat{c}_k^2 ((u_i^{(k)})^2 + (u_j^{(k)})^2) \leq 2 \sum_{k \in [d]} \hat{c}_k^2
\]

for all \( i, j \in [h] \). Hence,

component \( H \) \( \delta \)-nontrivial implies \( \sum_{k \in [d]} \hat{c}_k^2 > \delta^2/2 \).

Letting \( \tilde{\beta}(t) = \left( 1 - \sum_{k \in [m]} \beta_k^H(t), \ldots, 1 - \sum_{k \in [m]} \beta_k^H(t) \right)' \) and \( B(t) = \text{diag}(\tilde{\beta}(t))A(t) \) for \( A(t) \in \mathbb{R}^{h \times h} \) with \( A_{ij}(t) = \mathbb{1}(j \in \mathcal{N}^F_i(t)) / |\mathcal{N}^F_i(t)| \), we get

\[
x(t) - x(t + 1) = (I - B(t))x(t) - O(t) = [C(t) + F(t)L(t)]x(t) - O(t)
\]

where \( C(t) = I - \text{diag}(\tilde{\beta}(t)) \), \( F(t) = \text{diag}(\tilde{\beta}(t)) \text{diag}((d_i)_{i=1}^{d} + I)^{-1} \) with \( d_i \) the degree of vertex \( i \) in component \( H \), \( O(t) = \sum_{k \in [m]} \text{diag}((\beta_k^H(t))_{i \in [h]})(x_i^{(k)})'_{i \in [h]} \) and \( L(t) \) is the Laplacian of component \( H \). It follows from Lemmas 8 and 9 that

\[
\lambda_2(L(t)) \geq \frac{(2/h)^2}{2h} = 2/h^3,
\]

\[
\|F(t)L(t)x(t)\|^2 = \sum_{k \in [d]} \hat{c}_k^2 \|F(t)L(t)u^{(k)}\|^2 \geq \sum_{k \in [d]} \hat{c}_k^2 \lambda_2(L(t)F(t)\|L(t)\|^2 \geq \frac{(\delta^2/2)(\min_{i \in [h]} \tilde{\beta}_i(t)/h^2) \lambda_2^2(L(t)) \geq 2\delta^2(1 - \max_{i \in [h]} \sum_{k \in [m]} \beta_k^H(t)^2) / h^8.
\]

On the other hand, it follows from the triangle inequality that

\[
\|C(t)x(t)\| \leq \sum_{i \in [h]} \sum_{k \in [m]} \|\beta_k^H(t)x_i(t)\|
\]

\[
\leq mh \max_{i \in [h], k \in [m]} \beta_k^H(t) \left( \max_{i \in [h], k \in [m]} \|x_i(0)\| \vee \max_{k \in [m]} \|g_k\| \right),
\]

\[
\|O(t)\| \leq \sum_{i \in [h]} \sum_{k \in [m]} \|\beta_k^H x_i^{(k)}\|
\]

\[
\leq mh \max_{k \in [m], i \in [h]} \beta_k^H(t) \left( \max_{i \in [h], k \in [m]} \|x_i(0)\| \vee \max_{k \in [m]} \|g_k\| \right),
\]

therefore

\[
\sqrt{\sum_{i \in F} \|x_i(t) - x_i(t + 1)\|^2} \geq \sqrt{\sum_{i \in [h]} \|x_i(t) - x_i(t + 1)\|^2} = \|x(t) - x(t + 1)\|
\]

\[
= \|F(t)L(t) + C(t))x(t) - O(t)\| \geq \|F(t)L(t)y(t)\| - \|C(t)y(t) - O(t)\|
\]

\[
\geq \sqrt{2\delta(1 - \max_{i \in [h]} \sum_{k \in [m]} \beta_k^H(t)^2) / h^4 - 2mh \max_{k \in [m], i \in [h]} \beta_k^H(t)}
\]

\[
\times \left( \max_{i \in \cup_{k \in [m]} \cup \cup F} \|x_i(0)\| \vee \max_{k \in [m]} \|g_k\| \right)
\]

\[
\geq \sqrt{2\delta(1 - \max_{i \in F} \sum_{k \in [m]} \beta_k^H(t)^2 / |F|^4 - 2m|F| \max_{k \in [m], i \in F} \beta_k^H(t)}
\]

\[
\times \left( \max_{i \in \cup_{k \in [m]} \cup \cup F} \|x_i(0)\| \vee \max_{k \in [m]} \|g_k\| \right).
\]
Hsin-Lun Li

Proof of Theorem 4. We claim the following:

(1) All components of profile $G \cap \mathcal{G}$ on $F$ are $\delta$-trivial after some time for all $\delta > 0$.

(2) No components of profile $G \cap \mathcal{G}$ on $F$ interact with each other after some time.

Without loss of generality, we assume the social graph on $F$ remains constant over time, saying $G(t)|_F = G|_F = (F, E)$ for all $t \geq 0$. Observe that

$$
\sum_{t \geq 0} \max_{i \in F, k \in [m]} \beta_i^k(t) < \infty \implies \lim_{t \to \infty} \beta_i^k(t) = 0 \text{ for all } i \in F \text{ and } k \in [m].
$$

Hence, we derive

$$
a_t = \sqrt{2}\delta (1 - \max_{i \in F} \sum_{k \in [m]} \beta_i^k(t)/|F|^4) \to \sqrt{2}\delta /|F|^4,
$$

$$
b_t = 2m|F| \max_{k \in [m]; i \in F} \beta_i^k(t) \left( \max_{i \in \cup k \in [m]} \|x_i(0)\| \vee \max_{k \in [m]} \|g_k\| \right) \to 0 \text{ as } t \to \infty.
$$

There is $t_0 \geq 0$ such that

$$
a_t - b_t \geq \delta /|F|^4 \text{ for all } t \geq t_0.
$$

Assume that asymptotic stability does not hold in $\mathcal{G}$. Then, there are $\delta > 0$ and $(s_k)_{k \geq 0}$ increasing with $s_0 \geq t_0$ and some component in profile $G(t_k) \cap \mathcal{G}(t_k)$ on $F$ $\delta$-nontrivial for all $k \geq 0$. Letting

$$
M_0 = 4m|F|^2 \left( \max_{i, j \in \cup k \in [m]} \|x_i(0) - x_j(0)\| \vee \max_{i \in \cup k \in [m]} \|x_i(0) - g\| \right)^2,
$$

it turns out from Lemma 4 that

$$
X_0 + M_0 \sum_{t=0}^{\hat{m}} \max_{i \in F, k \in [m]} \beta_i^k(t) \geq \sum_{t=0}^{\hat{m}} (X_t - X_{t+1}) + M_0 \sum_{t=0}^{\hat{m}} \max_{i \in F, k \in [m]} \beta_i^k(t)
$$

$$
\geq 4 \sum_{t=0}^{\hat{m}} \sum_{i \in F} \|x_i(t) - x_i(t+1)\|^2 \text{ for all } \hat{m} \geq 0.
$$

As $\hat{m} \to \infty$, we derive

$$
\infty > W_0 + M_0 \sum_{t \geq 0} \max_{i \in F, k \in [m]} \beta_i^k(t) \geq 4 \sum_{t \geq 0} \sum_{i \in F} \|x_i(t) - x_i(t+1)\|^2
$$

$$
\geq 4 \sum_{k \geq 0} \sum_{i \in F} \|x_i(s_k) - x_i(s_k + 1)\|^2 \geq 4 \sum_{k \geq 0} \delta^2 /|F|^8 = \infty, \text{ a contradiction.}
$$

Hence, all components of profile $G \cap \mathcal{G}$ on $F$ are $\delta$-trivial after some time for all $\delta > 0$.

Next, we claim that no components of profile $G \cap \mathcal{G}$ on $F$ interact with each other after some time. It follows from claim 1 that all components of profile $G \cap \mathcal{G}$ on $F$ are $\epsilon/4$-trivial after some time $s_0$. Assume that claim 2 is not the case. By finiteness of the social graph, there are edge $(i, j)$ and $(t_k)_{k \geq 0}$ increasing with $t_0 \geq s_0$ such that vertices $i$ and $j$ belong to distinct components of profile $G \cap \mathcal{G}(t_k)$ on $F$,

$$
(i, j) \in E \cap \mathcal{G}(t_k) \text{ and } (i, j) \in E \cap \mathcal{G}(t_k + 1).
$$
Leader-Follower dynamics

Letting

\[
\hat{x}_i^F = \frac{1}{|\mathcal{N}_i^F(t_k)|} \sum_{j \in \mathcal{N}_i^F(t_k)} x_j(t_k), \quad x_i = x_i(t_k), \quad x_i^* = x_i(t_k + 1),
\]

\[
\hat{x}_i^L = \frac{1}{|\mathcal{N}_i^L(t_k)|} \sum_{j \in \mathcal{N}_i^L(t_k)} x_j(t_k), \quad \tilde{\beta}_i^j = \beta_i^j(t_k), \quad \tilde{\beta}_i = 1 - \sum_{j \in [m]} \beta_i^j(t_k)
\]

for all \(i \in F\) and \(k \geq 0\), it turns out from the triangle inequality that

\[
\epsilon < \|x_i - x_j\| \leq \|x_i - x_i^*\| + \|x_i^* - x_j^*\| + \|x_j^* - x_j\|.
\]

On top of that, we get

\[
\|x_i - x_i^*\| \leq \tilde{\beta}_i \|x_i - \hat{x}_i^F\| + \|\sum_{\beta_i^j} (x_i - \hat{x}_i^L)\|
\]

\[
\leq \tilde{\beta}_i \epsilon/4 + m \max_{j \in [m]} \max_{i \in F} \beta_i^j
\]

\[
\times \left( \max_{i,j \in [m]} \|x_i(0) - x_j(0)\| \vee \max_{i \in [m]} \max_{k \in F} \|x_i(0) - g\| \right).
\]

similarly for \(\|x_j - x_j^*\|\), therefore

\[
\liminf_{k \to \infty} \|x_i - x_i^*\| \leq \epsilon/4 \quad \text{and} \quad \liminf_{k \to \infty} \|x_j - x_j^*\| \leq \epsilon/4.
\]

This implies

\[
\epsilon/2 \leq \liminf_{k \to \infty} \|x_i^* - x_j^*\|, \quad \text{a contradiction.}
\]

It follows from Claims 1 and 2 that \(\sum_{j \in \mathcal{N}_j^F(t)} x_j(t)/|\mathcal{N}_j^F(t)|\) converges to some random variable \(\tilde{x}_i\) as \(t \to \infty\) for all \(i \in F\). Since \(\max_{k \in [m]} \beta_{i}^k(t) \to 0\) as \(t \to \infty\) and \(\sum_{j \in \mathcal{N}_j^L(t)} x_j(t)/|\mathcal{N}_j^L(t)|\) is bounded by \(\max_{k \in [m]} \|x_i(0)\| \vee \max_{k \in [m]} \|g_k\|\), we get \(x_i(t + 1) \to \tilde{x}_i\) as \(t \to \infty\) for all \(i \in F\). \(\square\)

ACKNOWLEDGMENT

The author is partially funded by NSTC grant.

REFERENCES

[1] L. W. Beineke, R. J. Wilson, P. J. Cameron, et al. Topics in algebraic graph theory, volume 102. Cambridge University Press, 2004.

[2] H.-L. Li. Mixed Hegselmann-Krause dynamics. Discrete and Continuous Dynamical Systems - B, 27(2):1149–1162, 2022.

[3] H.-L. Li. Mixed Hegselmann-Krause dynamics II. Discrete and Continuous Dynamical Systems - B, 28(5):2981–2993, 2023.

[4] Q. Zha, G. Kou, H. Zhang, H. Liang, X. Chen, C.-C. Li, and Y. Dong. Opinion dynamics in finance and business: a literature review and research opportunities. Financial Innovation, 6(1):1–22, 2020.

[5] Y. Zhao, G. Kou, Y. Peng, and Y. Chen. Understanding influence power of opinion leaders in e-commerce networks: An opinion dynamics theory perspective. Information Sciences, 426:131–147, 2018.

Email address: hsinlunl@asu.edu