A GEOMETRIC INVARIANT OF 6-DIMENSIONAL SUBSPACES OF 4 × 4 MATRICES

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Abstract. Let \( R \) be a 6-dimensional subspace of \( M_4(k) \), the ring of \( 4 \times 4 \) matrices over an algebraically closed field \( k \). We associate to \( R \) a closed subscheme \( X_R \) of the Grassmannian of 2-dimensional subspaces of \( k^4 \). To define \( X_R \) we write \( M_4(R) \) as \( V \otimes V^* \) where \( V \) is a 4-dimensional vector space. The reduced subscheme of \( X_R \) is the set of 2-dimensional subspaces \( Q \subseteq V \) such that \( (Q \otimes V^*) \cap R \neq 0 \). Our main result is that if \( \dim(X_R) = 1 \), the minimal possible dimension, then the degree of \( X_R \) as a subscheme of the ambient Plücker \( \mathbb{P}^5 \) is 20. We give two examples of \( X_R \) that involve elliptic curves: in one case \( X_R \) is a \( \mathbb{P}^1 \)-bundle over an elliptic curve; in the other it is a curve having 7 irreducible components, three of which are elliptic curves, and four of which are smooth conics.

1. Introduction

We work over a fixed algebraically closed field \( k \).

1.1. Let \( V \) be a 4-dimensional \( k \)-vector space. We identify the space of \( 4 \times 4 \) matrices over \( k \) with \( V \otimes V^* \). Since we never make use to the multiplicative structure of the matrix algebra we will replace \( V^* \) by \( V \).

We write \( G(2, V) \) for the Grassmannian of 2-planes in \( V \) and \( G(8, V^\otimes 2) \) for the Grassmannian of 8-dimensional subspaces of \( V \otimes V \). We will identify \( G(2, V) \) with its image in \( G(8, V^\otimes 2) \) under the closed immersion \( G(2, V) \to \to G(8, V^\otimes 2) \), \( Q \mapsto Q \otimes V \).

1.2. Let \( R \) be a 6-dimensional subspace of \( V \otimes V \). The set

\[ S_R := \{ W \in G(8, V^\otimes 2) \mid W \cap R \neq 0 \} \]

is a closed subvariety of \( G(8, V^\otimes 2) \). It is a special Schubert variety [5, p.146]. The geometric invariant we associate to \( R \) is the scheme

\[ X_R := S_R \cap G(2, V), \]

the scheme-theoretic intersection taken inside \( G(8, V^\otimes 2) \). Its reduced subscheme is

\[ (X_R)_{\text{red}} := \{ Q \in G(2, V) \mid (Q \otimes V) \cap R \neq 0 \} \subseteq G(2, V). \]

1.3. If \( R = \{ x \otimes y - y \otimes x \mid x, y \in V \} \), i.e., if \( R \) is the set of skew symmetric matrices, then \( X_R = G(2, V) \). More generally, if \( \theta \in \text{GL}(V) \) and \( R = \{ x \otimes y - y \otimes \theta(x) \mid x, y \in V \} \), then \( X_R = G(2, V) \).

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1.4. Our main result is

**Theorem 1.1** (Theorem 2.6 and Corollary 2.7). For all $R$, all irreducible components of $X_R$ have dimension $\geq 1$, and the $X_R$'s form a flat family over the locus of those $R$ where $\dim(X_R) = 1$.

Moreover, if $\dim(X_R) = 1$, then $\deg(X_R) = 20$, where its degree is computed by using the Plücker embedding $G(2, V) \to \mathbb{P}^5$ and viewing $X_R$ as a subscheme of $\mathbb{P}^5$.

1.5. Our reading (ignorance?) of the linear algebra literature suggests that $X_R$ is a new invariant. Sometimes, $X_R$ is isomorphic to the subscheme of $\mathbb{P}(R) \subseteq \mathbb{P}(V^\otimes 2)$ consisting of the rank-2 tensors that belong to $R$; i.e., the locally closed subscheme of $\mathbb{P}(R)$ where all $3 \times 3$ minors vanish.

Examples of that form prompted us to write this paper. Such examples appear in looking for non-commutative analogues of $\mathbb{P}^3$ or, almost equivalently, from non-commutative analogues of the polynomial ring in four variables. We say more about our motivation in §4.

2. The proof of Theorem 1.1

2.1. We will examine various families over the base scheme $G(6, V^\otimes 2)$. Define

\[
G := G(6, V^\otimes 2) \times G(8, V^\otimes 2),
\]

\[
G(1, 3) := G(6, V^\otimes 2) \times G(2, V),
\]

\[
S_R := \{W \in G(8, V^\otimes 2) \mid W \cap R \neq \{0\}\},
\]

\[
S := \{(R, x) \mid R \in G(6, V^\otimes 2), x \in X_R \} \subseteq G(1, 3),
\]

\[
X_R := S_R \cap G(2, V), \quad \text{the scheme-theoretic intersection taken inside } G(8, V^\otimes 2),
\]

\[
X := S \cap G(1, 3), \quad \text{the scheme-theoretic intersection taken inside } G(1, 3),
\]

\[
U := \{R \in G(6, V^\otimes 2) \mid \dim(X_R) = 1\} \subseteq G(6, V^\otimes 2),
\]

\[
X_U := \{(R, x) \mid \dim(X_R) = 1, x \in X_R \} \subseteq X.
\]

Thus $X_U = f^{-1}(U) = \pi^{-1}(U) \cap X$. In general, we denote the restriction of a family to a subscheme $U \subseteq G(6, V^\otimes 2)$ by the subscript $U$.

**Proposition 2.1.** The set $U$ is a dense open subset of $G(6, V^\otimes 2)$.

**Proof.** Each $S_R$ is a Schubert variety and, by [6, pp. 193-196], for example, its codimension in $G(8, V^\otimes 2)$ is 3. It follows that the codimension of $S$ in $G$ is 3. Since $\dim(G(2, V)) = 4$, the codimension of $G(1, 3)$ in $G$ is $\dim(G(8, V^\otimes 2)) = 4$. It follows that every irreducible component $X_i$, $i \in I$, of $S \cap G(1, 3) = X$ has codimension $\leq \dim(G(8, V^\otimes 2)) - 1$. In other words, the relative dimension, $\dim(X_i) - \dim(G(6, V^\otimes 2))$, is at least 1 for each $i \in I$.

We will now apply [7, Exercise II.3.22(d)] to the families obtained by restricting the projection $\pi$ to the reduced subschemes $(X_i)_{\text{red}}$ (and corestricting to the scheme-theoretic images of these restrictions). The conclusion of the cited exercise is that $Y := \{x \in X \mid \dim(X_{\pi(x)}) \geq$
2} is a closed subscheme of $X$. Since $\pi|_X$ is a projective morphism it is closed. Since $X_U = G(6, V^{\otimes 2}) - \pi(Y)$, $U$ is open. By [15], $U$ is non-empty, and therefore dense.

The following result is embedded in the proof of Proposition 2.1.

**Proposition 2.2.** Let $R \in G(6, V^{\otimes 2})$. If $\dim(X_R) = 1$, then every irreducible component of $X_R$ has dimension 1.

**Proposition 2.3.** Let $G$ be an algebraic group acting on $\mathbb{P}^n$. Let $T$ be an integral noetherian scheme endowed with a transitive action of $G$. Let $G$ act diagonally on $T \times \mathbb{P}^n = \mathbb{P}_T^n$. If $X \subseteq \mathbb{P}_T^n$ a $G$-stable closed subscheme, then $X$ is flat over $T$.

**Proof.** Let $t, t' \in T$. There is $g \in G$ such that $t' = g(t)$. The action of $g$ is such that the diagram

$$
\begin{array}{ccc}
X_t & \longrightarrow & \mathbb{P}^n_{k(t)} \\
\downarrow & & \downarrow \\
X_{t'} & \longrightarrow & \mathbb{P}^n_{k(t')}
\end{array}
$$

commutes. It follows that $X_t$ and $X_{t'}$ have the same Hilbert polynomial. It now follows from [7, Thm. III.9.9] that $X$ is flat over $T$.

**Corollary 2.4.** $S$ is flat over $G(6, V^{\otimes 2})$.

**Proof.** Let $G(8, V^{\otimes 2}) \to \mathbb{P}(\wedge^8(V^{\otimes 2}))$ be the Plücker embedding and consider $S$ as a closed subscheme of $G(6, V^{\otimes 2}) \times \mathbb{P}(\wedge^8(V^{\otimes 2}))$. Let $GL(V^{\otimes 2})$ act diagonally on the previous product in the obvious way. It is clear that the closed subscheme $S$ is stable under the action of $GL(V^{\otimes 2})$ and that the action of $GL(V^{\otimes 2})$ on $G(6, V^{\otimes 2})$ is transitive. The result now follows from Proposition 2.3.

**Lemma 2.5.** Both $X_U$ and $S$ are Cohen-Macaulay schemes.

**Proof.** Let $\pi : S \to G(6, V^{\otimes 2})$ be the restriction of the projection. Hochster proved that Schubert varieties in Grassmannians are Cohen-Macaulay [8]. Thus, $S_R$ is Cohen-Macaulay for all $R \in G(6, V^{\otimes 2})$. Let $x \in S$. By Corollary 2.4, the natural map $u : \mathcal{O}_{\pi(x), G(6, V^{\otimes 2})} \to \mathcal{O}_{x,S}$ is a flat homomorphism of noetherian local rings. The closed fibre of $u$, which is $\mathcal{O}_{x,S_{\pi(x)}}$, is Cohen-Macaulay. Since $\mathcal{O}_{\pi(x), G(6, V^{\otimes 2})}$ is also Cohen-Macaulay, [10, Cor 23.3] implies that $\mathcal{O}_{x,S}$ is Cohen-Macaulay. Thus $S$ is Cohen-Macaulay.

The Cohen-Macaulay property for $X_U$ will follow from [4, Prop. 18.13] applied to the following setup.

Let $A = \mathcal{O}_{x,X_U}$ be the local ring at a point $x \in X_U$. Recall that $X_U$ is the intersection $S_U \cap G(1, 3)_U$. Since $G(1, 3)_U$ is regular and hence a local complete intersection, we can find $n = \dim G(8, V^{\otimes 2}) - 4$ generators for the ideal $I$ of $A$ such that $\mathcal{O}_{x,S_U} = A/I$.

Since we are restricting our families to $U$, the codimension of $I$ in $A$ is precisely $n$. But then the hypotheses of [4, Prop. 18.13] are met, and the ring $A/I$ is Cohen-Macaulay.

**Theorem 2.6.** The scheme $X_U$ is flat over $U$.

**Proof.** Let $x \in X_U = f^{-1}(U)$. Let $B = \mathcal{O}_{f(x),U}$ and $A = \mathcal{O}_{x,X_U}$. We must show that $A$ is a flat $B$-module. Let $p$ be the maximal ideal in $B$.

Because $X_U$ is Cohen-Macaulay, $A$ is a Cohen-Macaulay ring. Since $\dim((X_U)_{\pi(x)}) = 1$, $Kdim(A) = Kdim(B) + Kdim(A/p)$. Taken together, this equality, the fact that $A$ is Cohen-Macaulay, and [4, Thm. 18.16(b)], imply that $A$ is a flat $B$-module.
Corollary 2.7. If \( \dim (X_R) = 1 \), then \( \deg (X_R) = 20 \) where the degree is taken inside the ambient “Plücker projective space” \( \mathbb{P}^5 \supseteq G(1, 3) \).

Proof. Since \( X_U \) is flat over \( U \) and \( k \) is algebraically closed, the Hilbert polynomial of \( X_R \), viewed as a closed subscheme of \( \mathbb{P}^5 \), is the same for all \( R \in U \) [7, Thm. III.9.9]. Since \( \dim (X_R) = 1 \), the leading coefficient of its Hilbert polynomial is the degree of \( X_R \) as a closed subscheme of \( \mathbb{P}^5 \). Thus, \( \deg (X_R) \) is the same for all \( b \in U \). We therefore need only exhibit a single \( R \) for which \( \deg (X_R) = 20 \). This is done in [1, Theorem 3.3].

This completes the proof of Theorem 1.1.

3. Examples

Throughout this section suppose the characteristic of \( k \) is not 2.

3.1. An example with \( X_R \) a surface. Let \( \{ \alpha_1, \alpha_2, \alpha_3 \} \subseteq k - \{ 0, \pm 1 \} \) and suppose that \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_1 \alpha_2 \alpha_3 = 0 \). Let \( \{ x_0, x_1, x_2, x_3 \} \) be a basis for a 4-dimensional vector space \( V \). Let \( E \) be the quartic elliptic curve in \( \mathbb{P}^3 \) given by the intersection of any two of the quadrics

\[
\begin{align*}
&x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0, \\
&x_0^2 - \alpha_2 \alpha_3 x_1^2 - \alpha_3 x_2^2 + \alpha_2 x_3^2 = 0, \\
&x_0^2 + \alpha_3 x_1^2 - \alpha_1 \alpha_3 x_2^2 - \alpha_1 x_3^2 = 0, \\
&x_0^2 - \alpha_2 x_1^2 + \alpha_1 x_2^2 - \alpha_1 \alpha_2 x_3^2 = 0.
\end{align*}
\]

Every elliptic curve is isomorphic to at least one of these \( E \)'s.

To save space we write \( x_i x_j \) for \( x_i \otimes x_j \in V^\otimes 2 \). Let \( R \) be the linear span of the 6 elements

\[
x_0 x_i - x_i x_0 - \alpha_i (x_j x_k + x_k x_j) \quad \text{and} \quad x_0 x_i + x_i x_0 - \alpha_i (x_j x_k - x_k x_j)
\]

where \((i, j, k)\) runs over the cyclic permutations of \((1, 2, 3)\).

By [9] and [20], \( X_R \) is isomorphic to a \( \mathbb{P}^1 \)-bundle over \( E \). More precisely, \( X_R \) consists of the lines in \( \mathbb{P}^3 \) whose scheme-theoretic intersection with \( E \) has multiplicity \( \geq 2 \), and hence \( = 2 \). In other words, \( X_R \) parametrizes the set of secant lines to \( E \).

3.2. An example with \( \dim (X_R) = 1 \). Retain the notation in §3.1, and let \( R' \) be the linear span of the 6 elements

\[
x_0 x_i - x_i x_0 - \alpha_i (x_j x_k - x_k x_j) \quad \text{and} \quad x_0 x_i + x_i x_0 - \alpha_i (x_j x_k + x_k x_j)
\]

where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\). By [3],

\[
X_{R'} = (C_1 \sqcup C_2 \sqcup C_3 \sqcup C_4) \cup (E_1 \sqcup E_2 \sqcup E_3),
\]

where

1. the \( C_i \)'s are disjoint smooth plane conics,
2. the \( E_j \)'s are degree-4 elliptic curves that span different 3-planes in \( \mathbb{P}^5 \), and
3. each \( E_j \) is isomorphic to \( E/(\xi_j) \) where \((\xi_1), (\xi_2), (\xi_3)\) are the three order-2 subgroups of \( E \), and
4. \(|C_i \cap E_j| = 2\) for all \((i, j) \in \{ 1, 2, 3, 4 \} \times \{ 1, 2, 3 \} \).
3.3. **The origin of the examples in §§3.1 and 3.2.** There is a point $\tau \in E$ such that the space $R$ given by (3.2) has the following description (see [20] for example).

Let $\mathcal{L}$ be a line bundle of degree four on $E$, and let $V = H^0(E, \mathcal{L})$. Then $R$ consists of those sections of the line bundle $\mathcal{L} \boxtimes \mathcal{L}$ on $E \times E$ whose divisor of zeros is of the form

$$D + \{(x, x + 2\tau) \mid x \in E\},$$

where $D$ is a divisor invariant under the involution $(x, y) \mapsto (y + 2\tau, x - 2\tau)$ such that

$$\{(x, x - 2\tau)\}$$

(the fixed-point set of the involution) occurs in $D$ with even multiplicity.

The quotient $TV/(R)$ of the tensor algebra on $V$ modulo the ideal generated by $R$ is a characteristic-free construction of the four-dimensional Sklyanin algebras of [16, 17, 11] introduced by Sklyanin to study certain solutions to the Yang-Baxter equation. By [18], the algebras $TV/(R)$ form a flat family of deformations of the symmetric algebra $SV$, the polynomial ring on four variables, with the data $(E, \tau)$ acting as deformation parameters.

The algebras $TV/(R')$ can be obtained from the Sklyanin algebras $TV/(R)$ by means of a cocycle twist construction, so they too have a geometric construction involving an elliptic curve. We refer to [2, 3] for details and the description of $X_{R'}$.

4. **Motivation**

4.1. **Non-commutative algebraic geometry.** This paper grew out of our interest in non-commutative analogues of $\mathbb{P}^3$.

Let $V$ be a 4-dimensional vector space. Then $\mathbb{P}^3 = \text{Proj}(SV)$, the projective scheme whose homogeneous coordinate ring is the polynomial ring on four variables. In [12], Serre proved that a certain quotient of the category of graded $SV$-modules is equivalent to $\text{Qcoh}(\mathbb{P}^3)$, the category of quasi-coherent sheaves on $\mathbb{P}^3$.

Since $SV = TV/(\text{Alt})$ where $\text{Alt}$ is the subspace of $V^{\otimes 2}$ consisting of skew-symmetric tensors one might perform the same quotient category construction on other algebras $A = TV/(R)$ when $R$ is any 6-dimensional subspace $V^{\otimes 2}$ and, if $A$ is “good”, one might hope that this quotient category behaves “like” $\text{Qcoh}(\mathbb{P}^3)$. This hope is a reality in a surprising number of cases and has led to a rich subject that goes by the name of non-commutative algebraic geometry—see [19] and the references therein.

From now on we will assume that $A$ is “good” without saying what that means.

We will denote the appropriate quotient of the category of graded $A$-modules by $\text{QGr}(A)$. We will write $\text{Proj}_{nc}(A)$ for the “non-commutative analogue of $\mathbb{P}^3$” that has $A$ as its homogeneous coordinate ring. This is a fictional object that is made manifest by declaring the category of quasi-coherent sheaves on $\text{Proj}_{nc}(A)$ to be $\text{QGr}(A)$.

4.2. **Points, lines, etc.** The most elementary geometric features of $\mathbb{P}^3$ are points, lines, planes, quadrics, and their incidence relations. There are non-commutative analogues of these and a first investigation of $\text{Proj}_{nc}(A)$ involves finding its “points” and “lines” and the “incidence relations” among them.

A non-commutative ring usually has far fewer two-sided ideals than a commutative ring. Even when $A$ is “good” it has far fewer two sided ideals than $SV$. As a consequence $\text{Proj}_{nc}(A)$ usually has far fewer “points” and “lines” than does $\mathbb{P}^3$.

For example, $\text{Proj}_{nc}(A)$ can have as few as 20 “points”, which is the case for $TV/(R')$ when $R'$ is the space in §3.2 [2], or even just one point with multiplicity 20 [13]. The “points”
in $\text{Proj}_{nc}(A)$ form a scheme that is called the point scheme. When $A$ is “good”, that point scheme is a closed subscheme of $\mathbb{P}^3$. When the point scheme has dimension zero its degree is 20.

By [14, 15], there is also a line scheme that classifies the “lines” in $\text{Proj}_{nc}(A)$. By [14], the line scheme is isomorphic to $X_R \subseteq G(2, V)$ and always has dimension $\geq 1$. Thus, the main result in this paper says that if $A$ is “good” and the line scheme has dimension 1, then its degree is 20.

4.3. Rank-2 elements in $R$. As before, $R$ denotes a 6-dimensional subspace of $V \otimes V$.

The rank of an element $t \in V^{\otimes 2}$ is the smallest $n$ such that $t = v_1 \otimes w_1 + \cdots + v_n \otimes w_n$ for some $v_i, w_i \in V$. The elements in $V^{\otimes 2}$ of rank $r$ form a closed subvariety of $V^{\otimes 2}$. This subvariety is a union of 1-dimensional subspaces so the lines through 0 and the non-zero elements of rank $\leq r$ form a closed subvariety $T_{\leq r}$ of $\mathbb{P}(V^{\otimes 2})$. We write $T_r$ for $T_{\leq r} - T_{\leq r-1}$.

Define $Z_R$ to be the scheme-theoretic intersection

$$Z_R := T_2 \cap \mathbb{P}(R).$$

Define $Y_R$ to be the scheme-theoretic intersection

$$Y_R := S_R \cap G'(2, V)$$

in $G(8, V^{\otimes 2})$ where $G'(2, V)$ is the image of the closed immersion of $G(2, V)$ in $G(8, V^{\otimes 2})$ given by $Q \mapsto V \otimes Q$. Thus, $Y_R = X_{\sigma(R)}$ where $\sigma : V^{\otimes 2} \to V^{\otimes 2}$ is the linear map $\sigma(u \otimes v) = v \otimes u$.

If $t \in Z_R$ and $t = a \otimes b + c \otimes d$, then the 2-dimensional subspace of $V$ spanned by $a$ and $b$ depends only on $t$ and not on its representation as $a \otimes b + c \otimes d$. Thus, there is a well-defined morphism $\phi : Z_R \to G(2, V)$, $\phi(a \otimes b + c \otimes d) := (ka + kc)$. The image of $\phi$ is $X_R$. There is a similar morphism $Z_R \to G(2, V)$, $(a \otimes b + c \otimes d) \mapsto (kb + kd)$. Thus, we have morphisms

$$\xymatrix{ & Z_R \\
X_R \ar[ru] & Y_R \ar[lu]}
$$

When $TV/(R)$ is “good”, the morphism $\phi$ is an isomorphism: our $\phi$ is the same as the morphism $\phi$ in [14, Lemma 2.5] where it is proved to be an isomorphism (our notation differs from that in loc. cit.).

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