SLIM CYCLOTONOMIC $q$-SCHUR ALGEBRAS

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Abstract. We construct a new basis for a slim cyclotomic $q$-Schur algebra $S_m(n, r)$ via symmetric polynomials in Jucys–Murphy operators of the cyclotomic Hecke algebra $H_m(r)$. We show that this basis, labelled by matrices, is not the double coset basis when $H_m(r)$ is the Hecke algebra of a Coxeter group, but coincides with the double coset basis for the corresponding group algebra, the Hecke algebra at $q = 1$. As further applications, we then discuss the cyclotomic Schur–Weyl duality at the integral level. This also includes a category equivalence and a classification of simple objects.

1. Introduction

Cyclotomic $q$-Schur algebras were first introduced by Dipper, James and Mathas (DJM) [20] as a natural generalisation of Dipper and James' $q$-Schur algebra [19] associated with the symmetric groups $S_r$ to a similar algebra associated with the wreath products $\mathbb{Z}_m \wr S_r$ of the cyclic group $\mathbb{Z}_m$ with $S_r$. See also [29] for the $q$-Schur$^2$ (or $q$-Schur$^3$) algebra associated with the type $B$ Weyl group $\mathbb{Z}_2 \wr S_r$ and [37, 49, 13] for the affine $q$-Schur algebra associated with the affine symmetric group $\mathbb{Z} \wr S_r$. Like the $q$-Schur algebras, the cyclotomic ones are quasi-hereditary with a cellular basis; see [20, (6.12), (6.18)].

As an endomorphism algebra of the direct sum of certain cyclic modules over the cyclotomic Hecke algebra [4], the construction of a cyclotomic $q$-Schur algebra involves all $m$-fold multipartitions of $r$, which index all irreducible characters of the finite group $\mathbb{Z}_m \wr S_r$. However, when the "cyclotomic Schur–Weyl duality" is under consideration, these algebras seem a bit too "fat". Thus, as suggested in [42], a certain centraliser algebra, involving only partitions of $r$, has been considered to play such a role. We will call these centraliser algebras slim cyclotomic $q$-Schur algebras in this paper.

A further study on slim cyclotomic $q$-Schur algebras is motivated by the following recent works: First, the investigation on affine $q$-Schur algebras and affine Schur–Weyl duality has made significant progress. This includes a full generalisation to the affine case [13, 24] of the original work [8] of Beilinson–Lusztig–MacPherson (BLM) and a classification of irreducible representations of affine $q$-Schur algebras. These works should have applications to slim cyclotomic $q$-Schur algebras. Second, there has been also a breakthrough in generalising the BLM work to types other than $A$ series by Bao–Kujawa–Li–Wang [7] for type $B/C$; see also [33] for type $D$ and [34] for
affine type $C$. The $q$-Schur algebra $S^i$ defined in \cite{7} \S 6.1 is used in linking the Hecke algebra of type $C$ with a certain quantum symmetric pair, where a coideal subalgebra is constructed to play the role required in the Schur–Weyl duality in this case. The algebra $S^i$ is the $m = 2$ slim cyclotomic $q$-Schur algebra. This algebra is also called the hyperoctahedral Schur algebra in \cite{37} or the $q$-Schur$^{1b}$ algebra in \cite{30}.

As a centraliser algebra of DJM’s cyclotomic $q$-Schur algebra, a slim cyclotomic $q$-Schur algebra inherits a cellular basis that does not provide enough information for the representations of the algebra. However, the most natural basis, analogous to the well-known orbital basis for the endomorphism algebra of a permutation module (see \cite{46}), is not available if the complex reflection groups $\mathbb{Z}_m \wr \mathfrak{S}_r$ is not a Coxeter group. In this paper, we will reveal a new integral basis for a slim cyclotomic $q$-Schur algebra. This basis is different from the usual double coset basis for the Hecke endomorphism algebra associated with a Coxeter group. However, it still reduces to a double coset basis for the group algebra. So we may regard our new basis as a new quantisation of the double coset basis, generalising the natural basis for the endomorphism algebra of a permutation module considered in \cite{46} to this cyclotomic world.

It is interesting to point out that the definition of this new basis involves symmetric polynomials in Jucys–Murphy operators and can be lifted to affine $q$-Schur algebras. In this way, we obtain an algebra epimorphism from an affine $q$-Schur algebra to a slim cyclotomic $q$-Schur algebra of the same degree. Thus, an epimorphism from the integral quantum loop algebra of $\mathfrak{gl}_n$ to a slim cyclotomic $q$-Schur algebra can be constructed. Thus, we may put slim cyclotomic $q$-Schur algebras in the context of a possible cyclotomic Schur–Weyl duality. In fact, we propose a candidate for a possible “finite type” quantum subalgebra which can be used to replace $U(\hat{\mathfrak{gl}_n})$ in the cyclotomic Schur–Weyl duality. We will also establish a Morita equivalence in the cyclotomic case and present a classification of irreducible modules for a slim cyclotomic $q$-Schur algebra. It should be noted that the Schur–Weyl duality for cyclotomic Hecke algebras has been investigated by many authors; see, for example, \cite{6, 45, 38}. However, the tensor spaces considered in these works are different from those used in the current paper.

In a forthcoming paper \cite{16}, we will determine which slim cyclotomic $q$-Schur algebras are quasi-hereditary and further investigate the representations of such an algebra at roots of unity.

We organize the paper as follows. In Section 2, we give a brief background on cyclotomic Hecke algebras and slim cyclotomic $q$-Schur algebras. In Section 3, we will see how the symmetric polynomials in Jucys–Murphy operators are used in the construction of an integral basis for $S_m(1, r)$, which is a centraliser algebra of $S_m(n, r)$ associated with the partition $(r)$, and prove that it is commutative. In Section 4, we generalise the construction to the entire algebra $S_m(n, r)$ by employing Mak’s double coset description in term of matrices \cite{43}. In Section 5, we will make a comparison of our new basis with the usual double coset basis for the Coxeter group, $\mathfrak{S}_{2, r}$ of type $B$. Sections 6 and 7 are devoted to the cyclotomic Schur–Weyl duality. Partial double centraliser property is established through affine $q$-Schur algebras and quantum loop algebra of $\mathfrak{gl}_n$. A certain category equivalence has also been lifted to the cyclotomic case. Finally, as a further application of our discovery, we classify all simple objects for the slim cyclotomic $q$-Schur algebras as an application of a similar classification.
for affine \( q \)-Schur algebras. The two appendices cover some technical proofs and a generalisation of Lemma 3.2 to the affine case.

2. Ariki–Koike algebras and slim cyclotomic \( q \)-Schur algebras

For any nonnegative integer \( m \geq 0 \), let \( \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z} \). We always identify \( \mathbb{Z}_m \) with the subset \( \mathbb{Z}_m = \{0, 1, \ldots, m-1\} \) of \( \mathbb{Z} \) if \( m \geq 1 \) (and, of course, \( \mathbb{Z}_0 = \mathbb{Z} \)). Thus, \[
\mathbb{Z}^r_m = \{ a = (a_i) \in \mathbb{Z}^r \mid a_i \in \mathbb{Z}_m, \forall 1 \leq i \leq r \}
\]
is a subset of \( \mathbb{Z}^r \).

Let \( \mathfrak{S}_r = \mathfrak{S}_{\{1,2,\ldots,r\}} \) be the symmetric group on \( r \) letters with the basic transpositions \( s_i = (i, i+1), 1 \leq i \leq r-1 \), as generators, and let \( \mathfrak{S}_{m,r} = \mathbb{Z}_m \wr \mathfrak{S}_r \) be the wreath product of the cyclic group \( \mathbb{Z}_m \) and \( \mathfrak{S}_r \). Then \( \mathfrak{S}_{0,r} = \mathfrak{S}_{\lambda,r} \) is the affine symmetry group, \( \mathfrak{S}_{1,r} = \mathfrak{S}_r \), \( \mathfrak{S}_{2,r} \) is the Weyl group of type \( B \), and \( \mathfrak{S}_{m,r} \) for \( m \geq 3 \) are the complex reflection groups of type \( G(m, 1, r) \).

From now onwards, \( \mathcal{R} \) always denotes a commutative ring (with 1) containing invertible element \( q \).

For \( m \geq 1 \) and an \( m \)-tuple \( \mathbf{m} = (u_1, \ldots, u_m) \in \mathcal{R}^m \), the \textit{cyclotomic Hecke algebra} (or the Ariki–Koike algebra) over \( \mathcal{R} \)

\[
\mathcal{H}_m(r) = \mathcal{H}_m(r)_{\mathcal{R}},
\]
associated with \( \mathfrak{S}_{m,r} \) (and parameters \( q, \mathbf{m} \)), is by definition the \( \mathcal{R} \)-algebra generated by \( T_i, L_j \) (\( 1 \leq i \leq r-1, 1 \leq j \leq r \)) with defining relations

\begin{align*}
(\text{CH1}) \quad & (T_i + 1)(T_i - q) = 0, \\
(\text{CH2}) \quad & T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i \quad (|i - j| > 1), \\
(\text{CH3}) \quad & L_iL_j = L_jL_i, \quad (L_1 - u_1) \cdots (L_1 - u_m) = 0, \\
(\text{CH4}) \quad & T_iL_iT_i = qL_{i+1}, \quad L_jT_i = T_jL_i \quad (j \neq i, i + 1).
\end{align*}

For \( m = 0 \), the affine Hecke algebra

\[
\mathcal{H}_0(r) = \mathcal{H}_0(r)_{\mathcal{R}},
\]
associated with \( \mathfrak{S}_{\lambda,r} \) (and parameter \( q \)), is presented by the generators \( T_i, X_j^\pm \) for \( 1 \leq i \leq r - 1 \) and \( 1 \leq j \leq r \) and relations (CH1), (CH2) together with

\begin{align*}
(\text{CH3}') \quad & X_iX_j = X_jX_i, \quad X_iX_i^{-1} = 1 = X_i^{-1}X_i, \\
(\text{CH4}') \quad & T_iX_iT_i = qX_{i+1}, \quad X_jT_i = T_jX_j \quad (j \neq i, i + 1).
\end{align*}

In both cases, the subalgebra generated by \( T_1, \ldots, T_{r-1} \) is the Hecke algebra \( \mathcal{H}(r) = \mathcal{H}(r)_{\mathcal{R}} \) of \( \mathfrak{S}_r \). If all \( u_i \in \mathcal{R} \) are invertible, then the \( L_i \) are invertible in \( \mathcal{H}_m(r) \). Thus, in this case, there is a natural surjective homomorphism\(^1\)

\[
\epsilon_\mathbf{m} : \mathcal{H}_0(r) \longrightarrow \mathcal{H}_m(r), \quad T_i \longmapsto T_i, \quad X_j \longmapsto L_j,
\]
which gives an algebra isomorphism

\[
\mathcal{H}_0(r)/\langle \langle X_1 - u_1 \rangle \cdots (X_1 - u_m) \rangle \cong \mathcal{H}_m(r).
\]

Thus, the algebra \( \mathcal{H}_m(r) \) is a cyclotomic quotient of the affine Hecke algebra \( \mathcal{H}_0(r) \) when all the parameters \( u_i \) are invertible.

\(^1\)When \( m = 1 \), \( \epsilon_\mathbf{m} \) is known as the evaluation map.
The algebra $\mathcal{H}_m(r)$ admits an anti-automorphism
\[ \tau: \mathcal{H}_m(r) \rightarrow \mathcal{H}_m(r), \quad T_i \mapsto T_i, \quad L_j \mapsto L_j. \tag{2.0.2} \]
Since $q$ is invertible in $\mathcal{R}$, all the $T_i$ for $1 \leq i \leq r - 1$ are invertible with
\[ T_i^{-1} = q^{-1}(T_i - (q - 1)). \]
By multiplying $T_i^{-1}$ and $T_i$ on both sides of $T_iL_iT_i = qL_i+1$, respectively, we get the following two equalities
\[ L_iT_i = T_iL_i+1 + (1 - q)L_i+1, \quad L_i+1T_i = T_iL_i + (q - 1)L_i+1. \tag{2.0.3} \]
For each $a = (a_1, \ldots, a_r) \in \mathbb{N}^r$, write
\[ L^a = L_1^{a_1} \cdots L_r^{a_r} \in \mathcal{H}_m(r), \quad X^a = X_1^{a_1} \cdots X_r^{a_r} \in \mathcal{H}_r(r). \]
Then, by an inductive argument, we have for each $1 \leq i \leq r - 1$ and $a \in \mathbb{N}^r$,
\[ L^aT_i = T_iL^{as_i} + (1 - q)\frac{L_i+1(L^a - L^{as_i})}{L_i - L_i+1}, \tag{2.0.4} \]
where $as_i$ is given by permuting the $i$-th and $(i + 1)$-th entries of $a$. The affine version of the formula (with $L^a$ replaced by $X^a$) is well known; see, e.g., [33 (3.3.0.2)].

For each $w \in \mathfrak{S}_r$ and $a \in \mathbb{Z}^r$, define the place permutation
\[ aw = (a_{w(1)}, a_{w(2)}, \cdots, a_{w(r)}). \tag{2.0.5} \]
For each $1 \leq i \leq r - 1$, set
\[ \alpha_i = (0, \ldots, 1, -1, 0, \ldots, 0) \in \mathbb{Z}^r \]
with 1 in the $i$-th position. Thus, $as_i = a + (a_{i+1} - a_i)\alpha_i$. Applying (2.0.4) gives the following lemma.

**Lemma 2.1.** Suppose that $1 \leq i \leq r - 1$ and $a \in \mathbb{N}^r$. Then in $\mathcal{H}_m(r)$
\[ L^aT_i = \begin{cases} T_iL^a = T_iL^{as_i}, & \text{if } a_i = a_{i+1}; \\ T_iL^{as_i} + (q - 1)\sum_{t=1}^{a_{i+1} - a_i} L^{as_i - t\alpha_i}, & \text{if } a_i < a_{i+1}; \\ T_iL^{as_i} + (1 - q)\sum_{t=0}^{a_i - a_{i+1} - 1} L^{as_i + t\alpha_i}, & \text{if } a_i > a_{i+1}. \end{cases} \]
A similar formula holds in $\mathcal{H}_r(r)$.

The following result is taken from [11, Thm 3.10].

**Lemma 2.2.** The algebra $\mathcal{H}_m(r)$ is a free $\mathcal{R}$-module with bases
\[ \{T_wL^a \mid w \in \mathfrak{S}_r, a \in \mathbb{Z}_m^r\} \quad \text{and} \quad \{L^aT_w \mid w \in \mathfrak{S}_r, a \in \mathbb{Z}_m^r\}. \]
Replacing $\mathbb{Z}_m^r$ by $\mathbb{Z}^r$ and $L^a$ by $X^a$ yields bases for $\mathcal{H}_r(r)$.

A sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of non-negative integers is called a composition (resp., partition) of $r$, if $|\lambda| = \sum \lambda_i = r$ (resp., if, in addition, $\lambda_1 \geq \lambda_2 \geq \ldots$). For a composition $\lambda$ of $r$, let $\mathfrak{S}_\lambda$ be the corresponding standard Young (parabolic) subgroup of $\mathfrak{S}_r$. Let $\mathfrak{S}_\lambda^{-1}$ be the set of minimal length right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_r$. Then $\mathfrak{S}_\lambda^{-1} = \{d^{-1} \mid d \in \mathfrak{S}_\lambda\}$ is the set of minimal length left coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_r$.

Set $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w \in \mathcal{H}(r) \in \mathcal{H}_m(r)$. Then (see, e.g., [14, Lem. 7.32])
\[ x_\lambda T_i = T_i x_\lambda = q x_\lambda, \quad \forall i \in J_\lambda, \tag{2.2.1} \]
where
\[ J_\lambda = \{1 \leq i \leq r - 1 \mid i \neq \sum_{j=1}^s \lambda_j; \forall 1 \leq s < n \}. \] (2.2.2)

We may use a similar condition to characterise the “permutation modules” \( x_\lambda \mathcal{H}_m(r) \) and \( \mathcal{H}_m(r) x_\lambda \).

**Lemma 2.3.** Let \( \lambda \in \Lambda(n,r) \). We have
\[
x_\lambda \mathcal{H}_m(r) = \{ h \in \mathcal{H}_m(r) \mid T_i h = q h, \forall i \in J_\lambda \},
\]
\[
\mathcal{H}_m(r) x_\lambda = \{ h \in \mathcal{H}_m(r) \mid h T_i = q h, \forall i \in J_\lambda \}.
\]
The two modules are \( \mathcal{R} \)-free with respective bases \( \{ x_\lambda T_d L^a \mid d \in \mathcal{D}_\lambda, \ a \in \mathbb{Z}_m^r \} \) and \( \{ L^a T_d x_\lambda \mid d \in \mathcal{D}_\lambda^{-1}, \ a \in \mathbb{Z}_m^r \} \). Moreover, we have an \( \mathcal{R} \)-module isomorphism
\[
\text{Hom}_{\mathcal{H}_m(r)}(x_\mu \mathcal{H}_m(r), x_\lambda \mathcal{H}_m(r)) \cong x_\lambda \mathcal{H}_m(r) \cap \mathcal{H}_m(r) x_\mu.
\]

**Proof.** By Lemma 2.2, \( \{ T_w L^a \mid w \in \mathcal{G}_r, \ a \in \mathbb{Z}_m^r \} \) is an \( \mathcal{R} \)-basis of \( \mathcal{H}_m(r) \). Suppose \( h = \sum w, a T_w L^a \in \mathcal{H}_m(r) \) and \( T_i h = q h, \forall i \in J_\lambda \). Equating gives \( T_i \sum w c_{w,a} T_w = q \sum w c_{w,a} T_w \) for all \( i \in J_\lambda, \ a \in \mathbb{Z}_m^r \). Now our assertion follows from the fact that \( x_\lambda \mathcal{H}(r) = \{ h \in \mathcal{H}(r) \mid T_i h = q h, \forall i \in J_\lambda \} \) (see [13] or [14 Lem. 7.33]). The basis assertion follows from the fact that \( \{ x_\lambda T_d \mid d \in \mathcal{D}_\lambda \} \) is an \( \mathcal{R} \)-basis of \( x_\lambda \mathcal{H}(r) \). Finally, if \( f \in \text{Hom}_{\mathcal{H}_m(r)}(x_\mu \mathcal{H}_m(r), x_\lambda \mathcal{H}_m(r)) \), then \( f(x_\mu) T_i = q f(x_\mu) \) for all \( i \in J_\lambda \). By the first assertion, \( f(x_\mu) \in \mathcal{H}_m(r) x_\mu \) and so \( f(x_\mu) = h f x_\mu \) for some \( h f \in \mathcal{H}_m(r) \). Hence, the right module homomorphism \( f \) is completely defined by the left multiplication by \( h_f \).

We now consider the set of all compositions of \( r \) into \( n \) parts:
\[
\Lambda(n,r) = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \sum_{1 \leq i \leq n} \lambda_i = r \}.
\]
For \( \bullet \in \{ \mathfrak{m}, \Delta \} \), let \( \mathcal{T}_\bullet(n,r) = \bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_\bullet(r) \mathcal{R} \). The endomorphism algebra
\[
\mathcal{S}_\mathfrak{m}(n,r) = \mathcal{S}_\mathfrak{m}(n,r) \mathcal{R} = \text{End}_{\mathcal{H}_m(r)}(\mathcal{T}_\mathfrak{m}(n,r))
\]
resp., \( \mathcal{S}_\Delta(n,r) = \mathcal{S}_\Delta(n,r) \mathcal{R} = \text{End}_{\mathcal{H}_\Delta(r)}(\mathcal{T}_\Delta(n,r)) \)
is called the **slim cyclotomic q-Schur algebra**, resp., the **affine q-Schur algebra**.

Note that \( \mathcal{S}_\mathfrak{m}(n,r) \) is indeed a centraliser algebra of the cyclotomic q-Schur algebra introduced in [20], which is called the q-Schur\(2^m\) algebra in [29] when \( \mathfrak{m} = (-1, q_0) \). Note also that the algebra \( \mathcal{S}_\mathfrak{m}(n,r) \) has been studied in [12]. See also [37, 29] and more recently in [7 §5.1] for the \( m = 2 \) case.

For each \( \mu \in \Lambda(n,r) \), let \( I_\mu \) denote the following composition of maps
\[
I_\mu : \bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_m(r) \xrightarrow{\pi} x_\mu \mathcal{H}_m(r) \xrightarrow{id} x_\mu \mathcal{H}_m(r) \xrightarrow{\iota} \bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_m(r) \quad (2.3.1)
\]
where \( \pi, \iota \) are the canonical projection and inclusion, respectively. This is an idempotent \( I_\mu^2 = I_\mu \) which defines a **centraliser algebra** of \( \mathcal{S}_\mathfrak{m}(n,r) \):
\[
I_\mu \mathcal{S}_\mathfrak{m}(n,r) I_\mu \cong \text{End}_{\mathcal{H}_m(r)}(x_\mu \mathcal{H}_m(r)). \quad (2.3.2)
\]

The next two sections are devoted to constructing explicitly an \( \mathcal{R} \)-basis of \( \mathcal{S}_\mathfrak{m}(n,r) \).
3. The commutative algebra $S_m(1, r)$

In this section we deal with the special composition $\lambda = (r, 0, \ldots, 0) \in \Lambda(n, r)$. Our aim is to construct an $R$-basis of $x_\lambda H_m(r) \cap H_m(r)x_\lambda$ and to prove that the centraliser algebra $I_\lambda S_m(n, r)I_\lambda \cong S_m(1, r)$ is commutative. For simplicity, we write

$$x(r) := x_\lambda = \sum_{w \in S_r} T_w$$

and $I(r) := I_\lambda \in S_m(n, r)$.

First of all, we have by (2.2.1) that

$$T_ix(r) = x(r)T_i = qx(r), \quad \forall 1 \leq i \leq r - 1.$$

This together with Lemma 2.2 implies the following.

Lemma 3.1. (1) The $H_m(r)$-modules $x(r)H_m(r)$ and $H_m(r)x(r)$ are $R$-free with, respectively, bases $\{x(r)L^a \mid a \in \mathbb{Z}^r_m\}$ and $\{L^a x(r) \mid a \in \mathbb{Z}^r_m\}$.

(2) Likewise, the $\mathcal{H}_d(r)$-modules $x(r)\mathcal{H}_d(r)$ and $\mathcal{H}_d(r)x(r)$ are $R$-free with, respectively, bases $\{x(r)X^a \mid a \in \mathbb{Z}^r\}$ and $\{X^a x(r) \mid a \in \mathbb{Z}^r\}$.

The following result provides a description of elements in $x(r)H_m(r) \cap H_m(r)x(r)$.

Lemma 3.2. Suppose that

$$z = \sum_{b \in \mathbb{Z}^r_m} c_b x(r)L^b \in x(r)H_m(r) \cap H_m(r)x(r), \quad \text{where } c_b \in R.$$

Then $c_b = c_{bw}$ for all $b \in \mathbb{Z}^r_m$ and $w \in S_r$.

Proof. It suffices to prove that $c_a = c_{a_i}$ for each fixed $a \in \mathbb{Z}^r_m$ and $1 \leq i \leq r$. Write $a = (a_1, \ldots, a_r)$. If $a_i = a_{i+1}$, then $a = a_is_i$ and hence, $c_a = c_{a_is_i}$, as desired. Now let $a_i \neq a_{i+1}$. We may suppose $a_i < a_{i+1}$ (Otherwise, we replace $a$ by $a_{si}$).

By Lemma 2.2, we obtain that

$$zT_i = \sum_{b \in \mathbb{Z}^r_m} c_b x(r)L^b T_i = \sum_{b \in \mathbb{Z}^r_m} c_b x(r)T_i L^{bs_i} + \sum_{b \in \mathbb{Z}^r_m} (q - 1)c_b x(r) \sum_{t=1}^{b_{i+1}-b_i} L^{b_{s_i}-t\alpha_i}$$

$$+ \sum_{b \in \mathbb{Z}^r_m} (1 - q)c_b x(r) \sum_{t=0}^{b_{i+1}-b_i-1} L^{bs_i+t\alpha_i}.$$

Since $z \in H_m(r)x(r)$ and $x(r)T_i = qx(r)$, it follows that $qz = zT_i$ which gives rise to the equality

$$\sum_{b \in \mathbb{Z}^r_m} qc_b x(r)L^b = \sum_{b \in \mathbb{Z}^r_m} qc_b x(r)L^{bs_i} + \sum_{b \in \mathbb{Z}^r_m} (q - 1)c_b x(r) \sum_{t=1}^{b_{i+1}-b_i} L^{b_{s_i}-t\alpha_i}$$

$$+ \sum_{b \in \mathbb{Z}^r_m} (1 - q)c_b x(r) \sum_{t=0}^{b_{i+1}-b_i-1} L^{bs_i+t\alpha_i}.$$
For the fixed $a \in \mathbb{Z}_m^r$, comparing the coefficients of $x(r)L^a$ on both sides implies that
\[ qc_a = qc_{as_i} + c' + c'', \quad (3.2.1) \]
where $c'$ (resp., $c''$) denotes the coefficient of $x(r)L^a$ in the second (resp., third) sum of the right hand side. In the following we calculate $c'$ and $c''$.

Consider those $b = (b_1, \ldots, b_r) \in \mathbb{Z}_m^r$ with $b_i < b_{i+1}$ such that
\[ bs_i - ta_i = a \text{ for some } 1 \leq t \leq b_{i+1} - b_i. \]
With $a$ fixed, it is equivalent to determine the precise range of $t$ for which $b = as_i-ta_i$. Since $b_i = a_{i+1} - t$ and $b_{i+1} = a_i + t$, it follows that $t < b_{i+1} - b_i = a_i - a_{i+1} + 2t$. Hence,
\[ a_{i+1} - a_i \leq t. \]
On the other hand, since $0 \leq b_i = a_{i+1} - t \leq m - 1$ and $0 \leq b_{i+1} = a_i + t \leq m - 1$, it follows that $t < a_{i+1}$ and $t \leq m - 1 - a_i$. Consequently,
\[ a_{i+1} - a_i \leq t \leq \min\{a_{i+1}, m - 1 - a_i\}. \]
Since $as_i - ta_i = a - (t - (a_{i+1} - a_i))a_i$, by substituting $t$ for $t - (a_{i+1} - a_i)$, we obtain that $b = a - ta_i$ with
\[ 0 \leq t \leq \min\{a_i, m - 1 - a_{i+1}\}. \]
Similarly, those $b = (b_1, \ldots, b_r) \in \mathbb{Z}_m^r$ with $b_i > b_{i+1}$ such that
\[ bs_i + ta_i = a \text{ for some } 0 \leq t \leq b_i - b_{i+1} - 1 \]
are simply $b = as_i + ta_i$ for all
\[ 0 \leq t \leq \min\{a_i, m - 1 - a_{i+1}\}. \]

We treat the following two cases.

**Case 1:** $a_i \leq m - 1 - a_{i+1}$. Then
\[ c' = (q - 1) \sum_{t=0}^{a_i} c_{a-ta_i} \quad \text{and} \quad c'' = (1 - q) \sum_{t=0}^{a_i} c_{as_i+ta_i}. \]
Hence, we obtain from (3.2.1) that
\[ qc_a = qc_{as_i} + (q - 1) \sum_{t=0}^{a_i} c_{a-ta_i} + (1 - q) \sum_{t=0}^{a_i} c_{as_i+ta_i} = qc_{as_i} + (q - 1)\left(c_a + \sum_{t=1}^{a_i} c_{a-ta_i}\right) + (1 - q)\left(c_{as_i} + \sum_{t=1}^{a_i} c_{as_i+ta_i}\right). \]
This together with the fact that $as_i + ta_i = (a - ta_i)s_i$ implies that
\[ c_a - c_{as_i} = \sum_{t=1}^{a_i} (q - 1)\left(c_{a-ta_i} - c_{(a-ta_i)s_i}\right). \]
If $a_i = 0$, then $c_a = c_{as_i}$. Let $1 \leq k \leq m - 1$ and suppose $c_a = c_{as_i}$ in case $a_i < k$. Now consider the case $a_i = k$. Since for each $1 \leq t \leq a_i$, $a - ta_i = (b^{(t)}_1, \ldots, b^{(t)}_r)$ satisfies $b^{(t)}_i = a_i - t < a_i = k$,
\[ b^{(t)}_i = a_i - t < a_{i+1} + t = b^{(t)}_{i+1} \quad \text{and} \quad b^{(t)}_i = a_i - t \leq m - 1 - a_{i+1} - t = m - 1 - b^{(t)}_{i+1}, \]
we have $c_{a-ta_i} = c_{(a-ta_i)s_i}$, $\forall 1 \leq t \leq a_i$. Hence, $c_a = c_{as_i}$. 
**Case 2:** $a_i > m - 1 - a_{i+1}$. In this case,

$$c' = (q - 1) \sum_{t=0}^{m-1-a_{i+1}} c_{a-ta_i} \quad \text{and} \quad c'' = (1 - q) \sum_{t=0}^{m-1-a_{i+1}} c_{as_i+ta_i}.$$ 

From (3.2.1) it follows that

$$qc_a = qc_{as_i} + (q - 1) \sum_{t=0}^{m-1-a_{i+1}} c_{a-ta_i} + (1 - q) \sum_{t=0}^{m-1-a_{i+1}} c_{as_i+ta_i}.$$ 

Using an argument similar to that in Case 1, we obtain that

$$c_a - c_{as_i} = \sum_{i=1}^{m-1-a_{i+1}} (q - 1)(c_{a-ta_i} - c_{a-ta_is_i}).$$ 

Finally, proceeding by induction on $m - 1 - a_{i+1}$ gives the equality $c_a = c_{as_i}$. □

**Remark 3.3.** In Lemma B.1 in the appendix, we will modify the proof to get the affine version of the lemma.

Let $\mathcal{R}[L_1, \ldots, L_r]_m$ be the free $\mathcal{R}$-submodule of $\mathcal{H}_m(r)$ with basis $\{L^a \mid a \in \mathbb{Z}_m^r\}$. Then the symmetric group $\mathfrak{S}_r$ acts on $\mathcal{R}[L_1, \ldots, L_r]_m$ defined by

$$w \cdot L^a = L^{aw^{-1}}, \quad \forall w \in \mathfrak{S}_r, \ a \in \mathbb{Z}_m^r.$$ 

By $\mathcal{R}[L_1, \ldots, L_r]_{m}^{\mathfrak{S}_r}$ we denote the set of fixed elements in $\mathcal{R}[L_1, \ldots, L_r]_m$.

Consider the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_r$ in $L_1, \ldots, L_r$, i.e.,

$$\sigma_i = \sigma_i(L_1, \ldots, L_r) = \sum_{1 \leq j_1 < \cdots < j_i \leq r} L_{j_1} \cdots L_{j_i}, \quad \forall 1 \leq i \leq r.$$ 

It is clear that $\sigma_1, \ldots, \sigma_r \in \mathcal{R}[L_1, \ldots, L_r]_{m}^{\mathfrak{S}_r}$.

Set

$$\Lambda(r, <m) = \bigcup_{k=1}^{m-1} \Lambda(r, k) = \{a = (a_1, \ldots, a_r) \in \mathbb{Z}_m^r \mid \sum_{i=1}^{r} a_i \leq m - 1\}. \quad (3.3.1)$$ 

Then for each $a = (a_1, \ldots, a_r) \in \Lambda(r, <m)$, the monomial

$$\sigma^a = \sigma^a(L_1, \ldots, L_r) = \sigma_1^{a_1} \cdots \sigma_r^{a_r}$$ 

lies in $\mathcal{R}[L_1, \ldots, L_r]_{m}^{\mathfrak{S}_r}$. Indeed, we have the following result.

**Lemma 3.4.** The set of fixed elements $\mathcal{R}[L_1, \ldots, L_r]_{m}^{\mathfrak{S}_r}$ is a free $\mathcal{R}$-module with basis

$$\{\sigma^a \mid a \in \Lambda(r, <m)\}.$$ 

Moreover, for each $1 \leq i \leq r - 1$ and $f \in \mathcal{R}[L_1, \ldots, L_r]_{m}^{\mathfrak{S}_r}$,

$$T_i f = f T_i.$$ 

**Proof.** By a standard method on symmetric polynomials (see [40 I, §2.13]), each $f \in \mathcal{R}[L_1, \ldots, L_r]_{m}^{\mathfrak{S}_r}$ can be written as an $\mathcal{R}$-linear combination of monomials

$$\sigma_1^{t_1} \sigma_2^{t_2} \sigma_3^{t_3} \cdots \sigma_{r-1}^{t_{r-1}} \sigma_r^{t_r},$$ 

where $t_1, \ldots, t_r \in \mathbb{N}$ satisfy

$$m - 1 \geq t_1 \geq t_2 \geq \cdots \geq t_r \geq 0.$$
Therefore, $R[L_1, \ldots, L_r]_{\mathfrak{m}}^{\mathfrak{e}}$ is spanned by the set $\{\sigma^a \mid a \in \Lambda(r, < m)\}$.

For each $a \in \Lambda(r, < m)$, define $\widehat{a} = (\widehat{a}_1, \ldots, \widehat{a}_r) \in \mathbb{Z}_m^r$ by setting

$$\widehat{a}_i = \sum_{j \in r} a_j, \quad \forall 1 \leq i \leq r.$$ 

Then the leading term of $\sigma^a$ as a polynomial in $L_1, \ldots, L_r$ is $L^\widehat{a}$ and, moreover, for $a, b \in \Lambda(r, < m)$,

$$a = b \iff \widehat{a} = \widehat{b}.$$ 

Consequently, the linear independence of the set $\{\sigma^a \mid a \in \Lambda(r, < m)\}$ follows from that of $\{L^a \mid a \in \mathbb{Z}_m^r\}$.

Let $1 \leq i, j \leq r$ with $i \neq r$. By summing up both sides of the formula (2.0.4) over $L^a = L_{k_1} \cdots L_{k_j}$ with $1 \leq k_1 < \cdots < k_j \leq r$, we obtain that $T_i \sigma_j = \sigma_j T_i$. Thus, the second assertion follows.

**Proposition 3.5.** As an $R$-module, $x_r(\mathcal{H}_m(r) \cap \mathcal{H}_m(r)x(r)$ is free with basis

$$B_{r,s} = B_{r,s}^{(m)} = \{x_r(\sigma^a = \sigma^a x(r) \mid a \in \Lambda(r, < m)\}.$$ 

**Proof.** By Lemma 3.4, each element in $x_r(\mathcal{H}_m(r) \cap \mathcal{H}_m(r)x(r)$ has the form $x_r f$ for some $f \in R[L_1, \ldots, L_r]_{\mathfrak{m}}^{\mathfrak{e}}$. This together with Lemma 3.4 implies that $x_r(\mathcal{H}_m(r) \cap \mathcal{H}_m(r)x(r)$ is spanned by the set $B_{r,s}$. Finally, the linear independence of $B_{r,s}$ follows from Lemma 3.4 and the fact that $\{L^a \mid a \in \mathbb{Z}_m^r\}$ is an $R$-basis of $\mathcal{H}_m(r)$.

Combining the results above gives the following corollary.

**Corollary 3.6.** The centraliser algebra $S_m(1, r) = I_r S_m(n, r)I_r$ is a commutative $R$-algebra. 

**Proof.** By (2.3.2), there is an algebra isomorphism

$$I_r S_m(n, r)I_r \cong \text{End}_{\mathcal{H}_m(r)}(x_r \mathcal{H}_m(r)) = S_m(1, r).$$ 

The latter as an $R$-module is isomorphic to $x_r(\mathcal{H}_m(r) \cap \mathcal{H}_m(r)x(r)$, by Lemma 2.3. For each $a \in \Lambda(r, < m)$, if we define a right $\mathcal{H}_m(r)$-module homomorphism

$$\Phi_a : x_r(\mathcal{H}_m(r) \longrightarrow x_r(\mathcal{H}_m(r), \quad x_r h \mapsto \sigma^a x(r) h, \quad \forall h \in \mathcal{H}_m(r),$$ 

then the proposition above implies that $\{\Phi_a | a \in \Lambda(r, < m)\}$ forms an $R$-basis for $\text{End}_{\mathcal{H}_m(r)}(x_r \mathcal{H}_m(r))$. Finally, it follows from Lemma 3.4 that $\Phi_a \circ \Phi_b = \Phi_b \circ \Phi_a$ for all $a, b \in \Lambda(r, < m)$. Hence, $S_m(1, r)$ is commutative, as desired.

For $t \in \mathbb{N}$ and $\nu = (\nu_1, \ldots, \nu_t) \in \Lambda(t, r)$, let

$$\Lambda(\nu, < m) = \Lambda(\nu_1, < m) \times \cdots \times \Lambda(\nu_t, < m)$$

$$= \{(a^{(1)}, \ldots, a^{(t)}) | a^{(i)} \in \Lambda(\nu_i, < m), \quad \forall 1 \leq i \leq t\}.$$ 

(3.6.1)

where $\Lambda(\nu_i, < m) = \{b \in \mathbb{N}^{\nu_i} | |b| < m\}$ as defined in (3.3.1).

**Definition 3.7.** For each $\nu = (\nu_1, \ldots, \nu_t) \in \Lambda(t, r)$ and each $1 \leq i \leq t$, let $\sigma^{(i)}_1, \ldots, \sigma^{(i)}_{\nu_i}$ denote the elementary symmetric polynomials in

$$L_{\nu_1 + \cdots + \nu_i - 1 + 1}, \ldots, L_{\nu_1 + \cdots + \nu_i - 1 + \nu_i}.$$
For every \( a(\nu) = (a^{(1)}, \ldots, a^{(t)}) \in \Lambda(\nu, < m) \) with \( a^{(i)} = (a_{i1}, \ldots, a_{it}) \), define
\[
\sigma^a(\nu) := (\sigma_1^{a_1})^{(1)} \cdots (\sigma_t^{a_t})^{(t)}
\]

Proposition 3.8. \( \text{next section. Recall the set} \)

For example, let \( m = 3 \), \( t = 4 \) and \( r = 7 \), and take \( \nu = (1, 2, 1, 3) \). Then, for \( a(\nu) = ((1), (1, 1), (1, 0, 1)), b(\nu) = ((1), (1, 0), (0), (0, 2, 0)) \in \Lambda(\nu, < 3) \), we have
\[
\sigma^a(\nu) = L_1 \cdot ((L_2 + L_3)L_2L_3) \cdot L_4 \cdot ((L_5 + L_6 + L_7)L_5L_6L_7)
\]
and
\[
\sigma^b(\nu) = L_1 \cdot (L_2 + L_3) \cdot (L_5L_6 + L_5L_7 + L_6L_7)^2.
\]

Let \( \nu = (\nu_1, \ldots, \nu_t) \in \Lambda(t, r) \) for some \( t \in \mathbb{N} \). The action of \( \mathfrak{S}_r \) on \( \mathcal{R}[L_1, \ldots, L_r] m \) restricts to an action of its subgroup \( \mathfrak{S}_m \). The following result will be needed in the next section. Recall the set \( J_\lambda \) defined in (2.2.2).

**Proposition 3.8.** (1) The set of fixed elements \( \mathcal{R}[L_1, \ldots, L_r] m \mathfrak{S}_m \) is a free \( \mathcal{R} \)-module with basis \( \{ \sigma^a(\nu) \mid a(\nu) \in \Lambda(\nu, < m) \} \). Moreover, for \( i \in J_\nu \) and \( f \in \mathcal{R}[L_1, \ldots, L_r] m \mathfrak{S}_m \), we have \( T_i f = f T_i \) in \( H_m(r) \).

(2) Suppose that \( z \in x_\nu H_m(r) \cap H_m(r)x_\nu \) has the “pure” form
\[
z = \sum_{a \in \mathbb{Z}_m} g_a x_\nu L^a = \sum_{b \in \mathbb{Z}_m} h_b L^b x_\nu, \quad \text{where} \quad g_a, h_b \in \mathcal{R}.
\]

Then \( z \) is an \( \mathcal{R} \)-linear combination of \( x_\nu \sigma^a(\nu) \) with \( a(\nu) \in \Lambda(\nu, < m) \).

**Proof.** (1) This statement follows from an argument similar to that in the proof of Proposition 3.5 together with an induction on \( t \).

(2) Clearly, for each \( i \in J_\nu \) (i.e., \( 1 \leq i \leq r - 1 \) with \( i \neq \sum_{j=1}^s \nu_j, \forall 1 \leq s \leq t - 1 \)), we have
\[
T_i x_\nu = x_\nu T_i = q x_\nu \quad \text{and} \quad z T_i = q z.
\]

By applying similar arguments to those in the proof of Lemma 3.2, we obtain that
\[
g_a = g_{a s}, \quad \forall i \in J_\nu.
\]

Thus, \( z = x_\nu f \) with \( f \in \mathcal{R}[L_1, \ldots, L_r] m \mathfrak{S}_m \). The statement then follows from (1). \( \square \)

**Remark 3.9.** By Lemma 3.1 and a similar argument, the \( \mathcal{R} \)-module \( x_\nu H_\nu(r) \cap H_\nu(r)x_\nu \) contains the linearly independent set \( \{ x_\nu \sigma^a(X_1, \ldots, X_r) \mid a \in \mathbb{N}^r \} \). Hence, there is an affine version of Proposition 3.8.

4. A NEW INTEGRAL BASIS FOR \( \mathcal{S}_m(n, r) \)

In this section we describe an \( \mathcal{R} \)-basis for \( x_\lambda H_m(r) \cap H_m(r)x_\mu \) \( (\lambda, \mu \in \Lambda(n, r)) \), all of which give rise to a new basis for \( \mathcal{S}_m(n, r) \). We keep all the notations in previous sections.

For two compositions \( \lambda \) and \( \mu \) of \( r \), let \( D_{\lambda, \mu} = D_\lambda \cap D_\mu^{-1} \). Then \( D_{\lambda, \mu} \) is the set of minimal length \( \mathfrak{S}_\lambda \mathfrak{S}_\mu \) double coset representatives in \( \mathfrak{S}_r \).

**Lemma 4.1.** ([10], [18] Lem. 1.6]) Let \( \lambda, \mu \) be two compositions of \( r \) and \( d \in D_{\lambda, \mu} \).

(1) There exist compositions \( \nu(d) = \lambda d \cap \mu, \nu(d^{-1}) = \lambda \cap \mu d^{-1} \) of \( r \) such that
\[
\mathfrak{S}_{\nu(d)} = d^{-1} \mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu, \quad \mathfrak{S}_{\nu(d^{-1})} = \mathfrak{S}_\lambda \cap d \mathfrak{S}_\mu d^{-1}.
\]
In particular, we have

\[
x_\lambda T_d = \left( \sum_{u \in D_{\nu(d)}^{-1} \cap S} T_u \right) T_d x_{\nu(d)} \quad \text{and} \quad T_d x_{\nu(d)} = T_d x_{\nu(d)} \left( \sum_{v \in D_{\nu(d)} \cap S} T_v \right)
\]

(4.1.1)

For \( \lambda, \mu \in \Lambda(n, r) \), \( d \in D_{\lambda, \mu} \), the double coset \( S_d \) \( S_{\mu} \) defines an \( n \times n \) matrix

\[
\theta(\lambda, d, \mu) = (|R_i^k \cap d(R_j^m)|),
\]

where, for \( \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \Lambda(n, r) \),

\[
R_i^\nu = \{ \tilde{v}_{i-1} + 1, \tilde{v}_{i-1} + 2, \ldots, \tilde{v}_{i-1} + \nu_i \} \quad \text{with} \quad \tilde{v}_0 = 0, \tilde{v}_k = \sum_{j=1}^{k} \nu_j
\]

(4.1.3)

see, e.g., [14] Thm 4.15. Let

\[
\Theta(n, r) = \{ \theta(\lambda, d, \mu) \mid \lambda, \mu \in \Lambda(n, r), \ d \in D_{\lambda, \mu} \} \quad \text{and} \quad \Theta(n) = \bigcup_{r \geq 0} \Theta(n, r).
\]

Then \( \Theta(n) \) is the set \( M_n(N) \) of \( n \times n \) matrices with nonnegative integer coefficients. For each \( A = (a_{ij}) \in \Theta(n) \), define its row and column vectors by

\[
\text{ro}(A) = (\sum_{j=1}^{n} a_{1j}, \ldots, \sum_{j=1}^{n} a_{nj}), \quad \text{co}(A) = (\sum_{i=1}^{n} a_{i1}, \ldots, \sum_{i=1}^{n} a_{in}) \in \mathbb{N}^n.
\]

Thus, if \( A = \theta(\lambda, d, \mu) \) and \( \nu(d) \) as defined in Lemma [14](1), then \( \lambda = \text{ro}(A), \mu = \text{co}(A) \), and

\[
\nu(d) = \nu(A) =: (a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, \ldots, a_{1n}, \ldots, a_{nn}) \in \Lambda(n^2, r).
\]

(4.1.4)

We also write \( d = d_A \). Hence, the inverse map of \( \theta \) has the form

\[
\theta^{-1} : \Theta(n, r) \longrightarrow \{ (\lambda, d, \mu) \mid \lambda, \mu \in \Lambda(n, r), \ d \in D_{\lambda, \mu} \}, \quad A \longmapsto (\text{ro}(A), d_A, \text{co}(A)),
\]

(4.1.5)

and the subset \( \Theta(n, r)_{\lambda, \mu} = \{ A \in \Theta(n, r) \mid \text{ro}(A) = \lambda, \text{co}(A) = \mu \} \) identifies the double cosets in \( \mathfrak{S}_\lambda \backslash \mathfrak{S}_r / \mathfrak{S}_\mu \).

C. Mak [43] Thm 4.1.2] has generalised this double coset correspondence from \( \mathfrak{S}_r \) to \( \mathfrak{S}_{m,r} (m \geq 1) \).

The group \( \mathfrak{S}_{m,r} = \mathbb{Z}_m \wr \mathfrak{S}_r \) admits a presentation with generators \( s_0, s_1, \ldots, s_{r-1} \) and relations:

\[
\begin{align*}
s_0^m &= 1, \quad s_i^2 = 1 \quad (1 \leq i \leq r - 1), \quad s_0s_1s_0s_1 = s_1s_0s_1s_0, \\
s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} \quad (1 \leq i < r - 1), \quad s_is_j &= s_js_i \quad (|i - j| > 1).
\end{align*}
\]

(4.1.6)

There is an alternative description of \( \mathfrak{S}_{m,r} \), following [43]. Let \( \xi \) be a primitive \( m \)-th root of unity. Set \( C_m = \{ 1, \xi, \ldots, \xi^{m-1} \} \) and \( R = \{ 1, 2, \ldots, r \} \). Consider the set

\[
C_mR = \{ \xi^a i \mid 0 \leq a \leq m - 1, \ 1 \leq i \leq r \}.
\]
Then $\mathcal{G}_{m,r}$ is identified with the group of all permutations $w$ of $C_m R$ satisfying

$$w(\xi^a i) = \xi^aw(i), \forall 1 \leq i \leq r.$$ 

In other words, each element $w$ in $\mathcal{G}_{m,r}$ is uniquely determined by the sequence $(w(1), \ldots, w(r))$. In particular, $s_0$ corresponds to the sequence $(\xi, 2, \ldots, r)$ and for $1 \leq i \leq r - 1$, $s_i$ corresponds to the sequence $(1, 2, \ldots, i - 1, i + 1, i, i + 2, \ldots, r)$.

Let $\Theta_m(n, r)$ be the set of $n \times nm$-matrices with nonnegative integer entries sum to $r$. We will write elements in $\Theta_m(n, r)$ as an $n \times n$ array with $m$-tuples of nonnegative integers as entries. More precisely,

$$\Theta_m(n, r) = \{ A = (a_{ij})_{n \times n} | a_{ij} \in \mathbb{N}^m, 1 \leq i, j \leq n; \sum_{i,j=1}^{n} |a_{ij}| = r \}. \quad (4.1.7)$$

Then each element $A \in \Theta_m(n, r)$ gives a matrix $|A| := (|a_{ij}|)_{n \times n} \in \Theta(n, r)$. Thus, we obtain a matrix-valued map

$$f = | : \Theta_m(n, r) \longrightarrow \Theta(n, r), \quad (a_{ij}) \longmapsto (|a_{ij}|).$$

For $\lambda, \mu \in \Lambda(n, r)$, let

$$\Theta_m(n, r)_{\lambda, \mu} = \{ A \in \Theta_m(n, r) | \text{ro}(|A|) = \lambda, \text{co}(|A|) = \mu \}.$$ 

The following can be seen from [43, Thm 4.1.2] and (4.1.5).

**Lemma 4.2.** Let $\lambda, \mu \in \Lambda(n, r)$.

1. If $A = \theta(\lambda, d, \mu)$ with $d \in \mathcal{G}_{\lambda, \mu}$, then the fibre of $f$ at $A$ is

$$f^{-1}(A) = \Theta_m(n, r)_{d, \lambda, \mu} := \{ A \in \Theta_m(n, r)_{\lambda, \mu} | d_{|A|} = d \}.$$ 

2. There is a bijection $\theta_{\lambda, \mu} = \theta_{\lambda, \mu}^{(m)} : \mathcal{G}_{\lambda} \backslash \mathcal{G}_{m,r} / \mathcal{G}_{\mu} \rightarrow \Theta_m(n, r)_{\lambda, \mu}$ sending a double coset $\mathcal{G}_{\lambda}w\mathcal{G}_{\mu}$ to $A = (a_{ij})_{n \times n}$, where $a_{ij} = (a_{ij}^{(1)}, \ldots, a_{ij}^{(m)})$ is defined by $a_{ij}^{(t)} = |c^t R_{ij}^t \cap w(R_{ij}^t)|$ for $1 \leq t \leq m$.

For each $A \in \Theta_m(n, r)_{\lambda, \mu}$, define a composition, as in (4.1.4), associated with the parabolic subgroup $\mathcal{G}_{\text{ro}(|A|)} \cap \mathcal{G}_{\text{co}(|A|)}$:

$$\nu(|A|) = \nu(d_{|A|}) = (|a_{11}|, \ldots, |a_{n1}|, |a_{12}|, \ldots, |a_{n2}|, \ldots, |a_{1n}|, \ldots, |a_{nn}|) \in \Lambda(n^2, r), \quad (4.2.1)$$

We also define a composition

$$\nu(A) = (a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, \ldots, a_{1n}, \ldots, a_{nn}) \in \Lambda(mn^2, r). \quad (4.2.2)$$

Then we have the associated Young subgroup $\mathcal{G}_{\nu(A)}$ of $\mathcal{G}_p$. The following lemma generalizes [18, Lem. 1.6] (for type $A$) and can be deduced from [43, Lem. 4.3.1(d)]. However, we provide a proof here for completeness and later use.

Let

$$\mathcal{I} = [1,n]^2, \quad \mathcal{J} = \mathcal{J}(A) = \{(i, j) \in \mathcal{I} | 1 \leq i, j \leq n, a_{ij}^{(m)} < |a_{ij}| \}. \quad (4.2.3)$$

We order $\mathcal{I}$ by setting

$$(i, j) \preceq (i', j') \text{ if } j < j' \text{ or } i < i' \text{ whenever } j = j'.$$ \quad (4.2.4)
For $1 \leq i, j \leq n$, $1 \leq k \leq m$, let
\[
a(i, j, k) = \tilde{a}_{ij} + \sum_{1 \leq x \leq k-1} a_{ij}^{(x)}, \quad \text{where} \quad \tilde{a}_{ij} = \sum_{1 \leq t' \leq n, t < j} |a_{ij}| + \sum_{1 \leq t' \leq i-1} |a_{i't'}| \quad (4.2.5)
\]
is the partial sum of the sequence in (4.2.1) up to $|a_{ij}|$, but not include $|a_{ij}|$.

For any $(i, j) \in I$ and $1 \leq k \leq m$, let
\[
t_1 = s_0, t_i = s_{i-1} \cdots s_1 t_1 s_1 t_2 \cdots s_{i-1}, \quad 2 \leq i \leq r;
\]
\[
t_{i,j}^{(k)} = \begin{cases} t_{a+1} t_{a+2} \cdots t_{a+a_{ij}}, & \text{if } (i, j) \in J, k \leq m - 1, a_{ij}^{(k)} \geq 1; \\ 1, & \text{otherwise}, \end{cases} \quad (4.2.6)
\]
where $a = a(i, j, k)$.

**Lemma 4.3.** For $\lambda, \mu \in \Lambda(n, r)$ and $w \in S_{m,r}$, if $\Lambda = (a_{ij}) = \theta_{\lambda, \mu}(S_{\lambda}wS_{\mu}) \in S_{m}(n, r)_{\lambda, \mu}$, then $w^{-1}S_{\lambda}w \cap S_{\mu} = S_{\nu(k)}$. Moreover, we may choose
\[
w = d_{|\lambda|} \prod_{(i,j) \in J} ((t_{ij}^{(1)})^{1/2} (t_{ij}^{(2)})^{1/2} \cdots (t_{ij}^{(m-1)})^{1/2 m}) \quad (4.3.1)
\]
(notating $(t_{ij}^{(m)}) = 1$).

**Proof.** Write $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$. For $1 \leq i \leq n$, let $S_{\lambda_{i+1}}^{(i)}$ be the subgroup of $S_{\lambda}$ generated by $s_{\lambda_{i+1}+1}, \ldots, s_{\lambda_{i+1}+\lambda_i-1}$ (by convention). Similarly, we obtain the subgroups $S_{\mu_{i+1}}^{(i)}$ of $S_{\mu}$. Then
\[
S_{\lambda} = S_{\lambda_{1}}^{(1)} \times \cdots \times S_{\lambda_{n}}^{(n)} \quad \text{and} \quad S_{\mu} = S_{\mu_{1}}^{(1)} \times \cdots \times S_{\mu_{n}}^{(n)}.
\]
This implies that $w^{-1}S_{\lambda}w = (w^{-1}S_{\lambda_{1}}^{(1)}w) \times \cdots \times (w^{-1}S_{\lambda_{n}}^{(n)}w)$. Hence,
\[
w^{-1}S_{\lambda}w \cap S_{\mu} = (w^{-1}S_{\lambda}w \cap S_{\mu_{1}}^{(1)}) \times \cdots \times (w^{-1}S_{\lambda}w \cap S_{\mu_{n}}^{(n)}) = \left( \prod_{i=1}^{n} w^{-1}S_{\lambda_{i}}^{(i)}w \cap S_{\mu_{i}}^{(i)} \right) \times \cdots \times \left( \prod_{i=1}^{n} w^{-1}S_{\lambda_{i}}^{(i)}w \cap S_{\mu_{i}}^{(i)} \right) \quad (4.3.2)
\]
By the definition, $S_{\lambda_{i}}^{(i)}$ is the subgroup of $S_{m,r}$ consisting of all the elements which fix the set $C_{m}(R \setminus R_{i}^{\lambda})$, and map $R_{i}^{\lambda}$ onto $R_{i}^{\lambda}$ (Thus, they map $\xi^{t}R_{i}^{\lambda}$ onto $\xi^{t}R_{i}^{\lambda}$ for all $1 \leq t \leq m$). Consequently, $w^{-1}S_{\lambda_{i}}^{(i)}w$ is a subgroup of $S_{m,r}$ consisting of all the elements which fix the set $C_{m}w^{-1}(R \setminus R_{i}^{\lambda})$ and map $w^{-1}(R_{i}^{\lambda})$ onto $w^{-1}(R_{i}^{\lambda})$ (They also map $\xi^{t}w^{-1}(R_{i}^{\lambda})$ onto $\xi^{t}w^{-1}(R_{i}^{\lambda})$ for $1 \leq t \leq m$).

Therefore, for $1 \leq i, j \leq n$, we can write
\[
w^{-1}S_{\lambda_{i}}^{(i)}w \cap S_{\mu_{j}}^{(j)} = \prod_{t=1}^{m} S_{ij_{t}}^{(t)}
\]
where $S_{ij_{t}}^{(t)}$ is the subgroup formed by all elements in $S_{m,r}$ which permute elements in $\xi^{t}w^{-1}(R_{i}^{\lambda}) \cap R_{j}^{\mu}$ and fix other elements in $C_{m}w^{-1}(R_{i}^{\lambda}) \cap R_{j}^{\mu}$.

By Lemma 4.2.2, for $1 \leq i, j \leq n$ and $1 \leq k \leq m$,
\[
|\xi^{k}w^{-1}(R_{i}^{\lambda}) \cap R_{j}^{\mu}| = |\xi^{k}R_{i}^{\lambda} \cap w(R_{j}^{\mu})| = a_{ij}^{(k)}.
\]
Therefore, $\mathcal{G}_{ij}^{(k)}$ is the subgroup of $\mathcal{G}_{m,r}$ generated by $s_{a+1}, s_{a+2}, \ldots, s_{a+a_{ij}^{(k)}-1}$, where $a = a(i, j, k)$ as defined in (4.2.5). This together with (4.3.2) implies that $w^{-1}\mathcal{G}_{\lambda}w \cap \mathcal{G}_{\mu} = \mathcal{G}_{\nu(\lambda)}$.

It remains to prove the last assertion. Assume now $w$ is given as in (4.3.1). For each element $u \in \mathcal{G}_{m,r}$, we write $u = (u(1), \ldots, u(r))$. Then, for $(i, j) \in J, 1 \leq k \leq m$ and $a = a(i, j, k)$,

$$(t_{ij})^{(k)} = (1, 2, \ldots, a, \xi^k(a+1), \xi^k(a+2), \ldots, \xi^k(a+a_{ij}^{(k)}), a+a_{ij}^{(k)}+1, a+a_{ij}^{(k)}+2, \ldots, r).$$

If we put $\beta_{ij}^{(k)} = (\xi^k(a+1), \xi^k(a+2), \ldots, \xi^k(a+a_{ij}^{(k)}))$, then (after concatenation)

$$d_{|\lambda|}^{-1}w := y = (\beta_{11}^{(1)}, \beta_{11}^{(m)}, \beta_{21}^{(1)}, \beta_{21}^{(m)}, \beta_{31}^{(1)}, \ldots, \beta_{n-1,n}^{(1)}, \beta_{n-1,n}^{(m)}, \beta_{n,n}^{(1)}, \ldots, \beta_{nn}^{(m)}).$$

Since $|a_{ij}| = |d_{|\lambda|}^{-1}R^\lambda \cap R^\mu|$, it follows that

$$a_{ij}^{(k)} = |\xi^k((d_{|\lambda|}^{-1}R^\lambda) \cap y(R^\mu))| = |\xi^kR^\lambda \cap w R^\mu|,$$

for all $1 \leq i, j \leq n$ and $1 \leq k \leq m$. Hence, $\theta_{\lambda\mu}(\mathcal{G}_{\lambda}w \mathcal{G}_{\mu}) = \mathbb{A}$. \hfill $\square$

We need more notations. For $k \in \mathbb{N}$ and $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda(m, k)$, let

$$b_i := \sharp\{t \in \{1, \ldots, m-1\} \mid \lambda_1 + \cdots + \lambda_t \geq i \} \quad (i = 1, 2, \ldots, k).$$

Then $m - 1 \geq b_1 \geq b_2 \geq \cdots \geq b_k$. In other words, with the notation in (4.1.3), $(b_1, \ldots, b_k)$ is the partition dual to $(\lambda_{m-1}, \ldots, \lambda_2, \lambda_1)$. Finally, define

$$\tilde{\lambda} = (b_1 - b_2, \ldots, b_{k-1} - b_k, b_k),$$

which clearly lies in $\Lambda(k, < m)$. Indeed, it is easy to check that the correspondence $\lambda \mapsto \tilde{\lambda}$ induces a bijection

$$g_{m,k} : \Lambda(m, k) \longrightarrow \Lambda(k, < m).$$

For example, let $m = 3, k = 6$ and take $\lambda = (2, 3, 1) \in \Lambda(3, 6)$. Then

$$(b_1, b_2, b_3, b_4, b_5, b_6) = (2, 2, 1, 1, 1, 0) \quad \text{and} \quad \tilde{\lambda} = (0, 1, 0, 0, 1, 0).$$

Let $\mathbb{A} = (a_{ij})_{n \times n} \in \Theta_m(n, r)$ with $a_{ij} = (a_{ij}^{(1)}, \ldots, a_{ij}^{(m)})$ for $1 \leq i, j \leq n$. By the construction above, each $a_{ij}$ as a composition in $\Lambda(m, |a_{ij}|)$ gives rise to a vector $\bar{a}_{ij} \in \Lambda(|a_{ij}|, < m)$ and a “matrix” $\mathbb{A} = (\bar{a}_{ij})$. Thus, by juxtaposition, $\mathbb{A}$ defines a composition

$$a(\mathbb{A}) := (\bar{a}_{11}, \ldots, \bar{a}_{1n}, \bar{a}_{21}, \ldots, \bar{a}_{2n}, \ldots, \bar{a}_{1n}, \ldots, \bar{a}_{mn}) \in \Lambda(\nu(|\mathbb{A}|), < m),$$

which further defines, in the notation of Definition 3.7, an element

$$\sigma^{\mathbb{A}} := \sigma^{a(\mathbb{A})} \in \mathcal{B}[L_1, \ldots, L_r]_{\mathcal{G}_{\nu(|\mathbb{A}|)}}.$$

(4.3.5) \hfill $\sigma^{\mathbb{A}}$

**Lemma 4.4.** Suppose $\lambda, \mu \in \Lambda(n, r)$, $d \in \mathcal{D}_{\lambda, \mu}$ and let $\nu = \lambda d \cap \mu$. Then for each $a(\nu) \in \Lambda(\nu, < m)$, there is a unique $\mathbb{A} \in \Theta_m(n, r)^d_{\lambda, \mu}$ such that $\sigma^{a(\nu)} = \sigma^{\mathbb{A}}$. \hfill $\mathbb{A}$

\footnote{Unlike the compositions $\nu(|\mathbb{A}|), \nu(\mathbb{A})$ defined in (4.2.1), (4.2.2), $a(\mathbb{A})$ is not a composition of $r$.}
Proof. Let $A = (a_{ij}) = \theta(\lambda, d, \mu) \in \Theta(n, r)$, i.e., $(\text{ro}(A), d_A, \text{co}(A)) = (\lambda, d, \mu)$. Then 
\[ \nu = \lambda d \cap \mu = \nu(A) = (a_{11}, \ldots, a_{n1}, a_{n2}, \ldots, a_{n1}, \ldots, a_{nn}) \in \Lambda(n^2, r). \]

Applying the bijections $g_{m,a_{ij}}$ defined as in (4.3.4) induces a bijection 
\[ g_A : \Theta_m(n, r)^\lambda_{\lambda, \mu} \longrightarrow \Lambda(\nu(A), < m), \ A \longmapsto a(\hat{A}), \]
noting that $|A| = A$. Hence, $A = g_A^{-1}(a(\nu))$. \qed

The lemma above together with Proposition 3.8(1) gives the following result.

**Proposition 4.5.** For $\lambda, \mu \in \Lambda(n, r)$ and $d \in D_{\lambda, \mu}$, the set 
\[ \{ \sigma^{\hat{A}} | A \in \Theta_m(n, r)^\lambda_{\lambda, \mu} \} \]
is linearly independent. Moreover, for each $A \in \Theta_m(n, r)$, 
\[ x_{\nu(d_{\lambda, \mu})} \sigma^{\hat{A}} = \sigma^{\hat{A}} x_{\nu(d_{\lambda, \mu})}. \quad (4.5.1) \]

**Example 4.6.** Let $n = 2$, $m = 3$, and $r = 11$. Choose 
\[ A = \begin{pmatrix} 1, 1, 1 & (1, 0, 2) \\ 1, 1, 0 & (1, 2, 0) \end{pmatrix} \in \Theta_3(2, 11). \]

Then $a_{11} = (1, 1, 1)$, $a_{21} = (1, 1, 0)$, $a_{12} = (1, 0, 2)$, $a_{22} = (1, 2, 0)$, and 
\[ |A| = \begin{pmatrix} 3 & 3 \\ 2 & 3 \end{pmatrix}. \]

By the definition above, we obtain 
\[ \hat{a}_{11} = (1, 1, 0), \ \hat{a}_{21} = (1, 1, 1), \ \hat{a}_{12} = (1, 0, 0), \ \hat{a}_{22} = (1, 0, 1), \]
and 
\[ \sigma^{\hat{A}} = \sigma^{a_{11}} \sigma^{a_{21}} \sigma^{a_{12}} \sigma^{a_{22}} \]
\[ = (L_1 + L_2 + L_3)(L_1 L_2 + L_2 L_3 + L_1 L_3)(L_4 + L_5)(L_4 L_5) \]
\[ = (L_6 + L_7 + L_8)(L_9 + L_{10} + L_{11})L_9 L_{10} L_{11}. \]

Moreover, $\nu(d_{\lambda, \mu}) = (3, 2, 3, 3)$, and the corresponding Young subgroup $G_{\nu(d_{\lambda, \mu})}$ is generated by $s_1, s_2, s_4, s_6, s_7, s_9, s_{10}$, where $s_i$ $(1 \leq i \leq 10)$ are the generators of $S_{11}$.

Using the element $\sigma^{\hat{A}}$ associated with $A \in \Theta_m(n, r)$ in (4.3.5), we further define with $\lambda = \text{ro}(|A|)$ and $\mu = \text{co}(|A|)$, 
\[ b_A = x_\lambda T_{d_{\lambda, \mu}} \sigma^{\hat{A}} \sum_{w \in G_{\nu(d_{\lambda, \mu})} \cap S_\mu} T_w \in x_\lambda H_m(r). \quad (4.6.1) \]

**Theorem 4.7.** For $\lambda, \mu \in \Lambda(n, r)$, the $R$-module $x_\lambda H_m(r) \cap H_m(r)x_\mu$ is free and the set 
\[ B_{\lambda, \mu} = B_{\lambda, \mu}(m) = \{ b_A \mid A \in \Theta_m(n, r)_{\lambda, \mu} \} \]
forms a basis.

The proof of the theorem is somewhat standard and will be given in Appendix A; compare [31].

We remark that the assertion that $R$-module $x_\lambda H_m(r) \cap H_m(r)x_\mu$ is $R$-free is not new. In [20] (6.3)], a basis that can give rise to a cellular basis for the cyclotomic $q$-Schur algebra is constructed. However, the basis $B_{\lambda, \mu}$ is new. We will show in §5
that it is a \( q \)-analogue of the usual double coset basis for the endomorphism algebra of a permutation module [26].

For given \( A \in \Theta_m(n, r) \), define
\[
\Phi_A \in S_m(n, r) = \text{End}_{H_m(r)} \left( \bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda H_m(r) \right)
\]
taking \( x_\mu h \mapsto \delta_{\mu, \text{co}(A)} b_A h \) for all \( \mu \in \Lambda(n, r) \) and \( h \in H_m(r) \).

Now we can state the main result of this section.

**Theorem 4.8.** The slim cyclotomic \( q \)-Schur algebra \( S_m(n, r) \) is a free \( \mathcal{R} \)-module with basis \( \{ \Phi_A \mid A \in \Theta_m(n, r) \} \) and rank \( \binom{mn^2 + r - 1}{r} \).

**Proof.** Since \( S_m(n, r) = \bigoplus_{\lambda \mu \in \Lambda(n, r)} \text{Hom}_{H_m(r)}(x_\mu H_m(r), x_\lambda H_m(r)) \), the first assertion follows from Lemma 2.3 and Theorem 4.7. The second assertion is clear as \( |\Theta_m(n, r)| = \binom{mn^2 + r - 1}{r} \). \( \square \)

**Remark 4.9.** (1) It would be interesting to generalise the construction to obtain a similar basis for the affine \( q \)-Schur algebra \( S_{\mathfrak{a}}(n, r) \). However, by simply replacing \( L_j \)’s by \( X_j \)’s in the definition, we obtain a linearly independent set \( \{ \Phi_A \mid A \in \Theta_m(n, r) \} \) for \( S_{\mathfrak{a}}(n, r) \). See Remarks 3.3 and 3.9.

(2) We also observe that the basis in Theorem 4.8 cannot be naturally extended to a basis for the cyclotomic \( q \)-Schur algebra discussed in [20].

View \( \Theta(n, r) \) as a subset of \( \Theta_m(n, r) \) via the following map:
\[
\iota^{(m)} : \Theta(n, r) \rightarrow \Theta_m(n, r), \quad A \mapsto A^{(m)}, \quad (4.9.1)
\]
where if \( A = (a_{ij}) \in \Theta(n, r) \), then \( A^{(m)} = (a_{ij}) \in \Theta_m(n, r) \) with all \( a_{ij} = (0, \ldots, 0, a_{ij}) \in \mathbb{N}^m \). Then by the definition, \( \sigma'A^{(m)} = 1 \) and, hence,
\[
\Phi_{A^{(m)}}(x_\mu h) = \delta_{\mu, \text{co}(A)} x_\lambda T_{d_A} \sum_{v \in \mathfrak{S}_A^{(m)}} T_v h, \ \forall h \in H_m(r).
\]

Note that the same rule with \( h \in H(r) \) defines an element \( \phi_A \) in the \( q \)-Schur algebra \( S_q(n, r) := \text{End}_{H(r)}(\oplus_{\lambda \in \Lambda(n, r)} x_\lambda H(r)) \). Thus, we have immediately the following:

**Proposition 4.10.** The \( q \)-Schur algebra \( S_q(n, r) \) can be embedded into \( S_m(n, r) \) via the map \( \phi_A \mapsto \Phi_{A^{(m)}} \) for \( A \in \Theta(n, r) \).

5. **Comparison with the Double Coset Basis of Type \( B/C \)**

Recall the presentation of \( \mathfrak{S}_{m, r} \) in (4.1.0). If \( m = 2 \), then
\[
W := \mathfrak{S}_{2, r}
\]
is a Coxeter (or Weyl) group of type \( B \) with Coxeter generators \( s_0, s_1, \ldots, s_{r-1} \). In this section we mainly deal with the cyclotomic Hecke algebra \( H_m(r) \) associated with \( W \). We will also choose
\[
m = 2 = (-1, q_0)
\]
with \( q_0 \) invertible in \( \mathcal{R} \) (\( q \) also invertible in \( \mathcal{R} \) as before) so that \( H_2(r) \) is isomorphic to the Hecke algebra of type \( B \) defined in [29, Sect. 3] by sending \( T_i \) to \( T_s \) and \( L_j \).
to $q^{1-i}T_{s_{j-1}} \cdots T_{s_1}T_{s_0}T_{s_1} \cdots T_{s_{j-1}}$. We will identify the two. Thus, $T_i = T_{s_i}$, $L_1 = T_0$, and $\mathcal{H}_2(r)$ has a basis $\{T_w\}_{w \in W}$, where $T_w = T_{i_1}T_{i_2} \cdots T_{i_l}$ for any reduced expression $w = s_{i_1}s_{i_2} \cdots s_{i_l}$. The associated slim cyclotomic $q$-Schur algebra is known as the $q$-Schur$^{18}$ algebra. See Remark \ref{rem:connections} for connections to other similar algebras.

For any subset $D$ of $W$, define

$$T_D = \sum_{w \in D} T_w \in \mathcal{H}_2(r), \quad D = \sum_{w \in D} w \in \mathcal{R}W.$$ 

For arbitrary $\lambda, \mu \in \Lambda(n, r)$, we shall make a comparison between the integral basis $\mathcal{B}_{\lambda, \mu}^{(2)}$ for $x_\lambda \mathcal{H}_2(r) \cap \mathcal{H}_2(r)x_\mu$ given in Theorem \ref{thm:main} and the double coset basis $\{T_D \mid D \in \mathcal{S}_\lambda \backslash W / \mathcal{S}_\mu\}$ given in \cite{29} Prop. 4.2.5 (and their counterparts in the non-quantum case).

Following \cite{29} Sect. 2], for each composition $\lambda$ of $r$, let $\mathcal{D}_\lambda$ be the set of shortest right coset representatives of the Young subgroup $\mathcal{S}_\lambda$ in $W$. Set $\mathcal{D}_\lambda^{-1} = \{d^{-1} \mid d \in \mathcal{D}_\lambda\}$. Then for two compositions $\lambda, \mu$ of $r$, $\mathcal{D}_{\lambda, \mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$ is the set of shortest $\mathcal{S}_\lambda$-$\mathcal{S}_\mu$ double coset representatives in $W$.

We will prove the following theorem in this section.

**Theorem 5.1.** For $\lambda, \mu \in \Lambda(n, r)$, let $\mathcal{B}_{\lambda, \mu}^{(2)}$ be the basis for the free $\mathcal{R}$-module $x_\lambda \mathcal{H}_2(r) \cap \mathcal{H}_2(r)x_\mu$ as given in Theorem \ref{thm:main}.

1. For $\lambda = \mu = (r) = (r, 0, \ldots, 0) \in \Lambda(n, r)$, write $\mathcal{D}_{r,r} := \mathcal{D}_{\lambda, \lambda}$. Then

$$\mathcal{B}_{r,r}^{(2)} = \{T_{\mathcal{S}_r \cdot d \mathcal{S}_r} \mid d \in \mathcal{D}_{r,r}\}.$$ 

2. If $\mathcal{H}_2(r)$ is the group algebra of $W$ over $\mathcal{R}$ (i. e., if $q = q_0 = 1$), then

$$\mathcal{B}_{\lambda, \mu}^{(2)} = \{\mathcal{S}_\lambda d \mathcal{S}_\mu \mid d \in \mathcal{D}_{\lambda, \mu}\}.$$ 

**Remark 5.2.** By this result, we see the basis constructed in Theorem \ref{thm:main} for the (non-quantum) Schur$^{18}$ algebra coincides with the standard basis for the endomorphism algebra of a permutation module \cite{46}. Thus, both the basis in Theorem \ref{thm:main} and the basis given in \cite{29} Thm 4.2.6 for the $q$-Schur$^{18}$ algebra are quantum analogues of the classical double coset basis for the Schur$^{18}$ algebra.

Thus, the basis constructed in Theorem \ref{thm:main} can be regarded as a generalisation of the double coset basis to the Hecke algebra of a complex reflection group.

We proceed to prove the theorem.

Let $d_0 = 1$ and, for $1 \leq i \leq r$, let

$$d_i = \tau_1 \cdot \tau_2 \cdots \tau_i, \quad \text{where} \quad \tau_j = s_{j-1}s_{j-2} \cdots s_1 s_0. \quad (5.2.1)$$

**Lemma 5.3.** We have $\mathcal{D}_{r,r} = \{d_i \mid 0 \leq i \leq r\}$ and $\ell(d_i) = i(i + 1)/2$.

**Proof.** This is clear since the double coset $\mathcal{S}_r d_i \mathcal{S}_r$ is the subset of $\mathcal{S}_{2r}$ consisting of elements satisfying the property that $s_0$ occurs exactly $i$ times in every reduced expression. \hfill $\Box$

\footnote{The (nonslim) cyclotomic $q$-Schur algebra in this case is called the $q$-Schur$^{28}$ (or $q$-Schur$^2$) algebra; see \cite{29}.}
Lemma 5.4. For $0 \leq i \leq r$, we have
\[ \mathcal{G}_{\nu(d_i)} := d_i^{-1} \mathcal{G}_s d_i \cap \mathcal{G}_r = \begin{cases} \mathcal{G}_r, & \text{if } i = 0; \\ \mathcal{G}_{\{1,2,\ldots,i\}} \times \mathcal{G}_{\{i+1,i+2,\ldots,r\}}, & \text{if } i \geq 1. \end{cases} \]

Proof. We first observe that $\mathcal{G}_{\nu(d_i)}$ is the subgroup of $\mathcal{G}_r$ generated by $\{s_2, s_3, \ldots, s_{r-1}\}$. If $2 \leq i \leq r$, then, for $1 \leq j \leq r - 1$,
\[ d_i^{-1}s_jd_i = \begin{cases} s_{i-j}, & \text{if } j \leq i - 1; \\ s_0s_1 \ldots s_is_{i-1} \ldots s_1s_0, & \text{if } j = i; \\ s_j, & \text{if } j \geq i + 1. \end{cases} \tag{5.4.1} \]
Therefore, $\mathcal{G}_{\nu(d_i)}$ is generated by $\{s_1, s_2, \ldots, s_{i-1}, s_i+1, \ldots, s_{r-1}\}$, as desired. \hfill \Box

We now compute $\mathcal{D}_{\nu(d_i)} = \mathcal{G}_{\nu(d_i)} \cap \mathcal{G}_r$. For $2 \leq i \leq r$, let
\[ \mathcal{E}_i = \{ j = (j_1, \ldots, j_i) \in \mathbb{N}^i \mid 1 \leq j_1 < \ldots < j_i \leq r \}. \]
Then, for each $j \in \mathcal{E}_i$, we have $j_k \geq k$ for all $1 \leq k \leq i$.

Lemma 5.5. We have $\mathcal{D}_{\nu(d_i)} = \{1\}$, $\mathcal{D}_{\nu(d_i)} = \{1, s_1, s_1s_2, \ldots, s_1s_2 \ldots s_{r-1}\}$, and, for $2 \leq i \leq r$,
\[ \mathcal{D}_{\nu(d_i)} = \{ v_j = (s_i \ldots s_{j_i-1})(s_{j_i-1} \ldots s_{j_{i-1}-1}) \ldots (s_1 \ldots s_{j_1-1}) \mid j = (j_1, \ldots, j_i) \in \mathcal{E}_i \}. \]
(If $j_k = k$, set $s_k \ldots s_{j_k-1} = 1$ by convention.)

Proof. Let $v_j^{-1} = (s_{j_1-1} \ldots s_1)(s_{j_1-1} \ldots s_2) \ldots (s_{j_i-1} \ldots s_1)$ act on the ordered sequence $1, \ldots, r$. Then we get an ordered sequence such that the first $i$ numbers are $j_1, \ldots, j_i$ and the remaining ones are in increasing order if and only if the number of inversions of the sequence is $\sum_{1 \leq k \leq i}(j_k - 1)$, which equals the length of $v_j^{-1}$. Hence, $v_j^{-1}$, as well as $v_j$, is reduced. The lemma then follows from the equalities $|\mathcal{E}_i| = |\mathcal{D}_{\nu(d_i)}| = \frac{r!}{i!(r-i)!}$ and the fact that $v_j \neq v_{j'}$ whenever $j \neq j'$. \hfill \Box

Thus, the double coset basis of $x(r)\mathcal{H}_2(r) \cap \mathcal{H}_2(r)x(r)$ consists of the following elements
\[ x(r), \quad x(r)T_0(T_1 + T_1T_2 + \cdots + T_1T_2 \cdots T_{r-1}), \quad x(r)T_{d_i} \sum_{j \in \mathcal{E}_i} T_{v_j}, \quad 2 \leq i \leq r. \]

By taking $m = 2 = (-1, q_0)$, the basis for $x(r)\mathcal{H}_2(r) \cap \mathcal{H}_2(r)x(r)$ in Proposition 3.3 becomes
\[ B_{x(r)}^{(2)} = \{ x(r)\sigma_i \mid 0 \leq i \leq r \} \quad (\text{where } \sigma_0 = 1). \]

Proof of Theorem 5.1(1). It is obvious for $i = 0$ as $x(r) = T_{\mathcal{E}_r}$, since $x(r)T_i = qx(r)$ for $1 \leq i \leq r - 1$ and $L_i = q^{-(i-1)}T_{i-1}T_{i-2} \cdots T_1T_0T_1 \cdots T_{i-1}$, it follows from Lemmas 5.5 and 4.1(2) that
\[ x(r)\sigma_1 = x(r)(L_1 + L_2 + \ldots + L_r) = x(r)T_0(1 + T_1 + T_1T_2 + \ldots + T_1T_2 \cdots T_{r-1}) = x(r)T_0T_{\mathcal{D}_{\nu(d_i)} \cap \mathcal{G}_r} = T_{\mathcal{E}_r \mathcal{D}_r} = T_{\mathcal{E}_r \mathcal{D}_r}. \]
We now assume $2 \leq i \leq r$. We first claim that for $j = (j_1, \ldots, j_i) \in \mathcal{E}_i$,
\[ x(r)T_iT_{v_j} = x(r)L_j \quad \text{where} \quad L_j = L_{j_1}, L_{j_2} \ldots L_{j_i}. \]
Indeed, by repeatedly applying (5.4.1) or a well-known fact, we have
\[ t_1 t_2 \cdots t_i = s_1(s_2 s_1) \cdots (s_{i-1} \cdots s_1) d_i = w_{0,i} d_i, \]
where \( w_{0,i} \) is the longest element of \( S_{1,2,\ldots,i} \), and \( t_1, \ldots, t_i \) are defined in (4.2.6). Thus, \( x(r)T_{d_i} = x(r)L_1 L_2 \cdots L_i \). By (2.0.3), for \( 2 \leq i \leq r - 1 \),
\[
x(r) L_1 \cdots L_{i-1} L_i T_i = x(r) L_1 L_2 \cdots L_{i-1} (L_i T_i)
\]
\[
=x(r) L_1 L_2 \cdots L_{i-1} (T_i L_{i+1} + (1 - q) L_{i+1})
\]
\[
=x(r) L_1 L_2 \cdots L_{i-1} (L_{i+1} + (1 - q) L_1 L_2 \cdots L_{i-1} L_{i+1})
\]
\[
=x(r) L_1 L_2 \cdots L_{i-1} L_{i+1} + (1 - q) L_1 L_2 \cdots L_{i-1} L_i
\]
\[
= x(r) L_1 L_2 \cdots L_{i-1} L_i + (1 - q) L_1 L_2 \cdots L_{i-1} L_i + (1 - q) L_1 L_2 \cdots L_{i-1} L_i
\]

Hence, inductively, we obtain
\[
x(r) L_1 L_2 \cdots L_i (T_j T_{j+1} \cdots T_{j-i}) = x(r) L_1 \cdots L_{i-1} L_j, \quad \forall 2 \leq i < j \leq r - 1.
\]

Since \( T_j = (T_j T_{j+1} \cdots T_{j-i}) (T_{j-1} T_j \cdots T_{j-i}) \cdots (T_1 T_2 T_{j-1}) \), repeatedly applying the formula gives
\[
x(r) T_{d_i} T_{j} = x(r) L_j L_{j+1} \cdots L_j
\]
proving the claim. Now, by Lemmas 4.1.2 and 5.5.1 and the claim,
\[
T_{e_{i},d_{i}} = x(r) T_{d_i} T_{e_{i}(d_i) e_{i}} = \sum_{j \in E_i} x(r) T_{d_i} T_{j} = x(r) \sum_{j \in E_i} L_j = x(r) \sigma_i,
\]
as desired. \( \square \)

For \( a, b \in \mathbb{N} \) with \( b \geq 1, a + b \leq r \), define the subgroups \( W^a_b, \mathcal{S}^a_b \) of \( W = \mathcal{S}_{2,r} \):
\[
W^a_b = \langle s_{a+1}, s_{a+2}, \ldots, s_{a+b-1}, t_{a+1} \rangle,
\]
\[
\mathcal{S}^a_b = \langle s_{a+1}, s_{a+2}, \ldots, s_{a+b-1} \rangle,
\]
(5.5.3)

where \( t_{a+1} = s_{a+1} \cdots s_{1} s_{a+1} \cdots s_{a-1} s_{a} \). Note that \( W^a_b \) is not a parabolic subgroup of \( W \). However, since \( t_{a+1}^2 = 1 \) and \( t_{a+1} s_{a+1} t_{a+1} s_{a+1} = s_{a+1} t_{a+1} s_{a+1} t_{a+1} \), there is a group embedding
\[
t : \mathcal{S}_{2,b} \longrightarrow W, \quad s_0 \mapsto t_{a+1}, s_i \mapsto s_{a+i} (1 \leq i \leq b - 1),
\]
(5.5.4)

which induces a group isomorphism \( \mathcal{S}_{2,b} \cong W^a_b \).

Note that \( \mathcal{S}^a_b \) is a parabolic subgroup of \( W \) and \( \mathcal{S}^a_b = \mathcal{S}_{a,b}^a \), the Young subgroup associated with \( \mu_{a,b} = (1^a, b, 1^{r-a-b}) \). Set
\[
\mathcal{G}^a_{b,b} = \mathcal{G}_{a,b} \cap W^a_b,
\]
the set of shortest double coset representatives of \( \mathcal{S}^a_b \) in \( W^a_b \). By (5.5.4) and Lemma 5.3.4, we have the following result.

**Lemma 5.6.** For \( a, b \in \mathbb{N} \) with \( b \geq 1, a + b \leq r \), we have
\[
\mathcal{G}^a_{b,b} = \{ d^{(0)}_{a,b} = 1, \quad d^{(i)}_{a,b} = \tau'_{1} \tau'_{2} \tau'_{3} \cdots \tau'_{i} | 1 \leq i \leq b \},
\]
where \( \tau'_{t} = s_{a+t-1} s_{a+t-2} \cdots s_{a+1} t_{a+1} \) for \( 2 \leq i \leq b \) and \( \tau'_{1} = t_{a+1} \).
Proof. By Lemma 5.5, the elements 1, \( \tau_1' \tau_2' \cdots \tau_i' \) \((1 \leq i \leq b)\) form a complete set of \((\mathcal{G}_b^a, \mathcal{G}_b^a)\)-double coset representatives in \(W_b^a\). Since, for every \(s_{a+i} \) \((1 \leq i < b)\), the product \(s_{a+i} \tau_1' \tau_2' \cdots \tau_i'\) has length increased, we conclude that
\[
\mathcal{D}_{b,b}^a = \{1, \tau_1' \tau_2' \cdots \tau_i' \mid 1 \leq i \leq b\}.
\]

If there is no confusion, we often drop the subscripts \(a, b\) in \(d_{a,b}^{(i)}\). For \(0 \leq i \leq b\), let \(\mathcal{G}_{b,\nu(d(i))}^a = d_{b,\nu(d(i))}^{(i)} \cap \mathcal{G}_b^a\) and let \(\mathcal{D}_{b,\nu(d(i))}^a\) be the set of shortest right coset representatives of \(\mathcal{G}_{b,\nu(d(i))}^a\) in \(\mathcal{G}_b^a\).

Set \(x_b^a = \sum_{w \in \mathcal{G}_b^a} T_w\). For \(1 \leq i \leq b\), let \(\sigma_{b,i}^a\) be the \(i\)-th elementary symmetric polynomial in \(L_{a+1}, \ldots, L_{a+b}\) and set \(\sigma_{b,0}^a = 1\). By a similar argument at the end of the proof of Lemma 3.4, we have
\[
\sigma_{b,i}^a T_j = T_j \sigma_{b,i}^a, \quad \forall 0 \leq i \leq b, \quad a + 1 \leq j < a + b - 1.
\] (5.6.1)

The following lemma is an analogue to (5.5.2).

Lemma 5.7. Let \(a, b \in \mathbb{N}\) with \(b \geq 1\), \(a + b \leq r\). Then for each \(0 \leq i \leq b\),
\[
x_b^a \sigma_{b,i}^a = q^{-ai}(\sum_{w \in \mathcal{G}_{b,\nu(d(i))}^a} T_w) = q^{-ai}x_b^a T_{d(i)}.
\]

Proof. Let \(\mathcal{H}(\mathcal{G}_{2,b})\) be the Hecke algebra associated with the Coxeter group with basis \(T_w, w \in \mathcal{G}_{2,b}\). The map \(\iota\) in (5.5.4) induces an \(R\)-module embedding
\[
\iota: \mathcal{H}(\mathcal{G}_{2,b}) \longrightarrow \mathcal{H}_2(r), T_w \longmapsto T_{\iota(w)}.
\]

This map satisfies the property: if \(w = s_{i_1} \cdots s_{i_l}\) is a reduced expression, then \(\iota(T_w') = T_{\iota(s_{i_1})} \cdots T_{\iota(s_{i_l})} = \iota(T_{s_{i_1}}') \cdots \iota(T_{s_{i_l}}')\), as \(\iota(s_{i_1}) \cdots \iota(s_{i_l})\) is reduced. Since \(\iota(L_i') = q^i L_{a+i}\), applying \(\iota\) to the formula in (5.5.2) for \(\mathcal{H}(\mathcal{G}_{2,b})\) yields the required one. \(\square\)

Consider a basis element \(b_\Lambda \in B_{\lambda\mu}^{(2)}\) with \(\Lambda = (a_{ij}) \in \Theta_2(n, r)_{\lambda\mu}\), where \(a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)})\). Then \(J = \{(i, j) \in I \mid a_{ij}^{(1)} \neq 0\}\); see (4.2.3). Note that \(\mathcal{D}_{\nu(\Lambda)} \subseteq \mathcal{D}_{\nu(|\Lambda|)} \subseteq \mathcal{G}_{\nu(|\Lambda|)} \subseteq \mathcal{G}_{\nu(\Lambda)}\). Thus,
\[
\mathcal{D}_{\nu(\Lambda)} \cap \mathcal{G}_\mu = (\mathcal{D}_{\nu(\Lambda)} \cap \mathcal{G}_{\nu(|\Lambda|)})(\mathcal{D}_{\nu(|\Lambda|)} \cap \mathcal{G}_\mu).
\] (5.7.1)

Every number \(\tilde{a}_{ij}\) defined in (4.2.5) defines a subgroup \(W_{\tilde{a}_{ij}}\) of \(W\) (by convention, \(W_{\tilde{a}_{ij}} = 1\) if \(|a_{ij}| = 0\)), as well as the subgroup \(\tilde{\mathcal{G}}_{\tilde{a}_{ij}}\) of \(W_{\tilde{a}_{ij}}\). By applying Lemma 5.6 to the case \(a = \tilde{a}_{ij}\) and \(b = |a_{ij}|\), we obtain an element
\[
d_{ij} := d_{\tilde{a}_{ij}, |a_{ij}|}^{(a_{ij}^{(1)})} \in \tilde{\mathcal{G}}_{\tilde{a}_{ij}, |a_{ij}|}.
\]

Proposition 5.8. For arbitrary \(\lambda, \mu \in \Lambda(n, r), \Lambda \in \Theta_m(n, r)_{\lambda\mu}\) and \(J = J(\Lambda)\), we have
\[
b_\Lambda = \left( \prod_{(i,j) \in J} q^{-\tilde{a}_{ij} a_{ij}^{(1)}} \right) x_\Lambda T_{d_{\tilde{a}_{ij}, |a_{ij}|}} \left( \prod_{(i,j) \in J} T_{d_{ij}} \right) \sum_{w \in \mathcal{D}_{\nu(\Lambda)} \cap \mathcal{G}_\mu} T_w,
\]
where the product of \(T_{d_{ij}}\) is taken over the order \(\leq\) given in (4.2.4).
Proof. Let \( d = d_{|A|} \). Then \( \nu(d) = \nu(|A|) \) satisfies \( \mathfrak{S}_\nu(|A|) = d^{-1} \mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu \). By (4.1.1) and (4.6.1),

\[
\mathfrak{b}_A = x_{\lambda} T_d \sigma^\Lambda \sum_{\nu \in \mathcal{D}_\nu(d) \cap \mathfrak{S}_\mu} T_{\nu} = \sum_{\nu \in \mathcal{D}_\nu^{-1}(d) \cap \mathfrak{S}_\lambda} T_u T_d x_{\nu(d)} \sigma^\Lambda \sum_{\nu \in \mathcal{D}_\nu^{-1}(d) \cap \mathfrak{S}_\mu} T_{\nu}. \tag{5.8.1}
\]

Then, by the definition,

\[
x_{\nu(d)} = \prod_{i,j=1}^{n} x_{[a_{ij}]}. \tag{5.8.2}
\]

Moreover, the terms \( x_{[a_{ij}]} \) commute with each other. Also, by definition,

\[
x_{\nu(d)} \sigma^\Lambda = \prod_{i,j=1}^{n} x_{[a_{ij}]} \sigma^{\tilde{a}_{ij}}. \tag{5.8.2}'
\]

Since \( \tilde{a}_{ij} \in A(|a_{ij}|, < 2) \) is computed by the partition dual to \( (a_{ij}^{(1)}) \), it follows that

\[
\tilde{a}_{ij} = \begin{cases} (0, \ldots, 0, 1, 0, \ldots, 0), & \text{if } a_{ij}^{(1)} \neq 0; \\ (0, \ldots, 0), & \text{otherwise}, \end{cases}
\]

where the 1 is at the \( a_{ij}^{(1)} \)-th position.

Note that, if \((i, j) \in \mathcal{J}\), then \( \sigma^{\tilde{a}_{ij}} = \sigma^{|a_{ij}|} a_{ij}^{(1)} \) is the \( a_{ij}^{(1)} \)-th elementary symmetric polynomial in \( L_{\tilde{a}_{ij}+1}, L_{\tilde{a}_{ij}+2}, \ldots, L_{\tilde{a}_{ij}+|a_{ij}|} \). We set \( \sigma^{\tilde{a}_{ij}} = 1 \) for \((i, j) \notin \mathcal{J}\). By Lemma 5.7 for \((i, j) \in \mathcal{J}\),

\[
x_{[a_{ij}]} \sigma^{\tilde{a}_{ij}} = x_{[a_{ij}]} |a_{ij}|^{a_{ij}} = q^{-a_{ij} a_{ij}^{(1)}} x_{[a_{ij}]} T_{d_{ij}} \sum_{x \in \mathcal{D}_{ij}} T_x,
\]

where \( \mathcal{D}_{ij} = \mathcal{D}_{[a_{ij}], \nu(d_{ij})} \) is the set of shortest right coset representatives of

\[
\mathfrak{S}_{\nu(d_{ij})} = \mathfrak{S}_{(a_{ij}, a_{ij}, 1, \ldots)} = d_{ij}^{-1} \mathfrak{S}_{[a_{ij}]} d_{ij} \cap \mathfrak{S}_{[a_{ij}]} \text{ in } A_{[a_{ij}]}.
\]

By substituting into (5.8.2)', we obtain that

\[
x_{\nu(d)} \sigma^{\tilde{a}_{ij}} = \prod_{(i, j) \in \mathcal{J}} \left( q^{-a_{ij} a_{ij}^{(1)}} x_{[a_{ij}]} T_{d_{ij}} \left( \sum_{x \in \mathcal{D}_{ij}} T_x \right) \right) \tag{5.8.2}'
\]

\[
= \left( \prod_{(i, j) \in \mathcal{J}} q^{-a_{ij} a_{ij}^{(1)}} \right) x_{\nu(d)} \prod_{(i, j) \in \mathcal{J}} T_{d_{ij}} \prod_{(i, j) \in \mathcal{J}} \left( \sum_{x \in \mathcal{D}_{ij}} T_x \right),
\]

where \((\leq)\) indicates the products are taken over the order \((\leq)\) defined in (4.2.4). The second equality is seen from the fact that if \((k, l) \leq (i, j)\), then \( T_{d_{kl}} \left( \sum_{x \in \mathcal{D}_{kl}} T_x \right) \)
commutes with \( \tilde{x}_{\lambda ij} \), and \( \sum_{x \in \mathcal{D}_{kl}} T_x \) with \( T_{d_{ij}} \). Then (5.8.1) becomes

\[
\mathbf{b}_{\lambda} = \sum_{u \in \mathcal{D}^{-1}_{\nu(d_{ij})} \cap \mathcal{G}_\lambda} T_u T_d x_{\nu(d)} \sigma^\lambda_{\nu(d)} \sum_{v \in \mathcal{D}_{\nu(d)} \cap \mathcal{G}_\mu} T_v
\]

\[
= \sum_{u \in \mathcal{D}^{-1}_{\nu(d_{ij})} \cap \mathcal{G}_\lambda} T_u T_d x_{\nu(d)} \prod_{i,j=1}^n q^{-\tilde{a}_{ij} a^{(1)}_{ij}} T_{d_{ij}} \prod_{i,j=1}^n (\sum_{x \in \mathcal{D}_{ij}} T_x) \sum_{v \in \mathcal{D}_{\nu(d)} \cap \mathcal{G}_\mu} T_v
\]

(5.8.3)

By Lemma 4.3,

\[
\prod_{i,j=1}^n (\sum_{x \in \mathcal{D}_{ij}} T_x) = \sum_{w \in \mathcal{D}_{\nu(d)} \cap \mathcal{G}_\mu} T_w,
\]

it follows by (5.7.1) that

\[
\prod_{i,j=1}^n (\sum_{x \in \mathcal{D}_{ij}} T_x) \sum_{v \in \mathcal{D}_{\nu(d)} \cap \mathcal{G}_\mu} T_v = \sum_{w \in \mathcal{D}_{\nu(d)} \cap \mathcal{G}_\mu} T_w \sum_{v \in \mathcal{D}_{\nu(d)} \cap \mathcal{G}_\mu} T_v = \sum_{w \in \mathcal{D}_{\nu(d)} \cap \mathcal{G}_\mu} T_w.
\]

Substituting into (5.8.3) gives the desired formula. \( \square \)

**Proof of Theorem 5.1(2).** Let \( w \) be the element given in (4.3.1) and assume that \( d_{\lambda} \) is the shortest representative of \( \mathcal{G}_\lambda w \mathcal{G}_\mu \). We claim that if \( d = d \cdot \prod_{(i,j) \in \mathcal{I}} d_{ij} \), then \( \mathcal{G}_\lambda \bar{d} \mathcal{G}_\mu = \mathcal{G}_\lambda w \mathcal{G}_\mu \). Indeed, for \( a = \tilde{a}_{ij}, b = |a_{ij}| \) and \( c = a^{(1)}_{ij} > 0 \), we have

\[
d_{ij} = t_{a+1} \cdot (s_{a+1} t_{a+1}) \cdot (s_{a+2} s_{a+1} t_{a+1}) \cdots (s_{a+c-1} s_{a+c-2} \cdots s_{a+1} t_{a+1})
\]

\[
= t_{a+1} \cdot (t_{a+2} s_{a+1}) \cdot (t_{a+3} s_{a+2} s_{a+1}) \cdots (t_{a+c} s_{a+c-1} s_{a+c-2} \cdots s_{a+1})
\]

\[
= t_{ij} \cdot s_{a+1} \cdot (s_{a+2} s_{a+1}) \cdot (s_{a+3} s_{a+2} s_{a+1}) \cdots (s_{a+c-1} s_{a+c-2} \cdots s_{a+1}),
\]

(thus, \( d_{ij} d_{kl} = d_{kl} d_{ij} \)) proving the claim. By Proposition 5.8 we have in the group algebra \( \mathbf{b}_{\lambda} = \mathcal{G}_\lambda \cdot d \cdot \mathcal{D}_{\nu(d)} \cap \mathcal{G}_\mu = \mathcal{G}_\lambda \cdot d_{\lambda} \cdot \mathcal{D}_{\nu(d)} \cap \mathcal{G}_\mu = \mathcal{G}_\lambda d_{\lambda} \mathcal{G}_\mu \). \( \square \)

**Example 5.9.** Let \( n = 2 \) and \( r = 3 \). For the matrix

\[
A = \begin{pmatrix}
(0,0) & (1,0) \\
(1,1) & (0,0)
\end{pmatrix} \in \Theta_2(2,3) \quad \text{with} \quad |A| = \begin{pmatrix}
0 & 1 \\
2 & 0
\end{pmatrix},
\]

we have \( \lambda = \text{ro}(|A|) = (1,2), \mu = \text{co}(|A|) = (2,1), \) and \( \nu(|A|) = (0,2,1,0) \). Then \( \mathcal{G}_\lambda = \{1, s_2\}, \mathcal{G}_\mu = \{1, s_1\}, d_{|A|} = s_1 s_2, \) and

\[
d = d_{|A|} d_{21} d_{12} = s_1 s_2 t_1 t_3 = s_0 s_1 s_0 s_2.
\]

But

\[
T_{|A|} T_{t_1} T_{t_3} = q^2 T_{0102} + q(q - 1) T_{1012} + (q - 1) T_{120102},
\]
where we write \( T_w = T_{i_1 \cdots i_r} \) whenever \( w = s_{i_1} \cdots s_{i_r} \) is reduced. Note that \( d_\lambda = s_0 s_1 s_0 s_2 \) with \( \nu(d_\lambda) = \nu(\lambda) = (0, 0, 1, 1, 0, 0, 0) \). By Proposition 5.8

\[
\begin{align*}
\mathfrak{b}_\lambda &= q^{-2}x_\lambda T_{|\lambda|} T_{i_1} T_{i_2} x_\mu \\
&= x_\lambda T_{10102} x_\mu + q^{-1}(q-1)x_\lambda T_{120102} x_\mu + q^{-1}(q-1)x_\lambda T_{120102} x_\mu \\
&= T_{\Theta_\Delta d \Theta_\mu} x_\mu + q^{-1}(q-1)T_{\Theta_\Delta d' \Theta_\mu} + q^{-1}(q-1)T_{\Theta_\Delta d'' \Theta_\mu},
\end{align*}
\]

where \( d' = s_1 s_0 s_1 s_0 s_2 \) and \( d'' = s_1 s_2 s_0 s_1 s_0 s_2 \).

**Remark 5.10.** The slim cyclotomic \( q \)-Schur algebra \( \mathcal{S}_2(n, r) \) is isomorphic to the hyperoctahedral Schur algebras (defined over \( \mathcal{B} = \mathbb{Z}[q, q^{-1}, Q, Q^{-1}] \) in indeterminates \( q, Q \)); see \([37\text{, Defs 3.2.1, 4.3.3}]\) and \([37\text{, Lem 4.3.4}]\). Note that our labelling \( 0, 1, 2, \ldots, r-1 \) of the type \( B/C \) Dynkin diagram corresponds to their labelling \( r, \ldots, 2, 1 \). The isomorphism can be induced from this correspondence. By \([7\text{, Rem.} 6.3]\), the type \( C \) Schur algebra \( \mathcal{S}_1 \) geometrically defined in \([7\text{, \S 6.1}]\) is isomorphic to the hyperoctahedral Schur algebra. Hence, \( \mathcal{S}_2(n, r) \) is isomorphic to \( \mathcal{S}_1 \).

6. **The cyclotomic Schur–Weyl duality: double centraliser property**

Let \( C = (c_{ij})_{n \times n} \) be the Cartan matrix of affine type \( A \); see, e. g., \([13\text{, (1.2.3.1)}]\).

**Definition 6.1.** Let \( \mathbf{U}_v(\mathfrak{sl}_n) \) be the quantum enveloping algebra over the field \( \mathbb{Q}(v) \) of rational functions in indeterminate \( v \) associated with the Cartan matrix \( C \). Then it can be presented by the generators \( E_i, F_i, K_i, K_i^{-1} \) \( (1 \leq i \leq n) \) and the relations (QGL1)–(QGL5) in \([13\text{, Thm 2.3.1}]\).

Define the quantum affine \( \mathfrak{g} \mathfrak{l}_n \) by

\[
\mathbf{U}_v(\widehat{\mathfrak{g} \mathfrak{l}_n}) = \mathbf{U}_v(\mathfrak{sl}_n) \otimes \mathbb{Q}(v)[c^+_s, c^-_s, c^+_2, c^-_2, \ldots, c^+_r, c^-_r],
\]

where \( c^+_s, c^-_s \) \( (s \in \mathbb{Z}^+) \) are commuting indeterminates.

The algebra \( \mathbf{U}_v(\widehat{\mathfrak{g} \mathfrak{l}_n}) \) is also a Hopf algebra with the structure maps \( \Delta, \varepsilon, \) and \( \sigma \) acting on the \( \mathfrak{sl}_n \) part as usual plus the following extra action on the \( c^+_s \):

\[
\Delta(c^+_s) = c^+_s \otimes 1 + 1 \otimes c^+_s, \quad \varepsilon(c^+_s) = 0, \quad \sigma(c^+_s) = -c^+_s.
\]

This Hopf algebra is also isomorphic to the quantum loop algebra of \( \mathfrak{g} \mathfrak{l}_n \), known as the Drinfeld new realisation in \([22]\); see \([13\text{, Thm 2.5.3}]\).

The quantum group \( \mathbf{U}_v(\widehat{\mathfrak{g} \mathfrak{l}_n}) \) has a second construction in terms of a double Ringel–Hall algebra. This new construction is crucial in the study of the (enhanced) affine quantum Schur–Weyl duality as presented in \([13]\). In this section, we investigate new applications of this theory to its cyclotomic counterpart. We first briefly review the definition of the double Ringel–Hall algebra associated with a cyclic quiver.

Let \( \Delta = \Delta(n) \) \( (n \geq 2) \) be the cyclic quiver with vertex set \( I = \mathbb{Z}/n\mathbb{Z} \) and arrow set \( \{ i \rightarrow i + 1 \}_{i \in I} \). Let \( \mathfrak{H}_0(n) = \text{span}\{ u_A \mid A \in \Theta_\Delta(n) \} \) be the Ringel–Hall algebra over \( \mathcal{Z} := \mathbb{Z}[v, v^{-1}] \) associated with \( \Delta(n) \) and let \( \mathfrak{H}_0(n) = \mathfrak{H}_0(n) \otimes \mathcal{Z} \mathbb{Q}(v) \). There are extended Ringel–Hall algebras over \( \mathbb{Q}(v) \)

\[
\mathfrak{H}_0(n)^{>0} = \mathfrak{H}_0(n) \otimes \mathbb{Q}(v)[K^{\pm 1}_1, \ldots, K^{\pm 1}_r]
\]

\footnote{The quantum enveloping algebra \( \mathbf{U}_v(\widehat{\mathfrak{g} \mathfrak{l}_n}) \) here is called the extended quantum affine \( \mathfrak{g} \mathfrak{l}_n \) defined in \([13\text{, p. 43}]\). The non-extended version is defined in \([13\text{, \S 1.3}]\).}
and, similarly, $\mathcal{H}_0(n)^0$ which possess the Green–Xiao Hopf algebra structure. By considering a certain skew–Hopf pairing $\psi : \mathcal{H}_0(n)^0 \times \mathcal{H}_0(n)^0 \to \mathbb{Q}(v)$, one defines the Drinfeld double $\hat{\mathcal{H}}_0(n)$ as the quotient algebra of the free product $\mathcal{H}_0(n)^0 \ast \mathcal{H}_0(n)^0$ by an ideal defined by $\psi$; see [13, §2.1]. The double Ringel–Hall algebra is a reduced version of $\hat{\mathcal{D}}_0(n)$:

$$\mathcal{D}_0(n) = \hat{\mathcal{D}}_0(n)/\mathfrak{J} \cong \mathcal{H}_0(n)^+ \otimes \mathbb{Q}(v)[K_1^{\pm 1}, \ldots, K_n^{\pm 1}] \otimes \mathcal{H}_0(n)^-$$

where $\mathfrak{J}$ denotes the ideal generated by $1 \otimes K_\alpha - K_\alpha \otimes 1$ for all $\alpha \in \mathbb{Z}I$ and $\mathcal{H}_0(n)^+ = \mathcal{H}_0(n)$, $\mathcal{H}_0(n)^- = \mathcal{H}_0(n)^{\text{op}}$.

Each vertex $i \in I$ of $\Delta(n)$ defines a simple representation which in turn defines generators $u_i^+$ and $u_i^-$. There are so-called Schiffmann–Hubery generators $z_i^+$ and $z_i^-$ for $\mathcal{D}_0(n)$ constructed in [13, §2.3]. By [13, Thm 2.3.1], there is a Hopf algebra isomorphism from $U_v(\mathfrak{g}_n)$ to $\mathcal{D}_0(n)$ given by

$$E_i \mapsto u_i^+, F_i \mapsto u_i^-, K_i^{\pm} \mapsto K_i^{\pm}, c_i^+ \mapsto z_i^+.$$

We identify the two algebras under this isomorphism.

Let $\Omega$ be a free $\mathbb{Z}$-module with basis $\{\omega_i \mid i \in \mathbb{Z}\}$ and let $\Omega = \Omega \otimes \mathbb{Z} \mathbb{Q}(v)$. Using the action given in [13, (3.5.0.1)], $\Omega$ becomes a left $\mathcal{D}_0(n)$-module. Hence, the Hopf algebra structure on $\mathcal{D}_0(n)$ induces a $\mathcal{D}_0(n)$-module structure on $\Omega^{\text{gr}}$. On the other hand, $\Omega^{\text{gr}}$ has a right $\mathcal{H}_0(r)_z$-module structure; see [13, (3.3.0.4)], commuting with the left $\mathcal{D}_0(n)$-module structure. Thus, we obtain a $\mathbb{Q}(v)$-algebra homomorphism

$$\xi_r : \mathcal{D}_0(n) \rightarrow \text{End}_{\mathcal{H}_0(r)}(\Omega^{\text{gr}}) = \mathcal{S}_0(n, r)_{\mathbb{Q}(v)}. \quad (6.1.1)$$

Let $\mathcal{D}_0(n)^0_Z$ denote the $\mathbb{Z}$-subalgebra of $\mathcal{D}_0(n)$ generated by $K_i^{\pm 1}$ and $[K_i^{0}]$ for $i \in I$ and $t > 0$ and put $\mathcal{D}_0(n)^\pm = \mathcal{H}_0(n)^\pm$; see [13, §2.4] for the details.

**Theorem 6.2.** Maintain the notations introduced above.

1. The $\mathbb{Z}$-submodule

$$\mathcal{D}_0(n)_Z := \mathcal{H}_0(n)^+ \mathcal{D}_0(n)^0 \mathcal{H}_0(n)^- \cong \mathcal{H}_0(n)^+ \otimes \mathcal{D}_0(n)^0 \otimes \mathcal{H}_0(n)^-$$

is a Hopf $\mathbb{Z}$-subalgebra of $\mathcal{D}_0(n)$.

2. There is a $\mathbb{Z}$-algebra isomorphism $\mathcal{S}_0(n, r)_Z \cong \text{End}_{\mathcal{H}_0(r)}(\Omega^{\text{gr}})$.

3. The restriction of $\xi_r$ to $\mathcal{D}_0(n)_Z$ induces a $\mathbb{Z}$-algebra epimorphism

$$\xi_r : \mathcal{D}_0(n)_Z \rightarrow \text{End}_{\mathcal{H}_0(r)}(\Omega^{\text{gr}}).$$

4. The map $\xi_r$ induces an $\mathcal{R}$-algebra epimorphism

$$\xi_r, \mathcal{R} : \mathcal{D}_0(n)_Z \otimes \mathcal{R} \rightarrow \mathcal{S}_0(n, r, \mathcal{R}).$$

Note that (1) is done in [25]; (2) is given in [13, Prop. 3.3.1]; (3) can be derived from [13, Thm 3.7.7] and the proof of [13, Thm 3.8.1], and (4) follows from (2). Note also that an explicit action of the generators of $\mathcal{D}_0(n)_Z$ on $\Omega$ can be found in [13, Prop. 3.5.3].

We now extend the algebra homomorphism to slim cyclotomic $q$-Schur algebras. Recall the canonical epimorphism $\epsilon_m : \mathcal{H}_0(r) \rightarrow \mathcal{H}_m(r)$ given in (2.1.1) under the
assumption that all the \( u_i \in \mathcal{R} \) in \( m \) are invertible. Regarding \( \mathcal{H}_m(r) \) as an \( \mathcal{H}_0(r) \)-module via this map, we obtain the canonical right \( \mathcal{H}_m(r) \)-module isomorphism

\[
\mathcal{T}_\lambda(n, r) \otimes_{\mathcal{H}_0(r)} \mathcal{H}_m(r) = \bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H}_0(r) \otimes_{\mathcal{H}_0(r)} \mathcal{H}_m(r) \cong \bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H}_m(r) = \mathcal{I}_m(n, r).
\]

Thus, we obtain an \( \mathcal{R} \)-algebra homomorphism

\[
\tilde{\varepsilon}_m : \mathcal{S}_\lambda(n, r) \rightarrow \mathcal{S}_m(n, r), \ f \mapsto f \otimes \text{id}.
\]

**Proposition 6.3.** Suppose that all the \( u_i \in \mathcal{R} \) are invertible. The \( \mathcal{R} \)-algebra homomorphism \( \tilde{\varepsilon}_m \) is surjective. In particular, the map \( \xi_{r, \mathcal{R}} \) extends to the \( \mathcal{R} \)-algebra epimorphism

\[
\tilde{\varepsilon}_m \circ \xi_{r, \mathcal{R}} : \mathcal{D}_\lambda(n) \otimes \mathcal{R} \rightarrow \mathcal{S}_m(n, r) \mathcal{R}.
\]

**Proof.** Consider the linearly independent elements \( \Phi^A, A \in \Theta_m(n, r) \), for \( \mathcal{S}_\lambda(n, r) \), which are defined similarly to the basis elements \( \Phi^A \) with all \( L_i \) replaced by \( X_i \); see Remarks 3.3, 3.9 and 4.9. By definition, we see easily that \( \tilde{\varepsilon}_m(\Phi^A) = \Phi^A \). In other words, the images of these elements form a basis for \( \mathcal{S}_m(n, r) \). Hence, \( \tilde{\varepsilon}_m \) is surjective.

**Remarks 6.4.**
1. The surjectivity of \( \tilde{\varepsilon}_m \) was given in \([42]\) Prop. 5.7(a).
2. For any \( t \geq 0 \), let \( \mathcal{D}_\lambda(n)^{(t)} = U'_v(\mathfrak{g}l_n) \otimes \mathbb{Q}(v)[c^+_1, c^-_1, c^+_2, c^-_2, \ldots, c^+_t, c^-_t] \). This is a Hopf subalgebra of \( \mathcal{D}_\lambda(n) \). In fact, by \([13]\) Thm 3.8.1(2)], if \( t \) satisfies \( r = tn + t0 \), where \( 0 < t0 < n \), then the restriction of the map \( \xi_r \) in \((6.1.1)\) gives an epimorphism

\[
\xi_r^{(t)} : \mathcal{D}_\lambda(n)^{(t)} \rightarrow \mathcal{S}_\lambda(n, r) \mathbb{Q}(v).
\]

This together with \( \tilde{\varepsilon}_m \) induces an epimorphism

\[
\xi_{r, m}^{(t)} : \mathcal{D}_\lambda(n)^{(t)} \rightarrow \mathcal{S}_m(n, r) \mathbb{Q}(v).
\]

It would be interesting to conjecture that there exists a \( t \), independent of \( r \), such that \( \xi_{r, m}^{(t)} \) is surjective for all \( r \geq 0 \).

Note that, if \( U'_v(\mathfrak{g}l_n) \) denotes the subalgebra generated by \( E_i, F_i, K_j^{\pm 1} \) for \( 1 \leq i, j \leq n, i \neq n \), then \( \mathcal{D}_\lambda(n)^{(t)} \) contains the subalgebra

\[
U_v^{(t)}(\mathfrak{g}l_n) := U'_v(\mathfrak{g}l_n) \otimes \mathbb{Q}(v)[c^+_1, c^-_1, c^+_2, c^-_2, \ldots, c^+_t, c^-_t].
\]

If the conjecture were true and the restriction of \( \xi_{r, m}^{(t)} \) to \( U_v^{(t)}(\mathfrak{g}l_n) \) remained surjective, then the new object \( U_v^{(t)}(\mathfrak{g}l_n) \) would be called the *cyclotomic quantum* \( \mathfrak{g}l_n \).

3. We will prove in \([16]\) that if \( \mathcal{H}_m(r) \) has a semisimple bottom in the sense of \([27]\), then the algebra homomorphism

\[
\xi_{r, m}^\vee : \mathcal{H}_m(r) \rightarrow \text{End}_{\mathcal{S}_m(n, r)}(\Omega^{\otimes r})
\]

is surjective. So it is natural to conjecture that the cyclotomic double centraliser property holds in general.
7. The cyclotomic Schur–Weyl duality: Morita equivalence

As part of the Schur–Weyl duality, it is well-known that, for any field $\mathcal{K}$, the $q$-Schur algebra $S_q(n, r)$ is Morita equivalent to the Hecke algebra $H_q(r)$ if $n \geq r$ and $q$ is not a root of the Poincaré polynomial of $S_r$. We now establish a similar result for the cyclotomic counterpart.

As in previous sections, $\mathcal{R}$ is a commutative ring and $q \in \mathcal{R}$ is invertible. Recall the notations $H_m(r) = H_m(r)_q$ and $S_m(n, r) = S_m(n, r)_q$ and from Proposition 4.10 that $S_q(n, r)$ is (isomorphic to) a subalgebra of $S_m(n, r)$.

Every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n, r)$ gives a diagonal matrix $D_{\lambda} = \text{diag}(\lambda)$ which in turn defines a matrix $D_{\lambda}^{(m)} \in \Theta_m(n, r)$ via the embedding $\iota^{(m)}$ in (4.9.1). Then $\Phi_{D_{\lambda}^{(m)}} = \iota_{\lambda}$ is the idempotent map defined in (2.3.1). Note that all $\iota_{\lambda}$ live in the $q$-Schur subalgebra $S_q(n, r)$ of $S_m(n, r)$; see Proposition 4.10. In particular, $\iota_{\lambda}^2 = \iota_{\lambda}$, $\iota_{\lambda} \iota_{\mu} = \delta_{\lambda, \mu} \iota_{\lambda}$, for $\lambda, \mu \in \Lambda(n, r)$ and

$$\sum_{\lambda \in \Lambda(n, r)} \iota_{\lambda} = 1. \quad (7.0.1)$$

Moreover, like the $q$-Schur algebra case, we have, for each $A \in \Theta_m(n, r)$, the following relations in $\mathcal{S}_m(n, r)$

$$\Phi_A \iota_{\lambda} = \begin{cases} \Phi_A, & \text{if } \lambda = \text{co}(\lambda) \iota_{\lambda}; \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \iota_{\lambda} \Phi_A = \begin{cases} \Phi_A, & \text{if } \lambda = \text{ro}(\lambda) \iota_{\lambda}; \\ 0, & \text{otherwise}. \end{cases}$$

In the rest of the section, we assume $n \geq r$ and let $\omega = (1, \ldots, 1, 0, \ldots, 0) \in \Lambda(n, r)$. Then $H_m(r)$ is a centraliser algebra of $\mathcal{S}_m(n, r)$ and (2.3.2) becomes

$$I_\omega \mathcal{S}_m(n, r) I_\omega \cong H_m(r).$$

We now consider some particular basis elements given in Theorem 4.8. For $\lambda, \mu \in \Lambda(n, r)$, let $\Phi_{\lambda, \mu} = \Phi_{A^{(m)}} \in \mathcal{S}_m(n, r)$, where $A = \theta(\lambda, 1, \mu)$ as defined in (4.1.2). In other words,

$$A = (|R_i^\lambda \cap R_j^\mu|)_{1 \leq i, j \leq n}$$

where $R_i^\lambda, R_j^\mu$ are defined in (4.1.3). Thus, by definition,

$$\Phi_{\lambda, \mu}(x_{\nu} h) = \delta_{\nu, \mu} x_{\lambda} \sum_{w \in S_\lambda \cap S_\mu} T_w h, \quad \forall h \in H_m(r), \quad (7.0.2)$$

where $S_{\lambda \cap \mu} = S_\lambda \cap S_\mu$. In particular, if $\lambda = \mu = \nu$, then

$$\Phi_{\lambda, \lambda}(x_{\lambda}) = x_{\lambda} = \iota_{\lambda}(x_{\lambda}) \quad \text{or} \quad \Phi_{\lambda, \lambda} = \iota_{\lambda}. \quad (7.0.3)$$

For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n, r)$, let

$$P_\lambda(q) = P_{S_\lambda}(q) = \sum_{w \in S_\lambda} q^{\ell(w)}.$$

Then $x_{\lambda}^2 = P_\lambda(q) x_{\lambda}$ for $\lambda \in \Lambda(n, r)$. Note that every $P_\lambda(q)$ is a factor of $P_{S_\lambda}(q)$.

**Proposition 7.1.** Keep the notations above and suppose that $n \geq r$. If $P_{S_\lambda}(q)$ is invertible in $\mathcal{R}$, then $\mathcal{S}_m(n, r)$ and $H_m(r)$ are Morita equivalent.
By (7.0.1), we only need to show
\[ P \quad \text{(7.0.1)} \]
Proof.
We apply the following general fact: for a ring \( S \) Since
\[ \Phi_{\lambda,\omega}(x_{\lambda}) = \Phi_{\lambda,\omega}(x_{\omega} \sum_{w \in \Theta_{\lambda}} T_w) = \Phi_{\lambda,\omega}(x_{\omega} \cdot x_{\lambda}) = x_{\lambda} \cdot x_{\lambda} = x_{\lambda}^2 = P_{\lambda}(q)x_{\lambda} = P_{\lambda}(q)\Phi_{\lambda,\lambda}(x_{\lambda}). \]
This implies that
\[ \Phi_{\lambda,\omega}(x_{\lambda}) = P_{\lambda}(q)\Phi_{\lambda,\lambda}. \quad \text{(7.1.2)} \]
Hence, by the hypothesis on \( P_{\lambda}(q) \), we obtain that
\[ I_{\lambda} = \Phi_{\lambda,\lambda} = P_{\lambda}(q)^{-1}\Phi_{\lambda,\omega}(x_{\lambda}). \]
This finishes the proof. \( \square \)

Remarks 7.2. (1) The proposition above is a (slim) cyclotomic analogue of \[17\] Appendix, Lem. 1.4 which states that if \( n \geq r \) and \( P_{\Theta}(q) \) is invertible in \( R \), then \( S(n,r)_{\Theta} \) is Morita equivalent to \( H_{\Theta}(r)_{\Theta} \). Note that this fact is proved through a category equivalence in \[13\] Thm 4.1.3 under the assumption that \( q \) is not a root of unity.

(2) For the cyclotomic \( q \)-Schur algebra, this Morita equivalence does not seem to be established in the literature. It is easy to see that the proof above does not work in this case, since, for the idempotent \( \Phi_{\lambda,\lambda} \) associated with a multipartition \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)}) \) of \( r \), formula \( \text{(7.1.2)} \) can not be established whenever \( |\lambda^{(m)}| < r \).

Let \( H_{\Theta}(r)_{\Theta} \)-Mod (resp., \( S_{\Theta}(n,r)_{\Theta} \)-Mod) be the category of left \( H_{\Theta}(r)_{\Theta} \)-modules (resp., \( S_{\Theta}(n,r)_{\Theta} \)-modules). By Proposition 7.1 we immediately have the following.

Corollary 7.3. Let \( R \) be a commutative ring in which \( P_{\Theta}(q) \) is invertible and assume \( n \geq r \). Then the categories \( H_{\Theta}(r)_{\Theta} \)-Mod and \( S_{\Theta}(n,r)_{\Theta} \)-Mod are equivalent.

By a standard argument for Morita equivalence, the category equivalence is given by the following two functors
\[ \mathcal{F} : \mathcal{H}\text{-Mod} \rightarrow \mathcal{S}\text{-Mod}, \quad N \mapsto Se \otimes_H N, \]
\[ \mathcal{G} : \mathcal{S}\text{-Mod} \rightarrow \mathcal{H}\text{-Mod}, \quad M \mapsto eS \otimes_S M, \quad \text{(7.3.1)} \]
where \( e = I_{\omega}, S = S_{\Theta}(n,r)_{\Theta} \) (\( n \geq r \)), and \( H = H_{\Theta}(r)_{\Theta} = eS e \).

Let \( \mathcal{K} \) be a field and let \( H_{\Theta}(r)_{\mathcal{K}}\text{-mod} \) (resp., \( S_{\Theta}(n,r)_{\mathcal{K}}\text{-mod} \)) be the category of finite dimensional left \( H_{\Theta}(r)_{\mathcal{K}} \)-modules (resp., \( S_{\Theta}(n,r)_{\mathcal{K}} \)-modules). Then the functors above induce a category equivalence between \( H_{\Theta}(r)_{\mathcal{K}}\text{-mod} \) and \( S_{\Theta}(n,r)_{\mathcal{K}}\text{-mod} \). In the next section, we will describe the simple objects in \( S_{\Theta}(n,r)_{\mathcal{K}}\text{-mod} \).
8. Irreducible objects in $\mathcal{S}_m(n,r)_\mathcal{K}$-mod

Throughout this section, let $\mathcal{K}$ be a field and let $0 \neq q \in \mathcal{K}$. We now use the Morita equivalence in the previous section and the classification of irreducible $\mathcal{H}_m(r)_\mathcal{K}$-modules to obtain a classification of irreducible $\mathcal{S}_m(n,r)_\mathcal{K}$-modules.

The classification of irreducible $\mathcal{H}_m(r)_\mathcal{K}$-modules is well known. In [33, 20], it is done by using a cellular basis, while more precise labelling in terms of Kleshchev multipartitions is given in [5, 3]. We now have a brief review.

Recall that a partition $\lambda$ of $r$ is a sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $|\lambda| = \sum \lambda_i = r$. By definition, an $m$-fold multipartition of $r$ (or simply, an $m$-multipartition) is a sequence of $m$ partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ such that $\sum_{1 \leq i \leq m} |\lambda^{(i)}| = r$. We use $\mathcal{P}_m(r)$ to denote the set of $m$-multipartitions of $r$.

The dominance order on $\mathcal{P}_m(r)$ is a partial order $\geq$ defined by setting, for $\lambda, \mu \in \mathcal{P}_m(r)$, $\lambda \geq \mu$ if, for all $j, k$,

$$\sum_{1 \leq i \leq k-1} |\lambda^{(i)}| + \sum_{1 \leq i \leq j} \lambda_i^{(k)} \geq \sum_{1 \leq i \leq k-1} |\mu^{(i)}| + \sum_{1 \leq i \leq j} \mu_i^{(k)}.$$ 

To each $\lambda \in \mathcal{P}_m(r)$ we can associate a left $\mathcal{H}_m(r)_\mathcal{K}$-module $S^\lambda = \mathcal{H}_m(r)_\mathcal{K} z_\lambda$, where $z_\lambda = \tau(z_\lambda) \in \mathcal{H}_m(r)_\mathcal{K}$ with $\tau, z_\lambda$ defined in (2.0.2) and [28, (2.1),(2.9)] (cf. [20, Def. 3.28]). These modules are called Specht modules. Each Specht module is naturally equipped with a bilinear form. Set $D^\lambda = S^\lambda / \text{rad} S^\lambda$, where $\text{rad} S^\lambda$ is the radical of the bilinear form. By the cellular algebra theory, those nonzero $D^\lambda$ form a complete set of irreducible $\mathcal{H}_m(r)_\mathcal{K}$-modules.

By determining Kashiwara’s crystal graphs, Ariki and Mathas gave in [5] another classification of irreducible $\mathcal{H}_m(r)_\mathcal{K}$-modules in terms of Kleshchev multipartitions. Let $\mathcal{K} \mathcal{P}_m(r) \subseteq \mathcal{P}_m(r)$ denote the subset of Kleshchev multipartitions of $r$. We now summarise these results in the following theorem.

Theorem 8.1. Suppose that $\mathcal{K}$ is a field with $q, u_1, \ldots, u_m$ all nonzero.

1. ([20, Thm 3.30]) The set $\{ D^\lambda \mid \lambda \in \mathcal{P}_m(r), D^\lambda \neq 0 \}$ forms a complete set of non-isomorphic irreducible left $\mathcal{H}_m(r)_\mathcal{K}$-modules. Further, these modules are absolutely irreducible.

2. ([5, Thm C, 3]) $D^\lambda \neq 0$ if and only if $\lambda \in \mathcal{K} \mathcal{P}_m(r)$.

3. Let $\lambda, \mu$ be $m$-multipartitions of $r$ with $\mu \in \mathcal{K} \mathcal{P}_m(r)$. If $[S^\lambda : D^\mu] \neq 0$, then $\lambda \geq \mu$. In particular, $[S^\lambda : D^\lambda] = 1$ for all $\lambda \in \mathcal{K} \mathcal{P}_m(r)$.

The map $\tau$ defined in (2.0.2) can be extended to an anti-automorphism of $\mathcal{S}_m(n,r)$ (see [20]). We may turn a module over these algebras to its opposite side module. For example, we may twist the $(\mathcal{S}_m(n,r), \mathcal{H}_m(r))$-bimodule structure on $\mathcal{T}_m(n,r)$ into the $(\mathcal{H}_m(r), \mathcal{S}_m(n,r))$-bimodule $\mathcal{T}_m(n,r)^\tau$ with the action $h \star x \star s = \tau(s)x\tau(h)$ for all $h \in \mathcal{H}_m(r), x \in \mathcal{T}_m(n,r), s \in \mathcal{S}_m(n,r)$. Note also that there is an obvious left $\mathcal{H}_m(r)$-module isomorphism $\mathcal{T}_m(n,r)^\tau \cong \bigoplus_{\lambda \in \Lambda(n,r)} \mathcal{H}_m(r)x_\lambda$.

---

5See [3] for the definition. If $q$ is not a root of unity, and the parameters $u_1, \ldots, u_m$ are powers of $q$, see, e.g., [32, 3.5.4] for a combinatorial definition.
For $\mu \in K\mathcal{P}_m(r)$, let
\[ L(\mu) = T_m(n, r) \otimes_{\mathcal{H}_m(r), x} D^\mu. \]
By Proposition 7.1 and Theorem 8.1, we have the following Corollary.

**Corollary 8.2.** Suppose $P_{\mathcal{S}_r}(q) \neq 0$ in $\mathcal{X}$. If $n \geq r$, then \( \{L(\mu) \mid \mu \in K\mathcal{P}_m(r)\} \) is a complete set of non-isomorphic irreducible $S_m(n, r)$-modules. Moreover, we have an $S_m(n, r, x)$-module isomorphism
\[ L(\mu) \cong \text{Hom}_{\mathcal{H}_m(r), x}(t_{\omega}S_m(n, r, x), D^\mu) \]

*Proof.* We only need to prove the second assertion. Let $S = S_m(n, r, x)$. Then, for $e = I_\omega$, $eSe \cong \mathcal{H}_m(r, x)$. Since $S$ is Morita equivalent to $eSe$, the multiplication map $Se \otimes_{eSe} eS \to S$ is an $(S, S)$-bimodule isomorphism (see, e.g., [9 Prop. 1.9(2)]). Then the following (left) $S$-module isomorphisms hold:
\[ L(\mu) \cong \text{Hom}_S(S, L(\mu)) \cong \text{Hom}_S(Se \otimes_{eSe} eS, L(\mu)) \]
\[ \cong \text{Hom}_{eSe}(eS, \text{Hom}_S(Se, L(\mu))) \cong \text{Hom}_{eSe}(eS, D^\mu), \]
since $\text{Hom}_S(Se, L(\mu)) \cong eL(\mu) \cong D^\mu$. \hfill \Box

Recall from [20] for the notion of semi-standard $\lambda$-tableaux of type $\mu$.

**Lemma 8.3.** For an $m$-multipartition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ and a composition $\mu \in \Lambda(n, r)$, if the set $T^ss(\lambda, \mu)$ of semistandard $\lambda$-tableaux of type $^\circ \mu := ((0), \ldots, (0), \mu)$ is not empty, then $\lambda^{(1)} + \cdots + \lambda^{(m)} \succeq \mu$.

*Proof.* If $S = (S^{(1)}, \ldots, S^{(m)}) \in T^ss(\lambda, \mu)$, then each $S^{(i)}$ is a semistandard tableau of type, say $\mu^{(i)}$. Thus, $\lambda^{(i)} \succeq \mu^{(i)}$ for all $i$. Since $\mu^{(1)} + \cdot \cdots + \mu^{(m)} = \mu$, it follows that $\lambda^{(1)} + \cdot \cdots + \lambda^{(m)} \succeq \mu$. \hfill \Box

For a partition $\lambda$, we use $l(\lambda)$ to denote the length of $\lambda$. Set
\[ K\mathcal{P}_m(n, r) = \{(\lambda^{(1)}, \ldots, \lambda^{(m)}) \mid l(\lambda^{(i)}) \leq n, 1 \leq i \leq m\}. \]

**Lemma 8.4.** Assume now that $q \in \mathcal{X}$ is not a root of unity. Suppose that $n < r$ and $\mu \in K\mathcal{P}_m(r)$. Then $T_m(n, r) \otimes_{\mathcal{H}_m(r), x} D^\mu \neq 0$ if and only if $\mu \in K\mathcal{P}_m(n, r)$.

*Proof.* Assume $N \geq r > n$ and consider the idempotent $f = \sum_{\lambda \in \Lambda(n, r)} l_{\tilde{\lambda}} \in S = S_m(N, r, x)$, where $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0) \in \Lambda(N, r)$. Then $S_m(n, r, x) \cong fSf$. By (8.2.1), we have
\[ L(\mu) \cong fL(\mu) \cong \text{Hom}_{\mathcal{H}_m(r)}(eSf, D^\mu) \cong \text{Hom}_{\mathcal{H}_m(r), x}(T_m(n, r, x)^\tau, D^\mu). \] (8.4.1)

Here we have used the $(\mathcal{H}_m(r, x), S_m(n, r, x))$-bimodule isomorphism $eSf \cong T_m(n, r, x)^\tau$.

By [20 Cor. 4.15], for each $\lambda \in \Lambda(n, r)$, there is a Specht module filtration of $\mathcal{H}_m(r, x)_{x\lambda}$ such that $[\mathcal{H}_m(r, x)_{x\lambda} : S^\nu]$ equals the number of semistandard $\nu$-tableaux of type $^\circ \lambda$, where $\nu \in P_m(n)$. By Lemma 8.3 if $S^\nu$ occurs in the filtration of $\mathcal{H}_m(r, x)_{x\lambda}$, then $\sum_{1 \leq i \leq m} \nu^{(i)} \succeq \lambda$ and, consequently, $\nu \in K\mathcal{P}_m(n, r)$.

Thus, if $T_m(n, r) \otimes_{\mathcal{H}_m(r), x} D^\mu \neq 0$, by (8.4.1) and (8.4.1), $D^\mu$ is a composition factor of $S^\nu$ for some $\nu \in K\mathcal{P}_m(n, r)$. Then, by Theorem 8.1(3), $\nu \succeq \mu$. Since $\nu \in K\mathcal{P}_m(n, r)$, we have $\mu \in K\mathcal{P}_m(n, r)$.

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\[ ^6 \text{See, e.g., [10, Prop. 3.5] for various left module structures on the Hom-space.} \]
Conversely, suppose that \( \mu = (\mu^{(1)}, \ldots, \mu^{(m)}) \in \mathcal{KP}_m(n, r) \). Then the irreducible \( \mathcal{H}_m(r)_\mathcal{X} \)-module \( D^\mu = \text{hd}(S^\mu) \) by Theorem 8.1 inflate \( D^\mu \) to an irreducible \( \mathcal{H}_\delta(r)_\mathcal{X} \)-module via (2.0.1) and, by [32, Lem. 4.1.1], there exists a multisegment \( s = s^c_{\mu,m} = (s_1, \ldots, s_t) \) consisting of column residual segments of \( \mu \) such that the simple \( \mathcal{H}_\delta(r)_\mathcal{X} \)-module \( V_s \) associated with \( s \) is isomorphic to \( D^\mu \). Since \( \mu \in \mathcal{KP}_m(n, r) \), each of the \( s_t \)'s has length at most \( n \). Now, by [12, Thm 6.6], we obtain a simple \( \mathcal{S}_\delta(n, r)_\mathcal{X} \)-module \( T_\delta(n, r) \otimes_{\mathcal{H}_\delta(r)_\mathcal{X}} V_s \). Hence,
\[
0 \neq T_\delta(n, r) \otimes_{\mathcal{H}_\delta(r)_\mathcal{X}} V_s \cong T_\delta(n, r) \otimes_{\mathcal{H}_\delta(r)_\mathcal{X}} \mathcal{H}_m(r)_\mathcal{X} \otimes_{\mathcal{H}_m(r)_\mathcal{X}} V_s \\
\cong T_\delta(n, r) \otimes_{\mathcal{H}_m(r)_\mathcal{X}} D^\mu,
\]
as desired. \( \square \)

Now we are ready to prove the main result of this section.

**Theorem 8.5.** Assume \( q \in \mathcal{X} \) is not a root of unity. For arbitrary positive integers \( n, r \), the following set
\[
\{ T_m(n, r) \otimes_{\mathcal{H}_m(r)_\mathcal{X}} D^\mu \mid \mu \in \mathcal{KP}_m(n, r) \}
\]
forms a complete set of non-isomorphic irreducible \( \mathcal{S}_m(n, r)_\mathcal{X} \)-modules.

**Proof.** If \( n \geq r \), then \( \mathcal{KP}_m(n, r) = \mathcal{KP}_m(r) \) and the result can be seen in Corollary 8.2.

Now suppose now \( n < r \leq N \). Then there exists an idempotent \( f \in \mathcal{S}_m(N, r)_\mathcal{X} \) such that
\[
\{ f(T_m(N, r) \otimes_{\mathcal{H}_m(r)_\mathcal{X}} D^\mu) \mid \mu \in \mathcal{KP}_m(N, r) = \mathcal{KP}_m(r) \} \setminus \{0\}
\]
forms a complete set of non-isomorphic irreducible \( \mathcal{S}_m(n, r) \)-modules. Since
\[
f(T_m(n, r) \otimes_{\mathcal{H}_m(r)_\mathcal{X}} D^\mu) \cong T_m(n, r) \otimes_{\mathcal{H}_m(r)_\mathcal{X}} D^\mu,
\]
which is nonzero if and only if \( \mu \in \mathcal{KP}_m(n, r) \) by Lemma 8.4. Hence, our assertion follows. \( \square \)

**Remark 8.6.** The hypothesis that \( q \) is not a root of unity in Theorem 8.5 is stronger than the hypothesis that \( P_{\mathcal{E}_\delta}(q) \neq 0 \) in \( \mathcal{X} \). This is required in order to use [12, Thm 6.6]. There should be a direct proof under the latter hypothesis.

We also note that when \( q \) is not a root of unity the algebra \( \mathcal{H}_m(r)_\mathcal{X} \), and hence \( \mathcal{S}_m(n, r)_\mathcal{X} \), can still be non-semisimple.

**Appendix A. Proof of Theorem 4.7**

Let \( \leq \) be the Bruhat order on \( \mathcal{S}_r \), that is, \( w' \leq w \) if \( w' \) is a subexpression of some reduced expression of \( w \). For any \( y, w \in \mathcal{S}_r \), there is an element \( y \ast w \in \mathcal{S}_r \) such that \( \ell(y \ast w) \leq \ell(y) + \ell(w) \) and
\[
T_y T_w = \sum_{z \leq y \ast w} f_{y, w, z} T_z,
\]
where \( f_{y,w} \in \mathcal{R} \); see [14, Prop. 4.30]. Also, by Lemma 2.1 and an induction on the length \( \ell(w) \) of \( w \), we have that, for \( a, b \in \mathbb{Z}_m \) and \( w, y \in \mathcal{S}_r \), there exist \( f_{y,w}^{a} \in \mathcal{R} \) such that
\[
L^a T_w = T_w L^a + \sum_{y < w, b \in \mathbb{Z}_m} f_{y,b}^{a} T_y L^b.
\]
Lemma A.1. The set
\[ \mathcal{X} = \bigcup_{\lambda, \mu \in \Lambda(n, r)} \{ T_u T_d L^a T_v \mid u \in \mathcal{S}_\lambda, d \in D_{\lambda, \mu}, a \in \mathbb{Z}_m, v \in D_{\nu(d)} \cap \mathcal{S}_\mu \} \]
forms an \( R \)-basis of \( H_m(r) \).

Proof. Recall form Lemma 2.2 that \( H_m(r) \) is a free \( R \)-module with basis
\[ \{ T_w L^a \mid w \in \mathcal{S}_r, a \in \mathbb{Z}_m \}, \]
which, by Lemma 4.1(2), has the same cardinality as \( \mathcal{X} \). Thus, it suffices to prove that each basis element \( T_w L^a \) of \( H_m(r) \) is an \( R \)-linear combination of elements in \( \mathcal{X} \). We proceed by induction on the length \( \ell(w) \) of \( w \).

Take arbitrary \( w \in \mathcal{S}_r \) and \( a \in \mathbb{Z}_m \). If \( \ell(w) = 0 \), then \( T_w L^a = L^a \in \mathcal{X} \). Suppose now \( \ell(w) \geq 1 \). By Lemma 4.1, there are uniquely determined elements \( u \in \mathcal{S}_\lambda, d \in D_{\lambda, \mu} \) and \( v \in D_{\nu(d)} \cap \mathcal{S}_\mu \) such that \( w = uv \) and \( \ell(w) = \ell(u) + \ell(d) + \ell(v) \).

Applying (A.0.2) and (A.0.1) gives the equalities
\[ T_w L^a = T_u T_d(T_v L^a) \]
\[ = T_u T_d(L^{a_{v-1}} T_v) + \sum_{y < v, b \in \mathbb{Z}_m} f_{y, b}^{a, v} T_y L^b \]
\[ = T_u T_d L^{a_{v-1}} T_v + \sum_{y < v, b \in \mathbb{Z}_m} f_{y, b}^{a, v} T_{ud} T_y L^b \]
\[ = T_u T_d L^{a_{v-1}} T_v + \sum_{y < v, b \in \mathbb{Z}_m} f_{y, b}^{a, v} \sum_{z \in (ud) y, b \in \mathbb{Z}_m} f_{ud, y, z} T_z L^b. \]

Since \( y < v \), we obtain that
\[ \ell(z) \leq \ell(u) + \ell(d) + \ell(y) < \ell(u) + \ell(d) + \ell(v) = \ell(w). \]

By induction, \( T_u L^a \) is an \( R \)-linear combination of elements in \( \mathcal{X} \).

Proof of Theorem 4.7. Take \( A \in \Theta_m(n, r)_{\lambda, \mu} \) and write \( d = d_{|A|} \) for notational simplicity. By (1.5.1), \( x_{\nu(d)} \sigma^A = \sigma^A x_{\nu(d)} \). This together with (4.1.1) implies that
\[ b_A = x_{\lambda} T_d \sigma^A \sum_{v \in D_{\nu(d)} \cap \mathcal{S}_\mu} T_v \]
\[ = \sum_{u \in D_{\nu(d)}^{-1} \cap \mathcal{S}_\lambda} T_u T_d x_{\nu(d)} \sigma^A \sum_{v \in D_{\nu(d)} \cap \mathcal{S}_\mu} T_v \]
\[ = \sum_{u \in D_{\nu(d)}^{-1} \cap \mathcal{S}_\lambda} T_u T_d \sigma^A (x_{\nu(d)} \sum_{v \in D_{\nu(d)} \cap \mathcal{S}_\mu} T_v) \]
\[ = \sum_{u \in D_{\nu(d)}^{-1} \cap \mathcal{S}_\lambda} T_u T_d \sigma^A x_{\mu} \in H_m(r) x_{\mu}. \]

Hence, \( b_A \in x_{\lambda} H_m(r) \cap H_m(r) x_{\mu} \).
Linear Independence of $B_{\lambda,\mu}$. Suppose that

$$
\sum_{A \in \Theta_m(n,r)_{\lambda,\mu}} \gamma_A b_A = 0, \text{ where } \gamma_A \in \mathcal{R}.
$$

By the definition, each $b_A$ is an $\mathcal{R}$-linear combination of elements of the form $T_u T_d L^a T_v$ for $u \in \mathcal{G}_\lambda$, $a \in \mathbb{Z}_m$, and $v \in D_{\nu(d_A)} \cap \mathcal{G}_\mu$. Since, by Lemma [A.2](1),

$$
\Theta_m(n,r)_{\lambda,\mu} = \bigcup_{d \in D_{\lambda,\mu}} \Theta_m(n,r)^d_{\lambda,\mu},
$$

it follows from Lemma [A.1] that for each fixed $d \in D_{\lambda,\mu}$,

$$
\sum_{A \in \Theta_m(n,r)^d_{\lambda,\mu}} \gamma_A b_A = 0. \quad (A.1.1)
$$

We now fix such $d \in D_{\lambda,\mu}$. Let $d_0$ be the (unique) longest element in $D_{\nu(d)} \cap \mathcal{G}_\mu$ such that $w^0_\mu = w^0(d_0)$, where $w^0_\mu$ (resp., $w^0(\nu(d))$) denotes the longest element of $\mathcal{G}_\mu$ (resp., $\mathcal{G}_{\nu(d)}$). If $A \in \Theta_m(n,r)^d_{\lambda,\mu}$, then, by (A.0.2) and (A.0.1),

$$
b_A = T_{w^0_\lambda} T_d \sigma^\lambda T_{d_0}^0 + \sum_{u \in \mathcal{G}_\lambda, v \in D_{\nu(d)} \cap \mathcal{G}_\mu} T_u T_d \sigma^\lambda T_v,
$$

where $f_A$ is an $\mathcal{R}$-linear combination of the elements $T_u T_d L^a T_v$ with $a \in \mathbb{Z}_m$, $u \in \mathcal{G}_\lambda$ and $v \in D_{\nu(d)} \cap \mathcal{G}_\mu$ satisfying $\ell(uvd) < \ell(w^0_\lambda d_0) = \ell(w^0_\lambda) + \ell(d) + \ell(d_0)$. Substituting $b_A$ in (A.1.1) gives that

$$
\sum_{A \in \Theta_m(n,r)^d_{\lambda,\mu}} \gamma_A T_{w^0_\lambda} T_d \sigma^\lambda T_{d_0}^0 + \sum_{A \in \Theta_m(n,r)^d_{\lambda,\mu}} \gamma_A f_A = 0.
$$

By Lemma [A.1] again, we obtain that

$$
\sum_{A \in \Theta_m(n,r)^d_{\lambda,\mu}} \gamma_A T_{w^0_\lambda} T_d \sigma^\lambda T_{d_0}^0 = 0.
$$

Since all $T_w$ are invertible in $H_m(r)$, the above equality gives that

$$
\sum_{A \in \Theta_m(n,r)^d_{\lambda,\mu}} \gamma_A \sigma^\lambda = 0.
$$

Applying Proposition [1.5] forces that $\gamma_A = 0$ for each $A \in \Theta_m(n,r)^d_{\lambda,\mu}$. We conclude that $B_{\lambda,\mu}$ is linearly independent.

Proof of the Spanning Condition. We now prove that $B_{\lambda,\mu}$ spans $x_\lambda H_m(r) \cap H_m(r)x_\mu$. Take an arbitrary element $z \in x_\lambda H_m(r) \cap H_m(r)x_\mu$. By Lemma [A.1], $z$ can be written as

$$
z = \sum_{d,a,v} \gamma_{(d,a,v)} x_\lambda T_d L^a T_v,
$$
where $\gamma_{(d,a,v)} \in \mathcal{R}$, and the sum is taken over all $d \in \mathcal{D}_{\lambda, \mu}$, $a \in \mathbb{Z}_m^\nu$, and $v \in \mathcal{D}_{\nu(d)} \cap \mathcal{S}_\mu$. By applying (A.1.1),

$$z = \sum_{d \in \mathcal{D}_{\lambda, \mu}} \left( \sum_{u \in \mathcal{D}^{-1}_{\nu(d-1)} \cap \mathcal{S}_\lambda} T_u \right) T_d \sum_{a,v} \gamma_{(d,a,v)} x_{\nu(d)} L^a T_v$$

$$= \sum_{d \in \mathcal{D}_{\lambda, \mu}} \left( \sum_{u \in \mathcal{D}^{-1}_{\nu(d-1)} \cap \mathcal{S}_\lambda} T_u \right) T_d z_d,$$

where

$$z_d = \sum_{a \in \mathbb{Z}_m^\nu, v \in \mathcal{D}_{\nu(d)} \cap \mathcal{S}_\mu} \gamma_{(d,a,v)} x_{\nu(d)} L^a T_v.$$

(A.1.2)

We claim that $z_d \in \mathcal{H}_m(r)x_\mu$ for all $d \in \mathcal{D}_{\lambda, \mu}$. Indeed, since $z \in \mathcal{H}_m(r)x_\mu$, we have by Lemma 2.3 that $z T_k = q z$ for $s_k \in J_\mu$. Thus,

$$\sum_{d \in \mathcal{D}_{\lambda, \mu}} \sum_{u \in \mathcal{D}^{-1}_{\nu(d-1)} \cap \mathcal{S}_\lambda} T_u T_d (z_d T_k - q z_d) = 0.$$

Further, by (A.0.2), we have for each fixed $d \in \mathcal{D}_{\lambda, \mu}$,

$$z_d = \sum_{a,v} \gamma_{(d,a,v)} x_{\nu(d)} L^a T_v = \sum_{a,v} \gamma_{(d,a,v)} x_{\nu(d)} \left( T_v L^a + \sum_{y < v, b \in \mathbb{Z}_m^\nu} f_{a,v}^y T_y L^b \right).$$

Since $\mathcal{D}_{\nu(d)} \subseteq \mathcal{S}_\mu$, $z_d$ can be written as an $\mathcal{R}$-linear combination of $T_w L^a$ with $w \in \mathcal{S}_\mu$ and $a \in \mathbb{Z}_m^\nu$. Thus, for each $s_k \in J_\mu$, $z_d T_k$ can also be written as an $\mathcal{R}$-linear combination of $T_w L^a$ with $w \in \mathcal{S}_\mu$ and $a \in \mathbb{Z}_m^\nu$. By Lemmas 1.1(2) and 2.2, the elements $T_u T_d T_w L^a$ for $u \in \mathcal{D}^{-1}_{\nu(d-1)} \cap \mathcal{S}_\lambda$, $w \in \mathcal{S}_\mu$ and $a \in \mathbb{Z}_m^\nu$ are linearly independent. This forces that $z_d T_k - q z_d = 0$ for all $s_k \in J_\mu$. By Lemma 2.3, $z_d \in \mathcal{H}_m(r)x_\mu$, proving the claim.

Let $d \in \mathcal{D}_{\lambda, \mu}$. If $w \in \mathcal{S}_{\nu(d)}$ and $w' < w$ (the Bruhat order of $\mathcal{S}_\nu$), then $w' \in \mathcal{S}_{\nu(d)}$. By applying (A.0.2), we obtain that for each $d \in \mathcal{D}_{\lambda, \mu}$ and $v \in \mathcal{D}_{\nu(d)} \cap \mathcal{S}_\mu$,

$$\sum_{a \in \mathbb{Z}_m^\nu} \gamma_{(d,a,v)} x_{\nu(d)} L^a = \sum_{b \in \mathbb{Z}_m^\nu, w \in \mathcal{S}_{\nu(d)}} h_{(b,w)} L^b T_w,$$

(A.1.3)

where $h_{(b,w)} \in \mathcal{R}$. Thus, we obtain that

$$z_d = \sum_{b \in \mathbb{Z}_m^\nu, w \in \mathcal{S}_{\nu(d)}} h_{(b,w)} L^b T_w \left( \sum_{v \in \mathcal{D}_{\nu(d)} \cap \mathcal{S}_\mu} T_v \right).$$

Further, for each $w \in \mathcal{S}_{\nu(d)}$ and $v \in \mathcal{D}_{\nu(d)} \cap \mathcal{S}_\mu$, we have $T_w T_v = T_{wv}$ with $wv \in \mathcal{S}_\mu$. Then by the fact that $z_d \in \mathcal{H}_m(r)x_\mu$ and Lemma 2.3, $z_d$ has the form

$$z_d = \sum_{c \in \mathbb{Z}_m^\nu} f_{(d,c)} L^c x_\mu$$

for some $f_{(d,c)} \in \mathcal{R}$. By further applying (4.1.1), we obtain that

$$z_d = \sum_{c \in \mathbb{Z}_m^\nu} f_{(d,c)} L^c x_\mu = \sum_{c \in \mathbb{Z}_m^\nu} f_{(d,c)} L^c x_{\nu(d)} \sum_{v \in \mathcal{D}_{\nu(d)} \cap \mathcal{S}_\mu} T_v.$$
In other words,
\[\sum_{a \in \mathbb{Z}_m} \gamma(d,a,v)x_{\nu(d)}L^aT_v = \sum_{c \in \mathbb{Z}_m^r} f(d,c)L^c x_{\nu(d)} \sum_{v \in \mathcal{D}_\nu(d) \cap \mathcal{S}_\mu} T_v,\]
that is,
\[\sum_{v \in \mathcal{D}_\nu(d) \cap \mathcal{S}_\mu} \left( \sum_{a \in \mathbb{Z}_m} \gamma(d,a,v)x_{\nu(d)}L^a - \sum_{c \in \mathbb{Z}_m^r} f(d,c)L^c x_{\nu(d)} \right)T_v = 0.\]
By \(\text{(A.0.2)}\), each term \(\sum_{a \in \mathbb{Z}_m} \gamma(d,a,v)x_{\nu(d)}L^a - \sum_{c \in \mathbb{Z}_m^r} f(d,c)L^c x_{\nu(d)}\) can be written as an \(\mathcal{R}\)-linear combination of \(L^\alpha T_w\) with \(w \in \mathcal{S}_\nu(d)\) and \(\alpha \in \mathbb{Z}_m^r\). By Lemma 2.2, the elements \(L^\alpha T_w\) for \(\alpha \in \mathbb{Z}_m^r\), \(w \in \mathcal{S}_\nu(d)\) and \(v \in \mathcal{D}_\nu(d) \cap \mathcal{S}_\mu\) are linearly independent. It follows that for each \(v \in \mathcal{D}_\nu(d) \cap \mathcal{S}_\mu\),
\[\sum_{a \in \mathbb{Z}_m} \gamma(d,a,v)x_{\nu(d)}L^a - \sum_{c \in \mathbb{Z}_m^r} f(d,c)L^c x_{\nu(d)} = 0,\]
and thus,
\[\sum_{a \in \mathbb{Z}_m} \gamma(d,a,v)x_{\nu(d)}L^a = \sum_{c \in \mathbb{Z}_m^r} f(d,c)L^c x_{\nu(d)} \in x_{\nu(d)}\mathcal{H}_m(r) \cap \mathcal{H}_m(r)x_{\nu(d)}.\]
Then by Proposition 3.3(2), \(\sum_a \gamma(d,a,v)x_{\nu(d)}L^a\) is an \(\mathcal{R}\)-linear combination of elements in \(\{x_{\nu(d)}\sigma^e | e \in \Gamma(m, \nu(d))\}\). Applying Lemma 4.4 implies that \(\sum_a \gamma(d,a,v)x_{\nu(d)}L^a\) is an \(\mathcal{R}\)-linear combination of elements in \(\{x_{\nu(d)}\sigma^\bar{A} | \bar{A} \in \Theta_m(n, r)_\lambda, \mu\}\).
We conclude that
\[z = \sum_{d \in \mathcal{D}_\nu} \left( \sum_{u \in \mathcal{D}_\nu^{-1}(d) \cap \mathcal{G}_\mu} T_u \right) T_d \sum_{a,v} \gamma(d,a,v)x_{\nu(d)}L^aT_v\]
\[= \sum_{d \in \mathcal{D}_\nu} \left( \sum_{u \in \mathcal{D}_\nu^{-1}(d) \cap \mathcal{G}_\mu} T_u \right) T_d \sum_{e,v} f(d,e)x_{\nu(d)}\sigma^eT_v\]
\[= \sum_{d \in \mathcal{D}_\nu} \sum_{A \in \Theta_m(n, r)_\lambda, \mu} f(d,A)x_{\nu(d)}T_d\sigma^\bar{A} \sum_{v \in \mathcal{D}_\nu(d) \cap \mathcal{S}_\mu} T_v\]
\[= \sum_{d \in \mathcal{D}_\nu} \sum_{A \in \Theta_m(n, r)_\lambda, \mu} f(d,A)b_A \in \sum_{A \in \Theta_m(n, r)_\lambda, \mu} \mathcal{R}b_A.\]
This finishes the proof. \(\square\)

APPENDIX B. THE AFFINE VERSION OF LEMMA 3.2

Lemma B.1. Suppose that
\[z = \sum_{b \in \mathbb{Z}^r} c_b x_{(r)}L^b \in x_{(r)}\mathcal{H}_\mu(r) \cap \mathcal{H}_\mu(r)\mathcal{R}x_{(r)}, \text{ where } c_b \in \mathcal{R}.\]
Then \(c_b = c_{bw}\) for all \(b \in \mathbb{Z}^r\) and \(w \in \mathcal{S}_r\).

Proof. For each \(a = (a_1, \ldots, a_r) \in \mathbb{Z}^r\), set
\[\|a\| = \max\{|a_i| \mid 1 \leq i \leq r\}.\]
It is clear that \( \|a\| = \|aw\| \) for each \( w \in G_r \). Since only finitely many \( c_b \) may not be zero, there is a positive integer \( N \) such that
\[
c_b = 0 \quad \text{whenever} \quad \|b\| > N.
\]

It suffices to prove that \( c_a = c_{as_i} \) for each fixed \( a \in \mathbb{Z}^r \) and \( 1 \leq i < r \). Write \( a = (a_1, \ldots, a_r) \). If \( a_i = a_{i+1} \), then \( a = as_i \) and hence, \( c_a = c_{as_i} \), as desired. Now let \( a_i \neq a_{i+1} \). We may suppose \( a_i < a_{i+1} \) (Otherwise, we replace \( a \) by \( as_i \)). We proceed by induction on \( a_{i+1} \). If \( |a_{i+1}| > N \), then \( \|a\| = \|as_i\| > N \). Thus, \( c_a = c_{as_i} = 0 \). Let \(-N \leq k \leq N \) and suppose that for each \( a \) with \( k + 1 \leq a_{i+1} \), the equality \( c_a = c_{as_i} \) holds. Now we take \( a \in \mathbb{Z}^r \) with \( a_i < a_{i+1} = k \).

By Lemma 2.1 we obtain that
\[
zT_i = \sum_{b \in \mathbb{Z}^r} c_b x_{(r)} X^b T_i
\]
\[
= \sum_{b \in \mathbb{Z}^r} c_b x_{(r)} T_i X^{bs_i} + \sum_{b \in \mathbb{Z}^r \atop b_i < b_{i+1}} (q - 1)c_b x_{(r)} \sum_{t=1}^{b_{i+1} - b_i} X^{bs_i - ta_i}
\]
\[
+ \sum_{b \in \mathbb{Z}^r \atop b_i > b_{i+1}} (1 - q)c_b x_{(r)} \sum_{t=0}^{b_i - b_{i+1} - 1} X^{bs_i + ta_i}.
\]

Since \( z \in H_{(r)} \) and \( x_{(r)} T_i = qx_{(r)} \), it follows that \( qz = zT_i \) which gives rise to the equality
\[
\sum_{b \in \mathbb{Z}^r} q c_b x_{(r)} X^b = \sum_{b \in \mathbb{Z}^r} c_b x_{(r)} X^{bs_i} + \sum_{b \in \mathbb{Z}^r \atop b_i < b_{i+1}} (q - 1)c_b x_{(r)} \sum_{t=1}^{b_{i+1} - b_i} X^{bs_i - ta_i}
\]
\[
+ \sum_{b \in \mathbb{Z}^r \atop b_i > b_{i+1}} (1 - q)c_b x_{(r)} \sum_{t=0}^{b_i - b_{i+1} - 1} X^{bs_i + ta_i}.
\]

For the \( a \in \mathbb{Z}^r \) chosen as above, comparing the coefficients of \( x_{(r)} X^a \) on both sides implies that
\[
qc_a = qc_{as_i} + c' + c'', \tag{B.1.1}
\]
where \( c' \) (resp., \( c'' \)) denotes the coefficient of \( x_{(r)} X^a \) in the second (resp., third) sum of the right hand side. In the following we calculate \( c' \) and \( c'' \).

Consider those \( b = (b_1, \ldots, b_r) \in \mathbb{Z}^r \) with \( b_i < b_{i+1} \) such that
\[
b_{s_i} - ta_i = a \quad \text{for some} \quad 1 \leq t \leq b_{i+1} - b_i.
\]
Then \( b_i = a_{i+1} - t \) and \( b_{i+1} = a_i + t \), and thus, \( t \leq b_{i+1} - b_i = a_i - a_{i+1} + 2t \). Hence,
\[
a_{i+1} - a_i < t.
\]
Since \( as_i - ta_i = a - (a_{i+1} - a_i) \alpha_i \), by substituting \( t \) for \( t - (a_{i+1} - a_i) \), we obtain that \( b = a - ta_i \) with \( t \geq 0 \).

Similarly, those \( b = (b_1, \ldots, b_r) \in \mathbb{Z}_m^r \) with \( b_i > b_{i+1} \) such that
\[
b_{s_i} + ta_i = a \quad \text{for some} \quad 0 \leq t \leq b_i - b_{i+1} - 1
\]
are simply \( b = as_i + ta_i \) for all \( t \geq 0 \).
We conclude that
\[ c' = (q - 1) \sum_{t \geq 0} c_{a - t \alpha_i} \quad \text{and} \quad c'' = (1 - q) \sum_{t \geq 0} c_{a s_t + t \alpha_i}. \]

Hence, we obtain from \((\text{B.1.1})\) that
\[
qc_a = qc_{as_i} + (q - 1) \sum_{t \geq 0} c_{a - t \alpha_i} + (1 - q) \sum_{t \geq 0} c_{as_i + t \alpha_i}
\]
\[
= qc_{as_i} + (q - 1) (c_a + \sum_{t \geq 1} c_{a - t \alpha_i}) + (1 - q) (c_{as_i} + \sum_{t \geq 1} c_{as_i + t \alpha_i}).
\]

This together with the fact that \(as_i + t \alpha_i = (a - t \alpha_i)s_i\) implies that
\[
c_a - c_{as_i} = \sum_{t \geq 1} (q - 1) (c_{a - t \alpha_i} - c_{(a - t \alpha_i)s_i}). \quad (\text{B.1.2})
\]

Since for each \(t \geq 1\), \(a - t \alpha_i =: a^{(t)} = (a_1^{(t)}, \ldots, a_r^{(t)})\) satisfies
\[
a_i^{(t)} = a_i - t < a_{i+1} + t = a_{i+1}^{(t)} \quad \text{and} \quad a_{i+1}^{(t)} = a_{i+1} + t \geq k + 1,
\]
we have by the induction hypothesis that \(c_{a - t \alpha_i} = c_{(a - t \alpha_i)s_i}\) for all \(t \geq 1\). Consequently, \(c_a = c_{as_i}\).

\[\square\]

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