On the volume of parent Hamiltonians

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We investigate the relative volume of parent Hamiltonians having a target ground state up to some fixed error $\epsilon$, a quantity which sets a benchmark on the performance of quantum simulators. For vanishing error, this relative volume is of measure zero, whereas for a generic $\epsilon$ we show that it increases with the dimension of the Hilbert space. We also analyse the complex task of computing the relative volume when Hamiltonians are restricted to be local. For translationally invariant Hamiltonians, we provide an upper bound to their relative volume. Finally, we estimate numerically the relative volume of parent Hamiltonians when the target state is the ground state of the Ising chain in a transverse field.

Quantum simulators aim at implementing non-trivial many-body Hamiltonians whose ground state, low energy physics, or dynamics are not well understood. Actually, the interactions embedded in such Hamiltonians give rise to highly complex quantum correlations, making analytical or numerical solutions often unfeasible. In some cases, the problem of interest is the inverse one. Namely, given a specific many-body quantum state with physically relevant properties, one is interested in knowing which are the parent Hamiltonians that generate it.

Very general properties of Hamiltonians can be derived from a measure theoretical approach, which does not require the knowledge of their explicit form. This is the approach that has been used in random matrix theory to study level repulsion [1], transport phenomena [2] or atomic spectra of complex atoms [3]. Also, in a somewhat different spirit, volumes of Hamiltonians have been employed to analyze storage capacities of attractor neural networks [4].

Inspired by these approaches, here we address the computation of the probability that by randomly sampling a Hamiltonian one obtains the parent Hamiltonian of a targeted ground state. That is, for a given set of Hamiltonians with some specifications (e.g., dimensions, symmetries, number of particles, locality, etc.), we estimate the proportion of them that have a ground state which is sufficiently close to the target one. For general Hamiltonians with the only restriction of constant dimensionality, this quantity can be viewed as the probability that a given quantum state of that dimension appears as a physically meaningful state. Furthermore, for a universal quantum simulator, such probability provides a benchmark on its minimal performance at implementing quantum states. In other words, it tells us how likely a target quantum state can be realized exactly or sufficiently well approximated.

This problem bears similarities to the one of calculating the volume of quantum states [5], or specific subsets of quantum states on a Hilbert space [6], the volume of quantum maps realizing a given task [7], or the volume of their corresponding Choi states [8]. What is different here is that Hamiltonians may present a richer internal structure arising, for instance, from locality constraints or frustration [9]. Moreover, the eigenstates of a Hamiltonian are endowed with a physical meaning, unlike those of a quantum state. Since a correspondence can be established between Hamiltonians and unitary evolution maps of isolated systems, we expect our results to be recoverable in the framework of volume computation of CPTP maps.

A particularly relevant set of many-body Hamiltonians comprises those whose interactions take place between a restricted number of parties. Such local structure of the Hamiltonian has profound implications on the entanglement and correlations of their corresponding ground states. Finding the ground state of such Hamiltonians, the so-called local Hamiltonian problem, is NP-hard [10, 11]. Notice that by analyzing the volume of local parent Hamiltonians, which is dual to the volume of such special ground states, we provide a novel perspective on the local Hamiltonian problem.

Before proceeding further, let us summarize our main results. Here, we restrict ourselves to non-degenerate bounded Hamiltonians in arbitrary finite dimensions. Despite the fact that gapless Hamiltonians have a clear physical relevance, its volume is of measure zero in the manifold of Hamiltonians. Under such premises, we first show that the relative volume of parent Hamiltonians with an exact target ground state is of measure zero. When allowing for some deviation from the target ground state, though, this volume is finite and increases with the dimension of the Hilbert space. This fact implies that implementing a ground state up to some fixed tolerance is more likely in higher-dimensional spaces than in lower-dimensional ones. We then address the problem of computing the relative volume of local Hamiltonians. The locality restriction renders the problem far more difficult. Nevertheless, we provide an upper bound for the specific case of $t$-local translationally invariant (TI) Hamiltonians. Finally, we numerically tackle the computation when considering the ground state of the quantum transverse Ising model. We compute, as a function of the number of spins, how many 2-local non-translationally invariant Hamiltonians are parent
to it up to some fidelity. For ease of exposition, we defer the proofs of Theorems and Propositions to the Supplementary Material (SM).

Volume of the manifold of Hamiltonians. Let $\mathcal{H}_N$ be an $N$-dimensional Hilbert space and $\mathcal{B}(\mathcal{H}_N)$ the set of its bounded operators. For convenience, which will become clear later on, we consider the manifold of $N$-dimensional (complex) non-degenerate Hamiltonians which are bounded, positively defined and have trace equal or smaller than $k > 0$, i.e., $\mathbf{H}_{N,k} := \{ H \in \mathcal{B}(\mathcal{H}_N) : H > 0; \text{Tr}H \leq k \}$. Note that any non-positive definite (bounded) Hamiltonian $H'$ can always be transformed into a positive one by freely shifting up its eigenenergies: $H' \rightarrow H = H' + \epsilon I$, for some finite $\epsilon \in \mathbb{R}^+$, making $H'$ equivalent to $H$. Thus, it suffices to calculate the volume of $\mathbf{H}_{N,k}$ for $k$ sufficiently large.

Any $H \in \mathbf{H}_{N,k}$ can be expressed as $H = UDU^\dagger$, where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, with $\lambda_i > 0 \forall i$, $\lambda_i \neq \lambda_j \forall i, j$, $\text{Tr}H = \sum_i \lambda_i \leq k$, and $U$ is a unitary matrix. The volume of this (convex) manifold can be computed with respect to several bona fide metrics, such as the ones induced by the Hilbert-Schmidt (HS), the Bures or the trace distance [5]. The main results of our work do not depend on the choice of the metric, as shown later. Here we choose the measure generated by the Hilbert-Schmidt (HS) distance, $d_{\text{HS}}(A, B) = \sqrt{\text{Tr}((A - B)^2)}$, for two Hermitian operators $A$ and $B$, inasmuch as the HS distance is simpler to deal with, it induces the Euclidean geometry into the manifold of Hermitian operators [12, 13], and it is widely used in quantum information tasks [14–17].

The first step to estimate the volume of the manifold is to obtain the infinitesimal distance $d_{\text{HS}}(H, H + dH)$, giving rise to its line element $ds^2 := d_{\text{HS}}^2(H, H + dH) = \text{Tr}((dH)^2)$, where $dH = U(dD + U^\dagger dUD - DU^\dagger dU)U^\dagger$, leading to

$$ds^2 = \sum_{i=1}^N (d\lambda_i)^2 + 2 \sum_{i < j}^N (\lambda_i - \lambda_j)^2 |(U^\dagger dU)_{ij}|^2,$$

where we have used $UdU^\dagger = -dUdU^\dagger$ (see [12] for more details). Notice that the two sets of variables $\{d\lambda_i\}$ and $\{\text{Re}(U^\dagger dU)_{ij}, \text{Im}(U^\dagger dU)_{ij}\}$ do not get mixed up in the line element, yielding a block-diagonal metric tensor whose determinant (in absolute value) corresponds to the squared magnitude of the Jacobian determinant of the transformation $H \rightarrow UD$. Hence, the volume element of the manifold reduces to the product form $dV = d\mu_{\lambda_1, \ldots, \lambda_N} \times dV_{\text{Haar}}$, where the first factor depends only on the eigenvalues of $H$ and the second one corresponds to the Haar measure on the $N$-dimensional complex flag manifold $F^{(N)}(\mathbb{C}) := U(N)/[U(1)^N]$, where $U(N)$ denotes the unitary group in dimension $N$. Indeed, a volume element of the referred form is specific to all unitarily-invariant measures, since the Haar measure is unitarily-invariant. After integration we arrive to

**Proposition 1** The HS volume of the manifold $\mathbf{H}_{N,k}$ ($N$-dimensional complex Hamiltonians with $H > 0$ and $\text{Tr}H \leq k$) amounts to:

$$\text{vol}_N(\mathbf{H}_{N,k}) = I_1(N,k) I_2(N),$$

where

$$I_1(N,k) = \frac{\sqrt{N}}{N^2} \xi_N \xi_{N-1} k^{N^2},$$

with $\xi_n = \frac{n!}{\pi^{n+1}}$, comes from the integration over the simplex of eigenvalues, and

$$I_2(N) = \text{vol}_N F^{(N)}(\mathbb{C}) = \frac{(2\pi)^{N(N-1)/2}}{\xi_{N-1}}$$

corresponds to the Haar volume of the unitaries over the complex flag manifold.

Note that the HS volume of the set of density matrices [12] $\rho \in \mathcal{B}(\mathcal{H}_N)$ s.t. $\rho \geq 0$ with $\text{Tr}(\rho) = 1$, is indeed the boundary surface of Eq. (2) for $k = 1$, i.e., $\partial_k \text{vol}_N(\mathbf{H}_{N,k})|_{k=1}$.

Relative volume of Hamiltonians with a target ground state. The relative volume indicates the probability of randomly sampling a Hamiltonian $H \in \mathbf{H}_{N,k}$ that is parent to a target state $|\psi_0\rangle$. To this aim, we must first calculate the volume of the manifold $\mathbf{H}^{(\psi_0)}_{N,k} \subset \mathbf{H}_{N,k}$. It corresponds to integrating over all unitaries in $U(N - 1)$. Since the volume of a manifold is basis-independent, one can always choose a basis where $|\psi_0\rangle := |0\rangle = (1, 0, \ldots, 0)^T$. As the columns of $U$ have to form an orthonormal basis, it follows that

$$U = \begin{pmatrix} 1 & 0 \\ 0 & U' \end{pmatrix},$$

where $U' \in F^{(N-1)}(\mathbb{C})$ (recall that $U$ is uniquely specified if it belongs to the complex flag manifold).

Thus, integrating over $U'$ leads to a volume in one dimension less, that is, a $(N - 1)$-dimensional hypersurface of $\mathbf{H}_{N,k}$. Analogously to Proposition 1, it follows that

**Proposition 2** The HS hypersurface of $N$-dimensional Hamiltonians with $\text{Tr}H \leq k$ and a target ground state $|\psi_0\rangle$ is given by

$$S^{(1)}(H^{(\psi_0)}_{\mathbb{C}}) = I_1(N,k) I_2(N - 1).$$

Accordingly, the volume of $N$-dimensional Hamiltonians with $L$ fixed eigenstates is actually a hypersurface $S^{(L)}(H^{(\psi_{L-1})}_{\mathbb{C}}) = I_1(N,k) I_2(N - L)$. From this result we also see that, by construction, the hypersurface of (unrestricted) Hamiltonians with a specified ground state does not depend on the choice of the latter. This will not be the case when imposing further structure on $H^{(\psi_0)}_{\mathbb{C}}$, e.g., when considering only the volume of local
Hamiltonians that are compatible with a given ground state. For instance, Matrix Product State (MPS) ground states are unique ground states of local, gapped, frustration-free Hamiltonians [9].

Still, Eq. (6) is an absolute volume, and as such it tells us little about the relative occurrence of Hamiltonians with a common ground state in a given dimension. A relative volume with respect to the volume of all Hamiltonians [cf. Eq. (2)] must be calculated. However, the volume of $H_{N,k}^{\psi_0}$ and the volume of $H_{N,k}$ refer to manifolds of different, and thus incomparable, dimensions. We address this issue by tolerating a small deviation $\epsilon$ from the ground state $|\psi_0\rangle$, that is, we consider the volume of the manifold $H_{N,k}^{\psi_0,\epsilon}$, corresponding to Hamiltonians with ground state $|\psi_0\rangle$ such that $|\langle \psi_0 | \psi_0^\prime \rangle| \geq 1 - \epsilon$. In doing so, we extend the hypersurface Eq. (6) into a volume in $N$ dimensions which is directly comparable with Eq. (2), enabling a proper definition of a relative volume (see Fig. 1). This relative volume can be regarded as the probability of randomly preparing Hamiltonians with sufficiently similar ground states, a more realistic scenario where exact implementations of $|\psi_0\rangle$ are neither feasible nor expected. The following result holds:

**Proposition 3** The HS volume of the manifold $H_{N,k}^{\psi_0}$ $(N$-dimensional positively defined Hamiltonians with $TrH \leq k$ and ground state $|\psi_0\rangle$ such that $|\langle \psi_0 | \psi_0^\prime \rangle| \geq 1 - \epsilon$ for sufficiently small $\epsilon$) is given by

\[
\text{vol}_N \left( H_{N,k}^{\psi_0} \right) = I_1(N,k) \int_{E_{\text{rad}}(N)} |\langle \psi_0 | U | 0 \rangle| \prod_{i<j} 2 \text{Re} (U^\dagger dU)_{ij} \text{Im} (U^\dagger dU)_{ij}^{\dagger} \\ \approx \epsilon I_1(N,k) I_2(N - 1),
\]

where $I_{[1-\epsilon,1]}$ is the indicator function.

Defining the relative volume of a subset $A_{N,k} \subset H_{N,k}$ with respect to $H_{N,k}$ as $\text{vol}_r(A_{N,k}) := \text{vol}_N(A_{N,k}) / \text{vol}_N(H_{N,k})$, one immediately obtains:

**Proposition 4** The relative volume of $H_{N,k}^{\psi_0}$ is:

\[
\text{vol}_r \left( H_{N,k}^{\psi_0} \right) \approx \epsilon I_2(N - 1) / I_2(N) \approx \epsilon (2\pi)^{-N} (N - 1)!.
\]
Volume of local Hamiltonians. Up to now we have considered Hamiltonians which, besides symmetry and dimensionality, do not present any specific feature. However, physically relevant Hamiltonians are usually local. A $n$-body $t$-local Hamiltonian is of the form $H = \sum_{i=1}^{M} h_i$, where $h_i$ is a Hamiltonian acting non-trivially on at most $t$ parties, and $M$ is some positive integer. Such $t$-local Hamiltonian can be viewed as a set of $M$ constraints on the $n$ parties, each involving at most $t$ of them.

According to the previous discussion, a way to calculate the volume of such manifold amounts to diagonalize each of the $M$ $t$-local Hamiltonians, $h_i = u_i \Lambda_i u_i^\dagger$, where $\Lambda_i$ are diagonal matrices of eigenvalues, and $u_i$ are the corresponding unitary matrices. Defining $dG_i := u_i^\dagger du_i$, the line element of this manifold becomes

$$ds^2 = \sum_{i=1}^{M} \left( \sum_{k=1}^{N} (d\Lambda_{ik})^2 + \sum_{k\neq l}^{N} (\Lambda_{ik} - \Lambda_{il})^2 \right) \left| (dG_i)_{kl} \right|^2 + \sum_{i \neq j} \text{Tr} \left( u_i \left( d\Lambda_i + dG_i \Lambda_i - \Lambda_i dG_i \right) u_i^\dagger \times u_j \left( d\Lambda_j + dG_j \Lambda_j - \Lambda_j dG_j \right) u_j^\dagger \right).$$

Although the first term of the line element can be treated in the same manner as in Eq. (1), the second one involves crossed terms $dh_i dh_j$, which turn out to be an involved function of the eigenvalues and eigenvectors both of $h_i$ and $h_j$, $\forall i \neq j$, preventing to obtain an evaluable expression for the volume of local Hamiltonians.

To simplify the calculation of the volume, one can restrict to translationally invariant (TI) Hamiltonians, i.e., those of the form $H = \sum h_i$, where all $h_i$ are locally equal, that is, $h_i = 1 \otimes \cdots \otimes 1 \otimes h^{(i)} \otimes 1 \otimes \cdots \otimes 1$, where $h$ is a $d^t$-dimensional Hamiltonian acting on $t$ $d$-dimensional parties, and the multi-index $i$ labels the set where $h$ acts. If we restrict to 1D models, the multi-index $i$ refers to the first particle in which $h$ acts.

To illustrate the complexity of the problem, let us consider a simple example of just 3 qubits ($d = 2$) in a 2-local TI-Hamiltonian, so that $H = h_1 + h_2 = h \otimes 1 + 1 \otimes h$ with $\text{dim} (h) = 2^2$. The line element of this manifold reads

$$ds^2 = 4 \sum_{k=1}^{4} (d\Lambda_k)^2 + 4 \sum_{k\neq l}^{4} (\Lambda_k - \Lambda_l)^2 \left| dG_{kl} \right|^2 + 2 \text{Tr} \left( u \left( d\Lambda + dG \Lambda - \Lambda dG \right) u^\dagger \right) \times P u \left( d\Lambda + dG \Lambda - \Lambda dG \right) u^\dagger P^l),$$

where $h_1 = uu^\dagger$, $h_2 = (Pu) \Lambda (Pu)^\dagger$ and $dG = u^\dagger du$, with $u$ a unitary matrix, $\Lambda$ a diagonal matrix of eigenvalues and $P$ a permutation of every row of $u$ except for the first and last ones. Even though the second term only depends on a single unitary $u$ and a permutation matrix $P$, the metric remains quite involved, and so a compact formula for the volume is not attainable even in this simple case [18].

One can find an upper bound to the volume of TI Hamiltonians by considering that the local terms $h_i$ are equal but act in disjoint subspaces, i.e., $H = \bigoplus h_i$. For such Hamiltonians, the line element would be given by the (corresponding) Eq. (1), permitting the computation of the volume. Locally non-overlapping TI Hamiltonians are subject to fewer constraints than their generic TI counterparts, and thus, the volume of the former should upper bound that of the latter. To see why, let us express the Hamiltonian in terms of the generators of the corresponding algebra. Any $t$-local TI Hamiltonian can be expressed as $H = \sum_{i=1}^{M} h_i$, with $h_i \equiv h = \sum_{i,j,k=0}^{d^2-1} \alpha_{ijk} \otimes \sigma_i \otimes \sigma_j \otimes \cdots \otimes \sigma_k$, $\forall i$. The set $\{\sigma_i\}$ denotes the generators of $SU (d)$ and the identity, forming a proper basis of $\mathcal{B} (H)$. The coefficients $\alpha_{ijk}$ are real and independent. Suppose that such a manifold is associated to some metric tensor $g$. Now, removing the crossed terms $dh_i dh_j$ from the line element in Eq. (9) results in a diagonal metric tensor $\tilde{g}$, such that $\tilde{g} = \hat{g} + X$, where $X$ is a matrix with vanishing diagonal. Due to the positiveness of metric tensors, Hadamard’s inequality [19] can be applied to show that $\det (\tilde{g}) \leq \det (\hat{g})$. Therefore, calculating the volume associated to the line element without crossed terms yields an upper bound for the volume of the manifold of $t$-local TI Hamiltonians of dimension $N (H_{N,k,t}^{M})$, as formally demonstrated in the SM.

**Theorem 5** The HS volume of the $t$-local manifold $H_{N,k,t}^{M}$ ($N$-dimensional $t$-local TI-Hamiltonians $H = \sum_{i=1}^{M} h_i$, with $n$-parties of dimension $d$) is upper bounded by:

$$\text{vol}_{d^t} (H_{N,k,t}^{M}) \leq \nu 2^t I_1 \left( d^t, \frac{k}{\nu} \right) I_2 \left( d^t \right),$$

where $\nu = Md^{n-t}$ and $\kappa = d^{2t} - 1$.

Like the absolute volume of generic Hamiltonians [Eq. (2)], this bound decreases with increasing number of parties $n$, dimension $d$, and locality $t$.

The $d^t$-dimensional volume in Eq. (11) upper bounding the volume of $t$-local TI Hamiltonians is of measure zero with respect to the $d^n$-dimensional volume of all Hamiltonians with the same number of parties. Thus, an upper bound for the relative volume cannot be defined under the TI restriction. To shed some light on this question, we now allow for locality to be broken up to a small extent. Consider a Hamiltonian of the form $H = h_{\text{TI}} + \delta h_{\text{NL}}$, where $h_{\text{TI}}$ is a TI Hamiltonian, $h_{\text{NL}}$ is a generic nonlocal Hamiltonian, and $\delta \ll 1$. Embedding the manifold of $t$-local TI Hamiltonians in such a $d^n$-dimensional manifold now permits the definition of a relative volume. Applying the same arguments leading to Theorem 5, one obtains
The relative volume of $d^n$-dimensional $\delta$-TI Hamiltonians $H = h_{TI} + \delta h_{NL}$, with $h_{TI}$ a $t$-local TI Hamiltonian such that $Tr h_{TI} \leq k$, $\delta \ll 1$ and $h_{NL}$ a general nonlocal Hamiltonian with $Tr h_{NL} \leq k'$, fulfills

$$\text{vol}_r \left( H_{N,k,k',\epsilon} \right) \leq \delta^{d^n/2} \frac{I_1(d^n, k) I_2(d^n) I_1(d^n, \epsilon k')}{I_1(d^n, k + \delta k')},$$

where $\nu = Md^{n-t}$, $\kappa = d^t - 1$, and $\kappa' = d^n - 1$.

The proof goes similarly to the one for Theorem 5. Again, this bound decreases with the number of parties and with the local dimension. Moreover, the factor $\delta^{d^n-1}$ makes the bound very small (e.g., for 2-local qubit Hamiltonians such that $\delta = 0.01$ and $k = k' = 100$ it is of the order of $10^{-14}$).

To analyze the performance of a quantum simulator using relative volumes under a more realistic scenario, we now take a numerical route. To this aim, similarly to Proposition 4, we fix as a target the ground state of a well known TI-parent Hamiltonian and explore how many Hamiltonians provide a ground state which is sufficiently close to it.

**Numerical study: transverse-field Ising chain.** We consider the transverse-field quantum Ising model in 1D with Hamiltonian $H = \sum_{i=1}^n J_i \sigma_i^z \sigma_{i+1}^z + g \sigma_i^x$, and ground state $|\psi_0\rangle$, which can be analytically obtained using a Jordan-Wigner transformation. Here $g$ is the magnitude of the external magnetic field, and $J_i$ the coupling between spins. Consider now that the spin-spin interactions deviate from the constant value $J_i$, so that the translational symmetry is broken and the Hamiltonian reads $H = \sum_{i=1}^n J_i \sigma_i^z \sigma_{i+1}^z + g \sigma_i^x$, with $H_{\text{Ising}}$ the manifold of all such Hamiltonians. Here we estimate the probability of randomly sampling a Hamiltonian with ground state $|\psi_0\rangle$ from the set $H_{\text{Ising}}$. The ratio between those and the total number of sampled $H$ gives us an estimation of the relative volume $\text{vol}_r \left( H_{\text{Ising}} \right) := \text{vol}_{2^n} \left( H_{\text{Ising}} \right) / \text{vol}_{2^n} \left( H_{\text{Ising}} \right)$. Such ratio is well approximated by a Beta cumulative distribution function, as the Beta distribution is well-suited for modeling the behavior of random variables that are limited to intervals of finite length, such as in our study where $J_i \in [0, 2]$. The relative volume as a function of $\epsilon$ naturally behaves as a cumulative distribution function, as depicted in Fig. 3. For small $\epsilon$, we can approximate the relative volume as $\text{vol}_r \left( H_{\text{Ising}} \right) \approx \frac{\Gamma(\alpha + \beta)}{\alpha \Gamma(\alpha) \Gamma(\beta)} \epsilon^\alpha$, where $\alpha \sim \text{poly}(n)$ and $\alpha, \beta > 0$, making it decrease with $n$. Interestingly, the probability of sampling the desired ground state up to a small error in this setting decreases with the number of spins, which is coherent with the observation that the TI constraint $J_i = J \forall i$ of the target ground state becomes more restrictive as $n$ increases. Although this behaviour seems a priori contradictory with what occurs when allowing for completely general Hamiltonians, notice that the relative volume here is defined w.r.t. the volume of the manifold $H_{\text{Ising}}$. Had the relative volume been estimated w.r.t all possible Hamiltonians in $N = 2^n$, we would have recovered the results obtained in Proposition 4.

**Discussion.** Measure theory is a powerful tool for tackling different aspects of Hamiltonians of which one has limited knowledge. Our work provides a novel application of this tool for the computation of volumes of parent Hamiltonians independently of their specific features. We have demonstrated that the HS measure, or any other unitarily invariant one, is appropriate to compute relative volumes of parent Hamiltonians of a target ground state up to some error. This quantity has a direct interpretation as a minimal benchmark to the performance of quantum simulators that aim at preparing a target ground state. We have also applied our method to the physically relevant class of local Hamiltonians, obtaining in this case an upper bound to the relative volume. The difficulty of computing an exact volume under locality constraints calls for the development of more convenient techniques, which could shed further light on the interplay between the physics of locality and the geometry of the underlying Hilbert space.

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Supplementary Material

I. PROOF OF PROPOSITIONS 1, 2 AND 3

Using $H = UDU^\dagger$ one obtains $dH = U (dD + U^\dagger dUD - DU^\dagger dU) U^\dagger$, which leads to

$$ds^2 = \sum_{i=1}^{N} (d\lambda_i)^2 + 2 \sum_{i<j} (\lambda_i - \lambda_j)^2 |(U^\dagger dU)_{ij}|^2.$$  \hfill (13)

Now, differentiating the condition $\sum_{i=1}^{N} \lambda_i = \hat{k} \leq k$, one gets $\sum_{i=1}^{N} d\lambda_i = 0$, which implies $d\lambda_N = - \sum_{i=1}^{N-1} d\lambda_i$. Then

$$\sum_{i=1}^{N} (d\lambda_i)^2 = \sum_{i=1}^{N-1} (d\lambda_i)^2 + \left( \sum_{i=1}^{N-1} d\lambda_i \right)^2 = \sum_{ij} d\lambda_ig^{(\lambda)}_{ij} d\lambda_j,$$ \hfill (14)

where $g^{(\lambda)} = \mathbb{I}_{N-1} + J_{N-1}$, with $J_N$ an $N$-dimensional matrix of ones and determinant $\det g^{(\lambda)} = N$.

Notice that, since the two sets of variables $\{d\lambda_i\}$ (with metric tensor $g^{(\lambda)}$) and $\{\Re (U^\dagger dU)_{ij}, \Im (U^\dagger dU)_{ij}\}$ (with metric tensor $g^{(U)}$) do not get mixed up in the line element, the global metric $g$ is block-diagonal and its determinant is given by $\det g = \det g^{(\lambda)} \det g^{(U)} = N \prod_{i<j} 2(\lambda_i - \lambda_j)^2$, which is positive since $H$ is a Riemannian manifold.

The volume element of a Riemannian manifold gains a factor $\sqrt{\det g}$ [20]. Thus,

$$dV = \sqrt{N} \prod_{i=1}^{N-1} d\lambda_i (\lambda_i - \lambda_j)^2 \prod_{i<j} 2\Re (U^\dagger dU)_{ij} \Im (U^\dagger dU)_{ij} |,$$ \hfill (15)

which has the form $dV = d\mu (\lambda_1, ..., \lambda_N) \times d\nu_{\text{Haar}}$, where $d\mu (\lambda_1, ..., \lambda_N)$ depends only on the eigenvalues of $H$ and $\nu_{\text{Haar}}$ is the Haar measure on the complex flag manifold $F^{(N)}_{\mathbb{C}} := U(N) /[U(1)^N]$. Indeed, the following invariant metric can be defined on the unitary group: $ds^2_U := d^2_{\text{HS}}(U, U + dU) = \Tr (d UdU^\dagger) = \Tr (U^\dagger dUDU^\dagger) = -\Tr (U^\dagger dU)^2$, where the last equality is obtained by noting that $U^\dagger U = 1$ implies $dU^\dagger U = -U^\dagger dU$. Then, $ds^2_U = \sum_{ij} |(U^\dagger dU)_{ij}|^2 + 2 \sum_{i<j} |(U^\dagger dU)_{ij}|^2$, which induces the Haar measure on $U(N)$. For unitaries with fixed diagonal, that is, $U \in F^{(N)}_{\mathbb{C}}$, only the second term is retrieved, yielding the Haar measure on $F^{(N)}_{\mathbb{C}}$ (which is present in our volume element). The Haar measure is invariant under unitary transformations, meaning that $\nu_{\text{Haar}} (V) = \nu_{\text{Haar}} (UV)$, where $V$ is a subset of $U(N)$.

Therefore, the volume of the set of Hamiltonians with $\Tr H \leq k$ amounts to

$$\text{vol}_N (H) := \int_{H, \Tr H > 0, \Tr H \leq k} dV = I_1(N, k) I_2(N),$$ \hfill (16)

where

$$I_1(N, k) = \frac{\sqrt{N}}{N!} \int_0^k \int_0^\infty \delta \left( \sum_{j=1}^{N} \lambda_j - \hat{k} \right) \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{N} d\lambda_i \, d\hat{k} = \frac{\sqrt{N}}{N!} \int_0^k \frac{\hat{k}^{N-2}}{\Gamma(N^2)} \prod_{j=1}^{N} \frac{\Gamma(j+1) \Gamma(j)}{\Gamma(2)} \, d\hat{k}$$

$$= \frac{\sqrt{N}}{N!} \frac{1}{\Gamma(N^2)} \prod_{j=1}^{N} \frac{\Gamma(j+1) \Gamma(j)}{\Gamma(2)} \frac{k^{N^2}}{N^2} = \left( \frac{\sqrt{N}}{N!} \prod_{j=1}^{N} \frac{\Gamma(j+1) \Gamma(j)}{\Gamma(2)} \right) k^{N^2}$$ \hfill (17)

(see Eqs. 3.37-3.44 in [21] and Eqs. 4.1-4.3 in [12]), and

$$I_2(N) = \int_{F^{(N)}_{\mathbb{C}}} \prod_{i<j} 2\Re (U^\dagger dU)_{ij} \Im (U^\dagger dU)_{ij} | = \text{vol}_N \left( F^{(N)}_{\mathbb{C}} \right) = \frac{(2\pi)^{N(N-1)/2}}{1!2!... (N-1)!}.$$ \hfill (18)

A few remarks are in order. Notice that the diagonalization transformation $H = UDU^\dagger$ needs to be unique; otherwise, the volume of $H$ would be overestimated. For that, one first has to fix the order of the eigenvalues (in our
case, $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_N$, since different permutations of the vector of eigenvalues pertain to the same unitary orbit. That is why we introduce the $1/N!$ factor in $I_1 (N, k)$. Second, since $H = UBDB^\dagger U^\dagger$, where $B$ is a diagonal unitary matrix, $U$ is generically determined up to the $N$ arbitrary phases present in $B$. Therefore, $U$ is uniquely specified if $U \in Fl_{C}^{(N)}$. The volume of this manifold w.r.t. the Haar measure is well-known and given by Eq. 18 [22].

Now, to calculate the volume of the Hamiltonians that have a specified ground state $|\psi_0\rangle$, one has to impose that one of the columns of the unitaries over which we integrate coincides with $|\psi_0\rangle$:

$$\int_{Fl_{C}^{(N)}} \delta (|\langle \psi_0 | U | 0 \rangle \rangle | - 1) | \prod_{i < j} 2 \text{Re} \left( U^\dagger d U \right)_{ij} \text{Im} \left( U^\dagger d U \right)_{ij} | = \text{vol}_{N−1} \left( Fl_{C}^{(N−1)} \right) = \left( \frac{2\pi \left( \frac{N−1}{2} \right)}{1 \pi (N−2)!} \right)$$

$$= I_2 (N − 1),$$

(19)

where $|0\rangle = (1, 0, \ldots, 0)^T$ and so $U |0\rangle$ denotes the first column of $U$.

The integration over the eigenvalues does not change, so we have

$$S_N^{(1)} (H|\psi_0\rangle) = I_1 (N, k) I_2 (N − 1).$$

(20)

Note that the volume of Hamiltonians with a target ground state is actually a hypersurface. In turn, fixing $L$ eigenstates implies $S_N^{(L)} = I_1 (N, k) I_2 (N − L)$.

If instead one wants to compute the volume of Hamiltonians with a given ground state $|\psi_0\rangle$ up to error $\epsilon$, one needs to impose that one of the columns of the unitaries in $I_2$ is approximately $|\psi_0\rangle$:

$$\int_{Fl_{C}^{(N)}} \mathbbm{1}_{[1−\epsilon, 1]} (|\langle \psi_0 | U | 0 \rangle \rangle |) | \prod_{i < j} 2 \text{Re} \left( U^\dagger d U \right)_{ij} \text{Im} \left( U^\dagger d U \right)_{ij} |$$

$$\approx \int_{Fl_{C}^{(N)}} \epsilon \delta (|\langle \psi_0 | U | 0 \rangle \rangle | - 1) | \prod_{i < j} 2 \text{Re} \left( U^\dagger d U \right)_{ij} \text{Im} \left( U^\dagger d U \right)_{ij} | = \epsilon I_2 (N − 1),$$

(21)

with $|0\rangle = (1, 0, \ldots, 0)^T$ and $\mathbbm{1}_{[1−\epsilon, 1]} (x)$ the indicator function being 1 for $x \in [1−\epsilon, 1]$ and 0 otherwise. Note that the approximation is valid for sufficiently small $\epsilon$.

The integral over the eigenvalues is the same, so finally

$$\text{vol}_{N} (H|\psi_0\rangle) \approx \epsilon I_1 (N, k) I_2 (N − 1).$$

(22)

As a consequence, the relative volume of Hamiltonians with a target state up to some error is given by

$$\text{vol}_{R} (H|\psi_0\rangle) := \frac{\text{vol}_{N} (H|\psi_0\rangle)}{\text{vol}_{N} (H)} = \epsilon (2\pi)^{N−1} (N − 1)!.$$  

(23)

Let us mention that all the previous results hold when considering complex Hamiltonians. However, it is also of interest to obtain the relative volume of the subset of real Hamiltonians: as recently argued in [23], it is experimentally easier to implement real states (rebits) and real operations in a single-photon interferometer set-up when compared to general states and operations. Knowing that a real $N$-dimensional Hamiltonian is diagonalized as $H = ODO^T$, where $O$ is an orthogonal matrix, suffices to extend our results to the domain of real Hamiltonians. In this case, $I_2 (N)$ corresponds to the volume of the real flag manifold [12], that is

$$I_2 (N) = \text{vol}_{N} \left( Fl_{R}^{(N)} \right) = \frac{(2\pi)^{N−1} / \sqrt{\pi}^{N/2}}{\Gamma \left( \frac{N}{2} \right) \Gamma \left( \frac{N}{2} \right)}.$$

(24)

implying

$$\text{vol}_{R}^{(1)} (H|\psi_0\rangle) \approx \epsilon \frac{\text{vol}_{N−1} \left( Fl_{R}^{(N−1)} \right)}{\text{vol}_{N} \left( Fl_{R}^{(N)} \right)} = \epsilon \frac{2^{4−N}}{\sqrt{\pi}^{N−1} \Gamma \left( \frac{N}{2} \right)}.$$  

(25)
II. TOWARDS CALCULATING THE VOLUME OF TRANSLATIONALLY IN Variant HAMILTONIANS

Translationally invariant (TI) Hamiltonians are the subset of local Hamiltonians, i.e. $H = \sum_{i=1}^{M} h_i$, for which $h_i \equiv h \forall i$. For simplicity, consider an array of 3 qubits, such that $d = 2$, $n = 3$ and $M = 2$. In this case, then, $H = h_1 + h_2 = h \otimes 1 + 1 \otimes h$, with dim $(h) = 4$. Thus, the matrix of eigenvalues is the same both for $h_1$ and $h_2$: $\Lambda = \text{diag}(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$, where $\{\Lambda_i\}_{i=1}^{4}$ are the eigenvalues of $h$. It can be checked that, if $h_1 = u\Lambda u^\dagger$, then $h_2 = v\Lambda v^\dagger$, where $v = Pu$ and $P$ is a permutation of every row of $u$, except for the first and last ones. Therefore, the matrix $dG$ is the same both for $h_1$ and $h_2$: $v^d v = u^d P^d P u = u^d u \equiv dG$, which makes the calculation of the line element simpler:

$$ds^2 = \text{Tr} (dh_1^2) + \text{Tr} (dh_2^2) + 2\text{Tr} (dh_1 dh_2) = 4 \sum_{k=1}^{4} (d\Lambda_k)^2 + 4 \sum_{k \neq l}^{4} (\Lambda_k - \Lambda_l)^2 |dG_{kl}|^2$$

$$+ 2\text{Tr} (u (d\Lambda + dG\Lambda - \Lambda dG) u^\dagger P u (d\Lambda + dG\Lambda - \Lambda dG) u^\dagger P^\dagger).$$

(26)

Although translational invariance has been imposed, the metric is still a complicated function of the eigenvalues and eigenvectors of $h$. One would need to compute the metric tensor in this case, and then integrate the corresponding volume element. In particular, this volume element would contain a function of the eigenvectors of $h$ to be integrated over the complex flag manifold, which in general will not correspond to the Haar measure on such manifold. Thus, a way to compute this integral would be to parametrize $U$ in terms of angles and phases and then integrate the resulting function over suitable domains.

A. Volume of translationally invariant Hamiltonians of the form $H = \bigoplus_{i=1}^{M} h_i$

The second term preventing the computation of the volume is not present in the metric of TI Hamiltonians of the form $H = \bigoplus_{i=1}^{M} h_i$, with $h_i \equiv h \forall i$ and dim $(h) = d$. Here the line element is given by

$$ds^2 = \sum_{i=1}^{M} \text{Tr} (dh_i^2) = M \left( \sum_{i}^{d} (d\Lambda_i)^2 + 2 \sum_{i<j}^{d} (\Lambda_i - \Lambda_j)^2 \left( \text{Re} (u^d du)_{ij} \right)^2 + 2 \sum_{i<j}^{d} (\Lambda_i - \Lambda_j)^2 \left( \text{Im} (u^d du)_{ij} \right)^2 \right),$$

(27)

where $h = u\Lambda u^\dagger$.

Now, since $\text{Tr} dh = \sum_{i=1}^{d} d\Lambda_i = 0$, we have that $d\Lambda_d = -\sum_{i=1}^{d-1} d\Lambda_i$, which implies

$$M \sum_{i=1}^{d} (d\Lambda_i)^2 = M \left( \sum_{i=1}^{d-1} (d\Lambda_i)^2 + \sum_{i=1}^{d-1} d\Lambda_i \right)^2 = \sum_{i,j=1}^{d-1} g_{ij}^{(\Lambda)} d\Lambda_i d\Lambda_j,$$

(28)

where $g^{(\Lambda)} = M (1_{d-1} + J_{d-1})$, with $J_d$ a $d$-dimensional matrix of ones and determinant $\det g^{(\Lambda)} = dM^{d-1}$. The determinant of the full metric is given by $\det g = dM^{d-1} |M^{(d)} \prod_{i<j}^{2} (\Lambda_i - \Lambda_j)^2 |^2$.

Thus, the corresponding volume element gains a factor $\sqrt{\det g}$:

$$dV = d^\frac{d}{2} M^{\frac{d^2}{2}} \prod_{i=1}^{d-1} d\Lambda_i \prod_{i<j}^{d-1} (\Lambda_i - \Lambda_j)^2 |\prod_{i<j}^{} 2\text{Re} (u^d du)_{ij} \text{ Im} (u^d du)_{ij} |.$$ 

(29)

Therefore, the volume of this subset of TI Hamiltonians with $\text{Tr}H \leq k$ amounts to

$$\text{vol}_d (H) = M^{\frac{d^2}{2}-1} I_1 \left( d, \frac{k}{M} \right) I_2 (d).$$

(30)

III. PROOF OF THEOREM 5

For the sake of clarity, we first demonstrate Theorem 5 for 2-local Hamiltonians and eventually generalize the proof to the $t$-local case:
Consider the manifold of 2-local $N$-dimensional TI Hamiltonians on a chain $H = \sum_{i=1}^{M} h_i$, with $h_i \equiv h \forall i$, describing an array of $n$ $d$-dimensional parties. For the purpose of this proof, we do not require $H > 0$, but only $\text{Tr} H \leq k \in \mathbb{R}$. Each subhamiltonian can be written as $h = \sum_{i,j=0}^{d^2-1} \alpha_{ij} \sigma_i \otimes \sigma_j$, where $\sigma_i$ are the generators of $SU(d)$ plus the identity and $\alpha_{ij} \in \mathbb{R}$. If $M = n - 1$, the line element of this manifold reads

$$ds^2 = \text{Tr} (dH^2) = \sum_{i=1}^{M} \text{Tr} (dh_i^2) + \sum_{i \neq j} \text{Tr} (dh_i dh_j)$$

$$= d^n \left( M \sum_{i,j=0}^{d^2-1} d\alpha_{ij}^2 + 2 \left( M - 1 \right) \sum_{j=1}^{d^2-1} d\alpha_{0j} d\alpha_{j0} + \left( M \frac{1}{2} \right) d\alpha_{00}^2 \right). \quad (31)$$

Now, since $\text{Tr} H = Md^n a_{00} = k' \leq k$, $d\alpha_{00} = 0$ and the metric becomes $\frac{g}{d^n} = \bigoplus_{i=1}^{d^2-1} \left( M - 1 \frac{M - 1}{M} \right) \otimes \bigoplus_{i=1}^{d^2-1} M - 2d^2 - 2d^2 + 1$, with determinant $\det \left( \frac{g}{d^n} \right) = (2M - 1)^{d^2 - 1} M^{d^2 - 2d^2 + 1}$. The volume of this manifold is then

$$\text{vol} (H) = \sqrt{\det g} \int \prod_{i,j=1}^{d} d\alpha_{ij} \int \prod_{j=1}^{d} d\alpha_{0j} d\alpha_{j0} \int \frac{d\ell}{\sqrt{-g}} \delta (\alpha_{00} - k') d\alpha_{00} dk'.$$

Consider now the previous line element without the term $\sum_{i \neq j} \text{Tr} (dh_i dh_j)$. Such line element corresponds to some manifold $\tilde{H}$:

$$ds^2 = \sum_{i=1}^{M} \text{Tr} (dh_i^2) = d^n \left( M \sum_{i,j=0}^{d^2-1} d\alpha_{ij}^2 \right). \quad (32)$$

Its metric is given by $\frac{\tilde{g}}{d^n} = \bigoplus_{i=1}^{d^2-1} M - 1$, with determinant $\det \left( \frac{\tilde{g}}{d^n} \right) = M^{d^2 - 1}$, yielding a volume $\text{vol} (\tilde{H}) = \sqrt{\det \frac{\tilde{g}}{d^n}} \int \prod_{i,j=1}^{d} d\alpha_{ij} \int \prod_{j=1}^{d} d\alpha_{0j} d\alpha_{j0} \int \frac{d\ell}{\sqrt{-\tilde{g}}} \delta (\alpha_{00} - k') d\alpha_{00} dk'$. Since $\det g \leq \det \tilde{g}$, it holds that $\text{vol} (H) \leq \text{vol} (\tilde{H})$.

Note that this argument can be extended to $t$-local TI Hamiltonians $H = \sum_{i=1}^{M} h_i$ in either 1D, 2D, or 3D, with $h_i \equiv h = \sum_{i,j \ldots k=0}^{d^2-1} \alpha_{ij \ldots k} \sigma_i \otimes \sigma_j \otimes \ldots \otimes \sigma_k$, and any value of $M$. Their associated $\tilde{g}$ metric is a diagonal matrix with repeated entry $Md^n$, whereas $g = \tilde{g} + X$, where $X$ is a matrix with vanishing diagonal. Now, since the metric is always positive definite, Hadamard’s inequality [19] can be applied to show that $\det (g) \leq \det (\tilde{g})$, implying $\text{vol} (H) \leq \text{vol} (\tilde{H})$.

In conclusion, calculating the volume associated to $\tilde{ds}^2$ will give an upper bound for the volume of $t$-local TI Hamiltonians. In order to do so, we now impose $\text{Tr} H > 0$ and rewrite the line element as $d\tilde{ds}^2 = \sum_{i=1}^{M} \sum_{k \neq l} (\Lambda_{ik} - \Lambda_{il})^2 \left| (dG_{kl}) \right|^2$, where $h_i$ is diagonalized as $h_i = u_i \Lambda_i u_i^\dagger$ with $\Lambda_i = \text{diag} (\Lambda_{1i}, \ldots, \Lambda_{Ni})$, $u_i$ an $N$-dimensional unitary matrix, and $dG_i = u_i^\dagger d u_i$. Now, since the Hamiltonian is TI, it holds that $\Lambda_i \equiv \Lambda \forall i$, where $\Lambda = \bigoplus_{i=1}^{d^2-1} \text{diag} (\Lambda_1, \ldots, \Lambda_{d^2})$, and $u_i = P_i u$ with $P_i$ a permutation matrix and $u$ an $N$-dimensional unitary. Therefore, $dG_i = u_i^\dagger d u_i = u_i^\dagger P_i^\dagger P_i d u = u_i^\dagger d u = d G \forall i$. Then we have $d\tilde{ds}^2 = Md^{n-t} \left( \sum_{i=1}^{d^2-1} (\Lambda_i)^2 + \sum_{k \neq l} (\Lambda_k - \Lambda_l)^2 \left| dG_{kl} \right|^2 \right)$.

Finally, imposing that the trace of $h$ is fixed, i.e. $\sum_{i=1}^{d^2-1} d\Lambda_i = 0$, one obtains $d\Lambda_{ij} = -\sum_{i=1}^{d^2-1} d\Lambda_i$ and so

$$d\tilde{ds}^2 = Md^{n-t} \left( \sum_{i=1}^{d^2-1} (\Lambda_i)^2 + \sum_{i=1}^{d^2-1} (\Lambda_i)^2 \right) \sum_{ij} q_{ij} \gamma_i \gamma_j, \quad (33)$$

with $q$ a metric tensor and $\gamma$ the vector of integration variables. The determinant of the metric tensor is $\det (q) = \nu^{d^2} \prod_{i<j}^4 (\Lambda_i - \Lambda_j)^4$, where $\nu = Md^{n-t}$ and $\kappa = d^t - 1 + \frac{d^{t+1}}{(d^2 - 2)!} = d^{2t} - 1$, so the volume element gains a factor $\sqrt{|\det (q)|}$: 

$$d\tilde{V} = \nu^t d^\frac{1}{2} \prod_{i<j}^4 (\Lambda_i - \Lambda_j)^2 \prod_{i=1}^{d^2-1} d\Lambda_i \prod_{i<j} 2 \text{Re} (dG_{ij}) \text{Im} (dG_{ij}) \text{.} \quad (34)$$
Recalling that $\text{Tr} H = M d^{n-t} \text{Tr} h \leq k$ and following the integration procedure in Section I, we obtain the claimed upper bound for the volume of $t$-local TI Hamiltonians:

$$\text{vol}_d(H) \leq \nu^2 I_1 \left( d^t, \frac{k}{M d^{n-t}} \right) I_2(d^t).$$

(35)