Phaseless Low Rank Matrix Recovery and Subspace Tracking

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Abstract—This work introduces the first simple and provably correct solution for recovering a low-rank matrix from phaseless (magnitude-only) linear projections of each of its columns. This problem finds important applications in phaseless dynamic imaging, e.g., Fourier ptychographic imaging of live biological specimens. We demonstrate the practical advantage of our proposed approach, AltMinLowRaP, over existing work via extensive simulation, and some real-data, experiments. Under a right incoherence (denseness of right singular vectors) assumption, our guarantee shows that, in the regime of small ranks, \( r \), the sample complexity of AltMinLowRaP is much smaller than what standard phase retrieval methods need; and it is only \( r^3 \) times the order-optimal complexity for low-rank matrix recovery. We also provide a solution for a dynamic extension of the above problem. This allows the low-dimensional subspace from which each image/signal is generated to change with time in a piecewise constant fashion.

I. INTRODUCTION

In recent years, there has been a resurgence of interest in the classical “phase retrieval (PR)” problem [1], [2]. The original PR problem involved recovering an \( n \)-length signal \( \mathbf{x} \) from the magnitudes of its Discrete Fourier Transform (DFT) coefficients. Its generalized version, studied in recent literature, replaces DFT by inner products with any arbitrary design vectors, \( \mathbf{a}_i \). Thus, the goal is to recover \( \mathbf{x} \) from \( y_i := |\langle \mathbf{a}_i, \mathbf{x} \rangle|, \) \( i = 1, 2, \ldots, m \). These are commonly referred to as phaseless linear projections of the unknown signal. While practical PR methods have existed for a long time, e.g., see [1], [2], the focus of the recent work has been on obtaining correctness guarantees for these and newer algorithms. This line of work includes convex relaxation methods [3], [4] as well as non-convex methods [5], [6], [7], [8], [9]. It is easy to see that, without extra assumptions, PR requires \( m \geq n \). The best known guarantees – see [7] and follow-up works – prove exact recovery with high probability (whp) with order-optimal number of measurements/samples: \( m = Cn \). Here and below, \( C \) is reused often to refer to a constant more than one. Most of the above work assumes that the \( \mathbf{a}_i \)'s are independent and identically distributed (iid) standard Gaussian vectors.

There is also very recent work on guarantees for PR methods that only need a random initialization (instead of a carefully designed spectral initialization which can be expensive); but the tradeoff is that these need more than \( O(n) \) samples. We do not discuss these here.

A natural approach to reduce the sample complexity is to impose structure on the unknown signal(s). In existing literature, with the exception of sparse PR which has been extensively studied, e.g., [10], [11], [12], there is little other work on structured PR. Low-rank is the other common structure. To our knowledge, this has not been explored in PR except for the references discussed in the next paragraph. It can be imposed in two ways. One is to assume that the unknown signal/image, whose phaseless linear projections are available, can be rearranged to form a low-rank matrix. A more practical and commonly used model is to consider the dynamic imaging setting and assume that a time sequence of signals/images is generated from a lower dimensional subspace of the ambient space. Equivalently, the matrix formed by arranging each signal/image as a column is low-rank. The same model also applies for imaging of a set of similar signals, e.g. a set of bacteria. It is a valid model whenever the set/sequence of signals is sufficiently similar (correlated). It forms the basis of many popular solution approaches, for example, eigen-faces for face recognition [13] (use of PCA for dimension reduction before classification), recommendation system design [14], and video background/foreground separation [15]. In the specific context of phaseless dynamic imaging, it allows us to reduce the sample complexity (and hence either the measurements’ acquisition time or the required number of imaging sensors) for dynamic Fourier ptychography: imaging of slowly changing dynamic scenes, such as live biological specimens, in vitro. The same idea can also be used for X-ray, sub-diffraction, or astronomical dynamic imaging.

In this work, we use the above model. Thus, our problem is to recover a low-rank matrix \( \mathbf{X}^* \in \mathbb{R}^{n \times q} \) from measurements of the form

\[
y_{ik} := |\langle \mathbf{a}_{ik}', \mathbf{x}_k^* \rangle|, \quad i = 1, \ldots, m, \quad k = 1, \ldots, q. \tag{1}
\]

Here and below, we use \( ' \) to denote vector or matrix transpose. Also, \( \mathbf{x}_k^* \) is the \( k \)-th column of \( \mathbf{X}^* \). Let \( \mathbf{X}^* \overset{SVD}{=} \mathbf{U}^* \mathbf{\Sigma}^* \mathbf{B}^* \) denote its singular value decomposition so that \( \mathbf{U}^* \in \mathbb{R}^{n \times r} \).
$B^* \in \mathbb{R}^{r \times q}$, and $\Sigma^* \in \mathbb{R}^{r \times r}$ is a diagonal matrix. Observe that this notation is a little non-standard, if the SVD was $U^* \Sigma^* V^*$, we are letting $B^* := V^*$. Thus, columns of $U^*$ and rows of $B^*$ are orthonormal. We use $\sigma^*_{\max}, \sigma^*_{\min}$ to denote the maximum and minimum singular values of $X^*$ and $\kappa = \sigma^*_{\max}/\sigma^*_{\min}$ to denote its condition number. Finally, let

$$
\hat{B}^* := \Sigma^* B^*.
$$

The only other assumption that we need on $X^*$ is right incoherence (incoherence/denseness of its right singular vectors) [14], [19]. In our notation, this means

$$
\max_k \|b^*_k\|^2 \leq \mu^2 r/q,
$$

with $\mu \geq 1$ being a constant. Clearly, this implies that

$$
\|\hat{b}^*_k\|^2 \leq \sigma^*_{\max} \frac{2\mu^2 r/q}{q} \text{ and } \|x^*_k\|^2 \leq \mu^2 \kappa^2 \|X^*\|_F^2/q.
$$

for each $k$. The converse is also true assuming we treat $\kappa$ as a constant; (3) also implies (2). Thus, right incoherence is the same as requiring that the energy in any one of the signals, $x^*_k$, is not much larger than its average value.

We use the above non-standard notation for SVD because (i) our solution approach will recover columns of $B^*$, $b^*_k$, individually by solving an $r$-dimensional standard PR problem (it is more intuitive to talk about recovery of column vectors than of rows); and (ii) it makes it easier to specify the dynamic problem setting. With the above notation, the QR decomposition of an estimate of $B^*$, denoted $\hat{B}$, will be written as $\hat{B} \leftarrow\rightleftharpoons R_B B$ with $B$ being an $r \times q$ matrix with orthonormal columns (or equivalently $\hat{B} \leftarrow\rightleftharpoons B(R_B')$).

For obtaining guarantees, we assume $a_{ik} \sim \mathcal{N}(0, 1)$ (iid standard Gaussian, real-valued).

Notation. We use $\|\cdot\|$ to denote the $l_2$-norm of a vector or the induced 2-norm matrix while using $\|\cdot\|_F$ to denote the Frobenius norm. We use $\hat{1}_{\text{statement}}$ to denote the indicator function; it takes the value one if statement is true and is zero otherwise. A tall matrix with orthonormal columns is referred to as a “basis matrix”. For two basis matrices $W, D$, we define the subspace error (distance) as $\text{SE}(W, D) = \|D \perp' W\| = \|W \perp D\|^2 = \|(I - WW')D\|$. This measures the largest principal angle between the two subspaces. We often use terms like “estimate $W$” when the goal is to really estimate its column span, $\text{Span}(W)$. Since we are working with real valued vectors and matrices, the phase-invariant distance is just the sign invariant distance and is defined as $\text{dist}(x^*, \hat{x}) = \min(\|x^* - \hat{x}\|, \|x^* + \hat{x}\|)$. Define the corresponding matrix distance as

$$
\text{mat-dist}(X^*, \tilde{X})^2 := \sum_{k=1}^q \text{dist}(x^*_k, \hat{x})^2.
$$

We reuse the letters $c, C$ to denote different numerical constants in each use, with the convention $C \geq 1$ and $c < 1$.

Contributions. This work provides the first provably correct solution for low-rank matrix recovery (LRMR) from column-wise phaseless linear projections. We henceforth refer to this problem as Phaseless Column-wise LRMR (Ph-Co-LRMR). We demonstrate the practical advantage of our proposed approach, AltMinLowRaP (Alt-Min for Phaseless Low Rank Recovery), over existing work via extensive simulation experiments; and a few experiments for recovering real videos from simulated coded diffraction pattern (CDP) measurements; e.g., see Fig. 1. This work also provides the first simple algorithm and guarantee for Phaseless Subspace Tracking. This allows the low-dimensional subspace from which each image/signal is generated to change with time in a piecewise constant fashion. It can be simply understood as a dynamic extension of Ph-Co-LRMR. Hence both the algorithm and the guarantee for it are simple, but useful, extensions of those for the static case.

AltMinLowRaP relies on three key ideas. The first is a clever spectral initialization for obtaining the first estimate of $\text{Span}(U^*)$. The second is the observation that, if $U^*$ were known, we only need to solve $q$ small ($r$-dimensional) standard PR problems to recover the $b^*_k$’s. Third, given a good estimate of $b^*_k$ and of $U^*$, we have a good estimate of $x^*_k$, and hence also of the phase (sign) of $(a_{ik} x^*_k)$. Thus, we can obtain a new improved estimate of $\text{Span}(U^*)$ by solving a least squares (LS) problem. The key insight that helps obtain a significant sample complexity reduction over standard PR is the observation that, for both the initialization and the update steps for $U^*$, conditioned on $X^*$, we have access to $mq$ mutually independent measurements. These are not identically distributed, however, the right incoherence assumption on $X^*$ ensures that the distributions are similar enough so that concentration holds with $mq$ samples.

Our guarantee for AltMinLowRaP shows that the sample complexity, $mq$, for recovering a rank $r$ matrix of size $n \times q$ to $\epsilon$ accuracy is just $C n^4 \mu^2 n^2 r^4 \log(1/\epsilon)$. If $q \approx n$, ignoring log factors and treating $\kappa$ and $\mu$ as constants, this means that only about $r^4$ samples per signal suffice when exploiting the low-rank property. For small $r$, this is a significant improvement over standard PR approaches which necessarily need $m \geq n$.

The minimum number of samples needed to recover an $n \times q$ matrix of rank $r$ is $(q + n)r \approx 2nr$. The AltMinLowRaP sample complexity is thus only $r^3$ times its order-optimal value, $Cnr$. We note here that, in problem settings like ours where the measurements are “not global” (no single measurement is a function of all the entries of the unknown quantity, here $X^*$), non-convex algorithms typically do need more than the order-optimal number of samples. Another LRMR problem from existing literature with non-global measurements is low-rank matrix completion (LRMC). The first non-convex LRMC solution, AltMinComplete [19], needed a sample complexity of order $\kappa/n^4 \mu^2 n^{4.5} \log(1/\epsilon)$ which is comparable to what we need. The best known guarantee is for a recent projected Gradient Descent solution from [20] and this needs $\Omega(nr^2 \log^2 n \log^2(\kappa/\epsilon))$ samples: this number is still $r^3$ times the order-optimal value of $nr$.

Paper Organization. We present our algorithm, guarantee, and simulation experiments for Ph-Co-LRMR in Sec. 11 given next. This section also includes a detailed discussion
of related work. The proof (sketch) is given in Sec. III. The lemmas used in Sec. III are proved in Sec. IV. We develop extensions to phaseless subspace tracking in Sec. V. We conclude in Sec. VI.

II. PHASELESS COLUMN-WISE LOW-RANK MATRIX RECOVERY

A. AltMinLowRaP: Alt-Min for Phaseless column-wise Low Rank Matrix Recovery

The goal is to recover $X^* = U^* B^*$ from measurements of the form (1). We adopt an alternating minimization (AltMin) approach [19]. Observe that we can rewrite $y_{ik}$ as $y_{ik} = |\langle a_{ik}, U^* b_k^* \rangle| = |\langle (U^*)^* a_{ik}, b_k^* \rangle|$ if $U^*$ were known, the problem of recovering each $b_k^*$ is an easy $r$-dimensional standard PR problem. If, instead, we have a good estimate of $\text{Span}(U^*)$, denoted $U$, we can still recover the $b_k^*$’s by solving a noisy version of the same problem. Any PR solution can be used, here we use reshaped Wirtinger flow (RWF) [8]. Since $U$ has orthonormal columns, the design vectors $(U^* a_{ik})$ are also standard Gaussian. By multiplying $U$ with the output of RWF, we get an estimate of $x_k^*$, and consequently also of the phase of $(a_{ik}, x_k^*)$. Using these phase estimates along with the measurements, one can obtain an updated (and hopefully better) estimate of $\text{Span}(U^*)$ by solving a standard LS problem. The output of the LS step may not have orthonormal columns; this is easily resolved by a QR decomposition. We summarize the complete algorithm in Algorithm 1.

Let $U^t$ denote the $t$-th estimate of $U^*$. In the $t$-th iteration, the PR step sees noise proportional to $\text{SE}(U^t, U^*)$. As a result the error in its estimate is of the same level. Thus, there is no advantage in running the full RWF in the initial few iterations. Under the assumption that the subspace recovery error decreases with $t$, one can obtain a speed-up by letting the number of RWF iterations at the $t$-th step, $T_{\text{RWF},t}$, grow with $t$. Since we prove geometric convergence, we can let $T_{\text{RWF},t}$ grow linearly with $t$.

To obtain the initialization, we develop a clever modification of the truncated spectral initialization idea from [7], [16]. First assume that $r$ is known. We initialize $\tilde{U}$ as the top $r$ left singular vectors of the following matrix:

$$Y_U = \frac{1}{mq} \sum_{k=1}^{m} \sum_{i=1}^{q} y_{ik}^2 a_{ik} a_{ik}' \mathbb{1}\{y_{ik} \leq \frac{\sigma}{\sqrt{2}} \sum_{i=1}^{q} y_{ik} \}.$$  \hspace{1cm} (4)

To understand why this works, consider the above matrix with the indicator function removed. Then it is not hard to see that its expected value equals $(1/q)[U^*(\Sigma^2)U' + 2\text{trace}\Sigma I]$, and so its span of top $r$ singular vectors equals $\text{Span}(U^*)$. Hence, with large enough $mq$, the same should approximately hold for the original matrix. However, when using $Y_U$ with the indicator function removed, a few “bad” measurements (those with very large magnitude $y_{ik}$ compared to their empirical mean over $i, k$) can heavily bias its value. To mitigate this effect and get a good initialization in spite of it, we will need a larger value of $mq$. Using the indicator function helps truncate the summation to only sum over the “good” measurements, and as a result a smaller value of $mq$ suffices. Mathematically, this helps ensure that $Y_U$ is close to a matrix that can be written as $\sum_{ik} w_{ik} w_{ik}'$ with $w_{ik}$’s being iid sub-Gaussian vectors (instead of sub-exponential in the case without truncation) [7].

We can also use $Y_U$ to correctly estimate $r$ whp by relying on the fact that when $m$ and $q$ are large the gap between its $r$-th and $(r + 1)$-th singular value is close to $\sigma_{min}^2 / q$. With this idea, we estimate $r$ as given in the first step of Algorithm 1.

By defining the $n \times m$ matrix $A_k := [a_{1,k}, a_{2,k}, \ldots, a_{m,k}]$ and $y_k := [y_{1,k}, y_{2,k}, \ldots, y_{m,k}]'$, and letting $|z|$ denote element-wise magnitude of a vector, we can rewrite (1) as $y_k = |A_k^* x_k^*|, \; k = 1, 2, \ldots, q$. This simplifies the writing of Algorithm 1. Also, as is commonly done in existing literature, e.g., [19], [5], in order to obtain a provable guarantee in a simple fashion, we use a new (independent) set of $m$ measurements in each new update of $U^*$ and of the $b_k^*$’s. Since we prove geometric convergence of the iterates, this increases the required sample complexity by a factor of only $\log(1/\epsilon)$. In our empirical evaluations, we do not do this.

B. Main Result

We have the following guarantee.
Algorithm 1 AltMin-LowRaP: Alt-Min for Phaseless Low Rank Recovery

1: Parameters: $T$, $T_{RW,F,t}$, $\omega$.
2: Partition the $m_{tot}$ measurements and design vectors for each $x_k^*$ into one set for initialization and $2T$ disjoint sets for the main loop.
3: Set $\hat{r}$ as the largest index $j$ for which $\lambda_j(Y_U) - \lambda_n(Y_U) \geq 0.25\sigma_{min}^2 / q$ where $Y_U$ is in (4).
4: $U^0 \leftarrow \hat{r}$ singular vectors of $Y_U$ defined in (4).
5: for $t = 0 : T$ do
6: \[ \hat{b}_k^t \leftarrow \text{RWF} \{Y_k^{(t)}, U^{(t)}, A_k^{(t)}\}, T_{RW,F,t} \] for each $k = 1, 2, \ldots, q$
7: Set $\hat{X}^t = U^t B^t$
8: Compute QR decomposition: $B^t \overset{QR}{=} R_B^t B^t$
9: $\hat{C}_k^t \leftarrow \text{Phase}(A_k^{(t)} \hat{b}_k^t)$ for each $k = 1, 2, \ldots, q$
10: $U^{t+1} \leftarrow \arg\min U \sum_{k=1}^q \| \hat{C}_k^t y_k^{(t)} - A_k^{(t)} U b_k^t \|^2$
11: Compute QR decomposition: $U^{t+1} \overset{QR}{=} U^{t+1} R_U^{t+1}$
12: end for

Theorem 2.1. Consider Algorithm 1. Assume that the $y_{ik}$’s satisfy (1) with $a_{ik}$ being iid standard Gaussian; $X^*$ satisfies right-incoherence with parameter $\mu$; and that the product $\mu \kappa$ is a constant. Set $T := C \log(1/\epsilon)$, $T_{RW,F,t} = C(\log t + \log \kappa + t \log(0.7) / \log(1 - c))$, and $\omega = 0.25 \sigma_{min}^2 / q$. Assume that, for the initialization step and for each new update, we use a new set of $m$ measurements with $m$ satisfying $mq \geq C \kappa \mu^2 n r^4$ and $m \geq C \max(r, \log q, \log n)$. Then, with probability (w.p.) at least $1 - \frac{C}{n^{10}}$,

\[ \text{SE}(U^*, U^T) \leq \epsilon, \quad \text{mat-dist}(\hat{X}^T, X^*) \leq \epsilon \| X^* \|_F \]

and dist$(\hat{x}_k^T, x_k^*) \leq \epsilon \| x_k^* \|_F$ for each $k$. Moreover, after the $t$-th iteration,

\[ \text{SE}(U^*, U^t) \leq 0.7^t \delta_{init}, t = 0, 1, 2, \ldots, T \]

where $\delta_{init} = \frac{\kappa}{2^2}$. Similar bounds also hold on the error in estimating $x_k^*$.

The time complexity is $mq r \log^2(1/\epsilon)$ and memory complexity is $m n q \log(1/\epsilon)$.

We prove this in Sec. III.

Remark 2.2. The requirement that $\mu \kappa$ be at most constant can be eliminated with the following simple changes: (i) multiply the threshold in the indicator function in the expression for $Y_U$ in (4) by $\mu^2 \kappa^2$; and (ii) multiply the required lower bound on $mq$ by $\mu^4 \kappa^4$.

Theorem 2.1 implies that the sample complexity $m_{tot} = (2T + 1) m$ needs to satisfy $m_{tot} q \geq C \kappa^4 \mu^2 n r^4 \log(1/\epsilon)$ along with $m_{tot} \geq C \max(r, \log q, \log n) \log(1/\epsilon)$. The required lower bound on just $m_{tot}$ is very small and essentially redundant. As discussed earlier, in the regime of small $r$, our sample complexity is not much larger than the optimal value of $(n + q) r$; and is significantly better than that of standard (unstructured) PR methods which necessarily need $m = C n$ samples per signal (matrix column).

The sample complexity gain over standard PR is to be expected because we are exploiting extra structure. But what is also expected is that time complexity increases when doing that. For a given value of $m$ and $q$, this is indeed true. AltMinLowRaP is about $r$ times slower than the best PR methods such as TWF or RWF. These need time of order $m q n \log(1/\epsilon)$ to recover a set of $q n$-length signals. However, if we instead consider the time needed if, for each method, we use the least number of measurements needed for the method to provably give an $\epsilon$-accurate estimate of the signals, then, in fact, we can argue that AltMinLowRaP is faster. More precisely, if we let $mq + C n r^4 \log^2(1/\epsilon)$ for AltMinLowRaP and $mq + C n q$ for TWF/RWF, then AltMinLowRaP is faster as long as $r^5 < q$.

A similar discussion applies for the memory complexity. In all experiments, we consider two settings (r known, and $r$ unknown). For the rank known case we use $m = 150$, $n = 400$, $q = 1000$, $r = 4$. In all experiments, we compare with LRPR2 [16] which is their best algorithm. The results are summarized in Fig. 2. Notice that our algorithm outperforms existing techniques in both scenarios. The complete details of experiments are provided in Appendix C. We also demonstrate the power of AltMinLowRaP for recovering a few real video sequences (these are only approximately low-rank) from simulated Coded Diffraction Pattern (CDP) measurements. We show the results on one video in Fig. 1. The experiment details and more video results are given in Appendix C.

C. Discussion of Related Work

Other work on phaseless low-rank recovery. The only other work that also studies Ph-Co-LMRR is [16]. This introduced a series of heuristics and evaluated them experimentally. It also attempted to provide a guarantee for obtaining an initial estimate of $X^*$. If we consider their main result (their Theorem 3.2) with ours, (i) it required the following
lower bound on just \( m \): \( m \geq C \max(\sqrt{n}, r^4)/\epsilon^2 \) in addition to a lower bound on \( mq \) that also depends on \( 1/\epsilon^2 \). The dependence on \( 1/\epsilon^2 \) is what makes this very large. (ii) Since it only analyzed initialization, a better way to compare its result with ours is to compare it with just our Lemmas 3.1 and 3.3. These together show that if \( mq \geq C \kappa^2 r^4/\delta_{\text{init}}^2 \), and \( m \geq C \max(r, \log q) \), then, whp, the initial estimate is \( \delta_{\text{init}} \)-accurate. To get a similar initialization guarantee, the result of [16] needed \( mq \geq \kappa^2 r^4/\delta_{\text{init}}^2 \) and \( m \geq C \max(\sqrt{n}, r^4) \). Their required lower bound on just \( m \) is a particularly strong requirement. (iii) Finally, its result assumed a bound on each entry of \( \hat{b}^* \), and this is a little stronger than just right incoherence.

Our approach for initializing \( U^* \) is taken from [16], but with a simple, but important, difference: the threshold in the indicator function used for defining \( Y_U^* \) in (4) now takes an average over all \( mq \) measurements (instead of over only the \( m \) measurements of the \( k \)-th column in [16]). This simple change allows us to use concentration over all the \( mq \) measurements (and design vectors) in every step of deriving the initialization guarantee for \( U^* \). This is what helps us eliminate the lower bound \( m \geq C r^4 \) on just \( m \) that was needed in [16]. A second, and most important, algorithmic difference is that, both for initialization and for later iterations, we recover \( \hat{b}^* \)'s by solving the full standard PR problem. This is what allows us to (i) eliminate the lower bound of \( m \geq C \sqrt{n} \) that [16] needed; and (ii) to get a complete guarantee for AltMinLowRaP. The reason is that, it allows us to show that the recovery error of \( \hat{b}^* \)'s is of the same order as that of subspace recovery at the current iteration. Instead, the algorithm in [16] only used one step (iteration) of AltMinPhase [5] for obtaining a new estimate of \( \hat{b}^* \)'s in each outer loop iteration.

There is no other work on phaseless ST except a recent short conference paper [17] that proposed a complicated algorithm which relied on impractical assumptions of subspace change. It does not contain any guarantees.

Another seemingly related work is [21]. This attempts to recover an \( n \times r \) matrix \( U^* \) from measurements \( y_i = \|a_i^*U^*\|^2 \). If \( r = 1 \), this is the standard PR problem. In the general case, this is related to covariance sketching, but not to our problem.

Existing work on low-rank matrix recovery. Low-rank matrix sensing (LRMS) involves recovering a low-rank \( X^* \) from \( y_i = \langle A_i, X^* \rangle \) with \( A_i \) being dense matrices. On the other hand, low-rank matrix completion (LRMC) involves recovering \( X^* \) from a subset of its entries, thus, in this case, the matrix \( A_i \) is one-sparse (it has a one in exactly one location). Both involve linear measurements (but very different kinds) and both have been extensively studied, see for example, [19], [22], [23] for LRMS solutions, and [19], [20] for LRMC solutions.

LRMS is the easier “global” measurements’ setting – each \( y_i \) contains information about the entire matrix \( X^* \). Many of the solution approaches for LRMS and their guarantees borrow ideas from the Compressive Sensing literature, which is another instance of a problem with global measurements (of the sparse vector). In both these cases, it is possible to prove a (sparse or low-rank) restricted isometry property (RIP) which simplifies the rest of the analysis. Our problem setting is different from, and more difficult than, LRMS or Compressive Sensing since our measurements are not global. In this sense, it can be compared to LRMC, however we should emphasize that our measurement model is not exactly like LRMC either. LRMC involves completely local measurements. Because of this, to allow for correct “interpolation”, it requires that \( X^* \) have dense rows and columns. This is imposed by putting denseness (incoherence) assumption on both its left and right singular vectors. In our setting, since we have global measurements of each column, but not of the entire matrix, only right incoherence suffices.

As noted above, even the linear version of our setting (suppose phase information was available) is different from both LRMS and LRMC. However, since no other complete guarantees exist for our problem setting even for its linear version, we compare our sample complexity with that of
LRMC methods (which is another low-rank recovery problem with non-global measurements). The first iterative LRMC solution, AltMinComplete [19], needed a sample complexity of about $C \kappa^2 \mu^2 n^2 4.5 \log (1/\epsilon)$. This is comparable to what we need. Our solution and overall proof approach both borrow ideas from this work. The best known LRMC guarantee is for a projected Gradient Descent solution from [20] and this needs $\Omega(nr^2 \log^2 n \log^2(\kappa/\epsilon))$ samples.

III. PROOF OF THEOREM 2.1

A. Overall lemmas and proof of Theorem 2.1

The proof is any easy consequence of the lemmas stated below. Observe that $U^0 = U^0$ since this initial estimate is obtained by SVD. We will also sometimes refer to it as $U_{\text{init}}$.

Claim 3.1 (Rank estimation and Initialization of $U^*$). Pick a $\delta_{\text{init}} < 0.25$. Assume $m q \geq \kappa^2 n^2 / \delta_{\text{init}}^2$. Set the rank estimation threshold $\omega = 0.3 \sigma_{\text{min}}^2 / q$. Assume also that $\mu_k \leq C$. Then, w.p. at least $1 - Cn^{-10}$, the rank is correctly estimated and

$$\text{SE}(U^*, U_{\text{init}}) \leq \delta_{\text{init}}$$

Definition 3.2. Define $g_k^t := (U^t)' x_k^t$ and $e_k^t := (I - U^t U^t') x_k^t$.

It is easy to see that $x_k^* = U^t g_k^t + e_k^t$ and so $y_{ik} = \|(U^t a_{ik}) g_k^t + a_{ik} e_k^t\|$. Thus, we have a noisy PR problem to solve with the noise magnitude proportional to $\|e_k^t\| \leq \text{SE}(U^t, U^*) \|x_k^t\|$. We use RWF to solve it. RWF provides an estimate of $\hat{g}_k^t = (U^t U^*) b_k^t$ which is just a rotated version of $b_k^t$. We show in the next lemma that the error dist$(\hat{g}_k^t, b_k^t)$, is proportional to $\text{SE}(U^t, U^*)$. By triangle inequality, the same is true for the error in $\hat{b}_k^t := U^t b_k^t$.

Lemma 3.3 (Recovery of $b_k^t$’s). At iteration $t$, assume that $\text{SE}(U^*, U^t) \leq \delta_t$. Pick a $\delta_b < 1$. If $m \geq C r$, and if we set $T_{\text{RWF}, t} = C \log \delta_t / \log (1 - c)$, then w.p. at least $1 - 2q \exp(-c \delta_b m)$, the following is true for each $k = 1, 2, \ldots, q$

$$\text{dist} \left( g_k^t, \hat{b}_k^t \right) \leq C \delta_b \| \hat{b}_k^t \| = C \delta_b \| x_k^t \|, $$

$$\text{dist} \left( \hat{x}_k^t, x_k^t \right) \leq (C + 1) \delta_b \| x_k^t \|, $$

$$\text{mat-dist} (G^t, B^t) \leq C \delta_b \| B^* \| = C \delta_b \| X^* \|$$

(5)

with $C = \sqrt{1 + \delta_b^2} + 1$.

Thus, if $m \geq C \max (r \log n, \log q) / \delta_b^2$, then the above holds w.p. at least $1 - n^{-10}$.

From the above lemma, $\hat{b}_k^t$ is close to $g_k^t$ (which is a rotated version of $\hat{b}_k^t$) for each $k$. We thus expect that $B^t$ also satisfies the incoherence assumption. We show next that this is indeed true if $\delta_b$ is small enough.

Lemma 3.4 (Incoherence of $B^*$ implies incoherence of $B^t$). Pick a $\delta_b < 1/10$ and assume that $m \geq C \max (r \log n, \log q) / \delta_b^2$. At iteration $t$, assume that $\text{SE}(U^*, U^t) \leq \delta_t$ with $\delta_t < \frac{\kappa}{C \sqrt{r \log n}}$. If $B^*$ is $\mu$-incoherent, then w.p. at least $1 - n^{-10}$, $B^t$ is $\mu$-incoherent with $\mu = C \kappa \mu$.

Finally, the next claim shows that the LS step to update $U$ reduces its error by a factor of $0.7$ at each iteration. Its proof relies on the previous two lemmas and the fact that $\hat{x}_k^t$ close to $x^*$ implies that, with large probability, the phases (signs) of $(a_{ik} \hat{x}_k^t)$ and $(a_{ik} x^*)$ are equal too.

Claim 3.5 (Descent Lemma). At iteration $t$, assume that $\text{SE}(U^*, U^t) \leq \delta_t$. If $\delta_t < \frac{\kappa}{C \sqrt{r}}$, $mq \geq C \kappa^3 \mu^2 n^2 / \delta_t^2$ and $m \geq C \max (r, \log n, \log q)$ then w.p. at least $1 - Cn^{-10}$,

$$\text{SE}(U^*, U^{t+1}) \leq 0.7 \delta_t := \delta_{t+1}.$$

Proof of Theorem 2.1. The $\text{SE}(U^*, U^t)$ bounds are an immediate consequence of Claims 3.1 and 3.5, along with setting $\delta_{\text{init}} = c / (C^5 r)$, $\delta_t = 0.7^t \delta_{\text{init}}$ and $\delta_b = 1/11$. The other bounds then follow by using Lemma 3.3.

We prove Claims 3.1 and 3.5 in the next two subsections (Sec. III-B and III-C). The proof of Lemmas 3.3 and 3.4 and of the lemmas needed for proving these two claims is postponed to Sec. IV.

B. Proof of Claim 3.1

The overall idea for proving this is borrowed from [16], [7]. But there are many important differences because we define $Y_{U}$ differently in this work: the threshold in the indicator function now takes an average over all $mq$ measurements (instead of over only the $m$ measurements of the $k$-th column as in [16]). This simple change enables us to get a significantly improved result. It uses let use concentration over all the $mq$ measurements (and design vectors) in each of the three steps of the proof. This is what helps eliminate the lower bound $m \geq C r^4$ on just $m$ that was needed in [16]. However, this also means that the proofs are much more involved (more quantities now vary with $k$). We give the proof next, and prove the three lemmas needed for the proof in Sec. IV-B.

Recall the expression for $Y_{U}$ from earlier, and define matrices $Y_{-}(e_1)$ and $Y_{+}(e_1)$ as given next. We will show that $Y_{U}$ is close to $Y_{-}$ and, hence, also to its expected value, $E [Y_{-}(e_1)]$. To do this we will first show that $Y_{U}$ is sandwiched between $Y_{-}$ and $Y_{+}$. Define

$$Y_{U} = \frac{1}{mq} \sum_{ik} |a_{ik} x_k|^2 a_{ik} a_{ik}^t \left\{ |a_{ik} x_k|^2 \leq \frac{\mu}{mq} \sum_{ik} |a_{ik} x_k|^2 \right\}$$

$$Y_{-}(e_1) = \frac{1}{mq} \sum_{ik} |a_{ik} x_k|^2 a_{ik} a_{ik}^t \left\{ |a_{ik} x_k|^2 \leq \frac{\mu(1-\epsilon_1)}{mq} \sum_{ik} |a_{ik} x_k|^2 \right\}.$$ 

Define $Y_{+}(e_1)$ similarly but with $(1-\epsilon_1)$ replaced by $(1+\epsilon_1)$.

Adapting the approach of [7], [16], we can show that (see Appendix B for a proof)

$$\text{E}[Y_{-}(e_1)] = \frac{1}{q} \left\{ \sum_{k} \beta_{1,k} x_k x_k^t + \left( \sum_{k} \beta_{2,k} \| x_k \|^2 \right) I \right\}$$

(6)
We get the exact same claim also for \( \beta_{1,k} \). Moreover, if \( \beta_{2,k} \) is a scalar standard Gaussian random variable. The expression for \( \mathbb{E}[Y_+ \langle \epsilon \rangle] \) is similar but with \( - \) replaced by \( + \). Notice that since \( Y_+ \) is different now compared to [16], the same is true for \( Y_-, Y_+ \) and their expected values. For example, \( \beta_{1,k} \) etc now depend on \( k \).

Observe that \( \mathbb{E}[Y_-] \) can be simplified as

\[
\mathbb{E}[Y_-] = \frac{1}{q}[U^*(\sum_k \beta_{1,k} b_k^* b_k^*)U^* + (\sum_k \beta_{2,k}\|b_k\|^2)I]
\]

Thus, the span of its top \( r \) eigenvectors (same as singular vectors) equals \( \text{Span}(U^*) \). Hence, we can use the \( \sin \theta \) theorem [24] in a fashion similar to [16] (Sec 6) to get

\[
\text{SE}(U^*, U_{\text{init}}) \leq \frac{\|Y_U - E[Y_-]\|}{\lambda_r(E[Y_-]) - \lambda_{r+1}(E[Y_-]) - \|Y_U - E[Y_-]\|} \leq \frac{1}{\text{min}_k \beta_{1,k}(\sigma_{min}^2/2)/q}
\]

Moreover,

\[
\lambda_r(E[Y_-]) - \lambda_{r+1}(E[Y_-]) = \frac{1}{q} \text{min}_k \sum_k \beta_{1,k} b_k^* b_k^*
\]

\[\geq \text{min}_k \beta_{1,k}(\sigma_{min}^2/2)/q\]

Now we just need to upper bound \( \|Y_U - E[Y_-]\| \) and lower bound \( \text{min}_k \beta_{1,k} \). Both these follow by combining the three lemmas given next and triangle inequality.

**Lemma 3.6.** We have that, w.p. at least \( 1 - \exp(-c_2^2 n q \sigma_{max}^2) \),

\[
Y_- \preceq Y_U \preceq Y_+
\]

and so \( \|Y_U - Y_-\| \leq \|Y_U - Y_+\| \).

**Lemma 3.7.** We have

\[
\|E[Y_+] - E[Y_-]\| \leq 9c_1 \|X^*\|^2_p \leq 9c_1 r \sigma_{max}^2 / q
\]

Moreover, if \( \mu \leq C \), then

\[
\text{min}_k \beta_{1,k} \geq 0.5.
\]

As noted in Remark 2.2, the requirement \( \mu \leq C \) can be eliminated with a simple change).

**Lemma 3.8.** We have, w.p. at least \( 1 - 2 \exp(n \log 9 - c_2^2 n q) \),

\[
\|Y_- - E[Y_-]\| \leq \frac{1.5c_2 r \sigma_{max}^2}{q}
\]

We get the exact same claim also for \( \|Y_+ - E[Y_+]\| \).

We prove the above lemmas in Sec. IV-B.

Using triangle inequality, \( \|Y_U - E[Y_-]\| \leq \|Y_U - Y_+\| + \|Y_- - E[Y_-]\| \). Moreover, using Lemma 3.6, \( \|Y_U - Y_-\| \leq \装入被遮挡的公式。
$c_{ik}, \hat{c}_{ik}$ are the phases (signs) of $a_{ik}'x_k^*$ and $a_{ik}'\hat{x}_k$.

We obtain high probability bounds on the three terms of MainTerm in the three lemmas that follow, Lemmas 3.10, 3.11, 3.12. All three lemmas first bound the terms for a fixed $W$, followed by using a carefully developed epsilon-net argument to extend the bounds for all unit Frobenius norm $W$'s. This is inspired by similar arguments in [25].

Consider a fixed $W$. To bound Term1, we first show $E[\text{Term}1] = 0$. Next we use Lemma 3.3 to show that $\|B'(B' - I)\| \leq C\delta_t\|X^*\|_F$. Finally, we use these two facts and a simple modification of Lemma 5.16 of [26] for sums of products of sub-Gaussian random variables (Lemma 4.1), along with careful linear algebra to show that, if $mq$ is large enough, w.p. $\|\text{Term}1\| \leq Cm\delta_t^2\|X^*\|_F$ for any $\delta_t < 0.1$. This is followed by a careful epsilon-net argument to extend the bound for all unit Frobenius norm $W$’s.

To bound Term2 for a fixed $W$, we first use Cauchy-Schwarz. This implies that

$$\|\text{Term2}(W)\| = \sqrt{\text{Term3}(W)\cdot\text{Term22}},$$

where

$$\text{Term22} := \sum_{ik} (c_{ik}\hat{c}_{ik} - 1)^2(a_{ik}'x_k^*)^2$$

Consider Term22. Notice that $(c_{ik}\hat{c}_{ik} - 1)^2$ takes only two values - zero or one. It is zero when the signs are equal, else it is one. Thus, to start bounding $E[\text{Term22}]$, we can use Lemma 1 of [8]. The idea is this: the probability that the signs are unequal is large only when $(a_{ik}'x_k^*)^2$ is comparable in magnitude to $\text{dist}^2(x_k^*, \hat{x}_k)$; and is very small otherwise. Moreover, on average, $(a_{ik}'x_k^*)^2$ is $1/\delta_t$ times larger $\text{dist}^2(x_k^*, \hat{x}_k)$, thus most of the time, they are not comparable. Using these ideas, we are able to show that $E[\text{Term22}] \leq m\delta_t^2\|X^*\|_F^2$. Careful use of concentration bounds then implies that, if $mq$ is large enough, the same order bound holds for $\|\text{Term22}\||\cdot\sqrt{\text{Term22}}$. Using this with an upper bound on $\|\text{Term22}\|$ helps bound $\|\text{Term22}\|$.

To bound Term3, notice first that $E[\text{Term3}] = m\|WB\|_F^2 = m$. Also, each summand in this term is sub-exponential with sub-exponential norm bounded by $\|WB_k\|^2$. Moreover, $\|WB_k\|^2 \leq \|b_k\|^2 \leq \mu^2r/q$ (by Lemma 3.4), and $\sum_{ik} \|WB_k\|^2 = m$. Using these facts and the Bernstein-like inequality for sums of sub-exponential random variables from [26], we can show that Term3 concentrates around $m$ w.p. $1/2$ and assuming that $\text{SE}(U^*, U) \leq \delta_t$, with $\delta_t < 1/20$, w.p. at least $1 - 2\exp\left(nr(\log 17) - c\frac{\delta_t^2mq}{\mu^2r}\right) - n^{-10}$.

$$\max_{W \in \mathcal{S}_W} \text{Term1}(W) \leq m\delta_t^2\|X^*\|_F$$

In proving the above, we also show that

$$\|B^*(B - B')\| \leq C\delta_t\|X^*\|_F$$

(we will use this in proving Lemma 3.13).

Lemma 3.12. Pick a $\delta_t < 1/10$ and assume that $m \geq C\max(r, n, q)/\delta_t^2$. Under the conditions of Theorem 2.1 and assuming $\text{SE}(U^*, U) \leq \delta_t$, with $\delta_t < 1/10$, w.p. at least $1 - 2\exp\left(nr(\log 17) - c\frac{\delta_t^2mq}{\mu^2r}\right) - 2\exp\left(-c\delta_t^2mq\right) - n^{-10}$.

$$\max_{W \in \mathcal{S}_W} \text{Term2}(W) \leq m\sqrt{1 + \delta_t^2}\|X^*\|_F$$

Finally, we lower bound the first denominator term of (8). This can be done by using the bound on $\|B^*(B - B')\|_F$ from Lemma 3.11. This, in turn, implies a lower bound on the minimum singular value of $B^*B'$, and hence the following.

Lemma 3.13. Pick a $\delta_t < 1/10$ and assume that $m \geq C\max(r, n, q)/\delta_t^2$. Under the conditions of Theorem 2.1, if $\text{SE}(U^*, U) \leq \delta_t$ with $\delta_t \leq \frac{1}{4r\sqrt{\mu}}$, then, w.p. at least $1 - n^{-10}$, $\sigma_{\min}(U^*\Sigma^*B^*B') \geq 0.9\sigma_{\min}$.

We prove the above lemmas in Sec. IV-E.

Proof of Claim 3.5. Combining Lemmas 3.9 and 3.13,

$$\text{SE}(U^{t+1}, U^*) \leq \frac{\text{MainTerm}}{0.9\sigma_{\min} - \text{MainTerm}}.$$ (9)

Set $\delta_t = 1/11$. Combining Lemmas 3.11, 3.12 and 3.10, and using $\|X^*\|_F \leq \sqrt{\sigma_{\max}}$, we conclude that

$$\text{MainTerm} \leq C(\delta_t + \sqrt{\delta_t})\delta_t\sqrt{\sigma_{\max}}$$

Using (9) and the above, and setting $\sqrt{\delta_t} = c/\sqrt{r}$,

$$\text{SE}(U^{t+1}, U^*) \leq \frac{C(\delta_t + \sqrt{\delta_t})\delta_t\sqrt{\sigma_{\max}}}{0.9\sigma_{\min} - C(\delta_t + \sqrt{\delta_t})\delta_t\sqrt{\sigma_{\max}}} \leq 0.7\delta_t \quad \Box$$

IV. PROOFS OF THE LEMMAS

A. Simple facts for various proofs

Our proofs will use the following facts: for two arbitrary matrices $A, H$,

1) $\sigma_{\max}(A + H) \leq \sigma_{\max}(A) + \|H\|

2) $\sigma_{\min}(A + H) \geq \sigma_{\min}(A) - \|H\|

3) $\sigma_{\min}(AH) \geq \sigma_{\min}(A)\sigma_{\min}(H).

4) For two basis matrices, $U_1, U_2$, $\sigma_{\min}^2(U_1'U_2) = 1 - \text{SE}^2(U_1, U_2).$

5) For any matrix $P$, $\|P\|_F \leq \|P\|_F \leq \|P\|_F^p = \frac{1}{\sigma_{\min}(P)}\|P\|_F.$

6) For an invertible matrix $P$,

$$\frac{1}{\|P\|_F} = \|P^{-1}\|_F \leq \|P^{-1}\|\|P\|_F = \frac{\|P\|_F}{\sigma_{\min}(P)}.$$
When $X_i = Y_i$, the above lemma simplifies to Lemma 5.16 of [26].

**B. Proof of the lemmas for Claim 3.1**

**Proof of Lemma 3.6.** Observe that we will be done if we can show that, whp, $\frac{1}{mq} \sum_{i,k} (a_{ik} x_k^*)^2$ lies in the interval $[(1 - \epsilon_1) \|X^*\|_F^2/q, (1 + \epsilon_1) \|X^*\|_F^2/q]$. Using Lemma 4.1, with $K_{X,k} = K_{Y,k} = \|x_k\|$, with probability more than $1 - 2 \exp (-\frac{C \epsilon_1^2 \log^2 m q}{\mu^2 k^2})$, we have

$$\sum_{i,k} (a_{ik} x_k^*)^2 - m \|X^*\|_F^2 \leq \epsilon_1 m \|X^*\|_F^2.$$

Details for obtaining this bound: using $\sum_k \|x_k^*\|^4 \leq \max_k \|x_k^*\|^2 \sum_k \|x_k\|^2$ and right incoherence,

$$\frac{t^2}{\max K_{X,k}^4} \geq \frac{\epsilon_1^2 m q}{\mu^2 k^2},$$

$$\frac{t}{\max K_{X,k}^4} m \|X^*\|_F^2 \geq \epsilon_1 m \|x_k^*\|^2 \geq \epsilon_1 m q \frac{\|x_k^*\|^2}{\mu^2 k^2}.$$

**Proof of Lemma 3.7.** Recall $\gamma_k = 9 \|X^*\|_F^2 / (q \|x_k^*\|^2)$. It is easy to see that

$$\|E[Y_+] - E[Y_-]\| \leq \frac{1}{q} \sum_k (\beta^+_{1,k} - \beta^-_{1,k}) \|x_k^*\|^2,$$

$$+ \frac{1}{q} \sum_k (\beta^+_{2,k} - \beta^-_{2,k}) \|x_k^*\|^2.$$

Now, using the fact that $x^3 e^{-x^2/2} \leq 3 \sqrt{3} e^{-3/2}$

$$\beta^+_{1,k} - \beta^-_{1,k} = \mathbb{E} [(\xi^4 - \xi^2) 1_{\{\xi^2 \leq (1 + \epsilon_1) \gamma_k \}}]$$

$$= \frac{2}{\sqrt{2\pi}} \int_{\sqrt{(1+\epsilon_1)\gamma_k}}^{\sqrt{(1+\epsilon_1)\gamma_k}} x^2 (x^2 - 1) e^{-x^2/2} dx$$

$$\leq \frac{2}{\sqrt{2\pi}} \int_{\sqrt{(1-\epsilon_1)\gamma_k}}^{\sqrt{(1+\epsilon_1)\gamma_k}} x^4 e^{-x^2/2} dx$$

$$\leq 6 \sqrt{3} e^{-3/2} \frac{1}{\sqrt{2\pi}} \int_{\sqrt{(1-\epsilon_1)\gamma_k}}^{\sqrt{(1+\epsilon_1)\gamma_k}} x dx$$

$$= 6 \sqrt{3} e^{-3/2} \frac{1}{\sqrt{2\pi}} \gamma_k \epsilon_1 \leq \gamma_k \epsilon_1.$$

Similarly, using $xe^{-x^2/2} \leq e^{-1/2}$

$$\beta^+_{2,k} - \beta^-_{2,k} = \mathbb{E} \left[ x^2 1_{\{x \leq (1 - \epsilon_1) \gamma_k \}} \right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_{\sqrt{(1-\epsilon_1)\gamma_k}}^{\sqrt{(1+\epsilon_1)\gamma_k}} x^2 e^{-x^2/2} dx$$

$$\leq \frac{2 e^{-1/2}}{\sqrt{2\pi}} \int_{\sqrt{(1-\epsilon_1)\gamma_k}}^{\sqrt{(1+\epsilon_1)\gamma_k}} x dx$$

$$= 2 e^{-1/2} \frac{1}{\sqrt{2\pi}} \gamma_k \epsilon_1 \leq \gamma_k \epsilon_1.$$
First consider a fixed unit vector \( z \). Observe that \( z' w_{ik} \) is sub-Gaussian with sub-Gaussian norm \( K = C \| X^* \|_F / \sqrt{q} \). Thus, using Lemma 4.1 with \( t = c_2 m \| X^* \|_F^2 \), and \( K x_{ik} = K y_{ik} = \| X^* \|_F^2 \sqrt{q} \), we can conclude that w.p. at least \( 1 - 2 \exp(c_2 m q) \),

\[
| z' \left( \sum_{ik} w_{ik} w_{ik}' - m q E [ Y_{ik}] \right) z | \leq c_2 m q \| X^* \|_F^2,
\]

After this, we can use a standard epsilon-net argument to extend the bound to all unit vectors \( z \). With it, we can conclude that, w.p. at least \( 1 - 2 \exp(n \log 9 - c_2 m q) \),

\[
\| Y_\cdot - E [ Y_\cdot] \| \leq \frac{1.5 \epsilon q \| X^* \|_F^2}{q}
\]

\( D \). Proof of Lemmas 3.3 and 3.4

In this section, \( a_{ik} \) := \( a_{ik}^{(t)} \) and same for \( y_{ik} \). Also, everywhere below, we remove the superscripts \( t \) for ease of notation. Recall that \( x_{ik} = U g_{ik} + e_{ik} \) with \( g_{ik} = U' x_{ik} = U' U b_{ik} \), \( e_{ik} := (I - UU') x_{ik} \). Also \( G = U' U^* B^* \).

Proof of Lemma 3.3. To estimate \( b_{k} \), we first need to estimate \( g_{k} \) which requires measurements of the form \( a_{ik} U g_{k} \). However, our measurements are of the form

\[
| a_{ik} U g_{k} | + \nu_{ik},
\]

where \( \nu_{ik} = | a_{ik} x_{ik}' | - | a_{ik} U g_{k} | \) is the noise. To solve this we use Theorem 2 of [8] which guarantees that, if \( m \geq C r \), w.p. at least \( 1 - \exp(-cm) \),

\[
\| \nu_{ik} \|^2 \leq m(1 + \delta_b) \| e_{ik} \|^2 \leq m(1 + \delta_b) \| b_{ik} \|^2.
\]

Clearly \( \| a_{ik}' U g_{k} + a_{ik} e_{ik} \| - | a_{ik} U g_{k} | \leq | a_{ik}' e_{ik} | | v_k |^2 = \sum_{i} \| a_{ik} \| e_{ik}^2 \).

\( \| v_k |^2 \leq \sum_{i} | a_{ik}' e_{ik} | | v_k |^2 \leq (1 + c_1) \| b_{ik} \|^2 \).

By setting \( T_{\text{RWF},t} \) so that \( (1 - c) \| b_{ik} \|^2 \leq \delta_b \), we get that \( \| b_{ik} \|^2 \leq \delta_b \| b_{ik} \|^2 \).

The above bound holds w.p. at least \( 1 - \exp(-cm) \). The above bound applies to each \( b_{ik} \).

To estimate \( b_{k} \) we use the above and \( | g_{k} | \leq \| b_{ik} \|^2 \).

The third claim is an immediate consequence of the first.

\( \square \)

Proof of Lemma 3.4. Recall that \( \tilde{B} \overset{QR}{=} R_B B \) and so \( b_{k} = R_B \tilde{b}_{ik} \). Using Lemma 3.3, \( | g_{k} | \leq \| b_{ik} \|^2 \), and right incoherence (which implies that \( \| b_{ik} \|^2 \leq \sigma_{\text{max}}^2 \mu^2 r/q \)).

\[
\| b_{ik} \|^2 \leq \| R_B^{-1} \| \| g_{ik} + C \| \| b_{ik} \|^2 \|
\]

\[
\leq \| R_B^{-1} \| \| g_{ik} + C \| \| b_{ik} \|^2 \|
\]

\[
\leq (1 + C \delta_b) \sigma_{\text{min}}^r \mu^2 r/q \leq \frac{1.5 \sigma_{\text{max}}^r \mu^2 r/q}{\sigma_{\text{min}}(R_B)}
\]

To lower bound \( \sigma_{\text{min}}(R_B) \), observe that \( \sigma_{\text{min}}(R_B) \geq \sigma_{\text{min}}(\tilde{B}) \). Using Lemma 3.3, the discussion of Sec. IV-C, facts from Sec. IV-A, and \( SE(U, U^*) \leq \delta_r \),

\[
\sigma_{\text{min}}(\tilde{B}) \geq \sigma_{\text{min}}(G) - \| G - B \|
\]

\[
\geq \sigma_{\text{min}}(U' U^*) \sigma_{\text{min}}(\tilde{B}^*) - \| G - B \|_F
\]

\[
\geq \sqrt{1 - SE^2(U, U^*)} \sigma_{\text{min}}(\tilde{B}^*) - C \delta_b \| B^* \|_F
\]

\[
\geq \sqrt{1 - \delta_r \| \sigma_{\text{max}}^r - C \delta_b \| B^* \|_F}
\]

\[
\square
Using $\delta_t \leq c/\kappa \sqrt{r}$, $\sigma_{\min}(R_B) = \sigma_{\min}(\hat{B}) \geq 0.9 \sigma_{\min}$.
Thus,
\[ \|b_k\| \leq \frac{1.5 \sigma_{\max}(R_B) \sqrt{r}/q}{0.9 \sigma_{\min}} \leq 2 \kappa \sqrt{r}/q := \tilde{\mu} \sqrt{r}/q. \]
All of the above bounds used the bound from Lemma 3.3. Thus the above bounds hold w.p. at least $1 - n^{-10}$ as long as $m \geq C \max(r, \log q, \log n)$.

E. Proof of the lemmas for Claim 3.5
In this section, $a_{ik} := a_{ik}^{(T+t)}$ and same for $y_{ik}$. Also, at almost all places, we remove the superscript $t$.

All the proofs in this section use incoherence of $B$ with parameter $\mu = \kappa \mu$. This holds w.p. at least $1 - n^{-10}$ as long as $m \geq C \max(r, \log n, \log q)/\delta_0^2$. Above follows by Lemma 3.4.

Proof of Lemma 3.10. Recall that $\text{Term}_3(W) := \sum_{ik} |a_{ik}^\prime W_{bk}|^2$. We have
\[ \mathbb{E} \left[ \left( \sum_{ik} |a_{ik}^\prime W_{bk}|^2 \right) \right] = m \|WB\|^2_F = m. \]
Let $X_{ik} = |a_{ik}^\prime W_{bk}|$. $X_{ik}$ is sub-Gaussian with sub-Gaussian norm $\|W_{bk}\|$. We use Lemma 4.1 with $Y_{ik} = X_{ik}$ and $t = \delta_t m$, along with the following facts to simplify the resulting expressions:
\[ \sum_k \|W_{bk}\|^4 \leq \max_k \|W_{bk}\|^2 \sum_k \|W_{bk}\|^2, \]
\[ \sum_k \|W_{bk}\|^2 = \|WB\|_F^2 = \text{trace}(WB^TW) = \|W\|_F^4 = 1, \]
\[ \max_k \|W_{bk}\|^2 \leq \|W\|^2 \max_k \|b_k\|^2 \leq \|W\|_F^2 \|b_k\|^2 \leq \max_k \|b_k\|^2 \leq \tilde{\mu}^2 r/q, \] w.p. at least $1 - n^{-10}$ as long as $m \geq C \max(r, \log n, \log q)/\delta_0^2$ (this follows by Lemma 3.4).

Using Lemma 4.1 and the above simplifications, for a fixed $W$,
\[ \Pr \left( \sum_{ik} |a_{ik}^\prime W_{bk}|^2 - m \geq \delta_t m \right) \leq 2 \exp \left( -c \frac{\delta_t^2 q}{\mu^2 r} \right), \]
\[ \Pr \left( \sum_{ik} |a_{ik}^\prime W_{bk}|^2 - m \leq (1 + \delta_t) m, \text{ for all } \tilde{W} \in \tilde{S}_W \right) \]
\[ \geq 1 - 2 |\tilde{S}_W| \exp \left( -c \frac{\delta_t^2 q}{\mu^2 r} \right), \]
\[ \geq 1 - 2 \exp \left( nr \log(17) - c \frac{\delta_t^2 q}{\mu^2 r} \right). \] (10)

Next we extend the above to obtain lower and upper bounds over the entire hyper-sphere, $S_W$. Define
\[ \theta_W = \max_{W \in S_W} \sum_{ik} |a_{ik}^\prime W_{bk}|^2, \]
as the maximum of Term$3(W)$ over $S_W$. We need to upper bound this.

Since $\Delta W = \sum_{ik} |a_{ik}^\prime W_{bk}|^2$, $\sum_{ik} |a_{ik}^\prime W_{bk}|^2 \leq \theta_W \|\Delta W\|^2_F \leq \theta_W \epsilon^2$.

Using this, (10), and Cauchy-Schwarz,
\[ \sum_{ik} |a_{ik}^\prime W_{bk}|^2 \]
\[ = \sum_{ik} |a_{ik}^\prime W_{bk}|^2 + \sum_{ik} |a_{ik}^\prime \Delta W_{bk}|^2 \]
\[ + 2 \sum_{ik} (a_{ik}^\prime W_{bk}) (a_{ik}^\prime \Delta W_{bk}) \]
\[ \leq (1 + \delta_t) m + c^2 \theta_W + 2 \sqrt{(1 + \delta_t) m \theta_W} \]
\[ = (1 + \delta_t) m + (1/4) \theta_W + (1/4) m \sqrt{1 + \sqrt{\theta_W/m}} \]
w.p. at least $1 - 2 \exp \left( nr \log(17) - c \frac{\delta_t^2 q}{\mu^2 r} \right)$. We used Cauchy-Schwarz and (10) to obtain the inequality. The last equality just used $c = 1/4$ and re-arranged the third term.

Now, if $\theta_W/m < 1$, we are done because then $\theta_W \leq m$. Otherwise, $\theta_W/m \geq 1$ and so $\sqrt{\theta_W/m} \leq \theta_W/m$. Therefore, using the definition of $\theta_W$, we have
\[ \theta_W \leq (1 + \delta_t) m + \theta_W ((1/4) m \sqrt{1 + \delta_t}). \]

By assumption, $\delta_t < 1/10$, and so the above implies that $\theta_W \leq 1.5 (1 + \delta_t) m$.
Thus, w.p. $1 - 2 \exp \left( nr \log(17) - c \frac{\delta_t^2 q}{\mu^2 r} \right)$,
\[ \theta_W := \max_{W \in S_W} \text{Term}_3 \leq 1.5 (1 + \delta_t) m \leq 1.7 m. \]

We now obtain the lower bound on the minimum of Term$3$ over the entire hyper-sphere. This uses (10), Cauchy-Schwarz,
and the upper bound on $\theta_W$ from above
\[
\sum_{ik} |a'_{ik}Wb_k|^2 \\
\geq \sum_{ik} |a'_{ik}Wb_k|^2 + 2 \sum_{ik} (a'_{ik}Wb_k) (a'_{ik}\Delta Wb_k) \\
\geq \sum_{ik} |a'_{ik}Wb_k|^2 - 2 \sum_{ik} a'_{ik}Wb_k a'_{ik}\Delta Wb_k| \\
\geq m(1 - \delta_k) - 2\sqrt{\sum_{ik} |a'_{ik}\Delta Wb_k|^2} \sqrt{\sum_{ik} |a'_{ik}Wb_k|^2} \\
\geq m(1 - \delta_k) - 2\sqrt{m(1 + \delta_k)} \sqrt{\theta_W \|\Delta W\|_F^2} \\
\geq m(1 - \delta_k) - 2m(1 + \delta_k)\sqrt{1.5e} \geq m(0.9 - 0.26)
\]

w.p. $1 - 2\exp\left( nr(\log 17) - c\sigma_{max}^2 / \mu_t \right)$. In the last line we substituted $\epsilon = 1/8$ and used $\delta_k < 1/10$.

All of the above bounds hold on the event in which $B$ is $\tilde{\mu}$ incoherent. This holds w.p. at least $1 - n^{-10}$ as long as $m \geq C \max( r, \log n, \log q ) / \delta_k^2$ (this follows by Lemma 3.4).

Thus, if $m \geq C \max( r, \log n, \log q ) / \delta_k^2$, w.p. $1 - 2\exp\left( nr(\log 17) - c\sigma_{max}^2 / \mu_t \right) - n^{-10}.

\[
\min_{W \in S_{W}} \text{Term 3}(W) \geq 0.64m.
\]

\[
\square
\]

**Proof of Lemma 3.11.** Define
\[
p_k := \tilde{B}^* B'b_k - \tilde{b}_k = \Sigma^* (B^* B' b_k - b_k)
\]

Recall that
\[
\text{Term 1}(W) = b_k^r W_{\tilde{a}_{ik}} a'_{ik} U^* p_k
\]

Define the matrix $P := [p_1, p_2, \ldots, p_q]$. Observe that
\[
P = \tilde{B}^* (B' B - I)
\]

We first upper bound $\|P\|_F$ by using Lemma 3.3. Recall that $g_k = \tilde{U}^* U^* b_k$ and $G = \tilde{U}^* U^* B$. In Lemma 3.3, we have bounded mat-dist($G, \tilde{B}$) = $\|G - \tilde{B}\|_F$ (by the discussion of Sec. IV-C). Recall also that $\tilde{B} QR_{\tilde{B}} B$ where $B$ is a matrix with orthonormal rows. So $B(I - B' B) = 0$. Using this and facts from Sec. IV-A,
\[
\|P\|_F = \|\tilde{B}^* (I - B' B)\|_F \\
\leq \frac{1}{\sigma_{min}(U^* U)} \|U' U^* \tilde{B}^* (I - B' B)\|_F \\
= \frac{1}{\sigma_{min}(U^* U)} \|(G - \tilde{B}) + \tilde{B} (I - B' B)\|_F \\
\leq \frac{1}{\sigma_{min}(U^* U)} \|G - \tilde{B}\|_F \\
\leq \frac{1}{\sigma_{min}(U^* U)} C\delta_l \|X^*\|_F.
\]

Furthermore $\sigma_{min}^2(U^* U) = 1 - \text{SE}(U, U^*)^2 \geq 1 - \delta_t^2$. By assumption, $\delta_t \leq 1/10$ and so

\[
\|P\|_F \leq \sqrt{10/9} C\delta_l \|X^*\|_F.
\]

Next we show that $\mathbb{E}[\text{Term 1}(W)] = 0$.
\[
\mathbb{E}[\text{Term 1}(W)] = m \sum_k b_k^r W'U^* \Sigma^* (B^* B' b_k - b_k) \\
= m \sum_k \text{tr}(W'U^* \Sigma^* (B^* B' b_k - b_k)) \\
= m \text{tr}(W'U^* \Sigma^* (B^* B' BB' - B^* B')) \\
= 0
\]

where we used $BB' = I_r$. Finally we will show that it also concentrates around zero. Let $X_{ik} = a'_{ik} W b_k$ and $Y_{ik} = a'_{ik} U^* p_k$. Both are sub-Gaussian, and so we can apply the Bernstein-type lemma, Lemma 4.1, for sums of products of sub-Gaussian r.v.’s. Observe that $K_{X_{ik}} = \|W b_k\| \leq \|W\|_F \|b_k\| \leq \|b_k\|$, and $K_{Y_{ik}} \leq \|p_k\|$.

Using Lemma 3.4, $b_k$’s are incoherent, i.e.,
\[
\|b_k\|_2^2 \leq \hat{\mu}^2 r/q = C \kappa^2 \mu^2 r/q.
\]

w.p. at least $1 - n^{-10}$ as long as $m \geq C \max( r, \log n, \log q ) / \delta_k^2$. Using the above, we can also show that $p_k$’s are incoherent as follows. Using $(a^2 + b^2)^2 \leq 2(a^2 + b^2)$, and $\|B\| = \|B'\| = 1$,
\[
\|p_k\|_2^2 \leq 2\sigma_{max}^2 (\|B^*\|_2^2 \|B\|_2^2) \leq 2\sigma_{max}^2 (\|b_k\|^2 + \|b_k\|^2) \\
\leq 2\sigma_{max}^2 (\|b_k\|^2 + \|b_k\|^2) \\
\leq 2\sigma_{max}^2 (\hat{\mu}^2 + \mu^2) r/q = C \kappa^2 \mu^2 \sigma_{max}^2 r/q
\]

We can now apply Lemma 4.1. Set $t = m \delta_t^2 \|X^*\|_F$. We have
\[
\sum_{ik} K_{X_{ik}} K_{Y_{ik}} \frac{\delta_t^2 \|X^*\|_F^2}{m} \geq \frac{\delta_t^2 \|X^*\|_F^2}{m} \\
\geq \frac{\max_k \|b_k\|_2^2 \|P\|_F^2}{m} \\
\geq \frac{\delta_t^2 \|X^*\|_F^2}{m} \\
\geq \frac{\delta_t^2 \|X^*\|_F^2}{m} \\
\geq \frac{\delta_t^2 \|X^*\|_F^2}{m}
\]

The second inequality used (11), the third used incoherence of $b_k$’s, the fourth used incoherence of $b_k$’s and $p_k$’s (proved above) and $\|X^*\|_F \geq \sqrt{\sigma_{min}}$. Thus
\[
\min \left( \frac{t^2}{\sum_{ik} K_{X_{ik}} K_{Y_{ik}}} ; \frac{t}{\max_k K_{X_{ik}} K_{Y_{ik}}} \right) \geq \frac{\delta_t^2 \|X^*\|_F}{C \kappa^2 \mu^2 r}
\]

and so
\[
\text{Pr}(|\text{Term 1}(W)| \leq m \delta_t^2 \|X^*\|_F) \geq 1 - \exp \left( -e \frac{\delta_t^2 \|X^*\|_F}{C \kappa^2 \mu^2 r} \right)
\]
Now we just need to extend our bound for all \( W \in S_W \). We first extend it to all \( W \) in an epsilon-net of \( S_W \). By [26](Lemma 5.2), there is an \( \epsilon \)-net, \( S_W' \), so that for any \( W \) in \( S_W' \), there is a \( W' \) in \( S_W \) such that \( \| W - W' \|_F \leq \epsilon \) and \( |S_W'| \leq (1 + \frac{\epsilon}{2})^{nr} n^r \). Pick \( \epsilon = 1/8 \). With this, \( |S_W'| \leq 17^{nr} \). Define \( \Delta W := W - W' \). We have \( \| \Delta W \|_F \leq \epsilon = 1/8 \).

Using union bound on the set \( S_W' \),

\[
\Pr(|\text{Term1}(W)| \leq m\delta_2^2\|X^*\|_F \text{ for all } W \in S_W') \\
\geq 1 - 2|S_W'| \exp \left(-\frac{m\delta_2^2}{\epsilon^2} \right) \\
\geq 1 - 2 \exp \left( -\frac{nr(\log 17) - cm\delta_2^2}{\epsilon^2} \right)
\]

(12)

To extend the claim to all \( W \in S_W \), define

\[
\theta_W := \max_{W \in S_W} \sum_{ik} (a'_{ik}Wb_{ik})(a'_{ik}U^*p_k).
\]

Since \( \|\Delta W\|_F \in S_W' \), \( \sum_{ik} (a'_{ik}\Delta Wb_{ik})(a'_{ik}U^*p_k) \leq \theta_W \|\Delta W\|_F \leq \theta_W \epsilon \). Thus, using (12), for any \( W \in S_W \),

\[
\sum_{ik} (a'_{ik}Wb_{ik})(a'_{ik}U^*p_k) = \sum_{ik} (a'_{ik}Wb_{ik})(a'_{ik}U^*p_k) + \sum_{ik} (a'_{ik}\Delta Wb_{ik})(a'_{ik}U^*p_k) \leq m\delta_2^2\|X^*\|_F + \theta_W \epsilon
\]

w.p. at least 1 $- 2 \exp \left( -\frac{nr(\log 17) - cm\delta_2^2}{\epsilon^2} \right)$.

Thus, with the above probability, \( \theta_W \leq m\delta_2^2\|X^*\|_F / (1 - \epsilon) = (8/7)m\delta_2^2\|X^*\|_F \) by setting \( \epsilon = 1/8 \).

All of the above bounds hold on the event in which \( B \) is incoherent. This holds w.p. at least \( 1 - n^{-10} \) as long as \( m \geq C\max(r, \log n, \log q)/\delta_2^2 \) (this follows by Lemma 3.4).

Thus, if \( m \geq C\max(r, \log n, \log q)/\delta_2^2 \), w.p. at least \( 1 - 2 \exp \left( -\frac{nr(\log 17) - cm\delta_2^2}{\epsilon^2} \right) - n^{-10} \), max\( W \in S_W |\text{Term1}(W)| \leq (8/7)m\delta_2^2\|X^*\|_F \).

**Proof of Lemma 3.12.** Recall that \( a_{ik} := (a_{ik}')^{(T+i)} \) and same for \( y_{ik}' \). Thus, these are independent of the current \( \hat{x}_k \)'s. By Cauchy-Schwarz,

\[
\text{Term2}(W) := \sum_{ik} (c_{ik}\hat{c}_{ik} - 1)(a'_{ik}Wb_{ik})(a'_{ik}x_k') \leq \sqrt{\sum_{ik} |a'_{ik}Wb_{ik}|^2} \sqrt{\sum_{ik} |c_{ik}\hat{c}_{ik} - 1|^2 |a'_{ik}x_k'|^2}.
\]

(13)

We will bound the first term using Lemma 3.10. Consider the second term. Since \( c_{ik} = \text{sign}(a_{ik}'x_k') \) and \( \hat{c}_{ik} = \text{sign}(a_{ik}'\hat{x}_k) \), it is easy to see that \( (c_{ik}\hat{c}_{ik} - 1)^2 = \mathbb{I}(c_{ik} \neq \hat{c}_{ik}) \). To bound this term we use the following result.

**Lemma 4.2** (Lemma 1 of [8]). Let \( a_i \) be standard Gaussian random vectors. For any given \( x^* \), and \( \hat{x} \) independent from \( a_i, i = 1, \ldots, m, \) that satisfy \( \|x^* - \hat{x}\| \leq 0.4 \), we have

\[
\Pr \left( (\mathbb{I}(\text{sign}(a_i'x^*) \neq \text{sign}(a_i'\hat{x})) | (a_i'x^*)^2 = z^2) \leq \text{erfc} \left( \frac{z}{2\|x^* - \hat{x}\|} \right),
\]

where \( \text{erfc}(u) := \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-t^2) \, dt \).

Let \( Q_{ik} := \mathbb{I}(c_{ik} \neq \hat{c}_{ik}) (a_{ik}'x_k')^2 \) and \( Z_{ik} := a_{ik}'x_k' \). Recall from Sec. IV-C that dist \( (x_k, x_k') = \|h_k\| \) with \( h_k = x_k' - \hat{x}_k \).

We first upper bound \( E(Q_{ik}) \). For simplicity, we remove the subscripts \( ik \) wherever these are not needed. Consider \( E(Q) = E[\mathbb{I}_{c \neq \hat{c}} Z^2] \). Observe that \( c = \text{sign}(Z) \) is a function of \( Z \) and \( \hat{c} \) depends on \( a \). Thus both of \( c, \hat{c} \) are dependent on \( Z \). We first bound \( E(Q|Z^2) \)

using the lemma stated above.

\[
E(Q|Z^2 = z^2) = E[\mathbb{I}_{c \neq \hat{c}} z^2 Z^2 = z^2] = z^2 \Pr(\mathbb{I}_{c \neq \hat{c}} Z^2 = z^2) \leq z^2 \text{erfc} \left( \frac{z}{2\|x^* - \hat{x}\|} \right) \leq z^2 \exp \left( -\frac{z^2}{4\|x^* - \hat{x}\|^2} \right).
\]

The first inequality follows using the lemma stated above, the second is a standard upper bound on the erf function [27]. Thus,

\[
E(Q) = E(E(Q|Z^2)) \leq E \left[ Z^2 \exp \left( -\frac{Z^2}{4\|x^* - \hat{x}\|^2} \right) \right].
\]

Since \( Z \) is zero mean Gaussian with variance \( \|x^*\|^2, Y := Z^2/\|x^*\|^2 \) is standard chi-squared with one degree of freedom. Thus, using \( \exp(-y/2) < 1 \), and \( E[Y] = 1 \), we get

\[
E(Q) \leq \int_0^{\infty} \exp \left( -\frac{y\|x^*\|^2}{4\|x^* - \hat{x}\|^2} \right) \exp(-y/2) 2y^{1/2} \, dy \leq \int_0^{\infty} \exp \left( -\frac{y\|x^*\|^2}{4\|x^* - \hat{x}\|^2} \right) \, dy = 2\sqrt{2} \text{dist}(x^*, \hat{x})^3 \leq 2\sqrt{2} \|x^*\|^3 \leq 4\delta_5^3 \|x^*\|^2.
\]

The last inequality used Lemma 3.3. Thus

\[
\sum_{ik} E(Q_{ik}) \leq mC\delta_5^3 \|x^*\|^2.
\]

(14)

Next we show that, whp, \( \sum_{ik} Q_{ik} \) is of the same order. As shown in the proof of Theorem 1 of [8], \( c_{ik} \neq \hat{c}_{ik} \) implies that \( (a_{ik}'x_k')^2 \leq (a_{ik}'h_k)^2 \). Here \( h_k = x_k' - \hat{x}_k \).

\[
\text{This follows from } (a'x)^2 = (a'h)^2 + (a'\hat{x})^2 + 2(a'h)(a'\hat{x}) = (a'h)^2 - (a'\hat{x})^2 + 2(a'x)(a'\hat{x}) \leq (a'h)^2.
\]

The inequality holds because \( c_{ik} \neq \hat{c}_{ik} \) means that the last term is negative.
Thus, using (14), w.p. at least 1

Thus, $Q_{ik} \leq \mathbb{E}(Q_{ik}) \geq m\delta_i \frac{\|X^\star\|^2_F}{\sigma_i^2}$.

Observe that

$$\frac{t^2}{\max_k \|h_k\|^2} \leq \frac{m^2 \delta_i^2 \|X^\star\|^2_F}{\sigma_i^2} \geq \frac{m \delta_i \|X^\star\|^4_F}{\sigma_i^2} \geq \frac{m \delta_i \|X^\star\|^4_F}{\sigma_i^2} \geq \frac{m \delta_i \|X^\star\|^4_F}{\sigma_i^2},$$

and

$$\frac{t \max_k \|h_k\|^2}{\max_k \|x_k\|^2} \geq \frac{m \delta_i \|x_k\|^2_F}{\sigma_i^2} \geq \frac{m \delta_i \|x_k\|^2_F}{\sigma_i^2}.$$

Here we have used the fact that right incoherence implies $\|x_k\|^2 F \leq \mu^2 \|x_k\|^2_F / q$. Thus, we have

$$\Pr \left\{ \sum_{ik} Q_{ik} - \mathbb{E}(Q_{ik}) \geq m \delta_i \frac{\|X^\star\|^2_F}{\sigma_i^2} \right\} \leq 2 \exp \left( -c \delta_i^2 mq / \mu^2 \right).$$

Thus, using (14), w.p. at least 1

$$\sum_{ik} Q_{ik} \leq (C + 1) m \delta_i \frac{\|X^\star\|^2_F}{\sigma_i^2}.$$

(15)

Finally, combining (13), (15), and Lemma 3.10, w.p. at least 1

$$\max_{W \in \mathcal{S}_W} \text{Term}2(W) \leq C m \sqrt{1 + \delta_i \sqrt{\delta_i} \|X^\star\|^2_F}. \tag*{\square}$$

Proof of Lemma 3.13. Using facts from Sec. IV-A, and using $\sigma_{ij}(U^\star) = 1$,

$$\sigma_{\min}(U^\star \Sigma^\star B^\star B^\star) \geq \sigma_{\min}(1 - SE^2(B^\star, B^\star)) \geq \sigma_{\min}(1 - SE^2(B^\star, B^\star)).$$

To upper bound $SE(B^\star, B^\star)$, first notice that $B^\star$ and $B^\star$ are basis matrices. Thus, $SE(B^\star, B^\star) = \|B^\star (I - B^\star B)\|_F$. We have upper bounded $\|B^\star (I - B^\star B)\|_F$ in Lemma 3.11. Also recall that $B^\star = \Sigma^\star B^\star$ and $\|B^\star\|^2_F = \|X^\star\|^2_F \leq \sqrt{\sigma_{\max}^\star}$. Thus,

$$\|B^\star (I - B^\star B)\|_F \leq \frac{C \delta_i \|X^\star\|^2_F}{\sigma_{\min}} \leq C \sqrt{r} \delta_i.$$

Thus, using $\delta_i \leq 0.7 / C \sqrt{r} \kappa$, $\sigma_{\min}(U^\star \Sigma^\star B^\star B^\star) \geq 0.95 \sigma_{\min}^\star$. \tag*{\square}

V. PHASELESS SUBSPACE TRACKING

Problem setting. The low-rank assumption is equivalent to assuming that $x_k^\star = U^\star b_k^\star$, where $U^\star$ specifies a fixed $r$-dimensional subspace. For long signal/image sequences, a better model (one that allows the required subspace dimension $r$ to be smaller) is to let the subspace change with time. As is common in time-series analysis, the simplest model for time-varying quantities is to assume that they are piecewise constant with time. We adopt this approach here. Moreover, in order to easily borrow ideas from the static setting, we will assume that we now have a total of $q_{\text{full}}$ signals (matrix columns) and we will denote the $n \times q_{\text{full}}$ matrix formed by all these columns by $X^\star_{\text{full}}$. Our algorithm will operate on measurements of $q$-consecutive-column sub-matrices of $X^\star_{\text{full}}$.

Let $k_0 = 1$, and let $k_j$ denote the $j$-th subspace change time, for $j = 1, 2, \ldots, J$ and let $k_{J+1} = q_{\text{full}}$. Assume that

$$x_k^\star = U_{\text{sub},k}^\star d_k^\star, \quad k = 1, 2, \ldots, q_{\text{full}}$$

(16)

with

$$U_{\text{sub},k}^\star = U_{\text{sub},k}^\star := U_{\text{sub},(j)}^\star, \quad \text{for all } k_j \leq k \leq k_{j+1}$$

(17)

where $U_{\text{sub},k}^\star$ is an $n \times r$ “basis matrix” for the subspace at time $k$, and $d_k^\star$ is the corresponding coefficients’ vector. The assumption (17) implies that the subspace is constant for a period of time $k \in [k_j, k_{j+1})$ and we use $U_{\text{sub},(j)}$ to denote the basis for this subspace.

The reason we use a different notation here (the subscript sub and use of $d_k^\star$ instead of $b_k^\star$) is as follows. Consider an $\alpha$-frame sub-matrix formed by $\alpha$ consecutive signals. Let us call it $X^\star$ and let $X^\star \in \mathcal{U}^\star \Sigma^\star B^\star$. If all the $x_k^\star$’s forming this matrix are generated from the same subspace, say $U_{\text{sub},(j)}^\star$, then $\text{Span}(U^\star) = \text{Span}(U_{\text{sub},(j)}^\star)$ and there is no need for a different notation. However, if a subspace change occurred inside this interval, then we cannot say anything simple like this. All we can say is that $X^\star = \bigcup_{\text{sub},(j+1)} U_{\text{sub},(j+1)}^\star D_{(j+1)}$ and so $\text{Span}(U^\star) \subseteq \bigcup_{\text{sub},(j+1)} \text{Span}(U_{\text{sub},(j+1)}^\star)$.

The goal is to track the subspaces $\text{Span}(U_{\text{sub},(j)}^\star)$ on-the-fly; of course, “on-the-fly” for subspace tracking means with a delay of at least $r$. Once this can be done accurately enough, it is easy to also recover the matrix columns $x_k^\star$ (by solving a simple r-dimensional PR problem to recover the $d_k^\star$’s).

The PhaST algorithm: detect and track large enough subspace changes. The measurements of the columns are obtained sequentially and hence there is benefit in developing an online (really a mini-batch) algorithm that works with measurements of short batches of $\alpha$ consecutive columns. Interestingly the algorithm that works for this purpose is a simple modification of the static case idea along with a carefully designed subspace change detection step. In the static case, in each iteration, we used a set of $mq$ measurements of the same data matrix $X^\star$. For obtaining the guarantees, we assumed a new (independent) set of $2mq$ measurements of the same matrix $X^\star$ were used in each iteration ($mq$ for updating the estimate of $B^\star$ and another $mq$ for $U^\star$). For the tracking setting, we assume that, each iteration uses $2mq$ measurements of a new $q$-consecutive-column sub-matrix of $X^\star_{\text{full}}$. However, it is getting the subspace estimate (estimate of $U^\star$) from the
previous iteration. Under the assumption that the subspace does not change for long enough, this does not create any problems.

We summarize the algorithm in Algorithm 2. This toggles between a “detect” and an “update” mode. It starts in the “update” mode (described above) and remains in it for the first $Tq$ frames. At this time it enters the “detect” mode. We are able to guarantee that, whp, in this mode, the previous subspace has been estimated to $\epsilon$ error. In the detect mode, the algorithm does not perform any subspace updates. This is the key to ensuring that, in the interval during which the subspace change occurs, the subspace is not updated. This is what allows us to use our previous two main claims (Claims 3.1 and 3.5) to analyze the update mode.

To understand the change detection strategy, let $k_j$ denote the estimated change times. Also, let $U_{\text{sub},(j-1)}$ denote the $\epsilon$-accurate estimate of $U_{\text{sub},(j-1)}^*$. Consider an $\alpha$ frame interval, $J_\alpha$, contained in $[k_j, k_{j+1})$. Assume that an $\epsilon$-accurate estimate of the previous subspace $U_{\text{sub},(j-1)}^*$ has been obtained by $k_{j-1} + T\alpha$ and that this time is before $k_j$.

Define the matrix

$$Y_{U, det, big} := (I - U_{\text{sub},(j-1)}U_{\text{sub},(j-1)\dagger})Y_U(I - U_{\text{sub},(j-1)}U_{\text{sub},(j-1)\dagger})$$

with $Y_U = Y_U(J_\alpha)$. This means that $Y_U$ is as defined earlier in (4) with the $k$ summation being over all $k \in J_\alpha$ (it is using measurements for signals (columns) within this $\alpha$-frame interval). With a little bit of work (see proof of Lemma 5.2), one can show that, in this interval, the matrix $Y_{U, det} := U_{\text{sub},(j-1)\dagger}Y_{U, det, big}U_{\text{sub},(j-1)\dagger}$ is close to a matrix $E_{det}$ whose eigenvalues satisfy

$$\lambda_{\text{max}}(E_{det}) - \lambda_{\text{min}}(E_{det}) \geq 0.5SE(U_{\text{sub},(j-1)}^*, U_{\text{sub},(j)}^*) - 2\sigma_{\text{min}}^2 / \alpha.$$

On the other hand, in an $\alpha$ frame interval contained in $[k_{j-1} + T\alpha, k_{j+1})$,

$$\lambda_{\text{max}}(E_{det}) - \lambda_{\text{min}}(E_{det}) \leq SE(U_{\text{sub},(j-1)}^*, U_{\text{sub},(j-1)})^2\sigma_{\text{max}}^2 / \alpha \leq c^2\sigma_{\text{max}}^2 / \alpha.$$

Thus, this quantity is small when the change has not occurred, and is large when the subspace has changed sufficiently. By using a large enough lower bound on the product $m\alpha$, and concentration bounds, the same can be shown for the difference between the max and min eigenvalues of $Y_{U, det}$. These are equal to the max and $(n - r)$-th eigenvalues of $Y_{U, det, big}$.

Once we have an $\epsilon$-accurate estimate of the current subspace, it is straightforward to also recover the corresponding subsignals $x_k$. This can simply be done by solving a standard PR problem to recover the coefficients vector. See last line of Algorithm 2. This borrows a similar idea from [28].

We can prove the following about Algorithm 2 (PhaST).

**Corollary 5.1** (PhaST algorithm). Consider Algorithm 2. Pick any value of $m \geq C \max(r, \log n, \log q_{\text{null}})$. For this $m$, set $\alpha = \frac{C\omega_{det}}{\sqrt{m}r^2}$. Set $T := C\log(1/\epsilon)$, and the detection threshold $\omega_{det} = c/(\sqrt{\alpha}r)$. Assume that $k_{j+1} - k_j \geq (T + 3)\alpha$ and that $SE(U_{\text{sub},(j-1)}^*, U_{\text{sub},(j)}^*) > 2c/(\sqrt{\alpha}r)$. Then, w.p. at least $1 - Cn^{-10}$,

1) we can detect the change with a delay of at most $2\alpha$, while ensuring no false detections: $k_j \leq k_j + 2\alpha$;

2) for any $\epsilon > 0$, we can get an $\epsilon$-accurate estimate with a delay of at most $(T + 3)\alpha$; and we have the following subspace error bounds:

$$SE(U_{\text{sub},k}, U_{\text{sub},(j)}) \leq \begin{cases} SE(U_{\text{sub},(j-1)^*}, U_{\text{sub},(j)}) + \epsilon & \text{if } k \in J_{-1} \\
(0.7)^{\ell}c/\kappa\sqrt{r} & \text{if } k \in J_{\ell} \\
\epsilon & \text{if } k \in J_{T+1} \end{cases}$$

Here, $J_{-1} := [k_j, k_j + 3\alpha], J_{\ell} := [k_j + (3 + \ell)\alpha, k_j + (3 + \ell + 1)\alpha]$ for $\ell = 0, 1, 2, \ldots, T$ and $J_{T+1} := [k_j + (T + 4)\alpha, k_{j+1}]$.

**Offline PST-large returns $\hat{X}$ that satisfies**

$$\text{sat-dist}(\hat{X}, X^*) \leq \epsilon.$$

The above result shows that, if the subspace remains constant for at least $\alpha \log(1/\epsilon)$ frames, and if the amount of subspace change (largest principal angle of subspace change) is of order $1/\sqrt{T}$ or larger, then we can both detect the change and track the changed subspace to $\epsilon$ error within a delay of order $\alpha \log 1/\epsilon$. Moreover, for only at most $3\alpha$ frames after a change, the subspace error does not reduce and is essentially bounded by the amount of change. After this, it decays exponentially every $\alpha$ frames.

Notice from the expression for $\alpha$ that, if we pick the smallest allowed value of $m$, then the required $\alpha$ (and hence the required delays) will be large. However, we are allowed to tradeoff $m$ and $\alpha$. If we let $m$ grow linearly with $n$, then we will only need $\alpha \approx r^4$, which is, in fact, close to the minimum required delay of $r$. This also matches what is seen in existing works on provable subspace tracking (ST) in other settings (e.g., robust ST, ST with missing data, or streaming PCA with missing data) [28], [29]. These are able to allow close to optimal detection and tracking delays but all these assume that $m$ increases linearly with $n$. Also, the only other works besides ours that can also provably handle time-varying (piecewise constant) subspaces are [28] and its precursors.

**Improved Phaseless ST: PST-all.** Notice from Theorem 5.1 that Algorithm 2 can only provably detect and track subspace changes that are larger than a small threshold. While this makes sense for detection, it should be possible to track all types of changes. By including a simple modification in Algorithm 2 (include the “update” step during the detection mode as well), we can empirically demonstrate that this is indeed true. See Fig 3. The proof that this is the case should also be possible to obtain, but needs a few careful changes.

**Numerical experiments.** Here we validate PST and PST-all algorithms. For this experiment we generate the data similarly as in the static case experiment. We use $n = 300$, $r = 2$, $k_1 = 2992$, $q_{\text{null}} = 6000$, $m = 100$, $\alpha = 250$ and consider two values of subspace change: $SE(U_0^*, U_1^*) = 0.01, 0.8$. 


Algorithm 2 PhaST: detect and track large subspace changes

1: Set $r$ equal to the largest index $j$ for which $\lambda_j(Y_U) - \lambda_n(Y_U) \geq \omega$.
2: $k_0 \leftarrow 0, j \leftarrow 0, \ell \leftarrow 0$
3: Mode $\leftarrow$ update
4: for $k \geq 0$ do
5: if Mode $= \text{update}$ then
6: if $k = k_j + (\ell + 1)\alpha$ then
7: if $\ell = 0$ then
8: $U_{\text{sub}, (j)} \leftarrow \text{top } r \text{ singular vectors of } Y_U$.
9: end if
10: $\tilde{b}_\tau \leftarrow RWF((y_\tau, U_{\text{sub}, (j)}^\ell A_{\tau}), T_{RWF})$, for $\tau \in [k-\alpha+1, k]$.
11: QR decomposition $\tilde{B} = QR R B$
12: $\tilde{C}_\tau \leftarrow \text{Phase } (A^\tau_{\text{sub}, (j)} \tilde{b}_\tau)$, for $\tau \in [k-\alpha+1, k]$.
13: $U_{\text{sub}, (j)}^{\ell+1} \leftarrow \text{arg min}_{U} \sum_{\tau \in [k-\alpha+1, k]} ||C_{\tau} y_{\tau} - A_{\tau} U_{\text{sub}, (j)}||^2$
14: QR decomposition $U_{\text{sub}, (j)}^{\ell+1} = QR R U_{\text{sub}, (j)}$
15: $\ell \leftarrow \ell + 1$
6: end if
7: if $\ell = T$ then
8: $U_{\text{sub}, (j)} \leftarrow U_{\text{sub}, (j)}^T$, Mode $\leftarrow$ detect
9: end if
20: end if
21: if Mode $= \text{detect}$ then
22: if $\lambda_{\text{max}}(Y_{\text{sub}, (j)}^{\ell} U_{\text{sub}, (j)}^{\ell} Y_{\text{sub}, (j)}^{\ell}) - \lambda_{n-r}(Y_{\text{sub}, (j)}^{\ell} Y_{\text{sub}, (j)}^{\ell}) \geq \omega_{\text{det}}$ then
23: $j \leftarrow j + 1, k_j \leftarrow k, \ell \leftarrow 0$, Mode $\leftarrow$ update
24: end if
25: end if
26: Output $U_{\text{sub}, k} \leftarrow U_{\text{sub}, (j)}^\ell$
27: end for

Offline PST: For each $k \in [k_j, k_{j+1})$, output $\hat{x}_k = U^\ell d_k^{\ast}$ where $d_k^{\ast}$ is an (at most) $2r$-length vector obtained by RWF applied on $\{y_{ik}, (U^i a_k), i = 1, 2, \ldots, m\}$ with $U = \text{basis}(U_{\text{sub}, (j)}, U_{\text{sub}, (j+1)})$. Here basis$(U_1, U_2)$ means a matrix with orthonormal columns that span the subspace spanned by the columns of $U_1$ and $U_2$. We need to use the union of both subspace estimates because the actual subspace change time, $k_{j+1}$, is not known. Corollary 5.1 implies that, whp, it is contained in $[k_j, k_{j+1})$.

The results are shown in Fig. 3. Notice that in the first case PST does not improve the estimation error while PST-all does and in the second case both algorithms succeed.

**Related Work.** Other problems subspace tracking (ST) problems that have been extensively studied include dynamic compressive sensing [30] (a special case of ST where the subspace is defined by the span of a subset of $k$ vectors from a known dictionary matrix), dynamic robust PCA (or robust ST), see [28] and references therein, streaming PCA with missing data [29], [31], and ST with missing data [32]. In terms of works with complete provable guarantees, there is the nearly optimal robust ST via recursive projected compressive sensing (ReProCS-NORST) approach [28] and its precursors, and recent papers on streaming PCA with missing data [29], [31]. For robust ST, the problem setting itself implies $m = n/2$, while in the streaming PCA case, the availability of $m = nm$ measurements, with $\rho < 1$, is assumed. This is why both achieve close to optimal tracking delays (at least when the added unstructured noise is nearly zero). As noted earlier, our method can also achieve a delay of order $r^\phi$ if we let $m$ grow with $n$. Unlike our work, most of these approaches (except ReProCS-NORST) cannot work with time-varying subspaces though.

A. Proof Sketch of Corollary 5.1

Suppose first that the subspace change times $k_j$ were known. By our assumption, $k_{j+1} - k_j > T\alpha$. Then the proof is almost exactly the same as that for the static case. The only difference is that, in the current case, every $\alpha$ frames, we are using measurements corresponding to a new set of $\alpha$ signals (columns of $X^{\text{full}}$) but we use the estimate of the subspace obtained from the measurements for the previous $\alpha$ frames. As long as the subspace has not changed between the two intervals, Claims 3.1 and 3.5 apply without change. Combining them, we can again conclude that $SE(U^0_{\text{sub}, (j)}, U^*_{\text{sub}, (j)}) \leq E_{\text{min}}$ at $k = k_j + \alpha$, and that the bound decreases $0.7$ times after each $\alpha$-frame epoch so that $SE(U^{T}_{\text{sub}, (j)}, U^*_{\text{sub}, (j)}) \leq \epsilon$ at $k = k_j + \alpha T$. By our assumption, $k_{j+1} > k_j + \alpha T$ so this happens before the next change.

The proof in the unknown $k_j$ case follows if we can show that, whp, $k_{j+1} \leq k_j + 2\alpha$. This can be done using Lemma 5.2 given below along with the following argument borrowed from [28], [33]. Consider the $\alpha$-frame interval in which $k_j$ lies. Assume that, before this interval, we have an $\epsilon$-accurate estimate of the previous subspace. In this interval, the first some data vectors satisfy $x^c_k = U_{\text{sub}, (j-1)} d^c_k$, while the rest satisfy $x^c_k = U_{\text{sub}, (j)} d^c_k$. By our assumption, this interval lies in the “detect phase”. We cannot guarantee whether the change will get detected in this interval. But it may. However, in the interval after this interval, all frames satisfy $x^c_k = U^*_{\text{sub}, (j)} d^c_k$. In this interval, Lemma 5.2 given below can be used to show that the change gets detected whp. Thus, either the change is detected in the first interval itself (the one that contains $k_j$), or it is not. If it is not, then, by Lemma 5.2, whp, it will get detected in the second interval (in which all signals are generated from the $j$-th subspace). Thus, $k_j \leq k_j + 2\alpha$. See Appendix A of [28] for a precise proof of this idea. The key point to note here is that we are never updating the subspace in the interval that contains $k_j$ and hence we do not have to prove a new descent lemma that deals with the interval in which the subspace changes.

We will replace $\alpha$ by $q$ in the following lemma and its proof, in order to able to use bounds from earlier proofs. Thus in this lemma, we are considering a $q$-frame epoch.
Assume that

\[ SE(U^*_k, U^*_t) = 0.01 \]

Also we have

\[ \lambda_{\text{max}}(Y_{U,det}) - \lambda_{\min}(Y_{U,det}) \]

Assume that \( \|Y_U - \mathbb{E}[Y_{-}]\| \leq \frac{\delta_{\text{init}} \sigma_{\min}^2}{q} \). This is true by (7).
Assume that \( SE(U_{\text{sub},(j-1)}, U^*_{\text{sub},(j-1)}) \leq \epsilon \). Then,

1) If \( J_q \subseteq [k_j, k_{j+1}) \) (change has occurred), then

\[
\begin{align*}
\lambda_{\max}(Y_{U,det}) - \lambda_{\min}(Y_{U,det}) & \geq \frac{\sigma_{\min}^2}{q} \left( 0.5 \text{SE} \left( U^*_{\text{sub},(j)}, U_{\text{sub},(j-1)} \right) - 2 \delta_{\text{init}} \right) \\
& \geq \frac{\sigma_{\min}^2}{q} \left( 0.5 \text{SE} \left( U^*_{\text{sub},(j)}, U^*_{\text{sub},(j-1)} \right) - 2 \epsilon \right)^2 - 2 \delta_{\text{init}} \end{align*}
\]

Thus using the facts from Sec. IV-A and \( \min_k \beta_{1,k} \geq 0.5 \) (proved while proving Claim 3.1 for initializing \( U^* \)),

\[
\begin{align*}
\lambda_{\max}(Y_{U,det}) - \lambda_{\min}(Y_{U,det}) & \geq \lambda_{\max} \left( \frac{1}{q} U_{\text{sub},(j-1)}, U^*_{\text{sub},(j)} \left( \sum_k \beta_{1,k} \hat{b}_k \hat{b}^*_k \right) U^*_{\text{sub},(j)}, U_{\text{sub},(j-1)} \right) \\
& - 2 \delta_{\text{init}} \sigma_{\min}^2 q, \\
& \geq \left( \frac{\sigma_{\min}^2}{q} \right) \left( 0.5 \text{SE} \left( U^*_{\text{sub},(j)}, U_{\text{sub},(j-1)} \right)^2 - 2 \delta_{\text{init}} \right).
\end{align*}
\]

Proof of item 2

2) If \( J_q \subseteq [k_j+1, T_q, k_{j+1}) \) (change has not occurred), then

\[
\begin{align*}
\lambda_{\max}(Y_{U,det}) - \lambda_{\min}(Y_{U,det}) & \leq \frac{1}{q} \sigma_{\max}^2 \text{SE} \left( U^*_{\text{sub},(j)}, U_{\text{sub},(j-1)} \right) + \frac{2 \delta_{\text{init}} \sigma_{\min}^2}{q} \\
& \leq \frac{1}{q} \sigma_{\max}^2 \epsilon^2 + 2 \delta_{\text{init}} \sigma_{\min}^2 q.
\end{align*}
\]

Proof. This proof uses the following fact: For basis matrices \( P_1, P_2, P_3 \) of the same size, \( \text{SE} (P_1, P_2) - 2 \text{SE} (P_2, P_3) \leq \text{SE} (P_1, P_3) \leq \text{SE} (P_1, P_2) + \text{SE} (P_2, P_3) \).

Define the \( (n-r) \times (n-r) \) matrix

\[ E_{\text{det}} = U_{\text{sub},(j-1)}, U^*_{\text{sub},(j-1), \perp}. \]

Proof of item 1

\[
\begin{align*}
\lambda_{\max}(Y_{U,det}) & \geq \lambda_{\max}(E_{\text{det}}) - \|Y_{U,det} - E_{\text{det}}\| \\
\|Y_{U,det} - E_{\text{det}}\| & \leq \|Y_{U,det} - E_{\text{det}}\|.
\end{align*}
\]

Also we have

\[
\begin{align*}
\lambda_{\min}(Y_{U,det}) & \leq \lambda_{\min}(E_{\text{det}}) + \|Y_{U,det} - E_{\text{det}}\|
\end{align*}
\]

VI. CONCLUSIONS AND FUTURE WORK

This work introduced the first set of simple, practically useful, and provable algorithms for (i) low-rank matrix recovery from column-wise phaseless linear projections, and (ii) for its dynamic extension, phaseless subspace tracking (PST). We showed that the required sample complexity is \( r^3 \) times its order-optimal value of \( nr \) and depends on the condition number \( \kappa \). We believe that both these requirements can be relaxed if we borrow ideas from the best results for LRMC such as [20]. Moreover, as discussed in Sec. V, the PST algorithm and guarantee can be significantly improved to also show that it can track slow subspace changes (ones that cannot be detected). Finally, the ideas developed here can potentially help obtain a phaseless robust subspace tracking (dynamic robust PCA) solution under a few extra assumptions similar to those needed for regular robust ST [28].

Fig. 3: Plot of subspace error versus time at each \( \alpha \) frames. Notice that for the cases where \( SE(U^*_k, U^*_t) = 0.01 \) both algorithms are able to detect and track changes whereas when \( SE(U^*_k, U^*_t) = 0.8 \) only the PST-all algorithm works.

Lemma 5.2. Consider the \( (n-r) \times (n-r) \) matrix

\[ Y_{U,det} := U_{\text{sub},(j-1), \perp} U(j_q) U_{\text{sub},(j-1), \perp}. \]
APPENDIX A
PROOF OF LEMMA 3.9

To prove the lemma we use the following definitions.

1) We use the subscript $M_{vec}$ to refer to the vectorized version of matrix $M$.

2) Scalars such as $b(q)$ refer to the $q$-th entry of vector $b$.

3) Define the diagonal matrix $C_k := \text{diag}(\text{sign}(a_{ik}', x_{ik}'))$, and let $\tilde{C}_k := \text{diag}(\text{sign}(a_{ik}, \tilde{x}_{ik}))$. Denote their $i$-th diagonal entry by $c_{ik}$ and $\tilde{c}_{ik}$ respectively.

4) For $p = 1, 2, \ldots, r$ and $q = 1, 2, \ldots, r$, define
   
   $D_{pq}^{(1)} = \sum_k b_k(p)b_k^*(q)A_kA_k'$,
   
   $D_{pq}^{(2)} = \sum_k b_k(p)b_k^*(q)\tilde{C}_kC_k - I)A_k'$,
   
   $S_{pq} = \left( \sum_k b_k(p)b_k^*(q) \right) I_r$, and
   
   $M_{pq} = \sum_k b_k(p)b_k^*(q)A_kA_k'$.

5) Observe that all the matrices above are of size $n \times n$. We now define a big matrix $D^{(1)}$ of size $nr \times nr$ with the matrices $D_{pq}^{(1)}$ as its $r^2$ blocks. Here $p = 1, 2, \ldots, r$ and $q = 1, 2, \ldots, r$. Thus, $D^{(1)} = [D_{pq}^{(1)}]_{pq} \in \mathbb{R}^{nr \times nr}$.

   Similarly define $S = [S_{pq}]_{p,q}, M = [M_{pq}]_{p,q}$ and $D^{(2)}$.

6) Define the “expanded” singular value matrix which is of size $nr \times nr$, $\Sigma_{big}^* := \text{diag}(\sigma_1^* I_n, \ldots, \sigma_n^* I_n)$ where $\sigma_1^*$ are the singular values of $\Sigma^*$.

7) Define an $nr$ length vector $F_{vec}$ as follows

   $F_{vec} = \left( M_{vec}^{-1}(MS - D^{(1)}) - M_{vec}^{-1}D^{(2)} \right) \Sigma_{big}^* U_{vec}^*$,

   and let $F \in \mathbb{R}^{n \times r}$ be the reshaped matrix formed from $F_{vec}$ (its first column is the first $n$ entries of $F_{vec}$, its second column is the second set of $n$ entries, and so on). Notice that $\|F_{vec}\| := \text{MainTerm}$.

**Lemma A.1.** Let $F$ be the matrix version of $M_{vec}^{-1}(MS - D^{(1)})\Sigma_{big}^* U_{vec}^* - M_{vec}^{-1}D^{(2)}\Sigma_{big}^* U_{vec}^*$. We have

$$\hat{U}_{t+1} = U^* \Sigma^* B^* B^* - F,$$

We prove this lemma after the current proof is completed.

Recall that $\hat{U}_{t+1}^\text{QR} = \hat{U}_{t+1}R_{U_t}$. Thus, $\hat{U}_{t+1} = \hat{U}_{t+1}^\text{QR}(R_{U_t})^{-1} = (U^* \Sigma^* B^* B^* - F)(R_{U_t})^{-1}$ and so

$$\text{SE}(U^*, \hat{U}_{t+1}^*) = \|U^* F(R_{U_t})^{-1}\| \leq \|F\| \|R_{U_t}^{-1}\| \leq \frac{\|F_{vec}\|}{\sigma_{\min}(R_{U_t})}$$

(18)

Since, $\sigma_{\min}(R_{U_t}) = \sigma_{\min}(U_{t+1}^*) = \sigma_{\min}(U^* \Sigma^* B^* B^* - F) \geq \sigma_{\min}(U^* \Sigma^* B^* B^* - \|F\|) \geq \sigma_{\min}(U^* \Sigma^* B^* B^*) - \|F_{vec}\|$. Thus,

$$\text{SE}(U^*, \hat{U}_{t+1}^*) \leq \frac{\|F_{vec}\|}{\sigma_{\min}(U^* \Sigma^* B^* B^*) - \|F_{vec}\|}$$

(19)

In the rest of this proof, we show that $\|F_{vec}\|$ is upper bounded by $\text{MainTerm}$. We have

$$\|F_{vec}\| = \|M_{vec}^{-1}||\|((MS - D^{(1)}) - D^{(2)}\Sigma_{big}^* U_{vec}^*)\| \leq \|M_{vec}^{-1}\| \left(\|MS - D^{(1)}\|\|\Sigma_{big}^* U_{vec}^*\| \right.$$  \( + \|D^{(2)}\|\Sigma_{big}^* U_{vec}^*\|)$$

(20)

Consider the first term, $M_{vec}^{-1}$. Since $M$ is a symmetric matrix

$$\sigma_{\max}(M) = \max_{w \in \mathbb{R}^{nr \times 1}, \|w\| = 1} w^* M w$$

$$\sigma_{\min}(M) = \min_{w \in \mathbb{R}^{nr \times 1}, \|w\| = 1} w^* M w$$

For all $w \in \mathbb{R}^{nr \times 1}, \|w\|^2 = 1$, we can write

$w^* M w = \sum_{pq} w_{pq}' M_{pq} w_q$

$$= \sum_{pq} w_{pq}' \left( \sum_k b_k(p)b_k^*(q)A_kA_k' \right) w_q$$

$$= \sum_{ik} \left( a_{ik}' W_b \right)^2 = \text{Term}_3(W)$$

where $W \in \mathbb{R}^{n \times r}$ is the matrix version of $w$. Finally,

$$\|M_{vec}^{-1}\| = \frac{1}{\sigma_{\min}(M)} = \frac{1}{\min_{w \in \mathbb{R}^{nr \times 1}, \|w\| = 1} \|w^* M w\|}$$

(22)

Now consider the second term inside the parenthesis,

$$\|MS - D^{(1)}\|\Sigma_{big}^* U_{vec}^* \|.$$ Using the variational definition,

$$\|MS - D^{(1)}\|\Sigma_{big}^* U_{vec}^* = \max_{w, \|w\| = 1} w^*(MS - D^{(1)})\Sigma_{big}^* U_{vec}^*$$

and from definitions we know that

$$\sum_{p,q} w_{pq}' (MS - D^{(1)})_{pq} \sigma_q^* U_q^*$$

$$= \sum_{p,q} w_{pq}' (MS)_{pq} \sigma_q^* U_q^* - \sum_{p,q} w_{pq}' (D^{(1)})_{pq} \sigma_q^* U_q^*$$

Consider the first term. It follows from definitions that

$$\sum_{p,q} w_{pq}' (MS)_{pq} \sigma_q^* U_q^*$$

$$= \sum_{p,q} \sum_{k,k'} w_{pq}' b_{k'}(p)b_{k'}(q) b_{k'} b_{k'} A_k A_k' \sigma_q^* U_q^*$$

$$= \sum_{k,k'} (b_{k'} b_{k'}) (\sum_{p} w_{pq}' b_{k'}(p)) A_k A_k' \sum_{q} (b_{k'}(q) \sigma_q^* U_q^*)$$

$$= \sum_{k,k'} (b_{k'} b_{k'}) b_{k'} W'A_k A_k' U^* \Sigma^* b_{k'}$$

$$= \sum_{k,k'} b_{k'} W'A_k A_k' U^* \Sigma^* b_{k'} b_{k'}$$

$$= \sum_{k} b_{k'} W'A_k A_k' U^* \Sigma^* b_{k'} b_{k'}$$

$$= \sum_{k} b_{k'} W'A_k A_k' U^* \Sigma^* b_{k'} b_{k'}$$

Therefore, we have

$$\|MS - D^{(1)}\|\Sigma_{big}^* U_{vec}^* \|$$

and

$$\|F_{vec}\| = \frac{1}{\sigma_{\min}(M)} \left( \|MS - D^{(1)}\|\Sigma_{big}^* U_{vec}^* \| + \|D^{(2)}\|\Sigma_{big}^* U_{vec}^*\| \right)$$

(21)
and similarly
\[
\sum_{p,q} w_p' (D^{(1)})_{pq} \Sigma_{big} (U_{vec}^*)^q \\
= \sum_{p,q} \sum_k w_p b_k(p) b_k(q) A_k A_k' \sigma_q^* (U_{vec}^*)^q \\
= \sum_k (\sum_p b_k(p) w_p') A_k A_k' (\sum_p b_k(q) \sigma_q^* (U_{vec}^*)^q)) \\
= \sum_k b_k' W'A_k A_k' U^* \Sigma^* b_k',
\]
and thus
\[
\|(MS - D^{(1)}) \Sigma_{big} U_{vec}^*\| = \max_{W \in S_W} |\text{Term} 1(W)|
\] (23)

Now consider the final term, \(\|D^{(2)} \Sigma_{big} U_{vec}^*\|.\) Using the variational definition,
\[
\|D^{(2)} \Sigma_{big} U_{vec}^*\| = \max_{W \in R^{n \times 1}} w' D^{(2)} \Sigma_{big} U_{vec}^* \]
and from definitions we know that
\[
\sum_{p,q} w_p' D^{(2)} (\sigma_q^* U_{vec}^*) \\
= \sum_{k,p,q} \sigma_q^* b_k(p) b_k(q) w_p' \left( A_k \left( \hat{C}_k C_k - I \right) A_k' \right) U_{vec}^* \\
= \sum_{i,k,p,q} (\hat{c}_{ik} c_{ik} - 1) \sigma_q^* b_k(p) b_k(q) w_p' a_{ik} a_{ik}' U_{vec}^* \\
= \sum_{i,k} (\hat{c}_{ik} c_{ik} - 1) \left( \sum_p b_k(p) w_p' \right) a_{ik} a_{ik}' \left( \sum_q \sigma_q^* b_k(q) U_{vec}^* \right) \\
= \sum_{i,k} (\hat{c}_{ik} c_{ik} - 1) (a_{ik}' W b_k) (a_{ik}' x_k') = \text{Term} 2(W)
\]
and thus
\[
\|D^{(2)} \Sigma_{big} U_{vec}^*\| = \max_{W \in S_W} |\text{Term} 2(W)|
\] (24)

From (20) - (24) it is easy to see that
\[
\|F_{vec}\| \\
\leq \frac{\max_{W \in S_W} |\text{Term} 1(W)| + \max_{W \in S_W} |\text{Term} 2(W)|}{\min_{W \in S_W} |\text{Term} 3(W)|}
\]

Proof of Lemma A.1. Recall from the algorithm that
\[
\hat{U}^{t+1} = \text{argmin}_U \sum_k \|\hat{C}_k \sqrt{y_k} - A_k' U b_k\|^2.
\]
Define \(j_k = \hat{C}_k \sqrt{y_k}.\) Then we have
\[
\|j_k - A_k' U b_k\|^2 = \|j_k\|^2 + b_k' U A_k A_k' U b_k - 2 b_k' U A_k j_k.
\]
Since \(j_k\) does not depend on \(U\), we have that
\[
\hat{U}^{t+1} = \text{argmin}_U \sum_k (b_k' U A_k A_k' U b_k - 2 b_k' U A_k j_k)
\]
\[
= \text{argmin}_U \sum_k (\sum_k b_k(p) b_k(q) A_k A_k' (\sum_q b_k(q) U_{vec}) - 2 \sum_k b_k(p) U_{vec}) A_k j_k
\]
\[
= \text{argmin}_U \sum_k \sum_k b_k(p) b_k(q) U_{vec} A_k A_k' U_{vec}
\]
\[
- 2 \sum_k b_k(p) U_{vec} A_k j_k.
\]
Here and below the \(\sum_p, \sum_q\) are short for \(\sum_{p=1}, \sum_{q=1}\).

Define
\[
E(U) = \sum_k \sum_k b_k(p) b_k(q) U_{vec} A_k A_k' U_{vec}
\]
Taking gradient of \(E(U)\) with respect to \(U_p\), and making it equal to zero, we will have
\[
\sum_q b_k(p) b_k(q) A_k A_k' U_{vec} = \sum_k b_k(p) A_k j_k.
\]
Since \(M_{pq} = \sum_k b_k(p) b_k(q) A_k A_k'\), we have
\[
\sum q M_{pq} U_{vec} = \sum_k b_k(p) A_k j_k.
\]
But
\[
j_k = \hat{C}_k y_k = \hat{C}_k C_k A_k A_k' U_{vec} \Sigma^* b_k',
\]
Hence
\[
\sum q M_{pq} U_{vec}^{t+1} = \sum_k b_k(p) A_k \hat{C}_k C_k A_k A_k' U_{vec} \Sigma^* b_k'
\]
\[
= \sum_k b_k(p) b_k(q) A_k \hat{C}_k C_k A_k' \sigma_q^* U_{vec}^* \\
= \sum_k b_k(p) b_k(q) A_k A_k' \sigma_q^* U_{vec}^* \\
+ \sum q b_k(p) b_k(q) A_k \left( \hat{C}_k C_k - I \right) A_k' \sigma_q^* U_{vec}^* \\
= \sum q D^{(1)} \Sigma_{vec}^* U_{vec}^* + \sum q D^{(2)} \Sigma_{vec}^* U_{vec}^* 
\]
which holds for all \(q = 1, \ldots, r\), and means that
\[
M \hat{U}_{vec}^{t+1} = D^{(1)} \Sigma_{vec}^* U_{vec}^* + D^{(2)} \Sigma_{vec}^* U_{vec}^* \\
= M S \Sigma_{vec}^* U_{vec}^* \\
+ D^{(1)} \Sigma_{vec}^* U_{vec}^* - M S \Sigma_{vec}^* U_{vec}^* + D^{(2)} \Sigma_{vec}^* U_{vec}^* \\
= M S \Sigma_{vec}^* U_{vec}^* + (D^{(1)} - M S) \Sigma_{vec}^* U_{vec}^* \\
+ D^{(2)} \Sigma_{vec}^* U_{vec}^* \\
\Rightarrow \hat{U}_{vec}^{t+1} \\
= \Sigma_{vec}^* U_{vec}^* - M^{-1} (M S - D^{(1)}) \Sigma_{vec}^* U_{vec}^* \\
+ M^{-1} D^{(2)} \Sigma_{vec}^* U_{vec}^*.
Now we define $F_{vec}^{(1)} = M^{-1} (MS - D^{(1)}) \Sigma_{big}^* U_{vec}^*$ and $F_{vec}^{(2)} = -M^{-1} D^{(2)} \Sigma_{big}^* U_{vec}^*$. Note that

$$[S \Sigma_{big}^* U_{vec}^*]_p = \sum_q \left( \sum_k b_k(p)b_k^*(q) \right) I_n \sigma_q^* U^*_q$$

$$= \sum_q \sigma_q^* U^*_q \sum_k b_k(p)b_k^*(q)$$

$$= \sum_k U^* \Sigma^* b_k^*(p) = U^* \Sigma^* \sum_k b_k^* b_k(p)$$

$$= U^* \Sigma^* B^* \begin{bmatrix} b_1(p) \\ b_2(p) \\ \vdots \\ b_n(p) \end{bmatrix},$$

and hence $\hat{U}^{t+1} = U^* \Sigma^* B^* B^t - F$ which completes the proof.

**APPENDIX B**

**SIMPLE PROOFS ADDED FOR COMPLETION**

**Proof of Lemma 4.1.** When $X_i, Y_i$ are sub-Gaussian random variables, then $X_iY_i$ is a sub-exponential random variable with sub-exponential norm less than $C(K_{X_i}^2 + K_{Y_i}^2)$. To prove this observe that $4XY = (X + Y)^2 - (X - Y)^2$. Now $(X + Y)$ is sub-Gaussian with sub-Gaussian norm $C(K_X + K_Y)$, and thus $(X + Y)^2$ is sub-exponential with sub-exponential norm $C(K_X + K_Y)^2 \leq C(K_X^2 + K_Y^2)$. Similar argument is true for $(X - Y)^2$. Thus $XY$ is sub-exponential with sub-exponential norm $C(K_X^2 + K_Y^2)$. Thus,

$$\|X_i Y_i\|_{\psi_1} \leq C(K_{X_i}^2 + K_{Y_i}^2).$$

Here $\| \cdot \|_{\psi_1}$ means the sub-exponential norm. Now if we multiply $X_i$ by $\lambda$, and $Y_i$ by $\frac{1}{\lambda}$, the left hand side does not change, but the right hand side changes and we have

$$\|X_i Y_i\|_{\psi_1} \leq C(\lambda^2 K_{X_i}^2 + K_{Y_i}^2).$$

Since this holds for any $\lambda$, we can get the best bound by minimizing over $\lambda$. This gives $\lambda = \frac{K_{Y_i}}{K_{X_i}}$ and thus

$$\|X_i Y_i\|_{\psi_1} \leq C K_{X_i} K_{Y_i} := K_{i}. $$

Thus, $X_i Y_i$ is a centered sub-exponential random variable with $\|X_i Y_i\|_{\psi_1} \leq C K_{X_i} K_{Y_i} = K_{i}$. Now applying Lemma 5.16 of [26] to $F_i = \bar{X}_{i} \bar{Y}_{i}$ with $K = 1$ and $a_i = K_{i}$ we get our result.

**Proof of (6).** Recall that

$$Y_{-}(\epsilon_1) = \frac{1}{mq} \sum_{i,k} |a_{ik}^* x_{k}^*|^2 a_{ik}^* a_{ik} \mathbb{I} \{ |a_{ik}^* x_{k}^*|^2 \leq \frac{\epsilon_1}{q} \|x^*\|_2^2 \}. $$

Now since standard Gaussian vectors are invariant due to any arbitrary rotation, without any loss of generality we can replace $a_{ik}$’s by $Ra_{ik} = a_{ik}$, where $R$ is any arbitrary rotation, and choose this rotation so that $Rx_{k}^* = e_1 \|x_{k}^*\|$, where $e_1$ is the first column of identity matrix. Now with these new replacements we compute the expectation of the inner part of the summation as what follows

$$E \left[ |a_{ik}^* e_1|^2 \|x_{k}^*\|^2 a_{ik}^* a_{ik}^* \right] = \begin{cases} \|x_{k}^*\|^2 E \left[ a_{ik}^* (1)^4 \mathbb{I} \{ a_{ik}^* (1)^2 \|x_{k}^*\|^2 \leq \frac{\epsilon_1}{q} \|X\|^2 \} \right], & i = k = 1 \\
\|x_{k}^*\|^2 E \left[ a_{ik}^* (1)^2 \mathbb{I} \{ a_{ik}^* (1)^2 \|x_{k}^*\|^2 \leq \frac{\epsilon_1}{q} \|X\|^2 \} \right], & i \neq k.
\end{cases}$$

So by defining $\xi$ as a standard gaussian variable and defining

$$\begin{align*}
\beta_1^{-1} &:= E[(\xi^4 - \xi^2) \mathbb{I} \{ \xi^2 \leq (1-\epsilon_1) \gamma_k \}] \\
\beta_2^{-1} &:= E[\xi^2 \mathbb{I} \{ \xi^2 \leq (1-\epsilon_1) \gamma_k \}], \\
\gamma_k &= \frac{9 \|X\|^2}{(q \|x\|_2^2)}
\end{align*}$$

we can get

$$E[Y_{-}(\epsilon_1)] = \frac{1}{q} \left\{ \sum_k \beta_{1,k} |x_{k}^*|^2 + \left( \sum_k \beta_{2,k} \|x_{k}^*\|^2 \right) I \right\},$$

which completes the proof.

**APPENDIX C**

**DETAILED NUMERICAL EVALUATION**

In this section we provide detailed description of the numerical evaluation of our algorithms on synthetic and real data. All time comparisons are performed on a Desktop Computer with Intel® Xeon E3-1240 8-core CPU @ 3.50GHz and 32GB RAM.

*Ph-Co-LRMR.* We first show that the rank estimation step works in practice. We generate the true data, $X^* = U^* B^*$ where $U^* \in \mathbb{R}^{n \times r}$ with $n = 200, r = 4$ is generated by orthonormalizing a i.i.d. standard Gaussian matrix. The entries of $B^* \in \mathbb{R}^{r \times q}$ with $q = 400$ are also chosen from i.i.d. standard Gaussian distribution. We generate the measurement matrices $A_i \in \mathbb{R}^{n \times m}$ with $m = 80$. We then set the observations, $y_i = |A_i^* x_i^*|$ for $i = 1, \cdots q$. We implement the practical version of Algorithm 1 in which we do not use new measurements at each time. This is because the theoretical algorithm requires “fresh” measurements in each iteration to update both $U, B$. We now consider the problem of recovering $X^*$ from $Y$. We consider two scenarios: (a) rank is unknown; and (b) rank is known. In the first scenario, notice that one would need a large value of $m, q$ to estimate $r$ provably but we are currently in the regime when $m \ll n$. Thus, our algorithm does not perform very well although it still outperforms existing methods. In the setting when $r$ is known, however, the proposed method significantly outperforms existing techniques.

We implement AltMinLowRaP with 10 outer iterations, and for the inner iterations, we linearly scaled the number of RWF iterations from [5, 30]. This is done because in the initial iterations, a low accuracy suffices for our example. For LRPR2 we used the default parameters mentioned in the
documentation, with the exception of 10 outer loop iterations to match AltMinLowRaP. For RWF we used the default parameters prescribed by the authors with the only exception that we used 300 iterations (instead of the default 85).

**Dynamic Ph-Co-LRMR.** In this experiment we validate PST-large and PST-all algorithms. We generate the true data for the first subspace $X_0* = U_0*B_0*$ where $U* \in \mathbb{R}^{n \times r}$ with $n = 300$, $r = 2$ is generated by orthonormalizing the columns of a $n \times r$ i.i.d. standard normal matrix. The entries of $B_0* \in \mathbb{R}^{r \times t_1}$ with $t_1 = 2992$ are also generated from an i.i.d. standard normal distribution. We generate the true data from the second subspace similarly and set $X_t* = U_t* B_t*$ and we set $q = 6000$. Notice that $\kappa \approx 1$. The subspace $U_t*$ is generated using the idea of [28] as $U_t* = e^{-\gamma M}U_0*$ in order to control the subspace error. Here $M$ is a skew-symmetric matrix and $\gamma$ controls the amount of subspace change. We study two cases in which we set $\gamma = 0.08, 0.001$ which roughly translates to $\text{SE}(U_0*, U_t*) = 0.8, 0.01$. We generate the measurement matrices $A_i(\in \mathbb{R}^{n \times m}) i.i.d. \sim N(0, I)$ with $m = 100$ for $i = 1, \cdots, q$. We then implement two versions of the PST algorithm. The first one is the theoretical version which requires large change in order to ensure good results, and the heuristic extension which works even with small changes. We refer to these algorithms as PST-large and PST-all respectively.

We chose the algorithm parameters as follows. We set $\alpha = 250$ and $L = 8$. For the detection, and initialization steps of both algorithms we set $m_{\text{init}} = n$ to ensure good concentration around expectation. We set the threshold for detection, $\omega = 0.6$ through cross-validation. The results for the two algorithms are shown in Fig. 3. Notice that for the small change case, since PST-large is always in the detect mode, it does not improve the estimation error whereas PST-all does. However, when the change is large enough, both algorithms converge to a small error. The results are averaged over 30 independent trials.

**Video Results.** We now present the results of recovering the low-rank matrix from Coded Diffraction Pattern (CDP) measurements. These measurements can be represented as $Y = |\mathcal{F}(Dx)|$ where $\mathcal{F}$ is the DFT operation and the matrix $D$ represents the mask whose entries are chosen uniformly at random from $\{1, -1, i, 1\}$ to modulate the intensity of the input. We generate CDP measurements of a video using the above setup. We compared our algorithm with LRPR2 and RWF. We present the quantitative results in Table 1. Notice that even with $m = 5n$ measurements, RWF is unable to accurately recover the video and our algorithm has a slightly better performance w.r.t. LRPR2. It is interesting to consider a different practical setup (instead of CDP) which ensures $m \ll n$ in which case we expect AltMinLowRaP to significantly outperform LRPR2 as well. We also provide a detailed visual example in Figs. 4 and 5. The algorithm parameters are set as done in the first experiment with the exception that we increased the number of outer loops from 10 to 30 for AltMinLowRaP and LRPR2 and from 85 to 30 to RWF.

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**TABLE 1: mat-dist($\hat{X}, X^*$) and time comparison for the mouse and plane videos.** We generate the measurements using the CDP model and consider two different number of settings. Notice that AltMinLowRaP is only slightly better than LRPR2 and we expect that considering a different measurement model such as Random Fourier to significantly improve since $m \ll n$.

| Algorithm          | mat-dist($\hat{X}, X^*$) | Time (seconds) |
|--------------------|--------------------------|----------------|
| m = 5n (mouse)     |                          |                |
| RWF                | 0.65                     | 0.35           |
| LRPR2              | 0.48                     | 81.8           |
| AltMinLowRaP       | 0.39                     | 297.6          |
| m = 5n (plane)     |                          |                |
| *RWF               | 0.65                     | 0.35           |
| LRPR2              | 0.10                     | 122.3          |
| AltMinLowRaP       | 0.09                     | 467.6          |
| m = 2n (mouse)     |                          |                |
| RWF                | 1.36                     | 0.25           |
| LRPR2              | 0.61                     | 31.0           |
| AltMinLowRaP       | 0.52                     | 122.1          |
Fig. 4: Comparison of visual performance for the plane video with $m = 2n$. The images are shown at $t = 20, 60, 100$.

Fig. 5: Comparison of visual performance for the plane video for $m = 5n$. The images are shown at $t = 20, 60, 100$. The RWF performance improves as compared to $m = 2n$ but is still bad compared to our algorithm.
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