WITT VECTORS AND TAMBARA FUNCTORS

M. BRUN

Abstract. We give, for every finite group $G$, a combinatorial description of the ring of $G$-Witt vectors on a polynomial algebra over the integers. Using this description we show that the functor, which takes a commutative ring with trivial action of $G$ to its ring of Witt vectors, coincides with the left adjoint of the algebraic functor from the category of $G$-Tambara functors to the category of commutative rings with an action of $G$.

Introduction

In [Wi] Witt constructed an endofunctor on the category of commutative rings, which takes a commutative ring $R$ to the ring $W_p(R)$ of $p$-typical Witt vectors. This construction can be used to construct field extensions of the $p$-adic numbers [SW], and it is essential in the construction of crystalline cohomology [Be]. In [DS] Dress and Siebeneicher constructed for every pro-finite group $G$ an endofunctor $\mathbb{W}_G$ on the category of commutative rings. In the case where $G = \hat{\mathbb{C}}_p$ is the pro-$p$-completion of the infinite cyclic group the functors $W_p$ and $\mathbb{W}_G$ agree. The functor $\mathbb{W}_G$ is constructed in such a way that $\mathbb{W}_G(\mathbb{Z})$ is an appropriately completed Burnside ring for the pro-finite group $G$. For an arbitrary commutative ring $R$, the ring $\mathbb{W}_G(R)$ is somewhat mysterious, even when $G = \hat{\mathbb{C}}_p$. The first aim of the present paper is to give a new description of the ring $\mathbb{W}_G(R)$ when $R$ is a polynomial algebra over the integers. In the special case where $G$ is finite our description is given in terms of a category $U^G$ introduced by Tambara in [Ta]. The category $U^G$ has as objects the class of finite $G$-sets, and the morphism set $U^G(X, Y)$ is constructed as follows: Let $U^G(X, Y)$ denote the set of equivalence classes of chains of $G$-maps of the form $X \leftarrow A \rightarrow B \rightarrow Y$, where two chains $X \leftarrow A \rightarrow B \rightarrow Y$ and $X \leftarrow A' \rightarrow B' \rightarrow Y$ are defined to be equivalent if there exist $G$-bijections $A \rightarrow A'$ and $B \rightarrow B'$ making the following diagram commutative:

\[
\begin{array}{ccc}
X & \leftarrow & A \\
\| & & \downarrow \\
X & \leftarrow & A'
\end{array}
\quad
\begin{array}{ccc}
 & \longrightarrow & B \\
\| & & \downarrow \\
 & \longrightarrow & B'
\end{array}
\quad
\begin{array}{c}
\rightarrow & Y
\end{array}
\]

Date: April 30, 2003.

1991 Mathematics Subject Classification. 13K05; 18C10; 19A22.

Key words and phrases. Witt vectors, Mackey functors, Tambara functors, coloured theories.

The author wishes to thank the University of Osnabrück for hospitality and Rainer Vogt for a number of helpful conversations. This work has been supported by the DFG.
The set $U^G(X,Y)$ has a natural semi-ring-structure described in [2]. By group-completing the underlying additive monoid of $U^G(X,Y)$ we obtain a commutative ring, and we define $U^G(X,Y)$ to be the set obtained by forgetting the ring-structure. For a subgroup $H$ of $G$ we let $G/H$ denote the $G$-set of left cosets of $H$ in $G$, and we let $G/e$ denote $G$ considered as a left $G$-set. The following theorem is a special case of theorem 7.5.

**Theorem A.** Let $G$ be a finite group and let $X$ be a finite set considered as a $G$-set with trivial action. The ring $U^G(X,G/e)$ is the polynomial ring $\mathbb{Z}[X]$ over $\mathbb{Z}$, with one indeterminate for each element in $X$, and the ring $U^G(X,G/G)$ is naturally isomorphic to the ring $\mathbb{W}_G(U^G(X,G/e)) = \mathbb{W}_G(\mathbb{Z}[X])$.

Our second aim is to advertise the category of $G$-Tambara functors, that is, the category $[U^G,\mathcal{Ens}]_0$ of set-valued functors on $U^G$ preserving finite products. Tambara calls such a functor a TNR-functor, an acronym for “functor with trace, norm and restriction”, but we have chosen to call them $G$-Tambara functors. This category is intimately related to the Witt vectors of Dress and Siebeneicher. In order to explain this relation we consider the full subcategory $U^{fG}$ of $U^G$ with free $G$-sets as objects, and we note that the category of $fG$-Tambara functors, that is, the category $[U^{fG},\mathcal{Ens}]_0$ of set-valued functors on $U^{fG}$ preserving finite products is equivalent to the category of commutative rings with an action of $G$ through ring-automorphisms. The categories $U^G$ and $U^{fG}$ are coloured theories in the sense of Boardman and Vogt [BV], and therefore they are complete and cocomplete, and the forgetful functor $[U^{fG},\mathcal{Ens}]_0 \to [U^G,\mathcal{Ens}]_0$ induced by the inclusion $U^{fG} \subseteq U^G$ has a left adjoint functor, which we shall denote by $L_G$.

**Theorem B.** Let $G$ be a finite group. If $R$ is a commutative ring with $G$ acting trivially, then for every subgroup $H$ of $G$, there is a natural isomorphism $\mathbb{W}_H(R) \cong L_GR(G/H)$.

We are also able to describe the ring $L_GR(G/H)$ in the case where $G$ acts nontrivially on $R$. In order to do this we need to introduce an ideal $\mathbb{I}_G(R)$ of $\mathbb{W}_G(R) = R\mathcal{Q}(G)$, where $\mathcal{Q}(G)$ is the set of conjugacy classes of subgroups of $G$. An element $a \in \mathbb{W}_G(R)$ is of the form $a = (a_H)_H \leq G$, where the prime means that $a_H = a_{gg^{-1}} \in R$ for $H \leq G$ and $g \in G$. We let $\mathbb{I}_G(R) \subseteq \mathbb{W}_G(R)$ denote the ideal generated by elements of the form $a - b$, where $a = (a_K)_K \leq G$ and $b = (b_R)_R \leq G$ satisfy that for every $K \leq G$ there exist $n \geq 1$ and $g_1,K,\ldots,g_n,K \in N_K(G)$ and $a_1,K,\ldots,a_n,K \in R$ such that firstly $g_1,K \cdots g_n,K = 1 = g_n,KK$ and secondly $a_K = a_1,K \cdots a_n,K$ and $b_K = (g_1,Ka_1,K) \cdots (g_n,Ka_n,K)$. Our description of $L_GR(G/H)$ is formulated in the following generalization of theorem 8.5.

**Theorem C.** Let $G$ be a finite group and let $R$ be a commutative ring with an action of $G$. There is a natural isomorphism $\mathbb{W}_H(R)/\mathbb{I}_H(R) \cong (L_GR(G/H))$.  

Let us stress that in our setup the above result only makes sense for a finite group. For a pro-finite group $G$ it might be possible to construct appropriate variants $\tilde{U}^G(X,Y)$ of the rings $U^G(X,Y)$ such that the the ring $\tilde{U}^G(X,G/G)$ is naturally isomorphic to $\mathbb{W}_G(\mathbb{Z}[X])$. We shall not go further into this here.
There have been given other combinatorial constructions of the Witt vectors by Metropolis and Rota [MR] and Graham [Gra]. The ring of Witt vectors has been related to the ring of necklaces by Dress and Siebeneicher in [DS2]. Our combinatorial description differs from the previous ones mostly in that it is forced by some additional structure on the Witt vectors.

It is our belief that the category of $G$-Tambara functors has applications yet to be discovered. There is a rich supply of them coming from equivariant stable homotopy theory. In fact, every $E_\infty$ ring $G$-spectrum gives rise to a $G$-Tambara functor by taking the zeroth homotopy group [Br]. In the case where $G = e$ is the trivial group, the category of $G$-Tambara functors is equivalent to the category of commutative rings. It is well known that every commutative ring can be realized as the zeroth homotopy group of an $E_\infty$ ring-spectrum. For an arbitrary group $G$ one may speculate whether $G$-Tambara functors can be realized as the zeroth homotopy group of an $E_\infty$ ring-spectrum with an action of $G$.

Some concepts from commutative algebra can be carried over to the setting of $G$-Tambara functors. Most notably, the notions of an ideals, modules and chain complexes have $G$-Tambara versions. We leave these concepts to future work.

The paper is organized as follows: In section 1 we have collected some of the results from the papers [DS] and [Ta] that we need in the rest of the paper. In section 2 we note that the category of Tambara functors is the category of algebras for a coloured theory. In section 3 we construct a homomorphism relating Witt vectors and Tambara functors, which we have chosen to call the Teichmüller homomorphism. In section 4 we prove the fundamental fact that the Teichmüller homomorphism is a ring-homomorphism. In section 5 we prove that for free Tambara functors, the Teichmüller homomorphism is an isomorphism, and finally in section 6 we prove theorem C.

1. Prerequisites

In this section we fix some notation and recollect results from [DS] and [Ta]. All rings are supposed to be both commutative and unital. Given a group $G$ we only consider left actions of $G$. A $G$-ring is a ring with an action of the group $G$ through ring-automorphisms.

Given a pro-finite group $G$ we let $\mathcal{O}(G)$ denote the $G$-set of open subgroups of $G$ with action given by conjugation and we let $\mathcal{O}(G)$ denote the set of conjugacy classes of open subgroups of $G$. For a $G$-set $X$ and a subgroup $H$ of $G$ we define $|X^H|$ to be the cardinality of the set $X^H$ of $H$-invariant elements of $X$. The following is the main result of [DS].

**Theorem 1.1.** Let $G$ be a pro-finite group. There exists a unique endofunctor $W_G$ on the category of rings such that for a ring $R$ the ring $W_G(R)$ has the set $R^{\mathcal{O}(G)}_2$ of maps from the set $\mathcal{O}(G)$ to $R$ as underlying set, in such a way that for every ring-homomorphism $h : R \to R'$ and every $x \in W_G(R)$ one has $W_G(h)(x) = h \circ x$, while for any subgroup $U$ of $G$ the family of $G$-maps

$$\phi_U^R : W_G(R) \to R$$
There is a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{p} & A \\
\downarrow{f} & & \downarrow{f'} \\
Y & \xleftarrow{\Pi_f p} & \Pi_f A
\end{array}
\]

where \( f' \) is the projection and \( e \) is the evaluation map \((x, s) \mapsto s(x)\). A diagram in \( \mathcal{F}in^G \) which is isomorphic to a diagram of the above form is called an exponential diagram.

We say that two diagrams \( X \xleftarrow{A} A \rightarrow B \rightarrow Y \) and \( X \xleftarrow{A'} A' \rightarrow B' \rightarrow Y \) in \( \mathcal{F}in^G \) are equivalent if there exist \( G \)-isomorphisms \( A \rightarrow A', B \rightarrow B' \) making the diagram

\[
\begin{array}{ccc}
X & \xleftarrow{A} & A' & \rightarrow & B & \rightarrow & Y \\
\downarrow & & & \downarrow & & & \downarrow \\
X & \xleftarrow{A'} & B' & \rightarrow & Y
\end{array}
\]

commutative, and we let \( U_+^G(X, Y) \) be the set of the equivalence classes \([X \xleftarrow{A} A \rightarrow B \rightarrow Y]\) of diagrams \( X \xleftarrow{A} A \rightarrow B \rightarrow Y \).
Tambara defines an operation \( \circ : U^G_+(Y, Z) \times U^G_+(X, Y) \to U^G_+(X, Z) \) by

\[
[Y \leftarrow C \to D \to Z] \circ [X \leftarrow A \to B \to Y] = [X \leftarrow A'' \to \tilde{D} \to Z],
\]

where the maps on the right are composites of the maps in the diagram

\[
\begin{array}{ccc}
X & \leftarrow & A \\
\downarrow & & \downarrow \\
B & \leftarrow & B' \\
\downarrow & & \downarrow \\
Y & \leftarrow & C \\
\downarrow & & \downarrow \\
Z & \leftarrow & D
\end{array}
\]

Here the three squares are pull-back diagrams and the diagram

\[
\begin{array}{ccc}
C & \leftarrow & B' \\
\downarrow & & \downarrow \\
D & \leftarrow & \tilde{D}
\end{array}
\]

is an exponential diagram. He verifies that \( U^G_+ \) is a category with \( \circ \) as composition and given \( f : X \to Y \) in \( \mathcal{F}in^G \) he introduces the notation

\[
R_f = [Y \leftarrow X \xrightarrow{f} X \xrightarrow{j} X],
\]

\[
T_f = [X \xleftarrow{\cong} X \xrightarrow{j} X \xrightarrow{f} Y] \quad \text{and}
\]

\[
N_f = [X \xleftarrow{\cong} X \xrightarrow{j} Y \xrightarrow{\cong} Y].
\]

Every morphism in \( U^G_+ \) is a composition of morphisms on the above form. He also shows that every object \( X \) of \( U^G_+ \) is a semi-ring object in the following natural way:

**Proposition 1.2.** Given objects \( X \) and \( Y \) in \( U^G_+ \), there is semi-ring-structure on \( U^G_+(X, Y) \) given as follows:

\[
0 = [X \leftarrow \emptyset \to \emptyset \to Y],
\]

\[
1 = [X \leftarrow \emptyset \to Y \to Y],
\]

\[
[X \leftarrow A \to B \to Y] + [X \leftarrow A' \to B' \to Y] = [X \leftarrow A \amalg A' \to B \amalg B' \to Y]
\]

and

\[
[X \leftarrow A \to B \to Y] \cdot [X \leftarrow A' \to B' \to Y] = [X \leftarrow B \times_Y A' \amalg A \times_Y B' \to B \times_Y B' \to Y].
\]

It is also shown in [Ta] that there is a unique category \( U^G \) satisfying the following conditions:

(i) \( \text{ob} U^G = \text{ob} U^G_+ \)
(ii) The morphism set \( U^G(X, Y) \) is the group completion of the underlying additive monoid of \( U^G_+(X, Y) \).

(iii) The group completion maps \( k : U^G_+(X, Y) \to U^G(X, Y) \) and the identity on \( \text{ob}(U^G_+) \) form a functor \( k : U^G_+ \to U^G \).

(iv) The functor \( k \) preserves finite products.

**Proposition 1.3.** (i) If \( X_1 \xrightarrow{i_1} X \xleftarrow{i_2} X_2 \) is a sum diagram in \( \text{Fin}^G \), then \( X_1 \xrightarrow{R_{i_1}} X \xrightarrow{R_{i_2}} X_2 \) is a product diagram in \( U^G \) and \( \emptyset \) is final in \( U^G \).

(ii) Let \( X \) be a \( G \)-set and \( \nabla : X \amalg X \to X \) the folding map, \( i : \emptyset \to X \) the unique map. Then \( X \) has a structure of a ring object of \( U^G \) with addition \( T_{\nabla} \), additive unit \( T_i \), multiplication \( N_{\nabla} \) and multiplicative unit \( N_i \).

(iii) If \( f : X \to Y \) is a \( G \)-map, then the morphisms \( R_f, T_f \) and \( N_f \) of \( U^G \) preserve the above structures of ring, additive group and multiplicative monoid on \( X \) and \( Y \), respectively.

Given a category \( C \) with finite products, we shall denote the category of set-valued functors on \( C \) preserving finite products by \( [C, \mathcal{E}ns]_0 \). The morphisms in \( [C, \mathcal{E}ns]_0 \) are given by natural transformations.

**Definition 1.4.** The category of \( G \)-Tambara functors is the category \( [U^G, \mathcal{E}ns]_0 \).

That is, a \( G \)-Tambara functor (called TNR-functor in [Tr]) is a set-valued functor on \( U^G \) preserving finite products, and a morphism of \( G \)-Tambara functors is a natural transformation.

Given a \( G \)-Tambara functor \( S \) and \( [X \leftarrow A \to B \to Y] \in U^G(X, Y) \) we obtain a function \( S[X \leftarrow A \to B \to Y] : S(X) \to S(Y) \). Since \( S \) is product-preserving, it follows from (ii) of proposition [3] that \( S(X) \) is a ring. Given a finite \( G \)-map \( f : X \to Y \) we shall use the notation \( S^*(f) = S(R_f), S_+(f) = S(T_f) \) and \( S_*(f) = S(N_f) \). It follows from (iii) of [3] that \( S^*(f) \) is a ring-homomorphism, that \( S_+(f) \) is an additive homomorphism and that \( S_*(f) \) is multiplicative. A \( G \)-Tambara functor \( S \) is uniquely determined by the functions \( S^*(f), S_+(f) \) and \( S_*(f) \) for all \( f : X \to Y \) in \( \text{Fin}^G \).

Given subgroups \( K \leq H \leq G \) we shall denote by \( \pi^K_H : G/K \to G/H \) the projection induced by the inclusion \( K \leq H \) and given \( g \in G \) we shall let \( c_g : G/H \to G/gHg^{-1} \) denote conjugation by \( g, c_g(\sigma H) = \sigma g^{-1}(gHg^{-1}) \).

2. **Coloured theories**

In this section we shall explain that the category \( U^G \) is an \( O(G) \)-coloured category in the sense of Boardman and Vogt [BV].

**Definition 2.1** ([BV 2.3]).

(i) Let \( O \) be a finite set. An \( O \)-coloured theory is a category \( \Theta \) together with a faithful functor \( \sigma_\Theta : (\text{Fin}/O)^{\text{op}} \to \Theta \) such that firstly \( \sigma_\Theta \) preserves finite products and secondly every object of \( \Theta \) is isomorphic to an object in the image of \( \sigma_\Theta \).

(ii) The category of algebras over a theory \( \Theta \) is the category \( [\Theta, \mathcal{E}ns]_0 \) of product-preserving set-valued functors on \( \Theta \).
(iii) A morphism $\gamma : \Theta \to \Psi$ of $\mathcal{O}$-coloured theories is a functor preserving finite products.

Other authors, e.g. AR and Re, use the name “sorted theory” for a coloured theory.

Given a finite group $G$, choosing representatives $G/H$ for the objects of $\mathcal{O}(G)$, we can construct a functor

$$\sigma_{(\text{Fin}^G)_{\text{op}}} : (\text{Fin}/\mathcal{O}(G))_{\text{op}} \to (\text{Fin}^G)_{\text{op}},$$

$$(z : Z \to \mathcal{O}(G)) \mapsto \coprod_{[G/H] \in \mathcal{O}(G)} G/H \times z^{-1}([G/H]).$$

This way we give $(\text{Fin}^G)_{\text{op}}$ the structure of an $\mathcal{O}(G)$-coloured theory. Composing $\sigma_{(\text{Fin}^G)_{\text{op}}}$ with the functor $R : (\text{Fin}^G)_{\text{op}} \to U^G$, $f \mapsto R_f$, we obtain by (i) of 1.3 a functor $\sigma_{U^G} : (\text{Fin}/\mathcal{O}(G))_{\text{op}} \to U^G$ preserving finite products, and we obtain a structure of $\mathcal{O}(G)$-coloured theory on $U^G$. The functor $R$ is a morphism of $\mathcal{O}(G)$-coloured theories.

Let $V^G \subseteq U^G$ denote the subcategory of $U^G$ with the same class of objects as $U^G$ and with $V^G(X,Y) \subseteq U^G(X,Y)$ given by the subgroup generated by morphisms of the form $[X \leftarrow A \Rightarrow A \to Y]$. The inclusion $V^G \subseteq U^G$ preserves finite products, and we have morphisms $(\text{Fin}^G)_{\text{op}} \to V^G \to U^G$ of $\mathcal{O}(G)$-coloured theories. The category $V^G$ is strongly related to the category spans considered by Lindner in LA, and in fact the category of Mackey functors is equal to the category of $V^G$-algebras.

Let $\text{Fin}^G$ denote the full subcategory of $\text{Fin}^G$ with finite free $G$-sets as objects. The functor

$$\sigma_{(\text{Fin}^G)_{\text{op}}} : (\text{Fin}/\mathcal{O}(G))_{\text{op}} \to (\text{Fin}^G)_{\text{op}}, \quad (z : Z \to \mathcal{O}(G)) \mapsto G/e \times z^{-1}([G/e])$$

gives $(\text{Fin}^G)_{\text{op}}$ the structure of an $\mathcal{O}(G)$-coloured theory. Similarly we can consider the $\mathcal{O}(G)$-coloured theories given by the full subcategories $U^G \subseteq U^G$ and $V^G \subseteq V^G$ with finite free $G$-sets as objects. We have the following diagram of morphisms of $\mathcal{O}(G)$-coloured theories:

$$\begin{array}{ccc}
(\text{Fin}^G)_{\text{op}} & \longrightarrow & V^G & \longrightarrow & U^G \\
\downarrow & & \downarrow & & \downarrow \\
(\text{Fin}^G)_{\text{op}} & \longrightarrow & V^G & \longrightarrow & U^G,
\end{array}$$

where the vertical functors are inclusions of full subcategories.

**Lemma 2.2.**

(i) The category $[U^G, \text{Ens}]_0$ of $fG$-Tambara functors is equivalent to the category of $G$-rings.

(ii) The category $[V^G, \text{Ens}]_0$ of $V^G$-algebras is equivalent to the category of left $\mathbb{Z}[G]$-modules.

(iii) The category $[(\text{Fin}^G)_{\text{op}}, \text{Ens}]_0$ of $(\text{Fin}^G)_{\text{op}}$-algebras is equivalent to the category of $G$-sets.

**Proof.** Since the statements have similar proofs we only give the proof of (i). Given an $fG$-Tambara functor $S$, we construct a $G$-ring-structure on $R = S(G/e)$. Indeed
by (ii) of \( S(G/e) \) is a ring, and given \( g \in G \) the right multiplication \( g : G/e \to G/e, x \mapsto xg \), induces a ring-automorphism \( S^*(g^{-1}) \) of \( R = S(G/e) \). From the functoriality of \( S \) we obtain that \( R \) is a \( G \)-ring. Conversely, given a \( G \)-ring \( R' \) we shall construct an \( fG \)-Tambara functor \( S' \). We define \( S'(X) \) to be the set of \( G \)-maps from \( X \) to \( R' \). Given \( [X \leftarrow d A \rightarrow f B \rightarrow g Y] \in U^G(X,Y) \), we define \( S'[X \leftarrow d A \rightarrow f B \rightarrow g Y] : S'(X) \to S'(Y) \) by the formula

\[
S'[X \leftarrow d A \rightarrow f B \rightarrow g Y](\phi)(y) = \sum_{b \in g^{-1}(y)}\left( \prod_{a \in f^{-1}(b)} \phi(d(a)) \right)
\]

for \( \phi \in S'(X) \) and \( y \in Y \). We leave it to the reader to check that \( S \to R \) and \( R' \to S' \) are inverse functors up to isomorphism. \( \square \)

We refer to [Re, Propositions 4.3 and 4.7.] for a proof of the following two results. Alternatively the reader may modify the proofs given in [Bo, 3.4.5 and 3.7.7] for their monocrome versions.

**Proposition 2.3.** Let \( \Theta \) be an \( O \)-coloured theory. The category of \( \Theta \)-algebras is complete and cocomplete.

**Proposition 2.4.** Given a morphism \( \gamma : \Theta \to \Psi \) of \( O \)-coloured theories, the functor \( \gamma^* : [\Psi, \mathcal{En}_s]_0 \to [\Theta, \mathcal{En}_s]_0, A \mapsto A \circ \gamma \) has a left adjoint \( \gamma_* : [\Theta, \mathcal{En}_s]_0 \to [\Psi, \mathcal{En}_s]_0 \).

**Definition 2.5.** The category of \( fG \)-Tambara functors is the category \([U^{fG}, \mathcal{En}_s]_0\) of \( U^{fG} \)-algebras.

**Definition 2.6.** We let \( L_G = i_* : [U^{fG}, \mathcal{En}_s]_0 \to [U^G, \mathcal{En}_s]_0 \) denote the left adjoint of the functor \( i^* : [U^G, \mathcal{En}_s]_0 \to [U^{fG}, \mathcal{En}_s]_0 \) induced by the inclusion \( i : U^{fG} \subseteq U^G \).

Let us note that \( L_G \) can be constructed as the left Kan extension along \( i \), and that for \( R \in [U^{fG}, \mathcal{En}_s]_0 \), we have for every finite free \( G \)-set \( X \) an isomorphism \( L_GR(X) \cong R(X) \) because \( U^{fG} \) is a full subcategory of \( U^G \).

3. The Teichmüller Homomorphism

We shall now give a connection between the category of \( G \)-Tambara functors and the category of rings with an action of \( G \). Recall that \( U^{fG} \) denotes the full subcategory of \( U^G \) with objects given by free \( G \)-sets. Below we use notation introduced in \[43\]

**Definition 3.1.** Let \( G \) be a finite group and let \( S \) be a \( G \)-Tambara functor. The unrestricted Teichmüller homomorphism \( t : \mathbb{W}_G(S(G/e)) \to S(G/G) \) takes \((x_U)_{U \subseteq G} \in \mathbb{W}_G(S(G/e))\) to

\[
t((x_U)_{U \subseteq G}) = \sum_{U \subseteq G} S_+(\pi_U^G)S_*(\pi_e U)(x_U).
\]

We shall prove the following proposition in the next section.

**Proposition 3.2.** The unrestricted Teichmüller homomorphism \( t : \mathbb{W}_G(S(G/e)) \to S(G/G) \) is a ring-homomorphism.
In general $t$ will neither be injective nor surjective. However, in the introduction, we have given an ideal $\mathbb{I}_G(S(G/e)) \subseteq \mathbb{W}_G(S(G/e))$ contained in the kernel of $t$:

**Definition 3.3.** Let $G$ be a finite group and let $R$ be a commutative ring with an action of $G$. We let $\mathbb{I}_G(R) \subseteq \mathbb{W}_G(R)$ denote the ideal generated by elements of the form $a - b$, where $a = (a_K)_{K \leq G}$ and $b = (b_K)_{K \leq G}$ satisfy that for every $K \leq G$ there exist $n \geq 1$ and $g_1, \ldots, g_n, K \in N_K(G)$ and $a_1, K, \ldots, a_n, K \in R$ such that firstly $g_1, K = \cdots = g_n, K$ and secondly $a_K = a_1, K \cdots a_n, K$ and $b_K = (g_1, K a_1, K) \cdots (g_n, K a_n, K)$.

**Lemma 3.4.** Let $S$ be a $G$-Tambara functor. The unrestricted Teichmüller homomorphism $t : \mathbb{W}_G(S(G/e)) \to S(G/G)$ maps the ideal $\mathbb{I}_G(S(G/e))$ to zero.

**Proof.** It suffices to consider the case where $a_U = 0$ for $U \neq K$. We have that

$$S_+\left(\pi^K G \right) S_\bullet(\pi^K G)(g_1 a_1 \cdots g_n a_n)$$

$$= S[\prod_{1}^{n} G/e \left< G/e \pi^K \circ \nabla \rightarrow G/K \rightarrow G/G \right> (a_1, \ldots, a_n)$$

$$= S[\prod_{1}^{n} G/e \left< G/e \pi^K \circ \nabla \rightarrow G/K \rightarrow G/G \right> (a_1, \ldots, a_n)$$

$$= S_+\left(\pi^K G \right) S_\bullet(\pi^K G)(a_1 \cdots a_n),$$

where $\nabla : \prod_{1}^{n} G/e \to G/e$ is the fold map. \hfill \Box

**Definition 3.5.** Let $G$ be a finite group and let $S$ be a $G$-Tambara functor. The Teichmüller homomorphism $\tau : \mathbb{W}_G(S(G/e))/\mathbb{I}_G(S(G/e)) \to S(G/G)$ is the ring-homomorphism induced by the unrestricted Teichmüller homomorphism $t$.

The following theorem, proved in section 5, implies the case $H = G$ of theorem C.

**Theorem 3.6.** Let $G$ be a finite group and let $R$ be an $fG$-Tambara functor. The Teichmüller homomorphism

$$\tau : \mathbb{W}_G(L_G R(G/e))/\mathbb{I}_G(L_G R(G/e)) \to L_G R(G/G)$$

is an isomorphism. In particular, if $G$ acts trivially on $R$, the Teichmüller homomorphism is an isomorphism on the form $\tau : \mathbb{W}_G(L_G R(G/e)) \to L_G R(G/G)$

In the above situation there is an isomorphism $L_G R(G/e) \cong R(G/e)$.

4. Witt polynomials

We shall prove the following version of a theorem of Dress and Siebenecrher [DS, Theorem 3.2]. As we shall see below, theorem 1.1 and proposition 3.2 are immediate consequences of it.

**Theorem 4.1.** Given a finite group $G$, there exist unique families $(s_U)_U \leq G$, $(p_U)_U \leq G$ of integral polynomials

$$s_U = S^G_U, \quad p_U = P^G_U \in \mathbb{Z}[a_V, b_V | U \leq V \leq G]$$
in two times as many variables $x_V, y_V$ ($U \leq V \leq G$) as there are conjugacy classes of subgroups $V \leq G$ which contain a conjugate of $U$ such that for every $G$-Tambara functor $S$ and for every $x = (x_U)_{U \leq G}$ and $y = (y_U)_{U \leq G} \in \mathbb{W}_G(S(G/e))$ we have that

$$\tau(x) + \tau(y) = \tau((s_U(x_V, y_V | U \leq V \leq G))_{U \leq G})$$

and

$$\tau(x) \cdot \tau(y) = \tau((p_U(x_V, y_V | U \leq V \leq G))_{U \leq G}).$$

Similarly, there exist polynomials $m_U = m_U^G \in \mathbb{Z}[a_U | U \leq V \leq G]$ such that for every $G$-Tambara functor $S$ and for every $x = (x_U)_{U \leq G} \in \mathbb{W}_G(S(G/e))$ we have that $-\tau(x) = \tau((m_U(x_V | U \leq V \leq G'))_{U \leq G})$. Further, for every subgroup $H$ of $G$ we have that

$$\phi_H(x) + \phi_H(y) = \phi_H((s_U(x_V, y_V | U \leq V \leq G))_{U \leq G})$$

and

$$\phi_H(x) \cdot \phi_H(y) = \phi_H((p_U(x_V, y_V | U \leq V \leq G))_{U \leq G}).$$

We shall call the polynomials $s_U, p_U$ and $m_U$ the Witt polynomials.

**Proof of theorem 1.1.** We first consider the case where $G$ is a finite. Given a ring $R$, we define operations $+$ and $\cdot$ on $\mathbb{W}_G(R)$ by defining $(a_U)_{U \leq G} + (b_U)_{U \leq G} := (s_U(a_U, b_V | U \leq V \leq G))_{U \leq G}$ and $(a_U)_{U \leq G} \cdot (b_U)_{U \leq G} := (p_U(a_U, b_V | U \leq V \leq G))_{U \leq G}$. In the case where $R$ has no torsion, the map $\phi : \mathbb{W}_G(R) \to \prod_{U \leq G} R$ with $U$-th component $\phi_U$ is injective, and hence $\mathbb{W}_G(R)$ is a subring of $\prod_{U \leq G} R$. In the case where $R$ has torsion, we can choose a surjective ring-homomorphism $R^\prime \to R$ from a torsion-free ring $R^\prime$. We obtain a surjection $\mathbb{W}_G(R^\prime) \to \mathbb{W}_G(R)$ respecting the operations $\cdot$ and $\cdot$. Since $\mathbb{W}_G(R^\prime)$ is a ring we can conclude that $\mathbb{W}_G(R)$ is a ring. Given a surjective homomorphism $\gamma : G \to G'$ of finite groups we obtain a ring-homomorphism $\text{restr}_G^{G'} : \mathbb{W}_G(R) \to \mathbb{W}_{G'}(R)$ with $\text{restr}_G^{G'}((a_U)_{U \leq G}) = ((b_U)_{U \leq G'})$, where $b_U = a_U \gamma^{-1}(V)$. (See [DS (3.3.11)].) The easiest way to see that $\text{restr}_G^{G'}$ is a ring-homomorphism is to note that $(\gamma^{-1}H : \gamma^{-1}U) = (H : U)$ and that $\phi_{\gamma^{-1}H}(G/\gamma^{-1}U) = \phi_H(G'/H)$. For the case where $G$ is a pro-finite group we note that for the ring $\mathbb{W}_G(R)$ can be defined to be the limit $\lim_N \mathbb{W}_{G/N}(R)$ taken over all finite factor groups $G/N$, with respect maps on the form $\text{restr}_G^{G'}$. □

**Proof of proposition 3.2.** Proposition 3.2 follows from the first part of theorem 1.1 because we use the Witt polynomials to define the ring-structure on the Witt vectors.

For the rest of this section we fix a finite group $G$. We shall follow [DS] closely in the proof of theorem 1.1. For the uniqueness of the Witt polynomials we consider the representable $G$-Tambara functor $\Omega := U^G(0, -)$. In [DS Theorem 2.12.7] it is shown that $\tau : \prod_{U \leq G} \mathbb{Z} = \mathbb{W}_G(\Omega(G/e)) \to \mathbb{W}_G(\mathbb{Z})$ is a bijection. Hence the Witt polynomials are unique. We shall use the following six lemmas to establish the existence of Witt polynomials with the properties required in theorem 1.1.

**Lemma 4.2.** For a subset $A$ of $G$, let $U_A := \{g \in G | Ag = A\}$ denote its stabilizer group and let $i_A := |A/U_A|$ denote the number of $U_A$-orbits in $A$. If the set $\mathcal{U}(G)$ of subsets of $G$ is considered as a $G$-set via $G \times \mathcal{U}(G) \to \mathcal{U}(G): (g, A) \mapsto Ag$, then...
for any \( s, t \in S(G/e) \) one has
\[
S_\bullet(\pi_e^G)(s + t) = \sum_{G, A \in G \cup (G)} S_\bullet(\pi_{U_A}^G)S_\bullet(\pi_e^{U_A})(s^{i_A}t^{G-A}).
\]

Proof. We let \( i_1, i_2 : G/e \to G/e \amalg G/e \) denote the two natural inclusions. We have an exponential diagram
\[
\begin{array}{ccc}
G/e & \xleftarrow{\nabla} & G/e \amalg G/e \\
\pi_e^G & \downarrow & \downarrow \text{pr} \\
G/G & \leftarrow & U(G) \relbar\leftrightsquigarrow \rightarrow U(G),
\end{array}
\]
where \( d(g, A) = i_1(g) \) if \( g^{-1} \in A \) and \( d(g, A) = i_2(g) \) if \( g^{-1} \notin A \). Since \( U(G) \cong \coprod_{G, A \in G \cup (G)} G \cdot A \), we have that
\[
S_\bullet(\pi_e^G)(s + t) = S_\bullet(\pi_e^G)S_\bullet(\nabla)(s, t)
= \sum_{G, A \in G \cup (G)} S[G/e \amalg G/e \xleftarrow{d} G/e \times U(G) \twoheadrightarrow U(G) \to G/G](s, t)
= \sum_{G, A \in G \cup (G)} S[G/e \amalg G/e \xleftarrow{d} G/e \times GA \to GA \to G/G](s, t)
= \sum_{G, A \in G \cup (G)} S[G/e \amalg G/e \xleftarrow{s, t} G/e \xrightarrow{\pi_{U_A}^G} G/U_A \xrightarrow{\pi_e^{U_A}} G/G](s^{i_A}t^{G-A})
= \sum_{G, A \in G \cup (G)} S_\bullet(\pi_{U_A}^G)S_\bullet(\pi_e^{U_A})(s^{i_A}t^{G-A}),
\]
where the maps without labels are natural projections.

Lemma 4.3. Let \( R \) be a commutative ring. With the notation of lemma 4.2, we have for every subset \( A \) and subgroup \( U \) of \( G \) and for every \( s, t \in R \), that
\[
(s + t)^{G:U} = \sum_{G, A \in G \cup (G)} |U(G/U_A)^U| \cdot (s^{i_A}t^{G-A})(U_A:U).
\]

Proof. We compute:
\[
(s + t)^{G:U} = \sum_{A \subseteq G/U} s^{|A|}t^{|G/U| - |A|} = \sum_{A \in U(G), U \subseteq U_A} s^{|A/U|}t^{|G - A)/U|}
= \sum_{G, A \in G \cup (G)} |(G/U_A)^U| \cdot (s^{i_A}t^{G-A})(U_A:U).
\]

The following lemma is a variation on [DS, lemma 3.2.5], and the proof we give here is essentially identical to the one given in [DS].
Lemma 4.4. For some \( k \in \mathbb{N} \) let \( V_1, \ldots, V_k \leq G \) be a sequence of subgroups of \( G \). Then for every subgroup \( U \leq G \) and every sequence \( \varepsilon_1, \ldots, \varepsilon_k \in \{ \pm 1 \} \) there exists a polynomial \( \xi_U = \xi_{U;V_1,\ldots,V_k;\varepsilon_1,\ldots,\varepsilon_k} = \xi_U(x_1,\ldots,x_k) \in \mathbb{Z}[x_1,\ldots,x_k] \) such that for every \( G \)-Tambara functor \( S \) and all \( s_1,\ldots,s_k \in S(G/e) \) one has

\[
\sum_{i=1}^{k} \varepsilon_i S_+^{(\pi_{V_i}^G)} S_+^{(\pi_e^U)}(s_i) = \sum_{U \leq G} S_+^{(\pi_{U}^G)} S_+^{(\pi_e^U)}(\xi_U(s_1,\ldots,s_k)).
\]

Proof. If \( \varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_k = 1 \) and if \( V_i \) is not conjugate to \( V_j \) for \( i \neq j \), then

\[
\sum_{i=1}^{k} \varepsilon_i S_+^{(\pi_{V_i}^G)} S_+^{(\pi_e^U)}(s_i) = \sum_{U \leq G} S_+^{(\pi_{U}^G)} S_+^{(\pi_e^U)}(s_U),
\]

with \( s_U = s_i \) if \( U \) is conjugate to \( V_i \) and \( s_U = 0 \) if \( U \) is not conjugate to any of the \( V_1,\ldots,V_k \). So in this case we are done: \( \xi_U(s_1,\ldots,s_k) = s_U \). We use triple induction to prove the lemma. First with respect to \( m_1 = m_1(U;V_1,\ldots,V_k;\varepsilon_1,\ldots,\varepsilon_k) := \max(|V_i| \mid \varepsilon_i = -1 \text{ or there exists some } j \neq i \text{ with } V_j \sim V_i) \), then with respect to \( m_2 := |\{i \mid |V_i| = m_1 \text{ and there exists some } j \neq i \text{ with } V_j \sim V_i\}| \) and then with respect to \( m_3 := |\{i \mid |V_i| = m_1 \text{ and } \varepsilon_i = -1\}| \).

We have just verified that the lemma holds in the case \( m_1 = 0 \). In case \( m_1 > 0 \) we have either \( m_2 > 0 \) or \( m_3 > 0 \). In case \( m_2 > 0 \), say \( |V_1| = m_1 \) and \( \varepsilon_1 = -1 \), we may use \ref{4.2} with \( G = V_1, s = -s_1, t = s_1 \) to conclude that

\[
0 = \sum_{V_1:A \in V_1 \setminus \mathcal{U}(V_1)} S_+^{(\pi_{V_1}^G)} S_+^{(\pi_e^U)}((-1)^{i_A} s_1^{(V_1:U_A)}).
\]

Therefore, considering the two special summands \( A = \emptyset \) and \( A = V_1 \) and putting \( \mathcal{U}_0(V_1) := \{ A \in \mathcal{U}(V_1) \mid A \neq \emptyset \text{ and } A \neq V_1 \} \), one gets

\[
-S_+^{(\pi_{V_1}^G)}(s_1) = S_+^{(\pi_{V_1}^G)}(-s_1) + \sum_{V_1:A \in V_1 \setminus \mathcal{U}_0(V_1)} S_+^{(\pi_{V_1}^G)} S_+^{(\pi_e^U)}((-1)^{i_A} s_1^{(V_1:U_A)}).
\]

Hence, if \( A_{k+1}, A_{k+2}, \ldots, A_{k'} \in \mathcal{U}_0(V_1) \) denote representatives of the \( V_1 \)-orbits \( V_1A \subseteq \mathcal{U}_0(V_1) \) and we let \( V_{k+1} := U_{A_{k+1}}, \ldots, V_{k'} := U_{A_{k'}} \) then \( V_i \leq V_1 \) and \( V \neq V_1 \) for \( i \geq k+1 \). If we put \( \varepsilon_{k+1} = \cdots = \varepsilon_{k'} = 1 \) and \( s_{k+1} := (-1)^{i_{A_{k+1}}} s_1^{(V_1:V_{k+1})}, \ldots, s_{k'} = (-1)^{i_{A_{k'}}} s_1^{(V_1:V_{k'})} \), then the polynomial

\[
\xi_{U;V_1,\ldots,V_{k+1};1,\varepsilon_2,\ldots,\varepsilon_k}(s_1,\ldots,s_k) := \xi_{U;V_1,\ldots,V_{k'};1,\varepsilon_2,\ldots,\varepsilon_{k'}}(-s_1,s_2,\ldots,s_{k'})
\]

makes the statement of the lemma hold. We can conclude that if the lemma holds for every \((n_1, n_2, n_3)\) with either \( n_1 < m_1 \) or \((n_1 = m_1, n_2 < m_2 \text{ and } n_3 \leq m_3)\), then it also holds for \((m_1, m_2, m_3)\).
Similarly, if \( m_2 = 0 \), but \( m_3 > 0 \), say \( V_1 = V_2 \), then we may use lemma 4.2 once more with \( G = V_1 \), \( s = s_1 \), and \( t = s_2 \) to conclude that

\[
S_\bullet (\pi_{e}^{V_1})(s_1 + s_2) = \sum_{V_1 \cdot A \in V_1 \cdot B(V_1)} S_\bullet (\pi_{U_A}^{V_1}) S_\bullet (\pi_{e}^{U_A})(s_1 s_2^{-1})
\]

so with \( V_{k+1}, \ldots, V_{k'} \) as above, but with \( \varepsilon_{k+1} = \cdots = \varepsilon_{k'} = -1 \) and with \( s_{k+1} := s_1 s_2^{-1} - 1 \), \( s_{k'} := s_1 s_2^{-1} - 1 \), the polynomial

\[
\xi^G_{(U; V_1, \ldots, V_{k+1}, \varepsilon_{k+1}; \varepsilon_{k+2}, \ldots, \varepsilon_{k'})} := \xi^G_{(U; V_2, \ldots, V_{k'}, \varepsilon_{k+2}, \ldots, \varepsilon_{k'})}(s_1 + s_2, s_3, \ldots, s_{k'})
\]

makes the statement of the lemma hold. We can conclude that if the lemma holds for every \((n_1, n_2, n_3)\) with either \( n_1 < m_1 \) or \((n_1 = m_1, n_2 < m_2 = 0 \) and \( n_3 < m_3 \)), then it also holds for \((m_1, 0, m_3)\). The statement of the lemma now follows by induction on first \( m_1 \), then on \( m_3 \) and finally on \( m_2 \). \( \square \)

**Lemma 4.5.** Let \( R \) be a ring. For some \( k \in \mathbb{N} \) let \( V_1, \ldots, V_k \leq G \) be a sequence of subgroups of \( G \) and let \( \varepsilon_1, \ldots, \varepsilon_k \in \{ \pm 1 \} \). For every subgroup \( H \) of \( G \) one has, with the notation of lemma 4.4 that

\[
\sum_{i=1}^{k} \varepsilon_i |(G/V_i)^H| \cdot (s_i)^{(V_i; H)} = \sum_{U \leq G} \|(G/U)^H| (\xi_U(s_1, \ldots, s_k))^{(U; H)}
\]

**Proof.** We use the same notation as in the above proof, and prove the lemma by induction on first \( m_1 \), then on \( m_3 \) and finally on \( m_2 \). Suppose that the lemma holds for every \((n_1, n_2, n_3)\) with \( n_1 < m_1 \) or \((n_1 = m_1, n_3 < m_3 \) or \((n_1 = m_1, n_3 = m_3, n_2 < m_2) \), and that we are given \((V_1, \ldots, V_k; \varepsilon_1, \ldots, \varepsilon_k)\) with \( m_i = m_i(V_1, \ldots, V_k; \varepsilon_1, \ldots, \varepsilon_k) \) with \( m_i \neq m_i(V_1, \ldots, V_k; \varepsilon_1, \ldots, \varepsilon_k) \). In the above proof we constructed \((W_1, \ldots, W_{k'}; \delta_1, \ldots, \delta_{k'}) \) and \( t_1, \ldots, t_{k'} \in R \) such that \( n_i = n_i(W_1, \ldots, W_{k'}; \delta_1, \ldots, \delta_{k'}) \), \( i = 1, 2, 3 \) are on the form covered by the induction hypothesis, and such that:

\[
\sum_{i=1}^{k} \varepsilon_i |(G/V_i)^H| (s_i)^{(V_i; H)} = \sum_{i=1}^{k'} \delta_i |(G/W_i)^H| (t_i)^{(W_i; H)}
\]

\[
= \sum_{(U \leq G)} \|(G/U)^H| (\xi^G_{(U; W_1, \ldots, W_{k'}; \delta_1, \ldots, \delta_{k'})}(t_1, \ldots, t_{k'}))^{(U; H)}
\]

\[
= \sum_{(U \leq G)} \|(G/U)^H| (\xi^G_{(U; V_1, \ldots, V_k; \varepsilon_1, \ldots, \varepsilon_k)}(s_1, \ldots, s_k))^{(U; H)}
\]

The other cases of the induction are similar. \( \square \)
Lemma 4.6. For every $G$-Tambara functor $S$, for every two subgroups $V, W \leq G$ and for every $s, t \in S(G/e)$ one has the modified Mackey formula

\[
S_+(\pi^G_V)S_*(\pi^V_e)(s) \cdot S_+(\pi^G_W)S_*(\pi^W_e)(t) = \sum_{VgW \in V \backslash G/W} S_+(\pi^G_{V \cap gWg^{-1}})S_*(\pi^V_{e \cap gWg^{-1}})(s^{(V \cap gWg^{-1})}, t^{(W^{-1}Vg \cap W)}).
\]

Proof. We use the diagram

\[
\begin{array}{ccccccc}
G/e \amalg G/e & \xrightarrow{pr_1 \amalg pr_2} & G/e \times G/W \amalg G/V \times G/e \\
\downarrow \pi^V_e \amalg \pi^W_e & & \downarrow \pi^V_e \times \id \amalg \id \times \pi^W_e \\
G/G \amalg G/G & \xrightarrow{pr_1 \amalg pr_2} & G/V \amalg G/W \amalg G/V \times G/W \\
\downarrow \nabla & & \downarrow \nabla \\
G/G & \leftarrow & G/V \times G/W & \xrightarrow{\gamma} & G/V \times G/W,
\end{array}
\]

where $\nabla$ is the fold map, the upper square is a pull-back and the lower rectangle is an exponential diagram. Concatenating with the diagram

\[
\begin{array}{ccccccc}
G/e \times G/W \amalg G/V \times G/e & \xrightarrow{\alpha_1 \amalg \alpha_2} & \coprod_{gW \in G/W} G/e \amalg \coprod_{Vg \in V \backslash G} G/e \\
\downarrow \pi^V \times \id \amalg \id \times \pi^W & & \|
\downarrow \gamma \\
G/V \times G/W \amalg G/V \times G/W & \xrightarrow{\gamma} & \coprod_{VgW \in V \backslash G/W} G/(V \cap gWg^{-1})
\end{array}
\]

with maps defined by

\[
\begin{align*}
\alpha_1(gW, \sigma) &= (\sigma, \sigma gW), \\
\alpha_2(Vg, \sigma) &= (\sigma g^{-1}V, \sigma), \\
\delta_1(gW, \sigma) &= (VgW, \sigma (V \cap gWg^{-1})), \\
\delta_2(Vg, \sigma) &= (VgW, \sigma g^{-1}(V \cap gWg^{-1})) \quad \text{and} \\
\gamma(VgW, \sigma (V \cap gWg^{-1})) &= (\sigma V, \sigma gW),
\end{align*}
\]

we get that

\[
S_+(\pi^G_V)S_*(\pi^V_e)(s) \cdot S_+(\pi^G_W)S_*(\pi^W_e)(t) = S[G/e \amalg G/e \leftarrow \coprod_{gW \in G/W} G/e \amalg \coprod_{Vg \in V \backslash G} G/e \rightarrow \coprod_{VgW \in V \backslash G/W} G/(V \cap gWg^{-1}) \rightarrow G/G](s, t)
\]

\[
= \sum_{VgW \in V \backslash G/W} S_+(\pi^G_{V \cap gWg^{-1}})S_*(\pi^V_{e \cap gWg^{-1}})(s^{(V \cap gWg^{-1})}, t^{(W^{-1}Vg \cap W)}).
\]

\[\square\]
Lemma 4.7 ([DS Lemma 3.2.13]). For every ring $R$, for every three subgroups $H, V, W \leq G$ and for every $s, t \in R$ one has the modified Mackey formula
\[
|(G/V)^H_s| |(G/W)^H_t| = \sum_{V g W \in V \setminus G/W} |(G/V \cap g W g^{-1})^H_s| |(s^{(V^{(V \cap g W g^{-1})})} g^{(W g^{-1} V g W)})^H_t| (V \cap g W g^{-1} H).
\]

Proof of theorem 4.7. Let $G = V_1, V_2, \ldots, V_k = U$ be a system of representatives of subgroups of $G$ containing a conjugate of $U$. We define
\[
s_U^G(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k}) := \xi_{(U; V_1, V_1, \ldots, V_k, V_k; 1, \ldots, 1)}^G(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k})
\]
and
\[
m_U^G(a_{V_1}, \ldots, a_{V_k}) := \xi_{(U; V_1, \ldots, V_k, U; 1, \ldots, 1)}^G(a_{V_1}, \ldots, a_{V_k}).
\]
By the lemmas 4.4 and 4.5 these are integral polynomials with the desired properties. For example, for $U \leq G$ we have that
\[
\phi_U(a) + \phi_U(b) = \sum_{i=1}^k |(G/V_i)^U| (a_{V_i}^{(V_i; U)} + b_{V_i}^{(V_i; U)}) = \phi_U(\xi_{(U; V_1, V_1, \ldots, V_k, V_k; 1, \ldots, 1)}(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k})) = \phi_U(s_U(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k})).
\]
To construct $p_U = p_U^G$ we first choose a system $x_1, x_2, \ldots, x_h$ of representatives of the $G$-orbits in
\[
X := \prod_{i,j=1}^k G/V_i \times G/V_j.
\]
Next we put $W_r := G_{x_r}$ and
\[
p_r = p_r(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k}) := a_{W_r}^{(V_i; W_r)} b_{W_r}^{(V_j; W_r)}
\]
in case $x_r = (g V_i, g' V_j) \in G/V_i \times G/V_j \subseteq X$. Using these conventions, we define
\[
p_r^G(a_{V_1}, b_{V_1}, \ldots, a_{V_k}, b_{V_k}) := \xi_{(U; W_1, \ldots, W_k, 1, \ldots, 1)}^G(p_1, \ldots, p_r).
\]
Using the lemmas 4.6 and 4.7 we see that $p_U$ has the desired properties. \hfill \Box

5. Free Tambara functors

In this section we give a proof of theorem 3.6. On the way we shall give a combinatorial description of the Witt vectors of a polynomial $G$-ring, that is, a $G$-ring of the form $U^G(X, G/e)$ for a finite $G$-set $X$. Recall from 2.2 that the functor $R \mapsto R(G/e)$ from the category $[U^G, \mathcal{Ens}]_0$ of $fG$-Tambara functors to the category of $G$-rings is an equivalence of categories, and that there are morphisms $(\mathcal{F}\text{in}^G)^{op} \subseteq (\mathcal{F}\text{in}^G)^{op} \to U^G$ of $\mathcal{O}(G)$-coloured theories. We let $F : \mathcal{Ens}^G \simeq ([\mathcal{F}\text{in}^G]^{op}, \mathcal{Ens})_0 \to [U^G, \mathcal{Ens}]_0$ denote the left adjoint of the forgetful functor induced by the above composition of morphisms of coloured theories.
Definition 5.1. Given finite $G$-sets $X$ and $Y$ we let $\tilde{U}^G(X, Y) \subseteq U^G(X, Y)$ denote those elements of the form $X \leftarrow A \rightarrow B \rightarrow Y$, where $G$ acts freely on $A$, and we let $\tilde{U}^G(X, Y) \subseteq U^G(X, Y)$ denote the abelian subgroup generated by $\tilde{U}^G(X, Y)$. The composition

$$U^G(Y, Z) \times \tilde{U}^G(X, Y) \cup \tilde{U}^G(Y, Z) \times U^G(X, Y) \subseteq U^G(Y, Z) \times U^G(X, Y) \xrightarrow{\phi} U^G(X, Z)$$

factors through the inclusion $\tilde{U}^G(X, Z) \subseteq U^G(X, Z)$. We obtain a functor $\tilde{U}^G : \text{Fin}^G \to [U^G, \text{Ens}]_0$ with $\tilde{U}^G(f : Y \to X) = \tilde{U}^G(R_f, -)$.

Lemma 5.2. Given a $G$-Tambara functor $S$ and a finite free $G$-set $A$, there is an isomorphism $\text{Ens}^G(A, S(G/e)) \iso S^*(A)$, which is natural in $A$.

Proof. Choosing an isomorphism $\phi : A \iso G/e \times A_0$ we obtain an isomorphism

$$\text{Ens}^G(A, S(G/e)) \xrightarrow{\phi^{-1} \ast} \text{Ens}^G(G/e \times A_0, S(G/e)) \iso \text{Ens}(A_0, S(G/e)) \iso S(G/e \times A_0) \xrightarrow{S^*(\phi)} S(A).$$

This isomorphism is independent of the choice of $\phi$. \qed

Lemma 5.3. Given a finite $G$-set $X$ we have that $FX \iso \tilde{U}^G(X, -)$.

Proof. For every $G$-Tambara functor $S$ we shall construct a bijection

$$\text{Ens}^G(X, S(G/e)) \iso [U^G, \text{Ens}]_0(\tilde{U}^G(X, -), S).$$

Given $f : X \to S(G/e) \in \text{Ens}^G(X, S(G/e))$ we let $\phi(f) \in [U^G, \text{Ens}]_0(\tilde{U}^G(X, -), S)$ take $x = [X \xrightarrow{a} A \xrightarrow{b} B \xrightarrow{c} Y] \in \tilde{U}^G(X, Y)$ to $\phi(f)(x) \in S(Y)$ constructed as follows: by 5.2 we obtain an element $a \in S(A)$, and we let $\phi(f)(x) = S_+(c)S_-(b)(a)$. Conversely, given $g \in [U^G, \text{Ens}]_0(\tilde{U}^G(X, x), S)$, we construct $\psi(g) \in \text{Ens}^G(X, S(G/e))$ by letting $\psi(g)(x) = [X \leftarrow G/e \to G/e \to G/e]$, where the map pointing left takes $e \in G$ to $x \in X$ and where the maps pointing right are identity maps. We leave it to the reader to check that $\phi$ and $\psi$ are inverse bijections. \qed

Corollary 5.4. For every finite $G$-set $X$ the functor $\tilde{U}^G(X, -) : U^G \to \text{Ens}$ is isomorphic to $L_GU^G(X, G/e)$.

Theorem 5.5. Let $X$ be a finite $G$-set and let $R = \tilde{U}^G(X, -)$. Then $R(G/e) = U^G(X, G/e)$, and the Teichmüller homomorphism

$$\tau : \mathbb{W}_G(L_GR(G/e))/\mathbb{I}_G(L_GR(G/e)) \to L_GR(G/e) = \tilde{U}^G(X, G/G)$$

is an isomorphism.

Proof of theorem 5.5. Let $X$ be a set with trivial action of $G$. The polynomial ring $\mathbb{Z}[X]$ has the universal property that determines the left adjunct of the forgetful functor from rings with an action of $G$ to $\text{Ens}^G$ up to isomorphism. Hence $\mathbb{Z}[X] \iso \tilde{U}^G(X, G/e) = U^G(X, G/e)$. By theorem 5.3, $\tilde{U}^G(X, G/G) = U^G(X, G/G)$ is isomorphic to $\mathbb{W}_G(\mathbb{Z}[X])$. \qed
Proof of theorem 3.6. Let \( R_0 = R(G/e) \). Given \( \alpha = [W \leftarrow C \rightarrow D \rightarrow X] \in U^{fG}(W, X) \) we have an \( fG \)-Tambara map \( \alpha^* : U^G(X, -) \to U^G(W, -) \) and we have the map \( R(\alpha) : R(W) \to R(X) \). Hence we obtain maps

\[
U^G(X, G/G) \times R(X) \leftarrow U^G(X, G/G) \times R(W) \to U^G(W, G/G) \times R(W).
\]

Recall from lemma 2.3 that \( L_GR(G/G) \) can be constructed by the coequalizer of the diagram

\[
\prod_{X, Y \in \text{ob} U^{fG}} U^G(X, G/G) \times R(Y) \xrightarrow{\rho} \prod_{X \in \text{ob} U^{fG}} U^G(X, G/G) \times R(X),
\]

induced by the above maps. We shall construct a map

\[
\rho : L_GR(G/G) \to \mathbb{W}_G(R_0)/\mathbb{I}_G(R_0)
\]

by specifying explicit maps \( \rho_X : U^G(X, G/G) \times R(X) \to \mathbb{W}_G(R_0)/\mathbb{I}_G(R_0) \). Given \( \underline{x} \in R(X) \), we have an \( fG \)-Tambara morphism \( \text{ev}_{\underline{x}} : U^{fG}(X, -) \to R \). In particular we have a \( G \)-homomorphism \( \text{ev}_{\underline{x}}(G/e) : U^G(X, G/G) \to R(G/e) = R_0 \). By theorem 5.3 we get an induced ring-homomorphism

\[
U^G(X, G/G) \cong \mathbb{W}_G(U^G(X, G/e))/\mathbb{I}_G(U^G(X, G/e)) \to \mathbb{W}_G(R_0)/\mathbb{I}_G(R_0),
\]

and hence by adjunction we obtain a map

\[
\rho_X : U^G(X, G/G) \times R(X) \to \mathbb{W}_G(R_0)/\mathbb{I}_G(R_0).
\]

We need to check that these \( \rho_X \) induce a map on the coequalizer \( L_GR(G/G) \) of the above coequalizer diagram, that is, for \( \alpha \) as above we need to show that the diagram

\[
\begin{array}{ccc}
U^G(X, G/G) \times R(W) & \xrightarrow{\alpha^* \times \text{id}} & U^G(W, G/G) \times R(W) \\
\downarrow \text{id} \times R(\alpha) & & \downarrow \rho_W \\
U^G(X, G/G) \times R(X) & \xrightarrow{\rho_X} & \mathbb{W}_G(R_0)/\mathbb{I}_G(R_0)
\end{array}
\]

commutes. In order to do this we note that the diagram

\[
\begin{array}{ccc}
\mathbb{W}_G(U^G(X, G/e)) & \xrightarrow{t} & U^G(X, G/G) \\
\downarrow \alpha^* & & \downarrow \alpha^* \\
\mathbb{W}_G(U^G(W, G/e)) & \xrightarrow{t} & U^G(W, G/G)
\end{array}
\]

commutes, and therefore it will suffice to show that the diagram

\[
\begin{array}{ccc}
\mathbb{W}_G(U^G(X, G/e)) \times R(W) & \xrightarrow{\alpha^* \times \text{id}} & \mathbb{W}_G(U^G(W, G/e)) \times R(W) \\
\downarrow \text{id} \times R(\alpha) & & \downarrow \\
\mathbb{W}_G(U^G(X, G/e)) \times R(X) & \longrightarrow & \mathbb{W}_G(R_0)
\end{array}
\]

commutes, where the arrows without labels are constructed using the homomorphisms \( \mathbb{W}_G(\text{ev}_{\underline{x}}(G/e)) \) for \( \underline{x} \) an element of either \( R(X) \) or \( R(W) \). Using diagonal
inclusions of the form
\[ \mathbb{W}_G(T) \times Z \rightarrow \mathbb{W}_G(T) \times \prod_{U \leq G} Z \approx \prod_{U \leq G} (T \times Z) \]
we see that it suffices to show note that the diagram
\[ \prod_{U \leq G}(U^G(X, G/e) \times R(W)) \xrightarrow{\prod_{U \leq G}(\alpha^* \times \text{id})} \prod_{U \leq G}(U^G(W, G/e) \times R(W)) \]
\[ \quad \downarrow \]
\[ \prod_{U \leq G}(U^G(X, G/e) \times R(X)) \xrightarrow{} \prod_{U \leq G}R(G/e) \approx \mathbb{W}_G(R_0) \]
commutes. This finishes the construction of \( \rho : LGR(G/G) \rightarrow \mathbb{W}_G(R_0)/\mathbb{W}_G(R_0) \). We
leave it to the reader to check that \( \rho \) and \( \tau \) are inverse bijections. For this it might
be helpful to note that
\[ \sum_{U \leq G} ([G/e \xrightarrow{\pi} G/U \rightarrow G/G] \circ [Y \leftarrow A_U \rightarrow B_U \rightarrow G/e]) \]
\[ = \prod_{U \leq G}[G/e \xrightarrow{\pi} \prod_{U \leq G} G/U \rightarrow G/G] \circ [Y \leftarrow \prod_{U \leq G}A_U \rightarrow \prod_{U \leq G}B_U \rightarrow \prod_{U \leq G} G/e]. \]
\[ \square \]

In order to begin the proof of theorem 5.5 we need to introduce filtrations of both
des.

**Definition 5.6.** Let \( R \) be a ring and let \( U \leq G \) be a subgroup of \( G \). We let \( I_U \subseteq \mathbb{W}_G(R) \) denote the ideal generated by those \( a \in (a_K)_{K \leq G} \in \mathbb{W}_G(R) \) satisfying that \( a_K \neq 0 \) implies that \( K \leq U \). We let \( \tilde{I}_U \subseteq I_U \) denote the subideal \( \tilde{I}_U = \sum_{V \leq U} J_V \subseteq I_U \).

**Definition 5.7.** Given a \( G \)-sets \( X \), we let \( J^+_U = J_U(X, G/G) \subseteq \tilde{U}^G(X, G/G) \) denote the subset of elements of the form \([X \leftarrow A \rightarrow B \rightarrow G/G] \in U^G(X, G/G) \subseteq \tilde{U}^G(X, G/G) \), where \( B\Kappa = \emptyset \) when \( U \leq K \) and \( U \neq K \). We let \( J_U \subseteq U^G(X, G/G) \) denote the ideal generated by \( J^+_U \) and we let \( \tilde{J}_U \subseteq J_U \) denote the subideal \( \tilde{J}_U = \sum_{V \leq U} J_V \subseteq J_U \).

**Lemma 5.8.** In the situation of the above definition the following holds:
(i) Any element in \( J_U \) is of the form \( x - y \) for \( x, y \in J^+_U \).
(ii) Every element \( x \) in the image of the map \( J^+_U \rightarrow J_U/\tilde{J}_U \) is of the form
\[ x = [X \xrightarrow{d} G/e \times A \xrightarrow{\pi^U_X f} G/U \times B \xrightarrow{q} G/G] + \tilde{J}_U, \]
with \( f : A \rightarrow B \) a map of (nonequivariant) sets and \( d \) a \( G \)-map, where \( q \) is the composition \( G/U \times B \xrightarrow{pr} G/U \xrightarrow{\pi_U} G/G \).
(iii) If
\[ x = [X \xrightarrow{d} G/e \times A \xrightarrow{\pi^U_X f} G/U \times B \xrightarrow{q} G/G] + \tilde{J}_U, \]
and
\[ x' = [X \xleftarrow{d'} G/e \times A' \xrightarrow{\pi^U_X f'} G/U \times B' \xrightarrow{q'} G/G] + \tilde{J}_U, \]
then \( x = x' \) if and only if there exist bijections \( \alpha : A \to A' \) and \( \beta : B \to B' \) and for every \( a \in A \) there exists \( g_a \in N_G(U) \) such that firstly \( d'(e, \alpha a) = d(g_a, a) \), and secondly, if \( a_1, a_2 \in A \) satisfy that \( f(a_1) = f(a_2) \), then \( g_{a_1}U = g_{a_2}U \).

**Proof.** A straight forward verification yields that multiplication in \( \tilde{U}^+_U(X, G/G) \) induces a map \( J^+_U \times \tilde{U}^+_U(X, G/G) \to J^+_U \) and that \( J^+_U \) is closed under sum. It follows that \( J^+_U \) is the abelian subgroup of \( U^G(X, G/G) \) generated by \( J^+_U \). The statement (i) is a direct consequence of this. For (ii) we note that for every element

\[
 r = [X \xleftarrow{d} D \xrightarrow{c} E \xrightarrow{f} G/G]
\]

in \( \tilde{U}^+_U(X, G/G) \), we have a decomposition \( E \cong \coprod_{K \leq G} E_K \), where \( E_K \cong G/K \times B_K \) for some \( B_K \). This decomposition induces an isomorphism

\[
 \tilde{U}^G(X, G/G) \cong \bigoplus_{K \leq G} J_K/\tilde{J}_K
\]

of abelian groups. Given an element \( x \) of the form

\[
 x = [X \xleftarrow{d} D \xrightarrow{c} G/U \times B \xrightarrow{g} G/G] + \tilde{J}_U,
\]

we can choose a \( G \)-bijection of the form \( c^{-1}(G/U \times \{b\}) \cong G/e \times A_b \) for every \( b \in B \). It follows that \( x \) is represented by an element of the form

\[
 x = [X \xleftarrow{d} G/e \times A \xrightarrow{\times f} G/U \times B \xrightarrow{g} G/G] + \tilde{J}_U.
\]

We leave the straight forward verification of part (iii) to the reader. \( \Box \)

**Lemma 5.9.** Let \( U \) be a subgroup of \( G \) and let \( a = (a_U)_{U \leq G} \in I_U \) have

\[
 a_U = [X \xleftarrow{d} A \times G/e \xrightarrow{f \times 1} B \times G/e \xrightarrow{pr} G/e].
\]

Then

\[
 \tau(a) \equiv [X \xleftarrow{d} A \times G/e \xrightarrow{f \times \pi_U} B \times G/U \to G/G] \mod \tilde{J}_U.
\]

**Proof.** The lemma follows from the diagram

\[
\begin{array}{cccccc}
X & \xleftarrow{d} & A \times G/e & \xleftarrow{W} & G/e \times A \\
& & \downarrow_{f \times \text{id}} & \downarrow & \\
G/e & \xleftarrow{\text{pr}} & G/e \times B & \xleftarrow{Z} & G/e \times B \\
& \downarrow_{\pi_U} & \downarrow & \downarrow & \\
G/U & \xleftarrow{\coprod_{gU \in G/U} \text{map}(gU, B)} & Y & \xleftarrow{G/U \times B},
\end{array}
\]

where the lower rectangle is an exponential diagram and the squares are pull-backs. We use that the map \( G/U \times B \to Y \) which takes \( (gU, b) \) to the constant map \( gU \to B \) with value \( b \) is an isomorphism on \( G/U \)-parts and that \( Y^H = \emptyset \) for \( U \leq H \) and \( U \neq H \). \( \Box \)
Corollary 5.10. Let $X$ be a $G$-set and let $R = \tilde{U}^G(X, -)$. The map
\[ \tau : \mathbb{W}_G(U^G(X, G/e)) \to L_G R(G/G) = \tilde{U}^G(X, G/G) \]
satisfies that $\tau(I_U) \subseteq J_U$ and that $\tau(\tilde{I}_U) \subseteq \tilde{J}_U$.

Proposition 5.11. Let $X$ be a $G$-set, let $R = \tilde{U}^G(X, -)$ and let $R_0 = R(G/e)$. For every $U \leq G$ the map $\tau : \mathbb{W}_G(R_0)/\mathbb{I}_G(R_0) \to L_G R(G/G) = \tilde{U}^G(X, G/G)$ induces an isomorphism $\tau_U : (\mathbb{I}_G(R_0) + I_U)/(\mathbb{I}_G(R_0) + \tilde{I}_U) \to J_U/\tilde{J}_U$.

Proof. Let $x \in I_U$ have $x_U = [X \not\cong A \times G/e \xmod{f} B \times G/e \xmapsto{g} G/e]$. Then by lemma 5.9 $\tau(x) \equiv [X \not\cong A \times G/e \xmod{f \times p} B \times G/U \xmapsto{g} G/G]$ mod $\tilde{J}_U$, with the notation introduced there, and it follows from lemma 5.8 that $\tau_U$ is onto. On the other hand, to prove injectivity, we pick $x_1, x_2 \in I_U$ with $\tau(x_1) \equiv \tau(x_2) \mod \tilde{J}_U$. Suppose that $x_{i,U}$ has the form
\[ x_{i,U} = [Z \not\cong A_i \times G/e \xmod{f_i} B_i \times G/e \xmapsto{g} G/e] \]
for $i = 1, 2$. Let
\[ y_i = [Z \not\cong A_i \times G/e \xmod{f_i \times p_i} B \times G/U \xmapsto{g} G/G] \]
for $i = 1, 2$. Then by lemma 5.9 $y_i \equiv \tau(x_i)$ mod $\tilde{J}_U$ for $i = 1, 2$. It follows from lemma 5.8 that there exist bijections $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ and for every $a \in A_1$ there exists $g_a \in N_G(U)$ such that firstly $d_1(aa) = g_a d_1(a)$ and secondly, if $a_1, a_2 \in A$ satisfy that $f(a_1) = f(a_2)$, then $g_{a_1} U = g_{a_2} U$. Given $a \in A_1$, let $z_{i,a} \in \tilde{U}^G(X, G/e)$ denote the element $[X \not\cong A \times G/e \xmapsto{g} G/e \xmod{z_{i,a}} G/G]$, where $d_i(a, e) = d_i(a, e)$. Then $z_{2,a} = g_a z_{1,a}$ and $x_{i,U} = \sum_{b \in B_i} \left( \prod_{a \in f_i^{-1}(b)} z_{i,a} \right)$ for $i = 1, 2$, where an empty product is 1 and an empty sum is 0. We can conclude that $x_{1,U} - x_{2,U} \in \mathbb{I}_G(R)$, and hence $x_1 - x_2 \in \mathbb{I}_G(R) + \tilde{I}_U$. In the general case $\tau(x_1 - x_2) \equiv \tau(x_1 - x_2) \mod \tilde{J}_U$ we easily obtain that $x_1 - x_2 \equiv x_1 - x_2 \mod \mathbb{I}_G(R) + \tilde{I}_U$ by collecting the positive terms. 

Proof of theorem 5.2. We first note that $\tilde{I}_V = \sum_{U \subseteq V} I_U \cong \colim_{U \subseteq V} I_U \subseteq I_V$ and that $\tilde{J}_V = \sum_{U \subseteq V} J_U \cong \colim_{U \subseteq V} J_U \subseteq J_V$. The result now follows by induction on the cardinality of $V$ using the above proposition and the five lemma on the following map of short exact sequences:
\[
\begin{align*}
\tilde{I}_G(R(G/e)) + I_V \longrightarrow & \tilde{I}_G(R(G/e)) + I_V \longrightarrow \frac{\tilde{I}_G(R(G/e)) + I_V}{(\mathbb{I}_G(R(G/e)) + I_V)} \\
\downarrow & \downarrow & \downarrow \\
\tilde{J}_V \longrightarrow & J_V \longrightarrow J_V/\tilde{J}_V.
\end{align*}
\]
6. The Witt Tambara-functor

In this section we finally prove theorem \[ \square \] Given a subgroup \( H \leq G \) and an \( H \)-set \( X \), we can construct a \( G \times H \)-set \( G/e \times X \), where \( G \) acts by multiplication on the left on \( G/e \), and where \( h \cdot (g,x) := (gh^{-1}, hx) \). We let \( \text{ind}^G_H X \) denote the \( G \)-set \( H \backslash (G/e \times X) \).

**Lemma 6.1.** Let \( H \) be a subgroup of \( G \). The functor \( \text{ind}^G_H : \mathcal{F} \mathcal{I} \mathcal{N}^H \rightarrow \mathcal{F} \mathcal{I} \mathcal{N}^G \) induces functors \( \text{ind}^G_H : U^H \rightarrow U^G \), and \( \text{ind}^G_H : U^f_H \rightarrow U^f_G \).

**Proof.** Since the functor \( \text{ind}^G_H : \mathcal{F} \mathcal{I} \mathcal{N}^H \rightarrow \mathcal{F} \mathcal{I} \mathcal{N}^G \) preserves pull-back diagrams and exponential diagrams it induces a functor \( \text{ind}^G_H : U^H \rightarrow U^G \) that takes \( X \leftarrow A \rightarrow B \rightarrow Y \) to \( \text{ind}^G_H X \leftarrow \text{ind}^G_A \rightarrow \text{ind}^G_B \rightarrow \text{ind}^G_Y \). \( \square \)

Given a \( G \)-Tambara functor \( S \) we can construct an \( H \)-Tambara functor \( \text{res}^G_H S = S \circ \text{ind}^G_H \). Similarly, given an \( fG \)-Tambara functor \( R \) we can construct an \( fH \)-Tambara functor \( \text{res}^{fG}_H S = S \circ \text{ind}^{fG}_H \).

**Theorem 6.2.** Given an \( fG \)-Tambara functor \( R \), the Teichmüller homomorphism

\[
\tau : \mathbb{W}_H(\text{res}^{fG}_{fH}(R(H/e))) / \mathbb{I}_H(\text{res}^{fG}_{fH}(R(H/e))) \rightarrow \text{res}^G_H L_G(H/H)
\]

is an isomorphism.

**Proof of theorem \[ \square \]** If we consider \( R_0 = R(G/e) \) as an \( H \)-ring, then

\[
\mathbb{W}_H(\text{res}^{fG}_{fH}(H/e)) / \mathbb{I}_H(\text{res}^{fG}_{fH}(H/e)) \cong \mathbb{W}_H(R_0) / \mathbb{I}_H(R_0),
\]

and \( \text{res}^G_H L_G(H/H) = L_G R(G/H) \). Combining these observations with \[ 6.2 \] we obtain the statement of theorem \[ \square \]

**Lemma 6.3.** Let \( H \) be a subgroup of \( G \). The forgetful functor \( i^* : \mathcal{F} \mathcal{I} \mathcal{N}^G \rightarrow \mathcal{F} \mathcal{I} \mathcal{N}^H \) which takes a \( G \)-set \( Y \) to the same set considered as an \( H \)-set induces functors \( i^* : U^G \rightarrow U^H \) and \( i_f^* : U^{fG} \rightarrow U^{fH} \).

**Lemma 6.4.** Let \( H \) be a subgroup of \( G \). The functor \( i^* : U^G \rightarrow U^H \) is left adjoint to the functor \( \text{ind}^G_H : U^H \rightarrow U^G \) and the functor \( i_f^* : U^{fG} \rightarrow U^{fH} \) is left adjoint to the functor \( \text{ind}^{fG}_H : U^{fH} \rightarrow U^{fG} \).

**Proof.** We prove only the first part of the lemma. Given \( X \leftarrow A \rightarrow B \rightarrow G \times_H Y \) in \( U^G(X, G \times_H Y) \) we construct an element in \( U^H(i^*X, Y) \) by the following diagram where the two squares furthest to the right are pull-back squares:

\[
i^*X \leftarrow A_H \longrightarrow B_H \longrightarrow H \times_H Y
\]

\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\]

\[
X \leftarrow A \longrightarrow B \longrightarrow G \times_H Y.
\]

Conversely, given \( i^*X \leftarrow E \rightarrow F \rightarrow Y \) in \( U^H(i^*X, Y) \) we construct the element \( X \leftarrow G \times_H E \rightarrow G \times_H F \rightarrow G \times_H Y \) in \( U^G(X, G \times_H Y) \). Here the arrow pointing to the left is the composite \( G \times_H E \rightarrow G \times_H i^*X \rightarrow X \). We leave it to the reader to check that the maps are inverse bijections in an adjunction. \( \square \)
We have the following commutative diagram of categories:

\[
\begin{array}{ccc}
U^f_G & \overset{i^*_G}{\longrightarrow} & U^f_H \\
\downarrow & & \downarrow \\
U^G & \overset{i^*}{\longrightarrow} & U^H,
\end{array}
\]

where the vertical functors are the natural inclusions. Since \( \text{ind}_H^G \) is right adjoint to \( i^* \), \( \text{res}_H^G \) is left adjoint to \([i^*, \mathcal{E}ns]_0\) (see for example [Sch], Proposition 16.6.3.) Similarly \( \text{res}_f^G \) is left adjoint to \([i^*_f, \mathcal{E}ns]_0\). From the commutative diagram of functor categories

\[
\begin{array}{ccc}
[U^f_G, \mathcal{E}ns]_0 & \overset{[i^*_G, \mathcal{E}ns]_0}{\longleftarrow} & [U^f_H, \mathcal{E}ns]_0 \\
\uparrow & & \uparrow \\
[U^G, \mathcal{E}ns]_0 & \overset{[i^*, \mathcal{E}ns]_0}{\longleftarrow} & [U^H, \mathcal{E}ns]_0,
\end{array}
\]

where the vertical maps are the forgetful functors induced by the inclusions \( U^f_G \subseteq U^G \) and \( U^f_H \subseteq U^H \) we can conclude that there is a natural isomorphism \( \text{res}_H^G L_G \cong L_H \text{res}_f^G \).

**Proof of theorem 6.2.** By theorem 3.6 we have an isomorphism

\[
\mathbb{W}_H(\text{res}_f^G R(H/e))/\mathbb{I}_H(\text{res}_f^G R(H/e)) \overset{\tau}{\longrightarrow} L_H \text{res}_f^G R(H/H) \cong \text{res}_H^G L_G R(H/H).
\]

\[
\square
\]

**References**

[AR] J. Adamek, J. Rosicky, *Locally Presentable and Accessible Categories*. London Mathematical Society Lecture Note Series 189, (1994).

[Be] P. Berthelot, *Cohomologie cristalline des schemas de caracteristique p > 0*. Lecture Notes in Mathematics. 407. (1974).

[BV] J. Boardman, R. Vogt, *Homotopy invariant algebraic structures on topological spaces*. Springer-Verlag, Berlin, (1973), Lecture Notes in Mathematics, Vol. 347.

[Bo] F. Borceux, *Handbook of categorical algebra 2*. Cambridge University Press, Cambridge (1994), Categories and Structures.

[Br] M. Brun, *Equivariant stable homotopy and Tambara functors*. In preparation.

[DS] A. Dress, C. Siebeneicher *The Burnside Ring of Profinite Groups and the Witt Vector Construction*, Advances in Math. 70 (1988).

[DS2] A. Dress, C. Siebeneicher *The Burnside ring of the infinite cyclic group and its relations to the necklace algebra, \( \lambda \)-rings, and the universal ring of Witt vectors*. Adv. Math. 78, No.1, 1-41 (1989).

[Gra] J. Graham, *Generalised Witt vectors*. Adv. Math. 99, No.2, 248-263 (1993).

[Li] H. Lindner, *A remark on Mackey-functors*. Manuscr. Math. 18, 273-278 (1976)

[MR] N. Metropolis, G-C. Rota, *Witt vectors and the algebra of necklaces*. Adv. Math. 50, 95-125 (1983)

[Re] C. Rezk, *Every homotopy theory of simplicial algebras admits a proper model*. Topology Appl. 119, No.1, 65-94 (2002).

[Sch] H. Schubert, *Categories*. Springer-Verlag (1972).

[SW] H. Schmid, E. Witt *Unverzweigte abelsche Körper vom Exponenten \( p^n \) über einem algebraischen Funktionenkörper der Charakteristik p*. J. Reine Angew. Math. 176, 168-173 (1936).
[Ta] D. Tambara On multiplicative transfer. Commun. Algebra 21, No.4, 1393-1420 (1993).
[Wi] E. Witt, Zyklische Körper und Algebren der Charakteristik p vom Grad $p^n$. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik p. J. Reine Angew. Math. 176, 126-140 (1936).

E-mail address: brun@mathematik.uni-osnabrueck.de

FB Mathematik/Informatik, Universität Osnabrück, 49069 Osnabrück, Germany