The algebra \( U_q(\mathfrak{sl}_2) \) in disguise

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Abstract

We discuss a connection between the algebra \( U_q(\mathfrak{sl}_2) \) and the tridiagonal pairs of \( q \)-Racah type. To describe the connection, let \( x, y^{\pm 1}, z \) denote the equitable generators for \( U_q(\mathfrak{sl}_2) \). Let \( U_q^\vee \) denote the subalgebra of \( U_q(\mathfrak{sl}_2) \) generated by \( x, y^{\pm 1}, z \). Using a tridiagonal pair of \( q \)-Racah type we construct two finite-dimensional \( U_q^\vee \)-modules. The constructions yield two nonstandard presentations of \( U_q^\vee \) by generators and relations. These presentations are investigated in detail.

Keywords. Quantum group; quantum enveloping algebra; equitable presentation; tridiagonal pair.

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1 Introduction

This paper is about a connection between the algebra \( U_q(\mathfrak{sl}_2) \) and a linear-algebraic object called a tridiagonal pair \([7]\). To describe the connection, we briefly recall the algebra \( U_q(\mathfrak{sl}_2) \) \([12,13]\). We will use the equitable presentation, which was introduced in \([11]\). Given a field \( \mathbb{F} \), the \( \mathbb{F} \)-algebra \( U_q(\mathfrak{sl}_2) \) has generators \( x, y^{\pm 1}, z \) and relations

\[
qxy - q^{-1}yx = 1, \quad qyz - q^{-1}zy = 1, \quad qzx - q^{-1}xz = 1.
\]

Let \( U_q^\vee \) denote the subalgebra of \( U_q(\mathfrak{sl}_2) \) generated by \( x, y^{\pm 1}, z \). We now briefly turn our attention to tridiagonal pairs. Let \( V \) denote a vector space over \( \mathbb{F} \) with finite positive dimension. Roughly speaking, a tridiagonal pair on \( V \) is a pair of diagonalizable \( \mathbb{F} \)-linear maps on \( V \), each acting on the eigenspaces of the other one in a block-tridiagonal fashion. There is a general family of tridiagonal pairs said to have \( q \)-Racah type \([8, \text{Definition 3.1}]\). Let \( A, A^* \) denote a tridiagonal pair on \( V \) that has \( q \)-Racah type. Associated with this tridiagonal pair are two additional \( \mathbb{F} \)-linear maps \( K : V \to V \) \([4, \text{Definition 3.1}]\) and \( B : V \to V \) \([4, \text{Definition 3.2}]\), which are roughly described as follows. The maps \( K, B \) are diagonalizable, and their eigenspace decompositions are among the split decompositions of \( V \) with respect to \( A, A^* \) \([7, \text{Section 4}]\). According to \([4, \text{Lemma 3.6, Theorem 9.9}]\), there exists \( a \in \mathbb{F} \setminus \{0, 1, -1\} \) such that

\[
qKA - q^{-1}AK = aK^2 + a^{-1}I, \quad qBA - q^{-1}AB = a^{-1}B^2 + aI,
\]

\[
aK^2 - a^{-1}q^{-1}KB = aq^{-1} - a^{-1}q^{-1}BK + a^{-1}B^2 = 0.
\]
The main purpose of this paper is to explain what the above three equations have to do with $U_q(\mathfrak{sl}_2)$. To summarize the answer, consider an $\mathbb{F}$-algebra defined by generators and relations; the generators are symbols $K, B, A$ and the relations are the above three equations. We are going to show that this algebra is isomorphic to $U_q^\vee$. The existence of this isomorphism yields a presentation of $U_q^\vee$ by generators and relations. There is another presentation of $U_q^\vee$ with similar features, obtained by inverting $q, a, K, B$. In our main results, we describe these two presentations in a comprehensive way that places them in a wider context. We will summarize our results shortly.

In the theory of $U_q(\mathfrak{sl}_2)$, it is convenient to introduce elements $\nu_x, \nu_y, \nu_z$ defined by

$$\nu_x = q(1 - yz), \quad \nu_y = q(1 - zx), \quad \nu_z = q(1 - xy).$$

One significance of $\nu_x, \nu_y, \nu_z$ is that

$$x\nu_y = q^2\nu_xy, \quad x\nu_z = q^{-2}\nu_zx,$$

$$y\nu_z = q^2\nu_yz, \quad y\nu_x = q^{-2}\nu_xy,$$

$$z\nu_x = q^2\nu_zx, \quad z\nu_y = q^{-2}\nu_yz.$$  

We now summarize our results. Fix a nonzero $a \in \mathbb{F}$ such that $a^2 \neq 1$. Define elements $X, Z$ of $U_q(\mathfrak{sl}_2)$ by

$$X = a^{-2}x + (1 - a^{-2})y^{-1}, \quad Z = a^2z + (1 - a^2)y^{-1}.$$

One readily verifies that

$$a^{-1}x + az = aX + a^{-1}Z. \quad (1)$$

Let $A$ denote the common value of (1). We consider two antiautomorphisms of $U_q(\mathfrak{sl}_2)$, denoted $\tau$ and $\dagger$. The map $\tau$ swaps $x \leftrightarrow z$ and fixes $y \pm 1$. The map $\dagger$ swaps $x \leftrightarrow X$, $z \leftrightarrow X$ and fixes each of $A, y \pm 1$. The composition of $\dagger$ and $\tau$ is an automorphism of $U_q(\mathfrak{sl}_2)$ which we denote by $\sigma$. The map $\sigma$ sends $x \mapsto X$, $z \mapsto Z$ and fixes $y \pm 1$. We give four presentations of $U_q(\mathfrak{sl}_2)$ by generators and relations. They are the Chevalley presentation, with generators $e, f, k \pm 1$; the equitable presentation, with generators $x, y \pm 1, z$; the modified Chevalley presentation, with generators $\nu_x, y \pm 1, \nu_z$; and the invariant presentation, with generators $A, y \pm 1$.

We show how these presentations are related. We show how the maps $\tau, \dagger, \sigma$ look in each presentation. We show how the Casimir element looks in each presentation. We then consider the subalgebra $U_q^\vee$. We give four presentations of $U_q^\vee$ by generators and relations. For these presentations the generators are (i) $x, y^{-1}, z$; (ii) $X, y^{-1}, Z$; (iii) $x, X, A$; (iv) $z, Z, A$. We explain how these presentations are related to each other and the four presentations of $U_q(\mathfrak{sl}_2)$. We show that $U_q^\vee$ is invariant under $\tau, \dagger, \sigma$. We show how $\tau, \dagger, \sigma$ act on the generating sets (i)–(iv). We compare the finite-dimensional modules for $U_q^\vee$ and $U_q(\mathfrak{sl}_2)$. Given a tridiagonal pair of $q$-Racah type we construct two finite-dimensional $U_q^\vee$-modules; in these constructions the presentations (iii), (iv) come up in a natural way. Presentation (iv) is the one we discussed at the outset, if we identify $z \mapsto K$, $Z \mapsto B$, $A \mapsto A$. Given a tridiagonal pair of $q$-Racah type, there is an associated linear map $\psi$ called the Bockting operator [34]. The map $\psi$ plays the following role in our theory. Consider our two $U_q^\vee$-modules.
constructed from the given tridiagonal pair. We show that each of these $U_q^\vee$-modules extends to a $U_q(\mathfrak{sl}_2)$-module. We show that on one of the $U_q(\mathfrak{sl}_2)$-modules $\nu_x$ acts as $a^{-1}\psi$, and on the other $U_q(\mathfrak{sl}_2)$-module $\nu_z$ acts as $a\psi$.

At the end of the paper we list some open problems concerning $U_q^\vee$ and related topics.

2 The algebra $U_q(\mathfrak{sl}_2)$

We now begin our formal argument. We adopt the following conventions. An algebra is meant to be associative and have a 1. A subalgebra has the same 1 as the parent algebra. Let $\mathbb{F}$ denote a field and fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$.

We now recall the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$. For background information on this topic we refer the reader to the books by Jantzen [12] and Kassel [13].

Definition 2.1 [12, Section 1.1]. Let $U_q(\mathfrak{sl}_2)$ denote the $\mathbb{F}$-algebra defined by generators $e, f, k^{\pm 1}$ and relations

\begin{align*}
kk^{-1} &= k^{-1}k = 1, \\
kek^{-1} &= q^2 e, \\
kfk^{-1} &= q^{-2} f, \\
ef - fe &= \frac{k - k^{-1}}{q - q^{-1}}.
\end{align*}

The presentation of $U_q(\mathfrak{sl}_2)$ given in Definition 2.1 is called the Chevalley presentation.

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Lemma 2.2 [12, Theorem 1.5]. The following is a basis for the $\mathbb{F}$-vector space $U_q(\mathfrak{sl}_2)$:

\[ e^r k^s f^t \quad r, t \in \mathbb{N}, \quad s \in \mathbb{Z}. \]

Definition 2.3 [12, Section 2.7]. Define

\[ \Lambda = (q - q^{-1})^2 ef + q^{-1} k + q k^{-1}. \]

$\Lambda$ is called the Casimir element of $U_q(\mathfrak{sl}_2)$.

Lemma 2.4 [12, Lemma 2.7, Proposition 2.18]. The center of $U_q(\mathfrak{sl}_2)$ contains $\Lambda$. This center has a basis $\{\Lambda^i\}_{i \in \mathbb{N}}$, provided that $q$ is not a root of unity.

Lemma 2.5 For all nonzero $a \in \mathbb{F}$ there exists a unique automorphism $\sigma_a$ of $U_q(\mathfrak{sl}_2)$ that sends

\[ e \mapsto a^{-2} e, \quad f \mapsto a^2 f, \quad k^{\pm 1} \mapsto k^{\pm 1}. \]

Moreover $\sigma_{a^{-1}} = \sigma_{a^{-1}}$.

Proof: Use (2)–(4).
Lemma 2.6 For all nonzero \( a \in \mathbb{F} \) the automorphism \( \sigma_a \) fixes \( \Lambda \).

Proof: Use Definition 2.3 and (6). □

The automorphisms \( \sigma_q \) and \( \sigma_{q^{-1}} \) act on \( U_q(\mathfrak{sl}_2) \) as follows.

Lemma 2.7 For \( \xi \in U_q(\mathfrak{sl}_2) \), \( \sigma_q \) sends \( \xi \mapsto k^{-1}\xi k \) and \( \sigma_{q^{-1}} \) sends \( \xi \mapsto k\xi k^{-1} \).

Proof: Use (3) and (6). □

3 The equitable presentation of \( U_q(\mathfrak{sl}_2) \)

In this section we recall the equitable presentation of \( U_q(\mathfrak{sl}_2) \) [11]. For background information on this topic we refer the reader to [11], [17], [18]. See also [1], [2], [19].

Lemma 3.1 [11, Theorem 2.1]. The \( \mathbb{F} \)-algebra \( U_q(\mathfrak{sl}_2) \) has a presentation by generators \( x, y^{\pm 1}, z \) and relations

\[
\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \quad \frac{yy^{-1} = y^{-1}y = 1}{q - q^{-1}}, \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.
\]

An isomorphism with the presentation in Definition 2.7 sends

\[
x \mapsto k^{-1} - k^{-1}eq(q - q^{-1}),
\]

\[
y^{\pm 1} \mapsto k^{\pm 1},
\]

\[
z \mapsto k^{-1} + f(q - q^{-1}).
\]

The inverse isomorphism sends

\[
e \mapsto (1 - yx)q^{-1}(q - q^{-1})^{-1},
\]

\[
k^{\pm 1} \mapsto y^{\pm 1},
\]

\[
f \mapsto (z - y^{-1})(q - q^{-1})^{-1}.
\]

The presentation for \( U_q(\mathfrak{sl}_2) \) given in Lemma 3.1 is called the equitable presentation.

Note 3.2 From now on, we identify the copy of \( U_q(\mathfrak{sl}_2) \) from Definition 2.1 with the copy of \( U_q(\mathfrak{sl}_2) \) from Lemma 3.1 via the isomorphism in Lemma 3.1.

Lemma 3.3 [18, Lemma 10.7]. The following is a basis for the \( \mathbb{F} \)-vector space \( U_q(\mathfrak{sl}_2) \):

\[
x^r y^s z^t \quad r, t \in \mathbb{N}, \quad s \in \mathbb{Z}.
\]

We now describe the Casimir element \( \Lambda \) from the equitable point of view.
Lemma 3.4 ([18] Lemma 2.15). The element $\Lambda$ is equal to each of the following:

\[
    \begin{align*}
        qx + q^{-1}y + qz - qxyz, & \quad q^{-1}x + qy + q^{-1}z - q^{-1}zyx, \\
        qy + q^{-1}z + qx - qyxz, & \quad q^{-1}y + qx + q^{-1}z - q^{-1}xzy, \\
        qz + q^{-1}x + qy - qzxy, & \quad q^{-1}z + qx + q^{-1}y - q^{-1}yxz.
    \end{align*}
\]

We now describe the automorphisms $\sigma_a$ of $U_q(\mathfrak{sl}_2)$ from the equitable point of view.

Lemma 3.5 For all nonzero $a \in \mathbb{F}$ the automorphism $\sigma_a$ sends

\[
    x \mapsto a^{-2}x + (1 - a^{-2})y^{-1}, \quad y^\pm \mapsto y^\pm, \quad z \mapsto a^2z + (1 - a^2)y^{-1}.
\]

Proof: Use (6) and the isomorphisms in Lemma 3.1.

Lemma 3.6 We have

\[
    \begin{align*}
        y^{-1}xy &= q^{-2}x + (1 - q^{-2})y^{-1}, \quad yxy^{-1} = q^2x + (1 - q^2)y^{-1}, \\
        yzy^{-1} &= q^{-2}z + (1 - q^{-2})y^{-1}, \quad y^{-1}zy = q^2z + (1 - q^2)y^{-1}.
    \end{align*}
\]

Proof: These equations are reformulations of the first two relations in (8). They can also be obtained from Lemma 2.7, by identifying $k = y$ and using Lemma 3.5.

We will be discussing antiautomorphisms, so let us recall that concept. Given an $\mathbb{F}$-algebra $\mathcal{A}$, a map $\gamma : \mathcal{A} \to \mathcal{A}$ is called an antiautomorphism whenever $\gamma$ is an isomorphism of $\mathbb{F}$-vector spaces and $(uv)\gamma = v\gamma u\gamma$ for all $u, v \in \mathcal{A}$.

Lemma 3.7 There exists a unique antiautomorphism $\tau$ of $U_q(\mathfrak{sl}_2)$ that sends

\[
    x \mapsto z, \quad y^\pm \mapsto y^\pm, \quad z \mapsto x.
\]

Moreover $\tau^2 = 1$.

Proof: Let $S$ denote the set of defining relations for $U_q(\mathfrak{sl}_2)$ given in Lemma 3.1. For each relation $r \in S$ let $r'$ denote the equation obtained by inverting the order of multiplication and swapping $x, z$. The map $r \mapsto r'$ permutes $S$, and therefore the antiautomorphism $\tau$ exists. The antiautomorphism $\tau$ is unique since $x, y^\pm, z$ generate $U_q(\mathfrak{sl}_2)$.

We now consider how $\tau$ acts on $e, f, k^\pm$.

Lemma 3.8 The antiautomorphism $\tau$ sends

\[
    e \mapsto -qkf, \quad f \mapsto -q^{-1}ek^{-1}, \quad k^\pm \mapsto k^\pm.
\]

Proof: Use (10) and the isomorphisms in Lemma 3.1.
Lemma 3.9  The antiautomorphism $\tau$ fixes $\Lambda$.

Proof: Use Lemma 3.4 and (10).

Lemma 3.10  For all nonzero $a \in \mathbb{F}$, $\tau \sigma_a = \sigma_a^{-1} \tau$.

Proof: Use Lemma 2.5 and Lemma 3.8.

We mention some identities for later use.

Lemma 3.11  We have

\[
\begin{align*}
    x^2y - (q^2 + q^{-2})xyx + yx^2 & = -x, \\
    y^2x - (q^2 + q^{-2})yxy + xy^2 & = -y, \\
    y^2z - (q^2 + q^{-2})yzy + zy^2 & = -y, \\
    z^2y - (q^2 + q^{-2})zxy + yz^2 & = -z, \\
    z^2 x - (q^2 + q^{-2})zxx + xz^2 & = -z, \\
    x^2z - (q^2 + q^{-2})xzx + zx^2 & = -x.
\end{align*}
\]  

(11) \( (12) \) \( (13) \)

Proof: We verify the equation on the left in (11). Observe that

\[
\frac{x^2y - (q^2 + q^{-2})xyx + yx^2}{q - q^{-1}} = q^{-1} x \frac{qxy - q^{-1} yx}{q - q^{-1}} - q \frac{qxy - q^{-1} yx}{q - q^{-1}} x.
\]

In the above equation, each fraction on the right is equal to 1, so the whole expression on the right is equal to \(-(q - q^{-1})x\). This yields the equation on the left in (11). The remaining equations are similarly verified.

\[\square\]

4  A $\mathbb{Z}_2$-grading of $U_q(\mathfrak{sl}_2)$

We will be discussing the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ of order 2. Given an $\mathbb{F}$-algebra $A$, by a $\mathbb{Z}_2$-grading of $A$ we mean a direct sum decomposition $A = \sum_{i \in \mathbb{Z}_2} A_i$ such that $A_i A_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}_2$. In this case $A_0$ is a subalgebra of $A$.

We now describe a $\mathbb{Z}_2$-grading of $U_q(\mathfrak{sl}_2)$.

Definition 4.1  Referring to the basis (9) of $U_q(\mathfrak{sl}_2)$, a basis element $x^r y^s z^t$ will be called even (resp. odd) whenever $r + s + t$ is even (resp. odd). Let $U_{even}$ (resp. $U_{odd}$) denote the subspace of $U_q(\mathfrak{sl}_2)$ spanned by the even (resp. odd) elements in (9). By construction

\[U_q(\mathfrak{sl}_2) = U_{even} + U_{odd}\]  

(14)

Lemma 4.2  The decomposition (14) is a $\mathbb{Z}_2$-grading of $U_q(\mathfrak{sl}_2)$. In particular $U_{even}$ is a subalgebra of $U_q(\mathfrak{sl}_2)$.
Proof: Recall the defining relations (7), (8) for the equitable presentation of $U_q(\mathfrak{sl}_2)$. In these relations every term has even degree. \qed

**Definition 4.3** Let $U'_q$ denote the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by $x, y, z$.

**Lemma 4.4** [18, Lemma 10.8]. The following is a basis for the $\mathbb{F}$-vector space $U'_q$:

$$x^r y^s z^t \quad r, s, t \in \mathbb{N}. \quad (15)$$

**Definition 4.5** Let $U'_{\text{even}}$ (resp. $U'_{\text{odd}}$) denote the subspace of $U'_q$ spanned by the even (resp. odd) elements in (15). By construction

$$U'_q = U'_{\text{even}} + U'_{\text{odd}} \quad \text{(direct sum).} \quad (16)$$

**Lemma 4.6** Referring to Definition 4.5,

$$U'_{\text{even}} = U'_q \cap U_{\text{even}}, \quad U'_{\text{odd}} = U'_q \cap U_{\text{odd}}. \quad (17)$$

Moreover the decomposition (16) is a $\mathbb{Z}_2$-grading of $U'_q$.

**Proof:** By construction. \qed

**Lemma 4.7** For all nonzero $a \in \mathbb{F}$ the automorphism $\sigma_a$ leaves invariant both $U_{\text{even}}$ and $U_{\text{odd}}$. However $\sigma_a$ does not leave $U'_q$ invariant, unless $a^2 = 1$.

**Proof:** Use Lemma 3.5. \qed

**Lemma 4.8** Each of the following is invariant under the antiautomorphism $\tau$:

$$U_{\text{even}}, \quad U_{\text{odd}}, \quad U'_q, \quad U'_{\text{even}}, \quad U'_{\text{odd}}.$$  

**Proof:** Use Lemma 3.7. \qed

**Lemma 4.9** The Casimir element $\Lambda$ is contained in $U'_{\text{odd}}$.

**Proof:** Use Lemma 3.4. \qed

**Corollary 4.10** Assume that $q$ is not a root of unity. Then the center of $U_q(\mathfrak{sl}_2)$ is contained in $U'_q$.

**Proof:** By Lemma 2.4 and Lemma 4.9. \qed
5 The elements $\nu_x, \nu_y, \nu_z$ of $U_q(\mathfrak{sl}_2)$

The elements $\nu_x, \nu_y, \nu_z$ of $U_q(\mathfrak{sl}_2)$ were introduced in [18]. In this section, we use these elements to obtain a generating set for the subalgebras $U_{even}$ and $U'_{even}$.

Reformulating the relations (8), we obtain

$$ q(1 - xy) = q^{-1}(1 - yx), \quad q(1 - yz) = q^{-1}(1 - yz), \quad q(1 - zx) = q^{-1}(1 - xz). $$

Following [18, Definition 3.1] we define

$$ \nu_x = q(1 - yz) = q^{-1}(1 - yz), \quad \nu_y = q(1 - zx) = q^{-1}(1 - zx), \quad \nu_z = q(1 - xy) = q^{-1}(1 - xy). $$

By Lemma 3.1 and Note 3.2,

$$ \nu_x = -q(q - q^{-1})kf, \quad \nu_z = (q - q^{-1})e. $$

Lemma 5.1 [18 Lemma 3.5]. We have

$$ xy = q^2 \nu_y x, \quad xy = q^{-2} \nu_x y, \quad \nu_x = q^2 \nu_x z, \quad \nu_y = q^2 \nu_y z. $$

We are going to show that $U_{even}$ (resp. $U'_{even}$) is generated by $\nu_x, \nu_y, \nu_z, y^{-2}$ (resp. $\nu_x, \nu_y, \nu_z$).

Lemma 5.2 The following relations hold in $U_q(\mathfrak{sl}_2)$:

$$ xy = 1 - q^{-1} \nu_z, \quad yx = 1 - q \nu_z, $$

$$ yz = 1 - q^{-1} \nu_x, \quad zy = 1 - q \nu_x, $$

$$ zx = 1 - q^{-1} \nu_y, \quad xz = 1 - q \nu_y. $$

Proof: These relations are reformulations of (18)–(20).

Lemma 5.3 [18 Lemma 3.10]. The following relations hold in $U_q(\mathfrak{sl}_2)$:

$$ x^2 = 1 - \frac{q \nu_y \nu_z - q^{-1} \nu_z \nu_y}{q - q^{-1}}, $$

$$ y^2 = 1 - \frac{q \nu_z \nu_x - q^{-1} \nu_x \nu_z}{q - q^{-1}}, $$

$$ z^2 = 1 - \frac{q \nu_x \nu_y - q^{-1} \nu_y \nu_x}{q - q^{-1}}. $$

Proposition 5.4 The following (i), (ii) hold.
(i) The $\mathbb{F}$-algebra $U_{\text{even}}$ is generated by $\nu_x, \nu_y, \nu_z, y^{-2}$;

(ii) The $\mathbb{F}$-algebra $U'_{\text{even}}$ is generated by $\nu_x, \nu_y, \nu_z$.

**Proof:** (i) Let $W$ denote the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by $\nu_x, \nu_y, \nu_z, y^{-2}$. We show that $W = U_{\text{even}}$. Certainly $W \subseteq U_{\text{even}}$, since each of $\nu_x, \nu_y, \nu_z, y^{-2}$ is contained in $U_{\text{even}}$ by Definition 4.1 and (18)–(20). We now show $W \supseteq U_{\text{even}}$. By Definition 4.1, $U_{\text{even}}$ is spanned by the even elements in the basis (9). Let $x^r y^s z^t$ denote an even element in this basis. Then $r + s + t = 2n$ is even. Write $x^r y^s z^t = g_1 g_2 \cdots g_n$ such that $g_i$ is among $x^2, xy, xy^{-1}, xz, y^2, y^{-2}, yz, y^{-1}z, z^2$ (25) for $1 \leq i \leq n$. Note that $xy^{-1} = xyy^{-2}$ and $y^{-1}z = y^{-2}yz$. Now by Lemmas 5.2, 5.3 we see that $W$ contains each term in (25). Therefore $W$ contains $g_i$ for $1 \leq i \leq n$, so $W$ contains $x^r y^s z^t$. Consequently $W \supseteq U_{\text{even}}$, so $W = U_{\text{even}}$.

(ii) Similar to the proof of (i).

We now consider how the automorphisms $\sigma_a$ act on $\nu_x, \nu_y, \nu_z$.

**Lemma 5.5** For all nonzero $a \in \mathbb{F}$ the automorphism $\sigma_a$ sends

$$\nu_x \mapsto a^2 \nu_x, \quad \nu_y \mapsto \nu_y + (a - a^{-1})(aq^2 \nu_x - a^{-1}q^{-2} \nu_z)y^{-2}, \quad \nu_z \mapsto a^{-2} \nu_z.$$

**Proof:** To get the action of $\sigma_a$ on $\nu_x$ and $\nu_z$, use Lemma 2.5 and (21). Now consider the action of $\sigma_a$ on $\nu_y$. By Lemma 3.4 and Lemma 5.2

$$\Lambda y = qzy + q^{-1}xy + q(1-\nu_z)y^2 = q(1-q\nu_x) + q^{-1}(1-q^{-1}\nu_z) + \nu_y y^2.$$

In this equation apply $\sigma_a$ to each side and use the fact that $\sigma_a$ fixes each of $\Lambda, y$. $\square$

We now consider how the antiautomorphism $\tau$ acts on $\nu_x, \nu_y, \nu_z$.

**Lemma 5.6** The antiautomorphism $\tau$ sends

$$\nu_x \mapsto \nu_z, \quad \nu_y \mapsto \nu_y, \quad \nu_z \mapsto \nu_x.$$

**Proof:** Use Lemma 3.7 and (18)–(20). $\square$

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6 A variation on the Chevalley presentation of $U_q(\mathfrak{sl}_2)$

In the previous section we saw that $U_q(\mathfrak{sl}_2)$ is not generated by $\nu_x, \nu_y, \nu_z$. In this section we show that $U_q(\mathfrak{sl}_2)$ is generated by $\nu_x, y^{\pm 1}, \nu_z$. Using these generators, we give a presentation of $U_q(\mathfrak{sl}_2)$ by generators and relations. As we will see, this presentation is quite close to the Chevalley presentation.
Lemma 6.1 The elements $\nu_x, y^{\pm 1}, \nu_z$ together generate $U_q(\mathfrak{sl}_2)$. Moreover
\[
x = y^{-1} - q y^{-1}\nu_x = y^{-1} - q^{-1}\nu_x y^{-1}, \quad (26)
\]
\[
z = y^{-1} - q\nu_x y^{-1} = y^{-1} - q^{-1}y^{-1}\nu_x. \quad (27)
\]

Proof: The relations (26), (27) follow from (22), (23). The first assertion follows from (26), (27) and since $x, y^{\pm 1}, z$ generate $U_q(\mathfrak{sl}_2)$. \hfill \Box

Theorem 6.2 The $\mathbb{F}$-algebra $U_q(\mathfrak{sl}_2)$ has a presentation by generators $\nu_x, y^{\pm 1}, \nu_z$ and relations
\[
\nu_x \mapsto -q(q - q^{-1})kf,
\]
\[
y^{\pm 1} \mapsto k^{\pm 1},
\]
\[
\nu_z \mapsto (q - q^{-1})e.
\]

The inverse isomorphism sends
\[
e \mapsto (q - q^{-1})^{-1}\nu_z,
\]
\[
k^{\pm 1} \mapsto y^{\pm 1},
\]
\[
f \mapsto -q^{-1}(q - q^{-1})^{-1}y^{-1}\nu_x.
\]

Proof: One checks that each map above is an $\mathbb{F}$-algebra homomorphism, and that these maps are inverses. Therefore each map is an $\mathbb{F}$-algebra isomorphism. \hfill \Box

Definition 6.3 The presentation of $U_q(\mathfrak{sl}_2)$ given in Theorem 6.2 will be called the modified Chevalley presentation.

7 The element $A$ of $U_q(\mathfrak{sl}_2)$

In Section 9 we will give another presentation of $U_q(\mathfrak{sl}_2)$ by generators and relations. To prepare for this, we introduce the element $A$ of $U_q(\mathfrak{sl}_2)$.

Up until the end of Section 11, fix a nonzero $a \in \mathbb{F}$ such that $a^2 \neq 1$. Referring to Lemma 2.5 we abbreviate $\sigma = \sigma_a$.

Definition 7.1 Let $X$ (resp. $Z$) denote the image of $x$ (resp. $z$) under $\sigma$. By Lemma 3.5
\[
X = a^{-2}x + (1 - a^{-2})y^{-1}, \quad Z = a^2z + (1 - a^2)y^{-1}. \quad (30)
\]
Lemma 7.2 We have
\[
\frac{qXy - q^{-1}yX}{q - q^{-1}} = 1, \quad \frac{qyZ - q^{-1}Zy}{q - q^{-1}} = 1, \quad \frac{qZX - q^{-1}XZ}{q - q^{-1}} = 1.
\]
Proof: For each relation in (8), apply $\sigma$ to each side.

Lemma 7.3 We have
\[
y^{-1} = \frac{aX - a^{-1}x}{a - a^{-1}} = \frac{az - a^{-1}Z}{a - a^{-1}}.
\]
Proof: Use (30).

Corollary 7.4 We have
\[
a^{-1}x + az = aX + a^{-1}Z.
\] (31)
Proof: Use Lemma 7.3.

Definition 7.5 Define
\[
A = a^{-1}x + az = aX + a^{-1}Z.
\] (32)
Our next general goal is to show that $A, y^\pm 1$ together generate $U_q(\mathfrak{sl}_2)$.

Lemma 7.6 We have
\[
Ay = a(1 - q\nu_x) + a^{-1}(1 - q^{-1}\nu_z), \quad (33)
yA = a(1 - q^{-1}\nu_x) + a^{-1}(1 - q\nu_z). \quad (34)
\]
Proof: Use $A = a^{-1}x + az$ and Lemma 5.2.

Lemma 7.7 We have
\[
\nu_x = a^{-1}a + a^{-1} - a^{-1}\frac{qAy - q^{-1}yA}{q^2 - q^{-2}}, \quad (35)

\nu_z = a^{-1}a + a^{-1} - a^{-1}\frac{qyA - q^{-1}Ay}{q^2 - q^{-2}}. \quad (36)
\]
Proof: Solve the equations (33), (34) for $\nu_x, \nu_z$.

Lemma 7.8 The elements $A, y^\pm 1$ together generate $U_q(\mathfrak{sl}_2)$. 

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Proof: By Lemmas 6.1 and 7.7.

We mention a variation on Lemma 7.6.

Lemma 7.9 We have

\[ yAy^{-1} = a^{-1}q^2x + aq^{-2}z + (q - q^{-1})(aq^{-1} - a^{-1}q)y^{-1}, \]

\[ y^{-1}Ay = a^{-1}q^{-2}x + aq^2z + (q - q^{-1})(a^{-1}q^{-1} - aq)y^{-1}. \]

Proof: To verify each equation, eliminate \( A \) using \( A = a^{-1}x + az \), and simplify the result using Lemma 3.6.

Lemma 7.10 We have

\[ yAy^{-1} - (q^2 + q^{-2})A + y^{-1}Ay = -(a + a^{-1})(q - q^{-1})^2y^{-1}. \]  \( (37) \)

Proof: To verify (37), evaluate the left-hand side using Lemma 7.9.

Corollary 7.11 We have

\[ y^2A - (q^2 + q^{-2})yAy + Ay^2 = -(a + a^{-1})(q - q^{-1})^2y. \]  \( (38) \)

Proof: In equation (37) multiply each term on the left and right by \( y \).

We now give \( x, z \) in terms of \( A, y^{\pm 1} \).

Lemma 7.12 We have

\[ x = A \frac{aq^{-2}}{q^2 - q^{-2}} + yAy^{-1} \frac{a}{q^2 - q^{-2}} + y^{-1}a \frac{(a^{-1}q - aq^{-1})}{q + q^{-1}} \]

\[ = A \frac{aq^2}{q^2 - q^{-2}} + y^{-1}Ay \frac{a}{q^2 - q^{-2}} + y^{-1}a \frac{(a^{-1}q^{-1} - aq)}{q + q^{-1}} \]

and

\[ z = A \frac{a^{-1}q^2}{q^2 - q^{-2}} + yAy^{-1} \frac{a^{-1}}{q^2 - q^{-2}} + y^{-1}a^{-1} \frac{(aq^{-1} - a^{-1}q)}{q + q^{-1}} \]

\[ = A \frac{a^{-1}q^{-2}}{q^2 - q^{-2}} + y^{-1}Ay \frac{a^{-1}}{q^2 - q^{-2}} + y^{-1}a^{-1} \frac{(aq - a^{-1}q^{-1})}{q + q^{-1}}. \]

Proof: To verify these equations, eliminate \( yAy^{-1} \) and \( y^{-1}Ay \) using Lemma 7.9.

We now describe how the automorphism \( \sigma \) acts on \( A, y^{\pm 1} \).
Lemma 7.13 The automorphism $\sigma$ sends

$$ A \mapsto A \frac{a^{-2}q^2 - a^2q^{-2}}{q^2 - q^{-2}} + y^{-1}A \frac{a^2 - a^{-2}}{q^2 - q^{-2}} + y^{-1} \frac{(a^2 - a^{-2})(a^{-1}q - aq^{-1})}{q + q^{-1}}, $$

$$ y^\pm \mapsto y^\pm. $$

Proof: By construction $\sigma$ fixes $y^\pm$. To obtain the image of $A$ under $\sigma$, in the equation $A = a^{-1}x + az$ first eliminate $x, z$ using (26), (27). In the resulting equation apply $\sigma$ to each term and evaluate using Lemmas 5.5, 7.7. \hfill \Box

We now describe how the antiautomorphism $\tau$ acts on $A, y^\pm$.

Lemma 7.14 The antiautomorphism $\tau$ sends

$$ A \mapsto A \frac{a^{-2}q^2 - a^2q^{-2}}{q^2 - q^{-2}} + yA \frac{a^2 - a^{-2}}{q^2 - q^{-2}} + y^{-1} \frac{(a^2 - a^{-2})(a^{-1}q - aq^{-1})}{q + q^{-1}}, $$

$$ y^\pm \mapsto y^\pm. $$

Proof: Consider the image of $A$ under $\tau$. Recall that $A = a^{-1}x + az$, and that $\tau$ swaps $x, z$. Therefore $\tau$ sends $A \mapsto ax + a^{-1}z$. Evaluating $ax + a^{-1}z$ using Lemma 7.12 we find that the image of $A$ under $\tau$ is as claimed. Our assertion about $y$ is clear. \hfill \Box

Lemma 7.15 The Casimir element $\Lambda$ looks as follows in terms of $A, y^\pm$:

$$ \Lambda = A^2y - (q^2 + q^{-2})AYA + yA^2 + (q^2 - q^{-2})y + (a + a^{-1})(q - q^{-1})A. $$

Proof: Using $A = a^{-1}x + az$ we find that $A^2y - (q^2 + q^{-2})AYA + yA^2$ is equal to $a^{-2}$ times

$$ x^2y - (q^2 + q^{-2})xyx + yx^2 $$

plus $a^2$ times

$$ z^2y - (q^2 + q^{-2})zyz + yz^2 $$

plus

$$ yzx + zyx - (q^2 + q^{-2})xyz + xzy + yxz - (q^2 + q^{-2})zyx. $$

By Lemma 3.11 the expressions (39) and (40) are equal to $-(q - q^{-1})^2x$ and $-(q - q^{-1})^2z$, respectively. By Lemma 3.4 the expression (41) is equal to

$$ \Lambda(q - q^{-1})(q^2 - q^{-2}) - x(q - q^{-1})^2 - y(q^2 - q^{-2})^2 - z(q - q^{-1})^2. $$

The result follows from these comments after a brief calculation. \hfill \Box

We mention several variations on Lemma 7.15.
Lemma 7.16 We have

\[ \Lambda = a \frac{q \nu_x A - q^{-1} A \nu_x}{q - q^{-1}} + (q + q^{-1}) y, \]  \tag{42} \\
\Lambda = a^{-1} q A \nu_z - q^{-1} \nu_z A \frac{q}{q - q^{-1}} + (q + q^{-1}) y. \tag{43} \]

Proof: To verify (42), eliminate \( \nu_x \) using (35) and compare the result with Lemma 7.15. Equation (43) is similarly verified, by eliminating \( \nu_z \) using (36). \( \square \)

In Corollary 7.11 we gave an equation relating \( A, y \). We now give some more equations relating \( A, y \). Recall the notation

\[ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{Z}. \]

Proposition 7.17 The elements \( A, y \) satisfy

\[ y^3 A - [3]_q y^2 A y + [3]_q y A y^2 - A y^3 = 0, \]  \tag{44} \\
\[ \frac{A^3 y - [3]_q A^2 y A + [3]_q A y A^2 - y A^3}{(q^2 - q^{-2})^2} = y A - A y. \]  \tag{45} \]

Proof: To get (44), take the commutator of \( y \) with each side of (38). To get (45), take the commutator of \( A \) with each side of the equation in Lemma 7.15. \( \square \)

Note 7.18 The equations (44), (45) are a special case of the tridiagonal relations [15].

Note 7.19 In [20] Section 23 C. Worawannotai shows how the elements \( A, y \) are related to Leonard pairs of dual \( q \)-Krawtchouk type.

Lemma 7.20 The elements \( A, y \) satisfy

\[ A y^2 A + A y A y + y A y A - [3]_q y A^2 y + (a + a^{-1})(q - q^{-1})^2 A y + y A = (q^2 - q^{-2})^2 y^2 + (aq - a^{-1} q^{-1})(aq^{-1} - a^{-1}) q (q - q^{-1})^2. \]

Proof: In the equation (29) eliminate \( \nu_x, \nu_z \) using Lemma 7.7 and simplify the result. \( \square \)

8 The antiautomorphism \( \dagger \)

In this section we introduce the antiautomorphism \( \dagger \) of \( U_q(\mathfrak{sl}_2) \), and describe it from several points of view.

Definition 8.1 Define a map \( \dagger : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \) to be the composition

\[ \dagger : \quad U_q(\mathfrak{sl}_2) \xrightarrow{\tau} U_q(\mathfrak{sl}_2) \xrightarrow{\sigma} U_q(\mathfrak{sl}_2) \]

where \( \tau \) is from Lemma 3.7 and \( \sigma \) is from above Definition 7.1.
Lemma 8.2  The map $\dagger$ is the unique antiautomorphism of $U_q(\mathfrak{sl}_2)$ that sends

$$
\begin{align*}
x & \mapsto Z, \\
y^{\pm 1} & \mapsto y^{\pm 1}, \\
z & \mapsto X.
\end{align*}
$$

(46)

Proof: The map $\dagger$ is an antiautomorphism since $\tau$ is an antiautomorphism and $\sigma$ is an automorphism. The antiautomorphism $\dagger$ satisfies (46) by Lemma 3.7 and Definition 7.1. The antiautomorphism $\dagger$ is unique since $x, y^{\pm 1}, z$ generate $U_q(\mathfrak{sl}_2)$.

Lemma 8.3  The map $\dagger$ is the unique antiautomorphism of $U_q(\mathfrak{sl}_2)$ that sends

$$
\begin{align*}
\nu_x & \mapsto a^{-2}\nu_z, \\
y^{\pm 1} & \mapsto y^{\pm 1}, \\
\nu_z & \mapsto a^2\nu_x.
\end{align*}
$$

(47)

Proof: By Lemma 8.2 the map $\dagger$ is an antiautomorphism and fixes $y^{\pm 1}$. The map $\dagger$ satisfies the rest of (47) by Lemmas 5.5, 5.6 and Definition 8.1. The antiautomorphism $\dagger$ is unique since $\nu_x, y^{\pm 1}, \nu_z$ generate $U_q(\mathfrak{sl}_2)$ by Lemma 6.1.

Lemma 8.4  The map $\dagger$ is the unique antiautomorphism of $U_q(\mathfrak{sl}_2)$ that fixes each of $A, y^{\pm 1}$.

Proof: By Lemma 8.2 the map $\dagger$ is an antiautomorphism and fixes $y^{\pm 1}$. By Definition 7.3 and Lemma 8.2, the image of $A = a^{-1}x + az$ under $\dagger$ is $a^{-1}Z + aX = A$. The antiautomorphism $\dagger$ is unique since $A, y^{\pm 1}$ generate $U_q(\mathfrak{sl}_2)$ by Lemma 7.8.

Lemma 8.5  The antiautomorphism $\dagger$ fixes $\Lambda$.

Proof: By Definition 8.1 and since each of $\tau, \sigma$ fixes $\Lambda$.

Lemma 8.6  We have $\dagger^2 = 1$.

Proof: Use Lemma 3.10 and Definition 8.1.

Lemma 8.7  We have the factorizations

$$
\begin{align*}
\sigma : & \quad U_q(\mathfrak{sl}_2) \xrightarrow{\tau} U_q(\mathfrak{sl}_2) \xrightarrow{\mathfrak{d}} U_q(\mathfrak{sl}_2), \\
\sigma^{-1} : & \quad U_q(\mathfrak{sl}_2) \xrightarrow{\mathfrak{d}} U_q(\mathfrak{sl}_2) \xrightarrow{\tau} U_q(\mathfrak{sl}_2).
\end{align*}
$$

Proof: Use Definition 8.1 along with Lemma 8.6 and the last assertion of Lemma 3.7.

In Lemma 7.12 we found each of $x, z$ in terms of $A, y^{\pm 1}$. We now do something similar for $X, Z$. 

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Lemma 8.8  The elements $X$, $Z$ look as follows in terms of $A,y^\pm$:

\[
X = A \frac{a^{-1}q^{-2}}{q^{-2} - q^2} + yAy^{-1} \frac{a^{-1}}{q^2 - q^{-2}} + y^{-1}a^{-1}(aq - a^{-1}q^{-1}) \frac{y}{q + q^{-1}}
\]

\[
= A \frac{a^{-1}q^2}{q^2 - q^{-2}} + y^{-1}Ay \frac{a^{-1}}{q^2 - q^{-2}} + y^{-1}a^{-1}(aq^{-1} - a^{-1}q) \frac{y}{q + q^{-1}}
\]

and

\[
Z = A \frac{aq^2}{q^2 - q^{-2}} + yAy^{-1} \frac{a}{q^2 - q^{-2}} + y^{-1}a(a^{-1}q^{-1} - aq) \frac{y}{q + q^{-1}}
\]

\[
= A \frac{aq^{-2}}{q^2 - q^{-2}} + y^{-1}Ay \frac{a}{q^2 - q^{-2}} + y^{-1}a(a^{-1}q - a^{-1}q^{-1}) \frac{y}{q + q^{-1}}
\]

Proof: In each of the four equations of Lemma 7.12 apply $\dagger$ to each side and evaluate the results using Lemmas 8.2, 8.4. □

9 The invariant presentation of $U_q(\mathfrak{sl}_2)$

In this section we give a presentation of $U_q(\mathfrak{sl}_2)$ by generators and relations, using the generators $A,y^\pm$.

Theorem 9.1  The $\mathbb{F}$-algebra $U_q(\mathfrak{sl}_2)$ has a presentation by generators $A,y^\pm$ and relations

\[
yy^{-1} = y^{-1}y = 1,
\]

\[
y^2A - (q^2 + q^{-2})yAy + Ay^2 = -(a + a^{-1})(q - q^{-1})^2y,
\]

\[
A^2y + AyAy + yAyA - [3]_q yA^2y + (a + a^{-1})(q - q^{-1})^2(Ay + yA)
\]

\[
= (q^2 - q^{-2})^2y^2 + (aq - a^{-1}q^{-1})(aq^{-1} - a^{-1}q)(q - q^{-1})^2.
\]

An isomorphism with the presentation in Theorem 6.2 sends

\[
A \mapsto a(1 - q\nu_x)y^{-1} + a^{-1}(1 - q^{-1}\nu_x)y^{-1}, \tag{48}
\]

\[
y^\pm \mapsto y^\pm. \tag{49}
\]

The inverse isomorphism sends

\[
\nu_x \mapsto a^{-1} \frac{a + a^{-1}}{q + q^{-1}} - a^{-1}qAy - q^{-1}yA \frac{q^2 - q^{-2}}{q^2 - q^{-2}}, \tag{50}
\]

\[
y^\pm \mapsto y^\pm, \tag{51}
\]

\[
\nu_z \mapsto a \frac{a + a^{-1}}{q + q^{-1}} - a^{-1}qyA - q^{-1}Ay \frac{q^2 - q^{-2}}{q^2 - q^{-2}}. \tag{52}
\]

Proof: The map (48), (49) gives an $\mathbb{F}$-algebra homomorphism by Corollary 7.11 and Lemma 7.20. One checks that the map (50)–(52) gives an $\mathbb{F}$-algebra homomorphism. One also checks that the above maps are inverses. Therefore each map is an $\mathbb{F}$-algebra isomorphism. □
Note 9.2 For the presentation of $U_q(sl_2)$ given in Theorem 9.1, the relations are invariant under each of the following moves: (i) $q \mapsto q^{-1}$; (ii) $a \mapsto a^{-1}$.

Definition 9.3 In view of Lemma 8.4 and Note 9.2, the presentation for $U_q(sl_2)$ given in Theorem 9.1 will be called the invariant presentation.

10 The algebra $U_q^\vee$

We turn our attention to a certain subalgebra of $U_q(sl_2)$, which is denoted $U_q^\vee$. We will define $U_q^\vee$ after some motivational comments. Recall the equitable generators $x, y, z$ of $U_q(sl_2)$ from Section 3, and the elements $X, Z, A$ from Section 7. Consider the subspace of $U_q(sl_2)$ spanned by $x, y^{-1}, z$. This subspace contains $X, Z, A$ by (30), (32), and is invariant under the maps $\sigma, \tau, \dagger$ by Lemmas 3.5, 3.7 and Definition 8.1.

Definition 10.1 Let $U_q^\vee$ denote the subalgebra of $U_q(sl_2)$ generated by $x, y^{-1}, z$.

Our next goal is to (i) display a basis for $U_q^\vee$; (ii) give several presentations of $U_q^\vee$ by generators and relations.

Lemma 10.2 The following relations hold in $U_q^\vee$:

$$\frac{qy^{-1}x - q^{-1}xy^{-1}}{q - q^{-1}} = y^{-2}, \quad \frac{qz^{-1}y - y^{-1}z}{q - q^{-1}} = y^{-2}, \quad \frac{qx - q^{-1}xz}{q - q^{-1}} = 1. \quad (53)$$

Proof: Adjust the relations (8). □

As we investigate $U_q^\vee$ it is useful to consider the following algebra.

Definition 10.3 Let $\overline{U}_q$ denote the $F$-algebra defined by generators $x, \overline{y}, z$ and relations

$$\frac{q\overline{y}x - q^{-1}x\overline{y}}{q - q^{-1}} = \overline{y}^2, \quad \frac{qz\overline{y} - \overline{y}^{-1}z}{q - q^{-1}} = \overline{y}^2, \quad \frac{qx - q^{-1}xz}{q - q^{-1}} = 1. \quad (54)$$

Lemma 10.4 There exists an $F$-algebra homomorphism $\iota: \overline{U}_q \rightarrow U_q(sl_2)$ that sends

$$x \mapsto x, \quad \overline{y} \mapsto y^{-1}, \quad z \mapsto z.$$

Proof: Compare (53) and (54). □

Note that $U_q^\vee$ is the image of $\overline{U}_q$ under $\iota$.

Lemma 10.5 The following is a basis for the $F$-vector space $\overline{U}_q$:

$$x^r\overline{y}^sz^t \quad r, s, t \in \mathbb{N}. \quad (55)$$
Proposition 10.6 gives an isomorphism of $\mathbb{F}$-algebras $\iota : \overline{U}_q \rightarrow U_q(\mathfrak{sl}_2)$ is injective.

**Proof:** For the basis vectors (55) their images under $\iota$ are linearly independent by Lemma 3.3. We show that the vectors (55) span $\overline{U}_q$. For $n \in \mathbb{N}$, by a word of length $n$ in $\overline{U}_q$ we mean a product $x_1x_2\cdots x_n$ such that $x_i$ is among $x, \overline{y}, z$ for $1 \leq i \leq n$. We interpret the word of length 0 to be the multiplicative identity of $\overline{U}_q$. By definition $\overline{U}_q$ is spanned by the words. Pick a word $w$ and write $w = x_1x_2\cdots x_n$. By the $(x,z)$-length of $w$ we mean the number of elements among $x_1, x_2, \ldots, x_n$ that are equal to $x$ or $z$. By an inversion in $w$ we mean an ordered pair of elements $(x_i, x_j)$ $(1 \leq i, j \leq n)$ such that $i < j$ and $(x_i, x_j)$ is $(\overline{y}, x)$ or $(z, \overline{y})$ or $(z, x)$. The word $w$ is called reducible whenever it has at least one inversion, and irreducible otherwise. The list (55) consists of the irreducible words. The words (55) span a subspace of $\overline{U}_q$ that we denote by $L$. We show $L = \overline{U}_q$. In order to do this, it suffices to show that each word is contained in $L$. Suppose there exists a word that is not contained in $L$. Let $N$ denote the minimal $(x, z)$-length among all such words. Consider the set of words that are not contained in $L$ and have $(x, z)$-length $N$. Pick a word $w$ in this set that has a minimal number of inversions. The word $w$ is reducible; otherwise $w$ is included in (55) and hence $w \in L$, for a contradiction. Write $w = x_1x_2\cdots x_n$. Since $w$ is reducible, there exists an integer $i$ $(2 \leq i \leq n)$ such that $(x_{i-1}, x_i)$ is an inversion. There are three cases. First assume that $(x_{i-1}, x_i) = (\overline{y}, x)$. Note that $\overline{y}x = q^{-2}x\overline{y} + (1 - q^{-2})\overline{y}^2$. Therefore $w = q^{-2}u + (1 - q^{-2})v$ where $u = x_1\cdots x_{i-2}x\overline{y}x_{i+1}\cdots x_n$ and $v = x_1\cdots x_{i-2}y^2x_{i+1}\cdots x_n$. The word $u$ has $(x,z)$-length $N$ and one fewer inversions than $w$. Therefore $u \in L$. The word $v$ has $(x,z)$-length $N-1$, so $v \in L$. Therefore $u,L \in L$ for a contradiction. Next assume that $(x_{i-1}, x_i) = (z, \overline{y})$. Note that $z\overline{y} = q^{-2}\overline{y}z + (1 - q^{-2})\overline{y}$. Therefore $w = q^{-2}u + (1 - q^{-2})v$ where $u = x_1\cdots x_{i-2}\overline{y}x_{i+1}\cdots x_n$ and $v = x_1\cdots x_{i-2}y^2x_{i+1}\cdots x_n$. The word $u$ has $(x,z)$-length $N$ and one fewer inversions than $w$. Therefore $u \in L$. The word $v$ has $(x,z)$-length $N-1$, so $v \in L$. Therefore $u \in L$ for a contradiction. In each of the three cases we obtained a contradiction. Therefore every word is contained in $L$, and consequently $L = \overline{U}_q$. The result follows. \hfill $\square$

**Proposition 10.6** The following is a basis for the $\mathbb{F}$-vector space $U_q^\vee$:

$$x^ry^{-s}z^t, \quad r, s, t \in \mathbb{N}.$$  

**Proof:** Apply the isomorphism $\iota : \overline{U}_q \rightarrow U_q^\vee$ to the basis (55). \hfill $\square$

**Corollary 10.7** The following (i), (ii) coincide:
(i) the intersection of $U_q^\lor$ and $U_q'$;

(ii) the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by $x$, $z$.

This common subalgebra has a basis

$$x^r z^t \quad r, t \in \mathbb{N}.$$ 

Proof: Compare the basis for $U_q'$ given in Lemma 4.4 with the basis for $U_q^\lor$ given in Corollary 10.7.

\[\Box\]

**Theorem 10.9** The $\mathbb{F}$-algebra $U_q^\lor$ has a presentation by generators $x, y^{-1}, z$ and relations

$$\frac{qy^{-1}x - q^{-1}xy^{-1}}{q - q^{-1}} = y^{-2}, \quad \frac{qzy^{-1} - q^{-1}y^{-1}z}{q - q^{-1}} = y^{-2}, \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$ (56)

Proof: By Definition 10.3 and since $\iota : U_q \rightarrow U_q^\lor$ is an isomorphism of $\mathbb{F}$-algebras.

We now clarify an aspect of $U_q^\lor$.

**Lemma 10.10** The element $y^{-1}$ is not invertible in $U_q^\lor$.

Proof: Suppose that $y^{-1}$ is invertible in $U_q^\lor$, and let $\xi \in U_q^\lor$ denote its inverse. In the algebra $U_q(\mathfrak{sl}_2)$, each of $y, \xi$ is an inverse of $y^{-1}$. The inverse of $y^{-1}$ is unique, so $y = \xi$. Comparing Lemma 3.3 and Corollary 10.7 we see that $y \notin U_q^\lor$. This is a contradiction, and the result follows.

\[\Box\]

**Corollary 10.11** The element $\mathbb{y}$ is not invertible in $\overline{U}_q$.

Proof: By Lemma 10.10 and since $\iota$ sends $\mathbb{y} \mapsto y^{-1}$.

\[\Box\]

In Theorem 10.9 we displayed a presentation of $U_q^\lor$ by generators and relations. Shortly we will display several more.

**Definition 10.12** Let $S$ denote the subspace of $U_q(\mathfrak{sl}_2)$ spanned by $x, y^{-1}, z$.

**Lemma 10.13** Each of the following (i)–(iv) is a basis for the $\mathbb{F}$-vector space $S$:

(i) $x, y^{-1}, z;$  
(ii) $X, y^{-1}, Z;$  
(iii) $x, X, A;$  
(iv) $z, Z, A.$

Proof: The vectors (i) are linearly independent, by Lemma 3.3 or Corollary 10.7. To finish the proof use (30) and $A = a^{-1}x + az$.

\[\Box\]

We now clarify the relationship between the four bases for $S$ given in Lemma 10.13. We will be discussing transition matrices and matrix representations, following the conventions of [19, Sections 10, 15].
Lemma 10.14 We refer to the bases (i)–(iv) in Lemma 10.13. The transition matrices from (i) to (ii) and (ii) to (i) are, respectively,
\[
\begin{pmatrix}
a^{-2} & 0 & 0 \\
1 - a^{-2} & 1 & 1 - a^{-2} \\
0 & 0 & a^2
\end{pmatrix},
\begin{pmatrix}
a^2 & 0 & 0 \\
1 - a^2 & 1 & 1 - a^{-2} \\
0 & 0 & a^{-2}
\end{pmatrix}.
\]

The transition matrices from (i) to (iii) and (iii) to (i) are, respectively,
\[
\begin{pmatrix}
1 & a^{-2} & a^{-1} \\
0 & 1 - a^{-2} & 0 \\
0 & 0 & a
\end{pmatrix},
\begin{pmatrix}
1 & \frac{a^{-1}}{a^{-1} - a} & -a^{-2} \\
0 & \frac{a}{a^{-1} - a} & 0 \\
0 & 0 & \frac{a^{-1}}{a^{-1} - a}
\end{pmatrix}.
\]

The transition matrices from (i) to (iv) and (iv) to (i) are, respectively,
\[
\begin{pmatrix}
0 & 0 & a^{-1} \\
0 & 1 - a^2 & 0 \\
1 & a^2 & a
\end{pmatrix},
\begin{pmatrix}
-a^2 & \frac{a}{a^{-1} - a} & 1 \\
0 & \frac{a}{a^{-1} - a} & 0 \\
0 & 0 & a
\end{pmatrix}.
\]

Proof: Use (30) and \( A = a^{-1}x + ax \). □

By our comments above Definition 10.1, the subspace \( S \) is invariant under \( \tau \) and \( \dagger \). We now describe the actions of \( \tau \) and \( \dagger \) on \( S \).

Lemma 10.15 We refer to the bases (i)–(iv) in Lemma 10.13. With respect to (i) the matrices representing \( \tau \) and \( \dagger \) are, respectively,
\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & a^{-2} \\
1 - a^2 & 1 & 1 - a^{-2} \\
a^2 & 0 & 0
\end{pmatrix}.
\]

With respect to (ii) the matrices representing \( \tau \) and \( \dagger \) are, respectively,
\[
\begin{pmatrix}
0 & 0 & a^4 \\
a^{-2}(a^2 - a^{-2}) & 1 & a^2(a^{-2} - a^2) \\
a^4 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & a^2 \\
1 - a^{-2} & 1 & 1 - a^2 \\
a^{-2} & 0 & 0
\end{pmatrix}.
\]

With respect to (iii) the matrices representing \( \tau \) and \( \dagger \) are, respectively,
\[
\begin{pmatrix}
-a^{-2} & -a^{-3}(a + a^{-1}) & a^{-1}(a^2 - a^{-2}) \\
0 & 1 & 0 \\
a^{-1} & a^{-3} & a^{-2}
\end{pmatrix},
\begin{pmatrix}
0 & -a^{-2} & 0 \\
-a^2 & 0 & 0 \\
a & a^{-1} & 1
\end{pmatrix}.
\]

With respect to (iv) the matrices representing \( \tau \) and \( \dagger \) are, respectively,
\[
\begin{pmatrix}
-a^2 & -a^3(a + a^{-1}) & a(a^2 - a^2) \\
0 & 1 & 0 \\
a & a^3 & a^2
\end{pmatrix},
\begin{pmatrix}
0 & -a^2 & 0 \\
-a^2 & 0 & 0 \\
a^{-1} & a & 1
\end{pmatrix}.
\]
Proof: Use Lemmas 3.7, 8.2, 10.14 and linear algebra.

The action of the automorphism \( \sigma \) on \( S \) is found using Lemmas 8.7 and 10.15.

**Lemma 10.16** The subalgebra \( U_q^\vee \) is invariant under each map \( \tau, \dagger, \sigma \).

*Proof:* The algebra \( U_q^\vee \) is generated by \( S \), and \( S \) is invariant under each of \( \tau, \dagger, \sigma \). \( \square \)

Consider the sets (i)–(iv) in Lemma 10.13. Each set is a generating set for \( U_q^\vee \). For case (i), the corresponding presentation of \( U_q^\vee \) by generators and relations was given in Theorem 10.9. For the cases (ii)–(iv) we now do something similar.

**Theorem 10.17** The \( \mathbb{F} \)-algebra \( U_q^\vee \) has a presentation by generators \( X, y^{-1}, Z \) and relations

\[
\frac{qy^{-1}X - q^{-1}XY^{-1}}{q - q^{-1}} = y^{-2}, \quad \frac{qZy^{-1} - q^{-1}y^{-1}Z}{q - q^{-1}} = y^{-2}, \quad \frac{qZX - q^{-1}XZ}{q - q^{-1}} = 1.
\]

*Proof:* The restriction of \( \sigma \) to \( U_q^\vee \) is an automorphism of \( U_q^\vee \) that sends the generators \( x, y^{-1}, z \) to the generators \( X, y^{-1}, Z \) respectively. The result follows from this and Theorem 10.9. \( \square \)

**Theorem 10.18** The \( \mathbb{F} \)-algebra \( U_q^\vee \) has a presentation by generators \( x, X, A \) and relations

\[
\frac{qAx - q^{-1}xA}{q - q^{-1}} = a^{-1}x^2 + a, \quad \frac{qAX - q^{-1}XA}{q - q^{-1}} = aX^2 + a^{-1},
\]

\[
a^{-1}x^2 - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}} xX - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}} XX + aX^2 = 0.
\]

An isomorphism with the presentation in Theorem 10.9 sends

\[
x \mapsto x,
\]

\[
X \mapsto a^{-2}x + (1 - a^{-2})y^{-1},
\]

\[
A \mapsto a^{-1}x + az.
\]

The inverse isomorphism sends

\[
x \mapsto x,
\]

\[
y^{-1} \mapsto \frac{aX - a^{-1}x}{a - a^{-1}},
\]

\[
z \mapsto a^{-1}A - a^{-2}x.
\]

*Proof:* One routinely checks that each map is an \( \mathbb{F} \)-algebra homomorphism, and that the two maps are inverses. Therefore each map is an \( \mathbb{F} \)-algebra isomorphism. \( \square \)
Theorem 10.19 The $\mathbb{F}$-algebra $U_q^\vee$ has a presentation by generators $z, Z, A$ and relations

\[
\begin{align*}
\frac{qzA - q^{-1}Az}{q - q^{-1}} &= az^2 + a^{-1}, \\
\frac{qZA - q^{-1}AZ}{q - q^{-1}} &= a^{-1}Z^2 + a, \\
az^2 - \frac{a^{-1}q - q^{-1}a}{q - q^{-1}}Z - \frac{aq - q^{-1}a}{q - q^{-1}}Zz + a^{-1}Z^2 &= 0.
\end{align*}
\]

An isomorphism with the presentation in Theorem 10.9 sends

\[
\begin{align*}
z &\mapsto z, \\
Z &\mapsto a^2z + (1 - a^2)y^{-1}, \\
A &\mapsto a^{-1}x + az.
\end{align*}
\]

The inverse isomorphism sends

\[
\begin{align*}
x &\mapsto aA - a^2z, \\
y^{-1} &\mapsto \frac{az - a^{-1}Z}{a - a^{-1}}, \\
z &\mapsto z.
\end{align*}
\]

Proof: Similar to the proof of Theorem 10.18.

We now consider $\nu_x, \nu_y, \nu_z$ and $\Lambda$ from the point of view of $U_q^\vee$.

Lemma 10.20 The subalgebra $U_q^\vee$ does not contain any of $\nu_x, \nu_z, \Lambda$.

Proof: Write these elements in the basis for $U_q(\mathfrak{sl}_2)$ from Lemma 3.3. This can be done using (18), (20) along with the top left equation in Lemma 3.4. Evaluate the results using Corollary 10.7.

In view of Lemma 10.20 we consider the following elements of $U_q(\mathfrak{sl}_2)$:

\[
\nu_x y^{-1}, \quad \nu_y, \quad \nu_z y^{-1}, \quad \Lambda y^{-1}.
\]

Proposition 10.21 Each of the elements (57) is contained in $U_q^\vee$. These elements look as follows in the basis for $U_q^\vee$ from Corollary 10.7.

\[
\begin{align*}
\nu_x y^{-1} &= q^{-1}(y^{-1} - z), \\
\nu_y &= q^{-1}(1 - xz), \\
\nu_z y^{-1} &= q(y^{-1} - x), \\
\Lambda y^{-1} &= q^{-1}(1 - xz) + q^{-1}xy^{-1} + q^{-1}y^{-1}z + (q - q^{-1})y^{-2}.
\end{align*}
\]

Proof: Use (18), (20) along with the second equation on the right in Lemma 3.4.

Proposition 10.21 shows how the elements (57) look in the presentation of $U_q^\vee$ from Theorem 10.9. We now show how the elements (57) look in our other three presentations of $U_q^\vee$.

Proposition 10.22 In the presentation of $U_q^\vee$ from Theorem 10.17 the elements (57) look as follows.
(i) $\nu_x y^{-1}$ is equal to
\[ q^{-1}a^{-2}(y^{-1} - Z). \]

(ii) $\nu_y$ is equal to $q^{-1}$ times
\[ 1 - XZ + (a - a^{-1})a^{-1}y^{-1}Z - (a - a^{-1})aXy^{-1} + (a - a^{-1})^{2}y^{-2}. \]

(iii) $\nu_z y^{-1}$ is equal to
\[ qa^2(y^{-1} - X). \]

(iv) $\Lambda y^{-1}$ is equal to
\[ q^{-1}(1 - XZ) + q^{-1}Xy^{-1} + q^{-1}y^{-1}Z + (q - q^{-1})y^{-2}. \]

Proof: (i)–(iii) By Lemma 10.14 we find $x = a^2X + (1 - a^2)y^{-1}$ and $z = a^{-2}Z + (1 - a^{-2})y^{-1}$. Use these equations to eliminate $x, z$ in (58).

(iv) In the equation (59), apply $\sigma$ to each side. Use the fact that $\sigma$ fixes each of $\Lambda, y^{-1}$ and sends $x, z$ to $X, Z$ respectively.

Proposition 10.23 In the presentation of $U_q^\vee$ from Theorem 10.18 the elements (57) look as follows.

(i) $\nu_x y^{-1}$ is equal to $q^{-1}a^{-1}$ times
\[ \frac{a^2 X - a^{-2} x}{a - a^{-1}} - A. \]

(ii) $\nu_y$ is equal to
\[ a^{-1}Ax - xA \quad \frac{q - q^{-1}}{q - q^{-1}}. \]

(iii) $\nu_z y^{-1}$ is equal to
\[ qa \quad \frac{X - x}{a - a^{-1}}. \]

(iv) $\Lambda y^{-1}$ is equal to each of
\[ qA \quad \frac{X - x}{a - a^{-1}} + a^{-1} \frac{q - q^{-1}}{a - a^{-1}}X^2 + \frac{aq - a^{-1} q^{-1}}{a - a^{-1}}Xx + q, \]
\[ q^{-1} \quad \frac{X - x}{a - a^{-1}AX} - a^{-1} \frac{q - q^{-1}}{a - a^{-1}}X^2 + \frac{aq - a^{-1} q^{-1}}{a - a^{-1}}Xx + q^{-1}, \]
\[ qA \quad \frac{X - x}{a - a^{-1}} + a^{-1} \frac{q - q^{-1}}{a - a^{-1}}X^2 + \frac{aq - a^{-1} q^{-1}}{a - a^{-1}}Xx + q^{-1}, \]
\[ q^{-1} \quad \frac{X - x}{a - a^{-1}AX} - a^{-1} \frac{q - q^{-1}}{a - a^{-1}}X^2 + \frac{aq - a^{-1} q^{-1}}{a - a^{-1}}Xx + q^{-1}. \]
Proof: By Lemma 10.14 we find \( y^{-1} = (aX - a^{-1}x)(a - a^{-1})^{-1} \) and \( z = a^{-1}A - a^{-2}x \). Use these equations to eliminate \( y^{-1}, z \) in (58), (59). Evaluate the results using the relations in Theorem 10.18.

\[ \bbox[3pt]{\text{Proposition 10.24 In the presentation of} \ U_q^{\vee} \ \text{from Theorem 10.19 the elements (57) look as follows.} } \]

(i) \( \nu_xy^{-1} \) is equal to
\[
q^{-1}a^{-1} \frac{z - Z}{a - a^{-1}}.
\]

(ii) \( \nu_y \) is equal to
\[
\frac{zA - Az}{a - a^{-1}}.
\]

(iii) \( \nu_zy^{-1} \) is equal to \( qa \) times
\[
\frac{a^2z - a^{-2}Z}{a - a^{-1}} - A.
\]

(iv) \( \Lambda y^{-1} \) is equal to each of
\[
\begin{align*}
q^{-1}A \frac{z - Z}{a - a^{-1}} + aq^{-1}a - a^{-1}q a^{-1} & \frac{z - Z}{a - a^{-1}}z + q^{-1}, \\
q \frac{z - Z}{a - a^{-1}} A - a^{-1}q^{-1}a^{-1} & \frac{z - Z}{a - a^{-1}}z + q, \\
q^{-1}A \frac{z - Z}{a - a^{-1}} - a^{-1}q^{-1}a^{-1} & \frac{z - Z}{a - a^{-1}}Z^2 + aq^{-1}a^{-1}zZ + q^{-1}, \\
q \frac{z - Z}{a - a^{-1}} A + a^{-1}q^{-1}a^{-1} & \frac{z - Z}{a - a^{-1}}Z^2 + aq^{-1}a^{-1}Zz + q.
\end{align*}
\]

Proof: By Lemma 10.14 we find \( x = aA - a^2z \) and \( y^{-1} = (az - a^{-1}Z)(a - a^{-1})^{-1} \). Use these equations to eliminate \( x, y^{-1} \) in (58), (59), and evaluate the results using the relations in Theorem 10.19. Part (iv) can also be obtained from Proposition 10.23(iv), by applying the antiautomorphism \( \dagger \) to each term in that part of the proposition. Use the fact that \( \dagger \) fixes each of \( \Lambda, y^{-1} \) and sends \( x, X \) to \( Z, z \) respectively.

\[ \bbox[3pt]{\text{11 Modules for} \ U_q(\mathfrak{sl}_2) \ \text{and} \ U_q^{\vee}} \]

In this section we compare the finite-dimensional modules for \( U_q(\mathfrak{sl}_2) \) and \( U_q^{\vee} \).

Throughout this section \( V \) denotes a vector space over \( \mathbb{F} \) with finite positive dimension. We will use the following fact from linear algebra.
Lemma 11.1 [5, p. 193]. Let $T : V \to V$ denote an invertible linear transformation. Then there exists a polynomial $f$ that has all coefficients in $\mathbb{F}$ and $T^{-1} = f(T)$.

A module for a given $\mathbb{F}$-algebra is called decomposable (resp. semisimple) whenever it is the direct sum of two nonzero submodules (resp. direct sum of irreducible submodules).

For the moment assume that $V$ is a $U_q(\mathfrak{sl}_2)$-module. If we restrict the $U_q(\mathfrak{sl}_2)$-action to $U^\vee_q$, then $V$ becomes a $U^\vee_q$-module.

Lemma 11.2 Assume that $V$ is a $U_q(\mathfrak{sl}_2)$-module. For any subspace $W$ of $V$ the following are equivalent:

(i) $W$ is a $U_q(\mathfrak{sl}_2)$-submodule of $V$;
(ii) $W$ is a $U^\vee_q$-submodule of $V$.

Proof: Assume $W \neq 0$; otherwise the result is trivial.

(i) $\Rightarrow$ (ii) Since $U_q(\mathfrak{sl}_2)$ contains $U^\vee_q$.

(ii) $\Rightarrow$ (i) By assumption $W$ is invariant under $y^{-1}$. Applying Lemma 11.1 to $T = y$, we see that $W$ is invariant under $y$. $\square$

Corollary 11.3 Assume that $V$ is a $U_q(\mathfrak{sl}_2)$-module.

(i) The $U_q(\mathfrak{sl}_2)$-module $V$ is irreducible if and only if the $U^\vee_q$-module $V$ is irreducible.
(ii) The $U_q(\mathfrak{sl}_2)$-module $V$ is decomposable if and only if the $U^\vee_q$-module $V$ is decomposable.
(iii) The $U_q(\mathfrak{sl}_2)$-module $V$ is semisimple if and only if the $U^\vee_q$-module $V$ is semisimple.

Proof: Use Lemma 11.2 $\square$

Note 11.4 Assume that $q$ is not a root of unity, and that $V$ is a $U_q(\mathfrak{sl}_2)$-module. If $y$ is diagonalizable on $V$ then $V$ is semisimple [12, Theorem 2.9]. Moreover, if the characteristic of $\mathbb{F}$ is not 2 then $y$ is diagonalizable on $V$ [12, Proposition 2.3].

For the moment assume that $V$ is a $U^\vee_q$-module. We now display some necessary and sufficient conditions for the $U^\vee_q$-action on $V$ to extend to a $U_q(\mathfrak{sl}_2)$-action on $V$.

Lemma 11.5 Assume that $V$ is a $U^\vee_q$-module. Then the following are equivalent:

(i) the action of $U^\vee_q$ on $V$ extends to an action of $U_q(\mathfrak{sl}_2)$ on $V$;
(ii) the element $y^{-1}$ is invertible on $V$;
(iii) the element $a^{-1}x - aX$ is invertible on $V$;
(iv) the element $az - a^{-1}Z$ is invertible on $V$. 

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We comment on linear algebra. Let $T: V \to V$ denote a linear transformation. Using $T$ we define the subspaces $V_{\text{inv}}$ and $V_{\text{nil}}$ of $V$ as follows. For all integers $n \geq 0$ let $V_n$ (resp. $V^{(n)}$) denote the image (resp. kernel) of $T^n$ on $V$. By construction $V_n$ and $V^{(n)}$ are $T$-invariant. By linear algebra $\dim V = \dim V_n + \dim V^{(n)}$. For $n \geq 1$ we have $V_n \subseteq V_{n-1}$ and $V^{(n-1)} \subseteq V^{(n)}$. Define $V_{\text{inv}} = \bigcap_{n=0}^\infty V_n$ and $V_{\text{nil}} = \cup_{n=0}^\infty V^{(n)}$. The subspaces $V_{\text{inv}}$ and $V_{\text{nil}}$ are $T$-invariant. By construction $T$ is invertible on $V_{\text{inv}}$ provided $V_{\text{inv}} \neq 0$. Moreover $T$ is nilpotent on $V_{\text{nil}}$. By these comments $V_{\text{inv}} \cap V_{\text{nil}} = 0$. Considering the dimensions we obtain $V = V_{\text{inv}} + V_{\text{nil}}$ (direct sum). We call $V_{\text{inv}}$ (resp. $V_{\text{nil}}$) the invertible part (resp. nilpotent part) of $V$ with respect to $T$.

**Lemma 11.6** Assume that $V$ is a $U_q^\vee$-module. Then the following are $U_q^\vee$-submodules of $V$:

(i) the image of $y^{-1}$ on $V$;

(ii) the kernel of $y^{-1}$ on $V$.

*Proof:* Use the relations (56). \(\square\)

**Lemma 11.7** Assume that $V$ is a $U_q^\vee$-module. Then for $n \in \mathbb{N}$ the following are $U_q^\vee$-submodules of $V$:

(i) the image of $y^{-n}$ on $V$;

(ii) the kernel of $y^{-n}$ on $V$.

*Proof:* (i) Denote this image by $V_n$. Our proof is by induction on $n$. The case $n = 0$ is trivial, so assume $n \geq 1$. By induction $V_{n-1}$ is invariant under $U_q^\vee$. For the $U_q^\vee$-module $V_{n-1}$ consider the image of $y^{-1}$. By Lemma 11.6(i) this image is invariant under $U_q^\vee$. By construction this image is equal to $V_n$. Therefore $V_n$ is a $U_q^\vee$-submodule of $V$.

(ii) Denote this kernel by $V^{(n)}$. Our proof is by induction on $n$. The case $n = 0$ is trivial, so assume $n \geq 1$. By induction $V^{(n-1)}$ is invariant under $U_q^\vee$. For the quotient $U_q^\vee$-module $V/V^{(n-1)}$ consider the kernel of $y^{-1}$. By Lemma 11.6(ii) this kernel is invariant under $U_q^\vee$. By construction this kernel is equal to $V^{(n)}/V^{(n-1)}$. Therefore $V^{(n)}/V^{(n-1)}$ is a $U_q^\vee$-submodule of $V/V^{(n-1)}$, and consequently $V^{(n)}$ is a $U_q^\vee$-submodule of $V$. \(\square\)

**Lemma 11.8** Assume that $V$ is a $U_q^\vee$-module. Then each of the following is a $U_q^\vee$-submodule of $V$:

(i) the invertible part of $V$ with respect to $y^{-1}$;

(ii) the nilpotent part of $V$ with respect to $y^{-1}$.

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Proof: Use Lemma \[\text{11.7}\] \qed

We now summarize the situation so far.

**Lemma 11.9** Assume that $V$ is a $U_q^\vee$-module. Let $V_{inv}$ (resp. $V_{nil}$) denote the invertible part (resp. nilpotent part) of $V$ with respect to $y^{-1}$. Then

$$V = V_{inv} + V_{nil} \quad \text{(direct sum of $U_q^\vee$-modules).} \quad (60)$$

Assume that $V_{inv}$ (resp. $V_{nil}$) is nonzero. Then on $V_{inv}$ (resp. $V_{nil}$) the action of $U_q^\vee$ does (resp. does not) extend to an action of $U_q(\mathfrak{sl}_2)$.

Proof: Line (60) is from the construction and Lemma \[\text{11.8}\]. To get the remaining assertions, apply parts (i), (ii) of Lemma \[\text{11.5}\] to $V_{inv}$ and $V_{nil}$. \qed

A module for a given $F$-algebra is called **indecomposable** whenever it is not decomposable.

**Corollary 11.10** Assume that $V$ is an indecomposable $U_q^\vee$-module. Then on $V$ the element $y^{-1}$ is either nilpotent or invertible.

Proof: Referring to (60), either $V_{inv} = 0$ or $V_{nil} = 0$. \qed

For the moment assume that $V$ is a $U_q(\mathfrak{sl}_2)$-module. Then on the $U_q^\vee$-module $V$ the element $y^{-1}$ is invertible. We now give some examples of a $U_q^\vee$-module on which $y^{-1}$ is nilpotent.

**Definition 11.11** Let $A$ denote the $F$-algebra defined by generators $u, v$ and one relation

$$\frac{quv - q^{-1}vu}{q - q^{-1}} = 1.$$ 

**Lemma 11.12** There exists an $F$-algebra homomorphism $U_q^\vee \to A$ that sends

$$x \mapsto v, \quad y^{-1} \mapsto 0, \quad z \mapsto u.$$

Proof: Use Theorem \[\text{10.9}\] \qed

**Example 11.13** Assume that $V$ is an $A$-module. If we pull back the $A$-module structure via the homomorphism $U_q^\vee \to A$ from Lemma \[\text{11.12}\] then $V$ becomes a $U_q^\vee$-module on which $y^{-1}$ is zero.
12 $U_{q}^{\vee}$-modules from tridiagonal pairs

In this section we use tridiagonal pairs of $q$-Racah type to construct examples of finite-dimensional $U_{q}^{\vee}$-modules.

We now recall the notion of a tridiagonal pair [7]. For background information on this topic, we refer the reader to [7,8,10,14,16].

**Definition 12.1** [7, Definition 1.1]. Let $V$ denote a vector space over $\mathbb{F}$ with finite positive dimension. By a tridiagonal pair on $V$, we mean an ordered pair of $\mathbb{F}$-linear maps $A : V \to V$ and $A^* : V \to V$ that satisfy the following conditions.

(i) Each of $A$, $A^*$ is diagonalizable on $V$.

(ii) There exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of $A$ such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \quad (61)$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V^*_i\}_{i=0}^{\delta}$ of the eigenspaces of $A^*$ such that

$$A V^*_i \subseteq V^*_{i-1} + V^*_i + V^*_{i+1} \quad (0 \leq i \leq \delta), \quad (62)$$

where $V^*_{-1} = 0$ and $V^*_{\delta+1} = 0$.

(iv) There does not exist a subspace $W \subseteq V$ such that $A W \subseteq W, A^* W \subseteq W$, $W \neq 0$, $W \neq V$.

**Note 12.2** According to a common notational convention $A^*$ denotes the conjugate-transpose of $A$. We are not using this convention. For a tridiagonal pair $A, A^*$ the $\mathbb{F}$-linear maps $A$ and $A^*$ are arbitrary subject to (i)–(iv) above.

In order to motivate our results, we summarize some facts about tridiagonal pairs. For the rest of this section, fix a tridiagonal pair $A, A^*$ on $V$ as in Definition [12.1]. By [7, Lemma 4.5] the integers $d$ and $\delta$ from (ii) and (iii) are equal; we call this common value the *diameter* of the pair. To avoid trivialities, we always assume that $d \geq 1$. An ordering of the eigenspaces of $A$ (resp. $A^*$) is said to be *standard* whenever it satisfies (61) (resp. (62)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^{d}$ denote a standard ordering of the eigenspaces of $A$. By [7, Lemma 2.4], the ordering $\{V_{d-i}\}_{i=0}^{d}$ is also standard and no further ordering is standard. A similar result holds for the eigenspaces of $A^*$. For the rest of this section fix a standard ordering $\{V_i\}_{i=0}^{d}$ (resp. $\{V^*_i\}_{i=0}^{\delta}$) of the eigenspaces of $A$ (resp. $A^*$).

For $0 \leq i \leq d$ let $\theta_i$ (resp. $\theta^*_i$) denote the eigenvalue of $A$ (resp. $A^*$) associated with $V_i$ (resp. $V^*_i$). By construction $\{\theta_i\}_{i=0}^{d}$ are mutually distinct scalars in $\mathbb{F}$, and $\{\theta^*_i\}_{i=0}^{\delta}$ are mutually distinct scalars in $\mathbb{F}$. By [7, Theorem 11.1] the expressions

\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}
\]
are equal and independent of $i$ for $2 \leq i \leq d - 1$.

We recall the split decomposition [7, Section 4]. For $0 \leq i \leq d$ define

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d).$$

For notational convenience define $U_{-1} = 0$ and $U_{d+1} = 0$. By [7, Theorem 4.6] we have $V = \sum_{i=0}^{d} U_i$ (direct sum). Moreover for $0 \leq i \leq d$,

$$(A - \theta_i I)U_i \subseteq U_{i+1}, \quad (A^* - \theta_i^* I)U_i \subseteq U_{i-1}.$$

For $0 \leq i \leq d$ define

$$U_i^\downarrow = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_0 + V_1 + \cdots + V_d).$$

For notational convenience define $U_{-1}^\downarrow = 0$ and $U_{d+1}^\downarrow = 0$. By [7, Theorem 4.6] we have $V = \sum_{i=0}^{d} U_i^\downarrow$ (direct sum). Moreover for $0 \leq i \leq d$,

$$(A - \theta_d I)U_i^\downarrow \subseteq U_{i+1}^\downarrow, \quad (A^* - \theta_d^* I)U_i^\downarrow \subseteq U_{i-1}^\downarrow.$$

We recall the definition of $q$-Racah type. Following [8, Definition 3.1] and [6, Definition 5.1], we say that $A, A^*$ has $q$-Racah type whenever there exist nonzero $a, b \in \mathbb{F}$ such that both

$$\theta_i = aq^{d-2i} + a^{-1}q^{2i-d}, \quad \theta_i^* = bq^{d-2i} + b^{-1}q^{-2i-d}$$

for $0 \leq i \leq d$. For the rest of this section assume that $A, A^*$ has $q$-Racah type. For $1 \leq i \leq d$ we have $q^{2i} \neq 1$; otherwise $\theta_0 = \theta_i$. Also $a^2 \neq 1$; otherwise $\theta_0 = \theta_d$. Similarly $b^2 \neq 1$. We now recall the maps $K, B$.

**Definition 12.3** [4, Definitions 3.1, 3.2]. Define an $\mathbb{F}$-linear map $K : V \to V$ such that for $0 \leq i \leq d$, $U_i$ is an eigenspace of $K$ with eigenvalue $q^{d-2i}$. Thus

$$(K - q^{d-2i} I)U_i = 0 \quad (0 \leq i \leq d).$$

Define an $\mathbb{F}$-linear map $B : V \to V$ such that for $0 \leq i \leq d$, $U_i^\downarrow$ is an eigenspace of $B$ with eigenvalue $q^{d-2i}$. Thus

$$(B - q^{d-2i} I)U_i^\downarrow = 0 \quad (0 \leq i \leq d).$$

**Lemma 12.4** [4, Lemma 3.6, Theorem 9.9]. The maps $K, B, A$ satisfy

$$\frac{qKA - a^{-1}AK}{q - q^{-1}} = aK^2 + a^{-1}I, \quad \frac{qBA - a^{-1}AB}{q - q^{-1}} = a^{-1}B^2 + aI,$$

$$aK^2 - \frac{q^{-1}aq - a^{-1}q^{-1}}{q - q^{-1}}KB - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}}BK + a^{-1}B^2 = 0.$$

We are done with our summary. We now use $A, A^*$ to construct two $U_q^\vee$-module structures on $V$. Here is the first one.
Theorem 12.5 There exists a unique $\mathcal{U}_q$-module structure on $V$ for which the generators $z, Z, A$ act as follows:

| generator | z | Z | A |
|-----------|---|---|---|
| action on $V$ | $K$ | $B$ | $A$ |

On the $\mathcal{U}_q$-module $V$ the elements $x, X, y^{-1}$ act as follows:

| element | x | X | $y^{-1}$ |
|---------|---|---|---------|
| action on $V$ | $aA - a^2K$ | $a^{-1}A - a^{-2}B$ | $\frac{aK - a^{-1}B}{a - a^{-1}}$ |

Proof: To see that the given $\mathcal{U}_q$-module structure exists, compare the three equations in Lemma 12.3 with the defining relations for $\mathcal{U}_q$ given in Theorem 10.19. The $\mathcal{U}_q$-module structure is unique since $z, Z, A$ generate $\mathcal{U}_q$. The actions of $x$ and $y^{-1}$ on $V$ are obtained using the identifications involving those elements given in Theorem 10.19. The action of $X$ on $V$ is found using $A = aX + a^{-1}Z$. \[\square\]

Proposition 12.6 On the $\mathcal{U}_q$-module $V$ from Theorem 12.5, the elements $\nu_x y^{-1}, \nu_y, \nu_z y^{-1}, \Lambda y^{-1}$ act as follows.

(i) $\nu_x y^{-1}$ acts as

$$q^{-1}a^{-1}K - B$$

(ii) $\nu_y$ acts as

$$a\left(\frac{KA - AK}{q - q^{-1}}\right).$$

(iii) $\nu_z y^{-1}$ acts as $qa$ times

$$a^2K - a^{-2}B$$

(iv) $\Lambda y^{-1}$ acts as each of

$$q^{-1}\frac{A}{a - a^{-1}} \frac{K - B}{a - a^{-1}} + a\frac{q - q^{-1}}{a - a^{-1}}K^2 + \frac{aq^{-1} - a^{-1}q}{a - a^{-1}}KB + q^{-1}I,$$

$$q\frac{A}{a - a^{-1}} \frac{K - B}{a - a^{-1}} - a\frac{q - q^{-1}}{a - a^{-1}}K^2 + \frac{aq - a^{-1}q^{-1}}{a - a^{-1}}BK + qI,$$

$$q^{-1}\frac{A}{a - a^{-1}} \frac{K - B}{a - a^{-1}} - a^{-1}\frac{q - q^{-1}}{a - a^{-1}}B^2 + \frac{aq - a^{-1}q^{-1}}{a - a^{-1}}BK + q^{-1}I,$$

$$q\frac{A}{a - a^{-1}} \frac{K - B}{a - a^{-1}} + a^{-1}\frac{q - q^{-1}}{a - a^{-1}}B^2 + \frac{aq^{-1} - a^{-1}q}{a - a^{-1}}KB + qI.$$
Proof: Apply Proposition 10.24 with \( z = K, \ Z = B, \ A = A \). \( \square \)

We have a comment.

Lemma 12.7 \([1, \text{Lemma } 9.7] \). The map \( aI - a^{-1}BK^{-1} \) is invertible.

Lemma 12.8 On the \( U_q \)-module \( V \) from Theorem 12.5, the element \( y^{-1} \) is invertible.

Proof: By Theorem 12.5 \( y^{-1} \) acts on \( V \) as a nonzero scalar multiple of \( aK - a^{-1}B \). We have \( aK - a^{-1}B = (aI - a^{-1}BK^{-1})K \), and \( aI - a^{-1}BK^{-1} \) is invertible by Lemma 12.7. The result follows. \( \square \)

We now bring in the Bockting operator \( \Psi \) associated with \( A, A^* \). This was introduced in \([3] \) and investigated further in \([4] \). Following \([4] \) we will work with the normalized version

\[
\psi = (q - q^{-1})(q^d - q^{-d})\Psi.
\]

One feature of \( \psi \) is that \( \psi U_i \subseteq U_{i-1} \) and \( \psi U_i^\circ \subseteq U_{i-1}^\circ \) for \( 0 \leq i \leq d \) \([3, \text{Lemma } 11.2, \text{Corollary } 15.3] \). By \([4, \text{Theorem } 9.8] \),

\[
\psi = q^{-1}(I - BK^{-1})(aI - a^{-1}BK^{-1})^{-1}. \quad (63)
\]

Proposition 12.9 The \( U_q^\circ \)-module \( V \) from Theorem 12.5 extends to a \( U_q(\mathfrak{sl}_2) \)-module. On the \( U_q(\mathfrak{sl}_2) \)-module \( V \) we have \( \nu_x = a^{-1}\psi \).

Proof: The first assertion follows from Lemma 11.5(i),(ii) and Lemma 12.8. We now prove the second assertion. Using Theorem 12.5 the following holds on \( V \):

\[
y^{-1} = \frac{aK - a^{-1}B}{a - a^{-1}} = \frac{aI - a^{-1}BK^{-1}}{a - a^{-1}}K. \quad (64)
\]

Using (63), (64) we find \( \psi y^{-1} = q^{-1}(K - B)(a - a^{-1})^{-1} \), which is equal to \( a\nu_x y^{-1} \) by Proposition 12.6(i). Now invoking Lemma 12.8 we see that \( \nu_x = a^{-1}\psi \) on \( V \). \( \square \)

Proposition 12.10 On the \( U_q(\mathfrak{sl}_2) \)-module \( V \) from Proposition 12.9, the action of \( y \) coincides with each of the following:

\[
K^{-1}(1 - a^{-1}q\psi), \quad (1 - a^{-1}q^{-1}\psi)K^{-1}, \quad B^{-1}(1 - aq\psi), \quad (1 - aq^{-1}\psi)B^{-1}.
\]

Proof: The four displayed expressions are equal by \([4, \text{Proposition } 9.2] \). We show that \( y = K^{-1}(1 - a^{-1}q\psi) \) on \( V \). Recall that \( \nu_x = q^{-1}(1 - zy) \). We have seen that on \( V \), \( \nu_x = a^{-1}\psi \) and \( z = K \). By these comments \( a^{-1}\psi = q^{-1}(1 - Ky) \) on \( V \). Solve this equation for \( y \) to find that \( y = K^{-1}(1 - a^{-1}q\psi) \) on \( V \). \( \square \)
Proposition 12.11  On the $U_q(\mathfrak{sl}_2)$-module $V$ from Proposition 12.9, the action of $\Lambda$ coincides with each of the following:

\begin{align*}
(A - aK - a^{-1}K^{-1})\psi + qK + q^{-1}K^{-1}, & \quad (65) \\
\psi(A - aK - a^{-1}K^{-1}) + q^{-1}K + qK^{-1}, & \quad (66) \\
(A - a^{-1}B - aB^{-1})\psi + qB + q^{-1}B^{-1}, & \quad (67) \\
\psi(A - a^{-1}B - aB^{-1}) + q^{-1}B + qB^{-1}. & \quad (68)
\end{align*}

Proof: The expressions (65)–(68) are equal by \cite{4, Lemma 9.1}. Using in order Proposition 12.10, Theorem 12.5, Proposition 12.9, and line (18), we find that on $V$ the expression (65) is equal to

\begin{align*}
(A - aK)\psi + qK + q^{-1}K^{-1}(1 - a^{-1}q\psi) \\
= (A - aK)\psi + qK + q^{-1}y \\
= qx + q^{-1}y + qz - qxyz,
\end{align*}

which is equal to $\Lambda$ by Lemma 3.4. The result follows. □

We are done discussing the $U_q^\vee$-module $V$ from Theorem 12.5. We now consider another $U_q^\vee$-module structure on $V$.

Lemma 12.12  The maps $K, B, A$ satisfy

\begin{align*}
\frac{qAK^{-1} - q^{-1}K^{-1}A}{q - q^{-1}} = a^{-1}K^{-2} + aI, \\
\frac{qAB^{-1} - q^{-1}B^{-1}A}{q - q^{-1}} = aB^{-2} + a^{-1}I.
\end{align*}

Proof: These are reformulations of the first two equations in Lemma 12.4. □

Lemma 12.13  \cite{4, Theorem 9.10}, The maps $K, B$ satisfy

\begin{align*}
a^{-1}K^{-2} - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}}K^{-1}B^{-1} - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}}B^{-1}K^{-1} + aB^{-2} = 0.
\end{align*}

Theorem 12.14  There exists a $U_q^\vee$-module structure on $V$ for which the generators $x, X, A$ act as follows:

| generator | $x$ | $X$ | $A$ |
|-----------|-----|-----|-----|
| action on $V$ | $K^{-1}$ | $B^{-1}$ | $A$ |

On this $U_q^\vee$-module $V$ the elements $z, Z, y^{-1}$ act as follows:

| element | $z$ | $Z$ | $y^{-1}$ |
|---------|-----|-----|---------|
| action on $V$ | $a^{-1}A - a^{-2}K^{-1}$ | $aA - a^2B^{-1}$ | $\frac{a^{-1}K^{-1} - aB^{-1}}{a^{-1} - a}$ |
Proof: To see that the given $U_q^\vee$-module structure exists, compare the equations in Lemmas 12.12, 12.13 with the defining relations for $U_q^\vee$ given in Theorem 10.18. The $U_q^\vee$-module structure is unique since $x, X, A$ generate $U_q^\vee$. The actions of $y^{-1}$ and $z$ on $V$ are obtained using the identifications involving those elements given in Theorem 10.18. The action of $Z$ on $V$ is found using $A = aX + a^{-1}Z$. 

Proposition 12.15 On the $U_q^\vee$-module $V$ from Theorem 12.14, the elements $\nu_x y^{-1}$, $\nu_y$, $\nu_z y^{-1}$, $\Lambda y^{-1}$ act as follows.

(i) $\nu_x y^{-1}$ acts as $q^{-1}a^{-1}$ times

$$a^2B^{-1} - a^{-2}K^{-1} - A.$$

(ii) $\nu_y$ acts as

$$a^{-1}AK^{-1} - K^{-1}A.$$ 

(iii) $\nu_z y^{-1}$ acts as

$$qaB^{-1} - K^{-1}.$$ 

(iv) $\Lambda y^{-1}$ acts as each of

$$qA\frac{B^{-1} - K^{-1}}{a - a^{-1}} + a^{-1}q^{-1}K^{-2} + \frac{aq^{-1} - a^{-1}q}{a - a^{-1}}K^{-1}B^{-1} + qI,$$

$$q^{-1}B^{-1} - K^{-1} - A - a^{-1}q^{-1}K^{-2} + \frac{aq^{-1} - a^{-1}q}{a - a^{-1}}B^{-1}K^{-1} + q^{-1}I,$$

$$qA\frac{B^{-1} - K^{-1}}{a - a^{-1}} - a^{-1}q^{-1}B^{-2} + \frac{aq^{-1} - a^{-1}q}{a - a^{-1}}B^{-1}K^{-1} + qI,$$

$$q^{-1}B^{-1} - K^{-1} - A + a^{-1}q^{-1}B^{-2} + \frac{aq^{-1} - a^{-1}q}{a - a^{-1}}K^{-1}B^{-1} + q^{-1}I.$$ 

Proof: Apply Proposition 10.23 with $x = K^{-1}$, $X = B^{-1}$, $A = A$. 

Lemma 12.16 On the $U_q^\vee$-module $V$ from Theorem 12.14, the element $y^{-1}$ is invertible.

Proof: Similar to the proof of Lemma 12.8. 

Proposition 12.17 The $U_q^\vee$-module $V$ from Theorem 12.14 extends to a $U_q(\mathfrak{sl}_2)$-module. On the $U_q(\mathfrak{sl}_2)$-module $V$ we have $\nu_z = a\psi$. 

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Proof: It suffices to show that $y^{-1}v_z = ay^{-1}\psi$ on $V$. To show this, use $y^{-1}v_z = q^{-2}v_zy^{-1}$ and an argument similar to the proof of Proposition 12.9.

Proposition 12.18 On the $U_q(sl_2)$-module $V$ from Proposition 12.17 the action of $y$ coincides with each of the following:

\[ K(1 - aq^{-1}\psi), \quad (1 - aq\psi)K, \]
\[ B(1 - a^{-1}q^{-1}\psi), \quad (1 - a^{-1}q\psi)B. \]

Proof: The four displayed expressions are equal by [4, Proposition 9.2]. To show that $y = K(1 - aq^{-1}\psi)$ on $V$, proceed as in the proof of Proposition 12.10.

Proposition 12.19 On the $U_q(sl_2)$-module $V$ from Proposition 12.17, the action of $\Lambda$ coincides with each of (65)–(68).

Proof: Similar to the proof of Proposition 12.11.

Corollary 12.20 The following coincide:

(i) the action of $\Lambda$ on the $U_q(sl_2)$-module $V$ from Proposition 12.9;

(ii) the action of $\Lambda$ on the $U_q(sl_2)$-module $V$ from Proposition 12.17.

Proof: Compare Proposition 12.11 and Proposition 12.17.

To summarize, in each of Theorems 12.5 and 12.14 we displayed a $U_q^{\vee}$-module structure on $V$. Using Propositions 12.9 and 12.17 we extended each $U_q^{\vee}$-module $V$ to a $U_q(sl_2)$-module $V$. Holding on these $U_q(sl_2)$-modules are all the $U_q(sl_2)$ relations from Sections 2–9, such as in Corollary 7.11, Lemma 7.15, Proposition 7.17 and Lemma 7.20. We expect that this information will be useful in future developments concerning the theory of tridiagonal pairs.

13 Directions for future research

In this section we make some suggestions for future research.

The following problem is motivated by Lemma 3.11. Note that the relations displayed in that lemma are invariant under the move $q \mapsto q^{-1}$.

Problem 13.1 Consider the $F$-algebra $U$ defined by generators $x, y, z$ and the relations from Lemma 3.11. Find a basis for the $F$-vector space $U$. Note that the symmetric group $S_3$ acts on $U$ as a group of automorphisms, by permuting $x, y, z$. Observe that there exists an $F$-algebra homomorphism $U \to U_q(sl_2)$ that sends $x \mapsto x$, $y \mapsto y$, $z \mapsto z$; let $J_q$ denote the kernel. Similarly there exists an $F$-algebra homomorphism $U \to U_{q^{-1}}(sl_2)$ that sends $x \mapsto x$, $y \mapsto y$, $z \mapsto z$; let $J_{q^{-1}}$ denote the kernel. Describe the ideals $J_q \cap J_{q^{-1}}$ and $J_q + J_{q^{-1}}$. 

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Problem 13.2 Find all the \(\mathbb{Z}_2\)-gradings of \(U_q(\mathfrak{sl}_2)\).

Problem 13.3 By Proposition 5.4(ii) the \(\mathbb{F}\)-algebra \(U'_{\text{even}}\) is generated by \(\nu_x, \nu_y, \nu_z\). Using these generators, find a presentation of \(U'_{\text{even}}\) by generators and relations. Investigate \(U'_{\text{odd}}\) as a \(U'_{\text{even}}\)-module.

To motivate the next problem we have some comments. Observe that for \(\varepsilon \in \{1,-1\}\) there exists an \(\mathbb{F}\)-algebra homomorphism \(U_q(\mathfrak{sl}_2) \to \mathbb{F}\) that sends \(x \mapsto \varepsilon, y \mapsto \varepsilon, z \mapsto \varepsilon\). For these two homomorphisms the intersection of their kernels will be denoted by \(J\). Note that \(J\) is a 2-sided ideal of \(U_q(\mathfrak{sl}_2)\).

Problem 13.4 Show that the above 2-sided ideal \(J\) is generated by \(\nu_x, \nu_y, \nu_z\). Also show that \(U_q(\mathfrak{sl}_2) = J + \mathbb{F}1 + \mathbb{F}y\) (direct sum of vector spaces).

Problem 13.5 Show that the center of \(U'_q\) is equal to \(\mathbb{F}1\), provided that \(q\) is not a root of unity.

The following problem is motivated by Corollary 11.10.

Problem 13.6 Classify up to isomorphism the finite-dimensional indecomposable \(U'_q\)-modules on which \(y^{-1}\) is nilpotent.

Note 13.7 We mention some infinite-dimensional \(U'_q\)-modules on which \(y^{-1}\) is nilpotent. Recall the basis (9) for \(U_q(\mathfrak{sl}_2)\). For \(i \in \mathbb{Z}\) let \(U_i\) denote the subspace of \(U_q(\mathfrak{sl}_2)\) spanned by those basis elements \(x^r y^s z^t\) such that \(s \leq i\). Thus \(U_0 = U'_q\). We have \(U_{i-1} \subseteq U_i\) for \(i \in \mathbb{Z}\). Moreover \(U_q(\mathfrak{sl}_2) = \bigcup_{i \in \mathbb{Z}} U_i\). One checks that
\[
x U_i \subseteq U_i, \quad z U_i \subseteq U_i, \quad y U_i = U_{i+1}, \quad y^{-1} U_i = U_{i-1}, \quad i \in \mathbb{Z}.
\]
Moreover \(U_i U_j \subseteq U_{i+j}\) for \(i, j \in \mathbb{Z}\). Thus the sequence \(\{U_i\}_{i \in \mathbb{Z}}\) is a \(\mathbb{Z}\)-filtration of \(U_q(\mathfrak{sl}_2)\). View \(U_q(\mathfrak{sl}_2)\) as a \(U'_q\)-module such that \(\xi\) sends \(v \mapsto \xi v\) for all \(\xi \in U'_q\) and \(v \in U_q(\mathfrak{sl}_2)\). For \(i \in \mathbb{Z}\) the subspace \(U_i\) is a \(U'_q\)-submodule. For \(i, j \in \mathbb{Z}\) such that \(j < i\), consider the quotient \(U'_q\)-module \(U_i/U_j\). For \(0 \leq r \leq i - j\) the image (resp. kernel) of \(y^{-r}\) is \(U_{i-r}/U_j\) (resp. \(U_{j+r}/U_j\)). In particular \(y^{i-j}\) is zero on \(U_i/U_j\), so \(y^{-1}\) is nilpotent on \(U_i/U_j\).

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