MULTIPLICATIVE DEPENDENCE AMONG ITERATED VALUES OF RATIONAL FUNCTIONS MODULO FINITELY GENERATED GROUPS

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Abstract. We study multiplicative dependence between elements in orbits of algebraic dynamical systems over number fields modulo a finitely generated multiplicative subgroup of the field. We obtain a series of results, many of which may be viewed as a blend of Northcott’s theorem on boundedness of preperiodic points and Siegel’s theorem on finiteness of solutions to $S$-unit equations.

1. Introduction and statements of main results

1.1. Motivation. Let $\mathbb{K}/\mathbb{Q}$ be a number field with algebraic closure $\overline{\mathbb{K}}$, and let $f(X) \in \mathbb{K}(X)$ be a rational function of degree at least 2. A famous theorem of Northcott [18] says that the set of $\overline{\mathbb{K}}$-preperiodic points of $f$ is a set of bounded Weil height. In particular, the set of $\mathbb{K}$-rational preperiodic points is finite. In [19] this result is extended to cover the case that points in an orbit are multiplicatively dependent, rather than forcing them to be equal. In this paper we extend this further to the case of points in an orbit that are multiplicatively dependent modulo a finitely generated subgroup $\Gamma$ of $\mathbb{K}^\ast$. Anticipating notation that is described in Section 1.2, we are interested in solutions $(n, k, \alpha, r, s)$ to the relation

$$f^{(n+k)}(\alpha)^r \cdot f^{(k)}(\alpha)^s \in \Gamma,$$

where $n \geq 1$, $k \geq 0$, $\alpha \in \mathbb{K}$, and $(r, s) \neq (0, 0)$. We view this as a combination of Northcott’s already cited theorem and Siegel’s theorem concerning integral points on affine curves, which at its heart deals with integral solutions to equations of the form $F(x, y) \in \Gamma$ for homogenous $F \in \mathbb{K}[X, Y]$. And indeed, a key tool in the proof of several of our main results is a theorem on dynamical Diophantine approximation (Lemma 2.5) that ultimately relies on Roth’s theorem and is very much

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analogous to the use of Diophantine approximation in the proof of Siegel’s theorem.

1.2. Notation and conventions. We now set the following notation, which remains fixed for the remainder of this paper:

- \( \mathbb{K} \) is a number field.
- \( \mathcal{O}_K \) is the ring of algebraic integers of \( \mathbb{K} \).
- \( \overline{\mathbb{K}} \) is an algebraic closure of \( \mathbb{K} \).
- \( f(X) \in \mathbb{K}(X) \) a rational function of degree \( d \geq 2 \).
- For \( n \geq 0 \), we write \( f^{(n)}(X) \) for the \( n \)th iterate of \( f \), i.e.,
  \[
  f^{(n)}(X) := f \circ f \circ \cdots \circ f(X) \quad \text{n copies}
  \]
- For \( \alpha \in \mathbb{P}^1(\mathbb{K}) \), we write \( \mathcal{O}_f(\alpha) \) for the (forward) orbit of \( \alpha \), i.e.,
  \[
  \mathcal{O}_f(\alpha) := \{ f^{(n)}(\alpha) : n \geq 0 \}
  \]
- \( \Gamma \) is a finitely generated subgroup of \( \mathbb{K}^* \).
- \( \text{PrePer}(f) \) is the set of preperiodic points of \( f \) in \( \mathbb{P}^1(\overline{\mathbb{K}}) \), i.e., the set of points \( \alpha \in \mathbb{P}^1(\overline{\mathbb{K}}) \) such that \( \mathcal{O}_f(\alpha) \) is finite.
- \( \text{Wander}_\mathbb{K}(f) \) is the complement of the set \( \text{PrePer}(f) \) in \( \mathbb{P}^1(\mathbb{K}) \), i.e., the set \( \mathbb{P}^1(\mathbb{K}) \setminus (\text{PrePer}(f) \cap \mathbb{K}) \) of \( \mathbb{K} \)-rational wandering points for \( f \).
- \( \mathbb{Z}_{\geq r} \) denotes the set of integers \( n \geq r \), where \( r \) is a real number.

It is also convenient to define the function
\[
\log^+ t = \log \max\{ t, 1 \}.
\]

We use \( M_\mathbb{K} \) to denote a complete set of inequivalent absolute values on \( \mathbb{K} \), normalized so that the absolute Weil height \( h : \mathbb{K} \to [0, \infty) \) is defined by
\[
(1.1) \quad h(\beta) = \sum_{\nu \in M_\mathbb{K}} \log^+ (\|\beta\|_\nu),
\]
and we write \( M_\mathbb{K}^\infty \) and \( M_\mathbb{K}^0 \) for, respectively, the set of archimedean and non-archimedean absolute values in \( M_\mathbb{K} \). See [11, 16] for further details on absolute values and height functions.

As in [23, Section 1.2], for a rational function \( f(X) \in \mathbb{C}(X) \) and \( \alpha \in \mathbb{C} \) with \( \alpha \neq \infty \) and \( f(\alpha) \neq \infty \), we define the ramification index of \( f \) at \( \alpha \) as the order of \( \alpha \) as a zero of the rational function \( f(X) - f(\alpha) \), i.e.,
\[
eq f(\alpha) = \text{ord}_\alpha(f(X) - f(\alpha)).
\]
In particular, we say that \( f \) is ramified at \( \alpha \) if \( e_f(\alpha) \geq 2 \), and totally ramified at \( \alpha \) if \( e_f(\alpha) = \deg f \). If \( \alpha = \infty \) or \( f(\alpha) = \infty \), we define \( e_f(\alpha) \).
by choosing a linear fractional transformation \( L \in \text{PGL}_2(\mathbb{C}) \) so that \( \beta = L^{-1}(\alpha) \) satisfies \( \beta \neq \infty \) and \( f_L(\beta) \neq \infty \), where \( f_L = L^{-1} \circ f \circ L \), and then we set \( e_f(\alpha) = e_{f_L}(\beta) \).

It is an exercise using the chain rule to show that \( e_f(\alpha) \) does not depend on the choice of \( L \); cf. [23, Exercise 1.5]. It is also an exercise to show that the ramification index is multiplicative under the composition, i.e., for rational functions \( f, g \in \mathbb{C}(X) \) and \( \alpha \in \mathbb{C} \cup \{\infty\} \), we have

\[
e_{g \circ f}(\alpha) = e_f(\alpha)e_g(f(\alpha)).
\]

1.3. Main results. In this section we describe the main results proven in this paper. In order to state our results, we define three sets of “exceptional values”. Our theorems characterize when these sets may be infinite.

**Definition 1.1.** With \( K, f, \Gamma \) as defined in Section 1.2, and for integers \( r, s \in \mathbb{Z} \) and real number \( \rho > 0 \), we define various sets of exceptional values:

\[
E_\rho(K, f, \Gamma, r, s) = \{(n, k, \alpha, u) \in \mathbb{Z}_{\geq \rho} \times \mathbb{Z}_{\geq 0} \times \text{Wander}_K(f) \times \Gamma : f^{(n+k)}(\alpha)^r = uf^{(k)}(\alpha)^s\};
\]

\[
F_\rho(K, f, \Gamma) = \{(n, \alpha) \in \mathbb{Z}_{\geq \rho} \times \text{Wander}_K(f) : f^{(n)}(\alpha) \in \Gamma\};
\]

\[
G(K, f, \Gamma) = \{\alpha \in K : f(\alpha) \in \Gamma\}.
\]

Clearly the finiteness of \( G(K, f, \Gamma) \) is equivalent to the finiteness of \( f(K) \cap \Gamma \). Thus, the first part of the following result is given in [15, Proposition 1.5(a)] in the form

\[
\#(f(K) \cap \Gamma) = \infty \quad \implies \quad \#f^{-1}(\{0, \infty\}) \leq 2,
\]

see also [20, Corollary 2.2]. We observe that the functions in Theorem 1.2(a) below are exactly the functions having this last property.

**Theorem 1.2.** We have:

(a) If the set \( G(K, f, \Gamma) \) is infinite, then \( f(X) \) has one of the following forms:

\[
f(X) = a(X - b)^{\pm d} \quad \text{with} \ a \neq 0,
\]

\[
f(X) = a(X - b)^d/(X - c)^d \quad \text{with} \ a(b - c) \neq 0.
\]

(b) If the set \( F_2(K, f, \Gamma) \) is infinite, then \( f(X) \) has the form \( f(X) = aX^{\pm d} \).

So next we only deal with the case \( rs \neq 0 \).
Theorem 1.3. Let \( r, s \in \mathbb{Z} \) with \( rs \neq 0 \), and set
\[
\rho = \frac{\log(|s|/|r|)}{\log d} + 1.
\]
Assume that 0 is not a periodic point of \( f \). Then
\[
\#E_\rho(\mathbb{K}, f, \Gamma, r, s) = \infty \implies \#f^{-1}(\{0, \infty\}) \leq 2.
\]

For notational convenience, let
\[
(1.3) \quad E_\rho(\mathbb{K}, f, \Gamma) := E_\rho(\mathbb{K}, f, \Gamma, 1, 1) \quad \text{and} \quad E(\mathbb{K}, f, \Gamma) := E_1(\mathbb{K}, f, \Gamma).
\]

Setting \( r = s = 1 \) in Theorem 1.3, so \( \rho = 1 \), gives a finiteness result for the set \( E(\mathbb{K}, f, \Gamma) \), which is the set of solutions to the equation
\[
f^{(n+k)}(\alpha) = uf^{(k)}(\alpha),
\]
\[
(n, k, \alpha, u) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0} \times \text{Wander}_K(f) \times \Gamma.
\]

In this situation we are able to give a full classification of the exceptional cases, i.e., a complete description of the maps \( f \) for which (1.4) may have infinitely many solutions.

Theorem 1.4. For the sets (1.3), we have:

(a) If \( E(\mathbb{K}, f, \Gamma) \) is infinite, then either \( f(\mathbb{X}) \) or \( f(\mathbb{X}^{-1})^{-1} \) has one of the following forms with \( abc(b - c) \neq 0 \):
\[
aX^{\pm d}, \quad aX^d/(X - b)^{d-1}, \quad aX(X - b)^{d-1},
\]
\[
aX/(X - b)^d, \quad aX(X - b)^{d-1}/(X - c)^{d-1}.
\]

(b) If \( E_2(\mathbb{K}, f, \Gamma) \) is infinite, then \( f(\mathbb{X}) \) has the form \( f(\mathbb{X}) = aX^{\pm d} \).

Remark 1.5. An analysis similar to the proof of Theorem 1.4 can be used to describe all \( f(\mathbb{X}) \) for which \( E(\mathbb{K}, f, \Gamma) \) may have infinitely many four-tuples \((1, k, \alpha, u)\) with \( k \geq 2 \).

Remark 1.6. If \( f(\mathbb{X}) \) is a polynomial, it suffices to assume in Theorem 1.4 that 0 is not a periodic point of \( f \) to ensure that \( E(\mathbb{K}, f, \Gamma) \) is finite. However, for rational functions there are examples such as \( f(\mathbb{X}) = (1 - \mathbb{X})^2/\mathbb{X} \) with 0 strictly preperiodic and
\[
(1, 0, 1/(u + 1), u^2) \in E(\mathbb{K}, f, \Gamma) \quad \text{for all} \ u \in \Gamma.
\]

In the case that \( f \in \mathbb{K}[\mathbb{X}] \) is a polynomial, we have the following broad extension of Theorem 1.3 in which the exponents \( r \) and \( s \) are not necessarily fixed. This allows us to bound all multiplicative dependences between \( f^{(m)}(\alpha) \) and \( f^{(n)}(\alpha) \) modulo \( \Gamma \).
Theorem 1.7. Let $f \in \mathbb{K}[X]$ be a polynomial without multiple roots, of degree $d \geq 3$ or, if $d = 2$, we also assume that $f^{(2)}$ has no multiple roots. Assume that 0 is not a periodic point of $f$. Let $\Gamma \subseteq \mathbb{K}^*$ be a finitely generated subgroup. Then there are only finitely many elements $\alpha \in \mathbb{K}$ such that for distinct integers $m, n \geq 1$, the values $f^{(m)}(\alpha)$ and $f^{(n)}(\alpha)$ are multiplicatively dependent modulo $\Gamma$.

Remark 1.8. Note that Theorem 1.7 fails if we allow $m$ or $n$ to be 0, since for any $u \in \Gamma$ and any $m \geq 1$, there is a multiplicative relation $(f^{(m)}(u))^0 \cdot f^{(0)}(u) = u \in \Gamma$.

Finally, we present an independence result of a slightly different type.

Definition 1.9. Let $k \geq 1$. A polynomial $F(T_1, \ldots, T_k) \in \mathbb{K}[T_1, \ldots, T_k]$ is said to be a multilinear polynomial with split variables if there are scalars $c_1, \ldots, c_k \in \mathbb{K}^*$ and a disjoint partition $J_1 \cup J_2 \cup \cdots \cup J_r = \{1, 2, \ldots, k\}$ of the set $\{1, \ldots, k\}$ so that $F$ has the form

$$F(T_1, \ldots, T_k) = \sum_{i=1}^{r} c_i \prod_{j \in J_i} T_j.$$

In other words, $F$ is a linear combination of monomials in the variables $T_1, \ldots, T_k$ with the property that each variable appears in exactly one monomial and to exactly the first power. We also define the height of $F$ to be

$$h(F) := \max_{1 \leq i \leq r} h(c_i).$$

Theorem 1.10. Let $F(T_1, \ldots, T_k) \in \mathbb{K}[T_1, \ldots, T_k]$ be a multilinear polynomial with split variables. Let $f(X) \in \mathbb{K}(X)$ be a rational function of degree $d \geq 2$.

(a) The set of $\alpha \in \overline{\mathbb{K}}$ for which there exists a $k$-tuple of distinct non-negative integers $(n_1, n_2, \ldots, n_k)$ satisfying

$$F \left( f^{(n_1)}(\alpha), f^{(n_2)}(\alpha), \ldots, f^{(n_k)}(\alpha) \right) = 0$$

is a set of bounded height.

(b) If $d \geq 3$, then for $\alpha$ as in (a), we have the explicit upper bound

$$h(\alpha) \leq \frac{2k}{d^k-1} h(F) + \frac{7}{3} C_1(f) + \frac{2}{9} \log 2,$$

where $C_1(f)$ is the constant appearing in Lemma 2.1(a).
(c) Let $\alpha \in \text{Wander}_K(f)$ have the property that $0 \notin \mathcal{O}_f(\alpha)$. Then there are only finitely many $k$-tuples of integers $n_1 > n_2 > \cdots > n_k \geq 0$ satisfying (1.5), and there is a bound for the number of such solutions that depends only on $K$, $f$ and $F$, independent of $\alpha$.

1.4. Multiplicative dependence and Zsigmondy-type results. Many of our results on multiplicative independence would follow from a sufficiently strong dynamical Zsigmondy theorem on primitive divisors, but despite considerable attention in recent years, there are no general unconditional result of the type that we would need.

We briefly expand on this remark. Let $f(X)$ be a rational function, and let $\alpha$ be a wandering point for $f$. We recall that a valuation $v$ is a primitive divisor of $f^{(n)}(\alpha)$ if $v(f^{(m)}(\alpha)) \neq 0$ and $v(f^{(m)}(\alpha)) = 0$ for all $m < n$. The dynamical Zsigmondy set associated to $(f, \alpha)$ is the set of $n$ such that $f^{(n)}(\alpha)$ does not have a primitive divisor. With appropriate conditions on $f$ and $\alpha$ to rule out trivial counterexamples, it is conjectured that the dynamical Zsigmondy set of $(f, \alpha)$ is always finite.

There are a few unconditional results. Among them we mention [6, 13], which prove Zsigmondy finiteness when 0 is a preperiodic point, and [4, 14], which prove Zsigmondy finiteness for unicritical binomials $f(X) = X^d + c$ over $\mathbb{Q}$. Unfortunately, results such as our Theorem 1.3 require that 0 not be periodic, so the former is not too helpful, and although the latter can be used in the context of Theorem 1.3, it applies only to a very restricted class of polynomials.

The only known general results on finiteness of the dynamical Zsigmondy set are conditional on very strong and difficult conjectures such as the ABC conjecture or Vojta’s conjecture; see, for example, [2, 9, 10, 24].

It is also interesting to recall that for slower growing sequences, finiteness of the Zsigmondy set may fail. For example, the Zsigmondy set of a collection of polynomial values $\{f(n) : n \in \mathbb{Z}\}$ has infinite Zsigmondy set; see [5].

2. Preliminaries

2.1. Background on heights and valuations of iterations. In this section we collect some useful facts from Diophantine geometry and arithmetic dynamics. We begin by recalling some standard properties of the canonical height function associated to $f$.

Lemma 2.1. There exists a unique function

$$\hat{h}_f : \mathbb{P}^1(K) \rightarrow [0, \infty),$$

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with the following properties:

(a) There is a constant \( C_1(f) \) such that

\[ -C_1(f) \leq \hat{h}_f(\gamma) - h(\gamma) \leq C_1(f) \quad \text{for all } \gamma \in \mathbb{P}^1(\mathbb{K}). \]

(b) For all \( \gamma \in \mathbb{P}^1(\mathbb{K}) \) we have, \( \hat{h}_f(f(\gamma)) = d \hat{h}_f(\gamma) \).

(c) For all \( \gamma \in \mathbb{P}^1(\mathbb{K}) \), we have \( \hat{h}_f(\gamma) = 0 \) if and only if \( \gamma \in \text{PrePer}(f) \).

(d) There is a strict inequality

\[ C_2(\mathbb{K}, f) := \inf \left\{ \hat{h}_f(\gamma) : \gamma \in \text{Wander}_\mathbb{K}(f) \right\} > 0. \]

Proof. For standard properties of dynamical canonical height functions, including proofs of (a,b,c), see, for example, [3] or [23, Section 3.4]. We mention that (a) and (b) suffice to determine \( \hat{h}_f \) uniquely. For (d), we first note from (a) that there is an inclusion

\[ \left\{ \gamma \in \mathbb{P}^1(\mathbb{K}) : \hat{h}_f(\gamma) \leq 1 \right\} \subseteq \left\{ \gamma \in \mathbb{P}^1(\mathbb{K}) : h(\gamma) \leq 1 + C_1(f) \right\}. \]

Next we use the fact that \( \mathbb{P}^1(\mathbb{K}) \) contains only finitely many points of bounded height [23, Theorem 3.7]. Hence we may take \( C_2(\mathbb{K}, f) \) to be the smaller of 1 and the minimum of \( \hat{h}_f(\gamma) \) over the finite set of points \( \gamma \in \mathbb{P}^1(\mathbb{K}) \) having infinite orbit and height \( h \) bounded by \( 1 + C_1(f) \), where (c) ensures that each of these finitely many values is strictly positive. \( \square \)

Definition 2.2. The function \( \hat{h}_f \) described in Lemma 2.1 is called the canonical height.

We make frequent use of the following elementary consequence of Lemma 2.1.

Lemma 2.3. Let \( \gamma \in \mathbb{P}^1(\mathbb{K}) \). Then the set

\[ \{(n, \alpha) \in \mathbb{Z}_{\geq 0} \times \text{Wander}_\mathbb{K}(f) : f^{(n)}(\alpha) = \gamma\} \]

is finite.

Proof. Let \( (n, \alpha) \) be in the indicated set. Then

\[ h(\gamma) = h(f^{(n)}(\alpha)) \quad \text{since } f^{(n)}(\alpha) = \gamma, \]

\[ \geq \hat{h}_f(f^{(n)}(\alpha)) - C_1(f) \quad \text{from Lemma 2.1(a)}, \]

\[ = d^n\hat{h}_f(\alpha) - C_1(f) \quad \text{from Lemma 2.1(b)}, \]

\[ \geq d^nC_2(\mathbb{K}, f) - C_1(f) \quad \text{from Lemma 2.1(d)}. \]
Thus $d^n \leq \left( h(\gamma) + C_1(f) \right) C_2(\mathbb{K}, f)^{-1}$ is bounded, since Lemma 2.1(d) says that $C_2(\mathbb{K}, f) > 0$. Hence $n$ is bounded. But for any given $n$, we have

$$h(\gamma) \geq d^n h_f(\alpha) - C_1(f) \geq d^n (h(\alpha) - C_1(f)) - C_1(f),$$

so $h(\alpha) \leq d^{-n}(h(\gamma) + C_1(f)) + C_1(f)$ is bounded, which implies that there are only finitely many possible values for $\alpha$. □

**Definition 2.4.** We recall that $\beta \in \mathbb{P}^1(\mathbb{Q})$ is an exceptional point for $f$ if its backward orbit $O_f^-(\beta) := \{ \gamma \in \mathbb{P}^1(\mathbb{Q}) : \beta \in O_f(\gamma) \}$ is finite.

It is a standard fact, see for example [23, Theorem 1.6], that

$$\#O_f^-(\beta) \in \{ 1, 2, \infty \},$$

and that, after a change of coordinates, the cases $\#O_f^-(\beta) = 1$ and $\#O_f^-(\beta) = 2$ correspond, respectively, to $f(X) \in \mathbb{K}[X]$ and $f(X) = cX \pm d$.

The key to proving Theorem 1.3 is the following dynamical Diophantine approximation result, whose proof ultimately relies on a suitably quantified version of Roth’s theorem.

**Lemma 2.5.** Let $\alpha \in \text{Wander}_{\mathbb{K}}(f)$. Assume that $0$ is not an exceptional point for $f$. Let $S$ be a finite set of places of $\mathbb{K}$, and let $1 \geq \epsilon > 0$. Then there is a constant $C_3(\mathbb{K}, S, f, \epsilon)$ such that

$$\max \left\{ n \in \mathbb{Z}_{\geq 0} : \sum_{v \in S} \log^+ \left( \| f^n(\alpha) \|_v^{-1} \right) \geq \epsilon h_f( f^n(\alpha) ) \right\} \leq C_3(\mathbb{K}, S, f, \epsilon).$$

**Proof.** The finiteness of the indicated set has originally been proven in [22] without an explicit upper bound. The quantified version that we quote here is [12, Theorem 11(c)]. More precisely, we apply [12, Theorem 11(c)] with $A = 0$ and $P = \alpha$. We note that for $A = 0$ and with our normalization of the $v$-adic absolute values, the local distance function $\lambda_v(Q, A)$ in [12] is given by $\lambda_v(\beta, 0) = \log^+ \| \beta \|_v^{-1}$. □

### 2.2. The genus of plane curves of the form $F(X) = cG(X)Y^m$.

There is a well-known formula for the genus of a superelliptic curve $f(X) = Y^m$, where $f(X) \in \mathbb{C}[X]$ is a polynomial; see for example [11, Exercise A.4.6]. In this section we find a similar formula in the case that $f(X) \in \mathbb{C}(X)$ is a rational function, modulo certain restrictions on $m$. We use square brackets $[\ast, \ldots, \ast]$ to denote points in projective space and parentheses $(\ast, \ldots, \ast)$ to denote points in affine space.
**Definition 2.6.** For a polynomial $F(X) \in \mathbb{C}[X]$, we define $\nu(F)$ to be the number of distinct complex roots of $F$, i.e., the number of roots counted without multiplicity.

We frequently apply Definition 2.6 to a product of two relatively primes polynomials $F(X), G(X) \in \mathbb{C}[X]$, in which case

$$\nu(FG) = \# f^{-1}(\{0, \infty\})$$

is equal to the total number of poles and zeros of the rational function $f(X) = F(X)/G(X) \in \mathbb{C}(X)$.

**Lemma 2.7.** Let $F(X), G(X) \in \mathbb{C}[X]$ be non-zero polynomials with no common roots, and assume that they are not both constant. Let $d_F = \deg F$ and $d_G = \deg G$, and let $m$ be an integer satisfying

$$(2.1) \quad m \geq d_F + 2 \quad \text{and} \quad \gcd(m, d_F! \cdot d_G!) = 1.$$

Let $C$ be the affine curve

$$C : F(X) = G(X)Y^m,$$

and let $\widetilde{C}$ be a smooth projective model of $C$.

(a) The curve $\widetilde{C}$ is irreducible.

(b) The genus of $\widetilde{C}$ is given by the formula

$$\text{genus}(\widetilde{C}) = \begin{cases} \frac{1}{2}(\nu(FG) - 1)(m - 1) & \text{if } d_F \neq d_G, \\ \frac{1}{2}(\nu(FG) - 2)(m - 1) & \text{if } d_F = d_G. \end{cases}$$

**Proof.** (a) We need to prove that the polynomial $F(X) - G(X)Y^m$ does not factor in $\mathbb{C}[X, Y]$. We apply [17, Chapter VI, Theorem 9.1] to the polynomial $Y^m - F(X)/G(X)$ in the variable $Y$ with coefficients in the field $\mathbb{C}(X)$. Since $m$ is odd, we see that $Y^m - F(X)/G(X)$ is irreducible in $\mathbb{C}(X)[Y]$ provided for every prime $p \mid m$, the rational function $F(X)/G(X)$ is not a $p$th power in $\mathbb{C}(X)$. But our choice of $m$ ensures that $p \nmid d_F!d_G!$, so $F(X)/G(X)$ cannot be a $p$th power.

(b) Write

$$F(X) = a \prod_{i=1}^{r} (X - \alpha_i)^{e_i} \quad \text{and} \quad G(X) = b \prod_{i=1}^{s} (X - \beta_i)^{\epsilon_i},$$

so $\nu(FG) = r + s$. We have assumed that $m > d_F$, so the Zariski closure $\bar{C}$ of $C$ in $\mathbb{P}^2$ is given by the homogeneous equation

$$\bar{C} : a \prod_{i=1}^{r} (X - \alpha_i Z)^{e_i} Z^{m+d_G-d_F} = cb \prod_{i=1}^{s} (X - \beta_i Z)^{\epsilon_i} Y^m.$$
An elementary calculation, which we give in Appendix A, shows that the singular points of $C$ in $\mathbb{P}^2$ are:

$[1,0,0]$ always, $[\alpha_i,0,1]$ if $e_i \geq 2$, $[0,1,0]$ if $\deg G \neq 1$.

We consider the map $\tilde{C} \to \bar{C}$. In an infinitesimal neighbourhood of the singular the equation of $\bar{C}$ looks like $Z^{m+d_G-d_F} = Y^m$, so there are $\gcd(d_G - d_F, m)$ points of $\bar{C}$ lying over $[1,0,0]$. This gives two cases. If $d_F = d_G$, then there are $m$ points of $\bar{C}$ lying over $[1,0,0]$. On the other hand, if $d_F \neq d_G$, then $1 \leq |d_F - d_G| \leq \max\{d_F, d_G\}$, so (2.1) implies that $\gcd(d_G - d_F, m) = 1$, and thus in this case there is 1 point of $\tilde{C}$ lying over $[1,0,0]$.

Similarly, we note that in an infinitesimal neighbourhood of a singular point $[\alpha_i,0,1]$, the equation of $C$ looks locally like $(X - \alpha_i)^{e_i} = Y^m$.

Blowing up this singularity yields $\gcd(e_i, m)$ points on $\tilde{C}$ lying over $[\alpha_i,0,1]$, and our assumption (2.1) implies that $\gcd(e_i, m) = 1$.

Finally, if $[0,1,0]$ is singular, then dehomogenising $Y = 1$ gives an affine equation for $\bar{C}$ of the form

\[ a \prod_{i=1}^{r} (X - \alpha_i Z)^{e_i} Z^{m+d_G-d_F} = cb \prod_{i=1}^{s} (X - \beta_i Z)^{e_i}. \]

We blow up the point $(0,0)$. We start with the chart on the blowup given by $X = SZ$. Substituting and canceling the common factor of $Z^{dg}$, we obtain the equation

\[ a \prod_{i=1}^{r} (S - \alpha_i)^{e_i} Z^m = cb \prod_{i=1}^{s} (S - \beta_i)^{e_i}. \]

The points on the blowup above $(X,Z) = (0,0)$ are the points $(S,Z) = (\beta_i,0)$. The point $(\beta_i,0)$ is singular if and only if $e_i \geq 2$, but just as in our earlier calculation, our choice of $m$ ensures that there is only one point of $\tilde{C}$ lying above each of these singular points. Thus we have found $s$ points of $\tilde{C}$ lying above the singular point $[0,1,0] \in \bar{C}$.

(We note that this is also true if $[0,1,0]$ is nonsingular, since that case occurs if $d_G = 1$, which implies also that $s = 1$.)

It remains to check the other chart on the blowup, which is given by $Z = TX$. Substituting and canceling a power of $X$ yields

\[ a \prod_{i=1}^{r} (1-\alpha_i T)^{e_i} T^{m+d_G-d_F} X^m - cb \prod_{i=1}^{s} (1-\beta_i T)^{e_i} = 0. \]

The only point in this chart that is not in the other chart is $(T,X) = (0,0)$, which is not a point on the blowup of the curve. So we obtain no further points in this chart.
To summarize, if we let \( \pi : \tilde{C} \to \bar{C} \) denote the map coming from the various blowups used to desingularize \( \bar{C} \), we have

\[
\#\pi^{-1}([1, 0, 0]) = \begin{cases} 
1 & \text{if } d_F \neq d_G, \\
m & \text{if } d_F = d_G,
\end{cases}
\]

\[
\#\pi^{-1}([\alpha_i, 0, 1]) = 1 \quad \text{for each } 1 \leq i \leq r,
\]

\[
\#\pi^{-1}([0, 1, 0]) = s.
\]

We now consider the covering map

\[
\varphi : \tilde{C} \longrightarrow \bar{C} \xrightarrow{[X, Y, Z] \mapsto [X, Z]} \mathbb{P}^1,
\]

where we note that \( \deg \varphi = m \). Indeed, the map \( \varphi \) is Galois with Galois group \( \mathbb{Z}/m\mathbb{Z} \). However, we need to be a bit careful, since the map from \( \bar{C} \to \mathbb{P}^1 \) is not defined at \([0, 1, 0] \in \bar{C}\), although it is defined at the \( s \) points of \( \tilde{C} \) lying over \([0, 1, 0]\). More precisely, on the chart \((2.2)\) for \( \tilde{C} \) with affine coordinates \((S, Z)\) and \( X = SZ\), the map \( \varphi \) is given by

\[
\varphi(S, Z) = [X, Z] = [SZ, Z] = [S, 1],
\]

so the \( s \) points \( \tilde{\beta}_1, \ldots, \tilde{\beta}_s \) on \( \tilde{C} \) lying over \([0, 1, 0] \in \bar{C}\) satisfy \( \varphi(\tilde{\beta}_i) = [\beta_i, 1] \), and indeed we have \( \varphi^{-1}([\beta_i, 1]) = \{\beta_i\} \) for each \( 1 \leq i \leq s \).

We use the Riemann–Hurwitz formula, see [23, Theorem 1.5],

\[
2 \text{ genus}(\tilde{C}) - 2 = (2 \text{ genus}(\mathbb{P}^1) - 2) \cdot \deg \varphi + \sum_{P \in \bar{C}} (e_{\varphi}(P) - 1).
\]

Substituting \( \deg \varphi = m \) and \( \text{genus}(\mathbb{P}^1) = 0 \), and applying [23, Corollary 1.3], this yields

\[
2 \text{ genus}(\tilde{C}) = 2(1 - m) + \sum_{P \in \bar{C}} (e_{\varphi}(P) - 1)
\]

\[
= 2(1 - m) + \sum_{Q \in \mathbb{P}^1} \sum_{P \in \varphi^{-1}(Q)} (e_{\varphi}(P) - 1)
\]

\[
= 2(1 - m) + \sum_{Q \in \mathbb{P}^1} (\deg \varphi - \#\varphi^{-1}(Q))
\]

\[
= 2(1 - m) + \sum_{i=1}^{r} (m - \#\varphi^{-1}([\alpha_i, 1]))
\]

\[
+ \sum_{i=1}^{s} (m - \#\varphi^{-1}([\beta_i, 1])) + (m - \#\varphi^{-1}([1, 0]))
\]

\[
= 2(1 - m) + r(m - 1) + s(m - 1) + \begin{cases} 
 m - 1 & \text{if } d_F \neq d_G \\
 0 & \text{if } d_F = d_G
\end{cases}
\]
\( (r + s - 1)(m - 1) \) if \( d_F \neq d_G \),
\( (r + s - 2)(m - 1) \) if \( d_F = d_G \).

This completes the proof. \( \square \)

2.3. **Generalised Schinzel-Tijdeman theorem.** We also need the following general version of the Schinzel-Tijdeman Theorem [21], which also extends [1, Theorem 2.3]. More precisely, the constant \( C \) in Lemma 2.8 stated below depends only on the prime ideal divisors of the coefficient \( b \), and not on its height as in [1]. For our purposes, this improvement is crucial.

For a set of places \( S \) of \( \mathbb{K} \), we write \( R_S \) for the ring of \( S \)-integers and \( R_S^* \) for the group of \( S \)-units.

**Lemma 2.8.** Let \( \mathbb{K} \) be a number field, and let \( S \) be a finite set of places of \( \mathbb{K} \) containing all infinite places. Let \( f \in R_S[X] \) be a polynomial without multiple roots and of degree at least 2. There is an effectively computable constant \( C(f, \mathbb{K}, S) \), depending only on \( f, \mathbb{K} \) and \( S \), so that the following holds: If \( b \in R_S^* \) and if the equation
\[
 f(x) = b \cdot y^m
\]
has a solution satisfying
\[ x, y \in R_S \quad \text{and} \quad y \notin \{0\} \cup R_S^*, \]
then
\[ m \leq C(f, \mathbb{K}, S). \]

**Proof.** The proof is nearly identical to that of [1, Theorem 2.3], with \( \hat{h} \) replaced by \( h(f) \). Here we only indicate the slight changes which are needed in the proof, under the assumption that \( b \in R_S^* \). Thus in this proof we use the notation from the proof of [1, Theorem 2.3], other than sticking with our notation for the sets of \( S \)-integers and \( S \)-units.

In [1, Lemmas 4.15 and 4.17], \( \hat{h} \) may be clearly replaced by \( h(f) \). Further, the estimates in [1, Equations (5.1)–(5.10)] remain valid if we replace \( \hat{h} \) by \( h(f) \), under the assumption that \( b \in R_S^* \).

Instead of [1, Equations (5.11) and (5.12)], we argue as follows. We may assume without loss of generality that
\[
 X \geq \max(C_3, m(4d)^{-1}(\log 3d)^{-3}),
\]
with \( C_3 \) being the constant specified in [1, Equation (5.12)]. Indeed, if
\[
 X \geq \max(C_3, m(4d)^{-1}(\log 3d)^{-3})
\]
then by
\[
\frac{1}{[K : \mathbb{Q}]} N_S(y^m) = \frac{1}{[K : \mathbb{Q}]} N_S(by^m) \leq h(by^m) \leq h(f(x)),
\]
we obtain
\[
m \leq [K : \mathbb{Q}] \frac{nX + h(a_0)}{\log 2}.
\]
The rest of the proof of [1, Theorem 2.3] follows without any changes other than replacing \(\hat{h}\) by \(h(f)\) at each occurrence. \(\square\)

**Remark 2.9.** It is crucial for our argument that the constant \(C(f, K, S)\) in Lemma 2.8 is independent of \(b \in \mathbb{R}_S\).

### 3. Proofs of main results

#### 3.1. Preliminary discussion.**

In this section we give the proofs of Theorems 1.2, 1.3, 1.4, 1.7 and 1.10, which are main results of this paper. Throughout these proofs we use the following common notation. We let \(S\) be the following set of absolute values on \(K\):
\[
S := M_\infty K \cup \{v \in M_0 K : \|\gamma\|_v \neq 1 \text{ for some } \gamma \in \Gamma\}.
\]
The set \(S\) is finite, since \(\Gamma\) is finitely generated, so in proving our main theorems, we may assume that \(\Gamma\) is the full group of \(S\)-units,
\[
\Gamma = R^*_S = \{\beta \in K^* : \|\beta\|_v = 1 \text{ for all } v \in M_K \setminus S\}.
\]
This is convenient because \(R^*_S\) is multiplicatively saturated in \(K^*\), i.e., if \(\gamma \in K^*\) and \(\gamma^n \in R^*_S\) for some \(n \neq 0\), then \(\gamma \in R^*_S\).

#### 3.2. Proof of Theorem 1.2.**

(a) As noted earlier, this statement has originally been proven in [15, Proposition 1.5 (a)], but as a convenience to the reader, we include the short proof. By assumption, the set
\[
\mathcal{G}(K, f, \Gamma) := \{\alpha \in K : f(\alpha) \in \Gamma\}
\]
is infinite. Let \(m = \max\{d! + 1, 5\}\). We fix coset representatives \(c_1, \ldots, c_t \in R^*_S\) for the finite group \(R^*_S/(R^*_S)^m\). Then for every \(\alpha \in \mathcal{G}(K, f, \Gamma)\), we can find an index \(i(\alpha) \in \{1, \ldots, t\}\) and some \(u \in R^*_S\) so that \(f(\alpha) = c_{i(\alpha)} u^m\). Hence the assumption that the set \(\mathcal{G}(K, f, \Gamma)\) is infinite means that we can find a \(c \in R^*_S\) so that the set
\[
\{(\alpha, u) \in K \times R^*_S : f(\alpha) = cu^m\}
\]
is infinite. It follows that every \((\alpha, u)\) in this set is on the algebraic curve
\[
C : f(X) = cY^m, \quad \text{and hence that } \#C(K) = \infty.
\]
Writing \( f(X) = F(X)/G(X) \) with \( F(X), G(X) \in \mathbb{K}[X] \) relatively prime, the curve \( C \) has the equation \( F(X) = cY^nG(X) \), so Lemma 2.7 tells us that the genus of a smooth projective model \( \tilde{C} \) for \( C \) is given by

\[
\text{genus}(\tilde{C}) = \frac{m-1}{2} \left( \nu(FG) - \begin{cases} 1 & \text{if } \deg(F) \neq \deg(G) \\ 2 & \text{if } \deg(F) = \deg(G) \end{cases} \right),
\]

where we recall that \( \nu(H) \) denotes the number of distinct complex roots of a polynomial \( H \). Faltings’ theorem \([7, 8]\) (Mordell conjecture) tells us that \( C(\mathbb{K}) \) is finite if genus(\( \tilde{C} \)) \( \geq 2 \), so the fact that \( \#C(\mathbb{K}) = \infty \) implies that one of the following cases is true.

**Case I:** \( \nu(FG) = 1 \), \( \deg(F) \neq \deg(G) \).

**Case II:** \( \nu(FG) = 2 \), \( \deg(F) = \deg(G) \).

In Case I, the fact that \( \nu(FG) = 1 \) means that one of \( F \) or \( G \) is constant and the other has only a single root. Since \( \deg f = d \), this means that \( f \) has the form \( f(X) = a(x - b)^{\pm d} \). In Case II, the fact that \( \nu(FG) = 2 \) and \( F \) and \( G \) both have degree \( d \) implies that they each have a single root, so \( F(X) = a_1(X - b)^d \) and \( G(X) = a_2(X - c)^d \), which proves that \( f \) has the form \( f(X) = a(X - b)^d/(X - c)^d \).

(b) We are assuming that the set

\[
\mathcal{F}_2(\mathbb{K}, f, \Gamma) = \{(n, \alpha) \in \mathbb{Z}_{\geq 2} \times \text{Wander}_\mathbb{K}(f) : f^{(n)}(\alpha) \in \Gamma \}
\]

is infinite. Since every pair \( (n, \alpha) \) in \( \mathcal{F}_2(\mathbb{K}, f, \Gamma) \) has \( n \geq 2 \), we have a well-defined map

\[
\mathcal{F}_2(\mathbb{K}, f, \Gamma) \longrightarrow \mathcal{G}(\mathbb{K}, f^{(2)}, \Gamma), \quad (n, \alpha) \longmapsto f^{(n-2)}(\alpha).
\]

If the set \( \mathcal{G}(\mathbb{K}, f^{(2)}, \Gamma) \) is finite, then there are only finitely many possible values for \( f^{(n-2)}(\alpha) \) as \( (n, \alpha) \) range over the set \( \mathcal{F}_2(\mathbb{K}, f, \Gamma) \), and then Lemma 2.3 tells us that there are only finitely many possibilities for \( n \) and \( \alpha \), contradicting the assumption that \( \#\mathcal{F}_2(\mathbb{K}, f, \Gamma) = \infty \).

Hence we must have \( \#\mathcal{G}(\mathbb{K}, f^{(2)}, \Gamma) = \infty \), and then (a) applied to the map \( f^{(2)}(X) \) tells us that \( f^{(2)}(X) \) has one of the following three forms:

\[
\begin{align*}
\varphi_{\pm}(X) := a(X - b)^{\pm d}, \\
\psi(X) := a(X - b)^d/(X - c)^d.
\end{align*}
\]

We claim that this forces \( f(X) \) to have the form \( aX^{\pm d} \).

We observe that \( \varphi_{\pm}(X) \) is totally ramified at \( b \) and \( \infty \), so in this case \( f \) is totally ramified at \( b, \infty, f(b), \) and \( f(\infty) \). Similarly, the map \( \psi(X) \) is totally ramified at \( b \) and \( c \), so in this case \( f \) is totally ramified at \( b, c, f(b), \) and \( f(c) \). The Riemann–Hurwitz formula \([23,\]

\]
Theorem 1.5] implies that a rational map has at most two points where it is totally ramified, and hence
\[
    f^{(2)}(X) = \varphi_{\pm}(X) \implies f(b), f(\infty) \in \{b, \infty\};
    
    f^{(2)}(X) = \psi(X) \implies f(b), f(c) \in \{b, c\}.
\]
This leads to several subcases, which are detailed in Figure 3.1, in which we use the symbol ‘→←’ to indicate a contradiction. Looking at Figure 3.1, we see that if \( f^{(2)} = \varphi_{+} \) and \( f(b) = b \), then 0 and \( \infty \) are totally ramified fixed points of \( f \), so \( f(X) = aXd \), while if \( f^{(2)} = \varphi_{+} \) and \( f(b) = \infty \), then 0 and \( \infty \) are totally ramified and form a 2-cycle, so \( f(X) = aX^{-d} \). Finally, all cases with \( f^{(2)} = \varphi_{-} \) and \( f^{(2)} = \psi \) lead to contradictions. This completes the proof of Theorem 1.2. □

3.3. **Proof of Theorem 1.3.** We are going to prove the contrapositive statement:

\[
    \#f^{-1}(\{0, \infty\}) \geq 3 \implies E_\rho(\mathbb{K}, f, \Gamma, r, s) \text{ is a finite set.}
\]

The assumption that \( f \) has at least 3 zeros and poles implies in particular that \( f \) does not have the form \( cX^{\pm d} \), so we know that at least one of 0 or \( \infty \) is not exceptional. If 0 is exceptional, then we claim that we can swap 0 and \( \infty \) and replace \( f(X) \) with the polynomial \( g(X) = f(X^{-1})^{-1} \in \mathbb{K}[X] \). To see this, we note that an initial wandering point \( \alpha \) for \( f \), which necessarily satisfies \( \alpha \neq 0 \) since 0 is exceptional, is mapped to the initial wandering point \( \alpha^{-1} \) for \( g \). Further, \( f^{(n)}(\alpha) = g^{(n)}(\alpha^{-1})^{-1} \), so

\[
    f^{(n+k)}(\alpha)^r/f^{(k)}(\alpha)^s \in \Gamma \iff g^{(k)}(\alpha^{-1})^s/g^{(n+k)}(\alpha^{-1})^r \in \Gamma \iff g^{(n+k)}(\alpha^{-1})^r/g^{(k)}(\alpha^{-1})^s \in \Gamma,
\]

where the first implication follows from the definition of \( g \), and the second from the fact that \( \Gamma \) is a group. Thus if 0 is exceptional for \( f \), then there is a bijection

\[
    E_\rho(\mathbb{K}, f(X), \Gamma, r, s) \longrightarrow E_\rho(\mathbb{K}, f(X^{-1})^{-1}, \Gamma, r, s),
\]

\[
    (n, k, \alpha, u) \mapsto (n, k, \alpha^{-1}, u^{-1}).
\]

Hence, without loss of generality, we may assume that 0 is not an exceptional point for \( f \), which in turn allows us to apply Lemma 2.5.

We want to study the set of triples \( (n, k, \alpha) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0} \times \text{Wander}_\mathbb{K}(f) \) such that

\[
    \|f^{(n+k)}(\alpha)\|_v^r = \|f^{(k)}(\alpha)\|_v^s \quad \text{for all} \; v \in M_\mathbb{K} \setminus S.
\]
\[ f(2) = \varphi_+, \ f(b) = b \quad \implies \quad \left( \begin{array}{l}
 b \xrightarrow{f} b \xrightarrow{f} 0 \\
 \infty \xrightarrow{f} f(\infty) \xrightarrow{f} \infty
\end{array} \right)
\]
\[ \implies \begin{cases}
 b = f(b) = 0, \text{ so} \\
f(\infty) \neq b, \text{ so} \\
f(\infty) = \infty.
\end{cases}
\]

\[ f(2) = \varphi_+, \ f(b) = \infty \quad \implies \quad \left( \begin{array}{l}
 b \xrightarrow{f} \infty \xrightarrow{f} 0 \\
 \infty \xrightarrow{f} f(\infty) \xrightarrow{f} \infty
\end{array} \right)
\]
\[ \implies \begin{cases}
 0 = f(2)(b) = f(\infty), \text{ so} \\
 \infty = f(2)(\infty) = f(0) \text{ and} \\
b = f^{-1}(\infty) = 0.
\end{cases}
\]

\[ f(2) = \varphi_-, \ f(b) = b \quad \implies \quad \left( \begin{array}{l}
 b \xrightarrow{f} b \xrightarrow{f} \infty \\
 \infty \xrightarrow{f} f(\infty) \xrightarrow{f} 0
\end{array} \right)
\]
\[ \implies \infty = f(b) = b. \quad \checkmark
\]

\[ f(2) = \varphi_-, \ f(b) = \infty \quad \implies \quad \left( \begin{array}{l}
 b \xrightarrow{f} \infty \xrightarrow{f} \infty \\
 \infty \xrightarrow{f} f(\infty) \xrightarrow{f} 0
\end{array} \right)
\]
\[ \implies \infty = f(\infty) = 0. \quad \checkmark
\]

\[ f(2) = \psi, \ f(b) = b \quad \implies \quad \left( \begin{array}{l}
 b \xrightarrow{f} b \xrightarrow{f} 0 \\
 c \xrightarrow{f} f(c) \xrightarrow{f} \infty
\end{array} \right)
\]
\[ \implies \begin{cases}
 0 = f(b) = b, \text{ so} \\
f(c) \neq b, \text{ so} f(c) = c, \text{ so} \\
\infty = f(c) = c. \quad \checkmark
\end{cases}
\]

\[ f(2) = \psi, \ f(b) = c \quad \implies \quad \left( \begin{array}{l}
 b \xrightarrow{f} c \xrightarrow{f} 0 \\
 c \xrightarrow{f} f(c) \xrightarrow{f} \infty
\end{array} \right)
\]
\[ \implies \begin{cases}
 f(c) \neq c \text{ (else } 0 = f(c) = \infty), \\
\text{ so } f(c) = b, \text{ so} \\
\infty = f(b) = c. \quad \checkmark
\end{cases}
\]

Figure 3.1. Case-by-case analysis for values of $f(2)$ and $f(b)$.
For an arbitrary choice of $\varepsilon$, to be specified later, we let $C_3(K, S, f, \varepsilon)$ be the constant from Lemma 2.5, and we split the proof into two cases, depending on the size of $n + k$.

**Case 1:** $n + k \geq C_3(K, S, f, \varepsilon)$.

In this case Lemma 2.5 tells us that $(n, k, \varepsilon)$ satisfies

$$\sum_{v \in S} \log^+ \left( \left\| f^{(n+k)}(\alpha) \right\|_v^{-1} \right) \leq \varepsilon \hat{h}_f \left( f^{(n+k)}(\alpha) \right).$$

Since $h(\gamma) = h(\gamma^{-1})$ and using (1.1), we compute

$$h \left( f^{(n+k)}(\alpha) \right) = h \left( f^{(n+k)}(\alpha)^{-1} \right) = \sum_{v \in \mathcal{M}_K} \log^+ \left( \left\| f^{(n+k)}(\alpha) \right\|_v^{-1} \right)$$

$$= \sum_{v \in S} \log^+ \left( \left\| f^{(n+k)}(\alpha) \right\|_v^{-1} \right) + \sum_{v \in \mathcal{M}_K \setminus S} \log^+ \left( \left\| f^{(n+k)}(\alpha) \right\|_v^{-1} \right).$$

Now, using (3.4) and (3.5)

$$h \left( f^{(n+k)}(\alpha) \right) \leq \varepsilon \hat{h}_f \left( f^{(n+k)}(\alpha) \right) + \sum_{v \in \mathcal{M}_K \setminus S} \log^+ \left( \left\| f^{(k)}(\alpha) \right\|_v^{-s/r} \right)$$

$$\leq \varepsilon \hat{h}_f \left( f^{(n+k)}(\alpha) \right) + \frac{|s|}{|r|} \hat{h}_f \left( f^{(k)}(\alpha)^{-1} \right)$$

$$= \varepsilon \hat{h}_f \left( f^{(n+k)}(\alpha) \right) + d^{p-1} h \left( f^{(k)}(\alpha) \right),$$

where for the last equality we have used the fact that $\rho$ is defined by the relation $|s|/|r| = d^{p-1}$. We next use Lemma 2.1(a) to replace the Weil height $h$ with the canonical height $\hat{h}_f$. This yields

$$\hat{h}_f \left( f^{(n+k)}(\alpha) \right) - C_1(f) \leq \varepsilon \hat{h}_f(f^{(n+k)}(\alpha)) + d^{p-1} \left( \hat{h}_f(f^{(k)}(\alpha)) + C_1(f) \right),$$

and a little algebra leads to

$$(1 - \varepsilon) \hat{h}_f(f^{(n+k)}(\alpha)) \leq d^{p-1} \hat{h}_f \left( f^{(k)}(\alpha) \right) + C_1(f) \left( 1 + d^{p-1} \right).$$

Using Lemma 2.1(b) gives

$$(1 - \varepsilon) d^{n+k} \hat{h}_f(\alpha) \leq d^{p-1+k} \hat{h}_f(\alpha) + C_1(f) \left( 1 + d^{p-1} \right),$$

and hence

$$d^k \left((1 - \varepsilon) d^m - d^{p-1} \right) \hat{h}_f(\alpha) \leq C_1(f) \left( 1 + d^{p-1} \right).$$

Taking $\varepsilon = 1/3$, we see that for $n \geq \rho$ we have

$$(1 - \varepsilon) d^m - d^{p-1} = \frac{2}{3} d^m - d^{p-1} \geq \frac{4}{3} d^{m-1} - d^{p-1} \geq \frac{1}{3} d^{m-1}.$$
and thus we derive from (3.6) that

\begin{equation}
(3.7) \quad d^{n+k-1} \hat{h}_f(\alpha) \leq 3C_1(f) (1 + d^{\rho-1}).
\end{equation}

Using \( d^{n+k-1} \hat{h}_f = \hat{h}_f \circ f^{(n+k-1)} \) and again Lemma 2.1(a), which gives \( \hat{h}_f \geq h - C_1(f) \), (3.7) yields

\begin{equation}
 h_f((n+k-1)(\alpha)) \leq 3C_1(f) (1 + d^{\rho-1}) + C_1(f).
\end{equation}

Thus \( f^{(n+k-1)}(\alpha) \) is in a set of bounded height, where the bound depends only on \( f \) and \( \rho \), so there are only finitely many possible values for \( f^{(n+k-1)}(\alpha) \). Then Lemma 2.3 tells us that there are only finitely many possible values of \( \alpha \).

**Case 2: \( n + k < C_3(\mathbb{K}, S, f, \frac{1}{3}) \).**

Replacing \( r \) and \( s \) with \(-r\) and \(-s\) if necessary, we may assume that \( r > 0 \). Further, since \( n + k \) is assumed bounded, we may assume that \( k \) and \( n \) are fixed, and we need to show that there are only finitely many \( \alpha \in \mathbb{P}^1(\mathbb{K}) \) satisfying

\begin{equation}
(3.8) \quad f^{(n+k)}(\alpha)^r / f^{(k)}(\alpha)^s \in \mathbb{K}^\times.
\end{equation}

By assumption we have \( s \neq 0 \). We let

\[ g(X) = f^{(n)}(X)^r / X^s \in \mathbb{K}(X). \]

Then (3.8) implies that

\[ g\left(f^{(k)}(\alpha)\right) = f^{(n+k)}(\alpha)^r / f^{(k)}(\alpha)^s \in \mathbb{K}^\times. \]

In the notation of Theorem 1.2(a), this says that \( f^{(k)}(\alpha) \in \mathcal{G}(\mathbb{K}, g, R_S^\times) \).

If \( \mathcal{G}(\mathbb{K}, g, R_S^\times) \) is finite, then \( f^{(k)}(\alpha) \) takes on only finitely many values, and Lemma 2.3 tells us that there are only finitely many values of \( \alpha \).

On the other hand, if \( \mathcal{G}(\mathbb{K}, g, R_S^\times) \) is infinite, then Theorem 1.2(a) says that \( g(X) \) has at most two zeros and poles. But the assumption that 0 is not periodic implies that 0 is a pole of \( g \), and the assumption that \( f(X) \neq cX^{\pm d} \) implies that \( f^{(n)}(X) \) has at least two poles or zeros distinct from 0. Hence \( g \) has at least three poles and zeros.

3.4. **Proof of Theorem 1.4.** We have shown in the proof of Theorem 1.3 that if

\[ \{ (n, k, \alpha, u) \in \mathcal{E}(\mathbb{K}, f, \Gamma) : n + k > C_3(\mathbb{K}, S, f, 1/3) \} \]

is infinite, then \( f(X) = cX^{\pm d} \) for some \( c \in \mathbb{K}^\times \). We are thus reduced to the situation that the following three conditions hold:
The function \( f(X) \in \mathbb{K}(X) \) is not of the form \( cX^d \) for any \( c \in \mathbb{K}^* \).

The integers \( n \geq 1 \) and \( k \geq 0 \) are fixed and satisfy
\[
n + k \leq C_3(\mathbb{K}, S, f, 1/3).
\]

There are infinitely many \( \alpha \in \text{Wander}_\mathbb{K}(f) \) satisfying the equation (1.4), i.e., there are infinitely many pairs \((\alpha, u) \in \text{Wander}_\mathbb{K}(f) \times R^*_S\) satisfying
\[
f^{(n+k)}(\alpha) = uf^{(k)}(\alpha).
\]
Since \( k \) is fixed, we can set \( \beta = f^{(k)}(\alpha) \), so we see that there are infinitely many pairs
\[
(\beta, u) \in \text{Wander}_\mathbb{K}(f) \times R^*_S \quad \text{satisfying} \quad f^{(n)}(\beta) = u\beta.
\]
We define
\[
m = \text{LCM}(2, 3, \ldots, d^n + 1) + 1,
\]
so in particular
\[
m \geq 7 \quad \text{and} \quad \gcd(m, (d^n + 1)! ) = 1.
\]

We fix coset representatives \( c_1, \ldots, c_t \in R^*_S \) for the finite group \( R^*_S/(R^*_S)^m \). Then each \( u \in R^*_S \) can be written in the form \( c_i \gamma^m \) for some \( i = i(u) \in \{ 1, \ldots, t \} \) and some \( \gamma = \gamma(u) \in R^*_S \). Hence in order to determine the number of pairs satisfying (3.9), it suffices to study, for each fixed \( c \in R^*_S \), the number of pairs
\[
(\beta, \gamma) \in \text{Wander}_\mathbb{K}(f) \times R^*_S \quad \text{satisfying} \quad f^{(n)}(\beta) = c\gamma^m \beta.
\]
We note that \( \beta \neq 0 \), since otherwise \( \beta \) is periodic.

In order to apply Lemma 2.7, we consider three possible ways to write the rational function \( f^{(n)}(X) \), depending on its order of vanishing at \( X = 0 \). More precisely, we choose \( F_n, G_n \in \mathbb{K}[X] \) satisfying \( \gcd(F_n, G_n) = 1 \) so that \( f^{(n)}(X) \) has one of the following forms:

**Case A:** \( f^{(n)}(X) = F_n(X)/G_n(X), \quad F_n(0) \neq 0, \)

**Case B:** \( f^{(n)}(X) = XF_n(X)/G_n(X), \quad F_n(0)G_n(0) \neq 0, \)

**Case C:** \( f^{(n)}(X) = X^eF_n(X)/G_n(X), \quad F_n(0)G_n(0) \neq 0, \quad e \geq 2. \)

(We note that by homogeneity, we may assume that either \( F_n \) or \( G_n \) is monic.) These three cases lead to studying points on the following three curves:

**Case A:** \( C : F_n(X) = cY^m XG_n(X), \)

**Case B:** \( C : F_n(X) = cY^m G_n(X), \)

**Case C:** \( C : X^{e-1}F_n(X) = cY^m G_n(X). \)
We let $\tilde{C}$ denote a smooth projective model of $C$. Applying Lemma 2.7, each of the three cases leads to two subcases, according to the relative degrees of $F_n$ and $G_n$. To ease notation, we let $d_{F_n} = \deg F_n$ and $d_{G_n} = \deg G_n$.

Then Lemma 2.7 yields:

$$\frac{2 \text{genus}(\tilde{C})}{m - 1} = \begin{cases} 
\nu(F_n G_n) & \text{in Case A, } d_{F_n} \neq d_{G_n} + 1, \\
\nu(F_n G_n) - 1 & \text{in Case A, } d_{F_n} = d_{G_n} + 1, \\
\nu(F_n G_n) - 1 & \text{in Case B, } d_{F_n} \neq d_{G_n}, \\
\nu(F_n G_n) - 2 & \text{in Case B, } d_{F_n} = d_{G_n}, \\
\nu(F_n G_n) & \text{in Case C, } d_{F_n} + e - 1 \neq d_{G_n}, \\
\nu(F_n G_n) - 1 & \text{in Case C, } d_{F_n} + e - 1 = d_{G_n},
\end{cases}$$

where, as before, $\nu(H)$ be the number of distinct complex roots of a polynomial $H$. Since $m \geq 7$, we see that either genus$(\tilde{C}) = 0$, or else genus$(\tilde{C}) \geq 3$. In the latter case, Faltings’ theorem [7, 8] (Mordell conjecture) tells us that $C(\mathbb{K})$ is finite. So it remains to analyze the six cases with genus$(\tilde{C}) = 0$, as described in Table 3.1. The remainder of the proof is a case-by-case analysis.

| Case | $C$ | $d_{F_n}$ and $d_{G_n}$ | $\nu(F_n G_n)$ |
|------|-----|-----------------|----------------|
| A1   | $F_n(X) = cY^m X G_n(X)$ | $d_{F_n} \neq d_{G_n} + 1$ | 0 |
| A2   | $F_n(X) = cY^m X G_n(X)$ | $d_{F_n} = d_{G_n} + 1$ | 1 |
| B1   | $F_n(X) = cY^m G_n(X)$ | $d_{F_n} \neq d_{G_n}$ | 1 |
| B2   | $F_n(X) = cY^m G_n(X)$ | $d_{F_n} = d_{G_n}$ | 2 |
| C1   | $X^{e-1} F_n(X) = cY^m G_n(X)$ | $d_{F_n} + e + 1 \neq d_{G_n}$ | 0 |
| C2   | $X^{e-1} F_n(X) = cY^m G_n(X)$ | $d_{F_n} + e + 1 = d_{G_n}$ | 1 |

Table 3.1. Cases with genus$(\tilde{C}) = 0$

Case A1: $\nu(F_n G_n) = 0$.
In this case $F_n$ and $G_n$ are constants, so $f^{(n)}(X)$ is constant, contradicting $\deg f \geq 2$.

Case C1: $\nu(F_n G_n) = 0$.
Again $F_n$ and $G_n$ are constants, so $f^{(n)}(X) = cX^e$ for some $e \geq 2$. This is only possible if $f$ has the form $f(X) = aX^{\pm d}$. 
**Case A2**: $\nu(F_n G_n) = 1 & d_{F_n} = d_{G_n} + 1$.

One of $F_n$ or $G_n$ is constant. The degree condition forces $G_n$ to be
constant, and hence $d_{F_n} = 1$. Therefore
\[
d^n = \deg f^{(n)}(X) = \deg F_n(X)/G_n(X) = \max\{d_{F_n}, d_{G_n}\} = 1.
\]
This contradicts $\deg f \geq 2$.

**Case C2**: $\nu(F_n G_n) = 1 & d_{F_n} + e - 1 = d_{G_n}$. 

One of $F_n$ or $G_n$ is constant. The degree condition forces $F_n$ to be
constant, and hence $d_{G_n} = e - 1$ and thus $d^n = \deg f^{(n)} = e$. Therefore
\[
f^{(n)}(X) = X^e F_n(X)/G_n(X) \text{ has the form } f^{(n)}(X) = aX^d / (X - b)^{d-1}.
\]
We are going to prove that this forces $n = 1$.

More generally, let
\[
\psi(X) = aX^D / (X - b)^{D-1} \text{ with } D \geq 2 \text{ and } ab \neq 0,
\]
and suppose that $g$ and $h$ are rational functions satisfying
\[
g \circ h(X) = \psi(X).
\]
We claim that $g$ or $h$ has degree 1.

To see this, we first note that
\[
e_{\psi}(\infty) = 1 \text{ and } e_{\psi}(b) = D - 1,
\]
since if we use the linear fractional transformation $L(X) = X^{-1}$ to
move $\infty$ to 0 and $b$ to $b^{-1}$, we have
\[
L^{-1} \circ \psi \circ L(X) = \psi(X^{-1})^{-1} = a^{-1} X (1 - bX)^{D-1},
\]
from which it is easy to read off the ramification indices. We next use
the fact that
\[
h^{-1} (g^{-1}(\infty)) = (g \circ h)^{-1}(\infty) = \psi^{-1}(\infty) = \{b, \infty\}
\]
to conclude that $\# g^{-1}(\infty)$ equals 1 or 2.

Suppose first that $g^{-1}(\infty) = \{c\}$ consists of a single point. We use
multiplicativity of ramification indices and (3.11) to compute
\[
1 = e_{\psi}(\infty) = e_h(\infty)e_g(c) \text{ and } D - 1 = e_{\psi}(b) = e_h(b)e_g(c).
\]
The first equality gives $e_g(c) = 1$, and then the second gives $e_h(b) = D - 1$. This gives the estimate
\[
\deg h \geq e_h(b) = D - 1 = \deg \psi - 1 = (\deg g)(\deg h) - 1.
\]
Therefore
\[
1 \geq (\deg h)(\deg g - 1),
\]
which forces either $\deg g = 1$ or $\deg h = 1$. 

We suppose next that $g^{-1}(\infty)$ consists of two points, i.e., suppose that $h(\infty) \neq h(b)$. Then $h^{-1}(h(b)) = \{b\}$, so $e_h(b) = \deg h$. Further, from

$$1 = e_\psi(\infty) = e_h(\infty)e_g(h(\infty)),$$

we see that $e_g(h(\infty)) = 1$, and hence using

$$\deg g = \sum_{c \in g^{-1}(\infty)} e_g(c) = e_g(h(\infty)) + e_g(h(b)),$$

we find that $e_g(h(b)) = \deg g - 1$. Hence

$$D - 1 = e_\psi(b) = e_h(b)e_g(h(b)) = (\deg h)(\deg g - 1) = D - \deg h.$$

Therefore $\deg h = 1$.

This completes the proof in Case C2 that $n = 1$, and hence that $f(X) = aX^d/(X - b)^{d-1}$.

**Case B1: $\nu(F_n, G_n) = 1$ & $d_{F_n} \neq d_{G_n}$.**

The assumption that $\nu(F_n, G_n) = 1$ implies that one of $F_n$ and $G_n$ is constant and the other has exactly one root. Further, since in Case B we have $f^{(n)}(X) = X F_n(X)/G_n(X)$, we see that $f^{(n)}$ has one of the following forms:

$$f^{(n)}(X) = aX(X - b)^{d^n-1} \quad \text{or} \quad f^{(n)}(X) = aX/(X - b)^{d^n}.$$  

In order to prove that $n = 1$, we suppose that $g$ and $h$ are rational functions satisfying

$$g \circ h = aX(X - b)^{D-1} \quad \text{or} \quad g \circ h = aX(X - b)^{-D}$$

with

$$b \neq 0 \quad \text{and} \quad D \geq 2.$$  

Our goal is to show that either $g$ or $h$ is linear.

We start with $g \circ h = aX(X - b)^{D-1}$, so $(g \circ h)^{-1}(0) = \{0, b\}$. Thus $\#g^{-1}(0) = 1$ or 2. Suppose first that $\#g^{-1}(0) = 1$, say $g^{-1}(0) = \{c\}$. Then $g$ is totally ramified at $c$, i.e., $e_g(c) = \deg g$. The form of $g \circ h$ implies that it is unramified at 0, so, by (1.2) we have

$$1 = e_{g \circ h}(0) = e_h(0)e_g(h(0)) = e_h(0)e_g(c) = e_h(0)\deg g.$$

Hence $\deg g = 1$.

Similarly, if $\#g^{-1}(0) = 2$, say $g^{-1}(0) = \{c_1, c_2\}$, then possibly after relabeling, we have $h^{-1}(c_1) = \{0\}$ and $h^{-1}(c_2) = \{b\}$. In particular, the map $h$ is totally ramified at 0, i.e., $e_h(0) = \deg h$. Again using the fact that $g \circ h$ is unramified at 0, using (1.2), we find that

$$1 = e_{g \circ h}(0) = e_h(0)e_g(h(0)) = e_h(0)e_g(c_1) = e_g(c_1)\deg h.$$  

Hence $\deg h = 1$. 


We next do the case that \( g \circ h = aX(X - b)^{-D} \), so \((g \circ h)^{-1}(0) = \{0, \infty\} \). Thus \#\(g^{-1}(0) = 1 \) or 2. Suppose first that \#\(g^{-1}(0) = 1 \), say \( g^{-1}(0) = \{c\} \). Then \( g \) is totally ramified at \( c \), i.e., \( e_g(c) = \deg g \). The form of \( g \circ h \) implies that it is unramified at \( 0 \), so by (1.2)

\[
1 = e_{g \circ h}(0) = e_h(0)e_g(h(0)) = e_h(0)e_g(c) = e_h(0)\deg g.
\]

Hence \( \deg g = 1 \).

Similarly, if \#\(g^{-1}(0) = 2 \), say \( g^{-1}(0) = \{c_1, c_2\} \), then possibly after relabeling, we have \( h^{-1}(c_1) = \{0\} \) and \( h^{-1}(c_2) = \{\infty\} \). In particular, the map \( h \) is totally ramified at \( 0 \), i.e., \( e_h(0) = \deg h \). Again using the fact that \( g \circ h \) is unramified at \( 0 \), using (1.2), we find that

\[
1 = e_{g \circ h}(0) = e_h(0)e_g(h(0)) = e_h(0)e_g(c_1) = e_g(c_1)\deg h.
\]

Hence \( \deg h = 1 \).

This completes the proof for Case B1 that \( n = 1 \), and either \( f(X) = aX(X - b)^{d-1} \) or \( f(X) = aX/(X - b)^d \).

**Case B2:** \( \nu(F_n G_n) = 2 \) & \( d_{F_n} = d_{G_n} \).

The fact that \( F_n \) and \( G_n \) have the same degree means that neither can be constant, so \( \nu(F_n G_n) = 2 \) implies that \( F_n \) and \( G_n \) each have exactly one root. Further, since \( f^{(n)}(X) = XF_n(X)/G_n(X) \) in Case B, we can read off the degrees of \( F_n \) and \( G_n \), so we find that

\[
f^{(n)}(X) = \frac{aX(X - b)^{d_n - 1}}{(X - c)^{d_n - 1}} \quad \text{with } b, c, 0 \text{ distinct.}
\]

In order to prove that \( n = 1 \), we suppose that \( g \) and \( h \) are rational functions satisfying

\[
g \circ h = \frac{aX(X - b)^D}{(X - c)^D} \quad \text{with } b, c, 0 \text{ distinct and } D \geq 1,
\]

and our goal is to show that either \( g \) or \( h \) is linear. We have \((g \circ h)^{-1}(\infty) = \{c, \infty\} \), so \#\(g^{-1}(\infty) = 1 \) or 2.

Suppose first that \#\(g^{-1}(\infty) = 1 \), say \( g^{-1}(\infty) = \{\tilde{c}\} \). Then \( g \) is totally ramified at \( \tilde{c} \), i.e., \( e_g(\tilde{c}) = \deg g \). The form of \( g \circ h \) implies that \( g \circ h \) is unramified at \( \infty \), hence, by (1.2)

\[
1 = e_{g \circ h}(\infty) = e_h(\infty)e_g(h(\infty)) = e_h(\infty)e_g(\tilde{c}) = e_h(\infty)\deg g.
\]

Therefore \( \deg g = 1 \).

Similarly, if \#\(g^{-1}(\infty) = 2 \), say \( g^{-1}(\infty) = \{c_1, c_2\} \), then possibly after relabeling, \( h^{-1}(c_1) = \{\infty\} \) and \( h^{-1}(c_2) = \{c\} \). In particular, \( h \) is totally ramified at \( \infty \), so \( e_h(\infty) = \deg h \). Again using the fact that \( g \circ h \) is unramified at \( \infty \), using (1.2), we find that

\[
1 = e_{g \circ h}(\infty) = e_h(\infty)e_g(h(\infty)) = e_h(\infty)e_g(c_1) = e_g(c_1)\deg h.
\]
Therefore \( \deg h = 1 \).

This completes the proof in Case B2 that \( n = 1 \), and hence that \( f(X) = aX(X - b)^{d-1}/(X - c)^{d-1} \). \( \square \)

3.5. Proof of Theorem 1.7. Let \( \alpha \in \mathbb{K} \) have the property that there exists a pair of non-negative integers \((m, n)\) with \( m > n > 0 \) and integers \( r \) and \( s \) and an \( S \)-unit \( u \in R_S^* \) such that

\[
(f(m)(\alpha))^r = u(f(n)(\alpha))^s. \tag{3.12}
\]

Since, by Northcott’s Theorem, the set \( \text{PrePer}(f) \cap \mathbb{K} \) is finite, it is enough to prove the finiteness of the set of points \( \alpha \in \text{Wander}_\mathbb{K}(f) \) satisfying (3.12).

We note that if \( r = 0 \) or \( s = 0 \), then the finiteness of \( \alpha \in \mathbb{K} \) satisfying (3.12) follows directly from Theorem 1.2 since \( f \) has \( d \geq 2 \) distinct roots. Thus, we may assume from now on that \( rs \neq 0 \).

From (3.12) and the power saturation of \( R_S^* \) in \( \mathbb{K}^* \), we see that

\[
u_p(u) = \nu_p(f(n)(\alpha))^s.
\]

This allows us to take the \( \gcd(r, s) \)-root of (3.12), so without loss of generality we may assume that

\[
\gcd(r, s) = 1.
\]

For a prime ideal \( \mathfrak{p} \) of the ring of integers \( R_\mathbb{K} \) of \( \mathbb{K} \), we denote the (normalized additive) valuation on \( \mathbb{K} \) at the place corresponding to \( \mathfrak{p} \) by \( \nu_\mathfrak{p} : \mathbb{K}^* \to \mathbb{Z} \). As usual, we say that a polynomial \( f(X) = c_0 + c_1X + \cdots + c_dX^d \) has bad reduction at \( \mathfrak{p} \) if either \( \nu_\mathfrak{p}(c_i) < 0 \) for some \( i \) or if \( \nu_\mathfrak{p}(c_d) > 0 \); otherwise we say it has good reduction. We let

\[
S_{f, \Gamma} := S \cup \{ \mathfrak{p} \in M_\mathbb{K}^0 : f \text{ has bad reduction at } \mathfrak{p} \},
\]

where we recall that \( S \) is the set of places (3.1) with \( \Gamma = R_S^* \). It is a standard fact, even for rational functions, that if \( f \) has good reduction at a prime, then so do all of its iterates; see [23, Proposition 2.18(b)]. (This is especially easy to see for polynomials.) Hence

\[
S_{f(m), \Gamma} \subseteq S_{f, \Gamma}, \text{ for all } m \geq 1. \tag{3.13}
\]

We let

\[
R_{S_{f, \Gamma}} := \{ \vartheta \in \mathbb{K} : \nu_\mathfrak{p}(\vartheta) \geq 0 \text{ for all } \mathfrak{p} \notin S_{f, \Gamma} \}
\]

be the ring of \( S_{f, \Gamma} \)-integers in \( \mathbb{K} \), and \( R_{S_{f, \Gamma}}^* \) denotes the group of \( S_{f, \Gamma} \)-units in \( \mathbb{K} \).

Recall that we assume \( rs \neq 0 \). Replacing \( r, s \) by \(-r, -s\) if necessary, we may assume that \( r > 0 \). We consider two cases depending on whether the initial point \( \alpha \) in (3.12) satisfies \( \alpha \in R_{S_{f, \Gamma}} \).
Case A: \( \alpha \in R_{S_{f,\Gamma}} \).

In this case, our definitions ensure that every iterate \( f^{(k)}(\alpha) \) is in \( R_{S_{f,\Gamma}} \), i.e.,

\[
(3.14) \quad v_p(f^{(k)}(\alpha)) \geq 0 \quad \text{for all } k \geq 0 \text{ and all } p \not\in S_{f,\Gamma}.
\]

We distinguish now the following four subcases:

**Case A.1: \( r > 0, \ s < 0 \).**

Then equation \((3.12)\) becomes

\[
(f^{(m)}(\alpha))^r(f^{(n)}(\alpha))^t = u \quad \text{with } t = -s > 0.
\]

Using the fact that \( u \in R_{S_{f,\Gamma}} \) and \( r, t > 0 \), and since \((3.14)\) tells us that the iterates have non-negative valuation, we conclude that

\[
v_p(f^{(m)}(\alpha)) = v_p(f^{(n)}(\alpha)) = 0 \quad \text{for all } p \not\in S_{f,\Gamma}.
\]

In particular, we have \( f^{(m)}(\alpha) \in R_{S_{f,\Gamma}}^* \). Since \( m > n > 0 \), we have \( m \geq 2 \), so Theorem 1.2(b) tells us that there are only finitely many \((m, \alpha)\) with this property unless \( f \) has the form \( f(X) = aX^{\pm d} \), which would contradict the assumption that \( 0 \) is not periodic for \( f \).

**Case A.2: \( r > 0 \) and \( s \geq 2 \).**

Since \( \gcd(r, s) = 1 \), we can choose integers \( a \) and \( b \) with \( ar + bs = 1 \). Then \((3.12)\) becomes

\[
(3.15) \quad f^{(m)}(\alpha) = u^a \left( (f^{(n)}(\alpha))^a \left( f^{(m)}(\alpha) \right)^b \right)^s.
\]

Since \( f^{(m)}(\alpha) \in R_{S_{f,\Gamma}} \) and \( u \in R_{S_{f,\Gamma}}^* \), we see from \((3.15)\) that

\[
(f^{(n)}(\alpha))^a \left( f^{(m)}(\alpha) \right)^b \in R_{S_{f,\Gamma}}^*.
\]

Clearly, if

\[
(f^{(n)}(\alpha))^a \left( f^{(m)}(\alpha) \right)^b \in R_{S_{f,\Gamma}}^*
\]

then \( f^{(m)}(\alpha) \in R_{S_{f,\Gamma}}^* \) as well. As in Case A.1, since \( m \geq 2 \), Theorem 1.2(b) tells us that there are only finitely many \((m, \alpha)\) with this property unless \( f \) has the form \( f(X) = aX^{\pm d} \), which would contradict the assumption that \( 0 \) is not periodic for \( f \). Thus we may assume that

\[
(3.16) \quad (f^{(n)}(\alpha))^a \left( f^{(m)}(\alpha) \right)^b \not\in R_{S_{f,\Gamma}}^*.
\]

Writing

\[
f^{(m)}(\alpha) = f \left( f^{(m-1)}(\alpha) \right),
\]
we see from Lemma 2.8 that under the condition (3.16) the exponent $s \geq 2$ in (3.16) is bounded above by a quantity depending only on $K$, $f$ and $S_{f, \Gamma}$.

We assume first that $\deg f \geq 3$. Applying [1, Theorem 2.2] to (3.15), we conclude that there are only finitely many values for $f^{(m-1)}(\alpha)$, and thus, Lemma 2.3 says that there are only finitely many possibilities for $m$ and $\alpha \in \overline{K}$.

Similarly, if $\deg f = 2$ and $s \geq 3$, then [1, Theorems 2.1] again says that there are only finitely many values for $f^{(m-1)}(\alpha)$, so we are done.

It remains to deal with the case that $\deg f = s = 2$. Since $m \geq 2$,

$$f^{(m)}(\alpha) = f^{(2)}\left(f^{(m-2)}(\alpha)\right),$$

and applying [1, Theorem 2.2] to (3.15) with $f^{(2)}$ (since we assume that $f^{(2)}$ has only simple roots), we conclude that there are only finitely many values for $f^{(m-2)}(\alpha)$, and therefore, by Lemma 2.3, finitely many possibilities for $m$ and $\alpha \in \overline{K}$.

**Case A.3: $r \geq 2$ and $s = 1$.**

We note that if $n \geq 2$, then the same discussion as above holds for this case too (where we replace $m$ be $n$ and $s$ by $r$). We therefore consider only the case $n = 1$, and (3.12) becomes

$$(3.17) \quad f(\alpha) = u^{-1}(f^{(m)}(\alpha))^r.$$

If $r \geq 3$, then [1, Theorems 2.1] again says that there are only finitely many values for $\alpha$ and $f^{(m)}(\alpha)$, so we are done.

We assume thus $r = 2$ in (3.17). Since $\deg f \geq 2$ and $f$ has only simple roots, we apply Theorem 1.3, where $\rho$ in this case satisfies $\rho < 1$, to conclude that there are finitely many $m$ and $\alpha$ satisfying (3.17).

**Case A.4: $r = s = 1$.**

This case leads to an equation of the form (1.4), which is covered by Theorem 1.3.

**Case B: $\alpha \not\in R_{S_{f, \Gamma}}$.**

We choose a prime ideal $q$ of $R_\mathbb{K}$ with

$$q \not\in S_{f, \Gamma} \quad \text{and} \quad v_q(\alpha) < 0.$$

Then, using (3.13), we see that $q \not\in S_{f^{(k)}, \Gamma}$ for all $k \geq 1$, and thus from the proof of [19, Theorem 4.11], we have

$$(3.18) \quad v_q\left(f^{(k)}(\alpha)\right) = d^k v_q(\alpha) \quad \text{for all } k \geq 0.$$

(In dynamical terms, this formula reflects the fact that $\alpha$ is in the $q$-adic attracting basin of the superattracting fixed point $\infty$ of $f$.)
Applying this to (3.12), we conclude that
\[(3.19) \quad r d^m v_q(\alpha) = s d^m v_q(\alpha).\]
We also know that
\[(3.20) \quad d \geq 2, \quad r > 0, \quad m > n > 0, \quad \gcd(r, s) = 1, \quad \text{and} \quad v_q(\alpha) \neq 0.\]
It follows from (3.19) and (3.20) that
\[r = 1 \quad \text{and} \quad s = d^{m-n},\]
and hence (3.12) becomes
\[(3.21) \quad f^{(m)}(\alpha) = u(f^{(n)}(\alpha))^{d^{m-n}}.\]
To facilitate a comparison with the proof of Theorem 1.3, we let \(m = n + k\), and then we reverse the roles of \(n\) and \(k\), which changes (3.21) into
\[(3.22) \quad f^{(n+k)}(\alpha) = u(f^{(k)}(\alpha))^{d^n}.\]
If \(n = 1\), i.e., \(f^{(k+1)}(\alpha) = u(f^{(k)}(\alpha))^d\), then we use Theorem 1.2(a), applied with the rational function \(f(X)/X^d\). Note that, since \(f\) has only simple roots and \(X \nmid f\), the rational function \(f(X)/X^d\) is not of one of the forms described in Theorem 1.2(a). Hence we have the desired finiteness in this case.

We now need to show that (3.22) has finitely many solutions
\[(3.23) \quad (n, k, \alpha, u) \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1} \times \text{Wander}_K(f) \times \Gamma.\]
We note that even for a fixed value of \(n\), we cannot apply Theorem 1.3 directly with \(r = 1\) and \(s = d^n\), because Theorem 1.3 only deals with solutions satisfying \(n \geq \rho\), while in our case
\[\rho = \log_d(s/r) + 1 = \log_d(d^n) + 1 = n + 1.\]
We replace \(\mathbb{K}\) by the extension field \(\mathbb{L}\) generated by the set of values \(\{\beta \in \overline{\mathbb{K}} : \beta^{d^2} \in \Gamma\}\).
Note that \(\mathbb{L}/\mathbb{K}\) is a finite extension depending only on \(\mathbb{K}\), \(d\), and \(\Gamma\), since \(\Gamma/\Gamma^{d^2}\) is a finite group.
We consider now two cases: \(d \geq 3\) and \(d = 2\).
If \(d \geq 3\) we consider the curve
\[C_1 : f(X) = Y^{d^2}.\]
Since \(f\) has only simple roots, applying the genus formula [11, Exercise A.4.6] for a smooth projective model \(\tilde{C}_1\) of the affine curve \(C_1\), we have
\[\text{genus}(\tilde{C}_1) = (d - 1)(d^2 - 2)/2 \geq 7\]
when \( d \geq 3 \). Therefore, by Faltings’ theorem \([7, 8]\), the set of \( \mathbb{L} \)-rational points \( C_1(\mathbb{L}) \) on \( C_1 \) is finite. However, every solution \((n, k, \alpha, u)\) of the form \((3.23)\) to the equation \((3.22)\) gives an \( \mathbb{L} \)-rational point \( C_1(\mathbb{L}) \) via the formula

\[
\left( f^{(n+k-1)}(\alpha), u^{1/d^2} (f^{(k)}(\alpha))^{d^n-2} \right) \in C_1(\mathbb{L}).
\]

(Note that \( u^{1/d^2} \in \mathbb{L} \) by the definition of \( \mathbb{L} \).) We conclude in particular that \( f^{(n+k-1)}(\alpha) \) takes on only finitely many values, and then Lemma 2.3 says that there are only finitely many possibilities for \( n, k, \) and \( \alpha \).

If \( d = 2 \), we consider the curve

\[
C_2 : f^{(2)}(X) = Y^{d^2}.
\]

In this case, since \( f^{(2)} \) has simple roots by hypothesis, the formula \([11, Exercise A.4.6]\) for the genus of a smooth projective model \( \tilde{C}_2 \) of the curve \( C_2 \) becomes

\[
\text{genus}(\tilde{C}_2) = (d^2 - 1)(d^2 - 2)/2 = 3.
\]

As above, Faltings’ theorem \([7, 8]\) implies that the set of \( \mathbb{L} \)-rational points \( C_2(\mathbb{L}) \) on \( C_2 \) is finite, and thus there are finitely many possibilities for \( n, k, \) and \( \alpha \).

In fact, we remark that if \( d = 2 \) and \( n \geq 3 \), one does not need the condition that \( f^{(2)} \) has simple roots. Indeed, in this case one can consider the curve

\[
C_3 : f(X) = Y^{d^3}.
\]

Since \( f \) has simple roots, the formula for the genus of a smooth projective model \( \tilde{C}_3 \) of \( C_3 \) is

\[
\text{genus}(\tilde{C}_3) = (d - 1)(d^3 - 2)/2 = 3,
\]

and thus, by Faltings’ theorem \([7, 8]\), we obtain the same finiteness conclusion as above.

This concludes the proof of Case B, and with it the proof of Theorem 1.7. \( \square \)

3.6. **Proof of Theorem 1.10.** Relabeling, we assume that the \( k \)-tuple of distinct integers is ordered so that

\[ n_1 > n_2 > \cdots > n_k \geq 0. \]

By assumption, the polynomial \( F \) has the form

\[
F(T_1, \ldots, T_k) = \sum_{i=1}^r c_i \prod_{j \in J_i} T_j
\]
for some disjoint partition \( J_1 \cup \cdots \cup J_r = \{1, 2, \ldots, k\} \). Relabeling, we may assume that \( 1 \in J_1 \).

We first note that \( \{ \alpha \in \overline{\mathbb{K}} : 0 \in \mathcal{O}_f(\alpha) \} \) is a set of bounded height. Explicitly, if \( f^{(n)}(\alpha) = 0 \), then
\[
\begin{align*}
h(\alpha) &\leq \hat{h}_f(\alpha) + C_1(f) = d^{-n}\hat{h}_f(f^{(n)}(\alpha)) + C_1(f) \\
&= d^{-n}\hat{h}_f(0) + C_1(f) \leq \hat{h}_f(0) + C_1(f) \\
&\leq h(0) + 2C_1(f) = 2C_1(f).
\end{align*}
\]

Thus for the proofs of (a) and (b), we may restrict attention to \( \alpha \in \overline{\mathbb{K}} \) such that \( 0 \notin \mathcal{O}_f(\alpha) \). Note that this also implies that \( r \geq 2 \).

(a) We rewrite (1.5) to isolate the \( n_1 \)-term,
\[
f^{(n_1)}(\alpha) = \sum_{i=2}^{r} (-c_i) \prod_{j \in J_i} f^{(n_j)}(\alpha) / c_1 \prod_{j \in J_1 \setminus \{1\}} f^{(n_j)}(\alpha).
\]

(This is where we use the fact that \( 0 \notin \mathcal{O}_f(\alpha) \).) We take the height of both sides and use the submultiplicativity and subadditivity properties of \( h \) [23, Exercise B.20], i.e.,
\[
h\left( \prod_{i=1}^{n} \beta_i \right) \leq \sum_{i=1}^{n} h(\beta_i), \quad h\left( \sum_{i=1}^{n} \beta_i \right) \leq \sum_{i=1}^{n} h(\beta_i) + \log n,
\]

together with the fact that \( h(\beta^{-1}) = h(\beta) \) for \( \beta \neq 0 \). This yields
\[
h\left( f^{(n_1)}(\alpha) \right) \leq h\left( \sum_{i=2}^{r} (-c_i) \prod_{j \in J_i} f^{(n_j)}(\alpha) \right) + h\left( c_1 \prod_{j \in J_1 \setminus \{1\}} f^{(n_j)}(\alpha) \right) \\
\leq \sum_{i=2}^{r} \left( h(c_i) + \sum_{j \in J_i} h\left( f^{(n_j)}(\alpha) \right) \right) + \log(r - 1) \\
+ h(c_1) + \sum_{j \in J_1 \setminus \{1\}} h\left( f^{(n_j)}(\alpha) \right) \\
= \sum_{j=2}^{k} h\left( f^{(n_j)}(\alpha) \right) + \sum_{i=1}^{r} h(c_i) + \log(r - 1).
\]

Using Lemma 2.1(a) to switch to canonical heights gives
\[
\hat{h}_f\left( f^{(n_1)}(\alpha) \right) - C_1(f) \\
\leq \sum_{j=2}^{k} \left( \hat{h}_f\left( f^{(n_j)}(\alpha) \right) + C_1(f) \right) + \sum_{i=1}^{r} h(c_i) + \log(r - 1),
\]
and then the transformation formula $\hat{h}_f \circ f^{(n)} = d^n \hat{h}_f$ of Lemma 2.1(b) gives

$$d^{n_1} \hat{h}_f(\alpha) \leq \left( \sum_{j=2}^{k} d^{n_j} \right) \hat{h}_f(\alpha) + kC_1(f) + \sum_{i=1}^{r} h(c_i) + \log(r - 1).$$

Hence

$$(d^{n_1} - \sum_{j=2}^{k} d^{n_j}) \hat{h}_f(\alpha) \leq kC_1(f) + \sum_{i=1}^{r} h(c_i) + \log(r - 1) \leq kC_1(f) + kh(F) + \log(k - 1),$$

where we have used $r \leq k$ and the definition of $h(F)$. We next use the fact that $n_1 > n_2 > \cdots > n_k \geq 0$ to estimate

$$d^{n_1} - \sum_{j=2}^{k} d^{n_j} \geq d^{n_1} - d^{n_1-1} - d^{n_1-2} - \cdots - d^{n_1-k+1} \geq \begin{cases} \dfrac{2^{n_1-k+1}}{d} \cdot d^{n_1} & \text{if } d = 2, \\ \dfrac{1}{2} d^{n_1} & \text{if } d \geq 3. \end{cases}$$

Using this bound in (3.24) gives a bound for $\hat{h}_f(\alpha)$ in terms of $F$ and $f$, and then using $h \leq \hat{h}_f + C_1(f)$ shows that $h(\alpha)$ is bounded, which completes (a).

(b) Continuing with the computation from (a), we note that $n_1 \geq k - 1$, so for $d \geq 3$, the inequalities (3.24) and (3.25) yield

$$\hat{h}_f(\alpha) \leq \frac{2kC_1(f)}{d^{k-1}} + \frac{2k}{d^{k-1}} h(F) + \frac{2\log(k - 1)}{d^{k-1}}.$$  

Again using $h \leq \hat{h}_f + C_1(f)$, together with the trivial estimates

$$2k/d^{k-1} \leq \frac{4}{3}$$

and

$$2 \log(k - 1)/d^{k-1} \leq \frac{2}{9} \log 2,$$

valid for $d \geq 3$ and $k \geq 2$, we find that

$$h(\alpha) \leq \frac{7}{3} C_1(f) + \frac{2k}{d^{k-1}} h(F) + \frac{2}{9} \log 2.$$

(c) We are assuming that $\alpha \in \mathbb{K}$ is wandering, so Lemma 2.1(d) says that $\hat{h}_f(\alpha) \geq C_2(\mathbb{K}, f) > 0$. Since we are further assuming that $0 \notin \mathcal{O}_f(\alpha)$, the estimate (3.24) in (a), combined with the lower bound (3.25) yields

$$d^{n_1} \leq \frac{d - 1}{d - 2 + d^{-k+1}} \cdot \frac{kC_1(f) + kh(F) + \log(k - 1)}{C_2(\mathbb{K}, f)}.$$
The right-hand side of (3.26) depends only on $K$, $f$, and $F$, so it gives a bound for $n_1$ depending only on these quantities. Since $n_1 > n_2 > \cdots > n_k \geq 0$, this completes the proof that there are only finitely many $k$-tuples $(n_1, \ldots, n_k)$ of distinct non-negative integers satisfying (1.5), and that the number of such $k$-tuples is bounded independently of $\alpha$. (This last statement would also follow from (a), which says that for all but finitely many $\alpha \in K$, the equation (1.5) has no solutions.) □

Appendix A. Computation of Singular Points of $\bar{C}$

On the chart where $Z \neq 0$, we dehomogenize to the equation $F(X) = cG(X)Y^m$. A point is singular if and only if

$$F'(X) = cG'(X)Y^m \quad \text{and} \quad 0 = cmG(X)Y^{m-1}. $$

If $G(X) = 0$, then $X = \beta_j$ for some $j$, and then from the equation of $C$ we must have $F(\beta_j) = 0$, contradicting the assumption that $F$ and $G$ have no common roots. So $Y = 0$, and the equation of $C$ forces $X = \alpha_k$ for some $k$. But we also have the condition $F'(X) = cG'(X)Y^m$, and evaluating at the point $(\alpha_k,0)$ shows that $F'(\alpha_k) = 0$. Thus $\alpha_k$ needs to be a double root of $F(X)$, i.e., $e_k \geq 2$.

We next consider the points on $\bar{C}$ satisfying $Z = 0$. Substituting $Z = 0$ into the equation of $\bar{C}$ gives $0 = dbX^sY^m$, so either $X = 0$ or $Y = 0$. So we are reduced to checking the two points $[1,0,0]$ and $[0,1,0]$. To ease notation, we write $\bar{F}(X,Z)$ and $\bar{G}(X,Z)$ for the homogenizations of $F$ and $G$. We also let $M = m + d_G - d_F$, where we note that $M \geq 2$ by the assumption that $m \geq d_F + 2$.

Around the point $[1,0,0]$, we dehomogenize $X = 1$, so $\bar{C}$ has the local affine equation $\bar{F}(1,Z)Z^M - c\bar{G}(1,Z)Y^m = 0$. Using the facts that $M \geq 2$ and $m \geq 2$, we see that both the $Y$ and $Z$ derivatives of this equation vanish at $[1,0,0]$, so $[1,0,0]$ is always a singular point.

Finally, around the point $[0,1,0]$, we denomogenize $Y = 1$, so $\bar{C}$ has the local affine equation $\bar{F}(X,1)Z^M - c\bar{G}(X,1) = 0$. Taking the $X$ and $Z$ derivatives and substituting $[0,1,0]$, we see that $[0,1,0]$ is a singular point if and only if $G_X(0,0) = G_Z(0,0) = 0$. Hence $[0,1,0]$ is singular if and only if $d_G \neq 1$. 

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References

[1] A. Bérczes, J.-H. Evertse and K. Győry, ‘Effective results for hyper- and superelliptic equations over number fields’, Publ. Math. Debrecen, 82 (2013), 727–756. (pp. 12, 13, and 26)
[2] A. Bridy and T. Tucker, ‘ABC implies a Zsigmondy principle for ramification’, J. Number Theory, 182 (2018), 296–310. (p. 6)
[3] G. S. Call and J. H. Silverman, ‘Canonical heights on varieties with morphisms’, Compositio Math., 89 (1993), 163–205. (p. 7)
[4] K. Doerksen and A. Haensch, ‘Primitive prime divisors in zero orbits of polynomials’, Integers, 12 (2012), 465–472. (p. 6)
[5] G. Everest and G. Harman, ‘On primitive divisors of \(n^2 + b\)’, Number theory and polynomials, London Math. Soc. Lecture Note Ser., 352, Cambridge Univ. Press, Cambridge, 2008, 142–154. (p. 6)
[6] X. Faber and A. Granville, ‘Prime factors of dynamical sequences’, J. Reine Angew. Math., 661 (2011), 189–214. (p. 6)
[7] G. Faltings, ‘Endlichkeitssätze für abelsche Varietäten über Zahlkörpern’, Invent. Math., 73 (1983), 349–366. (pp. 14, 20, and 28)
[8] G. Faltings, ‘Finiteness theorems for abelian varieties over number fields’, Arithmetic geometry, Storrs, Connecticut, 1984, Springer, New York, 1986. (pp. 14, 20, and 28)
[9] D. Ghioca, K. Nguyen and T. Tucker, ‘Squarefree doubly primitive divisors in dynamical sequences’, Math. Proc. Cambridge Philos. Soc., 164 (2018), 551–572. (p. 6)
[10] C. Gratton, K. Nguyen and T. Tucker, ‘ABC implies primitive prime divisors in arithmetic dynamics’, Bull. Lond. Math. Soc., 45 (2013), 194–1208. (p. 6)
[11] M. Hindry and J. H. Silverman, Diophantine geometry: An Introduction, Graduate Texts in Mathematics, 201, Springer-Verlag, New York, 2000. (pp. 2, 8, 27, and 28)
[12] L.-C. Hsia and J. H. Silverman, ‘A quantitative estimate for quasi-integral points in orbits’, Pacific J. Math., 249 (2011), 321–342. (p. 8)
[13] P. Ingram and J. H. Silverman, ‘Primitive divisors in arithmetic dynamics, Math. Proc. Cambridge Philos. Soc., 146 (2009), 289–302. (p. 6)
[14] H. Krieger, ‘Primitive prime divisors in the critical orbit of \(z^d + c\)’, Int. Math. Res. Not., 2013 (2013), 5498–5525. (p. 6)
[15] H. Krieger, A. Levin, Z. Scherr, T. Tucker, Y. Yasufuku and M. E. Zieve, ‘Uniform boundedness of \(S\)-units in arithmetic dynamics’, Pacific J. Math., 274 (2015), 97–106. (pp. 3 and 13)
[16] S. Lang, Fundamentals of Diophantine Geometry, Springer-Verlag, New York, 1983. (p. 2)
[17] S. Lang, Algebra, Graduate Texts in Mathematics, 211, Springer-Verlag, New York, 2002. (p. 9)
[18] D. G. Northcott, ‘Periodic points on an algebraic variety’, Ann. of Math., 51 (1950), 167–177. (p. 1)
[19] A. Ostafe, M. Sha, I. E. Shparlinski and U. Zannier, ‘On multiplicative dependence of values of rational functions and a generalisation of the Northcott theorem’, Michigan Math. J., (to appear). (pp. 1 and 26)
[20] A. Ostafe and I. E. Shparlinski, ‘Orbits of algebraic dynamical systems in subgroups and subfields’, Number Theory - Diophantine problems, uniform
distribution and applications, C. Elsholtz and P. Grabner (Eds.), Springer, 2017, 347–368. (p. 3)

[21] A. Schinzel and R. Tijdeman, ‘On the equation $y^m = P(x)$’, 
 Acta Arith., 31 (1976), 199–204. (p. 12)

[22] J. H. Silverman, ‘Integer points, Diophantine approximation, and iteration of rational maps’, 
 Duke Math. J., 71 (1993), 793–829. (p. 8)

[23] J. H. Silverman, The arithmetic of dynamical systems, Springer-Verlag, New York, 2007. (pp. 2, 3, 7, 8, 11, 15, 24, and 29)

[24] J. H. Silverman, ‘Primitive divisors, dynamical Zsigmondy sets, and Vojta’s conjecture’, 
 Number Theory, 133 (2013), 2948–2963. (p. 6)

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