A Resonance Problem in Relaxation of Ground States of Nonlinear Schrödinger Equations

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Abstract

In this paper we consider a resonance problem, in a generic regime, in the consideration of relaxation of ground states of semilinear Schrödinger equations. Different from previous results, our consideration includes the presence of resonance, resulted by overlaps of frequencies of different states. All the known key results, proved under non-resonance conditions, have been recovered uniformly. These are achieved by better understandings of normal form transformation and Fermi Golden rule. Especially, we find that if certain denominators are zeros (or small), resulted by the presence of resonances (or close to it), then cancellations between terms make the corresponding numerators proportionally small.

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1 Introduction

We consider the following 3-dimensional semilinear Schrödinger equations

\[ i\partial_t \psi(x,t) = -\Delta \psi(x,t) + V(x)\psi(x,t) + |\psi(x,t)|^2\psi(x,t), \]  
\[ \psi(x,0) = \psi_0(x) \in H^2(\mathbb{R}^3), \]  

where \( V : \mathbb{R}^3 \to \mathbb{R} \) is the external potential.

Such equations arise in the theory of Bose-Einstein condensation, nonlinear optics, theory of water waves and in other areas.

We start with formulating the problem. The potential \( V : \mathbb{R}^3 \to \mathbb{R} \) is a smooth, and rapidly decaying function. And if it is trapping potential, namely

\[ \inf_{\|f\|_2 = 1} \langle f, (-\Delta + V)f \rangle = -e_0 < 0, \]  

where \( e_0 \) is a constant.
then this linear unbounded self-adjoint operator $-\Delta + V$, mapping $L^2$ into $L^2$, has a ground state $\phi \in L^2$ with eigenvalue $-e_0$. Moreover the eigenvalue must be simple. Besides the ground states the linear operator might have some other finitely many neutral modes with nonpositive eigenvalues $-e_k$, $k = 1, \cdots, N$. Its continuous spectrum spans the interval $[0, \infty)$. It is well known that for the type of potential $V$ we chose, there is no positive eigenvalues, see e.g. [16].

In the nonlinear setting, the ground state bifurcates into a family of solitary wave solutions, see e.g. [24],

$$\psi(x, t) = e^{i\lambda t} \phi^\lambda(x)$$

with $\lambda \in \mathbb{R}$ being close to $e_0$ and $\phi^\lambda = C \sqrt{|e_0 - \lambda|} \phi + O(|e_0 - \lambda|^\frac{3}{2})$.

There is a rich literature on studying the orbital stability and asymptotic stability of the soliton manifold. By results in [25, 20, 13] it is well known that the ground state manifold is orbital stable in the $H^1$ space. After these, many attempts were made on proving the asymptotic stability of the ground state manifold, see e.g. [5, 18, 6, 24, 23, 20, 27, 15, 14].

In [22], multiple neutral modes was considered. Their eigenvalues $-e_k$, $k = 1, \cdots, N$ must satisfy two conditions:

1. $2e_k < e_0$,
2. The so-called non-resonance condition, namely there do not exist $n_k \in \mathbb{Z}$, $k = 0, 1, \cdots, N$ such that

$$\sum |n_k| \neq 0 \text{ and } \sum n_k e_k = 0,$$

see also the non-resonance conditions for multiple neutral modes in [8, 15].

On the technical level, the condition (4) was needed to prevent small denominator from appearing.

In [11, 12], the author, together with M. Weinstein, improved the above results by studying degenerate neutral modes, i.e.

$$e_k = e_1, k = 1, 2, \cdots, N,$$

or nearly degenerate.

The main purpose of the present paper is to include the presence of the resonance, specifically by removing the conditions (4) and (5), and to show all the proved results still hold, uniformly.

A graphic illustration is in Figure 1

On the technical level, we achieve this by re-defining normal form transformation and Fermi Golden rule. Especially we show that if some denominators are small, caused by the resonances, then their corresponding numerators are proportionally small, resulted by cancellations between terms. For the details, we refer to Sections 5 and 10, and Proposition 11.1.

To the best of our knowledge, our result and techniques are new.

**Related problems.** The main motivation of the present work is to understand certain types of small divisor problems. In many literatures, for example
in [3, 4, 2], in studying problems arising from statistical mechanics and PDE, the small divisor was avoided, by choosing initial conditions, for example.

The technical advantage in studying the present problem is that we only need to expand the solutions, in certain small parameters, finitely many times (two times), instead of infinitely many times as in [3]. We expect the normal form transformation invented here, which makes it easy to see the crucial cancellations between terms, together with algebraic structures observed in [3, 27, 8], for higher order iterations, can be applied to the other problems. These will be addressed in subsequent papers.

**Structure of the paper.** The paper is organized as follows. Some basic properties of the equation are studied in Section 3. The linearized operator $L(\lambda)$, obtained by linearizing the ground state solution, is studied in Section 4. The Fermi Golden rule condition is in Section 5. The main Theorem is stated in Section 6. The decomposition of the solutions, and the governing equations for various parameters and functions are in Section 7. The main Theorem is reformulated into Theorem 8.1 in Section 8. The proof of different parts of Theorem 8.1 are in Sections 9, 10 and 11. The main theorem is proved in Section 12.

Most parts of the paper follow the steps in the previous papers [11, 12]. Technically the main differences are in normal form transformation in Section 10 and subsequent sections.

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2 Notation

(1) \(\alpha_+ = \max\{\alpha, 0\}\), \([\tau] = \max_{\tilde{\tau} \in \mathbb{Z}} \{\tilde{\tau} \leq \tau\}\)

(2) \(?z = \text{real part of } z, \ ?z = \text{imaginary part of } z\)

(3) Multi-indices

\[z = (z_1, \ldots, z_N) \in C^N, \ \bar{z} = (\overline{z_1}, \ldots, \overline{z_N})\]

\[a \in \mathbb{N}^N, \ z^a = z_1^{a_1} \cdots z_N^{a_N}\]

\[|a| = |a_1| + \ldots + |a_N|\]

(4) \(\mathcal{H}^s = \text{Sobolev space of order } s\)

(5)

\[J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ H = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \ L = JH = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}\]

(6) \(\sigma_{\text{ess}}(L) = \sigma_c(L)\) is the essential (continuous) spectrum of \(L\), \(\sigma_d = \mathbb{C} - \sigma_c(L)\) is the discrete spectrum of \(L\).

(7) \(P_d(L)\) bi-orthogonal projection onto the discrete spectral part of \(L\)

(8) \(P_c(L) = I - P_d(L)\), bi-projection onto the continuous spectral part of \(L\)

(9) \(\langle f, g \rangle = \int f \bar{g}\)

(10) \(\|f\|_p = L^p\) norm, \(1 \leq p \leq \infty\).
3 Basic Properties

Equation (1) is a Hamiltonian system on Sobolev space $\mathcal{H}^1(\mathbb{R}^3, \mathbb{C})$ viewed as a real space $\mathcal{H}^1(\mathbb{R}^3, \mathbb{R}) \oplus \mathcal{H}^1(\mathbb{R}^3, \mathbb{R})$. The Hamiltonian functional is:

$$\mathcal{E}(\psi) := \int \left[ \frac{1}{2}(|\nabla \psi|^2 + V|\psi|^2) + \frac{1}{4}|\psi|^4 \right].$$

Equation (1) has the time-translational and gauge symmetries which imply the following conservation laws: for any $t \geq 0$, we have

- (CE) Conservation of energy: $\mathcal{E}(\psi(t)) = \mathcal{E}(\psi(0))$;
- (CP) Conservation of particle number:
  $$\mathcal{N}(\psi(t)) = \mathcal{N}(\psi(0)), \quad \text{where} \quad \mathcal{N}(\psi) := \int |\psi|^2.$$

In what follows we review the results of the existence of soliton and their properties.

The following arguments are almost identical to those in [18, 6, 24] except that here we have multiple neutral modes (or excited states), hence we state the results without proof. We assume that the linear operator $-\Delta + V$ has the following properties

- (NL) The linear operator $-\Delta + V$ has eigenvalues $-e_0 < -e_1 \leq \cdots \leq -e_N$ satisfying $e_0 < 2e_1$. $-e_0$ is the lowest eigenvalue with ground state $\phi > 0$, the eigenvalue $-e_1, \cdots, -e_N$ might be degenerate with eigenvectors $\xi_{lin}^1, \xi_{lin}^2, \ldots, \xi_{lin}^N$.

In the nonlinear setting the ground state bifurcates into a family of solitary wave solutions of (1), see e.g. [24],

$$\psi(x,t) = e^{i\lambda t} \phi^\lambda(x)$$  \hspace{1cm} (7)

and the function $\phi^\lambda > 0$ has the following properties, see e.g. [24].

**Lemma 3.1.** Suppose that the linear operator $-\Delta + V$ satisfies the conditions in (NL) above. Then there exists a constant $\delta_0 > 0$ such that for any $\lambda \in [e_0 - \delta, e_0)$ (1) has solutions of the form $\psi(x,t) = e^{i\lambda t} \phi^\lambda(x) \in L^2$ with

$$\phi^\lambda = \delta \phi + \mathcal{O}(\delta^3)$$  \hspace{1cm} (8)

and $\delta = (\int \phi^\lambda(x)dx)^{-\frac{1}{2}}(e_0 - \lambda)^{1/2} + o((e_0 - \lambda)^{1/2})$, moreover

$$\partial_\lambda \phi^\lambda = \mathcal{O}((e_0 - \lambda)^{-1/2})\phi + o((e_0 - \lambda)^{1/2}).$$

4 The Linearized Operator

After linearizing the solution around solitary wave solution (7), namely considering the solution of (1), $\psi(x, t) = e^{i\lambda t}[\phi^\lambda(x) + R(x, t)]$, then the linear part of the equation for $R(x, t)$ is

$$\partial_t R = L(\lambda) R$$  \hspace{1cm} (9)
with \( \vec{R} := (ReR, ImR)^T \), and the linearized operator \( L(\lambda) \) is defined as

\[
L(\lambda) := \begin{bmatrix} 0 & L_-(\lambda) \\ -L_+(\lambda) & 0 \end{bmatrix}
\]  

(10)

with \( L_\pm(\lambda) \) being linear Schrödinger operators defined as

\[
L_- (\lambda) := -\Delta + V + \lambda + (\phi^\lambda)^2, \quad \text{and} \quad L_+ (\lambda) := -\Delta + V + \lambda + 3(\phi^\lambda)^2.
\]

By general result (Weyl’s Theorem) on stability of the essential spectrum for localized perturbations of \( J(\Delta) \) [17],

\[
\sigma_{ess}(L(\lambda)) = ([0, -i\infty] \cap [i\lambda, i\infty])
\]

if the potential \( V \) in Equation (1) decays at \( \infty \) sufficiently rapidly.

Next we study the eigenvalues and eigenvectors of \( L(\lambda) \). The proof can be found in [11], hence we omit it.

Lemma 4.1. Let \( L(\lambda) \), or more explicitly, \( L(\lambda(\delta), \delta) \) denote the linearized operator about the bifurcating state \( \phi^\lambda, \lambda = \lambda(\delta) \). Note that \( \lambda(0) = e_0 \).

It has an eigenvector \( \begin{bmatrix} 0 \\ \phi^\lambda \end{bmatrix} \) and an associated eigenvector \( \begin{bmatrix} \partial_\lambda \phi^\lambda \\ 0 \end{bmatrix} \) with eigenvalue 0:

\[
L(\lambda) \begin{bmatrix} 0 \\ \phi^\lambda \end{bmatrix} = 0, \quad L(\lambda) \begin{bmatrix} \partial_\lambda \phi^\lambda \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi^\lambda \end{bmatrix}.
\]  

(11)

Corresponding to the (possibly degenerate) eigenvalue, \(-e_1, -e_2, \cdots, -e_N, \) of \(-\Delta + V,\) the matrix operator

\[
L(\lambda = e_0, \delta = 0)
\]

has eigenvalues

\[
\pm ie_n(e_0) = \pm i(e_0 - e_n), \quad n = 1, 2, \cdots, N.
\]  

(12)

For \( \delta > 0 \) and small, these bifurcate to (possibly degenerate) eigenvalues, of the operator \( L(\lambda), \pm ie_1(\lambda), \ldots, \pm ie_N(\lambda) \) with eigenvectors

\[
\begin{bmatrix} \xi_1 \\ \pm i\eta_1 \end{bmatrix}, \quad \begin{bmatrix} \xi_2 \\ \pm i\eta_2 \end{bmatrix}, \cdots, \begin{bmatrix} \xi_N \\ \pm i\eta_N \end{bmatrix}
\]  

(13)

with \( \xi_n, \eta_n \) being real functions and

\[
\langle \phi^\lambda, \xi_n \rangle = \langle \partial_\lambda \phi^\lambda, \eta_n \rangle = 0, \quad \langle \xi_n, \eta_m \rangle = \delta_{m,n}.
\]  

(14)

Moreover, for \( \delta \) sufficiently small \( 2E_n(\lambda) > 0, \ n = 1, 2, \cdots, N, \) (resonance at second order with radiation).

Furthermore we need the following condition on the threshold resonances.
Definition 1. A function $h$ is called a threshold resonance function of $-\Delta + V$ at 0, the endpoint of the essential spectrum, $|h(x)| \leq c|x|^{-1+}$ and $h$ is $C^2$ and solves the equation 
\[
(-\Delta + V)h = 0.
\]

A function $h$ is called a threshold resonance function of $L(\lambda)$ at $\mu = \pm i\lambda$, the endpoint of the essential spectrum, $|h(x)| \leq c|x|^{-1+}$ and $h$ is $C^2$ and solves the equation 
\[
(L(\lambda) - \mu)h = 0.
\]

In this paper we make the following assumption:

(SA) $-\Delta + V$ has no threshold resonance at 0.

This assumption is generic since it is known that the threshold resonance is unstable, see e.g. [16]. Based on this assumption it is well known that

Lemma 4.2. If $|\epsilon_0 - \lambda|$ is sufficiently small and the assumption (SA) holds, then $L(\lambda)$ has no threshold resonances at $\mu = \pm i\lambda$, and $L(\lambda)$ has no other eigenvectors and eigenvalues besides the ones listed in Lemma 4.1.

We denote the projection onto the essential spectrum of linear operator $L(\lambda)$ is 
\[
P_\lambda = 1 - P_d.
\]

In the following we give the explicit form of the projection $P_d$, whose proof for $N = 1$ can be found in [10], the proof of the general cases are similar, hence omitted.

Proposition 4.3. For the non self-adjoint operator $L(\lambda)$ the (Riesz) projection onto the discrete spectrum subspace of $L(\lambda)$, $P_d = P_d(L(\lambda)) = P_\lambda^d$, is given by

\[
P_d = \frac{2}{\partial_\lambda \|\phi^\lambda\|^2} \begin{pmatrix}
0 & 0 \\
\partial_\lambda \phi^\lambda & 0
\end{pmatrix}
+ \begin{pmatrix}
\partial_\lambda \phi^\lambda \\
0
\end{pmatrix}
\begin{pmatrix}
\phi^\lambda \\
0
\end{pmatrix}
- i \sum_{n=1}^{N} \begin{pmatrix}
\xi_n & i\eta_n \\
-i\eta_n & \xi_n
\end{pmatrix}
\begin{pmatrix}
\xi_n \\
i\eta_n
\end{pmatrix}.
\]

(15)

We define the projection onto the continuous spectral subspace of $L(\lambda)$ by

\[
P_c = P_c(L(\lambda)) = P_\lambda^c \equiv I - P_d.
\]

(16)

5 The Negativity of Fermi Golden Rule in Matrix Form

The Fermi Golden rule plays an essential role in determining the decay rate of the neutral modes. For simple neutral modes, as in [21, 24, 6, 19], the form is simple because the number is only one, or if more than one then they can be separated into independent ones by a near identity transformation (see [22, 8]). The problem of multiple neutral modes is more involved due to the fact that multiple coupled parameters appear and they can not be separated.
Next we define the new Fermi Golden Rule condition. Define a function \( e : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C} \) as
\[
e(x, k) := \left[ 1 + \left( -\Delta_x - |k|^2 - i0 \right)^{-1} V(x) \right]^{-1} e^{i k \cdot x}.
\]
(17)
It is known that for any fixed \( k \neq 0 \), this function is well defined by the type of potential \( V \) we chose, see e.g. [16]. And it satisfies the equation
\[
[-\Delta_x + V(x) - |k|^2] e(x, k) = 0.
\]
We define complex functions \( \Psi_{m,n} \) on the 2-dimensional unit sphere \( \mathbb{S}^2 \). For any \( \sigma \in \mathbb{S}^2 \),
\[
\Psi_{m,n}(\sigma) = \Psi_{n,m}(\sigma) := \int_{\mathbb{R}^3} e(x, |k|_{m,n} \sigma) \phi(x) \xi_m^{lin}(x) \xi_n^{lin}(x) \, dx
\]
with \( |k|_{m,n} \in \mathbb{R}^+ \) defined as
\[
|k|_{m,n} := (e_0 - e_m - e_n)^{\frac{1}{2}},
\]
where, recall that we assume that \( 2e_l < e_0 \), \( l = 1, 2, \cdots, N \), and \( \phi, \xi_m^{lin} \) are eigenvectors of \( -\Delta + V \) with eigenvalues \( -e_0 \) and \( -e_m \), \( m = 1, \cdots, N \).

Now we state our Fermi-Golden-rule condition.

(FGR) For any scalar vector \( z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^N \) satisfying \( |z| = 1 \), the functions defined on the unit sphere \( \mathbb{S}^2 \),
\[
\sum_{m,n=1}^N \Psi_{m,n}(\sigma) z_m z_n
\]
is not identically zero.

Its important ramification is that there exists some constant \( C > 0 \), such that for any \( z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^N \),
\[
\Gamma(z, \bar{z}) := \left\| \sum_{m,n=1}^N \Psi_{m,n}(\sigma) z_m z_n \right\|_{L^2(\mathbb{S}^2)}^2 \geq C |z|^4.
\]
(19)
Here we use that \( \sum_{m,n} \Psi_{m,n}(\sigma) z_m z_n \) is smooth in \( \sigma \), hence if it is not identically zero, we have the estimate above.

**Remark 1.** If the set of eigenvalues \( \{e_k | k = 1, \cdots, N\} \) can be grouped into well separated clusters, namely
\[
\{e_k | k = 1, \cdots, N\} = \cup_l A_l
\]
with properties that, for some constant \( c_0 = \mathcal{O}(1) \),
\[
|e_n - e_m| \geq c_0 \text{ if } e_n \in A_l, \; e_m \in A_k \text{ with } l \neq k.
\]
Then Fermi Golden rule assumption can be relaxed: for each fixed \( l \), the each of the functions
\[
\sum_{e_m, e_n \in A_l} \Psi_{m,n}(\sigma) z_m z_n \text{ is not identically zero, with } \sum_{e_m \in A_l} |z_m|^2 \neq 0.
\]
6 Main Theorem

In this section we state precisely the main theorem of this paper. The key fact is that despite of the possible presence of resonance, the main results remain the same as in [11, 12].

For technical reasons we impose the following conditions on the external potential $V$ of (1):

(VA) $V$ decays exponentially fast at $\infty$.

Recall the notations $\xi = (\xi_1, \cdots, \xi_N)$ and $\eta = (\eta_1, \cdots, \eta_N)$ for components of the neutrally stable modes of frequencies $\pm i \varepsilon_n(\lambda)$, $n = 1, \cdots, N$, of the linearized operator $L(\lambda)$ defined in (10).

Theorem 6.1. Suppose that Conditions (NL) in Section 3, (SA) in Section 4, (FGR) in Section 5 and (VA) above are satisfied. Let $\nu > 0$ be fixed and sufficiently large.

Then there exist constants $c, \epsilon_0 > 0$ such that, if

$$\inf_{\gamma \in \mathbb{R}} \left\| \psi_0 - e^{i\gamma} \left( \phi^{0\lambda} + (Re z^{(0)}) \cdot \xi + i (Im z^{(0)}) \cdot \eta \right) \right\|_{H^{3,\nu}} \leq c |z^{(0)}| \leq \epsilon_0,$$

then there exist smooth functions

$$\lambda(t) : \mathbb{R}^+ \to \mathcal{I}, \quad \gamma(t) : \mathbb{R}^+ \to \mathbb{R}, \quad z(t) : \mathbb{R}^+ \to \mathbb{C}^d,$$

$$R(x,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{C}$$

such that the solution of NLS evolves in the form:

$$\psi(x, t) = e^{i \int_0^t \lambda(s) ds} e^{i \gamma(t)} \times [\phi^{0\lambda} + a_1(z, \bar{z}) \partial_x \phi^{0\lambda} + ia_2(z, \bar{z}) \phi^{0\lambda} + (Re \ \tilde{z}) \cdot \xi + i(Im \tilde{z}) \cdot \eta + R],$$

where $\lim_{t \to \infty} \lambda(t) = \lambda_\infty$, for some $\lambda_\infty \in \mathcal{I}$.

Here, $a_1(z, \bar{z})$, $a_2(z, \bar{z}) : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{R}$ and $\tilde{z} - z : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N$ are polynomials of $z$ and $\bar{z}$, beginning with terms of order $|z|^2$. Their explicit definitions will be given in (58).

Moreover:

(A) $|z(t)| \leq c(1 + t)^{-\frac{1}{4}}$ and, there exists a polynomial $F(z, \bar{z}) = O(|z|^4) \in \mathbb{R}$ such that $z$ satisfies the initial value problem

$$\partial_t [|z|^2 + F(z, \bar{z})] = -C \Gamma(z, \bar{z}) + O((1 + t)^{-\frac{11}{4}})$$

where $C > 0$ is a constant, $\Gamma(z, \bar{z}) = O(|z|^4)$ is a positive quantity defined in (19).
\( \vec{R}(t) = (ReR(t), ImR(t))^T \) lies in the essential spectral part of \( L(\lambda(t)) \).

Equivalently, \( R(\cdot, t) \) satisfies the symplectic orthogonality conditions:

\[
\begin{align*}
\omega(R, i\phi^\lambda) &= \omega(R, \partial_\lambda \phi^\lambda) = 0 \\
\omega(R, i\eta_n) &= \omega(R, \xi_n) = 0, \quad n = 1, 2, \ldots, N,
\end{align*}
\]

where \( \omega(X, Y) = \text{Im} \int X \overline{Y} \) and satisfies the decay estimate:

\[
\| (1 + x^2)^{-\nu} \vec{R}(t) \|_2 \leq c(1 + |t|)^{-1}.
\]

The main theorem will be reformulated into Theorem 8.1 below.

### 7 The Effective Equations for \( \dot{z}, \dot{\lambda}, \dot{\gamma} \) and \( R \)

In this section we derive equations for \( \dot{z}, \dot{\lambda}, \dot{\gamma} \) and \( R \).

We decompose the solution as

\[
\psi(x, t) = e^{i \int_0^t \lambda(s) ds} e^{i\gamma(t)}
\]

\[
\times \left[ \phi^\lambda + a_1 \phi^\lambda + ia_2 \phi^\lambda + \sum_{n=1}^N (\alpha_n + p_n) \xi_n + i \sum_{n=1}^N (\beta_n + q_n) \eta_n + R \right]
\]

\[
= e^{i \int_0^t \lambda(s) ds} e^{i\gamma(t)} \left[ \phi^\lambda + a_1 \phi^\lambda + ia_2 \phi^\lambda + (\alpha + p) \cdot \xi + i(\beta + q) \cdot \eta + R \right]
\]

(26)

Here and going forward, we’ll use the notations:

\[
\alpha = (\alpha_1, \ldots, \alpha_N)^T, \quad \beta = (\beta_1, \ldots, \beta_N)^T,
\]

\[
\xi = (\xi_1, \ldots, \xi_N)^T, \quad \eta = (\eta_1, \ldots, \eta_N)^T.
\]

Introducing

\[
z = \alpha + i\beta,
\]

we have

\[
\alpha = \frac{1}{2}(z + \overline{z}), \quad \beta = \frac{1}{2i}(z - \overline{z}),
\]

and we seek \( a_j = a_j(z, \overline{z}) = O(|z|^2), \quad j = 1, 2 \) and \( p_j = p_j(z, \overline{z}) = O(|z|^2) \), polynomials in \( z \) and \( \overline{z} \), which are of degree larger than or equal to two, and are real. Substitution of the Ansatz (26) into NLS, equation (1), we have the following system of equations for \( \vec{R} \), defined as

\[
\vec{R} := \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \quad R_1 := ReR, \quad R_2 := ImR.
\]
\[
\partial_t \vec{R} = L(\lambda) \vec{R} + \dot{\gamma} J \vec{R} - J \vec{N}(\vec{R}, z) - \left( \partial_\lambda \phi^\lambda [\dot{\lambda} + \partial_t a_1] \right) / \phi^\lambda [\dot{\gamma} + \partial_t a_2 - a_1] \\
+ \left( \xi \cdot [E(\lambda)(\beta + q) - \partial_t (\alpha + p)] - \eta \cdot [E(\lambda)(\alpha + p) + \partial_t (\beta + q)] \right) \\
+ \dot{\gamma} \left( \begin{array}{c} (\beta + q) \cdot \eta \\ -(\alpha + p) \cdot \xi \end{array} \right) - \dot{\lambda} \left( \begin{array}{c} a_1 \partial_\lambda^2 \phi^\lambda + (\alpha + p) \cdot \partial_\lambda \xi \\ a_2 \partial_\lambda \phi^\lambda + (\beta + q) \cdot \partial_\lambda \eta \end{array} \right),
\]

(27)

Here,

\[
J \vec{N}(\vec{R}, z) := \begin{pmatrix} ImN(\vec{R}, z) \\ -ReN(\vec{R}, z) \end{pmatrix}
\]

(28)

with

\[
ImN(\vec{R}, z) := |\phi^\lambda + I_1 + iI_2|^2 I_2 - (\phi^\lambda)^2 I_2 \\
ReN(\vec{R}, z) := |\phi^\lambda + I_1 + iI_2|^2 - (\phi^\lambda)^2 (\phi^\lambda + I_1) - 2(\phi^\lambda)^2 I_1 \\
I_1 = A_1 + A_2 + R_1, \quad I_2 = B_1 + B_2 + R_2 \\
A_1 = \alpha \cdot \xi, \quad A_2 = a_1 \partial_\lambda \phi^\lambda + p \cdot \xi, \\
B_1 = \beta \cdot \eta, \quad B_2 = a_2 \phi^\lambda + q \cdot \eta.
\]

(29)

From the system of equations (27) and the orthogonality conditions (14) and (24) we obtain equations for \(\dot{\lambda}, \dot{\gamma}\) and \(z_n = \alpha_n + i\beta_n, \quad n = 1, \ldots, N:\)

\[
\partial_t (\alpha_n + p_n) - E_n(\lambda)(\beta_n + q_n) + \langle ImN(\vec{R}, z), \eta_n \rangle = F_{1n};
\]

(30)

\[
\partial_t (\beta_n + q_n) + E_n(\lambda)(\alpha_n + p_n) - \langle ReN(\vec{R}, z), \xi_n \rangle = F_{2n};
\]

(31)

\[
\dot{\gamma} + \partial_t a_2 - a_1 - \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} \langle ReN(\vec{R}, z), \phi^\lambda \rangle = F_3;
\]

(32)

\[
\dot{\lambda} + \partial_t a_1 + \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} \langle ImN(\vec{R}, z), \phi^\lambda \rangle = F_4.
\]

(33)
Finally, the scalar functions $F_{j,n}$, $j = 1, 2, F_3, F_4$, are defined as
\[
F_{1n} := \hat{\gamma} \langle \langle \beta + q \rangle \cdot \eta, \eta_n \rangle - \hat{\lambda} a_1 \partial_2^2 \phi^\lambda, \eta_n \rangle - \hat{\lambda} \langle \langle \alpha + p \rangle \cdot \partial_\lambda \xi, \eta_n \rangle
- \hat{\gamma} \langle R_2, \eta_n \rangle + \hat{\lambda} \langle R_1, \partial_\lambda \eta_n \rangle,
\]
\[
F_{2n} := - \hat{\gamma} \langle \langle \alpha + p \rangle \cdot \xi, \xi_n \rangle - \hat{\lambda} a_2 \phi^\lambda, \xi_n \rangle - \hat{\lambda} \langle \langle \beta + q \rangle \cdot \partial_\lambda \eta, \xi_n \rangle
+ \hat{\gamma} \langle R_1, \xi_n \rangle + \hat{\lambda} \langle R_2, \partial_\lambda \xi_n \rangle,
\]
\[
F_3 := \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} \times
\left[ \hat{\lambda} \langle R_2, \phi^\lambda \rangle - \hat{\gamma} \langle R_1, \phi^\lambda \rangle - \langle \hat{\gamma} \langle \alpha + p \rangle \cdot \xi + \hat{\lambda} a_2 \phi^\lambda + \hat{\lambda} \langle \beta + q \rangle \cdot \partial_\lambda \eta, \phi^\lambda \rangle \right],
\]
\[
F_4 := \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} \times
\left[ \hat{\lambda} \langle R_1, \phi^\lambda \rangle + \hat{\gamma} \langle R_2, \phi^\lambda \rangle + \langle \hat{\gamma} \langle \beta + q \rangle \cdot \eta - \hat{\lambda} a_1 \partial_2 \phi^\lambda - \hat{\lambda} \langle \alpha + p \rangle \cdot \partial_\lambda \xi, \phi^\lambda \rangle \right].
\]

To facilitate later discussions we cast (30) and (31) into a convenient form. Since $\alpha_n$ and $\beta_n$ are real parameters, it is equivalent to study the complex parameters $z_n := \alpha_n + i\beta_n$. Compute (30) + i (31) to find
\[
\partial_t z_n + i E_n(\lambda) z_n = - \partial_t (p_n + i q_n) - i E_n(\lambda) (p_n + i q_n) - \langle Im N(\bar{R}, z), \eta_n \rangle
+ i \langle Re N(\bar{R}, \xi), \xi_n \rangle + F_{1n} + i F_{2n}.
\]

Note that (30) and (31) can be recovered from the equation above by considering the real and imaginary parts of (35).

Remark 2. (a) Recall the estimate of Remainder in (48). By (30)–(33) we have
\[
\hat{\lambda}, \hat{\gamma}, \partial_t z_n + i E_n(\lambda) z_n = O(|z|^2) + \text{Remainder}.
\]

(b) The functions $a_j(z, \bar{z}), j = 1, 2$, $p_n(z, \bar{z}), q_n(z, \bar{z}), n = 1, \ldots, N$ will be chosen to eliminate “non-resonant” terms: $z^a \bar{z}^b$, $2 \leq a + b \leq 3$.

Finally, we derive an equation for
\[
\bar{R} = P_c^{\lambda(t)} \bar{R} = P_c \bar{R},
\]
the continuous spectral part of the solution, relative to the operator, $L(\lambda(t))$. Applying $P_c = P_c^{\lambda(t)}$ to (27) to use that (see (15))
\[
P_c \left( \begin{array}{c} \xi_n \\ \pm i \eta_n \end{array} \right) = P_c \left( \begin{array}{c} 0 \\ \phi^\lambda \end{array} \right) = P_c \left( \begin{array}{c} \partial_\lambda \phi^\lambda \\ 0 \end{array} \right) = 0
\]
to remove many terms on the right hand side, and using the commutator identity:
\[
P_c \partial_t \bar{R} = \partial_t \bar{R} - \hat{\lambda} \partial_\lambda P_c \bar{R}
\]

(37)
we obtain
\[ \partial_t \vec{R} = L(\lambda(t)) \vec{R} - P_c^\lambda(t) J \vec{N}(\vec{R}, \bar{z}) + L_{(\dot{\lambda}, \dot{\gamma})} \vec{R} + \mathcal{G}. \]  
(38)

Here the operator \( L_{(\dot{\lambda}, \dot{\gamma})} \) and the vector function \( \mathcal{G} \) are defined as
\[ L_{(\dot{\lambda}, \dot{\gamma})} = \dot{\lambda} \partial_\lambda P_c^\lambda(t) + \dot{\gamma} P_c^\lambda(t) J, \]
(39)
and
\[ \mathcal{G} = P_c^\lambda(t) \left( \dot{\gamma}(\beta + q) \cdot \eta - \dot{\lambda} \alpha + P_c^\lambda(t) \phi - \dot{\lambda}(\alpha + p) \cdot \partial_\xi \right). \]
(40)

8 Reformulation of The Main Theorem

The proof of Theorem 6.1 we use the following result, which gives a more detailed characterization of the terms in the decomposition.

Theorem 8.1. The function \( R \) in (22) of Theorem 6.1 can be decomposed as
\[ \vec{R} = \sum_{|m| + |n| = 2} R_{m,n}(\lambda) z^m \bar{z}^n + \vec{R} \]
(41)
where \( R_{m,n} \) are functions of the form
\[ R_{m,n} = \left[ L(\lambda) + iE(\lambda) \cdot (m - n) - 0 \right]^{-1} \phi_{m,n}, \]
\( \phi_{m,n} \) are smooth, spatially exponentially decaying functions. The function \( \vec{R} \) satisfies the equation
\[ \partial_t \vec{R} = L(\lambda) \vec{R} + M_2(z, \bar{z}) \vec{R} + P_c N_2(\vec{R}, \bar{z}) + P_c S_2(z, \bar{z}), \]
(42)
where

1. \( S_2(z, \bar{z}) = O(|z|^3) \) is a polynomial in \( z \) and \( \bar{z} \) with \( \lambda \)-dependent coefficients, and each coefficient can be written as the sum of functions of the form
\[ [L(\lambda) \pm i(E_m(\lambda) + E_n(\lambda)) - 0]^{-k} P_c \phi_{\pm k}(\lambda), \]
(43)
where \( k = 0, 1, 2 \) and the functions \( \phi_{\pm k}(\lambda) \) are smooth and decay exponentially fast at spatial \( \infty \);

2. \( M_2(z, \bar{z}) \) is an operator defined by
\[ M_2(z, \bar{z}) := \dot{\gamma} P_c J + \dot{\lambda} P_c + X, \]
(44)
where \( X \) is a \( 2 \times 2 \) matrix, satisfying the bound
\[ |X| \leq c|z| e^{-\epsilon_0|z|}. \]
(3) \( N_2(\vec{R}, z) \) can be separated into localized term and nonlocal term

\[
N_2 = \text{Loc} + \text{NonLoc}
\]

where Loc consists of terms spatially localized (exponentially) function of \( x \in \mathbb{R}^3 \) as a factor and satisfies the estimate

\[
\|\langle x \rangle^\nu (-\Delta + 1) \text{Loc} \|_2 + \| \text{Loc} \|_1 + \| \text{Loc} \|_4 \leq c (|z|^3(t) + |z(t)|\|\langle x \rangle^{-\nu} \vec{R} \|_2),
\]

and NonLoc is given by

\[
\text{NonLoc} := (R^2_1 + R^2_2)J\vec{R}.
\]

Here \( \nu \) is the same as in Theorem 6.1.

In the rest of the paper we denote by Remainder\((t)\) any quantity which satisfies the estimate:

\[
|\text{Remainder}(t)| \lesssim |z(t)|^4 + \|\langle x \rangle^{-\nu} \vec{R}(t)\|^2_2 + \|\vec{R}(t)\|_\infty^2 + |z(t)| \|\langle x \rangle^{-\nu} \vec{R}(t)\|_2.
\]

The functions \( \lambda, \gamma, z \) have the following properties

(A) \( \dot{\lambda} = \text{Remainder}(t) \);

(B) \( \dot{\gamma} = \Upsilon + \text{Remainder}(t) \)

with

\[
\Upsilon := \frac{\langle (\phi^\lambda)^2 [\frac{3}{2} |z \cdot \xi|^2 + \frac{1}{2} |z \cdot \eta|^2, \partial_\lambda \phi^\lambda] \rangle}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle};
\]

(C) there exists a polynomial \( F(z, \bar{z}) = O(|z|^4) \in \mathbb{R} \) such that the vector \( z \) satisfies the equation

\[
\partial_t [z^2 + F(z, \bar{z})] = -\Gamma(z, \bar{z}) + |z| \text{Remainder}(t)
\]

where \( \Gamma(z, \bar{z}) \) is a positive quantity defined in (19).

The definition of \( R_{m,n} \) in \( 41 \) will be in Section 7, the proof of \( 42 \) will be in Section 9, \( 49 \) and \( 50 \) will be reformulated into Proposition 10.1, \( 52 \) will be proved in Section 11.

9 Proof of \( 42 \)

Observe that in the equation for \( \vec{R} \) in \( 58 \), the term on the right hand side, specifically \( JN(\vec{R}, z) \), contains terms quadratic in \( z \) and \( \bar{z} \). Hence for fixed \( z(t) \in \mathbb{C}^N \), the equation for \( \vec{R}(t) \) is forced by terms of order \( O(|z(t)|^2) \).
In what follows, we extract the quadratic in \( z, \overline{z} \) part of \( \tilde{R}(t) \). Observe that the quadratic terms generated by the nonlinearity are of the form:

\[
\sum_{|m|+|n|=2} JN_{m,n} z^m \overline{z}^n = \left( \begin{array}{c} 2\phi^\lambda A_1 B_1 \\ -3\phi^\lambda A_1^2 - \phi^\lambda B_1^2 \end{array} \right). \tag{53}
\]

where \( A_1 = \alpha \cdot \xi, B_1 = \beta \cdot \eta \), and recall the definition of \( JN \) from (28). Substitute this into the equation for \( \tilde{R} \) in (38) and decompose \( \tilde{R} \) in the next results:

**Theorem 9.1.** Define

\[
R_{m,n} := \left[ L(\lambda) + iE(\lambda) \cdot (m - n) - 0 \right]^{-1} P_c JN_{m,n}, \tag{54}
\]

and decompose \( \tilde{R}(t) \) as

\[
\tilde{R} = \sum_{|m|+|n|=2} R_{m,n} z^m \overline{z}^n + \tilde{R} \tag{55}
\]

Then the vector-function \( \tilde{R}(x,t) \) satisfies (42).

The proof is the same to that in [11], and skipping it will not affect understanding the main part of this paper. Hence we choose to omit this part.

To facilitate later discussions, we further decompose

\[
J\tilde{N}_{>2} = J\tilde{N}(\tilde{R}, z) - \sum_{|m|+|n|=2} JN_{m,n} z^m \overline{z}^n.
\]

We extract the third order terms of \( J\tilde{N}_{>2} \):

\[
\sum_{|m|+|n|=3} JN_{m,n} z^m \overline{z}^n = \sum_{|m|+|n|=2} XR_{m,n} z^m \overline{z}^n + X \left( \begin{array}{c} A_2 \\ B_2 \end{array} \right) + \left( \begin{array}{c} (A_1^2 + B_1^2) B_1 \\ -(A_1^2 + B_1^2) A_1 \end{array} \right)
\]

where \( X \) is a \( 2 \times 2 \) matrix defined as

\[
X := \left( \begin{array}{cc} 2\phi^\lambda B_1 & 2\phi^\lambda A_1 \\ -6\phi^\lambda A_1 & -2\phi^\lambda B_1 \end{array} \right), \tag{57}
\]

and \( A_1, B_1 \) and \( A_2, B_2 \) are defined in (29).

**10 Normal Form Transformation, Proofs of (49) and (50)**

In this section we present the proofs of equations (49) and (50), governing \( \dot{\lambda} \) and \( \dot{\gamma} \), crucial to controlling the large time behavior. The main result is Proposition 10.1.
This part is different from [11, 12], in that we have to define a new normal form transformation, some of whose parameters are defined as solutions to systems of linear equations and their existence has to be justified. Moreover some small denominators will appear and we have to prove the numerators are also small.

Now we present the idea. Central to our claim about the large time dynamics of NLS, is that the solution settles into an asymptotic solitary wave, \( \phi^{\lambda_{\infty}} \), where \( \lambda(t) \to \lambda_{\infty} \). We achieve this by showing \( |\dot{\lambda}(t)| \lesssim \epsilon_0 (1 + t)^{-1-\delta} \) for some \( \delta > 0 \) and small \( \epsilon_0 > 0 \). Since we expect the neutral mode amplitudes, \( z(t) \), to decay with a rate \( t^{-\frac{1}{2}} \), we require that there be no \( O(|z(t)|^2) \) on the right hand side of the equation (33):

\[
\dot{\lambda}(t) = -\partial_t a_1(z, \bar{z}) - \frac{1}{\langle \phi^\lambda, \phi^\lambda \rangle} (\text{Im} N(\bar{R}, z), \phi^\lambda) + \ldots.
\]

The strategy is to choose the quadratic part of the polynomial \( a_1(z, \bar{z}) \) so as to eliminate all quadratic terms. There are two types of such terms, (1) the terms \( z_k \bar{z}_l \) and \( \bar{z}_k z_l \), and they are oscillatory with frequencies \( \sim -E_k - E_l \) or \( \sim E_k + E_l \), which stay away from zero. And the margins are large enough so that we can easily remove them by a normal form transformation, utilizing that \( z_k \bar{z}_l \approx \partial_t \frac{1}{(-E_k + E_l)} z_k \bar{z}_l \). (2) The terms \( z_k \bar{z}_m \) have frequencies \( \sim -E_k + E_m \), which might be of small, or zero, frequencies. The key observation is that is if the frequency of \( z_k \bar{z}_l \) is small, then the coefficient is proportionally small! This allows us to define normal form transformation. The calculation is carried out below; see Lemma 10.2, especially (61).

Similarly we choose \( p_n, q_n \), \( n = 1, 2, \cdots, N \), and \( a_2 \) to remove most of the lower order terms in the equations for \( \dot{\gamma}, \dot{\alpha}_n \) and \( \dot{\beta}_n \). It turns out some terms can not be removed, for example the term \( |z_n|^2 \) in the equation for \( \dot{\gamma} \), and \( |z_n|^2 z_n \) terms in the equation for \( \partial_t z \). On the other hand, there terms either play a favorable role in our analysis, or will not affect it, namely it does not matter if \( \gamma \) is not convergent, since \( e^{i\gamma(t)} \) is only a phase factor.

In defining \( p_n, q_n \), we have to solve system of linear equations, see e.g. (65). The existence and uniqueness of solutions have to be addressed.

In what follows we use the notations \( N_{m,n}^{I_m}, N_{m,n}^{Re} \) to stand for functions satisfying

\[
\begin{pmatrix}
  N_{m,n}^{I_m} \\
  -N_{m,n}^{Re}
\end{pmatrix} = J N_{m,n},
\]

where, recall the definition of \( J N_{m,n} \), \(|m| + |n| = 2, 3 \), from (63) and (66).

In what follows define the polynomials \( a_1, a_2, p_k \) and \( q_k \), \( k = 1, 2, \cdots, N \) in
by defining their coefficients:
\[ a_1(z, \bar{z}) := \sum_{|m|+|n|=2,3} A^{(1)}_{m,n}(\lambda) z^m \bar{z}^n, \]
\[ a_2(z, \bar{z}) := \sum_{|m|+|n|=2,3, |m| \neq |n|} A^{(2)}_{m,n}(\lambda) z^m \bar{z}^n, \]
\[ p_k(z, \bar{z}) := \sum_{|m|+|n|=2,3} P^{(k)}_{m,n}(\lambda) z^m \bar{z}^n, \]
\[ q_k(z, \bar{z}) := \sum_{|m|+|n|=2,3} Q^{(k)}_{m,n}(\lambda) z^m \bar{z}^n, \]
\[ (58) \]
with \( m, n \in (\mathbb{Z}^+ \cup \{0\})^N \). In what follows we use the notation
\[ m \cdot E(\lambda) = \sum_{k} m_k E_k(\lambda). \]

We start with defining \( A^{(1)}_{m,n} \). For \( |m| = 2,3 \)
\[ A^{(1)}_{m,0} := \frac{1}{i(m \cdot E(\lambda))} \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \langle N^{lm}_{m,0}, \phi^\lambda \rangle. \]
\[ (59) \]
For \( |m| = 2, |n| = 1 \),
\[ A^{(1)}_{m,n} := \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \frac{1}{i(m \cdot E(\lambda))} \times \]
\[ \left[ \langle N^{lm}_{m,n}, \phi^\lambda \rangle - \frac{i}{2} \sum_{|k|=|r|=|n|=1, k+r=m} \Upsilon_{k,n} \langle r \cdot \eta, \phi^\lambda \rangle \right], \]
\[ (60) \]
here \( \Upsilon_{m,n} \) is from the expansion of \( \Upsilon = \Upsilon(z, \bar{z}) \), defined in \( (51) \):
\[ \Upsilon = \sum_{|m|=|n|=1} \Upsilon_{m,n} z^m \bar{z}^n \]
and the vector \( r \) is in \( (\mathbb{Z}^+ \cup \{0\})^N \).
For \( |m| = |n| = 1 \), we define
\[ A^{(1)}_{m,n} := \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \frac{1}{i(m-n) \cdot E(\lambda)} \langle N^{lm}_{m,n}, \phi^\lambda \rangle \]
\[ = \frac{1}{4} \langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle \left[ \langle n \cdot \eta, \eta \cdot m \rangle + \langle n \cdot \xi, \xi \cdot m \rangle \right]. \]
\[ (61) \]
the denominator is small, then the numerator is proportionally small. Hence \( A_{m,n}^{(1)} \) are always well defined, a similar calculation was in [12].

After defining various terms above, the other terms in \( A_{m,n}^{(1)} \) are determined by the relations

\[
A_{k,l}^{(1)} := A_{l,k}^{(1)} \text{ for } |k| + |l| = 2, 3.
\]

Now we define various terms in \( A_{m,n}^{(2)} \), \(|m| + |n| = 2, 3, \ |m| \neq |n|\).

For \(|m| = 2, 3, |n| = 0\), we define \( A_{m,0}^{(2)} \) as

\[
- i m \cdot E(\lambda) A_{m,0}^{(2)} + A_{m,0}^{(1)} := \frac{1}{\langle \phi^\lambda, \partial \phi^\lambda \rangle} (N_{m,0}^{Re} \partial \phi^\lambda).
\]

For \(|m| = 2, |n| = 1\),

\[
- i (m - n) \cdot E(\lambda) A_{m,n}^{(2)} + A_{m,n}^{(1)} := \frac{1}{\langle \phi^\lambda, \partial \phi^\lambda \rangle} (N_{m,n}^{Re} \partial \phi^\lambda) - \frac{1}{2} \sum_{|k|=|n|=|r|=1,k+r=m} \Im k,n \langle r \cdot \xi, \partial \phi^\lambda \rangle],
\]

here the definition of \( \Im k,n \) and the convention on \( r \) are as in [60]. Especially in solving for \( A_{m,n}^{(2)} \) we need \( \frac{1}{(m-n) E(\lambda)} \) to be uniformly bounded, indeed this is true by the facts \( E_k(\lambda) \approx e_0 - e_k \) in [12] and \( 2e_k < e_0 \), see Condition (NL) in Section 3.

The other terms in \( A_{k,l}^{(2)} \) are determined by the relations

\[
A_{k,l}^{(2)} := A_{l,k}^{(2)} \text{ for } |k| + |l| = 2, 3.
\]

Next we define coefficients \( P_{m,n}^{(k)} \) and \( Q_{m,n}^{(k)} \) for the polynomials \( p_k \) and \( q_k \), \( k = 1, 2, \cdots, N \), see [52].

For \(|m| = 2, 3, |n| = 0\), we define \( P_{m,0}^{(k)} \) and \( Q_{m,0}^{(k)} \) to be solutions to the linear equations

\[
- i m \cdot E(\lambda) P_{m,0}^{(k)} - E_k(\lambda) Q_{m,0}^{(k)} := - \langle N_{m,0}^{Im}, \eta_k \rangle, \quad - i m \cdot E(\lambda) Q_{m,0}^{(k)} + E_k(\lambda) P_{m,0}^{(k)} := \langle N_{m,0}^{Re}, \xi_k \rangle.
\]

Here the solutions exist and are unique because the corresponding \( 2 \times 2 \) matrices

\[
D := \begin{bmatrix}
- i m \cdot E(\lambda) & - E_k(\lambda) \\
E_k(\lambda) & - i m \cdot E(\lambda)
\end{bmatrix}
\]

are uniformly invertible, or equivalently they determinants stay away from zero by a uniform margin. To see this, compute directly to obtain

\[
\det D = - [m \cdot E(\lambda)]^2 + E_k^2(\lambda) = - [m \cdot E(\lambda) + E_k(\lambda)] [m \cdot E(\lambda) - E_k(\lambda)].
\]

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Next we relate the quantities on the right hand side to \( e_0 \) and \( e_l \), \( l = 1, 2, \cdots, N \), by (12), namely

for any \( l = 1, 2, \cdots, N \), \( E_l(\lambda) \approx e_0 - e_l \).

Compute directly and use \( 2e_l < e_0 \) to see

\[
m \cdot E(\lambda) - E_k(\lambda) \approx \sum_{l=1}^{N} m_l(e_0 - e_l) - (e_0 - e_k) = (|m| - 1)e_0 + e_k - \sum_{l=1}^{N} m_le_l
\]

(68)

is positive and stays away from zero for any \( m \in \{ \mathbb{Z}^+ \cup \{0\}\}^N \) and \( |m| = 2, 3 \). This together with that \( m \cdot E(\lambda) + E_k(\lambda) \) is more positive implies the desired result: \( \text{Det}D \) stays away from zero.

For \( |m| = 1 \) and \( |n| = 2 \), we define \( P_m^{(k)} \) and \( Q_m^{(k)} \) to satisfy the equation

\[
iP_m^{(k)} - Q_m^{(k)} := \frac{-\langle N_m^{lm} \eta_k \rangle + i \langle N_m^{Re} \xi_k \rangle + i \sum_{|m|=|k|=|r|=1, k+r=n} \Upsilon_{m,k} \langle r \cdot \eta, \eta_k \rangle - \langle r \cdot \xi, \xi_k \rangle}{E_k - E(\lambda) \cdot (m - n)},
\]

(69)

here the denominator \( E_k - E(\lambda) \cdot (m - n) \) stays away from zero by a uniform margin, by the same justification as in (68), and the definition of \( \Upsilon_{m,n} \) and the convention on \( r \) is as in (60), \( r \cdot \eta := \sum_k r_k \eta_k \) and \( r \cdot \xi := \sum_k r_k \xi_k \). Note that at this moment (69) does not give unique solutions. This will be become clear in a moment.

For \( |m| = 2 \) and \( |n| = 1 \), we define

\[
iP_m^{(k)} - Q_m^{(k)} = 0.
\]

(70)

After defining \( iP_m^{(k)} - Q_m^{(k)} \) for \( |m|, |n| \) = (1, 2), (2, 1) above, it is not hard to see that these together with the relations \( P_m^{(k)} = P_m^{(k)} \) and \( Q_m^{(k)} = Q_m^{(k)} \) determine unique solutions for the linear equations.

We continue to define \( P_m, Q_m \) for \( |m| = |n| = 1 \),

\[
-i(m - n) \cdot E(\lambda) \cdot P_m^{(k)} - E_k(\lambda)Q_m^{(k)} := -\langle N_m^{lm} \eta_k \rangle, \\
-i(m - n) \cdot E(\lambda) \cdot Q_m^{(k)} + E_k(\lambda)P_m^{(k)} := \langle N_m^{Re} \xi_k \rangle.
\]

(71)

The solutions are well defined and unique since the matrix

\[
\left( \begin{array}{cc}
-i(m - n) \cdot E(\lambda) & -E_k(\lambda) \\
E_k(\lambda) & -i(m - n) \cdot E(\lambda)
\end{array} \right)
\]

(72)
is uniformly invertible by the same arguments in showing the invertibility of the matrix in (66).

We complete defining all the relevant terms by requiring that

\[ P_{m,n}^{(k)} := \overline{P_{m,n}^{(k)}} \quad Q_{m,n}^{(k)} := \overline{Q_{m,n}^{(k)}}. \]

By now we have finished defining the polynomials \( a_1, a_2, p_k, q_k, k = 1, 2, \ldots, N. \)

Next we study the equation for \( \dot{z}_k. \) By the definitions of coefficients \( P_{m,n}^{(k)}, Q_{m,n}^{(k)} \) we removed the following terms from the \( \dot{z}_k \)-equation: \( z^m \bar{z}^n \) if \( |m| + |n| = 2, 3 \) and \( (|m|, |n|) \neq (2, 1). \) The result is:

**Proposition 10.1.** Define the polynomials \( a_1(z, \bar{z}), a_2(z, \bar{z}), p_n(z, \bar{z}), q_n(z, \bar{z}) \) as above. Then, (49) - (50) holds and moreover for \( k = 1, 2, \ldots, N \)

\[
\partial_t z_k + i E_k(\lambda) z_k = -\left( \sum_{|m|=2, |n|=1} J N_{m,n} z^m \bar{z}^n, \left( \begin{array}{c} \eta_k \\ -i \xi_k \end{array} \right) \right) + \frac{1}{2} \Upsilon \sum_{m=1}^{N} z_m \left( \begin{array}{c} -i \eta_m \\ \xi_m \end{array} \right) \left( \begin{array}{c} \eta_k \\ i \xi_k \end{array} \right) + \text{Remainder}(t). \tag{73}
\]

**Proof.** Recall the convention that \( \text{Remainder} \) represents any quantity satisfying

\[
\lesssim |z(t)|^4 + \|\langle x \rangle^{-\nu} \tilde{R}(t)\|_2^2 + \|\tilde{R}(t)\|_2^2 + |z(t)| \|\langle x \rangle^{-\nu} \tilde{R}(t)\|_2. \tag{74}
\]

We start with casting the \( \dot{\lambda} \)- and \( \dot{\gamma} \)-eqns in (32), (33) into a matrix form

\[
[\text{Id} + M(z, \tilde{R}, p, q)] \begin{pmatrix} \dot{\lambda} \\ \dot{\gamma} - \Upsilon \end{pmatrix} = \Omega + \text{Remainder}, \tag{75}
\]

where, the vector \( \Omega \) is defined as

\[
\Omega := \begin{pmatrix} -\frac{1}{2} \left( \frac{1}{(\phi^* \cdot \partial_\lambda \phi^*)} \langle \text{Im} N, \phi^\lambda \rangle + i \frac{1}{2} \Upsilon \left( \langle z - \bar{z}, \eta, \phi^\lambda \rangle \right) \right) - \partial_t a_1 \\ \frac{1}{2} \Upsilon \left( \langle \text{Re} N, \partial_\lambda \phi^\lambda \rangle \right) - \frac{1}{2} \Upsilon \left( \langle z + \bar{z}, \xi, \partial_\lambda \phi^\lambda \rangle \right) - \Upsilon - \partial_t a_2 + a_1 \end{pmatrix}. \tag{76}
\]

The smallness of the matrix \( M(z, \tilde{R}, p, q) \) makes \( [\text{Id} + M]^{-1} \) uniformly bounded, hence

\[
|\dot{\lambda}|, |\dot{\gamma} - \Upsilon| \lesssim |\Omega| + \text{Remainder}. \tag{78}
\]
Next we estimate $\Omega$, and start with casting it into a convenient form.

The purpose of defining $a_1$ and $a_2$ in (58) is to remove the lower order terms, in $z$ and $\bar{z}$, from $(\text{Im}N, \phi^\lambda) - \frac{i}{2} \Upsilon ((z - \bar{z}) \cdot \eta, \phi^\lambda)$ and $(\text{Re}N, \partial_\lambda \phi^\lambda) + \frac{1}{2} \Upsilon ((z + \bar{z}) \cdot \xi, \partial_\lambda \phi^\lambda)$ to get

$$
\Omega = D_1 + D_2
$$

with

$$
D_1 := \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \left( -(\text{Im}N - \sum_{|m|+|n|=2,3} N_{m,n}^{\text{Im}} \bar{z}^m z^n, \phi^\lambda) \right)
$$

and

$$
D_2 := - \sum_{|m|+|n|=2,3} \left( \partial_t (A_{m,n}^{(1)} z^m \bar{z}^n) + iE(\lambda) \cdot (m - n) A_{m,n}^{(1)} z^m \bar{z}^n \right) \left( \partial_t (A_{m,n}^{(2)} z^m \bar{z}^n) + iE(\lambda) \cdot (m - n) A_{m,n}^{(2)} z^m \bar{z}^n \right).
$$

It is easy to see that

$$
D_1 = \text{Remainder}.
$$

To control $D_2$ we use a preliminary estimate from the $z_n$–equation in (50)

$$
\partial_t z_n + iE_n(\lambda) z_n = O(|z|^2) + \text{Remainder}
$$

(81)

to obtain

$$
D_2 = - \hat{\lambda} \sum_{|m|+|n|=2,3} \left( \partial_\lambda A_{m,n}^{(1)} z^m \bar{z}^n \right) + O(|z|^3) + \text{Remainder}
$$

(82)

$$
= O(|z|^3) + \text{Remainder}
$$

(83)

here the term $O(|z|^3)$ is from the term $O(|z|^2)$ in (50). Collect the estimates above to obtain

$$
\dot{\lambda}, \dot{\gamma} - \Upsilon = O(|z|^3) + \text{Remainder}.
$$

(84)

These estimates are still worse than the desired (49)-(50). The reason is that their derivations depend on the non-optimal (50). Next we improve it using (84).

Choose $p_n$ and $q_n$ as in (53) to remove the following lower order terms: $z^m \bar{z}^n$ satisfying $|m| + |n| = 2,3$ and $(|m|, |n|) \neq (2,1)$, to obtain

$$
\partial_t z_n + iE_n(\lambda) z_n = - \sum_{|m|+|n|=2,3} JN_{m,n} z^m \bar{z}^n + \frac{1}{2} \Upsilon \left( i z \cdot \eta \right) \left( \xi \cdot \eta_n \right)
$$

$$
+ D_3(n) + \text{Remainder}
$$

(85)
where $D_3(n)$ is defined as

$$
D_3(n) := - \sum_{|k|+|l|=2,3} \left[ \partial_t (P_{k,l}^{(n)} z^k \bar{z}^l) + i(k-l) \cdot E(\lambda) P_{k,l}^{(n)} z^k \bar{z}^l \right] - i \sum_{|k|+|l|=2,3} \left[ \partial_t (Q_{k,l}^{(n)} z^k \bar{z}^l) + i(k-l) \cdot E(\lambda) Q_{k,l}^{(n)} z^k \bar{z}^l \right] = - \dot{\lambda} \sum_{|k|+|l|=2,3} \left[ \partial \lambda P_{k,l}^{(n)} - \partial \lambda Q_{k,l}^{(n)} \right] z^k \bar{z}^l - \sum_{|k|+|l|=2,3} \left[ P_{k,l}^{(n)} - Q_{k,l}^{(n)} \right] \left[ \partial_t (z^k \bar{z}^l) + i(k-l) \cdot E(\lambda) z^k \bar{z}^l \right] = \Gamma_1 + \Gamma_2
$$

(85)

and $\Gamma_1$ and $\Gamma_2$ naturally defined.

For $\Gamma_1$ the estimate for $\dot{\lambda}$ in (83) implies that

$$
\Gamma_1 = \text{Remainder.}
$$

(86)

For $\Gamma_2$, the preliminary estimate in (81) implies

$$
\Gamma_2 = O(|z|^3) + \text{Remainder.}
$$

This, in turn, implies an estimate better than (81)

$$
\partial_t z_n + iE_n(\lambda) z_n = O(|z|^3) + \text{Remainder.}
$$

(87)

Since the estimates derived for $\Gamma_2$ depends on non-optimal (81), this optimal one enables us to find

$$
\Gamma_2 = \text{Remainder.}
$$

(88)

This together with $\Gamma_1 = \text{Remainder}$ in (86) implies

$$
D_3(n) = \Gamma_1 + \Gamma_2 = \text{Remainder.}
$$

Put this into the $\partial_t z_n$-eqn in (84) to obtain the desired estimate (73).

(87) also helps us to improve the estimate (82) for $D_2$

$$
D_2 = \text{Remainder.}
$$

(89)

This together with (80), (79) implies for $\Omega$ in (79)

$$
\Omega = D_1 + D_2 = \text{Remainder.}
$$

Put this into (78) to obtain the desired estimates (49), (50) for $\dot{\lambda}$ and $\dot{\gamma} - \Upsilon$.

The following result has been applied in (61) to show that the numerator is proportional to the denominator. Similar result can be found in [11].

23
Lemma 10.2. For $|m| = |n| = 1$, and $m, n \in (\mathbb{Z}^+ \cup \{0\})^N$

$$\langle N_{m,n}^{Im}, \phi^\lambda \rangle = \frac{1}{4i} (n - m) \cdot E(\lambda) \left[ (m \cdot \xi, n \cdot \xi) + (m \cdot \eta, n \cdot \eta) \right]. \quad (90)$$

Proof. We start with deriving an expression for $N_{m,n}^{Im}, |m| = |n| = 1$.

The explicit form of $\sum_{|m|+|n|=2} JN_{m,n} z^m \bar{z}^n$ in (53) implies that

$$\sum_{|m|+|n|=2} N_{m,n}^{Im} z^m \bar{z}^n = 2\phi^\lambda A_1 B_1 = \frac{1}{2i} \phi^\lambda \left( \sum_{n=1}^N z_n \xi_n + \sum_{n=1}^N \bar{z}_n \xi_n \right) \left( \sum_{m=1}^N z_m \eta_m - \sum_{m=1}^N \bar{z}_m \eta_m \right).$$

Take the relevant terms to obtain

$$N_{m,n}^{Im} = \frac{1}{2i} \phi^\lambda (\xi_n \eta_m - \xi_m \eta_n), \quad (91)$$

here we used the notation

$$\xi_n = n \cdot \xi, \quad \eta_n = n \cdot \eta, \text{ for } n \in (\mathbb{Z}^+ \cup \{0\})^N \text{ and } |n| = 1. \quad (92)$$

Hence the left hand side of (90) takes a new form,

$$\langle N_{m,n}^{Im}, \phi^\lambda \rangle = \frac{1}{2i} \int (\phi^\lambda)^2 (\xi_n \eta_m - \xi_m \eta_n)$$

$$= \frac{1}{4i} \left[ \langle [L_+(\lambda) - L_-(\lambda)] \xi_n, \eta_m \rangle - \langle [L_+(\lambda) - L_-(\lambda)] \xi_m, \eta_n \rangle \right],$$

where we used the fact that $L_+(\lambda) - L_-(\lambda) = 2(\phi^\lambda)^2$, see (10). Use the facts $L_+(\lambda)$ and $L_-(\lambda)$ are self-adjoint, and

$$L_-(\lambda) \eta_n = E_n(\lambda) \xi_n, \text{ and } L_+(\lambda) \xi_n = E_n(\lambda) \eta_n, \quad n = 1, 2, \ldots, N,$$

in (13), and hence the desired result, recall the notations in (92),

$$\langle N_{m,n}^{Im}, \phi^\lambda \rangle = \frac{1}{4i} (E_n(\lambda) - E_m(\lambda)) \left[ (\xi_m, \xi_n) + (\eta_m, \eta_n) \right]$$

$$= \frac{1}{4i} (n - m) \cdot E(\lambda) \left[ (m \cdot \xi, n \cdot \xi) + (m \cdot \eta, n \cdot \eta) \right]. \quad (93)$$

\[\square\]
11 Proof of the Normal Form Equation (52)

We expand the first two terms on the right hand side of the equations for \( z_n \) in (73) to obtain

\[
- \left\langle \sum_{|m|=2, |n|=1} J_{m,n} z^m \bar{z}^n, \left( \frac{\eta_k}{-i\xi_k} \right) \right\rangle + \frac{1}{2} \Upsilon \sum_{m=1}^N z_m \left\langle \left( -i\eta_m \right), \left( \frac{\eta_k}{i\xi_k} \right) \right\rangle
= \sum_{l=1}^5 \Theta_l(k),
\]

where, recall the definitions of \( J_{m,n} \), \(|m| + |n| = 3\), from (56), \( \Theta_1(k) \) is defined as

\[
\Theta_1(k) := - \left\langle \sum_{|m|=2} R_{m,0} z^m, \left( \frac{\eta_k}{-i\xi_k} \right) \right\rangle = \left\langle \sum_{|m|=2} R_{m,0} z^m, (X_1)^* \left( \frac{\eta_k}{-i\xi_k} \right) \right\rangle,
\]

where, recall the definition of \( 2 \times 2 \) matrix \( X \) in (57) and we divide it into two terms \( X = X_1 + X_1^\dagger \) with \( X_1 \) defined as

\[
X_1 := \begin{pmatrix}
-\phi^\lambda z \cdot \eta & \phi^\lambda z \cdot \xi \\
-3\phi^\lambda z \cdot \xi & i\phi^\lambda z \cdot \eta
\end{pmatrix},
\]

\[
\Theta_2(k) := - \left\langle X_1 \begin{pmatrix}
\sum_{n=1}^N \sum_{|m|=|l|=1} P_{m,l}^{(n)} z^m \bar{z}^l \xi_n \\
\sum_{n=1}^N \sum_{|m|=|l|=1} Q_{m,l}^{(n)} z^m \bar{z}^l \eta_n
\end{pmatrix}, \left( \frac{\eta_k}{-i\xi_k} \right) \right\rangle
\]

\[- \left\langle X_1 \begin{pmatrix}
\sum_{|m|=2} \sum_{n=1}^N P_{m,0}^{(n)} \xi_n + A_{m,0}^{(1)} \phi^\lambda \\
\sum_{|m|=2} \sum_{n=1}^N Q_{m,0}^{(n)} \eta_n + A_{m,0}^{(2)} \phi^\lambda
\end{pmatrix}, \left( \frac{\eta_k}{-i\xi_k} \right) \right\rangle,
\]

\[
\Theta_3(k) := - \frac{1}{8} \left\langle \left( (z \cdot \xi)^2 - (z \cdot \eta)^2 \right) \left( \frac{i\bar{z} \cdot \eta}{-\bar{z} \cdot \xi} \right), \left( \frac{\eta_k}{-i\xi_k} \right) \right\rangle
+ \frac{1}{4} \left\langle \left( |z \cdot \xi|^2 + |z \cdot \eta|^2 \right) \left( \frac{i\bar{z} \cdot \eta}{z \cdot \xi} \right), \left( \frac{\eta_k}{-i\xi_k} \right) \right\rangle,
\]

\[
\Theta_4(k) := \frac{1}{2} \Upsilon \left\langle \left( \frac{-i\bar{z} \cdot \eta}{z \cdot \xi} \right), \left( \frac{\eta_k}{i\xi_k} \right) \right\rangle,
\]

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\[ \Theta_5(k) := - \left( \sum_{|m|=|n|=1} R_{m,n} z^m \bar{z}^n, X_1^* \left( \frac{\eta k}{-i\xi_k} \right) \right), \]

where, recall the definition of real function \( \Upsilon \) from (50).

The result is the following

**Proposition 11.1.** \( \text{Re} \sum_{k=1}^N \bar{z}_k \Theta_1(k) \) can be decomposed into three terms,

\[ \text{Re} \sum_{k=1}^N \bar{z}_k \Theta_1(k) = - C(e_0 - \lambda) \Gamma(z, \bar{z}) \]
\[ + (e_0 - \lambda) \sum_{|m|=|n|=2} C_{m,n}(\lambda) E(\lambda) \cdot (m - n) z^m \bar{z}^n \]
\[ + |e_0 - \lambda|^2 \mathcal{O}(|z|^4) \]

where \( C \) is a positive constant, \( \Gamma(z, \bar{z}) \) is the positive term defined in Fermi-Golden-rule (19), \( C_{m,n}(\lambda) \) are uniformly bounded constants and recall that \( e_0 - \lambda > 0 \), in Lemma 3.1.

For \( \text{Re} \sum_{k=1}^N \bar{z}_k \Theta_2(k) \), there exist some constants \( D_{m,n}(\lambda) \) such that

\[ \text{Re} \sum_{k=1}^N \bar{z}_k \Theta_2(k) = (e_0 - \lambda) \sum_{|m|=|n|=2} D_{m,n}(\lambda) E(\lambda) \cdot (m - n) z^m \bar{z}^n; \]

For \( n = 3, 4, 5 \), we have

\[ \text{Re} \sum_{k=1}^N \bar{z}_k \Theta_n(k) = 0. \]

The proposition will be proved in subsequent subsections.

Next we prove the desired result (52).

**Proof of (52)** By the estimates above we have that

\[ \partial_t |z|^2 = -2C(e_0 - \lambda) \Gamma(z, \bar{z}) + (e_0 - \lambda) \sum_{|m|=|n|=2} C_{m,n} E(\lambda) \cdot (m - n) z^m \bar{z}^n \]
\[ + \mathcal{O}((e_0 - \lambda)^2 |z|^4) + |z| \text{Remainer}, \]

with \( C_{m,n} \) being some constant. Here we can take \( \sum_{|m|=|n|=2} C_{m,n} E(\lambda) \cdot (m - n) z^m \bar{z}^n \) to be real since it is in the equation for real parameter \( |z|^2 \) and \( CT(z, \bar{z}) \) is real. This implies

\[ \sum_{|m|=|n|=2} C_{m,n} E(\lambda) \cdot (m - n) z^m \bar{z}^n = \sum_{|m|=|n|=2} \overline{C_{m,n}} E(\lambda) \cdot (m - n) z^m \bar{z}^n, \]

26
and hence forces
\[ C_{m,n} = -\overline{C_{n,m}}. \]  

(102)

By observing that
\[ E(\lambda) \cdot (m-n)z^m \overline{z^n} = i\partial_t z^m \overline{z^n} + |z| \text{Remainder}, \]

the fact that \((e_0 - \lambda) \sum_{|m|=|n|=2} iC_{m,n} z^m \overline{z^n} \) is real implied by (102), we can define a new nonnegative parameter \(\tilde{z}\) satisfying
\[ \tilde{z}^2 = |z|^2 - (e_0 - \lambda) \sum_{|m|=|n|=2} iC_{m,n} z^m \overline{z^n} \]
such that
\[ \partial_t \tilde{z}^2 = -2C(e_0 - \lambda) \Gamma(z, \overline{z}) + O((|e_0 - \lambda|^2 |z|^4) + |z| \text{Remainder} \] 

(103)

which is the desired estimate (52).

□

Next we prove Proposition 11.1. In the proof the following results, from [12], will be used. Recall that the function \(\phi\) is the ground state for \(-\Delta + V\) with eigenvalue \(-e_0\), the functions \(\xi_{lin}^k, k = 1, 2, \cdots, N\), are neutral modes with eigenvalues \(-e_k\).

**Lemma 11.2.** There exist constants \(C_0, C_1 \in \mathbb{R}\) such that in the space \(\langle x \rangle^{-4} \mathcal{H}^2\)

\[ \phi^\lambda = C_0(e_0 - \lambda)^{\frac{1}{2}} \phi + O(|e_0 - \lambda|^\frac{1}{2}), \]
\[ \partial_\lambda \phi^\lambda = C_1(e_0 - \lambda)^{-\frac{1}{2}} \phi + O(|e_0 - \lambda|^\frac{1}{2}), \]
\[ \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \lesssim 1. \]

(104)

For the neutral modes we have
\[ \| \langle x \rangle^4 (\eta_m - \xi_{lin}^m) \|_{\mathcal{H}^2}, \| \langle x \rangle^4 (\xi_m - \xi_{lin}^m) \|_{\mathcal{H}^2}, \| \langle x \rangle^4 (\xi_m - \eta_m) \|_{\mathcal{H}^2} = O(|e_0 - \lambda|). \]

(105)

Recall \(P_{lin}^c\) is the orthogonal project onto the essential spectrum of \(-\Delta + V\), and \(P_\lambda^c\) of (16) is the Riesz projection onto the essential spectrum of \(L(\lambda)\)
\[ P_\lambda^c = P_{lin}^c \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O(|e_0 - \lambda|). \]

(106)

**11.1 Proof of (98)**

We start from the definition of \(\Theta_1(k)\) in (95). Compute directly to obtain
\[
\begin{pmatrix} \overline{X_1}^\lambda \end{pmatrix}^* \left( \begin{array}{c} \eta_k \\ -i\xi_k \end{array} \right) = \left( \begin{array}{c} -i\phi^\lambda z \cdot \eta_k + 3i\phi^\lambda z \cdot \xi_k \\ \phi^\lambda z \cdot \xi_k + \phi^\lambda z \cdot \eta_k \end{array} \right).
\]
Hence

\[ Re \sum_k \bar{z}_k \Theta_1(k) = -\left( \sum_{|m|=2} R_{m,0} z^m, \left( -i \phi^\lambda (z \cdot \eta)^2 + 3i \phi^\lambda (z \cdot \xi)^2 \right) \right). \]  

(107)

We extract its main part by define

\[ D := Re \sum_{|m|=2} \langle z^m \left[ (-\Delta + V + \lambda)J + i E(\lambda) m - 0 \right]^{-1} P c \phi(\xi^{lin})^m \left( \begin{array}{c} -i \\ 1 \end{array} \right), \times \phi(z \cdot \xi^{lin})^2 \left( \begin{array}{c} i \\ 1 \end{array} \right). \]  

(108)

Here \((\xi^{lin})^m\) is defined as, for \(m = (m_1, \, m_2, \, \ldots, \, m_N) \in (\mathbb{Z}^+ \cup \{0\})^N, \)

\[ (\xi^{lin})^m := (\xi_1^{lin})^{m_1} (\xi_2^{lin})^{m_2} \ldots (\xi_N^{lin})^{m_N}. \]

The result is

Lemma 11.3.

\[ Re \sum_{k=1}^N \bar{z}_k \Theta_1(k) = -C(e_0 - \lambda) \, D + O(|e_0 - \lambda|^2 |z|^4). \]  

(109)

The lemma will be proved in Part [11.1.1] below.

Next we study the term \(D\). To facilitate our estimate we diagonalize the matrix operator in (108). Define a unitary matrix \(U\) by

\[ U := \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right), \]  

(110)

then we have that

\[ J = i U \sigma_3 U^*, \]

with \(\sigma_3\) being the third Pauli matrix.

Insert the identity \(UU^* = U^*U = Id\) into appropriate places in the inner product of \(D\) to obtain, recall the convention that \(\langle f, \, g \rangle = \int f(x) \bar{g}(x) \, dx, \)

\[ D = 2 \sum_{|m|=|n|=2} Re \left[ i \left( -\Delta + V + \lambda - E(\lambda) m \right) + 0 \right]^{-1} P c \phi(\xi^{lin})^m, \phi(\xi^{lin})^n \, z^m \bar{z}^n \]

\[ = 2 \sum_{|m|=|n|=2} Im \left[ -\Delta + V + \lambda - E(\lambda) m - i0 \right]^{-1} P c \phi(\xi^{lin})^m, \phi(\xi^{lin})^n \, z^m \bar{z}^n. \]  

(111)

To cast the expression into a convenient form, we use the following two simple facts, for any functions \(f, \, g\) and real constant \(h, \)

\[ Im(f, \, g) = \frac{1}{2i}[(f, \, g) - (g, \, f)] \]
and \( \langle f, (-\Delta + V - h - i0)^{-1} g \rangle = \langle (-\Delta + V - h + i0)^{-1} f, g \rangle \). Compute directly to obtain

\[
D = \frac{1}{i} \sum_{|m| = |n| = 2} \langle L(m, n) P_c \phi(\xi^{lin})^m, \phi(\xi^{lin})^n \rangle z^m \bar{z}^n.
\]  

(112)

with \( L(m, n) \) being a linear operator defined as

\[ L(m, n) := [-\Delta + V + \lambda - E(\lambda) \cdot m - i0]^{-1} - [-\Delta + V + \lambda - E(\lambda) \cdot n + i0]^{-1} \]

In studying (112), the main tool is a well known fact that, see e.g. [16], for any constant \( h > 0 \),

\[
\frac{1}{i} \left[ [-\Delta + V - h^2 - i0]^{-1} - [-\Delta + V - h^2 + i0]^{-1} \right] P_c f, g \right]
\]

\[ = Ch \int_{S^2} \hat{f}(h \sigma) \overline{\hat{g}(h \sigma)} \, d\sigma \]

(113)

where \( C \) is a positive constant, and the complex function \( \hat{f} \) is defined as,

\[ \hat{f}(k) := \int_{\mathbb{R}^3} f(x) \, e(x, k) \, dx, \]

(114)

hence \( \hat{f}(h \sigma) \) is \( \hat{f} \) restricted to the sphere \( |k| = h \sigma, \ \sigma \in S^2 \). Here the complex function \( e : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{C} \) is defined as

\[ e(x, k) := [1 + (-\Delta - |k|^2 - i0)^{-1} V(x)]^{-1} e^{ix \cdot k}. \]

We continue to study (112). For the easiest cases \( E(\lambda) \cdot m = E(\lambda) \cdot n \), apply (113) directly to obtain

\[
\frac{1}{i} \langle L(m, n) P_c \phi(\xi^{lin})^m, \phi(\xi^{lin})^n \rangle = CC_m \int_{S^2} \hat{\phi}(\xi^{lin})^m(C_m \sigma) \overline{\hat{\phi}(\xi^{lin})^n(C_m \sigma)} \, d\sigma
\]

(115)

with \( C_m \in \mathbb{C} \) defined as

\[ C_m := \sqrt{E(\lambda) \cdot m - \lambda}, \]

here \( E(\lambda) \cdot m - \lambda \) is positive by the conditions that \( 2e_k < e_0 \) and \( E_k(\lambda) \approx e_0 - e_k, \ k = 1, 2, \ldots, N, \) and \( \lambda \approx e_0 \). Recall that \( m \in (\mathbb{Z}^+ \cup \{0\})^N \) and \( |m| = 2 \).

For the cases \( E(\lambda) \cdot m \neq E(\lambda) \cdot n \), we claim, for some constant \( C_{m,n} \),

\[
\frac{1}{i} \langle L(m, n) P_c \phi(\xi^{lin})^m, \phi(\xi^{lin})^n \rangle = CC_m C_n \int_{S^2} \hat{\phi}(\xi^{lin})^m(C_m \sigma) \overline{\hat{\phi}(\xi^{lin})^n(C_n \sigma)} \, d\sigma
\]

\[ + C_{m,n} E(\lambda) \cdot (m - n). \]

(116)

If the claim holds, this together with (115), Lemma [113] and the fact \( E_k(\lambda) = e_0 - e_k + O(e_0 - \lambda) \) by [12], implies the desired result [18].
What is left is to prove the claim (116). We start with decomposing the left hand side into two parts

\[
\frac{1}{i} \langle L(m,n)P_c \phi(\xi^{lin})^m, \phi(\xi^{lin})^n \rangle = A + B \tag{117}
\]

with

\[
A := \frac{1}{i} \langle L(m,m)P_c \phi(\xi^{lin})^m, \phi(\xi^{lin})^n \rangle,
\]

\[
B := \frac{1}{i} \langle [L(m,n) - L(m,m)]P_c \phi(\xi^{lin})^m, \phi(\xi^{lin})^n \rangle.
\]

By (113), it is easy to see that

\[
A = CC_m \int_{S^{L_2}} \hat{\phi}(\xi^{lin})^m(C_m \sigma) \overline{\hat{\phi}(\xi^{lin})^n(C_m \sigma)} \, d\sigma \tag{118}
\]

\[
= CC_m^2 \int_{S^{L_2}} \hat{\phi}(\xi^{lin})^m(C_m \sigma) \overline{\hat{\phi}(\xi^{lin})^n(C_n \sigma)} \, d\sigma + C_{m,n} E(\lambda) \cdot (m-n),
\]

where \(C_{m,n}\) is a constant, and in the second step we use that the functions \(\hat{\phi}(\xi^{lin})^m\) and the scalar \(C_m\) depend smoothly on \(E(\lambda) \cdot m\) and \(E(\lambda) \cdot n\).

For \(B\), it is easy to see that for some constant \(D_{m,n}\),

\[
B = D_{m,n} E(\lambda) \cdot (m-n).
\]

Collecting the estimates above, we prove the claim (116). Hence the proof is complete.

11.1.1 Proof of Lemma 11.3

We rewrite the expression in (107) as

\[
Re \sum_{k=1}^{N} \bar{z}_k \Theta_1(k) = - \left< \sum_{|m|=2} R_{m,0} z^m, A \right> \tag{119}
\]

with the vector function \(A\) defined as

\[
A := \begin{pmatrix} -i \phi^\lambda(z \cdot \eta)^2 + 3i \phi^\lambda(z \cdot \xi)^2 \\ 2 \phi^\lambda z \cdot \xi \cdot \eta \end{pmatrix}.
\]

Apply the estimates of \(\phi^\lambda\), \(\xi\) and \(\eta\) in (104) and (106) to obtain

\[
A = C_1(e_0 - \lambda)^{\frac{1}{2}} \phi(z \cdot \xi^{lin})^2 \begin{pmatrix} i \\ 1 \end{pmatrix} + O((e_0 - \lambda)^{\frac{3}{2}} |z|^2), \tag{120}
\]

here the expansion is in the space \(\langle x \rangle^{-4} L^2\) for some \(e_0 > 0, C_1 > 0\) is a constant.

Now we turn to \(R_{m,0}\), which is defined as

\[
R_{m,0} = [L(\lambda) + iE(\lambda) \cdot m - 0]^{-1} P_c J_{m,0};
\]
and for $JN_{m,0}$ we have, from (53),

$$
\sum_{|m|=2} JN_{m,0} z^m = \sum_{|m|=2} \left( \frac{Im N_{m,0}}{-Re N_{m,0}} \right) z^m = \frac{1}{4} \left( -2i\phi^\lambda z \cdot \xi z \cdot \eta - 3\phi^\lambda (z \cdot \xi)^2 + \phi^\lambda (z \cdot \eta)^2 \right)
$$

(121)

Now we extract the main part of $R_{m,0}$ and $JN_{m,0}$ by applying the estimates of $\phi^\lambda$, $\xi$, and $\eta$ and $P_c$ in (104) and (106), and use that

$$[L(\lambda) + iE(\lambda) \cdot m - 0]^{-1} = \left[ (-\Delta + V + \lambda)J + iE(\lambda) \cdot m - 0 \right]^{-1} + O(\epsilon_0 - \lambda)$$

to obtain

$$R_{m,0} = C_2(\epsilon_0 - \lambda)^{\frac{3}{2}} z^m \left[ (-\Delta + V + \lambda)J + iE(\lambda) \cdot m - 0 \right]^{-1} P_c \phi(\xi^{in}) m \left( \frac{-i}{1} \right)$$

$$+ O((\epsilon_0 - \lambda)^{\frac{3}{2}}) z^m$$

(122)

where $C_2 > 0$ is a constant, the expansion is in the space $\langle x \rangle^4 \mathcal{L}^2$.

Put this and (120) into (119) to obtain the desired result.

□

11.2 Proof of (99)

To illustrate the ideas we consider part of it, namely

$$\sum_k z_k \hat{\Theta}_2(k) := \left\langle X_1 \left( \sum_{|m|=2} z^m \sum_{n=1}^N p_{m,0}^{(n)} \xi_{m,0}^{(n)} z \right) \cdot \left( \frac{z \cdot \eta}{-iz \cdot \xi} \right) \right\rangle.$$

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The other part is similar, hence omitted. Compute directly to obtain

\[ \sum_k \bar{z}_k \tilde{\Theta}_2(k) \]

\[ = 4 \left( \sum_{|m|=2} z^m \sum_{n=1}^N P_{m,0}^{(n)} \xi_n \right) \sum_{|m|=2} z^m \sum_{n=1}^N Q_{m,0}^{(n)} \eta_n \]

\[ = 4 \left( \sum_{|m|=2} z^m \sum_{n=1}^N P_{m,0}^{(n)} \xi_n \right) \sum_{|m|=2} z^m \sum_{n=1}^N Q_{m,0}^{(n)} \eta_n \]

\[ \quad \cdot \left( \sum_{|m|=2} ReN_{l,0} \bar{z}^l \sum_{|m|=2} ImN_{l,0} \bar{z}^l \right) \]

\[ = -4i \sum_{n=1}^N \sum_{|m|=2} \sum_{|l|=2} \left[ P_{m,0}^{(n)} z^m \langle \xi_n, ReN_{l,0} \bar{z}^l \rangle + Q_{m,0}^{(n)} z^m \langle \eta_n, ImN_{l,0} \bar{z}^l \rangle \right] \quad (123) \]

where, in the second last step we used the definition of \( ReN_{l,0} \) and \( ImN_{l,0} \), \( |l|=2 \), in (124). Next, relate the definitions of \( P_{m,0}^{(n)} \) and \( Q_{m,0}^{(n)} \) in (63) to \( \langle \eta_n, ImN_{m,0} \rangle \) and \( \langle \xi_n, ReN_{m,0} \rangle \), and then take the real part for \( \sum_k \bar{z}_k \tilde{\Theta}_2(k) \)

to obtain

\[ Re \sum_k \bar{z}_k \tilde{\Theta}_2(k) = \sum_{n=1}^N \sum_{|m|=|l|=2} Re \left[ \left( P_{m,0}^{(n)} \bar{Q}_{l,0}^{(n)} - Q_{m,0}^{(n)} \bar{P}_{l,0}^{(n)} \right) l \cdot E(\lambda) z^m \bar{z}^l \right] \]

\[ = 2 \sum_{n=1}^N \sum_{|m|=|l|=2} Re \left[ \left( P_{m,0}^{(n)} \bar{Q}_{l,0}^{(n)} - Q_{m,0}^{(n)} \bar{P}_{l,0}^{(n)} \right) l \cdot E(\lambda) z^m \bar{z}^l \right] \]

\[ + 2 \sum_{n=1}^N \sum_{|m|=|l|=2} Re \left[ \left( P_{m,0}^{(n)} \bar{Q}_{l,0}^{(n)} - Q_{m,0}^{(n)} \bar{P}_{l,0}^{(n)} \right) l \cdot E(\lambda) z^m \bar{z}^l \right] \]

\[ = 2 \sum_{n=1}^N \sum_{|m|=|l|=2} Re \left[ \left( P_{m,0}^{(n)} \bar{Q}_{l,0}^{(n)} - Q_{m,0}^{(n)} \bar{P}_{l,0}^{(n)} \right) (l-m) \cdot E(\lambda) z^m \bar{z}^l \right] \]

\[ = \sum_{|m|=|l|=2} C_{m,l} (l-m) \cdot E(\lambda) z^m \bar{z}^l \quad (124) \]

where, in the second step we used a simple fact that \( Re \ a = Re \ \bar{a} \), for any parameter \( a \), for the second term in the second step, we interchange the indices \( m \) and \( l \) to obtain the third step, and in the last step \( C_{m,l} \) are constants naturally defined.

The proof is complete by observing that (124) is the desired result.
11.3 Proof of (100)

It is easy to see that for $j = 3, 4$, $\sum_{k=1}^{N} \bar{z}_k \Theta_j(k)$ are purely imaginary. Hence by taking the real part,

$$\text{Re} \sum_{k=1}^{N} \bar{z}_k \Theta_j(k) = 0. \quad (125)$$

Next we turn to $\Theta_5$. Use the definition of $X_1$ and compute directly to obtain

$$\sum_{k=1}^{N} \bar{z}_k \Theta_5(k) = -\left\langle \sum_{|m|=|n|=1} R_{m,n} z^m \bar{z}^n, \left( i\phi |z| z \cdot \eta + 3i\phi |z| \xi \cdot \eta \right) \right\rangle$$

By noticing that $\sum_{|m|=|n|=1} R_{m,n} z^m \bar{z}^n$ is real by definition, and that the other vector function is purely imaginary, we have that the inner product is purely imaginary, hence

$$\text{Re} \sum_{k=1}^{N} \bar{z}_k \Theta_5(k) = 0. \quad (126)$$

Collect the estimates above to complete the proof.

12 Proof of the Main Theorem 6.1

The proof is almost identical to the part in [11], hence we only sketch it.

We begin by introducing a family of space-time norms, $Z(T)$, $R_j(T)$, for measuring the decay of the $z(t)$ and $\tilde{R}(t)$ for $0 \leq t \leq T$, with $T$ arbitrary and large. We then prove that this family of norms satisfy a set of coupled inequalities, from which we can infer the desired large time asymptotic behavior. Define

$$T_0 := |z(0)|^{-1}, \quad (127)$$

where, recall that $|z(0)|$ is the initial amount of mass of neutral modes, see Theorem 6.1.

Now we define the controlling functions:

$$Z(T) := \max_{t \leq T} (T_0 + t)^{\frac{1}{2}} |z(t)|, \quad R_1(T) := \max_{t \leq T} (T_0 + t) \|\langle x \rangle^{-\nu} \tilde{R}(t)\|_{H^3},$$

$$R_2(T) := \max_{t \leq T} (T_0 + t) \|\tilde{R}(t)\|_{L^\infty}, \quad R_3(T) := \max_{t \leq T} (T_0 + t)^{\frac{1}{2}} \|\langle x \rangle^{-\nu} \tilde{R}(t)\|_{L^2},$$

$$R_4(T) := \max_{t \leq T} \|\tilde{R}(t)\|_{H^3}, \quad R_5(T) := \max_{t \leq T} \frac{(T_0 + t)^{\frac{1}{2}}}{\log(T_0 + t)} \|\tilde{R}(t)\|_{L^3}.$$  

\( (128) \)
To estimate $R_k$, $k = 1, 2, 3, 5$, or to control the dispersion $\tilde{R}$, we use the propagator estimate, namely for any function $g \in L^1$,
\[
\|e^{tL(x)}P_0^\pm g\|_\infty \lesssim t^{-\frac{1}{2}}\|g\|_1. \tag{129}
\]
To estimate the decay of $|z|$, we define a positive constant $q$ by
\[
q^2 = |z|^2 + F(z, \bar{z})
\]
where $F(z, \bar{z})$ is real and of order $|z|^4$, as stated in Statement C of Theorem 8.1. By Statement C of Theorem 8.1, we have
\[
\frac{d}{dt}|q|^2 \leq -C|q|^4 + \cdots. \tag{130}
\]
If the omitted part is small, then we have $|q|^2 = |z|^2 \lesssim (1 + t)^{-1}$.

The details are identical to the corresponding parts in [11], and the proof is tedious. Hence we omit the detail. The results are:

**Proposition 12.1.**

\[
R_1 \lesssim T_0 \|(x)^\nu \tilde{R}(0)\|_{\mathcal{H}^2} + R_4R_2 + Z^2 + T_0^{-\frac{1}{2}}[Z^3 + ZR_1 + R_4^2 + R_3^2].
\]
\[
R_2 \lesssim T_0 \|	ilde{R}(0)\|_1 + T_0 \|	ilde{R}(0)\|_{\mathcal{H}^2} + Z^2 + R_4^3R_2 + T_0^{-\frac{1}{2}}[Z^3 + ZR_1 + R_3^2].
\]
\[
R_3 \lesssim T_0 \|(x)^\nu \tilde{R}(0)\|_2 + T_0^\frac{3}{2}|Z|^2(0) + T_0^{-\frac{1}{4}}(Z^3 + ZR_3 + ZR_4 + R_3^2 + R_3^2R_4).
\]
\[
R_4^2 \lesssim \|	ilde{R}(0)\|^2_{\mathcal{H}^2} + T_0^{-1}[R_4^2 + Z^2R_1 + Z^2R_1^2 + R_3^2R_2^2].
\]
\[
R_5 \lesssim T_0 \|	ilde{R}(0)\|_1 + T_0 \|	ilde{R}(0)\|_{\mathcal{H}^2} + Z^2 + T_0^{-\frac{1}{2}}[R_5^3R_3 + Z^4 + ZR_3 + R_1^2 + R_2^2].
\]
\[
Z(T) \lesssim 1 + \frac{1}{T_0^\frac{3}{2}}Z(T) \ (Z(T) + R_4^2(T) + R_2^2(T) + Z(T)R_3(T))
\]

### 12.1 Proof of the Main Theorem [6.1]

Our goal is to prove that the functions $Z$ and $R_k$, $k = 1, 2, \cdots, 5$, are uniformly bounded, using the estimates in Proposition [12.1].

Define $M(T) := \sum_{n=1}^N R_n(T)$ and
\[
S := T_0 \|	ilde{R}(0)\|_{\mathcal{H}^2} + \|(x)^\nu \tilde{R}(0)\|_2 + \|	ilde{R}(0)\|_1 + \|	ilde{R}(0)\|_{\mathcal{H}^2} + T_0^{-\frac{1}{2}}|z_0|^2, \tag{131}
\]
where, recall the definition of $T_0$ after [128]. By the conditions on the initial condition, we have that $R_4(0)$ is small, $M(0)$, $Z(0)$ and $S$ are bounded. Then by the estimates in Proposition [12.1] we obtain
\[
M(T) + Z(T) \leq \mu(S), \text{ and } R_4 \ll 1, \tag{132}
\]
where $\mu$ is a bounded function if $S$ bounded. This together with the definitions of $R_k$, $k = 1, 2, \cdots, 5$, implies that

$$
\|\langle x \rangle^{-\nu} \vec{R} \|_2, \| \vec{R} \|_\infty \leq c(T_0 + t)^{-1}, \ |z(t)| \leq c(T_0 + t)^{-\frac{1}{2}} \tag{133}
$$

which is part of Statements (A) and (C) in Theorem 6.1. The rest of (A), Equation (23), is proved in (52).

□

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