A TROPICAL CHARACTERIZATION OF COMPLEX ANALYTIC VARIETIES TO BE ALGEBRAIC

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Abstract. In this paper we study a $k$-dimensional analytic subvariety of the complex algebraic torus. We show that if its logarithmic limit set is a finite rational $(k - 1)$-dimensional spherical polyhedron, then each irreducible component of the variety is algebraic. This gives a converse of a theorem of Bieri and Groves and generalizes a result proven in [MN14]. More precisely, if the dimension of the ambient space is at least twice of the dimension of the generic analytic subvariety, then these properties are equivalent to the volume of the amoeba of the subvariety being finite.

1. Introduction

Amoebas of complex varieties with their cousin coamoebas play a major role as a link between complex algebraic geometry and tropical geometry. Moreover, they are used in several areas of mathematics, in real algebraic geometry, mirror symmetry, algebraic statistics, complex analysis (see [MS], [Mil04], [FPT00], [NS13], [PR04], and [PS04]). We show in this paper that the logarithmic limit sets and the phase limit sets play a role as crucial as the role played by their relatives. Indeed, their role is a link between complex algebraic geometry and phase tropical geometry. More precisely, this is a quadruplet (logarithmic limit set, amoeba, coamoeba, phase limit set) and we cannot dissociate one of these objects from the others. We do believe that these objects are not yet fully exploited, and they contain more information about our original object which is the complex variety. This information can be apparently of different nature e.g., geometric, algebraic, topological, combinatorial. But they are often equivalent. We prove in this paper the equivalence between some properties of algebraic nature on one hand and some properties of combinatorial and topological nature, on the other hand. Another equivalence between geometric nature properties and of algebraic nature was proven in [MN14]. More precisely, we show that $k$-dimensional irreducible analytic subvarieties of the complex torus are algebraic, if their logarithmic limit sets are finite rational $(k - 1)$-dimensional complex polyhedrons. In addition, if the dimension of the ambient space is at least double of the dimension of the varieties, then, these properties are equivalent to the fact that the volume of the amoebas of the varieties are

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finite, which is a completely geometric property. The last particular case is proven in [MN14]. In [Ber71], Bergman introduced the notion of logarithmic limit set of a subvariety of the algebraic torus as the set of limiting directions of points in its amoeba. In [BG84], Bieri and Groves proved the following theorem, conjectured by Bergman.

**Theorem 1.1** (Bergman, Bieri–Groves). The logarithmic limit set $\mathcal{L}^\infty(V)$ of an algebraic variety $V$ in $(\mathbb{C}^*)^n$ is a finite union of rational spherical polyhedrons. The maximal dimension of a polyhedron $P$ in this union is such that $\dim_{\mathbb{R}} P = \dim_{\mathbb{R}} \mathcal{L}^\infty(V) = \dim_{\mathbb{C}} V - 1$.

The aim of this paper is to prove the converse of Theorem 1.1.

**Main Theorem.** Let $V$ be a $k$-dimensional irreducible analytic subvariety of the complex algebraic torus $(\mathbb{C}^*)^n$ and $\mathcal{A}(V)$ be its amoeba. Let $\mathcal{L}^\infty(V)$ be the logarithmic limit set of $V$. Assume that $\mathcal{L}^\infty(V)$ is a finite rational spherical polyhedron of dimension $k - 1$. Then $V$ is algebraic.

Chow’s theorem asserts that any analytic subvariety of the projective space is algebraic. Bieri-Grove’s theorem and the main theorem above give a necessary and sufficient condition for a subvariety in the complex algebraic torus to be algebraic.

The key ingredients in our proof are on one side, the topology and the combinatorial structure of the logarithmic limit set of our variety and on the other side, their link with the geometry of the amoeba.

The paper is organized as follows: In Section 2, we recall some basic definitions and introduce our notation. In Section 3, we prove the main theorem and give some consequences.

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2. **Preliminaries**

A subset $Q$ of $\mathbb{R}^n$ is said to be a rational convex polyhedron, if $P$ can be written as the intersection

$$Q = H_1 \cap H_2 \cap \cdots \cap H_l$$

of a finite number of closed affine half spaces in $\mathbb{R}^n$, where each $H_j$ can be defined in terms of inequalities of the form $\sum_{i=1}^n q_{ij} x_i \leq a_j$, with rational coefficients $q_{ij}$ and real numbers $a_j$. A subset $P \subset \mathbb{R}^k$ is said to be a rational complex polyhedron if it can be written as the union:

$$P = Q_1 \cup Q_2 \cup \cdots \cup Q_r$$

of a finite number of rational convex polyhedrons. $P$ is said to be homogeneous of dimension $k$ if the dimension of $Q_j$ is equal to $k$, for each $j = 1, \ldots, r$. A **rational spherical polyhedron** is a finite union of closed hemispheres which can be written in terms of a finite number of inequalities with integral coefficients.
Let $W$ be an analytic variety in $\mathbb{C}^n$ defined globally by an ideal $I$ of holomorphic functions on $\mathbb{C}^n$. We say that a subvariety $V$ of the complex algebraic torus $(\mathbb{C}^*)^n$ is analytic if there exists an analytic variety $W$ as above such that $V := W \cap (\mathbb{C}^*)^n$.

All the analytic varieties considered in this paper are defined as above such that $V$ contains an arrangement of $k$-dimensional real subtori.

The amoeba $\mathcal{A}$ of $V$ is by definition (see M. Gelfand, M.M. Kapranov and A.V. Zelevinsky [GKZ94]) the image of $V$ under the map:

$$\text{Log} : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|).$$

The amoeba of a variety of codimension one is closed and its complement components in $\mathbb{R}^n$ are convex (see [FPT00]). In [Hen03], Henriques generalized the notion of convexity as follows:

**Definition 2.1.** A subset $A \subset \mathbb{R}^n$ is called $l$-convex if for all oriented affine $(l + 1)$-planes $L \subset \mathbb{R}^n$, the induced homomorphism $H_1(L \cap A) \rightarrow H_1(A)$ does not send non-zero elements of $H_1^*(L \cap A)$ to zero, where $H_1(L \cap A)$ (resp. $H_1^*(L \cap A)$) denotes the reduced homology groups associated to the corresponding augmented complexes (resp. elements of $H_1(L \cap A)$) such that their image in $\bar{H}_1(L \setminus p) \sim \mathbb{Z}$ are non-negative for all $p \in L \cap A$.

When the subset $A$ is the complement of an amoeba, Henriques obtains the following result:

**Theorem 2.1** (Henriques [Hen03]). Let $V \subset (\mathbb{C}^*)^n$ be a variety of codimension $r$ and $\mathcal{A}$ be its amoeba. Let $L$ be an $r$-plane of rational slope and $c$ be a non-zero $(r - 1)$-cycle in $H_{r-1}(\mathbb{R}^n \setminus \mathcal{A})$. Then the image of $c$ in $H_{r-1}(\mathbb{R}^n \setminus \mathcal{A})$ is non-zero and $\mathbb{R}^n \setminus \mathcal{A}$ is $(r - 1)$-convex.

The logarithmic limit set $\mathcal{L}^\infty(V)$ of an analytic subvariety $V$ of the complex algebraic torus is the boundary of the closure of $\rho(\mathcal{A}(V))$ in the $n$-dimensional ball $B^n$, where $\rho$ is the map defined by (see Bergman [Ber74]):

$$\rho : \mathbb{R}^n \rightarrow B^n, \quad x \mapsto \rho(x) = \frac{x}{1 + |x|}.$$

If $V$ is algebraic of dimension $k$, then its logarithmic limit set is a finite rational spherical polyhedron of dimension $k - 1$.

The argument map is the map defined as follows:

$$\text{Arg} : (\mathbb{C}^*)^n \rightarrow (S^1)^n, \quad (z_1, \ldots, z_n) \mapsto (\arg(z_1), \ldots, \arg(z_n)).$$

where $\arg(z_j) = \frac{z_j}{|z_j|}$. The coamoeba of $V$, denoted by $\co \mathcal{A}$, is its image under the argument map (defined for the first time by Passare in 2004).

Sottile and the third author [NS13] defined the phase limit set of $V$, $\mathcal{P}^\infty(V)$, as the set of accumulation points of arguments of sequences in $V$ with unbounded logarithm. If $V$ is an algebraic variety of dimension $k$, $\mathcal{P}^\infty(V)$ contains an arrangement of $k$-dimensional real subtori.

Now, we introduce the notion of ends of an analytic subvariety of the complex algebraic torus. Let $V$ be an analytic variety in the complex algebraic torus $(\mathbb{C}^*)^n$, and let $\mathcal{L}^\infty(V)$ be its logarithmic limit set. Let $\{K_i\}_{i=0}^\infty$ be a family
of compact subsets of $V$ such that $K_l \subset K_{l+1}$ for $l = 0, \ldots, \infty$, and $V = \bigcup_{l=0}^{\infty} K_l$. We define the set of ends of $V$ as follows:

$$\mathcal{E}^\infty(V) := \bigcap_{l=0}^\infty (\overline{V} \setminus K_l),$$

where $\overline{V}$ denotes the closure of $V$ in $\mathbb{C}^n$, and each connected component of $\mathcal{E}^\infty(V)$ is called an end of $V$.

Let $x$ be a point in $\mathcal{L}^\infty(V)$, and $\mathcal{I}_x(V)$ be the set of sequences in $V$ defined as follows:

$$\mathcal{I}_x(V) := \{ s = \{z_l\}_{l=0}^\infty \subset V \mid x \in \mathcal{L}^\infty(\{s\}) \}.$$ 

Let $\mathcal{E}^\infty_x(V)$ be the subset of points in $\mathbb{C}^n$ defined as follows:

$$\mathcal{E}^\infty_x(V) := \bigcup_{s = \{z_l\}_{l=0}^\infty \in \mathcal{I}_x(V)} \left( \bigcap_{N=0}^\infty \left( \bigcap_{l=0}^N \{z_l\} \right) \right).$$

A connected component of $\mathcal{E}^\infty_x(V)$ is called an end of $V$ corresponding to $x$.

3. Proof of the main theorem

Before starting the proof, let us note that the assumption in the main theorem on the irreducibility of the algebraic subvariety is necessary. If $V$ has an infinite number of irreducible components and its logarithmic limit set is a finite rational spherical polyhedron of dimension $k - 1$, then in general it is not possible to conclude weather $V$ is algebraic or not. For example, the logarithmic limit set of the plane analytic curve $\mathcal{C} \subset (\mathbb{C}^*)^2$, with defining function $f(z_1, z_2) = \sin(\pi z_1 z_2)$, is $\mathcal{L}^\infty(\mathcal{C}) = \{ \pm(1, -1) \}$. This is a rational spherical polyhedron of dimension zero, but the curve $\mathcal{C}$ is not algebraic.

**Proof of main theorem.** From now on, we assume that $V \subset (\mathbb{C}^*)^n$ is a $k$-dimensional analytic variety, such that its logarithmic limit set is a finite rational spherical polyhedron of dimension $k - 1$. Moreover, we assume that the ideal $I(V)$ is generated by a set of entire functions $\{f_1, \ldots, f_q\}$, where each entire function $f_j$ can not be written as a product $f_j = h g$, with $h \neq 0$ non constant entire function and $g$ an entire function. Let $\text{Vert}(\mathcal{L}^\infty(V))$ be the set of vertices of $\mathcal{L}^\infty(V)$. We choose a vertex $v \in \text{Vert}(\mathcal{L}^\infty(V))$ with slope $(u_1, \ldots, u_n)$. We denote by $D_v$ the straight line in $\mathbb{R}^n$ directed by $v$ and asymptotic to the amoeba $\mathcal{A}(V)$. Let $\mathcal{H}(D_v)$ be the holomorphic cylinder which is the lifting of $D_v$ by $\text{Log}_1|_V$ and is asymptotic to the end of $V$ corresponding to $v$, such that $\mathcal{L}^\infty(\mathcal{H}(D_v)) \cap \mathcal{L}^\infty(V) = \{v\}$. The functions $f_j$’s are entire and their power series expansions are of the form $\sum_{\alpha} c_{j, \alpha} z^\alpha$ with $b_v + \sum_{i=1}^n u_i \alpha_i \leq 0$, where $b_v$ is a real number. In other words, the exponents of the power series expansion of the $f_j$’s are contained in some half space depending on the slope of the vertex $v$. By doing the same operation for all the vertices of $\mathcal{L}^\infty(V)$ and using Lemma 3.1 below, we conclude that the exponents of the power series expansion of $f_j$ are contained in a compact polytope. $\Box$

**Lemma 3.1.** Let $V$ be a subvariety in $(\mathbb{C}^*)^n$. If $S^{n-2}$ is a subsphere of $S^{n-1} = \partial B^n$, invariant under the involution $-\text{id}$, then $\mathcal{L}^\infty(V)$ intersects the interior of each connected component of $S^{n-1} \setminus S^{n-2}$. 


Proof. Set \( r := n - \dim V - 1 \). We know that the complement components of amoebas are \( r \)-convex by Theorem 2.1. Then the intersection of the closed half spaces in \( \mathbb{R}^n \) bounded by the hyperplanes normal to all the directions \( v \in \text{Vert}(\mathcal{L}(V)) \) is compact. \( \square \)

**Definition 3.1.** An analytic variety \( V \) is generic, if \( V \) is a finite union of irreducible components and each irreducible component of \( V \) contains an open dense subset \( U \), such that the Jacobian of the restriction of the logarithmic map to \( U \) has maximal rank.

We have the following proposition and corollaries:

**Proposition 3.1.** Let \( C \) be a generic analytic curve (not necessary algebraic) of \((\mathbb{C}^*)^n\). Then \( \mathcal{L}(C) \) is the union of a finite number of isolated points with rational slopes and a finite number of geodesic arcs with rational end slopes. In particular, if \( C \) is not algebraic, then the number of arcs in \( \mathcal{L}(C) \) is different than zero.

Proof. It is sufficient to show that any point in \( \mathcal{L}(C) \) with irrational slope is necessarily contained in the interior of the amoeba. Without loss of generality, we can assume that the curve \( C \) is irreducible. We suppose on the contrary that there exists a point with irrational slope \( s \), which is either isolated or is contained in the boundary of a connected component of \( \mathcal{L}(C) \). By Lemma 4.1 [MN14], the phase limit set \( \mathcal{P}(C) \) contains a subset of dimension at least two. More precisely, it contains an immersed circle \( \mathcal{S} \) of irrational slope \( s \) in the real torus \((S^1)^n\) such that its closure is at least 2-dimensional. Let \( U \) be an open subset of the torus \((S^1)^n\) such that \( U \cap \mathcal{S} \) is nonempty. Since the closure of the immersed circle \( \mathcal{S} \) is at least 2-dimensional, then the intersection \( U \cap \mathcal{S} \) has an infinite number of connected components. For each such connected component \( C_i \), we choose an open subset \( V_i \) of the regular part of the coamoeba such that \( \partial V_i \) contains \( C_i \) and is of area a constant \( A \) different than zero where \( \partial V_i \) denotes the closure of \( V_i \) (i.e., the area of \( V_i \) is equal to \( A \) for all \( i \)). We claim that the union of the following subsets of the amoeba \( \tilde{V}_i := (\text{Log}|_C) \circ (\text{Arg}|_C)^{-1}(V_i) \) is not bounded. Otherwise, if for every open set \( U \) of the real torus this union is bounded, and by compactness of the torus, this implies that the amoeba \( \mathcal{A}(C) \) has no tentacle of slope \( s \). Since the map \( \text{Log}|_C \circ (\text{Arg}|_C)^{-1} \) conserves the area and \( C \) is generic, then for any positive number \( R \gg 1 \), there exists an index \( i \) such that the intersection of the sphere \( S^1_R \) of radius \( R \) with \( \tilde{V}_i \) is of dimension one. Moreover, the length of \( I_i := \tilde{V}_i \cap S^1_R \) does not converge to zero. In fact, using the convexity (or higher convexity in the case of higher codimension) of the amoeba complement, the intersection \( I_i := \tilde{V}_i \cap S^1_R \) must converge to a point by hypothesis. But in this case, the area of \( \tilde{V}_i \) converges to zero too. This contradicts the fact that for any index \( i \) the area of \( V_i \) is equal to \( A \). This implies that if a point \( v \) in \( \mathcal{L}(C) \) has an irrational slope, then \( v \) must be in the interior of the logarithmic limit set. \( \square \)

The phase limit set version of Proposition 3.1 is the following:
Corollary 3.1. Let $C$ be a generic analytic curve (not necessarily algebraic) of $(\mathbb{C}^*)^n$. Then $\mathcal{P}_\infty(C)$ is an arrangement of a finite number of geodesic circles with rational slopes and a finite number of 2-dimensional flat tori. In particular, if $C$ is not algebraic, the number of 2-dimensional flat tori in $\mathcal{P}_\infty(C)$ is different than zero. Moreover, if $C$ is not algebraic and $n = 2$, then the closure of its coamoeba is the whole torus.

Proof. This corollary is a phase interpretation of Proposition 3.1. Indeed, if the curve is generic and not algebraic, then the dimension of its logarithmic set is equal to one. This implies that the phase limit set contains an immersed circle with closure a torus of dimension at least two.

Corollary 3.2. Let $V$ be a $k$-dimensional generic analytic subvariety of the complex algebraic torus $(\mathbb{C}^*)^n$. If $V$ is not algebraic, then the closure of the coamoeba $\co\mathcal{A}(V)$ contains a flat torus of dimension at least $k + 1$.

Proof. More precisely, the coamoeba $\co\mathcal{A}(V)$ of $V$ contains an immersed $k$-dimensional torus whose closure is of dimension at least $k + 1$. In fact, if $V$ is a $k$-dimensional generic analytic subvariety of the complex algebraic torus $(\mathbb{C}^*)^n$ and is not algebraic, then its logarithmic set is at least $k$-dimensional. In other words, its phase limit contains a torus of dimension at least $k + 1$.

(a) The amoeba $\mathcal{A}$ of the analytic curve parametrized by $z \mapsto (z, e^z)$.  
(b) The image of $\mathcal{A}$ by the retraction $\rho$.  
(c) The image by $\rho$ in the ball of the amoeba of the curve given by the parametrization $g(t) = (t, e^t, t + 1)$. 


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