Coherent Control of Multipartite Entanglement

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Quantum entanglement between an arbitrary number of remote qubits is examined analytically. We show that there is a non-probabilistic way to address in one context the management of entanglement of an arbitrary number of mixed-state qubits by engaging quantitative measures of entanglement and a specific external control mechanism. Both all-party entanglement and weak inseparability are considered. We show that for \(N \geq 4\), the death of all-party entanglement is permanent after an initial collapse. In contrast, weak inseparability can be deterministically managed for an arbitrarily large number of qubits almost indefinitely. Our result suggests a picture of the path that the system traverses in the Hilbert space.

Despite recent advances in experimental realization of multipartite entanglement [1], current schemes to preserve entanglement, such as the quantum Zeno effect [2], entanglement distillation [3] or weak measurements [4], lack an element of control and/or their success is probabilistic. We show here that there is a way to address all five aspects of the managed entanglement question, namely obtaining in one prescription simultaneous compatibility of (i) mixed states, (ii) arbitrary numbers of qubits, (iii) quantitative measure of entanglement, (iv) non-probabilistic success, and (v) external control. We show that the phenomenon of collapse and revival [5] offers a concrete example of a mechanism of deterministic control of multi-qubit mixed-state entanglement.

It is first needed to quantify the entanglement of multipartite mixed states, which remains a challenge. There have been advances in determining whether a state is entangled or not, but most lack a quantitative measure that can be used in a dynamical analysis [6–13] and numerical approaches are infeasible since the dimension of Hilbert space grows prohibitively large [14]. We pay particular attention to the special case of \(N\)-qubit X-states [15]. Hashemi Rafsanjani et al. [16] have developed an algebraic formula for their all-party entanglement. Remarkably the entanglement of the X-part of any \(N\)-qubit density matrix is a lower bound for the entanglement of the complete matrix [9, 17].

Here we combine knowledge about \(N\)-party entanglement with control of revival dynamics to demonstrate quantitative control of multipartite entanglement [15]. We present an example of multipartite entanglement that is initially shared by \(N\) remote qubits interacting with individual fields. Local control is managed by a coherent state of a resonant mode via collapse and revivals of the qubit coherences. By controlling the amplitude of the coherent states one controls the time of revivals and thus the recovery of the multi-qubit entanglement. We note that entanglement and revivals were previously discussed in studies focused on the entanglement of one qubit and its local field [18, 20]. Here, we mean the entanglement among the qubits and not inseparability from their local fields.

Multipartite entanglement can signal inseparability for different partitionings of the system. Here we examine two extreme kinds of entanglement: (i) all-party entanglement, also known as genuinely multipartite entanglement, which signals inseparability along all possible partitionings, and (ii) weak inseparability, defined as the lack of full separability. Full separability signals that the state is not entangled along any partitioning. We develop an approximation that allows us to obtain analytical expressions for both of these quantities.

The qubits are initially assumed to be in a Greenberger-Horne-Zeilinger (GHZ) state [21] and we explain below an approximation that reduces their density matrix to an X-state for all times. X-states are \(N\)-qubit density matrices whose non-zero elements are restricted to diagonal or anti-diagonal in an orthonormal product basis. They include important states such as GHZ and GHZ-diagonal states. Our approximation enables us to use the algebraic formula developed in [16] to quantify the all-party entanglement. We also utilize the distance from the set of fully-separable states as our measure of weak inseparability. We obtain an analytical formula for this quantity during dynamics. We observe that beyond three qubits the initial loss of all-party entanglement after collapse is permanent. We then examine weak inseparability, and demonstrate that, contrary to all-party entanglement, weak inseparability experiences revivals even for very large values of \(N\), although the strength of such revivals decreases with \(N\). Our result suggests a clear picture of the path that the \(N\)-qubit state traverses during the dynamics.

Each local field is described by a resonant mode of the field that is interacting with its local qubit. To observe revivals one has to prepare a resonator with a very small leakage constant. In our case this means \(N\) times smaller than the leakage constant needed to observe a revival in a single resonator. For a coherent state \(|\alpha\rangle\) with \(\alpha^2 = 100\) a ratio \(10^3\) of coupling constant to the decay rate is required, which is not outrageously higher than \(3 \times 10^2\), that was achieved in a circuit QED setup recently [22].
Thus only an order of magnitude improvement in this ratio leads to suitable condition for control of revival of multipartite entanglement in a setup with $N \geq 3$. We assume this condition is satisfied and ignore the resonator leakage altogether.

Each of $N$ remote and identical subsystems is made of a two-level system (a qubit) that is interacting with a single mode of the electromagnetic field through a Jaynes-Cummings interaction [23].

\[ H_i = \frac{\omega_0}{2} \sigma_{zi} + g (a_i^\dagger \sigma_i^- + a_i \sigma_i^+) + \omega_a^i a_i \]  

The JC Hamiltonian is integrable and one can analytically follow the evolution of the above model.

A coherent state can be written as $|\alpha\rangle = \sum_n A_n |n\rangle$ and $A_n = \exp (-\alpha^2/2) \alpha^n / \sqrt{n!}$ where $|n\rangle$ is a Fock state of $n$ excitations. For simplicity we assume that $\alpha$ is real and positive. Then we can write the coherently driven evolution of $|e\rangle$ and $|g\rangle$ states:

\[ U(t)|e, \alpha\rangle = |e, \alpha\rangle t = |e\rangle \otimes |\phi_0\rangle + |g\rangle \otimes |\phi_1\rangle \]  

\[ U(t)|g, \alpha\rangle = |g, \alpha\rangle t = |e\rangle \otimes |\phi_2\rangle + |g\rangle \otimes |\phi_3\rangle \]  

where $|\phi_0\rangle = \sum_n \phi_{0n}|n\rangle$ and their coefficients are $\phi_{0n} = A_n r_{n+1}$, $\phi_{1n} = A_{n-1} r_n$, $\phi_{2n} = A_{n+1} r_n$, $\phi_{3n} = A_n r_0$, $r_n = e^{i\omega t} \cos (gt/\sqrt{n})$, and $t_n = -ie^{-i\omega t} \sin (gt/\sqrt{n})$. All the dynamics we intend to investigate can be captured by the inner products of these four different $|\phi_i\rangle$'s. For $\alpha \geq 10$ excellent approximations are available to evaluate these inner products [24]. The summary of the results is given below.

\[ \langle \phi_i | \phi_i \rangle = \frac{1 + p_i x}{2}, \quad p_1 = \begin{cases} \frac{1}{2} + \frac{1}{4} \cos \omega t & \text{for } i = 0, 3, \\ \frac{1}{2} - \frac{1}{4} \cos \omega t & \text{for } i = 1, 2 \end{cases} \]  

\[ \langle \phi_i | \phi_{i+1} \rangle = \frac{e^{i\omega t}}{2} \left( I_2 + p_2 y \right), \quad p_2 = \begin{cases} \frac{1}{2} + \frac{1}{4} \cos \omega t & \text{for } i = 0, 3, \\ \frac{1}{2} - \frac{1}{4} \cos \omega t & \text{for } i = 1, 2 \end{cases} \]  

\[ \langle \phi_i | \phi_{3-i} \rangle = \frac{e^{i\omega t}}{2} \left( I_1 + p_3 x \right), \quad p_3 = \begin{cases} \frac{1}{2} + \frac{1}{4} \cos \omega t & \text{for } i = 0, 3, \\ \frac{1}{2} - \frac{1}{4} \cos \omega t & \text{for } i = 1, 2 \end{cases} \]  

\[ \langle \phi_0 | \phi_2 \rangle = -\langle \phi_1 | \phi_3 \rangle = -\frac{iy}{2} \]

where $x + iy = \sum_n A_n^2 \exp (2igt/\sqrt{n})$, and

\[ I_1 + iI_2 \simeq \exp (\frac{-g^2t^2}{24\alpha^2}) e^{igt/\pi}. \]  

These four quantities are evaluated in [24]. In Fig. 1 we present the time dependence of $x_1$, $I_1$, and $I_2$ for the coherent states with $\alpha = 10$. The plot for $y$ is similar to the plot for $x$ except that the fast Rabi oscillations in the revivals are $\pi$ out of phase. At the first revival the maximum of $|x|_1$ is $\frac{1}{2}$, and $I_1 \simeq -1$. At all revivals $I_2 \simeq 0$.

Density Matrices: X-states are $N$-qubit states whose non-zero elements are restricted to diagonal or anti-diagonal:

\[ \hat{X} = \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & \ddots & & \\ & & & a_d & \\ b_1 & & & & b_d \end{pmatrix} \]  

Here $d = 2^{N-1}$, and we require $|z_1| \leq \sqrt{a_1 b_1}$ and $\sum_i (a_i + b_i) = 1$ to ensure that $\hat{X}$ is positive and normalized. We also denote $w_i = \sum_j \sqrt{a_j b_j}$. It is shown in [10] that the all-party concurrence of an $N$-qubit X-matrix is

\[ C_N = 2 \max \{0, |z_i| - w_i\}, \quad i = 0, 1, \ldots, d. \]  

In deriving the density matrix, we follow the same approximation that leads to an X-state, and was developed by Yünaç and Eberly [18]. To decide which elements of the matrix can be safely discarded, we replace the coherent state $|\alpha\rangle$ by a Fock state $|n\rangle$ where $n = \alpha^2$, which is supported by the fact that for $n \gg 1$ the photon distribution of a coherent state is relatively narrowly peaked around $n$. One can instead assume a less extreme variant of this approximation where $|\alpha\rangle$ is replaced by a mixture of Fock states $|n\rangle$ that has the same photon distribution as $|\alpha\rangle$. Both of these approximations lead to an important consequence: the density matrix of the qubits becomes an X-state with only one non-zero off-diagonal element.

We then calculate the values of the non-zero elements using their values from a coherent state. In fact the simplification goes even further and we will show that $a_i = b_i$, i.e., the density matrix remains a GHZ-diagonal matrix. The above approximation has been shown to be an excellent choice in capturing the collapse and revivals [18, 25]. This approximation also enables us to calculate a proper measure of weak inseparability [24]. Finally we emphasize that the estimates of entanglement that we derive from the above approximation are lower bounds.
of the entanglement of the complete matrix, where no elements were discarded [10].

To simplify the calculations we choose the initial entangled state of the qubits to be a symmetric state with respect to the permutation of the qubits. Next we calculate the non-zero elements of the density matrix. We use the orthonormal product basis \(|{ee...}_i, {ee...}_g, ..., {gg...}_g\rangle\rangle\) to represent the density matrices of the \(N\) qubits and denote \(|e^{\otimes p}, g^{\otimes q}\rangle = \otimes^p|e\rangle \otimes^q|g\rangle\).

We first focus on the case of two qubits [18]. At any time the state of the system will read

\[|\Psi\rangle_t = \frac{1}{\sqrt{2}} (|\otimes^2 e, \alpha\rangle_t + |\otimes^2 g, \alpha\rangle_t).\]

The elements that we are interested in are

\[|X_{ee,gg}| = \frac{1}{4}(t_1^2 + x^2 - y^2 - t_2^2),\]
\[|X_{eg,eg}| = \frac{1}{4}(1 - x^2 + y^2).\]

We note that \(X_{eg,eg} = X_{ge,ge}\). Now we discuss the case of \(N \geq 3\). The state of the system is given by

\[|\Psi\rangle_t = \frac{1}{\sqrt{2}} (|\otimes^N e, \alpha\rangle_t + |\otimes^N g, \alpha\rangle_t).\]  

According to our approximation we only need to calculate one off-diagonal element:

\[2^{N+1}|X_{ee...e,gg...g}| = |(-i)^N((I_2 - y)^N + (I_2 + y)^N) + K|,\]
\[K = (I_1 - x)^N + (I_1 + x)^N.\] (7)

Next we calculate the diagonal elements:

\[2X_{e^{\otimes p}g^{\otimes q}e^{\otimes p}g^{\otimes q}} = \frac{1}{2^{n+m}}[(iy)^{n+m}((-1)^n + (-1)^m)] + \frac{1}{2^{n+m}}[(1 + x)^n(1 - x)^m + (1 - x)^n(1 + x)^m].\] (8)

This equation implies that \(|e^{\otimes p}, g^{\otimes q}|X|e^{\otimes p}, g^{\otimes q}\rangle = \langle g^{\otimes p}, e^{\otimes q}|X|g^{\otimes p}, e^{\otimes q}\rangle\). Using the above equations and also the permutation symmetry of the problem we can find all the diagonal elements of the density matrix. This simplification confirms that the X-part of the state will always remain a GHZ-diagonal state. This has two consequences. First the concurrence of a GHZ-diagonal state is directly proportional to the distance of that state to the set of biseparable states [20]. This enables us to draw a picture of the trajectory that the state traverses in the Hilbert space. Second, since these GHZ-diagonal states have only one non-zero anti-diagonal element, we can determine the full-separability of them [27].

**All-party entanglement:** For \(N = 2\) the result matches the result in [18], and the all-party concurrence is given by \(C_2 = \max\{0, Q_2\}\) where

\[Q_2 = \frac{1}{2}(I_1^2 + 2x^2 - 2y^2 - I_2^2 - 1).\] (9)

For \(N = 3\) the all-party concurrence is \(C_3 = \max\{0, Q_3\}\) where \(Q_3 = 2(|X_{ee,gg}| - 3|X_{eg,eg}|)\) and

\[|X_{ee,gg}| = \frac{1}{8}(I_1^2 + 3I_2^2x^2 + (I_2^2 + 3I_2^2y)^2),\]
\[|X_{eg,eg}| = \frac{1}{8}(1 - x^2).\] (10)

We plot \(C_2\) and \(C_3\) as a function of time in Fig. 2. As expected the entanglement dies out rapidly with the initial collapse. At \(gt = 2\pi\alpha\) entanglement revives to a small value. The maximum of this revival can be estimated from the above equations noting that the maximum value of \(x\) at first revival is \(1/2\), and \(I_1 \simeq -1\). At the revivals \(I_2 \simeq 0\) and since \(x, y\) are completely out of phase with each other when \(x\) is at maximum revival, \(y\) vanishes. Feeding these quantities to the above equations leads to the maximum height of the bipartite and tripartite entanglement revivals to be \(1/4\) and \(1/8\) respectively. The second revival does not occur for tripartite entanglement. For \(N > 3\) the sudden death of all-party entanglement is permanent.

**Weak Inseparability:** All-party entanglement can be thought of as the most exclusive kind of entanglement whose presence implies that system is inseparable along any possible partitioning. Full separability is the other extreme, implying that the system is separable along all possible partitionings. An \(N\)-party system is fully separable if it can be written as \(\sum_i p_i \rho_{i1} \otimes \rho_{i2} \otimes \cdots \otimes \rho_{Ni}\). The onset of full separability is the true end of entanglement. Now we examine what can be said under the looser requirement that the state is known to be inseparable without further constraints, which we refer to as weak inseparability. This will let us make a more complete picture of how the \(N\)-party system moves in the Hilbert space.

To study the dynamics of weak inseparability we use the distance of the state from the set of fully separable states as our measure. This distance \(S\) is a proper measure of weak inseparability and can be calculated for GHZ-diagonal states for which all but one of the anti-
diagonal elements vanish [23]:
\[ S = 2 \max \{0, |z| - c\}, \quad c = \min \{b_i\}. \] (11)

In Fig. 3 we plot the value of this distance for \( N = 3, 4, 5 \). In all cases we see that weak inseparability revives at multiples of \( gt = 2\pi \alpha \) until it decays away completely.

Finally we conclude with an attempt to capture the behavior of weak inseparability for large values of \( N \). To this end we replace \( c \) in the weak inseparability formula with the average values of all \( b_i \)'s. We also assume that \( N \) is odd. Thus we can approximate
\[ S_N \simeq \frac{1}{2^N} \left[(1-x)^N + (1+x)^N - 2\right]. \] (12)

This has the strong implication that at these revivals this inseparability revives even for very large \( N \). Yet its maximum distance from the boundary between fully separable and inseparable states decreases exponentially. For example at the first revival the maximum value of \( x \) is \( \frac{1}{2} \) and thus we can estimate \( S_N \simeq \left(\frac{1}{2}\right)^N \). Thus for large \( N \) the state follows a path to the boundary of inseparability and then stays below it, crossing it momentarily only at revivals.

The previous results and the fact that both of our measures have a geometrical interpretation can be combined to picture how the state moves in the Hilbert space [28]. In the Hilbert space there the convex set of biseparable states, \( BS \). Inside this set there is the convex set of fully-separable states \( FS \). For \( N = 2 \) these two sets match, and the initial state starts outside the \( BS \) and after five crossings it ends up inside \( BS \). For \( N = 3 \) the state starts outside \( BS \) and moves inside \( BS \) and also inside \( FS \). At the first revival the state goes outside of both these sets and then comes back inside and gets trapped inside \( BS \) permanently. The state then moves inside \( BS \), crossing \( FS \) several times until it ends up somewhere in \( FS \). For \( N \geq 4 \) once inside, the state never leaves \( BS \). It follows a trajectory that crosses \( FS \) several times and ends up in \( FS \). Fig. 4 provides a sketch of these dynamics.

In summary, we have provided an answer to a long-standing open question in application of the principles of quantum information, namely how to exert a form of determinstic control over a quantitative degree of entanglement shared among an unspecified number of mixed-state qubits. Our answer is admittedly not perfect, and probably beyond near-term laboratory realization, but it is a strong step forward because it shows by concrete example that there can be a prescription that at the same time addresses all five difficult aspects of the question: mixed states, arbitrary numbers, quantitative measure, deterministic success, and external control, for entanglement of qubits.

We have used the machinery of quantifiable measures of entanglement to controllably suppress and recover specified degrees of multipartite entanglement using the phenomenon of coherent-state revivals. Our results are limited because we explicitly studied only two extreme kinds of multipartite entanglement, namely all-party entanglement and weak inseparability. All-party entanglement is so fragile that beyond three qubits our method fails. But entanglement in the form of weak inseparability undergoes completely different dynamics and is almost indefinitely controllable. Even for a very large value of \( N \), weak inseparability repeatedly revives from zero to a substantial non-zero value before disappearing again, although the strength of the revivals shrinks with \( N \). Finally our results suggest a picture of system evolution in the Hilbert space – if the system starts in a genuinely \( N \)-partite entangled state its evolution takes it back and forth over the boundary of full-separability, ending up somewhere close to that border.

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