UPPER BOUNDS FOR THE DIAMETER OF A DIRECT POWER OF NON-ABELIAN SOLVABLE GROUPS

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Abstract. Let $G$ be a finite group with a generating set $A$. By the (symmetric) diameter of $G$ with respect to $A$ we mean the maximum over $g \in G$ of the length of the shortest word in $(A \cup A^{-1})A$ expressing $g$. By the (symmetric) diameter of $G$ we mean the maximum of (symmetric) diameter over all generating sets of $G$. Let $n \geq 1$, by $G^n$ we mean the $n$-th direct power of $G$. For $n \geq 1$ and finite non-abelian solvable group $G$ we find an upper bound, growing polynomially with respect to $n$, for the symmetric diameter and the diameter of $G^n$.

1. Introduction

Let $G$ be a finite group with a generating set $A$. By the (symmetric) diameter of $G$ with respect to $A$ we mean the maximum over $g \in G$ of the length of the shortest word in $(A \cup A^{-1})A$ expressing $g$. Producing a bound for the (symmetric) diameter of a finite group is an important area of research in finite group theory. It is worth to mention to the Babai’s conjecture [3]: every non-abelian finite simple group $G$ has diameter $\leq \log^k|G|$, where $k$ is an absolute constant; the conjecture is still open, despite great progress towards a solution both for alternating groups and for groups of Lie type. Asymptotic estimate of the symmetric diameters of non-abelian simple groups with respect to various types of generating sets can be find in the survey [2], which also lists related work, e.g., on the diameters of permutation groups. Furthermore, the area has progressed a lot over the last few years (see for instance [8, 5, 9]). A more modest question is that of finding bounds for the diameter of direct products of finite groups, depending on the diameter of their factors. In this direction the papers [10, 6] in which produced upper bounds for the diameter of direct product of non-abelian simple groups are significant.

Let $n \geq 1$, by $G^n$ we mean the $n$-th direct power of $G$. In [11] have been appeared the following question: How large can be the diameter of $G^n$ with respect to any generating set? There have been proved that if $G$ is abelian, then the diameter of $G^n$ with respect to any generating set is $O(n)$ and if $G$ is nilpotent, symmetric or dihedral, then there exist a generating set of minimum size which the diameter of
$G^n$ with respect to this generating set is $O(n)$. In [6] have been proved that if $G$ is a non-abelian simple group, then the diameter of $G^n$ with respect to any generating set is $O(n^3)$.

Our main goal here is to present upper bounds for the diameter and the symmetric diameter of $G^n$, in which $G$ is a non-abelian solvable group. In fact, we prove that if $G$ is a non-abelian solvable group, then

$$D^s(G^n) \leq \frac{1}{4}(4n)^l |G|$$

and

$$D(G^n) \leq n^l |G| \prod_{i=0}^{l-2} (|G^{(i)}| + 1),$$

in which $l$ is the length of derived series of $G$.

2. Preliminaries

Throughout the paper all groups are considered to be finite. The subset $A \subseteq G$ is a generating set of $G$, if every element of $G$ can be expressed as a sequence of elements in $A$. By the rank of $G$, denoted by rank($G$), we mean the cardinality of any of the smallest generating sets of $G$. By the length of a non identity element $g \in G$, with respect to $A$, we mean the minimum length of a sequence expressing $g$ in terms of elements in $A$. Denote this parameter by $l_A g$. Similarly we define the symmetric length of a non identity element $g \in G$, with respect to $A$, to be the minimum length of a sequence expressing $g$ in terms of elements in $A \cup A^{-1}$. Denote this parameter by $l^s_A g$.

**Convention 2.1.** We consider the (symmetric) length of identity to be zero, i.e. $l_A(1) = l^s_A(1) = 0$ for every generating set $A$.

**Definition 2.2.** Let $G$ be a finite group with generating set $A$. By the diameter of $G$ with respect to $A$ we mean

$$\text{diam}(G, A) := \max\{l_A(g) : g \in G\}.$$ And by the symmetric diameter of $G$ with respect to $A$ we mean

$$\text{diam}^s(G, A) := \max\{l^s_A(g) : g \in G\}.$$  

**Notation 2.3.** Let $G$ be a finite group with a generating set $A$. Let $S$ be a subset of $G$. Denote by $Ml_A(S)$ the maximum of $l_A(s)$ over all $s$ in $S$. Note that $Ml_A(G) = \text{diam}(G, A)$.

\[a\] Usually $A \subseteq G$ is considered to be a generating set, if every element of $G$ can be expressed as a sequence of elements in $A \cup A^{-1}$. When $G$ is finite the definitions coincide.
Notation 2.4. Denote by \((D^s(G))\) the maximum (symmetric) diameter over all generating sets of \(G\).

The following theorem has been proved by Wiegold in [15].

Theorem 2.5. [15] Let \(G\) be a finite non-trivial solvable group, and set \(\text{rank}(G) = \alpha, \text{rank}(G/G') = \beta\). Then
\[
\text{rank}(G^n) = \beta n,
\]
for \(n \geq \alpha/\beta\).

Lemma 2.6. [14] Let \(G\) be a finite group and \(k\) be a positive integer. Then
\[
(1) \quad k \text{rank}(G/G') \leq \text{rank}(G^k) \leq k \text{rank}(G),
\]
where \(G'\) is the commutator subgroup of \(G\).

Corollary 2.7. Let \(G\) be a finite group. If \(\text{rank}(G) = \text{rank}(G/G')\), then the following equality holds:
\[
(2) \quad \text{rank}(G^n) = n \text{rank}(G).
\]
In particular, nilpotent groups satisfy this property.

Proof. The first statement is an immediate consequence of Lemma 2.6. We prove the second statement. Note that, if \(H\) is a homomorphic image of a finite group \(G\), then \(\text{rank}(H) \leq \text{rank}(G)\). Therefore, it is enough to show that \(\text{rank}(G) \leq \text{rank}(G/G')\) for every finite nilpotent group \(G\). Let \(A = \{g_1G', g_2G', \ldots, g_kG'\}\) be a generating set of \(G/G'\) of minimum size. Consider an arbitrary element \(g \in G\). There exist some \(i_1, i_2, \ldots, i_l \in \{1, 2, \ldots, k\}\) such that \(gG' = g_{i_1}g_{i_2} \ldots g_{i_l}G'\). This shows that \(G\) is generated by \(\{g_1, g_2, \ldots, g_k\}\) together with some elements in \(G'\). Because \(G\) is nilpotent, it is generated by \(\{g_1, g_2, \ldots, g_k\}\) alone, see [12] page 350. Therefore, \(\text{rank}(G) \leq \text{rank}(G/G')\), which completes the proof. \(\square\)

3. Main results

We start by presenting an upper bound for the symmetric diameter of a direct power of a non-abelian solvable group. Let \(G\) be a non-abelian solvable group. Since solvable groups have a derived series of finite length our strategy is to find a relation between the diameter of a solvable group and the diameter of its derived subgroup. For this we need to establish a relation between the generating sets of the group and the generating sets of its subgroups. The following lemma, well known as Schreier Lemma, gives a generating set for a subgroup of a group with respect to a generating
set of the whole group. The generators of the subgroup are usually called Schreier generators. Using Schreier generators we derive a relation between the diameter of a group and the diameter of its subgroup.

**Definition 3.1.** Let $H$ be a subgroup of a group $G$. By a right transversal for $G$ mod $H$, we mean a subset of $G$ which intersects every right coset $Hg$ in exactly one element.

**Remark 3.2.** Let $G$ be a finite group with a generating set $X$ and a normal subgroup $H$. It is easy to see that the set $HX = \{ Hx : x \in X \}$ is a generating set of $G/H$. Given an arbitrary element $Hg \in G/H$, $Hg$ can be written as a product of at most $D(G/H)$ elements in $XH$. Hence, there exist $x_1, x_2, \ldots, x_{D(G/H)} \in X$ such that $gH = x_1Hx_2H \ldots x_{D(G/H)}H = x_1x_2 \ldots x_{D(G/H)}H$. It shows that there always exists a right transversal $T$ for $G$ mod $H$ such that

$$Ml_X(T) \leq D(G/H), \ 1 \in T.$$ 

**Lemma 3.3.** $\{1\} \leq G = \langle X \rangle$ and let $T$ be a right transversal for $G$ mod $H$, with $1 \in T$. Then the set

$$\{ txt_{t_1}^{-1} | t, t_1 \in T, x \in X, txt_{t_1}^{-1} \in H \}$$

generates $H$.

Using Schreie’s Lemma leads to the following Lemma which is [4, Lemma 5.1].

**Lemma 3.4.** If $1 \neq N$, $N \triangleleft G$, then the following inequalities hold:

$$D^s(G) \leq 2D^s(G/N)D^s(N) + D^s(G/N) + D^s(N) \leq 4D^s(G/N)D^s(N).$$

Now we are ready to prove the first main theorem.

**Theorem 3.5.** If $G$ is a non-abelian solvable group then

$$D^s(G^n) = \frac{1}{4}(4n)!|G|,$$

where $l$ is the length of the derived series of $G$.

**Proof.** Let

$$\{1\} = G^{(l)} \triangleleft G^{(l-1)} \triangleleft \cdots \triangleleft G'' \triangleleft G' \triangleleft G$$

be the derived series of the group $G$. Since for $1 \leq i \leq l$ we have

$$(G^{(i)})^n = (G^n)^{(i)},$$

the series

$$\{1\} = (G^{(l)})^n \triangleleft (G^{(l-1)})^n \triangleleft \cdots \triangleleft (G'')^n \triangleleft (G')^n \triangleleft G^n$$

is the derived series of the group $G^n$. Using the second inequality in Lemma 3.4, the maximum of the diameter of the group $G^n$ is bounded above by

\begin{equation}
4^{l-1} D^s(G^n/(G^n)) D^s((G')^n/(G'')^n) \cdots D^s((G^{(l-2)})^n/(G^{(l-1)})^n) D^s((G^{(l-1)})^n).
\end{equation}

Whereas, for $1 \leq i \leq l - 1$ we have

\begin{equation}
(G^{(i)})^n/(G^{(i+1)})^n \cong (G^{(i)}/G^{(i+1)})^n
\end{equation}

and the factors in a derived series are abelian, by [11, theorem 3.2] we get

\begin{equation}
D^s(G^{(i)})^n/(G^{(i+1)})^n \leq n |G^{(i)}/G^{(i+1)}| = n |G^{(i)}|/|G^{(i+1)}|
\end{equation}

for $1 \leq i \leq l - 1$ and

\begin{equation}
D^s((G^{(l)})^n) \leq n |G^{(l)}|.
\end{equation}

Substituting the inequalities (4) and (5) in (3), we get

\begin{equation}
D^s(G^n)^{1/4}(|G|^{1/4})\leq 2(|X|+1)(|X|+1)\ln|G|,
\end{equation}

which is the desired conclusion. \hfill \Box

In 2006, Babai and Seress has been presented a relation between diameter and symmetric diameter of a finite group (See [1, Corollary 2.2]). We apply this relation with the theorem 3.5 to find an upper bound for the diameter of $G^n$, where $G$ is a $p$-group.

**Lemma 3.6.** Let $G$ be a finite group and $X$ be a set of generators. The diameter and the symmetric diameter are related as follows:

\begin{equation}
diam(G, X) \leq 2(\diam^s(G, X) + 1)(|X| + 1)\ln|G|.
\end{equation}

**Proof.** See [1, Corollary 2.2]. \hfill \Box

**Theorem 3.7.** Let $G$ be a solvable group of derived length $l$ and let $A$ be a generating set of $G^n$ of minimum size. Set rank($G$) = $\alpha$, rank($G/G'$) = $\beta$. The following inequality holds,

\begin{equation}
D(G^n, A) \leq 2(1/4)(4n)^l|G| + 1)(n\beta + 1)n \ln|G|,
\end{equation}

for $n \geq \alpha/\beta$. In particular, if $G$ a $p$-group, then

\begin{equation}
D(G^n) \leq 2(1/4)(4n)^l|G| + 1)(n\beta + 1)n \ln|G|,
\end{equation}

for $n \geq 1$.\hfill \Box
Proof. By Lemma 3.6 we have,
\[ \text{diam}(G^n, A) \leq 2(\text{diam}^s(G^n, A) + 1)(|A| + 1)n \ln |G|. \]
In addition, \( \text{diam}^s(G^n, A) \leq D^s(G^n) \) by definition. Now by using theorem 3.5 and Theorem 2.5 we get the desired conclusion. The second statement follows from these two facts: First, if \( G \) is a \( p \)-group then every minimal generating set is a generating set of minimum size, which follows from the Burnside’s Basis Theorem [7]. Second, by Corollary 2.7, if \( G \) is a nilpotent group (note that every \( p \)-group is nilpotent) then \( \text{rank}(G) = \text{rank}(G/G') \). □

Now we prove a non symmetric version of Shereier Lemma (Lemma 3.4). This Lemma is essential in the proof of our main theorem.

**Lemma 3.8.** Let \( G \) be a finite group with a generating set \( X \) and a normal subgroup \( H \). Let \( T \) be a right transversal of \( G/H \) such that
\[ \text{ML}_X(T) \leq D(G/H), \ 1 \in T. \]
The following inequality holds:
\[ \text{diam}(G, X) \leq D(G/H) + (D(G/H) + 1 + \text{ML}_X(\{t^{-1} \mid t \in T\}))D(H). \]
Furthermore, we have
\[ D(G^n) \leq D(G^n/H^n) + (1 + |G|D(G^n/H^n))D(H^n). \]

**Proof.** Given \( g \in G \), we have \( g = ht \) for some \( h \in H \) and \( t \in T \). Hence
\[ l_X(g) \leq l_X(t) + l_X(h). \]
Since \( \text{ML}_X(T) \leq D(G/H) \), then \( l_X(g) \leq D(G/H) + l_X(h) \). Using Lemma 3.3 we get \( l_X(h) \leq (D(G/H) + 1 + \text{ML}_X(\{t^{-1} \mid t \in T\}))D(H) \). Combining these two facts gives the upper bound in the first inequality. Now we prove the second statement. Let \( X' \) be a generating set of \( G^n \) and let \( T' \) be a right transversal of \( G^n/H^n \) such that
\[ \text{ML}_{X'}(T') \leq D(G^n/H^n). \]
Proceeding as above for the case \( n = 1 \), it suffices to show that
\[ \text{ML}_{X'}(\{t^{-1} \mid t \in T'\}) \leq (|G| - 1)D(G^n/H^n). \]
For given \( t \in T' \) we have
\[ l_{X'}(t) \leq D(G^n/H^n). \]
Since
\[ t^{-1} = t^{o(t)-1}, \]
then we obtain
\[ l_{X'}(t^{-1}) \leq (o(t) - 1)l_{X'}(t). \]
Hence, we have

\[ l_{X'}(t^{-1}) \leq (|G| - 1)D(G^n/H^n), \]

since

\[ o(g) \leq |G|, \]

for every element \( g \in G^n \). The proof is complete. \( \square \)

Now we are ready to prove our main theorem.

**Theorem 3.9.** Let \( G \) be a non-abelian solvable group. Let

\[ \{1\} = G^{(l)} \triangleleft G^{(l-1)} \triangleleft \ldots \triangleleft G'' \triangleleft G' \triangleleft G \]

be the derived series of \( G \). For \( n \geq 2 \), the following inequality holds:

\[ D(G^n) \leq n^l|G| \prod_{i=0}^{l-2} (|G^{(i)}| + 1). \]

**Proof.** Since \((G^k)' = (G')^k\) for \( k \geq 1 \), then the derived series of \( G^n \) is

\[ \{1\} = (G^{(l)})^n \triangleleft (G^{(l-1)})^n \triangleleft \ldots \triangleleft (G'')^n \triangleleft (G')^n \triangleleft G^n. \]

Applying Lemma 3.8 to the group \( G^n \) with the subgroup \((G')^n\) gives

\[ D(G^n) \leq D(G^n/(G')^n) + 1 + |G|D(G^n/(G')^n)D((G')^n) \]

\[ = D(G^n/(G')^n) + 1 + |G|D(G^n/(G')^n)D((G')^n) \]

\[ \leq D(G^n/(G')^n)D((G')^n) + |G|D(G^n/(G')^n)D((G')^n) \]

\[ = D(G^n/(G')^n)D((G')^n)(1 + |G|), \]

the second inequality follows from the fact that \( D(G^n/(G')^n), D((G')^n) > 1 \) and this is because the quotient group \( G/G' \) and the commutator subgroup \( G' \) are not trivial. By repeating the process for the other subgroups in the series (6) we have

\[ D(G^n) \leq D(G^n/(G')^n)D((G')^n/(G''^n)) \ldots D((G^{(l-1)})^n) \prod_{i=0}^{l-2} (|G^{(i)}| + 1). \]

1 Since for every group \( G \) with a normal subgroup \( H \) we have \( G^n/H^n \cong (G/H)^n \), then

\[ D(G^n) \leq D((G/G')^n)D((G'/G'')^n) \ldots D((G^{(l-1)})^n) \prod_{i=0}^{l-2} (|G^{(i)}| + 1). \]

All the quotient groups in the inequality (8) and the group \( G^{(l-1)} \) are abelian. On the other hand, for any abelian group \( A \) we have

\[ D(A^n) \leq n(|A| - \text{rank}(A)) \leq n|A| \]
(see [11, theorem 3.2]). Then we get

\[ D(G^n) \leq nD(G/G')D(G'/G'') \cdots \]

\[ D(G^{(l-2)}/G^{(l-1)})D((G^{(l-1)}) \prod_{i=0}^{l-2}(|G^{(i)}| + 1)) \]

\[ \leq n|G/G'||G'/G''| \cdots |G^{(l-2)}/G^{(l-1)}||G^{(l-1)}| \prod_{i=0}^{l-2}(|G^{(i)}| + 1) \]

\[ = n|G| \prod_{i=0}^{l-2}(|G^{(i)}| + 1). \]

□

Note that the upper bound presented in theorem 3.7 is just satisfied for \( p \)-groups. While, the upper bound presented in theorem 3.9 not only is better, but also is satisfied for all non-abelian solvable groups.

As an example of a non-abelian solvable group which is also a 2-group we justify the upper bounds in theorems 3.7 and 3.9 for quaternion group \( Q_8 \). Let \( Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \} \) be the quaternion group in which

\[ i^2 = j^2 = k^2 = -1 \]

and

\[ ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j. \]

We have \( Q'_8 \cong Z_2 \) and \( Q_8/Q'_8 \cong Z_2 \times Z_2 \). The length of the derived series of \( Q_8 \) is 2. Hence, \( l = 2 \) and \( \beta = \text{rank}(Z_2 \times Z_2) = 2 \) in the notations of theorems 3.7 and 3.9. Therefore by theorem 3.7 we have

\[ D(Q_8^n) \leq 2n(32n^2 + 1)(2n + 1)ln(8), \]

and by theorem 3.9 we have

\[ D(Q_8^n) \leq 72n^2. \]

We now present a better upper bound for the diameter of the direct power of the quaternion group \( Q_8 \) by using Lemma 3.8 directly.

**Example 3.10.** For \( n \geq 1 \) we have \( D(Q_8^n) \leq 8n^2 + 3n. \)

**Proof.** Consider the normal subgroup \( H = \{1, -1\} \). Let \( X \) be a generating set of \( Q_8^n \). We have \( H^n \triangleleft Q_8^n \). Let \( T \) be a right transversal of \( Q_8^n \) mod \( H^n \) such that

\[ 1 \in T, \text{ML}_X(T \setminus \{1\}) \leq D(Q_8^n/H^n). \]

Using Lemma 3.8 we have

\[ \text{diam}(Q_8^n, X) \leq D(Q_8^n/H^n) + (D(Q_8^n/H^n) + 1 + \text{ML}_X(\{t^{-1} | t \in T\}))D(H^n). \]
On the other hand, since $H \cong Z_2$, $Q_8/H \cong Z_2 \times Z_2$, we have
\begin{equation}
(9) \quad \text{diam}(Q_8^n, X) \leq 2n + (2n + 1 + ML_X(\{t^{-1} \mid t \in T\})n).
\end{equation}
Since for every $g \in Q_8^n$, $g^4 = 1$, for every $t \in T$, $t^{-1} = t^3$. Hence, the following inequality holds:
\[l_X(t^{-1}) \leq 3l_X(t) \leq 3D(Q_8^n/H^n) \leq 6n.\]
Substituting $6n$ for $ML_X(\{t^{-1} \mid t \in T\}$ in (9) we get
\[D(Q_8^n) \leq 8n^2 + 3n. \quad \square\]

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