THE PRESCRIBED $Q$-CURVATURE FLOW FOR ARBITRARY EVEN DIMENSION IN A CRITICAL CASE

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ABSTRACT. In this paper, we study the prescribed $Q$-curvature flow equation on an arbitrary even dimensional closed Riemannian manifold $(M, g)$, which was introduced by S. Brendle in [3], where he proved the flow exists for long time and converges at infinity if the GJMS operator is weakly positive with trivial kernel and $\int_M Qd\mu < (n-1)!\text{Vol}(S^n)$. In this paper we study the critical case that $\int_M Qd\mu = (n-1)!\text{Vol}(S^n)$, we will prove the convergence of the flow under some geometric hypothesis. In particular, this gives a new proof of Li-Li-Liu’s existence result in [25] in dimension 4 and extend the work of Li-Zhu [26] in dimension 2 to general even dimensions. In the proof, we give a explicit expression of the limit of the corresponding energy functional when the blow up occurs.

1. INTRODUCTION

Consider a compact four dimensional manifold $(M, g)$. The $Q$-curvature is defined by

$Q = \frac{-1}{6} (\Delta R + R^2 - 3|\text{Ric}|^2),$

where $R$ denotes the scalar curvature and $\text{Ric}$ denotes the Ricci tensor. Under a conformal change of the metric $\tilde{g} = e^{2u}g$, the following equation holds:

$\tilde{Q} = e^{-4u}(Pu + Q), \quad (1.1)$

where $P$ denotes the Paneitz operator defined by

$Pu = \Delta^2 u + \text{div} \left( \frac{2}{3} Rg - 2\text{Ric} \right) \nabla u.$

The Paneitz operator is conformally covariant in the sense that if $\tilde{g} = e^{2u}g$, then the Paneitz operator with respect to $\tilde{g}$ is given by $\tilde{P} = e^{-4u}P$. The Gauss-Bonnet-Chern theorem asserts that

$\int_M Qd\mu + \frac{1}{4} \int_M |W|^2 d\mu = 8\pi^2 \chi(M), \quad (1.2)$

where $\chi(M)$ denotes the Euler characteristic of $M$, the letter $W$ denotes the Weyl curvature tensor of $g$ and $d\mu$ denotes the volume element of $g$. Since the Weyl curvature tensor is conformally invariant, it follows that

$\int_M Qd\mu = \int_M \tilde{Q}d\tilde{\mu}. \quad (1.1)$

We note here that the invariance (1.2) is also a direct consequence of equation (1.1).

So $Q$ and $P$ are the four dimensional analogues to the Gaussian curvature $K$ and the Laplace-Beltrami operator $-\Delta$ in the two dimensional case. We denote $Q = K$ and $P = -\Delta$ in two dimensional for simplicity.

In [22], it was introduced a family of conformally covariant differential operators on general even dimensional manifolds, whose leading term is $(-\Delta)^2$, were $n$ is the dimension of the manifold. These
operators are usually referred to as the GJMS operators. We denote them also by $P$. In [11], the corresponding $Q$-curvature was defined.

Furthermore as for the Laplace–Beltrami operator on compact surfaces and the Paneitz operator on compact four-dimensional manifolds, for every compact $n$-dimensional manifolds with $n$ even, we have that after a conformal change of metric $\tilde{g} = e^{2u}g$.

$$P = e^{-nu}P \quad Q = e^{-nu}(Pu + Q).$$

An important problem in conformal geometry is the construction of a conformal metric $\tilde{g} = e^{2u}g$ for which the $Q$-curvature $Q$ equals to a constant multiple of a prescribed function $f$. This is related to solve the equation

$$Pu + Q = lfe^{nu}. \quad (1.3)$$

From the equation (1.2), we know that $l = \frac{\tilde{Q}}{\tilde{f}}$, where

$$\tilde{Q} = \int_M Qd\mu; \quad \tilde{f} = \int_M f\tilde{d}\mu = \int_M fe^{nu}d\mu.$$ 

The equation (1.3) is the Euler-Lagrange equation of the functional

$$E[u] = \frac{n}{2} \int_M uPd\mu + n \int_M Qd\mu - \tilde{Q} \log\left(\int_M fe^{nu}d\mu\right). \quad (1.4)$$

S.-Y. Chang and P. C. Yang [10] first studied the equation (1.3) in the case $n = 4$ by minimizing the functional $E$. They constructed conformal metrics of constant $Q$-curvature when the Paneitz operator $P$ is weakly positive with trivial kernel and the total $Q$-curvature satisfies $\tilde{Q} < 16\pi^2$. The Paneitz operator $P$ is weakly positive means that for all $u \in C^\infty(M)$, $\int_M uPd\mu \geq 0$, and $P$ has trivial kernel means that its kernel consists only of constant functions. In view of the result of M. Gursky [20], $P$ is positive with trivial kernel whenever $\tilde{Q} > 0$ and the Yamabe constant of $(M, g)$ is positive. The key point in S.-Y. Chang and P. C. Yang’s proof is the following: Using Adams-Fontana inequality (see Proposition 4.3 below), supposing $P$ is weakly positive with trivial kernel and $\tilde{Q} < 16\pi^2$, they deduced that the functional $E$ is bounded from below and coercive. Then the solutions of equation (1.3) can be found as global minima of $E$.

Later, using the heat flow method, S. Brendle [3] extended S.-Y. Chang and P. C. Yang’s result to higher dimensions and to $f > 0$ which need not be constants. We now sketch S. Brendle’s proof in the four dimension case. First he introduced the flow equation:

$$\begin{cases}
\frac{\partial u}{\partial t} = -e^{-nu}(Pu + Q) + \frac{\tilde{Q}}{\tilde{f}}f \\
u(x, 0) = u_0(x) \in C^\infty(M),
\end{cases} \quad (1.5)$$ 

where $\tilde{Q} = \int_M Qd\mu$ and $\tilde{f}(t) = \int_M fe^{nu(t)}d\mu$. It is the negative gradient flow of the functional $E$ with respect to the inner product $\int_M (-, -)e^{nu(t)}d\mu$. One advantage of this flow is that it keeps the total volume constant, that is, $\int_M e^{nu(t)}d\mu \equiv \text{constant}$ along this flow. Then he derived various differential inequalities from the Gagliardo-Nirenberg inequality. Combining these inequalities and a stability argument, Brendle obtained that (notice that we state the result of S. Brendle in a form more convenient for us):

**Theorem 1.1.** Assume that the GJMS operator $P$ is weakly positive with trivial kernel, and $u(t)$ is the solution of the equation (1.5). Then:

(a) If we also assume that $\sup_{t \in [0, T_0]} \|u(t)\|_{W^{2, 4}(M)} \leq C(g, f, T_0) < +\infty$ for any $T_0 \in [0, T)$, where $T$ is the maximal existence time. Then $T = +\infty$, that is, $u(t)$ exists long time.

(b) If we also assume that $\sup_{t \in [0, T_0]} \|u(t)\|_{W^{2, 4}(M)} \leq C(g, f) < +\infty$ for any $T_0 \in [0, T)$, where $T$ is the maximal existence time and $C(g, f)$ is a constant which is independent of $T_0$. Then $u(t)$ exists long time and $u(t)$ converges smoothly to a smooth function $u_\infty$ satisfying the equation (1.3).
Then by using the coerciveness, he concluded that the condition of part (b) in above proposition is satisfied when \( \int_M Q d\mu < (n - 1)! \gamma_n \). Here we use \( \gamma_n \) to denote the volume of the standard round \( n \)-sphere. As a consequence, in the case that \( \int_M Q d\mu < (n - 1)! \gamma_n \), he proved long time existence and convergence of the flow equation (1.5) and the limit at infinity is a solution of the equation (1.3). In this way, S. Brendle generalized S.-Y. Chang and P. C. Yang’ existence result. For more information about convergence of the flow equation (1.5) and the limit at infinity is a solution of the equation (1.3). In this paper we study the flow equation (1.5) in the critical case for general Riemannian manifolds \( M \). Our results can apply to the mean field type equations. Notice that the new feature of our results is that we can show \( \lim_{t \to +\infty} E[u(t)] = \Lambda(g, \bar{Q}, p) \) for the alternative (b).

Theorem 1.2. Let \( (M, g) \) be a closed Riemannian manifold of even dimension \( n \geq 2 \), with \( \bar{Q} = (n - 1)! \gamma_n \). Suppose the GJMS operator \( P \) is weakly positive with trivial kernel and \( f > 0 \). Assume \( u(t) \) is the solution of the parabolic equation:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= -e^{-mu}(Pu + \bar{Q}) + \bar{f}f \\
u(x, 0) &= u_0(x) \in C^\infty(M),
\end{aligned}
\]

where \( \bar{Q} = \int_M Q d\mu \) and \( \bar{f}(t) = \int_M f e^{mu(t)} d\mu \). Then \( u(t) \) exists for long time and one of the following alternatives holds:

(a) either \( u(t) \) converges smoothly to a smooth function \( u_\infty \) satisfying the equation (1.3),
(b) or

\[
\lim_{t \to +\infty} E[u(t)] = -(n - 1)! \gamma_n \log \frac{n! \gamma_n f(p)}{2^n} - \frac{n! \gamma_n}{2} \left( H_p(p) + \frac{C_{log}}{C_0} \right) + \frac{n}{2} \int_M QG_p d\mu
\]

for some \( p \in M \), where \( \frac{C_{log}}{C_0} \) is a uniform constant, \( G_p \) is the Green’s function with a pole at \( p \), and the \( H_p \) is the regular part of the Green’s function as in Proposition 3.7.

Theorem 1.3. Under the assumption of Theorem 1.2 if

\[
\left( \Delta \left( H_p + \frac{1}{n} \log f \right) + n \left\| \nabla \left( H_p + \frac{1}{n} \log f \right) \right\|^2 - \frac{R_g}{6(n - 1)} \right)(p) > 0
\]

for some \( p \in M \), then \( \lim_{t \to +\infty} E[u(t)] = \Lambda(g, f, p) \). Even in the two dimension, only \( \lim_{t \to +\infty} E[u(t)] \geq \Lambda(g, f, p) \) has been known, see J. Li and C. Zhu in [26].

We can show the convergence of this flow with arbitrary initial data under some geometric conditions in the critical case. To the authors knowledge, similar results has only obtained when \( M \) is the standard \( S^n \) ( c.f. [6], [23] ).
for all \( p \in M \), and

\[
\Lambda(g, f, p) \equiv \Lambda
\]

for some constant \( \Lambda \) as a function of \( p \), then for arbitrary initial data \( u_0 \in C^\infty(M) \), \( u(t) \) exists for long time and converges smoothly to a smooth function \( u_\infty \) satisfying the equation (1.3).

Remark 1.1. When \( M \) is the two dimensional flat torus \( T^2 \), \( \bar{\mathcal{Q}} = 4\pi \) and \( f \equiv 1 \), using (3.6), it is easy to check the conditions in Theorem 1.3 are satisfied. This provides an interesting analogue of the Ricci flow on \( S^2 \).

We also have the convergence of (1.5) with small arbitrary initial data:

**Theorem 1.4.** Under the assumption of Theorem 1.2, if

\[
\left( \Delta \left( H_{p_0} + \frac{1}{n} \log f \right) + n \left\| \nabla \left( H_{p_0} + \frac{1}{n} \log f \right) \right\|^2 - \frac{R_g}{6(n-1)} \right)(p_0) > 0
\]

for some \( p_0 \) where \( \Lambda(g, f, p) \) achieves its global minimum, then there exists an initial data \( u_0 \in C^\infty(M) \) such that \( u(t) \) exists for long time and converges smoothly to a smooth function \( u_\infty \) satisfying the equation (1.3).

Theorem 1.4 extends the main result in [26] to general even dimensions and provides a new proof of the existence result of J. Li et al. (Theorem 1.2 in [25]).

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2. Preliminaries

In this section we collect some useful preliminary facts and analyze the volume concentration phenomena of the equation (1.5).

Actually we study the following perturbation equation:

\[
\begin{cases}
Pu + Q = (lf + h)e^{\alpha u}, \\
\int_M f e^{\alpha u} d\mu = \int_M Q d\mu = (n-1)! \gamma_n,
\end{cases}
\]

where \( l > 0 \) is a positive real number, \( f \in C^\infty(M) \) is a positive real function and \( h \) is a real function with

\[
E_\alpha = \int_M h e^{\alpha u} d\mu < \infty
\]

for some \( \alpha \in (1, \infty) \).

Notice that the equation (1.5) is a special case of (2.1) with \( h = \frac{\partial u}{\partial t} \).

We care about the behavior of \( u \) as \( E_\alpha \) tends to zero.

To study the equation (2.1), we begin with two propositions.

The following proposition concerning solutions of (2.1) is proved in [30] by using the asymptotic behavior of the Green’s function near the pole, see also [15], [18].

**Proposition 2.1.** Let \( (u, F) \in W^{n,1}(M) \times L^1(M) \) satisfy

\[
Pu = F
\]

with \( \|F\|_{L^1(M)} \leq K \) for some constant \( K \). Then for any \( x \in M \), for any \( r > 0 \), for any \( j \in [1, n-1] \cap \mathbb{N} \) and \( q \in [1, n/j) \), we have

\[
\int_{B_r(x)} |\nabla u|^q d\mu \leq C r^{n-j q},
\]

where \( C \) is a positive constant depending on \( K, M, q \).

Furthermore, we need the following proposition proved in [30]:

...
Proposition 2.2. Let \((u_k, F_k) \in W^{m,1}(M) \times L^1(M)\) satisfy
\[ Pu_k = F_k \]
with \(\|F_k\|_{L^1(M)} \leq K\) for some constant \(K\) independent of \(k\). Then:
(a) either
\[ \int_M e^{q(u_k - \bar{u}_k)} \, d\mu \leq C \]
for some \(q > 1\) and some \(C = C(n, K, q) > 0\),
(b) or there exists a point \(x \in M\) such that for any \(r > 0\), we have
\[ \liminf_{k \to +\infty} \int_{B_r(x)} |F_k| \, d\mu \geq \frac{1}{2} (n - 1)! \gamma_n. \]

Remark 2.1. Proposition 2.2 remains valid if one replaces the metric \(g\) on \(M\) by a family of metrics \((g_k)\) depending on \(k\) which is uniformly bounded in \(C^m(M)\) for any \(m \in \mathbb{N}\). The same result also holds if we replace \(M\) by any precompact open set in a noncompact manifold and assuming all the functions with compact support in this open set.

Using above two propositions, we have following estimate independent of specific form of \(f\) and \(h\):

Proposition 2.3. Let \(U\) be a precompact open set in \(M\), and \(u\) be the solution of the equation \((2.1)\). There is a uniform \(A \in \mathbb{R}\) such that when \(E_\alpha \leq A\), we have the following:
If there is some \(s > 0\), such that \(\int_{B_s(x)} |f e^{nu} d\mu | \leq \frac{1}{8} (n - 1)! \gamma_n\) for every \(x \in U\), then for any compact subset \(K\) of \(U\), we have
\[ \|u - a_U\|_{W^{\alpha,1}(K)} \leq C, \]
where \(C = C(K, U, n, \alpha, t)\) is a constant and \(a_U = \int_U u \, d\mu\).

Proof. Choose precompact open sets \(V\) and \(W\) satisfying
\[ K \subset W, \quad \overline{W} \subset V, \quad \overline{V} \subset U. \]
and a smooth cut-off function \(\eta\) satisfying
\[ \eta(x) = \begin{cases} 1 & x \in \overline{V}; \\ 0 & x \notin U. \end{cases} \]
We also set \(w = \eta(u - a_U)\). By Proposition 2.1, the precompactness of \(U\) and the boundness of \(l\) one finds
\[ \int_U |\nabla^l u|^q \, d\mu \leq C_U \quad q \in [1, n/j], \quad j \in [1, n - 1] \cap \mathbb{N} \]
and hence by the Poincaré inequality (recall that \(w\) has a uniform compact support and \((u - a_U)\) has mean value 0 on \(U\)) it follows that
\[ \|w\|_{W^{1,q}(U)} \leq C_U \quad q \in \left(1, \frac{n}{n - 1}\right), \quad (2.2) \]
and
\[ \|u - a_U\|_{W^{1,q}(U)} \leq C_U \quad q \in \left(1, \frac{n}{n - 1}\right). \quad (2.3) \]
By equation (2.1) there holds

\[ P_w = (-\Delta)^{n/2} w + O \left( \sum_{j=0}^{n-1} |\nabla^j w| \right) \]

\[ = \eta (-\Delta)^{n/2} u + O \left( \sum_{j=0}^{n-1} \left( |\nabla^j w| + |\nabla^j (u - a_U)| \right) \right) \]

\[ = \eta (lf + h) e^{nu} + O \left( \sum_{j=0}^{n-1} \left( |\nabla^j w| + |\nabla^j (u - a_U)| \right) \right) \]

\[ = \eta (lf + h) e^{nu} + G, \tag{2.4} \]

Using the equation (2.2) and (2.3), we can show

\[ ||G||_{L^q(U)} \leq C_U \quad q \in \left[ 1, \frac{n}{n-1} \right]. \]

Then by the Hölder inequality we can find \( s_1 > 0 \) such that

\[ ||G||_{L^1(B_{s_1}(x))} \leq \frac{1}{8} (n-1)! \gamma_n \quad x \in U. \]

Since \( \int_M |h e^{nu} d\mu| \leq \left( \int_M h^n e^{nu} d\mu \right)^{\frac{1}{n+1}} \left( \int_M e^{nq_1 u} d\mu \right)^{\frac{n}{n+1}} \), we can find \( A \in \mathbb{R} \) such that

\[ \int_M |h e^{nu} d\mu| \leq \frac{1}{8} (n-1)!, \]

for \( E_a \leq A \). Then the second alternative in Proposition 2.2 cannot hold. As a consequence of Remark 2.1:

\[ \int_U e^{nu} d\mu \leq C_U \quad \text{for some} \quad q_1 > 1. \]

On the other hand, using the Jensen’s formula, one can show

\[ e^{nu} \leq C_U \int_U e^{nu} d\mu \leq C_U. \]

Since \( w = u - a_U \) on \( V \), we have

\[ \int_V e^{nu} d\mu \leq e^{nu} \int_V e^{nu} d\mu \leq C_V. \]

Choosing \( q_2 = \frac{aq_1}{n+q_1-1} \), we can use the Hölder inequality to deduce that:

\[ \int_V (h e^{nu})^{q_2} d\mu \leq \left( \int_V h^n e^{nu} d\mu \right)^{\frac{n}{q_2(n-1)}} \left( \int_V e^{nu} d\mu \right)^{\frac{n}{n+1}} \leq C. \]

Thus the right hand of the equation (2.4) has a uniform \( L^\infty(V) \) bound. From standard elliptic estimates, we conclude

\[ ||w||_{W^{n/2}(V)} \leq C_W, \]

where \( C_W \) is a uniform constant. Using the Sobolev inequality we know \( w \) is pointwise bounded in \( W \). Then we can check that the right hand of the equation (2.4) has a uniform \( L^\infty(W) \) bound. Using elliptic estimates again, we can get

\[ ||u - a_U||_{W^{n/2}(K)} = ||w||_{W^{n/2}(K)} \leq C \]

for some uniform constant \( C \).

\[ \square \]

Observing the scaling invariance of \( L^\infty \) norm, we have the following corollary:
Corollary 2.4. Let $B_r(x)$ be a geodesic ball in $M$, and $u$ be the solution of the equation (2.1). There is a uniform $A \in \mathbb{R}$ such that when $E_\alpha \leq A$, we have the following:

If $\int_{B_r(x)} |f e^{mu} d\mu| \leq \frac{1}{8} (n-1)! \gamma_n$, then we have

$$\|u - a_{B_\delta(x)}\|_{L^\infty(B_\delta(x))} \leq C,$$

where $C = C(M, n, \alpha)$ is a constant and $a_{B_\delta(x)} = \frac{1}{\delta^n} \int_{B_\delta(x)} u d\mu$.

Inspired by Proposition 2.3, we give the following definition:

Definition 2.5. Let $u$ is a solution of the equation (2.1), we define the volume concentration radius function $s(x)$ to be the unique real function such that:

$$\int_{B_{s(x)}(x)} |f e^{mu} d\mu| = \frac{1}{8} (n-1)! \gamma_n. $$

Notice that $s(x)$ is a continous function of $x$ and it measure the scale of volume concentration. In the case that

$$\inf_{x \in M} s(x) \geq s$$

for some $s > 0$, we can apply Proposition 2.3 to the case $U = K = M$, thus we get

$$\|u - \tilde{u}\|_{W^{n,\alpha}(M)} \leq C$$

for some constant $C = C(n, \alpha, s)$ . In view of the Sobolev embedding theorem, we know $|u - \tilde{u}| \leq C$ pointwisely for some uniform constant $C$. Denoting $\int_M e^{mu} d\mu$ by $V$, we have

$$V = \int_M e^{\tilde{u}} e^{mu-\tilde{u}} d\mu \leq C \int_M e^{\tilde{\mu}} d\mu = C e^{\tilde{\mu}}.$$

On the other hand, by the Jensen’s formula, we know

$$e^{\tilde{\mu}} \leq C \int_M e^{mu} d\mu = CV.$$

Thus $|\tilde{u}| \leq C$ for some uniform $C$. Combining with $\|u - \tilde{u}\|_{W^{n,\alpha}(M)} \leq C$, we can conclude $\|u\|_{W^{n,\alpha}(M)} \leq C$ for some uniform $C$.

So when $s(x)$ has a uniform nozero lower bound, $u$ a has uniform $W^{n,\alpha}$ bound.

Now we care about the case where $\inf_{x \in M} s(x)$ is closed to 0.

In this case, we need to describe the behavior of $u$ locally. And we shall rescale $u$ first.

For any given pair $(p, s) \in M \times \mathbb{R}^+$, we set

$$\hat{u}_{p,s}(x) = u(exp_p(sx)) + \log s, \quad x \in B_{\delta/s}(0),$$

where $B_{\delta/s}(0) \subset \mathbb{R}^n$ is the Euclidean ball of center 0 and radius $\delta/s$, and $\delta$ is a positive number less than the injectivity radius of $M$. We note that the ball $B_{\delta/s}(0)$ approaches $\mathbb{R}^n$ when $s \to 0$. Let $T_{p,s}: B_{\delta/s}(0) \to M$ be defined by $T_{p,s}(x) = exp_p(sx)$, and define a metric on $B_{\delta/s}(0)$ by

$$g_{p,s} = s^{-2} T^* g.$$

Then $g_{p,s} \to g_0$ in $C^0(B_0(0))$ as $s \to 0$ for all $m \in \mathbb{N}$ and all $R > 0$, where $g_0$ is the standard Euclidean metric of $\mathbb{R}^n$. An easy calculation shows that $\hat{u}_{p,s}$ satisfies the following equation:

$$P_{g_{p,s}} \hat{u}_{p,s} + s^a \hat{Q}_{p,s} = (\hat{f}_{p,s} + \hat{h}_{p,s}) e^{\tilde{\mu}_{p,s}},$$

where $P_{g_{p,s}}$ is the GJMS operator of the metric $g_{p,s}$ in $B_{\delta/s}(0)$ and

$$\hat{Q}_{p,s}(x) = Q(exp_p(sx)), \quad x \in B_{\delta/s}(0),$$

$$\hat{f}_{p,s}(x) = f(exp_p(sx)), \quad x \in B_{\delta/s}(0),$$

$$\hat{h}_{p,s}(x) = h(exp_p(sx)), \quad x \in B_{\delta/s}(0).$$
and we also have \( \int_{B_0(0)} \tilde{h}^m \mu e^{\tilde{u}_v} \, d\mu = \int_{B(p)} h^m \mu e^{u} \, d\mu \) from a direct calculation, where \( d\mu \) is the volume element associated to \( g_{p,v} \).

Noticing that \( \int_{B_0(0)} e^{\tilde{u}_v} \, d\mu = \int_{B(p)} e^{u} \, d\mu \), we have

\[
\int_{B_1(x)} l_f \tilde{h}^m e^{\tilde{u}_v} \, d\mu \leq \frac{1}{8} (n-1)! \gamma_n.
\]

If we choose \( p \) as one of the global minimum point of \( s(x) \), and \( s = s(x) \), we can also have

\[
\int_{B_1(x)} l_f \tilde{h}^m e^{\tilde{u}_v} \, d\mu \leq \frac{1}{8} (n-1)! \gamma_n, \quad x \in B_{\alpha/4}(0)
\]

We have the following compactness result on \( \tilde{u}_{p,v} \):

**Proposition 2.6.** Assume \( u \) is a solution of the equation (2.1), \( p \) is one of the global minimum point of Proposition 2.6.

We also have

\[
\| \tilde{u}_{p,v} - U_{s_0,\varepsilon} \|_{W^{s,v}(B_{\alpha/4}(0))} \leq J(E_a, s),
\]

where the function \( U_{s_0,\varepsilon} \) is given by

\[
U_{s_0,\varepsilon} = -\log(1 + \frac{J^2}{\varepsilon^3} |x - x_0|^2) - \log \varepsilon, \quad \lambda = \left( \frac{l_f(p)}{2^n(n-1)!} \right)^\frac{1}{2},
\]

\( J \) and \( L \) are positive numbers satisfy

\[
\lim_{E_a,\varepsilon \to 0} J(E_a, s) = 0 \quad \lim_{E_a,\varepsilon \to 0} L(E_a, s) = \infty.
\]

Moreover, there exists a positive real number \( J_1(M, E_a, s) \), such that:

\[
\left| \int_{B_{\alpha/4}(p)} l f e^{u} \, d\mu - (n-1)! \gamma_n \right| \leq J_1(M, E_a, s),
\]

where \( J_1(M, E_a, s) \) is a positive real number satisfies

\[
\lim_{M \to \infty} \lim_{E_a,\varepsilon \to 0} J_1(M, E_a, s) = 0.
\]

We also have \( \varepsilon \leq c(n) \) for some uniform constant \( c(n) > 0 \), when \( E_a \) and \( s \) are both small enough.

**Proof.** We argue by contradiction. Assume there exists a sequence of function \( \{u_k\}_{k \in \mathbb{N}} \) satisfy:

\[
\begin{align*}
Pu_k + Q &= (l_k f_k + h_k) e^{u_k}, \\
\int_{
} h_k f_k e^{u} \, d\mu &= \int_{
} Qd\mu = (n-1)! \gamma_n, \\
\lim_{k \to \infty} E_{a,k} &= \lim_{k \to \infty} \int_{
} g_k^a e^{u} \, d\mu = 0,
\end{align*}
\]

and we denote the volume concentration radius function of \( u_k \) by \( s_k(x) \). Then we can choose \( p_k \) as a minimum point of \( s_k(x) \) and let \( \delta_k = s_k(x_k) \). We write \( \tilde{u}_k = \tilde{u}_{p_k,\varepsilon_k} \) as the rescaled function of \( u_k \).

We now assume there exist a \( L_1 > 0 \) and \( J_1 > 0 \) such that

\[
\| \tilde{u}_k - U_{s_0,\varepsilon} \|_{W^{s,v}(B_{\alpha/4}(0))} \geq J_1,
\]

for any \( k, \lambda_0 \) and \( \varepsilon \), since otherwise (2.8) will hold for all solution \( u \) of the equation (2.1).

By Proposition 2.1 and some scaling argument one finds

\[
\int_{B_{\alpha/4}(0)} |\nabla \tilde{u}_k|^q \, d\mu \leq C_U \quad q \in [1, n/j], \quad j \in [1, n-1] \cap \mathbb{N}
\]

for any \( R > 0 \). Then arguing as in Proposition 2.3 we obtain \( \| \tilde{u}_k - \tilde{u}_{p_k,\varepsilon_k} \|_{W^{s,v}(B_{\alpha/4}(0))} \leq C_R \), where \( C_R \) is independent of \( k \) and \( \tilde{u}_{p_k,\varepsilon_k} = \int_{B_{\alpha/4}(0)} \tilde{u}_k \, d\mu \).
From the Jensen’s inequality, we can deduce

\[
e^{\hat{u}_k \cdot \nu_0(0)} \leq C_R \int_{B_R(0)} e^{\hat{u}_k} d\mu_{\hat{u}_k} \leq C_R.
\]

By a direct calculation, we have that

\[
\frac{1}{8}(n-1)! \gamma_k \leq \int_{B_R(0)} e^{\hat{u}_k} e^{\nu - \hat{u}_k \cdot \nu_0(0)} d\mu_{\hat{u}_k} \leq C_R \int_{B_R(0)} e^{\hat{u}_k} d\mu_{\hat{u}_k} = C_R e^{\hat{u}_k \cdot \nu_0(0)}.
\]

Then we can conclude \(\|\hat{u}_k\|_{W^{n,n}(B_R(0))} \leq C_R\) for some \(C_R\) independent of \(k\). So after passing to a subsequence, we can assume \(\hat{u}_k \to \hat{u}_\infty\) in the local weak \(W^{n,n}\) topology, where \(\hat{u}_\infty\) satisfies the equation

\[
\begin{cases}
(-\Delta)^{\frac{n}{2}} \hat{u}_\infty = l_{\infty} f(p) e^{\hat{u}_\infty} \\
\int_{\mathbb{R}^n} e^{\hat{u}_\infty} dx \leq V.
\end{cases}
\]  

(2.12)

Moreover, using the local \(W^{n,n}\) bound we can rewrite the equation (2.6) as the following:

\[
(-\Delta)^{\frac{n}{2}} \hat{u}_k = l_{\infty} f(p) e^{\hat{u}_k} + H_k,
\]

where \(H_k\) satisfying \(\lim_{k \to +\infty} \int_{B_R(0)} H_k^2 d\mu_0 = 0\) for any \(R > 0\). As a consequence, we can improve the local weak convergence of \(\hat{u}_k\) to the local strong convergence by invoking elliptic estimates.

The solutions of the equation (2.12) have been classified in [40], [27] and [33] which tells us \(\hat{u}_\infty\) has the form

\[
\hat{u}_\infty = -\log(1 + \frac{\lambda^2_n}{\epsilon^2} |x - x_0|^2) - \log \epsilon, \quad \lambda = \left(\frac{l_{\infty} f(p)}{2^n (n-1)!}\right)^{\frac{1}{n}}.
\]

That is, \(\hat{u}_\infty = U_{\lambda, \epsilon, x_0}\) for some \(x_0\) and \(\epsilon\). Since \(\hat{u}_k\) converges to \(\hat{u}_\infty\) strongly in \(W^{n,n}_{\text{local}}(\mathbb{R}^n)\). And this is contract to our assumption. So we finish the proof of (2.8).

By a simple calculation we have \(\int_{\mathbb{R}^n} l_{\infty} f(p) e^{\hat{u}_\infty} dx = (n-1)! \gamma_k\). As a consequence, the inequality (2.10) follows from (2.8). The last assertion of this proposition comes from the inequality (2.7). \(\square\)

**Corollary 2.7.** Assume \(u\) is a solution of the equation (2.7). Then for \(E_\alpha\) and \(s_{\min} = \inf_{x \in M} s(\chi)\) are both small enough (that is, smaller than a given small real number), there exist \(s > 0\) and \(p \in M\) such that,

\[
\|\hat{u}_{p, s} - U_{0,1}(B_{s \gamma_n} (\mathbb{R}^n))\|_{W^{n,n}(B_{s \gamma_n}(0))} \leq J(E_\alpha, s_{\min}),
\]  

(2.13)

where the function \(\hat{u}_{0,1}\) is given by

\[
U_{0,1} = -\log(1 + \lambda^2 |x - x_0|^2), \quad \lambda = \left(\frac{l_{\infty} f(p)}{2^n(n-1)!}\right)^{\frac{1}{n}},
\]  

(2.14)

and \(J, L\) are positive numbers satisfy

\[
\lim_{E_\alpha, s_{\min} \to 0} J(E_\alpha, s_{\min}) = 0, \quad \lim_{E_\alpha, s_{\min} \to 0} L(E_\alpha, s_{\min}) = \infty.
\]

Moreover, there exists a positive real number \(J_1(M, E_\alpha, s_{\min})\), such that:

\[
\left|\int_{B_M(p)} l f e^{\hat{u}_\infty} d\mu - (n-1)! \gamma_n\right| \leq J_1(M, E_\alpha, s_{\min}),
\]  

(2.15)

where \(J_1(M, E_\alpha, s_{\min})\) is a positive real number satisfies

\[
\lim_{M \to \infty} \lim_{E_\alpha, s_{\min} \to 0} J_1(M, E_\alpha, s_{\min}) = 0.
\]

We also have \(s_{\min} \leq s \leq c(n)s_{\min}\).
Proof. We denote the $p$ and $s$ in Proposition 2.6 by $p_{\min}$ and $s_{\min}$, and choose
\[ p = \exp_{p_{\min}}(x_0) \quad s = s_{s_{\min}}. \]
Then this corollary follows easily. \[ \square \]

From now we fix $p$ and $s$ such that they fulfill the conclusion of Corollary 2.7.
We now describe the behavior of $u$ in $M \setminus \{p\}$ when $E_a$ and $s$ are both tend to 0.

Proposition 2.8. Assume $u$ is a solution of the equation (2.1) and $E_a$, $s$ as above. Then for $E_a$ and $s$ are both small enough (that is, smaller than a given small real number), we have:
\[ \|u - \bar{u} - G_p\|_{W^{\alpha}(M \setminus B_{E_a, \alpha}(p))} \leq J(E_a, s), \]
where $G_p$ is the Green’s function satisfying
\[ \begin{cases} 
PG_p + Q = (n - 1)!\gamma_0 \delta_p \\
\int_M G_p d\mu = 0.
\end{cases} \]
in the distribution sense. And $\delta(E_a, s)$ and $J(E_a, s)$ are positive real numbers such that
\[ \lim_{E_a, s \to 0} \delta(E_a, s) = 0 \quad \lim_{E_a, s \to 0} J(E_a, s) = 0. \]

Proof. We argue by contradiction. So we can assume there exists a sequence of function $\{u_k\}_{k \in \mathbb{N}}$ satisfy:
\[ \begin{aligned}
P_k u_k + Q &= (l_k f_k + h_k)e^{mu_k}, \\
\int_M l_k f_k e^{mu_k} d\mu &= \int_M Q d\mu = (n - 1)!\gamma_n, \\
\lim_{k \to \infty} E_{a,k} &= \lim_{k \to \infty} \int_M \delta_k e^{mu_k} d\mu = 0.
\end{aligned} \tag{2.16} \]
and there exists $\delta_1 > 0$ and $J_1 > 0$ such that
\[ \|u_k - \bar{u}_k - G_p\|_{W^{\alpha}(M \setminus B_{E_k, \alpha}(p_k))} \geq J_1. \]
By Proposition 2.4 using the compactness of $M$ and the Poincaré inequality, we have
\[ \|u_k - \bar{u}_k\|_{W^{1,q}(M)} \leq C_q, \quad q \in \left(1, \frac{n}{n - 1}\right) \]
for some $C_q$ independent of $k$.

Then after passing to a subsequence, we can assume $u_k - \bar{u}_k \to u_\infty$ weakly in $W^{1,q}(M)$ for some $u_\infty \in W^{1,q}(M)$.

In view of the formula (2.10), we know that
\[ \int_K l_k f_k e^{mu_k} d\mu \to 0 \]
for any compact set $K \subset M \setminus \{p\}$ as $k \to +\infty$. Then we can apply Proposition 2.3 to get the following estimate:
\[ \|u_k - a_{k,K}\|_{W^{\alpha}(K)} \leq C_K, \]
where $C_K$ is independent of $k$. Thus we obtain
\[ |a_{k,K} - \bar{u}_k| \leq C_K \left(\|u_k - a_{k,K}\|_{W^{\alpha}(K)} + \|u_k - \bar{u}_k\|_{W^{1,q}(M)}\right) \leq C_K. \]
So we can conclude that $\|u_k - \bar{u}_k\|_{W^{\alpha}(K)} \leq C_K$. By passing to a subsequence we can deduce
\[ u_k - \bar{u}_k \to u_\infty \quad \text{in} \quad W^{1,q}_{loc}(M \setminus \{p\}) \]
on the other hand, from the equation (2.1), we know $u_k - \bar{u}_k$ satisfies
\[ P(u_k - \bar{u}_k) + Q = e^{\bar{u}_k}(l_k f_k + h_k)e^{\bar{u}_k} - \bar{u}_k. \]
Let $k \to +\infty$ we get

$$\begin{cases}
P_{u_\infty} + Q = (n-1)!\gamma_2\delta_p \\
\int_M u_\infty d\mu = 0.
\end{cases}$$

So we can conclude $u_\infty = G_p$. This is contract to our assumption and we finish the proof of this proposition. □

In view of Proposition 2.7 and Proposition 2.8 we know the behavior of $u$ on $B_{L_s}(p)$ and $M \setminus B_\delta(p)$ pretty well.

To understand the behavior on $B_\delta(p) \setminus B_{L_s}(p)$, we need the following two lemmas:

**Lemma 2.9.** Assume $u$ is a solution of the equation (2.1). Then for $E_\alpha$ small enough, there exists a uniform constant $C > 0$, such that

$$d_\delta(x, p) \leq Cs(x).$$

**Proof.** By choose $M$ large enough and $\beta > 0$ small enough, we can deduce from the formula (2.15)

$$\int_{B_{\mu}(p)} |f| e^{\nu} d\mu \geq \frac{7}{8}(n-1)!\gamma_2,$$

for $s \leq \beta$ and $E_\alpha \leq \beta$. Then the definition of $s(x)$ tells us

$$s(x) \geq d_\delta(x, p) - Ms.$$

So when $d_\delta(x, p) \geq 2Ms$, we have

$$s(x) \geq 2Ms \geq \frac{d_\delta(x, p)}{2}.$$  

When $d_\delta(x, p) \leq 2Ms$, we know from $s(x) \geq s_{\min} \geq \frac{\delta}{c(\delta)}$ that

$$s(x) \geq \frac{d_\delta(x, p)}{2M\alpha(n)}.$$  

We now can set $C = \max\{2, 2Mc(n), Diam(M)/\beta\}$

□

**Lemma 2.10.** Assume $u$ is a solution of the equation (2.7) and $v \in [1, 2)$. Then for $E_\alpha$ and $s$ are both small enough (that is, $E_\alpha \leq \beta$, $s \leq \beta$ for some small real number $\beta > 0$ to be determined), there exist $\delta > 0$, $L > 0$ independent of $s$, such that $e^{\nu} e^{sn(r)}$ is monotonically decreasing on $[Ls, \delta]$, where

$$\tilde{u}_p(r) = \text{Vol}(\partial B_r(p))^{-1} \int_{\partial B_r(p)} u(x)d\sigma.$$  

**Proof.** We argue by contradiction. Assume there exists a sequence of function $\{u_k\}_{k \in \mathbb{N}}$ which satisfy:

$$\begin{cases}
P_{u_k} + Q = (l_kf_k + h_k)e^{nu_k}, \\
\int_M f_k e^{nu_k} d\mu = \int_M Qd\mu = (n-1)!\gamma_2, \\
\lim_{k \to \infty} E_{\alpha,k} = \lim_{k \to \infty} \int_M g_k^\alpha e^{nu_k} d\mu = 0.
\end{cases}$$

(2.17)

And $p_k$, $s_k$ is the parameter of $u_k$ determined by Proposition 2.7

$$A_\delta(r) = vr + \tilde{u}_p(r).$$

We know from Proposition 2.7 that $s_kA'_\delta(s_k r)$ converges to $\frac{v}{r} - \frac{2l \lambda_\nu}{1 + \lambda_\nu}$ in $W^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$, thus in $C^0_{\text{loc}}(\mathbb{R}^+)$. And this implies that for any $R \geq 2R_\nu := \frac{\delta}{4\sqrt{2-\nu}}$, there exists $k_0(R)$ such that

$$A'_\delta(s_k r) < 0 \text{ for } k \geq k_0(R), \text{ } r \in [2R_\nu, R].$$
Define
\[ r_k = \sup \{ r \in [2s_kR, \delta_0] : A'(r) < 0 \text{ in } [2s_kR, r) \} . \]
From above discussion we infer that
\[ \lim_{k \to \infty} r_k = \infty. \]
Now we can assume that
\[ \lim_{k \to \infty} r_k = 0. \]
Otherwise, \( \{u_k\}_{k \in \mathbb{N}} \) will obey the consequence of this Lemma.
Consider
\[ v_k(x) = (u_k)_{p_k,s_k} - C_k, \]
where \( C_k \) is a constant such that
\[ \int_{\partial B_1(0)} v_k d\sigma_k = 0. \]
Then we have
\[ \frac{d}{dr} \left( r^\nu e^{\delta_k} \right)(1) = 0. \quad (2.18) \]
Argue as in Proposition 2.8, after passing to a subsequence,
\[ v_k \rightharpoonup G_0 \quad \text{in } W^{\mu,\alpha}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \quad \text{as } k \to \infty. \]
Where \( G_0 \) satisfies
\[ (-\Delta_0)^{\frac{n}{2}} G_0 = (n-1)! \gamma_n. \]
Moreover, thanks to Corollary 2.4 and Lemma 2.9, we have
\[ \max_{B_{\frac{\delta}{\alpha}}(x)} G_0 - \min_{B_{\frac{\delta}{\alpha}}(x)} G_0 \leq C(M,n,\alpha). \]
Then we know from standard elliptic estimates:
\[ |\nabla^k G_0| \leq \frac{C(M,n,k,\alpha)}{|x|^k}, \]
hold for all \( k \in \mathbb{N} \). So
\[ G_0 = 2\log \frac{1}{|x|} + C. \]
Since \( \nu < 2 \), we get in particular that
\[ \frac{d}{dr} \left( r^\nu e^{\delta_k} \right)(1) < 0. \]
And this is contract to (2.18).

\[ \square \]

**Proposition 2.11.** Assume \( u \) is a solution of the equation (2.7). Then for \( E_{\alpha} \) and \( s \) are both small enough, there exist \( \delta > 0, L > 0 \) and \( \eta > 0 \) such that
\[ d_{\delta}(x,p)^\nu e^{a(x)} \leq \eta s^{a(v-1)} \quad \text{for } x \in B_{\delta}(p) \setminus B_L(p). \]

**Proof.** This result follows immediately from combing Proposition 2.10, Corollary 2.4 and Lemma 2.9.

Proposition 2.7 and Proposition 2.11 allow us to get control of \( \hat{u}_{p,s} \) on the whole \( B_{\frac{\delta}{\alpha}}(0) \).
Corollary 2.12. Assume $u$ is a solution of the equation (2.1). Then for $E_u$ and $s$ are both small enough, there exists a constant $C \geq 0$ independent of $s$ such that
\[ e^{\mu_{ps}} \leq C \| R^s \|_s. \]
In particular, we have
\[ \int_{B_{4/3}^s(0)} \hat{h}^p_s e^{\mu_{ps}} d\mu_{s^+} \leq C \int_{B_{4/3}^s(0)} \hat{h}^p_s e^{\mu_{ps}} d\mu_{s^+}. \]

Combing Proposition 2.11 and Proposition 2.8, we have the following corollary.

Corollary 2.13. Assume $u$ is a solution of the equation (2.1) and $v \in [1, 2)$. Then for $E_u$ and $s$ are both small enough, there exists a constant $C > 0$ independent of $s$, such that
\[ \tilde{u} \leq (v - 1) \log s + C. \]
In particular, we have
\[ \int_{M \setminus B(p)} e^{\mu_u} d\mu \leq C_1 s^{(v-1)}, \]
for some constant $C_1$ independent of $s$.

3. The Test Function $\varphi_{p,s}$

In this section, we will construct the test function $\varphi_{p,s}$ which plays a important role in the proof of our results. Our construction extends the test function in [13] and [25] to the general even dimension. A similar test function has been used by C.B. Ndiaye in [36].

Using the existence of conformal normal coordinates (see [7], [21]), we have that for any $p \in M$ there exists a function $u_p \in C^\infty(M)$ such that
\[ g_p = e^{2\nu_p} g \quad \text{verifies} \quad \det g_p(x) = 1 \quad \text{for} \quad x \in B, \]
with $0 < \rho_p < \frac{\rho_{j+1}(M)}{10}$. Moreover, we can take the families of functions $u_p$, $g_p$ and $\rho_p$ such that
the map $p \rightarrow u_p$ are $C^1$ and $\rho_p \geq \rho_0 > 0$,
for some small positive $\rho_0$ satisfying $\rho_0 < \frac{\rho_{j+1}(M)}{10}$, and
\[ \| u_p \|_{C^1(M)} = O(1), \quad \frac{1}{C_2 g} \leq g_p \leq C_2 g, \]
\[ u_p(x) = O\left( d^2_{g_p}(p, x) \right) = O\left( d^2_{g_p}(p, x) \right), \]
\[ u_p(p) = 0, \quad \nabla u_p(p) = 0, \quad R_{g_p}(p) = 0, \quad \text{Ric}_{g_p}(p) = 0, \]
for some large positive constant $C$ independent of $p$. Furthermore, using $\nabla u_p(p) = 0$ and the scalar curvature equation, namely
\[ -\frac{4(n-1)}{n-2} \Delta_{g_p}(e^{-\frac{2\nu_p}{n-2}}u_p) + R_{g_p} e^{-\frac{2\nu_p}{n-2}}u_p = R_g e^{-\frac{2\nu_p}{n-2}}u_p, \]
for $n \geq 3$, and
\[ 2\Delta_{g_p}(u_p) + R_{g_p} = R_g e^{-2\nu_p}, \]
for $n = 2$, it is easy to see that the following holds
\[ \Delta_{g_p} u_p(p) = \frac{R_g(p)}{2(n-1)} \]
(3.1)
On the other hand, using the properties of $g_p$, it is easy to check that for every $u \in C^2(M)$ there holds
\[ \nabla_{g_p} u(p) = \nabla u(p) = \nabla_{g_0} \tilde{u}(0), \quad \Delta_{g_p} u(p) = \Delta u(p) = \Delta_{g_0} \tilde{u}(0), \]
where
\[ \hat{u}(y) = u \left( \exp_p^y(y) \right), \quad y \in \mathbb{R}^n. \]

We have
\[ \Delta_{g_p} f(r_p) = \frac{\partial^2 f(r_p)}{r_p^2} + (n - 1) \frac{f'(r_p)}{r_p}, \tag{3.2} \]
where \( r_p(x) = d_{g_p}(x, p) \).

Using the conformal normal coordinate, we can give a simple proof of the following expansion of the Green’s function.

**Proposition 3.1.** Assume \( G_p \) is the solution of the equation (1.9). Then
\[ G_p(x) = H_p(x) + K_p(x) \]
is smooth on \( M \setminus \{p\} \), \( H_p \) extends to a twice differentiable function on \( M \) and
\[ K_p(x) = -2f(r) \log r, \]
where \( r(x) = d_q(p, x) \) is the geodesic distance from \( x \) to \( p \), \( f(r) \) is a \( C^\infty \) positive function such that \( f(r) = 1 \) on a neighborhood of \( r = 0 \), and \( f(r) = 0 \) for \( r \geq \text{inj}_q(M) \).

**Proof.** We denote \( r_p(x) = d_{g_p}(p, x) \). Then the knowledge from Riemannian geometry tells us:
\[ \Delta^2 r_p(x) = \Delta r^2 + \frac{1}{3} \left( |\nabla^2 u_p(x)|^2 \right) (X, X)r^2(x) + O(r^3), \tag{3.3} \]
where \( x = \exp_p(X) \). So we know that \( \Delta r \) is twice differentiable at \( p \).

Since the GJMS operator is a natural operator in Riemannian geometry, it will have the following form:
\[ P = (-\Delta)^2 + aR (-\Delta) + bRic_{ij} \nabla_i \nabla_j \Delta + \text{l.o.t.} \]
Here \( R \) is the scalar curvature, and \( Ric \) is the Ricci curvature tensor.

Since \( K_{g_p}(p) = 0 \) and \( Ric_{g_p}(p) = 0 \), we have
\[ P_{g_p} (2 \log r_p) = O \left( \frac{1}{r_p^3} \right) \in L^q \text{ for any } q < \frac{n}{n-3}. \]

On the other hand, from the conformally covariance, we have
\[ P_{g_p} (G_p - u_p) + Q_{g_p} = (n-1)! \gamma_p \delta_p. \]
The by basic elliptic estimate, we deduce
\[ G_p - u_p + 2 \log r_p \in W^{n,q} \subset C^{2,\alpha} \text{ for any } q < \frac{n}{n-3}. \]
That is
\[ G_p + 2 \log r + \left( \log \frac{r_p}{r} - u_p \right) \in C^{2,\alpha} \text{ for any } q < \frac{n}{n-3}. \]
Since \( \log \frac{r_p}{r} - u_p \) is twice differentiable at \( p \), the proof is finished. \( \square \)

The expansion (3.3) also tells us that:
\[ \Delta \left( \log \frac{r_p^2}{r^2} - u_p \right) = \frac{1}{3} \Delta u_p = \frac{R_x(p)}{6(n-1)} \tag{3.4} \]
We define the function as
Then we have
\[ \varphi_{p,s} = G_p + \eta_{p,s} \left( -\log \left( 1 + \frac{s^2}{\lambda^2 r_p^2} \right) + \sum_{k=0}^{n-2} B_k s \right) \]

Here \( \eta_{p,s} \) is a smooth function such that \( \eta_{p,s} = 1 \) on \( B^p_{r_p}(p) \) and \( \eta_{p,s} = 0 \) on \( M \setminus B^p_{r_p}(p) \) with \( |\nabla^k \eta_{p,s}| \leq \frac{C(n,k)}{L^{k+2}s^k} \), and
\[ B_k = \frac{1}{k!} \left. \frac{d^k}{dr^k} \log \left( 1 + \frac{s^2}{\lambda^2 r^2} \right) \right|_{r=r_p} \]

By a direct calculation, we find
\[ |B_k| \leq \frac{C(k)}{L^{k+2}s^k} \]

Then we have
\[ |\nabla^k (\eta_{p,s} (\varphi_{p,s} - G_p))| \leq \frac{C(n,k)}{L^{k+2}s^k} \]

So we can calculate that
\[
\begin{aligned}
&\left| p \varphi_{p,s} + Q - \frac{lf(p)^s \eta_{p,s} e^{\eta_{p,s}}}{(s^2 + \lambda^2 r_p^2)^n} \right| \\
= e^{\eta_{p,s}} \left| p \varphi_{p,s} + Q - \frac{lf(p)^s \eta_{p,s}}{(s^2 + \lambda^2 r_p^2)^n} \right| \\
\leq &\eta_{p,s} e^{\eta_{p,s}} \left| p \varphi_{p,s} + Q - \frac{lf(p)^s \eta_{p,s}}{(s^2 + \lambda^2 r_p^2)^n} \right| \\
\leq &\eta_{p,s} e^{\eta_{p,s}} \left| p \varphi_{p,s} + Q - \frac{lf(p)^s \eta_{p,s}}{(s^2 + \lambda^2 r_p^2)^n} \right| \\
\leq &C \eta_{p,s} \left( 1 + \frac{s^2}{\lambda^2 r_p^2} \right) + \sum_{k=1}^{n-2} \frac{k}{k!} \left( 1 + \frac{s^2}{\lambda^2 r_p^2} \right) + \frac{1}{L^{n+2}s^n} \right| \right) \\
\leq &\frac{C \eta_{p,s} e^{\eta_{p,s}}}{\lambda^2 r_p^2} + \frac{C \eta_{p,s} e^{\eta_{p,s}}}{L^{n+2}s^n} \\
\end{aligned}
\]

We now set \( E(x) = \sum_{k=0}^{n-2} B_{k,s} (r_p^2 - L^2 s^2) \), and compute \( E(\varphi_{p,s}) \). We have
\[
\begin{aligned}
&\frac{n}{2} \int_M \varphi_{p,s} P \varphi_{p,s} d\mu + n \int_M Q \varphi_{p,s} d\mu \\
= &\frac{n}{2} \int_M (\varphi_{p,s} P G_{p,s} + Q) d\mu + \frac{n}{2} \int_M (\varphi_{p,s} - G_{p,s}) (P \varphi_{p,s} + Q) d\mu + \frac{n}{2} \int_M Q G_{p,s} d\mu \\
= &\frac{n}{2} \int_M \left( \frac{\lambda^2}{s^2} + H_{p,s}(p) + E(p) \right) d\mu + \frac{n}{2} \int_M \left| Q G_{p,s} d\mu \right| \\
= &\frac{n}{2} \int_M \left( \frac{\lambda^2}{s^2} + H_{p,s}(p) + E(p) - \frac{C \log}{C_0} \right) + \frac{n}{2} \int_M \left| Q G_{p,s} d\mu \right| + \frac{n}{2} \int_M \left| Q G_{p,s} d\mu \right| + O \left( \frac{1}{s} + \frac{1}{L^{n+2}} \right) \\
= &\frac{n}{2} \int_M \left( \frac{\lambda^2}{s^2} + H_{p,s}(p) + 2E(p) - \frac{C \log}{C_0} \right) + \frac{n}{2} \int_M \left| Q G_{p,s} d\mu \right| + \frac{n}{2} \int_M \left| Q G_{p,s} d\mu \right| + O \left( \frac{1}{s} + \frac{1}{L^{n+2}} \right),
\end{aligned}
\]

where
\[
C_{log} = \int_{\mathbb{R}^n} \log \left( 1 + \frac{1}{\lambda^2 |x|^2} \right) \frac{1}{(1 + \lambda^2 |x|^2)^n} dx,
\]

and
\[
C_0 = \int_{\mathbb{R}^n} \frac{1}{(1 + \lambda^2 |x|^2)^n} dx.
\]
Notice that
\[
\frac{C_\log}{C_0} = \int_{\mathbb{R}^n} \log\left(1 + \frac{1}{|x|^2}\right) \frac{1}{(1 + |x|^2)^n} dx \leq \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^n} dx
\]
are constants depending only on \( n \). We can also calculate:
\[
\lambda^{-2n} s^n \int_{B_{L^2}(p)} f e^{\nu_{f,r}} d\mu = \lambda^{-2n} s^n \int_{B_{L^2}(p)} f e^{\nu_{f,r}} d\mu + O\left( \frac{1}{L^n} \right)
\]
\[
= \int_{B_{L^2}(p)} \frac{s^n}{(s^2 + \lambda^2 r^2)^n} e^{\nu_{f,r}} d\mu + O\left( \frac{1}{L^n} \right)
\]
\[
= \int_{B_{L^2}(p)} \frac{s^n}{(s^2 + \lambda^2 r^2)^n} e^{\nu_{f,r}} d\mu + O\left( \frac{1}{L^n} \right)
\]
\[
+ \frac{1}{n} \int_{B_{L^2}(p)} (s^2 + \lambda^2 r^2)^n \Delta g e^{\nu_{f,r}} d\mu + O\left( \frac{1}{L^n} \right)
\]
We denote
\[
C_2 = \int_{\mathbb{R}^n} \frac{|x|^2}{(1 + |x|^2)^n} dx.
\]
Then for \( n > 2 \), we have
\[
\lambda^{-2n} s^n \int_{B_{L^2}(p)} f e^{\nu_{f,r}} d\mu = C_0 e^{\nu_{f,r}} f + \frac{C_2 s^2}{n} \Delta g e^{\nu_{f,r}} f + O\left( \frac{1}{L^n} \right)
\]
\[
+ O\left( \frac{s^3 + 1}{L^2} + \frac{s^2}{L^{n-2}} + \frac{s^2}{L^n} \right)
\]
\[
+ O\left( \frac{C_2 s^2}{C_0} \left( H_p \left( \frac{1}{n} \log f \right) + n \left( H_p + \frac{1}{n} \log f \right) \right)^2 - \frac{R_g}{6(n-1)} \right)(p)
\]
\[
+ O\left( \frac{s^3 + 1}{L^2} + \frac{s^2}{L^{n-2}} + \frac{s^2}{L^n} \right).
\]
For \( n = 2 \), we have
\[
\lambda^{-2\alpha_s} s^n \int_{B_1(p)} f e^{\alpha_s r} \, d\mu = C_0 e^{\alpha_s(n+\log f)} \left( 1 + \frac{2\pi^2 \log L}{\lambda^2} \left( \Delta_{s} \left( H_p + \frac{1}{n} \log f \right) + n \left\| \nabla_{s} \left( H_p + \frac{1}{n} \log f \right) \right\|^2 - \frac{R_{\lambda}}{6(n-1)} \right)(p) \right) + O \left( s^2 + \frac{1}{L^2} + \frac{s^2 \log L}{L^2} \right).
\]

We set \( L = s^{-\frac{4}{n}} \left( \log \frac{4}{n} \right)^{\frac{1}{4}} \) when \( n > 2 \), so
\[
E[\varphi_{p,s}] = -(n-1)! \gamma_n \log \frac{\gamma_n}{2^n} \frac{\gamma_n}{2} \frac{C_{s} \log}{C_0} \left( \Delta \left( H_p + \frac{1}{n} \log f \right) + n \left\| \nabla \left( H_p + \frac{1}{n} \log f \right) \right\|^2 - \frac{R_{\lambda}}{6(n-1)} \right)(p) + O \left( s^2 \left( \log \frac{1}{s} \right)^{-1} \right).
\]

On the other hand, we set \( L = s^{-1} \left( \log \frac{4}{n} \right)^{-\frac{1}{4}} \) in the \( n = 2 \) case, then we have
\[
E[\varphi_{p,s}] = -4\pi \log f(p) - 4\pi \left( H_p(p) + 1 + \log \pi \right) - \frac{32\pi}{lf(p)} s^2 \log \frac{1}{s} \left( \Delta \left( H_p + \frac{1}{n} \log f \right) + 2 \left\| \nabla \left( H_p + \frac{1}{n} \log f \right) \right\|^2 - \frac{R_{\lambda}}{6} \right)(p) + O \left( s^2 \left( \log \frac{1}{s} \right)^{\frac{1}{4}} \right)
\]
\[
= - 4\pi \log f(p) - 4\pi \left( H_p(p) + 1 + \log \pi \right) - \frac{16\pi}{lf(p)} s^2 \log \frac{1}{s} \left( \Delta \log f + 4 \left\| \nabla \left( H_p + \frac{1}{2} \log f \right) \right\|^2 + 2Q - R_{\lambda} \right)(p) + O \left( s^2 \left( \log \frac{1}{s} \right)^{\frac{1}{4}} \right).
\]

In the last equality we have used the following identity for the two dimensional case (see [13]).
\[
\Delta H_p(p) + \frac{R_{\lambda}(p)}{3} = Q(p). \tag{3.6}
\]

This can be deduced directly by using
\[
\int_{\partial B_1(p)} \frac{\partial G_p}{\partial n} \, d\sigma = -4\pi + \int_{B_1(p)} Q \, d\mu.
\]

and
\[
\int_{\partial B_1(p)} \frac{\partial G_p}{\partial n} \, d\sigma = \int_{\partial B_1(p)} \left( -\frac{2}{r} + \frac{\partial H_p}{\partial n} \right) \, d\sigma = -\frac{2\text{Vol}(\partial B_1(p))}{r} + \int_{B_1(p)} \Delta H_p \, d\mu,
\]

and
\[
\text{Vol}(\partial B_1(p)) = 2\pi r - \frac{R_{\lambda}(p)\pi}{6} r^3 + O(r^4).
\]

From now on, we fix \( L = s^{-\frac{4}{n}} \left( \log \frac{4}{n} \right)^{\frac{1}{4}} \) for \( n > 2 \) and \( L = s^{-1} \left( \log \frac{4}{n} \right)^{-\frac{1}{4}} \) for \( n = 2 \).

With the help of \( \varphi_{p,s} \), we can now describe the global behavior of \( u \) as \( \inf_{x \in M} s(x) \) tend to zero.

**Theorem 3.2.** Assume \( u \) is a solution of the equation (2.7) and \( P \) has trivial kernel. Then for \( E_{\alpha} \) and \( s_{\min} = \inf_{x \in M} s(x) \) are both small enough (that is, smaller than a given small positive number), there exist \( s > 0 \) and \( p \in M \) such that,
\[
\|[u - \bar{u} - \varphi_{p,s}])_{\frac{4}{n} \cdot \lambda} \|_{L^2(M)} + \|[u - \bar{u} - \varphi_{p,s}]\|_{L^p(M)} \leq J(E_{\alpha}, s_{\min}),
\]
\( J \) is positive number satisfying 
\[
\lim_{E_a \to 0} J(E_a, s_{\min}) = 0.
\]

**Proof.** Combining the equation (2.1) and (3.5), we have that
\[
P(u - \varphi_{p,x}) = (lf + g) e^{m} - \frac{lf(p) s^{x} \chi_{B_x} e^{m_{p}}}{{L}^{n+2} s^{2} + \frac{s^{2} \chi_{B_x}^{r_{p}}}{r_{p}^{-3}}}
\]
Then by using Corollary 2.7, we have that
\[
\int_{M} |P(u - \varphi_{p,x})| d\mu \leq J_1(E_a, s_{\min}), \quad (3.7)
\]
where \( J_1 \) is a positive number such that
\[
\lim_{E_a \to 0} J_1(E_a, s_{\min}) = 0.
\]

Since \( P \) has trivial kernel, invoking the green representation formula, we have the following estimate for \( x \in B_\delta(p) \) (\( \delta \) is chosen as in Proposition 2.11).
\[
\begin{align*}
(u - \varphi_{p,x})(x) - (u - \varphi_{p,x}) &= \int_{M} G_p(x, y) P(u - \varphi_{p,x})(y) d\mu_y \\
&= \int_{B_\delta(p)} 2 \log \frac{1}{|x - y|} P(u - \varphi_{p,x})(y) d\mu_y + O(J_1(E_a, s_{\min})) \\
&= \int_{B_{\delta}(p)} 2 \log \frac{1}{|x - y|} \left( (lf + g) e^{m} - \frac{lf(p) s^{x} \chi_{B_x} e^{m_{p}}}{(s^{2} + \lambda^{2} r_{p}^{2})^{n}}(y) \right) d\mu_y + O(J_1(E_a, s_{\min})) \\
&+ O \left( \frac{1}{L_{2}} + Ls^{3} + L^{2} \frac{s^{2}}{L} \right) \\
&= \int_{B_{\delta}(p)} 2 \log \frac{s}{|x - y|} P(u - \varphi_{p,x})(y) d\mu_y + 2 \log \frac{1}{s} \int_{B_{\delta}(p)} P(u - \varphi_{p,x})(y) d\mu_y + O(J_1(E_a, s_{\min})).
\end{align*}
\]

To compute the first term in the right hand of above formula, we use the estimate of \( \hat{u}_{p,s} \). However, the \( \hat{u}_{p,s} \) is define by using \( exp_{p}^{r_{p}} \) instead of \( exp_{p} \) here.
One can see the $L^1$ terms tend to 0 as $E_\alpha$ and $s_{\text{min}}$ tend to 0 by using Corollary 2.7. The $L^\alpha$ terms also tend to 0 once we know the $L^1$ terms tend to 0, because we have the global $L^\infty$ control by Corollary 2.12.

It is remained to estimate the second term. By Corollary 2.13

$$\lim_{E_\alpha, s_{\text{min}} \to 0} \left\| (u - \varphi_{p,s}) - (\bar{u} - \varphi_{p,s}) \right\|_{L^\infty(B_r(\bar{u}, s))} = 0.$$  

On the other hand, we know from Proposition 2.8

$$\lim_{E_\alpha, s_{\text{min}} \to 0} \left\| (u - \varphi_{p,s}) - (\bar{u} - \varphi_{p,s}) \right\|_{L^\infty(B_\delta(\bar{u}))} = 0.$$  

Then we have

$$\lim_{s \to 0} \varphi_{p,s} = 0.$$  

Thus

$$\lim_{E_\alpha, s_{\text{min}} \to 0} \left\| u - \bar{u} - \varphi_{p,s} \right\|_{L^\infty(M)} = 0.$$  

From the property of the GJMS operator, we have

$$\left\| u - \bar{u} - \varphi_{p,s} \right\|_{W^{2,1}(M)} \leq C \left( \int_M (u - \bar{u} - \varphi_{p,s}) P(\bar{u} - \varphi_{p,s}) d\mu + \left\| u - \bar{u} - \varphi_{p,s} \right\|_{L^\infty(M)} \right).$$  

Then by the inequality (3.7),

$$\lim_{E_\alpha, s_{\text{min}} \to 0} \left\| u - \bar{u} - \varphi_{p,s} \right\|_{W^{2,1}(M)} \leq \lim_{E_\alpha, s_{\text{min}} \to 0} \left\| u - \bar{u} - \varphi_{p,s} \right\|_{L^\infty(M)} = 0.$$  

With the help of Theorem 3.2 we can compute the value of $E[u]$.

**Corollary 3.3.** Assume $u$ is a solution of the equation (2.7) and $P$ has trivial kernel. Then for $E_\alpha$ and $s_{\text{min}} = \inf_{x \in M} s(x)$ are both small enough (that is, smaller than a given small positive number), there exist $s > 0$ and $p \in M$ such that,

$$E[u] = E[\varphi_{p,s}] + J(E_\alpha, s_{\text{min}}) = \Lambda(g, f, p) + J_1(E_\alpha, s_{\text{min}}).$$  

Here $J$ and $J_1$ are positive numbers satisfying

$$\lim_{E_\alpha, s_{\text{min}} \to 0} J(E_\alpha, s_{\text{min}}) = \lim_{E_\alpha, s_{\text{min}} \to 0} J_1(E_\alpha, s_{\text{min}}) = 0.$$  

And

$$\Lambda(g, f, p) = -(n-1)! \gamma_n \log \frac{\gamma_n f(p)}{2^n} - n! \gamma_n \left( H_p(p) + C_0 \right).$$
Proof. A direct calculation shows
\[ |E[u] - E[\varphi_p]| \]
\[ = |E[u - \bar{u}] - E[\varphi_p]| \]
\[ \leq C \left( |u - \bar{u} - \varphi_p|_{W^{2,2}(M)} + |u - \bar{u} - \varphi_p|_{L^\infty(M)} \right). \]
This corollary follows from Theorem 3.2 and the explicit form of \( E[\varphi_p] \).

4. Proof of Theorem 1.2 and 1.4

In this section, we consider the convergence of the flow equation (1.5).

We begin with the Adams-Fontana inequality (see [2], [16]).

Proposition 4.1. Let \((M, g)\) be a compact \(n\)-dimensional manifold whose GJMS operator \(P\) is weakly positive with trivial kernel. Then for any \(u \in W^{2,2}(M)\), we have
\[
\log \int_M e^{u} \, d\mu \leq \frac{n}{2(n-1)!} \gamma_n \int_M u \, Pd\mu + \frac{n}{2} \int_M ud\mu + C
\]
for some constant \(C = C(M, g, n)\).

From the Adams-Fontana inequality, we know that the functional \(E\) has uniform lower bound. More precisely, let \(h\) be the solution of
\[
\begin{cases}
Ph + Q = (n-1)! \gamma_n \\
\int_M h \, d\mu = 0,
\end{cases}
\]
and denote \(u = h + v\). Then we have:
\[
E[u] = \frac{n}{2} \int_M uPd\mu + n \int_M Qvd\mu - \bar{Q} \log \left( \int_M e^{v} \, d\mu \right)
\]
\[
= \frac{n}{2} \int_M vPd\mu + n \int_M uPd\mu - \frac{n}{2} \int_M Phd\mu
\]
\[
+ n \int_M Qvd\mu - (n-1)! \gamma_n \log \left( \int_M e^{v} \, d\mu \right)
\]
\[
\geq \frac{n}{2} \int_M vPd\mu + n! \gamma_n \int_M vd\mu - (n-1)! \gamma_n \log \left( \int_M e^{v} \, d\mu \right)
\]
\[
- C \left( M, g, ||f||_{L^\infty(M)}, ||h||_{W^{2,2}(M)} \right)
\]
\[
\geq -L \left( M, g, ||f||_{L^\infty(M)}, ||h||_{W^{2,2}(M)} \right).
\]
Now we recall the following result of A. Fardoun and R. Regbaoui (Proposition 5.2 in [18]):

Proposition 4.2. Let \(u \in C^\infty(M \times [0, T])\) be the solution of the equation (1.5) defined on a maximal interval \([0, T_0]\), and assume that the GJMS operator \(P\) has trivial kernel. Then for any \(L > 0\) and any \(T_0 > 0\), there exists a positive constant \(C\) depending on \(L, g, f, T_0\) and \(||u_0||_{W^{2,2}(M)}\) such that, if
\[
\inf_{t \in [0, T_0]} E[u(t)] \geq -L,
\]
then we have
\[
\sup_{t \in [0, T_0]} ||u(t)||_{W^{2,2}(M)} \leq C.
\]

Remark 4. A. Fardoun and R. Regbaoui only handled the case \(f = \text{constant}\) and \(n = 4\). However, when \(f\) is not a constant and \(n\) is arbitrary, the proof is almost the same. For readers’ convenience, we shall include the proof in Section 6.
Combining Proposition 4.2 and part (a) of Theorem 1.1, we obtain the long time existence part in Theorem 1.2.

By a direct calculation, we know that
\[
\frac{d}{dt} E[u(t)] = n \int_M \left( \frac{\partial u}{\partial t} \right)^2 e^{u(t)} \, d\mu.
\]

Integrating this formula, we get
\[
n \int_0^T \int_M \left( \frac{\partial u}{\partial t} \right)^2 e^{u(t)} \, d\mu \, dt = E(u_0) - E(u(T)) \leq E[u_0] + L < +\infty.
\]

Since the right hand of the above formula is independent of \(T\), we can conclude that
\[
\int_0^\infty \int_M \left( \frac{\partial u}{\partial t} \right)^2 e^{u(t)} \, d\mu \, dt < \infty.
\]

As a consequence, we have
\[
\liminf_{t \to +\infty} \int_M \left( \frac{\partial u}{\partial t} \right)^2 e^{u(t)} \, d\mu = 0. \tag{4.4}
\]

We want to show

**Proposition 4.3.** Assume \(u(t)\) is a solution of the flow equation (1.5), and the GJMS operator \(P\) is weakly positive with trivial kernel, then
\[
\lim_{t \to +\infty} \int_M \left( \frac{\partial u}{\partial t} \right)^2 e^{u(t)} \, d\mu = 0. \tag{4.5}
\]

We need the following lemma.

**Lemma 4.4.** Assume \(u(t)\) is a solution of the flow equation (1.5), then there exits \(C = C(u(0), M, g)\) such that
\[
\left( \int_M v^6 \, d\mu_g \right)^{\frac{1}{6}} \leq C \left( \int_M v P_g \, d\mu_g + \left( \int_M Q_g \, d\mu_g \right)^{\frac{1}{2}} \right),
\]
for any \(v \in W^{2,2}(M)\), where \(g_t = e^{2u(t)} g\), \(P_g\) and \(Q_g\) is the GJMS operator and \(Q\) curvature associated to \(g_t\).

**Proof.** By a direct calculation, we have
\[
E_g[v] = \frac{n}{2} \int_M v P_g \, d\mu_g + n \int_M Q_g \, d\mu_g - (n - 1)! \gamma_n \log \left( \int_M f e^{nu} \, d\mu_g \right)
\]
\[
= \frac{n}{2} \int_M v P \, d\mu + n \int_M (P u(t) + Q) \, d\mu - (n - 1)! \gamma_n \log \left( \int_M f e^{nu(t)} \, d\mu \right)
\]
\[
= \frac{n}{2} \int_M (u(t) + v) P u(t) \, d\mu + n \int_M Q u(t) \, d\mu - (n - 1)! \gamma_n \log \left( \int_M f e^{nu(t)} \, d\mu \right)
\]
\[
- \frac{n}{2} \int_M u(t) P u(t) \, d\mu - n \int_M Q u(t) \, d\mu
\]
\[
= E[u(t) + v] - E[u(t)] - (n - 1)! \gamma_n \log \left( \int_M f e^{nu(t)} \, d\mu \right).
\]

Since \(u(t)\) is a solution of the flow equation, we have a uniform upper bound of
\[
E[u(t)] + (n - 1)! \gamma_n \log \left( \int_M f e^{nu(t)} \, d\mu \right),
\]
independent of $t$. So there exists a uniform positive constant $C$, such that

$$(n - 1)!\gamma_n \log \left( \int_M C f e^{nv} d\mu_{g_{\delta}} \right) \leq \frac{n}{2} \int_M v P_{g_{\delta}} v d\mu_{g_{\delta}} + n \int_M Q_{g_{\delta}} v d\mu_{g_{\delta}}.$$  

Then if we assume $\int_M w P_{g_{\delta}} v d\mu_{g_{\delta}} = 1$ and $\int_M Q_{g_{\delta}} v d\mu_{g_{\delta}} = 0$, we have

$$\int_M e^{nv} d\mu_{g_{\delta}} + \int_M e^{-nv} d\mu_{g_{\delta}} \leq C(n, f, u(0), M).$$

From element calculus, we know

$$|w|^6 \leq C_1 + C_2(e^{nw} + e^{-nw}),$$

for some $C_1, C_2$ depending only on $n$. Thus we can conclude that

$$\int_M |w|^6 d\mu_{g_{\delta}} \leq C(n, f, u(0), M). \quad (4.6)$$

Then for general $v \in \mathcal{W}_{2,2}^2(M)$, if

$$\int_M v P_{g_{\delta}} v d\mu_{g_{\delta}} = 0,$$

from the trivial kernel assumption, we know $v$ is a constant function. So we have

$$\left( \int_M v^6 d\mu_{g_{\delta}} \right)^{\frac{1}{6}} = \frac{Vol(M, g)^{\frac{1}{2}}}{(n - 1)!\gamma_n} \left( \int_M Q v d\mu_{g_{\delta}} \right)^{\frac{1}{2}}.$$

If

$$\int_M v P_{g_{\delta}} v d\mu_{g_{\delta}} \neq 0,$$

we let

$$w = \left( \int_M v P_{g_{\delta}} v d\mu_{g_{\delta}} \right)^{-\frac{1}{6}} \left( v - \frac{1}{\int_M Q_{g_{\delta}} v d\mu_{g_{\delta}}} \right).$$

Apply (4.6) to this $w$, we have

$$\left( \int_M \left( v - \frac{1}{\int_M Q_{g_{\delta}} v d\mu_{g_{\delta}}} \right)^6 d\mu_{g_{\delta}} \right)^{\frac{1}{6}} \leq C \left( \int_M v P_{g_{\delta}} v d\mu_{g_{\delta}} \right)^{\frac{1}{2}}.$$

This lemma then follows from the triangle inequality. \quad \Box

**Proof of Proposition 4.3**: A direct calculation shows

$$\frac{\partial}{\partial t} (Q_{g_{\delta}} - l_tf) = - P_{g_{\delta}}(Q_{g_{\delta}} - l_tf) + n Q_{g_{\delta}}(Q_{g_{\delta}} - l_tf) + n l_tf \int_M f (Q_{g_{\delta}} - l_tf) d\mu_{g_{\delta}}.$$

where

$$l_f = \frac{\int_M Q_{g_{\delta}} d\mu_{g_{\delta}}}{\int_M f d\mu_{g_{\delta}}} = \frac{(n - 1)!\gamma_n}{\int_M f d\mu_{g_{\delta}}}.$$
So
\[
\frac{\partial}{\partial t} \int_M (Q_\delta - l_0 f)^2 d\mu_{\delta_0} = -2 \int_M (Q_\delta - l_0 f) P_{\delta_0}(Q_\delta - l_0 f) d\mu_{\delta_0} + n \int_M (Q_\delta - l_0 f)^3 d\mu_{\delta_0} + 2n! y_n \left( \int_M f (Q_\delta - l_0 f) d\mu_{\delta_0} \right)^2 \tag{4.7}
\]
\[
\leq C \left( \int_M (Q_\delta - l_0 f)^2 d\mu_{\delta_0} + \left( \int_M (Q_\delta - l_0 f)^2 d\mu_{\delta_0} \right)^3 \right).
\]

Here we have used the following simple inequality:
\[
\int_M w^3 d\mu \leq \left( \int_M w^6 d\mu \right)^{\frac{1}{2}} \left( \int_M w^2 d\mu \right)^{\frac{1}{2}} \leq \varepsilon \int_M w^2 d\mu + C(\varepsilon) \left( \int_M w^2 d\mu \right)^3.
\]

Denote \( y(t) = \int_M (Q_\delta - l_0 f)^2 d\mu_{\delta_0} = \int_M \left( \frac{d\mu}{\mu} \right)^2 (t) d\mu_{\delta_0} \), then we have
\[
\frac{dy}{dt} \leq Cy(1 + y^2) \leq Cy(1 + y^2).
\]

Integrating it, we obtain
\[
C \int_{t_1}^{t_2} y(t) dt \geq \frac{1}{1 + y(t_1)} - \frac{1}{1 + y(t_2)}.
\]

Thus
\[
\liminf_{t \to \infty} \frac{1}{1 + y(t_2)} \geq \sup_{t \geq 0} \left( \frac{1}{1 + y(t_1)} + C \int_{t_1}^{t_2} y(t) dt \right) \geq \limsup_{t \to \infty} \left( \frac{1}{1 + y(t_1)} + C \int_{t_1}^{t_2} y(t) dt \right) = 1.
\]

So
\[
\lim_{t \to \infty} y(t) = \limsup_{t \to \infty} y(t) = 0.
\]

We denote the volume concentration radius function of \( u(t) \) by \( s_t \) in order to state our main result of this section.

**Theorem 4.5.** Assume \( u(t) \) is a solution of the flow equation (1.3), if
\[
\limsup_{t \to \infty} \inf_{x \in \mathcal{M}} s_t(x) > 0,
\]
then there exists a smooth function \( u_\infty \) satisfying the equation (1.3) such that
\[
\lim_{t \to \infty} ||u(t) - u_\infty||_{W^{2,2}(\mathcal{M})} = 0 \tag{4.8}
\]

**Proof.** Note that
\[
\limsup_{t \to \infty} \inf_{x \in \mathcal{M}} s_t(x) > 0,
\]
the argument below Definition 2.5 allows us to find a sequence \( \{t_k\}_{k \in \mathbb{N}} \), such that
\[
||u(t_k)||_{W^{2,2}(\mathcal{M})} \leq C,
\]
for some constant \( C \) independent of \( k \). Then there exists a function \( u_\infty \in W^{2,2}(\mathcal{M}) \), such that after passing to a subsequence, which we still denote by \( \{t_k\}_{k \in \mathbb{N}} \):
\[
\lim_{k \to \infty} \left( ||u(t_k) - u_\infty||_{L^2(\mathcal{M})} + ||u(t_k) - u_\infty||_{W^{2,2}(\mathcal{M})} \right) = 0.
\]

Then we argue as L. Sun et al. [37] by using the Łojasiewicz-Simon inequality. Actually, the general result proved in [17] says:
There are positive constants $\beta > 0$ and $\theta \in [1/2, 1)$ such that
\[ |E[u(t)] - E[u_\infty]|^\theta \leq \left( \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t)e^{2u(t)} d\mu \right)^{\frac{\theta}{2}}. \]
for $\|u(t) - u_\infty\|_{W^{2,2}(M)} \leq \beta$.

On the other hand, using Trudinger’s inequality, we have
\[ \int_M e^{\mu(t)} d\mu \leq C(n, q, \beta). \]
for all $q > 1$. Thus $u_t$ has a uniform lower bound as long as $\|u(t) - u_\infty\|_{W^{2,2}(M)} \leq \beta$. Combing this fact with Proposition 4.3 we have
\[ \|u(t)\|_{L^\infty(M)} \leq \|u(t)\|_{W^{2,2}(M)} \leq C(n, q, \beta), \]
for some $C(n, q, \beta)$ independent of $t$. So we have
\[ |E[u(t)] - E[u_\infty]|^\theta \leq \left( \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t)e^{u(t)} d\mu \right)^{\frac{\theta}{2}}. \]
for $\|u(t) - u_\infty\|_{W^{2,2}(M)} \leq \beta$.

A direct calculation shows
\[ -\frac{d}{dt}(E[u(t)] - E[u_\infty])^{1-\theta} = -(1 - \theta)(E[u(t)] - E[u_\infty])^{-\theta} \frac{d}{dt} E[u(t)] \]
\[ = (1 - \theta)n(E[u(t)] - E[u_\infty])^{-\theta} \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t)e^{u(t)} d\mu \]
\[ \geq C \left( \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t)e^{u(t)} d\mu \right)^{\frac{\theta}{2}} \geq \left( \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t) d\mu \right)^{\frac{\theta}{2}}. \]
Thus
\[ \frac{d}{dt}\|u(t) - u_\infty\|_{L^2(M)} \leq C \left( \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t) d\mu \right)^{\frac{\theta}{2}}. \]
So we have
\[ \|u(\tau) - u_\infty\|_{L^2(M)} \]
\[ \leq \|u(t_k) - u_\infty\|_{L^2(M)} + \int_{t_k}^{\tau} \left( \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t) d\mu \right)^{\frac{\theta}{2}} dt \]
\[ \leq \|u(t_k) - u_\infty\|_{L^2(M)} + (E[u(t_k)] - E[u_\infty])^{1-\theta}, \]
one $\|u(t) - u_\infty\|_{W^{2,2}(M)} \leq \beta$ for all $t \in [t_k, \tau]$.

We define
\[ \tau_k = \sup\{\tau : \|u(t) - u_\infty\|_{W^{2,2}(M)} \leq \beta \text{ for all } t \in [t_k, \tau]\}. \]
If $\tau_k < \infty$, we have
\[ \|u(\tau_k) - u_\infty\|_{W^{2,2}(M)} = \beta \]
\[ \lim_{k \to \infty} \|u(\tau_k) - u_\infty\|_{L^2(M)} = 0. \]
And this is contradic to $\|u(\tau_k)\|_{W^{2,2}(M)} \leq C(n, q, \beta)$. So we have $s_k = \infty$ for $k$ large enough.

Thus there exist a number $\tau_0 > 0$ such that
\[ \|u(\tau)\|_{W^{2,2}(M)} \leq C(n, q, \beta) \]
\[ \lim_{\tau \to \infty} \|u(\tau) - u_\infty\|_{L^2(M)} = 0, \]
for all \( \tau \in [\tau_0, \infty) \). This implies

\[
\lim_{t \to \infty} \left( \|u(t) - u_{\infty}\|_{L^\infty(M)} + \|u(t) - u_{\infty}\|_{W^{2,2}(M)} \right) = 0.
\]

The formula (4.8) then follows from element elliptic estimate as in the proof of Proposition 2.3.

Now we can give a proof of Theorem 1.2.

**Proof of Theorem 1.2:** If \( \lim_{t \to \infty} \inf_{x \in M} s_{i}(x) > 0 \), Theorem 4.3 shows the alternative (a) holds. If \( \lim_{t \to \infty} \inf_{x \in M} s_{i}(x) = 0 \), Corollary 3.3 tells the alternative (b) holds.

**Proof of Theorem 1.4:** From the assumption of this theorem and the explicit form of \( E[\varphi_{p, s}] \), we can find a positive number \( s_0 \) such that

\[
E[\varphi_{p, s_0}] < \inf_{p \in M} \Lambda (g, f, p).
\]

We now choose \( \varphi_{p, s_0} \) as the initial data, the alternative (b) in Theorem 1.2 cannot hold, and the flow converges.

5. **Proof of Theorem 1.3**

In this section, we study the flow equation (1.3) under the assumption

\[
\lim_{t \to \infty} \inf_{x \in M} s_{i}(x) = 0.
\]

By applying Theorem 5.2, we know there exist \( p_{i}, s_{i} \) such that

\[
\lim_{t \to \infty} \left( \|u(t) - \bar{u}(t) - \varphi_{p_{i}, s_{i}}\|_{W^{2,2}(M)} + \|u(t) - \bar{u}(t) - \varphi_{p_{i}, s_{i}}\|_{L^\infty(M)} \right) = 0.
\]

We also have \( \lim_{t \to \infty} s_{i} = 0 \).

We now choose \( (p_{i}, s_{i}) \in M \times \mathbb{R}^+ \) such that

\[
\int_{M} \left( u(t) - \varphi_{p_{i}, s_{i}} \right) P \left( u(t) - \varphi_{p_{i}, s_{i}} \right) d\mu \leq \inf_{(p, s) \in M \times \mathbb{R}^+} \int_{M} \left( u(t) - \varphi_{p, s} \right) P \left( u(t) - \varphi_{p, s} \right) d\mu.
\]

Thus we have

\[
\lim_{t \to \infty} \int_{M} \left( \varphi_{p_{i}, s_{i}} - \varphi_{p_{i}, s_{i}} \right) P \left( \varphi_{p_{i}, s_{i}} - \varphi_{p_{i}, s_{i}} \right) d\mu = 0.
\]

Using this, we can check

\[
\lim_{t \to \infty} \frac{d_{\epsilon}(p_{i}, p_{i})}{s_{i}} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{s_{i}}{s_{i}} = 1.
\]

Then

\[
\lim_{t \to \infty} \left( \|u(t) - \bar{u}(t) - \varphi_{p_{i}, s_{i}}\|_{W^{2,2}(\bar{M})} + \|u(t) - \bar{u}(t) - \varphi_{p_{i}, s_{i}}\|_{L^\infty(\bar{M})} \right) = 0.
\]

The inequality also implies

\[
\int_{M} \left( u(t) - \varphi_{p_{i}, s_{i}} \right) \frac{\partial \varphi_{p_{i}, s_{i}}}{\partial y} \bigg|_{y=0} d\mu = 0,
\]

and

\[
\int_{M} \left( u(t) - \varphi_{p_{i}, s_{i}} \right) \frac{\partial \varphi_{p_{i}, s_{i}}}{\partial \xi} \bigg|_{\xi=0} d\mu = 0.
\]

To proceed, we need the following Poincaré inequality by S. Brendle’s blow up argument (see [39]).
Lemma 5.1. Assume \( u \) is a solution of the equation (2.1). Then for \( E_{\alpha} \) and \( s_{\text{min}} = \inf_{x \in M} s(x) \) are both small enough (that is, smaller than a given positive real number), there exist a uniform constant \( \beta > 0 \), such that

\[
(1 - \beta) \int_M \omega P \omega \geq n! \gamma_n \frac{\int_M \omega^2 f e^{mu} d\mu}{\int_M f e^{mu} d\mu},
\]

for any \( \omega \in W^{2,2}(M) \) satisfying the following three constraints:

\[
\int_M \omega f e^{mu} d\mu = 0.
\]

\[
\int_M \omega \frac{\partial \varphi_{\alpha}(y), x}{\partial y} \bigg|_{y=0} d\mu = 0.
\]

And

\[
\int_M \omega \frac{\partial \varphi_{\alpha}(y), z}{\partial z} \bigg|_{z=0} d\mu = 0.
\]

Proof. Supposing this is not true. We can assume there exists a sequence of function \( \{u_k\}_{k \in \mathbb{N}} \) satisfying:

\[
\begin{align*}
Pu_k + Q &= (l_k f_k + h_k) e^{su_k}, \\
\int_M l_k f_k e^{mu} d\mu &= \int_M Qd\mu = (n-1)! \gamma_n, \\
\lim_{k \to \infty} E_{\alpha,k} &= \lim_{k \to \infty} \int_M g_k^2 e^{me} d\mu = 0, \\
\end{align*}
\]

(5.2)

and a sequence of function \( \{\omega_k\}_{k \in \mathbb{N}} \) satisfying the three constraints in the lemma such that:

\[
\int_M \omega_k P\omega_k d\mu \leq n! \gamma_n \frac{\int_M \omega_k^2 f e^{mu} d\mu}{\int_M f e^{mu} d\mu} = 1.
\]

Assume \( p_k, s_k \) is a pair of parameter of \( u_k \) satisfying the conclusion of Proposition 2.7 and denote \( \hat{u}_k = (u_k)_{p_k,s_k} \) and \( \hat{\omega}_k = (\omega_k)_{p_k,s_k} \). Then we have

\[
\int_{B_{\delta}(0)} \hat{u}_k P_{g_k} \hat{\omega}_k d\mu_{p,k} \leq C(n, g, f),
\]

and

\[
\int_{B_{\delta}(0)} \hat{\omega}_k^2 e^{mu} d\mu_{p,k} \leq C(n, g, f).
\]

Hence, if we take the weak limit as \( k \to \infty \), we obtain a function \( \hat{\omega} \in W^{2,2}(\mathbb{R}^n) \) such that

\[
\int_{\mathbb{R}^n} \hat{\omega} (-\Delta)^\frac{1}{2} \hat{\omega} dx \leq n! \gamma_n \int_{\mathbb{R}^n} \frac{\hat{\omega}^2}{(1 + x^2)^\frac{3}{2}} dx \int_{\mathbb{R}^n} \frac{1}{(1 + x^2)^\frac{3}{2}} dx = 1.
\]

(5.3)
Moreover, we have the following estimate from a direct calculation.

\[
P \frac{\partial}{\partial y} \varphi_{\varphi_p,y}(x) + \frac{2nf(p)s^2 \lambda^2 e^{-1}_p(x) \cdot y}{(s^2 + \lambda^2 r_p^2)^{n+1}}
\]

\[
= \frac{\partial}{\partial y} \left( (e^{2n\varphi_p(y)} - 1) P_{\varphi_p,y} \varphi_{\varphi_p,y} \right) + \frac{\partial}{\partial y} \left( P_{\varphi_p,y} \left( \frac{\varphi_{\varphi_p,y} + \log \left( \frac{s^2}{\lambda^2} + r_p^2 \varphi_{\varphi_p,y} \right)}{s^2 + \lambda^2 r_p^2} \right) \right)
\]

\[
+ \frac{\partial}{\partial y} \left( \left( -\Delta_{\varphi_p,y} \right) \varphi_{\varphi_p,y} \right) + \frac{\partial}{\partial y} \left( \frac{1}{(s^2 + \lambda^2 r_p^2)^{n+1}} \right)
\]

\[
= O \left( \frac{1}{r_p^{n-1}} + \frac{1}{L^{n+1} s^{n+1}} \right).
\]

and

\[
P \frac{\partial}{\partial s} \varphi_{\varphi_p,s}(x) + \frac{nlf(p)s^{n-1}(s^2 - \lambda^2 r_p^2)}{(s^2 + \lambda^2 r_p^2)^{n+1}}
\]

\[
= \frac{\partial}{\partial s} \left( (e^{2n\varphi_p} - 1) P_{\varphi_p,s} \varphi_{\varphi_p,s} \right) + \frac{\partial}{\partial s} \left( P_{\varphi_p,s} \left( \frac{\varphi_{\varphi_p,s} + \log \left( \frac{s^2}{\lambda^2} + r_p^2 \right)}{s^2 + \lambda^2 r_p^2} \right) \right)
\]

\[
+ \frac{\partial}{\partial s} \left( \left( -\Delta_{\varphi_p,s} \right) \varphi_{\varphi_p,s} \right) + \frac{1}{(s^2 + \lambda^2 r_p^2)^{n+1}} + O \left( \frac{1}{L^{n+1} s^{n+1}} \right)
\]

\[
= O \left( \frac{1}{r_p^{n-1}} + \frac{1}{L^{n+1} s^{n+1}} \right).
\]

Here we use \( e_p(y) \) to denote \( exp_{\varphi_p} (y) \).

Thus, we have the following constraints hold:

\[
\int_{\mathbb{R}^n} \frac{\hat{\omega}}{(1 + \lambda^2 |x|^2)^n} dx = 0,
\]

and

\[
\int_{\mathbb{R}^n} \frac{(1 - \lambda^2 |x|^2) \hat{\omega}}{(1 + \lambda^2 |x|^2)^{n+1}} dx = 0,
\]

and

\[
\int_{\mathbb{R}^n} \frac{\lambda \hat{\omega}}{(1 + \lambda^2 |x|^2)^{n+1}} dx = 0.
\]

By the conformal covariance of the GJMS operator, and the spectral property of the GJMS operator on the round \( S^n \), the equation (5.3) cannot hold. A contradiction.

\[ \square \]

We denote that

\[ \phi(t) = \varphi_{\varphi_p,y}^t + B_t, \quad \omega(t) = u(t) - \phi(t), \]

where

\[ B_t = \frac{\int_M (u(t) - \varphi_{\varphi_p,y}^t) f d\mu_{\varphi_p}}{\int_M f d\mu_{\varphi_p}}. \]

Thus

\[ \int_M \omega(t) f d\mu_{\varphi_p} = 0. \]
We also have
\[ \int_M \omega(t) \frac{\partial \varphi_{y(t)}(y, \cdot)}{\partial y} \bigg|_{y=0} d\mu = 0, \]
and
\[ \int_M \omega(t) \frac{\partial \varphi_{y(t)}(y, \cdot)}{\partial z} \bigg|_{z=0} d\mu = 0. \]

Then we can calculate
\[ E[u(t)] - E[\phi(t)] = \int_M e^{-\mu(t)} f d\mu_\omega, \]
and
\[ \int_M e^{-\mu(t)} f d\mu_\omega = \int_M e^{-\mu(t)} f d\mu_\omega. \]

Then we can calculate
\[ E[u(t)] - E[\phi(t)] = n \int_M \omega(t) (Pu(t) + Q) d\mu - \frac{n}{2} \int_M \omega(t) P\omega(t) d\mu + (n-1)! \gamma_n \log \left( \int_M e^{-\mu(t)} f d\mu_\omega \right) \]
\[ + \left( n - 1 \right) ! \gamma_n \log \left( \int_M \left( e^{-\mu(t)} - 1 + \omega(t) - \frac{n}{2} \omega^2(t) \right) f d\mu_\omega \right) \]
\[ \leq n \int_M \omega(t) \frac{\partial u}{\partial t} d\mu_\omega - \frac{n}{2} \left( \int_M \omega(t) P\omega(t) - n! \gamma_n \int_M \omega^2(t) f d\mu_\omega \right) \]
\[ + O \left( \int_M \omega^3 f d\mu_\omega \right) \leq C \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t)d\mu_\omega. \]

Notice that in the last inequality, we have used Lemma [5.1]

We now state the main result of this section.

**Theorem 5.2.** Assume \( u(t) \) is a solution of the flow equation (1.5), and denote \( E_{\infty} = \lim_{t \to \infty} E[u(t)] \), if
\[ \lim \inf_{\tau \to \infty} \omega(x) = 0, \]
then there does not exist a positive number \( T > 0 \), such that
\[ E[\phi(t)] \leq E_{\infty} \]
for all \( t \in [T, \infty) \).

**Proof:**
Supposing this is not true. We can assume there exists a positive number \( T > 0 \) such that (5.4) holds for all \( t \in [T, \infty) \). Thus we have
\[ E[u(t)] - E[u_{\infty}] \leq C \left( \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t)d\mu_\omega \right), \]
for all \( t \in [T, \infty) \).
\[ \int_T^\infty \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t)d\mu_\omega dt \leq -C \int_T^\infty \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t)d\mu_\omega dt. \]
So we have
\[ \int_T^\infty \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t)d\mu_\omega dt \leq A^2 e^{-2\beta \tau}, \]
for some uniform constants \( A > 0 \) and \( \beta > 0 \). Then
\[ \int_T^\infty \left( \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t)d\mu_\omega \right)^{\frac{1}{2}} dt = \sum_{i=0}^\infty \int_{\tau+i}^{\tau+i+1} \left( \int_M \left( \frac{\partial u}{\partial t} \right)^2 (t)d\mu_\omega \right)^{\frac{1}{2}} dt \leq \sum_{i=0}^\infty A e^{-\beta \tau} = A e^{-\beta \tau} \frac{1}{1 - e^{-\beta}}. \]
Thus
\[ \int_{B_r(p)} e^{m(t)} \, d\mu = \int_{B_r(p)} e^{m(t)} \, d\mu + n \int_0^{\tau_1} \int_{B_r(p)} \frac{\partial u}{\partial t} e^{m(t)} \, dt \]
\[ \leq \int_{B_r(p)} e^{m(t)} \, d\mu + \int_0^{\tau_1} \left( \int_M \left( \frac{\partial u}{\partial t} \right)^2 \, d\mu \right)^{\frac{1}{2}} \, dt \]
\[ \leq \int_{B_r(p)} e^{m(t)} \, d\mu + A e^{\beta \tau_1} \frac{e^{-\beta \tau}}{1 - e^{-\beta}}. \]

And this is contradict to the assumption \( \lim_{t \to \infty} \inf_{x \in M} s(x) = 0. \) □

Now we can give a proof of the Theorem 1.3.

**Proof of Theorem 1.3** By our assumption \( \Lambda(g, f, p) \equiv \Lambda, \) for some constant \( \Lambda. \) Then Corollary 3.3 tells us
\[ E_\infty = \lim_{t \to \infty} E[u(t)] = \Lambda. \]

By the explicit form of \( E[\varphi_{p, s}^t] \) and our assumption we can show
\[ E[\phi(t)] = E[\varphi_{p, s}^t] < \Lambda = E_\infty, \]
for \( t \) large enough. Thus we have
\[ \limsup_{t \to \infty} \inf_{x \in M} s_t(x) > 0, \]
by using Theorem 5.2. The convergence then follows from Theorem 4.5. □

**Remark 5.1.** When \( (M, g) \) is the standard round sphere \( (S^n, g_{\text{round}}), \) and \( f \equiv 1, \) we know the standard bubble function \( \Phi_{p,s} \) satisfying
\[ E[\Phi_{p,s}] \equiv \Lambda, \]
for some constant \( \Lambda. \) If \( \lim_{t \to \infty} \inf_{x \in M} s_t(x) = 0, \) Corollary 3.3 tells us \( E_\infty = \Lambda. \) Then we can argue as before by using Theorem 5.2 and Theorem 4.5 to show convergence in this special case. In this way, we recover the results of S. Brendle (see [6]) and P.T. Ho (see [23]).

### 6. Proof of Proposition 4.2

In this section we introduce the notation \( |A| = \int_A d\mu \) and assume \( \int_M e^{nu(t)} \, d\mu = \int_M d\mu = 1 \) for simplicity.

By a direct calculation, we have
\[ \frac{d}{dt} \int_M u e^{nu} \, d\mu = -n \int_M u P \, d\mu - n \int_M Q d\mu + n \frac{Q}{2} \int_M f u e^{nu} \, d\mu. \]

Denote \( U = u e^{nu} + \frac{1}{n} e^{-1}, \) then \( U \geq 0 \) from elementary calculus. By the assumption, we have
\[ \frac{n}{2} \int_M u P \, d\mu + n \int_M Q d\mu - \left( \int_M Q d\mu \right) \log \left( \int_M f e^{nu} \, d\mu \right) \geq -L. \]
Therefore,
\[
\frac{d}{dt} \int_M U d\mu \leq -\frac{n}{2} \int_M u P d\mu + \frac{\bar{Q}}{f} \int_M U d\mu - \frac{\bar{Q}}{e} \int_M f d\mu + \left( \int_M Q d\mu \right) \log \left( \int_M f e^u d\mu \right) + L \leq A(M, g, f) \int_M U d\mu + C(M, g, f, L).
\]

(6.1)

Then we can deduce:
\[
\int_M u e^u d\mu \leq C(M, g, T_0, f).
\]

Denote \(\varphi(z) = z \log z\). Then \(\varphi\) is convex on \((0, +\infty)\), and it satisfies that for each \(\lambda > 1\) and \(z > 0\)
\[
\varphi(z) \geq -e^{-1}
\]
which implies, because \(\varphi(z) \geq -e^{-1}\) for any \(z > 0\),
\[
z \leq \frac{\varphi(\lambda z)}{\varphi(\lambda)} + e^{-1} \log \lambda.
\]

For \(t \in [0, T_0]\), let \(A_t \subset M\) defined by
\[
A_t = x \in \{M : u(x, t) \geq \alpha_0\},
\]
where
\[
\alpha_0 = \frac{1}{n} \log \left( \frac{1}{2} \right).
\]

We shall demonstrate \(|A_t| \geq C > 0\) for some \(C\) independent of \(t\). If \(|A_t| \geq 1\), there is nothing to prove. So we assume \(|A_t| < 1\). From Jensen’s inequality it follows
\[
\varphi \left( \frac{1}{|A_t|} \int_{A_t} e^u d\mu \right) \leq \frac{1}{|A_t|} \int_{A_t} \varphi(e^u) d\mu \leq \frac{C_1}{|A_t|}.
\]

Choosing \(\lambda = \frac{1}{|A_t|}\) and \(z = \int_{A_t} e^u d\mu\), we have
\[
\int_{A_t} e^u d\mu \leq \frac{|A_t|}{\log \frac{1}{|A_t|}} \varphi \left( \frac{1}{|A_t|} \int_{A_t} e^u d\mu \right) + \frac{e^{-1}}{\log \frac{1}{|A_t|}} \leq (C_1 + e^{-1}) \frac{1}{\log \frac{1}{|A_t|}} \leq \frac{C_2}{\log \frac{1}{|A_t|}}.
\]

(6.2)

On the other hand, we have
\[
1 = \int_M e^u d\mu = \int_{A_t} e^u d\mu + \int_{M \setminus A_t} e^u d\mu.
\]

Since \(e^u < e^\alpha\) on \(M \setminus A_t\), we get
\[
\frac{1}{2} \leq \int_{A_t} e^u d\mu,
\]
which implies \(|A_t| \geq e^{-2C_2} = C_3\). So we have lower bound of \(|A_t|\) which is independent of \(t\).

By using the elementary inequality \(z \leq e^z\), we have
\[
\int_{A_t} u(t) d\mu \leq \frac{1}{n} \int_{A_t} e^u d\mu \leq \frac{1}{n}.
\]
Then by the definition of the set $A_t$, for any $t \in [0, T_0]$, we conclude
\[
\left| \int_{M} u(t) \, d\mu \right| \leq \left| \int_{A_t} u(t) \, d\mu \right| + \left| \int_{M \setminus A_t} u(t) \, d\mu \right|
\leq C_4 + \int_{M \setminus A_t} u(t) \, d\mu.
\] (6.3)

Using the Cauchy-Schwarz inequality and the Young’s inequality, one get
\[
\left| \int_{M \setminus A_t} u(t) \, d\mu \right| \leq |M \setminus A_t|^\frac{1}{2} \|u\|_{L^2(M)},
\]
and
\[
\left( \int_{M} u(t) \, d\mu \right)^2 \leq (1 + \varepsilon)|M \setminus A_t|\|u(t)\|_{L^2(M)}^2 + C_4(1 + \varepsilon^{-1}).
\]

Now, by Poincaré inequality we have
\[
\|u(t)\|_{L^2(M)}^2 \leq K \int_{M} u(t)Pu(t) \, d\mu + \frac{1}{|M|} \left( \int_{M} u(t) \, d\mu \right)^2.
\]
Combining above two inequality, we can conclude
\[
\left( 1 - \frac{1 + \varepsilon}{|M|} |M \setminus A_t| \right) \|u(t)\|_{L^2(M)}^2 \leq K \int_{M} u(t)Pu(t) \, d\mu + \frac{C_4}{|M|}(1 + \varepsilon^{-1}).
\]
That is,
\[
(|A_t| - \varepsilon|M \setminus A_t|)\|u(t)\|_{L^2(M)}^2 \leq K|M| \int_{M} u(t)Pu(t) \, d\mu + C_4(1 + \varepsilon^{-1}).
\]
Choosing $\varepsilon = \frac{C_4}{2|M|}$ and observing that $|M \setminus A_t| \leq |M|$, we obtain
\[
\|u(t)\|_{L^2(M)}^2 \leq C_5 \left( \int_{M} u(t)Pu(t) \, d\mu + 1 \right),
\] (6.4)

Since the functional $E$ is decreasing along the flow, we have
\[
\frac{n}{2} \int_{M} u(t)Pu(t) \, d\mu + n \int_{M} Qu(t) \, d\mu - \bar{Q} \log \left( \int_{M} f e^{u} \, d\mu \right) \leq E[u_0],
\]

hence
\[
\int_{M} u(t)Pu(t) \, d\mu \leq \frac{1}{2C_5} \|u(t)\|_{L^2(M)}^2 + C_6.
\] (6.5)

It is follows from (6.3) and (6.5) that
\[
\|u(t)\|_{L^2(M)} \leq C_7.
\] (6.6)
Combining (6.5) and (6.6), we get the estimate (4.3):
\[
\|u(t)\|_{W^{\frac{n}{2},2}(M)} \leq C_8.
\]

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THE PRESCRIBED $q$-CURVATURE FLOW FOR ARBITRARY EVEN DIMENSION IN A CRITICAL CASE

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