Bivariate asymptotics for eta-theta quotients with simple poles

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Abstract
We employ a variant of Wright’s Circle Method to determine the bivariate asymptotic behavior of Fourier coefficients for a wide class of eta-theta quotients with simple poles in \( \mathbb{H} \).

1 Introduction

Jacobi forms (see e.g., [14]) are ubiquitous throughout number theory and beyond. For example, they appear in string theory [19, 24], the theory of black holes [11], and the combinatorics of partition statistics [6]. The Fourier coefficients of Jacobi forms often encode valuable arithmetic information. To describe a motivating example, let \( \lambda \) be a partition of a positive integer \( n \), i.e., a list of non-increasing positive integers \( \lambda_j \) with \( 1 \leq j \leq s \) that sum to \( n \). We will denote the number of partitions of \( n \) by \( p(n) \), as usual. One of the most famous results in partition theory is due to Ramanujan, who proved in [23] that for \( n \geq 0 \) the following congruences hold

\[
p(5n + 4) \equiv 0 \pmod{5}, \quad p(7n + 5) \equiv 0 \pmod{7}, \quad p(11n + 6) \equiv 0 \pmod{11}.
\]

The rank [13] of \( \lambda \) is given by the largest part minus the number of parts. It offers a combinatorial explanation for the first and second congruence as conjectured by Dyson [13] and later proved by Atkin and Swinnerton-Dyer [5], since the partitions...
of $5n + 4$ (respectively, $7n + 5$) form 5 (respectively, 7) equal-sized groups when sorted by their ranks modulo 5 (respectively, 7). Dyson additionally conjectured the existence of another statistic, which he called the crank and which should explain all Ramanujan congruences. The *crank* of $\lambda$ was later found by Andrews and Garvan [3, 15] and is given by

$$
\begin{cases}
\lambda_1 & \text{if } \lambda \text{ contains no ones}, \\
\mu(\lambda) - \omega(\lambda) & \text{if } \lambda \text{ contains ones}.
\end{cases}
$$

Here, $\omega(\lambda)$ denotes the number of ones in $\lambda$, and $\mu(\lambda)$ denotes the number of parts greater than $\omega(\lambda)$. We denote by $M(m, n)$ (resp. $N(m, n)$) the number of partitions of $n$ with crank $m$ (resp. rank $m$). Throughout the rest of the paper we let $\zeta := e^{2\pi i z}$ for $z \in \mathbb{C}$, and $q := e^{2\pi i \tau}$ with $\tau \in \mathbb{H}$, the upper half-plane. It is well known that the generating function of $M$ is given by (see [6, equation (2.1)])

$$
\sum_{n \geq 0 \atop m \in \mathbb{Z}} M(m, n) \zeta^m q^n = \frac{i \left( \zeta^\frac{1}{2} - \zeta^{-\frac{1}{2}} \right) q^{\frac{1}{24}} \eta^2(\tau)}{\vartheta(z; \tau)},
$$

which is a weak Jacobi form (up to rational powers of $\zeta$ and $q$). Here, the Dedekind $\eta$-function is given by

$$
\eta(\tau) := q^{\frac{1}{24}} \prod_{n \geq 1} \left( 1 - q^n \right),
$$

and the Jacobi theta function is defined by

$$
\vartheta(z; \tau) := iq^\frac{1}{8} \zeta^{\frac{1}{2}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2 + n}{2}} \zeta^n. \quad (1.1)
$$

Note that a similar formula can be found for the generating function of $N$ as a mock Jacobi form involving an eta-theta quotient. In general Jacobi forms have a Fourier expansion of the form

$$
\sum_{n \geq 0 \atop m \in \mathbb{Z}} a(m, n) \zeta^m q^n.
$$

Many interesting examples of Jacobi forms arise as quotients of $\eta$- and $\vartheta$-functions. As an illuminating example, for $a_k, b_j \in \mathbb{N}$ and $n \in \mathbb{Z}$, consider the study of theta quotients [16, equation (13)],

$$
\frac{\vartheta(a_1 z; \tau) \vartheta(a_2 z; \tau) \cdots \vartheta(a_k z; \tau)}{\vartheta(b_1 z; \tau) \vartheta(b_2 z; \tau) \cdots \vartheta(b_j z; \tau)} \eta(\tau)^n
$$
which provide new constructions of (not necessarily holomorphic) Jacobi and Siegel modular forms. As highlighted by Gritsenko, Skoruppa, and Zagier, theta quotients also have deep applications to areas such as Fourier analysis over infinite-dimensional Lie algebras and the moduli spaces in algebraic geometry. In the present paper, we obtain the bivariate asymptotic behavior of the coefficients of a prototypical family of such theta quotients, while the steps presented here also offer a pathway to obtain similar results for more general families. Our framework covers theta quotients for \( k = j = 1, a_1 = 1, b_1 \in \mathbb{N}, \text{ and } n \in \mathbb{Z}. \)

In [6] Bringmann and Dousse pioneered the use of new techniques in the study of the bivariate asymptotic behavior of the Fourier coefficients and applied them to the partition crank function. In [12] Dousse and Mertens used these techniques to study the rank function. In particular, each of these papers used an extension of Wright’s Circle Method [26, 27] to obtain bivariate asymptotics of \( N(m, n) \) and \( M(m, n) \), with \( m \) in a certain range depending on \( n \).

Recently, the second author of the present paper extended these techniques to an example appearing in the partition function for entanglement entropy in string theory. In particular, [19, 20] considered the eta-theta quotient

\[
\frac{\vartheta(z; \tau)^4}{\eta(\tau)^9 \vartheta(2z; \tau)} =: \sum_{m \in \mathbb{Z}, n \geq 0} b(m, n) \zeta^m q^m
\]

with a simple pole at \( z = \frac{1}{2} \). Then bivariate asymptotic behavior of the coefficients \( b(m, n) \) is given by [20, Theorem 1.1].

**Theorem 1.1** For \( \beta := \pi \sqrt{n} \) and \( |m| \leq \frac{1}{6\beta} \log(n) \) we have that

\[
b(m, n) = (-1)^{m+\delta+1} \frac{i \beta^6 m}{8\pi^5 (2n)^{\frac{3}{4}}} e^{2\pi \sqrt{2n}} O \left( mn^{-\frac{15}{4}} e^{2\pi \sqrt{2n}} \right)
\]

as \( n \to \infty \). Here, \( \delta := 1 \) if \( m < 0 \) and \( \delta = 0 \) otherwise.

The present paper serves to extend these results to a large family of eta-theta quotients with multiple simple poles.\(^1\) Such eta-theta quotients appear in numerous places. For example, investigations into Vafa–Witten invariants [1, equation (2.5)] involve the functions

\[
\frac{i}{\eta(\tau)^N \vartheta(2\tau)}
\]

which also appear in investigations into the counting of BPS states via wall-crossing [25, equation (5.114)]. The asymptotics of this family of functions was studied in [8]. Other examples of similar shapes also arise as natural pieces of functions in

\(^1\) A similar framework exists for those without poles by simply extending the results of [6, 12].
investigations into BPS states, see e.g., [25, Section 5.6.2]. Similar functions also appear prominently in Watson’s well-known quintuple product formula

$$\vartheta^*(\tau; z) := \sum_{r \in \mathbb{Z}} \left( \frac{12}{r} \right) q^{r^2} \xi^{r_2} = \frac{\eta(\tau) \vartheta(2z; \tau)}{\vartheta(z; \tau)}$$

which has a plethora of applications in number theory and combinatorics, and our main theorem gives a bivariate asymptotic for the coefficients of $\vartheta^*(z; \tau)^{-1}$. Such asymptotics for inverse theta functions are a topic currently in vogue in the literature, see e.g., [8, 18] and the references contained therein.

Throughout, we consider an eta-theta quotient of the form

$$f(z; \tau) := \frac{\vartheta(z; \tau)}{\vartheta(bz; c\tau)} \prod_{j=1}^{N} \eta(a_j \tau)^{\alpha_j},$$

where $a_j, b, c \in \mathbb{N}$, $N \in \mathbb{N}_{>1}$, and $\alpha_j \in \mathbb{Z}$. Since we require asymptotic growth, we assume that $\sum_{j=1}^{N} \frac{\alpha_j}{a_j} < 0$. We omit the dependency on these parameters for notational ease. We assume that $b$ is even, $b \neq c$, and $b^2 > c$, and indicate the differences that would occur if $b$ were odd. In the language of [16], this is a family of theta quotients.

**Remarks**

1. Note that by the conditions from above we assume that we have exponential growth toward the cusp 0 and therefore ensure that the Circle Method works by choosing the major arc around $q = 1$.

2. The exposition presented here may be easily generalized to include products of theta functions in both the numerator and denominator of $f$, although this becomes lengthy to write out for the general case.

3. We include a theta function in the numerator to allow us to assume that there are no poles of $f$ at the lattice points 0 or 1. However, using the techniques presented here and shifting integrals to not have end-points at 0 or 1, a similar method holds for functions without a theta function in the numerator.

We define the coefficients $c(m, n)$ by

$$f(z; \tau) := \sum_{n \geq 0}^{m \in \mathbb{Z}} c(m, n) \xi^m q^n,$$

for some $z$ in a small neighborhood of 0 that is pole-free, and investigate their bivariate asymptotic behavior. To this end, we employ and extend the techniques of [6], which also appear in [12, 19, 20], using Wright’s Circle Method to arrive at the following theorem.

**Theorem 1.2** Define $\beta = \beta(n) := \pi \sqrt{2n}$ and $w := \frac{1}{2} \sum_{j=1}^{N} \alpha_j \in \frac{1}{2} \mathbb{Z}$, which is the weight of the eta quotient part of our function $f$, along with
\[ \Lambda_1 := (-1)^{2w+1} \left( \frac{1}{2\pi} \right)^{-w} \left( \frac{1}{4\pi^2 (2b^2 - b - c)} \right)^{\frac{3}{4}} \prod_{j=1}^{N} a_j^{-\frac{\alpha_j}{2}}, \]

and

\[ \Lambda_2 := \frac{b^2}{c} - \frac{b}{c} + \frac{1}{4c} - \frac{1}{4} - \sum_{j=1}^{N} \frac{\alpha_j}{12a_j}. \]

Assume that \( 0 < 1 - \sum_{j=1}^{N} \frac{\alpha_j}{12a_j} < \sqrt{\Lambda_2}, \sum_{j=1}^{N} \frac{\alpha_j}{a_j} < 0, b \) even with \( b \neq c, b^2 > c, \) and \( m = m(n) \) with \( |m| \leq \frac{1}{6\beta} n^{-\delta} \log(n) \) for some \( 0 < \delta < \frac{1}{2} \) such that \( m \to \infty \) as \( n \to \infty. \) Then

\[ c(m, n) = \frac{-i}{2\pi} \Lambda_1^{\beta-2w} \sqrt{\Lambda_2}^{-w} e^{2\pi\sqrt{2\Lambda_2 n}} \frac{e^{2\pi \sqrt{2\Lambda_2 n}}}{2\pi (2\Lambda_2 n)^{\frac{1}{4}}} + O \left( \frac{\beta^{3-w} e^{2\pi \sqrt{2\Lambda_2 n}}}{2\pi (2\Lambda_2 n)^{\frac{1}{4}}} \right) \]

as \( n \to \infty. \)

**Remark** Note that the restriction on \( \Lambda_2 \) still leaves infinitely many choices.

The paper is structured as follows. We begin in Sect. 2 by recalling results that are relevant to the rest of the paper. Section 3 deals with defining the Fourier coefficients of \( \zeta^m \) of \( f. \) In Sect. 4 we investigate the behavior of \( f \) toward the dominant pole \( q = 1. \) We follow this in Sect. 5 by bounding the contribution away from the pole at \( q = 1. \) In Sect. 6 we obtain the asymptotic behavior of \( c(m, n) \) and hence prove Theorem 1.2.

## 2 Preliminaries

Here, we recall relevant definitions and results which will be used throughout the rest of the paper.

### 2.1 Properties of \( \vartheta \) and \( \eta \)

When determining the asymptotic behavior of \( f \) we require the modular properties of both \( \vartheta \) and \( \eta. \) We will from now on define the square-root using the principal branch, which means that we exclude the negative reals and impose positive square roots for positive real numbers.

It is well known that \( \vartheta \) is a Jacobi form (see e.g., [22]).

**Lemma 2.1** The function \( \vartheta \) satisfies

\[ \vartheta(z; \tau) = -\vartheta(-z; \tau), \quad \vartheta(z; \tau) = -\vartheta(z + 1; \tau), \]

\[ \vartheta(z; \tau) = \frac{i}{\sqrt{-i\tau}} e^{-\pi i \frac{z^2}{\tau}} \vartheta \left( \frac{z}{\tau}; -\frac{1}{\tau} \right). \]
We also have the well-known triple product formula (see e.g., [28, Proposition 1.3] for this explicit formulation) that yields

$$\vartheta(z; \tau) = i \zeta^{\frac{1}{2}} q^{\frac{1}{8}} \prod_{n \geq 1} \left(1 - q^n \right) \left(1 - \zeta q^n \right) \left(1 - \zeta^{-1} q^{n-1} \right). \quad (2.1)$$

Furthermore, we have the following modular transformation formula of $\eta$ (see e.g., [17]).

**Lemma 2.2** We have that

$$\eta(\tau) = \sqrt{\frac{i}{\tau}} \eta \left(-\frac{1}{\tau}\right).$$

### 2.2 Integrals over segments of circles

Let $U_r(z_0) := \{z : |z - z_0| < r\}$ be the open disc around $z_0 \in \mathbb{C}$ with radius $r > 0$. Then we have the following result [10, page 263].

**Lemma 2.3** Let $g : U_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ be analytic and have a simple pole at $z_0$. Let $\gamma(\delta)$ be a circular arc with parametric equation $z = z_0 + \delta e^{i\theta}$, for $-\pi < \theta_1 \leq \theta \leq \theta_2 \leq \pi$ and $0 < \delta < r$. Then

$$\lim_{\delta \to 0} \int_{\gamma(\delta)} g(z) \, dz = i(\theta_2 - \theta_1) \text{Res}_{z_0}(g).$$

where $\text{Res}_{z_0}(g)$ denotes the residue of $g$ at $z_0$.

### 2.3 A particular bound

We require a bound on the size of

$$P(q) := \frac{q^{\frac{1}{2}}}{\eta(\tau)},$$

away from the pole at $q = 1$. For this we use [6, Lemma 3.5].

**Lemma 2.4** Let $M > 0$ be a fixed constant. Let $\tau = u + iv \in \mathbb{H}$ with $Mv \leq u \leq \frac{1}{2}$ for $u > 0$ and $v \to 0$. Then

$$|P(q)| \ll \sqrt{v} \exp \left[ \frac{1}{v} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left(1 - \frac{1}{\sqrt{1 + M^2}} \right) \right) \right].$$
In particular, with 
\[ v = \frac{\beta}{2\pi}, u = \frac{\beta m^{-\frac{1}{3}}}{2\pi}, \]
and 
\[ M = m^{-\frac{1}{3}}, \]
the bound
\[ |P(q)| \ll n^{-\frac{1}{4}} \exp \left[ \frac{2\pi}{\beta} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-\frac{2}{3}}}} \right) \right) \right]. \tag{2.2} \]

### 2.4 I-Bessel functions

Here, we recall relevant results on the \( I \)-Bessel function which for \( x > 0 \) may be written as (see e.g., [4, 6])

\[ I_\ell(x) := \frac{1}{2\pi i} \int_\Gamma t^{-\ell-1} e^{\frac{x}{2} (i + t)} dt, \]

where \( \Gamma \) is a contour which starts in the lower half-plane at \(-\infty\), surrounds the origin counterclockwise, and returns to \(-\infty\) in the upper half-plane. We are particularly interested in the asymptotic behavior of \( I_\ell \), given in the following lemma (see e.g., [2, (4.12.7)]).

**Lemma 2.5** For fixed \( \ell \) we have

\[ I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O \left( \frac{e^x}{x^\frac{3}{2}} \right) \]

as \( x \to \infty \).

### 3 Fourier coefficients of \( f \)

Note that \( f(-z; \tau) = f(z; \tau) \) by Lemma 2.1, and so \( c(-m, n) = c(m, n) \). For the case \( m = 0 \) one can use classical results (see e.g., [7, Theorem 15.10]) to calculate the Fourier coefficients. We therefore restrict our attention to the case \( m > 0 \).

We first define the Fourier coefficients of \( \zeta^m \) of \( f \). Since we focus only on the case \( z \in [0, 1] \), we let \( h_1, ..., h_s \in \mathbb{Q} \) denote the poles of \( f \) in this range. Note that the distribution of the poles is symmetric on the interval in question.

Define the path of integration \( \Gamma_{\ell,r} \) by

\[ \Gamma_{\ell,r} := \begin{cases} 0 \text{ to } h_1 - r & \text{if } \ell = 0, \\ h_\ell + r \text{ to } h_{\ell+1} - r & \text{if } 1 \leq \ell \leq s - 1, \\ h_s + r \text{ to } 1 & \text{if } \ell = s, \end{cases} \]

for some \( r > 0 \) sufficiently small. Following the framework of [11, 19, 20], we define

\[ f_m^\pm(\tau) := \sum_{\ell=0}^{s} \int_{\Gamma_{\ell,r}} f(z; \tau) e^{-2\pi i m z} dz + \sum_{\ell=1}^{s} G_{\ell,r}^\pm, \]
Fig. 1 The path of integration taking $\gamma^+_{\ell,r}$ at each pole

\[
\gamma^+_{\ell,r}
\]

Fig. 2 The contour $\gamma^+_{\ell,r}$ for a fixed $\ell$

where

\[
G^\pm_{\ell,r} := \int_{\gamma^\pm_{\ell,r}} f(z; \tau) e^{-2\pi imz} \, dz
\]

for a fixed pole $h_\ell$ ($1 \leq \ell \leq s$). Here, $\gamma^+_{\ell,r}$ is the semi-circular path of radius $r$ passing above the pole $h_\ell$ and $\gamma^-_{\ell,r}$ is the semi-circular path passing below the pole $h_\ell$, see Figs. 1 and 2.

Following [11] the Fourier coefficient of $\zeta^m$ of $f$, for fixed $m$, is given by

\[
f_m(\tau) := \lim_{r \to 0^+} \frac{f^+_m(\tau) + f^-_m(\tau)}{2} = \lim_{r \to 0^+} \frac{1}{2} \left( 2 \sum_{\ell=0}^s \int_{\Gamma_{\ell,r}} f(z; \tau) e^{-2\pi imz} \, dz + \sum_{\ell=1}^s G^+_{\ell,r} + G^-_{\ell,r} \right). \tag{3.1}
\]

For fixed $\ell$ we use Lemma 2.3 to see that

\[
\lim_{r \to 0^+} \left( G^+_{\ell,r} + G^-_{\ell,r} \right) = 0,
\]

since we only have simple poles.

The substitution $z \mapsto 1 - z$ gives us

\[
\sum_{\ell=0}^s \int_{\Gamma_{\ell,r}} f(z; \tau) e^{-2\pi imz} \, dz = - \sum_{\ell=0}^s \int_{\Gamma_{\ell,r}} f(z; \tau) e^{2\pi imz} \, dz,
\]

since $b$ is even and using that $f(1 - z; \tau) = (-1)^{b+1} f(z; \tau)$ by Lemma 2.1. Thus, (3.1) simplifies to
\( f_m(\tau) = -i \lim_{r \to 0^+} \sum_{\ell=0}^{s} \int_{\Gamma_{\ell,r}} f(z; \tau) \sin(2\pi mz)dz. \)  \( (3.2) \)

**Remark** For odd \( b \) one would obtain a similar formula with the integrand replaced by \( f(z; \tau) \cos(2\pi mz) \).

In the following two sections, we determine the asymptotic behavior of \( f \) toward and away from the dominant pole at \( q = 1 \), respectively. From now on we will let \( \tau = \frac{i\varepsilon}{2\pi}, \varepsilon := \beta(1 + ixm^{-\frac{1}{3}}), \beta = \pi\sqrt{2n}, \) and \( |m| \leq \frac{1}{6\beta^3} \log(n) \) for some \( 0 < \delta < \frac{1}{2} \) such that \( m \to \infty \) as \( n \to \infty \).

### 4 Bounds toward the dominant pole

In this section, we consider the behavior of \( f_m \) toward the dominant pole at \( q = 1 \). Remember that we have \( w \in \frac{1}{2} \mathbb{Z} \) by definition (see Theorem 1.2).

**Lemma 4.1** Let \( \tau = \frac{i\varepsilon}{2\pi} \), with \( 0 < \text{Re}(\varepsilon) \ll 1 \), let \( z \) be away from the poles, let \( M(z) \) be the function defined in (4.2) which is positive for all \( z \in (0, 1) \), and let

\[
C(z; \varepsilon) := (-1)^{\frac{1}{2}}e^{\frac{-\varepsilon}{4\pi^2 \varepsilon}} \left( \prod_{j=1}^{N} \frac{1 - e^{-\frac{4\pi^2 \varepsilon}{a_j \varepsilon}}}{1 - e^{-\frac{4\pi^2 \varepsilon}{2c \varepsilon}}} \right) \frac{\sinh(\frac{2\pi y}{c})}{\sinh(\frac{2\pi^2 b z}{c})} e^{\frac{2\pi^2 b z}{c}} \sum_{j=1}^{N} \frac{a_j}{a_j^2}.
\]

Then we have that

\[
f\left( z; \frac{i\varepsilon}{2\pi} \right) = C\left( z; \frac{i\varepsilon}{2\pi} \right) \left( 1 + O\left( e^{-\frac{4\pi^2 \varepsilon}{\varepsilon} M(z)} \right) \right)
\]

as \( n \to \infty \).

**Proof** Using Lemmas 2.1 and 2.2, the definition of \( \eta \), and (2.1), a lengthy but straightforward calculation shows that

\[
f\left( z; \frac{i\varepsilon}{2\pi} \right) = C\left( z; \frac{i\varepsilon}{2\pi} \right) \prod_{\kappa \geq 1} \left( 1 - e^{-\frac{4\pi^2 \varepsilon}{\varepsilon}} \right) \left( 1 - e^{-\frac{4\pi^2 \varepsilon}{2c \varepsilon}} \right) \left( 1 - e^{-\frac{4\pi^2 \varepsilon}{2c \varepsilon}} \right) \left( 1 - e^{-\frac{4\pi^2 \varepsilon}{c}} \right) \left( 1 - e^{-\frac{4\pi^2 \varepsilon}{c}} \right) \left( 1 - e^{-\frac{4\pi^2 \varepsilon}{c}} \right). \]

\( \square \)
In order to find a bound we inspect the asymptotic behavior of the product over $\kappa$. Splitting $a_j$ into positive and negative powers, labeled by $\gamma_j$, $\delta_j \in \mathbb{N}$ and $a_j$, into $x_j$ and $y_j$, respectively, we first rewrite this as

$$
\prod_{\kappa \geq 1} \frac{(1 - e^{-4\pi^2\kappa x/c}) (1 - e^{-4\pi^2\kappa (x+c)}) (1 - e^{-4\pi^2\kappa (x+c+z)}) \prod_{j=1}^{N_1} (1 - e^{-4\pi^2\kappa \gamma_j}) \prod_{j=1}^{N_2} (1 - e^{-4\pi^2\kappa \delta_j})}{(1 - e^{-4\pi^2\kappa (x-bz)}) (1 - e^{-4\pi^2\kappa (x-bz+c)}) (1 - e^{-4\pi^2\kappa (x-bz+c)})},
$$

(4.1)

since $|e^{-4\pi^2\kappa y/c}| < 1$ for all $\kappa \geq 1$. We also have that $|e^{-4\pi^2\kappa (x+c)}| < 1$ and $|e^{-4\pi^2\kappa (x+c+z)}| < 1$ for all $\kappa \geq 1$ since $b, c \in \mathbb{N}$. Therefore, we have that

$$
\frac{1}{(1 - e^{-4\pi^2\kappa (x-bz)}) (1 - e^{-4\pi^2\kappa (x-bz+c)})} = \sum_{\kappa \geq 0} e^{-4\pi^2\kappa \mu/c} \sum_{\xi \geq 0} e^{-4\pi^2\kappa \xi/c}.
$$

Up to this point our calculations are independent of the size of $z$. The remaining term is

$$
\frac{1}{1 - e^{-4\pi^2\kappa (x-bz)}}.
$$

Let $\kappa_0$ be the smallest $\kappa \geq 1$ such that $(\kappa - bz) \geq 0$. We may rewrite

$$
\prod_{\kappa \geq 1} \frac{1}{(1 - e^{-4\pi^2\kappa (x-bz)})} = \prod_{\kappa = 1}^{\kappa_0-1} \frac{1}{(1 - e^{-4\pi^2\kappa (x-bz)})} \prod_{\kappa \geq \kappa_0} \sum_{\mu \geq 0} e^{-4\pi^2\kappa \mu/c} (x-bz).
$$

The first product is

$$
\prod_{\kappa = 1}^{\kappa_0-1} \frac{1}{(1 - e^{-4\pi^2\kappa (x-bz)})} = \prod_{\kappa = 1}^{\kappa_0-1} (-e^{4\pi^2\kappa (x-bz)c}) \sum_{v \geq 0} e^{4\pi^2v (x-bz)}.
$$

Let

$$
\mathcal{M}(z) := \begin{cases} 
\min \left( 1 - z, \frac{1}{x_j}, \frac{1}{y_k}, c, \frac{\kappa_0-bz}{c}, \frac{bz+1-k_0}{c} \right) & \text{if } \kappa_0 \neq 1, \\
\min \left( 1 - z, \frac{1}{x_j}, \frac{1}{y_k}, \frac{1}{c}, \frac{\kappa_0-bz}{c} \right) & \text{if } \kappa_0 = 1,
\end{cases}
$$

(4.2)

running over all $1 \leq j \leq N_1$ and $1 \leq k \leq N_2$. Note that for $0 < \Re(\varepsilon) \ll 1$, and $z \in (0, 1)$ we have $\mathcal{M}(z) > 0$, so the product in (4.1) is of order

$$
1 + O(e^{-4\pi^2/c \mathcal{M}(z)}),
$$
which finishes the proof. \(\square\)

**Remark** By separating into cases, one is able to obtain more precise asymptotics. However, this is not required for what follows and we leave the details for the interested reader.

**Theorem 4.2** Let \(\Lambda_1\) and \(\Lambda_2\) be defined as in Theorem 1.2. For \(|x| \leq 1\) we have that

\[
f_m\left(\frac{i \varepsilon}{2\pi}\right) = \Lambda_1 e^{1-w} e^{\frac{2n^2}{i}} \Lambda_2 + O\left(\beta^{2-w} e^{\frac{2n^2}{i}} \Lambda_2\right)
\]

as \(n \to \infty\).

**Proof** Plugging Lemma 4.1 into (3.2) yields

\[
f_m\left(\frac{i \varepsilon}{2\pi}\right) = -i \sum_{\ell=0}^{s} \lim_{r \to 0^+} \int_{\Gamma_{L,r}} C(z) \left(1 + O\left(e^{-\frac{4\pi^2}{2\varepsilon} M(z)}\right)\right) \sin(2\pi mz) dz.
\]

We have that

\[
\frac{\sinh \left(\frac{2\pi^2 z}{\varepsilon}\right)}{\sinh \left(\frac{2\pi^2 \beta z}{c}\right)} = e^{\frac{2\pi^2 \beta z}{c}} \left(1 - e^{\frac{4\pi^2 \beta z}{c}}\right) \sum_{\lambda \geq 0} e^{-\frac{4\pi^2 \lambda \beta z}{c}} = e^{\frac{2\pi^2 \beta z}{c}} \left(1 + O\left(e^{-\frac{4\pi^2 \beta z}{c}}\right)\right),
\]

using \(|e^{-\frac{4\pi^2 \beta z}{c}}| < 1\). Additionally we see that

\[
e^{\frac{2\pi^2 \beta z}{c}} \left(\frac{k^2}{c} - 1\right) z^2 e^{\frac{2\pi^2 \beta z}{c}} = e^{\frac{2\pi^2 \beta z}{c}} \left(\frac{k^2}{c} - 1\right) z^2 \sum_{j=1}^{N} \frac{a_j}{12a_j}.
\]

Defining

\[
\Omega(m, n) := (-1)^w \left(\frac{\varepsilon}{2\pi}\right)^{-w} e^{\frac{1}{2}} \left(\prod_{j=1}^{N} a_j^{-a_j^2}\right) e^{\frac{2\pi^2}{c} \left(\frac{1}{4} - \frac{1}{3} + \sum_{j=1}^{N} \frac{a_j}{12a_j}\right)},
\]

we can then rewrite (4.3) as

\[
-i \Omega(m, n) \sum_{\ell=0}^{s} \lim_{r \to 0^+} \int_{\Gamma_{L,r}} e^{\frac{2\pi^2 (k^2 - 1) z^2}{c}} e^{\frac{2\pi^2 \beta z}{c}} \left(1 + O\left(e^{-\frac{4\pi^2 \beta z}{c}} N(z)\right)\right) \sin(2\pi mz) dz,
\]

where \(N(z) := \min(z, M(z))\).
We immediately see that this splits up into two integrals

$$\sum_{\ell = 0}^{s} \lim_{r \to 0^+} \int_{\Gamma_{\ell,r}} e^{\frac{2\pi^2}{r^2}(\ell^2-1)z^2} e^{2\pi^2(1-b/r)z} \sin(2\pi mz)dz$$

(4.4)

and

$$\sum_{\ell = 0}^{s} \lim_{r \to 0^+} \int_{\Gamma_{\ell,r}} e^{\frac{2\pi^2}{r^2}(\ell^2-1)z^2} e^{2\pi^2(1-b/r)z} O \left(e^{-\frac{4\pi^2}{r^2}N(z)}\right) \sin(2\pi mz)dz.$$  (4.5)

Let $\text{erf}(z) := \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2} dt$ denote the error function and note that $\frac{d}{dz} \text{erf}(z) = \frac{2e^{-z^2}}{\sqrt{\pi}}$. For arbitrary $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{C}$, with $\mathcal{H}_2 \neq 0$ a straightforward calculation, using the identity $\frac{1}{2i}(e^{2\pi imz} - e^{-2\pi imz}) = \sin(2\pi mz)$, gives us that

$$d\frac{\sqrt{\pi}}{4\sqrt{\mathcal{H}_2}} \left[e^{-\frac{1}{4}(\mathcal{H}_1+2\pi im)^2} \text{erf} \left(\frac{1}{2}\mathcal{H}_2 t + \mathcal{H}_1 + 2\pi im\right)\right] = e^{\mathcal{H}_1 z} e^{\mathcal{H}_2 z^2} \sin(2\pi mz).$$

Therefore the following formula holds

$$\int_{t}^{u} e^{\mathcal{H}_1 z} e^{\mathcal{H}_2 z^2} \sin(2\pi mz)dz$$

$$= \frac{\sqrt{\pi}}{4\sqrt{\mathcal{H}_2}} \left[e^{-\frac{1}{4}(\mathcal{H}_1+2\pi im)^2} \text{erf} \left(\frac{1}{2}\mathcal{H}_2 t + \mathcal{H}_1 + 2\pi im\right)\right]$$

$$+ e^{-\frac{1}{4}(\mathcal{H}_1+2\pi im)^2} \text{erf} \left(\frac{1}{2}\mathcal{H}_2 u - \mathcal{H}_1 + 2\pi im\right)$$

$$- e^{-\frac{1}{4}(\mathcal{H}_1+2\pi im)^2} \text{erf} \left(\frac{1}{2}\mathcal{H}_2 u + \mathcal{H}_1 + 2\pi im\right)$$

$$- e^{-\frac{1}{4}(\mathcal{H}_1+2\pi im)^2} \text{erf} \left(\frac{1}{2}\mathcal{H}_2 t - \mathcal{H}_1 + 2\pi im\right).$$

For arbitrary $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{C}$, with $\mathcal{H}_2 \neq 0$ we thus obtain

$$\sum_{\ell = 0}^{s} \lim_{r \to 0^+} \int_{\Gamma_{\ell,r}} e^{\mathcal{H}_2 z^2} e^{\mathcal{H}_1 z} \sin(2\pi mz)dz$$

$$= \frac{\sqrt{\pi}}{4\sqrt{\mathcal{H}_2}} \left[e^{-\frac{1}{4}(\mathcal{H}_1+2\pi im)^2} \text{erf} \left(\frac{1}{2}\mathcal{H}_1 + 2\pi im\right)\right] + e^{-\frac{1}{4}(\mathcal{H}_1+2\pi im)^2} \text{erf} \left(\frac{1}{2}\mathcal{H}_2 - \mathcal{H}_1 + 2\pi im\right)$$

$$+ e^{-\frac{1}{4}(\mathcal{H}_1+2\pi im)^2} \text{erf} \left(\frac{1}{2}\mathcal{H}_2 + \mathcal{H}_1 + 2\pi im\right) - e^{-\frac{1}{4}(\mathcal{H}_1+2\pi im)^2} \text{erf} \left(\frac{1}{2}\mathcal{H}_2 t - \mathcal{H}_1 + 2\pi im\right).$$
Using \( |\text{Arg}(\pm z)| < \frac{\pi}{4} \), we have that (see e.g., [9, page 10])

\[
erf(iz) = \frac{ie^{z^2}}{\sqrt{\pi}z} \left(1 + O\left(|z|^2\right)\right) = \frac{ie^{z^2}}{\sqrt{\pi}z} + O\left(e^{z^2}|z|^{-3}\right),
\]

as \( |z| \to \infty \). Note that taking the limit \( |z| \to \infty \) is equivalent to taking the limit \( n \to \infty \) in our setting.

Consider the integral (4.4), so set \( \mathcal{H}_1 = \frac{2\pi^2}{e} \left(1 - \frac{b}{c}\right) \) and \( \mathcal{H}_2 = \frac{2\pi^2}{e} \left(\frac{b^2}{c} - 1\right) \). In this case, since \( \frac{b^2}{c} > 1 \), we obtain

\[
\frac{1 \pm \mathcal{H}_1 + 2\pi im}{2 \sqrt{-\mathcal{H}_2}} = \pm \frac{\left(\frac{\pi}{e} \left(1 - \frac{b}{c}\right)\right) + im}{i \sqrt{\frac{2}{e} \left(\frac{b^2}{c} - 1\right)}} = iz_1,
\]

respectively,

\[
\frac{1 \pm 2\mathcal{H}_2 \pm \mathcal{H}_1 + 2\pi im}{2 \sqrt{-\mathcal{H}_2}} = \pm \frac{\left(\frac{\pi}{e} \left(\frac{b^2}{c} - 1\right)\right) \pm \left(\frac{\pi}{e} \left(1 - \frac{b}{c}\right)\right) + im}{i \sqrt{\frac{2}{e} \left(\frac{b^2}{c} - 1\right)}} = iz_2.
\]

Using \( \varepsilon = \beta(1 + ixm^{-\frac{1}{3}}) \) and \( \sqrt{z} = |z|\cos\left(\frac{1}{2} \text{Arg}(z)\right) + i |z| \sin\left(\frac{1}{2} \text{Arg}(z)\right) \) a straightforward calculation shows that

\[
z_1 = 2 \left(\frac{\frac{b^2}{c} - 1}{1 + x^2m^{-\frac{2}{3}}}\right) \sqrt{\frac{2}{e} \left(\frac{b^2}{c} - 1\right)} \times \left[ \left(\mp \pi \left(1 - \frac{b}{c}\right) + \beta x m^{\frac{2}{3}}\right) \cos\left(\frac{1}{2} \text{Arg} \left(1 - ixm^{-\frac{1}{3}}\right)\right) + \beta m \sin\left(\frac{1}{2} \text{Arg} \left(1 - ixm^{-\frac{1}{3}}\right)\right) + i \left(\mp \beta m \cos\left(\frac{1}{2} \text{Arg} \left(1 - ixm^{-\frac{1}{3}}\right)\right) + \left(\mp \pi \left(1 - \frac{b}{c}\right) + \beta x m^{\frac{2}{3}}\right) \sin\left(\frac{1}{2} \text{Arg} \left(1 - ixm^{-\frac{1}{3}}\right)\right)\right].
\]

respectively,

\[
z_2 = 2 \left(\frac{\frac{b^2}{c} - 1}{1 + x^2m^{-\frac{2}{3}}}\right) \sqrt{\frac{2}{e} \left(\frac{b^2}{c} - 1\right)} \times \left[ \left(\mp \pi \left(1 - \frac{b}{c}\right) + 2\pi \left(\frac{b^2}{c} - 1\right)\right) + \beta x m^{\frac{2}{3}} \cos\left(\frac{1}{2} \text{Arg} \left(1 - ixm^{-\frac{1}{3}}\right)\right) + \beta m \sin\left(\frac{1}{2} \text{Arg} \left(1 - ixm^{-\frac{1}{3}}\right)\right) + i \left(\mp \beta m \cos\left(\frac{1}{2} \text{Arg} \left(1 - ixm^{-\frac{1}{3}}\right)\right) + \left(\mp \pi \left(1 - \frac{b}{c}\right) + 2\pi \left(\frac{b^2}{c} - 1\right)\right) + \beta x m^{\frac{2}{3}} \sin\left(\frac{1}{2} \text{Arg} \left(1 - ixm^{-\frac{1}{3}}\right)\right)\right].
\]
Since $|x| < 1$ we see that $|\text{Arg}(1 - ixm^{-\frac{1}{2}})| < \frac{\pi}{4}$ and thus we have
\[
\left| \cos \left( \frac{1}{2} \text{Arg} \left( 1 - ixm^{-\frac{1}{2}} \right) \right) \right| > \left| \sin \left( \frac{1}{2} \text{Arg} \left( 1 - ixm^{-\frac{1}{2}} \right) \right) \right|.
\] (4.8)

From the assumption $|m| \leq \frac{1}{6\beta} n^{-\delta} \log(n)$ for some $0 < \delta < \frac{1}{2}$ we not only ensure that $m \to \infty$ as $n \to \infty$ but additionally that $\beta m \to 0$ and $\beta m^{\frac{2}{3}} \to 0$ as $n \to \infty$. Thus, together with (4.8), we see that $|\text{Re}(z_1)| > |\text{Im}(z_1)|$, respectively, $|\text{Re}(z_2)| > |\text{Im}(z_2)|$ for $n$ sufficiently large.

Therefore the arguments of the error functions in (4.6) satisfy the condition of (4.7).

Plugging in yields
\[
\sum_{\ell=0}^s \lim_{\epsilon \to 0^+} \int_{\gamma_{\ell,r}} e^{\mathcal{H}_2 z^2} e^{\mathcal{H}_1 z} \sin(2\pi m z) dz
\]
\[
= \frac{\sqrt{\pi}}{4\sqrt{\mathcal{H}_2}} \left( -\frac{ie^{\mathcal{H}_2+\mathcal{H}_1}}{\sqrt{\pi}} \right) + O \left( e^{\mathcal{H}_2+\mathcal{H}_1} \left| \frac{1}{2} \mathcal{H}_2 + \mathcal{H}_1 + 2\pi i m \right|^{-3} \right)
\]
\[
= \frac{e^{\mathcal{H}_2+\mathcal{H}_1}}{4i\mathcal{H}_2 + 2i\mathcal{H}_1 - 4\pi m} + O \left( \frac{|\sqrt{\mathcal{H}_2} e^{\mathcal{H}_2+\mathcal{H}_1}|}{\left| \frac{1}{2} \mathcal{H}_2 + \mathcal{H}_1 + 2\pi i m \right|^{-3}} \right).
\]

We thus obtain that (4.4) equals
\[
\frac{e^{2\pi^2 x} \left( \frac{b^2}{c} - \frac{b}{c} \right)}{4\pi^2 e^{-v_{\epsilon}} \left( \frac{2b^2}{c} - 1 - \frac{b}{c} \right) - 4\pi m + O \left( \frac{\sqrt{\pi}}{4} \left| \frac{e^{2\pi^2 x} \left( \frac{b^2}{c} - \frac{b}{c} \right)}{\sqrt{2\pi^2 e^{-v_{\epsilon}} \left( \frac{b^2}{c} - 1 \right)} - 4\pi m \right|^{-3} \right)}
\]

Combining this, along with the fact that $\mathcal{N}(z) > 0$ and recycling the same arguments for (4.5), yields
\[
f_m \left( \frac{i \epsilon}{2\pi} \right) = -i \Omega(m, n) \left( \frac{e^{2\pi^2 x} \left( \frac{b^2}{c} - \frac{b}{c} \right)}{4\pi^2 e^{-v_{\epsilon}} \left( \frac{2b^2}{c} - 1 - \frac{b}{c} \right) - 4\pi m + O \left( \frac{\sqrt{\pi}}{4} \left| \frac{e^{2\pi^2 x} \left( \frac{b^2}{c} - \frac{b}{c} \right)}{\sqrt{2\pi^2 e^{-v_{\epsilon}} \left( \frac{b^2}{c} - 1 \right)} - 4\pi m \right|^{-3} \right)} \right)
\]

Plugging in $\Omega(m, n)$ yields the claim. The main term here simplifies to
\[
\Lambda \epsilon^{1-w} e^{2\pi^2 x} \Lambda_2
\]
since $4\pi m \epsilon \to 0$ as $n \to \infty$. \qed
5 Bounds away from the dominant pole

In this section, we investigate the contribution of \( f_m \) away from the dominant pole at \( q = 1 \), and show that it forms part of the error term. Recall that from (3.2) we have

\[
 f_m(\tau) = -i \lim_{r \to 0^+} \left( \sum_{\ell=0}^{s} \int_{\Gamma_{\ell,r}} f(z; \tau) \sin(2\pi m z) \, dz \right).
\]

One immediately sees that

\[
 \left| \sum_{\ell=0}^{s} \int_{\Gamma_{\ell,r}} f(z; \tau) \sin(2\pi m z) \, dz \right| \ll \sum_{\ell=0}^{s} \left| \int_{\Gamma_{\ell,r}} f(z; \tau) \sin(2\pi m z) \, dz \right|.
\]

Consider

\[
 |f(z; \tau) \sin(2\pi m z)| = \left| \prod_{j=1}^{N} \eta(a_j \tau)^{\alpha_j} \frac{\vartheta(z; \tau)}{\vartheta(bz; c\tau)} \right| |\sin(2\pi m z)| 
\]

away from the dominant pole. We begin with the term \( \prod_{j=1}^{N} \eta(a_j \tau)^{\alpha_j} \). As in [19] we write

\[
 \prod_{j=1}^{N} \eta(a_j \tau)^{\alpha_j} = \prod_{j=1}^{N} \eta(x_j \tau)^{\alpha_j} \prod_{k=1}^{N_2} q^{-\frac{y_k \delta_k}{2\pi}} P(q^{y_k})^{\delta_k}.
\]

Using Lemma 2.2 we see that

\[
 \eta \left( \frac{ix_j \varepsilon}{2\pi} \right)^{y_j} \ll \left( \frac{2\pi}{x_j \beta} \right)^{y_j} e^{-\frac{\pi^2 y_j}{6\alpha_j \beta}}.
\]

By (2.2) we also obtain that

\[
 |P(q^{y_k})| \ll n^{-\frac{1}{4}} \exp \left[ \frac{2\pi}{\gamma_k \beta} \left( \pi \frac{12}{2\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-2/3}}} \right) \right) \right].
\]

Therefore we find

\[
 \prod_{j=1}^{N} \eta \left( \frac{ix_j \varepsilon}{2\pi} \right)^{\alpha_j} \ll \left( \prod_{j=1}^{N_1} \frac{2\pi}{x_j \beta} \right)^{y_j} e^{-\sum_{j=1}^{N_1} \frac{\pi^2 y_j}{6\alpha_j \beta}}.
\]
\[ \times \prod_{k=1}^{N_2} n^{-\frac{\delta_k}{y_k}} \exp \left[ \frac{2\pi \delta_k}{y_k \beta} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-\frac{2}{3}}}} \right) \right) \right], \]

and thus we obtain

\[
\left| \prod_{j=1}^{N} \eta(a_j \tau)^{a_j} \right| \left| \frac{\vartheta(z; \tau)}{\vartheta(bz; c\tau)} \right| \ll \left| \left( \prod_{j=1}^{N_1} \frac{2\pi}{x_j \beta} \right)^{y_j} e^{-\sum_{j=1}^{N_1} \frac{\pi^2 y_j}{6x_j \beta}} \prod_{k=1}^{N_2} n^{-\frac{\delta_k}{y_k}} \right| \times \exp \left[ \frac{2\pi \delta_k}{y_k \beta} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \left( 1 + m^{-\frac{2}{3}} \right)^{-\frac{1}{2}} \right) \right) \right]. \] (5.1)

Plugging in (1.1), using Lemma 2.1, and rearranging leads to

\[
\left| \frac{\vartheta(z; \tau)}{\vartheta(bz; c\tau)} \right| = \left| q^{-\frac{\delta}{y}} \right| \left| \frac{\vartheta(z; \tau)}{\vartheta(bz; c\tau)} \right| \ll \sum_{\kappa \in \mathbb{Z}} (-1)^\kappa q^{-\frac{\pi^2}{\beta} \left( \kappa^2 + (1-2z) \kappa \right)} \ll n^{\frac{1}{\beta}} e^{-\frac{2\pi^2}{\beta} \left( \kappa^2 + (1-2z) \kappa \right)} \ll n^{\frac{1}{\beta}}. \] (5.2)

Define

\[ B(m, n) := n^{\frac{1}{\beta}} \prod_{k=1}^{N_2} \frac{\delta_k}{y_k} \prod_{j=1}^{N_1} \left( \frac{2\pi}{x_j \beta} \right)^{y_j}. \]

Then equations (5.1) and (5.2) imply that for \( r \to 0^+ \)

\[
\left| \sum_{\ell=0}^{x} \int_{\Gamma_{\ell, r}} f(z; \tau) \sin(2\pi mz) \, dz \right| \ll \sum_{\ell=0}^{x} B(m, n) \exp \left[ \sum_{k=1}^{N_2} \frac{2\pi \delta_k}{y_k \beta} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-\frac{2}{3}}}} \right) \right) - \sum_{j=1}^{N_1} \frac{\pi^2 y_j}{6x_j \beta} \right].
\]

Hence, away from the dominant pole at \( q = 1 \) we have shown the following proposition.

**Proposition 5.1** For \( 1 \leq x \leq \frac{\pi m^{\frac{3}{4}}}{\beta} \) we have that
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\[ \left| f_m \left( \frac{i\epsilon}{2\pi} \right) \right| \ll (s + 1)B(m, n) \]

\[ \times \exp \left[ \sum_{k=1}^{N_2} \frac{2\pi \delta_k}{\gamma_k \beta} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1 + m^{-\frac{2}{3}}}} \right) \right) - \sum_{j=1}^{N_1} \frac{\pi^2 \gamma_j}{6x_j \beta} \right] \]

as \( n \to \infty. \)

6 The circle method

In this section, we use Wright’s variant of the Circle Method to complete the proof of Theorem 1.2. Cauchy’s Theorem implies that

\[ c(m, n) = \frac{1}{2\pi i} \int_C \frac{f_m(\tau)}{q^{n+1}} dq, \]

where \( C := \{ q \in \mathbb{C} : |q| = e^{-\beta} \} \) is a circle centered at the origin of radius less than 1 with the path taken in the counterclockwise direction, traversing the circle exactly once. Making a change of variables, reversing the direction of the path of integration, and recalling that \( \epsilon = \beta \left( 1 + ixm^{-\frac{1}{3}} \right) \) we have

\[ c(m, n) = \frac{\beta}{2\pi m^\frac{1}{3}} \int_{|x| \leq \frac{\pi m^\frac{1}{3}}{p}} f_m \left( \frac{i\epsilon}{2\pi} \right) e^{\epsilon x} dx. \]

Splitting this integral into two pieces, we have \( c(m, n) = M + E, \) where

\[ M := \frac{\beta}{2\pi m^\frac{1}{3}} \int_{|x| \leq 1} f_m \left( \frac{i\epsilon}{2\pi} \right) e^{\epsilon x} dx, \]

and

\[ E := \frac{\beta}{2\pi m^\frac{1}{3}} \int_{1 \leq |x| \leq \frac{\pi m^\frac{1}{3}}{p}} f_m \left( \frac{i\epsilon}{2\pi} \right) e^{\epsilon x} dx. \]

Next we determine the contributions of each of the integrals \( M \) and \( E, \) and see that \( M \) contributes to the main asymptotic term, while \( E \) is part of the error term.

6.1 The major arc

Considering the contribution of \( M, \) we obtain the following proposition.

Proposition 6.1 We have that

\[ M = \frac{-i}{2\pi} \Lambda_1 \beta^{2-w} \sqrt{\Lambda_2} \frac{e^{2\pi \sqrt{2\Lambda_2 n}}}{2\pi (2\Lambda_2 n)^{\frac{1}{3}}} + O \left( \beta^{3-w} \frac{e^{2\pi \sqrt{2\Lambda_2 n}}}{2\pi (2\Lambda_2 n)^{\frac{1}{3}}} \right) \]
as \( n \to \infty \).

**Proof** By definition we have that

\[
M = \frac{\beta}{2\pi m^3} \Lambda_1 \int_{|x| \leq 1} \epsilon^{1-w} e^{\frac{2\pi^2}{\epsilon} \Lambda_2} e^{\epsilon n} dx + \frac{\beta}{2\pi m^3} \int_{|x| \leq 1} O\left(\beta^{2-w} e^{\frac{2\pi^2}{\epsilon} \Lambda_2}\right) e^{\epsilon n} dx.
\]

(6.1)

Making the change of variables \( v = 1 + i x m^{-\frac{1}{2}} \) and then \( v \mapsto \sqrt{\Lambda_2} v \), we obtain that the first term equals

\[
\frac{-i}{2\pi} \Lambda_1 \beta^{2-w} \sqrt{\Lambda_2}^{-w} P_{1-w, 12\Lambda_2},
\]

(6.2)

where

\[
P_{s, k} := \int_{1-i m^{-\frac{1}{2}}/\sqrt{\Lambda_2}}^{1+i m^{-\frac{1}{2}}/\sqrt{\Lambda_2}} v^s e^{-\frac{k n}{6}} (v^{1/2}) dv.
\]

One may relate \( P_{s, k} \) to \( I \)-Bessel functions in exactly the same way as in [6, Lemma 4.2], making the adjustment for \( \sqrt{\Lambda_2} \) where necessary, to obtain that

\[
P_{s, k} = I_{-s-1} \left( \pi \sqrt{\frac{2kn}{3}} \right) + O \left( \exp \left( \pi \sqrt{\frac{kn}{6}} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}}} \right) \right) \right).
\]

Using the asymptotic behavior of the \( I \)-Bessel function given in Lemma 2.5 we obtain

\[
P_{1-w, 12\Lambda_2} = \frac{e^{2\pi \sqrt{2\Lambda_2 n}}}{2\pi (2\Lambda_2 n)^{\frac{1}{4}}} + O \left( \frac{e^{2\pi \sqrt{2\Lambda_2 n}}}{(8\pi^2 \Lambda^2 n)^{\frac{3}{4}}} \right) + O \left( e^{\pi \sqrt{2\Lambda_2 n} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}}} \right)} \right),
\]

and therefore (6.2) becomes

\[
\frac{-i}{2\pi} \Lambda_1 \beta^{2-w} \sqrt{\Lambda_2}^{-w} \frac{e^{2\pi \sqrt{2\Lambda_2 n}}}{2\pi (2\Lambda_2 n)^{\frac{1}{4}}}
\]

\[
+ O \left( \beta^{2-w} \frac{e^{2\pi \sqrt{2\Lambda_2 n}}}{(8\pi^2 \Lambda^2 n)^{\frac{3}{4}}} \right) + O \left( \beta^{2-w} e^{\pi \sqrt{2\Lambda_2 n} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}}} \right)} \right).
\]

Analogously, the second term of (6.1) is

\[
\frac{-i}{2\pi} \beta^{3-w} \sqrt{\Lambda_2}^{-1} P_{0, 12\Lambda_2} = O \left( \frac{\beta^{3-w} e^{2\pi \sqrt{2\Lambda_2 n}}}{2\pi (2\Lambda_2 n)^{\frac{1}{4}}} \right).
\]
This yields

\[
M = \frac{-i}{2\pi} \Lambda_1 \beta^{2-w} \sqrt{\Lambda_2} \cdot \frac{e^{2\pi \sqrt{2/\Lambda_2 n}}}{2\pi (2\Lambda_2 n)^{1/4}} + O \left( \beta^{2-w} \frac{e^{2\pi \sqrt{2/\Lambda_2 n}}}{(8\pi^2 \Lambda_2 n)^{3/4}} \right) \\
+ O \left( \beta^{2-w} e^{\pi \sqrt{2/\Lambda_2 n}} \left( 1 + \frac{1}{1+m^{-2/3}} \right) \right) + O \left( \beta^{3-w} \frac{e^{2\pi \sqrt{2/\Lambda_2 n}}}{2\pi (2\Lambda_2 n)^{1/4}} \right),
\]

This finishes the proof. \(\Box\)

### 6.2 The error arc

Finally, we bound \(E\) as follows.

**Proposition 6.2** We have \(E \ll M\) as \(n \to \infty\).

**Proof** By Proposition 5.1 we have

\[
E \ll \frac{1}{2\pi m^{3/4}} (s + 1) B(m, n) \\
\times \exp \left[ \sum_{k=1}^{N_2} \frac{2\pi \delta_k}{y_k \beta} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1+m^{-2/3}}} \right) \right) - \sum_{j=1}^{N_1} \frac{\pi^2 \gamma_j}{6x_j \beta} \right] \\
\times e^{\beta n} \int_{1 \leq |x| \leq \frac{\pi m^{3/4}}{\beta}} e^{\beta n i x m^{-1/3}} dx \\
\ll \frac{s + 1}{\pi} B(m, n) \exp \left[ \pi \sqrt{2n} \left( 1 - \sum_{j=1}^{N} \frac{\alpha_j}{12a_j} \right) - \sum_{k=1}^{N_2} \frac{\delta_k}{y_k \beta} \left( 1 - \frac{1}{\sqrt{1+m^{-2/3}}} \right) \right],
\]

where we trivially estimate the final integral. Using \(1 - \sum_{j=1}^{N} \frac{\alpha_j}{12a_j} < 2 \sqrt{\Lambda_2}\) the result follows immediately by comparing to \(M\) and therefore also finishes the proof of Theorem 1.2. \(\Box\)

### 7 Further questions

We end by briefly commenting on some related questions that could be the subject of further research.

(1) Here, we only discussed the case of eta-theta quotients with simple poles. A natural question to ask is: does a similar story hold for functions with higher order poles? The situation is of course expected to be more complicated, in particular finding Fourier coefficients with the method presented here seems to be much more
difficult. One could attempt to build a framework by following the definitions of Fourier coefficients given in [11, Section 8].

For example, in [21] Manschot and Zapata Rolón studied a Jacobi form with a double pole related to $\chi_y$-genera of Hilbert schemes on K3. They obtain bivariate asymptotic behavior in a similar flavor to those here. Can one extend this family?

(2) Although the functions considered in the present paper provide a wide family of results, it should be possible to extend the method to other related families of functions. In particular, it would be instructive to consider similar approaches for prototypical examples of mock Jacobi forms.

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