Persistence of lower dimensional invariant tori on sub-manifolds in Hamiltonian systems

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Abstract

Chow, Li and Yi in [2] proved that the majority of the unperturbed tori on sub-manifolds will persist for standard Hamiltonian systems. Motivated by their work, in this paper, we study the persistence and tangent frequencies preservation of lower dimensional invariant tori on smooth sub-manifolds for real analytic, nearly integrable Hamiltonian systems. The surviving tori might be elliptic, hyperbolic, or of mixed type.

keywords: Hamiltonian system; lower dimensional invariant tori; persistence on sub-manifolds; KAM theorem

1 Introduction

The persistence of quasi-periodic solutions or invariant tori to integrable Hamiltonian systems had puzzled scientists for long up to the appearance of the celebrated KAM [8, 1, 13] theory, which affirmed that the majority of invariant tori persist under small perturbations.

Later Melnikov [12] formulated a KAM type persistence result for elliptic lower dimensional tori of integrable Hamiltonian systems under so-called Melnikov’s non-resonance condition. More precisely, for a system with the following Hamiltonian

\[ H = N + P = \sum_{j=1}^{n} \omega_j y_j + \frac{1}{2} \sum_{j=1}^{m} \Omega_j (u_j^2 + v_j^2) + P, \]

Melnikov announced that the majority of invariant tori survive the small perturbations under the following conditions

\[ |\langle k, \omega \rangle + \langle l, \Omega \rangle| > \frac{\gamma}{|k|^r}, \quad |l| \leq 2 \]

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for \(k \in \mathbb{Z}^n, \ l \in \mathbb{Z}^m, |k| + |l| \neq 0\). The complete proof of his result was later carried out by Eliasson, Kuksin, and Pöschel \[4, 9, 15\]. In fact, Moser \[14\] had already noted the persistence of elliptic lower dimensional tori. He proved that the existence of the quasi-periodic solutions when the tori admit 2-dimensional elliptic equilibrium point. However the way he used can not be applied to higher dimension, because he requested the tangent frequency be fixed. Later, Eliasson \[4\] removed the restriction successfully by letting the frequency suffer small perturbations and later Pöschel \[15\] simplified the proof of Eliasson \[4\].

For Hamiltonian

\[
H = N + P = \langle \omega_0, y \rangle + \frac{1}{2} \langle u, Mu \rangle + P(x, y, u),
\]

where \((x, y, u) \in T^n \times R^n \times R^{2m}\), Moser \[14\] obtained the persistence of hyperbolic invariant tori when \(\omega_0 \in R^n\) is a fixed Diophantine toral frequency and the eigenvalues of \(JM\) (\(J\) being the standard symplectic matrix in \(R^{2m}\)) are real and distinct. Graff \[5\] generalized Moser’s result by allowing multiple eigenvalues of \(JM\). And the proof of Graff’s result was later given by Zehnder \[19\], who used implicit function techniques. More recently, Li and Yi \[11\] generalized the results of Graff and Zehnder on the persistence of hyperbolic invariant tori in Hamiltonian systems by allowing the degeneracy of the unperturbed Hamiltonians and they obtain the preservation of part or full components of tangent frequencies. They adopted the Fourier series expansion for normal form \(N\), which is a new technique.

Recently, Chow, Li and Yi \[2\] proved that the majority of the unperturbed tori on sub-manifolds will persist under a non-degenerate condition of Rüssmann type for standard Hamiltonian systems. Motivated by their work, in this paper, we shall show that lower dimensional tori also survive small perturbations on sub-manifolds under some assumptions. The surviving tori might be elliptic, hyperbolic, or of mixed type.

We consider a real analytic family of Hamiltonian systems of the following form

\[
H(x, y, u) = N(y, u) + P(x, y, u),
\]

where \((x, y, u)\) lies in a complex neighborhood \(\{ (x, y, u) : |\text{Im} x| \leq r, \text{dist}(y, G) \leq s, |u| \leq s \} \subset T^n \times G \times \{0\} \subset T^n \times R^n \times R^{2m}, G \subset R^n (n \geq 2)\) is a bounded closed region and \(P\) is small. Besides these, we also assume that

\[\text{A0)} \ N_u(y, 0) = 0, \ \text{det} N_{uu}(y, 0) \neq 0.\]

To prove the persistence of lower dimensional invariant tori of system (1.1), we first consider the following parameter-dependent, real analytic Hamiltonian
system
\[ H = e(\lambda) + \langle \omega(\lambda), y \rangle + \frac{1}{2} \langle A(\lambda) y, y \rangle + \frac{1}{2} \langle M(\lambda) u, u \rangle + P(x, y, u, \lambda), \tag{1.2} \]

where \((x, y, u)\) lies in a complex neighborhood \(D(r, s) = \{(x, y, u) : |\text{Im} x| \leq r, |y| \leq s, |u| \leq s\}\) of \(T^n \times \{0\} \times \{0\} \subset T^n \times \mathbb{R}^n \times \mathbb{R}^{2m}, \lambda\) is a parameter lying in a bounded closed region \(\Lambda \subset \mathbb{R}^n\) and \(M(\lambda)\) is nonsingular on \(\Lambda\). In the above, all \(\lambda\) dependency are of class \(C^{l_0}\) for some \(l_0 \geq n\).

We assume the following conditions:

**A1)** \(\text{rank}\{\frac{\partial^{|\alpha|} \omega}{\partial \lambda^{|\alpha|}} : |\alpha| \leq n - 1\} = n\) for all \(\lambda \in \Lambda\).

**A2)** For \((k, l)\) satisfying \(0 < |k| \leq K = \frac{4\sigma}{\sigma} \max_{1 \leq i \leq 2m, 0 \leq r \leq n-1} |\partial^r \Omega_i|\) and \(|l| \leq 2\), the following holds:

\[ \text{meas}\{\lambda : |i\langle k, \omega(\lambda)\rangle + \langle l, \Omega(\lambda)\rangle| = 0\} = 0, \]

where \(\Omega = (\Omega_1, \cdots, \Omega_{2m})^\top, \Omega_j, j = 1, \cdots, 2m\) are eigenvalues of \(JM, \sigma\) is a constant which will be determined later and “meas” denotes the Lebesgue measure in \(\mathbb{R}^{n_0}\).

**A3)** \(\text{rank} A(\lambda) = d\) on \(\Lambda\), and, there is a smoothly varying, nonsingular, \(d \times d\) principal minor \(\tilde{A}(\lambda)\) of \(A(\lambda)\).

Denote \(i_1, i_2, \cdots, i_d\) as the row indices (in the natural order) of \(\tilde{A}(\lambda)\) in \(A(\lambda)\).

Our main result states as follows.

**Theorem 1.1** Consider (1.2).

1) Assume A1), A2), A3) hold and let \(\tau > n(n-1) - 1\) be fixed. Then for a given \(\gamma\) there exists an \(\epsilon = \epsilon(r, s, l_0, \tau) > 0\) sufficiently small such that if

\[ |\partial^l \lambda P|_{D(r, s) \times \Lambda} \leq \epsilon s^2 \gamma^{4m^2(n+1)}, l \leq l_0, \tag{1.3} \]

then there exist Cantor sets \(\Lambda_\gamma \subset \Lambda\) with \(|\Lambda \setminus \Lambda_\gamma| = O(\gamma^{-\frac{1}{n-1}})\) and a \(C^{l_0-1}\) Whitney smooth family of symplectic transformations

\[ \Psi_\lambda : D\left(\frac{r}{2}, \frac{s}{2}\right) \longrightarrow D(r, s), \lambda \in \Lambda_\gamma \]

such that

\[ H \circ \Psi_\lambda(x, y, u) = e_*(\lambda) + \langle \omega_*(\lambda), y \rangle \]

\[ + \frac{1}{2} \langle A_*(\lambda) y, y \rangle + \frac{1}{2} \langle M_*(\lambda) u, u \rangle + P_*(x, y, u, \lambda) \] \(\tag{1.4}\)

and the following holds

\[ (\omega_*(\lambda))_{i_q} \equiv (\omega(\lambda))_{i_q}, q = 1, \cdots, d. \]
Thus, all unperturbed tori $T_\lambda = T^n \times \{0\} \times \{0\}$ with $\lambda \in \Lambda_{\gamma}$ will persist and preserve the frequency components $\omega_{i_1}, \cdots, \omega_{i_d}$ of the unperturbed tangent frequencies $\omega(\lambda)$.

2) Assume $A(\lambda)$ is nonsingular on $\Lambda$ and let $\tau > n - 1$ be fixed. Then there exists an $\epsilon = \epsilon(r, s, l_0, \tau) > 0$ sufficiently small such that if (1.3) holds, then each torus $T_\lambda = T^n \times \{0\} \times \{0\}$, $\lambda \in \Lambda_{\gamma}$, will persist with the normal form (1.4), and gives rise to an analytic, invariant perturbed torus which preserves its tangent frequencies.

The $n_0$ in the above theorem can be arbitrary positive integer. When $n_0 \leq n$, Theorem 1.1 has applications to Hamiltonian system (1.1) with respect to the persistence of invariant tori on sub-manifolds of $G$.

Consider (1.1) and let $S$ be an $n_0 \leq n$ dimensional, $C^l_0$ sub-manifold of $G$ which is either closed or with boundary. Denote

$$\omega(y) = \frac{\partial N}{\partial y}(y), \quad A(y) = \frac{\partial^2 N}{\partial y^2}(y), \quad y \in G.$$ 

We assume the following conditions:

A1)' For any coordinate chart $(\phi, U)$ of $S$, rank$\{\frac{\partial^\alpha(\omega \circ \phi^{-1})}{\partial x^\alpha} : |\alpha| \leq n - 1\} = n$ for all $\lambda \in \phi(U) \subset R^{n_0}$.

A2)' $\text{meas}\{\lambda : |i\langle k, \omega \circ \phi^{-1} \rangle + \langle l, \Omega \circ \phi^{-1} \rangle| = 0, 0 < |k| \leq K, |l| \leq 2, \text{ for all } \lambda \in \phi(U) \subset R^{n_0}, \text{ where } K \text{ is defined as in A2}\} = 0$.

A3)' rank$A(y) \equiv d$ on $S$, and, there is a smoothly varying, nonsingular, $d \times d$ principal minor $\tilde{A}(y)$ of $A(y)$ on $S$.

Corollary 1.1 Consider (1.1).

1) Assume $A0), A1)', A2)', A3)' and let $\tau > n(n - 1) - 1$ be fixed. Then there is an $\epsilon_0 = \epsilon_0(r, s, l_0, S, \tau) > 0$ and a family of Cantor sets $S_\epsilon \subset S, 0 < \epsilon \leq \epsilon_0$, with $|S \setminus S_\epsilon| = O(\gamma^{-n})$, such that for each $y \in S_\epsilon$, the unperturbed torus $T_y$ persists and gives rise to an analytic, invariant torus of the perturbed system whose tangent frequencies $\omega_\epsilon$ satisfies

$$(\omega_\epsilon(y))_{i_q} = (\omega(y))_{i_q}, \quad q = 1, \cdots, d,$$

where $i_1, \cdots, i_d$ are the row indices (in the natural order) of $\tilde{A}(y)$ located in $A(y)$.

Moreover, these perturbed tori form a Whitney smooth family.

2) Assume that $A(y)$ is nonsingular on $S$ and let $\tau > n - 1$ be fixed. Then each torus $T_y, y \in S_\epsilon$, will persist and gives rise to an analytic invariant perturbed torus with unchanged tangent frequencies.
3) Let \( y_0 \in S_\epsilon \) in 1). Then (1.1) admits the following normal form:

\[
H_{y_0}(x, y, u) = e_*(y_0) + \langle \omega_*(y_0), y - y_0 \rangle + \frac{1}{2} \langle A_*(y_0)(y - y_0), (y - y_0) \rangle + \frac{1}{2} \langle M_*(y_0)u, u \rangle,
\]

where \( \omega_*(y_0) \) is the tangent frequencies of the perturbed torus associated to \( y_0 \).

Similar to [2], to generalize the standard isoenergetic KAM theorem, we have to assume an additional sub-isoenergetic non-degenerate condition besides the Rüssmann non-degeneracy on an energy surface. More precisely, let \( S \) be a sufficiently smooth, relatively open, bounded subset of \( \{ N(y) = E \} \). We assume A1’) on \( S \) and also the following sub-isoenergetic non-degeneracy:

A1”) There is a smoothly varying \( d \times d \) principal minor \( \tilde{A}(y) \) of \( A(y) \) on \( S \) such that

\[
\det \begin{pmatrix}
\tilde{A}(y) & \omega^*(y) \\
\omega^*(y)^\top & 0
\end{pmatrix} \neq 0,
\]

where \( \omega^*(y) = \frac{\partial N_0}{\partial y}(y), y^* = (y_{i_1}, \ldots, y_{i_d})^\top \), and \( i_1, \ldots, i_d \) denote the row indices of \( \tilde{A}(y) \) in \( A(y) \).

**Theorem 1.2** Consider (1.1). Let \( S \) be a sufficiently smooth, relatively open, bounded subset of \( \{ N(y) = E \} \).

1) Assume A0), A1’), A2’) on \( S \) and let \( \tau > n(n - 1) - 1 \) be fixed. Then there is an \( \epsilon_0 = \epsilon_0(r, s, l_0, m, S, \tau) > 0 \) and a family of Cantor sets \( S_\epsilon \subset S, 0 < \epsilon \leq \epsilon_0 \), with \( |S \setminus S_\epsilon| = O(\gamma^{-\tau}) \), such that for each \( y \in S_\epsilon \), the unperturbed torus \( T_y \) persists and gives rise to an analytic, invariant torus \( T_{\epsilon,y} \) of the perturbed system on the energy surface \( \{ H(x, y, u) = E \} \). Moreover, these perturbed tori form a local Whitney smooth family.

2) If A1”) also holds on \( S \), then each perturbed torus \( T_{\epsilon,y} \) preserves the ratio of the \( i_1, \ldots, i_d \) components of its tangent frequencies \( \omega_\epsilon \), i.e.,

\[
[\omega_{\epsilon,i_1} : \cdots : \omega_{\epsilon,i_d}] = [\omega_{i_1} : \cdots : \omega_{i_d}],
\]

where \( \omega_{i,j}, \omega_{\epsilon,i} \) are the \( i_j \)-th components of unperturbed and perturbed tangent frequencies respectively, for \( j = 1, 2, \ldots, d \).

3) For \( y_0 \in S_\epsilon \), (1.1) admits the same normal form as in part 3) of Corollary 1.1.

**Remark 1.1.** 1) Our result is almost parallel to Chow, Li and Yi [2], i.e., we have the same results for lower dimensional invariant tori as that of the standard Hamiltonians which have been shown in [2].
2) In fact, in our case, we also can obtain arbitrarily prescribed high ordered normal form of Hamiltonian similar to Chow, Li and Yi [2]. To do so, we only need to change the iteration scheme a little, but we do not do it for brief.

2 KAM step

In this section, we describe the iterative scheme for the Hamiltonian (1.2) in one KAM step. For simplicity, we set $l_0 = n$.

Consider (1.2) and initially set

$$
\begin{align*}
& r_0 = r, \gamma_0 = \gamma, s_0 = s, \Lambda_0 = \Lambda, H_0 = H, e_0 = e, \omega_0 = \omega, \\
& A_0 = A, \tilde{A}_0 = \tilde{A}, M_0 = M, P_0 = P, \\
& N_0 = e_0(\lambda) + \langle \omega_0(\lambda), y \rangle + \frac{1}{2} \langle A_0(\lambda)y, y \rangle + \frac{1}{2} \langle M_0(\lambda)u, u \rangle.
\end{align*}
$$

Without loss of generality, we assume that $0 < r_0, s_0, \gamma_0 < 1$ and $\tilde{A}_0$ is the ordered $d \times d$ principal minor of $A_0$.

In what follows, the Hamiltonian without subscripts denotes the Hamiltonian in $\nu$-th step, while those with subscripts “+” denotes the Hamiltonian of $(\nu + 1)$-th step. And we shall use “$< \cdot$” to denote “$< c$” with a constant $c$ which is independent of the iteration step. To simplify the notations, we shall suspend the $\lambda$ dependence in most terms of this section.

Suppose at the $\nu$-th step, we have arrived at the following real analytic Hamiltonian:

$$
H = N + P,
$$
$$
N = e(\lambda) + \langle \omega(\lambda), y \rangle + \frac{1}{2} \langle A(\lambda)y, y \rangle + \frac{1}{2} \langle M(\lambda)u, u \rangle, \quad (2.1)
$$

which is defined on a phase domain $D(r, s)$ and depends smoothly on $\lambda \in \Lambda$, where $\Lambda \subset \Lambda_0$. Suppose that the $d \times d$ ordered principal minor $\tilde{A}$ of $A$ and $M$ are non-singular on $\Lambda$, and moreover, $P = P(x, y, u, \lambda)$ satisfies

$$
|\partial_{\lambda}^l P|_{D(r, s)} \leq \epsilon s^{2}\gamma^{4n^2(n+1)}, |l| \leq n. \quad (2.2)
$$

We will construct a symplectic transformation $\Phi_+$, which transforms the Hamiltonian (2.1) into the Hamiltonian of the next KAM cycle (the $(\nu + 1)$-th step), i.e.,

$$
H_+ = H \circ \Phi_+ = N_+ + P_+,
$$

where $N_+, P_+$ satisfy similar conditions as $N, P$ respectively on $D(r_+, s_+) \times \Lambda_+$. Define

$$
\epsilon_+ = \epsilon^{\frac{10}{9}},
$$

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\[ r_+ = \frac{r}{2} + \frac{r_0}{4}, \]
\[ s_+ = \frac{1}{8} \alpha s, \alpha = e^{\frac{i}{4}}, \]
\[ \gamma_+ = \frac{\gamma}{2} + \frac{\gamma_0}{4}, \]
\[ K_+ = (\lceil \log \frac{1}{\epsilon} \rceil + 1)^{a^* + 2}, \]
\[ D_{\alpha} = D(r_+ + \frac{i - 1}{8}(r - r_+), \frac{i}{8}\alpha s), i = 1, 2, \cdots, 8, \]
\[ D_+ = D_{\alpha} = D(r_+, s_+), \]
\[ \Lambda_+ = \{ \lambda \in \Lambda : |\langle k, \omega(\lambda) \rangle + \langle l, \Omega(\lambda) \rangle| > \frac{\gamma}{|k|^r}, \]
\[ |l| \leq 2, 0 < |k| \leq K_+, \]
\[ \Gamma(r - r_+) = \sum_{0 < |k| \leq K_+} |k|^{r(n+1)4m^2+4m^2n}e^{-|k|^{r-r_+}/8}, \]

where \( a^* \) is a constant such that \( (\frac{10}{9})^{a^*} > 2 \).

2.1 Truncation

Express \( P \) into Taylor-Fourier series
\[
P = \sum_{k \in \mathbb{Z}^n, l \in \mathbb{Z}^n_+, p \in \mathbb{Z}^2} p_{klp} y^l u^p e^{i<k,x>},
\]
and let \( R \) be the truncation of \( P \) with the form
\[
R = \sum_{|k| \leq K_+, |l|+|p| \leq 2} p_{klp} y^l u^p e^{i<k,x>}. \quad (2.3)
\]

Lemma 2.1 Assume that

H1) \( \int_{K_+} x^n e^{-r-x^{-r}} dx \leq \epsilon. \)

Then the following
\[
|\partial_\lambda (P - R)|_{D_{\alpha}} \leq \epsilon^2 s^2 \gamma 4m^2(n+1), \quad (2.4)
\]
\[
|\partial_\lambda R|_{D_{\alpha}} \leq \epsilon s^2 \gamma 4m^2(n+1)
\]
hold for all \( |l| \leq n, \lambda \in \Lambda. \)

Proof. Let
\[
I = \sum_{|k| > K_+, |l|+|p| \leq 2} p_{klp} y^l u^p e^{i<k,x>},
\]
\[
II = \sum_{|k| \leq K_+, |l|+|p| > 2} p_{klp} y^l u^p e^{i<k,x>}.
\]
Then

\[ P - R = I + II. \]

By Cauchy estimate and H1), we have

\[
|\partial^{l}_{\lambda} I\Rscr_{D(r+\frac{2}{3}(r-r_{+}),s)}| \leq \sum_{|k| > K_{+}} |\partial^{l}_{\lambda} P\Rscr_{D(r,s)}| e^{-|k| \frac{r-r_{+}}{s}}
\]

\[
\leq \epsilon s^{2} \gamma^{4m^{2}(n+1)} \sum_{x=K_{+}}^{\infty} x^{n} e^{-|x| \frac{r-r_{+}}{s}}
\]

\[
\leq \epsilon s^{2} \gamma^{4m^{2}(n+1)} \int_{K_{+}}^{\infty} x^{n} e^{-x \frac{r-r_{+}}{s}} dx \leq \epsilon s^{2} \gamma^{4m^{2}(n+1)}.
\]

It follows that

\[
|\partial^{l}_{\lambda} (P - I)\Rscr_{D(r+\frac{2}{3}(r-r_{+}),s)}| \leq |\partial^{l}_{\lambda} P\Rscr_{D(r,s)}| + |\partial^{l}_{\lambda} I\Rscr_{D(r+\frac{2}{3}(r-r_{+}),s)}|
\]

\[
\leq \epsilon s^{2} \gamma^{4m^{2}(n+1)}.
\]

For \(|q| = 3\), let \(\int\) be the obvious anti-derivative of \(\frac{\partial^{|l|+|p|}}{\partial y^{l} u^{p}}\), \(|l| + |p| = 3\). Then by Cauchy estimate, it follows that

\[
|\partial^{l}_{\lambda} II\Rscr_{D_{\xi_{a}}} = |\partial^{l}_{\lambda} \int \frac{\partial^{|l|+|p|}}{\partial y^{l} u^{p}} \sum_{|k| \leq K_{+}, |l|+|p| > 2} p_{klp} y^{l} u^{p} e^{i<k,x>} dy\Rscr_{D_{\xi},a}
\]

\[
\leq \frac{1}{s^{3}} \int |\partial^{l}_{\lambda} (P - I)\Rscr_{D_{\xi},a}| dy\Rscr_{D_{\xi},a}
\]

\[
\leq \frac{\alpha^{3} s^{3}}{s^{3}} |\partial^{l}_{\lambda} (P - I)\Rscr_{D_{\xi},a}|
\]

\[
\leq \epsilon s^{2} \gamma^{4m^{2}(n+1)}.
\]

Thus,

\[
|\partial^{l}_{\lambda} (P - R)\Rscr_{D_{\xi},a} \leq \epsilon s^{2} \gamma^{4m^{2}(n+1)},
\]

and therefore,

\[
|\partial^{l}_{\lambda} R\Rscr_{D_{\xi},a} \leq |\partial^{l}_{\lambda} (P - R)\Rscr_{D_{\xi},a} + |\partial^{l}_{\lambda} P\Rscr_{D(r,s)}| \leq \epsilon s^{2} \gamma^{4m^{2}(n+1)}. \quad \Box
\]

### 2.2 Averaging and solving homogeneous equation

To transform (2.1) into the Hamiltonian in the next KAM step, we shall construct a symplectic transformation as the time 1-map \(\phi^{1}_{F}\) of the flow generated by a Hamiltonian \(F\). To this end, suppose \(F\) has the following form:

\[
F = \sum_{0<|k| \leq K_{+}, |l|+|p| \leq 2} f_{klp} y^{l} u^{p} e^{i<k,x>} + \langle f_{001}, u \rangle + \langle f_{011}, u \rangle
\]

\[
= \sum_{0<|k| \leq K_{+}, |l| \leq 1} f_{klp} y^{l} e^{i<k,x>} + \sum_{0<|k| \leq K_{+}, |l| = 2} f_{klp} y^{l} e^{i<k,x>}
\]
\[
\sum_{0 \leq |k| \leq K_+} f_{klp} y^l e^{i(k,x)} + \sum_{0 \leq |k| \leq K_+} f_{klp} u^p e^{i(k,x)}
\]

\[
\sum_{0 < |k| \leq K_+} f_{klp} u^p e^{i(k,x)}
\]

\[
\equiv F_0 + F_1 + F_2 + F_3 + F_4,
\]

(2.5)

which satisfies the equation

\[
\{N, F\} + R - [R] + \langle p_{001}, u \rangle + \langle p_{011}, u \rangle = 0,
\]

(2.6)

where \([R] = \frac{1}{(2\pi)^n} \int_{T_n} R dx\). Substituting \(N\) and (2.5) into (2.6) and comparing coefficients yields that

\[
\Delta f_{kl0} = -p_{kl0}, 0 < |k| \leq K_+, |l| \leq 1,
\]

(2.7)

\[
\Delta f_{k20} = -p_{k20}, 0 < |k| \leq K_+, |l| = 2,
\]

(2.8)

\[
(\Delta I_{2m} + JM)f_{kl1} = -p_{kl1}, 0 \leq |k| \leq K_+, |l| = |p| = 1,
\]

(2.9)

\[
(\Delta I_{2m} + JM)f_{k01} = -p_{k001}, 0 \leq |k| \leq K_+, |l| = 0, |p| = 1,
\]

(2.10)

\[
(\Delta I_{2m} + JM)f_{k02} - f_{k02} JM = -p_{k02}, 0 < |k| \leq K_+, |l| = 0, |p| = 2,
\]

(2.11)

where \(\Delta = i\langle k, \omega(\lambda) + A(\lambda)y \rangle\), and \(J\) is the standard symplectic matrix in \(R^{2m}\).

2.3 Estimate on \(F\)

Let \(\Omega_j, j = 1, \cdots, 2m\) be the eigenvalues of \(JM\), where \(\Omega_j\) depends smoothly on \(\lambda\). Then by the non-degeneracy of \(M\) there exists a constant \(c\) such that

\[
|\Omega_j| \geq c, j = 1, \cdots, 2m.
\]

(2.12)

Lemma 2.2 Assume that

\(H2)\) \(2M_*, s \leq \frac{\gamma}{K_+^{s+1}},\)

where \(M_*\) is a constant defined to satisfy \(|A(\lambda)| \leq M_*\) on \(\Lambda_0\). Then we have the following result:

\[
\frac{1}{s^2} |\partial^j F|_{D(r_+ + r^-s, r- r_+, s) \times \Lambda_+} \leq \cdot (\epsilon \Gamma + \epsilon).
\]

(2.13)

Proof. By the definition of \(\Lambda\) and \(H2)\) we obtain that

\[
|\Delta| > \frac{\gamma}{2|k|^r}.
\]

(2.14)

And by the definition of \(\Lambda_+, H2), (2.12)\) and Lemma A.3, we have the following results:

\[
|det(\Delta I_{2m} - JM)|_{\Lambda_+} > \cdot \left(\frac{\gamma}{|k|^r}\right)^{2m}
\]

(2.15)
\[ |\text{det}(\Delta I_{4m^2} - I_{2m} \otimes (JM) - (JM) \otimes I_{2m})|_{\Lambda_+} > (\frac{\gamma}{|k|\tau})^{4m^2}. \]  

(2.16)

To estimate \( F \), we must estimate \( \partial^l_\lambda \Delta^{-1} \), \( \partial^l_\lambda ((\Delta I_{2m} + MJ)^{-1}) \), \( \partial^l_\lambda ((\Delta I_{4m^2} - I_{2m} \otimes (JM) - (JM) \otimes I_{2m})^{-1}) \) at first. For \( \partial^l_\lambda \Delta^{-1} \), we have

\[ |\partial^l_\lambda \Delta^{-1}|_{\Lambda_+} \leq |\Delta^{-1}|.|k|^l \leq \frac{|k|^\tau(l+1)+l}{\gamma^{l+1}}, |l| \leq n. \]  

(2.17)

We note by the definition of \( \Lambda_+ \), H2, (2.12) and Lemma A.3 that

\[ |\partial^l_\lambda (\Delta I_{2m} + MJ)^{-1}|_{\Lambda_+} \leq \frac{|k|^\tau}{\gamma}^{2m(l+1)} \times |k|^{2ml}, \]  

(2.18)

\[ |\partial^l_\lambda (\Delta I_{4m^2} - I_{2m} \otimes (JM) - (JM) \otimes I_{2m})^{-1}|_{\Lambda_+} \leq \frac{|k|^\tau}{\gamma}^{4m^2(l+1)} \times |k|^{4m^2l}. \]  

(2.19)

Thus on \( D(r, s) \times \Lambda_+ \), we have

\[ |\partial^l_{f_{k0}}| = |\partial^l_\lambda (\Delta^{-1} p_{k0})| \leq \frac{|k|^\tau(l+1)+l}{\gamma^{l+1}} e^{-|k|\tau s^2 - |l| \gamma 4m^2(n+1)}, \]  

(2.20)

\[ |\partial^l_{f_{k20}}| = |\partial^l_\lambda (\Delta^{-1} p_{k20})| \leq \frac{|k|^\tau(l+1)+l}{\gamma^{l+1}} e^{-|k|\tau s \gamma 4m^2(n+1)}, \]  

(2.21)

\[ |\partial^l_{f_{k11}}| = |\partial^l_\lambda [(\Delta I_{2m} + MJ)^{-1} p_{k11}]| \leq |\partial^l_\lambda [(\Delta I_{2m} - JM)^{-1} p_{k11}]| \leq \frac{|k|^\tau(l+1)2m+2ml}{\gamma^{l+1}2m} e^{-|k|\tau \gamma 4m^2(n+1)}, \]  

(2.22)

\[ |\partial^l_{f_{k01}}| = |\partial^l_\lambda [(\Delta I_{2m} + MJ)^{-1} p_{k01}]| \leq |\partial^l_\lambda [(\Delta I_{2m} - JM)^{-1} p_{k01}]| \leq \frac{|k|^\tau(l+1)2m+2ml}{\gamma^{l+1}2m} e^{-|k|\tau s \gamma 4m^2(n+1)}, \]  

(2.23)

\[ |\partial^l_{f_{k02}}| = |\partial^l_\lambda [(\Delta I_{4m^2} - I_{2m} \otimes (JM) - (JM) \otimes I_{2m})^{-1} p_{k02}]| \leq \frac{|k|^\tau(l+1)4m^2+4m^2l}{\gamma^{l+1}4m^2} e^{-|k|\tau \gamma 4m^2(n+1)}, \]  

(2.24)

where \( 0 < |k| \leq K_+ \). When \( k = 0 \), we have

\[ |f_{011}| \leq \epsilon \gamma 4m^2(n+1), |f_{001}| \leq \epsilon s \gamma 4m^2(n+1), \]  

(2.25)

on account of

\[ MJ f_{011} = -p_{011}, MJ f_{001} = -p_{001}. \]  

(2.26)

Therefore we obtain the estimate of \( F \):

\[ \frac{1}{s^2} |\partial^l_\lambda F|_{D(r + \frac{\varepsilon}{s}(r-r_+), s) \times \Lambda_+} \]
\[
\sum_{0 < |k| \leq K_+} \frac{|k|^r}{\gamma^{(n+1)4m^2}} e^{-|k| \frac{r-r_+}{n}} e^{\gamma 4m^2(n+1)}
\leq \cdot \epsilon \Gamma + \cdot \epsilon. \quad (2.27)
\]

Denote \(D_i = D(r_+ + \frac{i}{4}(r-r_+), \frac{i}{4}s), i = 1, 2, 3\). By (2.27) and Cauchy estimate, we obtain on \(D_3 \times \Lambda_+\) that:

\[
|\partial_{\lambda}^s F|, \; s|\partial_{\lambda}^s F_y|, \; s|\partial_{\lambda}^s F_u| \leq \cdot \epsilon (\Gamma + 1)s^2. \quad (2.28)
\]

Since \(F\) is a polynomial of \(y\) and \(u\) with order 2, by (2.28) we obtain

\[
|D^j F|_{D_2 \times \Lambda_+} \leq \cdot \epsilon (\Gamma + 1), \; |j| \geq 2. \quad (2.29)
\]

Let \(F\) be the Hamiltonian (2.5) with coefficients given by Lemma 2.2. Using \(\phi_F\) denotes the flow generated by \(F\), then

\[
H \circ \phi_F^t = (N + R) \circ \phi_F^t + (P - R) \circ \phi_F^t
\]

\[
= N + \{N, F\} + R + \int_0^1 \{R_t, F\} \circ \phi_F^t dt + (P - R) \circ \phi_F^t
\]

\[
= N + [R] - \langle p_{011} y, u \rangle + \{N, F\} + R + [R] + \langle p_{001}, u \rangle + \langle p_{011} y, u \rangle
\]

\[
+ \int_0^1 \{R_t, F\} \circ \phi_F^t dt + (P - R) \circ \phi_F^t
\]

\[
= (N + [R] - \langle p_{001}, u \rangle - \langle p_{011} y, u \rangle)
\]

\[
+ (\int_0^1 \{R_t, F\} \circ \phi_F^t dt + (P - R) \circ \phi_F^t)
\]

\[
= \tilde{N}_+ + \bar{P}_+ + \tilde{P}_+ + \tilde{P}_+, \quad (2.30)
\]

where \(R_t = (1-t)\{N, F\} + R\).

This completes the averaging process.

### 2.4 Translation and partial non-degeneracy

Denote by \(Y, P_{010}\) the vectors formed by the first \(d\) components of \(y, p_{010}\) respectively. Then it is easy to see that the equation

\[
\tilde{A} Y = -P_{010} \quad (2.31)
\]

has a unique solution \(Y^*\) on \(D(r, s)\) which depends smoothly on \(\lambda\). If we denote

\[
y^* = \begin{pmatrix} Y^* \\ 0 \end{pmatrix}
\]
then by (2.31), we obtain
\[ Ay^* = - \begin{pmatrix} P_{010} \\ 0 \end{pmatrix}. \] (2.32)

Consider the translation
\[ \phi : x \to x, \quad y \to y + y^*, \quad u \to u \]
and denote
\[ \Phi_+ = \phi_1^1 \circ \phi. \]

Then
\[ H \circ \Phi_+ = N_+ + P_+, \]
\[ N_+ = \bar{N}_+ \circ \phi - \psi \]
\[ = e_+ + \langle \omega_+, y \rangle + \frac{1}{2} \langle A_+ y, y \rangle + \frac{1}{2} \langle M_+ u, u \rangle, \]
\[ P_+ = \bar{P}_+ \circ \phi + \psi, \] (2.33)
where
\[ e_+ = e + \langle \omega, y^* \rangle + \frac{1}{2} \langle Ay^*, y^* \rangle + [R](y^*), \] (2.34)
\[ \omega_+ = \omega + p_{010} - \begin{pmatrix} P_{010} \\ 0 \end{pmatrix}, \] (2.35)
\[ M_+ = M + p_{002}, \] (2.36)
\[ A_+ = A + \partial_y^2 [R](y^*), \] (2.37)
\[ \psi = \langle \partial_y [R](y^*), y \rangle - \langle p_{010}, y \rangle = 2 \langle p_{020} y^*, y \rangle. \] (2.38)

2.5 Estimate on new normal form \( N_+ \)

**Lemma 2.3** We have the following holds for all \(|l| \leq n:\)

\[ |\partial_y y^*|_{A_+} \leq \epsilon s \gamma^{4m^2(n+1)}, \] (2.39)
\[ |\partial_y (e_+ - e)|_{A_+} \leq \epsilon s \gamma^{4m^2(n+1)}, \] (2.40)
\[ |\partial_y (\omega_+ - \omega)|_{A_+} \leq \epsilon s \gamma^{4m^2(n+1)}, \] (2.41)
\[ |\partial_y (A_+ - A)|_{A_+} \leq \epsilon \gamma^{4m^2(n+1)}, \] (2.42)
\[ |\partial_y (M_+ - M)|_{A_+} \leq \epsilon \gamma^{4m^2(n+1)}. \] (2.43)

**Proof.** It is very clear by (2.31) and (2.34)-(2.37). \( \square \)
Lemma 2.4 Assume that

H3) \( \varepsilon(\Gamma + 1)s < \frac{1}{8}(r - r_+); \varepsilon(\Gamma + 1)s < \frac{1}{8}\alpha s. \)

Then for all \( 0 \leq t \leq 1, \)

\[ \Phi_+ = \phi_F^1 \circ \phi : D_+ = D_{\frac{1}{8}^\alpha} \to D_{\frac{1}{8}^\alpha} \subset D(r, s), \tag{2.44} \]

more precise,

\[ \phi : D_{\frac{1}{8}^\alpha} \to D_{\frac{1}{8}^\alpha}, \tag{2.45} \]

\[ \phi_F^t : D_{\frac{1}{8}^\alpha} \to D_{\frac{1}{8}^\alpha} \tag{2.46} \]

are well defined, real analytic and depend smoothly on \( \lambda \in \Lambda_+. \)

Proof. (2.45) follows immediately from Lemma 2.3 and H3). To prove (2.46), we rewrite \( \phi_F^t = (\phi_{F_1}^t, \phi_{F_2}^t, \phi_{F_3}^t) \), where \( \phi_{F_1}^t, \phi_{F_2}^t, \phi_{F_3}^t \) are components of \( \phi_F^t \) in the directions \( x, y, u \) respectively. Let \( (x, y, u) \in D_{\frac{1}{8}^\alpha} \) and let \( t_* = \text{Sup}\{t \in [0, 1] : \phi_{F_1}^t(x, y, u) \in D_{\frac{1}{8}^\alpha} \}. \) Then for any \( 0 \leq t \leq t_* \),

\[
|\phi_{F_1}^t(x, y, u) - x| \leq \int_0^t |F_y \circ \phi_{F}^s|_{D_{\frac{1}{8}^\alpha}} ds \leq |F_y|_{D_{\frac{1}{8}^\alpha}} \leq \varepsilon(\Gamma + 1)s < \frac{1}{8}(r - r_+),
\]

\[
|\phi_{F_2}^t(x, y, u) - y| \leq \int_0^t |F_x \circ \phi_{F}^s|_{D_{\frac{1}{8}^\alpha}} ds \leq |F_x|_{D_{\frac{1}{8}^\alpha}} \leq \varepsilon(\Gamma + 1)s^2 < \frac{1}{8}\alpha s,
\]

\[
|\phi_{F_3}^t(x, y, u) - u| \leq \int_0^t |F_u \circ \phi_{F}^s|_{D_{\frac{1}{8}^\alpha}} ds \leq |F_u|_{D_{\frac{1}{8}^\alpha}} \leq \varepsilon(\Gamma + 1)s < \frac{1}{8}\alpha s.
\]

(2.47)

It follows that \( \phi_{F}^t(x, y, u) \in D_{\frac{1}{8}^\alpha} \subset D_\alpha. \) Thus, \( t_* = 1 \) and (2.46) holds. \( \square \)

Now we can give the estimate of \( \Phi_+ \).

2.6 Estimate on the transformation \( \Phi_+ \)

Lemma 2.5 For the transformation \( \Phi_+ \), we have the following estimates:

\[ |\Phi_+ - id|_{D_{\frac{1}{8}}} \leq \varepsilon(\Gamma + 1)s, |D\Phi_+ - Id|_{D_{\frac{1}{8}}} \leq \varepsilon(\Gamma + 1), \tag{2.48} \]

where \( id \) stands for the identity map, and \( Id \) stands for the elementary matrix.

Proof. By

\[ \phi_F^1 = id + \int_0^1 X_F \circ \phi_{F}^s ds, \tag{2.49} \]

we have

\[ |\phi_{F}^1 - id| \leq |X_F|_{D_{\frac{1}{8}}} \leq \varepsilon(\Gamma + 1)s. \]

For translation \( \phi \) we have

\[ |\phi - id| = |y^*| \leq \varepsilon s^{4\gamma^2(n+1)}. \tag{2.50} \]
so

$$|\phi| \leq 2.$$  

Since

$$\Phi_{+} - id = (\phi_{F}^{1} - id) \circ \phi + \begin{pmatrix} 0 \\ y^* \\ 0 \end{pmatrix}, \quad (2.51)$$

we have

$$|\Phi_{+} - id| \leq \cdot \epsilon (\Gamma + 1) s + \cdot \epsilon s \gamma^{4m^2(n+1)} \leq \cdot \epsilon (\Gamma + 1) s.$$  

By (2.49) and (2.50), it follows that

$$|D\phi_{F}^{1} - Id| \leq 2|D^2 F| \leq \cdot \epsilon (\Gamma + 1),$$  

$$|D\phi - Id| \leq \cdot \epsilon \gamma^{4m^2(n+1)}.$$  

So by (2.51), we obtain the estimate of $D\Phi_{+}$:

$$|D\Phi_{+} - Id| \leq |D(\phi_{F}^{1} - id)D\phi| + |Dy^*| \leq |D\phi_{F}^{1} - Id| \cdot |D\phi| + |Dy^*| \leq \cdot \epsilon (\Gamma + 1). \quad \square$$

### 2.7 Estimate on new perturbation $P_{+}$

**Lemma 2.6** Assume that

H4) \(\epsilon \frac{s}{\Gamma} \ll 1\),

then on $D_{+} \times \Lambda_{+}$,

$$|\partial_{\lambda}^{l} P_{+}| \leq \epsilon s^{2} \gamma_{+}^{4m^{2}(n+1)}, \quad |l| \leq n. \quad (2.52)$$

**Proof.** Since

$$R_{t} = (1 - t)\{N, F\} + R = tR + (1 - t)[R] - (1 - t)(< p_{001}, u > + < p_{011}, y >),$$

it is easy to see that

$$|\partial_{\lambda}^{l} R_{t}|_{D(r,s) \times \Lambda_{+}} \leq \cdot \epsilon s^{2} \gamma^{4m^{2}(n+1)}.$$  

By the estimate of $F$ and its derivative, we obtain that

$$|\partial_{\lambda}^{l} \{R_{t}, F\}|_{D_{+} \times \Lambda_{+}} \leq |\partial_{\lambda}^{l} R_{tx} F_x| + |\partial_{\lambda}^{l} R_{ty} F_y| + |\partial_{\lambda}^{l} R_{txu} F_u| \leq \cdot \epsilon s^{2} (\Gamma + 1) \gamma^{4m^{2}(n+1)}.$$  

By Lemma 2.1, (2.38) and (2.39), we have on $D_{+} \times \Lambda_{+}$ the following holds

$$|\partial_{\lambda}^{l} (P - R) \circ \phi_{F}^{1}| \leq \cdot \epsilon s^{2} \gamma^{4m^{2}(n+1)}$$
\[ |\partial^j_\lambda \phi| \leq \epsilon s \gamma^4 m^2 (n+1) \]
\[ |\partial^j_\lambda \psi| \leq \epsilon^2 s^2 \gamma^8 m^2 (n+1) .\]

So we have
\[ |\partial^j_\lambda P_+|_{D_+ \times A_+} \leq \epsilon^2 s^2 \gamma^4 m^2 (n+1) (\Gamma + 3) \]
by the above estimate and (2.33). Thus it is enough to verify
\[ \epsilon^2 s^2 \gamma^4 m^2 (n+1) (\Gamma + 3) \leq \epsilon_+ s^2 \gamma^4 m^2 (n+1). \]

By the definition of \( \epsilon_+, s_+, \gamma_+ \) and H4), it is clear that it does hold. \( \square \)

This completes one cycle of KAM steps.

3 Proof of main results

3.1 Iteration lemma

Considering (1.2), we define the following sequences inductively for all \( \nu = 1, 2, \cdots \):
\[
\begin{align*}
    r_\nu & = \frac{r_{\nu - 1}}{2} + \frac{r_0}{4}, \\
    s_\nu & = \frac{1}{8} \alpha_{\nu - 1} s_{\nu - 1}, \quad \alpha_{\nu} = \epsilon_\nu^{\frac{1}{3}}, \\
    \gamma_\nu & = \frac{\gamma_{\nu - 1}}{2} + \frac{\gamma_0}{4}, \\
    \epsilon_\nu & = \epsilon_{\nu - 1}, \\
    K_\nu & = (|\log \frac{1}{\epsilon_{\nu - 1}}| + 1)^{a^* + 2}, \\
    D_{\frac{1}{8}n} & = D(r_+ + \frac{i - 1}{8} (r_+ - r_+), \frac{i}{8} s), \quad i = 1, 2, \cdots, 8, \\
    D_\nu & = D(r_\nu, s_\nu), \\
    \Gamma_\nu & = \Gamma(r_\nu - r_{\nu + 1}), \\
    \Lambda_\nu & = \{ \lambda \in \Lambda_{\nu - 1} : |i \langle k, \omega_{\nu - 1}(\lambda) \rangle + \langle l, \Omega_{\nu - 1}(\lambda) \rangle| > \frac{\gamma_{\nu - 1}}{|k|^\tau}, \quad |l| \leq 2, \quad 0 < |k| \leq K_\nu \},
\end{align*}
\]

where \( a^* \) is a constant such that \( (\frac{10}{9})^{a^*} > 2 \).

**Lemma 3.1** If (1.3) holds for a sufficiently small \( \epsilon_0 \), then the following holds for all \( |l| \leq n; \nu = 1, 2, \cdots \).

1) \[
|\partial^j_\lambda (e_\nu - e_{\nu - 1})|_{\Lambda_\nu} \leq \epsilon_{\nu - 1} s_{\nu - 1} \gamma^4 m^2 (n+1),
\]
\[
|\partial_\lambda^k (\omega_\nu - \epsilon_0)|_{A_\nu} \leq \epsilon_0 s_0 \gamma_0^{4m^2(n+1)}, \tag{3.2}
\]
\[
|\partial_\lambda^k (\omega_\nu - \omega_{\nu-1})|_{A_\nu} \leq \epsilon_0 s_{\nu-1} \gamma_{\nu-1}^{4m^2(n+1)}, \tag{3.3}
\]
\[
|\partial_\lambda^k (\omega_\nu - \omega_0)|_{A_\nu} \leq \epsilon_0 s_0 \gamma_0^{4m^2(n+1)}, \tag{3.4}
\]
\[
|\partial_\lambda^k (A_\nu - A_{\nu-1})|_{A_\nu} \leq \epsilon_0 s_{\nu-1} \gamma_{\nu-1}^{4m^2(n+1)}, \tag{3.5}
\]
\[
|\partial_\lambda^k (A_\nu - A_0)|_{A_\nu} \leq \epsilon_0 s_0 \gamma_0^{4m^2(n+1)}, \tag{3.6}
\]
\[
|\partial_\lambda^k (M_\nu - M_{\nu-1})|_{A_\nu} \leq \epsilon_0 s_{\nu-1} \gamma_{\nu-1}^{4m^2(n+1)}, \tag{3.7}
\]
\[
|\partial_\lambda^k (M_\nu - M_0)|_{A_\nu} \leq \epsilon_0 s_0 \gamma_0^{4m^2(n+1)}, \tag{3.8}
\]
\[
|\partial_\lambda^k \Phi_{\nu}|_{D_\nu \times A_\nu} \leq \epsilon_0 s_0 \gamma_0^{4m^2(n+1)}. \tag{3.9}
\]

2) \((\omega_\nu(\lambda))_q = (\omega_{\nu-1}(\lambda))_q\) for all \(q = 1, 2, \ldots, d\) and \(\lambda \in A_\nu.\)

3) \(\Phi_\nu : D_\nu \times A_\nu \to D_{\nu-1}\) is symplectic for each \(\lambda \in \Lambda,\) and

\[
|\Phi_\nu - id|_{D_\nu \times A_\nu} \leq \epsilon_0 s_{\nu-1} (\Gamma_{\nu-1} + 1) s_{\nu-1}.
\]

Moreover, on \(D_\nu \times A_\nu,\)

\[
H_\nu = H_{\nu-1} \circ \Phi_\nu = N_\nu + P_\nu,
\]

where

\[
H_\nu = N_\nu + P_\nu,
\]

\[
N_\nu = \epsilon_\nu + (\omega_\nu, y) + \frac{1}{2} (A_\nu, y, y) + \frac{1}{2} (M_\nu, u, u),
\]

\(A_\nu\) is real symmetric with its \(d \times d\) ordered principal minor \(\tilde{A}_\nu\) being nonsingular on \(A_\nu.\)

**Proof.** We only have to verify H1)-H4) for all \(\nu.\) For simplicity, we let \(r_0 = 1.\)

First, we verify H1). By the choice of \(K_+ = ([\log \frac{1}{\epsilon}] + 1)a^* + 2,\) where \(a^*\) is a constant such that \((\frac{10}{\delta})^a > 2,\) we have

\[
\log(n+1)! + n(a^* + 2) \log([\log \frac{1}{\epsilon}] + 1) - \frac{1}{2^{\nu+5}} (\log \frac{1}{\epsilon}) a^* + 2
\]

\[
\leq (I^*) \log(n+1)! + n(a^* + 2) \log(\log \frac{1}{\epsilon} + 2) - (\log \frac{1}{\epsilon})^2
\]

\[
\leq - \log \frac{1}{\epsilon}
\]

if \(\epsilon_0\) is sufficiently small, where \((I^*)\) holds since \(\frac{1}{2^{\nu+5}} (\log \frac{1}{\epsilon}) a^* \geq 1\) by the choice of \(a^*.\) Thus H1) holds since \(\int_{K_+} x^n e^{-x - \frac{\gamma}{s} - \frac{\tau}{s}} dx \leq (n+1)! K_+ e^{-\frac{\gamma}{s} - \frac{\tau}{s}}.\)

Then we verify H2). We have

\[
2M_s K_+^{\tau+1} = 2M_s \left(\frac{1}{8}\right)^\nu s_0 \epsilon_0^{3\nu(\frac{10}{\delta})^{\nu-1}} (\log \frac{1}{\epsilon}) (a^* + 2)(\nu+1)
\]

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\[ \begin{align*}
&= 2M_s \left[ \left( \frac{10}{9} \right)^{\nu(a^*+2)(\tau+1)} \left( \frac{1}{8} \right)^{\nu} s_0 \epsilon_0^{3[\left( \frac{10}{9} \right)^{\nu}-1]} \left( \log \frac{1}{\epsilon_0} \right)^{(a^*+2)(\tau+1)} \right] \\
&\leq (II^*) C'' 2M_s s_0^{2[\left( \frac{10}{9} \right)^{\nu}-1]} \\
&\leq (III^*) s_0^{\nu} \epsilon_0^{3[\left( \frac{10}{9} \right)^{\nu}-1]} \\
&\leq (IV^*) \frac{\gamma_0}{2} ,
\end{align*}\]

where \( C = \left( \frac{10}{9} \right)^{(a^*+2)(\tau+1)} \frac{1}{8} \), \( (II^*) \), \( (III^*) \) can hold if \( \epsilon_0 \) is chosen sufficiently small such that \( \epsilon_0^{[\left( \frac{10}{9} \right)^{\nu}-1]} \left( \log \frac{1}{\epsilon_0} \right)^{(a^*+2)(\tau+1)} \leq 1 \) and \( 2M_s C'' \epsilon_0^{\left( \frac{10}{9} \right)^{\nu}-1} \leq 1 \), and \( (IV^*) \) is easily done, say, set \( s_0 = \frac{\gamma_0}{2} \).

To verify \( (V^*) \), we note that

\[ \begin{align*}
\epsilon^2 \Gamma &\leq \epsilon^2 \int_1^{\infty} \lambda^{\nu(n+1)4m^2+4m^2n+n} e^{-\lambda^{\frac{1}{2\nu}}} d\lambda \\
&\leq \epsilon^2 [\tau(n+1)4m^2+4m^2n+n+1]^{2(\nu+5)}[\tau(n+1)4m^2+4m^2n+n+1] \\
&\leq \epsilon^2 [\tau(n+1)4m^2+4m^2n+n+1] \\
&\leq \epsilon^2 [\frac{10}{9}]^{\nu} [\tau(n+1)4m^2+4m^2n+n+1] \\
&\leq (V^*) 1 ,
\end{align*}\]

where \( (V^*) \) holds if \( \epsilon_0 \) is chosen sufficiently small.

Now, the rest work is to prove \( H3 \). By \( H4 \), we have

\[ \epsilon(\Gamma+1)s \leq \epsilon^2 s \leq \epsilon^2 [\frac{10}{9}]^{\nu} s_0 \lesssim (VI^*) \frac{1}{2^{\nu+5}} = \frac{1}{8}(r-r_+) ,\]

where \( (VI^*) \) holds if \( \epsilon_0 \) is chosen sufficiently small. It is very clear that \( \epsilon(\Gamma+1)s < \frac{1}{8}(\text{as if } H4) \) holds.

In the process of proof of the lemma, we have used the sufficient smallness of \( \epsilon_0 \) in \( (I^*) - (VI^*) \). In fact, the existence of \( \epsilon_0 \) is obvious in \( (I^*) - (VI^*) \) in spite that we do not give the explicit form. So, in the end, we can choose the smallest \( \epsilon_0 \) of \( (I^*) - (VI^*) \) as the \( \epsilon_0 \) we need. This completes the proof of the lemma. □

### 3.2 Proof of Theorem 1.1

Denote

\[ \Psi^\nu = \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_\nu , \nu = 1, 2, \cdots . \]

Then \( \Psi^\nu : D_\nu \times \Lambda_\nu \rightarrow D_0 \), and,

\[ H_0 \circ \Psi^\nu = H_\nu = N_\nu + P_\nu , \nu = 1, 2, \cdots , \]
where $\Psi^0 = id$. Let

$$\Lambda_* = \bigcap_{\nu \geq 0} \Lambda_\nu.$$  

Then $\Lambda_*$ is a Cantor-like set. First, we show that we have the estimate

$$|\Lambda_0 \setminus \Lambda_*| = O(\gamma^{\frac{1}{n-1}}),$$

we will divide the proof of which into two cases.

**Case 1:** $n_0 = n$.

According to [17], $\{\partial^\beta \omega / \partial \lambda^\beta : \forall \beta, |\beta| = r\}$ and $\{D^r_\nu \omega : \forall V \in R^n\}$ are linearly equivalent, where $r > 0$ is an integer and $D^r_\nu \omega = d^r / dt^r \omega(\lambda + tV)|_{t=0}$. Since (3.4) is satisfied by the extended tangent frequencies $\omega_\nu$ on $\Lambda_0$, A1) implies that if $\epsilon_0$ is sufficiently small, then

$$\text{rank}\{\partial^\alpha \omega_\nu / \partial \lambda^\alpha : |\alpha| \leq n - 1\} = n$$

for all $\lambda \in \Lambda_0$, $\nu = 0, 1, \ldots$. In the following proof, we will omit the subscript $\nu$.

So there exist $n$ integers $0 \leq r_1, \ldots, r_n \leq n - 1$ and $n$ vectors $V_1, \ldots, V_n \in R^n$ such that

$$\text{rank}\{D^r_1 \omega, \ldots, D^r_n \omega\} = n, \forall \lambda \in \Lambda.$$

Denote $B = (D^r_1 \omega, \ldots, D^r_n \omega)$. Then there exist a constant $\sigma > 0$ such that for all $(\lambda, V) \in \Lambda \times U$,

$$|BV| \geq \sigma,$$

where $U = \{V \in R^n : |V_1| + \cdots + |V_n| = 1\}$. Then it follows that for some $1 \leq i \leq n$ and $\forall k \in Z^n\{0\}$,

$$|\langle D^r_i \omega, \frac{k}{|k|}\rangle| \geq \frac{\sigma}{n}.$$  

So by the definition of $K$, when $|k| > K$, we have

$$\left|D^r_i \left(\frac{k}{|k|}, \omega \langle l, \Omega \rangle + \frac{1}{|k|} l, \Omega \rangle\right)\right|$$

$$\geq \frac{\sigma}{n} - \frac{1}{|k|} |D^r_i \langle l, \Omega \rangle|$$

$$\geq \frac{\sigma}{n} - \frac{2}{|k|} \frac{K}{\sigma}$$

$$\geq \frac{\sigma}{2n}.$$  

Let

$$R_{kV_i} = \{t : \langle i\frac{k}{|k|}, \omega(\lambda + tV_i) + \frac{1}{|k|} l, \Omega(\lambda + tV_i)\rangle\}$$
\[
\leq \frac{\gamma}{|k|^\tau + 1}, \lambda \in \Lambda, \lambda + tV_i \in \Lambda, \]

\[
R_{k,l} = \{ \lambda \in \Lambda : |i\left(\frac{k}{|k|}, \omega(\lambda)\right) + \frac{1}{|k|}(l, \Omega(\lambda))| \leq \frac{\gamma}{|k|^\tau + 1}, \ |l| \leq 2. \}
\]

By Lemma A.1, when \( |k| > K \), we have

\[
|R_{kV_i}| \leq \cdot (\frac{\gamma}{|k|^\tau + 1})^{\frac{1}{n-1}} \cdot (\frac{\gamma}{|k|^\tau + 1})^{\frac{1}{n-1}}.
\]

Then it follows that

\[
|R_{k,l}| \leq \cdot (\text{diam } \Lambda)^{n-1} \frac{\gamma^{\frac{1}{n-1}}}{|k|^\tau + 1}.
\]

When \( |k| \leq K \), by A2) we have that \( |R_{k,l}| \to 0(\gamma \to 0) \), i.e., \( |R_{k,l}| = O(\gamma^{\frac{1}{n-1}}), (\gamma \to 0) \). So we obtain that

\[
|\Lambda_0 \setminus \Lambda_*| = |\bigcup_{k,l} R_{k,l}| \leq \sum_{k,l} |R_{k,l}|
\]

\[
\leq \cdot \gamma^{\frac{1}{n-1}} \sum_{|k| > K} \frac{1}{|k|^\tau + 1} + O(\gamma^{\frac{1}{n-1}}) \sum_{0 < l \leq K} l^n
\]

\[
= O(\gamma^{\frac{1}{n-1}}),
\]

which is the result we desired.

**Case 2:** \( n_0 < n \). Let \( \bar{\Lambda} = [1, 2]^{n-n_0} \) and define

\[
\check{\Lambda} = \Lambda_0 \times \bar{\Lambda},
\]

\[
\check{\Lambda}_* = \Lambda_* \times \bar{\Lambda},
\]

\[
\check{\lambda} = (\lambda, \bar{\lambda})^\top, \check{\lambda} \in \check{\Lambda},
\]

\[
\check{\omega}_\nu(\check{\lambda}) = \omega_\nu(\lambda), \nu = 0, 1, \cdots ; \check{\lambda} \in \check{\Lambda}.
\]

Then by A1) it is clear that

\[
\text{rank}\left\{ \frac{\partial^n \check{\omega}_\nu}{\partial \lambda^\alpha} : \alpha \leq n-1 \right\} = n
\]

on \( \check{\Lambda} \) for all \( \nu = 0, 1, \cdots \) as \( \epsilon_0 \) is sufficiently small. Similar to Case 1, we have that

\[
|\check{\Lambda} \setminus \check{\Lambda}_*| = O(\gamma^{\frac{1}{n-1}}).
\]

By Fubini’s theorem,

\[
|\Lambda_0 \setminus \Lambda_*| = O(\gamma^{\frac{1}{n-1}})
\]

as desired.

Since we mainly care about the persistence of invariant tori on sub-manifolds, the measure estimate’s case when \( n_0 > n \) is omitted. In fact the reader can also see the reference Chow, Li and Yi [2] or Li and Yi [10] for details.
Then we show the convergence of $H_\nu$ and $\Psi_\nu$. Similar to the argument in [2] and [10], in view of Lemma 2.5 and Lemma 3.1, it concludes that $\Psi_\nu$ converges uniformly to $\Psi_\infty$ on $D_\infty \times \Lambda_*$, and under the map $\Psi_\infty$, $N_\nu$ converges uniformly to $N_\infty$ on $D_\infty \times \Lambda_*$ with 

$$N_\infty = e_\infty(\lambda) + \langle \omega_\infty(\lambda), y \rangle + \frac{1}{2} \langle A_\infty(\lambda)y, y \rangle + \frac{1}{2} \langle M_\infty(\lambda)u, u \rangle.$$ 

Hence for each $\lambda \in \Lambda_*$, $T^n \times \{0\} \times \{0\}$ is an analytic invariant torus of $H_\infty$ with the frequencies $\omega_\infty(\lambda)$, which, by Lemma 3.1 2), satisfies 

$$(\omega_\infty(\lambda))_q \equiv (\omega_0(\lambda))_q, 1 \leq q \leq d.$$ 

Denote $\Psi_\lambda = \Psi_\infty(\cdot, \lambda)$, then $\{\Psi_\lambda : \lambda \in \Lambda_*\}$ is a $C^{n-1}$ Whitney smooth family of analytic symplectic transformations on $D(\frac{m}{2}, \frac{n}{2})$ (see [2] for details).

Similar to [2], following the Whitney extension of $\Psi_\nu$'s, all $e_\nu, \omega_\nu, A_\nu, M_\nu, P_\nu, \nu = 0, 1, \cdots$, admit uniform $C^{n-1+\sigma_0}$ extensions in $\lambda \in \Lambda_0$ with derivatives in $\lambda$ up to order $n - 1$ satisfying the same estimates (3.1)-(3.9). Thus, $e_\infty, \omega_\infty, A_\infty, M_\infty, P_\infty$, are $C^{n-1}$ Whitney smooth in $\lambda \in \Lambda_*$, and, the derivatives of $(e_\infty - e_0), (\omega_\infty - \omega_0), (A_\infty - A_0), (M_\infty - M_0)$ satisfy similar estimates as in (3.2), (3.4), (3.6), (3.8). Henceforth, the perturbed tori form a $C^{n-1}$ Whitney smooth family on $\Lambda_*$. 

This completes the proof of Theorem 1.1.  

\section*{3.3 Proof of Corollary 1.1}

\textbf{Proof.} Without loss of generality, we assume that $S$ admits a global coordinate, i.e., there is a bounded closed region $\Lambda \subset R^{n_0}$ and a $C^{l_0}$ diffeomorphism $y: \Lambda \to S$ such that $S = y(\Lambda)$. Let $\lambda \in \Lambda$ and consider the transformation 

$$y \to y + y(\lambda).$$

Then (1.1) gives rise to 

$$H(x, y, u, \lambda) = e(\lambda) + \langle \omega(\lambda), y \rangle + \frac{1}{2} \langle A(\lambda)y, y \rangle + \frac{1}{2} \langle M(\lambda)u, u \rangle + P(x, y, u, \lambda),$$

where 

$$e(\lambda) = N(y(\lambda), 0),$$ 

$$\omega(\lambda) = \frac{\partial N}{\partial y}(y(\lambda), 0),$$ 

$$A(\lambda) = \frac{\partial^2 N}{\partial y^2}(y(\lambda), 0),$$ 

$$M(\lambda) = \frac{\partial^2 N}{\partial u^2}(y(\lambda), 0),$$
By the analysis of the Hamiltonian and assumption A0), there is no $O(|yu|)$ in new perturbation $P$. Let $s_0 = \epsilon_0 \gamma_0^{-4m^2(n+1)}$. Then (1.3) holds and the Corollary follows immediately from the Theorem 1.1 as $\epsilon_0$ is sufficiently small. □

3.4 Proof of Theorem 1.2

Proof. By choosing $\lambda$, $\Lambda$ as in the Section 3.3 with the present $S$, the proof of Theorem 1.2 essentially follows that of Theorem 1.1, except the translation

$$\phi : x \rightarrow x, \ y \rightarrow y + y^*, \ u \rightarrow u$$

in Section 2.4 should be defined for purpose of eliminating the energy drift at each KAM step. The rest proof is similar to [2]. □

4 Some Examples

In this section we give some examples to illustrate our results.

Example 4.1. We consider the following unperturbed system:

$$N(y, u) = y_1 + \frac{1}{2} y_2^2 + \sqrt{2} \left( u^2 + v^2 \right),$$

where $y_1, y_2, u, v \in \mathbb{R}^1$, that is $n = 2, m = 1$. It is easy to see that:

$$\Omega = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}, \ \omega = \begin{pmatrix} 1 \\ y_2 \end{pmatrix}, \ A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$R = \begin{pmatrix} 1 & 0 \\ y_2 & \partial_\lambda y_2 \end{pmatrix}, \ M = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix},$$

where $R$ stands for the matrix $\{ \frac{\partial \omega}{\partial \alpha^\alpha} : \alpha \leq n - 1 \}$.

1) We consider the persistence of invariant tori on the line segment:

$$S_1 : \ y_1(\lambda) = a_1 \lambda, \ y_2(\lambda) = a_2 \lambda, \ \lambda \in [1, 2].$$

Obviously A1)' holds on $S_1$ if and only if $a_2 \neq 0$. We can easily verify that

$$\{ \lambda : |i(k, \omega) + (l, \Omega)| = 0, \ 0 < |k| \leq K, \ |l| \leq 2 \}$$

contains at most an isolated point, that is, A2)' holds. So by the expression of $A$ and our Corollary 1.1 1), the majority 2-tori on $S_1$ will persist with unchanged
second component of tangent frequencies. Since \( A \) is singular, part 2) of Corollary 1.1 is not applicable.

2) We consider the persistence of invariant tori on the parabola:

\[ S_2 : y_1(\lambda) = a_1 \lambda, \quad y_2(\lambda) = a_2 \lambda^2, \quad \lambda \in [1, 2]. \]

Similar to 1), we can verify that A1)' holds if and only if \( a_2 \neq 0 \). And similar to 1), we can verify that

\[ \{ \lambda : |i\langle k, \omega \rangle + \langle l, \Omega \rangle| = 0, \; 0 < |k| \leq K, \; |l| \leq 2 \} \]

contains at most two points, i.e., A2)' holds. Also we obtain that the majority 2-tori on \( S_2 \) will persist with unchanged second component of tangent frequencies.

As \( A \) is singular, part 2) of the Corollary 1.1 is not applicable.

Since the eigenvalues of \( J M \) are pure imaginary, the persistent tori are elliptic.

**Example 4.2.** We consider the following unperturbed system:

\[
N(y, u) = \frac{1}{2} y_1^2 + \frac{1}{3} y_2^3 + \frac{\sqrt{2}}{2} (u_1^2 + v_1^2) + \frac{\sqrt{3}}{2} (u_2^2 + v_2^2)
\]

with

\[
\omega = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2y_2 \end{pmatrix}, \quad R = \begin{pmatrix} \frac{\partial y_1}{\partial \lambda} & y_1 \\ y_2 & 2y_2 \frac{\partial y_2}{\partial \lambda} \end{pmatrix},
\]

\[
\Omega = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 \\ \sqrt{3} & 0 & \sqrt{3} & 0 \\ -\sqrt{3} & 0 & 0 & \sqrt{3} \end{pmatrix}, \quad M = \begin{pmatrix} \frac{\partial y_1}{\partial \lambda} & y_1 \\ y_2 & 2y_2 \frac{\partial y_2}{\partial \lambda} \end{pmatrix}.
\]

1) We consider the persistence of invariant tori on the line segment:

\[ S_1 : y_1(\lambda) = a_1 \lambda, \quad y_2(\lambda) = a_2 \lambda, \quad \lambda \in [1, 2]. \]

It is easy to see that \( R \) is nonsingular on \( S_1 \), i.e. A1)' holds, if and only if \( a_1 a_2 \neq 0 \). And it is obvious that

\[ \{ \lambda : |i\langle k, \omega \rangle + \langle l, \Omega \rangle| = 0, \; 0 < |k| \leq K, \; |l| \leq 2 \} \]

contains at most two points, that is, A2)' holds. So by the non-singularity of \( A \) and Corollary 1.1 2) we get the persistence of invariant 2-tori on \( S_1 \) with unchanged tangent frequencies.

2) We consider the persistence of invariant tori on the parabola:

\[ S_2 : y_1(\lambda) = a_1 \lambda, \quad y_2(\lambda) = a_2 \lambda^2, \quad \lambda \in [1, 2]. \]
It is easy to verify that $A1)'$ holds if and only if $a_1a_2 \neq 0$. And similar to $1)$, we obtain that $A2)'$ holds on $S_2$. So we get the same result on $S_2$ as in $1)$.

Since the eigenvalues of $JM$ are pure imaginary, the persistent tori are elliptic.

Example 4.3. We consider the following unperturbed system:

$$N(y, u) = \frac{1}{2} y_1^2 + \frac{1}{2} y_2^2 + \frac{1}{2} y_3^2 + \frac{1}{2} (u_1^2 - v_1^2) + \frac{1}{2} u_2^2 - \frac{3}{2} v_2^2$$

with

$$\omega = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R = \begin{pmatrix} y_1 & 1 & 0 \\ y_2 & 0 & 1 \\ y_3 & 0 & 0 \end{pmatrix},$$

$$\Omega = \begin{pmatrix} 1 \\ -1 \\ \sqrt{3} \end{pmatrix}, M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

where $R$ is obtained on the hyperplane $S: y_3 = a$, $a \neq 0$, i.e. the sub-manifold we will consider. It is easy to verify that $A1)'$ holds on $S$. And we easily observe that the set

$$\{ \lambda : |i<k, \omega> + <l, \Omega>| = 0, 0 < |k| \leq K, |l| \leq 2 \}$$

is a straight line in $S$, or empty set, that is, $A2)'$ also holds on $S$.

Since $A$ is always nonsingular on $S$, by Corollary 1.1 2) we obtain the persistence of invariant 3-tori with the same tangent frequencies as the unperturbed system. Besides, since all eigenvalues of $JM$ are real, the surviving tori are hyperbolic.

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Appendix A

Lemma A.1 Suppose that $g(x)$ is a m-th differentiable function on the closure $\bar{I}$ of $I$, where $I \subset \mathbb{R}^1$ is an interval. Let $I_h = \{ x : |g(x)| < h, x \in I \}, h > 0$. If on $I$, $|g^{(m)}(x)| \geq d > 0$, where $d$ is a constant, then $|I_h| \leq ch^{-m}$.

Proof. See Lemma 2.1 in [17].

Lemma A.2 Let $A, B, C$ be $n \times n, m \times m, n \times m$ matrices respectively. Then the equation

$$AX + XB = C,$$
where $X$ is an $n \times m$ unknown matrix, is solvable if and only if $I_m \otimes A^T + B \otimes I_n$ is nonsingular. Moreover

$$X = (I_m \otimes A^T + B \otimes I_n)^{-1}C.$$

**Proof.** See Appendix in [18]. □

**Lemma A.3** The eigenvalues of $i \langle k, \omega(\lambda) \rangle I_{2m} - JM, \ i \langle k, \omega(\lambda) \rangle I_{4m^2} - I_{2m} \otimes (JM) - (JM) \otimes I_{2m}$ are $i \langle k, \omega \rangle - \Omega_j, \ i \langle k, \omega \rangle - \Omega_j - \Omega_k, \ j, k = 1, \cdots, 2m$, respectively.

**Proof.** See Appendix in [18]. □

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