REACTION OF THE FLUID FLOW ON TIME-DEPENDENT BOUNDARY PERTURBATION

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Abstract. The aim of this paper is to investigate the effects of time-dependent boundary perturbation on the flow of a viscous fluid via asymptotic analysis. We start from a simple rectangular domain and then perturb the upper part of its boundary by the product of a small parameter $\varepsilon$ and some smooth function $h(x, t)$. The complete asymptotic expansion (in powers of $\varepsilon$) of the solution of the evolutionary Stokes system has been constructed. The convergence of the expansion has been proved providing the rigorous justification of the formally derived asymptotic model.

1. Introduction. Fluid flow through time–dependent domain is an important subject that has been studied by numerous authors with different motivations and using different approaches. Problems of that kind naturally appear in the study of physiological flows (see e.g. [7, 18]), flexible pipes [9], fishing nets [12], study of moving bodies in fluid [19], lubrication theory [20], etc.

The goal of the present paper is to study the effects of the moving boundary on the fluid flow in case when the displacement of the boundary is small. For the sake of simplicity, we take the rectangular domain and study the flow driven by the pressure drop from left to the right side of the domain. We modify the upper boundary with small perturbation $\varepsilon h(x, t)$ in vertical direction, causing the motion of the fluid in direction perpendicular to the pressure gradient as well as squeezing the fluid in the direction of the pressure gradient. It should be mentioned that introducing a small parameter as the perturbation quantity in the domain boundary makes analytical treatments very complicated. It is due to the fact that tedious change of a variable has to be performed. Consequently, not many analytical results on the subject can be found throughout the literature. In particular, the perturbation of the domain boundary has remained a rather neglected mathematical topic (see monograph [10]) and has been addressed mostly in the context of periodically corrugated boundaries (see e.g. [1], [11]), especially in the context of lubricated (thin-film) flows.

Before proceeding, we would like to refer to the works dealing with the analysis of the flows in a thin-film regime, being motivated by the engineering practice. Describing the boundary roughness by a periodic function, the flow of a viscous fluid between two very close surfaces has been investigated for various rugosity
profiles. The usual assumption where the size of the roughness is of the same small order as the film thickness, i.e.

\[ 0 < y < h(x) = \varepsilon \mathcal{H} \left( x, \frac{x}{\varepsilon} \right), \ 0 < \varepsilon \ll 1 \]

leads to the effective model in a form of the standard Reynolds approximation (see e.g. [3]). Consequently, to deduce the roughness-induced effects, one needs to compute the higher-order terms in the asymptotic expansion. As shown in [4], the similar result holds for \( h(x) = \varepsilon \mathcal{H} \left( x, \frac{x}{\varepsilon^\beta} \right) \) with \( \beta < 1 \). However, if one considers the specific rugosity profile given by

\[ h(x) = \varepsilon \mathcal{H} \left( x, \frac{x^{\beta}}{\varepsilon^2} \right), \]

it turns out that an extra term appears in the limit modifying the Reynolds equation at the main order. For the details, we refer the reader to [6], [8], [17].

Recently, the authors of this work have been investigating the stationary fluid flow in case of time-independent boundary perturbation. We refer the reader to [13] for classical Newtonian flow, [14] for micropolar fluid flow, [15, 16] for porous medium flow. The purpose of this paper is to generalize the techniques presented in those papers on the case of time-dependent boundary perturbation. Starting from evolutionary Stokes system, we manage to formally derive an asymptotic expansion of the flow giving an approximation of arbitrary order. The zero-order approximation, as expected, is not affected by the boundary perturbation. However, the correctors contain terms explicitly depending on the function \( h \) as well as on its derivatives (see Sections 4 and 5), allowing to closely study the reaction of the fluid flow on the motion of the boundary. Employing functional analysis techniques, we also prove the error estimate for the constructed asymptotic expansion in Section 6. By doing that, we provide the rigorous justification of our effective model and that represents our main contribution.

In the works [3, 4, 6, 8, 17] mentioned above, the oscillating part \( h \) describing the corrugations in \( \mathcal{H} \) is assumed to be a periodic function, independent of time. As a consequence, the limit system allows to be rigorously justified by the notion of the two-scale convergence. Here, we cannot proceed in a similar manner to derive the error estimates, due to the fact that we take boundary perturbation function \( h \) to be an arbitrary (not necessarily periodic) function which, additionally, depends on time. For that reason, we strongly believe that the findings presented here, though not put in the lubrication context, will prove useful in the engineering practice as well.

2. Description of the problem. We begin by introducing the simple rectangular domain

\[ \Omega = \{(x, y) \in \mathbb{R}^2; 0 < x < 1, 0 < y < 1\} \]

Then we take a small parameter \( 0 < \varepsilon \ll 1 \), a smooth function \( h \) and define our two-dimensional domain with perturbed boundary as follows:

\[ \Omega_\varepsilon(t) = \{(x, y) \in \mathbb{R}^2; 0 < x < 1, 0 < y < 1 - \varepsilon h(x, t)\}. \]

We consider the system

\[
\begin{align*}
\frac{\partial \mathbf{u}^\varepsilon}{\partial t} - \Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon &= 0, \quad \text{div} \mathbf{u}^\varepsilon = 0 \quad \text{in} \ \Omega_\varepsilon(t), \quad t > 0, \quad (1) \\
\mathbf{u}^\varepsilon(x, 0, t) &= 0, \quad \mathbf{u}^\varepsilon(x, 1 - \varepsilon h(x, t)) = -\varepsilon \frac{\partial h}{\partial t}(x, t)j, \quad (2)
\end{align*}
\]
The unknowns in the above system are \( \mathbf{u}^{\varepsilon} = (u_x^{\varepsilon}, u_y^{\varepsilon}) \) and \( p^{\varepsilon} \) representing the velocity and the pressure of the fluid, while \( p_0(t) \) and \( p_1(t) \) are prescribed pressures depending on time. We impose the following regularity and compatibility conditions on the given data:

- \( \mathbf{u}_0 \in L^2(\Omega(0))^2 \), \( p_k \in L^2(0, T), k = 0, 1 \), \( h \in C^1([0, T] \times [0, 1]) \),

- \( \frac{\partial h}{\partial t}(x, t) = 0 \) for \( x = 0, 1 \).

Following the approach from [2], we can prove the existence of a solution \((\mathbf{u}^{\varepsilon}, p^{\varepsilon}) \in [\mathbb{H}^{1}([0, T] \times \Omega(t))^2] \cap \mathbb{L}^\infty(0, T; \mathbb{L}^2(\Omega(t))^2)] \times H^{-1}(0, T; \mathbb{L}^2(\Omega(t))) \).

**Remark 1.** In fact, in [2] the existence was proved for the Navier-Stokes problem
\[
\frac{\partial \mathbf{u}^{\varepsilon}}{\partial t} - \Delta \mathbf{u}^{\varepsilon} + Re(\mathbf{u}^{\varepsilon} \cdot \nabla) \mathbf{u}^{\varepsilon} + \nabla p^{\varepsilon} = 0
\]
with boundary condition
\[
p^{\varepsilon} + Re \frac{2}{2} |\mathbf{u}^{\varepsilon}|^2 = p_k \text{ for } x = k, k = 0, 1
\]
and our case directly follows by taking \( Re = 0 \). The problem that they have considered is much more difficult since most of the proof is about handling the nonlinear inertial term.

### 3. Moving coordinates.
In the sequel, we assume that \( h \in C^\infty([0, 1] \times [0, +\infty]) \). As in [13], we introduce the new variable
\[
z = \frac{y}{1 - \varepsilon h},
\]
with the basic difference that \( h \) now depends on \( x \) and \( t \), and not only on \( x \). Thus, the coordinate \( z \) is now moving in time but in some prescribed way determined by \( h \). Obviously,
\[
\frac{\partial z}{\partial x} = \frac{\varepsilon y \frac{\partial h}{\partial x}}{(1 - \varepsilon h)^2} = \frac{\varepsilon z \frac{\partial h}{\partial x} \frac{\partial z}{\partial y}}{1 - \varepsilon h},
\]
\[
\frac{\partial^2 z}{\partial x^2} = \frac{\varepsilon y \frac{\partial^2 h}{\partial x^2}(1 - \varepsilon h) + 2y \varepsilon^2 \left( \frac{\partial h}{\partial x} \right)^2}{(1 - \varepsilon h)^3} = \frac{z}{1 - \varepsilon h} \frac{\partial^2 h}{\partial x^2} + 2 \varepsilon^2 \frac{z}{(1 - \varepsilon h)^2} \left( \frac{\partial h}{\partial x} \right)^2,
\]
\[ \frac{\partial z}{\partial t} = \frac{\varepsilon z}{1 - \varepsilon h} \frac{\partial h}{\partial t}. \]

It should be noted that now, in new variables \((x, z)\), the domain \(\Omega_\varepsilon(t)\) becomes square \(\Omega = [0, 1]^2\). Indeed, introducing

\[ \Psi(x, y) = (x, z) = \left( x, \frac{y}{1 - \varepsilon h(x, t)} \right) \]

we deduce \(\Psi(\Omega_\varepsilon(t)) = \Omega\).

Taking into account the above change of variables, the partial derivatives are changing as follows:

\[
\begin{align*}
\frac{\partial}{\partial x} &\rightarrow \frac{\partial}{\partial x} + \varepsilon \frac{\partial h}{\partial x} \frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \varepsilon \frac{y \frac{\partial h}{\partial x}}{(1 - \varepsilon h)^2} \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} &\rightarrow \frac{\partial}{\partial y} = \frac{1}{1 - \varepsilon h} \frac{\partial}{\partial z} = \sum_{k=0}^{\infty} \varepsilon^k h^k \frac{\partial}{\partial z}, \\
\frac{\partial^2}{\partial x^2} &\rightarrow \frac{\partial^2}{\partial x^2} + \varepsilon \frac{\partial^2 h}{\partial x^2} \frac{\partial}{\partial z} + 2 \frac{\partial z}{\partial x} \frac{\partial^2}{\partial x \partial z} + \left( \frac{\partial z}{\partial x} \right)^2 \frac{\partial^2}{\partial z^2} \\
&= \frac{\partial^2}{\partial x^2} + \varepsilon \frac{\partial^2 h}{\partial x^2} \frac{\partial^2}{\partial z^2} \\
&\quad + 2 \varepsilon \frac{\partial h}{\partial x} \frac{\partial^2}{\partial x \partial z} + \varepsilon^2 \frac{\left( \frac{\partial h}{\partial x} \right)^2}{(1 - \varepsilon h)^2} \frac{\partial^2}{\partial z^2} \\
&\quad + \frac{\partial^2}{\partial x^2} + 2 \varepsilon \frac{\partial^2 h}{\partial x^2} \frac{\partial}{\partial z} + \varepsilon^2 \frac{\left( \frac{\partial h}{\partial x} \right)^2}{(1 - \varepsilon h)^2} \frac{\partial^2}{\partial z^2} \\
&\quad + 2 \varepsilon \frac{\partial h}{\partial x} \frac{\partial^2}{\partial x \partial z} + \varepsilon^2 \frac{\left( \frac{\partial h}{\partial x} \right)^2}{(1 - \varepsilon h)^2} \frac{\partial^2}{\partial z^2}, \\
&\quad = \frac{\partial^2}{\partial x^2} + \frac{1}{(1 - \varepsilon h)^2} \frac{\partial^2}{\partial z^2}, \\
&\quad = \frac{\partial}{\partial t} + \varepsilon \frac{y \frac{\partial h}{\partial t}}{(1 - \varepsilon h)^2} \frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \varepsilon \frac{z \frac{\partial h}{\partial t}}{1 - \varepsilon h} \frac{\partial}{\partial z}.
\end{align*}
\]
System for
\[ U^\varepsilon(x, z, t) = u^\varepsilon(x, y, t) = u^\varepsilon(x, z(1 - \varepsilon h(x, t)), t), \]
\[ P^\varepsilon(x, z, t) = p^\varepsilon(x, y, t) = p^\varepsilon(x, z(1 - \varepsilon h(x, t)), t), \]
with \( U^\varepsilon = (U_x^\varepsilon, U_y^\varepsilon) \) now reads
\[
\begin{align*}
\frac{\partial U^\varepsilon}{\partial t} &- \left[ \frac{\partial^2 U^\varepsilon}{\partial x^2} + \frac{1 + \varepsilon^2 z^2 (\frac{\partial h}{\partial x})^2}{(1 - \varepsilon h)^2} \frac{\partial^2 U^\varepsilon}{\partial z^2} + \frac{2\varepsilon z \frac{\partial h}{\partial x} \frac{\partial^2 U^\varepsilon}{\partial z \partial x}}{(1 - \varepsilon h)^2} \right] + \\
&+ \frac{\varepsilon z}{1 - \varepsilon h} \left[ \frac{\partial h}{\partial x} \frac{\partial^2 U^\varepsilon}{\partial x^2} + \frac{2 \varepsilon (\frac{\partial h}{\partial x})^2}{1 - \varepsilon h} \frac{\partial h}{\partial t} \frac{\partial U^\varepsilon}{\partial z} \right] + \\
&+ \left( \frac{\partial P^\varepsilon}{\partial x} + \varepsilon \frac{\partial \varepsilon}{\partial x} \frac{\partial P^\varepsilon}{\partial z} \right) \frac{1}{1 - \varepsilon h} \frac{\partial P^\varepsilon}{\partial z} = 0, \\
\frac{\partial U_x^\varepsilon}{\partial x} &+ \frac{\varepsilon z \frac{\partial h}{\partial x} \frac{\partial U_y^\varepsilon}{\partial z}}{1 - \varepsilon h} + \frac{1}{1 - \varepsilon h} \frac{\partial P^\varepsilon}{\partial z} = 0,
\end{align*}
\]
\[ U^\varepsilon(x, 0, t) = 0, \quad U^\varepsilon(x, 1, t) = -\varepsilon \frac{\partial h}{\partial t}(x, t)j, \quad U_y^\varepsilon(0, z, t) = U_y^\varepsilon(1, z, t) = 0, \]
\[ P^\varepsilon(0, z, t) = p_0^\varepsilon, \quad P^\varepsilon(1, z, t) = p_1^\varepsilon, \quad U^\varepsilon(x, z, 0) = u_0^\varepsilon(x, z(1 - \varepsilon h(x, 0))). \]

In order to get the form more appropriate for further asymptotic analysis, we multiply (5) by \( (1 - \varepsilon h)^2 \) and (6) by \( 1 - \varepsilon h \). As a result, we get
\[
\begin{align*}
\frac{\partial U^\varepsilon}{\partial t} - \Delta_{xx} U^\varepsilon + \nabla_{xx} P^\varepsilon &+ \varepsilon \left\{ -h \left[ 2 \left( \frac{\partial U^\varepsilon}{\partial t} - \frac{\partial^2 U^\varepsilon}{\partial x^2} + \frac{\partial P^\varepsilon}{\partial x} \right) + \frac{\partial P^\varepsilon}{\partial z} \right] + \\
&+ \frac{\varepsilon}{1 - \varepsilon h} \left[ \frac{\partial h}{\partial x} \left( \frac{\partial P^\varepsilon}{\partial z} - \frac{\partial^2 U^\varepsilon}{\partial x \partial z} \right) + \frac{\partial h}{\partial x} \frac{\partial^2 U^\varepsilon}{\partial x^2} + \frac{\partial P^\varepsilon}{\partial x} \right] \right\} + \\
&+ \left\{ h^2 \left( \frac{\partial U^\varepsilon}{\partial t} - \frac{\partial^2 U^\varepsilon}{\partial x^2} + \frac{h \partial P^\varepsilon}{\partial x} \right) + h \frac{\partial U^\varepsilon}{\partial z} - \frac{z}{1 - \varepsilon h} \frac{\partial U^\varepsilon}{\partial t} \right\} + \\
&+ \frac{\varepsilon}{1 - \varepsilon h} \left( \frac{\partial U^\varepsilon}{\partial x} - \frac{\partial P^\varepsilon}{\partial z} \right) + \frac{h}{1 - \varepsilon h} \left( \frac{\partial^2 U^\varepsilon}{\partial x \partial z} - \frac{\partial U^\varepsilon}{\partial z} \right) \right\} = 0, \\
\text{div}_{xx} U^\varepsilon &+ \varepsilon \left( \frac{\partial U^\varepsilon}{\partial x} - \frac{h \partial^2 U^\varepsilon}{\partial x^2} \right) = 0.
\end{align*}
\]

4. **Asymptotic expansion.** To derive the complete asymptotic expansion, as usual, we assume that the given data \( h, u_0, \) and \( p_k \) are as smooth as needed. For computation we need \( p_k \) to be continuous, \( h \) to be a \( C^2 \) function in \( x \) and \( C^1 \) in \( t \). As for the initial condition \( u_0 \), since we use its Taylor expansion it has to be at least \( C^k \) to compute the approximation of order \( k \).

We now seek for the solution of (5)-(7) in the form of the asymptotic expansion:
\[
\begin{align*}
U^\varepsilon &\sim U^0 + \varepsilon U^1 + \varepsilon^2 U^2 + \varepsilon^3 U^3 + \cdots, \\
P^\varepsilon &\sim P^0 + \varepsilon P^1 + \varepsilon^2 P^2 + \cdots,
\end{align*}
\]
with \( U^k = (U_x^k(x, z, t), U_y^k(x, z, t)) \) and \( P^k = P^k(x, z, t) \). Plugging the above expansions in the governing system and collecting the terms with equal powers of \( \varepsilon \) give
\[
0 = \frac{\partial U^0}{\partial t} - \left( \frac{\partial^2 U^0}{\partial x^2} + \frac{\partial^2 U^0}{\partial z^2} \right) + \frac{\partial P^0}{\partial x} + \frac{\partial P^0}{\partial z} + \varepsilon \left( \frac{\partial U^1}{\partial x} - \frac{h \partial^2 U^0}{\partial x^2} \right) + \cdots
\]
we denote $\Delta$

As a result, we obtain a recurrent system of problems for $(U, U_0)$ endowed with the following boundary and initial conditions:

Finally, the higher-order terms $(U^k, P^k)$, $k \geq 2$ satisfy the following system:

For the zero-order approximation $(U^0, P^0)$ we have:

\begin{equation}
\frac{\partial U^0}{\partial t} - \Delta_x U^0 + \nabla_x P^0 = 0, \quad \text{div}_{xz} U^0 = 0 \quad \text{in}\ \Omega \times [0, +\infty],
\end{equation}

\begin{equation}
U^0(x, 0, t) = U^0(x, 1, t) = 0, \quad U^0_y(0, z, t) = U^0_y(1, z, t) = 0,
\end{equation}

\begin{equation}
P^0(0, z, t) = p_0, \quad P^0(1, z, t) = p_1, \quad U^0(x, z, t) = u_0(x, z).
\end{equation}

First-order approximation $(U^1, P^1)$ is given by the equations:

\begin{equation}
\frac{\partial U^1}{\partial t} - \Delta_x U^1 + \nabla_x U^0 + h \left[ 2 \left( \frac{\partial^2 U^0}{\partial x^2} - \frac{\partial U^0}{\partial t} - \frac{\partial P^0}{\partial x} \right) \right] +
\end{equation}

\begin{equation}
\text{div}_{xz} U^1 + z \frac{\partial U^0}{\partial x} - h \frac{\partial U^0}{\partial x} = 0
\end{equation}

duendowed with the following boundary and initial conditions:

\begin{equation}
U^1_y(0, z, t) = U^1_y(1, z, t) = 0, \quad U^1(0, z, t) = U^1(1, z, t) = 0,
\end{equation}

\begin{equation}
U^1(x, 0, t) = 0, \quad U^1(x, 1, t) = -\frac{\partial h}{\partial t}(x, t) j,
\end{equation}

\begin{equation}
U^1(x, z, 0) = -zh(0) \frac{\partial u_0}{\partial z}(x, z).
\end{equation}

Finally, the higher-order terms $(U^k, P^k)$, $k \geq 2$ satisfy the following system:

\begin{equation}
\frac{\partial U^k}{\partial t} - \Delta_x U^k + \nabla_x U^k + h \left[ 2 \left( \frac{\partial^2 U^{k-1}}{\partial x^2} - \frac{\partial U^{k-1}}{\partial t} - \frac{\partial P^{k-1}}{\partial x} \right) \right] +
\end{equation}

\begin{equation}
\text{div}_{xz} U^k + z \frac{\partial U^{k-1}}{\partial x} - h \frac{\partial U^{k-1}}{\partial x} = 0
\end{equation}

endowed with the following boundary and initial conditions:

\begin{equation}
U^k_y(0, z, t) = U^k_y(1, z, t) = 0, \quad U^k(0, z, t) = U^k(1, z, t) = 0,
\end{equation}

\begin{equation}
U^k(x, 0, t) = 0, \quad U^k(x, 1, t) = -\frac{\partial h}{\partial t}(x, t) j,
\end{equation}

\begin{equation}
U^k(x, z, 0) = -zh(0) \frac{\partial u_0}{\partial z}(x, z).
\end{equation}
of the same size, i.e. if $h < 0$, it is essential to observe that the right-hand side in the problems (16)-(20), (21)-(25) (the known part from the previous iteration) belongs to $L^2(0, T; V')$ so that the solution $(U^k, P^k) \in L^2(0, T; V') \times L^2(0, T; V^2(\Omega))$, for any $k \geq 0$. Furthermore, $U^k \in L^\infty(0, T; L^2(\Omega))$ and $\partial h \partial_t u^k \in L^2(0, T; V')$.

The functions $U^k$ are not divergence free but, if both outlets of the domain are of the same size, i.e. if $h(0, t) = h(1, t)$, we still have the equality of in and out flow, namely:

$$\int_0^1 U_x^k(0, z, t)dz = \int_0^1 U_x^k(1, z, t)dz.$$  

Since $u^\varepsilon$ is divergence free, it holds

$$\int_0^1 \varepsilon h(x, t) u_x^\varepsilon(x, y, t)dy = \text{const.}$$  

i.e. it is independent of $x$. Consequently,

$$[1 - \varepsilon h(x, t)] \int_0^1 U_x^\varepsilon(x, z, t)dz = \text{const..}$$

Substituting the expansion (11) yields

$$\int_0^1 U_x^k(x, z, t)dz - h(x, t) \int_0^1 U_x^{k-1}(x, z, t)dz = \text{const., } k > 0.$$  

5. Formal computation. Another approach would be to use the formal Taylor series approach. To keep the notation as simple as possible, we assume that $h < 0$ so that $\Omega \subset \Omega_x$. As a consequence, our solution $(u^\varepsilon, p^\varepsilon)$ is defined on $\Omega$ so we are in position to directly expand $u^\varepsilon$ in Taylor series with respect to $y$ near the upper boundary (assuming that the velocity is analytic). Otherwise, we would have to extend the solution to $\Omega$ and contaminate the notation. It should be emphasized that this is just a technical assumption, i.e. the obtained results are valid for an arbitrary (smooth enough) function $h$. It is due to the fact that, by the end of this section, we are going to prove that the approximation derived in the sequel is asymptotically the same as the one that we derived in the previous section (by passing to an $\varepsilon$-independent domain). In view of that, we expand as follows

$$u^\varepsilon(x, y, t) = \sum_{k=1}^{\infty} \frac{1}{k!} \partial_y^k u^\varepsilon(x, 1, t)(y - 1)^k.$$
For \( y = 1 - \varepsilon h \) we have
\[
-\varepsilon \frac{\partial h}{\partial t} = \mathbf{u}^\varepsilon(x, 1 - \varepsilon h) - \mathbf{u}^\varepsilon(x, 1, t) - \varepsilon \frac{\partial \mathbf{u}^\varepsilon}{\partial y}(x, 1, t) h + \varepsilon^2 \frac{1}{2} \frac{\partial^2 \mathbf{u}^\varepsilon}{\partial y^2}(x, 1, t) h^2 - \cdots .
\] (28)

Plugging an asymptotic expansion for the solution \((\mathbf{u}^\varepsilon, p^\varepsilon)\) of the form
\[
\mathbf{u}^\varepsilon = \mathbf{V}^0 + \varepsilon \mathbf{V}^1 + \varepsilon^2 \mathbf{V}^2 + \cdots ,
\] (29)
\[
p^\varepsilon = Q^0 + \varepsilon Q^1 + \varepsilon^2 Q^2 + \cdots
\] (30)

and collecting the terms with equal powers of \(\varepsilon\), we get using (28) and (29)
\[
-\varepsilon \frac{\partial h}{\partial t} = \mathbf{V}^0(x, 1, t) + \varepsilon \left( \mathbf{V}^1(x, 1, t) - h \frac{\partial \mathbf{V}^0}{\partial y}(x, 1, t) \right) + \cdots
\]
\[
\cdots + \varepsilon^\ell \left( \mathbf{V}^\ell(x, 1, t) - \sum_{k=1}^{\ell} h^k \frac{(-1)^{k-1}}{k!} \frac{\partial^k \mathbf{V}^0}{\partial y^k}(x, 1, t) \right) + \cdots
\] (31)

leading to the following effective boundary conditions:
\[
\mathbf{V}^0(x, 1, t) = 0,
\] (32)
\[
\mathbf{V}^1(x, 1, t) = h \frac{\partial \mathbf{V}^0}{\partial y}(x, 1, t) - \frac{\partial h}{\partial t} j,
\] (33)
\[
\mathbf{V}^0(x, 1, t) = 0 \Rightarrow \frac{\partial \mathbf{V}^0}{\partial x}(x, 1, t) = 0 \Rightarrow \frac{\partial \mathbf{V}^0}{\partial y}(x, 1, t) = 0 \Rightarrow \mathbf{V}^1_y(x, 1, t) = -\frac{\partial h}{\partial t} j,
\] (34)
\[
\mathbf{V}^\ell(x, 1, t) = -\sum_{k=1}^{\ell} h^k \frac{(-1)^{k-1}}{k!} \frac{\partial^k \mathbf{V}^0}{\partial y^k}(x, 1, t), \ell > 0.
\] (35)

The problem for \((\mathbf{V}^0, Q^0)\) now read:
\[
\frac{\partial \mathbf{V}^0}{\partial t} - \Delta \mathbf{V}^0 + \nabla Q^0 = 0, \quad \text{div} \mathbf{V}^0 = 0 \quad \text{in} \quad \Omega \times ]0, +\infty[,
\] (36)
\[
\mathbf{V}^0(x, 1, t) = 0, \quad \mathbf{V}^0(x, 0, t) = 0,
\] (37)
\[
V^0_y(0, y, t) = V^0_y(1, y, t) = 0, \quad Q^0(0, y, t) = p_0, \quad Q^0(1, y, t) = p_1,
\] (38)
\[
\mathbf{V}^0(x, 0, y) = \mathbf{u}_0(x, y).
\] (39)

The higher-order terms \((\mathbf{V}^k, Q^k), k \geq 1\) are given by:
\[
\frac{\partial \mathbf{V}^k}{\partial t} - \Delta \mathbf{V}^k + \nabla Q^k = 0, \quad \text{div} \mathbf{V}^k = 0 \quad \text{in} \quad \Omega \times ]0, +\infty[,
\] (40)
\[
\mathbf{V}^k(x, 1, t) = \sum_{j=1}^{k} \frac{(-1)^{j-1} h^j}{j!} \frac{\partial^j \mathbf{V}^0}{\partial y^j}(x, 1, t), \quad \mathbf{V}^k(x, 0, t) = 0,
\] (41)
\[
V^k_y(0, y, t) = V^k_y(1, y, t) = 0, \quad Q^k(0, y, t) = Q^k(1, y, t) = 0,
\] (42)
\[
\mathbf{V}^k(x, 0, y) = 0.
\] (43)

By comparing (13)-(15) and (36)-(39), it is obvious that \((\mathbf{U}^0, P^0) = (\mathbf{V}^0, Q^0)\). However, since \((\mathbf{V}^1, Q^1)\) satisfy the following problem:
\[
\frac{\partial \mathbf{V}^1}{\partial t} - \Delta \mathbf{V}^1 + \nabla Q^1 = 0, \quad \text{div} \mathbf{V}^1 = 0 \quad \text{in} \quad \Omega \times ]0, +\infty[,
\] (44)
\[ V^1(x,1,t) = h(x,t) \frac{\partial V^0}{\partial y}(x,1,t)i - \frac{\partial h}{\partial t}(x,t)j, \quad V^1(x,0,t) = 0, \quad (45) \]

\[ V^1_y(0,y,t) = V^1_y(1,y,t) = 0, \quad Q^1_y(0,y,t) = Q^1_y(1,y,t) = 0, \quad (46) \]

\[ V^1(x,y,0) = 0, \quad (47) \]

it is clear that \( V^1 \neq U^1 \). Nevertheless, from (16)-(20) and (44)-(47), it is straightforward to deduce

\[ V^1(x,y,t) = U^1(x,y,t) + yh(x,t) \frac{\partial U^0}{\partial y}(x,y,t), \quad (48) \]

\[ Q^1(x,y,t) = P^1(x,y,t) + yh(x,t) \frac{\partial P^0}{\partial y}(x,y,t). \quad (49) \]

Direct computation also yields

\[ V^2(x,y,t) = U^2(x,y,t) + yh(x,t) \frac{\partial U^1}{\partial y}(x,y,t) +
\]

\[ + (yh(x,t))^2 \left( \frac{1}{2} \frac{\partial^2 U^0}{\partial y^2}(x,y,t) + \frac{\partial U^0}{\partial z}(x,y,t) \right), \quad (48) \]

whereas

\[ Q^2(x,y,t) = P^2(x,y,t) + yh(x,t) \frac{\partial P^1}{\partial y}(x,y,t) +
\]

\[ + (yh(x,t))^2 \left( \frac{1}{2} \frac{\partial^2 P^0}{\partial y^2}(x,y,t) + \frac{\partial P^0}{\partial z}(x,y,t) \right). \quad (49) \]

Employing the expansion

\[ z = \frac{y}{1 - \varepsilon h} = y \sum_{k=0}^{\infty} \varepsilon^k h^k = y + \varepsilon hy + \varepsilon^2 h^2 + O(\varepsilon^3), \]

we obtain

\[ \begin{align*}
U^0(x,z,t) + \varepsilon U^1(x,z,t) + \varepsilon^2 U^2(x,z,t) &= \\
= U^0(x,y + \varepsilon hy(x,t) + \varepsilon^2 h(x,t)^2 y, t) + \\
+ \varepsilon U^1(x,y + \varepsilon hy(x,t), t) + \varepsilon^2 U^2(x,y,t) + O(\varepsilon^3) &= \\
= U^0(x,y,t) + \varepsilon \left[ U^1(x,y,t) + yh(x,t) \frac{\partial U^0}{\partial y}(x,y,t) \right] + \\
+ \varepsilon^2 \left[ U^2(x,y,t) + yh(x,t) \frac{\partial U^1}{\partial y}(x,y,t) + \\
+ (yh(x,t))^2 \left( \frac{1}{2} \frac{\partial^2 U^0}{\partial y^2}(x,y,t) + \frac{\partial U^0}{\partial z}(x,y,t) \right) \right] + O(\varepsilon^3) &= \\
U^0(x,y,t) + \varepsilon V^1(x,y,t) + \varepsilon^2 V^2(x,y,t) + O(\varepsilon^3). \quad (50)
\end{align*} \]

In view of that, we conclude that two approximations are asymptotically the same, as emphasized at the beginning of this section.

6. Error estimate. The main result of this section can be formulated as follows:

Theorem 1. Let

\[ R(k,\varepsilon) = U^k(x,z,t) - \sum_{i=0}^{k} \varepsilon^i U^i(x,z,t), \]
Then there exists $\varepsilon_0$ such that for any $\varepsilon < \varepsilon_0$ and $k \geq 0$:

\begin{align}
|R(k, \varepsilon)|_{L^2(0, T; V)} &\leq C\varepsilon^{k+1}, \\
|R(k, \varepsilon)|_{L^\infty(0, T; L^2(\Omega))} &\leq C\varepsilon^{k+1}, \\
|r(k, \varepsilon)|_{W^{-1,\infty}(0, T; L^2(\Omega)/R)} &\leq C\varepsilon^{k+1}.
\end{align}

Proof. By direct computation, we get

\begin{align}
(1 - \varepsilon h)^2 \left( \frac{\partial R(k, \varepsilon)}{\partial t} - \frac{\partial^2 R(k, \varepsilon)}{\partial x^2} \right) &= 1 + \varepsilon^2 z^2 \left( \frac{\partial h}{\partial x} \right)^2 \frac{\partial^2 R(k, \varepsilon)}{\partial z^2} - \\
-2\varepsilon(1 - \varepsilon h)\frac{\partial h}{\partial x} \frac{\partial^2 R(k, \varepsilon)}{\partial z \partial x} &= \\
-\varepsilon \left[ (1 - \varepsilon h) \left( \frac{\partial^2 h}{\partial x^2} - \frac{\partial h}{\partial x} \right) + 2\varepsilon \left( \frac{\partial h}{\partial x} \right)^2 \right] \frac{\partial R(k, \varepsilon)}{\partial z} &= \\
+(1 - \varepsilon h) \left[ (1 - \varepsilon h) \frac{\partial r(k, \varepsilon)}{\partial x} + \varepsilon^2 z \frac{\partial h}{\partial x} \frac{\partial r(k, \varepsilon)}{\partial z} \right] i + (1 - \varepsilon h) \frac{\partial r(k, \varepsilon)}{\partial z} j = E(k, \varepsilon),
\end{align}

\begin{align}
R(k, \varepsilon)(x, 0, t) &= 0, \quad R(k, \varepsilon)(x, 1, t) = 0, \\
\partial_r R(k, \varepsilon)(x, y, z, t) &= 0, \\
r(k, \varepsilon)(0, z, t) &= 0, \quad r(k, \varepsilon)(1, z, t) = 0, \quad R(k, \varepsilon)(x, z, 0) = b(k, \varepsilon). \quad (58)
\end{align}

For any $T > 0$ denoting $\Omega_T = ]0, T[ \times \Omega$, we have the following expressions for the right-hand side functions $E(k, \varepsilon), e(k, \varepsilon), b(k, \varepsilon)$:

\begin{align}
E(k, \varepsilon) &= \varepsilon^{k+1} \left\{ h \left[ 2 \left( \frac{\partial^2 U^k}{\partial t \partial x} - \frac{\partial U^k}{\partial t} - \frac{\partial P^k}{\partial x} \right) \right] - \frac{\partial P^k}{\partial z} j \right\} + \\
&+ \frac{\partial h}{\partial x} \left( \frac{\partial^2 U^k}{\partial x \partial z} - \frac{\partial P^k}{\partial z} \right) + z \left( \frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} \right) \frac{\partial U^k}{\partial z} + \\
&+ h^2 \left( \frac{\partial U^{k-1}}{\partial t} - \frac{\partial^2 U^{k-1}}{\partial x^2} + \frac{\partial P^{k-1}}{\partial x} \right) - h \frac{\partial U^{k-1}}{\partial z} - z h \frac{\partial h}{\partial t} \frac{\partial U^{k-1}}{\partial t} + \\
&+ z h \frac{\partial h}{\partial x} \left( \frac{\partial^2 U^{k-1}}{\partial z \partial x} - \frac{\partial P^{k-1}}{\partial z} \right) - z \left( \frac{\partial h}{\partial x} \right)^2 \left( \frac{\partial U^{k-1}}{\partial z} - z \frac{\partial U^{k-1}}{\partial z^2} \right) \right\} + \\
&+ \varepsilon^{k+2} \left\{ h^2 \left( \frac{\partial U^k}{\partial t} - \frac{\partial^2 U^k}{\partial x^2} + \frac{\partial P^k}{\partial x} \right) - h \frac{\partial U^k}{\partial z} - z h \frac{\partial h}{\partial t} \frac{\partial U^k}{\partial t} + \\
&+ z h \frac{\partial h}{\partial x} \left( \frac{\partial^2 U^k}{\partial z \partial x} - \frac{\partial P^k}{\partial z} \right) - z \left( \frac{\partial h}{\partial x} \right)^2 \left( \frac{\partial U^k}{\partial z} - z \frac{\partial U^k}{\partial z^2} \right) \right\}, \quad (59)
\end{align}

\begin{align}
e(k, \varepsilon) &= \varepsilon^{k+1} \left( h \frac{\partial U^k}{\partial x} - z \frac{\partial h}{\partial t} \frac{\partial U^k}{\partial z} \right), \\
b(k, \varepsilon) &= \frac{[z z h(0)]^{k+1}}{(k + 1)!} \left[ \frac{\partial^{k+1} u_{x0}}{\partial z^{k+1}}(x, \xi_x, 0)i + \frac{\partial^{k+1} u_{y0}}{\partial z^{k+1}}(x, \xi_y, 0)j \right]. \quad (61)
\end{align}
for some $\xi_x, \xi_y \in ]0, 1[$. Thus,
\[ |E(k, \varepsilon)|_{L^2(0, T; V')} \leq C \varepsilon^{k+1}, |e(k, \varepsilon)|_{L^2(\Omega_T)} \leq \varepsilon^{k+1}, |b(k, \varepsilon)|_{L^\infty(\Omega)} \leq \varepsilon^{k+1}. \]

Before we proceed, we have to correct the divergence by constructing the function $d^\varepsilon = d^\varepsilon_x \mathbf{i} + d^\varepsilon_y \mathbf{j}$ such that
\[
(1 - \varepsilon h) \frac{\partial d^\varepsilon_x}{\partial x} + \varepsilon z \frac{\partial d^\varepsilon_x}{\partial z} + \frac{\partial d^\varepsilon_y}{\partial z} = e(k, \varepsilon) \quad \text{in } \Omega, \tag{62}
\]
\[ d^\varepsilon = 0 \quad \text{for } z = 0, 1, \quad d^\varepsilon_y = 0 \quad \text{for } x = 0, 1. \tag{63} \]

Such function exists and can be chosen in a way that $d^\varepsilon \in L^2(0, T; H^1(\Omega))$ satisfying
\[ |d^\varepsilon|_{L^2(0, T; H^1(\Omega))} \leq C|e(k, \varepsilon)|_{L^2(\Omega_T)} \leq \varepsilon^{k+1}, \tag{64} \]
for some $C > 0$ independent on $\varepsilon$. We multiply (54) by
\[ (1 - \varepsilon h(x, t))^{-1} [R(k, \varepsilon) - d^\varepsilon] \]
and get the following integrals (abusing slightly the notations, because some of the integrals should be replaced by appropriate duality brackets due to the possible lack of regularity):
\[
1.) \quad \int_\Omega (1 - \varepsilon h)(R(k, \varepsilon) - d^\varepsilon) \frac{\partial R(k, \varepsilon)}{\partial t} = \frac{1}{2} \frac{d}{dt} \int_\Omega (1 - \varepsilon h)|R(k, \varepsilon)|^2 + \frac{\varepsilon}{2} \int_\Omega \frac{\partial h}{\partial t} |R(k, \varepsilon)|^2 - \int_\Omega (1 - \varepsilon h)d^\varepsilon \frac{\partial R(k, \varepsilon)}{\partial t},
\]
\[
2.) \quad - \int_\Omega (1 - \varepsilon h)(R(k, \varepsilon) - d^\varepsilon) \frac{\partial^2 R}{\partial x^2} = \]
\[
= - \int_\Omega (1 - \varepsilon h)(R(k, \varepsilon)_x - d^\varepsilon_x) \frac{\partial^2 R_x}{\partial x^2} + \int_\Omega (1 - \varepsilon h)(R(k, \varepsilon)_y - d^\varepsilon_y) \frac{\partial^2 R_y}{\partial x^2} = J_1 + J_2.
\]

The second integral $J_2$ can be treated as
\[
J_2 = \int_\Omega (1 - \varepsilon h) \left| \frac{\partial R(k, \varepsilon)_y}{\partial x} \right|^2 - \int_\Omega (1 - \varepsilon h) \frac{\partial R(k, \varepsilon)_y}{\partial x} \frac{\partial d^\varepsilon}{\partial x} - \varepsilon \int_\Omega \frac{\partial h}{\partial x} \frac{\partial R(k, \varepsilon)_y}{\partial x} (R(k, \varepsilon)_y - d^\varepsilon_y),
\]
whereas for $J_1$ we obtain:
\[
J_1 = \int_\Omega (1 - \varepsilon h)(R(k, \varepsilon)_x - d^\varepsilon_x) \frac{\partial}{\partial x} \left[ \frac{1}{1 - \varepsilon h} \left( \frac{\partial R(k, \varepsilon)_y}{\partial z} + \varepsilon z \frac{\partial h}{\partial x} \frac{\partial R(k, \varepsilon)_x}{\partial z} - e(k, \varepsilon) \right) \right]
\]
\[
= \int_\Omega (1 - \varepsilon h)(R(k, \varepsilon)_x - d^\varepsilon_x) \frac{\partial}{\partial x} \left[ \frac{1}{1 - \varepsilon h} \left( \frac{\partial R(k, \varepsilon)_y}{\partial z} \right) \right] + \varepsilon \int_\Omega (1 - \varepsilon h)(R(k, \varepsilon)_x - d^\varepsilon_x) \frac{\partial}{\partial x} \left( \frac{1}{1 - \varepsilon h} \frac{\partial h}{\partial x} \frac{\partial R(k, \varepsilon)_x}{\partial z} \right) - \int_\Omega (1 - \varepsilon h)(R(k, \varepsilon)_x - d^\varepsilon_x) \frac{\partial}{\partial x} \left( \frac{e(k, \varepsilon)}{1 - \varepsilon h} \right) = J^1_1 + J^1_1 + J^1_1. \]
For the sake of reader’s convenience, we split the above integral in three parts and present the estimation process of each part in details:

\[ J_1^1 = -\varepsilon \int_\Omega (1 - \varepsilon h) \frac{\partial}{\partial z} \left[ \varepsilon R(k, \varepsilon) x - d^2_x \right] \frac{\partial}{\partial x} \left( \frac{R(k, \varepsilon) x}{1 - \varepsilon h} \right) \leq \varepsilon \left( \left| \frac{\partial h}{\partial x} \right|_{L^\infty(0,1)} + \left| \frac{\partial^2 h}{\partial x^2} \right|_{L^\infty(0,1)} \right) |\nabla R(k, \varepsilon) x - d^2_x|_{L^2(\Omega)}, \]

\[ J_1^2 = -\varepsilon \int_\Omega (1 - \varepsilon h) \frac{\partial}{\partial z} \left[ \varepsilon R(k, \varepsilon) x - d^2_x \right] \frac{\partial}{\partial x} \left( \frac{R(k, \varepsilon) y}{1 - \varepsilon h} \right) \]

\[ = -\varepsilon \int_\Omega \frac{\partial R(k, \varepsilon) x}{\partial z} \frac{\partial R(k, \varepsilon) y}{\partial z} \]

\[ + \varepsilon \int_\Omega (1 - \varepsilon h) \frac{\partial d^2_x}{\partial x} \frac{\partial}{\partial x} \left( \frac{R(k, \varepsilon) y}{1 - \varepsilon h} \right) - \varepsilon \int_\Omega \frac{1}{1 - \varepsilon h} \frac{\partial R(k, \varepsilon) x}{\partial z} \frac{\partial h}{\partial x} R(k, \varepsilon) y = J_{11}^1 + J_{12}^1 + J_{13}^1, \]

where

\[ J_{13}^1 \leq \varepsilon \left| \frac{\partial h}{\partial x} \right|_{L^\infty(0,1)} (1 + 2\varepsilon |h|_{L^\infty(0,1)}) |\nabla R(k, \varepsilon) x|_{L^2(\Omega)}^2, \]

\[ J_{12}^1 \leq C \varepsilon^{k+1} |\nabla R(k, \varepsilon) x|_{L^2(\Omega)}. \]

Next,

\[ J_{11}^1 = -\int_\Omega \frac{\partial R(k, \varepsilon) x}{\partial z} \frac{\partial R(k, \varepsilon) y}{\partial z} = -\int_\Omega \frac{\partial R(k, \varepsilon) x}{\partial z} \frac{\partial R(k, \varepsilon) y}{\partial z} \]

\[ = \int_\Omega \frac{\partial R(k, \varepsilon) x}{\partial z} \left( (1 - \varepsilon h) \frac{\partial R(k, \varepsilon) x}{\partial z} + \varepsilon \frac{\partial h}{\partial z} \frac{\partial R(k, \varepsilon) x}{\partial z} - \varepsilon (k, \varepsilon) \right) \]

\[ = \int_\Omega (1 - \varepsilon h) \left| \frac{\partial R(k, \varepsilon) x}{\partial z} \right|^2 + J_{11}^{111}, \]

with

\[ J_{11}^{111} \leq C \varepsilon^{k+1} |\nabla R(k, \varepsilon) x|_{L^2(\Omega)} + \varepsilon \left| \frac{\partial h}{\partial x} \right|_{L^\infty(0,1)} |\nabla R(k, \varepsilon) x|_{L^2(\Omega)}^2. \]

Finally,

\[ J_1^3 = -\int_\Omega (1 - \varepsilon h)(R(k, \varepsilon) x - d^2_x) \frac{\partial}{\partial x} \left( \frac{e(k, \varepsilon)}{1 - \varepsilon h} \right) \]

\[ = -\int_\Omega (1 - \varepsilon h)^{-1}(R(k, \varepsilon) x - d^2_x) \frac{\partial h}{\partial x} \frac{\partial R(k, \varepsilon) x}{\partial x} - \int_\Omega (R(k, \varepsilon) x - d^2_x) \frac{\partial^2 e(k, \varepsilon)}{\partial x} \]

\[ = J_{11}^{31} + J_{11}^{32}, \]

where

\[ J_{11}^{31} \leq C \varepsilon^{k+1} |\nabla R(k, \varepsilon) x|_{L^2(\Omega)} + \varepsilon^{2k+2} \]

and

\[ J_{11}^{32} = -\varepsilon^{k+1} \int_\Omega (R(k, \varepsilon) x - d^2_x) \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \frac{\partial^2 U^k_x}{\partial x^2} - \frac{\partial h}{\partial x} \frac{\partial U^k_x}{\partial z} \right). \]

We have

\[ \varepsilon^{k+1} \int_\Omega (R(k, \varepsilon) x - d^2_x) \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \frac{\partial U^k_x}{\partial z} \right) = -\varepsilon^{k+1} \int_\Omega \frac{\partial (R(k, \varepsilon) x - d^2_x)}{\partial x} \frac{\partial h}{\partial x} \frac{\partial U^k_x}{\partial z}. \]
\[
\leq C \left( \varepsilon^{k+1} |\nabla R(k, \varepsilon, x)|_{L^2(\Omega)} + \varepsilon^{2k+2} \right),
\]
\[
\int_\Omega (R(k, \varepsilon) - d_x^\varepsilon) \frac{\partial h}{\partial x} \frac{\partial^2 U^k_x}{\partial x^2} \leq C \left( \varepsilon^{k+1} |\nabla R(k, \varepsilon, x)|_{L^2(\Omega)} + \varepsilon^{2k+2} \right).
\]

However, the last remaining term from \( J_1^{32} \), namely
\[
\varepsilon^{k+1} \int_\Omega h(R(k, \varepsilon, x) - d_x^\varepsilon) \frac{\partial^2 U^k_x}{\partial x^2}
\]
cannot be handled by partial integration since it only contains derivatives with respect to \( x \) and the two functions under the integral sign do not cancel for \( x = 0, 1 \). Therefore, we need the Lemma 1 from Appendix B, with \( \psi = R(k, \varepsilon, x) - d_x^\varepsilon \). It gives
\[
\varepsilon^{k+1} \int_\Omega h(R(k, \varepsilon, x) - d_x^\varepsilon) \frac{\partial^2 U^k_x}{\partial x^2} = \varepsilon^{k+1} \int_\Omega \left( h^2 (R(k, \varepsilon, x) - d_x^\varepsilon) \frac{\partial^2 U^k_x}{\partial x^2} + \mathcal{I} \right)
\]
\[
= \varepsilon^{k+1} \int_\Omega \left( h^2 (R(k, \varepsilon, x) - d_x^\varepsilon) \frac{\partial^2 U^k_x}{\partial x^2} + \mathcal{I} \right) = \cdots =
\]
\[
\leq \varepsilon^{k+1} \left( \mathcal{I} + \mathcal{I}_1 \mathcal{I}_2 + \mathcal{I}_3 \right) \leq C \left( \varepsilon^{k+1} |\nabla R(k, \varepsilon, x)|_{L^2(\Omega)} + \varepsilon^{2k+2} \right).
\]

We sum up the above computations to obtain
\[
2.) \quad - \int_\Omega (1 - \varepsilon h) R(k, \varepsilon) \frac{\partial^2 \mathbf{R}}{\partial x^2} = \int_\Omega (1 - \varepsilon h) \left| \frac{\partial \mathbf{R}(k, \varepsilon)}{\partial x} \right|^2 + \mathcal{I}_1,
\]
\[
\mathcal{I}_1 \leq C \left( \varepsilon^{k+1} |\nabla R(k, \varepsilon, x)|_{L^2(\Omega)} + \varepsilon^{2k+2} + C_0 |\nabla R(k, \varepsilon, x)|_{L^2(\Omega)}^2 \right),
\]
with
\[
C_0 = C_0 \left( \left| \frac{\partial h}{\partial \varepsilon} \right|_{L^\infty([0,1] \times [0,T])}, \left| h \right|_{L^\infty([0,1] \times [0,T])} \right).
\]

3.) \quad - \int_\Omega \frac{1 + \varepsilon z \frac{\partial R(k, \varepsilon)}{\partial x} \frac{\partial^2 R(k, \varepsilon)}{\partial x^2}}{1 - \varepsilon h} \frac{\partial^2 \mathbf{R}(k, \varepsilon)}{\partial x^2} = \int_\Omega \frac{1 + \varepsilon z \frac{\partial R(k, \varepsilon)}{\partial x} \frac{\partial^2 R(k, \varepsilon)}{\partial x^2}}{1 - \varepsilon h} \left| \frac{\partial \mathbf{R}(k, \varepsilon)}{\partial x} \right|^2
\]
\[
+ 2 \varepsilon \int_\Omega \frac{z \frac{\partial R(k, \varepsilon)}{\partial x} \frac{\partial^2 R(k, \varepsilon)}{\partial x^2} R(k, \varepsilon)}{1 - \varepsilon h}
\]

4.) \quad - \int_\Omega \varepsilon \frac{\partial R(k, \varepsilon)}{\partial x} \frac{\partial^2 R(k, \varepsilon)}{\partial x^2} \frac{\partial \mathbf{R}(k, \varepsilon)}{\partial x} = \int_\Omega \varepsilon \frac{\partial R(k, \varepsilon)}{\partial x} \frac{\partial^2 R(k, \varepsilon)}{\partial x^2} \frac{\partial \mathbf{R}(k, \varepsilon)}{\partial x} +
\]
\[
+ \varepsilon \int_\Omega \frac{\partial R(k, \varepsilon)}{\partial x} \frac{\partial \mathbf{R}(k, \varepsilon)}{\partial x} \mathbf{R}(k, \varepsilon)
\]

5.) \quad - \int_\Omega \varepsilon \frac{\partial R(k, \varepsilon)}{\partial x} \frac{\partial^2 R(k, \varepsilon)}{\partial x^2} \frac{\partial \mathbf{R}(k, \varepsilon)}{\partial x} = \int_\Omega \varepsilon \frac{\partial R(k, \varepsilon)}{\partial x} \frac{\partial^2 R(k, \varepsilon)}{\partial x^2} \frac{\partial \mathbf{R}(k, \varepsilon)}{\partial x} +
\]
\[
+ \varepsilon \int_\Omega \frac{\partial R(k, \varepsilon)}{\partial x} \frac{\partial^2 \mathbf{R}(k, \varepsilon)}{\partial x^2} \frac{\partial \mathbf{R}(k, \varepsilon)}{\partial x} -
\]
\[
- \frac{\varepsilon}{2} \int_0^1 \frac{\partial R(k, \varepsilon)}{\partial x} (0, t) R(k, \varepsilon, x) (0, t) + \frac{\varepsilon}{2} \int_0^1 \frac{\partial R(k, \varepsilon)}{\partial x} (1, t) R(k, \varepsilon, x) (1, t) \right) +
\]

6.) \quad - \int_\Omega \varepsilon \frac{\partial^2 \mathbf{R}(k, \varepsilon)}{\partial x^2} + \frac{\partial \mathbf{R}(k, \varepsilon)}{\partial x} \frac{\partial \mathbf{R}(k, \varepsilon)}{\partial x} \mathbf{R}(k, \varepsilon) =
To estimate the pressure we go back to the original variables:)

\[ \int_{\Omega} \frac{\partial^2 h}{\partial t^2} \partial R(k, \varepsilon) R(k, \varepsilon) - 2\varepsilon^2 \int_{\Omega} \left( \frac{\partial h}{\partial t} \right)^2 \partial R(k, \varepsilon) R(k, \varepsilon) - \frac{\varepsilon}{2} \int_{\Omega} \frac{\partial h}{\partial t} (R(k, \varepsilon))^2. \]

Summing up the above we get

\[ \int_{\Omega} (1 - \varepsilon h) |\nabla R(k, \varepsilon)|^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega} (1 - \varepsilon h)|R(k, \varepsilon)|^2 \]

\[ \leq C (\varepsilon^{k+1} |\nabla R(k, \varepsilon)|_{L^2(\Omega)} + \varepsilon^{2k+2}) + C_1 \varepsilon |\nabla R(k, \varepsilon)|_{L^2(\Omega)}^2, \tag{65} \]

with

\[ C_1 = C_1 \left( \left| \frac{\partial h}{\partial t} \right|_{L^\infty([0,1] \times [0,T])}, \left| \frac{\partial^2 h}{\partial x^2} \right|_{L^\infty([0,1] \times [0,T])}, \left| \frac{\partial h}{\partial x} \right|_{L^\infty([0,1] \times [0,T])}, |h|_{L^\infty([0,1] \times [0,T])} \right). \]

Integrating with respect to \( t \), for \( \varepsilon < \frac{1}{2C_1}, \frac{1}{|h|_{L^\infty}} \) we obtain

\[ |R(k, \varepsilon)|_{L^2(0,T;V^*)} \leq C\varepsilon^{k+1}. \tag{66} \]

\[ |R(k, \varepsilon)|_{L^2(0,T;L^2(\Omega))} \leq C\varepsilon^{k+1}. \tag{67} \]

To estimate the pressure we go back to the original variables \((x, y, t)\) and introduce

\[ \rho^\varepsilon(x, y, t) = \int_0^t r(k, \varepsilon)(x, z, s)ds, H^\varepsilon(x, y, t) = \int_0^t R(k, \varepsilon)(x, z, s)ds \]

\[ B^\varepsilon(x, y, t) = R(k, \varepsilon)(x, z, t). \]

Then for a.e. \( t \in [0, T] \)

\[ \nabla \rho^\varepsilon = \Delta H^\varepsilon + B^\varepsilon(x, y, 0) - B^\varepsilon(x, y, t) + M^\varepsilon, \]

where

\[ |M^\varepsilon(\cdot, t)|_{H^{-1}(\Omega(t))} \leq C\varepsilon^{k+1}, \]

\[ |B^\varepsilon(\cdot, t)|_{L^2(\Omega(t))} \leq C\varepsilon^{k+1} \text{ for (a.e.) } t \in [0, T]. \]

Testing the above equation with \( T \in H^1_0(\Omega(t)) \) we get

\[ |(\rho^\varepsilon(\cdot, t)|T)| \leq C\varepsilon^{k+1} |T|_{H^1(\Omega(t))}. \]

Now the Nečas inequality (see e.g. [5]) implies that

\[ \left| \rho^\varepsilon(\cdot, t) - \frac{1}{|\Omega(t)|} \int_{\Omega(t)} \rho^\varepsilon(x, y, t) dxdy \right|_{L^2(\Omega(t))} \leq C\varepsilon^{k+1}. \tag{68} \]

For \( \varepsilon < 1/|h|_{L^\infty} \), a simple change of variables (68) implies for (a.e.) \( t \in [0, T] \)

\[ \left| \int_0^t \rho^\varepsilon(\cdot, s)ds - \int \int_0^t \rho^\varepsilon(x, z, s)dsdz \right|_{L^2(\Omega)} \leq C\varepsilon^{k+1} \]

and thus

\[ \left| \int_0^t \rho^\varepsilon(x, y, s)ds - \int \int_0^t \rho^\varepsilon(x, z, s)dsdz \right|_{L^\infty(0,T;L^2(\Omega))} \leq C\varepsilon^{k+1}. \tag{69} \]
Finally, the continuity of the derivative \( \frac{\partial}{\partial t} : L^\infty(0,T;L^2(\Omega)) \rightarrow W^{-1,\infty}(0,T;L^2(\Omega)) \) implies the assertion (53).

7. Appendix A: Existence and a priori estimates. In this Appendix, for reader’s convenience, we provide the existence theorem and derive the a priori estimates for the Stokes system in a rectangle, with pressure boundary condition.

**Theorem 2.** Let \( f \in L^2(0,T;V') \), \( a \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1)) \), \( p_0, p_1 \in W^{1,\infty}(0,T) \) and \( v \in L^2(\Omega)^2 \) and the compatibility condition
\[
a(t,0) = a(t,1) = 0 \quad a(0,x) = v(x,1).
\]
holds. Then the problem
\[
\frac{\partial u}{\partial t} - \Delta u + \nabla p = f, \quad \text{div}\, u = 0 \quad \text{in} \ \Omega_T,
\]
\[
u(x,0,t) = 0, \quad u(x,1,t) = a(x,t)j,
\]
\[
w_y = 0 \quad \text{and} \quad p = p_k \quad \text{for} \ x = k, \ k = 0, 1, \ u(x,z,0) = v(x,z)
\]
has a unique solution \( u \in L^2(0,T;V), p \in W^{-1,\infty}(0,T;L^2(\Omega))/R \). Furthermore \( \frac{\partial u}{\partial t} \in L^2(0,T;V) \). If, in addition, \( f \in L^2(0,T;L^2(\Omega)^2) \) then \( u \in L^2(0,T;H^1(\Omega)^2) \) and \( \frac{\partial u}{\partial t} \in L^2(\Omega_T)^2 \).

**Proof.** The existence and uniqueness proof can be found in [5]. We scratch the proof and concentrate mainly on a priori estimates. We start by lifting the boundary condition on the upper boundary. We define the vector function
\[
A(x,z,t) = \left(6z(z-1) \int_0^x a(s,t)ds, z^2(3-2z)a(x,t)\right).
\]
By construction \( \text{div}\, A = 0 \) and
\[
A(x,0,t) = 0, \quad A(x,1,t) = a(x,t)j.
\]
Due to the assumption (70) we also have
\[
A_y(0,z,t) = A_y(1,z,t) = 0.
\]
Let \( w = u - A \). Then
\[
\frac{\partial w}{\partial t} - \mu \Delta w + \nabla p = g, \quad \text{div}\, w = 0 \quad \text{in} \ \Omega_T,
\]
\[
w(x,0,t) = 0, \quad w(x,1,t) = 0,
\]
\[
w_y = 0 \quad \text{and} \quad p = p_k \quad \text{for} \ x = k, \ k = 0, 1, \ w(x,z,0) = v(x,z) - A(x,z,0),
\]
with
\[
g = f - \frac{\partial A}{\partial t} + \mu \Delta A
\]
\[
= f - \left(6z(z-1) \left( \int_0^x \frac{\partial a}{\partial x}(s,t)ds - \mu \frac{\partial a}{\partial x} \right) - 12\mu \int_0^x a(s,t)ds \right) i -
\]
\[
- \left(z^2(3-2z) \left( \frac{\partial a}{\partial t} - \mu \frac{\partial^2 a}{\partial x^2} \right) - 6\mu(1-2z)a(s,t) \right) j.
\]
To solve that problem we use the Galerkin procedure. Let
\[
V = \{ v = (v_x, v_y) \in H^1(\Omega)^2; \text{div}\, v = 0, v(x,0) = v(x,1) = 0, v_y(0,z) = v_y(1,z) = 0 \}.
\]
be the Hilbert space with $H^1$ norm and let $V'$ be its dual space. Let $(b_j)_{j \in \mathbb{N}}$ be a basis in $V$. We look for an approximation of the solution to the problem (73) in the form
\[
w^m(x, z, t) = \sum_{j=1}^{m} c_{mj}(t)b_j(x, z).
\]

We choose the functions $c_{mj}$ such that for any $k \in \{1, \ldots, m\}$
\[
\int_{\Omega} \frac{\partial w^m}{\partial t} b_k + \mu \int_{\Omega} \nabla w^m \nabla b_k = \langle g | b^j \rangle_{V', V} + (p_1 - p_0) \int_{\Omega} b'_x
\]
where $P_m$ is an orthogonal projector on subspace span\{b^1, \ldots, b^m\}. That is a linear system of ordinary differential equations that admits a unique solution. Due to the assumptions on $f$, $p_0$, $p_1$ and $a$, we get that $w^m \in L^2(0, T; V) \cap L^\infty(0, T; L^2(\Omega)^2)$.

Furthermore, $\frac{\partial w^m}{\partial t} \in L^2(0, T; V')$.

Now, we multiply the equation (73) with $c_{mk}$ and take the sum with respect to $k$ from 1 to $m$. We obtain
\[
\int_{\Omega} \frac{\partial w^m}{\partial t} w^m + \mu \int_{\Omega} \nabla w^m \nabla w^m = \langle g | w^m \rangle_{V', V} + (p_1 - p_0) \int_{\Omega} w^m.
\]

We easily deduce that
\[
\int_{\Omega} \frac{\partial w^m}{\partial t} w^m = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |w^m|^2,
\]
\[\int_{\Omega} g w^m \leq C \left( |f|_{V'} + |a|_{H^2(0, 1)} + \frac{\partial a}{\partial t} \right) |w^m|_V.
\]

Thus,
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |w^m|^2 + \mu \int_{\Omega} |\nabla w^m|^2 \leq C \left( |f|_{L^2(0, T; V')} + |a|_{L^2(0, T; H^2(0, 1))} + \frac{\partial a}{\partial t} \right) |w^m|_V
\]
giving
\[
|w^m|_{L^2(0, T; V)} \leq C \left( |f|_{L^2(0, T; V')} + |a|_{L^2(0, T; H^2(0, 1))} + \frac{\partial a}{\partial t} \right).
\]

We conclude that there is a subsequence, denoted by the same symbol, $w^m$ that converges weakly in $L^2(0, T; V)$ and weak* in $L^\infty(0, T; L^2(\Omega)^2)$ to some $w$. Since the equation (74) is linear, we can pass to the limit as $m \to \infty$ and find that $w$ satisfies the equation (74). Furthermore, $w$ is unique (due to the linearity of the system) and, due to the weak lower semicontinuity of the norm, satisfies the estimate
\[
|w|_{L^2(0, T; V)} \leq C \left( |f|_{L^2(0, T; V')} + |a|_{L^2(0, T; H^2(0, 1))} + \frac{\partial a}{\partial t} \right).
\]

Next step is to use $\frac{\partial w}{\partial t}$ as the test function in (73). We notice that $\text{div} \frac{\partial w}{\partial t} = 0$ and
\[
\frac{\partial w}{\partial t}(x, 0, t) = 0, \frac{\partial w}{\partial t}(x, 1, t) = 0.
\]
Due to the assumption (70), we also have
\[ \frac{\partial w_y}{\partial t}(0, z, t) = \frac{\partial w_y}{\partial t}(1, z, t) = 0. \]

In view of that, after employing \( \frac{\partial w}{\partial t} \) as the test function in (73), we obtain the terms that can be treated as follows:

\[ -\mu \int_{\Omega} \Delta w \frac{\partial w}{\partial t} = -\mu \int_{\Omega} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) \frac{\partial w}{\partial t} \]

\[ = \mu \int_{\Omega} \nabla w \frac{\partial \nabla w}{\partial t} - \mu \int_{0}^{1} \frac{\partial w}{\partial x} \frac{\partial w}{\partial t} (0, z, t) dz + \mu \int_{0}^{1} \frac{\partial w}{\partial z} \frac{\partial w}{\partial t} (1, z, t) dz \]

\[ = \frac{\mu}{2} \int_{\Omega} |\nabla w|^2 - \mu \int_{0}^{1} \frac{\partial w_x}{\partial x} \frac{\partial w_x}{\partial t} (0, z, t) dz + \mu \int_{0}^{1} \frac{\partial w_x}{\partial z} \frac{\partial w_x}{\partial t} (1, z, t) dz \]

\[ = \frac{\mu}{2} \int_{\Omega} |\nabla w|^2, \]

\[ \int_{\Omega} \nabla \frac{\partial w}{\partial t} = - \int_{\Omega} \rho \frac{\partial w}{\partial t} + (p_0 - p_1) \int_{0}^{1} \frac{\partial w_x}{\partial t} (0, z, t) dz \]

\[ = \frac{d}{dt} \left( (p_0 - p_1) \int_{0}^{1} w_x(0, z, t) dz \right) - \frac{d}{dt}(p_0 - p_1) \int_{0}^{1} w_x(0, z, t) dz, \]

\[ \int_{\Omega} gw \leq C \left( |f|_{\text{Vr}} + |a|_{H^2(0,1)} + \left| \frac{\partial a}{\partial t} \right|_{L^2(0,1)} \right) |w|_{\text{Vr}}. \]

Since \( w \) is divergence free, the integral \( \int_{0}^{1} w_x(x, z, t) dz \) does not depend on \( x \) and, therefore, it is equal to \( \int_{\Omega} w_x(x, z, t) dx dz \). In particular,

\[ \int_{0}^{1} w_x(0, z, t) dz = \int_{\Omega} w_x(x, z, t) dx dz. \]

Summing up the above, we get for (almost) any \( t \):

\[ \int_{\Omega} \left| \frac{\partial w}{\partial t} \right|^2 + \frac{\mu}{2} \int_{\Omega} |\nabla w|^2 = \frac{d}{dt} \left( (p_0 - p_1) \int_{\Omega} w_x \right) - \frac{d}{dt}(p_0 - p_1) \int_{\Omega} w_x + \int_{\Omega} gw. \]

Integrating the above with respect to \( t \) over some interval \([0, s]\) we obtain (a.e) \( s \in [0, T]\):

\[ |\nabla w|_{L^2(\Omega)} \leq C \left( (|p_0 - p_1|(s)) ||w(s)||_{L^2(\Omega)} + (|p'_0 - p'_1|(s)) ||w||_{L^2(\Omega_T)} + |g||_{L^2(0,T;V')} \right) \]

and thus

\[ |\nabla w|_{L^\infty(0,T;L^2(\Omega))} \leq C \left( |p_0 - p_1|_{L^\infty(0,T)} ||w||_{L^\infty(0,T;L^2(\Omega))} + \\
+ |p'_0 - p'_1|_{L^\infty(0,T)} ||w||_{L^2(\Omega_T)} + |f|_{V'} + |a|_{L^2(0,T;H^2(0,1))} + \left| \frac{\partial a}{\partial t} \right|_{L^2(0,T;L^2(0,1))} \right). \]
Taking $s = T$ we also get that
\[
\left| \frac{\partial w}{\partial t} \right|_{L^2(0,T;L^2(\Omega))} \leq C \left( |p_0 - p_1|_{L^\infty(0,T)} |w|_{L^\infty(0,T;L^2(\Omega))} + |p'_1 - p'_1|_{L^\infty(0,T)} |w|_{L^2(\Omega)} + |f|_{L^2(\Omega)} + |a|_{L^2(0,T;H^2(0,1))} + \left| \frac{\partial a}{\partial t} \right|_{L^2(0,T;L^2(0,1))} \right).
\]
Finally we need to recover the pressure. We define
\[
W(x,y,t) = \int_0^t w(x,y,s)ds, \quad G = \int_0^t g(x,y,s)ds.
\]
Then $W \in L^\infty(0,T;V)$ and $G \in L^\infty(0,T;L^2(\Omega))$. Now for
\[
M(x,z,t) = \Delta W(x,z,t) + F(x,z,t) + v(x,z) - A(x,z,0) - W(x,z,t)
\]
and for any $V \in \{d \in H^1_0(\Omega)\}^2; \text{div} d = 0$ we have for (a.e) $t$:
\[
\langle M | V \rangle = 0.
\]
Due to the DeRham lemma, there exists $P \in L^\infty(0,T;L^2(\Omega))$ such that
\[
\nabla_{xz} P(x,z,t) = M(x,z,t), (x,z) \in \Omega, t \in [0,T].
\]
Furthermore,
\[
\int_0^T |\langle \nabla P | b \rangle| \leq C \left( |\nabla W|_{L^\infty(0,T;L^2(\Omega))} + |w|_{L^\infty(0,T;L^2(\Omega))} + |v|_{L^2(\Omega)} + |A|_{L^\infty(0,T;L^2(\Omega))} + |g|_{L^\infty(0,T;L^2(\Omega))} \right) |b|_{L^1(0,T;H^1(\Omega))}
\]
implicating
\[
|P|_{L^\infty(0,T;L^2(\Omega))} \leq C \left( |\nabla W|_{L^\infty(0,T;L^2(\Omega))} + |w|_{L^\infty(0,T;L^2(\Omega))} + |v|_{L^2(\Omega)} + |A|_{L^\infty(0,T;L^2(\Omega))} + |g|_{L^\infty(0,T;L^2(\Omega))} \right).
\]
We define $p(x,z,t) = \frac{\partial }{\partial t} P(x,z,t) \in W^{-1,\infty}(0,T;L^2(\Omega))$. Then
\[
|p|_{W^{-1,\infty}(0,T;L^2(\Omega))} \leq C \left( |\nabla w|_{L^2(0,T;L^2(\Omega))} + |w|_{L^\infty(0,T;L^2(\Omega))} + |v|_{L^2(\Omega)} + |A|_{L^\infty(0,T;L^2(\Omega))} + |g|_{L^\infty(0,T;L^2(\Omega))} \right) .
\]

8. Appendix B: Recursion related to the divergence corrector. In this Appendix, we prove a technical result needed to estimate the divergence truncation term $e(k, \varepsilon)$ defined by \eqref{eq:60}.

**Lemma 1.** Let $U^k$ be the sequence of functions defined recurrently by boundary value problems \eqref{eq:24}. Then for any $\psi \in Z = \{ \pi \in H^1(\Omega); \pi = 0 \text{ for } z = 0,1 \}$ the following relation holds for $k > 0$
\[
\left\langle \frac{\partial^2 U^k}{\partial x^2} \bigg| \frac{\partial}{\partial x^2} \bigg| h^m \psi \right\rangle_{Z',Z} = \left\langle \frac{\partial^2 U^{k-1}}{\partial x^2} \bigg| \frac{\partial}{\partial x^2} \bigg| h^{m+1} \psi \right\rangle_{Z',Z} + I^{k,m},
\]
with
\[
I^{k,m} \leq A_{k,m}|\nabla \psi|_{L^2(\Omega)}
\]
and
\begin{align*}
A_{k,m} = |h|^m L^\infty(\Omega) \left( \left| \frac{\partial h}{\partial x} \right|_{L^\infty(\Omega)} |\nabla U^{k-1}|_{L^2(\Omega)} + |\nabla U^k|_{L^2(\Omega)} \right) + \\
+ \left( \left| \frac{\partial h}{\partial x} \right|_{L^\infty(\Omega)} |\nabla U^{k-1}|_{L^2(\Omega)} \right) \left( \left| \nabla U^{k-1}|_{L^2(\Omega)} \right| \right).
\end{align*}
(82)

For \( k = 0 \)
\begin{equation*}
\left\langle \frac{\partial^2 U^0_x}{\partial x^2} h^m \psi \right\rangle_{z',z} = I^{0,m},
\end{equation*}
(83)

where (81) still holds for \( I^{0,m} \) and \( A^{0,m} \leq |h|^m L^\infty(\Omega) |\nabla U^0|_{L^2(\Omega)} \).

Proof. Using (24) we get
\begin{equation*}
\frac{\partial U^k_x}{\partial x} = -\frac{\partial U^k_y}{\partial z} - z \frac{\partial h}{\partial x} \frac{\partial U^{k-1}_x}{\partial z} + h \frac{\partial U^{k-1}_x}{\partial x}.
\end{equation*}

Deriving with respect to \( x \) and testing with \( h^m \psi \) we arrive at
\begin{align*}
\left\langle \frac{\partial^2 U^k_x}{\partial x \partial z} h^m \psi \right\rangle = -\left\langle \frac{\partial^2 U^k_y}{\partial x \partial z} + z \frac{\partial h}{\partial x} \frac{\partial U^{k-1}_x}{\partial z} + h \frac{\partial U^{k-1}_x}{\partial x} \right\rangle h^m \psi \\
= \int h^m \frac{\partial U^k_x}{\partial x} \frac{\partial \psi}{\partial z} + \int h^m \frac{\partial \psi}{\partial z} \left( \frac{\partial h}{\partial x} + \frac{\partial U^{k-1}_x}{\partial z} \right) + \\
+ \int h^m \frac{\partial h}{\partial x} \frac{\partial U^{k-1}_x}{\partial x} + \left\langle \frac{\partial^2 U^{k-1}_x}{\partial x^2} h^{m+1} \psi \right\rangle.
\end{align*}

First three integrals on the right-hand side can be directly estimated to obtain (80) and (81). The last assertion is proved following the same steps. \( \square \)

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