Spectral Bounds for the Torsion Function

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Abstract. Let $Ω$ be an open set in Euclidean space $\mathbb{R}^m$, $m = 2, 3, \ldots$, and let $v_Ω$ denote the torsion function for $Ω$. It is known that $v_Ω$ is bounded if and only if the bottom of the spectrum of the Dirichlet Laplacian acting in $L^2(Ω)$, denoted by $λ(Ω)$, is bounded away from 0. It is shown that the previously obtained bound $\|v_Ω\|_{L^\infty(Ω)} λ(Ω) \geq 1$ is sharp: for $m \in \{2, 3, \ldots\}$, and any $\epsilon > 0$ we construct an open, bounded and connected set $Ω_\epsilon \subset \mathbb{R}^m$ such that $\|v_{Ω_\epsilon}\|_{L^\infty(Ω_\epsilon)} λ(Ω_\epsilon) < 1 + \epsilon$. An upper bound for $v_Ω$ is obtained for planar, convex sets in Euclidean space $\mathbb{R}^2$, which is sharp in the limit of elongation. For a complete, non-compact, $m$-dimensional Riemannian manifold $M$ with non-negative Ricci curvature, and without boundary it is shown that $v_Ω$ is bounded if and only if the bottom of the spectrum of the Dirichlet–Laplace–Beltrami operator acting in $L^2(Ω)$ is bounded away from 0.

Mathematics Subject Classification. Primary 58J32, 58J35, 35K20.

Keywords. Torsion function, Dirichlet Laplacian, Riemannian manifold, Non-negative Ricci curvature.

1. Introduction

Let $Ω$ be an open set in $\mathbb{R}^m$, and let $Δ$ be the Laplace operator acting in $L^2(\mathbb{R}^m)$. Let $(B(s), s \geq 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be Brownian motion on $\mathbb{R}^m$ with generator $Δ$. For $x \in Ω$ we denote the first exit time, and expected lifetime of Brownian motion by

$$T_Ω = \inf \{s \geq 0 : B(s) \notin Ω\},$$

and

$$v_Ω(x) = E_x[T_Ω], \ x \in Ω,$$

respectively, where $E_x$ denotes the expectation associated with $\mathbb{P}_x$. Then $v_Ω$ is the torsion function for $Ω$, i.e. the unique solution of

$$−Δv = 1, \ v \in H^1_0(Ω).$$

The bottom of the spectrum of the Dirichlet Laplacian acting in $L^2(Ω)$ is denoted by
\[ \lambda(\Omega) = \inf_{\varphi \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |D\varphi|^2}{\int_{\Omega} \varphi^2}. \quad (3) \]

It was shown in [1,2] that \( \|v_\Omega\|_{L^\infty(\Omega)} \) is finite if and only if \( \lambda(\Omega) > 0 \). Moreover, if \( \lambda(\Omega) > 0 \), then

\[ \lambda(\Omega)^{-1} \leq \|v_\Omega\|_{L^\infty(\Omega)} \leq (4 + 3m \log 2)\lambda(\Omega)^{-1}. \quad (4) \]

The upper bound in (4) was subsequently improved (see [3]) to

\[ \|v_\Omega\|_{L^\infty(\Omega)} \leq \frac{1}{8} (m + cm^{1/2} + 8)\lambda(\Omega)^{-1}, \]

where

\[ c = (5(4 + \log 2))^{1/2}. \]

In Theorem 1 below we show that the coefficient 1 of \( \lambda(\Omega)^{-1} \) in the left-hand side of (4) is sharp.

**Theorem 1.** For \( m \in \{2, 3, \ldots \} \), and any \( \epsilon > 0 \) there exists an open, bounded, and connected set \( \Omega_\epsilon \subset \mathbb{R}^m \) such that

\[ \|v_{\Omega_\epsilon}\|_{L^\infty(\Omega_\epsilon)} \lambda(\Omega_\epsilon) < 1 + \epsilon. \quad (5) \]

The set \( \Omega_\epsilon \) is constructed explicitly in the proof of Theorem 1.

It has been shown by L. E. Payne (see (3.12) in [4]) that for any convex, open \( \Omega \subset \mathbb{R}^m \) for which \( \lambda(\Omega) > 0 \),

\[ \|v_\Omega\|_{L^\infty(\Omega)} \lambda(\Omega) \geq \frac{\pi^2}{8}, \quad (6) \]

with equality if \( \Omega \) is a slab, i.e. the connected, open set, bounded by two parallel \( (m-1) \)-dimensional hyperplanes. Theorem 2 below shows that for any sufficiently elongated, convex, planar set (not just an elongated rectangle) \( \|v_\Omega\|_{L^\infty(\Omega)} \lambda(\Omega) \) is approximately equal to \( \frac{\pi^2}{8} \). We denote the width and the diameter of a bounded open set \( \Omega \) by \( w(\Omega) \) (i.e. the minimal distance of two parallel lines supporting \( \Omega \)), and \( \text{diam}(\Omega) = \sup\{|x - y| : x \in \Omega, y \in \Omega\} \) respectively.

**Theorem 2.** If \( \Omega \) is a bounded, planar, open, convex set with width \( w(\Omega) \), and diameter \( \text{diam}(\Omega) \), then

\[ \|v_\Omega\|_{L^\infty(\Omega)} \lambda(\Omega) \leq \frac{\pi^2}{8} \left( 1 + 7 \cdot 3^{2/3} \left( \frac{w(\Omega)}{\text{diam}(\Omega)} \right)^{2/3} \right). \]

In the Riemannian manifold setting we denote the bottom of the spectrum of the Dirichlet–Laplace–Beltrami operator by (3). We have the following.

**Theorem 3.** Let \( M \) be a complete, non-compact, \( m \)-dimensional Riemannian manifold, without boundary, and with non-negative Ricci curvature. There
exists \( K < \infty \), depending on \( M \) only, such that if \( \Omega \subset M \) is open, and \( \lambda(\Omega) > 0 \), then

\[
\lambda(\Omega)^{-1} \leq \|v_\Omega\|_{L^\infty(\Omega)} \leq 2^{(3m+8)/4} \cdot 3^{m/2} K^2 \lambda(\Omega)^{-1},
\]

where \( K \) is the constant in the Li-Yau inequality in (35) below.

The proofs of Theorems 1, 2, and 3 will be given in Sects. 2, 3 and 4 respectively.

Below we recall some basic facts on the connection between torsion function and heat kernel. It is well known (see [5–7]) that the heat equation

\[
\Delta u(x; t) = \frac{\partial u(x; t)}{\partial t}, \quad x \in M, \quad t > 0,
\]

has a unique, minimal, positive fundamental solution \( p_M(x, y; t) \), where \( x \in M \), \( y \in M \), \( t > 0 \). This solution, the heat kernel for \( M \), is symmetric in \( x, y \), strictly positive, jointly smooth in \( x, y \in M \) and \( t > 0 \), and it satisfies the semigroup property

\[
p_M(x, y; s + t) = \int_M dz \, p_M(x, z; s)p_M(z, y; t),
\]

for all \( x, y \in M \) and \( t, s > 0 \), where \( dz \) is the Riemannian measure on \( M \). See, for example, [8] for details. If \( \Omega \) is an open subset of \( M \), then we denote the unique, minimal, positive fundamental solution of the heat equation on \( \Omega \) by \( p_\Omega(x, y; t) \), where \( x \in \Omega \), \( y \in \Omega \), \( t > 0 \). This Dirichlet heat kernel satisfies,

\[
p_\Omega(x, y; t) \leq p_M(x, y; t), \quad x \in \Omega, \quad y \in \Omega, \quad t > 0.
\]

Define \( u_\Omega : \Omega \times (0, \infty) \mapsto \mathbb{R} \) by

\[
u_\Omega(x; t) = \int_\Omega dy \, p_\Omega(x, y; t).
\]

Then,

\[
u_\Omega(x; t) = \mathbb{P}_x[T_\Omega > t],
\]

and by (1)

\[
u_\Omega(x) = \int_0^\infty dt \, \mathbb{P}_x[T_\Omega > t] = \int_0^\infty dt \int_\Omega dy \, p_\Omega(x, y; t).
\]

It is straightforward to verify that \( v_\Omega \) as in (8) satisfies (2).

2. **Proof of Theorem 1**

We introduce the following notation. Let \( C_L = (-\frac{L}{2}, \frac{L}{2})^m \) be the open cube with measure \( L^m \), and delete from \( C_L \), \( N^m \) closed balls with radii \( \delta \), where each ball \( B(c_i; \delta) \) is positioned at the centre of an open cube \( Q_i \) with measure \( (L/N)^m \). These open cubes are pairwise disjoint, and contained in \( C_L \). Let \( 0 < \delta < \frac{L}{2N} \), and put (Fig. 1)

\[
\Omega_{\delta,N,L} = C_L - \bigcup_{i=1}^N B(c_i; \delta).
\]

The set \( \Omega_{\delta,N,L} \) also features in [9], where the sharpness of an inequality due to Pólya has been established.
Below we will show that for any $\epsilon > 0$ we can choose $\delta, N$ such that

$$\|v_{\Omega_{\delta,N,L}}\|_{L^\infty(\Omega_{\delta,N,L})} \lambda(\Omega_{\delta,N,L}) < 1 + \epsilon.$$  

In Lemma 4 below we show that $\lambda(\Omega_{\delta,N,L})$ is approximately equal to the first eigenvalue, $\mu_{1,B(0;\delta),L/N}$, of the Laplacian with Neumann boundary conditions on $\partial C_{L/N}$, and with Dirichlet boundary conditions on $\partial B(0;\delta)$. The requirement $\mu_{1,B(0;\delta),L/N}$ not being too small stems from the fact that the approximation of replacing the Neumann boundary conditions on $C_L$ is a surface effect which should not dominate the leading term $\mu_{1,B(0;\delta),L/N}$.

**Lemma 4.** If $\delta \leq \frac{L}{4N}$, $N \geq 10$, and $\frac{N}{L^2} \leq \mu_{1,B(0;\delta),L/N}$, then

$$\lambda(\Omega_{\delta,N,L}) \leq \mu_{1,B(0;\delta),L/N} + 32m \left( \frac{5}{4} \right)^m \left( \frac{N}{L^2} + \frac{1}{N^{1/2}} \mu_{1,B(0;\delta),L/N} \right).$$

**Proof.** Let $\varphi_{1,B(0;\delta),L/N}$ be the first eigenfunction (positive) corresponding to $\mu_{1,B(0;\delta),L/N}$, and normalised in $L^2(C_{L/N} - B(0;\delta))$. In order to prove the lemma we construct a test function by periodically extending $\varphi_{1,B(0;\delta),L/N}$ to all cubes $Q_1, \ldots, Q_{N^m}$ of $\Omega_{\delta,N,L}$. We denote this periodic extension by $f$. We define

$$C_{L,N} = C_L(1 - \frac{z}{\pi}).$$

So $C_{L,N}$ is the sub-cube of $C_L$ with the outer layer of cubes of size $L/N$ removed. Let

$$\tilde{f} = \left( 1 - \frac{\text{dist}(x,C_{L,N})}{L/(4N)} \right)_+ f.$$
Then \( \tilde{f} \in H_0^1(\Omega_{\delta,N,L}) \), and

\[
\| \tilde{f} \|_{L^2(\Omega_{\delta,N,L})}^2 \geq \int_{C_{L,N}} f^2 = (N - 2)^m,
\]

(9)
since \( f \) restricted to any of the cubes \( Q_i \) in \( \Omega_{\delta,N,L} \) is normalised. Furthermore

\[
\left| D\tilde{f} \right|^2 \leq \left( 1 - \frac{\text{dist}(x,C_{L,N})}{L/(4N)} \right)^2 \left| Df \right|^2 + 1_{C_{L,C_{L,N}}} \left( \left( \frac{4N}{L} \right)^2 f^2 + \frac{8N}{L} f |Df| \right)
\]

\[
\leq \left| Df \right|^2 + \left( \frac{4N}{L} \right)^2 1_{C_{L,C_{L,N}}} f^2 + \frac{8N}{L} 1_{C_{L,C_{L,N}}} f |Df|.
\]

Hence

\[
\int_{\Omega_{\delta,N,L}} \left| D\tilde{f} \right|^2 \leq \int_{\Omega_{\delta,N,L}} \left| Df \right|^2 + \left( \frac{4N}{L} \right)^2 \int_{C_{L,C_{L,N}}} f^2
\]

\[
+ \frac{8N}{L} \left( \int_{C_{L,C_{L,N}}} |Df|^2 \right)^{1/2} \left( \int_{C_{L,C_{L,N}}} f^2 \right)^{1/2}
\]

\[
= N^m \mu_{1,B(0;\delta),L/N} + \left( N^m - (N - 2)^m \right) \left( \left( \frac{4N}{L} \right)^2 + \frac{8N}{L} \left( \mu_{1,B(0;\delta),L/N} \right)^{1/2} \right)
\]

\[
\leq N^m \mu_{1,B(0;\delta),L/N} + \left( N^m - (N - 2)^m \right) \left( \left( \frac{4N}{L} \right)^2 + 8N^{1/2} \mu_{1,B(0;\delta),L/N} \right),
\]

(10)

where we have used the last hypothesis in the lemma. By (9), (10), the Rayleigh-Ritz variational formula, and the hypothesis \( N \geq 10 \),

\[
\lambda(\Omega_{\delta,N,L}) \leq \mu_{1,B(0;\delta),L/N}
\]

\[
+ \frac{N^m - (N - 2)^m}{(N - 2)^m} \left( \left( \frac{4N}{L} \right)^2 + \left( 8N^{1/2} + 1 \right) \mu_{1,B(0;\delta),L/N} \right)
\]

\[
\leq \mu_{1,B(0;\delta),L/N} + 32m \left( \frac{5}{4} \right)^m \left( \frac{N}{L^2} + \frac{1}{N^{1/2} \mu_{1,B(0;\delta),L/N}} \right).
\]

(11)

\[ \square \]

To obtain an upper bound for \( \| v_{\Omega_{\delta,N,L}} \|_{L^\infty(\Omega_{\delta,N,L})} \), we change the Dirichlet boundary conditions on \( \partial C_L \) to Neumann boundary conditions. This increases the corresponding heat kernel, torsion function, and \( L^\infty \) norm respectively. By periodicity, we have that

\[
\| v_{\Omega_{\delta,N,L}} \|_{L^\infty(\Omega_{\delta,N,L})} \leq \| \tilde{v}_{C_{L/N} - B(0;\delta)} \|_{L^\infty(\Omega_{C_{L/N} - B(0;\delta)})}.
\]

(12)

where \( \tilde{v}_{C_{L/N} - B(0;\delta)} \) is the torsion function with Neumann boundary conditions on \( \partial C_{L/N} \), and Dirichlet boundary conditions on \( \partial B(0;\delta) \). Denote the spectrum of the corresponding Laplacian by \( \{ \mu_j := \mu_{j,B(0;\delta),L/N}, j = 1,2,\ldots \} \), and let \( \{ \varphi_j := \varphi_{1,B(0;\delta),L/N}, j = 1,2,\ldots \} \) denote a corresponding
orthonormal basis of eigenfunctions. We denote by \( \pi_{\delta,N/L}(x,y;t), x \in C_{L/N} - B(0;\delta), y \in C_{L/N} - B(0;\delta), t > 0 \) the corresponding heat kernel. Then

\[
\pi_{\delta,N/L}(x,y;t) = \sum_{j=1}^{\infty} e^{-t\mu_j} \varphi_j(x)\varphi_j(y),
\]

and

\[
\tilde{v}_{C_{L/N} - B(0;\delta)}(x) = \int_{0}^{\infty} dt \int_{C_{L/N} - B(0;\delta)} dy \ \pi_{\delta,N/L}(x,y;t) \left( \frac{\varphi_1(y)}{\|\varphi_1\|} + 1 - \frac{\varphi_1(y)}{\|\varphi_1\|} \right)
\]

\[
= \frac{1}{\mu_1} \|\varphi_1\| + \int_{0}^{T} dt \int_{C_{L/N} - B(0;\delta)} dy \ \pi_{\delta,N/L}(x,y;t) \left( 1 - \frac{\varphi_1(y)}{\|\varphi_1\|} \right)
\]

\[
\leq \frac{1}{\mu_1} + \int_{0}^{T} dt \int_{C_{L/N} - B(0;\delta)} dy \ \pi_{\delta,N/L}(x,y;t) \left( 1 - \frac{\varphi_1(y)}{\|\varphi_1\|} \right)
\]

\[
\leq \frac{1}{\mu_1} + T + \int_{T}^{\infty} dt \int_{C_{L/N} - B(0;\delta)} dy \ \pi_{\delta,N/L}(x,y;t) \left( 1 - \frac{\varphi_1(y)}{\|\varphi_1\|} \right),
\]

where \( \|\varphi_1\| = \|\varphi_1\|_{L^\infty(C_{L/N} - B(0;\delta))} \). By (13), we have that the third term in the right-hand side of (14) equals

\[
\sum_{j=1}^{\infty} \mu_j^{-1} e^{-t\mu_j} \varphi_j(x) \int_{C_{L/N} - B(0;\delta)} dy \ \varphi_j(y) \left( 1 - \frac{\varphi_1(y)}{\|\varphi_1\|} \right).
\]

The term with \( j = 1 \) in (15) is bounded from above by

\[
\mu_1^{-1} \|\varphi_1\| \int_{C_{L/N} - B(0;\delta)} \|\varphi_1\| \left( 1 - \frac{\varphi_1}{\|\varphi_1\|} \right)
\]

\[
= \mu_1^{-1} \|\varphi_1\| \int_{C_{L/N} - B(0;\delta)} (\|\varphi_1\| - \varphi_1)
\]

\[
\leq \mu_1^{-1} \left( \|\varphi_1\|^2 \left( \frac{L}{N} \right)^m - 1 \right),
\]

where we used the fact that \( 1 = \int_{C_{L/N} - B(0;\delta)} \varphi_1^2 \leq \|\varphi_1\| \int_{C_{L/N} - B(0;\delta)} \varphi_1 \).

It was shown on p.586, lines -3,-4, in [9] (with appropriate adjustment in notation) that

\[
\|\varphi_1\|^2 \leq \left( \frac{N}{L} \right)^m \left( 1 - s\mu_1 - \frac{mL^2}{3\varepsilon N^2} \right)^{-1}, \quad s \geq 0,
\]

provided the last term in the round brackets is non-negative. The optimal choice for \( s \) gives that

\[
\|\varphi_1\|^2 \leq \left( \frac{N}{L} \right)^m \left( 1 - \frac{(4m\mu_1)^{1/2}L}{(3\varepsilon)^{1/2}N} \right)^{-1}, \quad \mu_1 < \frac{3\varepsilon N^2}{4mL^2}.
\]
By further restricting the range for $\mu_1$, we have that the first term with $j = 1$ in (15) is then bounded from above by

$$
\mu_1^{-1} \frac{2L (m\mu_1/(3eN^2))^{1/2}}{1 - 2L (m\mu_1/(3eN^2))^{1/2}} \leq \frac{(2m)^{1/2}L}{\mu_1^{1/2}N}, \quad \mu_1 \leq \frac{3eN^2}{16mL^2}.
$$

(16)

The terms with $j \geq 2$ in (15) give, by Cauchy–Schwarz for both the series in $j$, and the integral over $C_{L/N} - B(0; \delta)$, a contribution

$$
\left| \sum_{j=2}^{\infty} \mu_j^{-1} e^{-T\mu_j} \varphi_j(x) \int_{C_{L/N} - B(0; \delta)} \varphi_j \left(1 - \frac{\varphi_1}{\|\varphi_1\|}\right) \right|
$$

$$
\leq \mu_2^{-1} \sum_{j=2}^{\infty} e^{-T\mu_j} \left|\varphi_j(x)\right| \int_{C_{L/N} - B(0; \delta)} \left|\varphi_j\right|
$$

$$
\leq \mu_2^{-1} \left(\frac{L}{N}\right)^{m/2} \left(\sum_{j=2}^{\infty} e^{-T\mu_j}\right)^{1/2} \left(\sum_{j=2}^{\infty} e^{-T\mu_j} \left|\varphi_j(x)\right|^2\right)^{1/2}
$$

$$
\leq \mu_2^{-1} \left(\frac{L}{N}\right)^{m/2} \left(\sum_{j=2}^{\infty} e^{-T\mu_j}\right)^{1/2} \left(\pi_{\delta,N/L}(x, x; T)\right)^{1/2}.
$$

(17)

To bound the first series in the right-hand side of (17), we note that the $\mu_j$’s are bounded from below by the Neumann eigenvalues of the cube $C_{L/N}$. So choosing $T = (L/N)^2$ we get that

$$
\left(\sum_{j=2}^{\infty} e^{-L^2\mu_j/N^2}\right)^{1/2} \leq \left(1 + \sum_{j=1}^{\infty} e^{-\pi^2 j^2}\right)^{m/2} \leq \left(\frac{4}{3}\right)^{m/2}.
$$

Similarly to the proof of Lemma 3.1 in [9], we have that

$$
\left(\pi_{\delta,N/L}(x, x; L^2/N^2)\right)^{1/2} \leq \left(\pi_{0,N/L}(x, x; L^2/N^2)\right)^{1/2}
$$

$$
\leq \left(\frac{N}{L}\right)^{m/2} \left(1 + 2 \sum_{j=1}^{\infty} e^{-\pi^2 j^2}\right)^{m/2}
$$

$$
\leq \left(\frac{4}{3}\right)^{m/2} \left(\frac{N}{L}\right)^{m/2}.
$$

(18)

Finally, $\mu_2 \geq \frac{\pi^2 N^2}{L^2}$, together with (12), (14), (16), (17), (18), and the choice $T = (L/N)^2$ gives that

$$
\|v_{\Omega_{\delta,N,L}}\|_{L^\infty(\Omega_{\delta,N,L})} \leq \mu_1^{-1} + \frac{(2m)^{1/2}L}{\mu_1^{1/2}N} + \left(\frac{4}{3}\right)^{m} \frac{L^2}{N^2}, \quad \mu_1 \leq \frac{3eN^2}{16mL^2}.
$$

(19)
Proof of Theorem 1. Let \(1 < \alpha < 2\). By (11) and (19), we have that
\[
\lambda(\Omega_{\delta,N,L}) \|v_{\Omega_{\delta,N,L}}\|_{L^\infty(\Omega_{\delta,N,L})} \leq \left(\mu_1 + 32m \left(\frac{5}{4}\right)^m \left(\frac{N}{L^2} + \frac{1}{N^{1/2}}\mu_1\right)\right) \times \left(\frac{\mu_1^{-1} + (2m)^{1/2}L}{\mu_1^{1/2}N} + \left(\frac{4}{3}\right)^m \frac{L^2}{N^2}\right),
\]
provided
\[
\frac{N}{L^2} \leq \mu_1 \leq \frac{3\epsilon N^2}{16mL^2}.
\]
First consider the planar case \(m = 2\). Recall Lemma 3.1 in [9]: for \(\delta < L/(6N)\),
\[
\frac{N^2}{100L^2} \left(\log \frac{L}{2\delta N}\right)^{-1} \leq \mu_{1,B(0;\delta),L/N} \leq \frac{8\pi N^2}{(4 - \pi)L^2} \left(\log \frac{L}{2\delta N}\right)^{-1}.
\]
Let
\[
\delta^* := \delta^*(\alpha, N, L) = \frac{L}{2N}e^{-N^{2-\alpha}},
\]
where \(1 < \alpha < 2\). Let \(N_1 \in \mathbb{N}\) be such that for all \(N \geq N_1\), \(\delta^* < L/(6N)\). We now use (21) to see that there exists \(C > 1\) such that
\[
C^{-1}N^{\alpha} \frac{L^2}{\mu_1,B(0;\delta^*),L/N} \leq C N^{\alpha} \frac{L^2}{
\]
provided
\[
\frac{N}{L^2} \leq C^{-1}N^{\alpha} \frac{L^2}{\mu_1,B(0;\delta^*),L/N} \leq C N^{\alpha} \frac{L^2}{
\]
By (20), (23), and all \(N \geq \max\{N_1, N_2\}\) we have that
\[
\lambda(\Omega_{\delta^*,N,L}) \|v_{\Omega_{\delta^*,N,L}}\|_{L^\infty(\Omega_{\delta^*,N,L})} \leq 1 + C \left(N^{1-\alpha} + N^{(\alpha-2)/2}\right),
\]
where \(C\) depends on \(C\) and on \(m\) only. Finally, we let \(N_3 \in \mathbb{N}\) be such that for all \(N \geq N_3\),
\[
\frac{N}{L^2} \leq C^{-1}N^{\alpha} \frac{L^2}{\mu_1,B(0;\delta^*),L/N} \leq C N^{\alpha} \frac{L^2}{
\]
We conclude that (5) holds with \(\Omega_\epsilon = \Omega_{\delta^*,N,L}\) with \(\delta^*\) given by (22), and \(N \geq \max\{N_1, N_2, N_3\}\).

Next consider the case \(m = 3, 4, \ldots\). We apply Lemma 3.2 in [9] to the case \(K = B(0;\delta)\), and denote the Newtonian capacity of \(K\) by \(\text{cap}(K)\). Then \(\text{cap}(B(0;\delta)) = \kappa_m \delta^{m-2}\), where \(\kappa_m\) is the Newtonian capacity of the ball with radius 1 in \(\mathbb{R}^m\). Then Lemma 3.2 gives that there exists \(C \geq 1\) such that
\[
C^{-1} \left(\frac{N}{L}\right)^m \delta^{m-2} \leq \mu_{1,B(0;\delta),L/N} \leq C \left(\frac{N}{L}\right)^m \delta^{m-2},
\]
provided
\[
\kappa_m \delta^{m-2} \leq \frac{1}{16} (L/N)^{m-2}.
\]
We choose
\[ \delta^* := \delta^*(\alpha, N, L) = LN^{(\alpha-m)/(m-2)}. \]  
(27)
This gives inequality (23) by (25). The requirement (26) holds for all \( N \geq N_1 \), where \( N_1 \) is the smallest natural number such that \( N_1^{2-\alpha} \geq 16\kappa_m \). The remainder of the proof follows the lines below (23) with the appropriate adjustment of constants, and the choice of \( \delta^* \) as in (27). \( \square \)

We note that the choice \( \alpha = \frac{4}{3} \) in either (22) or in (27) gives, by (24), the decay rate
\[ \lambda(\Omega_{\delta^*, N, L}) \| v_{\Omega_{\delta^*, N, L}} \|_{L^\infty(\Omega_{\delta^*, N, L})} \leq 1 + 2CN^{-1/3}. \]  
(28)

3. Proof of Theorem 2

In view of Payne’s inequality (6) it suffices to obtain an upper bound for \( \| v_\Omega \|_{L^\infty(\Omega)} \lambda(\Omega) \). We first observe, that by domain monotonicity of the torsion function, \( v_\Omega \) is bounded by the torsion function for the (connected) set bounded by the two parallel lines tangent to \( \Omega \) at distance \( w(\Omega) \). Hence
\[ \| v_\Omega \|_{L^\infty(\Omega)} \leq \frac{w(\Omega)^2}{8}. \]  
(29)
In order to obtain an upper bound for \( \lambda(\Omega) \), we introduce the following notation. For a planar, open, convex set, with finite measure, we let \( z_1, z_2 \) be two points on the boundary of \( \Omega \) which realise the width. That is there are two parallel lines tangent to \( \partial \Omega \), at \( z_1 \) and \( z_2 \) respectively, and at distance \( w(\Omega) \). Let the \( x \)-axis be perpendicular to the vector \( z_1 z_2 \), containing the point \( \frac{1}{2}(z_1 + z_2) \). We consider the family of line segments parallel to the \( x \)-axis, obtained by intersection with \( \Omega \), and let \( l_1, l_2 \) be two points on the boundary of \( \Omega \) which realise the maximum length \( L \) of this family. The quadrilateral with vertices, \( z_1, z_2, l_1, l_2 \) is contained in \( \Omega \). This quadrilateral in turn contains a rectangle with side-lengths \( h \), and \( 1 - \frac{h}{w(\Omega)} \) \( L \) respectively, where \( h \in [0, w(\Omega)] \) is arbitrary. Hence, by domain monotonicity of the Dirichlet eigenvalues, we have that
\[ \lambda(\Omega) \leq \pi^2h^{-2} + \pi^2 \left( 1 - \frac{h}{w(\Omega)} \right)^{-2} L^{-2}. \]
Minimising the right-hand side above with respect to \( h \) gives that
\[ h = \frac{(w(\Omega) L^2)^{1/3}}{1 + \left( \frac{L}{w(\Omega)} \right)^{2/3}}. \]
It follows that
\[ \lambda(\Omega) \leq \frac{\pi^2}{w(\Omega)^2} \left( 1 + \left( \frac{w(\Omega)}{L} \right)^{2/3} \right)^3. \]  
(30)
As \( w(\Omega) \leq L \) we obtain by (30) that

\[
\lambda(\Omega) \leq \frac{\pi^2}{w(\Omega)^2} \left( 1 + 7 \left( \frac{w(\Omega)}{L} \right)^{2/3} \right).
\] (31)

In order to complete the proof we need the following.

**Lemma 5.** If \( \Omega \) is an open, bounded, convex set in \( \mathbb{R}^2 \), and if \( L \) is the length of the longest line segment in the closure of \( \Omega \), perpendicular to \( z_1z_2 \), then

\[
\text{diam}(\Omega) \leq 3L.
\] (32)

**Proof.** Let \( d_1, d_2 \in \partial \Omega \) such that \( |d_1 - d_2| = \text{diam}(\Omega) \). We denote the projections of \( d_1, d_2 \) onto the line through \( z_1, z_2 \) by \( e_1, e_2 \) respectively. Let \( z \) be the intersection of the lines through \( z_1, z_2 \) and \( d_1, d_2 \) respectively. Then, by the maximality of \( L \), we have that \( |d_1 - e_1| \leq L, |d_2 - e_2| \leq L \). Furthermore, by convexity, \( |e_1 - z| + |e_2 - z| \leq w(\Omega) \). Hence,

\[
|d_1 - d_2| \leq |d_1 - e_1| + |e_1 - z| + |d_2 - e_2| + |e_2 - z| \leq 2L + w(\Omega) \leq 3L.
\]

\[\square\]

By (31), we have that

\[
\lambda(\Omega) \leq \frac{\pi^2}{w(\Omega)^2} \left( 1 + 7 \cdot 3^{2/3} \left( \frac{w(\Omega)}{\text{diam}(\Omega)} \right)^{2/3} \right).
\]

This implies Theorem 2 by (29).

\[\square\]

4. **Proof of Theorem 3**

We denote by \( d : M \times M \mapsto \mathbb{R}^+ \) the geodesic distance associated to \( (M, g) \). For \( x \in M, R > 0, B(x; R) = \{ y \in M : d(x, y) < R \} \). For a measurable set \( A \subseteq M \) we denote by \( |A| \) its Lebesgue measure. The Bishop–Gromov Theorem (see [10]) states that if \( M \) is a complete, non-compact, \( m \)-dimensional, Riemannian manifold with non-negative Ricci curvature, then for \( p \in M \), the map \( r \mapsto \frac{|B(p; r)|}{r^m} \) is monotone decreasing. In particular

\[
\frac{|B(p; r_2)|}{|B(p; r_1)|} \leq \left( \frac{r_2}{r_1} \right)^m, \quad 0 < r_1 \leq r_2.
\] (33)

Corollary 3.1 and Theorem 4.1 in [11], imply that if \( M \) is complete with non-negative Ricci curvature, then for any \( D_2 > 2 \) and \( 0 < D_1 < 2 \) there exist constants \( 0 < C_1 \leq C_2 < \infty \) such that for all \( x \in M, y \in M, t > 0 \),

\[
C_1 \frac{e^{-d(x,y)^2/(2D_1 t)}}{(|B(x; t^{1/2})||B(y; t^{1/2})|)^{1/2}} \leq p_M(x, y; t)
\]

\[
\leq C_2 \frac{e^{-d(x,y)^2/(2D_2 t)}}{(|B(x; t^{1/2})||B(y; t^{1/2})|)^{1/2}}.
\] (34)
Finally, since by (33) the measure of any geodesic ball with radius $r$ is bounded polynomially in $r$, the theorems of Grigor’yan in [6] imply stochastic completeness. That is, for all $x \in M$, and all $t > 0$,

$$\int_M dy p_M(x, y; t) = 1.$$ 

**Proof of Theorem 3.** We choose $D_1 = 1$, $D_2 = 3$ in (34), and define the corresponding number $K = \max\{C_2, C_1^{-1}\}$. Then

$$K^{-1}e^{-d(x, y)^2/(2t)} \le \left( |B(x; t^{1/2})||B(y; t^{1/2})| \right)^{1/2} \le Ke^{-d(x, y)^2/(6t)}.$$  

(35)

Let $q \in M$ be arbitrary, and let $R > 0$ be such that $\Omega(q; R) := B(q; R) \cap \Omega \neq \emptyset$. The spectrum of the Dirichlet Laplacian acting in $L^2(\Omega(q; R))$ is discrete. Denote the bottom of this spectrum by $\lambda(\Omega(q; R))$. Then $\lambda(\Omega(q; R)) \ge \lambda(\Omega)$. By the spectral theorem, monotonicity of Dirichlet heat kernels, and the Li-Yau bound (35), we have that

$$p_{\Omega(q; R)}(x, x; t) \le e^{-t\lambda(\Omega(q; R))/2}p_{\Omega(q; R)}(x, x; t/2) \le e^{-t\lambda(\Omega(q; R))/2}p_M(x, x; t/2) \le Ke^{-t\lambda(\Omega(q; R))/2}|B(x; (t/2)^{1/2})|^{-1}.$$  

(36)

By the semigroup property and the Cauchy–Schwarz inequality, for any open set $\Omega \subset M$, we have that

$$p_\Omega(x, y; t) = \int_\Omega dz p_\Omega(x, z; t/2) p_\Omega(z, y; t/2) \le \left( \int_\Omega dz p_\Omega^2(x, z; t/2) \right)^{1/2} \left( \int_\Omega dz p_\Omega^2(z, y; t/2) \right)^{1/2} = (p_\Omega(x, x; t) p_\Omega(y, y; t))^{1/2}.$$  

(37)

We obtain by (36), (37) (for $\Omega = \Omega(q; R)$), and $p_{\Omega(q; R)}(x, y; t) \le p_M(x, y; t)$, that

$$p_{\Omega(q; R)}(x, x; t) \le \left( p_{\Omega(q; R)}(x, x; t) p_{\Omega(q; R)}(y, y; t) \right)^{1/4} \le K^{1/2}e^{-t\lambda(\Omega(q; R))/4}(|B(x; (t/2)^{1/2})||B(y; (t/2)^{1/2})|)^{-1/4}p_M^{1/2}(x, y; t).$$  

(38)

By (38) and (35), we have that

$$p_{\Omega(q; R)}(x, y; t) \le Ke^{-t\lambda(\Omega(q; R))/4}(|B(x; (t/2)^{1/2})||B(y; (t/2)^{1/2})|)^{-1/4} \times (|B(x; t^{1/2})||B(y; t^{1/2})|)^{-1/4}e^{-d(x, y)^2/(12t)}.$$  

(39)
By the Li-Yau lower bound in (35), we can rewrite the right-hand side of (39) to yield,

\[
p_{\Omega(q;R)}(x, y; t) \leq K^2 e^{-t\lambda(\Omega(q;R))/4} p_M(x, y; 6t) \times \frac{\left( |B(x; (6t)^{1/2})||B(y; (6t)^{1/2})| \right)^{1/2}}{(|B(x; (t/2)^{1/2})||B(y; (t/2)^{1/2})||B(x; t^{1/2})||B(y; t^{1/2})|)^{1/4}}.
\]

By Bishop–Gromov (33), we have that the volume quotients in the right-hand side of (40) are bounded by \(2^{3m/4} \cdot 3^{m/2}\) uniformly in \(x\) and \(y\). Hence

\[
p_{\Omega(q;R)}(x, y; t) \leq 2^{3m/4} \cdot 3^{m/2} K^2 e^{-t\lambda(\Omega(q;R))/4} p_M(x, y; 6t).
\]

Since manifolds with non-negative Ricci curvature are stochastically complete, we have that

\[
\int_{\Omega(q;R)} dy \, p_{\Omega(q;R)}(x, y; t) \leq 2^{3m/4} \cdot 3^{m/2} K^2 e^{-t\lambda(\Omega(q;R))/4} \int_M dy \, p_M(x, y; 6t) = 2^{3m/4} \cdot 3^{m/2} K^2 e^{-t\lambda(\Omega(q;R))/4}.
\]

Integrating the inequality above with respect to \(t\) over \([0, \infty)\) yields,

\[
v_{\Omega(q;R)}(x) \leq 2^{(3m+8)/4} \cdot 3^{m/2} K^2 \lambda(\Omega(q;R))^{-1} \leq 2^{(3m+8)/4} \cdot 3^{m/2} K^2 \lambda(\Omega)^{-1}.
\]

Finally letting \(R \to \infty\) in the left-hand side above yields the right-hand side of (7).

The proof of the left-hand side of (7) is similar to the one in Theorem 5.3 in [1] for Euclidean space. We have that

\[
v_{\Omega(q;R)}(x) = \int_0^\infty dt \int_{\Omega(q;R)} dy \, p_{\Omega(q;R)}(x, y; t).
\]

We first observe that \(|\Omega(q;R)| < \infty\), and so the spectrum of the Dirichlet Laplacian acting in \(L^2(\Omega(q;R))\) is discrete and is denoted by \(\{\lambda_j(\Omega(q;R)), j \in \mathbb{N}\}\), with a corresponding orthonormal basis of eigenfunctions \(\{\varphi_j(\Omega(q;R)), j \in \mathbb{N}\}\). These eigenfunctions are in \(\mathcal{L}^\infty(\Omega(q;R))\). Then, by (41) and the eigenfunction expansion of the Dirichlet heat kernel for \(\Omega(q;R)\), we have that

\[
v_{\Omega(q;R)}(x) \geq \int_0^\infty dt \int_{\Omega(q;R)} dy \, p_{\Omega(q;R)}(x, y; t) \frac{\varphi_{1,\Omega(q;R)}(y)}{\|\varphi_{1,\Omega(q;R)}\|_{\mathcal{L}^\infty(\Omega(q;R))}}
= \int_0^\infty dt \, e^{-t\lambda_1(\Omega(q;R))} \frac{\varphi_{1,\Omega(q;R)}(x)}{\|\varphi_{1,\Omega(q;R)}\|_{\mathcal{L}^\infty(\Omega(q;R))}}
= \lambda_1(\Omega(q;R))^{-1} \frac{\varphi_{1,\Omega(q;R)}(x)}{\|\varphi_{1,\Omega(q;R)}\|_{\mathcal{L}^\infty(\Omega(q;R))}}.
\]

First taking the supremum over all \(x \in \Omega(q;R)\) in the left-hand side of (42), and subsequently taking the supremum over all such \(x\) in the right-hand side of (42) gives
\[ \|v_{\Omega(q;R)}\|_{L^\infty(\Omega(q;R))} \geq \lambda(\Omega(q;R))^{-1}. \] (43)

Observe that the torsion function is monotone increasing in \( R \). Taking the limit \( R \to \infty \) in the left-hand side of (43), and subsequently in the right-hand side of (43) completes the proof. \( \square \)

Acknowledgements

MvdB acknowledges support by The Leverhulme Trust through International Network Grant Laplacians, Random Walks, Bose Gas, Quantum Spin Systems.

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