The (ET4) axiom for extriangulated categories

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ABSTRACT
Extriangulated categories were introduced by Nakaoka and Palu, which is a simultaneous generalization of exact categories and triangulated categories. Axiom (ET4) for extriangulated categories is an analogue of octahedron axiom (TR4) for triangulated categories. In this paper, we use homotopy cartesian squares in pre-extriangulated categories to investigate axiom (ET4). We provide several equivalent statements of axiom (ET4) and find out conditions under which the axiom is self-dual.

ARTICLE HISTORY
Received 11 July 2023
Revised 06 November 2023
Communicated by Vanessa Miemietz

KEYWORDS
Extriangulated category; (ET4) axiom; homotopy cartesian square; shifted octahedron

2020 MATHEMATICS SUBJECT CLASSIFICATION
18E30; 18E10

1. Introduction

Exact categories and triangulated categories are two fundamental structures in mathematics. Nakaoka and Palu introduced the notion of extriangulated categories [8], which is a simultaneous generalization of exact categories and triangulated categories. The class of extriangulated categories contains not only extension-closed subcategories of triangulated categories, but also examples which are neither exact nor triangulated [3, 8, 10]. This new notion provides a convenient setup for writing down proofs which apply to both exact categories and triangulated categories.

Roughly speaking, a pre-extriangulated category is an additive category equipped with a class of E-triangles satisfying certain morphism axioms (ET3) and (ET3)op, where E : Eop × E → Ab is an additive bifunctor and E-triangles A → B → C are given by an additive realization s of E. A pre-extriangulated category (E, E, s) is called extriangulated if it satisfies axioms (ET4) and (ET4)op, which are similar to the octahedron axiom (TR4) for triangulated categories. To strengthen the connection with axiom (TR4), it is proved that four shifted octahedrons (ET4-1), (ET4-2), (ET4-3), and (ET4-4) hold in extriangulated categories [6, 8]; see Section 2.2 for details.

In this paper, we further investigate axiom (ET4). We don't know whether each shifted octahedron implies axiom (ET4) in pre-extriangulated categories. Our first aim is to discuss several equivalent statements of axiom (ET4). Note that axiom (TR4) is self-dual but axiom (ET4) is not. Our second aim is to find out conditions under which axiom (ET4) is self-dual.

Recall that the notion of a homotopy cartesian square in triangulated categories is the triangulated analogue of the notion of a bicartesian square in abelian categories. It is well-known that axiom (TR4) is equivalent to the homotopy cartesian axiom [5, 7]. In this paper we introduce the notion of a homotopy cartesian square in pre-extriangulated categories and propose a new shifted octahedron (ET4-5), which provides a useful method to achieve our two goals. We have the following main results.

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Theorem 1.1. Let \((\mathcal{C}, E, s)\) be a pre-extriangulated category. Then (ET4-1), (ET4-2), and (ET4-5) are equivalent and self-dual.

Theorem 1.2. Let \((\mathcal{C}, E, s)\) be a pre-extriangulated category. Then the following statements are equivalent:
(a) \(\mathcal{C}\) satisfies (ET4).
(b) \(\mathcal{C}\) satisfies (ET4-1) and (ET0).
(c) \(\mathcal{C}\) satisfies (ET4-2) and (ET0).
(d) \(\mathcal{C}\) satisfies (ET4-3) and (ET0).
(e) \(\mathcal{C}\) satisfies (ET4-5) and (ET0).

Recall that the first morphism in an \(E\)-triangle is called an inflation. We say \(\mathcal{C}\) satisfies (ET0) if the composition of two inflations is also an inflation.

Theorem 1.3. Let \((\mathcal{C}, E, s)\) be a pre-extriangulated category. If \(\mathcal{C}\) satisfies (ET0) and (ET0)\(^\text{op}\), then (ET4), (ET4-1), (ET4-2), (ET4-3), (ET4-4), and (ET4-5) are equivalent and self-dual.

The remaining part of this paper is organized as follows. In Section 2, we give some preliminaries on extriangulated categories. In Section 3, we introduce the notion of a homotopy cartesian square in pre-extriangulated categories and prove our main results.

2. Preliminaries

In this section, we first collect some definitions and facts on extriangulated categories, then provide several shifted octahedrons. The basic reference is [8, 9].

2.1. Definitions and facts on extriangulated categories

Let \(\mathcal{C}\) be an additive category equipped with an additive bifunctor \(E : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}\), where Ab is the category of abelian groups. For any objects \(A, C \in \mathcal{C}\), an element \(\delta \in E(C, A)\) is called an \(E\)-extension. For any \(a \in \mathcal{C}(A, A')\) and \(c \in \mathcal{C}(C, C')\), we denote by \(a_\delta c\) the \(E\)-extension \(E(C, a)(\delta) \in E(C, A')\) and by \(c^\delta\) the \(E\)-extension \(E(c, A)(\delta) \in E(C', A)\). The zero element \(0 \in E(C, A)\) is called the split \(E\)-extension. Let \(\delta \in E(C, A)\) and \(\delta' \in E(C', A')\). A morphism \((a, c) : \delta \rightarrow \delta'\) of \(E\)-extensions is a pair of morphisms \(a \in \mathcal{C}(A, A')\) and \(c \in \mathcal{C}(C, C')\) such that \(a_\delta c = c^\delta\delta'\). We denote by \(\delta \oplus \delta' \in E(C \oplus C', A \oplus A')\) the element corresponding to \((\delta, 0, 0, \delta')\) through the following isomorphism
\[
E(C \oplus C', A \oplus A') \cong E(C, A) \oplus E(C, A') \oplus E(C', A) \oplus E(C', A').
\]

Let \(A, C \in \mathcal{C}\) be any pair of objects. Two sequences of morphisms \(A \xrightarrow{x} B \xrightarrow{y} C\) and \(A \xrightarrow{x'} B' \xrightarrow{y'} C\) are said to be equivalent if there exists an isomorphism \(b \in \mathcal{C}(B, B')\) such that the following diagram
\[
\begin{array}{ccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C \\
\| & & \downarrow{\cong} & & \| \\
A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C
\end{array}
\]
is commutative. We denote the equivalence class of \(A \xrightarrow{x} B \xrightarrow{y} C\) by \([A \xrightarrow{x} B \xrightarrow{y} C]\). For any \(A, C \in \mathcal{C}\), we denote as
\[
0 = [A \xrightarrow{x} A \oplus C \xrightarrow{(0,1)} C].
\]
For any \([A \xrightarrow{x} B \xrightarrow{y} C]\) and \([A' \xrightarrow{x'} B' \xrightarrow{y'} C']\), we denote as
\[
[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'].
\]

**Definition 2.1.** [8, Definitions 2.9] Let \(s\) be a correspondence which associates an equivalence class \(s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]\) to any \(E\)-extension \(\delta \in E(C, A)\). If \(s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]\), then we say the sequence \(A \xrightarrow{x} B \xrightarrow{y} C\) realizes \(\delta\). This \(s\) is called a realization of \(E\), if it satisfies the following condition:

Let \((a, c) : \delta \rightarrow \delta'\) be a morphism of \(E\)-extensions, \(s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]\) and \(s(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']\). Then there exists a morphism \(b \in \mathcal{C}(B, B')\) which makes the following diagram commutative.

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C \\
\downarrow a & & \downarrow b & & \downarrow c \\
A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C'
\end{array}
\]

In the above situation, we say that the triplet \((a, b, c)\) realizes \((a, c)\).

**Definition 2.2.** [8, Definition 2.10] A realization \(s\) of \(E\) is called additive if it satisfies the following conditions.

(1) For any objects \(A, C \in \mathcal{C}\), the split \(E\)-extension \(0 \in E(C, A)\) satisfies \(s(0) = 0\).

(2) For any \(E\)-extensions \(\delta \in E(C, A)\) and \(\delta' \in E(C', A')\), we have \(s(\delta \oplus \delta') = s(\delta) \oplus s(\delta')\).

Let \(s\) be an additive realization of \(E\). A sequence \(A \xrightarrow{x} B \xrightarrow{y} C\) is called a conflation if realizes some \(E\)-extension \(\delta \in E(C, A)\). In this case, \(x\) is called an inflation, \(y\) is called a deflation and \(A \xrightarrow{x} B \xrightarrow{y} C\) \(\xrightarrow{\delta}\) is called an \(E\)-triangle.

Let \(A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \xrightarrow{\delta}\) and \(A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'} \xrightarrow{\delta'}\) be \(E\)-triangles. If \((a, c) : \delta \rightarrow \delta'\) is a morphism of \(E\)-extensions and \((a, b, c)\) realizes \((a, c)\), then we write it as

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \\
\downarrow a & & \downarrow b & & \downarrow c \\
A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{\delta'}
\end{array}
\]

and call \((a, b, c)\) a morphism of \(E\)-triangles.

**Definition 2.3.** [8, Definition 2.12] Let \(\mathcal{C}\) be an additive category. A triplet \((\mathcal{C}, E, s)\) is called a pre-ex triangulated category if it satisfies the following conditions.

(ET1) \(E : \mathcal{C}^\text{op} \times \mathcal{C} \rightarrow \text{Ab}\) is an additive bifunctor.

(ET2) \(s\) is an additive realization of \(E\).

(ET3) Each commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \xrightarrow{\delta} \\
\downarrow a & & \downarrow b & & \downarrow c \\
A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{\delta'} \xrightarrow{\delta'}
\end{array}
\]

whose rows are \(E\)-triangles, can be completed to a morphism of \(E\)-triangles.
(ET3)\(\text{op}\) Each commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow{b} & & \downarrow{c} \\
A' & \xrightarrow{x'} & B'
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow{y} & & \downarrow{\delta} \\
\downarrow{y'} & & \downarrow{\delta'} \\
C & \xrightarrow{\delta} & C'
\end{array}
\]

whose rows are \(\mathbb{E}\)-triangles, can be completed to a morphism of \(\mathbb{E}\)-triangles.

If the triplet \((\mathcal{C}, \mathbb{E}, \mathbb{s})\) satisfies the following axioms, then it is called an extriangulated category:

(ET4) Let \(A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{\delta} C \xrightarrow{g} F \xrightarrow{\delta'} \) be \(\mathbb{E}\)-triangles. There exists a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{d} \\
C & \xrightarrow{h'} & E \\
\downarrow{g'} & & \downarrow{e} \\
F & \xrightarrow{\delta'} & F
\end{array}
\]

such that the second row and the third column are \(\mathbb{E}\)-triangles, and moreover, \(d^* \delta'' = \delta\) and \(f^* \delta'' = e^* \delta'.\)

(ET4)\(\text{op}\) Let \(D \xrightarrow{d} A \xrightarrow{f} B \xrightarrow{\delta} C \xrightarrow{g} F \xrightarrow{\delta'} \) be \(\mathbb{E}\)-triangles. Then there exists a commutative diagram

\[
\begin{array}{ccc}
D & \xrightarrow{d} & E \\
\downarrow{f'} & & \downarrow{g'} \\
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
C & \xrightarrow{\delta'} & C
\end{array}
\]

such that the first row and the second column are \(\mathbb{E}\)-triangles, and moreover, \(\delta' = e^* \delta''\) and \(d^* \delta = g^* \delta''.\)

Lemma 2.4. [8, Propositions 3.3 and 3.11] Let \((\mathcal{C}, \mathbb{E}, \mathbb{s})\) be a pre-extriangulated category. Then for any \(\mathbb{E}\)-triangle \(A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \), the following sequences of natural transformations are exact:

\[
\begin{align*}
\mathcal{C}(C, -) & \xrightarrow{- \circ y} \mathcal{C}(B, -) \xrightarrow{- \circ x} \mathcal{C}(A, -) \xrightarrow{\delta^z} \mathbb{E}(C, -) \xrightarrow{y^*} \mathbb{E}(B, -) \xrightarrow{x^*} \mathbb{E}(A, -), \\
\mathcal{C}(-, A) & \xrightarrow{x_{op}} \mathcal{C}(-, B) \xrightarrow{y_{op}} \mathcal{C}(-, C) \xrightarrow{\delta_{op}} \mathbb{E}(-, A) \xrightarrow{x^*} \mathbb{E}(-, B) \xrightarrow{y^*} \mathbb{E}(-, C),
\end{align*}
\]

where \(\delta^z(f) = f^* \delta\) and \(\delta_{op}(g) = g^* \delta.\)
Lemma 2.5. [8, Corollary 3.5] Let \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) be a pre-extriangulated category. Assume that the following

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \xrightarrow{y} C \\
\downarrow{a} & & \downarrow{c} \\
A' & \xrightarrow{x'} & B' \xrightarrow{y'} C'
\end{array}
\]

is a morphism of \(\mathcal{E}\)-triangles. Then the following statements are equivalent.

1. \(a \circ f = 0\).
2. \(a^* \delta = c^* \delta' = 0\).
3. \(c \circ f = 0\).

Lemma 2.6. [8, Corollary 3.6] Let \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) be a pre-extriangulated category. Assume that \((a, b, c)\) is a morphism of \(\mathcal{E}\)-triangles. If two of \(a, b, c\) are isomorphisms, then so is the third.

Lemma 2.7. [8, Proposition 3.7] Let \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) be a pre-extriangulated category. Assume that \(A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) is an \(\mathcal{E}\)-triangle. If \(a \in \mathcal{C}(A, A')\) and \(c \in \mathcal{C}(C', C)\) are isomorphisms, then \(A' \xrightarrow{x'^{-1}} B \xrightarrow{c^{-1} y} C' \xrightarrow{a'^* c'^* \delta} \) is also an \(\mathcal{E}\)-triangle.

2.2. Shifted octahedrons

Axiom (ET4) is an analogue of octahedron axiom (TR4) for triangulated categories. For later use, we collect the shifted octahedrons in extriangulated categories from [6, 8].

Lemma 2.8. [8, Proposition 3.15] Let \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) be a pre-extriangulated category. If \(\mathcal{C}\) satisfies (ET4), then \(\mathcal{C}\) satisfies (ET4-1):

Let \(A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \xrightarrow{\delta_1} \) and \(A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \xrightarrow{\delta_2} \) be \(\mathcal{E}\)-triangles. Then there is a commutative diagram

\[
\begin{array}{ccc}
A_2 & \xrightarrow{x_2} & B_2 \\
\downarrow{m_2} & & \downarrow{y_2} \\
A_1 & \xrightarrow{x_1} & B_1 \\
\downarrow{e_1} & & \downarrow{y_1} \\
\end{array}
\]

whose second row and second column are \(\mathcal{E}\)-triangles such that \(m_1 \delta_1 + m_2 \delta_2 = 0\).

Lemma 2.9. [6, Proposition 1.20] Let \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) be a pre-extriangulated category. If \(\mathcal{C}\) satisfies (ET4-1), then \(\mathcal{C}\) satisfies (ET4-2):

Let \(A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) be an \(\mathcal{E}\)-triangle, \(f : A \rightarrow D\) be a morphism, and \(D \xrightarrow{d} E \xrightarrow{c} C \xrightarrow{\delta} \) be an \(\mathcal{E}\)-triangle realizing \(f \circ \delta\). Then there is a morphism \(g : B \rightarrow E\) such that the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow{a} & & \downarrow{b} \\
A' & \xrightarrow{x'} & B' \\
\downarrow{c} & & \downarrow{d} \\
E & \xrightarrow{e} & C \\
\downarrow{f} & & \downarrow{\delta} \\
E' & \xrightarrow{g} & C'
\end{array}
\]
is a morphism of $\mathbb{E}$-triangles such that $A \xrightarrow{\left(-f\right)\otimes f} D \oplus B \xrightarrow{\left(d \otimes g\right)} E \to$ is an $\mathbb{E}$-triangle.

**Lemma 2.10.** [8, Lemma 3.14] Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a pre-extriangulated category. If $\mathcal{C}$ satisfies (ET4), then $\mathcal{C}$ satisfies (ET4-3):

Let $A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{\delta_f} B \xrightarrow{g} C \xrightarrow{g'} F \xrightarrow{\delta_g} A \xrightarrow{h} C \xrightarrow{h_0} E_0 \xrightarrow{\delta_h} E_0$ be $\mathbb{E}$-triangles satisfying $h = gf$. Then there are morphisms $d_0 : D \to E_0$ and $e_0 : E_0 \to F$ such that the following

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
D & \xrightarrow{\delta_f} & D
\end{array}
\quad
\begin{array}{ccc}
F & \xrightarrow{\delta_g} & F \\
\downarrow & & \downarrow \\
F & \xrightarrow{\delta_g} & F
\end{array}
\]

is a commutative diagram whose third column is an $\mathbb{E}$-triangle, moreover, $d_0^* (\delta_h) = \delta_f$ and $f_0^* \delta_h = e_0^* (\delta_g)$.

**Lemma 2.11.** [8, Proposition 3.17] Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Then $\mathcal{C}$ satisfies (ET4-4):

Let $D \xrightarrow{f} A \xrightarrow{f'} C \xrightarrow{\delta_f} A \xrightarrow{g} B \xrightarrow{g'} F \xrightarrow{\delta_g} E \xrightarrow{h} B \xrightarrow{h'} C \xrightarrow{\delta_h} C$ be $\mathbb{E}$-triangles satisfying $h'g = f'$. Then there is an $\mathbb{E}$-triangle $D \xrightarrow{d} E \xrightarrow{e} F \xrightarrow{\theta} F$ such that the following diagram

\[
\begin{array}{ccc}
D & \xrightarrow{f} & A \\
\downarrow & & \downarrow \\
E & \xrightarrow{h} & B \\
\downarrow & & \downarrow \\
F & \xrightarrow{d} & C \\
\downarrow & & \downarrow \\
F & \xrightarrow{\theta} & C
\end{array}
\]

is commutative and satisfies the following three conditions.

1. $d_ e (\delta_f) = \delta_h$,
2. $f_ e (\theta) = \delta_g$,
3. $g_ e (\theta) + h_ e (\delta_f) = 0$.

We remark that the proof of Lemma 2.11 uses both (ET4) and (ET4)$^{\text{op}}$. 
3. Main results

In this section we introduce the definition of homotopy cartesian squares in pre-extriangulated categories and provide some equivalent statements of axiom (ET4).

Definition 3.1. (cf. [1, Definition 3.1]) The following commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
| & & | \\
\downarrow{g} & & \downarrow{\delta} \\
C & \xrightarrow{y} & D
\end{array}
\]

in a pre-extriangulated category \((\mathcal{C}, \mathbb{E}, s)\) is called homotopy cartesian if there is an \(\mathbb{E}\)-triangle \(A \xrightarrow{\delta} C \oplus B \xrightarrow{x \oplus y} D\), where \(\delta\) is called a differential.

Remark 3.2. We restate (ET4-2) in the language of homotopy cartesian squares:

Let \(A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) be an \(\mathbb{E}\)-triangle, \(f : A \rightarrow D\) be a morphism, and \(D \xrightarrow{d} E \xrightarrow{e} C \xrightarrow{\delta} \) be an \(\mathbb{E}\)-triangle realizing \(f \circ \delta\). Then there is a morphism \(g : B \rightarrow E\) which gives a morphism of \(\mathbb{E}\)-triangles

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \xrightarrow{y} C \xrightarrow{\delta} \\
\downarrow{f} & & \downarrow{\delta} \\
D & \xrightarrow{d} & E \xrightarrow{e} C \xrightarrow{\delta} \\
\end{array}
\]

and moreover, the leftmost square is homotopy catesian where \(e \circ \delta\) is a differential.

Theorem 3.3. Let \((\mathcal{C}, \mathbb{E}, s)\) be a pre-extriangulated category. Then \(\mathcal{C}\) satisfies (ET4-2) if and only if \(\mathcal{C}\) satisfies (ET4-5):

Let \(A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \xrightarrow{\delta} \) and \(A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \xrightarrow{\delta'}\) be \(\mathbb{E}\)-triangles, and \(g : B_1 \rightarrow B_2\) be a morphism such that \(y_2 g = y_1\). Then there is a morphism \(f : A_1 \rightarrow A_2\) such that \((f, g, 1)\) is a morphism of \(\mathbb{E}\)-triangles

\[
\begin{array}{ccc}
A_1 & \xrightarrow{x_1} & B_1 \xrightarrow{y_1} C \xrightarrow{\delta} \\
\downarrow{f} & & \downarrow{\delta} \\
A_2 & \xrightarrow{x_2} & B_2 \xrightarrow{y_2} C \xrightarrow{\delta'} \\
\end{array}
\]

and moreover, the leftmost square is homotopy cartesian where \(y_2 \delta\) is a differential.

Proof. (ET4-2) \(\Rightarrow\) (ET4-5). By (ET3)\(^{op}\), there exists a morphism \(f_1 : A_1 \rightarrow A_2\) which gives a morphism of \(\mathbb{E}\)-triangles

\[
\begin{array}{ccc}
A_1 & \xrightarrow{x_1} & B_1 \xrightarrow{y_1} C \xrightarrow{\delta} \\
\downarrow{f_1} & & \downarrow{\delta} \\
A_2 & \xrightarrow{x_2} & B_2 \xrightarrow{y_2} C \xrightarrow{\delta'} \\
\end{array}
\]
By (ET4-2), there exists a morphism \( g_1 : B_1 \to B_2 \) which gives a morphism of \( \mathbb{E} \)-triangles

\[
\begin{array}{ccc}
A_1 & \xrightarrow{x_1} & B_1 \\
\downarrow f_1 & & \downarrow g_1 \\
A_2 & \xrightarrow{x_2} & B_2
\end{array}
\]

\[
\begin{array}{c}
\downarrow f_1 \\
\downarrow y_1 \\
\downarrow y_2 \\
\end{array}
\]

\[
\begin{array}{c}
\rightarrow C - \delta \\
\rightarrow C - \delta'
\end{array}
\]

and moreover, the leftmost square is homotopy cartesian where \( y_2^\delta \delta \) is a differential. In this case, we have an \( \mathbb{E} \)-triangle \( A_1 \xrightarrow{\left(\frac{-f_1}{x_1}\right)} A_2 \oplus B_1 \xrightarrow{(x_2,g_1)} B_2 \xrightarrow{g_2^\delta} \). Since \( g_2^\delta(g - g_1) = 0 \), by Lemma 2.4 there is a morphism \( m : B_1 \to A_2 \) such that \( g - g_1 = x_2 m \). Set \( f = f_1 + mx_1 \), then \( x_2 f = x_2 f_1 + x_2 mx_1 = g_1 x_1 + (g - g_1)x_1 = gx_1 \). The following commutative diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\left(\frac{-f_1}{x_1}\right)} & A_2 \oplus B_1 \\
\downarrow & & \downarrow \left(\frac{1-m}{0 1}\right) \\
A_1 & \xrightarrow{\left(\frac{-f}{x_1}\right)} & A_2 \oplus B_1
\end{array}
\]

\[
\begin{array}{c}
\rightarrow B_2 \\
\rightarrow (x_2,g_1) \\
\rightarrow B_2
\end{array}
\]

implies that \( A_1 \xrightarrow{\frac{-f}{x_1}} A_2 \oplus B_1 \xrightarrow{(x_2,g_1)} B_2 \xrightarrow{g_2^\delta} \) is an \( \mathbb{E} \)-triangle. Note that \( (mx_1)_* \delta = 0 \) by Lemma 2.5. Thus \( f_* \delta = (f_1)_* \delta + (mx_1)_* \delta = (f_1)_* \delta = \delta' \) and (ET4-5) holds.

(ET4-5) \( \Rightarrow \) (ET4-2). Since \( (f, 1) : \delta \to f_* \delta \) is a morphism of \( \mathbb{E} \)-extensions, there exists a morphism \( g' : B \to E \) which makes the diagram below commutative.

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow f & & \downarrow g' \\
D & \xrightarrow{d} & E
\end{array}
\]

\[
\begin{array}{c}
y \rightarrow C - \delta \\
g' \rightarrow f_* \delta \\
e \rightarrow C - \delta
\end{array}
\]

By (ET4-5), there exists a morphism \( f' : A \to D \) which gives a morphism of \( \mathbb{E} \)-triangles

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow f' & & \downarrow g' \\
D & \xrightarrow{d} & E
\end{array}
\]

\[
\begin{array}{c}
y \rightarrow C - \delta \\
g' \rightarrow f_* \delta \\
e \rightarrow C - \delta
\end{array}
\]

and moreover, the leftmost square is homotopy cartesian where \( e^\delta \) is a differential. Since \( f'_* \delta = f_* \delta \), that is, \( (f - f')_* \delta = 0 \), there exists a morphism \( m : B \to D \) such that \( f - f' = mx \) by Lemma 2.5. Set \( g = g' + dm \), then \( gx = g'x + dm x = df' + d(f - f') = df \) and \( eg = e(g' + dm) = y + edm = y \). The following commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\left(\frac{-f'}{x}\right)} & D \oplus B \\
\downarrow \left(\frac{-f}{x}\right) & & \downarrow \left(\frac{1-m}{0 1}\right) \\
A & \xrightarrow{\left(\frac{-f}{x}\right)} & D \oplus B
\end{array}
\]

\[
\begin{array}{c}
\rightarrow E \\
\rightarrow (d, g) \\
\rightarrow E
\end{array}
\]

implies that \( A \xrightarrow{\left(\frac{-f}{x}\right)} D \oplus B \xrightarrow{(d, g)} E \xrightarrow{e^\delta} \) is an \( \mathbb{E} \)-triangle. Therefore, (ET4-2) holds. \( \square \)
Theorem 3.4. Let \((\mathcal{C}, \mathbb{E}, s)\) be a pre-extriangulated category. Then (ET4-1), (ET4-2), and (ET4-5) are equivalent and self-dual.

Proof. We note that (ET4-2) is equivalent to (ET4-5) by Theorem 3.3, and (ET4-1) implying (ET4-2) follows from Lemma 2.9. Since the notion of a pre-extriangulated category is self-dual, it remains to show that (ET4-2) implies (ET4-1)\(^\text{op}\).

Suppose there is a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow f & & \downarrow y \\
D & & C \\
\downarrow g & & \downarrow \delta_1 \\
E & & \\
\downarrow \delta_2 & & \\
\end{array}
\]

whose row and column are \(\mathbb{E}\)-triangles. By (ET4-2), there exists a morphism \(m_2 : B \rightarrow M\) such that the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow f & & \downarrow y \\
D & \xrightarrow{m_2} & M \\
\downarrow g & & \downarrow e_1 \\
E & \xrightarrow{f_\ast \delta_1} & C \\
\end{array}
\]

is commutative and \(A \rightarrow D \oplus B (m_1, m_2) \rightarrow M \xrightarrow{e_1 \delta_1} \) is an \(\mathbb{E}\)-triangle. By (ET4-2) again, there exists a morphism \(m_1' : D \rightarrow M'\) such that the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & D \\
\downarrow x & & \downarrow g \\
B & \xrightarrow{m_1'} & M' \\
\downarrow m_2' & & \downarrow e_2' \\
E & \xrightarrow{x_\ast \delta_2} & E \\
\end{array}
\]

commutes and \(A \rightarrow B \oplus D (m_2', m_1') \rightarrow M' \xrightarrow{e_2' \delta_2} \) is an \(\mathbb{E}\)-triangle. By (ET3) and Lemma 2.6, there is an isomorphism \(c : M' \rightarrow M\) such that the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & D \oplus B (m_1', m_2') \\
\downarrow x & & \downarrow 0 \\
A & \xrightarrow{f} & B \oplus D (m_2, m_1) \\
\downarrow c & & \downarrow e_1' \delta_1 \\
A & \xrightarrow{f} & B \oplus D (m_2, m_1) \\
\end{array}
\]

is commutative and \(c^\ast (e_1' \delta_1) = e_2' \delta_2\). Set \(e_2 = -e_2' c^{-1}\), then \(e_1' \delta_1 + e_2' \delta_2 = 0\) and we obtain a commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{m_2} & M \\
\downarrow m_2' & \cong & \downarrow e_2' \\
B & \xrightarrow{-c^{-1}} & M' \\
\end{array}
\]

\[
\begin{array}{ccc}
M & \xrightarrow{e_2} & E \\
\downarrow e_2' & \cong & \downarrow e_2' \\
E & \xrightarrow{e_2} & E. \\
\end{array}
\]
Since $e_2 m_1 = - e'_2 c^{-1} m_1 = e'_2 m'_1 = g$, we have the following commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow f & & \downarrow y \\
D & \xrightarrow{m_1} & M \\
\downarrow g & & \downarrow e_2 \\
E & \xrightarrow{\delta_2} & E
\end{array}
\]

whose second row and second column are $E$-triangles. Therefore, $(\text{ET4-1})^{\text{op}}$ holds.

Let $(\mathcal{C}, E, s)$ be a pre-extriangulated category. Consider the following conditions:

(ET0) Let $f : A \to B$ and $g : B \to C$ be morphisms in $\mathcal{C}$. If both $f$ and $g$ are inflations, then so is $gf$.

(ET0)$^{\text{op}}$ Let $f : A \to B$ and $g : B \to C$ be morphisms in $\mathcal{C}$. If both $f$ and $g$ are deflations, then so is $gf$.

**Theorem 3.5.** Let $(\mathcal{C}, E, s)$ be a pre-extriangulated category. Then the following statements are equivalent:

(a) $\mathcal{C}$ satisfies (ET4).

(b) $\mathcal{C}$ satisfies (ET4-1) and (ET0).

(c) $\mathcal{C}$ satisfies (ET4-2) and (ET0).

(d) $\mathcal{C}$ satisfies (ET4-3) and (ET0).

(e) $\mathcal{C}$ satisfies (ET4-5) and (ET0).

**Proof.** By Theorem 3.4, we have $(b) \iff (c) \iff (e)$. By Lemma 2.8, we have (a) implying (b). Thus we only need to show that (e) implies (d) and (d) implies (a).

(e) $\Rightarrow$ (d). We will show that (ET4-5) implies (ET4-3). The proof is an analogue of [4, 3.5]; see also [2, Lemma 4.2]. Assume that $A \overset{f}{\to} B \overset{f'}{\to} D \overset{\delta}{\to} B \overset{g}{\to} C \overset{g'}{\to} F \overset{\delta'}{\to}$ and $A \overset{h}{\to} C \overset{h'}{\to} E \overset{\delta''}{\to}$ are $E$-triangles satisfying $h = gf$. Note that (ET4-5) is equivalent to (ET4-5)$^{\text{op}}$ by Theorem 3.4. Thus there is a morphism $d : D \to E$ such that the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow g & & \downarrow d \\
A & \xrightarrow{h} & C \\
\downarrow & & \downarrow \\
D & \xrightarrow{\delta} & E
\end{array}
\]

is a morphism of $E$-triangles and the middle square is homotopy cartesian where $f_0 \delta''$ is a differential.

Thus $d^* \delta'' = \delta$ and $B \overset{\left(\frac{-f'}{g'}\right)}{\to} D \overset{(d, h')}{\to} C \overset{(d, h')}{\to} E \overset{\delta'}{\to}$ is an $E$-triangle. By (ET4-5)$^{\text{op}}$ again there is a morphism $e : E \to F$ such that the following diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\left(\frac{-f'}{g'}\right)} & D \overset{(d, h')}{\to} C \overset{\delta'}{\to} F \\
\downarrow g & & \downarrow (0,1) \\
B & \xrightarrow{g'} & C \\
\downarrow & & \downarrow e \\
D & \xrightarrow{f, \delta''} & E
\end{array}
\]
is a morphism of $\mathbb{E}$-triangles and the middle square is homotopy cartesian where $\left( \begin{array}{c} -f' \\ g' \end{array} \right)_* \delta' $ is a differential. Thus $e^* \delta' = f_* \delta''$ and

$$ D \oplus C \xrightarrow{\left( \begin{array}{c} -d \\ 0 \end{array} \right)} E \oplus C \xrightarrow{\left( e,g' \right)} F $$

is an $\mathbb{E}$-triangle. Note that we have the following morphisms of $\mathbb{E}$-triangles.

$$ D \oplus C \xrightarrow{\left( \begin{array}{c} -d \\ 0 \end{array} \right)} E \oplus C \xrightarrow{\left( e,g' \right)} F $$

Since $\left( -1,0 \right)_* \left( \begin{array}{c} -f' \\ g' \end{array} \right)_* \delta' = f'_* \delta'$, we have the following commutative diagram

$$ A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{\delta'} - \Rightarrow $$

such that the third column is an $\mathbb{E}$-triangle, $d^* \delta'' = \delta$ and $e^* \delta' = f'_* \delta''$.

$(d) \Rightarrow (a)$. Let $A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{\delta_f}$ and $B \xrightarrow{g} C \xrightarrow{\delta_g}$ be two $\mathbb{E}$-triangles. By (ET0), $h = gf : A \to C$ is an inflation. Assume that $A \xrightarrow{h} C \xrightarrow{h_0} E_0 \xrightarrow{\delta_h}$ is an $\mathbb{E}$-triangle. By (ET4-3), there are morphisms $d_0 : D \to E_0$ and $e_0 : E_0 \to F$ such that the following diagram

$$ A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{\delta_f} - \Rightarrow $$

is commutative and the third column is an $\mathbb{E}$-triangle, $d_0^* (\delta_h) = \delta_f$ and $f_* (\delta_h) = e_0^* (\delta_g)$. Therefore, (ET4) holds.
**Lemma 3.6.** Let \((\mathcal{C}, \mathbb{E}, s)\) be a pre-extriangulated category. If \(\mathcal{C}\) satisfies (ET4-4), then \(\mathcal{C}\) satisfies (ET4-5).

**Proof.** Suppose we have the following commutative diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{x_1} & B_1 & \xrightarrow{y_1} & C \\
& \downarrow{g} & & \downarrow{\delta} & \Rightarrow \\
A_2 & \xrightarrow{x_2} & B_2 & \xrightarrow{y_2} & C \\
\end{array}
\]

whose rows are \(\mathbb{E}\)-triangles. Consider the following commutative diagram

\[
\begin{array}{ccc}
A_2 \oplus B_1 & \xrightarrow{(x_2, 0)} & B_2 \oplus B_1 & \xrightarrow{(y_2, 0)} & C \\
& \downarrow{(1, 0)} & & \downarrow{-1} & \Rightarrow \\
A_2 \oplus B_1 & \xrightarrow{(x_2, g)} & B_2 \oplus B_1 & \xrightarrow{(-y_2, y_1)} & C \\
\end{array}
\]

Since the first row is an \(\mathbb{E}\)-triangle, the second row is also an \(\mathbb{E}\)-triangle by Lemma 2.7.

By (ET4-4), we have the following commutative diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{x_1} & B_1 & \xrightarrow{y_1} & C \\
& \downarrow{(x_1, 0)} & \downarrow{(0)} & \downarrow{-1} & \Rightarrow \\
A_2 \oplus B_1 & \xrightarrow{(x_2, g)} & B_2 \oplus B_1 & \xrightarrow{(-y_2, y_1)} & C \\
\end{array}
\]

such that the first column is an \(\mathbb{E}\)-triangle, \(\left(\frac{\vec{f}}{x_1}\right) \cdot \delta = -(\delta' \oplus 0), x_1 \cdot \delta'' = 0\) and \((1, 0)*\delta'' + (-y_2, y_1)*\delta = 0\). We have

\[
\delta'' = ((1, 0) (\frac{1}{0})) \cdot \delta'' = (\frac{1}{1}) \cdot (\frac{1}{0}) \cdot \delta'' = (\frac{1}{0}) \cdot (\frac{1}{0}) \cdot \delta = (y_2, -y_1) (\frac{1}{0}) \cdot \delta = y_2^* \delta.
\]

It remains to show that \(f_\delta \delta = \delta'\). In fact, by (ET3)* we have the following morphism of \(\mathbb{E}\)-triangles.

\[
\begin{array}{ccc}
A_2 \oplus B_1 & \xrightarrow{(x_2, g)} & B_2 \oplus B_1 & \xrightarrow{(-y_2, y_1)} & C \\
& \downarrow{(-1, b)} & \downarrow{-1, g} & \downarrow{-1} & \Rightarrow \\
A_2 & \xrightarrow{x_2} & B_2 & \xrightarrow{y_2} & C \\
\end{array}
\]

Note that \((-1, b)^* (-\delta' \oplus 0) = \delta'\) and \((bx_1)^* \delta = 0\) by Lemma 2.5, we have

\[
f_\delta \delta = f_\delta \delta + (bx_1)^* \delta = (-1, b)^* \delta = (-1, b)^* (-\delta' \oplus 0) = \delta'.
\]

Therefore, (ET4-5) holds. \(\square\)

**Theorem 3.7.** Let \((\mathcal{C}, \mathbb{E}, s)\) be a pre-extriangulated category. If \(\mathcal{C}\) satisfies (ET0) and (ET0)*, then (ET4), (ET4-1), (ET4-2), (ET4-3), (ET4-4), and (ET4-5) are equivalent and self-dual.
Proof. Since \( \mathcal{C} \) satisfies (ET0) and \( (ET0)^{op} \), by Theorems 3.4 and 3.5, (ET4), (ET4-1), (ET4-2), (ET4-3), and (ET4-5) are equivalent and self-dual. In this case, if \( \mathcal{C} \) satisfies (ET4), then \( \mathcal{C} \) is an extriangulated category. Thus (ET4) implying (ET4-4) follows by Lemma 2.11. We complete the proof by Lemma 3.6.

Acknowledgments

The authors sincerely thank the referee for careful reading and helpful comments and suggestions improving this paper.

Funding

This work was supported by the Natural Science Foundation of Fujian Province (Grant No. 2020J01075).

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