Optimal exact tests for composite alternative hypotheses on cross tabulated data

Daniel Yekutieli

December 12, 2013

Abstract

We present methodology for constructing exact significance tests for cross tabulated data for “difficult” composite alternative hypotheses that have no natural test statistic. We construct a test for discovering Simpson’s Paradox and a general test for discovering positive dependence between two ordinal variables. Our tests are Bayesian extensions of the likelihood ratio test, they are optimal with respect to the prior distribution, and are also closely related to Bayes factors and Bayesian FDR controlling testing procedures.

1 Introduction

We present Bayesian extensions of the likelihood ratio test that are optimal with respect to the prior distribution for testing composite alternative hypotheses that have no natural test statistic. As a motivating example, we present exact tests for discovering Simpson’s paradox.

Example 1.1 Table 1 displays data from a study on Death penalty in Florida (Agresti 2002, Table 2.13). The 326 subjects classified in Table 1 were the defendants in indictments involving cases with multiple murders in Florida. The goal of the analysis is to determine whether the probability of receiving death sentence depends on the defendant’s race.

The variables are $X$ – Race of Victim ("White", "Black"), $Y$ – Race of Defendant ("White", "Black"), and $Z$ – Death Penalty verdict ("Yes", "No"). $\pi_{ijk}$ is the probability that $X$ takes on its $i$th value and $Y$ takes on its $j$th value and $Z$ takes on its $k$th value. The conditional odds ratio between defendant’s race and death penalty for White victims is $\theta_{YZ|X=1} = (\pi_{111} \cdot \pi_{122})/(\pi_{112} \cdot \pi_{121})$ and for Black victims it is $\theta_{YZ|X=2} = (\pi_{211} \cdot \pi_{222})/(\pi_{212} \cdot \pi_{221})$. The marginal odds ratio between defendant’s race and death penalty is $\theta_{YZ} = (\pi_{+11} \cdot \pi_{+21})/(\pi_{+12} \cdot \pi_{+22})$. 
\[ \frac{\pi_{+22}}{\pi_{+12} \cdot \pi_{+21}} \] for \( \pi_{+jk} = \pi_{1jk} + \pi_{2jk} \). Similarly, \( \theta_{XZ} \) is the marginal odds ratio between victim’s race and death penalty and \( \theta_{XY} \) is the marginal odds ratio between defendant’s race and death penalty.

We used the R `fisher.test` function to test dependency between the pairs of variables. Defendant race and victim race are highly dependent, \( \hat{\theta}_{XY} = 27.1 \) with \( 0.95 \) CI [12.7, 64.8]; and risk of receiving death penalty is higher for white victims than for black victims, \( \hat{\theta}_{XZ} = 2.87 \) with \( 0.95 \) CI [1.13, 8.73]. Thus Victim’s race is a confounder: white defendants have higher probability of receiving death penalty just because they are more likely to kill a white victim. Indeed, we see that \( \hat{\theta}_{YZ} = 1.18 \) with \( 0.95 \) CI [0.56, 2.52].

To test the null hypothesis, for white victims we further condition on the observed values \( N_{11+} = 151, N_{12+} = 63, N_{1+1} = 30, N_{1+2} = 184 \), and for Black victims we further condition on the observed values \( N_{21+} = 9, N_{22+} = 103, N_{2+1} = 6, N_{2+2} = 106 \). Forming a conditional sample space with 217 points that can be expressed

\[ \Omega_a = \{(N_{111}, N_{211}) : N_{111} \in (0, 1, \ldots, 30), N_{211} \in (0, 1, \ldots, 6)\} \]

The observed data point is \( (N_{111} = 19, N_{211} = 0) \). Under \( H_0 \), \( N_{111} \) and \( N_{211} \) are independent and, using R notations, the probability of each data point is

\[ \Pr_{H_0}(N_{111} = x, N_{211} = y) = dhyper(x; 151, 63, 30) \cdot dhyper(y; 9, 103, 6). \]

Applying the R `fisher.test` function to the observed 2-by-2 tables corresponding to White and Black victims yields, \( \hat{\theta}_{YZ|X=1} = 0.68 \) with \( 0.95 \) CI [0.28, 1.70] and \( \hat{\theta}_{YZ|X=2} = 0 \) with \( 0.95 \) CI [0, 10.72]. To construct an exact test for \( H_0 \), the 217 data sample points are ordered according to a statistic that quantifies their strength of evidence in favor of Simpson’s paradox, and then the exact significance level of the observed table is the sum of the probabilities of the data points with greater or equal test statistic value. However, as Simpson’s paradox involves effects having conflicting signs, determining strength of evidence in favor of Simpson’s paradox is difficult. For example, does data point \((20, 0)\) with larger or equal conditional associations ( \( \hat{\theta}_{YZ|X=1} = 0.810, \hat{\theta}_{YZ|X=2} = 0 \) ) and larger marginal (\( \hat{\theta}_{YZ} = 1.34 \)) association offer more evidence in favor of Simpson’s paradox than the observed data point?
We propose two statistics for ordering the points in the data sample space. The first statistic is the posterior probability of the event corresponding to $H_1$

$$P_1 = \{ (\pi_{111} \cdots \pi_{222}) : \theta_{YZ|X=1} < 1, \theta_{YZ|X=2} < 1, 1 < \theta_{YZ} \}. $$

The second statistic is the ratio between the posterior probability of $P_1$ and the posterior probability of the event

$$P_0(\epsilon) = \{ (\pi_{111} \cdots \pi_{222}) : |\log(\theta_{YZ|X=1})| \leq \epsilon, |\log(\theta_{YZ|X=2})| \leq \epsilon \},$$

with $\epsilon = 0.1$. For our analysis we use a Dirichlet prior with concentration parameters $(0.5 \cdots 0.5)$. Thus for data point $(N_{111} \cdots N_{222})$, the posterior distribution of $(\pi_{111} \cdots \pi_{222})$ is Dirichlet with concentration parameters $(N_{111} + 0.5 \cdots N_{222} + 0.5)$. To compute the probability of $P_1$ and $P_0(0.1)$ for a given data point, we sample $(\pi_{111}, \cdots \pi_{222})$ from the posterior probability and count the proportion of samples that either events occurred.

Based on $2 \times 10^6$ samples from the posterior distribution, data point $(20, 0)$ with $Pr_{H_0}(20, 0) = 0.087$ has the largest posterior probability of $P_1$, 0.085954 ($s.e. < 0.0001$); the observed table with $Pr_{H_0}(19, 0) = 0.064$ has the second largest posterior probability, 0.0797 ($s.e. < 0.0001$); Data point $(21, 0)$ with $Pr_{H_0}(21, 0) = 0.101$ has the third largest posterior probability, 0.0795 ($s.e. < 0.0001$). Thus for the first statistic, the significance level of the observed table is $0.151 = 0.087 + 0.064$. To assess the posterior probability of $P_0(\epsilon)$ we sampled $10^6$ realizations from the posterior distribution. The posterior probability for the observed data point was 0.0054. Higher posterior probability was observed in 8 data points, among them $(20, 0)$ and $(21, 0)$. In 121 data points the ratio between the posterior probability of $P_1$ and $P_0(0.1)$ was at least as high as that of $(19, 0), 14.8 = 0.0797/0.0054$. The significance level of the observed table for the second statistic is 0.140, the sum of the probabilities under the null for these 121 data points.

| Victim | Defendant | Death Penalty | No Death Penalty |
|--------|-----------|---------------|------------------|
| White  | White     | 19            | 132              |
|        | Black     | 11            | 52               |
| Black  | White     | 0             | 9                |
|        | Black     | 6             | 97               |

Table 1: Death Penalty data

In Section 2 we present our general testing methodology and its conditional
variant, phrase and prove their optimality property, and explain the relation between our tests and Bayesian FDR controlling tests, Bayes factors and likelihood ratio tests. In Section 3 we demonstrate our methodology on a 4-by-4 contingency table, present an exact tests for discovering positive dependence between two ordinal variables, and perform a simulation that reveals that our methods may provide a slight power edge even for testing composite null hypothesis that have a natural statistic. We end the paper with a discussion.

2 Mean most powerful tests

We denote the parameter by $p \in P$, $\pi(p)$ is the prior distribution, the data is $N \in \Omega$, and the likelihood is $\Pr(n \mid p)$. The alternative hypothesis is $H_1 : p \in P_1$, for $P_1 \subseteq P$. Following Benjamini and Hochberg (1995) that referred to rejecting the null hypothesis as making a statistical discovery, $P_1$ is the discovery event and we call $P_0 \subseteq P - P_1$ the non-discovery event. The role of $P_0$ is to determine the optimality property of the test, given in Definition 2.1. We explain how to set $P_0$ in Remark 2.3. The null hypothesis $H_0$ does not have to correspond to an explicit subset of $P_0$, all we will need is that the null hypothesis specifies a null distribution $\Pr_{H_0}(N = n)$ on $\Omega$. Tests are mappings $T : \Omega \rightarrow \{0, 1\}$. For $S \subseteq \Omega$, let $T(S) := I(n \in S)$, where $T(S) = 1$ corresponds to declaring that $p \in P_1$. Thus the significance level of $T(S)$ is $\Pr_{H_0}(N \in S)$.

Our tests are Bayes rules for discriminating between $P_0$ and $P_1$ that minimize the average risk for the following loss function:

$$L(S; \lambda_1, \lambda_2) = \lambda_1 \cdot I(N \in S, \ P \in P_0) + \lambda_2 \cdot I(N \notin S, \ P \in P_1).$$

As the marginal distribution of $N$ is

$$\Pr(N = n) = \int_P \pi(p) \cdot \Pr(N = n \mid p) \, dp,$$

and the conditional distribution of $p$ given $N = n$ is

$$\pi(p \mid n) = \Pr(N = n \mid p) \cdot \pi(p) / \Pr(N = n),$$

the average risk can be expressed

$$\sum_{n \in \Omega} \Pr(n) \cdot \int_P \pi(p \mid n) \cdot [\lambda_1 \cdot I(n \in S, \ P \in P_0) + \lambda_2 \cdot I(n \notin S, \ P \in P_1)] \, dp$$

$$= \sum_{n \in S} \Pr(n) \cdot \lambda_1 \cdot \Pr(P \in P_0 \mid n) + \sum_{n \notin S} \Pr(n) \cdot \lambda_2 \cdot \Pr(P \in P_1 \mid n).$$


Thus for $\delta = \lambda_1/\lambda_2$, $S$ that minimizes the average risk in (2) is
\[
S^{Bayes}(\delta) = \{n : \delta \leq \frac{Pr(P \in P_1| n)}{Pr(P \in P_0| n)}\},
\] (3)
Similarly, the Bayes rule can be specified according to its significance level. For $\alpha \in [0,1]$, let $S^{Bayes}(\alpha) := S^{Bayes}(\delta_\alpha)$ for
\[
\delta_\alpha = \min\{\delta : Pr_{H_0}(N \in S^{Bayes}(\delta)) \leq \alpha\}.
\]

**Definition 2.1**

1. The mean significance level of $T(S)$ is $Pr(N \in S| p \in P_0)$.
2. The mean power of $T(S)$ is $Pr(N \in S| p \in P_1)$.
3. $T(S)$ is a mean most powerful test if all tests with less or equal mean significance level have less or equal mean power.

**Proposition 2.2** $\forall \delta$, $T(S^{Bayes}(\delta))$ is a mean most powerful test.

**Proof.** Let $T(\tilde{S})$ be a test with less or equal mean significance than $T(S^{Bayes})$
\[
Pr(N \in \tilde{S}| P \in P_0) \leq Pr(N \in S^{Bayes}| P \in P_0).
\] (4)
We begin by expressing
\[
Pr(N \in \tilde{S}| P \in P_0) = \sum_{n \in \tilde{S}} Pr(P_0| n) \cdot Pr(n)/Pr(P_0),
\] (5)
and expressing
\[
Pr(N \in S^{Bayes}| P \in P_0) = \sum_{n \in S^{Bayes}} Pr(P_0| n) \cdot Pr(n)/Pr(P_0).
\] (6)
Subtracting the summands in $S^{Bayes} \cap \tilde{S}$ from the sums in (5) and (6) and multiplying by $Pr(P_0)$, Inequality (4) implies that
\[
\sum_{n \in \tilde{S} - (S^{Bayes} \cap \tilde{S})} Pr(P_0| n) \cdot Pr(n) \leq \sum_{n \in S^{Bayes} - (S^{Bayes} \cap \tilde{S})} Pr(P_0| n) \cdot Pr(n).
\] (7)
According to the construction of $S^{Bayes}$, $\forall n_1 \in \tilde{S} - (S^{Bayes} \cap \tilde{S})$ and $\forall n_2 \in S^{Bayes} - (S^{Bayes} \cap \tilde{S})$,

$$\Pr(\mathcal{P}_1| n_1)/\Pr(\mathcal{P}_0| n_1) \leq \Pr(\mathcal{P}_1| n_2)/\Pr(\mathcal{P}_0| n_2).$$

(8)

Next, we express

$$\Pr(N \in \tilde{S} | p \in \mathcal{P}_1) = \sum_{n \in S^{Bayes} \cap \tilde{S}} \Pr(\mathcal{P}_1| n) \cdot \Pr(n)/\Pr(\mathcal{P}_1)$$

(9)

$$+ \sum_{n \in S - (S^{Bayes} \cap \tilde{S})} (\Pr(\mathcal{P}_0| n) \cdot \frac{\Pr(\mathcal{P}_1| n)}{\Pr(\mathcal{P}_0| n)} \cdot \frac{\Pr(n)}{\Pr(\mathcal{P}_1)}.$$  

(10)

and

$$\Pr(N \in \tilde{S} | p \in \mathcal{P}_1) = \sum_{n \in S^{Bayes} \cap \tilde{S}} \Pr(\mathcal{P}_1| n) \cdot \Pr(n)/\Pr(\mathcal{P}_1)$$

(11)

$$+ \sum_{n \in S^{Bayes} - (S^{Bayes} \cap \tilde{S})} (\Pr(\mathcal{P}_0| n) \cdot \frac{\Pr(\mathcal{P}_1| n)}{\Pr(\mathcal{P}_0| n)} \cdot \frac{\Pr(n)}{\Pr(\mathcal{P}_1)}.$$  

(12)

Note that Expression (10) is the left hand side of (7) and Expression (12) is the right hand side of (7), divided by $\Pr(\mathcal{P}_1)$ and multiplied by a factor, that according to (8), is larger in each summand of (12) than in all of the summands of (10). Therefore the sum in (12) is larger than the sum in (10), and as the sums in the right hand side of (9) and (11) are the same,

$$\Pr(N \in \tilde{S} | p \in \mathcal{P}_1) \leq \Pr(N \in S^{Bayes} | p \in \mathcal{P}_1).$$

Remark 2.3 Determining $\mathcal{P}_1$, $\mathcal{P}_0$, and $\pi(p)$, produces a family of mean most powerful tests. Per construction, $T(S^{Bayes}(\alpha))$ has significance level $\alpha$ and has more mean power than all mean most powerful tests with significance level $< \alpha$. According to Proposition 2.2, $T(S^{Bayes}(\alpha))$ also has more mean power than all tests with smaller or equal mean significance level. Note that in the examples in the paper we only compute the p-value for the observed data, applying $T(S^{Bayes}(\alpha))$ further entails rejecting $H_0$ if the p-value is $\leq \alpha$.

Ideally, the prior distribution captures the knowledge regarding the parameters that is available prior to the study. In the examples in the paper we used conjugate non-informative priors that provide easy test statistic computation and yield general optimal tests for each alternative null hypothesis. While the
choice of $\mathcal{P}_1$ is usually dictated by the application, $\mathcal{P}_0$ can just be a subset of $\mathcal{P} - \mathcal{P}_1$. We suggest either setting $\mathcal{P}_0$ to be a “small” set containing $p_0$, the parameter value under the null (we denoted this set by $\mathcal{P}_0(\epsilon)$ in Example 1.1), or setting $\mathcal{P}_0 = \mathcal{P} - \mathcal{P}_1$. If $\mathcal{P}_0 = \{p_0\}$, then the mean significance level would equal the significance level, thus $\mathcal{T}(S_{\text{Bayes}}(\alpha))$ would have more mean power then all tests with significance level $\leq \alpha$. As our choice of priors assigns zero probability to $\{p_0\}$, we resort to setting $\mathcal{P}_0 = \mathcal{P}_0(\epsilon)$ with small $\epsilon$ that produces a very similar family of mean most powerful tests. But note that using too small $\epsilon$ will make it very difficult to numerically assess $\Pr(\mathcal{P}_0(\epsilon)|n)$. The other option is setting $\mathcal{P}_0 = \mathcal{P} - \mathcal{P}_1$, that yields

$$\frac{\Pr(\mathcal{P} \in \mathcal{P}_1|n)}{\Pr(\mathcal{P} \in \mathcal{P}_0|n)} = \frac{\Pr(\mathcal{P} \in \mathcal{P}_1|n)}{1 - \Pr(\mathcal{P} \in \mathcal{P}_1|n)}.$$ 

This means that sorting the data points according to $Pr(\mathcal{P}_1|n)$ is equivalent to sorting the data points according to $Pr(\mathcal{P}_1|n)/Pr(\mathcal{P}_0|n)$. In this case the optimality property may be less appealing but it has the great technical advantage that to construct our test, for each data point, we only need to assess the posterior probability of $\mathcal{P}_1$.

### 2.1 Conditional mean most powerful tests

In this section we present mean most powerful tests for the conditional analysis of contingency tables, in which the sample space is partitioned according to the row and column sums and a separate level $\alpha$ test is conducted in each partition.

Let $a$ be the statistic that partitions the sample space $\Omega = \cup_{a \in \mathcal{A}} \Omega_a$, for $\mathcal{A} = \{a(N) : N \in \Omega\}$ the set of statistic values.

**Definition 2.4** A conditional level $\alpha$ test is $\mathcal{T}(\mathcal{S}_\mathcal{A}(\alpha))$ such that $\forall a \in \mathcal{A}$, $\Pr_{H_0}(N \in \mathcal{S}_\mathcal{A}(\alpha)|N \in \Omega_a) \leq \alpha$.

To construct $\mathcal{S}_\mathcal{A}(\text{Bayes})(\alpha)$, the rejection region of the conditional mean most powerful test, we repeat the following for each $a \in \mathcal{A}$: sort the data points $N \in \Omega_a$ according to $Pr(\mathcal{P} \in \mathcal{P}_1|N)/Pr(\mathcal{P} \in \mathcal{P}_0|N)$ and then following that order, as long as $\Pr_{H_0}(N \in \mathcal{S}_\mathcal{A}(\text{Bayes})(\alpha)|N \in \Omega_a) \leq \alpha$, sequentially add data points into $\mathcal{S}_\mathcal{A}(\text{Bayes})(\alpha)$.

**Remark 2.5** Per construction, $\mathcal{T}(\mathcal{S}_\mathcal{A}(\text{Bayes})(\alpha))$ is a conditional level $\alpha$ test and for all $a$, $\mathcal{T}(\mathcal{S}_\mathcal{A}(\text{Bayes})(\alpha) \cap \Omega_a)$ is a mean most powerful test on $\Omega_a$. Conditional
level $\alpha$ tests are also level $\alpha$ tests:

$$\Pr_{H_0}(N \in S_A(\alpha), N \in \Omega_a) = \sum_{a \in A} \Pr_{H_0}(N \in S_A(\alpha), N \in \Omega_a) \leq \sum_{a \in A} \alpha \cdot \Pr_{H_0}(N \in \Omega_a) = \alpha.$$ 

When $a$ assumes a single value then $S^\text{Bayes}_A(\alpha) = S^\text{Bayes}(\alpha)$. But in general, $T(S^\text{Bayes}_A(\alpha))$ is not a mean most powerful test and there may even be other conditional level $\alpha$ test with smaller mean significance level and larger mean power. However, if $P_0 = \{p_0\}$ and $\Pr_{H_0}(N \in S^\text{Bayes}_A(\alpha) | N \in \Omega_a) = \alpha$ for all $a$, then as $T(S^\text{Bayes}_A(\alpha) \cap \Omega_a)$ is a mean most powerful test on $\Omega_a$ and the mean significance level identifies with the significance level, any other conditional level $\alpha$ test, $T(S_A(\alpha))$, would have smaller mean significance level than $T(S^\text{Bayes}_A(\alpha))$ on $\Omega_a$ and thus it would also have smaller mean power on $\Omega_a$. Summing over all $\Omega_a$, $T(S_A(\alpha))$ would have smaller mean power than $T(S^\text{Bayes}_A(\alpha))$.

2.2 Relation between our tests and Bayesian FDR controlling tests, Bayes factors, and likelihood ratio tests

$Pr(P \in P_1 | n)$ is equal to one minus the local FDR (Efron et al., 2001). Thus setting $P_0 = P - P_1$ we follow Storey (2007), who suggested constructing optimal tests in which the local FDR is used for determining the order in which the data points are included into the rejection region. However, unlike the Bayesian FDR approach, in which $\pi(p)$ is the marginal parameter distribution in the population of parameters that is under study, and thus the Bayesian FDR can be used to determine the cutoff point of the rejection region (Heller and Yekutieli, 2012). In our tests the cutoff point is determined by the test’s significance level.

Expressing the statistic in (3)

$$\frac{Pr(P \in P_1 | N = n)}{Pr(P \in P_0 | N = n)} = \frac{Pr(N=n | P \in P_1) \cdot Pr(P \in P_1)}{Pr(N=n | P \in P_0) \cdot Pr(P \in P_0)} \propto \frac{Pr(N = n | P \in P_1)}{Pr(N = n | P \in P_0)}, \quad (13)$$

reveals that we actually order the data points according to the Bayes factor between “model” $P_1$ and “model” $P_0$. However, note that in our tests the cutoff point of the rejection region is not a nominal Bayes factor value (cf. Kass and Raftery, 1995).

Our tests are also closely related to likelihood ratio tests. For simple hypotheses, $H_0 : p = p_0$ for $p_0 \in P_0$ vs. $H_1 : p = p_1$ for $p_1 \in P_1$, our test
reduces to the likelihood ratio test if \( \mathcal{P}_0 = \{ \mathbf{p}_0 \} \) and \( \mathcal{P}_1 = \{ \mathbf{p}_1 \} \), or if the prior distribution assigns all its probability to the two hypotheses: \( \pi(\mathbf{p}_0) = \pi_0 \) and \( \pi(\mathbf{p}_1) = 1 - \pi_0 \), for \( 0 < \pi_0 < 1 \). The likelihood ratio statistic (Casella and Berger, 2001) for testing the composite hypotheses \( H_0 : \mathbf{p} \in \mathcal{P}_{null} \) vs. \( H_1 : \mathbf{p} \notin \mathcal{P}_{null} \) is

\[
\Lambda(n) = \frac{\sup_{\mathbf{p} \in \mathcal{P}_{null}} \Pr(N = n|\mathbf{p})}{\sup_{\mathbf{p} \in \mathcal{P}} \Pr(N = n|\mathbf{p})}.
\]

For \( \mathcal{P}_1 = \mathcal{P} - \mathcal{P}_{null} \), setting \( \mathcal{P}_0 = \mathcal{P} - \mathcal{P}_1 \) yields \( \mathcal{P}_0 = \mathcal{P}_{null} \) and thus \( \Lambda(n) \) orders the data points similarly to one minus our statistic, except that in our statistic we consider the average rather than the supremum of the likelihood, which according to our theoretical results yields tests with more power with respect to the prior distribution. However for \( \mathcal{P}_1 \subset \mathcal{P} - \mathcal{P}_{null} \) and setting \( \mathcal{P}_0 = \mathcal{P} - \mathcal{P}_1 \), our statistic, that orders the data points according to \( \mathcal{P}_1 \), yields considerably more powerful tests than \( \Lambda(n) \), that orders the data points according to the null hypothesis, especially for the case that \( \mathcal{P}_1 \) is a “small” subset of \( \mathcal{P} - \mathcal{P}_{null} \).

We illustrate this in the following example and it occurs in the two contingency table examples, where our tests yield considerably smaller p-values than the \( X^2 \) statistic, which is the likelihood ratio statistic for testing independence for cross-tabulated data.

**Example 2.6** The parameter is \( \mu = (\mu_1 \ldots \mu_K) \). The data is \( Y = (Y_1 \ldots Y_K) \) with \( Y_k \sim N(\mu_k, 1) \). The null hypothesis is \( H_0 : \mu = 0 \) and \( \mathcal{P}_1 = \{ \mu : 3 \leq \mu_1 \} \).

In the likelihood ratio test for \( H_0 : \mu = 0 \) vs. \( H_1 : \mu \neq 0 \), the data points are ordered according to their \( l_2 \) norm. Setting \( \mathcal{P}_0 = \mathcal{P} - \mathcal{P}_1 \) and using a flat prior for \( \mu \), our test sorts the data points are ordered according to \( Y_1 \). For \( K = 100 \) and \( \mu = (3.2, 0 \ldots 0) \), as 124.34 is the 0.95 quantile of the 100 degree of freedom \( \chi^2 \) distribution, the rejection region for the \( \alpha = 0.05 \) likelihood ratio test is \( \{ \mathbf{y} : 124.34 \leq \| \mathbf{y} \|^2 \} \) and the power of this test is 0.179, while for our \( \alpha = 0.05 \) test the rejection region is \( S_{Bayes}(0.05) = \{ \mathbf{y} : 1.64 \leq y_1 \} \) and its power is 0.940.

### 3 Job Satisfaction Example

The data in Table 2 was also taken from Agresti (2002, Table 2.8). A sample of 96 black males were classified by Income (“< 1500”, “15000 – 25000”, “25000 – 40000”, “> 40000”) and job satisfaction (“Very Dissatisfied”, “Little Dissatisfied”, “Moderately Satisfied”, “Very Satisfied”). For \( i = 1 \ldots 4 \) and
\( j = 1 \cdots 4, \pi_{ij} \) is the probability that a respondent has income level \( i \) and job satisfaction level \( j \). We assume that the number of respondents \( N = (N_{11} \cdots N_{44}) \) is multinom\((\pi_{11} \cdots \pi_{44})\). \( n_{ij} \) is the observed number of respondents recorded in Table 2. The null hypothesis is \( H_0 : \pi_{ij} = \pi_{i+} \pi_{+j} \), for \( \pi_{i+} = \pi_{i1} + \cdots + \pi_{i4} \) and \( \pi_{+j} = \pi_{1j} + \cdots + \pi_{4j} \). A pair of respondents is concordant if they have different income and job satisfaction and the respondent with higher income has higher job satisfaction. The probability that a pair of respondents is concordant is

\[
\Pi_C = 2 \sum_i \sum_j \pi_{ij} \left( \sum_{i<h} \sum_{j<k} \pi_{hk} \right).
\] (14)

A pair of respondents is discordant if they have different income and job satisfaction and the respondent with higher income has lower job satisfaction. The probability that a pair of respondents is discordant is

\[
\Pi_D = 2 \sum_i \sum_j \pi_{ij} \left( \sum_{i<h} \sum_{k<j} \pi_{hk} \right).
\] (15)

The degree of concordance is measured by Kendall’s gamma rank correlation coefficient, \( \gamma = (\Pi_C - \Pi_D)/\Pi_C + \Pi_D \). Which is the difference between the conditional probability of concordance and discordance given that the pair of respondents have different income and different job satisfaction.

We first test \( H_0 \) with tests implemented in R, whose significance levels are based on parametric approximations of the test statistics’ distribution under the null hypothesis. Pearson’s Chi-squared test \((\text{chisq.test} \text{ function})\) yielded \( X^2 = 5.97 \) with 9 degrees of freedom and p-value 0.743. Kendall’s rank correlation coefficient \((\text{cor.test} \text{ function})\), corresponding to alternative hypothesis of concordance between of income and job satisfaction, was \( \tau = 0.152 \) with p-value 0.043. Spearman’s rank correlation coefficient \((\text{cor.test} \text{ function})\), corresponding to alternative hypothesis of positive rank correlation, was \( \rho = 0.177 \) with p-value 0.042.

To construct the exact tests we condition on \( n_{i+} \) and \( n_{+j} \), the row and column sums of Table 2. There are 90,208,550 possible 4-by-4 tables with the same row and columns sums as Table 2. Under the null hypothesis, the distribution of these tables is multivariate hypergeometric.

The first exact test is based on Kendall’s gamma estimator, \( \hat{\gamma} = (\hat{\Pi}_C - \hat{\Pi}_D)/(\hat{\Pi}_C + \hat{\Pi}_D) \), for \( \hat{\Pi}_C \) and \( \hat{\Pi}_D \) computed by replacing \( \pi_{ij} \) with \( \hat{\pi}_{ij} = n_{ij}/96 \) in (14) and (15). The observed value is \( \hat{\gamma} = 0.221 \). Greater or equal \( \hat{\gamma} \) values

10
we computed for 21,101,151 tables. The sum of the probabilities under $H_0$ of these tables was 0.0415.

Our second statistic is the posterior probability of the concordance event, $P_{Cncrd}^1 = \{ (\pi_{11} \cdot \cdot \cdot \pi_{44}) : 0 \leq \gamma \}$. We use a Dirichlet prior distribution with concentration parameters $(0.5 \cdot \cdot \cdot 0.5)$ for $(\pi_{11} \cdot \cdot \cdot \pi_{44})$, for which the posterior probability is a dirichlet distribution with concentration parameters $(N_{11} + 0.5 \cdot \cdot \cdot N_{44} + 0.5)$. To compute the probability of the concordance event for a given table, we sample $(\pi_{11}, \cdot \cdot \cdot \pi_{44})$ from the posterior probability and record the proportion of times the concordance event occurs. The probability of concordance for $N_{ij} = n_{ij}$, based on a sample of $10^7$ draws from the posterior, was 0.9564 ($s.e. < 0.0001$). Computing this statistic for all 4-by-4 tables is too time consuming. Thus to assess the significance level for this statistic, we generated a sample of 50,000 4-by-4 contingency tables from the multivariate hypergeometric null distribution, and for each contingency table we sampled 10,000 $(\pi_{11}, \cdot \cdot \cdot \pi_{44})$ from the posterior probability and recorded the proportion of times the concordance event occurred. The estimated significance level was 0.036 ($s.e. < 0.001$), the proportion of contingency tables with estimated proportion of concordance $\geq 0.9564$.

Our statistic for the third exact test is the posterior probability that income and job satisfaction are positively dependent. This is a stronger property than concordance that corresponds to the event

$$P_{Pos}^1 = \{ (\pi_{11}, \cdot \cdot \cdot \pi_{44}) : \Pr(\pi_{j|i} \leq t) \geq \Pr(\pi_{j|i+1} \leq t) \ \forall t, \forall j, \forall i \},$$

(16)

for $\pi_{j|i} = \pi_{ij}/\pi_{i+}$. Based on a sample of $10^7$ draws, the posterior probability of positive dependence for the observed table is 0.0118 ($s.e. < 0.0001$). And again, to assess the significance level for this statistic we sampled 50,000 4-by-4 contingency tables from the multivariate hypergeometric null distribution and for each contingency table we sampled 10,000 $(\pi_{11}, \cdot \cdot \cdot \pi_{44})$ from the posterior probability. The estimated significance level was 0.0093 ($s.e. < 0.001$), which is the proportion of contingency tables with posterior probability of positive dependence $\geq 0.0118$.

Note that for the two Bayesian statistics we set $P_0 = P - P_1$. For $P_1 = P_{Pos}^1$, we had also experimented with setting $P_0$ to be a small subset containing the null, $P_0(\epsilon)$. However, with $\epsilon$ large enough to be able to estimate the posterior probability of $P_0(\epsilon)$ in comparable run time the p-value increased from less than 1% to more than 10%, suggesting that for this data setting $P_0 = P_0(\epsilon)$ is not a feasible option.
3.1 Job Satisfaction Simulation

The simulation compares the power of the conditional exact test whose test statistic is $\hat{\gamma}$ with the conditional exact test whose test statistic is $\Pr(0 \leq \gamma | N_{11} \cdots N_{44})$, on

$$
\Omega_a = \{(N_{11} \cdots N_{44}) : N_{1+} = n_{1+}, N_{2+} = n_{2+}, \ldots, N_{4+} = n_{4+}\}
$$

(17)

for which the null distribution of $N$ is the multivariate hypergeometric considered in the previous section. The alternative distribution is that $N$ is multinomial $(\hat{\pi}_{11} \cdots \hat{\pi}_{44})$, with $\hat{\pi}_{ij} = n_{ij}/96$, truncated to $\Omega_a$ in (17).

We use importance sampling to generate $N$ from the alternative distribution. We sample $10^6$ proposal realizations of $N$ from the multivariate hypergeometric null distribution; for each proposal realization we compute a sampling weight that is the probability of observing this realization under the alternative multinomial distribution divided by the probability of observing this realization under the multivariate hypergeometric null distribution; and use weighted with-replacement sampling of the $10^6$ proposal values to generate a sample of $10^5$ realizations from the alternative distribution. We then compute the two test statistic values for each of the $10^5$ realizations. Lastly, to assess the significance level of each realization for the two test statistics, we generate another sample of $10^5$ realizations of $(N_{11} \cdots N_{44})$ from the null distribution, and compute the two test statistic value for each null realization. The p-values assigned to each alternative distribution realization is the proportion of null realization for which the statistic values were larger than the realization’s test statistic values.

Recall that for the Table 2 data, the p-value for the exact test based on the $\hat{\gamma}$ statistic was 0.0415 and the p-value for the exact test for the probability of concordance statistic was 0.036. In our simulation, for the $\hat{\gamma}$ statistic computed for the $10^5$ alternative distribution realizations, the mean p-value was 0.0988
and the median p-value was 0.0399, 0.679 (s.e. < 0.005) of the p-values were smaller than 0.10 and 0.537 (s.e. < 0.005) of the p-values were smaller than 0.05. While for the p-values computed based on the probability of concordance statistics, the mean p-value was 0.0947 and the median p-value was 0.0370, 0.701 (s.e. < 0.005) of the p-values were smaller than 0.10 and 0.550 (s.e. < 0.005) of the p-values were smaller than 0.05.

4 Discussion

As we will usually need to assess our test statistic values and their significance levels numerically by simulation from the null hypothesis, followed by simulation from the parameter posterior distribution, our tests can be computationally intensive. We therefore suggest using our tests in “difficult” cases where the parameter space is high dimensional and we know how to express the alternative hypothesis as a subset of the parameter space, however it is not clear how to construct a test statistic for this hypothesis. We also suggest using our methods in cases where there is prior information on the parameter or for very high dimensional and very sparse tables in which the asymptotic results for the test statistic distribution fail and the usual statistics may be severely under powered.

We presented methodology for the analysis of contingency tables in which use of exact tests is well established. However note that our approach can also be used to construct optimal tests for other problems in which samples under the null hypothesis can be generated by permutations or bootstrapping.

References

[1] Agresti A. (2002) “Categorical Data Analysis“ John Wiley & Sons, 2002.
[2] Benjamini Y., Hochberg Y. (1995) ‘Controlling the false discovery rate: a practical and powerful approach to multiple testing” J. R. Statist. Soc. B, 57, 289-300.
[3] Casella G., Berger R. L. (1990) “Statistical inference” Belmont, CA: Duxbury Press.
[4] Efron B., Tibshirani R., Storey J. D. and Tusher V. (2001) “Empirical Bayes analysis of a microarray experiment”. J. Am. Statist. Ass., 96, 11511160.
[5] Heller R., Yekutieli D., “Replicability analysis for genome-wide association studies” arXiv:1209.2829.

[6] Kass R. E., Raftery A. E. (1995) “Bayes Factors” Journal of the American Statistical Association, 90, 773-795.

[7] Storey, J D. (2007) “The optimal discovery procedure: a new approach to simultaneous significance testing” J. R. Statist. Soc. B, 69, 347368.