DURFEE-TYPE INEQUALITY FOR HYPERSURFACE SURFACE SINGULARITIES

MAKOTO ENOKIZONO

Abstract. We prove a “strong” Durfee-type inequality for isolated hypersurface surface singularities, which implies Durfee’s strong conjecture for such singularities with non-negative topological Euler number of the exceptional set of the minimal resolution.

Introduction

Let \((X, 0)\) be an isolated hypersurface surface singularity, that is, \(X = \{h(x, y, z) = 0\} \subset \mathbb{C}^3\) for some analytic function \(h\) on a neighborhood at the origin \(0 \in \mathbb{C}^3\) with an isolated singularity \(0 \in X\). The geometric genus \(p_g\) of \((X, 0)\) is defined by \(\dim H^1(\mathcal{O}_X)\), where \(\mathcal{X} \to X\) is a resolution. Let \(M = X_\varepsilon \cap B\) be a (generic) Milnor fiber, where \(X_\varepsilon = \{h(x, y, z) = \varepsilon\}\) is a smoothing of \((X, 0)\) and \(B \subset \mathbb{C}^3\) is a small closed ball centered at the origin. The rank \(\mu\) of the second homology group \(H_2(M, \mathbb{Z})\) is called the Milnor number. Let \(\mu_+\) (resp. \(\mu_-\), \(\mu_0\)) be the number of positive (resp. negative, 0) eigenvalues of the natural intersection form \(H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \to \mathbb{Z}\). Then \(\mu = \mu_+ + \mu_- + \mu_0\) and \(\sigma = \mu_+ - \mu_-\) is called the signature. The original Durfee’s conjectures [1] for hypersurface singularities are as follows:

(Weak conjecture) \(\sigma \leq 0\).

(Strong conjecture) \(6p_g \leq \mu\).

From Durfee’s result \(2p_g = \mu_+ + \mu_0\) [1], the weak conjecture is equivalent to \(4p_g \leq \mu + \mu_0\). Thus the strong conjecture implies the weak conjecture. Kollár and Némethi showed in [3] that the weak conjecture is true. Moreover, they showed that the strong conjecture is true for hypersurface singularities with integral homology sphere link. In this paper, we attack Durfee’s conjectures by using the method of invariants of fibered surfaces and show that the strong conjecture is true for a large class of hypersurface singularities. Our main theorem is as follows:

Theorem 0.1. Let \((X, 0)\) be an isolated hypersurface surface singularity with Milnor number \(\mu\) and geometric genus \(p_g > 0\). Then we have

\[
6p_g \leq \mu - \chi_{\text{top}}(A),
\]

or equivalently,

\[
\sigma \leq -2p_g - 1 - s,
\]
where $\chi_{\text{top}}(A)$ is the topological Euler number of the exceptional set $A$ of the minimal resolution $\pi: \tilde{X} \to X$ and $s$ is the number of irreducible components of $A$. In particular, the strong conjecture holds if $\chi_{\text{top}}(A) \geq 0$ and the weak conjecture holds for any isolated hypersurface surface singularity.

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1. Proof of the main theorem

Let $f: S \to B$ be a surjective morphism from a non-singular projective surface $S$ to a non-singular projective curve $B$. If any general fiber $F$ is a non-singular plane curve of degree $d$, we call $f: S \to B$ a plane curve fibration of degree $d$. Let $K_f = K_S - f^*K_B$ be the relative canonical bundle of $f$ and put $\chi_f = \deg f_*\mathcal{O}_S(K_f)$. Let $f: S \to \Delta$ be a proper surjective holomorphic map from a non-singular complex surface $S$ to a small open disk $\Delta \subset \mathbb{C}$ centered at 0. If any fiber $f^{-1}(t)$ over $t \neq 0$ is a non-singular plane curve of degree $d$, we call the pair $(f: S \to \Delta, F_0 = f^{-1}(0))$ a fiber germ of plane curves of degree $d$, which we also denote simply by $F_0$ if there is no fear of confusion. A fiber germ $f: S \to \Delta$ of plane curves is relatively minimal if the central fiber contains no $(-1)$-curves. Two fiber germs $(f: S \to \Delta, F_0)$ and $(f': S' \to \Delta, F'_0)$ are holomorphically equivalent if there exist biholomorphic maps $\phi: S \to S'$ and $\psi: \Delta \to \Delta$ with $\psi(0) = 0$ such that $f' \circ \phi = \psi \circ f$ after shrinking $\Delta$ if necessary. Let $\mathcal{A}_d$ be the set of holomorphically equivalence classes of relatively minimal fiber germs of plane curves of degree $d$.

**Theorem 1.1** (Theorem 0.1 in [2]). There exists a non-negative function $\text{Ind}_d: \mathcal{A}_d \to \frac{1}{d-2}\mathbb{Z}_{\geq 0}$ (which is called a Horikawa index) such that for any relatively minimal plane curve fibration $f: S \to B$ of degree $d$, the value $\text{Ind}_d(F)$ equals to 0 for any general fiber $F$ of $f$ and

\[
K_f^2 = \frac{6(d-3)}{d-2} \chi_f + \sum_{p \in B} \text{Ind}_d(F_p)
\]

holds, where $F_p = f^{-1}(p)$ is regarded as the fiber germ over $p \in B$.

**Definition 1.2.** Let $(f: S \to \Delta, F_0)$ be a relatively minimal fiber germ of plane curves. Then we can take a line bundle $\mathcal{L}$ on $S$ such that the restriction $\mathcal{L}|_F$ to the general fiber $F$ defines the embedding $F \subset \mathbb{P}^2$ (cf. Theorem 1.1 in [2]). Thus the relative linear system $f_*\mathcal{L}$ defines a birational map onto the image $S \dashrightarrow X \subset \Delta \times \mathbb{P}^2$. If the image $X$ has only one isolated singularity $x$, we call the pair $(X, x)$ an isolated hypersurface singularity associated to a fiber germ $f: S \to \Delta$ of plane curves.

For an isolated hypersurface singularity $(X, x)$ associated to a fiber germ $(f, F_0)$ of plane curves of degree $d$, the Horikawa index $\text{Ind}_d(F_0)$ can be computed by some invariants of the singularity $(X, x)$:
Lemma 1.3. Let \((X, x)\) be an isolated hypersurface singularity associated to a fiber germ \((f: S \to \Delta, F_0)\) of plane curves of degree \(d\) with Milnor number \(\mu\) and geometric genus \(p_g\). Then we have

\[
\text{Ind}_d(F_0) = \mu - \left(6 + \frac{6}{d-2}\right)p_g - \chi_{\text{top}}(A) + 1 + \epsilon,
\]

where \(\chi_{\text{top}}(A)\) is the topological Euler number of the exceptional set \(A\) of the minimal resolution of \((X, x)\) and \(\epsilon\) is the number of blow-ups in the minimal desingularization of indeterminacy of the rational map \(S \to X\).

Proof. Let \((X, x)\) be an isolated hypersurface singularity associated to a fiber germ \((f: S \to \Delta, F_0)\) of plane curves of degree \(d\). Let \(\pi: \tilde{S} \to X\) be the minimal desingularization of indeterminacy of the rational map \(S \to X\), which is nothing but the minimal resolution of \((X, x)\). Taking algebraization of the fiber germ \(f: S \to \Delta\) in the sense of Lemma 4.2 in \([2]\), we may assume that \(\Delta = \mathbb{P}^1\). Let \(\tilde{f}: \tilde{S} \to \mathbb{P}^1\) and \(f: X \to \mathbb{P}^1\) denote the natural fibrations. Let \(K\) be the canonical cycle of the minimal resolution of \((X, x)\). Then we have

\[
p_g = \chi_f - \chi_{\tilde{f}} = \chi_f - \chi_f, \quad -K^2 = K_f^2 - K_{\tilde{f}}^2 = K_f^2 - K_{\tilde{f}}^2 + \epsilon.
\]

On the other hand, we have

\[
K_f^2 = \frac{6(d-3)}{d-2}\chi_f + \text{Ind}_d(F_0), \quad K_{\tilde{f}}^2 = \frac{6(d-3)}{d-2}\chi_{\tilde{f}},
\]

where the latter is obtained by a computation similar to that in the proof of Proposition 3.4 in \([2]\). Thus we get

\[
\text{Ind}_d(F_0) = (K_f^2 - K_{\tilde{f}}^2) - \frac{6(d-3)}{d-2}(\chi_f - \chi_{\tilde{f}}) = K^2 + \epsilon + \frac{6(d-3)}{d-2}p_g.
\]

Combining it with Laufer’s formula \(\mu = 12p_g + K^2 + \chi_{\text{top}}(A) - 1\) \([4]\), the desired equality holds. \(\Box\)

Lemma 1.4. Any isolated hypersurface surface singularity is holomorphically equivalent to some isolated hypersurface singularity \((X, x)\) associated to a fiber germ \((f: S \to \Delta, F_0)\) of plane curves with the birational morphism \(S = \tilde{S} \to X\) (i.e., \(\epsilon = 0\)).

Proof. Let \((X, 0), X = \{h(y, z_1, z_2) = 0\} \subset \mathbb{C}^3\) be any isolated hypersurface surface singularity. We may assume that the defining equation \(h(y, z_1, z_2)\) is a polynomial. Taking compactification \(\overline{X}\) of \(X\) in \(\mathbb{P}^1 \times \mathbb{P}^2\), the defining equation of \(\overline{X}\) can be written by the homogenization \(\overline{h}(Y_0, Y_1; Z_0, Z_1, Z_2)\) of \(h(y, z_1, z_2)\). Adding sufficiently higher terms to \(h(y, z_1, z_2)\), we may assume that \(0 \in X \subset \overline{X}\) is the unique singularity of \(\overline{X}\) and the central fiber \(\overline{X} \cap (0 \times \mathbb{P}^2) = \{\overline{h}(1, 0; Z_0, Z_1, Z_2) = 0\}\) is irreducible and non-rational. Thus the composite \(\overline{f} = p \circ \pi: \overline{S} \to \mathbb{P}^1\) of the minimal resolution \(\pi: \overline{S} \to \overline{X}\) and the projection \(p: \overline{X} \to \mathbb{P}^1\) is a relatively minimal plane curve fibration. Taking the fiber germ \(f: S \to \Delta\) of \(\overline{f}\) at the origin 0, the assertion follows. \(\Box\)
Proof of Theorem 0.1. Let \((X, x)\) be an isolated hypersurface surface singularity with Milnor number \(\mu\) and geometric genus \(p_g > 0\). Note that \((X, x)\) is not a rational double point. From Lemma 1.4, we may assume that \((X, x)\) is an isolated hypersurface singularity associated to a fiber germ \((f, F_0)\) of plane curves of degree \(d\) with \(\epsilon = 0\). From Lemma 1.3 and the positivity of the Horikawa index, we have

\[
\mu - \left(6 + \frac{6}{d-2}\right) p_g - \chi_{\text{top}}(A) + 1 = \text{Ind}_d(F_0) > 0.
\]

Thus we have

\[
\mu - 6p_g - \chi_{\text{top}}(A) > \frac{6}{d-2} p_g - 1 > -1.
\]

Since the left hand side of the above inequality is an integer, we get

\[
\mu - 6p_g - \chi_{\text{top}}(A) \geq 0.
\]

\[\square\]

References

[1] A.H. Durfee, The signature of smoothings of complex surface singularities, Math. Ann. 232 (1978), no.1, 85–98.
[2] M. Enokizono, Slopes and local invariants of surfaces fibered by plane curves, preprint.
[3] J. Kollár and A. Némethi, Durfee's conjecture on the signature of smoothings of surface singularities, to appear in Annales Sc. de l'Ecole Norm. Sup.
[4] H. Laufer, On \(\mu\) for surface singularities, in Proc. Symposia in Pure Math. 30 (1977), 45–49.

Makoto Enokizono, Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: m-enokizono@cr.math.sci.osaka-u.ac.jp