Classification of affine normal $SL_2$-varieties with a dense orbit

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Abstract

In this note we give a classification of all affine normal $SL_2$-varieties containing an open dense orbit over an algebraically closed field of characteristic zero. Such a classification was first obtained by Popov in [Pop73]. Here we provide an alternative approach.

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1 Introduction

In this note we give a classification of affine normal $SL_2$-varieties with a dense open orbit over an algebraically closed field of characteristic zero. Our proof is based on elementary representation theory and the Hilbert-Mumford criterion.

An affine algebraic normal $SL_2$-variety $X$ with a dense orbit is determined by a subalgebra $k[X]$ of the coordinate ring $k[SL_2]$ that is stable under the $SL_2$ action. This, in turn, is determined by an “admissible” subalgebra $k[X]^U$ of $k[SL_2]^U$ of invariants under the lower-triangular unipotent subgroup $U$ of $SL_2$ (see Definition 3.1).

The question is then to determine which subalgebras are admissible. In section 3 we study the multiplicative structure of the left regular representation $k[SL_2]$.
and give a criterion for admissibility (Proposition 3.6). This criterion is a central ingredient in the proofs of the results, and it has a straightforward graphical interpretation inside the lattice of monomials in $k[SL_2]^U$.

If $k[X]^U$ has dimension 1, then $X$ is spherical (Proposition 5.2). In section 4 we use the Hilbert-Mumford criterion (Lemma 4.4) and the admissibility criterion to obtain the classification in the case when $X$ is spherical (Theorem 4.10).

In section 5 we proceed to the case when $k[X]^U$ has dimension 2. In this case we use the admissibility criterion to show that $X$ admits an extra torus action induced by the right regular representation on $k[SL_2]$ (Lemma 5.4). This allows us to prove the main theorem of this paper:

**Theorem 5.10** Let $X$ be an affine normal $SL_2$-variety with an open dense orbit. Then $X$ is $SL_2$-isomorphic to exactly one of the following

1. A homogenous space $SL_2/H$, where $H$ is as in Proposition 5.1.
2. (See Definition 4.8) The spherical variety $SL_2/(\mu_f \ltimes \overline{U}) = \text{Spec} \left( R^{(f)}_{\infty} \right)$ for a unique positive integer $f$.
3. (See Definition 5.7) $\text{Spec} \left( R^{(f)}_q \right)$ for a unique positive integer $f$ and a unique rational number $q \geq 1$. In this case the stabilizer of the dense orbit is $\mu_f$.

This classification is in fact complete in the case where $X$ has dimension 2 without the assumption of normality. However, when $X$ has dimension 3 it is not. See [Pop73, §4] for examples.

The results presented here are known, and were first proved by Popov in [Pop73]. Popov considers an equivariant linear embedding $X \subset V$ ([Pop73, Theorem 2]), and then proceeds to deduce intrinsic invariants of $X$ from invariants of the embedding ([Pop73, Theorem 4]). See also Kraft’s book [Kra85, III.4] for a shorter proof in the spirit of Popov’s argument. Another approach to this classification can be found in the work of Luna and Vust [LV83, §9], which uses their more general theory of spherical embeddings.

Our approach, in contrast to the previous ones, is based on the combinatorics of the monomials of $k[X]^U$ under the multiplicative structure of the left regular representation. This lends itself to a straightforward graphical interpretation of the results, and of the classifying invariants $(q, f)$ inside the lattice of monomials in $k[SL_2]^U$. Even though the results are known, we hope that someone will find this new presentation to be useful.

See the work of Batyrev and Haddad [BH08] for applications of Theorem 5.10 above and further study of these varieties. In fact [BH08] gives an explicit categorical quotient description of the 3-dimensional $SL_2$-varieties appearing in the classification. This is used to compute their Cox rings and to construct $SL_2$-equivariant flips as GIT quotients.

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2 Notation

Let $k$ be an algebraically closed field of characteristic 0. All schemes will be assumed to be defined over $k$. In this note we study varieties $X$ equipped with a left action of the algebraic group $SL_2$. We will abbreviate this by saying that $X$ is a $SL_2$-variety.

We denote by $k[SL_2]$ the coordinate ring of the affine algebraic group $SL_2$. We think of a generic matrix in $SL_2$ as having entries $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we have $k[SL_2] = k[a, b, c, d]/(ad - bc - 1)$. The $k$-vector space $k[SL_2]$ acquires the structure of a $SL_2 \times SL_2$ representation. The first copy of $SL_2$ acts via the left regular representation, while the second copy of $SL_2$ acts via the right regular representation. We will employ the superscript $(-)^{op}$ whenever we are thinking of the second copy of $SL_2$. For example a $SL_2^{op}$-subrepresentation will be a subspace that is stable with respect to the right regular action.

We will denote by $B$ the Borel subgroup in $SL_2$ consisting of lower triangular matrices. We let $T$ denote the maximal torus $T \subset B$ defined by $T = \{ \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \} \cong \mathbb{G}_m$.

We will write $U$ for the unipotent radical of $B$.

3 Affine pointed $SL_2$-varieties with a dense orbit

In this section we describe a correspondence between affine normal pointed $SL_2$-varieties with a dense orbit and certain class of subalgebras of the polynomial ring $k[a, b]$. We call this class of subalgebras “admissible”.

Let $X$ be an affine normal $SL_2$-variety. Let $R$ denote the coordinate ring of $X$. Choose $p \in X(k)$. We can define a map $\phi : SL_2 \to X$ given by $\phi(g) = g \cdot p$. Suppose further that the $SL_2$-orbit of $p$ is dense in $X$. This means that $\phi$ is dominant, which implies that we can view $R$ as a subalgebra of $k[SL_2]$.

We can therefore understand such pointed $SL_2$-varieties once we determine the (left) $SL_2$-stable, normal, finitely generated subalgebras of $k[SL_2]$. By highest weight theory, these are determined by their $U$-invariants.

Note that the $U$-invariants of the left regular representation are $k[SL_2]^U = k[a, b]$. Here the torus $T$ acts with weight 1 on both $a$ and $b$. This motivates the following definition.

Definition 3.1. Let $A$ be a subalgebra of $k[a,b]$. We say that $A$ is admissible if

(a) $A$ is homogeneous for the grading by total degree (i.e. it is $T$-stable).

(b) $A$ is normal and finitely generated.

(c) Let $V_A$ denote the $SL_2$-subrepresentation of $k[SL_2]$ generated by $A$. Then $V_A$ is closed under multiplication (i.e. it is a subalgebra of $k[SL_2]$).
Now we state the promised correspondence.

**Lemma 3.2.** The map \( A \mapsto V_A \) sets up a bijection between admissible subalgebras and \( SL_2 \)-stable, normal, finitely generated \( k \)-subalgebras of the coordinate ring \( k[SL_2] \).

**Proof.** Let’s define an inverse map. For any \( SL_2 \)-stable subalgebra \( R \) of \( k[SL_2] \), we define \( A_R := R^T = R \cap k[a, b] \). This is a subalgebra of \( k[a, b] \). It is \( T \)-stable because \( R \) is \( T \)-stable. By highest weight theory, any representation of \( SL_2 \) is generated by its subspace of \( U \)-invariants. So we get \( V_{A_R} = R \).

This establishes a bijection between \( SL_2 \)-stable subalgebras of \( k[SL_2] \) and subalgebras of \( k[a, b] \) satisfying conditions (a) and (c) in Definition 3.1. In order to conclude the proof of the lemma, it suffices to show that \( R \) is finitely generated and normal if and only if \( A_R \) is finitely generated and normal.

Suppose that \( R \) is finitely generated. Then \( R^T \) is finitely generated by [Per][Thm. 4.1.12]. Conversely, suppose that \( A_R \) is finitely generated. Let \( \{v_i\}_{i \in I} \) be a finite set of generators of \( A_R \). Let \( W_{\{v_i\}} \) denote the finite dimensional \( SL_2 \)-submodule of \( k[SL_2] \) generated by the \( v_i \). Then we have that the algebra \( k[W_{\{v_i\}}] \) generated by \( W_{\{v_i\}} \) is a left \( SL_2 \)-stable subalgebra of \( R \). By assumption on the set \( \{v_i\} \), we know that \( k[W_{\{v_i\}}] \) contains \( A_R \). Therefore, we must have \( k[W_{\{v_i\}}] = R \) by highest weight theory. So \( R \) is finitely generated.

The fact that \( R \) is normal if and only if \( A_R \) is normal follows from the argument in [Per][Prop. 4.1.16 (ii)].

The rest of this section is dedicated to proving Proposition 3.6. This is a criterion for determining when condition (c) in Definition 3.1 is satisfied.

We will need to understand the multiplicative structure of the left regular representation \( k[SL_2] \). Consider the coordinate ring \( W = k[a, b, c, d] \) of the vector space of \( 2 \times 2 \) matrices. \( SL_2 \) acts on this ring by precomposing with the inverse of multiplication on the left. We have \( W = \bigoplus_{i,j \in \mathbb{N}} W^{i,j} \) where

\[
W^{i,j} = \text{Sym}^i (ka \oplus kc) \otimes \text{Sym}^j (kb \oplus kd)
\]

By representation theory of \( SL_2 \), we know that \( W^{(i,j)} \) breaks up into a direct sum of irreducibles

\[
W^{i,j} = \bigoplus_{s=0}^{\min(i,j)} L^{i,j}_{i+j-2s}
\]

Here \( L^{i,j}_{i+j-2s} \) has highest weight \( i+j-2s \) with respect to the Borel \( B \). It is apparent that \( L^{i,j}_{i+j} \) has highest weight vector \( a^ib^j \). In order to determine the highest weight vectors for the other summands, we will do a trick with the determinant. Note that multiplication by the invariant polynomial \( ad - bc \) induces an injective map \( W^{i,j} \hookrightarrow W^{i+1,j+1} \). By the decomposition above, this must induce isomorphisms \( L^{i,j}_{i+j-2s} \cong L^{i+1,j+1}_{i+j-2s} \). We conclude by using this map that \( L^{i,j}_{i+j-2s} \) has highest weight vector \( (ad - bc)^s a^{i-s}b^{j-s} \).

Thanks to the weight vectors obtained just above, we can describe explicitly the highest weight vectors of a tensor product of abstract irreducible representations of \( SL_2 \). But before we do that we need some notation.
Definition 3.3. Let $V$ be a representation of $SL_2$ and let $v \in V$ be nonzero weight vector with weight $n$. Then for every $e \in \mathbb{N}$, we define $\langle -e \rangle v$ to be the unique vectors determined by the equation

$$
\begin{bmatrix}
1 & -x \\
0 & 1
\end{bmatrix} \cdot v = \sum_{e \in \mathbb{N}} \left( \begin{array}{c}
n \\
e
\end{array} \right) x^e \langle -e \rangle v
$$

In other words, $\langle -d \rangle v$ is the component of weight $n - 2e$ in the vector

$$
\begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix} \cdot \left( \left( \begin{array}{c}
n \\
e
\end{array} \right)^{-1} v \right)
$$

Proposition 3.4. Let $V^1$, $V^2$ be two irreducible representations of $SL_2$ with highest weight $n_1$ and $n_2$ respectively. Let $v^1$ and $v^2$ be highest weight vectors of $V^1$ and $V^2$ respectively. The tensor product breaks up as a direct sum of irreducible representations

$$V^1 \otimes V^2 = \bigoplus_{s=0}^{\text{min}(n_1,n_2)} V_{n_1+n_2-2s}$$

where $V_{n_1+n_2-2s}$ is an irreducible representation of highest weight $n_1 + n_2 - 2s$ and highest weight vector

$$v_s = \sum_{e=0}^{s} (-1)^e \left( \begin{array}{c}
s \\
e
\end{array} \right) \langle -e \rangle v_1 \otimes \langle e \rangle v_2$$

Proof. It suffices to prove this theorem for a specific realization of the irreducible representations $V^1$ and $V^2$. For example, we can use $V^1 \cong \text{Sym}^{n_1} (ka \oplus kc)$ and $V^2 \cong \text{Sym}^{n_2} (kb \oplus kd)$. We can take $v^1 = a^{n_1}$ and $v^2 = b^{n_2}$. Now one can check that the vector $v_s$ described in this proposition reduces to $(ad - bc)^s a^{n_1-s} b^{n_2-s}$. We have seen in our discussion above that this is indeed the required highest weight vector.

Corollary 3.5. Let $p_1 = \sum_{i=0}^{n_1} z_{1,i} a^i b^{n_1-i}$ be a homogeneous polynomial of total degree $n_1$ in $k[a,b]$. Similarly, let $p_2 = \sum_{j=0}^{n_2} z_{2,j} a^j b^{n_2-j}$ be a homogeneous polynomial of degree $n_2$. Let $V^{p_1}$ (resp. $V^{p_2}$) be the irreducible subrepresentation of $W = k[a,b,c,d]$ with highest weight vector $p_1$ (resp. $p_2$). Then the tensor product decomposes into a direct sum

$$V^{p_1} \otimes V^{p_2} = \bigoplus_{s=0}^{\text{min}(n_2,n_2)} V_{n_1+n_2-2s}^{p_1+p_2}$$

where $V_{n_1+n_2-2s}^{p_1+p_2}$ is an irreducible representation of highest weight $n_1 + n_2 - 2s$.

The image of a highest weight vector $v^{p_1,p_2}_s$ of $V_{n_1+n_2-2s}^{p_1+p_2}$ under the multiplication map $m : V^{p_1} \otimes V^{p_2} \rightarrow W$ is

$$m(v^{p_1,p_2}_s) = \sum_{\alpha=s}^{n_1+n_2-s} g^{p_1,p_2}_{\alpha,s} (ad - bc)^s \cdot a^{\alpha-s} \cdot b^{n_1+n_2-\alpha-s}$$

where $g^{p_1,p_2}_{\alpha,s}$ is given by

$$g^{p_1,p_2}_{\alpha,s} = \sum_{i+j=\alpha} \sum_{e=0}^{s} (-1)^e \left( \begin{array}{c}
s \\
e
\end{array} \right) \left( \begin{array}{c}
n_1 - i \\
e
\end{array} \right) \left( \begin{array}{c}
n_2 - j \\
e
\end{array} \right) z_{1,i} z_{2,j}$$
Proof. We know that $m(v^{p_1,p_2})$ is a linear combination of highest weight vectors in $W$, because multiplication is $SL_2$-equivariant. We have seen that the highest weight vectors in $W$ of weight $n_1 + n_2 - 2s$ and total degree $n_1 + n_2$ in the variables $a, b, c, d$ are of the form $(ad - bc)^s \cdot a^{n_1 + n_2 - s}$ for some $s \leq \alpha \leq n_1 + n_2 - s$. So we only need to determine the coefficients $y^{p_1,p_2}_{s,\alpha}$. This can be done by using the formula in Proposition 3.4, expanding with the binomial theorem and looking at the coefficient of $a^\alpha \cdot d^s \cdot b^{n_1 + n_2 - \alpha - s}$.

We conclude this section with a useful criterion to determine if a subalgebra of $k[a, b]$ is admissible.

**Proposition 3.6.** Let $A \subset k[a, b]$ be a normal, finitely generated homogeneous subalgebra. Then $A$ is admissible if and only if for all pairs of homogeneous polynomials $p_1 = \sum_{i=0}^{n_1} z_{1,i} a^i b^{n_1 - i}$ and $p_2 = \sum_{j=0}^{n_2} z_{2,j} a^j b^{n_2 - j}$ in $A$, all of the polynomials $w^{p_1,p_2}_s$ defined for $0 \leq s \leq \min(n_1, n_2)$ by

$$w^{p_1,p_2}_s = \sum_{\alpha = s}^{n_1 + n_2 - s} y^{p_1,p_2}_{s,\alpha} a^{\alpha - s} b^{n_1 + n_2 - \alpha - s}$$

with

$$y^{p_1,p_2}_{s,\alpha} = \sum_{i+j=\alpha}^{s} \sum_{e=0}^{i} (-1)^e \binom{s}{e} \binom{n_1}{n_1 - i} \binom{n_2}{n_2 - j} \binom{e}{s - e} z_{1,i} z_{2,j}$$

also belong to the subalgebra $A$.

Proof. Let $V_A$ denote the $SL_2$ subrepresentation of $k[SL_2]$ generated by $A$. We can see that $V_A$ is closed under multiplication if and only if for all pairs of homogeneous polynomials $(p_1, p_2)$ we have that all of the highest weight vectors of the product $V^{p_1} \cdot V^{p_2}$ are also in $A$. By Proposition 3.5 these highest weight vectors are just the polynomials $w^{p_1,p_2}_s$ (recall that $ad - bc = 1$ in $k[SL_2]$). 

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**Figure 1:** Admissibility criterion for two homogeneous polynomials $p_1 = ab^2 + b^3$, $p_2 = a^2b^2 + ab^3 + b^4$. 

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This criterion has a simple graphical interpretation. Give two polynomials \( p_1 \) and \( p_2 \) in \( A \), we may consider their Newton polygons (i.e. the convex hull of points \( (u, v) \in \mathbb{N}^2 \) corresponding to monomials \( a^u b^v \) in their support) and draw a diagram as in Figure 1 above.

As long as the coefficients \( y_{p_1,p_2}^{\alpha,s} \) are nonzero, we can deduce that some elements of convenient form must be present in the algebra \( A \) starting from some known elements \( p_1, p_2 \) (for example, \( w_{3}^{p_1,p_2} = y_{3,3}^{p_1,p_2} b \) in the image).

4 Classification of affine spherical \( SL_2 \)-varieties

For this section, we will put an additional assumption on our affine pointed \( SL_2 \)-variety \( (X, p) \). Recall that we can use the point \( p \) to view \( X \) as \( \text{Spec}(R) \) for some \( SL_2 \)-stable subalgebra \( R \subset k[SL_2] \).

Lemma 4.1. Let \( X = \text{Spec}(R) \) be an affine normal pointed variety as above. Then the following statements are equivalent.

1. The orbit \( \overline{B} \cdot p \) is dense.
2. Every \( T \)-weight space of the algebra of invariants \( R^T = R \cap k[a,b] \) is one-dimensional.

Proof. See [Per] [Lemma 4.2.2].

As a consequence we get the following proposition.

Proposition 4.2. The correspondence \( A \mapsto V_A \) defines a bijection between the following two sets

(a) Admissible subalgebras \( A \) such that for all \( n \in \mathbb{N} \) the \( k \)-space of homogeneous polynomials of degree \( n \) in \( A \) has dimension at most 1.

(b) Affine normal pointed \( SL_2 \)-varieties \( X = \text{Spec}(V_A) \) such that \( \overline{B} \cdot p \) is dense.

Proof. This is an immediate consequence of Lemma 3.2 and Lemma 4.1.

Definition 4.3. An affine normal \( SL_2 \)-variety \( X \) is called spherical if it has a \( \overline{B} \)-orbit that is dense.

In order to classify spherical varieties, we will need a couple of lemmas. Recall that a one-parameter subgroup of \( SL_2 \) is a homomorphism of algebraic groups \( \gamma : \mathbb{G}_m \to SL_2 \). Let \( \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \) be the usual open immersion. Given a morphism \( \phi : \mathbb{G}_m \to X \) we say that \( \lim_{t \to 0} \phi(t) \) exists if the map \( \phi \) extends to a morphism \( \tilde{\phi} : \mathbb{A}^1 \to X \). If this is the case, we will use the notation \( \lim_{t \to 0} \phi(t) = \tilde{\phi}(0) \).

Lemma 4.4. Let \( X \) be an affine \( SL_2 \)-variety and let \( p \in X(k) \). Suppose that the orbit \( SL_2 \cdot p \) is open dense in \( X \). For any closed point \( x \in X(k) \setminus (SL_2(k) \cdot p) \), there exists a nonconstant one-parameter subgroup \( \gamma : \mathbb{G}_m \to SL_2 \) such that \( \lim_{t \to 0} \gamma(t) \cdot p \) is contained in the closure of the orbit \( SL_2 \cdot x \).
Proof. This is a version of the Hilbert-Mumford criterion. It follows from a modification of the argument in [MFK94][pg.53-54] after embedding $X$ into the total space of a linear representation of $SL_2$. \hfill \Box

Let $\gamma: \mathbb{G}_m \to SL_2$ be a nontrivial one-parameter subgroup. Recall that there is a Borel subgroup $B_\gamma$ associated to $\gamma$. It is defined by

$$B_\gamma = \{ g \in SL_2 \mid \lim_{t \to 0} \gamma(t) g \gamma(t)^{-1} \text{ exists in } SL_2 \}$$

The image $\text{Im}\gamma$ is a maximal torus inside $B_\gamma$. The dominant weights of $B_\gamma$ are the characters of $\text{Im}\gamma$ that pullback to a nonnegative weight of $\mathbb{G}_m$ under the homomorphism $\gamma$.

For the next lemma, we let $(X, p)$ be an affine pointed variety such that $SL_2 \cdot p$ is dense. As usual, we view the coordinate ring $R$ as a $SL_2$-stable subalgebra of $k[SL_2]$. Recall that we use the supercript $(-)^{op}$ to denote the right regular action.

**Lemma 4.5.** Let $\gamma: \mathbb{G}_m \to SL_2$ be a non-constant one-parameter subgroup. Let $B_\gamma$ denote the corresponding Borel subgroup. Then the following statements are equivalent

1. The limit $\lim_{t \to 0} \gamma(t) \cdot p$ exists in $X$
2. The subalgebra $R \subset k[SL_2]$ is contained in the sum of those $T^{op}$-weight spaces of $k[SL_2]$ that are dominant for the Borel $B_\gamma^{op}$.

**Proof.** Consider the composition

$$\varphi: SL_2 \times \mathbb{G}_m \xrightarrow{m} SL_2 \longrightarrow X$$

$$(g, t) \mapsto gt \mapsto g\gamma(t) \cdot p$$

At the level of coordinate rings, we have

$$\varphi^*: R \hookrightarrow k[SL_2] \xrightarrow{m^*} k[SL_2] \otimes k[T, T^{-1}]$$

Notice that $\mathbb{G}_m$ acts by multiplication on the right on $k[SL_2]$ via the homomorphism $\gamma$. The corresponding comodule map is $m^*$ above. If $f \in R$, we can break $f$ into a sum of weight vectors $f = \sum_{n \in \mathbb{Z}} f_n$ with $f_n \in k[SL_2]$ having weight $n$. By definition we have

$$\varphi^*(f) = \sum_{n \in \mathbb{Z}} f_n \otimes T^n$$

Assume that $\lim_{t \to 0} \gamma(t) \cdot p$ exists. By definition, this means that the map $\varphi|_{\{id\} \times \mathbb{G}_m}: \mathbb{G}_m \to X$ extends to a map $\tilde{\varphi}_0: \mathbb{A}^1 \to X$. Therefore, we get a $G$-equivariant map

$$\tilde{\varphi}: G \times \mathbb{A}^1 \longrightarrow X$$

$$(g, t) \mapsto g \cdot \tilde{\varphi}_0(t)$$

By construction this map agrees with $\varphi$ on $G \times \mathbb{G}_m$. We conclude that the map $\varphi^*: R \to k[SL_2] \otimes k[T, T^{-1}]$ factors through the subring $k[SL_2] \otimes k[T] \subset$
\( k[SL_2] \otimes k[T, T^{-1}] \). This means that for all \( f \in R \) we have \( f_n = 0 \) for \( n < 0 \). In other words, \( f \) belongs to the sum of weight spaces of the right regular representation \( k[SL_2] \) that are dominant for \( B_\gamma \).

Conversely, suppose that for any \( f \in R \) we have \( f_n = 0 \) for all \( n < 0 \). This implies that the morphism \( \varphi : G \times \mathbb{G}_m \rightarrow X \) extends to a morphism \( \overline{\varphi} : G \times \mathbb{A}^1 \rightarrow X \). We can restrict \( \overline{\varphi} \) to \( \{id\} \times \mathbb{A}^1 \) in order to deduce that the limit \( \lim_{t \to 0} \gamma(t) \cdot p \) exists.

The following is the main step in classifying affine spherical \( SL_2 \)-varieties.

**Proposition 4.6.** Let \( X \) be an affine spherical \( SL_2 \)-variety. Then \( X \) is \( SL_2 \)-isomorphic to one of the following

(a) A homogeneous space \( G/H \) for some closed algebraic subgroup \( H \subset G \).

(b) \( \text{Spec} (V_{k[b]}) \) for some positive integer \( f \).

(c) \( \text{Spec} (V_{k[ab]}) \).

(d) \( \text{Spec} (V_{k[(ab)^2]}) \).

**Proof.** Choose a point \( p \) in the dense \( \overline{B} \)-orbit. This allows us to view \( X \) as \( \text{Spec}(R) \) for some \( SL_2 \)-stable subalgebra \( R \subset k[SL_2] \).

Assume that \( X \) is not a homogeneous space. By the Hilbert-Mumford criterion (Lemma 4.3) there exists a nontrivial one-parameter subgroup \( \gamma : \mathbb{G}_m \rightarrow SL_2 \) such that \( \lim_{t \to 0} \gamma(t) \cdot p \) exists. Let \( B_\gamma \) be the corresponding Borel subgroup. Notice that right multiplying the algebra \( R \) by an element \( g \in SL_2(k) \) establishes an \( SL_2 \)-isomorphism \( \text{Spec}(R) \overset{\sim}{\rightarrow} \text{Spec}(R \cdot g) \). By Lemma 4.5 we have that \( \lim_{t \to 0} g^{-1} \gamma(t) g \cdot p \) exists in \( \text{Spec}(R \cdot g) \). The corresponding Borel subgroup will be \( g^{-1}B_\gamma g \). By choosing an appropriate element \( g \in SL_2(k) \) we can assume without loss of generality that \( B_\gamma = \overline{B} \).

Let \( A = R \cap k[a, b] \) be the associated admissible subalgebra of \( k[a, b] \). If \( A \) is the field of constants \( k \) then we have \( R = k \). So the corresponding variety \( X = \text{Spec}(k) \) is the homogeneous space \( SL_2/SL_2 \), which is not the case by assumption. Therefore \( A \) contains some homogeneous polynomial with positive degree in \( k[a, b] \).

Let \( p \) denote a homogeneous polynomial of smallest positive degree \( n \) in \( A \). We claim that \( A = k[p] \). Indeed let \( S \) denote the set of homogeneous polynomials in \( A \). It follows from the multiplicity condition (2) in Lemma 4.1 that all the weight spaces of the localization \( A[S^{-1}] \) have dimension 1. Let \( \mathbb{Z} \subset \mathbb{Z} \) denote the subgroup of weights of \( A[S^{-1}] \), where \( l \) is a positive integer. If \( l = n \) then multiplicity one implies that \( A[S^{-1}] = k[p^{\pm 1}] \). Therefore \( A = k[p] \). If \( l \neq n \) then \( l \) is a proper divisor of \( n \). Let \( p' \) be an element of \( A[S^{-1}] \) of weight \( l \). By multiplicity one for the weight spaces, we must have \( (p')^\ast \in k[p] \). Since \( A \) is normal, this implies \( p' \in A \). So we get a contradiction to the minimality of the degree \( n \). This concludes the proof of the claim \( A = k[p] \).

Suppose that \( p = \sum_{i=0}^{n} z_i a^i b^{n-i} \). By Lemma 4.5 \( p \) must be contained in a sum of \( \overline{B}^{op} \)-dominant weight spaces. Concretely, this means that \( z_i = 0 \) whenever
We have $n - i < i$. Let us denote by $m$ the largest index such that $z_m \neq 0$. We must have $m \leq \lfloor \frac{n}{2} \rfloor$.

Let’s apply the criterion in Proposition 3.6 to the pair $(p, p)$. For each $0 \leq s \leq n$, we want to determine whether the polynomial $w_{s}^{p,p}$ is in the subalgebra $A = k[p]$. Recall that

$$w_{s}^{p,p} = \sum_{\alpha = s}^{2n-s} y_{\alpha,s}^{p,p} \alpha^{-s} \cdot b^{\alpha + n_2 - \alpha - s}$$

with

$$y_{\alpha,s}^{p,p} = \sum_{i+j=\alpha}^{s} \sum_{e=0}^{n} (-1)^{e} \binom{n}{e} \binom{n-i}{s-e} \binom{n-j}{s-e} z_i z_j$$

Note that $y_{\alpha,s}^{p,p} = 0$ for all $\alpha > 2m$. This is because $z_i = 0$ whenever $i > m$. This means that for $s = 2m$ we have a single term

$$w_{2m}^{p,p} = y_{2m,2m}^{p,p} b^{2n-4m}$$

with

$$y_{2m,2m}^{p,p} = 2 z_m^2 \sum_{e=0}^{2m} (-1)^e \binom{2m}{e} \binom{n}{e} \binom{n-m}{2m-e} \binom{n-m}{2m-e} (n-m) (2m-e)$$

We have $z_m \neq 0$ by assumption. Lemma A.1 (with $i = m$ and $j = n - m$) implies that $y_{2m,2m}^{p,p} \neq 0$. This means that $b^{2n-4m}$ must be a constant multiple of a power of $p$. We have four possible cases.

(C1) $n \neq 2m$. Then $b^{2n-4m}$ is not a scalar. This means that $p$ can be taken to be a power of $b$. So $A = k[b^f]$ for some positive integer $f$. Therefore $X$ satisfies (b) in the proposition.

(C2) $n = 2$ and $m = 1$. Then up to multiplying $p$ by a scalar, we can write $p = ab + zb^2$ for some $z \in k$. Right multiplication by the element $\begin{bmatrix} 1 & 0 \\ -z & 1 \end{bmatrix}$ establishes a $SL_2$-isomorphism between $k[ab - zb^2]$ and the algebra $k[ab]$. This means that $X$ satisfies (c) in the proposition.

(C3) $n = 4$ and $m = 2$. Then $p$ can be chosen to be of the form $p = a^2 b^2 + z_1 ab^3 + z_0 b^4$. We can right multiply by $\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} z_1 & 1 \end{bmatrix}$ in order reduce to $z_1 = 0$. So $A = k[p] = k[a^2 b^2 + z_0 b^4]$. We claim that this algebra is admissible only if $z_0 = 0$. Indeed, the vector $w_2^{p,p}$ has a single nonzero summand in this case

$$w_2^{p,p} = y_{2,2}^{p,p} a^2 b^2$$

with

$$y_{2,2}^{p,p} = 2 \sum_{e=0}^{2} (-1)^e \binom{2}{e} \binom{4}{2} (2 \cdot 2 - e) z_0 + \sum_{e=0}^{2} (-1)^e \binom{4}{e} \binom{4}{2} (2 \cdot 2 - e) z_0$$

This simplifies to

$$y_{2,2}^{p,p} = \frac{1}{3} z_0$$

So we have $w_2^{p,p} = \frac{1}{3} z_0 a^2 b^2$. This can’t be in $A = k[p]$ unless $z_0 = 0$. We conclude that $A = k[(ab)^2]$. Therefore $X$ satisfies (d) in the proposition.
(C4) \( n = 2m \) and \( m > 2 \). We will show that this case can’t actually happen.

We can write \( p = z_m(ab)^m + \sum_{i=0}^{m-1} z_i a^i b^{2m-1} \) with \( z_m \neq 0 \). Let’s look at the highest weight vector \( w_{2m,2}^{p,p} \). Notice that the degree of \( w_{2m,2}^{p,p} \) is \( 4m - 4 \). Since \( m > 2 \), the only way \( w_{2m,2}^{p,p} \) can be a multiple of a power of \( p \) is if \( w_{2m,2}^{p,p} = 0 \) (by degree count). This means in particular that \( y_{2m,2}^{p,p} \) must be 0. Since \( z_i = 0 \) for \( i > m \), the expression for \( y_{2m,2}^{p,p} \) simplifies to

\[
y_{2m,2}^{p,p} = \sum_{e=0}^{2} (-1)^e \frac{(2m)}{e} \left( \frac{m}{e} \right) \left( \frac{m}{2} \right)^2 (2m)^2 - e \cdot \frac{z_m^2}{2m - 1}
\]

By Lemma 4.5 with \( i = 0 \), we have

\[
y_{2m,2}^{p,p} = z_m^2 \cdot \frac{-1}{2m - 1}
\]

So we see that \( y_{2m,2}^{p,p} = 0 \) implies that \( z_m = 0 \), a contradiction.

The following proposition shows that cases \((b), (c)\) and \((d)\) can actually arise.

**Proposition 4.7.** For any positive integer \( f \), the algebra \( k[bf] \) is admissible. The algebras \( k[(ab)] \) and \( k(ab)^2 \) are also admissible.

**Proof.** Set \( R_f = k[bf, dbf^{-1}, d^2bf^{-2}, ..., df] \) inside \( k[SL_2] \). Note that \( R_f \) is \( SL_2 \)-stable. We have \( k[bf] = R_f \cap k[a,b] \). This shows that \( V_{k[bf]} = R_f \) in part \((b)\) of the previous proposition.

It can be checked that we have \( k[ab] = k[ab, ad, bc, cd] \cap k[a,b] \). So in part \((c)\) we have \( V_{k[ab]} = k[ab, ad, bc, cd] \).

Finally, one can check in a similar way that \( V_{k[(ab)^2]} \) is the subalgebra \( k[a^2b^2, ab(ad + bc), cd(ad + bc), c^2d^2] \) inside \( k[SL_2] \).
Definition 4.8. Let $f$ be a positive integer. We define $R_\infty^{(f)}$ to be the $SL_2$-stable subalgebra of $k[SL_2]$ generated by $k[b^f]$. More concretely,

$$R_\infty^{(f)} = k[b^f, db^f-1, d^2b^f-2, ... , d^f]$$

Note that $R_\infty^{(f)}$ is the subalgebra of $(\mu_f \ltimes \overline{U})^{op}$-invariants $k[SL_2]^{(\mu_f \ltimes \overline{U})^{op}}$. Here $\mu_f \ltimes \overline{U} \subset T \ltimes \overline{U} = \overline{B}$.

Now it remains to classify the affine spherical homogeneous spaces.

**Proposition 4.9.** Up to isomorphism, the affine spherical homogeneous spaces for $SL_2$ are

(a) $SL_2/SL_2 = \text{Spec}(k)$.

(b) $SL_2/T$. This is isomorphic to $\text{Spec}(V_{k[ab]})$.

(c) $SL_2/N_T$. This isomorphic to $\text{Spec}(V_{k[(ab)^2]})$.

**Proof.** By Matsushima’s criterion [Ric77] the homogeneous space $SL_2/H$ is affine if and only if $H$ is reductive. If $SL_2/H$ has a dense $\overline{B}$-orbit, then by dimension count we must have $\dim H \geq 1$. By the classification of connected reductive groups, we must have $\dim H = 1$ or $3$. If $\dim H = 3$, then $H = SL_2$. If $\dim H = 1$, then the neutral component of $H$ must be a torus. By conjugating we can assume that the neutral component of $H$ is $T$. So we must have either $H = T$ or $H$ is the normalizer of the torus $N_T = \mathbb{Z}/2\mathbb{Z} \ltimes T$.

The explicit description of the algebras follows from taking the subalgebra of $H^{op}$-invariants of $k[a,b]$ in each case.

**Theorem 4.10.** Every affine spherical $SL_2$-variety is $SL_2$-isomorphic to exactly one of the following

(a) $SL_2/SL_2 = \text{Spec}(k)$.

(b) $SL_2/T$.

(c) $SL_2/N_T$.

(d) $(\text{See Definition 4.8}) \quad SL_2/(\mu_f \ltimes \overline{U}) = \text{Spec} \left( R_\infty^{(f)} \right)$ for a unique positive integer $f$.

**Proof.** The fact that this list is exhaustive follows from Proposition 4.6 and Proposition 4.9. We just need to show that no two of these spherical varieties are $SL_2$-isomorphic. $SL_2/SL_2$ is the only one that has dimension 0, so it can’t be isomorphic to the others.

In order to study the other ones, we can look at the algebras of $U$-invariants. Recall by Proposition 4.7

$$k[SL_2/T]^U \cong k[ab].$$
\[ k[SL_2/N_T]^T \cong k[(ab)^2]. \]

\[ (R^{(f)}_{\infty})^T \cong k[b^f]. \]

These are isomorphisms as algebras with a left \( T \)-action. Note that most of these algebras are not isomorphic even as \( T \)-representations. In fact, the only ones that are isomorphic as \( T \)-representations are \( k[b^2] \cong k[ab] \) and \( k[b^4] \cong k[(ab)^2] \).

We are therefore reduced to showing that \( SL_2/T \not\cong \text{Spec} \left( R^{(2)}_{\infty} \right) \) and \( SL_2/N_T \not\cong \text{Spec} \left( R^{(4)}_{\infty} \right) \) as \( SL_2 \)-varieties.

\( R^{(2)}_{\infty} = k[b^2, bd, d^2] \) is isomorphic as a ring to \( k[x, y, z]/(y^2 - xz) \). Note that this ring has a singularity at the origin \( x = y = z = 0 \). So \( \text{Spec} \left( R^{(2)}_{\infty} \right) \) is not smooth, and therefore it can’t be isomorphic to \( SL_2/T \).

A similar reasoning shows that \( \text{Spec} \left( R^{(4)}_{\infty} \right) \) is not smooth, and so it can’t possibly be isomorphic to \( SL_2/N_T \).

\[ \square \]

**Remark 4.11.** \( \text{Spec} \left( R^{(f)}_{\infty} \right) \) will have a singular point for \( f > 1 \). Indeed, \( \text{Spec} \left( R^{(f)}_{\infty} \right) \) is isomorphic to the affine cone over the degree \( f \) Veronese embedding of \( \mathbb{P}^1 \) into \( \mathbb{P}^f \). For \( f > 1 \), this will be an affine normal surface with a singular point (the cone point).

### 5 Classification of affine normal \( SL_2 \)-varieties with an open dense orbit

In this section we will apply similar techniques to classify all \( SL_2 \)-isomorphism classes of affine normal \( SL_2 \)-varieties with an open dense orbit. We start by dealing with the homogeneous ones.

**Proposition 5.1.** Let \( X = SL_2/H \) be an affine homogeneous \( SL_2 \)-variety. Then up to \( SL_2 \)-isomorphism, exactly one of the following holds

(a) \( X \) is spherical. We have either \( H = T \), \( H = N_T \) or \( H = SL_2 \).

(b) \((\text{Type} A)\) \( H \) is the \( f \)-torsion subgroup \( \mu_f \subset T \) for some positive integer \( f \).

(c) \((\text{Type} D)\) \( H \) is the subgroup \( \mathbb{Z}/2\mathbb{Z} \rtimes \mu_f \) inside \( N_T \).

(d) \((E_6)\) \( H \subset SL_2 \) is the binary tetrahedral group \( \mathbb{T} \).

(e) \((E_7)\) \( H \subset SL_2 \) is the binary octahedral group \( \mathbb{O} \).

(f) \((E_8)\) \( H \subset SL_2 \) is the binary icosahedral group \( \mathbb{I} \).

**Proof.** If \( \dim H \geq 1 \) then we are in case (a) by the proof of Proposition 4.9. If \( \dim H = 0 \), then \( H \) is a finite subgroup of \( SL_2 \). We are reduced to classifying finite subgroups of \( SL_2 \) up to conjugation. Such classification is well known,
see [Spr77][4.4] for a treatment. The classification includes exactly the cases (b) through (f) above.

For completeness, let’s recall explicit descriptions of the exceptional cases (d), (e) and (f) above. See [Slo80][§6.1] or [Spr77][4.4] for details.

(E6) $\mathbb{T}$ is generated by the group $\mathbb{Z}/2\mathbb{Z} \ltimes \mu_2 \subset N_T$ and the matrix $\frac{1}{\sqrt{2}} \begin{bmatrix} \epsilon^7 & \epsilon^7 \\ \epsilon^5 & \epsilon \end{bmatrix}$.
Here $\epsilon$ is a primitive 8-th root of unity.

(E7) $\mathbb{O}$ is generated by $\mathbb{T}$ and the matrix $\begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon^7 \end{bmatrix}$, where $\epsilon$ is the same 8th root of unity as above.

(E8) $\mathbb{I}$ is generated by the matrices $-\begin{bmatrix} \eta^3 & 0 \\ 0 & \eta^2 \end{bmatrix}$ and $\frac{1}{\eta^4-\eta} \begin{bmatrix} \eta + \eta^4 & 1 \\ -\eta - \eta^4 & 1 \end{bmatrix}$, where $\eta$ is a primitive 5th root of unity.

The admissible subalgebra $A \subset k[a, b]$ in each homogeneous case $SL/H$ can be computed as the ring of right invariants $k[a, b]^{Hop}$ (we take the distinguished point $p$ to be the identity). In Proposition 4.9 we have already described $A$ when $\dim H \geq 1$. See [Spr77] (Ch. 4) or [Dol] (Ch. 1) for details on the computation of these algebras when $H$ is a finite group.

It remains to classify the nonhomogeneous $SL_2$-varieties. We will split the classification into two cases, depending on the dimension of $k[X]^H$.

**Proposition 5.2.** Let $X$ be an affine normal $SL_2$-variety with an open dense orbit. Assume that $X$ is not a homogeneous $SL_2$-variety. Suppose that the ring $k[X]^H$ has dimension 1. Then $X$ is spherical. In particular $X$ is as in part (d) of Theorem 4.10.

**Proof.** Choose a closed point $p$ in the dense open orbit of $X$. This allows us to view $X$ as $Spec(R)$ for some left $SL_2$-subalgebra $R \subset k[SL_2]$. Since $X$ is not a homogeneous $SL_2$-variety, there is some closed point in the complement of the open dense orbit. By the Hilbert-Mumford criterion 4.4, we know that there exists a nonconstant one-parameter subgroup $\gamma$ such that $\lim_{t \to 0} \gamma(t) \cdot p$ exists in $Spec(R)$. After right multiplying by and element of $SL_2(k)$, we can assume that $B_\gamma = \overline{B}$.

Let $A = R^H = R \cap k[a, b]$ be the corresponding admissible algebra. By assumption, $\dim A = 1$. Therefore $A \neq k$. This means that there exist some homogeneous polynomial $p \in A$ with positive degree. If $k[p] = A$, then $X$ is spherical by Proposition 4.2.

Assume that $A \neq k[p]$. We claim that $b^f \in A$ for some $f \geq 1$. Up to a scalar multiple, we can write $p = a^n b^{n-m} + \sum_{i=0}^{m-1} z_i a^i b^{n-i}$ for some $m \leq n$. Since $B_\gamma = \overline{B}$, we conclude by Lemma 4.5 that $m \leq \left\lfloor \frac{n}{2} \right\rfloor$. If $m \neq \frac{n}{2}$, then the argument in Proposition 4.6 Case (C1) shows that the element $w_{mp}^n$ is of the form $zb^f$ for some $z \in k \setminus \{0\}$ and $f \geq 1$. So we would be done with the claim in this case. Suppose therefore that $m = \frac{n}{2}$.

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Since $k[p] \neq A$, there is a nonzero homogeneous polynomial $h \in A \setminus k[p]$. We can write $h = a^ib^{-i} + \sum_{i=0}^{s-1} z_i a^ib^{-i}$. If $s \neq \frac{n}{2}$, then by the same reasoning as above we can conclude that $b^f \in A$ for some $f \geq 1$. Otherwise, we can look at the nonzero polynomial $p^t - h^n$. Notice that this is a homogeneous polynomial of degree $nt$. The highest power of the variable $a$ must be strictly smaller than $\frac{nt}{2}$ because we are cancelling the monomials with the highest $a$-exponents in $p^t$ and $h^n$. Therefore, we can just apply the argument of Lemma 3.6 Case (C1) to deduce that $b^f \in A$ for some $f \geq 1$.

Let $f$ denote the minimal positive integer such that $b^f \in A$ (i.e. $k[b^f] = A \cap k[b]$). Since $\dim A = 1$, the localization $A \otimes_{k[b^f]} k(b^f)$ must be integral over the field $k(b^f)$. Note that $A \otimes_{k[b^f]} k(b^f) \subseteq k[b][a]$. The integral closure of the field $k(b^f)$ in the polynomial ring $k(b)[a]$ is just $k(b)$. Therefore we must have $A \subseteq k(b)$. This implies that $A = A \cap k[b] = k[b^f]$. It follows again that $X$ is spherical.

We are left with the case of dimension 2. We start with a useful lemma.

**Lemma 5.3.** Let $X$ be an affine variety equipped with a left action of a unipotent group $U$. Then we have equality between $U$-invariants $\text{Frac}(k[X]^U) = \text{Frac}(k[X])^U$.

**Proof.** See the argument in [Per] [4.1.16 (i)].

For the following lemma, recall that the choice of a closed point $X$ in the open dense orbit allows us to view $X = \text{Spec}(R)$ for some left $SL_2$-stable subalgebra $R \subseteq k[SL_2]$.

**Lemma 5.4.** Let $X$ be an affine normal $SL_2$-variety with an open dense orbit. Assume that $X$ is not a homogeneous $SL_2$-variety. Suppose that the ring $k[X]^U$ has dimension 2. Then $X$ admits a right $T$-action commuting with the $SL_2$-action.

More precisely, there exists a closed point $p \in X$ in the open dense orbit such that the corresponding $SL_2$-stable subalgebra $R \subseteq k[SL_2]$ is $T^\text{op}$-stable.

**Proof.** Let’s start by choosing a closed point $p$ in the open dense orbit of $X$. Let $R$ be the $SL_2$-stable subalgebra of $k[SL_2]$ such that $X = \text{Spec}(R)$. Since $X$ is not a homogeneous $SL_2$-variety, there is some closed point in the complement of the open dense orbit. By the Hilbert-Mumford criterion [4.4] we know that there exists a nonconstant one-parameter subgroup $\gamma$ such that $\lim_{t \to 0} \gamma(t) \cdot p$ exists in $\text{Spec}(R)$. After right multiplying by and element of $SL_2(k)$, we can assume that $B_\gamma = B_\gamma$.

Let $A = R^U = R \cap k[a, b]$ be the corresponding admissible subalgebra of $k[a, b]$. We just need to show that $A$ is right $T$-stable. This is because the $T^\text{op}$-action commutes with the left $SL_2$-action. So if $A$ is $T^\text{op}$-stable, then its $SL_2$-span $R$ will also be $T^\text{op}$-stable.

It suffices to prove that for every homogeneous polynomial $p \in A$, all of the $T^\text{op}$-weight components of $p$ are also in $A$.

We first claim that there exists some positive integer $f$ such that $b^f \in A$. Let $p \in A$ be a homogeneous polynomial of degree $n$. Up to a scalar multiple, we can write $p = a^mb^{m-n} + \sum_{i=0}^{m-1} z_i a^ib^{n-i}$ for some $m \leq n$. Since $B_\gamma = B_\gamma$, we conclude
by Lemma 4.5 that \( m \leq \left\lfloor \frac{n}{2} \right\rfloor \). If \( m \neq \frac{n}{2} \), then the argument in Proposition 4.6 Case (C1) shows that the element \( w_{2m}^{p^n} \) is of the form \( z b^f \) for some \( z \in k \setminus \{0\} \) and \( f \geq 1 \). So we would be done in this case. Suppose therefore that \( m = \frac{n}{2} \). Consider the subalgebra \( k[p] \subset A \). This inclusion can’t be an equality because \( A \) has dimension 2 by assumption. This means that there is a nonzero homogeneous polynomial \( h \in A \setminus k[p] \). We can write \( h = a^s b^{t-s} + \sum_{i=0}^{s-1} z_i a^i b^{t-i} \). If \( s \neq \frac{n}{2} \), then by the same reasoning as above we can conclude that \( b^f \in A \) for some \( f \geq 1 \).

Otherwise, we can look at the nonzero polynomial \( p^t - h^m \). Notice that this is a homogeneous polynomial of degree \( nt \). The highest power of the variable \( a \) must be strictly smaller than \( \frac{n^2}{2} \) because we are cancelling the monomials with the highest \( a \)-exponents in \( p^t \) and \( h^m \). Therefore, we can just apply the argument of 4.6 Case (C1) to deduce that \( b^f \in A \) for some \( f \geq 1 \).

Now let \( f \) be the smallest positive integer such that \( b^f \in A \). Let \( \text{Stab}_p \) denote the stabilizer of the distinguished point \( p \). Note that the algebra \( R \) must be invariant under (i.e. pointwise fixed by) the right action of \( \text{Stab}_p \). This in particular implies that \( A \) is invariant under the right action of \( \text{Stab}_p \). Since \( b^f \in A \), this means that \( b^f \) is fixed by the right action of \( \text{Stab}_p \). It can be checked that the right stabilizer of \( b^f \) is the subgroup \( \mu_f \ltimes U \) in \( \overline{B} = T \ltimes U \). Note that we must have \( \dim \text{Stab} = 0 \).

We claim that in fact \( \text{Stab}_p = \mu_f \). Say \( \text{Stab}_p = \mu_e \subset \mu_f \) for some \( e \) dividing \( f \). Note that \( \text{Spec}(R) \) contains \( SL_2/\text{Stab}_p \) as an open subscheme. The variety \( SL_2/\text{Stab}_p \) is \( \text{Spec} \) of the algebra of right \( \mu_e \)-invariants \( k[SL_2]^{\mu_e} \). Now we can use Lemma 5.3 in order to get the following chain of equalities.

\[
\text{Frac}(A) = \text{Frac}(R^\mu) = \text{Frac}(R)^\mu = \text{Frac} \left( k[SL_2]^{\mu_e} \right)^\mu = \text{Frac} \left( (k[SL_2]^{\mu_e})^\mu \right)
\]

Since the left and right \( SL_2 \)-actions commute, we can exchange the order of taking right and left invariants. This way we get

\[
\text{Frac}(A) = \text{Frac} \left( (k[SL_2]^{\mu_e})^\mu \right) = \text{Frac} \left( k[a^e, b^e, ab] \right)
\]

In particular \( b^e \in \text{Frac}(A) \). Since \( A \) is normal and \( b^f \in A \), we must have \( b^e \in A \). By minimality of \( f \) we have \( e = f \). This concludes the proof that \( \text{Stab}_p = \mu_f \). Since \( A \) is fixed by the right action of \( \text{Stab}_p \), we must have \( A \subset k[a, b]^{\mu_f} = k[a^f, b^f, ab] \).

We are now ready to finish the proof of the proposition. Let \( p \) be a homogeneous polynomial in \( A \). We will distinguish two cases.

(C1) \( f \) is odd. Since \( A \subset k[a^f, b^f, ab] \), we can write

\[
p = a^m b^{n-m} + \sum_{i=1}^{\lceil \frac{n}{2} \rceil} z_i a^{m-i} b^{n-m+i} f
\]

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for some $z_i \in k$. The $T^{op}$-weight components of $p$ are all the monomial summands $z_i a^{m-if} b^{n-m+if}$ and $a^m b^{n-m}$ in the sum above. We claim that we actually have $a^{m-if} b^{n-m+if} \in A$ for all $0 \leq i \leq \left\lfloor \frac{m}{f} \right\rfloor$. This would imply that all $T^{op}$-weight components are in $A$, thus concluding our proof.

Let’s prove this last claim. We will assume without loss of generality that $m \geq f$, otherwise the claim is obvious. Let us set $p_2 := b^{2f}$. We know that $b^{2f} \in A$ by our work above. Let’s look at the highest weight vector

$$w^{p,p_2}_f = \sum_{\alpha = f}^{n+f} y^{p,p_2}_{\alpha,f} a^{\alpha-f} b^{n+f-\alpha}$$

Notice that $y^{p,p_2}_{\alpha,f} = 0$ for all $\alpha > m$. This is because in the sum for $y^{p,p_2}_{\alpha,f}$ there are no nontrivial pairs $(i,j)$ with $i + j > m$. So we actually have

$$w^{p,p_2}_f = \sum_{\alpha = f}^{m} y^{p,p_2}_{\alpha,f} a^{\alpha-f} b^{n+f-\alpha}$$

The coefficient $y^{p,p_2}_{m,f}$ can be checked to be

$$y^{p,p_2}_{m,f} = \sum_{e=0}^{f} (-1)^e \frac{\binom{f}{e}}{\binom{n}{e}} \binom{n-m}{f-e} \left( \frac{2f}{f-e} \right)$$

This can be simplified to

$$y^{p,p_2}_{m,f} = \sum_{e=0}^{f} (-1)^e \binom{f}{e} \binom{n-m}{e} \frac{1}{\binom{n}{e}}$$

By Lemma A.3, we have $y^{p,p_2}_{m,f} \neq 0$. Therefore, we can divide by $y^{p,p_2}_{m,f}$ and write

$$\frac{1}{y^{p,p_2}_{m,f}} w^{p,p_2}_f = a^{m-f} b^{n-m+f} + \sum_{i=2}^{\left\lfloor \frac{m}{f} \right\rfloor} y^{p,p_2}_{m-i+1,f} a^{m-if} b^{n-m+if}$$

Let us call $q_0 := p$ and $q_1 := \frac{1}{y^{p,p_2}_{m,f}} w^{p,p_2}_f$. For every $0 \leq j \leq m$, we can define recursively $q_j := \frac{1}{y^{p,p_2}_{m-j+1,j+1,f}} w^{q_{j-1},p_2}_f$. The computation above shows that we can write

$$q_j = a^{m-jf} b^{n-m+jf} + \sum_{i=j+1}^{\left\lfloor \frac{m}{f} \right\rfloor} z_{j,i} a^{m-if} b^{n-m+if}$$

for some coefficients $z_{j,i} \in k$. By assumption $A$ is admissible, so $q_j \in A$ for all $j$. We will now show by descending induction that $a^{m-if} b^{n-m+if} \in A$ for all $0 \leq i \leq \left\lfloor \frac{m}{f} \right\rfloor$. For the base case we have $q_0 = b^{n-m+i} b^{n-m+if} \in A$. Let $0 \leq j \leq \left\lfloor \frac{m}{f} \right\rfloor$. Assume by induction that we know that $a^{m-if} b^{n-m+if} \in A$ for all $j+1 \leq i \leq \left\lfloor \frac{m}{f} \right\rfloor$. Then, we can write the monomial $a^{m-jf} b^{n-m+jf}$ as

$$a^{m-jf} b^{n-m+jf} = q_j - \sum_{i=j+1}^{\left\lfloor \frac{m}{f} \right\rfloor} z_{j,i} a^{m-if} b^{n-m+if}$$
Since all of the summands in the right-hand side are in \( A \), we conclude that \( a^{m-j}f b^{n-m+j} \in A \). So we are done by induction.

(C2) \( f \) is even. Since \( A \subset k[a^f, b^f, ab] \), we can write

\[
p = a^m b^{n-m} + \sum_{i=1}^{\frac{m}{2}} z_i a^{m-i \frac{f}{2}} b^{n-m+i \frac{f}{2}}\]

for some \( z_i \in k \). The \( T^{op} \)-weight components of \( p \) are all the monomial summands \( z_i a^{m-i \frac{f}{2}} b^{n-m+i \frac{f}{2}} \) and \( a^m b^{n-m} \) in the sum above. We can do the same argument as in case (C1) but replacing every instance of \( f \) by \( \frac{f}{2} \) (e.g. \( p_2 = b^f \) and we look at the highest weight vector \( w_{p_2}^{p_2} \)).

\[\square\]

![Figure 3: Application of the criterion in Proposition 3.6 to \((p, b^{2f})\) in the proof of Lemma 5.4, case (C1). The darker grid represents the sublattice induced by \( f = 3 \).](image)

**Definition 5.5.** Let \( q \geq 1 \) be a rational number. We define \( S_q \) to be the subalgebra of \( k[a, b] \) generated by all monomials \( a^i b^j \) satisfying \( j \geq qi \). We will denote by \( V_{S_q} \) the left \( SL_2 \)-subrepresentation of \( k[SL_2] \) spanned by \( S_q \).

**Lemma 5.6.** The algebra \( S_q \) is admissible in the sense of Definition 3.1.

**Proof.** \( \text{Spec}(S_q) \) is a toric variety for the torus \((T \times T^{op})/\mu_2\), where we are embedding \( \mu_2 \) diagonally. Motivated by this, we will use the notation for toric varieties as in [CLS11].

Let \( \Sigma_q \) be the semigroup in \( \mathbb{N}^2 \subset \mathbb{Z}^2 \) defined by

\[
\Sigma_q = \{(i, j) \in \mathbb{N}^2 \mid a^i b^j \in S_q\}
\]
We will denote by $R$.

It follows from the argument in the proof of Lemma 5.4 that the stabilizer $\text{Stab}(b)$.

This follows from combining the inequalities $j \geq q_i$.

Proof. Definition 5.7. For any positive integer $q$ and any rational number $q \geq 1$, we will denote by $S_q^{(f)}$ the subalgebra of $k[SL_2]$ defined by $S_q^{(f)} := S_q \cap k[a^f, b^f, ab]$. We will denote by $R_q^{(f)}$ the left $\text{SL}_2$-subrepresentation of $k[\text{SL}_2]$ spanned by $S_q^{(f)}$.

Lemma 5.8. (a) For every positive integer $f$ and rational number $q \geq 1$, the algebra $S_q^{(f)}$ is admissible. In particular $R_q^{(f)}$ is a subalgebra of $k[\text{SL}_2]$.

(b) Let $f_1, f_2$ be positive integers and let $q_1, q_2 \geq 1$ be rational numbers. If $f_1 \neq f_2$ or $q_1 \neq q_2$, then the varieties $\text{Spec}(R_q^{(f_1)})$ and $\text{Spec}(R_q^{(f_2)})$ are not $\text{SL}_2$-isomorphic.

Proof. (a) Notice that we have $S_q^{(f)} = (S_q)^{\mu_{1^f}}$ by definition. By Lemma 5.6, the algebra $S_q$ is finitely generated and normal. By [NS02], this implies that the algebra of invariants $(S_q)^{\mu_{1^f}}$ is finitely generated. Also $S_q$ being normal implies that $(S_q)^{\mu_{1^f}}$ is normal.

We are left to prove property (c) in Definition 3.1. This amounts to showing that $R_q^{(f)}$ is an algebra. But notice that $R_q^{(f)} = (V_{S_q})^{\mu_{1^f}}$. By Lemma 5.6 we know that $V_{S_q}$ is closed under multiplication. We conclude that the $\mu_{1^f}$-invariants $(V_{S_q})^{\mu_{1^f}}$ are also closed under multiplication, as desired.

(b) It follows from the argument in the proof of Lemma 5.4 that the stabilizer of the open dense orbit in $\text{Spec}(R_q^{(f_1)})$ (resp. $\text{Spec}(R_q^{(f_2)})$) is $\mu_{f_1}$ (resp. $\mu_{f_2}$). If $f_1 \neq f_2$, this shows that $\text{Spec}(R_q^{(f_1)})$ and $\text{Spec}(R_q^{(f_2)})$ can’t possibly be $\text{SL}_2$-isomorphic. Assume $f_1 = f_2 = f$. We can look at the algebras of $\overline{U}$-invariants

$$ \left( R_q^{(f)} \right)^{\overline{U}} = S_q^{(f)} $$

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\[(R_q^{(f)})^\mathbb{V} = S_{q2}^{(f)}\]

Suppose without loss of generality that \(q_1 > q_2\). Then we have \(S_{q1}^{(f)} \subsetneq S_{q2}^{(f)}\). This implies that \(S_{q1}^{(f)}\) and \(S_{q2}^{(f)}\) are not isomorphic even as \(T\)-representations. This is because for \(n \gg 0\), the dimension of the \(T\)-weight space \(S_{q1}^{(f)}\) must be strictly smaller than the dimension of the \(T\)-weight space \(S_{q2}^{(f)}\).

\[\square\]

**Proposition 5.9.** Let \(X\) be an affine normal \(SL_2\)-variety with an open dense orbit. Assume that \(X\) is not a homogeneous \(SL_2\)-variety. Suppose that the ring \(k[X]^T\) has dimension 2. Then there exists a unique positive integer \(f\) and a unique rational number \(q \geq 1\) such that \(X\) is \(SL_2\)-isomorphic to \(Spec\left(R_q^{(f)}\right)\).

**Proof.** By Lemma 5.4, we can choose a closed point \(p\) in the dense orbit such that

(a) \(\lim_{t \to 0} \gamma(t) \cdot p\) exists in \(X\) for some \(\gamma\) with \(B_\gamma = \overline{B}\).

(b) The stabilizer of \(p\) is the \(f\)-torsion subgroup \(\mu_f \subset T\).

We can use \(p\) to view \(X\) as \(Spec(R)\) for some \(SL_2\)-stable algebra \(R \subset k[SL_2]\). Let \(A = R \cap k[a, b]\) be the corresponding admissible subalgebra of \(k[a, b]\). It suffices to show that \(A = S_q^{(f)}\) for some \(q \geq 0\). By Lemma 5.4, \(A\) is generated by \(T \times T_{op}\)-weights in \(k[a, b]\). These are just the monomials in \(k[a, b]\), so \(A\) is determined by a semigroup on the set of monomials. Let \(\Sigma_A\) denote the semigroup of \(\mathbb{N}^2\) given by

\[\Sigma_A := \{(i, j) \in \mathbb{N}^2 \mid a^i b^j \in A\}\]

Also, let \(\Sigma^{(f)}\) be the semigroup of \(\mathbb{N}^2\) determined by \(k[a^f, b^f, ab]\)

\[\Sigma^{(f)} := \{(i, j) \in \mathbb{N}^2 \mid a^i b^j \in k[a^f, b^f, ab]\} = \{(i, j) \in \mathbb{N}^2 \mid f \mid (j - i)\}\]

We have seen during the proof of Lemma 5.4 that there exists a positive integer \(f\) such that \(A \subset k[a^f, b^f, ab]\) and that

\[\text{Frac}(A) = k(a, b)^{\mu_f} = k(a^f, b^f, ab)\]

This means that \(\Sigma_A \subset \Sigma^{(f)}\) and \(\mathbb{Z}\Sigma_A = \mathbb{Z}\Sigma^{(f)}\). Since \(A\) is normal, the proof of [CLS11][Thm. 1.3.5] implies that \(A\) is saturated in \(\mathbb{Z}\Sigma^{(f)}\). Also, \(\Sigma_A\) is generated by a finite set of monomial generators of \(A\). Since \(\Sigma_A\) is finitely generated and saturated in \(\mathbb{Z}\Sigma^{(f)}\), we must have that \(\Sigma_A = C \cap \Sigma^{(f)}\) for some rational cone \(C\) on the vector space \(\mathbb{R}^2 = \mathbb{Z}^2 \otimes \mathbb{R}\). In two dimensions, such a cone must be of the form \(\mathbb{R}^2 v_1 + \mathbb{R}^2 v_2\) for two (possibly equal) lattice vectors \(v_1, v_2 \in \mathbb{Z}^2\). Since \(\mathbb{Z}\Sigma_A = \mathbb{Z}\Sigma^{(f)}\), this means that \(v_1\) and \(v_2\) must be linearly independent.

By definition, we have \(\Sigma_A \subset \mathbb{N}^2\). Since \(B_\gamma = \overline{B}\), Lemma 4.5 implies that \(i \leq j\) for all \((i, j) \in \Sigma_A\). So the cone \(C\) must be contained in the cone \(\mathbb{R}^2 v_1 + \mathbb{R}^2 v_2\). Thus \(C\) must be contained in the cone \(\mathbb{R}^2 v_1 + \mathbb{R}^2 v_2\). By the proof of Lemma 5.4, we have \(b^f \in A\). This means that \((0, 1) \in C\), and so one of the vectors can be taken to be \(v_1 = (0, 1)\). The other linearly independent
vector is of the form $v_2 = (n_1, n_2)$ for some positive integers $n_1 \leq n_2$. So the cone is $C = \mathbb{R}_0^0(0,1) + \mathbb{R}^2(n_1, n_2)$. We conclude that

$$A = k[\Sigma_A] = k[C \cap \Sigma^{(f)}] = S_{n_1}^{(f)}$$

Uniqueness follows from part (b) of Lemma 5.8.

Now we can put everything together to get a classification of all affine normal $SL_2$-varieties with an open dense orbit.

Theorem 5.10. Let $X$ be an affine normal $SL_2$-variety with an open dense orbit. Then $X$ is $SL_2$-isomorphic to exactly one of the following

1. A homogeneous space $SL_2/H$, where $H$ is as in Proposition 5.1.
2. (See Definition 4.8) The spherical variety $SL_2/(\mu_f \ltimes U) = \text{Spec} \left( R_{\infty}^{(f)} \right)$ for a unique positive integer $f$.
3. (See Definition 5.7) $\text{Spec} \left( R_q^{(f)} \right)$ for a unique positive integer $f$ and a unique rational number $q \geq 1$. In this case the stabilizer of the dense orbit is $\mu_f$.

Proof. This follows by combining Proposition 5.1, Proposition 5.2 and Proposition 5.9.

We end this section by describing explicitly the subalgebra $R_q^{(f)} \subset k[SL_2]$. Let \( \{ a^{im} b^{jm} \}_{m \in I} \) be a finite set of monomial generators of $S_q^{(f)}$. This set can be obtained in practice by imitating the proof of Gordan’s lemma [CLS11][Prop. 1.2.17]. In other words, we look at $\Sigma^{(f)} \cap \Delta$, where $\Delta$ is a fundamental domain for the discrete subgroup of $\mathbb{R}^2$ generated by $(0,1)$ and $(n_1, n_2)$. Here $n_1, n_2$ are some positive integers such that $\frac{n_2}{n_1} = q$ and $n_2 - n_1$ is divisible by $f$.

For each $m \in I$, let us denote by $V^{(m)}$ the left $SL_2$-subrepresentation of $k[SL_2]$ spanned by the highest weight vector $a^{im} b^{jm}$. A simple $SL_2$ computation shows that $V^{(m)}$ has a basis \( \left( v_s^{(m)} \right)_{s=0}^{i_m+j_m} \), where

\[
v_s^{(m)} = \sum_{i+j=s} \binom{i}{i} \binom{j}{j} a^{im-i} c^i b^{jm-j} d^l \]

By the proof of the finite generation statement in Lemma 3.2, the corresponding $SL_2$-stable subalgebra $R_q^{(f)}$ is generated as a $k$-algebra by the elements $v_s^{(m)}$ for $m \in I$ and $0 \leq s \leq i_m + j_m$.

A Some lemmas about binomial sums

In this appendix we collect some lemmas on binomial sums that we use for our classification results.
Lemma A.1. Let \( i, j \) be nonnegative integers with \( i \leq j \). Then, we have

\[
\sum_{e=0}^{2i} (-1)^e \binom{2i}{e} \binom{j}{2i-e} = (-1)^i \binom{2i}{i}^2
\]

In particular this sum is never 0.

Proof. We can use the identity

\[
\binom{j}{i} \binom{j+e}{i} = \frac{1}{i+j} \binom{j+i-e}{i} \binom{j-i+e}{i}
\]
to write the sum as

\[
\sum_{e=0}^{2i} (-1)^e \binom{2i}{e} \binom{j+i-e}{i} \binom{j-i+e}{i}
\]

We are reduced to showing the identity

\[
\sum_{e=0}^{2i} (-1)^e \binom{2i}{e} \binom{j+i-e}{i} \binom{j-i+e}{i} = (-1)^i \binom{2i}{i}
\]

Let’s first show that the left hand side does not depend on \( j \). Let us fix \( i \) and view the expression as a polynomial in \( j \). We can expand

\[
\binom{j+i-e}{i} \binom{j-i+e}{i} = \sum_{s=0}^{2i} p_s(e) j^{2i-s}
\]

where \( p_s(e) \) is a polynomial on \( e \) with coefficients in \( \mathbb{Q}[i] \). Replacing this in the sum above, we get a polynomial in \( j \) given by

\[
\sum_{s=0}^{2i} \sum_{e=0}^{2i} (-1)^e \binom{2i}{e} p_s(e) j^{2i-s}
\]

In order to show that the sum does not depend on \( j \), it suffices to show that \( \sum_{e=0}^{2i} (-1)^e \binom{2i}{e} p_s(e) = 0 \) for \( s < 2i \). Notice that \( p_s(e) \) has degree at most \( s \) in the variable \( e \). So we can write

\[
p_s = \sum_{f=0}^{s} q_{s,f}(i) e^f
\]

for some \( q_{s,f} \in \mathbb{Q}[i] \). We can use this to get

\[
\sum_{e=0}^{2i} (-1)^e \binom{2i}{e} p_s(e) = \sum_{f=0}^{s} q_{s,f}(i) \sum_{e=0}^{2i} (-1)^e \binom{2i}{e} e^f
\]

Therefore we are reduced to showing that \( \sum_{e=0}^{2i} (-1)^e \binom{2i}{e} e^f = 0 \) for \( f \leq s < 2i \). In order to show this, we use the binomial theorem. Notice that the binomial theorem implies that

\[
\sum_{e=0}^{2i} x^e \binom{2i}{e} e^f = \left( x \frac{d}{dx} \right)^f (1 + x)^{2i}
\]
We can plug in $x = -1$ in the left hand side to get $\sum_{e=0}^{2i} (-1)^e \binom{2i}{e} e^f$. On the other hand, if we plug in $x = -1$ in the right hand side we get 0 (one can see this by using the product rule and $f < 2i$).

We conclude that the sum
$$\sum_{e=0}^{2i} (-1)^e \binom{2i}{e} \left( \frac{j + i - e}{i} \right) \left( \frac{j - i + e}{i} \right)$$
does not depend on $j$. So we can evaluate by plugging in any value of $j$ we wish. If we plug in $j = i$, then we see that the only nonzero summand occurs when $e = i$. Hence we get
$$\sum_{e=0}^{2i} (-1)^e \binom{2i}{e} \left( \frac{j + i - e}{i} \right) \left( \frac{j - i + e}{i} \right) = (-1)^i \binom{2i}{i}$$
as desired.  

Lemma A.2. Let $m$ be a positive integer and let $0 \leq i \leq m$. Then we have
$$\sum_{e=0}^{2} (-1)^e \binom{2}{e} \frac{(2m)}{(2m-e)} \frac{(m-e)}{(2-e)} \frac{(m+i)}{(2-i)} \frac{(m-i)}{(2+i)} \frac{(m+2)}{(2+2)} \frac{(2m-i)}{(2+i)} \frac{(2m-2)}{(2-i)} = \frac{mi^2 - m^2}{2m^2 \cdot (2m - 1)}$$

Proof. This is just tedious algebra. One needs to sum three terms and operate with the fractions to arrive at the expression in the right-hand side.  

The following sum is needed for some of the arguments in the last section.

Lemma A.3. Let $f, m, n$ be positive integers with $f \leq m \leq n$. Then
$$\sum_{e=0}^{f} (-1)^e \binom{f}{e} \binom{n-m}{e} \binom{m}{e} \binom{n}{f} = \binom{m}{f} \binom{n}{f}$$

In particular, this sum is not 0.

Proof. Fix the positive integer $f$. Let’s interpret $n, m$ as variables. For psychological purposes, let’s rename $n = x$ and $m = y$. The identity amounts to an equality of rational functions in the variables $x, y$

$$\sum_{e=0}^{f} (-1)^e \binom{f}{e} \prod_{i=0}^{e-1} (x-y-i) \prod_{i=0}^{e-1} (y-i) = \prod_{i=0}^{e-1} (y-i) \prod_{i=0}^{e-1} (x-i)$$

We can rewrite this using the Pochhammer symbol to simplify notation

$$\sum_{e=0}^{f} \binom{(-f)e}{n!(-x)e} = \frac{(-y)_f}{(-x)_f}$$

The left-hand side is just the finite hypergeometric function $\binom{2F_1}{-f, y-x; -x; 1}$ (evaluated at $z = 1$). The last equality of rational functions above is a direct consequence of the Chu-Vandermonde identity [AAR99][Cor. 2.2.3].
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