FOURIER QUASICRYSTALS AND DISCRETENESS OF THE DIFFRACTION SPECTRUM

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Abstract. We prove that a positive-definite measure in $\mathbb{R}^n$ with uniformly discrete support and discrete closed spectrum, is representable as a finite linear combination of Dirac combs, translated and modulated. This extends our recent results where we proved this under the assumption that also the spectrum is uniformly discrete. As an application we obtain that Hof’s quasicrystals with uniformly discrete diffraction spectra must have a periodic diffraction structure.

1. Introduction

1.1. By a Fourier quasicrystal one often means a discrete measure, whose Fourier transform is also a discrete measure. This concept was inspired by the experimental discovery in the middle of 80’s of non-periodic atomic structures with diffraction patterns consisting of spots. In this context, different versions of “discreteness” were discussed, see in particular [BT87], [CT87], [Mey95], [Dys09].

Let $\mu$ be a (complex) measure on $\mathbb{R}^n$, supported on a discrete set $\Lambda$:

$$\mu = \sum_{\lambda \in \Lambda} \mu(\lambda) \delta_{\lambda}, \quad \mu(\lambda) \neq 0. \quad (1.1)$$

We shall suppose that $\mu$ is a slowly increasing measure, which means that $|\mu|\{x : |x| < r\}$ grows at most polynomially as $r \to \infty$. Hence the Fourier transform

$$\hat{\mu}(t) := \sum_{\lambda \in \Lambda} \mu(\lambda) e^{-2\pi i \langle \lambda, t \rangle}$$

is well-defined as a temperate distribution on $\mathbb{R}^n$. Assume that also $\hat{\mu}$ is a slowly increasing measure, which is purely atomic and supported on a countable set $S$:

$$\hat{\mu} = \sum_{s \in S} \hat{\mu}(s) \delta_s, \quad \hat{\mu}(s) \neq 0. \quad (1.2)$$

The set $\Lambda$ is then called the support of the measure $\mu$, while $S$ is called the spectrum.

1.2. In [Lag00, Problem 4.1(a)] the following question was posed:

Suppose that $\mu$ is a positive measure, whose support $\Lambda$ and spectrum $S$ are both uniformly discrete sets. Is it true that $\Lambda$ can be covered by a finite union of translates of a certain lattice?

Recall that a set is uniformly discrete if the distance between any two of its points is bounded below by some positive constant.

In [LO13, LO15] we proved the following result, which answers the above question affirmatively, and moreover shows that the measure $\mu$ must have a periodic structure:

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Theorem 1.1. Let \( \mu \) be a positive (or positive-definite) measure on \( \mathbb{R}^n \) satisfying (1.1) and (1.2), and assume that \( \Lambda \) and \( S \) are both uniformly discrete sets. Then there is a lattice \( L \), such that \( \mu \) is representable in the form
\[
\mu = \sum_{j=1}^{N} \sum_{\lambda \in L + \tau_j} P_j(\lambda) \delta_\lambda
\]  
for certain vectors \( \tau_j \) in \( \mathbb{R}^n \) and trigonometric polynomials \( P_j \) (\( 1 \leq j \leq N \)). Moreover, for \( n = 1 \) the result holds even without the positivity assumption on the measure \( \mu \).

So the theorem says that under the conditions above, the measure \( \mu \) is a finite linear combination of Dirac combs, translated and modulated. Conversely, for any measure \( \mu \) of the form (1.3), the support \( \Lambda \) and spectrum \( S \) are both uniformly discrete sets.

It was also an open problem (see [Lag00, Problem 4.1(b)]) whether such a result can be proved in the more general situation when \( \Lambda, S \) are assumed to be just discrete closed sets. This problem was addressed in [LO16] where we proved the following:

Theorem 1.2. There is a (non-zero) measure \( \mu \) on \( \mathbb{R} \) satisfying (1.1) and (1.2), such that \( \Lambda \) and \( S \) are both discrete closed sets, but \( \Lambda \) contains only finitely many elements of any arithmetic progression.

The latter condition indicates that the measure \( \mu \) is “non-periodic” in a strong sense. In particular, it cannot be obtained as a finite linear combination of Dirac combs. A result of similar type is true also in \( \mathbb{R}^n, n > 1 \).

Moreover, in our construction both \( \mu \) and \( \hat{\mu} \) are translation-bounded measures (see Section 3 for the definition). This implies that \( \mu \) is an almost periodic measure, whose Fourier transform \( \hat{\mu} \) is also an almost periodic measure. By definition, a measure \( \mu \) on \( \mathbb{R}^n \) is an almost periodic measure if for every continuous, compactly supported function \( \varphi \) on \( \mathbb{R}^n \), the convolution \( \mu * \varphi \) is an almost periodic function in the sense of Bohr.

1.3. In the crystallography community, it seems to be commonly agreed that the support \( \Lambda \) should be a uniformly discrete set. We remind that Meyer’s quasicrystals [Mey95], which appeared first under the name “model sets” [Mey70], are uniformly discrete sets, and they support measures whose spectra are dense countable sets.

So it is a natural problem, to what extent can the spectrum \( S \) of a non-periodic quasicrystal be discrete, assuming that the support \( \Lambda \) is uniformly discrete? In the present paper we address this problem, and consider quasicrystals with non-symmetric discreteness assumptions on the support and the spectrum.

First we obtain several results which show that, under certain conditions, if the spectrum \( S \) is a discrete closed set, then in fact \( S \) must be uniformly discrete. These results thus reduce the situation to the setting in Theorem 1.1 above, which in turn allows us to conclude that the measure \( \mu \) is representable in the form (1.3).

On the other hand, we present an example of a non-periodic quasicrystal such that the spectrum \( S \) is a nowhere dense countable set.

We also apply our results to Hof’s quasicrystals. In this context, we prove that if a Delone set \( \Lambda \) of finite local complexity has a uniformly discrete diffraction spectrum, then the diffraction measure of \( \Lambda \) has a periodic structure.

Finally, we extend our results to the more general situation, where \( \hat{\mu} \) is a measure which has both a pure point component and a continuous one.
2. Results

2.1. Our first result deals with the case when $\mu$ is a positive-definite measure:

**Theorem 2.1.** Let $\mu$ be a positive-definite measure on $\mathbb{R}^n$ satisfying (1.1) and (1.2). Assume that the support $\Lambda$ is a uniformly discrete set, while the spectrum $S$ is a discrete closed set. Then also $S$ is uniformly discrete, and the measure $\mu$ has the form (1.3).

Actually, we will show that if a positive-definite measure $\mu$ with uniformly discrete support $\Lambda$ is not of the form (1.3), then its spectrum $S$ must have a relatively dense set of accumulation points (see Theorem 5.1).

2.2. In the next result, which holds for dimension $n = 1$, the positive-definiteness assumption is replaced by a stronger discreteness condition on the spectrum:

**Theorem 2.2.** Let $\mu$ be a measure on $\mathbb{R}$ satisfying (1.1) and (1.2). Assume that the support $\Lambda$ is a uniformly discrete set, while the spectrum $S$ satisfies the condition

$$\sup_{x \in \mathbb{R}} \#(S \cap [x, x + 1]) < \infty. \quad (2.1)$$

Then $S$ is a uniformly discrete set, and $\mu$ is of the form (1.3).

Notice that condition (2.1) means that $S$ is the union of a finite number of uniformly discrete sets.

2.3. In the following result, a stronger discreteness condition is imposed on the support:

**Theorem 2.3.** Let $\mu$ be a measure on $\mathbb{R}^n$ satisfying (1.1) and (1.2). Assume that the set $\Lambda - \Lambda$ is uniformly discrete, and that $S$ is a discrete closed set. Then the same conclusion as in the previous two theorems holds.

Moreover, we will prove that if $\Lambda - \Lambda$ is uniformly discrete, but the measure $\mu$ is not of the form (1.3), then again the spectrum $S$ must have a relatively dense set of accumulation points (see Theorem 7.1).

Theorem 2.3 implies the following result communicated to us by Y. Meyer: a “simple quasicrystal” cannot support a measure whose spectrum is a discrete closed set (for the definition of a simple quasicrystal, see [MM10]).

2.4. The previous results show that if the measure $\mu$ does not have the periodic structure (1.3), then the spectrum $S$ must have finite accumulation points. However, $S$ need not be dense in any ball, as the following result shows:

**Theorem 2.4.** There is a positive-definite measure $\mu$ on $\mathbb{R}^n$ satisfying (1.1) and (1.2), such that:

(i) The support $\Lambda$ is not contained in a finite union of translates of any lattice;
(ii) The set $\Lambda - \Lambda$ is uniformly discrete;
(iii) The spectrum $S$ is a nowhere dense (countable) set.

To prove this we base on Meyer’s construction, but with an additional modification.
2.5. Our results can be applied to quasicrystals in Hof’s sense [Hof95].

Recall that a set $\Lambda \subset \mathbb{R}^n$ is called a Delone set if it is uniformly discrete and also relatively dense. A Delone set $\Lambda$ is said to be of finite local complexity if the difference set $\Lambda - \Lambda$ is a discrete closed set. This means that $\Lambda$ has, up to translations, only a finite number of local patterns of any given size.

An autocorrelation measure $\gamma_\Lambda$ of the set $\Lambda$ is any weak limit point of the measures

$$
(2R)^{-n} \sum_{\lambda, \nu \in \Lambda \cap [-R,R]^n} \delta_{\nu - \lambda}
$$

as $R \to \infty$. An autocorrelation measure $\gamma_\Lambda$ is always positive-definite, and so its Fourier transform $\hat{\gamma}_\Lambda$ is a positive measure, called a diffraction measure of $\Lambda$. If the measure $\hat{\gamma}_\Lambda$ is purely atomic, then its support $S$ is called a diffraction spectrum of $\Lambda$.

The following result answers a question posed in [Lag00, Problem 4.2(a)]:

**Theorem 2.5.** Suppose that

(i) $\Lambda \subset \mathbb{R}^n$ is a Delone set of finite local complexity;

(ii) $\gamma_\Lambda$ is an autocorrelation measure of $\Lambda$; and

(iii) The diffraction spectrum $S$ (the support of $\hat{\gamma}_\Lambda$) is a uniformly discrete set.

Then $S$ is contained in a finite union of translates of some lattice, and the diffraction measure $\hat{\gamma}_\Lambda$ has the form (1.3).

A slightly more general version of this result will be given in Theorem 9.1. The same conclusion is true if the set $\Lambda - \Lambda$ is uniformly discrete, and the diffraction spectrum $S$ is a discrete closed set (see Theorem 9.2).

2.6. We also consider discrete measures $\mu$, whose Fourier transform $\hat{\mu}$ is a measure which has both a pure point component and a continuous one. We can extend our previous results to this more general situation using the following:

**Theorem 2.6.** Let $\mu$ be a measure on $\mathbb{R}^n$ with uniformly discrete support $\Lambda$, and assume that $\hat{\mu}$ is a (slowly increasing) measure. Then the discrete part of $\hat{\mu}$ is the Fourier transform of another measure $\mu'$, whose support $\Lambda'$ is also a uniformly discrete set.

Moreover, if $\Lambda - \Lambda$ is a uniformly discrete set, then also $\Lambda' - \Lambda'$ is uniformly discrete.

By applying the previous results to this new measure $\mu'$, one can obtain versions of the results for measures $\mu$ with non pure point Fourier transform (see Theorem 10.3).

3. Preliminaries

3.1. **Notation.** By $\langle \cdot, \cdot \rangle$ and $| \cdot |$ we denote the Euclidean scalar product and norm in $\mathbb{R}^n$. The open ball of radius $r$ centered at the origin is denoted $B_r := \{ x \in \mathbb{R}^n : |x| < r \}$.

A set $\Lambda \subset \mathbb{R}^n$ is said to be a discrete closed set if $\Lambda$ has only finitely many points in any bounded set. The set $\Lambda$ is called uniformly discrete if

$$
d(\Lambda) := \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0.
$$

The set $\Lambda$ is said to be relatively dense if there is $R > 0$ such that every ball of radius $R$ intersects $\Lambda$. 
By a (full-rank) lattice $L \subset \mathbb{R}^n$ we mean the image of $\mathbb{Z}^n$ under some invertible linear transformation $T$. The determinant $\det(L)$ is equal to $|\det(T)|$. The dual lattice $L^*$ is the set of all vectors $\lambda^*$ such that $\langle \lambda, \lambda^* \rangle \in \mathbb{Z}$, $\lambda \in L$.

By a “distribution” we will mean a temperate distribution on $\mathbb{R}^n$. By a “measure” we mean a complex, locally finite measure (usually infinite) which is assumed to be slowly increasing. By definition, a measure $\mu$ is slowly increasing if there is a constant $N$ such that $|\mu|(B_R) = O(R^N)$ as $R \to \infty$. The measure $\mu$ is called translation-bounded if

$$\sup_{x \in \mathbb{R}^n} |\mu|(x + B_1) < \infty. \quad (3.2)$$

Any translation-bounded measure is slowly increasing, and any slowly increasing measure is a temperate distribution. Remark that for a positive measure to be a temperate distribution, it is also necessary to be slowly increasing, but this is not true for complex (or real, signed) measures.

By the “support” of a pure point measure $\mu$ we mean the countable set of the non-zero atoms of $\mu$. This should not be confused with the notion of support in the sense of distributions, which is always a closed set.

The Fourier transform on $\mathbb{R}^n$ will be normalized as follows:

$$\hat{\varphi}(t) = \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i \langle t, x \rangle} dx.$$

We denote by $\text{supp}(\varphi)$ the closed support of a Schwartz function $\varphi$, and by $\text{spec}(\varphi)$ the closed support of its Fourier transform $\hat{\varphi}$.

If $\alpha$ is a temperate distribution, and $\varphi$ is a Schwartz function on $\mathbb{R}^n$, then $\langle \alpha, \varphi \rangle$ will denote the action of $\alpha$ on $\varphi$. The Fourier transform $\hat{\alpha}$ of the distribution $\alpha$ is defined by $\langle \hat{\alpha}, \varphi \rangle = \langle \alpha, \hat{\varphi} \rangle$.

A distribution $\alpha$ is called positive if $\langle \alpha, \varphi \rangle \geq 0$ for any Schwartz function $\varphi \geq 0$. It is well-known that if $\alpha$ is a positive distribution, then it is a positive measure. A distribution $\alpha$ is called positive-definite if $\hat{\alpha}$ is a positive distribution.

For a set $A \subset \mathbb{R}^n$ we denote by $\#A$ the number of elements in $A$, and by $\text{mes}(A)$ or $|A|$ the Lebesgue measure of $A$.

3.2. Measures. We will need some basic facts about measures in $\mathbb{R}^n$.

**Lemma 3.1.** Let $\mu$ be a measure on $\mathbb{R}^n$, whose support $\Lambda$ is a uniformly discrete set. Assume that $\hat{\mu}$ is a slowly increasing measure. Then

$$\sup_{\lambda \in \Lambda} |\mu(\lambda)| < \infty, \quad (3.3)$$

and so $\mu$ is a translation-bounded measure.

This can be proved in a similar way to [LO15, Lemma 2].

**Lemma 3.2.** Let $\mu$ be a measure on $\mathbb{R}^n$, whose support $\Lambda$ is a uniformly discrete set. Assume that $\hat{\mu}$ is a slowly increasing measure, with at least one non-zero atom. Then $\Lambda$ is a relatively dense set in $\mathbb{R}^n$.

**Proof.** By Lemma 3.1 the measure $\mu$ is translation-bounded. Let us suppose that $\Lambda$ is not relatively dense, and show that this implies that $\hat{\mu}(\{s\}) = 0$ for every $s \in \mathbb{R}^n$.
Choose a Schwartz function $\varphi$ whose Fourier transform $\hat{\varphi}$ has compact support, and $\hat{\varphi}(0) = 1$. For each $0 < \delta < 1$ define $\varphi_\delta(t) := \delta^n \varphi(\delta t)$. Then we have

$$\hat{\mu}(\{s\}) = \lim_{\delta \to 0} \int \varphi_\delta(t-s)e^{2\pi i (x,t-s)} d\hat{\mu}(t)$$

uniformly with respect to $x \in \mathbb{R}^n$. On the other hand,

$$\int \varphi_\delta(t-s)e^{2\pi i (x,t-s)} d\hat{\mu}(t) = \int \varphi(x-y)e^{-2\pi i (s,y)} d\mu(y).$$

If $\Lambda$ is not relatively dense, then for any $R > 0$ there is $x \in \mathbb{R}^n$ such that the ball $x + BR$ does not intersect $\Lambda$. Using the translation-boundedness of $\mu$ this implies that for any $\delta > 0$, there are values of $x$ for which the right-hand side of (3.5) is arbitrarily close to zero. Hence the limit in (3.4) must be zero, which proves the claim. □

### 3.3. Interpolation

For a compact set $\Omega \subset \mathbb{R}^n$, we denote by $\mathcal{B}(\Omega)$ the Bernstein space consisting of all bounded, continuous functions $f$ on $\mathbb{R}^n$ such that the distribution $\hat{f}$ is supported by $\Omega$. A set $\Lambda \subset \mathbb{R}^n$ is called an interpolation set for the space $\mathcal{B}(\Omega)$ if for every bounded sequence $\{c_\lambda\}$, $\lambda \in \Lambda$, there exists at least one $f \in \mathcal{B}(\Omega)$ satisfying $f(\lambda) = c_\lambda$ ($\lambda \in \Lambda$). It is well-known that such $\Lambda$ must be a uniformly discrete set.

The following result is due to Ingham for $n = 1$, and Kahane for $n > 1$, see [OU12].

**Theorem 3.3.** There is a constant $C$ which depends on the dimension $n$ only, such that if $\Lambda$ is a uniformly discrete set in $\mathbb{R}^n$, $d(\Lambda) \geq a > 0$, then $\Lambda$ is an interpolation set for $\mathcal{B}(\Omega)$ where $\Omega$ is any closed ball of radius $C/a$.

As a consequence we obtain:

**Corollary 3.4.** There is a constant $C$ which depends on the dimension $n$ only, such that if a measure $\mu$ is supported on a uniformly discrete set $\Lambda \subset \mathbb{R}^n$, $d(\Lambda) \geq a > 0$, and if the distribution $\hat{\mu}$ vanishes on a ball of radius $C/a$, then $\mu = 0$.

**Proof.** Suppose that $\hat{\mu}$ vanishes on the ball $BR$, where $R := (C + 1)/a$ and $C$ is the constant from Theorem 3.3 (we may assume that the ball is centered at the origin). Let $\Omega = \{x : |x| \leq C/a\}$. Given $\lambda \in \Lambda$, there is $f \in \mathcal{B}(\Omega)$ such that $f(\lambda) = 1$ and $f$ vanishes on $\Lambda \setminus \{\lambda\}$. Define $\varphi(x) := f(x)\psi(x)$, where $\psi$ is a Schwartz function such that $\psi(\lambda) = 1$ and $\text{spec}(\psi) \subset B_{1/a}$. Then $\varphi$ is a Schwartz function satisfying $\varphi(\lambda) = 1$, $\varphi(\lambda') = 0$ for all $\lambda' \in \Lambda \setminus \{\lambda\}$, $\text{spec}(\varphi) \subset BR$.

Hence

$$\mu(\lambda) = \int \overline{\varphi(x)}d\mu(x) = \langle \hat{\mu}, \varphi \rangle = 0.$$

This holds for any $\lambda \in \Lambda$, so we obtain $\mu = 0$. □

### 4. Auxiliary measures $\nu_h$

Let $\mu$ be a measure on $\mathbb{R}^n$ satisfying (1.1) and (1.2), and assume that the support $\Lambda$ is a uniformly discrete set. By Lemma 3.1 the atoms of $\mu$ are bounded, so $\mu$ is a translation-bounded measure.

For each $h \in S - S$ we denote

$$S_h := S \cap (S - h) = \{s \in S : s + h \in S\},$$

(4.1)
which is a non-empty subset of $S$, and we introduce a new measure
\[\nu_h := \sum_{s \in S_h} \hat{\mu}(s) \overline{\hat{\mu}(s + h)} \delta_s.\] (4.2)

Notice that it is a non-zero, slowly increasing measure, whose support is the set $S_h$.

**Lemma 4.1.** The Fourier transform $\hat{\nu}_h$ of the measure $\nu_h$ is also a measure, which is translation-bounded and supported by the closure of the set $\Lambda - \Lambda$.

This is an elaborated version of [LO15, Lemma 12]. That lemma stated that $\text{spec}(\nu_h)$ does not intersect the punctured ball $B_a \setminus \{0\}$, where $a := d(\Lambda) > 0$. However, the result there was formulated with the roles of $\Lambda$ and $S$ interchanged, and under the stronger assumption that $\Lambda, S$ are both uniformly discrete sets.

**Proof of Lemma 4.1.** Fix a Schwartz function $\varphi$, whose Fourier transform $\hat{\varphi}$ has compact support and $\hat{\varphi}(0) = 1$. For each $0 < \delta < 1$ we denote $\varphi_\delta(x) := \delta^n \varphi(\delta x)$, and define the measure
\[\nu_h^{(\delta)}(t) := (\hat{\mu} \ast \hat{\varphi_\delta})(t + h) \cdot \hat{\mu}(t).\] (4.3)

It is a slowly increasing measure, supported by $S$, which tends to $\nu_h$ in the space of temperate distributions as $\delta \to 0$. Hence $\hat{\nu}_h^{(\delta)}$ tends to $\hat{\nu}_h$ in the same sense as $\delta \to 0$.

We will show that $\hat{\nu}_h^{(\delta)}$ is a translation-bounded measure, and moreover
\[\sup_{x \in \mathbb{R}^n} |\hat{\nu}_h^{(\delta)}|(x + B_1)\]
is bounded by some constant $C(\mu, \varphi)$ which depends on $\mu$ and $\varphi$ only (in particular, it does not depend on $\delta$). Indeed, the Fourier transform of $\nu_h^{(\delta)}$ is the measure
\[\hat{\nu}_h^{(\delta)} = (\varphi_\delta \ast e_{-h} \cdot \mu)(x) \cdot \mu(-x),\] (4.4)
where $e_{-h}(x) := e^{-2\pi i (h \cdot x)}$. Hence, we have
\[\sup_{x \in \mathbb{R}^n} |\hat{\nu}_h^{(\delta)}|(x + B_1) \leq \left\{ \sup_{x \in \mathbb{R}^n} |\mu|(x + B_1) \right\} \left\{ \int |\varphi_\delta(x)| \, |d\mu(x)| \right\} \leq C(\mu, \varphi),\]
since $\mu$ is translation-bounded. Letting $\delta \to 0$ this implies that the limit $\hat{\nu}_h$ is also a translation-bounded measure, and in fact $|\hat{\nu}_h|(x + B_1) \leq C(\mu, \varphi)$ for all $x \in \mathbb{R}^n$.

Finally, it follows from (4.4) that the measure $\hat{\nu}_h^{(\delta)}$ is supported by $\Lambda - \Lambda$. Hence its limit as $\delta \to 0$ must be supported by the closure of $\Lambda - \Lambda$, which ends the proof. □

5. **Positive-definite measures**

5.1. In this section we consider positive-definite measures whose supports are uniformly discrete sets. We establish a dichotomy concerning the discreteness of the spectrum: either it is also uniformly discrete, or it is “non-discrete” in a strong sense.

**Theorem 5.1.** Let $\mu$ be a positive-definite measure on $\mathbb{R}^n$ satisfying (1.1) and (1.2), and assume that the support $\Lambda$ is a uniformly discrete set. Then, either

(i) $S$ is also uniformly discrete; or

(ii) $S$ has a relatively dense set of accumulation points.
In particular, it follows that if the spectrum $S$ is a discrete closed set, then it must be uniformly discrete. Hence Theorem 1.1 applies to this situation, and yields that the measure $\mu$ is representable in the form (1.3). So Theorem 2.1 follows.

**Remark.** Actually we will prove that there is a constant $C$ which depends on the dimension $n$ only, such that if the spectrum $S$ is not uniformly discrete, then every ball of radius $C/d(\Lambda)$ contains infinitely many points of $S$.

5.2. We will need the following auxiliary lemma:

**Lemma 5.2.** There is a real-valued Schwartz function $\varphi$ on $\mathbb{R}^n$ which has the following properties:

(i) There is $R$ such that $\varphi(x) > 0$ for $|x| \geq R$;
(ii) $\int \varphi(x) dx = 0$;
(iii) $\text{spec}(\varphi)$ is contained in $B_1$ (the open unit ball).

**Proof.** Choose a Schwartz function $\psi > 0$ whose Fourier transform $\hat{\psi}$ is supported in the ball $\{t : |t| \leq 1/3\}$. Define $\varphi := \alpha \psi - \beta \psi^2$, where $\alpha := (\int \psi)^{-1}$ and $\beta := (\int \psi^2)^{-1}$. It is easy to verify that all the properties (i), (ii) and (iii) are satisfied. □

5.3. **Proof of Theorem 5.1.** Assume that $S$ is not uniformly discrete. Let $\varphi$ be the function given by Lemma 5.2, and $R$ be the number from property (i) of that lemma. We will show that any closed ball of radius $C/a$ contains infinitely many points of $S$, where $C := R + 1$ and $a := d(\Lambda) > 0$ (notice that the constant $C$ indeed depends on the dimension $n$ only). In particular this will show that the set of accumulation points of $S$ must be relatively dense.

By multiplying $\mu$ by an exponential (which corresponds to translation of $\hat{\mu}$), it will be enough to show that the ball $\{ |t| \leq C/a \}$ contains infinitely many points of $S$. So, suppose to the contrary that this does not hold, namely this ball contains only finitely many points of $S$.

Since $S$ is not uniformly discrete, the set $S - S$ contains elements $h \neq 0$ arbitrarily close to zero. Hence we may choose $h \in S - S$ such that the set $S_h$ defined by (4.1) does not intersect the ball $\{|t| < R/a\}$. It follows that the measure $\nu_h$ in (4.2) is a non-zero positive measure, whose support is contained in $\{|t| \geq R/a\}$. Using property (i) from Lemma 5.2 this implies that

$$\int \varphi(ax) d\nu_h(x) > 0. \tag{5.1}$$

On the other hand, we have

$$\int \varphi(ax) d\nu_h(x) = a^{-n} \int \hat{\varphi}(-t/a) d\hat{\nu}_h(t). \tag{5.2}$$

By Lemma 1.1 $\hat{\nu}_h$ is a measure, supported by the closure of $\Lambda - \Lambda$. Since $\Lambda$ is uniformly discrete, this closure is contained in the set $\{0\} \cup \{|t| \geq a\}$. But from properties (ii) and (iii) in Lemma 5.2 it follows that the function $\hat{\varphi}(-t/a)$ vanishes on this set. Hence, the right-hand side of (5.2) must vanish, in contradiction with (5.1). □
6. Spectra with finite density

In this section we prove Theorem 2.2. We will show that under the conditions in the theorem, the spectrum $S$ of the measure $\mu$ must be a uniformly discrete set. Then the final conclusion that $\mu$ is of the form (1.3) can be deduced from Theorem 1.1.

6.1. For a set $\Lambda \subset \mathbb{R}$ we denote

$$\rho(\Lambda) := \sup_{x \in \mathbb{R}} \#(\Lambda \cap [x, x + 1]).$$

Notice that $\rho(\Lambda) < \infty$ if and only if $\Lambda$ is a finite union of uniformly discrete sets.

We will need the following notion of “lower density” of a set $\Lambda \subset \mathbb{R}$, defined by

$$D_\#(\Lambda) := \liminf_{R \to \infty} \frac{\#(\Lambda \cap (-R, R))}{2R}.$$

Clearly we have $D_\#(\Lambda) \leq \rho(\Lambda)$. It will be useful below to extend the definition of the density $D_\#$ also to multi-sets $\Lambda \subset \mathbb{R}$, that is, to the case when points in $\Lambda$ occur with multiplicities. Notice that $D_\#$ is super-additive in the sense that

$$D_\#(A \cup B) \geq D_\#(A) + D_\#(B),$$

where the union $A \cup B$ is understood in the sense of multi-sets.

6.2. The following result is a more general version of [LO15, Proposition 4].

**Proposition 6.1.** Let $\Lambda \subset \mathbb{R}$ be a set with $\rho(\Lambda) \leq M < \infty$. Assume that $\Lambda$ supports a non-zero, slowly increasing measure $\mu$, such that the distribution $\hat{\mu}$ vanishes on the open interval $(0, a)$ for some $a > 0$. Then

$$D_\#(\Lambda) \geq c(a, M),$$

where $c(a, M) > 0$ is a constant which depends on $a$ and $M$ only.

This can be deduced from the following lemma:

**Lemma 6.2.** Let $\Lambda$ be a finite subset of $(-R, R) \setminus (-1, 1)$, such that $\rho(\Lambda) \leq M$, and let $a > 0$. There is $c(a, M) > 0$ such that if $(\#\Lambda)/(2R) < c(a, M)$, then one can find a Schwartz function $\varphi$ with the following properties:

$$\varphi(0) = 1, \quad \varphi(\lambda) = 0 \ (\lambda \in \Lambda), \quad \text{spec}(\varphi) \subset (0, a), \quad \sup_{|x| \geq R} |\varphi(x)| \leq 1.$$ 

The proof of Lemma 6.2 as well as the deduction of Proposition 6.1 from this lemma, can be done in a way similar to [LO15, Section 4.1], and we omit the details.

6.3. **Lemma 6.3.** Let $S \subset \mathbb{R}$ be a set with $\rho(S) < \infty$. Suppose that there is $c = c(S) > 0$ such that $D_\#(S_h) > c$ for every $h \in S - S$. Then $\rho(S - S) < \infty$.

**Proof.** Let $x \in \mathbb{R}$, and suppose that $h_1, \ldots, h_N$ are distinct points in the set $(S - S) \cap [x, x + 1]$. Since the lower density $D_\#$ is super-additive, we have

$$cN \leq \sum_{j=1}^{N} D_\#(S_{h_j}) \leq D_\#\left(\bigcup_{j=1}^{N} S_{h_j}\right).$$
where the union is understood in the sense of multi-sets. Notice that each point in this union occurs with multiplicity not greater than $\rho(S)$. It follows that $cN \leq \rho(S)D_\#(S)$, which shows that the set $S - S$ cannot have more than $\rho(S)D_\#(S)/c$ elements in any closed interval of length 1. Hence $\rho(S - S) < \infty$, which proves the claim. \qed

6.4.

Proof of Theorem 2.2. We assume that $\mu$ is a measure on $\mathbb{R}$ satisfying (1.1) and (1.2), where $\Lambda$ is a uniformly discrete set, and $S$ is a set with $\rho(S) < \infty$.

For each $h \in S - S$, let $\nu_h$ be the measure defined by (4.2). By Lemma 4.1 the Fourier transform $\hat{\nu}_h$ is supported by the closure of the set $\Lambda - \Lambda$. Since $\Lambda$ is uniformly discrete this implies that $\hat{\nu}_h$ vanishes on the open interval $(0,a)$, where $a := d(\Lambda) > 0$. As the measure $\nu_h$ is supported by $S_h$, it follows from Proposition 6.1 that $D_\#(S_h) \geq c$, where $c > 0$ is a constant which depends on $d(\Lambda)$ and $\rho(S)$.

Since this holds for every $h \in S - S$, Lemma 6.3 allows us to deduce that $\rho(S - S) < \infty$. In particular, the set $S - S$ has no accumulation point at zero, so there is $\delta > 0$ such that $(S - S) \cap (-\delta,\delta) = \{0\}$. Hence $S$ must be uniformly discrete, and in fact $d(S) \geq \delta$.

Once we have concluded that $\Lambda$ and $S$ are both uniformly discrete sets, we can apply Theorem 1.1 which yields that the measure $\mu$ is representable in the form (1.3). \qed

7. Meyer sets

7.1. In this section we show that if the support $\Lambda$ satisfies a stronger discreteness condition than in Theorem 5.1, then the conclusion of this theorem remains true without any additional positivity restriction.

Definition. A set $\Lambda \subset \mathbb{R}^n$ is called a Delone set if $\Lambda$ is both a uniformly discrete and relatively dense set.

Lemma 3.2 implies that a uniformly discrete set $\Lambda$ which supports a measure $\mu$, whose Fourier transform $\hat{\mu}$ is a pure point measure, must be a Delone set.

Definition. A set $\Lambda \subset \mathbb{R}^n$ is called a Meyer set if the following two conditions are satisfied:

(i) $\Lambda$ is a Delone set;

(ii) There is a finite set $F$ such that $\Lambda - \Lambda \subset \Lambda + F$.

The concept of Meyer set was introduced in [Mey70, Mey72] in connection with problems in harmonic analysis. After the experimental discovery of quasicrystalline materials in the middle of 80’s, Meyer sets have been extensively studied as mathematical models of quasicrystals.

There are some equivalent forms of the definition of a Meyer set, see [Moo95]. In particular, the following is true (Lagarias [Lag96]):

A Delone set $\Lambda$ is a Meyer set if and only if $\Lambda - \Lambda$ is uniformly discrete

(a simplified version of the proof of this equivalence can be found in [LO15, Lemma 8]).
7.2. Now we show that if a measure $\mu$ is supported by a Meyer set $\Lambda$, then the dichotomy phenomenon for the spectrum $S$ is valid: either $S$ is uniformly discrete, or it is non-discrete with a relatively dense set of accumulation points.

**Theorem 7.1.** Let $\mu$ be a measure on $\mathbb{R}^n$ satisfying (1.1) and (1.2), and assume that the support $\Lambda$ is a Meyer set. Then the same conclusion as in Theorem 5.1 holds.

**Proof.** As in the proof of Theorem 5.1, it will be enough to prove that for every $h \in S - S$, the set $S_h$ must intersect any ball of radius $C/a$, but now we will take $C$ to be the number from Corollary 3.4, and $a := d(\Lambda - \Lambda) > 0$.

And indeed, by Lemma 4.1 the measure $\nu_h$ defined by (4.2) is a non-zero measure, whose Fourier transform $\hat{\nu}_h$ is a translation-bounded measure supported by the set $\Lambda - \Lambda$ (this set is uniformly discrete, so it is not necessary to consider its closure). Now Corollary 3.4 applied to the measure $\hat{\nu}_h$, implies that $\nu_h$ cannot vanish on a ball of radius $C/a$. Hence $S_h$ must intersect any such a ball, which completes the proof. □

**Remark.** The proof in fact shows that there is a constant $C$ which depends on the dimension $n$ only, such that if the spectrum $S$ is not uniformly discrete, then every ball of radius $C/d(\Lambda - \Lambda)$ contains infinitely many points of $S$.

7.3. Next, we deduce Theorem 2.3 from the above result. Suppose that $\Lambda - \Lambda$ is a uniformly discrete set, and that $S$ is a discrete closed set. Hence $S$ has no finite accumulation points, so it follows from Theorem 7.1 that $S$ must be uniformly discrete.

However, to complete the conclusion of Theorem 2.3 it still remains to show that $\mu$ is representable in the form (1.3). Here one cannot directly apply Theorem 1.1, since the measure was assumed to be neither positive nor positive-definite. In order to obtain (1.3) we use instead the following version of Theorem 1.1, proved in [LO13]:

**Theorem 7.2.** Let $\mu$ be a measure on $\mathbb{R}^n$ satisfying (1.1) and (1.2). Assume that the sets $\Lambda - \Lambda$ and $S$ are both uniformly discrete. Then the conclusion of Theorem 1.1 holds.

Combining Theorems 7.1 and 7.2 thus implies the full assertion of Theorem 2.3.

8. Nowhere dense spectra

In this section we prove Theorem 2.4. We will construct a non-periodic measure $\mu$ supported on a Meyer set $\Lambda \subset \mathbb{R}^n$, such that the spectrum $S$ is not dense in any ball. Moreover, the measure $\mu$ in the construction is positive-definite, and both $\mu$ and $\hat{\mu}$ are translation-bounded measures.

8.1. Let $\Gamma$ be a lattice in $\mathbb{R}^n \times \mathbb{R}^m$, and let $p_1$ and $p_2$ denote the projections onto $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Assume that the restrictions of $p_1$ and $p_2$ to $\Gamma$ are injective, and that their images are dense in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Let $\Gamma^*$ be the dual lattice, then the restrictions of $p_1$ and $p_2$ to $\Gamma^*$ are also injective and have dense images.

If $\Omega$ is a bounded open set in $\mathbb{R}^m$, then the set
$$
\Lambda(\Gamma, \Omega) := \{p_1(\gamma) : \gamma \in \Gamma, p_2(\gamma) \in \Omega\}
$$
(8.1)
is called the “model set”, or the “cut-and-project set”, associated to the lattice $\Gamma$ and to the “window” $\Omega$. It is well-known that any model set is a Meyer set.

Meyer observed [Mey70, p. 30] (see also [Mey95]) that model sets provide examples of non-periodic uniformly discrete sets, which support a measure $\mu$ such that the Fourier
transform $\hat{\mu}$ is also a pure point measure. Such a measure may be obtained by choosing a Schwartz function $\varphi$ on $\mathbb{R}^m$ such that $\text{supp}(\hat{\varphi}) \subset \Omega$, and taking

$$\mu = \sum_{\gamma \in \Gamma} \hat{\varphi}(p_2(\gamma)) \delta_{p_1(\gamma)}.$$  \hspace{1cm} (8.2)

It is not difficult to verify that this is a translation-bounded measure, whose Fourier transform is the (also translation-bounded) pure point measure

$$\hat{\mu} = \frac{1}{\det \Gamma} \sum_{\gamma \in \Gamma^*} \varphi(p_2(\gamma^*)) \delta_{p_1(\gamma^*)}.$$  \hspace{1cm} (8.3)

However, the compact support of $\hat{\varphi}$ implies that $\varphi$ is an entire function, and so $\varphi$ cannot also be supported on a bounded set. Hence the spectrum of the measure $\mu$ is only known to be contained in $p_1(\Gamma^*)$, and so it is generally everywhere dense in $\mathbb{R}^n$.

Nevertheless, we will see that one can construct a function $\varphi$ with sufficiently many zeros, in such a way that the non-zero atoms in (8.3) in fact lie in a nowhere dense set.

8.2. A set $\Lambda \subset \mathbb{R}^n$ is said to have a uniform density $D(\Lambda)$ if

$$\frac{\# (\Lambda \cap (x + B_R))}{|B_R|} \to D(\Lambda)$$

as $R \to \infty$ uniformly with respect to $x \in \mathbb{R}^n$.

Lemma 8.1. Let $\Lambda = \Lambda(\Gamma, \Omega)$ be a model set such that the boundary of $\Omega$ is a set of Lebesgue measure zero in $\mathbb{R}^m$. Then $\Lambda$ has uniform density

$$D(\Lambda) = \frac{\text{mes}(\Omega)}{\det(\Gamma)}.$$

A proof of this fact can be found e.g. in [MM10, Proposition 5.1].

8.3. We will now assume that $m = 1$, that is, $\Gamma$ is a lattice in $\mathbb{R}^n \times \mathbb{R}$. The following theorem is the main result of this section.

Theorem 8.2. For any $\varepsilon > 0$ there is a non-zero Schwartz function $\varphi \geq 0$ on $\mathbb{R}$, such that:

(i) The measure $\mu$ in (8.2) is supported by the model set $\Lambda(\Gamma, (-\varepsilon, \varepsilon))$;

(ii) The spectrum of $\mu$ is a nowhere dense set in $\mathbb{R}^n$.

Observe that the support of $\mu$ cannot be covered by a finite union of translates of any lattice, since it contains a model set. Hence this result implies Theorem 2.4. The condition $\varphi \geq 0$ guarantees that $\mu$ is a positive-definite measure.

The proof of Theorem 8.2 depends on the following:

Lemma 8.3. For each $j \geq 1$ let $Q_j \subset \mathbb{R}$ be a set with uniform density $D(Q_j)$. Assume that

$$\sum_{j \geq 1} D(Q_j) < a.$$  \hspace{1cm} (8.4)

Then one can find positive numbers $T_j$ and a non-zero Schwartz function $\varphi$ on $\mathbb{R}$ such that:

(i) $\text{spec}(\varphi) \subset (0, a)$;
(ii) \( \varphi \) vanishes on the set \( Q \) defined by
\[
Q := \bigcup_{j \geq 1} Q'_j, \quad Q'_j := Q_j \setminus (-T_j, T_j).
\] (8.5)

Before we prove Lemma 8.3, let us first show how to deduce Theorem 8.2 from it.

**Proof of Theorem 8.2.** Let \( \{x_j\} (j \geq 1) \) be a sequence of points which are dense in \( \mathbb{R}^n \). For each \( j \), choose an open ball \( B_j \) centered at the point \( x_j \), in such a way that
\[
\sum_{j \geq 1} \operatorname{mes}(B_j) < \frac{\varepsilon}{\det(\Gamma)}.
\] (8.6)

Consider the sets \( Q_j \subset \mathbb{R}^n \) defined by
\[
Q_j := \\{ p_2(\gamma^*) : \gamma^* \in \Gamma^*, \ p_1(\gamma^*) \in B_j \}.
\]
Then each \( Q_j \) is a model set, with uniform density
\[
D(Q_j) = \det(\Gamma) \operatorname{mes}(B_j)
\]
according to Lemma 8.1. Due to (8.6) this implies that
\[
\sum_{j \geq 1} D(Q_j) < \varepsilon.
\]

Lemma 8.3 therefore gives a sequence \( \{T_j\} \) of positive numbers, and a non-zero Schwartz function \( \psi \) on \( \mathbb{R} \), \( \text{spec}(\psi) \subset (0, \varepsilon) \), such that \( \psi \) vanishes on the set \( Q \) in (8.5). Hence \( \varphi := |\psi|^2 \geq 0 \) is also a Schwartz function vanishing on \( Q \), and \( \text{spec}(\varphi) \subset (-\varepsilon, \varepsilon) \).

For each \( j \geq 1 \) there are only finitely many points of the lattice \( \Gamma^* \) lying in the set \( B_j \times (-T_j, T_j) \), so we may choose an open ball \( \Omega_j \) contained in \( B_j \) such that \( \Omega_j \times (-T_j, T_j) \) has no points in common with \( \Gamma^* \). Notice that the set
\[
\Omega := \bigcup_{j \geq 1} \Omega_j
\]
is an open, dense set in \( \mathbb{R}^n \).

We claim that the spectrum of the measure (8.2) does not intersect the set \( \Omega \). Indeed, by (8.3), an element of the spectrum is a point of the form \( p_1(\gamma^*) \), where \( \gamma^* \in \Gamma^* \) and \( \varphi(p_2(\gamma^*)) \neq 0 \). If \( p_1(\gamma^*) \in \Omega_j \) for some \( j \), then we must have \( |p_2(\gamma^*)| \geq T_j \). Hence
\[
p_2(\gamma^*) \in Q_j \setminus (-T_j, T_j) \subset Q,
\]
which is not possible as \( \varphi \) vanishes on \( Q \).

We conclude that the spectrum \( S \) of the measure \( \mu \) is contained in the closed, nowhere dense set \( \mathbb{R}^n \setminus \Omega \). On the other hand, the support \( \Lambda \) of the measure is contained in the model set \( \Lambda(\Gamma, (-\varepsilon, \varepsilon)) \), so this completes the proof. \( \Box \)

**8.4.** It remains to prove Lemma 8.3. For this we will use the celebrated Beurling and Malliavin theorem, see \([BM67]\).

First we recall the definition of the *Beurling-Malliavin upper density* (there are several equivalent ways to define this density). By a *substantial system* of intervals we mean a system \( \{I_k\} \) of disjoint open intervals on \( \mathbb{R} \), such that \( \inf_k |I_k| > 0 \), and
\[
\sum_k \left( \frac{|I_k|}{1 + \text{dist}(0, I_k)} \right)^2 = \infty.
\]
If $\Lambda \subset \mathbb{R}$ is a discrete closed set, then its Beurling-Malliavin upper density $D^*(\Lambda)$ is defined to be the supremum of the numbers $d > 0$, for which there exists a substantial system $\{I_k\}$ satisfying

$$\frac{\#(\Lambda \cap I_k)}{|I_k|} \geq d$$

for all $k$. If for any $d > 0$ no such a system $\{I_k\}$ exists, then $D^*(\Lambda) = 0$.

**Theorem 8.4** (Beurling and Malliavin [BM67]). Let $\Lambda \subset \mathbb{R}$ be a discrete closed set. Then for any $a > D^*(\Lambda)$ one can find a non-zero function $\varphi \in L^2(\mathbb{R})$ such that:

(i) $\text{spec}(\varphi) \subset (0, a)$;

(ii) $\varphi$ vanishes on $\Lambda$.

By multiplying $\varphi$ by a Schwartz function with a sufficiently small spectrum, it is clear that one may assume the function $\varphi$ in this theorem to belong to the Schwartz class.

**Remark.** It was also proved by Beurling and Malliavin that if $a < D^*(\Lambda)$ then no such $\varphi$ exists; however we will not use this part of their result.

**Proof of Lemma 8.3.** Choose a sequence $\{\gamma_j\}$ and a number $d$ such that

$$D(Q_j) < \gamma_j \quad \text{and} \quad \sum_{j \geq 1} \gamma_j < d < a.$$  

For each $j$ find a number $M_j$, such that for any interval $I$ of length $|I| \geq M_j$ we have

$$\#(Q_j \cap I) \leq \gamma_j |I|. \quad (8.7)$$

Define

$$T_j := M_j^3 + M_j.$$  

We claim that if $Q$ is the set given by $(8.5)$, then $D^*(Q) \leq d$.

Let $\{I_k\}$ be a substantial system of intervals on $\mathbb{R}$. Observe first that there must exist at least one interval $I_k$ satisfying

$$\text{dist}(0, I_k) \leq |I_k|^3. \quad (8.8)$$

For otherwise, using the assumption that $\inf_k |I_k| > 0$, this would imply that

$$\sum_k \left( \frac{|I_k|}{1 + \text{dist}(0, I_k)} \right)^2 \leq \sum_k \left( 1 + \text{dist}(0, I_k) \right)^{-4/3} < \infty,$$

which is not possible since the system $\{I_k\}$ is substantial.

Now consider an interval $I_k$ from the system, satisfying $(8.8)$. We will show that

$$\frac{\#(Q \cap I_k)}{|I_k|} < d. \quad (8.9)$$

Indeed, we have

$$\#(Q \cap I_k) \leq \sum_{j \geq 1} \#(Q'_j \cap I_k). \quad (8.10)$$

Notice that for each $j$ such that $|I_k| < M_j$, it follows from $(8.8)$ that

$$I_k \subset (-M_j^3 - M_j, M_j^3 + M_j) = (-T_j, T_j),$$
and hence $I_k$ contains no points in common with $Q'_j$. On the other hand, for each $j$ such that $|I_k| \geq M_j$ we have 
\[ \#(Q'_j \cap I_k) \leq \gamma_j |I_k| \]
according to (8.7). Hence 
\[ \sum_{j \geq 1} \#(Q'_j \cap I_k) \leq \sum_{j : M_j \leq |I_k|} \gamma_j |I_k| < d |I_k|. \]  
Combining (8.10) and (8.11) confirms that (8.9) holds.

We have thus shown that any substantial system $\{I_k\}$ contains intervals for which (8.9) holds. Hence $D^*(Q) \leq d < a$. The proof is now concluded by Theorem 8.4. □

9. HOF’S QUASICRYSTALS

There exist also other approaches to the concept of quasicrystals. One of them, which is due to Hof [Hof95], was studied by many authors, see for example [Lag00], [BG13, Chapter 9] and the references therein. In this context, the diffraction spectrum of a point set $\Lambda$ is defined through the Fourier transform of an autocorrelation measure $\gamma_\Lambda$, which is associated to the set $\Lambda$ by a certain limiting procedure.

In this section we first recall Hof’s notion of diffraction, and then apply our previous results to analyze diffraction spectra of Delone sets with finite local complexity.

9.1. Let $\Lambda \subset \mathbb{R}^n$ be a Delone set. Hof proposed to understand the diffraction by $\Lambda$ using the following procedure. For each $R > 0$, consider the measure 
\[ \gamma^R_\Lambda := (2R)^{-n} \sum_{\lambda, \lambda' \in \Lambda \cap [-R, R]^n} \delta_{\lambda'-\lambda}. \]
It is a finite measure on $\mathbb{R}^n$ which is both positive and positive-definite. The uniform discreteness of $\Lambda$ implies that the measures $\gamma^R_\Lambda$ are all translation-bounded, with the constant in (3.2) bounded uniformly with respect to $R$. Hence there exists at least one weak limit point $\gamma_\Lambda$ of the measures $\gamma^R_\Lambda$ as $R \to \infty$. Any such limit point $\gamma_\Lambda$ is called an autocorrelation measure of the set $\Lambda$. The measure $\gamma_\Lambda$ is also translation-bounded, positive and positive-definite.

The positive-definiteness of $\gamma_\Lambda$ implies that its Fourier transform $\hat{\gamma}_\Lambda$ is a positive measure. It is called a diffraction measure of $\Lambda$. If the measure $\hat{\gamma}_\Lambda$ is purely atomic, then its support $S$ is called a diffraction spectrum of $\Lambda$, and $\Lambda$ is said to be a pure point diffractive set.

More generally, one can define diffraction by any translation-bounded measure $\mu$ on $\mathbb{R}^n$, in a similar way. Denote by $\mu_R$ the restriction of $\mu$ to the cube $[-R, R]^n$, and define a measure $\tilde{\mu}_R$ by 
\[ \tilde{\mu}_R(E) := \mu_R(-E). \]
Then the measures 
\[ \gamma^R_\mu := (2R)^{-n} \mu_R * \tilde{\mu}_R \]
are uniformly translation-bounded and so have at least one weak limit point $\gamma_\mu$ as $R \to \infty$, and any such limit point is called an autocorrelation measure of $\mu$. It is again a translation-bounded, positive-definite measure, and if $\mu$ is a positive measure then also $\gamma_\mu$ is positive. The diffraction measure $\hat{\gamma}_\mu$ and the diffraction spectrum $S$ (assuming that $\hat{\gamma}_\mu$ is purely atomic) are also defined in a similar way.
Notice that the diffraction by a Delone set $\Lambda$ described above, is included as a special case which corresponds to diffraction by the measure

$$\mu = \sum_{\lambda \in \Lambda} \delta_{\lambda}. \quad (9.1)$$

9.2. A Delone set $\Lambda \subset \mathbb{R}^n$ is said to be of **finite local complexity** if for every $R > 0$ there are only finitely many different sets of the form

$$(\Lambda - \lambda) \cap B_R, \quad \lambda \in \Lambda.$$ 

It is easy to verify that this condition is equivalent to the requirement that $\Lambda - \Lambda$ is a discrete closed set.

Notice that if $\mu$ is a translation-bounded measure supported by a Delone set $\Lambda$ of finite local complexity, then any autocorrelation measure $\gamma_{\mu}$ of $\mu$ must be supported by the set $\Lambda - \Lambda$. In particular, $\gamma_{\mu}$ is a discrete measure.

Model sets are well-studied examples of non-periodic Delone sets of finite local complexity, with pure point diffraction in Hof’s sense. More precisely, if $\Lambda = \Lambda(\Gamma, \Omega)$ is a model set defined by (8.1) and such that the boundary of the “window” $\Omega$ is a set of Lebesgue measure zero, then $\Lambda$ is a pure point diffractive set, with a dense countable diffraction spectrum (see for example [BG13, Section 9.4]).

9.3. Let $\Lambda \subset \mathbb{R}^n$ be a Delone set of finite local complexity. Assume that the diffraction spectrum $S$ is uniformly discrete. Is it true that $S$ must have a periodic structure?

The question was raised in [Lag00, Problem 4.2(a)]. It follows from our previous results that the answer is positive:

**Theorem 9.1.** Suppose that

(i) $\Lambda \subset \mathbb{R}^n$ is a Delone set of finite local complexity;

(ii) $\mu$ is a positive, translation-bounded measure supported by $\Lambda$;

(iii) $\gamma_{\mu}$ is an autocorrelation measure of $\mu$; and

(iv) the support $S$ of the diffraction measure $\hat{\gamma}_{\mu}$ is a uniformly discrete set.

Then $S$ is contained in a finite union of translates of a certain lattice, and the diffraction measure has the form (1.3).

**Proof.** The autocorrelation measure $\gamma_{\mu}$ is positive, and is supported by $\Lambda - \Lambda$. Hence the diffraction measure $\hat{\gamma}_{\mu}$ is a positive-definite measure on $\mathbb{R}^n$, whose support $S$ is a uniformly discrete set, and whose spectrum is contained in the discrete closed set $\Lambda - \Lambda$. Theorem 2.1 (applied to the measure $\hat{\gamma}_{\mu}$) therefore yields that $\hat{\gamma}_{\mu}$ is of the form (1.3). As a consequence, $S$ must be contained in a finite union of translates of a lattice. \[\square\]

In particular Theorem 9.1 applies to the measure (9.1). In this case the result shows that if a Delone set $\Lambda$ of finite local complexity is pure point diffractive, and if the diffraction spectrum $S$ is uniformly discrete, then $S$ is contained in a finite union of translates of a lattice, and the diffraction measure has the form (1.3). So we obtain Theorem 2.5.
9.4. If $\Lambda$ is a Meyer set, then the conclusion in the previous result remains true even if the spectrum is just a discrete closed set, and without the positivity of the measure:

**Theorem 9.2.** Suppose that

(i) $\Lambda$ is a Meyer set in $\mathbb{R}^n$;
(ii) $\mu$ is a translation-bounded measure supported by $\Lambda$;
(iii) $\gamma_{\mu}$ is an autocorrelation measure of $\mu$; and
(iv) the support $S$ of the diffraction measure $\hat{\gamma}_{\mu}$ is a discrete closed set.

Then the same conclusion as in Theorem 9.1 is true.

This can be deduced from either Theorem 2.1 or 2.3, using the fact that the autocorrelation measure $\gamma_{\mu}$ is a positive-definite measure, supported by the Meyer set $\Lambda - \Lambda$.

**Remarks.** 1. Similarly, one can prove a dichotomy result for the diffraction spectrum of a measure $\mu$ supported by a Meyer set: either the spectrum is uniformly discrete, or it has a relatively dense set of accumulation points (using Theorem 5.1 or 7.1).

2. In the latter case, the spectrum $S$ need not be dense in any ball. One can verify that the measure constructed in the proof of Theorem 8.2 is an autocorrelation of another measure whose support is also a Meyer set.

10. **Non pure point spectrum**

10.1. In crystallography it is often interesting to consider also discrete measures $\mu$, whose Fourier transform $\hat{\mu}$ is a measure which has both a pure point component and a continuous one. The pure point component is often referred to as “Bragg peaks”, while the continuous component is called “diffuse background”.

Let $\mu$ be a (slowly increasing) measure on $\mathbb{R}^n$ with discrete support $\Lambda$:

$$\mu = \sum_{\lambda \in \Lambda} \mu(\lambda) \delta_\lambda, \quad \mu(\lambda) \neq 0.$$  \hspace{1cm} (10.1)

Assume that $\hat{\mu}$ is also a slowly increasing measure, and consider its decomposition

$$\hat{\mu} = \hat{\mu}_d + \hat{\mu}_c$$  \hspace{1cm} (10.2)

into a sum of a pure point measure

$$\hat{\mu}_d = \sum_{s \in S} \hat{\mu}_d(s) \delta_s, \quad \hat{\mu}_d(s) \neq 0,$$  \hspace{1cm} (10.3)

and a continuous measure $\hat{\mu}_c$. The set $S$ is the support of the discrete part $\hat{\mu}_d$.

We can extend our previous results to this more general situation, using the following result (Theorem 2.6): If the support $\Lambda$ is uniformly discrete, then $\hat{\mu}_d$ is the Fourier transform of another measure $\mu'$, whose support $\Lambda'$ is also a uniformly discrete set.

**Remark.** We will see from the proof that the new measure $\mu'$ is a weak limit of translates of $\mu$. Hence, in particular, the following is true:

(i) If $\mu$ is a positive measure, then also $\mu'$ is positive;
(ii) If $\Lambda - \Lambda$ is a discrete closed set, then also $\Lambda' - \Lambda'$ is a discrete closed set;
(iii) If $\Lambda - \Lambda$ is uniformly discrete, then also $\Lambda' - \Lambda'$ is uniformly discrete.
Property (i) is obvious. Properties (ii) and (iii) follow from the fact that $\Lambda' - \Lambda'$ must be contained in the closure of the set $\Lambda - \Lambda$.

10.2. First we give the proof of Theorem 10.2. We will use the following lemmas:

**Lemma 10.1.** Let $\nu$ be a finite measure on $\mathbb{R}^n$. Then

$$
\lim_{R \to \infty} (2R)^{-n} \int_{[-R,R]^n} |\hat{\nu}(t)|^2 dt = \sum |\nu(\{a\})|^2,
$$

where $a$ goes through all the atoms of the measure $\nu$.

This is the well-known Wiener’s lemma in $\mathbb{R}^n$.

**Lemma 10.2.** Let $\nu$ be a (slowly increasing) continuous measure on $\mathbb{R}^n$. Then there exist vectors $\omega_k \in \mathbb{R}^n$ ($k \geq 1$) such that the measures

$$
\nu_k(x) := e^{-2\pi i \langle \omega_k, x \rangle} \nu(x)
$$

(10.4)
tend to zero as $k \to \infty$ in the space of temperate distributions.

**Proof.** Let $\{\varphi_j\}$, $j \geq 1$, be a sequence of functions dense in the Schwartz space. Define

$$
\Phi_k(t) := \sum_{j=1}^{k} \left| \int \varphi_j(x) e^{-2\pi i \langle t, x \rangle} d\nu(x) \right|^2, \quad t \in \mathbb{R}^n.
$$

For each $j$, the measure $\varphi_j \cdot \nu$ is finite and continuous, hence by Lemma 10.1 we have

$$
\lim_{R \to \infty} (2R)^{-n} \int_{[-R,R]^n} \Phi_k(t) dt = 0.
$$

This implies that for each $k$ one can find $\omega_k \in \mathbb{R}^n$ such that $\Phi_k(\omega_k) < 1/k^2$.

Now consider the measure $\nu_k$ defined by (10.4). We have

$$
|\langle \nu_k, \varphi_j \rangle| < \frac{1}{k} \quad (k \geq j)
$$

and hence $\langle \nu_k, \varphi_j \rangle \to 0$ as $k \to \infty$, for each $j$. Since $\nu$ is a slowly increasing measure, the sequence $\nu_k$ is uniformly bounded in the space of temperate distributions. Since the $\varphi_j$ are dense in the Schwartz space, we can conclude that $\langle \nu_k, \varphi \rangle \to 0$ as $k \to \infty$, for every Schwartz function $\varphi$. This proves the lemma. \hfill \Box

**Proof of Theorem 10.2** Use Lemma 10.2 to find vectors $\omega_k$ such that

$$
e^{-2\pi i \langle \omega_k, x \rangle} \hat{\mu_c}(x) \to 0, \quad k \to \infty
$$

in the space of temperate distributions. By taking a subsequence if necessary we can also assume that the sequence $e^{2\pi i \langle \omega_k, s \rangle}$ has a limit as $k \to \infty$, for each $s \in S$. Let $\{k_j\}$ be a sufficiently fast increasing sequence such that

$$
e^{2\pi i \langle \omega_j - \omega_{k_j}, x \rangle} \hat{\mu_c}(x) \to 0, \quad j \to \infty.
$$

(10.5)

Since the exponential in (10.5) tends to 1 on $S$, it follows that the measure

$$
e^{2\pi i \langle \omega_j - \omega_{k_j}, x \rangle} \hat{\mu}(x)
$$

(10.6)

tends to $\hat{\mu_d}$ as $j \to \infty$. The measure in (10.6) is the Fourier transform of the measure

$$
\mu_j(t) := \mu(t + \omega_j - \omega_{k_j}).
$$
Therefore, $\mu_j$ tends as $j \to \infty$ to a certain distribution $\mu'$, whose Fourier transform is $\hat{\mu}_d$. By Lemma 3.1 the measure $\mu$ is translation-bounded, and therefore also $\mu'$ is a translation-bounded measure. The measure $\mu_j$ is supported by the set

$$\Lambda_j := \Lambda - \omega_j + \omega_k,$$

which is uniformly discrete with $d(\Lambda_j) = d(\Lambda)$. Hence the support $\Lambda'$ of the measure $\mu'$ must satisfy $d(\Lambda') \geq d(\Lambda)$, so $\Lambda'$ is also a uniformly discrete set. □

10.3. Theorem 2.6 allows to extend our previous results to the case when the measure $\hat{\mu}$ has also a continuous component. For example, we have:

**Theorem 10.3.** Let $\mu$ be a measure on $\mathbb{R}^n$ satisfying (10.1)–(10.3). Assume that $\Lambda$ is uniformly discrete, $S$ is discrete and closed, and at least one of the following additional conditions is satisfied:

(i) $\mu$ is positive, and $S$ is uniformly discrete;

(ii) $\hat{\mu}_d$ is positive;

(iii) $n = 1$, and $S$ satisfies condition (2.1);

(iv) $\Lambda - \Lambda$ is uniformly discrete.

Then $S$ is a uniformly discrete set, contained in a finite union of translates of a lattice, and the measure $\hat{\mu}_d$ has the form (1.3).

To prove this we consider the measure $\mu'$ given by Theorem 2.6, and apply to this measure one of Theorems 1.1, 2.1, 2.2 or 2.3, according to which one of the conditions (i)–(iv) in Theorem 10.3 is satisfied.

In a similar way, one can extend the dichotomy results given in Theorems 5.1 or 7.1.

10.4. The same applies to Hof’s diffraction by measures supported on Meyer sets:

**Theorem 10.4.** Let $\mu$ be a translation-bounded measure on $\mathbb{R}^n$, supported by a Meyer set $\Lambda$. Let $\gamma_\mu$ be an autocorrelation measure of $\mu$, and denote by $S$ the support of the discrete part of the diffraction measure $\hat{\gamma}_\mu$. Then, either

(i) $S$ is a uniformly discrete set, contained in a finite union of translates of a lattice, and the discrete part of $\hat{\gamma}_\mu$ has the form (1.3); or

(ii) $S$ is not a discrete closed set, and moreover the set of accumulation points of $S$ is relatively dense.

To prove this one can first apply Theorem 2.6 to the measure $\gamma_\mu$ which is supported by the Meyer set $\Lambda - \Lambda$, and then use Theorem 1.1 and Theorem 5.1 or 7.1.

**Remark.** An alternative proof of Theorem 10.4 which does not rely on Theorem 2.6 can be given using the following:

**Lemma 10.5.** Let $\nu$ be a translation-bounded measure on $\mathbb{R}^n$, and assume that $\hat{\nu}$ is a slowly increasing measure. Then $\nu$ has a unique autocorrelation measure $\gamma_\nu$, and the diffraction measure $\hat{\gamma}_\nu$ is a pure point measure given by

$$\hat{\gamma}_\nu = \sum_a |\hat{\nu}(\{a\})|^2 \delta_a,$$

where $a$ goes through all the atoms of the measure $\hat{\nu}$.
To prove Theorem 10.4 using this lemma, let $\nu := \gamma \mu$. Then the autocorrelation measure $\gamma_\nu$ is a discrete measure, supported by the Meyer set $\Lambda + \Lambda - \Lambda - \Lambda$, and by Lemma 10.5 the measure $\hat{\gamma}_\nu$ is a pure point measure, with the same support as the discrete part of $\hat{\gamma}_\mu$. So our previous results can be applied to the measure $\gamma_\nu$.

We omit the proof of Lemma 10.5.

11. Remarks. Open problems

11.1. Very recently, Y. Meyer has found [Mey16] an interesting version of Theorem 1.2. Namely, he constructed measures $\mu$ whose supports and spectra are discrete closed sets, which can be described by simple effective formulas. He also proved that the parameters of this construction can be chosen so that both $\Lambda$ and $S$ are rationally independent sets. This is a stronger “non-periodicity” condition than in Theorem 1.2. However, such a measure cannot be translation-bounded, see [Mey16, Lemma 5].

It should be mentioned that the last paper contains some other examples of measures with discrete closed supports and spectra. See also [Kol16]. All these examples, in one way or another, are based on the classical Poisson summation formula. Question: can one construct an example which in no way is based on Poisson’s formula?

11.2. We mention some problems which are left open.

1. The first one concerns the positivity assumption in Theorem 1.1 in several dimensions. Let $\mu$ be a measure on $\mathbb{R}^n$, $n > 1$, satisfying (1.1) and (1.2). Assume that $\Lambda, S$ are both uniformly discrete sets. Is it true that $\Lambda$ can be covered by a finite union of translates of several, not necessarily commensurate, lattices? (an example in [Fav16] shows that $\Lambda$ need not be contained in a finite union of translates of a single lattice).

2. A second problem concerns the positive-definiteness assumption in Theorem 2.1 even in dimension one: Let $\mu$ be a measure on $\mathbb{R}$, with uniformly discrete support $\Lambda$ and discrete closed spectrum $S$. Does it follow that $S$ must be also uniformly discrete?

3. The following question is also open: can one get a positive measure in Theorem 1.2?

11.3. Our approach to prove Theorem 1.1 (see [LO15]) involved a combination of analytic and discrete combinatorial considerations. In the latter part, a conclusion about the arithmetic structure of a set $\Lambda \subset \mathbb{R}^n$ was derived from information on discreteness of $\Lambda - \Lambda$. In that point we relied on results which go back to Meyer [Mey72].

In this context, it is worth to mention Freiman’s theorem [Fre73], which states that a finite set $A$ such that $\#(A + A) \leq K \#A$, must be contained in a “generalized arithmetic progression” whose dimension and size are controlled in terms of the constant $K$. It might be interesting to see whether it can also be used for a proof of Theorem 1.1.

References

[BG13] M. Baake, U. Grimm, Aperiodic order, vol. 1. Cambridge University Press, 2013.

[BM67] A. Beurling, P. Malliavin, On the closure of characters and the zeros of entire functions. Acta Math. 118 (1967), 79–93.

[BT87] E. Bombieri, J. E. Taylor, Quasicrystals, tilings, and algebraic number theory: some preliminary connections. The legacy of Sonya Kovalevskaya, Contemp. Math., vol. 64, Amer. Math. Soc., Providence, RI, 1987, pp. 241–264.

[CT87] J. W. Cahn, J. E. Taylor, An introduction to quasicrystals. The legacy of Sonya Kovalevskaya, Contemp. Math., vol. 64, Amer. Math. Soc., Providence, RI, 1987, pp. 265–286.
[Dys09] F. Dyson, Birds and frogs. Notices Amer. Math. Soc. 56 (2009), no. 2, 212–223.

[Fav16] S. Favorov, Fourier quasicrystals and Lagarias’ conjecture. Proc. Amer. Math. Soc. 144 (2016), no. 8, 3527–3536.

[Fre73] G. A. Freiman, Foundations of a structural theory of set addition. Translated from the Russian. American Mathematical Society, Providence, 1973.

[Hof95] A. Hof, On diffraction by aperiodic structures. Comm. Math. Phys. 169 (1995), no. 1, 25–43.

[Kol16] M. Kolountzakis, Fourier pairs of discrete support with little structure. J. Fourier Anal. Appl. 22 (2016), no. 1, 1–5.

[Lag96] J. C. Lagarias, Meyer’s concept of quasicrystal and quasiregular sets. Comm. Math. Phys. 179 (1996), no. 2, 365–376.

[Lag00] J. C. Lagarias, Mathematical quasicrystals and the problem of diffraction. Directions in mathematical quasicrystals, 61–93, CRM Monogr. Ser., 13, Amer. Math. Soc., Providence, 2000.

[LO13] N. Lev, A. Olevskii, Measures with uniformly discrete support and spectrum. C. R. Math. Acad. Sci. Paris 351 (2013), no. 15–16, 599–603.

[LO15] N. Lev, A. Olevskii, Quasicrystals and Poisson’s summation formula. Invent. Math. 200 (2015), no. 2, 585–606.

[LO16] N. Lev, A. Olevskii, Quasicrystals with discrete support and spectrum. Rev. Mat. Iberoam. 32 (2016), no. 4, 1341–1352.

[MM10] B. Matei, Y. Meyer, Simple quasicrystals are sets of stable sampling. Complex Var. Elliptic Equ. 55 (2010), no. 8–10, 947–964.

[Mey70] Y. Meyer, Nombres de Pisot, nombres de Salem et analyse harmonique. Lecture Notes in Mathematics 117, Springer-Verlag, 1970.

[Mey72] Y. Meyer, Algebraic numbers and harmonic analysis. North-Holland, Amsterdam, 1972.

[Mey95] Y. Meyer, Quasicrystals, diophantine approximation and algebraic numbers. Beyond quasicrystals (Les Houches, 1994), 3–16, Springer, Berlin, 1995.

[Mey16] Y. Meyer, Measures with locally finite support and spectrum. Proc. Natl. Acad. Sci. USA 113 (2016), no. 12, 3152–3158.

[Moo95] R. V. Moody, Meyer sets and their duals. The mathematics of long-range aperiodic order (Waterloo, ON, 1995), 403–441, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 489, Kluwer Acad. Publ., Dordrecht, 1997.

[OU12] A. Olevskii, A. Ulanovskii, On multi-dimensional sampling and interpolation, Anal. Math. Phys. 2 (2012), no. 2, 149–170.

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