REPEATED GAMES WITH INCOMPLETE INFORMATION AND DISCOUNTING

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Abstract. We analyze discounted repeated games with incomplete information, and such that the payoffs of the players depend only on their own type (known-own payoff case). We assume that there exists an open set of payoffs in belief-free equilibria of Horner and Lovo (2009). The assumption is generically satisfied for games with one-sided incomplete information as well as important examples of games with multi-sided incomplete information. We prove a version of the folk theorem for this setting: When players become sufficiently patient, all Nash equilibrium payoffs can be approximated by payoffs in sequential equilibria in which information is revealed finitely many times. We describe an algorithm to construct the set of equilibrium payoffs. The results are illustrated on bargaining and duopoly examples.

1. Introduction

The goal of this paper is to describe all equilibrium payoffs in repeated games in which the players have permanent and private information about their own payoff types (known-own payoffs case). The players discount the future payoffs and we are interested in the limit when players get very patient. The model of repeated games with incomplete information was introduced in Aumann et al. (1966-68) (without discounting). The model has many applications ranging from the analysis of oligopoly with privately known costs, bargaining with incomplete information about preferences, nuclear disarmament, etc.

Our description of the equilibrium payoff set has two parts. In the first part, we characterize the set of payoffs in a class of finitely revealing equilibria: sequential equilibria in which players reveal their information (by taking partially or fully separating actions) at most finitely many times. In the second part, we show all payoffs attained.
in Nash equilibria of the repeated game can be approximated by the payoffs obtained from the first part.

The first part builds on Peski (2008) who studied the special case of games with one-sided incomplete information and two types. We begin with a well-known characterization of payoffs in equilibria in which no information is revealed. Next, we describe a sequence of steps. In each step, we construct a finitely revealing equilibrium profile with continuation payoffs that belong to one of the earlier steps. We alternate between three kinds of constructions in which the initial play involves either (a) the player types pooling their actions and not revealing any information, (b) the positive probability types revealing some substantial information, or (c) the zero probability types revealing information. Each step has a simple geometric characterization.

For the second part, we assume that there exists an open set of payoffs in sequential equilibria in which during the first period of the game, all players fully reveal their information (i.e., they take fully separating actions), and such that the players are \textit{ex post} indifferent between revealing their type truthfully or reporting any other type (i.e., they are indifferent conditionally on any type of the opponent). The assumption is equivalent to the existence of belief-free equilibria of Horner and Lovo (2009), i.e., equilibria in which players’ strategies form a Nash equilibrium in the complete information game with the true realized types. The assumption is generically satisfied in games with one-sided incomplete information, and in many important examples of games with multi-sided incomplete information (like oligopoly models). However, there are generic games with two players and incomplete information on both sides that do not satisfy the assumption.

The main advantage of our result is that the payoffs in finitely revealing profiles are relatively easy to describe. We illustrate it with examples. First, we discuss a model from Aumann et al. (1966-68) of bargaining over a pie with a cherry and with incomplete information about players’ fondness for cherry. This model belongs to a class of games in which all feasible payoffs are individually rational. We show that in such games, all equilibrium payoffs can be approximated by payoffs in equilibria in which all players immediately and fully reveal their information.
Next, we discuss a class of oligopoly games. That class includes a Bertrand oligopoly with privately known production costs from Athey and Bagwell (2008). In that important paper, the authors propose mechanism design methods for analyzing repeated games with incomplete information. Here, we explain that there is a relation between the mechanism design approach and equilibria in which all players fully and immediately reveal their information. We show that in oligopoly games all equilibrium payoffs can be attained by equilibria of such type, and can be described using the mechanism design approach. As an application, we argue that the “pooling” result from Athey and Bagwell (2008) is not robust to alternative demand specifications.

In the third example, we discuss a bargaining game with two players, one-sided incomplete information, and two types (normal and “strong”) of the informed player. We assume that the game between the normal type and the uninformed player has strictly conflicting interests (Schmidt (1993)). The strong type payoffs are parametrized as a convex combination between the payoffs of the normal type and the payoffs of a player who is committed to play a single action (i.e., for whom repetition of the single action is a dominant strategy in the repeated game). We describe the Pareto frontier of the equilibrium set as a solution to a system of differential equations. We show that there are efficient equilibria that require any arbitrarily large number of periods with information revelation. When the payoffs of the strong type converge to the payoffs of the committed player, all equilibrium payoffs converge to the Stackelberg outcome of the informed player.

The initial literature on repeated games with incomplete information focused on the no-discounting case (see Aumann and Maschler (1995)). This literature typically assumes the general payoff case, in which players’ payoffs may depend on their opponents’ type.\(^1\) Hart (1985) obtains a complete characterization of equilibrium payoffs with one-sided incomplete information, general payoff case, and no discounting. He

\(^1\)To avoid players learning about the other players’ types from their own payoffs, it is often assumed that the payoffs are not observed until the end of the (infinite) repeated game. This assumption is not needed in the known-own payoffs case.
shows that all such payoffs are equal to the expectations of certain type of bimartingales. Shalev (1994) and Koren (1992) present sharper results in the known-own payoffs case. This set is equal to the set of payoffs obtained in the sequential equilibria of games with discounting in which players immediately and completely reveal all their information (see Cripps and Thomas (2003)), or the set of payoffs obtained in the belief-free equilibria (Horner and Lovo (2009)).

Kreps and Wilson (1982) and Milgrom and Roberts (1982) introduced a model of reputation with one-sided incomplete information about the type of the long-run informed player: strategic or commitment (“reputational”) types. This literature was extended to equal discounting and patient players in Cripps and Thomas (1997), Chan (2000), and Cripps et al. (2005). Because, in the reputational model, the highest payoff of the commitment type is equal to his minimax payoffs, this model does not have an open set of payoffs. On the other hand, small perturbation of the reputational types’ payoffs may create an open thread and restore the assumption. We can use the “nearby” models to test the predictions of the reputational literature. Our third example illustrates the robustness of the result of Cripps et al. (2005). In the same vein, Horner and Lovo (2009) argue that Chan (2000) result is not robust.

Cripps and Thomas (2003) are the first to study payoffs in equal discounting and general repeated games with one-sided incomplete information. They look at the limit correspondence of payoffs when the probability of one of the types is close to 1.² They show that the set of payoffs of the uninformed player and the high probability type are close to the folk theorem payoffs in a complete information game. Cripps and Thomas (1997) and Chan (2000) ask similar questions within the framework of reputation games. All these results are proved by the construction of finitely revealing equilibria.

Horner and Lovo (2009) study the general payoff case with multi-sided incomplete information and they characterize the set of payoffs obtained in belief-free equilibria in general payoff case. (Horner et al. (2009)) describe detailed conditions for information

²Cripps and Thomas (2003) also discuss the limit of payoff sets when the two players become infinitely patient, but player $I$ becomes patient much more quickly than player $U$. Their characterization is closely related to Shalev and Koren’s results for the no-discounting case.
structures in $N$-player games under which the belief-free equilibria for all payoff functions.) Our main result is limited to games in which the belief-free equilibria exist. However, our characterization of equilibrium payoffs is not limited to such equilibria. In particular, even if the belief-free equilibria exist, they may not capture all equilibrium payoffs, or even, not all efficient equilibrium payoffs (see example at the end of Section 5.2 and in Section 5.3).

There are other related papers on repeated games with discounting but with different kind of incomplete information model. Wiseman (2005) considers the situation in which the payoffs are initially unknown by all players (i.e., there is no asymmetric incomplete information), and the players learn the payoff function from observing the realization of their payoffs over time. Fudenberg and Yamamoto (2010) and Fudenberg and Yamamoto (2011) study the case where the payoffs and the monitoring structure are initially unknown, and the players may start the game with private information about the state of the world. The players learn over time by observing signals. The authors find conditions on the informativeness of the signals that ensure that the complete information folk theorem for each state. In other words, in their setting, the set of payoffs is not affected by initially incomplete information.

The next section describes the model and preliminary results. Section 3 describes the construction of the limit set of payoffs in finitely revealing equilibria. Section 4 shows that given the existence of open set of payoffs in belief-free equilibria, each Nash equilibrium payoff can be approximated by a payoff in a finitely revealing equilibrium. We illustrate the result with examples in Section 5. Most of the proofs are postponed to the Appendix.

2. Model

2.1. Notation. For each set $X \subseteq \mathbb{R}^d$, we write $\text{int}X$, $\text{cl}X$, and $\text{con}X$ to denote the interior, closure, and convexification of $X$. For each $u \in \mathbb{R}^d$, each $\varepsilon > 0$, let $B(u, \varepsilon) = \{u' : \forall i \ |u_i - u_i'| < \varepsilon\}$ be an open ball in the 'city' metric.

Suppose that $X^\delta$ is a collection of sets for $\delta < 1$ such that $X^\delta \subseteq X$ for some compact set $X$. We are interested in the limits of payoffs when $\delta \to 1$. Define
• \( \limsup_{\delta \to 1} X^\delta \) as the set of all accumulation points of sequences \((\alpha_{n,})\), where \( x_{\delta_n} \in X_{\delta_n} \) and \( \delta_n \to 1 \). The supremum limit is the least upper bound on the set of accumulation points because it contains each one.

• \( \liminf_{\delta \to 1} X^\delta \) as the set of points \( x \) such that for each sequence \( \delta_n \to 1 \), there exists sequence \( x_{\delta_n} \to x \) and such that \( x_{\delta_n} \in X_{\delta_n} \). The infimum limit is the greatest lower bound on the set of accumulation points.

2.2. Repeated game. There are \( I \) players, \( i = 1, 2, ..., I \). Before the first period of the repeated game, each player \( i \) is privately informed about her payoff type \( \theta_i \). The types are chosen by Nature from finite set \( \Theta_i \). We write \( \Theta_{-i} = \times_{j \neq i} \Theta_j \) to denote the type tuples of all players but \( i \), and \( \Theta = \times_i \Theta_i \) to denote the type profile. We also write \( \Theta^* = \Theta_1 \cup ... \cup \Theta_I \) to denote the disjoint union of the sets of types for each player.

Each type \( \theta_i \) of player \( i \) starts the game with beliefs \( \pi^{\theta_i} \in \Delta \Theta_{-i} \) about the distribution of the other players’ types. The beliefs may differ across types and they may or may not be derived from a common prior. We assume that the belief system \( \pi = \left( \pi^{\theta_i} \right)_{i, \theta_i \in \Theta_i} \) satisfies common rectangular support property: there exists set \( \Theta_j^\pi \subseteq \Theta_j \) for each player \( j \) such that for each type \( \theta_i \) of each player \( i \), \( \pi^{\theta_i} (\theta_{-i}) > 0 \) if and only if \( \theta_{-i} \in \times_{j \neq i} \Theta'_j \). We refer to \( \Theta_j^\pi \) as the \( \pi \)-support of player \( j \). We say that type \( \theta_j \) has \( \pi \)-positive probability if \( \theta_j \in \Theta_j^\pi \), and \( \pi \)-zero probability otherwise. Let \( \Pi \) denote the space of belief systems with common rectangular support.

In each period \( t \geq 0 \), each player \( i \) takes an action \( a_i \in A_i \) and receives payoffs \( g_i (a_i, a_{-i}, \theta_i) \) that depend on the actions of all players and the type \( \theta_i \) of player \( i \) (known-payoff case).

We encode the payoffs of different types of different players as a tuple \( v = (v_i (\theta_i))_{i, \theta_i} \in R^{\Theta^*} \) with an interpretation that \( v_i (\theta_i) \) is the payoff of player type \( \theta_i \) of player \( i \) (possibly, after computing an expectation with respect to the distribution over other players’ types). Using this convention, we write \( g (a) = (g_i (a, \theta_i))_{i, \theta_i} \in R^{\Theta^*} \) to denote the vector of payoffs of all types of all players obtained with action profile \( a \in A \equiv \times_i A_i \).

\(^3\)The initial version of this paper was written for two players and common prior independent types.
Let 
\[ V = \text{con} \{ g(a) : a \in A \} \subseteq R^{\Theta^*}. \]
be the convex hull of payoff vectors \( g(a) \). Let \( M < \infty \) be an upper bound on the absolute value of payoffs.

Players discount the future with common discount factor \( \delta < 1 \). We refer to the game with discount factor \( \delta \) and initial beliefs \( \pi \) as \( \Gamma(\pi,\delta) \).

We assume that \( |A_i| \geq |\Theta_i| \) for each player \( i \). We also assume that players have access to public randomization. These assumptions are for simplicity only, and they can easily be removed.

Let \( H_t = A^t \) be the set of \( t \)-period histories \( h_t = (a_s)_{s=0}^{t-1} \) (in order to keep the notation as simple as possible, we omit the reference to public randomization). A (repeated game) strategy of player \( i \) is a mapping \( s_i : \bigcup_t H_t \to \Delta A_i \). Let \( S_i \) be the set of player \( i \)'s repeated game strategies. For any profile \( s = (s_i)_i \) of such strategies, let 
\[ v^\delta (s) = (1 - \delta) E_s \sum_t \delta^t g(s(h_t)) \in R^{\Theta^*} \]
denote the (normalized) expected payoff vector, where the expectation is computed with respect to distribution over histories induced by profile \( s \). Because
\[ v^\delta (s) = \sum_a g(a) \left[ E_s \sum_t (1 - \delta) \delta^t s_1 (a_1|h_t) s_2 (a_2|h_t) \right], \]
and the terms in the square bracket are non-negative and they sum up to 1, it must be that \( v^\delta (s) \in V \).

An (incomplete information game) strategy is a mapping \( \sigma_i : \Theta_i \to S_i \). A strategy profile is a tuple \( \sigma = (\sigma_i)_i \) of strategies \( \sigma_i \). Define the expected payoff of player \( i \) type \( \theta_i \) given a belief system \( \pi \) as
\[ v^\pi,\delta (\sigma_1, \sigma_2; \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \pi^\theta_{-i} (\theta_{-i}) v^\delta (\sigma_i (\theta_i), \sigma_{-i} (\theta_{-i}); \theta_i). \]

Let \( v^\pi,\delta (\sigma) = \left( v^\pi,\delta (\sigma; \theta_i) \right)_{i, \theta_i} \in R^{\Theta^*} \) be the payoff vector of player \( i \) in game \( \Gamma(\pi,\delta) \).

2.3. Equilibrium. A strategy profile \( \sigma \) is a (Bayesian) Nash equilibrium in game \( \Gamma(\pi,\delta) \) for some \( \pi \in \Pi \) if for each player \( i \) type \( \theta_i \), strategy \( \sigma_i (\theta_i) \) is the best response of type \( \theta_i \).
A strategy profile $\sigma$ is totally mixed if for each player $i$, type $\theta_i$, history $h_t$, action $a^i$, $\sigma_i(a_i|h_t, \theta_i) > 0$. Each totally mixed strategy profile $\sigma$ together with the initial belief system $\pi \in \Pi$ induces through the Bayes formula well-defined belief mapping $p^{(\sigma, \pi)} : \bigcup_t H_t \rightarrow \Pi$. (Notice that if the initial beliefs have common rectangular support, then the posterior beliefs also have common rectangular support.) For any strategy profile $\sigma$, say that belief mapping $p : \bigcup_t H_t \rightarrow \Pi$ is $(\sigma, \pi)$-consistent, if there exists a sequence of totally mixed strategy profiles $\sigma_n \rightarrow \sigma$ such that $p^{(\sigma_n, \pi)} \rightarrow p$. If history $h_t$ has a positive probability, i.e., if for each player $i$,

$$\prod_{s < t} \sigma_i(a_i^s | h_s, \theta_i) > 0,$$

then $p(h_t)$ does not depend on the choice of sequence $\sigma_n$. We use this observation without any further reminder.

A strategy profile $\sigma$ is a sequential equilibrium in game $\Gamma(\pi, \delta)$ if there exists $(\sigma, \pi)$-consistent belief mapping $p$ such that for each player $i$ type $\theta_i$, history $h_t$, continuation strategy $\sigma_i(h_t, .)$ is the best response to continuation strategy $\sigma_{-i}(h_t, .)$ given beliefs $p_i(h_t)$.

A sequential equilibrium is $n$-revealing if for any positive probability history $h$, there exists at most $n$ periods $t$ such that $p(h_t) \neq p(h_{t-1})$. A finitely revealing equilibrium is a sequential equilibrium profile $\sigma$ that is $n$-revealing for some $n$.

Let $NE^\delta(\pi)$, $SE^\delta(\pi)$, $FR^\delta_n(\pi)$, $FR^\delta(\pi) \subseteq \mathbb{R}^{\Theta^\ast}$ be the sets of expected payoff vectors $v^{\pi, \delta}(\sigma)$ in, respectively, Nash, sequential, $n$-revealing, or finitely revealing equilibrium $\sigma$.

Before we proceed further, it is worth pointing out a few differences between repeated games with and without incomplete information. In the latter, the set of feasible and individually rational payoffs is a natural candidate for the limit equilibrium set. The main difficulty lies in constructing subgame perfect equilibria that support each such payoff. With incomplete information, it is no longer true that all feasible and individually rational payoffs can be obtained in a Nash equilibrium of the repeated game and there is no natural candidate for the limit set of Nash payoffs. Moreover, the set of equilibrium payoffs typically depend on the initial beliefs. Because any equilibrium play in which information is revealed has continuation play
in a game with posterior beliefs that differ from the prior, the payoffs sets for different beliefs are related to each other. Thus, the characterization must simultaneously describe the entire equilibrium correspondence \( NE(.) \) (or \( FR(.) \), \( SE(.) \)).

2.4. Individual rationality. Let \( \sigma \) be a Nash equilibrium profile. The expected payoff of the weighted average of types of player \( i \) cannot be smaller than the weighted minimax of player \( i \): for each \( \phi \in \mathbb{R}^{\Theta_i} \)

\[
\sum_{\theta_i} \phi_i v_i (\sigma|\theta_i) \geq m_i (\phi) := \min_{\alpha \in \Theta, \Delta \Lambda_i} \max_{\alpha_i \in \Delta \Lambda_i} \sum_{\theta_i} \phi_i g (\alpha_i, \alpha_{-i}|\theta_i). \tag{2.1}
\]

If not, then at least one \( i \)'s type would have a profitable deviation. This is standard (Blackwell (1956); see also Hart (1985), and Peski (2008) or Horner and Lovo (2009) for games with discounting). Define the set of individually rational payoffs as

\[
IR = \{ v \in \mathbb{R}^{\Theta^*} : \forall \phi \in \Phi^{d_i}, \phi \cdot v_i \geq m_i (\phi) \}.
\]

2.5. “Complete information” game. We describe the payoffs in the “complete information” game with initial prior beliefs \( \pi^{\theta^*} \) that in which all types of all players assign full probability to type tuple \( \theta^* = (\theta^*_i)_{i} \in \Theta \). It turns out that each payoff can be interpreted as an outcome of the following mechanisms. First, players signal their types (either through actions, or, if it is available, using a costless communication device). If each player signals \( \theta^*_i \), the play continues with repeated game strategies that lead to payoff vector \( u^{\theta^*_i} = \ldots = u^{\theta^*_i} \in V \). If exactly one player signals a type \( \theta_i \neq \theta^*_i \) (i.e., \( \pi^\theta \)-zero probability type \( \theta_i \)), the continuation payoff vector is equal to \( u^{\theta_i} \in V \). The zero probability signals of two or more players are treated as if all players signals are equal to \( \theta^* \). To provide \( i \) incentives for the truthful revelation, it must be that \( u^{\theta_i} (\theta_i) \geq u^{\theta'_i} (\theta_i) \) for each player \( i \) and types \( \theta_i \) and \( \theta'_i \). The payoff of each type \( \theta_i \) in such a mechanism is equal to \( v^u (\theta_i) = u^{\theta_i} (\theta_i) \). We refer to \( v^u \in \mathbb{R}^{\Theta^*} \) as the value of allocation \( u \).

Let \( U (\theta^*) \) be the set of allocations \( (u^0) \) such that \( u^0 \in V \), \( u^{\theta_i} = u^{\theta_j} \) for all players \( i \) and \( j \), and that are incentive compatible. Let

\[
NE (\theta^*) = IR \cap \{ v^u : u \in U (\theta^*) \}.
\]
Theorem 1. If set $NE(\theta^*)$ has a nonempty interior, then
\[
\limsup_{\delta \to 1} NE^\delta (\pi^{\theta^*}) = \liminf_{\delta \to 1} FR_0^\delta (\pi^{\theta^*}) = NE(\theta^*).
\]

We omit the proof, because this result is well-known (Koren (1992), Shalev (1994), Peski (2008), and Horner and Lovo (2009)). The idea is to make players use actions to report their types and receive continuation payoffs $u^{\theta_i}$ following report $\theta_i$. The only difficulty is to make sure that the continuation payoffs are individually rational after one of the players reports zero-probability types. To address this difficulty, following zero-probability report $\theta_i \neq \theta_i^*$, we make players enter a short (in expectation) phase during which they choose actions that lead to an average payoff equal to $u^{\theta_i}$. At the (stochastic) end of this phase, the players restart the process by being offered yet another opportunity to signal their zero-probability types. In this way, we can ensure that the payoffs of the true types are individually rational, and, at the same time, player $i$ obtains the long-run payoff $u^{\theta_i}$ by repeatedly signaling the zero-probability type.

3. Finitely revealing payoffs

In this section, we construct a lower bound on the limit set of payoffs in finitely revealing equilibria, $\liminf_{\delta \to 1} FR^\delta (\pi)$. The construction goes through a number of steps. We start with the “complete information” payoffs:
\[
FR_0^C (\pi) = \begin{cases} 
\text{int} NE(\theta^*), & \text{if } \pi = \pi^{\theta^*}, \\
\emptyset, & \text{otherwise}.
\end{cases}
\]

By Theorem 1, $FR_0^C (\pi) \subseteq \liminf_{\delta \to 1} FR_0^\delta (\pi)$ for each $\pi$.

Next, we construct a sequence of sets:
\[
\ldots \subseteq FR_{n-1}^C (\pi) \subseteq FR_n^A (\pi) \subseteq FR_n^B (\pi) \subseteq FR_n^C (\pi) \subseteq \ldots
\]

Each set contains payoffs in equilibria with continuation payoffs that belong to the set obtained in the previous step. There are three kinds of constructions: either (A) equilibrium continuation play can be preceded by a sequence of non-revealing actions, or (B) positive probability types can reveal nontrivial information (B), or (C) players can signal their zero probability types.
We can define the limit of these sets as $n \to \infty$. Let

$$FR = \text{cl} \bigcup FR_n^C.$$  

The main result of this section shows that the limit in the limit set of the payoffs in finitely revealing equilibria.

**Theorem 2.** For each $\pi \in \Pi$, $FR(\pi) \subseteq \liminf_{\delta \to 1} FR^\delta(\pi)$.

### 3.1. Non-revealing actions

Suppose that $u$ is a payoff in a finitely revealing equilibrium $\sigma$. Take any $v \in V$ and assume that there exists a pure action profile $a \in A$ such that $g(a) = v$. (Such an assumption is without loss of generality due to the existence of public randomization.) Construct a new strategy profile $\sigma'$ such that in each period $s \leq t$ players chose actions $a$, and in period $t$, they start playing according to profile $\sigma$. The expected payoff of $\sigma'$ is a convex combination of payoffs $u$ and $v$ with weights corresponding to the length of the initial interval:

$$u' = (1 - \delta^t) v + \delta^t u.$$  

The new strategy profile is not necessarily an equilibrium as players may have incentives to deviate in the initial phase. However, as long as $u'$ is individually rational, one can modify $\sigma'$ to turn it into an equilibrium. Define

$$FR^A_n(\pi) = \text{int} \left( IR \cap \text{con} \left( FR^C_{n-1}(\pi) \cup V \right) \right).$$  

**Lemma 1.** If $FR^C_{n-1}(\pi) \subseteq \liminf_{\delta \to 1} FR^\delta_{n-1}(\pi)$, then $FR^A_n(\pi) \subseteq \liminf_{\delta \to 1} FR^\delta_{n-1}(\pi)$.

### 3.2. Revelation of information

Information is revealed (possibly, only partially) in each period in which different types of a player play different (possibly, mixed) actions. It is convenient to distinguish between two kinds of information revelations. First, we construct equilibria in which only positive probability types reveal information. Assume that the initial beliefs are equal to $\pi$. Let $\alpha = (\alpha_i)$ be a profile of the first-period stage game strategies $\alpha_i : \Theta_i \to \Delta A_i$. We assume that each action played with positive probability by some type is also played with positive probability by a $\pi$-positive probability type: for each $a_i$, if there exists $\theta_i$ such that $\alpha_i(a_i|\theta) > 0$, then there exists type $\theta_i' \in \Theta_i^\pi$ such that $\alpha_i(a_i|\theta_i') > 0$. 

Let \( u(a) \) be the continuation payoff vector after action profile \( a \) is played. To provide incentives for players to use strategies \( \alpha_i \), it must be that for each player \( i \), type \( \theta_i \), and any positive probability action \( a_i \),

\[
E_{\pi_{\theta_i}}^\theta_i u_i (a_i, \alpha_{-i} (\theta_{-i}), \theta_i) \leq E_{\pi_{\theta_i}}^\theta_i u_i (\alpha_i (\theta_i), \alpha_{-i} (\theta_{-i}), \theta_i) .
\] (3.1)

(We ignore the (small) payoffs from choosing actions \( \alpha_i \) in the first period.)

We refer to tuple \( l = (\alpha, u) \) as a continuation lottery. Define the value of the lottery \( l \) as a payoff vector \( v_{\pi,l} \in \mathbb{R}^{\Theta_i} \) such that for each player \( i \) and type \( \theta_i \)

\[
v_{\pi,l}^i (\theta_i) = E_{\pi_{\theta_i}}^\theta_i u_i (\alpha_i (\theta_i), \alpha_{-i} (\theta_{-i}), \theta_i) .
\]

If the lottery satisfies inequalities (3.1), we say that it is \( \pi \)-incentive compatible. Let \( L(\pi) \) be the set of \( \pi \)-incentive compatible lotteries and that satisfy the above restriction on the support of players’ strategies.

For each profile of actions \( a \) played with positive probability, let \( p_{\pi,l}^a (a) = \left( p_{\pi,l,a_{-i}}^i (a_{-i}) \right)_{i,\theta_i} \) denote the posterior belief system obtained through the Bayes formula. (Notice that the beliefs of player \( i \) depend only on the actions chosen by other players.) We show that if \( v_{\pi,l} \) is the value in the lottery \( l \in L(\pi) \) such that the continuation payoffs \( u(a) \) are finitely revealing equilibria in games with initial priors \( p_{\pi,l}^a (a) \), then \( v_{\pi,l} \) is the finitely revealing payoff in the game with initial beliefs \( \pi \). Define

\[
FR_B^B (\pi) = \left\{ v_{\pi,l} \mid l \in L(\pi), \text{ and } \forall \text{ pos. prob. } a, u(a) \in \text{int} \left( FR_A^A (p_{\pi,l}^a (a)) \right) \right\} .
\]

**Lemma 2.** If \( FR_A^A (\pi) \subseteq \liminf_{\delta \to 1} FR_{n-1}^B (\pi) \), then \( FR_B^B (\pi) \subseteq \liminf_{\delta \to 1} FR_{n-1}^B (\pi) \).

### 3.3. Zero-probability types.

Next, we describe equilibria in which only zero-probability types reveal information. The idea is similar to the construction described in the “complete information” case (see the discussion following Theorem 1).

For each finitely equilibrium payoff \( v \in FR_B^B (\pi) \), we can construct a new profile in the following way. In the first period, players have an opportunity to signal their type. If all players signal positive \( \pi \)-probability types, or two or more of the players signal \( \pi \)-zero probability types, the players play strategies that lead to the initial equilibrium payoff \( v \). To simplify the subsequent notation, we write \( u^{\theta_i} = v \) for any \( \pi \)-positive probability type \( \theta_i \). Otherwise, if player \( i \) signals a \( \pi \)-zero probability type \( \theta_i \), the players’ continuation payoffs are equal to \( u^{\theta_i} \in \text{intcon} \left( V \cup FR_B^B (\pi) \right) \). In such
a case, the outcome payoff can be any convex combination of non-revealing actions with finitely revealing payoffs. In order for players to signal their types truthfully, the allocation must be incentive compatible, i.e., $u^\theta_i (\theta_i) \geq u^{\theta'_i} (\theta_i)$ for each player $i$ and types $\theta_i$ and $\theta'_i$.

Let $U_n (\pi)$ be the set of incentive compatible allocations $u^\theta_i \in \text{intcon} \left(V \cup FR^B_n (\pi)\right)$ such that $u^\theta_i = u^{\theta_j} \in FR^B_n (\pi)$ for any $\pi$-positive probability types $\theta_i$ and $\theta_j$ of players $i$ and $j$. The payoff vector $v^u$ such that $v^u_i = u^\theta_i (\theta_i)$ for each player $i$ type $\theta_i$ is called the value of allocation $u$. Define

$$FR^C_n (\pi) = \text{int}IR \cap \{v^u : u \in U_n (\pi)\}$$

**Lemma 3.** If $FR^B_n (\pi) \subseteq \liminf_{\delta \to 1} FR_{n-1}^B (\pi)$, then $FR^C_n (\pi) \subseteq \liminf_{\delta \to 1} FR_{n-1}^B (\pi)$.

Theorem 2 follows from Theorem 1, and Lemmas 1, 2, and 3.

### 4. Equilibrium payoffs

In this section, we state our main assumption and show that under this assumption, all Nash equilibrium payoffs can be approximated by payoffs in the finitely revealing equilibria.

#### 4.1. Open thread assumption

A *thread* is an assignment $u^* : \Theta^* \to R^{\Theta^*}$ such that for each player $i$, each $\theta_i, \theta'_i \in \Theta_i$ and $\theta_{-i} \in \Theta_{-i}$,

$$u^* (\theta_i, \theta_{-i}) \in NE (\theta_i, \theta_{-i}) \quad \text{and}$$

$$u^*_i (\theta_i, \theta_{-i}) = u^*_i (\theta'_i, \theta_{-i}).$$

We say that there exists an *open thread* if $u^*$ can be chosen so that $u^*_i (\theta_i, \theta_{-i}) \in \text{int}NE (\theta_i, \theta_{-i})$.

Suppose that $u^*$ is an open thread. For each $\pi \in \Pi$, define $u^* (\pi) \in R^{\Theta^*}$ so that for each player $i$ type $\theta_i$,

$$u^*_i (\theta_i | \pi) = \sum_{\theta_{-i}} \pi^\theta_i (\theta_{-i}) u^* (\theta_i |., \theta_{-i}).$$

$(u^* (\theta_i |., \theta_{-i})$ is equal to $\theta_i$-coordinate of the payoff vector $u^* (\theta'_i, \theta_{-i})$ for some $\theta'_i$; by the assumption, this value does not depend on the choice of $\theta'_i$.)
Lemma 2 implies that $u^\ast (\pi)$ is a payoff in a fully and immediately revealing equilibrium of the game with initial beliefs $\pi$ and sufficiently high $\delta$. Thus, there exists a multi-linear thread of equilibrium payoff vectors that passes through game $\Gamma (\pi, \delta)$ for each $\pi \in \Pi$.

All games with two players and one-sided incomplete information have a thread. This follows from the analysis of the non-discounted games in Shalev (1994) (see also Peski (2008)). Additionally, in the case of two players, the threads are essentially equivalent to payoffs in belief-free equilibria of Horner and Lovo (2009), and the existence of a thread is a necessary and of an open thread is a sufficient condition for the existence of such equilibria (see Appendix A).

4.2. Main result. Our main result provides a characterization of the set of equilibrium payoffs.

**Theorem 3.** If there exists an open thread, then,

$$\limsup_{\delta \to 1} NE^\delta (\pi) = \liminf_{\delta \to 1} FR^\delta (\pi) = FR (\pi).$$

The Theorem has two main implications. First, a version of the folk theorem holds: All Nash equilibrium payoffs for a sufficiently high discount factor can be approximated by payoffs in sequential, finitely revealing equilibria.

Second, together with the results from Section 3, the Theorem provides an algorithm to find an explicit description of the set of equilibria payoffs. One should start with the sets of payoffs in the complete information games for each tuple of types and then follow a sequence of well-defined steps. In Section 5, we illustrate the construction of the equilibrium set with examples.

The proof is quite simple. We take a Nash equilibrium profile, and we modify it in order to pull the continuation payoff towards the multi-linear thread $u^\ast$. To see how it works in an example, suppose that $v$ is a payoff in a Nash profile in which during the first period the players choose non-revealing action profile $a$ (i.e., all types of each player $i$ play the same action $a_i$). Let $v (a)$ be the equilibrium continuation payoffs. Then, $v$ is a convex combination of instantaneous payoffs $g (a)$ and the equilibrium continuation payoffs $v (a)$, $v = (1 - \delta) g (a) + \delta v (a)$(see Figure 4.1).
Figure 4.1.

Suppose that $v'$ is a payoff vector that is a convex combination between $v$ and the value of the thread $u^*(\pi)$, $v' = \gamma v + (1 - \gamma) u^*(\pi)$. We can find a vector $v'(a)$ such that

- $v'$ is a convex combination between $v'(a)$ and $g(a)$, $v' = (1 - \delta') g(a) + \delta' v'(a)$.

Thus, we can interpret $v'$ as a payoff $w$ in a profile that starts with action $a$ followed by continuation payoffs $v'(a)$, in a game with discount factor $\delta' > \delta$.

- $v'(a)$ is a convex combination between $v(a)$ and the thread $u^*(\pi)$, $v'(a) = \gamma' v'(a) + (1 - \gamma') u^*(\pi)$.

Simple algebra shows that

$$\gamma = \frac{\gamma'}{\gamma' (1 - \delta) + \delta}$$

which implies that $\gamma' < \gamma$. Thus, the relative distance between $v'(a)$ and the thread $u^*(\pi)$ is smaller than the relative distance between $v'$ and the thread.

If $v$ is a payoff in an equilibrium in which some information is revealed in the first period, we use the multi-linearity of thread $u^*$ to show that the relative distance to the thread is preserved in games with new posterior beliefs. Once the continuation
payoffs get sufficiently close to thread $u^*$, we conclude the profile with one period of full revelation of information followed by an equilibrium of the “complete” information game.

Formally, Theorem 3 follows from two inclusions

$$FR(\pi) \subseteq \liminf_{\delta \to 1} FR^\delta(\pi) \subseteq \limsup_{\delta \to 1} NE^\delta(\pi)$$

and

$$\limsup_{\delta \to 1} NE^\delta(\pi) \subseteq FR(\pi). \quad (4.2)$$

The first inclusion is a consequence of Theorem 2. We need to show the other inclusion.

Suppose that $u^*(\pi)$ is an open thread. Let $r > 0$ be such that for all type profiles $\theta$,

$$B\left(u^*(\pi^\theta), r\right) \subseteq NE(\theta).$$

For each $\delta < 1$, define $\gamma_1^\delta = \frac{r}{2M}$. For each $n > 1$, inductively define

$$\gamma_n^\delta = \frac{\gamma_{n-1}^\delta}{\gamma_{n-1}^\delta (1-\delta) + \delta} \in \left(\gamma_{n-1}^\delta, 1\right). \quad (4.3)$$

Notice that $\gamma_n^\delta > \gamma_{n-1}^\delta$ and $\lim_{n \to \infty} \gamma_n^\delta = 1$. Inclusion (4.2) follows from the following result.

**Lemma 4.** For each $n$ such that $(1 - \gamma_n^\delta) r > (1 - \delta) M$, for each $\pi \in \Pi$, each $v \in NE^\delta(\pi)$,

$$\gamma_n^\delta v + \left(1 - \gamma_n^\delta\right) u^*(\pi) \subseteq \text{int} FR^C_n(\pi)$$

4.3. **Proof of Lemma 4.** The proof of Lemma 4 goes by induction on $n$. First, we show the inductive claim for $n = 1$. Because $\|v\| \leq M$ for each $v \in NE^\delta(\pi)$, we have

$$\frac{r}{2M} v + \left(1 - \frac{r}{2M}\right) u^*(\pi) \in B\left(u^*(\pi), r\right) \subseteq FR^B_n(\pi) \subseteq FR^C_n(\pi).$$

The first inclusion comes from Lemma 2 and the definition of an open thread. The second comes from the definition of set $FR^C_n(\pi)$.

Next, suppose that the argument holds for $n - 1$. Take any prior beliefs $\pi$ and Nash payoff vector $v \in NE^\delta(\pi)$. Find an equilibrium profile $\sigma$ that supports $v$. Say that action $a_i$ is played with positive probability by player $i$ in the first period if there
exists $\pi$-positive probability type $\theta_i$ such that $\sigma_i(a_i|\emptyset, \theta_i) > 0$. Let $A^0_i$ denote the set of actions played with positive probability.

We assume without loss of generality that the continuation strategies are the best responses for all players and all types after all positive probability histories. (If profile $\sigma$ does not have such a property, it can be easily modified without affecting the initial payoffs and equilibrium conditions.)

**Non-revealing payoffs.** For each positive probability action profile $a \in A^0_1 \times A^0_2$, each type $\theta_i$, let

$$v(a) = \left(v_i^{p(a)}(\sigma(a, \cdot); \theta_i)\right)_{i, \theta_i} \in R^{\Theta^*}$$

be the vector of continuation payoffs after $a$. Because $a$ occurs with positive probability, we can assume w.l.o.g. that the continuation profile $\sigma(a, \cdot)$ is a Nash equilibrium and that $v(a)$ is a Nash payoff in game $\Gamma(p(a), \delta)$. By the inductive assumption,

$$\gamma_{n-1}^\delta v(a) + \left(1 - \gamma_{n-1}^\delta\right)u^*(p(a)) \in \text{int} FR_{n-1}^C(p(a)).$$

Define

$$u(a) = (1 - \delta)g(a) + \delta v(a).$$

Using (4.3), we get

$$\gamma_n^\delta u(a) + \left(1 - \gamma_n^\delta\right)u^*(p(a))$$

$$= \gamma_n^\delta [\delta v(a) + (1 - \delta)g(a)] + \left(1 - \gamma_n^\delta\right)u^*(p(a)) \quad (4.4)$$

$$= \left(1 - (1 - \delta)\gamma_n^\delta\right)\left[\gamma_{n-1}^\delta v(a) + \left(1 - \gamma_{n-1}^\delta\right)u^*(p(a))\right] + (1 - \delta)\gamma_n^\delta g(a) \quad (4.5)$$

$$\in \text{int} \text{con} \left(FR_{n-1}^C(p(a)) \cup V\right).$$

Because $v(a)$ is a payoff in a Nash equilibrium, $v(a) \in IR$. Because $(1 - \delta)M \leq \left(1 - \gamma_{n-1}^\delta\right)r$, it must be that

$$\gamma_n^\delta [\delta v(a) + (1 - \delta)g(a)] + \left(1 - \gamma_n^\delta\right)u^*(p(a)) \in \text{int} IR. \quad (4.6)$$

Then, (4.5) and (4.6) imply that for each positive probability $a$,

$$\gamma_n^\delta u(a) + \left(1 - \gamma_n^\delta\right)u^*(p(a)) \in FR_n^A(p(a)) \quad (4.7)$$

**Revelation of information.** For each $\pi$-positive probability type $\theta_i$, let

$$\alpha_i(\theta_i) = \sigma_i(\emptyset, \theta_i) \in \Delta A^0_i.$$
For each $\pi$-zero probability type $\theta_i$, let

$$\alpha_i (\theta_i) \in \arg \max_{a_i \in A_i} u (a_i, \alpha_{-i}).$$

We claim that continuation lottery $l = (\alpha, u)$ is $\pi$-incentive compatible. Indeed, for positive probability types, inequality (3.1) follows from the fact that $\sigma$ is a Nash equilibrium profile; for the zero-probability types, the inequality follows from the choice of $\alpha_i (\theta_i)$.

Consider lottery $l' = \left(\alpha, \gamma_n^\delta u (.) + \left(1 - \gamma_n^\delta\right) u^* (p(.))\right)$. We show that lottery $l'$ is $\pi$-incentive compatible. Indeed, the properties of the thread $u^*$ imply that for each positive probability $a_i \in A_i$, all types $\theta_i, \theta_{-i}$,

$$E_{\pi_i} E_{\alpha_{-i}(\theta_{-i})} u^* (\theta_i | p_i (a_{-i}), p_i (a_i, a_{-i}))$$

$$= \sum_{\theta_{-i}, a_{-i}, \theta_{-i}' \pi_{-i} \alpha_{-i} (a_{-i}, \theta_{-i}) p_i (\theta_{-i}' | a_{-i}) u^*_i (\theta_i | \theta_{-i}', \theta_{-i}')$$

$$= \sum_{\theta_{-i}} \pi_{0i} \alpha_{-i} (\theta_{-i}) u^*_i (\theta_i | \theta_{-i}')$$

$$= u^*_i (\theta_i | \pi).$$

(4.8)

In particular, the first line of (4.8) does not depend on positive probability action $a_i$. Together with the fact that lottery $l$ is $\pi$-incentive compatible, the above implies that inequalities (3.1) hold for each type $\theta_i$.

The value of lottery $l'$ is equal to

$$v^{\pi,l'} = \gamma_n^\delta v^{\pi,l} + \left(1 - \gamma_n^\delta\right) u^* (\pi),$$

where $v^{\pi,l}$ is the value of lottery $l$. Then, (4.7) implies that

$$\gamma_n^\delta v^{\pi,l} + \left(1 - \gamma_n^\delta\right) u^* (\pi) = v^{\pi,l'} \in \text{int} FR^B_n (\pi).$$

For future reference, notice that

$$v^{\pi,l}_i (\theta_i) = v_i (\theta_i) \text{ for } \pi\text{-positive probability } \theta_i,$n$$

$$v^{\pi,l}_i (\theta_i) \leq v_i (\theta_i) = v^\pi_i (\sigma; \theta_i) \text{ for } \pi\text{-zero probability } \theta_i.$$

The latter follows from the fact that action $\alpha_i (\theta_i)$ is not necessarily the best response action of zero probability type $\theta_i$. 
Zero-probability types. Finally, we define an incentive compatible and \( u : \Theta^* \to \text{intcon}(V \cup B(u^*(\pi), r)) \):

- For each \( \pi \)-positive probability type \( \theta_i \), let
  \[
u^{\pi,d} = \gamma_n^\delta v^{\pi,l} + \left( 1 - \gamma_n^\delta \right) u^*(\pi) \in \text{int}FR_n^B(\pi) .\]

- For each \( \pi \)-zero probability type \( \theta_i \), let
  \[
u^{\theta_i} = \gamma_n^\delta \sum_{\theta_{-i} \in \Theta_{-i}} \pi_{-i}(\theta_{-i}) v_i^\delta(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})) + \left( 1 - \gamma_n^\delta \right) u^*(\pi) \]

Notice that \( u^{\theta_i} \in \text{intcon}(V \cup B(u^*(\pi), r)) \).

We check that \( u \) is incentive compatible. Because \( \sigma \) is a Nash equilibrium, type \( \theta_i' \) weakly prefers strategy \( \sigma_i(\theta_i') \) rather than strategy \( \sigma_i(\theta_i) \). It follows that

\[
u^{\theta_i}(\theta_i') \leq \gamma_n^\delta \sum_{\theta_{-i} \in \Theta_{-i}} \pi_{-i}(\theta_{-i}) v_i^\delta(\sigma_i(\theta_i'), \sigma_{-i}(\theta_{-i}); \theta_i') + \left( 1 - \gamma_n^\delta \right) u^*(\pi)
= u^{\theta_i}(\theta_i') .\]

Finally, notice that for each player \( i \), type \( \theta_i \),

\[
v_i(\theta_i) = v_i^{\nu}(\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \pi_{-i}(\theta_{-i}) v_i^\delta(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}); \theta_i) .\]

Thus,

\[
\gamma_n^\delta v + \left( 1 - \gamma_n^\delta \right) u^*(\pi) = \left( u^\theta(\theta) \right)_{\theta \in \Theta_1 \cup \Theta_2} \in FR_n^C(\pi) .\]

This ends the proof.

4.4. Quality of approximation. The proof of Theorem 3 leads to the following bounds on the quality of the approximation of the Nash equilibrium set by \( n \)-revealing sets \( FR_n^C \). Recall that \( M \) is an upper bound on the absolute value of the payoffs and \( r > 0 \) is the size of the open thread.

**Corollary 1.** Let \( A = \max \left\{ \frac{2M}{r}, 2 \right\} \). For each \( v \in NE^\delta(\pi) \), each \( \epsilon > (1 - \delta)A \), and either \( n \geq \left\lceil \frac{2 \log 2 A}{\epsilon (1 - \delta)} \right\rceil \), or \( n \geq \frac{1}{(1 - \delta)^2} \),

\[
(1 - \epsilon) v + \epsilon u^*(\pi) \in FR_n^C(\pi) .\]
Proof. We show first that for each $\delta \geq \frac{1}{2}$ and each $\epsilon > 0$, if $n \geq \left\lceil \frac{\log 2A}{\epsilon (1-\delta)} \right\rceil + 1$, then $\gamma^\delta_n \geq 1 - \epsilon$. If not, then $\gamma^\delta_1 \leq \ldots \leq \gamma^\delta_n \leq 1 - \epsilon$, and
\[ \gamma^\delta_n \geq \frac{1}{\delta + (1-\delta)(1-\epsilon)} \gamma^\delta_{n-1} = \frac{1}{1-(1-\delta)\epsilon} \gamma^\delta_n \geq \left(\frac{1}{1-(1-\delta)\epsilon}\right)^{n-1} \frac{1}{2A}, \]
where the last inequality follows from the definition of $\gamma^\delta_1 = \frac{r}{2M}$. Because $-\log (1-\epsilon(1-\delta)) \geq \epsilon(1-\delta)$, we have a contradiction:
\[ \gamma^\delta_n \geq e^{(n-1)\epsilon(1-\delta)} \frac{1}{2A} \geq 1 > 1 - \epsilon. \]

Fix $v \in NE^\delta(\pi)$. Take any $\epsilon > (1-\delta)A$. By Lemma 4 and the convexity of set $FR^C_n(\pi)$, $\gamma v + (1-\gamma) u* (\pi) \in FR^C_n(\pi)$ for each $n \geq \left\lceil \frac{\log 2A}{\epsilon (1-\delta)} \right\rceil + 1$ and any $\gamma \leq \gamma^\delta_n$ such that $1 - \gamma \leq 1 - (1-\delta)A$. Letting $\gamma = 1 - \epsilon$ establishes the first result.

For the second result, take $\epsilon = (1-\delta)A$, and observe that for $A \geq 2$, $\frac{\log 2A}{A} \leq 1$. The result follows from the first part.

5. Examples

Section 3 describes an algorithm for finding all the finitely revealing payoffs. In this section, we illustrate the algorithm with three examples. In the first two examples, bargaining over a pie with a cherry and a class of oligopoly games with privately known costs, all equilibrium payoffs can be approximated by the payoffs in fully and immediately revealing equilibria. In the third example, a bargaining game with one-sided incomplete information, the set of equilibrium payoffs is substantially larger than 1-revealing payoffs. In fact, the equilibria that yield the maximal payoff for the uninformed party typically involve a large number of revelation periods.

5.1. A pie with a cherry. In the first pages of their book, Aumann and Maschler (1995) describe repeated bargaining over a pie with a cherry. A version of this model goes like this. In each period, two players must divide a pie. The pie has two parts: with and without a cherry. In each stage, player 1 proposes a division of the two parts $(x, y)$, where $x, y \in (0, 1)$. Player 2 accepts or rejects the offer. If the offer is rejected, neither one gets anything, and the players’ stage payoffs are equal to 0. If
the offer is accepted, player 1 receives payoff $u_1(x) + \theta_1 u_1(y)$, and player 2 receives payoff $u_2(1-x) + \theta_2 u_2(1-y)$, where $\theta_i$ is privately known taste for the cherry and $u_i$ are strictly increasing utility functions.

It turns out that all equilibrium payoffs for positive probability types of patient players can be obtained by the full and immediate revelation of all private information. This result follows from a slightly more general observation. Notice that in the bargaining with a cherry, the individually rational set is equal to the set of all non-negative payoffs. Because all feasible payoffs are non-negative, they are also individually rational. Moreover, if the uncertainty over tastes for cherry is nontrivial, the feasible non-revealing set has nonempty interior.

**Theorem 4.** Suppose that set $V$ has a nonempty interior and that $V \subseteq \mathbb{IR}$. Then, for each belief system $\pi \in \Pi$ such that all types of each player have $\pi$-positive probability,

$$
\limsup_{\delta \to 1} NE^\delta(\pi) = \liminf_{\delta \to 1} FR^\delta(\pi) = \text{cl} FR^B(\pi).
$$

The proof of Theorem 4 is an application of the constructive characterization of the set of equilibrium payoffs from Theorem 3. We show that none of the stages of construction after stage $1B$ adds new vectors of payoffs for positive probability types (new payoffs may be added for zero-probability types). The details can be found in Appendix D.

5.2. Oligopoly. $I$ firms are competing on the same market. Let $M_i \subseteq \mathbb{R}^{\Theta_i}$ be the set of payoff vectors attainable by firm $i$ if firm $i$ was the only firm on the market. We refer to $M_i$ as the set of monopoly payoffs. We assume that $M_i$ is convex, that it contains the zero-payoff vector $0_i \in M_i$, and that the intersection of $M_i$ with the set of strictly positive payoff vectors has a nonempty interior.

Further, we assume that each payoff vector in the game between the firms is a convex combination of monopoly payoffs:

$$
V = \text{con} \{ M_1 \times \{0_2\} \cup \{0_1\} \times M_1 \}.
$$

Thus, for each $v$, for each player $i$, there exists monopoly payoff $m_i \in M_i$ and market share $\beta_i \geq 0$ such that $\sum_i \beta_i \leq 1$, and the vector of payoffs of player $i$ is equal to $v_i = \beta_i m_i$. The interpretation is that each payoff vector can be replicated by schemes so
that periods in which firm $i$ is monopolist and firm $-i$ is inactive, and possibly periods
in which both firms are inactive (or engaged in aggressive competition). Finally, we
assume that the set of individually rational payoffs is equal to the set of vectors with
non-negative coordinates, $IR=\{v : v_i(\theta_i) \geq 0 \text{ for each } i \text{ and } \theta_i \}$. Any game with such
a structure is called an oligopoly game.

If we interpret $\theta_i$ as the cost parameter, actions as quantities or prices, then the
above assumptions are satisfied in various oligopoly models.

**Example 1.** The firms play a Cournot duopoly w. The firms choose quantities $q_1$
and $q_2$. The payoff of firm $i$ with cost type $\theta_i$ is equal to $q_i(P(q_1 + q_2) - \theta_i)$
where $P(.)$ is an inverse demand function. The payoff is equal to the fraction $\frac{q_i}{q_1+q_2}$ of the
monopoly payoff obtained from producing quantity $q_1 + q_2$. By choosing quantity 0,
each firm can ensure that its payoff is not smaller than 0. Moreover, if we assume
that $\lim q \to \infty P(q) < \inf \Theta_i$, then, by choosing a sufficiently large quantity, firm $-i$
can ensure that the profits of firm $i$ are not higher than 0.

Another model is a Bertrand oligopoly with demand $D(.)$. The firms choose prices
$p_1$ and $p_2$. The payoff of each firm $i$ is equal to $D(p_i)(p_i - \theta_i)$ if the firm $i$’s price is
lower than the price of its competitor, $\frac{1}{k}D(p_i)(p_i - \theta_i)$, if the prices of $k$ firms are
equal to the lowest price, and 0 otherwise.

**Theorem 5.** For each oligopoly game, each belief system $\pi \in \Pi$ such that all types
of each player have $\pi$-positive probability to all types of all players,

$$\limsup_{\delta \to 1} NE^\delta(\pi) = \liminf_{\delta \to 1} FR^\delta(\pi) = clFR^B_1(\pi).$$

The Theorem provides a characterization of the Nash (and sequential) equilibrium
payoffs in oligopoly games. If the prior beliefs $\pi$ assign positive probability to each
type, then all equilibrium payoffs can be obtained (or, more precisely, approximated
by payoffs) in profiles in which all firms immediately reveal their costs.

The proof of Theorem 5 is an application of the constructive characterization of
the set of equilibrium payoffs from Theorem 3. We show that none of the stages of
construction after stage $1B$ adds new vectors of payoffs for positive probability types
(new payoffs may be added for zero-probability types). The details can be found in
Appendix D.
The proof of Theorem 5 implies a simple mechanism-design type-of characterization of the set of equilibrium payoffs (see Lemma 9 for details): For each payoff vector \( v \), \( v \in \text{cl} FR_{\pi}^B \) if and only if for each type profile \( \theta \), each firm \( i \), there exist monopoly payoffs \( m_i^\theta \in M_i \) and market shares \( \beta_i^\theta \geq 0 \), such that \( \sum_i \beta_i^\theta \leq 1 \) and the following conditions hold:

1. Individual rationality: \( m_i^\theta (\theta_i) \geq 0 \),
2. Incentive compatibility: for each type \( \theta_i' \),
   \[
   v (\theta_i) = \sum_{\theta_{-i}} \pi (\theta_{-i}) \beta_i^\theta m_i^\theta (\theta_i) \\
   \geq \sum_{\theta_{-i}} \pi (\theta_{-i}) \beta_i^\theta \max \left[ m_i^{\theta_i',\theta_{-i}} (\theta_i), 0 \right].
   \]

In particular, any equilibrium payoff \( v \) can be approximated by a payoff in a profile in which firms immediately reveal their costs and if \( \theta \) is the true type profile, then player \( i \)'s payoff is equal to \( \beta_i^\theta m_i^\theta (\theta_i) \). The first condition ensures that individual rationality is satisfied \( \text{ex post} \), and the second condition ensures that firms have incentives to reveal their types truthfully (although the incentives are not necessarily \( \text{ex post} \)).

**Non-revelation result of Athey and Bagwell (2008).** As an application, we perform a quick test of the robustness of a claim from Athey and Bagwell (2008). Athey and Bagwell (2008) analyze a Bertrand model with a demand that is constant and equal to one unit below some reservation price \( r > 0 \) and the demand disappears at prices higher than \( r \). They show that for a sufficiently large discount factor, and given some assumptions on the distribution of cost types, in the (ex ante) optimal symmetric equilibrium, all players choose the same price and receive the same market share regardless of their (privately known) costs. In other words, one can sustain the best payoff in equilibrium in which no player ever reveal any information. There is no contradiction between Athey and Bagwell (2008)'s result and Theorem 5.\(^4\)

\(^4\)There are other differences between Athey and Bagwell (2008)'s and our model. For example, the demand specification does not lead to nonempty interior and our result does not apply. However, it applies to “nearby” models in which the demand below price \( r \) is not completely inelastic. In addition, Athey and Bagwell (2008) work with the continuum type model, whereas this paper assumes that there are only finitely many types. These differences do not seem to be important for this discussion.
characterization of optimal equilibrium is tight for all sufficiently high \( \delta < 1 \), whereas ours simply says that any equilibrium payoff can be approximated by fully revealing payoffs. In fact, one can construct equilibria in which players fully reveal their costs in the first period and then they proceed to ignore the revealed information. Because revelation of information is costly, it should be avoided in optimal equilibrium of Athey and Bagwell (2008).

Nevertheless, the “equal price and market share” claim is not robust to modifications of the demand. Define the monopoly payoff vector that maximizes the payoffs of type \( \theta_i \) among all monopoly payoffs of player \( i \):

\[
m^*_i = \arg \max_{m \in M_i} m(\theta_i).
\]

In Athey and Bagwell (2008), the optimal monopoly price is equal to \( r \) and does not depend on the player’s type. In general, in both Cournot or Bertrand models, if the demand function is differentiable, then the optimal monopoly action depends on the cost.

**Corollary 2.** Suppose that the monopoly actions \( m^*_\theta_i \) are not the identical for all types. Then, for any \( \pi \) that assigns positive probability to all types, for all sufficiently high \( \delta < 1 \), there is no Pareto-optimal equilibrium in which players’ behavior does not depend on type.

**Proof.** Suppose that \( v \) is an efficient payoff in a profile in which, on the equilibrium path, the players’ behavior does not depend on the type. Then, there exists \( \beta \geq 0 \) and \( m_i \in M_i \) such that

\[
v = \beta (m_1, 0_2) + (1 - \beta) (0_1, m_2)
\]

is on the Pareto-frontier of \( FR^B_1(\pi) \). For each player, construct a payoff vector \( m^*_i \) such that for each type \( \theta_i \), \( m^*_i (\theta_i) = \max \{ m^*_\theta_i (\theta_i), m_i (\theta_i) \} \geq m_i (\theta_i) \) with some inequalities strict. Using the mechanism-design characterization, it is easy to check that

\[
v^* = \beta (m^*_1, 0_{-i}) + (1 - \beta) (0_{-i}, m^*_2) \in FR^B_1(\pi)
\]

and that there exists type \( \theta_i \) such that \( v^*_i (\theta_i) > v_i (\theta_i) \). \( \square \)
Belief-free vs. fully and immediately revealing equilibria. In the above characterization of equilibrium payoffs, firms have incentives to reveal their private information \textit{ex ante}, before they learn the true types of the other player. Next, we show on an example that we cannot improve the incentives to hold \textit{ex post} (i.e., conditionally on each of the type of the other player). In particular, we show that there exist efficient repeated equilibria that are fully and immediately revealing, but that are not belief-free.

Consider a symmetric Cournot model with two players and two cost types for each player, $\Theta = \{h, l\}$, where $h > l > 0$. Let $q_{\theta}$ be the quantity that maximizes the monopoly payoffs of type $\theta$ and let $m^q \in R^\Theta$ be the monopoly payoff vector from quantity $q$. Then, the monopoly profits are maximized by the firm with low costs and choosing quantity $q_l$. We assume that the optimum is strict:

$$m^{q_l}(l) > m^{q_h}(l), m^{q_l}(h).$$

(5.1)

Additonaly, we assume that the payoff of the high type from choosing quantity $q_l$ is strictly positive, but much smaller than the maximum payoff attainable by this type:

$$m^{q_h}(h) > 6m^{q_l}(h).$$

(5.2)

We are interested in strategies that maximize the ex ante expected sum of payoffs of both firms. Because of (5.1), the first best for interior beliefs is attained if and only if:

- conditionally on both types being equal to $\theta$, the strategies use public randomization to mix between action profiles in which one of the firms is inactive, and the other one produces quantity $q^\theta$. In symmetric equilibrium, the two action profiles are chosen with equal probability,
- conditionally on firm $i$ having low costs and firm $-i$ having high costs, only firm $i$ is active and it produces quantity $q^l$.

It is clear that the first best allocation cannot be attained in belief-free equilibrium. Indeed, notice that at least one firm $i$ must expect positive profits in state in which both firms report $l$. Because firm $i$ receives zero profits if it reports $l$ and the other firm reveals $h$, firm $i$ has no ex post incentives to reveal its true type conditionally on the other firm having low costs.
On the other hand, if \( \pi_i(h) = \frac{1}{2} \) for both players \( i \), then (5.2) implies that symmetric strategy profile satisfies the ex ante incentive compatibility. In particular, the first best expected payoff can be attained in repeated game equilibrium.

5.3. Labor union - firm bargaining. Consider the following class of games parametrized with \( x \in [0, 1] \). There are two players, a labor union \((U)\) and a firm \((F)\). The firm can be either a normal type, \( \theta_F = 1 \), or a strong type, \( \theta_F = 2 \). Each player chooses between two actions, Weak and Tough. The payoffs are given in Table 1.

- When \( x = 1 \), the payoffs of the normal and strong types of the firm are equal, and the firm and the union play a multi-period bargaining model with complete information.
- When \( x = 0 \), the union \( U \) and the normal type have payoffs as in the complete information game. The strong type has a (repeated-game) dominant action to play \( T \) in every period. This an example of a model of reputation with equal discount factors for two players. The complete information game has strictly conflicting interest (Schmidt (1993)): the normal type has a commitment action \( T \) such that the union’s best reply gives the union its minimax payoff of 0. Cripps et al. (2005) show a reputational result for this class of games: for any \( p < 1 \), and for \( \delta \) high enough, all Nash equilibrium payoffs of the union and the normal type are close to \((4, 0)\).
- For intermediate \( x \), the payoff of the strong type is a convex combination between the normal type and the completely strong type of the reputation case \( x = 0 \). The techniques of Cripps et al. (2005) do not apply. (In fact, as we show, the reputational result does not hold). On the other hand, the game has an open thread assumption, and we can use Theorem 3 to compute the set of equilibrium payoffs.

| \((u, f_1, f_2)\) | Weak          | Tough         |
|-------------------|---------------|---------------|
| Weak              | 2, 2, 2x      | 0, 4, 1 + 3x  |
| Tough             | 4, 0, 0       | −2, −2, 1 − 3x|

Table 1. Payoffs in bargaining game.
The goal of this section is to describe an “upper,” Pareto-optimal, part of the equilibrium set (the “lower” part can be described in an analogous way). To simplify the exposition, we assume that $x < \frac{1}{5}$.

**Notation.** Notice that the minimax strategy of each player is $T$, which implies that the set of individually rational payoffs is equal to

$$IR = \{(u, f_1, f_2) : u \geq 0, f_1 \geq 0, f_2 \geq 1 - 3x\}.$$

In order to describe the payoff sets, we need some notation. We use $\pi \in [0, 1]$ to denote the probability of the strong type. We write $f = (f_1, f_2) \in \mathbb{R}^2$ to denote the payoffs of the two types of player $F$, and $v = (u, f) \in \mathbb{R}^3$ to denote the vector of payoffs of both players. For any $f^a \neq f^b$, let $I\left(f^a, f^b\right)$ be the interval on a two-dimensional plane that connects $f^a$ and $f^b$. For any not co-linear $v^a, v^b, v^c \in \mathbb{R}^3$, for each $f \in \mathbb{R}^2$, let $H_{v^a,v^b,v^c}(f)$ be the unique value such that $\left(H_{v^a,v^b,v^c}(f) , f\right)$ belongs to the unique affine hyperplane that passes through points $v^x$.

Figure 5.1 illustrates the payoffs of the firm’s types. We find $f^* = (f_1^*, 1 - 3x)$ such that $f^* \in I\left(g_F(W, T), g_F(W, W)\right)$. Find $\hat{f} = \left(0, \hat{f}_2\right)$ such that $f \in I\left(g_F(W, T) , g_F(T, T)\right)$. Finally, we find $f^{**} = (f_1^{**}, 1 - 3x)$ so that $H_{g(T,W),g(T,W),g(T,T)}(f^{**}) = 0$. 

**Figure 5.1.** Payoffs of the firm’s types
Define sets \( A, B', B'' \subseteq R^2 \),

\[
A = \text{con} \left\{ f^*, f^*, g_F(W, T) \right\},
\]

\[
B' = \text{con} \left\{ f^*, (0, 1 - 3x), g_F(W, T) \right\},
\]

\[
B'' = \text{con} \left\{ \hat{f}, (0, 1 - 3x), g_F(W, T) \right\}.
\]

Sets \( A, B', \) and \( B'' \) are illustrated on Figure 5.1.

For each \( f \in B' \), choose \( j'(f) \in [0, f^*] \) so that \( f \) belongs to the interval \( I(g_F(W, T), (j'(f), 1 - 3x)) \). Similarly, for each \( f \in B'' \), choose \( j''(f) \in [1 - 3x, \hat{f}] \) so that \( f \) belongs to the interval \( I(g_F(W, T), (0, j''(f))) \).

**Complete information payoffs.** We say that function \( u^\pi \) describes the upper surface of equilibria if for each \( f \),

\[
u^\pi(f) = \sup \left\{ u : (u, f) \in FR(\pi) \right\}
\]

(we take \( u^\pi(f) = -\infty \), if the right-hand side set is empty). Then, using Theorem 1, we can describe the “upper” surface of the payoffs in the complete information case \( \pi \in \{0, 1\} \). Let

\[
u^1(f) = \begin{cases} 4 - f_1, & f \in A \cup B' \cup B'', \\ -\infty, & \text{otherwise.} \end{cases}
\]

\[
u^0(f) = \begin{cases} \min \left\{ Hg(T, W), g(T, T), g(W, T) \right\}(f), 4 - \frac{1}{1+3x}f_2 \right\} & f \in A, \\ -\infty, & \text{otherwise.} \end{cases}
\]

**Incomplete information payoffs.** We use the characterization from Section 3 to construct the upper surfaces of \( FR(\pi) \). First, we construct a sequence of payoff vectors \( v_n \) that belong to a finitely revealing set in game with initial belief \( p_n = \frac{n}{N} \), where \( N < \infty \). Next, we take \( N \to \infty \) and show that the constructed path of equilibria converges to the solution of a certain differential equation.

First, consider the game with initial belief \( p_0 = 0 \). Let \( j_0 = f^* \). Due to the above description of the upper surfaces in the complete information case, \( v_0 = (0, j_0, 1 - 3x) \in FR(0) \).

Next, consider the game with initial beliefs \( p_1 \). Vector

\[
v' = \frac{1 - p_1}{1 - p_0} (0, j_0, 1 - 3x) + \frac{p_1 - p_0}{1 - p_0} (u^1(j_0, 1 - 3x), j_0, 1 - 3x).
\]
is equal to the value of the $p_1$-incentive compatible lottery in which the firm’s normal type gets revealed with probability $\frac{p_1 - p_0}{1 - p_0}$, upon which the players’ continuation payoffs are equal to $(u^1(j_0, 1 - 3x), j_0, 1 - 3x)$. If the normal type is not revealed, the labor union updates its belief to $p_0$, and the play continues with payoffs $(0, j_0, 1 - 3x)$. Because of stage $B$ of the construction of the finitely revealing set (Lemma 1), $v' \in FR(p_1)$.

Further, construct a profile in which players play actions $(T, T)$ for some fraction $\alpha$ of time, and then continue with a profile that leads to payoffs $v'$. The payoffs in such a profile are equal to

$$v_1 = \alpha g(T, T) + (1 - \alpha) v'.$$

We choose $\alpha$ so that the payoff of the labor union in vector $v$ is equal to 0. Then, by stage $A$ (Lemma 2), $v_1 = (0, j_1, 1 - 3x) \in FR(p_1)$, where

$$j_1 = \frac{(p_1 - p_0) u^1(j_0, 1 - 3x)}{1 - p_0} \left(2 + \frac{p_1 - p_0}{1 - p_0} u^1(j_0, 1 - 3x)\right)^{-1} (2 + j_0).$$

Using the same argument, we show that if $v_n = (0, j_n, 1 - 3x) \in FR(p_n)$, and $j_n$ is not too close to 0, then $v_{n+1} = (0, j_{n+1}, 1 - 3x) \in FR(p_{n+1})$, where

$$j_{n+1} = j_n + \frac{(p_{n+1} - p_n) u^1(j_n, 1 - 3x)}{1 - p_n} \left(2 + \frac{p_{n+1} - p_n}{1 - p_n} u^1(j_n, 1 - 3x)\right)^{-1} (2 + j_n).$$

After some algebraic transformations, we obtain

$$\frac{p_{n+1} - p_n}{j_{n+1} - j_n} = \frac{2 + \frac{p_{n+1} - p_n}{1 - p_n} u^1(j_n, 1 - 3x)}{2 + j_n} \frac{1 - p_n}{u^1(j_n, 1 - 3x)}.$$

By taking limit $N \to \infty$, the above equation converges to the differential equation

$$\frac{dp}{dj} = -\frac{2}{2 + j} \frac{1 - p(j)}{u^1(j, 1 - 3x)}. \quad (5.3)$$

(The minus comes from the fact that $p_{n+1} - p_n = -\frac{1}{N}$.)

Suppose that $p' : [0, f^{**}] \to [0, 1]$ is a solution to (5.3) such that $p'(f^{**}) = 0$. Choose $\pi^*$ so that $p'(0) = \pi^*$. The above analysis implies that for each $\pi \leq \pi^*$, each $j \in [0, f^{**}]$,

$$(0, j, 1 - 3x) \in FR(p'(j)).$$
Because set $FR(p'(j))$ is convex and it contains vector $g(W,T)$, it must be that $(0,f)\in FR(p'(j'(f)))$ for each $f\in B'$.

Similar equations can be derived for the elements of set $B''$. Let $p'' : [1 - 3x, \hat{f}] \to [0,1]$ be a solution to the following differential equation: $p''(1 - 3x) = \pi^*$, and
\[
\frac{dp''}{dj} = -\frac{4/3 - p''(j)}{f - j} u^1(0,j) .
\] (5.4)

Then, for each $f \in B''$, we have $(0,f) \in FR(p''(j''(f)))$.

**Proposition 1.** The following functions describe the upper surfaces of $FR(\pi)$:

- if $\pi \leq \pi^*$, let
  \[
  u^*(f) = \begin{cases} 
  \pi u^1(f) + (1 - \pi) u^0(f), & f \in A, \\
  \frac{\pi - \rho'(j'(f))}{1 - \rho'(j'(f))} u^1(f), & f \in B' \text{ and } \pi \geq p'(j'(f)), \\
  -\infty, & \text{otherwise.}
  \end{cases}
  \]

- if $\pi > \pi^*$, let
  \[
  u^*(f) = \begin{cases} 
  \pi u^1(f) + (1 - \pi) u^0(f), & f \in A, \\
  \frac{\pi - \rho'(j'(f))}{1 - \rho'(j'(f))} u^1(f), & f \in B' \text{ and } \pi \geq p'(j'(f)), \\
  \frac{\pi - \rho''(j''(f))}{1 - \rho''(j''(f))} u^1(f), & f \in B'' \text{ and } \pi \geq p''(j''(f)), \\
  -\infty, & \text{otherwise.}
  \end{cases}
  \]

**Proof.** The above discussion shows that $(u^*(f),f) \in FR(\pi)$ for each $f \in R^2$ such that $u^*(f) > -\infty$. We are left with showing that for each $u > u^*(f)$, $(u,f) \notin FR(\pi)$.

Define correspondence $F(\pi) \supseteq \{(u,f) : u \leq u^*(f)\}$ for each $\pi$. We will show that none of the operations described in Section 3 adds any payoffs to correspondence $F$.

First, notice that $F(\pi) = IR \cap \text{con} \{V \cup F(\pi)\}$.

Second, we are going to show each $\pi$-incentive compatible lottery such that the continuation payoffs belong to correspondence $F(.)$ has its value in set $F(\pi)$. Indeed, suppose that $l = (\alpha, \psi)$ is such a lottery with value $v = (u,f)$ and continuation payoffs $\psi(a) = (u(a), f(a))$ after positive probability actions $a$ of the firm. Then, $f(a) \leq f$ with equality if action $a$ is played with positive probability by the two types of the firm. Moreover, if action $a$ is played with positive probability by only one type, we
can use the description of the upper surfaces in the “complete information” games to show that \( u^{p(a)} (f) \geq u^{p(a)} (f (a)) \).

Consider lottery \( l' = (\alpha, \psi') \), where \( \psi' (a) = (u^{p(a)}, f) \) for all actions \( a \). Then, the description of the upper surface \( u^\pi \) implies that

\[
    u \leq \sum_a p (a) u^{p(a)} \leq u^\pi (f),
\]

which, in turn, implies that \((u, f) \in F (\pi)\).

Finally, notice that for \( \pi \in (0, 1) \), both types of the firm have positive probability, and stage \( C \) does not add any payoffs. These three observations imply that \( FR (\pi) \subseteq F (\pi) \) and that \( u^\pi \) is the upper surface of equilibrium payoffs.

\[\square\]

**Equilibrium behavior.** One can use the above analysis to (approximately) predict the dynamics along the equilibria that support payoffs on the upper surfaces. As an example, we describe the equilibrium behavior that induces (approximately) payoff vector \((0, f_1, 1 - 3x)\) in the game with initial beliefs \( p' (f_1) \) for some \( f_1 \in [0, f^{**}] \).

Such a profile can be described by, roughly, three phases.

- In the **revelation phase**, the labor union and the strong type of player \( F \) play Tough. The normal type of \( F \) plays Tough almost all the time. Infrequently, the normal type plays Weak fully revealing himself. The phase ends either because the normal type plays \( W \), or because the posterior probability of the normal type becomes equal to 0. In the former case, the players continue with the “normal type” phase; in the latter, the players continue with the “strong type” phase. The continuation payoff of the normal type \( f_1 \) throughout the revelation phase gradually increases with the decreasing posterior probability \( p' (f_1) \) of the normal type. The rate with which the normal type chooses \( W \) is chosen so that the continuation payoff of the labor union is equal to 0 at each moment of the revelation phase.

- In the **“normal type” phase**, players play the “complete information” game equilibrium with payoffs equal to \((u^t (p' (f_1)), f_1, 1 - 3x)\), where \( f_1 \) is the expected continuation payoff of the normal type at the moment of revelation.

- In the **“strong type” phase**, players play the equilibrium of the “complete information” game with payoffs \((0, f^{**}, 1 - 3x)\).
\[ (u_1, u_2, f_1, f_2) \quad \text{Weak} \quad \text{Tough} \\
\text{Weak} \quad 2, 2x, 2, 2x \quad 0, 0, 4, 1 + 3x \\
\text{Tough} \quad 4, 1 + 3x, 0, 0 \quad -2, 1 - 3x, -2, 1 - 3x \\
\]

Table 2. Payoffs in bargaining game.

In a similar way, we can describe strategy profiles that induce any other payoff on the upper surface.

*Two-sided incomplete information.* We show that if \( x \leq \frac{3}{100} \), then a version of the above model with symmetric, two-sided incomplete information does not have any threads. Indeed, suppose that there are two types of each player, and the payoffs are given in Table 2.

On the contrary, suppose that \( u^* (\pi) \) is the thread. Let \( u^{ns} = u^* (\pi (\text{normal}_1, \text{strong}_2)) \) be the thread Nash equilibrium payoff vector given that the first player is revealed to be normal and the second player is revealed to be strong. Because the equilibrium payoffs must be individually rational, it must be that

\[
\begin{align*}
    u^{ns}_{1} (\text{normal}_1) &\geq 0 \\
    u^{ns}_{2} (\text{strong}_2) &\geq 1 - 3x.
\end{align*}
\]

By Theorem 1, there exists \( \alpha \in \Delta A \) such that

\[
\begin{align*}
    u^{ns}_{1} (\text{normal}_1) &= 2\alpha_{WW} + 4\alpha_{WT} - 2\alpha_{TT} \geq 0, \\
    u^{ns}_{2} (\text{strong}_2) &= 2x\alpha_{WW} + (1 + 3x)\alpha_{WT} + (1 - 3x)\alpha_{TT} \geq 1 - 3x,
\end{align*}
\]

and

\[
u^{ns}_{2} (\text{normal}_2) \geq 2\alpha_{WW} + 4\alpha_{WT} - 2\alpha_{TT}.
\]

The next result shows that \( u^{ns}_{2} (\text{normal}_2) > 2 \).

*Lemma 5.* Suppose that \( x \leq \frac{3}{100} \). Then, \( 2\alpha_{WW} + 4\alpha_{WT} - 2\alpha_{TT} > 2 \) for each \( \alpha \in \Delta A \) that satisfies inequalities (5.5).

The proof of Lemma 5 can be found in Appendix E.

A symmetric argument shows that \( u^{sn}_{1} (\text{normal}_1) > 2 \), where \( u^{sn} \) is the thread equilibrium payoff vector if the first player is strong, and the second player is normal.
Because players must be ex post indifferent about revealing their type truthfully, we have

\[ u_2^{nn}(\text{normal}_2) = u_2^{ns}(\text{normal}_2) > 2, \]
\[ u_1^{nn}(\text{normal}_1) = u_1^{sm}(\text{normal}_1) > 2, \]

where \( u^{nn} \) is the thread payoff vector if both players are revealed to be normal.

On the other hand, the sum of the payoffs of the normal types given any action profile is never higher than 4. This implies that for any equilibrium payoff vector \( u \in NE(\text{normal}_1, \text{normal}_2), u_1(\text{normal}_1) + u_2(\text{normal}_2) \leq 4 \). The contradiction shows that \( u^* \) cannot be a thread.

6. Conclusions

In this paper, we provide a characterization of the equilibrium payoffs in repeated games with incomplete information, with discounting, known-own payoffs, and permanent types. We assumed there exists an open multi-linear thread of payoffs in equilibria in which in the first period of the game, players fully reveal their information (i.e., all types of each players take separating actions), and such that the players are ex post indifferent between revealing their type truthfully or reporting any other type (i.e., they are indifferent conditionally on any type of the opponent). The assumption is generically satisfied in games with one-sided incomplete information as well as some important examples of games with multi-sided incomplete information.

Our characterization provides an algorithm to construct the equilibrium set through a sequence of geometric operations. This algorithm can be implemented numerically. In examples, we show the characterization can be used to find the exact description of the equilibrium sets analytically. Further work is required to build tools that allow for analytical description in general games. For instance, the equilibrium set in the bargaining problem from Section 5.3 is described as a solution to a certain ordinary differential equation. This method can be easily generalized to other games with one-sided uncertainty and two types. We suspect that differential equations play an important role in more general settings (with more types or with multi-sided uncertainty), but we do not know how to do it.
Other questions are left open by this paper. Most importantly, we would like to know whether a similar characterization holds for games in which an open thread assumption is not satisfied (see an example at the end of section 5.3 or Horner and Lovo (2009)). Our current methods do not allow us to form a hypothesis one way or the other. It would be interesting to check whether the current analysis extends in some way to the case of persistent types.\footnote{Athey and Bagwell (2008) introduce a model of persistent types. Escobar and Tolalka (2011) prove a folk theorem for limit $\delta \to 1$ and fixed rates of transitions. One can consider an alternative limit $\delta \to 1$ when the probability of transitions scales with $1 - \delta$.} We leave these questions for future research.

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Appendix A. Threads and belief-free equilibria with two players

Horner and Lovo (2009) give two necessary conditions for the existence of belief-free equilibria in the case of two players. We restate the conditions in our notation and in the known-payoff case. For each probability distribution $\alpha \in \Delta A$, let $g(\alpha) \in V$ be the expectation of payoff vectors $g(a)$ taken with respect to $\alpha$. Take a pair of vectors $v_i \in R^{\Theta_i \times \Theta_2}$ for each player $i$.

- Vectors $(v_1, v_2)$ satisfy Individual Rationality if for each player $i$, each type $\theta_{-i}$, the payoffs of player $i$ types are individually rational: $\forall \phi \in \Phi_i, \phi \cdot v_i^{\theta_{-i}} \geq m_i(\phi)$, where $m_i(\phi)$ is the value of the weighted minimax defined in (2.1).

- Vectors $(v_1, v_2)$ satisfy Incentive Compatibility if for each type profile $(\theta_1, \theta_2)$, there exists $\alpha_{\theta_i, \theta_2} \in \Delta A$ such that for each type profile $(\theta_1, \theta_2)$, player $i$, type $\theta_i'$,

$$v_i^{\theta_i', \theta_{-i}} = g_i(\alpha_{\theta_i, \theta_{-i}} | \theta_i) \geq g_i(\alpha_{\theta_i', \theta_{-i}} | \theta_i).$$

Say that $u^*: \Theta_1 \times \Theta_2 \rightarrow R^{\Theta^*}$ is a thread if conditions (4.1) for each player $i$, each $\theta_i, \theta_i' \in \Theta_i$ and $\theta_{-i} \in \Theta_{-i}$ are satisfied. The next result shows that the Threads are essentially equivalent to Individually Rational and Incentive Compatible payoff vectors.

Lemma 6. Suppose that $u^*: \Theta_1 \times \Theta_2 \rightarrow R^{\Theta^*}$ is a thread. Let $(v_1, v_2)$ be a pair of vectors $v_i \in R^{\Theta_i \times \Theta_2}$ such that $v_i^{\theta_i, \theta_{-i}} = u^*(\theta_i, \theta_{-i}|\theta_i)$ for each player $i$. Then, $(v_1, v_2)$ satisfy Individual Rationality and Incentive Compatibility.

Conversely, suppose that pair of vectors $v_i \in R^{\Theta_i \times \Theta_2}$ satisfies Individual Rationality and Incentive Compatibility. For each player $i$ types $\theta_i, \theta_i' \in \Theta_i$, and $\theta_{-i} \in \Theta_{-i}$, let

$$u^*(\theta_i', \theta_{-i}|\theta_i) = v_i^{\theta_i', \theta_{-i}}.$$

Then, $u^*$ is a thread.

Proof. Part I. Suppose that $u^*$ is a thread. By the definition of sets $NE(\theta_1, \theta_2)$ from Theorem 1, there exist probability distributions $\alpha_{\theta_i, \theta_{-i}} \in \Delta A$ such that for each type profile $(\theta_1^*, \theta_2^*)$, and for each $\theta_i, \theta_{-i}$,

$$u^*(\theta_i|\theta_1^*, \theta_2^*) = g_i\left(\alpha_{\theta_i, \theta_{-i}}^{\theta_i^*, \theta_2^*} | \theta_i\right),$$
and for each player $i$ and all types $\theta_i, \theta'_i$,
\[
g_i \left( \alpha_{\theta_i, \theta' \mid i} \right) \geq g_i \left( \alpha_{\theta'_i, \theta' \mid i} \right)
\]
Define
\[
v'_{i, \theta, \theta'} = u^* (\theta_i, \theta_{-i} | \theta_i).
\]
Because $u^*$ is a thread, for each player $i$, type $\theta_{-i}$, each type $\theta'_i$
\[
v'_{i, \theta, \theta'} = u^* (\theta'_i, \theta_{-i} | \theta_i).
\]
Because $u^* (\theta'_i, \theta_{-i}) \in IR$, the payoffs of types of player $i$ in the vector $u^* (\theta'_i, \theta_{-i})$ are individually rational. This shows that vectors $(v_1, v_2)$ satisfy Individual Rationality.

Next, we show that $(v_1, v_2)$ satisfies Incentive Compatibility. For each type profile $(\theta_1, \theta_2)$, define
\[
\alpha^*_{\theta_1, \theta_2} = \alpha_{\theta_1, \theta_2} \in \Delta A.
\]
Then,
\[
v'_{i, \theta, \theta'} = g \left( \alpha^*_{\theta_1, \theta_2} \right),
\]
and
\[
v'_{i, \theta, \theta'} = g \left( \alpha^*_{\theta_1, \theta_2} \right) = g \left( \alpha_{\theta_1, \theta_2} \right) = u^* (\theta_1, \theta_2) = u^* (\theta'_1, \theta_2)
\]
\[
= g \left( \alpha^*_{\theta_1, \theta_2} \right) = g \left( \alpha_{\theta_1, \theta_2} \right) = g_i \left( \alpha_{\theta_{-i} \mid i} \right).
\]

**Part II.** Suppose that pair of vectors $v_i \in R^{\Theta_1 \times \Theta_2}$ satisfies Individual Rationality and Incentive Compatibility. Let $\alpha_{\theta_1, \theta_2} \in \Delta A$ be as in the definition of Incentive Compatibility. For each profile $(\theta_1, \theta_2)$, each player $i$ type $\theta'_i$, define
\[
u^* (\theta_1, \theta_2 | \theta'_i) = v'_{i, \theta, \theta'} = g_i \left( \alpha_{\theta_{-i} \mid i} \right).
\]
Then, for each profile $(\theta'_1, \theta'_2)$, the vector of the payoffs of player $i$ types, $u^* (\theta_i, \theta'_i) = v'_{i, \theta, \theta'}$, is individually rational. Thus, $u^* (\theta'_1, \theta'_2) \in IR$. Moreover, for each profile $(\theta'_1, \theta'_2)$, and any two types $\theta_i, \theta'_i$,
\[
u^* (\theta'_1, \theta'_2 | \theta_i) = g_i \left( \alpha_{\theta_{-i} \mid i} \right) \geq g_i \left( \alpha_{\theta_{-i} \mid i} \right).
\]
This shows that $u^* (\theta'_1, \theta'_2) \in NE (\theta'_1, \theta'_2)$. \qed
Appendix B. Proofs of section 3

B.1. Preliminary results.

**Lemma 7.** For each $\varepsilon > 0$, there exists $\delta^\varepsilon < 1$, $m^\varepsilon < \infty$ such that for each player $i$, each $m \geq m^\varepsilon$, and each $v$ such that $B(v, \varepsilon) \subseteq \mathbb{R}$, there exists $m$-period strategies of players $j \neq i$, $\mu_{j, m^\varepsilon, \varepsilon}^v : \bigcup_{s < m^\varepsilon} (A_i)^{s-1} \rightarrow \Delta A_j$ such that for any sequence $\hat{a}^i = (a_{i0}^i, ..., a_{im^\varepsilon-1}^i)$ of actions of player $i$, each type $\theta_i$, each $\delta \geq \delta^\varepsilon$

$$M_i^{v, m^\varepsilon, \delta} (\hat{a}^i; \theta_i) := \frac{1 - \delta}{1 - \delta^m} \sum_{s=0}^{m^\varepsilon-1} \delta^s E g_i \left( a_{is}^i, \mu_{-i}^{v, \varepsilon} (a_{i0}^i, ..., a_{is-1}^i); \theta_i \right) \leq v_i (\theta_i),$$

where the expectation is taken over actions induced by strategies $\mu_{-i}^{v, m^\varepsilon, \varepsilon}$.

**Proof.** The Lemma is a discounted version of the Blackwell approachability argument (Blackwell (1956)). The proof follows the same line and an observation that when $\delta \rightarrow 1$, the discounted payoff criterion in a game with finitely many periods converges to the average payoff criterion.

B.2. Proof of Lemma 1. Before we present the details, we briefly describe the main idea. Take any $v^* \in FR_A^n (\pi) = \text{int} \mathbb{R} \cap \text{con} \left( \text{int} FR_{n-1}^C (\pi) \cup V \right)$. Find $\alpha^* \in (0, 1)$, $g^* \in V$, and $u^* \in \text{con}_{\text{int}} FR_{n-1}^C (\pi)$ such that $v^* = \alpha^* g^* + (1 - \alpha^*) u^*$. Assume that there exists a pure action profile $a^*$ such that $g(a^*) = g^*$. The assumption is without loss of generality due to public correlation.

Find a sequence of $t_\delta$ such that $\delta^t \rightarrow 0$ as $\delta \rightarrow 1$. We are going to construct a profile in which during the initial $t^\delta$ periods, players play action profile $a^*$ and then continue with a finitely revealing equilibrium with payoffs $u$ so that $v^* = \left( 1 - \delta^t \right) g^* + \delta^t u$. Any deviation by player $i$ during period $t$ triggers a punishment phase in which player $i$ is initially minimaxed using the strategy from Lemma 7, and then the players continue with a strategy profile that induces payoffs $v^i (\hat{a})$ that depend on the realized actions during the minimaxing. The continuation payoffs after the minimaxing are chosen so that all players are indifferent among all actions during the minimaxing phase and the overall payoff from the punishment of player $i$ phase is equal to $v^{i, t^\delta-t} = \left( 1 - \delta^{t^\delta-t} \right) g^* + \delta^{t^\delta-t} u^{i*}$. We choose $u$ and $u^{i*}$ so that they are sufficiently close to $u^*$.
and can be sustained by finitely revealing equilibria for sufficiently high $\delta$. Moreover, we need to choose $u^i*$ so that no player has proper incentives not to deviate.

Let $k^* = \frac{100}{1-\alpha^*}$ and find $\epsilon > 0$ so that $B(u^*, 2k\epsilon) \subseteq \text{conint} \mathcal{FR}^C_{n-1}(\pi)$. Using compactness, one can show that for sufficiently high $\delta$, for each $u \in B(u^*, k\epsilon)$, there exists a strategy profile $\sigma^u$ that induces payoff $u$ and such that is a $(n-1)$-revealing equilibrium of game $\Gamma(\pi, \delta)$. (It might be necessary to use public randomization if $u^* \notin \text{int} \mathcal{FR}^C_{n-1}(\pi)$.)

For each player $i$, find $u^i* \in B(u^*, k\epsilon)$ so that

\[ u^i*(\theta_i) \leq u^*(\theta) - \frac{2\epsilon}{1-\alpha^*} \quad \text{for each } \theta_i, \quad \text{and} \]

\[ u^i*(\theta_{-i}) \geq u^*(\theta_{-i}) \quad \text{for each } \theta_{-i}. \quad \text{(B.1)} \]

For each $t \leq t^\delta$ and each player $i$, let $v^i,t = (1-\delta^t) g^* + \delta^t u^i*$. Because of (B.1), for sufficiently high $\delta$, and each player $j \neq i$,

\[ v^i,t(\theta_{-i}) \geq v^i,t(\theta_i) - 2\epsilon. \quad \text{(B.2)} \]

Find $m^\epsilon$ and $\delta^\epsilon$ from Lemma 7. Assume that $m \geq m^\epsilon$ and the discount factor $\delta \geq \delta^\epsilon$ are high enough so that $m(1-\delta)M < \epsilon$, and $(1-\delta^m) \epsilon > 2(1-\delta) M$.

Let $\mu_j^* = \mu_j^{v^i,-\epsilon,m,\epsilon}$ be the minimax strategies of players $j \neq i$ from Lemma 7. Let $M_i^* (\hat{a}^i)$ be the associated payoff vector of player $i$ playing action sequence $\hat{a}^i = (a^i_0, \ldots, a^i_{m-1})$. For each sequence of actions $\hat{a}^i$ of player $i$ and $\hat{a}^{-i}$ of players $-i$, define $\hat{\alpha} = (\hat{a}^i, \hat{a}^{-i})$ and payoff vector $v^i(a)$ so that for each type $\theta_i$ of player $i$,

\[ (1-\delta^m) M_i^* (\hat{a}^i; \theta_i) + \delta^m v^i (\hat{\alpha}; \theta_i) = \hat{v}^i,t (\theta_i), \]

and for each type $\theta_j$ of player $j \neq i$,

\[ (1-\delta) \sum_{s=0}^{m-1} \delta^s g_j (a^i_s, a_j^{-i}; \theta_j) + \delta^m v^i (\hat{\alpha}, \theta_j) = \hat{v}^i,t (\theta_j). \]

Notice that because $M_i^* (a^i; \theta_i) \leq \hat{v}^i,t (\theta_i) - \epsilon$ for each type $\theta_i$ of player $i$,

\[ v^i (\hat{\alpha}, \theta_i) \geq \hat{v}^i,t (\theta_i) + (1-\delta^m) \epsilon = \hat{v}^i,t (\theta_i) + 2M (1-\delta). \]

Moreover, due to (B.2), for each type $\theta_i$ of player $j \neq i$,

\[ v^i (\hat{\alpha}, \theta_j) \geq \hat{v}^i,t (\theta_j) - (1-\delta^m) M \geq \hat{v}^i,t (\theta_j) - \epsilon > \hat{v}^{-i},t (\theta_j) + 2M (1-\delta). \]

Construct a strategy profile $\sigma$. There are two types of regimes:
• Normal\((v, t)\) for \(t \leq t^δ\) and \(v\) so that (a) there exists \(u \in B(u^∗, k\epsilon)\) such that 
\[v = (1 - δ^t) g^∗ + δ^t u,\]
and (b) \(v(θ_i) ≥ v^{i,t}(θ_i) + 2M (1 - δ)\) for each player \(i\) and type \(θ_i\). Players play action profile \(a^∗\) for \(t\) periods \(s = 0, 1, ..., t - 1\). If there is no deviation, player continue with strategy profile \(σ^u\). Simultaneous deviations of two or more players are ignored. A deviation by single player \(i\) in period \(s\) initiates regime Punishment\((i, t - s)\). The expected payoff in the beginning of regime Normal\((v, t)\) is equal to \(v\).

• Punishment\((i, t)\): The regime lasts \(m\) periods. Players \(-i\) play strategies \(μ^∗_{-i}\). Player \(i\) randomizes uniformly across all action sequences \((a^0_i, ..., a^{m-1}_i)\).

After \(m\) periods, regime Normal\((v^i(ā), t)\) is initiated, where \(ā\) are the actions played during the regime. The expected payoff in the beginning of regime Punishment\((i, t)\) is equal to \(v^{i,t}\).

The profile starts in regime Normal\((v^∗, t^δ)\). We check that the strategy profile \(σ\) is a \((n - 1)\)-revealing equilibrium. In each period of the Punishment\((i, t)\) regime, all players are indifferent between all actions. In particular, they do not have one-shot profitable deviations. Any one-shot deviation during the Normal\((v, t)\) period leads to the payoff not higher than \((1 - δ) M + δv^{i,t}.\) If \(v(θ_i) ≥ v^{i,t}(θ_i) + 2M (1 - δ)\), the deviation is not profitable.

Finally, because the strategies prescribe the same (possibly, mixed) actions for all types of each player, the beliefs do not get updated before \((n - 1)\)-revealing profile \(σ^u\) is started.

B.3. Proof of Lemma 2. Take any \(v \in FR^B_n(π)\) and find here exists \(ε > 0\) and an incentive compatible lottery \(l = (α, u)\) such that \(v = v^{π,l}\) and \(B(u(a), 2ε) ⊆ int FR^A_n\(p^{π,l}(a)\) for each positive probability action profile \(a\). We can assume w.l.o.g. that all actions have positive probability.

Using the compactness argument (and, possibly, public randomization), we can show that there exists \(δ_0\) such that for all \(δ ≥ δ_0\), each \(a\), and each \(u' ∈ B(u(a), ε)\), there exists a strategy profile that induces payoff \(u'\) and that is a \((n - 1)\)-revealing equilibrium of game \(Γ\(p^{π,l}(a), δ\).

For each action profile, let \(u^δ(a) = \frac{1}{δ} u(a) - (1 - δ) g(a) ∈ B(u(a), ε)\). For each \(a\), find \(n\)-revealing equilibrium profile \(σ^a\) that induces \(u^δ(a)\).
Let $\sigma$ be a strategy profile in which in the first period players play according to $\alpha$, and continue with $\sigma (a)$ after first period history $a$. Then, $\sigma$ is a $(n - 1)$-revealing equilibrium for sufficiently high $\delta$ with expected payoff $v$.

**B.4. Proof of Lemma 3.** Take any $v^* \in FR_n^C (\pi)$. By the definition of set $FR_n^C (\pi)$, there exists an allocation $u : \Theta^* \rightarrow (V \cup FR_n^B (\pi))$ such that $u^{\theta_i} = u^{\theta_j} \notin FR_n^B (\pi)$ for all $\pi$-positive probability types $\theta_i$ and $\theta_j$ of players $i$ and $j$, and $u^{\theta_i} (\theta_i) \geq u^{\theta_i} (\theta_i)$ for all players $i$ and types $\theta_i \neq \theta_i'$. 

- For each player $i$ and $\pi$-positive probability type $\theta_i$, let $u^* = u^{\theta_i} \in int FR_n^B (\pi)$.
- For each player $i$ and $\pi$-zero probability type $\theta_i$, find payoff vectors $\nu^{\theta_i} \in V$ and $u^{\theta_i} \in int FR_n^B (\pi)$, and real number $\beta^{\theta_i} \in [0, 1]$ such that

$$u^{\theta_i} = \beta^{\theta_i} \nu^{\theta_i} + \left(1 - \beta^{\theta_i}\right) w^{\theta_i}.$$

Due to public randomization, we can assume that there are pure action profiles $a^\theta$ such that $g \left(a^{\theta_i}\right) = v^{\theta_i}$.

Find $\epsilon > 0$ such that $B \left(u^*, 2\epsilon\right) \cup B \left(w^{\theta_i}, 2\epsilon\right) \subseteq FR_n^B (\pi)$. For each player $i$, find $u^{s_i} \in B \left(u^*, \epsilon/2\right)$ such that

$$u^{s_i} (\theta_i) \leq u^* (\theta_i) - \epsilon \text{ for each } \theta_i,$$

$$u^{s_i} (\theta_{-i}) \geq u^* (\theta_{-i}) \text{ for each } \theta_{-i}.$$

By the hypothesis, there exists $\delta^* < 1$ such that for all $\delta \geq \delta^*$, there are $n$-revealing equilibrium profiles $\sigma^{i, \delta}$ and $\sigma^{w, \delta}$ with expected payoffs, respectively, $u^*$, $u^{s_i}$, and $w$ for all players $i$ and all payoff vectors $w \in B \left(u^*, \epsilon/2\right) \cup \cup_{i, \theta_i} B \left(w^{\theta_i}, \epsilon/2\right)$.

For all sufficiently high $\delta$, we construct a strategy profile $\sigma^\delta$ with expected payoff $v^*$. There are two regimes:

- **Report:** The regime lasts one period, and the expected payoff vector at the beginning of the regime is equal to $v^*$. Each $\pi$-positive probability type of player $i$ plays action $s_i^* \in A_i$. Each $\pi$-zero probability type $\theta_i$ of player $i$ plays $s_i (\theta_i) \in A_i$ such that $s_i (\theta_i) \neq s_i^*$ and $s_i (\theta_i) \neq s_i (\theta_i')$ for any other $\pi$-zero probability type $\theta_i'$. Let $a$ be the realized action profile. If exactly one player chooses an action corresponding to a zero probability type $\theta_i$, regime
\text{Surprise}(\theta_i, a) \text{ is initiated. Otherwise, players continue with } n\text{-revealing equilibrium } \sigma^{w(a), \delta}, \text{ where } w(a) \text{ is chosen so that }

\[(1 - \delta) g(a) + \delta w(a) = u^*.
\]

For sufficiently high \(\delta\), \(w(a) \in B(u^*, \epsilon)\) and equilibrium profile \(\sigma^{w(a), \delta}\) exists.

- \text{Surprise}(\theta_i, a): In each period of the regime, players play action profile \(a^{\theta_i}\). At the end of each period, the players use the public randomization device to determine the continuation behavior. With probability \(x\), the players switch to \(n\)-revealing equilibrium profile \(\sigma^{w} \text{ with expected payoff vector } w \in B \left( w^{\theta_i}, \epsilon/2 \right) \text{ that we determine below; with probability } y\), the players initiate regime \text{Report}, and with the remaining probability \(1 - x - y\), the players remain in the regime \text{Surprise}(\theta_i, a). The expected payoff in the beginning of each period of the regime is equal to \(u^{\theta_i, a, \delta}\), where

\[(1 - \delta) g(a) + \delta u^{\theta_i, a, \delta} = \frac{\epsilon}{2M} u^{\theta_i} + \left(1 - \frac{\epsilon}{2M}\right) v^*.
\]

Any deviation from action profile \(a^{\theta_i}\) by player \(j\) results in abandoning the regime and switching to \(n\)-revealing equilibrium profile \(\sigma^{j, \delta}\).

We choose \(w \in B \left( w^{\theta_i}, \epsilon/2 \right) \) so that

\[\beta v^{\theta_i} + (1 - \beta) w = \frac{\epsilon}{2} u^{\theta_i} - (1 - \delta) v^{\theta_i}.
\]

We let \(x = \frac{1 - \delta - \beta}{\delta - \beta}\), and \(y\) so that

\[\frac{\delta y}{1 - \delta + \delta x + \delta y} = \frac{1}{\delta} \left(1 - \frac{\epsilon}{2M}\right).
\]

Then, for sufficiently high \(\delta\), \(x + y < 1\), and

\[u^{\theta_i, a, \delta} = \frac{1 - \delta}{1 - \delta + \delta x + \delta y} v^{\theta_i} + \frac{\delta x}{1 - \delta + \delta x + \delta y} w + \frac{\delta y}{1 - \delta + \delta x + \delta y} v^*
\]

satisfies equation (B.3).

The profile starts in the regime \text{Report}.

We use the one-shot deviation principle to check that the profile is an equilibrium. The expected payoff from playing action \(s_i(\theta_i)\) in the \text{Report} regime for any type \(\theta_i\) is equal to \(u^{\theta_i}(\theta_i)\), whereas the expected payoff from playing any other action is
not higher than $\max_{\theta_i} v_i^\theta_i(\theta_i)$. Thus, players have no profitable deviations in such a regime. A deviation during the Surprise regime is punished by a continuation utility loss of order at least $\epsilon/2$. Because the one-shot gain is not larger than $(1 - \delta) M$, the deviation is not profitable for a sufficiently high $\delta$. Finally, because all the $\pi$-positive probability types always play the same action, no information is revealed, and the beliefs do not change.

Appendix C. Proof of Theorem 4

For any two correspondences $F, G : \Pi \rightrightarrows R^{\Theta^*}$, say that $F$ is a good approximation of $G$ if for each $\pi \in \Pi$, each $v \in G(\pi)$, there exists $v' \in F(\pi)$ such that $v_i(\theta_i) \geq v'_i(\theta_i)$ for each type $\theta_i$, and $v_i(\theta_i) = v'_i(\theta_i)$ for each $\pi$-positive probability type $\theta_i$.

Theorem 4 follows from Lemma 8.

Lemma 8. Suppose that set $V$ has a nonempty interior and that $V \subseteq IR$. Then, correspondence $FR_1^B$ is a good approximation of $\bigcup_n FR_n^C(\pi)$.

Proof. The result follows from the fact that $FR_1^B$ is a good approximation of $FR_1^B$ and from the following three claims:

For each $n \geq 1$, if $FR_1^B(\pi)$ is a good approximation of $FR_n^A(\pi)$, then it is a good approximation of $FR_n^B(\pi)$. Indeed, take any $v \in FR_n^B(\pi)$ and find a lottery $l = (\alpha, u) \in L(\pi)$ such that $v^{\pi,l} = v$ and such that for any action profile $a \in A^I$, $u(a) \in FR_n^A\left(p^{\pi,l}(a)\right)$. Because $FR_1^B$ is a good approximation of $FR_n^A$, then we can find $u'(a) \in FR_1^B\left(p^{\pi,l}(a)\right)$ such that for each $a \in A^I$, $u_i(\theta_i|a) \geq u'_i(\theta_i|a)$ for each type $\theta_i$ with equality for positive probability types. Consider lottery $l' = (\alpha, u')$. Lottery $l'$ is clearly $\pi$-incentive compatible, and it has the same value as lottery $l$, $v^{\pi,l'} = v$. We use the definition of $FR_1^B$ to check that $v^{\pi,l'} \in FR_1^B(\pi)$.

For each $n \geq 1$, if $FR_1^B(\pi)$ is a good approximation of $FR_n^B(\pi)$, then it is a good approximation of $FR_n^C(\pi)$. This claim follows immediately from the construction of stage $nC$ and the definition of good approximation.

For each $n \geq 1$, if $FR_1^B(\pi)$ is a good approximation of $FR_n^C(\pi)$, then it is a good approximation of $FR_{n+1}^A(\pi)$. This claim follows from the fact that $V \subseteq IR$, which implies that int$V \subseteq FR_1^B(\pi)$.

□
Appendix D. Proof of Theorem 5

In this Appendix, we assume that the game has a structure described in Section 5.2. In particular,

\[ \text{int} IR = \left\{ v \in R^{\Theta^*} : v_i (\theta_i) > 0 \text{ for each } \theta_i \right\}, \]

and there exist sets \( M_i \subseteq R^{\Theta_i} \) such that \( 0_i \in M_i \) and the set

\[ \text{int} V = \text{intcon} \left\{ \bigcup_i M_i \times \{0_{-i}\} \right\} \]

is not empty.

We begin with a convenient characterization of set \( FR^B_1 (\pi) \). Lottery \( l = (\alpha, u) \) is fully-revealing, if all types of each player choose pure actions and any two different types \( \theta_i \neq \theta'_i \) play different actions, \( \alpha_i (\theta_i) \neq \alpha_i (\theta'_i) \).

**Lemma 9.** Let \( v \in R^{\Theta^*} \) be a payoff vector. Then, \( v \in FR^B_1 (\pi) \) if and only if there exists a fully revealing lottery \( l = (\alpha, u) \) such that

1. for each type profile \( \theta \), \( u (\alpha (\theta)) \in V \),
2. for each player \( i \), each type profile \( \theta = (\theta_i, \theta_{-i}) \in \Theta \)
   
   \[ u (\theta_i | \alpha (\theta)) \geq 0, \]

3. for each player \( i \), each type \( \theta_i \), and all types \( \theta_i \neq \theta'_i \),

\[ v (\theta_i) = \sum_{\theta_{-i}} \pi_{\theta_{-i}}^{\theta_i} (\theta_{-i}) u (\theta_i | \alpha (\theta)) \]

\[ \geq \sum_{\theta_{-i}} \pi_{\theta_{-i}}^{\theta_i} (\theta_{-i}) \max [u (\theta_i | \alpha_i (\theta'_i), \alpha_{-i} (\theta_{-i})), 0]. \]

**Proof.** By the definition of set \( FR^B_1 (\pi) \), for each \( v \in FR^B_1 (\pi) \), there exists a \( \pi \)-incentive compatible lottery \( l^0 = (\alpha^0, u^0) \) with value \( v \) and such that for each action profile \( a \), either beliefs \( p (a) \) are degenerate on the type tuple \( \theta \) and

\[ u^0 (a) \in FR^A_1 (\theta) = NE (\theta), \]

or the beliefs \( p (a) \) are non-degenerate, and

\[ u^0 (a) \in FR^A_1 (p (a)) = \text{int} IR \cap \text{int} V. \]
Let \( a \) be an action profile that is played with \( \pi \)-positive probability types \( \theta = (\theta_i, \theta_{-i}) \) in strategy profile \( a^0 \). Because
\[
\text{int} IR \cap \text{int} V \subseteq NE(\theta),
\]
we can assume that \( u^0(a) \in NE(\theta) \). Furthermore, we can find \( u^1(a) \in \text{int} V \) such that \( u^1(\theta_i|a) = u^0(\theta_i|a) \) and \( u^1(\theta'_i|a) \leq u^0(\theta'_i|a) \) for each player \( i \) and each type \( \theta'_i \neq \theta_i \). Also, notice that payoffs \( u^0(a) \) are individually rational, which implies that
\[
\max \{0, u^1(\theta'_i|a)\} \leq u^0(\theta'_i|a) \quad \text{for each type } \theta'_i.
\]

We construct a fully revealing lottery \( l = (a, u) \). Find a strategy profile \( \alpha \) such that all types of each player chooses pure actions and any two different types \( \theta_i \neq \theta'_i \) play different actions, \( \alpha_i(\theta_i) \neq \alpha_i(\theta'_i) \). For each type profile \( \theta \) define
\[
u(\theta_i) = \sum_{\alpha} \left( \prod_{i} \alpha_{i}^0(a_i|\theta_i) \right) u^1(a).
\]

We check that \( l \) satisfies the thesis of the Lemma. Notice that \( u(\alpha(\theta)) \) is equal to a convex combination of elements of \( V \), which implies that \( u(\alpha(\theta)) \in V \). Second, for each \( \theta \), \( u_i(\alpha(\theta)|\theta_i) \geq 0 \) as a convex combination of non-zero payoffs. Third, notice that because lottery \( l^0 \) is \( \pi \)-incentive compatible, for each action \( a_i \),
\[
u(\theta_i) \geq \sum_{\theta_{-i}} \pi_{i}^{\theta_i}(\theta_{-i}) u^0_i(\theta_i|a_i, \alpha_{-i}^0(\theta_{-i}))
\]
with equality when action \( a_i \) is played with positive probability by type \( \theta_i \), \( \alpha_i^0(a_i|\theta_i) > 0 \). It follows that
\[
u(\theta_i) = \sum_{\theta_{-i}} \pi_{i}^{\theta_i}(\theta_{-i}) u(\theta_i|\alpha_i(\theta_i), \alpha_{-i}(\theta_{-i}))
\]
\[
= \sum_{\theta_{-i}} \pi_{i}^{\theta_i}(\theta_{-i}) u^1(\theta_i|\alpha_i^0(\theta_i), \alpha_{-i}^0(\theta_{-i}))
\]
\[
\geq \sum_{\theta_{-i}} \pi_{i}^{\theta_i}(\theta_{-i}) u^0_i(\theta_i|\alpha_i^0(\theta'_i), \alpha_{-i}^0(\theta_{-i}))
\]
\[
\geq \sum_{\theta_{-i}} \pi_{i}^{\theta_i}(\theta_{-i}) \max \{0, u_i^1(\theta_i|\alpha_i^0(\theta'_i), \alpha_{-i}^0(\theta_{-i}))\}
\]
\[
= \sum_{\theta_{-i}} \pi_{i}^{\theta_i}(\theta_{-i}) \max \{0, u_i(\theta_i|\alpha_i^0(\theta'_i), \alpha_{-i}^0(\theta_{-i}))\}.
\]
\(\square\)
The next result is a key step in our argument. Take any vector of payoffs \( v^* \) that are individually rational for all positive \( \pi \)-probability types of all players and that can be obtained by a play of non-revealing actions followed by a payoff vector from stage 1B, \( v^* = \gamma g + (1 - \gamma) v' \) for some \( g \in V \) and \( v' \in FR_i^B(\pi) \). Then, the Lemma shows that there exist a corresponding fully revealing equilibrium \( v \), with the same payoffs as \( v^* \) for the positive probability types and not smaller, and individually rational payoffs for the zero-probability types.

**Lemma 10.** For each player \( i \), each \( \pi \in \Pi \), each payoff vector \( v^* \in \text{intcon} \left( FR_i^B(\pi) \cup V \right) \), if \( v^*_i(\theta_i) \geq 0 \) for each player \( i \) \( \pi \)-positive probability type \( \theta_i \), there exists \( v \in FR_i^B(\pi) \) st. (a) \( v^*_i(\theta_i) = v(\theta_i) \) for all \( \pi \)-positive probability types, and (b) \( v_i(\theta_i) = \max \{ 0, v^*_i(\theta_i) \} \) for all \( \pi \)-zero probability types \( \theta_i \).

The idea of the construction is to start with full and immediate revelation and delay the play of non-revealing actions after the revelation. We need to be careful so that the expected payoffs and the incentives to reveal information truthfully are not affected and that continuation payoffs after the revelation are individually rational. To see the difficulty, suppose that \( l' = (\alpha, u') \) is a fully-revealing lottery with value \( v' \) that satisfies the conditions from the above Lemma. Then, \( u'(\alpha(\theta)) \) is the continuation payoff after type profile \( \theta \) is revealed.

In a first attempt, we can construct a new lottery with continuation payoffs equal to \( u'(\alpha(\theta)) = \gamma g + (1 - \gamma) u'(\alpha(\theta)) \). In other words, we can ask players to reveal their information immediately, play non-revealing actions with payoffs \( g \) for fraction \( \gamma \) of time, and follow by the continuation payoffs from lottery \( u'(...) \). The value of such lottery is \( v^* \), and one can easily check that such lottery would be incentive compatible. However, in general, there is no guarantee that the payoffs \( u'(\alpha(\theta)) \) are individually rational for types \( \theta \).

For this reason, we distribute the fraction of time spent playing non-revealing actions across continuations following different realized types in an unequal way. We use the fact that we can find \( m_i^* \) and \( \gamma_i \geq 0 \) such that \( \gamma g = \sum_i \gamma_i (m_i^*, 0_{-i}) \) and \( \sum_i \gamma_i = \gamma \). We construct a new lottery with continuation payoffs equal to \( u'(\alpha(\theta)) = \sum_i \gamma_i (m_i^*, 0_{-i}) + (1 - \gamma) u'(\alpha(\theta)) \), where \( \gamma_i(\theta) \) is the fraction of time spent playing player \( i \)'s part of the non-revealing action following reported profile \( \theta \). We show that
we can choose \( \gamma_i(\theta) \)s so that the continuation payoffs are individually rational, and the value and the incentives are preserved. The proof contains the details.

**Proof.** Take \( v^* \in \text{intcon}(FR_1^B(\pi) \cup V) \) and such that \( v_i^*(\theta_i) > 0 \) for each player \( i \) \( \pi \)-positive probability types \( \theta_i \). Find \( \gamma_i \in [0, 1], \ m_i^* \in M_i, \) and \( v' \in FR_1^B(\pi) \) such that

\[
v^* = \sum_i \gamma_i (m_i^*, 0_{-i}) + (1 - \gamma_1 - \gamma_2) v'.
\]

Use Lemma 9 to find a fully revealing lottery \( l' = (\alpha, u') \) with value \( v' \) and such that \( l' \) satisfies the conditions from the Lemma. For each type profile \( \theta \in \Theta \), find \( \beta_i(\theta) \leq 1 \) and \( b_i(\theta) \in M_i \) for each player \( i \) such that \( \sum_i \beta_i(\theta) \leq 1 \), and

\[
u'(\alpha (\theta)) = \sum_i \beta_i(\theta) (b_i(\theta), 0_{-i}).
\]

(Here and below, \( (b_i(\theta), 0_{-i}) \) is a vector with 0s on coordinates of players \( j \neq i \) and the coordinates of player \( i \) equal to \( b_i(\theta) \).)

For each two types \( \theta_i \) and \( \theta_i' \) of player \( i \), define the expected payoff vector \( \hat{v}_{i, \theta_i}^* \) of player \( i \) who reports \( \theta_i \) during the lottery stage:

\[
\hat{v}_{i, \theta_i}^* = \gamma_i m_i^* + \left( 1 - \sum_i \gamma_i \right) \sum_{\theta_{-i}} \pi_i^\theta_i (\theta_{-i}) \beta_i(\theta_i, \theta_{-i}) b_i(\theta_i, \theta_{-i}).
\]

Thus, \( v_i^*(\theta_i) = \hat{v}_{i, \theta_i}^*(\theta_i) \). Moreover

We construct a fully revealing lottery \( \hat{l} = (\alpha, \hat{u}) \): For each type \( \theta_{-i} \),

- for each type \( \theta_i \) such that \( v_i^*(\theta_i) > 0 \), let

\[
\hat{u}_i(\alpha_i(\theta_i), \alpha_{-i}(\theta_{-i})) = \frac{\gamma_i + (1 - \sum_i \gamma_i \beta_i(\theta_i, \theta_{-i}))}{\gamma_i + (1 - \sum_i \gamma_i \sum_{\theta'_{-i}} \pi_{-i}(\theta'_{-i}) \beta_i(\theta_i, \theta'_{-i}))} \cdot \hat{v}_{i, \theta_i}^*.
\]

- for each type \( \theta_i \) such that \( v_i^*(\theta_i) \leq 0 \), let \( \hat{u}_i(\alpha_i(\theta_i), \alpha_{-i}(\theta_{-i})) = 0_i \).

We check that lottery \( \hat{l} \) satisfies the four conditions of Lemma 9 and that its value is \( v^* \).

For any profile \( \theta \), any player \( i \), the definition implies that \( \hat{u}(\theta_i|\alpha(\theta)) \geq 0 \).
We show that \( \hat{u}(\alpha(\theta)) \in V \) for each profile \( \theta \). Observe that

\[
\frac{\gamma_i(M_i^*, 0_{-i}) + (1 - \sum_i \gamma_i) \sum_{\theta'_{-i}} \pi_{-i}(\theta'_{-i}) \beta_i(\theta_i, \theta'_{-i}) (\theta_i, \theta_{-i}) \, (b_i(\theta_i, \theta_{-i}), 0_{-i})}{\gamma + (1 - \sum_i \gamma_i) \sum_{\theta'_{-i}} \pi_{-i}(\theta'_{-i}) \beta_i(\theta_i, \theta'_{-i})} \in M_i \times \{0_{-i}\}.
\]

(D.1)

If \( v_i^*(\theta_i) > 0 \) for the types of all players, then the claim follows from the observation and the fact that \( \hat{u}(\alpha(\theta)) \) is a convex combination of terms (D.1). If \( v_i^*(\theta_i) > 0 \) for exactly one player, then \( \hat{u}(\alpha(\theta)) \) is a convex combination of term (D.1) and \((0_i, 0_{-i})\). Finally, if \( v_i^*(\theta_i) \leq 0 \) for the types of all players, then \( \hat{u}(\alpha(\theta)) = (0_i, 0_{-i}) \in V \).

We compute the value \( v = v^{*\hat{u}} \) of the lottery \( \hat{L} \). For each type \( \theta_i \) such that \( v_i^*(\theta_i) > 0 \), the value is is equal to

\[
v_i(\theta_i) = \sum_{\theta_{-i}} \pi(\theta_{-i}) \hat{u}_i(\theta_i | \alpha_i(\theta_i), \alpha_{-i}(\theta_{-i}))
= \sum_{\theta_{-i}} \pi(\theta_{-i}) \frac{\gamma_i + (1 - \sum_i \gamma_i) \beta_i(\theta_i, \theta_{-i})}{\gamma_i + (1 - \sum_i \gamma_i) \sum_{\theta'_{-i}} \pi(\theta'_{-i}) \beta_i(\theta_i, \theta'_{-i})} \cdot v_i^*(\theta_i) = v_i^*(\theta_i).
\]

The value of the lottery \( \hat{L} \) for type \( \theta_i \) such that \( v_i^*(\theta_i) \leq 0 \) is equal to \( v_i(\theta_i) = 0 \).

For each player \( i \), take any two types \( \theta_i \neq \theta_i' \). For each type \( \theta_i' \) such that \( \hat{\alpha}^{*\theta_i'}(\theta_i) \leq 0 \),

\[
\sum_{\theta_{-i}} \pi(\theta_{-i}) \max \left\{ 0, u(\theta_i | \alpha_{-i}(\theta_i'), \alpha_{-i}(\theta_{-i})) \right\}
= \sum_{\theta_{-i}} \pi(\theta_{-i}) \max \left\{ 0, \frac{\gamma_i + (1 - \sum_i \gamma_i) \beta_i(\theta_i', \theta_{-i})}{\gamma_i + (1 - \sum_i \gamma_i) \sum_{\theta'_{-i}} \pi(\theta'_{-i}) \beta_i(\theta_i', \theta'_{-i})} \hat{\alpha}^{*\theta_i'}(\theta_i) \right\}
= 0 \leq v_i(\theta_i).
\]

For each type \( \theta_i' \) such that \( \hat{\alpha}^{*\theta_i'}(\theta_i) > 0 \),
\[
\sum_{\theta_{-i}} \pi (\theta_{-i}) \max \{0, u (\theta_i | \alpha_{-i} (\theta_i'), \alpha_{-i} (\theta_{-i}))\} = \sum_{\theta_{-i}} \pi (\theta_{-i}) \frac{\gamma_i + (1 - \sum_i \gamma_i) \beta (\theta_i', \theta_{-i})}{\gamma_i + (1 - \sum_i \gamma_i) \sum_{\theta'_{-i}} \pi (\theta'_{-i}) \beta (\theta_i', \theta'_{-i})} \hat{\theta}^* \theta_i' (\theta_i) = \gamma_i m^*_i (\theta_i) + \left(1 - \sum_i \gamma_i\right) \sum_{\theta'_{-i}} \pi (\theta'_{-i}) \beta (\theta_i' \theta_{-i}) b_i (\theta_i | \theta_i', \theta_{-i}) \leq \gamma_i m^*_i (\theta_i) + \left(1 - \sum_i \gamma_i\right) v^*_i (\theta_i) = v^*_i (\theta_i) = v_i (\theta_i).
\]

where the last inequality follows from the fact that lottery \( l \) satisfies condition 4 of the Lemma. The proof is concluded by an application of Lemma 9. \( \square \)

The last result uses the definition of good approximation introduced in Appendix C.

**Lemma 11.** \( FR_1^B \) is a good approximation of \( \bigcup_n FR_n^C (\pi) \).

**Proof.** As in Lemma 8, the result follows from the fact that \( FR_1^B \) is a good approximation of \( FR_1^B \) and from the following three claims. We omit the proofs of the first two claims, because the argument from Lemma 8 applies verbatim.

For each \( n \geq 1 \), if \( FR_n^B (\pi) \) is a good approximation of \( FR_n^A (\pi) \), then it is a good approximation of \( FR_n^B (\pi) \).

For each \( n \geq 1 \), if \( FR_n^B (\pi) \) is a good approximation of \( FR_n^B (\pi) \), then it is a good approximation of \( FR_n^C (\pi) \).

For each \( n \geq 1 \), if \( FR_n^B (\pi) \) is a good approximation of \( FR_n^C (\pi) \), then it is a good approximation of \( FR_{n+1}^A (\pi) \). Indeed, take any \( v \in \text{int} IR \cap \text{inter} \left( \bigcup_n FR_n^C (\pi) \right) \) and find \( \alpha \in [0, 1] \), \( a \in V \), and \( u \in FR_n^C (\pi) \) such that

\[ v = \alpha a + (1 - \alpha) u. \]

Find \( u' \in FR_1^B (\pi) \) such that \( u_i (\theta_i) \geq u_i' (\theta_i) \) for each type \( \theta_i \), and \( u (\theta_i) = u_i' (\theta_i) \) for each \( \pi \)-positive probability type \( \theta_i \). Let

\[ v' = \alpha a + (1 - \alpha) u'. \]
Then, \( v' \) satisfies the hypothesis of Lemma 10 and we can find \( v'' \in FR^B_i (\pi) \) such that (a) \( v'_i (\theta_i) = v'' (\theta_i) \) for all \( \pi \)-positive probability types \( \theta_i \), and (b) \( v''_i (\theta_i) = \max \{0, v' (\theta_i)\} \leq v_i (\theta_i) \) for all \( \pi \)-zero probability types \( \theta_i \).

\[ \square \]

Theorem 5 follows from Lemma 11.

**Appendix E. Proof of lemma 5**

The first inequality in (5.5) implies that

\[ \alpha_{TT} \leq \frac{2}{3} - \frac{1}{3} \alpha_{WW} - \frac{2}{3} \alpha_{WT}. \]

Substituting into the second inequality, we obtain

\[ 2x \alpha_{WW} + (1 + 3x) \alpha_{WT} \geq (1 - 3x) \left( \frac{1}{3} \alpha_{WW} + \frac{2}{3} \alpha_{WT} \right), \]

or, after some algebra,

\[ (9x - 1) \alpha_{WW} + (1 + 15x) \alpha_{WT} \geq 1 - 3x. \]

It follows that

\[ 2 \alpha_{WW} + 4 \alpha_{WT} - 2 \alpha_{TT} \]

\[ \geq \frac{8}{3} \alpha_{WW} + \frac{16}{3} \alpha_{WT} - \frac{4}{3} \]

\[ \geq \left( \frac{8}{3} - \frac{16}{3} \frac{9x - 1}{1 + 15x} \right) \alpha_{WW} + \frac{16}{3} \frac{1}{1 + 15x} (1 - 3x) - \frac{4}{3}. \]

If \( x < \frac{3}{100} \), then the above expression is strictly larger than 2 for each \( \alpha_{WW} \geq 0 \).

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