On the regularity of Dirichlet problem for fully non-linear elliptic equations on Hermitian manifolds

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Abstract
We derive the solvability and regularity of the Dirichlet problem for fully non-linear elliptic equations possibly with degenerate right-hand side on Hermitian manifolds, through establishing a quantitative version of boundary estimate under a subsolution assumption. In addition, we construct the subsolution when the background manifold is a product of a closed Hermitian manifold with a compact Riemann surface with boundary.

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1 Introduction

Let \((M, J, \omega)\) be a compact Hermitian manifold of complex dimension \(n \geq 2\) with boundary \(\partial M\), where \(\omega = \sqrt{-1}g_{ij}dz^i \wedge d\bar{z}^j\) denotes the Kähler form compatible with the complex structure \(J\). Suppose \(\chi = \sqrt{-1} \chi_{ij} dz^i \wedge d\bar{z}^j\) is a smooth real \((1, 1)\)-form on \(\bar{M} := M \cup \partial M\). Given a \(C^2\)-function \(u\) on \(\bar{M}\), one can obtain a new real \((1, 1)\)-form \(g[u] := \chi + \sqrt{-1} \partial \bar{\partial} u\).

This paper is devoted to investigating the following Dirichlet problem

\[ F(g[u]) = \psi \text{ in } M, \quad u = \varphi \text{ on } \partial M. \tag{1.1} \]

We assume the operator \(F(g[u])\) takes the form

\[ F(g[u]) = f(\lambda(g[u])), \]

where \(\lambda(g[u]) = (\lambda_1, \cdots, \lambda_n)\) denote the eigenvalues of \(g[u]\) with respect to \(\omega\), and \(f\) is a smooth symmetric function defined in an open symmetric convex cone \(\Gamma \subset \mathbb{R}^n\) containing the positive cone

\[ \Gamma_n := \{ \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n : \text{ each } \lambda_i > 0 \} \subseteq \Gamma \]
with vertex at the origin and with the boundary $\partial \Gamma \neq \emptyset$. In addition, we have the following hypotheses on $f$:

$$f_i(\lambda) := \frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \text{ in } \Gamma, \forall 1 \leq i \leq n,$$

(1.2)

$$f$$ is concave in $\Gamma$,

(1.3)

For any $\lambda \in \Gamma$, $\lim_{t \to +\infty} f(t, \lambda) > -\infty$, 

(1.4)

$$\lim_{t \to +\infty} f(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + t) = \sup_{\Gamma} f, \forall \lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma.$$ 

(1.5)

In real variables, equations of this type were investigated by Caffarelli-Nirenberg-Spruck [5], which extended the work of Ivochkina [28] on equations of Monge-Ampère type.

Notice that equation (1.1) includes many equations as special cases. For instance, if we set $f(\lambda) = \sum_{i=1}^{n} \log \lambda_i$, it reads the complex Monge-Ampère equation, which has played a significant role in Yau’s proof of Calabi’s conjecture [45]. In fact, over the past decades, there were many researches on complex Monge-Ampère equation. Below we list parts of these works. For $\chi = 0$ and $M = \Omega \subset \mathbb{C}^n$ a bounded strictly pseudoconvex domain, the Dirichlet problem for complex Monge-Ampère equation was solved by Caffarelli-Kohn-Nirenberg-Spruck [4] in the class of plurisubharmonic functions. Caffarelli-Kohn-Nirenberg-Spruck’s work was extended by Guan [16] to general bounded domains, replacing strictly pseudoconvex restriction to boundary by a subsolution assumption. Under the subsolution assumption, Guan-Li [18] solved the Dirichlet problem for complex Monge-Ampère equation on compact Hermitian manifolds with boundary. For $\psi \geq 0$, the Dirichlet problem for complex Monge-Ampère equation becomes degenerate, which is much more complicated. In [7], Chen solved the Dirichlet problem for homogeneous complex Monge-Ampère equation on $M = X \times A$ and proved the existence of $C^{1,\alpha}$-regularity (weak) geodesics in the space of Kähler potentials [11, 32, 35], where $A = S^1 \times [0, 1]$ and $X$ is a closed Kähler manifold. The Dirichlet problem for degenerate complex Monge-Ampère equation was further complemented by Błocki [2], Phong-Sturm [34] and Boucksom [3]. We refer to [8, 12] for complement and progress on understanding how geodesics and homogeneous complex Monge-Ampère equation are related to the geometry of $X$. 

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There are also several results on the Dirichlet problem (1.1) for other equations. For \( \chi = 0 \) and \( M = \Omega \) a bounded domain in \( \mathbb{C}^n \), the Dirichlet problem (1.1) was studied by Li [30]. For complex inverse \( \sigma_k \) equations on Hermitian manifolds, the Dirichlet problem was studied by Guan-Sun [21].

The above results mainly concentrate on special cases of (1.1) and the proof more or less relies on specific structures of equations or underlying manifolds, which seems not adaptive to more general cases. On the other hand, motivated by increasing interests from complex geometry and analysis, we are interested in investigating the Dirichlet problem (1.1) on curved complex manifolds, especially with degenerate right-hand side. Unfortunately, except the above-mentioned works regarding to complex Monge-Ampère equation [7, 3, 18, 34] and complex inverse \( \sigma_k \) equation [21], few progress has been made on the Dirichlet problem for general equations on curved Hermitian manifolds. The primary problem left open is to derive the gradient estimate as described below. To this end, in this paper we set up a quantitative version of boundary estimate of the form

\[
\sup_{\partial M} \Delta u \leq C \left( 1 + \sup_M |\nabla u|^2 \right)
\]

and then solve the Dirichlet problem.

First let us introduce some basic notions. We say a function \( w \in C^2(\bar{M}) \) is admissible if

\[
\lambda(g[w]) \in \Gamma \text{ in } \bar{M}.
\]

For an admissible function \( u \in C^2(\bar{M}) \), it is called an admissible subsolution of the Dirichlet problem (1.1), if

\[
f(\lambda(g[u])) \geq \psi \text{ in } \bar{M}, \quad u = \varphi \text{ on } \partial M.
\]

Moreover, \( u \) is called a strictly admissible subsolution, if

\[
f(\lambda(g[u])) > \psi \text{ in } \bar{M}, \quad u = \varphi \text{ on } \partial M.
\]

Meanwhile, we say the Dirichlet problem (1.1) is non-degenerate if

\[
\inf_M \psi > \sup_{\partial M} f.
\]

We say it is degenerate if

\[
\inf_M \psi = \sup_{\partial M} f.
\]
and
\[ f \in C^\infty(\Gamma) \cap C(\overline{\Gamma}). \] (1.11)

Here
\[ \sup f = \sup_{\partial \Gamma} \limsup_{\lambda \to \lambda_0} f(\lambda), \quad \overline{\Gamma} = \Gamma \cup \partial \Gamma. \]

Throughout this paper, we always assume the boundary data \( \varphi \) can be extended to a \( C^{2,1} \)-admissible function on \( \tilde{M} \). (Such an assumption is necessary for the solvability). For simplicity, we still denote it by \( \varphi \).

Remark 1.1. The perspective of subsolution was imposed by [16, 20, 26] as a vital tool to deal with the second order boundary estimates for Dirichlet problem of Monge-Ampère equation on bounded domains. The concept of subsolutions has a great advantage in applications to some geometric problem that it relaxes restrictions to the shape of boundary, see e.g. [7, 20, 22, 23].

Our main results are as follows.

Theorem 1.2. Let \((M,J,\omega)\) be a compact Hermitian manifold with smooth boundary. Let \( f \) satisfy the hypotheses (1.2)-(1.5). Assume the data \( \varphi \in C^{k,\alpha}(\partial M) \) and \( \psi \in C^{k,\alpha}(\tilde{M}) \) with \( k \geq 2, \ 0 < \alpha < 1 \) satisfy (1.9) and support a \( C^{2,1} \)-admissible subsolution. Then the Dirichlet problem (1.1) possesses a unique admissible solution \( u \in C^{k+2,\alpha}(\overline{M}) \).

Let \( \Gamma_\infty \) be the projection of \( \Gamma \) to the subspace of former \( n-1 \) subscripts. Namely, \((\lambda_1, \cdots, \lambda_{n-1}) \in \Gamma_\infty \) if and only if there exists \( \lambda_n > 0 \) such that \((\lambda_1, \cdots, \lambda_{n-1}, \lambda_n) \in \Gamma \).

Theorem 1.3. In Theorem 1.2 the hypothesis (1.5) on \( f \) can be dropped if the Levi form \( L_{\partial M} \) of \( \partial M \) satisfies
\[ (-\kappa_1, \cdots, -\kappa_{n-1}) \in \overline{\Gamma}_\infty \] (1.12)

where \( \kappa_1, \cdots, \kappa_{n-1} \) denote the eigenvalues of Levi form \( L_{\partial M} \) of \( \partial M \) with respect to \( \omega' = \omega|_{\partial M} \cap JT_{\partial M} \), and \( \overline{\Gamma}_\infty \) is the closure of \( \Gamma_\infty \).

To prove the above results, it remains to prove gradient estimate. However, it is still highly mysterious to prove gradient bound directly for (1.1) on complex manifolds, as Błocki [1], Hanani [25] and Guan-Li [18] did for complex Monge-Ampère equation, to be compared with the gradient estimate established by Li [31] and Urbas [43] for fully nonlinear elliptic equations on Riemannian manifolds. The complex version of Li’s result was obtained by the author [46] on Kähler manifolds with nonnegative orthogonal bisectional curvature; while the
trick used in [43] does not work in complex variables any more. The trouble comes from the lack of understanding of pure complex derivatives \( u_{ij}, u_{ij} \) when restricted to mixed complex derivatives \( u_{ij} \). Blow-up argument is an alternative approach to derive the gradient estimate. When \((M, J, \omega)\) is a closed Hermitian manifold, Székelyhidi [37] proved the second estimate for equation (1.1)

\[
\sup_M \Delta u \leq C \left( 1 + \sup_M |\nabla u|^2 \right)
\]

(1.13)

and then used it to derive gradient estimate via a blow-up argument. Such a blow-up argument using (1.13) appeared in literature that has been done by Chen [7], complemented by [3, 34], for Dirichlet problem of complex Monge-Ampère equation, and by Dinew-Kołodziej [10] for complex \( k \)-Hessian equations on closed Kähler manifolds using Hou-Ma-Wu’s second estimate [27, Theorem 1.1]. We also refer respectively to [38, 39, 40] and [47] for related works devoting to Gauduchon’s conjecture and complex \( k \)-Hessian equations on closed Hermitian manifolds.

To show Theorems 1.2 and 1.3, a specific problem that we have in mind is to establish the boundary estimate (1.6). Unfortunately, the proof in [7, 2, 34] relies heavily upon the specific structure of Monge-Ampère operator, which cannot be adapted to general equations. In this paper we set up Lemmas 3.2 and 3.6 in an attempt to bound the double normal derivative of solutions on the boundary in a quantitative form. Subsequently, we achieve the goal as follows.

**Theorem 1.4.** Let \((M, J, \omega)\) and \(f\) be as in Theorem 1.2. Assume that \( \varphi \in C^3(\partial M) \) and \( \psi \in C^1(\bar{M}) \) satisfy (1.9) and support a \( C^2 \)-admissible subsolution \( \underline{u} \). Then any admissible solution \( u \in C^3(M) \cap C^2(\bar{M}) \) to (1.1) satisfies

\[
\sup_{\partial M} \Delta u \leq C \left( 1 + \sup_M |\nabla u|^2 \right),
\]

where \( C \) is a uniform positive constant depending on \( |\varphi|_{C^1(M)}, |\nabla u|_{C^0(\partial M)}, |u|_{C^2(\bar{M})}, |\psi|_{C^1(M)}, \partial M \) up to third derivatives, and other known data under control (but not on \( \sup_M |\nabla u| \)).

Moreover, the hypothesis (1.5) can be dropped and the constant \( C \) is independent of \( (\delta_{\varphi, f})^{-1} \), provided \( \partial M \) satisfies (1.12), where

\[
\delta_{\varphi, f} = \inf_M \varphi - \sup_{\partial M} f.
\]

**Remark 1.5.** In the proof of estimates, it only requires the subsolution \( \underline{u} \in C^2(\bar{M}) \).
Note that the above estimate is fairly delicate. It does not depend on $(\delta_{\psi,f})^{-1}$ under assumption (1.12). As a result, together with Lemma 6.3, we can solve the Dirichlet problem for degenerate equations.

**Theorem 1.6.** Let $(M, J, \omega)$ be a compact Hermitian manifold with smooth boundary subject to (1.12), and let $f$ satisfy (1.2), (1.3) and (1.11). Assume $\varphi \in C^{2,1}(\partial M)$ and $\psi \in C^{1,1}(\tilde{M})$ satisfy (1.10) and support a strictly admissible subsolution $u \in C^{2,1}(\tilde{M})$. Then there exists a (weak) solution $u \in C^{1,\alpha}(\bar{M})$ to the Dirichlet problem (1.1) with $\forall 0 < \alpha < 1$ such that

$$\lambda(\varphi | u) \in \bar{\Gamma} \text{ in } \tilde{M}, \quad \Delta u \in L^\infty(\bar{M}).$$

**Remark 1.7.** To some sense, this may be the first breakthrough for Dirichlet problem for general degenerate fully nonlinear elliptic equations of the type (1.1) on complex manifolds.

**Remark 1.8.** When $\Gamma = \Gamma_\alpha$, Theorem 1.2 gives back a result of the author [46, Theorem 1] with a different method. The case $(f, \Gamma) = (\sigma_{\kappa}^{1/k}, \Gamma_k)$ was also proved by Collins-Picard [9] independently; while the right-hand side of equation considered there does not include degenerate case, to be compared with Theorem 1.6.

It is a remarkable fact that the Dirichlet problem is not always solvable without the subsolution assumption. A natural problem is to construct the subsolutions. Unfortunately, except on certain domains in Euclidean spaces [4, 5, 30], few progress has been made on general manifolds.

We confirm the subsolution assumption when the manifold is a product. Without specific clarification, $(X, J_X, \omega_X)$ is a closed Hermitian manifold of complex dimension $n-1$, and $(S, J_S, \omega_S)$ is a compact Riemann surface with boundary $\partial S$. Let $\pi_1 : X \times S \to X$ and $\pi_2 : X \times S \to S$ denote the natural projections. On the product $(M, J, \omega) = (X \times S, J, \omega)$, we are able to construct strictly admissible subsolutions for the Dirichlet problem, provided

$$\lim_{t \to +\infty} f(\lambda(\varphi) + t\pi_2^* \omega_S) > \psi \text{ in } \tilde{M}. \quad (1.14)$$

Here $J$ is the induced complex structure and $\omega$ is a Hermitian metric compatible with $J$ (but not necessary the standard product metric $\omega = \pi_1^* \omega_X + \pi_2^* \omega_S$). It is noteworthy that (1.14) automatically holds when $f$ satisfies (1.5).

As consequences, we deduce the following results.
**Theorem 1.9.** Let $(M, J, \omega) = (X \times S, J, \omega)$ be as above with $\partial S \in C^\infty$. Suppose in addition that (1.2)-(1.4) hold. Then for $\varphi \in C^\infty(\partial M)$, $\psi \in C^\infty(\bar{M})$ satisfying (1.9) and (1.14), the Dirichlet problem (1.1) admits a unique smooth admissible solution.

**Theorem 1.10.** Let $(M, J, \omega) = (X \times S, J, \omega)$ be as above with $\partial S \in C^{2,1}$, and let $f$ satisfy (1.2), (1.3) and (1.11). Given $\varphi \in C^{2-1}(\partial M)$ and $\psi \in C^{1-1}(\bar{M})$ satisfying (1.10) and (1.14), the Dirichlet problem (1.1) has a weak solution $u$ with

$$u \in C^{1,\alpha}(\bar{M}), \forall 0 < \alpha < 1, \Delta u \in L^\infty(\bar{M}), \lambda(\tilde{g}[u]) \in \tilde{\Gamma} \text{ in } \tilde{M}.$$ 

**Remark 1.11.** The degenerate fully nonlinear elliptic equations on the product $X \times S$ have many applications in geometry. When $S = \mathbb{S}^1 \times [0, 1]$ and $f = \sigma^1/n$, Theorem 1.10 immediately yields Chen’s existence and regularity of (weak) geodesics in the space of Kähler potentials $\mathcal{H}_{\omega_x}$. Moreover, as shown by Donaldson [11], for any compact Riemann surface $S$ with boundary, the homogeneous complex Monge-Ampère equation on $X \times S$ is closely related to the Wess-Zumino-Witten equation for a map from $S$ to $\mathcal{H}_{\omega_x}$. Our results regarding to the regularity of weak solutions and the construction of subsolutions apply to the Wess-Zumino-Witten equation and possibly enable one to attack some related problems.

In conclusion, in this paper we first derive the quantitative boundary estimate and then solve the Dirichlet problem for fully nonlinear elliptic equations, possibly with degenerate right-hand side. No matter the equations are degenerate or not, the existence results for Dirichlet problem of general fully nonlinear elliptic equations are rarely known until this work. In addition, we construct subsolutions on products, which are of numerous interests. Our results include the existence and regularity results of Chen [7] as a special case and extend extensively Székelyhidi’s work [37] to complex manifolds with boundary.

The paper is organized as follows. In Section 2 we sketch the proof of main estimates. In Section 3 we propose two lemmas which are key ingredients to understand the quantitative version of boundary estimate for double normal derivative. In Sections 4 and 5 we derive the quantitative boundary estimate for double normal and tangential-normal derivatives, respectively. The quantitative boundary estimate is then derived as a consequence. In Section 6 we complete the proof of Theorems 1.2, 1.3 and 1.6. In Section 7 the subsolutions are constructed when $M$ is a product of a closed complex manifold with a compact Riemann surface with boundary. In addition, we study Dirichlet problem on such products with
less regularity assumptions on boundary. The uniqueness of weak solutions and construction of subsolutions on more general products are discussed in Section 8.

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2 Sketch of the proof of main estimates

We first summarize the notations as follows.

- For $\sigma$, we denote
  \[\Gamma^{\sigma} = \{\lambda \in \Gamma : f(\lambda) > \sigma\}, \quad \partial \Gamma^{\sigma} = \{\lambda \in \Gamma : f(\lambda) = \sigma\}.\]  
  \label{2.1}

- $\vec{1} = (1, \cdots, 1) \in \mathbb{R}^n$.

- For the solution $u$ and subsolution $\underline{u}$, we denote $\underline{g} = g[u]$ and $\underline{g} = g[\underline{u}]$, respectively.

Condition (1.2) ensures (1.1) to be elliptic at admissible functions, while (1.3) implies that the operator $F(A) = f(\lambda(A))$ is concave with respect to $A$ subject to $\lambda(A) \in \Gamma$. Consequently, according to Evans-Krylov theorem [13, 29], adapted to complex setting (see e.g. [41]), together with Schauder theory, higher order estimates for admissible solutions can be derived from uniform bound of complex hessian

\[|\partial \overline{\partial} u| \leq C.\]  
\label{2.2}

The existence results then follow from standard continuity method.
Outline of the proof of Theorem 1.4

Before stating it, we denote
\[ \xi_1, \cdots, \xi_{n-1} \] (2.3)
an orthonormal basis of \( T_{\partial M}^{1,0} := T^{1,0}_M \cap T_{\partial M}^C \), \( \nu \) the unit inner normal vector along the boundary, and
\[ \xi_n = \frac{1}{\sqrt{2}}(\nu - \sqrt{-1}J\nu). \] (2.4)

Double normal case

Under the assumption either \( \partial M \) satisfies (1.12) or \( f \) satisfies (1.5), we derive the boundary estimate for double normal derivative in the following quantitative form
\[ g(\xi_n, J\xi_n)(p_0) \leq C \left( 1 + \sum_{\alpha=1}^{n-1} |g(\xi_\alpha, J\xi_\alpha)(p_0)|^2 \right), \forall p_0 \in \partial M. \] (2.5)

**Case 1: \( \partial M \) satisfies (1.12).** This assumption is used to compare \( g_{\alpha\beta} \) with \( g_{\alpha\beta} \) when restricted to boundary. We observe that it enables us to directly apply Lemmas 3.6 and 3.2 to understand the quantitative version of boundary estimate for double normal derivative (see Proposition 4.1).

**Case 2: \( f \) satisfies (1.5).** Without imposing any restriction to shape of boundary, following a strategy of Caffarelli-Nirenberg-Spruck [5], we construct delicate barrier functions based on a characterization of \( \Gamma_\infty \) to compare \( g_{\alpha\beta} \) with \( g_{\alpha\beta} \) on boundary. Subsequently, we apply Lemmas 3.2 and 3.6 to prove (2.5) in Proposition 4.2.

Tangential-Normal case

From the quantitative boundary estimate (2.5) for double normal derivative, it requires to prove quantitative boundary estimate for tangential-normal derivatives
\[ |g(\xi_\alpha, J\xi_n)(p_0)| \leq C \left( 1 + \sup_M |\nabla u| \right), \forall 1 \leq \alpha \leq n - 1, \forall p_0 \in \partial M. \] (2.6)

This is proved in Proposition 5.1, using local barrier technique going back at least to [26, 20, 16].
Sketch the proof of gradient estimate

The main obstruction to deriving (2.2) is the gradient estimate. Our strategy is blow-up argument. According to Lemma 3.2 below, in the presence of (1.2) and (1.3), condition (1.4) is in effect equivalent to

\[ \lim_{t \to +\infty} f(t \lambda) > f(\mu), \ \forall \lambda, \mu \in \Gamma. \quad (2.7) \]

Such a condition allows one to apply the blow-up argument in [37, Section 6].

As shown in Theorem 1.4, the admissible solutions satisfy

\[ \sup_{\partial M} \Delta u \leq C \left( 1 + \sup_M |\nabla u|^2 \right). \]

On the other hand, Székelyhidi’s second estimate [37] yields that any admissible solution to (1.1) satisfies

\[ \sup_M \Delta u \leq C \left( 1 + \sup_M |\nabla u|^2 + \sup_{\partial M} |\Delta u| \right). \]

Moreover, by \( \Gamma \subset \Gamma_1 \), one has \( \Delta u > -\text{tr} \omega \chi \). With those at hand, we obtain (1.13), i.e.

\[ \sup_M \Delta u \leq C \left( 1 + \sup_M |\nabla u|^2 \right). \]

Consequently, the gradient estimate can be derived from (1.13), using the Liouville type theorem [37, Theorem 20].

3 Lemmas

To prove Theorem 1.4, we propose Lemmas 3.2 and 3.6.

3.1 Criteria for symmetric concave functions

The concavity of \( f \) implies

\[ \sum_{i=1}^{n} f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda), \ \forall \lambda, \mu \in \Gamma. \quad (3.1) \]
This yields

\[ \text{For any } \lambda, \mu \in \Gamma, \sum_{i=1}^{n} f_i(\lambda) \mu_i \geq \limsup_{t \to +\infty} f(t\mu)/t. \]

Inspired by this observation, we introduce the following condition

\[ \text{For any } \lambda \in \Gamma, \limsup_{t \to +\infty} f(t\lambda)/t \geq 0. \] (3.2)

This leads to

\textbf{Lemma 3.1.} For \( f \) satisfying (1.3), the condition (3.2) is equivalent to each one of the following three conditions

\[ \sum_{i=1}^{n} f_i(\lambda) \mu_i \geq 0, \forall \lambda, \mu \in \Gamma, \] (3.3)

\[ f(\lambda + \mu) \geq f(\lambda), \forall \lambda, \mu \in \Gamma, \] (3.4)

\[ \sum_{i=1}^{n} f_i(\lambda) \lambda_i \geq 0, \forall \lambda \in \Gamma. \] (3.5)

If, in addition, \( \sum_{i=1}^{n} f_i(\lambda) > 0 \) then (3.2) is equivalent to each one of the following:

\[ \sum_{i=1}^{n} f_i(\lambda) \mu_i > 0, \forall \lambda, \mu \in \Gamma, \] (3.6)

\[ f(\lambda + \mu) > f(\lambda), \forall \lambda, \mu \in \Gamma. \] (3.7)

\textit{Proof.} Obviously, (3.2) implies (3.3). From (3.1), we have

\[ f(\lambda + \mu) - f(\lambda) \geq \sum_{i=1}^{n} f_i(\lambda + \mu) \mu_i. \] (3.8)

Thus (3.3) implies (3.4). It follows from (3.4) that

\[ f(t\lambda) \geq f(s\lambda), \forall \lambda \in \Gamma, \forall t > s \] (3.9)

which means \( \frac{d}{dt} f(t\lambda) \geq 0 \). Thus (3.4) yields (3.5). By (3.5) we have (3.9). And then it gives (3.2).

By the openness of \( \Gamma \), for \( \mu \in \Gamma \) there is \( \delta_\mu > 0 \) such that \( \mu - \delta_\mu \bar{1} \in \Gamma \). By (3.3), we have \( \sum_{i=1}^{n} f_i(\lambda) \mu_i \geq \delta_\mu \sum_{i=1}^{n} f_i(\lambda) \) and then (3.6) if \( \sum_{i=1}^{n} f_i(\lambda) > 0 \). By (3.8) and (3.6), we derive (3.7). \( \square \)
We now give criteria of symmetric concave functions satisfying (1.4).

Lemma 3.2. In the presence of (1.2) and (1.3), the following statements are equivalent.

1. $f$ satisfies (1.4).
2. $f$ satisfies (2.7).
3. $f$ satisfies (3.2).
4. $f$ satisfies (3.3).
5. $f$ satisfies (3.4).
6. $f$ satisfies (3.5).
7. $f$ satisfies (3.6).
8. $f$ satisfies (3.7).

Proof. Obviously, (2.7) $\Rightarrow$ (1.4) $\Rightarrow$ (3.2) $\Leftrightarrow$ (3.3) $\Leftrightarrow$ (3.6) $\Leftrightarrow$ (3.4) $\Leftrightarrow$ (3.5) $\Rightarrow$ (1.4).

It requires only to prove (3.7) $\Rightarrow$ (2.7). Fix $\lambda$, $\mu \in \Gamma$. Since $\Gamma$ is open, $t\lambda - \mu = t(\lambda - \mu/t) \in \Gamma$ for $t \geq t_{\lambda,\mu}$, depending only on $\lambda$ and $\mu$. Thus (3.7) yields

$$f(t\lambda) = f(\mu + (t\lambda - \mu)) > f(\mu).$$

□

Remark 3.3. Together with [5, Lemma 6.2] (a special case of a result of [33]), Lemma 3.2 implies that for any $n \times n$ Hermitian matrices $A = (A_{ij})$, $B = (B_{ij})$ with $\lambda(A), \lambda(B) \in \Gamma$,

$$\frac{\partial F}{\partial A_{ij}}(A)B_{ij} > 0.$$  \hfill (3.10)

Together with (3.7), we have

$$F(A + B) > F(A).$$  \hfill (3.11)

Remark 3.4. Condition (3.7) and (3.11) play a vital role in proof of quantitative boundary estimate for pure normal derivative.
Remark 3.5. According to Lemma 3.2, if (1.3) and (1.4) hold then
\[
\sum_{i=1}^{n} f_i(\lambda) > \frac{f(R\vec{1}) - f(\lambda)}{R} \text{ for any } R > 0.
\]
In particular, there is a uniform positive constant \( \kappa_\sigma \) depending on \( \sigma \) such that
\[
\sum_{i=1}^{n} f_i(\lambda) \geq \kappa_\sigma \text{ in } \partial \Gamma^\sigma.
\] (3.12)

3.2 Quantitative lemmas

A key ingredient for quantitative boundary for double normal derivative is how to follow the track of the behavior of the eigenvalues of the matrix \( g(\xi, J\vec{\xi}) \) as \( g(\xi, J\vec{\xi}) \) tends to infinity. To this end, we prove the following lemma.

**Lemma 3.6.** Let \( A \) be an \( n \times n \) Hermitian matrix
\[
\begin{pmatrix}
d_1 & a_1 \\
d_2 & a_2 \\
\vdots & \vdots \\
d_{n-1} & a_{n-1} \\
\bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{n-1} & a
\end{pmatrix}
\]
with \( d_1, \cdots, d_{n-1}, a_1, \cdots, a_{n-1} \) fixed, and with \( a \) variable. Denote the eigenvalues of \( A \) by \( \lambda = (\lambda_1, \cdots, \lambda_n) \). Let \( \epsilon > 0 \) be a fixed constant. Suppose that the parameter \( a \) satisfies the quadratic growth condition
\[
a \geq 2n - 3 \frac{\epsilon}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n - 1) \sum_{i=1}^{n-1} |d_i| + \frac{(n - 2)\epsilon}{2n - 3}.
\] (3.13)

Then the eigenvalues (possibly with a proper permutation) behave like
\[
d_\alpha - \epsilon < \lambda_\alpha < d_\alpha + \epsilon, \quad \forall 1 \leq \alpha \leq n - 1,
\]
\[
a \leq \lambda_n < a + (n - 1)\epsilon.
\]

This lemma asserts that if the parameter \( a \) satisfies the quadratic growth condition (3.13) then the eigenvalues concentrate near diagonal elements correspondingly. Consequently, it suggests an effective way to follow the track of the behavior of the eigenvalues as \( |a| \) tends to infinity. In fact, Lemma 3.6 can be viewed as a quantitative version of [5, Lemma 1.2].
Lemma 3.7 ([5, Lemma 1.2]). Consider the $n \times n$ symmetric matrix

$$A = \begin{pmatrix} d_1 & a_1 \\ d_2 & a_2 \\ & \ddots \\ d_{n-1} & a_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} & a \end{pmatrix}$$

with $d_1, \ldots, d_{n-1}$ fixed, $|a|$ tends to infinity and $|a_i| \leq C$, $i = 1, \ldots, n$.

Then the eigenvalues $\lambda_1, \ldots, \lambda_n$ behave like

$$\lambda_\alpha = d_\alpha + o(1), \ 1 \leq \alpha \leq n - 1,$$

$$\lambda_n = a \left(1 + O\left(\frac{1}{a}\right)\right),$$

where the $o(1)$ and $O\left(\frac{1}{a}\right)$ are uniform depending only on $d_1, \ldots, d_{n-1}$ and $C$.

This lemma was first used by Caffarelli-Nirenberg-Spruck [5] and later by [42, 30] to derive boundary estimates for double normal derivative of certain fully nonlinear elliptic equations on bounded domains $\Omega$ in Euclidean spaces. However, their boundary estimate is not a quantitative form. The reason is that Lemma 3.7 does not figure out how the eigenvalues of matrix $A$ concentrate explicitly near the corresponding diagonal elements when $|a|$ is sufficiently large.

In the rest of this subsection, we complete the proof of Lemma 3.6. We start with the case $n = 2$. For $n = 2$, the eigenvalues of $A$ are

$$\lambda_1 = \frac{a + d_1 - \sqrt{(a - d_1)^2 + 4|a_1|^2}}{2}, \quad \lambda_2 = \frac{a + d_1 + \sqrt{(a - d_1)^2 + 4|a_1|^2}}{2}.$$

We assume $a_1 \neq 0$; otherwise we are done. If $a \geq \frac{|a|^2}{\epsilon} + d_1$ then one has

$$0 \leq d_1 - \lambda_1 = \lambda_2 - a = \frac{2|a_1|^2}{\sqrt{(a - d_1)^2 + 4|a_1|^2} + (a - d_1)} < \frac{|a|^2}{a - d_1} \leq \epsilon.$$

Here we use $a_1 \neq 0$ to confirm the strictly inequality in the above formula.

The following lemma enables us to count the eigenvalues near the diagonal elements via a deformation argument. It is an essential ingredient in the proof of Lemma 3.6 for general $n$. 

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Lemma 3.8. Let $A$ be a Hermitian $n$ by $n$ matrix

\[
\begin{pmatrix}
  d_1 & a_1 \\
  d_2 & a_2 \\
  \vdots & \vdots \\
  d_{n-1} & a_{n-1} \\
  \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{n-1} & a
\end{pmatrix}
\]

with $d_1, \cdots, d_{n-1}, a_1, \cdots, a_{n-1}$ fixed, and with $a$ variable. Denote $\lambda = (\lambda_1, \cdots, \lambda_n)$ as the eigenvalues of $A$ with the order $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Fix a positive constant $\epsilon$. Suppose that the parameter $a$ in the matrix $A$ satisfies the following quadratic growth condition

\[
a \geq \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + \sum_{i=1}^{n-1} (d_i + (n - 2)|d_i|) + (n - 2)\epsilon.
\]  

Then for any $\lambda_\alpha$ ($1 \leq \alpha \leq n - 1$) there exists a $d_{i_\alpha}$ with lower index $1 \leq i_\alpha \leq n - 1$ such that

\[
|\lambda_\alpha - d_{i_\alpha}| < \epsilon,
\]  

\[
0 \leq \lambda_n - a < (n - 1)\epsilon + \left| \sum_{\alpha=1}^{n-1} (d_\alpha - d_{i_\alpha}) \right|.
\]

Proof. Without loss of generality, we assume $\sum_{i=1}^{n-1} |a_i|^2 > 0$ and $n \geq 3$ (otherwise we are done). Note that the eigenvalues have the order $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, as in the assumption of lemma. It is well known that, for a Hermitian matrix, any diagonal element is less than or equals to the largest eigenvalue. In particular,

\[
\lambda_n \geq a.
\]

We only need to prove (3.15), since (3.16) is a consequence of (3.15), (3.17) and

\[
\sum_{i=1}^{n} \lambda_i = \text{tr}(A) = \sum_{\alpha=1}^{n-1} d_\alpha + a.
\]

Let’s denote $I = \{1, 2, \cdots, n-1\}$. We divide the index set $I$ into two subsets:

- $B = \{\alpha \in I : |\lambda_\alpha - d_\alpha| \geq \epsilon, \ \forall i \in I\}$,

- $G = I \setminus B = \{\alpha \in I : \text{There exists } i \in I \text{ such that } |\lambda_\alpha - d_\alpha| < \epsilon\}$. 

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To complete the proof, it only requires to prove $G = I$ or equivalently $B = \emptyset$. It is easy to see that for any $\alpha \in G$, one has

$$|\lambda_\alpha| < \sum_{i=1}^{n-1} |d_i| + \epsilon. \quad (3.19)$$

Fix $\alpha \in B$, we are going to give the estimate for $\lambda_\alpha$. The eigenvalue $\lambda_\alpha$ satisfies

$$(\lambda_\alpha - a) \prod_{i=1}^{n-1} (\lambda_\alpha - d_i) = \sum_{i=1}^{n-1} |a_i|^2 \prod_{j \neq i} (\lambda_\alpha - d_j). \quad (3.20)$$

By the definition of $B$, for $\alpha \in B$, one then has $|\lambda_\alpha - d_i| \geq \epsilon$ for any $i \in I$. We therefore derive

$$|\lambda_\alpha - a| \leq \sum_{i=1}^{n-1} \frac{|a_i|^2}{|\lambda_\alpha - d_i|} \leq \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2, \quad \text{if } \alpha \in B. \quad (3.21)$$

Hence, for $\alpha \in B$, we obtain

$$\lambda_\alpha \geq a - \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2. \quad (3.22)$$

For a set $S$, we denote $|S|$ the cardinality of $S$. We shall use proof by contradiction to prove $B = \emptyset$. Assume $B \neq \emptyset$. Then $|B| \geq 1$, and so $|G| = n - 1 - |B| \leq n - 2$.

We compute the trace of the matrix $A$ as follows:

$$\text{tr}(A) = \lambda_n + \sum_{\alpha \in B} \lambda_\alpha + \sum_{\alpha \in G} \lambda_\alpha$$

$$> \lambda_n + |B|(a - \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2) - |G|(\sum_{i=1}^{n-1} |d_i| + \epsilon)$$

$$\geq 2a - \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 - (n-2)(\sum_{i=1}^{n-1} |d_i| + \epsilon) \quad (3.23)$$

$$\geq \sum_{i=1}^{n-1} d_i + a = \text{tr}(A),$$

where we use (3.14), (3.17), (3.19) and (3.22). This is a contradiction. We have $B = \emptyset$. Therefore, $G = I$ and the proof is complete. \qed
We consequently obtain

**Lemma 3.9.** Let $A(a)$ be an $n \times n$ Hermitian matrix

$$A(a) = \begin{pmatrix}
d_1 & a_1 \\
d_2 & a_2 \\
\vdots & \\
d_{n-1} & a_{n-1} \\
\bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{n-1} & a
\end{pmatrix}$$

with $d_1, \cdots, d_{n-1}, a_1, \cdots, a_{n-1}$ fixed, and with $a$ variable. Assume that $d_1, d_2, \cdots, d_{n-1}$ are distinct with each other, i.e. $d_i \neq d_j, \forall i \neq j$. Denote $\lambda = (\lambda_1, \cdots, \lambda_n)$ as the the eigenvalues of $A(a)$. Given a positive constant $\epsilon$ with $0 < \epsilon \leq \frac{1}{2} \min \{|d_i - d_j| : \forall i \neq j\}$, if the parameter $a$ satisfies the quadratic growth condition

$$a \geq \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n - 1) \sum_{i=1}^{n-1} |d_i| + (n - 2)\epsilon,$$  

(3.24)

then the eigenvalues behave like

$$|d_\alpha - \lambda_\alpha| < \epsilon, \forall 1 \leq \alpha \leq n - 1,$$

$$0 \leq \lambda_n - a < (n - 1)\epsilon.$$

**Proof.** The proof is based on Lemma 3.8 and a deformation argument. Without loss of generality, we assume $n \geq 3$ and $\sum_{i=1}^{n-1} |a_i|^2 > 0$ (otherwise we are done). Moreover, we assume in addition that $d_1 < d_2 \cdots < d_{n-1}$ and the eigenvalues have the order

$$\lambda_1 \leq \lambda_2 \cdots \leq \lambda_{n-1} \leq \lambda_n.$$

Fix $\epsilon \in (0, \mu_0]$, where $\mu_0 = \frac{1}{2} \min \{|d_i - d_j| : \forall i \neq j\}$. We denote

$$I_i = (d_i - \epsilon, d_i + \epsilon)$$

and

$$P_0 = \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n - 1) \sum_{i=1}^{n-1} |d_i| + (n - 2)\epsilon.$$

Since $0 < \epsilon \leq \mu_0$, the intervals disjoint each other

$$I_\alpha \cap I_\beta = \emptyset \text{ for } 1 \leq \alpha < \beta \leq n - 1.$$  

(3.25)
In what follows, we assume that the parameter $a$ satisfies (3.24) and the Greek letters $\alpha, \beta$ range from 1 to $n - 1$. We define a function

$$\text{Card}_a : [P_0, +\infty) \to \mathbb{N}$$

to count the eigenvalues which lie in $I_\alpha$. (Note that when the eigenvalues are not distinct, the function $\text{Card}_a$ means the summation of all the algebraic multiplicities of distinct eigenvalues which lie in $I_\alpha$). This function measures the number of the eigenvalues which lie in $I_\alpha$.

We are going to prove that $\text{Card}_a$ is continuous on $[P_0, +\infty)$ in an attempt to complete the proof.

First Lemma 3.8 asserts that if $a \geq P_0$, then

$$\lambda_\alpha \in \bigcup_{i=1}^{n-1} I_i, \ \forall 1 \leq \alpha \leq n - 1. \quad (3.26)$$

It is well known that the largest eigenvalue $\lambda_n \geq a$, while the smallest eigenvalue $\lambda_1 \leq d_1$. Combining it with (3.26) one has

$$\lambda_n \geq a > \sum_{i=1}^{n-1} |d_i| + \epsilon. \quad (3.27)$$

Thus $\lambda_n \in \mathbb{R} \setminus (\bigcup_{i=1}^{n-1} \overline{I_i})$ where $\overline{I_i}$ denotes the closure of $I_i$. Therefore, the function $\text{Card}_a$ is continuous (and so it is constant), since (3.26), (3.25), $\lambda_n \in \mathbb{R} \setminus (\bigcup_{i=1}^{n-1} I_i)$ and the eigenvalues of $A(a)$ depend on the parameter $a$ continuously.

The continuity of $\text{Card}_a(a)$ plays a crucial role in this proof. Following the line of the proof of Lemma 3.7 ([5, Lemma 1.2]), in the setting of Hermitian matrices, one can show that for $1 \leq \alpha \leq n - 1$,

$$\lim_{a \to +\infty} \text{Card}_a(a) \geq 1. \quad (3.28)$$

It follows from (3.27), (3.28) and the continuity of $\text{Card}_a$ that

$$\text{Card}_a(a) = 1, \ \forall a \in [P_0, +\infty), \ 1 \leq \alpha \leq n - 1.$$  

Together with (3.26), we prove that, for any $1 \leq \alpha \leq n - 1$, the interval $I_\alpha = (d_\alpha - \epsilon, d_\alpha + \epsilon)$ contains the eigenvalue $\lambda_\alpha$. We thus complete the proof. \qed
Suppose that there are two distinct indices $i_0, j_0$ ($i_0 \neq j_0$) such that $d_{i_0} = d_{j_0}$. Then the characteristic polynomial of $A$ can be rewritten as the following

$$(\lambda - d_{i_0}) \left[ (\lambda - a) \prod_{i \neq i_0} (\lambda - d_i) - |a_{i_0}|^2 \prod_{j \neq j_0, j \neq i_0} (\lambda - d_j) - \sum_{i \neq i_0} |a_i|^2 \prod_{j \neq j_0, j \neq i_0} (\lambda - d_j) \right].$$

So $\lambda_{i_0} = d_{i_0}$ is an eigenvalue of $A$ for any $a \in \mathbb{R}$. Noticing that the following polynomial

$$(\lambda - a) \prod_{i \neq i_0} (\lambda - d_i) - |a_{i_0}|^2 \prod_{j \neq j_0, j \neq i_0} (\lambda - d_j) - \sum_{i \neq i_0} |a_i|^2 \prod_{j \neq j_0, j \neq i_0} (\lambda - d_j)$$

is the characteristic polynomial of the $(n-1) \times (n-1)$ Hermitian matrix

$$
\begin{pmatrix}
  d_1 & a_1 & & \\
  & \ddots & & \\
  & & \tilde{d}_{i_0} & \\
  \tilde{a}_{i_0} & & \ddots & \ddots \\
  & \ddots & & \\
  \tilde{a}_{j_0} & & \ddots & |a_{j_0}|^2 + |a_{i_0}|^2 \frac{1}{2} \\
  a_1 & \cdots & \tilde{a}_{i_0} & \cdots & |a_{j_0}|^2 + |a_{i_0}|^2 \frac{1}{2} & \cdots & a
\end{pmatrix}
$$

where $\tilde{\ast}$ indicates deletion. Therefore, $(\lambda_1, \cdots, \tilde{\lambda}_{i_0}, \cdots, \lambda_n)$ are the eigenvalues of the above $(n-1) \times (n-1)$ Hermitian matrix. Hence, we obtain

**Lemma 3.10.** Let $A$ be as in Lemma 3.6 an $n \times n$ Hermitian matrix. Let

$$I = \begin{cases}
  \mathbb{R}^+ = (0, +\infty) & \text{if } d_i = d_1, \forall \leq i \leq n - 1; \\
  (0, \mu_0), \mu_0 = \frac{1}{2} \min\{|d_i - d_j| : d_i \neq d_j\} & \text{otherwise.}
\end{cases}$$

Denote $\lambda = (\lambda_1, \cdots, \lambda_n)$ by the the eigenvalues of $A$. Fix $\epsilon \in I$. Suppose that the parameter $a$ in $A$ satisfies (3.24). Then the eigenvalues behave like

$$|d_i - \lambda_\alpha| < \epsilon, \forall 1 \leq \alpha \leq n - 1,$$

$$0 \leq \lambda_n - a < (n - 1)\epsilon.$$

Applying Lemmas 3.8 and 3.10, we complete the proof of Lemma 3.6 without restriction to the applicable scope of $\epsilon$. 

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Proof of Lemma 3.6. We follow the outline of the proof of Lemma 3.9. Without loss of generality, we may assume

\[ n \geq 3, \sum_{i=1}^{n-1} |a_i|^2 > 0, \ d_1 \leq d_2 \leq \cdots \leq d_{n-1} \text{ and } \lambda_1 \leq \lambda_2 \leq \cdots \lambda_{n-1} \leq \lambda_n. \]

Fix \( \epsilon > 0 \). Let \( I'_\alpha = (d_\alpha - \frac{\epsilon}{2n-3}, d_\alpha + \frac{\epsilon}{2n-3}) \) and

\[ P'_0 = \frac{2n - 3}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n - 1) \sum_{i=1}^{n-1} |d_i| + \frac{(n - 2)\epsilon}{2n - 3}. \]

In what follows we assume (3.13) holds. The connected components of \( \bigcup_{\alpha=1}^{n-1} I'_\alpha \) are denoted as in the following:

\[ J_1 = \bigcup_{\alpha=1}^{j_1} I'_\alpha, \ J_2 = \bigcup_{\alpha=j_1+1}^{j_2} I'_\alpha, \ \cdots, \ J_i = \bigcup_{\alpha=j_{i-1}+1}^{j_i} I'_\alpha, \ \cdots, \ J_m = \bigcup_{\alpha=j_{m-1}+1}^{n-1} I'_\alpha. \]

Moreover,

\[ J_i \cap J_k = \emptyset, \text{ for } 1 \leq i < k \leq m. \]

It plays formally the role of (3.25) in the proof of Lemma 3.9.

As in the proof of Lemma 3.9, we let

\[ \overline{\text{Card}}_k : [P'_0, +\infty) \rightarrow \mathbb{N} \]

be the function that counts the eigenvalues which lie in \( J_k \). (Note that when the eigenvalues are not distinct, the function \( \overline{\text{Card}}_k \) denotes the summation of all the algebraic multiplicities of distinct eigenvalues which lie in \( J_k \)). By Lemma 3.8 and

\[ \lambda_n \geq a \geq P'_0 > \sum_{i=1}^{n-1} |d_i| + \frac{\epsilon}{2n - 3}. \]

we conclude that if the parameter \( a \) satisfies the quadratic growth condition (3.13) then

\[ \lambda_n \in \mathbb{R} \setminus \left( \bigcup_{k=1}^{n-1} \overline{I'_k} \right) = \mathbb{R} \setminus \left( \bigcup_{i=1}^{m} \overline{J_i} \right), \]

\[ \lambda_\alpha \in \bigcup_{i=1}^{n-1} I'_i = \bigcup_{i=1}^{m} J_i \text{ for } 1 \leq \alpha \leq n - 1. \]
Similarly, $\widehat{\text{Card}}_i(a)$ is a continuous function with respect to the variable $a$ when $a \geq P'_0$. So it is a constant. Combining it with Lemma 3.10, we see that
\[
\widehat{\text{Card}}_i(a) = j_i - j_{i-1}
\]
for $a \geq P'_0$. Here we denote $j_0 = 0$ and $j_m = n - 1$. We thus know that the $(j_i - j_{i-1})$ eigenvalues
\[
\lambda_{j_{i-1}+1}, \lambda_{j_{i-1}+2}, \ldots, \lambda_{j_i}
\]
lie in the connected component $J_i$. Thus, for any $j_{i-1} + 1 \leq \gamma \leq j_i$, we see $I'_\gamma \subset J_i$ and $\lambda_{\gamma}$ lies in the connected component $J_i$. Therefore,
\[
|\lambda_{\gamma} - d_{\gamma}| < \frac{(2(j_i - j_{i-1}) - 1)\epsilon}{2n - 3} \leq \epsilon.
\]
Here we use the fact that $d_{\gamma}$ is the midpoint of $I'_\gamma$ and $J_i \subset \mathbb{R}$ is an open subset. □

4 Quantitative boundary estimate: double normal case

This section is devoted to deriving quantitative version of boundary estimate for double normal derivative. The key ingredients in the proof are Lemmas 3.2 and 3.6.

Proposition 4.1. Let $(M, J, \omega)$ be a compact Hermitian manifold with $C^2$ boundary satisfying (1.12). Let $\xi_i$ be as in (2.3) and (2.4). Let $u \in C^2(\bar{M})$ be an admissible solution to Dirichlet problem (1.1). Suppose (1.2)-(1.4), (1.7) and (1.9) hold. Then
\[
g(\xi_n, J\bar{\xi}_n)(p_0) \leq C \left(1 + \sum_{\alpha=1}^{n-1} |g(\xi_\alpha, J\bar{\xi}_n)(p_0)|^2 \right), \quad \forall p_0 \in \partial M,
\]
where $C$ is a uniform positive constant depending only on $|u|_{C^0(M)}$, $|u|_{C^2(M)}$, $\partial M$ up to second order derivatives and other known data (but neither on $\sup_M |\nabla u|$ nor on $|\delta_{\varphi,f}|^{-1}$).

Without any restriction (1.12) to the shape of boundary, we obtain the following proposition when $f$ satisfies the unbounded condition (1.5).
Proposition 4.2. Let \((M, J, \omega)\) be a compact Hermitian manifold with \(C^3\) boundary, let \(\xi_i\) be as in (2.3) and (2.4). In addition to (1.2)-(1.4), (1.7) and (1.9), we assume \(f\) satisfies the unbounded condition (1.5). Then any admissible solution \(u \in C^2(\bar{M})\) to Dirichlet problem (1.1) satisfies

\[
\begin{align*}
g(\xi_n, J\xi_n)(p_0) & \leq C \left( 1 + \sum_{\alpha=1}^{n-1} |g(\xi_\alpha, J\xi_\alpha)(p_0)|^2 \right), \\
\forall p_0 & \in \partial M.
\end{align*}
\]

Here the constant \(C\) depends on \((\delta_{\phi, f})^{-1}, \sup_M \psi, |u|_{C^0(\partial M)}, |u|_{C^2(\partial M)}, \partial M\) up to third order derivatives and other known data (but not on \(\sup_M |\nabla u|\)).

4.1 Preliminaries

Given \(p_0 \in \partial M\), we can choose a local holomorphic coordinate systems

\[
(z_1, \cdots, z_n), \quad z_i = x_i + \sqrt{-1}y_i,
\]

centered at \(p_0\), so that \(g_{ij}(0) = \delta_{ij}, \frac{\partial}{\partial x_n}\) is the inner normal vector at origin, and \(T_{p_0, \partial M}^{1,0}\) is spanned by \(\frac{\partial}{\partial \bar{z}_\alpha}\) for \(1 \leq \alpha \leq n - 1\). For convenience we set

\[
t_{2k-1} = x_k, \quad t_{2k} = y_k, \quad 1 \leq k \leq n - 1; \quad t_{2n-1} = y_n, \quad t_{2n} = x_n.
\]

We also denote \(\sigma(z)\) by the distance function from \(z\) to \(\partial M\) with respect to \(\omega\). Near the origin \(p_0\),

\[
\sigma(z) = x_n + \sum_{i,j=1}^{2n} a_{ij}t_i t_j + O(|t|^3).
\]

In the computation we use derivatives with respect to Chern connection \(\nabla\) of \(\omega\), and write \(\partial_i = \frac{\partial}{\partial z_i}, \bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i}, \nabla_i = \nabla_{\partial_i}, \nabla_{\bar{i}} = \nabla_{\bar{\partial}_i}\). For a smooth function \(v\),

\[
v_i := \partial_i v, \quad v_{i} := \bar{\partial}_i v, \quad v_{ij} := \partial_i \bar{\partial}_j v, \quad v_{ij} := \nabla_i \nabla_j v = \partial_i \partial_j v - \Gamma_{ij}^k v_k, \cdots \text{etc},
\]

where \(\Gamma_{ij}^k\) are the Christoffel symbols defined by \(\nabla_{\partial_i} \frac{\partial}{\partial \bar{z}_j} = \Gamma_{ij}^k \frac{\partial}{\partial \bar{z}_k}\).

Now we derive the \(C^0\)-estimate, and boundary gradient estimates. Let \(\tilde{u}\) be the solution to

\[
\Delta \tilde{u} + \text{tr}_\omega(\chi) = 0 \text{ in } M, \quad \tilde{u} = \phi \text{ on } \partial M.
\]
The existence of \( \tilde{u} \) follows from standard theory of elliptic equations of second order. Such \( \tilde{u} \) is a supersolution of (1.1). By the maximum principle and boundary value condition, one derives

\[
\begin{align*}
u_{\alpha_n}(0) \leq u_{\alpha_n}(0) & \leq u_{\alpha_n}(0), \quad u(0) = u_{\alpha_n}(0), \quad \underline{u} \leq u \leq \bar{u} \text{ in } M. \quad (4.5)
\end{align*}
\]

This simply gives the following lemma.

**Lemma 4.3.** There is a uniform positive constant \( C \) such that

\[
\begin{align*}
\sup_M |u| + \sup_{\partial M} |\nabla u| & \leq C. \quad (4.6)
\end{align*}
\]

### 4.2 First ingredient of the proof

In the proof the Greek letters \( \alpha, \beta \) range from 1 to \( n-1 \). Let's denote

\[
A(R) = \begin{pmatrix}
g_{11} & g_{12} & \cdots & g_{1(n-1)} & g_{1n} 
g_{21} & g_{22} & \cdots & g_{2(n-1)} & g_{2n} 
\vdots & \vdots & \ddots & \vdots & \vdots 
g_{(n-1)1} & g_{(n-1)2} & \cdots & g_{(n-1)(n-1)} & g_{(n-1)n} 
g_{n1} & g_{n2} & \cdots & g_{n(n-1)} & R
\end{pmatrix}.
\]

Let \( \lambda = \lambda(g), \lambda' = \lambda(g), \lambda' = \lambda'_{\omega}(g_{\alpha \beta}), \lambda' = \lambda'_{\omega}(\underline{g}_{\alpha \beta}) \). Here as in Theorem 1.3 we denote \( \omega' = \omega|_{T_{\lambda M} \cap JT_{\lambda M}} \). We know

\[
\lambda', \lambda' \in \Gamma_{\infty}. \quad (4.7)
\]

The boundary value condition implies

\[
\begin{align*}
u_{\alpha \beta}(0) = u_{\alpha \beta}(0) + (u - u_{\alpha \beta})(0) & \sigma_{\alpha \beta}(0). \quad (4.8)
\end{align*}
\]

Let \( \eta = (u - u_{\alpha \beta})(0) \), then at \( p_0 (z = 0) \)

\[
\begin{align*}
g_{\alpha \beta} = \underline{g}_{\alpha \beta} + \eta \sigma_{\alpha \beta}. \quad (4.9)
\end{align*}
\]

We rewrite \( g_{\alpha \beta} \) as

\[
\begin{align*}
g_{\alpha \beta} = (1 - t)\underline{g}_{\alpha \beta} + \left( t\underline{g}_{\alpha \beta} + \eta \sigma_{\alpha \beta} \right). \quad (4.10)
\end{align*}
\]

For simplicity, we denote

\[
\begin{align*}
A_t = \sqrt{-1} \left[ t\underline{g}_{\alpha \beta} + \eta \sigma_{\alpha \beta} \right] d\zeta_{\alpha} \wedge d\bar{\zeta}_{\beta}. \quad (4.11)
\end{align*}
\]

Clearly, \( (A_1)_{\alpha \beta} = g_{\alpha \beta} \) so \( \lambda'_{\omega}(A_1) \in \Gamma_{\infty} \).
Lemma 4.4. Suppose there are constants $t_0 < 1$ and $R_0 > 0$ such that

$$\lambda_\omega'(A_{t_0}) \in \Gamma_\infty, \quad (4.12)$$

$$f \left( (1 - t_0)\lambda'_1, \cdots, (1 - t_0)\lambda'_{n-1}, R_0 \right) \geq f(\lambda). \quad (4.13)$$

Then there is a uniform positive constant $C$ depending on $(1 - t_0)^{-1}, |t_0|, R_0$ and other known data such that

$$g_{\hat{n}\hat{h}} \leq C \left( 1 + \sum_{\alpha=1}^{n-1} |g_{\alpha\hat{h}}|^2 \right).$$

Proof. It follows from (4.13) and the openness of $\Gamma$ that there are two uniform positive constants $R_1$ and $\varepsilon_0$ such that

$$f \left( (1 - t_0)(\lambda'_1 - \varepsilon_0/2), \cdots, (1 - t_0)(\lambda'_{n-1} - \varepsilon_0/2), R_1 \right) \geq f(\lambda), \quad (4.14)$$

$$\left( \lambda'_1 - \varepsilon_0, \cdots, \lambda'_{n-1} - \varepsilon_0, R_1/(1 - t_0) \right) \in \Gamma, \quad (4.15)$$

where $R_1$ depends on $R_0, \lambda'$, and possibly on $(1 - t_0)^{-1}$, and $\varepsilon_0$ depends on $\inf_{\partial M} \text{dist}(\lambda, \partial \Gamma), \lambda'$ and other known data.

Let

$$A(R) = \begin{pmatrix} g_{\alpha\beta} & g_{\alpha\hat{n}} \\ g_{\hat{n}\beta} & R \end{pmatrix}. \quad (4.16)$$

By (4.11) we can decompose $A(R)$ into

$$A(R) = A'(R) + A''(R)$$

where

$$A'(R) = \begin{pmatrix} (1 - t_0)(\tilde{g}_{\alpha\beta} - \frac{\varepsilon_0}{4}\delta_{\alpha\beta}) & g_{\alpha\hat{n}} \\ g_{\hat{n}\beta} & R/2 \end{pmatrix}, \quad A''(R) = \begin{pmatrix} (A_{t_0})_{\alpha\beta} + (1 - t_0)\varepsilon_0\delta_{\alpha\beta} & 0 \\ 0 & R/2 \end{pmatrix}. \quad (4.15)$$

Denote

$$\lambda_{\omega'}(A_{t_0}) := \lambda' = (\lambda'_1, \cdots, \lambda'_{n-1}). \quad (4.17)$$

By (4.5) there is a uniform constant $C_0 > 0$ depending on $|t_0|$, $\sup_{\partial M} |\nabla u|$ and other known data, such that $|\lambda'| \leq C_0$, that is $\lambda'$ is contained in a compact subset of $\Gamma_\infty$, i.e.

$$\lambda' \in K := \{ \lambda' \in \Gamma_\infty : |\lambda'| \leq C_0 \}. \quad (4.18)$$
Combining with (4.12) there is a uniform positive constant \( R_2 \) depending on \(((1 - t_0)\varepsilon_0)^{-1}, K \) and other known data, such that

\[
\lambda(A''(R)) \in \Gamma, \forall R > R_2. \tag{4.18}
\]

Let’s pick \( \varepsilon = \frac{(1 - t_0)\varepsilon_0}{4} \) in Lemma 3.6, and we set

\[
R_c = \frac{8(2n - 3)}{(1 - t_0)\varepsilon_0} \sum_{\alpha=1}^{n-1} |\varphi_{\alpha,n}|^2 + \frac{2(2n^2 - 4n + 1)(1 - t_0)\varepsilon_0}{4(2n - 3)} + 2(n - 1)|\varphi'| + 2R_1 + 2R_2
\]

where \( \varepsilon_0, R_1 \) and \( R_2 \) are fixed constants as we have chosen above.

According to Lemma 3.6, the eigenvalues \( \lambda(A'(R_c)) \) of \( A'(R_c) \) (possibly with an appropriate order) shall behave like

\[
\lambda_\alpha(A'(R_c)) \geq (1 - t_0)\left(\varphi' - \frac{\varepsilon_0}{2}\right), 1 \leq \alpha \leq n - 1,
\]

\[
\lambda_n(A'(R_c)) \geq R_c/2 - (n - 1)(1 - t_0)\varepsilon_0/4. \tag{4.19}
\]

In particular, \( \lambda(A'(R_c)) \in \Gamma. \) So \( \lambda(A(R_c)) \in \Gamma. \)

Next we will use Lemma 3.2. Precisely, together with (4.18), (3.11) gives

\[
f(\lambda(A(R_c))) \geq f(\lambda(A'(R_c))). \tag{4.20}
\]

From (4.14), (4.19) and (4.20), we deduce

\[
\varphi_{\alpha,n} \leq R_c.
\]

\[\square\]

### 4.3 Second ingredient of the proof

In order to prove Propositions 4.1 and 4.2, according to Lemma 4.4, it requires only to complete the following two steps:

- Confirm the assumptions imposed in Lemma 4.4.
- Prove that \((1 - t_0)^{-1}\) can be uniformly bounded from above, i.e.,

\[
(1 - t_0)^{-1} \leq C. \tag{4.21}
\]

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Case 1: $\partial M$ satisfies (1.12).

Note that $\eta = (u - \underline{u})_{\lambda_t}(0) \geq 0$. The assumption (1.12) yields $\lambda(\eta \sigma_{\eta \theta}) \in \overline{\Gamma}_\infty$ on $\partial M$. Since $\underline{u}$ is a subsolution, one can choose a constant $R$ sufficiently large, such that

$$f(\lambda', \underline{u}_{\lambda_t}) \geq f(\lambda).$$

Here we use (3.1). So the $t_0$ in Lemma 4.4 exists and

$$t_0 = 0.$$  

Thus (4.21) automatically holds. Consequently, we can deduce Proposition 4.1.

Case 2: $f$ satisfies the unbounded condition (1.5).

This case corresponds exactly to Proposition 4.2. We always assume $\Gamma$ is of type 1 in the sense of Caffarelli-Nirenberg-Spruck [5], then $\Gamma_\infty$ is an open symmetric convex cone in $\mathbb{R}^{n-1}$ and $\Gamma_\infty \neq \mathbb{R}^{n-1}$; otherwise, $\Gamma_\infty = \mathbb{R}^{n-1}$, then we have done as shown in Case 1.

We first confirm the assumptions of Lemma 4.4.

- Let $t_0$ be the first $t$ as we decrease $t$ from 1 such that

$$\lambda(\eta) \in \partial \overline{\Gamma}_\infty.$$  

Such $t_0$ exists, since $\lambda(1) \in \Gamma_\infty$ and $\lambda(A_t) \in \mathbb{R}^{n-1} \setminus \Gamma_\infty$ for $t \ll -1$. Furthermore, for a uniform positive constant $T_0$ under control,

$$-T_0 < t_0 < 1.$$  

- By the unbounded condition (1.5) there is a uniform positive constant $R_1$ depending on $(1 - t_0)^{-1}$, $e_0$ and $\lambda'$ such that (4.14) holds. Here is the only place where we use the unbounded condition (1.5). As a contrast, such an unbounded condition can be removed when $t_0 = 0$ as in Case 1.

To complete the proof of Proposition 4.2, it requires only to prove the following lemma.
Lemma 4.5. Let $t_0$ be as defined in (4.22), then

$$(1 - t_0)^{-1} \leq C$$

where $C$ is a uniform positive constant depending on $|u|_{C^0(M)}$, $|
abla u|_{C^0(\partial M)}$, $|u|_{C^2(M)}$, sup$_M\psi$, $(\delta_{\psi,f})^{-1}$, $\partial M$ up to third derivatives and other known data.

We assume $\eta > 0$ (otherwise we have done). In the proof we shall make use of some idea of Caffarelli-Nirenberg-Spruck [5], which was used by Li [30] to study the Dirichlet problem in $\mathbb{C}^n$. We use some notation of [5, 30]. Without loss of generality, we assume

$$t_0 > \frac{1}{2} \text{ and } \bar{\lambda}'_1 \leq \cdots \leq \bar{\lambda}'_{n-1}$$

where $\bar{\lambda}'$ is as we denoted in (4.17). Combining (4.23) with (4.5), we can deduce that there is a uniform constant $C'_0 > 0$ depending on sup$_{\partial M} |\nabla u|$ and other known data, such that $|\bar{\lambda}'| \leq C'_0$, i.e.

$$\bar{\lambda}' \in K' := \{\lambda' \in \partial \Gamma_{\infty} : |\lambda'| \leq C'_0\}.$$  \hspace{1cm} (4.24)

It was proved in [5, Lemma 6.1] that for $\bar{\lambda}' \in \partial \Gamma_{\infty}$ there is a supporting plane for $\Gamma_{\infty}$ and one can choose $\mu_j$ with $\mu_1 \geq \cdots \geq \mu_{n-1} \geq 0$ so that

$$\Gamma_{\infty} \subset \left\{ \lambda' \in \mathbb{R}^{n-1} : \sum_{a=1}^{n-1} \mu_a \lambda'_a > 0 \right\}, \quad \sum_{a=1}^{n-1} \mu_a = 1, \quad \sum_{a=1}^{n-1} \mu_a \lambda'_a = 0.$$ \hspace{1cm} (4.25)

According to [5, Lemma 6.2] (without loss of generality we assume $\bar{\lambda}'_1 \leq \cdots \leq \bar{\lambda}'_{n-1}$),

$$\sum_{a=1}^{n-1} \mu_a \bar{\sigma}_{\bar{\sigma}a} \geq \sum_{a=1}^{n-1} \mu_a \lambda'_a \geq \inf_{\rho \in \partial M} \sum_{a=1}^{n-1} \mu_a \lambda'(\rho) \geq a_0 > 0.$$ \hspace{1cm} (4.26)

Here we use (4.7), (4.24) and (4.25). We shall mention that $a_0$ depends on disc($A, \partial \Gamma$). Without loss of generality, we assume $(A_{\bar{\bar{a}}})_{\bar{\bar{a}}\bar{\sigma}} = t_0 \sum_{\bar{\sigma}a} a_{\bar{\sigma}a} + \eta \sum_{\bar{\sigma}a} \mu_a \sigma_{\bar{\sigma}a} \geq a_0$ and $\sum_{\bar{\sigma}a} \mu_a \sigma_{\bar{\sigma}a}$. From (4.25) one has at the origin

$$0 = t_0 \sum_{a=1}^{n-1} \mu_a \sigma_{\bar{\sigma}a} + \eta \sum_{a=1}^{n-1} \mu_a \sigma_{\bar{\sigma}a} \geq a_0 t_0 + \eta \sum_{a=1}^{n-1} \mu_a \sigma_{\bar{\sigma}a}.$$ \hspace{1cm} (4.27)

Together with (4.5), we see at the origin $|z = 0|$}

$$- \sum_{a=1}^{n-1} \mu_a \sigma_{\bar{\sigma}a} \geq \frac{a_0 t_0}{\sup_{\partial M} |\nabla(\bar{u} - u)|} =: a_1 > 0,$$ \hspace{1cm} (4.28)
where \( \tilde{u} \) and \( u \) are respectively supersolution and subsolution. Let

\[ \Omega_{\delta} = M \cap B_{\delta}(0), \]

where \( B_{\delta}(0) = \{ z \in M : |z| < \delta \} \). On \( \Omega_{\delta} \), we let

\[ d(z) = \sigma(z) + \tau|z|^2 \]

(4.29)

where \( \tau \) is a positive constant to be determined; and let

\[ w(z) = \tilde{u}(z) + (\eta/l_0)\sigma(z) + l(z)\sigma(z) + Ad(z)^2, \]

(4.30)

where \( l(z) = \sum_{i=1}^{n}(l_i z_i + \bar{l}_i \bar{z}_i) \), \( l_i \in \mathbb{C} \), \( \bar{l}_i = l_i \), to be chosen as in (4.33), and \( A \) is a positive constant to be determined. Furthermore, on \( \partial M \cap \Omega_{\delta} \), \( u(z) - w(z) = -A\tau^2|z|^4 \). On \( M \cap \partial B_{\delta}(0) \),

\[ u(z) - w(z) \leq |u - u|_{C^0(\Omega_{\delta})} - (2A\tau^2 + \frac{\eta}{l_0} - 2n \sup_i l_i |\delta|)\sigma(z) - A\tau^2\delta^4 \]

\[ \leq - \frac{A\tau^2\delta^4}{2} \]

provided \( A \gg 1 \).

Let \( T_1(z), \cdots, T_{n-1}(z) \) be an orthonormal basis for holomorphic tangent space of level hypersurface \( \{ w : d(w) = d(z) \} \) at \( z \), so that at the origin \( T_\alpha(0) = \frac{\partial}{\partial \alpha} \) for each \( 1 \leq \alpha \leq n - 1 \).

Such a basis exists: We see at the origin \( \partial d(0) = \partial \sigma(0) \). Thus for \( 1 \leq \alpha \leq n - 1 \),

we can choose \( T_\alpha \) such that at the origin \( T_\alpha(0) = \frac{\partial}{\partial \alpha} \).

By [5, Lemma 6.2], we have the following lemma.

**Lemma 4.6.** Let \( T_1(z), \cdots, T_{n-1}(z) \) be as above, and let \( T_n = \frac{\partial d}{\partial n} \). For a real \( (1,1) \)-form \( \Theta = \sqrt{-1} \Theta_{ij} dz_i \wedge d\bar{z}_j \) we denote by \( \lambda(\Theta) = (\lambda_1(\Theta), \cdots, \lambda_n(\Theta)) \) the eigenvalues of \( \Theta \) (with respect to \( \omega \)) with \( \lambda_1(\Theta) \leq \cdots \leq \lambda_n(\Theta) \). Then for any \( \mu_1 \geq \cdots \geq \mu_n \),

\[ \sum_{i=1}^{n} \mu_i \lambda_i(\Theta) \leq \sum_{i=1}^{n} \mu_i \Theta(T_i, J\bar{T}_i). \]

Let \( \mu_1, \cdots, \mu_{n-1} \) be as in (4.25), and set \( \mu_n = 0 \). Let’s denote \( T_\alpha = \sum_{k=1}^{n} T_k^{\alpha} \frac{\partial}{\partial \alpha} \). For \( \Theta = \sqrt{-1} \Theta_{ij} dz_i \wedge d\bar{z}_j \), we define

\[ \Lambda_{\mu}(\Theta) := \sum_{\alpha=1}^{n-1} \mu_\alpha T_\alpha^{ij} \bar{T}_\alpha^{ij} \Theta_{ij}. \]
Lemma 4.7. There are parameters $\tau, A, l, \delta$ depending only on $|u|_{C^4(M)}$, $|\nabla u|_{C^3(M)}$, $|\partial u|_{C^2(M)}$, $\partial M$ up to third derivatives and other known data, such that

$$\Lambda_{\mu}(g[w]) \leq 0 \text{ in } \Omega_{\delta}, \quad u \leq w \text{ on } \partial \Omega_{\delta}.$$  

Proof. By direct computation

$$\Lambda_{\mu}(g[w]) = \sum_{a=1}^{n-1} \mu_a T_a^i \bar{T}_a^j (\chi_{ij} + u_j + \frac{\eta}{t_0} \sigma_{ij}) + 2Ad(z) \sum_{a=1}^{n-1} \mu_a T_a^i \bar{T}_a^j d_{ij}$$

$$+ \sum_{a=1}^{n-1} \mu_a T_a^i \bar{T}_a^j (l(z) \sigma_{ij} + l_i \sigma_j + \sigma_j l_i).$$

- At the origin $|z| = 0$, $T_a^i = \delta_{ai}$,

$$\sum_{a=1}^{n-1} \mu_a T_a^i \bar{T}_a^j (\chi_{ij} + u_j + \frac{\eta}{t_0} \sigma_{ij})(0) = \frac{1}{t_0} \sum_{a=1}^{n-1} \mu_a (A_a)_{a\bar{a}} = 0.$$

So there are complex constants $k_i$ such that

$$\sum_{a=1}^{n-1} \mu_a T_a^i \bar{T}_a^j (\chi_{ij} + u_j + \frac{\eta}{t_0} \sigma_{ij})(z) = \sum_{i=1}^{n} (k_i \bar{z}_i + \bar{k}_i z_i) + O(|z|^3).$$

- $$2Ad(z) \sum_{a=1}^{n-1} \mu_a T_a^i \bar{T}_a^j d_{ij} \leq -\frac{a_1 A}{2} d(z).$$

since

$$\sum_{a=1}^{n-1} \mu_a T_a^i \bar{T}_a^j d_{ij} = \sum_{a=1}^{n-1} \mu_a \sigma_{a\bar{a}}(z) + \tau \sum_{a=1}^{n-1} \mu_a \alpha$$

$$+ \sum_{a=1}^{n-1} \mu_a \left( T_a^i \bar{T}_a^j (z) - T_a^i \bar{T}_a^j (0) \right) d_{ij}$$

$$= -a_1 + \tau + O(|z|) \leq -\frac{a_1}{4}$$

provided one chooses $0 < \delta, \tau \ll 1$. Here we also use (4.28),

$$\sum_{a=1}^{n-1} \mu_a T_a^i \bar{T}_a^j (z) = \sum_{a=1}^{n-1} \mu_a T_a^i \bar{T}_a^j (0) + O(|z|) = \sum_{a=1}^{n-1} \mu_a \alpha_{a\bar{a}} \delta_{a\bar{a}} + O(|z|), \quad (4.31)$$

and

$$\sum_{a=1}^{n-1} \mu_a \sigma_{a\bar{a}}(z) = \sum_{a=1}^{n-1} \mu_a \sigma_{a\bar{a}}(0) + O(|z|). \quad (4.32)$$
\begin{align*}
l(z) & \sum_{a=1}^{n-1} \mu_a T^i_a T^j_a \sigma_{ij} + \sum_{a=1}^{n-1} \mu_a T^i_d T^j_d (l_i \sigma_j + \sigma_i l_j) \\
& = l(z) \sum_{a=1}^{n-1} \mu_a \sigma_{a \alpha}(0) - \tau \sum_{a=1}^{n-1} \mu_a (z_a l_\alpha + \bar{z}_a \bar{l}_\alpha) + O(|z|^2)
\end{align*}

since by (4.31), \( \sum_{i=1}^n T^i_a \sigma_i = -\tau \sum_{i=1}^n T^i_a \bar{z}_i \) we have

\begin{align*}
l(z) & \sum_{a=1}^{n-1} \mu_a T^i_a T^j_a \sigma_{ij} = l(z) \sum_{a=1}^{n-1} \mu_a \sigma_{a \alpha}(0) + O(|z|^2), \\
& = \sum_{a=1}^{n-1} \mu_a T^i_a T^j_a (l_i \sigma_j + \sigma_i l_j) = -\tau \sum_{a=1}^{n-1} \mu_a (\bar{l}_a \bar{z}_\alpha + l_\alpha z_\alpha) + O(|z|^2).
\end{align*}

Putting these together,

\begin{align*}
\Lambda_\alpha(\beta[w]) & \leq \sum_{a=1}^{n-1} 2\Re \left\{ z_\alpha (k_\alpha - \tau \mu_\alpha l_\alpha + l_\alpha \sum_{\beta=1}^{n-1} \mu_\beta \sigma_\beta(0)) \right\} \\
& + 2\Re \left\{ z_n (k_n + l_n \sum_{\beta=1}^{n-1} \mu_\beta \sigma_\beta(0)) \right\} - \frac{A a_1}{2} d(z) + O(|z|^2).
\end{align*}

Let \( l_\alpha = -\frac{k_\alpha}{\sum_{\beta=1}^{n-1} \mu_\beta \sigma_\beta(0) - \tau \mu_\alpha} \). For \( 1 \leq \alpha \leq n - 1 \), we set

\[ l_\alpha = -\frac{k_\alpha}{\sum_{\beta=1}^{n-1} \mu_\beta \sigma_\beta(0) - \tau \mu_\alpha}. \quad (4.33) \]

From \( \mu_\alpha \geq 0 \) and (4.28), we see such \( l_i \) (or equivalently the \( l(z) \)) are all well defined and uniformly bounded.

We thus complete the proof if \( 0 < \tau, \delta \ll 1, A \gg 1 \). \( \square \)

**Completion of the proof of Lemma 4.5**

Let \( w \) be as in Lemma 4.7. From the construction above, we know that there is a uniform positive constant \( C_1 \) such that

\[ |\beta[w]|_{C_0(\Omega_\nu)} \leq C_1. \]

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Let $\lambda[w] = \lambda_\omega(g[w])$. Assume $\lambda_1[w] \leq \cdots \leq \lambda_n[w]$. Lemma 4.7, together with Lemma 4.6, implies

$$\sum_{a=1}^{n-1} \mu_a \lambda_a[w] \leq 0 \text{ in } \Omega_\delta.$$  

So $(\lambda_1[w], \cdots, \lambda_{n-1}[w]) \notin \Gamma_\infty$ by (4.25). In other words, $\lambda[w] \in X$, where

$$X = \{\lambda \in \mathbb{R}^n : \lambda' \in \mathbb{R}^{n-1} \setminus \Gamma_\infty \} \cap \{\lambda \in \mathbb{R}^n : |\lambda| \leq C_1\}.$$  

Let

$$\Gamma_{\inf} = \{\lambda \in \Gamma : f(\lambda) \geq \inf_{\tilde{M}} \psi\}$$  

be the closure of sublevel set $\Gamma_{\inf}$. Notice that $\Gamma_\infty$ is open so $X$ is a compact subset; furthermore $X \cap \Gamma_{\inf} = \emptyset$. So we can deduce that the distance between $\Gamma_{\inf}$ and $X$ is greater than some positive constant depending on $\delta, f$ and other known data. Therefore, there exists an $\epsilon > 0$ such that for any $z \in \Omega_\delta$

$$\epsilon \overline{1} + \lambda[w] \notin \Gamma_{\inf}.$$  

By (4.3) one can choose a positive constant $C'$ such that $x_n \leq C'|z|^2$ on $\partial M \cap \overline{\Omega}_\delta$. As a result, there is a positive constant $C_2$ depending only on $M$ and $\delta$ so that

$$x_n \leq C_2|z|^2 \text{ on } \partial \Omega_\delta.$$  

Let $\epsilon$ and $C_2$ be as above, we define $h(z) = w(z) + \epsilon(|z|^2 - \frac{\lambda}{C_2^2})$. Thus

$$u \leq h \text{ on } \partial \Omega_\delta.$$  

Moreover, $\chi_{ij} + h_{ij} = (\chi_{ij} + w_{ij}) + \epsilon \delta_{ij}$ so $\lambda[h] \notin \Gamma_{\inf}$. By [5, Lemma B], we have

$$u \leq h \text{ in } \Omega_\delta.$$  

Notice $u(0) = \varphi(0)$ and $h(0) = \varphi(0)$, we have $u_{x_n}(0) \leq h_{x_n}(0)$, and

$$t_0 \leq \frac{1}{1 + \epsilon/(\eta C_2)} \, \text{ i.e., } (1 - t_0)^{-1} \leq 1 + \frac{\eta C_2}{\epsilon}.$$  

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5 Quantitative boundary estimate: tangential-normal case

In this section we derive quantitative boundary estimate for tangential-normal derivatives.

**Proposition 5.1.** Let \((M, J, \omega)\) be a compact Hermitian manifold with \(C^3\)-smooth boundary. In addition we assume (1.2)-(1.4), (1.7) and (1.9) hold. Then for any admissible solution \(u \in C^3(M) \cap C^2(\overline{M})\) to the Dirichlet problem (1.1), there is a uniform positive constant \(C\) depending on \(|\phi|_{C^3(M)}\), \(|u|_{C^2(\overline{M})}\), \(|\psi|_{C^1(M)}\), \(|\nabla u|_{C^0(\partial M)}\), \(\partial_M u\) up to third derivatives and other known data (but neither on \((\delta_{\phi,f})^{-1}\) nor on \(\sup_M |\nabla u|\)) such that

\[
|\nabla^2 u(T, \nu)| \leq C \left(1 + \sup_M |\nabla u| \right)
\]  

(5.1)

for any \(T \in T_{\partial M}\) with \(|T| = 1\), where \(\nabla^2 u\) denotes the real Hessian of \(u\).

5.1 Tangential operators on the boundary

For a given point \(p_0 \in \partial M\), we choose local holomorphic coordinates (4.1) centered at \(p_0\) in a neighborhood which we assume to be contained in \(M_\delta := \{z \in M : \sigma(z) < \delta\}\), such that \(p_0 = \{z = 0\}\), \(g_{ij}(0) = \delta_{ij}\) and \(\frac{\partial}{\partial x_n}\) is the interior normal direction to \(\partial M\) at \(p_0\). As in (4.2) we set

\[
t_{2k-1} = x_k, \ t_{2k} = y_k, \ 1 \leq k \leq n - 1; \ t_{2n-1} = y_n, \ t_{2n} = x_n.
\]

We define the tangential operator on \(\partial M\)

\[
\mathcal{T} = \nabla_{\frac{\partial}{\partial x_\alpha}} - \eta \nabla_{\frac{\partial}{\partial x_n}}, \text{ for each fixed } 1 \leq \alpha < 2n,
\]  

(5.2)

where \(\eta = \frac{\sigma_{x_\alpha}}{\sigma_{x_n}}\), \(\sigma_{x_\alpha}(0) = 1\), \(\sigma_{x_n}(0) = 0\). One has \(\mathcal{T}(u - \phi) = 0\) on \(\partial M \cap \Omega_\delta\). The boundary value condition also gives for each \(1 \leq \alpha, \beta < n\),

\[
(u - \phi)_{t_\alpha t_\beta}(0) = (u - \phi)_{x_\alpha}(0)\sigma_{t_\alpha t_\beta}(0) \forall 1 \leq i, j < 2n.
\]  

(5.3)

Let’s turn our attention to the setting of complex manifolds with holomorphically flat boundary in the sense that, for any \(p_0 \in \partial M\), one can pick local holomorphic coordinates

\[
(z_1, \cdots, z_n), \ z_i = x_i + \sqrt{-1}y_i,
\]  

(5.4)
centered at $p_0$ such that $\partial M$ is locally of the form

$$\Re(z_n) = 0.$$ 

Furthermore we may assume $g_{ij}(p_0) = \delta_{ij}$. Under the holomorphic coordinate (5.4), we can take

$$T = D := \frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}, \quad 1 \leq \alpha \leq n - 1. \quad (5.5)$$

It would be worthwhile to note that such local holomorphic coordinate system (5.4) is only needed in the proof of Proposition 5.6. In addition, when $M = X \times S$, $D = \frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}$, where $z' = (z_1, \ldots, z_{n-1})$ is local holomorphic coordinate of $X$.

For simplicity we denote the tangential operator on $\partial M$ by

$$T = \nabla_{\frac{\partial}{\partial x_\alpha}} - \gamma \tilde{\eta} \nabla_{\frac{\partial}{\partial x_\alpha}} \quad (5.6)$$

where

$$\gamma = \begin{cases} 0 & \text{if } \partial M \text{ is holomorphically flat;} \\ 1 & \text{otherwise.} \end{cases}$$

By (4.3) one derives $|\tilde{\eta}| \leq C'|z|$ on $\Omega_\delta$. Since $(u - \varphi)|_{\partial M} = 0$ we obtain $T(u - \varphi)|_{\partial M} = 0$. Together with (4.5), one has

$$|(u - \varphi)_{\alpha}| \leq C|z| \text{ on } \partial M \cap \tilde{\Omega}_\delta, \forall 1 \leq \alpha < 2n. \quad (5.7)$$

### 5.2 Completion of proof of Proposition 5.1

Let’s denote

$$F^{ij}(A) = \frac{\partial F(A)}{\partial a_{ij}}, \quad A = (a_{ij}).$$

The linearized operator of equation (1.1) at $u$ is given by

$$\mathcal{L}v = F^{ij}(g[u])v_{ij}.$$ 

The following lemma plays an important role in the proof. The lemma of this type goes back at least to [17, Theorems 2.16, 2.17].

**Lemma 5.2** ([19, Lemma 2.2]). Suppose (1.2) and (1.3) hold. Let $K$ be a compact subset of $\Gamma$ and $\beta > 0$. There is a constant $\varepsilon > 0$ such that, for $\mu \in K$ and $\lambda \in \Gamma$, when $|\nu_\mu - \nu_\lambda| \geq \beta$,

$$\sum_{i=1}^{n} f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) + \varepsilon \left(1 + \sum_{i=1}^{n} f_i(\lambda)\right). \quad (5.8)$$

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Here $\nu_{x_{\lambda}} = D f(\lambda)/|D f(\lambda)|$ denotes the unit normal vector to the level set $\partial \Gamma f(\lambda)$, where $D f(\lambda) = (f_1(\lambda), \cdots, f_n(\lambda))$.

**Remark 5.3.** From the original proof of [19, Lemma 2.2], we know that the constant $\varepsilon$ in (5.8) depends only on $\mu, \beta$ and other known data.

We derive quantitative boundary estimates for tangential-normal derivatives by using barrier functions. This type of construction of barrier functions goes back at least to [20, 16]. We shall point out that, during the proof, the constants $C, C_0, C_1, C_1', C_2, A_1, A_2, A_3$, etc, depend on neither $|\nabla u|$ nor $(\delta_{\psi,f})^{-1}$. The constant $\gamma$ always stands for that from (5.6).

By direct calculations, one derives

$$u_{x_i} = u_{x_i} + \Gamma_{k_{\lambda}}^p u_p, \quad u_{y_i} = u_{y_i} + \sqrt{1} \Gamma_{k_{\lambda}}^p u_p,$$

$$(u_{x_i})_j = u_{x_i} + \Gamma_{k_{\lambda}}^j u_j, \quad (u_{y_i})_j = u_{y_i} - \sqrt{1} \Gamma_{k_{\lambda}}^j u_j,$$

$$(u_{x_i})_{ij} = u_{x_{ij}} + \Gamma_{k_{\lambda}}^i u_{ij} + \Gamma_{l_{\lambda}}^j u_{il} - g_{n_{l_{\lambda}}} R_{ijhm} u_{mi},$$

$$(u_{y_i})_{ij} = u_{y_{ij}} + \sqrt{1} \left( \Gamma_{k_{\lambda}}^i u_{ij} - \Gamma_{k_{\lambda}}^j u_{ii} \right) - \sqrt{1} g_{n_{l_{\lambda}}} R_{ijhm} u_{mi},$$

$$F_{ij} u_{x_{ij}} = F_{ij} u_{x_{ij}} + g_{n_{l_{\lambda}}} F_{ij} R_{ijhm} u_{mi} - 2 \Re \left\{ F_{ij} T_{ij}^p u_{p_{ij}} \right\},$$

$$F_{ij} u_{y_{ij}} = F_{ij} u_{y_{ij}} + \sqrt{1} g_{n_{l_{\lambda}}} F_{ij} R_{ijhm} u_{mi} + 2 \Im \left\{ F_{ij} T_{ij}^p u_{p_{ij}} \right\},$$

where

$$T_{ij}^k = g_{kl} \left( \frac{\partial g_{ji}}{\partial z_i} - \frac{\partial g_{ij}}{\partial z_j} \right), \quad R_{ijkl} = - \frac{\partial^2 g_{kl}}{\partial z_i \partial \bar{z}_j} + g_{pq} \frac{\partial g_{ka}}{\partial z_i} \frac{\partial g_{lp}}{\partial \bar{z}_j}.$$

As a consequence,

$$\mathcal{L}(\pm u_{x_{\lambda}}) \geq \pm \psi_{x_{\lambda}} - C \left( 1 + |\nabla u| \right) \sum_{i=1}^n f_i - C \sum_{i=1}^n f_i |\lambda_i|. \quad (5.9)$$

Denote

$$b_1 = 1 + \sup_M |\nabla u|^2.$$
Lemma 5.4. Given \( p_0 \in \partial M \), let \( u \) be a \( C^3 \) admissible solution to equation (1.1), and \( \Phi \) is defined as
\[
\Phi = \pm T (u - \varphi) + \frac{\gamma}{\sqrt{b_1}} (u_{\gamma_1} - \varphi_{\gamma_1})^2 \quad \text{in } \Omega_\delta.
\] (5.10)

Then there is a positive constant \( C_\Phi \) depending on \( |\varphi|_{C^1(\bar{M})}, |\chi|_{C^1(\bar{M})}, |\nabla \psi|_{C^1(\bar{M})} \) and other known data such that for some small positive constant \( \delta \),
\[
\mathcal{L} \Phi \geq -C_\Phi \sqrt{b_1} \sum_{i=1}^{n} f_i - C_\Phi \sum_{i=1}^{n} f_i |\lambda_i| - C_\Phi \text{ on } \Omega_\delta.
\]

In particular, if \( \partial M \) is holomorphically flat and \( \varphi \equiv \text{constant} \) then \( C_\Phi \) depends on \( |\chi|_{C^1(\bar{M})}, |\nabla \psi|_{C^1(\bar{M})} \) and other known data.

Proof. Together with (5.9) and Cauchy-Schwarz inequality, one obtains
\[
\mathcal{L} (\pm T u) \geq -C \sqrt{b_1} \sum_{i=1}^{n} f_i - C \sum_{i=1}^{n} f_i |\lambda_i| - \frac{\gamma}{\sqrt{b_1}} F_{ij} u_{\gamma_i} u_{\gamma_j} - C|\nabla \psi|,
\]

\[
F_{ij}(\tilde{\eta}_i)(u_{\gamma_j}) \leq C \sum_{i=1}^{n} f_i |\lambda_i| + \frac{1}{\sqrt{b_1}} F_{ij} u_{\gamma_i} u_{\gamma_j} + C \sqrt{b_1} \sum_{i=1}^{n} f_i,
\]

\[
\mathcal{L}((u_{\gamma_1} - \varphi_{\gamma_1})^2) \geq F_{ij} u_{\gamma_i} u_{\gamma_j} - C \left(1 + |\nabla u|^2\right) \sum_{i=1}^{n} f_i - C|\nabla u| \sum_{i=1}^{n} f_i |\lambda_i| - C \left(1 + |\nabla u|\right).
\]

Putting these inequalities together, we complete the proof. \( \square \)

To estimate the quantitative boundary estimates for mixed derivatives, we should employ barrier function of the form
\[
v = (u - u) - t\sigma + N\sigma^2 \quad \text{in } \Omega_\delta,
\] (5.11)

where \( t, N \) are positive constants to be determined.

Let \( \delta > 0 \) and \( t > 0 \) be sufficiently small with \( N\delta - t \leq 0 \), so that in \( \Omega_\delta, \)
\[
v \leq 0, \ \sigma \text{ is } C^2,
\] (5.12)

\[
\frac{1}{4} \leq |\nabla \sigma| \leq 2, \ \left| \mathcal{L} \sigma \right| \leq C_2 \sum_{i=1}^{n} f_i.
\] (5.13)
In addition, we can choose $\delta$ and $t$ small enough such that
\[
|2N\delta - t| \leq \min \left\{ \frac{\epsilon}{2C_2}, \frac{\beta}{16\sqrt{nC_2}} \right\},
\] (5.14)
where $\beta := \frac{1}{2} \min_{\Omega_n} \text{dist}(\nu_\omega, \partial \Gamma_n)$, $\epsilon$ is the constant corresponding to $\beta$ in Lemma 5.2, and $C_2$ is the constant in (5.13).

We construct in $\Omega_\delta$ the barrier function as follows:
\[
\tilde{\Psi} = A_1 \sqrt{b_1} v - A_2 \sqrt{b_1} |z|^2 + \frac{1}{\sqrt{b_1}} \sum_{\tau<\nu} |\tilde{u}_\tau|^2 + A_3 \Phi,
\] (5.15)
where and hereafter we denote $\tilde{u} = u - \varphi$.

Similar to [17, Proposition 2.19] one has the following lemma.

**Lemma 5.5.** There is an index $r$ so that
\[
\sum_{i \neq r} F_{i\tau}^j g_{i\tau} g_{r\tau} \geq \frac{1}{2} \sum_{i \neq r} f_i A_i^2.
\]

**Proof.** Let $U = (a_{ij})$ be an $n \times n$ unitary matrix that simultaneously diagonalizes $(F_{ij})$ and $(g_{ij})$ at a fixed point. That is
\[
(F_{ij}) = U^* \text{diag}(f_1, \ldots, f_n) U, \quad (g_{ij}) = U^* \text{diag}(\lambda_1, \ldots, \lambda_n) U.
\]
Here $U^* = (b_{ij})$, $b_{ij} = \overline{a_{ji}}$. Since $U$ is unitary, $U^* = U^{-1}$. Thus $(g_{ij}) \cdot (F_{ij}) \cdot (g_{ij}) = U^* \text{diag}(f_1 \lambda_1^2, \ldots, f_n \lambda_n^2) U$. For any fixed $1 \leq \tau \leq n$, we have
\[
\sum_{i,j=1}^{n} F_{ij}^\tau g_{i\tau} g_{r\tau} = \sum_{i=1}^{n} f_i A_i^2 |a_{ir}|^2.
\]
Consequently,
\[
\sum_{\tau=1}^{n-1} \sum_{i,j=1}^{n} F_{ij}^\tau g_{i\tau} g_{r\tau} = \sum_{i=1}^{n} f_i A_i^2 \left( 1 - |a_{ir}|^2 \right) \geq \frac{1}{2} \sum_{|a_{ir}|^2 \leq \frac{1}{2}} f_i A_i^2
\]
as required. \qed

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Proof of Proposition 5.1. If $A_2 \gg A_3 \gg 1$ then one has $\Psi \leq 0$ on $\partial \Omega_\delta$, here we use (5.7). Note $\Psi(p_0) = 0$. It suffices to prove

$$\mathcal{L}\Psi \geq 0 \text{ on } \Omega_\delta,$$

which yields $\Psi \leq 0$ in $\Omega_\delta$, and then $(\nabla, \Psi)(p_0) \leq 0$.

By a direct computation one has

$$\mathcal{L}v \geq F^{ij}(g_{ij} - g_{ij}) - C_2|2N\sigma - t| \sum_{i=1}^{n} f_i + 2NF^{ij}\sigma_i\sigma_j.$$

Applying [5, Lemma 6.2], with a proper permutation of $\lambda$ if necessary, one obtains

$$F^{ij}g_{ij} = F^{ij}(g)g_{ij} \geq \sum_{i=1}^{n} f_i(\lambda_i)\lambda_i = \sum_{i=1}^{n} f_i\lambda_i.$$

Since $F^{ij}g_{ij} = \sum_{i=1}^{n} f_i\lambda_i$, we have

$$F^{ij}(g_{ij} - g_{ij}) \geq \sum_{i=1}^{n} f_i(\lambda_i - \lambda_i).$$

Together with Lemma 5.5, some straightforward computations yield

$$\mathcal{L}\left(\sum_{t \neq n} |\bar{u}_t|^2\right) \geq \frac{1}{2} \sum_{t \neq n} F^{ij}g_{ij\sigma_{ij}} - C_1 \sqrt{\mathcal{L}} \sum_{i=1}^{n} f_i|\lambda_i| - C_1 b_1 \sum_{i=1}^{n} f_i - C_1' \sqrt{b_1}
\geq \frac{1}{4} \sum_{i \neq r} f_i\lambda_i^2 - C_1 \sqrt{\mathcal{L}} \sum_{i=1}^{n} f_i|\lambda_i| - C_1' b_1 \sum_{i=1}^{n} f_i - C_1' \sqrt{b_1}.$$

According to Lemma 3.2 and $\sum_{i=1}^{n} f_i(\lambda_i - \lambda_i) \geq 0$, we obtain the following inequalities respectively

$$\sum_{i=1}^{n} f_i|\lambda_i| = 2 \sum_{\lambda_i \geq 0} f_i\lambda_i - \sum_{\lambda_i < 0} f_i\lambda_i < 2 \sum_{\lambda_i \geq 0} f_i\lambda_i,$$

$$\sum_{i=1}^{n} f_i|\lambda_i| = \sum_{i=1}^{n} f_i\lambda_i - 2 \sum_{\lambda_i < 0} f_i\lambda_i \leq \sum_{i=1}^{n} f_i\lambda_i - 2 \sum_{\lambda_i < 0} f_i\lambda_i.$$

In conclusion, combining with Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{n} f_i|\lambda_i| \leq \frac{\epsilon}{4 \sqrt{b_1}} \sum_{i \neq r} f_i\lambda_i^2 + \left(\sup_{M} |\lambda| + \frac{4\sqrt{b_1}}{\epsilon}\right) \sum_{i=1}^{n} f_i.$$
Taking \( \epsilon = \frac{1}{C_1 + A_3 C \Phi} \), and putting the above inequalities together we have

\[
\mathcal{L} \tilde{\Psi} \geq A_1 \sqrt{b_1} \sum_{i=1}^{n} f_i (\lambda_i - \lambda_i) + 2A_1 N \sqrt{b_1} F^{ij} \sigma_i \sigma_j \\
- \{ C'_1 + A_2 + A_3 C \Phi + A_1 C_2 |2N\sigma - t| + 4(C'_1 + A_3 C \Phi)^2 \\
+ (C'_1 + A_3 C \Phi) \sup_{\bar{M}} |\lambda| / \sqrt{b_1} \} \sqrt{b_1} \sum_{i=1}^{n} f_i - (C'_1 + A_3 C \Phi) \}
\]

(5.16)

Let’s take \( \beta = \frac{1}{2} \min_{\bar{M}} \text{dist}(\nu_{\lambda}, \partial \Gamma_n) \) as above, and let \( \epsilon \) be the positive constant in Lemma 5.2 accordingly.

**Case I**: If \(|\nu_{\lambda} - \nu_{\bar{\lambda}}| \geq \beta\), then by Lemma 5.2 we have

\[
\sum_{i=1}^{n} f_i (\lambda_i - \lambda_i) \geq \epsilon \left( 1 + \sum_{i=1}^{n} f_i \right).
\]

(5.17)

Note that (5.14) implies \( A_1 C_2 |2N\sigma - t| \leq \frac{1}{2} A_1 \epsilon \). Taking \( A_1 \gg 1 \) we derive

\[
\mathcal{L} \tilde{\Psi} \geq 0 \text{ on } \Omega_\delta.
\]

**Case II**: Suppose that \(|\nu_{\lambda} - \nu_{\bar{\lambda}}| < \beta\) and therefore \( \nu_{\lambda} - \beta \bar{\lambda} \in \Gamma_n \) and

\[
f_i \geq \frac{\beta}{\sqrt{n}} \sum_{j=1}^{n} f_j.
\]

(5.18)

By (5.13), we have \(|\nabla \sigma| \geq \frac{1}{4} \) in \( \Omega_\delta \), then

\[
A_1 N \sqrt{b_1} F^{ij} \sigma_i \sigma_j \geq \frac{A_1 N \beta \sqrt{b_1}}{16 \sqrt{n}} \sum_{i=1}^{n} f_i \text{ on } \Omega_\delta.
\]

(5.19)

This term can control all the bad terms containing \( \sum_{i=1}^{n} f_i \) in (5.16). On the other hand, \( \mathcal{L}(u - u) \geq 0 \) and the bad term \(-(C'_1 + A_3 C \Phi)\) in the last term of (5.16) can be dominated by combining (5.19) with (3.12). Thus

\[
\mathcal{L}(\tilde{\Psi}) \geq 0 \text{ on } \Omega_\delta, \text{ if } A_1 N \gg 1.
\]

□
In the case when $\partial M$ is holomorphically flat and $\varphi$ is a constant, $\Phi = \pm Du$ in (5.10) and the local barrier function in (5.15) reads as follows

$$\tilde{\Psi} = A_1 \sqrt{b_1} v - A_2 \sqrt{b_1} |z|^2 + \frac{1}{\sqrt{b_1}} \sum_{\tau < n} |u_\tau|^2 \pm A_3 Du,$$

where $D$ is given in (5.5). So we have slightly delicate results.

**Proposition 5.6.** Suppose, in addition to (1.2)-(1.4), (1.7), (1.9) and $\psi \in C^1(\bar{M})$, that $\partial M$ is holomorphically flat and the boundary data $\varphi$ is of a constant. Then for any $T \in T_{\partial M} \cap JT_{\partial M}$ with $|T| = 1$, the admissible solution $u \in C^3(M) \cap C^2(\bar{M})$ to the Dirichlet problem satisfies

$$|\nabla^2 u(T, \nu)| \leq C \left( 1 + \sup_M |\nabla u| \right),$$

where $C$ depends on $|\psi|_{C^1(\bar{M})}$, $|\nabla u|_{C^2(\partial M)}$, $\partial M$ up to second derivatives and other known data (but neither on $(\delta_{\varphi, f})^{-1}$ nor on $\sup_M |\nabla u|$).

Together with Proposition 4.1, we obtain the following theorem.

**Theorem 5.7.** Suppose, in addition to (1.2)-(1.4), (1.7), (1.9) and $\psi \in C^1(\bar{M})$, that $\partial M$ is holomorphically flat and the boundary data $\varphi$ is of a constant. Then for any admissible solution $u \in C^3(M) \cap C^2(\bar{M})$ to Dirichlet problem (1.1), we have

$$\sup_{\partial M} \Delta u \leq C \left( 1 + \sup_M |\nabla u|^2 \right).$$

Here $C$ is a uniform positive constant depending only on $|\psi|_{C^1(\bar{M})}$, $|\nabla u|_{C^2(\partial M)}$, $\partial M$ up to second derivatives and other known data (but neither on $\sup_M |\nabla u|$ nor on $(\delta_{\varphi, f})^{-1}$).

## 6 Solving equations

### 6.1 Completion of the proof of Theorems 1.2 and 1.3

The following second order estimate is essentially due to Székelyhidi [37].

**Theorem 6.1.** Suppose, in addition to (1.2), (1.3), (2.7) and (1.9), that there is an admissible subsolution $u \in C^2(\bar{M})$. Then for any admissible solution $u \in C^4(M) \cap C^2(\bar{M})$ of Dirichlet problem (1.1) with $\psi \in C^2(M) \cap C^{1,1}(M)$, there exists a uniform positive constant $C$ depending only on $|u|_{C^2(\bar{M})}$, $|\psi|_{C^1(\bar{M})}$, $|\nabla u|_{C^2(\partial M)}$, $|\chi|_{C^2(\bar{M})}$ and other known data such that

$$\sup_M |\bar{\partial} u| \leq C \left( 1 + \sup_M |\nabla u|^2 + \sup_{\partial M} |\bar{\partial} u| \right).$$

(6.1)
Remark 6.2. Following the outline of proof of [37, Proposition 13], using Lemma 5.2 in place of [37, Proposition 6], we can check that Székelyhidi’s second order estimate still holds for the Dirichlet problem without assuming (2.7). Moreover, one can further verify that the constant $C$ in (6.1) does not depend on $(\delta_{\psi,f})^{-1}$.

The existence results follow from the standard continuity method and the above estimates. We assume $\psi, u \in C^\infty(\bar{M})$. The general case of $u \in C^3(\bar{M})$ and $\psi \in C^{k,\alpha}(\bar{M})$ follows by approximation process.

Let’s consider a family of Dirichlet problems as follows:

$$F(g[u']) = (1-t)F(g[u]) + t\psi \text{ in } M, \quad u' = \varphi \text{ on } \partial M. \tag{6.2}$$

We set

$$I = \{t \in [0,1] : \text{there exists } u' \in C^{4,\alpha}(\bar{M}) \text{ solving equation (6.2)} \}.$$ 

Clearly $0 \in I$ by taking $u^0 = u$. The openness of $I$ is follows from the implicit function theorem and the estimates.

We can verify that $u$ is the admissible subsolution along the whole method of continuity. Combining Theorem 1.4 with Theorem 6.1 and Lemma 4.3, we can conclude that there exists a uniform positive constant $C$ such that

$$\sup_M |\Delta u'| \leq C \left(1 + \sup_M |\nabla u'|^2\right).$$

One thus applies the blow-up argument used in [37], extending that of [7, 10], to derive the gradient estimate, and so $|\partial \bar{\partial} u'|$ has a uniform bound. Applying Evans-Krylov theorem [13, 29], adapted to the complex setting (cf. [41]), and Schauder theory, one obtains the required higher $C^{k,\alpha}$ regularities. Consequently, $I$ is closed.

Therefore, $I = [0,1]$. The proof is complete.

6.2 Completion of the proof of Theorem 1.6

Notice that in Theorem 1.6 we impose condition (1.11) rather than (1.4). Therefore, in order to apply Theorems 1.3 and 1.4, it requires to prove

Lemma 6.3. Suppose (1.2), (1.3) and (1.11) hold. Then $f$ satisfies (1.4).
Proof. The proof is based on Lemmas 3.2 and 6.4 below. By (1.11) we have
\[ \lim_{t \to 0^+} f(t \vec{1}) = f(\vec{0}) > -\infty. \]
Then we have (1.4) according to Lemma 6.4.

The following statement is standard and has been used in literature.

**Lemma 6.4.** Suppose $f$ satisfies (1.2) and (1.3). Then for any $\sigma$ with $\partial \Gamma^\sigma \neq \emptyset$, there exists $c_\sigma \in \partial \Gamma^\sigma$, $c_\sigma > 0$.

**Proof.** As in (2.1), we denote $\partial \Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) = \sigma \}$. The level set $\partial \Gamma^\sigma$ is a smooth convex noncompact hypersurface contained in $\Gamma$. Moreover, $\partial \Gamma^\sigma$ is symmetric with respect to diagonal $\{ t \vec{1} : t \in \mathbb{R} \}$.

Let $\lambda^0 \in \partial \Gamma^\sigma$ be the closest point to the origin. (Such a point exists, since $\partial \Gamma^\sigma$ is a closed set). The idea is to prove $\lambda^0$ is the one for what we seek.

Assume $\lambda^0$ is not contained in the diagonal. Then by the implicit function theorem, and the convexity and symmetry of $\partial \Gamma^\sigma$, one can choose $\lambda \in \partial \Gamma^\sigma$ so that the distance to origin is smaller than that of $\lambda^0$. It is a contradiction.

Theorem 1.6 follows as a conclusion of Theorems 1.3 and 1.4, Lemma 6.3 and the method of approximation.

### 7 Dirichlet problem on products

#### 7.1 Construction of subsolutions

According to the main theorems, the existence of subsolutions is a key ingredient to solve the Dirichlet problem.

Let $(M, J, \omega) = (X \times S, J, \omega)$ be a product of a closed complex manifold $X$ with a compact Riemann surface $S$ with sufficiently smooth boundary, and let $\nu$ be as above the unit inner normal vector along the boundary. We construct strictly admissible subsolutions with $\frac{\partial u}{\partial \nu} |_{\partial M} < 0$ on such a product. It is noteworthy that $J$ is the standard induced complex structure, and $\omega$ is not necessary the product metric $\omega = \pi_1^* \omega_X + \pi_2^* \omega_S$. 

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We begin with the solution $h$ to

$$\Delta_S h = 1 \text{ in } S, \quad h = 0 \text{ on } \partial S.$$  \hspace{1cm} (7.1)

Here $\Delta_S$ is the Laplacian operator of $(S, J_S, \omega_S)$. The existence can be found in standard monographs, see e.g. [15]. More precisely, when $\partial S \in C^\infty$, $h \in C^\infty(\bar S)$; while $\partial S \in C^{2,\beta}$, $0 < \beta < 1$, $h \in C^\infty(S) \cap C^{2,\beta}(\bar S)$. Moreover, $\frac{\partial h}{\partial \nu}|_{\partial S} < 0$.

We use such $h$ to construct subsolutions. Let

$$u = \varphi + t\pi_*^2 h$$  \hspace{1cm} (7.2)

for $t \gg 1$ ($\pi_*^2 h = h \circ \pi_2$, still denoted by $h$ for simplicity), then

$$g[u] = g[\varphi] + t\pi_*^2 \omega_S, \quad \frac{\partial u}{\partial \nu}|_{\partial M} < 0 \text{ for } t \gg 1.$$  

Therefore, $u$ is the subsolution if condition (1.14) holds. Furthermore, since the boundary data $\varphi$ is admissible, (1.14) always holds when imposed (1.5).

Significantly, according to Lemma 3.6, if $\omega = \pi_1^* \omega_X + \pi_2^* \omega_S$ and $\chi$ splits by $\chi = \pi_1^* \chi_1 + \pi_2^* \chi_2$, where $\chi_1$ is a real $(1,1)$-form on $X$, $\chi_2$ is a real $(1,1)$-form on $S$, then condition (1.14) reduces to

$$\lim_{t \to +\infty} f(\lambda'(\chi_1), t) > \psi \text{ and } \lambda_{\omega_X}(\chi_1) \in \Gamma_\infty \text{ in } \bar M,$$

where $\lambda'(\chi)$ are the eigenvalues of $\chi_1$ with respect to $\omega_X$. This indicates that, for $\chi = \pi_1^* \chi_1 + \pi_2^* \chi_2$, the solvability of Dirichlet problem is heavily determined by $\chi_1$ rather than by $\chi_2$.

### 7.2 The Dirichlet problem with less regular boundary

A somewhat remarkable fact to us is that the regularity assumptions on boundary can be further weakened. The motivation is primarily based on Theorem 5.7 which state that, when $\partial M$ is holomorphically flat and the boundary value is a constant, the quantitative boundary estimate (1.6) depends only on $\partial M$ up to second derivatives and other known data. Besides, a result due to Silvestre-Sirakov [36] allows one to derive $C^{2,\alpha}$ boundary regularity with only assuming
$C^{2,\beta}$ boundary. Together with the construction of subsolution, we can study the Dirichlet problem on the products with less regular boundary.*

**Theorem 7.1.** Let $(M, J, \omega) = (X \times S, J, \omega)$ be as above with $\partial S \in C^{2,\beta}, 0 < \beta < 1$. Suppose in addition that (1.2), (1.3) and (1.14) hold. Then we have two conclusions:

- Equation (1.1) has a unique $C^{2,\alpha}$ admissible solution with $u|_{\partial M} = 0$ for some $0 < \alpha \leq \beta$, provided that $\psi \in C^2(\bar{M})$, $\inf_M \psi > \sup_{\partial \Gamma} f$ and $f$ satisfies (1.4).

- Suppose in addition that $f \in C^\infty(\Gamma) \cap C(\bar{\Gamma})$, $\psi \in C^{1,1}(\bar{M})$ and $\inf_M \psi = \sup_{\partial \Gamma} f$. Then the Dirichlet problem (1.1) has a weak solution with $u|_{\partial M} = 0$, $u \in C^{1,\alpha}(\bar{M})$, $\forall 0 < \alpha < 1.\lambda(g[u]) \in \bar{\Gamma}$ and $\Delta u \in L^\infty(\bar{M})$.

**Proof.** It only requires to consider the nondegenerate case: For some $\delta_0 > 0$,

$$\psi > \sup_{\partial \Gamma} f + \delta_0 \text{ in } M. \quad (7.3)$$

The first step is to construct approximate Dirichlet problems with constant boundary value data. Let $h$ be the solution to (7.1). For $t \gg 1$, $u = th$ satisfies

$$f(\lambda(g[u])) \geq \psi + \delta_1 \text{ in } M \quad (7.4)$$

for some $\delta_1 > 0$. Note that

$$h \in C^\infty(S) \cap C^{2,\beta}(\bar{S}) \text{ and } \frac{\partial h}{\partial \nu}|_{\partial S} < 0,$$

we get a sequence of level sets of $h$, say $\{ h = -\alpha_k \}$ and a family of smooth Riemann surfaces $S_k$ enclosed by $\{ h = -\alpha_k \}$, such that $\cup S_k = S$ and $\partial S_k$ converge to $\partial S$ in the norm of $C^{2,\beta}$. Denote $M_k = X \times S_k$. For any $k \geq 1$, there exists a $\psi^{(k)} \in C^\infty(\bar{M}_k)$ such that

$$|\psi - \psi^{(k)}|_{C^1(\bar{M}_k)} \leq 1/k. \quad (7.5)$$

For $k \gg 1$ we have

$$f(\lambda(g[u])) \geq \psi^{(k)} + \delta_1/2 \text{ in } M_k, \quad u = -\alpha_k \text{ on } \partial M_k \quad (7.6)$$

*We emphasize that the geometric quantities of $(M, \omega)$ (curvature $R_{i\bar{j}k\bar{l}}$ and the torsion $T^i_{\bar{j}k}$) keep bounded as approximating to $\partial M$, and all derivatives of $\chi_{i\bar{j}}$ has continues extensions to $\bar{M}$, whenever $M$ has less regularity boundary. Typical examples are as follows: $M \subset M'$, $\dim_e M' = n$, $\omega = \omega_{M'}|_M$ and the given data $\chi$ can be smoothly defined on $M'$.}
which is a strictly admissible subsolution to approximate Dirichlet problem

\[ f(\lambda(g[u])) = \psi^{(k)} \text{ in } M_k, \quad u = -t\alpha_k \text{ on } \partial M_k. \quad (7.7) \]

According to Theorems 1.2 and 1.3, the Dirichlet problem (7.7) admits a unique smooth admissible solution \( u^{(k)} \in C^\infty(\bar{M}_k) \). Moreover, notice the boundary data of (7.7) is a constant, Theorem 5.7 applies. Therefore,

\[ \sup_{M_k} \Delta u^{(k)} \leq C_k \left(1 + \sup_{M_k} |\nabla u^{(k)}|^2 \right) \quad (7.8) \]

where \( C_k \) is a constant depending on \( |u^{(k)}|_{C^0(M_k)}, |\nabla u^{(k)}|_{C^0(\partial M_k)}, |\psi^{(k)}|_{C^2(\bar{M}_k)}, \partial M_k \) up to second order derivatives and other known data (but not on \( (\delta\phi_{ij})^{-1})). It requires to prove that

\[ |u^{(k)}|_{C^0(M_k)} + \sup_{\partial M_k} |\nabla u^{(k)}| \leq C, \text{ independent of } k. \quad (7.9) \]

Let \( w^{(k)} \) be the solution of

\[ \Delta w^{(k)} + \text{tr}_\omega \chi = 0 \text{ in } M_k, \quad w^{(k)} = -t\alpha_k \text{ on } \partial M_k. \]

By the maximum principle and the boundary value condition, one has

\[ u \leq u^{(k)} \leq w^{(k)} \text{ in } M_k, \quad \frac{\partial u}{\partial \nu} \leq \frac{\partial u^{(k)}}{\partial \nu} \leq \frac{\partial w^{(k)}}{\partial \nu} \text{ on } \partial M_k. \]

On the other hand, for \( t \gg 1 \), we have

\[ \Delta(-u - 2t\alpha_k) + \text{tr}_\omega \chi = -t\text{tr}_\omega(\pi^2_2\omega_S) + \text{tr}_\omega \chi \leq 0 \text{ in } M_k, \]

\[ -u - 2t\alpha_k = -t\alpha_k \text{ on } \partial M_k. \]

As a result, we have

\[ w^{(k)} \leq -u - 2t\alpha_k \text{ in } M_k, \quad w^{(k)} = -u - 2t\alpha_k \text{ on } \partial M_k, \]

which further implies

\[ \frac{\partial w^{(k)}}{\partial \nu} \leq -\frac{\partial u}{\partial \nu} \text{ on } \partial M_k \]

as required. Consequently, (7.8) holds for a uniform constant \( C' \) which does not depend on \( k \). Thus

\[ |u|_{C^2(M_k)} \leq C, \text{ independent of } k. \]
To complete the proof, we apply Silvestre-Sirakov’s [36] result to derive $C^{2,\alpha'}$ estimates on the boundary, while the convergence of $\partial M_k$ in the norm $C^{2,\beta}$ allows us to take a limit ($\alpha'$ can be uniformly chosen).

□

For the Dirichlet problem on $M = X \times S$ with homogeneous boundary value, according to Theorem 7.1, it is only requires to assume $\partial M \in C^{2,\beta}$. Such a regularity assumption on the boundary is impossible for homogeneous Monge-Ampère equation on certain bounded domains in $\mathbb{R}^n$ as shown by Caffarelli-Nirenberg-Spruck [6], where the $C^{3,1}$-regularity assumptions on boundary and boundary value are optimal for the $C^{1,1}$ global regularity of the weak solution to homogeneous real Monge-Ampère equation on $\Omega \subset \mathbb{R}^n$. Additionally, this is also different from the case for Dirichlet problem of nondegenerate real Monge-Ampère equation on certain bounded domains $\Omega \subset \mathbb{R}^2$, as shown by Wang [44] the optimal regularity assumptions on boundary and boundary value are both $C^3$-smooth. We refer to [14, 24] and references therein for more results regarding to Monge-Ampère equation with less regular right-hand side.

8 Other results

8.1 Uniqueness of weak solution

Following Chen [7], we define

**Definition 8.1.** A continuous function $u \in C(\bar{M})$ is a weak $C^0$-solution to the degenerate equation (1.1) with prescribed boundary data $\varphi$, if for any $\epsilon > 0$ there is a $C^2$-admissible function $\tilde{u}$ such that

$$|u - \tilde{u}| < \epsilon,$$

where $\tilde{u}$ solves

$$F(g[\tilde{u}]) = \psi + \rho_\epsilon \, \text{in} \, M, \, \tilde{u} = \varphi \, \text{on} \, \partial M.$$

Here $\rho_\epsilon$ is a function satisfying $0 < \rho_\epsilon < C(\epsilon)$, and $C(\epsilon) \to 0$ as $\epsilon \to 0$.

**Theorem 8.2.** Suppose $u^1$, $u^2$ are two $C^0$-weak solutions to the degenerate equation (1.1) with boundary data $\varphi^1$, $\varphi^2$. Then

$$\sup_M |u^1 - u^2| \leq \sup_{\partial M} |\varphi^1 - \varphi^2|.$$
The proof is almost parallel to that of [7, Theorem 4]. We omit it here.

**Corollary 8.3.** The weak $C^0$-solution to the Dirichlet problem (1.1) for degenerate equation is unique, provided the boundary data is fixed.

### 8.2 Construction of subsolutions revisited

Let $\Omega$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^m$, $2 \leq m \leq n - 1$, with smooth boundary $\partial \Omega$, let $(X, J_X, \omega_X)$ be a closed Hermitian manifold of complex dimension $n - m$. For the Dirichlet problem (1.1) satisfying

$$
\lim_{t \to +\infty} f(\lambda_1, \ldots, \lambda_{n-m}, \lambda_{n-m+1} + t, \ldots \lambda_n + t) = \sup_{\Gamma} f, \quad \forall \lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma,
$$

we can construct strictly admissible subsolutions on $(M, J, \omega) = (X \times \Omega, J, \omega)$ (not necessary to be the standard one $\omega = \pi_1^* \omega_X + \pi_2^* \omega_{\Omega}$).

Since $\Omega$ is a smooth bounded strictly pseudoconvex domain, there exists a smooth strictly plurisubharmonic function $w$ with

$$
\sqrt{-1} \partial \bar{\partial} w \geq \omega_{\Omega} \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega.
$$

Given an admissible boundary value $\varphi$, the subsolution is given by

$$
\underline{u} = tw + \varphi \text{ for } t \gg 1.
$$

As a consequence, we obtain the following theorem.

**Theorem 8.4.** Let $(M, J, \omega) = (X \times \Omega, J, \omega)$ be a product as above. Assume that $f$ satisfies (1.2), (1.3), (1.4) and (8.1). For $\varphi \in C^\infty(\partial M)$, $\psi \in C^\infty(\bar{M})$ satisfying $\inf_M \psi > \sup_{\partial \Gamma} f$, the Dirichlet problem (1.1) is uniquely solvable in class of smooth admissible functions.

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