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Supplementary materials for: Digitally tailoring arbitrary structured light of generalized ray-wave duality

In this supplementary material we provide additional information about eigenstates of harmonic oscillator in quantum optics, the classical trajectory and wave-packet of RWD modes and the simulation method of obtaining them. In addition, the phase mask design of generalized RWD modes is illustrated in detail.

1. SUPPLEMENTARY OF RWD THEORY

A. Eigenstates of harmonic oscillator in quantum optics

The eigenmodes of resonant cavity correspond to the eigenstates of Hamiltonian harmonic oscillator since paraxial wave equation has the same mathematical form as Schrodinger equation [1, 2]. The representation of a laser beam often includes transverse and longitudinal modes, yielded by the Hamiltonian $\mathcal{H}_0 = (a^+_x a_x + a^+_y a_y + 1) \hbar \omega_0 + (a^+_z a_z + 1/2) \hbar \omega_z$ for the 3D transversely symmetric harmonic oscillator in quantum optics [1], where $\omega_0$ and $\omega_z$ are the frequencies of transverse and longitudinal modes, $a^+_l$ and $a_l$ are the creation and annihilation operators for the transverse mode $(i = x, y)$ and longitudinal mode $(i = z)$, and $\hbar$ is the reduced Planck constant. The eigenstates $(n, m, l) (n, m, l \in \mathbb{N})$ of the Hamiltonian $\mathcal{H}_0$ can be generated by applying creation operators on the fundamental Gaussian mode as the ground state [1, 2]:

$$|n, m, l\rangle = \frac{(a^+_x)^n (a^+_y)^m (a^+_z)^l}{\sqrt{n!} \sqrt{m!} \sqrt{l!}} |0, 0, 0\rangle,$$

(S1)

with eigenfrequency $\omega_{n,m,l} = \omega_T + \omega_L$, where the transverse mode frequency $\omega_T = (n + m + 1)\omega_0$ and longitudinal mode frequency $\omega_L = (l + 1/2)\omega_z$. Eigenstates as described by Eq. (S1) are just corresponding to the well-known Hermite–Gaussian (HG) modes, with the transverse mode indices of $n$ and $m$ at $x$- and $y$-directions respectively, and the longitudinal mode index $l$ at $z$-direction. Nevertheless, recent advance of structured light unveiled many beams with complex patterns of singularities splitting are outdoing the transverse symmetry ($\omega_x \neq \omega_y$) [3]. As such, it should be generally yielded by the Hamiltonian $\mathcal{H} = (b^+_x b_x + 1/2) \hbar \omega_x + (b^+_y b_y + 1/2) \hbar \omega_y + (b^+_z b_z + 1/2) \hbar \omega_z$ of separable 3D harmonic oscillator [1] with different frequencies $\omega_x = \omega_0 - \Delta \omega_x$ and $\omega_y = \omega_0 + \Delta \omega_y$ along $x$- and $y$-axes. The general 3D harmonic oscillator can be transformed to a transversely symmetric harmonic oscillator by applying SU(2) unitary transformation on ladder operators [4]:

$$\begin{bmatrix}
  b^+_x \\
  b^+_y \\
  b^+_z
\end{bmatrix} =
\begin{bmatrix}
  e^{-in/2} \cos(\beta/2) & e^{in/2} \sin(\beta/2) & 0 \\
  -e^{-in/2} \sin(\beta/2) & e^{in/2} \cos(\beta/2) & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  a^+_x \\
  a^+_y \\
  a^+_z
\end{bmatrix},$$

(S2)

where $\alpha$ and $\beta$ are the two parametric rotation angles in SU(2) symmetry along $z$-axis, and the eigenstates are given by [1, 2]:

$$|\psi^{(a,\beta)}_{n,m,l}\rangle = \frac{(b^+_x)^n (b^+_y)^m (b^+_z)^l}{\sqrt{n!} \sqrt{m!} \sqrt{l!}} |0, 0, 0\rangle,$$

(S3)

where the eigenfrequency of $\omega_{n,m,l} = \omega_T + \omega_L$, in which the transverse mode frequency $\omega_T = (n + 1/2)\omega_x + (m + 1/2)\omega_y$ and longitudinal mode frequency $\omega_L = (l + 1/2)\omega_z$. Eigenstates in Eq. (S3) are corresponding to the Hermite–Laguerre–Gaussian (HLG) modes with transverse mode indices of $(n, m)$ and longitudinal mode index of $l$ under Cartesian coordinate representation. Of particular note, when $\beta = 0$ or $\beta = \pi$, $|\psi^{(a,\beta)}_{n,m,l}\rangle$ are reduced into HG modes; when $a = \beta = \pi/2$, $|\psi^{(a,\beta)}_{n,m,l}\rangle$ are reduced into LG modes. The eigenstates with SU(2) transformation can well represent various HLG eigenmodes.
B. Wave function of eigenstates under different coordinate representations

Under Cartesian coordinate representation, the wave function of eigenstate in Eq. (S1) is given by:

\[
\langle x, y, z| n, m, l \rangle = \psi_n^{(\text{HO})}(x) \psi_m^{(\text{HO})}(y) \times \exp \left[ ik_{n,m,s} z - i (m + n + 1) \theta \right] 
\]

\[
\psi_n^{(\text{HO})}(x) = \left( \frac{2^n n! \sqrt{\pi} w_x}{\omega_0} \right)^{-1/2} e^{-x^2/2 \omega_0^2} H_n(x) 
\]

where \( \psi_0^{(\text{HO})}(x) = \text{H} \), which can be gauged to fulfill the Gouy phase, \( \theta = \tan^{-1}(z/z_R) \).

In terms of the Wigner \( d \)-matrix, the eigenstates \( |\psi_{n,m,l}^{(a,b)}\rangle \) of the Hamiltonian \( \mathcal{H} \) can be analytically expressed as a linear combination of the eigenstates of the Hamiltonian \( \mathcal{H}_0 \), i.e.

\[
|\psi_{n,m,l}^{(a,b)}\rangle = e^{i(n+m)a/2} \sum_{k=0}^{n+m} e^{i k a} d_{k-n-m}^{a,b} \langle \beta | k, n + m - k, l \rangle, 
\]

where the elements of Wigner \( d \)-matrix are given by:

\[
d_{k-n-m}^{a,b} \langle \beta | = \sqrt{k!(n + m - k)!n!m!} \sum_{v=\max(0,k-n)}^{\min(m,k)} (-1)^v \left( \frac{\cos(\beta/2)^{m+k-2v}}{v!(m-v)!(k-v)!(n-k+v)!} \right)^{n-k+2v} \sin(\beta/2)^{m+k-2v}. 
\]

In terms of Wigner expression in Eq. (S6), the wave function of eigenstate in Eq. (S4) under Cartesian coordinate representation is given by:

\[
\langle x, y, z| \psi_{n,m,l}^{(a,b)} \rangle = e^{i(n+m)a/2} \times \sum_{k=0}^{n+m} e^{i k a} d_{k-n-m}^{a,b} \langle \beta | \langle x, y, z | k, n + m - k, l \rangle, 
\]

Based on Eqs. (S4) and (S8), we can numerically calculate the wave function of each eigenstate of SU(2) coupled linear oscillator.

C. Classical trajectory and wave-packets

Hereinafter we make further discussion on the classical trajectory and the corresponding wave-packets of typical RWD modes, i.e. the Lissajous-to-trochoidal parametric surface modes, multi-axis Hermite–Laguerre–Gaussian modes, and planar-to-vortex multi-path geometric modes. For a general case that the ratio of transverse and longitudinal frequency spacings can be expressed as \( \omega_0/\omega_z = P/Q \) where \( (P, Q) \) is a pair of coprime integers, the ratios of \( \omega_x/\omega_z \) and \( \omega_y/\omega_z \) are two rational numbers that can be expressed by:

\[
\frac{\omega_x}{\omega_z} = \frac{\omega_0 - \Delta \omega_x}{\omega_z} = \frac{\omega_0}{\omega_z} \left( 1 - \frac{q \Delta \omega}{\omega_0} \right) = \frac{P}{Q} \left( 1 - \frac{qM_1}{M_2} \right), 
\]

where \( q \) and \( M_1/M_2 \) are the rational numbers that can be expressed by:

\[
\frac{\omega_z}{\omega_x} = \frac{\omega_0 - \Delta \omega_x}{\omega_z} = \frac{\omega_0}{\omega_z} \left( 1 - \frac{q \Delta \omega}{\omega_0} \right) = \frac{P}{Q} \left( 1 - \frac{qM_1}{M_2} \right), 
\]

and \( q = \frac{\omega_y}{\omega_z} \).
\[
\frac{\omega_y}{\omega_z} = \frac{\omega_0 + \Delta \omega_y}{\omega_0 + \Delta \omega_z} = \frac{\omega_0}{\omega_z} \left(1 + \frac{p \Delta \omega}{\omega_0}\right) = \frac{P}{Q} \left(1 + \frac{pM_1}{M_2}\right),
\]

where \((M_1, M_2)\) is a pair of coprime integers satisfying \(\Delta \omega/\omega_0 = M_1/M_2\). We already know the classical trajectories of Lissajous-to-trochoidal parametric surface modes \(\left(x_{a,k}^{\pm}, y_{b,k}^{\pm}, z_{c,k}^{\pm}\right)\) are represented as a caustics cluster with limited rays:

\[
\begin{align*}
x_{a,k}^{\pm} &= \sqrt{N_x} w(z) \cos \left[2\pi k \frac{P}{Q} \left(1 - \frac{qM_1}{M_2}\right) + \phi_x \pm \theta(z)\right] \\
y_{b,k}^{\pm} &= \sqrt{N_y} w(z) \cos \left[2\pi k \frac{P}{Q} \left(1 + \frac{pM_1}{M_2}\right) + \phi_y \pm \theta(z)\right] \\
z_{c,k}^{\pm} &= z
\end{align*}
\]

The classical trajectories of Lissajous-to-trochoidal parametric surface modes \(\left(x_{a,k}^{\pm, \alpha, \beta}, y_{b,k}^{\pm, \alpha, \beta}, z_{c,k}^{\pm, \alpha, \beta}\right)\) are obtained by applying SU(2) transition on \(\left(x_{a,k}^{\pm}, y_{b,k}^{\pm}, z_{c,k}^{\pm}\right)\). And the wave-packets of Lissajous-to-trochoidal parametric surface mode are expressed as follows:

\[
\psi_{\alpha,\beta,\phi}^{(\alpha, \beta)} \mid_{n_0,m_0,l_0}^{N} = \frac{1}{2N/2} \sum_{k=0}^{N} \left(N\right)^{1/2} e^{iK\phi} \mid_{n_0+m_0+qK+K_0-K_0-PK}^{\psi_{\alpha,\beta}}.
\]

| \((\alpha, \beta)\) | \((0,0)\) | \((m/2, m/4)\) | \((m/2, m/2)\) |
|-----------------|---------|----------------|----------------|
| \((p, q)\)      |         |                |                |
| \(0\)           |         |                |                |
| \(\pi/4\)       |         |                |                |
| \(\pi/2\)       |         |                |                |
| \(3\pi/4\)      |         |                |                |
| \(\pi\)         |         |                |                |

Fig. S1. The contour lines of the classical trajectories of Lissajous-to-trochoidal parametric surface modes at \(z=0\).

The contour lines of some Lissajous-to-trochoidal parametric surface at \(z = 0\) are shown in Fig. S1. When \((\alpha, \beta)\) are \((0,0)\) and \((\pi/2, \pi/2)\), these contour lines turn to Lissajous and trochoidal curves, respectively.

For a specific case of \(\Delta \omega = 0\), i.e. \(q = 0\) and \(\omega_x/\omega_z = \omega_0/\omega_z = P/Q\), we can also get \(p\omega_0 + s\omega_z = 0\) based on the frequency-degenerate condition of Eq. (6) in the main text, then it can be deduced that \(p = Q\), \(q = 0\), and \(s = -P\) here, thus \(\omega_y/\omega_z = P/Q(1 + QM_1/M_2)\). Thus, the multi-axis HLG mode are obtained, and the classical trajectory of Eq. (S11) is reduced into:

\[
\begin{align*}
x_{a,k}^{\pm} &= \sqrt{N_x} w(z) \cos \left[2\pi k \frac{P}{Q} \phi_x \pm \theta(z)\right] \\
y_{b,k}^{\pm} &= \sqrt{N_y} w(z) \cos \left[2\pi k \frac{P}{Q} \left(1 + \frac{QM_1}{M_2}\right) + \phi_y \pm \theta(z)\right] \\
z_{c,k}^{\pm} &= z
\end{align*}
\]

which is a spatial caustics cluster with \(|QM_2|\) rays, while the dots distribution is no longer along a Lissajous curve but reduced to compose multiple linear oscillation orbits in a certain transverse
plane. The general SU(2) trajectory \((x_{\pm b,k}, y_{\pm b,k}, z_{\pm b,k})\) shows multi-axis elliptical orbits distributed on subset uniparted hyperboloid ruled surfaces, where the axes are located on a main uniparted hyperboloid ruled surface, composing a SU(2) symmetric structure. The corresponding coherent state wave-packet in Eq. (S12) is reduced into:

\[
|\Psi(a,b,\phi)\rangle^N_{m_0,m_0,\mu_0} = \frac{1}{2^{N/2}} \sum_{K=0}^{N} \left( \begin{array}{c} N \\ K \end{array} \right)^{1/2} e^{iK\phi} |\Psi(a,b)\rangle^{N}_{m_0+QNK0m_0-PK, \mu_0}\).
\]

For a more specific case of \(\Delta \omega_x = 0\) and \(\omega_y = 0\), naturally \(\omega_0 = \omega_x, N_y = m_0 = q = \Delta \omega = M_1 = 0, \omega_x/\omega_z = \omega_0/\omega_z = P/Q\). On this occasion, we have the planar-to-vortex multi-path geometric modes, with the classical trajectory Eq. (S13) further reduced as:

\[
\begin{align*}
\frac{x_{\pm b,k}}{\omega} &= \sqrt{N}\sqrt{\pi}(z) \cos \left[ 2\pi k \frac{P}{Q} + \phi_x \pm \theta(z) \right] \\
\frac{y_{\pm b,k}}{\omega} &= 0 \\
\frac{z_{\pm b,k}}{\omega} &= z 
\end{align*}
\]

where the reasonable values of \(k\) are \(k = 0, 1, 2, \cdots, Q - 1\), which corresponds to a cluster of \(Q\) rays in a planar hyperbola region. The corresponding general trajectory \((x_{\pm b,k}^d, y_{\pm b,k}^d, z_{\pm b,k}^d)\) after SU(2) transformation is actually the \(Q\) rays uniformly distributed on a uniparted hyperboloid ruled surface. At a certain transverse plane, the pattern illustrates multiple dots uniformly distributed on an ellipse orbit. The corresponding coherent state wave-packet is reduced from Eq. (S14) into:

\[
|\Psi(a,b,\phi)\rangle^N_{m_0,m_0,\mu_0} = \frac{1}{2^{N/2}} \sum_{K=0}^{N} \left( \begin{array}{c} N \\ K \end{array} \right)^{1/2} e^{iK\phi} |\Psi(a,b)\rangle^{N}_{m_0+QNK0m_0-PK}.
\]

which harnesses the multi-path geometric mode with the \(Q\) paths uniformly distributed on the main uniparted hyperboloid ruled surface, which can be obtained by replacing the sub-HLG beams in the multi-axis HLG mode by fundamental Gaussian beams. With the SU(2) transformation, the planar multi-path geometric mode is transformed into vortex multi-path geometric mode. Specially, the topological charge of the center vortex is equal to \(n\), and that for the partial phase singularities is \(QN\).

2. DIGITAL GENERATION OF RWD MODES

A. Phase mask design of generalized RWD modes

The generalized RWD states are scalar modes. The CGH method can be used for generating RWD modes in our experiment [5, 6]. The normalized generalized RWD field in any cross section can be represented by

\[
U(x, y) = A(x, y) \exp(i\Phi(x, y))
\]

where \(A\) and \(\Phi\) are amplitude and phase of generalized RWD mode, respectively, and \(A \subset [0, 1]\), \(\Phi \subset [0, 2\pi]\). We define a phase mask by

\[
M(x, y) = \exp \{i\Psi [A(x, y), \Phi(x, y)]\},
\]

where \(\Psi [A(x, y), \Phi(x, y)]\) is phase function of the pure phase mask. Here, the phase mask can be expressed by the Fourier series expansion.

\[
M(x, y) = \sum_{q=-\infty}^{\infty} m_q = \sum_{q=-\infty}^{\infty} c_q^A \exp [i\Phi(x, y)] \exp (-iq\Phi) \exp (-i\Phi) d\Phi,
\]

For the first order \((q = 1)\), we have the coefficient as \(m_1 = c_1^A \exp (i\Phi)\). Specifically when \(C\) is a positive constant, the target field can be well constructed by the first order. Substituting \(c_1^A = CA\) into Eq. (19), we have:

\[
\int^{-\pi}_{-\pi} \sin |\Psi (A, \Phi) - \Phi| d\Phi = 0,
\]

\[
\int^{-\pi}_{-\pi} \cos |\Psi (A, \Phi) - \Phi| d\Phi = 2\pi CA
\]
Therefore, we obtain that phase mask \( \Psi[A(x, y), \Phi(x, y)] \) has odd symmetry relative to the variable \( \Phi \), assuming the phase function is \( \Psi(A, \Phi) = \sin \Phi \times f(A) \). Using Eq. (S19), the Fourier series coefficients can be derived as \( c_{q}^{A} = I_{q}[f(A)] \), while the first order coefficient has been obtained as \( c_{1}^{A} = CA \). Therefore, the maximum value of \( C \) for fulfilling \( A = I_{1}[f(A)]/C \) is \( C = 0.5819 \). Thus the amplitude function can be expressed as \( f(A) = I_{1}^{-1}(CA) \). The phase mask is thus obtained as:

\[
M(x, y) = \exp \left\{ i \frac{1}{I_{1}} [CA(x, y)] \Phi(x, y) \right\}
\]

(S21)

Now that we have obtained the target RWD mode in the first order, it needs to be separated from other orders. A simple way of separating the mode is to add different spatial frequencies into different Fourier series, as given by:

\[
M_{s}(x, y) = \sum_{q=-\infty}^{\infty} m_{q}(x, y) \exp \left\{ i 2\pi \left( qu_{0}x + qu_{0}y \right) \right\} = \exp \left\{ i \Psi[A, \Phi + 2\pi (u_{0}x + v_{0}y)] \right\}
\]

(S22)

Some selected CGH phase masks of RWD modes used in the main text are shown in Fig. S2.

Fig. S2. The computer generated holograms of phase masks designed and used for Fig. 5(d), Fig. 5(e), Fig. 5(f), Fig. 6(a) and Fig. 6(c) in the main article are shown in sub-figure (a)-(e), respectively, along with the magnified views.

Fig. S3. Experimental results of eigenstates of 3D linear oscillator. (a) family of HG modes (HG_{00} – HG_{55}), (b) family of LG modes (LG_{00} – LG_{55}).

B. More experimental results of generalized RWD modes

It is worth noting that \( Q \) can be a negative integer in Eq. (2) in the main text, in which case, however, the SU(2) coherent state is equivalent to another state with \( Q \) as a positive integer. That
means both states have same features in terms of intensity pattern and mode evolution along the propagation. The key to the equivalence is that both SU(2) coherent states contain the same set of eigenstates, and every eigenstate has the same weight. The equivalent states can be expressed as:

$$|\Psi(\alpha, \beta, \phi)_{n, m, l}\rangle_N = |\Psi(\alpha, \beta, \phi)_{n', m', l}\rangle_N$$

(S23)

where \((p + q) = -(p' + q') < 0, n'_0 = n_0 + pN, m'_0 = m_0 + qN\).

Hereinafter, we supply more experimental result about digitally control of RWD mode. We already know the RWD mode would reduce into the eigenstates when \(Q \to \infty\). Besides, the eigenstates is sub-modes of SU(2) coherent state. Thus, the generation of eigenstate is crucial. The perfect eigenstates family of a 3D linear oscillator are obtained experimentally, as showed in Fig. S3. More experimental and theoretical results as examples of digital parameter control on RWD modes that break limitation of cavity are shown in Fig. S4.

![Fig. S4. Experimental and theoretical results of the propagation evolution of transverse patterns for \(Q = 2, Q = 1\) and \(Q = 0\) with different parameters. (a) \(|\Psi(\pi/2, \pi/2, \pi)_{20,10,l}\rangle_{5,-1,1}\). (b) \(|\Psi(\pi/2, \pi/2, \pi)_{20,5,1}\rangle_{5,-1,3}\). (c) \(|\Psi(0,0, \pi/2)_{10,10,1}\rangle_{5,-1,2,3}\). (d) \(|\Psi(\pi/2, \pi/2, \pi/2)_{20,10,1}\rangle_{5,-3,4}\). (e) \(|\Psi(0,0, \pi/2)_{20,10,1}\rangle_{5,-3,4}\). (f) |\Psi(\pi/2, \pi/2, 3\pi/2)_{20,10,1}\rangle_{5,-1,1}\).](image-url)
REFERENCES
1. R. Blumel, *Advanced quantum mechanics: the classical-quantum connection* (Jones & Bartlett Publishers, 2011).
2. J. G. Hartley and J. R. Ray, “Coherent states for the time-dependent harmonic oscillator,” Phys. Rev. D 25, 382 (1982).
3. Y. F. Chen, J. Tung, P. Tuan, and K. F. Huang, “Symmetry breaking induced geometric surfaces with topological curves in quantum and classical dynamics of the su (2) coupled oscillators,” Annalen der Physik 529, 1600253 (2017).
4. Y. F. Chen, “Geometry of classical periodic orbits and quantum coherent states in coupled oscillators with su (2) transformations,” Phys. Rev. A 83, 032124 (2011).
5. V. Arrizón, U. Ruiz, R. Carrada, and L. A. González, “Pixelated phase computer holograms for the accurate encoding of scalar complex fields,” J. Opt. Soc. Am. A 24, 3500–3507 (2007).
6. C. Rosales-Guzmán and A. Forbes, *How to shape light with spatial light modulators* (SPIE, 2017).