Reflected Backward SDEs with General Jumps

S. Hamadène
Laboratoire de Statistique et Processus
Département de Mathématiques
Université du Maine, 72085 Le Mans Cedex 9, France
e-mail: hamadene at univ-lemans.fr

and

Y. Ouknine
Département de Mathématiques,
Faculté des sciences Semlalia,
Université Cadi Ayyad, Marrakech, Maroc
e-mail: ouknine at ucam.ac.ma

Abstract

In the first part of this paper we give a solution for the one-dimensional reflected backward stochastic differential equation (BSDE for short) when the noise is driven by a Brownian motion and an independent Poisson point process. The reflecting process is right continuous with left limits (rcll for short) whose jumps are arbitrary. We first prove existence and uniqueness of the solution for a specific coefficient in using a method based on a combination of penalization and the Snell envelope theory. To show the result in the general framework we use a fixed point argument in an appropriate space. The second part of the paper is related to BSDEs with two reflecting barriers. Once more we prove the existence and uniqueness of the solution of the BSDE.

Key Words: Backward SDEs ; Penalization ; Poisson point process ; Martingale representation theorem ; Snell envelope ; Reflecting barriers ; Mokobodski’s hypothesis.

AMS Classification (1991): 60H10, 60H20, 60H99.

1 Introduction

Non-linear backward stochastic differential equations (BSDEs in short) were introduced by Pardoux & Peng [15] when the noise is driven by a Brownian motion. Their objective was to give a probabilistic interpretation of a solution of a second order quasi-linear partial differential equation. Since then, these equations have gradually became an important mathematical tool which is encountered in many fields of mathematics such as finance, stochastic optimal control and games, partial differential equations and so on (see e.g. [1, 3, 6, 7, 8, 16, 17, 18] and the references therein).

Later Tang & Li [18], considered standard BSDEs when the noise is driven not only by a Brownian motion but also by an independent Poisson random measure. They showed existence and uniqueness of the solution. Barles et al. [1] studied the link of those BSDEs with viscosity solutions of integral-partial differential equations.
One barrier reflected BSDEs have been introduced by El-Karoui et al. in [6]. In the setting of those BSDEs, one of the components of the solution is forced to stay above a given barrier which is a continuous adapted stochastic process. The main motivation in [6], is the pricing of American options especially in constrained markets (see also [7]). The generalization to the case of two reflecting barriers has been carried out by Cvitanic & Karatzas in [3].

Later, on the one hand, Hamadène & Oukine ([11]) have studied one reflecting barrier BSDEs when the noise is driven by a Brownian motion and an independent Poisson measure. They showed existence and uniqueness of the solution when the reflecting barrier has only inaccessible jumps, i.e., jumps which stem only from the Poisson part. On the other hand, S.Hamadène [8] has introduced BSDEs with one right continuous with left limits reflecting barrier in the case of Brownian noise. Since then there have been several works on BSDEs with discontinuous barriers when the noise comes only from a Brownian motion ([13, 14]).

So the main objective of this paper is to deal with reflected BSDEs when the noise comes from a Brownian motion and an independent Poisson process and the reflecting processes are just rcll. No more conditions are imposed on their jumps as it was e.g. in [11]. They could be predictable or inaccessible. In our study we consider the case of one reflecting barrier as well as the case of two reflecting barriers. For both cases we show existence and uniqueness of the solution when the coefficients of the BSDEs are Lipschitz.

This work completes the known results on the same subject since the jumps of the reflecting processes are arbitrary and the sources of noise are twice, Brownian and Poisson. A second motivation of our work is that the two barrier reflected BSDEs we consider here are much involved in finance, especially when we deal with convertible bonds in defautable markets. For more details on this latter subject one can see e.g. the paper by Mordecki et al. [2]. Finally, this work opens a window towards viscosity solutions of variational inequalities with discontinuous obstacles.

This paper is organized as follows.

Section 2 contains hypotheses and the setting of the problem. In Section 3, we show uniqueness of the solution of the BSDE with one reflecting rcll barrier \((S_t)_{t \leq 1}\), a Lipschitz coefficient \(f(t, \omega, y, z, v)\) and square integrable terminal value \(\xi\). In Section 4, we address the question of existence of the solution for the BSDE. Since there is a lack of comparison of solutions of standard BSDEs whose noise contains a Poisson part, especially when the coefficients depend on \(v\) (see [1] for a counterexample), we first assume that \(f\) does not depend on this latter variable. Then we show the existence and uniqueness of the solution of the BSDE. The method we used combines penalization with the general theory of Snell envelope of processes. Later, in order to obtain the result for general coefficients \(f(t, \omega, y, z)\) we have introduced a contraction in an appropriate Banach space of processes which then has a fixed point which is the unique solution of the BSDE. At the end of this section we focus on some properties of the reflecting process \(K\) of the solution. Finally in the last section, we address the problem with two reflecting barriers. Once more, under the well-known Mokobodski’s hypothesis, we show existence and uniqueness of the solution.

### 2 Setting of the problem and hypotheses

First for simplicity, we fix the horizon \(T\) of the problem equal to 1 and of course our results still valid if \(T \neq 1\). So let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq 1})\) be a stochastic basis such that \(\mathcal{F}_0\) contains all \(P\)-null sets of \(\mathcal{F}\),
\( F_{t+} = \bigcap_{\varepsilon > 0} F_{t+\varepsilon} = F_t, \forall t < 1 \), and we assume that the filtration is generated by the two following mutually independent processes:

- a \( d \)-dimensional Brownian motion \((B_t)_{t \leq 1}\),
- a random Poisson measure \( \mu \) on \( \mathbb{R}^+ \times U \), where \( U := \mathbb{R}^d \setminus \{0\} \) is equipped with its Borel fields \( U \), with compensator \( \nu(dt, de) = dt\lambda(de) \), such that \( \{\mu([0, t] \times A) = (\mu - \nu)([0, t] \times A)\}_{t \leq 1} \) is a martingale for every \( A \in U \) satisfying \( \lambda(A) < \infty \). The measure \( \lambda \) is assumed to be \( \sigma \)-finite on \((U, U)\) and such that the function \( e \in U \mapsto 1 + |e|^2 \) is \( \lambda \)-integrable.

Let us now introduce the following items:

- \( S^2 \) the set of \( F_t \)-adapted right continuous with left limit processes \((Y_t)_{t \leq 1}\) with values in \( \mathbb{R} \) and \( \mathbb{E}[\sup_{t \leq 1}|Y_t|^2] < \infty \). We denote by \( S^2_1 \) the subset of \( S^2 \) which contains non-decreasing processes \( K := (K_t)_{t \leq 1} \) with \( K_0 = 0 \).
- \( \mathcal{P} \) the \( \sigma \)-algebra of \( F_t \)-progressively measurable sets on \( \Omega \times [0, 1] \) and \( \mathcal{H}^{2,k} \) the set of \( \mathcal{P} \)-measurable processes \( Z := (Z_t)_{t \leq 1} \) with values in \( \mathbb{R}^k \) and \( dP \otimes dt \)-square integrable
- \( \mathcal{P}^d \) the \( \sigma \)-algebra of \( F_t \)-predictable sets on \( \Omega \times [0, 1] \) and \( \mathcal{L}^2 \) the set of mappings \( V : \Omega \times [0, 1] \times U \to \mathbb{R} \), \( \mathcal{P}^d \otimes \mathcal{U} \)-measurable and such that \( \mathbb{E}[\int_0^1 ds \int_U |V_s(e)|^2 \lambda(de)] < \infty \)
- \( T \) the set of \( F_t \)-stopping times with values in \([0, 1]\)
- for a given \( rcll \) process \((w_t)_{t \leq 1}\), for any \( t \leq 1 \), \( w_{t-} = \lim_{s \uparrow t} w_s \) \( (w_{0-} = w_0) \), \( \Delta_t w = w_t - w_{t-} \) and \( w_- := (w_{t-})_{t \leq 1} \).

We are now given three objects:

- a terminal value \( \xi \in L^2(\Omega, F_1, P) \)
- a map \( f : \Omega \times [0, 1] \times \mathbb{R}^{1+d} \times L^2(U, \mathcal{U}, \lambda; \mathbb{R}) \to \mathbb{R} \) which with \((t, \omega, y, z, v)\) associates \( f(t, \omega, y, z, v) \) and which is \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{1+d}) \otimes \mathcal{B}(L^2(U, \mathcal{U}, \lambda; \mathbb{R})) \)-measurable. In addition we assume that:
  
  (i) the process \((f(t, 0, 0, 0))_{t \leq 1}\) belongs to \( \mathcal{H}^{2,1} \)

  (ii) \( f \) is uniformly Lipschitz with respect to \((y, z, v)\), i.e., there exists a constant \( C_f \geq 0 \) such that for any \((y, z), (y', z') \in \mathbb{R} \times \mathbb{R}^d \) and \( v, v' \in L^2(U, \mathcal{U}, \lambda; \mathbb{R}) \) we have:
  
  \[ P - a.s., \quad |f(t, \omega, y, z, v) - f(t, \omega, y', z', v')| \leq C_f(|y - y'| + |z - z'| + ||v - v'||). \]

- an "obstacle" process \( S := (S_t)_{t \leq 1} \), which is real valued, \( rcll \) and \( \mathcal{P} \)-measurable process satisfying:

  \[ \mathbb{E}[\sup_{0 \leq t \leq 1} (S_t)^2] < +\infty. \]

Let us now introduce the reflected BSDE with general jumps associated with \((f, \xi, S)\). A solution is a quadruple \((Y, Z, K, V) := (Y_t, Z_t, K_t, V_t)_{t \leq 1}\) of processes with values in \( \mathbb{R}^{1+d} \times \mathbb{R}^+ \times L^2(U, \mathcal{U}, \lambda; \mathbb{R}) \) such that:

\[
\begin{aligned}
(i) & \quad Y \in S^2, \quad Z \in \mathcal{H}^{2,d}, \quad V \in \mathcal{L}^2 \text{ and } K \in S^2_1 \\
(ii) & \quad Y_t = \xi + \int_1^t f(s, Y_s, Z_s, V_s)ds + K_1 - K_t - \int_t^1 Z_s dB_s - \int_t^1 V_s(\bar{e})\tilde{\mu}(ds, de), \forall t \leq 1 \\
(iii) & \quad Y \geq S \\
(iv) & \quad \text{if } K^c \text{ (resp. } K^d \text{) is the continuous (resp. purely discontinuous) part of } K, \text{ then } K^d \text{ is } \mathcal{P}^d \text{-measurable, } \int_0^1 (Y_t - S_t)dK^c_t = 0 \text{ and } \forall t \leq 1, \Delta K^d_t = (S_{t-} - Y_t)1_{[Y_{t-} = S_{t-}]}. 
\end{aligned}
\]
The main reason for the second part of (iv) is that the process $Y$ has two types of jumps. The inaccessible ones which stem from the Poisson martingale part $(\int_0^t V_s(e)\tilde{\mu}(ds, de))_{t \leq 1}$ and the predictable ones which come from the predictable negative jumps of $S$. Those latter are the source of the predictable jumps of $Y$ and then also of $K$, which of course are the same. Thus the condition $\Delta K^d_t = (S_t - Y_t)^+1_{[Y_t = S_t]}$ is just a characterization of the predictable jumps of $Y$.

Remark 2.1: The second part of condition (iv) implies in particular that

$$\int_0^1 (Y_{t^-} - S_{t^-})dK_t = 0. \quad (2)$$

Actually

$$\int_0^1 (Y_{t^-} - S_{t^-})dK_t = \int_0^1 (Y_{t^-} - S_{t^-})dK^d_t + \int_0^1 (Y_{t^-} - S_{t^-})dK^d_t$$

$$= \int_0^1 (Y_t - S_t)dK_t^d + \sum_{t \leq 1} (Y_t - S_t)\Delta K^d_t = 0.$$

The last term of the second equality is null since $K^d$ jumps only when $Y_t = S_t$. ⊓⊔

To begin with, we are going to focus on the uniqueness of the solution of the BSDE (1).

3 Uniqueness

Proposition 3.1: Under the above assumptions on $f$, $\xi$, and $(S_t)_{t \leq 1}$, the reflected BSDE (1) associated with $(f, \xi, S)$ has at most one solution.

Proof: Let us consider two solutions $(Y, Z, K, V)$ and $(Y', Z', K', V')$ of (1). First let us assume that for any $t \leq 1$ we have $(Y_t - Y'_t)(dK_t - dK'_t) \leq 0$. Now for $t \leq 1$ we set

$$\Delta_t = |Y_t - Y'_t|^2 + \int_1^1 |Z_s - Z'_s|^2 ds + \int_1^1 (V_s(e) - V'_s(e))^2\lambda(de)ds.$$

Then applying Itô’s with $(Y - Y')^2$ and taking expectation yield:

$$\mathbb{E}[\Delta_t] \leq 2 \int_1^1 \mathbb{E}[(Y_s - Y'_s)(f(s, Y_s, Z_s, V_s) - f(s, Y'_s, Z'_s, V'_s))] ds, \forall t \leq 1.$$

Therefore because of the inequality $2|ab| \leq 2a^2 + \frac{b^2}{2}$ for any $a, b \in \mathbb{R}$ and since $f$ is uniformly Lipschitz we obtain: $\forall t \leq 1$,

$$\mathbb{E}[|Y_t - Y'_t|^2 + \frac{1}{2} \int_1^1 |Z_s - Z'_s|^2 ds + \frac{1}{2} \int_1^1 (V_s(e) - V'_s(e))^2 \lambda(de)ds]$$

$$\leq (2C_f + 4C_f^2) \mathbb{E}[\int_1^1 (Y_s - Y'_s)^2 ds]. \quad (3)$$

Then from Gronwall’s lemma and the right continuity of $Y - Y'$ we get $Y = Y'$. It follows from (3) that $(Z, V) = (Z', V')$ and finally $K = K'$. Whence the uniqueness of the solution of (1).

To complete the proof it remains to show that $\int_{[t, 1]} (Y_{s^-} - Y'_{s^-})(dK_s - dK'_s) \leq 0, \forall t \leq 1$. Actually for any $t \leq 1$ we have,

$$\int_{[t, 1]} (Y_{s^-} - Y'_{s^-})(dK_s - dK'_s) = \int_{[t, 1]} (Y_{s^-} - Y'_{s^-})(dK^c_s - dK'^c_s) + \int_{[t, 1]} (Y_{s^-} - Y'_{s^-})(dK^d_s - dK'^d_s). \quad (4)$$
But as \( Y \) and \( Y' \) belong to \( S^2 \) and their jumps \( \delta(\omega) := \{ t \in [0, 1], \Delta_t Y(\omega) \neq 0 \} \) and \( \delta'(\omega) := \{ t \in [0, 1], \Delta_t Y'(\omega) \neq 0 \} \) are at most countable then:

\[
\int_{[t,1]} (Y_{s-} - Y'_{s-})(dK^c_s - dK^{tc}_s) = \int_{[t,1]} (Y_s - Y'_s)(dK^c_s - dK^{tc}_s) = \int_{[t,1]} (Y_s - S_s)(dK^c_s - dK^{tc}_s) \\
+ \int_{[t,1]} (S_s - Y'_s)(dK^c_s - dK^{tc}_s) = - \int_{[t,1]} (Y_s - S_s)dK^c_s + \int_{[t,1]} (S_s - Y'_s)dK^{tc}_s \leq 0. \tag{5}
\]

Additionally we have:

\[
\int_{[t,1]} (Y_{s-} - Y'_{s-})(dK^d_s - dK^{rd}_s) = \int_{[t,1]} (Y_s - Y'_s) dK^d_s - \int_{[t,1]} (Y_s - Y'_s) dK^{rd}_s. \tag{6}
\]

However

\[
\int_{[t,1]} (Y_{s-} - Y'_{s-})dK^{rd}_s = \int_{[t,1]} (Y_{s-} - S_{s-})dK^{rd}_s \geq 0 \tag{7}
\]

since the jumps of \( K^{rd} \) occur only when \( Y' = S_- \). In the same way we have \( \int_{[t,1]} (Y_{s-} - Y'_{s-})dK^d_s \leq 0 \) and then \( \int_{[t,1]} (Y_{s-} - Y'_{s-})(dK^d_s - dK^{rd}_s) \leq 0 \). This inequality and (5) lead to the inequality in (4) which is the desired result, whence uniqueness. \( \square \)

### 4 Existence of the solution

We are now going to show that equation (1) has a solution. The method we used is a combination of the penalization method and the Snell envelope one. The penalization, as it has been used e.g. in [6], cannot be applied since we do not have an efficient comparison theorem for solutions of BSDEs whose noise contains a Poisson part and whose coefficients \( f \) depend on \( v \) (see e.g. [1] for a counter-example).

This is the main reason for which, in a first step, we suppose that the function \( f(t, \omega, y, z, v) \) does not depend on \( v \). However for the sake of simplicity we are going to suppose moreover that \( f \) does not depend also on \( (y, z) \), i.e., \( f(t, \omega, y, z, v) = g(t, \omega) \). But we should emphasize that if \( f \) depends on \( (y, z) \) the method still work in its main steps. We just need some minor adaptations. Later to obtain the result in the general setting we will use a fixed point argument with an appropriate mapping. Finally note that as a by-product of the penalization method is the approximation of the solution of the reflected equation by solutions of standard BSDEs, i.e. without reflection. This remark is important especially when we deal with the issue of numerical schemes.

To begin with let us assume that the function \( f \) does not depend on \( (y, z, v) \), i.e., P-a.s., \( f(t, \omega, y, z, v) \equiv g(t, \omega) \), for any \( t, y, z \) and \( v \). In the following result, we establish the existence of the solution of the BSDE associated with \((g, \xi, S)\).

**Theorem 4.1** : The BSDE with a reflecting barrier (1) associated with \((g, \xi, S)\) has a unique solution \((Y_t, Z_t, K_t, V_t)_{t \leq 1}\).

**Proof**: For \( n \geq 0 \), let \((Y^n_t, Z^n_t, V^n_t)_{t \leq 1}\) be the \( F_t \)-adapted process with values in \( \mathbb{R}^{1+d} \times L^2(U, \mathcal{U}; \Lambda; \mathbb{R}) \), unique solution of the BSDE associated with \((g(t, \omega) + n(y - S_t)^-, \xi) \) \((\xi^- \equiv \max(0, -x), \forall x \in \mathbb{R})\), which exists according to the results either by Tang & Li [18] or Barles et al. [1], i.e.,

\[
\begin{cases}
Y^n \in \mathcal{S}^2, Z^n \in \mathcal{H}^{2,d} 	ext{ and } V^n \in \mathcal{L}^2 \\
Y^n_t = \xi + \int_t^1 g(s)ds + \int_t^1 n (Y^n_s - S_s)^-ds - \int_t^1 Z^n_s dB_s - \int_t^1 V^n_s (e) \tilde{\mu}(ds, de), \forall t \leq 1.
\end{cases} \tag{8}
\]
From now on the proof will be divided into four steps.

**Step 1:** For any \(n \geq 0\), \(Y^n \leq Y^{n+1}\).

For any \(t \leq 1\), we have:

\[
Y^n_t - Y^{n+1}_t = \int_t^1 (Z^n_s - Z^{n+1}_s) dB_s + \int_t^1 \int_U (V^n_s(e) - V^{n+1}_s(e)) \mu(ds, de) \\
+ \int_t^1 \{n(Y^n_s - S_s) - (n+1)(Y^{n+1}_s - S_s)\} ds.
\]

But \(n(Y^n_s - S_s) - (n+1)(Y^{n+1}_s - S_s) = b^n_s + a^n_s(Y^n_s - Y^{n+1}_s)\) where \(b^n_s \leq 0\) and \(|a^n_s| \leq n+1, \forall s \leq 1\).

So let us set \(\Theta^n_t = e^{b^n_0 a^n ds}, t \leq 1\); then using Itô’s formula we obtain:

\[
d(\Theta^n_t (Y^n_t - Y^{n+1}_t)) = \Theta^n_t b^n dt + dM^n_t, \ t \leq 1
\]

where \((M^n_t)_{t \leq T}\) is a martingale. Taking now the conditional expectation w.r.t. \(\mathcal{F}_t\) we obtain \(Y^n \leq Y^{n+1}\) since \(Y^{n+1}_1 - Y^n_1 = 0\), \(\Theta^n \geq 0\) and \(b^n \leq 0\).

Finally let us point out that this is quite immediate since the function \(f\) does not depend on \((y, z)\). However if so, comparison of solutions of the penalization scheme still valid (see e.g. [11], pp.6) for the proof of this claim. \(\square\)

**Step 2:** For any \(n \geq 0\), the process \(Y^n\) satisfies:

\[
\forall t \leq 1, \ Y^n_t = \esssup_{s \geq t} \mathbb{E}\left[\int_t^s g(s)ds + (Y^n_s \wedge S_s)1_{[r<1]} + \xi 1_{[r=1]}|\mathcal{F}_t\right].
\]

Actually for any \(n \geq 0\) and \(t \leq 1\) we have:

\[
Y^n_t = \xi + \int_t^1 g(s)ds - \int_t^1 Z^n_s dB_s + \int_t^1 n(Y^n_s - S_s) - ds - \int_t^1 \int_U V^n_s(e) \mu(ds, de). \tag{9}
\]

Therefore for any stopping time \(\tau \geq t\) we have:

\[
Y^n_{\tau} = \mathbb{E}[(Y^n_{\tau} + \int_{\tau}^\infty g(s)ds + \int_{\tau}^\infty \{n(Y^n_s - S_s) - ds|\mathcal{F}_\tau\]
\geq \mathbb{E}[(S_{\tau} \wedge Y^n_{\tau})1_{[r<1]} + \xi 1_{[r=1]} + \int_{\tau}^\infty g(s)ds|\mathcal{F}_\tau] \tag{10}
\]

since \(Y^n_{\tau} \geq (S_{\tau} \wedge Y^n_{\tau})1_{[r<1]} + \xi 1_{[r=1]}\). On the other hand let \(\tau^*_t\) be the stopping time defined as follows:

\[
\tau^*_t = \inf\{s \geq t, K^n_s - K^n_t > 0\} \wedge 1
\]

where \(K^n_t = \int_0^t n(Y^n_s - S_s) - ds\). Let us show that \(1_{[\tau^*_t<1]}Y^n_{\tau^*_t} = 1_{[\tau^*_t<1]}Y^n_{\tau^*_t} \wedge S^n_{\tau^*_t}\).

Let \(\omega\) be fixed such that \(\tau^*_t(\omega) < 1\). Then there exists \(t_0(\omega) > \tau^*_t(\omega)\) such that \(\forall s \in [\tau^*_t(\omega), t_0(\omega)]\), \(K^n_s(\omega) > K^n_{\tau^*_t(\omega)}(\omega) = K^n_{t_0}(\omega)\) (this last equality is due to continuity of \(K^n\)). It implies that there exists a real sequence \((t_k)_{k \geq 0}\) which decreases strictly to \(\tau^*_t\) such that \((Y^n_{t_k} - S_{t_k})^- > 0\) i.e. \(Y^n_{t_k} \leq S_{t_k}\). Taking now the limit as \(k \rightarrow \infty\) we obtain that \(Y^n_{\tau^*_t} \leq S_{\tau^*_t}\) since \(Y^n\) and \(S\) are *cgl* processes. In other words we have also \(1_{[\tau^*_t<1]}Y^n_{\tau^*_t} = 1_{[\tau^*_t<1]}Y^n_{\tau^*_t} \wedge S_{\tau^*_t}\).

Now from (9) we deduce that:

\[
Y^n_{\tau^*_t} = \int_{\tau^*_t}^{\tau^*_t} g(s)ds - \int_{\tau^*_t}^{\tau^*_t} Z^n_s dB_s - \int_{\tau^*_t}^{\tau^*_t} \int_U V^n_s(e) \mu(ds, de) \\
= 1_{[\tau^*_t<1]}Y^n_{\tau^*_t} \wedge S_{\tau^*_t} + \xi 1_{[\tau^*_t=1]} + \int_{\tau^*_t}^{\tau^*_t} g(s)ds - \int_{\tau^*_t}^{\tau^*_t} Z^n_s dB_s - \int_{\tau^*_t}^{\tau^*_t} \int_U V^n_s(e) \mu(ds, de).
\]
Taking the conditional expectation and using inequality (10) we obtain: \( \forall n \geq 0 \) and \( t \leq 1 \),

\[
Y^n_t = \text{esssup}_{\tau \geq t} \mathbb{E} \left[ \int_t^\tau g(s) ds + (S_\tau \wedge Y^n_\tau) 1_{\{\tau < 1\}} + \xi 1_{\{\tau = 1\}} | \mathcal{F}_t \right].
\]

(11)

**Step 3:** There exists a rcll process \((Y_t)_{t \leq 1}\) of \( \mathcal{S}^2 \) such that: P-a.s.,

(i) \( Y = \mathcal{H}^{2,1} - \lim_{n \to \infty} Y^n \), \( S \leq Y \) and finally for any \( t \leq 1 \), \( Y^n_t \neq Y_t \).

(ii) for any \( t \leq 1 \),

\[
Y_t = \text{esssup}_{\tau \geq t} \mathbb{E} \left[ \int_t^\tau g(s) ds + S_\tau 1_{\{\tau < 1\}} + \xi 1_{\{\tau = 1\}} | \mathcal{F}_t \right] \quad (Y_1 = \xi).
\]

(12)

Actually for \( t \leq 1 \) let us set

\[
\tilde{Y}_t = \text{esssup}_{\tau \geq t} \mathbb{E} \left[ \int_t^\tau g(s) ds + S_\tau 1_{\{\tau < 1\}} + \xi 1_{\{\tau = 1\}} | \mathcal{F}_t \right].
\]

The process \( \tilde{Y} \) belongs to \( \mathcal{S}^2 \) since \( S \) is so, \( g \in \mathcal{H}^{2,1} \) and \( \xi \) is square integrable. On the other hand for any \( n \geq 0 \) and \( t \leq 1 \) we have \( Y^n_t \leq \tilde{Y}_t \). Thus there exits a \( \mathcal{P} \)-measurable process \( Y \) such that P-a.s. for any \( t \leq 1 \), \( Y^n_t \neq Y_t \) and then \( Y = \mathcal{H}^{2,1} - \lim_{n \to \infty} Y^n \). Besides the process \( (Y^n_t + \int_0^t g_s ds)_{t \leq 1} \) is a rcll supermartingale as a Snell envelope of \((f^n, \xi^n, S^n)\) to \( \mathcal{H}^{2,1} \). Henceforth the process \( Y \) is also rcll and belongs to \( \mathcal{S}^2 \) since it is dominated by \( \tilde{Y} \) which is an element of \( \mathcal{S}^2 \).

Next let us prove that \( Y \geq S \). Through (8) we have:

\[
\mathbb{E}[Y^n_0] = \mathbb{E}[\xi + \int_0^1 g(s) ds] + \mathbb{E}\left[ \int_0^1 n (Y^n_s - S_s)^- ds \right].
\]

Dividing the two hands-sides by \( n \) and taking the limit as \( n \to \infty \) to obtain \( \mathbb{E}[\int_0^1 (Y_s - S_s)^- ds] = 0 \). As the processes \( Y \) and \( S \) are rcll then P-a.s., \( Y_1 \geq S_t \) for \( t < 1 \). But \( Y_1 = \xi \geq S_1 \), therefore \( Y \geq S \).

Finally let us show that \( Y \) satisfies (12). But this a direct consequence of the continuity of the Snell envelope through sequences of increasing rcll processes (see Appendix [A1]). Actually on the one hand, the sequence of increasing rcll processes \((S_\tau \wedge Y^n_\tau) 1_{\{\tau < 1\}} + \xi 1_{\{\tau = 1\}}\) converges increasingly to the rcll \((S_\tau 1_{\{\tau < 1\}} + \xi 1_{\{\tau = 1\}})\) since \( Y \geq S \). Therefore because of (11) the sequence \((\int_0^t g_s ds + Y^n_t)_{t \leq 1} \) converges to \( \text{esssup}_{\tau \geq t} \mathbb{E} \left[ \int_t^\tau g(s) ds + S_\tau 1_{\{\tau < 1\}} + \xi 1_{\{\tau = 1\}} | \mathcal{F}_t \right] \) which then is equal to \((\int_0^t g_s ds + Y_t)_{t \leq 1} \) and which implies that \( Y \) satisfies (12). \( \square \)

**Step 4:** There exist three processes \( Z \in \mathcal{H}^{2,d}, V \in \mathcal{L}^2 \) and \( K \in \mathcal{S}^2 \) such that \((Y, Z, V, K)\) is a the solution of the BSDE associated with \((f, \xi, S)\). In addition \((Z^n)_{n \geq 0}\) (resp. \((V^n)_{n \geq 0}\)) converges in \( \mathcal{H}^{2,d} \) (resp. \( \mathcal{L}^2 \)) to \( Z \) (resp. \( V \)) and for any stopping time \( \tau \) the sequence of processes \((\int_0^\tau n (Y^n_s - S_s)^- ds)_{n \geq 0}\) converges in \( L^2(\Omega, dP) \) to \( K_\tau \).

We know from (12), that the process \((\int_0^t g_s ds + Y_t)_{t \leq 1}\) is a Snell envelope. Then through Appendix [A2], there exist a process \( K \in \mathcal{S}^2 \) \((K_0 = 0)\) and an \( \mathcal{F}_t \)-martingale \((M_t)_{t \leq 1}\) which belongs to \( \mathcal{S}^2 \) such that:

\[
\forall t \leq 1, \quad \int_0^t g_s ds + Y_t = M_t - K_t.
\]
Additionally $K = K^c + K^d$ where $K^c$ is continuous non-decreasing and $K^d$ non-decreasing purely discontinuous predictable and such that for any $t \leq 1$, $\Delta_t K^d = (S_{t-} - Y_t)^+ 1_{\{Y_t = S_{t-}\}}$.

Now the martingale $M$ belongs to $\mathcal{S}^2$ then the representation property (see e.g. [12]) implies the existence of two processes $Z$ and $V$ which belong respectively to $\mathcal{H}^{2,d}$ and $\mathcal{L}^2$ such that:

$$P\text{-a.s., } \forall t \leq 1, \ M_t = M_0 + \int_0^t \{Z_s dB_s + \int_U V_s(e) \tilde{\mu}(ds, de)\}.$$ 

Let us now show that $\int_0^1 (Y_s - S_s) dK^c_s = 0$. First let us remark that the Snell envelope of $(\int_0^t g_s ds + S_{1[t<1]} + \xi_{1[t=1]} + K^d_t)_{t \leq 1}$ is nothing else but $(\int_0^t g_s ds + Y_t + K^d_t)_{t \leq 1}$.

Actually for any $t \leq 1$ we have $\int_0^t g_s ds + Y_t = M_t - K^c_t - K^d_t$, therefore the process $(\int_0^t g_s ds + Y_t + K^d_t)_{t \leq 1}$ is also a right supermartingale which dominates the process $(\int_0^t g_s ds + S_{1[t<1]} + \xi_{1[t=1]} + K^d_t)_{t \leq 1}$. Besides if $(N_t)_{t \leq T}$ is a supermartingale of class $[D]$ which dominates this latter process then $(N_t - K^d_t)_{t \leq T}$ still a supermartingale of class $[D]$ which is greater than $(\int_0^t g_s ds + S_{1[t<1]} + \xi_{1[t=1]} + K^d_t)_{t \leq 1}$. Therefore $P\text{-a.s.}$ for any $t \leq 1$ we have $N_t - K^d_t \geq \int_0^t g_s ds + Y_t$ which implies that $\forall t \leq 1$, $N_t \geq \int_0^t g_s ds + Y_t + K^d_t$. It means that the process $(\int_0^t g_s ds + Y_t + K^c_t)_{t \leq T}$ is the smallest supermartingale of class $[D]$ which dominates $(\int_0^t g_s ds + S_{1[t<1]} + \xi_{1[t=1]} + K^d_t)_{t \leq 1}$ and then it is its Snell envelope.

Now the Snell envelope $(\int_0^t g_s ds + Y_t + K^c_t = M_t - K^c_t)_{t \leq 1}$ of the process $(\int_0^t g_s ds + S_{1[t<1]} + \xi_{1[t=1]} + K^d_t)_{t \leq 1}$ is regular (see Appendix A3) then for any $t \leq 1$, the stopping time $\tau_t = \inf\{s \geq t, K_s > K_t\} \wedge 1$ is optimal (see [A3]) therefore we have $\int_{\tau_t} (Y_s + K^d_t - S_s - K^c_t) dK^c_s = \int_{\tau_t} (Y_s - S_s) dK^c_s = 0$. As $t$ is arbitrary then we obtain $\int_0^1 (Y_s - S_s) dK^c_s = 0$.

Collecting now all those properties yields that the quadruple $(Y, Z, V, K)$ is a solution for the BSDE associated with $(f, \xi, S)$, i.e.,

$$\begin{cases}
Y \in \mathcal{S}^2, Z \in \mathcal{H}^{2,d}, V \in \mathcal{L}^2 \text{ and } K \in \mathcal{S}^2 \\
Y_t = \xi + \int_t^1 g(s) ds + K_1 - K_t - \int_t^1 Z_s dB_s - \int_t^1 U_s(e) \tilde{\mu}(ds, de), \forall t \leq 1 \\
Y \geq S \text{ and if } K^c \text{ (resp. } K^d) \text{ is the continuous (resp. purely discontinuous) part of } K, \text{ then } K^d \text{ is } \mathcal{P}^d\text{-measurable, } \int_0^1 (Y_t - S_t) dK^d_t = 0 \text{ and } \forall t \leq 1, \Delta K^d_t = (S_t - Y_t)^+ 1_{\{Y_t = S_t\}}.
\end{cases}$$

We now focus on the convergence of the sequences $(Z^n)_{n \geq 0}$ and $(V^n)_{n \geq 0}$. Applying Itô's formula with $(Y - Y^n)^2$ yields: $\forall t \leq 1$,

$$(Y_t - Y^n_t)^2 + \int_t^1 |Z_s - Z^n_s|^2 ds + \sum_{t<s\leq 1} (\Delta_s (Y - Y^n)^2) = 2 \int_t^1 (Y_s - Y^n_s)(dK^c_s - dK^n_s) - 2 \int_t^1 (Y_s - Y^n_s)((Z_s - Z^n_s)dB_s + \int_U (V_s(e) - V^n_s(e)) \tilde{\mu}(ds, de)).$$

Next taking expectation in both hand-sides and using the fact that $\int_t^1 (Y_s - Y^n_s)dK^n_s \geq 0$ we obtain:

$$\mathbb{E}\int_t^1 |Z_s - Z^n_s|^2 ds + \mathbb{E}|Y_t - Y^n_t|^2 + \sum_{t<s\leq 1} (\Delta_s (Y - Y^n)^2) \leq 2 \mathbb{E}\int_t^1 (Y_s - Y^n_s) dK^c_s.$$  \hspace{1cm} (13)

But for any $t \leq 1$,

$$\int_t^1 (Y_s - Y^n_s)dK^c_s = \int_t^1 (Y_s - Y^n_s)dK^c_s + \int_t^1 (Y_s - Y^n_s)dK^d_s.$$
and the first term in the right hand-side converges in $L^1(\Omega, dP)$ to 0 since $P$ - a.s., for any $t \leq 1$, $Y_t^n / \sim Y_t$. On the other hand

$$f_t^n(Y_{s-} - Y_{s-}^n) dK_s^d = f_t^n(Y_{s-} - S_{s-}) dK_s^d + f_t^n(S_{s-} - Y_{s-}^n) dK_s^d$$

$$= f_t^n(S_{s-} - Y_{s-}^n) dK_s^d$$

because of Remark 2.1. But $\lim_{n \to \infty} f_t^n(S_{s-} - Y_{s-}^n) dK_s^d = \lim_{n \to \infty} f_t^n(S_{s-} - Y_{s-}) dK_s^d = \sum_{t<\tau\leq 1} (\Delta_s (K^d)^2$ and then through monotone convergence theorem $\mathbb{E}[\int_t^1 (Y_{s-} - Y_{s-}^n) dK_s^d]$ converges to $\mathbb{E}[^{\sum_{t<\tau\leq 1} (\Delta_s (K^d)^2]}$ as $n \to \infty$. On the other hand,

$$\sum_{t<\tau\leq 1} (\Delta_s (Y - Y^n))^2 = \int_t^1 \int_{U} (V_s(e) - V_s^n(e))^2 \mu(ds, de) + \sum_{t<\tau\leq 1} (\Delta_s (K^d)^2)$$

since the jumps of $Y$ are of two types, the inaccessible ones which stem from the Poisson martingale part and the predictable ones which are the same as the ones of $K^d$. The process $Y^n$ has only inaccessible jumps. It implies that:

$$\mathbb{E}[\sum_{t<\tau\leq 1} (\Delta_s (Y - Y^n))^2] = \mathbb{E}[\int_t^1 ds \int_{U} (V_s(e) - V_s^n(e))^2 \lambda(de) + \sum_{t<\tau\leq 1} (\Delta_s (K^d)^2)].$$

Going back to (13) and taking the limit in both hand-sides to obtain that

$$\mathbb{E}[\int_0^1 ds \{ |Z_s - Z_s^n|^2 + \int_{U} (V_s(e) - V_s^n(e))^2 \lambda(de) \}] \to 0 \text{ as } n \to \infty. \quad (14)$$

Finally for any stopping time $\tau$ the convergence of the $(\int_0^\tau n(Y_{s-} - S_{s-}^{-} ds)_{n \geq 0})$ to $K_\tau$ in $L^2(\Omega, \mathcal{F}, dP)$ is a direct consequence of the convergence of $(Y_{r^n})_{n \geq 0}$ to $Y_\tau$ in the same space and Burkholder-Davis-Gundy inequality applied to the supremum of the martingale part of $Y - Y^n$ and the convergence property (14). \(\Box\)

**Remark 4.1** : (i) In the proof of the convergence of $(Z^n)_{n \geq 0}$ (resp. $(V^n)_{n \geq 0}$) to $Z$ (resp. $V$) in $H^{2,d}$ (resp. $\mathcal{L}^2$), the fact that $(Y_t + \int_0^t g_s ds)_{t \leq 1}$ is a Snell envelope of some specific process, and then its jumps are well characterized, plays a crucial role. In [14] or [13], in the context of Brownian noise only, the authors have been just able to show that $(Z^n)_{n \geq 0}$ converges to $Z$ in $L^p(\Omega \times [0, 1], dP \otimes dt)$ for $p \in [1, 2[$.

(ii) The previous method still work if the function $f$ depends on $y$ and non-negative. \(\Box\)

We are now ready to give the main result of this section.

**Theorem 4.2** : The reflected BSDE with generalized jumps (1) associated with $(f, \xi, S)$ has a unique solution $(Y, Z, K, V)$.

**Proof**: It remains to show existence which will be obtained via a fixed point argument. Actually let $\mathcal{D} := H^{2,1} \times H^{2,d} \times \mathcal{L}^2$ endowed with the norm

$$\| (Y, Z, V) \|_\alpha = \{ \mathbb{E}[\int_0^1 e^{\alpha s} (|Y_s|^2 + |Z_s|^2 + \int_{U} |V_s(e)|^2 \lambda(de)) ds] \}^{1/2} \; \alpha > 0.$$
On the other hand let Φ be the map from \( D \) into itself which with \((Y, Z, V)\) associates
\[
\Phi(Y, Z, V) = (\tilde{Y}, \tilde{Z}, \tilde{V}) \text{ where } (\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{V}) \text{ is the solution of the reflected BSDE associated with } (f(t,Y_t,Z_t,V_t), \xi, S).
\]
Let \((Y', Z', V')\) be another triple of \( D \) and \(\Phi(Y', Z', V') = (\tilde{Y}', \tilde{Z}', \tilde{V}')\). Using Itô’s formula we obtain: \( \forall t \leq 1 \),
\[
e^{\alpha t}(\tilde{Y}_t - \tilde{Y}'_t)^2 + \alpha \int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)^2 ds + \int_t^1 e^{\alpha s}|\tilde{Z}_s - \tilde{Z}'_s|^2 ds + \\
\sum_{t<s \leq 1} e^{\alpha s}(\Delta_s \tilde{Y} - \Delta_s \tilde{Y}')^2 = (M_1 - M_t) + 2 \int_t^1 e^{\alpha s}(\tilde{Y}_{s-} - \tilde{Y}'_{s-})(d\tilde{K}_s - d\tilde{K}'_s) + \\
2 \int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)(f(s,Y_s,Z_s,V_s) - f(s,Y'_s,Z'_s,V'_s)) ds
\]
where \(M_t)_{t \leq 1}\) is a martingale. But for any \( t \leq 1 \), \( \int_t^1 e^{\alpha s}(\tilde{Y}_{s-} - \tilde{Y}'_{s-})(d\tilde{K}_s - d\tilde{K}'_s) \leq 0 \). This can be shown as in the proof of uniqueness in Proposition 3.1. Therefore taking expectation in both hand-sides yields
\[
\alpha \mathbb{E}\left[ \int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)^2 ds \right] + \mathbb{E}\left[ \int_t^1 e^{\alpha s}|\tilde{Z}_s - \tilde{Z}'_s|^2 ds \right] + \mathbb{E}\left[ \int_t^1 e^{\alpha s} ds \int_U (\tilde{V}_s(e) - \tilde{V}'_s(e))^2 \lambda(de) \right] \\
\leq 2 \mathbb{E}\left[ \int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)(f(s,Y_s,Z_s,V_s) - f(s,Y'_s,Z'_s,V'_s)) ds \right] \\
\leq k \epsilon \mathbb{E}\left[ \int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)^2 ds \right] + \frac{k}{\epsilon} \mathbb{E}\left[ \int_t^1 e^{\alpha s}|Y_s - Y'_s|^2 + |Z_s - Z'_s|^2 + \\
\int_U |V_s(e) - V'_s(e)|^2 \lambda(de) \right] ds.
\]
It implies that
\[
(\alpha - k\epsilon) \mathbb{E}\left[ \int_t^1 e^{\alpha s}(\tilde{Y}_s - \tilde{Y}'_s)^2 ds \right] + \mathbb{E}\left[ \int_t^1 e^{\alpha s} ds \int_U (\tilde{V}_s(e) - \tilde{V}'_s(e))^2 \lambda(de) \right] \\
\leq \mathbb{E}\left[ \int_t^1 e^{\alpha s}(\tilde{Z}_s - \tilde{Z}'_s)^2 ds \right] + \\
\frac{k}{\epsilon} \mathbb{E}\left[ \int_t^1 e^{\alpha s}|Y_s - Y'_s|^2 + |Z_s - Z'_s|^2 + \int_U |V_s(e) - V'_s(e)|^2 \lambda(de) \right] ds.
\]
Now let \( \alpha \) be great enough and \( \epsilon \) such that \( k < \epsilon < \frac{\alpha - 1}{\alpha} \), then \( \Phi \) is a contraction on \( D \). Therefore there exists a triple \((Y, Z, V)\) such that \( \Phi(Y, Z, V) = (Y, Z, V) \) which, with \( K \), is the unique solution of the reflected BSDE associated with \((f, \xi, S)\) since \( Y \in \mathcal{S}^2 \). \( \square \)

### 4.1 Regularity of the process \( K \)

We now focus on the regularity of the process \( K \) of the solution of the BSDE (1).

**Proposition 4.1** Let \((Y, Z, K, V)\) be the unique solution of the reflected BSDE associated with \((f(t,y,z,v), \xi, S)\) and let us consider the following assertions:

(i) \( K \) is continuous \((K^d = 0)\)
(ii) \((Y^n)_{n \geq 0}\) converges uniformly to \( Y \)
(iii) \( \mathbb{E}\left[ \int_0^1 |Z^n_s - Z_s|^2 ds + \int_0^1 \int_U |V^n_s(e) - V_s(e)|^2 \lambda(de) \right] \to 0. \)

Then it holds true that (i) and (ii) are equivalent and the statement (ii) implies (iii).
Proof: (i) ⇒ (ii): Let us assume that (i) is fulfilled. Therefore the jumps of $Y$ are the same as the ones of its Poisson martingale part. It follows that for any $t \leq 1$, $pY_t = Y_{t-}$ where $pY$ is the predictable projection of $Y$. Because in that case $Y$ has only inaccessible jumps. But $Y^n \not\nearrow Y$, thus $pY^n \not\nearrow pY$, then for any $t \leq 1$ we have $Y^n_{t-} \not\nearrow Y_{t-}$. It follows from a generalized Dini theorem (see [4], pp.203) that P-a.s. the sequence $(Y^n)_{n\geq 0}$ converges uniformly to $Y$.

(ii) ⇒ (i): If $(Y^n)_{n\geq 0}$ converges uniformly to $Y$ then $Y^n \not\nearrow Y_-$ and $pY^n \not\nearrow pY$. But $Y^n$ is rcell and has only inaccessible jumps, then $pY^n = Y^n$. It follows that $pY = Y_-$. Therefore the Snell envelope $(\int_0^t f(s,Y_s,Z_s,V_s)ds + Y_t)_{t\leq 1}$ is regular, thus $K^d = 0$ (see Appendix [A3]) which yields the desired result.

(ii) ⇒ (iii): If the sequence $(Y^n)_{n\geq 0}$ converges uniformly to $Y$ then through dominated convergence theorem we have also $E[\sup_{t\leq 1} (Y - Y^n)^2] \rightarrow 0$. So to obtain the result, it is enough to apply Itô’s formula with $(Y - Y^n)^2$ after having remarked that $E[\sup_{t\leq 1}((S_t - Y^n)^+)^2] \rightarrow 0$ since $Y \geq S$. Therefore we have:

$E[\int_0^1 |Z^n_s - Z_s|^2 ds + \int_0^1 \int_U |V^n_s(e) - V_s(e)|^2 \lambda(de)ds] \rightarrow 0$. □

Remark 4.2 : let us point out that the implication (iii) ⇒ (i) is not true in general. Actually let us consider the following deterministic counter-example. Assume $\xi = \frac{1}{2}$, $f \equiv 0$ and $S_t = 1_{[t<\frac{1}{2}]}$. Therefore the solution of the BSDE associated with $(f,\xi,S)$ is $Z \equiv 0$, $V \equiv 0$, $Y_t = 1_{[t<\frac{1}{2}]} + \frac{1}{2} 1_{[t\geq \frac{1}{2}]}$ and $K_t = K^d_t = \frac{1}{2} 1_{[\frac{1}{2} \leq t \leq 1]}$. Indeed it is easily seen that for any $t \leq 1$, we have $Y_t = \frac{1}{2} + K^d_t - K^d_t$. So obviously the statement (iii) holds true since $Z^n = Z = 0$, $V^n = V = 0$ but $K$ is not continuous. □

5 BSDEs with two discontinuous reflecting barriers

We now consider the problem of reflection with respect to two barriers, an upper and a lower ones. So let us give two processes $L := (L_t)_{t \leq T}$ and $U := (U_t)_{t \leq}$ which stand for the barriers where ”the solution” is reflected and which satisfy:

(i) $L$ and $U$ belong to $S^2$ and $P$-a.s., $\forall t \leq 1$, $L_t \leq U_t$ and $L_1 \leq \xi \leq U_1$

(ii) there exist two non-negative supermartingales $(h_t)_{t \leq 1}$ and $(h'_t)_{t \leq 1}$ of $\mathcal{S}^2$ such that:

$$\forall t \leq 1, \ L_t \leq h_t - h'_t \leq U_t.$$  

This condition is the so-called Mokobodski’s hypothesis

(iii) $\forall t < 1, L_{t-} < U_{t-}$ and $L_t < U_t$, □

Let us now introduce the BSDE associated with $(f,\xi,L,U)$. A solution is a quintuple of processes $(Y_t, Z_t, K^+_t, K^-_t, V_t)_{t \leq 1}$ which satisfies:

$$\left\{ \begin{array}{l}
(i) \ Y \in S^2, K^\pm \in S^2, Z \in \mathcal{H}^{2,d} \text{ and } V \in \mathcal{L}^2 \\
(ii) \ -dY_t = f(t,Y_t,Z_t,V_t)dt + dK^+_t - dK^-_t - Z_t dB_t - \int_U V_t(e) \tilde{\mu}(dt,de), \ t \leq 1; \ Y_1 = \xi \\
(iii) \ \forall t \leq 1, L_t \leq Y_t \leq U_t \text{ and if } K^{\pm c} \text{ is the continuous part of } K^\pm \text{ then } (Y_t - L_t) dK^+_{t} = 0 \\
\text{ and } (U_t - Y_t) dK^-_{t} = 0 \\
(iv) \text{ if } K^{\pm d} \text{ denotes the purely discontinuous part of } K^\pm \text{ then } K^{\pm d} \text{ is } \mathcal{P}^d\text{-measurable} \\
\text{and } \forall t \leq 1, \Delta K^+_{t} = (L_{t-} - Y_{t-})^+ 1_{[Y_{t-} = L_{t-}]} \text{ and } \Delta K^-_{t} = (Y_{t-} - U_{t-})^+ 1_{[Y_{t-} = U_{t-}]}.
\end{array} \right.$$  

(15)
Note that the BSDE (15) may have not a solution. Actually if for example \( L \) is not a semimartingale, and \( L, U \) coincide then obviously the equation cannot have a solution since \( Y \) is a semimartingale. \( \square \)

First we are going to focus on the issue of uniqueness of the solution of (15).

**Proposition 5.1 : Uniqueness**

*The BSDE with two reflecting barriers associated with \((f, \xi, L, U)\) (15) has at most one solution.*

*Proof:* Assume that \((Y, Z, K^\pm, V)\) and \((Y', Z', K'^\pm, V')\) are two solutions of (15). First let us show that for any \( t \leq 1 \), \( \int_t^1 (Y_s - Y'_s)(dK_s^c - dK'_s^c) \leq 0 \) where \( K = K^+ - K^- \) and \( K' = K'^+ - K'^- \).

For any \( t \leq 1 \) we have,

\[
\int_{[t,1]} (Y_s - Y'_s)(dK_s^c - dK'_s^c) = \int_{[t,1]} (Y_s - Y'_s)(dK_s^{c+} - dK_s^{c'-c}) + \int_{[t,1]} (Y_s - Y'_s)(dK_s^{d} - dK'_s^{d}).
\]

The processes \( Y \) and \( Y' \) belong to \( S^2 \) and their jumps \( \delta(\omega) := \{ t \in [0,1], \Delta_s Y(\omega) \neq 0 \} \) and \( \delta'(\omega) := \{ t \in [0,1], \Delta_s Y'(\omega) \neq 0 \} \) are at most countable. Therefore

\[
\int_{[t,1]} (Y_s - Y'_s)(dK_s^{c+} - dK_s^{c'-c}) = \int_{[t,1]} (Y_s - Y'_s)(dK_s^{c+} - dK'_s^{c'+c}) - \int_{[t,1]} (Y_s - Y'_s)(dK_s^{c'-c} - dK'_s^{c'-c}).
\]

But since for any \( t \leq 1 \), \( (Y_t - L_t)dK_t^{c'-c} = 0 \) then

\[
\int_{[t,1]} (Y_s - Y'_s)(dK_s^{c+} - dK'_s^{c'+c}) = -\int_{[t,1]} (Y_s - L_s)dK_s^{c+} + \int_{[t,1]} (L_s - Y'_s)dK_s^{c+} \leq 0.
\]

In the same way, we can also show that \( \int_{[t,1]} (Y_s - Y'_s)(dK_s^{c'-c} - dK'_s^{c'-c}) \geq 0 \). Therefore we obtain:

\[
\forall t \leq 1, \int_{[t,1]} (Y_s - Y'_s)(dK_s^{c+} - dK'_s^{c'+c}) \leq 0.
\]

Let us now focus on the discontinuous parts of \( K - K' \). For any \( t \leq 1 \),

\[
\int_{[t,1]} (Y_s - Y'_s)(dK_s^{d} - dK'_s^{d}) = \int_{[t,1]} (Y_s - Y'_s)(dK_s^{d} - dK'_s^{d}) - \int_{[t,1]} (Y_s - Y'_s)(dK_s^{d} - dK'_s^{d}).
\]

But

\[
\int_{[t,1]} (Y_s - Y'_s)(dK_s^{d} - dK'_s^{d}) = \int_{[t,1]} (L_s - Y'_s)dK_s^{d} - \int_{[t,1]} (Y_s - L_s)(dK_s^{d} - dK'_s^{d}) \leq 0
\]

since \( Y \geq L \) (resp. \( Y' \geq L \)) and the jumps of \( K^{d} \) (resp. \( K'^{d} \)) occur only when \( Y_+ = L_- \) (resp. \( Y'_+ = L_- \)). In the same way we have \( \int_{[t,1]} (Y_s - Y'_s)(dK_s^{d} - dK'_s^{d}) \geq 0 \). It follows from (17) that \( \int_{[t,1]} (Y_s - Y'_s)(dK_s^{d} - dK'_s^{d}) \leq 0 \). Combining now this inequality with (16) we deduce that for any \( t \leq 1 \), we have \( \int_{[t,1]} (Y_s - Y'_s)(dK_s - dK'_s) \leq 0 \).

Now using Itô’s formula with \((Y - Y')^2\) and following the same steps as in the proof of uniqueness of BSDEs with one reflecting barrier (see Proposition 3.1) to obtain that \( Y = Y' \), \( Z = Z' \), \( V = V' \) and finally \( K = K' \). Thus, due to their expressions, we have also \( K^{+d} = K'^{+d} \) and \( K^{-d} = K'^{-d} \) and then \( K^{c} - K'^{c} = K'^{c} - K'^{-c} \). It remains to show that \( K^{+c} = K'^{+c} \) and \( K^{-c} = K'^{-c} \).
Therefore \( K^{-c} = K'^{-c} \) since for any \( t < T, L_t < U_t \) and then \( K^+c = K'^+c \). Thus we have uniqueness of the solution. □

Once again to show that equation (15) has a solution we are going first to suppose that \( f \) does not depend on \( y, z, v \), i.e., \( f(t, \omega, y, z, v) = f(t) \). Then we have the following:

**Theorem 5.1 :** There exists a unique 5-uple of processes \((Y_t, Z_t, K^+_t, K^-_t, V_t)_{t \leq T}\) solution of the backward stochastic differential equation with two reflecting barriers associated with \((f(t), \xi, L, U)\).

**Proof:** Even if the barriers have predictable jumps, the proof of this theorem, in its main steps, is classical (see e.g. [3], [10]).

Let us consider the following processes defined by: ∀\( t \leq 1, \)

\[
H_t = (h_t + \mathbb{E}[\xi^-|F_t])1_{[t < 1]} + \mathbb{E}[\int_t^1 f(s)ds|F_t],
\]

\[
\Theta_t = (h'_t + \mathbb{E}[\xi^+|F_t])1_{[t < 1]} + \mathbb{E}[\int_t^1 f(s)ds|F_t],
\]

\[
\tilde{L}_t = L_{1[1 < t]} + \xi_{1[1 < t]} - \mathbb{E}[\xi + \int_1^t f(s)ds|F_t],
\]

and \( \tilde{U}_t = U_{1[1 < t]} + \xi_{1[1 < t]} - \mathbb{E}[\xi + \int_1^t f(s)ds|F_t], \)

where \( f(t)^- = \max\{-f(t), 0\} \) and \( f(t)^+ = \max\{f(t), 0\} \). Since \( h \) and \( h' \) are non-negative supermartingales then \( H \) and \( \Theta \) are also non-negative supermartingales which moreover belong to \( S^2 \) and verify \( H_1 = \Theta_1 = 0 \). On the other hand, through Mokobodski’s hypothesis, we can easily verify that for any \( t \leq 1 \) we have:

\[
\tilde{L} \leq H - \Theta \leq \tilde{U}. \tag{18}
\]

Next let us consider the sequences \((N^+_n)_{n \geq 0}\) of processes defined recursively as follows:

\[
N^+_0 = 0 \quad \text{and for } n \geq 0, N^{+,n+1} = R(N^{-,n} + \tilde{L}) \quad \text{and} \quad N^{-,n+1} = R(N^{+,n} - \tilde{U})
\]

where \( R \) is the Snell envelope operator (see Appendix). Now by induction and using (18) we can easily verify that:

\[
\forall n \geq 0, 0 \leq N^{+,n} \leq N^{+,n+1} \leq H \quad \text{and} \quad 0 \leq N^{-,n} \leq N^{-,n+1} \leq \Theta.
\]

It follows that the sequence \((N^+_n)_{n \geq 0}\) (resp. \((N^-_n)_{n \geq 0}\)) converges pointwisely to a supermartingale \( N^+ \) (resp. \( N^- \)) (see e.g. [4], pp.86). In addition \( N^+ \) and \( N^- \) belong to \( S^2 \) and verify (see [A1]) :

\[
N^+ = R(N^- + \tilde{L}) \quad \text{and} \quad N^- = R(N^+ - \tilde{U}).
\]

Next the Doob-Meyer decompositions of \( N^\pm \) yield :

\[
\forall t \leq 1, N^\pm_t = M^\pm_t - K^\pm_t
\]

where \( M^\pm \) are rcll martingales and \( K^\pm \) non-decreasing processes such that \( K^\pm_0 = 0 \). Moreover since \( N^\pm \in S^2 \) then \( E[(K^\pm_T)^2] < \infty \) (see [A2]). Therefore \( M^\pm \) belong also to \( S^2 \) and then there exist processes \( Z^\pm \in \mathcal{H}^{2,d} \) and \( V^\pm \in \mathcal{L}^2 \) such that (see [12]):

\[
\forall t \leq 1, M^\pm_t = M^\pm_0 + \int_0^t \{Z^\pm_s dB_s + \int_U V^\pm_s(e)\hat{\mu}(ds, de)\}.
\]
Next let us denote by $K^{d}$ (resp. $K^{c}$) the purely discontinuous (resp. continuous) part of $K^{±}$. In the same way as shown for BSDEs with one reflecting barrier (see Section 4, Step 4) we have:

$$
\int_{0}^{1} (N^{+} - N^{-} - \tilde{L}) dK_{s}^{+ c} = \int_{0}^{1} (N^{-} - N^{+} + \tilde{L}) dK_{s}^{- c} = 0. \tag{19}
$$

On the other hand the processes $K^{±d}$ are predictable and if $\tau$ is a predictable stopping time (see [A2]) then

$$
\{\Delta K_{\tau}^{d} > 0\} \subset \{N_{\tau}^{+} = N_{\tau}^{-} + \tilde{L}_{\tau}\} \\
\{\Delta K_{\tau}^{-d} > 0\} \subset \{N_{\tau}^{-} = N_{\tau}^{+} - \tilde{U}_{\tau}\}.
$$

But $L_{-} < U_{-}$ and $\tau$ is predictable then we have $\tilde{L}_{\tau} < \tilde{U}_{\tau}$ since, the jumps of martingales with respect to $(\mathcal{F})_{t<1}$ are inaccessible because they come only from the Poisson part. Therefore the predictable processes $K^{d}$ and $K^{-d}$ cannot jump in the same time otherwise we would have $\tilde{L}_{\tau} = \tilde{U}_{\tau}$ which is impossible. Henceforth

$$
\{\Delta K_{\tau}^{d} > 0\} \subset \{N_{\tau}^{+} = N_{\tau}^{-} + \tilde{L}_{\tau}\} \\
\{\Delta K_{\tau}^{-d} > 0\} \subset \{N_{\tau}^{-} = N_{\tau}^{+} - \tilde{U}_{\tau}\}.
$$

It follows that for any $t \leq 1$ we have

$$
\Delta K_{t}^{d} = (N_{t}^{+} - N_{t}^{-}) + 1_{\{N_{t}^{+} = N_{t}^{-} + \tilde{L}_{t}\}} = (N_{t}^{-} + \tilde{L}_{t} - N_{t}^{+}) + 1_{\{N_{t}^{-} = N_{t}^{+} + \tilde{L}_{t}\}}.
$$

In the same way we obtain:

$$
\forall t \leq 1, \Delta K_{t}^{-d} = (N_{t}^{-} - N_{t}^{+}) + 1_{\{N_{t}^{-} = N_{t}^{+} - \tilde{U}_{t}\}} = (N_{t}^{+} - \tilde{U}_{t} - N_{t}^{-}) + 1_{\{N_{t}^{+} = N_{t}^{-} - \tilde{U}_{t}\}}.
$$

Finally for $t \leq 1$, let us set:

$$
Y_{t} = N_{t}^{+} - N_{t}^{-} + \mathbb{E}[\xi + \int_{0}^{1} f(s) ds | \mathcal{F}_{t}], \ Z_{t} = Z_{t}^{+} - Z_{t}^{-} + \eta_{t}, \ V_{t} = V_{t}^{+} - V_{t}^{-} + \rho_{t}
$$

where the processes $\eta$ and $\rho$ are such that

$$
\forall t \leq 1, \mathbb{E}[\xi + \int_{0}^{1} f(s) ds | \mathcal{F}_{t}] = \mathbb{E}[\xi + \int_{0}^{1} f(s) ds] + \int_{0}^{t} \eta_{s} dB_{s} + \int_{0}^{t} \rho_{s}(e) \tilde{\mu}(ds, de).
$$

Therefore the quintuple $(Y, Z, V, K^{+}, K^{-})$ is the solution of the BSDE with two reflecting barriers associated with $(f(t, y, z, v), \xi, L, U)$, i.e.,

$$
\left\{ \begin{array}{l}
Y \in \mathcal{S}^{2}, K^{±} \in \mathcal{S}_{c}^{2}, Z \in \mathcal{H}^{d, 2} \text{ and } V \in \mathcal{L}^{2} \\
-dY_{t} = f(t, Y_{t}, Z_{t}, V_{t}) dt + dK_{t}^{+} - dK_{t}^{-} - Z_{t} dB_{t} - \int_{U} V_{t}(e) \tilde{\mu}(dt, de), t \leq 1; \ Y_{1} = \xi \\
\forall t \leq 1, L_{t} \leq Y_{t} \leq U_{t} \text{ and if } K^{±c} \text{ is the continuous part of } K^{±} \text{ then } (Y_{t} - L_{t}) dK_{t}^{+ c} = 0 \\
\text{ and } (U_{t} - Y_{t}) dK_{t}^{- c} = 0 \\
\text{ if } K^{d} \text{ denotes the purely discontinuous part of } K^{±} \text{ then } K^{d} \text{ is } \mathcal{F}^{d}-\text{measurable} \\
\text{ and } \forall t \leq 1, \Delta K_{t}^{d} = (L_{t} - Y_{t}) + 1_{\{Y_{t} = L_{t}\}} \text{ and } \Delta K_{t}^{-d} = (Y_{t} - U_{t}) + 1_{\{Y_{t} = U_{t}\}}.
\end{array} \right.
$$

We are now ready to give the main result of this section.

**Theorem 5.2**: The reflected BSDE (15) associated with $(f(t, y, z, v), \xi, L, U)$ has a unique solution $(Y, Z, V, K^{+}, K^{-})$. 

14
Proof: We give a brief proof since once more it is somehow classical. Let \( \mathcal{H} := \mathcal{H}^{2,1} \times \mathcal{H}^{2,d} \times \mathcal{L}^2 \) and \( \Phi \) be the following application:

\[
\Phi : \quad \mathcal{H} \rightarrow \mathcal{H} \quad (y, z, v) : \mapsto \Phi(y, z, v) = (\bar{Y}, \bar{Z}, \bar{V})
\]

where \((\bar{Y}, \bar{Z}, \bar{V})\) is the triple for which there exists two other processes \( \bar{K}^\pm \) which belong to \( S^2 \) such that \((\bar{Y}, \bar{Z}, \bar{V}, \bar{K}^+, \bar{K}^-)\) is a solution of the BSDE with two reflecting barriers associated with \((f(t, y, z, v), \xi, L, U)\). Now let \( \alpha > 0 \), \((y', z', v') \in \mathcal{H} \) and \((\bar{Y}', \bar{Z}', \bar{V}') = \Phi(y', z', v')\). Using Itô’s formula and taking into account that \( e^{\alpha s} (\bar{Y}'_s - \bar{Y}'_s') d(\bar{K}^+_s - \bar{K}^-_s - \bar{K}'_s^+ + \bar{K}'_s^-) \leq 0 \) we show the existence of a constant \( C < 1 \), by an appropriate choice of \( \alpha \) (see e.g. [10, 11]), such that:

\[
\mathbb{E} \left[ \int_0^T e^{\alpha s} \left( |(\bar{Y}'_s - \bar{Y}'_s')^2 + |\bar{Z}'_s - \bar{Z}'_s|^2 + \int E |\bar{V}'_s(e) - \bar{V}'_s(e)|^2 \lambda(de) \right) ds \right] 
\leq \bar{C} \mathbb{E} \left[ \int_0^T e^{\alpha s} \left( |y_s - y'_s|^2 + |z_s - z'_s|^2 + |v_s - v'_s|^2 \right) ds \right].
\]

Then the mapping \( \Phi \) is a contraction which then has a unique fixed point \((Y, Z, V)\) which actually belongs to \( S^2 \times \mathcal{H}^{2,d} \times \mathcal{L}^2 \). Moreover there exists \( K^\pm \in S^2 (K^+_0 = 0) \) such that \((Y, Z, V, K^+, K^-)\) is solution for the reflected BSDE associated with \((f, \xi, L, U)\). □

5.1 Appendix

Throughout this appendix \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq 1}, P)\) is the same as in Section 2.

Let \( \eta := (\eta_t)_{t \leq 1} \) be a rcll, \( \mathcal{P} \)-measurable process with values in \( R \) and of a class \([D]\), i.e., the set of random variables \( \{\eta_t, \tau \leq 1\} \) is uniformly integrable. The Snell envelope of the process \( \eta \), which we denote \( R(\eta) := (R(\eta)_t)_{t \leq 1} \) is the lowest rcll \( \mathcal{F}_t \)-supermartingale of class \([D]\) which dominates \( \eta \), i.e., \( P - a.s., \forall t \leq 1, R(\eta)_t \geq \eta_t \). It has the following expression (see e.g. [5]):

\[
P - a.s., \forall t \leq 1, \quad R(\eta)_t = \text{esssup}_{\tau \geq t} \mathbb{E}[\eta_\tau | \mathcal{F}_t] \quad (R(\eta)_1 = \eta_1).
\]

We now give some properties of the Snell envelope of processes.

[A1]: Let \((U^n)_{n \geq 0}\) be a non-decreasing sequence of \( \mathcal{P} \)-measurable, rcll, \( R \)-valued processes of class \([D]\) which converges pointwise to \( U \) another rcll, \( R \)-valued, \( \mathcal{P} \)-measurable process of class \([D]\), then \( P - a.s., \forall t \leq 1, R(U^n)_t \nearrow R(U)_t \).

Proof: Actually for any \( n \geq 0, P - a.s. \forall t \leq 1, R(U^n)_t \leq R(U)_t \). Therefore \( P - a.s., \forall t \leq 1, \lim_{n \to \infty} R(U^n)_t \leq R(U)_t \). Note that the process \((\lim_{n \to \infty} R(U^n)_t)_{t \leq 1}\) is an rcll supermartingale of class \([D]\) since it is a limit of a non-decreasing sequence of supermartingales (see e.g. [4], pp.86). But \( U^n \leq R(U^n) \) implies that \( P - a.s., \forall t \leq 1, U_t \leq \lim_{n \to \infty} R(U^n)_t \) and then \( R(U)_t \leq \lim_{n \to \infty} R(U^n)_t \) since the Snell envelope of \( U \) is the lowest supermartingale which dominates \( U \). It follows that \( P - a.s., \forall t \leq 1, \lim_{n \to \infty} R(U^n)_t = R(U)_t \), whence the desired result. □

[A2]: Doob-Meyer decomposition of Snell envelopes

Let \( \eta := (\eta_t)_{t \leq 1} \) be a rcll, \( \mathcal{P} \)-measurable process with values in \( R \) and of a class \([D]\), and \( R(\eta) := (R(\eta)_t)_{t \leq 1} \) its Snell envelope. Then there exist an rcll \( \mathcal{F}_t \)-martingale \((M_t)_{t \leq 1}\) and a non-decreasing rcll \( \mathcal{F}_t \)-predictable process \((K_t)_{t \leq 1}\) (\( K_0 = 0 \)) such that:

\[
P - a.s., \forall t \leq 1, \quad R(\eta)_t = M_t - K_t.
\]
Moreover we have:

(i) if \( R(\eta) \) belongs also to \( \mathcal{S}^2 \) then \( E[K^2_\tau] < \infty \)

(ii) if \( K^c \) (resp. \( K^d \)) denotes the continuous (resp. purely discontinuous) part of \( K \) then \( K^d \) is \( \mathcal{F}_t \)-predictable and \( \{ \Delta K^d > 0 \} \subset \{ R(\eta)_- = \eta_- \} \) and \( \Delta \xi K^d = (\eta_- - R(\eta))_+ 1_{\{R(\eta)_t = \eta_-\}} \).

**Proof:** The existence of \( M \) and \( K \) is just the Doob-Meyer decomposition of supermartingales of class \([D]\) (see [4], pp.221). Besides if \( R(\eta) \) belongs to \( \mathcal{S}^2 \) then the process \( K \) is so. This a direct consequence of the dual predictable projection of \( K \) (see [4], pp. 221). The proof of \( \{ \Delta K^d > 0 \} \subset \{ R(\eta)_- = \eta_- \} \) is given in ([5], pp.131). Finally since the filtration is generated by a Brownian motion and an independent Poisson measure the jumps of \( M \) occur only at inaccessible stopping times. Therefore when \( K^d \) jumps, which is a predictable process, the process \( R(\eta) \) has the same jump. It follows that \( \Delta \xi K^d = (R(\eta)_t - R(\eta)_t)_+ 1_{\{R(\eta)_t = \eta_-\}} = (\eta_- - R(\eta)_t)_+ 1_{\{R(\eta)_t = \eta_-\}} \). □

Let \( X := (X_t)_{t \leq 1} \) be a process of class \([D]\). The predictable projection of \( X \), which we denote by \( X^p \), is an \( \mathcal{F}_t \)-predictable process which satisfies \( E[X_t|\mathcal{F}_-] = X^p_t \) for any predictable stopping time. The process \( X \) is called regular if it satisfies \( X^p_t = X_t^- \), for any \( t \leq 1 \).

The following result is related to the existence of an optimal stopping time when the Snell envelope is regular.

[A3] Let \( \eta \) be a process of \( \mathcal{S}^2 \) and \( R(\eta) \) its Snell envelope whose decomposition is \( M - K \). For \( t \leq 1 \), let \( \tau_t \) be the stopping time defined as follows:

\[
\tau_t = \inf\{ s \geq t, K_s - K_t > 0 \} \land 1.
\]

If \( R(\eta) \) is regular then \( K^d \equiv 0 \) and \( \tau_t \) is optimal after \( t \), i.e., it satisfies:

(i) \( E[\eta_{\tau_t}] = \sup_{r \geq t} E[\eta_r] \)

(ii) \( R(\eta)_{\tau_t} = \eta_{\tau_t} \) and \( (R(\eta)_{s \land \tau_t})_{s \geq t} \) is an \( \mathcal{F}_s \)-martingale.

A word about the proofs of those facts. The continuity of \( K \) when \( R(\eta) \) is regular is stated in ([4], pp.214). As for the optimality of \( \tau_t \), one can see e.g ([5], pp. 140). □

**Acknowledgement.** This paper has been carried out when the second author visited Université du Maine (Le Mans, France). Their hospitality was greatly appreciated. □

**References**

[1] G. Barles, R. Buckdahn, E. Pardoux: BSDEs and integral-partial differential equations. Stochastics 60, 57-83, 1997.

[2] T. Bielecki, S. Crepey, M. Jeanblanc, M. Rutkowski: Defautable options in a Markovian intensity model of credit risk. Mathematical Finance, Vol. 18, Issue 4, pp. 493-518, October 2008

[3] J. Cvitanić, I. Karatzas: Backward SDEs with reflection and Dynkin games, *Annals of Probability* 24 (4), pp. 2024-2056 (1996)

[4] C. Dellacherie, P.A. Meyer: Probabilités et Potentiel. Chap. V-VIII. Hermann, Paris (1980).

[5] N. El-Karoui: Les aspects probabilistes du contrôle stochastique, in Ecole d’été de Saint-Flour. Lecture Notes in Mathematics 876, 73-238. Springer Verlag Berlin.
[6] N.El-Karoui, C.Kapoudjian, E.Pardoux, S.Peng, M.C.Quenez: Reflected solutions of backward SDE’s and related obstacle problems for PDE’s. Annals of Probability 25 (2) (1997), pp.702-737.

[7] N.El-Karoui, E.Pardoux, M.-C.Quenez: Reflected backward SDEs and American options, in: Numerical Methods in Finance (L.Robers and D. Talay eds.), Cambridge U. P., 1997, 215-231

[8] S.Hamadène: Mixed Zero-sum differential game and American game options, SIAM JCO, Vol. 45 (2), pp.496-518 (2006).

[9] S.Hamadène: Reflected BSDE’s with discontinuous barriers and application. Stochastics and Stochastics Reports Vol.74(3-4), pp. 571-596 (2002).

[10] S.Hamadène, J.P.Lepeltier: Reflected Backward SDE’s and Mixed Game Problems. Stochastic Processes and their Applications 85 (2000) p. 177-188.

[11] S.Hamadène, Y.Ouknine Reflected Backward stochastic differential equation with jumps and random obstacle. EJP Vol. 8 pp. 1-20 (2003).

[12] N. Ikeda, S. Watanabe: Stochastic Differential Equations and Diffusion Processes, North Holland/Kodansha (1981).

[13] J.P. Lepeltier, X.Mingyu: Penalization method for reflected backward stochastic differential equations with one r.c.l.l. barrier. Statist. Probab. Lett. 75 (2005), no. 1, 58–66.

[14] X.Mingyu, S.Peng: The smallest g-supermartingale and reflected BSDE with single and double L2 obstacles, Annales de l'IHP (B), Probability and Statistics, Vol.41, Issue 3, May-June 2005, pp. 605-630.

[15] E.Pardoux, S.Peng: Adapted Solutions of Backward Stochastic Differential Equations. Systems and Control Letters 14, pp.51-61, 1990.

[16] E. Pardoux: BSDEs, weak convergence and homogenization of semilinear PDEs, in: F. Clarke and R. Stern (eds), Nonlin. Anal., Dif. Equa. and Control, 503-549 (1999), Kluwer Acad. Publi., Netherlands

[17] E. Pardoux, S. Peng: Backward Stochastic Differential Equations and Quasilinear Parabolic Partial Differential Equations, in: Stochastic Differential Equations and their Applications (B. Rozovskii and R. Sowers, eds.), Lect. Not. Cont. Inf. Sci., vol.176, Springer, 1992, 200-217

[18] S. Tang and X. Li: Necessary condition for optimal control of stochastic systems with random jumps, SIAM JCO 33 2, pp. 1447-1475, (1994).