On the reduction of the multidimensional Schrödinger equation to a first order equation and its relation to the pseudoanalytic function theory

Vladislav V. Kravchenko
Depto. de Telecomunicaciones, SEPI
Escuela Superior de Ingeniería Mecánica y Eléctrica
Instituto Politécnico Nacional
C.P.07738 México D.F., MEXICO

October 31, 2021

Abstract

Given a particular solution of a one-dimensional stationary Schrödinger equation this equation of second order can be reduced to a first order linear ordinary differential equation. This is done with the aid of an auxiliary Riccati differential equation. In the present work we show that the same fact is true in a multidimensional situation also. For simplicity we consider the case of two or three independent variables. One particular solution of the Schrödinger equation allows us to reduce this second order equation to a linear first order quaternionic differential equation. As in one-dimensional case this is done with the aid of an auxiliary quaternionic Riccati equation. The resulting first order quaternionic equation is equivalent to the static Maxwell system and is closely related to the Dirac equation. In the case of two independent variables it is the well known Vekua equation from theory of pseudoanalytic (or generalized analytic) functions. Nevertheless we show that even in this case it is very useful to consider not complex valued functions only, solutions of the Vekua equation but complete quaternionic functions. In this way the first order quaternionic equation represents two separate Vekua equations, one of which gives us solutions of the
Schrödinger equation and the other one can be considered as an auxiliary equation of a simpler structure. Moreover for the auxiliary equation we always have the corresponding Bers generating pair \((F, G)\), the base of the Bers theory of pseudoanalytic functions, and what is very important, the Bers derivatives of solutions of the auxiliary equation give us solutions of the main Vekua equation and as a consequence of the Schrödinger equation. Based on this fact we obtain an analogue of the Cauchy integral theorem for solutions of the Schrödinger equation. Other results from theory of pseudoanalytic functions can be written down for solutions of the Schrödinger equation. Moreover, for an ample class of potentials in the Schrödinger equation (which includes for instance all radial potentials), this new approach gives us a simple procedure allowing to obtain an infinite sequence of solutions of the Schrödinger equation from one known particular solution.

1 Introduction

Consider the one-dimensional static Schrödinger equation

\[ u'' + vu = 0 \quad (1) \]

and the associated Riccati equation

\[ y' + y^2 = -v. \quad (2) \]

Equation (1) is related to (2) by the easily inverted substitution

\[ y = \frac{u'}{u}. \]

Thus solutions of the Riccati equation (2) are simply logarithmic derivatives of solutions of the Schrödinger equation (1) and vice versa solutions of (1) are logarithmic antiderivatives of solutions of (2). The generalization of this fact for a multidimensional situation was obtained in [8] (see also [10]). Among the peculiar properties of the Riccati equation stands out an important theorem of Euler, dating from 1760. If a particular solution \(y_0\) of the Riccati equation is known, the substitution \(y = y_0 + z\) reduces (2) to a Bernoulli equation which in turn is reduced by the substitution \(z = \frac{1}{u}\) to a first order linear equation. Thus given a particular solution of the Riccati equation, it can be linearized and the general solution can be found in two integrations. As a consequence of this, given a particular solution of the Schrödinger equation (1) the general solution can be found from a first order
linear equation. This can be seen immediately from the factorization of the one-dimensional Schrödinger operator

\[ \partial^2 + v(x) = (\partial + y_0(x))(\partial - y_0(x)) \]  

(3)

which is valid if and only if \( y_0 \) is a solution of (2).

In the present work we show that given a particular solution of a multi-dimensional stationary Schrödinger equation this equation of second order can be reduced to a first order linear quaternionic differential equation. For doing this we use a quaternionic factorization of the Schrödinger operator proposed in [3], [4] (see also [7]) and the results on the quaternionic Riccati equation from [8] and [10] where it was shown that having a particular solution of the quaternionic Riccati equation one can reduce it to a second order linear equation. Here we show that the similarity with the one-dimensional situation is much closer, and one particular solution is sufficient to reduce the quaternionic Riccati equation to a first order linear equation. The resulting first order quaternionic equation is equivalent to the static Maxwell system and is closely related to the Dirac equation. In the case of two independent variables it is the well known Vekua equation from theory of pseudoanalytic (or generalized analytic) functions (see, e.g., [2], [5], [6], [14], [15]). We show that even in this case it is very useful to consider not only complex valued functions, solutions of the Vekua equation but complete quaternionic functions. In this way the first order quaternionic equation represents two separate Vekua equations, one of which gives us solutions of the Schrödinger equation and the other one can be considered as an auxiliary equation of a simpler structure. Moreover for the auxiliary equation we always have in explicit form the corresponding Bers generating pair \((F, G)\), the base of Bers’ theory of pseudoanalytic functions, and what is very important, the Bers derivatives of solutions of the auxiliary equation give us solutions of the main Vekua equation and as a consequence of the Schrödinger equation. Based on this fact, for example, we obtain an analogue of the Cauchy integral theorem for solutions of the Schrödinger equation. Other results from theory of pseudoanalytic functions can be written down for solutions of the Schrödinger equation. Moreover, for an ample class of potentials in the Schrödinger equation (which includes for instance all radial potentials), this new approach gives us a simple procedure allowing to obtain an infinite sequence of solutions of the Schrödinger equation from one known particular solution.

Besides this introduction the paper contains four sections. In Section 2 we introduce necessary notations from quaternionic analysis. In Section 3 we
prove a spatial generalization of the Euler theorem for the Riccati equation and show how a particular solution of the Schrödinger equation allows us to reduce it to a first order quaternionic equation. We observe that in the case of two independent variables this first order equation represents two separate Vekua equations. In order to apply theory of pseudoanalytic functions to the resulting Vekua equations, in Section 4 we introduce some necessary definitions and results from Bers’ theory. Finally in Section 5 we show how all the machinery of this quite forgotten mathematical theory allows us to obtain surprising results for the Schrödinger equation starting with an analogue of the Cauchy integral theorem and including infinite sequences of solutions generated by one particular solution.

2 Notations from quaternionic analysis

We will consider the algebra $\mathbb{H}(\mathbb{C})$ of complex quaternions or biquaternions which have the form $q = q_0 + q_1i + q_2j + q_3k$, where $\{q_k\} \subset \mathbb{C}$, and $i, j, k$ are the quaternionic imaginary units.

The vectorial representation of a complex quaternion will be used. Namely, each complex quaternion $q$ is a sum of a scalar $q_0$ and of a vector $\mathbf{q}$:

$$ q = \text{Sc}(q) + \text{Vec}(q) = q_0 + \mathbf{q}, $$

where $\mathbf{q} = q_1i + q_2j + q_3k$. The purely vectorial complex quaternions ($\text{Sc}(q) = 0$) are identified with vectors from $\mathbb{C}^3$. Note that $\mathbf{q}^2 = - <\mathbf{q}, \mathbf{q}>$ where $<\cdot, \cdot>$ denotes the usual scalar product.

By $M^p$ we denote the operator of multiplication by a complex quaternion $p$ from the right-hand side: $M^p q = q \cdot p$.

More information on the structure of the algebra of complex quaternions can be found for example in [10] or [12].

Let $q$ be a complex quaternion valued differentiable function of $\mathbf{x} = (x_1, x_2, x_3)$. Denote

$$ Dq = i \frac{\partial}{\partial x_1} q + j \frac{\partial}{\partial x_2} q + k \frac{\partial}{\partial x_3} q. $$

This expression can be rewritten in vector form as follows

$$ Dq = - \text{div} \mathbf{q} + \text{grad} q_0 + \text{rot} \mathbf{q}. $$

That is, $\text{Sc}(Dq) = - \text{div} \mathbf{q}$ and $\text{Vec}(Dq) = \text{grad} q_0 + \text{rot} \mathbf{q}$. Let us notice that $D^2 = -\Delta$.

If $q_0$ is a scalar function then $Dq_0$ coincides with $\text{grad} q_0$. The expression $Dq_0/q_0$ will be called the logarithmic derivative of $q_0$. 

4
3 Reduction of the Schrödinger equation to a first order quaternionic equation

Consider the equation
\[ (-\Delta + u) f = 0 \] (4)
where \( \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \), \( f \) and \( u \) are complex valued functions. We assume that \( f \) is twice differentiable. Together with (4) we introduce the following quaternionic equation
\[ Dq + q^2 = -u \] (5)
where \( q \) is a purely vectorial differentiable biquaternion valued function.

**Theorem 1** [3] For an arbitrary scalar twice differentiable function \( f \) the following equality holds
\[ (D + M^h)(D - M^h)f = (-\Delta + u)f \] (6)
if and only if \( h \) is a solution of (5).

Thus, given a particular solution of (5) the Schrödinger operator in (4) can be factorized.

**Theorem 2** [8] Solutions of (4) are related to solutions of (5) in the following way. For any nonvanishing solution \( f \) of (4) its logarithmic derivative
\[ q = \frac{Df}{f} \] (7)
is a solution of (5) and any solution \( q \) of (5) is a logarithmic derivative of the form (7) of a solution of (4).

**Proof.** A direct substitution into the equation (5) shows us that for a nonvanishing solution \( f \) of (4) its logarithmic derivative (7) is a solution of (5). Now let us suppose that \( q \) is a solution of (5). From the vector part of (5) we have that \( q \) is a gradient of some scalar function \( \xi \): \( q = \text{grad}\, \xi \). Then \( q \) can be represented in the form (7) where \( f = e^\xi \). Substituting (7) in (5) we obtain that \( f \) is a solution of (4).

**Remark 3** Theorems 1 and 2 show us that equation (5) is a generalization of the Riccati equation. We will call it quaternionic Riccati equation.
Lemma 4 \[\text{For a nonvanishing scalar differentiable function } \varepsilon \text{ there exists a one-to-one correspondence between solutions of the static Maxwell system}
\]

\[
\text{div}(\varepsilon \mathbf{E}) = 0, \quad (8)
\]

\[
\text{rot } \mathbf{E} = 0, \quad (9)
\]

and solutions of the equation

\[
(D + M^h)\mathbf{F} = 0 \quad (10)
\]

where

\[
h = \frac{D\sqrt{\varepsilon}}{\sqrt{\varepsilon}}.
\]

Vector \(\mathbf{E}\) is a solution of (8), (9) if and only if the vector \(\mathbf{F} = \sqrt{\varepsilon}\mathbf{E}\) is a solution of (10).

**Proof.** The system (8), (9) can be rewritten in the form

\[
D\mathbf{E} = \langle \frac{\text{grad } \varepsilon}{\varepsilon}, \mathbf{E} \rangle.
\]

Let us make a simple observation: the scalar product of two vectors \(\mathbf{p}\) and \(\mathbf{q}\) can be written as follows

\[
\langle \mathbf{p}, \mathbf{q} \rangle = -\frac{1}{2}(\mathbf{p} M + M \mathbf{p})\mathbf{q}.
\]

Then we have

\[
(D + \frac{1}{2} \frac{\text{grad } \varepsilon}{\varepsilon})\mathbf{E} = -\frac{1}{2} M \frac{\text{grad } \varepsilon}{\varepsilon} \mathbf{E}. \quad (11)
\]

Note that

\[
\frac{1}{2} \frac{\text{grad } \varepsilon}{\varepsilon} = \frac{\text{grad } \sqrt{\varepsilon}}{\sqrt{\varepsilon}}.
\]

Then equation (11) can be rewritten in the following form

\[
\frac{1}{\sqrt{\varepsilon}} D(\sqrt{\varepsilon}\mathbf{E}) + \mathbf{E} h = 0 \quad (12)
\]

where \(h = D\sqrt{\varepsilon}/\sqrt{\varepsilon}\). Introducing the notation \(\mathbf{F} = \sqrt{\varepsilon}\mathbf{E}\) and multiplying (12) by \(\sqrt{\varepsilon}\) we obtain the equivalence of the system (8)- (9) to (10). \(\blacksquare\)
Lemma 5 Let $h$ be an arbitrary particular solution of (5) (then as was mentioned above it is a gradient of some scalar function $\xi$). The general solution of (5) has the form
\[ q = h + g, \tag{13} \]
where $g = (\text{grad } \Psi)/\Psi$ and $\Psi$ is a general solution of the equation
\[ \Delta \Psi + 2 \langle \text{grad } \xi, \text{grad } \Psi \rangle = 0, \tag{14} \]
or equivalently of
\[ \text{div}(e^{2\xi} \text{grad } \Psi) = 0. \tag{15} \]

Proof. Substituting (13) in (5) gives
\[ Dg - 2 \langle h, g \rangle + g^2 = 0. \tag{16} \]
Note that the vector part of (16) is rot $g = 0$, so that
\[ g = \text{grad } \Phi \]
for some function $\Phi$. If $\Psi = e^\Phi$, this is equivalent to
\[ g = (\text{grad } \Psi)/\Psi. \]
Equation (16), written in terms of $\Psi$, is
\[ -\frac{1}{\Psi^2} (\text{grad } \Psi)^2 - \frac{1}{\Psi} \Delta \Psi - \frac{2}{\Psi} \langle \text{grad } \xi, \text{grad } \Psi \rangle + \frac{1}{\Psi^2} (\text{grad } \Psi)^2 = 0, \]
so that (16) is equivalent to
\[ \Delta \Psi + 2 \langle \text{grad } \xi, \text{grad } \Psi \rangle = 0. \]
Noting that
\[ \text{div}(e^{2\xi} \text{grad } \Psi) = 2e^{2\xi} \langle \text{grad } \xi, \text{grad } \Psi \rangle + e^{2\xi} \Delta \Psi = e^{2\xi}(\Delta \Psi + 2 \langle \text{grad } \xi, \text{grad } \Psi \rangle), \]
this equation can be rewritten in the form
\[ \text{div}(e^{2\xi} \text{grad } \Psi) = 0. \]

Now we are ready to prove a generalization of the Euler theorem for the quaternionic Riccati equation.
**Theorem 6** *(Euler’s theorem for the quaternionic Riccati equation)* Let \( h = \text{grad} \xi \) be a particular solution of (5). The general solution of the quaternionic Riccati equation has the form \( q = h + g \) where \( g = \frac{D\Psi}{\Psi} \), and \( \Psi \) is obtained from the equation

\[
\text{grad } \Psi = e^{-\xi} F
\]

(17)

where \( F \) is the general solution of (10).

**Proof.** According to Lemma 5 it is sufficient to prove that \( \Psi \) is a solution of (14) (or what is the same of (15)) if and only if the vector \( F = e^\xi \text{grad } \Psi \) is a solution of (10). Let us notice that if \( \Psi \) is a solution of (14) then the vector \( E = \text{grad } \Psi \) is a solution of the system \((8)-(9)\) where \( \varepsilon = e^{2\xi} \) and vice versa if \( E \) is a solution of \((8)-(9)\) then it is a gradient of some function \( \Psi \) which is necessarily a solution of (14). Now due to Lemma 3 \( \text{grad } \Psi \) is a solution of (8)-(9) if and only if \( F = e^\xi \text{grad } \Psi \) is a solution of the equation (10) where \( h = \frac{D\xi}{\varepsilon} = \text{grad } \xi \).

Thus, given a particular solution of the quaternionic Riccati equation, the general solution reduces to the linear first order equation (10), exactly as in one-dimensional situation.

Using Theorem 2 we immediately arrive at the following result for the Schrödinger equation.

**Theorem 7** Let \( f_0 \) be a nonvanishing particular solution of (4) and \( F \) be the general solution of (10) where \( h = \frac{Df_0}{f_0} \). Then the general solution \( f \) of (4) is the logarithmic antiderivative of \( q \): \( \frac{Df}{f} = q \), where \( q = h + g \), \( g = \frac{D\Psi}{\Psi} \) and \( \Psi \) is obtained from the equation

\[
\text{grad } \Psi = \frac{F}{f_0}.
\]

(18)

**Proof.** From Theorem 2 we have that \( f \) is the logarithmic antiderivative of \( q \) (equation (17)), where \( q \) is the general solution of (5). For a particular solution \( f_0 \) of (4) the vector

\[
h = \frac{Df_0}{f_0} = \text{grad } \ln f_0
\]

is a particular solution of (5). Due to Theorem 6 the general solution of (5) has the form \( q = h + g \), where \( g = \frac{D\Psi}{\Psi} \), and \( \Psi \) is obtained from the equation

\[
\text{grad } \Psi = e^{-\ln f_0} F = \frac{F}{f_0}
\]

where \( F \) is the general solution of (10). □
Remark 8 The logarithmic antiderivative of \( q \) always exists and can be obtained easily. Being a solution of (3) \( q \) is necessarily a gradient of some function \( \Phi \) which can be constructed analytically. Then \( f \) has the form \( f = Ce^\Phi \), where \( C \) is a complex constant.

Remark 9 From Theorem 7 it follows that for any vector \( F \), solution of (10) with \( h = Df_0/f_0 \), the vector \( F/f_0 \) must be a gradient of some scalar function \( \Psi \), and this is really true. Let us show that indeed \( \text{rot}(F/f_0) = 0 \). Note that this condition is equivalent to the equality \( \text{Vec}(D(F/f_0)) = 0 \). Consider

\[
D\left(\frac{F}{f_0}\right) = \frac{1}{f_0} DF - \frac{Df_0}{f_0} F = -(F \frac{Df_0}{f_0} + \frac{Df_0}{f_0} F) = 2 < F, \frac{Df_0}{f_0} > .
\]

Thus \( \text{Vec}(D(F/f_0)) = 0 \) and hence the vector \( F/f_0 \) is a gradient.

Remark 10 Let us summarize the results of this section in the following chain of actions which one should follow in order to obtain solutions of (4) from solutions of (10).

Given a nonvanishing particular solution \( f_0 \) of the Schrödinger equation (4) we construct the vector \( h = Df_0/f_0 \) and consider equation (10). Taking a solution \( F \) of (10) we find \( \Psi \) from (18). Then we construct the vectors \( g = D\Psi/\Psi \) and \( q = h + g \). \( q \) is necessarily a gradient of some scalar function \( \Phi \). Finding \( \Phi \) we finally obtain a solution of (4) as \( f = Ce^\Phi \), where \( C \) is a complex constant.

Thus given a particular solution of the Schrödinger equation (4), the general solution reduces to the first order equation (10). It is interesting to note that due to Lemma 4, equation (10) is equivalent to the static Maxwell system. It is closely related (see [7] and [11]) also to the Dirac equation as well as to the Beltrami fields which are solutions of the equation \( \text{rot} f + \alpha f = 0 \) (see, e.g., [1] and [13]).

The following statement gives us the way to transform solutions of the Schrödinger equation into solutions of (10).

Proposition 11 Let \( f_1 \) be another nonvanishing solution of the Schrödinger equation (4). Then the ratio \( \Psi = f_1/f_0 \) is a solution of the equation

\[
\text{div}(f_0^2 \text{grad} \Psi) = 0 \tag{19}
\]

and the vector \( F = f_0 D(f_1/f_0) \) is a solution of equation (10) where \( h = Df_0/f_0 \).

9
Proof. By Theorem 2 the vector $q = Df_1/f_1$ is a solution of (5). Then consider the vector $g$ from Lemma 5:

$$g = q - h = \frac{Df_1}{f_1} - \frac{Df_0}{f_0} = D\left(\frac{f_1}{f_0}\right)/(f_1/f_0).$$

Thus we have that $g = D\Psi/\Psi$ where $\Psi = f_1/f_0$, and by Lemma 5 $\Psi$ satisfies (19).

From (18) we obtain that $F = f_0 D\Psi = f_0 D(f_1/f_0)$ is a solution of (10).

Let us consider equation (10) for a biquaternion valued function $p$ whose scalar part is not necessarily zero

$$(D + M^h)p = 0 \quad (20)$$

and let us use the following representation for biquaternions:

$$p = P_1 + P_2 j,$$

where $P_1 = p_0 + p_3 k$ and $P_2 = p_2 - p_1 k$. Then $D = D_1 + D_2 j$, where $D_1 = \partial_3 k$, $D_2 = \partial_2 - \partial_1 k$ and $h = H_1 + H_2 j$, where $H_1 = h_3 k$, $H_2 = h_2 - h_1 k$. Using these notations, equation (20) can be rewritten as the following system

$$D_1 P_1 - D_2 \overline{P}_2 + H_1 P_1 - \overline{H}_2 P_2 = 0, \quad (21)$$

$$D_2 \overline{P}_1 + D_1 P_2 + H_2 P_1 + \overline{H}_1 P_2 = 0. \quad (22)$$

Now let us suppose that both $p$ and $h$ do not depend on $x_3$. Then system (21), (22) turns into the pair of decoupled equations:

$$\overline{D}_2 P_1 + \overline{H}_2 P_1 = 0 \quad (23)$$

and

$$\overline{D}_2 P_2 + \overline{H}_2 P_2 = 0 \quad (24)$$

which are nothing but Vekua’s equations describing pseudoanalytic or generalized analytic functions (see, e.g., [5] and [15]).

Remark 12 Here we should mention that in general the components $p_0, \ldots, p_3$ as well as $h_1, h_2$ can be complex valued functions, hence $P_1, P_2$ and $H_2$ can be bicomplex. Nevertheless this detail is insignificant for what follows, because all results from Bers’ theory which will be used in the subsequent sections are valid for bicomplex solutions also. Of course, when $u$ in (4) is a real valued function we can consider real valued solutions of (4) only. In that case (23) and (24) are usual Vekua’s equations.
Remark 13 In what follows we consider \( u \) and \( f \) in (4) being independent of \( x_3 \). Then given a particular solution \( f_0 \) of (4), the general solution reduces to equation (24) (which in this case is equivalent to (10)). Thus we are primarily interested in solutions of (24), and (23) can be considered as an auxiliary equation. Nevertheless as we will see in Section 5 equations (23) and (24) are closely related to each other. With the aid of Bers’ theory solutions of (24) can be obtained from solutions of (23) and it is interesting to notice that by construction we always have at least two solutions of (23) in explicit form. It is easy to see that the functions

\[
F = \frac{1}{f_0} \quad \text{and} \quad G = f_0 k
\]

are solutions of (20) where \( h = Df_0/f_0 \) and consequently they are solutions of (23).

4 Some definitions and results from Bers’ theory

Bers’ theory of pseudoanalytic functions was essentially developed in [5] (see also [6]). It is based on the so-called generating pair, a pair of complex functions \( F \) and \( G \) satisfying the inequality

\[
\text{Im}(FG) > 0 \quad (25)
\]

in some domain of interest \( \Omega \) which may coincide with the whole complex plane. \( F \) and \( G \) are assumed to possess partial derivatives with respect to the real variables \( x \) and \( y \). In this case the operators \( \partial_{\overline{z}} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \) and \( \partial_z = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \) can be applied (usually these operators are introduced with the factor 1/2, nevertheless here it is somewhat more convenient to consider them without it) and the following characteristic coefficients of the pair \((F,G)\) can be defined

\[
a_{(F,G)} = -\frac{FG_{\overline{z}} - F_{\overline{z}}G}{FG - F\overline{G}}, \quad b_{(F,G)} = \frac{FG_{\overline{z}} - F_{\overline{z}}G}{FG - F\overline{G}},
\]

\[
A_{(F,G)} = -\frac{FG_z - Fz\overline{G}}{FG - F\overline{G}}, \quad B_{(F,G)} = \frac{FG_z - Fz\overline{G}}{FG - F\overline{G}}.
\]

where the subindex \( \overline{z} \) or \( z \) means the application of \( \partial_{\overline{z}} \) or \( \partial_z \) respectively.

Every complex function \( w \) defined in a subdomain of \( \Omega \) admits the unique representation \( w = \phi F + \psi G \) where the functions \( \phi \) and \( \psi \) are real valued.
Sometimes it is convenient to associate with the function \( w \) the function \( \omega = \phi + i\psi \). The correspondence between \( w \) and \( \omega \) is one-to-one.

Bers introduces the notion of the \((F, G)\)-derivative of a function \( w \) which exists and has the form

\[
\dot{w} = \phi_z F + \psi_z G = w_z - A_{(F, G)} w - B_{(F, G)} \overline{w}
\]  

(26)

if and only if

\[
\phi_z F + \psi_z G = 0.
\]  

(27)

This last equation can be rewritten in the following form

\[
w_z = a_{(F, G)} w + b_{(F, G)} \overline{w}
\]

which we call the Vekua equation. Solutions of this equation are called \((F, G)\)-pseudoanalytic functions. If \( w \) is \((F, G)\)-pseudoanalytic, the associated function \( \omega \) is called \((F, G)\)-pseudoanalytic of second kind.

**Remark 14** The functions \( F \) and \( G \) are \((F, G)\)-pseudoanalytic, and \( \dot{F} \equiv \dot{G} \equiv 0 \).

**Definition 15** Let \((F, G)\) and \((F_1, G_1)\) - be two generating pairs in \( \Omega \). \((F_1, G_1)\) is called successor of \((F, G)\) and \((F, G)\) is called predecessor of \((F_1, G_1)\) if

\[
a_{(F_1, G_1)} = a_{(F,G)} \quad \text{and} \quad b_{(F_1, G_1)} = -B_{(F,G)}.
\]

The importance of this definition becomes obvious from the following statement.

**Theorem 16** Let \( w \) be an \((F, G)\)-pseudoanalytic function and let \((F_1, G_1)\) be a successor of \((F, G)\). Then \( \dot{w} \) is an \((F_1, G_1)\)-pseudoanalytic function.

**Definition 17** Let \((F, G)\) be a generating pair. Its adjoint generating pair \((F, G)^* = (F^*, G^*)\) is defined by the formulas

\[
F^* = -\frac{2F}{FG - \overline{FG}}, \quad G^* = \frac{2\overline{G}}{FG - \overline{FG}}.
\]

**Theorem 18**

\[
(F, G)^{**} = (F, G),
\]

\[
a_{(F^*, G^*)} = -a_{(F, G)}, \quad A_{(F^*, G^*)} = -A_{(F, G)},
\]

\[
b_{(F^*, G^*)} = -B_{(F, G)}, \quad B_{(F^*, G^*)} = -B_{(F, G)}.
\]
**Lemma 19** If \((F_1, G_1)\) is a successor of \((F, G)\) then \((F, G)^*\) is a successor of \((F_1, G_1)^*\).

The \((F, G)\) integral of \(w\) on a rectifiable curve \(\Gamma\) is, by definition,
\[
\int_{\Gamma} wd_{(F,G)} z = \text{Re} \int_{\Gamma} F^*wdz - i \text{Re} \int_{\Gamma} G^*wdz.
\]
Another important integral is also needed
\[
* \int_{\Gamma} wd_{(F,G)} z = \text{Re} \int_{\Gamma} G^*wdz + i \text{Re} \int_{\Gamma} F^*wdz
\]
(we follow the notations of L. Bers).

A continuous function \(w\) defined in a domain \(\Omega\) is called \((F, G)\)-integrable if for every closed curve \(\Gamma\) situated in a simply connected subdomain of \(\Omega\),
\[
\int_{\Gamma} wd_{(F,G)} z = 0.
\]

**Theorem 20** An \((F, G)\)-derivative \(\dot{w}\) of an \((F, G)\)-pseudoanalytic function \(w\) is \((F, G)\)-integrable and
\[
* \int_{z_0}^{z_1} wd_{(F,G)} z = \omega(z_1) - \omega(z_0).
\]

The integral \(* \int_{z_0}^{z_1} wd_{(F,G)} z\) is called \((F, G)\)-antiderivative of \(\dot{w}\).

**Theorem 21** Let \((F, G)\) be a predecessor of \((F_1, G_1)\). A continuous function is \((F_1, G_1)\)-pseudoanalytic if and only if it is \((F, G)\)-integrable.

## 5 Applications of Bers’ theory to the Schrödinger equation

Let us return to equations (23) and (24) which in a two dimensional case are equivalent to the quaternionic equation (20). In order to use Bers’ notations from the preceding section we rewrite (23) and (24) in the following form
\[
w_\tau = b\overline{w}, \quad (28)
\]
and
\[
v_\tau = \overline{b\overline{v}}, \quad (29)
\]
where \(z = x + iy, x = x_2, y = x_1\) and instead of the imaginary unit \(k\) we write \(i\). It is easy to see that \(b = -\overline{H}_2 = -\partial_{\overline{\tau}}f_0/f_0\) and \(w = P_1, v = P_2\).
As was mentioned above (Remark 13) for equation (28) we know always two solutions

\[ F = \frac{1}{f_0} \quad \text{and} \quad G = if_0, \tag{30} \]

which obviously fulfill (25). Thus \((F, G)\) is a generating pair corresponding to equation (28). We have

\[ a_{(F,G)} = 0, \quad b_{(F,G)} = b = - \frac{\partial_z f_0}{f_0}, \]

\[ A_{(F,G)} = 0, \quad B_{(F,G)} = - \frac{\partial_z f_0}{f_0}. \]

According to Definition 15 the characteristic coefficients for a successor of \((F, G)\) have the form

\[ a_{(F_1, G_1)} = 0, \quad b_{(F_1, G_1)} = \frac{\partial_z f_0}{f_0} = -b. \]

Then due to Theorem 16 if \(w\) is a solution of (28) then its \((F, G)\)-derivative is a solution of the equation

\[ W_z = -bW, \tag{31} \]

but solutions of the last equation multiplied by \(i\) become solutions of (29) and vice versa. Thus we obtain the following statement.

**Theorem 22** Let \(w\) be a solution of (28). Then the function

\[ v = \frac{1}{i} \cdot w = \frac{1}{i} (w + \frac{\partial_z f_0}{f_0} \overline{w}) \]

is a solution of (29).

It is easy to see that according to Definition 17

\[ F^* = -i \frac{1}{f_0}, \quad G^* = f_0 \]

and

\[ b_{(F^*, G^*)} = - \frac{1}{f_0} \frac{\partial_z f_0}{f_0} = -b. \]

Thus the \((F, G)\) integral of a function \(W\) is defined as follows

\[ \int_{\Gamma} Wd_{(F,G)}z = - \text{Re} \int_{\Gamma} \frac{i}{f_0} Wdz - i \text{Re} \int_{\Gamma} \overline{f_0} Wdz \]
From Theorems 20 and 21 we obtain the following result.

**Theorem 23** Let \( v \) be a solution of (29) in a domain \( \Omega \). Then for every closed curve \( \Gamma \) situated in a simply connected subdomain of \( \Omega \),

\[
\text{Re} \int_{\Gamma} \frac{v}{f_0} \, dz + i \text{Im} \int_{\Gamma} f_0 v \, dz = 0.
\]

(32)

**Proof.** For any solution \( v \) of (29) the function \( W = iv \) is a solution of (31). As (31) corresponds to a successor of \((F, G)\), by Theorem 20 \( W \) is \((F, G)\)-integrable. That is

\[
\text{Im} \int_{\Gamma} \frac{W}{f_0} \, dz - i \text{Re} \int_{\Gamma} f_0 W \, dz = 0.
\]

Now substituting \( iv \) instead of \( W \) we obtain (32). □

In order to analyze the meaning of this result for solutions of the Schrödinger equation let us rewrite some statements from Section 3 in our “two-dimensional” notations.

Consider the equation

\[
(-\partial_z \partial_{\bar{z}} + u) f = 0
\]

(33)

where \( u \) and \( f \) depend on \( x, y, \) and \( z = x + iy \). For simplicity we consider \( u \) and \( f \) being real valued functions. The corresponding Riccati equation (5) takes the form

\[
\partial_{\bar{z}} Q + |Q|^2 = u
\]

where \( q = Qj \). Equation (10) turns into (29), \( F \) from (10) and \( v \) from (29) are related by the equality \( F = vj \). Then Theorem 7 in a two-dimensional situation can be rewritten as follows.

**Theorem 24** Let \( f_0 \) be a nonvanishing particular solution of (33) and \( v \) be the general solution of (29) where \( b = -\partial_{\bar{z}} f_0/f_0 \). Then the general solution \( f \) of (33) is obtained from the equation \( \partial_z f = Q f \), where \( Q = \partial_z f_0/f_0 + \partial_z \Psi/\Psi \) and \( \Psi \) is obtained from the equation \( \partial_{\bar{z}} \Psi = v/f_0 \).

As was explained in Remarks 8-10, given a solution \( v \) of (29), the corresponding solution \( f \) of (33) can be constructed analytically. The procedure requires on two steps to reconstruct the potential function from its gradient.

From Proposition 11 we obtain the following statement.
Proposition 25 Let $f_1$ be another nonvanishing solution of \((33)\). Then the function
\[
v = f_0 \partial_z (f_1 / f_0)
\]
is a solution of \((33)\), where $b = -\partial_z f_0 / f_0$.

Having this precise relation between solutions of \((33)\) and \((29)\) we are able to prove the following result.

Theorem 26 (Cauchy’s integral theorem for the Schrödinger equation) Let $f_0$ and $f_1$ be two arbitrary nonvanishing solutions of \((33)\) in a domain $\Omega$. Then for every closed curve $\Gamma$ situated in a simply connected subdomain of $\Omega$,
\[
\text{Re} \int_\Gamma \partial_z (f_1 / f_0) dz + i \text{Im} \int_\Gamma f_0^2 \partial_z (f_1 / f_0) dz = 0.
\]
Proof. Substitution of \((34)\) into \((32)\) gives us the result.

In Example 31 we will give a nontrivial example illustrating this theorem.

From Theorem 21 and Theorem 26 we obtain an analogue of the Morera theorem for the Schrödinger equation \((33)\).

Theorem 27 Let $f_0$ be a nonvanishing particular solution of \((33)\). The function $f_1$ is a solution of \((33)\) also if \((35)\) is valid for every closed curve $\Gamma$ situated in a simply connected subdomain of $\Omega$.

Consider equation \((27)\) with the functions \((30)\). It takes the form
\[
\phi x + if_0^2 \psi = 0.
\]

Proposition 28 Let the function $w = \phi F + \psi G$ be $(F,G)$-pseudoanalytic corresponding to the functions \((27)\), that is
\[
w = \frac{\phi}{f_0} + if_0 \psi
\]
is a solution of \((28)\). Then
\[
< \text{grad} \phi, \text{grad} \psi >= 0.
\]

Proof. The function \((34)\) is a solution of \((28)\) if and only if \((36)\) is valid. Let us rewrite \((36)\) in the form
\[
\phi_x - f_0^2 \psi_y = 0
\]
(38)
\[ \phi_y + f_0^2 \psi_x = 0. \] (39)

From (39) we have \( f_0^2 = -\phi_y/\psi_x \). Substituting this expression into (38) we obtain

\[ \phi_x + \psi_y \frac{\phi_y}{\psi_x} = 0 \]

or

\[ \phi_x \psi_x + \phi_y \psi_y = 0. \]

In general, solution of (36) or equivalently of the system (38), (39) seems to be a difficult task. Nevertheless for a quite general class of functions \( f_0 \) we can obtain solutions of (38), (39) explicitly. Let us make the following suppositions.

**Condition 29** Let \( f_0 \) be a function of some variable \( \rho \):

\[ f_0 = f_0(\rho) \] such that

\[ \Delta \rho/|\text{grad } \rho|^2 \] is a function of \( \rho \). We denote it by \( s(\rho) = \frac{\Delta \rho}{|\text{grad } \rho|^2} \).

The simplest example of such \( \rho \) is of course any harmonic function. Another important example is \( \rho = \sqrt{x^2 + y^2} \).

Consider the system (38), (39) and look for \( \phi \) being a function of \( \rho \): \( \phi = \phi(\rho) \) (as we show below such solution always exists). Then

\[ \psi_x = -\frac{\rho_y}{f_0^2} \phi', \quad \psi_y = \frac{\rho_x}{f_0^2} \phi'. \] (40)

For the solubility of this system we obtain the following condition

\[ \frac{\partial}{\partial x} \left( \frac{\rho_x}{f_0^2} \phi' \right) + \frac{\partial}{\partial y} \left( \frac{\rho_y}{f_0^2} \phi' \right) = 0 \]

which can be written as an ordinary differential equation

\[ \phi'' + \left( s - 2 \frac{f_0'}{f_0} \right) \phi' = 0. \]

From here we have

\[ \phi'(\rho) = e^{-S(\rho)} f_0^2(\rho) \]

where \( S(\rho) = \int s(\rho) d\rho \).

With the aid of (40) we can reconstruct \( \psi \). Nevertheless we are interested nor in \( \phi \) neither in \( \psi \) but in \( \phi_z \) and \( \psi_z \) instead. Having them we construct
the function \( v = i(\phi_z F + \psi_z G) = i(\phi_z / f_0 + i\psi_z f_0) \) which gives us a solution of [29]. We have

\[
\phi_z = \phi' \rho_z = e^{-S} f_0^2 \rho_z \tag{41}
\]

and

\[
\psi_z = -\phi'(\rho_y + i\rho_x) = -i e^{-S} \rho_z. \tag{42}
\]

Then we obtain the following solution of [29]

\[
v_1 = i(\phi_z F + \psi_z G) = 2i f_0 e^{-S} \rho_z.
\]

In a much the same way we can construct another solution of [29] looking for \( \psi = \psi(\rho) \). Then

\[
\phi_x = f_0^2 \rho_y \psi', \quad \phi_y = -f_0^2 \rho_x \psi'.
\]

and \( \psi' = e^{-S} / f_0^2 \). Calculating \( \phi_z \) and \( \psi_z \) we obtain

\[
\phi_z = ie^{-S} \rho_z \tag{43}
\]

and

\[
\psi_z = e^{-S} f_0^2 \rho_z. \tag{44}
\]

Thus we arrive at the following solution of [29]

\[
v_2 = i(\frac{ie^{-S} \rho_z}{f_0} + i \frac{e^{-S} \rho_z f_0}{f_0}) = -2e^{-S} \rho_z.
\]

Denote

\[
F_I = \frac{v_1}{2} \quad \text{and} \quad G_I = \frac{v_2}{2}. \tag{45}
\]

Then \( \text{Im}(F_I G_I) = e^{-2S} |\text{grad } \rho|^2 > 0 \) and hence we have a generating pair for [29] in explicit form. Note that \( (F_I, G_I) \) is not a successor of \( (F, G) \) but a successor multiplied by \( i \): \( (F_I, G_I) = i(F_1, G_1) \).

It is interesting to see what are the new solutions \( f_1 \) and \( f_2 \) of the Schrödinger equation [33] corresponding to \( F_I \) and \( G_I \). As the procedure of converting solutions of [29] into solutions of [33] requires on two steps to find a potential function by its gradient, it will be more illustrative to consider here an example.
Example 30 Let \( u(x, y) = x^2 + y^2 \). Then a particular solution of (33) can be chosen in the form
\[
f_0(x, y) = e^{xy}.
\]
Obviously \( \rho(x, y) = xy \) being a harmonic function satisfies Condition 29.

Then \( F_I = ie^{xy}(y - ix) \) and \( G_I = -e^{-xy}(y - ix) \). Returning to the notations of Section 3 (see the beginning of Section 5) we have \( F_I = k e^{x_1x_2}(x_1 - kx_2) = e^{x_1x_2}(x_2 + kx_1) \) and \( G_I = -e^{-x_1x_2}(x_1 - kx_2) \). Then the corresponding pair of solutions of (10) has the form
\[
F_1 = F_I j = e^{x_1x_2}(-x_1 i + x_2 j) \quad \text{and} \quad F_2 = G_I j = -e^{-x_1x_2}(x_2 i + x_1 j).
\]

Next step (see Remark 10) consists in finding the corresponding functions \( \Psi_1 \) and \( \Psi_2 \) from equation (18). Thus we should reconstruct \( \Psi_1 \) and \( \Psi_2 \) from the equalities
\[
\text{grad } \Psi_1 = -x_1 i + x_2 j \quad \text{and} \quad \text{grad } \Psi_2 = -e^{-2x_1x_2}(x_2 i + x_1 j).
\]

Using the standard formula for finding the potential function from its gradient we obtain
\[
\Psi_1 = -\frac{1}{2}(x_1^2 - x_2^2 - C_1) \quad \text{and} \quad \Psi_2 = \frac{1}{2}(e^{-2x_1x_2} + C_2)
\]
where \( C_1 \) and \( C_2 \) are arbitrary constants.

Now we can construct the vectors \( g_1 = \text{grad } \Psi_1/\Psi_1 \) and \( g_2 = \text{grad } \Psi_2/\Psi_2 \):
\[
g_1 = \frac{2}{x_1^2 - x_2^2 - C_1}(x_1 i - x_2 j), \quad g_2 = 2\left(\frac{C_2}{e^{-2x_1x_2} + C_2} - 1\right)(x_2 i + x_1 j).
\]

Noting that \( h = Df_0/f_0 = x_2 i + x_1 j \) we obtain two solutions \( q_1 = h + g_1 \) and \( q_2 = h + g_2 \) for (3):
\[
q_1 = (x_2 + \frac{2x_1}{x_1^2 - x_2^2 - C_1})i + (x_1 - \frac{2x_2}{x_1^2 - x_2^2 - C_1})j
\]
and
\[
q_2 = (-x_2 + \frac{2C_2x_1}{e^{-2x_1x_2} + C_2})i + (-x_1 + \frac{2C_2x_1}{e^{-2x_1x_2} + C_2})j.
\]

Now we find the functions \( \Phi_1 \) and \( \Phi_2 \) which are solutions of the equations \( \text{grad } \Phi_{1,2} = q_{1,2} \):
\[
\Phi_1 = \ln \left| x_1^2 - x_2^2 - C_1 \right| + x_1 x_2 + C_3
\]

19
\[ \Phi_2 = x_1 x_2 + \ln |e^{-2x_1 x_2} + C_2| + C_4. \]

Then the corresponding solutions \( f_1 \) and \( f_2 \) of (32) have the form
\[ f_1 = e^{\Phi_1} = d_1 (x_1^2 - x_2^2 - C_1) e^{x_1 x_2} = d_1 (y^2 - x^2 - C_1) e^{xy} \]
and
\[ f_2 = e^{\Phi_2} = d_2 (e^{-x_1 x_2} + C_2 e^{x_1 x_2}) = d_2 (e^{-xy} + C_2 e^{xy}) \] (46)
where \( d_1 \) and \( d_2 \) are arbitrary constant.

Thus starting with a particular solution of (33) we constructed two classes of solutions for the same Schrödinger equation.

**Example 31** In order to illustrate the Cauchy integral theorem for the Schrödinger equation let us use the following two particular solutions from the preceding example. Let \( f_0(x, y) = e^{xy} \) and as \( f_1 \) we choose the function from (44) when \( C_2 = 0 \) and \( d_2 = 1 \), \( f_1(x, y) = e^{-xy} \). Both \( f_0 \) and \( f_1 \) are solutions of (33) with the same potential \( u \) in a whole plane. Thus we can apply Theorem 26 and consider \( \Gamma \) being for example a unitary circumference with centre at the origin. Then
\[
\Re \int_{\Gamma} \partial_z (f_1 f_0) dz + i \Im \int_{\Gamma} f_0^2 \partial_z (f_1 f_0) dz
\]
\[ = 2 \Re \int_{\Gamma} (-y + ix) e^{-2xy} dz + 2i \Im \int_{\Gamma} (-y + ix) dz \]
\[ = 2 \Re \int_0^{2\pi} i(- \sin \tau + i \cos \tau) e^{-2 \cos \tau \sin \tau} d\tau + 2i \Im \int_0^{2\pi} i(- \sin \tau + i \cos \tau) d\tau \]
\[ = -2 \int_0^{2\pi} \cos \tau \cdot e^{-2 \cos \tau \sin \tau} d\tau - 2i \int_0^{2\pi} \sin \tau d\tau. \]
It is easy to see that both integrals are really equal to zero.

Let us calculate the characteristic coefficients for the pair \((F_1, G_1)\) defined by (45). We have
\[ a_{(F_1, G_1)} = 0, \quad b_{(F_1, G_1)} = -\frac{\partial_z f_0}{f_0}, \]
\[ A_{(F_1, G_1)} = \frac{\rho_{xz}}{\rho_z} - \frac{\rho_x \rho_z}{\rho_z}, \quad B_{(F_1, G_1)} = -\frac{f_0 \rho_z^2}{f_0 \rho_z}. \]
Consider the equation
\[ \frac{1}{\rho_z} F_I + \frac{1}{\rho_x} G_I = 0 \]
where \( \phi \) and \( \psi \) are real valued functions. It has the form
\[ if_0^2 \frac{1}{\rho_z} \phi - \frac{1}{\rho_x} \psi = 0 \quad (47) \]
or as a system
\[ \frac{1}{\rho_x} \psi_x + f_0^2 \frac{1}{\rho_y} \phi_y = 0 \quad (48) \]
\[ \frac{1}{\rho_y} \psi_y - f_0^2 \frac{1}{\rho_x} \phi_x = 0 \quad (49) \]
Comparing with the system (48), (49), we note that solutions of that system can be transformed into solutions of the system (48), (49) in the following way
\[ \frac{1}{\rho} \phi = \psi \quad \text{and} \quad \frac{1}{\rho} \psi = -\phi. \]

Let us calculate \( F_I^* \) and \( G_I^* \) using Definition 17:
\[ F_I^* = -\frac{f_0}{\rho_z} e^S \quad \text{and} \quad G_I^* = -\frac{i e^S}{f_0 \rho_z}. \]
Consider the equation
\[ \frac{2}{\rho_z} \phi - \frac{2}{\rho_x} \psi = 0 \]
where \( \phi \) and \( \psi \) are real valued functions. It has the form
\[ if_0^2 \frac{2}{\rho_z} \phi - \frac{2}{\rho_x} \psi = 0 \quad (50) \]
Observe that it coincides with (47). Thus we obtain that the function \( u = \phi \phi_I + \psi \psi_I \) is \( (F_I, G_I) \)-pseudoanalytic iff the function \( W = \phi \phi_I^* + \psi \psi_I^* \) is \( (F_I^*, G_I^*) \)-pseudoanalytic.

Let us calculate
\[ B_{(F_I^*, G_I^*)} = \frac{\partial f_0}{f_0}. \]
That is the characteristic coefficient \( b \) of a successor of \((F_I^*, G_I^*)\) is equal to \(-\partial f_0 / f_0 = b_{(F,G)}\). Thus, \((F,G)\) is a successor of \((F_I^*, G_I^*)\). This important observation opens the way to obtain an infinite set of solutions of the original Schrödinger equation (48) if \( f_0 \) satisfies condition 29. Namely, we start with
a solution of (28), for instance with $F$. Then its $(F^*, G^*)$-antiderivative gives us an $(F^*, G^*)$-pseudoanalytic function, more precisely the corresponding functions $\phi$ and $\psi$ such that the function $W = \phi F^* + \psi G^*$ is $(F^*, G^*)$-pseudoanalytic. Then we take these real valued functions $\phi$, $\psi$ and consider the function $v = \phi F_I + \psi G_I$ which is $(F_I, G_I)$-pseudoanalytic. That is $v$ satisfies (29) and hence the vector $F = vJ$ with the aid of the chain of actions described in Remark 10 can be transformed into a solution of (33). Taking the $(F, G)$-antiderivative of $iv$ we obtain another solution of (28) and can start this cycle again. We can represent schematically this procedure for obtaining an infinite sequence of solutions of (29) and consequently of (33) as the following diagram.

\[
\begin{array}{ccc}
(F, G) & \leftarrow & (F_I, G_I) \\
& \uparrow & \\
& (F^*, G^*) &
\end{array}
\]

Let us consider how this procedure works on the following example.

**Example 32** Here we use the same $u$ and $f_0$ as in Example 30. Then $F^*_I = -e^{xy}/(y - ix)$, $G^*_I = -ie^{-xy}/(y - ix)$. Consider

\[
* \int_0^z Fd(F^*_I, G^*_I) = \text{Re} \int_0^z FGdz + i \text{Re} \int_0^z FF^*dz
\]

\[
= -\text{Re} \int_0^1 e^{-2xyt^2} (yt - ixt)(x + iy)dt + i \text{Re} \int_0^1 (xt + iyt)(x + iy)dt
\]

\[
= \frac{e^{-2xy} - 1}{2} + i \frac{(x^2 - y^2)}{2}.
\]

Thus we have $\phi = (e^{-2xy} - 1)/2$ and $\psi = (x^2 - y^2)/2$. It is easy to check that in fact $\phi F^*_I + \psi G^*_I = 0$, so the function

\[
W = \phi F^*_I + \psi G^*_I = -\frac{1}{2(y - ix)} (e^{-xy} - e^{xy} + ie^{-xy}(x^2 - y^2))
\]

is $(F^*_I, G^*_I)$-pseudoanalytic. Now we can construct an $(F_I, G_I)$-pseudoanalytic function as follows

\[
v = \phi F_I + \psi G_I = -\frac{(y - ix)}{2} (e^{-xy}(x^2 - y^2) - i(e^{-xy} - e^{xy})).
\]
Thus \( v \) is a solution of (29) and applying the procedure described above it can be transformed into a solution of (33).

Now multiplying the function \( v \) by \( i \) we obtain a solution of (31) the \((F,G)\)-antiderivative of which gives us a new solution of (28) and the cycle starts again.

Thus starting with a particular solution \( F \) of (28) we obtain an infinite sequence of solutions of (33).

References

[1] Athanasiadis C, Costakis G and Stratis I G 2000 On some properties of Beltrami fields in chiral media Reports on Mathematical Physics 45 257-71

[2] Begehr H 1985 Boundary value problems for analytic and generalized analytic functions. Oxford: North Oxford Academic, “Complex analysis: methods, trends, and applications”, Ed. by E. Lanckau and W. Tutschke, 150-165

[3] Bernstein S 1996 Factorization of solutions of the Schrödinger equation In: Proceedings of the symposium Analytical and numerical methods in quaternionic and Clifford analysis, Seiffen

[4] Bernstein S and Gürlebeck K 1999 On a higher dimensional Miura transform Complex Variables 38 307-19

[5] Bers L 1952 Theory of pseudo-analytic functions. New York University

[6] Bers L 1956 An outline of the theory of pseudoanalytic functions Bull. Amer. Math. Soc. 62 291–331

[7] Kravchenko V G and Kravchenko V V 2003 Quaternionic factorization of the Schrödinger operator and its applications to some first order systems of mathematical physics Journal of Physics A 36 11285-97

[8] Kravchenko V G, Kravchenko V V and Williams B D 2001 A quaternionic generalization of the Riccati differential equation. Kluwer Acad. Publ., “Clifford Analysis and Its Applications”, Ed. by F. Brackx et al., 143-54

[9] Kravchenko V V 2002 Quaternionic reformulation of Maxwell’s equations for inhomogeneous media and new solutions Zeitschrift für Analysis und ihre Anwendungen 21 21-6
[10] Kravchenko V V 2003 *Applied quaternionic analysis* Lemgo: Heldermann Verlag

[11] Kravchenko V V 2003 On Beltrami fields with nonconstant proportionality factor *Journal of Physics A* 36 1515-22

[12] Kravchenko V V and Shapiro M V 1996 *Integral representations for spatial models of mathematical physics.* Harlow: Addison Wesley Longman Ltd., Pitman Res. Notes in Math. Series, v. 351

[13] Lakhtakia A 1994 *Beltrami fields in chiral media* Singapore: World Scientific

[14] Tutschke W 2003 *Generalized analytic functions and their contributions to the development of mathematical analysis* Kluwer Acad. Publ., “Finite or Infinite Dimensional Complex Analysis and Applications” (Advances in Complex Analysis and Its Applications, 2), Ed. by Le Hung Son et al., 101-14

[15] Vekua I N 1962 *Generalized analytic functions.* Oxford: Pergamon Press