PROJECTIVE LINES IN THE AFFINE FLAG MANIFOLD WITH GIVVEN TANGENT ROOT VECTOR

CLAUDE EICHER

Abstract. We first describe the tangent space to the affine flag manifold associated to a simple algebraic group over \( \mathbb{C} \) at the distinguished point starting from standard definitions. We then construct projective lines in the affine flag manifold tangent to given root vectors associated to imaginary roots of the corresponding affine Kac-Moody algebra and describe in which Schubert varieties they lie.

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1. Notation and Conventions

An algebra over a field is understood to be commutative and unital. We denote by \( \text{Alg}/\mathbb{C} \), \( \text{Set} \), and \( \text{Gp} \) the category of \( \mathbb{C} \)-algebras, sets, and groups respectively. \( A^d \) denotes the affine space of dimension \( d \), \( \mathbb{G}_m \) and \( \mathbb{G}_a \) denotes the multiplicative and additive group.
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2. AFFINE KAC-MOODY LIE ALGEBRAS

Here we introduce some basic notation concerning affine Kac-Moody Lie algebras. When \( \mathfrak{g} \) is a simple \( \mathbb{C} \)-Lie algebra, we denote by \( \mathfrak{g}_{KM} \) the corresponding untwisted affine Kac-Moody Lie algebra. The set of roots of \( \mathfrak{g}_{KM} \) is denoted by \( \Phi_{KM} \) and the subset of positive, negative, and real roots by \( \Phi^+_{KM} \), \( \Phi^-_{KM} \), and \( \Phi_{re}^{KM} \) respectively. The indecomposable imaginary root in \( \Phi^+_{KM} \) is denoted by \( \delta \). We have triangular decompositions \( \mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+ \) and \( \mathfrak{g}_{KM} = n^-_{KM} \oplus \mathfrak{h}_{KM} \oplus n^+_{KM} \). Here, e.g. \( n^-_{KM} \) is the direct sum of the negative root spaces \( n^-_{KM} = \bigoplus_{\alpha \in \Phi^-_{KM}} \mathfrak{g}_{KM} \alpha \). The reflections \( s_\alpha, \alpha \in \Phi_{re}^{KM} \), generate the Weyl group \( W \) of \( \mathfrak{g}_{KM} \). It is the semidirect product of the finite Weyl group \( W_\mathfrak{g} \) and the coroot lattice \( \hat{Q} \) of \( \mathfrak{g} \), \( W = W_\mathfrak{g} \ltimes \hat{Q} \). We will denote the Bruhat order on \( W \) by \( \leq \) and the length function by \( \ell : W \to \mathbb{Z}_{\geq 0} \).

3. AFFINE FLAG MANIFOLD AND SCHUBERT VARIETIES

In §3.1-3.3 we recall the construction of the affine flag manifold and Schubert varieties following [PR08].

3.1. Affine flag manifold \( X \). Let \( G \) be a simple linear algebraic group over \( \mathbb{C} \) and \( B \supseteq T \) a choice of a Borel and maximal torus subgroup. Below we are going to assume the reader’s familiarity with the basic concepts of ind-schemes. An exposition of these can be found e.g. in [Lev13][Appendix A]. In contrast to this reference we are, however, going to consider an ind-scheme as a (covariant) functor \( \text{Alg}/\mathbb{C} \to \text{Set} \).

The affine flag manifold is defined as in [PR08][equation (1.6)] as the quotient of fpqc-sheaves \( L^+ G / L^+ I \), where the Iwahori group \( I \) is considered as a smooth affine group scheme over \( \text{Spec} \mathbb{C}[[t]] \). Below \( X \) will always denote the affine flag manifold. We recall that \( L^+ I \) is (represented by) an affine group scheme over \( \mathbb{C} \) not of finite type [PR08][section 1] that is the preimage of \( B \) under the natural morphism \( L^+ G \to G \). Here \( L^+ G \) denotes the similar construction where we consider the group scheme over \( \text{Spec} \mathbb{C}[[t]] \) induced by \( G \). Also, \( L G \) is a group ind-affine ind-scheme over \( \mathbb{C} \). We have a quotient morphism \( \pi : LG \to X \).

The basic theorem about \( X \) is

**Theorem 3.1.** [PR08][Theorem 1.4] \( X \) is a strict ind-scheme of ind-finite type over \( \mathbb{C} \).

The unit \( 1 \in G(\mathbb{C}( (t) )) \) defines the distinguished point of \( X \). We will also denote it by \( 1 \in X(\mathbb{C}) \).
3.2. Weyl group $W$. For the remaining article we assume $G$ to be simply connected. Let $N$ be the normaliser of $T$ in $G$. We have an isomorphism of groups

$$Q = \Hom(\mathbb{G}_m, T) \xrightarrow{\cong} T(\mathbb{C}((t)))/T(\mathbb{C}[[t]]) \ , \ \gamma \mapsto \gamma(t).$$

It extends to an isomorphism of the short exact sequences of groups

$$1 \to Q \to W \to W_\emptyset \to 1$$

and

$$1 \to T(\mathbb{C}((t)))/T(\mathbb{C}[[t]]) \to N(\mathbb{C}((t)))/T(\mathbb{C}[[t]]) \to N(\mathbb{C}((t)))/T(\mathbb{C}((t))) \to 1.$$ 

We define for $w \in W$ the point $\dot{w} \in X(\mathbb{C})$ by any preimage of $w$ under the map $N(\mathbb{C}((t))) \to N(\mathbb{C}((t)))/T(\mathbb{C}[[t]])$. We recover the distinguished point of $X$ as $\dot{1}$.

3.3. Schubert varieties $\overline{X}_w$. We define the (affine) Schubert cell $X_w$ and (affine) Schubert variety $\overline{X}_w$ as in [PR08][Definition 8.3] (denoted by $C_w$ and $S_w$ in loc. cit.). Then $X_w \cong \mathbb{A}^{l(w)}$. $\overline{X}_w$ is a closed subscheme of $X$ and an irreducible projective variety carrying an action of $L^+ I$. We have closed embeddings $\overline{X}_w \hookrightarrow \overline{X}_v$ when $w \leq v$.

**Theorem 3.2.** [PR08][Proposition 9.9] $X = \lim_{\gamma \in W} \overline{X}_w$.

3.4. Kac-Moody group $G_{KM}$. In fact, $X$ can be constructed in terms of the affine Kac-Moody group. We recall, cf. [BD][section 7.15.1], that the affine Kac-Moody group associated to $G$ can be described as $G_{KM} = \mathbb{G}_m^{rot} \ltimes \mathbb{L}G$, where the group of loop rotations $\mathbb{G}_m^{rot} = \mathbb{G}_m$ acts by scaling $t$. Here $\mathbb{L}G$ is a certain central extension of $L G$ by $\mathbb{G}_m^{cent} = \mathbb{G}_m$

$$1 \to \mathbb{G}_m^{cent} \to \mathbb{L}G \xrightarrow{p} LG \to 1$$

that splits canonically over $L^+ G$. We define $\mathbb{L}^+ I = p^{-1}(L^+ I)$ and $I_{KM} = \mathbb{G}_m^{rot} \ltimes L^+ I$.

Also we introduce $T_{KM}$ as the quotient of $I_{KM}$ by its pro-unipotent radical. Then $T_{KM} \cong \mathbb{G}_m^{rot} \ltimes \mathbb{G}_m^{cent} \times T$ is an algebraic torus over $\mathbb{C}$. We have a canonical identification $X = G_{KM}/I_{KM}$ of fpqc-sheaves. The $X_w$ and $\overline{X}_w$ are $I_{KM}$-invariant. It is known that the unique $T_{KM}$-fixed point in $X_w$ is $\dot{w}$.

3.5. Points of $X$ with values in a strictly Henselian local ring.

**Lemma 3.1.** Let $R$ be a $\mathbb{C}$-algebra and a strictly Henselian local ring. The map $G(\mathbb{R}(\!(t)\!))/I(\mathbb{R}(\![t]\!)) \to X(\mathbb{R})$ defined by sheafification is a bijection.

**Proof.** Our reference for the fpqc topology is [GW10], [Vis05]. Let $R$ be any $\mathbb{C}$-algebra. We have an “exact” sequence of pointed sets

$$0 \to (L^+ I)(R) \to (LG)(R) \xrightarrow{\pi} X(R) \xrightarrow{d} H^1((\Spec R)_{fpqc}, L^+ I).$$

Here $H^1((\Spec R)_{fpqc}, L^+ I)$ denotes the pointed set of isomorphism classes of right torsors over the group scheme $L^+ I$ on $\Spec R$ that become trivial on some fpqc cover of $\Spec R$. 

The coboundary map $d$ sends $s \in X(R)$ to the class of the torsor $\mathcal{P}$ defined by $\mathcal{P}(S) = \pi^{-1}(s|S)$ for $S \to \text{Spec} R$ an object of the small fpqc site $(\text{Spec} R)_{\text{fpqc}}$. We have a similar set $H^1((\text{Spec} R)_{\text{ét}}, L^+ I)$ for the étale topology and an inclusion $H^1((\text{Spec} R)_{\text{ét}}, L^+ I) \subseteq H^1((\text{Spec} R)_{\text{fpqc}}, L^+ I)$. It was proven in [HV11] that this inclusion is an equality (we can replace $\mathbb{F}_q$ by $\mathbb{C}$ in loc. cit.). If $R$ is a strictly Henselian local ring, then $H^1((\text{Spec} R)_{\text{ét}}, L^+ I)$ is trivial. Indeed, this follows from the fact that any surjective étale morphism $S \to \text{Spec} R$ has a section [Mil80][I, Theorem 4.2d)]. The statement of the lemma follows. \qed

4. Tangent space to $X$ at the distinguished point

4.1. **Tangent space to an ind-scheme.** Let $(Y_i)_{i \in I}$ be a directed family of schemes over $\mathbb{C}$ together with closed immersions $\iota_{ij} : Y_i \hookrightarrow Y_j$ for $i \leq j$. This defines a strict ind-scheme $Y = \lim_{i \in I} Y_i$ over $\mathbb{C}$. Then $Y(R) = \lim_{i \in I} Y_i(R)$ holds for any $\mathbb{C}$-algebra $R$. Let us furthermore assume that for each $i \in I$ we have a distinguished point $1 \in Y_i(\mathbb{C})$ such that $\iota_{ij}(1) = 1$. The tangent space $T_{1} Y_i$ is the preimage of 1 under the map $Y_i(\mathbb{C}[e]/(e^2)) \to Y_i(\mathbb{C})$ induced by the evaluation map $\mathbb{C}[e]/(e^2) \to \mathbb{C}$ (this is an example of a fiber product). In the same way we define $T_{1} Y$. We have induced embeddings of $\mathbb{C}$-vector spaces $T_{1} Y_i \hookrightarrow T_{1} Y_j$ for $i \leq j$ and $T_{1} Y = \lim_{i \in I} T_{1} Y_i$ holds since $\lim_{i \in I} \iota_{ij}$ commutes with finite inverse limits.

4.2. **Lie algebra of a group-valued functor.** Let $\mathcal{G} : \text{Alg} / \mathbb{C} \to \text{Gp}$ be a (co-)variant functor. We define $\text{Lie} \mathcal{G}$ as $\text{Lie} \mathcal{G} = \ker ev$, where the group homomorphism $ev : \mathcal{G}(\mathbb{C}[e]/(e^2)) \to \mathcal{G}(\mathbb{C})$ is induced by the evaluation map $\mathbb{C}[e]/(e^2) \to \mathbb{C}$. If $\mathcal{G}$ is the functor of points of a $\mathbb{C}$-scheme $G$, $\text{Lie} \mathcal{G}$ coincides with the tangent space of $G$ at the identity, i.e. with the $\mathbb{C}$-vector space $(\mathfrak{m}/\mathfrak{m}^2)^*$, where $\mathfrak{m}$ is the maximal ideal corresponding to $1 \in G(\mathbb{C})$, and is known to be a $\mathbb{C}$-Lie algebra. In particular, $\text{Lie} L^+ I$ is a $\mathbb{C}$-Lie algebra. More generally, according to [BBE02][2.10], $\text{Lie} \mathcal{G}$ is naturally endowed with the structure of a $\mathbb{C}$-Lie algebra when $\mathcal{G}$ commutes with finite inverse limits. As $LG$ is a group ind-affine ind-scheme, its functor of points indeed commutes with finite inverse limits, and hence $\text{Lie} LG$ is a $\mathbb{C}$-Lie algebra.

4.3. **Tangent space.** In this section we return to the affine flag manifold $X$ and abbreviate $\mathbb{C}[e]/(e^2) = D$. We define the tangent space $T_{1} X$ to $X$ at the distinguished point $1 \in X(\mathbb{C})$ as in § 4.1.

**Lemma 4.1.** We have a natural identification $T_{1} X = \mathfrak{n}_{\text{KM}}^{-} = \bigoplus_{\alpha \in \Phi_{\text{KM}}^{-}} g_{\text{KM} \alpha}$ of $T_{\text{KM}}$-modules.

**Proof.** We claim natural identifications

$$ev^{-1}(1) = \left( \frac{G_{\text{KM}}(D)}{I_{\text{KM}}(D)} \xrightarrow{ev} \frac{G_{\text{KM}}(\mathbb{C})}{I_{\text{KM}}(\mathbb{C})} \right)^{-1}(1) = \frac{\ker ev_{G_{\text{KM}}}}{\ker ev_{I_{\text{KM}}}} = \frac{\text{Lie} G_{\text{KM}}}{\text{Lie} I_{\text{KM}}}$$

of $T_{\text{KM}}$-modules. The action of $T_{\text{KM}}$ on $\text{Lie} G_{\text{KM}}$ is induced by the adjoint action of $T_{\text{KM}}$ on $G_{\text{KM}}$ and similarly for $I_{\text{KM}}$. We have introduced the homomorphisms $ev_{G_{\text{KM}}}$:
$G_{\text{KM}}(D) \to G_{\text{KM}}(\mathbb{C})$ and $\text{ev}_{\text{KM}} : I_{\text{KM}}(D) \to I_{\text{KM}}(\mathbb{C})$. In the first identification we use Lemma 3.1 for the strictly Henselian local rings $D$ and $\mathbb{C}$. Let us explain the second identification. There is clearly a natural map from the rhs to the lhs. Let us construct a map from the lhs to the rhs. Consider the assignment $g \mapsto g \text{ev}_{G_{\text{KM}}(g)^{-1}}$. This defines a map $G_{\text{KM}}(D) \to G_{\text{KM}}(D)$ by considering $G_{\text{KM}}(\mathbb{C}) \subseteq G_{\text{KM}}(D)$. $\text{ev}_{G_{\text{KM}}}$ restricts to the identity on this subgroup. Moreover $g \mapsto g \text{ev}_{G_{\text{KM}}(g)^{-1}}$ induces a map from the lhs to the rhs as $\text{ev}_{G_{\text{KM}}(g \text{ev}_{G_{\text{KM}}(g)^{-1}})} = 1$ and for $h \in I_{\text{KM}}(D)$

$$gh \text{ev}_{G_{\text{KM}}(g)^{-1}} = gh \text{ev}_{G_{\text{KM}}(h)^{-1}} \text{ev}_{G_{\text{KM}}(g)^{-1}} = g \text{ev}_{G_{\text{KM}}(g)^{-1}} \text{ev}_{G_{\text{KM}}(g)} \text{ev}_{G_{\text{KM}}(h)^{-1}} \text{ev}_{G_{\text{KM}}(g)^{-1}} = g \text{ev}_{G_{\text{KM}}(g)^{-1}} x.$$ 

Note that $\text{ev}_{G_{\text{KM}}(g)} \in I_{\text{KM}}(\mathbb{C})$ and hence $x \in I_{\text{KM}}(D)$. Also $\text{ev}_{I_{\text{KM}}(x)} = 1$. The two maps between the lhs and rhs are inverse to each other. Together with the description of $\text{Lie} G_{\text{KM}}$ following from the one of $\text{Lie} L G$ given in [PR08][Proposition 9.3] we conclude the statement of the lemma.

The fact that $\overline{X_w}$ is invariant under the action of $T_{\text{KM}}$ implies that $T_1 \overline{X_w} \subseteq T_1 X$ is a $T_{\text{KM}}$-submodule and hence has a basis of root vectors. In the remaining part of the article, we will study the following two questions.

**Question 4.1.** Given $w \in W$, which root vectors lie in $T_1 \overline{X_w}$?

**Question 4.2.** Given $w \in W$ and a root vector $v$ in $T_1 X$, does there exist a projective line contained in $\overline{X_w}$ such that $v$ is tangent to it?

Concerning the relation between these two questions, it is clear that an affirmative answer to the second question implies $v \in T_1 \overline{X_w}$, while the converse does not hold in general.

5. **Projective lines associated to real roots**

In [PR08][section 9.h] it is shown that the $\overline{X_w}$ are isomorphic to the Schubert varieties of the Kac-Moody theory as defined in [Kum02][7.1.13]. In this section we review the description of the tangent space to $\overline{X_w}$ given in [Kum02][Chapter XII] in order to provide a context for our result presented in § 6 below. According to [Kum02][12.1.7 Proposition] the one dimensional $T_{\text{KM}}$-orbit closures in $\overline{X_w}$ containing 1 are given by the closures of the images of the root groups ($\cong G_a$) associated to the elements of the set

$$\Phi_w = \{ \alpha \in \Phi_{\text{KM}} \mid s_{-\alpha} \leq w \}.$$ 

The orbit closure associated to $\alpha$ is a projective line contained in $\overline{X_w}$ and containing the $T_{\text{KM}}$-fixed points 1 and $s_{-\alpha}$. These orbit closures are all distinct. Thus, the answer to Question 4.2 is affirmative in case $v$ is a root vector associated to a root from $\Phi_w$. It
immediately follows [Kum02][12.1.10 Corollary]

\[ T_1 \mathcal{X}_w \supseteq \bigoplus_{\alpha \in \Phi_w} \mathfrak{g}_{\mathcal{K}_M} \alpha. \]  

(5.1)

This provides a partial answer to Question 4.1 in the sense that it describes root vectors contained in $T_1 \mathcal{X}_w$. In general, we have $|\Phi_w| \geq \ell(w)$ [Kum02][12.1.8 Corollary]. Since the singular locus of $\mathcal{X}_w$ is a union of Schubert subvarieties of $\mathcal{X}_w$, 1 is a smooth point of $\mathcal{X}_w$ if and only if $\mathcal{X}_w$ is smooth. Moreover, $\dim \mathcal{X}_w = \ell(w)$ implies that if $\mathcal{X}_w$ is smooth, then (5.1) is an equality and $|\Phi_w| = \ell(w)$. In particular, in this case $T_1 \mathcal{X}_w$ has a basis of root vectors associated to real roots. In §4.1 it was explained that $T_1 \mathcal{X} = \lim_{w \in W} T_1 \mathcal{X}_w$. In conjunction with Lemma 4.1 this shows that not all $\mathcal{X}_w$ can be smooth as $\Phi_{\mathcal{K}_M}$ contains imaginary roots. If 1 is only a rationally smooth point of $\mathcal{X}_w$, see [Kum02][12.2.7 Definition], then (5.1) is not in general an equality, but it is still known that $|\Phi_w| = \ell(w)$ [Kum02][12.2.14 Theorem].

6. Projective lines associated to imaginary roots

The following theorem is the main result of the article. Its proof constructs a projective line in the affine flag manifold of $\text{SL}_2$ that is tangent, at the distinguished point, to a root vector associated to an imaginary root. The statement of the theorem and its proof should be compared with [Hei10][Lemma 6].

**Theorem 6.1.** Let $G = \text{SL}_2$ and $n \in \mathbb{Z}_{>0}$. Let $\check{\alpha} \in \check{Q}$ be the positive coroot. Let $v \in T_1 X$ be a root vector associated to the imaginary root $-n\delta$ ($v$ is unique up to nonzero scalar). There is a one dimensional $\mathbb{G}_m^{\text{rot}}$-orbit $C$ in $\mathcal{X}_{-3n\check{\alpha}}$ such that $C \setminus C = 1 \cup -3n\check{\alpha}$ and $v$ is tangent to $C$. $C$ is isomorphic to a projective line.

**Proof.** 1. Construction of a morphism $f : \mathbb{A}^1 \to \mathcal{X}_w$.

Let $c = 1 + ct^{-n}$. We have the identity in $\text{SL}_2((\mathbb{C}[e]/(e^2))((t)))$

\[
\begin{pmatrix}
c & 0 \\
0 & c^{-1}
\end{pmatrix} = \begin{pmatrix}
1 & c \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & c \\
0 & 1
\end{pmatrix} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

(6.1)

with $cd = -1$. I.e. $d = -1 + ct^{-n}$. We can consider the image of the lhs in $T_1 X$. Then any root vector associated to the root $-n\delta$ is a nonzero multiple of it. We claim that if we consider $c, d$ instead as elements of $\mathbb{C}[e]((t))$, the rhs of this equation defines a morphism $f : \mathbb{A}^1 \to \mathcal{X}_w$, i.e. an element of $\mathcal{X}_w(\mathbb{C}[e])$, such that $f(0) = 1$ and the differential $df(0)$ at 0 is a root vector associated to the root $-n\delta$. Indeed, the entries of these matrices lie in $\mathbb{C}[e]((t))$ and hence the product of the matrices defines points in $\mathcal{X}_w$ for some $w \in W$ depending on $n$. Furthermore it is clear that the image of $f$ is the point 1 together with a one dimensional $\mathbb{G}_m^{\text{rot}}$-orbit $C$, see also 3. below.
2. We have \( \mathcal{f}(\infty) = \begin{pmatrix} e^{-3n} & 0 \\ 0 & t^{3n} \end{pmatrix} 1 = -3n\alpha \) and \( \mathcal{C} \subseteq X_{-3n\alpha} \).

Here \( \mathcal{f} : \mathbb{P}^1 \to X_w \) is the unique extension of \( f \), which exists because \( X_w \) is projective. The rhs of (6.1) considered as an element of \( \text{SL}_2((t))(\mathbb{C}[\epsilon]) \) is

\[
\begin{pmatrix}
1 + e_t^n + e^2t^{-2n} + e^3t^{-3n} & -e^2t^{-2n} \\
-e^2t^{-2n} & 1 - e^{-t^{-n}}
\end{pmatrix}
\]

It is easy to show that

\[
\begin{pmatrix}
1 + e_t^n + e^2t^{-2n} + e^3t^{-3n} & -e^2t^{-2n} \\
-e^2t^{-2n} & 1 - e^{-t^{-n}}
\end{pmatrix} 1 = \begin{pmatrix}
e^{-2} + e^{-1}t^{-n} + t^{-2n} + e^t - 3n & t^2n \\
-t^{-2n} & 0
\end{pmatrix} 1
\]

holds in \( X(\mathbb{C}[\epsilon, e^{-1}]) \) by multiplying with suitable elements in \((L^+) (\mathbb{C}[\epsilon, e^{-1}])\) from the right. From the second line in (6.2) we conclude \( \mathcal{C} \subseteq X_w \) for \( w = -3n\alpha \) using the explicit description of the Schubert cells in the case of \( \text{SL}_2 \) (we will not recall this well-known description here). From the third line in (6.2) we see

\[\mathcal{f}(\infty) = \begin{pmatrix} t^{-3n} & -t^{3n} \\ 0 & t^{3n} \end{pmatrix} 1 = \begin{pmatrix} t^{-3n} & 0 \\ 0 & t^{3n} \end{pmatrix} 1.\]

3. \( \mathcal{C} \) is isomorphic to a projective line.

The action of \( \mathbb{G}_m^{\text{rot}} \) on \( X \) defines an action morphism \( \mathbb{G}_m^{\text{rot}} \to C, \lambda \mapsto \lambda \cdot f(\epsilon) \), for fixed \( \epsilon \neq 0 \). In fact we have \( \lambda \cdot f(\epsilon) = f(\lambda^{1-n}) \). While this action morphism might not be an isomorphism, it yields an isomorphism \( \alpha : \mathbb{G}_m^{\cong} \to C \), which extends to a morphism \( \pi : \mathbb{P}^1 \to \mathcal{C} \). By 2. \( \pi \) is bijective. The differential of \( \mathcal{f} \) satisfies \( df(0) \neq 0 \) and by inspecting the \( \epsilon^{-1} \)-term in the third line of (6.2) we see \( d\mathcal{f}(\infty) \neq 0 \). Indeed, the automorphism of \( X \) defined by \( \begin{pmatrix} t^{3n} & 0 \\ 0 & t^{-3n} \end{pmatrix} \) induces an isomorphism \( \text{T}_{-3n\alpha} X \cong T_1 X \). Now one checks that the image of \( d\mathcal{f}(\infty) \) under this isomorphism is given by

\[
\left[ \begin{pmatrix} t^n & 0 \\ -t^{-5n} & -t^n \end{pmatrix} \right] = \left[ \begin{pmatrix} 0 & 0 \\ -t^{-5n} & 0 \end{pmatrix} \right] \in T_1 X,
\]

which is nonzero. This implies that \( d\pi \) is everywhere nonzero. By [Har92][Corollary 14.10] \( \pi \) is an isomorphism.

For general \( G \), we deduce the following theorem. It implies an affirmative answer to Question 4.2 when \( w = -3n\alpha \) and \( v \) is the coroot \( \check{\alpha} \) considered as a root vector associated to the imaginary root \( -n\delta \).
Theorem 6.2. Let \( n \in \mathbb{Z}_{>0} \). Let \( \check{\alpha} \in \mathfrak{h} \cong \mathfrak{g}_{\text{KM}}^{-n\check{\alpha}} \) be a positive coroot of \( G \). Then there is \( G_{\text{rot}} \)-orbit \( C \) in \( X_{-3n\check{\alpha}} \) such that \( \overline{C} \setminus C = 1 \cup -3n\check{\alpha} \) and \( \check{\alpha} \) is tangent to \( \overline{C} \). \( \overline{C} \) is isomorphic to a projective line.

Proof. Let \( \alpha \) be a positive root of \( \mathfrak{g} \). We have an embedding of Lie algebras \( \mathfrak{sl}_2 \hookrightarrow \mathfrak{g} \) that sends \( e \mapsto e\alpha \) and \( f \mapsto f\alpha \), where \( e\alpha \) and \( f\alpha \) is the standard generator in \( \mathfrak{g}_\alpha \) and \( \mathfrak{g}_{-\alpha} \) respectively. This map is the differential at the identity of a unique closed embedding \( \iota: \text{SL}_2 \hookrightarrow G \) of algebraic groups. Let us indicate that an object is constructed for \( \text{SL}_2 \) instead of \( G \) by a superscript \( ^{(\text{SL}_2)} \). The map \( \iota \) induces an injective group homomorphism \( \omega: W^{(\text{SL}_2)} \hookrightarrow W \), by the definition of the Weyl group given in § 3.2. \( \omega \) embeds the coroot lattice of \( \text{SL}_2 \) into the one for \( G \) by sending the positive coroot of \( \text{SL}_2 \) to \( \check{\alpha} \). Moreover \( \omega \) embeds the finite Weyl group of \( \text{SL}_2 \) into the one for \( G \) by sending the reflection to \( s_\alpha = e^{e\alpha} e^{-f\alpha} e^{e\alpha} \in N/T \). We claim that \( \iota \) induces a closed embedding of Schubert varieties \( X_w^{(\text{SL}_2)} \hookrightarrow X_{\omega(w)} \) for each \( w \in W^{(\text{SL}_2)} \). To prove this, we first note that \( \iota \) induces a closed embedding of ind-schemes \( \phi: X^{(\text{SL}_2)} \hookrightarrow X \). This follows from [BD][lemma in the proof of Theorem 4.5.1] and the fact that the quotient \( G/\iota(\text{SL}_2) \) is affine. Let \( S \) be a quasi-compact locally closed subscheme of \( X^{(\text{SL}_2)} \). Then \( S \) is a locally closed subscheme of \( X_w^{(\text{SL}_2)} \) for some \( w \in W^{(\text{SL}_2)} \). Furthermore \( \phi \) induces an isomorphism between the reduced subscheme structure on the Zariski closure of \( S \) in \( X^{(\text{SL}_2)} \) and the reduced subscheme structure on the Zariski closure of \( \phi(S) \) in \( X \). We take \( S = X_w^{(\text{SL}_2)} \), a Schubert cell, and note that the restriction of \( \phi \) to \( S \) defines a closed embedding \( S \hookrightarrow X_{\omega(w)} \). Hence the reduced subscheme defined on the closure of \( \phi(S) \) in \( X \) is a closed subvariety of the Schubert variety \( \overline{X_w^{(\text{SL}_2)}} \). It follows that \( \phi \) induces a closed embedding \( \overline{X_w^{(\text{SL}_2)}} \hookrightarrow \overline{X_{\omega(w)}} \). Theorem 6.1 now implies the statement. \( \square \)

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139 UNITED STATES

E-mail address: eicher@mit.edu