Abstract. The Riccati equation method is used to establish some new oscillatory criteria for the hamiltonian systems in a new direction, which is to break the positive definiteness restriction imposed on one of coefficients of the hamiltonian system. The obtained results are compared with some known oscillatory criteria.

Key words: Riccati equation, hamiltonian systems, prepared solution, oscillation, trace, non negative (positive) definiteness, the least eigenvalue of the hermitian matrix.
Definition 1.4. A prepared solution $(\Phi(t), \Psi(t))$ of the system (1.1) is called oscillatory on the interval $[a; b]$ (⊂ $[t_0; +\infty)$) if $\det \Phi(t)$ vanishes on $[a; b]$.

Definition 1.5. The system (1.1) is called oscillatory on the interval $[a; b]$ (⊂ $[t_0; +\infty)$) if its every prepared solution is oscillatory on $[a; b]$.

Study of the oscillatory behavior of the system (1.1) is an important problem of the qualitative theory of differential equations and many works are devoted to it (see [1 - 4] and cited works therein). In the works [1] and [2] some oscillatory criteria are proved for the system (1.1) in terms of the coefficients $B(t)$ and $C(t)$ and the fundamental matrix of the linear system $v' = A(t)v$, $t \geq t_0$. In the works [3] and [4] some oscillatory criteria are obtained for the system (1.1) in terms of its coefficients. In all these criteria the positive definiteness condition on $[t_0; +\infty)$ is imposed on the coefficient $B(t)$ of the system (1.1) (therefore $B(t)$ is invertible for all $t \geq t_0$). The goal of this paper is to obtain some oscillatory criteria for the system (1.1) in a new direction, which is to break the positive definiteness restriction imposed on $B(t)$ for all $t \geq t_0$.

§2. Oscillatory criteria

2.1. Main results. Let $a_{jk}(t), \ t \geq t_0, \ j, k = 1, 2$, be real valued continuous functions on $[t_0; +\infty)$. Along with the system (1.1) consider the following scalar one

$$
\begin{align*}
\phi' &= a_{11}(t)\phi + a_{12}(t)\psi; \\
\psi' &= a_{21}(t)\phi + a_{22}(t)\psi, \ t \geq t_0.
\end{align*}
$$

(2.1)

Definition 2.1. The system (2.1) is called oscillatory (on the interval $[a; b]$ (⊂ $[t_0; +\infty)$)) if for its every solution $(\phi(t), \psi(t))$ the function $\phi(t)$ has arbitrary large zeroes (the function $\phi(t)$ vanishes on $[a; b]$).

Hereafter we will assume that $[a; b]$ is a interval from the set $[t_0; +\infty)$. Denote by $\Omega_n$ the set of all normal matrices $M$ (i. e. $M^*M = MM^*$) of dimension $n \times n$ each of which have eigenvalues $\lambda_1 = \lambda_1(M), \ldots, \lambda_n = \lambda_n(M)$ with $Re\lambda_1 = \cdots = Re\lambda_n$ $\text{def} = W(M)$. Note that $M \in \Omega_n$ if in particular $M = \alpha I + iH$, where $\alpha \in (-\infty; +\infty)$, $I$ and $H$ are the identity and a hermitian matrices of dimension $n \times n$ respectively. Denote by $\lambda(H)$ the least eigenvalue of any hermitian matrix $H$, and the non negative (positive) definiteness of $H$ we denote by the symbol $H \geq 0$ ($> 0$). The trace of arbitrary square matrix $M$ we denote by $\text{tr}(M)$. For any continuously differentiable hermitian matrix - function $S(t)$ set:

$$DS(t) \equiv S'(t) + S(t)B(t)S(t) + A^*(t)S(t) + S(t)A(t) - C(t), \quad \sigma_S(t) \equiv W(A^*(t) + S(t)B(t)).$$

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Theorem 2.1. Let the following conditions be satisfied:

1) \( B(t) \geq 0, \ t \in [a; b] \);
2) there exists a continuously differentiable hermitian matrix - function \( S(t) \) on \([a; b]\) such that \( A^*(t) + S(t)B(t) \in \Omega_n, \ t \in [a; b] \);
3) the scalar system
\[
\begin{align*}
\phi' &= \sigma_S(t)\phi + \frac{\lambda(B(t))}{n}\psi; \\
\psi' &= -\text{tr}(DS(t))\phi - \sigma_S(t)\psi, \ t \in [a; b],
\end{align*}
\] is oscillatory on \([a; b] \).

Then the system (1) is also oscillatory on \([a; b] \). \( \square \)

Indicate some particular cases in which the condition 2) of Theorem 2.1 is satisfied:

I) \( A^*(t) \in \Omega_n, \ t \geq t_0 \) \( (S(t) \equiv 0) \);
II) \( A^*(t) = A_1(t) + \left( \begin{array}{cc} A_2(t) & 0 \\ 0 & A_3(t) \end{array} \right), \ B(t) = \left( \begin{array}{cc} B_1(t) & 0 \\ 0 & B_2(t) \end{array} \right), \ t \in [a; b], \) where \( A_2(t) \) and \( B_1(t) \) are some matrix - functions of dimension \( m \times m \) \((m < n)\), \( A_1(t) \in \Omega_n, \ A_3(t) \in \Omega_{n-m} \), \( W(A_3(t)) \equiv 0, \ B_1(t) > 0, \ t \in [a; b] \):

II_1) \( A_2(t)B_1(t) = B_1(t)A_2(t), \ t \in [a; b], \ A_2(t)B_1^{-1}(t) \) is a continuously differentiable matrix - function on \([a; b]\), \( S(t) \equiv \left( \begin{array}{cc} -A_2(t)B_1^{-1}(t) & 0 \\ 0 & 0 \end{array} \right), \ t \in [a; b] \);

II_2) \[A_2(t) + A_2^*(t)]B_1(t) = B_1(t)[A_2(t) + A_2^*(t)], \ t \geq t_0, \ [A_2(t) + A_2^*(t)]B_1^{-1}(t) \) is a continuously differentiable matrix - function on \([a; b] \);
\[
\left( S(t) \equiv \left( \begin{array}{cc} -\frac{A_2(t) + A_2^*(t)}{2}B_1^{-1}(t) & 0 \\ 0 & 0 \end{array} \right), \ t \in [a; b] \right);
\]

III) \( A^*(t) = A_1(t) + a(t)J, \ B(t) = b(t)J, \ J^2 = J = J^* = \text{const} \) \((e.g. \ J = \frac{1}{n}\left( \begin{array}{cccc} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{array} \right))\), \( A_1(t) \in \Omega_n, \ t \in [a; b], \) where \( a(t) \) and \( b(t) \) are some real valued continuous functions on \([a; b] \), \( b(t) \geq 0, \ t \in [a; b] \), \( \frac{a(t)}{b(t)} \) is a continuously differentiable function on \([a; b] \), \( (S(t) \equiv -\frac{a(t)}{b(t)}J) \);

IV) \( A^*(t) = A_1(t)B(t), \ t \in [a; b], \ A_1(t) = A_1^*(t), \ t \in [a; b], \) where \( A_1(t) \) is a continuously differentiable matrix - function on \([a; b] \), \( (S(t) \equiv -A_1(t), \ t \in [a; b]) \).

Remark 2.1. If the conditions of Theorem 2.1 are fulfilled on a countable set of intervals \([a_m; b_m], \ m = 1, 2, \ldots, \) and \( \lim_{m \to +\infty} a_m = +\infty, \) then the system (1.1) is oscillatory. In this case the condition \( B(t) \geq 0 \) may not be fulfilled outside of the set \( \bigcup_{m=1}^{+\infty}[a_m; b_m] \).
If the system (2.2) is oscillatory then from the Sturm type comparison Theorem 3.8 of work [5] (see [5], p. 1511) it follows that for any \( T \geq t_0 \) there exists \( T_1 > T \) such that the system (2.2) is oscillatory on \([T;T_1]\). Due to Remark 2.1 from here and from Theorem 2.1 we immediately get:

**Corollary 2.1.** Let the following conditions be satisfied:

1') \( B(t) \geq 0 \), \( t \geq t_0 \);
2') there exists a continuously differentiable hermitian matrix - function \( S(t) \) on \([t_0;+\infty)\) such that \( A^*(t) + S(t)B(t) \in \Omega_n, \ t \geq t_0 \);
3') the scalar system

\[
\begin{cases}
\phi' = \sigma S(t)\phi + \frac{\lambda(B(t))}{n}\psi; \\
\psi' = -tr(D_S(t))\phi - \sigma S(t)\psi, \ t \geq t_0,
\end{cases}
\]

(2.3)

is oscillatory.

Then the system (1.1) is also oscillatory. \( \square \)

Let \( \mu(t) \) be a real valued continuous function on \([t_0;+\infty)\). Consider the matrix equation

\[
B(t)X + XB(t) = 2\mu(t)I - A(t) - A^*(t), \quad t \geq t_0.
\]

(2.4)

This equation has a solution on \([a; b]\), if in particular \( B(t) \) and \(-B(t)\) have no common eigenvalue for all \( t \in [a; b] \) (e. g. \( B(t) > 0, \ t \in [a; b] \)) (see [6], p. 207). One can easily show that if \( B(t) \geq 0 \) and \( rankB(t) \geq n - 1 \) for \( t \in [a; b] \) (\( t \geq t_0 \)) then there exists some real valued continuous function \( \mu(t) \) on \([a; b] \) ([\( [t_0;+\infty) \)]) such that Eq. (2.4) has a solution on \([a; b] \) (on \([t_0;+\infty)\)). It is not difficult to verify that if \( X(t) \) is a solution of Eq. (2.4) then \( S(t) = \frac{X(t) + X^*(t)}{2} \) is a hermitian solution of Eq. (2.4).

**Theorem 2.2.** Let the following conditions be satisfied:

1) \( B(t) \geq 0, \ t \in [a; b] \);
4) Eq. (2.4) has a continuously differentiable hermitian solution \( S(t) \) on \([a; b]\);
5) the scalar system

\[
\begin{cases}
\phi' = \mu(t)\phi + \frac{\lambda(B(t))}{n}\psi; \\
\psi' = -tr(D_S(t))\phi - \mu(t)\psi, \ t \in [a; b],
\end{cases}
\]

(2.5)

is oscillatory on \([a; b]\).

Then the system (1.1) is also oscillatory on \([a, b]\). \( \square \)

Similar to Corollary 2.1 from here we obtain:
\textbf{Corollary 2.2.} Let the following conditions be satisfied:

1') $B(t) \geq 0, \ t \geq t_0$;
4') Eq. (2.4) has a continuously differentiable hermitian solution $S(t)$ on $[t_0; +\infty]$;
5') the scalar system
\begin{align*}
\phi' &= \mu(t)\phi + \frac{\lambda(B(t))}{n}\psi; \\
\psi' &= -\text{tr}(D_S(t))\phi - \mu(t)\psi, \ t \geq t_0,
\end{align*}

is oscillatory.

Then the system (1.1) is also oscillatory. $\square$

Obviously Theorem 2.1 and Theorem 2.2 as well as Corollary 2.1 and Corollary 2.2 are conditional results in that oscillation of the systems (2.2), (2.3), (2.5), (2.6) is only supposed rather than proved. The first of the following two assertions weakens the conditional character of Theorem 2.1 and Theorem 2.2 and the second one weakens the conditional character of Corollary 2.1 and Corollary 2.2. Set: $E(t) \equiv a_{11}(t) - a_{22}(t), \ t \geq t_0$.

\textbf{Theorem 2.3.} Let the following conditions be satisfied:

6') $a_{12}(t) \geq 0, \ t \in [a; b]$;
7') $\int_a^b \min_{a} a_{12}(t) \exp\left\{ -\int_a^t E(\tau)d\tau \right\}, -a_{21}(t) \exp\left\{ \int_a^t E(\tau)d\tau \right\} dt \geq \pi$.

Then the system (2.1) is oscillatory on $[a; b]. \square$

\textbf{Theorem 2.4.} Let the following conditions be satisfied:

6') $a_{12}(t) \geq 0, \ t \geq t_0$;
8') $\int_{t_0}^{+\infty} a_{12}(t) \exp\left\{ -\int_{t_0}^t E(\tau)d\tau \right\} = -\int_{t_0}^{+\infty} a_{21}(t) \exp\left\{ \int_{t_0}^t E(\tau)d\tau \right\} dt = +\infty$.

Then the system (2.1) is oscillatory. $\square$

\textbf{Remark 2.2.} Theorem 2.4 is a generalization of the Leighton’s oscillatory criterion (see [7], p. 70, Theorem 2.24).

\textbf{Remark 2.3.} Another oscillatory criteria for the system (2.1) are proved in [5], which are applicable to the systems (2.3) and (2.6).

Hereafter in this section we will assume that $B(t) \geq 0, \ t \geq t_0$, and $\sqrt{B(t)}$ is continuously differentiable on $[t_0; +\infty)$. Consider the matrix equation
\begin{equation}
\sqrt{B(t)}A^*(t) - \sqrt{B(t)} = \left[ \sqrt{B(t)}A^*(t) - \sqrt{B(t)} \right]X\sqrt{B(t)}, \ t \geq t_0.
\end{equation}

This equation has a solution on $[a; b]$ (on $[t_0; +\infty)$) if in particular $A^*(t) = \begin{pmatrix} A_1(t) & 0 \\ A_2(t) & 0 \end{pmatrix}$, $B(t) = \begin{pmatrix} B_1(t) & 0 \\ 0 & 0 \end{pmatrix}, \ t \geq t_0$, where $A_1(t)$ and $B_1(t)$ are some matrices of dimension
\[ m \times m \quad (m < n) \quad \text{and det} \, B(t) \neq 0, \quad t \in [a; b] \quad (t \geq t_0). \] In this case

\[ X(t) \equiv \begin{pmatrix} \sqrt{B_1(t)}^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad t \in [a; b] \quad (t \geq t_0), \]

is a solution of Eq. (2.7) on \([a; b]\) on \([t_0; +\infty)).\]

Let \( F(t) \) be a continuous matrix - function of dimension \( n \times n \) on \([t_0; +\infty)\). Denote:

\[
\mathcal{D}_F(t) \equiv -\left[ \frac{[\sqrt{B(t)}A^*(t) - \sqrt{B(t)}]F(t) + F^*(t)[A(t)\sqrt{B(t)} - \sqrt{B(t)}]}{2} \right] - \left[ \frac{[\sqrt{B(t)}A^*(t) - \sqrt{B(t)}]F(t) + F^*(t)[A(t)\sqrt{B(t)} - \sqrt{B(t)}]}{2} \right]^2 B(t)C(t), \quad t \geq t_0.
\]

**Theorem 2.5.** Let the following conditions be satisfied:

1) \( B(t) \geq 0, \quad t \in [a; b]; \)

9) Eq. (2.7) has a solution \( F(t) \) on \([a; b]\) such that \([\sqrt{B(t)}A^*(t) - \sqrt{B(t)}]F(t) + F^*(t)[A(t)\sqrt{B(t)} - \sqrt{B(t)}]\) is continuously differentiable on \([a; b];\)

10) the scalar equation

\[
\phi'' + \frac{\text{tr}(\mathcal{D}_F(t))}{n} \phi = 0, \quad t \in [a; b], \quad (2.8)
\]

is oscillatory on \([a; b].\)

Then the system (1.1) is also oscillatory on \([a; b] \quad \Box.\]

Taking into account Remark 2.1 on the strength of the Sturm’s comparison theorem (see [8], p. 334, Theorem 3.1) from here we immediately get:

**Corollary 2.3.** Let the following conditions be satisfied:

1') \( B(t) \geq 0, \quad t \geq t_0; \)

9') Eq. (2.7) has a solution \( F(t) \) on \([t_0; +\infty)\) such that \([\sqrt{B(t)}A^*(t) - \sqrt{B(t)}]F(t) \)

is continuously differentiable on \([t_0; +\infty);\)

10') the scalar equation

\[
\phi'' + \frac{\text{tr}(\mathcal{D}_F(t))}{n} \phi = 0, \quad t \geq t_0,
\]

is oscillatory.

Then the system (1.1) is also oscillatory. \( \Box.\)

**Remark 2.4.** If \( B(t) > 0, \quad t \in [a; b] \quad (t \geq t_0) \) and \( \sqrt{B(t)}A^*(t) - \sqrt{B(t)} \neq 0 \) then \( F(t) \equiv \sqrt{B^{-1}(t)}, \quad t \in [a; b] \quad (t \geq t_0) \) is the unique solution of Eq. (2.7), and if
in addition $\sqrt{B(t)}$ and $A(t)$ are permutable (e.g. $A(t) = \sum_{j=1}^{N} \alpha_j(t)K^j(t)$, $\sqrt{B(t)} = \sum_{j=1}^{N} \beta_j(t)K^j(t)$, $t \geq t_0$, where $\alpha_j(t)$, $\beta_j(t)$, $j = 1, N$, are some continuous functions on $[t_0; +\infty)$, $K(t)$ is a continuous square matrix function on $[t_0; +\infty)$; more detailed information about permutable matrices one can find in [6 pp. 199 - 207]) then it can be shown that

$$tr(D\sqrt{B^{-1}}(t)) = -tr \left[ \left( \frac{A(t) + A^*(t)}{2} \right) + \left( \frac{A(t) + A^*(t)}{2} \right)^2 + B(t)C(t) - (\sqrt{B(t)'}\sqrt{B^{-1}(t)})' + \frac{1}{2} \left( A(t) + A^*(t) \right)\sqrt{B(t)'}\sqrt{B^{-1}(t)} \right], \quad t \in [a; b], \quad (t \geq t_0).$$

2.2. Examples. In this section we present some examples demonstrating the capacities of the obtained results.

Example 2.1. Consider the matrix equation

$$\Phi'' + K(t)\Phi = 0, \quad t \geq t_0 > 0, \quad (2.10)$$

where $K(t) \equiv \left( \begin{array}{ccc} a_1 \sin \mu_1 t + a_2 \sin \mu_2 t & b \cos \mu_3 t & 0 \\ b \cos \mu_3 t & a_1 \sin \mu_1 t + a_2 \sin \mu_2 t & c \sin \mu_4 t \\ 0 & c \sin \mu_4 t & a_1 \sin \mu_1 t + a_2 \sin \mu_2 t \end{array} \right)$,

$a_1$, $a_2$, $\alpha$, $\beta$, $b$, $c$, $\mu_j$, $j = 1, 4$, are some real constants, $a_j \neq 0$, $j = 1, 2$, $\alpha > 1$, $\beta > 1$, $\mu_1/\mu_2$ is irrational. This Equation is equivalent to the system (1.1) for $A(t) \equiv 0$, $B(t) \equiv I$, where $I$ is the identity matrix of dimension $3 \times 3$, $C(t) \equiv -K(t)$. Therefore according to Theorem 2.1 Eq. (2.10) is oscillatory provided the scalar system

$$\begin{cases} 
\phi' = \frac{1}{3}\psi; \\
\psi' = -3(a_1 \sin \mu_1 t + a_2 \sin \mu_2(t))\phi, \quad t \geq t_0,
\end{cases}$$

is oscillatory, which is equivalent to the oscillation of the scalar equation

$$\phi'' + (a_1 \sin \mu_1 t + a_2 \sin \mu_2 t)\phi = 0, \quad t \geq t_0.$$ 

This equation is oscillatory (see [9], Corollary 1). Therefore the last system also is oscillatory. From here it follows that Eq. (2.10) is oscillatory. The eigenvalues of the matrix $K(t)$ are equal $\lambda_\pm(t) \equiv a_1 \sin \mu_1 t + a_2 \sin \mu_2 t \pm \sqrt{\frac{b^2 \cos^2 \mu_3 t}{\sin^2 \mu_3} + \frac{c^2 \sin^2 \mu_4}{\sin^2 \mu_4}}, \quad \lambda_1(t) \equiv
\[ a_1 \sin \mu_1 t + a_2 \sin \mu_2 t, \ t \geq t_0. \] This shows that the Theorems 5, 6 of work [10], the Theorems 1, 2, 3 of work [11] are not applicable to Eq. (2.10). The conditions of the remaining results of these works and the conditions of the results of the works [1 - 4, 12 - 14] contain arbitrary parameter-functions. Therefore it is very difficult to guess the applicability of these results to Eq. (2.10).

Example 2.2. Set:

\[ A_1^*(t) \equiv \begin{pmatrix} \cos t & a(t) \\ -a(t) & \cos t \end{pmatrix}, \quad B_1(t) \equiv \frac{1}{t} \begin{pmatrix} \sin t & 1 \\ -1 & \sin t \end{pmatrix}, \]

\[ C_1(t) \equiv \begin{pmatrix} -1/t + \alpha \cos t & c(t) \\ c(t) & \beta \sin t \end{pmatrix}, \ t \geq 1, \] where \( \alpha, \beta \in (-\infty; +\infty) \), \( a(t) \) and \( c(t) \) are some continuous functions on \([1; +\infty)\). Consider the system

\[
\begin{aligned}
\Phi' &= A_1(t)\Phi + B_1(t)\Psi; \\
\Psi' &= C_1(t)\Phi - A_1^*(t)\Psi, \ t \geq 1,
\end{aligned}
\] (2.11)

We will use Corollary 2.1 to show that this system is oscillatory. Set: \( S(t) \equiv 0, \ t \geq 1. \) Then it is not difficult to verify that \( A_1^*(t) + S(t)B(t) \in \Omega_n, \lambda(B_1(t)) = \frac{1 - |\sin t|}{t}, \ t \geq 1. \) \( \sigma_S(t) = \cos t, \ D_S(t) = -C_1(t), \ t \geq 1 \) and

\[
\int_1^{+\infty} \frac{1 - |\sin t|}{2t} \exp \left\{ -2 \int_1^t \cos \tau d\tau \right\} dt = +\infty.
\]

By Theorem 2.4 from here it follows that all conditions of Corollary 2.1 for the system (2.11) are fulfilled. Therefore the system (2.11) is oscillatory.

Example 2.3. Consider the system

\[
\begin{aligned}
\Phi' &= K_2(t)\Psi; \\
\Psi' &= -K_2(t)\Phi, \ t \geq 0,
\end{aligned}
\] (2.12)

where \( K_2(t) \equiv \text{diag}\{\nu \sin t, \ldots, \nu \sin t\}, \ t \geq 0, \ \nu \geq \frac{\pi}{2}. \) Obviously Corollary 2.1 is not applicable to this system (the condition 1) is not fulfilled). Note that By Theorem 2.3 for this system for \( a = 2\pi m, \ b = \pi(2m + 1) \) the conditions of Theorem 2.1 are satisfied for each \( m = 1, 2, \ldots. \) Due to Remark 2.1 from here it follows that the system (2.12) is oscillatory.

Remark 2.5. No result of works [1- 4, 12 - 14] is applicable to the systems (2.10) - (2.12).

Remark 2.6. Suppose \( A(t) \equiv 0, \ B(t) = -C(t) \equiv I, \ t \geq 0, \) where \( I \) is the identity matrix. It is evident that in this case for the system (1.1) the conditions 1) - 3) of
Theorem 2.1 are fulfilled on the arbitrary interval \([a; b](\subset [0; +\infty))\) and the condition 4) is fulfilled only if \(b - a \geq \pi\). It also is evident that for this case \((\Phi_0(t), \Psi_0(t))\), where \(\Phi_0(t) \equiv \text{diag}\{\sin t, \ldots, \sin t\}\), \(\Psi_0(t) \equiv \text{diag}\{\cos t, \ldots, \cos t\}\), is a prepared solution to the system (1.1). This solution is not oscillatory on \([\varepsilon; \pi - \varepsilon]\) for each \(\varepsilon \in (0; 1)\). Therefore in the inequality 7) we may not replace \(\pi\) by a number less than \(\pi\) (in this sense the condition 7) is sharp).

Example 2.4. Set \(M(t) \equiv \max\{\sin t, 0\}\), \(A_2(t) \equiv \begin{pmatrix} \mu(t) & 2\sin t & (1 + M(t)) \cos t \\ 0 & \mu(t) & (1 + M(t)) \sin t \\ 0 & 0 & \mu(t) \end{pmatrix}\), \(B_2(t) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & M(t) \end{pmatrix}\), \(C_2(t) \equiv \begin{pmatrix} -M(t) \sin^2 t & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2M(t) \end{pmatrix}\), \(t \geq t_0\), where \(\mu(t)\) is a continuous real valued function on \([t_0; +\infty)\). Consider the system

\[
\begin{align*}
\Phi' &= A_2(t)\Phi + B_2(t)\Psi; \\
\Psi' &= C_2(t)\Phi - A_2^*(t)\Psi, \quad t \geq 1,
\end{align*}
\]

(2.13)

One can readily check that \(S(t) \equiv \begin{pmatrix} 0 & -\sin t & -\cos t \\ -\sin t & 0 & -\sin t \\ -\cos t & -\cos t & 0 \end{pmatrix}\) is a solution to the matrix equation

\[B_2(t)X + XB_2(t) = 2\mu(t)I - A_2(t) - A_2^*(t), \quad t \geq t_0.\]

After some simple calculations we get: \(\lambda(B_2(t)) = M(t), \quad trD_2(t) = -2\sin^2 t, \quad t \geq t_0\). On the basis of Corollary 2.2 and Theorem 2.4 from here we conclude that if the function \(G(t) \equiv \int_{t_0}^t \mu(\tau)d\tau, \quad t \geq t_0,\) is bounded then the system (2.13) is oscillatory.

Example 2.5. Let \(a_{jk}(t), \quad c_{jk}(t), \quad j, k = 1, 3, \beta(t)\) be continuous functions on \([t_0; +\infty)\) such that \(a_{12}(t) + a_{22}(t) = a_{11}(t) + a_{21}(t), \quad a_{31}(t) = a_{32}(t), \quad c_{jk}(t) = c_{kj}(t), \quad j, k = 1, 3, \beta(t) \geq 0, \quad t \geq t_0; \quad \beta(t)\) and \(Re[a_{11}(t) + a_{21}(t) + a_{33}(t)] - \frac{\beta'(t)}{2\beta(t)}\) is continuously differentiable on \([t_0; +\infty)\). Set; \(A_3^*(t) \equiv (a_{jk}(t))_{j,k=1}^3, \quad C_3(t) \equiv (c_{jk}(t))_{j,k=1}^3, \quad B_3(t) \equiv \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \beta(t) \end{pmatrix}\), \(t \geq t_0\). Consider the system

\[
\begin{align*}
\Phi' &= A_3(t)\Phi + B_3(t)\Psi; \\
\Psi' &= C_3(t)\Phi - A_3^*(t)\Psi, \quad t \geq t_0,
\end{align*}
\]

(2.14)
It is not difficult to verify that $B_3(t) \geq 0$, $\sqrt{B_3(t)} = \begin{pmatrix} \frac{\sqrt{2}}{t} & \frac{\sqrt{2}}{t} & 0 \\ \frac{\sqrt{2}}{t} & \frac{\sqrt{2}}{t} & 0 \\ 0 & 0 & \sqrt{\beta(t)} \end{pmatrix}$, $t \geq t_0$, and the matrix function $F_3(t) \equiv \begin{pmatrix} \frac{\sqrt{2}}{t} & \frac{\sqrt{2}}{t} & 0 \\ \frac{\sqrt{2}}{t} & \frac{\sqrt{2}}{t} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\beta(t)}} \end{pmatrix}$, $t \geq t_0$, is a solution to the matrix equation 

$$\sqrt{B_3(t)} A_3(t) - \sqrt{B_3(t)} = [\sqrt{B_3(t)} A_3(t) - \sqrt{B_3(t)}] X \sqrt{B_3(t)}$$

on $[t_0; +\infty)$. So we have all data for calculation of $tr(D_F_3(t))$, $t \geq t_0$. After some simple arithmetic operations we obtain

$$tr(D_F_3(t)) = - \left[ Re(a_{11}(t)+a_{21}(t)+a_{33}(t))-\frac{\beta'(t)}{2\beta(t)} \right]' - \left[ Re(a_{11}(t)+a_{21}(t)+a_{33}(t))-\frac{\beta'(t)}{2\beta(t)} \right]^2 - c_{11}(t) - 2Re c_{12}(t) - c_{22}(t), \quad t \geq t_0.$$ 

By Theorem 2.5 (Corollary 2.3) the system (2.14) is oscillatory on $[a; b]$ (is oscillatory) provided the scalar equation

$$\phi'' + \frac{tr(D_F_3(t))}{3} \phi = 0, \quad t \geq t_0,$$

is oscillatory on $[a; b]$ (is oscillatory).

**Remark 2.7.** One can readily check that $\lambda(B_3(t)) \equiv 0$, $t \geq t_0$. Therefore Theorem 2.1 and Theorem 2.2 as well as Corollary 2.1 and Corollary 2.2 are not applicable to the system (2.14).

§3. Proof of the main results

3.1. Auxiliary propositions. Let $f(t)$, $g(t)$, $h(t)$, $f_1(t)$, $g_1(t)$, $h_1(t)$ be real valued continuous functions on $[t_0; +\infty)$. Consider the Riccati equations:

$$y' + f(t)y^2 + g(t)y + h(t) = 0, \quad t \in [a; b]; \quad (3.1)$$
$$y' + f_1(t)y^2 + g_1(t)y + h_1(t) = 0, \quad t \in [a; b]; \quad (3.2)$$

and the following inequalities

$$\eta' + f(t)\eta^2 + g(t)\eta + h(t) \geq 0, \quad t \in [a; b]; \quad (3.3)$$
\[ \eta' + f_1(t)\eta^2 + g_1(t)\eta + h_1(t) \geq 0, \quad t \in [a; b]; \quad (3.4) \]

**Remark 3.1.** If \( f(t) \geq 0 \) for \( t \in [a; b] \) then every solution of the linear equation
\[ y' + g(t)y + h(t) = 0 \]
on the interval \([a; b]\) is also a solution of the inequality (3.3).

**Remark 3.2.** Every solution of Eq. (3.2) on the interval \([a; b]\) is also a solution of the inequality (3.4).

**Theorem 3.1.** Let Eq. (3.2) has a real solution \( y_1(t) \) on \([a; b]\), and let the following conditions be satisfied: \( f(t) \geq 0 \) and \( \int_a^t \exp \left\{ \int_a^s (f_1(s) + g(s)) ds \right\} \left( f_1(t) - f(t) \right) dy_1^2(t) + (g_1(t) - g(t))y_1(t) + h_1(t) - h(t) \right\} d\tau \geq 0, \quad t \in [a; b], \) where \( \eta_0(t) \) and \( \eta_1(t) \) are solutions of the inequalities (3.3) and (3.4) respectively on \([a; b]\) such that \( \eta_j(a) \geq y_1(a), \quad j = 1, 2. \) Then for every \( \gamma_0 \geq Y_1(a) \) Eq. (3.1) has a real valued solution \( y_0(t) \) on \([a; b]\), satisfying the initial condition \( y_0(a) = \gamma_0. \)

Proof. By analogy of the proof of Theorem 3.1 from [15].

Consider the inequality
\[ y' + f(t)y^2 + g(t)y + h(t) \leq 0, \quad t \in [a; b]; \quad (3.5) \]

**Lemma 3.1.** If \( f(t) \geq 0, \quad t \in [a; b] \) then Eq. (3.1) has a solution on \([a; b]\) if and only if the inequality (3.5) has a solution on \([a; b]\).

Proof. Obviously every solution of Eq. (3.1) is also a solution of the inequality (3.5). Let \( y_1(t) \) be a solution to the inequality (3.5) on \([a; b]\). Set \( \tilde{h}(t) \equiv -y_1'(t) + f(t)y_1^2(t) + g(t)y_1(t), \quad t \in [a; b]. \) Since \( y_1(t)' + f(t)y_1^2(t) + g(t)y_1(t) + \tilde{h}(t) \leq 0, \quad t \in [a; b], \) we have
\[ h(t) \leq \tilde{h}(t), \quad t \in [a; b]. \quad (3.6) \]

Consider the equation
\[ y' + f(t)y^2 + g(t)y + \tilde{h}(t) = 0, \quad t \in [a; b]. \]

Using Theorem 3.1 to this equation and Eq. (3.1) and taking into account (3.6) we conclude that Eq. (3.1) has a solution on \([a; b]\). The lemma is proved.

**Lemma 3.2.** For any two square matrices \( M_1 \equiv (m^1_{ij}), \quad M_2 \equiv (m^2_{ij}) \) the equality
\[ tr(M_1M_2) = tr(M_2M_1) \]
is valid.
Proof. We have \( tr(M_1M_2) = \sum_{j=1}^{n} (\sum_{k=1}^{n} m_{jk}^1 m_{kj}^2) = \sum_{j=1}^{n} (\sum_{k=1}^{n} m_{jk}^2 m_{kj}^1) = tr(M_2M_1) \). The lemma is proved.

**Lemma 3.3.** Let the following conditions be satisfied:

1* \( f(t) \geq 0, \ t \geq t_0; \)  
2* \( h(t) \geq 0, \ t \geq t_0; \)  
3* \( \int_{t_0}^{+\infty} f(\tau) \exp\left\{ -\int_{t_0}^{\tau} g(s)ds \right\} d\tau = +\infty; \)  
4* For some \( t_1 \geq t_0 \) the equation

\[
y' + f(t)y^2 + g(t)y + h(t) = 0, \quad t \geq t_0
\]  

has a real valued solution on \( [t_1; +\infty) \).

Then Eq. (3.7) has a positive solution on \( [t_1; +\infty) \).

Proof. Let (according to the condition 4*) the function \( y(t) \) be a solution of Eq. (3.7) on \( [t_1; +\infty) \) for some \( t_1 \geq t_0 \) and let \( y_1(t) \) be another solution of Eq. (3.7) with

\[
y_1(t_1) > y(t_1). \tag{3.8}
\]

Then from 1* it follows that \( y_1(t) \) exists on \( [t_1; +\infty) \) (see [16]). Show that

\[
y_1(t) > 0, \quad t \geq t_1. \tag{3.9}
\]

Suppose for some \( t_2 \geq t_1 \)

\[
y_1(t_2) \leq 0. \tag{3.10}
\]

Consider the linear equation

\[
y' + \xi(t)y + h(t) = 0, \quad t \geq t_2,
\]

where \( \xi(t) \equiv f(t)y_1(t) + h(t), \ t \geq t_2. \) Obviously \( y_1(t) \) is a solution of this equation. Then by Cauchy’s formula we have:

\[
y_1(t) = \exp\left\{ -\int_{t_2}^{t} \xi(\tau)d\tau \right\} \left[ y_1(t_2) - \int_{t_2}^{t} \exp\left\{ \int_{t_2}^{\tau} \xi(s)ds \right\} h(\tau)d\tau \right], \quad t \geq t_2.
\]

From here from 2* and (3.10) it follows that

\[
y_1(t) \leq 0, \quad t \geq t_2. \tag{3.11}
\]

From 3* and from the easily verifiable equality

\[
\int_{t_2}^{+\infty} f(\tau) \exp\left\{ -\int_{t_2}^{\tau} g(s)ds \right\} d\tau = \exp\left\{ -\int_{t_2}^{t_0} g(s)ds \right\} \times
\]

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\[
\lambda A
\]

Since it follows that the equation 
\[
\text{(3.14)}
\]
follows contradiction proves (3.9). The lemma is proved.

3.1. Proof of Theorem 2.1. Let \((\Phi(t), \Psi(t))\) be a prepared solution of the system (1.1). Show that \(\det \Phi(t)\) vanishes on \([a;b]\). Suppose that it is not true. Then \(\det \Phi(t) \neq 0, \ t \in [a;b]\). It follows from here that the hermitian matrix \(Y_0(t) \equiv \Psi(t)\Phi^{-1}(t), \ t \in [a;b]\), is a solution to the matrix Riccati equation

\[
Y' + YB(t)Y + A^+(t)Y + YA(t) - C(t) = 0, \quad t \in [a;b].
\]  
(3.12)

Let \(S(t)\) satisfies the condition 2). In (3.12) make the substitution: \(Y = Z + S(t), \ t \geq T\). We obtain

\[
Z' + ZB(t)Z + [A^+(t) + S(t)B(t)]Z + Z[A(t) + B(t)S(t)] + D_S(t) = 0, \quad t \in [a;b].
\]

Obviously the hermitian matrix - function \(Z_0(t) \equiv Y_0(t) + S(t), \ t \in [a;b]\), is a solution to this equation on \([a;b]\). Therefore

\[
[trZ_0(t)]' + \frac{\lambda(B(t))}{n}[trZ_0(t)]^2 +
\]

\[
+tr[(A^+(t) + S(t)B(t))Z_0(t) + Z_0(t)[A(t) + B(t)S(t)]] + trD_S(t) \leq 0, \quad t \geq T. \quad (3.13)
\]

Since \(A(t) + S(t)B(t) \in \Omega_n, \ t \geq t_0\), it is not difficult to verify that \(tr[(A(t) + S(t)B(t))Z_0(t) + Z_0(t)[A^+(t) + B(t)S(t)]] = 2\sigma_S(t)trZ_0(t), \ t \geq T.\) From here and from (3.13) we get:

\[
[trZ_0(t)]' + \frac{\lambda(B(t))}{n}[trZ_0(t)]^2 + 2\sigma_S(t)Z_0(t) + trD_S(t) \leq 0, \quad t \geq T. \quad (3.14)
\]

By 1) we have \(\lambda(B(t)) \geq 0, \ t \in [a;b]\). By virtue of Lemma 3.1. from here and from (3.14) it follows that the equation

\[
y' + \frac{\lambda(B(t))}{n}y^2 + 2\sigma_S(t)y + trD_S(t) = 0, \quad t \in [a;b],
\]

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The obtained contradiction proves the theorem.

By virtue of Lemma 3.1 from here it follows that the equation
\[ t \in \left[a, b\right] \]
has a solution \( y(t) \) on \([a; b]\). Therefore (see [17]) the functions
\[ \phi(t) \equiv \exp\left\{ \int_{a}^{t} \left[ \frac{\lambda(B(\tau))}{n} y(\tau) + \sigma_{s}(\tau) \right] d\tau \right\}, \quad \psi(t) \equiv y(t)\phi(t), \quad t \in [a; b], \]
form a non oscillatory solution \((\phi(t), \psi(t))\) of the system (2.2) on \([a; b]\). Hence the system (2.2) is not oscillatory on \([a; b]\), which contradicts the condition 3) of the theorem. The obtained contradiction completes the proof of the theorem.

**3.2. Proof of Theorem 2.2.** Suppose the system (1.1) is not oscillatory on \([a; b]\). Then there exists a prepared solution \((\Phi(t), \Psi(t))\) of the system (1.1) such that \(\det\Phi(t) \neq 0\), \(t \in [a; b]\). Then for the hermitian matrix - function \(Y(t) \equiv \Psi(t)\Phi^{-1}(t), \quad t \in [a; b]\), the following equality takes place
\[ Y'(t) + Y(t)B(t)Y(t) + A^*(t)Y(t) + Y(t)A(t) - C(t) = 0, \quad t \in [a; b]. \]
In this equality make the substitution \(Y(t) \equiv U(t) + S(t), \quad t \in [a; b]\), where \(S(t)\) is a hermitian solution of Eq. (2.3) on \([a; b]\). Taking into account 4) we get:
\[ U'(t) + U(t)B(t)U(t) + (2\mu(t)I - L(t))U(t) + U(t)L(t) + D_S(t) = 0, \quad t \in [a; b], \]
where \(L(t) \equiv B(t)S(t) + A(t), \quad t \in [a; b]\). By Lemma 3.2 \(tr[L(t)U(t) - U(t)L(t)] \equiv 0, \quad t \in [a; b]\). From here and from (3.15) we obtain
\[ [tr(U(t))]' + \frac{\lambda(B(t))}{n}[tr(U(t))]^2 + 2\mu(t)tr(U(t)) + tr(D_S(t)) \leq 0, \quad t \in [a; b]. \]
By virtue of Lemma 3.1 from here it follows that the equation
\[ y' + \frac{\lambda(B(t))}{n} y^2 + 2\mu(t)y + tr(D_S(t)) = 0, \quad t \in [a; b], \]
has a solution \(y_1(t)\) on \([a; b]\). Therefore the functions
\[ \phi_1(t) \equiv \exp\left\{ \int_{a}^{t} \left[ \frac{\lambda(B(\tau))}{n} y_1(\tau) + \mu(\tau) \right] d\tau \right\}, \quad \psi_1(t) \equiv y(t)\phi_1(t), \quad t \in [a; b], \]
form a non oscillatory solution \((\phi_1(t), \psi_1(t))\) of the system (2.4) on \([a; b]\). Therefore the system (2.4) is not oscillatory on \([a; b]\), which contradicts the condition 4) of the theorem. The obtained contradiction proves the theorem.
3.3. Proof of Theorem 2.3. In the system (2.1) make the substitutions:

\[
\begin{align*}
\phi &= \exp\left\{ \int_{a}^{t} a_{11}(\tau) d\tau \right\} \rho \sin \theta; \\
\psi &= \exp\left\{ \int_{a}^{t} a_{22}(\tau) d\tau \right\} \rho \cos \theta, \quad t \in [a; b].
\end{align*}
\]  

(3.16)

We will get:

\[
\begin{align*}
\rho' \sin \theta + \theta' \rho \cos \theta &= A_{12}(t) \rho \cos \theta; \\
\rho' \cos \theta - \theta' \rho \sin \theta &= A_{21}(t) \rho \sin \theta, \quad t \in [a; b],
\end{align*}
\]  

(3.17)

where \( A_{12}(t) \equiv a_{12} \exp \left\{ - \int_{a}^{t} E(\tau) d\tau \right\} \), \( A_{21}(t) \equiv a_{21} \exp \left\{ \int_{a}^{t} E(\tau) d\tau \right\} \), \( t \in [a; b] \). This system is equivalent to the system (2.1) in the sense that to each nontrivial solution \((\phi(t), \psi(t))\) of the system (2.1) corresponds the solution \((\rho(t), \theta(t))\) of the system (3.17) with \( \rho(t) > 0 \), \( t \in [a; b] \), defined by (3.16). Let us multiply the first equation of the system (3.17) by \( \cos \theta \) and the second one by \( \sin \theta \) and subtract from the first obtained the second one. We get:

\[
\theta' \rho = \rho [A_{12}(t) \cos^2 \theta - A_{21}(t) \sin^2 \theta], \quad t \in [a; b].
\]  

(3.18)

Let \((\phi_0(t), \psi_0(t))\) be a nontrivial solution of the system (2.1) and let \((\rho_0(t), \theta_0(t))\) be the solution of the system (3.17) corresponding to \((\phi_0(t), \psi_0(t))\). Then \( \rho_0(t) \neq 0 \), \( t \in [a; b] \), and therefore by (3.18) the following equality takes place

\[
\theta'_0(t) = A_{12}(t) \cos^2 \theta_0(t) - A_{21}(t) \sin^2 \theta_0(t) = \frac{1}{2} \left[ A_{12}(t) - A_{21}(t) + (A_{12}(t) + A_{21}(t)) \cos 2\theta_0(t) \right],
\]  

t \in [a; b].

From here it follows

\[
\theta'_0(t) \geq \frac{1}{2} \left[ A_{12}(t) - A_{21}(t) - |A_{12}(t) + A_{21}(t)| \right] \geq \min\{A_{12}(t), A_{21}(t)\}, \quad t \in [a; b].
\]

Let us integrate this inequality from \( a \) to \( b \). Taking into account the conditions of the theorem we get:

\[
\theta_0(b) - \theta_0(a) \geq \int_{a}^{b} \min\{A_{12}(\tau), -A_{21}(\tau)\} d\tau \geq \pi.
\]
Due to (3.16) from here it follows that $\phi_0(t)$ has at least one zero on $[a; b]$. The theorem is proved.

**3.4. Proof of Theorem 2.4.** Suppose the system (2.1) is not oscillatory. Then (see [17]) the equation

$$y' + a_{12}(t)y^2 + E(t)y - a_{21}(t) = 0, \quad t \geq t_0,$$

(3.19)

has a solution $y(t)$ on $[t_1; +\infty)$ for some $t_1 \geq t_0$. Set: $u(t) \equiv a_{12}(t)\exp\left\{-\int_{t_1}^{t} E(\tau)d\tau\right\}$,

$w(t) \equiv -a_{21}(t)\exp\left\{\int_{t_1}^{t} E(\tau)d\tau\right\}$, $t \geq t_1$. In Eq. (3.19) make the substitution

$$y = z\exp\left\{-\int_{t_1}^{t} E(\tau)d\tau\right\}, \quad t \geq t_1.$$

We obtain

$$z' + u(t)z^2 + w(t) = 0, \quad t \geq t_1,$$

(3.20)

Show that

$$\int_{t_1}^{+\infty} u(\tau)\exp\left\{\int_{t_1}^{\tau} 2u(\tau)d\tau \int_{t_1}^{\tau} w(s)ds\right\}d\tau = +\infty.$$

(3.21)

By 8) we have $\int_{t_1}^{t} w(\tau)d\tau = -\int_{t_1}^{t} a_{21}(\tau)\exp\left\{-\int_{t_1}^{\tau} E(s)ds\right\}d\tau \geq 0, \quad t \geq t_2$, for some $t_2 \geq t_1$.

From here and from 8) it follows (3.21). In Eq. (3.20) make the substitution

$$z = U - \int_{t_1}^{t} w(\tau)d\tau, \quad t \geq t_1.$$ We get:

$$U' + u(t)U^2 - 2u(t)\int_{t_1}^{t} w(\tau)d\tau U + u(t)\left[\int_{t_1}^{t} w(\tau)d\tau\right]^2 = 0, \quad t \geq t_1.$$

(3.22)

Since Eq. (3.19) has a real valued solution on $[t_1; +\infty)$, from the substitutions of dependent variables, made above, it can be seen that Eq. (3.22) has a real valued solution on $[t_1; +\infty)$. On the strength of Lemma 3.3 from here from (3.21) and from the inequalities $a_{12}(t) \geq 0$, $u(t)\left[\int_{t_1}^{t} w(\tau)d\tau\right]^2 \geq 0, \quad t \geq t_1$, it follows that Eq. (3.22) has a positive solution $U_0(t)$ on
Then \( Z_0(t) \equiv U_0(t) - \int_{t_1}^{t} w(\tau)d\tau, \ t \geq t_1 \), is a solution to Eq. (3.20) on \([t_1; +\infty)\) such that
\[
Z_0(t) > -\int_{t_1}^{t} w(\tau)d\tau, \quad t \geq t_1. \tag{3.23}
\]
It follows from (3.20) that
\[
Z_0(t) = Z_0(t_1) - \int_{t_1}^{t} u(\tau)Z_0^2(\tau)d\tau - \int_{t_1}^{t} w(\tau)d\tau, \quad t \geq t_1. \tag{3.24}
\]
From here and from (3.23) it follows that
\[
0 \leq \int_{t_1}^{t} u(\tau)Z_0^2(\tau)d\tau < Z_0(t_1), \quad t \geq t_1 \tag{3.25}
\]
\((Z_0(t_1) = U_0(t_1) > 0)\). Taking into account 8) from here we get:
\[
\left[ Z_0(t_1) - \int_{t_1}^{t} u(\tau)Z_0^2(\tau)d\tau - \int_{t_1}^{t} w(\tau)d\tau \right]^2 \geq 1, \ t \geq T, \text{ for some } T \geq t_1. \]
From here and from (3.24) it follows that \( Z_0^2(t) \geq 1, \ t \geq T \). Therefore by 8) we have \( \int_{T}^{+\infty} u(\tau)Z_0^2(\tau)d\tau \geq \int_{T}^{+\infty} u(\tau)d\tau = +\infty \), which contradicts (3.25). The obtained contradiction completes the proof of the theorem.

3.5. Proof of Theorem 2.5. Suppose for some prepared solution \((\Phi(t), \Psi(t))\) of the system (1.1) \( \det \Phi(t) \neq 0, \ t \in [a; b] \). Then for the hermitian matrix - function \( Y(t) \equiv \Psi(t)\Phi^{-1}(t), \ t \in [a; b] \), the following equality holds
\[
Y'(t) + Y(t)B(t)Y(t) + A^*(t)Y(t) + Y(t)A(t) - C(t) = 0, \quad t \in [a; b].
\]
Multiplying both sides of this equality at left and at right by \( \sqrt{B(t)} \), and Taking into account the equality \( (\sqrt{B(t)}Y(t)\sqrt{B(t)})' = \sqrt{B(t)}Y'(t)\sqrt{B(t)} + \sqrt{B(t)}Y(t)\sqrt{B(t)} + +\sqrt{B(t)}Y(t)\sqrt{B(t)}', \ t \in [a; b] \), we get:
\[
Z'(t) + Z^2(t) + [\sqrt{B(t)}A(t) - \sqrt{B(t)}]Y(t)\sqrt{B(t)} + +\sqrt{B(t)}Y(t)[A^*(t)\sqrt{B(t)} - \sqrt{B(t)}] - \sqrt{B(t)}C(t)\sqrt{B(t)} = 0, \quad t \in [a; b],
\]
where $Z(t) \equiv \sqrt{B(t)}Y(t)\sqrt{B(t)}$, $t \in [a;b]$. From here and from the condition 2) we obtain

$$Z'(t) + Z^2(t) + L(t)Z(t) + Z(t)L^*(t) - \sqrt{B(t)}C(t)\sqrt{B(t)} = 0, \quad t \in [a;b].$$

where $L(t) \equiv \left[\sqrt{B(t)}A^*(t) - \sqrt{B(t)}\right]F(t)$, $t \in [a;b]$. Making substitution

$Z(t) \equiv V(t) - \frac{L(t)+L^*(t)}{2}$, $t \in [a;b]$, in this equality we get:

$$V'(t) + V^2(t) + D_F(t) + \frac{L(t) - L^*(t)}{2}V(t) - V(t)\frac{L(t) - L^*(t)}{2} + B(t)C(t) - \sqrt{B(t)}C(t)\sqrt{B(t)} = 0, \quad t \in [a;b]. \quad (3.26)$$

By Lemma 3.2 $tr\left[\frac{L(t)-L^*(t)}{2}V(t) - V(t)\frac{L(t)-L^*(t)}{2}\right] = tr\left[B(t)C(t) - \sqrt{B(t)}C(t)\sqrt{B(t)}\right] \equiv 0$, $t \in [a;b]$. From here and from (3.26) we obtain:

$$[tr(V(t))'] + \frac{1}{n}[tr(V(t))]^2 + tr(D_F(t)) \leq 0, \quad t \in [a;b].$$

By Lemma 3.1 from here it follows that the Riccati equation

$$y' + \frac{1}{n}y^2 + tr(D_F(t)) = 0, \quad t \in [a;b],$$

has a solution $y(t)$ on $[a;b]$. Then the function $\phi(t) \equiv \exp\left\{\int_a^t \frac{y(\tau)}{n}d\tau\right\}$, $t \in [a;b]$, is a non vanishing solution of Eq. (2.8) on $[a;b]$. Therefore Eq. (2.8) is not oscillatory on $[a;b]$, which contradicts the condition 10) of the theorem. The obtained contradiction completes the proof of the theorem.

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