LOW DEGREE HURWITZ STACKS IN THE GROTHENDIECK RING

AARON LANDESMAN, RAVI VAKIL, AND MELANIE MATCHETT WOOD

ABSTRACT. For $2 \leq d \leq 5$, we show that the class of the Hurwitz space of smooth degree $d$, genus $g$ covers of $\mathbb{P}^1$ stabilizes in the Grothendieck ring of stacks as $g \to \infty$, and we give a formula for the limit. We also verify this stabilization when one imposes ramification conditions on the covers, and obtain a particularly simple answer for this limit when one restricts to simply branched covers.

1. INTRODUCTION

The main results of this paper are Grothendieck ring analogues of classical theorems on the density of discriminants of number fields of degree at most 5 [DH69, Bha05, Bha10a]. Let $\text{Hur}_{d,g,k}$ be the moduli stack of degree $d$ covers of $\mathbb{P}^1$ with Galois group $S_d$ by smooth geometrically connected genus $g$ curves over a field $k$, see Definition 5.3. Let $\text{Hur}_{d,g,k}^s$ be the open substack of $\text{Hur}_{d,g,k}$ corresponding to simply branched covers, i.e., the open subset where the map to $\mathbb{P}^1$ has geometric fibers with at least $d - 1$ points. The main results of this paper are that for each $d \leq 5$, the classes of these moduli spaces converge in the Grothendieck ring as $g \to \infty$, to particularly nice limits. More precisely, we work in a suitably defined Grothendieck ring of stacks $\hat{\tilde{K}}_0(\text{Stacks}_k)$, see Definition 2.6, where as usual $\mathbb{L} := \{A^1\}$ is the class of the affine line.

**Theorem 1.1 (Theorem A).** Suppose $2 \leq d \leq 5$ and $k$ is a field of characteristic not dividing $d!$. In $\hat{\tilde{K}}_0(\text{Stacks}_k)$,

$$\lim_{g \to \infty} \frac{\{\text{Hur}_{d,g,k}^s\}}{\mathbb{L}^{\dim \text{Hur}_{d,g,k}}} = 1 - \mathbb{L}^{-2}.$$ 

**Theorem 1.2 (Theorem B).** Suppose $2 \leq d \leq 5$ and $k$ is a field of characteristic not dividing $d!$. In $\hat{\tilde{K}}_0(\text{Stacks}_k)$,

$$\lim_{g \to \infty} \frac{\{\text{Hur}_{d,g,k}\}}{\mathbb{L}^{\dim \text{Hur}_{d,g,k}}} = \begin{cases} 1 - \mathbb{L}^{-2} & \text{if } d = 2 \\ (1 + \mathbb{L}^{-1})(1 + \mathbb{L}^{-3}) & \text{if } d = 3 \\ \frac{1}{(1 - \mathbb{L}^{-1})\prod_{x \in \mathbb{P}^1_k}(1 + \mathbb{L}^{-2} - \mathbb{L}^{-3} - \mathbb{L}^{-4})} & \text{if } d = 4 \\ \frac{1}{(1 - \mathbb{L}^{-1})\prod_{x \in \mathbb{P}^1_k}(1 + \mathbb{L}^{-2} - \mathbb{L}^{-4} - \mathbb{L}^{-5})} & \text{if } d = 5. \end{cases}$$

The products on the right in the cases $d = 4$ and $d = 5$ are motivic Euler products in the sense of Bilu [Bil17, BF21], or more simply, power structures in the sense of Gusein-Zade, Luengo, and Melle–Hernández [GZLM04], see §2.10.
Theorem 1.1 is a special case of Corollary 10.5 while Theorem 1.2 is a special case of Corollary 10.6. Both are consequences of Theorem 10.4, describing the limits of branched covers with specified ramification, along with rates of convergence. These results lead to conjectures in higher degree, see §1.5.

Remark 1.3. The results Theorem 1.1 and Theorem 1.2 of this paper are stated above with the restriction that the Galois group of the cover is $S_d$. These results continue to hold when one removes this restriction, except that when $d = 4$, covers with Galois group $D_4$ must be removed. One can deduce these claims from Lemma 9.6.

1.4. Motivation. Motivations for Theorem 1.1 and Theorem 1.2 come from number theory, topology, and algebraic geometry.

1.4.1. Arithmetic motivation. One can also view results relating to counting number fields of bounded discriminant as “point counting analogs” of the stabilization of Hurwitz spaces. To spell this out, our main results on stabilization of the classes of Hurwitz spaces, suggest the number of $\mathbb{F}_q$ points of these Hurwitz spaces also stabilize in $g$. (This is not actually implied by our results, because we work in the dimension filtration of the Grothendieck ring, and so it is possible that high codimension substacks of these Hurwitz spaces contain many $\mathbb{F}_q$ points which could potentially alter the $g \to \infty$ limiting behavior of the $\mathbb{F}_q$ point counts.) In the degree 3 case, stabilization of the number of $\mathbb{F}_q$ points was shown by Datskovsky and Wright in [DW88]. Their results actually count $S_3$ covers of $\mathbb{A}^1$, and use that these are equivalent to covers of $\mathbb{P}^1$. However a proof over $\mathbb{P}^1$ has also been given by Gunther in [Gun]. These results have also been generalized to work in degrees 4 and 5 by Bhargava-Shankar-Wang in [BSW15]. Analogs over $\mathbb{Q}$ were known much earlier than these results over global function fields. That is, instead of counting $\mathbb{F}_q$ points of Hurwitz spaces, corresponding to $S_d$ covers of $\mathbb{P}^1_{\mathbb{F}_q}$, the arithmetic analog is to count $S_d$ extensions of $\mathbb{Q}$. When $d = 3$, these counts were carried out by Davenport and Heilbronn [DH69, DH71]. When $d = 4$ and $d = 5$, the number field counting was done by Bhargava in [Bha05, Bha10a, Bha14]. Our theorems can thus be viewed as Grothendieck ring analogs of these number field counting results. Indeed, the “Euler products” occurring in Theorem 1.2 with $L$ replaced by $p$ are exactly those that occur in the densities of discriminants of $S_d$-number fields of degree $d \leq 5$ [DH69, Bha05, Bha10a], which in particular demonstrates the great success of the notion of motivic Euler products. Similarly to our methods, the methods behind the number field counting results only apply when $d \leq 5$ because they rely on specific parametrizations [DF64, Bha04, Bha08] of low degree covers of Spec $\mathbb{Z}$.

1.4.2. Topological motivation. We now describe topological results demonstrating stabilization of Hurwitz spaces. One striking result is due to Ellenberg-Venkatesh-Westerland [EVW16], which has deep applications to number theory. Their result [EVW16, Thm. p. 732] implies that the dimension of the $i$th homology $h_i(\text{Hur}_{3g,C}, \mathbb{Q})$ stabilizes as $g \to \infty$. Unfortunately, although their methods apply in the case of degree 3 covers, they already fail to apply when $d = 4$, see the remarks in [EVW16 p. 732].
If, instead of working with covers of $\mathbb{P}^1$, one works with the full moduli stack of curves with marked points, $\mathcal{M}_{g,n}$, then these stacks satisfy certain homological stabilities, due to Harer, Madsen-Weiss, and others. See, for example, [MW07] and the survey article [Hat11].

1.4.3. Algebro-geometric motivation. Finally, from an algebraic geometric viewpoint, there are some further related unirationality results on objects of low degree and genus. For degrees $d \leq 5$ a simple parametrization of degree $d$ covers was originally given in [Mir85, Thm. 1.1], [CE96, Thm. 4.4], and [Cas96, Thm. 3.8], see also [Theorem 3.13, Theorem 3.14] and [Theorem 3.16] (as well as [Poo08, Prop. 5.1], [Woo11, Thm. 1.1]).

There have also been results proving stabilization of algebraic data relating to $\text{Hur}_{d,g,k}$. When $d = 3$, the rational Chow ring of the simply branched Hurwitz space is known to stabilize to $\mathbb{Q}$ [PV15, Thm. C]. It is also known that the rational Picard groups stabilize when $d \leq 5$, due to Deopurkar-Patel [DP15, Thm. A]. More recently, stabilization of the rational Chow groups for $d \leq 5$ (removing $D_4$ covers when $d = 4$) was demonstrated in [CL21a, Theorem 1.1].

There have also been some related stabilization results working in the Grothendieck ring. For example, the class of smooth hypersurfaces of degree $d$ in $\mathbb{P}^n$ stabilizes as $d \to \infty$ in the Grothendieck ring. This, and various related results are shown by the second and third authors in [VW15]. Building on this, Bilu and Howe prove more general stabilization results for sections of vector bundles in the Grothendieck ring [BH21, Thm. A]. We will use these results crucially in the present paper.

1.5. Conjectures and questions motivated by Theorems A and B. The most natural question following [Theorem 1.1] is whether the pattern continues for higher $d$. The continuation of analogies of this pattern have been conjectured in several different domains.

1.6. Arithmetic conjectures. In the context of counting degree $d$ number fields whose Galois closure has Galois group $S_d$, Bhargava [Bha10b, Conj. 1.2] has conjectured that an analog of [Theorem 1.2] holds for all $d$ (which, as mentioned above, is known for $d \leq 5$). Bhargava has given a specific conjectural expression for the Euler factors. It is natural to ask whether [Theorem 1.2] holds for $d \geq 6$ using the analogous Euler factors. That is, one may ask whether [Theorem 10.4] holds for $d \geq 6$ when all types of ramification are allowed. Further, the heuristics of [Bha10b] also predict the analog of [Theorem 1.1] in the number field counting setting for all $d$ (which again is a theorem for $d \leq 5$ [Bha14, Thm. 1.1]). Bhargava’s heuristics more generally apply to give a conjecture for counting $S_d$ degree $d$ fields with various ramification restrictions, and the analogy in the Grothendieck ring setting would be to conjecture that [Theorem 10.4] holds for $d \geq 6$.

The heuristics above are based on a mass formula proven by Bhargava [Bha10b, Theorem 1.1]. We prove an analogous mass formula in the Grothendieck ring in [Theorem 8.3] which we now state a consequence of. To make a precise statement,
let $\mathcal{X}_d$ denote the stack over $k$ whose $T$ points are finite locally free degree $d$ Gorenstein covers $Z$ of $T \times \text{Spec} \ k[\varepsilon]/(\varepsilon^2)$ so that for each geometric point $\text{Spec} \ k \to T$, $Z \times T \times \text{Spec} \ k[\varepsilon]/(\varepsilon^2)$ has $1$-dimensional Zariski tangent space at each point. Given a partition $R = (r_1, r_2, \ldots, r_n)$ of $d$, we define $r(R) := \sum_{i=1}^{n} (r_i - 1)t_i$ to be its ramification order. We can then deduce the following corollary of Theorem 8.3, also see Remark 8.8, by summing over partitions of $d$ in the same way that [Bha10b, Proposition 2.3] was deduced from [Bha10b, Proposition 2.2].

**Corollary 1.7.** For $d \geq 1$ and $k$ a field of characteristic not dividing $d!$, in $\widehat{\mathbb{K}}_0(\text{Stacks}_k)$,

$$\{\mathcal{X}_d\} = \sum_{R \vdash d} \mathbb{L}^{-r(R)} = \sum_{j=0}^{d-1} q(j, d - j)\mathbb{L}^{-j},$$

where $q(j, d - j)$ is the number of partitions of $j$ into at most $d - j$ parts.

The above heuristics can be expanded to make predictions when other finite groups replace $S_d$. These expanded heuristics are often called the Malle-Bhargava Principle (see [Woo16]), though in complete generality the predictions are not always correct. This principle, as long as one is imposing only geometric local conditions, (i.e. only local conditions on ramification,) naturally extends to the Grothendieck ring setting. Then, one can ask in what generality the predictions of the principle hold. Moreover, in the field counting setting, one naturally counts extensions of global fields other than $\mathbb{Q}$ or $\mathbb{F}_q(t)$, and the analog here would be replacing $\mathbb{P}^1$ with another fixed curve, which is another interesting direction to try to understand.

In addition to the above conjectures on $S_d$ extensions, there are also many open questions about Grothendieck ring versions of other extension counting problems. One particularly accessible problem may be that of counting $D_4$ extensions. In [CDyDO02, Corollary 1.4], the number of $D_4$ extensions of $\mathbb{Q}$ was computed when counted by discriminant, though the answer does not appear to have a simple closed form, and was expressed in terms of a sum over quadratic extensions of $\mathbb{Q}$. However, in [ASVW21, Theorem 1] these extensions were counted by conductor, and there was a closed form answer, expressed in terms of an Euler product.

**Question 1.8.** What is the class of the locus of $D_4$ covers of $\mathbb{P}^1$ in the Grothendieck ring $\widehat{\mathbb{K}}_0(\text{Stacks}_k)$ when counted by discriminant or conductor?

Similarly, it would be interesting to compute the class of abelian covers of $\mathbb{P}^1$.

**Question 1.9.** Fix an abelian group $G$. What is the class of the locus of $G$ covers of $\mathbb{P}^1$ in the Grothendieck ring $\widehat{\mathbb{K}}_0(\text{Stacks}_k)$ when counted by discriminant or conductor?

One way to approach this question could be to use that the moduli spaces of abelian covers of $\mathbb{P}^1$ can be described in terms of certain configuration spaces of (colored) points on $\mathbb{P}^1$. The classes of such configuration spaces can be extracted from [VW15, §5].
1.10. **Error terms and second order terms.** It would be interesting to understand the error terms in Theorem 10.4. More precisely, in Theorem 10.4, we show the equalities of Theorem 1.1 and Theorem 1.2 hold not just in the limit, but even hold for any fixed $g$ up to codimension $r_{d,G} := \min(\frac{2g+c_2}{d(d-1)}, \frac{2g+d-1}{d} - 4^{d-3})$, for $c_3 = 0, c_4 = -2, c_5 = -23$. We say two classes of dimension $d$ are equal modulo codimension $r$ in $\hat{K}_0(\text{Stacks}_k)$ if their difference lies in filtered part of $\hat{K}_0(\text{Stacks}_k)$ of dimension at most $d - r$. Concretely, in degree 3, a special case of Theorem 10.4 says:

**Corollary 1.11.** Suppose $k$ is a field of characteristic not dividing 6. Then

$$\frac{\{\text{Hur}_{3,g,k}\}}{L^{\dim \text{Hur}_{3,g,k}}} \equiv (1 + L^{-1}) \left(1 - L^{-3}\right)$$

are equal modulo codimension $\frac{g+2}{3}$ in $\hat{K}_0(\text{Stacks}_k)$.

Focusing on the degree 3 case, Roberts’ conjecture states that the number of degree 3 field extensions of $\mathbb{Q}$ of discriminant at most $X$ is $aX + \beta X^{5/6} + o(X^{5/6})$, for appropriate constants $a, \beta$. This was proved in [BST13] and [TT13] independently. Moreover, the error term was further improved to $aX + \beta X^{5/6} + O(X^{3/2+\varepsilon})$ in [BTT21].

In the context of function fields, one might similarly expect $a q^{\dim \text{Hur}_{d,g,k}} \beta q^{5/6 \dim \text{Hur}_{d,g,k}} + o(q^{5/6 \dim \text{Hur}_{d,g,k}})$ to count the number of extensions of $F_q(t)$ of genus $g$. Progress towards this was made in [Zha13]. In the context of the Grothendieck ring, as mentioned above, we were able to compute the class of the Hurwitz stack up to codimension $r_{3,g} := \min(\frac{g}{4}, \frac{g+2}{3}) = \frac{g}{4}$. Since $\dim \text{Hur}_{d,g,k} = 2g + 4$, we find $\frac{5}{6} \dim \text{Hur}_{d,g,k} = \dim \text{Hur}_{d,g,k} - \frac{g+2}{3}$, and so a weakened form of Roberts’ conjecture is the following:

**Conjecture 1.12.** Suppose $k$ is a field of characteristic not dividing 6. Then

$$\frac{\{\text{Hur}_{3,g,k}\}}{L^{\dim \text{Hur}_{3,g,k}}} \equiv (1 + L^{-1}) \left(1 - L^{-3}\right)$$

are equal modulo codimension $\frac{g+2}{3}$ in $\hat{K}_0(\text{Stacks}_k)$.

**Remark 1.13.** Note that $\frac{g+2}{3}$ is in fact the second term in the minimum defining $r_{3,g}$. There is only one step in our proof where the error term we introduce has codimension less than $\frac{g+2}{3}$, namely when we apply the sieve of [BH21] in Proposition 9.10 and Lemma 9.11. So if the sieving machinery could be improved, it may lead to a proof of Conjecture 1.12.

**Remark 1.14.** In the degree 3 case, it would be quite interesting to actually find the second order term, instead of just predicting the codimension of the error. Moreover, following [BTT21] it would be extremely interesting if it were possible to determine an asymptotic expression for $\{\text{Hur}_{3,g,k}\}$ up to codimension $1/3 \dim \text{Hur}_{d,g,k}$.

Additionally, it would be interesting, though likely more difficult, to determine the codimension of the error and the second order terms in degrees 4 and 5.
1.15. **Topological conjectures.** If $\text{Conf}_n$ denotes the configuration space of points on $\mathbb{P}^1$, i.e., the open subscheme of $\text{Sym}_n^{\mathbb{P}^1}$ parameterizing reduced degree $n$ subschemes of $\mathbb{P}^1$, then we have $\frac{\text{dim} \text{Conf}_n}{\text{dim} \text{Conf}_n} = 1 - \mathbb{L}^{-2}$ in the Grothendieck ring of varieties. This follows from [VW15, Lem. 5.9(a)] as we explain further toward the end of §11.3. There is a map $\text{Hur}_{d,g,k} \to \text{Conf}_{2g-2+2d}$ sending a curve to its branch locus, see [FP02]. Using this, Theorem 1.1 and the explicit formula for $\{\text{Conf}_{2g-2+2d}\}$ implies that the source and target of this map have equivalent classes in $\widetilde{K}_0(\text{Stacks}_k)$ (defined in [Definition 2.6]):

**Corollary 1.16.** For $2 \leq d \leq 5$ and $k$ a field of characteristic not dividing $d!$, 

$$
\lim_{g \to \infty} \frac{\{\text{Hur}_{d,g,k}\}}{\text{dim} \text{Hur}_{d,g,k}} = \frac{\{\text{Conf}_{2g-2+2d}\}}{\text{dim} \text{Hur}_{d,g,k}}
$$

in $\widetilde{K}_0(\text{Stacks}_k)$.

It was also conjectured in [EVW16, Conj. 1.5] that this map $\text{Hur}_{d,g,k} \to \text{Conf}_{2g-2+2d}$ induces an isomorphism on $i$th homology for fixed $d$ and sufficiently large $g$. This is in fact open for $d \geq 3$, though recent work of Zheng [Zhe21, Thm. 1.2] proves a closely related result in the $d = 3$ case, by finding the stable cohomology of $\text{Hur}_{3,g,C}$. Theorem 1.1 could be seen as an additional motivation for this conjecture, especially for $d \leq 5$. (Technically a slight variant of the above was conjectured in [EVW16, Conj. 1.5], with $\mathbb{A}^1$ base in place of $\mathbb{P}^1$.)

1.17. **Spelling out some questions.** Despite the numerous parametrizations mentioned above, the question of whether there exists simple parametrizations of covers of degree 6, or even whether the Hurwitz stack of genus $g$ degree 6 covers (for large $g$) is unirational, remains wide open.

Returning to the simply branched case for simplicity, we have now seen several ways in which we could ask whether the spaces $\text{Hur}_{d,g,k}$ and $\text{Conf}_{2g-2+2d}$ are similar as $g \to \infty$. The following questions have been raised:

1. Do they have the same points counts (asymptotically) over $\mathbb{F}_q$?
2. Do they have the same cohomology, in some stable limit?
3. Do they have the same normalized limit in the Grothendieck ring?

We also include:

4. Are they piecewise isomorphic up to pieces of codimension going to $\infty$?

Even though it is not technically about these spaces, in this sequence of questions one should also include:

1. Are the asymptotic counts of $S_d$ number fields as predicted by Bhargava in [Bha10b]?

For $d \geq 6$, it seems progressively harder to believe the questions (1) and (1'), (2), (3), and (4) could have positive answers, though for $d \leq 5$ the same parametrizations lead to positive answers to (1), (1'), (3), and (4) (and nearly to (2) for $d = 3$).
1.18. **Idea of the proof.** The idea of the proof of Theorem 1.1 and Theorem 1.2 is simplest to understand in the degree 3 case, so we describe this first. Miranda [Mir85] gave a parametrization of degree 3 covers of a base scheme, and we explain here how we can apply it for degree 3 overs of \( \mathbb{P}^1 \). Any degree 3 cover of \( \mathbb{P}^1 \) has a canonical embedding into a \( \mathbb{P}^1 \)-bundle \( \mathbb{P}E \) over \( \mathbb{P}^1 \). We can write \( E \cong \mathcal{O}(a) \oplus \mathcal{O}(b) \) where \( a + b = g + 2 \) and \( a \leq b \). We can therefore stratify the Hurwitz space by the isomorphism type of the bundle \( E \). The degree 3 curves lie in a particular linear series on \( \mathbb{P}E \). The idea is now to compute the locus of smooth curves in this linear system with particular ramification conditions, and then sum over all splitting types of bundles \( E \). The condition for a degree 3 cover of \( \mathbb{P}^1 \) to be smooth in a fiber over \( p \) can be checked over the preimage of the second order neighborhood of \( p \) in \( \mathbb{P}E \). We directly compute the classes of such curves in such an infinitesimal neighborhood. Using the notion of motivic Euler products, we can “multiply” these local classes to obtain the global class of smooth curves in \( \mathbb{P}E \) in the relevant linear system, at least up to high codimension. We then sum these resulting classes over allowed splitting types of \( E \). It turns out that we must have \( E \cong \mathcal{O}(a) \oplus \mathcal{O}(b) \) with \( a \leq b, 2a \geq b \), and a general member of the relevant linear system on any such bundle gives a smooth trigonal curve. Miraculously, in the simply branched case, this motivic Euler product exactly cancels out with the sum over splitting types of \( \mathbb{P}E \), weighted by their automorphisms. This follows from a motivic Tamagawa number formula for \( \text{SL}_2 \).

To generalize this idea to the cases of degrees 4 and 5 requires substantial additional work. First, it is no longer the case that curves of degrees 4 and 5 are elements of linear systems on a surface. Rather, there are parametrizations due to Casnati-Ekedahl [CE96, Cas96] describing covers of degree \( d \) in terms of pairs of vector bundles \( E \) and \( F \), where \( E \) has rank \( d - 1 \) and \( F \subset \text{Sym}^2 E \) corresponds to a certain family of quadrics determined by the curve. In the \( d = 4 \) case, \( F \) has rank 2, corresponding to 4 points in \( \mathbb{P}^2 \) being a complete intersection of two quadrics, while in the case \( d = 5 \), \( F \) has rank 5, corresponding to 5 points in \( \mathbb{P}^3 \) being the vanishing locus of the five \( 4 \times 4 \) Pfaffians of a certain \( 5 \times 5 \) matrix of linear forms. As in the degree 3 case, we can then stratify the Hurwitz stack in terms of the splitting types of \( E \) and \( F \), and compute the classes yielding curves of degree \( d \) as sections of a certain vector bundle \( \mathcal{H}(E, F) \) on \( \mathbb{P}^1 \), depending on \( E \) and \( F \). It is significantly more difficult to calculate the relevant local classes giving the smoothness conditions in fibers in degrees 4 and 5 than it is in degree 3. Nevertheless, we are able to do so by reformulating the question in terms of computing classes of certain classifying stacks for positive dimensional disconnected algebraic groups, and applying a number of results of Ekedahl. The result is Theorem 8.3, which can be viewed as a motivic analog of Bhargava’s mass formula [Bha10b] counting extensions of local fields in arbitrary degree. The specific splitting types of \( E \) and \( F \) which appear are not nearly so simple as in the degree 3 case, but it turns out that the expressions work out modulo high codimension. For this it is important not to count \( D_4 \) covers, i.e., degree 4 covers which factor through a hyperelliptic curve. As in the degree
3 case, it turns out that, at least in the simply branched case, the sum over splitting types of $\mathcal{E}$ and $\mathcal{F}$ cancel out with the local conditions we impose, again by the Tamagawa number formula.

1.19. **Outline of the paper.** The structure of the remainder of the paper is as follows. In §2 we give background on the Grothendieck ring of stacks, setup the precise variant we will work in, and recall the notion of Motivic Euler products. Then, in §3 we prove generalizations of parametrizations due to Miranda, Casnati-Ekedahl, and Casnati regarding Gorenstein covers of degree $d \leq 5$. In degrees 3 and 4, generalizations to arbitrary covers of an arbitrary base have been previous shown by Poonen [Poo08, Prop. 5.1] and the third author [Woo11, Thm. 1.1], but in degree 5 we require new arguments, and here we present a (mostly) uniform argument for degrees 3, 4, and 5. In §4 we upgrade the above mentioned parametrizations for $d \leq 5$ to describe simple presentations of the stack of degree $d$ Gorenstein covers as a global quotient stack. Having settled the above preliminaries, we define the Hurwitz stacks we will work with in §5 and prove they are algebraic. We then describe natural stratifications of these Hurwitz stacks that arise from the structure of the parametrizations in §6. Using these parametrizations, we give descriptions of these strata as quotient stacks in §7. We next begin our proof of the main theorem by computing the local conditions in the Grothendieck ring corresponding to a cover being smooth with specified ramification conditions in §8. In §9 we establish bounds on the codimension of the contributions to the Hurwitz stack from various strata, which will enable us to prove our main result in §10. The proof for the case of degree 2 is slightly different from that in degrees $3 \leq d \leq 5$, and we complete this in §11.

1.20. **Notation.** Let $X_Z$ denote the fibered product $X \times_Y Z$ of schemes, when $Y$ is clear from context. Similarly define $X_R := X \times_Y \text{Spec } R$.

Recall that for $G$ a group, the wreath product $G \wr S_n$ is the semidirect product $G^n \rtimes S_n$ where $S_n$ acts on $G^n$ by the permutation action on the $n$ copies of $G$. More generally, for $\mathcal{E}$ a category, let $\mathcal{E} \wr BS_j$ denote the corresponding wreath product of categories (see [Eke09b, p. 5]) so that in particular, $BG \wr BS_j = B(G \wr S_j)$.

For $\mathcal{X}$ a stack, and $G$ a group scheme acting on $\mathcal{X}$, we use $[\mathcal{X} \!/ G]$ to denote the quotient stack. To avoid confusion with this notation, for $\mathcal{X}$ a stack, we use $\{\mathcal{X}\}$ to denote its class in the Grothendieck ring of stacks, see Definition 2.6.

We call an algebraic group $G$ over a field $k$ special if every $G$-torsor over a $k$-scheme $X$ is trivial Zariski locally on $X$.

When working in $\widetilde{\mathbb{K}}_0(\text{Spaces}_k)$, defined in Definition 2.6 we say two classes $A, B \in \widetilde{\mathbb{K}}_0(\text{Spaces}_k)$ of dimension $d$ are equal modulo codimension $n$ to mean $A - B$ lies in the dimension $d - n$ filtered part of $\widetilde{\mathbb{K}}_0(\text{Spaces}_k)$.

Let $D := \text{Spec } k[\varepsilon]/(\varepsilon^2)$ be the dual numbers. For $X$ a projective scheme over $Y$, let $\text{Hilb}^d_{X/Y}$ denote the Hilbert scheme parameterizing degree $d$ dimension 0 subschemes of $X$ over $Y$. 
For $X \to Y$ a finite locally free map, and $Z$ an $X$-scheme, let $\text{Res}_{X/Y}(Z) \to Y$ denote the Weil restriction. Recall (e.g., [BLR90 §7.6]) that the Weil restriction is the functor defined on $T$ points by $\text{Res}_{X/Y}(Z)(T) = Z(T \times_Y X)$. For $Z$ quasi-projective over $X$, $\text{Res}_{X/Y}(Z)$ is representable [BLR90 §7.6, Thm. 4].

1.21. Acknowledgements. We thank Hannah Larson for carefully reading the paper and offering especially detailed comments. We also thank Manjul Bhargava, Margaret Bilu, Samir Canning, Gianfranco Casnati, Jordan Ellenberg, Sean Howe, Nikolas Kuhn, Anand Patel, Federico Scavia, Craig Westerland, Takehiko Yasuda for helpful discussions related to this paper. AL was supported by the National Science Foundation under Award No. DMS 2102955. RV was partly supported by NSF grant DMS-1601211. MMW was partly supported by a Packard Fellowship for Science and Engineering, a NSF Waterman Award DMS-2140043, and NSF CAREER grant DMS-2052036.

2. BACKGROUND: THE GROTHENDIECK RING OF STACKS AND MOTIVIC EULER PRODUCTS

In this section, we begin by defining useful variants of the Grothendieck ring. Ultimately, we will compute the classes of Hurwitz stacks in a ring we call $\widetilde{K}_0(\text{Spaces}_k)$, obtained from the usual Grothendieck ring of varieties by quotienting by universally bijective (i.e, radicial surjective) morphisms, inverting $\mathbb{L} = \{ \mathbb{A}^1 \}$, and then completing with respect to the dimension filtration. Following this, we recall basic definitions associated to motivic Euler products, following [Bil17] and [BH21]. We also prove these Euler products satisfy a multiplicativity property (Lemma 2.14).

2.1. Variations of the Grothendieck Ring. Recall that we are working over a fixed field $k$. We begin by introducing the Grothendieck ring of algebraic spaces.

Definition 2.2. Let $K_0(\text{Spaces}_k)$ denote the Grothendieck ring of algebraic spaces over $k$. This is the ring generated by classes $\{X\}$ of algebraic spaces $X$ of finite type over $k$ with relations given by $\{X\} = \{Y\}$ if there is an isomorphism $X \simeq Y$ over $k$ and $\{X\} = \{Z\} + \{X - Z\}$ for any closed sub-algebraic space $Z \subset X$. Applying this in the case $Z = X^{\text{red}}$, we have $\{X\} = \{X^{\text{red}}\}$. Multiplication is given by $\{X\} \cdot \{Y\} = \{X \times_k Y\}$.

Proposition 2.3. The ring $K_0(\text{Spaces}_k)$ is generated by integral, separated, finite type schemes with relations analogous to those in Definition 2.2.

In other words, the ring $K_0(\text{Spaces}_k)$ agrees with the usual Grothendieck ring of varieties over $k$.

Proof. Since finite type spaces are quasi-separated, they contain an dense open isomorphic to a scheme [Ols16, Thm. 6.4.1]. The result follows by a straightforward Noetherian induction, along with the fact that $\{X\} = \{X^{\text{red}}\}$ for all $X$. □

We next introduce the Grothendieck ring of algebraic stacks.
Definition 2.4. The Grothendieck ring of algebraic stacks (over $k$) is the ring $K_0(\text{Stacks}_k)$ generated by classes of algebraic stacks $\{\mathcal{X}\}$ of finite type over $k$ with affine diagonal, with the three relations:

1. $\{\mathcal{X}\} = \{\mathcal{Y}\}$ if there is an isomorphism $\mathcal{X} \simeq \mathcal{Y}$ over $k$,
2. $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{U}\}$ for any closed substack $\mathcal{Z} \subset \mathcal{X}$ with open complement $\mathcal{U} \subset \mathcal{X}$,
3. $\{\mathcal{E}\} = \{\mathcal{X} \times_k \mathcal{Y}\}$ for $\mathcal{E}$ any locally free sheaf $\mathcal{E}$ on $\mathcal{X}$ of rank $n$.

Multiplication in this ring is given by $\{\mathcal{X}\} \cdot \{\mathcal{Y}\} = \{\mathcal{X} \times_k \mathcal{Y}\}$.

Note that condition (3) above follows from the first two in the case of schemes, because vector bundles on schemes are Zariski locally trivial. However, vector bundles over stacks may fail to be Zariski locally trivial, as is the case for nontrivial vector bundles on $BG$.

Remark 2.5. Let $\mathcal{L} := \{A^1_k\}$ denote the class of the affine line. The natural map $K_0(\text{Spaces}_k) \to K_0(\text{Stacks}_k)$ induces an isomorphism

$$K_0(\text{Spaces}_k)[L^{-1},(L^n - 1)^{-1}] \xrightarrow{\sim} K_0(\text{Stacks}_k)$$

[Ek09a, Thm. 1.2]. Here $K_0(\text{Spaces}_k)[L^{-1},(L^n - 1)^{-1}]$ denotes the ring obtained from $K_0(\text{Spaces}_k)$ by inverting $L$, as well as $L^n - 1$ for all positive integers $n$. This isomorphism is motivated by Definition 2.4(3) and the fact that inverting the classes of $L$ and $L^n - 1$ for all $n$ is equivalent to inverting the classes of $GL_n$ for all $n$.

In order to apply the results of [BH21] to sieve out smooth covers from all covers, we will need to work in a slight modification of the Grothendieck ring of stacks where we invert universally bijective (i.e., radicial surjective) morphisms and then complete along the dimension filtration.

Definition 2.6. Let $k$ be a field and let $K_0(\text{Spaces}_k)$ denote the Grothendieck ring of algebraic spaces over $k$. From $K_0(\text{Spaces}_k)$, we will construct another ring, $\widehat{K}_0(\text{Spaces}_k)$, in three steps.

1. For any universally bijective map $f : X \to Y$ of finite type algebraic spaces over $k$, we impose the additional relation that $\{X\} = \{Y\}$. Call the result (only for the next paragraph) $K_0(\text{Spaces}_k)_{\text{RS}}$.
2. Define $\widetilde{K}_0(\text{Spaces}_k) := K_0(\text{Spaces}_k)_{\text{RS}}[L^{-1}]$. Like $K_0(\text{Stacks}_k)$, the ring $\widetilde{K}_0(\text{Spaces}_k)$ has a filtration given by dimension with the $i$th filtered part $F^i\widetilde{K}_0(\text{Spaces}_k) \subset \widetilde{K}_0(\text{Spaces}_k)$ denoting the subset of $\widetilde{K}_0(\text{Spaces}_k)$ spanned by classes of dimension at most $-i$.
3. Finally, let

$$\widehat{K}_0(\text{Spaces}_k) := \lim_{\leftarrow i \geq 0} \widetilde{K}_0(\text{Spaces}_k)/F^i\widetilde{K}_0(\text{Spaces}_k)$$

be the completion along the dimension filtration.
Remark 2.7. In characteristic 0, identifying classes along universally bijective morphisms does not alter the Grothendieck ring. See [BH21, Rem. 2.0.2, Rem. 7.3.2] for some justification of why we are inverting universally bijective morphisms.

But we do not know if inverting universally bijective morphisms alters the Grothendieck ring of spaces or stacks in positive characteristic.

Since Hurwitz stacks are not in general algebraic spaces, but the results of [BH21] apply to the completed Grothendieck ring of algebraic spaces $\hat{K}_0(Spaces_k)$, it will be useful to know that one can also obtain $\hat{K}_0(Spaces_k)$ from $K_0(Stacks_k)$ by inverting universally bijective maps and completing along the dimension filtration, as we next verify.

Lemma 2.8. Let $\hat{K}_0(Stacks_k)$ denote the ring obtained from $K_0(Stacks_k)$ by imposing the relations identifying all universally bijective morphisms of spaces and then completing along the dimension filtration, analogously to the construction of Definition 2.6. The natural map $\hat{K}_0(Spaces_k) \to \hat{K}_0(Stacks_k)$ is an isomorphism.

Proof. First note that although we constructed $\hat{K}_0(Spaces_k)$ from $K_0(Spaces_k)$ by first quotienting by universally bijective morphisms and then inverting $L$, we could have equally well first inverted $L$ and then inverted universally bijective morphisms. The two operations commute because both correspond to adding relations to the ring $K_0(Spaces_k)$, and the resulting commutative ring is independent of the order in which the relations were added.

As mentioned above, we may add in relations to $K_0(Spaces_k)$ in any order, and so using Remark 2.5 we can equivalently obtain $\hat{K}_0(Stacks_k)$ by identifying universally bijective morphisms of spaces and then inverting $L$, $L^n - 1$ and completing along the dimension filtration. To show $\hat{K}_0(Spaces_k) \to \hat{K}_0(Stacks_k)$ is an isomorphism, we wish to show that beginning with $\hat{K}_0(Spaces_k)$ and completing along the dimension filtration is equivalent to first inverting $L^n - 1$ for all $n \geq 0$ and then completing along the dimension filtration. Indeed, one may define a map $\hat{K}_0(Stacks_k) \to \hat{K}_0(Spaces_k)$ induced by the map $\hat{K}_0(Spaces_k)[(L^n - 1)_{n \geq 0}] \to \hat{K}_0(Spaces_k)$ extended by sending the class of $(L^n - 1)^{\sim} \mapsto \sum_{i \geq 1} L^{-in}$. Upon completing along the dimension filtration this defines the desired isomorphism $\hat{K}_0(Stacks_k) \to \hat{K}_0(Spaces_k)$ inverse to the natural map $\hat{K}_0(Spaces_k) \to \hat{K}_0(Stacks_k)$ given above. □

Remark 2.9. Due to the equivalence of Lemma 2.8 in what follows, we will work in $\hat{K}_0(Spaces_k)$, with the understanding that this ring also describes the completion along the dimension filtration of the Grothendieck ring of stacks with universally bijective morphisms inverted. In particular, it makes sense to speak of classes of stacks with affine diagonal in $\hat{K}_0(Spaces_k)$ by Lemma 2.8.
2.10. Motivic Euler Products. We recall the notion of motivic Euler products in the Grothendieck ring, which is crucial in our proof. See [Bil17] for an introduction to motivic Euler products, and [BH21, §6] for more details.

We begin by introducing notation to give the definition of motivic Euler products in the setting we will need. Let $X \rightarrow S$ be a map of finite type algebraic spaces over a field $k$. (In this paper, we will be primarily interested in the case $X = \mathbb{P}^1_k$ and $S = \text{Spec } k$.) For $I$ an indexing set, we use $P(I)$ to denote the set of \textit{generalized partitions} which are tuples of nonnegative integers $(m_i)_{i \in I}$ with only finitely many $i$ such that $m_i$ is nonzero. For any $\mu = (m_i)_{i \in I}$, there is a finite surjective map $p : \prod_{i \in I} \text{X}^{m_i} \rightarrow \prod_{i \in I} \text{Sym}^{m_i} X$. Let $U$ denote the open subscheme of $\prod_{i \in I} \text{X}^{m_i}$ where no two coordinates agree and let $C^\mu_{/S}(X)$ denote the open subscheme $p(U) \subset \prod_{i \in I} \text{Sym}^{m_i} X$. Informally speaking, $C^\mu_{/S}(X)$ parameterizes configurations of $\mu$-labeled points on $X$ mapping to the same point of $S$.

More generally, for $\mathcal{X} = (X_i)_{i \in I}$ a collection of varieties $X_i$ over $X$, and $\mu = (m_i)_{i \in I}$ a generalized partition, define $C_{X/S}(\mathcal{X})$ as the preimage of $C^\mu_{/S} \subset \prod_{i \in I} \text{Sym}^{m_i}(X)$ under the projection $\prod_{i \in I} \text{Sym}^{m_i} X_i \rightarrow \prod_{i \in I} \text{Sym}^{m_i}(X)$. As in [BH21, Defn. 6.1.8], one can extend this definition to make sense of $C^\mu_{X/S}(\mathcal{A})$ as an element of $K_0(\text{Spaces}_k)$ where $\mathcal{A} = (a_i)_{i \in I}$ with $a_i$ in $K_0(\text{Spaces}_k)$ over $X$.

Following [BH21, Defn. 6.1.11], for $\mathcal{A} = (a_i)_{i \in I}$ a collection of classes in $K_0(\text{Spaces}_k)$ over $X$, define the \textit{motivic Euler product}

\begin{equation}
(2.1) \prod_{x \in X/S} \left(1 + \sum_{i \in I} a_{i,x} t_i\right) := \sum_{\mu \in P(I)} C^\mu_{X/S}(\mathcal{A}) t^\mu \in K_0(\text{Spaces}_k)[[(t_i)_{i \in I}]],
\end{equation}

where $t^\mu = \prod_{i \in I} t_i^{m_i}$ for $\mu = (m_i)_{i \in I}$. Here, $a_{i,x}$ is notation for the restriction of $a_i$ to the point $x \in X$.

\textbf{Warning 2.11.} The left hand side of \textbf{(2.1)} is merely (evocative) notation, and has no intrinsic meaning beyond the right hand side.

To simplify matters in accordance with the situation we are in, we will always take $S = \text{Spec } k$. We will also primarily be concerned with the case that all $a_i$ are equal to a single element $a \in K_0(\text{Spaces}_k)$ and the indexing set $I$ has a single element. In such a scenario, motivic Euler products are the same as the power structures of Gusein-Zade, Luengo, and Melle–Hernández [GZLM04]. By abuse of notation, we refer to this collection $\mathcal{A}$ simply as $a$. We now specialize to this one variable case. Although our indexing set only has a single element $a$, to make sense of the motivic Euler product when $a$ is not effective, one must pass through motivic Euler products with infinite indexing sets, as seen in [BH21, Ex. 6.1.12].

In good circumstances, there is an \textit{evaluation map} at $t = 1$ sending a motivic Euler product, viewed as an element of $K_0(\text{Spaces}_k)[[t]]$ to an element of $K_0(\text{Spaces}_k)$, as in [BH21] Definition 6.4.1 and Notation 6.4.2]. This makes sense whenever the motivic Euler product “converges at $t = 1$”, meaning there are only finitely many terms $\mu$ so that $C^\mu_{X/S}(a)$ are nonzero in any given piece of the dimension filtration.
Notation 2.12. For a motivic Euler product \( \prod_{x \in X/S} (1 + a_x t) \) which converges at \( t = 1 \), we use
\[
\prod_{x \in X/S} (1 + a_x t) \big|_{t=1}
\]
or \( \prod_{x \in X/S} (1 + a_x) \), to denote the corresponding element in \( \hat{K}_0(\text{Spaces}_k) \). Whenever we write \( \prod_{x \in X/S} (1 + a_x) \), it denotes the evaluation of the motivic Euler product \( \prod_{x \in X/S} (1 + a_x t) \) at \( t = 1 \) in \( \hat{K}_0(\text{Spaces}_k) \).

Warning 2.13. Due to the extreme care with which one must handle motivic Euler products, we acknowledge that Notation 2.12 is not very good notation. It is likely best to think of motivic Euler products as power series in \( t \) which are being evaluated at values of \( t \), rather than actual elements in \( \hat{K}_0(\text{Spaces}_k) \), as the manipulations one wants to make have only primarily been established in terms of the power series, and not in terms of their evaluations in \( \hat{K}_0(\text{Spaces}_k) \). We choose to use this convention so as to shorten unwieldy formulas.

An important lemma will be that these Euler products in \( \hat{K}_0(\text{Spaces}_k) \) are multiplicative. We now verify this, the key input being multiplicativity of motivic Euler products in \( K_0(\text{Spaces}_k)[[((t_i))_{i \in I}]] \).

Lemma 2.14. Suppose \( a \) and \( b \) two classes in \( K_0(\text{Spaces}_k) \) such that the Euler products \( \prod_{x \in X/S} (1 + a_x t) \) and \( \prod_{x \in X/S} (1 + b_x t) \) converge at \( t = 1 \). Then,
\[
\prod_{x \in X/S} (1 + a_x) \cdot \prod_{x \in X/S} (1 + b_x) = \prod_{x \in X/S} ((1 + a_x)(1 + b_x))
\]
in \( \hat{K}_0(\text{Spaces}_k) \).

Proof. We would like to say this follows from multiplicativity of Euler products [Bil17 Prop. 3.9.2.4], but the issue is that when we write out the definitions, the left hand side of (2.2) is equal to
\[
\left( \prod_{x \in X/S} (1 + a_x t) \cdot \prod_{x \in X/S} (1 + b_x t) \right) \bigg|_{t=1} = \left( \prod_{x \in X/S} (1 + a_x t) \cdot (1 + b_x t) \right) \bigg|_{t=1} = \left( \prod_{x \in X/S} (1 + a_x t + b_x t + a_x b_x t^2) \right) \bigg|_{t=1}
\]
while the right hand side is by definition
\[
(2.4) \quad \left( \prod_{x \in X/S} (1 + a_x t + b_x t + a_x b_x t^2) \right) \bigg|_{t=1}.
\]
However, we claim that (2.4) is also equal to the final line of (2.3), when viewed as an element of \( \hat{K}_0(\text{Spaces}_k) \). To see this, we can view both as specializations of \( \prod_{x \in X/S} (1 + a_x t + b_x t + a_x b_x t^2) \), where the final line of (2.3) is obtained from this by
first evaluating at \( s = t^2 \) and subsequently evaluating at \( t = 1 \), while (2.4) is obtained by first evaluating at \( s = t \) and subsequently evaluating at \( t = 1 \). (Technically, here, we are working with motivic Euler products in two variables, but one can make sense of evaluation analogously to the one variable case.) Both of these operations are given by evaluating \( \prod_{x \in X/S} (1 + a_xt + b_xt + a_xb xs) \) at \( s = t = 1 \), and so they are equal as elements of \( \hat{\mathbb{K}}_0(\text{Spaces}_k) \).

\[ \square \]

3. Parametrizations of Low Degree Covers

The key to computing the class of Hurwitz stacks of low degree covers of \( \mathbb{P}^1 \) is the parametrization of covers of degree \( d \leq 5 \) of a general base scheme. In the case \( d = 3 \), the first such parametrization was given by Miranda [Mir85, Thm. 1.1], for arbitrary degree 3 covers of an irreducible scheme over an algebraically closed field of characteristic not equal to 2 or 3. Pardini [Par89] later generalized Miranda’s result to characteristic 3, and Casnati and Ekedahl [CE96, Thm. 3.4] generalized the result to Gorenstein degree 3 covers of an integral noetherian scheme. Poonen [Poo08, Prop. 5.1] gave a complete parametrization of degree 3 covers of an arbitrary base scheme (see also [Woo11, Thm. 2.1]). When \( d = 4 \), Casnati and Ekedahl [CE96, Thm. 4.4] gave a parametrization of Gorenstein degree 4 covers of an integral noetherian scheme. The third author [Woo11, Thm. 1.1] generalized this to a parametrization of arbitrary degree 4 covers along with the data of a cubic resolvent cover (which is unique in the Gorenstein case) over an arbitrary base scheme. When \( d = 5 \), Casnati [Cas96, Theorem 3.8] gave a parametrization of degree 5 covers, satisfying a certainly “regularity” condition (see Remark 3.6), of an integral noetherian scheme. (We also note that Wright and Yukie [WY92] gave these parametrizations for a covers of a field, and Delone and Faddeev [DF64] and Bhargava [Bha04, Bha08] gave these parametrizations for covers of \( \text{Spec} \mathbb{Z} \). Bhargava’s parametrizations require additional resolvent data for non-Gorenstein covers. Bhargava, Shankar and Wang [BSW15, Section 3] have refined Wright and Yukie’s work for covers of global fields.)

In this section, we will prove similar parametrizations, but suited for our particular application. For our purposes, we would like to parametrize only Gorenstein covers, but over an arbitrary base. For \( d = 3, 4 \), such a result could be deduced directly from [Poo08, Prop. 5.1] and [Woo11, Thm. 1.1] by specializing to Gorenstein covers. However, for the case \( d = 5 \) some new arguments are required both to obtain all Gorenstein covers and to generalize to an arbitrary base. For uniformity of exposition, we show how all of the parametrizations of Gorenstein covers can be obtained from the approach of Casnati and Ekedahl.

Casnati and Ekedahl [CE96] prove a structure theorem [CE96, Thm. 2.1] (a reformulation of [CE96, Thm. 1.3]), which describes a minimal resolution of covers of arbitrary degree of an integral scheme. We will need to extend this structure theorem from integral schemes to arbitrary (including non-reduced) bases. Essentially the same proof given in [CE96, Thm. 2.1] applies, suitably replacing Grauert’s theorem with cohomology and base change. We thank Gianfranco Casnati for helpful conversations confirming this. We will then apply this structure theorem to obtain our desired parametrizations of covers in degrees 3, 4, and 5, analogously to how it
was done by Casnati and Ekedahl in [CE96, Thm. 3.4, Thm. 4.4] and [Cas96, Thm. 3.8].

We also upgrade Casnati’s result in degree 5 in an additional way to deal with all Gorenstein covers, see Remark 3.6.

3.1. The main structure theorem from Casnati-Ekedahl. We next recall the main structure theorem and give its proof in the more general setting. In essence, it says that degree \( d \) Gorenstein covers are classified by linear-algebraic data. It is convenient to describe this as saying that a number of moduli stacks are isomorphic.

We first recall some terminology. We will consider degree \( d \) covers which are finite locally free. A finite locally free degree \( d \) cover is Gorenstein if the scheme-theoretic fiber \( X_y \) over \( \kappa(y) \) is Gorenstein for every \( y \in Y \). For \( k \) a field, a subscheme \( X \subseteq \mathbb{P}^n_k \) is arithmetically Gorenstein if the homogeneous coordinate ring \( \oplus_{i \geq 0} H^0(X, \mathcal{O}_X(i)) \) is Gorenstein. For \( \mathcal{E} \) a rank \( d + 1 \) locally free sheaf on \( \mathcal{O}_Y \), let \( \pi : \mathbb{P}\mathcal{E} \to Y \) denote the corresponding projective bundle \( \mathbb{P}\mathcal{E} := \text{ProjSym}^* \mathcal{E} \). We use the term projective bundle to describe the projectivization of a vector bundle. For \( \mathcal{G} \) a sheaf on a scheme or stack \( Z \), we use \( \mathcal{G}^\vee \) to denote its dual. Finally, for \( k \) a field, a subscheme of \( \mathbb{P}^n_k \) is nondegenerate if it is not contained in any hyperplane \( H \subseteq \mathbb{P}^n_k \).

Theorem 3.2 (Generalization of [CE96, Thm. 2.1], see also [CN07, Thm. 2.2]). Let \( X \) and \( Y \) be schemes and let \( \rho : X \to Y \) be a finite locally free surjective Gorenstein map of degree \( d \), for \( d \geq 3 \). Fix a vector bundle \( \mathcal{E}' \) of rank \( d - 1 \) on \( Y \) with corresponding projective bundle \( \mathbb{P} := \mathbb{P}\mathcal{E}' \), and fix an embedding \( i : X \to \mathbb{P}^n_k \) such that \( \rho = \pi \circ i \). We further require that \( \rho^{-1}(y) \subseteq \pi^{-1}(y) \cong \mathbb{P}^{d-2} \) is an arithmetically Gorenstein nondegenerate subscheme for each point \( y \in Y \). Any two such tuples \((\mathbb{P}, \pi, i)\) and \((\mathbb{P}_2, \pi_2, i_2)\) are uniquely isomorphic, meaning there is a unique isomorphism \( \psi : \mathbb{P} \cong \mathbb{P}_2 \) such that \( \pi_2 \circ \psi = \pi \) and \( \psi \circ i = i_2 \). Moreover, for any such triple \((\mathbb{P}, \pi, i)\) with \( \rho = \pi \circ i \), the following properties hold.

(i) Let \( \rho^\# : \mathcal{O}_Y \to \mathcal{O}_X \) denote the natural map induced by \( \rho : X \to Y \) and let \( \mathcal{E} := (\text{coker } \rho^\#)^\vee \). Then, \( \mathbb{P} \cong \mathbb{P}\mathcal{E} \).

(ii) The composition \( \phi : \mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E} \circ \mathcal{O}_{X/Y} \to \mathcal{O}_{X/Y} \) is surjective, and so induces a map \( j : X \to \mathbb{P}\mathcal{E} \). The ramification divisor \( R \subseteq X \) of \( \rho \) satisfies \( \mathcal{O}_X(R) \cong \mathcal{O}_{X/Y} \cong j^* \mathcal{O}_{X/Y}(1) \).

(iii) There is a sequence \( N_0, N_1, \ldots, N_{d-2} \) of finite locally free \( \mathcal{O}_{\mathbb{P}\mathcal{E}'} \) sheaves on \( \mathbb{P}\mathcal{E}' \) with \( N_0 := \mathcal{O}_{\mathbb{P}\mathcal{E}'} \) and an exact sequence

\[
\begin{align*}
0 & \longrightarrow N_{d-2}(-d) \xrightarrow{\alpha_{d-2}} N_{d-3}(-d + 2) \xrightarrow{\alpha_{d-3}} \cdots \\
\cdots & \xrightarrow{\alpha_2} N_1(-2) \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}\mathcal{E}'} \longrightarrow \mathcal{O}_X \longrightarrow 0,
\end{align*}
\]

unique up to unique isomorphism restricting to the identity map \( \mathcal{O}_X \to \mathcal{O}_X \) such that the restriction of (3.1) to the fiber \( (\mathbb{P}\mathcal{E}')_y := \pi^{-1}(y) \) over \( y \) is a minimal free resolution of the structure sheaf of \( X_y := \rho^{-1}(y) \) for every point \( y \in Y \). There are
finite locally free sheaves $\mathcal{F}_i$ on $Y$ so that $\mathcal{N}_i \simeq \pi^* \mathcal{F}_i$, so $\mathcal{N}_i$ is fiberwise trivial. Further $\mathcal{N}_{d-2}$ is invertible, and, for $i = 1, \ldots, d-3$, one has

$$\beta_i := \text{rk} \mathcal{N}_i = \text{rk} \mathcal{F}_i = \frac{i(d-2-i)}{d-1} \left( \frac{d}{i+1} \right).$$

Moreover, $X_y \subset \mathbb{P}_y$ is a nondegenerate arithmetically Gorenstein subscheme, $\pi^* \pi_* \mathcal{N}_i \simeq \mathcal{N}_i$ for $0 \leq i \leq d-2$, and $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^e}}(\mathcal{N}_i(-i-1), \mathcal{N}_{d-2}(-d)) \simeq \mathcal{N}_{d-2-i}(-(d-1-i))$, for $1 \leq i \leq d-3$. Additionally, the formation of $\pi_* \mathcal{N}_i$ commutes with base change on $Y$.

(iv) For $\mathcal{N}_{d-2}$ as in (3.1), we have $\mathcal{E}' \simeq \mathcal{E}$ if and only if $\mathcal{N}_{d-2} \simeq \pi^* \text{det} \mathcal{E}'$.

(v) The pushforward of the map $\alpha_1 : \mathcal{N}_1(-2) \to \mathcal{E}_{\mathbb{P}^1}$ along $\pi$ induces an injection $\mathcal{F}_1 \to \text{Sym}^2 \mathcal{E}$ and for $d-3 \geq i \geq 2$, the pushforward $\alpha_i : \mathcal{N}_i(-i-1) \to \mathcal{N}_{i-1}(-i)$ along $\pi$ induces an injection $\mathcal{F}_i \to \mathcal{F}_{i-1} \otimes \mathcal{E}$.

(vi) For any point $y \in Y$, no subscheme $X'_y \subset X_y$ of degree $d-1$ is sent under $\rho$ to a hyperplane of $\pi^{-1}(y)$.

Remark 3.3. The statement of Theorem 3.2 differs in several ways from the original statement in [CE96 Thm. 2.1]. As pointed out in [CN07 Thm. 2.2] we have added a nondegenerate hypothesis to the statement. We also do not work over noetherian or reduced bases, but to compensate, we have added a finite presentation hypothesis on the map $\rho$, i.e., we have required that it is locally free. We have also added property (v) and (vi).

Proof. As a first step, we reduce to the case $X$ and $Y$ are noethrian. In view of the asserted uniqueness, by Zariski descent, we may reduce to the case that $Y$ is affine. Because $\rho : X \to Y$ is locally finitely presented as it is finite locally free, we can write $Y$ as a limit of spectra of finite type $\mathbb{Z}$-algebras $Y_i$. We can then spread out all of the data described in the theorem to some such $Y_i$, and realize $\rho$ along with all of the above data as the base change of some $\rho_i : X_i \to Y_i$ along a map $Y \to Y_i$, for $Y_i$ finite type over $\text{Spec} \mathbb{Z}$. In particular, we can assume $Y$ and $X$ are noetherian.

The proof of [CE96 Thm. 2.1] in [CE96 Thm. 2.1], is broken up into steps A, B, C, and D. Step A has a minor inaccuracy which we next address. The only generalization needed occurs in step B, while steps C and D go through without change.

Next, the proof of this statement in the key case that $Y = \text{Spec} k$ is a field is given in [CE96 Step A, p. 443]. It appears this has a minor error as it is claimed that, given a finite $k$-algebra $A$ and a generalized trace map $\eta : A \to k$, i.e., a surjection whose kernel only contains the $0$ ideal, one can modify $\eta$ to assume $\eta(e_0) = 0$ for $e_0$ the unit of $A$. This is possible over an algebraically closed field, but is not possible over finite fields, such as in the case $Y = \text{Spec} F_2$ and $X = \bigcup_{i=1}^5 Y$. However, we can still construct the embedding $X \to \mathbb{P} \mathcal{E}$ over $k$ induced by the relative dualizing sheaf $\omega_{X/Y}$ and construct a minimal free resolution for this embedding. All the properties appearing in the theorem may be verified after base change to the algebraic closure of $k$, at which point the proof appearing in [CE96 Step A, p. 443] goes through. We also note that in the statement of [Sch86 Lem, p. 119] which is cited in [CE96 Step A, p. 443], the subscheme $D$ there should have degree $d$ and lie in $\mathbb{P}^{d-2}$, as opposed
to degree \( d - 2 \) in \( \mathbb{P}^{d-1} \). Note that in order to apply [Sch86, Lem., p. 119], we use the hypothesis that \( X \subset \mathbb{P}_e \) is nondegenerate, a hypothesis which was omitted in [CE96, Thm. 2.1]. At this point, (vi) follows from [Sch86, Lem., p. 119].

Having established the result when \( Y = \text{Spec} \, k \), it remains to carry out the proof for general bases following [CE96, Step B, C, and D, p. 445-447]. In what follows, we next recapitulate the argument for step B [CE96, p. 445], modifying the application of Grauer’s theorem to one of cohomology and base change.

Recall the statement of Step B: Suppose there is a factorization \( \rho = \pi \circ i \), for \( \pi : \mathbb{P} \to Y \) a projective \( \mathbb{P}^{d-2} \) bundle and \( i : X \to \mathbb{P} \) an embedding with \( X_y \) an arithmetically Gorenstein subscheme of \( \mathbb{P}_y \) for each \( y \in Y \). Then, (3.1) exists, is unique up to unique isomorphisms, restricts to a minimal free resolution of \( \mathcal{O}_{X_y} \) over each point \( y \in Y \), and \( \pi^* \mathcal{N} \simeq \mathcal{N} \).

For the remainder of the construction, we only handle the case \( d = 3 \) is quite analogous to the case \( d \geq 4 \), though significantly easier as the resolution has length 2.

Define maps \( j_y, i_y \) as in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & \mathbb{P} \\
\downarrow{\rho} & & \downarrow{\pi} \\
Y & \xleftarrow{i_y} & \mathbb{P}_y
\end{array}
\]

(3.3)

Letting \( \mathcal{I} \) denote the ideal sheaf of \( X \) in \( \mathbb{P} \), we claim that \( j_y^* \mathcal{I} \) is the ideal sheaf of \( X_y \) in \( \mathbb{P}_y \). To see this, we only need to verify that \( j_y^* \mathcal{I} \rightarrow j_y^* \mathcal{O}_\mathbb{P} \rightarrow j_y^* \mathcal{O}_X \) is exact. Since \( \mathcal{O}_X \) is flat over \( Y \), we will verify more generally that for \( \mathcal{H}, \mathcal{I}, \mathcal{F} \) three sheaves on \( X \) with \( \mathcal{F} \) flat over \( Y \), and an exact sequence \( 0 \rightarrow \mathcal{H} \rightarrow \mathcal{I} \rightarrow \mathcal{F} \rightarrow 0 \), the pullback sequence \( 0 \rightarrow j_y^* \mathcal{H} \rightarrow j_y^* \mathcal{I} \rightarrow j_y^* \mathcal{F} \rightarrow 0 \) is exact. Indeed, this holds because

\[
R^1 j_y^* \mathcal{F} = \mathcal{T}or_1^{\mathcal{O}_\mathbb{P}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}_y}) = \mathcal{T}or_1^{\mathcal{O}_Y}(\mathcal{F}, \kappa(y)) = 0.
\]

Here we are using that \( \mathcal{F} \) is flat over \( Y \) for the final vanishing and \( \mathcal{T}or_{\mathcal{O}_Y} \mathcal{O}_{\mathbb{P}_y} \simeq \mathcal{T}or_{\mathcal{O}_Y} \mathcal{O}_{\mathcal{F}} \kappa(y) \) for the equality of \( \mathcal{T}or \) sheaves.

Next, [CE96, Step A, p. 443] provides a resolution of \( \mathcal{I}_{X_y}/\mathcal{P}_y = j_y^* \mathcal{I} \) of the form

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}_y}(-d) \xrightarrow{\alpha_{d-2,y}} \mathcal{O}_{\mathbb{P}_y}(2-d) \xrightarrow{\alpha_{d-3,y}} \cdots
\]

(3.4)

\[
\cdots \xrightarrow{\alpha_{2,y}} \mathcal{O}_{\mathbb{P}_y}(-2) \xrightarrow{\alpha_{1,y}} j_y^* \mathcal{I} \longrightarrow 0.
\]

We claim \( j_y^* \mathcal{I} \) is 3-regular, in the sense of Castelnuovo-Mumford regularity, i.e., \( H^i(\mathbb{P}_y, j_y^* \mathcal{I}(3-i)) = 0 \) for \( i \geq 1 \). To verify this, it follows from the definition of regularity that for an exact sequence \( 0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0 \) of sheaves with \( \mathcal{F}' \) \( m + 1 \)-regular and \( \mathcal{F} \) \( m \)-regular, \( \mathcal{F}'' \) is also \( m \)-regular. Using this and the fact that \( \mathcal{O}_{\mathbb{P}_y}(-k) \) is \( k \)-regular (and hence it is also \( k + 1 \) regular by [FGI05, Lem. 5.1(b)]), it follows by induction that \( \text{im} \alpha_{d-i,y} \) is \( d - i + 2 \) regular. Therefore, \( j_y^* \mathcal{I} = \text{im} \alpha_{1,y} \) is
3-regular. By [FGI+05, Lem. 5.1(b)], we obtain \( H^1(Y, j^*_y \mathcal{I}(n)) = 0 \) for \( n \geq 2 \). Hence, by cohomology and base change, \( R^1 \pi_* \mathcal{I}(n) = 0 \) for \( n \geq 2 \).

For \( \mathcal{F} \) a sheaf, let us denote by \( \phi_y(\mathcal{F}) : R^i \pi_* \mathcal{F} \otimes \kappa(y) \to H^i(X_y, \mathcal{F}|_{X_y}) \) the natural base change map. Then we have seen above that, for \( n \geq 2 \), \( \phi_y(\mathcal{I}(n)) \) is an isomorphism at all \( y \). Further, \( R^1 \pi_* \mathcal{I}(n) \) is locally free (and in fact equal to 0) which implies by cohomology and base change that \( \phi_y(\mathcal{I}(n)) \) is an isomorphism for all \( n \geq 2 \). In other words, the formation of \( \pi_* \mathcal{I}(n) \) then commutes with base change on \( Y \). Further, again by cohomology and base change, \( \pi_* \mathcal{I}(n) \) is a locally free sheaf when \( n \geq 2 \) (since the condition from the theorem on cohomology and base change that \( \phi_y^{-1} \) be an isomorphism is vacuously satisfied).

Set \( \mathcal{F}_1 := \pi_* \mathcal{I}(2) \) and \( \mathcal{N}_1 := \pi^* \mathcal{F}_1 \). Let \( \alpha_1 : \mathcal{N}_1(-2) \to \mathcal{I} \) denote the evaluation map coming from the adjunction \( \pi^* \pi_* \mathcal{I}(2) \otimes \mathcal{O}_Y(-2) \to \mathcal{I}(2) \otimes \mathcal{O}_Y(-2) \to \mathcal{I} \).

As we have shown above, the formation of \( \mathcal{F}_1 \), and hence \( \mathcal{N}_1 \), commutes with base change. Further, naturality of the map \( \alpha_1 \), coming from the adjunction, also implies \( j^*_y(\alpha_1) = \alpha_{1,y} \). Therefore, \( \alpha_1 \) is surjective, as its cokernel has empty support.

We next construct sheaves \( \mathcal{F}_i \) and \( \mathcal{N}_i \) inductively, with \( \mathcal{N}_i = \pi^* \mathcal{F}_i \), for \( 2 \leq i \leq d - 3 \). Let \( \mathcal{A}_i := \mathcal{I} \). For \( i \geq 2 \), assume inductively we have constructed the map \( \alpha_{i-1} \) and define \( \mathcal{A}_i := \ker \alpha_{i-1} \). Analogously to the above verification that \( j^*_y \mathcal{I} \) is 3-regular, it follows that \( j^*_y \mathcal{A}_i \) is \( i + 2 \) regular. Therefore, by [FGI+05, Lem. 5.1(b)], \( H^1(Y, j^*_y \mathcal{A}_i(k)) = 0 \) for \( k \geq (i + 2) - 1 = i + 1 \). Analogously to the above case when \( i = 1 \), it follows from cohomology and base change that \( R^1 \pi_* \mathcal{A}_i(k) = 0 \) for \( k \geq i + 1 \), \( \pi_* \mathcal{A}_i(k) \) is locally free for \( k \geq i + 1 \), and the formation of \( \pi_* \mathcal{A}_i(k) \) commutes with base change for \( k \geq i + 1 \). Then, set \( \mathcal{F}_i := \pi_* \mathcal{A}_i(i + 1) \) and \( \mathcal{N}_i := \pi^* \mathcal{F}_i \).

We next construct the map \( \alpha_i : \mathcal{N}_i \to \mathcal{N}_{i-1} \). Begin with the inclusion \( \mathcal{A}_i(i + 1) \to \mathcal{N}_{i-1}(1) \) (obtained by twisting the inclusion \( \mathcal{A}_i \to \mathcal{N}_{i-1}(-i) \), coming from the definition of \( \mathcal{A}_i \), by \( i + 1 \)). Apply \( \pi^* \pi_* \) to obtain a map \( \pi^* \pi_* \mathcal{A}_i(i + 1) \to \pi^* \pi_* \mathcal{N}_{i-1}(1) \).

Twist by \( -i - 1 \) which yields the composite map

\[
\mathcal{N}_i(-i - 1) = (\pi^* \pi_* \mathcal{A}_i(i + 1))(-i - 1) \\
\to (\pi^* \pi_* \mathcal{N}_{i-1}(1))(-i - 1) \\
\simeq (\mathcal{N}_{i-1} \otimes \pi^* \mathcal{O}(1))(-i - 1) \\
\to \mathcal{N}_{i-1}(-i),
\]

which we call \( \alpha_i \). Since \( \mathcal{N}_i \) commutes with base change, and this map is obtained from adjunction, the formation of \( \alpha_i \) also commutes with base change. Also, since pushforward is left exact, we obtain condition (v) in the theorem from the above construction of \( \mathcal{F}_i \), provided we show the above construction is the unique such one as in the statement (which will be done later in the proof).

Finally, we similarly construct \( \mathcal{F}_{d-2}, \mathcal{N}_{d-2}, \) and \( \mathcal{A}_{d-2} \), assuming we have constructed \( \mathcal{A}_{d-3} \). Let \( \mathcal{A}_{d-2} := \ker \mathcal{A}_{d-3} \). By cohomology and base change, we find \( j^*_y \mathcal{A}_{d-2} \) is in fact \( d \)-regular (as opposed to only \( d - 1 \) regular, as was the case for \( \mathcal{A}_i \) with \( i < d - 2 \)). Therefore, by cohomology and base change, we find \( R^1 \pi_* \mathcal{A}_{d-2}(-d) = \)
0 and also that $\pi_\ast\mathcal{A}_{d-2}(-d)$ is locally free and commutes with base change. We set $\mathcal{F}_{d-2} := \pi_\ast\mathcal{A}_{d-2}(-d)$ and $\mathcal{M}_{d-2} := \pi^\ast\mathcal{F}_{d-2}$. Analogously to (3.5), there is a canonical map $a_{d-2} : \mathcal{M}_{d-2}(-d) \to \mathcal{M}_{d-2}(-d + 2)$ coming from adjunction which commutes with base change. Altogether, we have constructed a complex as in (3.1) which commutes with base change on $Y$ and restricts to the minimal free resolution (3.4) on each fiber $y \in Y$. It follows from Nakayama’s lemma that the complex (3.1) is exact, because it is exact when restricted to each fiber over $y \in Y$.

Further, because $\mathcal{N}_i = \pi^\ast\mathcal{F}_i$, it follows from the projection formula that $\pi_\ast\mathcal{N}_i \cong \pi_\ast(\mathcal{O}_U \otimes \pi^\ast\mathcal{F}_i) \cong \pi_\ast\mathcal{O}_U \otimes \mathcal{F}_i \cong \mathcal{F}_i$, and so $\pi^\ast\pi_\ast\mathcal{N}_i \cong \mathcal{N}_i$.

We next verify uniqueness of our constructed resolution $\mathcal{N}_\bullet$, up to unique isomorphism. Suppose $\mathcal{M}_\bullet$ is another such minimal free resolution. Over any local scheme $\text{Spec} \mathcal{O}_{y,Y} \subset Y$, there is an isomorphism $\phi_U : \mathcal{N}_\bullet|_{\text{Spec} \mathcal{O}_{y,Y}} \cong \mathcal{M}_\bullet|_{\text{Spec} \mathcal{O}_{y,Y}}$ by a sheafified version of [Eis95, Thm. 20.2]. Such an isomorphism spreads out to an isomorphism over some affine open $U \subset Y$. Further, this isomorphism is unique up to homotopy by a sheafified version of [Eis95, Lem. 20.3]. We claim there are no nonzero homotopies $s : \mathcal{N}_\bullet|_U \to \mathcal{M}_\bullet|_U$. Indeed, such an homotopy would yield a map $s_U : \mathcal{N}_i|_U \to \mathcal{M}_{i+1}|_U$. We wish to show this map is 0. To check it is 0, it suffices to show it is 0 over each $y \in Y$. Over a point $y \in Y$, this corresponds to a map $\mathcal{O}_{P_y}(a) \otimes \mathcal{O}_{Y_y}(b) \to \mathcal{O}_{P_y}(c) \otimes \mathcal{O}_{Y_y}(d)$ with $c < a$. It follows that there are no nonzero such maps, so the isomorphism $\phi_U$ is unique. Hence, by this uniqueness, we obtain via Zariski descent an isomorphism $\phi : \mathcal{N}_\bullet \cong \mathcal{M}_\bullet$. This isomorphism is unique because it is unique when restricted to each member of an open cover.

This concludes our update to the proof of [CE96, Thm. 2.1] since, as mentioned, the remaining steps C and D given in the proof of [CE96, Thm. 2.1] go through without change.

The following useful corollary tells us that any two “canonical embeddings” of a Gorenstein cover are related by an automorphism of $\mathbb{P}\mathcal{E}$ coming from $\mathcal{E}$. A special case of this was stated in [CN07, Corollary 2.3], though the proof there seems quite terse.

**Corollary 3.4.** With notation as in Theorem 3.2, suppose we are given $\rho : X \to Y$ and two embeddings $i_1 : X \to \mathbb{P}\mathcal{E}$ and $i_2 : X \to \mathbb{P}\mathcal{E}$ so that $\rho = \pi \circ i_1 = \pi \circ i_2$ and $\rho^{-1}(y)$ is arithmetically Gorenstein and nondegenerate under both embeddings $i_1$ and $i_2$. Then, the unique isomorphism $\psi : \mathbb{P}\mathcal{E} \to \mathbb{P}\mathcal{E}$ taking $i_1(X)$ to $i_2(X)$ is induced by an automorphism of $\mathcal{E}$.

**Proof.** For both maps $i_1$ and $i_2$, we know the restriction of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ (the line bundle receiving a surjection from $\rho^\ast\mathcal{E}$ in the statement of this corollary) to $X$ is isomorphic to $\omega_{X/Y}$ by Theorem 3.2(ii). Hence, we obtain that the automorphism is induced by some automorphism $\phi$ of $\pi_\ast\omega_{X/Y}$, determined up to unit. The maps $i_1$ and $i_2$ induce two surjections $q_1, q_2 : \pi_\ast\omega_{X/Y} \to \mathcal{O}_Y$ with the maps $i_1$ and $i_2$ coming via the linear subsystems $\ker(q_1)$ and $\ker(q_2)$. To show we have an induced map between $\ker(q_1)$ and $\ker(q_2)$, which are both abstractly isomorphic to $\mathcal{E}$, it is enough to show that, up to unit, $q_1 = q_2 \circ \phi$. We may verify this locally, and hence assume $Y$ is the spectrum of a local ring. Using Theorem 3.2(vi), in both of the maps $i_1$ and $i_2$, there
is no subscheme of degree $d - 1$ on the closed fiber contained in a hyperplane, and hence the same holds over the whole local scheme $Y$. We may rephrase this as the condition that the two relative hyperplane sections of $\mathcal{E}$ associated to $q_1$ and $q_2$ do not meet $i_1(X)$ and $i_2(X)$. Equivalently, the two hyperplane sections associated to $q_1$ and $q_2$ are nowhere vanishing on $X$, and therefore related by a unit. By modifying $\phi$ by this unit, we may assume $q_1 = q_2 \circ \phi$.

Under the above identifications, the image of $\mathcal{E} \to \pi_*\omega_{X/Y}$ is identified with the kernel of the natural map $\pi_*\omega_{X/Y} \to \mathcal{O}_X$ dual to $\rho^\#$. Since this map is also fixed by the resulting automorphism, the automorphism of $\pi_*\omega_{X/Y}$ restricts to an automorphism of $\mathcal{E}$ which induces the resulting automorphism of $\mathbb{P}\mathcal{E}$. □

3.5. Low degree parametrizations. We now apply Theorem 3.2 as in the work of Casnati and Ekedahl, to obtain parametrizations of Gorenstein covers of degrees 3, 4, and 5.

**Remark 3.6.** Our parametrization in degree 5, Theorem 3.16, is stronger than previous work in several ways. The similar result in degree 5 proven in [Cas96, Thm. 3.8] has certain additional restrictions on the covers and sections that Casnati refers to as being “regular.” This regularity condition amounts to the assumption that the map $\wedge^2 \mathcal{F} \otimes \det \mathcal{E} \to \mathcal{E}$ associated to a section $\eta \in \mathcal{H}(\mathcal{E}, \mathcal{F})$ is surjective. Additionally, [Cas96, Thm. 3.8] does not claim there is a bijection between covers and sections up to automorphisms of $\mathcal{E}$ and $\mathcal{F}$, but only gives constructions of maps in both directions. Further, [Cas96, Thm. 3.8] is stated for degree 5 finite flat surjective maps $X \to Y$ with $Y$ integral and noetherian, whereas ours hold for arbitrary schemes $Y$.

Although there are other ways of approaching the upcoming proofs, in order to prove the theorems in degrees 4 and 5, it will be convenient to appeal to smoothness of the algebraic stack of degree $d$ Gorenstein covers. To this end, let Covers$_d$ denote the fibered category over $\text{Spec} \mathbb{Z}$ whose $S$ points are finite locally free covers $X \to S$ of degree $d$ with Gorenstein fibers. Morphisms in this fibered category are morphisms of covers.

**Lemma 3.7.** For $d \leq 5$, Covers$_d$ is a smooth algebraic stack over $\text{Spec} \mathbb{Z}$.

**Proof.** The cases $d \leq 3$ follow from [Poo08, Prop. 8.4], so it only remains to deal with the cases $d = 4$ and $d = 5$. The basic input is [CN07, Rem. 5.5], which shows that $\text{Hilb}_{d,0}^{aG}$, the open subset of the Hilbert scheme of degree $d$ subschemes of $\mathbb{P}^{d-2}$ parameterizing arithmetically Gorenstein nondegenerate subschemes, has smooth geometrically irreducible fibers over $\text{Spec} \mathbb{Z}$ when $d \leq 5$. (We note that [CN07] makes a standing assumption that the characteristic is not 2 or 3, but it is not used in [CN07, Rem. 5.5] or the results leading to it. However, there is a minor error in the statement of [CN07, Rem. 3.5] as when $d = 4$, the dimension of the tangent space in that statement should be 8, which is different from the claimed value of 6.)

We next check $\text{Hilb}_{d,0}^{aG}$ is in fact smooth over $\text{Spec} \mathbb{Z}$, and not just over each residue field. For this it suffices to check it is flat. Because maps sending all associated points to the generic point of $\text{Spec} \mathbb{Z}$ are flat, it is enough to show $\text{Hilb}_{d,0}^{aG}$
is integral. Irreducibility follows because there is a dense open parameterizing étale degree \(d\) subschemes, which is dense in every fiber, as the fibers are geometrically irreducible. Granting this irreducibility, we then find that every point is regular, using the slicing criterion for regularity, and hence \(\text{Hilb}^{dG,0}\) is reduced.

Finally, we deduce smoothness and algebraicity of \(\text{Covers}_d\) over \(\text{Spec} \mathbb{Z}\) from smoothness of \(\text{Hilb}^{dG,0}\). By Theorem 3.2, the natural map \(\text{Hilb}^{dG,0} \to \text{Covers}_d\) is a \(\text{PGL}_{d-1}\) torsor, and so smoothness of the former implies smoothness of the later. \(\square\)

To introduce notation simultaneously in the cases of degrees 3, 4, and 5, we use the following notation.

**Notation 3.8.** Let \(d \in \{3, 4, 5\}\). Let \(Y\) be a scheme. Fix a locally free sheaf \(\mathcal{E}\) on \(Y\) of rank \(d - 1\). If \(d = 4\), let \(\mathcal{F}\) be a locally free sheaf on \(Y\) of rank 2 and if \(d = 5\), let \(\mathcal{F}\) be a locally free sheaf on \(Y\) of rank 5. We use the tuple \((\mathcal{E}, \mathcal{F})\) to denote the pair \((\mathcal{E}, \mathcal{F})\) when \(d = 4\) or \(d = 5\) and to denote \(\mathcal{E}\) when \(d = 3\). Define the associated sheaf

\[
\mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet) := \begin{cases} 
\text{Sym}^3 \mathcal{E} \otimes \det \mathcal{E}^\vee & \text{if } d = 3 \\
\mathcal{F}^\vee \otimes \text{Sym}^2 \mathcal{E} & \text{if } d = 4 \\
\wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee & \text{if } d = 5.
\end{cases}
\]

We will often use \(\mathcal{H}\) to denote \(\mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)\) when the data \((\mathcal{E}, \mathcal{F}_\bullet)\) is clear from context. We will see that sections of the above sheaf \(\mathcal{H}\) define subschemes of \(\mathbb{P} \mathcal{E}\). When these subschemes have dimension 0 in fibers, we will see they induce degree \(d\) locally free covers. The parametrizations for degrees 3, 4, and 5 essentially say that the resulting covers are in bijection with such sections, up to automorphisms of \((\mathcal{E}, \mathcal{F}_\bullet)\).

### 3.9. The resolutions in low degree

In order to state the parametrizations in degrees 3, 4, and 5, we now want a way of associating a subscheme of \(\mathbb{P} \mathcal{E}\) to a section. We will give a description of this association separately in the cases that \(d = 3, 4,\) and 5.

Renaming the sheaf \(\mathcal{E}'\) appearing in (3.1) as \(\mathcal{E}\) and renaming \(\mathcal{F}_1\) as \(\mathcal{F}\), in the cases \(d = 3, 4,\) and 5, (3.1) becomes respectively

(3.7) \[
0 \longrightarrow \pi^* \det \mathcal{E}(-3) \overset{\sigma}{\longrightarrow} \mathcal{O}_\mathcal{P} \longrightarrow \mathcal{O}_X \longrightarrow 0,
\]

(3.8) \[
0 \longrightarrow \pi^* \det \mathcal{E}(-4) \overset{\sigma}{\longrightarrow} \pi^* \mathcal{F}(-2) \longrightarrow \mathcal{O}_\mathcal{P} \longrightarrow \mathcal{O}_X \longrightarrow 0,
\]

(3.9) \[
0 \longrightarrow \pi^* \pi^* \det \mathcal{E}(-5) \overset{\sigma}{\longrightarrow} \pi^* \mathcal{F}^\vee \otimes \pi^* \det \mathcal{E}(-3) \overset{\sigma}{\longrightarrow} \pi^* \mathcal{F}(-2) \overset{\sigma}{\longrightarrow} \mathcal{O}_\mathcal{P} \longrightarrow \mathcal{O}_X \longrightarrow 0,
\]

with the rank of the locally free sheaves \(\mathcal{E}\) and \(\mathcal{F}\) in the degree 3, 4, and 5 cases given in Notation 3.8.
3.10. The maps $\Phi_d$ in low degree. In the above 3 cases, corresponding to degrees 3, 4, and 5 respectively, we have isomorphisms

$$\Phi_3 : H^0(Y, \text{Sym}^3 \mathcal{E} \otimes \det \mathcal{E}^\vee) \xrightarrow{\sim} H^0(\mathbb{P}\mathcal{E}, \pi^* \det \mathcal{E}^\vee(3)).$$

$$\Phi_4 : H^0(Y, \text{Sym}^2 \mathcal{E} \otimes \mathcal{F}^\vee) \xrightarrow{\sim} H^0(\mathbb{P}\mathcal{E}, \pi^* \mathcal{F}^\vee(2))$$

$$\Phi_5 : H^0(Y, \Lambda^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee) \xrightarrow{\sim} H^0(\mathbb{P}\mathcal{E}, \Lambda^2 \pi^* \mathcal{F} \otimes \pi^* \det \mathcal{E}^\vee(1)).$$

3.11. The maps $\Psi_d$ in low degree. For $\rho : X \to Y$ a finite locally free surjective Gorenstein map of degree $d$, we will use $\mathcal{E}^X$ to denote the Tschirnhausen bundle $\coker(\mathcal{O}_Y \to \rho_* \mathcal{O}_X)^\vee$ and $\mathcal{F}^X$ to denote the bundle $\mathcal{F}_1$ in the case we take $\mathcal{E}'$ in Theorem 3.2 (ii) to be the Tschirnhausen bundle $\mathcal{E}^X$.

Next, for $d \leq 3 \leq 5$, given a section $\eta \in H^0(Y, \mathcal{H}(\mathcal{E}, \mathcal{F}^\vee_1))$, we define an associated scheme $\Psi_d(\eta)$ over $Y$.

When $d = 3$, we begin with a section $\eta \in H^0(Y, \text{Sym}^3 \mathcal{E} \otimes \det \mathcal{E}^\vee)$, which, via $\Phi_3$ can be viewed as an element of $H^0(\mathbb{P}\mathcal{E}, \pi^* \det \mathcal{E}^\vee(3))$. Such a section corresponds to a map $\mathcal{O}_{\mathbb{P}\mathcal{E}} \to \pi^* \det \mathcal{E}^\vee(3)$, or equivalently a map $\pi^* \det \mathcal{E}^\vee(-3) \to \mathcal{O}_{\mathbb{P}\mathcal{E}}$. We let $\Psi_3(\eta)$ denote the support of the cokernel of this map. That is, we define $\Psi_3(\eta) \subset \mathbb{P}\mathcal{E}$ so that on $\mathbb{P}\mathcal{E}$ we have an exact sequence

$$\pi^* \det \mathcal{E}^\vee(-3) \longrightarrow \mathcal{O}_{\mathbb{P}\mathcal{E}} \longrightarrow \mathcal{O}_{\Psi_3(\eta)} \longrightarrow 0. \tag{3.10}$$

When $d = 4$, given $\eta \in H^0(Y, \mathcal{F}^\vee \otimes \text{Sym}^2 \mathcal{E})$, define $\Phi_4(\eta)$ to be the subscheme of $\mathbb{P}\mathcal{E}$, considered as the support of the cokernel of the map $\pi^* \mathcal{F}^\vee(-2) \to \mathcal{O}_{\mathbb{P}\mathcal{E}}(\mathcal{E})$ corresponding to $\Phi_4(\eta)$.

Finally, when $d = 5$, given $\eta \in H^0(Y, \Lambda^2 \mathcal{F} \otimes \pi^* \det \mathcal{E}^\vee \otimes \mathcal{E})$, from $\Phi_5(\eta)$ we obtain a corresponding alternating map $\pi^* \mathcal{F}^\vee \otimes \pi^* \det \mathcal{E}^\vee(-3) \to \pi^* \mathcal{F}^\vee(-2)$. The five $4 \times 4$ Pfaffians of this map determine a map of sheaves $\pi^* \mathcal{F}^\vee(-2) \to \mathcal{O}_{\mathbb{P}\mathcal{E}}(\mathcal{E})$, as may be computed locally. Define $\Psi_5(\eta)$ as the support of the cokernel of the map $\pi^* \mathcal{F}^\vee(-2) \to \mathcal{O}_{\mathbb{P}\mathcal{E}}(\mathcal{E})$ in $\mathbb{P}\mathcal{E}$.

**Definition 3.12.** Let $d \in \{3, 4, 5\}$, $\mathcal{Y}$ be a scheme, and $(\mathcal{E}, \mathcal{F}_1), \mathcal{H}(\mathcal{E}, \mathcal{F}_1)$ be sheaves on $Y$ as in Notation 3.8. We say $\eta \in H^0(Y, \mathcal{H}(\mathcal{E}, \mathcal{F}_1))$ has the right codimension at a point $y \in Y$ if the fiber of $\Psi_d(\eta)$ over $y$ has dimension 0. We say $\eta$ has the right codimension if it has the right codimension at every $y \in Y$.

Finally, we are ready to state the low degree parametrizations. The parametrization in degree 3 is as follows.

**Theorem 3.13** (Generalization of [CE96, Thm. 3.4], Specialization of [Poo08, Prop. 5.1]). Fix a scheme $Y$ and a rank 2 locally free sheaf $\mathcal{E}$ on $Y$. There is a bijection between finite locally free Gorenstein covers $\rho : X \to Y$ of degree 3 such that $\mathcal{E}^\vee \simeq \coker \rho^*$ and, up to automorphisms of $\mathcal{E}$, sections $\eta \in H^0(Y, \text{Sym}^3 \mathcal{E} \otimes \det \mathcal{E}^\vee)$ having the right codimension at every $y \in Y$. The bijection explicitly sends a section $\eta \in H^0(Y, \text{Sym}^3 \mathcal{E} \otimes \det \mathcal{E}^\vee)$ to $\Psi_3(\eta) \subset \mathbb{P}\mathcal{E}$.
The following proof extends that given in [CE96, Thm. 3.4]. We note that there the base is assumed to be reduced and noetherian, and the bijection is not explicitly stated. We outline the proof for the reader’s convenience.

**Proof.** Given such a \( p : X \to Y \), we obtain from [Theorem 3.2], a resolution of \( \mathcal{O}_{P^3} \) as in (3.7), unique up to unique isomorphism. The map \( \sigma \) in (3.7) defines a section \( \eta := \Phi_3^{-1}(\sigma) \in H^0(Y, \text{Sym}^3 \mathcal{E} \otimes \det \mathcal{E}^\vee) \). Note that the resulting \( \eta \) has the right codimension at every \( y \in Y \) because \( X \to Y \) is finite by assumption.

Conversely, given \( \eta \) of the right codimension at every \( y \in Y \), the resulting sequence (3.10) is then left exact as the kernel of \( \Phi_3^{-1}(\eta) \) has vanishing support. This presentation shows \( X \) is locally finitely presented over \( Y \). Further, \( X \) is finite as it is locally of finite presentation, proper, and quasi-finite [Gro66, 8.11.1]. Flatness of \( X \to Y \) may be verified locally, in which case it holds as \( X \) is cut out of \( P_Y^3 \) by a single equation of degree 3 not vanishing on any fibers. Therefore, \( X \) is a finite locally free degree 3 cover of \( Y \). Therefore, exactness of (3.10) implies \( \mathcal{E}^\vee \simeq \text{coker} \rho^\# \) from [Theorem 3.2](iii) and (iv).

It remains to see that these two maps we have defined establish a bijection. For this, we show the compositions of these maps in both orders are equivalent to the identity map. If we begin with a cover \( \rho : X \to Y \), (3.7) defines a resolution of \( X \to P^3 \) giving \( X \) as the vanishing locus \( \Psi_3(\eta) \subset P^3 \). To show the other composition is equivalent to the identity, begin with some \( \eta \in H^0(Y, \text{Sym}^3 \mathcal{E} \otimes \det \mathcal{E}^\vee) \), and let \( X \) denote the associated cover \( \Psi_3(\eta) \). The Tschirnhausen bundle \( \mathcal{E}^X \) as in §3.11 associated to \( X \) from [Theorem 3.2] is then isomorphic to \( \mathcal{E} \) using [Theorem 3.2](iv), as we may view \( \eta \) as a map \( \pi^* \det \mathcal{E}(-3) \to \mathcal{O}_{P^3} \). Upon choosing such an isomorphism \( \mathcal{E} \simeq \mathcal{E}^X \), we obtain a section \( \eta^X \in H^0(Y, \text{Sym}^3 \mathcal{E}^X \otimes \det(\mathcal{E}^X)^\vee) \simeq H^0(Y, \text{Sym}^3 \mathcal{E} \otimes \det \mathcal{E}^\vee) \). Using [Theorem 3.2](iv), there is an automorphism of \( P^3 \mathcal{E} \) taking \( \Psi_3(\eta) \) to \( \Psi_3(\eta^X) \). From [Theorem 3.2](iv) and the fact that the leftmost term of the resolution (3.7) is \( \pi^* \det \mathcal{E}(-3) \), we find \( \mathcal{E} \) is isomorphic to \( \text{ker}(\rho_* \omega_{X/Y} \to \mathcal{O}_Y) \). By [Corollary 3.4], this automorphism of \( P^3 \mathcal{E} \) is induced by an automorphism of \( \mathcal{E} \). Hence, after composing with the automorphism of \( \mathcal{E} \), we can assume \( \eta \) and \( \eta^X \) define isomorphic subschemes of \( P^3 \mathcal{E} \), and so differ by a scalar. By composing with an automorphism of \( \mathcal{E} \) multiplying by the inverse of this scalar, \( \eta \) and \( \eta^X \) are identified. \( \square \)

We next verify the parametrization in degree 4.

**Theorem 3.14** (Generalization of [CE96, Thm. 4.4], Specialization of [Woo11, Thm. 1.1]). Fix a scheme \( Y \), a rank 3 locally free sheaf \( \mathcal{E} \) on \( Y \), and a rank 2 locally free sheaf \( \mathcal{F} \) on \( Y \) such that there exists an unspecified isomorphism \( \det \mathcal{E} \simeq \det \mathcal{F} \). There is a bijection between

1. finite locally free Gorenstein maps \( \rho : X \to Y \) of degree 4 with associated sheaves \( \mathcal{E}^X, \mathcal{F}^X \) as in §3.11 which are isomorphic to \( \mathcal{E} \) and \( \mathcal{F} \),
2. and, up to automorphisms of \( \mathcal{E} \) and \( \mathcal{F} \), sections \( \eta \in H^0(Y, \mathcal{F}^\vee \otimes \text{Sym}^2 \mathcal{E}) \) having the right codimension at every \( y \in Y \).
First we construct the map from (1) to (2). The bijection explicitly sends a section \( \eta \in H^0(Y, \mathcal{F}^\vee \otimes \text{Sym}^2 \mathcal{E}) \) to \( \Psi_4(\eta) \), considered as a subscheme of \( \mathbb{P}\mathcal{E} \).

**Proof.** Beginning with a cover \( X \to Y \), we obtain a resolution (3.8), and, upon choosing isomorphisms \( \mathcal{E}^X \simeq \mathcal{E} \) and \( \mathcal{F}^X \simeq \mathcal{F} \), we obtain a section \( \eta \in H^0(Y, \mathcal{F}^\vee \otimes \text{Sym}^2 \mathcal{E}) \) having the right codimension at every \( y \in Y \).

To construct the map from (2) to (1), we first verify \( V(\Phi_4(\eta)) \) is a finitely presented Gorenstein cover of \( Y \). On fibers, \( V(\Phi_4(\eta)) \) is described as a dimension 0 intersection of two quadrics. Since \( \eta \) has the right codimension at \( y \in Y \), it has degree 4 over \( y \). Gorensteinness follows because \( V(\Phi_4(\eta)) \) is a local complete intersection. To deduce flatness, we may first reduce to the case \( Y \) is smooth. Indeed, we may work fppf locally on \( Y \), in which case \( X \to Y \) is pulled back from a cover \( X' \to Y' \) for \( Y' \) smooth using [Lemma 3.7](#). In this case, since \( Y \) is reduced, flatness follows from constancy of the degree (i.e., Hilbert polynomial under the canonical embedding of [Theorem 3.2](#)) of \( X \) over \( Y \).

To conclude the construction of the map from (2) to (1), we will show it is possible to choose identifications \( \mathcal{E}^X \simeq \mathcal{E} \) and \( \mathcal{F}^X \simeq \mathcal{F} \) so that we obtain an associated section \( \eta^X \in H^0(Y, \mathcal{F}^\vee \otimes \text{Sym}^2 \mathcal{E}) \simeq H^0(Y, (\mathcal{F}^X)^\vee \otimes \text{Sym}^2 \mathcal{E}^X) \).

First we show \( \mathcal{E}^X \simeq \mathcal{E} \). Indeed, there is a Koszul complex (3.11)

\[
0 \longrightarrow \pi^* \det \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(-2) \longrightarrow \pi^* \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}\mathcal{E}} \longrightarrow \mathcal{O}_X.
\]

It also follows from [Eis95](#) Theorem 20.15 [using the comments on [Eis95](#) p. 503] and the fact that Gorenstein schemes are Cohen-Macaulay) that (3.11) yields a minimal free resolution of \( X_y \) in \( \mathbb{P}\mathcal{E}_y \) for every \( y \in Y \). Because \( \det \mathcal{F} \simeq \det \mathcal{E} \) by assumption, [Theorem 3.2](#) implies \( \mathcal{E} \simeq \mathcal{E}^X \).

Using the isomorphism \( \mathcal{E} \simeq \mathcal{E}^X \), we also verify \( \mathcal{F} \simeq \mathcal{F}^X \). By pushing forward the twist of the above Koszul exact sequence by \( \mathcal{O}_{\mathbb{P}\mathcal{E}}(2) \) along \( \pi \), we find \( \mathcal{F} \simeq \ker(\text{Sym}^2 \mathcal{E} \to \rho_* \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(2)) \). The isomorphism \( \mathcal{E} \simeq \mathcal{E}^X \) is compatible with the above restriction map, and so induces an isomorphism \( \mathcal{F} \simeq \mathcal{F}^X \). This concludes the verification that the map we have produced indeed goes from (2) to (1).

It remains to prove the compositions of the above maps in both directions are equivalent to the identity. As in the degree 3 case, if we start with a cover, produce the associated section \( \eta^X \), \( \Psi_4(\eta^X) \) is isomorphic to \( X \) via the construction. For showing the reverse composition is equivalent to the identity, start with some section \( \eta \). Let \( X \) denote the resulting cover \( \Psi_4(\eta) \).

Given the above identifications \( \mathcal{E}^X \simeq \mathcal{E}, \mathcal{F}^X \simeq \mathcal{F} \), we wish to show \( \eta^X \) is related to \( \eta \) by automorphisms of \( \mathcal{E} \) and \( \mathcal{F} \). Note also here that any automorphism of \( \mathcal{E} \) and \( \mathcal{F} \) send \( \eta \) to another section defining an isomorphic cover. Using [Theorem 3.2](#) since there is an automorphism of \( \mathbb{P}\mathcal{E} \) taking the subscheme \( \Psi_4(\eta^X) \) to \( \Psi_4(\eta) \). From [Theorem 3.2](#) and the fact that the leftmost term of the resolution (3.8) is \( \pi^* \det \mathcal{E}(-4) \), we find \( \mathcal{E} \) is isomorphic to \( \ker(\rho_* \omega_{X/Y} \to \mathcal{O}_Y) \). By [Corollary 3.4](#) the above automorphism of \( \mathbb{P}\mathcal{E} \) is induced by an automorphism of the covering space.
By composing with the inverse of this automorphism, we may assume the resulting map is the identity on \( \mathbb{P} \mathcal{E} \), and so the automorphism of \( \mathbb{P} \mathcal{E} \) is then induced by some scalar automorphism of \( \mathcal{E} \). After adjusting this scalar, we may assume it is the identity. Since \( \mathcal{F} \) is subsheaf of Sym\(^2 \)\( \mathcal{E} \) by Theorem 3.2(v), the image of the induced map \( \mathcal{F} \to \text{Sym}^2 \mathcal{E} \) is uniquely determined by \( X \), but the precise map is only determined up to automorphism of \( \mathcal{F} \). Upon composing with such an automorphism, we may identify not just the images of \( \mathcal{F} \) in Sym\(^2 \)\( \mathcal{E} \), but further we may identify the maps. Under these identifications, \( \eta \) agrees with \( \eta^X \), when viewed as maps \( \mathcal{F} \to \text{Sym}^2 \mathcal{E} \).

We next state and prove the analogous parametrization in degree 5. As preparation, we will need the following application of the structure theorem for codimension 3 Gorenstein algebras due to Buchsbaum-Eisenbud.

**Lemma 3.15.** Let \( Y \) be a scheme, and let \( \mathcal{E} \) and \( \mathcal{F} \) be locally free sheaves on \( Y \) of ranks 3 and 5. A finite locally free Gorenstein map \( \rho : X \to Y \) of degree 5, described as \( \Psi^5(\eta) \) for \( \eta \in H^0(Y, \wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee) \), has a resolution of the form

\[
\begin{align*}
0 \quad &\longrightarrow \quad \pi^* \det \mathcal{E}^\vee \otimes \pi^* \det \mathcal{F}(-5) \quad &\xrightarrow{\beta_3} \quad \pi^* \det \mathcal{E}^\vee \otimes \pi^* \mathcal{F}^\vee(-3) \quad &\xrightarrow{\beta_2} \\
&\quad &\xrightarrow{\beta_2} \quad \pi^* \mathcal{F}(-2) \quad &\xrightarrow{\beta_1} \quad \mathcal{O}_{\mathcal{F},\mathcal{E}} \quad \longrightarrow \quad \mathcal{O}_X,
\end{align*}
\]

which restricts to a minimal free resolution over each \( y \in Y \), where \( \beta_2 \) is alternating and \( \beta_3 \) is identified with the dual of \( \beta_1 \) tensored with \( \pi^* \det \mathcal{E}^\vee \otimes \pi^* \det \mathcal{F}(-5) \).

**Proof.** In (3.12), the map \( \beta_2 \) is obtained from \( \eta \), interpreted as a section of \( H^0(Y, \wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee) \simeq H^0(\mathbb{P} \mathcal{E}, \pi^*(\wedge^2 \mathcal{F} \otimes \det \mathcal{E}^\vee)(1)) \). The map \( \beta_3 \) is obtained by taking five \( 4 \times 4 \) the Pfaffians of \( \beta_2 \). To make sense of this, one may first construct \( \beta_3 \) locally upon choosing trivializations of \( \mathcal{F} \) and \( \mathcal{E} \). One then obtains a global map \( \mathcal{F}(-2) \to \mathcal{O}_{\mathbb{P} \mathcal{E}} \) because the formation of the Pfaffians are compatible with restriction to an open subscheme of \( Y \). Finally, \( \beta_1 \) is obtained as the dual to \( \beta_3 \), tensored with \( \pi^* \det \mathcal{E}^\vee \otimes \pi^* \det \mathcal{F}(-5) \).

Since we have constructed the maps in (3.12) globally over \( \mathbb{P} \mathcal{E} \), it is enough to verify they furnish a minimal free resolution on geometric fibers. To this end, we may work locally on \( Y \) and choose a trivialization \( u : \det \mathcal{E} \simeq \mathcal{O}_Y \). Upon choosing this trivialization composing with the isomorphism \( u \) for the two left nonzero sheaves in (3.12), we obtain a sequence

\[
\begin{align*}
0 \quad &\longrightarrow \quad \pi^* \det \mathcal{F}(-5) \quad &\xrightarrow{\beta_3'} \quad \pi^* \mathcal{F}^\vee(-3) \quad &\xrightarrow{\beta_2'} \quad \pi^* \mathcal{F}(-2) \quad &\xrightarrow{\beta_1'} \quad \mathcal{O}_{\mathbb{P} \mathcal{E}} \quad \longrightarrow \quad \mathcal{O}_X,
\end{align*}
\]

where \( \beta_2' \) is still alternating, i.e., it corresponds to an element of \( H^0(Y, \wedge^2 \mathcal{F} \otimes \mathcal{E}) \), and \( \beta_3' \) remains identified with the dual of \( \beta_1' \), now tensored with \( \pi^* \det \mathcal{F}(-5) \). Since the sequence (3.13) commutes with base change on \( Y \), we may further restrict
to a geometric point \( y \in Y \), and hence assume \( Y \) is the spectrum of an algebraically closed field.

We wish to show (3.13) is a minimal free resolution. To do so, we wish to apply [BE77], and so we translate the above to the setting of commutative algebra. By Theorem 3.2(iii), \( X \to \mathbb{P}E \) is an arithmetically Gorenstein subscheme. Writing \( \mathbb{P}E = \text{Proj} \ k(y)[x_0, x_1, x_2, x_3] \), the cone over \( X \) defines a Gorenstein subscheme of \( \text{Spec} \ k(y)[x_0, x_1, x_2, x_3]_{(x_0, x_1, x_2, x_3)} \), the localization of \( \mathbb{A}^3_{x(y)} \) at the origin. Taking \( R := k(y)[x_0, x_1, x_2, x_3]_{(x_0, x_1, x_2, x_3)} \), we can identify \( \pi^* \mathcal{F} \) with a rank 5 free \( R \)-module \( F \). Let \( J \) denote the ideal of the cone over \( X \) in \( R \). The resolution (3.13) can then be reexpressed in the form

\[
0 \to R \xrightarrow{\beta_3''} F^\vee \xrightarrow{\beta_2''} F \xrightarrow{\beta_1''} R \to \mathcal{R} J / R,
\]

with \( \beta_2'' \in \Lambda^2 F \) alternating and \( \beta_3'' \) the dual of \( \beta_1'' \). By definition of \( \Psi_5(\eta) \) this sequence is exact at \( R \), so \( J \) is the image of \( \beta_1'' \). Since \( X \) has codimension 3 in \( \mathbb{P}E \) by assumption, \( J \) is of grade 3. Hence, (3.14) satisfies the hypotheses of [BE77] Theorem 2.1(1)] are satisfied. It is stated that any such resolution satisfying these hypotheses is a minimal free resolution of \( R / J R \) in the bottom paragraph of [BE77] p. 463] and the proof is given in [BE77] p. 464].

**Theorem 3.16 (Generalization of [Cas96, Thm. 3.8].)** Fix a scheme \( Y \), a rank 4 locally free sheaf \( \mathcal{E} \) on \( Y \), and a rank 5 locally free sheaf \( \mathcal{F} \) on \( Y \) such that there exists and unspecified isomorphism \( \det \mathcal{F} \cong (\det \mathcal{E}) \otimes^2 \). There is a bijection between

1. finite locally free Gorenstein maps \( \rho : X \to Y \) of degree 5 with associated sheaves \( \mathcal{E}^X, \mathcal{F}^X \) as in \( \S 3.11 \) which are isomorphic to \( \mathcal{E} \) and \( \mathcal{F} \), and
2. up to automorphisms of \( \mathcal{E} \) and \( \mathcal{F} \), sections \( \eta \in H^0(Y, \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee) \) having the right codimension at every \( y \in Y \).

The bijection explicitly sends a section \( \eta \in H^0(Y, \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee) \) to \( \Psi_5(\eta) \), considered as a subscheme of \( \mathbb{P}E \).

**Proof.** To start, we construct the map from (1) to (2). Beginning with a cover \( X \to Y \), we obtain a resolution (3.9). Upon choosing isomorphisms \( \mathcal{E}^X \simeq \mathcal{E} \) and \( \mathcal{F}^X \simeq \mathcal{F} \) we obtain a section \( \eta \in H^0(Y, \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee) \) having the right codimension at every \( y \in Y \). We wish to check next that this section actually lies in \( H^0(Y, \Lambda^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee) \). Viewing this as a map \( \pi^* \mathcal{F}^\vee \otimes \pi^* \det \mathcal{E} \to \pi^* \mathcal{F}^\vee(1) \) via (3.9), it is enough to verify the map is alternating locally on the base. Therefore, for this verification, we may assume \( Y \) is the spectrum of a local ring and \( \mathcal{E} \) is trivial. After this reduction, \( X \subset \mathbb{P}E \) is codimension 3 and arithmetically Gorenstein, and so the Buchsbaum-Eisenbud parametrization for codimension 3 Gorenstein schemes [BE77, Thm. 2.1(2)] applies. This produces a resolution of \( X \subset \mathbb{P}E \) as in Equation 3.12 which by Theorem 3.2 must agree with (3.9). Since the map corresponding to \( \pi^* \mathcal{F}^\vee \otimes \pi^* \det \mathcal{E} \to \pi^* \mathcal{F}^\vee(1) \) is alternating in the resolution of [BE77, Thm. 2.1(2)] it follows that \( \pi^* \mathcal{F}^\vee \otimes \pi^* \det \mathcal{E} \to \pi^* \mathcal{F}(1) \) is also alternating.

We next construct the map from (2) to (1). For this, we wish to verify \( V(\Phi_5(\eta)) \) is a finitely presented Gorenstein cover of \( Y \). The finite presentation condition follows
from the resolution given in (3.9). We may check the remaining conditions locally on $Y$, and hence assume $Y$ is the spectrum of a local ring. Observe that $X \to \mathbb{P}E$ is arithmetically Gorenstein and of codimension 3, using the assumption that $\eta$ has the right codimension at each $y \in Y$. Using [BE77, Thm. 2.1(1)], we find that $X$ is Gorenstein and is cut out scheme theoretically by the five $4 \times 4$ Pfaffians associated to $\eta$, thought of as a map $\pi^* \mathcal{F}^\vee \otimes \pi^* \det \mathcal{E} \to \pi^* \mathcal{F}(1)$. On fibers, $V(\Phi_5(\eta))$ is described as the vanishing of the five $4 \times 4$ Pfaffians of an alternating linear map.

The resolution [BE77, Thm. 2.1(1)], can be identified with one of the form (3.9), from which one may calculate that the Hilbert polynomial of every fiber is 5. Therefore, Theorem 3.2(iv) implies $E$ is the identity. By Theorem 3.2(iii), we obtain a unique isomorphism between the resolutions (3.8) is $\pi^* \det \mathcal{E}(-5)$, we find $\mathcal{E}$ is isomorphic to $\mathcal{E}^X$ and $\mathcal{F}^X \simeq \mathcal{F}$. To obtain the first identification, we use Lemma 3.15. Since $\det \mathcal{E}^X \simeq \det \mathcal{F}$, the leftmost nonzero term of the resolution in Lemma 3.15 becomes $\det \pi^* \mathcal{E}^\vee \otimes \pi^* \det \mathcal{F}(-5) \simeq \pi^* \det \mathcal{E}(-5)$. Hence, Theorem 3.2(iv) implies $\mathcal{E} \simeq \mathcal{E}^X$. By twisting (3.12) by $\omega_{\mathcal{E}^X}(2)$ and pushing forward, we find $\mathcal{F} \simeq \ker(\mathrm{Sym}^2 \mathcal{E} \to \rho_* \mathcal{O}_{\mathcal{E}^X}(2))$ and similarly $\mathcal{F}^X \simeq \ker(\mathrm{Sym}^2 \mathcal{E}^X \to \rho_* \mathcal{O}_{\mathcal{E}^X}(2))$. Hence, the isomorphism $\mathcal{E} \simeq \mathcal{E}^X$ induces the desired identification $\mathcal{F} \simeq \mathcal{F}^X$. This completes the construction of the map from (2) to (1).

It remains to prove the compositions of the above maps between (1) and (2) are equivalent to the identity. As in the degree 3 case, if we start with a cover, produce the associated section $\eta^X$, $\Psi_5(\eta^X)$ is isomorphic to $X$ via the construction.

For the reverse composition, start with some section $\eta$ and let $X$ denote the resulting cover $\Psi_5(\eta)$. Now, choose identifications $\mathcal{E}^X \simeq \mathcal{E}$, $\mathcal{F}^X \simeq \mathcal{F}$ as above so that we obtain an associated section $\eta^X \in H^0(Y, \wedge^2 \mathcal{F}^\vee \otimes \det \mathcal{E} \to \mathcal{E}) \simeq H^0(Y, \wedge^2 (\mathcal{F}^X)^\vee \otimes \det \mathcal{E}^X \to \mathcal{E}^X)$. We wish to show $\eta^X$ is related to $\eta$ by automorphisms of $\mathcal{E}$ and $\mathcal{F}$. Note also here that any automorphism of $\mathcal{E}$ and $\mathcal{F}$ send $\eta$ to another section defining an isomorphic cover. Using Theorem 3.2, since there is an automorphism of $\mathbb{P}E$ taking $\Psi_5(\eta^X)$ to $\Psi_5(\eta)$. From Theorem 3.2(iv) and the fact that the leftmost term of the resolution (3.8) is $\pi^* \det \mathcal{E}(-5)$, we find $\mathcal{E}$ is isomorphic to $\ker(\rho_* \omega_{X/Y} \to \mathcal{O}_Y)$. By Corollary 3.4, this automorphism of $\mathbb{P}E$ is induced by an automorphism of $\mathcal{E}$. By composing with the inverse of this automorphism, we may $\eta$ and $\eta^X$ define the same subscheme of $\mathbb{P}E$. Hence we may assume the automorphism of $\mathbb{P}E$ is then induced by some scalar automorphism of $\mathcal{E}$. After adjusting this scalar, we may assume it is the identity. By Theorem 3.2(iii), we obtain a unique isomorphism between the two resolutions of $X$ in $\mathbb{P}E$ (3.9) determined by $\eta$ and $\eta^X$. This isomorphism can be specified as a tuple of 5 maps between the nonzero terms of (3.9).

We next show we can apply an automorphism of $\mathcal{F}$ so as to assume the map $\pi^* \mathcal{F}(-2) \to \pi^* \mathcal{F}(-2)$ is the identity. By the above identifying the images $X \subset \mathbb{P}E$.
Since $\mathcal{F}$ is subsheaf of $\text{Sym}^2 \mathcal{E}$ by Theorem 3.2(v), the image of the induced map $\mathcal{F} \to \text{Sym}^2 \mathcal{E}$ coming from the Pfaffians associated to $\eta$ is uniquely determined by $X$, but the precise map is only determined up to automorphism of $\mathcal{F}$. Upon composing with such an automorphism, we may identify not just the images of $\mathcal{F}$ in $\text{Sym}^2 \mathcal{E}$, but further we may identify the maps. Under these identifications, $\eta$ agrees with $\eta^X$, when viewed as maps $\mathcal{F} \to \text{Sym}^2 \mathcal{E}$.

So far, we have constructed a map of the two resolutions (3.9) associated to $\eta$ and $\eta_X$. Upon choosing identifications $\mathcal{E} \simeq \mathcal{E}^X$ and $\mathcal{F} \simeq \mathcal{F}_X$ as above, we have enforced that the map of resolutions is given by the identity on the terms $\mathcal{O}_X \to \mathcal{O}_X$, $\mathcal{O}_P \to \mathcal{O}_P$, and $\pi^* \mathcal{F}(-2) \to \pi^* \mathcal{F}(-2)$. When we write the second nonzero term of (3.9) as $\pi^* \mathcal{F}^\vee \otimes \pi^* \det \mathcal{E}(-3)$, we have identified it via Grothendieck duality as pairing with the third nonzero term $\pi^* \mathcal{F}(-3)$ into $\pi^* \mathcal{E}(-5)$, and therefore the induced automorphism $\pi^* \mathcal{F}^\vee \otimes \pi^* \det \mathcal{E}(-3)$ must respect this duality. In particular, since we have reduced to the case where the automorphism of $\pi^* \mathcal{F}(-2)$ is the identity, we also obtain the induced automorphism of $\pi^* \mathcal{F}^\vee \otimes \pi^* \det \mathcal{E}(-3)$ is the identity. Using Theorem 3.2(v) to guarantee that the maps $\eta$ and $\eta^X$ from $\mathcal{F}^\vee \otimes \det \mathcal{E} \to \mathcal{F} \otimes \mathcal{E}$ are injective, we obtain the desired identification of $\eta$ with $\eta^X$. \qed

Finally, we recall a rather elementary criterion for when $\Psi_d(\eta)$ is geometrically connected.

**Theorem 3.17** (Part of [CE96, Thm. 3.6], [CE96, Thm. 4.5], [Cas96, Thm. 4.4]). Keeping notation as in Notation 3.8 assume that $Y$ is a geometrically connected and geometrically reduced projective scheme over a field $k$. If $h^0(Y, \mathcal{E}^\vee) = 0$, then $\Psi_d(\eta)$ is geometrically connected.

**Proof.** The proof is essentially given in [CE96, Thm. 3.6], and we repeat it for the reader’s convenience. Let $X := \Psi_d(\eta)$. If $h^0(Y, \mathcal{E}^\vee) = 0$ the exact sequence

\[(3.15) \quad 0 \longrightarrow \mathcal{O}_Y \longrightarrow \rho_*(\mathcal{O}_X) \longrightarrow \mathcal{E}^\vee \longrightarrow 0\]

induces an isomorphism $H^0(Y, \mathcal{O}_Y) \simeq H^0(Y, \rho_*(\mathcal{O}_X)) = H^0(X, \mathcal{O}_X)$. Since $Y$ is geometrically connected and geometrically reduced, we have $h^0(Y, \mathcal{O}_Y) = 1$. From this we find $H^0(X, \mathcal{O}_X) = 1$ as well, and therefore $X$ is necessarily geometrically connected. \qed

4. **Describing Stacks of Low-degree Covers as Quotients**

In this section, we give a description of the stack of degree $d$ Gorenstein covers as a global quotient stack for $3 \leq d \leq 5$. We now introduce the groups we will be quotienting by. Since the Hurwitz stack is closely related to the Weil restriction of the stack of degree $d$ covers along $\mathbb{P}^1 \to \text{Spec } k$, we will simultaneously define these automorphism groups along Weil restrictions.

**Definition 4.1.** Given a scheme $Y$ over a base $B$ and an integer $d$, let resolution data for $Y$ and $d$ denote a tuple of locally free sheaves $(\mathcal{E}, \mathcal{F}_\bullet)$ on $\overline{Y}$, where $\mathcal{E}$ is a locally free sheaf of rank $d - 1$ and $\mathcal{F}_\bullet$ denotes the sequence $\mathcal{F}_1, \ldots, \mathcal{F}_{\left\lfloor \frac{d-1}{2} \right\rfloor}$ where $\text{rk } \mathcal{F}_i = \beta_i$
as in (3.2). (We will typically take $B = \text{Spec } k$ for $k$ a field, except in Proposition 4.7 where we take $B = \text{Spec } \mathbb{Z}$.)

Let $3 \leq d \leq 5$, fix a scheme $Y$ over a field, and fix resolution data $(\mathcal{E}, \mathcal{F}_*)$ for a degree $d$ cover of $Y$. For $\mathcal{G}$ a locally free sheaf on $Y$, let $\Delta_{\mathcal{G}}^Y : G_m \to \text{Res}_{Y/B}(\text{Aut}_{\mathcal{G}/Y})$ denote the map adjoint to the central inclusion $(G_m \times_B Y) \to \text{Aut}_{\mathcal{G}/Y}$ on $Y$.

Then, define the automorphism sheaf of this resolution data to be the $B$-scheme

$$\text{Aut}_{\mathcal{G}/Y, \mathcal{F}_*}^Y := \begin{cases} \text{Res}_{Y/B}(\text{Aut}_{\mathcal{G}/Y}) & \text{if } d = 3 \\
\text{coker}(\Delta_{\mathcal{G}}^Y, (\Delta_{\mathcal{G}}^Y)^2) & \text{if } d = 4 \\
\text{coker}((\Delta_{\mathcal{G}}^Y)^2, (\Delta_{\mathcal{G}}^Y)^3) & \text{if } d = 5. \end{cases}$$

(4.1)

In the case $Y = B$, we notate $\text{Aut}_{\mathcal{G}/Y, \mathcal{F}_*}^Y$ simply by $\text{Aut}_{\mathcal{G}, \mathcal{F}_*}$. When $d = 4$ or 5, we will often denote $\mathcal{F}_1$ by $\mathcal{F}$.

**Remark 4.2.** Concretely, $\mathcal{E}$ and $\mathcal{F}_*$ in Definition 4.1 are (sequences of) sheaves of the following ranks. For $d = 3$, $\mathcal{E}$ is free of rank 2 and $\mathcal{F}_*$ is trivial (i.e., the sequence of sheaves has length 0). When $d = 4$, $\mathcal{E}$ is free of rank 3 and $\mathcal{F}_* = \mathcal{F}$ is free of rank 2. When $d = 5$, $\mathcal{E}$ is free of rank 4 and $\mathcal{F}_* = \mathcal{F}$ is free of rank 5.

In order to be able to calculate the class of quotients by the groups of Definition 4.1 in the Grothendieck ring, it will be useful to know these groups are often special. The following description of these quotients will allow us later to in Lemma 7.10 to easily deduce these groups are special.

**Lemma 4.3.** Maintaining the notation of Definition 4.1, we have an isomorphism of functors

$$\text{Aut}_{\mathcal{G}, \mathcal{F}_*}^Y \simeq \begin{cases} \ker(\text{det}, \text{det}^{-1}) : \text{Res}_{Y/B}(\text{Aut}_{\mathcal{G}/Y}) \times \text{Res}_{Y/B}(\text{Aut}_{\mathcal{F}/Y}) \to \text{Res}_{Y/B}(G_m) & \text{if } d = 4 \\
\ker(\text{det}^3, \text{det}^{-1}) : \text{Res}_{Y/B}(\text{Aut}_{\mathcal{G}/Y}) \times \text{Res}_{Y/B}(\text{Aut}_{\mathcal{F}/Y}) \to \text{Res}_{Y/B}(G_m) & \text{if } d = 5 \end{cases}$$

Here, by determinant we mean the map adjoint the the corresponding determinant map on $Y$.

**Proof.** We produce the claimed isomorphisms by constructing a section to the quotient map $q : \text{Res}_{Y/B}(\text{Aut}_{\mathcal{G}/Y}) \times \text{Res}_{Y/B}(\text{Aut}_{\mathcal{F}/Y}) \to \text{Aut}_{\mathcal{G}, \mathcal{F}_*}^Y$, defining $\text{Aut}_{\mathcal{G}, \mathcal{F}_*}^Y$.

To start, we cover the case $d = 4$. Given $(M, N) \in \text{Res}_{Y/B}(\text{Aut}_{\mathcal{G}/Y}) \times \text{Res}_{Y/B}(\text{Aut}_{\mathcal{F}/Y})$, for $\lambda \in G_m$, we can identify $q(M, N) = q(\lambda M, \lambda^2 N)$. For any such $(M, N)$ the key observation is that there is a unique $\lambda \in G_m$ such that $\text{det}(\lambda M) = \text{det}(\lambda^2 N)$. Indeed, $\text{det}(\lambda M) = \lambda^3 \text{det} M$ while $\text{det}(\lambda^2 N) = \lambda^4 \text{det} N$ and so the unique such $\lambda$ is $\lambda = \text{det} M/\text{det} N$. This gives the desired splitting realizing $\text{Aut}_{\mathcal{G}, \mathcal{F}_*}^Y$ as a subgroup of $\text{Res}_{Y/B}(\text{Aut}_{\mathcal{G}/Y}) \times \text{Res}_{Y/B}(\text{Aut}_{\mathcal{F}/Y})$ because the composition $\ker(\text{det}, \text{det}) \to \text{Res}_{Y/B}(\text{Aut}_{\mathcal{G}/Y}) \times \text{Res}_{Y/B}(\text{Aut}_{\mathcal{F}/Y}) \to \text{Aut}_{\mathcal{G}, \mathcal{F}_*}^Y$ is an isomorphism.

The $d = 5$ case is quite similar to the $d = 4$ case. Namely, in this case, for $(M, N) \in \text{Res}_{Y/B}(\text{Aut}_{\mathcal{G}/Y}) \times \text{Res}_{Y/B}(\text{Aut}_{\mathcal{F}/Y})$, there is again a unique $\lambda \in G_m$ so that $(\text{det}(\lambda^2 M))^2 = \text{det}(\lambda^3 N)$. Indeed, $(\text{det}(\lambda^2 M))^2 = \lambda^{16} \text{det} M^2$ and $\text{det}(\lambda^3 N) = \lambda^{15} \text{det} N$, so the unique desired $\lambda$ is $\text{det} N/(\text{det} M)^2$. As in the $d = 4$ case, this
provides a section to the given quotient map realizing $\text{Aut}_{\mathcal{E}, \mathcal{F}_*}^Y/B$ as the subgroup of $\text{Res}_{Y/B}(\text{Aut}_{\mathcal{E}/Y}) \times \text{Res}_{Y/B}(\text{Aut}_{\mathcal{F}/Y})$ given as those $(M, N)$ with $(\det M)^2 = \det N$.

We next describe a presentation of the stack parameterizing degree $d$ Gorenstein covers for $3 \leq d \leq 5$. To make our next definition, we will need to know the Gorenstein locus of a finite locally free map is open.

**Lemma 4.4.** Let $f : X \to Y$ be a finite locally free morphism of schemes. The locus of points of $Y$ on which the fiber of $f$ is Gorenstein is an open subscheme of $Y$.

**Proof.** First, by [Sta, Tag 00RH], the condition that the fiber be Cohen-Macaulay is an open condition. After restricting to such an open subscheme, by [Con00, Thm. 3.5.1], a dualizing sheaf exists, and the Gorenstein locus is then the locus where this dualizing sheaf is locally free, which again defines an open subscheme.

We are now ready to define the relevant Gorenstein loci. With notation as in Definition 4.1, we work over $B = \text{Spec } \mathbb{Z}$. As introduced in the beginning of §3.5, we use Covers$_d$ to denote the fibered category whose $S$ points are finite locally free covers $X \to S$ of degree $d$ with Gorenstein fibers.

**Definition 4.5.** For each $3 \leq d \leq 5$, fix free sheaves on $Y = B = \text{Spec } \mathbb{Z}$, $\mathcal{E}$ and $\mathcal{F}_*$ as in Definition 4.1 and Remark 4.2. Let $U_d \subset \text{Spec } \text{Sym}^\infty H^0(\text{Spec } \mathbb{Z}, \mathcal{H}(\mathcal{E}, \mathcal{F}_*))$ denote the open subscheme (using Lemma 4.4 functorially parameterizing those sections $\eta$ so that $\Psi_d(\eta)$ defines a degree $d$ locally free Gorenstein cover, for $\Psi_d$ the maps (depending on $3 \leq d \leq 5$) defined in §3.5.

**Definition 4.6.** For $d \leq 3 \leq 5$, the map $\Psi_d$ induces a map $\mu_d : U_d \to \text{Covers}_d$. There is a natural action of $\text{Aut}_{\mathcal{E}, \mathcal{F}_*}$ on $U_d$, induced by the action of $\text{Aut}_{\mathcal{E}} \times \text{Aut}_{\mathcal{F}_*}$ on $U_d$. The map $\mu_d$ is invariant under this action, since the resulting abstract degree $d$ cover is unchanged by such re-coordinatizations, we obtain an induced map from the quotient stack $\phi_d : [U_d/\text{Aut}_{\mathcal{E}, \mathcal{F}_*}] \to \text{Covers}_d$.

**Proposition 4.7.** For $3 \leq d \leq 5$, the map $\phi_d$ defined in Definition 4.6 is an isomorphism.

When $d = 3, 4$, Proposition 4.7 is the specialization of the isomorphisms of moduli stacks given in [Poo08, Prop. 5.1] and [Woo11, Thm. 1.1] to Gorenstein covers.

**Proof.** We will construct an inverse map using Theorem 3.2. Using Theorem 3.2 there is an $\text{Aut}_{\mathcal{E}} \times \text{Aut}_{\mathcal{F}_*}$ torsor torsor $T_d$ over Covers$_d$ whose $S$-points parameterize covers $X \to S$ together with specified trivializations $\mathcal{E}_X \cong \mathcal{E}, \mathcal{F}_X \cong \mathcal{F}_*$ of the sheaves $\mathcal{E}_X$ and $\mathcal{F}_X$ associated to $X$ coming from Theorem 3.2. Note here that $T_d$ maps surjectively to Covers$_d$ because for any $S$ point, there is an open cover of $S$ on which these vector bundles become isomorphic to trivial bundles. The parametrizations Theorem 3.13, Theorem 3.14, and Theorem 3.16 then give a section $\eta \in \mathcal{H}(\mathcal{E}, \mathcal{F}_*)$. This induces a map $T_d \to U_d$.

We wish to show this induced map is an isomorphism in degree 3 and a $\text{G}_m$ torsor in degrees 4 and 5, where $\text{G}_m$ is the copy of $\text{G}_m \subset \text{Aut}_{\mathcal{E}} \times \text{Aut}_{\mathcal{F}_*}$ as in Definition 4.1 whose quotient yields $\text{Aut}_{\mathcal{E}, \mathcal{F}_*}$. Once we verify this, the parametrizations Theorem 3.13.
Theorem 3.14 and Theorem 3.16 imply that the composition of this map with \( \phi_d \) is the structure map for the torsor \( T_d \to \text{Covers}_d \). Therefore, it will follow that the resulting map \([U_d / \text{Aut}_E, \mathcal{F}] \to [T_d / \text{Aut}_E \times \text{Aut}_F] \simeq \text{Covers}_d\) is an isomorphism.

First, we verify the map \( T_d \to U_d \) is invariant under the above mentioned \( \mathbb{G}_m \) action in the cases that \( d = 4 \) and \( 5 \). In the degree 4 case, scaling \( E \) by \( \lambda \) and \( F \) by \( \lambda^2 \) scales \( \mathcal{F}^\vee \otimes \text{Sym}^2 \mathcal{E} \) by \( \lambda^{-2} \cdot \lambda^2 = 1 \). In the degree 5 case, scaling \( \mathcal{E} \) by \( \lambda^2 \) and \( \mathcal{F} \) by \( \lambda^3 \) scales \( \wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \text{det} \mathcal{E}^\vee \) by \( \lambda^6 \cdot \lambda^2 \cdot \lambda^2 (-4) = 1 \).

Therefore, to conclude the verification, it is enough to show the only elements of \( \text{Aut}_E \times \text{Aut}_F \) fixing a given section are trivial when \( d = 3 \) and lie in \( \mathbb{G}_m \) when \( d = 4 \) or \( 5 \). To start, the map \( X \to \mathbb{P} \mathcal{E} \) realizes \( X \) as a nondegenerate subscheme of \( \mathbb{P} \mathcal{E} \), and therefore the only the trivial element of \( \text{PGL}_E \) fixes \( X \) as a subscheme of \( \mathbb{P} \mathcal{E} \). In the degree 3 case, scaling by \( \lambda \) in the central \( \mathbb{G}_m \subset \text{Aut}_E \) scales the resulting section by \( \lambda \), and so only the identity element of \( \text{Aut}_E \) preserves the section. This establishes the claim when \( d = 3 \).

We now consider the cases \( d = 4 \) and \( d = 5 \). We are seeking automorphisms of \( \mathcal{E} \) and \( \mathcal{F} \) preserving a given section \( \eta \in U_d \). We have seen above that any such automorphism must act on \( \mathcal{E} \) by some element \( \lambda \) in the central \( \mathbb{G}_m \subset \text{Aut}_E \). Since we are quotienting by a copy of \( \mathbb{G}_m \subset \text{Aut}_E \times \text{Aut}_F \) which maps surjectively to the central \( \mathbb{G}_m \) in \( \text{Aut}_E \), we may modify our given automorphism so as to assume it is trivial in \( \text{Aut}_E \). Note that when \( d = 5 \), we may have to pass to an fppf cover so as to extract a square root of \( \lambda \). We may now assume the automorphism is trivial on \( \mathcal{E} \) and wish to show it is also trivial on \( \mathcal{F} \). However, the given section \( \eta \) induces an injective map \( \mathcal{F} \to \text{Sym}^2 \mathcal{E} \), realizing \( \mathcal{F} \) as a subsheaf of \( \text{Sym}^2 \mathcal{E} \) by Theorem 3.2(v). Since we are assuming the automorphism acts as the identity on \( \mathcal{E} \) and it preserves this inclusion, it must also act as the identity on \( \mathcal{F} \).

\[ \square \]

5. Defining our Hurwitz stacks

In this section, we construct and define the Hurwitz spaces we will be working with. We will ultimately be interested in the Hurwitz space whose geometric points parameterize degree \( d \) \( S \)-covers of \( \mathbb{P}^1 \) which are smooth and connected. When one restricts to simply branched covers, such a Hurwitz scheme was constructed by Fulton [Ful69]. Another good reference is [Deo14, Theorem A], though this reference assumes characteristic 0. We were unable to find a reference that allows arbitrary branching and non-Galois covers in arbitrary characteristic, and so we give the construction here. To begin, we define a certain Hurwitz stack parameterizing covers of \( \mathbb{P}^1 \) which are not necessarily \( S \)-covers.

Definition 5.1. For \( S \) a base scheme, and \( d \geq 0 \) an integer, let \( \text{Hur}_{d,S} \) denote the category fibered in groups over \( S \)-schemes whose \( T \) points over a given map of
schemes $T \to S$ consisting of $(T, X, h : X \to T, f : X \to \mathbb{P}^1_T)$

\[
\begin{array}{c}
X \\
\downarrow f \\
\mathbb{P}_T^1 \\
\downarrow h \\
T
\end{array}
\]

where $X$ is a scheme, $f$ is a finite locally free map of degree $d$ and $h$ is a smooth proper relative curve. A map $(T, p, X, h, f) \to (T, X', h', f')$ consists of $T$-isomorphisms $\alpha : X \to X'$ and $\beta : \mathbb{P}_T^1 \to \mathbb{P}_T^1$ such that

\[
\begin{array}{c}
X \\
\downarrow f \\
\mathbb{P}_T^1 \\
\downarrow h \\
T \\
\alpha \\
\downarrow \\
X' \\
\downarrow f' \\
\mathbb{P}_T^1 \\
\downarrow h' \\
T \\
\beta
\end{array}
\]

commutes. For $g \geq 0$ an integer, let $\overline{\text{Hur}}_{d, g, S}$ denote the substack parameterizing those $T$-points of $\overline{\text{Hur}}_{d, S}$ such that $X \to T$ has arithmetic genus $g$.

**Lemma 5.2.** For $S$ a scheme, $\overline{\text{Hur}}_{d, S}$ and $\overline{\text{Hur}}_{d, g, S}$ are algebraic stacks.

**Proof.** First, we show $\overline{\text{Hur}}_{d, S}$ is an algebraic stack. It is enough to establish this in the universal case $S = \text{Spec} \mathbb{Z}$. Observe that $\overline{\text{Hur}}_{d, \mathbb{Z}}$ is a stack because descent for finite degree $d$ locally free morphisms is effective. To see it is algebraic, we construct it as a hom stack. Let $\mathcal{A}_d$ denote the stack parameterizing finite locally free degree $d$ covers, as constructed in [Poo08, Def. 3.2].

Next, we claim the mapping stack $\text{Hom}(\mathbb{P}^1, \mathcal{A}_d)$ is algebraic. This would follow from [Aok06b, Thm. 1.1], except the theorem there is not stated correctly, as mentioned in the erratum [Aok06a]. This erratum asserts that we only need verify the additional condition that for any complete local noetherian ring $A$ with maximal ideal $m$ and $A_n := A/m^n$, a collection of compatible maps $\text{Hom}(\mathbb{P}^1_{A_n}, (\mathcal{A}_d)_{A_n})$ for each $n$ lifts to a map $\text{Hom}(\mathbb{P}^1_A, (\mathcal{A}_d)_A)$. In our setting, this condition is indeed satisfied because specifying such maps over $A_n$ corresponds to specifying degree $d$ locally free covers $X_n \to \mathbb{P}^1_{A_n}$ over $A_n$ for each $n$. Then, by Grothendieck’s algebraization theorem [FGI+05, Thm. 8.4.10] such a family algebraizes to a family $X \to \mathbb{P}^1_A$ over $\text{Spec} A$, using the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ to $X$ as the relevant ample line bundle on $X$.

The stack $\overline{\text{Hur}}_{d, S}$ is then the open substack of the mapping stack $\text{Hom}(\mathbb{P}^1, \mathcal{A}_d)$ corresponding to those finite locally free covers $X \to \mathbb{P}^1$ which are smooth over the base.
Finally, $\text{Hur}_{d,g,S}$ is an open and closed substack of $\text{Hur}_{d,S}$ because, since the genus is locally constant in flat families. □

Having constructed the Hurwitz stack parameterizing all degree $d$ covers of $\mathbb{P}^1$, we next construct an open substack parameterizing $S_d$ covers, over geometric fibers. For the following definition, recall that $B_n$, the $n$th Bell number is the number of ways to partition a set of $n$ elements into subsets. So, for example, $B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15$.

**Definition 5.3.** Let $S$ be a scheme with $d!$ invertible on $S$. Let $\text{Hur}_{d,g,S}$ denote the substack of $\text{Hur}_{d,g,S}$ parameterizing those $(T, X, h : X \to T, f : X \to \mathbb{P}^1_T)$ such that $X^{d} := \underbrace{X \times_{\mathbb{P}^1} X \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1}}_{\text{d times}}X$ has $B_d$ irreducible components in each geometric fiber over $T$, where $B_d$ is the $d$th Bell number.

The above definition is a bit opaque, but the point is that it parameterizes degree $d$ covers $X \to \mathbb{P}^1$ so that the Galois closure of $K(X) \leftarrow K(\mathbb{P}^1)$ is an $S_d$ Galois extension, as we now verify.

**Lemma 5.4.** The fiber product $X^{d}$ as in Definition 5.3 always has at least $B_d$ irreducible components in each geometric fiber over $T$.

Further, it has exactly $B_d$ components if and only if $X \to \mathbb{P}^1_T$ is a degree $d$ cover whose Galois closure has Galois group $S_d$ on geometric fibers over $T$.

*Proof.* We may reduce to the case $T$ is a geometric point. First, we check $X$ has at least $B_d$ irreducible components. To see this, for any partition $U = \{S_1, \ldots, S_{\#U}\}$ of $\{1, \ldots, d\}$ into $\#U$ many subsets, let $X^U \subset X^{d}$ denote the subscheme of $X^{d}$ given as the image $X^U \to X^d$ sending the $i$th copy of $X$ via the identity to those copies of $X$ indexed by elements of $S_i$. For each partition $V$ of $\{1, \ldots, d\}$ such that $U$ refines $V$, the closure of $X^U \cup_{V \not\text{refines} V} X^V$ defines a nonempty union of irreducible components of $X^d$. We have therefore produced $B_d$ irreducible components of $X$, showing there are always at least $B_d$ irreducible components.

Conversely, $X^d$ has exactly $B_d$ geometric components if and only if each of the $B_d$ subschemes described in the previous paragraph are irreducible. Let us focus on the subscheme $Y$ corresponding to the partition $U = \{\{1\}, \{2\}, \ldots, \{d\}\}$ into singletons, which has degree $d!$ over $\mathbb{P}^1$ and is the closure of the complement of the “fat diagonal” in $X^d$. Observe that $X \to \mathbb{P}^1$ is generically étale because $X$ is smooth, and we are assuming the characteristic of $T$ does not divide $d!$. Therefore, $Y \to \mathbb{P}^1$ is also generically étale, and contains a component whose function field is the Galois closure of the extension of function fields $K(X) \leftarrow K(\mathbb{P}^1)$. Therefore, $Y$ is irreducible if and only if $K(Y)$ is the Galois closure of $K(X) \leftarrow K(\mathbb{P}^1)$. As $Y \to \mathbb{P}^1$ has degree $d!$, this in turn is equivalent to $X \to \mathbb{P}^1$ having Galois closure with Galois group $S_d$. In particular, for any cover $X \to \mathbb{P}^1$ whose Galois closure is smaller than $S_d X^d$ has strictly more than $B_d$ irreducible components.
Finally, we check that for any $S_d$ cover, each of the $B_d$ components described above are irreducible. As we have shown, even the component $Y$ of degree $d!$ over $\mathbb{P}^1$ is irreducible. Because all the other components correspond to intermediate extensions between $Y$ and $\mathbb{P}^1$, they are also irreducible. □

We next carry out the surprisingly tricky verification that $\text{Hur}_{d,g,S}$ is an open substack of $\text{Hur}_{d,g,S}$.

**Proposition 5.5.** For any integers $d, g \geq 0$, and $d!$ invertible on $S$, $\text{Hur}_{d,g,S}$ is an open substack of $\text{Hur}_{d,g,S}$, hence an algebraic stack. Further, if we have a family of curves $X \to \mathbb{P}^1_T \to T$ corresponding to a $T$-point of $\text{Hur}_{d,g,S}$, all fibers of $X$ over $T$ are geometrically irreducible.

**Proof.** It is enough to demonstrate $\text{Hur}_{d,g,S}$ is an open substack of $\text{Hur}_{d,g,S}$. Let $X \to \mathbb{P}^1_T \to T$ be a family of smooth curves, corresponding to a point of $\text{Hur}_{d,g,S}$. Let $X^d$ denote the $d$-fold fiber product of $X$ over $\mathbb{P}^1_T$. By [Lemma 5.4] any such point corresponds to an $S_d$ cover of $\mathbb{P}^1$ on geometric fibers, and therefore these geometric fibers are irreducible, verifying the final statement.

It remains to show that the locus where $X^d$ has $B_d$ irreducible fibers in geometric fibers is open on $T$. First, we will see in [Lemma 5.7] that the geometric fibers of $X^d$ over $T$ have no embedded points.

Because the fibers have no embedded points, we may apply [Gro65 12.2.1(xi)], which says that the total multiplicity (in the sense defined in [Gro65], following [Gro65 4.7.4], where total multiplicity is defined for integral schemes) is upper semicontinuous. From this, we conclude that the locus of geometric points in $T$ where the total multiplicity of $X^d$ is at least $B_d$ is open. By [Lemma 5.4] the total multiplicity of any geometric fiber is always at least $B_d$, and hence the locus where the total multiplicity is exactly $B_d$ is also open. To conclude, it remains to verify the total multiplicity of any geometric fiber is equal to the number of its irreducible components. Note that the radical multiplicity of any fiber is 1 because $X^d$ is generically reduced, since it has a generically separable map to $\mathbb{P}^1$ by assumption that $d! \nmid \text{char}(k)$. It follows that the total multiplicity is equal to the separable multiplicity. By definition, the separable multiplicity of a finite type scheme over a field is equal to 1 if and only if the scheme is geometrically irreducible, as desired. □

**Remark 5.6.** Later, in [Lemma 9.6] we will appeal to [Wew98] to construct substacks of $\text{Hur}_{d,g,S}$ parameterizing covers with specified Galois group $G \subset S_d$. One can also see using the method of proof of Proposition 5.5 that these form locally closed substacks, with partial ordering given by the partial ordering along inclusion of subgroups in $S_d$.

**Lemma 5.7.** Let $X \to \mathbb{P}^1$ be a degree $d$ map of smooth proper curves over an algebraically closed field $k$. If the characteristic of $k$ does not divide $d!$, then $X^d := X \times_{\mathbb{P}^1} X \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1} X$ is Cohen-Macaulay, and hence has no embedded points.
We wish to verify \( X^d \) is Cohen-Macaulay, as 1-dimensional Cohen-Macaulay schemes have no embedded points. To verify \( X^d \) is Cohen-Macaulay, we may do so étale locally on \( \mathbb{P}^1 \), and hence we may freely base change to the strict henselization of \( \mathbb{P}^1 \) at any given closed point. Using the assumption on the characteristic of \( k \) and the classification of prime to \( \text{char}(k) \) covers of the strict henselization of \( k[t] \), we may assume our cover is given by extracting roots of the uniformizer. Equivalently, it is enough to verify Cohen-Macaulayness in the case \( X^d \) is locally described as a localization of \( k[x_1] \otimes_{k[t]} k[x_2] \otimes_{k[t]} \cdots \otimes_{k[t]} k[x_m] \) where the maps \( k[t] \to k[x_i] \) are given by \( t \mapsto x_i^{s_i} \), for \( s_i \leq d \). We can equivalently write this tensor product as \( k[x_1] \otimes_{k[t]} k[x_2] \otimes_{k[t]} \cdots \otimes_{k[t]} k[x_m] \simeq k[x_1, x_2, \ldots, x_m]/(x_1^{s_1} - x_2^{s_2}, \ldots, x_1^{s_m} - x_m^{s_m}) := R \). We wish to verify \( R \) is Cohen-Macaulay. Observe that \( R \) is a 1 dimensional scheme, being a finite cover of \( k[t] \). Since it is defined by \( m - 1 \) equations in \( \mathbb{A}^m \), it is a complete intersection, and therefore Cohen-Macaulay.

□

The following remark will not be used in the remainder of the paper, but may be nice for the reader to keep in mind.

Remark 5.8. For \( d > 2 \) and \( g \geq 1 \), \( \text{Hur}_{d,g,S} \) is a scheme when \( d! \) is invertible on \( S \). We have seen above it is an algebraic stack. In order to see it is a scheme, one may first verify it is an algebraic space by checking any degree \( d \) cover of \( \mathbb{P}^1 \) with Galois group \( S_d \) for \( d > 2 \) has no nontrivial automorphisms [Sta, Tag 04SZ]. Indeed, if such a cover did have automorphisms, it would factor through an intermediate cover obtained by quotienting by some such nontrivial automorphism, forcing the Galois group to be smaller than \( S_d \).

Having established \( \text{Hur}_{d,g,S} \) is an algebraic space, we next wish to explain why it is a scheme. Observe this Hurwitz space has a map to the configuration space \( \text{Sym}^2_{\mathbb{P}^1} \) of \( 2g - 2 + 2d \) point on \( \mathbb{P}^1 \) given by “taking the branch locus.” This uses that \( d! \) is invertible on \( S \) and Riemann-Hurwitz. One may verify this map is separated (for example, using the valuative criterion) and quasi-finite (since the inertia data around the branch points determines the cover), hence quasi-affine [Sta, Tag 082J]. Therefore, it is quasi-affine over a scheme, and therefore a scheme.

6. Defining the Casnati-Ekedahl strata in Hurwitz stacks

For this section, we now fix a positive integer \( d \) and a base field \( k \) with \( d! \) invertible on \( k \). We parenthetically note that much of the following can be generalized to work over arbitrary base schemes. For \( T \) a \( k \)-scheme, given a Gorenstein finite locally free degree \( d \) cover \( X \to \mathbb{P}^1_T \), from Theorem 3.2 we obtain a canonical sequence of vector bundles \( (\mathcal{E}_X, \mathcal{F}_X^1, \mathcal{F}_X^2, \ldots, \mathcal{F}_X^{d-2}) \) on \( \mathbb{P}^1_T \). We next aim to define certain locally closed substacks of \( \text{Hur}_{d,g,S} \) corresponding to those covers \( X \to \mathbb{P}^1_T \) whose associated vector bundles are isomorphic to some specified sequence \( (\mathcal{E}, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{d-2}) \). To define this substack, we first define the corresponding stack of these vector bundles.

Recall that the stack of locally free rank \( n \) sheaves on \( \mathbb{P}^1_k \) is an algebraic stack, as is well known, see for example [Beh91, Prop. 4.4.6].
Definition 6.1. Let \( \text{Vect}_k^n \) denote the moduli stack of locally free rank \( n \) sheaves on \( \mathbb{P}^1_k \). For \( \bar{a} = (a_1, a_2, \ldots, a_n) \), with \( a_i \in \mathbb{Z} \), let \( \mathcal{O}_{\mathbb{P}^1_k}(\bar{a}) := \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1_k}(a_i) \) and let \( \text{Vect}^\bar{a}_{\mathbb{P}^1_k} \) denote the residual gerbe at the point corresponding to \( \text{the vector bundle} \ \mathcal{O}_{\mathbb{P}^1_k}(\bar{a}) \).

Remark 6.2. Note that this residual gerbe is indeed a locally closed substack by [Ryd11, Thm. B.2]. Alternatively, the residual gerbe is given concretely as the quotient stack \( B(\text{Res}_{\mathbb{P}^1_k/k}(\text{Aut}_{\mathbb{P}^1_k}(\bar{a}))) \).

In order to relate the genus of a cover of \( \mathbb{P}^1 \) to the associated vector bundle \( \mathcal{E} \) we need the following standard lemma.

Lemma 6.3. Suppose \( \rho : X \to \mathbb{P}^1_k \) is a degree \( d \) Gorenstein finite locally free cover and let \( \mathcal{E} := \ker(\rho_* \omega_X \to \mathcal{O}_{\mathbb{P}^1_k}) \). If \( h^0(X, \mathcal{O}_X) = 1 \), such as in the case that \( X \) is smooth and geometrically connected, then \( \deg(\det \mathcal{E}) = g + d - 1 \).

Proof. First, we claim \( \rho_* \mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{E}^\vee \). Indeed, by duality, we have a short exact sequence \( \mathcal{O}_{\mathbb{P}^1_k} \to \mathcal{O}_X \to \mathcal{E}^\vee \). Because all vector bundles on \( \mathbb{P}^1 \) split, and \( h^0(\mathbb{P}^1_k, \rho_* \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1 \), we find that \( \mathcal{E}^\vee \cong \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^1_k}(-a_i) \) for \( a_i > 0 \). Because there are no extensions of \( \mathcal{O}_{\mathbb{P}^1_k}(-a_i) \) by \( \mathcal{O}_{\mathbb{P}^1_k} \), the above exact sequence splits, yielding \( \rho_* \mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{E}^\vee \cong \mathcal{O}_{\mathbb{P}^1_k} \oplus \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^1_k}(-a_i) \). Then, for \( n \) sufficiently large and \( \mathcal{L} \) a degree \( n \) line bundle on \( \mathbb{P}^1_k \), Riemann Roch on the curve \( X \) implies \( h^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n) \oplus \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^1_k}(-a_i + n)) = h^0(\mathbb{P}^1_k, \rho_* \mathcal{O}_X \otimes \mathcal{L}) = h^0(X, \rho^* \mathcal{L}) = dn - g + 1 \). For \( n \) larger than the maximum of the \( a_i \), the left hand is equal to \( dn + d - \sum_{i=1}^{d-1} a_i \), and so we obtain \( -\sum_{i=1}^{d-1} a_i = -g - d + 1 \). Therefore, \( \deg(\det \mathcal{E}) = -\deg(\det \mathcal{E}^\vee) = \sum_{i=1}^{d-1} a_i = g + d - 1 \).

With the relation between \( g \) and \( \mathcal{E} \) of Lemma 6.3 established, we are ready to define the Casnati-Ekedahl strata. For the next definition, we will fix vectors \( \bar{a}^\mathcal{E}, \bar{a}^{\mathcal{F}_1}, \ldots, \bar{a}^{\mathcal{F}_{d-2}} \) and vector bundles \( \mathcal{E}, \mathcal{F}_1, \ldots, \mathcal{F}_{\lfloor \frac{d-2}{2} \rfloor} \) on \( \mathbb{P}^1 \) given by \( \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1_k}(\bar{a}^\mathcal{E}) \) and \( \mathcal{F}_i \cong \mathcal{O}_{\mathbb{P}^1_k}(\bar{a}^{\mathcal{F}_i}) \). Note that although \( d - 2 \) vector bundles appear in Theorem 3.2, the isomorphism classes of vector bundles \( \mathcal{F}_i \) for \( i > \lfloor \frac{d-2}{2} \rfloor \) are in fact determined by those with \( i \leq \lfloor \frac{d-2}{2} \rfloor \) because duality enforces the relation \( \mathcal{F}_{d-2} \cong \det \mathcal{E} \) and for \( 1 \leq i \leq d - 3 \), \( \mathcal{F}_{d-2-i} \cong \det \mathcal{E} \otimes \mathcal{F}_i^\vee \).

Definition 6.4. Let \( k \) be a field with \( d! \) invertible on \( k \), and fix a tuple of vectors \((\bar{a}^\mathcal{E}, \bar{a}^{\mathcal{F}_1}, \ldots, \bar{a}^{\mathcal{F}_{\lfloor \frac{d-2}{2} \rfloor}})\). Let \( g := 1 - d + \sum_{i=1}^{d-1} a_i^\mathcal{E} \) and define the Casnati-Ekedahl strata \( \mathcal{M}(\bar{a}^\mathcal{E}, \bar{a}^{\mathcal{F}_1}, \ldots, \bar{a}^{\mathcal{F}_{\lfloor \frac{d-2}{2} \rfloor}}) \) as the locally closed substack of \( \text{Hur}_{d,g,k} \) given as the fiber product

\[
\text{Hur}_{d,g,k} \times_{\text{Vect}^{d-1 \times \lfloor \frac{d-2}{2} \rfloor}_{\mathbb{P}^1_k}} \text{Vect}^{a^\mathcal{E}}_{\mathbb{P}^1_k} \times_{\prod_{i=1}^{\lfloor \frac{d-2}{2} \rfloor}} \text{Vect}^{a^{\mathcal{F}_i}}_{\mathbb{P}^1_k}.
\]
Here, $\beta_i$ are as in Theorem 3.2 and the map $\text{Hur}_{d,g,k} \to \text{Vect}_{\mathbb{P}^1_k}^{d-1} \times \prod_{i=1}^{\lfloor \frac{d-2}{2} \rfloor} \text{Vect}_{\mathbb{P}^1_k}^{\beta_i}$ is induced by Theorem 3.2. In other words, $\mathcal{M}(\vec{a}^\mathcal{E}, \vec{a}^{\mathcal{F}_1}, \ldots, \vec{a}^{\mathcal{F}_{\lfloor \frac{d-2}{2} \rfloor}})$ is the locally closed substack of the Hurwitz stack such that the associated morphism $T \to \text{Vect}_{\mathbb{P}^1_5}^{d-1} \times \prod_{i=1}^{\lfloor \frac{d-2}{2} \rfloor} \text{Vect}_{\mathbb{P}^1_5}^{\beta_i}$ factors through a map $T \to \text{Vect}_{\mathbb{P}^1_5}^{\mathcal{E}} \times \prod_{i=1}^{\lfloor \frac{d-2}{2} \rfloor} \text{Vect}_{\mathbb{P}^1_5}^{\mathcal{F}_i}$.

**Remark 6.5.** There is a natural generalization of the construction of Casnati-Ekedahl strata of covers of $\mathbb{P}^1$ to a version for covers of genus $g$ curves $C$ in place of the genus 0 curve $\mathbb{P}^1$. Namely, given a finite locally free cover $C' \to C$ over a base $T$, using Theorem 3.2, one can associate a sequence of vector bundles on on the relative curve $C \to T$. The Casnati-Ekedahl stratum would naturally be defined as the loci where these bundles have specific Harder-Narasimhan filtration, generalizing the notion of splitting type.

**Remark 6.6.** Since the substacks $\text{Vect}_{\mathbb{P}^1_5}^{\vec{a}}$ form a stratification of $\text{Vect}_{\mathbb{P}^1_5}^{\mathcal{E}, \mathcal{F}_1, \ldots, \mathcal{F}_{\lfloor \frac{d-2}{2} \rfloor}}$, it follows that the Casnati-Ekedahl strata, varying over all tuples $(\vec{a}^\mathcal{E}, \vec{a}^{\mathcal{F}_1}, \ldots, \vec{a}^{\mathcal{F}_{\lfloor \frac{d-2}{2} \rfloor}})$ form a stratification of $\text{Hur}_{d,g,S}$. This will enable us to write the class of $\text{Hur}_{d,g,S}$ in the Grothendieck ring as the sum of the classes of $\mathcal{M}(\vec{a}^\mathcal{E}, \vec{a}^{\mathcal{F}_1}, \ldots, \vec{a}^{\mathcal{F}_{\lfloor \frac{d-2}{2} \rfloor}})$ for $(\vec{a}^\mathcal{E}, \vec{a}^{\mathcal{F}_1}, \ldots, \vec{a}^{\mathcal{F}_{\lfloor \frac{d-2}{2} \rfloor}})$ varying over all integer tuples of vectors.

To conclude this section, we introduce some notation for objects we will associate with a Casnati-Ekedahl strata of the Hurwitz stack.

**Notation 6.7.** For $3 \leq d \leq 5$, $\mathcal{M}(\vec{a}^\mathcal{E}, \vec{a}^{\mathcal{F}_1}, \ldots, \vec{a}^{\mathcal{F}_{\lfloor \frac{d-2}{2} \rfloor}}) \subset \text{Hur}_{d,g,k}$ a Casnati-Ekedahl strata, for $\mathcal{E} := \bigoplus_j \mathcal{E}(\vec{a}^{\mathcal{E}_i}), \mathcal{F}_i := \bigoplus_j \mathcal{F}(\vec{a}^{\mathcal{F}_i})$, define $\text{Aut}_{\mathcal{M}} := \oplus (\text{Aut}_{\mathcal{P}^1/k}^{\mathcal{P}^1_5})$, as defined in Definition 4.1 depending on the value of $d$. Additionally, for $f : T \to \mathbb{P}^1_k$ denote $\text{Aut}_{f, \mathcal{M}} := \text{Aut}_{f, \mathcal{P}^1_5}^{\mathcal{P}^1_5}(\mathcal{F}, \mathcal{F}_i)$. When the map $f$ is understood, we also use $\text{Aut}_{\mathcal{M}|T}$ as notation for $\text{Aut}_{f, \mathcal{M}}$.

**Remark 6.8.** The construction $\text{Aut}_{\mathcal{M}|T}$ at the end of Notation 6.7 will primarily be used when $T = D$, the dual numbers, mapping to a point of $\mathbb{P}^1_1$. Note that, in this case $\mathcal{E}|D$ and $\mathcal{F}_i|D$ are free vector bundles because all locally free bundles over $D$ are free.

### 7. Presentations of the Casnati-Ekedahl Strata

We next aim to use the parametrizations from $\mathcal{M}(\mathcal{E}, \mathcal{F}_i)$ for $3 \leq d \leq 5$ as the quotient of an open in affine space by an appropriate group action. Because we will also want to parameterize simply branched covers, it will be useful to restrict the possible ramification types of these covers. We now introduce the notion of ramification profile, which describes the possible ramification types of a finite cover of $\mathbb{P}^1$ by a smooth curve.
Definition 7.1 (Ramification Profile). Fix a positive integer $d$ and let $R = (r_1^{i_1}, r_2^{i_2}, \ldots, r_n^{i_n})$ denote a partition of $d$, i.e., a collection of integers with $t_1, \ldots, t_n \geq 1$ so that $\sum_{i=1}^n t_i r_i = d$. Here, we think of $r_i$ as the part sizes appearing in the partition and $t_i$ as the corresponding multiplicity. A ramification profile of degree $d$ is a partition of $d$. For $X \to S$ a scheme, we say $X$ has ramification profile $R$ if for every geometric point $\text{Spec} \ k \in S$, the base change $X_k := X \times_S \text{Spec} \ k$ is isomorphic to $\bigsqcup_{i=1}^n \left( \bigsqcup_{j=1}^{t_i} \text{Spec} \ k[x]/(x^{r_i}) \right)$. We let $r(R) := \sum_{i=1}^n (r_i - 1) t_i$ denote the associated ramification order.

One way to think about ramification profiles as defined above is to think of each fiber $X_k$ of $X \to S$ having a partition into curvilinear schemes (i.e., schemes with cotangent spaces of dimension at most 1 at every point) of degrees determined by the partition $R$.

We next introduce the notion of an allowable collection of ramification profiles. The point of allowable collections are that covers of $\mathbb{P}^1$ whose ramification profiles lie in an allowable collection define an open substack of the Hurwitz stack with closed complement of high codimension. We use the notation $\lambda \vdash n$ to indicate that $\lambda$ is a partition of $n$.

Definition 7.2. Fix an integer $d$. Let $\mathcal{R}$ denote a collection of ramification profiles of degree $d$. We say $\mathcal{R}$ is an allowable collection of ramification profiles of degree $d$ if

1. $\mathcal{R}$ includes $(1^d)$ and $(2, 1^{d-2})$
2. Whenever $\lambda \vdash d$ lies in $\mathcal{R}$, and $\lambda' \vdash d$ is a partition refining $\lambda$, then $\lambda'$ also lies in $\mathcal{R}$.

We next define a certain space of sections of a vector bundle on $\mathbb{P}^1$ parameterizing smooth degree $d$ covers (for $3 \leq d \leq 5$) with specified ramification profiles in an allowable collection. We then make an analogous definition for Hurwitz spaces with restricted ramification in general.

Definition 7.3. For $3 \leq d \leq 5$, fix a choice of Casnati-Ekedahl strata $\mathcal{M} := \mathcal{M}(\mathbb{P}^1, \mathcal{F}_1, \ldots, \mathcal{F}_{\frac{d-2}{2}})$, with associated locally free sheaves on $\mathbb{P}^1_k$ given by $\mathcal{E}_\mathcal{M} := \bigoplus \mathcal{O}(\mathcal{F}_i)$ and, if $4 \leq d \leq 5$, $\mathcal{F}_\mathcal{M} := \bigoplus \mathcal{O}(\mathcal{F}_i)$. Let $g := \deg \det \mathcal{E}_\mathcal{M} - d + 1$. Let $\mathcal{H}_\mathcal{M}$ denote the associated locally free sheaf on $\mathbb{P}^1$ defined in (3.6). Let $\mathcal{R}$ denote an allowable collection of ramification profiles. Then, define $U^{\mathcal{R}}_{\mathcal{M}} \subset H^0(\mathbb{P}^1_k, \mathcal{H}_\mathcal{M})$ to be the open subscheme of $\text{Spec} \text{Sym} H^0(\mathbb{P}^1_k, \mathcal{H}_\mathcal{M})$ parameterizing $T$-points $\eta$ so that the associated closed subscheme of $\mathbb{P}^1_k \times_k T$ defines a smooth curve over $T$ with geometrically connected fibers such that over each geometric point $\text{Spec} \ k \to \mathbb{P}^1_T$, the pullback of $\Phi_d(\eta) \subset \mathbb{P}^1 \to \mathbb{P}^1_T$ along $\text{Spec} \ k \to \mathbb{P}^1_T$ has ramification profile lying in $\mathcal{R}$. Also, define $U^{\mathcal{R},S_d}_{\mathcal{M}} \subset U^{\mathcal{R}}_{\mathcal{M}}$ as the open subscheme parameterizing those sections $\eta$ for which $\Psi_d(\eta)$ is a smooth curve $X$ with geometrically connected fibers, such that over each fiber, the cover $X \to \mathbb{P}^1$ of degree $d$ has Galois closure which is an $S_d$ cover. In other words, $U^{\mathcal{R},S_d}_{\mathcal{M}}$ is the subset $U^{\mathcal{R}}_{\mathcal{M}}$ of for which the map $\Psi_d$ defining a point of $\text{Hur}_{d, g, k}$. 


Example 7.4. If we take $\mathcal{R}$ in Definition 7.3 to range over all possible ramification profiles (i.e., all partitions of $d$) then $U^{\mathcal{R}}_{/M}$ corresponds to all sections $\eta$ as in Definition 7.3 with $\Phi_{d}(\eta)$ a smooth geometrically connected degree $d$ covers of $\mathbb{P}^{1}_{k}$.

On the other hand, if we take $\mathcal{R}$ to be the union of two ramification profiles, the first given by $(1^{d})$ and the second given by $(2, 1^{d-2})$, we obtain all sections $\eta$ with $\Phi_{d}(\eta)$ defining a smooth geometrically connected curve which is simply branched over $\mathbb{P}^{1}_{k}$.

Definition 7.5. For $\mathcal{R}$ an allowable collection of ramification profiles of degree $d$, let $\text{Hur}_{d,g,k}^{\mathcal{R}} \subset \text{Hur}_{d,g,k}$ denote the open substack of $\text{Hur}_{d,g,k}$ whose $T$ points parameterize smooth curves $X \to \mathbb{P}^{1}_{T}$ over $T$ so that for each geometric point $\text{Spec} \kappa \to \mathbb{P}^{1}_{T}$, $X_{\kappa}$ has ramification profile in $\mathcal{R}$. Similarly, for $\mathcal{M} \subset \text{Hur}_{d,g,k}$ a Casnati-Ekedahl strata, let $\mathcal{M}^{\mathcal{R}} \subset \mathcal{M}$ denote the open substack $\mathcal{M} \times_{\text{Hur}_{d,g,k}} \text{Hur}_{d,g,k}^{\mathcal{R}} \subset \mathcal{M}$.

Remark 7.6. In the case $d = 2$, the only allowable $\mathcal{R}$ is $\mathcal{R} = \{(1^{2}), (2)\}$ and in this case $\text{Hur}_{2,g,k}^{\mathcal{R}} = \text{Hur}_{2,g,k}$.

7.7. Writing the class as a sum over Casnati-Ekedahl strata. Our goal for the remainder of the section is to express the class of the Hurwitz stack as a sum over the Casnati-Ekedahl strata, which will be somewhat more manageable due to their descriptions as quotients of opens in affine spaces by relatively simple algebraic groups.

Proposition 7.8. For $3 \leq d \leq 5$, and $\mathcal{R}$ an allowable collection of ramification profiles of degree $d$, we have an equality in $K_{0}(\text{Stacks}_{k})$

$$\{\text{Hur}_{d,g,k}^{\mathcal{R}}\} = \sum_{\text{Casnati-Ekedahl strata } \mathcal{M}} \frac{\{U^{\mathcal{R},S_{d}}_{/\mathcal{M}}\}}{\{\text{Aut}_{\mathcal{M}}\}}.$$ 

Assuming Proposition 7.9 and Lemma 7.10 We claim

$$\{\text{Hur}_{d,g,k}^{\mathcal{R}}\} = \sum_{\text{Casnati-Ekedahl strata } \mathcal{M}} \{\mathcal{M}^{\mathcal{R}}\} = \sum_{\text{Casnati-Ekedahl strata } \mathcal{M}} \left\{\left[\frac{U^{\mathcal{R},S_{d}}_{/\mathcal{M}}}{\text{Aut}_{\mathcal{M}}}\right]\right\} = \sum_{\text{Casnati-Ekedahl strata } \mathcal{M}} \{U^{\mathcal{R},S_{d}}_{/\mathcal{M}}\} \{\text{Aut}_{\mathcal{M}}\}.$$

The first equality holds because the Casnati-Ekedahl strata form a stratification of $\text{Hur}_{d,g,k}$ by locally closed substacks. The second holds by Proposition 7.9. The final equality holds by Lemma 7.10 using both that $\text{Aut}_{\mathcal{M}}$ is special so $\{\text{Aut}_{\mathcal{M}}\} \left\{\frac{U^{\mathcal{R},S_{d}}_{/\mathcal{M}}}{\text{Aut}_{\mathcal{M}}}\right\} = \{U^{\mathcal{R},S_{d}}_{/\mathcal{M}}\}$ by [Eke09a, Prop. 1.4(i)], and that $\text{Aut}_{\mathcal{M}}$ is invertible. □

To conclude our proof of Proposition 7.8, we need to verify Proposition 7.9 and Lemma 7.10. We omit the proof of Proposition 7.9 since it is analogous to Proposition 4.7.
where we additionally fix isomorphisms to fixed bundles $E, F$ on $\mathbb{P}^1_k$ (as opposed to trivial bundles on $\text{Spec} \mathbb{Z}$) and add in conditions associated to the ramification profiles in $R$ and lying in $\text{Hut}_{d, s}$ appropriately.

**Proposition 7.9.** For $3 \leq d \leq 5$, fix a choice of Casnati-Ekedahl strata $\mathcal{M} := \mathcal{M}(\tilde{\alpha}^E, \tilde{\alpha}^F, \ldots, \tilde{\alpha}^{\mathcal{F}_{d-2}})$ with associated sheaves $\mathcal{E}, \mathcal{F}$ and, if $4 \leq d \leq 5$, $\mathcal{F}$ as in [Definition 7.3](#). There are isomorphisms $[U_{\mathcal{M}}^R / \text{Aut}_{\mathcal{M}}] \simeq \mathcal{M}_R$, and $[U_{\mathcal{M}}^{R, S_d} / \text{Aut}_{\mathcal{M}}] \simeq \mathcal{M}_R^{R, S_d}$.

We now verify the relevant automorphism groups are special. Because later we will have to deal with an analogous constructions over the dual numbers $D$, we include that setting in the following lemma as well.

**Lemma 7.10.** For $\mathcal{V}$ any vector bundle on $Y$, for $Y = \mathbb{P}^1_k$ or $Y = D$, $\text{Res}_{Y/k}(\text{Aut}_Y)$ and $\text{Res}_{Y/k}(\ker(\det : \text{Aut}_\mathcal{V} \to \mathbb{G}_m))$ are special and their classes are invertible in $K_0(\text{Stacks}_k)$.

When $Y = \mathbb{P}^1_k$ or $Y = D$, the three group schemes appearing in Equation 4.1 in the cases $d = 3, 4, 5$ are special. Further, the classes of these groups are invertible in $K_0(\text{Stacks}_k)$.

**Proof.** We only explicate the proof in the case $Y = \mathbb{P}^1_k$, since the proof when $Y = D$ is analogous but simpler (noting that all vector bundles are trivial over $D$).

First we show that for any vector bundle $\mathcal{G}$ on $\mathbb{P}^1_k$, $\text{Res}_{\mathbb{P}^1_k/k}(\text{Aut}_{\mathcal{G}})$ is special. The reason for this is that when we write $\mathcal{G} = \bigoplus_i \mathcal{O}_\mathbb{P}^1_k(a_i)^{n_i}$ we can express $\text{Res}_{\mathbb{P}^1_k/k}(\text{Aut}_{\mathcal{G}}) \simeq \prod_i \text{GL}_{n_i} \ltimes \prod_{i<j} V_{ij}$ where $V_{ij}$ is the vector group $V_{ij} = \text{Res}_{\mathbb{P}^1_k/k}(\text{Hom}(\mathcal{O}_\mathbb{P}^1_k(a_i)^{n_i}, \mathcal{O}_\mathbb{P}^1_k(a_j)^{n_j})) \simeq \mathbb{G}_a^{(a_j-a_i+1)n_i n_j}$. It will also be useful to note that $\ker(\det) : \text{Res}_{\mathbb{P}^1_k/k}(\text{Aut}_{\mathcal{G}}) \to \text{Res}_{Y/k}(\mathbb{G}_m)$ is special, since it can be expressed as an extension of a power of $\mathbb{G}_m$ by $\prod_i \text{SL}_{n_i} \ltimes \prod_{i<j} V_{ij}$, both of which are special. These statements imply the first part of the lemma.

We now check the groups $\text{Aut}_{\mathcal{E}, \mathcal{F}}^{Y/k}$ are special when $d = 3, 4, 5$. The above observations immediately implies the claim when $d = 3$. To deal with the cases $d = 4$ and $d = 5$, we use Lemma 4.3. In both cases, the composition coming from $\text{Aut}_{\mathcal{E}, \mathcal{F}}^{Y/k} \to \text{Res}_{Y/k}(\text{Aut}_{\mathcal{E}, \mathcal{F}}) \times \text{Res}_{Y/k}(\text{Aut}_{\mathcal{F}}) \to \text{Res}_{Y/k}(\text{Aut}_{\mathcal{E}, \mathcal{F}})$ is surjective. From the description in Lemma 4.3, the kernel of this composition is identified with $\ker(\det) : \text{Res}_{Y/k}(\text{Aut}_{\mathcal{F}}) \to \mathbb{G}_m$. As mentioned above, this is special, and so $\text{Aut}_{\mathcal{E}, \mathcal{F}}^{Y/k}$ is an extension of special group schemes, hence special.

By the above explicit description of $\text{Aut}_{\mathcal{E}, \mathcal{F}}^{Y/k}$ in terms of classes of special linear groups, general linear groups, and vector groups, we conclude that $\text{Aut}_{\mathcal{E}, \mathcal{F}}^{Y/k}$ has class which is a product of powers of $\mathbb{L}$, and expressions of the form $\mathbb{L}^s - 1$ for varying $s$. Therefore, $\text{Aut}_{\mathcal{E}, \mathcal{F}}^{Y/k}$ is invertible in $K_0(\text{Stacks}_k)$.

8. Computing the Local Classes

The goal of this section is to compute the classes of sections over the dual numbers in [Theorem 8.9](#). These classes can be thought of as describing the “probability” that a curve is smooth at a point and has a certain ramification profile. We will then
use these classes to sieve for smoothness and ramification conditions by employing the work of Bilu and Howe [BH21] in Proposition 9.10. The condition of smoothness can be rephrased as a local condition over an infinitesimal neighborhood of the point in \( \mathbb{P}^1 \). We will first prove Theorem 8.3 which computes this “probability” for abstract covers, and from this deduce Theorem 8.9 which computes this “probability” for sections of \( \mathcal{H}(\mathcal{E}, \mathcal{P}). \) Theorem 8.3 can be thought of as a motivic analog of Bhargava’s mass formulas for counting local fields [Bha10b], though we note that the interesting part of [Bha10b] is when there is wild ramification, and our hypothesis eliminates that possibility. On the other hand, it is still interesting to upgrade even the (much easier) tame mass formula to a motivic statement.

The idea for computing these local classes seems one of the main new insights of this paper. In the arithmetic analogs of this work, one is able to directly count the number of sections over \( \mathbb{Z}/p^2\mathbb{Z} \), see [BST13, Lem. 18] for the degree 3 case, [Bha04, Lem. 23] for the degree 4 case, and [Bha08, Lem. 20] for the degree 5 case. In the Grothendieck ring, when working over infinite fields, there are infinitely many sections, and so to determine the relevant class, direct counting is no longer possible.

We relate computing the classes of these sections to computing the classes of the classifying stacks of abstract automorphism groups of the corresponding schemes. These classes can in turn be computed using stacky symmetric powers \( \text{Symm}^n \) (see Definition 8.15) and the class of \( BS_n \). An observation which is the key to the proof of Theorem 8.9 is that for \( G \) a group scheme, we have an isomorphism of stacks \( \text{Symm}^n(BG) \simeq B(G \wr S_n) \).

Throughout this section, we fix \( d \in \mathbb{Z}_{\geq 1} \) an integer and let \( k \) be a field with \( \text{char}(k) \nmid d! \). For later explicit calculations, it will be convenient to work with the following explicit scheme \( X_R \) over \( D \) which has ramification profile \( R \) over the closed point of \( D \) (so that \( (d_1, \ldots, d_m) \) is a partition of \( d \) agreeing with \( R \)). Define

\[
X_R := \coprod_{i=1}^m \text{Spec } k[x_i, \varepsilon] / (x^{d_i} - \varepsilon, \varepsilon^2).
\]

We hope the reader will be able to distinguish the meaning of \( X_R \) from the base change of \( X \) to \( R \) by context.

**Definition 8.1.** We let \( \mathcal{X}_{R,d} \subset \text{Res}_{D/k}(\text{Covers}_d) \) denote the residual gerbe at the \( k \)-point of \( \text{Res}_{D/k}(\text{Covers}_d) \) corresponding to the \( D \)-point of \( \text{Covers}_d \) given by \( X_D \).

**Remark 8.2.** Since we have an induced monomorphism \( B(\text{Res}_{D/k}(\text{Aut}_{X_{R,D}})) \rightarrow \text{Covers}_d \) and an epimorphism \( \text{Spec } k \rightarrow B(\text{Res}_{D/k}(\text{Aut}_{X_{R,D}})) \), it follows that \( \mathcal{X}_{R,d} \) is equivalent to \( B(\text{Res}_{D/k}(\text{Aut}_{X_{R,D}})) \) from the universal property for residual gerbes.

Our main result of this section is to compute the class of \( \mathcal{X}_{R,d} \) in \( K_0(\text{Stacks}_k) \), and we complete the proof at the end of the section in §8.17.

**Theorem 8.3.** Let \( R \) be a ramification profile which is a partition of \( d \). Let \( r(R) \) be the ramification order associated to the ramification profile \( R \), as defined in Definition 7.1. Then, for \( k \) a field with \( \text{char}(k) \nmid d! \), we have

\[
\{ \mathcal{X}_{R,d} \} = L^{-r(R)}
\]
in $K_0(\text{Stacks}_k)$.

The plan for the rest of the section is to first use Theorem 8.3 to deduce the local condition for a section of $\mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)$ to be smooth in Theorem 8.9. Following this, we devote the remainder of the section to proving Theorem 8.3. The main idea is to directly compute the automorphism group of $X_R$ in terms of its combinatorial data starting in §8.10 and culminating in Corollary 8.12. Using this, we will then be able to compute the class of the classifying stack of the resulting affine (but typically quite disconnected) group scheme in §8.13. For this, we appeal to a result of Ekedahl on stacky symmetric powers and another result of Ekedahl showing $\{BS_d\} = 1$. We complete the proof of Theorem 8.3 in §8.17.

Remark 8.4. With some additional work, one can also prove a variant of Theorem 8.3 which computes the class of the locally closed subscheme $\mathcal{Z}$ of the Hilbert scheme $\text{Res}_{D/k}(\text{Hilb}^d_{P_{d-2}/D})$ parameterizing curvilinear nondegenerate subschemes with ramification profile $\mathcal{R}$ so that on any geometric fiber, no degree $d-1$ subscheme is contained in a hyperplane. One can show, $\{\mathcal{Z}_{R,d}\} = \{\text{PGL}_{d-1}\} \mathcal{L}_{\text{dim PGL}_{d-1} - r(R)}$. Note there is some subtlety in verifying this because this Hilbert scheme is naturally a Res$_{D/k}(\text{PGL}_{d-1})$ torsor over Covers$_d$, and $\text{PGL}_{d-1}$ is not a special group. Nevertheless, one may prove this “linearizing the action” so as to construct this as a quotient of a $\text{Res}_{D/k}(\text{GL}_{d-1})$ torsor by $\text{Res}_{D/k}(\text{G}_m)$, both of which are special.

8.5. Using Theorem 8.3 to compute smooth sections. Before proving Theorem 8.3, we will see how it can be used to determine local conditions for a section in a given Casnati-Ekedahl stratum to be smooth. In order to apply Theorem 8.3 to our problem of computing the classes of Hurwitz stacks we want to relate it to sections of the sheaf $\mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)$ on $D$ (for $\mathcal{E}$ and $\mathcal{F}_\bullet$ trivial sheaves on $D$ of appropriate ranks as in Notation 3.8, depending on $d$ with $3 \leq d \leq 5$). For this we need a generalization of Proposition 4.7 where we take a Weil restriction from the dual numbers. More precisely, for $3 \leq d \leq 5$, the map $\mu_d : U_d \to \text{Covers}_d$ defined in Definition 4.6 induces a map $\text{Res}_{D/k}(\mu_d) : \text{Res}_{D/k}(U_d)_D \to \text{Res}_{D/k}(\text{Covers}_d)_D$. Since $\mu_d$ is invariant for the action of $\text{Aut}_{\mathcal{E}, \mathcal{F}_\bullet}$ as in Definition 4.6, we obtain a map $\phi_{d/k}^{D/k} : [\text{Res}_{D/k}(U_d)_D] / \text{Res}_{D/k}(\text{Aut}_{\mathcal{E}|D, \mathcal{F}_\bullet|D}) \to \text{Res}_{D/k}(\text{Covers}_d)_D$ induced by sending a section to its vanishing locus.

Lemma 8.6. For $3 \leq d \leq 5$, the map $\phi_{d/k}^{D/k} : [\text{Res}_{D/k}(U_d)_D] / \text{Res}_{D/k}(\text{Aut}_{\mathcal{E}|D, \mathcal{F}_\bullet|D}) \to \text{Res}_{D/k}(\text{Covers}_d)_D$ is an isomorphism.

This is proven via a nearly identical argument to Proposition 4.7 and we omit the proof. The one minor difference one must note is that, in order to show $\phi_{d/k}^{D/k}$ is surjective, for any $T \to \text{Spec}k$ and any vector bundle on $T \times_k D$, one may replace $T$ by an open cover which trivializes the bundle.

Definition 8.7. Let $\mathcal{Y}_{R,d} \subset \text{Res}_{D/k}(U_d)_D$ denote $(\phi_{d/k}^{D/k})^{-1}(\mathcal{Z}_{R,d})$.

Remark 8.8. We will implicitly use the following geometric description of the residual gerbe $\mathcal{Z}_{R,d}$ and its preimage $\mathcal{Y}_{R,d}$ in $\text{Res}_{D/k}(U_d)_D$. As a fibered category, $\mathcal{Z}_{R,d}$
has $T$ points given by finite locally free degree $d$ Gorenstein covers of $T \times_k D$ satisfying the following properties

1. $Z$ has ramification profile $R$ over each geometric point Spec $\kappa \to T_D$.
2. $Z$ is curvilinear in the sense that for each geometric point Spec $\kappa \to T$, the resulting scheme $Z \times_T \text{Spec} \kappa$ has 1-dimensional Zariski tangent space at each point.

Similarly, when $3 \leq d \leq 5$, we can describe $\mathcal{Y}_{R,d}$ as those sections $\eta \in \text{Res}_{D/k}(\mathcal{U}_{d})_D(T)$ for which the associated degree $d$ cover of $T \times_k D$, $\Psi_d(\eta)$ (as defined in §3.11) has the above properties. We note that $\mathcal{Y}_{R,d}$ is a locally closed subscheme of $\text{Res}_{D/k}(\mathcal{U}_{d})_D$ since the same holds for the residual gerbe $\mathcal{X}_{R,d}$ in $\text{Res}_{D/k}(\text{Covers}_d)$ [Ryd11, Thm. B.2]. One may also deduce this is locally closed directly from the above functorial description.

By combining [Theorem 8.3] with [Lemma 8.6] we can easily deduce the following.

**Theorem 8.9.** Let $R$ be a ramification profile which is a partition of $d$ and let $\mathcal{Y}_{R,d}$ be the scheme defined in [Definition 8.7] (with associated free sheaves $\mathcal{E}$, $\mathcal{F}_{\bullet}$ on $D$). Let $r(R)$ denote the ramification order associated to the ramification profile $R$, as defined in [Definition 7.1]. Then,

$$\{\mathcal{Y}_{R,d}\} = \{\text{Aut}_{\mathcal{E},\mathcal{F}_{\bullet}}^{D/k}\} L^{-r(R)}.$$

Let us now deduce this from [Theorem 8.3] and subsequently return to prove [Theorem 8.3].

**Proof of Theorem 8.9 assuming Theorem 8.3** Using [Lemma 8.6]

$$\{\mathcal{Y}_{R,d}/\text{Aut}_{\mathcal{E},\mathcal{F}_{\bullet}}^{D/k}\} = B(\text{Res}_{D/k}(\text{Aut}_{X_R/D})).$$

Since $\text{Aut}_{\mathcal{E},\mathcal{F}_{\bullet}}^{D/k}$ is special and has invertible class in $K_0(\text{Stacks}_k)$ by [Lemma 7.10], [Eke09a, Prop. 1.4(i), Prop. 1.1(ix)] implies that

$$\{\mathcal{Y}_{R,d}/\text{Aut}_{\mathcal{E},\mathcal{F}_{\bullet}}^{D/k}\} = \frac{\{\mathcal{Y}_{R,d}\}}{\{\text{Aut}_{\mathcal{E},\mathcal{F}_{\bullet}}^{D/k}\}}.$$

Then, by [Theorem 8.3]

$$\{\mathcal{Y}_{R,d}\} = \{\text{Aut}_{\mathcal{E},\mathcal{F}_{\bullet}}^{D/k}\} \cdot \{B(\text{Res}_{D/k}(\text{Aut}_{X_R/D}))\} = \{\text{Aut}_{\mathcal{E},\mathcal{F}_{\bullet}}^{D/k}\} \cdot L^{-r(R)}. \quad \square$$

**8.10. Computing the algebraic group** $\text{Res}_{D/k}(\text{Aut}_{X_R/D})$. Let $X_R$ denote the scheme as defined in [Equation 8.1]. Our next goal is to compute the group scheme $\text{Res}_{D/k}(\text{Aut}_{X_R/D})$, which we will carry out in [Corollary 8.12]. In order to do so, we first deal with the case that $X_R$ is connected.

**Lemma 8.11.** Let $d \in \mathbb{Z}_{\geq 1}$, and let $k$ be a field with char($k$) $\nmid d!$. Let $W := \text{Spec} k[y, \varepsilon]/(\varepsilon^2, \varepsilon - y^d)$. For $\text{Aut}_{W/D}$ the automorphism scheme of $W$ over $D$, we have $\text{Res}_{D/k}(\text{Aut}_{W/D}) \simeq \mu_d \times \mathbb{G}_m^{d-1}$, explicitly given by $a \in \mu_d$ sending $y \mapsto ay$ and $(a_1, \ldots, a_{d-1}) \in \mathbb{G}_m^{d-1}$ sending $y \mapsto y + \sum_{i=1}^{d} a_i y^{d+i}$.
Proof. For $T$ a $k$ algebra, a functorial $T$ point of $\text{Res}_{D/k}(\text{Aut}_{W/D})$ corresponds (upon taking global sections) to an isomorphism of $T$ algebras

$$\phi : T[y, \varepsilon]/(\varepsilon^2, \varepsilon - y^d) \simeq T[y, \varepsilon]/(\varepsilon^2, \varepsilon - y^d).$$

over $T[\varepsilon]/(\varepsilon^2)$. Such an automorphism is uniquely determined by where it sends $y$. To conclude the proof, it suffices to verify that any such $\phi$ is of the form $y \mapsto ay + \sum_{i=1}^{d-1} a_i y^{d+i}$ for $a \in \mu_d(T)$ and $a_i \in G_a(T)$, and conversely that any map of this form determines an automorphism.

Let $\phi_{a,a_1,\ldots,a_{d-1}}$ denote the map of $T$ algebras sending $y \mapsto ay + \sum_{i=1}^{d-1} a_i y^{d+i}$ as above. Under the isomorphism $T[y, \varepsilon]/(\varepsilon^2, \varepsilon - y^d) \simeq T[y]/y^{2d}$, any automorphism $\phi$ must induce an isomorphism on cotangent spaces, and hence send $y$ to some polynomial $p_\phi(y) = b_1 y + b_2 y^2 + \cdots + b_{2d-1} y^{2d-1}$, with $b_1 \neq 0$ and $b_i \in T$. The condition that $\phi$ determines a map $T[\varepsilon]/(\varepsilon^2)$ algebras precisely corresponds to $y^d = p_\phi(y)^d$. Comparing the coefficients of $y^d$ in this equation implies $b_1 \in \mu_d(T)$. Since $\text{char}(k) \nmid d!$, comparing the coefficients of $y^{d+1}, \ldots, y^{2d-1}$ in the equation $y^d = p_\phi(y)^d$ implies $b_2 = b_3 = \cdots = b_d = 0$. However, the coefficients $b_{d+1}, \ldots, b_{2d-1}$ can be arbitrary and $y^d = p_\phi(y)^d$ will be satisfied. So, any automorphism $\phi$ must be of the form $\phi_{a,a_1,\ldots,a_{d-1}}$ (where we take $a_i = b_{d+i}$ in the above notation).

To see any map $\phi_{a,a_1,\ldots,a_{d-1}}$ determines an automorphism of $T$ algebras, note first that it is well defined, because $(ay + \sum_{i=1}^{d-1} a_i y^{d+i})^d = y^d$, using that $y^{2d} = 0$. It is an automorphism as its inverse is explicitly given by $\phi_{-a^{-1},-a_1^{-1},\ldots,-a_{d-1}^{-1}}$. \hfill $\square$

Corollary 8.12. Choose a partition $(r_1, \ldots, r_n)$ of $d$, i.e., $d = \sum_{i=1}^n t_i \cdot r_i$. Let $W_i := \text{Spec} \prod_{j=1}^{t_i} k[y, \varepsilon]/(\varepsilon^2, \varepsilon - y^{r_i})$. Let $W := \prod_{i=1}^n W_i$, so that $W \simeq X_R$ when $R$ is the ramification profile associated to the above partition. We have an isomorphism $\text{Res}_{D/k}(\text{Aut}_{W/D}) \simeq \prod_{i=1}^n \left(G_a^{r_i-1} \rtimes \mu_{r_i}\right) \backslash S_{t_i}$, where each $G_a^{r_i-1} \rtimes \mu_{r_i}$ is explicitly realized acting on each component of $W_i$ as in Lemma 8.11 and the action of the wreath product with $S_{t_i}$ is obtained by permuting the $t_i$ components of $W_i$.

Proof. To compute the automorphism group of $W$, first observe that any automorphism must permute all connected components of a fixed degree, and therefore $\text{Aut}_{W/D} = \prod_{i=1}^n \text{Aut}_{W_i/D}$, and consequently

$$\text{Res}_{D/k}(\text{Aut}_{W/D}) = \text{Res}_{D/k} \left(\prod_{i=1}^n \text{Aut}_{W_i/D}\right) = \prod_{i=1}^n \text{Res}_{D/k}(\text{Aut}_{W_i/D}).$$

It therefore suffices to show $\text{Res}_{D/k}(\text{Aut}_{W_i/D}) \simeq \left(G_a^{r_i-1} \rtimes \mu_{r_i}\right) \backslash S_{t_i}$. As all connected components of $W_i$ are isomorphic, any automorphism is realized as the composition of an automorphism preserving each connected component, followed by some permutation of the connected components. Since there are $t_i$ connected components, the group of permutations of the components is the symmetric group $S_{t_i}$, while for $Z_i$ a connected component of $W_i$, we established $\text{Res}_{D/k} \text{Aut}_{Z_i/D} \simeq G_a^{r_i-1} \rtimes \mu_{r_i}$.
where the semidirect product $G_x \rtimes G_y$ is a well-defined operation on the Grothendieck ring of stacks by \[Eke09b,\ Prop. 2.5\].

Prop. 2.5]

8.15. **Computing** \{B \text{Res}_{D/k}(\text{Aut}_{X_k})\}. Our next goal is to prove [Theorem 8.3] by computing the class of \{B \text{Res}_{D/k}(\text{Aut}_{X_k})\} in $K_0(\text{Stacks}_k)$, which we carry out at the end of this section in \S 8.17. Of course, we will use our computation of \text{Res}_{D/k}(\text{Aut}_{X_k}) from Corollary 8.12. In order to set up our computation we need the following lemma.

**Lemma 8.14.** For $x$ and $y$ positive integers, \{B \{G^x \times \mu_y\}\} = \{B(G^x)\}$, where $\mu_y$ acts on $G^x_y$ by the scaling action $(\alpha, (a_1, \ldots, a_x)) \mapsto (\alpha a_1, \ldots, \alpha a_x)$.

**Proof.** Indeed, we have an inclusion

\[
(G^x_y \times \mu_y) \hookrightarrow (G^x \rtimes G_y)
\]

where the semidirect product $G^x_y \rtimes G_y$ is defined similarly to that in Lemma 8.11 so that $G_y$ acts on $G^x$ by

\[
G_y \times G^x \to G^x \quad (\alpha, (a_1, \ldots, a_x)) \mapsto (\alpha a_1, \ldots, \alpha a_x).
\]

The natural inclusion $\mu_y \to G_y$ then respects the constructed group structures. For simplicity of notation, temporarily define $K := G^x_y \rtimes \mu_y$ and $L := G^x \rtimes G_y$.

Since $L$ is special, and special groups are closed under extensions, it follows from \[Eke09a,\ Prop. 1.1(ix)\] that \{BK\} = \{L/K\}{BL}. However, since $G^x_y$ is a normal subgroup of both $L$ and $K$, the quotient $L/K$ is identified with

\[
L/K \simeq \frac{L/G^x_y}{K/G^x_y} \simeq G_y/\mu_y \simeq G_y.
\]

Since $L$ is special, using \[Eke09a,\ Prop. 1.4(i)] and \[Eke09a,\ Prop. 1.1(v)] we obtain that \{BL\} = $1/L^{-x}1$. Therefore,

\[
\{BK\} = \{L/K\}{BL} = (L - 1)L^{-x}1 = L^{-x} = \{B(G^x_y)\},
\]

using again that $L = G^x_y$ is special. \hfill \(\Box\)

Using Lemma 8.14, we next compute the class of $B \text{Res}_{D/k}(\text{Aut}_{X_k})$ in the case that the partition $R$ has a single part. To continue our computation, we need the notion of stacky symmetric powers:

**Definition 8.15.** For $\mathcal{X}$ a stack, define the stacky symmetric power $\text{Symm}^n \mathcal{X} := [\mathcal{X}^n/S_n]$ (where $[\mathcal{X}^n/S_n]$ denotes the stack quotient for $S_n$ acting on $\mathcal{X}^n$ by permuting the factors).

The key input to our next computation will be that taking the stacky symmetric powers is a well-defined operation on the Grothendieck ring of stacks by [Eke09b, Prop. 2.5].
Lemma 8.16. For integers s and t, 

\[ B \left( \left( G^{s-1}_a \rtimes \mu_s \right) \wr S_t \right) = L^{(s-1)t} \in K_0(\text{Stacks}_k). \]

Proof. By definition, 

\[ \left\{ B \left( \left( G^{s-1}_a \rtimes \mu_s \right) \wr S_t \right) \right\} = \left\{ B \left( G^{s-1}_a \rtimes \mu_s \right) \wr B(S_t) \right\} = \left\{ \text{Symm}^t \left( B \left( G^{s-1}_a \rtimes \mu_s \right) \right) \right\}. \]

Having computed \( \left\{ B \left( G^{s-1}_a \rtimes \mu_s \right) \right\} = \mathbb{L}^{-s} \) in Lemma 8.14, we therefore wish to next compute \( \left\{ \text{Symm}^t \left( B \left( G^{s-1}_a \rtimes \mu_s \right) \right) \right\}. \)

Since \( \left\{ \text{Symm}^n X \right\} \) only depends on \( \left\{ X \right\} \) by [Eke09b, Prop. 2.5], and we have shown \( \left\{ B \left( G^{s-1}_a \rtimes \mu_s \right) \right\} = \left\{ B(G^{s-1}_a) \right\} \) in Lemma 8.14, it follows that

\[ \left\{ \text{Symm}^t \left( B \left( G^{s-1}_a \rtimes \mu_s \right) \right) \right\} = \left\{ \text{Symm}^t \left( B(G^{s-1}_a) \right) \right\}. \]

Next, by [Eke09b, Lem. 2.4], we have

\[ \left\{ \text{Symm}^t \left( A^{s-1} \times BG^{s-1}_a \right) \right\} = \left\{ \text{Symm}^t \left( BG^{s-1}_a \right) \right\} \times A^{(s-1) t} \]

\[ = \left\{ \text{Symm}^t \left( BG^{s-1}_a \right) \right\} \cdot \mathbb{L}^{(s-1)t}. \]

However, since \( \left\{ A^{s-1} \times BG^{s-1}_a \right\} = 1 \), and \( \left\{ \text{Symm}^n \mathcal{X} \right\} \) only depends on \( \left\{ \mathcal{X} \right\} \) by [Eke09b, Prop. 2.5], we obtain

\[ \left\{ \text{Symm}^t \left( BG^{s-1}_a \right) \right\} = \left\{ \text{Symm}^t \left( A^{s-1} \times BG^{s-1}_a \right) \right\} \mathbb{L}^{-(s-1)t} \]

\[ = \left\{ \text{Symm}^t \left( 1 \right) \right\} \mathbb{L}^{-(s-1)t} \]

\[ = \left\{ BS_t \right\} \mathbb{L}^{-(s-1)t} \]

\[ = \mathbb{L}^{-(s-1)t}. \]

For the last step, we used Theorem A.1, which says that \( \left\{ BS_t \right\} = 1. \)

8.17. Completing the calculation of the local class. We now complete the proof of Theorem 8.3.

\textbf{Proof of Theorem 8.3} By Corollary 8.12 we equate

\[ \text{Res}_{D/k}(\text{Aut}_{X_R/D}) = \prod_{i=1}^{n} \left( G^{r_i-1}_a \rtimes \mu_{r_i} \right) \wr S_{t_i}. \]

Factoring this as a product, it suffices to compute the class of \( B \left( G^{r_i-1}_a \rtimes \mu_{r_i} \right) \wr S_{t_i}. \)

Using that \( \sum_{i=1}^{n} (r_i - 1)t_i = r(R) \), the result follows from Lemma 8.16. \qed

9. Codimension bounds for the main result

In this section, we establish various bounds on the codimension or certain bad loci we will want to weed out when computing the class of Hurwitz stacks in the Grothendieck ring.
9.1. **Weeding out the strata of unexpected codimension.** In order to compute the classes of Hurwitz stacks, we will stratify the Hurwitz stacks by Casnati-Ekedahl strata. The following lemma computes the codimension of these loci in the Hurwitz stack.

**Lemma 9.2.** Fix some $d$, resolution data $(\mathcal{E}, \mathcal{F}_*)$, and define $g$ by $\deg \det \mathcal{E} = g + d - 1$. Letting $\mathcal{H} := \mathcal{H}(\mathcal{E}, \mathcal{F}_*)$, the codimension of $\mathcal{M} := \mathcal{M}(\mathcal{E}, \mathcal{F}_*)$ in $\text{Hur}_{d,g,k}$, assuming it is nonempty, is

$$
\begin{cases}
  h^1(\mathbb{P}_k^1, \text{End} \mathcal{E}) & \text{if } d = 3 \\
  h^1(\mathbb{P}_k^1, \text{End} \mathcal{E}) + h^1(\mathbb{P}_k^1, \text{End} \mathcal{F}) - h^1(\mathbb{P}_k^1, \mathcal{H}) & \text{if } d = 4 \text{ or } 5
\end{cases}
$$

**Proof.** Let $\mathcal{M}^\circ := \mathcal{M}(\mathcal{E}^\circ, \mathcal{F}^\circ_*)$ denote the dense open stratum, corresponding to invertible sheaves $\mathcal{E}^\circ$ and $\mathcal{F}^\circ_*$ which are balanced (i.e., the degrees of all line bundle summands differ by at most 1), subject to the conditions that $\det \mathcal{F}^\circ \simeq \det \mathcal{E}^\circ$ when $d = 4$ and $\det \mathcal{F}^\circ \simeq (\det \mathcal{E}^\circ)^{\otimes 2}$ when $d = 5$, coming from Theorem 3.14 and Theorem 3.16. Using the description of $\mathcal{M}$ from Proposition 7.9 as a quotient of an open in the affine space associated to $H^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_*))$ by $\text{Aut}_\mathbb{A}$, it follows that the codimension of $\mathcal{M}$ in $\text{Hur}_{d,g,k}$ is $\dim \mathcal{M}^\circ - \dim \mathcal{M}$. Since $\dim \mathcal{M} = h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_*)) - \dim \text{Aut}_\mathbb{A}$, we are looking to compute

$$
(9.1) \quad (h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}^\circ, \mathcal{F}^\circ_*)) - \dim \text{Aut}_{\mathcal{M}^\circ} - (h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_*)) - \dim \text{Aut}_\mathbb{A}) - (h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_*)) - h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}^\circ, \mathcal{F}^\circ_*))) + (\dim \text{Aut}_\mathbb{A} - \dim \text{Aut}_{\mathcal{M}^\circ}).
$$

We will first identify $\dim \text{Aut}_\mathbb{A} - \dim \text{Aut}_{\mathcal{M}^\circ}$ with $h^1(\mathbb{P}_k^1, \text{End} \mathcal{E})$ when $d = 3$ and $h^1(\mathbb{P}_k^1, \text{End} \mathcal{E}) + h^1(\mathbb{P}_k^1, \text{End} \mathcal{F})$ when $d = 4$ or $d = 5$. Second, we will show $(h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}^\circ, \mathcal{F}^\circ_*)) - h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_*))$ vanishes when $d = 3$ and agrees with $h^1(\mathbb{P}_k^1, \mathcal{H})$ if $d$ is 4 or 5. Combining these with (9.1) will complete the proof.

To identify $\dim \text{Aut}_\mathbb{A} - \dim \text{Aut}_{\mathcal{M}^\circ}$, we may identify $\dim \text{Aut}_\mathbb{A}$ with the dimension of the tangent space to $\text{Aut}_\mathbb{A}$ at the identity, which is given by $H^0(\mathbb{P}_k^1, \text{End} \mathcal{E}) \times H^0(\mathbb{P}_k^1, \text{End} \mathcal{F}_*)$. It is then enough to show that

$$
h^0(\mathbb{P}_k^1, \text{End} \mathcal{E}^\circ) - h^0(\mathbb{P}_k^1, \text{End} \mathcal{E}) = h^1(\mathbb{P}_k^1, \text{End} \mathcal{E})
$$

and, when $d = 4$ or 5,

$$
h^0(\mathbb{P}_k^1, \text{End} \mathcal{F}^\circ) - h^0(\mathbb{P}_k^1, \text{End} \mathcal{F}) = h^1(\mathbb{P}_k^1, \text{End} \mathcal{F})
$$

We focus on the case of $\mathcal{E}$, as the case of $\mathcal{F}$ is completely analogous. By Riemann Roch, since the degrees and ranks of $\mathcal{E}$ and $\mathcal{E}^\circ$ are the same, we find

$$
h^0(\mathbb{P}_k^1, \text{End} \mathcal{E}^\circ) - h^0(\mathbb{P}_k^1, \text{End} \mathcal{E}) = h^1(\mathbb{P}_k^1, \text{End} \mathcal{E}^\circ) - h^1(\mathbb{P}_k^1, \text{End} \mathcal{E}) = 0.
$$

Because $\mathcal{E}^\circ$ is balanced, we find $h^1(\mathbb{P}_k^1, \text{End} \mathcal{E}^\circ) = 0$.

To complete the proof, it only remains to show $(h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}^\circ, \mathcal{F}^\circ_*)) - h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_*)))$ vanishes when $d = 3$ and agrees with $h^1(\mathbb{P}_k^1, \mathcal{H})$ if $d$ is 4 or 5. Similarly to our computation above for $h^0(\mathbb{P}_k^1, \text{End} \mathcal{E}^\circ) - h^0(\mathbb{P}_k^1, \text{End} \mathcal{E})$ we can see that $h^1(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}^\circ, \mathcal{F}^\circ_*)) =$
We will only have
\[ h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}^0, \mathcal{F}_\bullet)) - h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) = h^1(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}^0, \mathcal{F}_\bullet)) - h^1(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) \]
by Riemann Roch. Then, for \( d \) equal to either 4 or 5, using that \( h^1(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}^0, \mathcal{F}_\bullet)) = 0 \), by the relation between the determinants of \( \mathcal{F} \) and \( \mathcal{E} \), we find, \( h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) - h^0(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) = h^1(\mathbb{P}_k^1, \mathcal{H}) \). Indeed, in the cases \( d = 4 \) or 5, by writing out \( \mathcal{E} \) and \( \mathcal{F} \) as sums of line bundles on \( \mathbb{P}_k^1 \), we can see that when \( \mathcal{E} \) and \( \mathcal{F} \) are balanced, \( h^1(\mathbb{P}_k^1, \mathcal{H}) = 0 \).

In the case \( d = 3 \), it is no longer true that \( h^1(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}^0, \mathcal{F}_\bullet)) = h^1(\mathbb{P}_k^1, \mathcal{H}) \). However, in this case, we want to show \( h^1(\mathbb{P}_k^1, \text{Sym}^3 \mathcal{E} \otimes \mathcal{E}^\vee) \) is independent of the choice of \( \mathcal{E} \), subject to the constraint on the degree of \( \mathcal{E} \). We can write \( \mathcal{E} = \mathcal{O}_{\mathbb{P}_k^1}(a) \oplus \mathcal{O}_{\mathbb{P}_k^1}(b) \) for \( 0 < a \leq b \), and we find
\[
\text{Sym}^3 \mathcal{E} \otimes \mathcal{E}^\vee \simeq \mathcal{O}_{\mathbb{P}_k^1}(-3b) \oplus \mathcal{O}_{\mathbb{P}_k^1}(-2b-a) \oplus \mathcal{O}_{\mathbb{P}_k^1}(-2a-b) \oplus \mathcal{O}_{\mathbb{P}_k^1}(-3a).
\]
The first cohomology of such a sheaf has dimension \( 6(a+b) - 4 \) and therefore only depends on \( \text{deg det} \mathcal{E} = a + b \), and not \( a \) or \( b \) separately. \( \square \)

With the above lemma in hand, we may note that the codimension of the vector bundles \( (\mathcal{E}, \mathcal{F}_\bullet) \) in the stack of vector bundles is \( H^1(\mathbb{P}_k^1, \text{End}(\mathcal{E})) + H^1(\mathbb{P}_k^1, \text{End}(\mathcal{F})) \) (the latter interpreted as 0 when \( d = 3 \)).

**Remark 9.3.** We will think of a Casnati-Ekedhal strata as having the “expected codimension” when its codimension in the Hurwitz stack agrees with the corresponding codimension of \( (\mathcal{E}, \mathcal{F}_\bullet) \) in the stack of tuples of vector bundles. Using [Lemma 9.2], a strata is of the expected codimension precisely when \( H^1(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) = 0 \).

The next lemma bounds the codimension of strata not having the expected codimension.

**Lemma 9.4.** Suppose \( 3 \leq d \leq 5 \) and \( \mathcal{M} := \mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet) \) is a Casnati-Ekedhal strata containing a curve \( C \rightarrow \mathbb{P}_k^1 \) which does not factor through some intermediate cover \( C' \rightarrow \mathbb{P}_k^1 \) of positive degree. If \( H^1(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) \neq 0 \) or \( H^0(\mathbb{P}_k^1, \mathcal{E}^\vee) \neq 0 \), \( \text{codim}_{\mathcal{H}_{d,g,k}^\bullet} \mathcal{M} \geq \frac{g + d - 1}{d} - 4d - 3 \).

**Proof.** If \( H^1(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) \neq 0 \) then we have \( \text{codim}_{\mathcal{H}_{d,g,k}^\bullet} \mathcal{M} \geq \frac{g + d - 1}{d} - 4d - 3 \) by [CL21b, Lem. 5.8] when \( d = 4 \), [CL21b, Lem. 5.12] when \( d = 5 \) and [Mir85 (6.2)], for the cases that \( d = 3 \) (see also [BV12, Prop. 2.2]). It therefore remains to show that if \( H^1(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) = 0 \) but \( H^0(\mathbb{P}_k^1, \mathcal{E}^\vee) \neq 0 \), we will also have \( \text{codim}_{\mathcal{H}_{d,g,k}^\bullet} \mathcal{M} \geq \frac{g + d - 1}{d} - 4d - 3 \). In the case \( H^1(\mathbb{P}_k^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) = 0 \), the codimension of \( \mathcal{M} \) in \( \mathcal{H}_{d,g,k}^\bullet \) is simply \( h^1(\mathbb{P}_k^1, \text{End}(\mathcal{E})) + h^1(\mathbb{P}_k^1, \text{End}(\mathcal{F})) \), by [Lemma 9.2]. We will only have \( H^0(\mathbb{P}_k^1, \mathcal{E}^\vee) \neq 0 \) when some summand of \( \mathcal{E} \) is non-positive. Recall from [Lemma 6.3] \( \text{deg} \mathcal{E} = g + d - 1 \). Therefore, if \( \mathcal{E} = \bigoplus_{i=1}^{d-1} \mathcal{O}(e_i) \) with \( e_1 \leq 0 \),
then \( \sum_{i=2}^{d-1} (e_i - e_1) \geq g + d - 1 \) and hence
\[
h^1(\mathbb{P}^1, \text{End}(\mathcal{E})) \geq h^1(\mathbb{P}^1, \bigoplus_{i=2}^{d-1} \mathcal{E}_{\mathbb{P}^1}(e_i - e_1))
\geq g + d - 1 - (d - 2)
= g + 1
\geq \frac{g + d - 1}{d} - 4^{d-3}.
\]
\[\square\]

9.5. **Weeding out covers with smaller Galois groups.** In the next few results, culminating in [Lemma 9.8], we establish bounds on the codimension of degree \( d \) covers of \( \mathbb{P}^1 \) whose Galois closure has Galois group \( G \) strictly contained in \( S_d \).

For a group \( G \) and a base \( S \), with \(#G\) invertible on \( S \), we use Hur\(_{G,S}\) to denote the stack whose \( T \)-points are given by \( (T, X, h : X \to T, f : X \to \mathbb{P}^1_T) \) where \( X \) is a scheme, \( f \) is a finite locally free map of degree \(#G\) so that \( G \) acts on \( X \) over \( \mathbb{P}^1_T \), together with an isomorphism \( G \cong \text{Aut} f \). Note that Hur\(_{G,S}\) is an algebraic stack with an étale map to the configuration space of points in \( \mathbb{P}^1 \) given by taking the branch divisor, as follows from [Wew98, Thm. 4], (the key point of the construction being the algebraicity criterion in [Wew98, Thm. 1.3.3]). Upon specifying an embedding \( G \subset S_d \) for some \( d \), there is a natural map Hur\(_{G,S} \to \widehat{\text{Hur}}_{d,S}\) sending a given cover \( (T, X, h : X \to T, f : X \to \mathbb{P}^1_T) \) to an associated cover \( \bigsqcup_{g \in G \setminus S_d} (hX)/S_{d-1} \to \mathbb{P}^1_T \) where we take the disjoint union over cosets of \( G \setminus S_d \) and then quotienting the resulting \( S_d \) cover by \( S_{d-1} \). The image of this map is a substack of \( \widehat{\text{Hur}}_{d,S} \) whose geometric points parameterize degree \( d \) covers whose Galois group is \( G \) with the specified embedding \( G \subset S_d \). We note that we could have alternatively constructed Hur\(_{G,S}\) directly, as mentioned in [Remark 5.6], without appealing to [Wew98].

**Lemma 9.6.** Suppose \( G \subset S_d \) is a subgroup not containing a transposition. Then the closure of the image \( \text{Hur}_{G,S} \to \widehat{\text{Hur}}_{d,S} \cap \widehat{\text{Hur}}_{d,g,S} \) has dimension at most \( g - 1 + d \).

**Proof.** By [Wew98, Thm. 4], if the image \( \text{Hur}_{G,S} \to \widehat{\text{Hur}}_{d,S} \cap \widehat{\text{Hur}}_{d,g,S} \) parameterizes curves with \( n \) branch points, it has dimension \( n \). We therefore use \( n \) for the number of branch points. It is possible this image has multiple components, but because the Galois closure of a degree genus \( g \) degree \( d \) cover of \( \mathbb{P}^1_k \) is a curve of bounded genus, there can only be finitely many components. We now fix one of these components and wish to show \( n \leq g - 1 + d \).

Let \( X \to \mathbb{P}^1 \) be a degree \( d \) genus \( g \) cover corresponding to a point on this component with \( n \) branch points. If \( G \subset S_d \) has no transpositions, the inertia at any point of \( \mathbb{P}^1 \), which is tame by assumption, does not act as a transposition. Therefore the cover is not simply branched over that point, i.e., the ramification partition is not \((1^d)\) or \((2, 1^{d-2})\). Hence, the fiber over that point has total ramification degree at least \( 2 \). It follows from Riemann-Hurwitz that \( 2g - 2 \geq -2d + 2n \) so \( n \leq g - 1 + d \). \[\square\]

**Corollary 9.7.** If \( 2 \leq d \leq 5 \) the only proper conjugacy class of subgroups \( G \subset S_d \) acting transitively on \( \{1, \ldots, d\} \) and containing a transposition is \( D_4 \subset S_4 \), the dihedral group of order 8. In particular, for any \( G \subset S_d \) acting transitively on \( \{1, \ldots, d\} \) with \( G \) not
isomorphic to $D_4$, the image $(\text{Hur}_{G,S} \to \text{Hur}_{d,S}) \cap \text{Hur}_{d,g,S}$ has dimension at most $g - 1 + d$.

Proof. The first statement follows by a straightforward check of all subgroups of $S_d$. The second follows from Lemma 9.6 \qed

The next lemma shows that in any Casnati-Ekedahl strata having the expected codimension (see Remark 9.3) the locus of non $S_d$ covers has high codimension in the Hurwitz stack.

Lemma 9.8. Suppose $3 \leq d \leq 5$ and $\mathcal{M}(\mathcal{E}, \mathcal{F}_*)$ is a Casnati-Ekedahl strata with $H^1(\mathbb{P}^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_*)) = 0$. Suppose further that $[\mathcal{U}_{\mathcal{M}(\mathcal{E}, \mathcal{F}_*)}/\text{Aut}\mathcal{M}(\mathcal{E}, \mathcal{F}_*)]$ contains some geometrically connected cover $X \to \mathbb{P}^1_k$ whose Galois closure is not $S_d$. Then the codimension of this locus of covers in $\text{Hur}_{d,g,k}$ is at least $\frac{g+3}{2}$.

Note that the space of $D_4$ covers is typically of codimension 2 in $\text{Hur}_{d,g,k}$, but these covers will typically lie in Casnati-Ekedahl strata with $H^1(\mathbb{P}^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_*)) \neq 0$.

Proof. The most difficult case is when $d = 4$ and the Galois closure is $D_4$, the dihedral group of order 8, this was verified in [CL21b, Lem. 5.5]. Note here we are using that whenever the Galois group of a degree 4 cover is $D_4$, $C \to \mathbb{P}^1$ necessarily factors through an intermediate degree 2 cover.

It remains to verify that if we have a smooth geometrically connected curve $C \to \mathbb{P}^1$ whose Galois closure is not $D_4$, the codimension of such curves is at least $\frac{g+3}{2}$. The geometric connectedness condition guarantees that the action of $G$ on $\{1, \ldots, d\}$ is transitive. Note that the dimension of such a strata is at most $g - 1 + d$ by Corollary 9.7 and hence also codimension $g - 1 + d$ in the $2g + 2d - 2$ dimensional stack $\text{Hur}_{d,g,k}$. The lemma follows because $g - 1 + d > \frac{g+3}{2}$ \qed

9.9. Weeding out the singular sections. Our next goal is to show that for any given Casnati-Ekedahl strata, the sections defining smooth curves can be expressed in terms of a fairly simple motivic Euler product, away from high codimension. This is, in some sense, the key input to our approach, and draws heavily on the work of [BH21] while also making use of our computations of classes associated to sections with given ramification profiles over the dual numbers from [S8]. It will turn out that this codimension is the dominant term, in the sense that for large $g$, the codimension bound we obtain on these singular sections agrees with the codimension bound we find in our main result Theorem 10.4. At this point, it may be useful to recall notation for motivic Euler products from [S2.10]

Proposition 9.10. Let $3 \leq d \leq 5$, and let $\mathcal{R}$ be an allowable collection of ramification profiles of degree $d$. Suppose $t \geq 0$ is and $\mathcal{M} := \mathcal{M}(\mathcal{E}, \mathcal{F}_*)$ a Casnati-Ekedahl strata for which $\mathcal{H}(\mathcal{E}, \mathcal{F}_*)(-s)$ is globally generated and each entry of $\bar{a}^\mathcal{E}$ is positive. Then,

$$
(9.2) \quad \left\{ \mathcal{U}_{\mathcal{M}}^\mathcal{R} \right\} \equiv \mathbb{L}^{\dim \mathcal{U}_{\mathcal{M}}^\mathcal{R}} \prod_{x \in \mathbb{P}^1_k} \left( 1 - \left( 1 - \frac{\sum_{R \in \mathcal{R}} \mathbb{L}^{-r(R)}}{\mathbb{L}^{\mathcal{H}(D, \mathcal{H}(\mathcal{E}, \mathcal{F}_*)(D|_D))}} \right) t \right)_{t=1}^x
$$

AARON LANDESMAN, RAVI VAKIL, AND MELANIE MATCHETT WOOD
are equal modulo codimension $\lfloor \frac{s+1}{2} \rfloor$ in $K_0(Spaces_k)$

In the above product, the restriction to $D$ is understood to take place at the subscheme $D \subset \mathbb{P}^1_k$ supported at $x$.

Proof. First note that the geometric connectedness hypothesis follows from Theorem 3.17 and the assumption that each entry of $\bar{a}^\delta$ is positive, so $H^0(Y, \mathcal{E}^\vee) = 0$. The remainder essentially follows by applying [BH21, Thm. 9.3.1] with the local condition determined by the ramification profile $R$, as determined in Theorem 8.9, as we next explain. In some more detail, we take $(f : X \to \mathcal{S}, F, L, r, M)$ in [BH21, Thm. 9.3.1] to be $(\mathbb{P}^1 \to \text{Spec } k, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet), \mathcal{O}_{\mathbb{P}^1}(1), 1, 0)$ and the constructible Taylor conditions $T$ of [BH21, Thm. 9.3.1] to be that determined by $\mathcal{R}$. That is, in the fiber over a point $D \to \mathbb{P}^1_k$, we consider those sections $\eta \in H^0(\mathbb{P}^1_k, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet))$ so that $\Phi_d(\eta)$ defines a smooth curve whose ramification profile over $D$ lies in $\mathcal{R}$.

In order to apply [BH21, Thm. 9.3.1], we need to verify the above conditions are indeed admissible in the sense of [BH21, Def. 9.2.6]. Indeed, to see this, we need to check the Taylor conditions imposed by being smooth with ramification profile lying in $\mathcal{R}$ are the complement of a codimension 2 $= 1 + \dim \mathbb{P}^1_{\bar{k}}$ subset of the fiber of the first sheaf of principal parts associated to $\mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)$ over a field valued point of $\mathbb{P}^1_k$. First, one can first directly verify (for example, by using an incidence correspondence) that those sections $\eta$ for which $\Psi_d(\eta)$ are not curvilinear forms a locus of codimension at least 2 in $\text{Spec } \text{Sym}^\bullet H^0(D, \mathcal{H}(\mathcal{E}|_D, \mathcal{F}_\bullet|_D))$. (Note that non-curvilinear sections also include sections with $\Psi_d(\eta)$ of positive dimension.)

It remains to show those curvilinear sections having ramification profile not lying in $\mathcal{R}$ have codimension at least 2. This follows from knowledge of their class in the Grothendieck ring Theorem 8.9, which in particular shows the codimension of those sections having ramification profile $R$ is $r(R)$. Since the only ramification profiles with $r(R) \leq 1$ are $(1^d)$ and $(2, 1^{d-2})$, the claim follows from the first constraint in the definition of allowable Definition 7.2.

We next use [BH21, Ex. 5.4.6] to determine the value of $m$ appearing in [BH21, Thm. 9.3.1]. In place of the value $D$ used in [BH21, Ex. 5.4.6], we use $t$, since we are reserving $D$ for the dual numbers. Otherwise following the notation of [BH21, Ex. 5.4.6], since $\mathcal{O}(1) = \mathcal{L}$ is very ample and $\mathcal{L}^{\otimes 0} \cong \mathcal{O}_{\mathbb{P}^1}$ globally generated, we may take $A = 1$ and $B = 0$. It follows that, in their notation $H^0(X, \mathcal{F})$ (so, again, we are taking $\mathcal{F}$ to be $\mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)$) is $r$-infinitesimally $m$-generating whenever $s \geq 1 + (m - 1) \cdot (1 + 1) = 2m - 1$. Therefore, we may take $m = \lfloor \frac{s+1}{2} \rfloor$. Hence, we obtain the congruence (9.2), once we show that the class of the subschemes having specified ramification profile $R$ is $\mathbb{L}^{-r(R)} \{ \text{Aut } \mathcal{H}|_D \}$. This was computed as the class $\mathcal{Y}_{R,d}$ in Theorem 8.9. 

In order to get a good bound on the codimension up to which Proposition 9.10 holds, we need to show that the value of $s$ defined there is high whenever the codimension of the strata is low. We now establish this.
Lemma 9.11. For any Casnati-Ekedahl strata $\mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet)$ so that the minimum degree of a line bundle summand of $\mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)$ is $s$, and $H^1(\mathbb{P}^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) = 0$, we have $\text{codim}_{\text{H}_{\text{ur}}d,k} \mathcal{M} + \frac{s+1}{2} \geq \frac{g+c_d}{2(d-1)(d-2)}$, where $c_3 = 0, c_4 = -2, c_5 = -23$.

Proof. We can verify this in the case that $d = 3$ directly. Write $\mathcal{E} = \mathcal{O}(s) \oplus \mathcal{O}(g + 2 - s)$ with $s \leq g + 2 - s$, so that $\text{codim}_{\text{H}_{\text{ur}}d,k} \mathcal{M} = h^1(\mathbb{P}^1, \text{End}(\mathcal{E})) \geq (g + 2) - 2s - 1$. Then, $\text{codim}_{\text{H}_{\text{ur}}d,k} \mathcal{M} + \frac{s+1}{2} \geq (g + 2) - 2s + 1 + \frac{s+1}{2} = \frac{2g + 3 - 3s}{2}$. This is minimized when $s$ is maximized. Since we must have $s \leq \frac{g+2}{2}$, when $s = \frac{g+2}{2}$, we find $\frac{2g + 3 - 3s}{2} = \frac{g}{4}$.

We now concentrate on the cases $d = 4$ and $d = 5$. First, in the case that $\mathcal{E}$ and $\mathcal{F}$ are balanced, so that $\text{codim}_{\text{H}_{\text{ur}}d,k} \mathcal{M} = 0$, we claim that $\frac{s+1}{2} \geq \frac{g+1}{2}$. When $d = 4$, and $\mathcal{E}$ and $\mathcal{F}$ are balanced, the minimum line bundle summand of $\mathcal{E}$ has degree at least $\frac{g+1}{3}$, while the maximum line bundle summand of $\mathcal{F}$ has degree at most $\frac{g+1}{3}$ using Lemma 6.3 and the isomorphism $\det \mathcal{E} \simeq \det \mathcal{F}$ from Theorem 3.14. Hence, the minimum line bundle summand of $\mathcal{H}$ has degree $s \geq 2\frac{g+1}{3} - \frac{g+1}{2} = \frac{g - 8}{6}$. Therefore, $\frac{s+1}{2} \geq \frac{g - 2}{12}$.

When $d = 5$, and $\mathcal{E}$ and $\mathcal{F}$ are balanced, the minimum line bundle summand of $\mathcal{E}$ has degree at least $\frac{g+1}{3}$ by Lemma 6.3 and the minimum line bundle summand of $\mathcal{F}$ has degree at least $\frac{2(g+2)-4}{5}$ as $\det \mathcal{E} \otimes 2 = \det \mathcal{F}$ by Theorem 3.16. Therefore, the minimum degree of a line bundle summand of $\mathcal{H}$ is $s \geq 2\frac{2(g+2)-4}{5} - (g+4) + \frac{g+1}{2} = \frac{g - 43}{20}$ and $\frac{s+1}{2} \geq \frac{g - 23}{40}$.

In the case $d = 4$ or 5, it remains to see that these inequalities hold for non-general strata, supposing still that $H^1(\mathbb{P}^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) = 0$. In this case, by Lemma 9.2, the codimension of $\mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet)$ is $h^1(\mathbb{P}^1, \text{End}(\mathcal{E})) + h^1(\mathbb{P}^1, \text{End}(\mathcal{F}))$. Any Casnati-Ekedahl strata may be connected to the generic one by a sequence of strata $\mathcal{M}(\mathcal{E}_i, \mathcal{F}_i)$, each contained in the closure of the next. Further, we can assume that for any two adjacent indices $i$ and $i+1$, one of the following two cases occurs:

1. $\mathcal{E}_i \simeq \mathcal{E}_{i+1}$ and $\mathcal{F}_i$ differs from $\mathcal{F}_{i+1}$ only in two line bundles summands by a single degree.
2. $\mathcal{F}_i \simeq \mathcal{F}_{i+1}$ and $\mathcal{E}_i$ differs from $\mathcal{E}_{i+1}$ only in two line bundle summands by a single degree.

In order to show the claimed inequality holds for arbitrary strata, it suffices to show it remains true under such specializations. Because each such strata has codimension at least 1 in the next, it suffices to show the value of $s$ under such specializations decreases by at most 2. When $d = 4$ this is the case because $\mathcal{H}(\mathcal{E}, \mathcal{F}) = \text{Sym}^2 \mathcal{E} \otimes \mathcal{F}^\vee$ and increasing a summand of $\mathcal{F}$ by 1 only increases all summands of $\mathcal{H}(\mathcal{E}, \mathcal{F})$ by at most 1, while decreasing a summand of $\mathcal{E}$ decreases all summands of $\mathcal{H}(\mathcal{E}, \mathcal{F})$ by at most 2. Similarly, when $d = 5$, so $\mathcal{H}(\mathcal{E}, \mathcal{F}) = \wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee$, and decreasing a summand of $\mathcal{E}$ by 1 while maintaining $\det \mathcal{E}$ decreases all summands of $\mathcal{H}(\mathcal{E}, \mathcal{F})$ by at most 1, while decreasing a summand of $\mathcal{F}$ by 1 decreases all summands of $\mathcal{H}(\mathcal{E}, \mathcal{F})$ by at most 2. □
9.12. **Putting the codimension bounds together.** We now merge the bounds on codimension of various bad loci established earlier in this section to obtain the following result.

**Proposition 9.13.** For $3 \leq d \leq 5$, $k$ a field of characteristic not dividing $d!$, $\mathcal{R}$ an allowable collection of ramification profiles of degree $d$, let $c_d$ be as in [Lemma 9.11]. Define $n_{d,g} := \chi(\mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet))$. Then, $\{\text{Hur}_{d,g,k}^R\}$ is equal to

$$
\sum_{\text{Casnati-Ekedahl strata } \mathcal{M}} \frac{1}{\{\text{Aut } \mathcal{M}\}} L_n^{n_{d,g}} \prod_{x \in \mathbb{P}^1_k} \left( \sum_{R \in \mathcal{R}} L^{-r(R)} \right) \{\text{Aut } \mathcal{M}|D\} \prod_{x \in \mathbb{P}^1_k} L^{h^0(D, \mathcal{M}(\mathcal{E}|D, \mathcal{F}_\bullet|D))}
$$

modulo codimension $r_{d,g} := \min\left(\frac{g+c_d}{2(d-1)(d-2)}, \frac{g+d-1}{d}, 4^{d-3}\right)$ in $\widehat{K}_0(\text{Spaces}_k)$.

**Proof.** First, by [Proposition 7.8], it suffices to show

$$
\sum_{\text{nonempty Casnati-Ekedahl strata } \mathcal{M}} \{U_{d,g}^R, S_d\} \{\text{Aut } \mathcal{M}\}
$$

agrees with (9.3).

We next check (9.4) agrees with

$$
\sum_{\text{nonempty Casnati-Ekedahl strata } \mathcal{M}} \{U^R_{\mathcal{M}}\} \{\text{Aut } \mathcal{M}\}
$$

modulo codimension $\min\left(\frac{g+c_d}{2(d-1)(d-2)}, \frac{g+d-1}{d}, 4^{d-3}\right)$. Since we are working modulo codimension $\frac{g+d-1}{d} - 4^{d-3}$, we can assume $\mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet)$ has $H^1(\mathbb{P}^1, \mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet)) = 0$ and $H^1(\mathbb{P}^1, \mathcal{E}^\vee) = 0$, by [Lemma 9.4]. Note that the condition $H^1(\mathbb{P}^1, \mathcal{E}^\vee) \neq 0$ ensures all curves defined by sections of $U^R_{\mathcal{M}}$ are geometrically connected, by [Theorem 3.17]. Since we have now restricted ourselves to work with strata for which $H^1(\mathbb{P}^1, \mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet)) = 0$, it follows from [Lemma 9.11] with notation for $s$ as in [Lemma 9.11] that codim $\mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet) + s + 1 \geq \frac{g+c_d}{2(d-1)(d-2)}$. We also obtain from [Lemma 9.8] that the smooth geometrically connected curves in $U^R_{\mathcal{M}}$ which do not lie in Hur$_{d,g,k}$ (because they do not have Galois closure $S_d$) have codimension at least $\frac{g+3}{2}$ in Hur$_{d,g,k}$. Hence, as we are working modulo codimension $\frac{g+3}{2}$, we can freely ignore these, and so (9.4) agrees with (9.5).

We next claim (9.5) agrees with

$$
\sum_{\text{nonempty Casnati-Ekedahl strata } \mathcal{M}} \frac{1}{\{\text{Aut } \mathcal{M}\}} L^{\dim U^R_{\mathcal{M}}} \prod_{x \in \mathbb{P}^1_k} \left( \sum_{R \in \mathcal{R}} L^{-r(R)} \right) \{\text{Aut } \mathcal{M}|D\} \prod_{x \in \mathbb{P}^1_k} L^{h^0(D, \mathcal{M}(\mathcal{E}|D, \mathcal{F}_\bullet|D))}
$$

modulo codimension $\min\left(\frac{g+3}{2}, \frac{g+3}{2}, 4^{d-3}\right)$ in $\widehat{K}_0(\text{Spaces}_k)$. Since we have now restricted ourselves to work with strata for which $H^1(\mathbb{P}^1, \mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet)) = 0$, it follows from [Lemma 9.11] with notation for $s$ as in [Lemma 9.11] that codim $\mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet) + s + 1 \geq \frac{g+3}{2}$. We also obtain from [Lemma 9.8] that the smooth geometrically connected curves in $U^R_{\mathcal{M}}$ which do not lie in Hur$_{d,g,k}$ (because they do not have Galois closure $S_d$) have codimension at least $\frac{g+3}{2}$ in Hur$_{d,g,k}$. Hence, as we are working modulo codimension $\frac{g+3}{2}$, we can freely ignore these, and so (9.5) agrees with (9.6).
Indeed, this follows from Proposition 9.10 using the bounds on $s$ from Lemma 9.11. Next, we claim that for any $\mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet)$ as above, $\dim U^R_{\mathcal{M}}$ is independent of $\mathcal{M}$ whenever $\text{codim}_{\text{Hur}_{d,g,k}} \mathcal{M} \leq \min(\frac{s+c_d}{2(d-1)(d-2)}, \frac{s+d-1}{d} - 4d-3)$. Indeed, in this case, because $H^1(\mathbb{P}^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) = 0$, we find $\dim U^R_{\mathcal{M}} = h^0(\mathbb{P}^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) = \chi(\mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet))$ and indeed this Euler characteristic only depends on the degrees and ranks of $\mathcal{E}$ and $\mathcal{F}$. For notational convenience, we let $n_{d,g}$ denote this dimension $\chi(\mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet))$.

Then, up to codimension $\min(\frac{s+c_d}{2(d-1)(d-2)}, \frac{s+d-1}{d} - 4d-3)$, in $K_0(\text{Spaces}_k)$, we can rewrite (9.6) as (9.3).

To conclude the proof, we wish to remove the word “nonempty” in (9.6). That is, there may be certain strata which contain no $S_d$ covers, and we wish to show they do not contribute to (9.3) in low codimension. The summand in (9.3) associated to such an empty strata $\mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet)$ has codimension equal to the codimension of $(\mathcal{E}, \mathcal{F}_\bullet)$, considered as a point in the moduli stack of tuples of vector bundles on $\mathbb{P}^1$. Using Corollary 9.7, this is only potentially an issue in the case $d = 4$, where we must deal with strata $\mathcal{M}(\mathcal{E}, \mathcal{F}_\bullet)$ so that the generic members of $H^0(\mathbb{P}^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet))$ define $D_4$ covers. In [CL21b, Lem. 5.5] it is show that such strata are either codimension at least $\frac{s+3}{2}$ or else have $H^1(\mathbb{P}^1, \mathcal{H}(\mathcal{E}, \mathcal{F}_\bullet)) \neq 0$, and in the latter case, by [CL21b, Lem. 5.4] such strata have codimension at least $\frac{s+3}{2} - 4$ in the stack of vector bundles on $\mathbb{P}^1$. In either case, we may again we may ignore these contributions up to our codimension bounds, and so (9.6) agrees with (9.3).

\[ \square \]

10. Proving the main result

In this section, we prove our main result [Theorem 10.4] by massaging the formula for $\{\text{Hur}_{d,g,k}\}$ given in Proposition 9.13. We then deduce some corollaries.

In order to prove our main result we will need one of the simplest cases of the “motivic Tamagawa number conjecture” [BD07, Conj. 3.4]. To start this Tamagawa number formula, we employ the following notation.

**Notation 10.1.** For $\mathcal{G}$ a vector bundle on a scheme $X$, let $\text{Aut}^{SL,X}_\mathcal{G}$ denote the SL bundle over $X$ associated to $\mathcal{G}$ (i.e., the kernel of the determinant map of group schemes $\text{Aut}_\mathcal{G} \to G_m$). We use $\text{Aut}^{SL}_\mathcal{G}$ as notation for the Weil restriction $\text{Res}_{X/Spec(k)}(\text{Aut}^{SL,X}_\mathcal{G})$. For $(\mathcal{E}, \mathcal{F}_\bullet)$ resolution data, we use $\text{Aut}^{SL}(\mathcal{F}_\bullet) := \prod_{i=1}^{\frac{d-2}{2}} \text{Aut}^{SL}(\mathcal{F}_i)$.

**Lemma 10.2.** For any positive integer $n$,

\[ \sum_{\text{rank } n \text{ vector bundles } \mathcal{V} \text{ on } \mathbb{P}^1} \frac{1}{\{\text{Aut}^{SL}_\mathcal{V}\}} \prod_{x \in \mathbb{P}^1_k} \frac{1}{\text{dim } \text{Aut}^{SL}_{\mathcal{V}|_D}} = L^{-\text{rk}SL(\mathcal{E})} \in K_0(\text{Spaces}_k). \]

**Proof.** We will deduce this from the motivic Tamagawa number conjecture for SL$_n$ over $\mathbb{P}^1$ proven in [BD07, §7]. Let $\text{Bun}_{G,\mathbb{P}^1}$ denote the moduli stack of $G$-bundles on $\mathbb{P}^1$. It is shown in [BD07, §7], and also via a different argument in [BD07, §6], that, in
\( \tilde{K}_0(\text{Spaces}_k) \) (and even without inverting universally bijective morphisms) we have 
\[ \{ \text{Bun}_{\text{SL}_n, \mathbb{P}^1} \} = \mathbb{L}^{-\dim \text{SL}_n} \prod_{i=2}^n \mathbb{Z}(\mathbb{P}^1, \mathbb{L}^{-i}), \]
where \( \mathbb{Z}(\mathbb{P}^1, t) := \sum_{i=0}^{\infty} \text{Sym}^i_{\mathbb{P}^1} \{ \text{Sym}^i_{\mathbb{P}^1} \} t^i = \frac{1}{1-t} \) is the motivic Zeta function of \( \mathbb{P}^1 \).

Note that \( \mathbb{Z}(\mathbb{P}^1, \mathbb{L}^{-i}) = \frac{1}{1-\mathbb{L}^{-i}} \) is invertible in \( \tilde{K}_0(\text{Spaces}_k) \), with inverse equal to \( (1 - \mathbb{L}^{-i+1})(1 - \mathbb{L}^{-i+1}) \). To complete the proof, it is therefore enough to demonstrate the two equalities

\[(10.1)\quad \{ \text{Bun}_{\text{SL}_n, \mathbb{P}^1} \} = \sum_{\text{rank } n \text{ vector bundles } \mathcal{V} \text{ on } \mathbb{P}^1} \frac{1}{\det \mathcal{V} = \mathcal{O}_{\mathbb{P}^1}} \{ \text{Aut}_{\mathcal{V}}^{\text{SL}} \}
\]

\[(10.2)\quad \left( \prod_{i=2}^n \mathbb{Z}(\mathbb{P}^1, \mathbb{L}^{-i}) \right)^{-1} = \prod_{x \in \mathbb{P}^1_k} \frac{\{ \text{Aut}_{\mathcal{V} \mid \mathcal{D}}^{\text{SL}} \}}{\mathbb{L}^{-\dim \text{Aut}_{\mathcal{V} \mid \mathcal{D}}^{\text{SL}}}}.\]

We first verify \((10.1)\). Taking cohomology on \( \mathbb{P}^1 \) associated to exact sequence 
\[ \text{SL}_n \to \text{GL}_n \to \text{G}_m \] defining \( \text{SL}_n \) shows that \( \text{SL}_n \) torsors over \( \mathbb{P}^1 \) are in bijection with \( \text{GL}_n \) torsors of trivial determinant. We can then stratify \( \text{Bun}_{\text{SL}_n, \mathbb{P}^1} \) as a disjoint union of locally closed substacks corresponding to residual gerbes, as is explained for general \( G \) in place of \( \text{SL}_n \) [BD07, p. 636]. (Much of this argument can be verified more simply and directly in the case \( G = \text{SL}_n \).) Noting that \( \text{Aut}_{\mathcal{V}}^{\text{SL}} \) is special with invertible class in the Grothendieck ring by Lemma 7.10, we find 
\( \{ B(\text{Aut}_{\mathcal{V}}^{\text{SL}}) \} = \frac{1}{\{ \text{Aut}_{\mathcal{V}}^{\text{SL}} \}} \) and \((10.1)\) follows.

It remains only to prove \((10.2)\). First, note that \( \mathcal{V} \) is trivial Zariski locally and hence trivial over \( \mathcal{D} \), so \( \text{Aut}_{\mathcal{V} \mid \mathcal{D}}^{\text{SL}} \) is simply \( \text{Res}_{\mathcal{D}/\text{Spec} k}(\text{SL}_n) \) which is an extension of \( \text{SL}_n \) by \( \text{G}_a^{\dim \text{SL}_n} \). Therefore, for any vector bundle \( \mathcal{V} \), we may re-express \( \frac{\{ \text{Aut}_{\mathcal{V} \mid \mathcal{D}}^{\text{SL}} \}}{\mathbb{L}^{-\dim \text{Aut}_{\mathcal{V} \mid \mathcal{D}}^{\text{SL}}}} = \frac{\{ \text{SL}_n \}}{\mathbb{L}^{-\dim \text{SL}_n}} = \left( \prod_{i=2}^n (\mathbb{L}^{-i} - 1) \right) \mathbb{L}^{-\dim \text{SL}_n} = \prod_{i=2}^n (1 - \mathbb{L}^{-i}). \)

Using multiplicativity of Euler products [Lemma 2.14]
\[ \prod_{x \in \mathbb{P}^1_k} \frac{\{ \text{Aut}_{\mathcal{V} \mid \mathcal{D}}^{\text{SL}} \}}{\mathbb{L}^{-\dim \text{Aut}_{\mathcal{V} \mid \mathcal{D}}^{\text{SL}}}} = \prod_{x \in \mathbb{P}^1_k, i=2}^n (1 - \mathbb{L}^{-i}) = \prod_{i=2}^n \prod_{x \in \mathbb{P}^1_k} (1 - \mathbb{L}^{-i}).\]

Hence, to prove \((10.2)\), we only need check \( \mathbb{Z}(\mathbb{P}^1, \mathbb{L}^{-i})^{-1} = \prod_{x \in \mathbb{P}^1_k} (1 - \mathbb{L}^{-i}) \) for \( 2 \leq i \leq n \). The right hand side is by definition \( \prod_{x \in \mathbb{P}^1_k} (1 - \mathbb{L}^{-i}t) \). By [Bill17, §3.8, Property 4], we have \( \prod_{x \in \mathbb{P}^1_k} (1 - \mathbb{L}^{-i}t) \mid t = 1 = \prod_{x \in \mathbb{P}^1_k} (1 - t) \mid t = \mathbb{L}^{-i} \). (As a word of warning, it is important that the substitution we made here was via replacing \( t \) by its product with a power of \( \mathbb{L} \), see [BH21, Rem. 6.5.2 and 6.5.3]). Finally, by [BH21, Ex. 6.1.12] and multiplicativity of Euler products [Bill17, Prop. 3.9.2.4], \( \prod_{x \in \mathbb{P}^1_k} (1 - t) \mid t = \mathbb{L}^{-i} = \mathbb{Z}(\mathbb{P}^1, \mathbb{L}^{-i})^{-1}. \) \( \square \)
For our main theorem, we will also need the following elementary dimension comparison.

**Lemma 10.3.** For $d \leq 3 \leq 5$ and $n_{d,g}$ as in Proposition 9.13, $n_{d,g} - \rk \SL(\mathcal{E}) - \rk \SL(\mathcal{F}_\bullet) = \dim \Hur_{d,g,k} + 1$.

**Proof.** Indeed, this can be checked separately in the cases $d = 3, 4, \text{ and } 5$.

We now check the most difficult case that $d = 5$, leaving the other cases to the reader. In the case $d = 5$, one computes

\[
n_{d,g} = \chi(\wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee) = \rk \left( \wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee \right) + \deg \wedge^2 \mathcal{F} \otimes \mathcal{E} \otimes \det \mathcal{E}^\vee
= \binom{5}{2} \cdot 4 + 16 \deg \mathcal{F} - 30 \deg \mathcal{E} = 40 + 32 \deg \mathcal{E} - 30 \deg \mathcal{E}
= 40 + 2 \deg \mathcal{E}
= 40 + 2g + 2d - 2.
\]

Furthermore, still in the $d = 5$ case, $\rk \SL(\mathcal{E}) = 15$ and $\rk \SL(\mathcal{F}_\bullet) = 24$. Therefore,

\[
n_{d,g} - \rk \SL(\mathcal{E}) - \rk \SL(\mathcal{F}_\bullet) = 40 + 2g + 2d - 2 - 15 - 24
= (2g + 2d - 2) + 1
= \dim \Hur_{d,g,k} + 1
\]
as claimed. \qed

We are finally prepared to prove our main theorem. For the statement of our main theorem, recall we defined $r_{d,g} = \min \left( \frac{g + c_d}{2(d - 1)(d - 2)}, \frac{g + d - 1}{d} - 4^{d - 3} \right)$ in Proposition 9.13 with $c_3 = 0, c_4 = -2, \text{ and } c_5 = -23$. Note that for $g \gg 0$, $r_{d,g}$ is more than $\frac{g}{2(d - 1)(d - 2)} - 1$.

**Theorem 10.4.** Let $2 \leq d \leq 5$, $k$ a field of characteristic not dividing $d!$, $\mathcal{R}$ an allowable collection of ramification profiles of degree $d$. Then,

\[
\{ \Hur_{d,g,k}^\mathcal{R} \} \equiv \frac{L^{\dim \Hur_{d,g,k}}}{1 - L^{-1}} \left( \prod_{x \in \mathcal{P}_k} \left( \sum_{R \in \mathcal{R}} L^{-r(R)} \right) \left( 1 - L^{-1} \right) \right)
\]

are equal modulo codimension $r_{d,g}$ in $\widehat{K_0}(\Spaces_k)$. In the case $d = 2$, the left hand side and right hand side of (10.3) are actually equal in $K_0(\Stacks_k)$ (and not just equivalent in $\widehat{K_0}(\Spaces_k)$ modulo terms of a certain dimension).

**Proof.** The proof of the $d = 2$ case is of a different nature and we defer it to the end of §11. We now concentrate on the case $3 \leq d \leq 5$. By Proposition 9.13, our goal reduces to showing (9.3) agrees with the right hand side of (10.3).
Recall our notation for $n_{d,g}$ from Proposition 9.13. First, we claim we can rewrite (9.3) as

$$
\frac{1}{\mathcal{L} - 1} \sum_{\mathcal{E}, \mathcal{F}} \frac{1}{\{ \text{Aut}_{\mathcal{E}} \}} \frac{1}{\{ \text{Aut}_{\mathcal{F}} \}} \prod_{x \in \mathbb{P}^1} \left( \sum_{R \in \mathcal{R}} \mathcal{L}^{-r(R)} \right) \left( \mathcal{L} - 1 \right) \mathcal{L} \{ \text{Aut}_{\mathcal{E}|D} \} \{ \text{Aut}_{\mathcal{F}|D} \},
$$

with the summation over $\mathcal{E}, \mathcal{F}$ interpreted as follows: $\mathcal{E}$ ranges over all $\mathbb{P}^1$ bundles of rank $d - 1$ and degree $g + d - 1$; when $d = 3$, $\mathcal{F}_*$ is interpreted as being empty (so all classes associated to it are 1); when $d = 4$, $\mathcal{F}_* = \mathcal{F}$ has rank 2 and degree $g + d - 1$; when when $d = 5$, $\mathcal{F}_* = \mathcal{F}$ has rank 5 and degree $2(g + d - 1)$. To see this we proceed as follows. For $\mathcal{M} = \mathcal{M}(\mathcal{E}, \mathcal{F}_*)$, using the formula for $\text{Aut}_{\mathcal{F}|D}$ from Lemma 4.3 we can rewrite

$$
\frac{1}{\mathcal{L} - 1} \sum_{\mathcal{E}, \mathcal{F}} \frac{1}{\{ \text{Aut}_{\mathcal{E}} \}} \frac{1}{\{ \text{Aut}_{\mathcal{F}} \}} \prod_{x \in \mathbb{P}^1} \left( \sum_{R \in \mathcal{R}} \mathcal{L}^{-r(R)} \right) \left( \mathcal{L} - 1 \right) \mathcal{L} \{ \text{Aut}_{\mathcal{E}|D} \} \{ \text{Aut}_{\mathcal{F}|D} \},
$$

where we interpret $\{ \text{Aut}_{\mathcal{F}} \} = 1$ when $d = 3$. Similarly,

$$
\{ \text{Aut}_{\mathcal{F}|D} \} = \{ \text{Res}_{D/k}(G_m) \} \{ \text{Aut}_{\mathcal{E}|D} \} \{ \text{Aut}_{\mathcal{F}|D} \} = (\mathcal{L} - 1) \mathcal{L} \{ \text{Aut}_{\mathcal{E}|D} \} \{ \text{Aut}_{\mathcal{F}|D} \}.
$$

Hence, using (10.5) and (10.6), we can rewrite (9.3) as (10.4).

We next make a sequence of simplifications of (10.4). Then, summing over the same pairs $(\mathcal{E}, \mathcal{F}_*)$ as in (10.4), we can rewrite it as

$$
\frac{1}{\mathcal{L} - 1} \left( \sum_{\mathcal{E}} \left\{ \frac{1}{\{ \text{Aut}_{\mathcal{E}} \}} \right\} \right) \left( \sum_{\mathcal{F}_*} \left\{ \frac{1}{\{ \text{Aut}_{\mathcal{F}_*} \}} \right\} \right) \left( \sum_{R \in \mathcal{R}} \mathcal{L}^{-r(R)} \right) \left( \mathcal{L} - 1 \right) \mathcal{L} \{ \text{Aut}_{\mathcal{E}|D} \} \{ \text{Aut}_{\mathcal{F}_*|D} \},
$$

where the parenthesized sum of $\mathcal{F}_*$ is interpreted as 1 in the case $d = 3$, in this line and in the remainder of the proof.

Next, observe that $2 + \dim \text{Aut}_{\mathcal{E}|D} \cdot \dim \text{Aut}_{\mathcal{F}_*|D} = h^0(D, \mathcal{H}(\mathcal{E}|D, \mathcal{F}_*|D))$. Indeed, this can be checked separately in the cases $d = 3, 4,$ and 5. When $d = 3$, both sides equal 8, when $d = 4$, both sides equal 24, and when $d = 5$, both sides equal 80. Therefore, we can rewrite (10.4) as

$$
\frac{1}{\mathcal{L} - 1} \left( \sum_{\mathcal{E}} \left\{ \frac{1}{\{ \text{Aut}_{\mathcal{E}} \}} \right\} \right) \left( \sum_{\mathcal{F}_*} \left\{ \frac{1}{\{ \text{Aut}_{\mathcal{F}_*} \}} \right\} \right) \left( \sum_{R \in \mathcal{R}} \mathcal{L}^{-r(R)} \right) \left( \mathcal{L} - 1 \right) \mathcal{L} \{ \text{Aut}_{\mathcal{E}|D} \} \{ \text{Aut}_{\mathcal{F}_*|D} \},
$$

where
Using multiplicativity of Euler products [Lemma 2.14], this becomes
(10.9)
\[ \frac{1}{L-1} \left( \sum_{\mathcal{E}} \frac{1}{|\text{Aut}_{\mathcal{E}}|} \right) \left( \sum_{\mathcal{F}} \frac{1}{|\text{Aut}_{\mathcal{F}}|} \right) L^{n_{d,g}} \left( \prod_{x \in P^1_k} \left( \sum_{R \in \mathcal{R}} L^{-r(R)} \right) \left( 1 - L^{-1} \right) \right) \]
\[ \cdot \left( \prod_{x \in P^1_k} \frac{\text{dim Aut}_{\mathcal{E}|D}}{L} \right) \cdot \left( \prod_{x \in P^1_k} \frac{\text{dim Aut}_{\mathcal{F}|D}}{L} \right). \]

Then, by the Tamagawa number formula for $\text{SL}_n$, [Lemma 10.2],
\[ \sum_{\mathcal{E}} \frac{1}{|\text{Aut}_{\mathcal{E}}|} \prod_{x \in P^1_k} \frac{\text{dim Aut}_{\mathcal{E}|D}}{L} = L^{-\text{rk SL}(\mathcal{E})}, \]
\[ \sum_{\mathcal{F}} \frac{1}{|\text{Aut}_{\mathcal{F}}|} \prod_{x \in P^1_k} \frac{\text{dim Aut}_{\mathcal{F}|D}}{L} = L^{-\text{rk SL}(\mathcal{F})}, \]
where $\text{rk SL}(\mathcal{F})$ is interpreted as 0 in the case $d = 3$. Therefore, (10.9) simplifies to
(10.10)
\[ \frac{1}{L-1} L^{n_{d,g}-\text{rk SL}(\mathcal{E})-\text{rk SL}(\mathcal{F})} \prod_{x \in P^1_k} \left( \sum_{R \in \mathcal{R}} L^{-r(R)} \right) \left( 1 - L^{-1} \right). \]

Hence, using [Lemma 10.3], (10.11) simplifies to
(10.11)
\[ \frac{1}{L-1} L^{\text{dim Hur}_{d,g,k}+1} \prod_{x \in P^1_k} \left( \sum_{R \in \mathcal{R}} L^{-r(R)} \right) \left( 1 - L^{-1} \right). \]
which equals the right hand side of (10.3).

Specializing [Theorem 10.4] to the simply branched case gives the following corollary.

**Corollary 10.5.** For $2 \leq d \leq 5$, and $k$ a field of characteristic not dividing $d!$, in the case $\mathcal{R} = \{(1^d), (2, 1^{d-2})\}$ corresponding to simply branched curves, we have
\[ \{\text{Hur}^{\mathcal{R}}_{d,g,k}\} \equiv L^{\text{dim Hur}_{d,g,k}}(1 - L^{-2}). \]
in $\widehat{K}_0(\text{Stacks}_k)$ modulo codimension $r_{d,g}$ if $d \neq 2$, and in $K_0(\text{Stacks}_k)$ when $d = 2$.

Note that in the case $d = 2$, this corollary is equivalent to the statement of [Theorem 10.4] and is really proven in [§11].
Proof. Simply plug in $\mathcal{R} = \{(1^d), (2, 1^{d-2})\}$ into Theorem 10.4. Then, $\sum_{R \in \mathcal{R}} L^{-r(R)} = 1 + L^{-1}$ and so

$$
\prod_{x \in \mathbb{P}^1_k} (1 - L^{-1}) \left( \sum_{R \in \mathcal{R}} L^{-r(R)} \right) = \prod_{x \in \mathbb{P}^1_k} (1 - L^{-2})
$$

$$
= \prod_{x \in \mathbb{P}^1_k} (1 - L^{-2}t)_{|t=1}
$$

$$
= \prod_{x \in \mathbb{P}^1_k} (1 - t)_{|t=L^{-2}} \quad \text{by [BH11] §3.8, Property 4}
$$

$$
= \frac{1}{Z_{\mathbb{P}^1_k}(L^{-2})} \quad \text{by [BH21] Ex. 6.1.12}
$$

$$
= \left(1 - L^{-1}\right) \left(1 - L^{-2}\right).
$$

Therefore, modulo codimension $r_{d,8}$ in $\widehat{\text{K}}_0(\text{Spaces}_k)$ when $d \neq 2$, (and in $\text{K}_0(\text{Stacks}_k)$ when $d = 2$)

$$
\{\text{Hur}_{d,8}^{\mathcal{R}} \} \equiv \frac{L^{\dim \text{Hur}_{d,8,k}}}{(1 - L^{-1})} \prod_{x \in \mathbb{P}^1_k} (1 - L^{-1}) \left( \sum_{R \in \mathcal{R}} L^{-r(R)} \right)
$$

$$
= \frac{L^{\dim \text{Hur}_{d,8,k}}}{1 - L^{-1}} \left(1 - L^{-1}\right) \left(1 - L^{-2}\right)
$$

$$
= L^{\dim \text{Hur}_{d,8,k}} (1 - L^{-2}). \quad \square
$$

When we allow the ramification profile to be arbitrary in Theorem 10.4 we obtain the following corollary counting all degree $d$ $S_d$ Galois covers of $\mathbb{P}^1$. In the cases $d = 4$ and $d = 5$, there does not seem to be any obvious simplification of the motivic Euler product.

**Corollary 10.6.** For $k$ a field of characteristic not dividing $d!$,

$$
\text{Hur}_{d,g,k} \equiv \begin{cases} 
L^{\dim \text{Hur}_{2,g,k}} (1 - L^{-2}) & \text{if } d = 2 \\
L^{\dim \text{Hur}_{3,g,k}} (1 + L^{-1})(1 - L^{-3}) & \text{if } d = 3 \\
L^{\dim \text{Hur}_{4,g,k}} \prod_{x \in \mathbb{P}^1_k} (1 + L^{-2} - L^{-3} - L^{-4}) & \text{if } d = 4 \\
L^{\dim \text{Hur}_{5,g,k}} \prod_{x \in \mathbb{P}^1_k} (1 + L^{-2} - L^{-4} - L^{-5}) & \text{if } d = 5 
\end{cases}
$$

in $\widehat{\text{K}}_0(\text{Stacks}_k)$ modulo codimension $r_{d,8}$ if $d \neq 2$, and in $\text{K}_0(\text{Stacks}_k)$ when $d = 2$.

**Proof.** The case $d = 2$ is already covered in Corollary 10.5 since $\text{Hur}_{2,g,k}^{\{(1^2), (2)\}} = \text{Hur}_{2,g,k}$. Taking

$$
\mathcal{R} = \{(1^4), (2, 1^2), (3, 1), (2^2), (4)\}
$$
the $d = 4$ case follows from plugging $\mathcal{R}$ into \[\text{Theorem 10.4}\] and using $\text{Hur}_{4,g,k}^R = \text{Hur}_{4,g,k}$. Taking $\mathcal{R} = \{(1^5), (2, 1^3), (2^2, 1), (3, 2), (3, 1^2), (3, 1), (4, 1), (5)\}$ the $d = 5$ case follows from plugging $\mathcal{R}$ into \[\text{Theorem 10.4}\] and using $\text{Hur}_{5,g,k}^R = \text{Hur}_{5,g,k}$. Finally, let us check the $d = 3$ case. Here, for $\mathcal{R} = \{(1^3), (2, 1), (3)\}$, we have $\text{Hur}_{3,g,k}^R = \text{Hur}_{3,g,k}$. So, by \[\text{Theorem 10.4}\] using \[\text{[Bil17, §3.8, Property 4]}\] and by \[\text{[BH21, Ex. 6.1.12]}\] as in the proof of \[\text{Corollary 10.5}\],

\[
\{\text{Hur}_{d,g,k}^R\} = \prod_{x \in \mathbb{P}_k^1} \frac{L_{\dim \text{Hur}_{d,g,k}}}{1 - L^{-1}} \left(1 - L^{-1}\right) \left(\sum_{R \in \mathcal{R}} L^{-r(R)}\right) 
= \prod_{x \in \mathbb{P}_k^1} \frac{L_{\dim \text{Hur}_{d,g,k}}}{1 - L^{-1}} \left(1 - L^{-1}\right) \left(1 + L^{-1} + L^{-2}\right) 
= \prod_{x \in \mathbb{P}_k^1} \frac{L_{\dim \text{Hur}_{d,g,k}}}{1 - L^{-1}} \left(1 - L^{-1}\right) \left(1 - L^{-3}\right) 
= \prod_{x \in \mathbb{P}_k^1} \frac{L_{\dim \text{Hur}_{d,g,k}}}{1 - L^{-1}} \frac{1}{2} \left(1 - L^{-2}\right) \left(1 - L^{-3}\right) 
= \prod_{x \in \mathbb{P}_k^1} \frac{L_{\dim \text{Hur}_{d,g,k}}}{1 - L^{-1}} \left(1 - L^{-1}\right) \left(1 - L^{-3}\right),
\]

where we work in $\tilde{K}_0(\text{Stacks}_k)$ modulo codimension $r_{d,g}$.

\[\Box\]

11. Degree 2

Following the notation introduced in \[\text{[AV04]}\], let $\mathbb{A}_{sm}(1,n) \subset \text{SpecSym}^* H^0(\mathbb{P}_1, \mathcal{O}(n))$ denote the open subscheme parameterizing reduced degree $n$ divisors on $\mathbb{P}_1$.

**Lemma 11.1.** For $k$ a field with $\text{char } k \neq 2$, there is an isomorphism of stacks $\text{Hur}_{2,g,k} \simeq [\mathbb{A}_{sm}(1, 2g + 2)/G_m]$, for an appropriate action of $G_m$ on $\mathbb{A}_{sm}(1, 2g + 2)$.

**Remark 11.2.** This can be deduced from the proofs of \[\text{[AV04, Thm. 4.1, Cor. 4.7]}\], though there the authors work with a further quotient by the $\text{PGL}_2$ action on the base $\mathbb{P}_1$. The $G_m$ action on $\mathbb{A}_{sm}(1,n)$ in \[\text{Lemma 11.1}\] is explicitly given by $\alpha \cdot f(x) = \alpha^{-2} f(x)$, though we will not need this in what follows.

**Proof.** First, we verify that $\text{Hur}_{2,g,k}$ is equivalent to the fibered category whose $S$-points parameterize pairs $(\mathcal{L}, i : \mathcal{L} \otimes 2 \to \mathcal{O}_{\mathbb{P}_1^5})$, for $\mathcal{L}$ a degree $-g - 1$ invertible sheaf on $\mathbb{P}_1^5$, and $i$ an injective homomorphism of sheaves. Indeed, to connect this to our given definition of $\text{Hur}_{2,g,k}$, we follow \[\text{[AV04, Rem. 3.3 and Prop. 3.1]}\] given a cover $\rho : H \to \mathbb{P}_1^5$, we have a natural action of $\mu_2$ on $H$ over $\mathbb{P}_1$. This comes from the isomorphism $\mu_2 \simeq \mathbb{Z}/2\mathbb{Z}$ as we are assuming $\text{char}(k) \neq 2$. From this action,
we obtain an isomorphism \( \rho_*\mathcal{O}_H \simeq \mathcal{O}_{\mathbb{P}^1_S} \oplus \mathcal{L} \), for \( \mathcal{L} \) the subsheaf on which \( \mu_2 \) acts by \( (t,s) \mapsto t \cdot s \), i.e., \( \mathcal{L} \) is the weight 1 eigenspace of \( \mu_2 \), and \( \mathcal{O}_{\mathbb{P}^1_S} \) is the weight 0 eigenspace. The description of \( \mathcal{L} \) as the weight 1 eigenspace for the \( \mu_2 \) action yields a map \( i : \mathcal{L} \otimes \mathcal{L} \to \mathcal{O} \). In the other direction, given \( \mathcal{L} \), we can recover \( L = \text{Spec}_{\mathcal{O}_{\mathbb{P}^1_S}}(\mathcal{O}_{\mathbb{P}^1_S} \oplus \mathcal{L}) \). The given maps respect automorphisms over \( \mathbb{P}^1 \), as the only nontrivial automorphism in both cases is given by the hyperelliptic involution. Hence, they define an equivalence of algebraic stacks.

Next, consider the cover \( \text{Hur}_{g,2,k} \) of \( \text{Hur}_{2,g,k} \) given as the stackification of the fibered category whose \( S \) points parameterize triples \( (\mathcal{L}, \phi : \mathcal{L} \simeq \mathcal{O}(-g-1), i : \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}) \), with \( i \) injective. Note that \( \text{Hur}_{2,g,k} \to \text{Hur}_{g,2,k} \) is indeed surjective because \( \mathcal{L} \simeq \mathcal{E}^\vee \) is a degree \( -g-1 \) line bundle on \( \mathbb{P}^1 \) by Lemma 6.3. Observe that \( \text{Hur}_{2,g,k} \) has a natural action of \( G_m \) acting by automorphisms of \( \mathcal{L} \), so that \( \text{Hur}_{d,g,k} = [\text{Hur}_{d,g,k}/G_m] \). Said another way, quotienting by \( G_m \) forgets the data of the isomorphism \( \phi \).

It remains to identify \( \text{Hur}_{2,g,k} \) with \( \mathbb{A}_{sm}(1,2g+2) \). Indeed, this was done in the course of the proof of [AV04, Thm. 4.1]. Briefly, given an \( S \)-point \((\mathcal{L}, \phi, i)\), associate the map \( i \circ (\phi^{-1})^\otimes : \mathcal{O}_{\mathbb{P}^1_S}(-2g-2) \to \mathcal{O}_{\mathbb{P}^1_S} \) corresponding to a section of \( H^0(\mathbb{P}^1_S, \mathcal{O}(2g+2)) \). Conversely, given a section \( f \in H^0(\mathbb{P}^1_S, \mathcal{O}(2g+2)) \), associate the triple \((\mathcal{O}(-g-1), i : \mathcal{O}(-g-1) \to \mathcal{O}(-g-1), f : \mathcal{O}(-g-1)^\otimes \to \mathcal{O})\).

We are now ready to prove Theorem 10.4 in the case \( d = 2 \).

11.3. Proof of \( d = 2 \) case of Theorem 10.4. Note that the only allowable collection of ramification profiles is \( \mathcal{R} = \{(2), (1, 1)\} \). Since \( \text{Hur}_{2,g,k} \simeq \mathbb{A}_{sm}(1,2g+2)/G_m \) by Lemma 11.1 and \( G_m \) is special, we have \( \{G_m\} \{G_m\} = \mathbb{A}_{sm}(1,2g+2) \). Since

\[
\{G_m\} \frac{\dim \text{Hur}_{2,g,k}}{1 - L^{-1}} \left( \prod_{x \in \mathbb{P}^1_k} \left( \sum_{R \in \mathcal{R}} L^{-r(R)} \right) \left( 1 - L^{-1} \right) \right) = \frac{L - 1}{1 - L^{-1}} \cdot L^{2g+2} \prod_{x \in \mathbb{P}^1_k} \left( 1 - L^{-2} \right)
\]

\[
= L^{2g+3} \frac{1}{Z_{\mathbb{P}^1_k}(L^{-2})}
\]

\[
= L^{2g+3} \left( 1 - L^{-1} \right) \left( 1 - L^{-2} \right)
\]

(by [BH21, Ex. 6.1.12] and [Bil17, §3.8, Property 4], as in the proof of Corollary 10.5) it suffices to verify

\[
\mathbb{A}_{sm}(1,2g+2) = L^{2g+3} \left( 1 - L^{-1} \right) \left( 1 - L^{-2} \right)
\]

Indeed, this follows from [VW15, Lem. 5.9(a)]. In a bit more detail, taking \( a = 2 \) in [VW15, Lem. 5.9(a)], the expression \( K_{<2}(t) \) there is the generating function for which the coefficient of \( t^n \) is the class of \( w_{1^n} \) in the notation of [VW15, (5.1)]. Here,
\(w_{1^n}\) is the class of the space of degree \(n\) reduced divisors on \(\mathbb{P}^1\). Therefore, \(w_{1^n} = \{[\mathcal{A}_{sm}(1, n)/G_m]\}\), and so we only need check \(\{w_{1^n}\} = L^n - L^{n-2}\). But indeed, this is the coefficient of \(t^n\) in the expansion of

\[
\frac{Z_{\mathbb{P}^1}(t)}{Z_{\mathbb{P}^1}(t^2)} = \frac{(1 - t^2L)(1 - t^2)}{(1 - tL)(1 - t)} = (1 - t^2L)(1 + t) \left( \sum_{i=0}^{\infty} (tL)^i \right).
\]

**Remark 11.4.** The construction above used to compute the class of \(\text{Hur}_{2, g, k}\) is admittedly fairly ad hoc in the context of this paper. A similar construction, more in line with the themes of this paper could be obtained by realizing a given hyperelliptic curve \(\rho : H \to \mathbb{P}^1\) as a subscheme of \(\mathbb{P}((\rho_*\mathcal{O}_H)^\vee)\). One can verify that \(\mathbb{P}(\rho_*\mathcal{O}_H)^\vee\) is fpwp locally isomorphic to \(\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(g + 1)\right)\), and use this to deduce that \(\text{Hur}_{2, g, k}\) is the quotient of the smooth members of a certain linear series on \(\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(g + 1))\) by the automorphisms of \(\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(g + 1)\right)\) preserving the projection to \(\mathbb{P}^1\), and then use this description to compute \(\{\text{Hur}_{2, g, k}\}\), obtaining a formula similar to that of Theorem 10.4. However, such a proof would only calculate the class in \(K_0(\text{Stacks}_k)\) modulo a certain codimension, as opposed to the proof we give here, which actually calculates the class in \(K_0(\text{Stacks}_k)\).

## Appendix A. A Proof of a Theorem of Ekedahl

By Aaron Landesman and Federico Scavia

The main result of this appendix is a proof of the following Theorem of T. Ekedahl. We retain the notation for the Grothendieck ring of stacks described in §1.20.

**Theorem A.1 (Ekedahl).** [Eke09b] Thm. 4.3 *Let \(k\) be a field. Then, for all integers \(n \geq 1\), \(\{BS_n\} = 1\) in \(K_0(\text{Stacks}_k)\).*

Unfortunately, Ekedahl passed away prior to publishing [Eke09b], and so the article was never refereed. There are a number of typos and errors appearing in the proof of [Eke09b], Thm. 4.3. The objective of this appendix is to point out the fixes necessary.

Let \(k\) be a field, let \(G\) be a finite group, and let \(V\) be a \(G\)-representation of dimension \(d \geq 0\) over \(k\). If \(H\) is a subgroup of \(G\), we denote by \(V^H\) the subscheme of \(V\) fixed by \(H\), and by \(V_H\) the locally closed subscheme parameterizing the locus whose stabilizer is exactly \(H\). If there is a point of \(V\) whose stabilizer is exactly \(H\), we call \(H\) a stabilizer subgroup of \(G\). The normalizer \(N_G(H)\) of \(H\) acts on \(V^H\) and \(V_H\), and \(V_H\) is an open subscheme of \(V^H\). By definition, a stabilizer flag of length \(n\) is a sequence

\[
f = (\{e\} =: H_0 \subset H_1 \subset \cdots \subset H_n)
\]

of subgroups of \(G\) such that, for all \(0 \leq i \leq n - 1\), \(H_{i+1}\) is a stabilizer subgroup of the \(G\)-action on \(V\). We say that \(f\) is strict if \(H_i \nsubseteq H_{i+1}\) for all \(i\). We set \(n_f := n\), \(H_f := H_n\), \(d_f := \text{dim} V^H_f\) and \(N_G(f) := \cap_{0 \leq i \leq n} N_G(H_i)\).
Remark A.2. Our definition of stabilizer flag differs from the one used by Ekedahl [Eke09b, p. 10], as he required that $H_{i+1}$ be a stabilizer subgroup of the action of $\cap_{i\leq i} N_G(H_i)$ on $V^{H_i}$. In particular, in our definition it is not necessarily true that $H_f \subset N_G(f)$.

The conjugation action of $G$ on itself induces a $G$-action on the collection of all stabilizer flags. We say that two stabilizer flags are conjugate to each other if they belong to the same orbit under this action.

Proposition A.3. Let $K \subset G$ be the kernel of the $G$-action on $V$. We have:

\[(A.1) \quad \{BG\} L^d = \{[V_k/G]\} - \sum_f (-1)^{n_f} \{BN_G(f)\} L^{d_f},\]

where $f$ runs over a set of representatives of conjugacy classes of strict stabilizer flags of length $n_f \geq 1$.

Proposition A.3 corrects [Eke09b] Thm. 3.4. The formula there looks the same as ours (up to signs), but it is wrong as it is claimed with a different definition of stabilizer flag. The error there stems from the falsity of [Eke09b, Lem. 3.3(iv)], as illustrated by the following example.

Example A.4. The result [Eke09b] Lem. 3.3(iv)] claims that $V_H = (V^H)_H$, where $V^H$ is considered as an $N_G(H)$ representation. However, when $G = S_3$ and $H$ is the subgroup generated by (12), and $G$ acts as the 3-dimensional permutation representation, then $V_H = \{(a,a,b) : a \neq b\}$, while $N_G(H) = H$ and $V^H = \{(a,a,b)\}$. So here, when $V^H$ is considered as an $N_G(H) = H$ representation, we have that $H$ acts trivially and $(V^H)_H = V_H \neq V_H$.

Since [Eke09b, Lem. 3.3(iv)] is implicitly used in the proof of [Eke09b] Thm. 3.4, [Eke09b, Thm. 3.4] is also incorrect. To produce a counterexample to the statement of [Eke09b, Thm. 3.4] (even after correcting the + sign appearing in the statement there to the − sign of (A.1)), we can again take $G = S_3$. Then, the only strict stabilizer flags in the sense of [Eke09b, p. 10] (which are defined in a slightly different way in this appendix) up to conjugacy are $\{e\}$, $\{e\} \subset S_2$, $\{e\} \subset S_3$. In this case, with our knowledge that $\{BS_3\} = 1$, the formula of [Eke09b, Thm. 3.4] claims $L^3 = (L^3 - L^2) + (L^2) + (L)$. Of course, what is missing from this formula is that we should subtract off a term $L$ coming from the sequence of subgroups $\{e\} \subset S_2 \subset S_3$, which is a stabilizer flag in the sense of this appendix, but not in the sense of [Eke09b, p. 10].

Proof of Proposition A.3. Let $f$ be a strict stabilizer flag. The complement of $[V_{H_f}/N_G(f)]$ in $[V^{H_f}/N_G(f)]$ is the disjoint union of locally closed substacks

$$\left[ \bigcup_{H \subset G} V_{gHg^{-1}/N_G(f)} \right] \cong [V_H/N_G(f) \cap N_G(H)],$$

where $H$ runs among a set of representatives of $N_G(f)$-conjugacy classes of subgroups of $G$ acting on $V$ and properly containing $H_f$. For any such $H$, construct a
strict stabilizer flag \( f' \) by appending \( H \) at the end of \( f \). Then
\[ N_G(f) \cap N_G(H) = N_G(f'). \]

We conclude
\[
(A.2) \quad \{[V_{H'}/N_G(f)]\} = \{BN_G(f)\} \mathbb{L}^d - \sum_{f'} \{[V_{H'/N_G(f')}\},
\]
where \( f' \) runs over a set of representatives of conjugacy classes of strict stabilizer flags of length \( n_f + 1 \) and starting with \( f \).

We now wish to prove by induction on \( m \geq 1 \) that
\[
(A.3) \quad \{BG\} \mathbb{L}^d = \{[V_K/G]\} - \sum_{0<n_f<m} (-1)^{n_f} \{BN_G(f)\} \mathbb{L}^d - (-1)^m \sum_{n_f=m} \{[V_{H'/N_G(f')}\},
\]
where \( f \) runs among a set of representatives of conjugacy classes of strict stabilizer flags. When \( m = 1 \), \( A.3 \) coincides with \( A.2 \) for \( f = (K) \). Assume now that \( A.3 \) holds for some \( m > 1 \). One obtains the formula for \( m + 1 \), by starting from the formula for \( m \) and applying \( A.2 \) to every flag \( f \) of length \( m \).

Since \( G \) is finite, there are only finitely many strict stabilizer flags. The conclusion follows by choosing \( m \) to be larger than the length of every strict stabilizer flag. \( \square \)

Having replaced [Eke09b Thm. 3.4] by Proposition A.3, the proof of Theorem A.1 can be completed as in [Eke09b]. From now on, let \( G = S_n \) be the group of permutations of \( \Sigma := \{1, 2, \ldots, n\} \), and let \( V \) be the \( n \)-dimensional permutation representation of \( S_n \).

A flag is a pair \((S, R)\), where \( S \) is a finite set, and \( R \) is a sequence \( R_1 \subset R_2 \subset \cdots \subset R_n \) of equivalence relations \( R_i \subset S \times S \) on \( S \). An isomorphism of flags \((S', R') \rightarrow (S, R)\) is a bijection \( S' \xrightarrow{\sim} S \) sending \( R'_i \) to \( R_i \) for all \( i \). We denote by \( N_R(S) \) the automorphism group of \((S, R)\).

**Lemma A.5.** Assume that \( G = S_n \) and that \( V \) is the standard \( n \)-dimensional representation of \( S_n \). Let \( f \) be a strict stabilizer flag, and denote by \( H_i \) the stabilizer subgroups appearing in \( f \). For every \( i \), let \( R_i \) be the equivalence relation determined by the orbit partition of the \( H_i \)-action on \( \Sigma \), and let \( R \) be the flag on \( \Sigma \) given by the \( R_i \).

(a) We have \( N_{S_n}(f) = N_R(\Sigma) \).

(b) If \( N_{S_n}(f) = S_n \), then either \( f = (\{e\}) \) or \( f = (\{e\}) \subset S_n \).

(c) Assume that \( \{BS_m\} = 1 \) for all \( m < n \) and that \( N_{S_n}(f) \neq S_n \). Then \( BN_{S_n}(f) = 1 \).

**Proof.** (a) This follows from the fact that, for every \( i \), a bijection \( \sigma \) of \( \Sigma \) respects \( R_i \) if and only if it normalizes \( H_i \).

(b) If \( N_{S_n}(f) = S_n \), then for every \( i \), \( R_i \) is respected by every bijection of \( \Sigma \). It follows that either \( R_i \) is the diagonal in \( \Sigma \times \Sigma \) or \( R_i = \Sigma \times \Sigma \). Now (b) follows from (a).

(c) We may assume that \( \{BN_{S_n}(f')\} = 1 \) for all flags such that \( n_{f''} < n_f \). By [Eke09b Prop. 4.2], \( N_{S_n}(f) \) is a direct product of wreath products \( N' \cap S_r := (N')^r \times S_r \), where \( N' \) is the normalizer of a flag of smaller length, and \( S_r \) acts by permutation.
of the $r$ factors $N'$. Because, for $G$ and $H$ finite groups, $B(G \times H) \simeq BG \times BH$, it suffices to show $\{B(N' \int S_r)\} = 1$ We have $B(N' \int S_r) \simeq BN' \int BS_r \simeq \text{Symm}'(BN')$, as explained in [Eke09b, p. 5], where the symbols $\int$ and Symm for stacks are introduced in [Eke09b, p. 5]. By inductive assumption, $\{B(N' \int S_r)\} = \sigma_s^t(\{BN'\}) = \sigma_s^t(1) = 1$. For the symbol $\sigma_s^t$, see [Eke09b, Prop. 2.5].

Proof of Theorem A.7 Let $V$ be the $n$-dimensional permutation representation of $S_n$, and let $U := V_{(e)} \subset V$ be the free locus of the $S_n$-action. By Proposition A.3,

$$\{BS_n\}L^n = \{U/S_n\} - \sum_f (-1)^{n_f} \{BS_n(f)\} \L^{d_f},$$

where $f$ runs among conjugacy classes of strict stabilizer flags. By Lemma A.5(b), we may rewrite this as

$$\{BS_n\}(L^n - L) = \{U/S_n\} - \sum_f (-1)^{n_f} \{BS_n(f)\} \L^{d_f},$$

where now $f$ runs among conjugacy classes of strict stabilizer flags such that $N_{S_n}(f) \neq S_n$. By Lemma A.5(c), we have $\{BN_{S_n}(f)\} = 1$ for all such $f$.

We claim that $\{U/S_n\}$ is a polynomial in $L$ with integer coefficients. The stacks $V_H/N_{S_n}(H)$ are isomorphic to parts of a locally closed stratification of $V/S_n$. This is well known from general principles when char $k = 0$ or when char $k > 0$ does not divide $n$, but Ekedahl gave a proof in arbitrary characteristic in [Eke09b, Prop. 1.1(ii)].

To show $\{U/S_n\}$ is a polynomial in $L$, let $f$ be a strict stabilizer flag. Then, as in the proof of Proposition A.3, we have

$$\{V_{H_f}/N_{S_n}(f)\} = \{V^{H_f}/N_{S_n}(f)\} - \sum_{f'} \{V_{H_f}/N_{S_n}(f')\},$$

where $f'$ runs among conjugacy classes of strict stabilizer flags starting with $f$ and of length $n_f + 1$.

Applying the previous formula iteratively, we obtain

$$\{V/S_n\} = \{U/S_n\} - \sum_f (-1)^{n_f} \{V^{H_f}/N_{S_n}(f)\},$$

where $f$ runs among conjugacy classes of strict stabilizer flags of positive length. For every flag $f$, we claim that the quotient $W_f := N_{S_n}(f)/(H_f \cap N_{S_n}(f))$ is a product of symmetric groups, and $V^{H_f}$ is a permutation representation of $W_f$. To see this, note that $N_{S_n}(f)$ can be identified with $N_{R_f}(\Sigma)$ via Lemma A.5 for a sequence of equivalence relations $R_f$ given as $R_1 \subset R_2 \subset \cdots \subset R_{n_f}$. Under this identification and $H_f$ is identified with the subgroup of permutations acting trivially on the equivalence classes defined by $R_{n_f}$. Therefore, the action of $W_f$ on $V^{H_f}$ is generated by permutations switching two equivalence classes of $R_{n_f}$ for which there exists an isomorphism of those two classes respecting $R$. Therefore, $W_f$ is a product of symmetric groups acting by a permutation representation on $V^{H_f}$. Hence, by
the fundamental theorem for symmetric polynomials, $V^H_f / N_{S_n}(f) = V^H_f / W_f$ is an affine space over $k$. Since $V / S_n$ is also isomorphic to affine space, we deduce that $\{U / S_n\}$ is a polynomial in $L$, as claimed. We conclude that $\{BS_n\}$ can be written as a rational function in $L$ with integer coefficients, and with denominator $L^n - L$. By [Eke09b, Lem. 3.5], this implies that $\{BS_n\} = 1$. □

REFERENCES

[Aok06a] Masao Aoki. Erratum: “Hom stacks” [Manuscripta Math. 119 (2006), no. 1, 37–56; mr2194377]. Manuscripta Math., 121(1):135, 2006.

[Aok06b] Masao Aoki. Hom stacks. Manuscripta Math., 119(1):37–56, 2006.

[ASVW21] S. Ali Altug, Arul Shankar, Ila Varma, and Kevin H. Wilson. The number of $D_4$-fields ordered by conductor. J. Eur. Math. Soc. (JEMS), 23(8):2733–2785, 2021.

[AV04] Alessandro Arzie and Angelo Vistoli. Stacks of cyclic covers of projective spaces. Compos. Math., 140(3):647–666, 2004.

[BD07] Kai Behrend and Ajneet Dhillon. On the motivic class of the stack of bundles. Adv. Math., 212(2):617–644, 2007.

[BE77] David A. Buchsbaum and David Eisenbud. Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3. Amer. J. Math., 99(3):447–485, 1977.

[Beh91] Kai Achim Behrend. The Lefschetz trace formula for the moduli stack of principal bundles. ProQuest LLC, Ann Arbor, MI, 1991. Thesis (Ph.D.)–University of California, Berkeley.

[BH21] Margaret Bilu and Sean Howe. Motivic Euler products in motivic statistics. Algebra Number Theory, 15(9):2195–2259, 2021.

[Bha04] Manjul Bhargava. Higher composition laws III: The parametrization of quartic rings. Annals of Mathematics, 159(3):1329–1360, May 2004.

[Bha05] Manjul Bhargava. The density of discriminants of quartic rings and fields. Annals of Mathematics, 162(2):1031–1063, September 2005.

[Bha08] Manjul Bhargava. Higher composition laws. IV. the parametrization of quintic rings. Annals of Mathematics. Second Series, 167(1):53–94, 2008.

[Bha10a] Manjul Bhargava. The density of discriminants of quintic rings and fields. Annals of Mathematics, 172(3):1559–1591, October 2010.

[Bha10b] Manjul Bhargava. Mass formulae for extensions of local fields, and conjectures on the density of number field discriminants. International Mathematics Research Notices, July 2010.

[Bha14] Manjul Bhargava. The geometric sieve and the density of squarefree values of invariant polynomials. arXiv preprint arXiv:1402.0031v1, 2014.

[Bil17] Margaret Bilu. Produits euleriens motiviques. PhD thesis, Université Paris-Saclay (ComUE), 2017.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. Néron models, volume 21 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1990.

[BST13] Manjul Bhargava, Arul Shankar, and Jacob Tsimerman. On the davenport–heilbronn theorems and second order terms. Inventiones mathematicae, 193(2):439–499, August 2013.

[BSW15] Manjul Bhargava, Arul Shankar, and Xiaoheng Wang. Geometry-of-numbers methods over global fields I: Prehomogeneous vector spaces. arXiv preprint arXiv:1512.03035v1, 2015.

[BTT21] Manjul Bhargava, Takashi Taniguchi, and Frank Thorne. Improved error estimates for the davenport-heilbronn theorems. arXiv preprint arXiv:2107.12819, 2021.
[BV12] Michele Bolognesi and Angelo Vistoli. Stacks of trigonal curves. *Trans. Amer. Math. Soc.*, 364(7):3365–3393, 2012.

[Cas96] Gianfranco Casnati. Covers of algebraic varieties II. Covers of degree 5 and construction of surfaces. *Journal of Algebraic Geometry*, 5(3):461–478, 1996.

[CDyDO02] Henri Cohen, Francisco Díaz y Díaz, and Michel Olivier. Enumerating quartic dihedral extensions of Q. *Compositio Mathematica*, 133(1):65–93, 2002.

[CE96] Gianfranco Casnati and Torsten Ekedahl. Covers of algebraic varieties I. A general structure theorem, covers of degree 3, 4 and Enriques surfaces. *Journal of Algebraic Geometry*, 5(3):439–460, 1996.

[CL21a] Samir Canning and Hannah Larson. Chow rings of low-degree hurwitz spaces. *arXiv preprint arXiv:2110.01059v1*, 2021.

[CL21b] Samir Canning and Hannah Larson. Tautological classes on low-degree hurwitz spaces. *arXiv preprint arXiv:2103.09902v2*, 2021.

[CN07] Gianfranco Casnati and Roberto Notari. On some Gorenstein loci in *Hilb*6(ℙ4k). *J. Algebra*, 308(2):493–523, 2007.

[Con00] Brian Conrad. *Grothendieck duality and base change*, volume 1750 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2000.

[Deo14] Anand Deopurkar. Compactifications of Hurwitz spaces. *Int. Math. Res. Not. IMRN*, (14):3863–3911, 2014.

[DF64] B. N. Delone and D. K. Faddeev. *The theory of irrationalities of the third degree*. Translations of Mathematical Monographs, Vol. 10. American Mathematical Society, Providence, R.I., 1964.

[DH69] H. Davenport and H. Heilbronn. On the density of discriminants of cubic fields. *Bulletin of the London Mathematical Society*, 1(3):345–348, November 1969.

[DH71] H. Davenport and H. Heilbronn. On the density of discriminants of cubic fields. II. *Proceedings of the Royal Society. London. Series A. Mathematical, Physical and Engineering Sciences*, 322(1551):405–420, 1971.

[DP15] Anand Deopurkar and Anand Patel. The Picard rank conjecture for the Hurwitz spaces of degree up to five. *Algebra Number Theory*, 9(2):459–492, 2015.

[DW88] Boris Datskovsky and David J. Wright. Density of discriminants of cubic extensions. *Journal für die Reine und Angewandte Mathematik*, 386:116–138, 1988.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[Eke09a] T. Ekedahl. The Grothendieck group of algebraic stacks. *unpublished (arXiv:0903.3143v2)*, 2009.

[Eke09b] Torsten Ekedahl. A geometric invariant of a finite group. *arXiv:0903.3148v1*, March 2009.

[EVW16] Jordan S. Ellenberg, Akshay Venkatesh, and Craig Westerland. Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields. *Ann. of Math. (2)*, 183(3):729–786, 2016.

[FGI+05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck’s FGA explained.

[FP02] B. Fantechi and R. Pandharipande. Stable maps and branch divisors. *Compositio Math.*, 130(3):345–364, 2002.

[Ful69] William Fulton. Hurwitz Schemes and Irreducibility of Moduli of Algebraic Curves. *Annals of Mathematics*, 90(3):542–575, 1969.

[Gro65] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.*, (24):231, 1965.

[Gro66] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.
[Gun] Joseph Gunther. Counting Cubic Curve Covers over Finite Fields.

[GZLM04] S. M. Gusein-Zade, I. Luengo, and A. Melle–Hernández. A power structure over the grothendieck ring of varieties. *Mathematical Research Letters*, 11(1):49–57, 2004.

[Hat11] Allen Hatcher. A short exposition of the madsen-weiß theorem. *arXiv preprint arXiv:1103.5223*, 2011.

[Mir85] Rick Miranda. Triple covers in algebraic geometry. *Amer. J. Math.*, 107(5):1123–1158, 1985.

[MW07] Ib Madsen and Michael Weiss. The stable moduli space of Riemann surfaces: Mumford’s conjecture. *Ann. of Math.* (2), 165(3):843–941, 2007.

[Ols16] Martin Olsson. *Algebraic spaces and stacks*, volume 62 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2016.

[Par89] R. Pardini. Triple covers in positive characteristic. *Arkiv för Matematik*, 27(1-2):319–341, December 1989.

[Poo08] Bjorn Poonen. The moduli space of commutative algebras of finite rank. *Journal of the European Mathematical Society*, pages 817–836, 2008.

[PV15] A. Patel and R. Vakil. On the Chow ring of the Hurwitz space of degree three covers of $\mathbb{P}^1$. *arXiv:1505.04323v1*, May 2015.

[Ryd11] David Rydh. Étale dévissage, descent and pushouts of stacks. *J. Algebra*, 331:194–223, 2011.

[Sch86] F.-O. Schreyer. Syzygies of canonical curves and special linear series. *Math. Ann.*, 275(1):105–137, 1986.

[Sta] The Stacks Project Authors. *Stacks Project*. [http://stacks.math.columbia.edu](http://stacks.math.columbia.edu)

[TT13] Takashi Taniguchi and Frank Thorne. Secondary terms in counting functions for cubic fields. *Duke Mathematical Journal*, 162(13):2451–2508, October 2013.

[VW15] Ravi Vakil and Melanie Matchett Wood. Discriminants in the grothendieck ring. *Duke Mathematical Journal*, 164(6):1139–1185, April 2015.

[Wew98] Stefan Wewers. *Construction of Hurwitz spaces*. IEM, 1998.

[Woo11] Melanie Matchett Wood. Parametrizing quartic algebras over an arbitrary base. *Algebra & Number Theory*, 5(8):1069–1094, 2011.

[Woo16] Melanie Matchett Wood. Asymptotics for number fields and class groups. In *Directions in number theory*, volume 3 of *Assoc. Women Math. Ser.*, pages 291–339. Springer, [Cham], 2016.

[WY92] David J. Wright and Akihiko Yukie. Prehomogeneous vector spaces and field extensions. *Inventiones mathematicae*, 110(1):283–314, December 1992.

[Zha13] Yongqiang Zhao. *On sieve methods for varieties over finite fields*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–The University of Wisconsin - Madison.

[Zhe21] Angelina Zheng. Stable cohomology of the moduli space of trigonal curves. *arXiv preprint arXiv:2106.07245v1*, 2021.