Impact of energetic particle orbits on long range frequency chirping of BGK modes

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Abstract

Long range frequency chirping of Bernstein–Greene–Kruskal modes, whose existence is determined by the fast particles, is investigated in cases where these particles do not move freely and their motion is bounded to restricted orbits. A nonuniform equilibrium magnetic field is included into the bump-on-tail instability problem of a plasma wave. The parallel field gradients account for the existence of different orbit topologies of energetic particles. With respect to fast particles dynamics, the extended model captures the range of particles motion (trapped/passing) with energy and thus represents a more realistic 1D picture of the long range sweeping events observed for weakly damped modes, e.g. global Alfvén eigenmodes, in tokamaks. The Poisson equation is solved numerically along with bounce averaging the Vlasov equation in the adiabatic regime. We demonstrate that the shape and the saturation amplitude of the nonlinear mode structure depends not only on the amount of deviation from the initial eigenfrequency but also on the initial energy of the resonant electrons in the equilibrium magnetic field. Similarly, the results reveal that the resonant electrons following different equilibrium orbits in the nonuniform field lead to different rates of frequency evolution. As compared to the previous model (Breizman 2010 Nucl. Fusion 50 084014), it is shown that the frequency sweeps with lower rates. The additional physics included in the model enables a more complete 1D description of the range of phenomena observed in experiments.

Keywords: wave-particle-plasma interactions, nonlinear BGK waves, frequency sweeping in plasmas, long range chirping waves

(Some figures may appear in colour only in the online journal)
explosive formation stage is one of the key results in [11] to be taken into consideration. It should be mentioned that holes and clumps form not only in case of a weakly unstable mode but also with any amount of background dissipation [13]. The Berk–Breizman scenario has been proved to be successful in explaining the frequency chirping events observed in experiments with AEs [14, 15]. Moreover, the effect of different types of relaxation processes on the nonlinear evolution has been investigated in [16] and [17], with the BOT code introduced in the latter. All the mentioned models are based on the assumption that the range of frequency chirping is short and the mode structure is fixed.

However, experimental evidence exists for mode activities in which the frequency shifts are as large as the initial eigen-frequency itself [18–20]. As the mode amplitude saturates due to flattening of the distribution function of the energetic particles, the physical picture of each evolving phase-space structure is a BGK mode whose frequency changes in time and its structure is notably affected by the frequency shift. Recently, a nonperturbative model based on the adiabatic description of the fast particles contribution has been developed by Breizman [21] using a 1D BOT instability to interpret the long range chirping for an isolated nonlinear resonance. This approach is premised on the assumption that the width of the separatrix supported by the BGK mode is small compared with the characteristic width of the unperturbed distribution function. The Breizman model remains valid as long as the separatrix of the energetic particles inside the clump shrinks for a downward shift in the frequency. As an extension, the adiabatic description of treating an expanding separatrix which traps the ambient particles is presented in [22] by Nyqvist and Breizman.

In magnetized plasmas, e.g. magnetic confinement devices, the particles gyrate about the magnetic field lines and follow certain trajectories depending on their energy and the magnetic field inhomogeneity. Therefore, the impact of particle orbits on the long range frequency sweeping events, should also be investigated in order to better understand and control these instability-driven phenomena. A physical system where the energetic particles are not moving freely and their equilibrium motion is bounded to certain orbits, enables such an investigation through a 1D picture. This physical model is the subject of this paper. We add a fixed nonuniform equilibrium magnetic field to the BOT problem presented in [21], thus creating an energy-dependence of the particle oscillation frequency through the mirror effect of the parallel field gradients. In this new model, the equilibrium field is pointed mainly in the z-direction and has $B_0 = 0$. For a 1D representation, we consider only the axis ($r = 0$) of this magnetic mirror system and represent the magnetic field by

$$B = B_c - B_0 \cos(k_{eq}z),$$  

(1)

where $k_{eq}$ is the spatial frequency of the magnetic field. We assume that the energetic particle confinement is due to the confinement of a single magnetic moment ($\mu$) and treat the chirping of an unstable mode which has a low eigen-frequency compared to the ion cyclotron frequency and its wavelength is large compared to the electron Larmor radius of the resonant electrons. Therefore, the constants $B_0$ and $B_c$ are determined by

$$B_c - B_0 \gg \left(\frac{m_i\omega_{pe}}{\lambda_p}, \frac{m_e v_\perp}{e\lambda_p}\right),$$  

(2)

where $m_i$ and $m_e$ are the ion and electron mass, respectively, $\omega_{pe}$ is the electron plasma frequency, $Z_i$ is the number of ion charges, $e$ is the elementary charge, $v_\perp$ is the velocity of fast particle perpendicular to the magnetic field and $\lambda_p$ is the wavelength of the perturbed mode. In this new model, the unperturbed guiding center motion of the fast particles in the equilibrium field is governed by the following orbit-averaged Littlejohn’s Hamiltonian [23]:

$$H_0 = \frac{p_z^2}{2m_e} - \mu B_0 \cos(k_{eq}z) + \mu B_c,$$  

(3)

where $p_z$ is the energetic particles momentum in the z-direction and it is assumed $A_z = 0$ with $A$ the vector potential. The energetic particles interacting with the perturbed field are considered as trapped or passing in this magnetic mirror system, depending on their pitch angle. Figure 1, whose construction is detailed at the end of subsection 2.1, demonstrates the behavior of the equilibrium oscillation frequency of the fast electrons versus their energy. For each frequency of trapped particles motion in the equilibrium magnetic field, there exists a group of passing particles having the same frequency of the equilibrium motion. Hence, the mode can be simultaneously in resonance with both the trapped and passing electrons in this equilibrium field. This trapped and passing locus model resembles the trapped particles following the banana orbits and the passing particles in the magnetic field lines of a tokamak (see section 5). In addition to enabling the impact of particle orbits on the long range chirping of BGK modes, the contribution from different resonances can also be investigated through the energy dependence.

The nonlinear wave equation is expanded using Fourier decomposition which allows us to find an explicit expression
for the Hamiltonian of the fast particles motion in terms of the action-angle variables of the unperturbed motion. This expansion, together with treating the kinetic equation adiabatically, allows us to implement a numerical treatment to investigate the impact of particle orbits on the structure and the sweeping rate of the nonlinear wave.

In section 2, the basic system of equations adopted for the analysis and the dynamic equations of the unperturbed motion are presented, followed by the derivation of the linear growth rate, the equation for the BGK mode structure and the chirping rate. The numerical scheme used for solving the equations is assigned to section 3. Section 4 presents the results in the regions where the adiabatic invariant of the trapped particles in the BGK mode decreases (the separatrix shrinks) during chirping and the effect of the electrons equilibrium orbit on the nonlinear evolution of the mode. Finally, section 5 contains concluding remarks.

2. The model

In this extended 1D BOT model, we study a purely electrostatic mode in a plasma consisting of static background ions, cold electrons responding linearly to the mode and fast electrons which are trapped and co/counter-passing in the nonuniform magnetic field and are in resonance with the electrostatic mode. We focus on propagation and dynamics parallel to the form magnetic field and are in resonance with the electrostatic mode in a plasma consisting of static background ions, cold electrons responding linearly to the mode and fast electrons, which is treated through the Vlasov equation and the dynamic equations of the unperturbed motion (7a), (7b) and (7c) where \( \alpha \) is the energy parameter given by

\[
\alpha = \frac{E + \mu (B_0 - B_c)}{2 \mu B_0}
\]

and \( \pi(\zeta) \) and \( \Xi(\zeta) \) are the complete elliptic integral of the first and second kind, respectively, given by

\[
\pi(\zeta) = \int_0^\frac{\pi}{2} \frac{d\phi}{\sqrt{1 - \zeta \sin^2 \phi}},
\]

\[
\Xi(\zeta) = \int_0^\frac{\pi}{2} \sqrt{1 - \zeta \sin^2 \phi} d\phi.
\]

Using the canonical equations of motion, the frequency of the motion reads

\[
\Omega_{\alpha=T} = \frac{\partial H_{\alpha=T}}{\partial J_{\alpha=T}} = \frac{k_{eq} \pi \mu B_0}{2 \pi(\zeta) \sqrt{\mu B_0 k_{eq} B_0}},
\]

\[
\Omega_{\alpha=P} = \frac{k_{eq} \pi \mu B_0 \sqrt{\zeta}}{\pi(\zeta^{-1}) \sqrt{\mu B_0 k_{eq} B_0}}.
\]

The behavior of these frequencies (shown in figure 1) is similar to the bounce and transit frequency of the guiding center motion in tokamaks [24].

2.1. Fast particles orbits and dynamics

For the completely integrable system consisting of trapped and co/counter-passing particles whose motion is governed by the Hamiltonian presented in equation (3), it is possible to transform canonically from the variables \((z, p_z)\) to action–angle variables \((\theta, J_\alpha)\), written as

\[
J_{\alpha=T} = \frac{1}{2\pi} \int p_z dz,
\]

\[
= \frac{2}{\pi} \int_0^{\infty} \sqrt{2m_e (E - \mu B_c + \mu B_0 \cos (k_{eq} z))} dz,
\]

\[
= \frac{8 \sqrt{m_e \mu B_0}}{k_{eq} \pi} [\zeta - 1, \zeta + \Xi(\zeta)]
\]

\( J_{\alpha=P} = \frac{1}{2\pi} \int_0^\lambda p_z dz = \frac{4 \sqrt{\mu B_0}}{k_{eq} \pi} \Xi(\zeta^{-1}) \)

where \( J_\alpha \) is the action for the unperturbed motion of the fast particles, \( z_{max} \) is determined by \( p_z = 0 \) using equation (3), \( \lambda \) is the wavelength of the equilibrium field, \( E \) is the unperturbed energy denoting the orbits, \( \zeta \) is the energy parameter given by

\[
\zeta = \frac{E + \mu (B_0 - B_c)}{2 \mu B_0}
\]

and \( \pi(\zeta) \) and \( \Xi(\zeta) \) are the complete elliptic integral of the first and second kind, respectively, given by

\[
\pi(\zeta) = \int_0^\frac{\pi}{2} \frac{d\phi}{\sqrt{1 - \zeta \sin^2 \phi}},
\]

\[
\Xi(\zeta) = \int_0^\frac{\pi}{2} \sqrt{1 - \zeta \sin^2 \phi} d\phi.
\]

2.2. The linear growth rate

In this subsection, we investigate the linear interaction between the plasma mode and the fast particles that are trapped and co/counter-passing in the equilibrium magnetic
field. For a traveling wave solution, the general form of the physical quantities can be represented as

\[ U = \sum_{\omega} \exp \left[ \frac{i}{\hbar} \left( k_p z - \omega t \right) \right] + c.c. = \sum_{\omega} \exp \left[ \frac{i}{\hbar} \left( k_p z - \omega t \right) \right] + c.c. \]

where \( k_p \) is the wave-number of the plasma mode, \( \omega \) the complex frequency, and \( c.c. \) stands for the complex conjugate.

Now we substitute the relevant terms into (4c) and (4d) to find the dispersion relation of the mode given by

\[ \frac{c_0 k_p m_e}{e^2} \left( 1 - \frac{\omega^2}{\omega_p^2} \right) = \sum_\omega \int \sum_\beta \frac{\delta F_{eq,\alpha}}{\delta \Omega_{\nu,\alpha}} V_{\alpha,\nu}^{2} \right | \Omega_{\nu,\alpha} = \frac{\omega}{\omega_p} \right \}

Neglecting the small contribution of the principal value which does not modify the real part of the frequency significantly, allows us to set \( \omega = \omega_p \). Assuming \( \gamma = \omega_p \) (the wave evolves slowly compared with \( \omega_p^{-1} \)), equation (14) can be solved for \( \omega \). Consequently, the linear growth rate is found to be

\[ \gamma = \frac{\omega_p \pi e^2}{2c_0 k_p m_e} \sum_\alpha \sum_\beta \left[ \frac{\delta F_{eq,\alpha}}{\delta \Omega_{\nu,\alpha}} V_{\alpha,\nu}^{2} \right | \Omega_{\nu,\alpha} = \frac{\omega}{\omega_p} \right \}

which involves summing the contribution from all the resonances denoted by \( p \). Equation (15) is a function of the energy parameter (\( \zeta \)). This indicates the dependency of the linear growth rate on particle orbits (see figure 3). It should be noted that the contribution from the counter-passing electrons in the equilibrium field is much less than the co-passing ones. This can be shown by changing \( z \) to \(-z\) in equation (9) and evaluating the corresponding values of the coupling strength for counter-passing electrons numerically.

### 2.3 Nonlinear BGK modes

Nonlinear frequency chirping can occur in unstable systems both near or far from marginal stability, in the absence of collisions. For a near-threshold instability, the presence of dissipation leads to the formation of an unstable plateau in the distribution function of the energetic electrons which supports sideband oscillations that finally evolve into chirping modes [11, 13]. In this case, the chirping mode emerges nearly immediately near the marginal stability. However, for the case of a far from threshold instability, the system is so unstable that many modes are likely to be excited. If modes are comparable in frequency with overlapping eigenfunctions, this may lead to mode overlap. Simple chirping can however naturally occur in experiment when the system first goes unstable where there is only a discrete number of unstable modes that can arise from a near continuum of damped modes. Accordingly, we consider the case of a near-threshold instability. The condition \( \frac{k_p}{m_e} < \omega_p^2 \), with \( \omega \) the frequency of the BGK mode and \( \omega_p \) the bounce frequency of trapped electrons in this mode, ensures the existence of a trapping structure with a hole/clump in the phase-space of energetic particles. After development, the time scale of the motion of already established holes and clumps is much longer than the time scale of the energetic particles motion trapped in the BGK mode, i.e. \( \omega_p^{-1} \).
In the present model, we focus on the adiabatic description of nonlinear BGK modes and construct our formalism based on the limit \( \frac{\delta n_{\alpha}}{\delta t} \ll w_0^2 - \gamma^2 \sim \gamma^2, \) where the kinetic equation can be found to find the perturbed distribution function of the fast electrons. The adiabatic limit should, in general, be checked if it remains valid as the frequency deviates from the initial eigenfrequency [28–30].

Adopting a Fourier expansion for the periodic structure, the electrostatic energy of the nonlinear BGK mode can be written in the form

\[
U[z, t] = \sum_n A_n(t) \cos \left[ n (k_p z - \phi(t)) \right],
\]

where the Fourier coefficients \( A_n(t) \) evolve on a slow time scale but the periodic behavior of the BGK mode represents rapid oscillations with a time scale on the order of the inverse plasma frequency. The motion of the fast electrons can be investigated using the following Hamiltonian

\[
H_n = \mathcal{H}_{0, n} (J_\alpha) + \frac{1}{2} \sum_n \sum_p A_n(t) \times V_{\alpha,n,p} \exp \left[ i (p \theta - n \phi(t)) \right] + \text{c.c.},
\]

where \( \mathcal{H}_{0, n} (J_\alpha) \) is the extremum value of the BGK mode energy. The value of \( \frac{\partial^2 H_{n, \alpha}}{\partial J_{n, \alpha}} \) at \( J_{\alpha} = J_{\text{res}, \alpha}(t) \) (denoted by \( \Delta_\alpha \)) can be negative or positive for the trapped or passing electrons in the equilibrium field, respectively. Mathematically, this affects \( U_{\alpha, \text{ext}} \) in order to have a positive value under the square root in equation (20) and from the physical point of view, it shows that the passing electrons in the magnetic field are trapped in the energy well of the BGK mode, while the trapped electrons in this field are trapped in the energy hill of the BGK mode. This implies

\[
U_{\alpha, \text{ext}} = \begin{cases} \frac{\partial \mathcal{H}_{\alpha}}{\partial J_{\alpha}} \bigg|_{J_{\alpha} = J_{\text{res}, \alpha}(t)} = \frac{\partial \mathcal{H}_{\alpha}}{\partial J_{\alpha}} \bigg|_{J_{\alpha} = J_{\text{res}, \alpha}(t)} & \text{if } \Delta_\alpha > 0, \\ \frac{\partial \mathcal{H}_{\alpha}}{\partial J_{\alpha}} \bigg|_{J_{\alpha} = J_{\text{res}, \alpha}(t)} = \frac{\partial \mathcal{H}_{\alpha}}{\partial J_{\alpha}} \bigg|_{J_{\alpha} = J_{\text{res}, \alpha}(t)} & \text{if } \Delta_\alpha < 0. 
\end{cases}
\]

Phase-space trajectories of constant energy for the motion of energetic particles in the BGK mode are plotted in figure 2. It is shown that the separatrix supported by the nonlinear mode corresponding to the electrons trapped in the equilibrium magnetic field (figure 2(b)) has a phase shift of \( \pi \) with respect to the separatrix related to the passing group (figure 2(a)).

As the separatrix moves adiabatically, the phase-space area enclosed by the trajectories of the deeply trapped particles in the nonlinear wave, i.e. the shaded areas in figure 2, is conserved. Without trapping or detrapping over this region, the aforementioned conservation ensures that the value of the distribution function is conserved. The separatrix moves the trapped electrons in the BGK mode while the passing electrons are affected through the direction of their motion [21]. The adiabatic invariant of the motion of these electrons in the BGK mode reads (see appendix B for more details)

\[
I_\alpha = 2 \int_0^{2\pi} \left[ \left( \mathcal{K}_\alpha - \frac{1}{2} \sum_n A_n(t) V_{\alpha,n,n} \times \exp \left[ i p \theta + c.c. \right] \right)^2 \right] \frac{2}{\Delta_\alpha} \, d\theta.
\]

Substituting expression (16) into equation (4a) gives

\[
- \sum_n A_n(t) n^2 k_p^2 \cos [n (k_\perp - \phi(t))] = - \frac{e^2}{\epsilon_0} \times \left[ \frac{1}{m_e} \sum_\alpha \int_{-\infty}^{\infty} \hat{f}_\alpha(z, p_z) \, dp_z + \delta n_\alpha \right].
\]

where \( \delta n_\alpha \) can be derived under the linear response assumption of the bulk electrons. Similar to subsection 2.2, we multiply equation (23) by \( \cos [n (k_p z - \phi(t))] \) and integrate over one wavelength. We also write all the physical quantities in the fast particle term in terms of the new action-angle variables. The new Hamiltonian is then written in the form

\[
H_{n, \alpha} = \frac{1}{2} \sum_n \sum_p A_n(t) \times V_{\alpha,n,p} \exp \left[ i (p \theta - n \phi(t)) \right] + \text{c.c.},
\]

where \( \mathcal{H}_{0, n} (J_\alpha) \) is the extremum value of the BGK mode energy. The value of \( \frac{\partial^2 H_{n, \alpha}}{\partial J_{n, \alpha}} \) at \( J_{\alpha} = J_{\text{res}, \alpha}(t) \) (denoted by \( \Delta_\alpha \)) can be negative or positive for the trapped or passing electrons in the equilibrium field, respectively. Mathematically, this affects \( U_{\alpha, \text{ext}} \) in order to have a positive value under the square root in equation (20) and from the physical point of view, it shows that the passing electrons in the magnetic field are trapped in the energy well of the BGK mode, while the trapped electrons in this field are trapped in the energy hill of the BGK mode. This implies

\[
U_{\alpha, \text{ext}} = \begin{cases} \frac{\partial \mathcal{H}_{\alpha}}{\partial J_{\alpha}} \bigg|_{J_{\alpha} = J_{\text{res}, \alpha}(t)} = \frac{\partial \mathcal{H}_{\alpha}}{\partial J_{\alpha}} \bigg|_{J_{\alpha} = J_{\text{res}, \alpha}(t)} & \text{if } \Delta_\alpha > 0, \\ \frac{\partial \mathcal{H}_{\alpha}}{\partial J_{\alpha}} \bigg|_{J_{\alpha} = J_{\text{res}, \alpha}(t)} = \frac{\partial \mathcal{H}_{\alpha}}{\partial J_{\alpha}} \bigg|_{J_{\alpha} = J_{\text{res}, \alpha}(t)} & \text{if } \Delta_\alpha < 0. 
\end{cases}
\]
variables \((\tilde{\theta}, \tilde{J})\). After substituting the Fourier expansion of \(\cos \left[ n \left( k_z (\tilde{\theta}, \tilde{J}) - \phi (t) \right) \right]\) and neglecting the highly oscillating terms one finds

\[
A_n(t) = \frac{1}{2 \pi \theta_{kn_c} \omega_{pe}^2} \left[ \frac{\omega^2}{n^2 \omega^2 - 1} \right] \sum_{\alpha} \int_0^{2\pi} d\tilde{\theta} \int_0^{\infty} \left[ \tilde{f}_n(\tilde{\theta}, \tilde{J}) \times V_{\alpha,n,n} \exp (i \tilde{\theta}) + c.c. \right] |J| d\tilde{J},
\]

where the Jacobian of the canonical transformation \((z, p) \leftrightarrow (\tilde{\theta}, \tilde{J})\) is unity and \(\omega = \frac{\omega_{pe}}{\sqrt{n}}\) is the normalized frequency with respect to the initial electron plasma frequency.

In this model, the phase-space density of the fast electrons (the distribution function) is assumed to be the same inside the narrow shrinking separatrix supported by the BGK mode, the so-called top-hat model. The perturbed part of the fast electrons distribution function dominated by the trapped electrons inside the separatrix [21] is calculated using the bounce averaging method described in appendix B,

\[
\tilde{f}_n = \begin{cases} 
0, & \text{passing in BGK} \\
F_{eq, \alpha} \left( J_{res} (t = 0) \right) - F_{eq, \alpha} \left( J_{res} (t) \right), & \text{trapped in BGK} \end{cases}
\]

Using the above expression, equation (24) transforms into

\[
A_n(t) = \frac{\omega^2}{2 \pi \theta_{kn_c} \omega_{pe}^2} \left[ \frac{\omega^2}{n^2 \omega^2 - 1} \right] \sum_{\alpha} \int_0^{2\pi} d\tilde{\theta} \left[ V_{\alpha,n,n} \exp (i \tilde{\theta}) + c.c. \right] \Delta J_{\alpha,\max} (\tilde{\theta}),
\]

where \(\Delta J_{\alpha,\max} (\tilde{\theta})\) is the width of the separatrix. Using equation (20), we have

\[
A_n(t) = \frac{\omega^2}{\pi \theta_{kn_c} \omega_{pe}^2} \left[ \frac{\omega^2}{n^2 \omega^2 - 1} \right] \sum_{\alpha} \left[ F_{eq, \alpha} (t = 0) - F_{eq, \alpha} (t) \right] \times \int_0^{2\pi} \left[ \left( U_{\alpha,\text{ext}} - \frac{1}{2} \sum_n A_n(t) V_{\alpha,n,n} \exp (i \tilde{\theta}) + c.c. \right) \frac{2}{\Delta \alpha} \right] d\tilde{\theta}.
\]

The above equation can be solved numerically to derive the Fourier coefficients with which we can construct the structure of the plane wave. The numerical method used is presented in section 3.

The trapped electrons in the BGK mode travel in phase-space together with the nonlinear mode. Depending on whether the clumps are trapped or passing in the equilibrium field, their energy increases or decreases respectively with decreasing frequency of the mode and vice versa for the holes. Hence, formation of a hole in the distribution function of trapped particles in the equilibrium field accompanies a clump in the distribution of passing ones and vice versa. The change in the perturbed potential energy of the trapped electrons in the BGK mode is relatively small compared to the change in their equilibrium energy when the change in \(J_{res,\alpha} (t)\) is greater than the change in the separatrix width (see appendix C for more details). More energy is released by the fast particles via the motion of the phase-space structures than in the process of their formation and the released energy during chirping should compensate the dissipated energy in the bulk. The total amount of power released corresponding to the change of the structure energy is given by

\[
P_t = - \sum_{\alpha} N_{\alpha} \frac{dE_{\alpha}}{dt},
\]

where \(N_{\alpha}\) is the total number of each group of electrons in the hole/clump, \(\frac{dE_{\alpha}}{dt} = \Omega_{\alpha} \left( \frac{d\Omega_{\alpha}}{dt} \right) \frac{1}{\omega_{pe}^2} \frac{d\omega_{pe}^2}{dt}\) is the rate of change of the energy of each particle and the resonance condition allows setting \(\Omega_{\alpha} = \omega (t)\). Regarding to the definition of the adiabatic invariant of the trapped particles, \(N_{\alpha}\) can be calculated as

\[
N_{\alpha} = \frac{2}{m_e} \left[ F_{eq, \alpha} (t = 0) - F_{eq, \alpha} (t) \right] \times \int_0^{2\pi} \left[ \left( U_{\alpha,\text{ext}} - \frac{1}{2} \sum_n A_n(t) V_{\alpha,n,n} \exp (i \tilde{\theta}) + c.c. \right) \frac{2}{\Delta \alpha} \right] d\tilde{\theta}.
\]
equation of motion (4c) and considering the collisional term, we have

\[ P_d = \frac{2\pi \nu k_p^2 \langle \dot{U}^2 \rangle}{\omega^2 m_e}, \]  

(30)

where \( \langle \rangle \) denotes averaging over one wavelength and \( \langle \dot{U}^2 \rangle = \frac{1}{2} \sum_n A_n^2(t) \). The released power during the motion of the holes/clumps is equal to the power dissipated in the bulk through collisions. This power balance can be used to calculate the rate at which sweeping occurs, which results in

\[ \frac{d\omega(t)}{dt} = -\frac{\nu n_e \pi k_p^2}{\omega^3 m_e} \sum_n A_n^2(t) \frac{1}{\sum_{\alpha} N_{\alpha} \left( \frac{dA_n}{d\alpha} \right)^2}. \]  

(31)

3. Numerical scheme

In this section, we first derive the equation of the mode structure at early stage of chirping, say \( t_0 \), considering only the contribution from the trapped electrons in the equilibrium magnetic field. It is worth noting that here the contribution of passing electrons in the magnetic field to the equations of early stage is arbitrarily neglected just for the purpose of normalization. A simple evaluation of equation (27) at initial phase of sweeping when \( F_{eq,0}(t = 0) = F_{eq,0}(t = 0) \) and \( \omega = 1 \) (a sinusoidal mode structure) is and presented by

\[ A_{1,0} = -\left[ \frac{8\omega_e^2 P_{k_p} }{3\pi k_p n_e \sqrt{|\Delta T|}} \right] V_{T,1,1,0} \sqrt{A_{1,0} V_{T,1,1,0}}. \]  

(32)

Here, we have used the subscript 0 to denote evaluation at \( t = t_0 \). The term \( A_{1,0} \) can be expressed in terms of the linear growth rate to have

\[ A_{1,0} = \frac{16^2 \gamma}{9 |\Delta T|} V_{T,1,1,0} \pi. \]  

(33)

We also get \( \dot{A}_n(t) = A_n(t) / A_{1,0} \), \( \tilde{V}_{a,n}(t) = V_{a,n}(t) / V_{T,1,1,0} \), \( \tilde{U}_{a,ext} = U_{a,ext} / A_{1,0} V_{T,1,1,0} \), and \( F_{eq,0}(t) = c_{a} \tilde{V}_{a,0} \). Normalizing equation (27) with respect to \( A_{1,0} \) results in

\[ \dot{\tilde{A}}_n(t) = \left[ \frac{-3 \gamma^2}{8c_T \left( \frac{\partial \tilde{U}}{\partial T} \right)_{\omega = 1} (n^2 \tilde{\omega}^2 - 1)} \right] \sum_{\alpha} c_{a} \left[ \tilde{\omega}_0 - \tilde{\omega}_\alpha \right] \times \int_0^{2T} \left[ \tilde{U}_{a,ext} - \frac{1}{2} \sum_n \tilde{A}_n(t) \tilde{V}_{a,n} \exp \left( i n \tilde{\theta} \right) \right] + \text{c.c.} \frac{2}{T_{\alpha}} \left[ \tilde{V}_{a,n}(t) \exp \left( i n \tilde{\theta} \right) + \text{c.c.} \right] d\tilde{\theta}, \]  

(34)

which can be solved iteratively to derive the Fourier coefficients. In order to avoid the singularity in the numerical approach, a special treatment is applied to the first coefficient when the values of \( \omega \) are close to \( \omega_{pe} \). In this case, \( \tilde{\omega}_0(t) \) can be linear-approximated around the initial plasma frequency to cancel the effect of the pole in the denominator of equation (34).

Likewise, differential equation (31) can be investigated for the early phase of the structures motion in phase-space considering only the effect of trapped particles in the magnetic field. Substituting expression (29) into differential equation (31) and using equations (15) and (33), one finds

\[ \frac{d}{dt} \left( \frac{\omega - \omega_{pe}}{\omega_{pe}} \right)^2 = \frac{1}{3 \pi^2 \omega_{pe}} \frac{\nu}{\nu}. \]  

(35)

We define the dimensionless time \( \tau = \frac{3}{\pi^2} \left( \frac{16 T_{pe}}{\omega_{pe}} \right)^2 t \) and multiply differential equation (31) by \( \frac{1}{\nu} \) to have

\[ \frac{d\dot{\omega}}{d\tau} = -\left[ \frac{4}{\omega^3} \sum_{\alpha} \text{sgn}_{\alpha} \frac{\dot{A}_{\alpha,0}^2}{\rho_{\alpha,0}^0} \sum_n \tilde{A}_n^2 \right] \times \left\{ \int_0^{2\pi} \left[ (\nabla_{a,ext} - \frac{\dot{A}_n}{2}) \tilde{V}_{a,n}(t) \exp \left( i n \tilde{\theta} \right) + \text{c.c.} \right] \frac{2}{T_{\alpha}} \left[ \tilde{V}_{a,n}(t) \exp \left( i n \tilde{\theta} \right) + \text{c.c.} \right] d\tilde{\theta} \right\}, \]  

(36)

where \( \text{sgn}_{\alpha} = -1 \) and 1 for \( \alpha = T \) and \( P \), respectively. The above equation can be solved by a fourth-order Runge-Kutta method along with the iterative method used for solving the Fourier coefficients on the RHS.

If the electrons have small enough pitch angles (deeply passing electrons with \( \tilde{\omega} \gg 1 \)), their motion will not be affected by the equilibrium magnetic field and they move freely. In other words, \( \theta = k_{eq} \) (see figure 5(b) for \( \theta = 2 \)). Subsequently, only one resonance is non-zero and the orbit averaged mode amplitude is equal to unity (see figure 4(b)) under this condition. In this high energy range, one can find that \( k_{pz} = p \theta \) in the linear theory limit. Canonical equations of motion assure \( \tilde{\omega} = \Omega_{a,0} = 0 \) so using equation (12), the resonance condition becomes \( \omega = k_{pz} \), where \( v \) is the particle velocity. Consequently, solving equation (34) and differential equation (36) in the limit that \( \tilde{\omega} \gg 1 \), reproduces exactly the same results as in [21], which serves as the benchmark of the code and the numerical approach.

4. Results

For illustration, we have arbitrarily restricted attention to cases where \( k_p = k_{eq} \). In the linear regime, the plasma mode will grow at different rates depending on the initial orbits of the electrons interacting with the mode. Figure 3 demonstrates that the linear growth rate decreases to zero in the limit of having resonance with the particles close to the separatrix of the equilibrium motion.

As in subsection 2.3, the first resonance (\( l = 1 \)) is considered as the dominant resonance contributing to the interaction. The first four elements of the orbit averaged mode amplitude \( \tilde{V}_{a,n,p} \), indicating the coupling strength, corresponding to the
first ($\dot{V}_{\alpha, n, R}$) and the second ($\dot{V}_{\alpha, n, 2R}$) resonance are plotted in figure 4 versus energy parameter by numerical integrating of equation (9) over $\theta$. Investigation of figure 4 shows that there are regions (adjacent to $\zeta = 1$) where the values of the dominant element ($n = 1$) belonging to the second resonance overtake the values of the dominant element of the first resonance. In itself, this may indicate that the corresponding second resonance is dominant. However, consideration of the linear growth rate for different resonances shows that the first resonance is dominant. This can be understood by inspection of equation (15): the term $\frac{d\Omega}{d\zeta}$ increases with increasing the resonance, so $\gamma$ decreases with increasing resonance. In addition, evaluating the factors of equation (34) for higher resonances ($l \geq 2$) shows that it always the first resonance ($l = 1$) that has dominant contribution to the interaction in the hard nonlinear regime. Therefore, the submissive resonances are neglected. The other important point concerning the coupling strength is that all of its elements go asymptotically to zero as the energy parameter of the electrons approaches unity. Here, we explain this phenomenon in more detail: the equations (A.6) and (A.12) describe the equilibrium position ($z$) of the electrons in terms of the action–angle variables in the nonuniform magnetic field. Figure 5 illustrates this position at different times for different energy parameters. For the case of trapped (figure 5(a)) and passing (figure 5(b)) electrons, it is shown that for $\zeta \approx 1$, the electrons spend most of their period lingering at the two ends of the magnetic mirror system (the so-called magnetic bottle). This means that $z$ is almost $-\pi/k_{eq}$ during a half of the period and is almost $\pi/k_{eq}$ in the other half. Therefore, $\exp(ink_{eq}z)$ in the integrand of equation (9) is $\exp(ink_{eq}\pi/k_{eq})$ or $\exp(-ink_{eq}\pi/k_{eq})$ in each half period. For $k_p/k_{eq} = m$ with $m$ an integer, we have $\exp(ink_{eq}\pi/k_{eq}) = \exp(-ink_{eq}\pi/k_{eq}) = \text{cte}$ and consequently the value of the integral drops to zero as the energy parameter approaches one ($\zeta \approx 1$). These electrons barely move in $z$-direction, similar to the case where the electrons are deeply trapped ($\zeta \approx 0$).

Prior to solving the equations for the mode structure and the sweeping rate in the hard nonlinear regime, it is necessary to investigate the behavior of the adiabatic invariant (phase-space area) of the trapped electrons in the BGK mode that are trapped or passing in the equilibrium magnetic field. Figure 6 shows the values of the adiabatic invariant (equation (22)) at the separatrix determined by the BGK mode during frequency sweeping. For the case of upward frequency sweeping, the energy of the passing electrons in the equilibrium field decreases, so does the corresponding value of the adiabatic invariant (figure (6)). However, for trapped electrons, energy increases for downward frequency sweeping. Depending on the initial orbit, the adiabatic invariant can either initially increase ($\zeta < 0.4$ of figure (6(a)) or decrease ($\zeta \geq 0.4$ of figure (6(a))). Due to the assumption of a flat-top distribution function over the separatrix region, the model remains valid as long as the separatrix supported by the BGK modes shrinks and an expanding separatrix (an increasing adiabatic invariant) should be avoided. Therefore, the electrons in the following results have initial energy parameters $\zeta > 0.4$.

In this energy range and for the electrons trapped in the magnetic field, the coherent phase-space structure is a hole whose separatrix area (and the corresponding amplitude of the mode) is shrinking for a downsweeping frequency. For the case that new electrons are trapped into an expanding separatrix, it is required that the value of the distribution function of newly trapped particles is set to the value of the ambient distribution. The latter case is not the subject of this paper and the reader is referred to [22, 29, 31] where the subject of expanding separatrices is addressed.
4.1. The mode structure

Considering similar slopes for the initial distribution of both the trapped and passing electrons in the equilibrium magnetic field (simultaneously in resonance with the plasma mode), the structure of the BGK mode has been solved for different initial electron energy parameters, namely $\zeta = T_0(\tau = 0) = 0.4, 0.6$ and $0.8$. Figure 7 illustrates the mode structure for these initial energies in cases where $\hat{\omega} = 0.8$ and $0.6$, constructed by solving equation (34) iteratively for the Fourier coefficients. The results reveal that for a nonzero change in $\hat{\omega}$, the nonlinear behavior of the BGK mode is determined by the initial electron orbits. For constant $\hat{\omega}$, e.g. figures 7(a), (c) and (e), the maximum amplitude of the normalized mode structure (maximum value of $\sum_n \hat{A}_n \cos(nkp_0)$) changes with changing $\zeta = T_0$, and the absolute values of the normalized mode amplitude decrease with increasing $\zeta = T_0$. In other words, for higher values of initial energy parameter, the maximum amplitude drops to a lower fraction of its initial value. The shape of the nonlinear structure is not only affected by the amount of change in the frequency ($\hat{\omega}$) but also by the initial energy parameter ($\zeta = T_0$). In order to explain the observed behavior, we first calculate the contribution of the trapped and passing particles to the mode structure separately while they are simultaneously in resonance with the mode. Afterwards, the behavior of both the equilibrium frequency and the physical quantities appearing in equation (34) is investigated.

The Fourier coefficients are calculated by adding the two terms on the RHS of equation (34), corresponding to $\alpha = T$ and $P$. The separate contributions of these two groups of particles to the mode structure are shown in figure 8 for similar values of distribution function and in case of simultaneous resonance between the plasma mode and these two types of energetic particles. It is clear that the contribution of the passing electrons to the nonlinear behavior of the mode is relatively much smaller than the trapped ones. The reason being that the resonance occurs in a region where the equilibrium frequency of passing particles has much steeper gradient in energy (see figure 1). Therefore, for the purpose of investigating the parameters of equation (34), we only consider the dominant contribution from the trapped electrons in the equilibrium magnetic field.

At a constant value of the normalized frequency $\hat{\omega}$, a simple evaluation of equation (34) gives

$$\hat{A}_n(t) \propto \left( \frac{d\hat{\omega}}{d\zeta} \right)_{\alpha = T_0} \tilde{\zeta}_{\alpha = T_0} \left[ \tilde{\zeta}_{\alpha = T_0} - \tilde{\zeta}_{\alpha = T_0} \right]^{3/2} \tilde{V}_{\alpha = T_0} \tilde{V}_{\alpha = T_0}.$$

(37)

Starting from different initial energies, the trapped electrons in the equilibrium magnetic field should be moved on different energy increments by the nonlinear mode in order to have the same amount of change in the frequency. This results from the nonlinear dependency of the equilibrium frequency on the energy parameter (see figure 1). As an example for $\hat{\omega} = 0.8$, the fast electrons having the initial energy parameters of $\zeta_0 = T_0 = 0.4, 0.6$ and $0.8$ should be moved in phase-space to the points where $\zeta(t) = 0.783, 0.863$ and $0.94$, respectively and the energy increments become shorter for higher values of initial energy parameter. For a linear equilibrium
distribution, the difference in the energy increments will explicitly appear in the numerator of equation (34) through the perturbed density term, i.e.
\[
\zeta(t = 0) - \zeta(t)
\]
In general, the nonlinear dependency of the equilibrium frequency on the energy parameter will affect the values of all the physical parameters appearing in equation (34) for a fixed amount of frequency shift. Figure 9 shows the dependency of the factors \( \hat{A}_\alpha = T_{\alpha,0} \) on the initial energy parameter of the electrons (initial orbits) using the behavior of the factors illustrated in figure 9. Looking at the expression (36) for the sweeping rate at a constant value of \( \hat{\omega} \), it can be inferred that
\[
\frac{d\hat{\omega}}{d\tau} \propto \hat{A}_\alpha(t) \frac{\hat{\Gamma}_\alpha = T}{|\hat{\zeta}(t)|^{2}}.
\]
Using expression (37) one finds
\[
\frac{d\hat{\omega}}{d\tau} \propto \left( \frac{d\hat{\omega}}{d\zeta} \right)_{\alpha = T,0} \frac{|\hat{\zeta}(t)|^{2}}{\hat{\zeta}} \frac{\hat{V}_{n}^{2}_{\alpha = T,0}}{\hat{\Gamma}_{\alpha = T}.}
\]
Similar to subsection 4.1, one can consider figures 9(a) and (c) at a constant \( \hat{\omega} \) to investigate the value of the RHS of expression (39) for different electron orbits. It is clear that the RHS value becomes lower when the resonance occurs with the electrons (trapped in the fixed equilibrium magnetic field) having higher initial energy parameter \( \zeta_{\alpha = T,0} \). Therefore, we expect the mode frequency to chirp more slowly when the initial energy parameter of the electrons is higher. This can be verified by solving differential equation (36) using...
the numerical method stated in section 3 for different initial orbits. Figure 10 illustrates the time evolution of scaled values of for different initial conditions. However, it is shown in this model, the holes and clumps can move with much lower rates compared with the sweeping rates observed in [21]. On the other hand, as predicted above, we have expanded the square root dependency, plotted for comparison, and the result reported in [21], respectively.

\begin{equation}
\frac{d}{dt}(\tilde{\theta} - \pi) = \Delta_{\tilde{\alpha}} = P \sum_n A_n V_{\alpha, n}^2 \cos(n\tilde{\theta}) \times (\tilde{\theta} - \pi),
\end{equation}

with \( |\Delta_{\tilde{\alpha}} = P \sum_n A_n V_{\alpha, n}^2 \cos(n\tilde{\theta})| = \omega_{b, \tilde{\alpha}}^2 = P \), where we have expanded \( \sin(n\tilde{\theta}) \) about the O-point which is at \( \tilde{\theta} = 0 \) for trapped electrons in the magnetic field (see figure 2(b)). Similarly, for passing electrons in the magnetic field that are deeply trapped in the BGK mode, we have

\begin{equation}
\frac{d}{dt}(\tilde{\theta} - \pi) = \Delta_{\tilde{\alpha}} = P \sum_n A_n V_{\alpha, n}^2 \cos(n\tilde{\theta}) \times (\tilde{\theta} - \pi),
\end{equation}

with \( |\Delta_{\tilde{\alpha}} = P \sum_n A_n V_{\alpha, n}^2 \cos(n\tilde{\theta})| = \omega_{b, \tilde{\alpha}}^2 = P \), where we have expanded \( \sin(n\tilde{\theta}) \) about the O-point which is at \( \tilde{\theta} = 0 \) for trapped electrons in the magnetic field (see figure 2(a)).

We introduce the dimensionless variable \( \tilde{\omega}_{b, \alpha} = \frac{\omega_{b, \alpha}}{\omega_{b, \alpha} = T} \), with

\begin{equation}
\omega_{b, \alpha} = T = 0 \equiv \sqrt{\langle A_{\tilde{\alpha}} V_{\alpha, n}^2 \rangle},
\end{equation}

the bounce frequency of resonant electrons (trapped in the magnetic field) in their corresponding separatrix in the BGK mode at early stage of chirping denoted by \( \tilde{\theta} = 0 \). We can write the adiabatic limit, introduced in subsection 2.3, in the form

\begin{equation}
\frac{d}{dt} \left( \frac{\omega_{b, \alpha}}{\omega_{b, \alpha}} \right) = \frac{\omega_{b, \alpha}}{\omega_{b, \alpha} = T}. \end{equation}
The evolution of the normalized bounce frequency ($a$ and $b$), normalized time rate of change in the bounce frequency ($c$ and $d$) and the value of the RHS of inequality (45) ($e$ and $f$). Panels ($a$, $c$, $e$) and ($b$, $d$, $f$) correspond to trapped and passing electrons in the magnetic field, respectively. The dashed, dotted and dash-dotted curves represent an initial energy parameter value of 0.4, 0.6 and 0.8, respectively, for trapped electrons in the magnetic field. At $\tau = 0$, the values of the panels ($e$) and ($d$) go asymptotically to $-\infty$ and for the panels ($e$) and ($f$) the corresponding values are zero.

Using equation (33) and the expression for the dimensionless time introduced in section 3, one finds

$$\nu^2\frac{\omega_{b\alpha}}{\omega_{pe}^2} < \frac{9\pi^2 \omega_{b\alpha}^2}{16 \frac{d\omega_b}{dt}}$$

The time evolution of $\omega_b$ can be investigated in the same numerical code implemented to solve differential equation (36) for constructing figure 10. At each time step in the fourth-order Runge–Kutta method, the corresponding parameters can be used to derive $\omega_{b\alpha}(\tau)$. Afterwards, one can readily use numerical differentiation methods to find $d\omega_b/d\tau$.

Figures 11(a) and (b) show the normalized bounce frequency of the trapped electrons about the O-point of the separatrix inside the BGK mode corresponding to the trapped and passing electrons in the magnetic field, respectively, for the initial energy parameters considered in the previous subsections. The corresponding values of $\omega_{b\alpha}$ are demonstrated in figures 11(c) and (d), where the values decrease to $-\infty$ as we approach $\tau = 0$. Therefore, the value of the RHS of (45), illustrated in figures 11(e) and (f), drops to zero at the early stage of frequency chirping. This means that the adiabatic limit is never formally satisfied at initial stage of phase-space structures evolution. Nevertheless, we have $\gamma_1, \nu \ll \omega_{pe}$ and as a result the period during which the adiabatic condition is not satisfied is extremely short. The results reveal that as the system evolves while the adiabaticity limit is initially violated, for later evolution of phase–space structures the RHS value of (45) monotonically increases. Therefore, once the adiabatic limit (45) is satisfied with regards to the value of LHS, it will remain valid for later evolution. It should be noted that the adiabaticity condition is better satisfied for the passing electrons in the magnetic field compared to the trapped ones.

5. Concluding remarks

The more realistic 1D model shows that apart from the amount of deviation from the initial eigenfrequency during frequency sweeping, the initial orbit (initial energy parameter) of the particles in a nonuniform equilibrium magnetic field, determines both the linear and the hard nonlinear evolution behavior of a plasma mode. The model also resolves the simultaneous contributions from the two groups of particles having different orbit types as well as the contribution from higher resonances. We find however that the first resonance is dominant. We also identify different behavior of the adiabatic invariant in different energy regions. The model shows that for a constant trend in frequency sweeping, either upward or downward, the adiabatic invariant can have both positive and negative gradients in the energy parameter depending on the energy region considered. This behavior depends on factors such as the resonance number, the proportion of the plasma mode wave-number to the spatial frequency of the equilibrium mode wave-number to the spatial frequency of the equilibrium field ($k_p/k_{eq}$) and whether the particles were initially trapped or passing in the equilibrium field. This indicates that for realistic geometries where particles interacting with the mode can follow different equilibrium orbits, an extended approach is required to calculate the perturbed density inside the holes and clumps. The required approach should take into account that the adiabatic invariant (phase-space area) at the separatrix can initially expand followed by a shrinking behavior and vice versa, depending on the initial orbit of the energetic particles. This extension can highly benefit from the method presented in [22].

The presented model in this manuscript provides a more effective understanding of hard nonlinear wave-particle-plasma interactions in realistic geometries provided that the mode is subject to weak continuum damping (a global mode) i.e. its structure in the linear regime is not mainly determined by the energetic
particles. Two different orbit topologies of energetic particles created by adding a nonuniform magnetic field to the 1D bump-on-tail instability problem, bring it into analogy with tokamaks where trapped and passing topologies exist which can both resonate with modes with different coupling strength factors. In a high aspect ratio tokamak, the total magnetic field follows
\[
B \propto \frac{1}{R_0 + r \cos \theta} \propto \frac{1}{R_0} (1 - \epsilon \cos \theta),
\]
where \(B\) is the magnetic field, \(\epsilon\) is the inverse aspect ratio, \(\theta\) is the poloidal angle and \(R_0\) and \(r\) are the major and minor radius, respectively.

Using the orbit-averaged Littlejohn’s Hamiltonian, we have
\[
H_0 - \mu B_0 = \frac{1}{2} m v_\parallel^2 - \mu B_0 \cos (\theta),
\]
where \(H_0\) is the equilibrium Hamiltonian and \(v_\parallel\) is the velocity in the direction of the magnetic field. Taking into account the symmetry of the magnetic field in toroidal direction in realistic geometries and assuming that the deviation of the fast particles from the flux surface is infinitesimal, the above Hamiltonian is comparable to the equilibrium Hamiltonian presented in equation (3). Further restrictions on the perturbation such as symmetry in toroidal direction, being localized on one flux surface and the assumption that the perturbation on different flux surfaces are unlinked, might let the presented model to relax the assumption that the fast electron distribution function during long range frequency deviations requires the extension of the presented model, which is a part of our ongoing research. Another avenue for further research is to relax the assumption that the fast electron distribution function is linear.

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Appendix A. Calculation of \(z (J, \theta)\)

Using the equilibrium Hamiltonian (3) and \(p_z = m_e \frac{d \sigma}{d \tau}\), we find
\[
\frac{2}{k_{\text{eq}}} \frac{d \sigma}{d \tau} = \sqrt{\frac{2}{m_e} \left[ E + \mu (B_c + B_0 \cos (2\sigma)) \right]}
\]
(A.1)

where \(\sigma = \frac{t_0}{2}\). We take \(\sigma (t = 0) = 0\), to have
\[
\sqrt{\frac{2}{k_{\text{eq}}} \int_0^\sigma} \frac{d \sigma}{\sqrt{1 - \frac{2 \mu B_0}{E + \mu (B_c + B_0 \cos (\sigma))} \sin^2 (\sigma)}} = \int_0^t d \tau,
\]
(A.2)

where we have used \(\cos (2\sigma) = 1 - 2 \sin^2 \sigma\).

(I) For passing electrons in the nonuniform magnetic field, the coefficient of \(\sin^2 (\sigma)\) in integral equation (A.2) is less than unity. After changing the coordinates to action-angle variables in subsection 2.1, we can use the canonical equations of motion to find
\[
t = \frac{\theta}{\Omega_{\text{eq}, \alpha}},
\]
(A.3)

where \(\Omega_{\text{eq}, \alpha} = \Omega\). Substituting (A.3) into the RHS of (A.2), we find
\[
\int_0^\sigma \frac{d \sigma}{\sqrt{1 - \frac{2 \mu B_0}{E + \mu (B_c + B_0 \cos (\sigma))} \sin^2 (\sigma)}} = \frac{\theta \Omega (\zeta^{-1}, \zeta^{-1})}{\pi}.
\]
(A.4)

According to the definition of Jacobi elliptic functions, we find
\[
\sin \left( \frac{\theta \Omega (\zeta^{-1}, \zeta^{-1})}{\pi} \right) = \sin \sigma,
\]
(A.5)

which gives
\[
z_{\alpha} = \frac{1}{k_{\text{eq}}} \sin^{-1} \left[ \sin \left( \frac{\theta \Omega (\zeta^{-1}, \zeta^{-1})}{\pi} \right) \right].
\]
(A.6)

(II) For trapped electrons in the nonuniform magnetic field, the coefficient of \(\sin^2 (\sigma)\) in integral equation (A.2) is higher than unity. We implement a change of variables as follows,
\[
\sin (\eta) = \frac{\sin (\sigma)}{\sin (\sigma_{\text{max}})},
\]
(A.7a)

\[
d \sigma = \frac{\sin (\sigma_{\text{max}}) \cos (\eta)}{\sqrt{1 - \sin^2 (\sigma_{\text{max}}) \sin^2 (\eta)}} d \eta.
\]
(A.7b)

The maximum value of \(z\) is \(\frac{1}{k_{\text{eq}}} \cos^{-1} \left( \frac{\mu B_c - E}{\mu B_0} \right)\), derived from \(p_z = 0\), hence, \(\cos (2\sigma_{\text{max}}) = \frac{\mu B_c - E}{\mu B_0}\) and
\[
\sin^2 (\sigma_{\text{max}}) = \frac{E + \mu (B_c + B_0 \cos (\sigma))}{2 \mu B_0}.
\]
(A.8)

Now we substitute equations (A.7a) and (A.7b) into (A.2) and use (A.8) to have
\[
\sqrt{\frac{2}{k_{\text{eq}}} \sin (\sigma_{\text{max}})} \int_0^\eta \frac{d \eta}{\sqrt{1 - \sin^2 (\sigma_{\text{max}}) \sin^2 (\eta)}} = t,
\]
(A.9)

where we have used \(\frac{\cos \eta}{\sqrt{1 - \frac{2 \mu B_0}{E + \mu (B_c + B_0 \cos (\sigma))} \sin^2 (\sigma)}} = 1\). Using equation (A.3) for trapped electrons in the magnetic field and equation (8b), we find
\[
\int_0^{\infty} \frac{d\eta}{\sqrt{1 - \sin^2 (\sigma_{\text{max}}) \sin^2 (\eta)}} = \frac{2 \theta_0 K (\zeta)}{\pi}, \quad (A.10)
\]
which gives
\[
\frac{\sin \eta}{\sin \sigma} = \frac{\sin \sigma_{\text{max}}}{\pi}. \quad (A.11)
\]
We find
\[
z_{\alpha=\Gamma} = \frac{2}{k_{\text{eq}}} \sin^{-1} \left[ \sqrt{\frac{2 \theta_0 K (\zeta)}{\pi}} \right]. \quad (A.12)
\]
It should be mentioned that equations (A.6) and (A.12) can be inverted for the corresponding angle \(\theta(\zeta, \zeta)\) variables.

**Appendix B. Adiabatic invariant and bounce averaging method**

The adiabatic invariant for a Hamiltonian \(K(\hat{t}, \hat{J}, \lambda \equiv \beta t)\) with slow time dependency (\(\beta \ll \) typical orbit frequencies) is
\[
I^0 = I(q, p, \lambda) + \beta I_1(q, p, \lambda) + \beta^2 I_2(q, p, \lambda) + ..., \quad (B.1)
\]
where the lowest term is commonly taken to be the action,
\[
I(E, \lambda) = \oint \left( \hat{\theta}, E, \lambda \right) d\theta \quad \text{with} \quad K(\hat{\theta}, \hat{J}, \lambda) = E. \quad \text{We transform to action-angle variables using the generating function} \quad \Phi_2(\hat{\theta}, I, \lambda) = \int_{\Phi_0(I, \lambda)} \Phi_1(\hat{\theta}, I, \lambda) \quad \text{so the Hamiltonian transforms into} \quad K_{\text{new}}(\theta, I, \lambda) = K(I, \lambda) + \beta \frac{\partial \Phi_2}{\partial \lambda}. \quad \text{Now we consider the trapped electron Vlasov equation}
\]
\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \theta} \frac{\partial K_{\text{new}}}{\partial I} = \frac{\partial f}{\partial \theta} \frac{\partial K_{\text{new}}}{\partial \theta}. \quad (B.2)
\]
Using the equations of motion we have
\[
\hat{\theta} = \frac{\partial K_{\text{new}}}{\partial \theta} \quad \text{and} \quad \hat{I} = -\frac{\partial K_{\text{new}}}{\partial \theta} \quad \text{and} \quad \frac{\partial f}{\partial \theta} \frac{\partial K_{\text{new}}}{\partial I} = \frac{\partial f}{\partial \theta} \frac{\partial K_{\text{new}}}{\partial \theta} \quad (B.3a)
\]
\[
\frac{\partial f}{\partial \theta} \frac{\partial K_{\text{new}}}{\partial \theta} = \frac{\partial f}{\partial \theta} \frac{\partial K_{\text{new}}}{\partial \theta} = 0. \quad (B.3b)
\]
Substituting the above expressions in equation (B.2) gives
\[
\frac{\partial f}{\partial t} + \beta \frac{\partial f}{\partial \theta} \frac{\partial \Phi_2}{\partial \lambda} = 0. \quad (B.4)
\]
Following the same approach in [27], \(f\) can be expanded in terms of the small parameter \(\beta = \frac{2}{c}\) to have
\[
f = f_0 + \beta f_1 + \beta^2 f_2 + ..., \quad (B.5)
\]
where \(f_0\) is the bounce average of \(f\) over \(\Theta\). Using expression (B.5), we substitute for \(f\) in equation (B.4). To lowest order (\(\mathcal{O}(1)\)) in \(\beta\), one finds
\[
\frac{\partial f_0}{\partial \theta} = 0. \quad (B.6)
\]
To next order (\(\mathcal{O}(\beta)\)),
\[
\frac{\partial f_0}{\partial t} + \beta \frac{\partial f_1}{\partial \theta} + \frac{\partial f_0}{\partial \theta} \frac{\partial \Phi_2}{\partial \lambda} + \beta \frac{\partial f_1}{\partial \theta} = \frac{\partial f_0}{\partial \theta} \frac{\partial \Phi_2}{\partial \lambda} = 0. \quad (B.7)
\]
The second, sixth and eighth terms are on the order of \(\beta^2 (\mathcal{O}(\beta^2))\) and can be neglected at this stage. Equation (B.6) shows that \(f_0\) is independent of \(\Theta\), which allows us to set the fifth term to zero. Therefore, we reach
\[
\frac{\partial f_0}{\partial t} + \beta \frac{\partial f_1}{\partial \theta} = 0. \quad (B.8)
\]
After averaging (B.8) over \(\Theta\), the second and third terms vanish and we find
\[
\frac{\partial f_0}{\partial t} = 0. \quad (B.9)
\]
We define \(f_0 = \delta f + \left( f_{\text{eq}}(J_{\text{res}}(t)) \right)\), where \(\delta f\) denotes averaging over \(\Theta\) and \(f_0(t = 0) = f_{\text{eq}}(J_{\text{res}}(t = 0))\). The uniformity assumption of the distribution function over the separatrix region assures \(f_{\text{eq}}(J_{\text{res}}(t)) = f_{\text{eq}}(J_{\text{res}}(t))\). Hence, \(f_0(t) = \delta f + f_{\text{eq}}(J_{\text{res}}(t))\). According to (B.9), \(f_0\) should remain constant during frequency sweeping which gives
\[
\delta f = f_{\text{eq}}(J_{\text{res}}(t = 0)) - f_{\text{eq}}(J_{\text{res}}(t)). \quad (B.10)
\]

**Appendix C. Validation of the smallness of the perturbed potential energy change**

As illustrated in figure C1, we consider the case of a long range frequency chirping where the separatrix has approximately vanished. Therefore, we have \(J_{\alpha}(t) \approx 0\). The change in the equilibrium energy (\(E_{\text{eq},\alpha}\)) of the trapped electrons in the BGK mode is
\[
\Delta E_{\text{eq},\alpha} = \frac{\partial H_{\text{eq},\alpha}}{\partial J_{\alpha}} \Delta J_{\text{res},\alpha} = \omega_{p e} \Delta J_{\text{res},\alpha}. \quad (C.1)
\]
where \(\Delta J_{\text{res},\alpha} = J_{\text{res},\alpha}(t = t_0) - J_{\text{res},\alpha}(t)\). Using equation (20), we find
\[
\Delta J_{\alpha} = J_{\alpha}(t = t_0) - J_{\alpha+}(t) \approx \sqrt{\frac{A_{1,0} V_{\alpha,1,1,0}^2}{|\Delta_{\alpha,0}|}}, \quad (C.2)
\]
where, \(t = t_0\) is denoted by the subscript \(\alpha\) the change in the perturbed energy (\(E_{\text{perturbed,}\alpha}\)) of the electrons, which is the change in the perturbed potential energy, equals
\[
\Delta E_{\text{perturbed,}\alpha} = A_{1,0} V_{\alpha,1,1,0}. \quad (C.3)
\]
We have claimed that if the change in \(J_{\text{res,}\alpha}\) is greater than the change in the separatrix width, then \(\Delta E_{\text{perturbed,}\alpha} \ll \Delta E_{\text{eq},\alpha}\). Therefore, we have
\[
A_{1,0} V_{\alpha,1,1,0} \ll \omega_{p e} \Delta J_{\text{res},\alpha}. \quad (C.4)
Figure C1. Schematic of a separatrix shrinkage with $J_{\alpha,1}(t) \approx 0$ during long range frequency chirping.

The above inequality can be written into

$$\sqrt{A_{1,0}V_{0,1,1,0}/|\Delta_{0,0}|} \cdot \omega_B \ll \omega_{pe} \Delta J_{res,\alpha},$$

(C.5)

where $\omega_B = \sqrt{A_{1,0}V_{0,1,1,0}/|\Delta_{0,0}|}$ is the bounce frequency of trapped particles inside the separatrix in the BGK mode. Using equation (C.2) we find

$$\frac{\Delta J_{res,\alpha}}{\Delta J_{\alpha}} \gg \frac{\omega_B}{\omega_{pe}} \approx \frac{\gamma_l}{\omega_{pe}}.$$  

(C.6)

The RHS value is much less than unity ($\gamma_l \ll \omega_{pe}$). Therefore, if the change in $J_{res,\alpha}$ is greater than the change in the width of the separatrix, the condition (C.6) is sufficiently satisfied.

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