SOFICITY, SHORT CYCLES, AND THE HIGMAN GROUP

HARALD A. HELFGOTT AND KATE JUSCHENKO

Abstract. This is a paper with two aims. First, we show that the map from \( \mathbb{Z}/p\mathbb{Z} \) to itself defined by exponentiation \( x \to m^x \) has few 3-cycles—that is to say, the number of cycles of length 3 is \( o(p) \). This improves on previous bounds.

Our second objective is to contribute to an ongoing discussion on how to find a nonsofic group. In particular, we show that, if the Higman group were sofic, there would be a map from \( \mathbb{Z}/p\mathbb{Z} \) to itself, locally like an exponential map, yet satisfying a recurrence property.

1. Introduction

The questions treated in this paper are motivated in part by the study of sofic groups, and, more specifically, by the search for a nonsofic group. (Nonsic groups are not, as of the moment of writing, yet known to exist. See the survey [Pes08].) The same questions have been studied from a different angle in coding theory; see [GS10] and references therein.

As we shall discuss, it is natural to test whether the Higman group is sofic. We shall show that, if the Higman group were sofic, then there would have to exist a function \( f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \) that is, to say the least, odd looking: it would behave locally like an exponential map almost everywhere, but \( f \circ f \circ f \circ f \) would be equal to the identity.

We do not succeed here in showing that such a function does not exist. However, we do prove another kind of result. Just from the triviality of an analogue of the Higman group, we obtain easily that there is no map \( f \) that behaves locally like an exponential map and has \( f \circ f \circ f \) equal to the identity almost everywhere. If \( f \) is actually an exponential map—i.e., if \( f \) is given by \( f(x) = m^x \mod p \) for \( x = 0, 1, \ldots, p - 1 \) and some \( m \not\equiv 0, 1 \mod p \)—then we can actually prove that \( f \circ f \circ f \) is different from the identity almost everywhere; in other words, \( f \) has few 3-cycles. (Here “few” means “\( o(p) \)”.) This improves on the best result known on 3-cycles to prime moduli, namely, [GS10, Thm. 6]. (This is a matter of independent interest; it has been studied before, and, apparently, never in relation to sofic groups.)

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Returning to the function $f$ that has to exist if the Higman group is sofic: our strategy for extracting the existence of such a function from soficity involves the restriction of a so-called sofic approximation to amenable subgroups of the Higman group. All sofic approximations of an amenable group are, in an asymptotic sense, conjugate; this enables us to “rectify” them, i.e., conjugate them so as to take them to sofic approximations with geometric or arithmetic meaning.

In this way, we show that, if the Higman group is sofic, then there is a map $f$ that behaves locally like an exponential map almost everywhere and satisfies $f^4 = e$. Does this in fact suggest that the Higman group is not sofic? It would be premature to venture a definite answer. If $f$ behaves like an exponential map with too few exceptions, then we do arrive at a contradiction by $p$-adic arguments. This can be seen as a hint in the direction of nonsificity.

At the same time, there are two provisos. Shortly after the first draft of the present paper appeared, Glebsky [Gle] showed that analogous maps $f : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ do exist when $p|m - 1$, where $m$ is the base of the exponential. Such maps come from the fact that generalizations $H_{4,m}$ of the Higman group $H_4$ have finite $p$-quotients. The Higman group $H_4$ itself has no finite quotients, so the construction does not immediately apply to the function $f$ arising from it.

Second, very recent work by Kuperberg, Kassabov, and Riley [KKR]—motivated in part by the present paper—shows in a different way that functions with counterintuitive properties similar to those of our function $f$ exist. The proof has some elements in common with ours; the existence of a function resembling $f$ follows from the existence of a sofic group (not Higman’s) with certain properties. We will discuss this matter in §6. The question of the existence of our function $f$ remains open, and so does the soficity of the Higman group.

1.1. Main results. Let us start with the simple result on 3-cycles we just mentioned. Its proof uses almost no machinery.

**Definition 1.** Let $m, n \geq 2$ be coprime integers. We define $f_{m,n} : \{0, 1, \ldots, n - 1\} \to \{0, 1, \ldots, n - 1\}$ to be the map defined by

$$f_{m,n}(x) \equiv m^x \mod n.$$ 

**Theorem 1.** Let $m, n \geq 2$ be coprime integers. Then

$$f_{m,n}(f_{m,n}(f_{m,n}(x))) = x$$

can hold for at most $o_m(n)$ elements $x \in \{0, 1, \ldots, n - 1\}$.

Here, as usual, $o_m(n)$ means “a function $b_m(n)$ such that $\lim_{n \to \infty} b_m(n)/n = 0$”, where the subscript “$m$” warns us that the function $b_m(n)$ may depend on $m$. We also use the notation $O_m(n)$, meaning “a function $B_m(n)$ such that $B_m(n)/n$ is bounded”. As is standard, we also use $o(n)$ or $O(n)$ without subscripts when dependancies are nonexistent or obvious.

Compare Theorem 1 to [GS10, Thm. 6], which states that, for $p$ prime, the number of $x$ for which $f_{m,p}(f_{m,p}(f_{m,p}(x))) = x$ is $\leq 3p/4 + O_m(1)$. (The number of such $x$ is of interest in part because of the study of $f_{m,p}$ in the context of the generation of pseudorandom numbers; see the references [PS98], [GR03] given in [GST10].)

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1In the original version of our work, we had $m = 2$; then, of course, $m - 1$ has no prime divisors $p$. 

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While the results analogous to Theorem [1] with $f_{m,n}(x)$ or $f_{m,n}(f_{m,n}(x))$ instead of $f_{m,n}(f_{m,n}(f_{m,n}(x)))$ are very easy, the problem with more than three iterations is hard if $n$ is a prime. Indeed, for $n$ prime and $m$ such that $m \mod n$ generates $(\mathbb{Z}/n\mathbb{Z})^*$, we do not know how to prove that
\begin{equation}
\tag{1.1}
f_{m,n}(f_{m,n}(f_{m,n}(x))) = x
\end{equation}
can hold for at most $o(n)$ elements $x$ of $\{0, 1, \ldots, n - 1\}$, or even that it can hold for at most $n - 1$ elements of $\{0, 1, \ldots, n - 1\}$.

On the other hand, we do have good upper bounds (namely, $< p^k$) on the number of solutions to
\begin{equation}
\tag{1.2}
f_{m,n}(f_{m,n}(\cdots (f_{m,n}(x))\cdots)) = x \quad (k \text{ times})
\end{equation}
when $n = p^r$, $r \geq k + 1$ [Gle13 Cor. 3], [HR12 Thm. 5.7]. The argument in [HR12] is based on $p$-adic analysis, whereas [Gle13] is based on explicit matrix computations – though it arguably still has Hensel’s lemma at its core.

There turns out to be a relation between counting solutions to (1.1) and an important problem in asymptotic group theory, namely, that of constructing a nonsofic group. If (1.1) (with $m = 2$) could hold for $(1 - o(1))n$ elements of $\{0, 1, \ldots, n - 1\}$, $n$ odd, then a nontrivial quotient of the Higman group would be sofic, as one can easily see from the definitions (which we are about to give). More interestingly, as we are about to see, if the Higman group were in fact sofic, then there would be a map $f$, locally like $f_{2,n}$, such that
\[ f(f(f(f(x)))) = x \]
would hold for $(1 - o(1))n$ elements of $\{0, 1, \ldots, n - 1\}$ (where we can take $n$ to be a prime, or even a power $p^r$, $r \geq 5$, say).

Let us recall some standard notation. We write $\text{Sym}(n)$ for the symmetric group, i.e., the group of all permutations of a set with $n$ elements. The (normalized) Hamming distance $d_h(g_1,g_2)$ between two permutations $g_1,g_2 \in \text{Sym}(n)$ is defined to be the number of elements at which they differ, divided by $n$:
\[ d_h(g_1,g_2) = \frac{1}{n}|\{1 \leq i \leq n : g_1(i) \neq g_2(i)\}|, \]
where we write $|S|$ for the number of elements of a set $S$. It is clear that $d_h$ is an (left- and right-) invariant metric on $\text{Sym}(n)$. Write $\text{id}$ for the identity element of $\text{Sym}(n)$.

**Definition 2.** Let $G$ be a group. For $n \in \mathbb{Z}^+$, $\delta > 0$, and $S \subset G$ as a finite subset, an $(S, \delta, n)$-sofic approximation is a map $\phi : S \to \text{Sym}(n)$ such that
\begin{enumerate}
\item \(d_h(\phi(g)\phi(h), \phi(gh)) < \delta\) for all $g, h \in S$ such that $gh \in S$ (**“$\phi$ is an approximate homomorphism”**),
\item \(d_h(\phi(g), \text{id}) > 1 - \delta\) for all $g \in S$ distinct from the identity $e$ (i.e., the image of every $g \neq e$ has few fixed points).
\end{enumerate}

We say that the group $G$ is sofic if, for every finite subset $S \subset G$ and every $\delta > 0$, there is an $(S, \delta, n)$-sofic approximation for some $n \in \mathbb{Z}^+$.

It is easy to see that (2) implies that $d_h(\phi(e), \text{id}) < \delta$ for the identity $e \in G$ provided that $e \in S$.

The notion of sofic groups goes back to the work of Gromov, who used a different, but equivalent, definition. See [Pes08] for a survey. It is clear that, if a group is sofic, all of its subgroups are sofic as well. It is also immediate that, if a group is
not sofic, then it has a finitely generated (and, in particular, countable) subgroup that is not sofic either.

**Definition 3.** Let $n, m \geq 2$. We denote by $H_{n,m}$ the group generated by elements $a_1, a_2, \ldots, a_n$ subject to the following relations:

$$a_i^{-1}a_{i+1}a_i = a_{i+1}^m \quad \text{(for } 1 \leq i < n\text{)}, \quad a_n^{-1}a_1a_n = a_1^m.$$  

The group $H_4 = H_{4,2}$ is called the Higman group.

The group $H_{3,2}$ is trivial (this is easy). The group $H_{4,2}$ is trivial; this is shown at the end of [Hig51], where the proof is credited to K. A. Hirsch.

The Higman group was first constructed as an example of a group without finite quotients [Hig51]. It is not known whether it has amenable quotients. (We will go over the concept of amenability in §2; amenability implies soficity.) Because of this, as well as for other reasons (see [Tho12]), the Higman group $H_4 = H_{4,2}$ can be seen as a plausible candidate for a nonsofic group.

What is more, $H_4$ is SQ-universal [Sch71], meaning that every countable group is isomorphic to some subgroup of some quotient of $H_4$. This implies immediately that, if a nonsofic group exists, then there exists a nonsofic quotient $H_4/N$ of $H_4$.

**Theorem 2.** Let $m \geq 2$. Assume that the group $H_{4,m}$ is sofic. Then, for every $\epsilon > 0$, there is an $N > 0$ such that, for every $n \geq N$ coprime to $m$, there is a bijection $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ such that

$$f(x + 1) = mf(x) \quad \text{(1.3)}$$

for at least $(1 - \epsilon)n$ elements $x$ of $\mathbb{Z}/n\mathbb{Z}$, and

$$f(f(f(x)))) = x \quad \text{(1.4)}$$

for all $x \in \mathbb{Z}/n\mathbb{Z}$.

This result has a very easy almost-converse: if, for $\epsilon > 0$ arbitrary, there are $n$ and $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ such that (1.3) holds for at least $(1 - \epsilon)n$ elements of $\mathbb{Z}/n\mathbb{Z}$, then the Higman group has arbitrarily large sofic quotients. (That is, it has either an infinite sofic quotient or arbitrarily large finite quotients.) In fact, this is precisely what happens with $H_{4,m}$, $m > 2$, as Glebsky [Gle] first pointed out, and as we will discuss later.

If we assume that a given quotient, $H_{4,m}/N$, $m$ arbitrary, is sofic, then the proof of Theorem 2 can be modified trivially to give not just the same conclusion but a stronger one, including equalities other than (1.3) and (1.4). The same holds, in general, whenever we study a group $G$ into which the Baumslag–Solitar group $BS(1, m)$ embeds, whether or not this group $G$ is a quotient of a group $H_{4,m}$; the proof of Theorem 2 can be easily modified to show that the assumption that $G$ is sofic implies that there is a bijection $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ such that (1.3) holds together with some other conditions in the place of (1.4). We will discuss this at the end of §3.

We can choose to focus on $n$ prime or, instead, on $n$, a high power of a fixed prime, since then the statement goes in the opposite direction of the results from [Gle13, Cor. 3], [HR12, Thm. 5.7] we have mentioned, in the sense that the consequence of

\[\text{On the other hand, there is a sofic group for which there is no sequence of amenable groups converging to it in the Chabauty topology [Cor11].}\]

\[\text{Baumslag–Solitar groups are defined in [3].}\]
soficity it points out is the negation of a hypothetical stronger form of such results. We will use the same $p$-adic tools as in [HR12] to prove the following statement.

**Proposition 3.** Let $m > 1$, $p 
 (m - 1)$. Let a bijection $f : \mathbb{Z}/p^r\mathbb{Z} \rightarrow \mathbb{Z}/p^r\mathbb{Z}$, $r \geq 1$, be given. Then either

$$f(f(f(x)))) \neq x$$

for at least $p^r/2$ values of $x \in \mathbb{Z}/p^r\mathbb{Z}$ or

$$(1.5)\quad f(x+1) \neq mf(x)$$

for at least $p^r/2^1/4$ values of $x \in \mathbb{Z}/p^r\mathbb{Z}$.

Of course, this is unfortunately much too weak to contradict the conclusion of Theorem 2, for that, we would need (1.3) to hold for $> p^r - p^{r-1}/\sqrt{2}$ values of $x$, not just for $(1-\epsilon)p^r$ values of $x$. As we now know, the conclusion of Theorem 2 actually holds for $m > 2$.

It is easy to give a probabilistic argument (6) that the existence of a function $f$ such as that given by Theorem 2 is implausible (for $\epsilon < 1/4$), and hence that it is also implausible that the Higman group be sofic. Let us emphasize that this argument is merely heuristic; it is very far from a proof. The arithmetical flavor of Theorem 2 might seem to lend some credence to the heuristics, in so far as an assumption of independence of certain kinds of random variables underlies both the argument here and several classical conjectures in number theory. However, we are not in a context that is fully familiar to a number theorist, in particular, due to the large number of “exceptions to the rule” (namely, en).

Moreover—and this is important—the work of both Glebsky [Gle] and Kuperberg, Kassabov, and Riley [KKR] seems to cast doubt on these heuristics. Glebsky’s example assumes that $n = p^k$, $p | (m - 1)$. However, by the argument at the beginning of the proof of Theorem 2 (4), it implies the existence of a function just like that in the conclusion of Theorem 2 for every $m > 2$ and for every $n$ larger than a constant depending only on $m$ and $\epsilon$.

The idea behind Glebsky’s construction is to show that the groups $H_{4,m}$ have arbitrary large $p$-quotients for $m > 2$, $p | (m - 1)$. The intersection of the kernels of the quotient maps corresponding to these quotients has trivial intersection with the Baumslag–Solitar subgroups BS$(1,m)$ (generated by $a_i, a_{i+1}$). Using the strategy of the present paper, one can then derive the existence of functions with required approximation properties, as in the conclusion of Theorem 2.

The maps in [KKR] are also of the type in Theorem 2 though there $m$ is not constant but grows slowly with $n$. As [KKR] shows, these maps, like those coming from Glebsky’s work, go against the probabilistic heuristics discussed here. Such heuristics must thus be taken with extreme circumspection, to say the very least.

Returning to rigor: is it possible to give conclusions stronger than those of Theorem 2 if we make assumptions on “how sofic” $H_4$ is, i.e., assumptions on the dependence of $n$ on $S$ and $\delta$ in Definition 2? (Such assumptions have been formalized in different ways, as sofic dimension growth [AC] and as sofic profile [Cor13].) This question is related to that of strengthening the methods we are about to discuss [ES11], [KL13] or, more generally, to the problem of giving versions of results on stability and weak-stability [API15] with good bounds. We will address these matters in [17] but do not solve them.

To go back to Theorem 1, a result resembling Theorem 1 but with weaker conditions and conclusions, follows easily from the fact that $H_3$ is trivial. We will
go over this at the beginning of §4. It would be interesting if the triviality of $H_3$


could be used to prove Theorem 1 itself (or a statement with the same conclusions
but weaker conditions). As we said, the proof of Theorem 1 we give requires next
to no tools, though some will recognize the idea of Poincaré recurrence at work.

1.2. Methods. The main tool used toward the proof of Theorem 2 is the fact that
any two sofic representations of an amenable group are conjugate to each other.
What this means is that, if $G$ is an amenable group (we shall recall the definition)
generated by a finite set $S \subset G$, and $\phi, \phi'$ are two $(S', \delta, n)$-sofic approximations of
$G$ with $S' \supset S$ large enough and $\delta$ small enough, then there is a bijection $\tau$ from
\{1, \ldots, n\} to itself such that, for every $s \in S$, $\tau \circ \phi(s) \circ \tau^{-1}$ equals $\phi'(s)$ at almost
all points (i.e., the Hamming distance between $\tau \circ \phi(s) \circ \tau^{-1}$ and $\phi'(s)$ is small).

While the group $H_{4,m}$ is not amenable, we can take $G$ to be an amenable subgroup of $H_{4,m}$ (to wit, the Baumslag–Solitar group $BS(1,m)$). It is easy to see that
$G = BS(1,m)$ has a sequence of sofic approximations $\phi'_k : G \to \text{Sym}(n_k)$ with a
natural arithmetical description. Hence, if $\{\phi_k\}$ is a sequence of sofic approximations of $H_{4,m}$, the restriction of $\phi_k$ to $G$ must be conjugate to $\phi'_k$. This constrains $\phi_k$ severely; the same sequence of bijections $\tau'$ that show $\phi_k|_G$ to be conjugate to $\phi'_k$ shows that $\phi_k$ is conjugate to a sequence of maps having the properties given to $f$ in Theorem 2.

In fact, we will be working with $\left(\mathbb{Z}/4\mathbb{Z}\right) \ltimes H_{4,m}$ rather than $H_{4,m}$, so as to
strengthen the constraints on $\phi_k$. The reason we can proceed in this way is that, if $H_{4,m}$ is sofic, then so is $\left(\mathbb{Z}/4\mathbb{Z}\right) \ltimes H_{1,m}$, since any extension of a sofic group by an
amenable group (such as $\mathbb{Z}/4\mathbb{Z}$) is sofic [ES06].

The fact that any two sofic representations of an amenable group are conjugate
to each other is something that has been stated and proved in different ways. It
was proved by Elek and Szabó [ES11] using “infinitary” language (ultraproducts,
which depend on the axiom of choice). We will use a “finitary” statement (based
on a slight refinement of [KL13] Lem. 4.5) from which effective bounds could be
easily extracted.

2. Amenability and sofic approximations: Tools and background

Let $G$ be a group with a finite generating set $S$. In this case, one of the (mutually equivalent)
standard definitions of amenability is as follows: $G$ is said to be
amenable if there exists an infinite sequence (called a Følner sequence) of increasing
sets

\[ e \in F_1 \subset F_2 \subset \cdots \subset F_k \subset \cdots \subset G \]

such that $|sF_j \Delta F_j| \leq |F_j|/j$ for all $s \in S$ and all $j \geq 1$.

The aim of this section is to prove Proposition 5 which states, in effect, that all
sofic representations of an amenable group are conjugate. This is a key recent result
that is neither new nor ours; nevertheless, we will have to give a proof, since we
have not been able to find it in the literature in the concrete form we need (though
its meaning is identical to that of [ES11] Thm. 2], or rather the difficult direction
thereof).

The alternative would have been to derive Proposition 5 (a finitary statement) from [ES11] Thm. 2], which uses ultraproducts in its proof and formulation. This
would be much as in [AP15].
Now, given any $\eta > 0$, we can actually assume that
\begin{equation}
\|(F_{j-1}^{-1}F_j) \setminus F_j\| \leq \eta |F_j|
\end{equation}
for all $j > 1$, simply by replacing $\{F_j\}_{j \geq 1}$ with a subsequence, if necessary.

As it turned out, sofic approximations of amenable groups decompose particularly nicely: any such approximation has an almost-covering by trivial approximations of $F_j$. Let us give a precise statement. We take the following nomenclature from [OW87]: given $\epsilon > 0$ and a finite set $D$, we say a collection $\{A_i\}_{i \in I}$ of subsets of $D$ is $\epsilon$-disjoint if there exist pairwise disjoint subsets $A'_i \subset A_i$ such that $|A'_i| \geq (1-\epsilon)|A_i|$, and that it $(1-\epsilon)$-covers $D$ if $|\bigcup_{i \in I} A_i| \geq (1-\epsilon)|D|$.

**Lemma 4.** Let $G$ be a group. For any $\epsilon, \kappa > 0$, there are $k \geq 1$ and $\lambda_1, \ldots, \lambda_k \in (0,1]$ with $1-\epsilon \leq \lambda_1 + \cdots + \lambda_k \leq 1$ such that the following holds. For any infinite sequence of finite subsets
$$e \in F_1 \subset F_2 \subset \cdots \subset F_k \subset \cdots \subset G$$
satisfying (2.1) for $\eta = \kappa/(24/\epsilon)^{k-1}$, there are $\delta > 0$, $N \geq 1$, and a finite set $S \subset G$ such that, if $\phi : S \to \text{Sym}(n)$ is an $(S,\delta,n)$-sofic approximation with $n \geq N$, there exist $C_1, \ldots, C_k \subset \{1, \ldots, n\}$ such that
(a) the sets $\phi(F_1)C_1, \ldots, \phi(F_k)C_k$ are pairwise disjoint;
(b) for every $1 \leq j \leq k$ and every $c \in C_j$, the map $s \mapsto \phi(s)c$ from $F_j$ to $\phi(F_j)c \subset \{1, \ldots, n\}$ is injective;
(c) the family $\{\phi(F_j)c\}_{1 \leq j \leq k, c \in C_j}$ is $\epsilon$-disjoint and $(1-\epsilon)$-covers $\{1, \ldots, n\}$;
(d) $(1-\kappa)\lambda_j \leq |\phi(F_j)C_j|/n \leq (1+\kappa)\lambda_j$ for every $j = 1, 2, \ldots, k$.

This is essentially [KL13, Lemma 4.5]; we have added only conclusion (d), which will be crucial to our purposes. It was already pointed out in [DKPT11, Lemma 4.3] that the method of proof in [KL13, Lemma 4.5] can give conclusions like this one, but the version of conclusion (d) given there is unfortunately not quite strong for our purposes.

We remark in passing that the values of $\lambda_1, \ldots, \lambda_k$ given by the proof below depend only on $\epsilon$, not on $\kappa$, though we will not need this in what follows.

**Proof of Lemma 4** We will start with an $(S,\delta,n)$-sofic approximation $\phi$ and show how to construct the sets $C_1, \ldots, C_k$, in reverse order. It will become clear that the argument works provided that $k$ is larger than a constant depending only on $\epsilon$ (not on $\kappa$). We set $S = F_k^{-1}F_k$.

As in [KL13], we say that a collection $\{A_i\}$ of subsets of a finite set $X$ is a $\rho$-even covering of $X$ of multiplicity $M$ if (i) no element of $X$ is contained in more than $M$ elements of $\{A_i\}$, and (ii) $\sum_i |A_i| \geq (1-\rho)M|X|$. Since $S = F_k^{-1}F_k$, Definition [2] implies that there is a set $B \subset \{1,2,\ldots,n\}$ with $|B| \geq (1-\delta')n$, $\delta' = O_{|F_k|}(\delta)$, such that, for all $g,h \in F_k$,
(a) $\phi(g)^{-1}\phi(h)x = \phi(g^{-1}h)x$ for all $x \in B$,
(b) $\phi(g^{-1}h)x \neq x$ for all $x \in B$ unless $g = h$.

By (3) and (4), the map $s \mapsto \phi(s)x$ is injective on $F_k$ for $x \in B$. It is easy to see that this implies that the sets $\phi(F_k)x$ form a $\delta'$-even covering of $\{1, \ldots, n\}$ of multiplicity $|F_k|$.

By [KL13, Lemma 4.4], every $\rho$-even covering of a set $X$ contains an $\epsilon$-disjoint subcollection that $\epsilon(1-\rho)$-covers $X$. In our case, this means that there is a set
$C \subset B$ such that the sets $\phi(F_k)c$, $c \in C$, are $\epsilon$-disjoint and satisfy $|\bigcup_{c \in C} \phi(F_k)c| \geq \epsilon (1 - \delta')n$. We take $C_k$ to be a minimal such set $C$, and we set $\lambda_k = \epsilon$. Clearly,

$$(1 - \delta')\lambda_k n \leq |\phi(F_k)C_k| \leq (1 - \delta')\lambda_k n + |F_k|.$$ 

Since $n \geq N$ and we can assume that $N$ is larger than any given function of $\epsilon$, $\kappa$, and $|F_k|$, we may assume that $|F_k|$ is less than $n$ times an arbitrarily small constant that we may let depend on $\epsilon$ and $\kappa$. Since we can also take $\delta$ to be smaller than an arbitrary quantity depending on $\epsilon$, $\kappa$, and $|F_k|$, we can assume that $\delta'$ is smaller than any given quantity depending on $\epsilon$ and $\kappa$, and we conclude that

$$(1 - \kappa_k)\lambda_k n \leq |\phi(F_k)C_k| \leq (1 + \kappa_k)\lambda_k n,$$

where $\kappa_k > 0$ is an arbitrarily small constant that we are allowed to let depend on $k$, $\epsilon$, and $\kappa$; we will set it later. (Of course, since $k$ will be determined by $\epsilon$ and $\kappa$, a dependence on $k$, $\epsilon$, and $\kappa$ is the same as a dependence on just $\epsilon$ and $\kappa$.)

We can now set up an iteration. For $j = k, k - 1, \ldots, 1$, we will construct sets $C_j \subset B$ such that

(a) $\phi(F_j)C_j$ is disjoint from $\phi(F_{j+1})C_{j+1} \cup \cdots \cup \phi(F_k)C_k$;

(b) for every $c \in C_j$, the map $s \mapsto \phi(s)c$ defined on $F_j$ is injective;

(c) the family $\{\phi(F_j)c\}_{c \in C_j}$ is $\epsilon$-disjoint;

(d) $(1 - \kappa_j)\lambda_j \leq |\phi(F_j)C_j|/n \leq (1 + \kappa_j)\lambda_j$ for every $j = 1, 2, \ldots, k$, where $\kappa_j$ will depend only on $\epsilon$, $\kappa$, and $j$, and, moreover, $\kappa_j \geq \kappa_{j+1} \geq \cdots \geq \kappa_k$.

We have just constructed $C_k$; let $B_k = B$. We shall now construct $C_j$, $j < k$, assuming that we have already constructed $C_{j+1}, \ldots, C_k$. We define

$$B_j = \{s \in B : \phi(F_j)s \cap (\phi(F_{j+1})C_{j+1} \cup \cdots \cup \phi(F_k)C_k) = \emptyset\}.$$

By (2.1) (with $\eta = 1$) and the fact that $C_i \subset B$ for all $i \geq j$,

$$|\phi(F_j)^{-1}(\phi(F_{j+1})C_{j+1} \cup \cdots \cup \phi(F_k)C_k)|$$

$$= |\phi(F_j^{-1}F_{j+1}C_{j+1} \cup \cdots \cup F_j^{-1}F_kC_k)|$$

$$\leq \sum_{i=j+1}^{k} (|\phi(F_j^{-1}F_i \setminus F_i)| + |\phi(F_i)C_i|)$$

$$\leq \sum_{i=j+1}^{k} (|F_i||C_i| + |\phi(F_i)C_i|) \leq \sum_{i=j+1}^{k} \left(1 + \frac{\eta}{1 - \epsilon}\right) |\phi(F_i)C_i|$$

$$\leq \left(1 + \frac{\eta}{1 - \epsilon}\right) (1 + \kappa_{j+1}) \sum_{i=j+1}^{k} \lambda_i n.$$

Writing

$$\sigma_r = \sum_{i=r}^{k} \lambda_i,$$

$$\delta_j = \delta' + \left(1 + \frac{\eta}{1 - \epsilon}\right) (1 + \kappa_{j+1}) \sigma_{j+1},$$

we obtain that

$$|B_j| \geq |B| - \left(1 + \frac{\eta}{1 - \epsilon}\right) (1 + \kappa_{j+1}) \sigma_{j+1} n \geq (1 - \delta_j)n.$$

Hence, $\{\phi(F_j)c\}_{c \in B_j}$ is a $\delta_j$-even covering of $\{1, 2, \ldots, n\}$ with multiplicity $|F_j|$. Just as before, we apply [KL13] Lemma 4.4 and obtain a set $C_j \subset B_j$ such that
the sets $\phi(F_j)c$, $c \in C_j$ are $\epsilon$-disjoint and
\[
\epsilon(1 - \delta_j)n \leq |\phi(F_j)C_j| \leq \epsilon(1 - \delta_j)n + |F_j|.
\]
Since $C_j \subset B_j$, the set $\phi(F_j)C_j$ is disjoint from the sets $\phi(F_{j+1})C_{j+1}, \ldots, \phi(F_k)C_k$; since $B_j \subset B$, the map $s \mapsto \phi(s)c$ defined on $F_j$ is injective for all $c \in B_j$, and thus for all $c \in C_j$.

Note now that
\[
\epsilon(1 - \delta_j)n = \epsilon \left( 1 - \delta' - \left( 1 + \frac{\eta}{1 - \epsilon} \right) (1 + \kappa_{j+1})\sigma_{j+1} \right) n \geq \epsilon(1 - \sigma_{j+1})(1 - \kappa_j)n,
\]
where we set
\[
(2.2) \quad \kappa_j = \max \left( \frac{\delta' + \sigma_{j+1} \left[ \kappa_{j+1} \left( 1 + \frac{\eta}{1 - \epsilon} \right) + \frac{\eta}{1 - \epsilon} \right]}{1 - \sigma_{j+1}}, \kappa_{j+1} \right),
\]
and conclude that
\[
(2.4) \quad (1 - \kappa_j)\lambda_j \leq \frac{|\phi(F_j)C_j|}{n} \leq \lambda_j \leq (1 + \kappa_j)\lambda_j,
\]
as desired. We see that we have proved the four conditions that we stated $C_j$ would satisfy.

We continue this iteration until we reach $j = 1$. Conclusions (ii) and (iii) in the statement of the lemma are immediate, as is the first half of conclusion (i). We will have conclusion (i) directly from (2.4) provided that
\[
(2.5) \quad \kappa_1 \leq \kappa.
\]
Then
\[
\bigcup_{j=1}^{k} \phi(F_j)C_j = \sum_{j=1}^{k} |\phi(F_j)C_j| \geq n \cdot \sum_{j=1}^{k} (1 - \kappa_j)\lambda_j \geq n \cdot \sum_{j=1}^{k} (1 - \kappa)\lambda_j.
\]
Thus, the second half of conclusion (i) holds (i.e., \{\phi(F_j)C\}_{1 \leq j \leq k, c \in C_j} does $(1 - \epsilon)$-cover \{1, \ldots, n\}) provided that
\[
(2.6) \quad \lambda_1 + \cdots + \lambda_k \geq 1 - \epsilon/2
\]
and that $\kappa \leq \epsilon/2$ (as we may assume: if it is not the case, we reset $\kappa = \epsilon/2$ at the very beginning). Now, by (2.3), we have $(1 - \sigma_j) = (1 - \epsilon)(1 - \sigma_{j+1})$, so (2.5) holds provided that $(1 - \epsilon)^k \leq \epsilon/2$; for that, in turn, to be true, it is enough to set $k$ to be at least a constant times $\epsilon^{-1}\log \epsilon^{-1}$. We can, in fact, choose $k$ such that
\[
(2.7) \quad 1 - \epsilon/2 \leq \sigma_1 = \lambda_1 + \cdots + \lambda_k \leq 1 - \epsilon/4,
\]
since (2.5) holds and since we may assume $\epsilon < 1/2$. 

It remains to verify (2.5). By (2.2), for $1 \leq j \leq k - 1,$
\[
\kappa_j \leq \frac{1}{\epsilon^4} (\delta' + 2\eta + 3\kappa_{j+1}).
\]
Recall that $\delta' = O_1(F_j)(\delta);$ we can set $\delta$ small enough that $\delta' \leq \eta = \kappa/(24/\epsilon)^{k-1}.$
Recall also that we may set $\kappa_k$ to a very small value depending on $k,$ $\epsilon,$ and $\kappa;$ we set $\kappa_k = \eta$ and thus obtain that
\[
\kappa_{k-1} \leq \frac{1}{\epsilon^4} \cdot 6\eta = \frac{\kappa}{(4/\epsilon)^{k-2}},
\]
\[
\kappa_{k-2} \leq \frac{1}{\epsilon^4} (\delta' + 2\eta + 3\kappa_{k-1}) \leq \frac{1}{\epsilon^4} \cdot \frac{6\kappa}{(4/\epsilon)^{k-2}} = \frac{\kappa}{(4/\epsilon)^{k-3}},
\]
\[
\vdots
\]
\[
\kappa_1 \leq \kappa,
\]
where we assume, as we may, that $\epsilon \leq 1/4.$ Thus, (2.5) holds, and we are done. $\square$

The fact that sofic approximations of an amenable group are (asymptotically) conjugate is an easy corollary of Lemma 4 as the following shows. We thank D. Kerr for pointing us in this direction.

**Proposition 5.** Let $G$ be an amenable group. Then, for any $\epsilon > 0$ and any finite $S \subset G,$ there are a subset $S' \subset G$ with $S' \supset S$ and constants $N \in \mathbb{Z}^+,$ $\delta > 0$ such that, if $\phi_1$ and $\phi_2$ are $(S', \delta, n)$-sofic approximations of $G$ with $n \geq N,$ then there is a bijection $\tau : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that, for every $s \in S,$
\[
(\tau \circ \phi_1(s) \circ \tau^{-1})(y) = (\phi_2(s))(y)
\]
for $\geq (1 - \epsilon)n$ values of $x \in \{1, 2, \ldots, n\}.$

**Remark.** It is not actually necessary to require that $n \geq N.$ The statement is both true and relatively straightforward for $n < N;$ we can set $\delta \leq 1/N,$ and then there are no $(S', \delta, n)$-sofic representations with $n \leq N$ provided that $S'$ is large enough (why?); if $G$ is finite (so that we could be kept from choosing $S'$ large enough), we set $S' = G,$ thus making $\phi$ and $\phi'$ into injective homomorphisms from $G$ to $\text{Sym}(n)$ whose images are regular permutation subgroups. It is easy to show then that any two such homomorphisms must be conjugate.

**Proof.** We will use Lemma 4. Its application requires a consecutive choice of $\epsilon,$ $\kappa,$ and a Følner sequence $\{F_j\}_{j \geq 1}.$ We choose $\epsilon$ and $\kappa$ to both equal our $\epsilon/7.$ Lemma 3 then provides us with some values of $k \geq 1,$ $\lambda_1, \ldots, \lambda_k \in (0, 1].$ Since $G$ is amenable, there exists a Følner sequence $\{F_j\}_{j \geq 1}$ such that $|F_j \setminus (F_j \cap s^{-1}F_j)| \leq (\epsilon/7)|F_j|$ for all $s \in S,$ $j \geq 1,$ and such that, moreover, (2.1) holds with $\eta = \kappa/(24/\epsilon)^{k-1}.$ Then Lemma 4 gives us $\delta_0 > 0$ (called $\delta$ in the statement of the lemma) and $N \geq 1,$ as well as a finite set $S_0 \subset G$ (called $S$ in the statement of the lemma) such that conclusions (a)–(d) of Lemma 4 hold for both $\phi_1$ and $\phi_2$—with respect to some collection $C_{1,1}, \ldots, C_{1,k}$ of subsets of $\{1, 2, \ldots, n\},$ in the case of $\phi_1,$ and with respect to another collection $C_{2,1}, \ldots, C_{2,k}$ of subsets of $\{1, 2, \ldots, n\},$ in the case of $\phi_2.$ Set $S' = S_0 \cup S \cup SF_{F_1} \cup F_{F_1}^{-1}$ and $\delta = \min(\delta_0, 3\epsilon^2/49|F_k|).$

By conclusion (e) of Lemma 4, the families $\{\phi_1(F_j)c\}_{1 \leq j \leq k, c \in C_{1,i}}$ are $(\epsilon/7)$-disjoint. We can therefore choose $F_{j,c} \subset F_j$ such that $|F_{j,c}| \geq (1 - 2\epsilon/7)|F_j|,$ and both of the families $\{\phi_1(F_{j,c,c})c\}_{1 \leq j \leq k, c \in C_{1,j}}$ and $\{\phi_2(F_{j,c,c})c\}_{1 \leq j \leq k, c \in C_{2,j}}$ are disjoint.
families of disjoint subsets. Moreover, by conclusion (b), for $i = 1, 2, 1 \leq j \leq k$ and $c \in C_{i,j}$, the map $s \mapsto \phi_i(s)c$ from $F_{j,c}$ to $\phi_i(F_{j,c})$ is injective.

We choose subsets $C'_{i,j} \subset C_{i,j}$, for $i = 1, 2, 1 \leq j \leq k$, such that

$$|C'_{i,j}| = \min_{i=1,2} |C_{i,j}|.$$ 

Then, for $i = 1, 2$ and $1 \leq j \leq k$,

$$\left| \bigcup_{c \in C'_{i,j}} \phi_i(F_{j,c})c \right| = \sum_{c \in C'_{i,j}} |F_{j,c}| \geq \left( 1 - \frac{2\epsilon}{\eta} \right) |F_j| \cdot \min_{i=1,2} |C_{i,j}|$$

$$\geq \left( 1 - \frac{2\epsilon}{\eta} \right) \min_{i=1,2} |\phi_i(F_j)C_{i,j}|$$

$$\geq \left( 1 - \frac{2\epsilon}{\eta} \right) \left( 1 - \frac{\epsilon}{7} \right) \lambda_j n \geq \left( 1 - \frac{3\epsilon}{7} \right) \lambda_j n,$$

where we use conclusion (f) of Lemma [H]. Since $\lambda_1 + \cdots + \lambda_k \geq 1 - \epsilon/7$, this implies, by conclusion (b), that, for $i = 1, 2$, that the set

$$\Lambda_i = \bigcup_{1 \leq j \leq k} \bigcup_{c \in C'_{i,j}} \phi_i(F_{j,c})c$$

satisfies

$$|\Lambda_i| = \sum_{1 \leq j \leq k} \left| \bigcup_{c \in C'_{i,j}} \phi_i(F_{j,c})c \right| = \sum_{1 \leq j \leq k} \sum_{c \in C'_{i,j}} |F_{j,c}|$$

$$\geq \left( 1 - \frac{3\epsilon}{7} \right) \left( \sum_{j=1}^k \lambda_j \right) n \geq \left( 1 - \frac{4\epsilon}{7} \right) n.$$

Since $|C'_{1,j}| = |C'_{2,j}|$ for all $1 \leq j \leq k$, we also see that $|\Lambda_1| = |\Lambda_2|$.

Choose a bijection $\rho_j : C'_{1,j} \rightarrow C'_{2,j}$ for each $1 \leq j \leq k$. For every $x \in \Lambda_1$, there are uniquely determined elements $1 \leq j \leq k$, $c \in C_{1,j}$, $g \in F_{j,c}$ such that $x = \phi_1(g)c$. We define $\tau(x) = \phi_2(g)\rho_j(c) \in \Lambda_2$. We complete the definition of $\tau : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ by letting its restriction to $\{1, \ldots, n\} \setminus \Lambda_1$ be an arbitrary bijection to $\{1, \ldots, n\} \setminus \Lambda_2$.

Now let $s \in S$. For each $1 \leq j \leq k$ and each $c \in C'_{1,j}$,

$$|F_{j,c} \cap s^{-1}F_{j,c}| \geq |F_{j,c}| - |F_j \setminus (F_j \cap s^{-1}F_{j,c})|$$

$$\geq |F_{j,c}| - (|F_j \setminus (F_j \cap s^{-1}F_j)| + |s^{-1}F_j \setminus s^{-1}F_{j,c}|)$$

$$\geq \left( 1 - \frac{2\epsilon}{\eta} \right) |F_j| - \left( \frac{\epsilon}{\eta} |F_j| + \frac{2\epsilon}{\eta} |F_j| \right) = \left( 1 - \frac{5\epsilon}{\eta} \right) |F_j|,$$

where we use the assumption that $|F_j \setminus (F_j \cap s^{-1}F_j)| \leq (\epsilon/7)|F_j|$. Hence,

$$\left| \bigcup_{c \in C'_{1,j}} \phi_1(F_{j,c} \cap s^{-1}F_{j,c})c \right| = \sum_{c \in C'_{1,j}} |F_{j,c} \cap s^{-1}F_{j,c}| \geq \left( 1 - \frac{5\epsilon}{\eta} \right) |F_j| \min_{i=1,2} |C_{i,j}|$$

$$\geq \left( 1 - \frac{5\epsilon}{\eta} \right) \left( 1 - \frac{\epsilon}{7} \right) \lambda_j n = \left( 1 - \frac{6\epsilon}{7} \right) \lambda_j n.$$
Since \( \lambda_1 + \cdots + \lambda_k \geq 1 - \epsilon/7 \), this implies that the set

\[
\Lambda_{1,s} = \bigcup_{1 \leq j \leq k} \bigcup_{c \in C'_{1,j}} \phi_i(F_{j,c} \cap s^{-1}F_{j,c}) c
\]

has at least \((1 - \epsilon + 6\epsilon^2/49)n\) elements.

Since \( \phi_1 \) and \( \phi_2 \) are \((S',\delta,n)\)-sofic approximations, we see that, for \( i = 1,2 \), the set

\[
R_i = \{ x \in \{1,\ldots,n\} : \phi(s)x = \phi(sg)\phi(g)^{-1}x \quad \forall g \in F_k \}
\]

has at least \((1 - |F_k|\delta)n \geq (1 - 3\epsilon^2/49)n\) elements. Thus, the set

\[
\Lambda = \tau(\Lambda_{1,s}) \cap \tau(R_1) \cap R_2
\]

has at least \((1 - \epsilon + 6\epsilon^2/49 - 2 \cdot 3\epsilon^2/49)n = (1 - \epsilon)n\) elements.

Let \( y \in \Lambda \). Then \( \tau^{-1}(y) \in \Lambda_{1,s} \). As a consequence, there are uniquely determined elements \( 1 \leq j \leq k \), \( c \in C'_{1,j} \), \( g \in F_{j,c} \cap s^{-1}F_{j,c} \) such that \( \tau^{-1}(y) = \phi_1(g)c \); moreover, \( y = \phi_2(g)\rho_j(c) \). Since \( \tau^{-1}y \in R_1 \), we know that

\[
(\phi_1(s))(\tau^{-1}(y)) = (\phi_1(sg)\phi_1(g)^{-1})(\phi_1(g)c) = \phi_1(sg)c.
\]

Similarly, since \( y \in R_2 \), we know that

\[
(\phi_2(s))(y) = \phi_2(sg)\rho_j(c).
\]

Since \( g \in s^{-1}F_{j,c} \), we know that \( sg \in F_{j,c} \), so

\[
\tau(\phi_1(sg)c) = \phi_2(sg)\rho_j(c).
\]

In other words,

\[
(\tau \circ \phi_1 \circ \tau^{-1})(y) = (\phi_2(s))(y)
\]

for all \( y \in \Lambda \), as was desired. \( \square \)

This is a good point at which to emphasize the relation with the work of Arzhantseva and Paunescu [AP15], who introduced the concept of weakly stable groups. Translated into the language used here, their definition reads as follows: let \( G \) be a finitely generated group, and let \( A \) be a finite set of generators of \( G \). The group \( G \) is said to be weakly stable if, for every \( \epsilon > 0 \), there are a \( \delta > 0 \) and a finite subset \( S' \subset G \) with \( A \subset S' \) such that, for every \( n \geq 1 \) and every \((S',\delta,n)\)-sofic approximation \( \phi : S' \to \text{Sym}(n) \) of \( G \), there is a homomorphism \( \phi' : G \to \text{Sym}(n) \) such that \( d_h(\phi(g),\phi'(g)) < \epsilon \) for all \( g \in A \). (Actually, [AP15] requires \( S' \) to be the ball \( B(r) = \{ g_1 \cdots g_k : g_i \in A \cup A^{-1}, k \leq r \} \) for \( r = 1/\delta \), but it is easy to see that the definition thus obtained is equivalent to the one given here.)

It is stated by [AP15] Theorem 1.1 that a finitely generated amenable group \( G \) is weakly stable if and only if it is residually finite. Let us see how to prove this using Proposition 5. (We make no claim to originality here; we are simply showing how to do matters in elementary language, without using ultraproducts. In particular, the procedure in [AP15] [6] is very close to what we will do.) Let \( G \) be finitely generated and amenable. It is easy to show that weak stability implies residual finiteness. Let us prove the converse. Assume that \( G \) is residually finite. Let \( \epsilon > 0 \).

Proposition 5 (with \( S = A \)) gives us a finite subset \( S' \subset G \) with \( A \subset S' \) and constants \( N \in \mathbb{Z}^+ \), \( \delta > 0 \). Since \( G \) is residually finite, there exists a homomorphism \( \phi_0 : G \to H \), \( H \) finite, such that \( \phi_0(g) \neq e \) for every \( g \in S' \) with \( g \neq e \). We compose \( \phi_0 \) with the map \( \rho : H \to \text{Sym}(H) \sim \text{Sym}(n_0) \), \( n_0 = |H| \), defined by the action of \( H \) on itself by left multiplication, and obtain a homomorphism \( \rho \circ \phi_0 : G \to \text{Sym}(n_0) \) such that, for every \( g \in S' \) with \( g \neq e \), \( (\rho \circ \phi_0)(g) \) has no fixed points. Set
\( \delta' = \min(\epsilon, \delta/n_0, 1/N). \) Now let an \((S', \delta', n)\)-sofic approximation \( \phi : S' \to \text{Sym}(n) \) be given. If \( n < \max(N, n_0/\delta) \leq 1/\delta' \), then \( \phi \) is actually a homomorphism, and we are done. Assume that \( n \geq \max(N, n_0/\delta) \). Let \( \phi_1 : S' \to \text{Sym}(n) \) be the composition of the direct product \((\rho \circ \phi_0)^\ell : S' \to \text{Sym}(\ell n_0) \) \((\ell \text{ copies of } \rho \circ \phi_0, \ell = \lfloor n/n_0 \rfloor) \) with the embedding \( \text{Sym}(\ell n_0) \to \text{Sym}(n) \) induced by the inclusion \( \{1, 2, \ldots, \ell n_0\} \to \{1, 2, \ldots, n\} \). Then \( \phi_1 \) is a homomorphism such that \( \phi_1(g) \) has \( < n_0 \leq \delta n \) fixed points for every \( g \in S' \), \( g \neq e \); in particular, it is an \((S', \delta, n)\)-sofic approximation. Since \( \phi \) is also an \((S', \delta, n)\)-sofic approximation, we may apply Proposition 5 and we obtain a bijection \( \tau : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \) such that, for \( \phi' : G \to \text{Sym}(n) \) defined by \( \phi'(g) = \tau \circ \phi_1(g) \circ \tau^{-1} \) for \( g \in G \),

\[
d_h(\phi(g), \phi'(g)) < \epsilon
\]

for all \( g \in A, g \neq e \), and \( d_h(\phi(e), \phi'(e)) = d_h(\phi(e), \text{id}) < \delta' \leq \epsilon \). Since \( \phi' \) is a homomorphism, we have proved that \( G \) is weakly stable.

3. Baumslag–Solitar groups and the Higman group

The Baumslag–Solitar group \( \text{BS}(1, m) \) is defined by

\[
\text{BS}(1, m) = \langle a_1, a_2 : a_1^{-1}a_2a_1 = a_2^m \rangle,
\]

where \( m \geq 2 \). It is clear why we care about \( \text{BS}(1, m) \): the elements \( a_1, a_2 \) of the Higman group generate a group isomorphic to \( \text{BS}(1, 2) \).

Fix \( M \geq 1 \). Then the sets

\[
F_n = \{a_1^i a_2^j : 0 \leq i < 2n, 0 \leq j < 2Mm^{2n}\}, \quad n \geq 1,
\]

satisfy

\[
|sF_n \Delta F_n| \leq |F_n|/n \quad \text{for all } s \in \{a_1, a_2\},
\]

so \( \text{BS}(1, m) \) is amenable.

Proof of Theorem 2. Assume that the group \( H_{4,m} \) is sofic. We know that, for every sofic group \( H \), every semidirect product of the form \( F \ltimes H, F \) finite, is sofic. (This is true more generally for \( F \) amenable \cite{ES06}.) Hence, the following semidirect product is sofic:

\[
G = (\mathbb{Z}/4\mathbb{Z}) \ltimes H_{4,m},
\]

where we define the action of \( \mathbb{Z}/4\mathbb{Z} \) as follows, in terms of a generator \( t \) of \( \mathbb{Z}/4\mathbb{Z} \):

\[
ta_1 t^{-1} = a_2, \quad ta_2 t^{-1} = a_3, \quad ta_3 t^{-1} = a_4, \quad ta_4 t^{-1} = a_1.
\]

Set \( S' \subseteq G \) finite, \( \delta > 0 \); we will specify them later. Since \( G \) is sofic, there is an \((S', \delta/2, k)\)-sofic approximation \( \alpha \) of \( G \) for some \( k \geq 1 \). Then, for any \( r \geq 1 \), the direct product of \( r \) copies of \( \alpha \) is an \((S', \delta/2, rk)\)-sofic approximation of \( G \). Assume from now on that \( n \geq 2k/\delta \). Let \( r = \lfloor n/k \rfloor \). Then

\[
n \geq rk > n - k \geq \left( 1 - \frac{\delta}{2} \right) n.
\]

We embed \( \text{Sym}(rk) \) in \( \text{Sym}(n) \) and obtain an \((S', \delta, n)\)-sofic approximation \( \phi \) of \( G \).

Clearly, \( \phi \) restricts to an \((S'_0, \delta, n)\)-sofic approximation \( \phi|_{\text{BS}(1,2)} \) of \( \text{BS}(1, m) = \langle a_1, a_2 \rangle a_1^{-1}a_2a_1 = a_2^m \rangle < G \), where \( S'_0 = S' \cap \text{BS}(1, m) \). Since \( \text{BS}(1, m) \) is amenable, we will be able to use Proposition 5 ("any two sofic approximations of an amenable
group are conjugate”). Now, BS(1, m) has an \((S_0', \delta, n)\)-sofic approximation \(\psi\) that is easily described: identifying the set \(\{1, \ldots, n\}\) with \(\mathbb{Z}/n\mathbb{Z}\), we define
\[
\psi(a_2) = (x \mapsto x - 1 \mod n),
\]
\[
\psi(a_1) = (x \mapsto m^{-1}x \mod n).
\]
(We recall that \(n\) is coprime to \(m\).) This defines a homomorphism from BS(1, m) to \(\text{Sym}(\mathbb{Z}/n\mathbb{Z})\); i.e., condition (ii) in Definition 2 holds for \(\delta\), \(S_0'\) completely arbitrary (that is,\[
d_h(\phi(g)\phi(h), \phi(gh)) = 0
\]
for all \(g, h \in \text{BS}(1, m)\)). It remains to check rule (iii) in Definition 2; let us do this.

Given a nontrivial reduced word \(w = a_1^{i_1}a_2^{i_2} \cdots a_k^{i_k}, i_j = 1, 2\), the map \(\psi(w)\) is a linear polynomial map \(x \mapsto P(x) \mod n\) from \(\mathbb{Z}/n\mathbb{Z}\) to itself, where \(P(x) = m^ax + b\), \(a, b \in \mathbb{Z}\), with \(a\) and \(b\) depending only on \(w\) and not both equal to 0. An element \(x \in \mathbb{Z}/n\mathbb{Z}\) is a fixed point of \(x \mapsto P(x) \mod n\) if and only if \((m^a - 1)x = -b \mod n\). Thus, \(\psi(w)\) has at most \(m^a - 1\) \(O_w(1)\) fixed points unless \(m^a \equiv 1 \mod n\) and \(b \equiv 0 \mod n\). Clearly, if \(a \neq 0\) and \(m^a \equiv 1 \mod n\), then \(m^{|a|} \geq n + 1\); if \(b \neq 0\) and \(b \equiv 0 \mod n\), then \(|b| \geq n\). At the same time, \(|a| \leq \ell\) and \(|b| \leq m^\ell\), where \(\ell = \sum_j |r_j|\). It is also clear that \(a = 0, b = 0\) only when \(w\) equals the identity in BS(1, m). Hence, for \(S'\) given, \(\psi\) is an \((S_0', \delta, n)\)-sofic approximation provided that \(n\) is larger than a constant depending only on \(S_0'\) and \(\delta\), something that we can assume.

We can hence apply Proposition 5 and obtain a bijection \(\tau : \{1, 2, \ldots, n\} \rightarrow \mathbb{Z}/n\mathbb{Z}\) such that
\[
(\tau \circ \phi(s) \circ \tau^{-1})(x) = (\psi(s))(x)
\]
for all \(s \in S_0\) and \(\geq (1 - \epsilon)n\) values of \(x \in \mathbb{Z}/n\mathbb{Z}\), where \(S_0 \subset \text{BS}(1, m)\) and \(\epsilon > 0\) are arbitrary (and \(S_0'\) and \(\delta > 0\) are set in terms of \(S_0\) and \(\epsilon\)).

What remains is routine. Let \(S_0 = \{a_1, a_2\}\). Let
\[
S' = S_0' \cup \{e, a_1, a_2, a_2^{-1}, t a_1, t = t^{-1}, t^2, t^{-3}\}.
\]
(That is, we may add those elements to \(S'\) if needed.) Since \(ta_1t^{-1}a_2^{-1} = e\) in \(G\), we see that
\[
(\phi(t)\phi(a_1)\phi(t^{-1}))(x) = \phi(a_2)(x)
\]
for \(\geq (1 - O(\epsilon))n\) values of \(x \in \mathbb{Z}/n\mathbb{Z}\), so
\[
(\tau\phi(t^{-1})\tau^{-1})^{-1}\psi(a_1)(\tau\phi(t^{-1})\tau^{-1})(x)
= (\tau\phi(t^{-1})\tau^{-1})\tau\phi(a_1)\tau^{-1}\tau\phi(t^{-1})\tau^{-1})(x)
= (\tau\phi(t^{-1})\tau^{-1})(\tau\phi(a_1)\tau^{-1})(\tau\phi(t^{-1})\tau^{-1})(x)
= (\tau\phi(t^{-1})\tau^{-1})(x) = (\psi(a_2))(x)
\]
for \(\geq (1 - O(\epsilon))n\) values of \(x \in \mathbb{Z}/n\mathbb{Z}\). Defining \(g : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}\) by \(g = \tau\phi(t^{-1})\tau^{-1}\), we see that this means that
\[
g^{-1}(m^{-1}g(x)) = x - 1
\]
for \(\geq (1 - O(\epsilon))n\) values of \(x\), so
\[
g(y + 1) = mg(y)
\]
for \(\geq (1 - O(\epsilon))n\) values of \(y = x - 1 \in \mathbb{Z}/n\mathbb{Z}\).

At the same time, because \(t^{-4} = e\) in \(G\), we know that
\[
x = \phi(e)x = \phi(t^{-4})(x) = (\phi(t^{-1}))^4(x)
\]
for \( \geq (1 - O(\epsilon))n \) values of \( x \in \mathbb{Z}/n\mathbb{Z} \), so
\[
g^4(y) = (\tau \phi(t^{-1})\tau^{-1})^4(y) = y
\]
for \( \geq (1 - O(\epsilon))n \) values of \( y \in \mathbb{Z}/n\mathbb{Z} \). Let \( V \) be the set of all such values of \( y \); clearly, \( V' = V \cap g^{-1}V \cap g^{-2}V \cap g^{-3}V \) has size \( |V'| \geq (1 - O(\epsilon))n \).

Let \( f(y) = g(y) \) for \( y \in V' \) and \( f(y) = y \) for \( y \notin V' \). Then
\[
f^4(y) = y
\]
for all \( y \in \mathbb{Z}/n\mathbb{Z} \), and
\[
f(y + 1) = mf(y)
\]
for \( \geq (1 - O(\epsilon))n \) values of \( y \in \mathbb{Z}/n\mathbb{Z} \), as desired. \( \square \)

As we mentioned in the introduction, and as should be clear from above, the same proof works if one or more relations are added to the relations
\[
\begin{align*}
t^4 &= e, \quad ta_1t^{-1} = a_2, \quad ta_2t^{-1} = a_3, \\
ta_3t^{-1} &= a_4, \quad ta_4t^{-1} = a_1, \quad a_1^{-1}a_2a_1 = a_2^m
\end{align*}
\]
(3.4)
defining \( (\mathbb{Z}/4\mathbb{Z}) \ltimes H_{4,m} \). Assume that \( \text{BS}(1,m) \) embeds (by means of the map taking \( a_1 \) to \( a_1 \) and \( a_2 \) to \( a_2 \)) in the quotient of \( (\mathbb{Z}/4\mathbb{Z}) \ltimes H_{4,m} \) obtained in this way. (Just to give an example—a quick check via GAP software suggests that this is the case when the relation being added is \( (a_1a_3)^3 = e \).) We can then go through the proof above and obtain a result similar to Theorem [2]

For instance, assuming that \( \text{BS}(1,m) \) does embed in the group \( G \) given by the relation \( (a_1a_3)^3 = e \) and the relations in (3.4), we obtain that, if \( G \) is sofic, then, for any \( \epsilon > 0 \), there is an \( N > 0 \) such that, for every \( n \geq N \), there is a bijection \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \) for which, for \( g(x) = f^2(f^{-2}(x) + 1) \),
\[
f(x + 1) = mf(x), \quad g(g(x + 1) + 1) + 1 = x
\]
for at least \( (1 - \epsilon)n \) elements \( x \) of \( \mathbb{Z}/n\mathbb{Z} \) and, moreover,
\[
f(f(f(f(x)))) = x
\]
for all \( x \in \mathbb{Z}/n\mathbb{Z} \). Notice that \( g(x) \) behaves \( (m-)\)-locally like a constant times \( x^m \), meaning that, for at least \( (1 - O(\epsilon))n \) elements \( x \) of \( \mathbb{Z}/n\mathbb{Z} \),
\[
g(mx) = m^m g(x).
\]
This can be easily seen as follows: if \( f(y + 1) = my \) is valid for \( y = f^{-1}(x) \), \( y = f^{-1}(f^{-1}(x) + 1) \), \( y = f^{-2}(x) \), and \( y = mf^{-1} + k \), \( 0 \leq k \leq m - 1 \), then
\[
\begin{align*}
g(mx) &= f^2(f^{-2}(mx) + 1) = f^2(f^{-1}(f^{-1}(x) + 1) + 1) = f(mf^{-1}(x) + 1) \\
&= f(mf^{-1}(x) + m) = mf(mf^{-1}(x) + m - 1) = \cdots = m^m f(mf^{-1}(x)) \\
&= m^m f(f(f^{-2}(x) + 1)) = m^m g(x).
\end{align*}
\]
Of course, we obtain the same conclusion, without the condition \( f(f(f(f(x)))) = x \), if we remove the relation \( t^4 = 1 \); we did not use the condition \( f(f(f(f(x)))) = x \) in any step of [3].
4. Few cycles of length 3

The following statement resembles Theorem 1 but is neither weaker nor stronger. It will follow readily from the fact that the groups $H_{3,m}$ are finite. It should be clear that the proof would work just as well for analogous statements corresponding to any other finite presentation of the trivial group.

**Lemma 6.** There is a $\delta > 0$ such that, for every coprime integers $n, m > 1$ and every bijection $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, there are at least $\geq \delta n$ values of $x \in \mathbb{Z}/n\mathbb{Z}$ for which at least one of the equations

$$f(x + 1) = mf(x), \quad f(f(f(x))) = x$$

fails to hold.

**Proof.** Assume that the statement of the Lemma is false; that is, assume that equations (4.1) hold for $\geq (1 - \delta)n$ values of $x \in \mathbb{Z}/n\mathbb{Z}$, where $\delta > 0$ is arbitrarily small. We know that the group $H_{3,m}$ generated by elements $a_1, a_2, a_3$ satisfying the relations

$$a_1^{-1}a_2a_1 = a_2^m, \quad a_2^{-1}a_3a_2 = a_3^m, \quad a_3^{-1}a_1a_3 = a_1^m$$

is finite [Joh97, Ch. 7]. In other words, the normal closure of the subgroup of $\mathbb{F}_3$ generated by the words

$$w_1(a_1, a_2, a_3) = a_1^{-1}a_2a_1a_2^{-m}, \quad w_2(a_1, a_2, a_3) = a_2^{-1}a_3a_2a_3^{-m}, \quad w_3(a_1, a_2, a_3) = a_3^{-1}a_1a_3a_1^{-m}$$

has a finite index in $\mathbb{F}_3$. Let $k$ be the order of $a_1$ in $H_{3,m}$. Then $a_1^k$ is equal to a product of conjugates of $w_1, w_2, w_3$ and their inverses.

Let $g_1, g_2, g_3 : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be defined by

$$g_1(x) = x - 1, \quad g_3(x) = f(f^{-1}(x) - 1), \quad g_2(x) = f^2(f^2(x) - 1).$$

It is easy to check that, if (4.1) holds for $\geq (1 - \delta)n$ values of $x$, then

$$(g_1g_3)(x) = f^3f^{-1}(x) - 1 = \frac{1}{m}f^3f^{-1}(x) - 1 = \frac{x - m}{m} = f(f^{-1}(x - m) - 1) = (g_3g_1^m)(x)$$

holds for $\geq (1 - 2\delta)n$ values of $x$, so

$$d_h(w_3(g_1, g_2, g_3), e) \leq 2\delta.$$

Since $g_3 = f_1f^{-1}, \quad g_2 = f_3f^{-1} = f^2g_1f^{-2},$ and $g_1 = f^3g_1f^{-3} = f^2g_3f^{-2},$ we deduce that we also have

$$d_h(w_i(g_1, g_2, g_3), e) = O(\delta)$$

for $i = 1, 2$. Hence,

$$d_h(w \cdot w_i(g_1, g_2, g_3) \cdot w^{-1}, e) = O_w(\delta)$$

for any word $w \in \mathbb{F}_3$ and any $i = 1, 2, 3$. 


As we were saying, $a^k_1$ is equal to a product of conjugates of $w_1, w_2, w_3$ and their inverses; therefore,

$$d_n(g^k_1, e) = O(\delta).$$

At the same time, it is trivial that $d_n(g^k_1, e) = 1$ for large enough $p$. Setting $\delta$ smaller than a constant, we obtain a contradiction. □

Theorem I is a different matter. On the one hand, the map $f_{m,n}$ in the statement of Theorem I is actually an exponentiation map, instead of merely behaving locally like one. On the other hand, the conclusion of Theorem I is stronger than that of Lemma 6: we will see that the proportion of elements of $\mathbb{Z}/p\mathbb{Z}$ fixed by $f_{m,n}$ actually goes to 0, as opposed to just being bounded away from 1.

Let us first give a sketch of the proof of Theorem I. Suppose that

$$m^{m^x} = x$$

for a positive proportion of all $x$. Then there is a bounded $k$ such that (4.2) holds for both $x$ and $x + k$ for a positive proportion of all $x$. (This is a simple fact that we will derive explicitly; some readers will recognize the idea of Poincaré recurrence at work here.) For a positive proportion of all $x \in \{0,1,\ldots,p-1\}$, we should thus have

$$m^{m^{x+k}} = x + k = m^{m^x} + k;$$

writing $y = m^x$, we obtain that

$$m^{y^{m^k}} = m^y + k$$

for a positive proportion of all $y$. By the same argument as before, there is a bounded $\ell$ such that (4.3) is true for both $y$ and $\ell y$ for a positive proportion of all $y$. Writing $z = m^y$, we seem to obtain that

$$(z + k)^{e^{m^k}} = z^\ell + k$$

for a positive proportion of all $z$. However, (4.4) is a nontrivial polynomial equation and thus has a finite number of roots, giving us a contradiction.

We need to be a little more careful with this. For one thing, $f_{m,n}(x)$ is not exactly an exponentiation map; the exponentiation map $x \rightarrow m^x \mod n$ is defined from $\mathbb{Z}/\text{ord}_n(m)\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$, not from $\mathbb{Z}/n\mathbb{Z}$ to itself. Let us write out a correct proof in detail.

**Proof of Theorem I.** Let $f_{m,n}$ be as in the statement. Suppose that $f_{m,n}(f_{m,n}(f_{m,n}(x))) = x$ for all $x$ in a subset $X$ of $\{0,1,\ldots,n-1\}$ of size $|X| \geq \delta n$, where $\delta > 0$; we will show that this leads to a contradiction for $\delta$ sufficiently small and $n$ larger than a constant depending on $m$ and $\delta$. We can assume that the order of $m$ in $\left(\mathbb{Z}/n\mathbb{Z}\right)^*$ is at least $\delta n$, as otherwise the image of $f_{m,n}$ would be contained in a set of size $< \delta n$, and we would have a contradiction immediately.

If there were more than $\delta n/2$ elements $x$ such that the element $x'$ of $X$ immediately after $x$ were at a distance at least $2/\delta$ from $x$ in the natural cyclic ordering $(0,1,2,\ldots)$ for $\mathbb{Z}/n\mathbb{Z}$, we would get a contradiction: the total distance traversed by going through the elements $x_1, x_2, \ldots, x_m$ of $S$ and then from $x_m$ to $x_1$ would be more than $n$. Hence, there is a constant $1 \leq k < 2/\delta$ such that, for at least $\delta n/2$ elements $x$ of $X$, $x + k$ is also in $X$. The “$-1$” term is here simply because we want $x + k$, not just $x + k$ reduced modulo $n$, to
be in $X$.) We can assume that $n \geq \sqrt{20/\delta}$, so $\delta^2 n/4 - 1 \geq \delta^2 n/5$. We thus have $|X \cap (X - k)| \geq \delta^2 n/5$.

For all $x \in X \cap (X - k)$,
\[ f_{m,n}(f_{m,n}(f_{m,n}(x))) + k = x + k = f_{m,n}(f_{m,n}(f_{m,n}(x + k))) = f_{m,n}(f_{m,n}(m^k f_{m,n}(x))), \]
where, given $a \in \mathbb{Z}$, we write $\bar{a}$ for the element of $\{0, 1, \ldots, n - 1\}$ such that $\bar{a} \equiv a \mod n$. Obviously, $\bar{m^k f_{m,n}(x)} = \bar{m^k f_{m,n}(x)} - c \mod n$ for some $0 \leq c < m^k$. Writing $y$ for $f_{m,n}(f_{m,n}(x))$ and noting that $f_{m,n}(x) = \bar{m^x}$, we obtain that
\[ f_{m,n}(y) + k = f_{m,n}(\frac{m^{m^k f_{m,n}(x) - cn}}{\delta^m}) = f_{m,n}(\bar{c'(m^{f_{m,n}(x)^{m^k}})}), \]
\[ (4.5) \]
where $c' \in \{1, \ldots, n - 1\}$ is such that $c' \equiv m^{-cn} \mod n$. Because the order of $m$ is at least $\delta n$, the map $x \mapsto \bar{m^x} \mod n$ can send at most $1/\delta$ elements to the same element; hence, there are at least $\delta^2 \cdot \delta^2 n/5 = \delta^4 n/5$ values of $y$ for which (4.5) holds. Since there are $m^k$ possible values of $c$, there are at most $m^k$ possible values of $c'$; choose one such that
\[ f_{m,n}(y) + k = f_{m,n}(\bar{c' y^{m^k}}) \]
\[ (4.6) \]
holds for at least $\delta^4 n/5 m^k$ elements $y \in \{1, \ldots, n - 1\}$.

Now, by the same argument as before, this implies that there is a constant $\ell = m^r$, $1 \leq r < 10m^\delta/\delta^4$, such that there are at least $\delta^8 n/100 m^k$ elements $y \in \{1, \ldots, n - 1\}$ for which
\[ f_{m,n}(y) + k = f_{m,n}(\bar{c' y^{m^k}}) \quad \text{and} \quad f_{m,n}(\ell y) + k = f_{m,n}(\bar{c'(\ell y)^{m^k}}). \]
\[ (4.7) \]
(Note that no $-1$ term is needed now.)

Write $z$ for $f_{m,n}(y)$. Then $f_{m,n}(\ell y) = \bar{m^\ell y} \equiv z^\ell \mod n$, and, similarly,
\[
\frac{f_{m,n}(\bar{c'(\ell y)^{m^k}})}{m^\ell y^{m^k}} = \frac{m^\ell y^{m^k}}{m^{\ell m^k} c' y^{m^k} - \kappa n} = \kappa' \left( m^{c' y^{m^k}} \right)^{\ell m^k}
\]
\[ = \kappa' f_{m,n}(\bar{c' y^{m^k}})^{\ell m^k} \]
where $0 \leq \kappa < \ell m^k$ and $\kappa' \in \{1, \ldots, n - 1\}$ is such that $\kappa' = m^{-\kappa n} \mod n$. Hence, we obtain from (4.7) that
\[ z^\ell + k \equiv \kappa'(z + k)^{\ell m^k} \mod n \]
or, what is the same,
\[ (z + k)^{\ell m^k} - m^{\kappa n}(z^\ell + k) \equiv 0 \mod n. \]
\[ (4.8) \]
This is, of course, an equation of the form $P(z) \equiv 0 \mod n$, where $P$ is a nonconstant polynomial with integer coefficients.

There are two possible ways to proceed here. One would be to show that the discriminant of this polynomial is nonzero and then bound its common factor with $n$. We follow an alternative route\footnote{We thank I. Shparlinski for this suggestion.} by a result of Konjagin’s [Kon79b, Kon79a],
the number of roots mod \( n \) of a nonzero polynomial of degree \( d \) is at most \( c_d n^{1-1/d} \), where \( c_d = d/e + O(\log^2 d) \). Our polynomial \( P(z) \) is certainly nontrivial (it has leading coefficient 1); hence, the number of roots of \( P(z) \equiv 0 \mod n \) is

\[
O_{\ell,m,k} \left( n^{1-1/\ell^m k} \right).
\]

Of course, there are different possible choices of \( k \) and \( \ell \), but, since there are at most \( 2/\delta \) and \( 10m^k/\delta^4 \) such choices, respectively, and since both \( k \) and \( \ell \) are bounded in terms of \( m \) and \( \delta \) (really just in terms of \( \delta \), in the case of \( k \)), the total number of values of \( z \) that are roots of \( \text{(4.8)} \) for some possible \( z \) is

\[
O_{m,\delta} \left( n^{1-\eta_{m,\delta}} \right),
\]

where \( \eta_{m,\delta} > 0 \) depends only on \( m \) and \( \delta \).

At the same time, the number of elements \( z = f_{m,n}(y) \) we are considering is at least

\[
\delta \cdot \frac{\delta^8 n}{100m^k} = \frac{\delta^9 n}{100m^k} \geq \frac{\delta^9}{100m^{2/\delta}} n.
\]

Thus, we obtain a contradiction provided that \( n \) is larger than a constant depending only on \( m \) and \( \delta \). \( \square \)

5. Fixed points

Following the example of [HR12], we will use ideas from \( p \)-adic analysis to prove Proposition 3.

Proof of Proposition 3

Let \( f : \mathbb{Z}/p^r \mathbb{Z} \to \mathbb{Z}/p^r \mathbb{Z}, r \geq 5, \) be given. Assume that

\[
f(f(f(f(x)))) = x
\]

for at least \( p^r/2 \) values of \( x \), and that

\[
f(x + 1) = mf(x)
\]

for at least \( p^r - c \cdot p^{r/4} \) values of \( x \in \mathbb{Z}/p^r \mathbb{Z} \), where \( c > 0 \) will be set later. Define \( g : \mathbb{Z}/p^r \mathbb{Z} \to \mathbb{Z}/p^r \mathbb{Z} \) by \( g(x) = (p-1)^{-1}f((p-1)x) \). Then

\[
g(x + 1) = m^{p-1}g(x)
\]

for at least \( p^r - (p-1)c \cdot p^{r/4} \) values of \( x \in \mathbb{Z}/p^r \mathbb{Z} \).

Hence, there are \( c_1, \ldots, c_k \in \mathbb{Z}/p^r \mathbb{Z}, c_j \neq 0, k \leq (p-1)c\cdot p^{r/4} \), such that, for every \( x \in \mathbb{Z}/p^r \mathbb{Z} \), there is a \( 1 \leq j \leq k \) such that \( g(x) = g_j(x) \), where \( g_j : \mathbb{Z}/p^r \mathbb{Z} \to \mathbb{Z}/p^r \mathbb{Z} \) is defined by

\[
g_j(x) = c_j \cdot s^x
\]

for \( s = m^{p-1} \).

A remark on the definition of the maps \( g_j \) is in order. By Fermat’s little theorem, \( s \equiv 1 \mod p \). Since the kernel of the reduction mod \( p \) map \((\mathbb{Z}/p^r \mathbb{Z})^* \to \mathbb{Z}/p \mathbb{Z}\) has order \( p^{r-1} \), it follows that the map \( x \to s^x \) has a period dividing \( p^{r-1} \) and thus is well-defined as a map from \( \mathbb{Z}/p^r \mathbb{Z} \) to itself. (It is, in fact, well-defined as a function from the \( p \)-adic ring \( \mathbb{Z}_p \) to itself.) The maps \( g_j \) are thus well-defined.

It follows that, if \( g(g(g(g(x)))) = x \), there must be \( 1 \leq j_1, j_2, j_3, j_4 \leq k \) such that

\[
g_{j_4} (g_{j_3}(g_{j_2}(g_{j_1}(x)))) = x.
\]
This implies immediately that \((x, g_{j_1}(x), g_{j_2}(x), g_{j_3}(x))\) is a fixed point of the function \(G\) from \((\mathbb{Z}/p^r\mathbb{Z})^4\) to itself given by
\[
G(x_1, x_2, x_3, x_4) = (g_{j_4}(x_1), g_{j_1}(x_1), g_{j_2}(x_2), g_{j_3}(x_3))
= (c_{j_4} \cdot s^{x_4}, c_{j_1} \cdot s^{x_1}, c_{j_2} \cdot s^{x_2}, c_{j_3} \cdot s^{x_3}).
\]

Our aim is now to show that \(G\) can have at most one fixed point. Since \(G\) is actually well-defined on the \(p\)-adics, we could do this by an appeal to Hensel’s lemma and a brief argument involving the \(p\)-adic metric, but we can do without that (even though that is clearly the idea in what follows).

Let \((x_1, x_2, x_3, x_4) \in (\mathbb{Z}/p^r\mathbb{Z})^4\) be a fixed point of \(G\). Since \(s^x \equiv 1 \mod p\) for every \(x \in \mathbb{Z}/p^r\mathbb{Z}\), this implies that \((x_1, x_2, x_3, x_4) \equiv (c_{j_4}, c_{j_1}, c_{j_2}, c_{j_3}) \mod p\). Now the congruence class \(x \mod p\) determines \(s^x \mod p^2\); thus, we know that \(G(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) \mod p^2\). In general, the congruence class \(x \mod p^k\) determines \(s^x \mod p^{k+1}\), and so, iterating, we find that we know that \(G(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) \mod p^3, p^4, \ldots\) and, lastly, \(\mod p^r\). In other words, \(G\) has at most one fixed point.

Hence, the number of fixed points of all such \(G\) is at most \(k^4 \leq ((p-1)c)^4 \cdot p^r\). Hence, by assumption, \(((p-1)c)^4 \leq 1/2\). We get a contradiction for \(c = 1/2^{1/4}p\).

The same kind of argument can be applied to other relators. For instance, consider the function \(g\) discussed at the end of §6 a bijection \(g : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}\) such that \(g(mz) = m^n g(x)\) for \((1-c_n)n\) values of \(x \in \mathbb{Z}/n\mathbb{Z}\) and \(g(g(g(x)+1)+1)+1 = x\) for either \((1-c_n)n\) or even just (say) \(n/8\) values of \(x \in \mathbb{Z}/n\mathbb{Z}\). It is easy to see how this leads to a contradiction for \(\epsilon_n\) less than a constant \(c\) times \(n^{-2/3}\). Let us do this for \(n\) equal to a prime \(p\) not dividing \(m\), since \(\mathbb{Z}/p\mathbb{Z}\) gives us a nicer framework than \(\mathbb{Z}/p^r\mathbb{Z}\) for this sort of function \(g\).

There are \(c_1, \ldots, c_k \in \mathbb{Z}/p\mathbb{Z}\), \(c_j \neq 0\), \(k \leq \epsilon_p < c p^{1/3}\), such that, for every \(x \in \mathbb{Z}/p\mathbb{Z}\), there is a \(1 \leq j \leq k\) such that \(g(x) = g_j(x)\), where \(g_j : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}\) is defined by
\[
g_j(x) = c_j x^m.
\]
Thus, if \(g(g(g(x)+1)+1)+1 = x\), there must be \(1 \leq j_1, j_2, j_3 \leq k\) such that \((c_{j_4}(c_{j_2}(c_{j_1}x^m + 1)^m + 1)^m)) + 1 = x\).

There are at most \(m^3\) solutions to this equation for the \(j_1, j_2, j_3\) given. Hence, the total number of solutions is at most \(m^3 k^3 < (mc)^3 p\). We obtain a contradiction for \(c < 1/2m\) since then \((mc)^3 p < p/8\).

6. Heuristics

Let us now address the question: is the existence of a function \(f\) as in Theorem 2 plausible?

Let \(f\) be an element taken uniformly at random from the group \(\text{Sym}(n) = \text{Sym}(\mathbb{Z}/n\mathbb{Z})\) of all bijections from \(\mathbb{Z}/n\mathbb{Z}\) to itself. Let \(P_n\) be the probability that
\(f \circ f \circ f = e\). It is clear that, for \(n \geq 5\),

\[
P_n = \frac{1}{n} P_{n-1} + \frac{n-1}{n} \cdot \frac{1}{n-1} P_{n-2} + \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{1}{n-2} P_{n-4}
\]

\[
= \frac{P_{n-1} + P_{n-2} + P_{n-4}}{n}.
\]

(6.1)

It is easy to check that \(P_1 = 1, P_2 = 1, P_3 = 2/3, P_4 = 2/3, P_5 = 7/15\); in particular, \(P_1 \geq P_2 \geq P_3 \geq P_4 \geq P_5\). Using this as the base case of an induction and applying (6.1) for the inductive step, we obtain that \(P_n\) is nonincreasing for \(n \geq 1\). Hence, \(P_n \leq 3P_{n-4}/n\), so, for all \(n \geq 1\),

\[
P_n \leq \frac{3^{[n/4]}}{n(n-4)(n-8) \cdots (n-4[4n/4])} \leq \frac{1}{[n/4]!}
\]

\[
< \frac{1 + O(1/n)}{((n/4-1)/e)^{n/4-1}} \leq \frac{1}{(n/e)^{n/4-1/2}},
\]

where we are using Stirling’s formula.

How many maps \(f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}\) satisfy the condition \(f(x+1) = 2f(x)\) for \(\geq (1-\epsilon)n\) values of \(x\) in \(\mathbb{Z}/n\mathbb{Z}\)? Such a condition “forces” the value of \(f(x+1)\) wherever it is fulfilled; hence, we have the freedom to choose \(f(x+1)\) only at \(m = \lfloor en \rfloor\) places. We also have the freedom to choose where those places are.

Hence, the number of elements of the set \(S_n\) of all such maps \(f\) is bounded by

\[
|S_n| \leq \frac{n!}{(n-m)!} \leq \frac{n^m}{m^m} \cdot m^m \leq \left(\frac{1}{e(1-\epsilon)^{1-\epsilon}}\right)^n n^m.
\]

Hence, for any \(\epsilon < 1/4\), the product \(|S_n| \cdot P_n\) goes to 0 as \(n \to \infty\); indeed, it is \(o(e^{-cn})\) for any \(c > 0\). This implies that

\[
\lim_{N \to \infty} \sum_{n \geq N} |S_n| \cdot P_n = 0.
\]

How do we interpret this? If we model the event \(f \circ f \circ f \circ f = e\) as a random event with probability \(P_n\), independent of whether \(f \in S_n\) (and it is here, and not elsewhere, that the argument becomes a heuristic, rather than a proof), then the expected value of the number of elements \(f\) of \(S_n\) satisfying \(f \circ f \circ f \circ f = e\) is \(|S_n| \cdot P_n\). Hence,

\[
\sum_{n \geq N} |S_n| \cdot P_n
\]

is the expected value of the number of elements of \(f\) of \(S_n\), \(n \geq N\), satisfying \(f \circ f \circ f \circ f = e\), and thus is an upper bound on the probability that there is at least one \(n \geq N\) and at least one \(f \in S_n\) such that \(f \circ f \circ f \circ f = e\). As we just saw, this upper bound goes to 0 as \(N \to \infty\) (indeed, it goes to 0 faster than any exponential). In other words, the conclusion of Theorem 2 would seem to be made implausible by a simple probabilistic model.

There are two things to discuss: why this heuristic is no proof, and how much weight one should place on it, if any.

What keeps us from making this heuristic into a proof is the difficulty in ensuring mutual independence of our random events. It is easy to see that, if a map \(f\) satisfying \(f(x+1) = mf(x)\) for \(\geq (1-\epsilon)n\) values \(x \in \mathbb{Z}/n\mathbb{Z}\) is taken at random, the probability that \(f(f(f(f(x))))) = x_0\) for a given \(x_0\) is very close to \(1/n\). Pairwise independence is not difficult either: for \(x_0 \neq x_1\), the probability
that \( f(f(f(x)))) = x \) be true for \( x = x_0 \) or \( x = x_1 \) is very close to \( 1/n(n-1) \) (whether \( x_0 \) and \( x_1 \) are close to each other or not). The problem lies in ensuring the almost-independence of \( k \) such events for \( k \) rather large.

The strength of the heuristic, or rather its weakness, is a nonobvious matter. In number theory, arithmetic properties that have no reason to be correlated are generally believed to be independent in the limit. (Examples are the Hardy–Littlewood conjectures, Chowla’s conjecture, and statements on \( \sum_{n \leq N} \mu(n) \) equivalent to the Riemann hypothesis.) This would seem to support the heuristic. However, we are on unfamiliar terrain here: the functions we are working with behave locally like arithmetic functions almost everywhere but need not be close to them globally.

More importantly, in response to an earlier version of the current version, Kuperberg, Kassabov, and Riley have proven a result that goes against the heuristic.

**Theorem 7 ([KKR]).** For every \( \epsilon > 0 \) and every \( C > 0 \), there is an \( N \) such that, for any \( n \geq N \) and any \( (\log \log n)/C < b_n < C \log n \) coprime to \( n \), there is a bijection \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \) such that

\[
f(x + 1) = b_n f(x)
\]

for at least \( (1 - \epsilon)n \) values of \( x \in \mathbb{Z}/n\mathbb{Z} \), and

\[
f(f(f(f(x)))) = x
\]

for all \( x \in \mathbb{Z}/n\mathbb{Z} \).

This theorem does not necessarily mean that the functions \( f \) in Theorem 2 are likely to exist, or that \( H_{4,m} \) is likely to be sofic. However, it does mean that the naïve heuristic examined in this section should be distrusted.

Moreover, as we said, the work of Glebsky [Gle] implies the following.

**Corollary 8 (To [Gle]).** For every \( m > 2 \) and every \( \epsilon > 0 \), there is an \( N \) such that, for any \( n \geq N \) coprime to \( m \), there is a bijection \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \) such that

\[
f(x + 1) = m f(x)
\]

for at least \( (1 - \epsilon)n \) values of \( x \in \mathbb{Z}/n\mathbb{Z} \), and

\[
f(f(f(f(x)))) = x
\]

for all \( x \in \mathbb{Z}/n\mathbb{Z} \).

While Glebsky works with \( n = p^k \), a statement for such \( n \) implies a statement for all large enough \( n \): just choose \( N \) large enough that, for every \( n \geq N \), there is a multiple of \( p^k \) between \( (1 - \epsilon/2)n \) and \( n \). Then proceed as we did right after (3.3).

Incidentally, from the perspective of a heuristic model such as the one we have just discussed, it is wholly unsurprising that Proposition 8 gives only that the number of points at which \( f(x + 1) \neq m f(x) \) or \( f(f(f(f(x)))) \) is at least \( m^{r/4-1} = n^{1/4}/m \). The proof of Proposition 8 bounds, in effect, the number of \( f \) satisfying a much looser condition than \( f(x + 1) = m f(x) \) outside \( m \) points: it allows \( f(x) \) to be \( c(x)m^x \), where \( c(x) \) can be any of \( m \) distinct values. It is easy to see that there are very roughly \( n^m m^n \) maps satisfying such a condition; this is larger than \( 1/P_n \) whenever \( m \geq n^{1/4} \).
7. FURTHER QUESTIONS

7.1. Bounds for stability. We have chosen to use, for the most part, finitary versions of existing tools to solve problems with finitary formulations. This is not the only option: for example, the work of Arzhantseva and Păunescu [AP15] applies [ES11] (which uses ultraproducts heavily) to prove finitary statements.

Arzhantseva and Păunescu undertook a general study of the following question: given two (or more) elements of Sym(n) that almost satisfy a relation, must they be close to elements that actually satisfy it? To take a particular case—they showed that the answer is “yes” (stability) for the commutator relation

\[ [x, y] = e \]

or, for that matter, for the set of all commutator relations \([x_i, x_j] = e\) (1 ≤ i, j ≤ k) among k elements \(x_i\). (This is [AP15, Main Thm.].)

A line of further inquiry suggests itself: how close is close? That is, the stability of the commutator means that for every \(\epsilon > 0\) there is a \(\delta > 0\) such that, if \(d_h([x, y], e) < \delta\), then there are \(x', y'\) such that \(d_h(x, x'), d_h(y, y') < \epsilon\), and \([x', y'] = e\); the question is: how does \(\delta\) depend on \(\epsilon\)?

While the method based on [KL13] that we follow here, being finitary and explicit, can in principle be made to give a bound in answer to this question, it seems likely that such a bound would be far from optimal. It has been pointed out to us by E. Hrushovski (in a different context) that methods based on ultraproducts can at least sometimes give computable bounds for some problems; still, it seems very likely that the usage of such methods here would give even worse bounds.

7.2. Sofic profile and sofic dimension growth. Cornulier [Cor13] defines the sofic profile of a group \(G\) in terms of the growth as \(\delta \to 0\), for \(S\) fixed, of the least \(n\) such that \(G\) has an \((S, \delta, n)\)-sofic approximation. In particular, a group \(G\) has at most linear sofic profile if for every finite \(S\) there is a constant \(c_S > 0\) such that, for every \(\delta > 0\), there is an \(n \leq c_S/\delta\) such that \(G\) has an \((S, \delta, n)\)-sofic approximation.

The definition of sofic profile was preceded by that of sofic dimension growth [AC]. For \(S\) fixed, define \(\phi_S(r)\) to be the least \(n\) such that \(G\) has an \((S^r, 1/r, n)\)-sofic approximation. Then the question is the growth of \(\phi_S(r)\) as \(r \to \infty\).

Strangely enough, it is unknown whether it is the case that all groups have at most linear sofic profile; that question is [Cor13, Problem 3.18]. We know that there are groups that have a linear sofic profile (meaning: the least \(n\) such that \(G\) has an \((S, \delta, n)\)-sofic approximation obeys \(c_S' \delta \leq n \leq c_S/\delta\) for some \(c_S, c_S' > 0\)). As [Cor13] shows, a group that has at most linear sofic profile either has linear sofic profile or is an LEF group. The concept of an LEF group was introduced by Vershik and Gordon [VG97]. In brief, a group is LEF if one can set \(\delta = 0\); for finitely presented groups, being LEF is equivalent to being residually finite (to see this, choose \(S\) large enough to include all relations).

Since the Higman group has no proper subgroups of finite index [Hig51], it is not residually finite, and thus it is not LEF. Does the Higman group have a linear sofic profile? It is natural to venture that it does not; the question remains tantalizingly open.

It does not seem viable to address this question with the tools in §2 since they worsen dramatically the dependence of \(n\) on \(\delta\). It is not clear whether one can

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5We note that [KL13 §4] credits [OW87] as the source of some of the main ideas in the method.
proceed similarly with other tools. The goal would be to show that, if the Higman group has linear sofic profile, then, for every \( n_0 \geq 1 \), there is an \( n \) with \( |n - n_0| = O(1) \) such that

\[
(f_{2,n}(f_{2,n}(f_{2,n}(x)))) = x
\]

holds for all but \( O(1) \) values of \( x \in \{0, 1, \ldots, n - 1\} \); here \( f_{m,n} \) is as in Definition 1. The converse is clear: if there are enough \( n \) for which (7.1) holds for all but \( O(1) \) values of \( x \in \{0, 1, \ldots, n - 1\} \), then the Higman group does have a linear sofic profile.

As we said in the introduction, we know that (7.1) has a few solutions for some specific \( n \), but such \( n \) are rare (fifth and higher powers of primes) \cite{Gle13}. It does seem very unlikely that there are infinitely many \( n \) for which (7.1) holds for all but \( O(1) \) values of \( x \in \{0, 1, \ldots, n - 1\} \), but showing that this is the case is an open problem.

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Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstrasse 3-5, D-37073 Göttingen, Germany; and IMJ-PRG, UMR 7586, 58 avenue de France, Bâtiment Sophie Germain, Case 7012, 75013 Paris CEDEX 13, France

Email address: helfgott@math.univ-paris-diderot.fr

Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, Illinois 60208

Email address: kate.juschenko@gmail.com

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