VELOCITY MODIFICATION OF THE POWER SPECTRUM FROM AN ABSORBING MEDIUM

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ABSTRACT

A quantitative description of the statistics of intensity fluctuations within spectral line data cubes introduced in our earlier work is extended to the absorbing media. The possibility of extracting three-dimensional velocity and density statistics from both integrated line intensity and the individual channel maps is analyzed. We find that absorption enables the velocity effects to be seen even if the spectral line is integrated over frequencies. This regime, which is frequently employed in observations, is characterized by a nontrivial relation between the spectral index of velocities and the spectral index of intensity fluctuations. For instance, when density is dominated by fluctuations at large scales, i.e., when correlations scale as \( r^{-\gamma} \), \( \gamma < 0 \), the intensity fluctuations exhibit a universal spectrum of fluctuations \( \sim K^{-3} \) over a range of scales. When small-scale fluctuations of density contain most of the energy, i.e., when correlations scale as \( r^{-\gamma} \), \( \gamma > 0 \), the resulting spectrum of the integrated lines depends on the scaling of the underlying density and scales as \( K^{-3+\gamma} \). We show that if we take spectral line slices that are sufficiently thin, we recover our earlier results for thin-slice data without absorption. As a result, we extend the velocity channel analysis (VCA) technique to optically thick lines, enabling studies of turbulence in molecular clouds. In addition, the mathematical machinery developed enables a quantitative approach to solving other problems that involved statistical description of turbulence within emitting and absorbing gas.

Subject headings: ISM: general — ISM: structure — MHD — radio lines: ISM — turbulence

1. INTRODUCTION

There is little doubt that the interstellar medium (ISM) is turbulent (see reviews by Armstrong et al. 1995; Lazarian 1999b; Lazarian et al. 2002). Turbulence proved to be ubiquitous in molecular clouds (Dickman 1985) and diffuse ionized (Cordes 1999) and neutral (Lazarian 1999a) media. Magnetohydrodynamic (MHD) turbulence controls many essential astrophysical processes, including star formation, transport and acceleration of cosmic rays, and transport of heat and mass (see Schlickeiser 1999; Vázquez-Semadeni et al. 2000; Narayan & Medvedev 2001; Wolfire et al. 2003; Cho et al. 2003; see recent reviews by Cho et al. 2002c, hereafter CLV02c; Elmegreen & Scalo 2004; Lazarian & Cho 2004). Unfortunately, we are still groping for the basic properties of interstellar turbulence. For instance, it is unclear whether turbulence in molecular clouds is part of a global turbulent cascade or has a local origin.

Recent years have been marked by substantial progress in theoretical and numerical description of both incompressible and compressible MHD turbulent cascade (Goldreich & Sridhar 1995; Lithwick & Goldreich 2001; Cho & Lazarian 2002b, 2003a, 2003b; CLV02c). It is encouraging that the measured statistics of electron density fluctuations (see Armstrong et al. 1995) and synchrotron and polarization degree fluctuations (see Cho & Lazarian 2002a) are in rough correspondence with the theoretical expectations (see more discussion in Cho & Lazarian 2003b). Further testing requires better input data, and this makes the determination of the actual power spectra of interstellar turbulence extremely important and meaningful. Indeed, it presents a chance to distinguish between different pictures of turbulence, e.g., shocks versus eddies, and to put theoretical constructions developed for various interstellar processes, e.g., cosmic-ray propagation (see Yan & Lazarian 2002), grain dynamics (Lazarian & Yan 2002; Yan & Lazarian 2003), and stochastic reconnection (Lazarian & Vishniac 1999; Lazarian et al. 2004), on solid ground. Even within the picture of MHD cascade that is advocated by the simulations in Cho & Lazarian (2002b), which does not contain shocks, the distribution of energy between Alfvén, slow, and fast modes does depend on the turbulence driving. Therefore, the spectra measured from observations can be used to constrain the driving. In addition, the particular features of the power spectrum, which are associated with energy injection and dissipation, are important. Identification of such features and the spatial variations of their properties will shed light on the dynamics of interstellar gas and processes of star formation (see Lazarian & Cho 2004).

We should stress that while the statistics of density fluctuations can only be considered as an indirect way of testing turbulence, the spectral surveys contain information about the direct measure, namely, the velocity statistics. This statistics is extremely valuable, provided that we can extract it from the observational data. As MHD turbulence is indeed an interdisciplinary subject, high-resolution studies of interstellar turbulence can test contemporary ideas of MHD cascade. The implications of an improved understanding would affect description of a wide range of astrophysical phenomena, from solar flares to gamma-ray bursts (see Lazarian et al. 2003b).

Additional motivation for studying interstellar turbulence stems from advances in direct numerical simulations of the ISM (see review by Vázquez-Semadeni et al. 2000). Unlike MHD simulations focused on obtaining fundamental properties of turbulence (see CLV02c), those of the ISM attempt to include various aspects of interstellar physics. This usually limits the
inertial range covered. However, testing of the numerical results against observations is becoming more interesting as numerical resolution increases. While other techniques for comparing numerics with observations have their problems (see review by Ostriker 2003; Brunt et al. 2003), power spectra and second-order correlation functions look like a promising tool for the future (see Lazarian 1995; Lazarian & Pogosyan 2000, hereafter LP00; Vestuto et al. 2003). The potential for combining different techniques in order to get the properties of turbulence is also great.

Although in this paper we primarily refer to interstellar processes, the technique we discuss has a broader application. For instance, recent advances in X-ray astronomy have allowed the gathering of information on turbulence in intracluster gas (see Inogamov & Sunyaev 2003; Sunyaev et al. 2003). The issues that researchers face in that field are similar to those dealt with in interstellar turbulence studies.

For insight into turbulent processes, a statistical approach is useful (see Monin & Yaglom 1975; Dickey 1995). So far, in the astrophysical context the most tangible progress has been achieved via scintillations and scattering technique (see Spangler 1999). Those measurements are limited to probing fluctuations of electron density on rather small scales, $10^9 - 10^{13}$ cm. This research profited greatly from an adequate theoretical understanding of processes of scintillations and scattering (Goodman & Narayan 1985), which made it very different from other branches of interstellar turbulence research. In comparison, the formation of emission-line profiles in turbulent media has not been properly described until very recently, and it is natural that numerous attempts to study turbulence in the diffuse ISM, H II regions, and molecular clouds using emission lines (see Munch 1958; O'Dell 1986; O'Dell & Castaneda 1987; Miesch & Bally 1994; reviews by Scalo 1987; Lazarian 1992) were only partially successful. This is very unfortunate, as the line profiles contain unique information about the turbulent velocity field. We reiterate that studies of stochastic density provide only indirect insight into turbulence and cannot distinguish between active and fossil turbulence pictures.

Studies of the velocity field have been attempted at different times with velocity centroids (e.g., Munch 1958; Miesch et al. 1999; Miville-Deschênes et al. 2003b). However, it has long been realized that the centroids are affected, in general, by both velocity and density fluctuations (see Stenholm 1990). An important criterion for when centroids indeed reflect the velocity statistics was obtained in Lazarian & Esquivel (2003, hereafter LE03). LE03 showed that when the criterion is not satisfied, the centroids are dominated by density. LE03 pointed out that the velocity centroids can be modified to extract the velocity contribution, but the resulting measures may be pretty noisy. As a result, velocity centroids taken alone may not be a sufficiently robust and reliable technique. A recent study by Esquivel & Lazarian (2004) revealed that for Mach numbers larger than 2 or 3 the traditional velocity centroids fail. As Mach numbers as high as 10 are expected for molecular clouds, the applicability of the traditional centroids to molecular clouds is highly questionable.

Principal component analysis (PCA) of the emission data (Heyer & Schloerb 1997; Brunt & Heyer 2002a, 2002b) has been suggested as a new way to study interstellar turbulence. However, recent testing showed that it provides statistics different from power spectra (M. H. Heyer 2003, private communication; Brunt et al. 2003). What sort of statistics we can obtain using the PCA requires further study.

Wavelet analysis (see Gill & Henriksen 1990), spectral correlation functions¹ (see Rosolowsky et al. 1999; Padoan et al. 2001), and genus analysis (see Lazarian et al. 2002) are other important statistical tools. They can provide statistics and insight complementary to power spectra (see review by Lazarian 1999b). Synergy of different techniques should provide the necessary insight into turbulence and enable the comparison of observational statistics with theoretical expectations (see Cho & Lazarian 2003a for a review).

Velocity statistics contribute to intensity variations observed in channel maps. Power-law spectra suggestive of underlying turbulence were obtained at different times by a number of researchers (see Kalberla & Mebold 1983; Green 1993). The problem that plagued those studies was an inability to separate contributions due to velocity and density. Indeed, both velocity and density fluctuations affect the small-scale emissivity fluctuations observed at a given velocity. Therefore, the separation of contributions to velocity and density requires a quantitative description of spectral line data cube statistics.

This problem was addressed in LP00, where the fluctuations of intensity in channel maps were related to the statistics of velocity and density. LP00 introduced a new statistical technique that was termed in Lazarian et al. (2002) “velocity channel analysis” (VCA). Within VCA the separation of velocity and density contributions is obtained by changing the thickness of the analyzed slice of the position-position-velocity (PPV) data cube. The theoretical description of the intensity statistics for both thick and thin velocity slices enabled LP00 to disentangle velocity and density contributions to intensity fluctuations in PPV cubes.

The VCA was successfully tested numerically in Lazarian et al. (2001) and Esquivel et al. (2003). Those tests employed the velocity and density obtained via simulations of compressible MHD turbulence to obtain synthetic maps, which were analyzed using VCA to recover underlying velocity statistics. Since then the VCA has been successfully applied to obtain velocity spectra of turbulence (see § 5).

The VCA in its existing form, however, does not account for absorption of radiation. LP00 argued that for the 21 cm emission data, the effect of absorption is marginal in the direction of the Galactic anticenter for which the technique was applied. However, the data in Dickey et al. (2001) show substantial absorption in H I toward the inner part of the Galaxy. Moreover, the absorption is definitely essential for other (e.g., CO) transitions.

The purpose of this paper is to extend the earlier theoretical description of the Doppler-shifted spectral line data cubes by including the treatment of absorption effects and thus to obtain a quantitative tool to study turbulence in various interstellar conditions, including molecular clouds. We introduce the three-dimensional anisotropic statistics of fluctuations within PPV data cubes in § 2, describe absorption effects in § 3, provide the statistics for thin and thick slices in § 4, explain some of our findings in physical terms in § 5, and discuss our results and their implications for observational data in § 6. The summary is provided in § 7. The lengthy derivations are

¹ Spectral correlation functions, if they are measured for the same velocity, present just another way of dealing with fluctuations within channel maps. Therefore, all the VCA results are applicable to them. If one tries to generalize spectral correlation functions to the velocity direction (see Lazarian 1999b), only limited information about turbulence can be available using the tool (see Appendix C).
collected in the Appendices, which are an important part of the paper.

2. STATISTICS OF VELOCITY AND DENSITY

2.1. Basics of Turbulence Statistics

Statistics in real space.—In the presence of the magnetic field, MHD turbulence gets axisymmetric in the system of reference related to the local direction of the magnetic field. In this system of reference magnetic fields can be easily mixed by turbulence in the direction perpendicular to magnetic fields. Those mixing motions generate wavelike perturbations propagating along magnetic field lines. As the kinetic energy decreases with the decrease of the scale, the weak mixing motions bend magnetic field lines less and less and the eddies get more and more elongated.

The Goldreich & Sridhar (1995) model of incompressible turbulence prescribes Kolmogorov scaling of mixing motions perpendicular to magnetic field lines (see a simplified discussion in CLV02c) and the elongation of the velocity fluctuations that increases as the scale decreases. Further research in Lithwick & Goldreich (2001) and Cho & Lazarian (2002b) has shown that the basic features of the Goldreich-Sridhar turbulence carry on for Alfvénic perturbations to the compressible regime.2

Observations, however, are usually unable to identify the local orientation of the magnetic field and deal with the magnetic field projection integrated over the line of sight. As a result, the locally defined perpendicular and parallel directions are mixed together in the process of observations (see CLV02c). The fluctuations with more power dominate the signal for both the direction of the averaged \( \mathbf{B} \) and perpendicular to it. There is some residual anisotropy, but this anisotropy is scale-independent and is determined by the rate of the meandering of the large-scale field. In other words, from the observational point of view the spectra of intensity fluctuations obtained from the Goldreich & Sridhar turbulence and the isotropic turbulence are similar. Observations in Green (1993) and numerical studies in Esquivel et al. (2003) support this picture.

The facts above allow for a substantial simplification of the turbulence description; namely, they permit us to use standard isotropic statistics (Monin & Yaglom 1975). Obtaining this statistics would correspond to obtaining the scaling of density and velocity fluctuations perpendicular to magnetic field lines.

The statistically homogeneous and isotropic in \( (x, y, z) \)-space density field \( \rho(x) \) has the correlation function

\[
\xi(r) = \langle \rho(x)\rho(x+r) \rangle.
\]

We also use the correlation function of density fluctuations

\[
\tilde{\xi}(r) = \langle \delta \rho(x)\delta \rho(x+r) \rangle = \xi(r) - \langle \rho \rangle^2.
\]

The structure function

\[
d(r) = \left\langle \left[ \rho(x+r) - \rho(x) \right]^2 \right\rangle
\]

is another way of describing turbulence. Here \( \tilde{d}(r) \) for fluctuations coincides with \( d(r) \).

Statistical descriptors of fluctuations can often be assumed to have power-law dependence on a scale \( \sim r^{-\gamma} \). If the correlation is dominated by large scales, the structure functions are used and \( \gamma < 0 \), while for small-scale-dominated statistics the correlation functions are usually used and \( \gamma > 0 \). We show in Appendix A that the two cases in the asymptotic regime of small \( r \) can be treated very similarly. Therefore, we frequently refer to correlation functions having in mind both \( \gamma > 0 \) and \( \gamma < 0 \).

An isotropic velocity field \( \mathbf{u}(x) \) is fully described by the structure tensor \( \langle \Delta u_i \Delta u_j \rangle \), which can be expressed via longitudinal \( D_{LL} \) and transverse \( D_{NN} \) components (Monin & Yaglom 1975),

\[
\langle \Delta u_i \Delta u_j \rangle = [D_{LL}(r) - D_{NN}(r)] \frac{\delta_{ij}}{r^2} + D_{NN}(r) \delta_{ik},
\]

where \( \delta_{ik} = 1 \) for \( i = k \) and zero otherwise. We define the \( z \)-projection of the velocity structure function as

\[
D_z(r) \equiv \langle \Delta u_i \Delta u_j \rangle \hat{z}_i \hat{z}_j = D_{NN}(r) + [D_{LL}(r) - D_{NN}(r)] \cos^2 \theta, \quad \cos \theta \equiv \hat{r} \cdot \hat{z},
\]

which for the power-law velocity gives

\[
D_z(r) = C \rho^m \left[ 1 + \frac{m}{2} \left( 1 - \cos^2 \theta \right) \right],
\]

if we assume that the velocity is solenoidal.3 For this paper it is only important that \( D_z \sim C \rho^m \). A discussion of the shallow and steep spectra of density, as well as of the velocity statistics within finite-size clouds, is given in Appendix A.

Spectra and correlation functions.—Spectra and correlation/structure functions are two complementary ways of describing turbulence. Given an \( N \)-dimensional correlation function \( \xi_N(r) \), one can obtain the spectrum

\[
P(k) = \int d^N r e^{ik \cdot r} \xi_N(r),
\]

where the integration is performed in the \( N \)-dimensional space.

In LP00 the Fourier space statistics, namely spectra, were widely used, with the correlation functions playing an auxiliary role. In the present paper we deal with absorption defined in real space. Therefore, we use real-space statistics, namely, correlation and structure functions. It is obvious from equation (7) that for the power-law correlation function (CF), i.e., \( \xi_N \sim r^{-\gamma} \), the spectral index is also a power law, i.e., \( P \sim k^n \), where \( n = -N - \gamma \). In other words,

\[
\text{spectral index} = -\text{dimensions of space} - \text{CF index}.
\]

In Kolmogorov turbulence, passive scalar density correlations scale as \( r^{2/3} \), which corresponds to \( \gamma = -5/3 \) in our notations. Thus, the Kolmogorov spectrum index is \(-11/3\). In turbulence literature \( E(k) = 4\pi k^2 P(k) \) is usually used. In this notation the

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2 Magnetosonic fast modes that arise in compressible fluid are, however, isotropic (Cho & Lazarian 2002b, 2003a, 2003b). This entails many important astrophysical consequences (see review by Lazarian et al. 2003a). When both fast and Alfvén modes are present in the fluid, the total anisotropy of magnetic and velocity fluctuations decreases.

3 Separating solenoidal and potential components of the velocity is an important problem that we do not address here. It was suggested in LE03 that this separation could be done by combining VCA and velocity centroids.
Kolmogorov spectrum is $E(k) \sim k^{-5/3}$, and thus the Kolmogorov law is often referred to as the $-5/3$ law.

### 2.2. Statistics in PPV

One does not observe the gas distribution in the real-space galactic coordinates $(x, y, z)$ where the three-dimensional vector $x$ is defined. Rather, the intensity of the emission in a given spectral line is defined in PPV cubes toward some direction on the sky and at a given line-of-sight velocity $v$.\(^4\) In the plane-parallel approximation, the direction on the sky is identified with the $(x, y)$-plane where the two-dimensional spatial vector $X$ is defined, so that the coordinates of PPV cubes available through observations are $(X, v)$. The relation between the real-space and PPV descriptions is defined by a map $(X, z) \rightarrow (X, v)$.

The central object for our study is a turbulent cloud in PPV coordinates. The statistical properties of the PPV density $\rho_v(X, v)$ depend on the density of gas in real galactic coordinates, but also on velocity distribution of gas particles. Hereafter we use the subscript $s$ to distinguish the quantities in $(X, v)$ coordinates from those in $(x, z)$ coordinates. We always assume two-dimensional statistical homogeneity and isotropy of $\rho_v(X, v)$ in the $X$-direction over the image of a cloud. However, homogeneity along the velocity direction can only be assumed after additional considerations, if at all. Naturally, there is no symmetry between $v$ and $X$.

We use the following notation for statistical descriptors of the density in PPV space: the mean density is $\bar{\rho}_s(v_1) = \langle \rho_s(X_1, v_1) \rangle$, and the correlation functions of the density and (closely related) density fluctuations $\delta \rho_s(X_1, v_1) = \rho_s(X_1, v_1) - \bar{\rho}_s(v_1)$ are

$$
\xi_s(R, v_1, v_2) \equiv \langle \delta \rho_s(X_1, v_1) \delta \rho_s(X_2, v_2) \rangle,
$$

$$
\tilde{\xi}_s(R, v_1, v_2) \equiv \langle \delta \rho_s(X_1, v_1) \delta \rho_s(X_2, v_2) \rangle
= \xi_s(R, v_1, v_2) - \bar{\rho}_s(v_1)\bar{\rho}_s(v_2).
$$

Here we have maintained notation that highlights the symmetries of the statistics. Homogeneity and isotropy in position-orientation coordinates $X$ lead the mean density to depend only on velocity, while the correlation function depends only on the magnitude of separation between two sky directions $R = |R| = |X_1 - X_2|$. Dependence on the velocity is retained for the time being in the general form.

For PPV statistics it is more convenient to use the structure functions

$$
d_s(R, v_1, v_2) = \langle [\delta \rho_s(X_1, v_1) - \rho_s(X_2, v_2)]^2 \rangle,
$$

$$
\tilde{d}_s(R, v_1, v_2) = \langle [\delta \rho_s(X_1, v_1) - \delta \rho_s(X_2, v_2)]^2 \rangle
= d_s(R, v_1, v_2) - \bar{\rho}_s(v_1)\bar{\rho}_s(v_2).
$$

Let us derive the two-point correlation function in PPV space without using the power spectrum formalism employed in LP00. The line-of-sight component of velocity $v$ at the position $x$ is a sum of the regular gas flow (e.g., due to Galactic rotation) $v_{\text{gal}}(x)$, the turbulent velocity $u(x)$, and the residual component due to thermal motions. This residual thermal velocity $v - v_{\text{gal}}(x) - u(x)$ has a Maxwellian distribution

$$
\phi_v(x) \, dv = \frac{1}{(2\pi\beta)^{3/2}} \exp \left\{ -\frac{(v - v_{\text{gal}}(x) - u(x))^2}{2\beta} \right\} \, dv,
$$

where $\beta = k_BT/m$, $m$ being the mass of atoms. The temperature $T$ can, in general, vary from point to point.

The density of the gas in the PPV space $\rho_v(X, v)$ can then be written as

$$
\rho_v(X, v) \, dX \, dv = \int_0^S d\rho(X, v) \phi_v(x) \, dX \, dv,
$$

where $\rho(x)$ is the (random) density of gas in real space. This expression just counts the number of atoms along the line of sight that have a $z$-component of velocity in the interval $[v, v + dv]$. The limits of integration are defined by the spatial extent $S$ of the emitting gas distribution.

To compute the correlation function $\xi_v$, we disregard the effect of correlations between density and velocity fluctuations,\(^5\)

$$
\langle \rho_s(X_1, v_1) \rho_s(X_2, v_2) \rangle = \int_0^S dz_1 \int_0^S dz_2 \langle \rho(x_1) \rho(x_2) \rangle \langle \phi_v(x_1) \phi_v(x_2) \rangle,
$$

$$
\langle \rho_s(X, v) \rangle = \int_0^S dz \langle \rho(x) \rangle \langle \phi_v(x) \rangle.
$$

The brackets $\langle \cdot \cdot \rangle$ in equation (16) denote statistical averaging over realizations of the random density $\rho(x)$ and the turbulent velocity $u(x)$ of gas. In particular, for the density we can write $\langle \rho(x) \rangle = \rho$ and $\langle \rho(x_1) \rho(x_2) \rangle = \xi_v$, without any assumptions on density statistics. For the Gaussian velocity field $u$, described by structure functions given by equation (4), the statistical average of the Maxwellian functionals given by equation (13) is given in Appendix B (see eq. [B8], [B9], and [B4]).

The general expression for the correlation function given in Appendix B (see eq. [B3]) can be simplified if one neglects the edge effects associated with the wings\(^6\) of spectral lines (see derivation of eq. [B7]). In this case we have

$$
\xi_v(R, v) \sim \frac{1}{2\pi(\sigma v)^2} \int_S^{\infty} dz \int_{-\sigma v/2}^{\sigma v/2} dz_+ \xi_v(R, z, v) + 2\beta^{-1/2} \left\{ -\frac{(v - v_{\text{gal}})^2}{2\beta} \right\},
$$

where equation (A2) was used to write the prefactor explicitly. For a homogeneous statistics, all correlation functions depend only on the separation between points. Here and further on we use variables without indices to denote separation, as in $R = X_1 - X_2, z = z_1 - z_2, v = v_1 - v_2$, and $v_{\text{gal}} = v_{\text{gal},1} - v_{\text{gal},2}$.

\[^5\] A similar assumption was used in LP00. Tests by Lazarian et al. (2001) and Esquivel et al. (2003) have shown that it does not cause problems in realistic simulated cases.

\[^6\] Edge effects make the image of turbulence inhomogeneous. It is intuitively clear that those effects should not affect the statistics on the scale much less than the spectral line width.
In the absence of the turbulent and thermal motions, $D_z$, $\beta \to 0$, the kernel in equation (17) reduces to the delta function $\delta(r - v_{gal})$, and for monotonic $v_{gal}(z)$ one can fully recover the line-of-sight position of an emitting atom from its velocity. In a general case, while thermal broadening just smooths fluctuations out, the effect of the turbulent velocity fluctuations is scale dependent and leads to a change of the correlation function slope.

Equation (17) allows for an arbitrary form of the regular flow $v_{gal}$. A variable regular flow produces a complex map from galactic to velocity space by itself. Two tractable cases are when the regular flow is of a simple linear form or when it is possible to attribute all motions to turbulence, thus setting $v_{gal} = 0$. In LP00 we have considered the case of linearized shear flow $v_{gal} = f^{-1}z$ due to Galactic rotation:

$$
\xi_i(R, v) \sim \int_{-\infty}^{\infty} dz \xi(r) [D_z(r) + 2\beta]^{-1/2} \times \exp \left\{ -\frac{(v - f^{-1}z)^2}{2[D_z(r) + 2\beta]} \right\},
$$

where we have omitted an unimportant dimensional prefactor (see eq. [B9] for the explicit form of the prefactor). Galactic shear introduces a natural scale $\lambda = (f^2C)^{1/12-m}$, at which the velocity dispersion $\sim C\lambda^m$ becomes equal to the squared difference of the regular velocities determined by Galactic rotation (i.e., $f^{-2}z^2$). Asymptotics obtained in LP00 correspond to the scales $l$, which are much smaller than $\lambda$. Although at these scales the Galactic rotational velocity is much larger than the turbulent one, its gradient is much smaller. It is because of this that the LP00 results in the presence of Galactic rotation and without coincidence. The regime when the small-scale turbulent shear exceeds the regular one is natural. Indeed, flows with high Reynolds number produce turbulence with shear larger than that of the original flow. This is true, for instance, for Couette flows in the presence of the moderately strong magnetic field (Velikhov 1959; Chandrasekhar 1960; Balbus & Hawley 1991). The corresponding criterion on the magnetic field is usually satisfied in the ISM.

If gas is confined in an isolated cloud of size $S$ and the galactic shear over this scale is neglected, we get

$$
\xi_i(R, v) \sim \int_{-S}^{S} dz \left( 1 - \frac{|z|}{2S} \right) \frac{\xi(r)}{[D_z(r) + 2\beta]^{1/2}} \times \exp \left\{ -\frac{v^2}{2[D_z(r) + 2\beta]} \right\},
$$

where again a somewhat ugly prefactor entering equation (B9) is omitted. Turbulent effects remain important up to the scale of the cloud $S$ at which the turbulent structure function saturates at the value $\sim (CS^m)^{1/2}$. The cloud size $S$ now plays the role of the scale $\lambda$. This observation allows us to translate the LP00 results, mostly written for Galactic H i with rotation curve mapping, to a case of an individual cloud. Rigorous calculations provided in LP00 prove that for scales much smaller than $S$ one can use the results obtained for an infinite medium in the presence of shear and substitute $\lambda$ for $S$.

When the amplitude of fluctuations grows with separation, one should use structure functions in PPV. The transfer from one type of statistics to another is similar to that discussed in Appendix A.

An important feature of the PPV space is that $\rho_i(X, v)$ exhibits fluctuations even if the flow is incompressible and no density fluctuations are present. Indeed, when one substitutes the expanded expression given by equation (A4), $\xi(r) = \tilde{\xi}^2 + \tilde{\rho}^2(r_0/r)^2$, into equations (17)–(19), both terms will give rise to nontrivial contributions to $\xi_i(R, v)$. We therefore split the result correspondingly,

$$
\xi_i(R, v) = \tilde{\xi}_i(R, v) + \tilde{\rho}_i(R, v),
$$

with the $\nu$-term describing pure velocity effects, while the $\rho$-term arises from the actual real-space density inhomogeneities that are modified by velocity mapping. To simplify the notation, we have dropped the index $s$ from the quantities on the right-hand side, since this split is only meaningful in PPV space. In Appendix C we discuss asymptotic small-$R$ scalings of different contributions to PPV structure and correlation functions.

3. TURBULENT STATISTICS AND RADIATIVE TRANSFER

We start with the standard equation of radiative transfer (Spitzer 1978),

$$
dI_{\nu} = -g_{\nu}I_{\nu} ds + j_{\nu} ds.
$$

In the case of self-adsorbing emission in spectral lines, this is proportional to the first power of density,

$$
g_{\nu} = \alpha(z)\rho(z)\phi_{\nu}(z),
$$

$$
j_{\nu} = \epsilon(\nu)\phi_{\nu}(z),
$$

where $\phi_{\nu}(z)$ is given by equation (13).

A solution of this equation, if no external illumination is present, is

$$
I_{\nu} = \int_{0}^{S} dz \rho(z)\phi_{\nu}(z) \exp \left[ -\int_{0}^{z} \alpha(z')\rho(z')\phi_{\nu}(z') dz' \right].
$$

To integrate equation (24), we assume that $\alpha$ is constant. This constancy is the essence of the Sobolev approximation, which has been found to be useful in many astrophysical applications. In this approximation we can use an integration variable

$$
\nu(z) = \int_{0}^{z} \rho(z')\phi_{\nu}(z') dz',
$$

the value of which at $x = S$ coincides with the density in PPV coordinates, $\nu(S) = \rho_i(X, v)$ and $\nu(0) = 0$, to integrate equation (24),

$$
\nu(X) = \int_{0}^{\nu} d\nu_{\nu} e^{-\alpha\nu_{\nu}} = \frac{\nu_{\nu}}{\alpha} \left( 1 - e^{-\alpha\rho_i(X, v)} \right).
$$

In the case of vanishing absorption, the intensity is given by the linear term in the expansion of the exponent in equation (26),

$$
I_{\nu}(X) = \nu_{\nu} \rho_i(X, v),
$$

and reflects the PPV density of the emitters. If, however, the absorption is strong, the intensity of the emission is saturated at the value $\nu/\alpha$ wherever $\rho_i(X, v) \gg 1/\alpha$. Identification of the low-contrast residual fluctuations may be difficult in practice.
The approximate expression given by equation (26) can be used to find the statistics of the expected emissivity fluctuations. First of all, the mean profile of the line is given by

\[ \langle I_e(X) \rangle = \frac{\epsilon}{\alpha} \left( 1 - \langle e^{-\alpha \rho(X,v)} \rangle \right). \]  

(28)

Our main goal is to calculate the structure function of the observed emissivity,

\[ \mathcal{D}(R) \equiv \left\langle \left[ \int I_e(X_1)W_v dv - \int I_e(X_2)W_v dv \right]^2 \right\rangle. \]

\[ R = X_1 - X_2. \]  

(29)

Here the window function \( W_v \) describes how the integration over velocities is performed. When \( W_v \equiv 1 \), the integration is being performed over the whole line, as is a frequent case for CO turbulence studies (see Falgarone et al. 1998; Stutzki et al. 1998), while measurements in velocity slices of the PPV data cube (channel maps) correspond to \( W_v \) strongly peaked at a particular velocity. Within the VCA technique, by varying the width of velocity channels one can obtain statistics of turbulent velocities and density inhomogeneities of the emitting medium. Note that the minimal width of the velocity channel is determined by the resolution of an instrument.

Using elementary identities to transfer from averaging over two-dimensional intensities to averaging the underlying three-dimensional fluctuations (see Lazarian 1995), in the limit when the absorption can be neglected, we rewrite the expression given by equation (29) in the form

\[ \mathcal{D}(R) = \epsilon^2 \int dv_1 W(v_1) \]

\[ \times \int dv_2 W(v_2) [d_e(R, v_1, v_2) - d_e(0, v_1, v_2)], \]  

(30)

where homogeneity in physical directions is assumed. For an infinite emitting medium, homogeneous turbulence produces a homogeneous image in the velocity space. For a finite emitting cloud, the PPV image is approximately homogeneous over velocity separations much less than the Doppler line width. Notably, the combination in brackets does not depend on the mean Doppler-broadened profile of the spectral line, \( d_e(R, v_1, v_2) = d_e(0, v_1, v_2) \). Thus, it is sufficient to assume that in the \( v \)-direction fluctuations of \( \rho_0 \) are homogeneous. In this case,

\[ \mathcal{D}(R) = \epsilon^2 \int dv W(v) [d_e(R, v) - d_e(0, v)], \]  

(31)

with \( W(v) = \int dv_\perp W(v_\perp - v)W(v_\perp + v) \). Equation (31) together with Appendix B can be taken as a starting point to reproduce the results in LP00 using structure functions rather than spectra (compare with LP00). This is demonstrated in Appendix D.

Substitution of equation (26) into equation (29) gives

\[ \mathcal{D}(R) = \frac{\epsilon^2}{\alpha^2} \int dv_1 W(v_1) \int dv_2 W(v_2) \]

\[ \times \left\langle e^{-\alpha (\rho_{11} + \rho_{12})} - e^{-\alpha (\rho_{21} + \rho_{22})} - e^{-\alpha (\rho_{11} + \rho_{22})} - e^{-\alpha (\rho_{21} + \rho_{12})} \right\rangle, \]  

(32)

where we have used a shorthand notation \( \rho(X_1, v_1) = \rho_{ij} \). Homogeneity in spatial directions dictates that after averaging the first term is equal to the second one and the third is equal to the fourth (e.g., for the density itself \( \langle \rho_{11} + \rho_{12} \rangle = \langle \rho_{22} + \rho_{21} \rangle \) and \( \langle \rho_{12} + \rho_{21} \rangle = \langle \rho_{11} + \rho_{12} \rangle \).

We now derive an approximate expression for \( \mathcal{D}(R) \), applicable for small separations \( R \). Let us rewrite equation (32) as

\[ \mathcal{D}(R) = \frac{\epsilon^2}{\alpha^2} \int dv_1 W(v_1) \int dv_2 W(v_2) \]

\[ \times \left\langle e^{-\alpha (\rho_{11} + \rho_{12})} (1 + e^{-\alpha (\rho_{21} + \rho_{22} - \rho_{11} - \rho_{12})}) \right\rangle, \]  

(33)

It is easy to see that the term in angle brackets depends only on the differences in density taken between two lines of sight at the same velocity, i.e., \( \rho_{11} - \rho_{21} \) and \( \rho_{22} - \rho_{12} \). The point of rearranging the terms in this way is that for small separations \( R \) these differences are small, so we can retain only the leading order in the power series expansion for the corresponding exponentials,

\[ \mathcal{D}(R) \sim \epsilon^2 \int dv_1 W(v_1) \int dv_2 W(v_2) \]

\[ \times \left\langle e^{-\alpha (\rho_{11} + \rho_{12})} [ (\rho_{11} - \rho_{21}) (\rho_{12} - \rho_{22}) ] \right\rangle. \]  

(34)

The structure of the resulting expression is as follows. The term in brackets is similar to the integrand in equation (30) [indeed, \( \langle \rho_{11} - \rho_{21} \rangle (\rho_{12} - \rho_{22}) = d_0(R, v_1, v_2) - d_0(0, v_1, v_2) \)], while the effect of the absorption is manifested in the exponential term, which depends on the density distribution in the velocity direction along the line of sight. As we see, the series expansion we have performed at small separation \( R \) is indeed an expansion that assumes \( \alpha^2[d_0(R, v_1, v_2) - d_0(0, v_1, v_2)] \) to be small. Similar expansion is then applicable for any combination of structure functions that vanishes at \( R = 0 \), and this opens a way of dealing with non-Gaussian statistics.

Although there is a cross-correlation between the two terms in the integrand and statistical averaging cannot, in a general case, be factorized, the correction for cross-correlation is of higher order of smallness than the leading term. This can be seen by expanding the remaining exponential term in power series. We can illustrate this statement by specifying Gaussian statistical distribution for \( \rho_0 \) and performing explicit averaging, which then gives

\[ \mathcal{D}(R) \sim \epsilon^2 \int dv_1 W(v_1) \int dv_2 W(v_2) e^{(v_1/2)^2} \alpha^2 (\rho_{11} + \rho_{12})^2 \]

\[ \times [d_0(R, v_1, v_2) - d_0(0, v_1, v_2) + \alpha^2 d_{12}]. \]  

(35)

The cross-correlation combination \( d_{12} \) is vanishing as \( R \to 0 \), and \( \alpha^2 d_{12} \) is small, as expected. For non-Gaussian statistics the irreducible higher order correlations appear. However, it is suggestive that these correlations provide a vanishing contribution for \( R \to 0 \). Without providing a formal derivation for
the general case, we propose that equation (34) can be rewritten as

$$\mathcal{D}(R) \sim e^2 \int dv_1 W(v_1) \int dv_2 W(v_2) e^{-\alpha (p_{11} + p_{12})} \times [d_i(R, v_1, v_2) - d_i(0, v_1, v_2)],$$

or when approximation of homogeneity in the velocity direction is warranted,

$$\mathcal{D}(R) \sim e^2 \int dv_1 W(v_1) \int dv_2 W(v_2) e^{-\alpha (p_{11} + p_{12})} \times [d_i(R, v) - d_i(0, v)].$$

(Equations (36) and (37) show that for sufficiently small $R$ the structure functions of intensity differ from the earlier studied case by the window function determined by the absorption $W_{\text{absorption}} = e^{-\alpha (p_{11} + p_{12})}$. Two important conclusions directly follow from this observation. First, it is clear that if the window function determined by data slicing is much narrower than that given by the absorption, we get results indistinguishable from the earlier studied case of no absorption. Second, in the case of integration over the whole line of sight, the results will differ from the earlier studied case because the window function is no longer equal to unity. The criterion when the absorption is not important for velocity studies is straightforward. Equation (37) transfers into equation (31) when $W_{\text{absorption}}$ is of the order of unity over the range of scales studied.

4. SCALING REGIMES OF THE EMISSIVITY STATISTICS

In this section we study emissivity statistics in the regimes where it exhibits power-law scaling. Such behavior is possible at the small scales where the effect of the absorption is limited $\alpha^2 [d_i(R, v_1, v_2) - d_i(0, v_1, v_2)] < 1$ and equation (37) provides a good approximation. We also restrict our consideration to the scales smaller than the cloud size $R/S < 1$ so that inhomogeneous boundary effects can be neglected.

4.1. Integrated Lines

The presence in equation (37) of the window function determined by absorption, i.e., $e^{-\alpha (p_{11} + p_{12})}$, changes results substantially compared to the case of no absorption discussed in LP90. Indeed, the integration over the whole spectral line in the latter case removes the dependence of intensity statistics on velocity. The window function determined by absorption defines to what extent the integration over velocities is performed.

Let us estimate the form of the window assuming Gaussian statistics and homogeneity of the density fluctuations $\delta \rho_s = \rho_s - \langle \rho_s \rangle$. Then (see eq. [E3])

$$W_{\text{absorption}} = e^{-\alpha \rho_s (X, v_1)} e^{-\alpha \rho_s (X, v_2)} e^{\alpha^2 (\delta \rho_s^2 (X, v_1))} \times e^{\alpha^2 (\delta \rho_s^2 (X, v_2))} e^{-\alpha^2/2} d_i(0, s).$$

For homogeneous statistics the variance of the fluctuations does not depend on velocity; thus, the corresponding factors are constant and affect only the normalization of the result. However, their presence reveals limitations of the Gaussian approximation when the absorption is large. Indeed, a similar term appears in the mean line profile equation (28) computed for Gaussian $\rho_s$, namely,

$$\langle h(X, v_1) \rangle = \frac{e}{\alpha} \left[ 1 - \exp \left[ -\alpha \langle \rho_s (X_1, v_1) \rangle + \frac{\alpha^2}{2} \langle \delta \rho_s^2 (X_1, v_1) \rangle \right] \right].$$

(Clearly the answer is unphysical when $\alpha > 2 \langle \rho_s (X_1, v_1) \rangle / \langle \delta \rho_s^2 (X_1, v_1) \rangle$. The problem arises because the Gaussian fluctuations do not obey the constraint that the density $\rho_s$ is positive. Therefore, for high absorption, negative density excursions, however rare, dominate the result. In view of this, we select in equation (38) the factors that describe the variable part of the mean intensity profile and write them in nonexpanded form,

$$W_{\text{absorption}} = e^{-\alpha \rho_s (X, v_1)} e^{-\alpha \rho_s (X, v_2)} e^{\alpha^2/2} (\delta \rho_s^2 (X_1, v_1) + \delta \rho_s^2 (X_1, v_2)) e^{-\alpha^2/2} d_i(0, s).$$

This approximate formula does not suffer from the defect we have mentioned. The residual dependence on $\delta \rho_s$ variance can be traced to the real effect of increase of intensity contrast between two points if absorption is present. However, this factor is constant in our treatment, and we normalize it out. We summarize our considerations in the formula

$$\mathcal{D}(R) \propto \int dv \tilde{W}(v) e^{-\alpha^2/2} d_i(0, s) [d_i(R, v) - d_i(0, v)].$$

where

$$\tilde{W}(v) \equiv \int dv_1 W(v_1) W(v_2) e^{-\alpha \rho_s (X, v_1)} e^{-\alpha \rho_s (X, v_2)}.$$
where we have omitted numerical coefficients of order unity and estimated the mean density in PPV space as  
\[ \langle \rho_s \rangle = \frac{\bar{\delta} S}{D_s(S)^{1/2}}. \]  
(45)

Absorption effects become negligible if \( v_{ab}/D_s(S)^{1/2} \gg 1. \)

### 4.1.1. Effect of Velocity Fluctuations

Let us consider first the \( \delta \rho_s \) fluctuations that arise from velocity fluctuations only. Indeed, even if the underlying density is constant, random velocities do produce caustics that were shown in LP00 to be very important for the analysis.

In Figure 1 we plot the correspondent kernel \( [d_s(R, v) - d_s(0, v)] \) as a function of \( v \) for a fixed sample value of \( R \) and, simultaneously, the window function \( e^{-v^2/2}d_s(0, v) \) for the range of absorption amount characterized by \( \alpha \rho_s \). The kernel is highly peaked at the zero velocity separation \( v = 0 \), has a region of negative values, and approaches zero at large \( v \). The window is unity for zero absorption but also peaks at \( v = 0 \) as absorption increases.

Integration of the velocity kernel over the whole line in the absence of absorption and \( W(v) = 1 \) gives zero, in concordance with the expectation of canceling all the velocity effects. In the presence of high absorption one might hope that downweighting all nonzero \( v \) contribution will act as an effective thin slice reproducing thin slice power-law asymptotics of \( D(R) \), which is achieved when the width of the slice becomes small compared to rms velocity at the scale of the study \( R \) (LP00). However, we are restricted by the fact that our linearized approximation given by equation (34) is valid only when absorption is moderate \( \alpha^2 d_s(R, 0) < 1 \), while higher absorption (or larger scales at fixed \( \alpha \)) will induce nonlinear modifications of the result. Thus, we find that thin slice scaling is never realized for \( m \geq \frac{3}{2} \). Instead, the new universal intermediate power-law scaling is exhibited for moderate absorption. Dependence of the two-dimensional structure function on the underlying velocity index \( m \) is lost in this regime. For \( m < \frac{3}{2} \) there is a range of scales when absorption can modify the scaling to that of the thin slice, in which case sensitivity to \( m \) is present.

Analytical estimates can provide some insight into the expected behavior of \( D_s(R) \) after integration as \( R \to 0 \). The value of the kernel at the peak, \( d_s(R, 0) \), is given by the asymptotic expression given by equation (C4) (see also LP00 for asymptotics of the correlation functions in the velocity space),  
\[ d_s(R, 0) \sim \frac{\sqrt{\pi} \Gamma(m/4 - 1/2)}{\Gamma(m/4)} R^{3-m/2}, \]  
(46)

where we are more interested in the \( R \) dependence of the expression than in the numerical prefactor. Incidentally, this gives a criterion on the validity of the linear expansion of the velocity kernel. With dimensional quantities restored the condition  
\[ \alpha^2 \langle \rho_s \rangle^2 < \frac{\Gamma(m/4)}{\sqrt{\pi} \Gamma(m/4 - 1/2)} \left( \frac{S}{R} \right)^{1-m/2} \]  
(47)

sets the scale range where our treatment is applicable.

Let us first consider the case \( m \geq \frac{3}{2} \). With the width of the absorption-induced window given by equation (43), the condition given by equation (47) requires \( v_{ab}^2 \gtrsim D_s(S)(R/S)^m \), i.e.,

[Diagram showing velocity kernel and density kernel]

**Figure 1.**—Velocity kernel \( d_s(R, v) - d_s(0, v) \) (thick solid line) and density kernel \( d_s(R, v) - d_s(0, v) \) (dotted line) for \( R/S = 0.01 \) and \( m = \frac{3}{4} \) as a function of the line-of-sight velocity separation \( v/D_s^2(S) \) normalized over velocity dispersion. The thin solid lines correspond to the window function \( \exp(-\alpha^2 d_s(0, v)/2) \) and are drawn for the values \( \alpha \rho_s \) from 0.3 to 30 (top to bottom). For the displayed value of \( R \), the linearized treatment of the kernels is applicable up to \( \alpha \rho_s \approx \sqrt{(R/S)/(m+3)} \approx 6 \); the corresponding line is highlighted.

velocity cutoff exceeds rms turbulent velocity at scale \( R \) and thus is insufficient to reproduce thin slices, which require the opposite relation to be true (see LP00).

This is, however, sufficient to suppress the negative large \( v \) tail, in which case the dominant contribution to the integral scales as the height of the kernel peak times its width. The scaling of the width of the kernel peak with \( R \) can be readily obtained from the following consideration. If we integrate the kernel over \( v \) choosing fixed integration range \( dv \) but varying \( R \) from large values to small ones, then the projected result will scale with \( R \) according to thin slice asymptotics while \( D_s(R)^{1/2} \) dominates. The change to the thick slice behavior takes place when the condition is reversed. However, formally in the thin slice regime the integral is determined by the zero-lag peak behavior only, while in the thick slice integration encompasses most of the kernel. Therefore, \( dv = D_s(R)^{1/2} \), where \( D_s(R) \) is the \( z \) projection of the correlation tensor of velocity (see eq. [5]), represents the required estimate of the width of the kernel. This scaling is valid for both the zero crossing point of the velocity kernel and the width of the density kernel that we consider later.

The total scaling of the two-dimensional emissivity structure function is thus \( D_s(R) \sim d_s(0, 0) dv \propto R^{3-m/2} R^{m/2} \propto R \), with dependence on \( m \) canceled out. The corresponding two-dimensional spectra scale as \( k^{-3} \).

Numerical integration using equation (35) confirms that indeed the velocity statistics exhibit a new intermediate asymptotic regime, which is characterized by a universal spectral index of \( -3 \). In Figure 2 we show structure functions of intensity for two different indices \( m = \frac{3}{2} \) and 1. The range for the universal asymptotics is clearly seen.

Figure 2 also shows that for very small \( R \) the structure function grows faster than \( R \), which entails spectra steeper than \( k^{-3} \). Indeed, for sufficiently small \( R \) the \( \exp(\alpha^2 d_s(0, v)) \) window does not cut out even the distant region where the kernel gets negative, in which case we expect to see the thick slice asymptotics \( \sim R^{1+m/2} \) (i.e., \( k^{-3-m/2} \) for the spectrum; see LP00) in accordance with our numerical calculations. Thermal effects (not included in numerical experiment) at scales where turbulence is subsonic \( D_s(R) < \beta \) lead to even faster falloff of
the structure function at small $R$ and indeed make any coherent scaling of turbulent velocity hard to observe in emissivity statistics at these scales.

For $m < \frac{2}{3}$, on the other hand, absorption may lead to thin slice scaling $\propto R^{1-m/2}$ for some intermediate range of scales before $R$ is so large as to cause nonlinear deviations. Indeed, in this case $(\bar{\rho}/D_S) \approx v_{ab}/D_S(R)^{1/2}$ and equation (47) can be rearranged to read $D_2(R)/v_{ab}(R)/S)^{1-3m/2} < 1$, which can be satisfied even for relatively high absorption $v_{ab}^2/D_2(R) > 1$ corresponding with thin slice velocity cutoff. For a given absorption parameterized by $v_{ab}$, we summarize the behavior at the different scales in Table 1. All four scaling regimes are present for $m < \frac{2}{3}$, although how pronounced thin slice scaling actually is depends on the extent of the scale range $[v_{ab}^2/D_2(S)]^{1/m} < [v_{ab}^2/D_2(S)]^{2(2-m)}$, which grows with absorption strength if $m < \frac{2}{3}$. This interval does not exist for $m \geq \frac{2}{3}$. We note that the two-dimensional emissivity statistics of integrated lines $D_2(R)$ restore $m$ sensitivity for $m < \frac{2}{3}$ where the line-of-sight statistics $d_0(0, v)$ is saturated at $\propto v^2$. The latter behavior is not surprising. Indeed, structure functions $d_0(0, v)$ cannot grow faster than $v^2$, and this explains the saturation. However, the information along the line of sight is not lost. The analysis of the spectra of fluctuations along the velocity coordinate performed in LP00 confirms this.

### 4.1.2. Effect of Density Fluctuations

Density fluctuations modify the statistics of emissivity. Mathematically, the density fluctuations are imprinted on the $\xi_\rho$ part of the three-dimensional correlation function. This correlation function depends on the statistics of both density and velocity. As we discussed earlier, density correlations may be short-wave or long-wave dominated. Within the power-law model for overdensity statistics they are described, respectively, by correlation $\xi_\rho(r) = \bar{\rho}^2(\rho_0/r)^{3}$ or structure $d_3(r) = \bar{\rho}^2(\rho_0/r)^{1}$ functions, where $\gamma$ is positive in the former and negative in the latter case (see Appendix A). The amplitude of the density effect is encoded in the correlation scale $r_0$, which has a meaning of the scale at which density inhomogeneities have an amplitude equal to the mean density of the gas (i.e., $\delta\rho \sim \bar{\rho}$ for $\gamma > 0$ or $\rho_1 - \rho_2 \sim \bar{\rho}$ for $\gamma < 0$). This gives rise to the amplitude factor $(r_0/S)^{1/2}$ in the asymptotic expressions of Appendix C (e.g., eq. [C5]).

Contribution from density to PPV statistics combines linearly with the pure velocity term, and an important question of which term dominates arises. Note that both terms contribute to the absorption window at the same time in nonlinear fashion.

Table 2 summarizes at which scales the density inhomogeneity contribution dominates velocity effects in PPV space. For steep spectra, the density contribution dominates large scales, above the characteristic scale determined by $r_0$. For shallow spectra it is dominant at small scales. Density inhomogeneities also dominate observed statistics at small scales where velocity is subsonic.

What are the reasonable values for $r_0$? We argue that the amplitude of density perturbations of the scale of the cloud itself should not significantly exceed the mean density; otherwise, the applicability of the notion of a cloud is suspect. For a shallow spectrum $\gamma > 0$ this means $r_0 < S$. For a steep

### TABLE 1

| Scale Range | Intensity Scaling | Regime |
|-------------|------------------|--------|
| $R/S < [v_{ab}/D_2(S)]^{1/m}$ | $D(R) \propto R^{1-m/2}$ | Velocity effects erased |
| $R/S < [v_{ab}/D_2(S)]^{1/m}$ | $D(R) \propto R^{1}$ | Subsonic regime |
| $[v_{ab}/D_2(S)]^{1/m} \leq R < S < [v_{ab}/D_2(S)]^{2(1-m)}$ | $D(R) \propto R^{1-m/2}$ | Thick slice |
| $[v_{ab}/D_2(S)]^{1/m} \leq R < S < [v_{ab}/D_2(S)]^{2(1-m)}$ | $D(R) \propto R^{1-m/2}$ | Intermediate scaling |
| $S < [v_{ab}/D_2(S)]^{2(1-m)}$ | $D(R) \propto R^{1-m/2}$ | Thin slice |
| $S < [v_{ab}/D_2(S)]^{2(1-m)}$ | $D(R) \propto R^{1-m/2}$ | Not a power law |
| $S < [v_{ab}/D_2(S)]^{2(1-m)}$ | $D(R) \propto R^{1-m/2}$ | Strong absorption regime |

Notes.—Scalings of structure functions of intensity fluctuations arising from velocity fluctuations for the power law underlying three-dimensional velocity statistics. In the strong absorption regime $D$ does not follow a simple power law. The thin slice regime does not exist for $m \geq \frac{2}{3}$. $D_2(S)$ is given by eq. (5).
having energy sources only weakly correlated at larger scales. As seen from the expansion $\tilde{d}(r)$, when $r > r_c$, the effective correlation scale in this case is $r_0 = r_c p^2/\tilde{d}(\infty)$. In the range of scales $r_0 < R < r_c$, one will observe the density effects. The physical picture, leading to such a situation, is to have strong energy injection at a scale smaller than a cloud size, with turbulent cascade establishing a steep spectrum to smaller scales, while having energy sources only weakly correlated at larger scales.

Table 2 shows that for a shallow spectrum the density term is important for PPV statistical descriptors in both $R$- and $v$-directions at small scales $R < r_0$ and $r^2 < D_2(r_0)$ (for $m \geq \frac{5}{4}$).

The marked effect of absorption on intensity fluctuations in integrated lines is that the density contribution cannot always be recovered by increasing the slice thickness, which is the case of no or small absorption. On the contrary, we are able to observe the density impact on the emissivity only in the restricted scale range. The reason for this is that for a wide range of scales above the thermal dispersion scale the velocity term is not suppressed in the presence of the absorption even when we integrate over the whole line.

We now discuss how the density contribution scales, depending on separation $R$ and conditions on $\gamma$ and $m$. Whether the density term will dominate the overall PPV statistics depends on the amplitude of density inhomogeneities given by $r_0$ according to Table 2. Our results are summarized in Table 3.

Scaling estimations for density are obtained in a similar way to the case of the pure velocity term in the previous section. The main ingredients are density kernel and absorption windows, exemplified in Figure 1. The complication is that the absorption window is determined by $d_0(v)$, with contributions from both velocity and density giving rise to two multiplicative windows (see eq. [42]). The one with the smallest width of the two is the most important one. We explicitly note hereafter when the details of the absorption window are important.

The peak of the density kernel now scales as $R^{1-\gamma-\gamma_m/2}$ (see eq. [C5]), while its width is $dV \sim D_2(R)^{1/2}$, as we discussed before. Integrating over the whole line in the absence of absorption gives an estimate $D(R) \approx R^{1-\gamma-\gamma_m/2}$ $R^{m/2} \sim R^{1-\gamma}$ (this refers to a correlation, rather than to a structure function of emissivity if $\gamma > 1$), which is exactly the thick slice regime of LP00. If absorption is not too strong, $r_{ab} > D_2(R)^{1/2}$, it will not affect this result. This is different from the pure velocity term, since the velocity term exactly vanishes when integrated over the whole velocity range in the absence of absorption, and small absorption leads to qualitatively new intermediate "universal" scaling.

The condition of weak absorption in the presence of density, which replaces the one in equation (47), is given by

$$\alpha^2 \mu_2^2 < \frac{\Gamma(\gamma/2 + m/4)}{\sqrt{\pi} \Gamma(\gamma/2 + m/4 - 1/2)} \left( \frac{S}{R} \right)^{1-\gamma-m/2}. \quad (48)$$

Using equations (43) and (44) to express absorption via $v_{ab}$, we find familiar constraints when absorption effects are not strongly nonlinear:

$$v_{ab}^2 > D_2(S) R^{(1-\gamma)/m} \sim D_2(R), \quad m > \frac{2}{\gamma} (1 - \gamma), \quad (49)$$

$$v_{ab}^2 > D_2(S) R^{(1-\gamma-m/2)/m} \sim D_2(R), \quad m < \frac{2}{\gamma} (1 - \gamma). \quad (50)$$

### Table 2

| Condition | Case $\gamma < 0$ | Case $\gamma > 0$ | Equations |
|-----------|--------------------|--------------------|-----------|
| $d_s(R, 0) > d_s(R, 0)$ | $R > r_0$ | $R < r_0$ | (C4), (C5) |
| $d_s(0, v) > d_s(0, v)$ | $m \geq \max \left[ \frac{3}{4} \left( 1 - \gamma \right) \right]$ | $r^2 > D_2(S)(r_0/S)^m$ | $r^2 < D_2(S)(r_0/S)^m$ | (C8), (C12) |
| $\frac{5}{4} (1 - \gamma) < m < \frac{3}{4} (1 - \gamma)$ | $r^2 > D_2(S)(r_0/S)^{5/3}$ | Not applicable | (C8), (C13) |
| $m \leq \min \left[ \frac{3}{4} \left( 1 - \gamma \right) \right]$ | $r_0/S < 1$ | $r_0/S > 1$ | (C9), (C12) |

**Notes.**—Range of the scales where the impact of density inhomogeneities to the PPV statistics exceeds the velocity contribution. For steep spectra $\gamma > 0$, we need $r_0 < S$ for density effects to be seen. This is feasible in a model where high-amplitude density inhomogeneities are saturated at scales $r_0 < S$. For the validity of the asymptotic scaling used, $R < S$ (or $r_c$) is required. When $d_s(0, v) > d_s(0, v)$, the window function is determined by $d_s(0, v)$ (see eq. [42]). In addition, the window function is determined by density fluctuations for subsonic velocities.

### Table 3

| Scale Range | Intensity Scaling | Regime |
|-------------|------------------|--------|
| $R/S < \left[ \beta / D_2(S) \right]^{1/m}$ | $D(R) \propto R^{1-\gamma}$ | Subsonic |
| $R/S < \left[ \frac{v^2_{ab}}{D_2(S)} \right]^{1/m}$ | $D(R) \propto R^{1-\gamma}$ | Thick slice |
| $\left[ \frac{v^2_{ab}}{D_2(S)} \right]^{1/m} < R/S < \left[ \frac{v^2_{ab}}{D_2(S)} \right]^{1/(2-2m)}$ | $D(R) \propto R^{1-\gamma-m/2}$ | Thin slice |
| $\left[ \frac{v^2_{ab}}{D_2(S)} \right]^{1/(2-2m)} < R/S$ | Not a power law | Strong absorption |

**Notes.**—Scales of structure functions of intensity fluctuations arising from density inhomogeneities for the power law underlying three-dimensional velocity statistics. The thin slice behavior requires $\gamma < 1 - 3m/2$. In the strong absorption regime $D$ does not follow a simple power law.
The first case, \( m > \frac{3}{2}(1 - \gamma) \), is applicable for the most interesting values of \( m \) if the density spectrum is shallow, \( \gamma > 0 \). Here linearity and thick slice conditions coincide and one cannot achieve the thin slice regime before the absorption effects become nonlinear. The second case, \( m < \frac{3}{2}(1 - \gamma) \), has a wide range of applicability for steep spectra \( \gamma < 0 \). Here the linearity condition is less stringent and in the range of scales \( [v_{ab}^2/D_2(r_c)]^{1/m} < R/r_c < [v_{ab}^2/D_2(r_c)]^{1/1 - \gamma - \gamma/m} \) absorption effects modify the emissivity scaling toward the thin slope \( R^{1 - \gamma - \gamma/m} \), while nonlinear effects come into play only at larger scales.\(^9\)

To summarize, for shallow spectra one expects line-integrated statistics for density perturbations to follow thick slice scaling until absorption effects become too strong. Density terms are visible in the overall emissivity statistics at \( R < r_0 \). For steep spectra, the density contribution may exhibit, in addition to thick slice behavior at small scales, the thin slice scaling at intermediate scales. However, the density contribution can be dominant in emissivity statistics only when \( R > r_0 \), which requires high-amplitude density inhomogeneities saturated at scales smaller than a cloud size. Table 3 and Figure 3 illustrate how the statistics of emissivity changes with the change of the spectral index of density and velocity.

Can we ever observe density fluctuations in the absorbing gas outside the ranges set by correlation scales, for example, when turbulent velocities are small? While one decreases the amplitude of supersonic turbulence, the absorption effects increase as a result of atoms concentrating at similar velocities, which is expressed by PPV density \( \bar{\rho}_s \approx \rho_S/[D_2(S)] + 2\beta^{1/2} \) growth, making recovering turbulence scaling more difficult. However, velocity effects disappear as turbulence becomes subsonic at all scales \( D_2(S) \ll \beta \). The power-law index of density inhomogeneities can then be obtained over the range of scales if \( \alpha_0S/\sqrt{\beta} < 1 \). Under similar conditions we can see density fluctuations at small scales \( r < r_{\text{thermal}} \), where the turbulence is subsonic even if it is supersonic at large scales. The scale \( r_{\text{thermal}} \) can be determined as \( D_2(r_{\text{thermal}}) = \beta \).

It may look somewhat counterintuitive that even in the absence of velocity we never see a thin skin of the cloud exhibiting the two-dimensional spatial\(^{10}\) slice of density fluctuations. Since absorption is proportional to density in our model, the optical depth changes with the change of density. As a result, the emission fluctuations are not dominated by the atoms within the thin skin depth from the cloud surface. The origin of the effect can be traced to original equations of radiative transfer (see eq. [26]). For instance, in the absence of velocity effects \( \rho_s \) is proportional to the integral over the whole line of sight (see eq. [25]).

4.2. Thin Velocity Slices of Spectral Line Data

Within the VCA technique introduced in LP00, the change of the spectral index of emission fluctuations with the thickness of the channel maps was employed to separate the contributions from density and velocity. In particular, to see the thin slice scaling that reflects the statistics of turbulent velocities, the effective window of the velocity slice should be narrower than the turbulent velocity dispersion at the scale of the study.

The thermal broadening (i.e., \( \beta \) factor in the previous section) of the line can be considered as a convolution with the window \( W_{\text{slice}} \) describing the channel slicing adopted in the data analysis. The LP00 study showed that adopting the slice thickness less than the thermal width \( \delta V < \beta \) does not provide new information about turbulence.\(^{11}\) In our discussion we treat thermal effects as adding (in quadrature) the minimum width to the data slice, symbolically \( \delta V^2 + \beta \).

Absorption results in the appearance of the second window function \( W_{\text{absorption}} \). Two cases can take place. If \( v_{ab}^2 < 6\delta V^2 + \beta \), the regime is not different from that discussed in the earlier subsection. Indeed, it is the absorption that limits the integration, and the results of integration over the slice thickness and over the whole emission line should be similar. According to the previous discussion, absorption smoothing reveals the underlying velocity statistics only for \( m < \frac{3}{2} \). Since the thermal

\(^9\) To be consistent, velocity effects of Table 1 have to be reevaluated in this case as a result of a larger contribution from density perturbations to the absorption.

\(^{10}\) We may note that a thin velocity slice is very different from a thin slice in space. Integration over the entire spatial extent of the cloud is performed while the velocity slice is defined.

\(^{11}\) It does provide the information about the gas temperature, however. This point will be elaborated elsewhere.
sorption is strong, and therefore is fixed, we get the following picture. If broadening is given by the properties of the media under study, the resolution of the instruments is poor. Instruments with good spectral resolution even if the spatial resolution is poorer than the velocity spectral index. Potentially, one may attempt to remove the universal statistics in this case, but this may be very challenging in practical terms. It is also evident that for subsonic turbulence the thin slice regime is not attainable.

5. ACCOMPLISHMENTS AND QUALITATIVE UNDERSTANDING OF RESULTS

Let us start with an explanation of what has been done so far in different parts of the present paper. We started by introducing radiative transfer in § 3 and obtaining general expressions for the statistics of intensity in the presence of the radiative transfer. We showed that absorption plays the role of an additional window function in those expressions. Then in §4 we discussed the two limiting and observationally important cases. One is related to the way most present-day observations of fluctuations are performed, namely, through measuring fluctuations of the integrated intensity. In this situation we show that the information about the underlying statistics in most cases is lost. We show, however, that the information can be recovered if fluctuations in velocity slices are analyzed. These are two major results of the paper.

A good deal of the paper and most of the Appendices deal with the statistics of the fluctuations without absorption. This was necessary because the earlier machinery was formulated in terms of power spectra, while the transformation from the PPV power spectra to correlation functions happens to be not trivial. In terms of our earlier work, this provides not only an independent check of our results in LP00 but also a very important extension of the technique. For instance, we obtain the predictions for the statistics of fluctuations along the velocity direction. These fluctuations can be studied using instruments with good spectral resolution even if the spatial resolution of the instruments is poor.

With the mathematical machinery in the paper being sufficiently involved, the issue of qualitative understanding of our results is of particular interest. This is important especially for our results, which may look somewhat counterintuitive.

Consider first the formulae for thin slices. The spectrum of intensity in a thin slice gets shallower as the underlying velocity gets steeper. To understand this effect, consider turbulence in incompressible optically thin media. The intensity of emission in a slice of data is proportional to the number of atoms per velocity interval given by the thickness of the data slice. Thin slice means that the velocity dispersion at the scale of study is larger than the thickness of a slice. The increase of the velocity dispersion at a particular scale means that less and less energy is being emitted within the velocity interval that defines the slice. As a result, the image of the eddy gets fainter. In other words, the larger the dispersion at the scale of the study, the less intensity is registered at this scale within the thin slice of spectral data. This means that steep velocity spectra that correspond to the flow with more energy at large scales should produce an intensity distribution within the thin slice for which more brightness will be at small scales. This is exactly what our formulae predict for thin slices (see also LP00).

The result above becomes obvious when one recalls that the largest intensities within thin slices are expected from the regions that are the least perturbed by velocities. If density variations are also present, they modify the result. When the amplitude of density perturbation becomes larger than the mean density, both the statistics of density and velocity are imprinted in thin slices. For small-scale asymptotics, with which we are mostly interested in this paper, this happens, however, only when the density spectrum is shallow, i.e., dominated by fluctuations at small scales.

Understanding the results for the integrated spectral line in the presence of absorption is a bit more involved. Again, it is advisable to consider incompressible flows first. If no absorption is present, the integrated spectral line images are not affected by velocity fluctuations. Indeed, the velocity field just redistributes atoms along the line of sight, and this redistribution cannot affect the total intensity in the absence of absorption.

If absorption is present, the fact that velocity redistributes atoms along the line of sight causes fluctuations of the integrated intensity. Two effects, however, are present simultaneously as the velocity field at a particular scale spreads atoms in PPV space. It is easy to see that the most absorption is expected when the atoms have the same velocities. Therefore, velocity dispersion provided by turbulence decreases the absorption and therefore increases the signal. However, the decrease of the optical thickness makes the media behave more like their optically thin counterpart, in which the velocity fluctuations are averaged out. In other words, while the mean level of intensity increases as a result of turbulence, the contrast of the fluctuations caused by the velocity fluctuations decreases. The range of the universal asymptotics $K^{-3}$ is the range at which the two effects exactly compensate each other.

Density fluctuations get important only when their amplitude gets larger than the mean density. This is quite analogous to the case of thin slices discussed above. The outcome is also analogous, namely, for the small-scale asymptotics only density with a shallow spectrum is important. Density with a steep spectrum may affect the large-scale asymptotics.
6. DISCUSSION

6.1. Description of Turbulence in Absorbing Media

In LP00 we provided the statistical description of spectral line data cubes obtained via observations of turbulent media. In the present paper we generalized this description for the presence of the absorption effects.

If we compare our present treatment of PPV statistics with that in LP00, we see that it differs in terms of the statistical tools used. As the absorption takes place in real and not Fourier space, the use of structure functions is advantageous. In Appendix D we rederive LP00 results for optically thin media using structure functions. Our present treatment extends the VCA technique for studies of turbulent absorbing media. How good is our treatment of absorption?

Above we used a one-dimensional model of radiative transfer. We would, however, argue that it is adequate for our purposes. Indeed, our aim is to study velocity and density fluctuations before the effects of optical depth distort the underlying power-law spectra. On the contrary, when the three-dimensional radiative transfer becomes important, we expect the nonlinear flattening of the observed emissivity spectra.

For the sake of simplicity in our treatment, we made major calculations assuming that the fluctuations within PPV data cubes are Gaussian. However, we showed that the structure of our expressions does not depend on this assumption. Whatever is the statistics of emissivity fluctuations in PPV space, the absorption introduces a new window function that truncates the velocity range over which the integration is provided. Possible changes in the window function arising from non-Gaussian fluctuations are expected to result in the unimportant changes of the numerical prefactors of the expressions.

We can test our result in the limiting cases of high and low absorption. In the latter case our formulae can be expanded and we get the LP00 result (see also Appendix C) for \( \alpha \to 0 \). For high absorption, it is only very small fluctuations that are still in the linear regime. Those small fluctuations are likely to be Gaussian.

There are a number of issues that are relevant to studies of turbulence in both absorbing and nonabsorbing media. One of them is the issue of velocity-density correlations. While the issue, in general, requires further research, the case of velocity-density correlations has been studied analytically in LP00 and numerically (Lazarian et al. 2001; Esquivel et al. 2003). Our discussion above reveals that when the effective thickness of the slice is less than the window function given by absorption \( W_{\text{abs}} \), the effects of absorption are unimportant. For such slices, if they are thin, the effect of velocity-density correlations was shown to be marginal.

For the sake of simplicity both in our present paper and in LP00 we assumed that the emissivity is proportional to the first power of density. If the dependence is nonlinear, e.g., proportional to \( n^2 \) as in the case of H\(_\alpha\) emission, for small-amplitude perturbations it is possible to show that linear terms still dominate and therefore our results above are applicable (see discussion in Cho & Lazarian 2002a).\(^{12}\)

An additional simplification shared by this work and LP00 is that the thermal velocity of media is assumed constant.

discussed above, thermal velocities affect the effective window function over which the turbulent velocities are smeared. As a result, if observations sample regions with different temperatures of gas, the thermal smearing will provide higher weighting to the regions where thermal velocity is less than the slice thickness. This stays true in the presence of absorption.

Another nonlinearity disregarded in the treatment above, as well as in LP00, is related to the nonlinearities in the Galactic rotation curve. The effects of the rotation curve shear were studied in Esquivel et al. (2003). There it was shown that effects of the shear can be ignored unless the galactic shear and the shear provided by the eddies are comparable. The latter is a very artificial situation, as the large-scale shear in most cases produces turbulence. For Kolmogorov turbulence the local shear \( \nu/l \) scales as \( l^{-2/3} \). Magnetorotational instability (see Chandrasekhar 1960; Balbus & Hawley 1991) ensures that the global galactic shear creates turbulence. If the shear provides marginal influence, the higher order corrections provided by the nonlinearity of the shear are also negligible.

Our treatment above assumes that the absorption happens within the emitting gas. What will happen when the radiation source is outside the turbulent cloud? If turbulence is studied using absorption lines, the registered intensity is

\[
I = I_0 \exp \left[- \int_o^L \alpha(z) n(z) d z \right],
\]

where \( I_0 \) is the intensity coming from the external source. The easiest way is to correlate logarithms of \( I/I_0 \). For constant \( \alpha \) these quantities are evidently proportional to the integrals of emissivities and the analysis in LP00 is directly applicable.

Is it always true that we have to deal with velocity fluctuations? If the velocity turbulence gets damped, while density statistical structure persists, e.g., in the form of entropy fluctuations, the effects of velocity modification become marginal.\(^{13}\)

6.2. Interpretation of Observations

In a number of instances the power-law spectra of interstellar turbulence were reported for the spectral line data. They include H\(_1\) (Green 1993; Stanimirović et al. 1999), CO in molecular clouds (Stutzki et al. 1998), H\(_2\) in the Reynolds layer (Minter & Spangler 1996), and H\(_1\) absorption (Deshpande et al. 2000). In all cases the reported index was close to \(-2.7\), which is really amazing as in all those cases very different quantities were measured.\(^{14}\) In comparison, fluctuations of synchrotron emission and starlight polarization show very different indices, although underlying turbulence is likely to be Kolmogorov (Cho & Lazarian 2002b).

Results of the earlier applications of VCA to the H\(_1\) data can be summarized as follows: (1) the observed scaling of

\(^{12}\) One may wonder whether complications arise at large scales. We believe that this is unlikely for steep densities. For instance, for H\(_\alpha\) the emissivity is proportional to density squared. The correlation functions of density \( 2(\delta n)^2 \) correspond to steeper spectra and therefore their contribution within thin slices is subdominant.

\(^{13}\) An important case of this regime is the viscosity-damped MHD turbulence reported in Cho et al. (2002b). This new type of MHD turbulence produces small-scale magnetic fluctuations through stretching of magnetic field by marginally damped eddies (Lazarian et al. 2004). The slowly evolving fluctuations of magnetic field result in the fluctuations of density that are essentially stationary (Cho & Lazarian 2003a, 2003b).

\(^{14}\) The spectral index obtained in Stanimirović et al. (1999) was somewhat steeper, namely, \(-3\). However, it was shown in Stanimirović & Lazarian (2001) that the difference is due to the slice thickness. As slice thickness was reduced, the index became approximately \(-2.8\) over the whole range of scales under study.
two-dimensional power spectra within thin slices is consistent with arising from velocity caustics, (2) the density power spectrum is steep at least in the case of Small Magellanic Cloud data, and (3) H I data are consistent with the velocity spectral index being approximately Kolmogorov.

The interpretation of Hα data may be done along the same line of reasoning. Earlier we argued that nonlinearities in the Hα emissivities are not important for studies of small-scale fluctuations. This would also indicate the correspondence with the Kolmogorov spectrum of turbulence of velocity.

Consider now H I absorption. Instead of the overdensity, the absorption measures deal with the ratio of density over temperature $n(x)/T(x)$. Both quantities fluctuate, and a priori one expects the spectrum to differ from that of H I emission. However, Deshpande et al. (2000) found that the index of the absorption spectrum in slices of their data is similar to that in Green (1993), but over the range from 3 to 0.07 pc. However, at scales of approximately 0.2 pc a change of the index of the power spectrum was observed. Therefore, the observed power law covers approximately 1 order of magnitude on scales from 3 to 0.2 pc. If at those scales turbulent velocities are larger than thermal velocities, the slices used may be thin, and the coincidence of the spectral index in the emission and absorption would mean the continuation of the turbulent cascade to small scales. This, however, may be a bit surprising, as the velocity turbulence at those scales is likely to be damped through viscosity by neutrals unless the turbulence is generated at very small scales.

To determine what is going on, Deshpande et al. (2000) used VCA prescriptions to evaluate the effects of the velocity and did not detect any change of the spectral index as they changed the thickness of the data slice. The fact that Deshpande et al. (2000) could not see changes of the spectral index can be interpreted to mean that the fluctuations arise exclusively from density variations with the shallow spectral index (roughly −2.7).

If further research confirms that the density spectrum is indeed shallow, this may be the signature of the magnetic energy cascade simulated in Cho et al. (2002a) and Cho & Lazarian (2003a) and described in Lazarian et al. (2004). In this new regime of turbulence, fluctuations of density arise from fluctuations of magnetic field, while velocity fluctuations are marginal. The expected power spectrum of density scales as $k^{-3}$ and is roughly consistent with the observations. Note that such a shallow spectrum of density, according to Deshpande (2000), can explain the detected mysterious tiny scale structures in H I (see Heiles 1997). It is obvious that a further study of the statistics of H I fluctuations at small scales is necessary. Synergy of different techniques might be called for. For instance, modified velocity centroids (LE03) could provide an independent test of whether velocity fluctuations are important.

All the data above were obtained by studies of fluctuations within velocity slices of spectroscopic data. On the other hand, CO data of Stutzki et al. (1998) present the power spectrum of fluctuations obtained via integrating the intensity over the whole emission line. According to Falgarone et al. (1998), the turbulent velocities even at the smallest scales are still supersonic. Thus, we expect to see a velocity modification of the spectrum. Stutzki et al. (1998) find an index of order −2.8. Accepting possible criticism that in the case of H I we assumed that the deviation of −2.7 from −3 is meaningful and informative, we may speculate that the measured CO spectral index is close to the universal asymptotics of −3 predicted in the paper above. We may speculate that the flattening of the spectrum compared to the expected −3 may be caused in the case of CO by instrumental noise (this possibility was mentioned to us by our referee).

Alternatively, one expects to see the power index $n + m/2$ if the density spectral index $n > −3$. If we want to account for the −2.8 index, we would have to accept not only that the density is shallow but also that the velocity is shallower than the Kolmogorov value. More studies utilizing recent theoretical insights are clearly necessary.

What are the goals that can be achieved by studying properties of interstellar turbulence? Establishing turbulence spectra by applying VCA to different emission and absorption lines could help to answer fundamental questions of interstellar physics. For instance, it could answer the question of whether the turbulence in molecular clouds is a part of a bigger cascade from the large scales or is locally generated. It could also answer the question of to what extent self-gravity is important in determining the spectra of fluctuations at different scales.

In addition, VCA states that the intensity of fluctuations measured in a velocity channel depends on whether the channel width is larger or smaller than the thermal velocity dispersion of the atoms. This potentially allows us to find the distribution of temperatures of emitters along the line of sight. This may provide an independent testing of theories of the ISM thermal structure and answer the question of the role of thermal instability for the ISM (see Heiles & Troland 2003).

6.3. Range of Applicability and Numerical Testing

The VCA technique is most efficient for restoring the underlying spectral indices when the turbulence obeys a power law and the turbulent velocities are larger than the thermal velocities of atoms. While the first requirement is usually fulfilled in interstellar turbulence (see Stanimirović & Lazarian 2001), numerical simulations provide a limited inertial range if any. This presents problems when the testing is involved. For instance, in Lazarian et al. (2001) and Esquivel et al. (2003) the spectral slope was corrected to fit a power law. While the power-law requirement is not an essential constraint for most of the interstellar situations, the ratio of the turbulent to thermal velocity may be a constraint. To use the VCA for the subsonic turbulence, one should use the heavier species, in which thermal velocities are less than that of turbulence if the thin slice asymptotics is intended to be observed. Note that in the case of turbulence that is supersonic at large scales, this potentially limits the range of the small scales that can be studied through the VCA technique. Indeed, at some small scale the turbulence gets subsonic and the slope of the emissivity fluctuations within a slice should change.\footnote{This change can be used to determine at what scale turbulent and thermal velocities are equal. If the dispersion of turbulent velocities is known at the large scales, this allows us to measure the temperature of gas. If, alternatively, the temperature of gas is known, this allows us to calibrate the power spectra in terms of absolute values of turbulent velocities.}

Another problem of numerical testing is the appearance of shot noise related to the limited number of data points available through simulations. Unless understood, this noise produces a bias in the power spectral indices obtained through synthetic observations based on the numerically generated data cubes. This problem was analyzed in Esquivel et al. 15
understanding of the statistics of spectral line data cubes goes
finite. When absorption is taken into account, our preliminary
observations, where the number of emitters is essentially in-
(2003), who concluded that it is not important for actual
observations, where the number of emitters is essentially in-
-16. The good news is that by varying the thickness of the velocity slices,
one can find the ratio of the thermal and turbulent velocities (LP00).

The nontrivial issues above may cause confusion. For in-
stance, in Miville-Deschênes et al. (2003b) the utility of the
VCA was questioned. We have to reiterate that the effects that
the authors encountered, e.g., shot noise, are related to the
synthetic data they used and not to the actual observational
data. None of the problems they discuss were present when
supersonic turbulence of the SMC was studied in Stanimirović
& Lazarian (2001).

In another paper, Miville-Deschênes et al. (2003a) mention
limitations of the VCA related to the thermal motions of
atoms. They point out that the thermal line width of atoms can
entail asymptotic behavior that can be mistakenly identified
with the actual saturation of the power-law index in a thin
slice. In fact, the observed saturation in the two cases is dif-
derent. If the thermal line width gets larger than the thickness
of the slice, a further decrease in the slice thickness does not
change the shape of the slice emissivity spectrum. It consists
of two parts: a shallow part corresponding to the scales at
which turbulent velocity is larger than \( v_{\text{thermal}} \) (thin slice), and a
steeper part corresponding to the scales at which the tur-
bulent velocity is larger than \( v_{\text{thermal}} \) (thick slice). On the other
hand, if the turbulence is supersonic on the scales under study,
the change of the slice thickness makes the change from thin
to thick slice move to smaller and smaller scales. In other
words, the slice is thin at large scales and thick at small scales,
with the border between these two regimes moving as the
thickness of the slice changes, provided that the experimen-
tally determined thickness of the slice is larger than the ther-
mal broadening of the line.

Such a study has not been done in Miville-Deschênes et al.
(2003a), nor do they give estimates of the thermal and tur-
bulent velocities of gas in the region Ursa Major Galactic
cirrus they studied. Instead, they pointed out the discrepancy
between the spectral index obtained using the velocity cen-
troid analysis and those the VCA would give if the asymptotic
regime were not due to thermal broadening. By itself, this
shows an important synergy of different techniques. Unfor-
nately, the authors did not check whether their centroids are
dominated by velocity or density. The application of the cri-
terion obtained in LE03 to the data seems essential for an-
swering this question. A high degree of correlation between
the averaged density and the averaged centroid maps, as well
as a close similarity of the spectra deduced from averaged
density and the centroid maps, calls for further testing of what
exactly the velocity centroids measure in each particular case
(see LE03). A study in Esquivel & Lazarian (2004) testifies
that velocity centroids are dominated by density if magnetic
turbulence has Mach numbers higher than 2. This also sug-
gests that centroids obtained for the hypersonic turbulence in
\( \text{H} \alpha \) are dominated by density.

We would like to stress that the importance of quantitative
understanding of the statistics of spectral line data cubes goes
far beyond the particular recipes of the VCA technique. LP00
and this paper provide for the first time the machinery suitable
for dealing with the statistics of spectral line data. This ma-
chinery not only should provide advances when analysis of
spectra in channels is involved but also suggests new techni-
ques. For instance, our treatment of velocity correlations in the
\( v \)-direction allows a new way of measuring the spectral index
of velocity fluctuations.

7. SUMMARY

In this paper we have studied effects of absorption on the
statistics of spectral line data. In particular, we discuss how the
three-dimensional velocity and density statistics of turbulence
can be recovered from the spectral line data. We subjected to
scrutiny two regimes of studying turbulence, namely, (1) when
integrated spectral line intensity is used and (2) when the
channel maps are used. For our treatment we assumed the
power law underlying velocity and density statistics. Our re-
sults are summarized below:

1. For integrated spectral line data we find that:

(a) Absorption introduces nonlinear effects that distort
the power-law behavior of measured intensity. The nonlinear
effects are marginal when fluctuations are studied at sufficiently
small scales.

(b) While in the absence of absorption the velocity effects
were washed out for the integrated spectral data, absorption
makes velocity modification prominent. The recovery of the
underlying velocity spectrum from integrated spectral data is
nontrivial and sometimes impossible. For a range of scales, the
resulting power spectrum of intensity is caused by velocity
fluctuations but does not depend on the spectral index of ve-
locity fluctuations.

(c) Density spectra can be recovered in the limit of small-
scale asymptotics that we mostly deal with here only when the
density has a shallow spectrum, i.e., is dominated by fluctua-
tions at small scales or at scales where turbulence gets subsonic.

2. For spectral data in channel maps we find that if slices are
sufficiently thin, we recover thin slice asymptotics obtained
earlier in LP00. However, the choice of slice thickness is
limited by absorption, which introduces the minimal thickness
of the slice for which nonlinearity is negligible.

3. We incorporated our results into the VCA technique
intended for the observational studies of turbulence. The de-
tailed description of the spectral line data cube statistics that we
obtained should enrich the set of tools available for advancing
quantitative insight into interstellar turbulence.

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APPENDIX A

STATISTICS OF VELOCITY AND DENSITY IN XYZ COORDINATES

Statistics of a finite-size cloud.—If we deal with a cloud of size $S$, the turbulence has a maximal scale. In the absence of additional physics the size $S$ also serves as a structure function cutoff scale $D(r) \sim D(\infty) r^m/(r^m + S^m)$. We notice that if this law is adopted for $D_{LL}(r)$, the $z$-component structure function in the case of solenoidal turbulence is

$$D_z(r) = D(\infty) \frac{r^m}{r^m + S^m} \left[ 1 + \frac{m/2}{1+ (r/S)^m} (1 - \cos^2 \theta) \right], \quad \cos \theta = r \cdot \hat{z}/r,$$

$$D_z(\infty) = D(\infty) = C S^m.$$  \hfill (A1)

We further assume that the turbulent velocity field is Gaussian, in which case its properties are fully determined by the two-point probability distribution function $P(u_1,u_2)$, where $u_1 = \mathbf{u}_t(x_1), u_2 = \mathbf{u}_t(x_2)$. Using variables $u = u_1 - u_2, u_+ = (u_1 + u_2)/2$, this function is conveniently expressed [we write down the distribution of the $z$-component of the velocity field $u(x)$ only] as

$$P(u_1, u_2) = P(u, u_+) = \frac{1}{\pi \sqrt{2D_z(\infty) - D_z(r)\sqrt{D_z(r)}}} \exp \left[ -\frac{u^2}{2D_z(r)} \right] \exp \left[ -\frac{u_+^2}{2D_z(\infty) - D_z(r)/2} \right].$$ \hfill (A3)

Validity of equation (A3) is immediately checked by observation that $u, u_+$ are uncorrelated $\langle uu_+ \rangle \sim \langle u^2 \rangle \sim \langle u_+^2 \rangle = 0$ Gaussian quantities with dispersions

$$\langle u^2 \rangle = \langle (u_1 - u_2)^2 \rangle = D_z(r),$$

$$\langle u_+^2 \rangle = \frac{1}{4} \langle (u_1 + u_2)^2 \rangle = \frac{1}{4} \left[ 2\langle u_1^2 \rangle + 2\langle u_2^2 \rangle - \langle (u_1 - u_2)^2 \rangle \right] = \frac{1}{4} [D_z(\infty) - D_z(r)/2],$$

and the Jacobian of transformation between $u, u_+$ and $u_1, u_2$ is unity.

Shallow density spectrum.—Here $u_1 = \mathbf{u}_t(x_1), u_2 = \mathbf{u}_t(x_2)$. When statistical properties of the density fluctuations are dominated by short wavelengths, we use power-law correlation functions of overdensity:

$$\xi(r) = \langle \rho \rangle^2 \left[ 1 + \frac{\langle \rho \rangle}{\rho} \right]^\gamma, \quad \gamma > 0.$$ \hfill (A4)

Note that the power-law part of the correlation corresponds to the three-dimensional power-law spectrum $\propto k^n$, where $n = \gamma - 3$, which for $\gamma > 0$ means that $n$ is shallow, i.e., less than $-3$.

Steep density spectrum.—To describe the density statistics for a steep (i.e., $n < -3$) power spectrum, one should use a structure function description. For $\gamma < 0$ we write

$$\xi(r) = \frac{1}{2} d(\infty) [1 - d(r)/d(\infty)],$$ \hfill (A5)

where $d(r)$ is a structure function of density given by equation (3).

Real-world structure functions do not grow infinitely, and therefore we have to introduce a cutoff at some large scale. If the cutoff happens at $r_c$, then for the long-wave–dominated turbulence ($\gamma < 0$)

$$d(r) = d(\infty) \frac{r^{-\gamma}}{r^{-\gamma} + r_c},$$ \hfill (A6)

and the correlation function is

$$\xi(r) = \frac{d(\infty) r_c^{-\gamma}}{2 (r^{-\gamma} + r_c)}.$$ \hfill (A7)

For sufficiently small $r \ll r_c$, equation (A7) gives

$$\xi(r) \approx \frac{1}{2} d(\infty) [1 - (r/r_c)^{-\gamma}].$$ \hfill (A8)

Note that in equation (A8) $r_c[(\rho^2/d(\infty))]^{-1/\gamma}$ plays the role of $r_0$ in equation (A4).

When using both correlation and structure functions, one essential difference should be kept in mind: while the value of the structure function at some scale $r_c, d(r_c)$, is determined by the power of fluctuations at smaller scales $r \leq r_c$, the value of the correlation function $\xi(r_c)$ reflects the integral power at scales $r \geq r_c$. For the power-law statistics, the correspondent integral contributions are dominated by the power at scale $r_c$ itself, localizing the information provided by $\xi(r_c)$ and $d(r_c)$ values.
APPENDIX B

STATISTICS OF DENSITY IN PPV

In this appendix we calculate some statistical properties of the velocity space density

$$\rho_s(X, v) = \int_0^S dz \rho(x) \phi_s(x),$$

(B1)

where $\phi_s(x)$ is given by equation (13), and random fields $\rho(x)$ and $u(x)$ over distribution of which we average are assumed to be uncorrelated. We assume the turbulent velocity field to be described by the Gaussian two-point probability distribution function given in equation (A3). We also consider statistical properties of the density distribution in the galactic coordinates to be homogeneous.

The mean PPV density is given by

$$\langle \rho_s(X_1, v_1) \rangle = \int_0^S dz_1 \langle \rho(x_1) \rangle \phi_s(x_1)$$

and the two-point correlation function is

$$\langle \rho_s(X_1, v_1) \rho_s(X_2, v_2) \rangle = \int_0^S dz_1 \int_0^S dz_2 \langle \rho(x_1) \rangle \langle \rho(x_2) \rangle \phi_s(x_1) \phi_s(x_2)$$

$$= \frac{1}{2\pi} \int_{-S}^S dz \int_{|z|/2}^{S-|z|/2} dz_+ \xi(r) [D_s(r) + 2\beta]^{1/2} \exp \left[ -\frac{(v - v_{gal})^2}{2[D_s(r) + 2\beta]} \right]$$

$$\times \sqrt{\frac{2}{|\beta + D_s(\infty) - D_s(r)/2|^{1/2}}} \exp \left[ -\frac{(v - v_{gal})^2}{2[D_s(\infty) - D_s(r)/2]} \right].$$

(B3)

A particular case of equation (B3) is the second moment

$$\langle \rho_s(X_1, v_1)^2 \rangle = \int_0^S dz_1 \int_0^S dz_2 \langle \rho(x_1, z_1) \rho(x_1, z_2) \rangle \phi_s(x_1) \phi_s(x_1)$$

$$= \frac{1}{2\pi} \int_{-S}^S dz \int_{|z|/2}^{S-|z|/2} dz_+ \xi(|z|) [D_s(|z|) + 2\beta]^{1/2} \exp \left[ -\frac{v_{gal}^2}{2[D_s(|z|) + 2\beta]} \right]$$

$$\times \sqrt{\frac{2}{|\beta + D_s(\infty) - D_s(|z|)/2|^{1/2}}} \exp \left[ -\frac{(v - v_{gal})^2}{2[D_s(\infty) - D_s(|z|)/2]} \right].$$

(B4)

Here we have used our standard notation $v = v_1 - v_2, v_+ = (v_1 + v_2)/2, z = z_1 - z_2, z_+ = (z_1 + z_2)/2, r = x_1 - x_2, v_{gal} = v_{gal}(x_1) - v_{gal}(x_2)$, and $v_{gal, +} = [v_{gal}(x_1) + v_{gal}(x_2)]/2$.

We observe that the density in the velocity space is statistically inhomogeneous, which is reflected first of all in a residual dependence of the quantities in equations (B2)–(B4) on the absolute velocity $v_1$ or $v_+$ and not only on velocity difference $v$. In particular, equation (B2) describes the mean velocity profile of the density, which is not, in general, uniform.

There can also be dependence of the statistics on sky coordinates $X$, if the regular flow pattern $v_{gal}(x)$ is complex. We do not consider this possibility, restricting our attention to either $v_{gal}$ described by the linear shearing pattern or the case in which regular flow can be neglected altogether as is the case for isolated clouds.

Equations (B2)–(B4) are quite complex. They are significantly simplified in several astrophysically important cases. First of all, when one considers small enough scales so that turbulent velocities are smaller than $D_s(\infty)^{1/2}$, all exponential terms that contain $D_s(\infty)$ can be taken equal to unity and we have

$$\langle \rho_s(X_1, v_1) \rangle = \text{const},$$

(B5)

$$\langle \rho_s(X_1, v_1)^2 \rangle = \text{const},$$

(B6)

$$\langle \rho_s(X_1, v_1) \rho_s(X_2, v_2) \rangle \propto \int_{-\infty}^{\infty} dz \left( 1 - \frac{|z|}{S} \right) \frac{\xi(r)}{[D_s(r) + 2\beta]^{1/2}} \exp \left[ -\frac{(v - v_{gal})^2}{2[D_s(r) + 2\beta]} \right].$$

(B7)
These equations are applicable when the velocity line is extended compared with the turbulent scales under study, which is the case, for example, for the 21 cm emission of interstellar H I. Henceforth we do not follow precise values of the proportionality factors, which depend on the extent of the line-of-sight integration $S$ and the details of the integral cutoff. For the analysis of the scaling laws at small separations $R$ the boundary effects can be neglected and the range of integration extended to infinity.

In another limit, when dealing with lines from individual clouds, one can consider regular flow to be absent. In this case

$$
\langle \rho_s(\mathbf{X}_1, \ v_1) \rangle \propto \frac{\bar{\rho}^2 S^2}{[D_z(\infty) + \beta]^{3/2}} \exp \left[ -\frac{v_i^2}{D_z(\infty) + \beta} \right],
$$

which is equivalent to equation (C1) of LP00 if temperature broadening is taken to be small. These equations are applicable when the velocity line is extended compared with the turbulent scales under study, which is the case, for example, for the 21 cm emission of interstellar H I.

Again, for small separations $R$, the integration can be extended to an infinite range, $S \to \infty$.

An image of an individual cloud in PPV space has an extension $\sim D_z^{1/2}(S)$. This stems from the fact that for an individual cloud the motions on the large scale determine the velocity dispersion. The assumed distribution, e.g., the Gaussian distribution, of large-scale cloud velocities is present for an ensemble of clouds, while for an individual cloud under study its extension in the velocity space is a fixed realization from this distribution. Typically, for an individual cloud, the mean density is fixed. As a result, the terms like $\exp \left\{ -v_i^2/2[D_z(\infty) + \beta] \right\}$ in equations (B8)–(B10) represent numerical factors and are omitted in our further discussion.

APPENDIX C

CORRELATION AND STRUCTURE FUNCTIONS IN PPV

The study of the three-dimensional correlation function in PPV space has been performed in LP00, with an emphasis on the case of Galactic turbulence characterized by the presence of the coherent flow due to Galactic rotation. These results follow from equation (B7), which can be written in the form equivalent to equation (C1) of LP00 if temperature broadening is taken to be small (or considered later as a part of the velocity slice width):

$$
\xi_s(\mathbf{R}, \ v) = \xi_s(\mathbf{R}, \ v) + \xi_{s\rho}(\mathbf{R}, \ v),
$$

$$
\xi_s(\mathbf{R}, \ v) \propto \int_{-\infty}^{\infty} \frac{\bar{\rho}^2 S}{[D_z(r) + \beta]^{1/2}} \exp \left[ -\frac{(v - v_{\text{gal}})^2}{2D_z(r)} \right],
$$

$$
\xi_{s\rho}(\mathbf{R}, \ v) \propto \int_{-\infty}^{\infty} \frac{\bar{\xi}(r)}{[D_z(r) + \beta]^{1/2}} \exp \left[ -\frac{(v - v_{\text{gal}})^2}{2D_z(r)} \right].
$$

In this appendix we present a complimentary set of results for the PPV structure functions, suitable for description of the gas confined to isolated clouds. Our starting point is equation (B9). We are interested only in the small-scale $r/S \ll 1$ behavior, for which we use the simplified expression

$$
\xi_s(\mathbf{R}, \ v) \propto \frac{\bar{\rho}^2 S}{D_z(S)^{1/2}} \int_{-S}^{S} \frac{1}{D_z(r)^{1/2}} \exp \left[ -\frac{v^2}{2D_z(r)} \right],
$$

where boundary inhomogeneous effects are neglected. The quantity $\bar{\rho}_s \approx \bar{\rho}S/D_z(S)^{1/2}$ is the approximate mean density in PPV space. At small amplitudes of turbulence velocity one should restore the thermal term in this expression, $\bar{\rho}_s \approx \bar{\rho}S/D_z(S) + 2\beta^{1/2}$.

Observations of interstellar turbulence testify that the emissivity obeys power laws (see Armstrong et al. 1995; Lazarian 1999b; CLV02c). This makes it natural to assume that the underlying velocity and density statistics are also power laws. Power laws for the statistics of turbulence are also expected from theoretical considerations (see CLV02c). In what follows we use normalized variables, which for a cloud of extent $S$ would amount to using spatial coordinates normalized by $S$ and the velocities normalized by the velocity dispersion at the cloud scale $CS^m$. For simplicity, we also write $D_z(r) = Cr^m$, ignoring noncritical additional
angular dependence exhibited by pure solenoidal or potential flows. Asymptotics for \( \tilde{d}_s(R, 0), \tilde{d}_p(R, 0), R/S \to 0 \), as well as \( d_s(0, v), d_p(0, v), v^2/C S^m \to 0 \), follow from analysis of the integral given by equation (C.3). Namely,

\[
\tilde{d}_s(R, 0) = 2[\xi_s(0, 0) - \xi_s(R, 0)]
\]

\[
\propto \frac{\tilde{\rho}^2 S^2}{D_s(S)} \int_{-1}^{1} dz \left[ \frac{1}{2m/2} - \frac{1}{[(R/S)^2 + z^2]^{m/2}} \right]
\]

\[
\propto -\frac{\tilde{\rho}^2 S^2}{D_s(S)} \sqrt{\pi \Gamma(m/4 - 1/2)} (R/S)^{1-m/2},
\]

\[ (C.4) \]

\[
\tilde{d}_p(R, 0) = 2[\xi_p(0, 0) - \xi_p(R, 0)]
\]

\[
\propto \frac{\tilde{\rho}^2 S^2(r_0/S)^{7}}{D_s(S)} \int_{-1}^{1} dz \left[ \frac{1}{2^{\gamma+m/2}} - \frac{1}{[(R/S)^2 + z^2]^{\gamma+m/2}} \right]
\]

\[
\propto -\frac{\tilde{\rho}^2 S^2(r_0/S)^{7}}{D_s(S)} \sqrt{\pi \Gamma(\gamma/2 + m/4 - 1/2)} (R/S)^{1-\gamma-m/2}.
\]

\[ (C.5) \]

The latter estimation is valid for \( \gamma + m/2 < 1 \), i.e., for sufficiently steep density spectra in real space. The Kolmogorov density spectrum \( \gamma = -\frac{2}{3} \) belongs to this class. For shallow spectra that have positive \( \gamma \), one should use the correlation function \( \xi_p \) in place of \( d_p \), but one finds (see LP00) that \( \xi_p(R, 0) \) still obeys the scaling given by equation (C.5).

Somewhat more complex is the velocity dependence of the structure functions. For simplicity, we describe all the steps in detail for the velocity term:

\[
\tilde{d}_s(0, v) = 2[\xi_s(0, 0) - \xi_s(v)]
\]

\[
\propto \frac{\tilde{\rho}^2 S^2}{D_s(S)} \int_{-1}^{1} dz \left[ 1 - \exp \left( -\frac{v^2}{2am} \right) \right]
\]

\[
\propto \frac{\tilde{\rho}^2 S^2}{D_s(S)} \frac{1}{mp} \left[ 1 - p^{2-p} v^2 \Gamma \left( -p, \frac{v^2}{2} \right) \right],
\]

\[ (C.6) \]

where \( p = 1/m - \frac{1}{2} > 0 \) and \( \Gamma \) is the incomplete gamma function. For small argument, the series expansion of the incomplete gamma function is

\[
\Gamma \left( -p, \frac{v^2}{2} \right) \sim \Gamma[-p] + v^{-2p} \left[ \frac{2^p}{p} + \frac{2^{p-1}}{1-p} v^2 + O(v^4) \right].
\]

\[ (C.7) \]

Substituting equation (C.7) into equation (C.6), we see that the most divergent term in the \( \Gamma(-p, v^2/2) \) cancels out, while all the residual terms are convergent. The leading behavior for the velocity structure function is \( \tilde{d}_s(0, v) \sim \Gamma(-p) v^2p \) for \( p < 1 \), which changes to \( \tilde{d}_s(0, v) \sim (2p + 1)/4(p - 1) v^2 \) for \( p > 1 \). Thus, the structure function scaling depends on \( m \) for \( m > \frac{3}{2} \), \( \tilde{d}_s(0, v) \sim v^{(2-m)/m} \) and is fixed to \( \tilde{d}_s(0, v) \sim v^2 \) at \( m < \frac{3}{2} \).

\[
\tilde{d}_s(0, v) \propto \frac{\tilde{\rho}^2 S^2}{D_s(S)} \left( \frac{1}{2} - \frac{1}{m} \right) \left( \frac{v^2}{D_s(S)} \right)^{1/m-1/2}, \quad m > \frac{2}{3},
\]

\[ (C.8) \]

\[
\tilde{d}_s(0, v) \propto \frac{\tilde{\rho}^2 S^2}{D_s(S)} \frac{1}{2 - 3m} \frac{v^2}{D_s(S)}, \quad m < \frac{2}{3}.
\]

\[ (C.9) \]

Notably, the Kolmogorov value \( m = \frac{3}{2} \) is the boundary case here. One should note that representation given by equation (C.7) is incorrect for integer \( p \), i.e., for \( m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \). Series expansion of the incomplete gamma function is irregular in these cases, which manifests itself in the appearance of logarithmic terms. For example,

\[
\Gamma(-1, v^2/2) \sim 2v^2 + \log v^2 + O(v^0), \quad m = \frac{2}{3},
\]

\[ (C.10) \]

\[
\Gamma(-2, v^2/2) \sim 2v^{-4} - 2v^2 - O(\log v), \quad m = \frac{2}{5}.
\]

\[ (C.11) \]

Thus, quadratic scaling of the structure function with velocity is preserved even in these cases, except that for the Kolmogorov turbulence, logarithmic correction enters the leading behavior.
As in the case of the radial dependence given by equation (C4), the velocity term can be considered a special case $\gamma = 0$ of the density term. Indeed, $\bar{d}_\rho(0, v)$ is obtained analogous to just completed analysis by substituting $p = 1/m - \gamma/m - \frac{1}{2}$ and restoring the $(r_0/S)^\gamma$ amplitude factor:

\[
\bar{d}_\rho(0, v) \propto \frac{\tilde{\rho}^2 S^2}{D_2(S)} \left( \frac{r_0}{S} \right)^\gamma \left( \frac{1}{2} - \frac{1 - \gamma}{m} \right) \left[ \frac{v^2}{D_2(S)} \right]^{\frac{(1 - \gamma)/m - 1/2}{D_2(S)}} \cdot m > \frac{2}{3}(1 - \gamma),
\]

\[
\bar{d}_\rho(0, v) \propto \frac{\tilde{\rho}^2 S^2}{D_2(S)} \left( \frac{r_0}{S} \right)^\gamma \frac{1}{2(1 - \gamma) - 3m D_2(S)} \cdot m < \frac{2}{3}(1 - \gamma).
\]

Notably, the range of $m$ where velocity scaling is sensitive to $m$ decreases for the long-range–dominated density spectra, $\gamma < 0$. For example, for Kolmogorov density spectra $\gamma = -\frac{2}{3}$, we should use the second scaling solution for all $m < 10/9$. Thus, the quadratic scaling solution given by equation (C13) is the one of predominant interest in the study of such density inhomogeneities. On the other hand, for the short-wave–dominated density spectra $\gamma > 0$ the $m$ sensitivity is retained over a much wider range and one should use equation (C12).\(^{17}\) The exact transitional value of $m = \frac{2}{3}(1 - \gamma)$ points again to the presence of the slow-varying logarithmic term in the asymptotic scaling.

One may think that there is a contradiction of our present results with those in LP00. Indeed, spectra along the velocity coordinate do not have this property. This contradiction resolves in a very simple way. The structure functions cannot grow faster than $v^2$. As a result, they do saturate at this value. At the same time spectra do not have such a limitation and properly reflect very steep spectra that arise. Therefore, to identify the power index of fluctuations along the velocity coordinate, it is necessary to measure spectra along the velocity axis. This provides a new way\(^{18}\) to study turbulence that we currently work on with A. Chepurnov and shall elaborate elsewhere. Here we only note that the steepness of the spectrum prevents a useful generalization of the spectral correlation function approach (see Rosolowsky et al. 1999) to the velocity direction (compare to a discussion in Lazarian 1999b).

**APPENDIX D**

**OPTICALLY THIN CASE: STRUCTURE FUNCTION APPROACH**

In this appendix we revisit the problem of the intergalactic turbulence in the case of optically thin lines, which was the focus of LP00. With the correlation function in velocity space given by equation (18), we are now able to directly derive the asymptotic expressions for the correlation (or structure) functions of the intensity, bypassing rather cumbersome calculations of the three-dimensional power spectrum in the velocity space.

The intensity of the emission in an optically thin line is proportional to the density of the atoms in the velocity space

\[
I(X_1, v_1) = \epsilon \rho_s(X_1, v_1).
\]

(D1)

The mean line profile follows from equation (C3),

\[
\langle I(X_1, v_1) \rangle = \frac{\epsilon \tilde{\rho}}{\{2\pi [\beta + D_2(\infty)]\}^{1/2}} \int_0^S \! dz \exp \left\{ - \frac{[v_1 - v_{\text{gal}}(x_1)]^2}{2[\beta + D_2(\infty)]} \right\}.
\]

(D2)

Total intensity in a velocity channel of width $\delta V$ centered at velocity $V$ and described by the shape function $W_C(v - V)$, such that $\int dv W_C(v - V) = \delta V$, is

\[
I_C(X_1) = \epsilon \int \! dv_1 W_C(v_1 - V) \rho_s(X_1, v_1).
\]

(D3)

The mean of the channel-integrated intensity is

\[
\langle I_C(X_1) \rangle = \frac{\epsilon \tilde{\rho}}{\{2\pi [\beta + D_2(\infty)]\}^{1/2}} \int_0^S \! dz \int \! dv W_C(v - V) \exp \left\{ - \frac{[v - v_{\text{gal}}(x_1)]^2}{2[\beta + D_2(\infty)]} \right\}.
\]

(D4)

For the correlation function,

\[
\xi_i(R) \equiv \langle I_C(X_1)I_C(X_2) \rangle = \epsilon^2 \int \! dv_1 W_C(v_1) \int \! dv_2 W_C(v_2) \langle \rho_s(X_1, v_1)\rho_s(X_2, v_2) \rangle,
\]

(D5)

\(^{17}\) However, for $\gamma + m/2 > 1$ our analysis should be replaced by the PPV correlation function formalism.

\(^{18}\) Note that it is not necessary to have good spatial resolution to get turbulence if spectra along the velocity coordinate are taken. Therefore, velocity turbulence studies may become an important part of extragalactic research. The corresponding asymptotics can be found in Table 3 of LP00.
we use equation (B7) applicable to the case of intergalactic turbulence with extended gas distribution and coherent galactic flow with a linearized shear:

$$
\xi_\nu(\mathbf{R}) \propto \frac{\epsilon^2 \beta^2}{2\pi} \int_{-\infty}^{\infty} dz \left[ 1 + \tilde{\xi}(r) \right] [D_\nu(r) + 2\beta]^{-1/2} \int dv W(v) \exp \left\{ -\frac{(v - f - 1\gamma)^2}{2[D_\nu(r) + 2\beta]} \right\}.
$$  \hspace{1cm} (D6)

Here two velocity channel windows are combined into $W(v) = \int dv_+ W_C(v_+/2 - v)W_C(v_+/2 + v)$. Equation (D6) describes the effect of the line-of-sight projection of three-dimensional turbulent distributions of emitters. The velocity kernel both modifies the projection of the density correlation function $[\xi(r)]$ term in the first bracket, which now exhibits different scaling than just a column density correlation function would, and also introduces a new phenomenon: even uniform spatial distribution of emitters following incompressible turbulent flow will cause fluctuating intensity if observations are carried out in a sufficiently narrow velocity channel. The pure velocity-induced fluctuations are defined by the first (namely, unity) term in the first bracket. Furthermore, we discuss two effects separately, using the obvious split of the full correlation function into two terms $\xi_\nu(\mathbf{R}) = \xi_\nu(\mathbf{R}) + \xi_\nu(\mathbf{R}).$

As expected, the intensity integrated over the whole line contains no information about the velocity field. Indeed, integration with a constant window $W(v)$ over the entire velocity range in equation (D6) gives

$$
\xi_\nu(\mathbf{R}) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \xi(r) \sim R^{1-\gamma},
$$  \hspace{1cm} (D7)

while $\xi_\nu$ is reduced to a constant that describes the uniform mean intensity distribution and can be dropped.

In LP00 we presented a detailed discussion of the velocity influence on the projected intensity statistics in optically thin lines. Here we reproduce the scaling solutions at small sky separations $R$, which was shown in LP00 to be found for observations in thin velocity channels, using the correlation function formalism of equation (D6) as the starting point.

For the very narrow channel $W(v) = \delta(v)$ we have

$$
\xi_\nu(\mathbf{R}) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \left[ 1 + \tilde{\xi}(r) \right] [D_\nu(r) + 2\beta]^{-1/2} \exp \left\{ -\frac{(f - 1\gamma)^2}{2[D_\nu(r) + 2\beta]} \right\}.
$$  \hspace{1cm} (D8)

The asymptotic behavior of the density term at the small scales $R \ll (C^2 f)^1/(2-m)$ is obtained straightforwardly:

$$
\xi_\nu(\mathbf{R}) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \tilde{\xi}(r)[D_\nu(r) + 2\beta]^{-1/2} \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} dz \tilde{\xi}(r)/[D_\nu(r)]^{-1/2} \sim R^{1-\gamma-m/2}.
$$  \hspace{1cm} (D9)

The latter expansion, valid when $R$ is still large enough for the turbulent dispersion to exceed the thermal one $D(R) > \beta$, is the one of the thin slice regime. Importantly, the slope is modified by the term $m$, reflecting the statistical properties of the turbulent velocity.

Asymptotic analysis of the pure velocity contribution

$$
\xi_\nu(\mathbf{R}) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} dz [D_\nu(r) + 2\beta]^{-1/2} \exp \left\{ -\frac{(f - 1\gamma)^2}{2[D_\nu(r) + 2\beta]} \right\}
$$  \hspace{1cm} (D10)

is more delicate, since the integral $\int_{-\infty}^{\infty} dz [D_\nu(r) + 2\beta]^{-1/2}$ does not converge and the exponential factor cannot be just set to unity in the leading order. The following manipulations lead to the correct result.

Using $D(r) = C(R^2 + z^2)^m/2$, omitting the thermal term for brevity, and introducing a new variable $y = \lambda^{m/2-1/2}/(R^2 + z^2)^{m/4}$, $\lambda = (C^{1/2} f)^2/(2-m)$, we obtain

$$
\xi_\nu(\mathbf{R}) \sim \int_{-\infty}^{\infty} dy \left\{ \frac{R^2 + z^2}{R^2 + z^2(1 - m/2)} \right\} e^{-y^2/2},
$$  \hspace{1cm} (D11)

where we have left $z$ as the implicit function of $y$. Now we integrate by parts:

$$
\xi_\nu(\mathbf{R}) \sim \int_{-\infty}^{\infty} \left[ \text{erf} \left( \frac{y}{\sqrt{2}} \right) \frac{R^2 + z^2}{R^2 + z^2(1 - m/2)} \right]_0^\infty - \int_{-\infty}^{\infty} dy \text{erf} \left( \frac{y}{\sqrt{2}} \right) \frac{d}{dy} \frac{R^2 + z^2}{R^2 + z^2(1 - m/2)}. \hspace{1cm} (D12)
$$

The first term is finite, while in the second one we replace the integration variable back to $\bar{z} = z/R$ and expand the error function in power series in $y = (R/\lambda)^{1-m/2}/(1 + \bar{z}^2)^{m/4}$, keeping the dominant first linear term. The remaining integral is convergent, and the scale-dependent term in the correlation function can be expressed through gamma and hypergeometric functions

$$
\xi_\nu(\mathbf{R}) \sim \frac{2}{2 - m} - (R/\lambda)^{1-m/2} \Gamma[m/4 - 3/2] \frac{2^{5/2} \Gamma[m/4]}{2^{5/2} \Gamma[m/4]} \frac{1}{\frac{1}{2} \frac{5}{2} 1 - m/2}. \hspace{1cm} (D13)
$$
Actually, only the \( m/(2-m) \) part of the second moment \( \xi_2(0) = 2/(2-m) = 1 + m/(2-m) \) describes the variance of the fluctuations of intensity, while the extra unity (in our normalization) comes from a nonzero mean value of \( I(X) \). The structure function, insensitive to the mean value of the field, is

\[
D_s(R) \sim \langle R/\lambda \rangle^{1-m/2} \frac{1}{2} \Gamma[m/4 - 3/2] \frac{1}{2} \Gamma[m/4] \frac{2}{2} F_1 \left( \frac{3}{2}, 2, \frac{5}{2} - \frac{m}{4}, 1 - \frac{m}{2} \right).
\]

Table 4 summarizes these small-\( R \) asymptotics of correlation functions and the corresponding behavior of the power spectra.

The correlation function formalism, developed in this appendix, also makes self-evident the distinction between thin and thick channels, introduced in LP00. Direct inspection of equation (D6) shows that one can recover thin slice behavior if the width of the channel window \( \delta V \) combined with thermal broadening does not exceed the turbulent velocity amplitude at the scale of consideration \( \delta V^2 + 2\beta \ll D(R) \), while if the opposite is true, the slice is thick. Naturally, this criterion coincides with that derived in LP00.

### APPENDIX E

GAUSSIAN FLUCTUATIONS: GENERAL CASE

Whereas in § 3 we attempted to be as general as possible while dealing with correlations at small angular separations, here we deal with arbitrary angular separations. The price that we pay for that is that we have to make an assumption about the statistics from the very beginning. Namely, we assume that fluctuations are Gaussian with the probability function:

\[
P = \frac{1}{2\pi C^{1/2}} \exp \left[ -\frac{1}{2C} (\delta \rho_1^2 A - 2\delta \rho_1 \delta \rho_2 \xi + \delta \rho_2^2 A) \right],
\]

where \( A \) is the dispersion and the information about the field correlation is contained in \( \xi \).

The exponent in equation (26) can be rewritten in terms of fluctuations \( \exp (\alpha \rho_1) \equiv \exp (\alpha \bar{\rho}_1) \exp (\alpha \delta \rho_1) \). Then equation (32) can be rewritten:

\[
D(R) = \frac{e^2}{\alpha^2} \int dv_1 W(v_1) \int dv_2 W(v_2)e^{-\alpha(\delta \rho_1 + \delta \rho_2)} \left( e^{-\alpha(\delta \rho_1 + \delta \rho_2)} - e^{-\alpha(\delta \rho_1 + \delta \rho_2)} + e^{-\alpha(\delta \rho_2 + \delta \rho_1)} - e^{-\alpha(\delta \rho_2 + \delta \rho_1)} - e^{-\alpha(\delta \rho_1 + \delta \rho_2)} + e^{-\alpha(\delta \rho_2 + \delta \rho_1)} \right).
\]

To evaluate the averages in equation (29) with probability distribution given by equation (E1), we observe\(^{19}\) that

\[
\langle \exp[-\alpha(\delta \rho_i + \delta \rho_j)] \rangle = \exp\left[ \frac{\alpha^2}{2} \right] \left( \delta \rho_i + \delta \rho_j \right)^2, \tag{E3}
\]

where quantities \( \delta \rho_i \) and \( \delta \rho_j \) are measured at points \( X_i, v_i \) and \( X_j, v_j \) of the data cube.

Assuming homogeneity in the velocity distribution, we get correlations that depend only on the velocity difference\(^{20}\) \( v \) between the points and the separation between the points:

\[
D(R) \propto \frac{e^2}{\alpha^2} \int dv W_2(v) \left\{ \exp[\alpha^2 \xi_2(0, v)] - \exp[\alpha^2 \xi_2(R, v)] \right\}. \tag{E4}
\]

The expansion of the exponent for small separations \( R \) results in equation (37), taking into account that \( 2[\xi_2(0, v) - \xi_2(R, v)] = d_s(R, v) - d_s(0, v) \).

\( \quad \text{\cite{19}} \) The procedure of averaging with distribution given by eq. (E1) is straightforward (see LP00) and amounts to taking Gaussian integrals over \( v_1 \) and \( v_2 \).

\( \quad \text{\cite{20}} \) A complication arises from the fact that the emission line is finite in the velocity space and therefore the velocity distribution, in general, is not homogeneous. The finite velocity extent of the emission line guarantees that the contribution coming from \( v \), which is much larger than the velocity dispersion, tends to zero. An analogous effect arising from finite boundaries of an emitting region was treated in LP00, where it was shown that for sufficiently small scales under study the distortions introduced by boundaries are marginal.
