Function Approximation Based Control for Non-Square Systems

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Abstract: A generic control method is proposed for the non-square systems where the number of system inputs is not equal to that of the states. The non-square system to be controlled is first restructured in form of the combination of a square system and the variation from the original non-square system. This variation term is treated as a time-varying uncertainty to the restructured square system. Thus the stabilization for a non-square system is reformulated as an adaptive control problem for a square system. In this paper we address this adaptive control problem by applying the function approximation technique. Specifically, we can parameterize the variation with a chosen basis function weighted by unknown constant parameters. Then we define an update law such that the parameters of the weighted basis function can be automatically determined and the variation between the auxiliary square system and the original non-square system can then be eliminated. The asymptotic stability is established for the closed loop system formulated by the non-square system and the constructed controller. The feasibility of the proposed control method is verified under simulations for linear system, nonlinear underactuated system, and nonholonomic system.

Key Words: non-square system, underactuated system, nonholonomic system.

1. Introduction

The need for control of mechanical systems arises in many practical applications that include the use of ships, underwater vehicles, helicopters, aircraft, satellites, mobile robots, space platforms, flexible joint robots, hyper-redundant and snake-like manipulator, walking robots, capsule robots, and hybrid machines [1]–[4], and control methods have been proposed [1]–[16]. The proposed methods have certain requirements on some special features of mechanical systems and often are not feasible for those systems that fail to satisfy the requirements. Instead of classifying the systems by their features and developing corresponding techniques to tackle the control problem, we intend to propose a generic method that works on a wide class of mechanical systems. This constitutes the main goal of this research.

For control systems with equal number of inputs and states, the control matrix that weights the system inputs, is square such that a state feedback law is easily defined. However in most cases, the control matrix for mechanical systems represented in the state space form, is not square due to that the number of inputs and of states are not always matched. These systems refer to the non-square systems. In this paper, we place emphasis on a class of non-square systems where number of inputs is less than that of states.

The control problem for non-square systems is difficult because the control matrix is not invertible. The most straightforward way is to square the system by applying the pseudo inverse. However, the idea of using the pseudo inverse is to approximate the original non-square system by a square system, and the performance of this approximation requires certain properties of the pseudo inverse. These properties of the pseudo inverse for its application on the control problems of non-square system are studied by [17]. Other studies on the control problem of non-square systems either apply fuzzy control [18] or are only suitable under certain situations [19],[20]. In this paper, we propose an exact control algorithm for systems expressed by non-square state space form.

We construct the algorithm based on the function approximation technique, inspired by its application on the adaptive control [21]–[24]. The main idea of the construction of the control law is as follows. Note that all non-square systems can be expressed by the combination of an approximated square system, referring to the auxiliary system, and the variation from it. This variation term is treated as a time-varying uncertainty to the restructured square system.

Then the construction of the function approximation based controller is formulated by two parts. The first part is to control the auxiliary system, which is trivial. The second part is to eliminate the influence of the time varying uncertainty to the control process. We parameterize the uncertainty term with a set of chosen basis functions weighted by unknown parameters. These parameters are required to be constants for the sake of applying the Lyapunov design. Then define update laws such that the parameters of the weighted basis functions can be automatically determined and the variation between the auxiliary square system and the original non-square system can be eliminated. By combining these two parts, we obtain a feedback controller that makes the non-square system asymptotically stable.

The rest of this paper is organized as follows. First, in Section 2 we state the control problem for the non-square system and propose the control method based on the function approximation technique. Also, in this section, we establish the asymptotic stability for the closed loop system combined by the non-square system and the constructed feedback controller. The feasibility of the proposed control method is verified under simula-
tions in Section 3. Finally, conclusions are drawn in Section 4.

2. Function Approximation Technique Based Feedback Control

A non-square control system can be written in the state-space form as
\[ \dot{x} = f(x) + G(x)u, \] (1)
where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m \) and \( m > n \). By selecting the state \( x \) to be the system output, the control problem addressed in this paper can be stated as constructing an input \( u \) such that \( \lim x = 0 \).

For a square system where \( G \) is an invertible matrix, a feedback law can be easily formulated as
\[ u = -G^{-1}(f + Kx), \] (2)
where \( K \) is a positive definite matrix. However for a non-square system where \( G \) is an \( m \times n \) matrix, \( G^{-1} \) does not exist. Hence the control law (2) is not applicable.

2.1 Design of Controller

To square the control system, one introduces the auxiliary input \( u^* \in \mathbb{R}^m \) where
\[ u = G^*u^*, \] (3)
and the auxiliary matrix \( G^* \) is chosen to be a full rank \( n \times m \) matrix, the weighted pseudoinverse of matrix \( G \), that is
\[ G^* = (G^TWG)^{-1}G^*W, \] (4)
where \( W \) is a constant positive definite symmetric matrix. Then the non-square system (1) is rewritten as
\[ \dot{x} = f + GG^*u^*, \] (5)
which is a square system. However as matrix \( GG^* \) is singular and not invertible, a feedback control law for the auxiliary input \( u^* \) in form of (2) is not easily defined. For the control purpose, we restructure the system (5) as
\[ \dot{x} = f + GG^*u^* + u^* - u^* = u^* + d, \] (6)
where
\[ d(x, u^*, t) = f + (GG^* - I)u^*. \] (7)
Note that through the above rearrangement, the original non-square system is restructured as the combination of two parts, a square system
\[ \dot{x} = u^*, \] (8)
referring to the auxiliary system, and \( d \), which can be viewed as an uncertainty term. Thus the control problem for the non-square system is reformulated to the adaptive control problem for a square system with time-varying uncertainties, which is stated as constructing \( u^* \) such that \( \lim x = 0 \), with \( d \) unknown.

Note that compared with the adaptive control problem for a system with parametric uncertainty, it is challenging for a system with time-varying uncertainty where traditional techniques such as the model reference adaptive control (MRAC) are not feasible. To tackle this problem, we adopt the function approximation technique (FAT) based adaptive control [21]–[24]. Compared with the traditional MRAC, the advantage of the FAT-based control is in the representation of the time-varying uncertainties by a set of given basis functions weighted by a set of unknown constant parameters. Thus the problem of eliminating the influence of the uncertainty terms is transformed to the estimation of parametric errors. Then Lyapunov designs are applied to derive proper update laws adjusting the estimates of the unknown parameters. In what follows, the construction of the controller based on the FAT approach is shown in details.

To control (6), the effect of the uncertainty term \( d \) to the system needs to be eliminated. Note that the uncertainty term varies with respect to time \( t \) and the state \( x \) (the input \( u \) can also be expressed by \( t \) and \( x \)). However, at any moment (i.e., \( t = 1, t = 2, \ldots \)) \( d \) is a constant. We use weighted basis functions to approximate \( d \) at each moment as
\[ d(x, t, u^*) = \sum_{i=0}^{N} d_i \psi_i(x, t) + \epsilon, \] (9)
where \( d_i \), referring to the plant parameter, is constant, \( \psi_i \), referring to the basis function, consists of \( x \) and \( t \), and \( \epsilon \), referring to the approximation error, describes the deviation between the uncertainty \( d \) and the weighted basis functions. Hence by substituting (9) into the state equation (6) yields
\[ \dot{x} = u^* + \sum_{i=0}^{N} d_i \psi_i(x, t) + \epsilon \] (10)
at each moment.

The control of (10) requires the unknown plant parameters \( d_i \) to be identified. To reach this goal, these plant parameters \( d_i \) at each time \( t \) are estimated by \( \hat{d}_i(t) \), referring to the control parameter. The change of \( \hat{d}_i(t) \), with respect to time, is governed by an update law that we are to define in the construction of a feedback controller \( u^* \) from the following process.

To construct \( u^* \) that steers \( x \) to zero and an update law that defines \( \hat{d}_i(t) \), a feasible Lyapunov candidate function would be
\[ V = \frac{1}{2} x^\top x + \frac{1}{2} \sum_{i=0}^{N} (\hat{d}_i(t) - d_i)^\top (\hat{d}_i(t) - d_i), \] (11)
which combines both the state \( x \) and the estimation error \( \hat{d}_i(t) - d_i \) between the plant parameters \( d_i \) and their estimates \( \hat{d}_i(t) \). Note that in (11), the parameters \( d_i \) are constants, which implies \( \hat{d}_i = 0 \). Therefore the derivative of the Lyapunov function candidate is calculated as
\[ \dot{V} = x^\top \dot{x} + \sum_{i=0}^{N} (\hat{d}_i(t) - d_i)^\top \dot{\hat{d}}_i(t). \] (12)
To further simplifying the above equation, substituting the vector \( x \) expressed by (10) into (12) gives
\[ \dot{V} = x^\top (u^* + \sum_{i=0}^{N} d_i \psi_i(x, t) + \epsilon) \]

\[ + \sum_{i=0}^{N} (\hat{d}_i(t)^\top \hat{d}_i(t) - d_i^\top \hat{d}_i(t)) \]

\[ = x^\top (u^* + \epsilon) \]

\[ + \sum_{i=0}^{N} d_i^\top \hat{d}_i(t) + \sum_{i=0}^{N} d_i^\top (x\psi_i(x, t) - \hat{d}_i(t)). \]
As the control law to be constructed cannot contain unmeasur-
able elements, the unknown parameters $d_i$ in the derivative
of the Lyapunov function need to be excluded. To cancel the terms
with $d_i$, define the update law
\begin{equation}
\dot{d}_i(t) = x \psi_i(x, t), \tag{13}
\end{equation}
which leads to
\begin{equation}
\dot{V} = -x^T (u^* + \sum_{i=0}^{N} \dot{d}_i(t) \psi_i(x, t) + \epsilon).
\end{equation}
The approximation error $\epsilon$ needs to be considered in the con-
struction of the auxiliary input $u^*$. Select $u^*$ as
\begin{equation}
u^* = u^*_1 + u^*_2,
\end{equation}
where $u^*_1$ is to cover the effect of $\epsilon$. Thus the derivative of the
Lyapunov candidate function $V$ becomes
\begin{equation}
\dot{V} = -x^T (u^*_1 + \sum_{i=0}^{N} \dot{d}_i(t) \psi_i(x, t) + \epsilon).
\end{equation}
Constructing
\begin{equation}
u^*_1 = -Kx - \sum_{i=0}^{N} \dot{d}_i(t) \psi_i(x, t), \tag{16}
\end{equation}
where $K$ is a positive matrix, yields
\begin{equation}
\dot{V} = -x^T K x - x^T (u^*_1 + \epsilon).
\end{equation}
Then one designs a robust control law for $u^*_1$ to cover the effect
of $\epsilon$. Denote the components of $\epsilon \in \mathbb{R}^m$ to be $\epsilon_j$, where $j = 1, 2, ..., m$. Suppose $\epsilon_j$ is bounded and its variation bound is
available, i.e., there exists $\delta_j > 0$ such that $\| \epsilon_j \| \leq \delta_j$. Then selecting $u^*_1$ the same as (23) and $u^*_2$ as
\begin{equation}
u^*_2 = -\delta_j \text{sgn}(x_j) \tag{18}
\end{equation}
yields
\begin{equation}
\dot{V} = -x^T K x + \sum_{j=0}^{m} \left( x_j \epsilon_j - \delta_j \| x_j \| \right).
\end{equation}
Note that
\begin{equation}
\sum_{j=0}^{m} \left( x_j \epsilon_j - \delta_j \| x_j \| \right) \leq 0 \tag{20}
\end{equation}
holds true when $\epsilon_j$ is bounded by $\delta_j$. Therefore, $\dot{V} \leq 0$ is guar-
anteed under the control law
\begin{equation}
u = G^* (u^*_1 + u^*_2). \tag{21}
\end{equation}
The convergence of the state $x$ is achieved while the effect of the
approximation error $\epsilon$ is covered. Thus, (21) together with the
update law (13) formulates the FAT based controller for non-
square systems.

Note that if the number of the basis functions $\psi_i$ is chosen to be sufficiently large such that $\epsilon = 0$, the designed controller
could be simplified while guaranteeing the convergence of the state $x$. Specifically, by selecting $u^*_1 = 0$, the derivative of the
Lyapunov function candidate (21) becomes
\begin{equation}
\dot{V} = -x^T (u^*_1 + \sum_{i=0}^{N} \dot{d}_i(t) \psi_i(x, t)). \tag{22}
\end{equation}
Constructing the auxiliary input
\begin{equation}
u^* = u^*_1 = -Kx - \sum_{i=0}^{N} \dot{d}_i(t) \psi_i(x, t), \tag{23}
\end{equation}
where $K$ is a positive matrix, yields
\begin{equation}
\dot{V} = -x^T K x \leq 0, \tag{24}
\end{equation}
which implies the convergence of $x$. Hence the system input $u$
can be simplified from (21) as
\begin{equation}
u = -G^* (K x + \sum_{i=0}^{N} \dot{d}_i(t) \psi_i(x, t)) \tag{25}
\end{equation}
with the update law defined by (13).

2.2 Proof of Stability
To prove the asymptotic stability of the closed-loop system
formulated by (1), (13), and (25) around the equilibrium point,
we follow the Lyapunov-like analysis [25] based on the follow-

Lemma 1 If a scalar function $V$ satisfies the following conditions
\begin{enumerate}
\item $V$ is lower bounded
\item $V$ is negative-semidefinite
\item $\dot{V}$ is uniformly continuous in time
\end{enumerate}
then $\dot{V} \to 0$ as $t \to \infty$.

The Lyapunov function $V$ expressed by (11) satisfies the first
and second conditions that $V$ is lower bounded and $V$ is negative
semi-definite. To follow the Lyapunov-like analysis, we need to
show the uniform continuity of $\dot{V}$. This can be done by proving
that $\dot{V}$ is bounded, where
\begin{equation}
\dot{V} = -2x^T K \dot{x}. \tag{26}
\end{equation}
As $\dot{V} \leq 0$, $V$ is bounded and thus so is its component $x$. Ac-
cording to (24), we see that $\dot{V}$ is only composed of variables $x$,
$V$ is bounded, and, thus, so is its component $x$. Therefore $\dot{V}$
is bounded since it only contains the elements $x$ and $\dot{x}$ which
are shown above to be bounded. Hence, $V$ is uniformly con-

2.3 Discussion
Note that the constructed control law (25) varies from a clas-
sical feedback that it synthesizes a adaptation mechanism
\begin{equation}
u = -G^* \sum_{i=0}^{N} \dot{d}_i(t) \psi_i(x, t), \tag{27}
\end{equation}
which deals with the variation $d$ between the non-square system
(1) and the auxiliary square system (8). However the influence of
the adaption mechanism $u_a$ to the control process is vanishing
when the coefficients in the gain matrix $K$ are chosen large
and the controller is approaching a pure state feedback
\[ u = -G^*Kx. \]  

(28)

This feature is significant when applying this function approximation based algorithm on the nonholonomic systems since according to well-known Brockett’s condition [26], there exists no state feedback for nonholonomic systems that achieves asymptotic stability. Therefore for the control of this specific type of systems, the following points need to be aware of.

First the gain coefficients \( k_1, k_2, \ldots, k_m \) corresponding to the state variables \( x_1, x_2, \ldots, x_n \) are required to be small to weaken the influence of non-adaptive portion (28) in the controller. However, the small value of feedback coefficients leads to the low converging speed for the system output. This problem can be solved by assigning positive functions to the feedback coefficients \( k_1, k_2, \ldots, k_m \), which start from small values but keep growing with respect to time, instead of using positive constants.

Second, the initial value of the control parameters \( \hat{d}_i(t) \) cannot be chosen zero. If so, the controller (25) yields to a state feedback in from of (28) at the instant \( t = 0 \) and the nonholonomic system is not controllable via (28). The closer the initial values of the state variables to the equilibrium are given, the smaller value of feedback coefficient \( k_i \) will be required to assign to the low converging speed for the system output. This problem can be solved by assigning positive functions to the feedback coefficients \( k_1, k_2, \ldots, k_m \), which start from small values but keep growing with respect to time, instead of using positive constants.

Note that a linear system is a special case of nonlinear system

\[ f(x) = Ax, \quad G(x) = B. \]  

(35)

Then to apply the constructed control law (31), we need to select a suitable matrix \( B^* \). As \( B^* \) cannot contain zero columns, it is chosen to be a weighted pseudoinverse \((B^*WB)^{-1}B^*W\) of matrix \( B \), calculated as

\[ B^* = \begin{bmatrix} 1 & 1 \end{bmatrix}. \]

To know whether the selected \( B^* \) is proper, let us check the controllability for the reformulated system

\[ \dot{x} = Ax + B^*u^*, \quad B = BB^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \]

(36)

after introducing the auxiliary input \( u = B^*u^* \). For the new system (36) with auxiliary input \( u^* \), the controllability is easily proved. As system (36) is controllable, the controller (31) is applicable and simulation is conducted. In the simulation, the initial values of the control parameters \( \hat{d}_i(t) \) are chosen to be zero. The gain matrix is selected positive definite, as

\[ K = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}. \]

and the initial conditions of the system output \( x_1 \) and \( x_2 \) are specified as 2 m and 1 m/s.

Under the constructed control law, the system responses \( x_1 \) and \( x_2 \) converge to desired values, as illustrated in Fig. 1, where the left and right graphs stand for the trajectories for \( x_1 \) and \( x_2 \) respectively. The input signal is shown by Fig. 2.

\[ \ddot{x} = u, \]  

(32)

construct \( u \) such that \( x \) converges to zero. Select the state vector to be

\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \]  

(33)

where \( x_1 = x, x_2 = \dot{x}, \) and rewrite the system equations (32) in the state space form as

\[ \dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \]  

(34)

The control problem is then redescribed as constructing an input \( u \) such that \( \lim_{t \to \infty} x = 0 \).

In all the cases, we choose the second order polynomial function to estimate the system variation \( d \), namely, \( N = 2 \).

3. Case Study

In this section, we test the control law for three cases, a linear system (double integrator system), a nonlinear underactuated system (cart-pole system), and a nonholonomic system (unicycle system). We choose a set of polynomial basis functions for estimating the variation \( d \) expressed by (7), that is,

\[ \psi_i(x, t) = t^i. \]  

(29)

By assuming \( i \) is large enough so that \( \epsilon \) is ignorable, one can estimate the variation \( d \) as

\[ d \approx \sum_{i=0}^{N} d_i(t) t^i. \]  

(30)

Note that the estimation of \( d \), formulated by (30), may diverge because the polynomials are proportional to time \( t \). For finite \( t \) however, this does not represent a problem — the output signal will still converge as shown in Section 2.2, even if the estimation (30) does not approach to \( d \).

The feedback controller and the update law corresponding to (30) are defined as

\[ u = -G^*(Kx + \sum_{i=0}^{N} \hat{d}_i(t) t^i), \quad \dot{\hat{d}}_i(t) = x^i. \]  

(31)

3.1 Linear System

Let us start from a linear example, the double integrator system. The control problem is stated as given a double integrator system
3.2 Nonlinear System

Then let us test the controller for a nonlinear example, the well-known cart-pole system. To describe the system under consideration, we introduce the following coordinate frames (see Fig. 3). The base coordinates $\Sigma$ and pendulum coordinates $\Sigma_p$ are fixed on the ground and on the pendulum respectively. The rotation angle between $\Sigma_p$ and $\Sigma$ is denoted by the rotation angle of the pendulum $\theta$, increasing in the counterclockwise direction. The displacement of the cart is denoted by $s$. The length from the pin joint to the gravitational center of the pendulum with respect to its center of mass is $l$. The masses of the cart the pendulum are represented by $m$ and $m_p$, and $J_p$ stands for the inertia moment of the pendulum with respect to its center of mass.

Based on the above settings, the dynamic model, which is derived from the Lagrange equation, is expressed as follows [27]:

$$
\begin{bmatrix}
H_{cc} & H_{cp} \\
H_{pc} & H_{pp}
\end{bmatrix}
\begin{bmatrix}
\ddot{s} \\
\dot{\theta}
\end{bmatrix}
+
\begin{bmatrix}
h_c \\
h_p
\end{bmatrix}
= 1u,
$$

(37)

where

$$
H_{cc} = m + m_p, \quad H_{cp} = -m_p l \cos \theta, \quad h_c = m_p l \sin \theta \dot{\theta}^2,
$$

$$
H_{pc} = -m_p l \cos \theta, \quad H_{pp} = J_p, \quad h_p = m_p g l \sin \theta,
$$

and $J_p = J_0 + m_p l^2$ represents for the moment inertia of the pendulum with respect to the pin point. The dynamic equations govern the relation between the input torque and corresponding motion of the cart-pole system. The constructed control law (31) is tested under the above model. To apply the control algorithm, we choose the state vector and the system input to be

$$
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
s \\
\dot{s} \\
\theta \\
\dot{\theta}
\end{bmatrix},
\quad
u = f,
$$

(38)

and rewrite the dynamic equations of the cart-pole system (37) in the state space form as

$$
\dot{x} = f(x) + G(x)u
= \begin{bmatrix}
x_2 \\
\frac{H_{pc} h_c - H_{cp} h_p}{H_{cc} H_{pp} - H_{cp} H_{pc}} x_1 \\
\frac{H_{pc} h_c - H_{cp} h_p}{H_{cc} H_{pp} - H_{cp} H_{pc}} x_2 \\
\frac{H_{cc} h_p - H_{cp} h_p}{H_{cc} H_{pp} - H_{cp} H_{pc}} x_3 \\
\frac{H_{cc} h_p - H_{cp} h_p}{H_{cc} H_{pp} - H_{cp} H_{pc}} x_4
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
u.
$$

The control problem is then defined as constructing an input $u$ such that $\lim_{t \to \infty} x = 0$.

To apply the constructed control law (31), we select the weight matrix $W$ to be a symmetric positive definite matrix as

$$
W = \begin{bmatrix}
2 & 2 & 0 & 0 \\
2 & 5 & 0 & 0 \\
0 & 0 & 15 & 5 \\
0 & 0 & 5 & 2
\end{bmatrix}.
$$

Note that the off-diagonal elements of $W$ affect the converging speed and the pattern of convergence (less oscillatory) of the state vector $x$, while the diagonal elements were adjusted to keep the weight matrix $W$ positive definite. Thus, $G^*$ is calculated as

$$
G^* = \begin{bmatrix}
\frac{2H_{pc} h_c - H_{cp} h_p}{2H_{cc} H_{pp} - H_{cp} H_{pc}} x_1  \\
\frac{2H_{pc} h_c - H_{cp} h_p}{2H_{cc} H_{pp} - H_{cp} H_{pc}} x_2  \\
\frac{2H_{cc} h_p - H_{cp} h_p}{2H_{cc} H_{pp} - H_{cp} H_{pc}} x_3 \\
\frac{2H_{cc} h_p - H_{cp} h_p}{2H_{cc} H_{pp} - H_{cp} H_{pc}} x_4
\end{bmatrix}.
$$

To know whether the selected $G^*$ is proper, the controllability for the reformulated system

$$
\dot{x} = f + G^* u^* = f + g_1 u_1 + g_2 u_2 + g_3 u_3 + g_4 u_4
$$

(39)

after introducing the auxiliary input $u = G^* u^*$ is checked in Appendix A.

As (39) is locally controllable, the control law (31) is applicable and simulation is conducted. In the simulation, the gain matrix is selected as

$$
K = \begin{bmatrix}
5 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 \\
0 & 0 & 5 & 0 \\
0 & 0 & 0 & 5
\end{bmatrix},
$$

and the initial conditions of the system output $x_1, x_2, x_3,$ and $x_4$ are specified as $2, 0, 0, 0$. The numerical values used in the simulation are $m = 1 \text{ kg}$, $m_p = 1 \text{ kg}$, $l = 0.1 \text{ m}$, $g = 9.8 \text{ m/s}^2$, and $T = 15 \text{ s}$. The initial state is $(2, 0, 0, 0)$ and the initial values of the control parameters $\delta(t)$ are chosen to be zero.

Under the constructed controller, the system outputs $x_1, x_2, x_3$, and $x_4$ converge to zero, as illustrated in Fig. 4. The solid curves stand for the trajectories of the displacement $x_3$ and velocity $x_4$ of the cart, whereas the dashed curves for the angular displacement $x_2$ and angular velocity $x_4$ of the pendulum. The input signal is shown by Fig. 5.

3.3 Nonholonomic System

In this section, let us check the validity of the constructed controller (31) applied on a nonholonomic system, the unicycle.
system. A unicycle is a vehicle with a single orientable wheel. Its configuration is described by

\[
q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix},
\]

(40)

where \(x\) and \(y\) are the Cartesian coordinates of the contact point of the wheel with the ground, and the angle \(\theta\) is the orientation of the wheel with respect to the \(x\)-axis (see Fig. 6). The pure rolling constraint for the wheel, which is nonholonomic, is expressed as

\[
\dot{x} \sin \theta - \dot{y} \cos \theta = 0,
\]

(41)

implying that the velocity of the contact point is zero in the direction orthogonal to the local \(y\)-axis of the vehicle. From the nonholonomic constraint, the kinematic model of the unicycle is then derived as

\[
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & 0 \\ \sin \theta & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix},
\]

(42)

where the inputs \(v\) and \(\omega\) have a clear physical interpretation. In particular, \(v\) is the linear velocity of the vehicle whereas the steering velocity \(\omega\) is the wheel angular speed around the vertical axis. To apply the control algorithm, we choose the state vector and the system input to be

\[
x = q, \quad u = \begin{bmatrix} v \\ \omega \end{bmatrix},
\]

and rewrite the system equations in state space form as

\[
\dot{x} = G(x)u = \begin{bmatrix} \cos \theta & 0 & 0 \\ \sin \theta & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}.
\]

(43)

The control problem is then described as constructing an input \(u\) such that \(\lim_{t \to \infty} x = 0\). For the construction of input \(u\), let us introduce the auxiliary input \(u^*\), where

\[
u = G^*u^*.
\]

(44)

By selecting

\[
W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

one calculates matrix \(G^*\) as

\[
G^* = G^\top = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

and the state equation is rewritten as

\[
\dot{x} = GG^\top u^* = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta & 0 \\ \cos \theta \sin \theta & \sin^2 \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} u^*.
\]

(45)

To test the validity of the chosen \(G^*\) matrix, the local controllability around the equilibrium point \(x = (0, 0, 0)\) for the following system

\[
\dot{x} = g_1u_1 + g_2u_2 + g_3u_3
\]

is checked in Appendix B.

Knowing that the unicycle system is locally controllable, we apply the constructed control law (31), where the gain matrix is given as

\[
K = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & k_\theta \end{bmatrix}.
\]

The system responses \(x_1, x_2,\) and \(x_3\) converge to desired values, as illustrated in Fig. 7, where the solid and dashed curves in the left picture stand for the trajectories displacement of the vehicle \(x_1\) and \(x_2\) respectively, while the solid curve in the right picture for the orientation \(x_3\) of the system. The initial conditions of the system output \(x_1, x_2,\) and \(x_3\) are specified as 5 m, 5 m, and \(\pi\) rad. The initial values of the control parameters \(u(t)\) are chosen to be 1. The input signals are shown by Fig. 8, where the linear and angular velocity inputs are respectively represented by solid and dashed curves.

As introduced in Section 2, there are two essential points to be noted for the control of the nonholonomic unicycle system. First, the gain coefficient \(k_\theta\) corresponding to the orientational angle \(\theta\) of the unicycle, which is the only state variable appears in the control matrix \(G\), is required to be small. However, the small value of \(k_\theta\) leads to the low converging speed for the system output \(\theta\). To address this problem, in the simulation \(k_\theta\) is selected to be \(\frac{1}{20}\), which keeps increasing with respect to time.
Second, the initial value of the control parameters $\dot{d}(0)$ cannot be chosen small. The reason is formulated as follows. Although the restructured system (45) with auxiliary input $u^*$ is proved to be controllable, its linearized system near the equilibrium

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} u^*$$  \hfill (46)

is not controllable. At the equilibrium, the selection of the initial value for the control parameters $\dot{d}(0) = 0$ leads to a condition that the auxiliary input

$$u^* = -Kx$$  \hfill (47)

becomes a state feedback law. According to well-known Brock- eit’s condition [26], for those systems, of which the linearized system is not controllable, there exists no pure state feedback. Therefore, for the nonholonomic systems, the initial value of the control parameters cannot be chosen as zero. Moreover, to enhance the influence of the adaptation term $u_a$ in form of (27) to the control process, the smaller the initial values of the state variables are given, the larger values of $\dot{d}(0)$ are required to assign in order to weight the adaptation mechanism $u_a$.

4. Conclusions

A generic control method has been proposed for the non-square systems where the number of system inputs is not equal to that of the states. By introducing an auxiliary input the dimension of which equals that of the system state, the non-square system was restructured in the form of the combination of a square system and the variation from the original non-square system, which is treated as system uncertainties.

In the control process, the uncertainty term was replaced by its approximation as a chosen basis function weighted by constant parameters to be determined. These unknown plant parameters are estimated at each instant, denoted by the adjustable control parameters using a defined update law. Thus the influence to the control process caused by the variation between the auxiliary square system and the original non-square system can be eliminated.

The asymptotic stability was established for the closed loop system formulated by the non-square system and the constructed controller. The feasibility of the proposed control method was verified under simulations for the point stabilization problem of a linear system, a nonlinear underactuated system, and a nonholonomic system, respectively.

The following issues need to be clarified in the future research. First, the relation between the actual input $u$ and the auxiliary input $u^*$ is expressed by (3). Note that in (3) there exists a matrix $G^*$ which was assigned to be a weighted pseudo-inverse of $G$ in this paper. However, the features of a good candidate for matrix $G^*$ are not specifically analyzed and a more systematic way of selecting $G^*$ needs to be developed.

Second, in this paper, we chose polynomials to be the basis functions for estimating the variation term $d$ of the auxiliary square system from the original non-square system. However, the advantages and disadvantages of polynomials as basis function are not analyzed and more candidate functions need to be investigated in the future study.

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References

[1] R. Lozano and I. Fantoni: Non-Linear Control for Underactuated Mechanical Systems, Springer, 2001.
[2] A. Bloch, D. Chang, N. Leonard, and J. Marsden: Controlled lagrangians and the stabilization of mechanical systems II: Potential shaping, IEEE Transactions on Robotics and Automation, Vol. 46, No. 10, pp. 1556–1571, 2001.
[3] N. Aneke: Control of Under actuated Mechanical Systems, Ph.D. dissertation, TU Eindhoven, Eindhoven, The Netherlands, 2003.
[4] D. Chang: On the method of interconnection and damping assignment and passivity-based control for the stabilization of mechanical systems, Regular and Chaotic Dynamics, Vol. 19, No. 5, pp. 556–575, 2014.
[5] A. Shiriav, H. Ludvigsen, O. Egeland, and A. Pogromsky: On global properties of passivity based control of the inverted pendulum, Proc. 38th IEEE Conference on Decision and Control, Phoenix, Arizona, USA, pp. 2513–2518, December 7–10 1999.
[6] X. Xin and M. Kanaeda: Analysis of the energy-based control for swinging up two pendulums, IEEE Transactions on Automatic Control, Vol. 50, No. 5, pp. 679–684, May 2005.
[7] B. Gao, X. Zhang, H. Chen, and J. Zhao: Energy-based control design of an underactuated 2-dimensional tora system, Proc. IEEE Int. Conference on Intelligent Robots and Systems, St. Louis, USA, pp. 1296–1301, October 10–15 2009.
[8] M. Spong: Partial feedback linearization of underactuated mechanical systems, Proc. IEEE Int. Conference on Intelligent Robots and Systems, Munich, Germany, pp. 314–321, September 12–16 1994.
[9] K. Pettersen and H. Nijmeijer: Tracking control of an underactuated surface vessel, Proc. 37th IEEE Conference on Decision and Control, Tampa, Florida, USA, pp. 4561–4566, December 16–18 1998.
[10] R. Olfati-Saber: Trajectory tracking for a flexible one-link robot using a nonlinear noncollocated output, Proc. 39th IEEE Conference on Decision and Control, Sydney, Australia, pp. 4024–4029, December 12–15 2000.
[11] D. Qian, J. Yi, and D. Zhao: Robust control using sliding mode for a class of underactuated systems with mismatched uncertainties, Proc. IEEE Int. Conference on Robotics and Automation, Rome, Italy, pp. 1449–1454, April 10–14 2007.
[12] R. Xu and U. Özgüner: Sliding mode control of a class of underactuated systems, Automatica, Vol. 44, No. 1, pp. 233–241, January 2008.
[13] V. Sankaranarayanan and A. Mahindrakar: Control of a class of underactuated mechanical systems using sliding modes, IEEE Transactions on Robotics, Vol. 25, No. 2, pp. 459–467, February 2009.
[14] J. Huang, Z. Guan, T. Matsumo, T. Fukuda, and K. Sekiyama: Sliding-mode velocity control of mobile-wheeled inverted-pendulum systems, IEEE Transactions on Robotics, Vol. 26, No. 4, pp. 750–758, August 2010.
[15] S. Nersesov, H. Ashrafuon, and P. Ghorbani: On the stability of sliding mode control for a class of underactuated nonlinear systems, Proc. American Control Conference, Baltimore, MD, USA, pp. 3446–3451, June 30–July 2 2010.
[16] L. McNinch and H. Ashrafuon: Predictive and sliding mode cascade control for unmanned surface vessels, Proc. American Control Conference, San Francisco, CA, USA, pp. 184–189, June 29–July 01 2011.
[17] S. Li and J. Zhang: Some properties of generalized inverse of non-square systems, Proc. 52nd IEEE Conference on Decision and Control 2013, Florence, Italy, pp. 6446–6451, December
Appendix A  Proof of Controllability for the Reformulated Cart-Pole System

To know whether the selected $G$ matrix is proper, let us check the controllability, following to [28], [29], for the reformulated cart-pole system (39) after introducing the auxiliary input $u = G^Tu'$. Noting that the drift term $f$ of the reformulated system (39) becomes zero at the equilibrium point $x = x_o = (0, 0, 0, 0)$, we verify that the system is considered to be locally controllable at $x = x_o$ if

a) it is locally accessible at $x = x_o$,

b) all the so-called bad Lie brackets calculated for (39) can be represented by the linear combinations of the good ones having lesser value of $p$,

d) $L$ is bad if $\delta_0$ is an odd number and $\delta_1$ is an even number.

e) $L$ is good if it is not bad.

Let us first check whether the reformulated cart-pole system (39) is locally accessible at the equilibrium point $x = x_o$. The Lie brackets calculated for the control system (39) are

$$L_1 = [f, f], L_2 = [g_1, g_1], L_3 = [g_2, g_2], L_4 = [g_3, g_3],$$

where

$$L_6 = [f, g_1], L_7 = [f, g_2], L_8 = [f, g_3], L_9 = [g_1, g_3],$$

$$L_{10} = [g_1, g_2], L_{11} = [g_1, g_3], L_{12} = [g_2, g_3], L_{13} = [g_2, g_1],$$

$$L_{16} = [f, f, g_1], L_{19} = [g_1, f, g_1], L_{21} = [f, g_2, g_1],$$

$$L_{20} = [g_2, f, g_1], L_{22} = [g_1, [f, g_1]], L_{23} = [g_1, [f, g_1]],$$

and by picking all the non-zero Lie brackets, we construct the accessibility matrix $A = \begin{bmatrix} L_1 & L_2 & \cdots & L_{22} \end{bmatrix}$. As $\text{Rank}(A) = 4$ at the equilibrium point, the system under consideration is locally accessible at the equilibrium point. The first condition for the local controllability is satisfied. Furthermore, in our case, all the bad Lie brackets can be represented by the linear combinations of the good ones having lesser value of $p$. Thus the system is locally controllable at the equilibrium point.

Appendix B  Proof of Controllability for the Reformulated Unicycle System

To test the validity of the chosen $G$ matrix, let us then check the local controllability around the equilibrium point $x_o = (0, 0, 0, 0)$ for the reformulated unicycle system (45). The Lie brackets calculated for (45) around the equilibrium point $x = x_o$ are

$$L_1 = g_1, L_2 = g_2, L_3 = g_3,$$

$$L_4 = [g_1, g_2], L_5 = [g_1, g_3], L_6 = [g_2, g_1],$$

where

$$L_7 = [g_1, g_2], L_8 = [g_2, g_1], L_9 = [g_3, g_2],$$

$$L_{10} = [g_3, g_1], L_{11} = [g_2, g_3], L_{12} = [g_2, g_3], L_{13} = [g_2, g_1],$$

$$L_{14} = [g_2, g_3], L_{15} = [g_3, g_2], L_{16} = [g_3, g_1],$$

$$L_{17} = [g_3, g_2], L_{18} = [g_1, [f, g_1]], L_{19} = [g_1, [f, g_1]],$$

$$L_{20} = [g_2, [f, g_1]], L_{21} = [g_1, [f, g_1]], L_{22} = [g_2, [f, g_1]].$$

Then, by picking all the non-zero Lie brackets at the equilibrium point, we construct the accessibility matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

As $\text{Rank}(A) = 3$, the reconstructed unicycle system (45) is locally accessible at the equilibrium point. Because the system (45) is driftless, local accessibility implies local controllability.
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