A Numerical Study of Spectral Flows of Hermitian Wilson-Dirac Operator and the Index Theorem in Abelian Gauge Theories on Finite Lattices

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Abstract

We investigate the index of the Neuberger’s Dirac operator in abelian gauge theories on finite lattices by numerically analyzing the spectrum of the hermitian Wilson-Dirac operator for a continuous family of gauge fields connecting different topological sectors. By clarifying the characteristic structure of the spectrum leading to the index theorem we show that the index coincides to the topological charge for a wide class of gauge field configurations. We also argue that the index can be found exactly for some special but nontrivial configurations in two dimensions by directly analyzing the spectrum.

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It is generally believed that the nontrivial topological gauge field configurations are responsible for nonperturbative aspects such as the large $\eta-\eta'$ mass splitting in QCD and the fermion number violation in the standard model. Lattice gauge theories are the most promising approach to such nonperturbative phenomena. However, the configuration space of lattice gauge fields is topologically trivial since any configuration can be continuously deformed into the trivial one. This is not quite disappointing because physically interesting gauge fields are smooth and we can impose a kind of smoothness conditions that restrict plaquette variables within a small neighborhood of unity [1]. The space of smooth link variables acquires a nontrivial topological structure and it is possible to define topological invariants geometrically in terms of lattice gauge fields [1, 2, 3, 4]. The index theorem plays a role of a touchstone in lattice QCD [5, 6, 7, 8].

In the case of Ginsparg-Wilson (GW) Dirac systems [9, 10, 11] it is possible to define exact chiral symmetry [12] and the index theorem on the lattice relating the index of the GW Dirac operator to chiral anomaly is known by Hasenfratz, Laliena and Niedermayer (HLN) [13, 12]. Their index theorem is applicable not only for any gauge field configurations but also for any GW Dirac operators. Since the topological structure of the gauge fields is reflected on the very construction of the GW Dirac operator rather implicitly, the HLN index theorem tells very few about the relationship between the index and the topological invariant of gauge fields for strictly finite lattices. Nevertheless we expect that a lattice extension of the index theorem relating the index directly with the topological invariants of gauge fields can be established for sufficiently smooth configurations and physically acceptable GW Dirac operators [14]. In fact it is possible to relate the axial anomaly (the index density) to the topological charge density for abelian theories on the infinite lattice quite generally by invoking general principles such as the gauge invariance and the locality [15]. On strictly finite lattices, however, we cannot appeal to the locality. This makes the differential geometrical method developed in ref. [15] inapplicable.

In this note we focus our attention to the index of Neuberger’s Dirac operator [11] for compact U(1) theories on finite periodic lattices in two and four dimensions and aim at establishing the index theorem for smooth background gauge fields. To achieve the goal we investigate the spectrum of the related hermitian Wilson-Dirac operators numerically for a family of link variables connecting configurations with distinct topological charges and clarify the characteristic structure of the spectrum leading to the index theorem. Such an analysis has already been reported in ref. [18] within the framework of the overlap formalism. We shall extend the analysis from the point of view of Neuberger’s Dirac operator to convincing extent. We then show some analytic results concerning the spectrum and the index. They are very useful to understand the numerical results.

Let us begin with the definition of Neuberger’s Dirac operator on a $d = 2N$ dimensional euclidean hypercubic regular lattice $L^d$ with periodic boundary conditions. For simplicity we choose the lattice spacing $a = 1$ and take the lattice size $L$ to be a positive integer. We

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† The chiral anomalies have been established in ref. [16] in the continuum limit. See also ref. [17].
first introduce the hermitian Wilson-Dirac operator $H$ by

$$H\psi(x) = \gamma_{d+1}\left\{ (d-m)\psi(x) - \sum_{\mu=1}^{d} \left( \frac{1-\gamma_{\mu}}{2} U_{\mu}(x)\psi(x + \hat{\mu}) + \frac{1+\gamma_{\mu}}{2} U_{\mu}(x - \hat{\mu})\psi(x - \hat{\mu}) \right) \right\} , \quad (1)$$

where $m$ is the fermion mass and the Wilson parameter $r = 1$ is assumed. The $\gamma$-matrices are taken to be hermitian and satisfy $\{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu}$. We use $\gamma_{d+1} = (-i)\gamma_1 \cdots \gamma_d$. The link variables $U_{\mu}(x)$ are subject to the periodic boundary conditions $U_{\mu}(x + L\hat{\nu}) = U_{\mu}(x)$. We also employ the periodic boundary conditions for $\psi(x)^\dagger$. The Neuberger’s Dirac operator $D$ is then given by

$$D = 1 + \gamma_{d+1} \frac{H}{\sqrt{H^2}}. \quad (2)$$

The $m$ is chosen in the range $0 < m < 2$ to avoid species doubling. We shall often use $m = 1$ in the following analysis and denote $H(1)$ simply by $H$. Whenever the $m$ dependence is concerned, we write it explicitly as $H(m)$.

For the Neuberger’s Dirac operator (2) the index theorem of HLN can be stated as

$$\text{index } D = \text{Tr} \gamma_{d+1} \left( 1 - \frac{1}{2} D \right) = -\frac{1}{2} \text{Tr} \frac{H}{\sqrt{H^2}}. \quad (3)$$

This equation relates the index of $D$ to the spectral asymmetry of $H$ since the trace on the rhs is nothing but the number of positive eigenvalues of $H$ minus that of negative ones $\text{Tr}$. It enables us to analyze the index of the complicated operator $D$ in terms of the much simpler operator $H$ that is ultra local on the lattice.

On finite lattices $D$ is well-defined unless $\text{det } H = 0$. By excising gauge field configurations leading to $\text{det } H = 0$ the space of the link variables becomes disconnected. We shall refer to the decomposition of the configuration space as the analytic. The index theorem (3) associates $\text{index } D$ to each connected component of the space of link variables.

The lattice index theorem (3) is valid for any gauge field configuration with $\text{det } H \neq 0$. However, we can restrict ourselves to smooth configurations by imposing

$$\sup_{x,\mu,\nu} ||1 - P_{\mu\nu}(x)|| \leq \eta, \quad (4)$$

where $P_{\mu\nu}$ is the standard plaquette variable and $\eta$ is a positive constant to be mentioned later. The norm $||A||$ of a matrix $A$ is defined by

$$||A|| = \sup_{\varphi, ||\varphi||=1} ||A\varphi||, \quad (5)$$

where $||\varphi||$ is the standard norm for vectors. It is well-known that the space of such smooth configurations still contains any physically interesting gauge field configurations with non-trivial topological structure if the size of the lattice is taken to be sufficiently large $\text{[4, 20]}$.

$^1$We may take anti-periodic boundary condition in the $d$-th lattice coordinate. This does not lead to any significant change in the numerical analysis.
For suitable choices of $\eta$ the space of link variables becomes disconnected and one can define topological invariants geometrically in terms of the link variables. We shall refer to the decomposition of the space of link variables as the geometric. We describe this in detail for compact $U(1)$ theories with which we shall be exclusively concerned.

In the case of compact $U(1)$ theories (4) can be cast into the simpler form:

$$\sup_{x,\mu,\nu} |F_{\mu\nu}(x)| < \eta' \quad \left( \eta' = 2 \sin^{-1} \frac{\eta}{2} \right) \quad (6)$$

We may choose $0 < \eta' < \pi$ for $d = 2$ and $0 < \eta' < \pi/3$ for $d \geq 4 \quad [2, 15, 20, 4]$. The space of link variables is decomposed into a finite number of connected components characterized by a set of magnetic fluxes $\phi_{\mu\nu} = 2\pi m_{\mu\nu}$ through $\mu\nu$-plane defined by [20]

$$\phi_{\mu\nu} = \sum_{s,t=0}^{L-1} F_{\mu\nu}(x + s\hat{\mu} + t\hat{\nu}) \quad (7)$$

where $m_{\mu\nu} = -m_{\nu\mu}$ is an integer and $F_{\mu\nu}$ is the abelian field strength given by

$$F_{\mu\nu}(x) = \frac{1}{i} \ln P_{\mu\nu}(x) \quad (|F_{\mu\nu}(x)| < \pi) \quad (8)$$

Note that the rhs of (6) is independent of $x$ by virtue of the Bianchi identity $\partial_{\lambda} F_{\mu\nu} = 0$. Here $\partial_{\mu}$ is the forward difference operator defined by $\partial_{\mu} f(x) = f(x + \hat{\mu}) - f(x)$. The $\phi_{\mu\nu}$ is not only a smooth function of the link variables but also is a topological invariant. This implies that any two gauge field configurations with distinct sets of magnetic fluxes cannot be deformed into each other continuously and, hence, belong to different connected components. Conversely, any gauge field configurations satisfying (6) can be continuously deformed into one another if they have the same set of magnetic fluxes [20]. This implies that the space of gauge fields with a given set of magnetic fluxes that satisfy (6) are connected. In particular any gauge field with a given set of magnetic fluxes $\phi_{\mu\nu} \equiv 2\pi m_{\mu\nu}$ can be continuously deformed to a configuration with constant field strengths $F_{\mu\nu}(x) = \phi_{\mu\nu}/L^2$. For later convenience we denote the space of link variables that satisfy (6) and give a set of magnetic fluxes $\phi_{\mu\nu}$ by $U_{\eta,\phi}$.

We can define a topological charge by the lattice analog of Chern character [15, 4] as

$$Q_N = \frac{1}{(4\pi)^N N!} \sum_x \epsilon_{\mu_1\nu_1\cdots\mu_N\nu_N} F_{\mu_1\nu_1}(x) F_{\mu_2\nu_2}(x + \hat{\mu}_1 + \hat{\nu}_1) \times \cdots \times F_{\mu_N\nu_N}(x + \hat{\mu}_1 + \hat{\nu}_1 + \cdots + \hat{\mu}_{N-1} + \hat{\nu}_{N-1}) \quad (9)$$

where $\epsilon_{\mu_1\cdots\mu_d}$ is the Levi-Civita symbol in $d$ dimensions. $Q_N$ is a smooth function of the link variables within a connected component and takes an integer value given by [2, 4]

$$Q_N = \frac{1}{2^N N!} \epsilon_{\mu_1\nu_1\cdots\mu_N\nu_N} m_{\mu_1\nu_1} m_{\mu_2\nu_2} \cdots m_{\mu_N\nu_N} \quad (10)$$
A bound on $|Q_N|$ can be easily found from (9) by noting (6) as

$$|Q_N| < L^{2N} \frac{\eta' N (2N - 1)!!}{(2\pi)^N}.$$  \hspace{1cm} (11)

We thus find two different definitions for decomposition of the space of link variables, the analytic and the geometric. The analytic decomposition is based on the requirement $\det H \neq 0$ and index $D$ is associated to each connected component. The geometric decomposition is based on the condition (6) and $Q_N$ can be assigned similarly to each connected component. In general they are not mutually consistent and one can find gauge field configurations that satisfy (6) and $\det H = 0$ simultaneously. However, if we choose $\eta$ within the range

$$0 < \eta < \frac{2 - \sqrt{2}}{d(d - 1)},$$

(12)

$H(= H(1))$ cannot have zero-modes [21, 14] and any connected component of the space of link variables satisfying (8) is contained in some connected component obtained by the analytic decomposition. Therefore it is very natural to expect that the index of $D$ given by (8) and the topological charge (8) coincide with each other. The precise form of the index theorem for abelian gauge theories can be stated as:

For the link variables satisfying (8) and (12) the index of Neuberger’s Dirac operator $D$ and the topological charge $Q_N$ are related by

$$\text{index} D = (-1)^N Q_N.$$  \hspace{1cm} (13)

To establish (13) on an arbitrary connected component $U_{\eta,\phi}$ it suffices to show the equality for a particular gauge field configuration with constant field strengths $F_{\mu\nu}(x) = \phi_{\mu\nu}/L^2$.

One way to find $\text{index} D$ for an arbitrary configuration is to count the net number of spectral flows of $H(m)$ as $m$ varies from 0 to 1 [13, 24]. Since $H(m)$ is known to have a symmetric spectrum for $m \leq 0$ [13], the net number of eigenvalues that changes the sign from plus to minus just coincides to the index of $D$ at $m = 1$ as can be easily seen from (8).

In what follows we shall take another path since we intend to understand the behaviors of $\text{index} D$ under continuous changes of link variables that include topological jumps. We consider a one-parameter family of link variables $\{U^{(t)}_{\mu}\}_{0 \leq t \leq T}$ connecting two gauge field configurations $\{U^{(0)}_{\mu}\}$ and $\{U^{(T)}_{\mu}\}$, where both topological charge and index are known for $\{U^{(0)}_{\mu}\}$ but only topological charge is known for $\{U^{(T)}_{\mu}\}$. If $\{U^{(0)}_{\mu}\}$ and $\{U^{(T)}_{\mu}\}$ belong to two distinct connected components, $U^{(t)}_{\mu}$ goes through configurations violating $\det H \neq 0$ or (8). We expect jumps in $\text{index} D$ and $Q_N$ at some values of $t$. In the actual analysis we shall choose $U^{(0)}_{\mu} = 1$. The index $D$ for $\{U^{(T)}_{\mu}\}$ can then be found by counting the net number of the eigenvalues of $H$ that change the sign from plus to minus as $t$ varies from 0 to $T$.

In two dimensions we consider the one-parameter family of link variables given by

$$U^{(t)}_1(x) = \exp \left[ -it \frac{2\pi}{L} x_2 \delta_{x_1 L - 1} \right], \quad U^{(t)}_2(x) = \exp \left[ it \frac{2\pi}{L^2} x_1 \right]. \quad (0 \leq x_{\mu} \leq L - 1)$$  \hspace{1cm} (14)
The field strength is then computed as

\[ F^{(i)}_{12}(x) = t \frac{2\pi}{L^2} - 2\pi(t-n)\delta_{x_1,L-1}\delta_{x_2,L-1} \quad \text{for} \quad \frac{(2n+1)L^2}{2(L^2-1)} < t < \frac{(2n-1)L^2}{2(L^2-1)}, \quad (15) \]

where \( n \) is an integer with \(|n| \leq L^2/2 - 1\) and is chosen to satisfy \(|F^{(i)}_{12}(L-1,L-1)| < \pi\).

The topological charge \( Q_1 \) is nothing but \( m_{12} \) and is given by \( Q_1 = n \). The link variable \( \{L\} \) connects uniform field strength configurations \( F^{(i)}_{12}(x) = 2\pi n/L^2 \) of topological charge \( Q_1 = n \) \((n = \pm 1, \pm 2, \cdots)\) continuously to the trivial one \( F^{(i)}_{12}(x) = 0 \) for which \( H \) has a symmetric spectrum. As a function of \( t \), \( Q_1 \) has a discontinuity at \( t = t^{\text{top}}_j \equiv \frac{(2j+1)L^2}{2(L^2-1)} \) \((j = 0, \pm 1, \pm 2, \cdots)\).

Before going into detail about the numerical results, it is helpful to note the following facts: (1) The eigenvalues \( \lambda \) of \( H(m) \) are bounded by \(|\lambda| \leq d + |d - m| [21]\). (2) The eigenvalue spectrum of \( H(m) \) is \( L^2 \) periodic in \( t \). (3) The eigenvalue spectrum of \( H(m) \) is symmetric with respect to the point \( t = 0 \), i.e., \(-\lambda \) is an eigenvalue of \( H(m)|_{t \to -t} \) if \( \lambda \) is an eigenvalue of \( H(m) \). From (2) and (3) we see that index \( D \) is an \( L^2 \) periodic odd function of \( t \) and vanishes at \( t = 0, \pm L^2/2 \), where the spectrum is symmetric.

We have analyzed the eigenvalue spectrum of \( H \) over the range \(-L^2/2 \leq t \leq L^2/2\) for \( 2 \leq L \leq 15\) by searching for the values of \( \lambda \) such that \( \text{Re}[\det(H-\lambda)]\text{Re}[\det(H-\lambda-\Delta\lambda)] < 0 \) for small enough \( \Delta\lambda \). The eigenvalues that are degenerate or approximately degenerate with an even multiplicity cannot be detected in this method. This, however, causes no problem since for most values of \( t \) the eigenvalues are not degenerate.

In Figure \( \text{II} \) the whole spectrum of \( H \) is shown for \( L = 6 \). The characteristic structure of the spectrum is not changed with the lattice size \( L \). It can be summarized as follows:

- Most of the eigenvalues are contained in the upper and the lower trapezoid regions symmetrically separated by the parallelogram region.
- Every time \( t \) increases by unity from an integer an eigenvalue belonging to the lower trapezoid crosses the parallelogram upward and moves to the upper trapezoid. In particular \( H \) has zero-modes for a sequence of values of \( t \) on the interval of the horizontal axis cut by the parallelogram. They are almost equally spaced.
- For integer values of \( t \) a kind of clustering of the spectrum occurs on the trapezoids and there appear large gaps on the parallelogram. In particular, there are \( 2sL \) eigenvalues with degeneracy \( r \) at \( t = r \) if \( L \) is a product of two positive integers \( r \) and \( s \).

Let \( \kappa(L) \) be the minimum value of \( t \) \((> 0)\) where an eigenvalue changes the sign from plus to minus, then \( \kappa(L) \) is larger than \( L^2/4 \). This can be qualitatively understood by noting that

\footnote{In the actual computation the approximated values of \( \det(H-\lambda) \) have imaginary parts. However, the ratios of the imaginary part to the real part are very small and the real part is considered to well approximate the determinant.}
the smoothed boundary of the parallelogram is convex and goes through the points $(0, 1)$ and $(L^2/2, -1)$. As we shall mention, it is possible to show analytically that for even $L$ the eigenvalues of $H$ at $t = L^2/2$ satisfy

$$\sqrt{2} - 1 \leq |\lambda| \leq 1, \quad \sqrt{5} \leq |\lambda| \leq \sqrt{2} + 1.$$  

(16)

Hence the point $(L^2/2, -1)$ can be regarded as a lower right corner point of the upper trapezoid. We see that there is a large gap $-\sqrt{5} < \lambda < -1$ on which $H$ has no eigenvalue. It is analogous to the gap $-1 < \lambda < 1$ for the free theory and can be observed at integer $t$. As for the lower bound on $\kappa(L)$ we also note the exact result $\text{index } D = -L^2/4$ for $L$ a multiple of 4. They support the aforementioned observations based on the numerical analysis.

On the interval $-\kappa(L) < t < \kappa(L)$, $2n(L)$ eigenvalues flow upward and change the sing at $t_j^{\text{ind}}$ ($j = -n(L) + 1, -n(L) + 2, \ldots, n(L) - 1, n(L)$) from minus to plus, where $t_j^{\text{ind}} < t_k^{\text{ind}}$ for $j < k$. The index $D$ jumps by $-1$ at these points and monotonically decreases from the maximum $n(L)$ to the minimum $-n(L)$. Since $n(L) \approx \kappa(L)$ for large $L$, the maximal index that can be realized by $(2)$ grows faster than $L^2/4$ as $L$ increases. In Table 1, $n(L)$ is given for $2 \leq L \leq 15$. We find numerically $L^2/n(L) \approx 3.6$ for large $L$.

Numerically $t_j^{\text{ind}}$ is approximately equal to $t_j^{\text{top}}$ for $|j| \leq n(L)$ and $t_j^{\text{ind}} > t_j^{\text{top}}$ for $j > 0$. In the case of $L = 4$, for instance, $t_j^{\text{ind}}$ ($t_j^{\text{top}}$) for $j = 1, \ldots, 4$ are given by $0.5387$ ($0.5333 \cdots$), $1.6203$ ($1.6$), $2.7161$ ($2.666 \cdots$), $3.8447$ ($3.7333 \cdots$). The difference $t_j^{\text{ind}} - t_j^{\text{top}}$ is monotonically increasing in $j$ for a fixed $L$, while it is monotonically decreasing in $L(\geq 4)$ for a fixed $j(> 0)$. We have found that the maximal deviations $t_{n(L)}^{\text{ind}} - t_{n(L)}^{\text{top}}$ are scattered at around 0.2 and $t_j^{\text{ind}}$ for $|j| << n(L)$ approaches to $t_j^{\text{top}}$ as $L$ becomes large. The numerical values of $t_j^{\text{ind}} - t_j^{\text{top}}$
for fixed $j$ can be well-fitted to a curve $\alpha_j L^{-2}$ for large $L$, where the coefficient $\alpha_j$ depends on $j$. In the limit $L \to \infty$, both $t^\text{ind}_j$ and $t^\text{top}_j$ approach to $j + \frac{1}{2}$. This is in accord with the observation that on the infinite lattice the topological invariants can be investigated quite generally by using the geometric concepts [13]. In Figure 4 index $D$ and $-Q_1$ are plotted over the region $0 \leq t \leq L^2/2$ for $L = 5$. The correspondence between the index and the topological charge is very excellent on the region $|t| < \kappa(L)$. We thus conclude that index $D$ coincides to $-Q_1$ for gauge fields sufficiently close to configurations with the constant field strength $|F_{12}| \leq 2\pi n(L)/L^2$. This is consistent with the numerical results given in ref. [23].

The jumps of index $D$ and those of $-Q_1$, however, do not exactly synchronize and the equality between index $D$ and $-Q_1$ is violated near the discontinuities, where the field strength (15) at the corner point $x_1 = x_2 = L - 1$ takes a large value $\approx \pm \pi$. The finite deviations between $t^\text{ind}_j$ and $t^\text{top}_j$ imply that the analytic decomposition of the space of link variables into connected components by imposing $\det H \neq 0$ and the geometric one by imposing (6) are not consistent with each other for a general choice of $\eta$. If $\eta$ is chosen to satisfy $0 < \eta < 1 - 1/\sqrt{2}$ as given by (12), the condition (6) excludes uniformly the regions where the jumps both in the index and in the topological charge occur. We thus find that index $D$ coincides with $-Q_1$ for any gauge fields contained in $U_{\eta,\phi}$ and the index theorem (13) holds true in two dimensions. On the configuration space $U_{\eta,\phi}$ the maximal index is given by $[\eta' L^2/2\pi]$. In Table 1 we give bounds $n_\eta(L)$ on the index for $2 \leq L \leq 15$ for comparison with $n(L)$.

The spectral flow crossing $\lambda = 0$ is complicated on the interval $\kappa(L) \leq t \leq L^2/2$. The eigenvalues flow both downward and upward. The downward flows are dense and concentrated in narrow regions, where the index increases rapidly. Then the index decreases slowly due to the sparse upward flows until the next dense downward flows approach to the horizontal axis. The downward flows and the upward flows are repeated alternately depending on the size of the lattice. Finally we have index $D = 0$ at $t = L^2/2$. Similar thing also happens on the interval $-L^2/2 \leq t \leq -\kappa(L)$. We should consider, however, such peculiar behaviors on the outer regions $\kappa(L) \leq |t| \leq L^2/2$ as a kind of lattice artifacts and the corresponding field configurations as unphysical. In other words the field configurations should be restricted not to involve such lattice artifacts in order to keep proper connection with continuum theories.

We have carried out a similar analysis in four dimensions by taking the link variables

| $L$ | 2 3 4 5 6 7 8 9 10 11 12 13 14 15 |
|-----|------------------|
| $n(L)$ | 1 2 4 7 10 13 18 22 28 34 39 47 55 62 |
| $n_\eta(L)$ | 0 0 0 1 1 2 2 3 4 5 6 7 9 10 |

Table 1:
Figure 2:

(14) and

\[ U_3^{(s)}(x) = \exp\left[-is\frac{2\pi}{L}\delta_{x_3,L-1}\right], \quad U_4^{(s)}(x) = \exp\left[is\frac{2\pi}{L^2}x_3\right], \tag{17} \]

where \( s \) is an integer (\(|s| \leq L^2/2\)) corresponding to \( m_{34} = s \). Only the link variables along the first two lattice coordinates are changed. Since the spectrum of \( H \) is symmetric at \( t = 0 \), we can find index \( D \) as in two dimensions. The characteristic features of the spectral flows observed in two dimensions can be seen also in four dimensions. In Figure 3, we indicate the spectrum of \( H \) for \( L = 4, |\lambda| \leq 1, s = 1 \) and \(|t| \leq 8\). Most of the eigenvalues are distributed outside the parallelogram region and an eigenvalue crosses the parallelogram downward as \( t \) increases by unity from an integer. When it crosses the horizontal axis, index \( D \) increases by one unit. Note that the direction of the eigenvalue flows on the parallelogram region is consistent with the factor \((-1)^N\) on the rhs of (13). The characteristic feature of the spectrum is not changed for \( s > 1 \). The parallelogram region becomes smaller as \( s \) increases. In general, \( s \) adjacent eigenvalues flow downward and index \( D \) increases by \( s \) when they cross the \( t \)-axis. This is consistent with (10). We found that the index and the topological charge coincide with each other for the values of \((s, t) (0 < s \leq t)\) given in Table 2. Though our analysis is restricted to rather small lattice sizes \( 2 \leq L \leq 4 \), we anticipate that the parallelogram region expands rapidly enough as \( L \) increase and the condition (6) with \( 0 < \eta < (2 - \sqrt{2})/12 \) is sufficient to ensure the lattice index theorem (13) in four dimensions.

\[ * \] For gauge fields satisfying (11) and (12), nonvanishing topological charges can be realized only for \( L \geq 9 \).
Coming back to two dimensions, we give some analytic results on the spectrum of $H$ and index $D$ at $t = r, L^2/r$, where $r$ is an arbitrary integer factor of $L$. They are helpful to understand our numerical results.

The eigenvalues $\lambda$ of $H$ at $t = L^2/r$ with $L$ a multiple of $r$ are parametrized by $p$ and $q$

$$p = \frac{2\pi k}{L}, \quad q = \frac{2r\pi l}{L}, \quad \left(\left[-\frac{L}{2}\right] < k \leq \left[-\frac{L}{2}\right], \quad 0 \leq l < \left[\frac{L}{r}\right]\right) \quad (18)$$

and satisfy the secular equation [24]

$$\det \begin{pmatrix} B(p, q) - \lambda & C(p, q) \\ C(p, q)^\dagger & -B(p, q) - \lambda \end{pmatrix} = 0 \quad (19)$$

where $B(p, q)$ and $C(p, q)$ are $r \times r$ matrices defined by

$$(B(p, q))_{kl} = -\frac{1}{2} \delta_{k,l+1}^{(q)} + \left[1 - \cos \left(p + \frac{2\pi k}{r}\right)\right] \delta_{k,l}^{(q)} - \frac{1}{2} \delta_{k+1,l}^{(q)}$$

$$(C(p, q))_{kl} = -\frac{1}{2} \delta_{k,l+1}^{(q)} + \sin \left(p + \frac{2\pi k}{r}\right) \delta_{k,l}^{(q)} + \frac{1}{2} \delta_{k+1,l}^{(q)} \quad (0 \leq k, l \leq r - 1) \quad (20)$$
The $\delta_{k,l}^{(q)}$ is the Kronecker’s $\delta$-symbol for $0 \leq k, l \leq r - 1$ and satisfies the twisted boundary conditions $\delta_{k,r}^{(q)} = e^{-iq\delta_{k,0}}$ and $\delta_{r,k}^{(q)} = e^{iq\delta_{0,k}}$. It can be shown that (19) takes the following form [24]

$$f_r(\lambda) = \frac{(-1)^{r-1}}{2r-4} \sin^2 \frac{rp}{2} \sin^2 \frac{q}{2}, \quad (21)$$

where $f_r = \lambda^{2r} + \cdots$ is a polynomial of degree $2r$ and is independent of $p$ and $q$. For $1 \leq r \leq 6$ it is given by

- $f_1 = \lambda^2 - 1$,
- $f_2 = \lambda^4 - 6\lambda^2 + 1$,
- $f_3 = \lambda^6 - 9\lambda^4 + \frac{69}{4}\lambda^2 - 3\sqrt{3}\lambda - \frac{1}{4}$,
- $f_4 = \lambda^8 - 12\lambda^6 + 42\lambda^4 - 8\lambda^3 - 44\lambda^2 + 24\lambda - 3$,
- $f_5 = \lambda^{10} - 15\lambda^8 + \frac{595 + 5\sqrt{5}}{8}\lambda^6 - \frac{5\sqrt{10 + 2\sqrt{5}}}{2}\lambda^4 - \frac{1095 + 65\sqrt{5}}{8}\lambda^4 + 15\sqrt{10 + 2\sqrt{5}}\lambda^3 + \frac{1115 + 270\sqrt{5}}{16}\lambda^2 - (85 + 35\sqrt{5})\sqrt{10 + 2\sqrt{5}}\lambda + \frac{109 + 50\sqrt{5}}{16}$,
- $f_6 = \lambda^{12} - 18\lambda^{10} + \frac{237}{2}\lambda^8 - 6\sqrt{3}\lambda^7 - 360\lambda^6 + 54\sqrt{3}\lambda^5 + \frac{8145}{16}\lambda^4 - \frac{297\sqrt{3}}{2}\lambda^3 - \frac{2007}{8}\lambda^2 + \frac{255\sqrt{3}}{2}\lambda - \frac{719}{16}. \quad (22)$

The case of $r = 1$ corresponds to the free spectrum since the link variables become trivial.

The eigenvalues must satisfy the inequality

$$0 \leq (-1)^{r-2}2^{r-4}f_r(\lambda) \leq 1. \quad (23)$$

This gives rise to $2r$ allowed intervals $\lambda_{s}^{(0)} \leq \lambda \leq \lambda_{s}^{(1)}$ ($s = 1, \cdots, 2r$), where $\lambda_{s}^{(0)}$ and $\lambda_{s}^{(1)}$ are the $s$-th largest roots of either $f_r(\lambda) = 0$ or $f_r(\lambda) = (-1)^{r-1}2^{-r-4}$. In general the allowed intervals become shorter and shorter as $r$ increases. This is the reason why eigenvalues form clusters around the point where $t = L^2/r$ takes an integer. In particular we find (16) from [23] for $r = 2$. Similar mechanism of clustering is also at work for other integer points $t (\neq 1)$ as is seen in Figure [1].

If $f_r$ is known, it is possible to find index $D$ at $t = L^2/r$. As an example, we consider the case that $L$ is a multiple of 4 and take $r = 4$. For any $p, q$ the eight roots of (21) are separately located in the following narrow intervals

- $-2.4142 \cdots \leq \lambda \leq -2.4087 \cdots$,
- $-1.9429 \cdots \leq \lambda \leq -1.9032 \cdots$,
- $-1.7320 \cdots \leq \lambda \leq -1.6914 \cdots$,
- $0.1029 \cdots \leq \lambda \leq 0.1939 \cdots$,
- $0.4142 \cdots \leq \lambda \leq 0.5609 \cdots$,
- $0.9260 \cdots \leq \lambda \leq 1$,
- $1.7320 \cdots \leq \lambda \leq 1.7448 \cdots$,
- $2.7082 \cdots \leq \lambda \leq 2.7092 \cdots. \quad (24)$
It has three negative roots and five positive roots for each $p,q$, giving the exact index $-1 \times L \times L/4 = -L^2/4$. Note also that there is a large gap between the largest negative root and the smallest positive root. In general (21) has $r - 1$ negative roots and $r + 1$ positive roots for $r \geq 4$ and, hence, $\text{index} D = -L^2/r$ at $t = L^2/r$. On the other hand it is vanishing for $1 \leq r \leq 3$ since the number of positive roots and that of negative ones are always equal.

These results can be extended to integers $t = r = L/s \leq L$ by noting the relation

$$\det(H - \lambda)|_{t=r} = (f_{sL}(\lambda))^r,$$

where $r$ and $s$ are positive integers satisfying $L = rs$. This gives at $t = r$ the index $-r$ for $sL \geq 4$ and 0 for $sL \leq 3$. These are completely consistent with the numerical results.

We have confirmed that the equality (13) between the index and the topological charge holds true for a wide class of gauge fields on the finite periodic lattice in two and four dimensions. The condition (6) with (12) excludes uniformly the configurations for which the discrepancy between the index and the topological charge appears and ensures the index theorem (13). We should give a remark on the possibility of extending our analysis to nonabelian theories with nontrivial topological charges. Link variables with nontrivial topological charges can be easily found within the commuting subgroup. The analysis can be reduced to that of a multi-U(1) cases and our results can be applied straightforwardly. The index theorem (13) also has an implication on chiral gauge theories on finite periodic lattices. We will argue them elsewhere.

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References

[1] M. Lüscher, Commun. Math. Phys. 85 (1982) 39.
[2] A. Phillips, Ann. Phys. 161 (1985) 399.
[3] M. Göckeler, A.S. Kronfeld, M.L. Laursen, G. Schierholz and U.-J. Wiese, Nucl. Phys. B292 (1987) 349.
[4] T. Fujiwara, H. Suzuki and K. Wu, IU-MSTP/39, hep-th/0001029.
[5] F. Karsch, E. Seiler ans I.O. Stamatescu, Nucl. Phys. B271 (1986) 349; J. Smit and J.C. Vink, Nucl. Phys. B286 (1987) 485.
[6] S. Itoh, Y. Iwasaki and T. Yoshié, Phys. Rev. D36 (1987) 527.
[7] R. Narayanan and P. Vranas, Nucl. Phys. B506 (1998) 373.
[8] W. Bardeen, A. Duncan, E. Eichten, G. Hockney and H. Thacker, Phys. Rev. D57 (1998) 1633; 
   C. Gattringer and I. Hip, Nucl. Phys. B536 (1998) 363; 
   P. Hernández, Nucl. Phys. B536 (1998) 345.

[9] P.H. Ginsparg and K.G. Wilson, Phys. Rev. D25 (1982) 2649.

[10] P. Hasenfratz, Nucl. Phys. (Proc. Suppl.) 63 (1998) 53; Nucl. Phys. B525 (1998) 401.

[11] H. Neuberger, Phys. Lett. B417 (1998) 141; B427 (1998) 353.

[12] M. Lüscher, Phys. Lett. B428 (1998) 342.

[13] P. Hasenfratz, V. Laliena and F. Niedermayer, Phys. Lett. B427 (1998) 125.

[14] P. Hernández, K. Jansen and M. Lüscher, Nucl. Phys. B552 (1999) 363.

[15] M. Lüscher, Nucl. Phys. B538 (1999) 515; 
   T. Fujiwara, H. Suzuki and K. Wu, Phys. Lett. B463 (1999) 63; Nucl. Phys. B569 (2000) 643; IU-MSTP/37, hep-lat/9910030.

[16] Y. Kikukawa and A. Yamada, Phys. Lett. B448 (1999) 265; 
   D.H. Adams, hep-lat/9812003; 
   H. Suzuki, Prog. Theor. Phys. 102 (1999) 141.

[17] D. H. Adams, Chin. J. Phys. 38 (2000) 633.

[18] R. Narayanan and H. Neuberger, Phys. Rev. Lett. 71 (1993) 3251; Nucl. Phys. B443 (1995) 305.

[19] R. Narayanan and H. Neuberger, Nucl. Phys. B412 (1994) 574.

[20] M. Lüscher, Nucl. Phys. B549 (1999) 295.

[21] H. Neuberger, Phys. Rev. D61 (2000) 085015.

[22] R. Edwards, U. Heller and R. Narayanan, Nucl. Phys. B522 (1998) 285.

[23] T.W. Chiu, hep-lat/9911010.

[24] T. Fujiwara, in preparation.