PUZZLES AND THE FATOU–SHISHIKURA INJECTION FOR RATIONAL NEWTON MAPS

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Abstract. We establish a principle that we call the Fatou–Shishikura injection for rational Newton maps: there is a dynamically natural injection from the set of non-repelling periodic orbits of any rational Newton map to the set of its critical orbits. This injection obviously implies the classical Fatou–Shishikura inequality, but it is stronger in the sense that every non-repelling periodic orbit has its own critical orbit.

Moreover, for every rational Newton map we associate a forward invariant graph (a puzzle) which provides a dynamically defined partition of the Riemann sphere into closed topological disks (puzzle pieces). The puzzle construction is, for the purposes of this paper, just a tool for establishing the Fatou–Shishikura injection; it also provided the foundation for future work on rigidity of Newton maps, in the spirit of the celebrated work by Yoccoz and others.

1. Introduction

Over the past several decades, substantial progress has been made in the understanding of the dynamics of iterated rational maps, with a particular focus on the dynamics of polynomials: the invariant basin of infinity, and the resulting dynamics in Böttcher coordinates, provide strong tools for global coordinates of the dynamics and subsequently for very deep studies of the dynamical fine structure. In comparison, the understanding of the dynamics of non-polynomial rational maps is lagging behind in a number of ways. We believe that Newton maps of polynomials are not only dynamically well-motivated as root finders (see for instance [HSS, BAS, Sch2, RSS] for recent progress) but are also a most suitable class of rational maps for which significant progress in analogy to polynomials is possible. This paper provides some results — in particular the Fatou–Shishikura injection — as well as foundations for subsequent results.

Among these results that extend our understanding of polynomial maps to Newton maps (and possibly further rational maps) are the following:

• a classification of all postcritically finite Newton maps; this is given in [LMS1, LMS2]. This establishes the first large family of rational maps of all degrees, beyond polynomials [Poi], for which such a classification is available;
• the rational rigidity principle for Newton maps: any two orbits in the Julia set of a Newton map can be combinatorially distinguished, except when they are related

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to actual polynomial dynamics. Such results are often described in terms of “local connectivity” of the Julia set, while a stronger and more precise way to express this is “triviality of fibers” in the Julia set (compare [Sch1]). These results will be described in [DS].

Results like these might lead one to think that, contrary to frequent belief, rational dynamics may not be much more complicated than polynomial dynamics as soon as a good combinatorial structure is established. An earlier application of this philosophy could be found in the case of cubic Newton maps — a one-dimensional family of maps with only one “free” critical point. The latter fact helped to develop a deep understanding of this family and its parameter space: [Tan], building on [Hea], produced combinatorics leading to a fruitful study of local connectivity and rigidity in [Roe, RWY] (see also [AR]). Our results, in this and subsequent manuscripts, extend previous work by moving from degree three to arbitrary degrees, and hence by moving from complex one-dimensional to arbitrary finite-dimensional parameter spaces. This is in analogy to the development from iterated quadratic polynomials (a fairly well understood one-dimensional family of unicritical maps) to polynomials of arbitrary degrees (the multicritical case is much less understood).

1.1. The main results. For a given polynomial \( p : \mathbb{C} \to \mathbb{C} \), the Newton map of \( p \) is a rational map \( N_p : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) defined as \( N_p(z) := z - p(z)/p'(z) \). Such maps naturally arise in Newton’s iterative method for finding roots of \( p \). We will assume that the degree of \( N_p \) is at least 3.

Newton maps form a large and important family of rational maps. The goal in this paper is to establish the following theorem:

Theorem A (The Fatou–Shishikura injection for rational Newton maps). There is a dynamically natural injection from the set of non-repelling periodic orbits of any rational Newton map to the set of critical orbits. More precisely, every non-repelling periodic orbit has (at least) one associated critical orbit that is not well separated from the non-repelling orbit, while any two non-repelling periodic orbits are well separated from each other. Therefore the critical orbits associated to two different non-repelling periodic orbits are disjoint.

Here we say that two points \( z, z' \in \mathbb{C} \) are well separated if one of the following conditions is satisfied: either there is a finite graph \( \Gamma \subset \hat{\mathbb{C}} \) that is forward invariant under the dynamics of \( N_p \) up to homotopy within individual Fatou components and so that \( z \) and \( z' \) are in two different components of \( \hat{\mathbb{C}} \setminus \Gamma \), or these two points belong to the filled Julia set of a polynomial-like restriction of \( N_p \) with connected Julia set and the images of \( z \) and \( z' \) under a straightening map for this polynomial-like map are separated by a periodic ray pair landing together. Both of these conditions are dynamically natural.

Remark. One can, at least in many cases, realize the preimages of polynomial dynamic rays under the straightening map by actual forward invariant objects in the dynamical plane of the Newton map, called “bubble rays” consisting of a countable string of bounded Fatou components. In the case of postcritically finite Newton maps, the construction of such rays is described in [LMS1].
Note that in Theorem A we do not say that there exists a unique such injection: there may be several infinite critical orbits associated to any non-repelling orbit.

Remark. It is a kind of art to include ever more dynamical features in the Fatou–Shishikura inequality and even the Fatou–Shishikura injection. This has been developed in [BCLOS] in the case of polynomials: for instance, one can take into account (combinatorial) “wandering triangles” (triplets of dynamic rays that land or accumulate at a common non-periodic point) or recurrent critical orbits. One can extend this injection also in our case; for instance, the polynomial results from [BCLOS] can be “imported” into our setting in a straightforward way. Another observation is that this injection can include, for instance, infinitely renormalizable polynomial-like maps (or arbitrary renormalizable polynomial-like maps): a renormalizable polynomial-like map is by definition a periodic domain that contains the entire orbits of all critical points it contains, and one of our results is that every indifferent periodic point generates such a periodic domain. Our result thus “factors through” such renormalizable domains. This observation is helpful in further work [DS] where we establish that the main difficulty for triviality of fibers (and hence local connectivity of the Julia set) is the existence of indifferent periodic orbits or infinitely renormalizable domains. The underlying idea that for polynomials every indifferent orbit has its own critical orbit is due to Kiwi [Ki]; some of our arguments are close in spirit to his.

The proof of Theorem A exemplifies the philosophy mentioned in the beginning of the introduction, and here is an outline of the arguments involved.

1. The Fatou–Shishikura injection holds true for polynomials.
2. For every rational Newton map \( N_p \), every non-repelling periodic point is either a (super-) attracting fixed point (a root of \( p \)), or it is contained in a domain of renormalization (defined below).
3. The process of renormalization preserves the Fatou–Shishikura injection.

The key step in this chain of arguments is to establish (2), and will be done by using a properly defined (Newton) puzzle partition. This is not an obvious task: unlike in the polynomial case, where the basin of a superattracting fixed point at \( \infty \) is partitioned in a straightforward way by equipotentials and rays landing together (the classical Yoccoz jigsaw puzzle construction for polynomials [McM, Chapter 8]), the rational case, in general, lacks such a global combinatorial structure. Our second main result of this paper — Theorem B — shows that for Newton maps, in fact, this difficulty can be taken care of, and roughly says that every rational Newton map gives rise to a well-defined puzzle partition of any given depth. More precisely it says the following (all undefined terms will be defined in the later sections).
Theorem B (Newton puzzles for rational Newton maps). Every Newton map $N_p$, possibly after a slight quasiconformal perturbation in the basins of the roots of $p$, has an iterate $g := N_p^M$ for which there exists a forward invariant graph $\Gamma \subset \hat{\mathbb{C}}$ such that for every $n \geq 0$ the complementary components of $g^{-n}(\Gamma)$ in $\hat{\mathbb{C}}$ exhibit a Markov property under $g$ and are topological disks with Jordan boundary (subject to a suitable truncation in the basins of the roots of $p$). The latter components define a (Newton) puzzle partition of depth $n$.

Note that our construction of puzzles in Theorem B, restricted to cubic Newton maps, is different from the one in [Roe]. Theorem B provides the foundation for much of our subsequent work using puzzles theory on rigidity of Newton maps; see in particular [DS].

The paper is organized as follows. We start by reviewing some known facts about Newton maps in Section 2. After that, in Section 3 we establish Theorem B (in fact, in a slightly more general form, see Theorem 3.9). The Newton puzzle partition is then exploited to identify renormalization domains for points with periodic itineraries with respect to this partition, and to extract corresponding polynomial like maps. The results of Section 3 will be then used in Section 4 to derive Theorem A. In order to make the paper self-contained, in Section 4 we describe a proof of the Fatou–Shishikura injection for polynomials; it is based on a perturbation argument and the Goldberg–Milnor fixed point portraits, similar as in [Ki].

2. Background on Newton maps

Let $N_p$ be the Newton map of a polynomial $p$, and $d$ be the degree of $N_p$. It is straightforward to check that the fixed points of $N_p$ in $\mathbb{C}$ are exactly the distinct roots of $p$. Every such fixed point is attracting with multiplier $(m - 1)/m$, where $m > 1$ is the multiplicity of this point as a root of $p$. In particular, simple roots are superattracting fixed points of $N_p$. Every Newton map has one more fixed point at $\infty$; it is repelling with multiplier $\deg p / (\deg p - 1)$.

A polynomial $p$ and its Newton map $N_p$ have the same degree if and only if all roots of $p$ are simple. In general, the degree of the Newton map equals the number of distinct roots of $p$. Since the case $d = 2$ is trivial, we will assume that $d \geq 3$ for all further construction without explicit mention.

For simplicity of notation, we will refer to the Newton map of a polynomial simply as a “Newton map”.

The rational maps that arise as Newton maps can be described explicitly as follows (see [Hea, Proposition 2.1.2], as well as [RS, Proposition 2.8] for the proof):

Proposition 2.1 (Head’s theorem). A rational map $f$ of degree $d \geq 3$ is a Newton map if and only if $\infty$ is a repelling fixed point of $f$ and for each fixed point $\xi \in \mathbb{C}$, there exists an integer $m \geq 1$ such that $f'(\xi) = (m - 1)/m$. $\square$

The Fatou components of $N_p$ containing roots play a fundamental role in the study of Newton maps.

Definition 2.2 (Immediate basin). Let $N_p$ be a Newton map and $\xi \in \mathbb{C}$ a fixed point of $N_p$. Let $B_\xi = \{ z \in \mathbb{C} : \lim_{n \to \infty} N_p^n(z) = \xi \}$ be the basin (of attraction) of $\xi$. The
connected component of \( B_\xi \) containing \( \xi \) is called the immediate basin of \( \xi \) and denoted \( U_\xi \).

Clearly, \( B_\xi \) is open. By a theorem of Przytycki [Prz], \( U_\xi \) is simply connected and \( \infty \in \partial U_\xi \) is an accessible boundary point; in fact, a result of Shishikura [Sh2] implies that every component of the Fatou set of \( N_\rho \) is simply connected.

**Definition 2.3** (Access to \( \infty \)). Let \( U_\xi \) be the immediate basin of the attracting fixed point \( \xi \in \mathbb{C} \). Consider an injective curve \( \Gamma : [0, 1] \to U_\xi \cup \{ \infty \} \) with \( \Gamma(0) = \xi \) and \( \Gamma(1) = \infty \). Its homotopy class within \( U_\xi \cup \{ \infty \} \), fixing endpoints, defines an access to \( \infty \) for \( U_\xi \).

In topologically simple cases, a simpler definition suffices.

**Definition 2.4** (Accesses to vertices of graphs). For a finite graph \( \Gamma \) embedded in the sphere, an access to a vertex \( x \in \Gamma \) is given in terms of a (sufficiently small) disk \( D \) around \( x \): an access is then represented by a component of \( D \setminus \Gamma \) that contains \( x \) on the boundary.

Most of the time we will use a simple definition of an access to a vertex of a finite graph. However, topological accesses to infinity within the immediate basins provides the important first-level combinatorial data due to the following proposition.

**Proposition 2.5** (Accesses to infinity; [HSS, Prop. 6]). Let \( N_\rho \) be a Newton map of degree \( d \geq 3 \) and \( U_\xi \) an immediate basin for \( N_\rho \). Then there exists \( k_\xi \in \{1, \ldots, d-1\} \) such that \( U_\xi \) contains \( k_\xi \) critical points of \( N_\rho \) (counting multiplicities), \( N_\rho|_{U_\xi} \) is a branched covering map of degree \( k_\xi + 1 \), and \( U_\xi \) has exactly \( k_\xi \) accesses to \( \infty \). \( \square \)

Let \( N_\rho \) be a Newton map. A point \( z \in \mathbb{C} \) is called a pole if \( N_\rho(z) = \infty \), a prepole if \( N_\rho^k(z) = \infty \) for some \( k > 1 \), and a pre-fixed point if \( N_\rho^k(z) \) is a finite fixed point for some \( k > 1 \).

**Definition 2.6** (Finite graph). A vertex is a point in \( S^2 \). Let \( V \) be a finite set of distinct vertices. An edge is a subset of \( S^2 \) of the form \( \lambda(I) \) where \( I = [0, 1] \) and

- \( \lambda : I \to S^2 \) is injective on \( (0, 1) \), and
- \( \lambda(x) \in V \iff x \in \partial I \).

Let \( E \) be a finite set of edges that (pairwise) intersect only at vertices. A finite graph (in \( S^2 \)) is a pair of the form \((V, E)\).

We sometimes omit the reference to the ambient space \( S^2 \) though it is always implicit.

**Definition 2.7** (Subgraphs). Let \( \Gamma_1 = (V_1, E_1) \) and \( \Gamma_2 = (V_2, E_2) \) be finite graphs. We say that \( \Gamma_1 \) is a subgraph of \( \Gamma_2 \) (denoted \( \Gamma_1 \subset \Gamma_2 \)) if \( V_1 \subset V_2 \) and \( E_1 \subset E_2 \).

Now we want to define a graph which will encode important combinatorial data about our Newton map. Although this graph can be defined for general Newton maps, it the case of attracting-critically-finite maps it can be done in a forward invariant way.

**Definition 2.8** (Attracting-critically-finite). We say that a Newton map \( N_\rho \) of degree \( d \) is attracting-critically-finite if all critical points in the basins of the roots have finite orbits, or equivalently, all attracting fixed points are superattracting and all critical orbits in their basins eventually terminate at the fixed points.
The following observation is well known, and its proof is standard and omitted.

**Lemma 2.9** (Only critical point). Let $N_p$ be a Newton map that is attracting-critically-finite and let $ξ ∈ C$ be a fixed point of $N_p$ with immediate basin $U_ξ$. Then $ξ$ is the only critical point in $U_ξ$. □

A postcritical set of $N_p$ is the closure of the union of all forward iterates of critical points of $N_p$. A map is called postcritically finite if its postcritical set is finite. Clearly, every postcritically finite Newton map is attracting-critically-finite, but the latter class of maps is much larger. In fact, using Head’s theorem (Proposition 2.1) and a quasiconformal surgery in the basins of roots, one can prove the following (see [DMRS, Section 3]):

**Proposition 2.10** (Turning Newton map into attracting-critically-finite). For every polynomial $p$ there exists a polynomial $\tilde{p}$ and a quasiconformal homeomorphism $τ: \hat{C} → \hat{C}$ with $τ(∞) = ∞$ such that:

1. $\deg N_\tilde{p} = \deg N_p = \deg \tilde{p} ≤ \deg p$;
2. the Newton map $N_\tilde{p}$ is attracting-critically-finite;
3. $τ$ conjugates $N_p$ and $N_\tilde{p}$ in some neighborhood of the Julia sets of $N_p$ and $N_\tilde{p}$ union all Fatou components (if any) that do not belong to the basins of roots; moreover, $τ$ has zero dilatation on the latter components and the Julia set. □

Using Proposition 2.10 we can focus our attention on the attracting-critically-finite Newton maps. For such maps all finite fixed points are superattracting, and are the only critical points in their respective immediate basins (Lemma 2.9). Denote these points by $ξ_1, ξ_2, ..., ξ_d$. For such maps we can define a Newton graph (see [DMRS, Section 2]), the construction of which we will now recall.

Let $U_ξ$ be the immediate basin of $ξ_i$. Then $U_ξ$ has a global Böttcher coordinate $φ_i : (D, 0) → (U_ξ, ξ_i)$ (here $D$ is the standard unit disk) with the property that $N_p(φ_i(z)) = φ_i(z^{k_i})$ for each $z ∈ D$, where $k_i − 1 ≥ 1$ is the multiplicity of $ξ_i$ as a critical point of $N_p$. The map $z → z^{k_i}$ fixes $k_i − 1$ rays in $D$. Under $φ_i$, these are mapped to $k_i − 1$ pairwise disjoint (except for endpoints) simple curves $Γ^1_i, Γ^2_i, ..., Γ^{k_i−1}_i ⊂ U_i$ that connect $ξ_i$ to $∞$, are pairwise non-homotopic in $U_i$ (with homotopies fixing the endpoints) and are invariant under $N_p$. They represent all accesses to $∞$ of $U_ξ$ (see Proposition 2.5).

**Definition 2.11** (Channel diagram of a Newton map). The channel diagram $Δ$ associated to $N_p$ is the finite connected graph with vertex set $\{∞, ξ_1, ξ_2, ..., ξ_d\}$ and edge set

$$\bigcup_{i=1}^{d} k_i−1 \bigcup_{j=1}^{Γ^j}_i \big\}.$$

It follows from the definition that $N_p(Δ) = Δ$. The channel diagram records the mutual locations of the immediate basins of $N_p$ and provides a first-level combinatorial information about the dynamics of the Newton map. It turns out that the channel diagram carries more information about location of the roots.

**Proposition 2.12** (Complement of immediate basin; [RS]). For every immediate basin $U_ξ$ of a Newton map, every component of $C \setminus U_ξ$ contains at least one fixed point. □
For attracting-critically-finite maps with a well-defined channel diagram, the last proposition implies that each of the two complementary components of every 2-cycle in $\Delta$ contains at least one vertex of $\Delta$. (In fact, one can prove a much stronger result: see [DMRS, Theorem 2.2].)

**Definition 2.13** (Level $n$ Newton graph). For any $n \geq 0$, denote by $\Delta_n$ the connected component of $N_p^{-n}(\Delta)$ that contains $\Delta$ (with $\Delta_0 := \Delta$). The graph $\Delta_n$ is called the *Newton graph* of $N_p$ at level $n$.

By construction, the Newton graph is forward invariant, that is $N_p(\Delta_{n+1}) = \Delta_n \subset \Delta_{n+1}$ for every $n \geq 0$. Every edge of $\Delta_n$ is an internal ray of a component of some basin $B_{\xi_i}$, while every vertex is either $\xi_i$, or $\infty$, or an iterated preimage of these. Observe that vertices in $\Delta_n$ are alternating points in the Fatou and the Julia set of $N_p$.

We next state one of the main results in [DMRS] ([DMRS, Theorem 3.4]; this result is the core of [DMRS, Theorem A], which explains the structure of the Fatou set of general Newton maps).

**Theorem 2.14** (Poles connect to $\infty$). There exists a positive integer $N$ so that $\Delta_N$ contains all poles of $N_p$. □

As an immediate corollary we see that each prepole is contained in a Newton graph of sufficiently high level (see [DMRS, Corollary 3.5]).

**Corollary 2.15** (Prepoles in Newton graph). Let $m \geq 0$ be an integer, and let $N$ be as in Theorem 2.14. Then every point in $N_p^{-m}(\infty)$ is a vertex of $\Delta_{m+N}$. □

### 3. Renormalization of rational Newton maps

In this section we develop a combinatorial tool called a *Newton puzzle partition*, which leads to the proof of Theorem B. Using this tool we then identify renormalization domains for rational Newton maps, as a key preparatory step towards the proof of Fatou–Shishikura injection (Theorem A). As we mentioned earlier, without loss of generality (due to Proposition 2.10) we work with attracting-critically-finite Newton maps, for which the notions of a channel diagram and a Newton graph are well-defined.

#### 3.1. Circle separation property of Newton graphs

This subsection contains the key technical result of the section – Proposition 3.1. This proposition shows that, in fact, at all sufficiently high levels $n$ all critical points of $N_p$ are separated from $\infty$ by a subset of $\Delta_n$ given by an *almost* disjoint union of topological circles (where *almost* means that all mutual intersections of these circles coincide with the set of finite fixed points of $N_p$). This result will prove its importance in Subsection 3.2, where it will be used to resolve boundary pinching problems for would-be puzzle pieces.

**Proposition 3.1** (Newton graphs circle-separated). For every attracting-critically-finite Newton map $N_p$ there exists an index $K \geq 1$ such that for every $n \geq K$ and for every component $V$ of $\hat{\mathbb{C}} \setminus \Delta$ there exists a topological circle $X_V \subset \Delta_n \cap V$ that passes through all finite fixed points in $\partial V$ and separates $\infty$ from all critical values of $N_p$ in $V$. 

Remark. By using Proposition 2.10, it is not hard to state the corresponding result for general Newton maps in terms of touching Fatou components.

Proposition 3.1 immediately implies the following result for attracting-critically-finite Newton maps, which was shown in [DMRS, Lemma 4.9] in the special case of postcritically fixed Newton maps.

**Corollary 3.2** (Connectivity in \( \mathbb{C} \)). If \( N_p \) is an attracting-critically-finite Newton map, then for all sufficiently large \( n \), the graph \( \Delta_n \setminus \Delta \) is connected. \( \square \)

We start our preparation for the proof of Proposition 3.1 by introducing some notations.

Let \( \Gamma \subset \hat{\mathbb{C}} \) be a connected graph embedded into \( \hat{\mathbb{C}} \), and let \( x \) be a vertex of \( \Gamma \). Denote by \( \mathcal{A}_x(\Gamma) \) the set of all accesses to \( x \) for \( \Gamma \) (see Definition 2.4). Likewise, if \( G \) is a component of \( \hat{\mathbb{C}} \setminus \Gamma \) with \( x \in \partial G \), we define \( \mathcal{A}_x(G) \) to be the number of accesses to \( x \) in \( G \).

Since \( \Delta \subset \Delta_n \), and \( \Delta \) is invariant under \( N_p \), we have \( \mathcal{A}_\infty(\Delta) = \mathcal{A}_\infty(\Delta_n) \) for every \( n \geq 0 \) (we shorten this notation by setting \( \mathcal{A}_\infty(\Delta) \)). Every access \( a \in \mathcal{A}_\infty(\Delta) \) corresponds to a unique pair of adjacent fixed rays \( \Gamma^i \neq \Gamma^j \) intersecting the boundary of \( a \) (here adjacent means with respect to the cyclic order of edges at \( \infty \)). If \( \xi_i \) and \( \xi_j \) are the starting points of these rays, then \( \xi_i \neq \xi_j \), as it follows from Proposition 2.12. Note that a pair of fixed points can be starting points for a pair of rays realizing up to two distinct accesses in \( \mathcal{A}_\infty \).

The proof of Proposition 3.1 heavily exploits the ideas used in the proof of [DMRS, Theorem 3.4]. Before going into details with the proof, let us recall some constructions from [DMRS, Section 3].

Recall that by [DMRS, Theorem 3.4] (cited above as Theorem 2.14), there exists an integer \( N \geq 1 \) such that \( \Delta_N \) contains all the poles of \( N_p \).

The first construction we will need is a map between boundary parametrizations. Let \( V_n \) be a component of \( \hat{\mathbb{C}} \setminus \Delta_n \), and \( V_{n+1} \) be a component of \( N_p^{-1}(V_n) \); assume, for simplicity, that both \( V_n \) and \( V_{n+1} \) are topological disks (for \( V_n \) this is automatic (\( \Delta_n \) is connected), for \( V_{n+1} \) we can assume \( n > N \)). The boundary \( \partial V_n \) can be viewed as an image of \( \mathbb{S}^1 \) under a piecewise analytic surjection \( \gamma_n \) \( \mathbb{S}^1 \to \partial V_n \) (called a boundary parametrization of \( \partial V_n \)) such that for every edge \( e \) of \( \partial V_n \) the preimage \( \gamma_n^{-1}(e) \) is one or two disjoint intervals in \( \mathbb{S}^1 \) which are mapped diffeomorphically onto \( e \). We also assume that the winding number of \( \gamma_n \) around an inner point of \( V_n \) is equal to one. If \( \gamma_{n+1} : \mathbb{S}^1 \to \partial V_{n+1} \) is the boundary parametrization for \( V_{n+1} \), then the proper map \( N_p : V_{n+1} \to V_n \) gives rise to a proper continuous orientation-preserving map \( \tau_n : \mathbb{S}^1 \to \mathbb{S}^1 \) such that \( N_p \circ \gamma_{n+1} = \gamma_n \circ \tau_n \); moreover, \( \deg \gamma_n = \deg(N_p : V_{n+1} \to V_n) \). We will call \( \tau_n \) a map between boundary parametrizations (of \( V_{n+1} \) and \( V_n \)). We have \( \deg(N_p : V_{n+1} \to V_n) \) choices for the map \( \tau_n \); for a fixed choice, \( \tau_n \) is unique up to pre-composition by a (orientation-preserving) piecewise analytic circle homeomorphism subject to a re-parametrization of \( \partial V_{n+1} \).

In the notations above, we can prove the following lemma, which will serve as a key step in proving Proposition 3.1.

**Lemma 3.3** (Pole-free bridge). If \( \mathcal{A}_\infty(V_n) = \mathcal{A}_\infty(V_{n+1}) =: A \neq \emptyset \), then there exists a Jordan arc \( X_{n+1} \subset \Delta_{n+1} \setminus \Delta \), which we call a bridge, such that \( \overline{X_{n+1}} \) connects two fixed points in \( \partial V_n \cap \partial V_{n+1} \) and \( N_p|_{X_{n+1}} \) is a homeomorphism.
Proof. Let $|A| = m \geq 1$. Since $A_\infty(V_n) = A_\infty(V_{n+1})$, and since the Newton graph is forward invariant, it follows that $V_{n+1} \subset V_n$. Moreover, there exist $3m$ distinct points $t_1, s_1, t_1', t_2, s_2, t_2', \ldots, t_m, s_m, t_m' \in \mathbb{S}^1$ (listed in a cyclic order) such that: 1) $\{s_1, \ldots, s_m\} = \gamma_n^{-1}(\{\infty\})$; 2) $\gamma_n(t_i)$ and $\gamma_n(t_i')$ are finite fixed points of $N_p$; 3) $\gamma_n|_{[t_i, t_i']}$ is an embedding; 4) all these points are fixed by $\tau_n$. Properties 1)-3) follows directly from the definition of a boundary parametrization, whereas the last property is due to the fact that we have same accesses to $\infty$ of a boundary parametrization, whereas the last property is due to the fact that we have same accesses to $\infty$ in $V_n$ and $V_{n+1}$, after, perhaps, a suitable re-parametrization of $\partial V_{n+1}$. Property 4) also guarantees that $s_1, \ldots, s_m$ are the only points on $\mathbb{S}^1$ that are mapped to $\infty$, and for $1 \leq i \leq m$ the images $\gamma_n([t_i, s_i]) = \gamma_{n+1}([t_i, s_i])$, $\gamma_n((s_i, t_i']) = \gamma_{n+1}((s_i, t_i'])$ are pairs of the fixed rays realizing all $m$ accesses in $A$.

Since finite fixed points are critical for $N_p$, we have $t_i' \neq t_{i+1}',$ and thus $I_i := [t_i', t_{i+1}]$ are non-degenerate intervals. From [DMRS, CLAIM 1 in the proof of Theorem 3.4] (which, in turn follows from the key lemma of [DMRS, Section 3] — [DMRS, Lemma 3.3]) it follows that there exists an $i$ such that $\gamma_n(I_i)$ contains no poles of $N_p$ (hence so does $\gamma_{n+1}(I_i)$), and thus $\tau_n|_{I_i}$ is an orientation-preserving homeomorphism of $I_i$. For simplicity, assume $i = 1$, and put $\gamma_n(t_1) = \xi_1$, $\gamma_n(t_1') = \xi_2$ (note that it is a priori possible that $\xi_1 = \xi_2$).

Let $\xi \in \gamma_{n+1}(I_1)$ be a fixed point of $N_p$ (then automatically $\xi \in \gamma_n(I_1)$). The point $\xi$ might be equal $\xi_1, \xi_2$, or be distinct from them (a priori, we can’t exclude the latter case).

Since $\tau_n|_{I_1}$ is a homeomorphism that maps every $x \in I_1$ with $\gamma_{n+1}(x) = \xi$ to the point $y \in I_1$ with $\gamma_n(y) = \xi$ preserving the order of such points on the interval, we see that for every $b_{n+1} \in A_\xi(V_{n+1})$ there exists a unique $b_n \in A_\xi(V_n)$ such that $N_p(b_{n+1}) \subset b_n$ (after possibly shrinking domains defining accesses, if needed; we are free to do that since $b_{n+1}$ and $b_n$ defined locally around $\xi$). However, $V_{n+1} \subset V_n$. Therefore, for every such $\xi$ there exists at least one pair of accesses $b_{n+1}$ and $b_n$ such that $b_{n+1} \subset b_n$ and $N_p(b_{n+1}) = b_n$. This is only possible when $\overline{b_{n+1}} \cap \overline{b_n}$ contains a fixed ray (indeed, locally around $\xi$ the Newton map is conjugated to a power map $z \mapsto z^{k_{\xi}}$, with $k_{\xi} > 1$, which implies the claim).

From the arguments in the previous paragraph it follows that if $\xi \in \gamma_{n+1}(I_1)$, then $\xi = \xi_1$ or $\xi = \xi_2$, that is $\gamma_{n+1}(I_1)$ cannot contain fixed points other than $\xi_1$ or $\xi_2$.

If $\xi_1 \neq \xi_2$, then the lemma follows immediately with $X_{n+1}$ being a unique Jordan arc in $\gamma_{n+1}(I_1)$ connecting $\xi_1$ and $\xi_2$.

If $\xi_1 = \xi_2 =: \xi$, then we should be more careful: it is not a priori clear that such arcs exist. Let us show that this is indeed the case. Since $\xi$ is a critical point, the claim is clear if there at least two fixed rays starting at $\xi$ in the corresponding immediate basin, that is, when $\xi$ is a critical point of local degree at least 3. If $\xi$ is of local degree 2, then if the claim was not true, we would have $A_\xi(\Delta_{n+1}) = A_\xi(V_{n+1})$, which contradicts the fact that $\tau_n|_{I_1}$ is a homeomorphism. Therefore, we have at least one Jordan arc in $\gamma_{n+1}(I_1)$, which yields the claim of the lemma. \[\square\]
In the notation of Lemma 3.3, let \( v_1, v_2 \in I_1 \) be a pair of points (\( t'_1 < v_1 < v_2 < t_2 \)) such that \( \gamma_{n+1}(v_i) = \partial U_i \cap X_{n+1} \), where \( U_i \) is the immediate basin for \( \xi_i, \ i \in \{1, 2\} \). Put

\[
\Gamma_1 := \gamma_{n+1} \left( (s_1, t'_1) \right), \quad \Gamma_2 := \gamma_{n+1} \left( (t_2, s_2) \right), \\
e^1_{n+1} := \gamma_{n+1} \left( (t'_1, v_1) \right), \quad e^2_{n+1} := \gamma_{n+1} \left( (v_2, t_2) \right), \\
e_n := N_p \left( e^1_{n+1} \right) = \gamma_n \left( (t'_1, \tau(v_1)) \right), \quad e^2_n := N_p \left( e^2_{n+1} \right) = \gamma_n \left( (\tau(v_2), t_2) \right).
\]

Here \( \Gamma_1, \Gamma_2 \) is a pair of fixed rays starting at \( \xi_1 \) resp. \( \xi_2 \), \( e^1_{n+1}, e^2_{n+1} \) is a pair of terminal edges of \( X_{n+1} \), and \( e^1_n, e^2_n \) is the image of this pair under \( N_p \) (note the third line in definition above is consistent since \( \tau_1 1 \) is a homeomorphism and \( N_p \circ \gamma_{n+1} = \gamma_n \circ \tau_n \)).

Finally, for \( i = 1 \) resp. \( i = 2 \) denote by \( a^i_n, e^i_{n+1} \) the angle between \( \Gamma_i \) and \( e^i_n \) measured around \( \xi_i \) within the access defined by \( t'_1 \) resp. \( t_2 \); define \( a^i_{n+1} \) in a similar way.

**Lemma 3.4 (Terminal edges of pole-free bridge).** In the notations above, the arc \( X_{n+1} \) can be uniquely chosen in such a way that \( \alpha^i_{n+1} = \alpha^i_n / k_i \), where \( k_i \) is the local degree of the critical point \( \xi_i, \ i \in \{1, 2\} \).

**Proof.** This lemma is trivial if \( \xi_1 \neq \xi_2 \), and follows since \( N_p \) on \( U_i \) is conjugate to \( z \mapsto z^{k_i} \) on \( \mathbb{D} \).

For \( \xi_1 = \xi_2 \) the conclusion of the lemma follows from the same argument (by contradiction) as we used in Lemma 3.3 to prove that \( \gamma_{n+1}(I_1) \) contains no fixed points except \( \xi_1 \) and \( \xi_2 \).

**Proof of Proposition 3.1.** Assume \( n \geq N \). For a given access \( a \in \mathcal{A}_\infty \), let \( V_n \) be the unique unbounded component of \( \widehat{\mathbb{C}} \setminus \Delta_n \) with \( a \in \mathcal{A}_\infty(V_n) \). Define \( V_{n+1} \) to be the component of \( N_p^{-1}(V_n) \) having the same access \( a \). Such a component necessarily exists since the inverse branch of \( N_p \) fixing \( \infty \) preserves \( a \). Moreover, since for all \( n \) greater \( N \) the Newton graph \( \Delta_n \) contains all the poles (by Theorem 2.14), the component \( V_{n+1} \) is a topological disk. And since, by definition, the Newton graph is forward invariant, we also have \( V_{n+1} \subset V_n \).

Inductively, we construct a sequence of nested topological disks \( V_N \supset V_{N+1} \supset \ldots \) with \( a \in \mathcal{A}_\infty(V_n) \) and \( N_p: V_{n+1} \to V_n \) proper for every \( n \geq N \). Hence, \( (\mathcal{A}_\infty(V_n))^{\infty}_{n=N} \) is a non-increasing (under the inclusion relation) sequence of finite sets (bounded below by \( \{a\} \)), and thus starting from some \( n = k_a \) this sequence is constant. We claim that

\[
(3.1) \quad \mathcal{A}_\infty(V_n) \equiv \{a\} \quad \text{for all } n \geq k_a.
\]

To see this, we will apply Lemma 3.3 to each consecutive pair of domains \( V_n, V_{n+1} \) (with \( n \geq k_a \)). We can do that inductively in a consistent way: if \( X_{n+1} \) is an arc given by Lemma 3.3 for a pair \( V_n, V_{n+1} \), then the arc \( X_{n+2} \) for the pair \( V_{n+2}, V_{n+1} \) can be chosen in such a way that \( X_{n+2} \) connects the same fixed points as \( X_{n+1} \) and \( N_p \) maps \( X_{n+2} \) homeomorphically onto \( X_{n+1} \). Indeed, (in the notations of Lemma 3.3) if \( \gamma_{n+1}(I_1) \) contains no poles (and thus \( \tau_n|I_1 \) is a homeomorphism), then, re-parametrizing \( \partial V_{n+2} \) if necessary, \( \gamma_{n+2}(I_1) \) will also be pole-free, hence \( \tau_{n+1}|I_1 \) will be a homeomorphism onto \( I_1 \). In this way we will obtain a sequence of Jordan arcs \( (X_n)_{n=k_a+1}^{\infty} \) disjoint from \( \Delta \), closures of which connect the same fixed points \( \xi_1 \) and \( \xi_2 \), and such that \( N_p: X_n \to X_n \) is an
orientation-preserving homeomorphism for every \( n > k_a \). Moreover, these arcs will be chosen to satisfy the conclusion of Lemma 3.4.

Finally, define \( W_n \) to be the component of \( \hat{C} \setminus (X_n \cup \Gamma_1 \cup \Gamma_2) \) containing \( V_n \). By the ‘sweeping argument’ [DMRS, Claim 3 in the proof of Theorem 3.4] and Lemma 3.4, the domains \( W_n \) with \( n > k_a \) form a nested sequence of topological disks such that

\[
(3.2) \quad \bigcap_{n > k_a} W_n = \Gamma_1 \cup \Gamma_2.
\]

Equality (3.2) implies (3.1) (and, in particular, shows that \( \xi_1 \neq \xi_2 \)). If we denote the arc \( X_n \) (for \( n > k_a \)) as \( X_{a,n} \), then Proposition 3.1 follows by setting

\[
K' := \max_{a \in A_\infty} k_a,
\]

\[
K := \min \{ n > K' : X_{a,n} \cap X_{a',n} = \emptyset \text{ for every pair } a, a' \in A_\infty \},
\]

where the last minimum is well-defined since any particular \( X_{a,n} \) can intersect only finitely many elements from the sequence \( (X_{a',n})_{n > k_a} \). Finally,

\[
X_V := \bigcup_{a \in A_\infty (V)} X_{a,n}
\]

is the required circle in the component \( V \) of \( \hat{C} \setminus \Delta \); a topological circle \( X_V \) is defined for all \( n \geq K \). \qed

3.2. Puzzle construction for rational Newton maps. In this subsection we will present a construction of puzzles for an iterate of a (attracting-critically-finite) Newton map. A prototype for this construction is the Newton graph \( \Delta_n \) itself — it is a forward invariant graph, and thus might serve to define a Markov partition of \( \hat{C} \). However, components of \( \hat{C} \setminus \Delta_n \) are not necessarily Jordan disks, which is a problem (we want puzzle pieces, that is components in the complement of a puzzle, to be Jordan disks in order to apply the standard theory of polynomial-like maps, see below). Proposition 3.1 will help us to resolve this problem. We will refine a Newton graph by adding to it all separating circles (existence of which is proven in the proposition). The details of the construction are as follows.

Define \( \Delta^+ \) to be the union of \( \Delta \) and all circles in all of the components of \( \hat{C} \setminus \Delta \) given by Proposition 3.1, and let \( \Delta^+_n \) to be the component of \( N_p^{-n}(\Delta^+) \) containing \( \infty \) (we assume \( \Delta^+_0 := \Delta^+ \)).

A point \( w \) is a \((pre-)pole at level \( l \) if \( w \in N_p^{-l}(\infty) \), and \( l \) is the smallest among such iterates. If \( X \) is the union of all the circles given by Proposition 3.1, let \( \nu \) be the largest level of \((pre-)poles in \( X \). Recall that \( N \) was a level of the Newton graph such that \( \Delta_n \) contains all the poles of \( N_p \) for all \( n \geq N \) (existence of \( N \) was guaranteed by Theorem 2.14). The following lemma summarizes the essential properties of \( \Delta^+_n \) in terms of indices \( N \) and \( \nu \).

Lemma 3.5 (Properties of \( \Delta^+_n \)). In the notations above,

1. \( \Delta_n \subset \Delta^+_n \) for all \( n \);
(2) $\Delta_{(i-1)(N-1+\nu)}^+ \subset \Delta_{(N-1+\nu)}^+$ for every positive integer $i$;
(3) $N_p^{-l}(\Delta_n^+) = \Delta_{n+l}^+$ for all positive integers $n$ and $l$ such that $n \geq N - 1 + l$;
(4) for every $n \geq 0$ each component of $\hat{\mathcal{C}} \setminus \Delta_n^+$ is a disk that might be pinched only at (finitely many) pre-fixed points.

Proof. (1): Since $\Delta \subset \Delta^+$, the first property immediately follows from the definition of $\Delta_n^+$.

(2): Observe that since $\Delta_N$ contains all the poles of $N_p$, inductively $\Delta_{N-1+\nu}$ contains all the (pre-)poles of $N_p$ at level at most $\nu$ (see Corollary 2.15), thus $\Delta_{N-1+\nu} \supset \Delta^+$. Therefore, by definition of $\nu$ and using (1), we obtain (2) for $i = 1$. The rest follows by induction.

(3): If $n \geq N - 1 + l$, then $\Delta_n$ (and hence $\Delta_n^+$, by (1)) contains all the (pre-)poles at level at most $l$. However, every component of $N_p^{-l}(\Delta_n^+)$ must contain a prepole of level $l$ (since $\Delta_n^+ \supset \Delta$; see Corollary 2.15), and therefore, every such component is connected to $\Delta_n^+$. From definition of $\Delta_n^+$ it then follows that $N_p^{-l}(\Delta_n^+) = \Delta_{n+l}^+$.

(4): To begin with, by construction, every component of $\hat{\mathcal{C}} \setminus \Delta_0^+$ is a Jordan disk. Assume $n > 0$. Every (pre-)pole on the graph $\Delta_n^+$ comes in one of two types: preimages of vertices on the bridges given by Proposition 3.1 and preimages of vertices on the channel diagram $\Delta$ (that is preimages of $\infty$). The points of the first type are always uniquely accessible from every domain with such points on its boundary. Every access to each (pre-)pole of the second type must be separated from any another access to the same (pre-)pole by a pair of preimages on the corresponding bridges in $\Delta_n^+$. Therefore, each (pre-)pole of the first type cannot be multiply accessible. The claim follows.

Lemma 3.5 gives us almost all we need to construct renormalization domains for (an iterate of) the Newton map $N_p$. However, by property (4) in Lemma 3.5, a component of $\hat{\mathcal{C}} \setminus \Delta_n^+$ can be pinched at finitely many pre-fixed points. We want to avoid such pinchings by considering a truncation of $\Delta_n^+$. This a standard procedure, and is being carried out as follows.

For every fixed point $\xi_i$ of $N_p$ take an equipotential $E(\xi_i)$ at level 1/2 in the immediate basin $U_i$ (by definition, $E(\xi_i)$ is the preimage of the circle $\{z: |z| = 1/2\}$ under a Riemann map $\varphi$ normalized at $\varphi(\xi) = 0$ that uniformizes $U_i$ to the unit disk $\mathbb{D}$).

Let $z_{i,k} \in \Delta_0^+$ be such that $N_p^{-k}(z_{i,k}) = \xi_i$, where $\xi_i$ is some fixed point (here we take the smallest possible $k$). Define $E(z_{i,k})$ to be the component of $N_p^{-k}(E(\xi_i))$ lying in the Fatou component containing $z_{i,k}$. Suppose $E_0$ is a union of all $E(z_{i,k})$, where $z_{i,k}$ runs over all described above points on $\Delta_0^+$ (which includes all the fixed points of $N_p$), and let $Q_0$ be the unique unbounded component of $\hat{\mathcal{C}} \setminus E_0$. Put

$$\Delta_0^+ := (\Delta_0^+ \cap Q_0) \cup E_0.$$

**Definition 3.6** (Truncated puzzle of depth $n$). A truncated puzzle of depth $n$ (with $n \geq 0$) is a component $\Delta_0^+$ of $N_p^{-n}(\Delta_0^+)$ containing $\infty$.

Using truncated puzzles we can finally define Newton puzzles for an iterate of $N_p$.  

Definition 3.7 (Puzzles and puzzle pieces for iterate of Newton map). Set $M := N - 1 + \nu$. For $n \geq 0$, a puzzle of depth $n$ is a finite graph $\Delta^t_{(n+2)M}$. A puzzle piece of depth $n$ is the closure of a component $P_n$ of $\hat{\mathbb{C}} \setminus \Delta^t_{(n+2)M}$ intersecting the Julia set of $N_p$.

For every point $x$ which is not attracted to one of the roots of $p$ and for a given depth $n$, define $P_n(x)$ to be the union of all puzzle pieces of depth $n$ containing $x$.

From the definition above it is clear that if $x$ is not $\infty$ or a (pre-)pole, then $P_n(x)$ is a unique puzzle piece of depth $n$ containing the point $x$. Otherwise, $P_n(x)$ is a finite union of puzzle pieces with $x$ as their common boundary point. (Note that, even though in the latter case $P_n(x)$ is a union of puzzle pieces, we will keep calling this set a puzzle piece; this should not cause ambiguity.)

Definition 3.8 (Fiber). Let $x \in \hat{\mathbb{C}}$ be a point that is not attracted to any of the roots of $p$; the set

$$\text{fib}(x) := \bigcap_{n \geq 0} P_n(x)$$

is called the fiber of $x$ (with respect to the partition given by the set of puzzles $(\Delta^t_{(n+2)M})^\infty_{n=0}$).

The following theorem (quoted in the introduction in a weaker and less precise form as Theorem B) summarizes the constructions we have done so far, and establishes properties of puzzle pieces and fibers for general (attracting-critically-finite) Newton maps. We view this theorem (together with Theorem B) as one of the two main results of the paper.

Theorem 3.9 (Newton puzzles for Newton maps). Every attracting-critically-finite Newton map $N_p$ has an iterate $g := N^\circ M$ for which there exists a puzzle partition of any given depth $n$ with the following properties:

1. every puzzle piece is a closed topological disk (that is a topological disk with Jordan boundary);
2. any two puzzle pieces are either nested or disjoint;
3. if $x$ is a point that is not attracted to a root of $p$, then for all $n, k \geq 0$ the map
   $$g^k : P_{n+k}(x) \to P_n(g^k(x))$$
   is a branched covering;
4. if $x$ is $\infty$, a pole or a prepole, then $\text{fib}(x) = \{x\}$.
5. if $x$ is not eventually mapped to $\infty$ or attracted to one of the roots of $p$, then the fiber $\text{fib}(x)$ is a closed connected set such that $\text{fib}(x) \subset \hat{P}_n(x)$ for every $n \geq 0$.

Proof. The first part of the theorem is a summary of Definition 3.7: for $M = N - 1 + \nu$ the puzzle partition for $g$ of depth $n$ is given by the graph $\Delta^t_{(n+2)M}$. Let us show that the listed properties hold true.

By property (4) of Lemma 3.5, each component of the complement of $\Delta^t_n$ is a Jordan disk that might be pinched only at finitely many pre-fixed points. By construction of $\Delta^t_n$, any such pinching point must be separated from $\infty$ by the preimage of the curve in $E_0$, and thus must be non-accessible from any puzzle piece of depth $n$. This proves property (1) of the theorem.
Property (2) was encoded into the definition of a puzzle piece and directly follows from property (2) of Lemma 3.5. Similarly, property (3) follows from property (3) of Lemma 3.5.

Property (4) is a corollary of Proposition 3.1. Indeed, for every $\epsilon > 0$ and for every component $V$ of $\hat{\mathbb{C}} \setminus \Delta$ the circle $X_V$ given by Proposition 3.1 can be chosen to be $\epsilon$-close in the spherical metric on $\hat{\mathbb{C}}$ to $\partial V$. This implies $\text{fib}(\infty) = \{\infty\}$. Let us show that the same holds for poles and prepoles. First observe that there exists an integer $k \geq 1$ such that $P_{n+k}(\infty) \setminus P_n(\infty)$ is a non-degenerate annulus; call this annulus $A_n(\infty)$. Indeed, we know that a bridge and its image (in Proposition 3.1) can intersect at most the number of times equals to the length of these bridges. From this the existence of $k$ follows. Let $z$ be a (pre-)pole at level $l$; since $\text{fib}(\infty) = \{\infty\}$, for all large enough $n$ the map $N_p^l$ has no critical points in $P_n(z)$ other then possibly at $z$. Therefore, $N_p^l : P_n(z) \to P_{n+l}(\infty)$ has bounded (in $n$) degree, and hence if $(A_{n_i}(\infty))$ is a sequence of nested annuli around $\infty$ such that

$$\sum_i \text{mod } A_{n_i}(\infty) = \infty,$$

then for the pull-back sequence $(A_{n_i}(z))$ at $z$ under $N_p^{-l}$ of those annuli we will have

$$\sum_i \text{mod } A_{n_i}(z) = \infty,$$

thus $\text{fib}(z) = \{z\}$.

Finally, let us prove property (5). Assume the contrary, and there is a point $z \in \text{fib}(x)$ that belongs to $\partial P_n(x)$ for all sufficiently large $n$. Observe that $z$ must be a pole or a prepole (we can exclude $\infty$). But we know by property (4) that $\text{fib}(z) = \{z\}$. Therefore, there must be a large index $k$ such that $P_k(z)$ and $P_k(x)$ are disjoint, a contradiction. \qed

### 3.3. Combinatorially recurrent points and renormalization of periodic orbits.

Let us fix $g := N_p^0$ to be an iterate of the Newton map $N_p$ for which we have well-defined puzzles of any depth.

We say that $x$ is a **combinatorially recurrent point** if $x$ is not eventually mapped to $\infty$ or attracted to the roots of $p$, and for every well-defined puzzle piece $P_n(x)$ of depth $n$ there exists $m \geq 1$ ($m$ depends on $n$) such that $g^{om}(x) \in P_n(x)$. By pulling back along the orbit of $x$ we can define an infinite sequence $P_{n_0}(x) \supset P_{n_1}(x) \supset P_{n_2}(x) \supset \ldots$, $k_i := n_i - n_{i+1} > 0$, of puzzle pieces (which we will call a nest) such that $k_i$ is the first return time from $P_{n_{i+1}}$ to $P_{n_i} = g^{k_i}(P_{n_{i+1}})$ (here $n_0$ is arbitrary, while the increasing sequence $(n_i)_{i \geq 0}$ depends on $n_0$ and $x$).

**Proposition 3.10** (Combinatorially recurrent points are well inside). For every $n_0 \geq 0$, if $x$ is combinatorially recurrent, and $(P_n(x))_{i \geq 0}$ is the nest built by pulling back $P_{n_0}(x)$ along the orbit of $x$, then for sufficiently deep level $i$ the puzzle piece $P_{n_{i+1}}(x)$ is compactly contained in $P_{n_i}(x)$.

**Proof.** To prove this proposition, first observe that if the boundaries of two puzzle pieces intersect, then they have a common (pre-)pole. Moreover, the only periodic (fixed) point of $g$ that may lie on the boundary of a puzzle piece is $\infty$. Since $\infty$ has a trivial fiber
(property (4) of Theorem 3.9), we can assume that the boundaries of the elements in our
nest are disjoint from $\infty$, thus have no periodic points of $g$ for all $i$.

If for some $i$ the boundaries $\partial P_{n_i}(x)$ and $\partial P_{n_{i+1}}(x)$ has a common (pre-)pole, say $z$, then $g^{\circ k_i}(z) \in \partial P_{n_i}(x) \setminus \{z\}$ ($z$ cannot be periodic), and hence there must be two poles or
prepoles on $\partial P_{n_0}(x)$ such that one is mapped to the other by $g^{\circ k_i}$ (here, same as above,
$k_i = n_i - n_{i+1}$). If we assume that any two consecutive puzzle pieces in the nest have
intersecting boundaries, then (pre-)poles on $\partial P_{n_0}(x)$ must be arranged in pairs of points
with one being mapped to the other by the corresponding $g^{\circ k_i}$. However, $\partial P_{n_0}(x)$ has only
finitely many (pre-)poles, thus the infinite directed graph with vertices being (pre-)poles,
and two vertices $w$ and $z$ being connected by a directed edge from $w$ to $z$ if $w$ is mapped
to $z$ by some $g^{\circ k_i}$, must have a directed cycle, which implies that $\partial P_{n_0}(x)$ has a periodic
point which is a (pre-)pole, a contradiction. □

A periodic point serves as an obvious example of a combinatorially recurrent point. Now
we want to show how to extract a polynomial-like map for some periodic orbits. Let us
briefly review the standard theory of polynomial-like maps. These were introduced by
Douady and Hubbard [DH] and have played an important role in complex dynamics ever
since.

**Definition 3.11.** A polynomial-like map of degree $d \geq 2$ is a triple $(f, U, V)$ where $U$, $V$
are open topological disks in $\hat{\mathbb{C}}$, the set $\overline{U}$ is a compact subset of $V$, and $f : U \to V$
is a proper holomorphic map such that every point in $V$ has $d$ preimages in $U$ when counted
with multiplicities.

**Definition 3.12 (Filled Julia set).** Let $f : U \to V$ be a polynomial-like map. The filled
Julia set of $f$ is the set of points in $U$ that never leave $V$ under iteration of $f$, i.e.
$$K(f) = K(f, U, V) = \bigcap_{n=1}^{\infty} f^{-n}(V).$$

As with polynomials, we define the Julia set as $J(f) = \partial K(f)$.

The simplest example of a polynomial-like map comes from restricting an actual poly-
nomial: let $p$ be a polynomial of degree $d \geq 2$, let $V = \{z \in \mathbb{C} : |z| < R\}$ for sufficiently
large $R$ and $U = f^{-1}(V)$. Then $p : U \to V$ is a polynomial-like mapping of degree $d$.

Two polynomial-like maps $f$ and $g$ are hybrid equivalent if there is a quasiconformal
conjugacy $\psi$ between $f$ and $g$ that is defined on a neighborhood of their respective filled
Julia sets so that $\partial \psi = 0$ on $K(f)$.

The crucial relation between polynomial-like maps and polynomials is explained in the
following theorem, due to Douady and Hubbard [DH].

**Theorem 3.13 (The straightening theorem).** Let $f : U \to V$ be a polynomial-like map of
degree $d$. Then $f$ is hybrid equivalent to a polynomial $P$ of degree $d$. Moreover, if $K(f)$ is
connected and $d \geq 2$, then $P$ is unique up to affine conjugation. □

As an immediate consequence of the first part of the theorem, it follows that $K(f)$ is
connected if and only if $K(f)$ contains the critical points of $f$. 

Now we define the notion of renormalization of rational maps. Let $R$ be a rational map of degree $d$.

**Definition 3.14 (Renormalization).** $R^n$ is called renormalizable if there exist open disks $U, V \subset \mathbb{C}$ such that $(R^n, U, V)$ is a polynomial-like map whose critical points are contained in the filled Julia set of $(R^n, U, V)$.

Such a triple $g := (R^n, U, V)$ is called a renormalization, and $n$ is the period of the renormalization $g$.

The filled Julia set of $g$ is denoted by $K(g)$, and the critical and postcritical sets by $C(g)$ and $P(g)$ respectively. The $i$th small filled Julia set is given by $K(g, i) = R^i(K(g))$. The $i$th small critical set is $C(g, i) = K(g, i) \cap C_R$ for $1 \leq i \leq n$.

The following result shows that a renormalization does not depend on domains provided that a small critical set is fixed [McM, Theorem 7.1].

**Theorem 3.15 (Uniqueness of renormalization).** Let $g = (R^n, U, V)$ and $g' = (R^n, U', V')$ be two renormalizations of the same period. If $C(g, i) = C(g', i)$ for all $1 \leq i \leq n$, then the filled Julia sets are the same, i.e. $K(g) = K(g')$.

Finally, we are in the position to prove the key proposition about the renormalization of periodic points whose fiber contains a critical point.

Recall that the only possible fixed points of a Newton map $N_p$ are $\infty$ or roots of the polynomial $p$. The local dynamics at these points is well-understood, and the following proposition describes the dynamics at higher period points in terms of renormalizations.

**Remark.** Note that repelling periodic postcritical points are also considered in the statement below, and though they are not needed for the Fatou–Shishikura injection (Theorem A), which concerns only with non-repelling cycles, we include them for the later application (see Proposition 5.1).

**Proposition 3.16 (Renormalization at periodic points).** For each attracting-critically-finite Newton map $N_p$, there are a large enough index $n$ and an iterate $M > 0$ so that

1. if $q$ is a non-fixed periodic point and $\text{fib}(q)$ contains at least one critical point, the map $N^{skM}_p : P_{n+k}(q) \to P_n(q)$ is a polynomial-like map of degree $d \geq 2$ with connected filled Julia set for some $k > 0$, and
2. for any two non-fixed periodic points $q$ and $q'$, either $\text{fib}(q) = \text{fib}(q')$ or $P_n(q) \cap P_n(q') = \emptyset$.

**Proof.** Suppose $q$ is a $k$-periodic point of $N_p$ with $k \geq 2$, and $\text{fib}(q)$ contains a critical point. Then $q \notin \Delta_n^t$ for all possible $n$, and hence for every such $q$, by property (5) of Theorem 3.9, the fiber $\text{fib}(q)$ is a closed connected set disjoint from the boundary of any puzzle piece $P_n(q)$.

Choose $n$ sufficiently large so that for any two non-fixed periodic postcritical points $q$ and $q'$, either $\text{fib}(q) = \text{fib}(q')$ or $P_n(q) \cap P_n(q') = \emptyset$, where in the former case it is evident that $P_n(q) = P_n(q')$. By properties (2) and (3) of Theorem 3.9, $N^{skM}_p$ with $M = N + \nu - 1$ (recall Theorem 2.14 for the definition of $N$, and the paragraph before Lemma 3.5 for the
definition of \( \nu \) maps \( P_{n+k}(q) \) onto \( P_n(N^k_p(q)) = P_n(q) \), and \( P_{n+k}(q) \subset P_n(q) \), and by property (1) of Theorem 3.9 all such puzzle pieces are closed topological disks.

Suppose \( \text{fib}(q) \) contains at least one critical point. Obviously, \( q \) is a combinatorially recurrent point, and therefore by Proposition 3.10 we can choose \( n \) big enough so that \( P_{n+k}(q) \) is contained in \( \hat{P}_n(q) \). For such \( n \), the mapping \( N^{okM}_p : P_{n+k}(q) \to P_n(q) \) is a polynomial-like map with filled Julia set equal to \( \text{fib}(q) \), where the existence of some critical point in \( \text{fib}(q) \) guarantees that the degree of \( N^{okM}_p \) is at least two. □

Remark. It is immediate from Proposition 3.16 that every non-repelling periodic point of period at least two is contained in a polynomial-like restriction specified by the proposition, and every two such non-repelling periodic points either have the same fibers, or can be separated by a truncated puzzle of some deep level.

4. Proof of the Fatou–Shishikura injection (Theorem A)

We start by proving the Fatou–Shishikura injection for polynomials. This result was initially established in [Ki] and later extended in much stronger form in [BCLOS]. Since it is not as well known as it might be, we provide here a self-contained proof, similar as in [Ki], based on fixed point portraits of polynomials as introduced in [GM].

Proposition 4.1 (The Fatou–Shishikura injection for polynomials). The Fatou–Shishikura injection holds for every iterated polynomial: for every polynomial there is a dynamically meaningful injection from the set of non-repelling periodic points to the set of critical values so that each periodic orbit is not well separated from the corresponding critical value and is well-separated from the remaining non-repelling periodic orbits, and thus from the corresponding critical values.

Proof. We do not assume the fact that every polynomial has only finitely many non-repelling periodic orbits; this will follow once we show that every finite subset of the set of non-repelling periodic orbits of a degree \( d \) polynomial has cardinality at most \( d - 1 \). (The Fatou–Shishikura inequality in its sharp form was established by Shishikura in [Sh1]; there is a simple proof for the special case of polynomials, using the statement of polynomial-like maps but not the straightening theorem: see [D]).

So consider any polynomial \( p \) of degree \( d \) and let \( S \) be any finite subset of the set of its non-repelling periodic points. Let \( L \) be the least common multiple of its periods and consider the polynomial \( P := p^L \). Clearly, \( p \) and \( P \) have the same non-repelling periodic points, but all of these in \( S \) are fixed points of \( P \).

All the fixed rays of \( P \) land at repelling or parabolic fixed points; some of these rays might land alone, while others might land in pairs or larger groups. Let \( \mathcal{R} \) be the union of all these fixed rays, together with their landing points. A basic region is defined to be a component of \( \mathbb{C} \setminus \mathcal{R} \).

By [GM, Theorem 3.3], every basic region contains exactly one interior fixed point or interior virtual fixed point (where a virtual fixed point is an invariant Fatou petal of a parabolic fixed point). Therefore, any pair of non-repelling non-parabolic fixed points in \( S \) is separated by a pair of fixed rays landing at a common fixed point. For \( p \), this means any
pair of non-repelling non-parabolic periodic points in $S$ is separated by a pair of periodic rays landing at a common periodic point.

Now construct a small perturbation of $p$ within the space of polynomial-like maps (in the sense of Definition 3.11) with the property that all parabolic orbits in $S$ retain their multipliers, and so that the landing points of all the repelling separating ray pairs remain repelling. More precisely, restrict $p$ to a polynomial-like map $p: U \to V$ and construct a polynomial $q$ (of arbitrary degree) so that $q(z) = 0$ for all $z \in S$ and $q'(z) = 0$ for all parabolic $z \in S$; moreover, require that $q'(z) = p'(z)$ for all non-parabolic $z \in S$. These finitely many conditions can obviously be satisfied by a polynomial of finite degree.

Then $p_\varepsilon := p + \varepsilon q$ is always a polynomial-like map $p_\varepsilon: U_\varepsilon \to V$ for $|\varepsilon|$ sufficiently small and $U_\varepsilon$ the component of $p_{\varepsilon}^{-1}(V)$ intersecting $U$, and it has the same degree as $p$; denote this degree by $d$. All parabolic cycles of $p$ remain parabolic with equal multiplier for $p_\varepsilon$, all attracting cycles remain attracting (if $\varepsilon$ is small enough), and all non-parabolic indifferent cycles, say with multiplier $\mu$ and period $n$, are still periodic orbits of period $n$ with multiplier $(1 + \varepsilon)^n$. Therefore, if $\varepsilon < 0$ then all non-parabolic indifferent periodic orbits of $p$ are attracting for $p_\varepsilon$.

Since the polynomial-like map $p_\varepsilon$ has degree $d - 1$, it has $d - 1$ critical points (counting multiplicities). Every attracting and every parabolic periodic orbit of $p_\varepsilon$ attracts at least one critical orbit, so $p_\varepsilon$ has at most $d - 1$ attracting or parabolic periodic orbits; hence the set $S$ contains at most $d - 1$ orbits in total. This shows that $p$ has at most $d - 1$ non-repelling orbits; this is the proof from [D]. It is thus no loss of generality to assume that the finite set $S$ contains all non-repelling periodic orbits of $p$.

Every attracting or parabolic orbit of $p_\varepsilon$ has the property that at least one infinite critical orbit converges to the attracting or parabolic orbit (in the special case of a superattracting periodic orbit, the critical orbit is already periodic). In particular, the attracting or parabolic orbit is not separated from an infinite critical orbit for $p_\varepsilon$ with $\varepsilon < 0$.

As $\varepsilon \nearrow 0$, the critical orbits of $p_\varepsilon$ converge to the critical values of $p$. There is thus (at least) one critical value of $p$ associated to every non-repelling periodic orbit of $p$; this is obvious for attracting and parabolic orbits of $p$, and for the non-parabolic indifferent ones there is a critical value by continuity. This critical value cannot be separated from one of the non-repelling periodic points by a ray pair landing at a repelling or parabolic periodic point: these separations would be preserved under small perturbations of $p$ (using the assumption that parabolics remain parabolic). The same is true for the entire critical orbit, and the claim follows.

Proof of Theorem A. All fixed points of $N_p$ are either at $\infty$ and repelling, or roots of the polynomial $p$ and attracting or superattracting. The latter have at least one critical point in their immediate basins, and thus the existence of the Fatou–Shishikura injection is thus obvious for fixed points (if there are several critical points in any given immediate basin, then there is a choice for this injection). We can thus restrict attention to non-repelling periodic orbits of periods 2 or higher.

Using the result of Proposition 2.10, we can assume that $N_p$ is attracting-critically-finite. Indeed, the surgery of Proposition 2.10 preserves a possible injection away from the
basins of roots (basins of roots are already taken care of) and relates forward invariance of graphs for attracting-critically-finite Newton maps to forward invariance up to homotopy rel. postcritical set and the grand orbit of ∞ for general Newton maps. Hence we can use the whole toolbox developed in Section 3. Set $g := N_p^M$. It follows that $g$ and $N_p$ have the same non-repelling periodic points. Therefore, we can focus on this iterate of $N_p$.

By Proposition 3.16 (see also the follow-up remark), every non-repelling periodic point of $g$ of period at least two is contained in a polynomial-like map with connected Julia set, and different non-repelling periodic points are either separated by $\Delta^n_t$ for sufficiently large $n$, or lie in the same small filled Julia set of the same polynomial-like map. We can consider the smallest $m = m(n)$ such that $\Delta^m_\infty \supset \Delta^n_\infty$. It is clear that $\Delta^m_\infty$ separates those orbits as well. Since the Newton graph is forward invariant, any two non-repelling periodic points are well separated (in the sense we described in the very beginning of the paper).

It therefore suffices to prove the claim for any polynomial-like map with connected Julia set.

The Fatou–Shishikura injection is true for the polynomials resulting from straightening by Proposition 4.1: these polynomials have connected Julia sets, so if their degree is $d$ then they contain $d - 1$ critical points, all their non-repelling periodic points are separated by ray pairs landing at repelling periodic points, and every non-repelling periodic orbit has at least one associated infinite critical orbit that is not separated from it. These properties are obviously respected by the straightening map, and we again obtain the well separation property, as desired. □

5. Postcritically finite Newton maps and other future directions

The results on combinatorics of puzzles (obtained in Section 3) will be used in the subsequent work [LMS1] to derive combinatorial properties of postcritically finite (pcf) rational Newton maps. It will be shown that every pcf Newton map has an associated forward invariant graph comprised of three parts: the Newton graph, Hubbard trees, and bubble rays. The Newton graph (in the terminology of the present paper) captures the behavior of all critical points that are eventually fixed, and relying on Corollary 3.2 is taken at a sufficiently high level so that the ever-important connectedness in $\mathbb{C}$ holds. Hubbard trees will capture the dynamics of all eventually periodic postcritical points which are not fixed relying on Proposition 3.16 of the present work. Thirdly, so-called bubble rays will connect the Newton graph to Hubbard trees. The combinatorial data provided by such a tri-partite graph will be just enough to reconstruct a Newton map using W. Thurston’s characterization of rational maps. Moreover, a complete classification of pcf Newton maps will be given in the forthcoming manuscript [LMS2].

To obtain a classification of pcf Newton maps in [LMS2], Hubbard trees will be extracted from polynomial-like restrictions around non-fixed postcritical points. However, it is essential for classification purposes that the polynomial-like restrictions have smallest possible period. Taken by itself, Proposition 3.16 would only allow us to extract Hubbard trees from an iterate of the desired polynomial-like mapping, and so with slight modification we
strenthen the proposition so that renormalizations have lowest period. Even though this result will be mainly used in the postcritically finite setting, we will formulate and prove it in the general (attracting-critically-finite) case (see Definition 2.8; recall also that every pef Newton map is automatically attracting-critically-finite).

Let \( q \) be a non-fixed periodic point of \( N_p \). The period of the fiber \( \text{fib}(q) \) is defined to be the minimal integer \( i \) so that \( N_p^i(\text{fib}(q)) = \text{fib}(q) \). Note that in the trivial case \( \text{fib}(q) = \{ q \} \) this period is equal to the period of the repelling cycle containing \( q \). In the case when \( \text{fib}(q) \) is nontrivial, it is a consequence of Proposition 3.16 that \( q \) is renormalizable with filled Julia set (defined in Definition 3.12) given by \( \text{fib}(q) \).

For a given renormalization \( q \) with the filled Julia set \( \text{fib}(q) \), the period of \( \text{fib}(q) \) divides the period of \( q \) (see Definition 3.14 for the definition of the period of a renormalization). A lowest period renormalization is a renormalization whose period coincides with the period of the corresponding fiber.

Let \( Q \) denote the set of critical and postcritical points of \( N_p \) that have finite orbit and are not in the basin of any root of \( p \).

**Proposition 5.1** (Separable lowest period renormalization). Let \( N_p \) be an attracting-critically-finite Newton map. There is a level of the Newton graph so that for all \( q,q' \in Q \), either \( \text{fib}(q) \) and \( \text{fib}(q') \) are in different complementary components of the Newton graph or \( \text{fib}(q) = \text{fib}(q') \). Moreover, if \( \text{fib}(q) \) is not a point and \( q \) is periodic, then \( q \) is lowest period renormalizable with filled Julia set \( \text{fib}(q) \).

**Proof.** There is a sufficiently deep puzzle so that for all \( q,q' \in Q \), either \( \text{fib}(q) \) and \( \text{fib}(q') \) are equal or are in different puzzle pieces. Some level of the Newton graph contains this puzzle as a subset, and so the Newton graph of this level enjoys the same separation property.

Let \( q \) be a periodic point of \( N_p \) of period \( k \geq 2 \) so that \( \text{fib}(q) \) contains a critical point (if not, then \( \text{fib}(q) = \{ q \} \), and we are done). From Proposition 3.16, there exists a pair of Jordan disks \( U,V \) with \( \overline{U} \subset V \), such that \( q \in U \) and \( N_p^{kM}: U \to V \) is a polynomial-like mapping (with non-escaping critical points). We want to extract a polynomial-like map with connected filled Julia set containing \( q \) given by a polynomial-like restriction of exactly \( N_p^{k'} \), where \( k' \) is the period of the Julia set \( \text{fib}(q) \). As we mentioned above, \( k'|k \), and let \( M' := kM/k' \). In order to extract the lowest period renormalization, we will modify \( U \) and \( V \) as follows.

Define inductively \( V_0 := V \), \( V_{i+1} \) to be the component of \( N_p^{-k'}(V_i) \) containing \( \text{fib}(q) \) (and hence \( q \), \( i \in \{ 0, \ldots, M' - 1 \} \) (we will obtain \( V_{M'} = U \)). Put

\[
U' := \bigcap_{i=0}^{M'} V_i.
\]

By construction, \( U' \) is non-empty (it contains \( q \)) Jordan disk (as an intersection of finitely many Jordan disks; note that we can assume by passing if necessary to a deeper level of puzzles, that \( \partial V_j \) for all \( j \) are disjoint from the critical set of \( N_p \) and contain no poles of \( N_p \), see property 4 of Theorem 3.9). Moreover, \( U' \) is exactly a subset of \( U \) containing all points that do not escape \( U \) under \( (k')^{1\text{th}} \) iterate of \( N_p \). The map \( N_p^{k'}: U' \to N_p^{k'}(U') \)
is a proper map between two Jordan disks $U'$ and $V' := N_p^{\delta k'}(U')$, and almost gives us a desired polynomial-like restriction for $q$. The only thing we need to ensure is that $\overline{U'} \subset V'$.

It is clear that $U' \subset V'$. Indeed, if $z \in \partial U'$, then there exists a $j \in \{1, \ldots, M'\}$ such that $z \in \partial V_j$. Therefore, $N_p^{\delta k'}(z) \in \partial V_{j-1}$, and thus $N_p^{\delta k'}(z)$ can not lie in $U'$, since the latter domain is the intersection of all $V_j$.

By the earlier construction of truncated puzzle pieces (see property 4 of Lemma 3.5 and Definition 3.6), $\partial U' \cap \partial V'$ may only consist of prepoles (that is, points in the Julia set of $N_p$). With a slight modification, this nontrivial intersection may be eliminated by a ‘thickening construction’. For some $\varepsilon > 0$, there is a neighborhood $W$ of $\infty$ so that the diameter of $W$ in the spherical metric is less than $\varepsilon$, the set $W$ does not intersect the critical or postcritical set of $N_p$ (which is finite by assumption), and $W \subset N_p(W)$ (for instance one could choose $W$ using linearizing coordinates for the repelling fixed point $\infty$). Each prepole $z \in U' \cap \partial V'$ satisfies $N_p^{m}(z) = \infty$ for a minimal choice of $m$ that depends on $z$. Denote by $W(z)$ the component of $N_p^{-m}(W)$ that contains $z$. After possibly passing to a smaller choice of $\varepsilon$, the neighborhoods $W(z)$ for each $z \in U' \cap \partial V'$ are pairwise disjoint, and let $\widehat{U}$ denote the union of $U'$ with the neighborhoods $W(z)$ for each $z \in U' \cap \partial V'$. Let $\widehat{V} := N_p^{\delta k'}(\widehat{U})$. Evidently $N_p^{\delta k'} : \widehat{U} \to \widehat{V}$ is a polynomial-like mapping, and for small enough $\varepsilon$ its filled Julia set is equal to $\text{fib}(q)$.

The fact that filled Julia sets are either identical or separated by the Newton graph of some level is an immediate consequence of the second statement of Proposition 3.16, using the fact that filled Julia sets are fibers. \qed

5.1. Concluding remarks. One might wonder in which generality of rational maps the Fatou–Shishikura injection holds true. We believe that Newton maps could lead the way to results for more general maps than rational Newton maps coming from polynomials. A first natural class of rational maps for which the results should be true are Newton maps of entire transcendental functions of the form $f(z) = p(z) \exp(q(z))$ with polynomials $p$ and $q$. For such $f$ the corresponding Newton maps are rational, and in fact their dynamics is very close to the dynamics of Newton maps coming from polynomials. Such Newton maps were studied by Khudoyor Mamayusupov in [Ma1, Ma2, Ma3]. For these maps, $\infty$ is no longer a repelling, but a parabolic fixed point. It is quite plausible that the results and constructions established in our work can be carried over to rational Newton maps of transcendental functions using the methods studied by Mamayusupov.

Moreover, there is current work in progress to decompose large classes of rational maps into Newton-like components and Sierpiński-like maps. Such a decomposition would allow one to extend results on Newton maps to the respective more general rational maps.

References

[AR] Magnus Aspenberg, Pascale Roesch, Newton maps as matings of cubic polynomials. Proc. Lond. Math. Soc. (3) 113 (2016), no. 1, 77–112.

[BAS] Todor Bilarev, Magnus Aspenberg, Dierk Schleicher, On the speed of convergence of Newton’s method for complex polynomials. Mathematics of Computation, 85:298 (2016), 693–705.
Alexander Blokh, Doug Childers, Genadi Levin, Lex Oversteegen, Dierk Schleicher, *An extended Fatou–Shishikura inequality and wandering branch continua for polynomials*. Advances in Mathematics **288** (2016) 1121–1174.

Adrien Douady, *Systèmes dynamiques holomorphes*. Séminaire Bourbaki, Astérisque **599** (1983), 39–63.

Adrien Douady and John Hubbard, *On the dynamics of polynomial-like mappings*. Ann. Sci. École. Norm. Supér. 4e series **18** (1985), 287–343.

Kostiantyn Drach, Dierk Schleicher, *The rational rigidity principle for Newton maps*. in preparation.

Kostiantyn Drach, Yauhen Mikulich, Johannes Rückert, Dierk Schleicher, *A combinatorial classification of postcritically fixed Newton maps*, to appear in Ergod. Theor. Syst, 2018.

Lisa Goldberg, John Milnor, *Fixed points of polynomial maps, Part II. Fixed point portraits*. Ann. Sci. École. Norm. Supér. (4) **26** (1993) 51–98.

Janet Head, *The combinatorics of Newton’s method for cubic polynomials*, PhD thesis, Cornell University, 1987.

John Hubbard, Dierk Schleicher, Scott Sutherland, *How to find all roots of complex polynomials by Newton’s method*, Invent. Math. **146** (2001), 1–33.

Jan Kiwi, *Non-accessible critical points of Cremer polynomials*. Ergodic Theory Dynam. Systems **20** 5 (2000), 1391–1403.

Russell Lodge, Yauhen Mikulich, Dierk Schleicher, *Combinatorial properties of Newton maps*, in preparation, arXiv:1510.02761.

Russell Lodge, Yauhen Mikulich, Dierk Schleicher, *A classification of postcritically finite Newton maps*, in preparation, arXiv:1510.02771.

Khudoyor Mamayusupov, *On Postcritically Minimal Newton Maps*. PhD thesis, Jacobs University Bremen (2015).

Khudoyor Mamayusupov, *Newton maps of complex exponential functions and parabolic surgery*. Fundamenta Mathematicae, to appear.

Khudoyor Mamayusupov, *A characterization of postcritically minimal Newton maps of entire functions*. Ergod. Theor. Dyn. Syst., to appear.

Curtis McMullen, *Complex dynamics and renormalization*, Annals of Math Studies, 135, 1994.

Alfredo Poirier, *On postcritically finite polynomials, part 2: Hubbard trees*. Stony Brook IMS preprint 93/7, 1993.

Pascale Roesch, *On local connectivity for the Julia set of rational maps: Newton’s famous example*. Ann. Math. **168** (2008) 1–48.

Pascale Roesch, Xiaoguang Wang, Yongcheng Yin, *Moduli space of cubic Newton maps*. Adv. Math. **322** (2017), 1–59.

Dierk Schleicher, *On fibers and local connectivity of Mandelbrot and Multibrot sets*. In: M. Lapidus, M. van Frankenhuyzen (eds): *Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot*. Proceedings of Symposia in Pure Mathematics **72**, American Mathematical Society (2004), 477–507.

Dierk Schleicher, *On the efficient global dynamics of Newton’s method for complex polynomials*. Preprint, arXiv:1108.5773; submitted.
[Sh1] Mitsuhiro Shishikura, *On the quasiconformal surgery of rational functions*, Ann. Sci. École. Norm. Supéér. 20 (1987) 1–29.

[Sh2] Mitsuhiro Shishikura, *The connectivity of the Julia set and fixed points*, In D. Schleicher, editor, *Complex dynamics: families and friends*, pp. 257–276. A K Peters, Wellesley, MA, 2009.

[Tan] Tan Lei, *Branched coverings and cubic Newton maps*, Fundamenta mathematicae, 154 (1997) 207–259.

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