ERRATUM FOR “NONHOLONOMIC AND CONSTRAINED VARIATIONAL MECHANICS”

ANDREW D. LEWIS
Department of Mathematics and Statistics
Queen’s University
Kingston, ON K7L 3N6, Canada

There is an error in the statement of Theorem 4.25 in [1], a somewhat related typographical error in Remark 4.26, and an error in Remark 4.27 following directly from that in Theorem 4.25. Footnote 8 is also now obsolete. In order to ensure that the errors are unambiguously fixed, what appears below should replace the original text starting from just before the statement of Theorem 4.25 and ending at the end of Section 4.

Next we consider the case of affine vector fields. Here we wish to obtain conditions on a defining subbundle \( \Delta \) that ensure that its corresponding affine subbundle variety \( A(\Delta) \) remains in a given subbundle \( F \). However, because \( A(\Delta) \) may be empty, we would like instead to make the problem into one that always has a solution, and then leave the matter of checking whether \( A(\Delta) \) is nonempty to something one can do afterwards. To this end, we note that, if \( A(\Delta) \subseteq E \) is flow-invariant under the affine vector field \( X_{\text{aff}} \), then

\[
\{(e,1) \in E \oplus \mathbb{R}_M \mid e \in A(\Delta)\} = \Lambda(\Delta) \cap (E \times \{1\})
\]
is flow-invariant under \( \hat{X}_{\text{aff}} \), according to Lemma 4.18. Clearly

\[
A(\Delta) \subseteq F \iff \{(e,1) \in E \oplus \mathbb{R}_M \mid e \in A(\Delta)\} \subseteq \hat{F} \triangleq \{(e,1) \in E \oplus \mathbb{R}_M \mid e \in F\}.
\]

Therefore, we seek conditions on a defining bundle \( \Delta \subseteq E^* \oplus \mathbb{R}_M \) that is flow-invariant under \( X_{\text{aff}} \) (meaning, by definition, that it is flow-invariant under \( \hat{X}_{\text{aff},*} \)) and satisfies

\[
\hat{\Lambda}(\Delta) \triangleq \Lambda(\Delta) \cap (E \times \{1\}) \subseteq \hat{F}.
\]

The following result gives conditions to this end, recalling from (10) the definition of \( \Delta_1 \simeq \text{pr}_1(\Delta) \).

**Theorem 4.25** (Defining subbundles invariant under an affine vector field and annihilating a cogeneralised subbundle). Let \( r \in \{\infty, \omega\} \), let \( \pi: E \to M \) be a \( C^r \)-vector bundle, let \( \nabla \) be a \( C^r \)-linear connection in \( E \), let \( F \subseteq E \) be a \( C^r \)-cogeneralised subbundle, let \( \Delta \) be a \( C^r \)-defining subbundle, let \( X_0 \in \Gamma^r(M) \), let \( b \in \Gamma^r(E) \), and let \( A \in \Gamma^r(\text{End}(E)) \). Denote

\[
X_{\text{aff}} = X_0^h + A^e + b^v
\]

and suppose that \( \Delta \) is flow-invariant under \( X_{\text{aff}} \). Consider the following statements:

(i) \( \Lambda(\Delta) \subseteq \hat{F} \);

(ii) the following conditions hold:

(a) \( A(\Lambda(\Delta_{1,x})) \subseteq F_x \) for \( x \in M \);
Proof. Let us first explore the linear algebra associated with the objects in the statement of the theorem.

**Lemma 1.** The following statements hold:

1. \( \tilde{F} = \{(e, a) \in E \oplus R_M \mid F(\lambda,a)(e,a) = 0, \lambda \in \Gamma^r(\Lambda(F)), g \in C^r(M) \} \), where
   \[ F(\lambda,a) = (\lambda,g)^{e}\pi^*g, \lambda \in \Gamma^r(E^*), g \in C^r(M), \]
   and where \( \pi : E \oplus R_M \to M \) is the vector bundle projection;
2. \( L(\Lambda(\tilde{\Delta})) = \Lambda(\Delta) \cap (E \oplus 0) = \Lambda(\Delta_1) \oplus 0; \)
3. \( \Lambda(L(\tilde{\Delta})) = \Delta + (0 \oplus R_M) = \Delta_1 \oplus R_M; \)
4. condition (i) of the theorem holds if and only if \( L(\Lambda(\tilde{\Delta})) \subseteq F \oplus 0. \)

**Proof.**

(i) We note that
   \[ \tilde{F} = (0,1) + F \oplus 0 \subseteq E \oplus R_M. \]

This part of the result then follows from Lemma 2.26.

(ii) We have
   \[ \Lambda(\tilde{\Delta}) \subseteq E \times \{1\} \]
   and
   \[ L(E \times \{1\}) = E \oplus 0. \]

Therefore,
   \[ L(\Lambda(\tilde{\Delta})) \subseteq E \oplus 0. \]

Now suppose that \( (e, 0) \in L(\Lambda(\tilde{\Delta})) \). Then
   \[ (e,0) + (e',1) = (e+e',1) \in \Lambda(\Delta), \quad (e',1) \in \Lambda(\Delta). \]

Therefore, since \( \Lambda(\tilde{\Delta}) \subseteq \Lambda(\Delta) \),
   \[ (e,0) = (e+e',1) - (e',1) \in \Lambda(\Delta). \]

Thus \( L(\Lambda(\tilde{\Delta})) \subseteq \Lambda(\Delta) \cap (E \oplus 0) \). Conversely, suppose that \( (e,0) \in \Lambda(\Delta) \). Then, for every \( (e',1) \in \Lambda(\tilde{\Delta}) \),
   \[ (e',1) + (e,0) = (e'+e,1) \in \Lambda(\Delta), \]

again since \( \Lambda(\tilde{\Delta}) \subseteq \Delta(\lambda) \). Thus we have \( \Lambda(\Delta) \cap (E \oplus 0) \subseteq L(\Lambda(\tilde{\Delta})). \)

Now let us show that \( \Lambda(\Delta) \cap (E \oplus 0) = \Lambda(\Delta_1) \). To do so, let us denote by
   \[ pr_1 : E^* \oplus R_M \to E^* \]
the projection and by
   \[ i_1 : E \to E \oplus R_M \]
the inclusion. Note that \( \Delta_1 = pr_1(\Delta) \) and that \( i_1^* = pr_1 \).

Let \( e \in \Lambda(\Delta_1) \). Then \( (e,0) \in E \oplus 0 \), obviously. Also, if \( (\lambda,a) \in \Delta \), then
   \[ \langle (\lambda,a); (e,0) \rangle = \langle (\lambda,a); i_1(e) \rangle = \langle pr_1(\lambda,a); e \rangle = 0, \]
and so \( (e,0) \in \Lambda(\Delta) \). Thus \( \Lambda(\Delta_1) \subseteq \Lambda(\Delta) \cap (E \oplus 0) \).

Next let \( (e,0) \in \Lambda(\Delta) \cap (E \oplus 0) \). Let \( (\lambda,a) \in \Delta \) so that \( \lambda = pr_1(\lambda,a) \in \Delta_1 \). Then
   \[ \langle (\lambda,0); (e,0) \rangle = \langle pr_1(\lambda,a); e \rangle = \langle (\lambda,a); i_1(e) \rangle = \langle (\lambda,a); (e,0) \rangle = 0, \]
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and so $\Lambda(\Delta) \cap (E \oplus 0) \subseteq \Lambda(\Delta_1) \oplus 0$.

(iii) We have

\[
\Lambda(L(\widehat{\Lambda}(\Delta))) = \Lambda(\Lambda(\Delta) \cap (E \oplus 0)) \\
= \Lambda(\Lambda(\Delta)) + \Lambda(E \oplus 0) \\
= \Delta + (0 \oplus R_M).
\]

By (ii) we also have $\Lambda(L(\widehat{\Lambda}(\Delta))) = \Lambda(\Delta_1) \oplus R_M$.

(iv) As in part (i) of the lemma, the condition (i) of the theorem is equivalent to

\[
\langle \lambda(\pi(e)); e \rangle + (a - 1)g(\pi(e)) = 0, \quad (e, a) \in \widehat{\Lambda}(\Delta_x), \quad \lambda \in \Gamma^r(\Lambda(F)), \quad g \in C^r(M).
\]

By letting $\lambda = 0$ and $g$ be arbitrary, we see that this implies that $a = 1$ (of course). Thus we arrive at the conclusion that condition (i) is equivalent to

\[
(e, 1) \in \widehat{\Lambda}(\Delta_x) \implies e \in F,
\]
as claimed. \n
\[\n\]

Note that $\Lambda(\Delta)$ is flow-invariant under $\widehat{X}^\text{aff}$ by Lemma 4.7, since $\Delta$ is flow-invariant under $\widehat{X}^\text{aff}$, and $E \times \{1\}$ is flow-invariant under $\widehat{X}^\text{aff}$ (by Lemma 4.18(iii), taking $\Delta = \{0\}$ and so $\Lambda(\Delta) = E$, we conclude that the cogeneralised affine subbundle $\Lambda(\Delta)$ is flow-invariant under $\widehat{X}^\text{aff}$, being the intersection of two flow-invariant sets. Thus, by Proposition 4.13 (specialised to linear vector fields), we have

1. $\widehat{A}(\Lambda(F)) \subseteq L(\widehat{\Lambda}(\Delta))$,
2. $\widehat{\nabla}_{X_0}(0, 1) - (\widehat{A}, b)(0, 1) \in L(\widehat{\Lambda}(\Delta))$, and
3. $\widehat{\nabla}_{X_0}(\zeta_{L(\widehat{\Lambda}(\Delta))}) \subseteq \zeta_{L(\widehat{\Lambda}(\Delta))}$.

With the preceding observations in place, we can prove the theorem.

(i) $\implies$ (ii) Condition 1 and part (iv) of the lemma immediately gives

\[
(A, b)(L(\widehat{\Lambda}(\Delta))) \subseteq L(\widehat{\Lambda}(\Delta)) \subseteq F \oplus 0.
\]

Bearing in mind part (i) of the lemma, we have

\[
\widehat{(A, b)}(e, 0) = (A(e), 0).
\]

and so (iia) of the theorem holds, keeping in mind part (ii) of the lemma.

Now note that $\widehat{\nabla}_{X_0}(0, 1) = 0$ since $\widehat{\nabla}_{X_0}$ acts by Lie differentiation in the second component. Thus 2 gives

\[
\widehat{(A, b)}(0, 1) = (b, 0) \in L(\widehat{\Lambda}(\Delta)),
\]

which gives (iib) of the theorem, keeping in mind part (iv) of the lemma.

Finally, we have

\[
L(\widehat{\Lambda}(\Delta)) \subseteq F \oplus 0 \implies \Lambda(F) \oplus 0 = \Lambda(F) \oplus R \subseteq \Lambda(L(\widehat{\Lambda}(\Delta)))
\]

\[
\implies \text{pr}_1(\Lambda(F) \oplus R) \subseteq \text{pr}_1(\Lambda(L(\widehat{\Lambda}(\Delta))))
\]

\[
\implies \Lambda(F) \subseteq \Delta_1,
\]

by part (iii) of the lemma and noting that $\Delta_1 = \text{pr}_1(\Delta)$. By 3, we have

\[
\widehat{\nabla}_{X_0}(\zeta_{\Lambda(F) \oplus R}) \subseteq \widehat{\nabla}_{X_0}(\zeta_{\Lambda(L(\widehat{\Lambda}(\Delta)))}) \subseteq \zeta_{\Lambda(L(\widehat{\Lambda}(\Delta)))}.
\]
Now let \( \mathcal{U} \subseteq \mathcal{M} \) be open, and let \( \lambda \in \mathcal{G}^r_{\mathcal{L}(\mathcal{F})}(\mathcal{U}) \) and \( g \in \mathcal{G}^r_{\mathcal{M}}(\mathcal{U}) \) and compute

\[
\tilde{\nabla}_{X_0}(\lambda, g) = (\nabla_{X_0}\lambda, \mathcal{L}_{X_0}g) \in \mathcal{G}^r_{\mathcal{L}(\mathcal{L}(\Delta))}(\mathcal{U}).
\]

Applying \( \text{pr}_1 \) to this inclusion gives \( \nabla_{X_0}\lambda \in \mathcal{G}^r_{\Delta_1}(\mathcal{U}) \), which is part (iiic) of the theorem.

(ii) \( \implies \) (i) As in part (i) of the lemma, we can differentiate as in Lemma 4.16 to get

\[
\mathcal{L}_{\tilde{X}^{\text{aff}}}(\tilde{F}_{\lambda,g} - \hat{\pi}_*g)(e,a) = (\nabla_{X_0}\lambda(\pi(e)); e) + (\lambda(\pi(e)); A(e)) + a(dg(\pi(e)) ; X_0(\pi(e))) + a(\lambda(\pi(e)) ; b(\pi(e))) - (dg(\pi(e)) ; X_0(\pi(e))).
\]

We consider this formula with \( (e,a) = (e,1) \in \Lambda(\Delta) \), with \( \lambda \in \Gamma^r(\Lambda(\mathcal{F})) \), and with the active hypotheses. Since \( (e,0) \in L(\Lambda(\Delta)) \) by part (ii) of the lemma, we have \( A(e) \in \mathcal{F} \) by (iia) of the theorem and part (ii) of the lemma, and so

\[
(\lambda(\pi(e)); A(e)) = 0.
\]

By (iib) of the theorem, we have \( b(\pi(e)) \in \mathcal{F} \), and so

\[
(\lambda(\pi(e)); b(\pi(e))) = 0.
\]

By (iiic) of the theorem we have

\[
\nabla_{X_0}\lambda \in \Gamma^r(\Delta_1).
\]

Therefore,

\[
(\nabla_{X_0}\lambda, 0) \in \Gamma^r(\Delta_1 + \mathbb{R}) = \Gamma^r(\Delta + (0 + \mathbb{R})) = \Gamma^r(\Lambda(\Lambda(\Delta))),
\]

by part (iii) of the lemma. Thus

\[
(\nabla_{X_0}\lambda; e) = (\tilde{\nabla}_{X_0}(\lambda, 0); (e,0)) = 0
\]

by part (ii) of the lemma. Thus, for \( (e,1) \in \Lambda(\Delta) \), we have

\[
\mathcal{L}_{\tilde{X}^{\text{aff}}}(\tilde{F}_{\lambda,g} - \hat{\pi}_*g)(e,1) = 0, \quad \lambda \in \Gamma^r(\Lambda(\mathcal{F})), \quad g \in C^r(\mathcal{M}).
\]

We conclude, from Proposition 4.13, that, when \( r = \omega \) or when \( r = \infty \) and \( \mathcal{F} \) is a subbundle, that all integral curves of \( \tilde{X}^{\text{aff}} \) with initial conditions in \( \Lambda(\Delta) \) remain in \( \tilde{\mathcal{F}} \). Since \( \Lambda(\Delta) \) is flow-invariant under \( \tilde{X}^{\text{aff}} \) (as we pointed out in the preamble to the proof), this implies (i). \( \square \)

One can combine the previous results with Proposition 4.13 to obtain the following procedure for finding invariant affine subbundles contained in a given subbundle.

We first consider the linear case.

Remark 4.26 (Finding invariant cogeneralised subbundles contained in a cogeneralised subbundle). Let \( r \in \{\infty, \omega\} \), let \( \pi : \mathcal{E} \to \mathcal{M} \) be a \( C^r \)-vector bundle, let \( \nabla \) be a \( C^r \)-linear connection in \( \mathcal{E} \), let \( \mathcal{F} \subseteq \mathcal{E} \) be a \( C^r \)-cogeneralised subbundle, let \( X_0 \in \Gamma^r(\mathcal{M}) \) be complete, and let \( A \in \Gamma^r(\text{End}(\mathcal{E})) \). Denote

\[
X^{\text{lin}} = X_0^h + A^r.
\]

Find a flow-invariant cogeneralised subbundle \( \mathcal{L} \subseteq \mathcal{F} \) satisfying the following algebraic/differential conditions:

1. \( A(\mathcal{L}) \subseteq \mathcal{F}; \)
2. \( \nabla_{X_0}(\mathcal{G}^r_{\mathcal{L}(\mathcal{F})}) \subseteq \mathcal{G}^r_{\mathcal{L}(\mathcal{L})}. \)
We shall say that \( L \) satisfying these conditions is \((X^{\text{lin}}, F)\)-admissible. The resulting cogeneralised subbundle \( L \) is then flow-invariant under \( X^{\text{lin}} \) and is contained in \( F \).

In the affine case, we have the following.

**Remark 4.27** (Finding invariant affine subbundle varieties contained in a cogeneralised subbundle). Let \( r \in \{\infty, \omega\} \), let \( \pi: E \to M \) be a \( C^r \)-vector bundle, let \( \nabla \) be a \( C^r \)-linear connection in \( E \), let \( F \subseteq E \) be a \( C^r \)-cogeneralised subbundle, let \( X_0 \in \Gamma^r(M) \) be complete, let \( b \in \Gamma^r(E) \), and let \( A \in \Gamma^r(\text{End}(E)) \). Denote

\[
X^{\text{aff}} = X_0^{\text{h}} + A^v + b^v.
\]

Find a flow-invariant defining subbundle \( \Delta \subseteq E^* \oplus \mathbb{R}_M \) satisfying the following algebraic/differential conditions:

1. \( A(\Lambda(\Delta_{1,x})) \subseteq F_x \) for \( x \in M \);
2. \( b(x) \in F_x \) for \( x \in M \);
3. \( \nabla_{X_0}(G_{\Lambda(F)}) \subseteq G_{\Delta} \).

We shall say that \( \Delta \) satisfying these conditions is \((X^{\text{aff}}, F)\)-admissible. Having found such a \( \Delta \), check the following:

4. the set \( S(\Lambda(\Delta)) = \{ x \in M \mid (0, 1) \notin \Delta_x \} \) is nonempty.

The resulting affine subbundle variety \( \Lambda(\Delta) \) is then flow-invariant under \( X^{\text{aff}} \) and is contained in \( F \).

The methodology outlined in the preceding constructions involve some interesting partial differential equations with algebraic constraints. With some effort, it might be possible to apply the integrability theory for partial differential equations [23, 24] to arrive at the obstructions to solving these equations. An application of the resulting conditions to the setup of Section 7 would doubtless lead to some interesting answers to the central questions of this paper.

**REFERENCES**

[1] Andrew D. Lewis, Nonholonomic and constrained variational mechanics, *Journal of Geometric Mechanics*, 12 (2020), 165–308.

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E-mail address: andrew.lewis@queensu.ca