Difference Sets Disjoint from a Subgroup II: Groups of Order $4p^2$

Stephen P. Humphries$^1$ · Nathan L. Nicholson$^1$

Received: 27 January 2020 / Revised: 2 June 2021 / Accepted: 8 June 2021 / Published online: 26 June 2021
© The Author(s), under exclusive licence to Springer Japan KK, part of Springer Nature 2021

Abstract
We study finite groups $G$ having a normal subgroup $H$ and $D \subseteq G \setminus H, D \cap D^{-1} = \emptyset$, such that the multiset $\{xy^{-1} : x, y \in D\}$ has every non-identity element occur the same number of times (such a $D$ is called a \textit{DRAD difference set}). We show that there are no such groups of order $4p^2$, where $p$ is an odd prime.

Keywords Difference set · Subgroup · DRAD

Mathematics Subject Classification Primary 05B10 · Secondary 20C05

1 Introduction

For a group $G$ we will identify a finite subset $X \subseteq G$ with the element $\sum_{x \in X} x \in \mathbb{Q}G$ of the group algebra. We also let $X^{-1} = \{x^{-1} : x \in X\}$. Write $C_n$ for the cyclic group of order $n$.

A $(v,k,\lambda)$ difference set is a subset $D \subseteq G, |D| = k$, such that every element $1 \neq g \in G$ occurs $\lambda$ times in the multiset $\{xy^{-1} : x, y \in D\}$. Here $|G| = v$.

Then a $(v,k,\lambda)$ difference set is a $(v,k,\lambda)$ \textit{DRAD difference set (with subgroup $H$ and difference set $D$)} if it also satisfies the conditions: there is a subgroup $1 \neq H \triangleleft G$ such that

1. $D \cap D^{-1} = \emptyset$;
2. $G \setminus (D \cup D^{-1}) = H$.

$^1$ Department of Mathematics, Brigham Young University, Provo, UT 84602, USA
A group $G$ will be called a **DRAD difference set group** if there is a DRAD difference set over $G$. See [5, 8, 9] for more on DRADs. DRAD difference sets are examples of Hadamard (or Menon) difference sets [4]. Let

$$h = |H|, \quad u = |G : H|. \quad \text{(1)}$$

We proved the following result in [6]:

**Theorem 1.1** Let $G$ be a $(v, k, \lambda)$ DRAD difference set group with subgroup $H$ and difference set $D$. Then

(i) $u = h \geq 4$ is even, $v = |G| = h^2$, and

$$\lambda = \frac{1}{4} h(h-2), \quad k = \frac{1}{2} h(h-1); \quad \text{(2)}$$

(ii) each non-trivial coset $Hg \neq H$ meets $D$ in $h/2$ points;

(iii) $H$ contains the subgroup generated by all the involutions in $G$;

(iv) any abelian $(v, k, \lambda)$ DRAD difference set group is a 2-group.

All known DRAD difference set groups are 2-groups. There is one such group of order 16 and at least 33 of order 64. In [6] a bi-infinite family of DRAD groups is constructed. In this paper we show

**Theorem 1.2** There are no $(v, k, \lambda)$ DRAD difference set groups of order $4p^2$, for an odd prime $p$.

Theorem 1.2 is related to Question 9 of [4], where Davis and Jedwab ask if there are any Hadamard difference sets of order $4p^2$, where $p > 5$ is an odd prime. Note that when $h = 2p$, then by Theorem 1.1 we have $(v, k, \lambda) = (4p^2, 2p^2 - p, p^2 - p)$, the parameters for a Hadamard difference set.

For the proof of Theorem 1.2 we make use of a result of Iiams [7], who showed that any group of order $4p^2$ (where $p > 3$ is a prime) that has $C_p \times C_2^2$ as a factor group, does not have a $(4p^2, 2p^2 - p, p^2 - p)$ difference set. See Theorem 2.1 below for a full statement of the result of Iiams. This result was extended by Wan [17] to show that any group of order $4p^4$ that has $C_{p^2} \times C_2^2$ as a factor group does not have a $(4p^4, 2p^4 - p^2, p^4 - p^2)$ difference set. Lastly, the authors of [1] show that if $q = p^n, p > 3$ a prime, where $G$ has $C_q \times C_2^2$ as a factor group, then $G$ does not admit a $(4q^2, 2q^2 - q, q^2 - q)$ difference set. We note that Smith [15] was the first to find non-abelian examples with these parameters, and that McFarland [14] shows that no nontrivial difference set exists in abelian groups of order $4p^2$ where $p > 3$ is a prime. We also note that Davis and Polhill construct an infinite family for abelian groups [5].

### 2 The Result of Iiams and the Cases $G_4, G_{13}, G_{16}$

The result of Iiams [7] that we use is
Theorem 2.1 Let $p \geq 5$ be prime, and let $G$ be a group of order $4p^2$ that contains a Menon-type difference set. Then either

(i) $p \equiv 1 \mod 4$ and $G \cong G_{11}$;

(ii) $G$ is isomorphic to one of $G_4, G_{13}, G_{14}, G_{15}, G_{16}$.

Here the groups $G_{11}, G_4, G_{13}, G_{14}, G_{15}, G_{16}$ are defined below. We now consider these six cases. (In general we have replaced $z$ by $z^{-1}$ in the presentations that liams gives, thus allowing conjugation to be written $y^z = z^{-1}yz$.)

Case $G_4$: Let $p$ be an odd prime and let $f \in \mathbb{N}$ such that $f^2 \equiv -1 \mod p^2$. Then $G_4$ has the following presentation:

$$G_4 = \langle x, z | x^{p^2}, z^4, x^z = x^f \rangle.$$ 

Note that $x^z = x^{-1}$, which gives $(z^2)^x = x^{-2}z^2$, and so $(z^2)^{x^k} = x^{-2k}z^2$.

Thus each $x^{-2k}z^2$ has order 2, and since $x$ has order $p^2$ (with $p$ odd) we see that all these elements of order 2 generate the subgroup $\langle x, z^2 \rangle$ of order $|G_4|/2 = h^2/2$. By Theorem 1.1 we have $|H| = h$ and $H$ has to contain this subgroup, so we have $h^2/2 \leq |H| = h$, a contradiction to Theorem 1.1 (i).

Case $G_{13}$: Here $G_{13}$ has presentation:

$$G_{13} = \langle x, y, z | x^p, y^p, (x, y), z^4, x^z = y^{-1}, y^z = x \rangle.$$ 

Again we have $x^z = x^{-1}, y^z = y^{-1}$, which gives $(z^2x^ay^b)^2 = 1$ for $0 < a, b < p$. Thus, as in the $G_4$ case, the subgroup $\langle z^2x^ay^b : 0 \leq a, b < p \rangle \leq G_{13}$ generated by the involutions, has index 2. This rules out this case.

Case $G_{16}$: Let $f \in \mathbb{N}$ such that $f^2 \equiv -1 \mod p$. Then

$$G_{16} = \langle x, y, z | x^p, y^p, (x, y), z^4, x^z = x^f, y^z = y^f \rangle.$$ 

Again we have $x^z = x^{-1}, y^z = y^{-1}$, and so, as in the $G_{13}$ case, the subgroup of $G_{16}$ generated by the involutions has index 2, and we obtain a contradiction.

3 The Groups $G_{11}$ and $G_{14}$

For these cases we will need the following Lemma:

Lemma 3.1 Suppose that $G$ has a non-principal linear character $\chi$.

If $\chi(H) = 0$ and $\chi$ takes values in a field $K$ where $i = \sqrt{-1} \not\in K$, then $G$ is not a DRAD group with subgroup $H$.

Proof Since $\chi$ is linear and non-principal we have $\chi(G) = 0$. By hypothesis we have $\chi(H) = 0$, so that $G = H + D + D^{-1}$ gives $\chi(D^{-1}) = -\chi(D)$. Then

$$DD^{-1} = \lambda G + (k - \lambda) = \lambda G + \frac{1}{4}h^2$$

and the linearity of $\chi$ gives $\chi(D)^2 = -\frac{1}{4}h^2$, which gives $\chi(D) = \pm i\frac{h}{2} \in K$, a contradiction. □
Case $G_{11}$: Let $p$ be a prime where $p \equiv 1 \mod 4$. Then
\[ G = G_{11} = \langle x, y, z | x^p, y^p, (x, y), z^4, x^z = x, y^z = y^{-1} \rangle \cong \langle x \rangle \times \langle y, z \rangle. \]

One checks that there are two possibilities for $H$: $Z = Z(G) = \langle x, z^2 \rangle$ and $Y = \langle y, z^2 \rangle$. Note that $Z, Y \trianglelefteq G$.

Let $\zeta_p = \exp 2\pi i/p$.

First consider $H = Z = \langle x, z^2 \rangle$ and let $\chi$ be the linear character
\[ \chi(x) = \zeta_p, \quad \chi(y) = 1, \quad \chi(z) = -1. \quad (3.1) \]

Then $\chi(H) = 0$ (since $x \in H$ and $\chi(x) \neq 1$) and $\chi(g) \in \mathbb{Q}(\zeta_p)$ for all $g \in G$, from which Lemma 3.1 shows that this case cannot happen, since $i \not\in \mathbb{Q}(\zeta_p)$.

So now consider $H = Y = \langle y, z^2 \rangle \trianglelefteq G$.

Elements of $G$ will have normal form $g_{j,k,w} = x^j y^k z^w$, $0 \leq j, k < p$, $0 \leq w < 4$. Here elements of $H$ have the form $g_{0,k,w} = y^k z^w$, $0 \leq k < p, w \in \{0, 2\}$. Now we let
\[ D = \sum_{j,k,w} \varepsilon_{j,k,w} g_{j,k,w}, \]
where $\varepsilon_{j,k,w} \in \{0, 1\}$, and $\sum_{j,k,w}$ means we sum over $0 \leq j, k < p, w \in \{0, 1, 2, 3\}$.

For some fixed $m \in \mathbb{Z}$ we consider the linear character
\[ \chi_m : x \mapsto \zeta_p^m, \quad y \mapsto 1, \quad z \mapsto i = \sqrt{-1}. \quad (3.2) \]

Then we have $\chi_m(H) = 0$, since $z^2 \in H$ and $\chi_m(z^2) = -1$. We also have $\chi_m(G) = 0$. Thus $D + D^{-1} = G - H$ gives $\chi_m(D^{-1}) = -\chi_m(D)$ and so $DD^{-1} = \lambda G + (k - \lambda)$ gives
\[ -\chi_m(D)^2 = k - \lambda = \frac{1}{4} h^2 = p^2. \]

So we have $\chi_m(D) = \delta_m p i$, for some $\delta_m \in \{ \pm 1 \}$. Thus
\[ \delta_m p i = \chi_m(D) = \chi_m \left( \sum_{j,k,w} \varepsilon_{j,k,w} x^j y^k z^w \right) \]
\[ = \sum_{j,k,w} \varepsilon_{j,k,w} \chi_m(x^j y^k z^w) = \sum_{j,k,w} \varepsilon_{j,k,w} \zeta_p^{jm} t^w. \quad (3.3) \]

We write (3.3) as follows, splitting off the $j = 0$ part:
\[ \delta_m p i = \sum_{k,w} \varepsilon_{0,k,w} t^w + \sum_{j=1}^{p-1} \sum_{k,w} \varepsilon_{j,k,w} \zeta_p^{jm} t^w. \quad (3.4) \]

Summing over $0 \leq m < p$ we obtain
\[ p 
abla \sum_{m=0}^{p-1} \delta_m = p \sum_{k,w} \epsilon_{0,k,w} \lambda^w + \sum_{j=1}^{p-1} \sum_{k,w} \epsilon_{j,k,w} \lambda^w \sum_{m=0}^{p-1} r_m \psi_p = p \sum_{k,w} \epsilon_{0,k,w} \lambda^w. \]

Thus

\[ \sum_{k,w} \epsilon_{0,k,w} \lambda^w = i \sum_{m=0}^{p-1} \delta_m. \]

Substituting this value of \( \sum_{k,w} \epsilon_{0,k,w} \lambda^w \) in (3.4) we get

\[ \delta_m p i = i \sum_{h=0}^{p-1} \delta_h + \sum_{j=1}^{p-1} \sum_{k,w} \epsilon_{j,k,w} \psi_{p,w} \lambda^w, \]

or

\[ 0 = \left( -\delta_m p + \sum_{h=0}^{p-1} \delta_h \right) + \sum_{j=1}^{p-1} \sum_{k,w} \epsilon_{j,k,w} \psi_{p,w} \lambda^{w-1}. \] (3.5)

Now (since \( p \) is an odd prime) we have \( \mathbb{Q}(i) \cap \mathbb{Q}(\zeta_p) = \mathbb{Q} \), and looking at the part of (3.5) with \( w = 0, 2 \), gives

\[ 0 = \sum_{j=1}^{p-1} \sum_k (\epsilon_{j,k,0} - \epsilon_{j,k,2}) \psi_{p,w}. \] (3.6)

Now we use the fact that for \( a_u \in \mathbb{Q}, 0 \leq u < p \), if \( \sum_{u=0}^{p-1} a_u \psi_p = 0 \), then \( a_0 = a_1 = \cdots = a_{p-1} \). Thus (3.6) gives

\[ 0 = \sum_k (\epsilon_{j,k,0} - \epsilon_{j,k,2}), \quad \text{for all } j > 0. \]

Now from Theorem 1.1 we have \( |D \cap \chi H| = h/2 = p \) for all \( 0 < j < p \). Thus for \( j \neq 0 \) we have

\[ \sum_k \epsilon_{j,k,0} + \epsilon_{j,k,2} = p. \]

But from the above we have: \( \sum_k (\epsilon_{j,k,0} - \epsilon_{j,k,2}) = 0 \), and adding we get

\[ 2 \sum_k \epsilon_{j,k,0} = p, \]

a contradiction since \( p \) is odd.

\textbf{Case G}_{14}: Let \( f \in \mathbb{N} \) satisfy \( f^2 \equiv -1 \mod p \). Then

\[ G = G_{14} = \langle x, y, z | x^p, y^p, (x, y), z^4, x^2 = x, y^2 = y \rangle. \]

Thus \( G = \langle x \rangle \times \langle y, z \rangle \). Now there are \( p \) elements of order 2, namely \( (z^2)^{u} \) and they generate the normal subgroup \( \langle y, z^2 \rangle \) of order \( 2p \), which must then be \( H \). We also
have the linear character $\chi_m$ as in (3.1). Then, using this character, the same argument as in the $G_{11}$ case gives a contradiction. Thus $G_{14}$ is not a DRAD group. □

4 The Group $G_{15}$

Let $f \in \mathbb{N}$ satisfy $f^2 \equiv -1 \mod p$. Since $p$ is prime, this implies that $p \equiv 1 \mod 4$.

Then

$$G = G_{15} = \langle x, y, z \mid x^p, y^p, (x, y), z^4, x^z = x^{-1}, y^z = y^i \rangle.$$  

Here $G = \langle x \rangle \times \langle y, z \rangle$. Now there are $p$ involutions $(z^2)^i, 0 \leq i < p$, and they generate $\langle y, z^2 \rangle \cong D_{2p}$, so we must have $H = \langle y, z^2 \rangle \cong D_{2p}, H \triangleleft G$.

This group does not succumb to any of the character-theoretic arguments that have worked in previous cases.

The proof will be by contradiction, so that we will assume that we have such a difference set $D \subset G$, where we write

$$D = \sum_{g \in G} \varepsilon_g g \in \mathbb{Z}[G],$$

with $\varepsilon_g \in \{0, 1\}$ and $\varepsilon_g = 0$ if $g \in H$. Then we have

$$G = H + D + D^{-1}, \quad D \cap D^{-1} = D \cap H = \emptyset, \quad (4.1)$$

and

$$D^{-1} = \sum_{g \in G} \varepsilon_g g^{-1}.$$  

The set-up for the proof of this case will then be as follows: we assume that

$$D = \sum_{g \in G} \varepsilon_g g,$$

where we can now think of the $\varepsilon_g$s as polynomial indeterminates generating the polynomial ring $R = \mathbb{Z}[\varepsilon_g]_{g \in G}$. Then $D$ and $D^{-1}$ will belong to the group ring $R[G]$.

From (4.1) we see that the $\varepsilon_g$s satisfy

$$\varepsilon_g^2 = \varepsilon_g, \quad \varepsilon_g + \varepsilon_g^{-1} = 1, \quad \text{for } g \not\in H, \quad \varepsilon_g = 0 \text{ for } g \in H. \quad (4.2)$$

Let $I$ denote the ideal of $R = \mathbb{Z}[\varepsilon_g]_{g \in G}$ generated by the relations in (4.2) and $2\mathbb{Z}$.

We now consider the relations that we obtain from the difference set equation. Ultimately we will consider these relations mod $I$. Let

$$E = DD^{-1} - (\lambda(G - 1) + k) \in RG, \quad (4.3)$$

and for $k \in G$ let $E_k \in R$ denote the coefficient of $k$ in $E$. Since
\[ E = DD^{-1} - (\lambda(G - 1) + k) = \sum_{g \in G} e_g g \sum_{h \in G} e_h h^{-1} - (\lambda(G - 1) + k), \]

we see that for \( k \in G, k \neq 1 \), we have (from (4.3)):

\[ E_k = \sum_{h \in G} e_{kh} e_h - \lambda. \quad (4.4) \]

Using the relations in \( G \) we see that a normal form for elements of \( G \) is \( g = x^i y^j z^k \), where \( 0 \leq i, j < p, 0 \leq k < 4 \).

In what follows we will need the following group-theoretic facts about \( G = \langle x, y, z \rangle \):

\[
\begin{align*}
(x^i y^j)^{-1} &= x^{-i} y^{-j}, \\
(x^i y^j z)^{-1} &= x^{-i} y^{-j} z^3,
\end{align*}
\]

From now on most of the equations that we write down will be considered to be in the quotient ring \( R = \mathbb{Z}[e_g]_{g \in G}/\mathcal{I} \). We will sometimes indicate this by writing \( \text{mod } \mathcal{I} \) at the end of the equation.

Thus, for example, we know from (4.4) that, since \( \lambda = p^2 - p \) is even, we have

\[ E_k = \sum_{h \in G} e_{kh} e_h \text{ mod } \mathcal{I} \]

for \( k \in G, k \neq 1 \).

Define subgroups

\[ Y = \langle y \rangle, \quad N = \langle x, y, z^2 \rangle, \quad A = \langle x, y \rangle, \]

so that \( Y, N, A \triangleleft G \).

Let \( p_2 = (p - 1)/2 \).

Now for \( k \in G \), summing the \( E_g \) over the elements of the coset \( Yk = kY \) we define:

\[ Z_k = \sum_{i=0}^{p-1} E_{y^i k} = \sum_{i=0}^{p-1} \sum_{h \in G} e_{y^i kh} e_h \in \mathbb{Z}[e_g]. \quad (4.5) \]

For \( g \in G \) we define

\[ \mathcal{Y}(g) = \{ y^i g : 0 \leq i < p \} = \{ gy^i : 0 \leq i < p \}. \]

We also define

\[ \Sigma \mathcal{Y}(g) = \sum_{u \in \mathcal{Y}(g)} e_u = \sum_{i=0}^{p-1} e_{y^i g}. \]

Let \( \mathcal{E} = \mathcal{E}_p \) be the ideal of \( \mathbb{Z}[e_g]_{g \in G} \) generated by \( \mathcal{I} \) and all the \( E_k, k \in G, k \neq 1 \). We note that if there is a DRAD difference set for \( G \), then the quotient ring \( R/\mathcal{E} \) would be non-trivial. Thus the following result will conclude the \( G_{15} \) case:
Theorem 4.1  As an equation in $\mathbb{Z}[\epsilon] / \mathcal{I}$ we have
\[ Z_x + Z_{xz^2} + Z_{x^{(p-1)/2}z^2} = 1 \mod \mathcal{I}. \]

Proof Define $Z_1, \ldots, Z_6 \in R$ as follows:
\[
\begin{align*}
Z_1 &= \sum_{i=0}^{p-1} \sum_{h \in G \setminus N} e^{y_i x h} \epsilon_h; \\
Z_2 &= \sum_{i=0}^{p-1} \sum_{h \in G \setminus N} e^{y_i x z^2 h} \epsilon_h; \\
Z_3 &= \sum_{i=0}^{p-1} \sum_{h \in G \setminus N} e^{y_i x^2 z^2 h} \epsilon_h; \\
Z_4 &= \sum_{i=0}^{p-1} \sum_{h \in N \setminus H} e^{y_i x h} \epsilon_h; \\
Z_5 &= \sum_{i=0}^{p-1} \sum_{h \in N \setminus H} e^{y_i x z^2 h} \epsilon_h; \\
Z_6 &= \sum_{i=0}^{p-1} \sum_{h \in N \setminus H} e^{y_i x^2 z^2 h} \epsilon_h.
\end{align*}
\]

Then we have:
\[
Z_x = Z_1 + Z_4, \quad Z_{xz^2} = Z_2 + Z_5, \quad Z_{x^{p-1/2}z^2} = Z_3 + Z_6,
\]
so that
\[ Z_x + Z_{xz^2} + Z_{x^{(p-1)/2}z^2} = \sum_{i=1}^{6} Z_i. \]

Thus we will now find $Z_1, \ldots, Z_6$, and Theorem 4.1 will follow by showing that $\sum_{i=1}^{6} Z_i = 1$. Our first goal is simply stated:

Lemma 4.1  $Z_1 = 1, \ Z_2 = 0, \ Z_3 = 0$.

Proof We partition $G \setminus N$ into $P \cup P^{-1}$ where
\[ P = Az = \{ x^a y^b z : 0 \leq a, b < p \}. \]

One can check that $\{P, P^{-1}\}$ is a partition of $G \setminus N$.

We consider the pairs $(y^i k h, h) \in (G \setminus N)^2$. We wish to pair up a pair $(y^i k h, h)$ (see the definition of $Z_k$ in (4.5) above to motivate this) with another pair of the form $(y^i k h_1, h_1)$, where $(y^i k h_1, h_1) = ((y^i k h)^{-1}, h^{-1})$. This second pair is called the dual
of \((y^j k, h)\). We now explain how to do this pairing for the situation where \(h \in G \setminus N\) and \(k \in \{x, xz^2, x^p z^2\}\), so that \(k\) has the form \(k = x^d z^{2d}\).

Suppose that \(h \in P \subset G \setminus N\), so that \(h = x^j z^{2d}\). Then for \(k = x^d z^{2d}\) we want the pair dual to \((y^j k, h)\) to be \(((y^j k)^{-1}, h^{-1})\); but we want it to have the correct form i.e. to also look like \((y^j k_1, h_1)\) for some \(j\) and some \(h_1 \in G \setminus N\). Solving \(((y^j k)^{-1}, h^{-1}) = (y^j k_1, h_1)\) we get \(h_1 = h^{-1}\) and \((y^j k)^{-1} = y^j k_1\), and so we need to find \(j\) so that \(h^{-1} k^{-1} y^{-i} \cdot h_1^{-1} k^{-1} = y^j\) (if possible). Well from these equations we have

\[
\begin{align*}
    h^{-1} k^{-1} y^{-i} \cdot h_1^{-1} k^{-1} &= h^{-1} k^{-1} y^{-i} h k^{-1} \\
    &= z^{-1} x^{-a} y^{-b} \cdot x^{-c} z^{2d} \cdot y^{-i} \cdot x^d y^b z \cdot x^{-c} z^{2d} \\
    &= z^{-1} y^{-b} \cdot z^{2d} \cdot y^{-i} \cdot y^b z \cdot z^{2d} \\
    &= z^{-1} y^{-b} \cdot z^{2d} \cdot y^{-i} \cdot y^b \cdot z^{2d} \in Y.
\end{align*}
\]

Thus for \(h \in P\) and \(k = x^d z^{2d}\), with the value of \(j\) determined by (4.6), we pair \((y^j k, h)\) with \(((y^j k)^{-1}, h^{-1}) = (y^j k_1, h_1)\).

For \(h \in P \subset G \setminus N, k \in \{x, xz^2, x^p z^2\}\), then, as described above, we pair \((y^j k, h)\) with \(((y^j k)^{-1}, h^{-1})\). The point here is that the part of the sum for \(Z_k\) coming from the pairs \((y^j k, h)\) (for \(h \in P\) and summing over \(0 \leq i < p\)) is

\[
\sum_{i=0}^{p-1} \epsilon_{y^j k^i} \epsilon_h, \tag{4.7}
\]

while the sum coming from the dual pairs \(((y^j k)^{-1}, h^{-1})\) (where \(h^{-1} \in P^{-1}\)) is

\[
\sum_{i=0}^{p-1} \epsilon_{(y^j k)^{-1}} \epsilon_h^{-1} = \sum_{i=0}^{p-1} (1 - \epsilon_{y^j k^i}) (1 - \epsilon_h). \tag{4.8}
\]

Adding (4.7) and (4.8) we get (considering these elements modulo the ideal \(I\))

\[
\sum_{i=0}^{p-1} (1 + \epsilon_{y^j k^i} + \epsilon_h), \text{ which is equal to } 1 + \sum_{i=0}^{p-1} (\epsilon_{y^j k^i} + \epsilon_h). \tag{4.9}
\]

Now we add (4.9) over all \(h = x^a y^b z \in P\) to get (since \(p\) is odd)

\[
\sum_{a, b=0}^{p-1} \left(1 + \sum_{i=0}^{p-1} (\epsilon_{y^j k^i} + \epsilon_{y^j k^i}) + \epsilon_{y^j k^i} + \epsilon_{y^j k^i}) \mod I. \tag{4.10}
\]

Now if \(k = x\), then the sum (4.10) is \(Z_1\); but since, with \(k = x\), we have
\[ \sum_{a,b=0}^{p-1} (e_{y'k} e_{y'z} + e_{y'z}) = \sum_{a,b=0}^{p-1} (e_{y'k} e_{y'z} + e_{y'z}) = 0 \mod I, \]

we see that \( Z_1 = 1 \).

On the other hand, if \( k = x^2 z^2, c \in \{1, p_2\} \), then for (4.10) we get

\[ 1 + \sum_{i=0}^{p-1} \sum_{a,b=0}^{p-1} (e_{y'k} e_{y'z} + e_{y'z}) = 1 + \sum_{i=0}^{p-1} \sum_{a,b=0}^{p-1} (e_{y'z} + e_{y'z}) = 1 + \sum_{i=0}^{p-1} \sum_{a,b=0}^{p-1} 1 = 0 \mod I. \]

Thus \( Z_2 = Z_3 = 0 \). This concludes the proof of Lemma 4.1. \( \square \)

Having found \( Z_1, Z_2, Z_3 \) we now show how to determine \( Z_4, Z_5, Z_6 \).

For \( g \in G \) recall that \( \Sigma_y(g) = \sum_{i=0}^{p-1} e_{y^i g} = \sum_{i=0}^{p-1} e_{y^i g} \).

**Lemma 4.2** (i) For \( g \in G \setminus H \) we have \( \Sigma_y(g) + \Sigma_y(g^{-1}) = 1 \mod I \).

(ii) For \( h \in H \) we have \( \Sigma_y(h) = 0 \mod I \).

(iii) For \( g \in G \) we have \( \Sigma_y(g) \cdot \Sigma_y(g^{-1}) = 0 \mod I \).

**Proof** (i) For \( g \in G \setminus H \) we have, using the fact that \( p \) is odd:

\[ \Sigma_y(g) + \Sigma_y(g^{-1}) = \sum_{i=0}^{p-1} e_{y^i g} + e_{(y^i g)^{-1}} = \sum_{i=0}^{p-1} e_{y^i g} + e_{(y^i g)^{-1}} = \sum_{i=0}^{p-1} 1 = 1 \mod I. \]

(ii) is clear since for \( g \in H \) we have \( y^i g \in H \), so that \( e_{y^i g} = 0 \).

(iii) For \( g \in H \) this follows from (ii), while for \( g \in G \setminus H \) we have, using (i):

\[ \Sigma_y(g) \cdot \Sigma_y(g^{-1}) = \Sigma_y(g) \cdot (1 + \Sigma_y(g)). \quad (4.11) \]

Letting \( E = \Sigma_y(g) \) we have \( E^2 = E \mod I \) and so (4.11) is \( E(1 + E) = 0 \mod I \). \( \square \)

We are now in a position to prove:

**Proposition 4.1** (i) \( Z_4 = \Sigma_y(x) + \Sigma_y(x^{p^2}) + \Sigma_y(xz^2) + \Sigma_y(x^{p^2}z^2) \mod I \).

(ii) \( Z_5 = Z_{x^2} = 1 + \sum_{a=1}^{p^2-1} (\Sigma_y(x^a) + \Sigma_y(x^{a+1}z^2)) \mod I; \)

(iii)

\[ Z_6 = Z_{x^{p^2}} = 1 + \Sigma_y(x^{p^2}z^2) + \sum_{a=1}^{p^2} \Sigma_y(x^a) + \Sigma_y(x^{a+p^2}z^2) \]

\[ + \sum_{a=1}^{p^2} \Sigma_y(x^{a+1+p^2}) + \Sigma_y(x^{a}z^2) \mod I. \]

**Proof** So that the reader may more easily see the structure of the main argument we have relegated the proof of this result to the Appendix. \( \square \)
Corollary 4.1 \( Z_4 + Z_5 + Z_6 = 0. \)

**Proof** From Proposition 4.1 we have

\[
Z_4 + Z_5 + Z_6 = \sum_\gamma y(x) + \sum_\gamma y(x^{p_2}) + \sum_\gamma y(xz^2)
\]

\[
+ 1 + \sum_{a=1}^{p-2} \left( \sum_\gamma y(x^a) + \sum_\gamma y(x^{a+1}z^2) \right)
\]

\[
+ \sum_{a=1}^{p_2} \sum_\gamma (x^{p_2}z^2) + \sum_{a=1}^{p_2} \sum_\gamma (x^{a+p_2}z^2) + \sum_{a=1}^{p_2} \sum_\gamma \left( x^{a+1+p_2} + \sum_\gamma (x^a z^2) \right)
\]

\[
= \sum_\gamma (x) + \sum_\gamma (x^{p_2}) + \sum_\gamma (xz^2) + \sum_\gamma (x^{p_2}z^2) + \sum_{a=1}^{p-2} \sum_\gamma y(x^a) + \sum_\gamma (x^{a+1}z^2)
\]

\[
+ \sum_{a=1}^{p_2} \sum_\gamma \left( x^{a+p_2}z^2 \right) \text{right} + \sum_{a=1}^{p_2} \sum_\gamma \left( x^{a+1+p_2} + \sum_\gamma (x^a z^2) \right).
\]

\[
(4.12)
\]

Now one checks that

\[
\sum_\gamma (x) + \sum_\gamma (x^{p_2}) + \sum_{a=1}^{p-2} \sum_\gamma (x^a) + \sum_{a=1}^{p_2} \sum_\gamma (x^{a+1}z^2) = 0,
\]

so that (4.12) now gives

\[
Z_4 + Z_5 + Z_6 = \sum_\gamma (xz^2) + \sum_{a=1}^{p-2} \sum_\gamma (x^{a+1}z^2) + \sum_{a=1}^{p_2} \sum_\gamma (x^a z^2) + \sum_{a=1}^{p_2} \sum_\gamma (x^{a+1+p_2} z^2),
\]

which one can similarly see is equal to zero.

Thus we have \( Z_4 + Z_5 + Z_6 = 0 \), concluding the proof of Corollary 4.1.

So from Lemma 4.1 and Corollary 4.1 we have

\[
Z_x + Z_{xz^2} + Z_{x^2z^2} = \sum_{i=1}^6 Z_i = (Z_1 + Z_2 + Z_3) + (Z_4 + Z_5 + Z_6) = 1 + 0 = 1,
\]

concluding the proof of Theorem 4.1.

Theorem 1.2 follows as we have now eliminated each of the six groups not covered by the paper of Iiams [7].

5 Appendix: The Proof of Proposition 4.1

For Proposition 4.1(i), we have \( Z_4 = \sum_{i=0}^{p-1} \sum_{h \in N \setminus H} e_{y} e_{h} \) and \( N \setminus H \) is a union of \( Y \)-cosets, where
\[(N \setminus H)/Y = \{x, x^2, \ldots, x^{p-1}\} \cup \{x^2z^2, x^2z^2, \ldots, x^{p-1}z^2\}\]

is a set of coset representatives for \(N \setminus H\). Thus we can write

\[
Z_4 = \sum_{i=0}^{p-1} \sum_{h \in N \setminus H} \epsilon_y x^i h = \sum_{i=0}^{p-1} \sum_{h \in (N \setminus H)/Y} \sum_{b=0}^{p-1} \epsilon_y x^i h^b
\]

\[
= \sum_{i=0}^{p-1} \sum_{b=0}^{p-1} \sum_{a=1}^{p-1} \epsilon_y x^i h^b \epsilon_x x^a + \epsilon_y x^i h^b \epsilon_x x^a z^2
\]

\[
= \sum_{i=0}^{p-1} \left( \sum_{b=0}^{p-1} \epsilon_y x^i h^b \sum_{a=1}^{p-1} \epsilon_x x^a + \epsilon_y x^i h^b \sum_{a=1}^{p-1} \epsilon_x x^a z^2 \right)
\]

\[
= \sum_{a=1}^{p-1} \sum_{i=0}^{p-1} \epsilon_y x^i h^b \sum_{a=1}^{p-1} \epsilon_x x^a
\]

\[
= \sum_{a=1}^{p-1} \sum_{i=0}^{p-1} \frac{\epsilon_y x^i h^b \epsilon_x x^a}{C_0/C_1/C_0/C_1} + \frac{\epsilon_y x^i h^b \epsilon_x x^a z^2}{C_0/C_1/C_0/C_1}
\]

\[
= \sum_{a=1}^{p-1} \epsilon_y x^a + \sum_{a=1}^{p-1} \epsilon_y x^a z^2
\]

Let the two sums in (5.1) be denoted by \(Z_{41}, Z_{42}\). Then using Lemma 4.2 and the fact that \(p_2\) is even, we have:

\[
Z_{41} = \sum_{a=1}^{p-1} \epsilon_y x^a + \sum_{a=p_2+1}^{p-1} \epsilon_y x^a
\]

\[
= \sum_{a=1}^{p_2-1} \epsilon_y x^a + \epsilon_y x^{p_2+1} + \sum_{a=p_2+1}^{p-1} \epsilon_y x^a
\]

\[
= \sum_{a=1}^{p_2-1} \epsilon_y x^a + \epsilon_y x^{p_2+1} + \sum_{a=1}^{p_2-1} \epsilon_y x^{p_2} + \sum_{a=1}^{p_2-1} \epsilon_y x^{p_2+1}
\]

\[
= \sum_{a=1}^{p_2-1} \epsilon_y x^a + \epsilon_y x^{p_2+1} + \sum_{a=1}^{p_2-1} \epsilon_y x^{p_2+1}
\]

\[
= \sum_{a=1}^{p_2-1} \epsilon_y x^a + \epsilon_y x^{p_2+1} + \sum_{a=1}^{p_2-1} \epsilon_y x^{p_2+1}
\]

\[
= \sum_{a=1}^{p_2-1} \epsilon_y x^a + \sum_{a=1}^{p_2-1} \epsilon_y x^{p_2+1}
\]

\[
= 1 + \sum_{a=1}^{p_2-1} \epsilon_y x^a + \sum_{a=1}^{p_2-1} \epsilon_y x^{p_2+1}
\]

For \(Z_{42}\) we similarly obtain:

\[
= 1 + \sum_{a=1}^{p_2-1} \epsilon_y x^a + \sum_{a=1}^{p_2-1} \epsilon_y x^{p_2+1}
\]
Thus
\[ Z_4 = Z_{41} + Z_{42} \]
\[ = \sum_{a=1}^{p-1} \left( \Sigma_y(x^a) + \Sigma_y(x^{a+1}) \right) + \sum_{a=1}^{p-1} \left( \Sigma_y(x^{a-2}) + \Sigma_y(x^{a+1}z^2) \right) \]
\[ = \Sigma_y(x) + \Sigma_y(x^{p^2}) + \Sigma_y(xz^2) + \Sigma_y(x^{p+1}z^2), \]
as required for Proposition 4.1 (i).

For \( Z_5 \) we again sum over cosets of \( Y \), so that we have
\[ Z_5 = \sum_{i=0}^{p-1} \sum_{h \in N\setminus H} E_{x^a h} E_{h} \]
\[ = \sum_{i,b=0}^{p-1} \sum_{a=1}^{p-1} E_{x^a z^2 x^a y^b} E_{x^a y^b} + \sum_{i,b=0}^{p-1} \sum_{a=1}^{p-1} E_{y x^a z^2 x^a y^b} E_{x^a y^b} z^2 \]
\[ = \sum_{i,b=0}^{p-1} \sum_{a=1}^{p-1} E_{y x^a z^2 x^a y^b} + \sum_{i,b=0}^{p-1} \sum_{a=1}^{p-1} E_{y x^a z^2 x^a y^b} E_{x^a y^b} z^2 \]
\[ = \sum_{a=1}^{p-1} \Sigma_y(x^{a+1}z^2) \Sigma_y(x^a) + \sum_{a=1}^{p-1} \Sigma_y(x^{a+1}) \Sigma_y(x^{a-1}). \]

Now when \( a = p-1 \) the corresponding terms of each sum of (5.2) are zero, while all other terms are non-zero. Thus we have
\[ Z_5 = \sum_{a=1}^{p-2} \Sigma_y(x^{a+1}z^2) \Sigma_y(x^a) + \sum_{a=1}^{p-1} (1 + \Sigma_y(x^{a-1})) (1 + \Sigma_y(x^{a-1})) \]
\[ = \sum_{a=1}^{p-2} \Sigma_y(x^{a+1}z^2) \Sigma_y(x^a) \]
\[ + \sum_{a=1}^{p-2} (1 + \Sigma_y(x^{a-1})) + \Sigma_y(x^{a-2}) + \Sigma_y(x^{p-a}z^2) \Sigma_y(x^{p-a-1}) \]
\[ = \sum_{a=1}^{p-2} \Sigma_y(x^{a+1}z^2) \Sigma_y(x^a) \]
\[ + 1 + \sum_{a=1}^{p-2} (\Sigma_y(x^{a-1}) + \Sigma_y(x^{a-1}) + \Sigma_y(x^{p-a}z^2) \Sigma_y(x^{p-a-1}) \]
Now one checks that \( \sum_{a=1}^{p-2} \Sigma_y(x^{a+1}z^2) \Sigma_y(x^a) = \sum_{a=1}^{p-2} \Sigma_y(x^{p-a}z^2) \Sigma_y(x^{p-a-1}) \), so that we have
\[ Z_5 = 1 + \sum_{a=1}^{p-2} \left( \Sigma_y(x^{-a-1}) + \Sigma_y(x^{-a}z^2) \right) = 1 + \sum_{a=1}^{p-2} \left( \Sigma_y(x^a) + \Sigma_y(x^{a+1}z^2) \right). \]

This gives Proposition 4.1(ii).

For \( Z_6 \) we again sum over cosets of \( Y \), so that we have

\[ Z_6 = \sum_{i=0}^{p-1} \sum_{h \in \mathbb{Z} / N \setminus H} e^{rac{2\pi i a}{p} z^i h} e^h \]

\[ = \sum_{i=0}^{p-1} \sum_{b=0}^{p-1} \sum_{a=1}^{p-1} \left( e^{\frac{2\pi i a}{p} z^i} e^{2\pi i a} + \sum_{i=0}^{p-1} \sum_{b=0}^{p-1} e^{\frac{2\pi i a}{p} z^{i+b}} e^{2\pi i a} \right) \]

\[ = \sum_{i=0}^{p-1} \sum_{b=0}^{p-1} \sum_{a=1}^{p-1} \left( e^{\frac{2\pi i a}{p} z^i} e^{2\pi i a} + \sum_{i=0}^{p-1} \sum_{b=0}^{p-1} e^{\frac{2\pi i a}{p} z^{i+b}} e^{2\pi i a} \right) \]

\[ = \sum_{a=1}^{p-1} \Sigma_y(x^{a+p_2}z^2) \Sigma_y(x^a) + \sum_{a=1}^{p-1} \Sigma_y(x^{a+p_2}) \Sigma_y(x^{a}z^2) \]

Now in the sums of (5.3) the only term of (5.3) having the form \( \Sigma_y(*) \Sigma_y(*) \) that is zero, occurs when \( a = p_2 + 1 \), there being two such occurrences in (5.3). So (5.3) is equal to

\[ = \sum_{a=1}^{p-1} \Sigma_y(x^{a+p_2}z^2) \Sigma_y(x^a) + \sum_{a=1}^{p-1} \Sigma_y(x^{a+p_2}) \Sigma_y(x^{a}z^2) \]

\[ = 1 + \sum_{a=1}^{p-1} \Sigma_y(x^{a+p_2}z^2) \Sigma_y(x^a) + \sum_{a=1}^{p-1} \left( \Sigma_y(x^{-a-p_2}) + \Sigma_y(x^{-a}z^2) + \Sigma_y(x^{p-a}z^2) \Sigma_y(x^{p-a}z^2) \right) \]

\[ = \frac{Z_6}{2} \]

We split (5.4) into three parts, so that \( Z_6 = 1 + Z_{61} + Z_{62} \). Taking \( a = 1, \ldots, p_2 \) in (5.4) will determine \( Z_{61} \):

\[ Z_{61} = \sum_{a=1}^{p_2} \Sigma_y(x^{a+p_2}z^2) \Sigma_y(x^a) + \sum_{a=1}^{p_2} \left( \Sigma_y(x^{-a-p_2}) + \Sigma_y(x^{-a}z^2) + \Sigma_y(x^{p-a}z^2) \Sigma_y(x^{p-a}z^2) \right) \]

\[ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qanda; Springer
\[ \sum_{a=1}^{p_2} f(a) = \sum_{a=1}^{p_2} f(1 + p_2 - a), \]

it follows that the degree 2 part of (5.5) is equal to

\[ \sum_{a=1}^{p_2} \Sigma_{y}(x^{a+p_2} z^2) \Sigma_{y}(x^a) + \sum_{a=1}^{p_2} \Sigma_{y}(x^{p-a} z^2) \Sigma_{y}(x^{p-a-p_2}) \]

\[ = \sum_{a=1}^{p_2} \Sigma_{y}(x^{a+p_2} z^2) \Sigma_{y}(x^a) + \sum_{a=1}^{p_2} \Sigma_{y}(x^{p-(p_2+1)-a}) z^2 \Sigma_{y}(x^{p_2+1-(p_2+1)-a}) \]

\[ = \sum_{a=1}^{p_2} \Sigma_{y}(x^{a+p_2} z^2) \Sigma_{y}(x^a) + \sum_{a=1}^{p_2} \Sigma_{y}(x^{a+p_2} z^2) \Sigma_{y}(x^a) = 0. \]

With this we now have

\[ Z_{61} = \sum_{a=1}^{p_2} \Sigma_{y}(x^{-a-p_2}) + \Sigma_{y}(x^{-a} z^2) \]

\[ = \sum_{a=1}^{p_2} \Sigma_{y}(x^{-(1+p_2-a)-p_2}) + \Sigma_{y}(x^{-(1+p_2-a)} z^2) \]

\[ = \sum_{a=1}^{p_2} \Sigma_{y}(x^a) + \Sigma_{y}(x^{a+p_2} z^2). \]

Taking \( a = p_2 + 2, \cdots, p - 1 \) in (5.4) gives \( Z_{62} \):

\[ Z_{62} = \sum_{a=2+p_2}^{p-1} \Sigma_{y}(x^{a+p_2} z^2) \Sigma_{y}(x^a) + \]

\[ \sum_{a=2+p_2}^{p-1} (\Sigma_{y}(x^{-a-p_2}) + \Sigma_{y}(x^{-a} z^2) + \Sigma_{y}(x^{p-a} z^2) \Sigma_{y}(x^{p-a-p_2})). \]

Again we look at the degree 2 part of (5.5):

\[ \sum_{a=2+p_2}^{p-1} \Sigma_{y}(x^{a+p_2} z^2) \Sigma_{y}(x^a) + \Sigma_{y}(x^{p-a} z^2) \Sigma_{y}(x^{p-a-p_2}), \]

but it is easy to see that the terms in each of the sums of (5.8) are the same (only listed in reverse order in the second sum) so that (5.8) is zero. Thus the degree 2 part of (5.5) is zero, and we have:
\[
Z_{62} = \sum_{a=2}^{p-1} \Sigma_y(x^{-a-p_2}) + \Sigma_y(x^{-a-z^2})
\]
\[
= \sum_{a=2}^{p_2} \Sigma_y(x^{-a-2p_2}) + \Sigma_y(x^{-a-p_2-z^2})
\]
\[
= \sum_{a=2}^{p_2} \Sigma_y(x^{1-a}) + \Sigma_y(x^{1+p_2-a-z^2})
\]
\[
= \Sigma_y(x^{p_2-z^2}) + \sum_{a=1}^{p_2} \Sigma_y(x^{1-a}) + \Sigma_y(x^{1+p_2-a-z^2})
\]
\[
= \Sigma_y(x^{p_2-z^2}) + \sum_{a=1}^{p_2} \Sigma_y(x^{a+1+p_2}) + \Sigma_y(x^{a-z^2}).
\]

From (5.6) and (5.9) we get
\[
Z_6 = 1 + Z_{61} + Z_{62} = 1 + \Sigma_y(x^{p_2-z^2}) + \sum_{a=1}^{p_2} \Sigma_y(x^a) + \Sigma_y(x^{a+p_2-z^2})
\]
\[
+ \sum_{a=1}^{p_2} \Sigma_y(x^{a+1+p_2}) + \Sigma_y(x^{a-z^2}),
\]
which completes the proof of Proposition 4.1.

\[\square\]

Acknowledgements We thank a referee for pointing out an error in an earlier version of this paper. All calculations made in writing this paper were accomplished using Magma [2].

References

1. AbuGhneim, O.A., Smith, K.W.: Tightening Turyn’s bound for Hadamard difference sets. J. Algebr. Combin. 27(2), 187–203 (2008)
2. Bosma, W., Cannon, J.: MAGMA. University of Sydney, Sydney (1994)
3. Cohen, H.: A Course in Computational Algebraic Number Theory, GTM, vol. 138. Springer, Berlin (1996)
4. Davis, J., Jedwab, J.: A survey of Hadamard difference sets, HPL-94-14. HP Laboratories, Bristol (1994)
5. Davis, J.A., Polhill, J.: Difference set constructions of DRADs and association schemes. J. Combin. Theory Ser. A 117(5), 598–605 (2010)
6. Hoagland, C., Humphries, S.P., Nicholson, N.: Seth Poulsen difference sets disjoint from a subgroup. Graphs Combin. 35, 579–597 (2019). https://doi.org/10.1007/s00373-019-02017-2
7. Iiams, J.: On difference sets in groups of order 4p^2. J. Comb. Theory A 1996, 256–276 (1996)
8. Ito, N., Raposa, B.P.: Nearly triply regular DRADs of RH type. Graphs Combin. 8(2), 143–153 (1992)
9. Ito, N.: Automorphism groups of DRADs. Group theory (Singapore, 1987), pp. 151–170, de Gruyter, Berlin (1989)
10. Jedwab, J.: Perfect Arrays, Barker Arrays, and Difference Sets. University of London, London (1991). Ph.D. thesis
11. Kesava Menon, P.: On difference sets whose parameters satisfy a certain relation. Proc. Am. Math. Soc. 13, 739–745 (1962)
12. Kibler, R.E.: A summary of noncyclic difference sets, k < 20. J. Combin. Theory Ser. A 25(1), 62–67 (1978)
13. Kraemer, R.: A result on Hadamard difference sets. J. Combin. Theory (A) 63, 1–10 (1993)
14. McFarland, R.L.: Difference sets in abelian groups of order $4p^2$. Mitt. Math. Sem. Giessen No. 192(i—iv), 1–70 (1989)
15. Smith, K.W.: Non-abelian Hadamard difference sets. J. Combin. Theory Ser. A 70(1), 144–156 (1995)
16. Turyn, R.J.: Character sums and difference sets. Pac. J. Math. 15, 319–346 (1965)
17. Wan, Z.: Difference sets in groups of order $4p^4$. Beijing Daxue Xuebao Ziran Kexue Ban 36(3), 331–341 (2000)
18. Webster, J.D.: Reversible difference sets with rational idempotents. Arab. J. Math. (Springer) 2(1), 103–114 (2013)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.