Existence of Periodic Solutions for Nonlinear Fully Third-Order Differential Equations

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1. Introduction

In this paper, we consider the existence of periodic solutions for nonlinear fully third-order differential equation

\[ x'''(t) + f(t, x(t), x'(t), x''(t)) = 0, \quad t \in \mathbb{R}, \tag{1}\]

where \( f: \mathbb{R}^4 \to \mathbb{R} \) is continuous and \( T \)-periodic with respect to \( t \).

The third-order periodic problem arises in many areas of applied mathematics and physics, and so it has been extensively studied by many authors via various methods, for instance, see [1–22] and the references therein. Among a substantial number of works, we mention that the upper and lower solutions method is used in [5, 6, 15, 16], Leray–Schauder continuation theorem is used in [1, 10, 20, 21], Leray–Schauder degree theory or the Schauder-fixed-point theorem is used in [10, 12, 13, 18], Mawhin coincidence degree theory is used in [2, 3, 8, 17], and fixed-point theorem in cone or fixed-point index theory is used in [7, 9, 11, 19, 22]. However, to the best of our knowledge, there is no work that refers to periodic solutions of equation (1) using the topological transversality method.

Recently, Kelevedjiev and Todorov [23] have used the topological transversality method and barrier strip technique to study various third-order two-point boundary value problems. But, they did not consider the third-order periodic boundary value problem.

Motivated and inspired by the aforementioned works, the aim of this paper is to establish new existence results of periodic solutions to equation (1) by using the topological transversality method together with barrier strip technique. It is worth mentioning that our results do not need any growth restrictions on the nonlinearity.

In addition, compared with the corresponding ones in the known literature, the barrier strip technique we use to estimate a prior bounds of periodic solutions is essentially new.

This work is organized as follows. In Section 2, we first introduce some notations and lemmas and then estimate a prior bounds of periodic solutions of equation (1) by using barrier strip technique. Finally, by using the topological transversality method, we establish the existence results of periodic solutions to equation (1). As applications of our main results, an example is given in the last section.

2. Main Results

Throughout this section, the following assumptions are used:

\((H_0)\): there exists \( M > 0 \) such that, for any \( T \)-periodic function \( x \in C^2(\mathbb{R}) \),
We say that \( x \) is a solution of PBVP (7) and (8) for some \( \lambda \in (0,1] \). We may assume that \( x'(t_1) > M_1 \). Let \( t_2 \in [0,T] \) be such that
\[
\begin{align*}
x'(t_2) &= \max_{t \in [0,T]} x'(t) > M_1.
\end{align*}
\]

Without loss of generality, we assume that \( t_2 \in (0,T) \); then, \( x''(t_2) = 0 \) and \( x'''(t_2) \leq 0 \). It follows from condition (\( H_1 \)) that
\[
0 \geq x'(t_2)x'''(t_2) = -x'(t_2)\lambda f(t_2,x(t_2),x'(t_2),0) > 0,
\]
which is a contradiction. This means that (10) holds.

Next, we prove that (9) holds. Indeed, integrating the equation in (7) from 0 to \( T \), we obtain that
\[
\int_0^T f(t,x(t),x'(t),x''(t))dt = 0.
\]
This together with the condition (\( H_0 \)) implies that there exists \( \xi \in [0,T] \) such that \( |x(\xi)| \leq M \). It follows from (10) that, for \( t \in [0,T] \),
\[ |x(t)| = |x(\xi) + \int_{\xi}^{t} x'(s)ds| \leq M + \int_{0}^{T} |x'(s)|ds \leq M_0, \]

which means that (9) holds. This completes the proof of the lemma.

**Lemma 4.** Suppose that \((H_0)\) or \((H'_0)\), \((H_1)\), and \((H_2)\) hold. Let \(x(t)\) be a solution of PBVPs (7) and (8) for some \(\lambda \in (0, 1)\). Then,

\[ |x''(t)| \leq M_2 := \max\{L_1, -L_4\}, \quad \forall t \in [0, T]. \]  

**Proof.** We estimate \(x''(t)\) by using the barrier strip technique. From condition \((H_2)\), \(f(t, x, y, z)\) does not change its sign for \((t, x, y, z) \in D_1\) and for \((t, x, y, z) \in D_2\), respectively. For the sake of certainty, without loss of generality, we assume that

\[
\begin{align*}
  f(t, x, y, z) &\leq 0, \quad \text{on } D_1, \\
  f(t, x, y, z) &\geq 0, \quad \text{on } D_2.
\end{align*}
\]  

Notice that \(x'(0) = x'(T)\), and from Rolle’s mean value theorem, there exists \(\eta \in (0, T)\) such that \(x''(\eta) = 0\). Let

\[
\begin{align*}
  S_0 &= \{ t \in [0, \eta] : L_1 < x''(t) \leq L_2 \}, \\
  S_1 &= \{ t \in [0, \eta] : L_3 < x''(t) \leq L_4 \}.
\end{align*}
\]  

We now assert that the sets \(S_0\) and \(S_1\) are empty. We shall complete the proof in two steps.

**Step 1.** Show that \(S_0 = \emptyset\). Suppose on the contrary that there exist some \(t_0 \in S_0\). Then, \(L_1 < x''(t_0) \leq L_2\), and \(t_0 \in [0, \eta]\). Since \(x''(t)\) is continuous on \([0, \eta]\), there exist \(t_0 \leq t_1 < t_2 < \eta\) such that

\[
\begin{align*}
  L_1 < x''(t_2) &< x''(t_1) \leq L_2, \\
  x''(t_2) &\leq x''(t) \leq x''(t_1), \quad \forall t \in [t_1, t_2],
\end{align*}
\]

and so \([t_1, t_2] \subset S_0\). Consequently, from assumption (16), we have

\[
\begin{align*}
  x''(t) &= -\lambda f(t, x(t), x'(t), x''(t)) \geq 0, \quad \forall t \in [t_1, t_2] \subset S_0,
\end{align*}
\]

and thus,

\[
\begin{align*}
  x''(t_2) &\geq x''(t_1),
\end{align*}
\]

which contradicts (18). This implies that \(S_0 = \emptyset\).

**Step 2.** Prove that \(S_1 = \emptyset\). By contradiction, assume that there exist some \(t_0' \in S_1\). Then, \(L_3 < x''(t_0') \leq L_4\), and \(t_0' \in [0, \eta]\). We now assert that

\[
\begin{align*}
  x''(0) &= L_4.
\end{align*}
\]

Indeed, if \(x''(0) \geq L_4\), then from the fact \(x''(0) = x''(T)\), it follows that \(x''(T) \geq L_4\). Notice that \(x''(\eta) = 0\), and it follows from the continuity of \(x''(t)\) on \([\eta, T]\) that there exist \(\eta < \eta_1 < \eta_2 \leq T\) such that

\[
\begin{align*}
  L_3 &\leq x''(\eta_2) < x''(\eta_1) < L_4, \\
  x''(\eta_1) &\leq x''(t) \leq x''(\eta_1), \quad \forall t \in [\eta_1, \eta_2].
\end{align*}
\]

Hence, from Lemma 3, we have

\[
\begin{align*}
  \left( t, x(t), x'(t), x''(t) \right) \in D_2, \quad \forall t \in [\eta_1, \eta_2],
\end{align*}
\]

and so from (16), it follows that

\[
\begin{align*}
  x''(t) &= -\lambda f(t, x(t), x'(t), x''(t)) \geq 0, \quad \forall t \in [\eta_1, \eta_2].
\end{align*}
\]

Therefore,

\[
\begin{align*}
  x''(\eta_2) &\geq x''(\eta_1),
\end{align*}
\]

which contradicts (22). This means that (21) holds.

Thus, from (21) and the continuity of \(x''(t)\) on \([0, \eta]\), there exist \(0 < t_1' < t_2' < \eta\) such that

\[
\begin{align*}
  L_3 &\leq x''(t_2') < x''(t_1') < L_4, \\
  x''(t_1') &\leq x''(t) \leq x''(t_1'), \quad \forall t \in [t_1', t_2'],
\end{align*}
\]

and thus, \([t_1', t_2'] \subset S_1\). It follows from assumption (16) that

\[
\begin{align*}
  x''(t) &= -\lambda f(t, x(t), x'(t), x''(t)) \geq 0, \quad \forall t \in [t_1', t_2'] \subset S_1,
\end{align*}
\]

and hence,

\[
\begin{align*}
  x''(t_2') &\geq x''(t_1'),
\end{align*}
\]

which contradicts (26). This implies that \(S_1 = \emptyset\).

Therefore, by the facts that \(x''(\eta) = 0\) and the continuity of \(x''(t)\) on \([0, \eta]\), we obtain

\[
\begin{align*}
  L_4 &\leq x''(t) \leq L_1, \quad \forall t \in [0, \eta]. \quad (29)
\end{align*}
\]

In particular, \(L_4 \leq x''(0) \leq L_1\). Notice \(x''(0) = x''(T)\), and we have

\[
\begin{align*}
  L_4 &\leq x''(T) \leq L_1. \quad (30)
\end{align*}
\]

We now let

\[
\begin{align*}
  S_2 &= \{ t \in [\eta, T] : L_1 < x''(t) \leq L_2 \}, \\
  S_3 &= \{ t \in [\eta, T] : L_3 < x''(t) \leq L_4 \}.
\end{align*}
\]

Notice that, from (30), using the similar arguments on \(S_0 = \emptyset\) and \(S_1 = \emptyset\), we can show that

\[
\begin{align*}
  S_2 &= \emptyset, \\
  S_3 &= \emptyset.
\end{align*}
\]

Hence,

\[
\begin{align*}
  L_4 &\leq x''(t) \leq L_1, \quad \forall t \in [\eta, T]. \quad (33)
\end{align*}
\]

From this, together with (29), it follows that

\[
\begin{align*}
  L_4 &\leq x''(t) \leq L_1, \quad \forall t \in [0, T], \quad (34)
\end{align*}
\]

which implies that
\[ |x''(t)| \leq \max\{L_1, -L_4\} = M_2, \quad \forall t \in [0, T]. \tag{35} \]

This completes the proof of the lemma.

Now, denote \( Y = C^2[0, T] \times \mathbb{R} \) the Banach space equipped with the norm \( \| (x, r) \| = \| x \|_{\infty} + \| x' \|_{\infty} + \| x'' \|_{\infty} + |r| \). Set
\[
U = \{(x, r) \in Y : x(0) = x(T) = 0, r \in \mathbb{R}\}, \\
\Omega = \{(x, r) \in U : \| x \|_{\infty} < 2M_0 + 1, \| x' \|_{\infty} < M_1 + 1, \| x'' \|_{\infty} < M_2 + 1, |r| < M_0 + 1\}. \tag{36} \]

Then, \( U \) is a closed and convex subset of \( Y \), and \( \Omega \) is an open subset of \( U \).

We now give two lemmas which will be used in the proof of our main theorem.

**Lemma 5.** Suppose that \((H_0)\) holds. Let the operator \( F_1 : \Omega \to U \) be defined by
\[
F_1(x, r) = \left( 0, r - \int_0^T f(t, x(t) + r, x'(t), x''(t)) \, dt \right). \tag{37} \]

Then, \( F_1 \) is essential.

**Proof.** Define \( H : \Omega \times [0, 1] \to U \) by
\[
H(x, r, \lambda) = \left( 0, \lambda r - \lambda \int_0^T f(t, x(t) + r, x'(t), x''(t)) \, dt \right). \tag{38} \]

Then, \( H(\cdot, 1) = F_1(\cdot, \cdot) \) and \( H(x, r, 0) = (0, 0) \in \Omega \) for \((x, r) \in \Omega\). Thus, it follows from Lemma 1 that \( H(x, r, 0) \) is essential. Meanwhile, by a standard argument, it is easy to show that \( H(x, r, \lambda) \) is compact.

We now show that
\[
H(x, r, \lambda) \neq (x, r), \quad \forall (x, r) \in \partial \Omega, \lambda \in [0, 1]. \tag{39} \]

Obviously, \( H(x, r, 0) \neq (x, r) \) for all \((x, r) \in \partial \Omega\). Suppose that \( H(x_0, r_0, \lambda_0) = (x_0, r_0) \) for some \((x_0, r_0) \in \partial \Omega\) and \( \lambda_0 \in (0, 1] \). Then, \( x_0(t) \equiv 0 \) on \([0, T]\), and so
\[
\int_0^T f(t, r_0, 0, 0) \, dt = \left( 1 - \frac{1}{\lambda_0} \right) r_0. \tag{40} \]

Hence, from \((H_0)\), we can deduce that \(-M \leq r_0 \leq M\), which contradicts \((0, r_0) \in \partial \Omega\). This implies that (39) holds. Therefore, from Lemma 2, \( F_1(\cdot, \cdot) = H(\cdot, 1) \) is essential. This completes the proof of the lemma. \(\square\)

**Lemma 6.** Suppose that \((H'_0)\) holds. Let the operator \( F_2 : \Omega \to U \) be defined by
\[
F_2(x, r) = \left( 0, r + \int_0^T f(t, x(t) + r, x'(t), x''(t)) \, dt \right). \tag{41} \]

Then, \( F_2 \) is essential.

**Proof.** The proof is similar to the proof of Lemma 5 and hence is omitted. \(\square\)

**Theorem 1.** Suppose that \((H_0), (H_1), \) and \((H_2)\) hold. Then, equation (1) has at least one \(T\)-periodic solution \( x = x(t) \) satisfying (9), (10), and (15).

**Proof.** At first, we define operator \( A : \Omega \times [0, 1] \to C^2[0, T] \) by
\[
A(x, r, \lambda) = \lambda \int_0^T K(t, s) \int_0^s f\left( \tau, x(\tau) + r, x'(\tau), x''(\tau) \right) \, d\tau - \frac{1}{T} \int_0^T (T - \tau) f\left( \tau, x(\tau) + r, x'(\tau), x''(\tau) \right) \, d\tau \right] \, ds, \tag{42} \]
where
\[
K(t, s) = \frac{1}{T} \left[ \begin{array}{l} t(T - s), \quad 0 \leq t \leq s \leq T; \\
\eta(T - t), \quad 0 \leq s \leq t \leq T. \end{array} \right. \tag{43} \]

It is easy to check that, for each \((x, r, \lambda) \in \Omega \times [0, 1]\), \( A(x, r, \lambda) \) is the unique solution of the following boundary value problem:
\[
\begin{cases} u'' + \lambda f(t, x(t) + r, x'(t), x''(t)) = 0, & t \in [0, T], \\
u(0) = u(T) = 0, \\
u'(0) = u'(T). \end{cases} \tag{44} \]

Furthermore, by a standard argument, it is easy to show that the operator \( A(x, r, \lambda) \) is completely continuous. We now define operator \( G_1 : \Omega \times [0, 1] \to U \) by
\[
G_1(x, r, \lambda) = \left( A(x, r, \lambda), r - \int_0^T f\left( \tau, x(\tau) + r, x'(\tau), x''(\tau) \right) \, d\tau \right). \tag{45} \]

Suppose that \((x_1, r_1)\) is a fixed point of \( G_1(\cdot, \cdot, 1) \). Then,
\[
x_1(t) = \int_0^T K(t, s) \left[ \int_0^s f\left( \tau, x_1(\tau) + r_1, x'_1(\tau), x''_1(\tau) \right) \, d\tau \right] \, ds - \frac{1}{T} \int_0^T (T - \tau) f\left( \tau, x_1(\tau) + r_1, x'_1(\tau), x''_1(\tau) \right) \, d\tau \right] \, ds, \tag{46} \]

It follows that \( x_1(0) = x_1(T) = 0, x'_1(0) = x'_1(T) \), and
and so is essential by Lemma 5, for the existence of a fixed point of

\[ x(t) = \int_0^t f\left(\tau, x(t) + r_1, x'(t), x''(t)\right) d\tau, \quad t \in [0, T]. \]

and thus, by (46),

\[ x_1(0) = \frac{1}{T} \int_0^T (T - r)f\left(\tau, x(t) + r_1, x'(t), x''(t)\right) d\tau = x_1'(T). \]

Set \( x_2(t) = x_1(t) + r_1 \) for \( t \in [0, T] \). It is easy to see that \( x_2(t) \) is a solution of PBVP (6), and validity of (9), (10), and (15) now follows from Lemmas 3 and 4. Therefore, to prove the existence of solutions of PBVP (6) satisfying (9), (10), and (15), it is sufficient to show that the operator \( G_1(x, r, \lambda) \) has at least one fixed point. Since \( G_1(x, r, 0) = F_1(x, r) \) and \( F_1 \) is essential by Lemma 5, for the existence of a fixed point of \( G_1(x, r, 1) \), it is sufficient to verify (i) and (ii) of Lemma 2. Notice that operator \( A \) is completely continuous, then operator \( G_1 \) is continuous, and also \( G_1(\Omega \times [0, 1]) \) is compact in \( U \). Thus, (i) of Lemma 2 is satisfied. Let \( G_1(x_0, r_0, \lambda_0) = (x_0, r_0, \lambda_0) \) for some \( (x_0, r_0) \in \partial\Omega \) and \( \lambda_0 \in [0, 1] \). If \( \lambda_0 = 0 \), then \((x_0, r_0) \notin \partial\Omega \), which has been proved in the proof of Lemma 5. Let \( \lambda_0 \in [0, 1] \). Then,

\[ x_0(t) = \lambda_0 \int_0^T K(t, s) \left[ \int_0^s f\left(\tau, x_0(\tau) + r_0, x'_0(\tau), x''_0(\tau)\right) d\tau \right] ds, \]

\[ x'_0(0) + \frac{1}{T} \int_0^T (T - r)f\left(\tau, x_0(\tau) + r_0, x'_0(\tau), x''_0(\tau)\right) d\tau = 0. \]

Hence,

\[ -x''_0(t) = \lambda_0 f\left(t, x_0(t) + r_0, x'_0(t), x''_0(t)\right), \quad t \in [0, T], \]

\[ x_0(0) = x_0(T) = 0, \]

\[ x'_0(0) = x'_0(t) \]

and so

\[ x''_0(0) = \frac{1}{T} \int_0^T (T - r)f\left(\tau, x_0(\tau) + r_0, x'_0(\tau), x''_0(\tau)\right) d\tau = x''_0(T). \]

Set \( x(t) = x_0(t) + r_0 \) for \( t \in [0, T] \). We can see that \( x(t) \) is a solution of PBVPs (7) and (8) with \( \lambda = \lambda_0 \). It follows from Lemmas 4 and 5 that

\[ \|x_0 + r_0\|_{\infty} = \|x\|_{\infty} \leq M_0, \]

\[ \|x_0\|_{\infty} = \|x'_0\|_{\infty} \leq M_1 \leq 1, \]

\[ \|x'_0\|_{\infty} = \|x''_0\|_{\infty} \leq M_2 \leq 1. \]

Since \( x_0(0) = 0 \), the first inequality of (52) yields

\[ r_0 \leq \|x_0\|_{\infty} \leq M_0 < M_0 + 1, \]

and thus,

\[ \|x_0\|_{\infty} \leq M_0 + \|r_0\| \leq 2M_0 + 1. \]

Hence, \((x_0, r_0) \notin \partial\Omega\), and so (ii) of Lemma 2 is verified. This completes the proof of the theorem.

**Theorem 2.** Suppose that \((H'_0), (H_1), \) and \((H_2)\) hold. Then, equation (1) has at least one \( T \)-periodic solution \( x = x(t) \) satisfying (9), (10), and (15).

**Proof.** The proof is similar as that for Theorem 1 except that

\[ G_2(x, r, \lambda) = \left( A(x, r, \lambda), r + \int_0^T f\left(\tau, x(t) + r, x'(t), x''(t)\right) d\tau \right) \]

and \( F_2 \) are used in place of \( G_1(x, r, \lambda) \) and \( F_1 \), respectively, and hence is omitted. This completes the proof of the theorem.

**3. An Example**

In this section, we give an example to demonstrate the applications of the our results.

**Example 1.** Consider the nonlinear third-order differential equation

\[ x'' + \left(\frac{x'^2}{2} + 1\right)x'' - \left(1 - \frac{1}{2} \cos x\right)x' + \frac{x - 1}{x^2 + 1} = \sin t, \quad t \in \mathbb{R}. \]

(56)

Let

\[ f(t, x, y, z) = (y^2 + 1)z - \left(1 - \frac{1}{2} \cos x\right)y + \frac{x - 1}{x^2 + 1} - \sin t, \]

\[ (t, x, y, z) \in \mathbb{R}^4. \]

(57)

Then, for any \( 2\pi \)-periodic function \( x \in C^2(\mathbb{R}) \), we have

\[ (\text{sign } x(t)) \int_0^{2\pi} f(t, x(t), x'(t), x''(t)) dt = (\text{sign } x(t)) \int_0^{2\pi} \frac{x(t) - 1}{x^2(t) + 1} dt > 0, \quad \text{if } \min_{t \in [0, 2\pi]} |x(t)| > 1 = M. \]

(58)
and so condition \((H_0)\) holds. Notice that

\[
\lim_{y \to \pm \infty} y f(t, x, y, 0) = \lim_{y \to \pm \infty} y \left(-\frac{1}{2} \cos x - y + \frac{x - 1}{x^2 + 1 - \sin x}\right) = -\infty,
\]

(59)

uniformly in \((t, x, y) \in [0, 2\pi] \times \mathbb{R}\); then, there exists \(M_1 > 0\) such that

\[
y f(t, x, y, 0) < 0, \quad \forall (t, x) \in [0, 2\pi] \times \mathbb{R}, |y| > M_1,
\]

(60)

that is, condition \((H_1)\) holds. In addition, since

\[
\lim_{z \to \pm \infty} f(t, x, y, z) = \pm \infty,
\]

(61)

uniformly in \((t, x, y) \in [0, 2\pi] \times [-M_0, M_0] \times [-M_1, M_1]\), where \(M_0 = M + 2\pi M_1\), it follows that condition \((H_2)\) holds. Hence, by Theorem 1, third-order differential equation (56) has at least one \(2\pi\)-periodic solution.

**Data Availability**

There are no data in this paper.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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