A note on the novel 4D Einstein–Gauss–Bonnet gravity

Wen-Yuan Ai

Centre for Cosmology, Particle Physics and Phenomenology, Université catholique de Louvain, Louvain-la-Neuve B-1348, Belgium

E-mail: wenyuan.ai@uclouvain.be

Received 22 April 2020, revised 27 May 2020
Accepted for publication 11 June 2020
Published 5 August 2020

Abstract

Recently, a novel 4D Einstein–Gauss–Bonnet gravity has been proposed by Glavan and Lin (2020 Phys. Rev. Lett. 124 081301) by rescaling the coupling $\alpha \to \alpha/(D - 4)$ and taking the limit $D \to 4$ at the level of equations of motion. This prescription, though was shown to bring non-trivial effects for some spacetimes with particular symmetries, remains mysterious and calls for scrutiny. Indeed, there is no continuous way to take the limit $D \to 4$ in the higher $D$-dimensional equations of motion because the tensor indices depend on the spacetime dimension and behave discretely. On the other hand, if one works with 4D spacetime indices the contribution corresponding to the Gauss–Bonnet term vanishes identically in the equations of motion. A necessary condition (but may not be sufficient) for this procedure to work is that there is an embedding of the 4D spacetime into the higher $D$-dimensional spacetime so that the equations in the latter can be properly interpreted after taking the limit. In this note, working with 2D Einstein gravity, we show several subtleties when applying the method used in (2020 Phys. Rev. Lett. 124 081301).

Keywords: Einstein–Gauss–Bonnet gravity, modified gravity theories, general relativity

1. Introduction

Although general relativity (GR) is the most established and successful theory of gravity, it must be modified [1–5]. This is partially because GR is not theoretically complete and partially because several experimental observations, which are closely related to gravitational interaction, cannot be explained by it. Quantum effects, as typically shown in String theory, are believed to generate higher-order curvature terms in the low-energy effective theory of gravity. The most general metric theory of gravity which yields conserved second-order equations of motion in an arbitrary $D$-dimensional spacetime is given by Lovelock theory [6]. The Lagrangian of Lovelock theory is given by a sum of terms with each term, $\mathcal{L}_n (2n < D)$, being the (generalized) Euler density in $2n$-dimensional spacetime. For the critical dimension of spacetime $D = 2n$, $\mathcal{L}_n$ becomes topological and does not contribute to local dynamics. For example, when $D = 4$, the Gauss–Bonnet term, $\mathcal{L}_{(2)}$, has no local dynamics and Lovelock theory reduces to GR. For studies of gravity with the Gauss–Bonnet term, see e.g. [7–15].

Very recently, Glavan and Lin [16] proposed a novel 4D Einstein–Gauss–Bonnet gravity where the Gauss–Bonnet term does contribute to local dynamics. To extract the local dynamics, they first rescale the coupling associated with the Gauss–Bonnet term in $D$-dimensional spacetime, $\alpha \to \alpha/(D - 4)$, and then take the limit $D = 4$ in the equations of motion. Through this process, they were able to obtain finite contributions from the Gauss–Bonnet term in the local equations of motion for some 4D spacetimes with particular symmetries. This prescription has also been used earlier in [17, 18] and below we shall refer to it as the dimensional-regularization prescription. Although the proposed 4D Einstein–Gauss–Bonnet gravity has already intrigued a large amount of work in applications (see [19–51]), the dimensional-regularization prescription has not been justified with the matched rigor. In this note, we examine it for the simplest case, i.e. 2D Einstein gravity, and show that, without care, one may give incorrect interpretations in the obtained equations of motion. In the next section, we review the dimensional-regularization prescription used in [16]. In section 3, we apply the dimensional-regularization prescription to 2D Einstein gravity. Section 4 is left for discussions and conclusions.
2. Dimensional-regularization prescription

It is well known that the Gauss–Bonnet term is topological in 4D spacetime while it becomes local in higher-dimensional spacetime. The integral of the Gauss–Bonnet invariant over a 4D spacetime $M_4$ (properly compactified) gives the Euler characteristic $\chi(M_4)$ via
\[
\chi(M_4) = \frac{1}{32\pi^2} \int_{M_4} d^4x \sqrt{-g} \mathcal{G},
\]
where
\[
\mathcal{G} = R^\mu_{\rho\sigma} R^\rho_{\mu\sigma} - 4 R^\mu_{\kappa\lambda} R^\kappa_{\mu\lambda} + R^2 = 6 R^\mu_{[\mu\rho|R^\rho_{\nu\sigma}|^\nu]}_{[\sigma],},
\]
is the Gauss–Bonnet invariant. The action from a topological term is thus invariant under the variation of the metric field whose boundary values are fixed. Thus, the Gauss–Bonnet term does not contribute to local dynamics. Further, in classical gravity, the topology of the spacetime is fixed which makes the Gauss–Bonnet term totally unobservable. (While in quantum gravity, one may sum over geometries with different topologies which might lead to observable effects.) However, the Gauss–Bonnet term becomes local when going beyond 4D spacetime and does contribute to local dynamics in the equations of motion. One may expect that all the contributions from the Gauss–Bonnet term in Einstein’s equations in higher $D$-dimensional spacetime carry a factor of $D - 4$ so that they vanish when $D = 4$.

To extract finite contributions to the local dynamics from the Gauss–Bonnet term, the authors in [16] rescale the coupling $\alpha$ to $\alpha/(D - 4)$ which leads to the following action
\[
S_{\text{EGB}} = \int d^4x \sqrt{-g} \left[ \frac{M_5^2}{2} R - \Lambda + \frac{\alpha}{D - 4} \mathcal{G} \right].
\]
While one still keeps the general $D$ in deriving the equations of motion (obviously, it makes no sense to put $D = 4$ in the action), one may be able to take the limit $D \to 4$ finally. It was shown explicitly that for some highly symmetric spacetimes, the factor $1/(D - 4)$ from the new action will cancel out all the factors $D - 4$ in Einstein equations, giving rise to finite contributions after taking $D = 4$ [16]. The cancellation was also expected for more general cases.

If the dimensional-regularization prescription does work in general, it would be astonishing because one can then apply the same trick to other Lovelock densities in Lovelock gravity in $D$-dimensional spacetime and obtain finite contributions to the local dynamics in lower-dimensional spacetime which would be otherwise vanishing without playing the trick [32]. To see it, we recall the general Lovelock Lagrangian
\[
\mathcal{L} = \sum_{n=0}^{\ell} \mathcal{L}_{(n)} = \sqrt{-g} \sum_{n=0}^{\ell} \alpha_n \mathcal{R}_{(n)},
\]
where
\[
\mathcal{R}_{(n)} = \frac{1}{2^n} \epsilon_{\alpha_1 \beta_1 \ldots \alpha_n \beta_n} \prod_{i=1}^{n} R_{\alpha_i \beta_i} \ldots R_{\alpha_n \beta_n},
\]
when multiplied by $\sqrt{-g}$, are the generalized Euler densities in $2n$-dimensions (also called the Lovelock densities).

Here
\[
\delta^\mu_{\alpha_1 \beta_1 \ldots \alpha_p \beta_p} \delta^\nu_{\alpha_p \beta_p} \ldots \delta^\psi_{\alpha_n \beta_n} \delta^\xi_{\alpha_n \beta_n} = n! \delta^\mu_{\alpha_1 \beta_1} \ldots \delta^\psi_{\alpha_p \beta_p} \delta^\xi_{\alpha_n \beta_n},
\]
is the generalized Kronecker delta symbol. For example, $\mathcal{R}_{(0)} = 1$, $\mathcal{R}_{(1)} = R$, $\mathcal{R}_{(2)} = \mathcal{G}$, giving the cosmological constant, the Hilbert–Einstein and the Gauss–Bonnet terms, respectively. In equation (4), $t = D/2$ for even $D$ and $t = (D - 1)/2$ for odd $D$. This is simply because there is no nonvanishing $p$-form for $p > D$ in $D$-dimensional spacetime (the generalized Kronecker delta vanish for $2n > D$).

Every term $\mathcal{L}_{(n)}$ has a corresponding geometrical interpretation as the Gauss–Bonnet invariant does in 4D spacetime. In $2n$-dimensional compact spacetime $M_{2n}$ we have
\[
\chi(M_{2n}) = \frac{1}{(4\pi)^{n/2}} \int_{M_{2n}} d^{2n}x \sqrt{-g} \mathcal{R}_{(n)},
\]
where $\chi(M_{2n})$ is the Euler characteristic. Thus, the term $\mathcal{L}_{(n)}$ is topological in $2n$-dimensional spacetime (and hence does not contribute to any local dynamics) while it becomes local in $D > 2n$ dimensional spacetime. Therefore one may also expect that the contributions in the equations of motion in $D > 2n$ dimensional spacetime from $\mathcal{L}_{(n)}$ are proportional to $D - 2n$. One can then absorb $D - 2n$ into $\alpha_n$ to generate finite local dynamics from the topological term in $2n$-dimensional spacetime. In particular, the method may be applied for Einstein gravity, with which we are very familiar, in 2D spacetime.

One can further extend this procedure to gauge theories when there are terms that are topological in certain dimensional spacetime but cease to be so in higher-dimensional spacetime. For instance, the second Chern-form $\tau F_{\mu
u} \tilde{F}^{\mu\nu}$, which plays an important role in quantum chromodynamics, is topological in 4D spacetime.

Since the consequences of the dimensional-regularization prescription, i.e. generating local dynamics from topological terms, are so novel, it perhaps requires more rigorous scrutiny. The very crucial condition for the dimensional-regularization prescription to work is that the continuous limit $D \to 4$ can be properly taken in the higher-dimensional equations of motion. Although this is believed and assumed by Glavan and Lin, in our opinion, it has not been proved explicitly. (Otherwise, one can immediately apply the dimensional-regularization prescription in 4D Einstein–Gauss–Bonnet gravity for any spacetime, without the help of particular symmetries.) In the next section, we examine the dimensional-regularization prescription in the simplest case, 2D Einstein gravity.

3. 2D Einstein gravity

We consider 2D Einstein gravity
\[
S = \int d^2x \sqrt{-g} [\alpha R - \Lambda + L_{\text{matter}}].
\]
\footnote{As noted in [16], one support on this assumption is the Einstein–Lovelock equations in terms of differential forms [53], presented in equation (5) in that paper. However, that differential-form equation can only be read correctly for $D > 2p$, as indicated by the indices and by itself cannot be applied to the limit $D = 2p$.}
It has been known that 2D Einstein gravity, unless additional non-minimal coupling to other fields is included, is trivial. This is because $\sqrt{-g}R$ is a total derivative in 2D spacetime, precisely as is the case for the Gauss–Bonnet term in 4D spacetime. Taking the variation with respect to the metric, one obtains
\begin{equation}
\Lambda g_{\mu\nu} = T_{\mu\nu},
\end{equation}
where
\begin{equation}
T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_{\text{matter}})}{\delta g_{\mu\nu}}.
\end{equation}
Taking the trace, one has $\Lambda = T/2$ where $T$ is the trace of the energy-momentum tensor. For a 2D conformal field, whose energy-momentum tensor is traceless, a cosmological term is thus inconsistent.

Now we want to generate local dynamics from the Einstein–Hilbert term in 2D spacetime, following [16]. Quite interestingly, the obtained theory is equivalent to that obtained much earlier by Mann and Ross [54, 55]. Although they also suggested deriving the theory via rescaling the coupling and taking the limit $D \to 2$ for GR, their theory can be formulated as one in which there is a scalar field non-minimally coupled to gravity and no divergent coupling at the action level. Thus their theory is solidly based and does not crucially rely on the dimensional-regularization prescription. More details can be found in [56–60]. Here we will take this simple example to draw some caution on the application of the dimensional-regularization prescription where one takes the $D \to 2$ limit at the level of equations of motion.

We now consider the $D$-dimensional theory
\begin{equation}
S = \int d^Dx \sqrt{-g} \left[ \frac{\kappa}{D-2} R - \Lambda + L_{\text{matter}} \right].
\end{equation}
Einstein field equations read
\begin{equation}
\frac{2\kappa}{D-2} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \Lambda g_{\mu\nu} = T_{\mu\nu}.
\end{equation}
Now taking the limit $D \to 2$ from above, i.e. from $D > 2$, in the Einstein equation is subtle, if not ill-defined. Even the limit $D \to 2$ can be understood in a continuous sense for some explicit factors of $D - 2$ appearing in the equations of motion, the indices behave discretely. For example, there are $D(D-1)/2$ independent components\(^2\) in the Einstein equation and the metric tensor. How do these equations continuously evolve from the higher $D$-dimensional case to the 2D case in which we have only one independent metric component? For instance, what does it mean by taking $D = 2.1$ in equation (12) if we are concerned about the indices?\(^3\) If we assume that the indices take values as in a 2D spacetime before taking the limit, then the Einstein tensor vanishes identically. Anyway, the indices cannot take a continuous value. The problem pointed out here is general for the dimensional-regularization prescription, not only specific to 2D Einstein gravity. We refer to it as the index problem.

Whether or not one can extract a total factor $D - 2$ in the Einstein tensor in $D$-dimensional spacetime is unimportant. For an integer $D \geq 3$, defining a tensor $H_{\mu\nu}$ through $G_{\mu\nu} \equiv (D - 2)H_{\mu\nu}$ is trivial; $H_{\mu\nu}$ has exactly the same tensor structure as the Einstein tensor. Even if we ignore the index problem when we take the limit $D \to 2$, it is mysterious if $G_{\mu\nu}$ vanishes while $H_{\mu\nu}$ does not for $D = 2$ when they have the same tensor structure.

Then in which cases may the limit have physical interpretations? First, for scalar equations, e.g. the trace of the Einstein equation, there is not the index problem. For the general tensor equations, we have to first embed the 2D spacetime into the $D$-dimensional spacetime so that there is a clear map between the components of the metric tensors and the Einstein equations in the higher- and lower-dimensional cases. Consider a 2D spacetime with coordinates $\{x^0, x^1\}$ embedded in a $D$-dimensional spacetime with coordinates $\{x^a, x^1, x^n\}$ where $a = 2, ..., D - 1$. Then one may take the $D \to 2$ limit for the zero–zero, zero–one and one–one components in the Einstein equation. After taking the limit, one may simply discard the equations for the components from the extra dimensions. Note that, the dimensional-regularization prescription differs from the dimensional reduction through Kaluza–Klein compactification. In the former the extra dimensions have no physical meaning and only serve to define the limit [16]. For the example spacetimes that have been worked out in [16] in which the metric is given through an Ansatz which is valid for both the higher- and lower-dimensional cases, the embedding is automatically assumed. For example, when constructing a black hole solution in the 4D Einstein–Gauss–Bonnet gravity from the 5D solution, it was assumed that the three-space of the 4D spacetime lies on the equatorial hyperplane in the four-space of the 5D spacetime. Our question is, does the obtained solution through the dimensional-regularization prescription describe a 4D black hole faithfully or an effective one viewed on the equatorial hyperplane in the 5D Schwarzschild spacetime? The full answer will not be pursued in this note but we will show the possibility for the latter interpretation through 2D Einstein gravity.

Taking the trace of equation (12), we obtain
\begin{equation}
-\kappa R + D \Lambda = T.
\end{equation}
This equation is correct for $D > 2$ but now the limit $D = 2$ can be taken and gives
\begin{equation}
-\kappa R + 2 \Lambda = T,
\end{equation}
without causing any trouble. Do we obtain a non-trivial local dynamics for 2D Einstein gravity? Yes and no. Yes: because in this case, one may give a certain interpretation for the obtained equation of motion, which we shall show how later. No: because when we trace back to the origin, equation (14) is really inherited from the higher-dimensional dynamics. To see it more clearly, assume we live in a $D = 3$ dimensional world where the Einstein–Hilbert term contributes to the
\(^2\) $D$ of them are constraint equations.
\(^3\) In dimensional-regularization in quantum field theory, one also encounters quantities carrying indices, e.g. momenta. But there one typically takes analytic continuation of $D$ in the results of the loop integrals which carry no tensor index. For a rigorous treatment, see e.g. [61].
equation of motion (but no propagating gravitational modes). (3D spacetime may not be a good example because 3D Einstein gravity can be formulated as Chern–Simons theory and hence is also topological, see [62].) Then equation (13) gives the 3D trace equation
\[ -\alpha R + 3\Lambda^{(3)} = T. \] (15)
If we pretend to take the limit \( (D = 3) \to 2 \), we are simply redefining the cosmological constant \( \Lambda = 3\Lambda^{(3)}/2 \).

Now if we restrict \( R(t, x, y), T(t, x, y) \) (with \( \{ t, x, y \} \) being a chosen coordinate frame) on the 2D subspace \( \{ t, x \} \) and interpret \( \Lambda = 3\Lambda^{(3)}/2 \) as an (effective) cosmological constant in this 2D spacetime \( \{ t, x \} \), we can interpret equation (14) as describing 2D local gravity dynamics from the Einstein–Hilbert term. However, this is an illusion because we are really working with the higher-dimensional dynamics but look at a sub-space-time embedded in it. If one wanders on a wire and gets surprised by the local gravity he/she sees, he/she may explain all this by just taking a step off that wire.

The trace equation (13) describes the full dynamics for the maximally symmetric spacetime. Taking the limit \( D = 2 \) for the vacuum case, one obtains \( R = 2\Lambda/\alpha \). As we argued, this maximally symmetric 2D spacetime might be interpreted as a time-like two-surface in higher \( D \)-dimensional spacetime. Different \( D \) gives a different relation between \( \Lambda^{(D)} \), the cosmological constant in \( D \)-dimensional spacetime, and the effective \( \Lambda \). We can also work with the Friedmann–Lemaître–Robertson–Walker (FLRW) universe and the Schwarzschild spacetime, precisely as the authors of [16] did for the 4D Einstein–Gauss–Bonnet gravity. With Ansätze for the metric, one can reduce Einstein equations to scalar equations.

For a \( D \)-dimensional FLRW metric \( ds^2 = -dt^2 + a^2(t)[(1/(1 - kr^2))dr^2 + r^2d\Omega_{D-2}^2] \) with \( d\Omega_{D-2}^2 \) being the metric of the unit \( (D - 2) \)-sphere, Einstein equations (12) lead to the Friedmann equation
\[ \alpha(D - 1)\left( H^2 + \frac{k}{a^2} \right) = \rho + \Lambda, \] (16)
and the continuity equation
\[ \rho + (D - 1)(\rho + p)H = 0, \] (17)
where \( \rho = T_{00}, p = T_{rr} \) and \( H = \dot{a}/a \) with \( \cdot \) representing the derivative with respect to \( t \). The derivation of the above equations is tedious and the details are given in the appendix. One indeed can take the limit \( D = 2 \). However, in this case even the embedding picture should be interpreted with care because the equations for \( D > 2 \) is not related to \( D = 2 \) by simple redefinitions of parameters because there is no other parameter except for the dimension \( D \) in the continuity equation.

For Schwarzschild spacetime, we consider the following metric
\[ ds^2 = -e^{2\omega(r)}dr^2 + e^{-2\omega(r)}d\tau^2 + r^2d\Omega_{D-2}^2. \] (18)
The dynamic equations could be derived similarly as in the appendix. For example, we consider the vacuum Einstein equation with vanishing cosmological constant, \( G_{\mu\nu} = 0 \), in \( D \)-dimensional spacetime. With the metric Ansatz (18), the component equations \( G_{00} = 0 \) and \( G_{11} = 0 \) give the same equation while \( G_{ii} = 0 \) with \( i \geq 2 \) give another equation [63]. We finally have
\[ \frac{(D - 2)e^{2\omega}(2\omega' + D - 3)}{r^2} + \frac{(D - 2)(D - 3)}{2r^2} = 0, \] (19a)
\[ \frac{e^{2\omega} - 2\omega'' + 2\omega' + \frac{(D - 4)(D - 3)}{r^2}}{2r^2} = 0, \] (19b)
where the prime denotes the derivative with respect to \( r \). We see that for the \( t-t \) and \( r-r \) components of the Einstein equation (equation (19a)), the factors \( D - 2 \) will be canceled out by the factor \( 1/(D - 2) \) in equation (12). However for the other component equations (equation (19b)), the divergent factor \( 1/(D - 2) \) in equation (12) will not be canceled out. This may by itself again indicate an inconsistency of the dimensional-regularization prescription. However, as we mentioned above, one may also simply discard the other component equations beyond the \( \{ t, r \} \) dimensions when taking the \( D \to 2 \) limit. One could also construct 2D black hole solutions [64].

We believe the above analysis also applies to the 4D Einstein–Gauss–Bonnet gravity and therefore more checks beyond the highly symmetric spacetimes are needed. We emphasize that to define a theory, one shall be able to fully determine the dynamical equations without symmetry constraints on the spacetime. Otherwise, the embedding picture may be preferred. Note that the Gauss–Bonnet term does play a role in 4D anti-de Sitter (AdS) spacetime in the context of AdS/CFT duality [65]. Its topological nature, i.e. being a boundary term, in the holographic renormalization correctly leads to the standard thermodynamics for AdS black holes [66].

4. Conclusions and discussions

The novel 4D Einstein–Gauss–Bonnet gravity, recently proposed by Glavan and Lin, has sparked many discussions in the community of gravity. In this theory, the topological Gauss–Bonnet term was shown to have local dynamics. The way to extract the local dynamics, however, relies on an unusual action principle where they have a coupling \( \alpha/(D - 4) \) associated with the Gauss–Bonnet term in the action, and take the limit \( D = 4 \) in the equations of motion. One then expects that the divergent factor \( D - 4 \) will be canceled out by the factors \( D - 4 \) in the equations of motion. In this note, we first argue that, if this dimensional-regularization prescription does work, then it can be applied as a general method to extract local dynamics from topological terms, even beyond gravity theories. Second, we point out the index problem for this procedure. Specifically, even though the factors \( D - 4 \) can be

4 In [63], the metric signature is \((+, -, - , \cdots, -)\). But the equation of motion for \( \omega(r) \) shall not depend on this choice of the metric signature.
taken to the zero limit continuously, the tensor indices take
discrete values and it is unclear how the equations of motion in
a general $D$-dimensional spacetime will converge to the 4D
dynamical equations. One condition we argue is that the 4D
spacetime must be able to be embedded into the higher
$D$-dimensional spacetime. Third, working with 2D Einstein
gravity, we show a different interpretation on the obtained
equations of motion from the dimensional-regularization
prescription.

If one constructs the 4D dynamics through dimensional
reduction, there must be additional degrees of freedom
introduced in the effective 4D theory. Indeed the effective 4D
Einstein–Gauss–Bonnet theory can be reformulated as a
scalar-tensor theory [67, 68] in which an extra scalar field
appears. It was shown that the linear perturbation of the
lead to extra degrees of freedom, see
dimensional-regularization prescription, which also typically
asymptotic structure for such a scalar-tensor theory has been
studied independently, the author
noticed that the observation on 2D Einstein gravity from the
highly symmetric spacetimes.

At last, we note that, after the appearance of this note, more and more authors
have raised doubts on the original 4D Einstein–Gauss–Bonnet
gravity obtained from the dimensional-regularization
prescription, see [73–79].

Acknowledgments

We are grateful to Dražen Glavan and Carlos Tamarit for
helpful discussions. We also thank Robert Mann, Rodrigo
Olea and Julio Oliva for kind comments. WYA is supported
by the Incoming Postdoc fellowship of UC Louvain.

Note added

After this work was finished independently, the author
noticed that the observation on 2D Einstein gravity from the
dimensional-regularization prescription has also been made by
Nojiri and Odintsov [64], whose paper came out on arXiv
on 03 April 2020.

Appendix. Dynamic equations in 2D FLRW universe
from the dimensional-regularization prescription

In this appendix we derive the dynamic equations in the 2D
FLRW universe using the dimensional-regularization
prescription. As we shall see, the dimensional-regularization
prescription indeed yields regular equations of motion in this
highly symmetric spacetime. But this by no means proves the
validity of the dimensional-regularization prescription to
define a theory. And the problems and subtleties mentioned in
the main text should appear immediately if we go beyond the
highly symmetric spacetimes.

Our main task is to compute the Einstein tensor in the
$D$-dimensional FLRW universe. Let us first consider the
simpler flat case, $k = 0$, with the metric $ds^2 = -dr^2 +
a^2(t)(dx_1^2 + \ldots + dx_{D-1}^2)$. It is easy to show that the non-
vanishing Christoffel symbols are

$$\Gamma^0_0 = \delta_0^i a \dot{a}, \quad \Gamma^i_0 = \delta_i^j \frac{\dot{a}}{a},$$

(A1)

where both $\delta_0$ and $\delta_i^j$ are the Kronecker symbol and
$i, j = 1, \ldots, D - 1$. Using the formula

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu\alpha} - \Gamma^\alpha_{\mu\alpha\nu} + \Gamma^\alpha_{\nu\beta\mu} \Gamma^\beta_{\mu\alpha} - \Gamma^\alpha_{\nu\alpha\mu} \Gamma^\beta_{\mu\beta},$$

(A2)

one obtains the nonvanishing components of the Ricci tensor

$$R_{00} = -(D - 1) \frac{\ddot{a}}{a}, \quad R_{ii} = (D - 2) \dot{a}^2 + a \ddot{a}. \quad (A3)$$

The Ricci scalar is then

$$R = (D - 1) [ (D - 2) \left( \frac{\ddot{a}}{a} \right)^2 + \frac{2 \dot{a}^2}{a} \right], \quad (A4)$$

Finally, one obtains the nonvanishing components of the Einstein tensor

$$G_{00} = \frac{(D - 1)(D - 2)}{2} \left( \frac{\ddot{a}}{a} \right)^2, \quad (A5a)$$

$$G_{ii} = \frac{(D - 2)(D - 3)}{2} \dot{a}^2 - (D - 2) a \ddot{a}. \quad (A5b)$$

Substituting the above expressions into equation (12) and
recalling $T_{00} = \rho, T_{ii} = p_{00}$, we obtain

$$\alpha(D - 1) H^2 = \rho + \Lambda, \quad (A6a)$$

$$-\alpha(D - 3) \left( \frac{\dot{a}}{a} \right)^2 - 2 \alpha \ddot{a} \frac{\dot{a}}{a} = p - \Lambda. \quad (A6b)$$

Now taking the limit $D \to 2$, we obtain the dynamic
equations in the 2D flat FLRW universe in Einstein gravity
from the dimensional-regularization prescription. Note that
from equations (A6a), (A6b), one can obtain the continuity
equation (17).

Next we consider the general case $ds^2 = -dr^2 +
a^2(t)(dr^2/(1 - kr^2) + r^2 d\Omega_{D-2}^2)$ where we used the
coordinates $x^\mu = \{t, r, \theta_1, \ldots, \theta_{D-2}\}$. In this case, all the non-
vanishing Christoffel symbols are

$$\Gamma^0_0 = \frac{\dot{a}}{a} r_0, \quad \Gamma^i_0 = \Gamma^0_i = \frac{\dot{a}}{a}, \quad (A7a)$$

$$\Gamma^{i}_{11} = \frac{kr}{1 - kr^2}, \quad \Gamma^{i}_{22} = -(1 - kr^2), \quad (A7b)$$

$$\Gamma^{i}_{ii} = -(1 - kr^2)(\sin^2 \theta_1 \cdot \ldots \cdot \sin^2 \theta_{i-2}), \quad \text{for } i \geq 3, \quad (A7c)$$

$$\Gamma^{i}_{i1} = \frac{1}{r}, \quad \text{for } i > 2, \quad (A7d)$$

$$\Gamma^{i}_{ik} = \Gamma^{i}_{ki} = \cot \theta_{k-1}, \quad \text{for } i > k \geq 2, \quad (A7e)$$

$$\Gamma^{i}_{(i+1)(i+1)} = -\sin \theta_{i-1} \cos \theta_{i-1}, \quad \text{for } i \geq 2, \quad (A7f)$$

$$\Gamma^{i}_{(i+k)(i+k)} = -\sin \theta_{i-1} \cos \theta_{i-1} \times (\sin^2 \theta_1 \cdot \ldots \cdot \sin^2 \theta_{i+k-2}), \quad \text{for } i \geq 2, \quad k \geq 2. \quad (A7g)$$

Now calculating the components of the Ricci tensor is
straightforward but tedious. First, one can show that $R_{\mu\nu} = 0$
if $\mu = \nu$. For $R_{00}$, we obtain the same result as in
equation (A3). Although $R_{ii}$ have different expressions with
different $i$, one can show that $g^{00}R_0$ (here there is no summation over $i$) has the same expression for any $i$. The result is

$$g^{00}R_0 = (D - 2) \frac{\dddot{a}}{a^2} + \frac{\ddot{a}}{a} + (D - 2) \frac{k}{a^2}, \quad (A8)$$

where there is no summation over $i$. This leads to that all the (ii)-component equations of equation (12) are equivalent. From the expression of $R_{00}$ in equation (A3) and equation (A8), one obtains the Ricci scalar

$$R = (D - 1) \left[ (D - 2) \frac{\dddot{a}^2}{a^2} + 2 \frac{\ddot{a}}{a} + (D - 2) \frac{k}{a^2} \right]. \quad (A9)$$

This finally leads to

$$G_{00} = \frac{(D - 1)(D - 2)}{2} \left( H^2 + \frac{k}{a^2} \right), \quad (A10a)$$

$$g^{00}G_0 = - \frac{(D - 2)(D - 3)}{2} \left( H^2 + \frac{k}{a^2} \right) - (D - 2) \frac{\ddot{a}}{a}, \quad (A10b)$$

where in the second equation there is no summation over $i$. Substituting the above equations into equation (12), one then obtains the dynamic equations (16) and (17). (Again, the continuity equation can be derived from the original two dynamic equations after some simple algebra.)

**ORCID iDs**

Wen-Yuan Ai &https://orcid.org/0000-0002-6042-7407

**References**

[1] Nojiri S and Odintsov S D 2011 Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models Phys. Rep. 505 59

[2] Clifton T, Ferreira P G, Padilla A and Skordis C 2012 Modified gravity and cosmology Phys. Rep. 513 1–189

[3] Capozziello S and De Laurentis M 2011 Extended theories of gravity Phys. Rep. 509 171–321

[4] Nojiri S, Odintsov S D and Oikonomou V K 2017 Modified gravity theories on a nutshell: in dimensionally continued gravity and cosmology Eur. J. Phys. 38 588

[5] Ishak M 2019 Testing general relativity in cosmology Living Rev. Relativ. 22 1

[6] Lovelock D 1971 The Einstein tensor and its generalizations J. Math. Phys. 12 498–501

[7] Boulware D G and Deser S 1985 String generated gravity models Phys. Rev. Lett. 55 2656

[8] Wiltshire D 1986 Spherically symmetric solutions of Einstein–Maxwell theory with a Gauss–Bonnet term Phys. Lett. B 169 36–40

[9] Wheeler J T 1986 Symmetric solutions to the Gauss–Bonnet extended Einstein equations Nucl. Phys. B 268 737–46

[10] Cai R and Soh K 1999 Topological black holes in the dimensionally continued gravity Phys. Rev. D 59 044013

[11] Cai R 2002 Gauss–Bonnet black holes in AdS spaces Phys. Rev. D 65 084014

[12] Cvetiˇc M, Nojiri S and Odintsov S D 2002 Black hole thermodynamics and negative entropy in de Sitter and anti-de Sitter Einstein–Gauss–Bonnet gravity Nucl. Phys. B 628 295–330

[13] Odintsov S and Oikonomou V 2019 Inflationary phenomenology of Einstein–Gauss–Bonnet gravity compatible with GW170817 Phys. Lett. B 797 134874

[14] Odintsov S, Oikonomou V and Fronimos F Rectifying Einstein–Gauss–Bonnet inflation in view of GW170817 [arXiv:2003.13724 [gr-qc]]

[15] Odintsov S D and Oikonomou V K 2020 Swampland implications of GW170817-compatible Einstein–Gauss–Bonnet gravity Phys. Lett. B 805 135437

[16] Glavan D and Lin C 2020 Einstein–Gauss–Bonnet gravity in 4-dimensional space–time Phys. Rev. Lett. 124 081301

[17] Tomozawa Y Quantum corrections to gravity [arXiv:1107.1424 [gr-qc]]

[18] Cognola G, Myrzakulov R, Sebastiani L and Zerbini S 2013 Einstein gravity with Gauss–Bonnet entropic corrections Phys. Rev. D 88 024006

[19] Konoplya R and Zinhailo A Quasinormal modes, stability and shadows of a black hole in the novel 4D Einstein–Gauss–Bonnet gravity [arXiv:2003.01188 [gr-qc]]

[20] Guo M and Li P C 2020 The innermost stable circular orbit and shadow in the novel 4D Einstein–Gauss–Bonnet gravity Eur. J. Phys. C 80 588

[21] Fernandes P G S 2020 Charged black holes in AdS spaces in 4D Einstein–Gauss–Bonnet gravity Phys. Lett. B 805 135468

[22] Wei S and Liu Y Testing the nature of Gauss–Bonnet gravity by four-dimensional rotating black hole shadow [arXiv:2003.07769 [gr-qc]]

[23] Konoplya R A and Zhidenko A 2020 Black holes in the four-dimensional Einstein–Lovelock gravity Phys. Rev. D 101 084038

[24] Hegde K, Naveena Kumara A, Rizwan C A, M A K and Ali M S Thermodynamics, phase transition and joule Thomson expansion of novel 4D Gauss–Bonnet AdS black hole [arXiv:2003.08778 [gr-qc]]

[25] Kumar R and Ghosh S G Rotating black holes in the novel 4D Einstein–Gauss–Bonnet gravity [arXiv:2003.08927 [gr-qc]]

[26] Ghosh S G and Maharaj S D Radiating black holes in the novel 4D Einstein–Gauss–Bonnet gravity [arXiv:2003.09841 [gr-qc]]

[27] Doneva D D and Yazadjiev S S Relativistic stars in 4D Einstein–Gauss–Bonnet gravity [arXiv:2003.10284 [gr-qc]]

[28] Zhang Y, Wei S and Liu Y Spinning test particle in four-dimensional Einstein–Gauss–Bonnet black hole [arXiv:2003.10960 [gr-qc]]

[29] Konoplya R and Zhidenko A BTZ black holes with higher curvature corrections in the 3D Einstein–Lovelock theory [arXiv:2003.12171 [gr-qc]]

[30] Singh D V and Siwach S Thermodynamics and P-v criticality of Bardeen-AdS black hole in 4D Einstein–Gauss–Bonnet gravity [arXiv:2003.11754 [gr-qc]]

[31] Ghosh S G and Kumar R Generating black holes in the novel 4D Einstein–Gauss–Bonnet gravity [arXiv:2003.12291 [gr-qc]]

[32] Konoplya R and Zhidenko A (In)stability of black holes in the 4D Einstein–Gauss–Bonnet and Einstein–Lovelock gravities [arXiv:2003.12492 [gr-qc]]

[33] Kumar A and Kumar R Bardeen black holes in the novel 4D Einstein–Gauss–Bonnet gravity [arXiv:2003.13104 [gr-qc]]

[34] Zhang C, Li P and Guo M Greybody factor and power spectra of the Hawking radiation in the novel 4D Einstein–Gauss–Bonnet–de Sitter gravity [arXiv:2003.13068 [hep-th]]

[35] Hosseini Mansoori S A Thermodynamic geometry of novel 4D Gauss Bonnet AdS black hole [arXiv:2003.13382 [gr-qc]]

[36] Wei S and Liu Y X 2020 Extended thermodynamics and microstructures of four-dimensional charged Gauss–Bonnet black hole in AdS space Phys. Rev. D 101 104018

[37] Singh D V, Ghosh S G and Maharaj S D Clouds of string in 4D novel Einstein–Gauss–Bonnet black holes [arXiv:2003.14136 [gr-qc]]
[38] Churilova M Quasinormal modes of the Dirac field in the novel 4D Einstein–Gauss–Bonnet gravity [arXiv:2004.00513 [gr-qc]]

[39] Islam S U, Kumar R and Ghosh S G Gravitational lensing by black holes in 4D Einstein–Gauss–Bonnet gravity [arXiv:2004.01038 [gr-qc]]

[40] Kumar A and Ghosh S G Hayward black holes in the novel 4D Einstein–Gauss–Bonnet gravity [arXiv:2004.01131 [gr-qc]]

[41] Liu C, Zhu T and Wu Q Thin accretion disk around a four-dimensional Einstein–Gauss–Bonnet black hole [arXiv:2004.01662 [gr-qc]]

[42] Mishra A K Quasinormal modes and strong cosmic censorship in the novel 4D Einstein–Gauss–Bonnet gravity [arXiv:2004.01243 [gr-qc]]

[43] Li S, Wu P and Yu H Stability of the Einstein static universe in 4D Gauss–Bonnet gravity [arXiv:2004.02080 [gr-qc]]

[44] Konoplya R A and Zinhailo A F Grey-body factors and Hawking radiation of black holes in 4D Einstein–Gauss–Bonnet gravity [arXiv:2004.02248 [gr-qc]]

[45] Heydari-Fard M, Heydari-Fard M and Sepangi H Bending of light in novel 4D Gauss–Bonnet–de Sitter black holes by Rindler–Ishak method [arXiv:2004.02140 [gr-qc]]

[46] Jin X, Gao Y and Liu D Strong gravitational lensing of a 4D Einstein–Gauss–Bonnet black hole in homogeneous plasma [arXiv:2004.02261 [gr-qc]]

[47] Zhang C, Zhang S, Li P and Guo M Superradiance and stability of the novel 4D charged Einstein–Gauss–Bonnet black hole [arXiv:2004.03141 [gr-qc]]

[48] Eslam Panah B and Jafarzade K 4D Einstein–Gauss–Bonnet AdS black holes as heat engine [arXiv:2004.04058 [hep-th]]

[49] Naveena Kumara A, Rizwan C A, Hegde K, Ali M S and M A K Rotating 4D Gauss–Bonnet black hole as particle accelerator [arXiv:2004.04521 [gr-qc]]

[50] Aragón A, Bécar R, González P and Vásquez Y Perturbative and nonperturbative quasinormal modes of 4D Einstein–Gauss–Bonnet black holes [arXiv:2004.05632 [gr-qc]]

[51] Malafarina D, Toshmatov B and Dadhich N 2020 Dust collapse in 4D Einstein–Gauss–Bonnet gravity Phys. Dark Univ. 30 100598

[52] Casalino A, Colleaux A, Rinaldi M and Vicentini S Regularized Lovelock gravity [arXiv:2003.07068 [gr-qc]]

[53] Mardones A and Zanelli J 1991 Lovelock–Cartan theory of gravity Class. Quant. Grav. 8 1545–58

[54] Mann R B 1992 Lower dimensional black holes Gen. Rel. Grav. 24 433–49

[55] Mann R B and Ross S 1993 The D → 2 limit of general relativity Class. Quant. Grav. 10 1405–8

[56] Mann R B, Morsink S, Sikkema A and Steele T 1991 Semiclassical gravity in (1+1)-dimensions Phys. Rev. D 43 3948–57

[57] Ohta T and Mann R B 1996 Canonical reduction of two-dimensional gravity for particle dynamics Class. Quant. Grav. 13 2585–602

[58] Mann R B and Ohta T 1997 Exact solution for the metric and the motion of two bodies in (1+1)-dimensional gravity Phys. Rev. D 55 4723–47

[59] Mureika J and Nicolini P 2013 Self-completeness and spontaneous dimensional reduction Eur. Phys. J. Plus 128 78

[60] Frassino A M, Mann R B and Mureika J R 2015 Lower-dimensional black hole chemistry Phys. Rev. D 92 124069

[61] Collins J C 1984 Renormalization (Cambridge: Cambridge University Press) [https://doi.org/10.1017/CBO9780511622656]

[62] Witten E 1988 (2+1)-dimensional gravity as an exactly soluble system Nucl. Phys. B 311 46

[63] Tangherlini F 1963 Schwarzschild field in n dimensions and the dimensionality of space problem Nuovo Cim. 27 636–51

[64] Nojiri S and Odintsov S D 2020 Novel cosmological and black hole solutions in Einstein and higher-derivative gravity in two dimensions EPL 130 10004

[65] Maldacena J M 1999 The large N limit of superconformal field theories and supergravity Int. J. Theor. Phys. 38 1113–33

[66] Miskovic O and Olea R 2009 Topological regularization and self-duality in four-dimensional anti-de Sitter gravity Phys. Rev. D 79 124020

[67] Lu H and Pang Y Horndeski gravity as D → 4 limit of Gauss–Bonnet [arXiv:2003.11552 [gr-qc]]

[68] Kobayashi T 2020 Effective scalar-tensor description of regularized Lovelock gravity in four dimensions J. Cosmol. Astropart. Phys. JCAP07(2020)013

[69] Ma L and Lu H Vacua and exact solutions in Lower-D limits of EGB [arXiv:2004.14738 [gr-qc]]

[70] Lu H and Mao P Asymptotic structure of Einstein–Gauss–Bonnet theory in lower dimensions [arXiv:2004.14400 [hep-th]]

[71] Fernandes P G S, Carrilho P, Clifton T and Mulryne D J 2020 Derivation of regularized field equations for the Einstein–Gauss–Bonnet theory in four dimensions Phys. Rev. D 102 024025

[72] Aoki K, Gorji M A and Mukohyama S A consistent theory of D → 4 Einstein–Gauss–Bonnet gravity [arXiv:2005.03859 [gr-qc]]

[73] Gurses M, Sisman T C and Tekin B Is there a novel Einstein–Gauss–Bonnet theory in four dimensions? [arXiv:2004.03390 [gr-qc]]

[74] Shu F W Vacua in novel 4D Einstein–Gauss–Bonnet gravity: pathology and instability? [arXiv:2004.09339 [gr-qc]]

[75] Mahapatra S A Note on the total action of 4D Gauss–Bonnet theory [arXiv:2004.09214 [gr-qc]]

[76] Hemnagar R A, Kubiznak D, Mann R B and Pollack C 2020 On taking the D → 4 limit of Gauss–Bonnet gravity: theory and solutions J. High Energy Phys. 124069 [gr-qc]

[77] Tian S and Zhu Z H Comment on ‘Einstein–Gauss–Bonnet gravity in four-dimensional spacetime’ [arXiv:2004.09954 [gr-qc]]

[78] Bonifacio J, Hinterbichler K and Johnson L A 2020 Amplitudes and 4D Gauss–Bonnet theory Phys. Rev. D 2 024029

[79] Arrechea J, Delhom A and Jiménez-Cano A Yet another comment on four-dimensional Einstein–Gauss–Bonnet gravity [arXiv:2004.12998 [gr-qc]]