Transport theory of carbon nanotube Y junctions

R Egger\textsuperscript{1}, B Trauzettel\textsuperscript{2}, S Chen\textsuperscript{1} and F Siano\textsuperscript{1}

\textsuperscript{1} Institut für Theoretische Physik, Heinrich-Heine-Universität, D-40225 Düsseldorf, Germany
\textsuperscript{2} Physikalisches Institut, Albert-Ludwigs-Universität, D-79104 Freiburg, Germany
E-mail: egger@thphy.uni-dusseldorf.de

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Abstract. We describe a generalization of Landauer–Büttiker theory for networks of interacting metallic carbon nanotubes. We start with symmetric starlike junctions and then extend our approach to asymmetric systems. While the symmetric case is solved in closed form, the asymmetric situation is treated by a mixture of perturbative and non-perturbative methods. For \( N > 2 \) repulsively interacting nanotubes, the only stable fixed point of the symmetric system corresponds to an isolated node. Detailed results for both symmetric and asymmetric systems are shown for \( N = 3 \), corresponding to carbon nanotube Y junctions.

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1. Introduction

Landauer–Büttiker (LB) scattering theory [1] marks a powerful approach to multi-terminal transport in conventional (Fermi liquid) mesoscopic devices, but is well known to be inapplicable to strongly correlated electron systems. Progress along this direction has recently been reported for interacting metallic 1D systems, where a good description in terms of a Luttinger liquid (LL) [2] is often available. Single-walled carbon nanotubes (SWNTs) represent a prime example for such a system [3]–[5]. For a two-terminal set-up, the generalization of the Landauer formalism to a LL (containing an impurity) coupled to noninteracting leads has been formulated and solved previously [6, 7]. For related work, see [8]–[15]. Recently, these ideas were generalized to also treat \(N\)-terminal starlike LL/nanotube networks \((N > 2)\) [16]. Such a transport theory is then able to make concrete predictions for experiments, e.g. on carbon nanotube Y junctions. First steps towards the production and experimental analysis of SWNT networks are under way. Template-based chemical vapour deposition [17] and electron beam welding methods [18] allow one to fabricate and contact multi-terminal nanotube junctions. Recently, intrinsically nonlinear current–voltage characteristics of multi-walled nanotube Y junctions have been reported [19]. Furthermore, four-terminal devices have been realized by several groups using two crossed NTs [20, 21]. Such nanotube-based (or related) set-ups are of growing interest to many theorists working in the field of interacting mesoscopic systems. To mention some recent work, in [22] a detailed renormalization group analysis of a model similar to ours was performed. Other authors used perturbation theory on the hopping [23, 24] or on the interaction strength [25]. Furthermore, the conductance of a junction of three LLs enclosing a magnetic flux was investigated [26]. The cited activity underlines the need for a generalization of the LB approach to correlated systems. With this work we hope to contribute a significant step towards this aim. Some of our results have been published before [16], but most of the material is new.

The paper is organized as follows. In section 2 we describe our model for the interacting multi-terminal nanotube device, and outline its general solution in the symmetric case. As the step from \(N = 3\) to \(N > 3\) is straightforward [16], we concentrate on nanotube Y junctions \((N = 3)\) in the following sections. In section 3 the nonlinear conductance matrix of a symmetric Y junction is discussed. The case of asymmetric junctions is then studied in section 4. Finally, we conclude and point out some open questions in section 5.

2. General model

We shall first discuss the simplest case of a spinless single-channel LL. In fact, in the first part (section 2.1), we even ignore the interactions and briefly review some aspects of LB theory relevant to our purposes. The case of a symmetric junction is then treated using various boundary conditions for the interacting system in section 2.2, and the extensions necessary to treat nanotubes are discussed in section 2.3. These include the electronic spin and an additional flavour degree of freedom due to the presence of two K points in the first Brillouin zone [4]. The basic set-up studied here is shown schematically in figure 1.

2.1. Fermi liquid case

When one ignores the Coulomb interaction among electrons, transport in a starlike set-up of \(N\) nanotubes can be described by the scattering matrix approach, employing a unitary \(S\) matrix.
Figure 1. Basic set-up: \( N \) nanotubes form a starlike junction (shown for \( N = 3 \)). Each nanotube is supposed to be adiabatically contacted, with chemical potential \( \mu_i \) of the respective reservoir. The cross indicates an additional impurity.

with entries \( s_{ij} \) determining the transmission coefficients \( T_{ij} = |s_{ij}|^2 \) of the system. The current through nanotube \( i \) may we written as

\[
I_i = \frac{e^2}{h} \int_{-\infty}^{\infty} dE \left\{ [1 - T_{ii}(E)] f(E - \mu_i) - \sum_{j \neq i} T_{ij}(E) f(E - \mu_j) \right\},
\]

where \( f(E) \) is the Fermi function (we consider only a single mode here, while a true SWNT has four). Unitarity of the \( S \) matrix implies (1) the Kirchhoff rule (charge conservation), \( \sum_i T_{ij} = 1 \), and (2) gauge invariance under a uniform potential shift in all the reservoirs, \( \sum_j T_{ij} = 1 \). When all transmission coefficients are energy independent, the nonlinear conductance matrix normalized to \( e^2 / h \) reads

\[
G_{ij} = \frac{h}{e^2} \frac{\partial I_i}{\partial \mu_j} = \delta_{ij} - T_{ij}.
\]

Without an applied magnetic field, time-reversal symmetry implies \( T_{ij} = T_{ji} \). For concrete calculations of the \( S \) matrix for nanotube Y junctions, we refer the reader to [27]–[29]. The \( N \times N \) matrix \( S \) relates the outgoing and incoming scattering states at the node,

\[
\Psi_I(0) = S \Psi_R(0).
\]

To describe the \( N \) (for now semi-infinite) tubes, we use \( \Psi(x) = (\psi_1, \ldots, \psi_N) = \Psi_L + \Psi_R \) with \( x \leq 0 \), which contains the outgoing (left-moving, L) components as well as the incoming (right-moving, R) states, with the junction at \( x = 0 \). Unfortunately, the boundary condition (2) is exceedingly difficult to handle for correlated electrons.

To make progress, we shall first consider only symmetric junctions. A wide class of such junctions can be parametrized by a symmetric \( S \) matrix of the form

\[
S = \begin{pmatrix}
\frac{2}{N+\lambda} & -1 & \frac{2}{N+\lambda} & \cdots & \frac{2}{N+\lambda} \\
\frac{2}{N+\lambda} & \frac{2}{N+\lambda} & \cdots & \frac{2}{N+\lambda} & \frac{2}{N+\lambda} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{2}{N+\lambda} & \frac{2}{N+\lambda} & \cdots & \frac{2}{N+\lambda} & -1
\end{pmatrix},
\]

where \( \lambda \geq 0 \) serves as arbitrary parameter. This is the most general \( S \) matrix that implies wavefunction matching at the node [30],

\[
\psi_1(0) = \cdots = \psi_N(0),
\]
for the components of $\Psi = \Psi_L + \Psi_R$. The $S$ matrix (3) leads to
\[
T_{ii} = \frac{(N - 2)^2 + \lambda^2}{N^2 + \lambda^2}, \quad T_{i\neq j} = \frac{4}{N^2 + \lambda^2},
\]
which, for $\lambda = 0$, describes a maximally transmitting symmetric junction, $T_{ii} = (N - 2)^2/N^2$ and $T_{i\neq j} = 4/N^2$. On the other hand, for $\lambda \to \infty$, a very bad junction forms, $T_{ii} = 1$ and $T_{i\neq j} = 0$. For arbitrary $\lambda$, the conductance matrix then follows from equation (1). It is worth noting that no symmetric starlike junction can act as an ideal beam splitter, since backscattering at the node is unavoidable. The only possibility for having an ideal beam splitter, $T_{ii} = 0$, arises in asymmetric junctions [28].

Since phases in $\psi_i(0)$ can be gauged away, a density matching condition arises for any symmetric junction described by equation (3),
\[
\rho_i(0) = \cdots = \rho_N(0),
\]
where $\rho_i(0)$ denotes the total electronic density in tube $i$ at the node. The condition (6) is crucial in our approach for interacting systems, as it avoids the explicit wavefunction matching. The density in equation (6) refers to the sum of the slow (long-wavelength) part, $\rho_i^S(x) = \Psi_{R,i}^\dagger \Psi_{R,i} + \Psi_{L,i}^\dagger \Psi_{L,i}$, and the Friedel oscillation, $\delta \rho_i(x) = \Psi_{R,i}^\dagger \Psi_{L,i} + \Psi_{L,i}^\dagger \Psi_{R,i}$. Since the parameter $\lambda$ will affect the Friedel oscillation, there is a hidden dependence on $\lambda$ not directly visible in equation (6).

2.2. Luttinger liquid case

Next we include the bulk interaction in the tubes, but still restrict ourselves to a symmetric junction. We assume that all nanotubes can be characterized by the same interaction strength, and for now still consider just the spinless single-channel version. Following [4], we may neglect backscattering and therefore obtain a LL, where interactions are encoded in the standard Luttinger Hamiltonian is (we put $\hbar = 1$)
\[
H = \frac{v_F}{2} \sum_{i=1}^{N} \int_{-L}^{0} dx \left[ \Pi_i^2 + \frac{1}{g^2} (\partial_x \theta_i)^2 \right],
\]
where $\Pi_i(x)$ is the momentum canonical to the standard boson field $\theta_i(x)$ for tube $i$, and $v_F$ is the Fermi velocity. Similarly to the two-terminal case, adiabatically coupled external voltage reservoirs (at $x \approx -L$) lead to radiative boundary conditions for the long-wavelength part of the chiral electron densities [6],
\[
(g^{-2} + 1) \rho_{i,R}^0(-L) + (g^{-2} - 1) \rho_{i,L}^0(-L) = \mu_i/\pi v_F.
\]
These conditions only depend on the injected currents, which are independent of any backscattering happening at the node or within the nanotube. To fulfill equation (8), the chiral densities must combine to give
\[
I_i = e v_F (\rho_{i,R}^0 - \rho_{i,L}^0) = \frac{e^2}{2\pi} \left( U_i - \sum_j \tilde{T}_{ij} U_j \right),
\]
\[
\rho_i^0 = \rho_{i,R}^0 + \rho_{i,L}^0 = \frac{e g^2}{2\pi v_F} \left( U_i + \sum_j \tilde{T}_{ij} U_j \right),
\]
which defines effective transmission matrix elements $\tilde{T}_{ij}(g, \{\mu_k\}, k_B T)$. The matrix $\tilde{T}$ should not be confused with the LB transmission matrix $T$ of the noninteracting system. Although $\tilde{T} = T$ for $g = 1$, the effective transmission matrix $\tilde{T}$ cannot be interpreted as the single-electron transmission matrix for $g < 1$. The applied voltages in equations (9) and (10) are defined as

$$eU_i = \mu_i - \bar{\mu}, \quad \bar{\mu} = \frac{\sum_i \mu_i}{N}. \quad (11)$$

Under this definition, gauge invariance is automatically fulfilled if the transport properties of the system only depend on the $U_i$ and not explicitly on the $\{\mu_k\}$. Charge conservation (Kirchhoff’s rule) can be incorporated by imposing

$$\sum_{i=1}^N \tilde{T}_{ij} = 1. \quad (12)$$

Finally, the density matching conditions (6) have to be obeyed.

Due to the pinning amplitude $\delta\rho_i(0)$, i.e. the value of the Friedel oscillation at the node, $\rho_i(0)$ will deviate from $\rho_i^0$ in equation (10). To compute the pinning amplitude, we first note that the Friedel oscillation $\delta\rho_i(x)$ in tube $i$ arises due to interference of the incoming right-movers and the left-movers that are backscattered at the node. Left-movers transmitted from the other $N - 1$ tubes into tube $i$ cannot interfere with the incoming right-movers and will therefore not contribute to $\delta\rho_i(x)$. To verify this argument explicitly, it suffices to analyse the corresponding two-terminal case with one impurity at $x = 0$. Without loss of generality, let us take a $k$-independent bare transmission amplitude $t$ for the impurity. Taking $k > 0$, there is a right-scatterer eigenstate $\phi_k(x)$,

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} e^{ikx} + re^{-ikx} & (x < 0), \\ te^{ikx} & (x > 0), \end{cases}$$

where $|t|^2 + |r|^2 = 1$. Similarly, the left-scatterer is $\phi_{-k}(x) = \phi_k(-x)$. We now show that there is no contribution to $\delta\rho(x)$ from interfering right- and left-scatterers. While this statement is immediately seen to be correct for noninteracting electrons, it can also be confirmed for $g < 1$. To that end, let us expand the electron operator in terms of the right- and left-scatterers instead of the standard R/L movers,

$$\Psi(x) \sim \sum_{k>0} a_k \phi_k(x) + b_k \phi_{-k}(x). \quad (13)$$

Of course, the kinetic part of the Hamiltonian decouples in the Fermi operators $a_k$ and $b_k$. Moreover, when inserting equation (13) into the interaction Hamiltonian, no terms containing only one $a_k$ or $b_k$ can appear because the right- and left-scattering states $\phi_k(x)$ and $\phi_{-k}(x)$ are orthonormal. As a consequence, in the expectation value of the density operator, there can be no terms involving $\langle a_k^\dagger b_k \rangle$ or $\langle b_{-k}^\dagger a_k \rangle$. Hence there is no interference between scattering states originating in different reservoirs, a finding that directly translates to the $N$-terminal set-up under study here.

As a consequence of the above discussion, $\delta\rho_i(0)$ must be identical to the pinning amplitude in a related two-terminal set-up defined by an impurity whose bare reflection probability is determined by $T_{ii}$. Using this mapping, we can thus compute the pinning amplitude from

$$\langle \delta\rho_i(0) \rangle = -g^2 eV_i / \pi v_F, \quad (14)$$

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where the four-terminal voltage $V_t = V_t(U_i)$ has to be determined through a self-consistency relation; see below. The validity of equation (14) can be seen as follows. The long-wavelength density fluctuation in the equivalent two-terminal set-up, due to an applied voltage $U_i$ as given by equation (11), is [6]

$$\langle \rho^0(x) \rangle = -g^2 e V_i \text{sgn}(x)/2\pi v_F.$$  

With respect to a density corresponding to the chemical potential $\bar{\mu}$, taking $x < 0$, we thus have $\rho^0_i = g^2 e V_i(U_i)/\pi v_F$. For the two-terminal problem, antisymmetry of $\langle \rho(x) \rangle$ under $U \to -U$ implies that the net change in density due to $U$ is zero at $x = 0$. Therefore $\rho^0_i$ must be cancelled by the pinning amplitude; see equation (14). Note that this reasoning does not assume the validity of the density matching conditions (6).

For $g = 1/2$, the self-consistency relation determining $V(U)$ can be obtained using refermionization, and takes a rather simple form. Results for this special interaction strength have been discussed in [16]. Using integrability concepts, however, the respective self-consistency relation can also be derived and solved for arbitrary $g$ [7]. Let us write down the necessary equations for $g = 1/p$ (with integer $p$), keeping in mind that arbitrary values of $g$ can also be treated. The bare reflection probability of the impurity determines the energy scale $k_B T_B \propto \exp \theta_B$, see below, and the elementary excitations scattered one by one at the impurity are kinks, antikinks, and $p - 2$ neutral breathers. The two-terminal current $I$ then follows as an integral over rapidities $\theta$,

$$I = \int |T_{++}|^2 (\sigma_+ - \sigma_-) d\theta,$$

where $|T_{++}|^2 = 1/[1 + \exp[-2(p-1)(\theta - \theta_B)]]$. The occupation probabilities are $\sigma_\pm = n f_\pm$ for the kinks/antikinks, where $2\pi n = \epsilon \, d\theta$ with the eigenenergy $\epsilon$ of the kink/antikink. Explicit expressions for the eigenenergies can be found in [7]. Furthermore, the filling fractions read $f_\pm(\theta) = 1/[1 + \exp[(\epsilon(\theta) - \epsilon W/2)/T]]$, where $\pm W/2$ is an effective chemical potential for the kink/antikink. The parameter $W$ has to be determined self-consistently through

$$\int (|T_{++}|^2 + p|T_{--}|^2)(\sigma_+ - \sigma_-) d\theta = \frac{geU}{2\pi v_F},$$

where $U$ is the applied voltage and $|T_{++}|^2 = 1 - |T_{--}|^2$. Since $(e^2/2\pi)(U - W) = (1 - p) I$, see [7], the solution of equations (15) and (16) then yields $W = W(g, k_B T, U)$, and finally the four-terminal voltage follows from $V = W - 2\pi I/g e^2$.

In these equations, $T_B$ appears as the effective impurity strength of the related two-terminal problem. With $T_{ii}$ given in equation (5), an effective dimensionless impurity strength $v_0$ can be determined by examining the noninteracting limit; see below. Given $v_0$, the scale $T_B$ then follows as

$$k_B T_B/D = c_g v_0^{1/(1-g)},$$

where $D$ is the electronic bandwidth and $c_g$ a numerical prefactor of order unity whose precise value is given in [7]. Using this correspondence, the complete transport problem for a symmetric junction described by the bare $S$ matrix (3) can be solved explicitly. Using equation (14), the density matching conditions (6) read $(1 + \tilde{T}_{jj})U_j - 2V_j = \tilde{T}_{kj} U_j$ for all pairs $k \neq j$. Exploiting Kirchhoff’s rule (12), this set of equations admits only the solution

$$\tilde{T}_{ii} = \frac{2 - N}{N} + \frac{2(N-1)V_i}{NU_i}, \quad \tilde{T}_{k\neq i} = \frac{2}{N} - \frac{2V_i}{NU_i}.$$
Evidently, the $\tilde{T}_{ij}$ depend only on the applied voltages $U_i$ and not separately on the chemical potentials $\mu_i$. Thus gauge invariance is automatically fulfilled. The conductance matrix then follows immediately from equation (9). In section 3, we will discuss this solution in some detail for the case $N = 3$ (nanotube Y junction). It then only remains to fix the impurity scale $v_0$ employed in determining $V_i(U_i)$.

As the impurity strength parameter $v_0$ can be obtained in the noninteracting limit, a convenient way to fix $v_0$ is as follows. For $g = 1$, as pointed out above, $\tilde{T}_{ii}$ should simply reproduce the LB transmission coefficient $T_{ii}$ specified in equation (5). In addition, in the noninteracting limit, the ratio $V_i/U_i$ is just the bare reflection coefficient $R$ of the impurity in the two-terminal problem [6]. The latter can be expressed in terms of $v_0$ according to

$$R = \frac{v_0^2}{(1 + v_0^2/4)^2},$$

where $v_0 = 2$ corresponds to the unitary limit; larger values for $v_0$ lead to unphysical results. Combining above equations then leads to

$$v_0(\lambda, N) = 2\left(\sqrt{N^2 + \lambda^2} - \sqrt{2}\right)/\sqrt{N(N - 2) + \lambda^2}. \quad (19)$$

For a very bad junction, $\lambda \to \infty$, we are indeed in the unitary limit, $v_0 = 2$. For a perfect junction with $\lambda = 0$, however, $v_0 = 2(\sqrt{N} - \sqrt{2})/\sqrt{N - 2}$. In particular, for $N = 3$, this value is $v_0 \approx 0.63567$.

2.3. Extensions

Real SWNTs are characterized by four channels, resulting from the electron spin and the additional flavour degree of freedom. Including these degrees of freedom as additional indices $(\sigma, \alpha)$, equation (2) now involves a $(4N \times 4N)$-dimensional $S$ matrix, and the electron state $\Psi = \Psi_R + \Psi_I$ has $4N$ entries $\psi_{i\sigma\alpha}(x)$. Assuming that at the node neither flavour mixing nor spin-flip processes are present, the previous analysis can largely be carried over. Since the $S$ matrix factorizes, the Kirchhoff rule as well as the density matching conditions hold separately for each component $(\sigma\alpha)$. The Hamiltonian (7) then corresponds to a four-channel boson model [4] in terms of symmetric/antisymmetric charge/spin fields. The interaction parameter in the symmetric charge mode is then $g_{c+} < 1$, while the neutral modes are essentially at the noninteracting value. The radiative boundary conditions (8) are only imposed in the symmetric charge sector, and our above discussion then directly leads to an expression very similar to equation (9). However, when computing the effective transmission matrix $\tilde{T}$, the four-terminal voltage parameter $V_i$ is now nontrivially affected by the additional degrees of freedom. This is due to the fact that backscattering at the node couples the four boson modes diagonalizing the Hamiltonian. At the moment, we are not aware of an exact solution to this problem. Nevertheless, two remarks can be made even in this situation. First, the prefactor in the current (9) is changed from $e^2/h$ to $4e^2/h$, reflecting the presence of four channels. Second, we expect the typical power law exponents to lowest order to be modified according to $g \to (g_{c+} + 3)/4$. Therefore, the qualitative physics seems to be captured already by the one-channel version discussed further on. However, we wish to stress that the finite value for $v_0$ obtained under the mapping does not seem to allow for simple perturbative access to the full four-channel problem.
3. Conductance of a symmetric Y junction

In this section we discuss the conductance defined by $G_{ij} = (h/e)\partial I_i / \partial \mu_j$ for the case of a symmetric Y junction ($N = 3$), based on the solution of the general problem in section 2.2. Using equation (18) gives the nonlinear conductance matrix in the form

$$G_{ii} = \frac{8}{9} \left(1 - \frac{\partial V_i}{\partial U_i}\right) + \frac{2}{9} \sum_{j \neq i} \left(1 - \frac{\partial V_j}{\partial U_j}\right),$$

$$G_{j \neq i} = \frac{4}{9} \left(\frac{\partial V_i}{\partial U_i} - 1\right) + \frac{4}{9} \left(\frac{\partial V_j}{\partial U_j} - 1\right) - \frac{2}{9} \left(\frac{\partial V_k}{\partial U_k} - 1\right),$$

where $k \neq i \neq j$ in the second equation. Equation (20) holds for any interaction strength $g$. To obtain explicit results, one only needs to compute $V(U)$ for the corresponding two-terminal problem. We shall show explicit results now for several values of $\lambda$, the special interaction parameter $g = 1/3$, and applied voltages $\mu_1 = E_F + eU$, $\mu_2 = \mu_3 = E_F$. In figure 2, the linear conductance is shown, and in figure 3 the nonlinear conductance is plotted for $k_B T / D = 0.01$.

As illustrated in figures 2 and 3 for $G_{12}$ at $g = 1/3$, at low energies the system always flows to a fixed point corresponding to disconnected nanotubes,

$$G_{ij} = 0.$$  

(21)

To the best of our knowledge, all theoretical work on this problem [16, 22]–[26] agrees that for symmetric junctions, equation (21) represents the only stable fixed point in the case of repulsive electron–electron interaction. This fixed point represents an isolated node weakly coupled to $N$ broken-up quantum wires. It is then of interest to study the energy dependence of the $G_{ij}$ at

**Figure 2.** The temperature dependence of the linear conductance coefficient $G_{12}$ for $g = 1/3$ on a double-logarithmic scale, for several different values of the parameter $\lambda$ in equation (3).
Figure 3. The voltage dependence of the nonlinear conductance $G_{12}$ (double-logarithmic scale) at $k_B T / D = 0.01$, for several values of $\lambda$.

low energies, where a typical LL $g$-dependent power law behaviour arises; see also [25]. The leading corrections to equation (21) can therefore be formulated in terms of the well-known end tunnelling density of states, with associated power law exponents given in textbooks [2].

4. Conductance of an asymmetric Y junction

Next, we construct more general $S$ matrices starting from the solution for the symmetric situation discussed in the two previous sections, focusing for clarity on the case $N = 3$. The basic idea to make the system asymmetric is to add impurities of dimensionless strength $v_i$ to tube $i$. To avoid the appearance of spurious resonances, these are taken sufficiently close to the node, at, say, $x \approx -1/k_F$. This modelling of an asymmetric junction still allows for $g < 1$, e.g. by using perturbation theory in the $v_i$ around the above solution for a symmetric junction ($v_i = 0$). The prototypical situation for exactly one impurity, $v_1 > 0$ but $v_2 = v_3 = 0$, is illustrated in figure 1. Since the impurities are displaced away from $x = 0$, the junction itself is characterized by the symmetric $S$ matrix (3), and therefore the density matching conditions (6) can still be used. Nevertheless, effectively this procedure allows one to generate asymmetric $S$ matrices.

The free real-time action then reads in the framework of bosonization [2]

$$S_0[\theta_i] = \frac{v_F}{2} \sum_{i=1}^{3} \int_{-L}^{0} dx \int dt \left[ \frac{1}{v_F} (\partial_t \theta_i)^2 - \frac{1}{g^2} (\partial_x \theta_i)^2 \right],$$

where the bosonic phase field $\theta_i(x, t)$ is decomposed according to $\theta_i = \theta_i^h + \theta_i^p$. Here, $\theta_i^h$ obeys homogeneous ($U_i = 0$) boundary conditions (8), and $\theta_i^p$ is a particular solution to the equations of motion obeying equation (8). Written in terms of the boson fields, one has $I_0^h(x, t) = (e/\sqrt{\pi}) \triangledown \theta_i(x, t))$ and $\rho_0^h(x, t) = (1/\sqrt{\pi}) \triangledown \theta_i(x, t))$, and one checks easily that the equations of motion as well as equation (8) are satisfied by

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\[ \theta^b_i(x, t) = \sqrt{\pi} I^0_i t/e + \sqrt{\pi} \rho^0_i x, \]

where \( I^0_i \) and \( \rho^0_i \) are given by equations (9) and (10), respectively. Here we assume that the applied voltages are chosen such that \( \bar{\mu} = 0 \).

Adding an impurity of strength \( \Lambda_1 = Dv_1/\pi \) (with bandwidth \( D \)) into tube 1 then yields the action contribution

\[ S_I = \Lambda_1 \int dt \cos[\sqrt{4\pi} \theta^b_i(0, t) + 2\pi I^0_i t/e]. \]

From now on, \( L \to \infty \) to allow for analytical progress. Inserting equation (23) into (22), we obtain

\[ S_0[\theta_i] = S_0[\theta^b_i] - \sqrt{\pi} v_F \sum_{i=1}^{3} (\rho_i(0)/g^2) \int dt \theta^b_i(0, t). \]

The second term can be dropped because of equation (6) and Kirchhoff’s rule, which implies that \( \sum_i \theta^b_i(0, t) \) equals an arbitrary constant (which can be put to zero). The current \( I_i \) is then given by

\[ I_i = \frac{e^2}{2\pi} \left( U_i - \sum_j T_{ij} U_j \right) + \frac{e}{\sqrt{\pi}} \langle \partial_i \theta^b_i(0, t) \rangle. \]

4.1. Weak asymmetry: perturbation theory

The impurity strength \( \Lambda_1 \) measures the degree of asymmetry of the junction. First, we treat the case of weak asymmetry under a perturbative calculation in \( \Lambda_1 \), using the Keldysh technique. In this subsection, we write \( \theta^b_i \to \theta \) and restrict ourselves to \( T = 0 \). Then the current (25) follows from

\[ \langle \partial_i \theta_i(0, t) \rangle = \frac{1}{Z} \sum_{\omega} \int D\theta_i e^{i\omega \theta_i} \delta \left( \sum_{\omega} \theta_i(0, t) \right) \partial_i \theta_i(0, t), \]

where \( S[\theta_i] = S_0[\theta_i] + S_I \). The \( \delta \)-function reflecting Kirchhoff’s rule is enforced using a Lagrange multiplier field \( \gamma(t) \), yielding the additional action contribution \( S_\gamma = \int dt \gamma(t) \sum_i \theta_i(0, t) \). Switching to Fourier space, the action \( S[\theta_i, \gamma] = S_0[\theta_i] + S_\gamma \) reads

\[ S[\theta_i, \gamma] = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{d\omega}{2\pi} \sum_{i=1}^{3} \left[ \frac{v_F}{4} \left( \frac{\omega^2}{v_F^2} - \frac{k^2}{g^2} \right) |\theta_i(k, \omega)|^2 + \gamma(-\omega)\theta_i(k, \omega) \right]. \]

When switching to \( k \)-space, a principal value contribution has been dropped, which is justified in the low-energy regime of interest here [31]. We then introduce rotated phase fields, \( \theta_\pm = \theta_2 \pm \theta_3 \), with conjugate momenta \( \Pi_\pm = (\Pi_2 \pm \Pi_3)/2 \), and subsequently integrate out the fields \( \theta_1 \) and \( \gamma \) by standard Gaussian integration. This procedure leads to the (impurity-free) effective action \( S_0[\theta_+] + S_0[\theta_-] \), where

\[ S_0[\theta_\pm] = \frac{v_\pm}{2} \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} \left( \frac{\omega^2}{v_\pm^2} - \frac{k^2}{g^2} \right) |\theta_\pm(k, \omega)|^2, \]

with \( v_+ = 4v_F/3, g_+ = 4g/3, v_- = 4v_F, \) and \( g_- = 4g \). Fourier transformation back to real space gives

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where the rotated boson fields $\theta_{\pm}(x)$ ‘live’ on the full line. As we shall see below, only a certain linear combination of the right- and left-moving components of $\theta_{\pm}$ couples to the impurity, and hence the extension to the positive half-line ($x > 0$) does not introduce spurious effects while simplifying the subsequent analysis considerably.

Expectation values of the form (26) are then calculated by first evaluating

$$
\langle \partial_t \theta_{\pm}(t) \rangle = \frac{1}{Z} \int D\theta_{\pm} D\theta_{\mp} e^{i(S[\theta_{+}]+S[\theta_{-}])} \partial_t \theta_{\pm}(t),
$$

where the impurity in tube 1 acts only in the $+\overline{+}$ sector,

$$
S[\theta_{+}] = S_0[\theta_{+}] + \Lambda_1 \int \mathrm{d}t \cos[-4\pi \theta_{+}(0, t) + 2\pi I_1^0 t / e].
$$

Since $\langle \partial_t \theta_{-}(t) \rangle = 0$, we are left with the relations $\langle \partial_t \theta_{1} \rangle = -\langle \partial_t \theta_{+} \rangle$ and $\langle \partial_t \theta_{2,3} \rangle = \langle \partial_t \theta_{+} \rangle / 2$.

Equation (29) can now be evaluated perturbatively in $\Lambda_1$, and equation (25) then gives

$$
I_1 = I_1^0 - \delta I_{\Lambda_1}, \quad I_{2,3} = I_{2,3}^0 + \delta I_{\Lambda_1} / 2,
$$

where $I_1^0$ denotes the currents computed in section 2.2 in the absence of the impurity, and the lowest-order perturbative correction due to the impurity is

$$
\delta I_{\Lambda_1} = e g_+(\Lambda_2^2 / D) \sin (2\pi g_+) \Gamma (1 - 2 g_+) \text{sgn} (I_1^0) (2\pi |I_1^0| / e D)^{2\epsilon_{1}-1}.
$$

To treat more general $S$ matrices, one can add a second (weak) impurity $\Lambda_2$, say, into tube 2. Then, within order $\Lambda_2^2$, again correction (32) arises, the correction of order $\Lambda_1 \Lambda_2$ vanishes, and corrections of order $\Lambda_2^3$ may be calculated in the same way as above. This gives

$$
\delta I_2 \pm = \frac{e}{\sqrt{\pi}} \langle \partial_t \theta_{\pm}(0, t) \rangle
$$

$$
= -e g_\mp (\Lambda_2^2 / 2D) \sin (2\pi g_+) \Gamma (1 - 2 g_+) \text{sgn} (I_2^0) (2\pi |I_2^0| / e D)^{2\epsilon_{1}-1},
$$

where we replaced most of the $g_-$ that actually appear in equation (33) by $g_+$, utilizing $g_+ + g_- = 4g_+$, and the currents $I_i$ are

$$
I_1 = I_1^0 - \delta I_{\Lambda_1} - \delta I_{\Lambda_2}^+, \quad I_{2,3} = I_{2,3}^0 + (\delta I_{\Lambda_1} + \delta I_{\Lambda_2}^+ \pm \delta I_{\Lambda_2)^0} / 2.
$$

Evidently, a wide class of asymmetric Y junctions are accessible by this construction method even for interacting SWNTs.

4.2. Arbitrary asymmetry: refermionization

In the case of just one impurity ($\Lambda_1$), it is possible to solve the full transport problem in a nonperturbative manner at the special interaction strength $g = 3 / 8$. This point corresponds to the Toulouse limit $g_+ = 1 / 2$ in the rotated picture used in section 4.1, and allows one to use refermionization [2]. In fact, one can solve this transport problem for arbitrary $g$ using integrability methods, see [7], but for clarity, in this section we only discuss the simplest nontrivial case, $g = 3 / 8$, which admits the use of standard techniques. We only have to study the $\theta_{\pm}$ sector, as the $\theta_{-}$ field decouples from the impurity and can be ignored.
To sketch the $g = 3/8$ solution, let us first define new chiral boson fields ($\phi_+(x)$ is the conjugate field, $\Pi_+ = \partial_x \phi_+$)

$$\varphi_{R/L}(x) = \sqrt{\pi} \left( \sqrt{g_+} \phi_+(x) \pm \theta_+(x) / \sqrt{g_+} \right),$$

which obey the algebra $[\varphi_+(x), \varphi_+(x')] = -i\pi \delta_{pp'} \text{sgn}(x - x')$. Next we switch to linear combinations of these chiral fields, $\varphi_{\pm} = (\varphi_R \mp \varphi_L) / \sqrt{2}$, with associated densities $\rho_{\pm} = \partial_x \varphi_{\pm} / 2\pi$. Then the effective Hamiltonian reads

$$H = \frac{v_+}{4\pi g_+} \int_{-\infty}^{\infty} dx \left[ (\partial_x \varphi_+)^2 + (\partial_x \varphi_-)^2 \right] + \Lambda_1 \cos(\sqrt{2g_+} \varphi_+(0)), \quad (34)$$

where the radiative boundary conditions are $3 \rho_+(L) - \rho_+(-L) = 2g_+ I_1^0 / e v_+$, and $3 \rho_-(L) + \rho_-(-L) = 0$. Therefore, only the $\varphi_+$ boson contributes to transport. Next we introduce effective fermion fields, $\tilde{\psi}(x) = \sqrt{k_F / 2\pi} \exp(i\varphi_+(x))$. It is convenient to introduce yet another fermion $\psi(x)$ via $\tilde{\psi}(x) = (c + c^\dagger) \psi(x)$, where $c$ is an auxiliary fermion. This standard procedure is explained in detail in the textbook [2]. In the fermionized framework, equation (34) becomes

$$H = -\frac{iv_+}{g_+} \int dx \psi^\dagger \partial_x \psi + \sqrt{\frac{v_+ \Lambda_B}{2g_+}} (c + c^\dagger)(\psi(0) - \psi^\dagger(0)), \quad (35)$$

with $\Lambda_B = \pi \Lambda_1^2 / D$ determining the degree of asymmetry of the junction. In this form, the Hamiltonian can be diagonalized immediately, and following [2, 6, 7], a self-consistency relation determining $\delta I_{\Lambda_1}$ follows:

$$\delta I_{\Lambda_1} = (e \Lambda_B / \pi) \text{Im} \psi \left( \frac{1}{2} + \frac{\Lambda_B + 2\pi i(I_1^0 - \delta I_{\Lambda_1}/2)}{2\pi k_B T} \right), \quad (36)$$

where $\psi$ is the digamma function. The currents $I_i$ are then given by equation (31), with $\delta I_{\Lambda_1}$ obtained as the self-consistent solution of equation (36). One checks easily that the $T = 0$ corrections for small $e \Lambda_B / I_1^0$ predicted by this solution coincide with our previous perturbative results; see equation (32) for $g_+ = 1/2$. In the opposite limit of strong asymmetry, we find the expansion

$$\delta I_{\Lambda_1}/I_1^0 = 1 - (\pi^2 / 6)(I_1^0 / e \Lambda_B)^2 + \cdots. \quad (37)$$

This corresponds to the highly asymmetric case with almost no current through tube 1 and the currents through tubes 2 and 3 given by $I_{2/3} = I_{2/3}^0 + I_1^0 / 2$.

Remarkably, this procedure allows one to nonperturbatively treat the full crossover between a symmetric Y junction and a single SWNT that is approached by the tip of another SWNT at some point in its bulk. The corresponding asymptotic power law exponents therefore correspond to the sum of end and bulk exponents in the tunnelling density of states [2]. Finally, we note that the situation of a perfect beam splitter cannot be described by our model. This is due to the initial symmetry of our set-up without impurities, while a beam splitter can only be realized if transport between two of the three tubes is totally suppressed [28].

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5. Conclusions

In summary, we have proposed a generalization of Landauer-type transport theory to $N$-terminal starlike interacting nanotubes. Due to the formulation of three different boundary conditions, reflecting the coupling to external leads, charge conservation, and density matching conditions at the node, it has been possible to find an analytical solution for the nonlinear conductance matrix of the symmetric system for arbitrary electron–electron interaction. Based on the symmetric solution, we have constructed a wide class of asymmetric transmission matrices by adding impurities close to the node in the individual nanotubes. The asymmetric problem can be also be solved in a nonperturbative fashion if just one impurity is put into a nanotube arm. For simplicity, we have discussed this solution for the special interaction strength $g = 3/8$, corresponding to the Toulouse limit in a folded picture.

A largely open issue is the question of noise and particle statistics in such a nonchiral multi-terminal LL network. For low energies, the disconnected-node fixed point seems to hinder the observation of fractional quasiparticle cross-correlations, because tunnelling of electrons from one tube to the other is the first allowed process once the applied voltage is raised. However, for intermediate energies, which could be temperature and/or applied voltage in experiments, the disconnected fixed point opens up and quasiparticle tunnelling between different tubes might give features distinct from the current cross-correlations and therefore allow one to study quasiparticle statistics. These questions will be studied in future work.

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