Nonlinear Electromagnetic Quasinormal Modes and Hawking Radiation of A Regular Black Hole with Magnetic Charge

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Based on a regular exact black hole (BH) from nonlinear electrodynamics (NED) coupled to General Relativity, we investigate its stability of such BH though the Quasinormal Modes (QNMs) of electromagnetic (EM) field perturbation and its thermodynamics through Hawking radiation. In perturbation theory, we can deduce the effective potential from nonlinear EM field. The comparison of potential function between regular and RN BHs could predict their similar QNMs. The QNMs frequency tell us the effect of magnetic charge $q$, overtone $n$, angular momentum number $l$ on the dynamic evolution of NED EM field. Furthermore we also discuss the cases near extreme condition (called as strong charged cases) of such magnetically charged regular BH, the corresponding QNMs spectrum illuminates some special properties in the strong charged cases. For the thermodynamics, we employ Hamilton-Jacobi method to calculate the near-horizon Hawking temperature of the regular BH and reveal the relationship between classical parameters of black hole and its quantum effect.

Keywords: regular black hole; nonlinear electrodynamics; Quasinormal Modes

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I. INTRODUCTION

The research of singularity-free black holes has attracted a considerable attention during the last decades. Fortunately physicists have found a new type of black-hole solution without singularity in General Relativity and more general gravity theories to construct solutions without a singularity, such as string theory or exact conformal field theory. Especially when gravitation coupling to a suitable nonlinear electrodynamics (NED) field, some regular metrics would be obtained. Ayon-Beato, García and Bronnikov successively found some static, spherically symmetric non-singular solutions. It is also remarkable that all the NED satisfy the zeroth and first laws of BH mechanics. In the theory of NED coupling to Einstein equations, it is important to seek a suitable gauge-invariant Lagrangians $\mathcal{L}(F)$ ($F = F_{\mu\nu}$ is the electromagnetic tensor), and its energy-momentum tensor (EMT)

$$T_{\nu\mu} = \frac{1}{4\pi} \left[ \frac{d\mathcal{L}(F)}{dF} F_{\mu\nu} F_{\rho\sigma} - \frac{1}{4} g_{\mu\nu} \mathcal{L}(F) \right],$$

satisfying the symmetry $T^\alpha_\beta = T^\beta_\alpha$. In this paper, we consider the stability of a specific regular BH solution, supposing

$$\mathcal{L}(F) = F \cosh^{-2} \left( \frac{|q|^{3/2}}{2M} \left| \frac{F}{2} \right|^{1/4} \right),$$

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in NED theory, where $M$ is the mass of BH. Without loss the generality, we set $M = 1$ in this paper. Then the static, spherically symmetric metric is described as \[ ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \] where \[ f(r) = 1 - \frac{2M}{r} \left[ 1 - \tanh \left( \frac{q^2}{2Mr} \right) \right], \] where $q$ should obey Theorem 1 discussed in \[ \] (i.e., $q = q_m \neq 0, q_c = 0$, where $q_m$ is magnetic charge).

A black hole can be described completely under perturbation, so that it is necessary to analyze the perturbations once you want to know something about its stability \[ \]. After perturbation, BH will experience three stages (Initial oscillation - QNMs ringing - Ringdown), which can give us a glimpse into the interior region of black holes. The second stage named “quasinormal mode (QNM) ringing” contributes to gravitational wave (GW) detection. From a theoretical point of view, perturbations of a black hole space-time can be performed in two ways: by adding fields to the black hole space-time or by perturbing the black hole metric (the background) itself. The former way refers to physical particle fields such as scalar, Dirac and electromagnetic (EM) field; The latter one is the gravitational perturbation resulting in GWs. Many researches into regular BH QNMs have made great progress, focusing on scalar field \[ ] and Dirac perturbations \[ \]. Usually the singularity of a regular black hole can be vanished under the condition when gravitation coupling to a suitable nonlinear electrodynamics (NED) field, therefore we consider the QNMs of EM perturbations in specific regular space-times will be more meaningful.

The beginning of a QNMs calculation is to reduce the perturbation equations (which vary between spin of perturbation fields) to the two-dimensional wavelike form with decoupled angular variables. Once the variables are decoupled, a equation for radial and time variables usually has the Schrödinger-like form for stationary backgrounds. Then the corresponding potential functions $V(r)$ can be determined, which is the key to numerical computation of QNMs frequency (QNFS). The numerical methods for calculation of QNFS have been developed for several years, and now mainly consist of time domain methods \[ ] , expansion method \[ ] , direct integration in the frequency domain \[ ], WKB method \[ \] , finite differential method \[ \] etc. Since the WKB scheme has been shown to be more accurate for both the real and imaginary parts of the dominant QNMs with $n \leq l$ \[ \], we apply WKB method to the QNFS calculation and compare the results with the ones from expansion method.

As important as the stability of BH, the thermodynamics of black holes is thought to be the connection between black hole physics and quantum theory. Hawking radiation can be seen as quantum tunneling around the horizon, and the Hawking temperature of black hole can be deduced through tunneling rate \[ ] . In order to find the effect of their classical parameters on Hawking radiation, we also study this topic for the magnetically charged regular BH in this paper.

The paper is organized as follows. In section \[ \] we describe the nonlinear electromagnetic field equations in regular spacetimes and determine the shape of potential. Since the regular space-time asymptotically behaves as the Reissner Nordström (RN) BH, we compare them. In section \[ \] we use a 6th-order WKB method and expansion method to compute the QNFS of the regular solution and RN BH. Furthermore we apply a finite differential method to display intuitive images of the QNM perturbation. In section \[ \] we investigate the strong charged cases for the spherically symmetric regular black hole and evaluate the QNMs through 3rd-order WKB method, since the 6th order WKB method may break down if the potential is complex \[ \]. In order to further understand the magnetically charged regular BH, Section \[ \] shows the Hawking radiation of such BH. Conclusions and future work are presented in section \[ \].

## II. THE NONLINEAR ELECTROMAGNETIC FIELD PERTURBATION TO THE REGULAR BH

The action proposed in Einstein-dual nonlinear electromagnetic theory is \[ ]

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} |R - \mathcal{L}(F)|,
\]

where $R$ is the scalar curvature, $F = F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field. $\mathcal{L}(F)$ is the Lagrangian function of this theory, and reduces to the linear case (i.e., Maxwell field) at small $F$: $\mathcal{L}(F) \simeq F$ as $F \to 0$. The tensor $F_{\mu\nu}$ obey the equation \[ \]

\[
\nabla_\mu (\mathcal{L}_F F^{\mu\nu}) = 0,
\]
The following replacement $A_3 = \tilde{A}_3 + \delta A_3$, \hspace{1cm} (7)

where the NED electromagnetic perturbation, $\delta A_3$, has a small value, which can be expressed as $\psi(t,r) P_l$ after separation of variables. According to $F_{\mu\nu} = \partial_{\mu} A_\nu - \partial_{\nu} A_\mu$, $F^{\mu\nu} = F_{\alpha\beta} g^{\alpha\mu} g^{\beta\nu}$, we can calculate the perturbed $F$,

$$F = \frac{2q^2}{r^4} + \frac{4q \csc \theta \psi(t,r) \partial P_l}{r^4} + \mathcal{O}(\psi^2).$$ \hspace{1cm} (8)

Substituting Eq.(8) into Eq.(2) yields $\mathcal{L}(F)$ and $\mathcal{L}_F$.

Therefore, substituting above quantities into Eq.(6) and only considering the terms to first order in the perturbation, we obtain the main equation and find the potential function $V(r)$ after changing it into Schrödinger-like form

$$V_{\text{regular}}(r) = \frac{l(l+1)f(r)}{r^2} - \frac{f^3 \mathcal{L}_F'(\mathcal{L}_F f' + f \mathcal{L}_F') (-3\mathcal{L}_F^2 + 2\mathcal{L}_F \mathcal{L}_F'')} {8\mathcal{L}_F^2} - \frac{f(r)}{4r^4 \mathcal{L}_F} \left\{ q^2 q(l+1) \text{sech}^2 \left( \frac{a\sqrt{q}}{r} \right) \left[ -2 + 3 \text{sech}^2 \left( \frac{a\sqrt{q}}{r} \right) + 5 \frac{r}{a\sqrt{q}} \tanh \left( \frac{a\sqrt{q}}{r} \right) \right] \right\}.$$ \hspace{1cm} (9)

where “’” represents $d/dr$.

The RN BH is a result from the linear electromagnetic field, in which $q$ represents the electric charge $q_e$. So it is meaningful to compare the effective potential function $V(r)$ with the RN solution. We use the same principle under the following replacement $A_0 = q/r$; $A_3 = \delta A_3$; $F = -2q^2/r^3; \mathcal{L}(F) = F$, yielding

$$V_{\text{RN}}(r) = \frac{l(l+1)f_{\text{RN}}(r)}{r^2} - \frac{f_{\text{RN}}^3 \mathcal{L}_F f_{\text{RN}} + f_{\text{RN}} \mathcal{L}_F' (-3\mathcal{L}_F^2 + 2\mathcal{L}_F \mathcal{L}_F'')} {8\mathcal{L}_F^2}.$$ \hspace{1cm} (10)

Note for Eq.(10) due to $\mathcal{L}(F) = F$, $\mathcal{L}_F = 1$, the above equation can be simplified as

$$V_{\text{RN}}(r) = \frac{l(l+1)f_{\text{RN}}(r)}{r^2}.$$ \hspace{1cm} (11)

As predicted, the potential behavior of the regular solution is quite similar to RN BH although the regular metric comes from NED coupling to the Einstein equation while the RN solution is deduced from linear EM theory. FIG[11] displays the effective potential of the regular metric and the RN solution. We can find that their difference mainly concentrated in the area near $r = 0$, and they always tend to be overlapped by each other once away from $r = 0$. Therefore the QNMs frequencies for them would also be much similar, which can be reflected by FIG[2] and Table[1].

### III. QNMS FOR THE NONLINEAR ELECTROMAGNETIC FIELD PERTURBATION

The WKB approximation can be used for effective potentials that have the form of a potential barrier and take constant values at the event horizon and spatial infinity. The method is based on matching the asymptotic WKB
solutions at spatial infinity and the event horizon with a Taylor expansion near the top of the potential barrier through two turning points. Using the above potential functions, we calculate the QNM frequencies through a 6th order WKB approximation, which has been proven to be the most accurate for finding lower overtones of the quasinormal spectrum [33].

Firstly, in order to illuminate the similarities of the QNMs from RN and such regular BHs, it is necessary to compare their QNFs with some typical values. From FIG.2, it can be found that their discrepancy keeps in a small range, which is consistent with the results in Table I and more importantly varying the parameters their QNFs curves have the same shape. So the relationship between QNMs of the regular metric and parameters (such as $q$, $l$ and $n$) can stand for the cases of RN solution. Therefore in the following text, we only figure out the QNMs of the regular BH.

Secondly, we calculate the QNMs frequencies with $n = 0, 1, 2$, and for each overtone $n$ we choose $l = n + 1, n + 2, n + 3$ to determine how the magnetic charges impact the QNMs. FIG. 3. shows the shapes of QNMs with different $n$ and $l$. For the real part of $\omega$, it increases with larger $q$ and $l$, however decreases slightly with higher overtone. This means the perturbation in the fundamental mode (i.e., $n = 0$) with larger $q$ and $l$ leads to more intense QNMs oscillation. For the imaginary part of $\omega$, we can find that 1) given $n, l$, $|\text{Im}(\omega)|$ increases at higher charges. 2) given $q$, in $n = 0$ case the higher $l$ enhances $|\text{Im}(\omega)|$ while in $n = 1, 2$ case $l$ has an inverse effect on $|\text{Im}(\omega)|$. This can be seen as a property of fundamental QNMs oscillation. Overall, for such regular BH the fundament mode with lower charge and angular momentum number plays the dominant role in QNMs oscillation since it has the longest existence.

Next, we employ the expansion method to evaluate the QNMs frequencies. It has been known for many years that QNMs are intimately linked to the existence and properties of unstable null orbits, and then Dolan and Ottewill developed a simple method for determining the QNMs frequency $\omega$, which can be expanded in inverse powers of $l = l + 1/2$.

The NED EM perturbation is governed by the wave equation,

$$\left[ \frac{d^2}{dr_*^2} + \omega^2 - V(r, t) \right] \psi(t, r) = 0,$$

where $d/dr_* = f(r)d/dr$, $\psi(t, r) = \Phi(r)e^{-i\omega t}$. The solution satisfies the following boundary conditions:

FIG. 2: The comparison of QNMs frequencies from the regular BHs and RN BHs.
FIG. 3: The QNMs frequencies of regular metric

1. Pure ingoing waves at the event horizon $\Phi(r) \sim e^{-i\omega r^*}$, $r^* \to -\infty$.

2. Pure outing waves at the spatial infinity $\Phi(r) \sim e^{i\omega r^*}$, $r^* \to \infty$.

According to the expansion method proposed by Dolan and Ottewill, the radial function $\Phi(r)$ can be redefined as

$$\Phi(r) = u(r)e^{\int_{r^*}^{r} \alpha(r)dr^*},$$

where $\alpha(r) = i\omega b_c k_c(r)$ and

$$k_c(r) = (r-r_c)\sqrt{\frac{k^2(r, b_c)}{(r-r_c)^2}}, \quad k^2(r, b) = \frac{1}{b^2} - \frac{f(r)}{r^2},$$

(13)
with the condition
\[ k^2(r_e, b_e) = \left. \frac{\partial k^2(r, b_c)}{\partial r} \right|_{r=r_e} = 0, \]  
(15)
yielding the wave equation as
\[ \frac{d^2 v(r)}{d r^2} + 2\alpha(r)f(r)f'(r) + \left[ \omega^2 + \alpha^2(r) - V(r) + f(r)\alpha'(r) \right] v(r) = 0. \]  
(16)

For the fundamental mode \( n = 0 \), \( \omega \) and \( v(r) \) can be expanded as
\[ \omega = \sum_{i=-1}^{\infty} \left( \frac{a_i}{i} \right), \quad \ln v(r) = \sum_{i=0}^{\infty} \left[ i^{i-1} S_i(r) \right]. \]  
(17)

So the key advantage of the expansion method over many other approaches is that it furnishes us with simple approximations for the wave function (i.e., eq. (13)) and it is more accurate in the case \( l > n \). In order to illustrate the accuracy of the expansion method, we calculate some QNMs frequencies with specific parameters and angular momentum numbers for the regular and RN metrics (cf. Table I). The results show good agreement with those from the WKB approximation.

**TABLE I: QNMs frequencies evaluated by expansion method \((n = 0, M = 1 \text{ ignoring } L^{-3} \text{ term})\)**

| \( q \) | \( l \) | Regular metric | RN metric |
|-------|-------|----------------|-----------|
| 0.1   | 1     | 0.250538 - 0.0930387i | 0.252657 - 0.0897465i |
|       | 2     | 0.458758 - 0.0951121i | 0.460029 - 0.0939269i |
|       | 3     | 0.65815 - 0.0956833i | 0.659059 - 0.0950786i |
| 0.2   | 1     | 0.251969 - 0.093213i | 0.254115 - 0.0898996i |
|       | 2     | 0.461174 - 0.0952761i | 0.462472 - 0.0940833i |
|       | 3     | 0.661545 - 0.0958445i | 0.662484 - 0.0952359i |
| 0.3   | 1     | 0.254405 - 0.0934975i | 0.256624 - 0.0901488i |
|       | 2     | 0.465284 - 0.0955432i | 0.46667 - 0.0943376i |
|       | 3     | 0.667317 - 0.0961069i | 0.668367 - 0.0954916i |
| 0.4   | 1     | 0.257929 - 0.0938805i | 0.260315 - 0.0904819i |
|       | 2     | 0.471218 - 0.0959013i | 0.472834 - 0.0946768i |
|       | 3     | 0.675649 - 0.096458i | 0.677001 - 0.0958326i |
| 0.5   | 1     | 0.262671 - 0.0943383i | 0.265402 - 0.0908729i |
|       | 2     | 0.479186 - 0.096326i | 0.481308 - 0.095074i |
|       | 3     | 0.686829 - 0.0968736i | 0.688863 - 0.0962315i |

Finally in order to determine the QNMs oscillation shape, we adopt a finite difference method to study the dynamical evolution of the NED field perturbation in the time domain and examine the stability of the regular black hole.

Firstly, we re-write the wave equation in terms of the variables \( \mu = t - r_* \) and \( \nu = t + r_* \) \((dr_* = dr/f(r))\):
\[ 4 \frac{\partial^2 \Psi}{\partial \mu \partial \nu} + V(r)\Psi = 0. \]  
(18)
This two-dimensional wave equation can be integrated numerically by using the finite difference method suggested in [22, 23]. It can be discretized as
\[ \psi(\mu + \delta \mu, \nu + \delta \nu) = \psi(\mu, \nu) + \psi(\mu + \delta \mu, \nu) - \psi(\mu, \nu) - \delta \mu \delta \nu V \left( \frac{2\nu - 2\mu + \delta \nu - \delta \mu}{4} \right) \psi(\mu + \delta \mu, \nu) + \psi(\mu, \nu + \delta \nu) + O(\epsilon^4), \]  
(19)
where \( \epsilon \) is an overall grid scale factor (i.e., \( \delta \mu \sim \delta \nu \sim \epsilon \)). Set a boundary condition \( \psi(\mu, \nu = \nu_0) = 0 \) and suppose an initial perturbation as a gaussian pulse, centered on \( \nu_c \) and with width \( \sigma \) on \( \mu = \mu_0 \) as
\[ \psi(\mu = \mu_0, \nu) = e^{-\frac{(\nu - \nu_c)^2}{2\sigma^2}}. \]  
(20)
FIG. 4: The dynamical evolution of NED EM field in the background of the regular black hole spacetime. The constants in the gaussian pulse $\nu_c = 1$ and $\sigma = 1$.

The above equations indicate the key steps for numerical calculation of the field dynamical evolution. FIG.4 is just the dynamical evolution of the NED EM field in the background of the regular space-time. As expected, the QNMs oscillation of RN and regular solutions are almost the same (cf. the left subplot). The oscillation frequency increases with $q$ and $l$, and decay speed becomes slightly faster with larger $q$ and $l$, which are in accordance with the results of WKB method.

IV. STRONG CHARGED CASES

In this section, we consider the strong charged cases around the extreme condition. When the inner horizon $r_+$ coincides with the outer horizon $r_0$, the extreme condition occurs. In order to maintain cosmic censorship \cite{34, 35}, the mass $M$ and $q$ should be confined by certain relationship in order to keep $r_+ \leq r_0$. So we have

$$f(r) = 1 - \frac{2M}{r} \left(1 - \tanh \left(\frac{|q|^2}{2Mr}\right)\right) = 0,$$

$$f'(r)|_{r=r_0} = 0,$$

The second equation comes from requiring that the temperature of a BH in the extreme condition should be zero. Through Eq. (21) and Eq. (22), the location of the event horizon can be expressed explicitly as $r_0 = 4Mq'/q' - W(-q'\exp(q'))$, where $W(x)$ is Lambert function and $q' = q^2/(2M)^2$. So we find one threshold quantity as $(q_c \simeq 1.0554, r_0 \simeq 0.8712)$. Then we plan to analyze the QNMs around the threshold charge $q_c$. Due to the special function tanh in $f(r)$, the potential function has a complicated form causing endless computation and inaccurate 6th order WKB results. Therefore we employ a Taylor expansion around $q_c$ and consider the QNMs by 3rd order WKB approximation. Given the common parameters, FIG.5 compares the QNMs of the regular BH and RN BH around extreme condition, and indicates that in the strong charged cases the regular BH after the NED EM perturbation tends to stability more faster than the RN BH.

FIG.6 describes the QNMs frequencies of the regular BH around the extreme charge $q_c$. Like the behavior of weak charged case, the real part of $\omega$ is enhanced by the increase of $q$ and $l$. But the imaginary part of $\omega$ behaves much differently from the weak charged condition. It is found that the absolute value of $\omega$ decreases with larger $q$ and lower overtone. The most important property is the effect of the angular momentum number, which increases the decay rate for the $n = 0$ case, but reduces the decay when $n = 2$. Especially in the $n = 1$ case, $|\omega|$ has the similar shape as the one with $n = 2$ at lower charge. As the charge approaches $q_c$, the decay behavior is different from the cases of $n = 0, 2$. This may become the characteristic of such strong charged BH’s perturbation.

For the strong charged cases, we also employ the finite difference method to determine the dynamical evolution of NED EM perturbation. The same calculation procedures as mentioned in the previous section produce the results in FIG.7. This figure shows that as $q$ increase the decay rate decreases while the oscillation frequency increases, which is also agreement with the WKB results.

V. HAWKING RADIATION OF THE MAGNETICALLY CHARGED REGULAR BLACK HOLE

As an important issue of black hole physics, Hawking radiation has been proven to support a new way to understand the thermodynamics of black holes. Then, we apply a charged Hamilton-Jacobi equation, which is considered as
an effective and simpler method for studying black hole’s radiation, to study such regular black hole’s Hawking temperature.

The charged Hamilton-Jacobi equation is given as \[36\]:

\[
g^{\mu\nu} \left( \frac{\partial S}{\partial x^\mu} + eA_\mu \right) \left( \frac{\partial S}{\partial x^\nu} + eA_\nu \right) + m^2 = 0, \tag{23}\]

Where \( S \) is the action and the \( e \) is the charge of a Hawking radiation particle, especially, in regular charged spacetime, 
\( A_\mu = \delta_\mu^3 A_3 = -q_m \delta_3^\mu \cos \theta \). Through variable separation the action can be written as

\[
S = -\omega t + R(r) + Y(\theta, \phi). \tag{24}\]

So in the regular black hole spacetime, eq.(23) can be rewritten as

\[
-\frac{\omega^2}{f(r)} + f(r) \left( \frac{dR}{dr} \right)^2 + m^2 + \frac{\lambda}{r^2} = 0, \tag{25}\]

\[
\left( \frac{\partial Y}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial Y}{\partial \phi} - eq_m \cos \theta \right)^2 = \lambda. \tag{26}\]

According to eq.(25) and eq.(26), the radial function can be deduced from the partial equation

\[
\frac{dR(r)}{dr} = \pm \sqrt{\omega^2 - f(r)(m^2 + \frac{\lambda}{r^2})} \frac{f(r)}{f(r)}. \tag{27}\]

Since our concern is around the horizon \( r_0 \), we can expand the \( f(r) \) as

\[
f(r_0) = f'(r_0)(r - r_0) + f''(r_0)\frac{(r - r_0)^2}{2} + .... \tag{28}\]
Then the radial function is deduced as
\[ R_{\pm}(r) = \pm \int \frac{\sqrt{\omega^2 - f'(r_0)(r - r_0)(m^2 + \frac{1}{r^2})}}{f'(r_0)(r - r_0)} dr. \] (29)

Apply residue theorem to eq. (29), the radial function \( R(r) \) should be
\[ R_{\pm}(r) = \pm \frac{i \pi \omega}{f'(r_0)}. \] (30)

So the tunneling rate of Hawking radiation is \[ 44, 45 \]
\[ \Gamma = \exp(2\text{Im}R_+) = \exp \left( -4\pi \frac{\omega}{f'(r_0)} \right), \] (31)
FIG. 7: The dynamical evolution of NED EM field in the background of the extreme regular black hole spacetime. We set $M = 1$. The constants in the gaussian pulse $\nu_c = 1$ and $\sigma = 1$.

and the Hawking temperature should be

$$T_h = \frac{f'(r_0)}{4\pi} = \frac{2M + (r_0 - 4M) \tanh^{-1} \left(1 - \frac{r_0}{2M}\right)}{8M\pi r_0}. \tag{32}$$

where $r_0$ can be determined by

$$q_m^2 = 2Mr_0 \tanh^{-1} \left(\frac{2M - r_0}{2M}\right). \tag{33}$$

For the extreme BH, the Hawking temperature is zero. But in the weak charged case, the Hawking temperature illuminates how the parameters, such as charge and mass of BH, impact the thermodynamics of a black hole. FIG 8 shows the shapes of $T_h(M)$. In order to make the function $T_h(M, r_0)$ meaningful, the domain of the mass shifts to the right with increasing $r_0$. For the cases with $r_0 = 1, 1.5, 2$, the Hawking temperature has a maximum value. Meanwhile, the figure tells us the relationship among $q_m, M$ and $T_h$.

VI. CONCLUSIONS

The regular BH discussed by us is one solution of nonlinear electrodynamics coupling with Einstein theory. Choosing different Lagrangians functions $\mathcal{L}(F)$, some other various regular solutions can be deduced [13]. All of them can return to Schwarzschild black hole when $q = 0$. For non-zero $q$, they have no singularity—even at $r = 0$, $f(r)$ is finite. Furthermore, it can be expanded into

$$f(r) = 1 - \frac{2M}{r} + \frac{q^2}{r^2} - \frac{q^6}{12(M^2r^4)} + \mathcal{O}(q^7). \tag{34}$$

In fact, all regular BHs can be expanded into a similar polynomial, so it can be noted that the regular metric asymptotically behaves as the RN BH. FIG 9 indicates the difference between the metric of regular BH and RN BH. It is apparent that the regular BHs with lower charge would be much more similar to RN space-time except in the area around the singular point $r = 0$.

For the weak charged condition, the magnetic charge $q$ should be confined by $q < q_c$ to make sure that the regular spacetime has physical meaning. Considering the complicated expression of potential function, we try to seek more accurate QNMs frequency through higher order Taylor expansion. The FIG 3 from 6th WKB results shows the behavior of the perturbation field evolution in detail.

For the strong charged cases, it is meaningful to reveal how the perturbation behaves when the magnetic charge $q$ approaches $q_c$. We find the distinguishing properties between near extreme case and weak charged conditions, which are mainly be reflected in the relationship between magnetic charge and decay rate. There is a curve intersection in the $n = 1$ case, and the effect of angular momentum number $l$ on perturbation field decay appears opposite between $n = 0$ and $n = 2$. 
FIG. 8: The Hawking temperature varies different mass of black hole. The dashing, solid, dash-dotted lines are $r_0 = 1, 2, 3$ respectively.

FIG. 9: The figure illustrates $f(r)$ of the regular metric and RN BH. From left to right $q = 1, 0.5, 0.1$ respectively. Note for regular BH $q = q_m$ while $q = q_e$ in RN condition.

But in each case, we can find the fundamental mode with low angular momentum number plays dominant role on perturbation field decay rate. This is also supported by the intuitive image from finite difference method.

From the discussion of Hawking radiation, we find another useful property of regular BH. Although we do not determine how the magnetic charge impacts on the temperature evidently, that can be deduced from Eq.(33).

As an important application of NED theory, we choose this solution to understand the stability of initially global regular configurations. In one sense, it provides a new approach to improve general relativity. However, we also notice that there are some deficiencies of this metric. For instance the Lagrangians $\mathcal{L}(F)$ is not a monotonic function in the whole range of invariant $F$, but rather having some junctions where singularities appear [37]-[39]. That can be well revealed using the effective metric[41]. Furthermore some researchers put forward more generic conditions [42] where the inner horizons of such regular black holes would be instable due to a mass-inflation [43]. Therefore we plan to study these issues in our future works.
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