Cyclotron Emission with a Helical Wavefront from an Electron Accelerated by the Circularly Polarized Wave

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Abstract. In this paper, we calculated the radiation from the state where an electron in cyclotron motion under the uniform magnetic field $B_{ex}$ was accelerated by the externally applied circularly polarized wave with strong $E_{in}$ and $B_{in}$. The electron was accelerated in the parallel direction by the term $\beta_\perp \times B_{in}$ ($\beta_\perp$: perpendicular velocity of the electron) as well as in the perpendicular direction by the electric field of the wave $E_{in}$. Then, the electron can be resonantly accelerated when the $B_{ex}$ is set to the magnetic field strength which is equivalent to frequency of the externally applied circularly polarized wave. The radiation from such an accelerated electron was confirmed to have helical wavefront at harmonics cyclotron frequency.

Keywords: Electron Cyclotron Emission, Helical Wavefront, Circularly Polarized Wave

1. Introduction

It is well known that a photon has Spin Angular Momentum (SAM) and Orbital Angular Momentum (OAM). The SAM is characterized by a polarization of waves. A right-handed polarized wave and a left-handed polarized wave have $\pm 1$ SAM, respectively. The presence of a photon OAM was shown by Allen [1] in 1992. The paraxial solution of the wave equation in the cylindrical coordinate system shows the presence of an angular momentum which does not depend on the polarization. This is called an orbital angular momentum in order to distinguish the momentum from a spin angular momentum. At present, one of the waves with OAM is called the Laguerre-Gauss (LG) beam. In particular, the LG beams with frequency band of the visible light are called an optical vortex. The reason why the LG beam is called a vortex is that there is a phase term that depends on the azimuth direction and forms a helical wavefront. Furthermore, the beam with helical wavefront has a singular point which cannot be defined by phase on the optical axis. Then the intensity on the optical...
axis is defined by zero.

The beam with helical wavefront is primarily converted from beam without an OAM via the passive components such as spiral phase plate, holography, and q-plate [2–4]. These indirect methods generate the vortex beam from Gaussian beams with a plane or a spherical phase front by conversion elements. However, it was shown by Katoh [5] in 2017 that the radiation field created by charged particles in circular motion has a helical wavefront. At present, the actively generated beam with helical wavefront has been studied in a shorter wavelength regime such as γ-rays and extreme ultraviolet. For example, it has been reported that the radiation from the helical undulator has helical wavefront in extreme ultraviolet regime [6]. In addition, it is theoretically shown that the inverse Compton scattering using a high intensity circularly polarized laser also has helical wavefront [7, 8].

However, the studies of Electron Cyclotron Emission (ECE) with helical wavefront in the longer wavelength regime such as millimeter-wave have not been reported yet. This is because the measurement focused on the helical wavefront has never been carried out. In addition, the phase of the ECE from the multi-electron system is usually cancelled out due to the random rotation phase of each electron. The measurement system for millimeter-wave with helical wavefront was recently reported [9]. In order to demonstrate the ECE with helical wavefront, we designed and developed an experimental device that generates the ECE with helical wavefront in multi-electron system [10]. In this experiment, we attempt to control the rotation phase of electrons with cyclotron motion by externally applied circularly polarized wave so that the coherent radiation can be generated. This experiment is equivalent to the principle of Electron Cyclotron Resonance Heating (ECRH) in fusion plasma, and the generation and measurement of ECE in millimeter-wave regime with helical wavefront greatly contribute to the understanding of plasma heating physics in fusion research.

In this study, we calculated the radiation from the state where an electron in cyclotron motion under the uniform magnetic field $B_{ex}$ was accelerated by the externally applied circularly polarized wave with strong $E_{in}$ and $B_{in}$. The electron was accelerated in the parallel direction by the term $\beta_\perp \times B_{in}$ ($\beta_\perp$: perpendicular velocity of the electron) as well as in the perpendicular direction by the electric field of the wave $E_{in}$. Then, the electron can be resonantly accelerated when the $B_{ex}$ is set to the magnetic field strength which is equivalent to frequency of the externally applied circularly polarized wave. In this calculation, we consider one electron for simplification. However, the same phenomena will happen to each electron in multi-electron system. That is, we can generate the state where the phase relationship between acceleration in perpendicular direction of each electron and the direction of electric field $E_{in}$ is the same at the give time and place. Therefore, we can obtain the coherent radiation with helical wavefront even in multi-electron system.

This paper is composed of four sections. In section 2, the relativistic equation of motion of an electron describing what happens when the electron with cyclotron motion in uniform magnetic field is accelerated by externally applied electromagnetic wave is briefly derived. In section 3, calculation results of the trajectory information of the electron are discussed, and the calculation results of the radiation with helical wavefront from the electron are also shown. The summary of this paper is provided in section 4.
2. Cyclotron Motion in the Externally Applied Electromagnetic Field

The relativistic equation of motion regarding an electron in the electromagnetic field is represented by

\[
\frac{d}{dt}\left[\gamma m_e u(t)\right] = -e\left[E_{in}(r, t) + u(t) \times (B_{in}(r, t) + B_{ex})\right] = F_{E_{in}} + F_{B_{in}} + F_{B_{ex}}
\]

where \(m_e, u(t),\) and \(e\) are represented by the rest mass of an electron, the velocity of the electron, and the elementary charge, respectively. \(B_{ex}\) is the static external magnetic field. In addition, the \(E_{in}(r, t)\) and \(B_{in}(r, t)\) are the electromagnetic fields applied from outside to the electron. Lorentz force is defined as \(F_{E_{in}}, F_{B_{in}},\) and \(F_{B_{ex}}\) on the last equal sign below,

\[
F_{E_{in}} \equiv -eE_{in}(r, t)
\]

\[
F_{B_{in}} \equiv -eu(t) \times B_{in}(r, t)
\]

\[
F_{B_{ex}} \equiv -eu(t) \times B_{ex}.
\]

The \(\gamma\) is the relativistic factor as follows:

\[
\gamma \equiv \frac{1}{\sqrt{1 - |\beta|^2}}
\]

where the \(\beta\) is velocity normalized to the speed of light \(c\) defined by

\[
\beta \equiv \frac{u(t)}{c}.
\]

Although the \(\gamma\) and the \(\beta\) are dependent on time, we have omitted the variable for simplification. From now, we will expand and deform the left-hand side (LHS) of eq.(1), \(F_{E_{in}}, F_{B_{in}},\) and \(F_{B_{ex}}\).

(i) LHS of eq.(1) (see Appendix A.)

By expanding the LHS of eq.(1), we can obtain

\[
\frac{d[\gamma m_e u(t)\]}{dt} = \gamma m_e c \left(I + \gamma^2 (\beta \otimes \beta)\right) \cdot \dot{\beta}
\]

where \(\dot{\beta}\) and \(I\) are the acceleration of the normalized velocity and the unit matrix. In addition, the symbol \(\otimes\) means dyadic, which is defined by using two vectors \(A = (A_x, A_y, A_z)\) and \(B = (B_x, B_y, B_z)\) as follows:

\[
A \otimes B = \begin{pmatrix}
A_x B_x & A_x B_y & A_x B_z \\
A_y B_x & A_y B_y & A_y B_z \\
A_z B_x & A_z B_y & A_z B_z
\end{pmatrix}
\]
Also we use the identity below to obtain the relationship eq.(7),

\[(\beta \cdot \dot{\beta})\beta = (\beta \otimes \beta) \cdot \dot{\beta}.\]  

(9)

(ii) \(F_E\)  
We can rewrite by using unit matrix \(I\) below

\[F_E = -eI \cdot E_{in}(r, t).\]  

(10)

(iii) \(F_B\)  
In Maxwell’s equations for the curl of an electric field, \(E_{in}(r, t)\) and \(B_{in}(r, t)\) are related to each other,

\[\nabla \times E_{in}(r, t) = -\frac{\partial B_{in}(r, t)}{\partial t}.\]  

(11)

Fourier transformation regarding \(E_{in}(r, t)\) and \(B_{in}(r, t)\) are represented by

\[E_{in}(r, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} dk \tilde{E}_{in}(k, t)e^{i(k \cdot r - \omega_k t)}\]  

(12)

\[B_{in}(r, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} dk \tilde{B}_{in}(k, t)e^{i(k \cdot r - \omega_k t)}\]  

(13)

By substituting eq.(12) and eq.(13) into eq.(11), we can obtain the relationship of Fourier component between \(\tilde{E}_{in}(k, t)\) and \(\tilde{B}_{in}(k, t)\) as follows,

\[\tilde{B}_{in}(k, t) = \frac{k}{\omega_k} \times \tilde{E}_{in}(k, t).\]  

(14)

Furthermore, by substituting eq.(13) with eq.(14) into eq.(3), \(F_B\) is deformed as follows,

\[F_B = v(t) \times \frac{-e}{(2\pi)^3} \int_{-\infty}^{\infty} dk \frac{k}{\omega_k} \times \tilde{E}_{in}(k, t)e^{i(k \cdot r - \omega_k t)}\]  

\[= -e\beta \times n \times E_{in}(r, t)\]  

(15)

where \(n\) is the unit vector with (0, 0, 1). That is, \(n\) represents the propagation direction for the electromagnetic wave with the relationship below,

\[k = \frac{\omega_k}{c} n.\]  

(16)

Note that the externally applied electromagnetic wave with \(E_{in}(r, t)\) and \(B_{in}(r, t)\) is considered as plane wave because the direction of \(k\) is regarded to be independent of frequency. Furthermore, eq.(15) is rewritten by vector identity

\[\beta \times n \times E_{in}(r, t) = (n \otimes \beta) \cdot E_{in}(r, t) - ((n \cdot \beta)I) \cdot E_{in}(r, t)\]  

(17)
that is,

\[ F_{B_{in}} = ((-n \cdot \beta)I + n \otimes \beta) \cdot E_{in}(r, t). \]  

(18)

(iv) \( F_{B_{ex}} \)

We can rewrite as follows,

\[ F_{B_{ex}} = -m_e c \omega_{ce}(\beta \times \hat{e}_z) \]  

(19)

where the \( \hat{e}_z \) indicates the unit vector along the z-direction. Here, we consider the situation where the static magnetic field \( B_{ex} = (0, 0, B_{ex}) \) is applied. And the cyclotron angular frequency is defined by

\[ \omega_{ce} \equiv \frac{eB_{ex}}{m_e}. \]  

(20)

By substituting eq.(7), eq.(10), eq.(18), and eq.(19) into eq.(1), and multiplying the inverse matrix for \( I + \gamma^2(\beta \otimes \beta) \) by both sides, we can obtain

\[ \dot{\beta} = -\frac{e}{m_e c \gamma} \left( I + \gamma^2(\beta \otimes \beta) \right)^{-1} \cdot ( (1 - n \cdot \beta)I + n \otimes \beta ) \cdot E_{in}(r, t) \]

\[ -\left( I + \gamma^2(\beta \otimes \beta) \right)^{-1} \cdot \frac{\omega_{ce}}{\gamma}(\beta \times \hat{e}_z). \]  

(21)

The inverse matrix in eq.(21) is represented as follows (see Appendix B.),

\[ \left( I + \gamma^2(\beta \otimes \beta) \right)^{-1} = I - \beta \otimes \beta. \]  

(22)

Furthermore, the second term of eq.(21) is calculated by

\[ (I - \beta \otimes \beta) \cdot (\beta \times \hat{e}_z) = \beta \times \hat{e}_z. \]  

(23)

Therefore, the motion of the electron is represented by the relativistic equation of motion as follows

\[ \dot{\beta} = \frac{-e}{m_e c \gamma} (I - \beta \otimes \beta) \cdot ((1 - n \cdot \beta)I + n \otimes \beta) \cdot E_{in}(r, t) - \frac{\omega_{ce}}{\gamma}(\beta \times \hat{e}_z) \]  

(24)

This equation shows that if the external electric field satisfies with \( E_{in}(r, t) = 0 \), this equation represents pure cyclotron motion. The orbit is distorted by the external electric field.
3. Calculation Results

We consider a situation where the electromagnetic wave is applied to an electron with cyclotron motion in a static magnetic field. As shown in Fig.1, an electron is injected from $z = -415$ mm with perpendicular velocity that corresponds to $20$ keV. Then, the momentum of the electron only has the $x$-$y$ plane, that is, the electron initially rotates on the $x$-$y$ plane. In addition, the electromagnetic wave applied from the outside is considered as Gaussian beam with right-handed circular polarization as follows

$$E_{in} = E_0 \frac{\omega_0}{\omega_c} \exp\left(\frac{-r^2}{\omega_z^2}\right) \exp\left(i(kz - \omega t + \frac{kr^2}{2R_c} + \zeta)\right) e_+$$  

with

$$e_+ = e_x + ie_y$$

where $E_0$ is the amplitude of the electric field; $r$ and $z$ are parameters in the cylindrical coordinate system; $\omega_0$, $\omega_z$, $k$, $R_c$, and $\zeta$ are waist size, spot size, wave-number, radius of curvature, and Gouy phase shift, respectively. $\omega$ and $t$ are angular frequency and time. The waist size is $30$ mm, and is located at the origin. The frequency of the Right-handed Circularly Polarized (RHCP) wave is $82.7$ GHz. And the power of the RHCP wave is $100$ kW. The external applied magnetic field $B_{ex}$ is set to $2.954$ T which is the magnetic field strength equivalent to frequency for external applied RHCP wave.
In this calculation, the ordinary differential equation is numerically solved to obtain the electron trajectory. After that, the calculated trajectory information is substituted into the Liénard-Wiechert potential as shown below to calculate the radiation field [11]:

\[
E(R, t) = \frac{-e}{4\pi \varepsilon_0} \frac{n' \times (n' - \beta(t')) \times \beta(t')}{c|r|[(1 - n' \cdot \beta(t'))^3]
\]

with

\[
t = t' + \frac{|r|}{c}
\]

where \(E(R, t)\) is an electric field on the observer point, \(\varepsilon_0\) is dielectric constant of the vacuum, \(r\) is the vector toward observer from electron position, and \(n'\) is the unit vector toward observer from position of the electron. \(t'\) is retarded time. Here, the retarded time and the observer time are explicitly distinguished in eq.(27). Note that the retarded time \(t'\) is the time frame of electron motion and this is calculated at first on the \(t'\) time frame. And then, the advanced time \(t\) is calculated by eq.(28) as observer time frame. The radiation field is observed on the upper hemisphere where the distance from the origin is \(|R| = 15\) m. Since the radiation field includes higher harmonics radiation other than the fundamental wave, the observed radiation signal is spectrally decomposed by Fast Fourier Transform (FFT). Before executing FFT, it is necessary to equalize the sample step by spline interpolation regarding the result of the calculated radiation field. This is because when the radiation field is calculated by Liénard-Wiechert potential, the time step becomes not uniform due to the retarded time. Thus, constant time step data set are required for the FFT. The electron cyclotron period \(T_{ce} \approx B_{ex} = 2.954\) T is 12 p sec. The calculation time step \(\Delta t = 150\) f sec. is sufficiently smaller than \(T_{ce}\).

3.1. Electron Trajectory in the Electromagnetic Wave

An electron trajectory can be calculated by eq.(24) when the electromagnetic wave with the power of 100 kW and frequency of 82.7 GHz is applied to an electron with cyclotron motion. Figure 2 shows the trajectory and some parameters of the electron. Figure 2 (a) shows the part of the electron trajectory. It can be seen that the Lamar radius gradually becomes larger with time. However, the increase of the Lamar radius does not continue constantly. The Lamar radius settles to a certain value with time. This is a phenomenon caused by a phase relationship between the electron and the externally applied RHCP wave. Figure 2 (g) purple line shows the phase relationship between the velocity of the electron in the \(x-y\) direction \(\beta_\perp\) and the electric field \(E_{in}\). Furthermore, Fig.2 (g) green line shows the phase relationship between the velocity of the electron in the \(x-y\) direction \(\beta_\perp\) and the magnetic field \(B_{in}\). These are defined as follows

\[
\cos \phi_E = \frac{E_{in} \cdot \beta_\perp}{|E_{in}| |\beta_\perp|}
\]

\[
\sin \phi_B = \frac{B_{in} \cdot \beta_\perp}{|B_{in}| |\beta_\perp|}
\]
Figure 2: Trajectory information and some parameters. (a): Part of the electron trajectory when the RHCP wave is applied into an electron with cyclotron motion in static uniform magnetic field from the outside. (b)-(d): Trajectory information concerning positions, velocities, and accelerations. The Larmor radius $r$ is defined by $r = c|\beta_\perp|/\omega_{ce,\gamma}$. (e): Relativistic electron cyclotron frequency and Doppler-shifted frequency of RHCP wave seen from the reference frame of electron with relativistic velocity. (f): Kinetic energy of the electron. (g): Phase relationships between the acceleration of the electron in the $x$-$y$ direction $\dot{\beta}_\perp$ and the electric field $E_{in}$, also between the velocity of the electron in the $x$-$y$ direction $\beta_\perp$ and the magnetic field $B_{in}$, which are defined by eq.(29) and eq.(30).

where $\phi_E$ indicates the angle between $E_{in}$ and $\beta_\perp$ and $\phi_B$ indicates the angle between $B_{in}$ and $\beta_\perp$. When $\cos \phi_E$ changes from $-1$ to $1$, the electron is in the acceleration phase, while $\cos \phi_E$ changes from $1$ to $-1$, the electron is in the deceleration phase. In addition, when $\sin \phi_B$ has negative value, the electron is accelerated in the $z$-direction by the effect of $\beta_\perp \times B_{in}$, while $\sin \phi_B$ has positive value, the electron is decelerated in the reverse direction. As
shown in Figs.2 (b) to (g), the region I of the diagonal line is the acceleration phase in both perpendicular and parallel direction, and the region II of gray hatching is the deceleration phase in both perpendicular and parallel direction. In the acceleration phase, an increase of the Larmor radius and a decrease of the cyclotron frequency can be found with an increase of the kinetic energy as shown in Fig.2 (b), (e) and (f). This is because the kinetic energy is changed by the relativistic effect. Here, as shown in the following equation, the relativistic cyclotron frequency is represented by

$$\omega_{ce,\gamma} = \frac{\omega_{ce}}{\gamma}.$$  \hspace{1cm} (31)

Also, it can be seen that the electron has a linear motion with a constant velocity, but the motion to z-direction includes small oscillation due to the effect of $\beta_\perp \times B_0$. In addition, regarding velocity and acceleration as shown in Fig.2 (c) and (d), the velocity in perpendicular direction is increased by the acceleration in the same direction, while the velocity in parallel direction (z-direction) is also increased by acceleration in z-direction with positive value. The opposite motion occurs in the deceleration phase. At this time, since the electron is moving with relativistic velocity in the z-direction, the electron sees the RHCP wave with Doppler-shifted frequency $\omega_D$ (defined below), which is lower than the original frequency of the RHCP wave [12].

$$\omega_D = \omega_E \sqrt{\frac{1 - \beta_\parallel}{1 + \beta_\parallel}}.$$  \hspace{1cm} (32)

where it is assumed that the electron has only z component. Eq.(32) describes the Doppler effect of light as seen from the reference frame of a particle (observer) which is traveling with a relativistic velocity. Figure 2 (e) green line shows the result of plotting eq.(32). As a result, it can be seen that the relativistic electron cyclotron frequency and the frequency of the RHCP wave seen from the reference frame of electron are satisfied with a relationship of $\omega_{ce,\gamma} \simeq \omega_D$. Such a system is called “cyclotron auto-resonance acceleration” and has been well studied by Kuramitsu et al. [13, 14]. In other words, when a circularly polarized wave has the same polarization direction as the rotation direction of a charged particle, the electron can be almost always maintained in the resonance state.

### 3.2. Observation of the ECE with Helical Wavefront

Since we have obtained the trajectory information of electron when the circularly polarized wave is applied to the electron with cyclotron motion in a uniform magnetic field, we can calculate the radiation field by substituting the trajectory information into Liénard-Wiechert potential. Figure 3 (a) shows the time variation of the spectrum observed at $\theta = 0$ deg., $\theta = 30$ deg., and $\theta = 70$ deg. on $\phi = 0$ deg. Considering the retarded time, the radiation arrives around 50 n sec. As can be seen, only the fundamental radiation appears at $\theta = 0$, while higher harmonics appear outside $\theta = 0$. Since $\theta = 0$ corresponds to the optical axis, this suggests that higher harmonics do not have intensity on the optical axis, which is one of the characteristics of the radiation with a helical wavefront. It can also be seen that these spectra change with time. This is related to the Doppler effect in which the electron travels
Figure 3: (a): Time variation of the spectrum observed on the upper hemisphere at the distance from the origin \( |\mathbf{R}| = 15 \) m, \( \theta = 0, 30, \) and 70 deg. on \( \phi = 0 \) deg. The intensity with linear scale was normalized by being maximum value 1 (b): Time variation of the frequency calculated by eq.(33). (c): Time variation of the cosine value of the phase difference between \( \phi = 0 \) deg. and \( \phi = 90 \) deg. calculated by eq.(34). (d): Time variation of the cosine value of the phase difference between \( \phi = 0 \) deg. and \( \phi = 180 \) deg. calculated by eq.(34).

to the observer with relativistic velocity while maintaining cyclotron motion.

The Doppler effect depends on the relative position between the observer and the light source. The Doppler effect including the case where there is an observer in the transverse direction is given by the following equation [12].

\[
\omega_{T,D} = \frac{\omega_{ce,\gamma}}{\gamma_{||}(1 + \beta_{||}\cos \alpha)}
\]  

(33)

where \( \alpha \) is the angle between the line of sight of the observer and the propagation direction as shown in Fig.1. Here, it is assumed that \( |\mathbf{r}| \ll |\mathbf{R}| \), that is, the \( \gamma_{||} \) is considered as the parallel component of the eq.(5). This equation represents that the observer located in transverse direction with \( \theta \sim 90 \) deg. observes the radiation with frequency which corresponds to \( \omega_{ce,\gamma} \). Figure 3 (b) shows the result of plotting eq.(33). As you can see, when \( \theta \) is small, the blue-shifted frequency which is higher than \( \omega_{ce,\gamma} \) (Fig.2 (e)) is observed at the observer position. On the other hand, when \( \theta \) is large, the red-shifted frequency which is lower than \( \omega_{ce,\gamma} \) is observed at the observer position. That is, the time variation of the spectra in the
Fig. 3 (a) can be explained by the transverse Doppler effect.

In order to demonstrate that this radiation has helical wavefront as well as the donut-shaped intensity distribution, we calculate the phase difference of the measured radiation at two spatial points. Since the beam with helical wavefront has spatial phase structure, we can detect the phase difference between two spatial points. Thus, we compare the phase of the radiation observed between \( \phi = 0 \) deg. and \( \phi = 90 \) deg. or \( \phi = 180 \) deg. If the observed radiation has helical wavefront, that radiation should have a phase difference of \( 90(n - 1) \) deg. or \( 180(n - 1) \) deg., where \( n \) is the harmonic number. To find the phase difference between two spatial points, we use the following equation:

\[
 f_{ph}(\phi) \equiv \cos \left( \arg \left( \frac{E_{\phi}}{E_{\phi=0}} \right) \right).
\]

Figure 3 (c) and (d) show the cosine value of the phase difference calculated by eq. (34). As can be seen in both figures, because the spatial location between \( \phi = 0 \) deg. and \( \phi = 90 \) deg. at \( \theta = 0 \) deg. or between \( \phi = 0 \) deg. and \( \phi = 180 \) deg. at \( \theta = 0 \) deg. represents a same location, the phase difference is zero, that is, the cosine value is 1. In other \( \theta \), it can be seen that the cosine value also changes along the spectrum of fundamental and harmonics radiation. In the case of fundamental radiation, it can be seen in both figures that the cosine value is 1 at all \( \theta \). That is, the phase difference is zero in fundamental radiation. This indicates that the fundamental radiation is just a Gaussian beam with a planar or spherical equiphase front. In the case of second harmonic radiation, it can be seen that when \( f_{ph}(90) \), which is shown in Fig.3 (c), the cosine value is 0 at all \( \theta \) except for \( \theta = 0 \) deg., which indicates that the phase difference is 90 deg., while when \( f_{ph}(180) \), which is shown in Fig.3 (d), the cosine value is \(-1\) at all \( \theta \) except for \( \theta = 0 \) deg., which indicates that the phase difference is 180 deg. That is, the phase in second harmonic radiation changes 360 deg. around the optical axis. In addition, in the case of third harmonic radiation, when \( f_{ph}(90) \), which is shown in Fig.3 (c), the cosine value of the third harmonic is \(-1\), which indicates that there is a phase difference of 180 deg., while when \( f_{ph}(180) \), which is shown in Fig.3 (d), the cosine value is 1 again at all \( \theta \) except for \( \theta = 0 \) deg., which indicates that the phase difference is 360 deg. That is, the phase in third harmonic radiation changes by 720 deg. around the optical axis. In this way, it was shown that the harmonics radiation has the phase difference with \( 90(n - 1) \) between \( \phi = 0 \) deg. and \( \phi = 90 \) deg. or \( 180(n - 1) \) between \( \phi = 0 \) deg. and \( \phi = 180 \) deg. Therefore, considering the discussion in the previous result where higher harmonics do not appear on the optical axis, the fundamental radiation has no helical wavefront, but only the higher harmonics have a helical wavefront.

Finally, we show the intensity distribution and the phase structure of the fundamental and the second harmonic radiation observed on the upper hemisphere at a given time. Figure 4 shows the intensity distribution and the phase structure of the fundamental radiation and second harmonic radiation at \( t = 70.4 \) n sec. Figure 4 (a) shows the fundamental radiation, and Fig.4 (b) shows the second harmonics radiation. In both figures, the calculated radiation fields are projected on the \( x-y \) plane. The arrows indicate the direction of the electric field at a given time and place, and the color bars indicate the power of the electric field, which is normalized by the maximum value. The powers are time-averaged. As can be seen, in the left figure, the intensity profile of the electric field is Gaussian-shaped with an intensity
Figure 4: Intensity distribution and phase structure of the fundamental radiation and second harmonic radiation at $t = 70.4$ n sec. (a): Fundamental radiation. (b): Second harmonic radiation. Both figures are projected on the $x$-$y$ plane. The arrows indicate the direction of the electric field at a given time and place, and the color bars indicate the power of the electric field, which is normalized by the maximum value. The power is time-averaged.

peak on the beam axis. The electric field is circularly polarized in the paraxial area and is facing the same direction at the same polar angle. This beam is merely a Gaussian beam, thus, it is not characteristic of the vortex beam. However, due to the transverse Doppler effect mentioned above, the radiation frequency depends on the polar angle $\theta$. Next in the right figure, the intensity profile of the electric field is donut-shaped with no intensity on the beam axis, and although the electric field is circularly polarized in the paraxial area, the direction is reversed at the symmetric point with respect to the beam axis. However, due to the transverse Doppler effect, the radiation frequency depends on the polar angle $\theta$. These are characteristics of the vortex beam. Thus, we have successfully calculated the vortex radiation from cyclotron motion.

4. Summary

In this study, we carried out a numerical calculation regarding cyclotron emission from an electron in static magnetic field accelerated by externally applied circularly polarized wave. In the electron trajectory, we showed that the electron was accelerated/decelerated repeatedly, and then it was converged to a certain value with time. In the steady state, the autoresonance state was maintained with almost same frequencies between relativistic cyclotron frequency and the frequency of the externally applied circularly polarized wave seen from the reference frame of electron. In addition, we showed that the energy of electrons could be increased from low-energy to high-energy for obtaining high power radiation. In the ECE calculation, only spectrum of the fundamental radiation appeared at $\theta = 0$, while spectra of the higher harmonics appeared at other $\theta$ with small frequency fluctuation with time. Since $\theta = 0$ corresponds to the optical axis, this showed that higher harmonics do not have inten-
sity on the optical axis. Also, in the phase difference between two spatial points, we showed the phase difference with $90(n - 1)$ deg. at between $\phi = 0$ deg. and $\phi = 90$. Furthermore, we showed the phase difference with $180(n - 1)$ at between $\phi = 0$ deg. and $\phi = 180$ deg. These results showed that only higher harmonics radiation have the helical wavefront. Also, we have successfully shown the intensity distribution and the phase structure of the fundamental and the second harmonic radiation observed on the upper hemisphere.
Appendix A.

By expanding the LHS of eq.(1), we can obtain
\[
\frac{d[\gamma m_e v(t)]}{dt} = m_e c \left\{ \gamma \frac{d\beta}{dt} + \frac{d\gamma}{dt} \beta \right\}.
\] (35)

Here, the differential calculus of the \( \gamma \) in the second term on the Right-hand side (RHS) of eq.(35) is
\[
\frac{d\gamma}{dt} = \gamma^3 \beta \cdot \dot{\beta}.
\] (36)

Therefore, since \( \frac{d\beta}{dt} = \dot{\beta} \), eq.(35) becomes
\[
\frac{d[\gamma m_e v(t)]}{dt} = m_e c \left\{ \gamma + \gamma^2 (\beta \otimes \beta) \right\} \cdot \dot{\beta}
\] (37)

where we have used the relationship as follows to further simplify on the left side of eq.(37),
\[
(\beta \cdot \dot{\beta})\beta = (\beta \otimes \beta) \cdot \dot{\beta}.
\] (38)

Eq.(38) is demonstrated as follows

(i) LHS of eq.(38)

LHS of eq.(38) = \( (\beta \cdot \dot{\beta})\beta \)

\[
= (\beta_x \dot{\beta}_x + \beta_y \dot{\beta}_y + \beta_z \dot{\beta}_z)
\begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \end{pmatrix}
\]

\[
= \begin{pmatrix} \beta_x^2 \dot{\beta}_x + \beta_y \dot{\beta}_y + \beta_z \dot{\beta}_z \\ \beta_x \beta_y \dot{\beta}_x + \beta_x \dot{\beta}_y \beta_z + \beta_y \beta_y \dot{\beta}_z \\ \beta_x \beta_z \dot{\beta}_x + \beta_y \dot{\beta}_y \beta_z + \beta_z \dot{\beta}_z \end{pmatrix}
\] (39)

(ii) RHS of eq.(38)

RHS of eq.(38) = \( (\beta \otimes \beta) \cdot \dot{\beta} \)

\[
= \begin{pmatrix} \beta^2_x & \beta_x \beta_y & \beta_x \beta_z \\ \beta_x \beta_y & \beta^2_y & \beta_y \beta_z \\ \beta_x \beta_z & \beta_y \beta_z & \beta^2_z \end{pmatrix}
\begin{pmatrix} \dot{\beta}_x \\ \dot{\beta}_y \\ \dot{\beta}_z \end{pmatrix}
\]

\[
= \begin{pmatrix} \beta_x^2 \dot{\beta}_x + \beta_y \dot{\beta}_y + \beta_z \dot{\beta}_z \\ \beta_x \beta_y \dot{\beta}_x + \beta_x \dot{\beta}_y \beta_z + \beta_y \beta_y \dot{\beta}_z \\ \beta_x \beta_z \dot{\beta}_x + \beta_y \dot{\beta}_y \beta_z + \beta_z \dot{\beta}_z \end{pmatrix}
\] (40)

Therefore, from eq.(39) and eq.(40), we can easily find the relationship of eq.(38)
Appendix B.

The inverse matrix of the $M$ with each component $m_{ij}$ can be represented by using the cofactor $\Delta_{ij}$ of the matrix and the determinant as follows:

$$
M^{-1} = \frac{\begin{pmatrix}
\Delta_{11} & \Delta_{21} & \Delta_{31} \\
\Delta_{12} & \Delta_{22} & \Delta_{32} \\
\Delta_{13} & \Delta_{23} & \Delta_{33}
\end{pmatrix}}{|M|} \quad (41)
$$

Here, the matrix $M$ is defined by as follows:

$$
M \equiv I + \gamma^2(\beta \otimes \beta) = \begin{pmatrix}
1 + \gamma^2\beta_x^2 & \gamma^2\beta_y\beta_z & \gamma^2\beta_y\beta_z \\
\gamma^2\beta_x\beta_y & 1 + \gamma^2\beta_y^2 & \gamma^2\beta_y\beta_z \\
\gamma^2\beta_x\beta_z & \gamma^2\beta_y\beta_z & 1 + \gamma^2\beta_z^2
\end{pmatrix} \quad (42)
$$

To solve the inverse matrix of $M$, we will find each cofactor $\Delta_{ij}$ as follows.

$$
\Delta_{11} = (-1)^{1+1} \begin{vmatrix}
1 + \gamma^2\beta_y^2 & \gamma^2\beta_y\beta_z \\
\gamma^2\beta_x\beta_y & 1 + \gamma^2\beta_z^2
\end{vmatrix} = 1 + \gamma^2(\beta_y^2 + \beta_z^2)
= \gamma^2(1 - \beta_z^2) \quad (43)
$$

$$
\Delta_{12} = (-1)^{1+2} \begin{vmatrix}
\gamma^2\beta_x\beta_y & \gamma^2\beta_y\beta_z \\
\gamma^2\beta_x\beta_y & 1 + \gamma^2\beta_z^2
\end{vmatrix} = -\gamma^2\beta_x\beta_y \quad (44)
$$

$$
\Delta_{13} = (-1)^{1+3} \begin{vmatrix}
\gamma^2\beta_x\beta_y & \gamma^2\beta_y\beta_z \\
\gamma^2\beta_x\beta_z & \gamma^2\beta_y\beta_z
\end{vmatrix} = -\gamma^2\beta_x\beta_z \quad (45)
$$

$$
\Delta_{21} = (-1)^{2+1} \begin{vmatrix}
\gamma^2\beta_x\beta_y & \gamma^2\beta_y\beta_z \\
\gamma^2\beta_y\beta_z & 1 + \gamma^2\beta_z^2
\end{vmatrix} = -\gamma^2\beta_x\beta_y \quad (46)
$$

$$
\Delta_{22} = (-1)^{2+2} \begin{vmatrix}
1 + \gamma^2\beta_x^2 & \gamma^2\beta_x\beta_z \\
\gamma^2\beta_x\beta_z & 1 + \gamma^2\beta_z^2
\end{vmatrix} = 1 + \gamma^2(\beta_x^2 + \beta_z^2)
= \gamma^2(1 - \beta_y^2) \quad (47)
$$
\[ \Delta_{23} = (-1)^{2+3} \begin{vmatrix} 1 + \gamma^2 \beta_x^2 & \gamma^2 \beta_y \beta_z \\ \gamma^2 \beta_4 \beta_z & \gamma^2 \beta_y \beta_z \end{vmatrix} = -\gamma^2 \beta_y \beta_z \]  
(48)

\[ \Delta_{31} = (-1)^{3+1} \begin{vmatrix} \gamma^2 \beta_y \beta_z & \gamma^2 \beta_4 \beta_z \\ 1 + \gamma^2 \beta_y^2 & \gamma^2 \beta_y \beta_z \end{vmatrix} = -\gamma^2 \beta_y \beta_z \]  
(49)

\[ \Delta_{32} = (-1)^{3+2} \begin{vmatrix} 1 + \gamma^2 \beta_x^2 & \gamma^2 \beta_4 \beta_z \\ \gamma^2 \beta_4 \beta_y & \gamma^2 \beta_y \beta_z \end{vmatrix} = -\gamma^2 \beta_y \beta_z \]  
(50)

\[ \Delta_{33} = (-1)^{3+3} \begin{vmatrix} 1 + \gamma^2 \beta_x^2 & \gamma^2 \beta_y \beta_z \\ \gamma^2 \beta_4 \beta_y & 1 + \gamma^2 \beta_y^2 \end{vmatrix} = 1 + \gamma^2 (\beta_x^2 + \beta_y^2) \]  
(51)

The determinant of the matrix \( M \) can be found by the cofactor expansion of the first row as follows:

\[
|M| = \sum_{k=1}^{3} m_{1k} \Delta_{1k} \\
= (1 + \gamma^2 \beta_x^2)(1 + \gamma^2 (\beta_y^2 + \beta_z^2)) + \gamma^2 \beta_4 \beta_y (-\gamma^2 \beta_y \beta_z) + \gamma^2 \beta_4 \beta_z (-\gamma^2 \beta_y \beta_z) \\
= 1 + \gamma^2 (\beta_x^2 + \beta_y^2 + \beta_z^2) \\
= \gamma^2. 
\]  
(52)

Therefore, the inverse matrix of the \( M \) can be given by eq.(41) as follows:

\[ M^{-1} = I - \beta \otimes \beta. \]  
(53)

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