New 3-mode squeezing operator and squeezed vacuum state in 3-wave mixing *

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Abstract

In a 3-wave mixing process occurring in some nonlinear optical medium when \(a_1\) interacts with both \(a_2\) and \(a_3\) modes, we theoretically study the squeezing effect generated by the operator \(S_3 \equiv \exp[j(\mu a_1 a_2 - a_1^\dagger a_2^\dagger) + \nu (a_1 a_3 - a_1^\dagger a_3^\dagger)]\). The new 3-mode squeezed vacuum state in Fock space is derived, and the uncertainty relation for it is demonstrated. It turns out that \(S_3\) may exhibit enhanced squeezing. By virtue of the technique of integration within an ordered product (IWOP) of operators, we also derive \(S_3\)'s normally ordered expansion. The Wigner function of new 3-mode squeezed vacuum state is calculated by using the Weyl ordering invariance under similar transformations.

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1 Introduction

Nowadays quantum entanglement is the focus of quantum information research and attracts many interests due to its wide applications in quantum communication [1, 2]. Entangled states have brought much attention and interests of physicists [3, 4]. The usual two-mode squeezed state, generated from a parametric amplifier [5], not only exhibits squeezing, but also quantum entanglement between the idle-mode and the signal-mode in frequency domain. Therefore, it is simultaneously a typical entangled state of continuous variable. Theoretically, the two-mode squeezed state is constructed by acting a two-mode squeezing operator \(S_2 = \exp[\lambda (a_1 a_2 - a_1^\dagger a_2^\dagger)]\) on the two-mode vacuum state |00\>, i.e. \(S_2 |00\rangle = \text{sech} \lambda \exp[-a_1^\dagger a_2^\dagger \tanh \lambda] |00\rangle\), where \(\lambda\) is a squeezing parameter, and \(a_i (a_i^\dagger)\) Bose annihilation (creation) operator satisfying \([a_i, a_j^\dagger] = \delta_{ij}\). Using the relation between Bose operators \((a_i, a_i^\dagger)\) and the coordinate and momentum operators

\[
Q_i = \frac{a_i + a_i^\dagger}{\sqrt{2}}, \quad P_i = \frac{a_i - a_i^\dagger}{\sqrt{2}},
\]

one can recast \(S_2\) into the form

\[
S_2 = \exp[i\lambda (Q_1 P_2 + Q_2 P_1)],
\]

noting

\[
[Q_1 P_2, Q_2 P_1] = i (Q_2 P_2 - Q_1 P_1),
[Q_1 P_2, i (Q_2 P_2 - Q_1 P_1)] = 2Q_1 P_2,
[Q_2 P_1, i (Q_2 P_2 - Q_1 P_1)] = -2Q_2 P_1,
\]

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thus there involves a $SU(1, 1)$ algebraic stricture. In the state $S_2 |00\rangle$, the variances of the two-mode quadrature operators of light field,
\[
\hat{x} = \frac{Q_1 + Q_2}{2}, \quad \hat{p} = \frac{P_1 + P_2}{2},
\]
satisfying the commutation relation $[\hat{x}, \hat{p}] = \frac{i}{\hbar}$, exhibiting the standard squeezing, i.e.,
\[
\langle 00 | S_2^4 \hat{x}^2 S_2^4 |00\rangle = \frac{1}{4} e^{-2\lambda}, \quad \langle 00 | S_2^4 \hat{p}^2 S_2^4 |00\rangle = \frac{1}{4} e^{2\lambda},
\]
which satisfy $(\Delta \hat{x})(\Delta \hat{p}) = \frac{1}{4}$.

An interesting question naturally arises: if $a_i$ mode in a nonlinear optical medium, interacting with both $a_2$ mode and $a_3$ mode (e.g., a three-wave mixing), and the corresponding three-mode exponential operator is introduced as
\[
S_3 \equiv \exp[\mu(a_1a_2 - a_1^\dagger a_2^\dagger) + \nu(a_1a_3 - a_1^\dagger a_3^\dagger)].
\]
Using Eq. (6) we can recast $S_3$ into the form
\[
S_3 \equiv \exp \left[ i \mu (Q_1 P_1 + Q_1 P_2) + i \nu (Q_2 P_2 + Q_1 P_3) \right],
\]
where $\mu, \nu$ are two different interaction parameters, then what is its squeezing effect for the 3-mode quadratures of light field?

To answer this question we must know what is the state $S_3 |000\rangle$ ($|000\rangle$ is the 3-mode vacuum state) in Fock space, for this aim, we should know what is the normally ordered expansion of $S_3$. But how to disentangle the exponential operator $S_3$? Because there is no simple $SU(1, 1)$ algebraic structure among $Q_2 P_1, Q_1 P_2, Q_3 P_3$ and $Q_1 P_3$, the disentangling seems hard. Thus we turn to appeal to Dirac’s coordinate representation and the technique of integration within an ordered product (IWOP) of operators [8, 9, 10, 11] to solve this problem. Our work is arranged as follows: firstly we derive the explicit form of $S_3 |000\rangle$, then we demonstrate that it really satisfies the Heisenberg uncertainty relation and may exhibit squeezing enhancement. We also employ the technique of integration within an ordered product (IWOP) of operators to derive the normally ordered expansion of $S_3$. The Wigner function of $S_3 |000\rangle$ is calculated by using the Weyl ordering invariance under similar transformations [12, 13, 14].

# 2 New 3-mode squeezed vacuum state

For the sake of convenience, we rewrite $S_3$ in Eq. (7) as the following compact form,
\[
S_3 = \exp[iQ_i \Lambda_{ij} P_j], \quad i, j = 1, 2, 3,
\]
where the repeated indices imply the Einstein summation notation, and
\[
\Lambda = \begin{pmatrix}
0 & \mu & \nu \\
\mu & 0 & 0 \\
\nu & 0 & 0
\end{pmatrix},
\]
thus
\[
e^\Lambda = \begin{pmatrix}
\cosh r & \cos \theta \sinh r & \sin \theta \sinh r \\
\cos \theta \sinh r & \sin^2 \theta + \cosh^2 \theta \cosh r & \sinh \theta \cosh r \\
\sin \theta \sinh r & \sinh \theta \cosh r & \cosh \theta \cosh r - 1
\end{pmatrix},
\]
its inverse is
\[
e^{-\Lambda} = \begin{pmatrix}
\cosh r & -\cos \theta \sinh r & \sin \theta \sinh r \\
-\cos \theta \sinh r & \sin^2 \theta + \cosh^2 \theta \cosh r & \sinh \theta \cosh r \\
-\sin \theta \sinh r & \sinh \theta \cosh r & \cosh \theta \cosh r - 1
\end{pmatrix}.
\]
where we have set
\[ r = \sqrt{\mu^2 + \nu^2}, \cos \theta = \frac{\mu}{r}, \sin \theta = \frac{\nu}{r}, \]
noting that \( \Lambda \) is a symmetric matrix. Using the Baker-Hausdorff formula,
\[
e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \cdots ,
\]
we see that \( S_3 \) causes the following transformation
\[
S_3^{-1} Q_k S_3 = (e^{-\Lambda})_{ki} Q_i, \quad S_3^{-1} P_k S_3 = (e^{\Lambda})_{ki} P_i.
\]
It then follows that \( S_3^{-1} a_k S_3 = (e^{-\Lambda})_{ki} a_i, \) i.e.
\[
S_3^{-1} a_1 S_3 = a_1 \cosh r - a_3^\dagger \cos \theta \sinh r - a_3^\dagger \sin \theta \sinh r,
\]
\[
S_3^{-1} a_2 S_3 = -a_1^\dagger \cos \theta \sinh r + a_2 (\sin^2 \theta + \cos^2 \theta \cosh r)
+ \frac{1}{2} a_3 (\cosh r - 1) \sin 2\theta,
\]
\[
S_3^{-1} a_3 S_3 = -a_1^\dagger \sin \theta \sinh r + \frac{1}{2} a_2 (\cosh r - 1) \sin 2\theta
+ a_3 (\sin^2 \theta \cos r + \cos^2 \theta).
\]
Noticing that \( S_3^\dagger = S_3^{-1} \) and \( S_3^\dagger (\mu, \nu) = S_3 (-\mu, -\nu) \), from Eq. (15) we also have
\[
S_3 a_1 S_3^{-1} = a_1 \cosh r + a_3^\dagger \cos \theta \sinh r + a_3^\dagger \sin \theta \sinh r,
\]
\[
S_3 a_2 S_3^{-1} = a_1^\dagger \cos \theta \sinh r + a_2 (\sin^2 \theta + \cos^2 \theta \cosh r)
+ \frac{1}{2} a_3 (\cosh r - 1) \sin 2\theta
\]
\[
S_3 a_3 S_3^{-1} = a_1^\dagger \sin \theta \sinh r + \frac{1}{2} a_2 (\cosh r - 1) \sin 2\theta
+ a_3 (\sin^2 \theta \cos r + \cos^2 \theta).
\]
For convenience to write, we set \( S_3 |000\rangle = |000\rangle \). In order to obtain the explicit form of \( |000\rangle \), using Eq. (15) and \( a_1 |000\rangle = 0 \), we operate \( a_1 \) on \( |000\rangle \) and obtain
\[
a_1 |000\rangle = S_3 S_3^{-1} a_1 S_3 |000\rangle
= S_3 (a_1 \cosh r - a_3^\dagger \cos \theta \sinh r - a_3^\dagger \sin \theta \sinh r) |000\rangle
= S_3 (-a_3^\dagger \cos \theta \sinh r - a_3^\dagger \sin \theta \sinh r) S_3^{-1} S_3 |000\rangle
= -S_3 (a_2^\dagger \cos \theta \sinh r + a_3^\dagger \sin \theta \sinh r) S_3^{-1} |000\rangle ,
\]
then we continue to use Eq. (16) to derive
\[
a_1 |000\rangle = -\{ [a_1 \cos \theta \sinh r + a_3^\dagger (\sin^2 \theta + \cos^2 \theta \cosh r)]
+ \frac{1}{2} a_3 (\cosh r - 1) \sin 2\theta \cos \theta \sinh r
+ [a_1 \sin \theta \sinh r + \frac{1}{2} a_2^\dagger (\cosh r - 1) \sin 2\theta
+ a_3^\dagger (\sin^2 \theta \cos r + \cos^2 \theta)] \sin \theta \sinh r \} |000\rangle
= -(a_1 \sin^2 r + a_3^\dagger \cos \theta \cosh r \sinh r
+ \frac{1}{2} a_3 \sin \theta \sinh 2r) |000\rangle ,
\]
so we reach the equation

$$a_1 \|000\rangle = -\tanh r(a_2^\dagger \cos \theta + a_3^\dagger \sin \theta) \|000\rangle .$$

(19)

Similarly, operating $a_2$ on $\|000\rangle$ and using Eqs. (15) and (16) yields

$$a_2 \|000\rangle = S_3 S_3^{-1} a_2 \|000\rangle = S_3 (-a_2^\dagger \cos \theta \sinh r) \|000\rangle$$

$$= S_3 (-a_3^\dagger \cos \theta \sinh r) S_3^{-1} \|000\rangle$$

$$= -(a_3^\dagger \cosh r + a_2 \cos \theta \sinh r) \cos \theta \sinh r \|000\rangle ,$$

(20)

which leads to

$$[a_2 (1 + \cos^2 \theta \sinh^2 r) + \frac{1}{2} a_3 \sin 2\theta \sinh^2 r] \|000\rangle$$

$$= -\frac{1}{2} a_3^\dagger \cos \theta \sinh 2r \|000\rangle .$$

(21)

On the other hand, operating $a_3$ on $\|000\rangle$ and using Eqs. (15) and (16) yields

$$a_3 \|000\rangle = S_3 S_3^{-1} a_3 \|000\rangle = S_3 (-a_3^\dagger \sin \theta \sinh r) \|000\rangle$$

$$= S_3 (-a_3^\dagger \sin \theta \sinh r) S_3^{-1} \|000\rangle$$

$$= -(a_3^\dagger \cosh r + a_2 \cos \theta \sinh r) \sin \theta \sinh r \|000\rangle ,$$

(22)

i.e.,

$$[a_3 (1 + \sin^2 \theta \sinh^2 r) + \frac{1}{2} a_2 \sin 2\theta \sinh^2 r] \|000\rangle$$

$$= -\frac{1}{2} a_3^\dagger \sin \theta \sinh 2r \|000\rangle .$$

(23)

Combining Eqs. (21) and (23) we have

$$a_2 \|000\rangle = -a_1^\dagger \tanh r \cos \theta \|000\rangle ,$$

(24)

and

$$a_3 \|000\rangle = -a_1^\dagger \tanh r \sin \theta \|000\rangle .$$

(25)

From Eqs. (19), (24) and (25), we may predict that $\|000\rangle$ has the following explicit form:

$$\|000\rangle = N \exp[-(a_2^\dagger \cos \theta + a_3^\dagger \sin \theta) a_1^\dagger \tanh r] \|000\rangle ,$$

(26)

where $N$ is the normalization constant, which can be determined by $\langle 000 \|000\rangle = 1$, and we calculate $N = \text{sech} r$.

3 Squeezing property and quantum fluctuation in $\|000\rangle$

Squeezing is an important phenomenon in quantum theory and has many applications in various areas in quantum optics and quantum information [15]. In this section, we examine the quadrature squeezing effects of $\|000\rangle$. The quadratures in the 3-mode case are defined as

$$X_1 = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} Q_i, \quad X_2 = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} P_i,$$

(27)
which satisfy the relation $[X_1, X_2] = \frac{i}{2}$. Their variances are $(\Delta X_i)^2 = \langle X_i^2 \rangle - \langle X_i \rangle^2$, $i = 1, 2$. Noting the expectation values of $X_1$ and $X_2$ in the state $|000\rangle$ is $\langle X_1 \rangle = \langle X_2 \rangle = 0$. With the help of Eq. (15), we can calculate that the corresponding variances in the state $|000\rangle$: (noting $\Lambda$ is symmetric)

\[
(\Delta X_1)^2 = \langle 000| S_3^{-1} X_1^2 S_3 |000\rangle = \frac{1}{6} \sum_{i=1}^{3} \sum_{j=1}^{3} (e^{-\Lambda})_{ki} (e^{-\Lambda})_{jl} \langle 000| Q_k Q_l |000\rangle \\
= \frac{1}{12} \sum_{i=1}^{3} \sum_{j=1}^{3} (e^{-\Lambda})_{ki} (e^{-\Lambda})_{jl} \langle 000| a_k a_l^\dag |000\rangle \\
= \frac{1}{12} \sum_{i=1}^{3} \sum_{j=1}^{3} (e^{-\Lambda})_{ki} (e^{-\Lambda})_{jl} \delta_{kl} \\
= \frac{1}{12} \sum_{i,j} (e^{-2\Lambda})_{ij}, \quad (28)
\]

and

\[
(\Delta X_2)^2 = \langle 000| S_3^{-1} X_2^2 S_3 |000\rangle = \frac{1}{12} \sum_{i,j} (e^{2\Lambda})_{ij}. \quad (29)
\]

The explicit form of the matrices $e^{2\Lambda}$ and $e^{-2\Lambda}$ can be derived from Eq. (10) and (11), so we can obtain

\[
(\Delta X_1)^2 = \frac{1}{12} [(2 \cosh 2r + 1) + \sin 2\theta(\cosh 2r - 1) \\
+ 2 (\cos \theta + \sin \theta) \sinh 2r], \quad (30)
\]

and

\[
(\Delta X_2)^2 = \frac{1}{12} [(2 \cosh 2r + 1) + \sin 2\theta(\cosh 2r - 1) \\
- 2 (\cos \theta + \sin \theta) \sinh 2r]. \quad (31)
\]

We can successfully verify

\[
(\Delta X_1)(\Delta X_2) = \frac{1}{12} \sqrt{(4 \cosh 2r + 4) + (1 - 2 \sinh^2 r \sin 2\theta)^2} \\
\geq \frac{1}{12} \sqrt{(4 \cosh 2r + 4) + (1 - 2 \sinh^2 r)^2} \\
= \frac{1}{12} \sqrt{\frac{1}{2} \cosh 4r + \frac{17}{2} \geq \frac{1}{4}}, \quad (32)
\]

which confirms the uncertainty relation of quantum mechanics.

To see the trend of squeezing effects in the $X_1$– or $X_2$–direction, we plot $(\Delta X_1)^2$ and $(\Delta X_2)^2$ as the function of parameter $\mu$ for different $\nu$ in Fig.1. When $\nu = 0$, it exhibits the usual two mode squeezing effect depending on the varying $\mu$, $(\Delta X_1)^2$ increases accompanying $(\Delta X_2)^2$ decreases; when $\nu = 0.5$, $(\Delta X_1)^2$ increases more than the case of $\nu = 0$, which exhibits enhanced squeezing in certain domain of $\mu$. In Fig.2, we plot the uncertainty value $(\Delta X_1)(\Delta X_2)$ as the function of $r$ for different $\theta$. 

\[\text{Fig.1, Fig.2}\]
4 Normally ordered form of $S_3$

We calculate the normally ordered form (denoted by $\hat{\cdot} \hat{\cdot} \hat{\cdot}$) of $S_3$ by inserting the completeness relation of coherent state

$$S_3 = \int \frac{d^2 z_1 d^2 z_2 d^2 z_3}{\pi^3} S_3 |z_1 z_2 z_3\rangle \langle z_1 z_2 z_3|,$$

(33)

where $|z_1 z_2 z_3\rangle$ is the three-mode coherent state and $|z_i\rangle = \exp[- \frac{|z_i|^2}{2} + z_i a_i^\dagger]|0_i\rangle$, $i = 1, 2, 3$.

Using the relations in Eqs. (15) and (16), we have the explicit relation of $S_3 |z_1 z_2 z_3\rangle$

$$S_3 |z_1 z_2 z_3\rangle = \exp(-\sum_{i=1}^3 |z_i|^2) S_3 \exp \left( z_1 a_1^\dagger + z_2 a_2^\dagger + z_3 a_3^\dagger \right) S_3^{-1} S_3 |000\rangle$$

$$= \frac{1}{\cosh r} \exp \left( -\sum_{i=1}^3 |z_i|^2 + z_1 z_2 \cos \theta \tanh r + z_1 z_3 \sin \theta \tanh r \right)$$

$$\times \exp \left( \frac{1}{\cosh r} [a_1^\dagger z_1 + (a_2^\dagger \sin^2 \theta \cos r + \cos^2 \theta) - \frac{1}{2} a_3^\dagger (\cosh r - 1) \sin 2\theta] z_2 \right)$$

$$+ (a_3^\dagger \sin^2 \theta + \cos^2 \theta \cos r) - \frac{1}{2} a_2^\dagger (\cosh r - 1) \sin 2\theta) z_3 \right)$$

$$\times \exp[-a_1^\dagger \tanh r (a_2^\dagger \cos \theta + a_3^\dagger \sin \theta)] |000\rangle.$$

(34)

Substituting Eq. (34) into Eq. (33), noticing that $|000\rangle \langle 000| = \exp(-a_1^\dagger a_1 - a_2^\dagger a_2 - a_3^\dagger a_3)$, and using the following formula

$$\int \frac{d^2 z}{\pi} \exp \left( \zeta |z|^2 + \xi \bar{z} + \eta z^* \right) = -\frac{1}{\zeta} e^{-\frac{\xi^2}{2\eta}}, \quad \text{Re} (\zeta) < 0,$$

(35)

as well as the IWOP technique, we can obtain the explicit normally ordered expansion of $S_3$:

$$S_3 = \frac{1}{\cosh r} \exp[-a_1^\dagger (a_2^\dagger \cos \theta + a_3^\dagger \sin \theta) \tanh r]$$

$$\times \exp \left[ \frac{1}{\cosh r} (a_1^\dagger a_1 + a_2^\dagger a_2 \cos^2 \theta + a_3^\dagger a_3 \sin^2 \theta + \frac{1}{2} a_2^\dagger a_2^\dagger a_3 \sin 2\theta + \frac{1}{2} a_3^\dagger a_3^\dagger a_3 \sin 2\theta) \right].$$

(36)

5 Wigner function of $\|000\rangle$

Wigner distribution function of quantum states [16, 17, 18] is widely studied in quantum statistics and quantum optics and is very important tool for a global description of nonclassical effect in the quantum system, which can be measured by various means such as photon counting experiment and homodyne tomography. Now we derive the Wigner function of $\|000\rangle$ by using a new method.

Recalling that in Ref. [14] we have introduced the Weyl ordering form of single-mode Wigner operator $\Delta_1 (q_1, p_1)$,

$$\Delta_1 (q_1, p_1) = \delta (q_1 - Q_1) \delta (p_1 - P_1),$$

(37)

where the symbols $\hat{\cdot} \hat{\cdot} \hat{\cdot}$ denote the Weyl ordering, while its normal ordering form is

$$\Delta_1 (q_1, p_1) = \frac{1}{\pi} \exp \left[- (q_1 - Q_1)^2 - (p_1 - P_1)^2 \right].$$

(38)
Thus the Wigner function for $|0\rangle$ can be easily expressed as $\langle 0 | \Delta_1 (q_1, p_1) | 0 \rangle = \frac{1}{\pi} \exp (-q_1^2 - p_1^2)$. Note that the order of Bose operators $a_1$ and $a_1^\dagger$ within a normally ordered product (or a Weyl ordered product) can be permuted. That is to say, even though $[a_1, a_1^\dagger] = 1$, we can have $: a_1 a_1^\dagger : = a_1^\dagger a_1^\dagger$ and $: a_1^\dagger a_1 : = a_1^\dagger a_1^\dagger$. The Weyl ordering of operators has a remarkable property, i.e., the Weyl-ordering invariance of operators under similar transformations, which means

$$U : (\circ \circ \circ ) U^{-1} = : U (\circ \circ \circ ) U^{-1} :,$$

(39)

as if the “fence” $: :$ did not exist when $U$ operates.

For 3-mode case, the Weyl ordering form of the Wigner operator is

$$\Delta_3 (q, p) = : \delta (q - Q) \delta (p - P) :,$$

(40)

where $Q = (Q_1, Q_2, Q_3)^T$, $P = (P_1, P_2, P_3)^T$, $q = (q_1, q_2, q_3)^T$ and $p = (p_1, p_2, p_3)^T$. Then according to the Weyl ordering invariance under similar transformations and using Eq.(14), we have

$$S_3^{-1} \Delta_3 (q, p) S_3$$

$$= S_3^{-1} : \delta (q - Q) \delta (p - P) : S_3$$

$$= : \delta (q_k - (e^{-\Lambda})_{ki} Q_i) \delta (p_k - (e^{\Lambda})_{ki} P_i) :$$

$$= : \delta ((e^{\Lambda})_{ki} q_i - Q_k) \delta ((e^{-\Lambda})_{ki} p_i - P_k) :$$

$$= : \delta (q' - Q) \delta (p' - P) :$$

$$= \Delta_3 (q', p'),$$

(41)

where $q'_k = (e^{\Lambda})_{ki} q_i$, $p'_k = (e^{-\Lambda})_{ki} p_i$. Thus the Wigner function of $|000\rangle$ is

$$\langle 000 | S_3^{-1} \Delta_3 (q, p) S_3 | 000 \rangle$$

$$= \langle 000 | \Delta_3 (q', p') | 000 \rangle$$

$$= \frac{1}{\pi^3} \exp \left( -q^T e^{2\Lambda} q - p^T e^{-2\Lambda} p \right),$$

(42)

where $e^{2\Lambda}$ and $e^{-2\Lambda}$ are given by $e^{\Lambda}$ in Eq.(10) and $e^{-\Lambda}$ in Eq.(11), respectively.

In summary, we have shown that the operator $S_3 \equiv \exp [\mu (a_1 a_2 - a_1^\dagger a_2^\dagger) + \nu (a_1 a_3 - a_1^\dagger a_3^\dagger)]$ is a new 3-mode squeezed operator by calculating the quantum fluctuation for 3-mode quadratures. We have obtained the new 3-mode squeezed vacuum state and derived the normally ordered expansion of $S_3$. The IWOP technique brings convenience in our derivation.

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Figure 1: (Colour online) The quantity \((\Delta X_1)^2\) and \((\Delta X_2)^2\) as the function of squeezing parameter \(\mu\) for different case \(\nu = 0\) and \(\nu = 0.5\).
Figure 2: (Colour online) The uncertainty value \((\Delta X_1)(\Delta X_2)\) as the function of \(r\) for different \(\theta\).