Quasiparticle Spectrum in Mesoscopic Superconducting Junctions with Weak Magnetization

S.-I. Suzuki\textsuperscript{1,2}, A. A. Golubov\textsuperscript{2}, Y. Asano\textsuperscript{3}, and Y. Tanaka\textsuperscript{1}

\textsuperscript{1}Department of Applied Physics, Nagoya University, Nagoya 464-8603, Japan,
\textsuperscript{2}MESA+ Institute for Nanotechnology, University of Twente, 7500 AE Enschede, The Netherlands, and
\textsuperscript{3}Department of Applied Physics, Hokkaido University, Sapporo 060-8628, Japan,
(Dated: April 28, 2022)

We theoretically investigate the effects of the weak magnetization on the local density of states of mesoscopic proximity structures, where two superconducting terminals are attached to a side surface of the diffusive ferromagnet wire with a phase difference. When there is no phase difference, the local density of states is significantly modified by the magnetization in both spin-singlet $s$-wave and spin-triplet $p$-wave cases. When the phase difference is $\pi$, the local density of stets is less modified by the magnetization compared with the in-phase case because of the destructive interference of Cooper pairs.

I. INTRODUCTION

The proximity effect, the penetration of Cooper pairs, is a characteristic phenomenon observed in superconducting junctions. Cooper pairs penetrating into a non-superconducting material show superconducting-like phenomena, for example, the suppression of the local density of states (LDOS) at the Fermi level (zero energy) and the screening of magnetic fields. The penetration length of Cooper pairs is limited to $\xi = (\mathcal{D}/2\pi T)^{1/2}$ with $\mathcal{D}$ and $T$ being the diffusion constant and the temperature. Although $\xi$ is the characteristic length scale of the proximity effect, Volkov and Takayanagi have shown that the conductance of a two-superconductor (SC) junction [see Fig. 1(a)] depends on the phase difference between the two SCs even when the distance between two SC is much longer than $\xi_T$ [1]. This effect is named the long-range phase-coherent effect.

The theory of the phase-coherence effect has been extended to spin-triplet $p$-wave junctions [2]. The $p$-wave SC is known to host Majorana bound states which is essential to realize the quantum computation. The so-called Tanaka-Nazarov boundary condition enables to study the energy spectrum of junctions with an unconventional pairing [3]. The Majorana zero-energy peak (ZEP) in the LDOS has been found to vanish when the phase difference is $\pi$ because of the destructive interference of Cooper pairs.

The ZEP can appear also in the junction of a diffusive weak-ferromagnetic metal (DF) and an $s$-wave SC. The energy shift due to the magnetization $M$ can result in an accidental ZEP when $|M|$ is comparable to the minigap size [4–7]. In the $p$-wave case, the magnetization known to change the Majorana ZEP to the V-shaped LDOS [8]. However, we have not known how the accidental ZEP for the $s$-wave junction and the V-shaped LDOS are modulated by the interference of Cooper pairs. The interplay between the magnetization and the phase coherence has not been elucidated yet.

In this paper, we study the LDOS in a DF wire by solving numerically the quasiclassical Usadel equation. We consider the Volkov-Takayanagi (VT) junction as shown in Fig. 1(a), where two SCs are attached to the side surface of the DF with the phase difference $\delta \Phi$. The order parameter is assumed the spin-singlet $s$-wave and triplet $p$-wave. The magnetization is assumed $M = M \hat{z}$ with $M \sim \Delta_0$. In the $s$-wave case, the accidental ZEP caused by $M$ is flattened when $\delta \Phi = \pi$ due to the destructive interference among Cooper pairs injected from different SC wires. In the $p$-wave junction with $\delta \Phi = 0$, the LDOS is modified by $M$ from the ZEP to the V-shaped. When $\delta \Phi = \pi$, on the other hand, the LDOS is less modified by $M$ because the LDOS structure is less prominent due to the destructive interference.

II. SYSTEM AND FORMULATION

A. Usadel equation

In this paper, we consider the junctions of a DF where two superconducting (S) wires are attached to a side surface of the DF as shown in Fig. 1 [i.e., Volkov-Takayanagi (VT) junctions]. Narrow S wires with the width $w$ are attached to the DF wire at $|x \mp L_1| < w/2$ and $y = 0$ with an interface resistance $R_0$, where $w \ll \xi_0$ with $\xi_0 = \sqrt{\mathcal{D}/2\pi T}$. The DF is connected to clean normal-metal wires at $x = \pm L$ which are sufficiently narrow and thin in the $y$ and $z$ directions (i.e., $L_{y(z)} \ll \xi_0$).

The Green’s function in the DF obeys the Usadel equation\textsuperscript{[9]}:

\[ \mathcal{D} \vec{\nabla} \cdot \left( \hat{g}^X \mathcal{D} \nabla \hat{g}^X \right) + i \left[ H^X, \hat{g}^X \right]_- = 0, \] (1)

\[ \hat{g}^X(r, \varepsilon) = \begin{pmatrix} \hat{g}^X_{fX} & \hat{f}^X_{fX} \\ -\hat{f}^X_{fX} & -\hat{g}^X_{fX} \end{pmatrix}, \] (2)

where $\mathcal{D}$ is the diffusion constant in the DF, $\hat{g}^X(r, \varepsilon)$ with $X = R$, and $A$ are the retarded and advanced components of the Usadel Green’s function, and $H^X$ is the Hamiltonian-like matrix. In this paper, the accents $\cdot$ and $\cdot$ means matrices in particle-hole space and spin space.
The identity matrices in particle-hole and spin space are respectively denoted by $\hat{\tau}_0$ and $\hat{\sigma}_0$. The Pauli matrices are denoted by $\hat{\tau}_j$ and $\hat{\sigma}_j$ with $j \in \{1, 3\}$. The Usadel equation is supplemented by the so-called normalization condition: $\tilde{g}^X \tilde{g}^X = 1$. Assuming the width of the DF is much narrower than $\xi_0$, we can ignore the spatial variation of the Green’s function in the $y$ direction in the DF. Namely, one need to consider a one-dimensional diffusive system where the Usadel equation is reduced to

$$\partial_x \left( \tilde{g}^X \partial_x \tilde{g}^X \right) + i \left[ \hat{H}^X, \tilde{g}^X \right] + \tilde{S}^X \Theta_S(x) = 0,$$

where the last term $\tilde{S}^X(x, \varepsilon)$ represents effects of the S wires and $\tilde{g}^X = \tilde{g}^X(x, \varepsilon)$. The source term $\tilde{S}^X(x, \varepsilon)$ is reduced from the boundary condition in the $y$ direction [1, 2]. The function $\tilde{S}_S(x)$ is unity only beneath the S wires; $\tilde{S}_S(x) = \Theta(w/2 - |x-L_1|) + \Theta(w/2 - |x+L_1|)$. The LDOS $\nu(x, \varepsilon)$ and its deviation from the normal-state value $\delta \nu$ can be obtained from the $\tilde{g}^R$ and $\tilde{g}^A$ as

$$\nu(x, \varepsilon) = \frac{\nu_0}{S} \text{Tr} \left[ \hat{\tau}_3 (\tilde{g}^R - \tilde{g}^A) \right],$$

$$\delta \nu(x, \varepsilon) = \frac{\nu(x, \varepsilon) - \nu_0}{\nu_0},$$

where $\nu_0$ is the density of states per spin at the Fermi level in the normal states.

In the presence of the weak magnetization, the matrix $\hat{H}^X$ is given by [10]:

$$\hat{H}^X = \varepsilon^X \sigma_0 \hat{\tau}_3 - M \hat{\sigma}_3 \hat{\tau}_0,$$

where $M$ is magnetization of the DF which is assumed much smaller than the chemical potential [11]. The factor $\varepsilon^X$ depends on $X$: $\varepsilon^R = \varepsilon + i \gamma$ and $\varepsilon^A = \varepsilon - i \gamma$, where $\varepsilon$ and $\gamma$ being the energy and the depairing ratio due to inelastic scatterings [12] (i.e., Dynes formulation). In this case, it is convenient to reduce the $4 \times 4$ matrix into $2 \times 2$ one as

$$\partial_x \left( \tilde{g}^X \partial_x \tilde{g}^X \right) + i \left[ (\varepsilon^X - \alpha M) \hat{\tau}_3, \tilde{g}^X \tilde{g}^A \right] + \tilde{S}^X \Theta_S(x) = 0,$$

$$\tilde{g}^X(\varepsilon, x, \varepsilon) = \begin{pmatrix} g^X & f^X \\ -f^X & -g^X \end{pmatrix},$$

where the symbol $\tilde{\cdot}$ meaning a $2 \times 2$ matrix in spin-reduced particle-hole space, the factor $\alpha = +1$ and $-1$ is for the majority and the minority spin, respectively. Here we assumed $\delta \Phi = 0$ or $\pi$ by which one can express $\tilde{\cdot}$ in terms of $\tilde{\cdot}X$. In what follows, we make $\alpha$ explicit only when necessary. The Green’s function can be simplified by the so-called $\theta$ parameterization [13–15]:

$$\tilde{g}^X = \tilde{g}_3 \cosh \theta + i \tilde{g}_2 \sinh \theta,$$

where we omit the index $X$ from $\theta = \theta^X(x, \varepsilon)$. The Usadel equation is reduced to

$$\partial^2 \partial^2 \alpha + 2i(\varepsilon^X - \alpha M) \sinh \theta + \Theta_S(x) S(x, \varepsilon) = 0.$$

The Usadel equation (9) is supplemented by the boundary conditions. The boundary conditions for $\delta \Phi = 0$ is given by $\theta(x, \varepsilon)_{|x=\pm L} = 0$ and $\partial_x \theta(x, \varepsilon)_{|x=0} = 0$, whereas that for $\delta \Phi = 0$ is $\theta(x, \varepsilon)_{|x=\pm L} = 0$ and $\theta(x, \varepsilon)_{|x=0} = 0$.

### B. Effects of superconducting terminals

The source term $S(x, \varepsilon)$ in Eq. (9) represents the effect of the S arms [13, 15]. The typical boundary conditions [16] are no longer available for unconventional pairings. Therefore, one must employ the so-called Tanaka-Nazarov condition [3], an extension of the circuit theory [17]. The boundary condition in the $y$ direction is reduced to the source term $S$:

$$\frac{d\theta}{dy}igg|_{y=0} = \frac{R_N}{R_b L_y} \langle F \rangle_\phi,$$

$$F = \frac{-2T_N(f_S \cosh \theta_0 - g_S \sinh \theta_0)}{(2 - T_N)\Xi + T_N(g_S \cosh \theta_0 - f_S \sinh \theta_0)},$$

where $R_N = \rho_N L_y/\xi_{0y}$, $\rho_N$ and $R_b$ are the specific resistance of the DF and the interface resistance at the DF/S interface, $\tilde{g} = y/\xi_0$, and $L_y = L_{y}/\xi_0$. The angle-dependent function $T_N(\phi) = \cos^2 \phi/(\cos^2 \phi + z^2_b)$ is the transmission coefficient of the N/N interface with the $\delta$-function barrier potential $h v_F z_b \delta(y)$, $\phi$ is the angle of the momentum measured from the $k_x$ axis, and $\theta_0(x) = \theta(x)_{|y=0}$. The angular bracket means the angle average:

$$\langle \cdots \rangle_\phi \equiv \left( \int_{-\pi/2}^{\pi/2} \cdots \cos \phi \, d\phi \right) \left( \int_{-\pi/2}^{\pi/2} T_N \cos \phi \, d\phi \right)^{-1}.$$

The functions $g_S$ and $f_S$ can be obtained from the Green’s...
functions in a homogeneous ballistic SC:

\[ g_S = g_{S+} + g_{S-}, \]
\[ f_S = \begin{cases} f_{S+} + f_{S-} & \text{for singlet SCs,} \\ i(g_{S+}f_{S-} - g_{S-}f_{S+}) & \text{for triplet SCs,} \end{cases} \]

where \( g_{S\pm}(\phi) = \frac{\epsilon}{\sqrt{\epsilon^2 - |\Delta_\pm|^2}}, \) \( f_{S\pm}(\phi) = \frac{\Delta_\pm}{\sqrt{\epsilon^2 - |\Delta_\pm|^2}}, \) \( \Xi = 1 + g_{S+}g_{S-} - f_{S+}f_{S-}, \) the symbol \( X \) has been omitted, and \( \Delta_+(\phi) = \Delta_-(\pi - \phi). \) The pair potential depends on the pairing symmetry of the SC: \( \Delta_+(\phi) = \Delta_0 \) for an \( s \)-wave SC and \( \Delta_0 \cos \phi \) for a \( p \)-wave one, where \( \Delta_0 \in \mathbb{R} \) characterizes the amplitude of the pair potential and the \( d \)-vector is assumed \( d \parallel M \parallel z. \) In this paper, the spin-orbit coupling is not taken into account for simplicity[11]. The boundary condition (11) is transformed into the source term:

\[ S(x, \epsilon) = \frac{\partial}{\xi_0} \gamma_B^{-1}(F(x, \epsilon, \phi))\phi, \]

where \( \gamma_B = R_0 L_0^2 / R_N \) is the dimensionless parameter. The parameters \( \gamma_B \) and \( \zeta_0 \) are independent. The function \( R_0 \) is a function of \( \zeta_0 \) because it determines \( T_N, \) whereas \( \gamma_B \) determines how strong the proximity effect is.

### III. NUMERICAL RESULTS

We first show the results for the spin-singlet \( s \)-wave junctions in Figs. 2 and 3 where the phase difference is set to \( \delta \Phi = 0 \) and \( \pi \) respectively. The parameters are set to \( \zeta_0 = 1, \) \( \gamma_B = 1, \) \( L_1 = 5\xi_0, \) and \( L = 6\xi_0 \) throughout this paper. Beneath the SC wire, the proximity effect results in the peaks at \( \epsilon = \Delta_0 \) and \( |x| = L_1 \) (i.e., the coherence peak) and the zero-energy dip. As shown in the left panel of Fig. 2, the zero-energy dip appears between the two SC terminals when \( \delta \Phi = 0 \) and \( M = 0. \) On the other hand, the low-energy spectrum is a peak [7] when \( M = 0.2\Delta_0 \) as in the center panel of Fig. 2. The peak becomes higher as with increasing the distance from the SC terminal. The energy shift by \( M, \) which depends on the quasiparticle spin, results in this accidental ZEP as shown in the right panel of Fig. 2, where we show \( \delta \nu \) at \( x = 2\xi_0. \)

When \( \delta \Phi = \pi, \) the Cooper-pair interference is destructive. Therefore the pair amplitude from each SC terminal perfectly compensates each other, leading \( \delta \nu|_{x=0} = 0 \) as shown in the left and center panels of Fig. 3. Moreover, comparing Figs 2 and 3, we see that the zero-energy dip and ZEP become less prominent compared with those with \( \delta \Phi = 0. \) Therefore, the destructive interference is concluded to diminish the zero-energy dip and the ZEP by \( M \) in the DF.

The results for the spin-triplet \( p \)-wave junctions are shown in Figs. 4 and 5, where the phase difference is set to \( \delta \Phi = 0 \) and \( \pi \) respectively. The topologically-protected ZEP characterises the spin-triplet \( p \)-wave junction as shown in the left panel of Fig. 4 [18, 19]. This ZEP is caused by the induced odd-frequency Cooper pairs [20, 21]. The ZEP is robust against the weak magnetization beneath the SC wire (i.e., \( |x| \sim L_1 \)) but fragile between them as shown in the center panel of Fig. 4.

The LDOS deviation \( \delta \nu \) at \( x = 2\xi_0 \) is shown in the right panel.
of Fig. 4. The LDOS changes from the peak to the dip with increasing $M$ because of the spin-dependent energy shift. The split LDOS peak appears V-shaped at the low energy when $M = 0.1\Delta_0$ and $0.2\Delta_0$. In the $M \gg \Delta_0$ limit, $\delta\nu$ reaches to zero.

When $\delta\Phi = \pi$, the ZEP around $x = 0$ vanishes due to the deconstructive interference as discussed in [2]. As a result, the peak splitting becomes much less prominent as shown in the center panel of Fig. 5, where the peak height is less than 0.2. The LDOS deviation at $x = 2\xi_0$ is shown in the right panel of Fig. 5. Even when $M = 0$, the LDOS peak is much smaller compared with that for $\delta\Phi = 0$. The magnetization $M$ makes this LDOS peak much smaller.

IV. SUMMARY

We have investigated the effects of the weak magnetization on the LDOS of mesoscopic proximity structures, where two SC terminals with a finite phase difference $\delta\Phi$ are attached to the side surface of the DF.

In spin-singlet $s$-wave junctions with $\delta\Phi = 0$, the accidental ZEP of the LDOS appears due to the energy shift by the magnetization, whereas the LDOS structures become less prominent when the phase difference is $\delta\Phi = \pi$ because of the destructive interference of Cooper pairs injected from different SC terminals.

In spin-triplet $p$-wave junctions are characterised by the topologically-protected ZEP in the LDOS, corresponding to the Majorana bound state. When $\delta\Phi = 0$, the LDOS is split and becomes V-shaped at the low energy by the magnetization. On the contrary, when $\delta\Phi = \pi$, the effects of the magnetization is not significant because the ZEP is much smaller even when $M = 0$ due to the destructive interference.

ACKNOWLEDGMENTS

This work was supported by Grants-in-Aid from JSPS for Scientific Research on Innovative Areas “Topological Materials Science” (KAKENHI Grant Numbers JP15H05851, JP15H05852, JP15H05853 and JP15K21717), Scientific Research (B) (KAKENHI Grant Number JP18H01176), Japan-RFBR Bilateral Joint Research Projects/Seminars number 19-52-50026, JSPS Core-to-Core Program (A. Advanced Research Networks). A. A. G. acknowledges supports by the European Union H2020-WIDESPREAD-05-2017-Twinning project “SPINTECH” under grant agreement Nr. 810144.
Although odd-parity superconductivity is often found in 5f-electron systems such as uranium compounds where the spin-orbit coupling (SOC) plays roles, we ignore the effects of SOC for simplicity. Therefore, in this paper, the magnetization is assumed to interact simply with electron spin.