Constructive Semigroups with Apartness:
Towards a New Algebraic Theory

M. Mitrović, S. Silvestrov, S. Crvenković, D. A. Romano

1 Faculty of Mechanical Engineering, University of Niš, Serbia
2 Mälardalen University, Västerås, Sweden
3 Department of Mathematics and Informatics, University of Novi Sad, Serbia
4 International Mathematical Virtual Institute, Banja Luka, Republic of Srpska, Bosnia and Herzegovina

E-mail: melanija.mitrovic@masfak.ni.ac.rs, sergei.silvestrov@mdh.se, sima@dmi.uns.ac.rs, bato49@hotmail.com

Abstract. The theory of constructive semigroups with apartness is a new approach to semigroup theory, and not a new class of semigroups. Of course, our work is partly inspired by classical semigroup theory, but, on the other hand, it is distinguished from it by two significant aspects: first, we use intuitionistic logic rather than classical, secondly, our work is based on the notion of apartness (between elements, elements and sets). Here, the focus is on E. Bishop’s approach to constructive mathematics (BISH), [4].

1. Introduction
This paper is an introduction to constructive development of the semigroups with apartness. Constructive in our case means Errett Bishop’s approach to constructive mathematics (BISH), [4]. Following Bishop we make every effort to follow classical case as closely as possible, but our work distinguishes from classical case by two significant aspects: we use intuitionistic logic rather than classical through, and our work is based on the notion of apartness (between elements, elements and sets). In the context of semigroups with apartness the basic notions of special subsets and special relations as well as constructive analogues of classical isomorphism theorems will be presented in Section 2. Some concluding remarks will be given in Section 3. More background on constructive mathematics can be found in [1], [3], [4], [7], [22]. The standard reference for constructive algebra is [19]. For classical semigroups see [17], [20]. Examples of applications of these ideas can be found in [2], [6], [9], [14], [18], [21].
between elements of $S$ – which is a meter of convention, except that it must be an equivalence. A set $(S, =)$ is an inhabited set if we can construct an element of $S$. Distinction between notions of nonempty set and inhabited set is a key in constructive set theories. A property $P$ which is applicable to the elements of a set $S$ is determined subset of $S$ denoted by $\{x \in S : P(x)\}$. Furthermore, we will be interested only in properties $P(x)$ which are extensional in the sense that for all $x_1, x_2 \in S$ with $x_1 = x_2$, $P(x_1)$ and $P(x_2)$ are equivalent. The notion of equality of different sets is not defined. The only way in which elements of two different sets can be regarded as equal is by requiring them to be subsets of a third set. For this reason the operation of union and intersection is defined only for sets which are given as subsets of a given set. There is another problem more to be faced with when we consider families of sets that are closed under a suitable operation of complementation. Following [5] “we do not wish to define complementation in the terms of negation; but on the other hand, this seems to be the only method available. The way out of this awkward position is to have a very flexible notion based on the concept of a set with apartness” #, the second most important relation in constructive mathematics, [22].

Let $(S, =)$ be an inhabited set. By an apartness on $S$ we mean a binary relation $#$ on $S$ which satisfies the axioms of irreflexivity, symmetry and cotransitivity:

$$\neg(x \# x), \ x \# y \Rightarrow y \# x, \ x \# z \Rightarrow \forall y(x \# y \lor y \# z).$$

If $x \# y$, then $x$ and $y$ are different, or distinct. Roughly speaking, $x = y$ means that we have a proof that $x$ equals $y$ while $x \# y$ means that we have a proof that $x$ and $y$ are different. Therefore, the negation of $x = y$ does not necessarily implies that $x \# y$ and vice versa: given $x$ and $y$, we may have neither a proof that $x = y$ nor a proof that $x \# y$. Negation of apartness is an equivalence ($\cong$) $=_{def} (\neg \#)$ called weak equality on $S$. By extensionality we have $x \# y \land y = z \Rightarrow x \# z$.

The apartness on a set $S$ is tight if $\neg(x \# y) \Rightarrow x = y$. Apartness is tight just when $\cong$ and $=$ are the same, i.e. $\neg(x \# y) \iff x \cong y$. In what follows we will denote tight apartness by $\sharp$. Set with tight apartness will be denoted by $(S, \cong, \sharp)$, or, shortly by $(S, \sharp)$. Tight apartness on the real numbers was introduced by L. Brouwer in the early 1920s. Brouwer introduced the notion of apartness as a positive intuitionistic basic concept. A formal treatment of apartness relations began with A. Heyting’s formalization of elementary intuitionistic geometry in [15]. Intuitionistic axiomatizations of apartness is given in [16].

A set with apartness $(S, =, \#)$ is the starting point for our further considerations, and will be simply denoted by $S$. The existence of an apartness relation on a structure often gives rise to apartness relation on another structure. For example, given two sets with apartness $(S, =, \#_S)$ and $(T, =, \#_T)$, it is permissible to construct the set of mappings between these. Let $f : S \rightarrow T$ be a mapping (function) of sets with apartness. The well-definedness of $f$, i.e.

$$\forall_{x,y\in S} (x =_S y \Rightarrow f(x) =_T f(y)),$$

follows by extensionality. A mapping $f : S \rightarrow T$ is:

- onto $S$: $\forall_{y\in T} \exists_{x\in S} (y =_T f(x))$;
- one-one or injection: if $\forall_{x,y\in S} f(x) =_T f(y) \Rightarrow x =_S y$);
- bijection between $S$ and $T$: it is a one-one and onto;
- strongly extensional mapping, shortly se-mapping if

$$\forall_{x,y\in S} (f(x) \#_T f(y) \Rightarrow x \#_S y);$$

- apartness injective, shortly a-injective: $\forall_{x,y\in S} (x \#_S y \Rightarrow f(x) \neq_T f(y));$
- apartness bijection: it is an injective se-bijection.

Presence of apartness implies appearance of different types of substructures connected to it. We give a connection between two elementary relations: “to be a subset” (in all sorts of
Proposition 2.1. Let $Y$ be a subset of $S$. Then:

(i) any se-subset is an SE-subset of $S$;

(ii) any SE-subset $Y$ of $S$ satisfies $\sim Y = \neg Y$;

(iii) any SE-subset is a d-subset of $S$.

Proof. (i). Let $Y$ be an se-subset of $S$. Then, applying definition and certain logical axiom we have

$$\forall x \in S \ (x \in Y \lor x \in \neg Y);$$

- an strongly extensional subset of $S$, shortly an se-subset of $S$, if

$$\forall x \in S \ (x \in Y \lor x \in \sim Y),$$

- an SE-subset of $S$, if

$$\forall x \in S \forall y \in Y \ (x \in Y \lor x \# y).$$

(ii). Let $Y$ be an SE-subset, and let $a \in \neg Y$. By the assumption we have

$$\forall x \in S \forall y \in Y \ (x \in Y \lor x \# y),$$

so substituting $a$ for $x$ we get $\forall y \in Y \ (a \in Y \lor a \# y)$, and since, by assumption, $\neg (a \in Y)$, it follows that $a \# y$ for all $y \in Y$. Hence $a \in \sim Y$.

(iii). Follows immediately by (ii) and the definition of d-subsets. 

Given two sets with apartness $S$ and $T$ it is permissible to construct the set of ordered pairs $(S \times T, =, \#)$ of these sets defining apartness by $(s, t) \# (u, v) \iff s \# u \lor t \# T v$. Let $(S \times S, =, \#)$ be a set with apartness. An inhabited subset of $S \times S$ is called a (binary) relation on $S$.

Example 2.2. Let $S = \{1, 2, 3\}$ be a set with apartness $\# = \{(1, 3), (3, 1), (2, 3), (3, 2)\}$. Let $\alpha = \{(1, 3), (3, 1)\}$ be a relation on $S$. Its a-complement

$$\sim \alpha = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$

is a proper subset of its logical complement $\neg \alpha$. 
A binary relation $\tau$ defined on semigroup with apartness $S$ is
- **consistent** if $\tau \subseteq \#$;
- **cotoransitive** if $(x, z) \in \tau \Rightarrow \forall y \ ((x, y) \in \tau \lor (y, z) \in \tau)$;
- **coquasordering** if it is consistent and cotransitive.

**Proposition 2.3.** Let $\tau$ be a coquasordering on $S$. Then:

(i) $\tau$ is an SE-subset of $S \times S$;

(ii) $\sim \tau = \neg \tau$.

**Proof.** (i). Let $(x, y) \in S \times S$. Then, for all $(a, b) \in \tau$,

$$a\tau x \lor x\tau b \implies a\tau x \lor x\tau y \lor y\tau b$$

$$\implies a\# x \lor x\tau y \lor y\# b$$

$$\implies (a, b)\#(x, y) \lor x\tau y,$$

that is, $\tau$ is an SE-subset.

(ii). Follows by (i) and Proposition 2.1(ii).

Coquasorders are the building blocks of the order theory of semigroups with apartness we develop.

Quotient structures are not part of **BISH**. Quotient structure does not have, in general, a natural apartness relation. For most purposes we overcome this problem using a **coequivalence**—symmetric coquasiorder—instead of an equivalence. Existing properties of a coequivalence guarantees its a-complement be an equivalence as well as quotient set of that equivalence will inherit an apartness. For what follows we will need the following definition. For any two relations $\alpha$ and $\beta$ on $S$ we say that $\alpha$ **defines apartness on** $S/\beta$ if $x\beta y \iff_{def} (x, y) \in \alpha$. Next theorem is the key for the solution of the quotient structure problem, **QSP**, for sets with apartness.

**Theorem 2.4.** If $\kappa$ is a coequivalence on $S$, then the relation $\sim\kappa(= \neg\kappa)$ is an equivalence on $S$, and $\kappa$ defines apartness on $S/\sim\kappa$.

**Proof.** Reflexivity of $\sim\kappa$ is almost obvious from the consistency of $\kappa$. As $\kappa$ is symmetric, then

$$(x, y) \in \sim\kappa \iff \forall (a, b) \in \kappa ((x, y)\#(a, b))$$

$$\Rightarrow \forall (b, a) \in \kappa ((x, y)\#(b, a))$$

$$\Rightarrow \forall (b, a) \in \kappa ((x, y)\#(b, a))$$

$$\Rightarrow \forall (a, b) \in \kappa ((y, x)\#(a, b))$$

$$\Leftrightarrow (y, x) \in \sim\kappa.$$  

If $(x, y) \in \sim\kappa$ and $(y, z) \in \sim\kappa$, then, by the definition of $\sim\kappa$, we have that $(x, y) \triangleright \kappa$ and $(y, z) \triangleright \kappa$. For an element $(a, b) \in \kappa$, by cotransitivity of $\kappa$, we have $(a, x) \in \kappa$ or $(x, y) \in \kappa$ or $(y, z) \in \kappa$ or $(z, b) \in \kappa$. Thus $(a, x) \in \kappa$ or $(z, b) \in \kappa$, which implies that $a\#x$ or $b\#z$, i.e. $(x, z)\#(a, b)$. So $(x, z) \triangleright \kappa$ and $(x, z) \in \sim\kappa$. Therefore $\sim\kappa$ is an equivalence on $S$.

Let $a(\sim\kappa)\#b(\sim\kappa)$; then $(a, b) \in \kappa$ implies that $(b, a) \in \kappa$, that is $b(\sim\kappa)\#a(\sim\kappa)$.

Let $a(\sim\kappa)\#b(\sim\kappa)$ and $u(\sim\kappa) \in S/\sim\kappa$. Then $(a, b) \in \kappa$, and, by the cotransitivity of $\kappa$, we have $(a, u) \in \kappa$ or $(u, b) \in \kappa$. Finally we have that $a(\sim\kappa)\#u(\sim\kappa)$ or $u(\sim\kappa)\#b(\sim\kappa)$, so the relation $\#$ is cotransitive.

The irreflexivity of $\#$ is implied by its definition and by the irreflexivity of $\kappa$. Therefore $\kappa$ defines apartness on $S/\sim\kappa$.

**Corollary 2.5.** The quotient mapping $\pi : S \to S/\sim\kappa$, defined by $\pi(x) = x(\sim\kappa)$, is an onto se-mapping.
Proof. Let \( \pi(x) \# \pi(y) \), i.e. \( x(\sim \kappa) \# y(\sim \kappa) \), which, by what we have just proved, means that \( (x, y) \in \kappa \). Then, by the consistency of \( \kappa \), we have \( x \# y \). So \( \pi \) is an se-mapping.

Let \( a(\sim \kappa) \in S/\sim \kappa \) and \( x \in a(\sim \kappa) \). Then \( (a, x) \in \sim \kappa \), i.e. \( a(\sim \kappa) = x(\sim \kappa) \), which implies that \( a(\sim \kappa) = x(\sim \kappa) = \pi(x) \). Thus \( \pi \) is an onto mapping.

Now we can formulate one of the main results of this section - Apartness isomorphism theorem for sets with apartness.

**Theorem 2.6.** Let \( f : S \to T \) be an se-mapping between sets with apartness. Then

(i) the relation \( \ker f = \{ (x, y) \in S \times S : f(x) \# f(y) \} \) is a coequivalence on \( S \) (which we call the cokernel of \( f \)) if and only if \( \ker f \subseteq \sim \ker f \);

(ii) the relation \( \theta : S/\ker f \to T \), defined by \( \theta(x(\ker f)) = f(x) \), is a one-one, a-injective se-mapping such that \( f = \theta \circ \pi \); and

(iii) if \( f \) maps \( S \) onto \( T \), then \( \theta \) is an apartness bijection.

Proof. (i). The consistency of \( \ker f \) is easy to prove: if \( (x, y) \in \ker f \), then \( (f(x) \# f(y)) \) and therefore \( x \# y \).

If \( (x, y) \in \ker f \), then, by the symmetry of apartness in \( T \), \( f(y) \# f(x) \); so \( (y, x) \in \ker f \).

If \( (x, y) \in \ker f \) and \( z \in S \) — i.e. \( f(x) \# f(y) \) and \( f(z) \in T \) — then \( (f(x) \# f(z) \) or \( f(z) \# f(y) \); that is, either \( (x, z) \in \ker f \) or \( (z, y) \in \ker f \). Hence \( \ker f \) is a coequivalence on \( S \).

Let \( (x, y) \in \ker f \) and \( (y, z) \in \ker f \); then \( f(x) \# f(y) \) and \( f(y) \# f(z) \). Hence \( f(x) \# f(z) \) — that is, \( (x, z) \in \ker f \) — and \( \ker f \) defines an apartness on \( S/\ker f \).

Now let \( (x, y) \in \ker f \), so \( f(x) = f(y) \). If \( (u, v) \in \ker f \), then, by the cotransitivity of \( \ker f \), it follows that \( (u, x) \in \ker f \) or \( (x, y) \in \ker f \) or \( (y, v) \in \ker f \). Thus either \( (u, x) \in \ker f \) or \( (y, v) \in \ker f \), and, by the consistency of \( \ker f \), either \( u \# x \) or \( y \# v \); whence we have \( (x, y) \neq (u, v) \). Thus \( (x, y) \in \ker f \) or, equivalently \( (x, y) \in \sim \ker f \).

(ii). Let us first prove that \( \theta \) is well defined. Let \( x(\ker f) \), \( y(\ker f) \in S/\ker f \) be such that \( x(\ker f) = y(\ker f) \); that is, \( (x, y) \in \ker f \). Then we have \( f(x) = f(y) \), which, by the definition of \( \theta \), means that \( \theta(x(\ker f)) \neq \theta(y(\ker f)) \). Now let \( \theta(x(\ker f)) = \theta(y(\ker f)) \); then \( f(x) = f(y) \). Hence \( (x, y) \in \ker f \), which implies that \( x(\ker f) = y(\ker f) \). Thus \( \theta \) is one-one.

Next let \( \theta(x(\ker f)) \neq \theta(y(\ker f)) \); then \( f(x) \# f(y) \). Hence \( (x, y) \in \ker f \), which, by (i), implies that \( x(\ker f) \neq y(\ker f) \). Thus \( \theta \) is an se-mapping.

Let \( x(\ker f) \# y(\ker f) \); that is, by (i), \( (x, y) \in \ker f \). So we have \( f(x) \# f(y) \), which, by the definition of \( \theta \), means \( \theta(x(\ker f)) \# \theta(y(\ker f)) \). Thus \( \theta \) is injective. On the other hand, by the Corollary 2.5 and the definition of \( \theta \), for each \( x \in S \) we have

\[
(\theta \circ \pi)(x) = \pi(\theta(x)) = \pi(x(\ker f)) = f(x).
\]

(iii). Taking into account (ii), we have to prove only that \( \theta \) is onto. Let \( y \in T \). Then, as \( f \) is onto, there exists \( x \in S \) such that \( y = f(x) \). On the other hand \( \pi(x) = x(\ker f) \). By (ii), we now have

\[
y = f(x) = (\theta \circ \pi)(x) = \theta(\pi(x)) = \theta(x(\ker f)).
\]

Thus \( \theta \) is onto.

A tuple \((S, =, \# , \cdot)\) is a semigroup with apartness with \((S, =, \#)\) as a set with apartness, \( \cdot \) an associative binary operation on \( S \) which is strongly extensional, i.e.

\[
\forall a, b, x, y \in S \quad (a \cdot x \# b \cdot y \Rightarrow (a \# b \lor x \# y)).
\]
As it is shown in [10], apartness does not have to be tight. As usual, we are going to write \( ab \) instead of \( a \cdot b \). Hereinafter we will consider only semigroups with apartness, calling them, in short, semigroups, and denoting them by \( S \).

Let us remember that in CLASS the compatibility property is an important condition for providing the semigroup structure on quotient sets. Now we are looking for the tools for introducing apartness relation on a factor semigroup. Our starting point, beside the above given results (for a set with apartness), is the next definition. A coequivalence \( \kappa \) is cocongruence if it is cocompatible, i.e.

\[
\forall_{a,b,x,y \in S} ((ax, by) \in \kappa) \Rightarrow (a, b) \in \kappa \lor (x, y) \in \kappa.
\]

**Theorem 2.7.** If \( \kappa \) is a cocongruence on \( S \), then the relation \( \sim_{\kappa} (= \sim_{\kappa}) \) is an congruence on \( S \), and \( \kappa \) defines apartness on \( S/ \sim_{\kappa} \).

**Proof.** By Theorem 2.4, \( \sim_{\kappa} \) is an equivalence on \( S \) such that \( \kappa \) defines apartness on \( S/ \sim_{\kappa} \). If \( (a, b), (x, y) \in \sim_{\kappa} \) then for any \( (u, v) \in \kappa \) we have both \( (a, b)\#(u, v) \) and \( (x, y)\#(u, v) \). Now, we also have \((u, ax) \in \kappa \) or \((ax, by) \in \kappa \) or \((by, v) \in \kappa \). If \((ax, by) \in \kappa \), then by the cocompatibility of \( \kappa \), either \((a, b) \in \kappa \) or \((x, y) \in \kappa \), which is impossible. Thus \((u, ax) \in \kappa \) or \((by, v) \in \kappa \); so either \( u\#ax \) or \( by\#v \), and therefore \((ax, by)\#(u, v) \). Hence \((ax, by) \in \sim_{\kappa} \). Thus \((a, b), (x, y) \in \sim_{\kappa} \), and \( \sim_{\kappa} \) is a congruence on \( S \).

Let \( a(\sim_{\kappa})x(\sim_{\kappa}) \neq b(\sim_{\kappa})y(\sim_{\kappa}) \); then \((ax)\sim_{\kappa})(by)\sim_{\kappa}) \). By Theorem 2.4, we have that \((ax, by) \in \kappa \). But \( \kappa \) is a congruence, so either \((a, b) \in \kappa \) or \((x, y) \in \kappa \). Thus, by the definition of \( \# \) in \( S/ \sim_{\kappa} \), either \( a(\sim_{\kappa})\#b(\sim_{\kappa}) \) or \( x(\sim_{\kappa})\#y(\sim_{\kappa}) \). So \((S/ \sim_{\kappa}, =, \#, \cdot) \) is a semigroup with apartness.

The Apartness isomorphism theorem for semigroups with apartness follows.

**Theorem 2.8.** Let \( f : S \rightarrow T \) be an se-homomorphism between semigroups with apartness. Then:

(i) \( \text{coker } f \) is a cocongruence on \( S \) which defines apartness on \( S/ \ker f \), and \( \ker f \subseteq \sim_{\text{coker } f} \).

(ii) the mapping \( \theta : S/ \ker f \rightarrow T \), defined by \( \theta(x(\ker f)) = f(x) \), is an apartness embedding such that \( f = \theta \circ \pi \); and

(iii) if \( f \) maps \( S \) onto \( T \), then \( \theta \) is an apartness isomorphism.

**Proof.** (i) Taking into account Theorem 2.6 and Theorem 2.7, it is enough to prove that coker \( f \) is cocompatible with multiplication in \( S \). Let \((ax, by) \in \text{coker } f \) — i.e. \(f(ax)\#f(by) \). Since \( f \) is a homomorphism, we have \( f(a)f(x)\#f(b)f(y) \). The strong extensionality of multiplication implies that either \( f(a)\#f(b) \) or \( f(x)\#f(y) \). Thus either \((a, b) \in \text{coker } f \) or \((x, y) \in \text{coker } f \), and therefore coker \( f \) is a cocongruence on \( S \).

(ii) Using Theorem 2.7 and the assumption that \( f \) is a homomorphism, we have

\[
\theta(x(\ker f))y(\ker f)) = \theta((xy)(\ker f)) = f(xy) = f(x)f(y) = \theta(x(\ker f))\theta(y(\ker f)).
\]

By Theorem 2.6, \( \theta \) is a one-one, a-injective se-homomorphism — that is, an apartness embedding.

(iii) This follows by Theorem 2.6 and (ii).
3. Conclusion Remarks

In an intuitionistic setting it is, however, appropriate to consider semigroups with an extra structure - apartness. Semigroup with apartness satisfies a number of extra conditions, firstly the well known axioms of apartness, and secondly the semigroup operation has to be strongly extensional.

It is well known that formalization is a general method in science. Although it has been created as a technique in logic and mathematics it has entered into engineering as well. Formal engineering methods can be understood as mathematically-based techniques for the functional specification, development and verification in the engineering of software and hardware systems. Despite some initial suspicion it has been proved that formal methods are powerful enough to deal with real life systems. For example, it is shown that “software of the size and complexity as we find in modern cars today can be formally specified and verified by applying computer based tools for modeling and interactive theorem proving,” [8].

Proof assistants are computer systems which give a user the possibility to do mathematics on a computer: from (numerical and symbolical) computing aspects to the aspects of defining and proving. The latter ones, doing proofs, are the main focus. It is believed that, besides their great future within the area of formalization of mathematics, their applications within computer supported modelling and verification of the systems (are and) will be more important. More about proof assistants - their history, ideas and future, can be found in [12]. One of the most popular, with intuitionistic background, is the proof assistant computer system Coq.

Coq is, also, used for formal proving of a well known mathematical theorems, such as, for example is the Fundamental Theorem of Algebra, FTA, [13]. For that purpose constructive algebraic hierarchy for Coq was developed, [14]. That algebraic hierarchy consists of constructive basic algebraic structures (semigroups, monoids, groups, rings, fields) with tight apartness. In addition, all these structures are limited to the commutative case. As it is noticed in [14] “that algebraic hierarchy has been designed to prove FTA. This means that it is not rich as one would like. For instance, we do not have noncommutative structure because they did not occur in our work.” ... So, a question which arose from here is:

What can be done in connection with noncommutative semigroups with apartness which is “only” apartness - not the tight one?

We put noncommutative constructive semigroups with “ordinary” apartness in the centre of our study, proving first, of course, that such ones do exist, [10]. Although the presentation given in Section 2 is based on material given in [10], [11], it is, by no means an attempt to give a complete overview of our existing results. Results of several years long investigation, presented in [10], [11], present a semigroup facet of some relatively well established direction of constructive mathematics. In what follows some possible connections between semigroup with apartness with computer science are given.

\[
\begin{array}{ll}
\text{semigroups with apartness} & \text{semigroups with apartness} \\
\downarrow & \downarrow \\
\text{bisimulation} & \text{automated theorem proving} \\
\downarrow & \uparrow \\
\text{formal reasoning about processes} & \text{knowledge representation and automated reasoning} \\
\downarrow & \downarrow \\
\text{process algebra} & \text{artificial intelligence} \\
\downarrow & \downarrow \\
\text{transactions and concurrency} & \downarrow \\
\downarrow & \\
\text{databases} & \\
\end{array}
\]
We hope that we or someone else will be able to say more about this link. In the meantime, our work (or someone else) will be focused on further developing the theory of semigroups with apartness. At the very end we can say that our experience of doing constructive algebra suggests that we are dealing with “normal” mathematical objects, and we are working only with intuitionistic logic. The study of constructive semigroups with apartness as well as constructive algebra as a whole can have an effect on development of other areas of constructive mathematics. On the other hand, it can make both proof engineering and programming more flexible.

Acknowledgments
Melanija Mitrović is supported by the Ministry of Education, Science and Technological Development of Serbia, Grant 174026.

References
[1] Bauer A 2017 Five stages of accepting constructive mathematics, Bull. (New Series) of the Amer. Math. Soc. 54 3 481-498
[2] Baroni M and Bridges D S 2008 Continuity properties of preference relations Math. Log. Quart. 54 5 454-459
[3] Beeson M J 1985 Foundations of Constructive Mathematics (Springer-Verlag)
[4] Bishop E 1967 Foundations of Constructive Analysis (New York: McGraw-Hill)
[5] Bishop E and Bridges D S 1985 Constructive Analysis (Berlin: Grundlehren der mathematischen Wissenschaften 279, Springer)
[6] Bridges D S and Havea R 2001 A Constructive version of the spectral mapping theorem Math. Log. Quart. 47 3 299-304
[7] Bridges D S and Vițâ L S 2011 Apartness and Uniformity - A Constructive Development (Springer, CiE series on Theory and Applications of Computability)
[8] Broy M 2009 Seamless model driven systems engineering based on formal models ICFEM 2009 (LNCS 5885) Ed K Breitman and A Cavalcanti 1-19
[9] Calderón G 2017 Formalizing constructive projective geometry in Agda The Proc. of The 12th Workshop on Logical and Semantic Frameworks, with Appl., LSFA 2017 (Brasília, 23-24 September 2017 - in preparation)
[10] Crvenković S, Mitrović M and Romano D A 2013 Semigroups with apartness Math. Log. Quart. 59 6 407-414
[11] Crvenković S, Mitrović M and Romano D A 2016 Basic notions of (constructive) semigroups with apartness Semigroup Forum 92 3 659-674
[12] Guevers H 2009 Proof assistants: history, ideas and future Sādhanā J. Acad. Proc. in Eng. Sciences - Special Issue on Interactive Theorem Proving and Checking (Indian Academy of Science) 34 1 3-25
[13] Guevers H, Wiedijk F and Zwanenburg J 2002 A constructive proof of the Fundamental theorem of algebra without using the rationals Proc. of TYPES 2000 Workshop Durham UK Callaghan P, Luo Z, McKinna J and Pollack R (Eds.) (Springer-Verlag) LNCS 2277 96-111
[14] Geuvers H, Pollack R, Wiedijk F and Zwanenburg J 2002 A Constructive algebraic hierarchy in Coq J. Symb. Comput. 34 271-286
[15] Heyting A 1927 Zur intuitionistischen axiomatik der projektiven geometrie Mathematische Ann. 98 491-538
[16] Heyting A 1956 Intuitionism, an Introduction (North - Holland)
[17] Howie J M 1995 Fundamentals of Semigroup Theory (London Mathematical Society Monographs, New Series, Oxford; Clarendon Press)
[18] Gunther E, Gadea A and Pagano M 2017 Formalization of universal algebra in Agda The Proc. of The 12th Workshop on Logical and Semantic Frameworks, with Appl., LSFA 2017 (Brasília, 23-24 September 2017 - in preparation)
[19] Mines R, Richman F and Ruitenburg W 1988 A Course of Constructive Algebra (New York: Springer-Verlag)
[20] Mitrović M 2003 Semilattices of Archimedean Semigroups (Niš: University of Niš - Faculty of Mechanical Engineering)
[21] Moshier M A 1995 A rational reconstruction of the domain of feature structures J. of Logic, Language and Information 4 2 111-143
[22] Troelstra A S and van Dalen D 1988 Constructivism in Mathematics, An Introduction (two volumes) (Amsterdam: North - Holland)