On the Conditions to Extend Ricci Flow

Bing Wang

Abstract

Consider \{((M^n, g(t)), 0 \leq t < T < \infty)\} as an unnormalized Ricci flow solution: \( \frac{d}{dt} g_{ij} = -2R_{ij} \) for \( t \in [0, T) \). Richard Hamilton shows that if the curvature operator is uniformly bounded under the flow for all \( t \in [0, T) \) then the solution can be extended over \( T \). Natasa Sesum proves that a uniform bound of Ricci tensor is enough to extend the flow. We show that if Ricci is bounded from below, then a scalar curvature integral bound is enough to extend flow, and this integral bound condition is optimal in some sense.

1 When can Ricci flow be extended?

In (7), R. Hamilton introduces Ricci flow which deforms Riemannian metrics in the direction of the Ricci tensor. One hopes that the Ricci flow will deform any Riemannian metric to some canonical metrics, such as Einstein metrics. One can even understand geometric and topological structure of the underlying differential manifold by this sort of deformation. The idea is best illustrated in (8) where Hamilton proves that in any simply connected 3 manifold without boundary, any Riemannian metric with positive Ricci curvature can be deformed into a positive space form (up to scaling). Consequently, R. Hamilton proves that the underlying manifold is indeed diffeomorphic to \( S^3 \). This fundamental work sparks a great interest of many mathematicians in Ricci flow. In a series of work, R. Hamilton introduces an ambitious program to prove the Poincaré conjecture via Ricci flow (cf. (9) for Hamilton’s program and early references in Ricci flow.). The celebrated work of G. Perelman (14), (15) and (16) indeed proves the Poincaré conjecture which states that every simply connected 3 manifold is \( S^3 \). We refer the readers to (12), (13) for more information.

After Perelman’s work in the Ricci flow, there is a renewed interest in Ricci flow and its application around the world. We will refer readers to the book (4) for more updated references. In this note, we want to concentrate in studying some basic issue on Ricci flow: the maximal existence time of Ricci flow and the geometric conditions that might affect the maximal existence time.

One notes that Ricci flow is a weak Parabolic flow. R. Hamilton first proves that for any smooth initial data, the flow will exist for a short time in (7). In (8), Hamilton’s proof is simplified greatly by a clever choice of gauge. The next immediate question is the so called “maximal existence time” for the Ricci flow (with respect to initial metric). In (8), Hamilton proves that if \( T < \infty \) is the maximal existence time of a closed Ricci flow solution \(((M^n, g(t)), 0 \leq t <\)
$T < \infty$, then Riemannian curvature is unbounded as $t \to T$. In other words, a uniform bound for Riemannian curvature on $M \times [0, T)$ is enough to extend Ricci flow over time $T$. In (18), by a blowup argument, Sesum shows that Ricci curvature uniformly bounded on $M \times [0, T)$ is enough to extend Ricci flow over $T$. Sesum's surprising work uses the no local collapsing theorem of Perelman. A natural question arises: what is the optimal condition for the Ricci flow to be extended? In many ways, we believe that the scalar curvature bound shall be enough to extend the flow. In this note, we first prove (See Definition 2.1 for notations),

**Theorem 1.1.** \{$(M^n, g(t)), 0 \leq t < T < \infty$\} is a closed Ricci flow solution. If

1. $\text{Ric}(x, t) \geq -A$ for all $(x, t) \in M \times [0, T)$, $A$ is a positive constant,
2. $\|R\|_{\alpha, M \times [0,T)} < \infty$, $\alpha \geq \frac{n+2}{2}$,

then this flow can be extended over time $T$.

and

**Theorem 1.2.** \{$(M^n, g(t)), 0 \leq t < T < \infty$\} is a closed Ricci flow solution. If

$$\|Rm\|_{\alpha, M \times [0,T)} < \infty, \quad \alpha \geq \frac{n+2}{2},$$

then this flow can be extended over time $T$.

These two theorems are optimal in some aspects as illustrated by Example 2.1 in the next section.

**Remark 1.1.** In theorem 1.1, 1.2, let $\alpha = \infty$, we can recover Sesum’s and Hamilton’s results.

**Organization** Let’s sketch the outline of this note. We first fix some notations in section 2. Then, in section 3, we prove Theorem 1.2 for all $n \geq 2$. In section 4, we use no local collapsing theorem and Croke’s argument to establish a local Sobolev constant control. Then we use this control to develop a general parabolic Moser iteration under Ricci flow in section 5. Applying Moser iteration to $R$ in section 6, we prove Theorem 1.1 for $n \geq 3$.

**Acknowledgements:** I would like to express my gratitude to my advisor, professor Xiuxiong Chen. He directed me to this subject and brought this problem to my attention and even showed me the main tools to handle this problem. I’m grateful to professor Dan Knopf and professor Sigurd Angenent for their helpful discussions. I would like to thank Haozhao Li for pointing out some errors in the earlier version of this note.

### 2 Preliminary

Let $M^n$ be a connected compact manifold without boundary. $(M^n, g(t))$ is called a closed Ricci flow solution if the metric satisfies the equation:

$$\frac{dg_{ij}}{dt} = -2R_{ij}. \quad (1)$$
By direct calculation, we have the evolution equations for curvatures under Ricci flow:

\[
\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2, \tag{2}
\]

\[
\frac{\partial R_{ij}}{\partial t} = \Delta R_{ij} + 2R_{ikl}R_{kl} - 2R_{ik}R_{kj}, \tag{3}
\]

\[
\frac{\partial R_{ijkl}}{\partial t} = \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\
- (R_{ip}R_{pqkl} + R_{jp}R_{ipkl} + R_{kp}R_{ijpl} + R_{lp}R_{ijkp}), \tag{4}
\]

where \(B_{ijkl} \equiv -R_{ipqj}R_{kpql} \).

The evolution equation of volume element is

\[
\frac{\partial d\mu}{\partial t} = -Rd\mu. \tag{5}
\]

For convenience, we define some norm of the space time manifold \(M \times [0, T)\) below.

**Definition 2.1.** Suppose \(N \subset M\), for any measurable function \(F\) defined on \(N \times [0, T)\) and \(\alpha \geq 1\), we define

\[
\|F\|_{\alpha, N \times [0, T)} \equiv (\int_0^T \int_N |F|^\alpha d\mu dt)^{\frac{1}{\alpha}},
\]

\[
\|F(\cdot, t)\|_{\alpha, N} \equiv (\int_N |F|_{g(t)}^\alpha d\mu_{g(t)})^{\frac{1}{\alpha}},
\]

\[
F_+ \equiv \max\{F, 0\}, \quad F_- \equiv \max\{-F, 0\}.
\]

Now we are ready to give example to illustrate that Theorem 1.1 is sharp in some aspects.

**Example 2.1.** Let \((S^n, g_s)\) be the space form of constant sectional curvature 1. Now we start Ricci flow from metric \((S^n, g_s)\). By direct calculation, \(g(t) = (1-2(n-1)t)g_s\) is the Ricci flow solution. Therefore, \(T = \frac{1}{2(n-1)}\) is the maximal existence time. However, we compute

\[
\|R\|_{\alpha, M \times [0, T)} = \left\{ \begin{array}{ll}
\int_0^T \int_M |R|^\alpha d\mu dt \right\}^{\frac{1}{\alpha}} \\
= \left\{ \int_0^T V(t)(\frac{n}{2(T-t)})^\alpha dt \right\}^{\frac{1}{\alpha}} \\
= \frac{n}{2}V(0)^{\frac{1}{\alpha}}T^{-\frac{n}{2\alpha}} \left\{ \int_0^T (T-t)^{-\alpha} dt \right\}^{\frac{1}{\alpha}},
\]

therefore,

\[
\|R\|_{\alpha, M \times [0, T)} \begin{cases}
= \infty, & \alpha \geq \frac{n}{2} + 1, \\
< \infty, & \alpha < \frac{n}{2} + 1.
\end{cases}
\]
Moreover, Ric ≥ 0. This suggests us that Theorem 1.1 cannot be improved to α < \(\frac{n+2}{2}\).

Since \(S^n\) is space form, \(|Rm|^2 = C(n)^2|R|^2\), then
\[
\|Rm\|_{\alpha, M \times [0,T)} = C(n)\|R\|_{\alpha, M \times [0,T)}.
\]
Hence,
\[
\|Rm\|_{\alpha, M \times [0,T)} \begin{cases} = \infty, & \alpha \geq \frac{n}{2} + 1, \\ < \infty, & \alpha < \frac{n}{2} + 1. \end{cases}
\]
This implies Theorem 1.2 cannot be improved to \(\alpha < \frac{n+2}{2}\).

The uniform Sobolev constant control will play an important role in our proof.

**Definition 2.2.** Suppose \(\{(M^n, g(t)), 0 \leq t < T < \infty\}\) is a closed Ricci flow solution, \(N \subseteq M\). We say \(\sigma\) is a uniform Sobolev constant for \(N\) at each time slice, if
\[
(\int_N |v|^{2}\mu_{g(t)})^{\frac{2}{n-2}} \leq \sigma \int_N |\nabla v|^{2} \mu_{g(t)},
\]
for every function \(v \in W^{1,2}(N)\) and \(0 \leq t < T\).

If Ricci is bounded from below, we can control \(\frac{\partial R}{\partial t}\) by \(R\).

**Property 2.1.** Suppose Ric ≥ −B, let \(\hat{R} = R + nB\), then
\[
\frac{\partial \hat{R}}{\partial t} \leq \triangle \hat{R} + 2(\hat{R} - 2B)\hat{R} + 2nB^2.
\]

**Proof.** Choose an orthonormal basis to diagonalize Ricci such that \(\text{Ric} = \text{diag}\{\lambda_1, \cdots, \lambda_n\}\), then
\[
\text{Ric} + BI = \text{diag}\{\lambda_1 + B, \cdots, \lambda_n + B\},
\]
where each term is nonnegative. Therefore,
\[
(\lambda_1 + B)^2 + \cdots (\lambda_n + B)^2 \leq (\lambda_1 + B + \cdots + \lambda_n + B)^2,
\]
consequently,
\[
\lambda_1^2 + \cdots + \lambda_n^2 \leq (\lambda_1 + \cdots + \lambda_n)^2 + 2(n-1)B(\lambda_1 + \cdots + \lambda_n) + n(n-1)B^2;
\]
i.e.
\[
|Ric|^2 \leq R^2 + 2(n-1)BR + n(n-1)B^2
\]
\[
= \hat{R}^2 - 2B\hat{R} + nB^2.
\]
From inequality \(2\), we have
\[
\frac{\partial \hat{R}}{\partial t} = \frac{\partial R}{\partial t}
\]
\[
= \triangle R + 2|Ric|^2
\]
\[
\leq \triangle \hat{R} + 2(\hat{R} - 2B)\hat{R} + 2nB^2
\]
\[
= \triangle \hat{R} + 2(\hat{R} - 2B)\hat{R} + 2nB^2.
\]
In [14], Perelman proves the fundamental no local collapsing Theorem:

**Theorem 2.1.** \{\( (M^n, g(t)), 0 \leq t < T < \infty \) is a closed Ricci flow solution. Then there exists a \( \kappa > 0 \), such that for any \( (x, t) \in M \times [0, T), r > 0 \), if \( \sup_{y \in B_{g(t)}(x, r)} |Rm|(y, t) \leq r^{-2} \), then\]

\[
\frac{\text{Vol}_{g(t)}(B_{g(t)}(x, r))}{r^n} \geq \kappa.
\]

Actually, Perelman has already noticed that the same conclusion still holds if we replace the Riemannian curvature by scalar curvature. That is the next theorem.

**Theorem 2.2.** \{\( (M^n, g(t)), 0 \leq t < T < \infty \) is a closed Ricci flow solution. Then there exists a \( \kappa > 0 \), such that for any \( (x, t) \in M \times [0, T), r > 0 \), if \( \sup_{y \in B_{g(t)}(x, r)} |R(y, t)| \leq r^{-2} \), then\]

\[
\frac{\text{Vol}_{g(t)}(B_{g(t)}(x, r))}{r^n} \geq \kappa.
\]

The proof of Theorem 2.2 can be found in [12], [14]. We will use Theorem 2.2 to get Sobolev constant control.

## 3 Proof of Theorem 1.2 for \( n \geq 2 \)

**Proof.** By Hölder’s inequality, \( \|Rm\|_{L^\alpha(M \times [0, T])} < \infty \) implies \( \|Rm\|_{L^{\frac{n+2}{n}+2, M \times [0, T]} < \infty \) if \( \alpha > \frac{n+2}{2} \). So we only need to prove Theorem 1.2 for \( \alpha = \frac{n+2}{2} \).

We argue by contradiction.

Suppose \( T \) is the maximal existence time. Then there is a sequence \( (x^{(i)}, t^{(i)}) \) with \( \lim_{i \to \infty} t^{(i)} = T \) and \( \lim_{i \to \infty} |Rm|(x^{(i)}, t^{(i)}) = \infty \). Moreover,

\[
|Rm|(x^{(i)}, t^{(i)}) = \max_{(x, t) \in M \times [0, t^{(i)})} |Rm|(x, t).
\]

Let

\[
Q^{(i)} = |Rm|(x^{(i)}, t^{(i)}),
\]

\[
g^{(i)}(t) = Q^{(i)}g((Q^{(i)})^{-1} t + t^{(i)}).
\]

By Theorem 2.1 we have uniform lower bound of injectivity radius at points \( (x^{(i)}, t^{(i)}) \) for the sequence \( \{((M^n, x^{(i)}), g^{(i)}(t)), -Q^{(i)}t^{(i)} \leq t \leq 0 \} \). So it subconverges to an ancient Ricci flow solution \( \{((M, \tilde{x}), \tilde{g}(t)), -\infty \leq t \leq 0 \} \).

Therefore, by the scaling invariance of \( \int_0^T \int_M |Rm|^{\frac{n+2}{2}} d\mu dt \), we have

\[
\int_{-1}^0 \int_{B_{g^{(i)}(x, 1)}} |Rm|^{\frac{n+2}{2}} d\mu dt \leq \lim_{i \to \infty} \int_{-1}^0 \int_{B_{g^{(i)}(x, 1)}} |Rm|^{\frac{n+2}{2}} g^{(i)}(t) d\mu g^{(i)}(t) dt
\]

\[
= \lim_{i \to \infty} \int_{t^{(i)}}^0 \int_{B_{g^{(i)}(x, 1)}} |Rm|^{\frac{n+2}{2}} d\mu dt
\]

\[
\leq \lim_{i \to \infty} \int_{t^{(i)}}^0 \int_M |Rm|^{\frac{n+2}{2}} d\mu dt
\]

\[
= 0.
\]

(9)
The last equality holds since \( \int_0^T \int_M |Rm|^\frac{n+2}{2} d\mu dt < \infty \) and \( \lim_{t \to -\infty} (Q^{(i)})^{-\frac{1}{2}} = 0 \). Since \((M, \bar{g}(t))\) is a smooth Riemannian manifold for each \( t \leq 0 \), equality \( \mathfrak{m} \) implies that \( |Rm| \equiv 0 \) on the parabolic ball \( B_{\bar{g}(0)}(\bar{x}, 1) \times [-1, 0] \). In particular, \( \lim_{t \to -\infty} (|Rm|^{(i)}(\bar{x}, 0), 0) = 0 \). On the other hand,

\[
|Rm|(\bar{x}, 0) = \lim_{i \to \infty} |Rm|^{(i)}(x^{(i)}, 0) = 1.
\]

So we get a contradiction.

When dimension is 2, \( Rm = R \). Thus Theorem 1.1 and Theorem 1.2 are the same. So we have already proved Theorem 1.1 for \( n = 2 \). When \( n \geq 3 \), \( R \) and \( Rm \) are different. Accordingly we have to develop some new techniques to prove Theorem 1.1. Moser iteration will play a critical role in our proof. In order to apply Moser iteration, we need to get a local Sobolev constant control first.

### 4 Local Sobolev Constant Control

In this section, we discuss how to control isoperimetric constant locally. By the equivalence of isoperimetric constant and Sobolev constant, we get the local control for Sobolev constant. The following argument comes from Croke’s paper [5].

**Definition 4.1.** Suppose \((N, \partial N, g)\) be a smooth compact manifold with smooth boundary and Riemannian metric \( g \).

\[
\Phi(N) \triangleq \inf_{\Omega \in N} \frac{\text{Area}(\partial \Omega)^n}{\text{Vol}(\Omega)^{n-1}}.
\]

Let \( UN \to N \) represent the unit sphere bundle with the canonical measure. For \( v \in UN \), let \( \gamma_v \) be the geodesic with \( \gamma'_v(0) = v \), let \( \zeta^t(v) \) represent the geodesic flow, i.e. \( \zeta^t(v) = \gamma'_v(t) \). Let \( l(v) \) be the smallest value of \( t > 0 \) (possibly \( \infty \)) such that \( \gamma_v(t) \in \partial N \). Note \( \zeta^t(v) \) is defined for \( t \leq l(v) \). Let

\[
\hat{l}(v) \triangleq \sup \{ t | \gamma_v \text{ minimizes up to } t \text{ and } t \leq l(v) \},
\]

\( \hat{U}M \triangleq \{ v \in UM | \hat{l}(v) = l(v) \}, \quad \hat{U}_p \triangleq \pi_{U_M}^{-1}(p), \)

\[
\tilde{\omega}_p \triangleq \frac{\text{Area} \hat{U}_P}{\text{Area} U_p}, \quad \tilde{\omega} \triangleq \inf_{p \in \partial N} \tilde{\omega}_p,
\]

\[
\alpha(n) \triangleq \text{volume of unit sphere of dimension } n.
\]

For \( p \in \partial N \), let \( N_p \) be the inwardly pointing unit normal vector. Let \( U^+ \partial N \to \partial N \) be the bundle of inwardly pointing unit vectors. That is,

\[
U^+ \partial N = \{ u \in UN |_{\partial N} | (u, N_{\pi(u)}) \geq 0 \}.
\]

\( U^+ \partial N \) has natural metric structure.

This \( \tilde{\omega} \) is related to \( \Phi(N) \) closely. If we have a control over \( \tilde{\omega} \), then it’s easy to get a control for \( \Phi(N) \).
Proposition 4.1. For \((N, \partial N, g)\) we have

\[
\int_{\tilde{U}} f(v) dv = \int_{U+\partial N} \int_0^{\bar{l}(u)} f(\zeta^s(u)) < u, N_{\pi(\bar{u})} > drdu,
\]
where \(f\) is any integrable function. In particular for \(f \equiv 1\), we have

\[
\text{Vol}(\tilde{U}M) = \int_{U+\partial N} \bar{l}(u) < u, N_{\pi(\bar{u})}> du.
\]

This formula occurs in [1], p. 286, and [17], pp. 336-338.

Proposition 4.2. Let \(N^n\) be a Riemannian manifold and \(u \in U N\). Then for every \(l \leq C(u)\) (the distance to the cut locus in the direction \(u\)):

\[
\int_{x=0}^{x=l} \int_{z=0}^{z=l-x} F(\zeta^x(z), z) dz dx \geq C_1(n) \frac{n+1}{\pi^{n+1}},
\]
where \(C_1(n) = \frac{\sigma(n) \pi^{\frac{n}{2}}}{2 \alpha(n-1)}\), \(F(v, z)\) is the volume form in polar coordinates, i.e.,

\[
\int_{U^r} \int_0^{C(v)} F(v, z) dz dv = \text{Vol}(M).
\]

The proof can be found in Berger’s work [2] (Appendix D).

Lemma 4.1. For \((N, \partial N, g)\) we have the isoperimetric inequality:

\[
\frac{\text{Area}(\partial N)^n}{\text{Vol}(N)^{n-1}} \geq C_2(n) \bar{\omega}^{n+1},
\]
where \(C_2(n) = 2^{n-1} \frac{\alpha(n-1)^n}{\alpha(n)^{n-1}}\).

Proof.

\[
\text{Vol}(N)^2 = \int_N \text{Vol}(N) dp
\geq \int_N \int_{U^r} \int_0^{\bar{l}(u)} F(u, t) dt du dp
\geq \int_{U^r} \int_0^{\bar{l}(u)} F(u, t) du
\geq \int_{U^r} \int_0^{\bar{l}(v)} \bar{l}(\zeta^s(v)) F(\zeta^s(v), t) < v, N_{\pi(\bar{u})}> dt ds dv
\geq \int_{U^r} \int_0^{\bar{l}(v)} \bar{l}(v) - s F(\zeta^s(v), t) dt dv
\geq \frac{C_1(n)}{\pi^{n+1}} \int_{U^r} (\bar{l}(v))^{n+1} < v, N_{\pi(\bar{u})}> dv.
\]
By Hölder inequality,
\[
\int_{U^+ \partial N} \tilde{l}(v, N_{\pi(v)}) dv = \int_{U^+ \partial N} (\tilde{l}(v, N_{\pi(v)})^{\frac{n+1}{n}}(v, N_{\pi(v)})^{\frac{n}{n+1}} dv
\leq \left( \int_{U^+ \partial N} \tilde{l}^{n+1}(v, N_{\pi(v)}) dv \right)^{\frac{n}{n+1}} \left( \int_{U^+ \partial N} (v, N_{\pi(v)}) dv \right)^{\frac{n}{n+1}},
\]
then,
\[
\int_{U^+ \partial N} \tilde{l}^{n+1}(v, N_{\pi(v)}) dv \geq \frac{\left( \int_{U^+ \partial N} \tilde{l}(v, N_{\pi(v)}) dv \right)^{n+1}}{\left( \int_{U^+ \partial N} (v, N_{\pi(v)}) dv \right)^n}.
\]

Put inequality (15) into inequality (14), we get
\[
\text{Vol}(N)^2 \geq C_1(n) \frac{\left( \int_{U^+ \partial N} \tilde{l}(v, N_{\pi(v)}) dv \right)^{n+1}}{\left( \int_{U^+ \partial N} (v, N_{\pi(v)}) dv \right)^n},
\]
therefore,
\[
\text{Vol}(N)^2 \left( \int_{U^+ \partial N} (v, N_{\pi(v)}) dv \right)^n \geq C_1(n) \frac{\text{Vol}(\tilde{U} M)^{n+1}}{\pi^{n+1}} \text{Vol}(\tilde{U} M)^n
\geq C_1(n) \frac{\text{Vol}(N)^{n+1}}{\pi^{n+1}} \left( \tilde{\omega}(n-1) \text{Vol}(N) \right)^{n+1}.
\]
Note that
\[
\int_{U^+ \partial N} (v, N_{\pi(v)}) dv = \frac{\alpha(n)}{2\pi} \text{Area}(\partial N),
\]
consequently,
\[
\frac{\text{Area}(\partial N)^n}{\text{Vol}(N)^{n-1}} \geq C_1(n) \frac{\tilde{\omega}^{n+1} \alpha(n-1)^{n+1} (2\pi)^n}{\alpha(n)^n}
= 2^{n-1} \frac{\alpha(n-1)^n}{\alpha(n)^{n-1}} \tilde{\omega}^{n+1}
\triangleq C_2(n) \tilde{\omega}^{n+1}.
\]

**Lemma 4.2.** M is a complete Riemannian manifold with \(\text{Ric} \geq -(n-1)K^2\). \(\Omega \subset N_1 \subset N_2 \subset M\), \(\Omega\) is a domain with smooth boundary, and \(\text{diam}(N_2) \leq D\). Then
\[
\tilde{\omega}(\Omega) \geq \frac{\text{Vol}(N_2) - \text{Vol}(N_1)}{\alpha(n-1) \int_0^D \left( \sinh K r \right)^{n-1} dr}.
\]

**Proof.** Choose \(p \in \Omega\). Then \((\Omega, \partial \Omega, g)\) is a smooth Riemannian manifold with boundary. We look \((\Omega, \partial \Omega, g)\) as \((N, \partial N, g)\) in our previous argument. Let
\[
O_p \triangleq \{ q \in M \mid q = \exp_p t u, u \in \hat{U}_p, t \leq C(u) \},
\]
where $C(u)$ is the cut radius at direction $u$. Since $u \in \tilde{U}_p$, $\tilde{l}(u) = l(u)$. Therefore $M \setminus \Omega \subset O_p$, in particular, $N_2 \setminus N_1 \subset O_p$. And also we know, $N_2 \setminus N_1 \subset N_2 \subset B(p, D)$. Then

$$\text{Vol}(N_2 \setminus N_1) \leq \text{Vol}(O_p \cap B(p, D))$$

$$= \int_{\tilde{U}_p} \int_0^D F(u, r)drdu$$

$$\leq \tilde{\omega}_p \alpha (n-1) \int_0^D \left( \frac{\sinh Kr}{K} \right)^{n-1} dr.$$

Consequently,

$$\tilde{\omega}_p \geq \frac{\text{Vol}(N_2) - \text{Vol}(N_1)}{\alpha (n-1) \int_0^D \left( \frac{\sinh Kr}{K} \right)^{n-1} dr}.$$

Since $p$ is an arbitrary point in $\Omega$, we have

$$\tilde{\omega} = \inf_{p \in \Omega} \tilde{\omega}_p \geq \frac{\text{Vol}(N_2) - \text{Vol}(N_1)}{\alpha (n-1) \int_0^D \left( \frac{\sinh Kr}{K} \right)^{n-1} dr}.$$

**Theorem 4.1.** Suppose $\{(M^n, g(t)), \ 0 \leq t \leq 1\}, \ n \geq 3$ is a Ricci flow solution.
\( p \in M \), and

\[
Ric(x, t) \geq -(n-1), \quad \forall (x, t) \in M \times [0, 1];
\]

\[
Ric(x, t) \leq (n-1), \quad \forall (x, t) \in B_{g(t)}(p, 1) \times [0, 1];
\]

\[\text{Vol}_{g(t)}(B_{g(t)}(p, 1)) \geq \kappa.\]

Let \( r(\kappa) \) be the solution of

\[
\int_0^{r(\kappa)} \left( \sinh s \right)^{n-1} ds = \frac{\kappa}{2\alpha(n-1)e^{n(n-1)}}.
\]

Then there is a uniform Sobolev constant \( \sigma(n, \kappa) \) for \( B_{g(t)}(p, r(\kappa)) \) on each time slice, i.e., for any \( f \in W^{1, 2}_0(B_{g(t)}(p, r(\kappa))) \),

\[
\| f \|_{L^2}^2 \leq \sigma(n, \kappa) \| \nabla f \|_{L^2}^2.
\]  

(17)

Proof. Let \( N_1 \triangleq B_{g(1)}(p, r(\kappa)), \quad N_2 \triangleq B_{g(1)}(p, 1) \). Calculating the evolution equation for volume:

\[
\frac{d\text{Vol}_{g(t)}(N_2)}{dt} = -\int_{N_2} Ric d\mu
\]

\[
\leq n(n-1) \text{Vol}_{g(t)}(N_2), \quad (Ric \geq -(n-1))
\]

hence,

\[
\text{Vol}_{g(t)}(N_2) \geq e^{n(n-1)(t-1)} \text{Vol}_{g(1)}(N_2)
\]

\[
\geq e^{-n(n-1)} \text{Vol}_{g(1)}(N_2) \quad (0 \leq t \leq 1)
\]

\[
\geq e^{-n(n-1)} \kappa.
\]  

(18)

Similarly, by the condition \( Ric \leq (n-1) \),

\[
\text{Vol}_{g(t)}(N_1) \leq e^{n(n-1)(1-t)} \text{Vol}_{g(1)}(N_1)
\]

\[
\leq e^{n(n-1)} \text{Vol}_{g(1)}(N_1)
\]

\[
= e^{n(n-1)} \int_{B_{g(1)}(p, r(\kappa))} d\mu
\]

\[
\leq e^{n(n-1)} \alpha(n-1) \int_0^{r(\kappa)} (\sinh s)^{n-1} ds
\]

\[
\leq \frac{\kappa}{2} e^{-n(n-1)}.
\]  

(19)

Now we consider the diameter change under Ricci flow. Suppose \( \{ \gamma(s), 0 \leq s \leq \rho \} \) is a normalized shortest geodesic contained in \( N_2 \) at time \( t \), then

\[
\frac{dL_{g(t)}(\gamma)}{dt} = -\int_0^{\rho} Ric(\gamma', \gamma') ds
\]

\[
\geq -(n-1)L_{g(t)}(\gamma).
\]

Let \( D(t) \) be the diameter of \( N_2 \) at time \( t \), we have

\[
\frac{dD(t)}{dt} \geq -(n-1)D(t),
\]

10
hence,
\[ D(t) \leq D(1)e^{(n-1)(1-t)} \leq 2e^{(n-1)}. \] (20)

Choose an arbitrary domain \( \Omega \subset N_1 \) with smooth boundary. By inequalities (18), (19) and (20), from lemma 4.2, we know

\[ \tilde{\omega}_{\gamma(g)}(\Omega) \geq \frac{\text{Vol}_{\gamma(g)}(N_2) - \text{Vol}_{\gamma(g)}(N_1)}{\alpha(n - 1)} \int_0^{D(t)} \sinh^s \eta ds \]
\[ \geq \frac{2(\alpha - 1)}{\kappa e^{-n(n-1)}} \int_0^{2\gamma(g)} \sinh^s \eta ds \]
\[ \equiv C_3(n, \kappa). \]

Then, from lemma 4.1, we have

\[ \frac{\text{Area}(\partial \Omega)}{\text{Vol}(\Omega)^{n-1}} \geq C_2(n)\alpha(n-1) \int_0^D \sinh^s \eta ds \]
\[ \equiv C_4(n, \kappa). \]

Since we can approximate any domain by domains with smooth boundary, we actually get

\[ \Phi(N_1) = \inf_{\Omega \subset N_1} \frac{\text{Area}(\partial \Omega)}{\text{Vol}(\Omega)^{n-1}} \geq C_4(n, \kappa). \] (21)

Accordingly, by the equivalence of isoperimetric constant and Sobolev constant, for any \( f \in W^{1,1}_0(N_1) \),

\[ C_4(n, \kappa) \int_{N_1} |f|^{\frac{n}{n-1}} d\mu \leq \int_{N_1} |\nabla f|. \] (22)

We refer the readers to (20) for a detailed proof for the equivalence of inequality (21) and inequality (22). Let \( \gamma > 0 \), then

\[ \|f\|_{\frac{n}{n-1}, N_1} \leq \frac{1}{C_4} \|\gamma f^{-1}\|_{p, N_1} \|\nabla f\|_{p, N_1}, \]

therefore,

\[ \|f\|_{\frac{n}{n-1}, N_1} \leq \frac{1}{C_4} \|\gamma f^{-1}\|_{p, N_1} \|\nabla f\|_{p, N_1}. \]

Choose \( \gamma = \frac{p(n-1)}{n-p} \), we have

\[ \|f\|_{\frac{n}{n-1}, N_1} \leq \frac{1}{C_4} \cdot \frac{p(n-1)}{n-p} \|\nabla f\|_{p, N_1}. \]

In particular, choose \( p = 2 \), let

\[ \sigma(n, \kappa) = \left( \frac{2(n-1)}{C_4(n, \kappa)(n-2)} \right)^2. \]
we obtain
\[ \|f\|_{2,N_1}^2 \leq \sigma(n, \kappa) \|\nabla f\|_{2,N_1}^2 \]
for any \( f \in W^{1,2}_0(N_1) \). \( \square \)

After we get the local Sobolev constant control, we are able to get some Moser iteration formula under Ricci flow.

5 Moser Iteration of Scalar curvature \((n \geq 3)\)

We will give a detailed construction of local Moser iteration under Ricci flow in this section. The idea comes from the Moser iteration in (3). Let us fix notation first.

**Definition 5.1.** \( \{(M^n, g(t)), 0 \leq t \leq 1\} \) is a closed Ricci flow solution. Fixing \( p \in M, r > 0 \), we define
\[
\Omega \triangleq B_g(1)(p, r), \quad \Omega' \triangleq B_g(1)(p, \frac{r}{2}),
\]
\[
D \triangleq \Omega \times [0, 1], \quad D' \triangleq \Omega' \times \left[ \frac{1}{2}, 1 \right].
\]

Inequality \( \Box \) is only Sobolev inequality for time slices. In order to apply Moser iteration on the parabolic domain \( D \), we need a parabolic version of Sobolev inequality.

**Property 5.1.** Suppose there is a uniform Soblev constant \( \sigma \) for \( \Omega \) at each time slice, \( v \in C^1(D) \), and \( v(\cdot, t) \in C^1_0(D) \), \( \forall t \in [0, 1] \), we have
\[
\int_D v^{\frac{2(n+2)}{n}} d\mu dt \leq \sigma \max_{0 \leq t \leq 1} \|v(\cdot, t)\|_{2,\Omega}^\frac{2}{n} \int_D |\nabla v|^2 d\mu dt. \tag{23}
\]

**Proof.** By Hölder inequality and inequality \( \Box \), we have
\[
\int_D v^{\frac{2(n+2)}{n}} d\mu dt = \int_0^1 dt \int_\Omega v^2 \cdot v^\frac{2}{n} d\mu
\]
\[
\leq \int_0^1 dt \left( \int_\Omega v^\frac{2(n+2)}{n} d\mu \right)^\frac{n}{2(n+2)} \cdot \left( \int_\Omega v^\frac{2}{n} d\mu \right)^\frac{n}{n}
\]
\[
= \int_0^1 dt \left\| v(\cdot, t) \right\|_{2,\Omega}^\frac{2}{n} \left( \int_\Omega v^\frac{2(n+2)}{n} d\mu \right)^\frac{n}{2(n+2)}
\]
\[
\leq \sigma \max_{0 \leq t \leq 1} \left\| v(\cdot, t) \right\|_{2,\Omega}^\frac{2}{n} \int_D |\nabla v|^2 d\mu dt.
\]

\( \square \)

Then we start the main Lemmas in this section.

**Lemma 5.1.** \( \{(M^n, g(t)), 0 \leq t \leq 1\} \) is a closed Ricci flow solution with \( \text{Ric} \geq -B \). Suppose there is a uniform Soblev constant \( \sigma \) for \( \Omega \) at each time slice. If \( u \in C^1(D) \) and \( u \geq 0 \),
\[
\frac{\partial u}{\partial t} \leq \triangle u + fu + h, \tag{24}
\]
in distribution sense, and \( \|f\|_{q,D} + \|R_-\|_{q,D} + 1 \leq C_0 \) for some \( q > \frac{n}{2} + 1 \). Then there is a constant \( C_a = C_a(n, q, \sigma, C_0, r, B) \) such that

\[
\|u\|_{\infty,D'} \leq C_a (\|u\|_{q+2,D} + \|h\|_{q,D} \cdot \|1\|_{q+2,D}).
\]  

(25)

Proof. Choose a cutoff function \( \eta \in C^\infty(D) \) such that \( \eta(\cdot,t) \in C_0^\infty(\Omega), \forall t \in [0,1], \) and \( \eta(x,0) = 0 \). Moreover, \( \eta(x,\cdot) \) is a nondecreasing function for every \( x \in \Omega \).

Define

\[
\kappa \triangleq \|h\|_{q,D'}, \quad v \triangleq u + \kappa.
\]

Fix \( \beta > 1 \), use \( \eta^2(u + \kappa)^{\beta-1} \) as a test function, from inequality (24),

\[
-\Delta v + \frac{\partial v}{\partial t} \leq f u + h.
\]

Then, for any \( s \in (0,1] \), we have

\[
\begin{align*}
\int_0^s \int_\Omega (-\Delta v) \eta^2 v^{\beta-1} d\mu dt &+ \int_0^s \int_\Omega \frac{\partial v}{\partial t} \eta^2 v^{\beta-1} d\mu dt \\
&\leq \int_0^s \int_\Omega (fu + h) \eta^2 (u + \kappa)^{\beta-1} d\mu dt \\
&\leq \int_0^s \int_\Omega (|f| + \frac{|h|}{\kappa}) \eta^2 v^{\beta} d\mu dt.
\end{align*}
\]
Note that $\frac{\partial d\mu}{\partial t} = -Rd\mu$, integrating by parts yields

$$\int_0^s \int_\Omega (2\eta < \nabla \eta, \nabla v > v^{\beta-1}) \, d\mu \, dt + \frac{1}{\beta} \int_\Omega \eta^2 v^\beta \, d\mu \, |s| \, - \int_0^s \int_\Omega 2\eta \frac{\partial \eta}{\partial t} v^\beta \, d\mu \, dt + \int_\Omega \eta^2 v^\beta Rd\mu \, dt \leq \int_0^s \int_\Omega (|f| + \frac{|h|}{\kappa}) \eta^2 v^\beta \, d\mu \, dt. \quad (26)$$

By Schwartz inequality,

$$\int_0^s \int_\Omega 2\eta < \nabla \eta, \nabla v > v^{\beta-1} \, d\mu \, dt \geq -\epsilon^2 \int_0^s \int_\Omega \eta^2 v^{\beta-2} |\nabla v|^2 - \frac{1}{\epsilon^2} \int_0^s \int_\Omega v^\beta |\nabla \eta|^2. \quad (27)$$

Plugging inequality (27) into (26), we get

$$(\beta - 1 - \epsilon^2) \int_0^s \int_\Omega \eta^2 v^{\beta-2} |\nabla v|^2 \, d\mu \, dt + \frac{1}{\beta} \int_\Omega \eta^2 v^\beta \, d\mu \, |s| \leq \int_0^s \int_\Omega (|f| + \frac{|h|}{\kappa} + R) \eta^2 v^\beta \, d\mu \, dt + 2(\beta - 1 + \beta - 1 \beta) \int_0^s \int_\Omega v^\beta |\nabla \eta|^2 \, d\mu \, dt + \int_0^s \int_\Omega 2\eta \frac{\partial \eta}{\partial t} v^\beta \, d\mu \, dt.$$

Let $\epsilon^2 = \frac{\beta - 1}{2}$, since $|\nabla v^\beta|^2 = \frac{\beta^2}{\beta-1} v^{\beta-2} |\nabla v|^2$, we know

$$2(1 - \frac{1}{\beta}) \int_0^s \int_\Omega \eta^2 |\nabla v^\beta|^2 \, d\mu \, dt + \int_\Omega \eta^2 v^\beta \, d\mu \, |s| \leq \beta \int_0^s \int_\Omega ((|f| + \frac{|h|}{\kappa} + R) \eta^2 v^\beta \, d\mu \, dt + \frac{2\beta}{\beta - 1} \int_0^s \int_\Omega v^\beta |\nabla \eta|^2 \, d\mu \, dt \, + \int_\Omega 2\eta \frac{\partial \eta}{\partial t} v^\beta \, d\mu \, dt.$$

Since

$$|\nabla (\eta v^\beta)|^2 \leq 2\eta^2 |\nabla v^\beta|^2 + 2v^\beta |\nabla \eta|^2,$$

we have

$$(1 - \frac{1}{\beta}) \int_0^s \int_\Omega |\nabla (\eta v^\beta)|^2 \, d\mu \, dt + \int_\Omega \eta^2 v^\beta \, d\mu \, |s| \leq \beta \int_0^s \int_\Omega ((|f| + \frac{|h|}{\kappa} + R) \eta^2 v^\beta \, d\mu \, dt + 2(\frac{\beta}{\beta - 1} + \frac{\beta - 1}{\beta}) \int_0^s \int_\Omega v^\beta |\nabla \eta|^2 \, d\mu \, dt + \int_\Omega 2\eta \frac{\partial \eta}{\partial t} v^\beta \, d\mu \, dt.$$
Therefore,

\[
\int_0^s \int \Omega \left| \nabla \left( \eta v^n \right) \right|^2 \, d\mu dt + \int_\Omega \eta^2 v^\beta \, d\mu_s \leq \Lambda(\beta) \left( \int_0^s \int \Omega \left( |f| + \frac{|h|}{k} + R_- \right) \eta^2 v^\beta \, d\mu dt + \int_\Omega \eta^2 |\nabla \eta| \, d\mu + \int_\Omega 2 \eta \frac{\partial \eta}{\partial t} v^\beta \, d\mu dt \right) \\
\leq \Lambda(\beta) \left( \left( \int_0^s \int \Omega \left( |f| + \frac{|h|}{k} + R_- \right) \eta^2 v^\beta \, d\mu dt \right)^{\frac{\sigma}{\sigma-1}} \left( \int_\Omega \left( \eta^2 v^\beta \right)^{\frac{\sigma}{\sigma-1}} \, d\mu \right)^{\frac{\sigma-1}{\sigma}} + \int_0^s \int \Omega \eta^2 |\nabla \eta|^2 \, d\mu dt + \int_0^s \int \frac{\partial \eta}{\partial t} \eta^2 v^\beta \, d\mu dt \right)
\]

\leq \Lambda(\beta) \left\{ \left( \int_0^s \int \Omega \left( |f| + \frac{|h|}{k} + R_- \right) \eta^2 v^\beta \, d\mu dt \right)^{\frac{\sigma}{\sigma-1}} + \int_0^s \int \Omega \eta^2 |\nabla \eta|^2 \, d\mu dt + \int_0^s \int \frac{\partial \eta}{\partial t} \eta^2 v^\beta \, d\mu dt \right\}

\leq \Lambda(\beta) \left( C_0 \left( \int_\Omega \left( \eta^2 v^\beta \right)^{\frac{\sigma}{\sigma-1}} \, d\mu \right)^{\frac{\sigma-1}{\sigma}} + \int_\Omega \eta^2 |\nabla \eta|^2 \, d\mu + \int_0^s \int \frac{\partial \eta}{\partial t} \eta^2 v^\beta \, d\mu dt \right).

We can choose \( \Lambda(\beta) = 6 \beta \) if \( \beta \geq 2 \). In particular,

\[
\max_{0 \leq \beta \leq 1} \int_\Omega \eta^2 v^\beta \, d\mu_s \leq \Lambda(\beta) \left( \left( \int_\Omega \left( |\nabla \eta|^2 + 2 \eta \frac{\partial \eta}{\partial t} \right) v^\beta \, d\mu \right)^{\frac{\sigma}{\sigma-1}} + C_0 \left( \eta^2 v^\beta \right)^{\frac{\sigma}{\sigma-1}} \right),
\]

\[
\int_0^s \int \Omega \eta^2 |\nabla \eta|^2 \, d\mu dt \leq \Lambda(\beta) \left( \left( \int_\Omega \left( |\nabla \eta|^2 + 2 \eta \frac{\partial \eta}{\partial t} \right) v^\beta \, d\mu \right)^{\frac{\sigma}{\sigma-1}} + C_0 \left( \eta^2 v^\beta \right)^{\frac{\sigma}{\sigma-1}} \right).
\]

The Sobolev inequality [28] on the parabolic domain \( D \) yields

\[
\| \eta^2 v^\beta \|_{2+ \frac{\sigma}{\sigma-1},D} \leq \sigma^{\frac{\sigma}{\sigma-1}} \Lambda(\beta) \left( \left( \int_\Omega \left( |\nabla \eta|^2 + 2 \eta \frac{\partial \eta}{\partial t} \right) v^\beta \, d\mu \right)^{\frac{\sigma}{\sigma-1}} + C_0 \left( \eta^2 v^\beta \right)^{\frac{\sigma}{\sigma-1}} \right). \tag{28}
\]

Since \( q > \frac{n+2}{2} \), \( \frac{n+2}{n-1} \leq \frac{n+2}{n} \), by interpolation inequality,

\[
\| \eta^2 v^\beta \|_{\frac{n+2}{n-1},D} \leq \epsilon' \| \eta^2 v^\beta \|_{\frac{n+2}{n},D} + (\epsilon')^{-\nu} \| \eta^2 v^\beta \|_{1,D},
\]

where \( \nu = \frac{n+2}{2n-1-\frac{n+2}{n}} \). Therefore,

\[
(1 - \Lambda(\beta) \sigma^{\frac{\sigma}{\sigma-1}} C_0 \epsilon') \| \eta^2 v^\beta \|_{\frac{n+2}{n-1},D} \leq \Lambda(\beta) \sigma^{\frac{\sigma}{\sigma-1}} \left( \left( \int_\Omega \left( |\nabla \eta|^2 + 2 \eta \frac{\partial \eta}{\partial t} \right) v^\beta \, d\mu \right)^{\frac{\sigma}{\sigma-1}} + C_0 \cdot (\epsilon')^{-\nu} \| \eta^2 v^\beta \|_{1,D}. \right)
\]

Let \( \epsilon' = \frac{1}{2 \Lambda(\beta) \sigma^{\frac{\sigma}{\sigma-1}} C_0} \), we get

\[
\| \eta^2 v^\beta \|_{\frac{n+2}{n-1},D} \leq 2 \Lambda(\beta) \sigma^{\frac{\sigma}{\sigma-1}} \left( \left( \int_\Omega \left( |\nabla \eta|^2 + 2 \eta \frac{\partial \eta}{\partial t} \right) v^\beta \, d\mu \right)^{\frac{\sigma}{\sigma-1}} + C_0 \cdot (2 \Lambda(\beta) \sigma^{\frac{\sigma}{\sigma-1}} C_0)^\nu \| \eta^2 v^\beta \|_{1,D}. \right)
\]

Since we can always choose \( \Lambda(\beta) \geq 1 \), we obtain

\[
\| \eta^2 v^\beta \|_{\frac{n+2}{n-1},D} \leq C_1 \left( n, q, C_0 \right) \Lambda(\beta)^{1+\nu} \int_D \left( |\nabla \eta|^2 + 2 \eta \frac{\partial \eta}{\partial t} + \eta^2 \right) v^\beta \, d\mu dt. \tag{29}
\]
Then we construct cutoff functions and domains. Define
\[ t_k \triangleq \frac{1}{2} - \frac{1}{2^{k+1}}, \quad r_k \triangleq \left( \frac{1}{2} + \frac{1}{2^{k+1}} \right) r, \quad k \geq 0, \]
\[ \Omega_k \triangleq B_{g(1)}(p, r_k), \quad D_k \triangleq \Omega_k \times [t_k, 1], \quad k \geq 0. \quad (30) \]

Let \( \gamma \in C^\infty(\mathbb{R}, \mathbb{R}) \), \( 0 \leq \gamma' \leq 2 \), and
\[ \gamma(t) = \begin{cases} 
0, & t \leq 0, \\
1, & t \geq 1.
\end{cases} \]

Define \( \gamma_k(t) \triangleq \gamma(\frac{t - t_{k-1}}{r_k}), \quad k \geq 1. \)
Let \( \rho \in C^\infty(\mathbb{R}, \mathbb{R}) \), \( -2 \leq \rho' \leq 0 \), and
\[ \rho(s) = \begin{cases} 
1, & s \leq 0, \\
0, & s \geq 1.
\end{cases} \]

Define \( \rho_k(s) \triangleq \rho(\frac{s - r_k}{r_k - r_{k-1}}), \quad k \geq 1. \) Then let
\[ \eta_k(x, t) = \gamma_k(t) \rho_k(d_{g(1)}(x, p)). \]

Therefore, \( 0 \leq \eta_k \leq 1 \), and
\[ \eta_k(x, t) = \begin{cases} 
0, & (x, t) \in D/D_{k-1}, \\
1, & (x, t) \in D_k.
\end{cases} \]

Moreover,
\[ \left| \frac{\partial \eta_k}{\partial t} \right| = \left| \frac{\partial \gamma_k(t)}{\partial t} \rho_k(r(x)) \right| = \left| \frac{\gamma'}{t_k - t_{k-1}} \rho_k(d_{g(1)}(x, p)) \right| \leq 2^{k+2}, \]
\[ |\nabla \eta_k|_{g(1)} = |\gamma_k(t) \nabla \rho_k(d_{g(1)}(x, p))|_{g(1)} \]
\[ = |\gamma_k(t) \rho_k'(d_{g(1)}(r, p)) \nabla d_{g(1)}(x, p)|_{g(1)} \]
\[ \leq |\rho_k'(d_{g(1)}(x, p))|_{g(1)} \]
\[ \leq \frac{\rho'}{r_{k-1} - r_k} \leq 2^{k+2} r^{-1}. \]
Figure 4: basic cutoff functions

Note that
\[
\frac{d}{dt} \left| \nabla \eta_k \right|^2_{g(t)} = 2 \text{Ric}_{g(t)}(\nabla \eta_k, \nabla \eta_k) \geq -2B \left| \nabla \eta_k \right|^2_{g(t)},
\]
hence
\[
\left| \nabla \eta_k \right|^2_{g(t)} \leq e^{2B(1-t)} \left| \nabla \eta_k \right|^2_{g(1)} \leq e^{2B} \left| \nabla \eta_k \right|^2_{g(1)}.
\]
Therefore, we know
\[
\left| \frac{\partial \eta_k}{\partial t} \right| \leq 2k + 2,
\]
\[
\left| \nabla \eta_k \right|_{g(t)} \leq e^{B} 2k + 2r - 1, \quad \forall t \in [0, 1].
\]
(31)

If \( \beta \geq 2, \Lambda(\beta) = 6\beta \), by inequality (29), we have
\[
\| v^\beta \|_{\frac{n+2}{n} D_k} \leq \| \eta_k^2 v^\beta \|_{\frac{n+2}{n} D_k} \leq \| \eta_k^2 v^\beta \|_{\frac{n+2}{n} D_{k-1}} \leq C_2(n, q, \sigma, C_0) \beta^{1+\nu} \int_{D_{k-1}} (\left| \nabla \eta_k \right|^2 + 2\eta_k \frac{\partial \eta_k}{\partial t} + \eta_k^2) v^\beta d\mu dt \leq 4^{k+2} C_3(r, B) C_2(n, q, \sigma, C_0) \beta^{1+\nu} \int_{D_{k-1}} v^\beta d\mu dt \triangleq C_4(n, q, \sigma, C_0, r, B) \cdot 4^{k+2} \beta^{1+\nu} \| v^\beta \|_{1, D_{k-1}}.
\]
consequently,
\[
\| v \|_{\frac{n+2}{n} \beta, D_k} \leq C_4^{\frac{1}{\beta}} \cdot 4^{\frac{k+2}{\beta}} \cdot \beta^{1+\nu} \| v \|_{\beta, D_{k-1}}.
\]
(32)

Let \( \lambda \triangleq \frac{n+2}{n} \), then
\[
\| v \|_{\lambda k, D_k} \leq C_4 \frac{\frac{1}{\lambda} + \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^{k_0}}}{\lambda^{k_0}} + \cdots + \frac{k_0}{\lambda^{k_0}} \lambda^{(1+\nu)\left(\frac{1}{\lambda} + \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^{k_0}}\right)} \| v \|_{\lambda^{k_0}, D_{k_0}} \triangleq C_5(n, q, \sigma, C_0, r, B) \| v \|_{\lambda^{k_0}, D_{k_0}}.
\]
Here \( k_0 = k_0(n) \) is the smallest integer such that \( \lambda^{k_0} \geq 2 \). If \( \beta < 2 \), since (29) is true, we can still do iteration. Starting from \( \| v \|_{\lambda, D_1} \), in \( k_0 \) steps, we can get a control of \( \| v \|_{\lambda^{k_0}, D_{k_0}} \). That is,
\[
\| v \|_{\lambda^{k_0}, D_{k_0}} \leq C_6(n, q, \sigma, C_0, r, B) \| v \|_{\lambda, D_1}.
\]
17
Lemma 5.2. \(\|v\|_{\lambda^k, D_k} \leq C_7(n, q, \sigma, C_0, r, B)\|v\|_{\lambda, D_1}, \quad \forall \ k \geq 0. \quad (33)\)

Actually, what we get is
\[
\|v\|_{\lambda^{k_1}, D_k} \leq C_7(n, q, \sigma, C_0, r, B)\|v\|_{\lambda^{k_1}, D_{k_1}}, \quad \forall \ 0 \leq k_1 \leq k_2. \quad (34)
\]
From inequality (33), and \(D' \subset D_k, \forall \ k \geq 0\), we get
\[
\|v\|_{\lambda^{k_1}, D'} \leq \|v\|_{\lambda^{k_1}, D_k} \leq C_7\|v\|_{\lambda, D_k} \leq C_7\|v\|_{\lambda, D}.
\]
Let \(k \to \infty\), \(C_a \triangleq C_7(n, q, \sigma, C_0, r, B)\), we get
\[
\|v\|_{\infty, D'} \leq C_a(n, q, \sigma, C_0, r, B)\|v\|_{\lambda, D}.
\]
Since \(u \geq 0\), we have
\[
\|u\|_{\infty, D'} \leq \|v\|_{\infty, D'} \leq C_a(n, q, \sigma, C_0, r, B)\|v\|_{\lambda, D}
\]
\[
\leq C_a(n, q, \sigma, C_0, r, B)(\|u\|_{\lambda, D} + \kappa\|1\|_{\lambda, D})
\]
\[
= C_a(n, q, \sigma, C_0, r, B)(\|u\|_{\lambda, D} + \|h\|_{q, D}\|1\|_{\lambda, D}).
\]

\[
\square
\]

Remark 5.1. From our proof, in order inequality (34) to be true, we only need \(\|f\|_{q, D_{k_1}} + \|R - |\|q, D_{k_1} + 1 \leq C_0.\) Consequently, inequality (25) is true for the same constant if \(D\) is replaced by \(D_k\), i.e.,
\[
\|u\|_{\infty, D'} \leq C_a(n, \sigma, C_0, r, B)(\|u\|_{\lambda^{k_1}, D_k} + \|h\|_{q, D}\|1\|_{\lambda^{k_1}, D_k}).
\]

Lemma 5.2. \(\{(M^n, g(t)), \ 0 \leq t \leq 1\}\) is a closed Ricci flow solution with \(\text{Ric} \geq -B\). There is a uniform Soblev constant \(\sigma\) for \(\Omega\) at each time slice. If \(u \in C^1(D)\) and \(u \geq 0,\)
\[
\frac{\partial u}{\partial t} \leq \Delta u + fu + h,
\]
in distribution sense. Here \(f \in L^{\frac{n+2}{n-2}}(D)\). Fix \(\beta > 1\). Then there are two constants \(\delta_b(n, \sigma, \beta), C_b(n, \sigma, r, B, \beta)\) such that if \(\|f\|_{\frac{n+2}{n-2}, D} + \|R - |\|\geq \delta_b, then
\[
\|u\|_{\frac{n+2}{n-2}, \beta, D_k} \leq C_b(n, \sigma, r, B, \beta)(\|u\|_{\beta, D} + \|h\|_{\frac{n+2}{n-2}, D}\|1\|_{\beta, D}).
\]

Here \(D_1\) is defined by equation (35).

Proof. Let \(\eta = \eta_1\), then we do the calculation as in the proof of lemma 5.1. Instead of \(\kappa = \|h\|_{q, D}\) in the previous lemma, we let \(\kappa = l \cdot \|h\|_{\frac{n+2}{n-2}, D}\) for some positive number \(l\). We can get a similar inequality as inequality (28),
\[
\|\eta_1^2 v\|_{\frac{n+2}{n-2}, D} \leq \sigma \frac{n+2}{n-2}\Lambda(\beta)\int_D (|\nabla \eta_1|^2 + 2\eta_1 \frac{\partial \eta_1}{\partial t})v^2 d\mu dt + (\|f\|_{\frac{n+2}{n-2}, D} + \|R - |\|\frac{n+2}{n-2}, D + \frac{1}{l})\|\eta_1^2 v\|_{\frac{n+2}{n-2}, D}\.\]

18
If \( \|f\|_{\frac{n+2}{n+1},D} + \|R\|_{\frac{n+2}{n+1},D} \leq \frac{1}{4\sigma \lambda(B)} \), choose \( l = 4\sigma \lambda(B) + 1 \), we obtain

\[
\|\eta^2_1 v^\beta\|_{\frac{n+2}{n+1},D} \leq 2\sigma \lambda(B) C_8(r,B) \|v^\beta\|_{1,D}.
\]

Consequently,

\[
\|v\|_{\frac{n+2}{n+2},D} = \|v^\beta\|_{\frac{n+2}{n+1},D} \leq 2\sigma \lambda(B) C_8(r,B) \|v\|_{1,D}.
\]

Let \( C_9(n,\sigma,r,B,\beta) \equiv \frac{2\sigma \lambda(B) C_8(r,B)^\frac{1}{n+2}}{n+2} \), we get

\[
\|v\|_{\frac{n+2}{n+2},D} \leq C_9(n,\sigma,r,B,\beta) \|v\|_{\beta,D}.
\]

Since \( v = u + \kappa \), \( u \geq 0 \),

\[
\|u\|_{\frac{n+2}{n+2},D} \leq \|v\|_{\beta,D} \leq C_9(n,\sigma,r,B,\beta) \|u\|_{\beta,D} \leq C_9(n,\sigma,r,B,\beta) (\|u\|_{\beta,D} + \|\kappa\|_{\beta,D}) = C_9(n,\sigma,r,B,\beta) (\|u\|_{\beta,D} + \|h\|_{\frac{n+2}{n+2},D} \|1\|_{\beta,D}) \leq C_9(n,\sigma,r,B,\beta) (\|u\|_{\beta,D} + \|h\|_{\frac{n+2}{n+2},D} \|1\|_{\beta,D}).
\]

Therefore, we finish the proof if we choose

\[
\delta_b(n,\sigma,\beta) = \frac{1}{4\sigma \lambda(B)}.
\]

\[
C_b(n,\sigma,r,B,\beta) = C_9(n,\sigma,r,B,\beta) \cdot (4\sigma \lambda(B) + 1).
\]

\[\square\]

Before we use Moser iteration for \( R \), we need some volume control.

**Property 5.2.** \( \{(M^n,g(t)), 0 \leq t \leq 1\} \) is a closed Ricci flow solution.

\[
|Ric(x,t)| \leq (n-1), \quad \forall (x,t) \in \Omega \times [0,1].
\]

Then there exists a constant \( \bar{V}(n,r) \geq 1 \) such that

\[
\|1\|_{q,D} \leq \frac{\bar{V}}{q} \leq \bar{V}, \quad \forall q \geq 1.
\]

**Proof.** Since \( \Omega = B_{g(t)}(p,r), Ric \leq (n-1) \), by the evolution of geodesic length under Ricci flow, we have

\[
\Omega \subset B_{g(t)}(p,e^{(n-1)r}), \quad \forall t \in [0,1].
\]
On the other hand, $\text{Ric} \geq -(n - 1)$, by volume comparison theorem, we obtain

$$\int_{B_{g(t)}(p,e^{-r})} d\mu \leq \alpha(n - 1) \int_0^{e^{-r}} (\sinh r)^{n-1} dr \triangleq C_{10}(n,r),$$

where $\alpha(n - 1)$ is the area of $S^{n-1}$ with canonical metric. Hence,

$$\|1\|_{1,D} = \int_0^1 \int_\Omega d\mu d\tau \leq \int_0^1 \int_{B_{g(t)}(p,e^{-r})} d\mu d\tau \leq C_{10}.$$

Let $\tilde{V}(n,r) \triangleq \max\{C_{10}, 1\}$, then

$$\|1\|_{q,D} = \|1\|_{1,D}^{\frac{q}{2}} \leq \tilde{V}^{\frac{q}{2}} \leq \tilde{V}, \quad \forall \ q \geq 1.$$

Now we can apply Moser iteration to $R$.

**Theorem 5.1.** \{(M',g(t)), \ 0 \leq t \leq 1\} is a closed Ricci flow solution. Suppose $\text{Ric}(x,t) \geq -B$, $\forall (x,t) \in M \times [0,1]$, $0 \leq B \leq 1$.

There is a uniform Soblev constant $\sigma$ for $\Omega$ at each time slice. Then there are constants $\delta(n,\sigma,r), C(n,\sigma,r)$ such that if $\|R\|_{\frac{n+2}{2},D} + B \leq \delta$, then

$$\|R\|_{\infty,D} \leq C(\|R\|_{\frac{n+2}{2},D} + B).$$  \hspace{1cm} (38)

**Proof.** Since $\text{Ric} \geq -B$, define $\hat{R} \triangleq R + nB$, we get inequality [4].

$$\frac{\partial \hat{R}}{\partial t} \leq \triangle \hat{R} + 2(\hat{R} - 2B)\hat{R} + 2nB^2.$$

Because $0 \leq B \leq 1$, in $D = \Omega \times [0,1]$, $|\text{Ric}| \leq (n - 1)$, by Property 5.2,

$$\|1\|_{q,D} = \|1\|_{1,D}^{\frac{q}{2}} \leq \tilde{V}^{\frac{q}{2}} \leq \tilde{V}, \quad \forall \ q \geq 1.$$

Let $u = \hat{R}, f = 2(\hat{R} - 2B), h = 2nB^2$. As in lemma 5.2 let

$$\beta = \frac{n + 2}{2};$$

$$\delta_b = \delta_b(n,\sigma,\beta);$$

$$C_b = C_b(n,\sigma,r,1,\beta).$$

If $\|R\|_{\frac{n+2}{2},D} + B$ is very small, say,

$$\|R\|_{\frac{n+2}{2},D} + B \leq \delta(n,\sigma,r) \triangleq \frac{\delta_b}{3n\tilde{V}},$$

where
\[ \|2(\hat{R} - 2B)\|_{\frac{n+2}{2},D} + \|R_\omega\|_{\frac{n+2}{2},D} \]
\[ = \|2(R + (n-2)B)\|_{\frac{n+2}{2},D} + \|R_\omega\|_{\frac{n+2}{2},D} \]
\[ \leq 3\|R\|_{\frac{n+2}{2},D} + 2(n-2)B\|1\|_{\frac{n+2}{2},D} \]
\[ \leq 3n\hat{V}^{\frac{n+4}{4}}(\|R\|_{\frac{n+2}{2},D} + B) \]
\[ \leq \frac{\delta_b}{\hat{V}^{\frac{n+4}{4}}} \leq \delta_b, \]

hence, by lemma 5.2

\[ \|\hat{R}\|_{\frac{n+2}{2},D} \leq C_b(\|\hat{R}\|_{\frac{n+2}{2},D} + 2nB^2\|1\|_{\frac{n+2}{2},D}) \]
\[ \leq C_b(\|R\|_{\frac{n+2}{2},D} + nB\|1\|_{\frac{n+2}{2},D} + 2nB^2\|1\|_{\frac{n+2}{2},D}) \]
\[ \leq C_b(\|R\|_{\frac{n+2}{2},D} + 3nB\hat{V}^{\frac{n+4}{4}}) \]
\[ \leq C_b3n\hat{V}^{\frac{n+4}{4}}(\|R\|_{\frac{n+2}{2},D} + B) \]
\[ \leq C_b\delta_b. \tag{39} \]

Now let \( q = \frac{n+2}{n} > \frac{n+2}{2} \), then from inequality (40),

\[ \|2(\hat{R} - 2B)\|_{q,D_1} + \|R_\omega\|_{q,D_1} + 1 \leq 3\|\hat{R}\|_{q,D_1} + (n + 4)B\|1\|_{q,D_1} + 1 \]
\[ \leq 3C_b\delta_b + (n + 4)B\hat{V}^{\frac{n+4}{4}} + 1 \]
\[ \leq 3C_b\delta_b + \delta_b + 1. \]

Note that \( 0 \leq B \leq 1 \), by the definition of \( C_b \) in Lemma 5.1 we get

\[ C_b(n, \frac{(n+2)^2}{2n}, (3C_b + 1)\delta_b + 1, \sigma, r, B) \leq C_b(n, \frac{(n+2)^2}{2n}, (3C_b + 1)\delta_b + 1, \sigma, r, 1). \]

Let \( C_a = C_a(n, \frac{(n+2)^2}{2n}, (3C_b + 1)\delta_b + 1, \sigma, r, 1). \)

From Remark 5.1 we have

\[ \|\hat{R}\|_{\infty,D'} \leq C_a(n, \frac{(n+2)^2}{2n}, (3C_b + 1)\delta_b + 1, \sigma, r, B)(\|\hat{R}\|_{\frac{n+2}{2},D_1} + \|h\|_{q,D}1\|_{\frac{n+2}{2},D_1}) \]
\[ \text{by H"older inequality} \]
\[ \leq C_a(\|\hat{R}\|_{\frac{(n+2)^2}{2n},D_1}1\|_{\frac{n+2}{2},D_1} + 2nB^2\|1\|_{\frac{n+2}{2},D_1}) \]
\[ \text{from inequality (39)} \]
\[ \leq C_a(3nC_b\hat{V}^{\frac{n+4}{4}}(\|R\|_{\frac{n+2}{2},D} + B)\|1\|_{\frac{n+2}{2},D_1} + 2nB^2\|1\|_{\frac{(n+2)^2}{2n},D}1\|_{\frac{n+2}{2},D_1}) \]
\[ \text{since } \hat{V} \geq 1 \]
\[ \leq C_a\hat{V}^{\frac{n+4}{4}}(3nC_b(\|R\|_{\frac{n+2}{2},D} + B) + 2nB^2) \]
\[ \text{note that } B^2 \leq B \]
\[ \leq 3n(C_b + 1)C_a\hat{V}^{\frac{n+4}{4}}(\|R\|_{\frac{n+2}{2},D} + B). \tag{41} \]
Note that \(\|R_t\|_{\infty,D'} \leq \|\hat{R}\|_{\infty,D'}\). Let \(C(n,\sigma,r) \triangleq 3n(C_b+1)C_\sigma \hat{V}^{n+\frac{\alpha-2}{2}}\), from inequality (41), we have
\[
\|R_t\|_{\infty,D'} \leq C(n,\sigma,r)(\|R\|_{\infty,D} + B).
\]

\[\square\]

6 Proof of Theorem 1.1 for \(n \geq 3\)

Proof. Since \(\|R\|_{\alpha,M \times [0,T)} < \infty\) implies \(\|R\|_{\alpha,M \times [0,T)} < \infty\) if \(\alpha \geq \frac{n+2}{2}\), so we only need to prove Theorem 1.1 for \(\alpha = \frac{n+2}{2}\). We shall argue by contradiction.

Suppose the flow cannot be extended, then \(|Ric|\) is unbounded by Sesum’s result. Since \(Ric \geq -A\), we know
\[
\sup_{(x,t) \in M \times [0,T)} R(x,t) = \infty.
\]
Therefore, there exists a sequence \((x^{(i)},t^{(i)})\) such that \(\lim_{i \to \infty} t^{(i)} = T\), and
\[
R(x^{(i)},t^{(i)}) = \max_{(x,t) \in M \times [0,t^{(i)}]} R(x,t).
\]
Consequently, \(\lim_{i \to \infty} R(x^{(i)},t^{(i)}) = \infty\). Define
\[
Q^{(i)} \triangleq R(x^{(i)},t^{(i)}), \quad P^{(i)} \triangleq B_{g(t^{(i)})}(x^{(i)},(Q^{(i)})^{-\frac{1}{2}}) \times [t^{(i)}-(Q^{(i)})^{-1},t^{(i)}],
\]
then for any \((x,t) \in D^{(i)}\), \(R(x,t) \leq Q^{(i)}\).

Now, let \(g^{(i)}(t) \triangleq Q^{(i)}g((Q^{(i)})^{-1}(t-1) + t^{(i)})\). We have a sequence of Ricci flow solutions: \(\{(M^n,g^{(i)}(t),)\}, 0 \leq t \leq 1\}. Moreover,
\[
\begin{align*}
R^{(i)}(x,t) &\leq 1, \quad \forall (x,t) \in B_{g^{(i)}(1)}(x^{(i)},1) \times [0,1]; \\
Ric^{(i)}(x,t) &\geq \frac{A_{Q^{(i)}}}{Q^{(i)}}, \quad \forall (x,t) \in M \times [0,1].
\end{align*}
\]

(42)

Since \(Ric^{(i)} + \frac{A}{Q^{(i)}} > 0\), so
\[
Ric^{(i)} + \frac{A}{Q^{(i)}} \leq tr(Ric^{(i)} + \frac{A}{Q^{(i)}}) = R^{(i)} + \frac{nA}{Q^{(i)}}.
\]
Consequently, \(Ric^{(i)} \leq R^{(i)} + \frac{(n+1)A}{Q^{(i)}}\). Note that \(\lim_{i \to \infty} \frac{A}{Q^{(i)}} = 0\), \(n \geq 3\), by inequalities (42), we get
\[
\begin{align*}
Ric^{(i)}(x,t) &\leq n-1, \quad \forall (x,t) \in B_{g^{(i)}(1)}(x^{(i)},1) \times [0,1]; \\
Ric^{(i)}(x,t) &\geq -\frac{A}{Q^{(i)}}, \quad \forall (x,t) \in M \times [0,1].
\end{align*}
\]

(43)

Since for any \(x \in B_{g^{(i)}(1)}(x^{(i)},(Q^{(i)})^{-\frac{1}{2}})\), \(-nA \leq R(x,t) \leq Q^{(i)}\), for large \(i\), we have \(|R(x,t)| \leq Q^{(i)}\). By Theorem 2.22 there exists a \(\kappa\) such that
\[
\text{Vol}_{g^{(i)}(1)}(B_{g^{(i)}(1)}(x^{(i)},1)) = \frac{\text{Vol}_{g^{(i)}(1)}(B_{g^{(i)}(1)}(x^{(i)},(Q^{(i)})^{-\frac{1}{2}}))}{(Q^{(i)})^{-\frac{1}{2}}} \geq \kappa.
\]

(44)
From inequalities \((\ref{13})\) and \((\ref{14})\), we are able to use Theorem \((\ref{5.1})\). Therefore, we get a constant \(r(\alpha, n)\) such that for large \(i\), on the geodesic ball \(B_{g^{(i)}}(p, r)\), there is a uniform Sobolev constant \(\sigma(n, r)\) for every time slice \(t \in [0, 1]\).

Then we collect conditions to use Theorem \((\ref{5.1})\). Define
\[
\Omega^{(i)} \triangleq B_{g^{(i)}}(p, r), \quad \Omega^{(i)}' \triangleq B_{g^{(i)}}(p, \frac{r}{2}),
\]
\[
D^{(i)} \triangleq \Omega^{(i)} \times [0, 1], \quad D^{(i)}' \triangleq \Omega^{(i)}' \times [\frac{1}{2}, 1].
\]

Since \(\int_0^T \int_M |R|^\frac{n+2}{2} d\mu dt\) is a scale invariant,
\[
\lim_{i \to \infty} \|R^{(i)}\|_{\nabla^2 D^{(i)}}, \frac{A}{Q^{(i)}} = \lim_{i \to \infty} \|R^{(i)}\|_{\nabla^2 D^{(i)}}
\]
\[
= \lim_{i \to \infty} \int_{t^{(i)}-(Q^{(i)})^{-1}} \int B_{g^{(i)}}(p, r(Q^{(i)})^{-\frac{1}{2}}) |R|^{\frac{n+2}{2}} d\mu dt
\]
\[
\leq \lim_{i \to \infty} \int_{t^{(i)}-(Q^{(i)})^{-1}} \int_M |R|^{\frac{n+2}{2}} d\mu dt
\]
\[
= 0.
\]

The last step comes from \(\int_0^T \int_M |R|^{\frac{n+2}{2}} d\mu dt < \infty\) and \(\lim_{i \to \infty} (Q^{(i)})^{-1} = 0\). Consequently, for large \(i\), \(\|R^{(i)}\|_{\nabla^2 D^{(i)}}, \frac{A}{Q^{(i)}} \leq \delta(n, \sigma, r)\). From Theorem \((\ref{5.1})\) we know
\[
\|R^{(i)}\|_{\infty, D'} \leq C(n, \sigma, r)(\|R^{(i)}\|_{\nabla^2 D^{(i)}}, \frac{A}{Q^{(i)}}).
\]
Taking limit on both sides, we get
\[
\lim_{i \to \infty} \|R^{(i)}\|_{\infty, D'} \leq \lim_{i \to \infty} C(n, \sigma, r)(\|R^{(i)}\|_{\nabla^2 D^{(i)}}, \frac{A}{Q^{(i)}}) = 0.
\]

On the other hand,
\[
\lim_{i \to \infty} \|R^{(i)}\|_{\infty, D'} \geq \lim_{i \to \infty} R^{(i)}(x^{(i)}, 1) = 1.
\]

Therefore we get a contradiction. \(\square\)

**Remark 6.1.** From the proof, we know the condition \(\text{Ric} \geq -A\) is used only to assure that after blowup, Ricci curvature becomes almost nonnegative. However, when \(\dim = 3\), this can be achieved automatically. Actually, by Hamilton-Allen’s pinch\(\text{cf.}(\ref{14}),\) Theorem\(4.1),\)
\[
R \geq |\nu|(|\nu| + \log(1 + t) - 3).
\]
Here \(\nu(x, t)\) is the smallest eigenvalue of the curvature operator and we have normalized the initial metric such that \(\inf_{x \in M} \nu(x, 0) \geq -1\). This tells us that Ricci curvature must be nonnegative after blowup. Therefore, we can get the following Corollary.

**Corollary 6.1.** \(\{M^3; g(t), 0 \leq t < T < \infty\}\) is a closed Ricci flow solution. If \(\|R\|_{0, \infty(M \times [0, T])} < \infty, \alpha \geq \frac{3}{2}\), then this flow can be extended over time \(T\).

A natural question is whether the Ricci lower bound condition superfluous in higher dimension. To be conservative, can we substitute the condition \(\text{Ric} \geq -A\) by a weaker one?
References

[1] M. Berger, Lectures on Geodesics in Riemannian Geometry, Tata Institute, Bombay, 1965.

[2] A. Besse, Manifolds All of Whose Geodesics are Closed, Ergebnisse der Mathematik, Vol. 93, Springer, Berlin-Heidelberg-New York, 1978.

[3] Xiuxiong Chen, Gang Tian, Ricci flow on Kähler-Einstein manifolds, Duke Math. J. 131 (2006), no. 1, 17–73.

[4] Bennett Chow, Peng Lu, Lei Ni, Hamilton’s Ricci Flow, preprint.

[5] Christopher B. Croke, Some Isoperimetric Inequalities and Eigenvalue Estimates, Ann. scient. Éc. Norm. Sup., 1980, 419-435.

[6] D. DeTurck: Deforming metrics in the direction of their Ricci tensors, J. Differential Geometry. 18(1983), no.1, 157-162.

[7] R. S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geometry. 17(1982), no.2, 255-306.

[8] R. S. Hamilton, Four-manifolds with positive curvature operator, J. Differential Geometry. 24(1986), no.2, 153-179.

[9] R. S. Hamilton, The formation of singularities in the Ricci flow, Surveys in Differential Geometry, vol.2, International Press, Cambridge, MA(1995) 7-136.

[10] R. S. Hamilton, A compactness property for solutions of the Ricci flow, Amer. J. Math. 117(1995) 545-572.

[11] R. S. Hamilton, Non-singular solutions of the Ricci flow on three manifolds, Commun. Anal. Geom. 7(1999), 695-729.

[12] Bruce Kleiner, John Lott, Notes on Perelman’s papers, arXiv: math.DG/0605667.

[13] John W. Morgan, Gang Tian, Ricci Flow and the Poincaré Conjecture, arXiv: math.DG/0607607.

[14] Grisha Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv: math.DG/0211159.

[15] Grisha Perelman, Ricci flow with surgery on three-manifolds, arXiv: math.DG/0303109.

[16] Grisha Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, arXiv: math.DG/0307245.

[17] L. A. Santalo, Integral Geometry and Geometric Probability, Encyclopedia of Mathematics and Its Applications, Addison-Wesley, London-Amsterdam-Don Mills, Ontario-Sydney-Tokyo, 1976.

[18] Natasa Sesum, Curvature tensor under the Ricci flow, arXiv: math.DG/0311397.
[19] Natasa Sesum, Gang Tian, Bounding scalar curvature and diameter along the Kaehler-Ricci flow (after Perelman) and some applications, http://www.math.lsa.umich.edu/~lott/ricciflow/perelman.html

[20] R. Schoen, S.-T. Yau, Lectures on differential geometry, Cambridge, MA, International Press, c1994.

University of Wisconsin Madison, Department of Mathematics
480 Lincoln Drive, Madison, WI, 53706, U.S.A

E-mail address: bwang@math.wisc.edu