Branched Hamiltonians and time translation symmetry breaking in equations of the Liénard type

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Abstract

Shapere and Wilczek (Phys. Rev. Lett. 109, 160402 and 200402 (2012)) have recently described certain singular Lagrangian systems which display spontaneous breaking of time translation symmetry. We begin by considering the standard Liénard equation for which a Lagrangian is constructed by using the method of Jacobi Last Multiplier. The velocity dependance of the Lagrangian is such that the momentum may exhibit multivaluedness thereby leading to the so called branched Hamiltonian. Next with a quadratic velocity dependance in the Liénard equation one can construct a Hamiltonian description involving a position dependent mass. We compute the Lagrangian and Hamiltonian of this system and show that the canonical Hamiltonian is single valued. However, we find that up to a constant shift, the square of this Hamiltonian describes systems giving rise to spontaneous time translation symmetry breaking provided the potential function is negative.

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1 Introduction

Recently Shapere and Wilczek [1, 2] have shown that for certain special Lagrangian systems the time translation symmetry can be spontaneously broken in the lowest energy or ground state. This has revived interest in the study of systems with non-standard and/or non-convex Lagrangians especially with regard to spontaneous breaking of time translation symmetry. A direct consequence of the spontaneously broken time translation symmetry in the ground states is the multivaluedness of the Hamiltonian.

A common feature shared by all the models considered by Shapere and Wilczek [1, 2] is that the energy function (Hamiltonian) or Lagrangian systems become multivalued in terms of the canonical phase space variables. Recently it has become clear that, for special kinds of mechanical systems, there are choices of Hamiltonian structures in which certain fundamental aspects of classical canonical Hamiltonian mechanics are changed. It has been explored in [3, 4, 5], one can change the phase space variables which makes the Hamiltonian and symplectic structures on the phase space simultaneously well defined at the price of introducing a non-canonical symplectic structure. Curtright and Zachos [6] displayed some simple unified Lagrangian prototype systems which, by virtue of non-convexity in their velocity dependence, branch into double-valued (but still self-adjoint) Hamiltonians.

It is noteworthy that for systems possessing multiple Hamiltonian descriptions, there have been discussions in the literature as to find the proper choice of Hamiltonian functions. Furthermore an analysis of such models has even led to speculations about the possibility of perpetual motion. Shapere and Wilczek papers triggered a new interest on the systems with branched Hamiltonians.

The issue of time independent classical dynamical systems exhibiting motion in their lowest energy states has been instrumental in the introduction of a time analogue of spatial order as in a crystalline substance [1] (the so called time crystals) and its spontaneous breaking. It is therefore natural to investigate the issue of time translation breaking from the perspective of second-order differential equations within the general framework of Lagrangian/Hamiltonian mechanics [3, 4].

Motivation and result: The motivation for the present work arose originally from Shapere and Wilczek’s observation that the Lagrangians of some mechanical systems display spontaneous time translation symmetry breaking properties in their lowest energy state, and the Hamiltonian descriptions of certain singular models involving multi-valuedness and branching point singularities. In a previous article we obtained the Chiellini integrability criterion for the Liénard equation by using Jacobi’s last multiplier [15] and derived the bi-Hamiltonian structure of those equations of the Liénard type satisfying this particular criterion. Moreover we also constructed certain non-natural Lagrangians and Hamiltonians for the Liénard equation using Jacobi’s last multiplier; consequently it is only natural that we investigate the possible existence of time translation symmetry breaking of the ground state for such systems. The first case we deal with is that of a second-order ordinary differential equation (ODE) of the usual Liénard type viz

\[ \ddot{x} + f(x)\dot{x} + g(x) = 0, \]  

(1.1)
for which we present specific cases of a double valued Hamiltonian and its branches. This is followed up with a quadratic version (the Liénard-II equation) [7], namely

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0.$$  \hspace{1cm} (1.2)

The latter naturally emerges from Newton’s second law when dealing with a system characterized by a variable mass (depending on the position coordinate) and also frequently arises in the context of isochronous systems [8, 9, 10]. By a suitable modification of the Hamiltonian of this equation we obtain the locus of the curve of the singular points for which the energy is less than the minimum value indicating the spontaneous breaking of time translation symmetry.

This paper is organized as follows. We present the branched Hamiltonian description of the Liénard equation in Section 2. We also illustrate the double valuedness of the Hamiltonian description. Section 3 is devoted to the hamiltonization of an equation of Lienard type with a quadratic dependence on the velocity, dubbed as Liénard II equation. We demonstrate how the time translation symmetry spontaneously broken for Liénard II system in Section 4.

## 2 The Liénard-I equation and branched Hamiltonians

There exists an extensive literature on the Liénard-I equation (for example, [13, 14]) and in this section our attempt is to incorporate the Liénard-I equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$  \hspace{1cm} (2.1)

into the branched Hamiltonian framework. It has been shown in [8, 9] how a system of the Liénard type as given by (1.2) can be embedded into the Hamiltonian formalism. We briefly recapitulate the procedure below. Given a second-order ordinary differential equation (ODE)

$$\ddot{x} = F(x, \dot{x})$$  \hspace{1cm} (2.2)

we define the Jacobi last multiplier $M$ as a solution of the following ODE

$$\frac{d}{dt} \log M + \frac{\partial F(x, \dot{x})}{\partial \dot{x}} = 0.$$  \hspace{1cm} (2.3)

Assuming (2.2) to be derivable from the Euler-Lagrange equation one can show that the JLM is related to the Lagrangian by the following equation

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}.$$  \hspace{1cm} (2.4)

From (2.3) a formal solution of the Jacobi last multiplier for (2.1) may be written as

$$M(t, x) = \exp \left( \int f(x)dt \right) := u^{1/\ell},$$  \hspace{1cm} (2.5)

where $u$ is a new nonlocal variable and $\ell$ is a parameter whose value is fixed by the following lemma once $f$ and $g$ are given.
Lemma 2.1  The Liénard equation (2.1) can be written as the following system

\[ \dot{u} = \ell u f(x), \quad \dot{x} = u + W(x) \]

where \( W = g/f\ell \) with the parameter \( \ell \) being determined by the following condition

\[ \frac{d}{dx} \left( \frac{g}{f} \right) = -\ell(\ell + 1)f(x). \]  

\[ (2.6) \]

**Proof:** From (2.5) we have \( \log u = \ell \int f(x)dt \), which implies \( \dot{u} = \ell uf(x) \). Setting \( \dot{x} = u + W(x) \) we find by differentiating with respect to \( t \)

\[ \ddot{x} = \dot{u} + W'(x)\dot{x}. \]

Inserting the expression for \( \dot{u} \) from the previous equation and after eliminating \( u \) we find that

\[ \ddot{x} = \ell f(x)(\dot{x} - W) + W'(x)\dot{x}. \]

Comparison with (2.1) then shows \( W'(x) = -(\ell + 1)f(x) \) and \( W(x) = g/\ell f \). Consistency now requires that

\[ \frac{d}{dx} \left( \frac{g}{f} \right) = -\ell(\ell + 1)f(x), \]

which represents actually the Cheillini integrability condition for (2.1) (see [15], for Cheillini integrability condition in the context of Liénard equation).

Since the transformation is nonlocal so a mapping to the \((x, u)\)-plane is not possible and therefore one cannot really analyse the problem in the local manner of point transformations.

However, from (2.4) and (2.5) we have

\[ \frac{\partial^2 L}{\partial \dot{x}^2} = \left( \dot{x} - \frac{1}{\ell} \frac{g}{f} \right)^{1/\ell}, \]

and it may be shown that (2.1) can be derived from the following Lagrangian

\[ L = \frac{\ell^2}{(\ell + 1)(2\ell + 1)} \left( \dot{x} - \frac{1}{\ell} \frac{g}{f} \right)^{(2\ell+1)/\ell}, \]  

\[ (2.7) \]

provided the functions \( f \) and \( g \) satisfy the Cheillini integrability condition (2.6).

### 2.1 A class of double-valued Hamiltonians

Before proceeding to a determination of the Hamiltonian for (2.1) from the above Lagrangian we note that the curvature \( \partial^2 L/\partial \dot{x}^2 \) changes sign at the points where \( \dot{x} = g/f\ell \) provided \( \ell \) is an odd integer or \( 1/\ell \) is an odd integer. The conjugate momentum is given as usual by

\[ p = \frac{\ell}{\ell + 1} \left( \dot{x} - \frac{1}{\ell} \frac{g}{f} \right)^{(\ell+1)/\ell}. \]
The inversion of this relation to determine $\dot{x}$ as a function of $p$ and $x$ presents us with difficulty and is the source of the double valuedness of the resulting Hamiltonian. Formally the Hamiltonian is

$$H = p^{2\ell + 1/\ell + 1}K(\ell) - (g/\ell f)p$$

where $K(\ell)$ is just a scaling factor.

By enlarging the phase space and making use of Dirac’s theory on constrained Hamiltonian systems Zhao et al [4] presented the Hamiltonian description and formulated a method to avoid the multivaluedness and the branching point singularities.

We consider the following example to illustrate our point.

**Example**

\[\ddot{x} + x\dot{x} + x - x^3 = 0\]

Here $f(x) = x$ and $g(x) = x - x^3$. One can easily verify that the Cheillini condition is satisfied with $\ell = 1$ and $-2$. For $\ell = 1$ we obtain $p = (\dot{x} - 1 + x^2)^2/2$. A plot of the variation of the conjugate momentum with $x$ and $\dot{x} = y$ is shown below in Fig. 1. On the other hand upon inversion we have $\dot{x} = 1 - x^2 \pm \sqrt{2p}$ and a plot of the variation of $\dot{x}$ with $x$ and $p$ is depicted in Fig. 1. It is observed that $\dot{x} = 1 - x^2 \pm \sqrt{2p}$ whence the Hamiltonian is double valued with the branches:

$$H_{\pm} = p(1 - x^2 \pm 2\sqrt{2p})$$

The variation of the Hamiltonians are depicted below in Fig 2.

![3D plot showing the variation $\dot{x} = 1 - x^2 \pm \sqrt{2p}$ when $\ell = 1$, the lower (upper) one is the negative ne, both meet at $p = 0$](image)

However, when $\ell = -2$ then $p = 2\sqrt{\dot{x} + \frac{1}{2}(1 - x^2)}$ leading to $\dot{x} = p^2/4 - (1 - x^2)/2$ and leads to the Hamiltonian $H = p^3/12 - p(1 - x^2)/2$, i.e., we have a single valued Hamiltonian.

We illustrate the variation of velocity and Hamiltonian when $l = -2$ in figure 3 and 4 diagrams respectively.
Figure 2: 3D plot showing the variation of the Hamiltonian $H_\pm$ when $\ell = 1$

Figure 3: 3D plot showing the variation $\dot{x} = p^2/4 - (1 - x^2)/2$ when $\ell = -2$

Figure 4: 3D plot showing the variation of Hamiltonian when $\ell = -2$

3 Hamiltonian aspects of Liénard-II equation

For the equation

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \quad (3.1)$$

one can show that a solution of the JLM is given by

$$M(x) = e^{2F(x)}, \quad F(x) := \int^x f(s)ds. \quad (3.2)$$
Furthermore it follows from (2.4) that its Lagrangian is

\[ L(x, \dot{x}) = \frac{1}{2} e^{2F(x)} \dot{x}^2 - V(x), \]  

where the potential term

\[ V(x) = \int^x e^{2F(s)} g(s) ds. \]  

Clearly the conjugate momentum

\[ p := \frac{\partial L}{\partial \dot{x}} = \dot{x} e^{2F(x)} \]  

implies \( \dot{x} = p e^{-2F(x)} \),

so that the final expression for the Hamiltonian is

\[ H = \frac{p^2}{2M(x)} + \int^x M(s) g(s) ds, \]  

where \( p = M(x) \dot{x} \) and \( M(x) = \exp(2F(x)) \) with \( F(x) = \int^x f(s) ds \). The canonical variables are \( x \) and \( p \) and they satisfy the standard Poisson brackets \( \{x, p\} = 1 \). In terms of the canonical Poisson brackets the equations of motion appear as

\[ \dot{x} = \{x, H\} = \frac{p}{M(x)}, \quad \dot{p} = \{p, H\} = \frac{M'(x)}{2M(x)} p^2 - M(x) g(x) \]

from which we can recover (3.1) upon elimination of the conjugate momentum \( p \). Here we have purposely written the Hamiltonian \( H \) in terms of the last multiplier \( M(x) \) to highlight the latter’s role as a position dependent mass term. From (3.6) it is natural that the potential \( V(x) \) be identified with

\[ V(x) = \int^x M(s) g(s) ds. \]  

As for the existence of a minima of \( H \), considered as a function of \( x \) and \( p \), it is necessary that

\[ \frac{\partial H}{\partial x} = 0 \quad \text{and} \quad \frac{\partial H}{\partial p} = 0 \]

whose solutions then define the stationary points. The former yields

\[ -p^2 \frac{M'(x)}{2M^2(x)} + M(x) g(x) = 0 \]

while the latter implies \( p/M(x) = 0 \). Therefore the stationary points are characterized by \( p = 0 \) and the value(s) of \( x \) for which \( g(x) = 0 \). If \( x = x^* \) denotes a root of \( g(x) = 0 \) then \( (x^*, p = 0) \) is a stationary point (s.p). For the s.p to be a minimum one requires that the principal minors of

\[ \Delta = \begin{vmatrix} H_{xx} & H_{xp} \\ H_{px} & H_{pp} \end{vmatrix}_{s.p} = 0. \]
be positive definite, i.e.,

\[ g'(x^*) > 0 \quad \text{and} \quad M(x^*)g'(x^*) > 0, \]

and consistency therefore requires \( M(x^*) > 0 \). Note that \( M(x) \), which may be thought of as some kind of 'effective mass' such as within a spatial crystal, may be negative for \( x \neq x^* \). Clearly the fact that \( p = 0 \) in the minimum energy state (ground state) of the system precludes the possibility of any motion.

4 A modified Hamiltonian and spontaneously broken time translation symmetry

Consider a one-dimensional generalized Hamiltonian system \( \tilde{H} = \mathcal{F}(H) \) with Hamiltonian vector field given in terms of the canonical form

\[ \mathcal{X}_{\tilde{H}} = \frac{\partial \tilde{H}}{\partial p} \frac{\partial}{\partial x} - \frac{\partial \tilde{H}}{\partial x} \frac{\partial}{\partial p}, \quad \{G, \tilde{H}\} = \dot{G}. \]

In the symplectic coordinates \((x, p)\) this is equivalent to canonical Hamiltonian equations

\[ \dot{x} = \mathcal{F}(H)'\{x, H\}, \quad \dot{p} = \mathcal{F}(H)'\{p, H\}, \quad \text{where} \quad \mathcal{F}(H)' > 0. \]

It may be easily verified that the above set of Hamiltonian equations may be obtained from the modified symplectic form \( \omega = \mathcal{F}(H)'dx \wedge dp \). Moreover this change of Hamiltonian structure will not change the partition function, hence all thermodynamic quantities will remain unchanged.

Let us consider a new Hamiltonian [4] defined by

\[ \tilde{H} = \left( \frac{p^2}{2M(x)} + \int^x M(s)g(s)ds \right)^2 + E_0 = H^2 + E_0, \tag{4.1} \]

where \( E_0 \) is an arbitrary constant. As the New Hamiltonian is anticipated to generate a dynamics which is distinct from that of \( H \), let us also introduce the following Poisson structure \( \{x, p\} = \xi(x, p) \) so that the equations of motion which follow from

\[ \dot{x} = \{x, \tilde{H}\}, \quad \dot{p} = \{p, \tilde{H}\} \tag{4.2} \]

give

\[ \dot{x} = 2\xi H \frac{p}{M(x)} \tag{4.3} \]
\[ \dot{p} = -2\xi H \left( \frac{M'(x)}{2M^2(x)}p^2 + M(x)g(x) \right). \tag{4.4} \]

At this point we need to make a clear distinction regarding the two Poisson structures we have introduced. It will be noticed that if one assumes \( \{x, p\} = \xi(x, p) = \frac{1}{2H(x,p)} \) then we get
back the original Liénard-II equation \((3.1)\), if however we persist with \(\xi = 1\), i.e., assume \(x\) and \(p\) are canonical then the equation of motion resulting from the Hamiltonian \(\tilde{H}\) is of the form

\[
\ddot{x} + 2H(f(x)\dot{x}^2 + g(x)) = 0.
\] (4.5)

Although \((4.5)\) appears to be different from \((3.1)\) it is interesting to note that \((4.5)\) can be mapped to the original set of Hamiltonian equations by using a (nonlocal) Sundman transformation \([12]\) through a transformation of the independent temporal variable \(t\) to a new independent variable \(s\) given by \(ds = 2Hdt\), whence we obtain

\[
x' = \frac{p}{M(x)}, \quad p' = -\left(\frac{M'(x)}{2M^2(x)}p^2 + M(x)g(x)\right),
\] (4.6)

where \(\prime = \frac{d}{ds}\). In fact such transformations were used by Sundman while attempting to solve the restricted three body problem.

As for the stationary points of the Hamiltonian \(\tilde{H}\), these follow from the solutions of \(\partial \tilde{H}/\partial x = 0\) and \(\partial \tilde{H}/\partial p = 0\). The latter yields either \(p = 0\) or \(H = 0\). If \(p = 0\) then the former condition gives either \(H = 0\) or \(g(x) = 0\), i.e \(x = x^*\). The pair \((x^*, p = 0)\) leads by the previous analysis to the case

\[
\tilde{H}_{\text{min}} = \left(\int_{x^*}^x M(s)g(s)ds\right)^2 + E_0.
\] (4.7)

From the above equation it is clear that the local minimum of \(\tilde{H}\) is in general greater than the constant \(E_0\) because the potential \(V(x^*)\) is not required to vanish at \(x = x^*\). As the stationary point corresponds to \(p = 0\) the time translation symmetry is not broken and we have the same situation as previously discussed in section 2.

However one also has now the possibility wherein \(H = 0\) which implies that the locus of the stationary points lie on the curve

\[
\frac{p^2}{2M(x)} + \int_{x^*}^x M(s)g(s)ds = 0.
\] (4.8)

This condition obviously implies that \(\tilde{H}\) has a minima with \(\tilde{H}_{\text{min}} = E_0\) which is less than that given by \((4.7)\). Now for real values of \(p\) it is then necessary that

\[
V(x) = \int_{x^*}^x M(s)g(s)ds < 0.
\]

The force \(dV/dx\) is clearly not necessarily zero and motion can therefore occur in the ground state. The existence of motion under such circumstances is indicative of the spontaneous breaking of the time-translation symmetry \([1]\).

To investigate the possible nature of the motion in this scenario let us demand that

\[
V(x) = \int_{x^*}^x M(s)g(s)ds = -\frac{1}{2}X(x)^2,
\] (4.9)
where \( X(x) = \int \sqrt{M(x)} \, dx \). Such a choice is consistent with the view expressed in [3] that time translation symmetry may be present in almost all Newtonian mechanical systems with a conservative potential provided the potential can be shifted to acquire a negative value. Furthermore such symmetry breaking occurs in a non-standard Hamiltonian description where the new Hamiltonian is the square of the canonical Hamiltonian together with Poisson brackets which are nonlinear. Differentiating (4.9) we get

\[
M(x)g(x) = -X(x)X'(x) \quad \text{with} \quad X'(x) = \sqrt{M(x)} = e^{F(x)}
\]

so that \( e^{F(x)} g(x) = -X(x) \) which after another differentiation with respect to \( x \) leads to the condition

\[
g'(x) + f(x)g(x) = -1, \tag{4.10}
\]

in view of the fact that \( f(x) = M'(x)/2M(x) \). Notice that this basically represents motion in an inverted oscillator potential and it is therefore not surprising that the last condition on the functions \( f \) and \( g \) is just the ‘inverted isochronicity’ condition [7]. The notion of an inverted oscillator also appears in the context of de-Sitter gravity. To arrive at concrete models for the function \( f \) in this case, we note that one may solve (4.10) for \( f \) to get

\[
f = -\frac{1 + g'}{g}, \quad \text{which then implies} \quad M(x) = \frac{1}{g^2(x)} \exp \left( -2 \int \frac{dx}{g} \right). \tag{4.11}
\]

from (4.9) it follows that

\[
X(x) = \int \frac{1}{g(x)} \exp \left( - \int \frac{dx}{g} \right) \, dx \tag{4.12}
\]

The points of minima therefore lie on the curve

\[
p = \pm \sqrt{M(x)}X(x) = \pm \frac{1}{g(x)} \exp \left( - \int \frac{dx}{g} \right) \int \frac{1}{g(x)} \exp \left( - \int \frac{dx}{g} \right) \, dx.
\]

We end this section with a couple of examples:

**Example 1**

Let \( g(x) = x \) then we have

\[
M(x) = \frac{1}{x^4}, \quad X(x) = -x, \quad \text{and} \quad p = \mp \frac{1}{x^3},
\]

the singular nature of \( M(x) \) at \( x = 0 \) forces us to confine ourselves to the half line. It is evident that the particle can at any instant of time have only one of the two possible values for the momentum. The particular choice of any one of these two possible values therefore breaks the time translation symmetry.

**Example 2**

If \( g(x) = 1/x \) then we obtain

\[
M(x) = x^2 e^{-x^2}, \quad X(x) = -e^{-x^2} \quad \text{and} \quad p = \mp xe^{-x^2}.
\]
5 Conclusion

We have shown that nonlinear ODEs of the Liénard type it is easy to recast them into the Lagrange/Hamilton formalism and the basic results of Shapere-Wilczek are apparently applicable to such a differential system. In particular, we have studied the branched Hamiltonian and multivaluedness of momentum of this equation. Our analysis is based on $\tilde{H} = H^2 + E_0$. Actually when we consider such kind of generalized Hamiltonian the number of critical points is changed drastically, and most of the critical points of the generalized Hamiltonian are not the images of the critical point of the original Hamiltonian. Careful readers might have noticed that this (quadratic) Hamiltonian connected to (exotic) Lagrangian via Legendre transformation injected the multivaluedness of the momentum.

Shapere and Wilczek found that the direct consequence of this multivaluedness is that the time translation symmetry is spontaneously broken in the ground states. The phenomenon of spontaneous symmetry breaking was hitherto mostly restricted to the quantum domain. The most outstanding example being that of the Higgs boson besides superconductors, ferromagnets and liquid crystals. The fact that such a phenomenon may also occur in the classical regime is tantalizing at least from the theoretical point of view if nothing else. The introduction of the associated concept of time crystals by Shapere and Wilczek is not without controversy especially regarding their experimental realization. While the examples considered by them as also by L. Zhao et al are drawn from classical mechanics and field theory our motivation in this note is to extend this notion to nonlinear ordinary differential equations. We have shown that nonlinear ODE of the Liénard type it is easy to recast them into the Lagrange/Hamilton formalism and the basic results of Shapere-Wilczek are apparently applicable to such a differential system.

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References

[1] A. Shapere and F. Wilczek, Phys. Rev. Lett. 109, 160402 (2012).
[2] A. Shapere and F. Wilczek, Phys. Rev. Lett. 109, 200402 (2012).
[3] L. Zhao, W. Xu and P. Yu, Landau meets Newton: time translation symmetry breaking in classical mechanics, arXiv:1208.3974v2.
[4] L. Zhao, P. F. Yu and W. Xu, Hamiltonian description of singular Lagrangian systems with spontaneously broken time translation symmetry, Mod. Phys. Lett. A Vol. 28, No. 5 (2013) 1350002, arXiv: 1206.2983.

[5] L. Zhao, Strange Lagrangian systems and statistical mechanics, J. Phys. A: Math. Theor. 46 (2013) 265002.

[6] T L Curtright and C K Zachos, Branched Hamiltonians and supersymmetry, J. Phys. A: Math. Theor. 47 (2014) 145201.

[7] M Sabatini, On the period function of $x'' + f(x)x'^2 + g(x) = 0$. J. Differential Equations 196 (2004), no. 1, 151–168.

[8] A. Ghose Choudhury and P. Guha, J. Phys. A: Math. Theor. 43 (2010) 125202.

[9] P. Guha and A. Ghose Choudhury, Rev. Math. Phys. 25 (6) (2013) 1330009.

[10] A. Ghose Choudhury and P. Guha, J. Phys. A: Math. Theor. 46 (2013) 165202.

[11] A. Goriely, Integrability and Nonintegrability of Dynamical Systems. Advanced Series in Nonlinear Dynamics, 19. World Scientific Publishing Co., Inc., River Edge, NJ, 2001. xviii+415 pp.

[12] K.F. Sundman, Mémoire sur le Problème des trois corps, Acta Mathematica 36 (1912) 105-179.

[13] L. Perko, Differential equations and dynamical systems. Third edition. Texts in Applied Mathematics, 7. Springer-Verlag, New York, 2001. xiv+553 pp.

[14] C. Chicone, Ordinary differential equations with applications, Second edition. Texts in Applied Mathematics, 34. Springer, New York, 2006. xx+636.

[15] A. Ghose Choudhury and P. Guha, Discrete Contin. Dyn. Syst. Ser. B 22 (2017), no. 6, 2465-2478.