Width Hierarchies for Quantum and Classical Ordered Binary Decision Diagrams with Repeated Test

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Abstract. We consider quantum, nondeterministic and probabilistic versions of known computational model Ordered Read-\textit{k}-times Branching Programs or Ordered Binary Decision Diagrams with repeated test (\textit{k}-QOBDD, \textit{k}-NOBDD and \textit{k}-POBDD). We show width hierarchy for complexity classes of Boolean function computed by these models and discuss relation between different variants of \textit{k}-OBDD.

Keywords: quantum computing, quantum OBDD, OBDD, Branching programs, quantum vs classical, quantum models, hierarchy, computational complexity, probabilistic OBDD, nondeterministic OBDD

1 Introduction

Branching programs are one of the well known models of computation. These models have been shown useful in a variety of domains, such as hardware verification, model checking, and other CAD applications (see for example the book by I. Wegener [Weg00]). It is known that the class of Boolean functions computed by polynomial size branching programs coincide with the class of functions computed by non-uniform log-space machines.

One of important restrictive branching programs are oblivious read once branching programs, also known as Ordered Binary Decision Diagrams (OBDD) [Weg00]. It is a good model of data streaming algorithms. These algorithms are actively used in industry, because of rapidly increasing of size of data which should be processed by programs. Since a length of an OBDD is at most linear (in the length of the input), the main complexity measure is “width”, analog of size for automata. And it can be seen as nonuniform automata (see for example [AG05]).

In the last decades quantum model of OBDD came into play [AGK01, NHK00, SS05a, Sau00]. Researchers also interested in read-\textit{k}-times quantum model of OBDD (\textit{k}-QOBDD). \textit{k}-QOBDD can be explored from automata point of view.
of view. And in that case we can found good algorithms for two way quantum classical automata in paper [?] of Ambainis and Watrous. Other automata models, that have relation with k-QOBDD are restart and reset quantum automata [?].

One of the interesting questions, which researchers explored is hierarchy of complexity classes for classical and quantum k-OBDDs. These models have two main characteristics of complexity: number of layers (k) and width. Hierarchy for numbers of layers was investigated in papers [BSSW98], [Kha10], [?], [?].

In same time, there are only few work on width hierarchy. For example, width hierarchy for deterministic k-OBDD is presented in [Kha15]. Width hierarchies for classical and quantum 1-OBDDs are discussed in [AGKY14], [AGKY16], [?].

In this paper we prove width hierarchy for nondeterministic, probabilistic k-OBDD and quantum k-OBDD with natural order of input bits. We considered sub linear width and hierarchies based on results and lower bounds from [?], [Kha10], [?].

The paper is organized in following way. In Section 2 we present definitions. Section 3 contains Hierarchy results for classical models and Section 4 for quantum one. We discuss relation between different models in Section 5.

2 Preliminaries

Ordered read ones Branching Programs (OBDD) are well known model for Boolean functions computation. A good source for different models of branching programs is the book by I. Wegener [Weg00].

A branching program over a set X of n Boolean variables is a directed acyclic graph with two distinguished nodes s (a source node) and t (a sink node). We denote such program Ps,t or just P. Each inner node v of P is associated with a variable x ∈ X. Deterministic P has exactly two outgoing edges labeled x = 0 and x = 1 respectively for such node v.

The program P computes the Boolean function f(X) (f : {0, 1}^n → {0, 1}) as follows: for each σ ∈ {0, 1}^n we let f(σ) = 1 if and only if there exists at least one s − t path (called accepting path for σ) such that all edges along this path are consistent with σ.

A branching program is leveled if the nodes can be partitioned into levels V1, . . . , Vℓ and a level Vℓ+1 such that the nodes in Vℓ+1 are the sink nodes, nodes in each level Vj with j ≤ ℓ have outgoing edges only to nodes in the next level Vj+1. For a leveled Ps,t the source node s is a node from the first level V1 of nodes and the sink node t is a node from the last level Vℓ+1.

The width w(P) of a leveled branching program P is the maximum of number of nodes in levels of P. w(P) = max1≤j≤ℓ |Vj|. The size of branching program P is a number of nodes of program P.

A leveled branching program is called oblivious if all inner nodes of one level are labeled by the same variable. A branching program is called read once if each variable is tested on each path only once. An oblivious leveled read once branching program is also called Ordered Binary Decision Diagram (OBDD).
OBDD $P$ reads variables in its individual order $\pi = (j_1, \ldots, j_n)$, $\pi(i) = j_i$, $\pi^{-1}(j)$ is position of $j$ in permutation $\pi$. We call $\pi(P)$ the order of $P$. Let us denote natural order as $id = (1, \ldots, n)$. Sometimes we will use notation $id$-OBDD $P$, it means that $\pi(P) = id$. Let width($f$) = $\min_{\nu} w(P)$ for OBDD $P$ which computes $f$ and $id$–width($f$) is the same but for $id$-OBDD.

The Branching program $P$ is called $k$-OBDD if it consists from $k$ layers, where $i$-th ($1 \leq i \leq k$) layer $P^i$ of $P$ is an OBDD. Let $\pi_i$ be an order of $P^i$, $1 \leq i \leq k$ and $\pi_1 = \cdots = \pi_k = \pi$. We call order $\pi(P) = \pi$ the order of $P$.

Nondeterministic OBDD (NOBDD) is nondeterministic counterpart of OBDD. Probabilistic OBDD (POBDD) can have more than two edges for node, and choose one of them using probabilistic mechanism. POBDD $P$ computes Boolean function $f$ with bounded error $0.5 - \varepsilon$ if probability of right answer is at least $0.5 + \varepsilon$.

Let us discuss a definition of quantum OBDD (QOBDD). It is given in different terms, but you can see that it is equivalent. You can see [AGK +05, AGK01] for more details.

For a given $n > 0$, a quantum OBDD $P$ of width $w$, defined on $\{0, 1\}^n$, is a 4-tuple $P = (T, |\psi\rangle_0, \text{Accept}, \pi)$, where

- $T = \{T_j : 1 \leq j \leq n\}$ and $T_j = (G^0_j, G^1_j)$ are ordered pairs of (left) unitary matrices representing the transitions is applied at the $j$-th step, where $G^0_j$ or $G^1_j$, determined by the corresponding input bit, is applied.
- $|\psi\rangle_0$ is initial vector from $w$-dimensional Hilbert space over field of complex numbers. $|\psi\rangle_0 = |q_0\rangle$ where $q_0$ corresponds to the initial node.
- Accept $\subset \{1, \ldots, w\}$ is accepting nodes.
- $\pi$ is a permutation of $\{1, \ldots, n\}$ defining the order of testing the input bits.

For any given input $\sigma \in \{0, 1\}^n$, the computation of $P$ on $\sigma$ can be traced by a vector from $w$-dimensional Hilbert space over field of complex numbers. The initial one is $|\psi\rangle_0$. In each step $j$, $1 \leq j \leq n$, the input bit $\sigma_{x(j)}$ is tested and then the corresponding unitary operator is applied: $|\psi\rangle_j = G^x_{\pi^{-1}(j)}(|\psi\rangle_{j-1})$, where $|\psi\rangle_{j-1}$ and $|\psi\rangle_j$ represent the state of the system after the $(j-1)$-th and $j$-th steps, respectively, where $1 \leq j \leq n$.

In the end of computation program $P$ measure qubits. The accepting (return 1) probability $Pr_{|\psi\rangle_0}(\sigma)$ of $P_n$ on input $\sigma$ is $Pr_{\text{Accept}}(\nu) = \sum_{\nu \in \text{Accept}} \nu^2_{\nu}$, for $|\psi\rangle_n = (v_1, \ldots, v_w)$. We say that a function $f$ is computed by $P$ with bounded error if there exists an $\varepsilon \in (0, \frac{1}{2}]$ such that $P$ accepts all inputs from $f^{-1}(1)$ with a probability at least $\frac{1}{2} + \varepsilon$ and $P_n$ accepts all inputs from $f^{-1}(0)$ with a probability at most $\frac{1}{2} - \varepsilon$. We can say that error of answer is $\frac{1}{2} - \varepsilon$.

Let $k$-QOBDD$_w$ be a set of Boolean functions which can be computed by bounded error $k$-QOBDDs of width $w$. $k$-id-QOBDD$_w$ is same for bounded error $k$-QOBDDs with order $id = (1, \ldots, n)$. $k$-NOBDD$_w$ and $k$-POBDD$_w$ is similar classes for $k$-NOBDD and bounded error $k$-POBDD.

3 Width Hierarchies on Classical $k$-OBDD

Firstly, let us discuss required definitions.
Let \( \pi = (X_A, X_B) \) be a partition of the set \( X \) into two parts \( X_A \) and \( X_B = X \setminus X_A \). Below we will use equivalent notations \( f(X) \) and \( f(X_A, X_B) \). Let \( f|_\rho \) be a subfunction of \( f \), where \( \rho : X_A \to \{0, 1\}^{\vert X_A \vert} \). Function \( f|_\rho \) is obtained from \( f \) by applying \( \rho \). Let \( N^\pi(f) \) be number of different subfunctions with respect to partition \( \pi \). Let \( \Theta(n) \) be the set of all permutations of \( \{1, \ldots, n\} \).

Let partition \( \pi(\theta, u) = (X_A, X_B) = (\{x_{j_1}, \ldots, x_{j_u}\}, \{x_{j_{u+1}}, \ldots, x_{j_n}\}) \), for permutation \( \theta = (j_1, \ldots, j_n) \in \Theta(n), 1 \leq u < n \). We denote \( \Pi(\theta) = \{\pi(\theta, u) : 1 \leq u < n\} \). Let \( N^\theta(f) = \max_{\pi \in \Pi(\theta)} N^\pi(f) \), \( N(f) = \min_{\theta \in \Theta(n)} N^\theta(f) \).

Secondly, let us present existing lower bounds for nondeterministic and probabilistic \( k \)-OBDDs.

**Lemma 1 ([?]).** Let function \( f(X) \) is computed by \( k \)-OBDD \( P \) of width \( w \), then \( N(f) \leq w^{(k-1)w+1} \).

**Lemma 2 ([Kha16]).** Let function \( f(X) \) is computed by \( k \)-NOBDD \( P \) of width \( w \), then \( N(f) \leq 2^w(k-1)w+1 \).

**Lemma 3 ([Kha16]).** Let function \( f(X) \) be computed by bounded error \( k \)-POBDD \( P \) of width \( w \), then

\[
N(f) \leq (C_1 k (C_2 + \log_2 w + \log_2 k))^{(k+1)w^2}
\]

for some constants \( C_1 \) and \( C_2 \).

Thirdly, let us define **Shuffled Address Function** \( (SAF_{k, w}) \) from [Kha15] based on definition of well known Pointer Jumping Function [BSSW98], [NW91].

**Definition 1 (Shuffled Address Function).** Let us define Boolean function \( SAF_{k, w}(X) : \{0, 1\}^n \to \{0, 1\} \) for integer \( k = k(n) \) and \( w = w(n) \) such that

\[
2kw(2w + \lfloor \log k \rfloor + \lfloor \log 2w \rfloor) < n.
\]

We divide input variables to \( 2kw \) blocks. There are \( \lceil n/(2kw) \rceil \) a variables in each block. After that we divide each block to address and value variables. First \( \lfloor \log k \rfloor + \lfloor \log 2w \rfloor \) variables of block are address and other \( a - \lfloor \log k \rfloor - \lfloor \log 2w \rfloor = b \) variables of block are value.

We call \( x^p_0, \ldots, x^p_{b-1} \) value variables of \( p \)-th block and \( y^p_0, \ldots, y^p_{\lfloor \log k \rfloor + \lfloor \log 2w \rfloor} \) are address variables, for \( p \in \{0, \ldots, 2kw - 1\} \).

Boolean function \( SAF_{k, w}(X) \) is iterative process based on definition of following six functions:

Function \( AdrK : \{0, 1\}^n \times \{0, \ldots, 2kw - 1\} \to \{0, \ldots, k - 1\} \) obtains first part of block’s address. This block will be used only in step of iteration which number is computed using this function:

\[
AdrK(X, p) = \sum_{j=0}^{\lfloor \log k \rfloor - 1} y^p_j \cdot 2^j \pmod{k}.
\]
Function $AdrW : \{0,1\}^n \times \{0,\ldots,2kw-1\} \rightarrow \{0,\ldots,2w-1\}$ obtains second part of block’s address. It is the address of block within one step of iteration:

$$AdrW(X,p) = \sum_{j=0}^{[\log 2w] - 1} y_j^p \cdot 2^j \pmod{2w}.$$ 

Function $Ind : \{0,1\}^n \times \{0,\ldots,2w-1\} \times \{0,\ldots,k-1\} \rightarrow \{0,\ldots,2kw-1\}$ obtains number of block by number of step and address within this step of iteration:

$$Ind(X,i,t) = \begin{cases} p, & \text{where } p \text{ is minimal number of block such that } AdrK(X,p) = t \text{ and } AdrW(X,p) = i, \\ -1, & \text{if there are no such } p. \end{cases}$$

Function $Val : \{0,1\}^n \times \{0,\ldots,2w-1\} \times \{1,\ldots,k\} \rightarrow \{-1,\ldots,w-1\}$ obtains value of block which have address $i$ within $t$-th step of iteration:

$$Val(X,i,t) = \begin{cases} \sum_{j=0}^{b-1} x_j^p \pmod{w}, & \text{where } p = Ind(X,i,t), \text{ for } p \geq 0, \\ -1, & \text{if } Ind(X,i,t) < 0. \end{cases}$$

Two functions $Step_1$ and $Step_2$ obtain value of $t$-th step of iteration. Function $Step_1 : \{0,1\}^n \times \{0,\ldots,k-1\} \rightarrow \{-1,\ldots,2w-1\}$ obtains base for value of step of iteration:

$$Step_1(X,t) = \begin{cases} -1, & \text{if } Step_2(X,t-1) = -1, \\ 0, & \text{if } t = -1, \\ Val(X,Step_2(X,t-1),t) + w, & \text{otherwise.} \end{cases}$$

Function $Step_2 : \{0,1\}^n \times \{0,\ldots,k-1\} \rightarrow \{-1,\ldots,w-1\}$ obtain value of $t$-th step of iteration:

$$Step_2(X,t) = \begin{cases} -1, & \text{if } Step_1(X,t) = -1, \\ 0, & \text{if } t = -1, \\ Val(X,Step_1(X,t),t), & \text{otherwise.} \end{cases}$$

Note that address of current block is computed on previous step. Result of Boolean function $SAF_{k,w}(X)$ is computed by following way:

$$SAF_{k,w}(X) = \begin{cases} 0, & \text{if } Step_2(X,k-1) \leq 0, \\ 1, & \text{otherwise.} \end{cases}$$

Let us discuss complexity properties of the function:

**Lemma 4 (Kha15).** For integer $k = k(n)$, $w = w(n)$ and Boolean function $SAF_{k,w}$, such that inequality (1) holds, the following statement is right: $N(SAF_{k,w}) \geq w^{(k-1)(w-2)}$. 

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Lemma 5 [Kha15]. There is $2k$-OBDD $P$ of width $3w + 1$ which computes $SAF_{k,w}$.

Let us present Lemma 4 in more useful form:

Corollary 1. For integer $k = k(n)$, $w = w(n)$ and Boolean function $SAF_{k,w}$, such that inequality (1) holds, the following statement is right: $N(SAF_{k,w}) \geq w^{kw/6}$.

3.1 Hierarchy Results for Classical Models

Hierarchy for deterministic OBDD is already known:

Theorem 1 [Kha15]. For integer $k = k(n)$, $w = w(n)$ such that $2kw(2w + \lfloor \log k \rfloor + \lfloor \log 2w \rfloor) < n$, $k \geq 2$, $w \geq 64$ we have $k$-OBDD$_{w/16} \not\subset k$-OBDD$_w$.

Let us discuss hierarchies for nondeterministic and probabilistic models.

Theorem 2. For $w \geq 8$ we have $k$-NOBDD$_{\sqrt{w}/2} \not\subset k$-NOBDD$_{3w+1}$

Proof. It is clear that $k$-NOBDD$_{\sqrt{w}/2} \subset k$-NOBDD$_{3w+1}$. Let us proof inequality of these classes. By Lemma 3 we have $SAF_{k,w} \in 2k$-NOBDD$_{3w+1}$. Let us show that $SAF_{k,w} \not\in 2k$-NOBDD$_{\sqrt{w}/2}$.

$$N(SAF_{k,w}) \geq \frac{N(SAF_{k,w})}{2 \sqrt{w}/2(1+(2k-1)\sqrt{w}/2)} \geq$$

$$\geq \frac{w^{kw/6}}{2 \sqrt{w}/2(1+(2k-1)\sqrt{w}/2)} =$$

$$= 2^{\frac{k}{2k} \log w - \frac{\sqrt{w}}{4} - \frac{1}{4} w(2k-1)} =$$

$$= 2^{\frac{k}{2k} \log w - \frac{\sqrt{w}}{4} - \frac{1}{4} w(2k-1)} > 1$$

Therefore $SAF_{k,w} \not\in 2k$-OBDD$_{\sqrt{w}/2}$, due to Lemma 2 and $k$-NOBDD$_{\sqrt{w}/2} \not\subset k$-NOBDD$_{3w+1}$. □

Theorem 3. For $\sqrt{w}/(\log_2 k \log_2 w) \geq 1$ we have $k$-POBDD$_{\sqrt{w}/(\log_2 k \log_2 w)} \not\subset k$-POBDD$_{3w+1}$

Proof. It is clear that $k$-POBDD$_{\sqrt{w}/(\log_2 k \log_2 w)} \subset k$-POBDD$_{3w+1}$. Let us proof inequality of these classes. By Lemma 5 we have $SAF_{k,w} \in 2k$-POBDD$_{3w+1}$. Let us show that $SAF_{k,w} \not\in 2k$-POBDD$_{\sqrt{w}/(\log_2 k \log_2 w)}$.

$$N(SAF_{k,w}) \geq \frac{N(SAF_{k,w})}{(C_1 k(C_2 + 0.5 \log_2 w - \log_2 \log_2 k - \log_2 \log_2 w + \log_2 k))^{(2k+1)w/(\log_2 k \log_2 w)^2}} \geq$$

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Let us present existing lower bound for $k$-$OBD\!\!\!\ddot{D}$

\[ N^*(f) \leq w^{C(kw)^2} \text{ for some } C = \text{const}. \]

Then let us discuss Boolean function Matrix XOR Pointer Jumping, which complexity property allow to show hierarchy.

**Definition 2.** Firstly, let us present version of PJ function which works with integer numbers. Let $V_A, V_B$ be two disjoint sets (of vertices) with $|V_A| = |V_B| = d$ and $V = V_A \cup V_B$. Let $F_A = \{ f_A : V_A \rightarrow V_B \}$, $F_B = \{ f_B : V_B \rightarrow V_A \}$ and $f = (f_A, f_B) : V \rightarrow V$ defined by $f(v) = f_A(v)$, if $v \in V_A$ and $f = f_B(v)$, $v \in V_B$. For each $j \geq 0$ define $f^{(j)}(v)$ by $f^{(0)}(v) = v$, $f^{(j+1)}(v) = f(f^{(j)}(v))$. Let $v_0 \in V_A$. The functions we will be interested in computing is $g_{k,d} : F_A \times F_B \rightarrow V$ defined by $g_{k,d}(f_A, f_B) = f^{(k)}(v_0)$.

**Definition of Matrix XOR Pointer Jumping function** looks like Pointer Jumping function.

Firstly, we introduce definition of $\text{Matrix PJ}_{2k,d}$ function. Let us consider functions $f_{A,1}, f_{A,2}, \cdots f_{A,k} \in F_A$ and $f_{B,1}, f_{B,2}, \cdots f_{B,k} \in F_B$.

On iteration $j + 1$ function $f^{(j+1)}(v) = f_{j+1}(f^{(j)}(v))$, where

\[
\begin{align*}
f_i(v) &= \begin{cases} 
    f_{A,i}(v) & \text{if } i \text{ is odd} \\
    f_{B,i}(v) & \text{if } i \text{ is even}
\end{cases} 
\end{align*}
\]

Matrix $\text{PJ}_{2k,d}(f_{A,1}, f_{A,2}, \cdots f_{A,k}, f_{B,1}, f_{B,2}, \cdots f_{B,k}) = f^{(k)}(v_0)$.

Secondly, we add XOR-part to $\text{Matrix PJ}_{2k,d}$ (note it $\text{XMPJ}_{2k,d}$). Here we take $f^{(j+1)}(v) = f_{j+1}(f^{(j)}(v)) \oplus f^{(j-1)}(v)$, for $j \geq 0$

Finally, we consider boolean version of these functions. Boolean function $\text{PJ}_{t,n} : \{0,1\}^n \rightarrow \{0,1\}$ is boolean version of $g_{k,d}$, where we encode $f_A$ in a binary string using $d \log d$ bits and do it with $f_B$ as well. The result of function is parity of binary representation of result vertex.
In respect to boolean $MXPJ_{2k,d}$ function we encode functions in input in following order $f_{A,1}, \ldots, f_{A,k}, f_{B,1}, \ldots, f_{B,k}$. Let us describe process of computation on Figure 1. Function $f_{A,i}$ is encoded by $a_{i,1}, \ldots, a_{i,d}$, for $i \in \{1 \cdots k\}$. And $f_{B,i}$ is encoded by $b_{i,1}, \ldots, b_{i,d}$, for $i \in \{1 \cdots k\}$. Typically, we will assume that $v_0 = 0$.

![Figure 1. Boolean function $XMPJ_{k,d}$](image)

Let us discuss complexity properties of $MXPJ_{2k,d}$.

**Lemma 7 ([?]).** For $kd \log d = o(n)$ following is right: $N^{id}(MXPJ_{2k,d}) \geq d^{\left\lfloor \frac{d}{3} - 1 \right\rfloor (k-3)}$.

**Lemma 8 ([?]).** There is exact $k$-id-QOBDD $P$ of width $d^2$ which computes $MXPJ_{2k,d}$.

Let us present Lemma 7 in more useful form:

**Corollary 2.** For integer $k = k(n)$, $w = w(n)$, $kd \log d = o(n)$ and Boolean function $MXPJ_{2k,d}$, the following statement is right: $N^{id}(MXPJ_{2k,d}) \geq d^{\frac{2k}{16}}$.

### 4.1 Hierarchy Results for Quantum Models

Now we can prove hierarchy results for $k$-QOBDDs.

**Theorem 4.** We have $k$-id-QOBDD $\frac{\sqrt{d/C_1k}}{C_1k} \subseteq k$-id-QOBDD $\frac{d}{C_1k}$ for some $C_1 = \text{const}$.

**Proof.** It is clear that $k$-id-QOBDD $\frac{\sqrt{d/C_1k}}{C_1k} \subseteq k$-id-QOBDD $\frac{d}{C_1k}$. Let us prove inequality of these classes.

Due to Lemma 8, $MXPJ_{2k,d} \in k$-id-QOBDD $\frac{d}{C_1k}$. Let us show that $MXPJ_{2k,d} \notin k$-id-QOBDD $\frac{d}{C_1k}$:

$$\frac{N^{id}(MXPJ_{2k,d})}{\sqrt{d/C_1k}} \geq C(k \sqrt{d/C_1k})^2$$
Let \( C \) be from Lemma [6], then we have

\[
\frac{N^{id}(MXPJ_{2k,d})}{\sqrt{d/C_1 k^2}} \geq 2^{\frac{k}{\log 2} \log C_1 k} > 1.
\]

Therefore \( MXPJ_{2k,d} \notin k\text{-id-OBDD}_{\sqrt{d/C_1 k}} \), due to Lemma [6]

And \( k\text{-id-QOBDD}_{\sqrt{d/C_1 k}} \notin k\text{-id-QOBDD}_{d} \). \( \Box \)

5 Discussion on Relation Between Models

You can see some existing discussion between models in [?], AGKY16, AGK+05, Gail19, [?], [?]. Here we will present some relations on fixed \( k \).

Relation between models follows from Theorems [2, 3]

Theorem 5. There are Boolean function \( f \), such that:

\[
f \in k\text{-OBDD}_{3w+1,k}\text{–NOBDD}_{3w+1,k}\text{–POBDD}_{3w+1};
\]

\[
f \notin k\text{-OBDD}_{\lfloor w/16 \rfloor -3,k}\text{–NOBDD}_{\lfloor w/2 \rfloor -3,k}\text{–POBDD}_{\lfloor w/ \log_4 k \log_2 w \rfloor}.
\]

Let us compare classical models and quantum models. Firstly, let us discuss classical complexity properties of \( MXPJ_{2k,d} \) function.

Lemma 9. There is \( k\text{-id-OBDD} P \) of width \( d^2 \) which computes \( MXPJ_{2k,d} \).

Proof. Let us construct such \( k\text{-id-OBDD} P \).

By the definition of function \( MXPJ_{2k,d} \) input separated into \( 2dk \) blocks by \( t = \lfloor \log_2 d \rfloor \) bits. Blocks encode integers \( a_{i1}, a_{i2}, \ldots a_{id} \) for \( i \in \{1, \ldots, k\} \) in the first part of input; and \( b_{i1}, b_{i2}, \ldots, b_{id} \) for \( i \in \{1, \ldots, k\} \) in the second part (see Figure [?]). Let elements of block representing \( a_{ij} \) be \( X^{i,j} = (x_0^{i,j}, \ldots, x_{d-1}^{i,j}) \) for \( i \in \{1, \ldots, k\}, j \in \{1, \ldots, d\} \) and elements of block representing \( b_{ij} \) be \( Y^{i,j} = (y_0^{i,j}, \ldots, y_{d-1}^{i,j}) \) for \( i \in \{1, \ldots, k\}, j \in \{1, \ldots, d\} \).

Let us discuss \( i \)-th layer. On the first level we have \( d^2 \) nodes, each of them corresponds to pair \( (u, v) \), for \( u, v \in 0, \ldots, d - 1 \) for storing \( f^{(2i-3)} \) and \( f^{(2i-2)} \). At first \( P \) skips all blocks except \( x_i f^{(2i-2)} \). Then it will compute XOR of bits of the block and \( u \) of pair. In the end of the block \( P \) leads node corresponding to \( (f^{(2i-1)}, f^{(2i-2)}) \). After that the program skip all other blocks of first part and all blocks of first part except \( y^i f^{(2i-1)} \). Then it computes XOR with of bits of the block and \( v \) of pair. In the end of the block \( P \) leads node corresponding to \( (f^{(2i-1)}, f^{(2i)}) \).

On the last layer after computing \( f^{(2k)} \), all nodes which XOR result is 1 leads 1-sink. \( \Box \)
Lemma 10. $\text{MXPJ}_{2k,d} \notin k\text{-id-OBDD}_{d/32}$.

Proof. Let us apply Lemma 1

$$\frac{\text{MXPJ}_{2k,d}}{(d/32)^{(k-1)d/32+1}} \geq \frac{d^{dk/16}}{(d/r)^{(k-1)d/32+1}} \geq 2^{\frac{dk}{16} \log d - 2 \log(d/32)kd/(32)} = 2^{kd/16 - 2(\log d - 5)/32} > 1$$

Therefore $\text{MXPJ}_{2k,d} \notin k\text{-id-OBDD}_{d/32}$, due to Lemma 1.

Lemma 11. $\text{MXPJ}_{2k,d} \notin k\text{-id-NOBDD}_{\sqrt{(d \log d)/33}}$.

Proof. Let us apply Lemma 2. Let $r = \sqrt{33d/\log d}$

$$\frac{\text{MXPJ}_{2k,d}}{2^{((k-1)d/r+1)d/r}} \geq \frac{d^{dk/16}}{2^{((k-1)d/r+1)d/r}} \geq 2^{\frac{dk}{16} \log d - 2(kd/r)d/r} = 2^{kd/16 - 2d/r^2} > 1$$

Therefore $\text{MXPJ}_{2k,d} \notin k\text{-id-NOBDD}_{\sqrt{(d \log d)/33}}$, due to Lemma 2.

Lemma 12. $\text{MXPJ}_{2k,d} \notin k\text{-id-POBDD}_{d/\log k}$.

Proof. Let us apply Lemma 3. Let $r = \sqrt{d/\log k}$

$$\frac{\text{MXPJ}_{2k,d}}{(C_1k(C_2 + \log_2 d - \log_2 r + \log_2 k))^{(k+1)d^2/r^2}} \geq \frac{d^{dk/16}}{(C_1k(C_2 + \log_2 d - \log_2 r + \log_2 k))^{(k+1)d^2/r^2}} \geq 2^{\frac{dk}{16} \log d - 2(kd^2/r^2)(C_3 + \log_2 k + \log_2(C_2 + \log_2 d + \log_2 k))} = 2^{2kd/16 - d(C_3 + \log_2 k + \log_2(C_2 + \log_2 d + \log_2 k)/r^2)} > 1$$

Therefore $\text{MXPJ}_{2k,d} \notin k\text{-id-POBDD}_{d/\log k}$, due to Lemma 3.

Then base on these lemmas we can get following result:

Theorem 6. There is Boolean function $f$, such that:

- $f \in k\text{-id-OBDD}_{d/2}$, $k\text{-id-NOBDD}_{d/2}$, $k\text{-id-POBDD}_{d/2}$, $k\text{-id-QOBDD}_{d/2}$;
- $f \notin k\text{-OBDD}_{d/32}$, $k\text{-id-NOBDD}_{\sqrt{(d \log d)/33}}$, $k\text{-id-POBDD}_{\sqrt{d/\log k}}$, $k\text{-id-QOBDD}_{\sqrt{d/C_1k}}$.
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