Density of a minimal submanifold and total curvature of its boundary

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Given a piecewise smooth submanifold $\Gamma^{n-1} \subset \mathbb{R}^m$ and $p \in \mathbb{R}^m$, we define the vision angle $\Pi_p(\Gamma)$ to be the $(n-1)$-dimensional volume of the radial projection of $\Gamma$ to the unit sphere centered at $p$. If $p$ is a point on a stationary $n$-rectifiable set $\Sigma \subset \mathbb{R}^m$ with boundary $\Gamma$, then we show the density of $\Sigma$ at $p$ is at most the density at its vertex $p$ of the cone over $\Gamma$. It follows that if $\Pi_p(\Gamma)$ is less than twice the volume of $S^{n-1}$, for all $p \in \Gamma$, then $\Sigma$ is an embedded submanifold.

As a consequence, we prove that given two $n$-planes $R^n_1, R^n_2$ in $\mathbb{R}^m$ and two compact convex hypersurfaces $\Gamma_i$ of $R^n_i$, $i = 1, 2$, a nonflat minimal submanifold spanned by $\Gamma := \Gamma_1 \cup \Gamma_2$ is embedded.

1. Introduction

Fenchel [F1] showed that the total curvature of a closed space curve $\gamma \subset \mathbb{R}^m$ is at least $2\pi$, and it equals $2\pi$ if and only if $\gamma$ is a plane convex curve. Fáry [Fa] and Milnor [M] independently proved that a simple knotted regular curve has total curvature larger than $4\pi$. These two results indicate that a Jordan curve which is curved at most double the minimum is isotopically simple. But in fact minimal surfaces spanning such Jordan curves must be simple as well. Indeed, Nitsche [N] showed that an analytic Jordan curve in $\mathbb{R}^3$ with total curvature at most $4\pi$ bounds exactly one minimal disk. Moreover, Ekholm, White and Wienholtz [EWW] proved that a minimal surface spanning such a Jordan curve in $\mathbb{R}^m$ is embedded.

Given an $n$-dimensional submanifold $M$ of $\mathbb{R}^m$, there are two well-studied ways of defining the total curvature of $M$: the higher-dimensional Gauss-Bonnet integral $\int_M \Omega$ as defined in [AW] and [C1]; and the total absolute curvature of $M$, $\int_M K^* dV_M$ as defined by Chern and Lashof in [CL] (see section 2 below). Chern and Lashof proved that $\int_M K^* dV_M \geq 2$, with equality if and only if $M$ is a convex hypersurface in an $(n+1)$-dimensional plane.

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Eells and Kuiper have shown that if \( \int_M K^*dV_M < 3 \) then \( M \) is homeomorphic to \( S^n \) and that if \( \int_M K^*dV_M < 4 \) then \( M \) is homeomorphic to \( S^n, \mathbb{R}P^n, \mathbb{C}P^{n/2}, \mathbb{H}P^{n/4} \) or to \( \text{CayP}^2 \) (for \( n = 16 \)). [EK].

In the light of Ekholm-White-Wienholtz’s theorem, it is quite natural to conjecture that an \( n \)-dimensional minimal submanifold \( \Sigma \subset \mathbb{R}^m \) spanning a compact connected submanifold \( \Gamma^{n-1} \) with total absolute curvature \( < 4 \) is embedded. In this paper we prove a theorem in the spirit of this conjecture: given two \( n \)-planes \( R^n_1, R^n_2 \) in \( \mathbb{R}^m \) and two compact convex hypersurfaces \( \Gamma_i^{n-1} \) of \( R^n_i, i = 1, 2 \), a nonflat minimal submanifold spanned by \( \Gamma := \Gamma_1 \cup \Gamma_2 \) is embedded.

In [Fa] Fáry showed that the total curvature of a space curve \( \gamma \) in \( \mathbb{R}^m \) is equal to the average over all 2-planes \( R^2 \subset \mathbb{R}^m \) of the total curvature of the orthogonal projection of \( \gamma \) onto the \( R^2 \). We shall use an extension of Fáry’s theorem, due to Langevin and Shifrin [LS], which shows that given an \( (n-1) \)-dimensional submanifold \( \Gamma \) of \( \mathbb{R}^m \), the total absolute curvature of \( \Gamma \) equals the average over all \( n \)-planes \( R^n \subset \mathbb{R}^m \) of the total absolute curvature of the orthogonal projection of \( \Gamma \) into the \( n \)-plane \( R^n \).

2. Total absolute curvature

Consider a submanifold \( M^n \) of Euclidean space \( \mathbb{R}^m \). As discussed above, in high dimension and codimension we discuss two types of total curvature: one intrinsic (Allendörfer-Weil-Chern-Gauss-Bonnet), and one extrinsic (Chern-Lashof). In this section we shall review Chern-Lashof’s total absolute curvature. This total curvature may be understood in terms of Gauss-Kronecker curvature of hypersurfaces.

Let \( M^n \) be an oriented hypersurface immersed in \( \mathbb{R}^{n+1} \). A unit normal vector \( \nu \) to \( M \) at \( p \in M \) defines the Gauss map \( G_1 : M \to S^n \). The determinant of the differential \( G_1^* \), or of the second fundamental form of \( M \), is called the \textit{Gauss-Kronecker curvature} of \( M \), which we shall denote \( GK_M \).

It follows that for \( M \) compact,

\[
\int_M GK_M dV_M = c_n \deg(G_1), \quad c_n := \text{Vol}(S^n).
\]

Furthermore, if \( n \) is even, H. Hopf [H] showed

\[
(1) \quad \int_M GK_M dV_M = \frac{1}{2} c_n \chi(M).
\]
Now let $M$ be an $n$-dimensional submanifold of $\mathbb{R}^m$. The volume form of the unit normal bundle $N_1 M$ of $M$ is $dV_M \wedge d\sigma_{m-n-1}$ where the restriction of $d\sigma_{m-n-1}$ to a fiber of $N_1 M$ at $p$ is the volume form of the sphere of unit normal vectors at $p \in M$. Define the Gauss map $G_1 : N_1 M \to \mathbb{S}^{m-1}$ by $G_1(p, \nu) = \nu$ and let $d\sigma_{m-1}$ be the volume form of $\mathbb{S}^{m-1}$. Then the Lipschitz-Killing curvature $G(p, \nu)$ of $M$ at $(p, \nu)$ is defined to be the scalar $G(p, \nu)$ such that

$$G_1^*(d\sigma_{m-1}) = G(p, \nu) dV_M \wedge d\sigma_{m-n-1}.$$ 

Then $G(p, \nu)$ is exactly the volume expansion ratio of $G_1$, that is,

$$G(p, \nu) = \lim_{D \to \{p\}} \frac{\text{Vol}(G_1(D))}{\text{Vol}(D)},$$

where $\text{Vol}(G_1(D))$ denotes the signed volume of $G_1(D)$. In fact, $G(p, \nu)$ has the following geometric interpretation [CL]: $G(p, \nu)$ is equal to the Gauss-Kronecker curvature at $p$ of the orthogonal projection of $M$ onto the $(n + 1)$-dimensional plane $L(\nu)$ spanned by $T_p M$ and $\nu$.

Let $\pi$ be the canonical projection of $N_1 M$ into $M$. The integrals

$$K(p) := \frac{1}{c_{m-1}} \int_{\pi^{-1}(p)} G(p, \nu) d\sigma_{m-n-1} \quad \text{and}$$

$$K^*(p) := \frac{1}{c_{m-1}} \int_{\pi^{-1}(p)} |G(p, \nu)| d\sigma_{m-n-1}$$

are called the *total curvature* and the *total absolute curvature* of $M$ at $p$, respectively. The integrals

$$\tau(M) := \int_M K dV_M, \quad \text{and} \quad \tau^*(M) := \int_M K^* dV_M$$

are called the *total curvature* and the *total absolute curvature* of $M$, respectively. Lipschitz and Killing have shown that $K(p)$ is an intrinsic quantity of $M$ at $p$ for $n$ even (see [SS] for a more general result). However, $K(p) = 0$ for $n$ odd. Both $\tau(M)$ and $\tau^*(M)$ remain unchanged even if the ambient space $\mathbb{R}^m$ is embedded into $\mathbb{R}^k$, $k > m$.

For $M^n \subset \mathbb{R}^m$, Fenchel [F2] generalized Hopf’s theorem (1):

$$(2) \quad \int_M K dV_M = \chi(M).$$
In contrast, Chern and Lashof [CL] proved that

\[ \int_M K^* dV_M \geq 2, \]

with equality if and only if \( M \) is a convex hypersurface in an \((n+1)\)-dimensional plane, and that if \( \int_M K^* dV_M < 3 \) then \( M \) is homeomorphic to \( S^n \). Moreover, Morse theory tells us that

\[ \int_M K^* dV_M \geq \sum_i \beta_i, \]

where \( \beta_i \) is the \( i \)-th Betti number of \( M \) ([W], Theorem 28).

### 3. Vision angle versus average density

A minimal submanifold \( \Sigma^n \) in \( \mathbb{R}^m \) has the remarkable property that the density of \( \Sigma \) at \( p \in \Sigma \) is bounded above by that of the cone \( C = p \times \partial \Sigma \) at its vertex \( p \). (We assume that \( \Sigma \) with its boundary is compact.) Recall that the density of \( \Sigma \) is defined as

\[ \Theta_{\Sigma}(p) = \lim_{r \to 0} \frac{\text{Vol}(\Sigma \cap B^m_r(p))}{\text{Vol}(B^n_r(p))}. \]

Further, the density of a cone \( C \) has the interesting property that it equals the average of the densities of the orthogonal projections of \( C \) onto \( n \)-planes in \( \mathbb{R}^m \). These properties will be verified in this section.

In what follows, we shall write \( \nabla \) for the Euclidean connection on \( \mathbb{R}^m \), and \( \nabla = \nabla_M \) for the induced connection on a submanifold \( M \).

**Lemma 1.** Let \( \Sigma \) be an \( n \)-dimensional minimal submanifold of \( \mathbb{R}^m \), \( p \) a point of \( \mathbb{R}^m \), and \( C \) an \( n \)-dimensional piecewise smooth cone with vertex \( p \). Define the Euclidean distance function \( r(x) = \text{dist}(p, x), x \in \mathbb{R}^m \). Let \( Y_1 = r \nabla r \) and \( Y_2 = r^{1-n} \nabla r \), and define \( \text{div}_{\Sigma} Y_i = \text{tr}_{\Sigma} \nabla Y_i = \sum_j \langle \nabla e_j Y_i, e_j \rangle, \{ e_1, \ldots, e_n \} \) being an orthonormal frame of \( \Sigma \). Then

1. On \( \Sigma \), \( \text{div}_{\Sigma} Y_1 = n \) and \( \text{div}_{\Sigma} Y_2 \geq 0 \);
2. On \( C \), \( \text{div}_{C} Y_1 = n \) and \( \text{div}_{C} Y_2 = 0 \).
We require that $C$ be piecewise smooth, that is, a topological manifold which has a triangulation into simplices that are $C^2$ up to their boundaries.

**Proof.** Given an $n$-dimensional submanifold $M \subset \mathbb{R}^m$, it is well known that

$$\Delta_M x := (\Delta_M x_1, \ldots, \Delta_M x_m) = \vec{H},$$

where $\vec{H}$ is the mean curvature vector of $M$, the trace of its second fundamental form. Hence the orthogonal coordinate functions $x_1, \ldots, x_m$ of $\mathbb{R}^m$ are harmonic on a minimal submanifold $\Sigma^n$ of $\mathbb{R}^m$. If we take $p$ as the origin, then since $\vec{H} = 0$ on $\Sigma$,

$$\text{div}_\Sigma(Y_1) = \text{div}_\Sigma(r\nabla r) = \frac{1}{2} \Delta_\Sigma r^2 + \langle r\nabla r, \vec{H} \rangle = \frac{1}{2} \sum \Delta_\Sigma x_i^2 = \sum x_i \Delta_\Sigma x_i + \sum |\nabla x_i|^2 = n.$$

On the cone $C$, since $\vec{H}$ is perpendicular to $r\nabla r = x \in C$, we have

$$\text{div}_C(Y_1) = \text{div}_C(r\nabla r) = \frac{1}{2} \Delta_C r^2 + \langle r\nabla r, \vec{H} \rangle = \frac{1}{2} \sum \Delta_C x_i^2 = \langle x, \vec{H} \rangle + \sum |\nabla x_i|^2 = n.$$

On the other hand, for $M = \Sigma$ or $C$,

$$\text{div}_M Y_2 = \text{div}_M (r^{-n}Y_1) = -nr^{-n-1}\langle \nabla r, Y_1 \rangle + r^{-n} \text{div}_M(Y_1) = nr^{-n} \left( -|\nabla r|^2 + 1 \right).$$

Note that $|\nabla r| \leq 1$ on $M = \Sigma$ and $|\nabla r| \equiv 1$ on $M = C$. This completes the proof. 

**Theorem 1.** Let $\Sigma$ be a stationary $n$-rectifiable set with boundary $\Gamma$ in $\mathbb{R}^m$, an open dense subset of $\Sigma$ being a smooth minimal submanifold. Let $C$ be the cone $p \times \Gamma$, $p \in \mathbb{R}^m$. Then

$$\Theta_\Sigma(p) \leq \Theta_C(p),$$

with equality if and only if $\Sigma = C$ and $C$ is star-shaped with respect to $p$. 

Proof. Compute the first variation of volume with respect to the (Lipschitz continuous) variation vector field

\[ Y := r^{1-n} \nabla r \quad \text{for} \quad r \geq \varepsilon \]

and

\[ Y := \varepsilon^{-n} r \nabla r \quad \text{for} \quad r \leq \varepsilon. \]

Then the first variation of \( \Sigma \) with respect to the flow with velocity field \( Y \) [Si, p. 80] is

\[ \int_{\Sigma} \text{div}_\Sigma Y \, dV_\Sigma, \]

which must equal

\[ \int_{\Gamma} \langle Y, \nu_\Sigma \rangle \, dV_{\Gamma}, \]

where \( \nu_\Sigma \) is the outward unit normal vector to \( \Gamma \) tangent to \( \Sigma \).

Computing the divergence on smooth subsets of the stationary set \( \Sigma \), we find by Lemma 1 (a)

(4) \quad \text{div}_\Sigma Y \geq 0 \quad \text{for} \quad r \geq \varepsilon,

with equality at points where \( \nabla r \) lies in the tangent space, and

\[ \text{div}_\Sigma Y = n \varepsilon^{-n} \quad \text{for} \quad r \leq \varepsilon. \]

It follows that for each small \( \varepsilon \),

(5) \quad \frac{\text{Vol}(\Sigma \cap B_\varepsilon(p))}{|B^n_1| \varepsilon^n} \leq \frac{1}{n |B^n_1|} \int_{\Gamma} r^{1-n} \langle \nabla r, \nu_\Sigma \rangle \, dV_{\Gamma}, \quad |B^n_1| := \text{Vol}(B^n_1(0)).

Now apply Stokes’ theorem to the integral of \( \text{div}_C Y \) on \( C \):

\[ \int_C \text{div}_C Y \, dV_C = \int_{\partial C} \langle Y, \nu_C \rangle = \int_{\Gamma} \langle Y, \nu_C \rangle, \]

where \( \nu_C \) is the outward unit conormal to \( \Gamma \) on \( C \). Therefore, by Lemma 1(b)

(6) \quad \frac{\text{Vol}(C \cap B_\varepsilon(p))}{|B^n_1| \varepsilon^n} = \frac{1}{n |B^n_1|} \int_{\Gamma} r^{1-n} \langle \nabla r, \nu_C \rangle \, dV_{\Gamma}.

Note here that

\[ 0 \leq \langle \nabla r, \nu_C \rangle \]
and

\[ \langle \nabla r, \nu_\Sigma \rangle \leq \langle \nabla r, \nu_C \rangle. \tag{7} \]

Thus, letting \( \varepsilon \to 0 \) in inequality (5) and equation (6), we get the desired density estimate. If equality holds, then we must have equality in inequalities (4) and (7), which implies \( \Sigma = C \) and \( \partial r / \partial \nu \geq 0 \).

\[ \square \]

**Definition 1.** Let \( \pi_p \) be the radial projection of \( \mathbb{R}^m \setminus \{p\} \) onto \( \partial B_1(p) \), the unit sphere centered at \( p \in \mathbb{R}^m \). Define the *vision angle* at \( p \) of an \( (n-1) \)-rectifiable set \( \Gamma \subset \mathbb{R}^m \) by

\[ \Pi_p(\Gamma) = \text{Vol}(\pi_p(\Gamma)), \]

and the *vision angle* of \( \Gamma \) by

\[ \Pi(\Gamma) = \sup_{p \in \mathbb{R}^m} \Pi_p(\Gamma). \]

Here the volume \( \text{Vol}(\pi_p(\Gamma)) \) counts multiplicity.

Clearly we have for any \( p \in \mathbb{R}^m \) and \( C := p \times \Gamma \)

\[ c_{n-1} \Theta_C(p) = \Pi_p(\Gamma^{n-1}) \leq \Pi(\Gamma), \quad c_{n-1} := \text{Vol}(S^{n-1}), \]

and hence we get the following corollaries to Theorem 1.

**Corollary 1.** If \( \Gamma \subset \mathbb{R}^m \) is an \( (n-1) \)-dimensional compact manifold, then any stationary rectifiable set \( \Sigma \) spanning \( \Gamma \) satisfies

\[ c_{n-1} \Theta_\Sigma(p) \leq \Pi_p(\Gamma) \]

for all \( p \in \Sigma \).

**Corollary 2.** If \( \Gamma \subset \mathbb{R}^m \) is an \( (n-1) \)-dimensional compact manifold with \( \Pi(\Gamma) < 2c_{n-1} \), then any immersed minimal submanifold \( \Sigma \) spanning \( \Gamma \) is embedded.

**Proof.** An immersed submanifold \( \Sigma \) with density \( \Theta_\Sigma(q) < 2 \) at each point \( q \in \mathbb{R}^m \) has no self-intersection.

\[ \square \]

**Remark.** It may appear inappropriate to view \( \Pi(\Gamma) \) as a total curvature. But it has its own merit, as the following example demonstrates. Define an
immersed closed $C^1$ curve $\gamma \subset \mathbb{R}^2$ (the unit square plus four small loops at the corners) by

$$\gamma = \partial([-1, 1]^2) \cup \{(x, y) : |x| > 1, |y| > 1, \left[\left(|x| - 1\right)^2 + \left(|y| - 1\right)^2\right]^{3/2} = \varepsilon(|x| - 1)(|y| - 1)\}$$

and define a Jordan curve $\Gamma \subset \mathbb{R}^n$ to be an embedded $C^2$ curve $C^1$-close to $\gamma$. Then for small $\varepsilon$,

$$\int_\Gamma |\vec{k}| ds > 6\pi, \quad \text{however,} \quad \Pi(\Gamma) \approx 3\pi.$$ 

Hence by Corollary 2 any immersed minimal surface $\Sigma$ spanning $\Gamma$ is embedded since $2c_1 = 4\pi$, although the Ekholm-White-Wienholtz theorem [EWW] cannot give the same conclusion.

Let $G_n(\mathbb{R}^m)$ denote the Grassmann manifold of $n$-planes through the origin in $\mathbb{R}^m$, equipped with the unique $\mathbb{O}(m)$-invariant probability measure, and let $\text{Ave}_{P \in G_n(\mathbb{R}^m)}$ be the average over all $P \in G_n(\mathbb{R}^m)$. Denote by $\psi_P$ the orthogonal projection of $\mathbb{R}^m$ onto $P \in G_n(\mathbb{R}^m)$.

**Lemma 2.** Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n \subset \mathbb{R}^m$ centered at the origin $O$ of $\mathbb{R}^m$ and let $D$ be a domain in $S^{n-1}$. Then

$$\text{Ave}_{P \in G_n(\mathbb{R}^m)} \{\Theta_{\psi_P(O \times D)}(O)\} = \Theta_{O \times D}(O).$$

**Proof.** Assume that $a(D) > 0$ is a positive real number such that

$$\text{Ave}_{P \in G_n(\mathbb{R}^m)} \{\Theta_{\psi_P(O \times D)}(O)\} = a(D) \cdot \Theta_{O \times D}(O).$$

Letting $D$ shrink to a point $x \in S^{n-1}$, one can define a function $a : S^{n-1} \to \mathbb{R}$ given by

$$a(x) := \lim_{D \to \{x\}} \frac{\text{Ave}_{P \in G_n(\mathbb{R}^m)} \{\Theta_{\psi_P(O \times D)}(O)\}}{\Theta_{O \times D}(O)}.$$ 

Then, by means of a partition of unity by functions of small support, one can see that

$$a(D) = \frac{\int_D a(x) dV_{S^{n-1}}}{\text{Vol}(D)}.$$ 

Note here that $\mathbb{O}(n)$ is transitive on $S^{n-1}$ and that the elements of $\mathbb{O}(n)$ preserve the volume form $dV_{S^{n-1}}$ on $S^{n-1}$. Therefore one concludes that for
all \( x \in S^{n-1} \),

\[
a(x) \equiv c \quad \text{for a positive constant } c
\]

and hence for any domain \( D \subset S^{n-1} \),

\[
a(D) \equiv c.
\]

Therefore it follows from equation (8) that

\[
(9) \quad \text{Ave}_{P \in G_n(\mathbb{R}^m)} \{ \Theta_{\psi_P(O \times D)}(O) \} = c \cdot \Theta_{O \times D}(O)
\]

for any domain \( D \subset S^{n-1} \). However, for almost all \( P \in G_n(\mathbb{R}^m) \),

\[
\Theta_{\psi_P(O \times S^{n-1})}(O) = \Theta_{O \times S^{n-1}}(O) = 1.
\]

Thus \( c = 1 \) in equation (9), which completes the proof. \( \square \)

**Theorem 2.** Let \( \Gamma^{n-1} \subset \mathbb{R}^m \) be a compact submanifold. Then

\[
\Pi_q(\Gamma^{n-1}) = \text{Ave}_{P \in G_n(\mathbb{R}^m)} \{ \Pi_{\psi_P(q)}(\psi_P(\Gamma)) \}.
\]

**Proof.** The cone \( q \times \Gamma \) can be thought of as a union of infinitesimal cones \( q \times \Delta \Gamma_i \) and then one can apply Lemma 2 to each \( q \times \Delta \Gamma_i \). Hence

\[
\Pi_q(\Gamma) = c_{n-1} \Theta_{q \times \Gamma}(q) = c_{n-1} \text{Ave}_{P \in G_n(\mathbb{R}^m)} \{ \Theta_{\psi_P(q \times \Gamma)}(\psi_P(q)) \} = \text{Ave}_{P \in G_n(\mathbb{R}^m)} \{ \Pi_{\psi_P(q)}(\psi_P(\Gamma)) \}.
\]

\( \square \)

We shall also require the following generalization of Fáry’s theorem to any dimension \( n \) and to any codimension \( m - n \), which was proved by Langevin and Shifrin ([LS], Proposition 2.15):

**Theorem LS.** Let \( \Gamma^{n-1} \) be a smooth submanifold of \( \mathbb{R}^m \), \( m \geq n \). Then

\[
\frac{c_{n-1}}{2} \int_{\Gamma} K^* dV_{\Gamma} = \text{Ave}_{P \in G_n(\mathbb{R}^m)} \int_{\psi_P(\Gamma)} |GK_{\psi_P(\Gamma)}| dV_{\psi_P(\Gamma)}.
\]
4. Embeddedness of minimal submanifolds

It is tempting to propose a higher-dimensional extension of Ekholm-White-Wienholtz’s theorem as follows:

**Conjecture.** If \( q \in \Sigma \), a minimal submanifold of \( \mathbb{R}^m \) spanning an \( (n-1) \)-dimensional compact manifold \( \Gamma \), then

\[
\Theta_{\Sigma}(q) \leq \frac{1}{2} \int_{\Gamma} K^* dV_{\Gamma}.
\]

If this were known, one could prove the following as well:

*If an \((n-1)\)-dimensional compact connected manifold \( \Gamma \) satisfies \( \int_{\Gamma} K^* dV_{\Gamma} < 4 \), then any immersed minimal submanifold \( \Sigma \) spanning \( \Gamma \) is embedded.*

Conjecture seems to be hard to prove as yet.

However, if we let \( \Gamma_i \) be a compact convex hypersurface of an affine \( n \)-plane \( \mathbb{R}^n_i \subset \mathbb{R}^m \), \( i = 1, 2 \), and define \( \Gamma = \Gamma_1 \cup \Gamma_2 \), then we may prove Conjecture for this case. Our proof uses the vision angle of \( \Gamma \) from a point of \( \Sigma \), and averages over projections onto all \( n \)-dimensional subspaces \( P \) of \( \mathbb{R}^m \). Namely, for \( i = 1, 2 \),

\[
(10) \quad \Pi_{\psi_P(q)}(\psi_P(\Gamma_i)) \leq c_{n-1} = \int_{\psi_P(\Gamma_i)} |GK_{\psi_P(\Gamma_i)}| dV_{\psi_P(\Gamma_i)},
\]

since \( \psi_P(\Gamma_i) \) is a convex hypersurface in \( \psi_P(\mathbb{R}^n_i) \). Here equality holds for all \( P \) if and only if \( q \) is in \( \mathbb{R}^n_i \) and inside \( \Gamma_i \). Thus we have the following:

**Theorem 3.** Given two \( n \)-planes \( \mathbb{R}^n_i, \mathbb{R}^n_2 \) in \( \mathbb{R}^m \), let \( \Gamma_i \) be a compact convex hypersurface in \( \mathbb{R}^n_i \), \( i = 1, 2 \). If \( \Gamma = \Gamma_1 \cup \Gamma_2 \), then any \( n \)-dimensional minimal submanifold \( \Sigma \) spanning \( \Gamma \) is either a union of two flat domains of \( \mathbb{R}^n_i \) or is nonflat and has no self intersection.

**Proof.** We may compute that

\[
\int_{\Gamma} K^* dV_{\Gamma} = \sum_{i=1,2} \int_{\Gamma_i} K^* dV_{\Gamma_i} = 4.
\]
Thus by inequality (10) and Corollary 1 we have $\Theta_\Sigma \leq 2$. If $\Theta_\Sigma = 2$, inequality (10) and Corollary 1 imply $\Sigma$ is flat. If $\Theta_\Sigma < 2$, $\Sigma$ is nonflat and has no self intersection.  

**Remark.** It should be mentioned that R. Schoen [Sc] proved a theorem which implies a special case of Theorem 3:

If $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1, \Gamma_2$ are $(n-1)$-spheres in parallel $n$-planes with the line $\ell$ joining their centers being orthogonal to these hyperplanes, then any immersed minimal submanifold $\Sigma^n$ spanning $\Gamma$ is a hypersurface of revolution with axis $\ell$. In particular, $\Sigma$ is a catenoid or a pair of plane disks.

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