Anisotropy of the semi-classical gluon field of a large nucleus at high energy

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The McLerran-Venugopalan model describes a highly boosted hadron/nucleus as a sheet of random color charges which source soft classical Weizsäcker-Williams gluon fields. We show that due to fluctuations, individual configurations are azimuthally anisotropic. We compute the first four azimuthal Fourier amplitudes of the S-matrix of a fundamental dipole in such background fields.

I. INTRODUCTION

To explain azimuthal asymmetries observed in high-energy pA collisions [1–5] Refs. [6–8] argued that individual configurations of the light-cone electric fields of the target should be anisotropic, leading to a non-trivial azimuthal distribution of a projectile parton scattered off such a target. That is, configuration by configuration, two-dimensional rotational symmetry is broken by E-field “domains” of finite size in the impact parameter plane. These, in contrast to Weiss magnetic domains separated by domain walls, arise purely due to fluctuations of the valence (large-x) random color charge sources for the soft, small-x E field.

Assuming said azimuthal anisotropy of the light-cone electric fields several features of the data could be described, at least qualitatively [7, 8]. On the other hand, a direct calculation of the anisotropic distributions, in particular for a large nucleus and small x (i.e. high energy), has so far been lacking. It is our goal here to compute scattering of a dipole off a large nucleus, and specifically, to determine its angular dependence. That is, we compute the (first four) Fourier amplitudes of the dipole S-matrix with respect to the azimuthal orientation of the dipole. We should stress that we do not address the fluctuations of \( S(r, b) \) in impact parameter space \( b \) (see Ref. [9] for a recent study) but rather its dependence on the size and orientation of the dipole vector \( r \) which is the variable conjugate to the transverse momentum of the parton in the final state.

II. THE MODEL

In the McLerran-Venugopalan model [10] the large-x valence partons are viewed as random, recoilless color charges \( \rho^a(x) \) which source the semi-classical small-x gluon fields. We first provide a brief description of how these color charge configurations are generated on a lattice; more detailed discussions can be found in the literature [11, 12].

The effective action describing color charge fluctuations is taken to be quadratic,

\[
S_{\text{eff}}[\rho^a] = \int \! d^2 x \frac{\rho^a(x^-) \rho^a(x^-, x)}{2\mu^2}
\]

(1)

with \( \mu^2 \sim g^2 A_{1/3} \) proportional to the thickness of a nucleus [10]; here \( A \) denotes the number of nucleons in the nucleus. The variance of color charge fluctuations determines the saturation scale \( Q_s^2 \sim g^4 \mu^2 \) [13]. The coarse-grained effective action (1) applies to (transverse) area elements containing a large number of large-x “valence” charges, \( \mu^2 \Delta A_{\perp} \sim Q_s^2 / g^4 \gg 1 \).

Hence, in the first step we construct a random configuration of color charges on a 2D lattice according to the distribution \( \exp(-S[\rho]) \). Their (non-Abelian) Weizsäcker-Williams fields are pure gauges; in covariant gauge,

\[
A^{\mu a}(x^-, x) = -\delta^{\mu+} \frac{g}{\sqrt{2}} \rho^a(x^-, x).
\]

(2)
This also satisfies $A^- = 0$ and thus the only non-vanishing field strength is $F^{+i} = -\partial^j A^+$. The (light-cone) electric field is

$$E^i = \int dx^- F^{+i} = -\partial^i \int dx^- A^+ \ .$$

The propagation of a fast charge in this field is described by an eikonal phase given by a light-like SU(3) Wilson line $V(x)$:

$$V(x) = \mathbb{P} \exp \left\{ ig^2 \int dx^- \frac{1}{\nabla^2} \rho^a(x, x^-) \right\},$$

where $\mathbb{P}$ denotes path-ordering in $x^-$. The absolute value squared of this amplitude gives the S-matrix for scattering of this charge off the given target field configuration,

$$S_\rho(r, b) \equiv \frac{1}{N_c} \text{tr} V^\dagger(x) V(y) , \quad r \equiv x - y , \quad 2b \equiv x + y \ .$$

Thus, following the ideas leading to the MV model we assume that every particular scattering event probes one particular configuration in the target, i.e. that the S-matrix is computed with a frozen $\rho^a(x)$. The main purpose of this paper is to analyze the dependence of the S-matrix on the angular orientation of the dipole vector $r$, conjugate to the transverse momentum, at fixed transverse impact parameter (coordinate) $b$.

The S-matrix for a fundamental charge is complex (for three or more colors). Its real (imaginary) part corresponds to C-even (C-odd) exchanges [14]:

$$1 - D_\rho(r) \equiv \text{Re} S_\rho(r) = \frac{1}{2N_c} \text{tr} \left[ V^\dagger(x) V(y) + V^\dagger(y) V(x) \right] ,$$

$$O_\rho(r) \equiv \text{Im} S_\rho(r) = \frac{-i}{2N_c} \text{tr} \left[ V^\dagger(x) V(y) - V^\dagger(y) V(x) \right] .$$

C conjugation transforms $\rho^a(x) \to -\rho^a(x)$ and $V(x) \to V^\dagger(x)$. The dipole scattering amplitude $D_\rho(r) = D_\rho(-r)$ is even under $r \to -r$ and generates even azimuthal $v_{2n}$ harmonics while the odderon $O_\rho(r) = -O_\rho(-r)$ generates the odd $v_{2n+1}$ [7].

It is useful to consider the limit of small dipoles, $rQ_s \ll 1$. Then the real part of the S-matrix from Eq. (6) is

$$\text{Re} S_\rho(r) - 1 = \frac{(ig)^2}{2N_c} \text{tr} (r \cdot \mathbf{E})^2 + \mathcal{O}(r^4) \ .$$

To compute the elliptic (dipole) asymmetry, Refs. [6] [8] considered the following angular dependence of the two-point function

$$\frac{g^2}{2N_c} \left\langle \text{tr} E^i(b_1) E^j(b_2) \right\rangle = \frac{1}{4} Q_s^2 \Delta(b_1 - b_2) \left( \delta^{ij} + 2a \left( \hat{\alpha}^i \hat{\alpha}^j - \frac{1}{2} \delta^{ij} \right) \right) ,$$

where $\mathbf{a}$ corresponds to the “event plane” orientation, and $\Delta(b_1 - b_2)$ describes the E-field correlations in the transverse impact parameter plane. It is implicit that for each configuration $\mathbf{E}(b)$ is rotated to point in a particular, fixed direction $\hat{\mathbf{a}}$ before performing the ensemble average. In fact, Eq. (9) is the MV model analogue of the gluon TMD for an unpolarized target [15] [16],

$$\delta^{ij} f_1^g(x, k^2) + \left( \hat{k}^i \hat{k}^j - \frac{1}{2} \delta^{ij} \right) h_{1+}^g(x, k^2) .$$

Thus, the amplitude $A$ from Eq. (9), which we shall denote $A_2(r)$ below, is basically $h_{1+}^g$ at small $x$. However, beyond the MV model the relation between these functions may be more involved.

The action [14] is C-even and so $\langle O_\rho(r) \rangle = 0$ while $\langle D_\rho(r) \rangle \sim r^2 Q_s^2$ (at small $r$) is proportional to the thickness of the nucleus, $A^+$. A C-odd operator

$$\frac{1}{k_3} a^{abc} \rho^a \rho^b \rho^c,$$
with $\kappa_3 \sim g^3 A^{2/3}$ could be added to the action\textsuperscript{1} which would then induce an expectation value $\sim A^{1/3}$ for the odderon \cite{17}. This is beyond the scope of the present paper, we focus here on azimuthal anisotropies due to fluctuations of the charge densities $\rho^a(x)$ and their associated electric fields $E^a(x)$.

### III. IMPLEMENTATION

To generate the random configurations $\rho^a(x^-, x)$ via Monte-Carlo techniques we discretize the longitudinal and transverse coordinates. The number of sites in the longitudinal direction is taken to be $N_x = 100$ while the number of sites in either transverse direction is $N_{\perp} = 1024$. All of our results presented here have been obtained with $g^2\mu a = 0.05$, hence $g^2\mu L = 51.2$, where $a \equiv L/N_{\perp}$ denotes the transverse lattice spacing. We have determined numerically that $Q_s \approx 0.7125 g^2 \mu$ as defined from $\langle S_0 \rangle (r = \sqrt{2}/Q_s) = \exp(-1/2)$. The physical value for the lattice spacing could be determined by assigning a physical value to $Q_s$; instead, we choose to measure distance scales in units of $1/Q_s$ or $1/g^2\mu$ and so this step is not required.

We use periodic boundary conditions in the transverse directions and solve the Poisson equation \cite{2} by Fast Fourier Transform. The amplitude of the zero mode of $\rho^a(k)$ is set to zero before inversion which ensures color neutrality of each configuration.

We have generated about $10^4$ configurations; for each of them we measured $D_\rho(r)$ and $O(r)$ at $b = 0$. Both functions were decomposed into their Fourier series to extract the amplitudes of azimuthal anisotropy:

\begin{align}
D_\rho(r) &= N(r) \left( 1 + \sum_{n=1}^{\infty} A'_{2n}(r) \cos(2n\phi_r) \right), \\
O_\rho(r) &= N(r) \sum_{n=0}^{\infty} A'_{2n+1}(r) \cos((2n+1)\phi_r). 
\end{align}

The function $N(r)$ is the isotropic part of the dipole S-matrix, see for example Ref. \cite{12}. Each amplitude $A'_n$ contains a random phase $\exp(iN\psi)$ which fluctuates from configuration to configuration. This corresponds to a random global rotation of the charge distribution $\rho^a(x)$ from configuration to configuration. Azimuthal harmonics $\delta_n$ are defined from multi-particle correlation functions in such a way that they are invariant under a global shift of the azimuthal angles of all particles by the same amount. Consequently, we discard this random phase by defining $A_n = |A'_n|$. Averaging over configurations we finally obtain $\langle A_1 \rangle, \cdots, \langle A_4 \rangle$ as well as the variances of $A_1$ and $A_2$.

### IV. RESULTS

Before presenting our results for the azimuthal amplitudes we show two examples for $S_\rho(r)$ in Figs. 1 and 2. Either of these corresponds to one particular (random) configuration of color charges. The real parts display predominantly a $\sim \cos(2\phi)$ angular dependence, with $\phi$ the angle between $r$ and $E(b = 0)$. On the other hand, the imaginary part for the configuration shown in Fig. 1 is predominantly $\sim \cos(\phi)$ while that from Fig. 2 is mainly $\sim \cos(3\phi)$, modulo a random phase shift as mentioned above. The figures show, also, that the angular structures appear at a resolution on the order of $rg^2\mu \sim 1$; this is consistent with the requirement $\mu^2 \Delta A_\perp \gg 1$ mentioned above (which sets the regime of applicability of the effective theory) at weak coupling: $1/g^2 \gg 1$.

Figure 3 shows our results for the averaged amplitudes of the first four azimuthal harmonics. As expected, the biggest one is the quadrupole amplitude $\langle A_2 \rangle$ which reaches $\gtrsim 12\%$ at $r \lesssim 1/Q_s$. Such values are in the range of the asymmetries extracted phenomenologically \cite{7} for high-multiplicity p+Pb collisions at LHC energies. However, here we have not made any attempts to bias the configurations towards “high multiplicities”. The fact that the variance $\sqrt{\langle (\delta A_2)^2 \rangle}$ is not much smaller than $\langle A_2 \rangle$ indicates that some configurations generate much larger elliptic asymmetries than others. Also, we observe that $\langle A_2 \rangle$ is approximately constant for $r < 1/Q_s$ since up to quadratic order the real part of the S-matrix is

\begin{equation}
D(r) = \frac{g^2}{2N_c} \text{tr} (r \cdot E)^2 - \frac{1}{2} \frac{g^4}{4N_c^2} \left[ \text{tr} (r \cdot E)^2 \right]^2 + \cdots
\end{equation}

\textsuperscript{1} Beyond a perturbative treatment of the cubic Casimir one would have to add the quartic Casimir, too, so that the action is bounded from below $\cite{18}$.
FIG. 1: The S-matrix in the fundamental representation as a function of the dipole vector \( \mathbf{r} = (r_x, r_y) \) at fixed impact parameter \( b = 0 \) for one particular random configuration of color charges \( \rho^a(x) \).

FIG. 2: Same as Fig. 1 for a second configuration of color charges \( \rho^a(x) \).

at small \( r \). To derive this expression one performs a gradient expansion of \( \text{Re} \ tr \ V(x)V^\dagger(y) \), assuming that the electric field is smoothly varying over scales of order \( r \). The leading term on the r.h.s., if scaled by \( 1/r^2 \), is independent of \( r \). For not too large dipoles our numerical result agrees well with the behavior derived in Ref. [16]:

\[ h_1^{+g}(x, r^2) \propto \frac{1}{r^2 Q_s^2} \left[ 1 - \exp \left( -\frac{r^2 Q_s^2}{4} \right) \right]. \tag{15} \]

Equation (15) provides a perfect fit of \( \langle A_2 \rangle \) for \( r Q_s \lesssim 3 \), as shown in Fig. 3.

The second term in Eq. (14) generates a hexadecupole asymmetry at the next to leading order in \( r^2 \). However, the numerical result for \( A_4(r) \) shown in Fig. 3 is essentially constant at small \( r \). We interpret this as due to corrections to the gradient expansion which leads to Eq. (14): a \( \sim \cos(4\phi) \) angular component appears already at \( O(r^2) \) albeit with a much smaller amplitude than the \( \sim \cos(2\phi) \) harmonic.

We now turn to the odd amplitudes \( A_1 \) and \( A_3 \). As already mentioned above, the expectation value of the odderon over a C-even ensemble such as that generated by the action [1] is of course zero. Nevertheless, each particular
Fig. 3: The averaged amplitudes $\langle A_n \rangle(r)$ vs. the dipole size $r$ for $n = 1, \cdots, 4$. The fit for $\langle A_2 \rangle(r)$ is based on Eq. (15).

configuration of semi-classical small-$x$ fields \cite{2} does contain a $C$-odd component and $iO(r)$ as defined in Eq. (7) is non-zero. This is due to fluctuations of the saturation momentum $Q_s$ in impact parameter space \cite{19},

$$iO(r) \sim i \alpha_s r \cdot \nabla_b (1 - D(r, b)) \simeq i \alpha_s r^3 Q_s^2 \cos \phi_r \left[ 1 - \frac{r^2}{4} \left( \frac{Q_s^2 \cos^2 \phi_r}{3} + Q_s^2 \right) \right].$$

(16)

The expression on the r.h.s. corresponds to an expansion in powers of $r$; $Q_c$ is a cutoff for the spectrum of fluctuations of $Q_s(b)$ which was otherwise assumed to be scale invariant, and $B$ is their amplitude \cite{7}. Eq. (16) shows that for small dipoles, after we divide by the isotropic normalization factor $N(r) \sim r^2$, that we should expect $A_1 \sim r$ as well as a smaller $A_3 \sim r^3$. The lattice results appear consistent with $\langle A_1 \rangle \sim r$ at $r \ll 1/Q_s$ but so is $\langle A_3 \rangle$, albeit with a smaller slope. Future simulations on larger lattices may be able to push to smaller $r$, and the analytical derivation of Eq. (16) based on a simple fluctuation spectrum could perhaps be refined as well.

Just as for the elliptic asymmetry we have also computed the standard deviation of the amplitude $A_1$. Again, we find that $\sqrt{\langle (\delta A_1)^2 \rangle}$ is not much smaller than $\langle A_1 \rangle$, i.e. that some configurations generate much larger dipole asymmetries than others.

We have also analyzed the effect of “smearing” the impact parameter of the projectile over a region corresponding to its size \cite{20}. If the $E$-field anisotropy exhibits a non-zero correlation length in the impact parameter plane \cite{6–8}, specifically a correlation length that exceeds the size of the dipole, then the azimuthal moments should remain approximately the same.

Hence, we have also computed the azimuthal amplitudes $A_n$ from “smeared” configurations:

$$\overline{D}_\rho(r, b) = \int \frac{d^2 b'}{\pi r'^2} \Theta(r - |b - b'|) D_\rho(r, b'),$$

(17)

and similarly for $i\overline{O}_\rho(r, b)$. On the r.h.s. the points $x = b' + r/2$ and $y = b' - r/2$ are now determined by $r$ and $b'$. Equation (17) averages the $S$-matrix over an area $\pi r'^2$. The result is shown in Fig. 4 which can be compared to Fig. 3 from above. Except for a slight suppression of their magnitudes, we do not observe any substantial modification of the amplitudes $\langle A_n \rangle$.

The behavior for large dipoles is different, c.f. Fig. 5. For a fixed impact parameter the harmonic amplitudes approach a common non-zero function at large $r \gg 1/Q_s$. This is consistent with universal (angular) scale invariant fluctuations of the azimuthal dependence of the $S$-matrix. Indeed, if $D(r, b)$ and $O(r, b)$ are first averaged over an area $\pi r'^2$, see Eq. (17), then the resulting $\langle A_n \rangle$ are strongly suppressed. This shows that the direction of $E$ is not correlated over distances much beyond $\sim 1/Q_s$. Also, we note that the resummed analytical result $\langle A_2 \rangle$ written in Eq. (15) does not provide a good fit for $r Q_s \gtrsim 3$. This is not unexpected since the derivation involves adhoc infrared cutoffs which need to be introduced by hand (c.f. related discussion in Ref. [12]). On the other hand, the non-perturbative lattice computation does not require IR cutoffs beyond imposing global color neutrality.
V. SUMMARY

Following the conjecture by Kovner and Lublinsky [6], we have analyzed azimuthal anisotropies of the S-matrix \( \mathcal{S}(r) \) for scattering of a dipole off a large nucleus. They arise due to fluctuations of the configuration of color charges \( \rho_a(x) \), described here within the Gaussian McLerran-Venugopalan model [10]; an alternative picture in terms of classical fluctuations of the energy-momentum tensor of a holographic shock wave has been discussed in Ref. [21].

For a projectile in the fundamental representation of color SU(3), these fluctuations generate both \( C \)-even as well as \( C \)-odd target field configurations which correspond to \( \cos(n\phi) \) moments. For small dipoles, \( r \lesssim 1/Q_s \), we find that \( \langle A_2 \rangle \) and \( \langle A_4 \rangle \) are approximately constant and that the amplitude of the elliptic harmonic is much larger than that of the quadrangular harmonic, \( \langle A_2 \rangle \gg \langle A_4 \rangle \). Odd harmonics appear at higher order in \( r \) [7, 19] and so their amplitudes decrease with decreasing \( r \). The fluctuations of both \( A_1 \) and \( A_2 \) are comparable to their mean values, indicating that some configurations exhibit much larger anisotropies than others.

For large dipoles, \( r \gtrsim 1/Q_s \), we find that all amplitudes \( \langle A_1 \rangle(r), \ldots, \langle A_4 \rangle(r) \) asymptotically approach a universal function if the S-matrix is evaluated at fixed impact parameter. This points at angular scale invariant fluctuations of the direction of \( \mathbf{E} \) over large distances. Accordingly, if the S-matrix is averaged over an area \( \pi r^2 \) the resulting \( \cos(n\phi) \)
amplitudes are strongly suppressed.

Our calculations confirm that individual small-$x$ target field configurations do exhibit angular dependence which would play an important role in understanding azimuthal $v_n$ harmonics in pp and pA collisions [6–8]. In particular, the amplitude of elliptic anisotropies $\langle A_2 \rangle \sim 10 – 15\%$ is on the order of the $v_2$ harmonic observed in p+Pb collisions at the LHC. In other words, our result supports the conjecture that the $h_1^{\perp}(x, k^2)$ gluon distribution of a nucleus at small $x$ is significant [13, 16].

We have here considered the classical fields of a large nucleus at moderately small $x$ within the McLerran-Venugopalan model [10]. In the future it will be important to incorporate QCD evolution effects. Mean-field evolution of the dipole has been shown to wash out initial elliptic anisotropies after several units of rapidity [6]. The effects of fluctuations in small-$x$ evolution on the anisotropies deserves further study.

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