Finite size scaling of current fluctuations in the totally asymmetric exclusion process

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Abstract

We study the fluctuations of the current \(J(t)\) of the totally asymmetric exclusion process with open boundaries. Using a density matrix renormalization group approach, we calculate the cumulant generating function of the current. This function can be interpreted as a free energy for an ensemble in which histories are weighted by \(\exp(-sJ(t))\). We show that in this ensemble the model has a first-order spacetime phase transition at \(s = 0\). We numerically determine the finite size scaling of the cumulant generating function near this phase transition, both in the non-equilibrium steady state and for large times.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Heat or particle currents arise in systems that are driven away from equilibrium by bringing them in contact with reservoirs at different temperatures or chemical potentials. In macroscopic systems, fluctuations of these currents can often be neglected and a description using non-equilibrium thermodynamics is then appropriate [1]. However, current fluctuations can become important in mesoscopic systems and in the vicinity of non-equilibrium critical points.

In recent years, the interest in current fluctuations has increased considerably. From the experimental side, it is nowadays possible to measure third- and higher-order cumulants of the current in problems of charge transport [2, 3]. In quantum point contacts, current fluctuations can be used as an entanglement meter [4]. Theoretically, one is interested in the entropy produced in the presence of a current, since its fluctuations have been shown to obey various kinds of fluctuation theorems [5–11].

In this paper, we study current fluctuations in the totally asymmetric exclusion process (TASEP). In this classical, stochastic model, particles on a one-dimensional lattice jump to
the right (provided that site is empty), thus giving rise to a time-dependent particle current. The TASEP was originally introduced to describe the biological process of translation, in which ribosomes move along a messenger RNA (mRNA) and use the information stored in it to build proteins [12]. In this context, one can relate current fluctuations to fluctuations in protein production as we recently showed [13]. Nowadays, the TASEP, and some of its variants, have become paradigmatic for non-equilibrium statistical mechanics [14]. Current fluctuations in the TASEP have recently been investigated as a model for the full counting statistics in quantum-dot chains [15].

Fluctuations in the current can be determined from the cumulant generating function (CGF). In the large time limit, the TASEP evolves to a non-equilibrium steady state (NESS) for which the CGF equals the largest eigenvalue of a generalized generator [14]. This largest eigenvalue has been determined exactly for the TASEP on a ring [16, 17] using the Bethe ansatz and was found to have an interesting scaling form, in which the KPZ dynamical exponent \(z = 3/2\) [18, 19] appears. Much less is known about a configuration with open boundaries where particles can enter from the left, and leave from the right. The average and variance of the current have been determined using a matrix approach [20, 21], but no results are known for higher cumulants or for the full generating function. Recently, we showed how the latter can be determined numerically using a density matrix renormalization group (DMRG) approach [22]. In this paper, we further analyse our results and show that in the thermodynamic limit, the CGF has a non-analytic behaviour which corresponds to a spacetime phase transition of a type that has been discovered recently in models of glassy dynamics [23–26]. We establish the scaling form of the CGF near this phase transition.

Using the same DMRG approach we also investigate the gap in the spectrum of the generalized generator. This gap determines the approach of the current fluctuations to their NESS value. We investigate the scaling properties of the gap and conjecture a scaling form for the time-dependent current fluctuations which so far have been hardly studied, but which could be relevant, for example, in applications to protein production [13]. The scaling form is verified with simulations based on the Gillespie algorithm.

This paper is organized as follows. In section 2, we introduce the model, the CGF of the current and the reinterpretation of this function as a free energy in the so called \(s\)-ensemble. In section 3, we show that the CGF must be a non-analytical function of the variable \(s\). In section 4, we discuss the DMRG approach. In sections 5 and 6, we present the results of our numerical calculations on the CGF and the time dependence of the current fluctuations respectively. We pay particular attention to the scaling behaviour of these quantities. Finally, in section 7 we discuss our results.

### 2. The CGF and the \(s\)-ensemble for the TASEP

In the TASEP, each site of a one-dimensional lattice of \(L\) sites can be empty or occupied by at most one particle. The dynamics of the TASEP is that of a continuous time Markov process for which the probability \(P(C, t)\) that the system is in a configuration \(C\) at time \(t\) evolves according to the master equation

\[
\frac{\partial}{\partial t} P(C, t) = \sum_{C' \neq C} \left[ w(C' \rightarrow C) P(C', t) - w(C \rightarrow C') P(C, t) \right].
\]

Here \(w(C \rightarrow C')\) is the transition rate from configuration \(C\) to \(C'\). In the TASEP, particles can only jump to the right (provided that site is empty) with unit rate. We impose open boundary conditions for which particles can enter the lattice on the left side with rate \(\alpha \leq 1\), and leave
it on the right with rate $\beta \leq 1$. For further reference, we introduce the inverse lifetime, or escape rate, $r(C)$ of the state $C$, $r(C) = \sum_{C \neq C'} w(C \rightarrow C')$.

As a consequence of these dynamics, a current flows from left to right through the system. The current at time $t$, $j(i, t)$, gives the number of particles passing per unit of time through the bond between sites $i$ and $i+1$. In a particular realization, or history, of the TASEP, we can count the total integrated current $J(t, L) = \sum_{i=0}^{L} \int_{0}^{t} j(i, t') dt' \geq 0$ through all bonds up to time $t$ (where the particles entering respectively leaving the system correspond with $i = 0$ and $i = L$). This is a stochastic variable whose properties can be obtained from the CGF

$$\mu(s, t, L) = \frac{1}{t} \ln \langle e^{-s J(t, L)} \rangle,$$

where the average $\langle \cdot \rangle$ is taken over the histories of the process. From (2), the average current per unit time $\bar{J}(t, L)$, its variance $\Delta^2 J(t, L)$ and higher cumulants can be found by taking derivatives:

$$\bar{J}(t, L) = \frac{1}{t} \langle J(t, L) \rangle = -\frac{\partial \mu}{\partial s}(0, t, L)$$

$$\Delta^2 J(t, L) = \frac{1}{t} \left[ \langle J^2(t, L) \rangle - \langle J(t, L) \rangle^2 \right] = \frac{\partial^2 \mu}{\partial s^2}(0, t, L).$$

Equation (2) has a large mathematical similarity with the definition of the dimensionless free energy in equilibrium situations in which the inverse temperature $\beta = 1/k_B T$ is replaced by $s$ and the sum over microstates is replaced by a sum over histories. We can therefore interpret (2) as a thermodynamics of histories [27] in which realizations of the stochastic process are weighted by $\exp(-s J(t, L))$. For negative $s$, (2) will be dominated by histories with a large current, whereas for positive $s$ it will mainly get contributions from those with a small current.

There is no clear physical meaning to the variable $s$, but in the context of glassy dynamics, it has been shown that extending the parameter space with the variable $s$ can lead to new and interesting insights [23–25].

In the long time limit, the TASEP evolves to a unique NESS. The average current $J^*(L) = \lim_{t \to \infty} \bar{J}(t, L)$ in this state can be obtained exactly with a matrix technique [20]. If one also takes the thermodynamic limit, the current per bond $J^* = \lim_{L \to \infty} J^*(L)/(L+1)$ is found to behave non-analytically as a function of $\alpha$ and $\beta$ leading to the recognition of three phases [20]. In the low-density (LD) phase ($\alpha < 1/2, \beta > \alpha$), $J^* = \alpha(1-\alpha)$, while in the high-density (HD) phase ($\beta < 1/2, \alpha > \beta$), $J^*$ equals $\beta(1-\beta)$. Finally, in the maximal current (MC) phase ($\alpha > 1/2, \beta > 1/2$), $J^* = 1/4$. The TASEP therefore has three (boundary-driven) nonequilibrium phase transitions.

It is useful to also introduce the $s$-weighted average current in the NESS together with its variance which is defined as

$$J^*(s, L) = -\lim_{t \to \infty} \frac{\partial \mu}{\partial s}(s, t, L)$$

$$\Delta^2 J^*(s, L) = \lim_{t \to \infty} \frac{\partial^2 \mu}{\partial s^2}(s, t, L).$$

For clarity, we now repeat a standard argument [27] that shows that in the NESS, the CGF equals the largest eigenvalue of a matrix $H(s)$, and that the approach to the asymptotic value is determined by the lowest gap of that matrix. We therefore introduce first the probability

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3 Overlined quantities represent averages over time, and quantities with a * represent time averages in the NESS.
that the system is in a configuration \( C \) and has an integrated current \( J \) at time \( t \). Using (1), we immediately find that

\[
\partial_t P(C, J, t) = \sum_{C' \neq C} \left[ w(C' \rightarrow C) P(C', J - 1, t) - w(C \rightarrow C') P(C, J, t) \right].
\]

(7)

Consequently, the discrete Laplace transform \( \hat{P}(C, s, t) = \sum_{J=0}^{\infty} e^{-sJ} P(C, J, t) \) evolves according to

\[
\partial_t \hat{P}(C, s, t) = \sum_{C' \neq C} \left[ w(C' \rightarrow C) e^{-sJ} \hat{P}(C', s, t) - w(C \rightarrow C') \hat{P}(C, s, t) \right].
\]

(8)

To continue, it is convenient to introduce a matrix notation as is common in the so-called quantum approach to stochastic particle systems [28]. We therefore introduce a set of basis vectors \( |C⟩ \) each corresponding to a configuration \( C \) and a vector \( |\hat{P}(s, t)⟩ \) with components \( \hat{P}(C, s, t) = ⟨C|\hat{P}(s, t)⟩ \). Using this notation, the set of equations (8) can be rewritten as

\[
\partial_t |\hat{P}(s, t)⟩ = H(s)|\hat{P}(s, t)⟩.
\]

(9)

The diagonal elements of the matrix \( H(s) \) are equal to minus the inverse lifetimes of the states, while the off-diagonal elements are given by the transition rates multiplied by \( e^{-s} \). For \( s = 0 \), (9) reduces to the master equation (1) and \( H(0) \) corresponds to the generator of the stochastic process. We will therefore refer to \( H(s) \) as the generalized generator.

Using the ‘quantum’ notation of [28] in which an empty (occupied) site is associated with an up (down) spin, one can easily show that for the TASEP

\[
H(s) = \sum_{i=1}^{L-1} \left[ e^{-s_i s_{i+1}} - n_i v_i + \alpha [e^{-s_i s_{i-1}} - v_i] + \beta [e^{-s_i s_{i+2}} - n_L] \right].
\]

(10)

Here \( n_i, v_i, s^+_i \) and \( s^-_i \) are the standard particle number, vacancy number, particle annihilation and creation operators at site \( i \):

\[
n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad s^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad s^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

(11)

The formal solution to (9) is

\[
|\hat{P}(s, t)⟩ = e^{H(s)t}|\hat{P}(s, 0)⟩.
\]

(12)

Therefore, we have

\[
\langle e^{-sJ(t, L)}⟩ = \sum_C \sum_J e^{-sJ} P(C, J, t) = \sum_C \hat{P}(C, s, t) = \sum_C ⟨C|e^{H(s)t}|\hat{P}(s, 0)⟩.
\]

(13)

Using the spectral theorem, the sum in (13) can be written in terms of the eigenvalues and eigenvectors of \( H(s) \). In the long time limit, this sum is dominated by the largest eigenvalue \( \lambda_1(s, L) \) while the first correction term involves the gap \( G(s, L) = \lambda_1(s, L) - \lambda_2(s, L) \) with the second largest eigenvalue \( \lambda_2(s, L) \). One has

\[
\langle e^{-sJ(t, L)}⟩ = A_1 e^{\lambda_1(s, L)t} [1 + A_2 e^{-G(s, L)t} + \cdots],
\]

(14)

where \( A_1 \) and \( A_2 \) are time-independent factors depending on the initial conditions and the eigenvectors associated with the two largest eigenvalues. Therefore, for \( t \rightarrow \infty \)

\[
\mu^*(s, L) = \lim_{t \rightarrow \infty} \mu(s, t, L) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln\langle e^{-sJ(t, L)}⟩ = \lambda_1(s, L)
\]

(15)

which shows that \( \mu^* \) is intensive in time.
3. Spacetime phase transition

In this section, we give a simple argument that shows that in the NESS and in the thermodynamic limit the CGF is non-analytic at $s = 0$. Such phase transitions in the properties of histories have been called spacetime transitions.

First, we observe that $\mu(s, t, L)$ is a non-increasing function of $s$ and by definition is zero at $s = 0$. Hence we obtain the bounds

$$0 \geq \mu(s, t, L) \geq \mu(s \rightarrow \infty, t, L) \quad s \geq 0.$$  \hspace{1cm} (16)

For $s \rightarrow \infty$, only histories with $J(t, L) = 0$ contribute to $\mu$. This implies that the system at time $t$ is still in the configuration $C_0$ in which it was initially. The probability for this to happen decays exponentially with waiting time $1/r(C_0)$. Taking an average over possible initial conditions, one obtains

$$\lim_{s \rightarrow \infty} \langle e^{-sJ(t, L)} \rangle = \sum_{C_0} p_0(C_0) P(C_0, J = 0, t) = \sum_{C_0} p_0(C_0) e^{-r(C_0)t},$$  \hspace{1cm} (17)

where $p_0(C_0)$ is the probability that the system is at $t = 0$ in $C_0$. In the long time limit, this sum will be determined by the configurations with the largest lifetime. It is not too difficult to realize that these are the completely empty configuration $C_e$ with $r(C_e) = \alpha$ and the fully occupied configuration $C_f$ with $r(C_f) = \beta$. Therefore, one has for very large times

$$\lim_{s \rightarrow \infty} \langle e^{-sJ(t, L)} \rangle = p_{\min} e^{-\min[\alpha, \beta] t} + \cdots$$  \hspace{1cm} (18)

(19)

Comparing with (14) we have that $\lambda_1(s \rightarrow \infty, L) = -\min[\alpha, \beta]$ and $G(s \rightarrow \infty, L) = |\alpha - \beta|$. Inserting (18) in (16) and taking $t \rightarrow \infty$, one obtains

$$0 \geq \mu^*(s, L) \geq -\min[\alpha, \beta].$$  \hspace{1cm} (19)

The inequalities imply that the space-intensive quantities $\lim_{L \rightarrow \infty} \mu^*(s, L)/(L + 1)$ and $\lim_{L \rightarrow \infty} J^*(s, L)/(L + 1)$ are equal to zero for any strictly positive $s$. On the other hand, as discussed above, it is known from the exact solution [20] that $J^* = \lim_{L \rightarrow \infty} J^*(0, L)/(L + 1)$ is non-zero. Hence, it follows that in the $s$-ensemble, the TASEP has a first-order spacetime phase transition at $s = 0$, and this for every $\alpha$ and $\beta$.

In the remainder of this paper, we are interested in the finite size scaling properties of the CGF near this transition. For the TASEP on a ring it has been shown exactly [16, 17] that the CGF in the NESS scales as

$$\mu^*_{\text{ring}}(s, L) = -sL\rho(1 - \rho) + \sqrt{\frac{\rho(1 - \rho)}{2\pi L^3}} H(s\sqrt{2\pi\rho(1 - \rho)L^3}).$$  \hspace{1cm} (20)

Here $\rho$ is the density of particles and $H$ is a scaling function. The first term on the right-hand side of this equation equals $-sLJ^*$ since in the ring case $J^* = \rho(1 - \rho)$. In the remainder of this paper, we will use a numerical approach to investigate the scaling properties of $\mu$ for the case of open boundaries.

4. The DMRG approach

The behaviour of the CGF of the current at large times in a finite system is determined by the two largest eigenvalues of the generalized generator (10). If we interpret this generator as a ‘Hamiltonian’ [28], we realize that calculating these eigenvalues is mathematically similar to determining the ground state energy and the gap of a quantum spin chain. The
main difference from a standard quantum problem comes from the non-Hermiticity of the
generalized generator.

One of the most precise numerical procedures for calculating ground state properties of
quantum chains is the DMRG introduced by White [29–31]. Later, this technique was adapted
to generators of stochastic processes [32, 33] and in this context a study of the gap in the
generator (at \( s = 0 \)) of the TASEP was made [34]. More recently, we applied the DMRG for
the first time to the generalized generators associated with current fluctuations in driven lattice
gases and activity fluctuations in the contact process [22, 35].

The application of the DMRG to these generalized generators is not fundamentally
different from the standard approach used for quantum systems. For some of the technical
aspects, we refer to [35].

We have used the DMRG to calculate the two largest eigenvalues of the generator (10) as
a function of \( s \) for various points in the phase diagram of the TASEP. We are typically able
to reach reliable results for \( L \) up to 60. At \( s = 0 \) it is possible to go up to \( L \approx 100 \). The upper
limit for \( L \) that can be reached is essentially set by the stability of the Arnoldi algorithm used
to diagonalize large non-Hermitian matrices.

Once the largest eigenvalue has been calculated with sufficient numerical accuracy,
the average current and its variance in the \( s \)-ensemble are determined using numerical
differentiation. Because of numerical round-offs, it is not possible to obtain cumulants beyond
the second with sufficient accuracy.

5. Current fluctuations in the stationary state

In this section, we present our results for the current fluctuations in the NESS.

In figure 1 we show a typical result for the CGF \( \mu^*(s, L) = \lambda_1(s, L) \). The data are for
a point in the LD phase with \( \alpha = 0.35, \beta = 2/3 \). In the inset, it can be seen that already for
rather small \( s \)-values the asymptotic value \( \min[\alpha, \beta] = -0.35 \) is reached, as predicted in
the previous section. When \( L \) increases this limiting value is reached for ever smaller \( s \). The
behaviour shown here is typical for all values of \( \alpha \) and \( \beta \) that we investigated.

In figure 2 we show the result for the average current per bond \( J^*(s, L)/(L + 1) \) as a
function of \( s \) at \( \alpha = .5, \beta = 2/3 \) (LD–MC transition line). In this figure, the spacetime
transition at \( s = 0 \), rounded by finite size effects, is clearly visible. Qualitatively similar
behaviour is found for other values of \( \alpha \) and \( \beta \).

We next propose a scaling form for the CGF. The fact that such a scaling exists can be
expected from analogy with the ring case (20). Moreover, the similarity between the CGF
and the equilibrium free energy leads us to expect that such a scaling can hold near the
nonequilibrium phase transitions in the TASEP. In writing down a scaling relation for the CGF
a natural variable will therefore be \( \Delta \alpha = \alpha - 1/2 \), the distance from the transition between
the LD and the MC phase (or equivalently \( \Delta \beta = \beta - 1/2 \) for the HD to MC transition)\(^4\).
These considerations lead us to make the finite size scaling ansatz

\[
\mu^*(s, L, \Delta \alpha) = -s(L + 1)J^* + L^{-z}H(sL^{\gamma_s}, \Delta \alpha L^{\gamma_y}),
\]

where \( H \) is a scaling function and \( \gamma_s \) and \( \gamma_y \) are two critical exponents that we will determine
below. The factor \( L + 1 \) in the first term on the right-hand side equals the number of bonds
through which the total current passes (including incoming and outgoing particles). It replaces
the factor \( L \) in the ring case. The appearance of the dynamical exponent \( z \) in the prefactor \( L^{-z} \)

\(^4\) We could not study the so-called coexistence line, which is the first-order transition line between the LD and HD
phases, because of numerical instabilities in the DMRG procedure.
is a consequence of the time intensivity of $\mu^\star$. It replaces the space-dimension $d$ in the factor $L^{-d}$ appearing in the scaling of the equilibrium free energy.

From (21) it follows that the average current at $s = 0$ scales as

$$J^\star(s = 0, L, \Delta \alpha) = (L + 1) J^\star + L^{-z / h} H'(\Delta \alpha L^\nu),$$  

$$\text{Figure 1.} \text{ Cumulant-generating function as a function of } s \text{ for } \alpha = 0.35, \beta = 2/3 \text{ for } L = 10, 20, 30, 40, 50 \text{ and } 60 \text{ (increasing from bottom to top for } s < 0). \text{ The region with } 0 \leq s \leq 1 \text{ is shown in more detail in the inset } (L = 20, 40 \text{ and } 60, \text{ increasing from top to bottom). On the right we show the CGF per bond for the same system sizes.}$$

$$\text{Figure 2.} \text{ Average current per bond as a function of } s \text{ at } \alpha = 0.50, \beta = 2/3 \text{ for } L = 10, 20, 30, 40, 50 \text{ and } 60 \text{ (increasing (decreasing) from bottom to top for } s \text{ negative (positive)).}$$
where $H'$ is another scaling function. We can compare this scaling prediction with the exact results for large $L$. First, at the LD–MC transition one has [20]

$$J^*(s = 0, L, \Delta \alpha = 0) = \frac{L + 1}{4} \left( 1 + \frac{1}{2L} + \cdots \right). \tag{23}$$

Comparing with (22), one obtains $y_s = z = 3/2$. To also determine $y_\alpha$ we take the derivative of (22) with respect to $\Delta \alpha$ at $\alpha = 1/2$. We obtain

$$\frac{dJ^*}{d\alpha} (s = 0, L, \Delta \alpha = 0) \sim L^{y_\alpha}. \tag{24}$$

From the exact asymptotic results in the LD phase [20], one can easily find that asymptotically in $L$

$$\frac{dJ^*}{d\alpha} (s = 0, L, \Delta \alpha = 0) = \frac{\sqrt{\pi}}{2} L^{1/2} (1 + \cdots) \tag{25}$$

so that we conclude $y_\alpha = 1/2$. Inserting the exponent values into (21), we obtain the final form for the scaling of the CGF near the LD–MC transition:

$$\mu^*(s, L, \Delta \alpha) = -s(L + 1)J^* + L^{-3/2}H(sL^{3/2}, \Delta \alpha L^{1/2}). \tag{26}$$

We have performed several tests of this ansatz with the DMRG approach and where possible using exact results.

We start with the scaling for the current at $s = 0$ as given in (22) since it can be checked using the results in [20]. In figure 3 we present data obtained by numerical evaluation of the exact expression for the current for $L$ up to 200. As can be seen the scaling is well satisfied, especially from $L \approx 80$ onwards.

Next we turn to the scaling of the CGF itself. Figure 4 shows a scaling plot of $\mu^*(s, L, 0) + s(L + 1)(4)L^{3/2}$ as a function of $x = sL^{3/2}$. These data are determined from the DMRG. For the range of $x$-values shown, the collapse is perfect from $L = 30$ onwards. For larger $x$-values the scaling becomes less good, indicating that our values of $L$ are not asymptotic enough in this regime.

In figure 5 we present the scaling plot for $J^*(s, L)$ as a function of $s$ at $\Delta \alpha = 0$ from the DMRG data. There is again a good collapse of the data, especially for $x < 5$. 

\[ \text{Figure 3. Scaling of } J^*(s = 0, L, \Delta \alpha) - (L + 1)J^* \text{ as a function of } \Delta \alpha L^{1/2} \text{ at } \beta = 2/3. \]
Figure 4. Scaling of $\gamma^*(s, L, 0) + s(L+1)/4L^{3/2}$ as a function of $sL^{3/2}$ at $\alpha = 1/2, \beta = 2/3$ (for $L = 30, 40, 50$ and $60$). Within the scale of the figure, all results collapse perfectly on one curve.

Figure 5. Scaling of $J^*(s, L, \Delta \alpha = 0) - (L+1)J^*$ as a function of $sL^{3/2}$ (for $L = 50, 52, 54, 56, 58$ and $60$ increasing from top to bottom).

Finally, we turn to the variance of the current, which in this non-equilibrium context plays a role similar to the specific heat or the susceptibility in equilibrium. How does the variance of the current behave near the LD–MC transition? From (26), we derive the scaling form at $s = 0$

$$\Delta^* J(0, L, \Delta \alpha) = L^{3/2} H_2(\Delta \alpha L^{1/2})$$

(27)

with $H_2$ a scaling function. An exact formula for the variance of the current was derived in [21] using the matrix product technique. Due to the complexity of this formula, closed expressions for the variance could only be derived at the point $\alpha = \beta = 1$ in the MC phase, and along
the line $\alpha + \beta = 1$ (LD and HD phases). For these situations it was found that $\Delta^*J$ grows respectively as $L^{3/2}$ and as $L^2$. We calculated the variance of the current in several points in the phase diagram using the DMRG technique and verified that these results are universal for each phase. Moreover, at the LD–MC transition line we found that $\Delta^*J(0, L, \Delta\alpha) \sim L^{1.50\pm.02}$, consistent with (27) [22]. To be in agreement with the scalings in the LD and MC phases just discussed, $H_2(x)$ should be constant for $x \gg 0$ and $H_2(x) \sim x$ for $x \ll 0$. In figure 6 we show our results for $\Delta^*J(0, L, \Delta\alpha)L^{z-2\gamma}$ as a function of $\Delta\alpha L^{\gamma}$. Here we used the algorithm presented in [38] to collapse the data for the largest $L$-values. As the figure shows, this is possible but with exponents that deviate 10% from the conjectured ones. This is probably because our data are not yet in the asymptotic regime where scaling is expected to hold. Indeed, from the results of figure 3, we expect the scaling in $\Delta\alpha$ to hold from $L \approx 80$ on, and such large $L$-values cannot be reached with the DMRG. The prediction of linear behaviour for $H_2(x)$ when $x$ is negative enough is however clearly verified.

From (26) one can also obtain a scaling form for the variance of the current as a function of $s$. We have also checked that this form is consistent with the DMRG results as already shown in figure 3 of [22].

In conclusion, the scaling assumption (26) allows a consistent description of all exact and numerical data on the average current and its fluctuations. Due to the symmetries of the TASEP we expect a completely similar scaling to hold near the HD–MC transition.

6. Time-dependent current fluctuations

We first discuss our results for the gap $G(s, L)$ near the LD–MC transition and then investigate their implications for the time-dependent current fluctuations.

The behaviour of the gap for $s = 0$ was first determined numerically using the DMRG approach to stochastic operators [34]. Later exact results were obtained from an analysis of the Bethe ansatz equations [36, 37]. In the MC phase, it was found that $G(0, L)$ goes to zero as $L^{-3/2}$, consistent with the criticality of the phase. In the LD and HD phases on the other
hand, the gap goes to a constant indicating a finite relaxation time. Here we study for the first time the gap as a function of $s$.

In figure 7 we show a typical result for different $L$-values at $\alpha = 1/2$, $\beta = 2/3$. As can be seen, the gap approaches $|\alpha - \beta| = 1/6$, the value predicted for large $s$ in section 3, already for rather small $s$-values.

We have found that all the exact and numerical results for the gap can again be described by a scaling function. In analogy with (26) we propose the form

$$G(s, L, \Delta \alpha) = L^{-3/2} F(sL^{3/2}, \Delta \alpha L^{1/2}),$$

where $F$ is a scaling function. This form can describe the exact results at $s = 0$ if $F(0, y)$ goes to a constant for $y \gg 0$ and goes as $y^3$ for $y \ll 0$. Figure 8 shows our numerical results. At $s = 0$ reliable values for the gap can be obtained up to $L \approx 100$. The scaling is already satisfied very well for $L \approx 50$ and the data also show the expected behaviour for large $|y|$.

In figure 9 we show the scaling of the gap at $\Delta \alpha = 0$ as a function of $s$. There is again a good collapse, especially for negative $s$-values.

The scalings that we have found for $\mu^*$ and $G$ imply a scaling for the late time behaviour of the time-dependent current fluctuations. Inserting (26) and (28) into (14), we obtain

$$\langle e^{-sJ(t)} \rangle = A_1 e^{-s(L+1)J^*} [e^{-L^{3/2}tH(sL^{3/2}, \Delta \alpha L^{1/2})}(1 + A_2 e^{-L^{-3/2}tF(sL^{3/2}, \Delta \alpha L^{1/2})} + \cdots)].$$

The whole term within square brackets depends on time through the combination $tL^{-3/2}$, so that we obtain

$$\mu(s, L, t) = -s(L + 1) J^* + L^{-3/2} K(sL^{3/2}, \Delta \alpha L^{1/2}, tL^{-3/2}),$$

where $K$ is once more a scaling function. This equation gives insight into how the time-dependent current and its fluctuations reach their asymptotic NESS value. For the average current at the LD–MC transition, we obtain for example

$$\overline{J}(t, L, \Delta \alpha = 0) = (L + 1)/4 + K_1(tL^{-3/2})$$

for the LD–MC transition.
where $K_1$ is another scaling function. Comparison with (23) gives $K_1(x) \to 1/8$ for $x \to \infty$.

For the variance of the current one finds

$$
\Delta J(t, L, \Delta \alpha) = L^{3/2} K_2(\Delta \alpha L^{1/2}, tL^{-3/2}).
$$

(31)

The time-dependent current and its variance cannot be obtained directly from our DMRG method. Here instead, we have verified relations like (30) and (31) by simulations based on the Gillespie algorithm for $L$ up to 100. In figure 10 we show our data for $J(t, L, \Delta \alpha = 0) - (L + 1)/4$ as a function of $tL^{-3/2}$. The initial condition was an empty lattice.

We simulated $10^4$ histories for $L \leq 50$. For larger system sizes the number of realizations decreased up to 2000 for $L = 100$. As can be seen, the scaling is well satisfied. The inset shows that for $t$ large, the difference $J(t, L, \Delta \alpha = 0) - (L + 1)/4$ approaches the value $1/8$ for increasing $L$-values.
A similar picture arises for the time dependence of the variance of the current though the data are somewhat noisier for the number of histories that we could simulate (see figure 11). Within their accuracy they are consistent with the prediction (31). The variance of the current in the NESS cannot be obtained very precisely from these simulations. While consistent with the DMRG values, the latter are much more precise. This is another advantage of the DMRG in comparison with simulations.
7. Discussion

In this paper we have studied the current fluctuations of the TASEP with open boundaries, both in the NESS and as a function of time.

We have shown that, in the NESS and in the thermodynamic limit, the CGF must be non-analytical at $s = 0$ implying the existence of a spacetime phase transition in the TASEP.

It is interesting to note that in the partially asymmetric exclusion process (PASEP) the behaviour of the CGF may be more complicated. In that model, particles can also jump to the left with a rate $q < 1$, provided that site is empty. Hence, the argument of section 3, which is based on a lower bound for the total current, does not apply. Closely related to this is the property of the PASEP that each history can be time reversed. This implies the existence of a fluctuation theorem for the CGF of the current which for the model on a ring [8] states that $\mu^*(s) = \mu^*(2E - s)$ where $E = -\ln(q)/2$. The CGF is therefore symmetric around $s = E$. This implies that if there is a spacetime transition at some $s_c$, there should be a second one at $2E - s_c$. Behaviour of this kind was observed recently in the CGF for the entropy production of a particle driven in a periodic potential [39]. Whether such a scenario holds for the PASEP deserves further study.

We have proposed the finite size scaling form (29) for the (time-dependent) CGF. Important variables herein are the distance $\Delta \alpha$ from the non-equilibrium phase transition line and $s$. The form proposed is a natural extension of that near a nonequilibrium phase transition and that found for the TASEP on the ring.

One can question how general our scaling ansatz is. Turning to other lattice gases, we note that in the simple symmetric exclusion process (SEPP) on a ring, where particles can jump to the left and the right with equal rates ($q = 1$), it was found recently [40] that the CGF has a form that is rather similar to (20). More precisely, using our notation, one has

$$\mu^*_{\text{SSEP, ring}}(s, L) = \frac{s^2}{2} \Delta^* J(s, L) + \frac{1}{L^2} F\left[\sqrt{\rho(1 - \rho)sL}\right],$$

(32)

where $F$ is a scaling function and $\rho$ is the density of particles. The term in the average current in (20), which is zero in the SSEP on a ring, is replaced here by a term containing the variance of the current. For the critical exponents, one finds $y_s = 1$ and $z = 2$, the latter value being consistent with the diffusive character of this model. For the SSEP with open boundaries, there is again a current flowing through the system. The CGF for the current through the first bond was calculated in [41]. Using a similarity transformation, it is possible to show that for exclusion processes the CGF through all bonds at a given value of $s$ equals that through the first bond at $(L + 1)s$ [48]. Using the results of [41], it then follows that for $L$ large

$$\mu^*_{\text{SSEP}}(s, L) = \frac{1}{L} R(Ls),$$

(33)

where $R$ is a scaling function that also depends on the particle density at the two boundaries. This result seems to be different from the other cases since there is no regular term. One would also expect the same exponent values for the ring and open boundary case. This holds for $y_s$ but not for $z$. It therefore remains an open question how general our finite size scaling ansatz is, and how it can be understood a priori without having to do explicit calculations. A renormalization approach to current fluctuations could give further insight into this issue.

Finally, we observe that a scaling form similar to (21) was found for the cumulant generating form of the activity in the contact process [22, 35]. It would be of interest to investigate the scaling near the finite temperature critical point that was recently observed in a model of a glass-forming liquid with softened kinetic constraints [42].

A similar but more complicated expression can be written down in the open boundary case [37].
We have verified our ansatz using available exact results and numerical approaches based on the DMRG and the Gillespie algorithm. In recent years, several other methods have been introduced to study current fluctuations and the CGF. Among these we mention perturbation techniques [43], renormalization methods [44] and fully numerical approaches such as cloning algorithms [45, 46] and the transition path sampling method [47]. In comparison with the DMRG, these numerical approaches allow larger systems to be studied and they can also more easily be extended to higher dimensions. However, in the cloning method one needs to do simulations for large enough times in order to be in the NESS, whereas the DMRG calculation of the largest eigenvalue immediately gives the CGF in the NESS. Due to the nature of the DMRG algorithm, a calculation for the system size $L$ also gives results for all smaller system sizes. Furthermore, the DMRG approach gives numerically exact results, while simulation methods always include some statistical errors. Finally, the DMRG is the only method (apart from the Bethe ansatz) we know that allows a calculation of the gap, necessary for an understanding of the dynamical scaling of the model.

The TASEP was originally introduced as a model for mRNA translation [12]. We have recently shown how fluctuations in the number of proteins produced by one mRNA can be related to the time-dependent current fluctuations in a model for translation [13]. From the scaling proposed here, it is then possible to derive how, for example, the variance in the number of proteins produced depends on mRNA length. This opens a road to a possible experimental verification of our scaling ansatz.

In this paper we have only studied the two largest eigenvalues of the generalized generator $H(s)$. The DMRG also allows the calculation of the associated left- and right eigenvectors. From these it is possible to obtain expectation values, such as the density at a given site [26]. By tuning the parameter $s$, it is then possible to see how the typical density profile changes for a current that deviates from its average value. We are currently calculating these properties. This is another application where we believe that the DMRG is superior in comparison with simulation methods. The results will be published elsewhere.

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