Chance Constrained Covariance Control for Linear Stochastic Systems With Output Feedback

Jack Ridderhof Kazuhide Okamoto Panagiotis Tsiotras

Abstract—We consider the problem of steering, via output feedback, the state distribution of a discrete-time linear stochastic system from an initial Gaussian distribution to a terminal Gaussian distribution with prescribed mean and maximum covariance, subject to probabilistic path constraints on the state. The filtered state is obtained by a Kalman filter, and the problem is formulated as a deterministic convex program in terms of the distribution of the filtered state. We observe that in the presence of constraints on the state covariance, and in contrast to classical LQG control, the optimal feedback control depends on both the process noise and the observation model. The effectiveness of the proposed approach is verified using a simple numerical example.

I. INTRODUCTION

The problem of covariance steering is a stochastic optimal control problem aiming to design a controller that steers the state covariance of a stochastic system to a target terminal value, while minimizing the expectation of a quadratic function of the state and the input. In this work, we focus on a discrete-time linear time-varying stochastic system with partially observable state, a given Gaussian initial state distribution, and an independent and identically distributed (i.i.d.) standard Gaussian additive diffusion to the dynamics and the measurement.

The infinite horizon covariance control problem for linear time invariant systems has been researched since the late 80’s [1], [2], where the authors investigated the full state-feedback gains that assign a state covariance value to the system, i.e., the system state covariance converges asymptotically to the assigned value. The infinite horizon case has gained recent attention [3], [4], [5], [6], [7], [8]. Chance constraints, which are probabilistic constraints that impose a maximum probability of constraint violation, were first introduced to the covariance control problem in [9]. The latter work draws connections between covariance control and a large class of stochastic control problems for which chance constraints are utilized in order to guarantee performance under uncertainty [10], [11], such as stochastic model predictive control (SMPC) [12], [13] and vehicle path planning in belief space [14], [15].

The majority of covariance control research only considers controlling the state covariance. However, as discussed in [9], when state chance constraints exist, the mean and the covariance are coupled. The authors in [9] introduced the first covariance steering controller that simultaneously deals with the mean and the covariance dynamics such that the resulting trajectories satisfy the state chance constraints.

The approach was further modified to be computationally more efficient in [15], which was eventually extended to deal with input hard constraints [16] and nonlinear dynamics [17]. Furthermore, covariance control theory was applied to autonomous vehicle control in [18] and spacecraft control in [19], [20]. Finally, in [21], the authors applied covariance control theory to SMPC for linear time-invariant systems under unbounded additive disturbance, and they showed the guaranteed stability and recursive feasibility of the controlled system.

The above-mentioned research on covariance control assumes full state feedback. This paper, in contrast, is concerned with covariance steering for the case that the state is only implicitly accessible via noisy measurements.

The problem of output-feedback covariance steering has been visited in [22], [23], where the problem had no constraints other than a terminal boundary constraint. Thus, these works only dealt with the control of the state covariance and did not consider mean dynamics. The proposed approach deals with state chance constraints and simultaneously steers the mean and the covariance of the system state, and thus, can be applied to more realistic scenarios.

A similar problem setup has also been visited from the SMPC community [24], where an output feedback controller was designed to deal with chance constraints. Although the approach in [24] successfully computes control commands that satisfy all the constraints, the control policy suffers from conservativeness due to the convex relaxation of the covariance dynamics. The approach in our work extends the covariance steering controller in [15], which allows a direct assessment of the covariance at each time step and eliminates the need to conduct conservative convex relaxations of the covariance dynamics.

The main contribution of this work is the development of a novel covariance steering control policy for linear systems with Gaussian process and measurement noise. The proposed approach is a nontrivial extension of the full-state feedback covariance control policy proposed in [15], which allows one to directly assess the covariance value at each time step, while converting the original stochastic control problem to a deterministic convex programming problem. We observe that, as a direct consequence of the constraints on the state
covariance, and in contrast to the classical LQG solution, the optimal feedback control depends on both the process noise and the observation model.

Notation

For a sequence \( x = (x_1, \ldots, x_N) = (x_k)_{k=1}^N \), we use the shorthand \((x_k)\) to refer to the sequence and write \(x_k\) to refer to an element of that sequence. We write \(\sigma(z)\) to denote the \(\sigma\)-algebra generated by the random variable \(z\). For a symmetric matrix \(A\), we write \(A > 0\) (\(\geq 0\)) if \(A\) is positive (semi-)definite.

II. PROBLEM STATEMENT

Consider the stochastic discrete-time linear system given by

\[
x_{k+1} = A_k x_k + B_k u_k + G_k w_k,
\]

for \(k = 0, 1, \ldots, N - 1\), where \(x_k \in \mathbb{R}^{n_x}\) and \(u_k \in \mathbb{R}^{n_u}\) are the state and control, and \(A_k \in \mathbb{R}^{n_x \times n_x}\), \(B_k \in \mathbb{R}^{n_x \times n_u}\), and \(G_k \in \mathbb{R}^{n_x \times n_v}\) are system matrices. Steps of the disturbance process \(w_k \in \mathbb{R}^{n_v}\) are i.i.d. standard Gaussian random vectors. The state is measured through the observation process

\[
y_k = C_k x_k + D_k v_k,
\]

where \(y_k \in \mathbb{R}^{n_y}\) is the measurement and \(v_k \in \mathbb{R}^{n_v}\) is measurement noise, and \(C_k \in \mathbb{R}^{n_y \times n_x}\) and \(D_k \in \mathbb{R}^{n_y \times n_v}\) are given. Steps of the measurement noise \(v_k\) are i.i.d. standard Gaussian random vectors, and the matrix \(D_k\) is assumed invertible. The state is initially estimated to be \(\hat{x}_0\) with estimation error \(\tilde{x}_0 = x_0 - \hat{x}_0\). We assume that \(\hat{x}_0\) and \(\tilde{x}_0\) are independent and that \(\hat{x}_0\) and \(\tilde{x}_0\) are distributed as

\[
\hat{x}_0 \sim \mathcal{N}(\hat{x}_0, \bar{P}_0), \quad \tilde{x}_0 \sim \mathcal{N}(0, \bar{P}_0),
\]

where the positive semi-definite matrices \(\bar{P}_0\), \(\bar{P}_0\) and the vector \(\tilde{x}_0\) are all fixed and known. We also assume that \(\hat{x}_0\), \(\tilde{x}_0\), \((u_k)\), and \((v_k)\) are independent.

Define the filtration \((\mathcal{F}_k)_{k=1}^N\) by \(\mathcal{F}_0 = \sigma(\hat{x}_0)\) and \(\mathcal{F}_k = \sigma(\hat{x}_0, y_i : 0 \leq i \leq k)\) for \(0 \leq k \leq N\). This filtration represents the information available to the controller, in the sense that the feedback control input at step \(k\) is \(\mathcal{F}_k\)-measurable random variable. The initial \(\sigma\)-algebra \(\mathcal{F}_0\) is defined for logical consistency, since the initial state estimate is known before any measurements are taken.

Let \(\bar{x}_k = \mathbb{E}(x_k)\) be the mean state, and define the estimated (filtered) state as \(\hat{x}_k = \mathbb{E}(x_k|\mathcal{F}_k)\) and the estimation error as \(\tilde{x}_k = x_k - \hat{x}_k\). The estimated state has mean

\[
\mathbb{E}(\hat{x}_k) = \mathbb{E}(\mathbb{E}(x_k|\mathcal{F}_k)) = \mathbb{E}(x_k) = \bar{x}_k,
\]

and hence the estimation error has zero mean. Define the estimated state and estimation error covariances as

\[
\bar{P}_k = \mathbb{E}((\hat{x}_k - \bar{x}_k)(\hat{x}_k - \bar{x}_k)^\top), \quad \bar{P}_k = \mathbb{E}((\tilde{x}_k - x_k)(\tilde{x}_k - x_k)^\top),
\]

The estimated state is uncorrelated with the estimation error, since

\[
\mathbb{E}(\hat{x}_k \tilde{x}_k) = \mathbb{E}(\hat{x}_k(x_k - \hat{x}_k)) = \mathbb{E}([\hat{x}_k(x_k - \hat{x}_k)|\mathcal{F}_k]) = \mathbb{E}(\tilde{x}_k\mathbb{E}(x_k|\mathcal{F}_k) - \tilde{x}_k) = 0,
\]

and from this expression it can be shown that the state covariance satisfies \(\bar{P}_k = \bar{P}_k + \bar{P}_k\). Define the prior estimated state and prior estimation error as \(\bar{x}_k = \mathbb{E}(x_k|\mathcal{F}_k)\) and \(\tilde{x}_k = x_k - \bar{x}_k\), with the covariances \(\bar{P}_k\) and \(\tilde{P}_k\) as above. It follows that the initial state is distributed as

\[
x_0 \sim \mathcal{N}(\bar{x}_0, \bar{P}_0),
\]

where \(\bar{P}_0 = \bar{P}_0 + \tilde{P}_0\). We require that

\[
\mathbb{P}(x_k \notin \chi) \leq p_{\text{fail}}, \quad k = 0, 1, \ldots, N,
\]

where \(0 < p_{\text{fail}} < 0.5\) is fixed, and where

\[
\chi = \bigcap_{j=1}^{N_\chi} \{x : \alpha_i^j x \leq \beta_j \} \subseteq \mathbb{R}^{n_x},
\]

where \(\alpha_i^j \in \mathbb{R}^{n_x}\) and \(\beta_j \in \mathbb{R}\). This paper is concerned with the following stochastic optimal control problem.

Problem 1: We seek an \((\mathcal{F}_k)\)-adapted (nonanticipative) control sequence \(u = (u_k)_{k=0}^N\) such that the chance constraints (9) are satisfied, and such that the state at step \(N\) is distributed according to \(\mathcal{N}(\bar{x}_f, \bar{P}_N)\) for \(\bar{P}_N = \bar{P}_N + \bar{P}_N \leq P_f\), where \(\bar{x}_f\) and \(P_f\) are given, while minimizing the cost functional

\[
J(u) = \mathbb{E}\left(\sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k\right),
\]

for a given sequence of matrices \((Q_k \geq 0)\) and \((R_k > 0)\).

Remark 1: For simplicity, we do not consider chance constraints on the control. However, the method developed in this work may be easily extended to include chance constraints on the control. See, for instance, [20], [21].

A. Admissible Controls

In the analysis that follows, we restrict the control input at step \(k\) to be an affine function of an \(\mathcal{F}_k\)-measurable random variable which is jointly Gaussian distributed with the state \(x_k\). We say that a control sequence \((u_k)\) is admissible if it satisfies this property for every step \(k\). This ensures that \(x_k\) and \(u_k\) are jointly Gaussian, and thus the state \(x_{k+1}\) will also be Gaussian. Since we assume that the initial state to be Gaussian, it follows by induction that the state remains Gaussian distributed over the entire problem horizon, even in the presence of the chance constraints.

B. Separation of the Observation and Control Problems

Since the system is linear and the state is Gaussian distributed, the estimated state may be obtained by the Kalman Filter, that is, the filtered state satisfies [25]

\[
\hat{x}_k = \bar{x}_k + L_k(y_k - C_k\hat{x}_k),
\]

for a given sequence of matrices \((Q_k \geq 0)\) and \((R_k > 0)\).
where
\[ L_k = \hat{P}_k^{-1} C_k^T (C_k \hat{P}_k^{-1} C_k^T + D_k D_k^T)^{-1} \] (14)
is the Kalman gain, and the error covariances are
\[ \hat{P}_k = (I - L_k C_k) \hat{P}_k (I - L_k C_k)^T + L_k D_k D_k^T L_k^T, \] (15)
\[ \hat{P}_{k-1} = A_k \hat{P}_k A_k^T + G_k G_k^T. \] (16)
We see that the estimation error covariance \( \hat{P}_k \) is deterministic and does not depend on the control. Using properties of conditional expectation, it is easy to show that
\[ \mathbb{E}(x_k^T Q_k x_k) = \text{tr} \hat{P}_k Q_k + \mathbb{E}(\hat{x}_k^T Q_k \hat{x}_k), \] (17)
and therefore the objective may be rewritten as
\[ J(u) = \sum_{k=0}^{N-1} \text{tr} \hat{P}_k Q_k + \hat{J}(u), \] (18)
where
\[ \hat{J}(u) = \mathbb{E} \left( \sum_{k=0}^{N-1} \hat{x}_k^T Q_k \hat{x}_k + u_k^T R_k u_k \right). \] (19)
Since the estimation error covariance \( \hat{P}_k \) is determined by the Kalman Filter and not by the control, optimizing over the objective \( \hat{J}(u) \) is equivalent to optimizing over \( J(u) \). Furthermore, we can determine the distribution of the state as a function of the mean and covariance of the estimated state process, that is,
\[ x_k \sim \mathcal{N}(\bar{x}_k, P_k) \iff \hat{x}_k \sim \mathcal{N}(\tilde{x}_k, P_k - \hat{P}_k). \] (20)
It follows that, in order for the final state covariance to satisfy
\[ 0 < P_N \leq P_f, \]
the maximum final covariance \( P_f \) must satisfy \( P_f > \hat{P}_N \). Define the innovation process \( \tilde{x}_k \) by
\[ \tilde{x}_k = x_k - \mathbb{E}(x_k | \mathcal{F}_{k-1}), \] (21)
for \( k = 0, 1, \ldots, N \). Since
\[ \mathbb{E}(y_k | \mathcal{F}_{k-1}) = \mathbb{E}(C_k x_k + D_k v_k | \mathcal{F}_{k-1}) = C_k \hat{x}_k, \] (22)
we obtain, by substituting the observation model (2) in (21),
\[ \tilde{y}_k = y_k - C_k \tilde{x}_k = C_k \hat{x}_k + D_k v_k. \] (23)
Since \( \tilde{x}_k \) and \( v_k \) are independent, we can compute the covariance of the innovation process as
\[ P_{\tilde{y}_k} = \mathbb{E}(\tilde{y}_k \tilde{y}_k^T) = C_k \hat{P}_k C_k^T + D_k D_k^T. \] (24)
Thus, the distribution of the innovation process is determined by the estimation error covariance \( \hat{P}_k \), and therefore may be computed prior to solving for the control inputs. We rewrite the estimated state process as
\[ \hat{x}_{k+1} = A_k \hat{x}_k + B_k u_{k-1} + L_{k+1} \tilde{y}_{k+1}, \] (25)
where \( \hat{x}_0 = \hat{x}_{\hat{y}} + L_0 \hat{y}_0 \). We have thus replaced the state process (1) with noise term \( G_k b_k \) with a corresponding filtered state process with noise \( L_{k+1} \tilde{y}_{k+1} \). The stochastic optimal control problem may now be posed entirely in terms of the filtered state process (25).

C. Block-Matrix Formulation

The filtered state process (25) may be written in matrix notation as
\[
\begin{bmatrix}
\hat{x}_0 \\
\hat{x}_1 \\
\vdots \\
\hat{x}_N
\end{bmatrix} =
\begin{bmatrix}
I & & & \\
A_0 & B_0 & 0 & \\
A_1 & B_1 & & \\
& & \ddots & \\
& & & B_{N-1}
\end{bmatrix}
\begin{bmatrix}
\hat{y}_0 \\
\hat{y}_1 \\
\vdots \\
\hat{y}_N
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
\ddots \\
0
\end{bmatrix}
\begin{bmatrix}
\hat{u}_0 \\
\hat{u}_1 \\
\vdots \\
\hat{u}_N
\end{bmatrix}.
\] (26)

Let \( \hat{X} \) and \( \tilde{Y} \) be column vectors constructed by stacking \( \hat{x}_k \) and \( \tilde{y}_k \) for \( k = 0, 1, \ldots, N \), and, similarly, let \( U \) be the column vector constructed by stacking \( u_k \) for \( k = 0, 1, \ldots, N \). Formally, we have that the column vector \( \hat{X} \) is isomorphic to the sequence \( (\hat{x}_k) \), which we denote by \( (\hat{x}_k) \cong \hat{X} \). For appropriately constructed block matrices \( A, B, \) and \( L \) as in (26), the filtered state process can be written as the linear matrix equation
\[ \hat{X} = A \hat{y} + BU + L \tilde{Y}. \] (27)

Let \( \hat{X} \) and \( \tilde{Y} \) be column vectors constructed by stacking \( \hat{x}_k \) and \( \tilde{y}_k \) for \( k = 0, 1, \ldots, N \), and, similarly, let \( U \) be the column vector constructed by stacking \( u_k \) for \( k = 0, 1, \ldots, N \). Formally, we have that the column vector \( \hat{X} \) is isomorphic to the sequence \( (\hat{x}_k) \), which we denote by \( (\hat{x}_k) \cong \hat{X} \). For appropriately constructed block matrices \( A, B, \) and \( L \) as in (26), the filtered state process can be written as the linear matrix equation
\[ \hat{X} = A \hat{y} + BU + L \tilde{Y}. \] (27)

See [15], [9] for details on this construction. We may then rewrite the cost function (19) in matrix form as
\[ \hat{J}(u) = \mathbb{E}(\hat{X}^T Q \hat{X} + U^T RU), \] (28)
where \( Q = \text{blkdiag}(Q_0, \ldots, Q_{N-1}, 0) \geq 0 \) and \( R = \text{blkdiag}(R_0, \ldots, R_{N-1}) > 0 \). Let \( E_k \) be a matrix defined so that \( E_k \hat{X} = \hat{x}_k \), and denote the mean state by \( \bar{X} = \mathbb{E}(\hat{X}) \cong (\hat{x}_k) \). We can then write the terminal state distribution constraints as
\[ E_N \bar{X} = \bar{x}_f, \] (29a)
\[ E_N \mathbb{E}[(\hat{X} - \bar{X})(\hat{X} - \bar{X})^T] E_N^T \leq P_f - \hat{P}_N. \] (29b)
Finally, in terms of the column vector \( \tilde{X} \cong (\hat{x}_k) \), the chance constraints (9) may be written as
\[ \mathbb{P}(E_k(\tilde{X} + \tilde{X}) \notin \chi) \leq p_{\text{fail}}, \quad k = 0, 1, \ldots, N. \] (30)

The distribution of \( \tilde{X} + \tilde{X} \) is determined, per (20), by the filtered process (27) and the sequence \( (\hat{P}_k) \), and therefore the probability in (30) depends solely, for fixed problem parameters, on the control sequence \( U \). In summary, we have reformulated the original stochastic optimal control problem (1) in terms of the inaccessible state into the following problem in terms of the accessible filtered state.

Problem 2: Find the nonanticipative admissible control sequence \( U^* \cong (u_k) \) that minimizes the objective (28) subject to the terminal constraints (29), for \( P_f > \hat{P}_N \), and chance constraints (30).
III. CONTROL OF THE FILTERED STATE

Define the process \( z = (z_k) \) by

\[
z_{k+1} = A_k z_k + L_k \hat{y}(k+1),
\]
for \( k = 0, \ldots, N - 1 \), where \( z_0 = z_{00} + L_0 \hat{y}_{00} \), with \( z_{00} = \bar{x}_0 - \tilde{x}_0 \). We have that \( z \) is \( (F_k) \)-adapted and that \( z_0 \) is distributed as \( N(0, P_0) \). The following lemma establishes that the optimal control for Problem 2 may be constructed as a function of this process.

**Lemma 3.1:** Let the control sequence \( u^* = (u^*_k) \) be a solution to Problem 2. Then there exist deterministic feedback gains \( K_{k,i}^* \in \mathbb{R}^{n_x \times n_u} \) and feedforward controls \( m_k^* \in \mathbb{R}^{n_u} \) such that, for all \( 0 \leq k \leq N - 1 \),

\[
u_k^* = \sum_{i=0}^{k} K_{k,i}^* z_i + m_k^*.
\]

**Proof:** Let the control sequence \( u^* = (u^*_k) \) be a solution to Problem 2. In general, the optimal control at step \( k \) may be written as [26]

\[
u_k^* = u_k^*[p(x_k|F_k)],
\]
where \( p(x_k|F_k) \) is the conditional probability density of \( x_k \) on \( F_k \). Since \( u^* \) is admissible, the state is Gaussian and thus its conditional probability density is determined by its mean and covariance. Furthermore, the conditional covariance \( P_k \) of \( x_k \) is deterministic, and therefore only the conditional mean is required to characterize the conditional density. The optimal control can thus be written as a function of the conditional mean \( \bar{x}_k \).

To complete the proof, it suffices to show that \( \bar{x}_k \) is an affine function of \( (z_i^*)_{i=0}^k \), since the composition of affine functions is affine. For the initial step, \( \bar{x}_0 = z_{00} + \bar{x}_0 \) is affine since \( \bar{x}_0 \) is deterministic. Proceeding by induction, suppose that \( \bar{x}_i \) is an affine function of \( (z_j^*)_{j=0}^i \). Substituting (31) into (25), we obtain

\[
\bar{x}_{i+1} = A_i \bar{x}_i + B_i u_i^* + z_{i+1} - A_i z_i.
\]

Since \( u_i^* \) can be given as an affine function of \( x_i \), we have that \( \bar{x}_{i+1} \) is the sum of \( z_{i+1} \) and terms which are affine in \( (z_j^*)_{j=0}^i \). Therefore, by induction, \( \bar{x}_k \) is affine in \( (z_j^*)_{j=0}^k \) for all \( 0 \leq k \leq N - 1 \). It follows that there exist \( K_{k,i}^* \in \mathbb{R}^{n_x \times n_u} \) and \( m_k^* \in \mathbb{R}^{n_u} \) such that \( u^* \) is given as in (32).

From this lemma we have that Problem 2 may be solved by identifying the deterministic sequences \( (K_{k,i}) \) and \( (m_k) \). Furthermore, as we show in the following theorem, this optimization problem is convex.

**Theorem 3.2:** Problem 2 may be solved as a deterministic convex optimization over \( (K_{k,i}) \) and \( (m_k) \), provided a conservative relaxation in the chance constraints (30).

**Proof:** Let \( Z \cong (z_k) \) and \( M \cong (m_k) \) be column vectors defined as \( \bar{X} \) and \( U \), respectively, and let

\[
K = \begin{bmatrix} K_{0,0} & 0 & 0 & \cdots & 0 \\
K_{1,0} & K_{1,1} & \ddots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
K_{N-1,0} & K_{N-1,1} & \cdots & K_{N-1}\end{bmatrix}.
\]

We may then write the processes \( u \) and \( z \) in terms of the matrix equations

\[
U = KZ + M, \quad Z = A \bar{z}_0 + L\bar{Y}.
\]

The filtered state process (27) is thus given by

\[
\bar{X} = A\bar{x}_0 + B(KZ + M) + L\bar{Y}.
\]

Since \( Z \) and \( \bar{Y} \) have zero mean, it follows that

\[
\bar{X} = \mathbb{E}(\bar{X}) = A\bar{x}_0 + BM,
\]
and thus the terminal constraint \( \mathbb{E}(x_N) = \bar{x}_f \) is written as

\[
E_N\bar{X} = E_N(A\bar{x}_0 + BM) = \bar{x}_f,
\]
which is affine in \( M \), and hence convex. Since \( z_{00} = \bar{x}_0 - \tilde{x}_0 \) and \( U - \mathbb{E}(U) = KZ \), we have

\[
\bar{X} - \bar{X} = A(\bar{x}_0 - \tilde{x}_0) + BKZ + L\bar{Y} = (I + BK)Z.
\]

By assumption, \( \bar{x}_0 \) is independent from both \( \tilde{x}_0 \) and \( v_0 \), and therefore by (23) we have that \( \bar{x}_0 \) is independent from \( \bar{Y} \). It follows that \( Z \) has covariance

\[
P_Z = \mathbb{E}(ZZ^\top) = \tilde{A}\tilde{P}_Y A^\top + LP_Y L^\top,
\]
where, since steps of \( (\bar{y}_k) \) are independent [27], the covariance of \( \bar{Y} \) is the block diagonal matrix

\[
P_Y = \mathbb{E}(\bar{Y}\bar{Y}^\top) = \text{blkdiag}(P_{\bar{y}_0}, \ldots, P_{\bar{y}_N}),
\]
where \( P_{\bar{y}_k} \) as in (24). The covariance of the filtered process is thus

\[
\tilde{P} = \mathbb{E}[(\bar{X} - \bar{X})(\bar{X} - \bar{X})^\top] = (I + BK)P_Z(I + BK)^\top,
\]
and the covariance of the control is given by

\[
P_U = \mathbb{E}[(U - \mathbb{E}(U))(U - \mathbb{E}(U))^\top] = KP_Z K^\top.
\]

We can then rewrite the objective (28) as

\[
\bar{J}(K, M) = (A\bar{x}_0 + BM)^\top Q(A\bar{x}_0 + BM) + M^\top RM + \text{tr}[[I + BK]^\top Q(I + BK) + K^\top R K]P_Z],
\]
which is convex in \( K \) and \( M \), because \( Q \geq 0 \) and \( R > 0 \). The terminal covariance constraint (29b) may be written as

\[
E_N(I + BK)P_Z(I + BK)^\top E_N^\top \leq P_f - \tilde{P}_N,
\]
or, equivalently [9],

\[
\|P_Z^{1/2}(I + BK)^\top E_N(P_f - \tilde{P}_N)^{-1/2} \| - 1 \leq 0,
\]
where \( P_Z^{1/2} \) denotes a matrix satisfying \( P_Z = (P_Z^{1/2})^2 P_Z^{1/2} \). The matrix \( (P_f - \tilde{P}_N)^{-1/2} \) exists since \( P_f > \tilde{P}_N \). Next, consider the chance constraint (9). It has been shown in [10] that if constants \( p_j \) for \( j = 1, \ldots, N \) satisfy \( \sum_{j=1}^{N} p_j \leq p_{\text{fail}} \), and if, for all \( j \), \( \mathbb{P}(\alpha_j^x x_k > \beta_j) \leq p_j \), then the chance constraint (9) is satisfied. Therefore, we will proceed to enforce a single half-plane constraint, which has been shown in [15] to be equivalent to the constraint

\[
\text{cdfn}^{-1}(1 - p_j)\|P_f^{1/2}(\alpha_j x_k - \beta_j)\| - 1 \leq 0.
\]
where cdfn1 is the inverse of the cumulative normal distribution function and \( P_k \) is the state covariance at time step \( k \), which we can write as

\[ P_k = E_k \hat{P} k + \hat{P}_k. \]  

(49)

In addition, \( P^{1/2}_k \) satisfies \( (P^{1/2}_k)^T P^{1/2}_k = P_k \) and is obtained by

\[ P^{1/2}_k = \left[ P^{1/2}_k (I + BK)^T E_k \right]. \]  

(50)

Notice that, because each \( p_j < 0.5 \), it follows that cdfn1(1 - \( p_j \)) > 0. Finally, substituting into the chance constraint, we obtain the second order cone constraint

\[ cdfn^{-1}(1 - p_j) \left[ \left[ P^{1/2}_k (I + BK)^T E_k \right] \alpha_j \right] + \alpha_j E_k (A \bar{x}_0 + BM) - \beta_j \leq 0, \quad j = 1, \ldots, N_x. \]  

(51)

The resulting convex problem formulation for Problem 2 is summarized below.

**Problem 3:** Find \( K^* \) and \( M^* \) that minimize the objective (45) subject to constraints (39), (47), (51) for given constants \( p_j \) for \( j = 1, \ldots, N_x \) that sum to less than or equal to \( p_{\text{fail}} \).

A. Discussion on Separation of Estimation and Control

The separation theorem is argued on a similar basis as the present paper [26]. First, a quadratic performance criterion as (11) is separated as in (18). Then the Kalman filter is introduced to compute the conditional expectation \( \mathbb{E}(x_k|\mathcal{F}_k) \), and it is observed that the estimation error covariance is deterministic and does not depend on the control. The resulting filtered state process (25) has the same structure as the state process (1), except that the disturbance term is replaced by the innovation process, which depends on the observation model. Finally, for the LQG problem, the separation principle follows from the observation that the optimal control for a quadratic criterion does not depend on the intensity of the noise; rather, only the optimal cost depends on the noise intensity. Writing the LQG cost in terms of (11) as

\[ J_{\text{LQG}}(u) = J(u) + \mathbb{E}(x_N Q_N x_N) \]

(52)

\[ = J(u) + \text{tr}(Q_N P_N) + \bar{x}_N^T Q_N \bar{x}_N, \]  

(53)

we see that the terminal covariance is penalized by the fixed weight \( Q_N \). On the other hand, if we fix \( P_N = P_f \), then \( Q_N \) would need to take a particular value in order for the resulting cost to be equal to the cost of the corresponding unconstrained problem. In other words, \( Q_N \) acts as a Lagrange multiplier in this optimization problem. Specifically, it has been shown in [7] that, in the absence of path constraints, there exists a \( Q_N \) determined by the initial covariance, desired final covariance, and disturbance terms such that the LQG solution satisfies \( P_N = P_f \). But by this procedure the control cost is made a function of the noise [3], [15], and hence, when considering the control of the filtered process, the control depends on the observation model through the statistics of the innovation process.

IV. NUMERICAL EXAMPLE

Consider for \( \Delta t = 0.2 \) the following double integrator system with the horizon \( N = 20 \) given by, for all \( k \),

\[ A_k = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_k = \begin{bmatrix} \Delta t^2/2 & 0 \\ 0 & \Delta t^2/2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix}, \]  

(54)

and \( G_k = 0.01 \times I_3, \quad C_k = [0_{3 \times 1} \quad I_3], \quad D_k = \text{diag}(0.1, 0.003, 0.003), \quad G_k = 0.01 \times I_3 \). The initial state distribution is described by \( P_0 = \text{diag}(2, 1, 1.4, 1.4) \times 10^{-2} \), \( P_0 = \text{diag}(8, 9, 0.6, 0.6) \times 10^{-2} \), and the target state distribution is constrained to have mean \( \bar{x}_f = [6.5, 1.5, 0, 0]^T \) and maximum covariance \( P_f = \text{diag}(6, 6, 0.6, 0.6) \times 10^{-2} \). Finally, the region \( \chi \) is defined for \( N_\chi = 2 \) half plane constraints as in (10) given by \( \alpha_1 = [1, 5, 0, 0]^T, \quad \alpha_2 = [1, 1, 0, 0]^T, \quad \beta_1 = 27.5, \quad \beta_2 = 9, \quad p_1 = p_2 = 5 \times 10^{-4}, \quad p_{\text{fail}} = p_1 + p_2 \). The resulting trajectory of the distribution of the position coordinates are shown in Figure 1. In this example it is clear that the resulting control depends on the observation model. Since the first position coordinate is not directly measured, there is a larger uncertainty in the estimated value of the first position coordinate compared to the second position coordinate. The controller compensates accordingly by using sufficient control effort along the first position coordinate so that the chance constraints are satisfied. We can see this by observing the shape of the 3\( \sigma \) ellipse of the filtered state covariance \( \hat{P}_k \) in the bottom plot of Figure 1.

V. CONCLUSIONS

In this paper, we have developed a covariance steering control policy for discrete-time linear stochastic systems with Gaussian process and measurement noise. The filtered state was obtained by a Kalman filter, and then, in terms of the filtered state, the covariance steering problem was posed as a deterministic convex optimization problem. It was observed that, due to the covariance-based constraints on the state distribution, the resulting optimal control depends on both the process noise and measurement model. Finally, the developed theory was demonstrated for a numerical example.

In future work, the proposed approach can be extended to vehicle path planning problems under Gaussian disturbance and measurement uncertainty [28], [29]. In particular, the proposed approach can be extended to handle a non-convex feasible region as in [15], where the authors converted the original stochastic vehicle path planning problem to a deterministic mixed integer convex programming problem.

ACKNOWLEDGMENT

This work of the first author was supported by a NASA Space Technology Research Fellowship. The work of the second and third authors was supported by NSF award CPS-1544814, by ARL under DCIST CRA W911NF-17-2-0181, and by ONR award N00014-18-1-2828. The authors would also like to thank Dipankar Maity for many helpful discussions and suggestions.
Fig. 1. On the top, $3\sigma$ covariance ellipses drawn at each step; on the bottom, detail of $3\sigma$ covariance ellipses for steps $k = 11$ and $k = N = 20$. Arrows indicate the direction of motion and the compliment of $\chi$ is marked by diagonal lines. $x_k^{(1)}$ and $x_k^{(2)}$ denote the first two coordinates of the state.

REFERENCES

[1] A. F. Hotz and R. E. Skelton, “A covariance control theory,” in IEEE Conference on Decision and Control, vol. 24, (Fort Lauderdale, FL), pp. 552–557, Dec. 11 – 13, 1985.

[2] A. Hotz and R. E. Skelton, “Covariance control theory,” International Journal of Control, vol. 46, no. 1, pp. 13–32, 1987.

[3] Y. Chen, T. T. Georgiou, and M. Pavon, “Optimal steering of a linear stochastic system to a final probability distribution, Part I,” IEEE Transactions on Automatic Control, vol. 61, pp. 1158–1169, May 2016.

[4] E. Bakolas, “Optimal covariance control for discrete-time stochastic linear systems subject to constraints,” in IEEE Conference on Decision and Control, (Las Vegas, NV), pp. 1153–1158, Dec. 12 – 14, 2016.

[5] E. Bakolas, “Finite-horizon separation-based covariance control for discrete-time stochastic linear systems,” in IEEE Conference on Decision and Control, (Miami Beach, FL), pp. 3299–3304, Dec. 17 – 19, 2018.

[6] A. Beghi, “On the relative entropy of discrete-time Markov processes with given end-point densities,” IEEE Transactions on Information Theory, vol. 42, no. 5, pp. 1529–1535, 1996.

[7] M. Goldshtein and P. Tsioftras, “Finite-horizon covariance control of linear time-varying systems,” in IEEE 56th Annual Conference on Decision and Control, (Melbourne, Australia), pp. 3606–3611, Dec. 12 – 15, 2017.

[8] A. Halder and E. D. Wendel, “Finite horizon linear quadratic Gaussian density regulator with Wasserstein terminal cost,” in American Control Conference, (Boston, MA), pp. 7249–7254, July 6 – 8, 2016.

[9] K. Okamoto and P. Tsioftras, “Optimal stochastic vehicle path planning using covariance steering,” IEEE Robotics and Automation Letters, vol. 4, no. 3, pp. 2276–2281, 2019.

[10] L. Blackmore and M. Ono, “Convex chance constrained predictive control without sampling,” in AIAA Guidance, Navigation, and Control Conference, (Chicago, IL), Aug. 10 – 13, 2009.

[11] K. Okamoto, M. Goldshtein, and P. Tsioftras, “Optimal covariance control for stochastic systems under chance constraints,” IEEE Conference on Decision and Control, vol. 27, no. 6, pp. 1080–1084, 2011.

[12] K. Okamoto, M. Goldshtein, and P. Tsioftras, “Optimal covariance control for stochastic systems under chance constraints,” IEEE Conference Systems Letters, vol. 2, pp. 266–271, April 2018.

[13] K. Okamoto and P. Tsioftras, “Input hard constrained optimal covariance steering,” in IEEE Conference on Decision and Control, (Nice, France), Dec 11 – 13, 2019. Preprint available: arXiv: 1903.10964.

[14] Y. Chen, T. Georgiou, and M. Pavon, “Steering state statistics with output feedback,” in IEEE Conference on Decision and Control, (Nice, France), Dec 11 – 13, 2019. Preprint available: arXiv: 1903.10919.

[15] N. Chohan, M. A. Nazari, H. Wymeersch, and T. Charalambous, “Robust trajectory planning of autonomous vehicles at intersections with communication impairments,” in Allerton Conference on Communication, Control, and Computing, (Monticello, IL), Sept 24 – 27, 2019.

[16] J. Ridderhof and P. Tsioftras, “Uncertainty quantification and control during Mars powered descent and landing using covariance steering,” in IEEE Conference on Decision and Control, (Kissimmee, FL), Jan. 8 – 12, 2018.

[17] J. Ridderhof and P. Tsioftras, “Minimum-fuel powered descent in the presence of random disturbances,” in AIAA Guidance, Navigation, and Control Conference, (San Diego, CA), Jan. 7 – 11, 2019.

[18] K. Okamoto and P. Tsioftras, “Stochastic model predictive control for constrained linear systems using optimal covariance steering,” preprint arXiv:1905.13296, 2019. (Under Review).

[19] E. Bakolas, “Covariance control for discrete-time stochastic linear systems with incomplete state information,” in American Control Conference, (Seattle, WA), pp. 432–437, May 24 – 26, 2017.

[20] K. Okamoto and P. Tsioftras, “Stochastic linear model predictive control with chance constraints—a review,” Journal of Process Control, vol. 24, (Fort Lauderdale, FL), pp. 432–437, May 24 – 26, 2017.

[21] K. Okamoto and P. Tsioftras, “Optimal covariance control for stochastic systems subject to constraints,” in IEEE Conference on Decision and Control, (Seattle, WA), Jan. 7 – 11, 2019.

[22] K. Okamoto and P. Tsioftras, “Stochastic model predictive control for constrained linear systems using optimal covariance steering,” arXiv preprint arXiv:1905.13296, 2019. (Under Review).

[23] K. Okamoto and P. Tsioftras, “Stochastic linear model predictive control with chance constraints—a review,” Journal of Process Control, vol. 24, (Seattle, WA), pp. 432–437, May 24 – 26, 2017.