The classical Kepler problem and geodesic motion on spaces of constant curvature

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Abstract. In this paper we clarify and generalise previous work by Moser and Belbruno concerning the link between the motions in the classical Kepler problem and geodesic motion on spaces of constant curvature. Both problems can be formulated as Hamiltonian systems and the phase flow in each system is characterised by the value of the corresponding Hamiltonian and one other parameter (the mass parameter in the Kepler problem and the curvature parameter in the geodesic motion problem). Using a canonical transformation the Hamiltonian vector field for the geodesic motion problem is transformed into one which is proportional to that for the Kepler problem. Within this framework the energy of the Kepler problem is equal to (minus) the curvature parameter of the constant curvature space and the mass parameter is given by the value of the Hamiltonian for the geodesic motion problem. We work with the corresponding family of evolution spaces and present a unified treatment which is valid for all values of energy continuously. As a result, there is a correspondence between the constants of motion for both systems and the Runge-Lenz vector in the Kepler problem arises in a natural way from the isometries of a space of constant curvature. In addition, the canonical nature of the transformation guarantees that the Poisson bracket Lie algebra of constants of motion for the classical Kepler problem is identical to that associated with geodesic motion on spaces of constant curvature.

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1. Introduction

It is well known that the classical Kepler problem is an example of a dynamical system with a “hidden” dynamical symmetry: The system is obviously spherically symmetric but the existence of the Runge-Lenz vector means the Lie algebra of constants of motion can be extended to so(4), iso(3) and so(3, 1) for negative, zero and positive energy orbits respectively. Thus there is a one-to-one correspondence between the dynamical symmetry algebras of motion in the classical Kepler problem (with Hamiltonian function $H$) and geodesic motion on three-dimensional spaces of constant curvature (with Hamiltonian function $G$) for each energy surface (i.e., the three-sphere $S^3$ with positive curvature, Euclidean three-space $E^3$ with zero curvature and the three-hyperboloid $H^3$ with negative curvature).
Moser [1] addressed the geometrical nature of the energy surface for the Kepler problem for negative values of energy and claims that for a negative constant $E$, the energy surface $H = E$ can be mapped topologically one-to-one into the unit tangent bundle of the $n$-dimensional sphere $S^n$ with the north pole excluded. The flow defined by the Kepler problem is mapped into the geodesic flow on the punctured sphere after a change of independent variable. This is extended by Belbruno [2] to include positive energy orbits $E > 0$ in the Kepler problem and the three-hyperboloid $H^3$ and also for the case with $E = 0$ and three-dimensional Euclidean space $E^3$. Osipov [3] also tackles the case of positive energies $E > 0$. More recently, Mesón and Vericat [4] have considered the case of a repulsive field.

In this paper we extend the results of Moser [1] and Belbruno [2] by introducing a family of evolution spaces rather than considering a single Kepler problem. We then present a transformation allowing us to relate this space of all classical Kepler problems with distinct mass parameters $\alpha$ to the space of all Hamiltonian systems of free particle motion on spaces of constant curvature $k$. Thus the phase flow in each system is characterised by two parameters, the constant energy $H = E$ (or $G = C$) and an additional constant $\alpha$ (or $k$). We demonstrate (via a convenient choice of canonical coordinates) that the phase flows for the two systems can be considered parallel. Within this framework the energy of the Kepler problem is equal to (minus) the curvature parameter of the constant curvature space and the mass parameter is given by the value of the Hamiltonian for the geodesic motion problem. The fact that the Hamiltonian vector fields can be considered to be parallel ensures the correspondence of the constants of motion for both systems. This result gives the Runge-Lenz vector a “purely geometrical” interpretation in that it arises from isometries of a three-dimensional manifold associated with the Kepler Hamiltonian system. In addition, the canonical nature of the transformation guarantees the correspondence between the Poisson bracket Lie algebras of constants of motion on the constant energy hypersurfaces of the two systems. Our analysis is valid for all values of energy $E$ continuously and is thus a unified treatment which is independent of the sign of the energy. This is in contrast to the work of Moser [1] and Belbruno [2] which dealt with a single Kepler problem and necessitated the use of a different procedure for each sign of energy.

The dynamical features of the Kepler problem can be analysed using alternative techniques. Iwai [5] defines a four-dimensional conformal Kepler problem in order to associate the three-dimensional Kepler problem with the four-dimensional harmonic oscillator. The Hamiltonian vector fields corresponding to the motion in the four-dimensional conformal Kepler problem and the four-dimensional harmonic oscillator are shown to be parallel on appropriate energy surfaces and then the four-dimensional conformal Kepler problem is reduced to the three-dimensional Kepler problem. Thus the (negative) energy surface in the three-dimensional Kepler problem is obtained together with the $SO(4)$ symmetry from an appropriate energy surface of the four-dimensional harmonic oscillator. Mladenov [6] applies this technique to the MIC-Kepler problem (motion in the dual charged Coulomb field modified by a centrifugal term) and then
applies a geometric quantization scheme to the extended phase space of the MIC-Kepler problem. Guillemin and Sternberg [7] show that the Kepler motion can be enlarged to geodesic flow on a curved Lorentzian five-dimensional manifold. In their work the mass parameter $\alpha$ is directly related to a conjugate momentum coordinate in the cotangent bundle. The advantage of the method described in the present work is that the dynamical features of the Kepler problem are derived directly from those of a system with the same dimension. In addition, the features of the classical Kepler problem have a direct geometrical origin in that they arise from the properties of the geodesics of a three-dimensional manifold of constant curvature.

There is an extensive amount of material in the literature relating to the geometry of the Kepler problem and we refer the reader to Guillemin and Sternberg [7] and Milnor [8] for an overview.

We briefly outline some concepts necessary for the description of Hamiltonian systems in section 2. In section 3 we discuss the classical Kepler problem and its symmetries and constants of motion. Geodesic motion on constant curvature spaces and canonical transformations of the associated phase space are considered in section 4. Then in section 5 we construct a map between these two dynamical systems and clarify their relationship.

2. Hamiltonian systems

A symplectic manifold $N$ endowed with a closed nondegenerate symplectic two-form $\tilde{\omega}$ is denoted $(N, \tilde{\omega})$. Let us introduce the $2n$ coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n)$. Then the canonical one-form $\tilde{p}$ on the symplectic manifold $(N, \tilde{\omega})$ is defined as

$$\tilde{p} = p_i \, dx^i,$$

and the canonical symplectic two-form

$$d\tilde{p} = dp_i \wedge dx^i.$$

The symplectic two-form $\tilde{\omega}$ can always be written locally as $\tilde{\omega} = d\tilde{p}$. For a symplectic manifold $(N, \tilde{\omega})$, the Hamiltonian vector field $\tilde{X}_f$ corresponding to the function $f$ is defined as the unique smooth vector field on $N$ satisfying

$$\tilde{\omega}(\tilde{X}_f) = -df.$$

A Hamiltonian system is a symplectic manifold $(N, \tilde{\omega})$ endowed with a Hamiltonian function $H$ and denoted $(N, \tilde{\omega}, H)$.

Consider the direct product space $W = \mathbb{R} \times N$ which is a $2n + 1$ dimensional manifold locally described by the coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n, \tau)$. Then we can define a closed two-form on $W$ by

$$\Omega = dp_i \wedge dx^i - dH \wedge d\tau,$$

where $H$ is a function on $W$. Then $(W, \Omega)$ is said to be an evolution space, see Andrié [9] for further details.
3. The classical Kepler Problem

The classical Kepler problem is the motion of a particle in Euclidean (configuration) space $E^3$ under a central inverse square force $-\alpha/|x|^2$. The singularity $|x| = 0$ is removed from the manifold $E^3$: The configuration space is taken to be $E^3 - \{0\}$. The corresponding phase space is the cotangent bundle $(E^3 - \{0\}) \times \mathbb{R}^3$. Thus the classical Kepler problem is the Hamiltonian system $(N, \tilde{\omega}, H)$ where $N$ is the cotangent bundle $(E^3 - \{0\}) \times \mathbb{R}^3$ and $H$ is the Hamiltonian function

$$H = \frac{|p|^2}{2} - \frac{\alpha}{|x|}$$

with $\alpha$ a constant. The Hamiltonian vector field corresponding to the phase flow is

$$\hat{X}_H = p_i \frac{\partial}{\partial x^i} - \alpha \frac{x^j}{|x|^3} \frac{\partial}{\partial p_j}$$

where $i, j = 1, 2, 3$. The system $(N, \tilde{\omega}, H)$ only shares the rotational symmetry of the underlying base manifold $E^3$, however it does admit an extra set of constants of motion $A_i$. These quantities are the components of the Runge-Lenz vector which determines the orientation of the major axis in the orbital plane. The quantities

$$J_i = \varepsilon_{ij} x^j p_k,$$

$$\sqrt{2}A_i = x^i \left( |p|^2 - \frac{\alpha}{|x|} \right) - p_i (x \cdot p)$$

are constants of motion for the system $(N, \tilde{\omega}, H)$, i.e., $\hat{X}_H(J_i) = \hat{X}_H(A_i) = 0$. The quantities $A_i$ are quadratic in the momenta and so do not arise as a result of any simple Killing vector symmetry of the manifold $E^3$.

Finally, we note that the Poisson brackets of the constants of motion $J_i$ and $A_i$ are

$$\{J_i, J_j\} = -\varepsilon_{ij} J_k,$$

$$\{J_i, A_j\} = -\varepsilon_{ij} A_k,$$

$$\{A_i, A_j\} = H \varepsilon_{ij} J_k.$$

Therefore, on constant energy hypersurfaces $H = E$ the Lie algebra given by the Poisson brackets is isomorphic to $so(4)$ for the case $E < 0$, $so(3,1)$ for $E > 0$, and $iso(3)$ for $E = 0$. Thus there is a one-to-one correspondence between the dynamical symmetry algebras of motion in the classical Kepler problem and geodesic motion on three-dimensional spaces of constant curvature.

4. Geodesic motion on spaces of constant curvature

We shall now present the Hamiltonian function, phase flow and constants of motion for geodesic motion on spaces of constant curvature and indicate how they are related to those corresponding to classical Kepler motion. We can write the line element for a three-dimensional space of constant curvature $k$ in terms of the stereographic coordinates $\{x^i\}, i = 1, 2, 3$ as

$$ds^2 = \left( 1 + \frac{k|x|^2}{4} \right)^{-2} ds_E^2$$
where $ds^2 = \delta_{ij} dx^i dx^j$ and $|x|^2 = \delta_{ij} x^i x^j$. We denote the generic three-geometry as $G^3(k)$.

Geodesic motion on such a space can be described by a Hamiltonian

$$G = \frac{1}{2} \left(1 + \frac{k|x|^2}{4}\right)^2 |p|^2$$

where the three-vectors $x, p$ represent the position and velocity vectors, respectively, of the particle. Thus we label this Hamiltonian system $(\mathcal{N}, \bar{\omega}, G)$ where $\bar{\omega} = dp_i \wedge dx^i$.

The homogeneity and isotropy of these three-dimensional spaces give rise to a six-dimensional group of isometries and the corresponding Killing vectors form a six-dimensional Lie algebra. The subgroup $SO(3)$ is common to all three types of geometry $S^3, E^3$ and $H^3$. A basis for the associated Lie algebra of Killing vectors is

$$R_i = \epsilon_{ij} x^j \frac{\partial}{\partial x^k}.$$  

The homogeneity is granted through invariance of the metric (6) under transitive motions. However, the transitive subgroup is different according to the value of $k$. A basis for the associated Lie algebra of Killing vector fields is

$$P_i = \left(1 - \frac{k|x|^2}{4}\right) \frac{\partial}{\partial x^i} + \frac{k}{2} x^i \left(x^j \frac{\partial}{\partial x^j}\right).$$

The associated constants of motion $L_i = R_i \cdot p$ and $D_i = P_i \cdot p$ are respectively

$$L_i = \epsilon_{ij} x^j p_k,$$

$$D_i = \left(1 - \frac{k|x|^2}{4}\right) p_i + \frac{k}{2} x^i (x \cdot p),$$

satisfying $\dot{X}_G(L_i) = \dot{X}_G(D_i) = 0$. The constants of motion have Poisson brackets with structure constants identical to those of the Killing vectors and are as follows

$$\{L_i, L_j\} = -\epsilon_{ij}^k L_k, \quad \{L_i, D_j\} = -\epsilon_{ij}^k D_k,$$

$$\{D_i, D_j\} = -k \epsilon_{ij}^k L_k.$$  

Thus, the functions $L_i, D_j$ form a Lie algebra under the Poisson bracket operation depending on the value of $k$. It can be seen that the Lie algebra given by the Poisson brackets is isomorphic to the Lie algebra $so(4)$ for the case $k > 0$, $so(3,1)$ for $k < 0$, and $iso(3)$ for $k = 0$.

We now implement a canonical transformation on the phase space as described in the Appendix. The Hamiltonian function becomes

$$G = \frac{1}{4} \left(k + \frac{|\bar{p}|^2}{2}\right)^2 |\bar{x}|^2$$

and the Hamiltonian vector field corresponding to the phase flow is

$$\dot{X}_G = G^\frac{1}{2} |\bar{x}| \left(p_i \frac{\partial}{\partial \bar{x}^i} - 2 G^\frac{1}{2} \frac{\bar{x}^j}{|\bar{x}|^2} \frac{\partial}{\partial \bar{p}_j}\right).$$


We note that $\hat{X}_G$ has a similar form to that for Kepler motion (3), except for the factor $G^{\frac{1}{2}}|\vec{x}|$. The constants of motion are

$$L_i = \epsilon_{ij}^{\frac{k}{2}} \bar{x}^i \bar{p}_k,$$

$$\sqrt{2}D_i = \bar{x}^i \left( |\bar{p}|^2 - \frac{2G^{\frac{1}{2}}}{|\vec{x}|} \right) - \bar{p}_i (\bar{x} \cdot \bar{p}),$$

(15)

which are similar to those associated with Kepler motion presented in (4). At this point we note that (13) can easily be rearranged to give

$$-k = \frac{|\bar{p}|^2}{2} - \frac{2G^{\frac{1}{2}}}{|\vec{x}|},$$

(16)

which resembles the Kepler problem Hamiltonian function. Expressions (13) - (16) form the basis of the mapping discussed in section 5.

5. Related dynamical systems

Consider for the moment the constant energy surface $G = C$ in system $(\bar{N}, \bar{\omega}, G)$. If we identify the constants $\alpha = 2C^{\frac{1}{2}}$ then the Hamiltonian vector fields (3) and (14) have the same form, apart from a factor $C^{\frac{k}{2}}|\vec{x}|$. Thus if we can identify the phase space coordinates under some map then, for the constant energy surface $H = E$ in the Hamiltonian system $(N, \tilde{\omega}, H)$, we can say that $E = -k$ and we can see that the Hamiltonian vector fields can be considered parallel. We point out that the case where $\alpha = 2C^{\frac{1}{2}} = 0$ must be excluded from this analysis. If we write $\hat{X}_G = d/d\lambda$ and $\hat{X}_H = d/d\tau$, then the relationship between the Hamiltonian vector fields (3) and (14) can be written

$$\frac{d}{d\lambda} = C^{\frac{k}{2}}|\vec{x}| \frac{d}{d\tau}$$

(17)

and so the time parameters are related by the following $d\tau/d\lambda = C^{\frac{k}{2}}|\vec{x}|$. This parameter change is part of Moser’s transformation, see equation (2.8) in [1]. It can also be seen that upon this identification, the quantities $A_i$ and $D_i$ have precisely the same form. It is clear that the quantities $J_i$ and $L_i$ have the same form.

Thus we can see from (16) that such a mapping will result in the energy $E$ in the Kepler problem being given by $E = -k$, i.e., the whole of phase space for motion on a space with curvature $k$ will be mapped to energy surfaces for Kepler problems with different mass parameters $\alpha$ but with the same energy $E$. It then follows from this that energy surfaces $G = C$ for motion on a space of fixed $k$ will be mapped to energy surfaces $E = -k$ for Kepler motion with fixed mass parameter $\alpha = 2C^{\frac{k}{2}}$, i.e., energy surfaces are mapped to energy surfaces as in Moser [1] and Belbruno [2].

We shall now formulate these notions rigourously and consider the implications of this result. First we shall construct a map $\psi$ between the $2n$-dimensional phase spaces $N$ and $\bar{N}$

$$\psi : \bar{N} \mapsto N$$

(18)
such that $x^i = \bar{x}^i$ and $p_j = \bar{p}_j$. Thus we have that $dp_i \wedge dx^i = d\bar{p}_j \wedge d\bar{x}^j$. Thus the manifolds $N$ and $\bar{N}$ have the same symplectic structure.

Now we can construct the $2n + 1$ dimensional evolution space $W^\alpha = \mathbb{R} \times N$ with closed two-form

$$\Omega = dp_i \wedge dx^i - dH \wedge d\tau,$$

where $\tau$ is the time parameter for the system. This evolution space is characterised by the parameter $\alpha$ via the Hamiltonian function $H$. Thus we can construct the family of evolution spaces, i.e., the fibre bundle $\mathcal{F} = \mathbb{R} \times W^\alpha$. This fibre bundle can be regarded as the space of all Kepler problems, each characterised by the value of $\alpha$. Similarly we can construct the evolution space $\bar{W}^k = \mathbb{R} \times \bar{N}$ with closed two-form

$$\bar{\Omega} = d\bar{p}_i \wedge d\bar{x}^i - dG \wedge d\lambda$$

characterised by the constant $k$. The corresponding family of evolution spaces is $\bar{\mathcal{F}} = \mathbb{R} \times \bar{W}^k$. This fibre bundle can be regarded as the space of all systems of geodesic motion on spaces of constant curvature.

Now, $\bar{\mathcal{F}}|_k = \bar{W}^k$ is a hypersurface in $\bar{\mathcal{F}}$ corresponding to a particular value of $k$. We can define an energy hypersurface via the map $\bar{\phi} : \bar{\mathcal{F}}|_C \mapsto \bar{\mathcal{F}}|_k$ to be the surface defined by $G = C$, i.e.,

$$-k = |\bar{p}|^2/2 - 2C^{1/2}/|\bar{x}|.$$  

(21)

Similarly, $\mathcal{F}|^\alpha = W^\alpha$ is a hypersurface in the fibre bundle $\mathcal{F}$ and we can define the energy surface via the map $\phi : \mathcal{F}|_E \mapsto \mathcal{F}|^\alpha$ to be the surface $H = E$, i.e.,

$$E = |p|^2/2 - \alpha/|x|.$$  

(22)

Alternatively, we can define a hypersurface in the fibre bundle $\mathcal{F}$ corresponding to the space of all orbits with energy $H = E$ via a map $\pi : \mathcal{F}|_E \mapsto \mathcal{F}$.

We then define a map $\Phi$ between energy hypersurfaces in the respective evolution spaces as follows

$$\Phi : \mathcal{F}|_k \mapsto \mathcal{F}|_E$$  

(23)

such that $E = -k$. Thus under the map $(\Phi \circ \psi)$ we find that $\alpha = 2C^{1/2}$ and that the surface $\bar{\mathcal{F}}|_C^\alpha$ is mapped into the surface $\mathcal{F}|_E^\alpha$. Under the the map $\psi$ (18) we find that $\bar{\omega} = \bar{\omega}$ and so $(\Phi \circ \psi)^* \bar{\omega} = \bar{\omega}$ and under the map $\Phi$ we have as a result of (17) that $dH \wedge d\tau = dG \wedge d\lambda$ and so

$$(\Phi \circ \psi)^* \bar{\Omega} = \Omega.$$  

(24)

Now consider the surface $\bar{\mathcal{F}}|_C^\alpha$, in $\bar{\mathcal{F}}$. We have established that the quantities $L_i$ and $D_j$ have Poisson brackets given by (12) and that the symplectic two-form $\bar{\omega}$ is preserved under the map $(\Phi \circ \psi)$. Since the form of the quantities $L_i$ and $D_j$ are preserved under the map $(\Phi \circ \psi)$, these quantities have the same poisson brackets on the surface $\mathcal{F}|_E^\alpha$ in $\mathcal{F}$. Under the map we can write

$$(\Phi \circ \psi)_* \hat{X}_H = \frac{1}{C_1^{1/2}/|x|} \hat{X}_G.$$  

(25)
and so the quantities $L_i$ and $D_j$ will also be constants of the motion on the energy surface $H = E = -k$, $\alpha = 2C^\frac{1}{2}$ in the system $(N, \tilde{\omega}, H)$. Since the poisson structure is preserved then their Poisson bracket Lie algebra will be identical to that in (12). It is indeed the case that the quantities $J_i = L_i$ and $A_i = D_i$ under the map $(\Phi \circ \psi)$.

Thus, given any point $(\bar{x}^i, \bar{p}_j, \lambda, k)$ in the fibre bundle $\bar{F}$, we have defined the corresponding point $(x^i, p_j, \tau, \alpha)$ in the Kepler problem fibre bundle $F$ by $x^i = \bar{x}^i$, $p_j = \bar{p}_j$, $\alpha = +2C^\frac{1}{2}$ and $\tau = \tau(\lambda, x^i, p_j)$ from relation (17). This mapping is injective. The cases $\alpha > 0$ and $\alpha < 0$ must be treated separately and in the latter case we would define a map with $\alpha = -2C^\frac{1}{2}$. In a sense, the Kepler problem is the “square root” of geodesic motion on spaces of constant curvature in the same way as spinors are the “square root” of tensors. We note that this suggests considering the Kepler problem in a complex space. Two-body motion with a central repelling field, that is, the case where $\alpha < 0$, necessarily implies that $H > 0$ and so $k < 0$ which corresponds to the hyperbolic geometry $H^3$.

Thus the phase flows in the surfaces $\bar{F}_{|E}^k$ are mapped into those in $F|_E^C$ where $E = -k$ and $\alpha = +2C^\frac{1}{2}$ (or $\alpha = -2C^\frac{1}{2}$) and so the constants of motion in the first will be mapped into constants of the motion in the second. The formalism presented above explains why the Poisson bracket Lie algebra of constants of motion on constant energy surfaces in the classical Kepler problem is identical to that for the constants for geodesic motion on spaces of constant curvature. In addition it is clear that replacing the constant $k$ by the function $-H$ in the constants of motion $D_i$ gives the components of the Runge-Lenz vector $A_i$. Hence, the constants of motion $A_i$ in the classical Kepler problem arise in a natural way from the transitive isometries of the associated spaces of constant curvature. Furthermore, the commutation relations amongst the constants of motion in the Kepler problem $(N, \tilde{\omega}, H)$ can be thought of as being inherited from those in $(\bar{N}, \bar{\omega}, G)$, i.e., the commutation relations (12), when restricted to the appropriate energy surface.

So far we have excluded the singularity at the origin $|x| = 0$ in the Kepler problem. This is equivalent to exclusion of certain points in the corresponding spaces $G^3(k)$. It is now straightforward to reinstate these points thereby regularising the problem. For example, the $E < 0$ energy surface in the Kepler problem is mapped to the tangent bundle of the sphere $S^3$ punctured at one point, the north pole. The $E < 0$ Kepler problem energy surface is compactified when we include this point and so the geodesics through the north pole are transformed into collision orbits.

6. Conclusions

The formalism we have presented generalises and we hope, clarifies previous work. The main difference is that there are two parameters in each problem: the mass parameter $\alpha$ and the value of the Hamiltonian $H$ in the Kepler problem and the curvature $k$ and the value of the Hamiltonian $G$ in the system of geodesic motion on spaces of constant curvature. In this paper we have constructed the family of evolution spaces
corresponding to both problems and via a suitable map shown the equivalence of the flows in the two problems [equations (24) and (25)]. Within this framework the energy of the Kepler problem is equal to (minus) the curvature parameter of the constant curvature space, i.e., $H = -k$, and the mass parameter is given by the value of the Hamiltonian for the geodesic motion problem via the relation $\alpha = 2G^2$. This ensures the correspondence of the constants of motion for both systems. This result gives the Runge-Lenz vector a “purely geometrical” interpretation in that it arises from isometries of a three-dimensional manifold associated with the Kepler Hamiltonian system. In addition, the canonical nature of the transformation guarantees the correspondence between the Poisson bracket Lie algebras of constants of motion on the constant energy hypersurfaces of the two systems. We have presented a unified treatment which is valid for all values of energy continuously. This is in contrast to the work of Moser [1] and Belbruno [2] which dealt with a single Kepler problem and necessitated the use of a different procedure for each sign of energy. Our method highlights the relationship between the two phase flows in a geometrical way and we have used clear and simple canonical transformations to obtain the result. Finally we note that we have said only that the two systems are related and we have not transformed one system into the other. In particular, $H$ and $G$ are not the same Hamiltonian.

It is apparent from this result that the curvature parameter $k$ may be considered to be a coordinate in some higher dimensional space and that this may provide further information about the dynamics of the classical Kepler problem. In addition, it is well known that the spectrum generating Lie algebra and Lie algebra corresponding to the dynamical group of the classical Kepler problem is the Lie algebra $so(4,2)$. This Lie algebra also appears as the conformal symmetry group of Minkowski spacetime and our formalism suggests a connection between the two. These issues are currently being investigated by the authors.

This type of procedure may have applications to other dynamical systems with dynamical symmetries, for example, the harmonic oscillator and systems which admit Killing tensors and associated quadratic constants of motion. That is, there may be a geometrical interpretation available for the dynamical constants of motion which exist in these systems.

Appendix: Canonical transformation on $T^* G^3(k)$

It turns out that for our purposes it is simplest to transform from the natural coordinate system of (6) to a new coordinate system on configuration space. We perform an inversion of the coordinates

$$x'^i = \frac{x^i}{|x|^2};$$

(A.1)

The corresponding canonical momenta ($p'_i = \partial x^i / \partial x'^j p_j$) are

$$p'_i = |x|^2 p_i - 2x^i (x \cdot p).$$

(A.2)
Since this is just a coordinate transformation on configuration space, it is obviously canonical. Using these inverted coordinates makes the relationship between the classical Kepler motion and geodesic motion on surfaces of constant curvature much more transparent. For flat Euclidean space $E^3$, i.e., the case $k = 0$, this coordinate inversion takes points near the origin to points near infinity and vice versa, but of course the intrinsic geometry of the space is unaltered. A similar argument holds for the case $H^3$, $k < 0$ as shown by Belbruno [2]. For the three-sphere $S^3$ with $k > 0$ this corresponds to an antipodal mapping which is an isometry of $S^3$. This is why Moser’s method [1] does not require the inversion. Then we implement a further (canonical) transformation

$$\bar{x}^i = p_i' / 2\sqrt{2}, \quad \bar{p}_i = -2\sqrt{2}x'^i. \quad (A.3)$$

This transformation relates the momentum space of the classical Kepler problem to a configuration space of constant curvature, that is, the velocity hodographs can be regarded as geodesics on such spaces, as discussed in Moser [1] and Belbruno [2].

Combining these two transformations the Hamiltonian function becomes

$$G = \frac{1}{4} \left( k + \frac{|\bar{p}|^2}{2} \right)^2 |\bar{x}|^2 \quad (A.4)$$

and the conserved quantities are

$$L_i = \epsilon_{ij}^k \bar{x}^j \bar{p}_k, \quad (A.5)$$

$$\sqrt{2}D_i = \bar{x}^i \left( |\bar{p}|^2 - \frac{2G^2}{|\bar{x}|} \right) - \bar{p}_i (\bar{x} \cdot \bar{p}). \quad (A.6)$$

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