A TERRIBLE EXPANSION OF THE DETERMINANT

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Abstract. From a transfer formula in multivariate finite operator calculus, comes an expansion for the determinant similar to Ryser’s formula for the permanent. Although this one contains many more terms than the usual determinant formula. To prove it, we consider the poset of ordered partitions, properties of the permutahedron, and some good old fashioned combinatorial techniques.

1. Introduction

One of the foundational concepts of linear algebra is the determinant. At the most basic level, this matrix parameter is celebrated for its intricate ties to the set of eigenvalues and as a similarity invariant. However, the determinant still surprises us as the solution to a varying array of problems.

In addition to solving systems of linear equations and performing a change of variables in calculus, the determinant can help us count! Benjamin and Cameron [1] recently showed the determinant will calculate the number of nonintersecting \( n \)-paths in certain nonpermutable digraphs, where an \( n \)-path is a set of \( n \) paths from \( n \) distinct source vertices to \( n \) distinct sink vertices. In fact, the permanent will count the number of all \( n \)-paths.

The determinant of a matrix can be found recursively, as an alternating sum of minors. Often the determinant of an \( n \times n \) matrix \( A \) is defined compactly using the Leibniz formula, precisely

\[
\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}.
\]

Similarly, the permanent can be defined as a sum over subsets of \([n] := \{1, 2, \ldots, n\}\) using Ryser’s formula [3]

\[
\text{perm}(A) = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^{n} \sum_{j \in S} a_{i,j}.
\]

In this paper, we prove a much messier expansion of the determinant by instead indexing our terms using the set of ordered partitions of \([n]\). Aptly, we call this the terrible expansion of the determinant. This expansion is analogous to Ryser’s formula for the permanent. Section 5 explains the origins of this expansion as it relates to multivariate finite operator calculus, a branch of mathematics that has proven useful in enumerating ballot (generalized Dyck) paths containing certain patterns [4, 7].

Before stating the formula for our expansion of the determinant, we introduce it with two fundamental examples. When \( n = 2 \) we see that Equation (1) provides the following expansion for the determinant.

\[
\begin{vmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{vmatrix}
= a_{11}(a_{12} + a_{22}) + a_{22}(a_{11} + a_{21}) - (a_{11} + a_{21})(a_{12} + a_{22})
\]

\[1\]
Likewise when \( n = 3 \) we get the following expansion:

\[
|A| = a_{11}(a_{12} + a_{22})(a_{13} + a_{23} + a_{33}) + a_{11}(a_{12} + a_{22} + a_{32}) + a_{22}(a_{11} + a_{21})(a_{13} + a_{23} + a_{33}) + a_{22}(a_{12} + a_{22} + a_{32}) + a_{33}(a_{12} + a_{22} + a_{32}) + a_{33}(a_{22} + a_{32}) + a_{33}(a_{12} + a_{22} + a_{32}) + a_{11}(a_{12} + a_{22} + a_{32}) + a_{33}(a_{12} + a_{22} + a_{32}) - a_{22}(a_{11} + a_{21} + a_{31})(a_{13} + a_{23} + a_{33}) - a_{33}(a_{11} + a_{21} + a_{31})(a_{12} + a_{22} + a_{32}) - (a_{11} + a_{21})(a_{12} + a_{22})(a_{13} + a_{23} + a_{33}) - (a_{11} + a_{31})(a_{13} + a_{33})(a_{12} + a_{22} + a_{32}) - (a_{22} + a_{32})(a_{23} + a_{33})(a_{11} + a_{21} + a_{31}) + (a_{11} + a_{21} + a_{31})(a_{12} + a_{22} + a_{32})(a_{13} + a_{23} + a_{33}) \]

Our main theorem gives a general description of the terrible expansion of the determinant.

**Theorem 1.** Let \( A = (a_{ij})_{n \times n} \). The following formula is an expansion for the determinant of \( A \):

\[
\det(A) = \sum_{B \in [n]} (-1)^{|B|-|B'|} \prod_{a_{ij} \in B} \sum_{\beta_k \in B \setminus \beta_k'} \sum_{i \in \beta_k'} a_{ij},
\]

where the outer summation runs over all ordered partitions \( B = (\beta_1, \beta_2, \ldots, \beta_r) \) of the set \([n]\) and the inner summation runs over all integers \( i \) in the union of first \( k \) parts \( \beta_k = \bigcup_{j=1}^k \beta_j \) of the partition \( B \).

The following example serves to clarify the notation in Theorem 1.

**Example 2.** If \( B = (\beta_1, \beta_2, \beta_3) = (\{2\}, \{1, 3\}, \{4, 5\}) \),

then

\( B' = (\beta'_1, \beta'_2, \beta'_3) = (\{2\}, \{1, 2, 3\}, \{1, 2, 3, 4, 5\}) \),

and the corresponding expression in Equation (2) for the ordered partition \( B \) is

\[
a_{22}(a_{11} + a_{21} + a_{31})(a_{13} + a_{23} + a_{33})(a_{14} + a_{24} + a_{34} + a_{44} + a_{54})(a_{15} + a_{25} + a_{35} + a_{45} + a_{55}).
\]

notice that the first index runs through \( B' \), while the second index runs through the partition \( B \).

The rest of this paper proceeds as follows: In Section 2 we analyze the functions \( f : [n] \to [n] \) indexing the terms in the terrible expansion. After setting notation and proving a few fundamental lemmas, we end that section with Corollary 1 which proves that when \( f : [n] \to [n] \) is a bijective function, or a permutation, then the coefficients \( c_f = sgn(f) \). In other words, they are precisely the nonzero coefficients appearing in the determinant. In Section 3 we study the poset of ordered partitions and identify the importance of singleton partitions so that we can formulate our problem in more geometric terms as Euler characteristics of convex polytopes relating to the permutahedron. In Section 4 we prove that \( c_f = 0 \) for all non-bijective functions \( f : [n] \to [n] \) by analyzing Euler characteristics of subsets of the permutahedron. This proves that the terrible expansion does indeed give a formula for the determinant. In Section 5 we give an extremely brief introduction to multivariate finite operator calculus, and state a more general open conjecture that motivated this paper.

2. Flattening Functions

Upon expanding the expression in Theorem 1 many terms will cancel. In this section, we set up the groundwork to keep track of each of the terms and show how they cancel. Now consider any
function \( f : [n] \rightarrow [n] \) and define the monomial

\[
a_f := \prod_{j=1}^{n} a_{f(j), j}.
\]

Expanding the terms in Equation (2) results in a sum of the form

\[
\sum_{B \vdash [n]} (-1)^{n-|B|} \prod_{k \in B} \prod_{j \in \beta_k} \prod_{i \in \beta_k^j} a_{ij} = \sum_f c_f a_f,
\]

where each term \( c_f a_f \) corresponds to a function from the set \([n]\) to itself. For instance, in Equation (1) the four functions \( f_i : [2] \rightarrow [2] \) are

\[
f_1(1) = 1, \quad f_1(2) = 1; \quad f_2(1) = 1, \quad f_2(2) = 2; \quad f_3(1) = 2, \quad f_3(2) = 2; \quad \text{and} \quad f_4(1) = 2, \quad f_4(2) = 1.
\]

The terms corresponding to each function are labeled below

\[
a_{11}(a_{12} + a_{22}) + a_{22}(a_{11} + a_{21}) - (a_{11} + a_{21})(a_{12} + a_{22})
\]

\[
= (a_{11}a_{12} - a_{11}a_{12}) + (a_{11}a_{22} + a_{11}a_{22} - a_{11}a_{22})
\]

\[
= (a_{21}a_{22} - a_{21}a_{22}) + (-a_{12}a_{21})
\]

\[
= 0 + a_{11}a_{22} + 0 - a_{12}a_{21}
\]

\[
= c_{f_1} a_{f_1} + c_{f_2} a_{f_2} + c_{f_3} a_{f_3} + c_{f_4} a_{f_4},
\]

where first we expand the terms and then we simplify. Hence we see that \( c_{f_1} = 0, \ c_{f_2} = 1, \ c_{f_3} = 0, \) and \( c_{f_4} = -1. \)

Thus, the goal of this paper is to combinatorially identify the coefficients \( c_f \) for each such function, and show that they agree with the coefficients of \( a_f \) in the determinant. To do so, we must identify the set \( S_f = \{ B \vdash [n] : a_f \text{ appears as a summand of the product indexed by } B \} \). The following definition and lemma describe criteria for when an ordered partition \( B \vdash [n] \) appears in \( S_f \).

**Definition 3.** Let \( B = (\beta_1, \beta_2, \ldots, \beta_r) \) be an ordered partition of \([n]\). For each \( i \in [n] \), define \( \beta(i) = k \) if \( i \in \beta_k \). We say \( i \) **precequals** \( j \) in \( B \), denoted \( i \preceq j \), if \( i \) appears in an earlier part or the same part as \( j \) in the ordered partition \( B \), i.e.,

\[
\beta(i) \leq \beta(j).
\]

**Lemma 4.** Let \( B \vdash [n] \) be an ordered partition of \([n]\). The term \( a_f \) appears in the product

\[
\prod_{k \in B} \prod_{j \in \beta_k} \prod_{i \in \beta_k^j} a_{ij}
\]

iff \( f(j) \preceq j \) in \( B \) for all \( 1 \leq j \leq n \).

**Proof.** Let \( B \vdash [n] \) be an ordered partition of \([n]\). Since \( \beta_k' = \bigcup_{j=1}^{k} \beta_j \), we see that \( a_f \) appears in a term in the product

\[
\prod_{k \in B} \prod_{j \in \beta_k} \prod_{i \in \beta_k^j} a_{ij}
\]

precisely when \( i = f(j) \preceq j \) in the ordered partition \( B \) for all \( 1 \leq j \leq n \).

Now that Lemma 4 has established that an ordered partition \( B \) appears in the set \( S_f \) if and only if \( f(j) \preceq j \) in \( B \) for all integers \( 1 \leq j \leq n \), we are ready to analyze which functions \( f : [n] \rightarrow [n] \) correspond to nonzero terms in the terrible expansion. To do so, we start by introducing some notation regarding the structure of such functions.

**Definition 5.** Let \( f : [n] \rightarrow [n] \) be any function. We say that the function \( f \) is **acyclic** if \( f^k(i) = i \) implies \( k = 1 \) for all \( i \in [n] \). Otherwise, we say \( f \) contains a cycle.

Henceforth we will prefer to work with acyclic functions. The next definition, describes how each function \( f : [n] \rightarrow [n] \) which contains a cycle can be simplified or flattened to an acyclic function \( \bar{f} \) on a different set of elements.
**Definition 6.** Let \( f : [n] \to [n] \) be any function, \( C_f = \{ \sigma_1, \sigma_2, \ldots, \sigma_r \} \) represent the cycles of \( f \), \( N_f \subseteq [n] \) be the elements not in a cycle, and \( D_f = C_f \cup N_f \). Define the flattened function \( \overline{f} : D_f \to D_f \) as follows

\[
\overline{f}(i) = \begin{cases} 
  f(i) & \text{if } f(i) \text{ is not in a cycle of } f \\
  \sigma_j & \text{if } f(i) \text{ belongs to the cycle } \sigma_j \\
  i & \text{if } i \in C_f
\end{cases}
\]

Intuitively, \( \overline{f} \) acts just like \( f \), but shrinks each cycle of \( f \) to a fixed point, and thus it is an acyclic function. The following example illustrates Definitions 5 and 6.

**Example 7.** Let \( f : [6] \to [6] \) be defined by \( f(1) = 1, f(2) = 3, f(3) = 2, f(4) = 3, f(5) = 6, \) and \( f(6) = 5 \). Then \( C_f = \{ \sigma_1, \sigma_2 \} \) where \( \sigma_1 = (23) \) and \( \sigma_2 = (56) \). The sets \( N_f = \{ 1, 4 \} \) and \( D_f = N_f \cup C_f = \{ 1, 2, 3, 4 \} \). The function \( \overline{f} : D_f \to D_f \) is defined by \( \overline{f}(1) = 1 \), and \( \overline{f}(4) = 1 \).

The following lemma shows that we can reduce the problem of calculating the coefficients \( c_f \) in the terrible expansion, to that of calculating the coefficients \( c_{\overline{f}} \) corresponding to acyclic functions.

**Lemma 8.** If \( f : [n] \to [n] \) is any function, then the coefficients \( c_f \) and \( c_{\overline{f}} \) are related by the equation

\[
c_f = (-1)^{n-|D_f|} c_{\overline{f}}.
\]

**Proof.** Let \( S_f := \{ B \upharpoonright [n] : a_f \text{ appears as a summand of the product indexed by } B \} \), and similarly let \( S_{\overline{f}} := \{ A \upharpoonright D_f : a_{\overline{f}} \text{ appears as a summand of the product indexed by } A \} \).

If \( B \) is an ordered partition for which \( a_f \) appears as a summand, then Lemma 4 implies that for each cycle

\[
\sigma_j = (i, f(i), f^2(i), \ldots, f^k(i))
\]

of \( f \), the following precedence relation must hold

\[
i \preceq f^k(i) \preceq f^{k-1}(i) \preceq \cdots \preceq f(i) \preceq i
\]

in the ordered partition \( B \). Therefore \( c_j \) must be contained in the same part \( \beta \) of the ordered partition \( B \). From here, it is clear to see that \( S_f \) is equivalent to the set of ordered partitions of \( D_f \), and so each term in Equation (3) has the form

\[
c_f a_f = \sum_{B \in S_f} (-1)^{n-|B|} a_f
\]

\[
= (-1)^{n-|D_f|} \sum_{A \in D_f} (-1)^{|D_f|-|A|} a_f
\]

\[
= (-1)^{n-|D_f|} c_{\overline{f}} a_f,
\]

and the lemma follows. \( \square \)

The number of ordered partitions of \([n]\) with \( k \) parts is well known to be \( k!S(n, k) \), where \( S(n, k) \) are the Stirling numbers of the second kind. We obtain an important corollary to Lemma 8 from the following well-known result about Stirling numbers of the second kind.

**Lemma 9.** The following identity holds for ordered partitions:

\[
\sum_{k=0}^{n} (-1)^{n-k} k!S(n, k) = 1.
\]
Proof. The result follows immediately upon setting \( x = -1 \) in the following identity on Stirling numbers of the second kind [6, p. 35]:

\[
\sum_{k=0}^{n} S(n, k)(x)_k = x^n.
\]

Corollary 10. If \( f : [n] \to [n] \) is bijective, i.e., \( f = \pi \) for some \( \pi \in \mathfrak{S}_n \), then \( c_f = \text{sgn}(\pi) \).

Proof. Since \( f \) is bijective, it consists only of cycles. Thus \( D_f = C_f = \{\sigma_1, \ldots, \sigma_r\} \) and \( |D_f| = r \). By Lemma 8 and Lemma 9 we see

\[
c_f = (-1)^{|D_f|} \sum_{A \in D} (-1)^{|D_f| - |A|} = (-1)^{n-r} \sum_{B \vdash [r]} (-1)^{|B|} = (-1)^{n-r} \sum_{k=0}^{n} (-1)^{r-k} k! S(r, k) = (-1)^{n-r}.
\]

We leave it to the reader to verify that \( \text{sgn}(\pi) = (-1)^{n-r} \).

With Corollary 10 we have that the terrible expansion contains every term of the determinant with the correct coefficient. It remains to show that whenever \( f \) is not bijective, \( c_f = 0 \). By Lemma 8 it suffices to show this for acyclic functions.

3. The Poset of Ordered Partitions

We next consider the poset of ordered partitions in order to show that the set \( S_f \) has a nice structure when \( f \) is acyclic. This will allow us to eventually switch to a more geometric viewpoint.

Let \( \mathcal{P}_n \) denote the poset of ordered partitions of the set \([n]\). In Figure 1 we see the poset \( \mathcal{P}_3 \) of ordered partitions on 3 elements. At the top of the poset \( \mathcal{P}_n \), we have the \( n! \) ordered partitions consisting of singletons. These partitions correspond bijectively with the elements of \( \mathfrak{S}_n \). Because of their importance later, we will refer to them as singleton partitions.

![Figure 1. Poset of Ordered Partitions on 3 elements](image)

Directly below a given ordered partition \( B \) in \( \mathcal{P}_n \) are ordered partitions formed by taking the union of two consecutive parts in \( B \). For example, directly below the singleton partition \( 3/1/2/4 \) are the ordered partitions \( 13/2/4, 3/12/4 \), and \( 3/1/24 \). All ordered partitions under a singleton partition creates an \((n-1)\)-cube. An example is given in Figure 2.
An acyclic function can be viewed as a rooted forest, where the fixed points are the roots. An example is given in Figure 3. Given an acyclic function \( f \), if a path exists from \( p \) to \( q \), with \( p \) closer to the root than \( q \), then \( f^k(q) = p \) for some \( k \). Thus, \( p \preceq q \), and so we say \( f \) has the rule \( p \preceq q \). In this way, each function \( f \) stipulates a set of rules

\[
R_f = \left\{ p \preceq q \mid f^k(q) = p \text{ for } q, k \in [n] \right\}.
\]

The following lemma and corollary will show that the set of ordered partitions \( S_f \) has a nice structure in \( \mathcal{P}_n \).

**Figure 2.** The Cube Under the Singleton Partition 1/2/3/4

**Figure 3.** Acyclic Function Represented by a Rooted Forest

**Lemma 11.** If a singleton partition \( A \) in \( \mathcal{P}_n \) satisfies the rules \( R_f \) given by an acyclic function \( f \) then every ordered partition \( B \leq A \) in \( \mathcal{P}_n \) also satisfies the rules \( R_f \). If an ordered partition \( B \) in \( \mathcal{P}_n \) satisfies the rules \( R_f \) then there exists at least one singleton partition \( A \geq B \) in \( \mathcal{P}_n \) that satisfies the precedence rules \( R_f \).

**Proof.** The first statement is obvious. If \( A \) is a singleton partition with the rule \( p \preceq q \) in \( A \), and \( B \leq A \) is an ordered partition in \( \mathcal{P}_n \), then the parts of \( B \) are unions of consecutive parts of \( A \). Hence \( p \preceq q \) in \( B \) as well.

The second statement is slightly less obvious. Let \( f \) be an acyclic function, and suppose that \( B \) in \( \mathcal{P}_n \) is a nonsingleton ordered partition that satisfies the rules \( R_f \). We must show there is a singleton partition \( A \geq B \) above it in \( \mathcal{P}_n \) that also satisfies \( R_f \). Consider a part \( \beta \) of \( B \) that is not a singleton. If no pair of elements in \( \beta \) has a rule associated with it, then the elements of \( \beta \) can be
ordered arbitrarily. Otherwise, there are elements of $\beta$ that have rules imposed on them. Consider the elements in the intersection of $\beta$ and a rooted tree associated with $f$. We order those elements by their distance from the root. (Those elements having the same distance from the root can be put in any order with respect to each other.) We do this for every rooted tree associated with $f$ to impose an order on all of $\beta$. Doing the same to each part will result in a singleton partition $A$ above $B$ satisfying the rules of $f$. □

Corollary 12. The set of ordered partitions of $[n]$ satisfying the rules of an acyclic function is a union of $(n-1)$-cubes in $P_n$.

Proof. By the above lemma, we can account for all the ordered partitions by only considering the singleton partitions satisfying the acyclic function, and all the ordered partitions below them. The result follows since the ordered partitions below a singleton partition form an $(n-1)$-cube. □

Lemma 11 and Corollary 12 tell us that once we know which singleton partitions appear in $S_f$, then we know that $S_f$ is precisely those singletons and all the ordered partitions under them in $P_n$. Because of their importance we will start to label the singleton partitions without slashes, e.g. $1/2/3/4 \rightarrow 1234$, unless we need to distinguish them from the ordered partition with one part. In the next section, we turn our attention to a geometric object isomorphic to $P_n$, the permutahedron.

4. The Permutahedron

In this section we show how the alternating sums giving $c_f$ when $f$ is acyclic are related to the Euler characteristic of the permutahedron and use this correspondence to show that $c_f = 0$. It is well-known that the poset of the ordered partitions is isomorphic to the face lattice of the permutahedron [5, Fact 4.1]. Specifically, each vertex on the permutahedron represents a singleton partition, the edges incident to a vertex represent the ordered partitions just below that singleton partition in the poset, the faces adjacent to those edges represent the ordered partitions just below again, and so on, until the permutahedron itself represents the ordered partition with one part at the bottom of the poset. Note that two vertices are adjacent if one can be obtained by a single swap of consecutive elements. For example, 315624 is adjacent to 351624. Figure 4 shows the transformation from the poset on 3 element to the permutahedron on 3 elements, which in this case is a hexagon. Figure 5 shows the poset on 4 elements as the ordinary permutahedron (truncated octahedron).

![Figure 4. Poset of Ordered Partitions on 3 elements: Top View (left) and as a Permutahedron (right)](image)

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Figure 5. Poset of Ordered Partitions on 4 elements as a Permutahedron

In Figure 5 we only label the singleton partitions at the vertices, but the labeling of the other ordered partitions would be similar to Figure 4.

The permutahedron \( \Pi_n \) is often defined as the convex hull of the points

\[ P_\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \]

for every \( \sigma \in S_n \). It is a convex polytope, and in particular, it is contractible to a point. Thus, the permutahedron has Euler characteristic 1 [5]. Since the ordered partitions are in bijection with the faces of the permutahedron, the alternating sum of the ordered partitions is precisely the Euler characteristic of the permutahedron, and this gives us a second proof of Lemma 9.

We adopt a slightly different convention, relabeling the vertices of the \( \Pi_n \) to \( P_{\sigma^{-1}} \). We will denote this relabeled permutahedron by \( \Pi'_n \). Figure 6 shows \( \Pi_n \) in \( \mathbb{R}^3 \), the relabeled \( \Pi'_n \), and the correspondence between \( x_1 \leq x_2 \) and \( 1 \preceq 2 \). Figure 6 also shows the fact that \( \Pi_n \), and thus \( \Pi'_n \), is an \((n-1)\)-dimensional object, since all the points lie in the hyperplane \( x_1 + x_2 + \cdots + x_n = \frac{n+1}{2} \).

In general \( x_i \leq x_j \) in \( \Pi_n \) corresponds to \( i \preceq j \) in \( \Pi'_n \).

**Lemma 13.** The singleton partitions in \( \Pi'_n \) satisfying the precedence rule \( i \preceq j \) are contained in the corresponding half-space \( x_i \leq x_j \) in \( \Pi_n \).

**Proof.** A permutation \( \sigma^{-1} \) satisfies \( \sigma^{-1}(i) \preceq \sigma^{-1}(j) \) precisely when \( \sigma(i) \leq \sigma(j) \). Hence we see that if the permutation (or singleton partition) \( \sigma^{-1} \) satisfies the precedence relation \( i \preceq j \) then the vertex \( (\sigma(1), \sigma(2), \ldots, \sigma(n)) \) is in the half-space \( x_i \leq x_j \) and vice versa. \( \square \)

We are now ready to prove the main result of this section.

**Proposition 14.** If \( f : [n] \to [n] \) is not bijective, then the coefficient \( c_f = 0 \) is zero.

**Proof.** Let \( f : [n] \to [n] \) be an acyclic function. Let \( R_f = \{ f(j) = i \preceq j \} \) denote the set of precedence rules determined by \( f \). To each precedence rule \( f(j) = i \preceq j \) in \( R_f \) we can assign a half-space \( H_{ij} := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_{f(j)} = x_i \leq x_j \} \), and the intersection of these half-spaces with the permutahedron \( \Pi'_n \) defines a convex polytope, which we will denote by \( \Pi_f \).

The faces of \( \Pi_f \) fall into two disjoint subsets: those faces that correspond to the ordered partitions in \( S_f \) and those that do not. Let \( \Gamma_f \) denote the faces of \( \Pi_f \) which correspond to elements of \( S_f \) and let \( \Delta_f \) denote the faces of \( \Pi_f \) that do not. The set of faces \( \Delta_f \) are precisely the faces of \( \Pi_f \)
Figure 6. The permutahedra $\Pi_3$ (left) and $\Pi'_3$ (right)

which lie entirely on the boundary of at least one half-space $x_i = x_j$ because they resulted from intersecting $\Pi'_n$ with one of the half-spaces $H_{ij}$. Thus each face of $\Delta_f$ is a convex polytope, and we see that $\Delta_f$ is a union convex polytopes. Each of these convex polytopes contains the point $(x_1, x_2, \ldots, x_n)$ where $x_1 = x_2 = \cdots = x_n$, so $\Delta_f$ is a contractible space. Hence $\Delta_f$ has Euler characteristic $\chi(\Delta_f) = 1$.

Since $\Pi_f$ is a convex polytope and $\Pi_f = \Gamma_f \sqcup \Delta_f$ we see that

$$1 = \chi(\Pi_f) = \chi(\Gamma_f) + \chi(\Delta_f) = 0 + 1.$$ 

It follows that $\chi(\Gamma_f) = 0$. Since the summands of the alternating sum $\sum_{B \in S_f} (-1)^{n-|B|}$ correspond with the faces of $\Gamma_f$ we see that $c_f = \sum_{B \in S_f} (-1)^{n-|B|} = \chi(\Gamma_f) = 0$ as desired. \qed

We end this section with an example demonstrating the proof of Proposition 14.

**Example 15.** Figure 7 shows $\Pi_f$ with $\Gamma_f$ bold and $\Delta_f$ shaded for the terms $a_f = a_{11}a_{12}a_{13}$, which has the rules $1 \preceq 2$ and $1 \preceq 3$ and $a_f = a_{11}a_{12}a_{33}a_{34}$, which has the rules $1 \preceq 2$ and $3 \preceq 4$.

5. Multivariate Finite Operator Calculus

This terrible expansion of the determinant came from a conjecture about a transfer formula in multivariate finite operator calculus (MFOC). In this section we give a very brief overview of the objects of study in MFOC and the conjecture that gives this expansion. The interested reader is encouraged to read [8] for a more comprehensive description of this subject matter.

Let $k$ be a field. Let $\{e_i\}_{1 \leq i \leq \ell}$ denote the standard $\ell$-dimensional basis of $k^\ell$. The main objects of study in MFOC are polynomials $p \in k[x_1, \ldots, x_\ell]$ and operators $T \in k[[D_1, \ldots, D_\ell]]$, where $D_i$ is the partial derivative with respect to $x_i$. A sequence of polynomials $b_n(x) = b_{n_1, \ldots, n_\ell}(x_1, \ldots, x_\ell)$ is called a Sheffer sequence if there is a set of operators $B = (B_1, \ldots, B_\ell)$ with $B_i = D_iP_i$ where

$$b_{n_1, \ldots, n_\ell}(x_1, \ldots, x_\ell) = \sum_{|\alpha| = n_1} \cdots \sum_{|\alpha| = n_\ell} \frac{1}{\alpha!} P_{\alpha_1}(x_1) \cdots P_{\alpha_\ell}(x_\ell).$$
Figure 7. $a_f = a_{11} a_{12} a_{13}$ (left) and $a_f = a_{11} a_{12} a_{33} a_{34}$ (right)

Each $B_i : b_n \to b_{n-e_i}$ and each $P_i$ an invertible operator. Such a set of operators is called a delta $\ell$-tuple. The power series for an operator $T$ is written as

$$T = \sum_{n \geq 0} a_n D^n = \sum_{n_1, \ldots, n_\ell \geq 0} a_{n_1, \ldots, n_\ell} D_1^{n_1} \cdots D_\ell^{n_\ell}. \tag{4}$$

These are all standard notations in any multivariate theory. However, the following notation is not completely standard in MFOC. Given a subset $A \subseteq [\ell]$ we define $X_A := \prod_{i \in A} X_i$.

The following is the Transfer Theorem from MFOC, and is essentially Theorem 1.3.6 in [8] with different notation.

**Theorem 16** (Transfer Theorem). Let $J$ denote the usual Jacobian matrix of a collection of polynomials. Suppose $\mathbf{B} = (B_1, B_2, \ldots, B_\ell)$ is a delta $\ell$-tuple where $B_i = D_i P_i^{-1}$, then

$$b_n(x) = P_{n+1} J(B_1, B_2, \ldots, B_\ell) \frac{x^n}{n!}$$

is the basic sequence for $\mathbf{B}$ written in terms of $\frac{x^n}{n!}$.

Within the Jacobian, we have Pincherle derivatives $\frac{\partial B_i}{\partial D_j} = B_i \theta_j - \theta_j B_i$, where $\theta_j : p \to x_j p$ is the $j$th umbral shift operator that does not commute with the delta operators. Thus, there are many ways this transfer formula could be expanded. The following conjecture (based on the examples provided below) gives one such way.

**Conjecture 17.** The basic sequence from the Transfer Theorem can also be calculated as

$$b_n(x) = \sum_{B \vdash [\ell]} (-1)^{\ell - |B|} (\theta_\beta P_\beta) B \frac{x^{n-1}}{n!},$$

where $B$ runs through all ordered partitions of $[\ell]$ and $\beta$ runs through the partitions of $B$.

Because of our abuse of some notation, we give some examples.

$$b_{m,n}(u, v) = (u P_1^m v P_2^n + v P_2^n u P_1^m - uv P_1^m P_2^n) \frac{u^{m-1} v^{n-1}}{m! n!}$$
\[
\begin{align*}
    b_{m,n,p}(a,b,c) &= (aRbScT + aRcTbS + bSaRcT + bScTaR + cTaRbS + cTbSaR \\
    &\quad -aRbST - bSacRT - cTabRS - abRScT - acRTbS - bcSTaR \\
    &\quad + abcRST) \frac{a^{m-1}b^{n-1}c^{p-1}}{m!n!p!}
\end{align*}
\]

\[
    B = (\{2\}, \{1,3\}) \Rightarrow (\theta_\beta P_\beta)_B = bSacRT
\]

The terrible expansion of the determinant comes from setting each \( B_i = D^{\alpha_i} = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} \), or in other words, it is one term of the power series in Equation (4).

6. Conclusions and Open Questions

We end with few open questions stemming from our work.

**Q1.** Now that Theorem 1 shows that Conjecture 17 is true for one term of an operator’s power series, can Conjecture 17 be proven by a linearity argument?

**Q2.** Can the proof of Ryser’s formula given by Horn and Johnson [3] be modified to give another proof of Theorem 1?

**Q3.** Can our proof of Theorem 1 be modified to prove Ryser’s formula by using the topological/combinatorial properties of the cube instead of the permutahedon?

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