Instantaneous Bethe–Salpeter Equation and Its Analog: Breit-like Equation

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(Received March 10, 2005)

Abstract We take the \((\mu^\pm e^\mp)\) system as an example, but restrict ourselves to highlight the states with quantum number \(J^P = 0^-\), to explore the different contents of the instantaneous Bethe–Salpeter (BS) equation and its analog, the relativistic version of Breit equation, by solving them exactly. The results show that the two equations are not equivalent, although they are analogous. Furthermore, since the Breit equation contains extra un-physical solutions, we point out that it should be abandoned if one wishes to have an accurate description of the bound states for the instantaneous interacting binding systems.

PACS numbers: 11.10.St, 36.10.Dr, 12.20.Ds

Key words: instantaneous BS equation, Breit equation, exact solutions

In Ref. [1], we presented a way to solve the Bethe–Salpeter (BS) equation,[2] which has an instantaneous kernel exactly. In the meantime we also touched the relativistic version of the Breit equation\textsuperscript{[1,4]} as an analog of it a little. In this paper we will mainly explore the different contents of the two equations (BS and Breit) by taking the bound states with quantum number \(J^P = 0^-\) (S-wave) of the \((\mu^\pm e^\mp)\) system as an example to solve the two equations exactly, and discuss their consequences.

BS equation for a fermion-antifermion system has the general formulation,[2]

\[
(p_1 - m_1)\chi_r(q)(p_2 + m_2) = \int \frac{d^3k}{(2\pi)^3} V(P, q, k)\chi_r(k),
\]

where \(\chi_r(q)\) is the BS wave function, \(P\) is the total momentum, \(q\) is relative momentum, and \(V(P, q, k)\) is the kernel of the equation, \(p_1\) and \(p_2\) are the momenta of the constituents 1 and 2, respectively. The total momentum \(P\) and the relative momentum \(q\) are related to the momenta \(p_1, p_2\) by

\[
p_1 = \alpha_1 P + q, \quad \alpha_1 = \frac{m_1}{m_1 + m_2},
\]
\[
p_2 = \alpha_2 P - q, \quad \alpha_2 = \frac{m_2}{m_1 + m_2}.
\]

If the kernel \(V(P, q, k)\) of the BS equation has the behavior \(V(P, q, k)|_{P=0} = V(q, k)\) (\(\vec{P} = 0\) in center mass frame of the concerned bound state), the BS equation is called as an “instantaneous BS equation”.

For instance, in Coulomb gauge, the terms corresponding to the possible transverse-photon exchange between the two components in BS kernel for \((\mu^\pm e^\mp)\) systems are considered as higher-order, so the “lowest order” BS equation kernel for the systems has the form

\[
V(P, q, k)|_{\text{photons}} = \gamma^0 V_e \gamma^0 = -\gamma^0 \frac{4\pi\alpha}{(q - k)^2} \gamma^0.
\]

Namely, there are some physical systems with reasonable approximation, which may be described by an instantaneous BS equation.

Firstly, let us follow the way to solve the equation as done by the authors of Refs. [5] and [6] but without approximation, and introduce the “instantaneous BS wave function” \(\varphi_r(q)\) as

\[
\varphi_r(q) \equiv i \int \frac{d^4q}{2\pi} \chi_r(q^0, \vec{q}),
\]

then the BS equation (1) can be re-written as

\[
\chi_r(q^0, \vec{q}) = S_r^{(1)}(p_1^0)\eta(q) S_r^{(2)}(-p_2^0).
\]

Here \(S_r^{(1)}(p_1)\) and \(S_r^{(2)}(-p_2)\) are the propagators of the fermion and anti-fermion respectively, and

\[
\eta(q) \equiv \int \frac{d^3k}{(2\pi)^3} V(q, k) \varphi_r(k).
\]

The propagators (in C.M.S. i.e., \(\vec{P} = 0\)) can be decomposed as

\[
-iJS_r^{(1)}(Jp_i^\mu) = \frac{\Lambda_r^+(q)}{Jq^0 + \alpha_i M - \omega_i + i\epsilon},
\]

\[
+ \frac{\Lambda_r^-(q)}{Jq^0 + \alpha_i M + \omega_i - i\epsilon}
\]

with

\[
\omega_i = \sqrt{m_i^2 + q^2}, \quad \Lambda_r^\pm(q) = \gamma^0 \omega_i \gamma_{j} (m_i + \gamma_j q),
\]

where \(J = 1\) for the fermion (\(i = 1\)) and \(J = -1\) for the anti-fermion (\(i = 2\)). It is easy to check that

\[
\Lambda_r^\pm(q) + \Lambda_r^\pm(q^\pm) = \gamma^0, \quad \Lambda_r^\pm(q^\parallel)\gamma^0 \Lambda_r^\parallel(q^\parallel) = 0,
\]

\footnotetext{\textsuperscript{1}The project supported in part by National Natural Science Foundation of China
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\[ \Lambda_{\pm}^\pm (\vec{q}) \gamma^0 \Lambda_{\pm}^\pm (\vec{q}) = \Lambda_{\pm}^\pm (\vec{q}). \]  

Thus \( \Lambda_{\pm} \), being “complete”, can be considered as “energy” projection operators. For below discussions let us introduce the notations \( \varphi_{\pm}^\pm (\vec{q}) \) as
\[
\varphi_{\pm}^\pm (\vec{q}) \equiv \Lambda_{\pm}^\pm (\vec{q}) \gamma^0 \varphi_{\pm} (\vec{q}) \gamma^0 \Lambda_{\pm}^\pm (\vec{q}).
\]

Because of the completeness of the projection for \( \Lambda_{\pm} \), we have
\[
\varphi_{\pm} (\vec{q}) = \varphi_{\pm}^+ (\vec{q}) + \varphi_{\pm}^- (\vec{q}) \]

for the BS wave function \( \varphi_{\pm} (\vec{q}) \). If we further carry out a contour integration for the time-component \( q^0 \) on both sides of Eq. (4), then we obtain
\[
\varphi_{\pm} (\vec{q}) = \frac{\Lambda_{\pm}^+ (\vec{q}) \eta_{\pm}^r (\vec{q}) \Lambda_{\pm}^2 (\vec{q})}{M - \omega_1 - \omega_2} - \frac{\Lambda_{\pm}^- (\vec{q}) \eta_{\pm}^r (\vec{q}) \Lambda_{\pm}^2 (\vec{q})}{M + \omega_1 + \omega_2},
\]

(M is the eigenvalue). To apply the complete set of the projection operators \( \Lambda_{\pm}^\pm (\vec{q}) \) to Eq. (8), we then obtain the four equations:
\[
(M - \omega_1 - \omega_2) \varphi_{\pm}^+ (\vec{q}) = \Lambda_{\pm}^+ (\vec{q}) \eta_{\pm}^r (\vec{q}) \Lambda_{\pm}^2 (\vec{q}),
\]
\[
(M + \omega_1 + \omega_2) \varphi_{\pm}^- (\vec{q}) = -\Lambda_{\pm}^- (\vec{q}) \eta_{\pm}^r (\vec{q}) \Lambda_{\pm}^2 (\vec{q}),
\]
\[
\varphi_{\pm}^+ (\vec{q}) = \varphi_{\pm}^- (\vec{q}).
\]

Note that the two equations in Eq. (11) do not contain eigenvalue \( M \), so essentially they are “constraints” for the BS wave function \( \varphi_{\pm} (\vec{q}) \). Because of the completeness of the projectors \( \Lambda_{\pm} \), only the four equations contained in Eqs. (9) ~ (11) are equivalent to Eq. (8).

In the earlier days in literature the relativistic version of Breit equation for bound states in C.M.S. of the bound state \( (\vec{P} = 0) \) was proposed[3,4] as
\[
[M \varphi (\vec{q}) - \gamma^0 H_1 (\vec{q}) \gamma^0 \varphi (\vec{q}) - \varphi (\vec{q}) \gamma^0 H_2 (\vec{q}) \gamma^0] = \Lambda_1^+ (\vec{q}) \eta (\vec{q}) \Lambda_2^2 (\vec{q}) - \Lambda_1^- (\vec{q}) \eta (\vec{q}) \Lambda_2^2 (\vec{q}),
\]
\[
H_1 (\vec{q}) \equiv m_1 \beta + \hat{q} \cdot \vec{\alpha},
\]
\[
H_2 (\vec{q}) \equiv m_2 \beta + \hat{q} \cdot \vec{\alpha},
\]
with the definitions \( \beta = \gamma^0, \vec{\alpha} = \beta \vec{\gamma} \). In fact, the non-relativistic version of the Breit equation, which is based on the expansion of the relativistic one, is still used sometimes. In Ref. [1], it is pointed out that equation (12) is not equivalent to the instantaneous BS equation, i.e. the coupled-equations Eqs. (9) ~ (11). Namely when applying the project operator \( \Lambda_1^+ (\vec{q}) \gamma^0 \otimes \gamma^0 \Lambda_2^2 (\vec{q}) \) to Eq. (12), equation (9) is obtained, and when applying the project operator \( \Lambda_1^- (\vec{q}) \gamma^0 \otimes \gamma^0 \Lambda_2^2 (\vec{q}) \) to Eq. (12), equation (10) is obtained, whereas when applying \( \Lambda_1^+ (\vec{q}) \gamma^0 \otimes \gamma^0 \Lambda_2^2 (\vec{q}) \) to Eq. (12), then we obtain the homogeneous equations:
\[
[M - \omega_1 (\vec{q}) + \omega_2 (\vec{q})] \varphi^+ (\vec{q}) = 0,
\]
\[
[M + \omega_1 (\vec{q}) - \omega_2 (\vec{q})] \varphi^- (\vec{q}) = 0,
\]
where \( \omega_{1,2} = \sqrt{m_{1,2}^2 + \hat{q}^2} \). The two equations in Eq. (13) contain the eigenvalue \( M \), so they are not “constrains” any longer but homogeneous equations. In general, equation (11) can be considered as their “trivial solutions”, but additional “non-trivial solutions” may exist. Note that \( m_1 = m_2 \) also means \( \omega_1 = \omega_2 \), so from Eq. (13) only \( M = 0 \) corresponds to the “non-trivial solutions” of Eq. (13), whereas when \( M = 0 \), there is not \( \vec{P} = 0 \) frame at all, so equations (9) ~ (13) written in \( \vec{P} = 0 \) system will not be correct. Instead of re-deriving the equations and to avoid the difficulty in the cases of \( m_1 = m_2 \), below we restrict ourselves to consider the cases \( m_1 \neq m_2 \) only. It is why here we take \( (\mu^\pm e^\mp) \) system as an example.

It is interesting to explore the differences of the two equations: the instantaneous BS equations (9) ~ (11) and the Breit one Eq. (12) or say Eqs. (9), (10), and (13), because it is not only more precisely to prove the un-equivalence of the two equations than the above argument about the homogeneous equations (13), but also is useful for understanding the two equations: the instantaneous BS equation and the Breit equation respectively. Specifically, to do it, we restrict ourselves on the \( (\mu^\pm e^\mp) \) system for the bound states with quantum number \( J^P = 0^- \) as examples, and solve Eqs. (9) ~ (11) and Eqs. (9), (10), and (13) respectively without any approximations. In fact, for the present purpose, at the moment it is enough only to focus the bound states with quantum number \( J^P = 0^- \).

As pointed out in Ref. [1], in general, the wave functions for \( J^P = 0^- \) states in C.M.S. (\( \vec{P} = 0 \)) have the following formulation:
\[
\phi = (\gamma^0 g_1 + \hat{g} g_2 + \hat{g}^\prime \gamma^0 g_4) \gamma^5,
\]
where \( \hat{g} = -\hat{q} \cdot \vec{\gamma}/|\vec{q}| \). For convenience, let us introduce the functions \( f_i \) (\( i = 1, 2, 3, 4 \)), which relate the functions \( g_i \) as follows:
\[
g_1 = f_1, \quad g_2 = f_2,
\]
\[
g_3 = \frac{- (m_1 - m_2) |\vec{q}|}{m_1 m_2 + \omega_1 \omega_2 + \hat{q}^2} f_2 + f_3,
\]
\[
g_4 = \frac{(\omega_1 + \omega_2) |\vec{q}|}{m_1 m_2 + \omega_1 \omega_2} f_1 - f_4.
\]

From Eq. (11), we may straightforward obtain the requirements \( f_3 = f_2 = 0 \) and then from Eqs. (9) and (10) and having the angular integration done, we obtain
\[
M f_1 = (m_1 + m_2) f_2 - \frac{\alpha_s}{\pi} \int \frac{|\vec{k}|}{|\vec{q}|} d|\vec{k}| \left\{ \frac{m_1 + m_2}{2 \omega_1 \omega_2 (\omega_1 + \omega_2)} \right\} \left\{ (m_1 m_2 + \omega_1 \omega_2 + \hat{q}^2) Q_0 f_2 + \frac{(m_1 - m_2)^2 |\vec{q}|^2 |\vec{k}|}{m_1 m_2 + E_1 E_2 + \hat{k}^2} Q_1 f_2 \right\},
\]
\[
M_{f_2} = \frac{(\omega_1 + \omega_2)^2}{m_1 + m_2} f_1 - \frac{\alpha_s}{\pi} \int \frac{|\vec{k}|}{|\vec{q}|} d|\vec{k}| \frac{m_1 + m_2}{2 \omega_1 \omega_2} \left\{ (m_1 \omega_2 + m_2 \omega_1)Q_0 f_1 + \frac{(E_1 + E_2)(\omega_1 + \omega_2)|\vec{q}|}{m_1 E_2 + m_2 E_1} |\vec{k}| Q_1 f_1 \right\}, \tag{16}
\]

where \( E_i \equiv \omega_i(|\vec{k}|) = \sqrt{m_i^2 + |\vec{k}|^2} \) and the angular integrations of the equations have been carried out already, so \( Q_n \equiv [n|\vec{q}|^2 + |\vec{k}|^2]/2|\vec{q}| |\vec{k}| \) \((n = 0, 1, \ldots)\), the \( n \)-th Legendre functions of the second kind appear due to the precise BS kernel Eq. (2).

Equations (15) and (16) are a complete set of coupled equations about \( f_1 \) and \( f_2 \), and exactly equivalent to the instantaneous BS equation.

While from equations Eqs. (9), (10), and (13), i.e., the Breit equation, with straightforward derivation, we “directly” obtain the coupled equations:

\[
M_{f_1} = (m_1 + m_2) f_2 - \frac{\alpha_s}{\pi} \int \frac{|\vec{k}|}{|\vec{q}|} d|\vec{k}| \frac{m_1 + m_2}{2 \omega_1 \omega_2} \left\{ (m_1 \omega_2 + m_2 \omega_1 + \omega_2^2)Q_0 f_2 \\
+ \frac{(m_1 - m_2)^2|\vec{q}|}{m_1 m_2 + E_1 E_2 + |\vec{k}|^2} Q_1 f_2 - (m_1 - m_2)|\vec{q}| Q_1 f_3 \right\},
\tag{17}
\]

\[
M_{f_2} = \frac{(\omega_1 + \omega_2)^2}{m_1 + m_2} f_1 - 2|\vec{q}| f_4 - \frac{\alpha_s}{\pi} \int \frac{|\vec{k}|}{|\vec{q}|} d|\vec{k}| \frac{1}{2 \omega_1 \omega_2} \left\{ (m_1 \omega_2 + m_2 \omega_1)Q_0 f_1 \\
+ (E_1 + E_2)(\omega_1 + \omega_2)|\vec{q}| |\vec{k}| Q_1 f_1 - (\omega_1 + \omega_2)|\vec{q}| Q_1 f_4 \right\},
\tag{18}
\]

\[
M_{f_3} = \frac{(\omega_1 - \omega_2)^2}{m_1 - m_2} f_4,
\tag{19}
\]

\[
M_{f_4} = (m_1 - m_2) f_3.
\tag{20}
\]

Namely equation (17) \sim (20) are fully equivalent to Eq. (12), the Breit equation. It is easy to realize that if one sets \( f_3 = f_4 = 0 \), equations (17) \sim (20) for the Breit equation will “return” to Eqs. (15) and (16), i.e., the instantaneous BS equation.

The normalization condition for the solutions of the equations reads

\[
\int \frac{|\vec{q}|^2 d|\vec{q}|}{(2\pi)^2 |\omega_1|^2} \left\{ 4g_2[(m_2 - m_1)|\vec{q}| g_3 |(\omega_1 + \omega_2)|\vec{q}| g_4 + (m_1 \omega_2 + m_2 \omega_1) g_3 g_4] \right\} \right\} = 2M.
\tag{21}
\]

Now, let us solve Eqs. (17) \sim (20) numerically by transforming the coupled equations into an eigenvalue problem of a matrix one, i.e., by expanding \( f_i \) \((i = 1, 2, 3, 4)\), in terms of the bases of the exact \( S \)-wave solutions of the Schrödinger equation (in momentum representation) \( R_{nl}(|\vec{k}|)\)\[^8\]

\[
R_{nl}(|\vec{k}|) = \sqrt{\frac{2}{\pi}} \frac{(n - l - 1)!}{(n + l)!} n^2 2^{2l+1} \Gamma[n^2 2^{2l+1} + 1] C_{i,m}^{j}(|\vec{k}|),
\tag{22}
\]

where \( C_{i,m}^{j}(x) \) is the Gegenbauer function, defined as the coefficient of \( h^n \) in the expansion of \((1 - 2h x + h^2)^{-\nu}\) in powers of \( h \). To be practicable, as general cases, the expansion is truncated according to the request accuracy, so the present problem becomes an eigenvalue problem of a finite matrix. If we truncate the expansion up to \( j = 5 \),

\[
f_i^{(j)}(|\vec{q}|) = \sum_{j=1}^{5} C_{i,n}^{j} R_{nl}(|\vec{q}|),
\tag{23}
\]

the results (eigenvalues and eigenfunctions accordingly). If ignoring the negative energy eigenvalue solutions which have similar physics meanings as those of Dirac equations, we may organize the solutions into two types: type-A (the so-called ‘trivial’ solutions \( f_3 = f_4 = 0 \)) and type-B (the so-called ‘non-trivial’ solutions \( f_3 \neq 0, f_4 \neq 0 \)).

**Type A: the so-called “trivial” solutions**

The “trivial”-eigenvalues (in unit eV) for \((\mu^\pm e^\mp)\) systems are

- \(-13.541 \, 086 \, 6\), \(-3.386 \, 311 \, 2\), \(-1.504 \, 820 \, 7\),
- \(-0.846 \, 791 \, 2\), \(-0.542 \, 061 \, 0\), \ldots

The eigenfunctions corresponding to the eigenvalues have the expansion coefficients as those in Table. 1.
The overall renormalisation constants for the “trivial” solutions are 45 778.501, 45 778.730, 45 778.772, 45 778.787, 45 778.794, ... accordingly.

In fact, as expected and from Table 1 one may realize that the “trivial solutions” of Eqs. (17) ~ (20) have $f_3 = f_4 = 0$ exactly, and indeed they just correspond to solutions of Eqs. (15) and (16).

**Table B: the so-called “non-trivial” solutions**

The “non-trivial”-eigenvalues (in unit eV) for ($\mu^5e^+\tau$) systems are

- $-1.022 015 012 \times 10^6$, $-1.021 999 529 \times 10^6$, $-1.021 998 308 \times 10^6$, $-1.021 997 962 \times 10^6$, $-1.021 997 837 \times 10^6$, ...

It means that the energy levels have so great binding energy almost as $2m_e$ (two times of the electron mass). They are “extra” to the instantaneous BS equation. It seems that in certain sense, the Breit equation is a “bi-product” of two Dirac equations, so there are extra solutions, which still fall in the positive energy region for the binding systems, i.e., would correspond to those negative energy solutions if the lighter component were free. Thus the gap of the non-trivial solutions from the trivial ones can be so great as two times of the electron mass for the systems ($\mu^5e^+\tau$). To understand the fact, we have done several exercises, and find that it is a general feature for the Breit equation that the trivial and non-trivial solutions have so deep a gap as the two times mass of the lighter component in the bound states.

The overall renormalisation constants for the “trivial” solutions are 45 778.501, 45 778.730, 45 778.772, 45 778.787, 45 778.794, ... accordingly.

In fact, as expected and from Table 1 one may realize that the “trivial solutions” of Eqs. (17) ~ (20) have $f_3 = f_4 = 0$ exactly, and indeed they just correspond to solutions of Eqs. (15) and (16).

**Table 2.** Expansion coefficients for the eigenfunctions.

| {nl} | WF | $R_{10}$ | $R_{20}$ | $R_{30}$ | $R_{40}$ | $R_{50}$ |
|------|----|---------|---------|---------|---------|---------|
| {10} | $f_1$ | 2.284 × 10$^{-3}$ | 1.067 9 × 10$^{-3}$ | 5.770 × 10$^{-4}$ | 3.615 × 10$^{-4}$ | 2.407 × 10$^{-4}$ |
|      | $f_2$ | 2.262 × 10$^{-3}$ | 1.057 7 × 10$^{-3}$ | 5.714 × 10$^{-4}$ | 3.580 × 10$^{-4}$ | 2.384 × 10$^{-4}$ |
|      | $f_3$ | -0.615 535 | -0.284 194 | -0.157 829 | -0.101 564 | -0.071 540 2 |
|      | $f_4$ | 0.615 535 | 0.284 194 | 0.157 829 | 0.101 564 | 0.071 540 2 |
| {20} | $f_1$ | 4.169 × 10$^{-4}$ | -3.866 × 10$^{-4}$ | -5.112 × 10$^{-4}$ | -3.862 × 10$^{-4}$ | -2.437 × 10$^{-4}$ |
|      | $f_2$ | 4.129 × 10$^{-4}$ | -3.829 × 10$^{-4}$ | -5.063 × 10$^{-4}$ | -3.825 × 10$^{-4}$ | -2.414 × 10$^{-4}$ |
|      | $f_3$ | -0.320 488 | 0.306 022 | 0.403 615 | 0.309 568 | 0.211 896 |
|      | $f_4$ | 0.320 488 | -0.306 022 | -0.403 615 | -0.309 568 | -0.211 896 |
| {30} | $f_1$ | 6.420 × 10$^{-5}$ | -3.422 × 10$^{-5}$ | -2.655 × 10$^{-5}$ | -2.547 × 10$^{-5}$ | -2.267 × 10$^{-5}$ |
|      | $f_2$ | 6.367 × 10$^{-5}$ | -3.389 × 10$^{-5}$ | -2.630 × 10$^{-5}$ | -2.434 × 10$^{-5}$ | -2.245 × 10$^{-5}$ |
|      | $f_3$ | -0.119 399 | 0.469 465 | 1.961 10$^{-2}$ | -0.364 445 | -0.363 513 |
|      | $f_4$ | 0.119 399 | -0.469 465 | -1.961 10$^{-2}$ | 0.364 445 | 0.363 513 |
| {40} | $f_1$ | -2.126 × 10$^{-5}$ | 1.133 10$^{-4}$ | -1.822 10$^{-4}$ | -3.558 10$^{-5}$ | 1.545 10$^{-4}$ |
|      | $f_2$ | -2.106 10$^{-5}$ | 1.122 10$^{-4}$ | -1.805 10$^{-4}$ | -3.524 10$^{-5}$ | 1.531 10$^{-4}$ |
|      | $f_3$ | -0.292 977 | 0.446 233 | 7.971 10$^{-2}$ | -0.453 219 | -0.453 219 |
|      | $f_4$ | -0.292 977 | 0.446 233 | 7.971 10$^{-2}$ | 0.453 219 | 0.453 219 |
| {50} | $f_1$ | 1.786 10$^{-7}$ | 2.763 10$^{-5}$ | -5.647 10$^{-5}$ | 9.173 10$^{-5}$ | -3.973 10$^{-5}$ |
|      | $f_2$ | 1.769 10$^{-7}$ | 2.677 10$^{-5}$ | -5.593 10$^{-5}$ | 9.085 10$^{-5}$ | -3.934 38 |
|      | $f_3$ | 2.300 10$^{-2}$ | -0.140 3 | 0.335 672 | -0.504 656 | 0.335 311 |
|      | $f_4$ | -2.300 10$^{-2}$ | 0.140 3 | -0.335 672 | 0.504 656 | -0.335 311 |
Since there is no evidence in experiments for the systems \((\mu^+e^-)\), so exactly to say, the relativistic Breit equation is not physical. Only when one restricts oneself to consider the week binding solutions and ignores the (deep) non-trivial solutions is the Breit equation meaningful. In literature, fortunately, people used the Breit equation with a strong implication that only the very weak binding solutions are pursued. Indeed there is no problem only when applying the Breit equation to the weak binding spectrum studies, whereas if applying it for complete energy spectrum and/or (12) is not equivalent to the instantaneous BS equation, and the Breit equation (12) should be abandoned for all kinds of instantaneous interacting binding systems, if one wishes to apply it to finding exact and complete solutions and/or uses it for computing the relativistic correction effects without restriction. In contrary, the instantaneous BS equation does not have such un-physical solutions for the instantaneous interacting binding systems at all, therefore one may freely apply the BS equation to various problems of the systems.

The eigenfunctions corresponding to the eigenvalues have the expansion coefficients as those in Table 2.

The overall renormalization constants for the ‘non-trivial’ solutions are \(1.240 \ 240 \ 535 \times 10^7\), \(3.652 \ 200 \ 818 \times 10^7\), \(6.728 \ 317 \ 282 \times 10^7\), \(1.251 \ 161 \ 253 \times 10^8\), \(2.757 \ 689 \ 877 \times 10^8\), \ldots, accordingly.

To test the “stability” of the numerical solutions, we also try to truncate the expansion up to \(j=10\), and obtain the eigenvalues for non-trivial and trivial as follows:

\[
\begin{align*}
-1.022 \ 015 \ 269 \times 10^6, &\quad -1.021 \ 999 \ 737 \times 10^6, &\quad -1.021 \ 998 \ 500 \times 10^6, &\quad -1.021 \ 998 \ 133 \times 10^6, \\
-1.021 \ 997 \ 979 \times 10^6, &\quad -1.021 \ 997 \ 900 \times 10^6, &\quad -1.021 \ 997 \ 856 \times 10^6, &\quad -1.021 \ 997 \ 830 \times 10^6, \\
-1.021 \ 997 \ 815 \times 10^6, &\quad -1.021 \ 997 \ 807 \times 10^6, &\quad \ldots
\end{align*}
\]

From the eigenvalues obtained by a different truncation, one may see that the low-laying numerical solutions for \(j=5\) are quite accurate for our observation. The table for \(j=10\) eigenfunction coefficients is too big to present here, thus we omit it.

From the example, we have learnt that the precise numerical solutions of Eqs. (17) \sim (20) have confirmed the arguments on Eq. (13) about the homogeneous equations. Generally, the Breit equation (12) contains not only all the solutions of the instantaneous BS equation, but also the un-physical solutions, the non-trivial solutions, although we have the lesson only for \(J^p = 0^-\) states of the \((\mu^+e^-)\) systems. Therefore, we can conclude that the Breit equation (12) is not equivalent to the instantaneous BS equation, and the Breit equation (12) should be abandoned for all kinds of instantaneous interacting binding systems, if one wishes to apply it to finding exact and complete solutions and/or uses it for computing the relativistic correction effects without restriction. In contrary, the instantaneous BS equation does not have such un-physical solutions for the instantaneous interacting binding systems at all, therefore one may freely apply the BS equation to various problems of the systems.

Acknowledgments

One of the authors (C.H. Chang) would like to thank Stephen L. Adler for valuable suggestions and discussions.

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