Critical behaviour of dynamic analogues of equilibrium lattice models

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Abstract. It is not difficult to construct chaotic Coupled Map Lattices (CMLs) that display the type of synchronization usually associated to Phase Transitions (PTs) in equilibrium statistical models. These CMLs show the emergence of an order parameter that signals long-range order in the lattice, without losing its chaotic character (the existence of a positive sector in the Lyapunov spectrum). This order parameter remains null for small couplings, and changes to non-zero values for couplings above some critical value. Around this critical coupling one also has strong fluctuations in both long- and short-range correlations. In this work two examples are reviewed, the Miller-Huse CML, which is a dynamical analogue to the Ising model, and a recently constructed dynamical analogue to the $q=3$ Potts Model. Both models are constructed using fully hyperbolic continuous piece-wise linear mappings, and a diffusive coupling. It is shown for both examples how the symmetries of the models allows one to connect to the corresponding equilibrium classes. The critical behaviour of Miller-Huse in two dimensions (2D) under simultaneous updating is revisited, and new results for this model in a triangular lattice are shown, including a careful evaluation of the effects of the leading correction to scaling. It is concluded that this model does belong to the 2D Ising Universality Class. For the dynamical analogue to the $q=3$ Potts Model a well defined order-disorder PT is also found. The phase diagrams for both simultaneous and sequential updating of the model are obtained. For simultaneous updating a line of continuous PTs is found. The corresponding critical exponents —again, including a leading correction to scaling— are consistent with those of the equilibrium Potts Model.

1. Introduction

The Miller-Huse (MH) model [1] was introduced some 10 years ago as a dynamical model expected to show critical behaviour in the two-dimensional (2D) Ising Universality Class. It is given by a Coupled Map Lattice (CML) with scalar local variables that follow a chaotic evolution rule, coupled via a simple diffusive interaction. The first results [1] gave in fact some evidence for the expected Ising-like behaviour, including the presence of a continuous order-disorder phase transition for a rather large value of the coupling. But subsequent and much more detailed simulations [2, 3] found a small but very consistent discrepancy: it was found that, under simultaneous updating of the model, the correlation-distance exponent $\nu$ fell short by some 10% of the Ising value $\nu = 1$, while the ratios $\beta/\nu$ and $\gamma/\nu$ keep their Ising values. This result was extended to some other models with the same symmetries, giving as a result a proposal for the existence of a new class of universality for these CMLs, different from that...
of the Ising Model [3]. Posterior work by other authors (including the present one) verified the existence of this anomaly in related models given by Cellular Automata [4], and Coupled Stochastic Lattices [5].

The simulation of the same model under sequential updating is reported to yield critical exponents in the 2D Ising Universality Class [3], implying therefore that the updating scheme is a relevant variable for the model. For extended dynamical systems this does not look too surprising, since they frequently show very different behaviour when different updating rules are applied. This issue has been explored for extended systems with local interactions [6], where it was found that simultaneous updating is essential for them to exhibit non-trivial collective behaviour, and also for globally coupled systems [7, 8, 9]. But on the other hand, simultaneous updating of the Metropolis Algorithm for the 2D Ising Model [10] did not show any departure from 2D Ising universality.

In view of the mixed results found up to now, taking into account that the discrepancy in $\nu$ for the MH model is relatively small, and that the evaluation of critical exponents in all these models is hampered by the presence of strong corrections to scaling, one should not really consider this issue as settled. It is therefore worthwhile to revisit it, taking advantage of the present availability of faster and cheaper computers. Thus, this work reconsiders this problem in two ways: first, the MH model is explored again, now in a triangular lattice, generating much larger statistics which allows a more precise evaluation of critical exponents. It is found that the model does indeed fall in the universality class of the 2D Ising Model. Second, the ideas behind the MH model are extended to a dynamical analogue to the $q = 3$ Potts Model [11]. Again, for simultaneous updating one finds a line of continuous phase transitions with critical behaviour consistent with the corresponding equilibrium universality class.

2. The Miller Huse Model
The MH model is given by a lattice of odd-symmetric piecewise-linear chaotic maps with diffusive coupling. The local map takes the $[-1, 1]$ interval into itself, and is given by

$$F(x) = \begin{cases} 
-2 - 3x & \text{if } -1 \leq x < -1/3, \\
3x & \text{if } -1/3 \leq x < 1/3, \\
2 - 3x & \text{if } 1/3 \leq x < 1,
\end{cases} \quad (1)$$

This local map has a slope $m = \pm 3$ everywhere, making it strongly chaotic (see figure (1)). The diffusive coupling is implemented via

$$x_r(t+1) = F(x_r(t)) + \frac{\epsilon}{Z} \sum_{r'} [F(x_{r'}(t)) - F(x_r(t))], \quad (2)$$

where the sum is over the $Z$ nearest neighbors $r'$ of the site $r$, and $0 \leq \epsilon \leq 1$. The lattice has side $L$, and periodic boundary conditions are applied. Notice that all quantities on the right-hand side of (2) are taken at time $t$; this is what simultaneous updating means in the present context.

The analogy between this model and the equilibrium Ising model is based on the presence, in both cases, of two conflicting tendencies: on the one hand, the chaotic evolution of the local variables tends to uncorrelate their values, on the other, diffusion among neighboring sites has the opposite effect. One can therefore establish at least a heuristic connection between chaos –measured, say, by the Lyapunov exponent of the local maps– and thermal fluctuations, and a corresponding connection between the strength of the Hamiltonian coupling and the diffusive coupling $\epsilon$. Now, the symmetries for the MH mapping correspond clearly to those of the Ising Model: in both cases one has two equivalent states, which differ only in their signs, such that a simultaneous change in sign applied to all sites of the lattice does not affect at all the evolution
of any measurable quantity (except of course for an immaterial change in sign of odd moments of the magnetization). It is clear, however, that there is no actual exact mapping between the two models: were it so, there would be no question about which universality class the MH model belongs to.

An instantaneous order parameter $m_t$ is defined by the average of $x$ over the lattice

$$m_t = \frac{1}{N} \sum_r x_r(t),$$

where $N$ is the total number of sites. Stationary-state averaged moments of this order parameter are defined by

$$\langle m^k \rangle = \frac{1}{T} \sum_{t=1}^{T} m_t^k,$$

with $T$ large (that is, much larger than any correlation time of the model), and with the proviso that $t$ is set to 1 only after a large enough transient has passed. Most often one prefers to use the magnitude of the order parameter, defining

$$\langle |m|^k \rangle = \frac{1}{T} \sum_{t=1}^{T} |m_t|^k.$$

This because for finite lattices there is always a non-zero probability of flipping the sign of $m_t$, which for $T \to \infty$ takes the averages of odd moments to zero. For actual, finite $T$ runs, these random flippings of $m_t$ make the values of odd moments of $m$ so noisy as to be useless.

From these moments one calculates the quantities of interest, which are, among others, a “magnetization” $M = \langle |m| \rangle$ and a “susceptibility” $\chi' = N(\langle m^2 \rangle - \langle |m| \rangle^2)$. One can also introduce an “energy” $e_t$, which is just a measure of the short-distance correlation, given by

$$e_t = \frac{2}{NZ} \sum_{r,r'} x_r(t) x_{r'}(t).$$

**Figure 1.** Chaotic map for the Miller-Huse model, (1).
with \((\mathbf{r}, \mathbf{r}')\) nearest neighbors. From the averaged moments of this energy one can construct an “specific heat” of the form \(c = N((\epsilon^2) - (\epsilon)^2)\). All these quantities are just heuristic analogues of their counterparts in equilibrium models; even so, from now on these names will be used without quotation marks.

Of special interest is the Fourth Order Cumulant \([12, 13]\)

\[
U^4 = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2},
\]

which is commonly used to locate the critical coupling when using Finite Size Scaling (FSS). Besides \(U_4\), here two extra cumulants of different order are used. These quantities are not necessarily related to those obtained from the expansion of \(\log(\exp(-\beta H))\), here we are just looking for quantities for which FSS does not give an \(L\) dependent prefactor. These are given by

\[
U^3 = 1 - \frac{\langle |m|^3 \rangle}{2\langle m^2 \rangle \langle |m| \rangle},
\]

\[
U^2 = 1 - \frac{2\langle m^2 \rangle}{\pi \langle |m| \rangle^2}.
\]

They are constructed in an analogous way to \(U^4\), so that they go to zero in the disordered phase, and to some constant in the ordered phase. With respect to scaling they behave in all ways as \(U^4\), and therefore provide no new information. On the other hand, when Finite Size Corrections are included, they show different, non-universal coefficients for these corrections. Advantage will be taken of these coefficients in the next section.

In a square lattice this CML shows a continuous PT at \(\epsilon = 0.82136(8) \) \([1, 2, 3]\) (digits in parentheses give the uncertainty in the corresponding last digits of the quantity). This transition is affected by strong corrections to scaling, probably related to the fact that the square lattice favors the formation of antiferromagnetic domains for very large coupling. For a triangular lattice the increase in coordination reduces the critical coupling, and also frustrates the formation of antiferromagnetic clusters.

3. Finite Size Scaling and Leading Correction to Scaling

For the evaluation of critical parameters one resorts to the well known approach of Finite Size Scaling (FSS), as described, for instance, in \([14]\), including an estimation of the leading Finite Size Corrections (FSCs). For completeness we give here some details of the techniques involved, following mainly \([3]\).

Even if this is not a genuine equilibrium system, one makes the heuristic assumption that equilibrium results apply. This means that the standard formulation of scaling for equilibrium models can be followed, and so one writes a free energy density for a system of linear size \(L\) as

\[
F(\epsilon, H, S; L) \approx L^{-d} \hat{F}((\epsilon - \epsilon_c)L^{1/\nu}, HL^{(\beta+\gamma)/\nu}, SL^{-\omega/\nu}),
\]

allowing for only one irrelevant scaling field \(S\) (the leading correction). Here \(H\) is an external symmetry-breaking field, \(d\) is the dimension of the lattice and \(\hat{F}\) is universal. A non-singular background is ignored. The critical coupling \(\epsilon_c\) and the critical exponents \(\nu, \beta\) and \(\gamma\) take their infinite-size lattice values, as does the leading correction exponent \(\omega\). Here we are assuming simple power-law corrections; the presence of irrelevant operators that generate logarithmic scaling would make the analysis of numerically generated data exceedingly difficult.
From here standard procedures allow one to find that, for finite \( L \) and in the absence of an external field, the magnetization \( M \), the susceptibility \( \chi' \) and the cumulants \( U^a \), \( a = 2, 3, 4 \), behave in the critical region as

\[
M(\epsilon, S; L) \approx L^{-\beta/\nu} \left\{ \tilde{M}_0(L^{1/\nu}(\epsilon - \epsilon_c)) + \frac{S L^{-\omega/\nu} \tilde{M}_1(L^{1/\nu}(\epsilon - \epsilon_c))}{\chi_0(L^{1/\nu}(\epsilon - \epsilon_c))} + \ldots \right\},
\]

(11)

\[
\chi'(\epsilon, S; L) \approx L^\gamma/\nu \left\{ \chi_0'(L^{1/\nu}(\epsilon - \epsilon_c)) + \frac{S L^{-\omega/\nu} \chi_1'(L^{1/\nu}(\epsilon - \epsilon_c))}{\chi_0(L^{1/\nu}(\epsilon - \epsilon_c))} + \ldots \right\},
\]

(12)

\[
U^a(\epsilon, S; L) \approx \hat{U}_0^a(L^{1/\nu}(\epsilon - \epsilon_c)) + SL^{-\omega/\nu} \hat{U}_1^a(L^{1/\nu}(\epsilon - \epsilon_c)) + \ldots,
\]

(13)

with \( \tilde{M}_0, \tilde{M}_1, \chi_0', \chi_0, \) etc., universal functions.

Assuming large enough \( L \), these expressions are usually truncated to just one correction. Setting the arguments of the universal functions to zero one gets the standard forms

\[
M(\epsilon_c; L) \approx L^{-\beta/\nu} \left\{ M^* + AM^{-\omega/\nu} \right\},
\]

(14)

\[
\chi'(\epsilon_c; L) \approx L^\gamma/\nu \left\{ \chi'' + A\chi'L^{-\omega/\nu} \right\},
\]

(15)

\[
U^a(\epsilon_c; L) \approx U^a + AUaL^{-\omega/\nu},
\]

(16)

where \( M^* \), \( \chi'' \) and the different \( A'\)'s are now non-universal constants. The behaviour of the specific heath \( c \) will be discussed later on.

The starting point in the determination of the critical parameters is the location of the critical coupling. If there were no FSCs, this would be quite a simple matter: it would suffice to look for the crossing points of the \( U^a \) \( (a = 2, 3, 4) \) curves for different values of \( L \). In actual practice, but under favorable situations, these corrections become negligible (that is, smaller than the statistical dispersion of the data) for large enough lattices, and then one usually talks about having reached scaling, and proceeds to take a simple average of the crossings. Unfortunately scaling is still out of reach for the MH model, where one is limited to sizes up to \( L \approx 100 \). This because the updating scheme is quite inefficient, in the sense of having a large dynamical exponent \( z \approx 1.9 \). There is no way of going around this problem, since the updating scheme is in itself the subject of study.

Therefore it is necessary to deal directly with the FSCs. For this one may resort to some simple prescriptions, like choosing the crossings between the two largest lattices [15], but this clearly ignores some of the information contained in the numerical trends. An standard way to take into account a leading FSC has been known for a long time [12, 16], and is based on the following consideration: accepting the form given in (13), keeping only two terms, one notices that it can be taken as a first order expansion of a universal function \( \hat{U}^a(L^{1/\nu}(\epsilon - \epsilon_c) + B_a L^{-\omega/\nu}) \), where \( B_a \) is some nonuniversal constant. Then, if \( \epsilon^a_{ij} \) is the crossing point between cumulants of type \( a \) for lattice sizes \( L_j \) and \( L_j \), one gets

\[
L_i^{1/\nu}(\epsilon^a_{ij} - \epsilon_c) - L_j^{1/\nu}(\epsilon^a_{ij} - \epsilon_c) = -B_a(L_i^{-\omega/\nu} - L_j^{-\omega/\nu}).
\]

(17)

from where it is immediate that

\[
\epsilon^a_{ij} = \epsilon_c - B_a L_i^{-\omega/\nu} - L_j^{-\omega/\nu} \frac{L_j^{1/\nu} - L_i^{1/\nu}}{L_i^{1/\nu} - L_j^{1/\nu}}.
\]

(18)
From here one gets that, adjusting $\nu$ and $\omega$, the crossings between the curves $U^a(\epsilon; L)$ for different $L$ should fall on a straight line when plotted against a variable $x_{ij} = (L_i^{-\omega/\nu} - L_j^{-\omega/\nu})/(L_1^{1/\nu} - L_j^{1/\nu})$. Better yet, the intercept of this line in the vertical axis should give $\epsilon_c$.

This approach is occasionally used, but since one needs to do a nonlinear fit over two parameters (with some extra approximations [12, 16], one parameter), in the presence of noise, this yields less than reliable results. There is, however, another way of handling this formula, one that avoids completely the nonlinear fits. Start by writing (18) for two different types of cumulant, say $U^p$ and $U^q$:

$$\epsilon_{ij}^p = \epsilon_c - B_p x_{ij}; \quad \epsilon_{ij}^q = \epsilon_c - B_q x_{ij}. \tag{19}$$

Adding and subtracting these two equations it is elementary to show that

$$\frac{\epsilon_{ij}^p + \epsilon_{ij}^q}{2} = \epsilon_c + \frac{B_p + B_q}{2(B_p - B_q)}(\epsilon_{ij}^p - \epsilon_{ij}^q) \tag{20}$$

and so, as long as the coefficients in front of the correction terms in (16) are different, one can use the small difference in behaviour of two distinct cumulants to perform an estimation of the critical coupling that avoids the use of non-linear fits.

Once an estimate of $\epsilon_c$ as been achieved, we proceed to estimate $\omega/\nu$, using (16) at $\epsilon_c$. This necessarily involves a nonlinear fit. The value encountered here for $\omega/\nu$ is assumed to be a good estimate of the actual correction exponent and is used in all subsequent numerical fits. (We have chosen not to use effective exponents, as was done in [2, 3]. The use of these exponents, one for every measurable quantity, introduces extra degrees of freedom in the nonlinear fits, and may end up producing less reliable results.) Next, $\nu$ is determined by fitting the data to the customary expression [3, 5]:

$$\partial_\epsilon U^a(\epsilon_c; L) \simeq L^{1/\nu}(C_0^{U^a} + C_1^{U^a} L^{-\omega/\nu}), \tag{21}$$

which comes from (13). Here the $C$s are again non-universal parameters. For $\beta/\nu$ and $\gamma/\nu$ one uses (14) and (15).

4. Results for the Miller-Huse Model

The raw data are the measurements of the first four moments of magnetization and the first two moments of energy, realized over many repetitions so as to generate estimates of statistical errors. The lattice sizes used here are $L = 24, 28, 32, 36, 40, 46, 52, 60, 68, 78, 90, 102$ and 114. For each value of $L$ the correlation time $\tau$ for $\epsilon$ in the critical region is estimated. After that, for each value of $\epsilon$ and $L$, 50 or more samples are taken, each with a transient of $8\tau$ and a running time of $500\tau$ or longer. In all cases, and for each value of $L$, polynomial fits are performed. For the cumulants, the $\chi^2$ per degree of freedom becomes small in both the original fit and in the fit for (20) when using a 3rd degree polynomial, and therefore this is the degree chosen for all fits. The range of $\epsilon$ used corresponds roughly to $|L^{1/\nu}(\epsilon - \epsilon_c)| < 0.5$, where initial rough estimates for $\nu$ and $\epsilon_c$ are good enough in this expression. Changes due to a different degree in the polynomial fit or to modifications of the fitting range are small, and are included in the error estimates. Where derivatives are needed, the fit is implemented first and the resulting polynomial is then differentiated. Due to the convoluted dependence of the final results on the raw data, statistical errors have been calculated via bootstrapping [17].

For the triangular lattice a critical coupling $\epsilon_c = 0.743548(86)$ was found, as shown in figure (2). It should be noticed that for this fit (20) has been relaxed a bit by allowing for a quadratic fit, taking into account the weak curvature still apparent in the graph. This amounts
Figure 2. Estimation of the critical coupling $\epsilon_c$, using the sums $\sigma(\epsilon) = (\epsilon_{i,j}^p + \epsilon_{i,j}^q)/2$ and differences $\delta(\epsilon) = \epsilon_{i,j}^p - \epsilon_{i,j}^q$, where $\epsilon_{a,i,j}$ is the crossing point between cumulants of type $a$ for lattice sizes $L_i$ and $L_j$. Here the fit has been done using a quadratic. The three lines correspond to $p = 4, q = 3$, $p = 4, q = 2$ and $p = 3, q = 2$. Notice the extraordinary convergence of the three lines for $\delta(\epsilon) = 0$.

Table 1. Critical exponents for the MH model in a triangular lattice. In the second column $\omega/\nu$ is set to 2. The third column gives the well-known equilibrium Ising 2D values.

| $\omega/\nu$ estimated | $\omega/\nu$ fixed | Equilibrium |
|-------------------------|---------------------|-------------|
| $\omega/\nu$           |                      |             |
| 1.85(13)                | 2.0                 | 2.0         |
| $1/\nu$                 | 0.991(21)           | 0.998(16)   | 1.0         |
| $\beta/\nu$            | 0.1227(22)          | 0.1226(21)  | 0.125       |
| $\gamma/\nu$           | 1.7574(35)          | 1.7461(74)  | 1.75         |

It is important to notice that $\omega$ falls within error bars from the known value for the leading correction in the Ising Model, $\omega = 2$ [18]. It then becomes an interesting exercise to repeat the calculation of other critical exponents fixing $\omega/\nu = 2$. This, by eliminating one degree of freedom from the problem, reduces the error margins a bit. The results of this calculation are given in the second column of table 1, which now shows an even better agreement between the MH and Ising models. Some examples of the fits used to extract these exponents are given in figures (3) and (4), which show some of the Log-Log plots from where $1/\nu$ and $\gamma/\nu$ were extracted.
Figure 3. Log-Log plot for the derivative of the fourth order cumulant, $\partial U^4(\epsilon_c, L)/\partial \epsilon$ vs $L$. The continuous line gives the expected $L \to \infty$ limit, and its slope is $1/\nu$, which for this cumulant evaluates to $1/\nu = 1.0004$. The dotted line includes the leading correction to scaling. Here the correction exponent $\omega/\nu$ has been set to 2.

Figure 4. Log-Log plot for the susceptibility $\chi'(\epsilon_c, L)$ vs $L$. The continuous line gives the expected $L \to \infty$ limit, and its slope is $\gamma/\nu = 1.7461$. The dotted line includes the leading correction to scaling. Here the correction exponent $\omega/\nu$ has been set to 2.
5. Critical Behavior of the Specific Heat

Some special attention is required for the analysis of the specific heat of the model. As can be seen in figure (5), this quantity grows quite slowly, and this may very well be a signature of the expected logarithmic divergence in the thermodynamic limit. Assuming therefore that the growth of \( c \) is logarithmic, one needs to find a way to insert FSCs in its fit, these corrections being particularly large in this case. Here, simply following the form given in (14) and (15), the following equation for \( c \) is proposed

\[
c(\epsilon_c; L) \approx \ln\left(\frac{L}{L_0}\right) \left\{ \epsilon_c^* + A_c L^{-\omega/\nu} \right\},
\]

where \( L_0 \) is some scale factor. The results are given in figure (6), which shows the raw data, the best fit including corrections, and the linear trend. Here the correction exponent has been set to \( \omega/\nu = 2 \). As can be seen in this figure, the fit is excellent. It could still be possible that this is just an artifact caused by the simultaneous adjustment of 3 parameters, but the good agreement in all other exponents makes this very unlikely.

On the other hand, a blind application of the same exponent-extracting protocol that has been used for \( 1/\nu, \beta/\nu \) and \( \gamma/\nu \) gives for the \( \alpha/\nu \) exponent a value of 0.02541(23) when the correction exponent is also calculated, and of 0.024957(85) when \( \omega/\nu \) is set to 2. These very small values are probably meaningless, since, as mentioned, corrections to scaling are quite large. The resulting fit is neither better nor worse than that shown if figure (6). This is a simple consequence of the fact that the spread of values for \( c \) obtained in the simulations is very narrow, compared to the minimum value of \( c \) found. The logarithm over such a narrow range is therefore very well approximated by a linear expansion. The graph one obtains for \( \ln(c) \) vs. \( \ln(L) \) is then identical to figure (6), except for an affine transformation of the \( y \) axis. Given that the same fitting protocol is applied, the fitted lines are the same.

One has then that a linear-plus-FSCs fit to the data works equally well for both \( c \propto \ln(L) \) and \( c \propto L^{\alpha/\nu} \), but the value obtained for \( \alpha/\nu \) is exceedingly small. One then may conclude that a logarithmic scaling is a more natural choice for the growth of \( c \) in the MH model.
Figure 6. Specific heat at critical coupling $c(\epsilon_c)$ vs. $\log(L)$. The continuous line gives the expected $L \to \infty$ limit. The dotted line includes the leading correction to scaling, as defined by (22). Here the correction exponent $\omega/\nu$ has been set to 2. Due to the very narrow spread in $c$, taking its log just effects a shift plus change of scale in the vertical axis, and therefore the Log-Log and Function-Log plots look identical.

6. A dynamical analogue of the q=3 Potts Model

The second example to be studied [11] is given by a 2D square lattice, whose elements, with individual dynamics given by the nonlinear chaotic map $x(t+1) = F(x(t))$ evolve in time according to a diffusive rule similar to the one used for the MH model (2):

$$x_r(t+1) = F(x_r(t)) + \frac{\epsilon}{Z} \sum_{r'} \text{dist}(F(x_{r'}(t)), F(x_r(t)))$$

(23)
as before $\epsilon$ is the coupling intensity and $Z$ is the number of nearest neighbors. Here a square lattice is used, so $Z = 4$, and dist $(x, y)$ stands for the shortest signed distance between $x$ and $y$, to be defined later. As for the local map, the goal is to generate one that is: a) chaotic (actually, hyperbolic), so that the exponential separation of orbits generates the disorder needed in the model (and no periodic windows appear), and b) with the same symmetries of the Potts $q = 3$ model.

A map that fulfills these requirements can be obtained as follows: take the simple linear map, defined in the $[0,1)$ real interval

$$f(x) = \begin{cases} 
  gx & \text{if } x \leq x_0, \\
  a - gx & \text{if } x_0 < x \leq x_1, \\
  b + gx & \text{if } x_1 < x \leq 1,
\end{cases}$$

(24)

where $g$ is the slope of the map, and one sets $g > 3$ so that the map is both chaotic and takes the $[0,1)$ interval outside of itself. The constants $a$, $b$, $x_0$ and $x_1$ are such that $f(1/2) = 1/2$, $f(1) = 1$ and the map is single-valued and continuous. The corresponding values are

$$a = \frac{g + 1}{2}; \quad b = 1 - g; \quad x_0 = \frac{g + 1}{4g}; \quad x_1 = \frac{3g - 1}{4g}.$$  

(25)
Now, put 3 of these maps together in the plane, joining the points (1,1) of one map with the point (0,0) of the next. This defines the segment \([0,3)\) as the domain. Finally, apply a modulo 3 operation to the outcome so that the range is also confined to the segment \([0,3)\). All together, the mapping becomes (see figure (7))

\[
F(x) = (\text{int}(x) + f(x \mod 1)) \mod 3,
\]

with \(f\) given in (24). The model is completed by implementing a diffusive coupling as given in (23). The function ‘dist’ is the shortest signed distance on a circumference of perimeter 3:

\[
\text{dist}(x, y) = \begin{cases} 
  x - y & \text{if } |x - y| < 3/2, \\
  y - x - 3/2 & \text{if } y - x > 3/2, \\
  y - x + 3/2 & \text{if } y - x < -3/2.
\end{cases}
\]

With these two goals fulfilled, one can then extend here the observations made about the MH model. First, there are as before two opposing tendencies: to disorder, due here both to the chaotic character of the maps and —of crucial importance for this case—, to the fact that the condition \(g > 3\) mixes the three states of the model; and to order, simply due to diffusion. These two characteristics of the model are the ones that generate an order-disorder transition. Moreover, one finds that the dynamical model has precisely the symmetries of the \(q = 3\) Potts model. In fact, the mapping (26) with the periodicity induced by the modulo operations, is invariant under the transformation \(x \rightarrow x + 1\), which generates the even permutations of the 3 states. Since it is also odd-symmetric around the point \(x = 2/3\), the change \(x \rightarrow 3 - x\) allows for the generation of the odd permutations. One has then that if at some given time one changes the integer part of \(x\) in all sites according to any of the 5 nontrivial permutations of the set of three objects, all measurable quantities in the model remain the same for all subsequent iterations, just as in the equilibrium Potts Model. Again, there is no perfect equivalence of the two models, and so the critical behaviour of the dynamical model remains an open question.

The quantities to be studied are based in the local ‘spins’ \(S_r(t)\), defined as the integer part of \(x_r(t)\) and taking therefore the values 0, 1 and 2. From these spins an instantaneous order
Figure 8. Phase diagram for the dynamical analogue of the $q = 3$ Potts Model. Here both simultaneous (squares) and sequential (triangles) updating have been explored. The lines are given only as a guide to the eye.

parameter is constructed, following the usual definition: for a lattice with $N$ sites, let $n_q(t)$ be the number of sites whose spin $S = q$ at time $t$, and let $n(t)$ be the greatest among them, i.e. $n(t) = \max(n_1(t), n_2(t), n_3(t))$; then the instantaneous order parameter $m'^L_t$ is given by:

$$m'^L_t = \frac{3n(t) - N}{2N}. \quad (28)$$

Once this order parameter is defined, all other quantities are defined by the expressions already given in section 2. Some care has to be taken about the energy, which in this case is defined as in the original Potts Model, by

$$e_t = \frac{2}{NZ} \sum_{(r,r')} \delta(S_r(t)S_{r'}(t)). \quad (29)$$

FSS tools similar to those used for the MH model are used here to estimate the critical coupling and exponents.

7. Results for Simultaneous Updating

In this model, under simultaneous updating, one finds an order-disorder PT that depends upon the coupling intensity $\epsilon$ and the slope $g$ of the chaotic map, as can be seen in figure (8), where a phase diagram is presented. A reentrance behaviour is observed within a range of values for $g$, where the system can go from a disordered phase to an ordered one and back to the disordered phase as the coupling intensity increases. This is a common phenomenon for these type of CMLs in square lattices, but gets suppressed in triangular lattices. For simultaneous updating these PTs are found to be continuous transitions, as no hysteresis has been observed.

For the evaluation of critical exponents, the continuous PT that the system presents when the slope of the chaotic map varies has been studied, using a fixed coupling intensity ($\epsilon = 0.8$).
Simulations were carried out for lattice sizes $L = 16, 20, 26, 32, 40, 50, 64, 80,$ and 102; for one sample (a given value of $L$ and slope) the transient time is taken to be 4 times the correlation time for $g$ in the critical region, time intervals for averages are set at 500 times the correlation time; and no less than 200 samples are collected for each point.

From the fourth order cumulants the critical point is found to be located at $g_c = 3.9574(2)$, with a correction exponent $\omega/\nu = 1.2(2)$. The values obtained for the critical exponents are: $\nu = 0.82(2), \beta/\nu = 0.15(1)$ and $\gamma/\nu = 1.68(3)$. These values are to be compared with the reported critical exponents for the 2D, $q = 3$ Pots model: $\nu = 5/6, \beta/\nu = 2/15$, and $\gamma/\nu = 26/15$ [19]. As can be seen in table 2, there is a good agreement for the correlation-distance exponent $\nu$, while $\beta/\nu$ and $\gamma/\nu$ show some marginal agreement. Still, these two exponents do not deviate enough from their equilibrium values as to really suggest that the dynamical model does not follow universality. It is more plausible that the strong corrections to scaling also present for this model originate the deviations found.

8. Conclusions

This work shows two dynamical analogues to well-known equilibrium models, the 2D Ising Model and the 2D $q = 3$ Potts Model, and finds that their critical behaviour, under simultaneous updating, agrees with their corresponding equilibrium universality classes. One gets then a particular type of synchronization in extended dynamical systems that joins smoothly with the old field of equilibrium Statistical Mechanics.

The question of whether these models do follow the expected universality is not moot, since, as mentioned in the introduction, there are several results in the literature pointing to non-universal behaviour, and some other results do show that there are indeed differences between equilibrium models and their associated CMLs. An interesting question that emerges here is how these simple models behave under other types of approximation. In particular, it should be mentioned that the Mean Field (MF) limit of the MH Model offers different behaviours depending on how it is implemented: an imposed breakdown of closest-neighbor correlations gives a regular continuous transition, in the MF universality class [20]. An implementation of the same MF limit using fully connected graphs gives a first order transition [21]. In the dynamical analogue of the Potts Model mentioned in this work it has also been found that sequential updating gives origin to a tricritical point [11]. It is therefore clear that for these dynamical systems, symmetries and range of interactions are not enough to predict their critical behaviour.

A particular difficulty with these systems is that the FSCs are always so large that it becomes a very delicate operation to separate the signal from the corrections. An attempt was done to reduce these corrections via the introduction of a second coupling, following the equivalent idea for equilibrium models given in [22], but with negative results: this second coupling increased the corrections [23].
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