Geometric measures of discordlike quantum correlations based on Tsallis relative entropy

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In this paper we formulate a kind of new geometric measure of quantum correlations. This new measure is in terms of the quantum Tsallis relative entropy and can be viewed as a one-parameter extension quantum discordlike measure that satisfies all requirements of a good measure of quantum correlations. It is of an elegant analytic expression and contains several existing good quantum correlation measures as special cases.

I. INTRODUCTION

Quantum correlation beyond entanglement [1, 2] has many important applications in the quantum information theory [3], so finding physically meaningful and mathematically rigorous quantifiers of the quantum correlations are a long-standing central problem among the community of quantum information science, and various useful measures have been established so far [4, 5, 6, 7]. Historically, the discordlike quantum correlation measures can be categorized roughly into two different families, namely, those based on quantum entropy theory, and those based on various distance measures of quantum states (also called the geometric quantum discord measures). For the first category one can see [7] and [8] for a detailed overview. And the recent paper [9] provided an elegant review of the second category of discordlike quantum correlation measures.

It has been shown that there are quantum correlations that may arise without entanglement, such as quantum discord [10] and measurement-induced nonlocality [11]. For recent progresses and its applications one can see [12] and [13].

Quantum coherence [12], on the other hand, is another important resource in the quantum mechanics [13]. And it turns out that there is an intimate relationship between the quantum coherence and quantum correlations [8]. Recently, the coherence measure based on quantum Tsallis relative entropy [14] had been formulated and its modified version [15] is showed to be a bona fide coherence quantifier. So it is naturally to ask whether there is or not a similar correlation measure. In this paper, we study the geometric measure of discordlike quantum correlations in terms of quantum Tsallis relative entropy and formulate a class of new definitions of quantum correlation measure which satisfies all the requirements of a good quantum correlation measure. Furthermore, this kind of measures enjoy an elegant analytic expression. Moreover, our new measure can be viewed as a one-parameter extension of von Neumann relative entropy quantum correlation measure and contains the geometric discord measures based on Hellinger distance [5] and skew information [6] as special cases.

II. TSALLIS RELATIVE ENTROPY

For the convenience of the readers, we simply review some basic properties about Tsallis relative entropy. For two probability distributions $p$ and $q$ on index set $I$, and for $0 < \alpha \neq 1$, the Tsallis relative $\alpha$-entropy is defined as [16, 17]

$$D_\alpha(p||q) = \frac{1}{\alpha - 1} \left( \sum_{j \in I} p_j^\alpha q_j^{1-\alpha} - 1 \right).$$

For $0 < \alpha \neq 1$ and real $\xi > 0$ the $\alpha$-logarithm is defined as [16]:

$$\log_\alpha(\xi) = \frac{\xi^{1-\alpha} - 1}{1 - \alpha}.$$

For $\alpha \to 1$, the $\alpha$-logarithm reduces to the usual logarithm. It is easy to see that $\log_\alpha(\xi)$ is strictly concave and

$$D_\alpha(p||q) = -\sum_j p_j \log_\alpha \left( \frac{q_j}{p_j} \right) \geq 0,$$

with equality if and only if $p_j = q_j$ for all $j \in I$.

Now we consider the quantum case. Given a Hilbert space $\mathcal{H}$ and two quantum density operators $\rho$ and $\sigma$ on $\mathcal{H}$, for $\alpha \in \left[\frac{1}{2}, 1\right] \cup (1, 2]$, the Tsallis relative entropy is defined as [14]

$$D_\alpha(\rho||\sigma) = \frac{\text{Tr}(\rho^\alpha \sigma^{1-\alpha}) - 1}{\alpha - 1}. \quad (1)$$

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When $\alpha \to 1$, $D_\alpha (\rho)$ reduces to the usual von Neumann relative entropy $S(\rho | \sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma)$. One can see that the Tsallis relative entropy is nonnegative \[14\]: $D_\alpha (\rho | \sigma) \geq 0$ with equality if and only if $\rho = \sigma$. Furthermore, the quantum Tsallis relative entropy $D_\alpha (\rho | \sigma)$ is monotone under TPCC maps for $\alpha \in (0,1) \cup (1,2] \ [14]$. That is,

$$D_\alpha (\Phi (\rho) | \Phi (\sigma)) \leq D_\alpha (\rho | \sigma),$$  

(2)

for any TPCC map $\Phi$.

At the meanwhile, $D_\alpha (\rho | \sigma)$ is jointly convex for $\alpha \in (0,1) \cup (1,2] \ [14]$, i.e., if $\{p_j\}$ is a probability distribution and $\rho_j$ and $\sigma_j$ are quantum states, then

$$D_\alpha \left( \sum_j p_j \rho_j \| \sum_j p_j \sigma_j \right) \leq \sum_j p_j D_\alpha (\rho_j | \sigma_j).$$  

(3)

See \[18\] for an elegant proof of the jointly concavity and monotonicity of quantum Tsallis relative entropy.

In \[14\], the author introduced a coherence measure based on the quantum $\alpha$ divergence and obtained an analytic expressions as following.

For a fixed basis $\{|i\rangle\}$ of $\mathcal{H}$, $\mathcal{I}$ denotes the set of all incoherence states which are diagonal in basis $\{|i\rangle\}$. The coherence measure based on Tsallis relative entropy is defined as \[14\]

$$C_\alpha (\rho) = \min_{\delta \in \mathcal{I}} D_\alpha (\rho | \delta)$$

$$= \frac{1}{\alpha - 1} \left\{ \left( \sum_j \langle j | \rho^\alpha | j \rangle^\frac{1}{\alpha} \right)^\alpha - 1 \right\}.$$  

(4)

Although this measure enjoys some good properties such as elegant expression, the explicitly optimal incoherence state, convexity and modified strong monotonicity under incoherent operations, it violates the strong monotonicity at some special examples. For this reason, Zhao and Yu \[15\] formulated the following modified coherence measure:

$$\tilde{C}_\alpha (\rho) = \min_{\delta \in \mathcal{I}} \frac{1}{\alpha - 1} \left( \text{Tr}(\rho^\alpha \delta^{1-\alpha}) \right)^\frac{1}{\alpha} - 1$$

$$= \frac{1}{\alpha - 1} \left( \sum_j \langle j | \rho^\alpha | j \rangle^\frac{1}{\alpha} - 1 \right),$$  

(5)

for $\alpha \in (0,1) \cup (1,2]$. This modified measure inherits some good properties of (4) and does satisfy the strong monotonicity. Moreover, the proof method provided in \[15\] is very insightful and for our use we only recall one of the observations in \[15\]:

**Observation:** If $f_\alpha (\rho, \sigma) = \text{Tr}(\rho^\alpha \sigma^{1-\alpha})$, then $f_\alpha (\rho, \sigma) \geq 1$ for $\alpha \in (1,2]$ and $f_\alpha (\rho, \sigma) \leq 1$ for $\alpha \in (0,1)$ with equality if and only if $\rho = \sigma$.

The main result of this paper is to characterize the quantum correlations in bipartite system in terms of the above Tsallis type of relative entropy.

### III. GEOMETRIC MEASURES OF QUANTUM CORRELATIONS IN BIPARTITE SYSTEM

Given an Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Recall that for a contractive distance $D$, the geometric measure of quantum correlations induced by $D$ is defined as

$$D^G (\rho) = \min_{\chi \in \mathcal{A}} D(\rho, \chi),$$

(6)

where $\rho$ is a bipartite state and $\mathcal{A}$ is the set of all classical-quantum state and for any $\chi \in \mathcal{A}$ is of the form $\chi = \sum_j q_j | j \rangle \langle j | \otimes \sigma_j$ in some basis $\{|j\rangle\}$.

Similarly for any bipartite state $\rho_{AB}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ we define the geometric measure of quantum correlation for $\alpha \in (0,1) \cup (1,2]$ as

$$Q_\alpha (\rho_{AB}) = \min_{\chi \in \mathcal{A}} D_\alpha (\rho | | \chi),$$  

(7)

where $\chi$ ranges over all classical-quantum state $\mathcal{A}$. For a fixed basis $\{|i\rangle\}$, $\chi$ is of the form

$$\chi = \sum_i | i \rangle \langle i | \otimes Y_i,$$

where $Y_i \geq 0$ are positive operators on $\mathcal{H}_B$ and $\sum_i \text{Tr}(Y_i) = 1$. In the same way, $\rho^\alpha$ can be written as

$$\rho^\alpha = \sum_{i,j} | i \rangle \langle j | \otimes R_{ij}$$

where $R_{ij}$ are operators on $\mathcal{H}_B$ which need not all to be positive but $R_{ii}$ are exactly positive. Then

$$\text{Tr}(\rho^\alpha \chi^{1-\alpha}) = \sum_i \text{Tr} \left( R_{ii} Y_i^{1-\alpha} \right).$$

Set $X_i = R_{ii}^{1/\alpha} \geq 0$,

$$D_\alpha (\rho | | \chi) = \frac{1}{\alpha - 1} \left( \sum_i \text{Tr} \left( X_i^\alpha Y_i^{1-\alpha} \right) - 1 \right),$$

by Hölder’s inequality for matrix and its equality condition \[19\] and the observation in the previous section, for $0 < \alpha < 1$, we have

$$D_\alpha \geq \frac{1}{\alpha - 1} \left( \sum_i \text{Tr} (X_i^\alpha \text{Tr}(Y_i)^{1-\alpha}) - 1 \right),$$

with equality if and only if $X_i = k_i Y_i$ for some $k_i \geq 0$. Then if we set $\text{Tr}(X_i) = r_i$ and $\text{Tr}(Y_i) = s_i$, we have

$$D_\alpha (\rho | | \chi) \geq \frac{1}{\alpha - 1} \left( \sum_i r_i^\alpha s_i^{1-\alpha} - 1 \right) \geq \frac{N^\alpha - 1}{\alpha - 1},$$

$$\geq \frac{N^\alpha - 1}{\alpha - 1},$$

for $\alpha$ large in $\mathcal{A}$.
where $N = \sum_i \text{Tr}(X_i) = \sum_i r_i$ and the second inequality is due to $D_\alpha(\chi||s_i) \geq 0$ with equality if and only if $\frac{r_i}{s_i} = s_i$. The first inequality with equality if and only if $X_i = k_i Y_i$ for some $k_i \geq 0$, which means that $r_i = N s_i = k_i s_i$. This is equivalently to say that $D_\alpha(\rho||\chi) = \frac{N^\alpha-1}{\alpha-1}$ if and only if $X_i = \frac{1}{\alpha} Y_i$, thus we can conclude that the minimum can be obtained if $Y_i = \frac{1}{\alpha} X_i$. Then for $Q_\alpha(\rho_{AB})$, we obtain the analytic expression as

$$Q_\alpha(\rho_{AB}) = \min_{\chi \in A} D_\alpha(\rho_{AB}||\chi) = \min_{\chi \in A} \frac{N^\alpha-1}{\alpha-1},$$

where $N = \sum_i \text{Tr}(X_i)$ and $X_i = |i\rangle\langle i|\rho^\alpha |i\rangle^0$. Furthermore, the optimal classical-quantum state $\chi$ is of the form $\chi = \sum_i |i\rangle\langle i| \otimes Y_i$ where $Y_i = \frac{1}{\alpha} |i\rangle\langle i|\rho^\alpha |i\rangle^0$.

Note that the above derivation is only focused on $0 < \alpha < 1$, but for $1 < \alpha < 2$, by the reverse Hölder’s inequality and its equality condition and the observation in preliminaries, our conclusion preserves unchanged. Moreover, if we set $\alpha = \frac{1}{2}$, we obtained the correlation measure based on the Hellinger distance, see [3] for more details. Thus our work generalized the results in [3] $\alpha = \frac{1}{2}$ to $\alpha \in (0, 1) \cup (1, 2]$ case.

In the next we show that $Q_\alpha(\rho_{AB})$ satisfies some properties:

1. $Q_\alpha(\rho_{AB}) \geq 0$ with $Q_\alpha(\rho) = 0$ if and only if $\rho$ is a classical-quantum state;
2. $Q_\alpha(\rho_{AB}) = Q_\alpha(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger)$;
3. $Q_\alpha(\rho_{AB}) \geq Q_\alpha(\Phi_B(\rho_{AB}))$ for local TPCP map $\Phi_B$ on system $B$.

For property 1, $Q_\alpha(\rho_{AB}) \geq 0$ by definition. If $\rho_{AB}$ is a classical-quantum state, from the deduction of equality (8) we can find an optimal state in $A$. On the other hand, if $Q_\alpha(\rho_{AB}) = 0$, since $Q = \min_{\chi \in A} D_\alpha(\rho_{AB})$, there exists a state $\chi_0 \in A$ such that $D_\alpha(\rho_{AB}||\chi_0) = 0$. The Tsallis relative entropy $D_\alpha(\rho||\sigma)$ is of the form $\chi_0 \in A$. Property 3 can be obtained from the monotonicity of $D_\alpha(\rho||\sigma)$, so we only need to prove property 2. Since

$$D_\alpha(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger) = D_\alpha(\rho||\chi'),$$

where $\chi' = U_A^\dagger \otimes U_B^\dagger \chi U_A \otimes U_B$ is also a classical-quantum state in another local basis $\{|i^{A}\rangle\}$. After the minimization over all local basis $\{|i^{A}\rangle\}$, the results follows.

Thus we formulate a new class of bona fide geometric measures of quantum correlations in bipartite system based on Tsallis relative entropy.

IV. CORRELATION MEASURES BASED ON THE MODIFIED QUANTUM TSALLIS RELATIVE ENTROPY

Inspired by the work in [13], in this section, we show that the modified quantum Tsallis relative entropy can also be used to quantify quantum correlations. Denote $f(\rho, \sigma) = \text{Tr}(\rho^\alpha \sigma^{1-\alpha})$, a new coherence measure can be defined as

$$C_\alpha(\rho) = \min_{\delta \in \mathcal{L}} \frac{f^{1/\alpha}(\rho, \delta) - 1}{\alpha - 1}. \tag{9}$$

Similarly, we can define a correlation measure as

$$\tilde{Q}_\alpha(\rho_{AB}) = \frac{f^{1/\alpha}(\rho_{AB}, \chi) - 1}{\alpha - 1}, \tag{10}$$

by virtue of the same proof method in section III and the fact of the monotonicity of function $g(x) = x^t$ for any fixed $t > 0$, we have

$$\tilde{Q}_\alpha(\rho_{AB}) = \min_{\chi \in A} \frac{N - 1}{\alpha - 1}. \tag{11}$$

Moreover, when $\alpha = \frac{1}{2}$, our correlation measure is just the correlation measure defined in [3], which is induced by skew information, up to a factor 2. By the way, if $\rho_{AB}$ is a pure bipartite state $\rho_{AB} = |\psi_{AB}\rangle\langle \psi_{AB}|$ where $|\psi_{AB}\rangle = \sum_j \sqrt{\lambda_j}|j_A \rangle |j_B\rangle$ is its Schmidt decomposition, by simple calculation,

$$\tilde{Q}_\alpha(|\psi_{AB}\rangle) = \sum_j \sqrt{\lambda_j^{1/\alpha} - 1} \frac{1}{\alpha - 1},$$

which is a measure of entanglement in terms of Tsallis entropy [20], up to a constant factor.

V. SOME ILLUMINATED EXAMPLES

In this section, we present the analytic form of the geometric measure for Werner state and isotropic state in arbitrary dimensions.

Example 1. The $d \otimes d$ Werner state is

$$\rho_W = \frac{d - x}{d^3 - d} |I_d \otimes I_d + dx^3 - 3 dx^2 - 3dV|, \quad x \in [-1, 1], \tag{12}$$

with $V = \sum_{ij} |ij\rangle\langle ji|$ the swap operator. From the description previous, the key point is the calculation of $N = \sum_i \text{Tr}(|i\rangle\langle i|\rho^\alpha|a\rangle)$. In this case,

$$N_W = 1 + \frac{x}{d} + (d - 1) \left\{ \frac{1}{2} \left( \frac{1 + x}{d + 1} \right)^\alpha + \frac{1}{2} \left( 1 - \frac{x}{d - 1} \right)^\alpha \right\}^{1/\alpha}. \tag{13}$$

$Q_\alpha(\rho_W) = 0$ when $x = \frac{1}{d}$. Also if $\alpha = \frac{1}{2}$, our result coincides with [3] [21].

Example 2. The $d \otimes d$ isotropic state is

$$\rho_I = \frac{1 - x}{d^2 - 1} |I_d \otimes I_d + dx^2 - 1 dx^2 - 1|\Phi\rangle\langle \Phi|, \quad x \in [0, 1], \tag{14}$$

where $|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_k |kk\rangle$. The key quantity $N$ can be obtained as

$$N_I = \frac{d(1 - x)}{d^2 - 1} + \left\{ (m - 1) \left( \frac{1 - x}{d^2 - 1} \right)^\alpha + x^\alpha \right\}^{1/\alpha}. \tag{15}$$

And when $x = \frac{1}{d}$, $Q_\alpha(\rho_I) = 0$. 
VI. QUANTUM COHERENCE MEASURE IN LÜDERS MEASUREMENT PICTURE

Recently, the coherence measures are generalized to Lüders measurement picture as a partial coherence quantification \([21] [22] [23] [24]\). In this section we show that the coherence measure in terms of quantum Tsallis relative entropy can be also generalized into this setting.

For a Hilbert space \(\mathcal{H}\) and a fixed Lüders measurement \(\{\Pi_j, j = 1, \cdots, m\}\), the incoherent states with respect to this Lüders measurement can be written as

\[
\mathcal{I}_L = \{\delta \mid \sum_j \Pi_j \delta \Pi_j = \delta\}. \tag{16}
\]

For any density operator \(\rho\) on \(\mathcal{H}\) and any \(\alpha \in (0, 1) \cup (1, 2]\), we define the partial coherence measure as

\[
C_{\alpha}(\rho) = \min_{\delta \in \mathcal{I}_L} D_{\alpha}(\rho|\delta). \tag{17}
\]

\(\rho^a\) can be written as

\[
\rho^a = \begin{pmatrix}
R_{11} & \cdots & R_{1m} \\
\vdots & \ddots & \vdots \\
R_{m1} & \cdots & R_{mm}
\end{pmatrix},
\]

where the partition is in conformable with the Lüders measurement \(\{\Pi_j\}\), and \(R_{ii}\) are positive. Similarly, for any \(\delta \in \mathcal{I}_L\),

\[
\delta = \begin{pmatrix}
Y_1 & 0 & \cdots & 0 \\
0 & Y_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & Y_m
\end{pmatrix}.
\]

Set \(X_i = R_{ii}^{1/\alpha}\), and by similar calculation in previous section, we have

\[
C_{\alpha}(\rho) = \frac{N^\alpha - 1}{\alpha - 1}, \tag{18}
\]

where \(N = \sum_j \text{Tr}\{(\Pi_j \rho^a \Pi_j)^{1/\alpha}\}\).

Note that recently Rastegin \([25]\) had calculated the same quantity as ours, but we think our results is more precise, since we consider the general incoherent state, or equivalently, the block diagonal states with respect to the given Lüders measurement.

Furthermore, for \(\alpha \in (0, 1) \cup (1, 2]\), if we define

\[
\tilde{C}_{\alpha}(\rho) = \min_{\delta \in \mathcal{I}_L} \tilde{D}_{\alpha}(\rho|\delta), \tag{19}
\]

we can obtain the analytic expression:

\[
\tilde{C}_{\alpha}(\rho) = \frac{N - 1}{\alpha - 1}, \tag{20}
\]

where \(N = \sum_j \text{Tr}\{(\Pi_j \rho^a \Pi_j)^{1/\alpha}\}\). And the optimal incoherent state is

\[
\tilde{\delta}_L = \frac{1}{N} \sum_j (\Pi_j \rho^a \Pi_j)^{1/\alpha}.
\]

What is more, if we set \(\alpha = \frac{1}{2}\), we reproduce the quantum uncertainty measure formulated in \([22]\), up to a factor 2.

VII. CONCLUDING REMARKS

In this paper we formulated a class of new quantum correlation measures which based on quantum Tsallis relative entropy and its modified version, which has been shown not only enjoy elegant analytic expressions, but also contains many novel known quantum correlation measures as special cases. It is also can be viewed as an one-parameter extension of the von Neumann relative entropy correlation measure. The future work is to find some applications and operational interpretations in quantum information processing tasks.

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[1] Leah Henderson and Vlatko Vedral. Classical, quantum and total correlations. *Journal of physics A: mathematical and general*, 34(35):6899, 2001.
[2] Marco Genovese. Research on hidden variable theories: A review of recent progresses. *Physics Reports*, 413(6):319–396, 2005.
[3] Ryszard Horodecki, Pawel Horodecki, Michal Horodecki, and Karol Horodecki. Quantum entanglement. *Reviews of modern physics*, 81(2):685, 2009.
[4] Anindita Bera, Tamoghna Das, Debasis Sadhukhan, Sudipto Singha Roy, Aditi Sen De, and Ujjwal Sen. Quantum discord and its allies: a review of recent progress. *Reports on Progress in Physics*, 81(2):024001, 2017.
[5] Wojciech Roga, Dominique Spehner, and Fabrizio Illuminati. Geometric measures of quantum correlations: characterization, quantification, and comparison by distances and operations. *Journal of Physics A: Mathematical and Theoretical*, 49(23):235301, 2016.
[6] Chang-shui Yu, Shao-xiong Wu, Xiaoguang Wang, XX Yi, and He-shan Song. Quantum correlation measure in arbitrary bipartite systems. *EPL (Europhysics Letters)*, 107(1):10007, 2014.
[7] Kavan Modi, Aharon Brodutch, Hugo Cable, Tomasz Paterek, and Vlatko Vedral. The classical-quantum bounda-
ry for correlations: discord and related measures. *Reviews of Modern Physics*, 84(4):1655, 2012.

[8] Gerardo Adesso, Thomas R Bromley, and Marco Ciancianuso. Measures and applications of quantum correlations. *Journal of Physics A: Mathematical and Theoretical*, 49(47):473001, 2016.

[9] Ming-Liang Hu, Xueyuan Hu, Jieci Wang, Yi Peng, Yu-Ran Zhang, and Heng Fan. Quantum coherence and geometric quantum discord. *Physics Reports*, 2018.

[10] Harold Ollivier and Wojciech H Zurek. Quantum discord: a measure of the quantumness of correlations. *Physical review letters*, 88(1):017901, 2001.

[11] Shunlong Luo and Shuangshuang Fu. Measurement-induced nonlocality. *Physical review letters*, 106(12):120401, 2011.

[12] Tillmann Baumgratz, Marcus Cramer, and MB Plenio. Quantifying coherence. *Physical review letters*, 113(14):140401, 2014.

[13] Alexander Streltsov, Gerardo Adesso, and Martin B Plenio. Colloquium: quantum coherence as a resource. *Reviews of Modern Physics*, 89(4):041003, 2017.

[14] Alexey E Rastegin. Quantum-coherence quantifiers based on the tsallis relative α entropies. *Physical Review A*, 93(3):032136, 2016.

[15] Haiqing Zhao and Chang-shui Yu. Coherence measure in terms of the tsallis relative α entropy. *Scientific reports*, 8(1):299, 2018.

[16] Lisa Borland, Angel R Plastino, and Constantino Tsallis. Information gain within nonextensive thermostatistics. *Journal of Mathematical Physics*, 39(12):6490–6501, 1998.

[17] Shigeru Furuichi, Kenjiro Yanagi, and Ken Kuriyama. Fundamental properties of tsallis relative entropy. *Journal of Mathematical Physics*, 45(12):4868–4877, 2004.

[18] Anna Jenčová and Mary Beth Ruskai. A unified treatment of convexity of relative entropy and related trace functions, with conditions for equality. *Reviews in Mathematical Physics*, 22(09):1099–1121, 2010.

[19] Gabriel Larotonda. The case of equality in hölder’s inequality for matrices and operators. In *Mathematical Proceedings of the Royal Irish Academy*, volume 118, pages 1–4. JSTOR, 2018.

[20] Yu Luo, Tian Tian, Lian-He Shao, and Yongming Li. General monogamy of tsallis q-entropy entanglement in multiqubit systems. *Physical Review A*, 93(6):062340, 2016.

[21] Yuan Sun, Yuanyuan Mao, and Shunlong Luo. From quantum coherence to quantum correlations. *EPL (Europhysics Letters)*, 118(6):60007, 2017.

[22] Shunlong Luo and Yuan Sun. Quantum coherence versus quantum uncertainty. *Physical Review A*, 96(2):022130, 2017.

[23] Shunlong Luo and Yuan Sun. Partial coherence with application to the monotonicity problem of coherence involving skew information. *Physical Review A*, 96(2):022136, 2017.

[24] Shunlong Luo and Yuan Sun. Coherence and complementarity in state-channel interaction. *Physical Review A*, 98(1):012113, 2018.

[25] Alexey E Rastegin. Coherence quantifiers from the viewpoint of their decreases in the measurement process. *Journal of Physics A: Mathematical and Theoretical*, 2018.