BOUNDED COHOMOLOGY, CROSS RATIOS AND COCYCLES

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Abstract. We use cross ratios to describe second real continuous bounded cohomology for locally compact σ-compact topological groups. We also investigate the second continuous bounded cohomology group of a closed subgroup of the isometry group Iso(X) of a proper hyperbolic geodesic metric space X and derive some rigidity results for Iso(X)-valued cocycles.

1. Introduction

A geodesic metric space X is called δ-hyperbolic for some δ > 0 if it satisfies the δ-thin triangle condition: For every geodesic triangle in X with sides a, b, c the side a is contained in the δ-neighborhood of b ∪ c. If X is proper then X can naturally be compactified by adding the Gromov boundary ∂X. The isometry group Iso(X) of X, equipped with the compact open topology, is a locally compact σ-compact topological group which acts on X ∪ ∂X as a group of homeomorphisms. The limit set of a closed subgroup G of Iso(X) is the set of accumulation points in ∂X of an orbit of the action of G on X; this limit set is a compact G-invariant subset of ∂X. The group G is called elementary if its limit set consists of at most two points.

Let S be a standard Borel space and let µ be a Borel probability measure on S. Let Γ be a countable group which admits a measure preserving ergodic action on (S, µ). An Iso(X)-valued cocycle for this action is a measurable map α : Γ × S → Iso(X) such that α(gh, x) = α(g, hx)α(h, x) for all g, h ∈ Γ and µ-almost every x ∈ S. The cocycle α is cohomologous to a cocycle β : Γ × S → Iso(X) if there is a measurable map ϕ : S → Iso(X) such that ϕ(gx)α(g, x) = β(g, x)ϕ(x) for all g ∈ G, µ-almost every x ∈ S. Recall that a compact extension of a locally compact topological group G is a locally compact topological group H which contains a normal compact subgroup K such that G = H/K as topological groups. Extending earlier results of Monod and Shalom [14] we show in Section 4.

Theorem A: Let G be a semi-simple Lie group with finite center, no compact factors and of rank at least 2. Let Γ < G be an irreducible lattice which admits a mixing measure preserving action on a probability space (S, µ). Let X be a proper hyperbolic geodesic metric space and let α : Γ × S → Iso(X) be a cocycle; then one of the following two possibilities holds.

(1) α is cohomologous to a cocycle into an elementary subgroup of Iso(X).

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(2) \( \alpha \) is cohomologous to a cocycle into a subgroup \( H \) of \( \text{Iso}(X) \) which is a compact extension of a simple Lie group \( L \) of rank one, and there is a continuous surjective homomorphism \( G \to L \).

The proof of this result uses second bounded cohomology for closed subgroups of the isometry group \( \text{Iso}(X) \) of \( X \). Here the second bounded cohomology group of a locally compact topological group \( G \) with coefficients in a Banach module \( E \) for \( G \), i.e. a Banach space \( E \) together with a continuous homomorphism of \( G \) into the group of linear isometries of \( E \), is defined as follows. For every \( i \geq 1 \), the group \( G \) naturally acts on the vector space \( C_b(G_i, E) \) of continuous bounded maps \( G_i \to E \). If we denote by \( C_b(G_i, E) \) the linear subspace of all \( G \)-invariant such maps, then the second continuous bounded cohomology group \( H^2_{cb}(G, E) \) of \( G \) with coefficients \( E \) is defined as the second cohomology group of the complex

\[
0 \to C_b(G, E)^G \xrightarrow{d} C_b(G^2, E)^G \xrightarrow{d} \ldots
\]

with the usual homogeneous coboundary operator \( d \) (see [13]). If \( G \) is a countable group with the discrete topology then we write \( H^2_{cb}(G, E) = H^2_c(G, E) \). Of particular importance is the case that \( E = \mathbb{R} \) with the trivial \( G \)-action which yields real continuous second bounded cohomology. There is a natural map of \( H^2_{cb}(G, \mathbb{R}) \) to the ordinary continuous second cohomology group \( H^2_c(G, \mathbb{R}) \) of \( G \) which in general is neither injective nor surjective.

If \( G \) is a non-elementary closed subgroup of the isometry group of a proper hyperbolic geodesic metric space \( X \) with limit set \( \Lambda \subset \partial X \) then the kernel of the natural map \( H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R}) \) is infinite dimensional [9]. In contrast, we show in Section 3.

**Theorem B**: Let \( X \) be a proper hyperbolic geodesic metric space and let \( G < \text{Iso}(X) \) be a non-elementary closed subgroup with limit set \( \Lambda \). If \( G \) acts transitively on the complement of the diagonal in \( \Lambda \times \Lambda \) then the kernel of the natural map \( H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R}) \) is trivial.

In Section 2 we identify for an arbitrary locally compact \( \sigma \)-compact topological group \( G \) the group \( H^2_{cb}(G, \mathbb{R}) \) with the vector space of measurable anti-symmetric \( G \)-cross ratios on a strong boundary for \( G \). As an application we describe the kernel of the natural map \( H^2_{cb}(\Gamma, \mathbb{R}) \to H^2_c(\Gamma, \mathbb{R}) \) for a lattice \( \Gamma \) in \( \text{PSL}(2, \mathbb{R}) \). Namely, recall that \( \Gamma \) acts as a group of orientation preserving isometries on the hyperbolic plane \( \mathbb{H}^2 \) and hence it acts on the boundary \( S^1 = \partial \mathbb{H}^2 \) of \( \mathbb{H}^2 \) preserving the measure class of the Lebesgue measure \( \lambda \). We show that the kernel of the natural map \( H^2_{cb}(\Gamma, \mathbb{R}) \to H^2_c(\Gamma, \mathbb{R}) \) is naturally isomorphic to the vector space of all finite \( \Gamma \)-invariant reflection-anti-invariant finitely additive signed measures \( \mu \) on the complement of the diagonal in \( S^1 \times S^1 \) with the additional property that the map \( (a, b, c, d) \in S^1 \to \mu(a, b) \times [c, d) \) is \( \lambda \)-measurable.
2. Cross-ratios and bounded cohomology

Let $X$ be an arbitrary set with infinitely many points. We define an anti-symmetric cross ratio for $X$ to be a bounded function $[ ]$ on the space of quadruples of pairwise distinct points in $X$ with the following additional properties.

1. $[\xi, \xi', \eta, \eta'] = -[\xi', \xi, \eta, \eta']$.
2. $[\eta, \eta', \xi, \xi'] = -[\xi, \xi', \eta, \eta']$.
3. $[\xi, \xi', \eta, \eta'] + [\xi', \xi'', \eta, \eta'] = [\xi, \xi'', \eta, \eta']$.

Note that $[ ]$ can naturally be extended by 0 to the set of quadruples of points in $X$ of the form $(\xi, \xi, \eta, \zeta)$ where $\xi, \eta, \zeta$ are pairwise distinct.

Let $\Gamma$ be an arbitrary countable group which acts on $X$. We define a $\Gamma$-cross ratio on $X$ to be an anti-symmetric cross ratio on $X$ which is invariant under the action of $\Gamma$. Let $C(\Gamma, X)$ be the vector space of all anti-symmetric $\Gamma$-cross ratios on $X$, equipped with the supremums-norm $\| \|$.

The second real bounded cohomology group $H^2_b(\Gamma, \mathbb{R})$ of $\Gamma$ is a vector space of equivalence classes of bounded functions on $\Gamma \times \Gamma \times \Gamma$ which are invariant under the free left action of $\Gamma$. It can be equipped with a natural pseudo-norm which assigns to a cohomology class the infimum of the supremums-norms of any representative of the class. For the second bounded cohomology, this pseudo-norm turns out to be a norm which provides $H^2_b(\Gamma, \mathbb{R})$ with the structure of a Banach space. We have.

**Proposition 2.1:** There is a continuous linear map $\Omega : C(\Gamma, X) \to H^2_b(\Gamma, \mathbb{R})$.

**Proof:** Let $[ ] \in C(\Gamma, X)$ and let $(\xi, \zeta, \eta)$ be a triple of pairwise distinct points in $X$. Choose an arbitrary point $\nu \in X$ which is distinct from any of the points in the triple and define

$$\varphi(\xi, \eta, \zeta) = \frac{1}{2}([\xi, \zeta, \eta, \nu] + [\zeta, \eta, \xi, \nu] + [\eta, \xi, \zeta, \nu]).$$

We claim first that this does not depend on the choice of $\nu$. Namely, let $\nu' \in X - \{\xi, \eta, \zeta\}$ be arbitrary. We compute

$$[\xi, \zeta, \eta, \nu'] + [\zeta, \eta, \xi, \nu'] + [\eta, \xi, \zeta, \nu'] - [\xi, \zeta, \eta, \nu] - [\zeta, \eta, \xi, \nu] - [\eta, \xi, \zeta, \nu]$$

$$= [\xi, \zeta, \nu', \eta] + [\zeta, \eta, \nu', \xi] + [\eta, \xi, \nu', \zeta] + [\eta, \xi, \zeta, \nu] + [\eta, \xi, \nu', \zeta] + [\eta, \xi, \zeta, \nu]$$

$$= [\xi, \zeta, \nu', \nu] + [\zeta, \eta, \nu', \nu] + [\eta, \xi, \nu', \nu] = 0$$

by the defining properties of an anti-symmetric cross ratio. This shows our claim. As a consequence, the function $\varphi$ is well defined, moreover it is invariant under the action of the group $\Gamma$. Its supremums norm is bounded in absolute value by $3\| [ ] \| / 2$. We extend $\varphi$ by 0 to the set of all triples of points in $X$ for which at least two of the points coincide. Then $\varphi$ induces a bounded function $\mu$ on $\Gamma^3$ which is invariant under the free left action of $\Gamma$ as follows. Choose any point $\xi \in X$ and define $\mu(\gamma_1, \gamma_2, \gamma_3) = \varphi(\gamma_1 \xi, \gamma_2 \xi, \gamma_3 \xi)$. 


We next show that this function \( \mu \) is a cocycle, i.e. that it is alternating and its image under the natural coboundary map \( L^\infty(\Gamma^3, \mathbb{R}) \rightarrow L^\infty(\Gamma^4, \mathbb{R}) \) vanishes. For this it is enough to show the corresponding properties for the function \( \varphi \). Namely, observe that for a triple \((a, b, c)\) of pairwise distinct points in \( X \) and any \( d \in X - \{a, b, c\} \) we have

\[
2(\varphi(a, c, b) + \varphi(a, b, c)) = [a, b, c, d] + [b, c, a, d] + [c, a, b, d] + [a, c, b, d] + [b, a, c, d] = 2(\varphi(a, c, b) + \varphi(a, b, c)).
\]

which shows that \( \varphi \) is alternating. Next we have to show that

\[
(5) \quad \varphi(b, c, d) - \varphi(a, c, d) + \varphi(a, b, d) - \varphi(a, b, c) = 0
\]

for any quadruple \((a, b, c, d)\) of points in \( X \). By our observation \((4)\), equation \((5)\) is valid for every quadruple \((a, b, c, d)\) of points in \( X \) for which at least 2 of the points coincide. On the other hand, for every quadruple \((a, b, c, d)\) of pairwise distinct points in \( X \) we have

\[
(6) \quad 2(\varphi(a, b, c) - \varphi(b, c, d)) = [a, c, b, d] + [c, b, a, d] + [b, a, c, d] - [d, c, b, a] - [c, b, a, d] = 2[c, b, a, d]
\]

by the defining properties of an anti-symmetric cross ratio. Together with equation \((4)\) we deduce with the same calculation as before that \( \varphi(a, c, d) - \varphi(a, b, d) = -\varphi(c, a, d) + \varphi(a, d, b) = \) \(-[d, a, c, b] = -[c, b, a, d] \) and consequently \( \varphi \) satisfies the cocycle equation \((5)\). Therefore the function \( \mu \) on \( \Gamma \) defines a bounded cocycle and hence a second bounded cohomology class \( \Omega([[\cdot]]) \in H^2_b(\Gamma, \mathbb{R}) \).

We next observe that the cohomology class \( \Omega([[\cdot]]) \) does not depend on the choice of the point \( \xi \in X \) which we used to define the function \( \varphi \). Namely, let \( \eta \) be a different point in \( X \) and define \( \mu'(\gamma_1, \gamma_2, \gamma_3) = \varphi(\gamma_1\eta, \gamma_2\eta, \gamma_3\eta) \). For \( \gamma \in \Gamma \) write

\[
\nu(\gamma) = [\gamma \xi, \eta, \xi, \gamma \eta].
\]

By equation \((4)\), \((6)\) we have

\[
\nu(\gamma) = \varphi(\xi, \eta, \gamma \xi) + \varphi(\gamma \xi, \eta, \gamma \eta)
\]

and consequently \( \mu(e, \gamma_1, \gamma_2) - \mu'(e, \gamma_1, \gamma_2) - \nu(\gamma_1) - \nu(\gamma_2) \) is the value of \( \varphi \) on the boundary of a singular polyhedron of dimension 3 whose vertices consist of the points \( \xi, \gamma_1 \xi, \gamma_1 \eta, \gamma_2 \xi, \gamma_2 \eta \) and whose sides contain the simplices with vertices \( \xi, \gamma_1 \xi, \gamma_2 \xi \) and \( \eta, \gamma_1 \eta, \gamma_2 \eta \) as well as three quadrangles with the same set of vertices. Since \( \varphi \) is a cocycle, the evaluation of \( \varphi \) on this polyhedron vanishes and therefore the function \( \mu - \mu' \) is the image of a \( \Gamma \)-invariant bounded function on \( \Gamma^2 \) under the coboundary map \( L^\infty(\Gamma^3) \rightarrow L^\infty(\Gamma^4) \). As a consequence, for every \([\cdot] \in C(\Gamma, X)\) the bounded cohomology class \( \Omega([[\cdot]]) \in H^2_b(\Gamma, \mathbb{R}) \) only depends on the cross ratio \([\cdot]\) but not on the choice of a point \( \xi \in X \). The resulting map \( \Omega : C(\Gamma, X) \rightarrow H^2_b(\Gamma, \mathbb{R}) \) is clearly linear and continuous with respect to the supremaums norm on \( C(\Gamma, X) \) and the Gromov norm on \( H^2_b(\Gamma, \mathbb{R}) \).

\[
\square
\]

For a topological space \( X \) define a continuous anti-symmetric cross ratio to be a continuous function \( [\cdot] \) on the space of quadruples of pairwise distinct points in \( X \) which is an anti-symmetric cross ratio in the sense of our definition. If \( G \) is any locally compact \( \sigma \)-compact topological group which acts on a locally compact \( \sigma \)-compact topological space \( X \) as a group of homeomorphisms, then we define a continuous anti-symmetric \( G \)-cross-ratio to be a continuous anti-symmetric cross ratio on \( X \) which is invariant under the diagonal action of \( G \).
Every locally compact σ-compact group \( G \) acts freely on itself by left translations. Thus we can use Proposition 2.1 for \( X = G \) to deduce that there is a continuous linear map from the vector space of all continuous anti-symmetric \( G \)-cross ratios on \( G \) into \( H^2_{cb}(G, \mathbb{R}) \). Namely, we have.

**Corollary 2.2:** Let \( G \) be a locally compact σ-compact group; then \( H^2_{cb}(G, \mathbb{R}) \) is the quotient of the space of continuous anti-symmetric \( G \)-cross ratios on \( G \) under the coboundary relation.

**Proof:** By Proposition 2.1 and its proof there is a linear map \( \Omega \) from the space of all continuous anti-symmetric \( G \)-cross ratios on \( G \) into \( H^2_{cb}(G, \mathbb{R}) \). We have to show that \( \Omega \) is surjective. For this recall first that every second continuous bounded cohomology class for \( G \) is the class of a \( G \)-invariant continuous alternating function \( \varphi \) on \( G^3 \) which satisfies the cocycle equation (5) (see [10, 13]). For such a function \( \varphi \) define \( [g, g', h, h'] = \varphi(h, g', g) - \varphi(g', g, h') \) as in the proof of Proposition 2.1. Then \( [\cdot] \) is continuous and \( G \)-invariant; we claim that it satisfies the requirements of an anti-symmetric \( G \)-cross ratio.

Namely, first of all we have

\[
(7) \quad [b, c, a, d] = \varphi(a, c, b) - \varphi(c, b, d) = -\varphi(a, b, c) + \varphi(b, c, d) = -[c, b, a, d]
\]

since \( \varphi \) is alternating and similarly

\[
(8) \quad [a, d, c, b] = \varphi(c, d, a) - \varphi(d, a, b) = -\varphi(a, b, d) + \varphi(a, c, d) = -[c, b, a, d].
\]

In the same way we obtain from the cocycle equation (5) for \( \varphi \) that

\[
(9) \quad [a, b, c, d] + [b, b', c, d] = [a, b', c, d]
\]

which shows that \( [\cdot] \) is indeed a \( G \)-invariant anti-symmetric cross ratio. By the proof of Proposition 2.1, the cohomology class of \( \varphi \) is just \( \Omega([\cdot]) \). Thus the map \( \Omega \) is surjective. \( \square \)

A locally compact σ-compact topological group \( G \) admits a strong boundary [11] which is a standard measure space \((B, \mu)\) together with a measure class preserving action of \( G \) such that the following two conditions are satisfied.

1. The \( G \)-action on \( B \) is amenable.
2. For any separable Banach-\( G \)-module \( E \), any measurable \( G \)-equivariant map \( B \times B \to E \) is essentially constant.

Define a measurable anti-symmetric cross ratio on \( B \) to be a measurable essentially bounded \( G \)-invariant function on the space of quadruples of distinct points in \( B \) which satisfies almost everywhere the defining equation for a cross ratio. Our above considerations immediately imply.

**Lemma 2.3:** Let \((B, \mu)\) be a strong boundary for \( G \); then \( H^2_{cb}(G, \mathbb{R}) \) is naturally isomorphic to the space of \( G \)-invariant \( \mu \)-measurable anti-symmetric cross ratios on \( B \).
Proof: Let $G$ be a locally compact $\sigma$-compact group and let $(B, \mu)$ be a strong boundary for $G$. Denote for $k \geq 1$ by $L^\infty(B^k, \mu^k)$ the Banach space of essentially bounded $\mu^k$-measurable functions on $B^k$; it contains the closed subspace $L^\infty(B^k, \mu^k)^G$ of all $G$-invariant such functions. By the results of Burger and Monod [6], $H^2_{cb}(G, \mathbb{R})$ coincides with the second bounded cohomology group of the resolution

$$\mathbb{R} \to L^\infty(B, \mu)^G \to L^\infty(B^2, \mu^2)^G \to \ldots$$

with the usual homogeneous coboundary operator [13]. By Proposition 2.1 and its proof, there is a natural bijection between the space of $\mu^4$-measurable essentially bounded anti-symmetric $G$-cross ratios on $B$ and the space of $\mu^3$-measurable bounded cocycles on $B$, i.e. the space of alternating functions in $L^\infty(B^3, \mu^3)^G$ which satisfy the cocycle equation. Any two such bounded cocycles are cohomologous if and only if they differ by the image under the coboundary operator of an element of $L^\infty(B^2, \mu^2)^G$. However, by ergodicity of the measure class $\mu^2$ under the diagonal action of $\Gamma$, the vector space $L^\infty(B^2, \mu^2)^G$ only contains essentially constant functions. This means that the natural map which assigns to a $\mu^4$-measurable $G$-cross ratio its corresponding bounded cohomology class is injective. $\square$

We next look at a specific example. Namely, the ideal boundary of the oriented hyperbolic plane $\mathbb{H}^2$ can naturally be identified with the oriented circle $S^1$. The action on $S^1$ of the group $PSL(2, \mathbb{R})$ of orientation preserving isometries of $\mathbb{H}^2$ preserves the measure class of the Lebesgue measure $\lambda$. Moreover, $(S^1, \lambda)$ is a strong boundary for $PSL(2, \mathbb{R})$. A triple $(a, b, c)$ of pairwise distinct points in $S^1$ will be called ordered if the point $b$ is contained in the interior of the oriented subinterval $(a, c)$ of $S^1$. Similarly we define a quadruple $(a, b, c, d)$ of pairwise distinct points on $S^1$ to be ordered. Ordered quadruples define a $PSL(2, \mathbb{R})$-invariant subset of the space of quadruples of pairwise distinct points in $S^1$. There is a unique antisymmetric cross ratio $[]_0$ on $S^1$ with the following properties.

1. $[a, b, c, d]_0 = 0$ if $(a, b, c, d)$ is ordered.
2. $[a, b, c, d]_0 = 1$ if $(a, c, b, d)$ is ordered.

Clearly this cross ratio is measurable and invariant under the action of $PSL(2, \mathbb{R})$. We have.

Lemma 2.4:

1. Up to a constant, $[]_0$ is the unique non-trivial antisymmetric cross ratio on $S^1$ which is invariant under the full group $PSL(2, \mathbb{R})$ of orientation preserving isometries of $\mathbb{H}^2$.
2. If $\mu$ is any Borel measure on $S^1$ without atoms and if $[]$ is any $\mu$-measurable antisymmetric cross ratio on $S^1$ which satisfies $[a, b, c, d] = 0$ for $\mu$-almost every ordered quadruple of pairwise distinct points in $S^1$ then $[]$ is a multiple of $[]_0$.

Proof: If $[]$ is any antisymmetric cross ratio which is invariant under the full group $PSL(2, \mathbb{R})$ then the same is true for the induced function $\varphi$ on the set of triples of pairwise distinct points in $S^1$. Recall that the group $PSL(2, \mathbb{R})$ acts
transitively on the space of ordered triples of distinct points in $S^1$ and consequently we have \( \varphi(a, b, c) = \varphi(b, c, d) \) for every ordered quadruple \((a, b, c, d)\) of pairwise distinct points in $S^1$. Hence equation (6) in the proof of Proposition 2.1 implies that \([a, b, c, d] = 0\) whenever the quadruple \((a, b, c, d)\) of pairwise distinct points in $S^1$ is ordered. As a consequence, if \([\\cdot]\) $\neq 0$ then after possibly multiplying \([\\cdot]\) with a constant we conclude from the definition of an anti-symmetric cross ratio that \((a, b, c, d) = 1\) whenever \((a, c, b, d)\) is ordered. In other words, up to a constant we have \([\\cdot] = [\\cdot]\) which shows the first part of the lemma.

To show the second part of the lemma, let \(\mu\) be a Borel measure on $S^1$ without atoms and let \([\\cdot]\) be any \(\mu\)-measurable antisymmetric cross ratio on $S^1$ such that \([a, b, c, d] = 0\) for almost every ordered quadruple of pairwise distinct points in $S^1$. Let \((a, b, c, d)\) be a quadruple of pairwise distinct points in $S^1$ such that \([a, b, c, d] \neq 0\). By the symmetry relation of an antisymmetric cross ratio we may assume that \((a, c, b, d)\) is ordered. Thus the point \(b\) is contained in the oriented open subinterval \((c, d)\) of $S^1$. Now if \(b' \in (c, d)\) is another such point then by the defining properties of an antisymmetric cross ratio we have \([a, b, c, d] + [b, b', c, d] = [a, b', c, d]\). Assume first that \(b' \in (b, d)\); then the quadruple \((b, b', d, c)\) is ordered and consequently we have \([b, b', c, d] = 0\) by assumption and \([a, b, c, d] = [a, b', c, d]\). Via exchanging \(b\) and \(b'\) we conclude that \([a, b, c, d]\) does not depend on \(b \in (c, d)\). Similarly we deduce that in fact \([a, b, c, d]\) only depends on the order of the points in the quadruple \((a, b, c, d)\). In other words, \([\\cdot]\) is a multiple of \([\\cdot]\). \(\square\)

Lemma 2.4 shows in particular that \([\\cdot]\) corresponds to a generator \(\alpha\) of the group \(H^2_{cb}(PSL(2, \mathbb{R}), \mathbb{R})\).

Now let $\Gamma < PSL(2, \mathbb{R})$ be any finitely generated torsion free discrete group. Then $\Gamma$ acts freely on the hyperbolic plane $\mathbb{H}^2$ as a properly discontinuous group of isometries, and $S = \mathbb{H}^2/\Gamma$ is an oriented surface. The unit circle $S^1 = \partial \mathbb{H}^2$ with the $\Gamma$-invariant measure class of the Lebesgue measure $\lambda$ is a strong boundary for $\Gamma$. If $\Gamma$ is cocompact then we have $H^2(\Gamma, \mathbb{R}) = H^2(M, \mathbb{R}) = \mathbb{R}$ and there is a natural surjective restriction homomorphism $H^2_b(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$. The diagram

$$
\begin{array}{ccc}
H^2_b(\Gamma, \mathbb{R}) & \longrightarrow & H^2(\Gamma, \mathbb{R}) \\
\uparrow & & \uparrow \\
H^2_b(PSL(2, \mathbb{R}), \mathbb{R}) & \longrightarrow & H^2_b(PSL(2, \mathbb{R}), \mathbb{R})
\end{array}
$$

commutes and hence in our identification of $H^2_b(\Gamma, \mathbb{R})$ with the space of $\Gamma$-invariant antisymmetric cross ratios on the unit circle $S^1$, the $PSL(2, \mathbb{R})$-invariant cross ratio \([\\cdot]\) from Lemma 2.4 maps to a multiple of the Euler class of the tangent bundle of $S$. An easy calculation shows that the multiplication factor is in fact 1 (compare [10] for a detailed explanation of this fact). If $\Gamma$ is not cocompact then $\hat{S}$ is homeomorphic to the interior of a compact oriented surface $\hat{S}$ whose boundary $\partial \hat{S}$ is a finite union of circles. In particular, the fundamental group of each of the boundary circles is amenable. Now the bounded cohomology of any countable cellular space $X$ is naturally isomorphic to the bounded cohomology of its fundamental group $\pi_1(X)$ [10]. Thus $H^2_b(\partial \hat{S}, \mathbb{R}) = \{0\}$ and hence the relative group $H^2_b(\hat{S}, \partial \hat{S}, \mathbb{R})$ is isomorphic to $H^2_b(\hat{S}, \mathbb{R}) = H^2_b(\Gamma, \mathbb{R})$. The image under the
natural map \( i : H^2_b(\hat{S}, \partial \hat{S}, \mathbb{R}) \to H^2(\hat{S}, \partial \hat{S}, \mathbb{R}) \) of the pull-back \( i^*\alpha \) of \( \alpha \) under the inclusion \( i : \Gamma \to PSL(2, \mathbb{R}) \) is just the Euler class of the (bordered) surface \( \hat{S} \). In other words, the evaluation of \( i(i^*\alpha) \) on the fundamental class of the pair \((\hat{S}, \partial \hat{S})\) just equals the Euler characteristic of \( S \) (we refer to [3] for a detailed discussion of a slightly different viewpoint on these facts).

To describe the kernel \( Q \) of the natural map \( H^2_b(\Gamma, \mathbb{R}) \to H^2(\hat{S}, \partial \hat{S}, \mathbb{R}) \), let \( \Delta \) be the diagonal in \( S^1 \times S^1 \) and recall that a finitely additive signed measure on \( S^1 \times S^1 - \Delta \) is a function \( \mu \) which assigns to each oriented rectangle of the form \([a, b] \times [c, d]\) where \( a, b, c, d \) are pairwise distinct a number \( \mu[a, b] \times [c, d] \in \mathbb{R} \) such that

\[
\mu[a, b] \times [c, d] + \mu[b, b'] \times [c, d] = \mu[a, b'] \times [c, d]
\]

whenever \([a, b'] = [a, b] \cup [b, b']\) is disjoint from \([c, d]\). This measure is flip-anti-invariant if \( \mu[a, b] \times [c, d] = -\mu[c, d] \times [a, b] \). Let \( \lambda \) be the Lebesgue measure on \( S^1 \) which defines the unique \( PSL(2, \mathbb{R}) \)-invariant measure class. The collection of all finitely additive flip-anti-invariant signed measures \( \mu \) on \( S^1 \times S^1 \) with the additional property that the function \((a, b, c, d) \to \mu[a, b] \times [c, d]\) is \( \lambda \)-measurable is clearly a vector space; we call it the space of \( \lambda \)-measurable flip-antiinvariant signed measures. We have (compare Section 2 of [3]).

**Corollary 2.5:** There is a linear isomorphism \( \Psi \) of the kernel of the map \( H^2_b(\Gamma, \mathbb{R}) \to H^2(\hat{S}, \partial \hat{S}, \mathbb{R}) \) onto the vector space of all \( \lambda \)-measurable finite flip-anti-invariant \( \Gamma \)-invariant finitely additive signed measures on \( S^1 \times S^1 - \Delta \).

**Proof:** Every \( \Gamma \)-invariant \( \lambda \)-measurable anti-symmetric cross ratio \([]\) on \( S^1 \) defines a finite finitely additive flip-anti-invariant signed measure \( \mu = \Psi([]) \) by assigning to an ordered quadruple \((a, b, c, d)\) of pairwise distinct points in \( S^1 \) the value \( \mu[a, b] \times [c, d] = [a, b, c, d] \). The map \( \Psi : [\ ] \to \Psi([\ ]) \) is clearly linear, moreover by Lemma 2.4 its kernel is spanned by the cross ratio \([\ ]\) which defines the Euler class of \( S \) viewed as a class in \( H^2(\hat{S}, \partial \hat{S}, \mathbb{R}) \). As consequence, the restriction of the map \( \Psi \) to the kernel \( Q \) of the natural map \( H^2_b(\Gamma, \mathbb{R}) \to H^2(\hat{S}, \partial \hat{S}, \mathbb{R}) \) is injective.

Now let \( \mu \) be any \( \lambda \)-measurable finitely additive \( \Gamma \)-invariant flip anti-invariant signed measure on \( S^1 \times S^1 - \Delta \). Define an antisymmetric cross ratio \([\ ]\) as follows. First, if \((a, b, c, d)\) is ordered and if \( a, b, c, d \) are typical points for \( \lambda \) such that \( \mu[a, b] \times [c, d] \) is defined then we define \([a, b, c, d] = \mu[a, b] \times [c, d]\). This determines \([a, b, c, d]\) whenever \((a, b, c, d)\) or \((b, a, c, d)\) or \((a, b, c, d)\) is ordered. Choose an arbitrary fixed typical ordered quadruple \((a, b, c, d)\) and define \([a, c, b, d] = 0 \). As in the proof of Lemma 2.4, if \( b' \in (a, b) \) is arbitrary (and typical) then \((c, a, b', b)\) is ordered and we define \([a, c, b', d] = [a, c, b', b] + [a, c, b, d] = \mu[c, a] \times [b', b] \). Similarly, for a point \( b' \in (a, b) \) and every \( a' \in (d, a) \) define \([a', c, b', d] = [a', a, b', d] + [a, c, b', d] = \mu[a', a] \times [b', d] + [a, c, b, d] \). Successively we can define in this way an anti-symmetric measurable \( \Gamma \)-cross ratio on \( S^1 \). As consequence, the map \( \Psi \) is surjective. This shows the corollary.

\[\square\]

It follows from our consideration that there is a direct decomposition \( H^2_b(\Gamma, \mathbb{R}) = Q \oplus \ker(\Psi) \) where \( Q \) is the kernel of the natural map \( H^2_b(\Gamma, \mathbb{R}) \to H^2(\hat{S}, \partial \hat{S}, \mathbb{R}) \) and
Ψ is the map which associates to a $\lambda$-measurable anti-symmetric $\Gamma$-cross ratio $[,]$ its corresponding finitely additive signed measure on $S^1 \times S^1 - \Delta$.

Remark: Let $\Gamma$ be a cocompact torsion free lattice in $PSL(2, \mathbb{R})$. Then $S = H^2 / \Gamma$ is a closed surface of genus $g \geq 2$. The geodesic flow $\Phi^t$ acts on the unit tangent bundle $T^1 S$ of $S$. Call two Hölder continuous functions $f, u : T^1 S \to \mathbb{R}$ cohomologous if the integrals of $f, u$ over every periodic orbit for $\Phi^t$ coincide. This defines an equivalence relation on the vector space of all Hölder function tangent bundle $T^1 S$ and induces a map on the space of cohomology classes. We showed in [8] that the space of cohomology classes of Gromov boundary $\partial X$ is defined as follows. For a fixed point $x \in X$, define

\begin{equation}
(y, z)_x = \frac{1}{2}(d(y, x) + d(z, x) - d(y, z)).
\end{equation}

Call two sequences $(y_i, z_i) \subset X$ equivalent if $(y_i, z_i)_x \to \infty (i \to \infty)$. By hyperbolicity of $X$, this notion of equivalence defines an equivalence relation for the collection of all sequences $(y_i) \subset X$ with the additional property that $(y_i, y_j)_x \to \infty (i, j \to \infty)$ [3]. The boundary $\partial X$ of $X$ is the set of equivalence classes of this relation.

There is a natural topology on $X \cup \partial X$ which restricts to the given topology on $X$. With respect to this topology, a sequence $(y_i) \subset X$ converges to $\xi \in \partial X$ if and only if we have $(y_i, y_j)_x \to \infty$ and the equivalence class of $(y_i)$ defines $\xi$. Since $X$ is proper by assumption, the space $X \cup \partial X$ is compact and metrizable. Every isometry of $X$ acts naturally on $X \cup \partial X$ as a homeomorphism. Moreover, for every $x \in X$ and every $a \in \partial X$ there is a geodesic ray $\gamma : [0, \infty) \to X$ with $\gamma(0) = x$ and $\lim_{t \to \infty} \gamma(t) = a$.

For every proper metric space $X$ the isometry group $\text{Iso}(X)$ of $X$ can be equipped with a natural locally compact $\sigma$-compact metrizable topology, the so-called compact open topology. With respect to this topology, a sequence $(g_i) \subset \text{Iso}(X)$ converges to some isometry $g$ if and only if $g_i \to g$ uniformly on compact subsets of $X$. In this topology, a closed subset $Y \subset \text{Iso}(X)$ is compact if and only if there is
a compact subset $K$ of $X$ such that $gK \cap K \neq \emptyset$ for all $g \in Y$. In particular, the action of $\text{Iso}(X)$ on $X$ is proper. In the sequel we always assume that subgroups of $\text{Iso}(X)$ are equipped with the compact open topology.

The limit set $\Lambda$ of a subgroup $G$ of the isometry group of a proper hyperbolic geodesic metric space $X$ is the set of accumulation points in $\partial X$ of one (and hence every) orbit of the action of $G$ on $X$. If $G$ is non-compact then its limit set is a compact non-empty $G$-invariant subset of $\partial X$. The group $G$ is called elementary if its limit set consists of at most two points. In particular, every compact subgroup of $\text{Iso}(X)$ is elementary. If $G$ is non-elementary then its limit set $\Lambda$ is uncountable without isolated points, and $G$ acts as a group of homeomorphisms on $\Lambda$. An element $g \in \text{Iso}(X)$ is called hyperbolic if its action on $\partial X$ admits an attracting fixed point $a \in \Lambda$ and a repelling fixed point $b \in \Lambda - \{a\}$ and if it acts on $\Lambda$ with north-south-dynamics with respect to these fixed points.

Assume that $G < \text{Iso}(X)$ is a closed non-elementary subgroup of $\text{Iso}(X)$ with limit set $\Lambda \subset \partial X$. If the group $G$ does not act transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then the kernel of the natural map $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ is infinite dimensional [9]. Extending Lemma 6.1 of [5], we prove Theorem B from the introduction and show that this is not true for groups $G$ which act transitively on the complement of the diagonal in $\Lambda \times \Lambda$.

**Proposition 3.1:** Let $G < \text{Iso}(X)$ be a closed non-elementary subgroup with limit set $\Lambda$. If $G$ acts transitively on the complement of the diagonal in $\Lambda \times \Lambda$ then the kernel of the natural map $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ is trivial.

**Proof:** Let $G < \text{Iso}(X)$ be a closed non-elementary subgroup which acts transitively on the space $A$ of pairs of distinct points in $\Lambda$. Then for every fixed point $a \in \Lambda$ the stabilizer $G_a$ of $a$ acts transitively on $\Lambda - \{a\}$.

Following [5], an element in the kernel $H^2_{cb}(G, \mathbb{R}) \to H^2_c(G, \mathbb{R})$ can be represented by a continuous quasi-morphism, i.e. a continuous function $q : G \to \mathbb{R}$ which satisfies

\begin{equation}
\sup_{g,h \in G} |q(g) + q(h) - q(gh)| < \infty.
\end{equation}

Such a continuous quasi-morphism is bounded on every compact subset of $G$ and is bounded on each fixed conjugacy class in $G$. The quasi-morphism $q$ defines a non-trivial element of $H^2_{cb}(G, \mathbb{R})$ only if $q$ is unbounded.

Since $G$ is non-elementary by assumption, $G$ contains a hyperbolic element $g \in G$. Let $a \in \Lambda$ be the attracting fixed point of $g$ and let $b \in \Lambda - \{a\}$ be the repelling fixed point. Then $g$ preserves the set of geodesics connecting $b$ to $a$; note that such a geodesic exists since $X$ is proper. For $x, y \in X$ and a geodesic ray $\gamma : [0, \infty) \to X$ connecting $x$ to $a$ write $\beta(y, \gamma) = \limsup_{t \to \infty} (d(y, \gamma(t)) - t)$ and define the Busemann function

\begin{equation}
\beta_a(y, x) = \sup \{ \beta(y, \gamma) \mid \gamma \text{ is a geodesic ray connecting } x \text{ to } a \}.
\end{equation}
Then we have $|\beta_a(y, x)| \leq d(y, x)$, moreover there is a number $c > 0$ only depending on the hyperbolicity constant for $X$ such that
\begin{equation}
|\beta_a(\cdot, y) - \beta_a(\cdot, x) + \beta_a(y, x)| \leq c
\end{equation}
for all $x, y \in X$ (Proposition 8.2 of [7]). Let $\gamma : [0, \infty) \to X$ be a geodesic ray connecting $x$ to $a$. By hyperbolicity, for every $t > 0$ every geodesic ray connecting $x$ to $a$ passes through a neighborhood of $\gamma(t)$ of uniformly bounded diameter and therefore via possibly enlarging $c$ we may assume that
\begin{equation}
|\beta_a(\gamma(t), x) + d(\gamma(t), x)| \leq c
\end{equation}
independent of $x, t$. Define the horosphere at $a$ through $x$ to be the set $H_a(x) = \beta_a(\cdot, x)^{-1}(0)$. For all $x, y \in X$ the distance between the horospheres $H_a(x), H_a(y)$ is not smaller than $|\beta_a(x, y)| - c$. Moreover, the estimates (15), (16) show that for all $x, y$ with $\beta_a(x, y) \leq 0$ the distance between $y$ and $H_a(x)$ is not bigger than $|\beta_a(x, y)| + 2c$.

Let $W(a, b) \subset X$ be the closed non-empty subset of all points in $X$ which lie on a geodesic connecting $b$ to $a$. The isometry $g \in G$ is hyperbolic with fixed points $a, b \in \Lambda$ and therefore it preserves $W(a, b)$. If we denote by $\Gamma$ the infinite cyclic subgroup of $G$ generated by $g$ then $W(a, b)/\Gamma$ is compact. As a consequence, there is a number $\nu > 0$ and for every $x \in W(a, b)$ and every $t \in \mathbb{R}$ there is a number $k(t) \in \mathbb{Z}$ with $|\beta_a(g^{k(t)}x, x) - t| < \nu$. It follows from this and (16) that for every $y \in X$ there is some $k = k(y) \in \mathbb{Z}$ such that the distance between $g^kx$ and the horosphere $H_a(y)$ is at most $\delta_0 = \nu + 2c$.

By assumption, the stabilizer $G_a < G$ of the point $a \in \Lambda$ acts transitively on $\Lambda - \{a\}$. Thus there is for every $\zeta \in \Lambda - \{a\}$ an element $h_\zeta \in G_a$ with $h_\zeta(b) = \zeta$. Let $x \in W(a, b)$. Since $h_\zeta \in G_a$ we have $h_\zeta^{-1}(H_a(x)) = H_a(h_\zeta^{-1}(x))$. By our above consideration, there is a number $\ell \in \mathbb{Z}$ such that the distance between $g^\ell(x)$ and $H_a(h_\zeta^{-1}(x))$ is at most $\delta_0$ and therefore the distance between $h_\zeta \circ g^\ell(x)$ and $H_a(x)$ is at most $\delta_0$. Hence via replacing $h_\zeta$ by $h_\zeta \circ g^\ell \in G_a$ we may assume that the distance between $h_\zeta(x)$ and $H_a(x)$ is at most $\delta_0$. In particular, we have
\[|\beta_a(\cdot, x) - \beta_a(\cdot, h_\zeta(x))| \leq \delta_0 + c.\]
Now $h$ maps the geodesic $\gamma$ connecting $x$ to $a$ to a geodesic $h_\zeta \gamma$ connecting $h_\zeta x$ to $a$ and hence by the definition of the Busemann functions, for every $t \in \mathbb{R}$ the distance between $h_\zeta(\gamma(t))$ and $H_a(\gamma(t))$ is bounded from above by $\delta_0 + 3c$. Since $\Gamma$ preserves the set of geodesics connecting $b$ to $a$ we conclude that there is a universal constant $\delta_1 > 0$ such that for every $\ell \geq 0$ the distance between $h_\zeta(g^\ell x)$ and $H_a(g^\ell(x))$ is at most $\delta_1$.

For $x \in X$ denote by $N_{a, x} \subset G_a$ the set of all elements $h \in G_a$ with the property that the distance between $H_a(x)$ and $hx$ is at most $\delta_0$. Our above consideration shows that for every $x \in W(a, b)$ and every $\zeta \in \Lambda - \{a\}$ there is some $h_\zeta \in N_{a, x}$ which maps $b$ to $\zeta$. For every $h \in N_{a, x}$ the sequence $(g^{-\ell} \circ h \circ g^\ell) x$ is contained in $G_a$ and by the above consideration, it maps $x$ into the $\delta_1$-neighborhood of $H_a(x)$. Moreover, the sequence $(g^{-\ell} \circ h \circ g^\ell(b))_\ell \subset \Lambda$ converges as $\ell \to \infty$ to $b$. This means that there is a number $\delta_2 > \delta_1$ such that for sufficiently large $\ell$ the element $g^{-\ell} \circ h \circ g^\ell \in G_a$ maps the point $x$ into the closed $\delta_2$-neighborhood $B_x$ of $x$. The group $\Gamma$ acts on $W(a, b)$ cocompactly and hence if $C \subset W(a, b)$ is a compact fundamental domain for this action then $B = \cup_{x \in C} B_x$ is compact, moreover for
every $x \in W(a,b)$, every element $h \in N_{a,x}$ is conjugate in $G$ to an element in the compact subset $K = \{ u \in G \mid uB \cap B \neq \emptyset \}$ of $G$. As a consequence, the restriction to $N_{a,b} = \bigcup_{x \in W(a,b)} N_{a,x}$ of any continuous quasi-morphism $q$ on $G$ is uniformly bounded. By our assumption on $G$ the sets $N_{a,b} ((a,b) \in A)$ are pairwise conjugate in $G$ and hence $q$ is uniformly bounded on $\bigcup_{(a,b) \in A} N_{a,b} = N$.

Next we show that the restriction of a quasi-morphism $q$ to the subgroup $G_{a,b}$ of $G$ which stabilizes both points $a,b \in \Lambda$ is bounded. For this consider an arbitrary element $u \in G_{a,b}$. Let $x \in W(a,b)$ and let $uβ_a(ux,x) = τ$; we may assume that $u \notin N_{a,b}$ and hence after possibly exchanging $b$ and $a$ that $τ < 0$. Since $u$ preserves $W(a,b)$, by (16) we have $|d(ux,x) + τ| \leq c$ where $c > 0$ is a universal constant. Choose some $h \in G$ with $h(a,b) = (b,a)$; then $h$ preserves the set $W(a,b)$. Let $g \in G_{a,b}$ be the fixed hyperbolic element as before. By our above consideration, via composing $h$ with $g^\ell$ for a number $\ell \in \mathbb{Z}$ depending on $h$ we may assume that $d(x,hx) \leq δ_2$ for a universal constant $δ_2 > 0$. We have $huhx \in W(a,b)$ and $|β_b(ux,x) - β_a(ux,x)| \leq |β_b(hux,hx) - β_a(ux,x)| + d(hx,x) \leq δ_2$. On the other hand, from another application of the estimate (16) we obtain the existence of a constant $δ_3 > 0$ such that $|β_a(z,y) - β_a(y,z)| \leq δ_3$ for all $y,z \in W(a,b)$. But this just means that the distance between $huhx$ and $u^{-1}x$ is uniformly bounded and hence the distance between $uhux$ and $x$ is uniformly bounded as well. As a consequence, the element $uuhu$ is contained in a fixed compact subset of $G$. As before, we conclude from this that $|q(u)|$ is bounded from above by a universal constant not depending on $u$ and hence the restriction of $q$ to $G_{a,b}$ is uniformly bounded.

Now let $h \in G$ be arbitrary and assume that $ha = x, hb = y$. We showed above that there are $h_y \in N \cap G_x, h_x \in N \cap G_b$ with $h_y(y) = b$ and $h_x(x) = a$; then $h' = h_xh_yh \in G_{a,b}$ and $|q(h') - q(h)|$ is uniformly bounded. As a consequence, $q$ is bounded and hence it defines the trivial bounded cohomology class. This completes the proof of the proposition.

From now on we assume that $X$ is of bounded growth which means that there is a number $b > 1$ such that for every $R > 1$, every metric ball of radius $R$ contains at most $b^{\log R}$ disjoint metric balls of radius 1. Let $G < \text{Iso}(X)$ be a closed non-elementary group with limit set $\Lambda$. Let $(B,μ_0)$ be a strong boundary for $G$. By Lemma 2.2 of [9] there is a $G$-equivariant map $ϕ : B \to Λ$. The image $μ$ of $μ_0$ under the map $ϕ$ is a measure on $Λ$ whose measure class is invariant under the action of $G$ and such that the diagonal action of $G$ on $Λ \times Λ$ is ergodic with respect to the class of $μ \times μ$. Since $X$ is of bounded growth by assumption, the stabilizer in $\text{Iso}(X)$ of each point in $∂X$ is amenable and therefore the measure space $(Λ,μ)$ is a strong boundary for $G$ (compare the discussion in [11]). In particular, Corollary 2.3 implies that $H^*_b(G,\mathbb{R})$ is isomorphic to the space of $μ$-measurable anti-symmetric $G$-cross ratios on $Λ$.

Now assume that $G$ acts transitively on the complement of the diagonal in $Λ \times Λ$. The stabilizer $G_{a,b}$ of $(a,b)$ acts on $Λ - \{a,b\}$ as a group of homeomorphisms. Moreover, there is some $g \in G$ which maps $(a,b)$ to $(b,a)$. Define the group $G$ to be directed if there is a $G_{a,b}$-invariant subset $U(a,b)$ of $Λ - \{a,b\}$ of positive $μ$-mass such that $gU(a,b) \cap U(a,b) = \emptyset$. We have.
Lemma 3.2: Let $G < \text{Iso}(X)$ be a closed subgroup with limit set $\Lambda$. Assume that $G$ acts transitively on the complement $A$ of the diagonal in $\Lambda \times \Lambda$; if $H_2^{cb}(G, \mathbb{R}) \neq \{0\}$ then $G$ is directed.

Proof: Let $G < \text{Iso}(X)$ be a closed subgroup with limit set $\Lambda$ which acts transitively on the complement $A$ of the diagonal in $\Lambda \times \Lambda$. If $H_2^{cb}(G, \mathbb{R}) \neq \{0\}$ then there is some $G$-invariant measurable nontrivial alternating function $\varphi$ on the space of triples of pairwise distinct points in $\Lambda$ which satisfies the cocycle identity \[ \varphi(a, b, u) = \varphi(a, b, gu). \]

Let $\varphi$ be any $G$-invariant $\mu$-measurable alternating function on the space of triples of pairwise distinct points in $\Lambda$. Let $(a, b) \in A$; since the action of $G$ on $A$ is transitive, there is a set $U \subset \Lambda$ of positive $\mu$-mass such that $\varphi(a, b, u) > 0$ for every $u \in U$. If $v = hu$ for some $h \in G_{a,b}$ then $\varphi(a, b, u) = \varphi(a, b, v)$. Let $g \in G$ be such that $g(a, b) = (b, a)$. Then $g^{-1}G_{a,b}g = G_{a,b}$ and if $G$ is not directed then for every $G_{a,b}$-invariant measurable subset $U$ of $\Lambda - \{a, b\}$ with $\mu(U) > 0$ we have $\mu(U \cap gU) > 0$. Now for every $c \in \mathbb{R}$ and every $\epsilon > 0$ the set $U_{c,\epsilon} = \{u \mid \varphi(a, b, u) \in (c - \epsilon, c + \epsilon)\}$ is $G_{a,b}$-invariant. By assumption, if $\mu(U_{c,\epsilon}) > 0$ then we have $\mu(gU_{c,\epsilon} \cap U_{c,\epsilon}) > 0$ as well and therefore there is some $u \in U_{c,\epsilon}$ with $|\varphi(a, b, gu) - c| < \epsilon$. Since $\varphi$ is alternating we obtain that $\varphi(a, b, gu) = -\varphi(b, a, gu) = -\varphi(ga, gb, gu) = -\varphi(a, b, u) \in (-c - \epsilon, -c + \epsilon)$ and consequently $|c| < 2\epsilon$. On the other hand, for $\mu$-almost every $u \in \Lambda$ and every $\epsilon > 0$ we have $\mu(U_{\varphi(a, b, u),\epsilon}) > 0$. From this we deduce that $\varphi$ vanishes almost everywhere and hence $H_2^{cb}(G, \mathbb{R}) = \{0\}$. \[ \square \]

In the case that $G$ is a simple Lie group of non-compact type and rank 1 we have $H_2^{cb}(G, \mathbb{R}) \neq \{0\}$ if and only if $G = SU(n, 1)$ for some $n \geq 1$. That this condition is necessary is immediate from Lemma 3.2 and the following observation.

Lemma 3.3: A simple Lie group $G$ of non-compact type and rank one is directed only if $G = SU(n, 1)$ for some $n \geq 1$.

Proof: Let $G$ be a simple Lie group of non-compact type and rank one. Then $G$ is the isometry group of a symmetric space $X$ of non-compact type and negative curvature with ideal boundary $\partial X$. The action of $G$ on $\partial X$ preserves the measure class of the Lebesgue measure $\lambda$ and $(\partial X, \lambda)$ is a strong boundary for $G$. The group $G$ acts transitively on the space $A$ of pairs of distinct points in $\partial X$. If $G = SO(n, 1)$ for some $n \geq 3$ then $G$ acts transitively on the space of triples of pairwise distinct points in $\partial X$ and hence by the above, $G$ is not directed.

Now let $G = Sp(n, 1)$ for some $n \geq 1$. Then for every pair $(a, b) \in A$ there is a unique totally geodesic embedded quaternionic line $L \subset X$ of constant curvature $-4$ whose boundary $\partial L = S^3 \subset \partial X$ contains $a$ and $b$. The stabilizer $G_{a,b}$ of $(a, b)$ is contained in the stabilizer $G_L$ of $L$ in $G$ which is conjugate to the quotient of the group $Sp(1,1) \times Sp(n-1) < Sp(n, 1)$ by its center. Moreover, $G_L$ acts transitively on the space of triples of pairwise distinct points in $\partial L$. In particular, for every $u \in \partial L$ and every $g \in G$ with $g(a, b) = (b, a)$ the $G_{a,b}$-orbit of $u$ coincides with the $G_{a,b}$-orbit of $gu$. More generally, this is true for every point $u \in \partial X - \{a, b\}$. Namely, let $P : \partial X \to L$ be the shortest distance projection. The subgroup $G_{Pu}$ of $G_L$ which stabilizes $Pu$ is the quotient of the group $Sp(1) \times Sp(n-1)$ by its
center. The factor subgroup \( Sp(n - 1) \) acts transitively on \( P^{-1}(Pu) \) while the orbit of \( Pu \) under the group \( G_{a,b} \) consists of the set of all points in \( L \) whose distance to the geodesic connecting \( b \) to \( a \) coincides with the distance of \( Pu \). As above we conclude that for every \( u \in \partial X \) and every \( g \in G \) with \( g(a, b) = (b, a) \) the \( G_{a,b} \)-orbit of \( u \) coincides with the \( G_{a,b} \)-orbit of \( gu \). In other words, \( G \) is not directed. In the same way we obtain that the exceptional Lie group \( F_{20}^4 \) is not directed as well. \( \square \)

Note that for every \( n \geq 1 \) the group \( SU(n, 1) \) admits a non-trivial continuous second bounded cohomology class \( \alpha \in H^2_{cb}(SU(n, 1), \mathbb{R}) \) induced by the Kähler form \( \omega \) of the complex hyperbolic space \( X = SU(n, 1)/S(U(n)U(1)) \), and this class generates \( H^2_{cb}(SU(n, 1), \mathbb{R}) \). The anti-symmetric cross ratio \([\ ]\) on \( \partial X \) defining the class \( \alpha \) is a continuous function on \( \partial X \) with values in \([\pi, \pi]\). We have \( |[a, b, c, d]| = \pi \) if and only if \((a, c, b, d) \) or \((c, a, b, d) \) is an ordered quadruple of points in the boundary of a complex line in \( X \) (for a discussion of this fact, references and applications to rigidity, see the recent preprint of Burger and Iozzi [4]).

4. RIGIDITY OF COCYCLES

In this section we consider as before a proper hyperbolic geodesic metric space \((X, d)\). We equip the isometry group \( \text{Iso}(X) \) with the compact open topology; with respect to this topology, \( \text{Iso}(X) \) is a locally compact \( \sigma \)-compact topological group. Let \( S \) be a standard Borel space and let \( \mu \) be a Borel probability measure on \( S \). Assume that \( G \) is a locally compact \( \sigma \)-compact topological group which admits a measure preserving ergodic action on a standard probability space \((S, \mu)\). This action then defines a natural continuous unitary representation of \( G \) into the Hilbert space \( L^2(S, \mu) \) of square integrable functions on \( S \). Let \( \alpha : G \times S \to \text{Iso}(X) \) be a cocycle, i.e. \( \alpha : G \times S \to \text{Iso}(X) \) is a Borel map which satisfies \( \alpha(gh, x) = \alpha(g, hx)\alpha(h, x) \) for all \( g, h \in G \) and \( \mu \)-almost every \( x \in S \). The cocycle is cohomologous to a cocycle \( \beta : G \times S \to \text{Iso}(X) \) if there is a measurable function \( \psi : S \to \text{Iso}(X) \) such that \( \psi(gx)\alpha(g, x) = \beta(g, x)\psi(x) \) for all \( g \in G \) and almost all \( x \in S \). We use Theorem 4.1 of 9 to show (compare [11], [12]).

**Lemma 4.1:** Let \( G \) be a locally compact \( \sigma \)-compact group and let \((S, \mu)\) be an ergodic \( G \)-probability space. Let \( X \) be a proper hyperbolic geodesic metric space and let \( \alpha : G \times S \to \text{Iso}(X) \) be a cocycle. Let \( H < \text{Iso}(X) \) be a closed subgroup with limit set \( \Lambda \) and assume that \( \alpha(G \times S) \supset H \) but that \( \alpha \) is not cohomologous to a cocycle into a proper subgroup of \( H \). If \( H^2_{cb}(G, L^2(S, \mu)) \) is finite dimensional then either \( H \) is elementary or \( H \) acts transitively on the complement of the diagonal in \( \Lambda \times \Lambda \).

**Proof:** Let \( G \) be a locally compact \( \sigma \)-compact group which admits an ergodic measure preserving action on a standard Borel probability space \((S, \mu)\). Let \( \alpha : G \times S \to \text{Iso}(X) \) be a cocycle and assume that \( \alpha(G \times S) \) is contained in a closed subgroup \( H \) of \( \text{Iso}(X) \) and that \( \alpha \) is not equivalent to a cocycle with values in a proper closed subgroup of \( H \). Assume moreover that \( H \) is not elementary and does not act transitively on the complement of the diagonal in \( \Lambda \times \Lambda \). By Theorem 4.1 of [11], \( H^2_{cb}(H, \mathbb{R}) \) is infinite dimensional. More precisely, if \( T \subset \Lambda^3 \) is the space of
triples of pairwise distinct points in \( \Lambda \) then there is an infinite dimensional vector space of continuous \( H \)-invariant cocycles on \( T \), i.e. continuous \( H \)-invariant maps \( T \to \mathbb{R} \) which satisfy the cocycle equation (5).

Let \((B, \nu)\) be a strong boundary for \( G \). By Lemma 2.2 of [19] there is an \( \alpha \)-equivariant measurable map \( \varphi : B \times S \to \Lambda \). This means that for almost every \((w, \sigma) \in B \times S\) and every \( g \in G \) we have \( \varphi(gw, g\sigma) = \alpha(g, \sigma) \varphi(w, \sigma) \). We claim that the image of \( \varphi \) is dense in \( \Lambda \). Namely, otherwise there is a proper closed subset \( A \) of \( \Lambda \) which contains the image of \( \varphi \). Then for every \( g \in G \) and almost every \( u \in S \) the element \( \alpha(g, u) \in H \) stabilizes \( A \), and \( \alpha \) is cohomologous to a cocycle with values in the intersection of \( H \) with the stabilizer of \( A \). Since \( A \) is a closed proper subset of \( \Lambda \), the stabilizer of \( A \) intersects \( H \) in a closed subgroup \( H \) of infinite index which contradicts our assumption on \( \alpha \).

We now follow Monod and Shalom [14]. Namely, choose a continuous \( H \)-invariant bounded cocycle \( \omega : T \to \mathbb{R} \). Define a measurable map \( \eta(\omega) \) from the space \( \tilde{B} \) of triples of pairwise distinct points in \( B \) into \( L^\infty(S, \mu) \) by \( \eta(x_1, x_2, x_3)(z) = \omega(\varphi(x_1, z), \varphi(x_2, z), \varphi(x_3, z)) \); then \( \eta \) can be viewed as a measurable \( G \)-equivariant bounded cocycle with values in the Hilbert space of square integrable function on \( S \). Since \( \omega \) is continuous and \( \varphi(B \times S) \) is dense in \( \Lambda \), this cocycle does not vanish identically. As a consequence, the space of \( G \)-equivariant measurable bounded cocycles \( \tilde{B} \to L^2(S, \mu) \) is infinite dimensional. On the other hand, \( B \) is a strong boundary for \( G \) and therefore \( H^2_\text{cb}(G, L^2(S, \mu)) \) is isomorphic to the space of measurable \( G \)-equivariant bounded cocycles \( \tilde{B} \to L^2(S, \mu) \). In other words, \( H^2_\text{cb}(G, L^2(S, \mu)) \) is infinite dimensional. This shows the lemma.

Now consider a semi-simple Lie group \( G \) with finite center and no compact factors. Let \( \Gamma < G \) be an irreducible lattice in \( G \), i.e. \( \Gamma \) is a discrete subgroup of \( G \) such that the volume of \( G/\Gamma \) is finite and that moreover if \( G = G_1 \times G_2 \) is a non-trivial product then the projection of \( \Gamma \) to each factor group \( G_i \) \((i = 1, 2)\) is dense. The group \( G \) acts on \( G/\Gamma \) preserving the projection of the Lebesgue measure \( \lambda \) on \( G \).

Let \((S, \mu)\) be a standard probability space with a measure preserving action of \( \Gamma \). Then \( \Gamma \) admits a measure preserving action on the product space \( G \times S \). Since \( \Gamma < G \) is a lattice, the quotient space \((G \times S)/\Gamma \) can be viewed as a bundle over \( G/\Gamma \) with fibre \( S \). If \( \Omega \subset G \) is a finite measure Borel fundamental domain for the action of \( \Gamma \) on \( G \) then \( \Omega \times S \subset G \times S \) is a finite measure Borel fundamental domain for the action of \( \Gamma \) on \( G \times S \) and up to normalization, the product measure \( \lambda \times \mu \) projects to a finite measure \( \nu \) on \((G \times S)/\Gamma \). The action of \( G \) on \( G \times S \) by left translation commutes with the action of \( \Gamma \) and hence it projects to an action on \((G \times S)/\Gamma \) preserving the measure \( \nu \) (see p.75 of [16]). We have.

**Lemma 4.2:** Let \( \Gamma \) be an irreducible lattice in a product \( G = G_1 \times G_2 \) of two semi-simple non-compact Lie groups \( G_1, G_2 \). If the action of \( \Gamma \) on \((S, \mu)\) is mixing then the induced action of \( G_1 \) on \((G \times S)/\Gamma, \nu)\) is ergodic.

**Proof:** Let \( \Gamma \) be an irreducible lattice in a product \( G = G_1 \times G_2 \) of two semi-simple non-compact Lie groups \( G_1, G_2 \) with finite center. Let \((S, \mu)\) be a mixing
Γ-probability space. The induced action of $G$ on $(G \times S)/\Gamma$ preserves the measure $\nu$ and restricts to an action of $G_1$. Since the measure $\nu$ is finite, this action is ergodic if and only if every $G_1$-invariant function $f \in L^2((G \times S)/\Gamma, \nu)$ is constant.

We argue by contradiction and we assume that there is a $G_1$-invariant function $f \in L^2((G \times S)/\Gamma, \nu)$ with $\int \|f\|^2 \, d\nu = 1$ and zero mean $\int f \, d\nu = 0$. Let $\tilde{f}$ be the lift of $f$ to a $\Gamma$-invariant function on $G \times S$; then $\tilde{f}$ is locally square integrable with respect to the product measure $\lambda \times \mu$. For $x \in G$ define a function $\varphi(x)$ on $S$ by $\varphi(x)(y) = \tilde{f}(x, y)$; by Fubini's theorem we have $\varphi(x) \in L^2(S, \mu)$ for $\lambda$-almost every $x$, moreover the function $x \to \int_S \varphi(x) \, d\mu$ is measurable with respect to $\lambda$. We claim that $\int_S \varphi(x) \, d\mu = 0$ and $\int_S |\varphi(x)|^2 \, d\mu = 1$ for $\lambda$-almost every $x \in G$.

For this recall that by definition of the action of $G_1$, the functions $x \to \int_S \varphi(x) \, d\mu$ and $x \to \int_S |\varphi(x)|^2 \, d\mu$ are invariant under the action of $G_1$ on $G/\Gamma$ since the action of $\Gamma$ on $(S, \mu)$ is measure preserving. By Moore's ergodicity theorem the action of $G_1$ on $G/\Gamma$ is ergodic and hence these functions are constant almost everywhere. If $\Omega \subset G$ is a Borel-fundamental domain for the action of $\Gamma$ on $G$ and if the Lebesgue measure $\lambda$ on $G$ is normalized in such a way that $\lambda(\Omega) = 1$ then $\int_{\Omega} \int_S \varphi(x) \, d\mu \, d\lambda = 0$, $\int_{\Omega} \int_S |\varphi(x)|^2 \, d\mu \, d\lambda = 1$ and hence these constants equal 0, 1. In particular, if we denote by $\nu$ the orthogonal complement in $L^2(S, \mu)$ of the constant functions then the assignment $x \to \varphi(x)$ is a $\nu$-valued measurable function on $G/\Gamma$.

Let $L^{[1]}(G, \mathcal{V})^\Gamma$ be the space of all $\Gamma$-invariant measurable maps $h : G \to \mathcal{V}$ for which the function $\|h\|^2$ is locally integrable with respect to the Lebesgue measure $\lambda$ on $G$. Note that $L^{[1]}(G, \mathcal{V})^\Gamma$ naturally has the structure of a separable Banach space. The group $G_1$ acts isometrically on $L^{[1]}(G, \mathcal{V})^\Gamma$ by left translation. By our assumption, there is a non-zero $G_1$-invariant vector $\varphi \in L^{[1]}(G, \mathcal{V})^\Gamma$. Then $\varphi : G \to \mathcal{V}$ is measurable and essentially bounded. Thus the restriction of $\varphi$ to suitable compact sets of large measure is continuous; in particular, we can find a compact set $K = K_1 \times K_2 \subset G_1 \times G_2$ of positive measure, Borel sets $D, E \subset S$ of positive measure and a number $\epsilon > 0$ such that the following is satisfied.

\begin{enumerate}
  \item $\varphi(z)(u) \geq \epsilon$ for every $z \in K, u \in D$.
  \item $\varphi(z)(v) \leq 0$ for every $z \in K, v \in E$.
\end{enumerate}

Since $\varphi$ is invariant under the left action of $G_1$, for any $z_0 \in K_1, z_2 \in K_2$ and $z_1 \in G_1$, $u \in S$ we have $\varphi(z_1, z_2)(u) = \varphi(z_0, z_2)(u)$. On the other hand, $\varphi$ is also invariant under the right action of $\Gamma$ and hence $\varphi(z_1, z_2)(u) = \varphi((z_1, z_2)\eta)(\eta u)$ for every $\eta \in \Gamma$. In particular, if $(z_1, z_2)\eta \in K$ then we deduce that $\mu(\eta D \cap E) = 0$.

The right action of $\Gamma$ on $G_1 \setminus G$ is ergodic and therefore there are infinitely many elements $\eta \in \Gamma$ such that the measure of $(G_1 \times K_2)\eta \cap K$ is positive. Moreover, the action of $\Gamma$ on $(S, \mu)$ is mixing by assumption and hence for every $\eta \in \Gamma$ which is sufficiently far away from the identity the image of $D$ under $\eta$ intersects $E$ in a set of positive measure. In particular, there is some $\eta \in \Gamma$ with $\lambda((G_1 \times K_2)\eta \cap K) > 0$ and $\mu(\eta D \cap E) > 0$ which contradicts the above consideration. Thus the vector space of $G_1$-invariant maps in $L^{[1]}(G, \mathcal{V})^\Gamma$ is trivial and the action of $G_1$ on $(G \times S)/\Gamma$ is ergodic. \hfill $\square$
The following corollary completes the proof of Theorem A from the introduction and follows as in [13] from Lemma 4.2, the rigidity results for bounded cohomology of Burger and Monod [3, 6] and the work of Zimmer [16].

**Corollary 4.3:** Let $G$ be a semi-simple Lie group with finite center, no compact factors and of rank at least 2. Let $\Gamma < G$ be an irreducible lattice and let $(S, \mu)$ be a mixing $\Gamma$-space. Let $X$ be a proper hyperbolic geodesic metric space and let $\alpha : \Gamma \times S \to \text{Iso}(X)$ be a cocycle; then either $\alpha$ is cohomologous to a cocycle into an elementary subgroup of $\text{Iso}(X)$ or there is a closed subgroup $H$ of $\text{Iso}(X)$ which is a compact extension of a simple Lie group $L$ of rank one and there is a surjective homomorphism $G \to L$.

**Proof:** Let $G$ be a semi-simple Lie group of non-compact type with finite center and of rank at least 2 and let $\Gamma$ be an irreducible lattice in $G$. Let $(S, \mu)$ be a mixing $\Gamma$-space with invariant Borel probability measure $\mu$. Let $\alpha : \Gamma \times S \to \text{Iso}(X)$ be a cocycle into the isometry group of a proper hyperbolic geodesic metric space $X$. Let $H < \text{Iso}(X)$ be a closed subgroup such that $\alpha$ is cohomologous to an $H$-valued cocycle but not to a cocycle with values in a proper subgroup of $H$. We may assume that $\alpha(\Gamma \times S) \subset H$. Assume that $H$ is not elementary, with limit set $A$. By Theorem A in [13], there is a continuous $L^2(H)$-valued bounded cocycle $\varphi : \Lambda^3 \to L^2(H)$.

Let $\Omega \subset G$ be a Borel fundamental domain for the action of $\Gamma$ on $G$. Let $\nu$ be the $G$-invariant Borel probability measure on $(G \times S)/\Gamma \sim \Omega \times S$. We obtain a $\nu$-measurable function $\beta : G \times (G \times S)/\Gamma \to H$ as follows. For $z \in \Omega$ and $g \in G$ let $\eta(g, z) \in \Gamma$ be the unique element such that $gz \in \eta(g, z)\Omega$ and define $\beta(g, (z, \sigma)) = \alpha(\eta(g, z), \sigma)$. By construction, $\eta$ satisfies the cocycle equation for the action of $G$ on $(G \times S)/\Gamma$. Let $(B, \mu)$ be a strong boundary for $G$; we may assume that $B$ is also a strong boundary for $\Gamma$. By Lemma 4.2 the action of $G$ on $(G \times S)/\Gamma$ is ergodic and hence Lemma 2.2 of [12] show that we can find a measurable $\beta$-equivariant Fürstenberg map $\psi_0 : (G \times S)/\Gamma \times B \to \Lambda$ and hence a $\beta$-equivariant map $\psi : (G \times S)/\Gamma \times B^3 \to \Lambda^3$ defined by $\psi(x, a, b, c) = (\psi_0(x, a), \psi_0(x, b), \psi_0(x, c)) \ (x \in (G \times S)/\Gamma)$. Let $L^2[\{(G \times S)/\Gamma, L^2(H)\}]$ be the space of all measurable maps $(G \times S)/\Gamma \to L^2(H)$ with the additional property that for each such map $\varphi$ the function $x \to \|\varphi\|$ is square integrable on $(G \times S)/\Gamma$. Then $L^2[\{(G \times S)/\Gamma, L^2(H)\}]$ has a natural structure of a separable Hilbert space, and the group $G$ acts on $L^2[\{(G \times S)/\Gamma, L^2(H)\}]$ as a group of isometries. In other words, $L^2[\{(G \times S)/\Gamma, L^2(H)\}]$ is a Hilbert module for $G$ and the cocycle $\varphi$ can be composed with the map $\psi$ to a $\beta$-invariant measurable bounded map $B^3 \to L^2[\{(G \times S)/\Gamma, L^2(H)\}]$. Since $B$ is a strong boundary for $G$ we conclude that this map defines a nontrivial cohomology class in $H^3_G((G \times S)/\Gamma, L^2(H))$.

Now if $G$ is simple then the results of Monod and Shalom [14] show that there is a $\beta$-equivariant map $(G \times S)/\Gamma \to L^2(H)$. Since the action of $G$ on $(G \times S)/\Gamma$ is ergodic, by the cocycle reduction lemma of Zimmer [16] the cocycle $\beta$ and hence $\alpha$ is cohomologous to a cocycle into a compact subgroup of $H$ which is a contradiction. On the other hand, if $G = G_1 \times G_2$ for semi-simple Lie groups $G_1, G_2$ with finite center and without compact factors, then the results of Burger and Monod [3, 6] show that via possibly exchanging $G_1$ and $G_2$ we may assume that there is an
equivariant map \((G \times S)/\Gamma \to L^2(H)\) for the restriction of \(\beta\) to \(G_1 \times (G \times S)/\Gamma\), viewed as a cocycle for \(G_1\). By Lemma 4.2 the action of \(G_1\) on \((G \times S)/\Gamma\) is ergodic and therefore the cocycle reduction lemma of Zimmer [16] shows that the restriction of \(\beta\) to \(G_1\) is equivalent to a cocycle into a compact subgroup of \(H\). We now follow the proof of Theorem 1.2 of [14] and find a minimal such compact subgroup \(K\) of \(H\). The cocycle \(\beta\) and hence \(\alpha\) is cohomologous to a cocycle into the normalizer of \(K\) in \(H\) which then coincides with \(H\) by our assumption of \(H\). Moreover, there is a continuous homomorphism of \(G\) onto \(H/K\). Since \(G\) is connected, the image of \(G\) under this homomorphism is connected as well and hence by Theorem A of [9], \(H/K\) is a simple Lie group of rank one. This completes the proof of the corollary. \(\square\)

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