Strong Interactions in the Three Black Holes Problem

M. Imbrogno, C. Meringolo, and S. Servidio
Dipartimento di Fisica, Università della Calabria, I-87036 Cosenza, Italy
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We study the three-body problem from different perspectives, going from classical Newtonian physics to general relativity. In the classical case, we modeled the interactions in a typical chaotic configuration, identifying strong interaction times (SITs), namely transients in which the system manifests complex, highly energetic dynamics. By selecting one of such SITs as an initial condition for the general relativistic case, we perform simulations of numerical relativity, showing net differences with the Newtonian case. In the relativistic case, comparing also with the more “quiet” two-body inspiral, we observed strong nonlinear emission of gravitational waves. The spectrum of the three-body signals during a SIT might be a guidance for future gravitational waves observations.

I. INTRODUCTION

The three-body problem represents the non-linear problem par excellence [1]. There is a very large number of applications of this problem, going from the microphysics to the extraordinarily large-scale case of galaxy clusters. In this cross-scale variety of examples, the problem remains puzzling, despite years of studies, since the bodies have an unpredictable behavior (see [2] for a review.) From the classical perspective, in problems of celestial mechanics, one may almost always identify a binary and a third body. A binary can be treated as a single entity with certain internal properties (like a molecule). The couple interacts with a third body, once or more frequently, with a resultant change of the internal properties of the binary. Over all the interaction, the system conserves the total energy, the mass, the momentum and the angular momentum, even though the trajectories of the single bodies are particularly unpredictable [1][3].

Solutions of the three-body problem in astrophysics are characterized by an high level of complexity. The importance of the process lies in its ability to redistribute energy and angular momentum efficiently; the population of binaries may become more and more tightly bound as time and three-body scattering goes on, while the population of single stars may gain speed and become “heated”. This has profound consequences on the structure and evolution of star clusters.

Three (or more)-body interactions are expected to be common in clusters and galactic cores [4]. Multiple black holes might also be formed in galactic nuclei undergoing sequential mergers [2][5], triple quasar systems [6], in globular clusters and galactic disks [7][9]. Therefore there is a good chance of having, at some point of the interaction, the three bodies strongly dominated by their mutual gravitational attraction. In this regime, when masses are huge, the chaotic three-body dynamics should be treated via general relativistic models and merging events might be expected [10][11]. In globular clusters, indeed, N-body interactions are important in the formation of massive black holes [12][14]. In such system the conservation of the classic invariants need to be revised in terms of general relativity [15][16]. The mergers of compact massive objects such as black holes and neutron stars represent the most suggestive and extraordinary events in astrophysics. According to general relativity, these systems are powerful sources of gravitational waves. However, the gravitational radiation, detected through modern experimental settings, cannot completely characterize what happens in the vicinity of the merging region and complementary studies of numerical relativity are needed [17][21]. The understanding of such strong, nonlinear interactions represents a new challenge for the comprehension of cosmological problems and need to be tackled via appropriate models, following the Einstein ideas.

We will start with the study the behavior of three point-like bodies subject to mutual gravitational attraction according to the classical laws of Newtonian physics (specifically the Burrau three-body problem), by using a simple (Lagrangian) numerical model [10]. Subsequently, we will study the same problem from the point of view of Einstein general relativity. In the latter case, we will inspect the waveform of the gravitational radiation emitted from the system, comparing it with the two-body spinning case. This radiation consists of gravitational waves, namely perturbations of spacetime generated by the interaction of three black holes. These studies might be useful for the description and the understanding gravitational signals, as observed by novel observational campaigns [22][24].

The paper is organized as follows. In Section II we summarize the classical results of the chaotic motion of the three body problem, by solving numerically the equation of motion and by introducing measurements of strong body interactions. In Section III we will tackle the problem from the point of view of general relativity, by introducing a 3+1 formalism to integrate the Einstein field equations, presenting the Spectral-Filtered Numerical Gravity code (SFINGE). Simulations of multi-black hole dynamics will be presented in Section IV, comparing the outcome of two-black hole inspiraling with the three-black hole dynamics initiated via the Newtonian code. Finally, discussions and conclusions will be presented in Section V.
II. THREE-BODIES IN THE NEWTONIAN CASE

Despite the success in the understanding of simple (reduced) cases [1, 25], the solution of the general three-body problem remained elusive for about two hundred years after the publication of Newton’s *Principia*. In the general three-body problem all three masses are non-zero and their initial positions and velocities are not arranged in any particular way. The difficulty of the general three-body problem derives from the fact that there are no coordinate transformations which simplify the problem. This is in contrast to the two-body problem where the solutions are found most easily in the centre of mass coordinate system. In the three-body case, such transformation does not alleviate the problem, a difficulty which made the system analytically rather intractable, up to the modern age of computers. The numerical approach, indeed, revealed that the orbits are good examples of chaos in nature [26–28]. As follows, we run through again the main numerical results of the classical problem.

The system of equations describing the multi-body system of $N$ point-like masses $m_i$, affected only by their mutual gravitational forces, is given by:

$$\dot{x}_i = v_i,$$

$$m_i \dot{v}_i = -\sum_{j \neq i}^N \frac{m_i m_j (r_i - r_j)^2}{|r_i - r_j|^3},$$  \(1, 2\)

where we used the geometrical units $c = G = 1$, $r_i$ is the position of the $i^{th}$ body, $v_i$ its velocity, and $N = 3$.

We numerically solved the above equations, by using a forth-order Runge-Kutta technique, with quadruple precision. Hereafter, we will concentrate on a particular initial configuration of the three-body problem that immediately leads to a chaotic motion, namely the so-called Pythagorean problem proposed by Burrau [29]. The three bodies are initially at the corners of a Pythagorean right triangle, with the masses that are respectively $M_1 = 3$ units, $M_2 = 4$ units and $M_3 = 5$ units. These are placed on a $xy$ plane ($z = 0$), at the corners of a triangle, as reported in Fig. 1 (a).

At the beginning the bodies are at rest. Burrau’s calculation revealed the typical behaviour of a three-body system: two bodies approach each other, have a close encounter, and then recede again. Subsequently, other two-body encounters were calculated by Burrau until he came to the end of his calculating capacity. Later work has shown that the solution of the Pythagorean problem is quite typical of initially bound three-body systems. After many close two-body approaches a configuration arises which leads to an escape of one body and the formation of a binary by the other two bodies [1], as represented in Fig. 1 (a), where we report the trajectories of the three bodies up to $t = 80$.

We measured the kinetic $T$, the potential $U$ and the total energy $E$ of the system, as a function of time, until the escaper and the binary leave apart, which happens at about $t \approx 80$. These global quantities are reported in Fig. 1 (b). These interactions are extraordinarily intermittent, revealing spikes of kinetic and potential energy. We call these bursts Strong Interaction Times (SITs), where the three bodies are particularly close and the system has very large kinetic and potential energy. Analogously, to characterize the system configuration in such periods, we measured the moment of inertia $I = \sum m_i r_i^2$ and its acceleration $\ddot{I}$, as reported in Fig. 1 (c), which peaks at the SIT, as expected. Differentiating the moment of inertia twice with respect to time, in fact, is easy to obtain the Lagrange-Jacobi identity $\ddot{I} = 4T + 2U$. The latter can be viewed as unique identifier of a SIT event.

The system is long-living and chaotic, but the infinitely small size of the masses makes such results unrealistic when it comes to massive bodies. In particular, is is nat-
III. NUMERICAL RELATIVITY

The loss of energy and angular momentum from the system via gravitational radiation changes the basic constants of the three-body problem, and therefore general relativity effects need to be incorporated. Gravitational radiation leads to a decay of a binary orbit, and finally to a collapse of the binary black hole system into a single black hole. The final stages of the decay are very rapid and this might change dramatically the evolution of the three-body problem in the case of black holes.

What happens to three black holes when they strongly interact? The end result might be an escape of one of the black holes, and the recoil of the binary in the opposite direction, or it can be the subsequent merging of the three massive objects. We will focus on the interaction of the three black holes with the aim of extracting the gravitational waves generated due to the loss of energy and angular momentum from the system. In order to understand such emission of information, we will compare the 2 body inspiraling of the 3 body case, the latter particularly during strong interactions. In order to treat precisely such interactions, we will use a small system, concentrating on transients of times that just precedes a SIT. The pre-SIT configuration will be taken from the classical Newtonian simulation described in Section II

We will introduce numerical techniques adequate for stable and accurate evolution of multiple black-hole spacetimes. In particular, the techniques will be based on the moving puncture approach [30], treated by a novel pseudo-spectral method. The whole system will be evolved on the basis of the Baumgarte-Shapiro-Nakamura-Oohara-Kojima formulation of Einstein’s equations (hereafter BSSN for brevity) [31], with a highly structured and precise numerical code.

The BSSN decomposition is a formalism based on the work by Baumgarte and Shapiro [32] and by Shibata and Nakamura [33]. This formulation starts from a conformal rescaling of the physical metric

\[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}, \]

where \( \psi \) is a conformal factor. We follow the approach of Campanelli et al. [30], where this factor is written as \( \chi = \psi^{-4} \). This choice has been demonstrated to be a better alternative when considering singular spacetimes for which \( \psi \) typically has a \( r^{-1} \) singularity, while \( \chi \) is a \( C^4 \) function at the singularity. By following this particular choice, the Ricci tensor can be separated into two contributions, \( R_{ij} = \tilde{R}_{ij} + \chi^{-6} R^\chi_{ij} \), where

\[
\tilde{R}_{ij} = -\frac{1}{2} \tilde{\gamma}^{lm} \partial_l \partial_m \tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i} \partial_{j)} \tilde{\Gamma}^k + \tilde{\Gamma}^l \tilde{\gamma}_{ij} \]

\[
+ \tilde{\gamma}^{lm}(2 \tilde{\Gamma}^r_{k(l} \tilde{\gamma})^{km} + \tilde{\Gamma}^r_{lm} \tilde{\gamma}_{kj})
\]

is the Ricci tensor related to the conformal metric and

\[
R^\chi_{ij} = \frac{1}{2\chi} \left\{ \partial_i \partial_j \chi - \frac{\partial_i \partial_j \chi}{2\chi} - \tilde{\Gamma}_{ij} \right\}
\]

is the part that depends on the conformal factor \( \chi \). In the BSSN approach, the extrinsic curvature \( K_{ij} \) is divided into two independent variables, the trace \( \Gamma \) and its trace-free parts \( A_{ij} \). The last one is subjected to the same conformal transformation described by equation (3), i.e. \( \tilde{K}_{ij} = \psi^{-4}(K_{ij} - \frac{1}{3} \delta_{ij} K) \). Finally, a new field is introduced, namely the contracted Christoffel symbols associated with the conformal metric \( \tilde{\Gamma}^{ij} = \tilde{\gamma}^{ij} \tilde{\Gamma}^{ij}_{jk} \).

With the above change of variables, the system of BSSN equations reads:

\[
\partial_0 \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij}, \quad (4)
\]

\[
\partial_0 \chi = \frac{2}{3} \chi \alpha K, \quad (5)
\]

\[
\partial_0 \tilde{A}_{ij} = \chi \left[ -D_i D_j \alpha + \alpha R_{ij} \right]^{TF} + \alpha \left( K \tilde{A}_{ij} - 2 \tilde{A}_{ik} \tilde{A}^k_{ij} \right), \quad (6)
\]

\[
\partial_0 \tilde{\Gamma}^{ij} = \tilde{\gamma}^{lm} \partial_l \partial_m \beta^i + \frac{1}{3} \tilde{\gamma}^{ij} \partial_l \partial_m \beta^m + \beta^k \partial_k \tilde{\Gamma}^{ij} - \tilde{\Gamma}^k \partial_k \beta^i + \frac{2}{3} \tilde{\Gamma}^{ij} \partial_k \beta^k - 2 \tilde{A}_{ik} \partial_k \alpha + \alpha \left( 2 \tilde{\Gamma}^l_{im} \tilde{A}^m_{jl} - \frac{3}{\chi} \tilde{A}_{ik} \partial_k \chi - \frac{4}{3} \tilde{\gamma}^{ik} \partial_k K \right). \quad (7)
\]

In the above equations note that \( \partial_0 \alpha = \partial_0 - L_\beta \), where \( L_\beta \) is the Lie derivative with respect to the shift \( \beta^k \), \( \alpha \) is the lapse, \( D_i \) is the covariant derivative associated with the physical three-metric \( \gamma_{ij} \) and \( D^2 = \gamma^{ij} D_i D_j \). As usual, “\( TF \)” indicates the trace-free part of a tensor (i.e. for a generic tensor \( T_{ij} \) one has \( [T_{ij}]^{TF} = T_{ij} - \frac{1}{3} \delta_{ij} T_{lm} \)). We now need the evolution equation for the lapse and the shift, choosing therefore the slicing conditions. In the framework of the Bona-Masso formalism [32] [34] [35], we choose

\[
\partial_0 \alpha = -\alpha^2 f(\alpha) K, \quad (9)
\]
In all the above equations, \( \eta \) is a positive constant and is set to \( \eta = 2.8 \) [36], and the factor \( 3/4 \) is somewhat arbitrary but leads to good numerical results.

In addition to the evolution equations, as discussed in [37][41], the BSSN variables must satisfy the vacuum constraints

\[
\mathcal{H} = R - \tilde{A}_{lm} \tilde{A}^{lm} + \frac{2}{3} K^2 = 0, \tag{12}
\]

\[
\mathcal{M}^i = \partial_k \tilde{A}^{ik} + \tilde{\Gamma}^i_{lm} \tilde{A}^{lm} - \frac{3}{2\chi} \tilde{A}^{ik} \partial_k \chi - \frac{2}{3} \tilde{\gamma}^{ik} \partial_k K = 0, \tag{13}
\]

\[
\mathcal{G}^i = \tilde{\Gamma}^i + \partial_j \tilde{\gamma}^{ij} = 0, \tag{14}
\]

\[
\tilde{\gamma} - 1 = 0, \tag{15}
\]

\[
\tilde{A} = 0, \tag{16}
\]

where \( \tilde{\gamma} \) is the determinant of the conformal metric, and \( \tilde{A} \) is the trace of \( \tilde{A}_{ij} \). During the numerical simulations, we enforce the algebraic constraints in equations (15) and (16). The remaining constraints, \( \mathcal{H} \), \( \mathcal{M}^i \) and \( \mathcal{G}^i \) are not actively enforced and are used to monitor the accuracy of the numerical solutions [36]. In all the above equations, \( \tilde{\Gamma}^i \) is replaced by \( -\partial_j \tilde{\gamma}^{ij} \), wherever it is not differentiated.

A. The SFINGE code

The numerical procedure is described in [42] and has been tested against all classical testbeds. The spatial derivatives are computed in Cartesian geometry, via standard FFTs. For each BSSN field \( f(x,t) = f(x,t) \), the equations have been discretized on an equally-spaced lattice of \( N_x \times N_y \times N_z \) mesh points. At the collocation points, \( f(x,t) = \sum_k \tilde{f}_k(t) \exp(ik \cdot x) \), with \( x \) being the positions of the nodes, \( \tilde{f}_k(t) \in \mathbb{Z} \) the Fourier coefficients and the wavevectors \( k = (k_x, k_y, k_z) \). Along each box side \( L_0 \), the wavevector is \( k = 2\pi n/L_0 \), with \( m = 0, \pm 1, \pm 2, \cdots \pm N_k \), where \( N_k = N/2 \) the Nyquist mode. The code is based on a parallel architecture and makes use of MPI directives and FFTW libraries.

In the spectral space, the nonlinear terms become a convolution and there are several transform-based techniques for evaluating it efficiently [13][15]. Numerical problems might arise because of the so-called aliasing instabilities [46]: any product among fields creates higher \( k \)'s, causing an alias when these new modes become larger than the maximum size \( N_k \) of sampled Fourier modes. A Fourier mode with a wavenumber out of the size range is aliased to another wavenumber in the domain and, in a time-evolving system, it creates growing numerical problems. The importance of eliminating such pathology (aliasing error) has been studied since pioneering works by Orszag et al. [47]. We use a strategy to mitigate the above problems, based on an analogy with compressible hydrodynamics. We inspect the role of the aliasing truncation [48][49], by using a filtering technique. In practice, for each representation one has

\[
f(x,t) = \sum_k \tilde{f}_k(t) e^{ik \cdot x} \Phi_{k^*}(k), \tag{17}
\]

where the spectral anti-aliasing filter is given by

\[
\Phi_{k^*}(k) = e^{-a |\frac{2\pi k}{N}|^2}. \tag{18}
\]

Here \( a \) is a free parameter (we set \( a = 20 \)) that gives the sharpness of the filter and \( k^* \) is again the cutoff. Different values of \( k^* \) have been chosen, depending on the difficulty of the simulation and on the initial data. Hereafter we set \( k^* = N/2.5, \) where \( N = 512 \) is number of grid points along each spatial direction.

For the time integration we adopt a second-order Runge-Kutta method, with a time-step can be changed during the evolution, as described in [12]. Finally, we introduce an implicit, viscous strategy to absorb boundary disturbances, typical of the numerical tests that have a small violation of the periodicity. Loosely speaking, we use Cartesian geometries in order to simulate spherical domains and dissipate the regions close to the corners, with a radial envelope [50][52]. Hyperviscous terms of the type \( \nu_n \nabla^n f \) (where \( \nu_n \) is a numerical coefficient) are able to dissipate quickly ripples and numerical artifacts. For any dynamical BSSN variable, we advance in time the
solution \( f^{\\text{ideal}}(\mathbf{x}, t) \) by using the classical second-order Runge-Kutta method. Simultaneously, we evolve in time an accompanying, twin-field \( f^{H}(\mathbf{x}, t) \) which obeys to the same BSSN equation but is also subject to hylperscous dissipation, being therefore highly damped. We advance in time this hylperscous field \( f^{H}(\mathbf{x}, t) \) by using a Crank-Nicolson, semi-implicit method. With this procedure, the internal ideal region, advanced via the Runge-Kutta method, is matched with the outer diffusive layer integrated via the implicit scheme. Overall, the price to pay is that we have to double the time of integration, slowing down the computation. On the other hand, the method gives clear benefits on the stability and the goodness of the solutions.

IV. MULTI-BLACK HOLES DYNAMICS

Before treating the complex three-body problem via the Einstein field equations, we first calibrate our numerical experiments via the classic two-body inspiraling, which will provide us the classical waveform. We focus on the problem of a black-hole binary, with the scope of measuring the gravitational waves generated by such system and then compare it to the 3-body case.

The startiing point for multiple black hole data is the Schwarzschild solution in isotropic coordinates. In the case of time symmetry, these conformally flat data have been shown to generalize to an arbitrary number of black holes by adding the individual contribute in the conformal factor \( \psi = \chi^{-1/4} \). This time-symmetric initial configuration of multiple black holes is known as Brill-Lindquist data [54]. For boosted black holes, the conformal factor is written as \( \psi = 1 + \sum \frac{m_i}{r_i} + u \), where \( m_i \) and \( r_i \) are respectively the mass parameter and the distance from the \( i^{th} \) black hole, and \( u \) is a corrective term [54]. In order to build the initial condition for multiple black holes with arbitrary boost (hereafter we set null spin for all the single black holes), one has to solve an elliptic equation for \( u \). We have used an iterative Gauss-Seidel algorithm, starting from the initial guess specified in [55].

The numerical experiment describes the evolution of two black holes of equal masses which at the initial time are placed symmetrically with respect to the \( x \)-axis and have a small initial velocity which is the same for both bodies but directed in opposite directions. The grid is represented by a cube consisting of 512 points along each direction, while the physical dimension is \( 50M \) along each direction. The time step initially chosen corresponds to \( \Delta t = 8 \cdot 10^{-3} \). Furthermore, we require that hamiltonian and momentum constraints are well satisfied.

A simple, fast way to identify a gravitational disturbance is to look at a scalar measure of the value of the extrinsic curvature, especially in the plane of the ecliptic. As can be seen from the panels depicted in Fig. [2] and as expected, waves of small amplitude begin to propagate during the merger of the two black holes. Just after the merger, a larger amplitude modulation propagates away from the merging region, once the two event horizons come into contact with each other [56–60]. At this point several waves of smaller amplitude begin to propagate until the system, for long times, “relax” after giving rise to a single “Kerr” type black hole.

As the second, central simulation, we describe the evolution of a system composed of three black holes, referring to the simulations already carried out for the Pythagorean problem in the classical context. To this end, we take from the classical case a configuration such that the three bodies are to interact strongly. Specifically, referring to Fig. [1] (b), we are interested in the configuration of the three bodies immediately preceding the first larger SIT, at time \( t \approx 16 \) units.

Following the above procedure, we identify the instants of time that precedes the strongest three-body interactions. With this strategy we can study via spectral methods a relatively short period of time and save computational resources. The initial configuration chosen for the three black holes as regards initial positions and velocities have positions \( r_1 = (0.932, 2.695, 0.0) \), \( r_2 = (-1.544, -0.931, 0.0) \), \( r_3 = (0.676, -0.872, 0.0) \), with velocities \( \mathbf{v}_1 = (-0.037, -0.039, 0.0) \), \( \mathbf{v}_2 = (0.251, 0.037, 0.0) \) and \( \mathbf{v}_3 = (-0.179, 0.066, 0.0) \). The value of the masses of the three black holes are chosen as in the classical case (\( M_1 = 3 \) units, \( M_2 = 4 \) units, \( M_3 = 5 \)), only that in the relativistic context they are normalized so that the sum of all masses makes unity.

Compared to the case of the binary system, the run for the three black holes system contains more conditions since we have to take into account the initial position

![3D representation of the three-black hole interaction, at time t = 26.](image)
and the initial velocity of the third object. This could have important implications for the timing of the implementation of the initial conditions of the problem for the iterative Gauss-Seidel technique of the elliptic equation. To accelerate convergence for the initial data problem, we use a successively increasing resolution technique: we started from low resolution initial cube (64³) and then recursively adapted the relaxed solution for finer grids, arriving to the initial data on the 512³ lattice.

Focusing on the two-dimensional analysis of $K$, reported in Fig.2 (c) and (d), we can see how, unlike the case of the two black holes, here no spiral is visible. Furthermore, it is interesting to note how the front of the gravitational wave generated by the two mergers presents at the bottom a different amplitude compared to the rest of the wave front. In the last panel, after the system has been allowed to “relax”, it is possible to see a different pattern for the extrinsic curvature, suggesting a different kind of solution for the final black hole. In particular, this latter does not have angular momentum, because of the initial configuration, and therefore is different from the Kerr-type metric. In Fig.3 we report a full-dimensional rendering of the three black holes problem. Here we represent the gravitational radiation after the subsequent merging events. The (red and blue) shaded contours represent the isosurfaces of constant $K$, at two different values, showing how the outgoing waves are irregular and skewed. In the middle, the region close to the final event horizon is represented with a black sphere.

At this point, we analyze the waveform obtained by such a binary system defined by the projection of the Newman-Penrose scalar $Ψ_4$, which quantifies the outgoing radiation from the source of gravitational waves and which represents the relativistic analogue of the Poynting vector [31, 61–65]. We interpolated $Ψ_4$ from the Cartesian to a spherical grid, on which we calculated the outgoing radiation via the spin-weighted spherical harmonic $Y^{(−2)}_{22}(θ, ϕ)$. We compare therefore the gravitational signal measured away from the sources, for both the two-body inspiraling and the three-body SIT, as reported in Fig.4. In the $xy$ plane in the middle of the box ($z = 25$) represents the color contour of the Newman-Penrose scalar $Ψ_4$ for the three body case.

Contrary to the case of the two black holes, the real part and the imaginary part of the waveform are almost perfectly in phase, due to the fact that in the three body case the signal comes from subsequent non-spinning (null angular momentum), head-on collisions of the bodies. Such a configuration depends on the fact that the initial positions and velocities derive from the classical case of the Pythagorean problem, in which the three bodies start at rest.

Secondly, if in the case of the two black holes only one peak is observed corresponding to the merger of the two event horizons, in the multi-black hole case two peaks are visible. The first, which occurred around time $t ≈ 40M$, corresponds to the merging of the first two black holes (those aligned along approximately the same $x$ coordinate) and, except for the phase agreement, recalls the waveform seen for the binary system; the second, which occurred later around time $t ≈ 80M$, corresponds to the merging with the third black hole to give life to a single final black hole. In the time interval between the two peaks it is possible to notice a transient during which non-linear effects prevail.

Since the three-body problem is evidently highly non-linear, it could be more complicated to distinguish a signal that derives from the merger of three black holes and compare with database of numerical relativity. In this regard, we have compared the power spectra of the waveforms, in the case of the two and three black holes, as shown in Fig.5. As can be noticed from the spectra, the merger of three black holes during extreme SIT involves the presence of higher energy harmonics at medium-high frequencies while at low frequencies in both cases the behavior is about the same. Energy is more coherently distributed at a single peak for the two-body inspiraling, while the complex three body interactions transfer energy to higher frequencies. This could represent a discriminant that allows us to distinguish in the future a signal that suggests that somewhere in the Universe a merger involving three black holes happened. It is interesting to notice that in the three body case the spectrum manifests a power law behaviour for normalized frequencies between 0.6 and 3 stimulating interesting speculations about the possibility of a gravitational turbulent cascade.
V. CONCLUSIONS

The three-body problem is a classical example of complex dynamics, where the three interacting masses experience a chaotic behavior. We concentrated on two different approaches, namely (1) the Newtonian physics and (2) the Einstein’s theory of general relativity. Regarding the first part, we summarized the classical results and developed a very accurate Lagrangian code that solves the trajectories of the bodies, starting from a so-called Pythagorean configuration. Such configuration immediately evolves in a chaotic way, showing complex patterns. During the evolution, the system experiences quiet transients as well as very bursty behavior, typical of chaos intermittency [66]. We solved the trajectories accurately and identified such most intense spikes, named SITs. In the second (and more challenging) part of the paper, we built a gravitational model in order to study the dynamics of SITs in terms of Einstein gravitation. By using a sequence of conformal transformations, we briefly introduced the BSSN equations. Finally, with equations in hand, we presented the SFINGE code, based on a filtered pseudospectral scheme and a time-adapting Runge-Kutta. We used this code by initiating the movement of the multiple-body problems, in fully nonlinear, 3D regimes.

As a guidance for our general relativistic experiments, we considered the problem of the binary coalescence of two black holes, in their final stage of their coalescence, inside the innermost stable circular orbit (ISCO), solving the inspiraling dynamics. We introduced the problem of wave extraction, in order to measure the gravitational radiation emitted by the binary system. By using a spherical interpolation, and following the Newman-Penrose formalism, we measured such gravitational waves, which are qualitatively in agreement with typical gravitational waves signals.

After the two-body simulations we finally considered the three black holes interaction. We extracted from the Newtonian simulations the configuration preceding a strong SIT, when the bodies are sufficiently apart. Therefore we used such SIT as an initial condition for the Einstein field equations. We monitored several quantities, such as the metric tensor and the extrinsic curvature of the system, revealing a very high energy emission of gravitational waves, with subsequent coalescence events. In analogy with the binary case, we used the numerical extraction technique and reconstructed the emitted waveform, which is highly irregular and nonlinear. We compared finally the binary-merger and the strong three-body interaction, by computing the power spectra of the gravitational wave emitted in the two cases. We found a net difference, essentially at high frequencies, where the three-black hole system exhibits higher energies and wider distribution of the power. The latter seems consistent with a power law which might suggest that gravitation, similarly to fluid mechanics, is subject to a turbulent cascade. In future works, we will inspect such possibility, by performing larger and longer simulations, in different regimes.

These results could be of interest for very recent observational campaigns, since we expect that with more modern technologies we should be able to have better signal-to-noise measurements and identify the imprint of such complex, multi-body dynamics, confirming whether such spectacular events sometimes occur in the Universe. While the two-body inspiraling signal has been largely investigated in literature [55, 59, 67-73], in the three black hole SIT there is much less documentation [15, 16] and at the present no observational evidence, probably due to a weaker and broader signal. In future work we plan to study more SITs events and other initial configuration, such as those with initial angular momentum or with non-planar motion of the three bodies.

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