MULTIPLE BOUNDARY REPRESENTATIONS OF $\lambda$-HARMONIC FUNCTIONS ON TREES

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Abstract. We consider a countable tree $T$, possibly having vertices with infinite degree, and an arbitrary stochastic nearest neighbour transition operator $P$. We provide a boundary integral representation for general eigenfunctions of $P$ with eigenvalue $\lambda \in \mathbb{C}$, under the condition that the oriented edges can be equipped with complex-valued weights satisfying three natural axioms. These axioms guarantee that one can construct a $\lambda$-Poisson kernel. The boundary integral is with respect to distributions, that is, elements in the dual of the space of locally constant functions. Distributions are interpreted as finitely additive complex measures. In general, they do not extend to $\sigma$-additive measures: for this extension, a summability condition over disjoint boundary arcs is required. Whenever $\lambda$ is in the resolvent of $P$ as a self-adjoint operator on a naturally associated $\ell^2$-space and the diagonal elements of the resolvent ("Green function") do not vanish at $\lambda$, one can use the ordinary edge weights corresponding to the Green function and obtain the ordinary $\lambda$-Martin kernel.

We then consider the case when $P$ is invariant under a transitive group action. In this situation, we study the phenomenon that in addition to the $\lambda$-Martin kernel, there may be further choices for the edge weights which give rise to another $\lambda$-Poisson kernel with associated integral representations. In particular, we compare the resulting distributions on the boundary.

The material presented here is closely related to the contents of our "companion" paper [17].

1. Introduction

Let $T$ be a countable tree, i.e., a connected graph without cycles. We allow vertices with infinite degree, but for simplicity, we exclude leaves (vertices with degree 1). Here, the degree $\text{deg}(x)$ of a vertex $x$ is the number of its neighbours. We tacitly identify $T$ with its vertex set.

On $T$, we consider the stochastic transition matrix $P = (p(x, y))_{x,y \in T}$ of a nearest neighbour random walk. This means that $p(x, y) > 0$ if and only if $x \sim y$ (i.e., $x$ and $y$ are neighbours). $P$ acts on functions $f : T \to \mathbb{C}$ by

$$(1.1)\quad Pf(x) = \sum_y p(x, y)f(y),$$

where in case when $\text{deg}(x) = \infty$ we postulate that the sum converges absolutely. For $\lambda \in \mathbb{C}$, a $\lambda$-harmonic function is a function $h : T \to \mathbb{C}$ which satisfies $Ph = \lambda \cdot h$.

For "good" values of $\lambda$, every $\lambda$-harmonic function has a boundary integral representation over the geometric boundary at infinity of the tree. This is analogous to the Poisson integral formula for classical harmonic functions on the open unit disk, where the boundary is the unit circle. The Poisson kernel of the disk has to be replaced by the $\lambda$-Martin kernel, and the integral is with respect to a distribution on the boundary. The good

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values include in particular $\lambda = 1$, when the random walk is transient. More generally, they comprise at least all $\lambda \in \mathbb{C}$ where $|\lambda| > \rho$ with $\rho = \rho(P)$, the \textit{spectral radius} of the random walk (the definitions will be given in more detail further on). For \textit{positive} $\lambda$-harmonic functions – whose existence necessitates that $\lambda \geq \rho$ is real – the representing distribution on the boundary is a finite ($\sigma$-additive) Borel measure.

The results that we have mentioned in this last paragraph are due to Cartier [5] for the case when $\lambda \geq \rho$ and the tree is locally finite, and the extension to the non-locally finite case can be found in the book of Woess [22, Ch. 9]. For general complex $\lambda$, these results are proved in our recent paper [17], when $\lambda$ is in the resolvent set of $P$ and the diagonal elements of the \textit{Green kernel} (Green function) do not vanish at $\lambda$. This was preceded by a result of Figà-Talamanca and Steger [8] for the locally finite case, when $P$ is the transition matrix of a group invariant random walk on a free group, or a close relative of that group.

All this comprises the long known example of the \textit{simple random walk} on $T = T_q$, the regular tree with degree $q + 1 \geq 3$, where $p(x,y) = 1/(q + 1)$ when $x \sim y$. In this case, it follows from the results of Mantero and Zappa [13] that, besides the ordinary $\lambda$-Martin kernel, there is a second kernel which gives rise to a boundary integral representation of $\lambda$-harmonic functions. Indeed, this plays an important role in the context of the representation theory of free groups. Since then, this phenomenon has remained the object of repeated discussions, in particular between the first author and David Singman (George Mason University, Fairfax).

The purpose of the present note is to shed more light on these multiple boundary integral representation by approaching them from a wider viewpoint. Thereby, part of our presentation lays out in detail several proofs which take up and generalize previous work.

We first (§2) recall the construction of the boundary at infinity $\partial T$ of $T$ and the corresponding compactification. We introduce distributions on $\partial T$ and explain how locally constant functions on $\partial T$ are integrated against a distribution.

Then (§3) we start with an arbitrary $\lambda \in \mathbb{C}$ and put weights on the oriented edges of $T$. They are required to satisfy certain axioms (this might not be possible for all $\lambda$) and then they can be used to define a general $\lambda$-potential kernel and subsequently a $\lambda$-Poisson kernel $k(x,y), x, y \in T$. This kernel extends in the second variable to a locally constant function on $\partial T$, and we use it to prove a general Poisson-Martin boundary integral representation theorem for $\lambda$-harmonic functions.

Let us write $\text{res}^*(P)$ for the set of all elements in the resolvent set of $P$ as a self-adjoint operator for which the diagonal matrix elements of the resolvent ($\lambda$-Green function) do not vanish. For $\lambda \in \text{res}^*(P)$, the classical weights satisfying the needed axioms are suitable quotients of the $\lambda$-Green function, which we call the \textit{Green weights}. This leads to the above mentioned representation proved in [17] and the preceding work.

Later on (§4), we restrict attention to the case when $P$ is invariant under a transitive group of automorphisms of $T$. In this situation, we discuss the cases where in addition to the classical ones, one can find different sets of weights which also lead to boundary integral representations for the same space of $\lambda$-harmonic functions. In this case, however,
we show that the distribution which arises for a given λ-harmonic function does typically not extend to a (σ-additive) Borel measure on the boundary, even when this is true with respect to the Green weights.

2. Boundary and distributions

A. The end compactification

For two vertices \(x, y \in T\), the geodesic or geodesic path from \(x\) to \(y\) is the unique shortest path \(\pi(x, y)\) from \(x\) to \(y\), and the distance \(d(x, y)\) is the length (number of edges) of \(\pi(x, y)\).

A ray or geodesic ray in \(T\) is a sequence \([x_0, x_1, x_2, \ldots]\) such that \(x_{i-1} \sim x_i \) and \(x_{i+1} \neq x_{i-1}\) for all \(i\). Two rays are equivalent, if they differ by finitely many initial vertices. An end of \(T\) is an equivalence class of rays. If \(x\) is a vertex and \(\xi\) an end, there is a unique geodesic ray \(\pi(x, \xi)\) which starts at \(x\) and represents \(\xi\). The boundary \(\partial T\) of \(T\) is the set of all ends of \(T\). For \(x, y \in T\) with \(x \neq y\), the branch or cone \(T_{x,y}\) is the subtree spanned by all vertices \(w\) with \(y \in \pi(x, w)\), and the boundary arc \(\partial T_{x,y}\) is the set of all ends which have a representative ray in \(T_{x,y}\).

We set \(\hat{T} = T \cup \partial T\) and \(\hat{T}_{x,y} = T_{x,y} \cup \partial T_{x,y}\). We put the following topology on \(\hat{T}\): it is the topology on the vertex set, and a neighbourhood base of \(\xi \in \partial T\) is given by the collection of all \(\hat{T}_{x,y}\) which contain a ray that represents \(\xi\). (Here, we may fix \(x\) and vary only \(y \neq x\)) The resulting space is metrizable. It is compact precisely when \(T\) is locally finite, but otherwise, it is not complete. This can be overcome as follows. For each vertex \(x\) with infinite degree – following an idea of Soardi [4] – we add a boundary point as follows: we introduce a new improper vertex \(x^*\), the shadow of \(x\), and we set \(T^* = \{x^* : x \in T, \ deg(x) = \infty\}\), as well as \(\partial^* T = T^* \cup \partial T\). Analogously, \(\partial^* T_{x,y} = T^*_{x,y} \cup \partial T_{x,y}\).

Let us write \(\bar{T} = T^* \cup \hat{T}\) and \(\bar{T}_{x,y} = T^*_{x,y} \cup \hat{T}_{x,y}\). Again, \(T\) is discrete in \(\bar{T}\). A neighbourhood base of end \(\xi \in \partial T\) is now provided by all \(\bar{T}_{x,y}\) which contain a geodesic that represents \(\xi\). A neighbourhood sub-base of \(x^* \in T^*\) is given by all \(\bar{T}_{x,x}\), where \(v \sim x\).

We now describe convergence of sequences in \(\bar{T}\) in this topology. We choose a root vertex \(o \in T\) and write \(T_x = T_{o,x}\) for any \(x \in T\); in particular, \(T_o = T\). Throughout everything which follows, it will be useful to define the predecessor \(x^-\) of a vertex \(x \neq o\). This is the neighbour of \(x\) on the geodesic \(\pi(o, x)\), and \(x\) is a called a forward neighbour of \(x^-\). For \(x \in T\), set \(|x| = d(o, x)\), the graph distance. For \(\xi \in \partial T\), set \(|\xi| = \infty\).

For any pair of elements \(v, w \in \hat{T}\) (i.e., not in \(T^*\)), their confluent \(v \wedge w\) with respect to \(o\) is the last common element on the geodesics \(\pi(o, v)\) and \(\pi(o, w)\). It is a vertex, unless \(v = w \in \partial T\), in which case the confluent is that end. Now, if \((w_n)\) is a sequence in \(\hat{T}\), then

- \(w_n \rightarrow x \in T\) when \(w_n = x\) for all but finitely \(n\).
- \(w_n \rightarrow \xi \in \partial T\) when \(|w_n \wedge \xi| \rightarrow \infty\).
- \(w_n \rightarrow x^* \in T^*\) when \(w_n\) “rotates” around \(x\), that is, any \(y \sim x\) lies on at most finitely many geodesics \(\pi(x, w_n)\).

Finally, if \((x_n^*)\) is a sequence of improper vertices, then
• $x_n^* \to x^* \in T^*$ or $x_n^* \to \xi \in \partial T$ when $x_n \to x^*$, resp. $x_n \to \xi$ in the above sense.

Now $\overline{T}$ is compact, and $T$ is an open-discrete subset, so that also $\partial^* T$ is compact. For the understanding of distributions, the next considerations will be useful. They follow CARTWRIGHT, SOARDI AND WOESS [4], see also [22, Thm. 7.13].

Let $X$ be a countable set. By a compactification of $X$ we mean a compact Hausdorff space into which $X$ embeds as a discrete, open, dense subset. Now let $\mathcal{F}$ be a countable family of bounded functions $f : X \to \mathbb{R}$. Then there is a unique minimal compactification $\overline{X}_{\mathcal{F}}$ of $X$ such that each $f \in \mathcal{F}$ extends to a continuous function on $\overline{X}_{\mathcal{F}}$. Here, “minimal” refers to the partial order on compactifications where one is smaller than the other if the identity mapping on $X$ extends to a continuous surjection from the bigger to the smaller one, and two compactifications are considered equal, if that extension is a homeomorphism.

Now let $T$ be a countable tree (or any connected, countable graph) with edge set $E = \{(x, y) \in T^2 : x \sim y\}$. A function $f : T \to \mathbb{C}$ is called locally constant, if the set of edges along which $f$ changes its value,

$$\{e = [x, y] \in E : f(x) \neq f(y)\},$$

is finite. The vector space $V$ of all locally constant functions is spanned by the countable set $\mathcal{F}$ of all those functions in $V$ which take values in $\{0, 1\}$. Therefore, in the corresponding compactification $\overline{T}_{\mathcal{F}}$, every locally constant function on $T$ has a continuous extension. Now, as explained in [4], when the tree (graph) is locally finite, then one gets the well-known end compactification. When the tree $T$ is not locally finite, we just get the compactification $\overline{T} = T^* \cup \hat{T}$ described above.

For the purposes of the present note, the improper vertices remain an artifact which provides compactness, but will not be used in a specific way, except to clarify the view on the subject.

B. Distributions on the boundary

The following material is adapted and extended from [17]. Consider a function $f \in V$. Let $E_f$ be the finite set of edges along which $f$ changes value. We can choose a finite subtree $\tau$ of $T$ which contains all those edges as well as the chosen root $o$. For a vertex $x \in \tau$, write $S_x(\tau)$ for the set of forward neighbours $y$ of $x$ in $\tau$ (it may be empty). The boundary $\partial \tau$ of $\tau$ in $T$ consists of those $x \in \tau$ which have a neighbour outside $\tau$. For each $x \in \partial \tau$, the function $f$ is constant on the part of the tree which branches off at $x$, which is

$$T_x(\tau) = T_x \setminus \bigcup\{T_y : y \in S_x(\tau)\}$$

Now let $\partial V$ be the trace of the vector space $V$ on $\partial T$, and define $\partial \mathcal{F}$ correspondingly. By the above considerations, each element of $\partial \mathcal{F}$ is the indicator function of a subset of $\partial T$ which can be written as a finite disjoint union of sets of the form

$$\partial T_x \setminus \bigcup\{\partial T_y : y \in S_x\},$$

where $x \in T$ and $S_x$ is a finite collection of forward neighbours of $x$ (possibly empty). If $\nu$ is an element in the dual space of $\partial V$ then it can be seen as a complex-valued set
function on the collection of all those sets, and we call it a distribution. The following is now obvious.

(2.1) Lemma. Any distribution \( \nu \) is characterized by the property that, for every \( x \in T \) and finite set \( S_x \) of forward neighbours of \( x \),

\[
\nu(\partial T_x) = \sum_{y \in S_x} \nu(\partial T_y) + \nu\left( \partial T_x \setminus \bigcup \{ \partial T_y : y \in S_x \} \right).
\]

In particular, if \( T \) is locally finite, then \( \nu \) can be described as a set function on all boundary arcs such that

(2.2) \[
\nu(\partial T_x) = \sum_{y : y = x} \nu(\partial T_y) \text{ for every } x \in T.
\]

In [17], we have defined distributions analogously in the non-locally finite case, requiring in that case that the sum in (2.2) converges absolutely. In this case, let us call \( \nu \) a strong distribution here. For all results of [17] as well as the present note, the distributions actually involved are always strong.

In particular, when \( \nu \) is non-negative real, then it is not only strong, but extends to a finite, \( \sigma \)-additive Borel measure on \( \partial T \), as explained in [17, 3.10]. As mentioned there, when \( \nu \) is a complex-valued distribution, a necessary and sufficient condition for its extendability to a \( \sigma \)-additive, signed measure on the Borel \( \sigma \)-algebra of \( \partial T \) is that there is \( M < \infty \) such that for any sequence of pairwise disjoint boundary arcs \( \partial T_{x_n} \), one has

(2.3) \[
\sum_n |\nu(\partial T_{x_n})| \leq M.
\]

This is an easy extension of the corresponding condition in the locally finite case of Cohen, Colonna and Singman [6].

For any distribution \( \nu \), we now write

\[
\nu(\varphi) = \int_{\partial T} \varphi \, d\nu \quad \text{for} \quad \varphi \in \partial \mathcal{V}.
\]

By the above, given \( \varphi \), there are a finite subtree \( \tau \) of \( T \) containing \( o \) and constants \( \varphi_x \), \( x \in \partial \tau \), such that

\[
\varphi \equiv \varphi_x \text{ on } \partial T_x(\tau) = \partial T_x \setminus \bigcup \{ \partial T_y : y \in S_x(\tau) \}, \quad \text{and}
\]

(2.4) \[
\int_{\partial T} \varphi \, d\nu = \sum_{x \in \partial \tau} \varphi_x \nu(\partial T_x(\tau)).
\]

By construction, this does not depend on the specific choice of the finite tree \( \tau \) associated with \( \varphi \). If \( \nu \) extends to a \( \sigma \)-additive complex Borel measure on \( \partial T \), then the integral is an ordinary one in the sense of Lebesgue.

C. Self-adjointness of the transition operator

With the action defined by (1.1), the transition operator \( P \) is self-adjoint on the Hilbert space \( \ell^2(T, m) \) of all functions \( f : T \to \mathbb{C} \) with \( \langle f, f \rangle < \infty \), where

\[
\langle f, g \rangle = \sum_x f(x)\overline{g(x)} \, m(x),
\]
with the measure \( m \) on \( T \) as follows:

\[
\text{for } x \in T \text{ with } \pi(o, x) = [x_0, x_1, \ldots, x_k], \quad m(x) = \frac{p(x_0, x_1) \cdots p(x_{k-1}, x_k)}{p(x_1, x_0) \cdots p(x_k, x_{k-1})}.
\]

In particular, \( m(o) = 1 \). Self-adjointness is a consequence of reversibility: \( m(x)p(x, y) = m(y)p(y, x) \) for all \( x, y \). The norm (spectral radius) of \( P \) is

\[
\rho = \rho(P) = \limsup_{n \to \infty} p^{(n)}(x, y)^{1/n}
\]

(independent of \( x \) and \( y \)), where \( p^{(n)}(x, y) \) is the \((x, y)\)-element of the matrix power \( P^n \). Since trees are bipartite, the spectrum \( \text{spec}(P) \subset [-\rho, \rho] \) is symmetric around the origin.

Positive \( \lambda \)-harmonic functions exist if and only if \( \lambda \geq \rho \) (real). At this point, we state a warning: when viewing \( \lambda \)-harmonic functions as “eigenfunctions” of \( P \), they are not considered as eigenfunctions of the above self-adjoint operator on \( \ell^2(T, m) \). As a matter of fact, besides possibly for \( \lambda = \pm \rho \) in very specific situations, our \( \lambda \)-harmonic functions will usually not belong to \( \ell^2(T, m) \). In a variety of known cases, \( \text{spec}(P) \) contains no eigenvalues, that is, there is no point spectrum on \( \ell^2(T, m) \). In any case, our methods and results do not cover the case where \( \lambda \in \text{spec}(P) \setminus \{\pm \rho\} \).

3. The general integral representation

We now fix a candidate eigenvalue \( \lambda \in \mathbb{C} \) and we suppose that we can equip the oriented edge set \( E(T) = \{(x, y) \in T^2 : x \sim y\} \) of \( T \) with \( \lambda \)-weights \( f(x, y) \in \mathbb{C} \) satisfying the following properties for every \( x \in T \) and every \( y \) with \( x \sim y \).

\[
(3.1) \quad f(x, y)f(y, x) \neq 1 \quad \text{for all pairs of neighbours } x, y,
\]

\[
(3.2) \quad u(x, x) \neq \lambda, \quad \text{where} \quad u(x, x) = \sum_v p(x, v)f(v, x),
\]

\[
(3.3) \quad \lambda f(x, y) = p(x, y) + \left( u(x, x) - p(x, y)f(y, x) \right)f(x, y).
\]

If \( \deg(x) = \infty \) then we require that the sum in (3.2) converges absolutely. Note that it follows from (3.3) that \( f(x, y) \neq 0 \) for all pairs of neighbours. The above three axioms arise by mimicking the main properties of the natural Green weights, which will be discussed at the end of this section.

Using these weights, for arbitrary \( x, y \in T \) we define

\[
(3.4) \quad f(x, y) = f(x_0, x_1)f(x_1, x_2) \cdots f(x_{k-1}, x_k), \quad \text{if} \quad \pi(x, y) = [x_0, \ldots, x_k].
\]

In particular, \( f(x, x) = 1 \).

(3.5) Lemma. For fixed \( y \), the function \( x \mapsto f(x, y) \) satisfies

\[
P f(x, y) = \lambda f(x, y) \quad \text{if } x \neq y, \quad \text{and} \quad P f(y, y) = u(y, y).
\]
Proof. The second identity is the definition (3.2) of \( u \). For the first identity, let \( \pi(x, y) \) be as in (3.4). Consider the neighbours \( x = x_0 \) and \( x_1 \). Then (3.2) and (3.3) yield
\[
P f(x, y) = p(x, x_1) f(x_1, y) + \sum_{v \neq x_1} p(x, v) f(v, x) f(x, y)
\]
\[
= p(x, x_1) f(x_1, y) + \left( u(x, x) - p(x, x_1) f(x_1, x) \right) f(x, x_1) f(x_1, y)
\]
\[
= \lambda f(x, x_1) f(x_1, y) = \lambda f(x, y),
\]
as stated. □

Note that absolute convergence of the sum in (3.2) is crucial for the Lemma. It is this property that further on will give us strong distributions. Thanks to (3.2) we can set (3.6)
\[
g(x, y) = \frac{f(x, y)}{\lambda - u(y, y)}, \quad x, y \in T.
\]
Then we see from Lemma 3.5 that the function \( x \mapsto g(x, y) \) satisfies the resolvent equation
\[
P g(x, y) = \lambda g(x, y) - \delta_x(y).
\]

The following Lemma shows how the transition probabilities can be recovered from the weights \( f(x, y) \), compare with [5] for the locally finite case with standard non-negative Green weights.

(3.8) Lemma. For \( x \in T \) and \( y \sim x \),
\[
g(x, x)p(x, y) = \frac{f(x, y)}{1 - f(x, y)f(y, x)},
\]
\[
g(x, x)g(y, y) = g(x, y) \left( \frac{1}{p(x, y)} + g(y, x) \right), \quad \text{and}
\]
\[
\lambda g(x, x) = 1 + \sum_{y : y \sim x} \frac{f(x, y)f(y, x)}{1 - f(x, y)f(y, x)}.
\]
When \( \deg(x) = \infty \), the last sum converges absolutely.

Proof. We can rewrite (3.3) as
\[
p(x, y)\left( 1 - f(x, y)f(y, x) \right) = f(x, y) \left( \lambda - u(x, x) \right).
\]
Since \( \lambda - u(x, x) = 1/g(x, x) \), the first identity follows, and with \( g(x, y) = f(x, y)g(y, y) \) as well as \( g(y, x) = f(y, x)g(x, x) \), we get
\[
g(x, y)\left( \frac{1}{p(x, y)} + g(y, x) \right) = g(x, x)g(y, y) \left( \frac{1}{g(x, x)p(x, y)} + f(y, x) \right).
\]
This is the second identity. We now multiply the first identity with \( f(y, x) \). The sum over all neighbours \( y \) of \( x \) is absolutely convergent by assumption, so that we have indeed
absolute convergence of the right hand side of the third identity, and
\[ \sum_{y \sim x} \frac{f(x, y) f(y, x)}{1 - f(x, y) f(y, x)} = g(x, x) u(x, x) = \lambda g(x, x) - 1 \]
by the definition of \( g(x, x) \).

We define the \( \lambda \)-Poisson kernel associated with our weights by
\[ k(x, w) = \frac{f(x, x \wedge w)}{f(o, w)} = \frac{f(x, x \wedge o)}{f(o, o)} = g(x, x \wedge o) g(o, o) \]
for every vertex \( w \in \hat{T} \).

Thus,\( (3.9) \)
\[ k(x, w) = f(x, v) f(o, v) g(x, v) g(o, v) \]
for every vertex \( v \in \pi(x \wedge w, w) \).

By our assumptions, \( Ph(\cdot, w) \) is well defined as a function of the first variable. That is, even at vertices with infinite degree, the involved sum is absolutely convergent, and for \( \xi \in \partial T \),
\[ (3.10) \]
\[ \sum_{y \sim x} p(x, y) k(y, \xi) = \lambda k(x, \xi) \]
for every \( x \in T \).

Now let \( x \in T \) and \( \pi(o, x) = [o = x_0, x_1, \ldots, x_k = x] \). Then \( x \wedge \xi \in \{x_0, x_1, \ldots, x_k\} \)
for every \( \xi \in \partial T \), and
\[ k(x, \xi) = k(x, x_i) \]
when
\[ \begin{cases} 
\xi \in \partial T_{x_i} \setminus \partial T_{x_{i+1}}, & i \leq k - 1, \\
\xi \in \partial T_{x_k}, & i = k.
\end{cases} \]

Thus, \( \varphi = k(x, \cdot) \) is locally constant on \( \partial T \), and we can use \( \pi(o, x) \) as a tree \( \tau \) to which
\[ (2.4) \]
applies. Then we have the following.

\[ (3.11) \text{Proposition. If } \nu \text{ is a strong distribution } \nu \text{ on } \partial T, \text{ its Poisson transform} \]
\[ h(x) = \int_{\partial T} k(x, \xi) d\nu(\xi) \]
is a \( \lambda \)-harmonic function, and
\[ h(x) = \sum_{i=0}^{k-1} k(x, x_i) \left( \nu(\partial T_{x_i}) - \nu(\partial T_{x_{i+1}}) \right) + k(x, x) \nu(\partial T_x) \]
\[ (3.12) \]
\[ = k(x, o) \nu(\partial T) + \sum_{i=1}^{k} \left( k(x, x_i) - k(x, x_{i-1}) \right) \nu(\partial T_{x_i}) . \]

\[ \text{Proof.} \] The proof of \( \lambda \)-harmonicity of \( h \) is obvious when \( T \) is locally finite. Otherwise, some care is needed, and we go through the details in order to show the necessity of the assumption that the distribution \( \nu \) be strong. First of all, we show that \( Ph(o) = \lambda h(o) \).

By \[ (3.12) \], if \( x \sim o \),
\[ h(x) = f(x, o) \nu(\partial T) + \left( \frac{1}{f(o, x)} - f(x, o) \right) \nu(\partial T_x) . \]

\[ (3.13) \]
Similarly, let

\[ f(3.14) \]

we also need (3.14) for

\[ y \]

by (3.3). In case \( \text{deg}(x) = \infty \), we needed absolute convergence of the involved series. Similarly, let \( x \neq o \). Then (3.12) yields the formula

\[
(3.14) \quad h(x) = f(x, x^-)h(x^-) + \frac{1 - f(x, x^-)f(x^-)}{f(o, x)} \nu(\partial T_x),
\]

that will also be important further below. To prove (3.14), we first observe that it is the same as (3.13) when \( x \sim o \). Now let \( k \geq 2 \) in (3.12), and note that for \( i \leq k - 1 \) we have

\[
k(x, x_i) = f(x, x_i)/f(o, x_i) = f(x, x^-)k(x^-, x_i),
\]

with \( x^- = x_{k-1} \). Then, using the first of the two identities of (3.12),

\[
h(x) = f(x, x^-) \sum_{i=0}^{k-2} k(x^-, x_i) \left( \nu(\partial T_{x_i}) - \nu(\partial T_{x_{i+1}}) \right)
\]

\[
+ f(x, x^-)k(x^-, x^-) \left( \nu(\partial T_{x^-}) - \nu(\partial T_x) \right) + k(x, x) \nu(\partial T_x)
\]

\[
= f(x, x^-)h(x^-) - \frac{f(x, x^-)}{f(o, x^-)} \nu(\partial T_x) + \frac{1}{f(o, x)} \nu(\partial T_x).
\]

Since \( f(o, x) = f(o, x^-)f(x^-, x) \), this reduces to the desired formula. For the following, we also need (3.14) for \( y \) with \( y^- = x \). Absolute convergence in the first of the following identities is justified a posteriori, and the first identity of Lemma 3.8 is used for the underbraced as well as for the overbraced term, and again in the very last step.

\[
Ph(x) = p(x, x^-)h(x^-) + \sum_{y^- = x} p(x, y)f(y, x)h(x)
\]

\[
+ \frac{1}{f(o, x)} \sum_{y^- = x} p(x, y) \frac{1 - f(x, y)f(y, x)}{f(x, y)} \nu(\partial T_y)
\[
1/g(x, x)
\]

\[
= p(x, x^-)h(x^-) - \frac{1}{f(o, x)} p(x, x^-) \frac{1 - f(x, x^-)f(x^-)}{f(x, x^-)} \nu(\partial T_x)
\]

\[
+ u(x, x)h(x) - p(x, x^-)f(x^-, x)h(x) + \frac{1}{g(o, x)} \nu(\partial T_x)
\]

\[
= \left( p(x, x^-) \frac{1 - f(x, x^-)f(x^-)}{f(x, x^-)} + u(x, x) \right) h(x) = \lambda h(x).
\]

In the second identity we made use of the assumption that \( \nu \) is strong. \[ \square \]
The proof of the following is very similar to [22, Thm. 9.37]: we rewrite its main part here to take care of absolute convergence in the non-locally finite case.

**Theorem.** Suppose that we have edge weights \(f(x, y)\) which satisfy (3.1) – (3.3). A function \(h : T \to \mathbb{C}\) satisfies \(Ph = \lambda \cdot h\) if and only if it is of the form

\[
h(x) = \int_{\partial T} k(x, \xi) \, d\nu(\xi),
\]

where \(\nu\) is a strong complex distribution on \(\partial T\). The distribution \(\nu\) is determined by \(h\), that is, \(\nu = \nu^h\), where

\[
\nu^h(\partial T) = h(o) \quad \text{and} \quad \nu^h(\partial T_x) = f(o, x) \frac{h(x) - f(x, x^-)h(x^-)}{1 - f(x, x^-)f(x, x^-)}, \quad x \neq o.
\]

**Proof.** We first show that if \(h\) is \(\lambda\)-harmonic, then \(\nu^h\) as defined in the theorem is indeed a strong distribution, and \(h\) is its Poisson transform. We start with the identity

\[
\lambda g(x, x)h(x) = \sum_{y : y \sim x} g(x, x)p(x, y)h(y),
\]

and recall that the sum on the right hand side is assumed to converge absolutely when \(\text{deg}(x) = \infty\). Using Lemma 3.3 we rewrite this as

\[
\left(1 + \sum_{y : y \sim x} \frac{f(x, y)f(y, x)}{1 - f(x, y)f(y, x)}\right) h(x) = \sum_{y : y \sim x} \frac{f(x, y)}{1 - f(x, y)f(y, x)} h(y).
\]

Since the involved sums converge absolutely, we can regroup the terms and get

\[
(3.16) \quad h(x) = \sum_{y : y \sim x} f(x, y) \frac{h(y) - f(y, x)h(x)}{1 - f(x, y)f(y, x)}.
\]

Convergence is again absolute when \(\text{deg}(x) = \infty\).

For \(x = o\), the last identity says that \(\nu^h(\partial T) = \sum_{y \sim o} \nu^h(\partial T_y)\). If \(x \neq o\), then by (3.10),

\[
\sum_{y : y \sim x} \nu^h(\partial T_y) = f(o, x) \sum_{y : y \sim x} f(x, y) \frac{h(y) - f(y, x)h(x)}{1 - f(x, y)f(y, x)} = f(o, x) \left( h(x) - f(x, x^-)h(x^-) - f(x^-, x)h(x) \right) \frac{1 - f(x, x^-)f(x^-, x)}{1 - f(x, x^-)f(x^-, x)} = f(o, x) h(x) - f(x, x^-)h(x^-) = \nu^h(\partial T_x).
\]

So \(\nu^h\) is indeed a signed distribution on \(F_o\). We verify that \(\int_{\partial T} k(x, \xi) \, d\nu^h(\xi) = h(x)\). For \(x = o\) this is clear, so let \(x \neq o\). With notation as in (3.12), we simplify

\[
(k(x, x_i) - k(x, x_{i-1})) \nu^h(\partial T_{x_i}) = f(x, x_i)h(x_i) - f(x, x_{i-1})h(x_{i-1}),
\]

whence we obtain

\[
\int_{\partial T} K(x, \xi) \, d\nu^h(\xi) = k(x, o)h(o) + \sum_{i=1}^k \left( f(x, x_i)h(x_i) - f(x, x_{i-1})h(x_{i-1}) \right) = f(x, x)h(x) = h(x),
\]
as stated.

Second, we need to verify that given \( \nu \) and its Poisson transform \( h \), we have \( \nu = \nu^h \). This part of the proof is nothing but the identity (3.14) in the proof of Proposition 3.11. \( \square \)

The natural Green weights

We now “reveal” the origin of the axioms (3.1) – (3.3) for the edge weights. Let \( \text{res}(P) \) be the resolvent set of the self-adjoint operator \( P \) acting on \( \ell^2(T, \mathfrak{m}) \) according to §2.C. For \( \lambda \in \text{res}(P) \), we write \( \mathcal{G}(\lambda) = (\lambda \cdot I - P)^{-1} \) for resolvent operator. Its matrix element

\[
G(x, y|\lambda) = (\lambda \cdot I - P)^{-1}1_y(x), \quad x, y \in T,
\]

is the Green function. It is an analytic function of \( \lambda \in \text{res}(P) \supset \mathbb{C} \setminus [-\rho, \rho] \), and for \( |\lambda| > \rho \),

\[
G(x, y|\lambda) = \sum_{n=0}^{\infty} p^{(n)}(x, y) \lambda^{-n-1}.
\]

At \( \lambda = \rho \), the latter series converge or diverge simultaneously for all \( x, y \). If they converge, i.e., \( G(x, y|\rho) < \infty \) for all \( x, y \), then \( P \), resp. the associated random walk, is called \( \rho \)-transient, and otherwise it is called \( \rho \)-recurrent. Set

\[
\text{res}^*(P) = \{ \lambda \in \text{res}(P) : G(x, x|\lambda) \neq 0 \text{ for all } x \in T \}. \tag{3.18}
\]

For \( \lambda \in \text{res}^*(P) \),

\[
F(x, y|\lambda) = G(x, y|\lambda)/G(y, y|\lambda), \quad x, y \in T, \tag{3.19}
\]

is an analytic function of \( \lambda \). For \( |\lambda| \geq \rho \),

\[
F(x, y|\lambda) = \sum_{n=0}^{\infty} f^{(n)}(x, y)/\lambda^n, \tag{3.20}
\]

where \( f^{(n)}(x, y) \) is the probability that the random walk starting at \( x \) hits \( y \) at time \( n \geq 0 \) for the first time. Also,

\[
U(x, x|\lambda) = \sum_{y \sim x} p(x, y)F(y, x|\lambda) = \sum_{n=1}^{\infty} u^{(n)}(x, x)/\lambda^n,
\]

where \( u^{(n)}(x, x) \) is the probability that the random walk starting at \( x \) returns to \( x \) at time \( n \geq 1 \) for the first time. Now it is well known, and also explained in [5], [22] as well as in [17], that the edge weights

\[
f(x, y) = F(x, y|\lambda), \quad x, y \in T, \quad x \sim y
\]

are \( \lambda \)-weights which fulfill the requirements (3.1) – (3.3) for \( \lambda \in \text{res}^*(P) \), and for arbitrary \( x, y \in T \),

\[
F(x, y|\lambda) = F(x_0, x_1|\lambda) \cdots F(x_{k-1}, x_k|\lambda), \quad \text{where } \pi(x, y) = [x = x_0, x_1, \ldots, x_k = y]. \tag{3.21}
\]
With notation as in §3, we also have \( u(x, x) = U(x, x|\lambda) \) and \( g(x, y) = G(x, y|\lambda) \). The associated kernel according to (3.9), called the \( \lambda \)-Martin kernel, is

\[
(3.22) \quad k(x, w) = K(x, w|\lambda) = \frac{G(x, v|\lambda)}{G(o, v|\lambda)} \quad \text{for every vertex } v \in \pi(x \land w, w),
\]

where \( x \in T \) and \( w \in \hat{T} \). All this also works for \( \lambda = \pm \rho \) in the \( \rho \)-transient case. Thus, Theorem 3.15 yields the following, which we restate here once again.

\[
(3.23) \quad \text{Corollary. For } \lambda \in \text{res}^*(P), \text{ as well as for } \lambda = \pm \rho \text{ in the } \rho \text{-transient case, every } \lambda \text{-harmonic function } h \text{ has an integral representation}
\]

\[
h(x) = \int_{\partial T} K(x, \xi|\lambda) \, d\nu(\xi).
\]

The strong complex distribution \( \nu = \nu^h \) on \( \partial T \) is determined by \( h \),

\[
\nu^h(\partial T) = h(o) \quad \text{and } \quad \nu^h(\partial T_x) = F(o, x|\lambda) \frac{h(x) - F(x, x^-|\lambda)h(x^-)}{1 - F(x^-, x|\lambda)F(x, x^-|\lambda)} \quad \text{for } x \neq o.
\]

As already mentioned, this general result of [17] was preceded by various earlier ones, starting with the seminal paper [5] (that deals with locally finite trees and positive \( \lambda > \rho \), and also \( \lambda = \rho \) in the \( \rho \)-transient case), and another proof in [16]. In [8], one finds the result for complex \( \lambda \) in the locally finite case corresponding to nearest neighbour group invariant random walks on free groups (resp. closely related groups freely generated by involutions): the special case of the simple random walk in this environment goes back to [13]. A first proof for the non-locally finite case and \( \lambda = 1 \) (transient case) is in [22 §9.D].

\[
(3.24) \quad \text{Remark. If } \lambda \geq \rho, \text{ or if } \lambda = \rho \text{ in the } \rho \text{-transient case, it is a well-known fact that for any positive } \lambda \text{-harmonic function } h, \text{ one has}
\]

\[
F(x, y|\lambda) h(y) \leq h(x) \quad \text{for all } x, y.
\]

(This holds for any irreducible Markov chain.) In particular, the distribution \( \nu^h \) of Corollary 3.23 is non-negative, whence it extends to a \( \sigma \)-additive measure on \( \partial T \), and Corollary 3.23 leads to the classical Poisson-Martin representation. Furthermore, in that case, the real \( \lambda \)-harmonic functions which are Poisson transforms of \( \sigma \)-additive signed Borel measures on \( \partial T \) are precisely the differences of non-negative \( \lambda \)-harmonic functions. For the complex-valued case, the situation is analogous.

There are many analogies between the structure, group actions, harmonic analysis and potential theory on trees (in particular, regular trees) and the Poincaré disk, that is, the open unit disk with the hyperbolic metric. The discrete Laplacian \( P - I \) arising from a random walk on \( T \) is an analogue of the hyperbolic Laplace-Beltrami operator on the disk. See e.g. Boiko and Woess [4] for a mostly potential theoretic “dictionary” regarding the correspondences. In this sense, our representation theorem 3.15 should be seen as a discrete analogue of a result of Helgason [11] for a Poisson-type integral representation of all harmonic functions on rank 1 symmetric spaces, and in particular, the hyperbolic disk: see the beautifully written exposition by Eymard [7]. There, the
integral representation is with respect to analytic functionals on the boundary (the unit circle), of which our strong distributions are the analogues in the tree setting.

4. Twin kernels for affine and simple random walks

As we have seen above, the natural version of Theorem 3.15 is the one where the $\lambda$-weights are $f(x, y) = F(x, y|\lambda)$, where $\lambda \in \text{res}^*(P)$, resp. $\lambda = \pm \rho$ in the $\rho$-transient case.

Now, there are cases where one has another choice for the collection of $\lambda$-weights $f(x, y)$ satisfying (3.1) – (3.3), leading to another kernel which can also be used to describe the $\lambda$-harmonic functions of $P$. The main aim of this section is to obtain a better understanding of such twin kernels and the different integral representations for a class of random walks which includes the simple random walk on a homogeneous tree.

We consider $T = T_q$, the homogeneous tree with degree $q + 1$, where $q \geq 1$. In case $q = 1$, this is just the bi-infinite integer line $\mathbb{Z}$.

For any end $\xi$ of $T$, we define the associated horocycle index $h(x, \xi) = d(x, x \wedge \xi) - d(o, x \wedge \xi) \in \mathbb{Z}$, (we recall that $\wedge$ stands for taking the confluent with respect to $o$). In addition to the root vertex, we choose and fix a reference end $\mathring{\xi}$ and write $h(x) = h(x, \mathring{\xi})$. The horocycles are the resulting level sets: $\mathcal{H}_k = \{x \in T : h(x) = k\}, k \in \mathbb{Z}$. Thus, (following Cartier) one can imagine the tree as an infinite genealogical tree, where $\mathring{\xi}$ is the mythical ancestor, and the horocycles are the successive generations. Each of them is infinite, and each $x \in \mathcal{H}_k$ has precisely one neighbour (parent) in $\mathcal{H}_{k-1}$ and $q$ neighbours (children) in $\mathcal{H}_{k+1}$ (see Figure 1). The subgroup of $\text{Aut}(T_q)$ which preserves this genealogical order, i.e., the group of automorphisms which fix $\mathring{\xi}$, is called the affine group $\text{Aff}(T_q)$ of $T_q$. It was shown to be amenable by Nebbia [15], but non-unimodular for $q \geq 2$, see Trofimov [19]. We note that the indexing of the horocycles here is opposite to the one which is commonly used in the unit disk, resp. hyperbolic upper half plane. The reason lies in the opposite behaviour of absolute values and $q$-adic norms. Very general random walks on $\text{Aff}(T_q)$ were studied in detail by Cartwright, Kaimanovich and Woess [3].

Here we only consider nearest neighbour walks which are invariant under that group. Their transition probabilities are parametrized by an $\alpha \in (0, 1)$ as follows:

\begin{equation}
\text{for } x \sim y, \quad p(x, y) = \begin{cases}
\alpha/q, & \text{if } h(y) = h(x) + 1, \\
1 - \alpha, & \text{if } h(y) = h(x) - 1.
\end{cases}
\end{equation}

The simple random walk arises when $\alpha = q/(q + 1)$. It is easy to see, and a consequence of the next computations, that the spectral radius is

$$\rho = \rho(P) = 2\sqrt{\alpha(1 - \alpha)}.$$ 

\textbf{(4.2) Remark.} In the group invariant case, $G(\lambda) = G(x, x|\lambda)$ is independent of $x$. (Do not confuse this with the resolvent operator $\mathfrak{G}(\lambda)$, of which $G(\lambda) = \mathfrak{G}(\lambda)1_x(x)$ is the diagonal matrix element.) In the present example, we can use the argument at the end of [17] Remark 2.8] to see that $G(\lambda) \neq 0$ for any $\lambda \in \text{res}(P)$. Indeed, as stated there,
if \(G(\lambda) = 0\) then some and thus every \(x \in T\) would have a unique neighbour \(y\) such that \(p(x, y)G(y, x|\lambda) = p(y, x)G(x, y|\lambda) = -1\). But since \(\text{Aff}(\mathbb{T}_q)\) acts transitively on the edges (preserving orientation, hence the “parent relation”), this would hold for all pairs of neighbours, a contradiction. \(\square\)

We shall of course see this via explicit computation in a moment. By group-invariance, there are only two types of functions \(F(x, y|\lambda)\) for neighbours \(x, y\). We set \(F_+(\lambda) = F(x, y|\lambda)\) when \(\mathfrak{h}(y) = \mathfrak{h}(x) + 1\), and \(F_-(\lambda) = F(x, y|\lambda)\) when \(\mathfrak{h}(y) = \mathfrak{h}(x) - 1\). As we have mentioned in \([3, C]\), these functions, as \(\lambda\)-weights on the edges, satisfy \([3.1] - [3.3]\): see \([\text{[17]}\) Lemma 2.3]. A priori, this is true for \(|\lambda| > \rho\), and for other \(\lambda \in \text{spec}(P)\), one uses analytic continuation. Now \([3.3]\) yields the following equations.

\[(4.3)\quad \lambda F_-(\lambda) = (1 - \alpha) + \alpha F_-(\lambda)^2, \quad \text{and}\]
\[(4.4)\quad \lambda F_+(\lambda) = \frac{\alpha}{q} + (1 - \alpha)F_+(\lambda)^2 + \frac{q - 1}{q} \alpha F_-(\lambda)F_+(\lambda).\]

Throughout this paper we make the following habitual choice.

\[(4.5)\quad \text{Convention.}\] Our usual choice for the analytic continuation of the square root is the one on the slit plane without the negative half-axis, that is \(\sqrt{r} e^{i\theta} = \sqrt{r} e^{i\theta}/2\) for \(r > 0\) and \(-\pi < \theta < \pi\).

With this in mind, equation \([4.3]\) has the two solutions

\[(4.6)\quad F_-(\lambda) = \frac{\lambda}{2\alpha} \left(1 - \sqrt{1 - 4\alpha(1 - \alpha)/\lambda^2}\right), \quad \tilde{F}_-(\lambda) = \frac{\lambda}{2\alpha} \left(1 + \sqrt{1 - 4\alpha(1 - \alpha)/\lambda^2}\right).\]

The solution that gives rise to the function defined in \([3.20]\), and thus is associated with the resolvent \(G(x, y|\lambda)\), is given by the convergent series \([3.20]\) in powers of \(1/\lambda\). It
must be analytic for \( \lambda \in \mathbb{C} \setminus [-\rho, \rho] \) and decreasing for real \( \lambda > \rho \), so it is the former in (4.6). If we insert it into (4.4) then we get once more two solutions,

\[
F_+(\lambda) = \frac{\alpha}{(1 - \alpha) q} F_- (\lambda) \quad \text{and} \quad \tilde{F}_+(\lambda) = \frac{\alpha}{(1 - \alpha) F_- (\lambda)}.
\]

Again, the solution corresponding to the resolvent is \( F_+ (\lambda) \). On the other hand, if in (4.4) we insert \( \tilde{F}_- (\lambda) \) instead of \( F_- (\lambda) \), then we get the following two other solutions of that equation:

\[
\tilde{F}_+(\lambda) = \frac{\alpha}{(1 - \alpha) q} \tilde{F}_-(\lambda) \quad \text{and} \quad \tilde{F}_+ (\lambda) = \frac{\alpha}{(1 - \alpha) F_- (\lambda)}.
\]

A priori, we might consider to use any of the four pairs \( (F_- , F_+), (\tilde{F}_-, \tilde{F}_+), (F_- , \tilde{F}_+) \) and \( (\tilde{F}_-, F_+) \) for defining weights \( f(x,y) \) on the edges in a way which remains invariant under \( \text{Aff}(\mathbb{T}_q) \). But \( F_- (\lambda) \tilde{F}_+(\lambda) = \tilde{F}_-(\lambda) \tilde{F}_+(\lambda) = -1 \), and this is not compatible with (3.1).

Thus, we have the natural choice \( (F_- , F_+) \) and the “twin” \( (\tilde{F}_-, \tilde{F}_+) \). The weights provided by \( (F_- (\lambda), F_+(\lambda)) \) in the sense of Section 3 are the Green weights, \( f(x,y) = F_\pm(\lambda) \) for neighbours \( x, y \) with \( h(y) = h(x) \pm 1 \). An easy consequence of (3.6) is

\[
G(x,x|\lambda) = G(\lambda) = \frac{2q/\lambda}{(q - 1) + (q + 1) \sqrt{1 - 4\alpha(1 - \alpha)/\lambda^2}}.
\]

We remark that from this one can deduce by classical spectral methods that \( \text{spec}(P) = [-\rho, \rho] \), where \( \rho = 2\sqrt{\alpha(1 - \alpha)} \). Namely, \( G(x,x|\lambda) \) is the Stieltjes transform of the Plancherel or spectral measure, also called KNS-measure by Grigorchuk and Žuk [9]. That measure is the diagonal element of the resolution of the identity of the operator \( P \); in the context of infinite graphs, see e.g. Mohar and Woess [14]. Some more details will be considered in [5]. The measure, and in this case, its density with respect to Lebesgue measure, can be computed via the inversion formula of Stieltjes–Perron; see Wall [20]. The spectrum is the support of that measure.

We also observe that our random walk is \( \rho \)-transient precisely when \( q \geq 2 \). We see that the Green weights fulfill the requirements (3.1) – (3.3) for any \( \lambda \in \mathbb{C} \setminus [-\rho, \rho] \), as well as for \( \lambda = \pm \rho \) when \( q \geq 2 \).

On the other hand, the only value of \( \lambda \) for which (3.1) does not hold, i.e. \( \tilde{F}_-(\lambda) \tilde{F}_+(\lambda) = 1 \), is

\[
\lambda_0 = \frac{q + 1}{2\sqrt{q}} \rho = \frac{\rho}{\rho(\text{SRW})},
\]

where \( \rho(\text{SRW}) \) is the spectral radius of the simple random walk on \( \mathbb{T}_q \), that is the random walk that arises for \( \alpha = q/(q + 1) \). It is also easy to check that

\[
\tilde{U}(\lambda) = \tilde{U}(x,x|\lambda) := \sum_{y \sim x} p(x,y) \tilde{F}(y,x|\lambda) = \frac{q + 1}{2q} \lambda \left( 1 + \sqrt{1 - 4\alpha(1 - \alpha)/\lambda^2} \right)
\]
satisfies \( \tilde{U}(\lambda) = \lambda \) precisely when \( \lambda = \lambda_0 \). Thus, using \( (\tilde{F}_-(\lambda), \tilde{F}_+(\lambda)) \), the weights \( \tilde{f}(x,y) = \tilde{F}_\pm(\lambda) \) for \( x \sim y \) with \( h(y) = h(x) \pm 1 \) fulfill the requirements (3.1) – (3.3) for any \( \lambda \in \mathbb{C} \setminus (-\rho, \rho) \), with the exception of \( \lambda_0 \).
The associated Poisson kernels are
\[ v_c \in \ell \text{x} \] then let \( v_d \). This formula arises as follows: first, \( \ell \) apply to (4.12) Corollary. Then it is natural to write \( \partial \) function with value 1 at the origin. According to (3.21), resp (3.4), for \( \lambda \) If in addition \( q \) then there also is a unique strong distribution \( \nu \). Consider the case when \( q \) are distinct, and every \( \lambda \)-eigenfunction arises as a unique \( \ell \)-harmonic function. Its
\[ K(x, \xi | \lambda) = F_-(\lambda)^{\cdot b(x, \xi)} \bigg( \frac{(1 - \alpha)q}{\alpha} \bigg) \ell(x, \xi) \quad \text{and} \]
\[ \tilde{K}(x, \xi | \lambda) = \tilde{F}_-(\lambda)^{\cdot b(x, \xi)} \bigg( \frac{(1 - \alpha)q}{\alpha} \bigg) \ell(x, \xi) , \quad \text{where} \]
\[ \ell(x, \xi) = d(x \wedge \xi, \pi(\alpha, \varpi)). \]
This formula arises as follows: first, \( \ell(x, \varpi) = 0 \) so that \( K(x, \varpi) = F_-(\lambda)^{b(x)} \). If \( \xi \neq \varpi \) then let \( v = \varpi \wedge \xi \) and \( c = x \wedge \xi \in \pi(v, \varpi) \cup \pi(v, \xi) \); see Figure 1. If \( c \in \pi(v, \varpi) \), then \( F(x, c | \lambda) = F_-(\lambda)^{d(x, c)} \), \( F(o, c | \lambda) = F_-(\lambda)^{d(o, c)} \), and \( \ell(x, \xi) = 0 \). On the other hand, if \( c \in \pi(v, \xi) \) then still \( F(x, c | \lambda) = F_-(\lambda)^{d(x, c)} \), but \( F(o, c | \lambda) = F_-(\lambda)^{d(o, v)} F_+(\lambda)^{d(v, c)} \), and \( d(v, c) = \ell(x, \xi) \). Now the first identity in (4.10) follows from (4.7). The same arguments apply to \( \tilde{K} \) and \( \tilde{F} \). Note that, for \( \lambda = \pm \rho \), we have \( \tilde{K} = K \).

(4.11) Remark. Consider the case when \( q = 1 \) and the random walk is on \( \mathbb{T}_2 \equiv \mathbb{Z} \). Its non-zero transition probabilities are
\[ p(x, x + 1) = \alpha \quad \text{and} \quad p(x, x - 1) = 1 - \alpha , \quad x \in \mathbb{Z} . \]
Then it is natural to write \( \partial \mathbb{T}_2 = \partial \mathbb{Z} = \{ \pm \infty \} \), with \( \varpi = -\infty \). Note that \( \lambda_0 = \rho \). When \( \lambda \in \mathbb{C} \setminus [-\rho, \rho] \), we have
\[ \tilde{K}(x, +\infty | \lambda) = K(x, -\infty | \lambda) \quad \text{and} \quad \tilde{K}(x, -\infty) = K(x, +\infty | \lambda) , \]
the kernels at \( +\infty \) and \( -\infty \) are distinct, and every \( \lambda \)-eigenfunction arises as a unique linear combination of those two kernels.

When \( \lambda = \rho \), the function \( K(x, +\infty | \rho) = K(x, -\infty | \rho) \) is the unique positive \( \rho \)-harmonic function with value 1 at the origin. \( \square \)

This settles the special case \( q = 1 \). We are more interested in \( q \geq 2 \), where we get the following.

(4.12) Corollary. For \( q \geq 2 \), let \( \lambda \in \mathbb{C} \setminus (-\rho, \rho) \), and let \( h \) be a \( \lambda \)-harmonic function. Then there is a unique strong distribution \( \nu^h \) on \( \partial \mathbb{T} \) such that
\[ h(x) = \int_{\partial \mathbb{T}} K(x, \xi | \lambda) \, dv^h(\xi) . \]
If in addition \( \lambda \neq \lambda_0 \) then there also is a unique strong distribution \( \tilde{\nu}^h \) on \( \partial \mathbb{T} \) such that
\[ h(x) = \int_{\partial \mathbb{T}} \tilde{K}(x, \xi | \lambda) \, d\tilde{\nu}^h(\xi) . \]
Of course, when \( \lambda = \pm \rho \), we have \( \tilde{K} = K \) and \( \tilde{\nu}^h = \nu^h \), but otherwise we shall see that the kernels and the representing distributions are distinct.
To our knowledge, this twin representation of $\lambda$-harmonic functions was first observed and used for the simple random walk in the context of the representation theory of free groups by Mantero and Zappa [13].

If $\lambda \geq \rho$, then it is well-known that the functions $x \mapsto K(x, \xi|\lambda)$, $\xi \in \partial T$, are the minimal $\lambda$-harmonic functions, that is, the extremal elements of the convex set

$$\mathcal{H}_o(\lambda) = \{ h : T \to (0, \infty) \mid h(o) = 1, Ph = \lambda \cdot h \}.$$  

(When $T$ is locally finite, this set is compact in the topology of pointwise convergence.)

The index $o$ stands for normalization at the reference point $o$.

**Theorem (4.14)** Assume that $q \geq 2$. For $\lambda \in \mathbb{C} \setminus [-\rho, \rho]$, $\lambda \neq \lambda_0$, and for $\xi \in \partial T$, let $\nu^\xi$ and $\tilde{\nu}^\xi$ be the strong distributions on $\partial T$ in the sense of Corollary 4.12 such that

$$\tilde{K}(x, \xi|\lambda) = \int_{\partial T} K(x, \cdot|\lambda) d\tilde{\nu}^\xi \quad \text{and} \quad K(x, \xi|\lambda) = \int_{\partial T} \tilde{K}(x, \cdot|\lambda) d\tilde{\nu}^\xi.$$  

Then $\nu^\xi$ extends to a complex ($\sigma$-additive) Borel measure on $\partial T$, while this does not hold for $\tilde{\nu}^\xi$.

If, in particular, $\lambda > \rho$ is real, then the Borel probability measure $\nu^\xi$ is supported by all of $\partial T$, so that $\tilde{K}(\cdot, \xi|\lambda)$ is not minimal in $\mathcal{H}_o(\lambda)$.

**Proof.** We start with an inequality that will be needed below:

$$\left| F_-(\lambda) \right| = \left| 1 - \sqrt{1 - \frac{\rho^2}{\lambda^2}} \right|^2 < 1 \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus [-\rho, \rho].$$  

Recalling Convention 4.5 we obtain (4.15) by a few elementary computations.

Now let $x \in T \setminus \{ o \}$. Noting that $\mathfrak{h}(x^-) = \mathfrak{h}(x) \pm 1$, we can use the first ones of the respective identities (4.17) and (4.18) plus (4.3) to compute

$$F(x, x^-|\lambda) \frac{F(x, x^-|\lambda)}{\tilde{F}(x, x^-|\lambda)} = F(x^-|\lambda)$$

$$\frac{F(x, x^-|\lambda)}{\tilde{F}(x, x^-|\lambda)} = F_-(\lambda) \tilde{F}_-(\lambda) = F_-(\lambda) \tilde{F}_-(\lambda) \frac{\alpha}{(1 - \alpha)q} = \frac{1}{q},$$

because either $F(x, x^-|\lambda) = F_-(\lambda)$ and $\tilde{F}(x, x^-|\lambda) = \tilde{F}_+(\lambda)$, or $F(x, x^-|\lambda) = F_+(\lambda)$ and $\tilde{F}(x, x^-|\lambda) = \tilde{F}_-(\lambda)$. In particular,

$$|F_+(\lambda)F_-(\lambda)| = \frac{F_-(\lambda)^q}{q F_-(\lambda)} < \frac{1}{q}.$$  

By Theorem 3.15

$$\nu^\xi(\partial T_x) = F(o, x|\lambda) \frac{\tilde{K}(x, \xi|\lambda) - F(x, x^-|\lambda)\tilde{K}(x^-, \xi|\lambda)}{1 - F_+(\lambda)F_-(\lambda)}.$$  

Case 1: $x \in \pi(o, \xi)$.

Then $\tilde{K}(x, \xi|\lambda) = 1/\tilde{F}(o, x|\lambda)$ and $\tilde{K}(x^-, \xi|\lambda) = \tilde{F}(x^-, x|\lambda)/\tilde{F}(o, x|\lambda)$, and (4.16) yields

$$\nu^\xi(\partial T_x) = \frac{1 - 1/q}{1 - F_+(\lambda)F_-(\lambda)} F(o, x|\lambda) = \frac{1 - 1/q}{1 - F_+(\lambda)F_-(\lambda)} \left( \frac{F_-(\lambda)}{F_-(\lambda)} \right)^d(o,x).$$
We note immediately that this is strictly positive when \( \lambda > \rho \), because in view of (3.20), combined with (4.6) and (4.7), we then have \( F_+(\lambda) F_-(\lambda) \leq F_+(\rho) F_-(\rho) = \left( \frac{\rho}{2\alpha} \right)^2 \frac{\alpha}{(1-\alpha)^q} = \frac{1}{q} < 1 \).

**Case 2:** \( x \notin \pi(o, \xi) \).

Let \( v = x \wedge \xi = x^- \wedge \xi \). Then \( \tilde{K}(x, \xi|\lambda) = \tilde{F}(x, v|\lambda)/\tilde{F}(o, v|\lambda) \) and \( \tilde{K}(x^-, \xi|\lambda) = \tilde{K}(x, \xi|\lambda)/\tilde{F}(x, x^-, \xi|\lambda) \). Now, (4.16) yields that \( F(v, x|\lambda) \tilde{F}(x, v|\lambda) = q^{-d(x, v)} \), because it is the product of \( d(x, v) \) terms of the form \( F(x_i, x_i|\lambda) \tilde{F}(x_i, x_i|\lambda) \).

Therefore

\[
\nu^\xi(\partial T_x) = \frac{1 - F_-(\lambda)/\tilde{F}_-(\lambda)}{1 - F_+(\lambda)/\tilde{F}_-(\lambda)} F(o, x|\lambda) \tilde{F}(x, v|\lambda) F(v, x|\lambda) \tilde{F}(x, v|\lambda) = C(\lambda) \left( \frac{F_-(\lambda)}{\tilde{F}_-(\lambda)} \right)^{d(o, v)} \left( \frac{1}{q} \right)^{d(x, v)}.
\]

Again, this is strictly positive when \( \lambda > \rho \), and we obtain that in this case the Borel probability measure \( \nu^\xi \) is supported by all of \( \partial T \).

We now prove that for any \( \lambda \in \mathbb{C} \setminus (-\rho, \rho) \), the distribution \( \nu^\xi \) extends to a \( \sigma \)-additive Borel measure on \( \partial T \). Let \((x_n)_{n \geq 0}\) be a sequence of vertices such that the arcs \( \partial T_{x_n} \) are pairwise disjoint.

Write \( \pi(o, \xi) = [v_0, v_1, \ldots] \). There can be at most one \( x_n \) on that geodesic ray. In that case, suppose it is \( x_0 \), that is, \( x_0 = v_k \) for some \( k \geq 0 \). By (4.15), (4.17) and (4.18),

\[
|\nu^\xi(\partial T_{x_0})| < \left| \frac{1}{1 - F_+(\lambda)/\tilde{F}_-(\lambda)} \right| < 1.
\]

Next, let \( A_k = \{ n \geq 1 : x_n \wedge \xi = v_k \} \). We claim that, using (4.19), one has

\[
\sum_{n : x_n \in A_k} |\nu^\xi(\partial T_{x_n})| \leq |\nu^\xi(\partial T_{x_0})| \leq |\nu^\xi(\partial T_{x_0})|^{k}. \]

Indeed, consider the equidistribution \( \nu \) on \( \partial T \), that is, \( \nu(\partial T_x) = 1/((q + 1)q^{d(o, x) - 1}) \) for \( x \neq 0 \). It extends to a Borel probability measure on \( \partial T \), and for \( k \geq 1 \),

\[
\sum_{n : x_n \in A_k} q^{-d(x_n, v_k)} = (q + 1)q^{k-1} \sum_{n : x_n \in A_k} \nu(\partial T_{x_n}) \leq (q + 1)q^{k-1} \nu(\partial T_{v_k} \setminus \partial T_{v_{k+1}}) = q^{-1}. \]

For \( k = 0 \), the analogous computation yields the upper bound 1. By (4.15),

\[
\sum_{n=0}^{\infty} |\nu^\xi(\partial T_{x_n})| \leq \left| \frac{1}{1 - F_+(\lambda)/\tilde{F}_-(\lambda)} \right| + \sum_{k=0}^{\infty} \sum_{n : x_n \in A_k} |\nu^\xi(\partial T_{x_n})| \leq 1 + \frac{|\nu^\xi(\partial T_{x_0})|}{1 - |\nu^\xi(\partial T_{x_0})|}. \]

So condition (2.3) is satisfied, and \( \nu^\xi \) has a \( \sigma \)-additive extension, as stated.

To obtain the analogous formulas to (4.17) and (4.19) for \( \nu^\xi \), we just have to exchange \( F \) and \( \tilde{F} \) in each occurrence. We write \( C(\lambda) \) for the resulting constant in the analogue of (4.19). In this case, let the sequence \((x_n)\) consist of all the neighbours of the \( v_k \), \( k \geq 1 \).
which do not lie on \( \pi(o, v) \). Thus, the set \( A_k \) defined above consists of the neighbours of \( v_k \), and by the same computation we obtain

\[
\sum_{n: x_n \in A_k} |\tilde{\nu}^\xi(\partial T_{x_n})| = |\tilde{C}(\lambda)| \cdot |\tilde{F}_-(\lambda)|/F_-(\lambda)|^k.
\]

The sum over all \( k \) diverges by (4.15), so that \( \tilde{\nu}^\xi \) does not satisfy the bounded variation condition (2.3).

5. General transitive group actions

After the detailed study of multiple integral representations in \([11]\) we now turn to general transitive group actions in the place of \( \text{Aff}(\mathbb{T}_q) \). Once more, we take up material from our “companion” paper \([17, \S4]\): we assume that the transition probabilities are invariant under a general group \( \Gamma \) of automorphisms of the tree which acts transitively on the vertex set. That is,

\[
p(\gamma x, \gamma y) = p(x, y) \quad \text{for all} \quad x, y \in T \quad \text{and} \quad \gamma \in \Gamma.
\]

Let \( I = \Gamma \setminus E(T) \) be the set of orbits of \( \Gamma \) on the set of oriented edges of \( T \). If \( j \in I \) is the orbit (type) of \((x, y) \in E(T) \) then we write \( p_j = p(x, y) \) and \(-j \) for the orbit of \((y, x) \). Then \(-j \) is independent of the representative \((x, y) \), and \(-(-j) = j \). In particular, \(-j = j \) if and only if there is \( \gamma \in \Gamma \) for which \( \gamma x = y \) and \( \gamma y = x \). For each \( j \in I \) and fixed \( x \in T \), we set \( d_j = |\{y \sim x : (x, y) \in I\}| \). This is finite because \( d_j \leq 1/p_j \), and independent of \( x \) by transitivity of \( \Gamma \). For example, when \( \Gamma = \text{Aut}(\mathbb{T}_q) \) then \( I = \{1\} \) with \( d_1 = q + 1 \), while when \( \Gamma = \text{Aff}(\mathbb{T}_q) \) then \( I = \{\pm 1\} \) with \( d_{-1} = 1 \) and \( d_1 = q \). Thus, \( \sum_{j \in I} d_j p_j = 1 \), and \( \deg(x) = \sum_{j \in I} d_j \).

As clarified in \([17, \text{Remark 4.4, second half}] \), one can start with a finite or countable set \( I \) with an involution \( j \mapsto -j \) and a collection \((d_j)_{j \in I}\) of natural numbers. Then for the regular tree \( T \) with degree \( \sum_j d_j \leq \infty \), there is a group \( \Gamma \leq \text{Aut}(T) \) which acts transitively and such that \( I \) is in one to one correspondence with its set of orbits and the associated cardinalities are \( d_j \).

For example, when \( d_j = 1 \) for all \( j \), then we can choose \( \Gamma \) as the discrete group

\[
\Gamma = \langle a_j, j \in I \mid a_j^{-1} = a_{-j} \text{ for all } j \in I \rangle.
\]

Then, when \( j \neq -j \), we can choose just one out of \( a_j \) and \( a_{-j} \) as a free generator. Instead, when \( j = -j \), then \( a_j \) is a generator whose square is the group identity. In this example, \( \Gamma \) acts transitively with trivial stabilizers, and the fact that this provides all possible groups which act in this way on a countable tree is a well-known basic part of Bass–Serre theory (see SERRE \([18]\)). In all other cases, \( \Gamma \) will have non-discrete closure in \( \text{Aut}(T) \).

In the general situation of a transitive group action which leaves the transition probabilities invariant, it is shown in \([17, \text{Thm. 4.2}] \) that \( \text{res}(P) \setminus \text{res}^*(P) \subset \{0\} \). That is, \( G(\lambda) \neq 0 \) for all \( \lambda \in \text{res}(P) \setminus \{0\} \), where \( G(\lambda) = G(x, x|\lambda) \), which is independent of \( x \) by transitivity. We remark that it may happen that \( 0 \) is part of the resolvent set of \( P \) \([8]\).

Here, we shall always assume that the vertex degree is \( \geq 3 \), so that our random walk has to be \( \rho \)-transient by a result of GUIVARC’H \([10]\). When \( I \) is infinite we make the
additional assumption that

\[(5.2) \sum_{j \in I} d_j p_{-j} < \infty.\]

Note that this is the sum over all neighbours of any vertex \(x\) of the incoming probabilities \(p(y, x)\). The assumption is satisfied, for instance, if the quotients \(p_{-j}/p_j\) are bounded.

If \((x, y)\) is an edge of type \(j\), then \(g(x, y) = G(x, y|\lambda) = G_j(\lambda)\) depends only on \(j\). By reversibility, we have

\[p_j G_{-j}(\lambda) = p_{-j} G_j(\lambda),\]

and the second identity of Lemma 3.8 becomes

\[(5.3) p_{-j} G_j(\lambda)^2 + G_j(\lambda) - p_j G(\lambda)^2 = 0.\]

When \(\lambda > \rho\) is real, among the two solutions of this equation the meaningful one is

\[(5.4) G_j(\lambda) = \frac{1}{2p_{-j}} \left( \sqrt{1 + 4p_j p_{-j} G(\lambda)^2} - 1 \right),\]

because the functions \(G(\lambda)\) and \(G_j(\lambda)\) are decreasing in this range of \(\lambda\). In other regions of the plane, there may be a minus sign in front of the root.

\[(5.5) \text{Proposition. Let } \kappa = \max\{2\sqrt{p_j p_{-j}} : j \in I\}. \text{ Then the identity \(5.4\) holds for all } \lambda \text{ in the set } U = \{\lambda \in \mathbb{C} : |\lambda| > \rho\} \setminus \{\pm it : \rho < t \leq \kappa\}. \text{ (When } \kappa \leq \rho \text{ the last part is empty.)}\]

\[\text{Proof. Each of the functions}\]

\[(5.6) \Phi_j(t) = \frac{1}{2} \left( \sqrt{1 + 4p_j p_{-j} t^2} - 1 \right)\]

is analytic in the slit plane

\[(5.7) W = \mathbb{C} \setminus \{\pm it : t \geq 1/\kappa\}.\]

We shall show that the function \(G(\lambda)\) maps \(U\) into \(W\). This implies that the functions appearing in \(5.4\) are all analytic, so that the identity must hold on all of \(U\) by analytic continuation.

We use some well-known spectral theory. Let \(\mu\) be the Plancherel measure of our random walk, introduced in \([1]\). Recall that \(\mu\) is a probability measure concentrated on \(\text{spec}(P)\), and is the diagonal matrix element at \((x, x)\) (independent of \(x \in T\) by group invariance) of the spectral resolution of the self-adjoint operator \(P\) on \(\ell^2(T, m)\). In more classical terms, it is the measure on \([-\rho, \rho]\) whose \(n\text{th}\) moments are the return probabilities \(p^{(n)}(x, x)\) for \(n \geq 0\). Since in the present case, these probabilities are \(0\) when \(n\) is odd, \(\mu\) is symmetric (invariant under the reflection \(t \mapsto -t\)). Thus

\[G(\lambda) = \int_{[-\rho, \rho]} \frac{1}{\lambda - t} d\mu(t), \quad \lambda \in \mathbb{C} \setminus \text{spec}(P).\]
Now let \(|\lambda| > \rho\) be such that \(\Re(\lambda) \neq 0\), and write \(\bar{\lambda}\) for its complex conjugate. Then
\[
\overline{G(\lambda)} = \int_{[-\rho, \rho]} \frac{1}{\lambda - t} \, d\mu(t) = \int_{[-\rho, \rho]} \frac{1}{\lambda + t} \, d\mu(t), \quad \text{whence}
\]
\[
\Re(G(\lambda)) = \frac{1}{2} \int_{[-\rho, \rho]} \left( \frac{1}{\lambda - t} + \frac{1}{\lambda + t} \right) \, d\mu(t) = \Re(\lambda) \int_{[-\rho, \rho]} \frac{|\lambda|^2 - t^2}{(|\lambda|^2 - t^2)^2 + 4t^2\Im(\lambda)^2} \, d\mu(t).
\]
The last integral is \(> 0\), so that also \(\Re(G(\lambda)) \neq 0\). Therefore \(G(\lambda) \in \mathcal{W}\).

Next, let \(\lambda = i\beta\), where \(\beta \in \mathbb{R}\) and \(|\beta| > \max\{\rho, \kappa\}\). Then, using again that \(\mu\) is symmetric (so that odd functions integrate to 0),
\[
G(i\beta) = \int_{[-\rho, \rho]} \frac{-i\beta - t}{\beta^2 + t^2} \, d\mu(t) = -\frac{i}{\beta} \int_{[-\rho, \rho]} \frac{1}{1 + (t/\beta)^2} \, d\mu(t).
\]
Therefore \(|G(i\beta)| \leq 1/|\beta| < 1/\kappa\), and also \(G(i\beta) \in \mathcal{W}\).

We now obtain the following.

\((5.8)\) \textbf{Theorem.} For \(\lambda \in \mathcal{U}\),
\[
\lambda G(\lambda) = \Phi(G(\lambda)), \quad \text{where} \quad \Phi(t) = 1 + \sum_{j \in I} \frac{d_j}{2} \left( \sqrt{1 + 4p_jp_{-j}t^2} - 1 \right).
\]
The function \(\Phi(t)\) is analytic in the domain \(\mathcal{W}\) of \([5.7]\). Furthermore,
\[
\rho = \min\{\Phi(t)/t : t > 0\} = \Phi(\theta)/\theta,
\]
where \(\theta\) is the unique positive real solution of the equation \(\Phi'(t) = \Phi(t)/t\).

\textbf{Proof.} First of all, observe that for \(t \in \mathbb{C} \setminus \{i s : s \in \mathbb{R}; \ |s| \geq 1\}\),
\[
|\sqrt{1 + t^2} - 1| < |t|.
\]
Therefore, summing over all \(j \in I\),
\[
\sum_{j \in I} \frac{d_j}{2} \sqrt{1 + 4p_jp_{-j}t^2} - 1 < |t| \sum_{j \in I} d_j \sqrt{p_jp_{-j}} \leq |t| \sqrt{\sum_{j \in I} d_j p_{-j}},
\]
which is finite by assumption \([5.2]\). Consequently, even when \(I\) is infinite, the defining series of \(\Phi(t)\) converges absolutely and locally uniformly on \(\mathcal{W}\), so that \(\Phi(t)\) is indeed analytic on that set. Now we can use \([5.4]\) and Proposition \([5.5]\) for \(\lambda \in \mathcal{U}\),
\[
\lambda G(\lambda) - 1 = \sum_{y} p(x, y)G(y, x|\lambda) = \sum_{j \in I} d_j p_j G_{-j}(\lambda) = \Phi(G(\lambda)) - 1.
\]
The remaining statements of the theorem follow well-known lines, compare e.g. with \([22]\) Ex. 9.46], where the variable \(z = 1/\lambda\) is used instead of \(\lambda\), and see also below. \(\square\)

\((5.9)\) \textbf{Remarks.} For the free group with (finitely or) infinitely many generators, the equation for \(G(\lambda)\) of Theorem \((5.8)\) was first deduced and used for finding the asymptotics of \(p^{(n)}(x, x)\) by Woess \([21]\). Its validity was restricted to a complex neighbourhood of the real half-line \([\rho, +\infty)\). There, computations are performed in the variable \(z = 1/\lambda\). A previous variant (for \(z\), resp. \(\lambda\) positive real) is inherent in work of Levit and Molchanov \([12]\). Later on, Aomoto \([1]\) considered equations of the same nature as \((5.3)\) for the case of
finely generated free groups plus reasonings of algebraic geometry to study the nature of the involved functions and the spectrum of $P$. Similarly, Figà-Talamanca and Steger [8] considered the case when the group is discrete as in (5.1), $I$ is finite, and $j = -j$ for all $j$. This served for an in-depth study of the associated harmonic analysis.

What is new here is

- the extension to the general group-invariant case, with $I$ finite or infinite,
- the validity of the equation for $G(\lambda)$ in the large domain $U$.

This domain can be further extended a bit by additional estimates, but for complex $\lambda$ close to $\text{spec}(P)$, the situation is more complicated. Indeed, in such regions, the correct solution of (5.3) may be the one where one has to use the negative branch of the square root in (5.4). The general formula instead of the one of Theorem 5.8 is then

$$
\lambda G(\lambda) = 1 + \sum_{j \in I} \frac{d_j}{2} \left( \pm \sqrt{1 + 4p_j p_{-j} G(\lambda)^2} - 1 \right),
$$

where the signs may vary according to the region to which $\lambda$ belongs. This requires some subtle algebraic geometry beyond the focus of the present paper [1], [8].

In the general group-invariant set-up, and even for non-locally finite $T$, we obtain the integral representation of Theorem 3.15 with respect to the Martin kernel $k(x, \xi) = K(\cdot, \cdot | \lambda)$ for any $\lambda$-harmonic function, whenever $0 \neq \lambda \in \mathbb{C} \setminus \text{spec}(P)$, for $\lambda = \pm \rho$, and possibly also for $\lambda = 0$.

The study of twin kernels and the resulting integral representation of $\lambda$-harmonic functions becomes more delicate in view of the fact that $G(\lambda)$, and thus also the functions $F_j(\lambda) = G_j(\lambda) / G(\lambda)$, are only given via the implicit equation for $G(\lambda)$ of Theorem 5.8. Therefore we limit attention to the case when $\lambda \in (\rho, +\infty)$ is real. For real $t$, each function $\Phi_j$ of (5.6) describes the upper branch of a hyperbola. Thus, the function $\Phi$ has the following properties: it is strictly increasing and strictly convex,

$$
\Phi(0) = 1, \quad \Phi'(0) = 0, \quad \text{and} \quad \lim_{t \to \infty} \Phi'(t) = \lambda_0, \quad \text{where} \quad \lambda_0 = \sum_{j \in I} d_j \sqrt{p_j p_{-j}}.
$$

We have $\lambda_0 < \infty$ by assumption (5.2). Note that in the case of the affine random walks of [4] this is the same $\lambda_0$ as in (4.9). For $\lambda \geq \lambda_0$, the equation $\lambda t = \Phi(t)$ has a unique positive solution. This is $t = G(\lambda)$. See Figure 2, where we assume that $\text{deg}(x) = q + 1$ is finite. With $\theta$ and $\rho$ as in Theorem 5.8, it is clear from the shape of $\Phi$ that $\lambda_0 > \rho$, and for $\lambda_0 > \lambda > \rho$, there are precisely two solutions of the equation $\lambda t = \Phi(t)$. One is smaller than $\theta$ and the other is larger than $\theta$. By continuity of $G(\cdot)$, the correct solution for $G(\lambda)$ is the one for which $G(\lambda) < \theta$: this is the solution that leads to the ordinary $\lambda$-Martin kernel $K(\cdot, \cdot | \lambda)$ and the resulting integral representation of any $\lambda$-harmonic function over $\partial T$. But we also have the second solution $G(\lambda) > \theta$. Working with this one, we also find that for all $\lambda \in (\rho, \lambda_0)$ one has

$$
\tilde{G}(\lambda) \neq 0 \quad \text{and} \quad \tilde{G}_j(\lambda) = \Phi_j(\tilde{G}(\lambda)) / p_{-j} \neq 0,
$$
whence $\tilde{F}_j(\lambda) = \tilde{G}_j(\lambda)/\tilde{G}(\lambda) \neq 0$. Also,

$$\tilde{F}_j(\lambda) \tilde{F}_{-j}(\lambda) = \frac{\Phi_j(\tilde{G}(\lambda))^2}{p_j p_{-j} \tilde{G}(\lambda)^2} < 1,$$

since $\sqrt{1 + t^2} - 1 < t$ for $t > 0$. Thus, (3.1) holds for the weights $f(x,y) = \tilde{F}_j(\lambda)$, when $(x,y)$ is an oriented edge of type $j$. Let us verify (3.2):

$$\sum_{x \in T} p(x,v) f(v,x) = \sum_{j \in I} d_j p_j \frac{\tilde{G}_{-j}(\lambda)}{\tilde{G}(\lambda)} = \frac{\Phi(\tilde{G}(\lambda)) - 1}{\tilde{G}(\lambda)} = \frac{\lambda \tilde{G}(\lambda) - 1}{\tilde{G}(\lambda)} < \lambda.$$

Finally, (3.3) reduces to equation (5.3), which holds for $\tilde{G}_j(\lambda)$ as well as for $G_j(\lambda)$. We conclude that these edge weights lead to a second kernel $k(x,\xi) = \tilde{K}(x,\xi|\lambda)$, $x \in T$, $\xi \in \partial T$, $\lambda \in (\rho,\lambda_0)$,

so that $x \mapsto \tilde{K}(x,\xi|\lambda)$ is positive $\lambda$-harmonic. Thus, every $\lambda$-harmonic function has a second integral representation as in Theorem 3.15 in addition to the one with respect to the ordinary Martin kernel $K(\cdot,\cdot|\lambda)$.

Again, for any $\xi \in \partial T$, there is a positive ($\sigma$-additive!) Borel probability measure $\nu^\xi$ on $\partial T$ such that

$$\tilde{K}(x,\xi|\lambda) = \int_{\partial T} K(x,\cdot|\lambda) \, d\nu^\xi.$$

We omit the computation which shows that $\nu^\xi$ is supported by all of $\partial T$, which is a consequence of the fact that $T$ has degree $\geq 3$. In particular, $\tilde{K}(x,\xi|\lambda)$ cannot be a minimal $\lambda$-harmonic function, i.e., an extremal point of the set $H_0$ of (4.13). Therefore the converse representing distribution $\nu^\xi$, that by Theorem 3.15 gives the integral representation

$$K(x,\xi|\lambda) = \int_{\partial T} \tilde{K}(x,\cdot|\lambda) \, d\nu^\xi,$$

cannot have a $\sigma$-additive extension.

We may ask how to proceed for $\lambda > \lambda_0$, while we exclude the case $\lambda = \lambda_0$, since we have already seen in §4 that for affine random walks there is no natural choice for a second
family of weights for $\lambda_0$. We choose to proceed as follows, requiring here that $I$ be finite and $\sum_j d_j = q + 1$.

The second solution of (5.3) is

$$\tilde{G}_j(\lambda) = \frac{1}{p_{-j}} \tilde{\Phi}_j(\tilde{G}(\lambda)),$$

where

$$\tilde{\Phi}_j(t) = \frac{1}{2} \left( -\sqrt{1 + 4p_{-j} t^2} - 1 \right).$$

Then we set

$$\tilde{\Phi}(t) = \sum_{j \in I} d_j \tilde{\Phi}_j(t).$$

(When $I$ is infinite, the series does not converge.) While $\Phi(t)$ is a sum of upper branches of hyperbolic functions, $\tilde{\Phi}(t)$ is the sum of the associated lower branches. The two asymp-
totes of $\Phi(t)$ and $\tilde{\Phi}(t)$ are

$$y = \pm \lambda t - (q - 1)/2.$$ Thus, any line $y = \lambda t$ has exactly two intersection points with the “twin curve” $(\Phi(t), \tilde{\Phi}(t))$, except for $\lambda = \pm \rho$, in which cases there is only one double solution, and $\lambda = \pm \lambda_0$, in which case there is only one simple solution. Thus, for $\lambda > \lambda_0$, we choose $\tilde{G}(\lambda)$ as the unique solution of

$$\lambda \tilde{G}(\lambda) = \tilde{\Phi}(\tilde{G}(\lambda)),$$

which is negative. The associated solution for $\tilde{G}_j(\lambda)$ is

$$\tilde{G}_j(\lambda) = \frac{1}{p_{-j}} \tilde{\Phi}_j(\tilde{G}(\lambda)),$$

so that indeed

$$\sum_{j \in I} d_j p_{-j} \tilde{G}_{-j}(\lambda) = \tilde{\Phi}(\tilde{G}(\lambda)) - 1 = \lambda \tilde{G}(\lambda) - 1$$

Note that also $\tilde{G}_j(\lambda) < 0$, so that $\tilde{F}_j(\lambda) = \tilde{G}_j(\lambda)/\tilde{G}(\lambda) > 0$. The associated edge weights are again given by $f(x,y) = \tilde{F}_j(\lambda)$, when $(x,y)$ is an oriented edge of type $j$. It is straightforward to see that they also satisfy the requirements (5.1) – (5.3), so that we also obtain a positive kernel $k(x,\xi) = \tilde{K}(x,\xi|\lambda)$ with the same properties as above.

By symmetry, analogous properties hold for negative $\lambda \in (-\infty, -\rho) \setminus \{-\lambda_0\}$.

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