Chordal Graphs are Fully Orientable

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Abstract
Suppose that $D$ is an acyclic orientation of a graph $G$. An arc of $D$ is called dependent if its reversal creates a directed cycle. Let $d_{\min}(G)$ ($d_{\max}(G)$) denote the minimum (maximum) of the number of dependent arcs over all acyclic orientations of $G$. We call $G$ fully orientable if $G$ has an acyclic orientation with exactly $d$ dependent arcs for every $d$ satisfying $d_{\min}(G) \leq d \leq d_{\max}(G)$. A graph $G$ is called chordal if every cycle in $G$ of length at least four has a chord. We show that all chordal graphs are fully orientable.

Keyword: acyclic orientation; full orientability; simplicial vertex; chordal graph.

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1 Introduction

Let $G$ be a finite graph without multiple edges or loops. We use $|G|$ and $\|G\|$ to denote the number of vertices and the number of edges of $G$, respectively. An orientation $D$ of a graph $G$ is obtained by assigning a fixed direction, either $x \to y$ or $y \to x$, on every edge $xy$ of $G$. The original undirected graph is called the underlying graph of any such orientation.

An orientation $D$ is called acyclic if there does not exist any directed cycle. A directed graph having no directed cycle is commonly known as a directed acyclic graph, or DAG for short. DAGs provide frequently used data structures in computer science for encoding dependencies. An equivalent way of describing a DAG is the existence of a particular type of ordering of the vertices called a topological ordering. A topological ordering of a directed graph $G$ is an ordering of its vertices as $v_1, v_2, \ldots, v_{\|G\|}$ such that for every arc $v_i \to v_j$, we have $i < j$. The reader who is interested in knowing more about DAGs is referred to reference [1] which supplies a wealth of information on DAGs.

Suppose that $D$ is an acyclic orientation of $G$. An arc of $D$, or its underlying edge, is called dependent (in $D$) if its reversal creates a directed cycle in the resulted orientation. Note that $u \to v$ is a dependent arc if and only if there exists a directed walk of length at least two from $u$ to $v$. Let $d(D)$ denote the number of dependent arcs in $D$. Let $d_{\text{min}}(G)$ and $d_{\text{max}}(G)$ be, respectively, the minimum and the maximum values of $d(D)$ over all acyclic orientations $D$ of $G$. It is known (\cite{8}) that $d_{\text{max}}(G) = \|G\| - |G| + c$ for a graph $G$ having $c$ connected components.

An interpolation question asks whether $G$ has an acyclic orientation with exactly $d$ dependent arcs for every $d$ satisfying $d_{\text{min}}(G) \leq d \leq d_{\text{max}}(G)$. Following West (\cite{20}), we call $G$ fully orientable if its interpolation question has an affirmative answer. Note that forests
are trivially fully orientable. It is also easy to see ([13]) that a graph is fully orientable if all of its connected components are. West [20] showed that complete bipartite graphs are fully orientable. Let \( \chi(G) \) denote the chromatic number of \( G \), i.e., the least number of colors to color the vertices of \( G \) so that adjacent vertices receive different colors. Let \( g(G) \) denote the girth of \( G \), i.e., the length of a shortest cycle of \( G \) if there is any, and \( \infty \) if \( G \) is a forest. Fisher, Fraughnaugh, Langley, and West [8] showed that \( G \) is fully orientable if \( \chi(G) \leq g(G) \). They also proved that \( d_{\min}(G) = 0 \) when \( \chi(G) < g(G) \). In fact, \( d_{\min}(G) = 0 \) if and only if \( G \) is a cover graph, i.e., the underlying graph of the Hasse diagram of a partially ordered set. ([16], Fact 1.1).

A number of graph classes have been shown to consist of fully orientable graphs in recent years. Here, we give a brief summary of some results.

A graph is called 2-degenerate if each of its subgraphs contains a vertex of degree at most two. Lai, Chang, and Lih [12] have established the full orientability of 2-degenerate graphs that generalizes a previous result for outerplanar graphs ([15]). A Halin graph is a plane graph obtained by drawing a tree without vertices of degree two in the plane, and then drawing a cycle through all leaves in the plane. A subdivision of an edge of a graph is obtained by replacing that edge by a path consisting of new internal vertices. A subdivision of a graph is obtained through a sequence of subdivisions of edges. Lai and Lih [13] showed that subdivisions of Halin graphs and graphs with maximum degree at most three are fully orientable. In [15], Lai, Lih, and Tong proved that a graph \( G \) is fully orientable if \( d_{\min}(G) \leq 1 \). This generalizes the results in [8] mentioned before.

The main purpose of this paper is to show that the class of fully orientable graphs includes the important class of chordal graphs.

Let \( C \) be a cycle of a graph \( G \). An edge \( e \) of \( G \) is called a chord of \( C \) if the two endpoints of \( e \) are non-consecutive vertices on \( C \). A graph
A graph $G$ is called chordal if each cycle in $G$ of length at least four possesses a chord. Chordal graphs are variously known as triangulated graphs \cite{2}, rigid-circuit graphs \cite{6}, and monotone transitive graphs \cite{17} in the literature. Chordal graphs can be characterized in a number of different ways. (For instance, \cite{3}, \cite{6}, \cite{9}, \cite{10}, and \cite{17}).

Chordal graphs have applications in areas such as the solution of sparse symmetric systems of linear equations \cite{18}, data-base management systems \cite{19}, knowledge based systems \cite{7}, and computer vision \cite{5}. The importance of chordal graphs primarily lies in the phenomenon that many NP-complete problems can be solved by polynomial-time algorithms for chordal graphs.

We need the following characterization of chordal graphs to prove our main result. A complete subgraph of a graph $G$ is called a clique of $G$. A vertex $v$ of a graph $G$ is said to be simplicial if $v$ together with all its adjacent vertices induce a clique in $G$. An ordering $v_1, v_2, \ldots, v_n$ of all the vertices of $G$ forms a perfect elimination ordering of $G$ if each $v_i$, $1 \leq i \leq n$, is simplicial in the subgraph induced by $v_i, v_{i+1}, \ldots, v_n$.

**Theorem 1** \cite{18} A graph $G$ is a chordal graph if and only if it has a perfect elimination ordering.

The reader is referred to Golumbic’s classic \cite{11} for more information on chordal graphs.

**2 Results**

Up to the naming of vertices, any acyclic orientation $D$ of $K_n$ produces the topological ordering $v_1, \ldots, v_n$ such that the arc $v_i \to v_j$ belongs to $D$ if and only if $i < j$. Moreover, $v_i \to v_j$ is a dependent arc in $D$ if and only if $j - i > 1$. A vertex is called a source (or sink) if it has no ingoing (or outgoing) arc. The following observation is very useful in the sequel. Let $D$ be an acyclic orientation of the
complete graph $K_n$ where $n \geq 3$. The number of dependent arcs in $D$ incident to a vertex $v$ is $n - 2$ if $v$ is the source or the sink of $D$ and is $n - 3$ otherwise.

In this section, we assume that the clique $Q$ of a graph $G$ has $q$ vertices. Let $G'$ be the graph obtained from $G$ by adding a new vertex $v$ adjacent to all vertices of $Q$. We see that $d_{\text{max}}(G') = \|G'\| - |G'| + 1 = (\|G\|+q) - (|G|+1) + 1 = (\|G\| - |G| + 1) + q - 1 = d_{\text{max}}(G) + q - 1$. Furthermore, we have the following.

**Lemma 2**

(1) If $G$ has an acyclic orientation $D$ with $d(D) = d$, then $G'$ has an acyclic orientation $D'$ with $d(D') = d + q - 1$.

(2) We have $d_{\text{min}}(G') = d_{\text{min}}(G) + q - 2$ or $d_{\text{min}}(G) + q - 1$.

**Proof.** The statements hold trivially when $q = 1$. Assume $q \geq 2$.

(1) Let $D'$ be the extension of $D$ into $G'$ by making $v$ into a source. Clearly, $D'$ is an acyclic orientation. Let $v_1, \ldots, v_q$ be the topological ordering of vertices of $Q$ with respect to $D$. Suppose that $x \rightarrow y$ is a dependent arc in $D'$.

Case 1. If this arc is in $D$, then it is already dependent in $D$ since $v$ is a source in $D'$.

Case 2. If this arc is $v \rightarrow v_1$, then, for some $2 \leq i \leq q$, a directed path $v \rightarrow v_i \rightarrow z_1 \rightarrow \cdots \rightarrow z_i \rightarrow v_1$ of length at least three would be produced such that $z_1, \ldots, z_i$ are all vertices in $G$. It follows that $v_1 \rightarrow v_i \rightarrow z_1 \rightarrow \cdots \rightarrow z_i \rightarrow v_1$ is a directed cycle in $D$, contradicting to the acyclicity of $D$.

Case 3. If this arc is $v \rightarrow v_k$ for $2 \leq k \leq q$, then it is a dependent arc in $D'$ since $v \rightarrow v_{k-1} \rightarrow v_k$ is a directed path of length two.

Therefore, $d(D') = d + q - 1$.

(2) By statement (1), we have $d_{\text{min}}(G') \leq d_{\text{min}}(G) + q - 1$. Let $D'$ be an acyclic orientation of $G'$ with $d(D') = d_{\text{min}}(G')$. Since the subgraph induced by $Q$ and $\{v\}$ is a clique of order $q + 1$, the number of dependent arcs in $D'$ incident to $v$ is $q - 1$ or $q - 2$. Let $D$ be
the restriction of $D'$ to $V(G)$. Then we have $d_{\min}(G') = d(D') \geq d(D) + q - 2 \geq d_{\min}(G) + q - 2$.

Since every number $d$ satisfying $d_{\min}(G) + q - 1 \leq d \leq d_{\max}(G) + q - 1$ is achievable as $d(D')$ for some acyclic orientation $D'$ of $G'$ by (1), the following is a consequence of (2).

**Corollary 3** If $G$ is fully orientable, so is $G'$.

The above theorem amounts to preserving full orientability by the addition of a simplicial vertex. Hence, by successively applying it to the reverse of a perfect elimination ordering of a connected chordal graph, every such graph is fully orientable. Our main result thus follows.

**Theorem 4** If $G$ is a chordal graph, then $G$ is fully orientable.

**Remark.** Adding a simplicial vertex may not increase the maximum and the minimum numbers of dependent edges by the same amount. For instance, any acyclic orientation of a triangle gives rise to exactly one dependent arc. However, the graph $K_4$ minus an edge, which is obtained from a triangle by adding a simplicial vertex, has minimum value one and maximum value two.

Now we want to give a characterization to tell which case in (2) of Lemma 2 will happen. A dependent arc in $Q$ is said to be non-trivial with respect to the acyclic orientation $D$ if it is dependent in $D$ but not in the induced orientation $D[Q]$. Equivalently, any directed cycle obtained by reversing that arc contains vertices not in $Q$.

**Lemma 5** Assume $q \geq 2$. There is an acyclic orientation $D$ of $G$ such that $Q$ has a dependent arc that is non-trivial with respect to $D$ if and only if $D$ can be extended to an acyclic orientation $D'$ of $G'$ with $d(D') = d(D) + q - 2$. 

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Proof. ($\Rightarrow$) Assume that $D$ is an acyclic orientation of $G$ such that $Q$ has a dependent arc that is non-trivial with respect to $D$. Let $v_1, \ldots, v_q$ be the topological ordering of the vertices of $Q$ with respect to $D$. The arcs in the set $\{v_i \to v_j \mid j - i > 1\}$ are dependent arcs that are not non-trivial with respect to $D$.

By our assumption, we can find $1 \leq k < q$ such that $v_k \to v_{k+1}$ is a dependent arc in $D$. We obtain an extension $D'$ of $D$ into $G'$ by defining $v_a \to v$ for all $a < k$ and $v \to v_b$ for all $b > k$. This $D'$ must be acyclic for otherwise a directed path would be produced in $D$, contradicting the acyclicity of $D$. The set $\{vv_r \mid r \neq k, k+1\}$ gives rise to a set of dependent arcs in $D'$ and both $vv_k$ and $vv_{k+1}$ are not dependent in $D'$. Moreover, an edge of $G$ is dependent in $D$ if and only if it is dependent in $D'$. Therefore, $d(D') = d(D) + q - 2$.

($\Leftarrow$) Assume that $D$ can be extended to an acyclic orientation $D'$ of $G'$ with $d(D') = d(D) + q - 2$. If the vertex $v$ is a source or a sink, then $d(D') = d(D) + q - 1$, contradicting our assumption. Without loss of generality, we may suppose that, for some $1 \leq k < q$, $v_a \to v$ for all $1 \leq a \leq k$ and $v \to v_{k+1}$. The acyclicity of $D'$ implies that $v \to v_b$ for all $b > k$. Hence, the arc $v_k \to v_{k+1}$ is dependent in $D'$ for $v_k \to v \to v_{k+1}$ is a directed path of length two. Since the $q - 2$ arcs $vv_r (r \neq k, k+1)$ incident to $v$ are already dependent in $D'$, it forces $v_k \to v_{k+1}$ to be a dependent arc in $D$. Therefore, $v_k \to v_{k+1}$ is non-trivial with respect to $D$.

Corollary 6 Assume $q \geq 2$. There is an acyclic orientation $D$ of $G$ such that $d(D) = d_{\min}(G)$ and $Q$ has a dependent arc that is non-trivial with respect to $D$ if and only if $d_{\min}(G') = d_{\min}(G) + q - 2$.

Remark. For the complete graph $K_n$ on $n$ vertices, $d_{\min}(K_n) = d_{\max}(K_n) = (n - 1)(n - 2)/2$ is a well-known fact (20). Hence, the condition in Theorem 5 and Corollary 6 that $Q$ has a dependent arc that is non-trivial with respect to $D$ can be replaced by the condition that $Q$ has more than $(q - 1)(q - 2)/2$ arcs that are dependent in $D$. 7
In contrast to the addition of a simplicial vertex, the deletion of a simplicial vertex may destroy full orientability. The following example attests to this possibility.

Let $K_{r(n)}$ denote the complete $r$-partite graph each of whose partite sets has $n$ vertices. It is proved in [4] that $K_{r(n)}$ is not fully orientable when $r \geq 3$ and $n \geq 2$. Any acyclic orientation of $K_{3(2)}$ has 4, 6, or 7 dependent arcs. Figure 1 shows an acyclic orientation of $K_{3(2)}$ with 6 dependent arcs. Two dependent arcs appear in the innermost triangle 146. Let $K'$ be the graph obtained from $K_{3(2)}$ by adding a vertex $v$ adjacent to vertices 1, 4, and 6. By Lemma 2 there exist acyclic orientations of $K'$ with 6, 8, or 9 dependent arcs. Actually, $d_{\text{max}}(K') = 9$. Applying Lemma 5 to Figure 1 we obtain an acyclic orientation of $K'$ with 7 dependent arcs. Any acyclic orientation of $K_{3(2)}$ with 4 dependent arcs cannot have two dependent arcs from the triangle 146 since there are three triangles each of which is edge-disjoint from the triangle 146 and we know that every triangle must have one dependent arc. It follows from Corollary 6 that $d_{\text{min}}(K') = 6$. Hence, $K'$ is fully orientable. The deletion of the simplicial vertex $v$ from $K'$ produces $K_{3(2)}$ that is not fully orientable.
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