Abstract

Let $M$ be a smooth manifold, and let $\mathcal{O}(M)$ be the poset of open subsets of $M$. Let $\mathcal{C}$ be a category that has a zero object and all small limits. A homogeneous functor (in the sense of manifold calculus) of degree $k$ from $\mathcal{O}(M)$ to $\mathcal{C}$ is called very good if it sends isotopy equivalences to isomorphisms. In this paper we show that the category $\text{VGHF}_k$ of such functors is equivalent to the category of contravariant functors from the fundamental groupoid of $F_k(M)$ to $\mathcal{C}$, where $F_k(M)$ stands for the unordered configuration space of $k$ points in $M$. As a consequence of this result, we show that the category $\text{VGHF}_k$ is equivalent to the category of representations of $\pi_1(F_k(M))$ in $\mathcal{C}$, provided that $F_k(M)$ is connected. We also introduce a subcategory of vector bundles that we call very good vector bundles, and we show that it is abelian, and equivalent to a certain category of very good functors.

1 Introduction

Let $M$ and $\mathcal{O}(M)$ as in the abstract. Manifold calculus, due to Goodwillie and Weiss, is a calculus of functors suitable for studying good contravariant functors $F: \mathcal{O}(M) \to \text{Top}$ from $\mathcal{O}(M)$ to the category of spaces. The philosophy of calculus of functors is to take a functor $F$ and replace it by its Taylor tower \{\(T_k(F) \to T_{k-1}(F)\)\}_{k \geq 1}, which converges to the original functor in good cases, very much like the approximation of a function by its Taylor series. Each $T_k(F)$ is called polynomial approximation to $F$ of degree $k$. The “difference” $L_k F$ between $T_k F$ and $T_{k-1} F$, or more precisely the homotopy fiber of the canonical map $T_k F \to T_{k-1} F$, belongs to a nice class of functors called homogeneous functors of degree $k$. In [11, Theorem 8.5], Weiss proves a deep result about the classification of homogeneous functors of degree $k$. More precisely, he shows that any such functor is equivalent to a functor $G$ constructed from a fibration over the unordered configuration space $F_k(M)$ of $k$ points in $M$.

In this paper we look at the category $\mathcal{F}_k(\mathcal{O}(M); \mathcal{C})$ of homogeneous functors $F: \mathcal{O}(M) \to \mathcal{C}$, into a “nice” category $\mathcal{C}$, that send isotopy equivalences to isomorphisms. Such functors, which we call very good, have been never considered before. Our main result, Theorem 1.1 below, roughly classifies objects of $\mathcal{F}_k(\mathcal{O}(M); \mathcal{C})$.

1.1 Statements of the main results and motivation

Let $\mathcal{F}(\Pi(F_k(M)); \mathcal{C})$ denote the category of contravariant functors from the fundamental groupoid $\Pi(F_k(M))$ of $F_k(M)$ to $\mathcal{C}$. At first glance, this latter category and $\mathcal{F}_k(\mathcal{O}(M); \mathcal{C})$ appear quite different, but, somewhat miraculously, they turn out to be related. Specifically, we have the following result.

**Theorem 1.1.** Let $\mathcal{C}$ be a category that has a zero object and all small limits. Then the category $\mathcal{F}_k(\mathcal{O}(M); \mathcal{C})$ of very good homogeneous functors of degree $k$ is equivalent to the category $\mathcal{F}(\Pi(F_k(M)); \mathcal{C})$. That is,

$\mathcal{F}_k(\mathcal{O}(M); \mathcal{C}) \simeq \mathcal{F}(\Pi(F_k(M)); \mathcal{C})$.


This result has a strong consequence. When $F_k(M)$ is connected, the category $F_k(O(M);C)$ turns out to be deeply related to a certain category of representations that we now recall. Let $\text{Rep}_C(G)$ denote the following category of representations of a group $G$ in $C$. An object of $\text{Rep}_C(G)$ is a pair $(A,\rho)$ where $A$ is an object of $C$, and $\rho: G \to \text{Aut}(A)$ is a homomorphism of groups. A morphism from $(A,\rho)$ to $(A',\rho')$ consists of a morphism $\varphi: A \to A'$ in $C$ such that for all $x \in G$, $\varphi\rho(x) = \rho'(x)\varphi$.

**Corollary 1.2.** Let $C$ as in Theorem 1.1. Assume that $F_k(M)$ is connected. Then the category $F_k(O(M);C)$ is equivalent to the category of representations of the fundamental group $\pi_1(F_k(M))$ in $C$. That is,

$$F_k(O(M);C) \simeq \text{Rep}_C(\pi_1(F_k(M))). \quad (1.1)$$

**Remark 1.3.** Let $G = \pi_1(F_k(M))$. If $C = \mathcal{R}\text{-Mod}$, the category of modules over a ring $\mathcal{R}$, then $\text{Rep}_C(G)$ is nothing but the standard category $\mathcal{R}[G]\text{-Mod}$ of modules over the group ring $\mathcal{R}[G]$, that is, the category of representations of $G$ over $\mathcal{R}$. In that case (1.1) becomes $F_k(O(M);\mathcal{R}\text{-Mod}) \simeq \mathcal{R}[\pi_1(F_k(M))]\text{-Mod}$.

As a quick consequence of Corollary 1.2, if $F_k(M)$ happens to be simply connected then the category $F_k(O(M);C)$ is equivalent to $C$. In particular the category $F_1(O(S^n);C)$ of very good linear functors on the $n$-sphere is equivalent to $F_1(O(\mathbb{R}^n);C)$ when $n \geq 2$.

We also prove Theorem 6.8, which roughly states that the category of very good contravariant functors into finite dimensional vector spaces is equivalent to a nice subcategory of vector bundles, which we call very good vector bundles (see Definition 6.1). We let $\text{VGVB}$ denote the category of such bundles, and we let $\text{VB}$ denote the traditional category of vector bundles over $M$. By definition the category $\text{VGVB}$ is a subcategory of $\text{VB}$. It is well known that the latter category is not abelian as there is a technical issue with the existence of all kernels and cokernels.

**Theorem 1.4 (Theorem 6.6).** The category $\text{VGVB}$ of very good vector bundles is abelian.

We are working on a project that consists of studying polynomial functors $F: O(M) \to \text{Ch}_*$ into chain complexes in the setting of triangulated categories [7]. Let $P$ denote the category of such functors, and let $\mathcal{H} \subset P$ denote the category of homogeneous functors. It turns out that [10] the associated “derived” categories, denoted $\mathcal{DP}$ and $\mathcal{DH}$, are triangulated categories. If $F$ is polynomial of degree $\leq 2$, then it fits into the triangle $L_2F \to F \to T_1F$, where $L_2F$ and $T_1F$ are indeed objects of $\mathcal{DH}$. So by induction on $k$, one can show that every object of $\mathcal{DP}$ can be written as extension of objects of $\mathcal{DH}$. This reduces the study of polynomial functors to the study of homogeneous functors. Moreover, one can show that [10] the category $\mathcal{DH}$ is generated, in the triangulated categorical language, by the category $\text{VGHF}$ of very good homogeneous functors. This explains why we have started our project by an investigation of $F_k(O(M);C)$.

Our Theorem 6.8 might be helpful to establish a connection between homogeneous functors and sheaves. More precisely, let $S$ denote the category of sheaves on $M$ with values in (finite) dimensional vector spaces, and let $\mathcal{DS}$ denote its “derived” category, which is a triangulated category. As mentioned above, one has $\text{VGVB} \subseteq \text{VB}$. It is well known that the the natural functor from $\text{VB}$ to $S$ turns out to be an embedding functor. So, by using the obvious inclusion functor $S \subseteq \mathcal{DS}$, the category $\text{VGVB}$ can be viewed as a subcategory of $\mathcal{DS}$. We claim that $\mathcal{DS}$ is generated by $\text{VGVB}$. If this is true, then one natural question arises: since the category $\mathcal{DH}$ is generated by $\text{VGHF}$ [10], and since $\text{VGHF} \simeq \text{VGVB}$ by Theorem 6.8, one may ask the question to know whether the categories $\mathcal{DS}$ and $\mathcal{DH}$ are equivalent or how they are related. That question is interesting and will be addressed in [10] as well as the claim of course.

### 1.2 Overview of the proof of Theorem 1.1

We first need some notation. For a subposet $A \subseteq O(M)$, we let $F(A;C)$ denote the category of very good contravariant functors $F: A \to C$. We let $O_k(M)$ denote the subposet of $O(M)$ consisting of open subsets
diffeomorphic to the disjoint union of at most $k$ open balls. Let $\mathcal{B}(M)$ be a basis (consisting of open subsets diffeomorphic to a ball) for the topology of $M$. We let $\mathcal{B}^{(k)}(M) \subseteq \mathcal{O}(M)$ denote the subposet whose objects are exactly the disjoint union of $k$ elements of $\mathcal{B}(M)$, and whose morphisms are isotopy equivalences.

The proof goes through three steps in which all of our constructions are explicit.

1. The first thing we need is Theorem 3.8 which states that any very good homogeneous functor $F: \mathcal{O}(M) \rightarrow \mathcal{C}$ of degree $k$ is determined by its values on $\mathcal{B}^{(k)}(M)$. More precisely, the category $\mathcal{F}_k(\mathcal{O}(M); \mathcal{C})$ is equivalent to $\mathcal{F}(\mathcal{B}^{(k)}(M); \mathcal{C})$. That is,

$$\mathcal{F}_k(\mathcal{O}(M); \mathcal{C}) \simeq \mathcal{F}(\mathcal{B}^{(k)}(M); \mathcal{C}).$$

To prove (1.2), we essentially use the right Kan extension functor $\text{Ran}_i(-)$ along the inclusion $i: \mathcal{B}^{(k)}(M) \hookrightarrow \mathcal{O}(M)$, and show that it is the “inverse” for the restriction functor. One of the key points is to prove that $\text{Ran}_i(-)$ preserves the very goodness property. To do this, we introduce the concept of admissible family of open balls (see Definition 3.9). As an example, the family $\{B, B_1, A, B_2, B'\}$ from Figure 1 is admissible. Associated with that family is the isomorphism

$$(F(i_1))^{-1}F(i_2)(F(i_3))^{-1}F(i_4): F(B') \rightarrow F(B),$$

where $i_1, i_2, i_3, i_4$ fit into the poset $B \xhookrightarrow{i_1} B_1 \xhookrightarrow{i_2} A \xhookrightarrow{i_3} B_2 \xhookrightarrow{i_4} B'$ of inclusions.

Figure 1: An example of an admissible family of open balls (here $k = 1$)

Isomorphisms like (1.3) play a crucial role here. Along the way we use the fact that the category $\mathcal{C}$ has a zero object and all small limits. Note that those requirements about $\mathcal{C}$ are used only in this step.

Equation (1.2) has two nice consequences. If $\mathcal{B}^{(1)}(F_k(M)) \subseteq \mathcal{O}(F_k(M))$ denotes the subposet whose objects are exactly the product of $k$ elements of $\mathcal{B}(M)$, then $\mathcal{B}^{(1)}(F_k(M)) \cong \mathcal{B}^{(k)}(M)$. And therefore (1.2) becomes

$$\mathcal{F}_k(\mathcal{O}(M); \mathcal{C}) \simeq \mathcal{F}(\mathcal{B}^{(1)}(F_k(M)); \mathcal{C}).$$

This latter equation and (1.2) imply that the category of very good homogeneous functors of degree $k$ is equivalent to the category of linear functors $\mathcal{O}(F_k(M)) \rightarrow \mathcal{C}$. (In the follow up paper [9, Theorem 1.3] we show that the same result holds for good homogeneous functors.) So it is enough to work with $k = 1$. The second consequence, which will be used in the next step, is the fact that the righthand side of (1.4) does not depend on the choice of the basis $\mathcal{B}(M)$ for the topology of $M$.

2. Let $T^M$ be a triangulation of $M$, that is, a simplicial complex homeomorphic to $M$. By first taking two barycentric subdivisions of $T^M$, and then the interior $U_\sigma$ of the star of each simplex $\sigma$, we obtain a subposet $\mathcal{U}(T^M) := \{U_\sigma\}_{\sigma \in T^M} \subseteq \mathcal{O}(M)$, which covers $M$. Such a poset is a very good cover (see Definition 4.1

\footnotetext{In the follow up paper [9, Theorem 1.4], we use that concept to prove a certain result about the homotopy right Kan extension of good functors.}
and Proposition [1.4]. It is in particular what Weiss calls a good 1-cover. To any very good cover, one can associate a basis \( B_{U(T^M)} \) for the topology of \( M \):

\[
B_{U(T^M)} = \{ B \text{ diffeomorphic to an open ball} | B \subseteq U_\sigma \text{ for some } U_\sigma \in U(T^M) \}.
\]

Taking \( B^{(1)}(M) := B_{U(T^M)} \), the second thing we need in proving Theorem [1.1] is the following equivalence (Proposition [1.7]):

\[
\mathcal{F}(B^{(1)}(M); \mathcal{C}) \simeq \mathcal{F}(U(T^M); \mathcal{C}). \tag{1.5}
\]

The key point in the proof of (1.5) is the fact that the poset \( U(T^M) \) turns out to be a very good cover of \( M \).

(3) Lastly, we need the following equivalence of categories (Theorem [4.9])

\[
\mathcal{F}(U(T^M); \mathcal{C}) \simeq \mathcal{F}(\Pi(M); \mathcal{C}). \tag{1.6}
\]

To prove (1.6), we first construct a functor \( \Psi: \mathcal{F}(U(T^M); \mathcal{C}) \to \mathcal{F}(\Pi(M); \mathcal{C}) \) as follows. First of all, it is well known that the fundamental groupoid \( \Pi(M) \) can be viewed as the category whose objects are vertices of \( T^M \), and whose morphisms are homotopy classes of edge-paths. (Recall that an edge-path is a chain \( f = (v_0, \cdots, v_r) \) of vertices connected by edges in \( T^M \).) Now define \( \Psi(F)(v) := F(U(v)) \). On morphisms \( f \) of \( \Pi(M) \), we explain the idea of the definition of \( \Psi(F) \) through the following example. Consider the triangulation of \( M = S^3 \) with three vertices \( \langle v_0, v_1, v_2 \rangle \) and three edges \( \langle v_0v_1 \rangle, \langle v_1v_2 \rangle, \langle v_0v_2 \rangle \). Let \( f = (v_0, v_1, v_2) \) be an edge-path from \( v_0 \) to \( v_2 \). Associated with \( f \) is the natural poset \( \mathcal{U}_f \) as shown (1.7).

\[
\mathcal{U}_f = \left\{ U(v_0) \xrightarrow{i_1} U(v_0v_1) \xrightarrow{i_2} U(v_1) \xrightarrow{i_3} U(v_1v_2) \xrightarrow{i_4} U(v_2) \right\}. \tag{1.7}
\]

The isomorphism \( \Psi(F)(f): F(U(v_0)) \leftrightarrow F(U(v_2)) \) is then defined as the “composition along \( \mathcal{U}_f \)”. That is, \( \Psi(F)(f) := F(i_1)(F(i_2))^{-1}F(i_3)(F(i_4))^{-1} \). On morphisms \( \eta: F \to F' \) of \( \mathcal{F}(U(T^M); \mathcal{C}) \), we define \( \Psi \) as the component of \( \eta \) at \( U(v_0) \).

We also construct a functor \( \Phi: \mathcal{F}(\Pi(M); \mathcal{C}) \to \mathcal{F}(U(T^M); \mathcal{C}) \), and show that it is the “inverse” for \( \Psi \). Given \( G \in \mathcal{F}(\Pi(M); \mathcal{C}) \), the idea of the construction of \( \Phi(G) \) is to proceed by induction on the skeletons of \( T^M \) (see Subsection 1.3).

Combining now (1.4), (1.5), and (1.6), and after replacing \( M \) by \( F_k(M) \) in (1.5) and (1.6), we deduce Theorem [1.1].

1.3 Outline of the paper

In Section [2] we fix some notation.

In Section [3] we first define some basic concepts. Next we introduce the notion of admissible family in Subsection 3.2. The next subsection defines an important isomorphism out of an isotopy and some other data. A typical example of that isomorphism is given by (1.3). We show in Subsection 3.4 that it does not depend on the choice of the isotopy. Lastly, we prove (1.2) or Theorem [3.8] at the end of Subsection 3.5.

In Section [4] we first prove (1.5) or Proposition [4.7]. Next, in Subsection 4.2 we construct the functor \( \Psi \), while the functor \( \Phi \) is constructed in Subsection 4.3 as mentioned before. In the last subsection, we prove (1.6) or Theorem [4.9] and Theorem [1.1].

In Section [5] we first prove Proposition [5.1] which says that the category \( \mathcal{F}(\Pi(M); \mathcal{C}) \) is equivalent to the category of representations of \( \pi_1(M) \) in \( \mathcal{C} \) provided that \( M \) is connected. Then, combining Theorem [1.1] and Proposition [5.1] we deduce Corollary [1.2].
In Section 6 we first introduce the notion of very good vector bundles (vgvb), and provide two examples. In the next subsection, we prove Theorem 1.4 or more precisely Theorem 6.6. In Subsection 6.3, we prove Theorem 6.8 which says that the category of vgvb is equivalent to the category of very good contravariant functors into finite dimensional vector spaces.

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2 Setup of notation

In this section we fix some notation.

• A smooth manifold $M$ will be fixed. As in Weiss’s work [11], we let $\mathcal{O}(M)$ denote the poset of open subsets of $M$, morphisms being inclusions.

• For $k \geq 0$, we let $\mathcal{O}_k(M) \subseteq \mathcal{O}(M)$ denote the full subcategory of $\mathcal{O}(M)$ whose objects are open subsets diffeomorphic to the disjoint union of at most $k$ balls.

• For subsets $A$ and $B$ of $M$ such that $A \subseteq B$, we let $AB: A \hookrightarrow B$ denote the inclusion map.

• Given two objects $U, V \in \mathcal{O}(M)$ such that $U \subseteq V$, we use the notation $U \subseteq_{ie} V$ to mean that the inclusion of $U$ inside $V$ is an isotopy equivalence.

• We let $F_k(M)$ denote the unordered configuration space of $k$ points in $M$.

• We let $T^M$ denote a triangulation of $M$ with the maximal tree denoted $mT^M$. We write $T^M_p$ for the $p$-skeleton of $M$.

• An $r$-simplex of $T^M$ generated by vertices $v_0, \ldots, v_r$ is denoted $\langle v_0 \ldots v_r \rangle$. If $r = 0$, we will sometimes write $v_0$ for $\langle v_0 \rangle$.

• We let $f\text{Vect}_K$ denote the category of finite dimensional vector spaces over a field $K$.

• Our functors are contravariant unless stated otherwise.

• If $\beta: F \longrightarrow G$ is a natural transformation, we denote by $\beta[A]: F(A) \longrightarrow G(A)$ the component of $\beta$ at $A$.

• If $F$ and $G$ are two functors, we will use the notation $F \cong G$ to mean that $F$ is naturally isomorphic to $G$.

• A category $\mathcal{C}$ that has a zero object, denoted 0, and all small limits will be fixed.

• If $\mathcal{A} \subseteq \mathcal{O}(M)$ is a subposet, we let $\mathcal{F}(\mathcal{A}; \mathcal{C})$ denote the category of very good contravariant functors (see Definition 3.3) from $\mathcal{A}$ to $\mathcal{C}$.

• We use the notation $\mathcal{A} \simeq \mathcal{B}$ to mean that a category $\mathcal{A}$ is equivalent to another category $\mathcal{B}$.

3 Characterization of very good homogeneous functors

In this section, we show that similar results to those of Weiss-Pryor [11,8] hold for very good homogeneous functors (see Definition 3.3 below). Specifically, we prove Theorem 3.8 which states that very good homogeneous functors of degree $k$ are determined by their restriction to the subposet $B^{(k)}(M) \subseteq \mathcal{O}(M)$ (see Definition 3.6 below). As a consequence, we prove Corollary 3.30 which states that the category of very good homogeneous functors $\mathcal{O}(M) \longrightarrow \mathcal{C}$ of degree $k$ is equivalent to the category of linear functors $\mathcal{O}(F_k(M)) \longrightarrow \mathcal{C}$.
3.1 Definition of basic concepts

We define the notion of very good functor and that of very good homogeneous functor of degree $k$. We also state the main result, Theorem 3.8 of the section.

**Definition 3.1.** (i) Let $i: U \hookrightarrow W$ and $i': U' \hookrightarrow W$ be morphisms of $\mathcal{O}(M)$. We say that $i$ is isotopic to $i'$ (or that $U$ is isotopic to $U'$) if there exists a continuous map $L: U \times [0, 1] \rightarrow W$, $(x, t) \mapsto L_t(x)$ satisfying the following three conditions: (a) $L_0 = i$, (b) $L_1(U) = U'$, and (c) for all $t$, $L_t: U \rightarrow W$ is a smooth embedding. Such a map $L$ is called an isotopy equivalence from $U$ to $U'$.

(ii) An inclusion $i: U \rightarrow W$ in $M$ is said to be an isotopy equivalence, and we denote $U \subseteq_{\text{ie}} W$, if $i$ is isotopic to the identity $\text{id}: W \rightarrow W$.

The following well known result will be extensively used in this paper.

**Proposition 3.2.** [Chapter 8] (i) If $i: U \hookrightarrow W$ is isotopic to $i': U' \hookrightarrow W$, then $\pi_0(U) \cong \pi_0(U')$. (ii) Let $U, W \in \mathcal{O}(M)$ be diffeomorphic to a disjoint union of open balls such that $U \subseteq W$. Then the inclusion map $i: U \hookrightarrow W$ is an isotopy equivalence if and only if the induced map $\pi_0(i)$ is an isomorphism.

**Definition 3.3.** For a subcategory $A \subseteq \mathcal{O}(M)$, a contravariant functor $F: A \rightarrow C$ is called very good if it sends isotopy equivalences to isomorphisms.

**Definition 3.4.** Let $F: \mathcal{O}(M) \rightarrow C$ be a contravariant functor.

(i) We say that $F$ is polynomial of degree $\leq k$ if it is determined by its values on $\mathcal{O}_k(M)$. That is, for any $U \in \mathcal{O}(M)$, $F(U) \cong \lim_{V \in \mathcal{O}_k(U)} F(V)$.

(ii) The $k$th polynomial approximation of $F$, denoted $T_kF$, is the contravariant functor $T_kF: \mathcal{O}(M) \rightarrow C$ defined as $T_kF(U) = \lim_{V \in \mathcal{O}_k(U)} F(V)$.

**Definition 3.5.** (i) A very good homogeneous functor of degree $k$ is a contravariant functor $F: \mathcal{O}(M) \rightarrow C$ that satisfies the following three conditions. (a) $F$ is very good; (b) $F$ is polynomial of degree $\leq k$; (c) $T_{k-1}F(U)$ is isomorphic to $0$ for all $U$. (ii) A linear functor is a homogeneous functor of degree 1.

For the next definition, we let $B(M)$ denote a basis, that consists of open subsets diffeomorphic to a ball, for the topology of $M$.

**Definition 3.6.** (i) Define $\mathcal{B}^{(k)}(M), k \geq 0$, to be the subcategory of $\mathcal{O}(M)$ whose objects are exactly the disjoint union of $k$ objects of $B(M)$, and whose morphisms are isotopy equivalences.

(ii) Define $\mathcal{O}^{(k)}(M), k \geq 0$, to be the subcategory of $\mathcal{O}(M)$ whose objects are diffeomorphic to the disjoint union of exactly $k$ open balls, morphisms being isotopy equivalences.

By definition $\mathcal{B}^{(k)}(M)$ is a full subcategory of $\mathcal{O}^{(k)}(M)$. To point out the difference between the objects of $\mathcal{B}^{(k)}(M)$ and of $\mathcal{O}^{(k)}(M)$, let us consider the following example. Take $M$ to be a smooth codimension zero submanifold of $\mathbb{R}^N$, and take $B(M)$ to be the subsets of $M$ consisting of open balls (with respect to the euclidean metric). Then an object of $\mathcal{B}^{(k)}(M)$ is exactly the disjoint union of $k$ genuine open balls, while an object of $\mathcal{O}^{(k)}(M)$ is diffeomorphic to the disjoint union of $k$ open balls.

**Definition 3.7.** Define $\mathcal{F}_k(\mathcal{O}(M); C)$ as the category of very good homogeneous functors $\mathcal{O}(M) \rightarrow C$ of degree $k$. Also define $\mathcal{F}(\mathcal{B}^{(k)}(M); C)$ to be the category of very good functors $\mathcal{B}^{(k)}(M) \rightarrow C$.

**Theorem 3.8.** Let $C$ be a category that has a zero object and all small limits. Then the category $\mathcal{F}_k(\mathcal{O}(M); C)$ of very good homogeneous functors of degree $k$ is equivalent to the category $\mathcal{F}(\mathcal{B}^{(k)}(M); C)$. That is, $\mathcal{F}_k(\mathcal{O}(M); C) \simeq \mathcal{F}(\mathcal{B}^{(k)}(M); C)$. 

6
We will prove Theorem 3.8 in Subsection 3.5. We first need to introduce some terminology. Also we need to establish a certain amount of intermediate results.

### 3.2 Admissible family of open subsets

We introduce the concept of admissible family (see Definition 3.9 below), which is crucial for the paper. We also derive a couple of results (Propositions 3.10, 3.13) that will be used in next subsections. For this subsection, consider the following data: \( W \in \mathcal{O}(M) \), \( U, V \in \mathcal{O}_k(W) \), and \( L: U \times [0,1] \to W \) is an isotopy from \( U \) to \( V \).

Recall the notation \( \subseteq_{ic} \) from Definition 3.1(ii). Also recall the notation \( \text{AB} \) from Section 2.

**Definition 3.9.** Let \( K \subseteq U \) be a nonempty compact subset such that \( \pi_0(KU) \) is surjective. A family \( a = \{a_0, a_1, \ldots, a_m, a_{m+1}\} \subseteq [0,1] \) such that \( a_0 = 0, a_{m+1} = 1 \), and \( a_i \leq a_{i+1}, 0 \leq i \leq m \), is called admissible with respect to \( \{L, K\} \) (or just admissible) if there exists a collection \( \mathcal{U}_a = \{U_{01}, U_{12}, \ldots, U_{m(m+1)}\} \) of objects of \( \mathcal{O}_k(M) \) such that for all \( i \), for all \( s \in [a_i, a_{i+1}] \), one has

\[
L_s(K) \subseteq U_{i(i+1)} \subseteq_{ic} L_s(U),
\]

and

\[
\overline{U}_{i(i+1)} \subseteq L_s(U),
\]

where \( \overline{U}_{i(i+1)} \) stands for the closure of \( U_{i(i+1)} \). Such a collection \( \mathcal{U}_a \) is said to be \( \{a, K, L\}\)-admissible.

**Proposition 3.10.** Let \( K \) as in Definition 3.9. Then there exists an admissible family \( a = \{a_0, \ldots, a_{m+1}\} \) with respect to \( \{K, L\} \).

To prove Proposition 3.10 we will need two lemmas, the first being a matter of point-set topology.

**Lemma 3.11.** Let \( X \) be a Hausdorff space, and let \( K, K' \) be two compact spaces. Let \( f: K \to X \), and \( g: K' \to X \) be continuous such that \( f(K) \cap g(K) = \emptyset \). Consider a homotopy \( H: K \times [0,1] \to X \) such that \( H_t = f \) for some \( t \in [0,1] \). Then there exists \( \epsilon > 0 \) such that for all \( s \in (t - \epsilon, t + \epsilon) \), \( H_s(K) \cap g(K') = \emptyset \).

**Lemma 3.12.** Assume that \( U \) is diffeomorphic to an open ball, and let \( j: U \to W \) be the inclusion map. Let \( K \subseteq U \) as before. Consider an isotopy \( H: U \times [0,1] \to W \) such that \( H_t = j \) for some \( t \in [0,1] \). Then there exist \( \epsilon > 0 \) and \( V_t \) diffeomorphic to an open ball such that for all \( s \in (t - \epsilon, t + \epsilon) \), we have \( H_s(K) \subseteq V_t \subseteq H_s(U) \), and \( V_t \subseteq H_s(U) \).

**Proof.** Let \( n \) be the dimension of \( M \). For \( r > 0 \), we let \( B_r = \{x \in \mathbb{R}^n | \|x\| < r\} \) and \( S_r = \overline{B_r \setminus B_r} \). Consider a diffeomorphism \( \theta: B_1 \to U \). Since \( K \) is compact, there exists \( \delta > 0 \) such that \( K \subseteq \theta(B_{\delta}) \).

Also consider the inclusion maps \( f: \theta(S_{\frac{1}{2} + \delta}) \to W \) and \( g: \theta(S_{\delta}) \to W \). Clearly, one has \( f(\theta(S_{\frac{1}{2} + \delta})) \cap g(\theta(S_{\delta})) = \emptyset \). So by applying Lemma 3.11, there is \( \epsilon' > 0 \) such that for all \( s \in (t - \epsilon', t + \epsilon') \), \( H_s(\theta(S_{\frac{1}{2} + \delta})) \cap \theta(S_{\delta}) = \emptyset \). This implies

\[
\forall s \in (t - \epsilon', t + \epsilon'), (B_{\delta}) \subseteq H_s(\theta(B_{\frac{1}{2} + \delta})) \subseteq H_s(\theta(B_1)) = H_s(U),
\]

and

\[
\forall s \in (t - \epsilon', t + \epsilon'), \theta(B_{\delta}) \subseteq H_s(\theta(B_{\frac{1}{2} + \delta})) \subseteq H_s(\theta(B_1)) = H_s(U).
\]

Similarly, by Lemma 3.11 there exists \( \epsilon'' > 0 \) such that \( H_s(K) \cap \theta(S_{\delta}) = \emptyset \) for all \( s \in (t - \epsilon'', t + \epsilon'') \). This implies

\[
\forall s \in (t - \epsilon'', t + \epsilon''), H_s(K) \subseteq \theta(B_{\delta}).
\]

Letting \( \epsilon = \min(\epsilon', \epsilon'') \) and \( V_t = \theta(B_{\delta}) \), the desired result follows from (3.3), (3.4) and (3.5).

\[\Box\]
Proof of Proposition 3.10. Let $p$ denote the number of connected components of $U$. Since $U \in \mathcal{O}_k(M)$, for $1 \leq r \leq p$ each component, $U^r$, of $U$ is diffeomorphic to an open ball. Let $K^r = K \cap U^r$, and let $t \in [0,1]$. ($K^r \neq \emptyset$ since by assumption $\pi_0(KU)$ is surjective.) By Lemma 3.12 there exist $\epsilon > 0$ and $V_t^r$ diffeomorphic to an open ball such that for all $s \in (t-\epsilon, t+\epsilon)$, $L_s(K^r) \subseteq V_t^r \subseteq L_s(U^r)$, and $V_t^r \subseteq L_s(U^r)$. Varying $t$, we get an open cover $\{(t-\epsilon, t+\epsilon)\}_t$ of $[0,1]$. Using now the compactness of $[0,1]$, there exists a finite family $a^r = \{a_{0}^r, \cdots, a_{m^r}^r+1\}$ such that $a_0^r = 0, a_{m^r+1}^r = 1$, $a_i \leq a_{i+1}$ for all $i$, and each interval $[a_i^r, a_{i+1}^r]$ is contained in one of the open subsets of the cover. By construction, such a family $a^r$ comes indeed together with a collection $\{U_t^r(i+1)\}_{i=0}^{m^r} \subseteq \{V_t^r\}_t$ that satisfies (3.1). So $a^r$ is admissible with respect to $\{K^r, L\}$. Defining $a := \cup_{r=1}^{\infty} a^r$, one can see that $a$ is admissible with respect to $\{\cup_{r=1}^{\infty} K^r, L\} = \{K, L\}$. □

Proposition 3.13. (i) Let $T$ be a finite subset of $[0,1]$. If $\{a_0, \cdots, a_{m+1}\}$ is admissible with respect to $\{L, K\}$, then so is $\{a_0, \cdots, a_{m+1}\} \cup T$ with respect to the same set.

(ii) Let $K$ and $K'$ be two nonempty compact subsets of $U$ such that $K \subseteq K'$. If a family $\{a_0, \cdots, a_{m+1}\}$ is admissible with respect to $\{L, K'\}$, then it is also admissible with respect to $\{L, K\}$.

Proof. (i) By induction on the cardinality of $T$. If $T = \{t\}$ for some $t \in [0,1]$, then there exists $0 \leq j \leq m$ such that $t \in [a_j, a_{j+1}]$. Let $b = a \cup \{t\}$, and define $\mathcal{U}_b = \{U_0^b, \cdots, U_{(m+1)(m+2)}^b\}$ as

$$U_i^b = \begin{cases} U_i & \text{if } i \leq j - 1 \\ U_{j+1} & \text{if } i \in \{j, j+1\} \\ U_i & \text{if } i \geq j + 2, \end{cases} \quad 0 \leq i \leq m + 1. \quad (3.6)$$

Manifestly $\mathcal{U}_b$ satisfies (3.1), which proves the base case. The inductive step is handled in the same way as the base case. (ii) This follows directly from the definition. □

3.3 The isomorphism $\text{Iso}(\mathcal{U}_a, a, K, L)$

We continue to use the same data $(W, U, V, L)$ as in Subsection 3.2. The very first goal here is to define an important isomorphism, $\text{Iso}(\mathcal{U}_a, a, K, L) : F(U) \leftrightarrow F(V)$, out of an admissible family and a very good functor $F : \mathcal{O}_k(M) \rightarrow \mathcal{C}$. Next we show that this isomorphism is independent of the choice of $\mathcal{U}_a, a$, and $K$ in Lemmas 3.15, 3.17, and 3.18 respectively. These lemmas are part of ingredients needed for the proof of Theorem 3.8.

Let $c(U)$ denote the number of components of $U$. Of course $c(U) = c(V)$ since $U$ is isotopic to $V$. Let $\{U^1, \cdots, U^{c(U)}\}$ be the set of components of $U$. For each $1 \leq r \leq c(U)$, we let $K^r \subseteq U^r$ denote a nonempty compact subset, and $K := \cup_{r=1}^{c(U)} K^r$. Also let $a = \{a_0, \cdots, a_{m+1}\}$ be an admissible family with respect to $\{K, L\}$, and $F : \mathcal{O}_k(M) \rightarrow \mathcal{C}$ be a very good functor. We want to define an isomorphism $F(U) \leftrightarrow F(V)$ out of these data. First of all, for each $0 \leq i \leq m + 1$, define $U_i$ as the image of $U$ under $L_{a_i}$. That is, $U_i := L_{a_i}(U)$. Clearly, one has $U_0 = U$ and $U_{m+1} = V$. Since $a$ is admissible, there is a collection $\{U_i(i+1)\}_{i=0}^{m}$ which is $\{a, K, L\}$-admissible in the sense of Definition 3.9. Certainly, for every $i$, the inclusions $U_i(i+1)U_i(i+1)U_i(i+1)U_i(i+1)U_i(i+1)U_i(i+1)$ are isotopy equivalences by (3.1). So $F(U_i(i+1)U_i)$ and $F(U_i(i+1)U_i)$ are both isomorphisms for all $i$ since $F$ is very good. Applying now $F$ to the zigzag

$$U_0 \Leftarrow U_0 \leftarrow U_1 \cdots \leftarrow U_m \leftarrow U_{m+1},$$

we get the following diagram of isomorphisms.

$$F(U_0) \rightarrow F(U_1) \rightarrow \cdots \rightarrow F(U_m) \rightarrow F(U_{m+1}) \Leftarrow F(U_{m+1})$$
Definition 3.14. Define $\text{Iso}(\mathcal{U}_a, a, K, L) : F(U) \mapsto F(V)$ as the composition

$$
\text{Iso}(\mathcal{U}_a, a, K, L) = (F(U_0 U_0))^{-1} \cdots (F(U_{i(i+1)} U_i))^{-1} F(U_{i+1}) \cdots F(U_{m(m+1)} U_{m+1}).
$$

Lemma 3.15. Let $\mathcal{U}_a' = \{U_i'\}_{i=0}^{m}$ be another collection which is $\{a, K, L\}$-admissible. Then $\text{Iso}(\mathcal{U}_a, a, K, L) = \text{Iso}(\mathcal{U}_a', a, K, L)$.

Proof. Since $L_{a_i}(K) \subseteq U_i$ and $L_{a_i}(K) \subseteq U_i'$, $0 \leq i \leq m$, by (3.1) it follows that $L_{a_i}(K) \subseteq U_i \cap U_i'$. Since $K$ is nonempty, there exists $D_{i(i+1)} \in \mathcal{O}_k(M)$ contained in $L_{a_i}(K)$ such that $D_{i(i+1)} \subseteq \text{t} U_i$ and $D_{i(i+1)} \subseteq \text{t} U_i'$. Consider now the following commutative diagram of isotopy equivalences.

By applying $F$ to it, and by using the fact that each square of the resulting diagram commutes, and the fact that every vertical map $F(U_i) \mapsto F(U_i)$ is the identity, we get $\text{Iso}(\mathcal{U}_a, a, K, L) = \text{Iso}(\mathcal{U}_a', a, K, L)$, which is the required result.

Lemma 3.16. Let $T \subseteq [0, 1]$ be a finite set, and let $a = \{a_0, \ldots, a_{m+1}\}$ be admissible with respect to $\{K, L\}$. Consider another family $b = a \cup T$, which is indeed admissible with respect to $\{K, L\}$ by Proposition 3.13(i). Then for any collection $\mathcal{U}_b$ which is $\{b, K, L\}$-admissible, one has $\text{Iso}(\mathcal{U}_a, a, K, L) = \text{Iso}(\mathcal{U}_b, b, K, L)$.

Proof. By induction on the cardinality of $T$. Assume that $T = \{t\}$ for some $t \in [0, 1]$. Then there exists $j \in \{0, \ldots, m\}$ such that $a_j \leq t \leq a_{j+1}$. Since $\text{Iso}(\mathcal{U}_b, b, K, L)$ does not depend on the choice of $\mathcal{U}_b$ by Lemma 3.15, to make things easier, we take $\mathcal{U}_b = \{U_i'\}_{i=0}^{m}$ as defined in (3.6). Note that the inclusions $U_j(U) \mapsto L_t(U)$ and $U_j(U(t+1)U) \mapsto L_t(U)$ coincide since $U_j(U) = U_j(U(t+1)U) = U_j(t+1)$ by (3.6). This implies that $F(U_j(U(t+1)U)) \circ (F(U_j(U(t+1)U)))^{-1} = \text{id}$. Using Definition 3.14 and this latter equation, one can easily see that $\text{Iso}(\mathcal{U}_b, b, K, L) = \text{Iso}(\mathcal{U}_a, a, K, L)$, which proves the base case. The inductive step works in the same way as the base case.

The following lemma follows directly from Lemma 3.16.

Lemma 3.17. Let $a = \{a_0, \ldots, a_{m+1}\}$ and $b = \{b_0, \ldots, b_{n+1}\}$ be admissible with respect to the same set $\{K, L\}$. Then $\text{Iso}(\mathcal{U}_a, a, K, L) = \text{Iso}(\mathcal{U}_b, b, K, L)$.

Lemma 3.18. Let $K$ and $K'$ be nonempty compact subsets of $\mathcal{U}$ such that $\pi_0(KU)$ and $\pi_0(K'U)$ are both surjective. Let $a$ (respectively $b$) be admissible with respect to $\{K, L\}$ (respectively $\{K', L\}$). Then one has $\text{Iso}(\mathcal{U}_a, a, K, L) = \text{Iso}(\mathcal{U}_b', a', K', L)$.

Proof. By Proposition 3.10 there exists an admissible family $b$ with respect to $\{K \cup K', L\}$. Let $c := b \cup \alpha u$. Certainly $c$ is admissible with respect to $\{K \cup K', L\}$ by Proposition 3.13(i). Again by the same proposition (but part (ii)), $c$ is admissible with respect to both $\{K, L\}$ and $\{K', L\}$. So, by Definition 3.14 one has $\text{Iso}(\mathcal{U}_c, c, K \cup K', L) = \text{Iso}(\mathcal{U}_c, c, K, L)$ and $\text{Iso}(\mathcal{U}_c, c, K \cup K', L) = \text{Iso}(\mathcal{U}_c, c, K', L)$. Moreover, by Lemma 3.16 one has $\text{Iso}(\mathcal{U}_c, c, K, L) = \text{Iso}(\mathcal{U}_a, a, K, L)$ and $\text{Iso}(\mathcal{U}_c, c, K', L) = \text{Iso}(\mathcal{U}_a', a', K', L)$. Combining these equations, we get the desired result.
3.4 Dependence of Iso(\(U_a, a, K, L\)) on the choice of the isotopy \(L\)

In Subsection 3.3, we showed that the isomorphism Iso(\(U_a, a, K, L\)) introduced in Definition 3.14 does not depend on \(U_a, a\), and \(K\). Here the goal is to prove Proposition 3.21 which says that under certain conditions Iso(\(U_a, a, K, L\)) is independent of the choice of \(L\) as well. This result is one of the key ingredients in proving Theorem 3.8.

Let \(\mathcal{B}(M)\) as in Subsection 3.1. Recall the categories \(\mathcal{O}(k)(M)\) and \(\mathcal{B}(k)(M)\) from Definition 3.6. We continue to use the same data as in Subsection 3.2 except that here we make the following restrictions: \(W \in \mathcal{O}(k)(M)\), \(U, V \in \mathcal{B}(k)(M)\) are such that the inclusions \(U \hookrightarrow W\) and \(V \hookrightarrow W\) are both isotopy equivalences, \(L: U \times [0,1] \to W\) is an isotopy from \(U\) to \(V\) such that for all \(t\), \(L_t(U) \in \mathcal{B}(M)\), and \(F: \mathcal{B}(k)(M) \to \mathcal{C}\) is a very good functor.

To state Proposition 3.21, we need to make a definition. As in the preceding subsection, let \(\{U^r\}_{r=1}^k, \{V^r\}_{r=1}^k\), and \(\{W^r\}_{r=1}^k\) denote the set of components of \(U, V\), and \(W\) respectively. For each \(1 \leq r \leq k\), let \(x_r \in U^r\) be a point, and consider the compact set \(K = \{x_1, \ldots, x_k\}\). Let \(\lambda_{Lx}: [0,1] \to W^r\) be the path defined as \(\lambda_{Lx}(t) := L(x_r, t)\), and let \(\lambda_{Lx} := \{\lambda_{Lx}\}_r\). Consider an admissible family \(a = \{a_0, \ldots, a_m\}\) with respect to \((x, L)\). By (3.1), there exists a collection \(U_a = \{U_{i(i+1)}\}\) such that

\[
\lambda_{Lx}(a_i) = L(a_i, x_r) \in U_{i(i+1)}, 1 \leq i \leq m
\]

(3.7)

Remark 3.19. Looking closer at the proof of Lemma 3.12 and using the fact that \(\mathcal{B}(M)\) is a basis for the topology of \(M\), one can always assume that each \(U_{i(i+1)}\) is an object of \(\mathcal{B}(M)\) since each component, \(K^r\), of \(K\) is a single point here.

By Lemmas 3.15, 3.17, 3.18, the isomorphism Iso(\(U_a, a, x, L\)) is independent of the choice of \(U_a, a\), and \(x\) so that one can rewrite it just in term of \(L\).

Definition 3.20. Define \(Iso(\lambda_{Lx}) = Iso(\lambda_L): F(U) \to F(V)\) as \(Iso(\lambda_L) := Iso(U_a, a, x, L)\).

Proposition 3.21. Let \(L': U \times [0,1] \to W\) be another isotopy from \(U\) to \(V\) such that \(L'_t(U) \in \mathcal{B}(M)\) for all \(t\). Then \(Iso(\lambda_{L'}) = Iso(\lambda_{L})\).

To prove this result, we need two lemmas.

Lemma 3.22. Let \(A, B, C, D \in \mathcal{B}(k)(M)\) such that for any \(1 \leq r \leq k\), \(C^r \subseteq A^r \cap B^r\) and \(D^r \subseteq A^r \cap B^r\). Suppose there is a path \(\beta': [0,1] \to A^r \cap B^r\) such that \(\beta'(0) \in C^r\) and \(\beta'(1) \in D^r\). Then \(Iso(A, C, B) = Iso(\beta', B)\), where \(Iso(A, C, B): F(A) \to F(CB)\) is defined as

\[
Iso(A, C, B) = (F(CA))^{-1} \circ F(CB).
\]

(3.8)

Proof. By the compactness of \([0,1]\), there exists a collection \(\{X^r_i\}_{i=0}^{n+1}\) of objects of \(\mathcal{B}(M)\) such that \(n\) is independent of \(r\), \(X^r_i \subseteq A^r \cap B^r\) for all \(r, i\), \(X^r_0 = C^r, X^r_{n+1} = D^r\), and \(X^r_i \cap X^r_{i+1} \neq \emptyset\) for any \(r, i\), and the family \(\{X^r_i\}_{i=0}^{n+1}\) forms an open cover of \(\beta'\) for all \(r\). By the fact that \(\mathcal{B}(M)\) is a basis for the topology of \(M\), there exists another collection \(\{X^r_{i(i+1)}\}_{i=0}^{n}\) of objects of \(\mathcal{B}(M)\) such that \(X^r_{i(i+1)} \subseteq X^r_i \cap X^r_{i+1}\) for all \(r, i\). Define \(X_i := \cup_{r=1}^k X^r_i\), and \(X_{i(i+1)} := \cup_{r=1}^k X^r_{i(i+1)}\), and consider the following commutative diagram of isotopy equivalences.
Applying \( F \) to it, and using the fact that each square of the resulting diagram commutes, we get the desired result.

One can extend the definition of \( \text{Iso}(\lambda_L) \) to any path (which does not necessarily come from an isotopy) in the following way.

**Definition 3.23.** For \( 1 \leq r \leq k \), let \( \gamma^r : [0,1] \rightarrow W^r \) be a path such that \( \gamma^r(0) \in U^r \) and \( \gamma^r(1) \in V^r \), and let \( \gamma := \{\gamma^r\}_r \). Let \( c = \{c_0, \ldots, c_{m+1}\} \) be a family of points of \([0,1]\) such that \( c_0 = 0, c_{m+1} = 1 \), and \( c_i \leq c_{i+1} \) for all \( i \). A collection \( \{A_i, A_{j(i+1)}\}, 0 \leq i \leq m \) of objects of \( \mathcal{B}(k)(M) \) is called admissible with respect to \( \{\gamma, c\} \) or just \( \{\gamma, c\}-\text{admissible} \) if the following four conditions are satisfied for all \( r \):

1. \( A_0^r = U^r \) and \( A_{m+1}^r = V^r \);
2. \( A_{i(i+1)}^r \subseteq A_i^r \cap A_{i+1}^r \);
3. \( \gamma^r([c_i-1, c_i]) \subseteq A_i^r \);
4. \( \gamma^r(c_i) \in A_i^r \).

**Example 3.24.** Consider the collection of paths \( \lambda_{Lx} = \{\lambda_{Lx}\}, \) and the family \( U_a = \{U_{j(i+1)}\} \) as before. If we set \( U_i = L_{a_i}(U) \), then one can easily check that the collection \( \{U_i, U_{j(i+1)}\}_{i,j} \) is \( \{\lambda_{Lx}, a_\lambda\}-\text{admissible} \).

Associated with an \( \{\gamma, c\}-\text{admissible} \) family \( \{A_i, A_{j(i+1)}\}_{i,j} \) is an isomorphism from \( F(V) \) to \( F(U) \) denoted \( \text{Iso}(\gamma, c, A_i, A_{j(i+1)}) \) or just \( \text{Iso}(\gamma, c, A_i, A_{j(i+1)}) \) and defined as

\[
\text{Iso}(\gamma, c, A_i, A_{j(i+1)}) := (F(A_0 A_0)^{-1}) \cdots (F(A_{i(i+1)} A_{i+1}))^{-1} F(A_{i(i+1)} A_{i+1}) \cdots F(A_{m(i+1)} A_{m+1}).
\]

Notice that \( \text{Iso}(\gamma, c, A_i, A_{j(i+1)}) \) is defined in the same way as the isomorphism from Definition 3.14. The following lemma says that \( \text{Iso}(\gamma, c, A_i, A_{j(i+1)}) \) does not depend on the choice of \( \{c, A_i, A_{j(i+1)}\} \).

**Lemma 3.25.**

(i) If \( \{A_i, A_{j(i+1)}\}_{i,j} \) and \( \{A_i, A_{j(i+1)}\}_{i,j} \) are both \( \{\gamma, c\}-\text{admissible} \), then one has \( \text{Iso}(\gamma, c, A_i, A_{j(i+1)}) = \text{Iso}(\gamma, c, A_i, A_{j(i+1)}) \).

(ii) If \( \{A_i, A_{j(i+1)}\}_{i,j} \) and \( \{B_i, B_{j(i+1)}\}_{i,j} \) are both admissible with respect to \( \{\gamma, c\} \), then \( \text{Iso}(\gamma, c, A_i, A_{j(i+1)}) = \text{Iso}(\gamma, c, B_i, B_{j(i+1)}) \).

(iii) Let \( T \subseteq [0,1] \) be finite. Let \( \{A_i, A_{j(i+1)}\}_{i,j} \) be an \( \{\gamma, c\}-\text{admissible} \) family. Let \( \{B_s, B_{i(s+1)}\}_{s,t} \) be admissible with respect to \( \{\gamma \cup T\} \). Then we have \( \text{Iso}(\gamma, c, A_i, A_{j(i+1)}) = \text{Iso}(\gamma, c \cup T, B_s, B_{i(s+1)}) \).

(iv) Let \( d = \{d_0, \ldots, d_{i+1}\} \) be another family of points of \([0,1]\) such that \( d_0 = 0, d_{m+1} = 1 \), and \( d_i \leq d_{i+1} \) for all \( i \). If \( \{A_i, A_{j(i+1)}\}_{i,j} \) is \( \{\gamma, c\}-\text{admissible} \), and if \( \{B_s, B_{i(s+1)}\}_{s,t} \) is \( \{\gamma, d\}-\text{admissible} \), then \( \text{Iso}(\gamma, c, A_i, A_{j(i+1)}) = \text{Iso}(\gamma, d, B_s, B_{i(s+1)}) \).

**Proof.** The proof of (i) works exactly in the same way as that of Lemma 3.15. For (ii), its proof follows from (i) and (ii) by induction on the cardinality of \( T \) (this is similar to the proof of Lemma 3.16). The last part, (iv), is an immediate consequence of (iii). Now we prove the second part. For all \( r, i \) one has \( A_{i(i+1)}^r \cap B_{i(i+1)}^r \neq \emptyset \) by the property (d) above. So, since \( \mathcal{B}(M) \) is a basis for the topology of \( M \), there exists \( C_{i(i+1)}^r \in \mathcal{B}(M) \) such that \( \gamma^r(c_i) \in C_{i(i+1)}^r \) and \( C_{i(i+1)}^r \subseteq A_{i(i+1)}^r \cap B_{i(i+1)}^r \). Using Lemma 3.22, we get the following three equations. \( \text{Iso}(A_i, A_{i(i+1)}; A_{i+1}) = \text{Iso}(A_i, C_{i(i+1)}^r, A_{i+1}) \), and \( \text{Iso}(B_i, B_{i(i+1)}; B_{i+1}) = \text{Iso}(B_i, C_{i(i+1)}^r, B_{i+1}) \) for \( 0 \leq i \leq m \), and \( \text{Iso}(A_i, C_{i(i+1)}^r, B_{i+1}) = \text{Iso}(A_i, C_{i(i+1)}^r, B_{i+1}) \), for \( 1 \leq i \leq m \). The desired result easily follows from those equations.

We can now prove the main result of this subsection.

**Proof of Proposition 3.21** Our goal is to show that \( \text{Iso}(\lambda_L) = \text{Iso}(\lambda_{L'}) \). Since \( W^r \) is diffeomorphic to an open ball, and then contractible, there exists a homotopy \( H' : [0,1] \times [0,1] \rightarrow W^r \), \( (s, t) \mapsto H'_t(s) \) such that \( H'_0 = \lambda_{Lx} \) and \( H'_1 = \lambda_{L'x} \). Let \( H_t := (H'_t)^k \). Certainly the data involved in the definition of \( \text{Iso}(\lambda_L) \) determine a family \( \{U_0, U_{i+1}, \ldots, U_{m(m+1)+1}, U_{m+1}\} \), which is \( \{\lambda_{Lx}, a\}-\text{admissible} \). So \( \text{Iso}(\lambda_L) = \text{Iso}(H_0, a, U_s, U_{s(i+1)}) \) and \( \text{Iso}(\lambda_{L'}) = \text{Iso}(H_1, a', U'_s, U'_{s(i+1)}) \), where \( a' \) is an admissible family with respect to \( \{x, L'\} \). By the
compactness of \([0, 1] \times [0, 1]\), for each \(1 \leq r \leq k\), there exist a subdivision \(\{[c_{i-1}, c_i] \times [d_{j}, d_{j+1}]\}_{i,j}, 1 \leq i \leq m_r + 1, 0 \leq j \leq n_r\), of the rectangle \([0, 1] \times [0, 1]\) (of course \(c_0 = d_0 = 0\), and \(c_{m+1} = d_{n+1} = 1\)), and a family \(\{A^i_p\}_{p,q}, 0 \leq p \leq m_r + 1, 0 \leq q \leq n_r + 1\) of objects of \(B(W')\) that satisfy the following two conditions: (a) for all \(q\), \(A^i_{m+1} = U\) and \(A^i_{n+1} = V\); (b) \(H^r([c_{i-1}, c_i] \times [d_{j}, d_{j+1}]) \subseteq A^i_j\) for any \(i, j\). Since the objects of \(B(M)\) form a basis for that topology of \(B\), there exists \(A^i_{i(i+1)} \subseteq B(W')\) such that \(A^i_{i(i+1)} \subseteq A^i_j \cap A^i_{i+1}\), \(A^i_{i(i+1)} \subseteq A^i_j \cap A^i_{i+1}\), and \(H^r(c_i) \subseteq A^i_{i(i+1)}\). We can assume that all \(m_r\)'s are equal since one can refine the subdivision as many times as possible. The same assumption can be made for \(n_r\). So from now on, we will write \(m\) for \(m_r\) and \(n\) for \(n_r\). As usual, let \(A^i_j := \cup_{r=1}^k A^i_j\), and \(A^i_{i(i+1)} := \cup_{r=1}^k A^i_{i(i+1)}\). Also let \(c = \{c_0, \cdots, c_{m+1}\}\). Clearly, for each \(0 \leq j \leq n + 1\), the collection \(\{A^i_j, A^i_{i(i+1)}\}_{i,j}\) with \(0 \leq i \leq m + 1\) and \(0 \leq l \leq m\), is \((H_{d_j}, c)\)-admissible, and one has the equation

\[
\text{Iso}\left(H_{d_j}, c, A^i_j, A^i_{i(i+1)}\right) = \text{Iso}\left(H_{d_{i+1}}, c, A^i_{i+1}, A^i_{i(i+1)}\right),
\]

which is obtained by combining the equations \(\text{Iso}(A^i_j, A^i_{i(i+1)}) = \text{Iso}(A^i_i, A^i_{i(i+1)})\), \(0 \leq i \leq m\), \(\text{Iso}(A^i_{i(i-1)}, A^i_{i-1}) = \text{Iso}(A^i_{i-1}, A^i_{i-1})\), \(1 \leq i \leq m\), \(\text{Iso}(A^i_{i(i+1)}, A^i_{i+1}) = \text{id}\), and \(\text{Iso}(A^i_{i(i+1)}, A^i_{i(i+1)}) = \text{id}\). The first two equations come from Lemma 3.22 while the other ones come from condition (a) above. Now the desired result follows from (3.10) and Lemma 3.25(iv).

### 3.5 Characterization of very good homogeneous functors

The aim here is to prove Theorem 3.8 announced earlier at the end of Subsection 3.1.

We will need the results obtained in the previous subsections, and three more lemmas. For the first one, we need the following definition. A category \(\mathcal{I}\) is said to be **connected** if for any objects \(a, b \in \mathcal{I}\) there exists a zigzag \(b = b_0 \leftarrow b_1 \leftarrow \cdots \leftarrow b_m = a\) of morphisms of \(\mathcal{I}\). If \(F: \mathcal{I} \rightarrow \mathcal{C}\) is a contravariant functor that sends every morphism to an isomorphism, then to any such a zigzag one can replace \(m\) by \(m+1\) of objects of \(\mathcal{I}\) such that \(b_0 = a\) is a diffeomorphism. Our goal is to show that the canonical map

\[
\text{Iso}(b, b_1, \cdots, b_m, a): F(a) \rightarrow F(b)\text{ defined in the same way as (3.9)}.
\]

The following lemma is straightforward.

**Lemma 3.26.** If for any \(a, b \in \mathcal{I}\), \(\text{Iso}(b, b_1, \cdots, b_m, a)\) does not depend on the choice of the zigzag between \(a\) and \(b\), then the limit of the \(F\) over \(\mathcal{I}\) is isomorphic to \(F(c)\) for any \(c \in \mathcal{I}\).

**Lemma 3.27.** Let \(F: \mathcal{B}^{(k)}(M) \rightarrow \mathcal{C}\) be a very good functor, and let \(W \in \mathcal{O}^{(k)}(M)\). Consider the full subcategory \(\mathcal{B}^{(k)}(W) \subseteq \mathcal{B}^{(k)}(W)\) whose objects \(U\) have the property that the canonical inclusion \(U \hookrightarrow W\) is an isotopy equivalence. Then the limit of the restriction \(F|\mathcal{B}^{(k)}(W)\) is isomorphic to \(F(U)\) for any \(\tilde{U} \in \mathcal{B}^{(k)}(W)\). Of course the same result holds when the domain of \(F\) is replaced by \(\mathcal{O}^{(k)}(M)\).

**Proof.** Thanks to Lemma 3.22 and Proposition 3.21 one can see that the restriction \(F|\mathcal{B}^{(k)}(W)\) satisfies the hypothesis of Lemma 3.26 which completes the proof.

**Lemma 3.28.** Let \(F: \mathcal{O}(M) \rightarrow \mathcal{C}\) be a very good functor. Then the functor \(F^!:\mathcal{O}(M) \rightarrow \mathcal{C}\) defined as \(F^!(U) = \lim_{V \in \mathcal{O}_p(M)} F(V)\) is also very good.\(^2\)

**Proof.** Let \(f: U \hookrightarrow U'\) be an isotopy equivalence of \(\mathcal{O}(M)\). Then there exists an isotopy \(L: U \times [0, 1] \rightarrow U'\) such that \(L_0 = f\) and \(L_1: U \rightarrow U'\) is a diffeomorphism. Our goal is to show that the canonical map

\[^2\]The functor \(F^!\) is nothing but the right Kan extension of \(F\) along the inclusion \(\mathcal{O}_k(M) \hookrightarrow \mathcal{O}(M)\).
\[ \psi : \lim_{V \in \mathcal{O}_k(U')} F(V) \rightarrow \lim_{V \in \mathcal{O}_k(U)} F(V) \] is an isomorphism. To do this, we will write \( \psi \) as a composition \( \psi = \lambda \phi \) of two isomorphisms:

\[ \phi: \lim_{V \in \mathcal{O}_k(U')} F(V) \rightarrow \lim_{V \in \mathcal{O}_k(U)} F(L_1(V)) \quad \text{and} \quad \lambda: \lim_{V \in \mathcal{O}_k(U)} F(L_1(V)) \rightarrow \lim_{V \in \mathcal{O}_k(U)} F(V). \]

We proceed in three steps.

- **Construction of \( \phi \).** Let \( \iota: \mathcal{O}_k(U) \rightarrow \mathcal{O}_k(U') \) be the functor defined as \( \iota(V) = L_1(V) \). Clearly \( \iota \) has an inverse since \( L_1 \) is a diffeomorphism. This implies that the triangle

\[ O_k(U) \xrightarrow{G} C \]

\[ \xrightarrow{\iota \cong} \]

\[ \xrightarrow{F} O_k(U'), \]

in which \( G := F \iota \), induces an isomorphism from the limit of \( F \) to that of \( G \). This isomorphism is nothing but \( \phi \).

- The map \( \lambda \) is induced by a natural isomorphism \( \beta: G \rightarrow F|O_k(U) \) defined in the following way. Let \( V \in \mathcal{O}_k(U) \), and let \( K \subseteq V \) be a compact subset such that \( \pi_0(KV) \) is surjective. Then, by Proposition 3.10 there exists an admissible family \( a = \{a_0, \ldots, a_{m+1}\} \) with respect to \( \{K, L|V \times [0, 1]\} \). By Definition 3.9 such a family comes together with a collection \( V_a = \{V_0, \ldots, V_{m+1}\} \) that satisfies (3.1). Now define \( \beta[V]: F(L_1(V)) \rightarrow F(V) \) as \( \beta[V] := \text{Iso}(V_a, a, K, L) \) where \( \text{Iso}(V_a, a, K, L) \) is the isomorphism introduced in Definition 3.14. The naturality of \( \beta \) is rather technical. Let \( g: V \leftrightarrow V' \) be a morphism of \( \mathcal{O}_k(U) \). The idea is to find an open cover, \( \{(s - \epsilon, s + \epsilon_s)\}_{s \in I} \) of \( I \), for which there is a commutative square

\[ F(V_{s-\epsilon_s}) \xleftarrow{\cong} F(V_{s+\epsilon_s}) \]

\[ F(V'_{s-\epsilon_s}) \xleftarrow{\cong} F(V'_{s+\epsilon_s}) \]

for each open interval \((s - \epsilon, s + \epsilon)\). Let \( s \in [0, 1] \) and let \( V_s := L(V, s) \). Also let \( 0 \leq i \leq m + 1 \) such that \( s \in [a_i, a_{i+1}] \), and let \( K'_s := V_{i(i+1)} \). Certainly \( K'_s \) is a compact subset of \( V'_s \) by (3.2) and the fact that \( V \subseteq V' \). Without loss of generality we can assume that \( \pi_0(K'_sV'_s) \) is surjective (otherwise, if a component of \( V'_s \) does not intersect \( K'_s \), it suffices to add to \( K'_s \) one point from that component.) Now let \( b = \{b_s, \ldots, b_{i+1}\} \), with \( b_s = s \) and \( b_{i+1} = 1 \), be an admissible family with respect to \( \{K'_s, L|V'_s \times [s, 1]\} \), and let \( \mathcal{V}' = \{V'_{s(i+1)}, \ldots, V'_{s(t+1)}\} \) be an associated collection satisfying (3.1). Define \( \epsilon_1 := \min(b_{i+1}, a_{i+1}) - s \).

By applying \( F \) to the commutative diagram

\[ V_s \xleftarrow{\text{Iso}(V_s,V_{i(i+1)},V_{i+1})} V_{s+\epsilon_1} \]

\[ V'_s \xleftarrow{\text{Iso}(V'_s,V'_{s(i+1)},V'_{s+1})} V'_{s+\epsilon_1}, \]

and by recalling the definition of \( \text{Iso}(A, C, B) \) from (3.8), we get the following commutative square

\[ F(V_s) \xleftarrow{\text{Iso}(V_s,V_{i(i+1)},V_{i+1})} F(V_{s+\epsilon_1}) \]

\[ F(V'_s) \xleftarrow{\text{Iso}(V'_s,V'_{s(i+1)},V'_{s+1})} F(V'_{s+\epsilon_1}). \]
Similarly, there exist $\epsilon_2$ and a commutative square.

\[
\begin{array}{ccc}
F(V_{s-\epsilon_2}) & \xleftarrow{\text{Iso}(V_{s-\epsilon_2},V_{t+1}),V_s} & F(V_s) \\
\uparrow & & \uparrow \\
F(V_{s'_{-\epsilon_2}}') & \xleftarrow{\text{Iso}(V_{s'_{-\epsilon_2}'}),V_{t'-1}},V_{s'} & F(V_{s'}').
\end{array}
\] (3.13)

Taking $\epsilon_s = \min(\epsilon_1, \epsilon_2)$, and merging (3.12) and (3.13), we get (3.11). Now, by using the compactness of $I$, we have a finite subcover of $I$ and this produces a finite sequence of squares. Merging these squares, we get the obvious commutative square involving $F(V), F(V'), F(L_1(V))$, and $F(L_1(V'))$, which proves the naturality of $\beta$.

- By construction, it is straightforward to check that $\psi = \lambda \phi$, which completes the proof. \qed

Remark 3.29. In (11), Lemma 3.8 asserts the same thing as our Lemma 3.28 but for good functors $O(M) \rightarrow \text{Top}$ into spaces instead. One might then ask the question to know why we provided another proof here, or why we did not adapt the proof of Weiss to our case. The main reason is the fact that Weiss' proof uses geometric realizations of categories, which lie naturally in spaces (and not in $\mathcal{C}!$).

We can now prove the main result of the section.

Proof of Theorem 3.8. We want to prove that the categories $\mathcal{F}(\mathcal{B}^{(k)}(M); C)$ and $\mathcal{F}_k(O(M), C)$ are equivalent. Our strategy consists of doing that through two new categories. The first one, denoted $\mathcal{F}(O^{(k)}(M); C)$, is the category of very good functors from $O^{(k)}(M)$ to $C$. And the second, denoted $\mathcal{F}_k(O_k(M); C)$, is the category of very good functors $F: O_k(M) \rightarrow C$ such that $F|O_{k-1}(M) = 0$, where 0 denotes the zero object of $C$.

These categories fit into the diagram

\[
\begin{array}{ccc}
\mathcal{F}(\mathcal{B}^{(k)}(M); C) & \xrightarrow{\psi_1} & \mathcal{F}(O^{(k)}(M); C) \\
\phi_1 & & \phi_2 \\
\mathcal{F}_k(O_k(M); C) & \xrightarrow{\psi_3} & \mathcal{F}_k(O(M), C)
\end{array}
\]

in which $\phi_1, \phi_2,$ and $\phi_3$ are the restriction functors, $\psi_1$ and $\psi_3$ are defined as $\psi_1(F)(U) = \lim_{B \in \mathcal{B}^{(k)}(U)} F(B)$ and $\psi_3(F) = F^i$, where $F^i$ is the functor defined in the statement of Lemma 3.28 and $\psi_2$ is defined as

\[
\psi_2(F)(U) = \begin{cases} F(U) & \text{if } U \in O^{(k)}(M) \\ 0 & \text{otherwise.} \end{cases}
\]

Here $\mathcal{B}^{(k)}(U)$ is the category introduced in the statement of Lemma 3.27. From now on, our goal is to prove the following three claims. For $1 \leq i \leq 3$, the $i$th claim says that the functor $\psi_i$ is an equivalence of categories with inverse $\phi_i$.

For the first claim, let $F: \mathcal{B}^{(k)}(M) \rightarrow C$ be an object of $\mathcal{F}(\mathcal{B}^{(k)}(M); C)$. We first need to show that $\psi_1(F)$ is very good. This easily follows from two applications of Lemma 3.27. Certainly one has $\phi_1 \phi_1 \cong id$ and $\phi_1 \psi_1 \cong id$. The second claim follows immediately from the definitions. For the third claim, let $F \in \mathcal{F}_k(O_k(M); C)$. By Lemma 3.28 the functor $\psi_3(F) = F^i$ is very good. Moreover $\psi_3(F)$ is polynomial of degree $\leq k$ by Definition 3.4(ii). Furthermore, recalling the functor $T_k$ from Definition 3.4(iii), one has

\[
T_{k-1}F^i(U) = \lim_{V \in O_{k-1}(U)} F^i(V) = \lim_{V \in O_{k-1}(U)} \left( \lim_{W \in O_{k}(V)} F(W) \right) \\
\cong \lim_{V \in O_{k-1}(U)} F(V) \quad \text{since } V \text{ is a terminal object of } O_{k}(V) \\
\cong 0 \quad \text{since } F|O_{k-1}(M) = 0.
\]

Hence $\psi_3(F)$ is a very good homogeneous functor of degree $k$. Certainly one has natural isomorphisms $\phi_3 \psi_3 \cong id$ and $\phi_3 \psi_3 \cong id$, which completes the proof of the theorem. \qed

\[14\]
As a consequence of Theorem 3.8 we have the following in which $F_k(M)$ denotes the unordered configuration space of $k$ points in $M$ with the subspace topology.

**Corollary 3.30.** The category of very good homogeneous functors of degree $k$ is equivalent to the category of very good linear functors $\mathcal{O}(F_k(M)) \to \mathcal{C}$. That is, there is an equivalence between $\mathcal{F}_k(\mathcal{O}(M); \mathcal{C})$ and $\mathcal{F}_1(\mathcal{O}(F_k(M)); \mathcal{C})$, provided that $\mathcal{C}$ has a zero object and all small limits.

**Proof.** Recall $\mathcal{B}(M)$ from the paragraph just before Definition 3.6. Let $\mathcal{B}^{(1)}(F_k(M)) \subseteq \mathcal{O}^{(1)}(F_k(M))$ be the full subcategory whose objects are the product of exactly $k$ pairwise disjoint objects of $\mathcal{B}(M)$. Clearly objects of $\mathcal{B}^{(1)}(F_k(M))$ form a basis for the topology of $F_k(M)$. It is also clear that the category $\mathcal{B}^{(k)}(M)$ is canonically isomorphic to $\mathcal{B}^{(1)}(F_k(M))$. This implies that the categories $\mathcal{F}(\mathcal{B}^{(k)}(M); \mathcal{C})$ and $\mathcal{F}(\mathcal{B}^{(1)}(F_k(M)); \mathcal{C})$ are equivalent. Moreover, the categories $\mathcal{F}(\mathcal{B}^{(k)}(M); \mathcal{C})$ and $\mathcal{F}(\mathcal{B}^{(1)}(F_k(M)); \mathcal{C})$ are equivalent as well as $\mathcal{F}(\mathcal{B}^{(1)}(F_k(M)); \mathcal{C})$ and $\mathcal{F}_1(\mathcal{O}(F_k(M)); \mathcal{C})$ by Theorem 3.8. This proves the corollary. 

## 4 Very good functors

The goal of this section is to prove the main result of the paper: Theorem 1.1. The proof, which will be done at the end of Subsection 4.4, goes through two big steps. The first step (Theorem 3.8) has been already accomplished in Section 3. The second one is Theorem 4.9 below, which roughly says that the category of very good functors is equivalent to the category of functors from the fundamental groupoid of $M$ to $\mathcal{C}$. Both steps are connected by Proposition 4.7, which involves the concept of very good covers that we now explain.

### 4.1 Very good covers

In this subsection, we introduce the notion of very good cover (see Definition 4.1 below), $U$, of $M$. We show that such a cover produces a natural basis, $\mathcal{B}_U$, of open balls for the topology of $M$. As posets, $U$ is smaller than $\mathcal{B}_U$, but the categories $\mathcal{F}(U; \mathcal{C})$ and $\mathcal{F}(\mathcal{B}_U; \mathcal{C})$ of very good functors are equivalent by Proposition 4.7, whose proof is the main goal here.

**Definition 4.1.** An open cover $U = \{U_\sigma\}_\sigma$ of $M$ is called very good if it satisfies the following four conditions:

(C0) Each $U_\sigma$ is diffeomorphic to an open ball.

(C1) For every $\sigma, \lambda$, the intersection $U_\sigma \cap U_\lambda$ is either the emptyset or a finite union of elements of $U$.

(C2) For every $\lambda$, the set $\{U_\sigma | U_\sigma \subseteq U_\lambda\}$ is finite.

(C3) For every open subset $B$ diffeomorphic to an open ball such that $B$ is contained in some $U_\lambda$, there exists a smallest (with respect to the order $U_\sigma \subseteq U_\lambda$ if and only if $U_\sigma \subseteq U_\lambda$) $U_\sigma_B \in U$ such that $B \subseteq U_\sigma_B$.

**Remark 4.2.** If one replaces (C1) by

($\tilde{C}1$) for every $\sigma, \lambda$, the intersection $U_\sigma \cap U_\lambda$ is either the emptyset or an element of $U$,

then (C2) will imply (C3). Indeed, suppose we have ($\tilde{C}1$) and (C2), and let $B \subseteq U_\lambda$ for some $\lambda$. Then the set $A = \{\sigma | B \subseteq U_\alpha \subseteq U_\lambda\}$ is finite because of (C2). Take then $U_\sigma_B = \cap_{\alpha \in A} U_\alpha$. This latter intersection lies in $U$ because of ($\tilde{C}1$). We thus get another definition of a very good cover with only three axioms: (C0), ($\tilde{C}1$) and (C2).

**Example 4.3.** Take $M = S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. 

\[ \frac{3}{15} \]
Proposition 4.7.

While the cover $\mathcal{U}$ is not determined by a triangulation. In general, given a triangulation $\mathcal{T}^M$ of a smooth manifold $M$, one can always define a cover $\mathcal{U}((\mathcal{T}^M)) = \{U_\sigma\}_{\sigma \in \mathcal{T}^M}$ of $M$ such that each $U_\sigma$ is diffeomorphic to an open ball, and

$$\sigma_1 \cap \sigma_2 = \sigma \quad \text{if and only if} \quad U_{\sigma_1} \cap U_{\sigma_2} = U_\sigma. \quad (4.1)$$

Such a cover can be obtained in the following way. First take two barycentric subdivisions of $\mathcal{T}^M$, and then define $U_\sigma$ as the interior of the star of $\sigma$. This is indeed homeomorphic to an open ball $B^n$, $n := \dim(M)$. To see that $U_\sigma$ is diffeomorphic to $B^n$, we need to deal with two cases. If $n \neq 4$, then there exists a unique smooth structure on $\mathbb{R}^n$ (see Section 2.4), and this implies that the smooth manifold $U_\sigma$ is diffeomorphic to $B^n$. If $n = 4$, one can see that the smooth manifolds $U_\sigma$ and $B^n$ are combinatorially equivalent, which implies the desired result by Corollary 6.6 from [9]. From now on, a triangulation $\mathcal{T}^M$ is fixed once and for all.

Proposition 4.4. The cover $\mathcal{U}((\mathcal{T}^M))$ is very good.

Proof. The axioms (C0) and (C1) are satisfied by construction, while (C2) follows from (4.1).

Proposition 4.5. Let $\mathcal{U}$ be any very good cover of $M$. Then the set

$$\mathcal{B}_\mathcal{U} = \{B \text{ diffeomorphic to an open ball such that } B \subseteq U_\sigma \text{ for some } U_\sigma \in \mathcal{U}\}$$

forms a basis for the topology of $M$.

Proof. This follows immediately from the axiom (C1), and the fact that $U \subseteq \mathcal{B}_\mathcal{U}$.

Remark 4.6. Recall the notation “$\mathcal{B}(M)$” introduced right before Definition 3.6. If one takes $\mathcal{B}(M) = \mathcal{B}_{\mathcal{U}((\mathcal{T}^M))}$, then by Definition 3.6(i) it is easy to see that the posets $\mathcal{B}(1)(M)$ and $\mathcal{B}_{\mathcal{U}((\mathcal{T}^M))}$ coincide. That is, $\mathcal{B}(1)(M) = \mathcal{B}_{\mathcal{U}((\mathcal{T}^M))}$.

Recall the notation $\mathcal{F}(\mathcal{A};\mathcal{C})$ from Section 2.

Proposition 4.7. Let $\mathcal{C}$ be any category. Then the categories $\mathcal{F}(\mathcal{B}_{\mathcal{U}((\mathcal{T}^M))};\mathcal{C})$ and $\mathcal{F}(\mathcal{U}((\mathcal{T}^M));\mathcal{C})$ are equivalent.

Proof. Define $\phi: \mathcal{F}(\mathcal{B}_{\mathcal{U}((\mathcal{T}^M))};\mathcal{C}) \rightarrow \mathcal{F}(\mathcal{U}((\mathcal{T}^M));\mathcal{C})$ as the restriction to $\mathcal{U}((\mathcal{T}^M))$. That is, $\phi(G) := G|\mathcal{U}((\mathcal{T}^M))$. To define a functor $\psi$ in the other way, let $F: \mathcal{U}((\mathcal{T}^M)) \rightarrow \mathcal{C}$ be an object of $\mathcal{F}(\mathcal{U}((\mathcal{T}^M));\mathcal{C})$. For $B \in \mathcal{B}_{\mathcal{U}((\mathcal{T}^M))}$ define $\psi(F)(B) := F(U_{\sigma_B})$, where $U_{\sigma_B}$ is provided by the axiom (C3) from Definition 4.4. Again from the same axiom, one can easily define $\psi(F)$ on morphisms. If $\eta: F \rightarrow F'$ is a morphism of $\mathcal{F}(\mathcal{U}((\mathcal{T}^M));\mathcal{C})$, we define $\psi(\eta)(B) := \eta[U_{\sigma_B}]$. It is straightforward to check that $\phi \psi = id$ and $\psi \phi \equiv id$.

We close this subsection with the statement of Theorem 4.9.

---

3Two smooth manifolds are said to be combinatorially equivalent if they possess isomorphic $C^2$ triangulations.
Theorem 4.9. Let $\Psi$ and $\Phi$ be any category. Then the categories $\Psi$ and $\Phi$ are contravariant functors $F : \Pi(M) \to C$, and whose morphisms are natural transformations.

Theorem 4.11. The fundamental groupoid of $M$ is isomorphic to the category $P \mathcal{T}^\sim$.

From now on, one should think $\Pi(M)$ as the category $P \mathcal{T}^\sim$. Consider the poset $\mathcal{U}(\mathcal{T}^M)$ from Subsection 4.2 and let $f = (v_0, \ldots, v_r)$ be an edge-path. For $0 \leq i \leq r - 1$, one has the inclusions $U_{(v_i)} \subseteq U_{(v_i, v_{i+1})}$ and $U_{(v_{i+1})} \subseteq U_{(v_{i}, v_{i+1})}$, which we denote $p_{i(i+1)} : U_{(v_i)} \hookrightarrow U_{(v_i, v_{i+1})}$ and $q_{i(i+1)} : U_{(v_{i+1})} \hookrightarrow U_{(v_{i}, v_{i+1})}$.

Remark 4.12. The edge-path $f = (v_0, \ldots, v_r)$ is canonically oriented from $v_0$ to $v_r$. (i) If $(v_i, v_j, v_k)$ is an edge-path, then the corresponding edges $\langle v_i, v_j \rangle$ and $\langle v_j, v_k \rangle$ differ by their orientations. But, by the definition of $U_\sigma$, one has $U_{(v_i, v_j)} = U_{(v_j, v_k)}$. (ii) Clearly, one has $p_{ij} = q_{ji}$ and $q_{ij} = q_{ji}$ since $U_{(v_i, v_j)} = U_{(v_j, v_i)}$. So $p_{ij} \neq p_{ji}$ and $q_{ij} \neq q_{ji}$ whenever $i \neq j$.

The following definition is that of $\Psi$ on objects. We will define it on morphisms at the end of this subsection.

Definition 4.13. Define $\Psi(F)(v) := F(U_{(v)})$. If $f = (v_0, \ldots, v_r)$ is an edge-path, define $\Psi(F)(f) : \Psi(F)(v_r) \to \Psi(F)(v_0)$ as the composite

$$
\Psi(F)(f) := F(p_0)(F(q_0))^{-1} \cdots F(p_{(i+1)})(F(q_{(i+1)}))^{-1} \cdots F(p_{(r-1)r})(F(q_{(r-1)r}))^{-1}.
$$

For the sake of simplicity, we will write $\Psi_F$ for $\Psi(F)$. To check that $\Psi_F(f)$ is well defined, we need the following lemma.

Lemma 4.14. Let $v_0, v_1$, and $v_2$ be the vertices of a 2-simplex $\langle v_0v_1v_2 \rangle$, and let $v_0, v_1, v_2, v_0$ be an edge-loop. Then

$$
F(p_0)(F(q_0))^{-1}F(p_1)(F(q_1))^{-1}F(p_{20})(F(q_{20}))^{-1} = id
$$
Proof. Consider the inclusions $d^0 : U(v_1v_2) \hookrightarrow U(v_0v_1v_2)$, $d^1 : U(v_2v_0) \hookrightarrow U(v_0v_1v_2)$, and $d^2 : U(v_0v_1) \hookrightarrow U(v_0v_1v_2)$. Also consider the diagram

$$
\begin{array}{c}
\xymatrix{
F(U(v_1v_2)) & F(U(v_1v_2)) & F(U(v_1v_2)) \\
F(U(v_1v_2)) & F(U(v_1v_2)) & F(U(v_1v_2)) \\
F(U(v_1)) & F(U(v_1)) & F(U(v_1)) \\
F(p_{01}) & F(q_{01}) & F(p_{01}) \\
} \ar^{F(p_{20})}[rr] & & \ar^{F(q_{12})}[rr] & & \ar^{F(p_{12})}[rr] & & \ar_{F(d^0)}[rr] & & \ar_{F(d^0)}[rr] & & \ar_{F(d^0)}[rr] & & \\
\end{array}
$$

The desired result follows from the fact that each square of that diagram commutes.

Proposition 4.15. $\Psi_F(f) = \Psi_F(g)$ when $f \sim g$.

Proof. This follows immediately from the definition of $\Psi_F$, Remark 4.12 and Lemma 4.14.

Now we define $\Psi$ on morphisms.

Definition 4.16. If $\eta : F \to F'$ is a morphism of $\mathcal{F}(U(T^M); C)$, define $\Psi_\eta : \Psi_F \to \Psi_F'$ as $\Psi_\eta[v] := \eta[U(v)]$.

The map $\Psi_\eta$ is a member of $\hom_{\mathcal{F}(U(M); C)}(\Psi_F, \Psi_{F'})$ because of the following. For an edge-path $f = (v_0, v_1, \cdots, v_r)$, one can consider the following diagram

$$
\begin{array}{c}
\xymatrix{
F(U(v_0)) & F(U(v_1)) & \cdots & F(U(v_{r-1})) & F(U(v_r)) \\
\ar_{F(p_{01})}[rr] & & \ar_{F(q_{01})}[rr] & & \ar_{F(q_{12})}[rr] & & \\
\ar_{F(p_{01})}[rr] & & \ar_{F(q_{01})}[rr] & & \ar_{F(q_{12})}[rr] & & \\
\ar_{\eta[U(v_0)]}[rr] & & \ar_{\eta[U(v_1)]}[rr] & & \ar_{\eta[U(v_{r-1})]}[rr] & & \ar_{\eta[U(v_r)]}[rr] & & \\
\ar_{\eta[U(v_0)]}[rr] & & \ar_{\eta[U(v_1)]}[rr] & & \ar_{\eta[U(v_{r-1})]}[rr] & & \ar_{\eta[U(v_r)]}[rr] & & \\
}\end{array}
$$

in which the horizontal arrows are obtained by applying $F$ and $F'$ to $p_{i(i+1)}$ and $q_{i(i+1)}$. Since each square of that diagram commutes by the naturality of $\eta$, it follows that $\Psi_\eta$ is a natural transformation.

4.3 Construction of the functor $\Phi : \mathcal{F}(\Pi(M); C) \to \mathcal{F}(U(T^M); C)$

We begin with some notation. For a subcomplex $\mathcal{K} \subseteq T^M$, we will write $U(\mathcal{K}) \subseteq U(T^M)$ for the full subcategory whose objects are $U_\sigma$ with $\sigma$ running over the set of simplices of $\mathcal{K}$. For a category $A$ we will write $ob(A)$ for the collection of objects, and $mor(A)$ for the collection of morphisms of $A$. Recall the notation $T^M_p$ from Section 2.

Let $G : \Pi(M) \to C$ be a contravariant functor, that is, $G \in \mathcal{F}(\Pi(M); C)$. We wish to define a very good functor $\Phi_G : U(T^M) \to C$, where $\Phi_G$ stands for $\Phi(G)$. Our strategy is to first define $\Phi_G$ on $U(T^0_M)$, then on $U(T^M_1)$, and $U(T^M_2)$, and lastly, by using Proposition 4.19 below, we will extend the definition on the rest of the triangulation.
On the 0-skeleton $T_0^M$, define $\Phi_G(U_{(v)}) := G(v)$. On the 1-skeleton $T_1^M$, let $\langle v_0v_1 \rangle$ be an edge, and let $d^0 := q_0 : U_{(v_1)} \rightarrow U_{(v_0v_1)}$ and $d^1 := p_0 : U_{(v_0)} \rightarrow U_{(v_0v_1)}$. Define

$$\Phi_G(U_{(v_0v_1)}) := G(v_1), \quad \text{and} \quad \Phi_G(d^i) := \begin{cases} id & \text{if } i = 0 \\ G((v_0, v_1)) & \text{if } i = 1. \end{cases}$$

Of course $G((v_0, v_1))$ is an isomorphism since every morphism of $\Pi(M)$ is invertible. So $\Phi_G$ thus defined is a very good functor. By definition it satisfies the following two conditions: (a) $\Phi_G(U_{(v)}) = G(v)$ for all $v \in T_0^M$; (b) $\Phi_G(d^1)(\Phi_G(d^0))^{-1} = G((v_0, v_1))$ for any edge $\langle v_0v_1 \rangle$.

**Proposition 4.17.** Given any other very good functor $F : \mathcal{U}(T_1^M) \rightarrow \mathcal{C}$ satisfying (a) and (b), there exists a natural isomorphism $\beta : \Phi_G \rightarrow F$.

**Proof.** Define $\beta$ as

$$\beta[U_\sigma] = \begin{cases} id & \text{if } \sigma \in T_0^M \\ (F(d^0))^{-1} & \text{if } \sigma = \langle v_0v_1 \rangle. \end{cases}$$

It is straightforward to check the naturality of $\beta$. \qed

Now we define $\Phi_G$ on the 2-skeleton $T_2^M$. Let $\tau = \langle v_0v_1v_2 \rangle$ be a 2-simplex, and let $\partial\mathcal{U}(\tau) \subseteq \mathcal{U}(\tau)$ denote the full subposet whose $\text{ob}(\partial\mathcal{U}(\tau)) = \text{ob}(\mathcal{U}(\tau)) \setminus \{U_1\}$. Let $F : \partial\mathcal{U}(\tau) \rightarrow \mathcal{C}$ be a very good functor, and let $\iota : \partial\mathcal{U}(\tau) \hookrightarrow \mathcal{U}(\tau)$ be the inclusion functor. A contravariant functor $\mathcal{F} : \mathcal{U}(\tau) \rightarrow \mathcal{C}$ is called an extension of $F$ if (i) $\mathcal{F}$ is very good, and (ii) $\mathcal{F} \circ \iota = F$.

**Proposition 4.18.** Such an extension $\mathcal{F}$ exists if and only if

$$F(p_{01})(F(q_{01}))^{-1}F(p_{12})(F(q_{12}))^{-1}F(p_{20})(F(q_{20}))^{-1} = id. \quad (4.3)$$

Moreover $\mathcal{F}$ is unique up to isomorphism.

**Proof.** If $\mathcal{F}$ exists, then the equation (4.3) holds by Lemma 4.14. Now assume that we have (4.3), and define $\mathcal{F} := F$ on the poset $\partial\mathcal{U}(\tau)$. Also define $\mathcal{F}(U_\tau) := F(U_{(v_0v_1v_2)})$. On morphisms $d^i : U_{(v_0...\hat{v}_i...v_2)} \hookrightarrow U_{(v_0...\hat{v}_i...v_2)}$, define $\mathcal{F}(d^i) := id$, $\mathcal{F}(d^i) := (F(p_{20}))^{-1}F(q_{12})$, and $\mathcal{F}(d^i) := (F(q_{12}))^{-1}F(p_{12})$.

If $a_1$ and $a_2$ are two composable morphisms of $\mathcal{U}(\tau)$, define $\mathcal{F}(a_2a_1) := \mathcal{F}(a_1)\mathcal{F}(a_2)$. To see that $\mathcal{F}$ is well defined on compositions, we need to show that equations $\mathcal{F}(p_{12})\mathcal{F}(d^0) = \mathcal{F}(q_{01})\mathcal{F}(d^2)$, $\mathcal{F}(q_{12})\mathcal{F}(d^0) = \mathcal{F}(p_{20})\mathcal{F}(d^1)$, and $\mathcal{F}(p_{01})\mathcal{F}(d^2) = \mathcal{F}(q_{20})\mathcal{F}(d^1)$ hold. The first two follow directly from the definition of $\mathcal{F}(d^0), \mathcal{F}(d^2)$ and $\mathcal{F}(d^1)$, while the latter follows from (4.3). Thus $\mathcal{F}$ is a well defined functor, which is clearly very good and satisfies $\mathcal{F} \circ \iota = F$.

To prove the uniqueness, let $\mathcal{F}' : \mathcal{U}(\tau) \rightarrow \mathcal{C}$ be another extension of $F$. Define $\beta : \mathcal{F} \rightarrow \mathcal{F}'$ as

$$\beta[U_\sigma] = \begin{cases} (F'((d^0)))^{-1} & \text{if } \sigma = \tau \\ id & \text{otherwise.} \end{cases}$$

By the definitions, it is straightforward to check that $\beta$ is a natural isomorphism. \qed

The functor $\Phi_G : \mathcal{U}(T_1^M) \rightarrow \mathcal{C}$ certainly satisfies (4.3) because of the following reason. Since $\tau = \langle v_0v_1v_2 \rangle$ is a 2-simplex, it follows that $\langle (v_0, v_1, v_2, v_3) \rangle \sim \langle v_0 \rangle$. Therefore one has $G((v_0, v_1))G((v_1, v_2))G((v_2, v_3)) = id$. Hence, thanks to Proposition 4.18, we can extend $\Phi_G$ to $\mathcal{U}(T_2^M)$ up to isomorphism, and the new functor is still denoted $\Phi_G$. 19
Now we define \( \Phi_G \) on the rest of the triangulation. Let \( \Delta[k], k \geq 0 \), denote the poset whose objects are nonempty subsets of \( \{0, 1, \ldots, k\} \), and whose morphisms are inclusions. Consider the dual category \( \Delta[k]^{\text{op}} \). Given an object \( S \) of \( \Delta[k] \) and \( i \notin S \), the inclusion \( S \mapsto S \cup \{i\} \) gives rise to a unique morphism \( d_i: S \cup \{i\} \rightarrow S \) in \( \Delta[k]^{\text{op}} \). In fact \( d_i \) consists of taking out \( i \). We will write \( \{a_1, \ldots, \hat{i}, \ldots, a_p\} \) for \( \{a_1, \ldots, a_p\} \setminus \{i\} \). The main observation here is the fact that every morphism of \( \Delta[k]^{\text{op}} \) can be written as a composition of \( d_i \)'s. Let \( \partial \Delta[k]^{\text{op}} \subseteq \Delta[k]^{\text{op}} \) be the full subposet of all objects except \( \{0, 1, \ldots, k\} \), and let \( \iota: \partial \Delta[k]^{\text{op}} \rightarrow \Delta[k]^{\text{op}} \) be the inclusion functor. The following result gives us a way to extend \( \Phi_G \) to \( \mathcal{U}(\mathcal{T}_k^M) \) when \( k \geq 3 \).

**Proposition 4.19.** Assume \( k \geq 3 \), and let \( \phi: \partial \Delta[k]^{\text{op}} \rightarrow \mathcal{C} \) be a covariant functor that sends every morphism to an isomorphism. Then there exists a unique functor (up to isomorphism) \( \overline{\phi}: \Delta[k]^{\text{op}} \rightarrow \mathcal{C} \) such that (i) the image of any morphism under \( \overline{\phi} \) is an isomorphism, and (ii) \( \overline{\phi} \circ \iota = \phi \).

**Proof.** We begin by showing that \( \overline{\phi} \) exists. On the poset \( \partial \Delta[k]^{\text{op}} \), define \( \overline{\phi} := \phi \). Also define \( \overline{\phi}(\{0, \ldots, k\}) := \phi(\{1, \ldots, k\}) \). On morphisms, we first define \( \overline{\phi}(d_0) := id \), where \( d_0: \{0, \ldots, k\} \rightarrow \{1, \ldots, k\} \). Next define \( \overline{\phi}(d_i), 1 \leq i \leq k \), in such a way that the following square commutes.

\[
\begin{array}{ccc}
\overline{\phi}(\{0, \ldots, k\}) & \overline{\phi}(d_i) & \overline{\phi}(\{1, \ldots, k\}) \\
\overline{\phi}(d_0) = id & & \overline{\phi}(d_0) = \phi(d_0) \\
\overline{\phi}(\{1, \ldots, k\}) & \overline{\phi}(d_i) = \phi(d_0) & \overline{\phi}(\{1, \ldots, k\} - \{1\})
\end{array}
\]

Namely, define \( \overline{\phi}(d_i) := (\phi(d_0))^{-1} \phi(d_j) \overline{\phi}(d_0) \). On the compositions, we define \( \overline{\phi} \) in the obvious way. Since there could be many different ways to go from one object of \( \Delta[k]^{\text{op}} \) to another one, one needs to check that \( \overline{\phi} \) is well defined on compositions. To do that, it is enough to show that the equations \( \phi(d_i) \overline{\phi}(d_j) = \phi(d_j) \overline{\phi}(d_i) \), \( 0 \leq i < j \leq k \), hold. The case \( i = 0 \) follows immediately from the definition of \( \overline{\phi}(d^0) \), while the other cases follow from the case \( i = 0 \) and the equations \( \phi(d_i) \phi(d_j) = \phi(d_j) \phi(d_i), i \neq j \). Thus \( \overline{\phi} \) is a functor which by definition satisfies conditions (i) and (ii) of the proposition.

To prove the uniqueness part, let \( F: \Delta[k]^{\text{op}} \rightarrow \mathcal{C} \) be another functor satisfying (i) and (ii). By the definitions, it is straightforward to show that the map \( \beta: \Phi \rightarrow F \) defined as

\[
\beta[S] := \begin{cases} 
(F(d_0))^{-1} & \text{if } S = \{0, \ldots, k\} \\
\text{id} & \text{if } S \in \partial \Delta[k]^{\text{op}}
\end{cases}
\]

is a natural isomorphism. This ends the proof. \( \square \)

Now we define by induction \( \Phi(\eta) := \Phi_\eta \) where \( \eta: G \rightarrow G' \) is a morphism of \( \mathcal{F}(\Pi(M); \mathcal{C}) \). On the 1-skeleton, define

\[
\Phi_\eta[U_\sigma] := \begin{cases} 
\eta[v] & \text{if } \sigma = v \in \mathcal{T}_0^M \\
\eta[v_1] & \text{if } \sigma = (v_0 v_1).
\end{cases}
\]

Assume that we have defined \( \Phi_\eta \) on \( \mathcal{U}(\mathcal{T}_k^{M-1}) \), for some \( k \geq 2 \). For a \( k \)-simplex \( \langle v_0 \cdots v_k \rangle \), we define \( \Phi_\eta[U_{\langle v_0 \cdots v_k \rangle}] := \Phi_\eta[U_{\langle v_1 \cdots v_k \rangle}] \). By induction on the skeletons, one can easily verify that the collection \( \{\Phi_\eta[U_\sigma]\}_{U_\sigma \in \mathcal{U}(\mathcal{T}_k^M)} \) is a natural transformation from \( \Phi_G \) to \( \Phi_{G'} \).

### 4.4 Proof of the main result of the paper

The goal of this subsection is to prove the main result of the paper: Theorem 4.1.3 Before we do this, we will first prove Theorem 4.1.2 announced earlier at the end of Subsection 4.1. We will need two lemmas.
Lemma 4.20. For any very good functor \( F: \mathcal{U}(\mathcal{T}^M) \to \mathcal{C} \), there is a natural isomorphism \( \beta[F]: \Phi \Psi \overset{\cong}{\to} F \).

Proof. By induction. On the 1-skeleton, define
\[
\beta[F][U_\sigma] := \begin{cases}
    \text{id} & \text{if } \sigma = v \in T^M_0 \\
    (F(d^\sigma))^{-1} & \text{if } \sigma = \langle v_0v_1 \rangle.
\end{cases}
\]
Assuming that \( \beta[F] \) is defined on \( \mathcal{U}(\mathcal{T}^M_{k-1}) \) for some \( k \geq 2 \), we define \( \beta[F][U_{\langle v_0 \cdots v_k \rangle}] := (F(d^\sigma))^{-1}\beta[F][U_{\langle v_1 \cdots v_k \rangle}] \), where \( d^\sigma: U_{\langle v_1 \cdots v_k \rangle} \to U_{\langle v_0 \cdots v_k \rangle} \). By the definitions, one can easily check that \( \beta[F] \) is a natural isomorphism. \( \square \)

Again by induction, one can prove the following lemma just by using the definitions.

Lemma 4.21. The collection \( \beta = \{\beta[F]\}_{F \in \mathcal{F}(\mathcal{T}^M); \mathcal{C}} \) defines a natural isomorphism from \( \Phi \Psi \) to \( \text{id} \).

We can now prove Theorem 4.9 and Theorem 1.1.

Proof of Theorem 4.9. From the definitions, one has \( \Psi \Phi = \text{id} \), and from Lemma 4.21, we have \( \Phi \Psi \cong \text{id} \). \( \square \)

Proof of Theorem 1.1. First of all, by Corollary 3.30, the category \( \mathcal{F}_k(\mathcal{O}(M); \mathcal{C}) \) of very good homogeneous functors \( \mathcal{O}(M) \to \mathcal{C} \) of degree \( k \) is equivalent to the category \( \mathcal{F}_1(\mathcal{O}(F_k(M)); \mathcal{C}) \) of very good linear functors \( \mathcal{O}(F_k(M)) \to \mathcal{C} \). Let \( \mathcal{T}^{F_k(M)} \) denote a triangulation of \( F_k(M) \). Consider the associated cover \( \mathcal{U}(\mathcal{T}^{F_k(M)}) \), which is very good by Proposition 4.4. Also consider the basis \( \mathcal{B}_{\mathcal{U}(\mathcal{T}^{F_k(M)})} \) for the topology of \( F_k(M) \) (see Proposition 4.5). By Theorem 3.8, one has the equivalence \( \mathcal{F}_1(\mathcal{O}(F_k(M)); \mathcal{C}) \simeq \mathcal{F}(\mathcal{B}^{(1)}(F_k(M)); \mathcal{C}) \). Since \( \mathcal{B}^{(1)}(F_k(M)) = \mathcal{B}_{\mathcal{U}(\mathcal{T}^{F_k(M)})} \) by Remark 4.6, this latter equivalence becomes \( \mathcal{F}_1(\mathcal{O}(F_k(M)); \mathcal{C}) \simeq \mathcal{F}(\mathcal{B}_{\mathcal{U}(\mathcal{T}^{F_k(M)})}; \mathcal{C}) \). Applying now Proposition 4.7, we get \( \mathcal{F}(\mathcal{B}_{\mathcal{U}(\mathcal{T}^{F_k(M)})}; \mathcal{C}) \simeq \mathcal{F}(\mathcal{U}(\mathcal{T}^{F_k(M)}); \mathcal{C}) \). Lastly, we have the equivalence \( \mathcal{F}(\mathcal{U}(\mathcal{T}^{F_k(M)}); \mathcal{C}) \simeq \mathcal{F}(\Pi(F_k(M)); \mathcal{C}) \) by Theorem 4.9. Combining all these equivalences, we get the desired result. \( \square \)

5 Connection to representation theory

The goal of this short section is to prove Corollary 1.2, which establishes a connection between very good homogeneous functors and representation theory. Specifically, it says that the category of very good homogeneous functors of degree \( k \) is equivalent to that of representations of \( \pi_1(F_k(M)) \) provided that \( F_k(M) \) is connected.

To prove Corollary 1.2, we will need Proposition 5.1 below in which it is important to view \( \Pi(M) \) as in Section 3 (see Theorem 1.1). It is also important to view the fundamental group \( \pi_1(M) \) as the set of equivalence classes of edge-loops starting and ending at the same vertex, namely \( w \). For a group \( G \), recall the category \( \text{Rep}_G(G) \) from the introduction.

Proposition 5.1. Assume that \( M \) is connected. Then the category \( \mathcal{F}(\Pi(M); \mathcal{C}) \) from Definition 4.8 is equivalent to the category \( \text{Rep}_G(\pi_1(M)) \) of representations of \( \pi_1(M) \) in \( G \). That is, \( \mathcal{F}(\Pi(M); \mathcal{C}) \simeq \text{Rep}_G(\pi_1(M)) \).

Proof. We define two functors
\[
\mathcal{F}(\Pi(M); \mathcal{C}) \xrightarrow{\Theta} \text{Rep}_G(\pi_1(M)) \xleftarrow{\Lambda} \mathcal{F}(\Pi(M); \mathcal{C}).
\]
as \( \Theta(F) := (F(w), \rho_F) \) where \( \rho_F(f) := F(f) \) for any edge-loop \( f \). If \( \eta: F \to F' \) is a natural transformation, we define \( \Theta(\eta): (F(w), \rho_F) \to (F'(w), \rho_{F'}) \) as \( \eta[w] \). Now define \( \Lambda(V, \rho)(v) := V \). To define \( \Lambda(V, \rho) \) on morphisms of \( \Pi(M) \), we need to introduce the following definition. An edge-path \( f = (v_0, \cdots, v_r) \) is called reduced if \( v_i \neq v_j \) whenever \( i \neq j \). Given two different vertices \( v \) and \( v' \), there exists a unique reduced edge-path, denoted \( g_{vv'} = (v, v_1, \cdots, v_{r-1}, v') \), in the maximal tree \( m\mathcal{T}_1^M \). This is because \( m\mathcal{T}_1^M \) is contractible since \( M \) is connected by assumption. If \( f = (v_0, \cdots, v_r) \) is an edge-path we define \( \Lambda(V, \rho)(f) := \rho(f) \) where \( f := g_{v_0w}g_{v,w} \). On morphisms of \( \text{Rep}_C(\pi_1(M)) \), we define \( \Lambda \) in the obvious way. That is, \( \Lambda(\varphi)[v] := \varphi \). By the definitions, it is straightforward to verify that \( \Theta \Lambda = \text{id} \) and \( \Lambda \Theta = \text{id} \), which completes the proof. 

**Proof of Corollary 1.2.** This follows immediately from Theorem 1.1 and Proposition 5.1. □

## 6 Very good vector bundles

We prove Theorem 6.8 below which states that the category of very good functors, studied in the previous sections, is equivalent to a particular class of vector bundles (which we call very good vector bundles (see Definition 6.1 below)). We also prove Theorem 6.6 which states that our category of very good vector bundles is abelian. Throughout this section, we will write \( \mathcal{fVect}_K \) for the category of finite dimensional vector spaces over a field \( K \).

### 6.1 Definition and examples

Roughly speaking, a very good vector bundle is a vector bundle in the classical sense endowed with an extra structure (which is a very good covariant functor) that satisfies the axioms for a vector bundle, and an additional axiom (which is some kind of compatibility between local trivializations). To be more precise, we have the following definition.

**Definition 6.1.** Let \( \mathcal{B}^{(1)}(M) \) as in Definition 3.6. The category \( \text{VGVB}(\mathcal{B}^{(1)}(M)) \) of very good vector bundles is defined as follows. An object of \( \text{VGVB}(\mathcal{B}^{(1)}(M)) \) is a triple \((E, \pi, F_\pi)\) where \( E \) is a topological space, \( \pi: E \to M \) is a continuous surjection, and \( F_\pi: \mathcal{B}^{(1)}(M) \to \mathcal{fVect}_K \) is a very good covariant functor. Such a triple is endowed with a family of homeomorphisms \( \{\varphi^x_U: U \times F_\pi(U) \to \pi^{-1}(U)\}_{U \in \mathcal{B}^{(1)}(M)} \) that satisfy the following four axioms:

1. **(VB0)** For every \( x \in M \), the preimage \( \pi^{-1}(x) \) is a vector space;
2. **(VB1)** For all \( U \in \mathcal{B}^{(1)}(M) \), and for all \((x, v) \in U \times F_\pi(U)\), \( (\pi \circ \varphi^x_U)(x, v) = x \);
3. **(VB2)** For all \( U \in \mathcal{B}^{(1)}(M) \), and for all \( x \in U \), the map \( \varphi^x_U: F_\pi(U) \to \pi^{-1}(x) \) defined by \( \varphi^x_U(v) = \varphi^x_U(x, v) \) is an isomorphism of vector spaces.
4. **(VB3)** For any morphism \( j: U \to V \) of \( \mathcal{B}^{(1)}(M) \), the following square commutes.

\[
\begin{array}{ccc}
U \times F_\pi(U) & \xrightarrow{\varphi^x_U} & \pi^{-1}(U) \\
\downarrow{j \times F_\pi(j)} & & \downarrow{\pi^{-1}(U)} \\
V \times F_\pi(V) & \xrightarrow{\varphi^x_U} & \pi^{-1}(V).
\end{array}
\]

A morphism from \((X, p, F_p)\) to \((Y, q, F_q)\) consists of a pair \((f, \eta)\) where \( f: X \to Y \) is a map making the obvious triangle commute, and \( \eta: F_p \to F_q \) is a natural transformation such that for every \( U \in \mathcal{B}^{(1)}(M) \),
the following square commutes.

\[
\begin{array}{ccc}
U \times F_p(U) & \xrightarrow{id \times \eta[U]} & U \times F_q(U) \\
\varphi^p_U & \downarrow & \varphi^q_U \\
p^{-1}(U) & \xrightarrow{f} & q^{-1}(U).
\end{array}
\]

By definition, any very good vector bundle is a vector bundle in the classical sense. But the converse is not true as shown Example 6.3 below.

**Example 6.2.** Consider the Mobius bundle \((E, \pi)\) where \(E = [0, 1] \times \mathbb{R} / \sim\) and \(\sim\) is the equivalence relation \(\sim\) is generated by \((0, t) \sim (1, -t)\) for all \(t \in \mathbb{R}\). Of course \(\pi: E \to M = S^1 \equiv [0, 1]/0 \sim 1\) is the canonical projection. Also consider the contravariant functor \(F_\pi: \mathcal{B}(1)(M) \to f \text{Vect}_{\mathbb{R}}\) defined as \(F_\pi(U) = \mathbb{R}\), and

\[
F_\pi(U \hookrightarrow V) = \begin{cases} 
-\text{id}_\mathbb{R} & \text{if } U \subseteq (0, 1) \text{ and } [0] = [1] \in V \\
\text{id}_\mathbb{R} & \text{otherwise}.
\end{cases}
\]

One can easily verify that \((E, \pi, F_\pi)\) is a very good vector bundle.

Many other examples of classical vector bundles are very good including line bundle over the real projective space \(\mathbb{RP}^n\).

**Example 6.3.** The canonical complex line bundle over \(\mathbb{CP}^1 \cong S^2 = M\) is not a very good vector bundle essentially because of the fact that the second component of the local trivializations \(\varphi^p_U(x, z)\) depends on both variables \(x\) and \(z\).

**Remark 6.4.** Consider the Mobius bundle \((E', \pi')\) where \(E' = [0, 1] \times \mathbb{R} / (0, t) \sim (1, -2t)\), and \(\pi': E' \to S^1\) is defined by \(\pi'([x, t]) = [x]\). Also consider the vector bundle \((E, \pi)\) from Example 6.2. Certainly the bundle \((E', \pi')\) is very good as in Example 6.2. The point is that \((E', \pi', F_{\pi'})\) and \((E, \pi, F_\pi)\) are isomorphic as vector bundles, but not as very good vector bundles.

### 6.2 Proving that the category \(\text{VGVB} (\mathcal{B}(1)(M))\) is abelian

To prove that \(\text{VGVB} (\mathcal{B}(1)(M))\) is abelian, we need the following lemma.

**Lemma 6.5.** Let \((f, \eta): (X, p, F_p) \to (Y, q, F_q)\) be a morphism of \(\text{VGVB} (\mathcal{B}(1)(M))\). Let \(U, V \in \mathcal{B}(1)(M)\) such that \(U \cap V \neq \emptyset\). Then for any \(x \in U\), and \(y \in V\) \((x\text{ and } y\text{ need not lie in } U \cap V)\), there are isomorphisms \(\ker f_p \cong F(U) \cong F(V) \cong \ker f_q\), and \(\coker f_x \cong G(U) \cong G(V) \cong \coker f_y\), of vector spaces. Here \(f_x: p^{-1}(x) \to q^{-1}(x)\) is the canonical linear map induced by \(f\), \(F(U)\) and \(G(U)\) are the kernel and cokernel of \(\eta[U]: F_p(U) \to F_q(U)\) respectively.

**Proof.** We will prove the first set of isomorphisms; the proof of the second set is similar. The first isomorphism, \(\ker f_p \cong F(U)\), is readily obtained from the following commutative diagram in which the isomorphisms \(\varphi^p_U\) and \(\varphi^q_U\) are provided by the axiom (VB2).

\[
\begin{array}{ccc}
F(U) = \ker[\eta[U]] & \xrightarrow{\varphi^p_U} & F_p(U) \\
\downarrow \cong & \downarrow \cong & \downarrow \cong \\
\ker f_x & \xrightarrow{p^{-1}(x)} & q^{-1}(x).
\end{array}
\]

The second isomorphism follows from the facts (i) the intersection of \(U\) and \(V\) is not empty, (ii) objects of \(\mathcal{B}(1)(M)\) form a basis for the topology of \(M\), (iii) every morphism of \(\mathcal{B}(1)(M)\) is an isotopy equivalence, and
Theorem 6.6. The category $\text{VGB} \left( \mathcal{B}^{(1)}(M) \right)$ of very good vector bundles over a manifold $M$ is an abelian category.

Proof. First recall that a category is abelian if it satisfies the following four axioms: (Ab1) it has a zero object, (Ab2) it has all binary products and binary coproducts, (Ab3) it has all kernels and cokernels, and (Ab4) every monomorphism (respectively epimorphism) is a kernel (respectively cokernel) of a map.

- **Axiom (Ab1).** The zero object is $(X_0, p_0, F_{p_0})$ where $X_0 = M \times \{0\}$, for all $x \in M, p_0(x, 0) = x$, and for all $U \in \mathcal{B}^{(1)}(M), F_{p_0}(U) = \{0\}$, the trivial vector space.

- **Axiom (Ab2).** Let $(X, p, F_p)$ and $(Y, q, F_q)$ be two objects of $\text{VGB} \left( \mathcal{B}^{(1)}(M) \right)$. Define a new object $(E, \pi, F_\pi)$ as follows. The total space is the pullback of $X \xrightarrow{p} M \xleftarrow{q} Y$. The projection is $\pi := pf_{\pi} = qf_{\pi}$, where $f_{\pi} : E \to X$ and $f_q : E \to Y$ are the projections on the first and second component respectively. The covariant functor $F_\pi$ is defined as the product $F_\pi = F_p \times F_q$. For every $U$, define $\varphi_U^p(x, v_1, v_2) := (\varphi_U^p(x, v_1), \varphi_U^q(x, v_2))$. It is clear that $\varphi_U^p$ is an homeomorphism because so are $\varphi_U^p$ and $\varphi_U^q$. It is also clear that $(E, \pi, F_\pi)$ satisfies the axioms of a very good vector bundle. One can easily verify that $(E, \pi, F_\pi)$ is the product as well as the coproduct of $(X, p, F_p)$ and $(Y, q, F_q)$.

- **Axiom (Ab3).** Let $(f, \eta) : (X, p, F_p) \to (Y, q, F_q)$ be a morphism of $\text{VGB} \left( \mathcal{B}^{(1)}(M) \right)$. We wish to construct a new object $(E, \pi, F_\pi)$, which will represent the kernel of $(f, \eta)$. First of all, define $E$ as $E = \{(x, v) \mid x \in M, v \in \ker f_x \}$. Next define $\pi : E \to M$ by $\pi(x, v) = x$, and $F_\pi : \mathcal{B}^{(1)}(M) \to \text{fVect}_k$ by $F_\pi(U) = \ker[f]$. Recalling the first isomorphism of Lemma 6.5, we define the local trivialization $\varphi_U^\pi : U \times F_p(U) \to \pi^{-1}(U)$ by the formula $\varphi_U^\pi(x, v) = (x, \lambda_{xU}(v))$. It is clear that the triple $(E, \pi, F_\pi)$ thus defined is an object of $\text{VGB} \left( \mathcal{B}^{(1)}(M) \right)$. Moreover $(E, \pi, F_\pi)$ is the desired kernel. Similarly, one has the cokernel of $(f, \eta) : (X, p, F_p) \to (Y, q, F_q)$, which is obtained by replacing in the preceding construction “ker” by “coker”.

- **Axiom (Ab4).** Since we defined the kernel and the cokernel fibrewise, and since the category of vector spaces is abelian, it follows that every monomorphism in $\text{VGB} \left( \mathcal{B}^{(1)}(M) \right)$ is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Remark 6.7. The traditional category of vector bundles over a fixed base is not abelian as there is an issue with the kernel of a morphism, which is not a bundle in any natural way. For instance, consider the trivial bundle $(X, p)$ over $\mathbb{R}$ with $X = \mathbb{R} \times \mathbb{R}$, and $p(x, y) = x$. Also consider the map $f : X \to X$ defined by $f(x, y) = (x, xy)$. Fibrewise, the kernel of $f$ is $\mathbb{R}$ over $0$, and is reduced to the trivial vector space otherwise. So the map $x \mapsto \dim(\ker f_x)$ is not locally constant, and therefore the kernel of $f$ is not a vector bundle neither in the classical sense nor in the sense of Definition 6.4. A similar issue happens to the cokernel. The crucial thing that turns our category of vector bundles into an abelian category is Lemma 6.5.

### 6.3 Equivalence between $\text{VGB} \left( \mathcal{B}^{(1)}(F_k(M)) \right)$ and $\mathcal{F} \left( \mathcal{B}^{(1)}(F_k(M)); \text{fVect}_k \right)$

The goal of this subsection is to prove Theorem 6.8 which says that the category of very good vector bundles is equivalent to the category of very good functors. Recall the notation $F_k(M)$ from Section 2.

Theorem 6.8. The category $\text{VGB} \left( \mathcal{B}^{(1)}(F_k(M)) \right)$ of very good vector bundles over $F_k(M)$ is equivalent to the category $\mathcal{F} \left( \mathcal{B}^{(1)}(F_k(M)); \text{fVect}_k \right)$ of very good contravariant functors from $\mathcal{B}^{(1)}(F_k(M))$ to $\text{fVect}_k$.

Proof. It suffices to work with $k = 1$; the proof remains unchanged for other values of $k$. Let $\mathcal{F}^* \left( \mathcal{B}^{(1)}(M); \text{fVect}_k \right)$ denote the category of very good covariant functors $F : \mathcal{B}^{(1)}(M) \to \text{fVect}_k$. 
Since every object of $f\text{Vect}_K$ is a finite dimensional vector space, the functor $F \left( B^{(1)}(M); f\text{Vect}_K \right) \rightarrow F^* \left( B^{(1)}(M); f\text{Vect}_K \right)$ that sends a contravariant functor to its dual (defined objectwise) is an equivalence of categories. Now we prove that the categories $F^* \left( B^{(1)}(M); f\text{Vect}_K \right)$ and $VGVB \left( B^{(1)}(M) \right)$ are equivalent.

To proceed, we define a pair of functors

$$\theta : F^* \left( B^{(1)}(M); f\text{Vect}_K \right) \rightleftarrows VGVB \left( B^{(1)}(M) \right) \colon \psi$$

For $\theta$, let $G$ be an object of $F^* \left( B^{(1)}(M); f\text{Vect}_K \right)$. Define $\theta(G) = (E_G, \pi_G, F_{\pi_G})$ as

$$E_G := \colim_{U \in B^{(1)}(M)} U \times G(U) = \left( \coprod_{U \in B^{(1)}(M)} U \times G(U) \right) / \sim,$$

where $\sim$ is the usual equivalence relation for colimits. The map $\pi_G$ is defined by $\pi_G([x, v]) = x$, and the functor $F_{\pi_G}$ is the same as $G$. Now define the local trivialization $\varphi_U^G : U \times G(U) \rightarrow \pi_G^{-1}(U)$ by $\varphi_U^G(x, v) = [(x, v)]$. It is straightforward to check that $(E_G, \pi_G, F_{\pi_G})$ is a very good vector bundle. Now define $\psi$ as $\psi(E, \pi, F_{\pi}) = F_{\pi}$. By the definitions, one has $\psi \theta = id$ and $\theta \psi \cong id$, which completes the proof.

We close this section with a corollary. In the particular case when $B^{(1)}(M)$ is constructed from a triangulation $T^M$ of $M$, the category of very good vector bundles is deeply related to the category of representations of $\pi_1(M)$. Specifically, one has the following result.

**Corollary 6.9.** Let $T^M$ be a triangulation of $M$. As in Remark 4.6 take $B_{U(T^M)}$ as the basis for the topology of $M$. Assume that $M$ is connected. Then one has $VGVB \left( B^{(1)}(M) \right) \simeq fK[\pi_1(M)]$-Mod, where $fK[\pi_1(M)]$-Mod denotes the category of finite dimensional representations of $\pi_1(M)$ over $K$.

**Proof.** This follows from Theorem 6.8, Proposition 4.7 and Theorem 4.9.

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