INEQUALITIES ON GEOMETRICALLY CONVEX FUNCTIONS

M. EMİN ÖZDEMİR

Abstract. In this paper, we obtain some new upper bounds for differentiable mappings whose q-th powers are geometrically convex and monotonically decreasing by using the Hölder inequality, Power mean inequality and properties of modulus.

1. INTRODUCTION

The following double inequality is well known in the literature as Hadamard’s inequality:

Let \( f: I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function defined on an interval \( I \) of real numbers, \( a, b \in I \) and \( a < b \), we have

\[
\frac{f\left(\frac{a+b}{2}\right)}{b-a} \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]

Both inequalities hold in the reversed direction if \( f \) is concave.

It was first discovered by Hermite in 1881 in the Journal Mathesis (see [8]). The inequality (1.1) was nowhere mentioned in the mathematical literature until 1893. Beckenbach, a leading expert on the theory of convex functions, wrote that inequality (1.1) was proven by Hadamard in 1893 (see [9]). In 1974 Mitrinović found Hermite’s note in Mathesis. That is why, the inequality (1.1) was known as Hermite-Hadamard inequality.

A function \( f: [a, b] \subseteq \mathbb{R} \to \mathbb{R} \) is said to be convex if whenever \( x, y \in [a, b] \) and \( t \in [0, 1] \), the following inequality holds:

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).
\]

We say that \( f \) is concave if \( -f \) is convex. This definition has its origins in Jensen’s results from [7] and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them.

In [6], the concept of geometrically convex functions were introduced as following:

Definition 1. A function \( f: I \subset \mathbb{R}_+ \to \mathbb{R}_+ \) is said to be a geometrically convex function if

\[
f(x^t y^{1-t}) \leq [f(x)]^t [f(y)]^{1-t}
\]

for all \( x, y \in I \) and \( t \in [0, 1] \).

For some recent results connected with geometrically convex functions, see [3]-[6].
Definition 2. Let $a, b \in \mathbb{R}$, $a, b \neq 0$ and $|a| \neq |b|$. Logarithmic mean for real numbers was introduced as follows:

$$L(a, b) = \frac{a - b}{\ln|a| - \ln|b|}.$$

In [10], Özdemir and Yıldız established the following Theorem:

Theorem 1. Let $f : I^o \subset \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function on $I^o$, $a, b \in I$ with $a < b$ and $f' \in L_1[a, b]$. If $|f'|^q$ is geometrically convex and monotonically decreasing on $[a, b]$ and $t \in [0, 1]$, then we have the following inequality:

$$\left(1 + \frac{1}{p} + \frac{1}{q}ight) = \frac{1}{p} + \frac{1}{q} = 1. L(, )$$

is Logarithmic mean for real numbers.

In order to prove our main results we need the following lemma (see [1]).

Lemma 1. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^o$ where $a, b \in I$ with $a < b$. If $f'' \in L_1[a, b]$, then the following equality holds:

$$\frac{1}{b - a} \int_a^b f(u) \, du - f(x) = \left(1 + \frac{1}{b - a} \int_a^b f(u) \, duight) - f(x)$$

$$= \frac{(x - a)^3}{2(b - a)} \int_0^1 t^2 f''(tx + (1 - t)a) \, dt + \frac{(b - x)^3}{2(b - a)} \int_0^1 t^2 f''(tx + (1 - t)b) \, dt$$

for each $x \in [a, b]$.

In [2], in order to prove some inequalities related to Hermite-Hadamard inequality, Kavurmacı et al. used the following lemma.

Lemma 2. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^o$ where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:

$$\frac{(b - x)f(b) + (x - a)f(a)}{b - a} = \frac{1}{b - a} \int_a^b f(u) \, du$$

$$= \frac{(x - a)^2}{b - a} \int_0^1 (1 - t)f'(tx + (1 - t)a) \, dt + \frac{(b - x)^2}{b - a} \int_0^1 (1 - t)f'(tx + (1 - t)b) \, dt.$$

The main aim of this paper is to establish new inequalities for geometrically convex functions.

2. MAIN RESULTS

Theorem 2. Let $f : I^o \subset \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function on $I^o$, $a, b \in I$ with $a < b$ and $f'' \in L_1[a, b]$. If $|f''|^q$ is geometrically convex and monotonically
Therefore, we have the following inequality:

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left( x - \frac{a+b}{2} \right) f'(x) \right| \leq \frac{1}{(2p+1)^\frac{1}{q}} \left\{ \frac{(x-a)^3}{2(b-a)} \left[ L \left( |f''(x)|^q, |f''(a)|^q \right) \right]^{\frac{1}{q}} + \frac{(b-x)^3}{2(b-a)} \left[ L \left( |f''(x)|^q, |f''(b)|^q \right) \right]^{\frac{1}{q}} \right\}
\]

where \( 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1, L(\cdot, \cdot) \) is Logarithmic mean for real numbers.

**Proof.** From Lemma 1 with properties of modulus and using the H"older inequality, we have

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left( x - \frac{a+b}{2} \right) f'(x) \right| \leq \frac{(x-a)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)a)| dt + \frac{(b-x)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)b)| dt
\]

\[
\leq \frac{(x-a)^3}{2(b-a)} \left( \int_0^1 t^{2p} dt \right)^\frac{1}{p} \left( \int_0^1 |f''(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}}
\]

\[
+ \frac{(b-x)^3}{2(b-a)} \left( \int_0^1 t^{2p} dt \right)^\frac{1}{p} \left( \int_0^1 |f''(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}
\]

\[
= \frac{1}{(2p+1)^\frac{1}{q}} \left\{ \frac{(x-a)^3}{2(b-a)} \left( \int_0^1 |f''(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}}
\]

\[
+ \frac{(b-x)^3}{2(b-a)} \left( \int_0^1 |f''(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right\}.
\]

Since \( |f''|^q \) is geometrically convex and monotonically decreasing on \([a, b]\), we obtain

\[
x^t a^{1-t} \leq tx + (1-t)a
\]

\[
|f''(tx + (1-t)a)|^q \leq |f''(x^t a^{1-t})|^q.
\]

Therefore, we have

\[
I = \int_0^1 |f''(tx + (1-t)a)|^q dt
\]

\[
\leq \int_0^1 |f''(x^t a^{1-t})|^q dt
\]

\[
\leq \int_0^1 \left[ |f''(x)|^t |f''(a)|^{1-t} \right]^q dt
\]

\[
= L \left( |f''(x)|^q, |f''(a)|^q \right)
\]

and

\[
\int_0^1 t^{2p} dt = \frac{1}{2p + 1}.
\]
By making use of inequalities (2.4) and (2.3) in (2.2), we obtain (2.1). This completes the proof. □

Corollary 1. Since \( \frac{1}{3} < \frac{1}{(2p+1)^{\frac{1}{p}}} < 1 \), if we choose \( |f''(a)| = |f''(b)| \) in Theorem 2 we have

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left( x - \frac{a+b}{2} \right) f'(x) \right| \leq \frac{(x-a)^3 + (b-x)^3}{2(b-a)} \left[ L \left( |f''(x)|^q, |f''(a)|^q \right) \right]^{\frac{1}{q}}.
\]

Theorem 3. Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R}_+ \) be differentiable function on \( I^* \), \( a, b \in I \) with \( a < b \) and \( f'' \in L[a, b] \). If \( |f''|^q \) is geometrically convex and monotonically decreasing on \([a, b]\) for \( q \geq 1 \) and \( t \in [0, 1] \), then the following inequality holds:

\[
(2.5) \quad \left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left( x - \frac{a+b}{2} \right) f'(x) \right| \leq \left\{ \frac{k}{3} \right\}^{1-\frac{1}{q}} \left\{ \frac{(x-a)^3}{2(b-a)} \left( \frac{k}{\ln k} - \frac{2k}{(\ln k)^2} + \frac{2k}{(\ln k)^3} \right)^{\frac{1}{q}} + \frac{2(b-x)^3}{2(b-a)} \left( \frac{l}{\ln l} - \frac{2l}{(\ln l)^2} + \frac{2l}{(\ln l)^3} \right)^{\frac{1}{q}} \right\}.
\]

where

\[
k = \frac{|f''(x)|}{|f''(a)|^q} \quad \text{and} \quad l = \frac{|f''(x)|}{|f''(b)|^q}.
\]

Proof. From Lemma 1 and using the well-known power-mean inequality, we have

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left( x - \frac{a+b}{2} \right) f'(x) \right| \leq \frac{(x-a)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)a)| dt + \frac{(b-x)^3}{2(b-a)} \int_0^1 t^2 |f''(tx + (1-t)b)| dt
\]

\[
\leq \left\{ \frac{k}{3} \right\}^{1-\frac{1}{q}} \left\{ \frac{(x-a)^3}{2(b-a)} \left( \int_0^1 t^2 dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t^2 |f''(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} + \frac{2(b-x)^3}{2(b-a)} \left( \int_0^1 t^2 dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t^2 |f''(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right\}.
\]

\[
= \left\{ \frac{k}{3} \right\}^{1-\frac{1}{q}} \left\{ \frac{(x-a)^3}{2(b-a)} \left( \int_0^1 t^2 |f''(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} + \frac{2(b-x)^3}{2(b-a)} \left( \int_0^1 t^2 |f''(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right\}.
\]
Since $|f''|^q$ is geometrically convex and monotonically decreasing on $[a, b]$, we have

\[
\left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left( x - \frac{a+b}{2} \right) f'(x) \right| 
\leq \left( \frac{1}{3} \right)^{1-\frac{1}{q}} \left( \frac{(x-a)^3}{2(b-a)} \left( \int_0^1 t^2 \left[ |f''(x)|^t |f''(a)|^{1-t} \right]^q dt \right) \right)^{\frac{1}{q}} 
+ \frac{(b-x)^3}{2(b-a)} \left( \int_0^1 t^2 \left[ |f''(x)|^t |f''(b)|^{1-t} \right]^q dt \right)^{\frac{1}{q}}.
\]

By integration by parts, we have the inequality (2.5). □

**Corollary 2.** From Theorem 3 and Theorem 3, we have

\[
\left| \frac{1}{b-a} \int_a^b f(u) du - f(x) + \left( x - \frac{a+b}{2} \right) f'(x) \right| \leq \min \{v_1, v_2\}
\]

where

\[
v_1 = \frac{1}{(2p+1)\frac{1}{2}} \left\{ \frac{(x-a)^3}{2(b-a)} \left[ L \left( |f''(x)|^q, |f''(a)|^q \right) \right]^{\frac{1}{q}} + \frac{(b-x)^3}{2(b-a)} \left[ L \left( |f''(x)|^q, |f''(b)|^q \right) \right]^{\frac{1}{q}} \right\}
\]

and

\[
v_2 = \left( \frac{1}{3} \right)^{1-\frac{1}{q}} \left\{ \frac{(x-a)^3}{2(b-a)} \left( \frac{k - \frac{2k}{\ln k}}{(\ln k)^2} + \frac{2k}{\ln k} \right)^{\frac{1}{q}} + \frac{(b-x)^3}{2(b-a)} \left( \frac{l - \frac{2l}{(\ln l)^2} + \frac{2l}{(\ln l)^2}}{(\ln l)^2} \right)^{\frac{1}{q}} \right\}
\]

\[k = \frac{|f''(x)|^q}{|f''(a)|^q} \text{ and } l = \frac{|f''(x)|^q}{|f''(b)|^q}.
\]

**Theorem 4.** Let $f : I^0 \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be differentiable function on $I^0$, $a, b \in I$ with $a < b$ and $f' \in L_1[a, b]$. If $|f''|^q$ is geometrically convex and monotonically decreasing on $[a, b]$ and $t \in [0, 1]$, then we have the following inequality:

\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| 
\leq \frac{1}{(p+1)\frac{1}{2}} \left\{ \frac{(x-a)^2}{b-a} \left[ L \left( |f''(x)|^q, |f''(a)|^q \right) \right]^{\frac{1}{q}} + \frac{(b-x)^2}{b-a} \left[ L \left( |f''(x)|^q, |f''(b)|^q \right) \right]^{\frac{1}{q}} \right\}
\]

where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $L(\ , \ )$ is Logarithmic mean for real numbers.
Proof. From Lemma 2 with properties of absolute value and using the Hölder inequality, we obtain

\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right| \\
\leq \frac{(x-a)^2}{b-a} \int_0^1 |t| |f'(tx + (1-t)a)| dt + \frac{(b-x)^2}{b-a} \int_0^1 |t-1||f'(tx + (1-t)b)| dt \\
\leq \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
+ \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\
= \frac{1}{(p+1)^{\frac{1}{p}}} \left\{ \frac{(x-a)^2}{b-a} \left( \int_0^1 |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right. \\
+ \left. \frac{(b-x)^2}{b-a} \left( \int_0^1 |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right\}.
\]

Since \(|f|^q\) is geometrically convex and monotonically decreasing on \([a, b]\), we obtain

\[
x^t a^{1-t} \leq tx + (1-t)a \\
|f''(tx + (1-t)a)|^q \leq |f''(x^t a^{1-t})|^q.
\]

Therefore, we have

\[
K = \int_0^1 |f'(tx + (1-t)a)|^q dt \\
\leq \int_0^1 |f'(x^t a^{1-t})|^q dt \\
\leq \int_0^1 \left[ |f'(x)|^t |f'(a)|^{1-t} \right]^q dt \\
= L \left( |f'(x)|^q, |f'(a)|^q \right)
\]

and

\[
\int_0^1 (1-t)^p dt = \frac{1}{p+1}.
\]

This completes the proof.

Remark 1. In Theorem 4, if we take \(f(x) = \frac{(b-x)f(b) + (x-a)f(a)}{b-a}\), we obtain the inequality (1.2).

Corollary 3. Since \(\frac{1}{(p+1)^{\frac{1}{p}}} < 1\) for \(1 < p < \infty\), if we choose \(|f'(a)| = |f'(b)|\) in Theorem 4, we have

\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right| \\
\leq \frac{(x-a)^2 + (b-x)^2}{b-a} \left[ L \left( |f'(x)|^q, |f'(a)|^q \right) \right]^{\frac{1}{q}}.
\]
Theorem 5. Let $f : I \subset \mathbb{R} \to \mathbb{R}_+$ be differentiable function on $I^\circ$, $a, b \in I$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is geometrically convex and monotonically decreasing on $[a, b]$ for $q \geq 1$ and $t \in [0, 1]$, then the following inequality holds:

$$
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \left( \frac{1}{2} \right)^{1 - \frac{2}{q}} \left\{ \frac{(x-a)^2}{b-a} \left( \frac{k - \log k - 1}{(\log k)^2} \right)^{\frac{1}{q}} + \frac{(b-x)^2}{b-a} \left| f'(b) \right| \left( \frac{l - \log l - 1}{(\log l)^2} \right)^{\frac{1}{q}} \right\}
$$

where

$$
k = \frac{|f'(x)|^q}{f'(a)} \quad \text{and} \quad l = \frac{|f'(x)|^q}{f'(b)}.
$$

Proof. From Lemma 2 and using the well-known power-mean inequality, we have

$$
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{(x-a)^2}{b-a} \int_0^1 \left| f'(tx + (1-t)a) \right| dt + \frac{(b-x)^2}{b-a} \int_0^1 \left| f'(tx + (1-t)b) \right| dt
$$

$$
\leq \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t) dt \right)^{1 - \frac{2}{q}} \left( \int_0^1 (1-t) \left| f'(tx + (1-t)a) \right|^q dt \right)^{\frac{1}{q}}
$$

$$
+ \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t) dt \right)^{1 - \frac{2}{q}} \left( \int_0^1 (1-t) \left| f'(tx + (1-t)b) \right|^q dt \right)^{\frac{1}{q}}.
$$

Since $|f'|^q$ is geometrically convex and monotonically decreasing on $[a, b]$, we have

$$
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \left( \frac{1}{2} \right)^{1 - \frac{2}{q}} \left\{ \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t) \left| f'(x a^{1-t}) \right|^q dt \right)^{\frac{1}{q}}
$$

$$
+ \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t) \left| f'(x b^{1-t}) \right|^q dt \right)^{\frac{1}{q}} \right\}
$$

$$
\leq \left( \frac{1}{2} \right)^{1 - \frac{2}{q}} \left\{ \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t) \left( |f'(x)|^q |f'(a)|^{1-t} \right)^q dt \right)^{\frac{1}{q}}
$$

$$
+ \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t) \left( |f'(x)|^q |f'(b)|^{1-t} \right)^q dt \right)^{\frac{1}{q}} \right\}.
$$

This completes the proof. \qed
Corollary 4. Since $\frac{1}{2} < \left(\frac{1}{2}\right)^{1-\frac{1}{p}} < 1$, if we choose $|f'(a)| = |f'(b)|$ in Theorem 3, we have

$$\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \left( \frac{k - \log k - 1}{(\log k)^2} \right)^{\frac{1}{q}} |f'(a)| + \frac{(x-a)^2 + (b-x)^2}{b-a}.$$  

Corollary 5. From Theorem 4 and Theorem 5, again we have Corollary 5.

$$\left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left( x - \frac{a+b}{2} \right) f'(x) \right| \leq \min \{\eta_1, \eta_2\}$$

where

$$\eta_1 = \frac{1}{(p+1)^\frac{1}{p}} \left\{ \frac{(x-a)^2}{b-a} \left[ L \left( |f'(x)|^q, |f'(a)|^q \right) \right]^\frac{1}{q} + \frac{(b-x)^2}{b-a} \left[ L \left( |f'(x)|^q, |f'(b)|^q \right) \right]^\frac{1}{q} \right\}$$

and

$$\eta_2 = \left( \frac{1}{2} \right)^{1-\frac{1}{p}} \left\{ \frac{(x-a)^2}{b-a} |f'(a)| \left( \frac{k - \log k - 1}{(\log k)^2} \right)^{\frac{1}{q}} + \frac{(b-x)^2}{b-a} |f'(b)| \left( \frac{l - \log l - 1}{(\log l)^2} \right)^{\frac{1}{q}} \right\}$$

$k = \frac{|f'(x)|^q}{|f'(a)|^q}$ and $l = \frac{|f'(x)|^q}{|f'(b)|^q}$.

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