GROWTH OF HEAT TRACE COEFFICIENTS
FOR LOCALLY SYMMETRIC SPACES

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Abstract. We study the asymptotic behavior of the heat trace coefficients $a_n$ as $n \to \infty$ for the scalar Laplacian in the context of locally symmetric spaces. We show that if the Plancherel measure of a noncompact type symmetric space is polynomial, then these coefficients are $O\left(\frac{1}{n!}\right)$. On the other hand, for even dimensional locally rank 1-symmetric spaces, one has $|a_n| \approx C_n \cdot n!$; we conjecture this is the case in general if the associated Plancherel measure is not polynomial. These examples show that growth estimates conjectured by Berry and Howls [3] are sharp. We also construct examples of locally symmetric spaces which are not irreducible, which are not flat, and so that only a finite number of the $a_n$ are non-zero.

1. Introduction and Statement of Results

Let $\Delta_M$ be the Laplace-Beltrami operator of a compact connected Riemannian manifold $M := (M, g)$ without boundary of dimension $m \geq 2$. The fundamental solution of the heat equation, $e^{-t\Delta_M}$, is of trace class in $L^2$ and as $t \downarrow 0$ there is a complete asymptotic expansion with locally computable coefficients of the form:

$$\text{Tr}_{L^2}(e^{-t\Delta_M}) = (4\pi t)^{-m/2} \sum_{n=0}^{N} a_n(M)t^n + O(t^{-m/2+N}) \text{ for any } N.$$

These coefficients are well known; see, for example, the discussion in [1, 6] and the references therein. In particular $a_0(M) = \text{Vol}(M)$.

1.1. Growth estimates for heat trace asymptotics. Berry and Howls [3] examined the heat trace coefficients for a real analytic domain $\Omega$ in $\mathbb{R}^2$ where $D$ was the Dirichlet Laplacian and conjectured there were growth estimates of the nature $a_n(\Omega) \approx C(\Omega)^n \cdot n!$ in that context. Inspired by the work of Howls and Berry, Travêncel and Samaj [12] got similar factorial growth for the heat content asymptotics of real analytic domains in Euclidean space in certain settings.

We shall assume for the remainder of this paper that $M$ is a compact connected Riemannian manifold without boundary of dimension $m \geq 2$. The following result was established by van den Berg et. al [2] using the Seeley calculus [10, 11]:

Theorem 1.1. If $M$ is real analytic, then there exists a constant $C_1 = C_1(M)$ so that $|a_n(\Delta_M)| \leq C_1^n \cdot n! \cdot \text{Vol}(M)$ for any $n$.

The existence of manifolds where a similar lower bound held was left open in [2] and formed the initial focus of investigation for this paper; it will follow from Theorem 1.4 below that Theorem 1.1 is sharp in this regard. We note there are no universal growth estimates available in the smooth context. If $h \in C^\infty(M)$, let $M_h := (M, e^{2h}g)$ be the conformally adjusted Riemannian manifold. One has [2]:

Theorem 1.2. Let constants $C_n > 0$ for $n \geq 3$ be given. If $M$ is only assumed to be smooth, then there exists $h \in C^\infty(M)$ so that $|a_n(M_h)| \geq C_n$ for any $n \geq 3$. 

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1.2. Symmetric spaces. We shall assume henceforth that $\mathcal{M} = (M, g)$, is locally a symmetric space, i.e. that the covariant derivative $\nabla R$ of the curvature tensor vanishes. Consequently, $\mathcal{M}$ is real analytic and $\mathcal{M}$ is locally isometric to a quotient of the form $G/K$ for some suitable subgroup $K$ of a Lie group $G$. Thus $\mathcal{M}$ is locally homogeneous and we shall say that $\mathcal{M}$ is modeled on $G/K$. There is a discrete, cocompact subgroup $\Gamma$ of $G$ so that $\mathcal{M} = \Gamma \backslash G/K$. Let

$$A_n(\mathcal{M}) := \frac{a_n(\Delta \mathcal{M})}{\text{Vol}(\mathcal{M})}.$$  

With this normalization, $A_0(\mathcal{M}) = 1$ and $A_n(\mathcal{M})$ only depends on the local isometry type of $\mathcal{M}$ and is determined by the model, $G/K$. We consider the formal power series

$$\mathcal{H}_\mathcal{M}(t) := \sum_{n=0}^{\infty} A_n(\mathcal{M})t^n. \quad (1.a)$$

1.3. Rank 1-symmetric spaces. The complete simply connected rank 1-symmetric spaces are classified; see Proposition 3.1 for details. The only odd dimensional examples are modeled on the spheres and hyperbolic spaces. Before stating the next result, we define the following equivalence relation:

**Definition 1.3.** Let $\Xi := \{\Xi_n\}_{n \geq 0}$ and $\tilde{\Xi} := \{\tilde{\Xi}_n\}_{n \geq 0}$ be two infinite sequences of real numbers which are positive for $n$ large. We say that $\Xi \sim \tilde{\Xi}$ if given any $0 < \varepsilon < 1$, there exists $N(\varepsilon) \in \mathbb{N}$ so that

$$\Xi_n(1 - \varepsilon)^n < \tilde{\Xi}_n < \Xi_n(1 + \varepsilon)^n \quad \text{for} \quad n \geq N(\varepsilon).$$

We note that Definition 1.3 gives a fairly crude growth estimate since, for example, $\{14n^2 \cdot n!\} \approx \{n!\}$; thus multiplicative constants and finite powers of $n$ are suppressed; this will simplify the exposition. The following result will be proved in Section 3; it shows that Theorem 1.1 is sharp:

**Theorem 1.4.** Let $\mathcal{M}$ be modeled on an even dimensional irreducible rank 1 symmetric space. There exists $C = C(\mathcal{M}) > 0$ so that

$$\left\{|A_n(\mathcal{M})|\right\}_{n \geq 0} \approx \left\{C^n \cdot n!\right\}_{n \geq 0}.$$  

1.4. Properties of the heat trace asymptotics. We will establish the following result in Section 2. Assertion (1) will let us assume $\mathcal{M}$ is modeled on an irreducible symmetric space, Assertion (2) will permit us to rescale the metric, and Assertion (3) is the duality result of Cahn and Wolf [4] that will let us pass between models of non-compact type and models of compact type:

**Lemma 1.5.** Let $\mathcal{M} = (M, g)$ be modeled on a symmetric space.

1. If $\mathcal{M}$ is modeled on a product $\mathcal{M}_1 \times \ldots \times \mathcal{M}_k$ of symmetric spaces, then $H_\mathcal{M}(t) = H_{\mathcal{M}_1}(t) \cdots H_{\mathcal{M}_k}(t)$.

2. If $0 \neq c \in \mathbb{R}$, then $H_{(\mathcal{M}, c^2 g)}(t) = H_{(\mathcal{M}, g)}(c^2 t)$.

3. If $\mathcal{M}$ is modeled on a symmetric space of non-compact type and if $\tilde{\mathcal{M}}$ modeled on the dual symmetric space of compact type, then $H_{\tilde{\mathcal{M}}}(t) = H_{\mathcal{M}}(-t)$.

1.5. Locally symmetric spaces where the heat trace asymptotics decay. The Plancherel measure plays a crucial role in the analysis. Section 4 is devoted to the proof of the following result:

**Theorem 1.6.** Let $G/K$ be an irreducible symmetric space type of non-compact type and let $\hat{G}/\hat{K}$ be the associated dual symmetric space of compact type. Assume the Plancherel measure of $G$ is polynomial. Let $\mathcal{H}_G(t)$ be as in Equation (1.a). Then there exists $\kappa_{G/K} < 0$ and a polynomial $P_{G/K}(t)$ so that:
(1) If $\mathcal{M}$ is modeled on $G/K$, then $\mathcal{H}_\mathcal{M}(t) = e^{kG/Kt} \cdot P_{G/K}(t)$.
(2) If $\tilde{\mathcal{M}}$ is modeled on $\tilde{G}/\tilde{K}$, then $\mathcal{H}_{\tilde{\mathcal{M}}}(t) = e^{-kG/Kt} \cdot P_{G/K}(-t)$.

Proposition 4.1 will list the models which can occur in Theorem 1.6 and will give the associated Plancherel measures; we postpone the discussion until Section 4 to establish the requisite notation. The odd dimensional spheres and projective spaces are of the form given in Theorem 1.6 so this completes our discussion of manifolds which are modeled on the simply connected irreducible rank 1-symmetric spaces.

We can construct examples of non-flat manifolds so that $a_n(\mathcal{N}) = 0$ for $n$ large. We apply Lemma 1.5 to see:

**Corollary 1.7.** Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be as in Theorem 1.6. Set $\mathcal{N} := \mathcal{M} \times \tilde{\mathcal{M}}$; this is, of course, not irreducible. Then:

$$\mathcal{H}_\mathcal{N}(t) = e^{k_{\mathcal{M}}t}e^{-k_{\tilde{\mathcal{M}}}t}P_{\mathcal{M}}(t)P_{\mathcal{M}}(-t) = P_{\mathcal{M}}(t)P_{\mathcal{M}}(-t).$$

In particular, we could take $\mathcal{M} = \Gamma \setminus H^3$ to have constant curvature $-1$ and $\tilde{\mathcal{M}} = S^3$ to have constant curvature $+1$. We shall see presently that $H_{S^3}(t) = e^{t/4}$ and $H_{H^3}(t) = e^{-t/4}$. Consequently $H_\mathcal{N} = 1$, so $a_n(\mathcal{N}) = 0$ for $n \geq 1$.

Theorem 1.4 and Theorem 1.6 lead us to make the following:

**Conjecture 1.8.** Let $\mathcal{M}$ be modeled on an irreducible simply connected symmetric space $G/K$ of non-compact type. If the Plancherel measure of $G$ is not polynomial, then there exists $C = C(\mathcal{M})$ so that $|\mathcal{A}_n(\mathcal{M})| \geq C^n \cdot n!$ for $n$ sufficiently large.

### 2. Proof of Lemma 1.5

We have that $\Delta_{\mathcal{M}_1 \times \mathcal{M}_2} = \Delta_{\mathcal{M}_1} \oplus \Delta_{\mathcal{M}_2}$. Consequently:

$$\text{Tr}_{L^2} \left\{ e^{-t(\Delta_{\mathcal{M}_1 \times \mathcal{M}_2})} \right\} = \text{Tr}_{L^2} \left\{ e^{-t(\Delta_{\mathcal{M}_1} \oplus \Delta_{\mathcal{M}_2})} \right\} = \text{Tr}_{L^2} \left\{ e^{-t\Delta_{\mathcal{M}_1}} \right\} \cdot \text{Tr}_{L^2} \left\{ e^{-t\Delta_{\mathcal{M}_2}} \right\}.$$

Since $(4\pi t)^{-(m_1+m_2)/2} = (4\pi t)^{-m_1/2}(4\pi t)^{-m_2/2}$ and since $dx = dx_1 \cdot dx_2$, we may prove Lemma 1.5 (1) by equating terms in the asymptotic expansions. Because $\Delta_{(M,e^{-2g})} = c^2 \Delta_{(M,g)}$, we have that:

$$\text{Tr}_{L^2} \left\{ e^{-t\Delta_{(M,e^{-2g})}} \right\} = \text{Tr}_{L^2} \left\{ e^{-c^2 t\Delta_{(M,g)}} \right\}.$$

Since $(4\pi t)^{-m/2}dx_g = (4\pi c^2 t)^{-m/2}dx_{e^{-2g}}$, Lemma 1.5 (2) follows by equating terms in the asymptotic expansions.

The heat trace coefficients are given by integrating local invariants $a_n(\cdot, \Delta)$. These are invariant expressions which are homogeneous of order $2n$ in the derivatives of the metric or, equivalently, in the curvature tensor $R$, in the covariant derivative of the curvature tensor $\nabla R$, and so forth. If $\mathcal{M}$ is modeled on a symmetric space, then $\nabla^k R = 0$ for any $k > 0$ and thus $a_n(\cdot, \Delta) = a_n(R_{ijkl})$ is a polynomial which is homogeneous of degree $n$. Since $R_\mathcal{M} = -R_{\tilde{\mathcal{M}}}$ where $\tilde{\mathcal{M}}$ is modeled on the dual symmetric space, Lemma 1.5 (3) follows.

### 3. The proof of Theorem 1.4

The difference between $A_n$ and $a_n$, lies in the multiplicative constant $\text{Vol}(\mathcal{M})$. Since the equivalence relation of Definition 1.3 is not sensitive to multiplicative constants, we will work with $a_n$ rather than $A_n$. 

\[ \square \]
3.1. The classification of rank 1-symmetric spaces.

**Proposition 3.1.** Let \( M \) be an irreducible simply connected rank 1-symmetric space. Then \( M \) is, up to homothety, one of the following examples:

1. **Compact type:**
   - (a) The sphere \( S^m \) of radius 1 in \( \mathbb{R}^{m+1} \).
   - (b) The complex projective space \( \mathbb{C}P^m \) with the Fubini-Study metric.
   - (c) The quaternionic projective space \( \mathbb{H}P^m \) with the Fubini-Study metric.
   - (d) The Cayley projective plane \( \mathbb{O}P^2 \) with the canonical metric.

2. **Non-compact type:**
   - (a) The real hyperbolic space \( H^m \) of constant sectional curvature \(-1\). This is the non-compact type dual of the \( m \)-sphere.
   - (b) The complex hyperbolic space \( \mathbb{C}H^m \), non-compact type dual of the complex projective \( m \)-space.
   - (c) The quaternionic hyperbolic space \( \mathbb{H}H^m \), non-compact type dual of the quaternionic projective \( m \)-space.
   - (d) The Cayley hyperbolic plane \( \mathbb{O}H^2 \), non-compact type dual of the Cayley projective plane.

We may apply Lemma 1.5 (2) to see that the estimates of Theorem 1.4 are unchanged by homothety and therefore to assume that the curvature of \( M \) is standard. We use Lemma 1.5 (3) to assume \( M \) is of compact type. We will then proceed on a case by case basis to prove Theorem 1.4 using the classification of Proposition 3.1. Section 3.2 deals with the even dimensional spheres, Section 3.3 deals with the complex projective spaces, Section 3.4 deals with the quaternionic projective spaces, and Section 3.5 deals with the Cayley plane; the odd dimensional spheres are treated in Theorem 1.6. We shall use results of [5] in Section 3; results of [9] will play a prominent role in the analysis of Section 4.

3.2. **Even dimensional spheres.** Theorem 1.4 for the round sphere will follow in this case from the following result:

**Lemma 3.2.** Let \( M \) be the sphere of radius 1 in \( \mathbb{R}^{2m+1} \). Then

\[
\left\{ (-1)^{m-1}a_n(M) \right\}_{n \geq 0} \approx \left\{ \frac{(m - \frac{1}{2})^n}{\pi^{2n} \cdot 4^n \cdot n!} \right\}_{n \geq 0}.
\]

**Proof.** Recall that the Bernoulli numbers were expressed by Euler in terms of the Riemann zeta function in the form [13]:

\[
B_{2n} = (-1)^{n+1}2(2\pi)^{-2n}(2n)! \left\{ 1 + 2^{-2n} + 3^{-2n} + 4^{-2n} + \ldots \right\}
\]

Following [5] (see page 12) one defines

\[
c_n := (-1)^n(n + 1)^{-1}B_{2n+2}(1 - 2^{-2n-1}).
\]

Stirling’s formula [14] yields

\[
\left\{ n! \right\}_{n \geq 1} \approx \left\{ \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \right\}_{n \geq 1} \approx \left\{ \left( \frac{n}{e} \right)^n \right\}_{n \geq 1}.
\]

Combining these results permits us to compute:

\[
\left\{ c_n \approx (2\pi)^{-2n}(2n)! \right\}_{n \geq 0} \approx \left\{ (2\pi)^{-2n}2^{2n} \left( \frac{n}{e} \right)^n \cdot \left( \frac{n}{e} \right)^n \right\}_{n \geq 0} \approx \left\{ \pi^{-2n} \cdot n! \cdot n! \right\}_{n \geq 0}.
\]
Following [5] (see page 16), set \( \beta_{0,1} := 1 \) and for \( \bar{m} > 1 \) and for \( 0 \leq j \leq \bar{m} - 1 \), define constants \( \beta_{j,\bar{m}} \) to satisfy the identity:

\[
\prod_{k=1}^{2\bar{m}-2} \left( s + k - \bar{m} + \frac{1}{2} \right) = \prod_{j=\frac{1}{2}}^{\bar{m}-\frac{1}{2}} (s^2 - j^2) = \sum_{j=0}^{\bar{m}-1} \beta_{j,\bar{m}} s^{2j}
\]

where the product runs through the half integers that are not integers. We have:

\[
(-1)^{j+\bar{m}-1} \beta_{j,\bar{m}} > 0.
\]

Following [5] (see page 17), we may express \( a_n(\Delta_{\mathbb{S}^{2m},g}) \) for any \( n \geq \bar{m} \) in the form:

\[
a_n = \frac{(4\pi)^{\bar{m}} 4^{m-n}}{(2\bar{m}-1)!} \sum_{k=0}^{\bar{m}-1} \frac{(-1)^{n-\bar{m}}+2k+2k!}{(n-\bar{m}+k+1)!} \beta_{k,\bar{m}}
\]

\[
+ \frac{(4\pi)^{\bar{m}} 4^{m-n}}{(2\bar{m}-1)!} \sum_{k=0}^{\bar{m}} \sum_{j=0}^{\bar{m}-1} (-1)^j c_{j+k} \beta_{j,\bar{m}} \frac{\bar{m}-1}{k!(n-\bar{m}-k)!}.
\]

There are \( \bar{m} - 1 \) terms in the first sum and they tend to zero as \( n \to \infty \). They play no role in establishing either the lower bound or the upper bound. The terms in the double sum \( \{0 \leq k \leq n - \bar{m}, 0 \leq j \leq \bar{m} - 1\} \) all have the same sign and thus do not cancel; this is a crucial point. They are positive if \( \bar{m} \) is odd and negative if \( \bar{m} \) is even. We may bound \( k!(n-\bar{m}-k)! \leq n! \). On the other hand, if we take \( k = n - \bar{m} \) and \( j = \bar{m} - 1 \), then

\[
\frac{c_{n-1}}{(n-\bar{m})!(n-1)!} \geq \frac{c_{n-1}}{n!} \geq \pi^{-2n} n!.
\]

Consequently the growth of the terms in equation (3.a) is at least \( \pi^{-2n} n! \) and at most \( \kappa_m n^2 \pi^{-2n} n! \) where we bound the \( \beta_{j,\bar{m}} \) by \( \kappa_m \) for some universal constant \( \kappa_m \). The desired estimate now follows. \( \square \)

3.3. Complex projective space.

Lemma 3.3. Let \( \mathcal{M} \) be complex projective space \( \mathbb{CP}^{\bar{m}} \) where \( \bar{m} \geq 2 \). Then

\[
\left\{ (-1)^{\bar{m}-1} a_n(\mathcal{M}) \right\}_{n \geq 0} \approx \left\{ \pi^{-2n}(\bar{m} + 1)^n \cdot n! \right\}_{n \geq 0}.
\]

Proof. We follow the discussion in [5] pages 18-19. We distinguish two cases:

**Case I.** Let \( \bar{m} \) be odd. We define constants \( \gamma_{\ell,\bar{m}} \) using the relation:

\[
\prod_{k=1}^{\bar{m}-1} \left( s + k - \frac{\bar{m}}{2} \right)^2 = \prod_{j=\frac{1}{2}}^{\bar{m}-\frac{1}{2}} (s^2 - j^2)^2 = \sum_{\ell=0}^{\bar{m}-1} \gamma_{\ell,\bar{m}} s^{2\ell}.
\]

Again there is a parity constraint on these variables. Suppose we set \( \bar{s} := \sqrt{-1}s \). We would then have

\[
\prod_{j=\frac{1}{2}}^{\bar{m}-\frac{1}{2}} (\bar{s}^2 + j^2)^2 = \sum_{\ell=0}^{\bar{m}-1} \gamma_{\ell,\bar{m}} (-1)^{\ell} \bar{s}^{2\ell}.
\]

It is clear that the coefficients of \( \bar{s}^{2\ell} \) must all be positive; consequently

\[
(-1)^{\ell} \gamma_{\ell,\bar{m}} \geq 1 \quad \text{for} \quad 0 \leq \ell \leq \bar{m} - 1.
\]
Proof. We now consider the generating function

\[ \mathcal{M} = \sum_{j=0}^{\bar{m}-1} \frac{j! \cdot \gamma_{j,m}}{(n - \bar{m} + 2 + j)!} \left( \frac{2}{n} \right)^{2(n-\bar{m}+2+j)} \]

\[ + \frac{(4\pi)^{\bar{m}-1}(\bar{m}+1)^2 \bar{m}^{n-\bar{m}+1}}{\bar{m}!(\bar{m}-1)!} \sum_{k=0}^{\bar{m}-1} \frac{\bar{m}^{2k}}{4^{k}(\bar{m}+1)^{2k}} \]

\[ \times \sum_{j=0}^{\bar{m}-1} (-1)^j \gamma_{j,m} d_{n-\bar{m}+1-k+j} k!(n - \bar{m} + 1 - k)! . \]

The terms in the sum \( 0 \leq k \leq m - 2 \) tend to zero as \( n \to \infty \) and play no role. The terms in the sum \( \{0 \leq k \leq n - \bar{m} + 1, 0 \leq j \leq \bar{m} - 1\} \) are all positive and do not cancel. The dominant term arises when \( k = 0 \) and \( j = \bar{m} - 2 \). The desired estimate now follows exactly as in the case of the even dimensional spheres.

**Case II.** Let \( \bar{m} \) be even. We again consider the generating function:

\[ \prod_{k=1}^{\bar{m}-1} \left( s + k - \frac{\bar{m}}{2} \right) = \prod_{j=0}^{\bar{m}-1} \left( \frac{s^2 - j^2}{2} \right) = \sum_{k=0}^{\bar{m}-1} \gamma_{k,m} s^{2k} . \]

The same argument as that used in Case I shows that \( (-1)^{k-1} \gamma_{k,m} \geq 1 \) for all \( k \).

Following [5] (see page 14), we set

\[ d_n = (-1)^n B_{2n+2}/(n+1) . \]

If \( n \geq \bar{m} - 1 \), we have (see [5] page 19) that:

\[ a_n = \frac{(4\pi)^{\bar{m}-1}(\bar{m}+1)^2 \bar{m}^{n-\bar{m}+1}}{\bar{m}!(\bar{m}-1)!} \sum_{k=0}^{\bar{m}-1} \frac{k! \cdot \bar{m}^{2(n-\bar{m}+2+k)}}{(n - \bar{m} + 2 + k)!4^{n-\bar{m}+2+k}} \]

\[ + \frac{(4\pi)^{\bar{m}-1}(\bar{m}+1)^2 \bar{m}^{n-\bar{m}+1}}{\bar{m}!(\bar{m}-1)!} \sum_{k=0}^{\bar{m}-1} \frac{\bar{m}^2}{4^{k}(\bar{m}+1)^{2k}} \]

\[ \times \sum_{j=0}^{\bar{m}-1} (-1)^j (\bar{m}+1)^{-k} \gamma_{j,m} d_{n-\bar{m}+1-k+j} k!(n - \bar{m} + 1 - k)! . \]

As before, the terms in the first summation contribute nothing to the analysis. The terms in the second summation are all negative; the dominant term is obtained by taking \( k = 0 \) and \( j = \bar{m} - 1 \) to derive the desired lower bound. \( \Box \)

### 3.4. Quaternionic projective space.

**Lemma 3.4.** Let \( \mathcal{M} = \mathbb{HP}^m \) be quaternionic projective space where \( m \geq 2 \). Then

\[ \left\{ -a_n(\mathcal{M}) \right\}_{n \geq 0} \approx \left\{ \pi^{-2n} \cdot \kappa_n \right\}_{n \geq 0} . \]

**Proof.** We now consider the generating function

\[ \prod_{j=\bar{m}}^{\bar{m}+\frac{1}{2}} (s^2 - j^2) \cdot \prod_{j=1}^{\bar{m}} (s^2 - j^2) = \sum_{k=0}^{2\bar{m}-3} \delta_{k,m} s^{2k} . \]
If we set $\tilde{s} = \sqrt{-1}s$, we can rewrite this in the form:

$$\prod_{j=\frac{1}{2}}^{2\tilde{m}-3} (-\tilde{s}^2 - j^2) \cdot \prod_{j=\frac{1}{2}}^{2\tilde{m}} (-\tilde{s}^2 - j^2) = - \prod_{j=\frac{1}{2}}^{2\tilde{m}-3} (\tilde{s}^2 + j^2) \cdot \prod_{j=\frac{1}{2}}^{2\tilde{m}} (\tilde{s}^2 + j^2)$$

$$= \sum_{k=0}^{2\tilde{m}-3} \delta_{k,\tilde{m}} \tilde{s}^{2k} \quad \text{so} \quad (-1)^{k+1} \delta_{k,\tilde{m}} \geq 1.$$ 

We then have (see [5] page 20) for $n \geq 2\tilde{m} - 2$ that:

$$a_n(\mathcal{M}) = \frac{(4\pi)^{2\tilde{m}-2}}{(2\tilde{m}-1)!(2\tilde{m}-3)!} \sum_{k=0}^{2\tilde{m}-3} \left( \frac{(\tilde{m} - \frac{1}{2})^2}{2(\tilde{m}+1)} \right)^{2(n+2\tilde{m}-3-k)}$$

$$\times \frac{k!}{(n+2\tilde{m}-3-k)!} \delta_{k,\tilde{m}}$$

$$+ \frac{(4\pi)^{2\tilde{m}-2}}{(2\tilde{m}-1)!(2\tilde{m}-3)!} \sum_{k=0}^{n-2\tilde{m}+2} \frac{(\tilde{m} - \frac{1}{2})^{2k}}{2^k(\tilde{m}+1)^k}$$

$$\times \sum_{j=0}^{2\tilde{m}-3} (-1)^j \delta_{j,\tilde{m}} \frac{c_j+n-k}{k!(n-k)!}.$$

The terms in the first sum play no role; the terms in the double summation are all negative and thus do not cancel. We take $k = 0$ and $j = 2\tilde{m} - 3$ to obtain the desired estimate as before; this is the dominant term.

3.5. Cayley Plane.

**Lemma 3.5.** Let $\mathcal{M}$ be the Cayley plane $\mathbb{O}P^2$. Then $$\left\{ -a_n(\Delta_{\mathcal{M}}) \right\}_{n \geq 0} \approx \left\{ \pi^{-2n} \cdot n! \right\}_{n \geq 0}.$$ 

**Proof.** Following [5] (page 20), we define constants $\eta_i$ for $i = 0, 1, ..., 7$ by setting:

$$\eta_7 := 1, \quad \eta_6 := -\frac{170}{4}, \quad \eta_5 := \frac{10437}{16}, \quad \eta_4 := -\frac{262075}{64},$$

$$\eta_3 := \frac{2858418}{256}, \quad \eta_2 := -\frac{13020525}{1624}, \quad \eta_1 := \frac{18455239}{4096}, \quad \eta_0 := -\frac{8037225}{16384}.$$ 

The crucial point is that $(-1)^{i+1} \eta_i \geq 1$. For $n \geq 7$, one has [5] that:

$$a_n = \frac{3!}{7!!11!} (4\pi)^8 \sum_{k=0}^{7} \left( \frac{121}{72} \right)^{n+7-k} \frac{\eta_k k!}{(n+7-k)!}$$

$$+ \frac{3!}{7!!11!} (4\pi)^8 \sum_{k=0}^{n-8} \left( \frac{121}{72} \right)^k \sum_{j=0}^{7} (-1)^j \frac{\eta_j c_j+n-k}{k!(n-k)!}.$$ 

We argue as before to complete the proof of Lemma 3.5 and thereby also complete the proof of Theorem 1.4 as well.

4. Symmetric spaces with polynomial Plancherel measure

Lemma 1.5 (3) permits us to pass between $\mathcal{M}$ and the dual manifold $\mathcal{M}$. Thus it suffices to consider symmetric spaces of non-compact type to establish Theorem 1.6. The heat trace coefficients were determined in [9] for quite general operators of Laplace type acting on the space of smooth sections of a locally homogeneous vector bundle over an arbitrary locally symmetric space $\mathcal{M}$ of strictly negative curvature. These results were extended [7], in the spherical case, to all irreducible, non compact, symmetric spaces of higher rank and classical type. Let $\mathcal{M} = \Gamma \backslash G/K$ where $G$ is a non compact semi-simple Lie group, $K$ is a maximal compact subgroup.
and $\Gamma$ is a uniform lattice in $G$, that is, a discrete, co-compact subgroup of $G$. We also restrict to the scalar Laplacian although in principle these methods could treat the bundle Laplacian as well.

We adopt the following notational conventions. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of $G$ and $K$ respectively. We take a Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ of $\mathfrak{g}$; let $\theta$ be the Cartan involution. We fix a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. The Killing form $B_{\mathfrak{g}}(.,.)$ of $\mathfrak{g}$ induces an inner product on $\mathfrak{g}$ given by $\langle X, Y \rangle := -B_{\mathfrak{g}}(X, Y)$. We take the dual inner product on the dual space, $\mathfrak{a}^*$.

The Plancherel theorem and the Selberg trace formula play a main role in the proof of the results in [9] and in [7]. Let $\mu_G$ denote Plancherel measure of $G$. If $G$ has rank 1, then $\mu_G(\lambda) = p_G(\lambda)f_G(\lambda)$ where $p_G$ is a polynomial of degree $m - 1$ with $m = \dim(G/K)$ and $f_G(\lambda)$ is either 1, or $\tanh(\eta(\lambda))$ or $\coth(\eta(\lambda))$ where $\eta \in \mathfrak{a}^*$. For groups of arbitrary rank, $\mu_G$ is a product of Plancherel measures associated to rank one subgroups corresponding to each indivisible restricted root of $\mathfrak{g}$. We restrict to the case $f_G = 1$, i.e. $\mu_G$ is a polynomial function. We have:

**Proposition 4.1.** Let $G/K$ be a simply connected irreducible symmetric space of non-compact type where $\mu_G$ is a polynomial. Then $G/K$ is one of the following:

1. Let $\mathcal{M} = H^{2\bar{m} + 1} = \SO(2\bar{m} + 1, 1)/\SO(2\bar{m} + 1)$ so $G = \SO(2\bar{m} + 1, 1)$. Let $\lambda = \lambda_1\alpha$, and let $\alpha$ be the simple restricted root of $(\mathfrak{g}, \mathfrak{a})$. Then:
   $$p_G(\lambda) = C_G \prod_{0 \leq h \leq \bar{m}} \left( \lambda_1^2 + h^2 \right).$$

2. Let $\mathcal{M} = \SU^*(2\bar{m})/\Sp(\bar{m})$ so $G = \SU^*(2\bar{m})$. Adopt the notation of [7]. If $\lambda \in \mathfrak{a}^*$, then:
   $$p_G(\lambda) = C_G \prod_{1 \leq i < j \leq \bar{m} + 1} \left( \lambda - \lambda_j \right)^2 \left( (\lambda_i - \lambda_j)^2 + 1 \right).$$

3. Let $\mathcal{M} = E_6^{IV}/F_4$ so $G = E_6^{IV}$. If $\lambda \in \mathfrak{a}^*$, then:
   $$p_G(\lambda) = C_G \prod_{1 \leq i < j \leq 3} \prod_{0 \leq h \leq 3} \left( \lambda - \lambda_j \right)^2 \left( (\lambda_i - \lambda_j)^2 + h^2 \right).$$

4. Let $\mathcal{M} = G/G_u$ where $G$ is a complex simple Lie group looked on as a real Lie group and where $G_u$ is a compact real form of $G$. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ and let $\Delta^+$ be the set of positive roots of $\mathfrak{g}$. If $\lambda \in \mathfrak{a}^*$, then:
   $$p_G(\lambda) = C_G \prod_{\alpha \in \Delta^+} \left( \lambda + \rho, \alpha \right)^2.$$

**Remark 4.2.** Let $d_G$ be the degree of the polynomial $p_G(\lambda)$;

$$d_G = \dim(G/K) - \text{rank}(G/K).$$

In cases (1)-(4) above, $\deg p_G(\lambda)$ equals $2\bar{m} + 2$, $2\bar{m}(\bar{m} + 1)$, 24 and $\#\Delta^+$, respectively. In particular $d_G$ is always even.

Proposition 4.1 can be established by using the classification of real simple Lie algebras (see for instance [8] p. 518 and p. 532). The explicit form of the Plancherel measure follows by reduction to the rank one case, by using the Gindikin-Karpelevic formula. However we will not make use of the precise form of the polynomial in what follows. Theorem 1.6 will follow from the following result:

**Theorem 4.3.** If the Plancherel measure $\mu_G(\lambda)$ is polynomial, then there is a polynomial $P_M(t)$ of degree $\frac{m_\nu}{r}$ with $P_M(0) = 1$ so that $H_M(t) = e^{-t\langle \rho, \rho \rangle}P_M(t)$. Consequently,

$$\left\{ \left\langle a_n(\mathcal{M}) \right\rangle \right\}_{n \geq 0} \approx \left\{ \langle \rho, \rho \rangle^n \frac{1}{n!} \right\}_{n \geq 0}.$$
Proof. The approach (see [7] or [9]) can be summarized as follows. By using the Selberg trace formula, \( \text{Tr}_{L^2}\{e^{-t\Delta_M}\} \) can be expressed as a sum of orbital integrals of a function \( h_t \) on \( G \) defined by means of spherical inversion. Up to a multiplicative constant,

\[
h_t(x) = \int_{a^*} \phi_\lambda(x)e^{-t(\lambda, \lambda)+(\rho, \rho)} \mu_G(\lambda) \, d\lambda.
\]

Now, by the Selberg trace formula, one has that:

\[
\text{Tr}_{L^2}\{e^{-t\Delta}\} = \text{Vol}(\Gamma\backslash G/K)h_t(1) + \sum_{[\gamma] \in \Gamma'}\text{Vol}(\Gamma_\gamma \backslash G_\gamma) I_\gamma(h_t)
\]

where the sum is over the conjugacy classes of \( \Gamma, G_\gamma \) and \( \Gamma_\gamma \) are the centralizers of \( \gamma \) in \( G \) and \( \Gamma \) respectively, and \( I_\gamma(h_t) \) is the \( \gamma \)-orbital integral of \( h_t \). One first shows that the infinite sum in the right hand-side is asymptotic to 0 so it suffices to determine the asymptotic expansion of

\[
h_t(1) = e^{t(\rho, \rho)} \int_{a^*} e^{-t(\lambda, \lambda)} \mu_G(\lambda) \, d\lambda.
\]

Fix an orthonormal basis \( \{f_1, \ldots, f_r\} \) for \( a^* \). Expand

\[
p_G(\lambda) = p_G(\sum_{j=1}^r \lambda_j f_j) = \sum a_{i_1, \ldots, i_r} \lambda_{i_1}^{i_1} \ldots \lambda_{i_r}^{i_r}.
\]

If \( i_j \) is even we put \( i_j = 2h_j \) with \( h_j \in \mathbb{N}_0 \). Since \( \int_{\mathbb{R}} \lambda^h e^{-\lambda^2} \, d\lambda = \Gamma\left(\frac{h}{2} + \frac{1}{2}\right) \), we have that:

\[
e^{-t(\rho, \rho)} \int_{a^*} p_G(\lambda)e^{-t(\lambda, \lambda)} \, d\lambda
\]

\[
= e^{-t(\rho, \rho)} \sum_{i_1, \ldots, i_r \text{ even}} a_{i_1, \ldots, i_r} \prod_{1 \leq j \leq r} \int_{\mathbb{R}} \lambda_{i_j}^{i_j} e^{-t\lambda_{i_j}^2} \, d\lambda_j
\]

\[
= t^{-r/2} e^{-t(\rho, \rho)} \sum_{i_1, \ldots, i_r \text{ even}} a_{i_1, \ldots, i_r} \prod_{1 \leq j \leq r} t^{-i_j} \int_{\mathbb{R}} \lambda_{i_j}^{i_j} e^{-\lambda_{i_j}^2} \, d\lambda_j
\]

\[
= \pi^{\frac{r}{2}} t^{-m/2} e^{-t(\rho, \rho)} \sum_{h_1, \ldots, h_r} a_{2h_1, \ldots, 2h_r} \prod_{1 \leq j \leq r} \Gamma(h_j + \frac{1}{2}) t^{\frac{m-r}{2} - \sum h_j}
\]

\[
= \pi^{\frac{r}{2}} t^{-m/2} e^{-t(\rho, \rho)} \sum_{h=0}^{m-r} \left( \sum_{h_1 + \ldots + h_r = h} a_{2h_1, \ldots, 2h_r} \prod_{1 \leq j \leq r} \Gamma(h_j + \frac{1}{2}) \right) t^{h}
\]

By making the change of variables \( h' = \frac{m-r}{2} - h \), we may complete the proof by setting:

\[
P_M(t) := \pi^{\frac{r}{2}} \sum_{h=0}^{m-r} \left( \sum_{h_1 + \ldots + h_r = h} a_{2h_1, \ldots, 2h_r} \prod_{1 \leq j \leq r} \Gamma(h_j + \frac{1}{2}) \right) t^{h}. \quad \Box
\]

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