A pure Dirac’s method for Husain-Kuchar theory

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A pure Dirac’s canonical analysis, defined in the full phase space for the Husain-Kuchar model
is discussed in detail. This approach allows us to determine the extended action, the extended
Hamiltonian, the complete constraint algebra and the gauge transformations for all variables that
occur in the action principle. The complete set of constraints defined on the full phase space allow
us to calculate the Dirac algebra structure of the theory and a local weighted measure for the path
integral quantization method. Finally, we discuss briefly the necessary mathematical structure to
perform the canonical quantization program within the framework of the loop quantum gravity
approach.

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I. INTRODUCTION

The construction of a consistent quantum theory of gravity is a difficult task. It has been a long
way since the first quantized geometric models, including topological field theories, lower dimensional
gravity and minisuperspace homogeneous cosmological models [1, 2]. Loop quantum gravity (LQG)
has emerged in recent years as one of the most important candidates for describing the unification
between gravity and quantum mechanics. The theory has a mathematical rigorous basis for its
quantum kinematics [3] given by the measure defined in the configuration space [4]. It also has
achieved several promising physical results, at the theoretical level it provides a detailed microscopic
picture of black hole entropy and the big bang scenario in the context of homogeneous quantum
cosmologies. Nevertheless, the problem of the dynamics of the full theory has remained unsettled.
In this respect, there are open issues as for instance how general relativity (GR) arises from LQG as
a semiclassical limit of the quantum theory, and the fact that the algebra of quantum constraints,
though free of anomalies, does not correspond to the algebra of classical constraints completely [5].
Furthermore, other problems are centered about the Hamiltonian constraint which generates

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the dynamics of the system, unlike the gauge constraints associated with spatial diffeomorphisms and the internal gauge group remaining under control. Thus, the problem with the Hamiltonian constraint within the quantization procedure remains controversial. In this manner, to understand and solve the problems found in the quantization of gravity, has been necessary to study models with a close relation with GR, such is the case of gravity in three dimensions, topological field theories, $BF$ field theories, or an interesting model the so-called Husain-Kuchar (HK) model [8].

HK theory is a background independent theory, it share relevant symmetries used in the quantum regime of GR as for instance the diffeomorphisms covariance, however, it lacks of the hamiltonian constrain. Hence, HK model is a four-dimensional background-independent system with local degrees of freedom, describing equivalence classes of metrics in the spatial slices of a 3+1 foliation of space-time, although without time evolution [9]. In other words, it is like trying to obtain information from the whole space-time manifold by studying the embedded hypersurfaces. Even so, it is believed that a quantization of this kind of covariant field theories would shed light towards the quantization of GR. With respect to the hamiltonian analysis, HK theory is very close to 3+1 GR expressed in terms of Ashtekar variables [10]. In fact, the solutions to Einstein’s equations with SO(3) as the internal group, can be seen as a subset of the solutions to HK theory.

In the present letter, we perform a pure Dirac’s analysis to HK model. By this we mean that we will consider all the variables that occur in the Lagrangian density as dynamical variables and not only those ones that occur in the action with temporal derivatives [11, 12]. One could naively realize that a pure Dirac’s analysis is not mandatory, however it is not true at all. The approach developed in this work, is quite different to the standard Dirac’s analysis; this means that in agreement with the background independence structure that presents the theory under study, we will develop the Hamiltonian framework by considering all the fields defining our theory as dynamical ones; this fact will allow us to find the complete structure of the constraints, the equations of motion, gauge transformations, the extended action as well as the extended Hamiltonian. Generally, in theories just like GR we are able to realize that developing the Hamiltonian approach on a smaller phase space context, the structure obtained for the constraints is not right. In fact, we observe in [13] that the Hamiltonian constraint for Palatini theory does not has the required structure to form a closed algebra with all constraints; this problem emerges because by working on a smaller phase space context we do not have control on the constraints, in order to obtain the correct structure of them they have usually fixed by hand. Moreover, in three dimensional tetrad gravity, in despite of the existence of several articles performing the hamiltonian analysis on the reduced phase space, in some papers it is written that the gauge symmetry is Poincare symmetry [1, 14], in others that is Lorentz symmetry plus diffeomorphisms [15], or that there exist various ways to define the constraints leading to different gauge transformations. We think that a pure Dirac’s formalism is the best tool for solving these problems and for [HK] theory we have performed an a complete analysis. Nevertheless, by working with the full phase space it is possible to find the full set of constraints defined on the full phase space, allowing us to obtain for instance, a closed algebra among the constraints just as is required [11]. For these reasons, we believed that a complete Dirac’s approach
applied to HK theory would be useful for understanding the problems found in GR. Finally, it is worth mentioning that once the full set of constraints is calculated, this procedure could shed light on the search of observables in the context of covariant field theories, specifically in the case of strong-Dirac observables, which must be defined on the complete phase space and not on the reduced one. Finally, our approach will allow us determine the measure in the path integral quantization approach, all those ideas will be clarified along the paper.

The paper is organized as follows: In Section II, we perform the pure Dirac’s method for the HK model, and we find the extended Hamiltonian, the extended action, the complete constraint algebra, the full phase space gauge transformations, the Dirac bracket structure and the path integral measure. In Section III we discuss on remarks and conclusions.

II. A PURE DIRAC’S METHOD FOR THE HUSAIN-KUCHAR MODEL

In this section, we carry out a pure Dirac’s analysis. As was commented above, the approach developed in this work consists in consider that all the variables that occurs in the action will be treated as dynamical ones, of course, our approach is fully in agreement with the background independence of the theory because all the fields that define our theory are fully dynamical, none are fixed.

We start with the following action principle \[ S[e, A] = \int_\mathcal{M} \epsilon_{IJK} e^I \wedge e^J \wedge F^K[A], \] (1)

where \( F^I[A] = dA^I + \epsilon^I_{JK} A^J \wedge A^K \) is the curvature of the \( su(2) \)-valued connection 1-form \( A^I = A^I_a dx^a \), which defines a covariant derivative acting on the gauge group \( D_a \lambda_b^I = \partial_a \lambda_b^I + \epsilon^I_{JK} A^J_a e^K_b \), and \( e^I = e^I_I dx^\mu \) is a \( su(2) \)-valued 1-form field. Here \( \mathcal{M} \), is a four dimensional manifold \( \mathcal{M} = \mathbb{R} \times \Sigma \), with \( \Sigma \) (which we take to be compact and without boundary) corresponding to Cauchy’s surfaces and \( \mathbb{R} \) representing an evolution parameter. Capital letters indices represent the internal space, greek indices are spacetime indices running over 0,1,2,3, and \( x^\mu \) are the coordinates that label the points on the manifold \( \mathcal{M} \). The internal group manifold carries an \( su(2) \)-invariant metric, \( \delta_{IJ} \), and a volume element given by the Levi-Civita tensor, \( \epsilon_{IJK} \). It is important to remark that there exist alternative proposals to action \[ S[e, A] = \int_\mathcal{M} \eta^{ijk} \epsilon_{IJK} [e^I_i e^J_j F^K_{0k} + e^I_i e^J_j F^K_{ik}], \] (2)

In this sense, both \( e \) and \( A \) are considered as dynamical variables and we will take this fact into account below. By performing the 3 + 1 decomposition, we can write the action as

\[ S[e, A] = \int_{\mathbb{R} \times \Sigma} \eta^{ijk} \epsilon_{IJK} [e^I_i e^J_j F^K_{0k} + e^I_i e^J_j F^K_{ik}], \] (3)
where $\eta^{ijk} = \epsilon^{0ijk}$. From this action, we identify the Lagrangian density

$$\mathcal{L} = \epsilon_{IJK} \left[ \epsilon_0^i e_j^j F_{jk}^K + \epsilon_j^i F_{0k}^K \right].$$

(4)

By determining the set of dynamical variables, the application of the pure Dirac’s method calls for the definition of the momenta $(p_A^\alpha, \pi_A^\alpha)$,

$$p_A^\alpha = \frac{\delta \mathcal{L}}{\delta \dot{e}^A_\alpha}, \quad \pi_A^\alpha = \frac{\delta \mathcal{L}}{\delta \dot{A}_A^\alpha},$$

(5)

canonically conjugate to $(e_A^A, A_A^A)$. The matrix elements of the Hessian,

$$\frac{\partial^2 \mathcal{L}}{\partial (\partial_\mu A_A^\alpha) \partial (\partial_\mu A_B^\beta)}, \quad \frac{\partial^2 \mathcal{L}}{\partial (\partial_\mu e_A^\alpha) \partial (\partial_\mu e_B^\beta)}, \quad \frac{\partial^2 \mathcal{L}}{\partial (\partial_\mu e_A^\alpha) \partial (\partial_\mu e_B^\beta)},$$

(6)

vanish, which means that the rank of the Hessian is equal to zero, so that, 24 primary constraints are expected. From the definition of the momenta, it is possible to identify the following 24 primary constraints:

$$\phi^0_A : p^0_A \approx 0,$$

$$\phi^a_A : p^a_A \approx 0,$$

$$\psi^0_A : \pi^0_A \approx 0,$$

$$\psi^a_A : \pi^a_A - \eta^{abc} \epsilon_{ABC} e^B_e e^C_e \approx 0.$$  

(7)

By neglecting the terms on the frontier, the canonical Hamiltonian for the HK model is expressed as

$$H_c = - \int_{\Sigma} dx^3 \left[ A_0^J D_i \pi^i_J + \eta^{ijk} \epsilon_{IJK} \epsilon_0^i e_j^j F_{jk}^K \right].$$

(8)

By adding the primary constraints to the canonical Hamiltonian, we obtain the primary Hamiltonian

$$H_P = H_c + \int_{\Sigma} dx^3 \left[ \lambda_0^0 \phi^0_0 + \lambda^i_0 \phi^i_0 + \gamma_0^I \psi^0_I + \gamma_i^I \psi^i_I \right],$$

(9)

where $\lambda_0^0$, $\lambda^i_0$, $\gamma_0^I$ and $\gamma_i^I$ are Lagrange multipliers enforcing the constraints. The non-vanishing fundamental Poisson brackets for the theory under study are given by

$$\left\{ e_A^\alpha(x^0, x), p^\beta_I(y^0, y) \right\} = \delta^\alpha_\beta \delta^A_I \delta^3(x, y),$$

$$\left\{ A_A^\alpha(x^0, x), \pi^\beta_I(y^0, y) \right\} = \delta^\alpha_\beta \delta^A_I \delta^3(x, y).$$

(10)

Now, we need to identify if the theory has secondary constraints. For this aim, we compute the $24 \times 24$ matrix whose entries are the Poisson brackets among the primary constraints.
\[
\begin{align*}
\{\phi_A^0(x), \phi_A^0(y)\} &= 0, \quad \{\phi_A^\alpha(x), \phi_A^0(y)\} = 0 \\
\{\phi_A^\alpha(x), \phi_I^\beta(y)\} &= 0, \quad \{\phi_A^\alpha(x), \psi_I^\beta(y)\} = 0, \\
\{\psi_A^\alpha(x), \phi_I^\beta(y)\} &= 0, \quad \{\psi_A^\alpha(x), \psi_I^\beta(y)\} = 0, \\
\{\dot{\phi}_A^0(x), \phi_I^\beta(y)\} &= 0, \quad \{\dot{\phi}_A^\alpha(x), \phi_I^\beta(y)\} = 2\eta^{\alpha\beta} \epsilon_{AIK} e^K(y) \delta(x,y), \\
\{\dot{\phi}_A^\alpha(x), \psi_I^\beta(y)\} &= 0, \quad \{\dot{\psi}_A^\alpha(x), \phi_I^\beta(y)\} = 0, \\
\{\dot{\psi}_A^\alpha(x), \psi_I^\beta(y)\} &= 0, \quad \{\dot{\psi}_A^\alpha(x), \psi_I^\beta(y)\} = 0.
\end{align*}
\]

This matrix has rank=18 and 6 linearly independent null-vectors, which implies that there are 6 secondary constraints. By requiring consistency of the temporal evolution of the constraints and using the 6 null vectors, the following 6 secondary constraints arise

\[
\begin{align*}
\dot{\phi}_A^0 &= \{\phi_A^0, H_P\} \approx 0 \quad \Rightarrow \quad F_A := \eta^{\alpha\beta} \epsilon_{A[K} e^K_{i\beta]} \approx 0, \\
\dot{\psi}_A^\alpha &= \{\psi_A^\alpha, H_P\} \approx 0 \quad \Rightarrow \quad G_A := D_a \pi_A^a \approx 0,
\end{align*}
\]

and the following Lagrange multipliers are fixed

\[
\begin{align*}
\dot{\phi}_A^\alpha &= \{\phi_A^\alpha, H_P\} \approx 0 \quad \Rightarrow \quad \eta^{\alpha\beta} \epsilon_{A[K} \left( e^K_{i\beta]} F^i_l \gamma^l - \eta^{\ell K} e^K_{i\ell} \right) = 0, \\
\dot{\psi}_A^\alpha &= \{\psi_A^\alpha, H_P\} \approx 0 \quad \Rightarrow \quad 2\eta^{\alpha\gamma} \epsilon_{A[I} \left( D_k \left( e^K_{0} e^I_0 \right) - \lambda^I_{\alpha} \right) - \epsilon^J_{AI} A^I_0 \pi^a_A = 0.
\end{align*}
\]

This theory does not have tertiary constraints. By following the study, we determine which ones constraints are first class and which are second class. To accomplish such a task we calculate the Poisson brackets between the primary and secondary constraints. To complete the constraint matrix, we add to the algebra shown in Eq. (11) the following expressions

\[
\begin{align*}
\{\phi_A^0(x), G_I(y)\} &= 0, \quad \{\phi_A^\alpha(x), G_I(y)\} = 0, \\
\{\phi_A^\alpha(x), F_I(y)\} &= 0, \quad \{\phi_A^\alpha(x), F_I(y)\} = 0, \\
\{\psi_A^\alpha(x), G_I(y)\} &= 0, \quad \{\psi_A^\alpha(x), G_I(y)\} = \epsilon^K_{AI} \pi^K_A, \\
\{\psi_A^\alpha(x), F_I(y)\} &= 0, \quad \{\psi_A^\alpha(x), \phi_I^\beta(y)\} = -2\eta^{\alpha\beta} \left[ \epsilon_{A[I} e^K_{I} \delta(y) + \epsilon_{AKM} e^K_{I} A^M_{\alpha I} \right] \delta(x,y), \\
\{F_A(x), F_I(y)\} &= 0, \quad \{G_A(x), F_I(y)\} = \epsilon^K_{AI} F^C = 0, \\
\{G_A(x), G_I(y)\} &= \epsilon^K_{AI} G^C = 0.
\end{align*}
\]

The matrix formed by the Poisson brackets between all the constraints exhibited in Eqs. (11) and
(14) has rank=18 and 12 null-vectors. The contraction of the null vectors with the matrix formed by the constraints, one obtains the following 12 first class constraints

\[ \Phi^0_A : p^0_A, \]
\[ \Psi^0_A : \pi^a_A, \]
\[ G_A : D_a \pi^a_A + \epsilon_{AB} C^c \epsilon^a_A p^c_C, \]
\[ F_A : \epsilon_{ABC} \eta^{abc} \epsilon^B A^c + D_a p^a_A. \]  

(15)

On the other hand, the rank allow us to find the following 18 second class constraints

\[ \chi^a_A : p^a_A \approx 0, \]
\[ \xi^a_A : \pi^a_A - \eta^{abc} \epsilon_{ABC} \epsilon^B A^c \approx 0. \]  

(16)

The correct identification of the constraints is a very important step because they are used to carry out the counting of the physical degrees of freedom and to identify the gauge transformations if there exist first class constraints. On the other hand, the constraints are the guideline to make the best progress for the quantization of the theory. Hence, the counting of degrees of freedom is carry out follows: there are 48 canonical variables, 12 independent first class constraints and 18 independent second class constraints, which leads to determine, that theory under study has 3 degrees of freedom per space-time point. Of course, by considering the second class constraints Eq. (10) as strong equations, the above relations are reduced to the usual constraints obtained in [8], so this analysis extends and completes those results found in the literature.

By calculating the algebra among the constraints, we find that

\[ \{ \Phi^0_A (x), \Phi^0_B (y) \} = 0, \]
\[ \{ \chi^a_A (x), \chi^a_B (y) \} = 0, \]
\[ \{ \Phi^0_A (x), \chi^i_B (y) \} = 0, \]
\[ \{ \chi^a_A (x), \Psi^0_B (y) \} = 0, \]
\[ \{ \Phi^0_A (x), \xi^i_B (y) \} = 0, \]
\[ \{ \chi^a_A (x), \xi^i_B (y) \} = 2 \eta^{aik} \epsilon_{AIK} \epsilon^K (y) \delta (x, y), \]
\[ \{ \Phi^0_A (x), \xi^i_B (y) \} = 0, \]
\[ \{ \Psi^0_A (x), \Psi^0_B (y) \} = 0, \]
\[ \{ \xi^a_A (x), G^a_B (y) \} = \epsilon_{AIK} C^a \approx 0, \]
\[ \{ \xi^a_A (x), G^i_B (y) \} = \epsilon_{AIK} C^a \approx 0, \]
\[ \{ \Phi^0_A (x), F^i_B (y) \} = 0, \]
\[ \{ \chi^a_A (x), G^i_B (y) \} = \epsilon_{AIK} C^a \approx 0, \]
\[ \{ \Phi^0_A (x), G^i_B (y) \} = 0, \]
\[ \{ \chi^a_A (x), F^i_B (y) \} = F^i_A p^a + \chi^i_A D_i p^a \approx 0, \]
\[ \{ \Phi^0_A (x), G^i_B (y) \} = 0, \]
\[ \{ \xi^a_A (x), F^i_B (y) \} = G^i_A p^a - \epsilon_{IB} C^a \epsilon^a_B \epsilon^B A^c \approx 0, \]
\[ \{ \Psi^0_A (x), F^i_B (y) \} = 0, \]
\[ \{ G^a_A (x), F^i_B (y) \} = \epsilon_{AIK} F^i C^a \approx 0, \]
\[ \{ F^a_A (x), F^i_B (y) \} = 0, \]

(17)

\[ \{ G^a_A (x), G^i_B (y) \} = \epsilon_{AIK} F^i C^a \approx 0, \]

\[ \{ F^a_A (x), F^i_B (y) \} = 0, \]

\[ \{ G^a_A (x), G^i_B (y) \} = \epsilon_{AIK} F^i C^a \approx 0, \]

\[ \{ F^a_A (x), F^i_B (y) \} = 0, \]

(16)

\[ \{ G^a_A (x), G^i_B (y) \} = \epsilon_{AIK} F^i C^a \approx 0, \]

\[ \{ F^a_A (x), F^i_B (y) \} = 0, \]

(17)

\[ \{ G^a_A (x), G^i_B (y) \} = \epsilon_{AIK} F^i C^a \approx 0, \]

\[ \{ F^a_A (x), F^i_B (y) \} = 0, \]

(17)

1 The null vector space generated by the complete constraint hypersurface is a subset of a quotient vector space \( \mathbb{R}^{30}/G \), where \( G \) is the set given by all the primary and secondary constraints.
from where we appreciate that the constraints form a set of first and second class constraints, as is expected. The obtention of the constraints defined on the full phase space, will allow us to find the extended action. By employing the first class constraints \([15]\), the second class constraints \([16]\) and the Lagrange multipliers \([13]\), we find that the extended action takes the form

\[
S_E\{e^A, p_A^\alpha, A^A, \pi_A^\alpha, \lambda^A, \gamma^A, \lambda_0^A, \gamma_0^A, \lambda^A, \gamma^A\} = \int \left[ \dot{e}^A p_A^\alpha + \dot{A}^A \pi_A^\alpha - H - \lambda_0^A \Phi_0^0 - \gamma_0^A \Psi_0^0 - \lambda^A G_A - \gamma^A F_A \right] - \gamma^A F_A - \lambda_0^A \lambda^A - \gamma_0^A \gamma^A |d^3 x|
\]

where \(H\) is a linear combination of first class constraints, and is given by

\[
H = A_0^A \left[ D_0 \pi_0^\alpha + \epsilon_{AB}^C e_0^B p_0^C \right] + \epsilon_0^A \left[ \epsilon_{ABC} \eta^{abc} e_0^B F_0^C + D_0 p_0^A \right]
\]  

(18)

and \(\lambda_0^A, \lambda^A, \gamma_0^A, \gamma^A, \lambda^A, \gamma^A\) are the Lagrange multipliers enforcing the first and second class constraints respectively. We are able to observe, by considering the second class constraints as strong equations, that the Hamiltonian \([18]\) is reduced to the usual expression found in the literature \([8]\), which is defined on a reduced phase space context. From the extended action, we identify the extended Hamiltonian which is given by

\[
H_E = H - \lambda_0^A \Phi_0^0 - \gamma_0^A \Psi_0^0 - \lambda^A G_A - \gamma^A F_A.
\]  

(19)

By using our expressions for the complete set of constraints, it is possible to obtain the gauge transformations acting on the full phase space. For this important step, we shall use Castellani’s formalism \([21]\), which allows us to define the following gauge generator in terms of the first class constraints:

\[
G = \int \Sigma \left[ D_0 \epsilon_0^A p_A^0 + D_0 \zeta_0^A + \epsilon^A G_A + \zeta^A F_A \right] d^3 x
\]  

(20)

where \(\epsilon_0^A, \epsilon^A, \zeta_0^A\) and \(\zeta^A\) are arbitrary continuum real parameters. Thus, we find that the gauge transformations in the phase space are

\[
\begin{align*}
\delta_0 e^I_0 &= D_0 e^I_0, \\
\delta_0 e^I_i &= \epsilon^I AB \bar{e}^A e^B_i - D_i \zeta^I, \\
\delta_0 A^I_0 &= D_0 \zeta_0^A, \\
\delta_0 A^I_i &= -D_i \epsilon^I, \\
\delta_0 p_0^I &= 0, \\
\delta_0 p_0^I_i &= \epsilon_1 A^C \left( \epsilon^A p^i_C + \eta^{abc} \zeta^A F_{bc}^C \right), \\
\delta_0 \pi_0^I &= 0, \\
\delta_0 \pi_0^I_i &= \epsilon_1 A^C \left( \epsilon^A \pi^i_C - 2\eta^{abc} D_b \left( \zeta^A e_{C_a} \right) - \zeta^A p^i_C \right).
\end{align*}
\]  

(21)

In order to recover the diffeomorphisms symmetry, one can redefine the gauge parameters as \(-\epsilon^I_0 = \epsilon^I = -v^\alpha A^I_\alpha\), and \(-\zeta^I_i = \zeta^I = -v^\alpha e_\alpha^I\). With this election, the gauge transformations take the form

\[
\begin{align*}
\epsilon^I_\alpha &\rightarrow \epsilon^I_\alpha + L_v e^I_\alpha + D_{[\alpha}^I e_{\mu]}^I v^\mu, \\
A^I_\alpha &\rightarrow A^I_\alpha + L_v A^I_\alpha + v^\mu F_{\mu \alpha}^I,
\end{align*}
\]  

(22)
corresponding to diffeomorphism gauge invariance \[8, 22\]. Some interesting features follow from the
gauge orbits. There exist a vector density, namely \(n^\alpha = \dot{n}^\alpha/\dot{e}\) \[22\], where \(\dot{e}\) represents an auxiliary
foliation defined by a scalar function \(t\) as \(\dot{e} = \dot{n}^\alpha \partial_\alpha t\). The flow lines of \(n^\alpha\) define a privileged reference
frame through \(\dot{\alpha}^\alpha = \frac{1}{\pi} \epsilon^{\alpha\beta\mu\nu} \epsilon_{IJK} e^I_\beta e^J_\mu e^K_\nu\). Then, by using \(\xi^\alpha\) \[22\], it is possible to notice that, in fact,
there is no dynamics in the model. The projections of the field equations onto the direction normal
to the spatial slices are zero. One can observe, that the space-time (degenerate) metric does not
change. In other words, every transverse hypersurface has the same intrinsic geometry, which means
that every solution to GR is a solution of the HK model.

Classically, it is possible to write down explicitly an infinite number of constants of motion, since the
Hamiltonian constraint vanishes identically. This implies that the theory is a Liouville integrable covariant field
theory with local degrees of freedom. On the other hand, the analysis developed above, shows that
the geometry is a constant of motion. This implies that the theory is a Liouville integrable covariant field
theory with local degrees of freedom. On the other hand, the analysis developed above, shows that
the theory is dimensional reduced, so that, the dynamical evolution of the system is determined by
the spatial diffeomorphisms. From Eq. \(\text{(20)}\), the complete spatial diffeomorphism generator acting
in the full phase space is given by

\[
H_i = A_i^A \left( D_a \pi^a_A + \epsilon_{AB} C^B e^a_B p^a_c \right) + \epsilon^a_i \left( \epsilon_{ABC} \eta^{abc} e^B_a F^C_e + D_a p^a_A \right). \tag{23}
\]

In order to recovering the usual spatial diffeomorphism generator reported in \[8\], it is simple to see
that once the second class constraints \(10\) are solved, there exists an homomorphism between the
Poisson algebra \(17\), and the usual constraint algebra reported in \[8\], thorough a dreibein field.

From the constraint analysis, it is important to determine the Dirac brackets among our canonical
variables. We need to remember that after Dirac's brackets are constructed, second class constraints
can be treated as strong equations \[20\], hence it is an important step in our calculations. For this
aim, let \(D\) the matrix formed by the Poisson brackets between the second class constraints

\[
D_{\alpha\beta} = \begin{pmatrix}
0 & \{\xi_i^A(x), \chi^A(y)\} \\
\{\chi^A(x), \xi_i^A(y)\} & 0
\end{pmatrix}. \tag{24}
\]

The Dirac bracket between two phase space functionals \(F\) and \(G\) is defined as

\[
\{F(x), G(y)\}_D \equiv \{F(x), G(y)\} - \int du dv \{A(x), \Psi^\alpha(u)\} D^{-1}_{\alpha\beta} \{\Psi^\beta(v), G(y)\}, \tag{25}
\]

where \(\{F(x), G(y)\}\) is the usual Poisson bracket between the functionals \(F\) and \(G\), \(\Psi^\alpha = (\chi^A, \xi^A)\)
and \(D^{-1}\) are the components of the inverse of the matrix \(D\), which take the values, \(D^{-1}_{ai} = \frac{1}{2\pi} (\frac{1}{2} e^A_a e^i_l - e^A_i e^i_l)\). By using the definition of the Dirac bracket, the non-trivial canonical relations,
\text{i.e.} those phase space variables that do not Poisson commute with the second class constraints, are
given by

\[
\{e^A_a, \pi_i^A\}_D = 0, \quad \{A^A_a, \pi_i^A\}_D = \delta_i^A \delta^A_a, \tag{26}
\]

\[
\{e^A_a, p^A_i\}_D = 0, \quad \{A^A_a, e_i^A\}_D = D^{-1}_{ai} A^A_i. \tag{26}
\]
Briefly, we turn now to the observables issue. An observable in a theory with first class and second class constraints is defined to be a phase space function whose Dirac’s brackets commutes with all the first class constraints. In this respect, in the HK model there are not scalar constraint so that, the observables will be those phase space functionals whose Dirac’s brackets commute with Gauss and spatial diffeomorphisms constraints.

We will finish this section with some comments about the path integral quantization procedure. An important aspect for defining a path integral quantum theory, is the determination of the correct measure. For theories with constraints and some interacting theories, this aim is non-trivial to obtain, and usually is not given by the heuristic Lebesgue measure. The obtention of the measure, has been an relevant step in developing of spin-foam models, which can be thought of as a path integral version of LQG 5, 26. To cut a long story short (see e.g. 20, 27), the central ingredient for most applications of the path integral is the generating functional, which in our case takes the form

\[ Z = \int \mathcal{D}A_{\mu}^{I} \mathcal{D}\pi_{I}^{\mu} \mathcal{D}c_{I}^{\mu} \mathcal{D}p_{J}^{\mu} \delta(\chi_{A}^{I}(x)) \delta(\xi_{J}^{I}(y)) \sqrt{|D|} \delta(\Phi_{A}^{I}) \delta(\Psi_{A}^{I}) \delta(G_{A}) \delta(F_{A}) \sqrt{|E|} \prod_{\alpha} \delta(\zeta_{\alpha}) \exp i \int dt d^{3}x \pi_{I}^{\mu} \dot{A}_{\mu}^{I} + p_{I}^{\mu} \dot{c}_{\mu}^{I}, \]

(27)

here \( D \) denotes the determinant of the matrix given in (24), \( \zeta_{\alpha} \) any choice of gauge fixing conditions, \( E \) is the square of the determinant of the Poisson brackets between first class constraints and the gauge fixing conditions. In addition \( \mathcal{D}q = \prod_{\alpha} dq(t) \) for all phase space variables. We will drop the exponential of the current in what follows, since it does not affect any of our manipulations, hence we will deal with the partition function \( Z = Z[0] \). At the end we will be really interested in \( Z[j]/Z \), then we can drop overall constant factors from all subsequent formulas. Moreover, we will assume for simplicity that all the gauge fixing conditions are functions independent of the connection \( A_{\mu}^{I} \). In the equation (27), \( D \) is the determinant of the Dirac matrix, equation (24), formed by the second class constraints. Therefore \( |D| = |\det\{\chi_{A}^{I}(x), \xi_{J}^{I}(y)\}|^{2} \), let \( C_{A}^{IJ} \) denote this matrix. From the singular value decomposition theorem 28, there exist orthogonal matrices \( O_{a}^{I}, O_{j}^{I} \) such that \( O_{a}^{I}O_{j}^{I}e_{b}^{I} \) is diagonal, that is \( O_{a}^{I}O_{j}^{I}e_{b}^{I} = \lambda_{b}\delta_{a}^{I} \). Let \( O_{b}^{I} = O_{a}^{I}O_{b}^{I} \), also an orthonormal matrix, we use it to define

\[ \hat{C}_{IJ}^{ab} = O_{a}^{I}C_{IJL}^{abc}O_{b}^{I} = \sum_{c} \eta^{abc} \epsilon_{IJc} \lambda_{c}. \]

(28)

Then \( \hat{C}_{IJ}^{ab} = 0 \) when \( (I = J) \) or \( (a = b) \) or \( \{a, b\} \neq \{I, J\} \). Reducing by minors we obtain

\[ \det C_{IJ}^{ab} = 2(\lambda_{1}\lambda_{2}\lambda_{3})^{3} = 2(\det e_{I}^{I})^{3} = 2e^{3}, \]

(29)

thus, up to an overall factor, \( \sqrt{|D|} = e^{3} = V^{3} \). It is so difficult to perform the integrations of equation (27) in order to compute transition amplitudes. However if we transform the integral of the HK Lagrangian in terms of the configuration variables, it would be easy to handle. So, by using the reduce phase space technique 20, integrating the second class constraints (they are in fact, primary second class constraints) and taking into account the bijection between \( \pi_{I}^{I} \) and \( e_{I}^{I} \) when \( \det e_{I}^{I} \neq 0 \). In terms of the extended Hamiltonian (19), it is straightforward to obtain

\[ Z = \int \mathcal{D}A_{\mu}^{I} \mathcal{D}c_{I}^{\mu} V^{6} \sqrt{|E|} \prod_{\alpha} \delta(\zeta_{\alpha}) \exp i \int \epsilon_{IJK} e_{I} \wedge e_{J} \wedge F^{K}[A]. \]

(30)
It would be desirable to follow the usual way by employing perturbation theory. However, there is no expansion that disentangle the free theory from the interaction term. Then, it is not possible to construct a perturbative quantum theory in the usual way. However, Spin foams intend to be a path integral formulation of LQG and theories covariant under diffeomorphisms [26], mainly motivated from Feynman’s ideas but appropriately suited to background independence symmetry. In this paper, we have developed all the necessary elements to quantize the HK model within the framework of LQG. Due to the fact the HK model has not Hamiltonian constraint, HK theory does not share the usual ambiguities that are present in the quantum scalar constraint of GR. From the covariant quantization perspective, (the spin foams formalism [26]), it might be possible to study the different simplicial constraints defined by the HK model when it is written as a constrained BF theory [8, 18, 33]. Such analysis should shed light on the relation between canonical and covariant quantizations for covariant field theories, as in the case of loop quantum gravity and spin foam models approaches.

III. CONCLUSIONS AND PROSPECTS

In this paper, we have consistently performed a pure Dirac’s method of the HK theory. The analysis was carried out in the full phase space, enabled us to identify the extended action, the extended Hamiltonian and the complete set of constraints. Once the constraints were classified as first and second class by means of the null vector space defined by the constraints hypersurface, this procedure allowed us to carry out the counting of the degrees of freedom and to calculate the gauge transformations acting in the complete phase space. From the gauge orbits, we realized that the dynamical evolution of the HK model is given by the full phase space spatial diffeomorphism generator. This means that, classically, it is possible to write down an infinite number of constants of motion, implying that the model is a complete Liouville integrable covariant field theory with local degrees of freedom. One of the main purposes for working on the complete phase space, lies on the full identification of the complete set of constraints, allowing us to calculate the Dirac algebra structure of the theory and a local weighted measure for the path integral quantization approach which will be useful in the spin foam formulation. Finally, we have introduced the necessary mathematical structure to complete the canonical quantization program. We can observe that the resulting physical Hilbert space will be constituted by the so called spin network states, which are defined on equivalence classes of graphs under diffeomorphisms. Finally, we would comment that our approach allow us know the complete structure of the constraints, thus, we are able to perform any discrete approach such as spin foam quantization of gravity [26]. It is pointed out by Gambini et. al. [17], the inconsistencies arising in discretizations methods are present in spin foam approaches. When one discretizes the action in order to compute the path integral, one is left with a discrete action whose equations yield an inconsistent theory. In order to avoid these problems, all the variables that occur in the action must be taken into account.
and no Hamiltonian reduction is allowed, resulting the presence of second class constraints just as our approach. In the same way, the discrete path integral measure obtained in the full phase space would be different to that calculated in the reduced phase-space approach (which usually do not have second class constraints), resulting new terms originated by parentheses between second class constraints. It is believed that these terms could cure some divergences that arise from the remaining gauge symmetry of the fields. Hence, we believe that our approach could shed light in the continuum limit of the discrete covariant methods reported in the literature. All these ideas are in progress and will be reported in forthcoming works.

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