Generalizations of the Kovalevskaya case and quaternions

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Abstract. This paper provides a detailed description of various reduction schemes in rigid body dynamics. Analysis of one of such nontrivial reductions makes it possible to order the cases already found and to obtain new generalizations of the Kovalevskaya case to $e(3)$. We note that the above reduction allows one to obtain in a natural way some singular additive terms which were proposed earlier by D. N. Goryachev.

Keywords rigid body dynamics, quaternions, reduction, cyclic coordinates, Kovalevskaya top
1 Introduction

Two possible integrable generalizations of the classical Kovalevskaya top are well known from rigid body dynamics. One of them involves the introduction of additional terms to the system with two degrees of freedom on the algebra \(e(3)\), for example, a gyrostatic parameter or terms added to the potential, which do not break the symmetry of the field relative to the fixed axis. These additive terms were examined in detail by Yehia \[23, 24\], Valent \[21\] and others.

The other type of generalization involves the introduction of additional fields (along with the gravitational field), for example, a magnetic and homogeneous electric field. In the general case, the fields are assumed to be transversal to each other. In contrast to the gravitational field, additional fields do not allow a usual reduction by the precession angle (which is cyclic due to the invariance of the system under rotations about a fixed axis), in this case it is necessary to investigate a system with three degrees of freedom. A general integrable case for this system was obtained by A. G. Reyman and M. A. Semenov-Tianshansky \[16\], and a generalization of this case was obtained by A. V. Tsiganov and V. V. Sokolov \[19\]. This system possesses a quadratic integral and a fourth-degree integral (for bifurcation analysis of this case see the work of M. P. Kharlamov \[13\]). In some cases presented in \[22, 6\], the quadratic integral reduces to a linear one, and a constructive order reduction is possible (see also \[3\]).

An interesting fact was that for a constructive reduction it is convenient to use quaternions, which were introduced and advocated by W. Hamilton to solve various mechanics problems. In the paper by Borisov and Mamaev \[6\] (1997), an explicit process of reduction using quaternions for the equations of rigid body dynamics was described and isomorphisms between different integrable systems were revealed. However, these results remained little-known (apparently because they were published only in the Russian language), and, as a consequence, publications still appear in which the connections between systems, as described in \[6\], are ignored. For example, the author of \[20\] presents a “new” integrable system which, as it turns out, can be obtained using reduction from a general integrable system as obtained earlier in \[19\]. Moreover, the author of \[14\] presents a separation of variables which can be obtained using the above isomorphism \[6\] and from the separation found earlier in \[17\].

The goal of this paper is to give, once again, a more detailed description of various reduction schemes in rigid body dynamics, which are of interest in themselves and are presented only in the book \[4\], which has also been published only in the Russian language. Analysis of one nontrivial reduction makes it possible to order the cases already found and to obtain new generalizations of the Kovalevskaya case to \(e(3)\). We note that the above reduction allows one to obtain for this case in a natural way some singular additive terms which were proposed earlier in the work of D. N. Goryachev \[11, 12\]. To conclude, we note that the quaternion equations presented in \[6\] and \[4\]

\[1\] There is an extensive literature on the classical Kovalevskaya top (see, e.g., \[14, 2\])

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are still poorly understood, although they can be used for various algebraic and geometric methods of integration, for example, in \cite{5} they were used to construct an L-A pair of the Goryachev–Chaplygin top.

2 Equations of motion

Consider a rigid body rotating in a potential force field about a fixed point \( O \). The configuration space, which is a set of all positions of the rigid body, is the Lie group \( SO(3) \), and we can take, for example, the Euler angles \( \theta, \phi, \psi \) \cite{4} as (local) coordinates specifying the position of the rigid body.

To specify them, we define two coordinate systems with origin at the fixed point \( O \):

- a fixed coordinate system \( OX_1X_2X_3 \),

- a moving coordinate system \( Ox_1x_2x_3 \) rigidly attached to the rotating rigid body (Fig. 1).

The transformation from the fixed axes to the moving axes is given by the orthogonal matrix \( Q \in SO(3) \) (matrix of direction cosines), which is defined by three successive rotations:

\[
Q_\theta = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix}, \quad Q_\phi = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad Q_\psi = \begin{pmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
\[
Q = \begin{pmatrix}
\cos \varphi \cos \psi - \cos \theta \sin \psi \sin \varphi & - \sin \varphi \cos \psi - \cos \theta \sin \psi \cos \varphi & \sin \theta \sin \psi \\
\cos \varphi \sin \psi + \cos \theta \cos \psi \sin \varphi & - \sin \varphi \sin \psi + \cos \theta \cos \psi \cos \varphi & - \sin \theta \cos \psi \\
\sin \varphi \sin \theta & \cos \varphi \sin \theta & \cos \theta
\end{pmatrix} = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{pmatrix}.
\]

(1)

The rows of this matrix, \( \alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3), \gamma = (\gamma_1, \gamma_2, \gamma_3) \), are the unit vectors of the fixed axes \( OX_1X_2X_3 \) projected onto the moving axes \( Ox_1x_2x_3 \). Since they have a clear geometric meaning, they are used in what follows to elucidate the physical meaning of forces acting on the body.

Supplementing these position variables with the corresponding canonical momenta \( p_\theta, p_\varphi, p_\psi \), we write the equations of motion of the body in the canonical Hamiltonian form

\[
\dot{\theta} = \frac{\partial H}{\partial p_\theta}, \quad \dot{\varphi} = \frac{\partial H}{\partial p_\varphi}, \quad \dot{\psi} = \frac{\partial H}{\partial p_\psi},
\]

\[
\dot{p}_\theta = -\frac{\partial H}{\partial \theta}, \quad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi}, \quad \dot{p}_\psi = -\frac{\partial H}{\partial \psi}.
\]

As a rule, it is not convenient to use equations in this form to search for and analyze integrable cases, so we transform them to an appropriate form. To do this, we shall use as configuration variables the quaternions \( \lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \) with the unit norm

\[
\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1,
\]

which are also called the Rodrigues–Hamilton parameters. Their relation with the Euler angles is given by

\[
\lambda_0 = \cos \frac{\theta}{2} \cos \frac{\psi + \varphi}{2}, \quad \lambda_1 = \sin \frac{\theta}{2} \cos \frac{\psi - \varphi}{2}, \quad \lambda_2 = \sin \frac{\theta}{2} \sin \frac{\psi - \varphi}{2}, \quad \lambda_3 = \cos \frac{\theta}{2} \sin \frac{\psi + \varphi}{2}.
\]

(2)

Multiplication in the group \( SO(3) \) is consistent with multiplication of the quaternions, and the matrix of direction cosines is written as

\[
Q = \begin{pmatrix}
\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 & 2(\lambda_1 \lambda_2 - \lambda_0 \lambda_3) & 2(\lambda_0 \lambda_2 + \lambda_1 \lambda_3) \\
2(\lambda_0 \lambda_3 + \lambda_1 \lambda_2) & \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2 & 2(\lambda_2 \lambda_3 - \lambda_0 \lambda_1) \\
2(\lambda_1 \lambda_3 - \lambda_0 \lambda_2) & 2(\lambda_0 \lambda_1 + \lambda_2 \lambda_3) & \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2
\end{pmatrix}.
\]

Remark 1. From the geometrical point of view, quaternions with unit norm form a three-dimensional sphere \( S^3 \), which doubly covers the group \( SO(3) \). It is this fact that relations (2) express.
It is more convenient to use, instead of canonical momenta, the projections of angular momentum $M$ onto the moving axes $Ox_1x_2x_3$, which are defined as

$$
M_1 = \frac{\sin \varphi}{\sin \theta} (p_\psi - p_\varphi \cos \theta) + p_\theta \cos \varphi, \quad M_2 = \frac{\cos \varphi}{\sin \theta} (p_\psi - p_\varphi \cos \theta) - p_\theta \sin \varphi,
$$

$$
M_3 = p_\varphi.
$$

In this case, the kinetic energy of the rigid body is a quadratic form with constant coefficients

$$
T = \frac{1}{2} (M, AM), \quad A = I^{-1},
$$

where $I$ is the tensor of inertia of the body relative to the point $O$ in the moving coordinate system $Ox_1x_2x_3$; if $Ox_1, x_2, x_3$ are the principal axes of inertia, then $I = \text{diag}(I_1, I_2, I_3)$ and hence $A = \text{diag}(a_1, a_2, a_3)$.

In the new variables the Poisson structure turns out to be linear (Lie–Poisson bracket)

$$
\{ M_i, M_j \} = -\varepsilon_{ijk} M_k, \quad \{ \lambda_0, \lambda_i \} = \{ \lambda_i, \lambda_j \} = 0,
$$

$$
\{ M_i, \lambda_0 \} = \frac{1}{2} \lambda_i, \quad \{ M_i, \lambda_j \} = -\frac{1}{2} (\varepsilon_{ijk} \lambda_k + \delta_{ij} \lambda_0) \quad i, j, k = 1, 2, 3 \quad (3)
$$

It turns out to be degenerate and possesses the unique Casimir function

$$
C_0(\lambda) = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2.
$$

The equations of motion of the body in the new variables can be represented in vector form:

$$
\dot{M} = M \times \frac{\partial H}{\partial M} + \frac{1}{2} \lambda \times \frac{\partial H}{\partial \lambda} + \frac{1}{2} \frac{\partial H}{\partial \lambda_0} - \frac{1}{2} \lambda_0 \frac{\partial H}{\partial \lambda},
$$

$$
\dot{\lambda}_0 = -\frac{1}{2} \left( \lambda, \frac{\partial H}{\partial M} \right), \quad \dot{\lambda} = \frac{1}{2} \lambda \times \frac{\partial H}{\partial M} + \frac{1}{2} \lambda_0 \frac{\partial H}{\partial M}, \quad (4)
$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. In what follows we consider Hamiltonians of the form

$$
H = \frac{1}{2} (M, AM) + (M, W(\lambda)) + U(\lambda), \quad (5)
$$

where $U(\lambda)$ and $W(\lambda)$ are, respectively, the scalar and vector potentials describing the interaction of the body with the external fields.

**Remark 2.** There is a connection between the Rodrigues–Hamilton parameters and the direction cosines $\alpha, \beta, \gamma$:

$$
\lambda_0^2 = \frac{1 + \alpha_1 + \beta_2 + \gamma_3}{4}, \quad \lambda_1^2 = \frac{1 + \alpha_1 - \beta_2 - \gamma_3}{4},
$$

$$
\lambda_2^2 = \frac{1 - \alpha_1 + \beta_2 - \gamma_3}{4}, \quad \lambda_3^2 = \frac{1 - \alpha_1 - \beta_2 + \gamma_3}{4}, \quad (6)
$$
3 Reduction

We now consider three cases in which the system is invariant under the (Hamiltonian) action of the rotation group $S^1$. The corresponding Hamiltonians generating these actions have the form

$$H_1 = p_\psi, \quad H_2 = p_\varphi, \quad H_3 = p_\psi - p_\varphi.$$ 

In the matrix representation the action on $SO(3)$, which is generated by the Hamiltonian $H_1$, is given by multiplication on the left by the rotation matrix:

$$g_t(Q) = S_t Q, \quad S_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

For the Hamiltonian $H_2$ this action is a multiplication on the left:

$$g_t(Q) = QS_t$$

For the Hamiltonian $H_3$ the corresponding action is defined by the relation

$$g_t(Q) = S_t QS_{-t}.$$ 

As is well known, a reduction (of order) of the system is possible in this case. Moreover, in all three cases the reduced system can be represented in a natural way in Hamiltonian form on the zero orbit of the coalgebra $e^*(3)$. In other words, one can choose the variables of the reduced system $L = (L_1, L_2, L_3)$, $n = (n_1, n_2, n_3)$ in such a way that they form a Lie–Poisson bracket of the form

$$\{L_i, L_j\} = -\varepsilon_{ijk} L_k, \quad \{L_i, n_j\} = -\varepsilon_{ijk} n_k, \quad \{n_i, n_j\} = 0,$$

and the level set of the Casimir functions is fixed as follows:

$$C_1 = (n, n) = 1, \quad C_2 = (L, n) = 0.$$ 

The equations of motion in the new variables have the well-known form

$$\dot{L} = L \times \frac{\partial H}{\partial L} + n \times \frac{\partial H}{\partial n}, \quad \dot{n} = n \times \frac{\partial H}{\partial L}.$$ 

This will allow us to establish a relation between different integrable cases.

3.1 The area integral $H_1 = p_\psi$

Symmetries leading to such an integral are natural: they are due to the invariance of the external field under rotations about some fixed axis. Such axisymmetric fields include homogeneous fields, in particular, a gravitational field. The precession angle $\psi$ is a cyclic variable.
Let us choose the invariants of this action as follows:

\[ n_1 = 2(\lambda_1 \lambda_2 - \lambda_0 \lambda_2) = \sin \theta \sin \varphi, \quad n_2 = 2(\lambda_0 \lambda_1 + \lambda_2 \lambda_3) = \sin \theta \cos \varphi, \]
\[ n_3 = \lambda_2^2 - \lambda_1^2 - \lambda_3^2 + \lambda_0^2 = \cos \theta, \]
\[ L_1 = M_1 - \frac{(M, n)n_1}{n_1^2 + n_2^2}, \quad L_2 = M_2 - \frac{(M, n)n_2}{n_1^2 + n_2^2}, \quad L_3 = M_3, \]

where \( n = (n_1, n_2, n_3) = \gamma \) is the last column of the matrix \( Q \). This column has a simple physical meaning: it is a unit vector directed along the symmetry axis in the coordinate system \( O x_1 x_2 x_3 \). In this case, the area integral is represented as

\[ H_1 = p_\psi = (M, n). \]

A straightforward verification shows that the variables \( L, n \) satisfy relations (7) and (8).

Using the relation \( \{ H, H_1 \} = 0 \), we find conditions under which the system with the Hamiltonian (4) admits this symmetry group. Expressed in terms of Euler angles, these conditions have the simplest form:

\[ \frac{\partial W_i}{\partial \psi} = 0, \quad \frac{\partial U}{\partial \psi} = 0, \quad i = 1, 2, 3. \]

Thus, at a fixed value of the area integral \( p_\psi = c \) we obtain a Hamiltonian of the reduced system in the form

\[ H = \frac{1}{2}(L, A L) + (L, W_r) + U_r, \]
\[ W_r = W + cA \tau, \quad U_r = U + c(r, W) + \frac{c^2}{2} (\tau, A \tau), \]
\[ \tau = \left( \frac{n_1}{n_1^2 + n_2^2}, \frac{n_2}{n_1^2 + n_2^2}, 0 \right). \]

As can be seen, when \( c \neq 0 \), this system has singularities on the Poisson sphere \( n^2 = 1 \) at the points \( n_+ = (0, 0, 1) \) and \( n_- = (0, 0, -1) \).

### 3.2 The Lagrange integral \( H_2 = p_\varphi \)

This integral is a projection of the angular momentum on the body-fixed axis \( O x_3 \):

\[ H_2 = M_3. \]

In this case, the rigid body must be dynamically symmetric:

\[ I_1 = I_2, \]

where without loss of generality we set \( I_1 = I_2 = 1 \). Moreover, it is necessary that the force field also be invariant under rotation about the axis of dynamical
symmetry, and so in the general form the Hamiltonian of the system can be represented as

\[ H = \frac{1}{2}(M_1^2 + M_2^2 + a_3 M_3^2) + \tilde{W}_1(\theta, \psi)(M_1 \gamma_1 + M_2 \gamma_2 + M_3 \gamma_3) + \tilde{W}_3(\theta, \psi)M_3 + U(\theta, \psi). \]  

(10)

As the invariants of the action of the symmetry group we have to choose the following variables:

\[ n_1 = \alpha_3 = \sin \theta \sin \psi, \quad n_2 = \beta_3 = -\sin \theta \cos \psi, \quad n_3 = \gamma_3 = \cos \theta; \]

\[ L_1 = M_3 \frac{\alpha_3}{\alpha_3^2 + \beta_3^2} - (M, \alpha), \quad L_2 = M_3 \frac{\beta_3}{\alpha_3^2 + \beta_3^2} - (M, \beta), \quad L_3 = -(M, \gamma). \]

As above, these variables satisfy relations (7) and (8).

On the fixed level set of the Lagrange integral \( M_3 = c \), from (10) we obtain a Hamiltonian of the reduced system in the form

\[ H = \frac{1}{2} L^2 - L_3 \left( \tilde{W}_1(n) - c \frac{n_3}{n_1^2 + n_2^2} \right) + \frac{c^2}{2} \frac{n_3^2}{n_1^2 + n_2^2} + U(n), \]

where the insignificant constants have been omitted.

### 3.3 The integral \( H_3 = p_\psi - p_\phi \)

In quaternion variables this integral is represented as

\[ H_3 = 2M_3(\lambda_1^2 + \lambda_2^2) - 2(M_1 \lambda_1 + M_2 \lambda_2) \lambda_3 + 2(M_1 \lambda_2 - M_2 \lambda_1) \lambda_0. \]

**Remark 3.** We note that in the direction cosines this integral has the form

\[ H_3 = M_3 - (M, \gamma). \]

First of all, we find out under what conditions a system with the Hamiltonian (5) admits this integral. From the relation \( \{H, H_3\} = 0 \) we find that the most general form of the Hamiltonian is

\[ H = \frac{1}{2}(M_1^2 + M_2^2 + a_3 M_3^2) + \tilde{W}_1(\lambda_0, \lambda_3)(M_1 \lambda_1 + M_2 \lambda_2) - \tilde{W}_2(\lambda_0, \lambda_3)(M_1 \lambda_2 - M_2 \lambda_1) - \tilde{W}_3(\lambda_0, \lambda_3)M_3 + U(\lambda_0, \lambda_3), \]  

(11)

where \( a \) is an arbitrary constant and \( \tilde{W}_i(\lambda_0, \lambda_3), \ i = 1, 2, 3 \) and \( U(\lambda_0, \lambda_3) \) are the functions characterizing the vector and scalar potentials of the external field. The cyclic variable is \( \psi - \varphi \).

This implies, in particular, that for the existence of this symmetry group it is necessary that the body have dynamical symmetry: \( I_1 = I_2 \). (Without loss of generality, in (11) we have set \( I_1 = 1, \ I_3 = a_3^{-1} \).)
The invariants of the action generated by the Hamiltonian $H_2$ can be chosen as follows:

\[ n_1 = \lambda_3 = \cos \frac{\theta}{2} \sin \frac{\psi + \varphi}{2}, \quad n_2 = \lambda_0 = \cos \frac{\theta}{2} \cos \frac{\psi + \varphi}{2}, \quad n_3 = \sqrt{\lambda_1^2 + \lambda_2^2} = \sin \frac{\theta}{2} \]

\[ L_1 = \frac{2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \left( \frac{H_3}{2C_0} \lambda_3 - M_1 \lambda_1 - M_2 \lambda_2 \right), \]

\[ L_2 = \frac{2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \left( \frac{H_3}{2C_0} \lambda_0 + M_1 \lambda_2 - M_2 \lambda_1 \right) \]

\[ L_3 = \frac{H_3}{C_0} + 2M_3. \]  

(12)

For the variables (12) relations (7), (8) and (9) hold, i.e., as in the previous case the reduced system is represented only on the orbit of the coalgebra $e^*(3)$ on the zero level set of the area integral.

On the fixed level set of the first integrals $H_3 = c$ and $C_0 = 1$, in the new variables the Hamiltonian (11) can be rewritten, up to insignificant constants, as follows:

\[ H = \frac{1}{8}(L_1^2 + L_2^2 + a_3 L_3^2) + \frac{1}{2}(L, \widetilde{W}_r) + U_r, \]

\[ \widetilde{W}_r = \left( n_1 \widetilde{W}_1 - \frac{c}{2n_3} n_1, \quad n_2 \widetilde{W}_2 - \frac{c}{2n_3} n_2, \quad \widetilde{W}_3 - \frac{c}{2} a_3 \right) \]  

\[ U_r = U(n_1, n_2) - \frac{c}{2}(n_1 \widetilde{W}_1 + n_2 \widetilde{W}_2 + \widetilde{W}_3) + \frac{c^2}{8n_3^2}. \]  

(13)

where $\widetilde{W} = (\widetilde{W}_1, \widetilde{W}_2, \widetilde{W}_3)$. As we can see, the reduced system has singularities on the equator $n_3 = 0$ of the Poisson sphere $n^2 = 1$.

We refer the reader to the recent and extensive work [8], which is devoted to general methods of reduction in nonholonomic systems (describing, for example, the rolling motion of rigid bodies). It would be interesting to investigate a nonholonomic (non-Hamiltonian) analog of the reduction described in the above-mentioned work. Such an analog arises, for example, in the analysis of the rolling motion of a rigid body in the presence of homogeneous force fields. Such problems have not yet been considered in nonholonomic mechanics.

4 The Kovalevskaya case

Consider a particular case of the Hamiltonian (11) that corresponds to the Kovalevskaya top lying in a potential field and that in the direction cosines $\alpha, \beta, \gamma$ has the form

\[ H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) + c_1 M_3 + c_2(M_1 \alpha_3 + M_2 \beta_3) - c_2(\alpha_1 + \beta_2) M_3 + c_3(\alpha_2 - \beta_1). \]

(14)
In this case, the equations of motion possess, along with \( H_2 \), an additional integral of degree 4 in momenta, which can be obtained from the L-A pair presented by V. V. Sokolov and A. V. Tsiganov [19] and generalizing the earlier L-A pair of A. G. Reyman, M. A. Semenov-Tian-Shansky [16].

**Remark 4.** We note that, generally speaking, the authors of [19] presented a more general case of the Kovalevskaya top in which the equations of motion possess an additional integral of degree 2 and 4 in momenta (the explicit form of the integral is presented in [15]).

After passing to the quaternions we find
\[
\begin{align*}
\tilde{W}_1 &= 2c_2\lambda_3, \\
\tilde{W}_2 &= -2c_2\lambda_0, \\
\tilde{W}_3 &= 2(\lambda_0^2 - \lambda_3^2)c_2 - c_1, \\
U &= -4c_3\lambda_0\lambda_3.
\end{align*}
\]

Further, by applying the reduction procedure described in the previous section, we obtain an (integrable) system on \( e(3) \) on the zero level set of the area integral.

In order to compare the resulting system with those obtained earlier, we make the canonical change of variables
\[
\begin{align*}
L_1 &\to \frac{L_1 + L_2}{\sqrt{2}}, \\
L_2 &\to \frac{L_1 - L_2}{\sqrt{2}}, \\
L_3 &\to L_3, \\
n_1 &\to \frac{n_1 + n_2}{\sqrt{2}}, \\
n_2 &\to \frac{n_1 - n_2}{\sqrt{2}}, \\
n_3 &\to n_3.
\end{align*}
\]

As a result, the Hamiltonian can be represented as
\[
H = \frac{1}{8}(L_1^2 + L_2^2 + 2L_3^2) - \frac{1}{4}(c + 2c_1)L_3 + \\
+ c_2(n_2n_3L_1 + n_1n_3L_2 - 2n_1n_2L_3) - 2c_3(n_1^2 - n_2^2) + \frac{c^2}{8n_3^2},
\]
\[\text{(15)}\]

In this case, the additional integral has the form
\[
F = \left(\frac{L_1^2 - L_2^2 + 8c_2n_3(L_1n_2 - L_2n_1) - 16c_3n_3^2 + \frac{c^2(n_1^2 + n_2^2)}{n_3^2}}{n_3}\right)^2 + \\
+ 4\left(L_1L_2 - \frac{c^2n_1n_2}{n_3^2}\right)^2 + 4(2c_1 + c)(L_3 - 2c_1 - c)\left(L_1^2 + L_2^2 + c^2\left(1 + \frac{1}{n_3^2}\right)\right) + \\
+ 64c_3(2c_1 + c)(L_1n_1 - L_2n_2) + \\
+ 32c_2(2c_1 + c)\left(L_1n_1n_2 + L_1L_2n_3^2 + L_2n_1n_2 + c^2\frac{n_1n_2}{n_3^2}\right).
\]

Various particular cases of the Hamiltonian \[\text{(15)}\] were presented earlier.

- If the relations
\[
c_2 = 0, \\
c = -2c_1
\]
hold, then the Hamiltonian \[\text{(15)}\] reduces to the Goryachev case, for which a separation of variables was performed in [17]. A separation of variables in the initial system (i.e., with the Hamiltonian \[\text{(14)}\]), also under the above condition, but without the above analogy, was performed in [14]. It is obvious that in this case the separation in \[\text{(14)}\] coincides with that in \[\text{(17)}\].
The last section of [13] discusses the case
\[ c = 0, \quad c_1 = 0, \quad c_2 = 0. \]
It follows from the above isomorphism that it is equivalent to the Chaplygin case integrated by Chaplygin himself. This isomorphism was found for the first time in [6].

Under the conditions
\[ c_1 = 0, \quad c = 0 \]
the additional integral was found earlier by A.V. Tsiganov in [20]. As can be seen, this integral is not new and can be obtained from the system [19] using the reduction procedure described above.

Remark 5. There are a lot of publications by H.M. Yehia (and his colleagues) on the generalization of the Kovalevskaya case. A weak point of these publications is the absence of Hamiltonian formalism of the problem. Because of this many generalizations presented by Yehia are dependent and can be obtained by the simplest algebraic transformation. This aspect of his work is discussed in the book [4] and in the recent paper [1].

5 Quaternion Euler–Poisson equations

Let us consider the case of equations of motion of a rigid body with a potential that is linear not in the direction cosines, but in the quaternions

\[ H = \frac{1}{2}(M, AM) + \sum_{i=0}^{3} r_i \lambda_i, \quad A = \text{diag}(a_1, a_2, a_3), \quad r_i = \text{const}, \quad (16) \]
assuming that the equations of motion have the form (4). We note that such potentials are not encountered in mechanics, since their dependence on the position of the body is ambiguous [4]. Problems of quantum mechanics, dynamics of point masses in curved space \( S^3 \) [7], as well as some formal methods for constructing \( \mathbf{L} \)-\( \mathbf{A} \)-pairs [7] can be regarded as a motivation for considering such equations. Moreover, it turns out that the order reduction of the system (16) leads to the standard Euler–Poisson equations with additional terms having different physical interpretations.

An interesting singularity of the system (16) is that using transformations linear in \( \lambda_i \), the general form of the potential

\[ V = \sum_{i=0}^{3} r_i \lambda_i \quad (17) \]
can be reduced to the form

\[ V = r_0 \lambda_0. \quad (18) \]
Indeed, linear transformations of the quaternion space $\lambda_i$ (which do not change the commutator relations and the norm of the quaternion) of the form

$$
\bar{\lambda}_0 = R^{-1}(r_0 \lambda_0 + r_1 \lambda_1 + r_2 \lambda_2 + r_3 \lambda_3),
\bar{\lambda}_1 = R^{-1}(r_0 \lambda_1 - r_1 \lambda_0 - r_2 \lambda_3 + r_3 \lambda_2),
\bar{\lambda}_2 = R^{-1}(r_0 \lambda_2 + r_1 \lambda_3 - r_2 \lambda_0 - r_3 \lambda_1),
\bar{\lambda}_3 = R^{-1}(r_0 \lambda_3 - r_1 \lambda_2 + r_2 \lambda_1 - r_3 \lambda_0),
$$

reduce the potential (17) to the form (18). The existence of such a linear transformation is a remarkable singularity of the quaternion variables and of the bracket (3); it has no analogs for the brackets of the algebra $e(3)$ and so(4).

In the dynamically asymmetric case $a_1 \neq a_2 \neq a_3 \neq a_1$ the system (16) is apparently nonintegrable and neither of the two necessary additional integrals exists. It would be interesting to use the Kovalevskaya method and other methods for finding additional first integrals to investigate (16).

For $a_1 = a_2$ there always exists the linear integral

$$
H_3 = M_3(r_0^2 + r_1^2 + r_2^2 + r_3^2) + N_3(r_1^2 + r_2^2 - r_0^2 - r_3^2) + 2N_2(r_1 r_0 - r_3 r_2) - 2N_1(r_1 r_2 - r_0 r_3),
$$

where $N_i$ are the projections of angular momentum onto the fixed axes

$$
N_1 = (M, \alpha), \quad N_2 = (M, \beta), \quad N_1 = (M, \gamma).
$$

Under the conditions $r_1 = r_2 = r_3 = 0$ this integral takes the natural form

$$
H_3 = M_3 - N_3
$$

and corresponds to the cyclic variable $\varphi + \psi$. The reduction described above leads to a Hamiltonian system on the algebra $e(3)$ with zero value of the area integral $(M, \gamma) = 0$ and with the Hamiltonian

$$
H = \frac{1}{2}(M_1^2 + M_2^2 + a_3 M_3^2) + c(a_3 - 1)M_3 + r_0 \gamma_2 + \frac{1}{2} \frac{c^2}{\gamma_3}.
$$

We present here integrable cases of the system (16) which turn out to be equivalent to the integrable cases of the system (22).

**The spherical top** ($a_1 = a_2 = a_3$). The Hamiltonian has the form

$$
H = \frac{1}{2} M^2 + r_0 \lambda_0,
$$

and, as shown in [7], the system is equivalent to the problem of the motion of a material point over a three-dimensional sphere $S^3$. Since the potential depends
only on $\lambda_0$, it can be assumed that the material point moves in the field of a fixed center placed in the northern (southern) pole, and the force of interaction depends only on the distance to it (analog of the problem in the central field for $\mathbb{R}^3$). As in the planar case, the angular momentum vector of the particle is preserved:

$$L = \frac{1}{2} (N - M) = \text{const}, \quad (23)$$

where $N$ is the angular momentum vector in the fixed axes.

The components of the vector $L$ form the algebra $so(3)$: $\{L_i, L_j\} = \varepsilon_{ijk} L_k$, and the integrability is noncommutative; moreover, the system possesses a redundant set of integrals and its three-dimensional tori are foliated by two-dimensional tori.

**The Kovalevskaya case.** The Hamiltonian and the additional integral involutive to $F_1$ (of degree 4) have the form

$$H = \frac{1}{2} (M_1^2 + M_2^2 + 2M_3^2) + r_0 \lambda_0,$$

$$F = (M_1N_1 + M_2N_2 + 2r_0\lambda_0)^2 + (N_1M_2 - N_2M_1 - 2r_0\lambda_3)^2 +$$

$$+(N_3 - M_3)(M_3(M_2^2 - M_3N_3) + 2r_0(M_2\lambda_1 - M_1\lambda_2 + \frac{\lambda_0}{2}(M_3 - N_3))). \quad (24)$$

**The Goryachev – Chaplygin case.** The Hamiltonian and the additional integral have the form

$$H = \frac{1}{2} (M_1^2 + M_2^2 + 4M_3^2) + r_0 \lambda_0,$$

$$F = M_3(M_1^2 + M_2^2) + r_0(M_2\lambda_1 - M_1\lambda_2). \quad (25)$$

Somewhat unexpected is the circumstance that the Lagrange and Hess cases cannot be generalized to the system [19]. We note that for the quaternion Euler–Poisson equations both the Kovalevskaya case and the Goryachev–Chaplygin case are general integrable cases.

### 6 Generalization of the quaternion cases of Kovalevskaya and Goryachev–Chaplygin

Let us consider a generalization of the quaternion case of Kovalevskaya

$$H = \frac{1}{2} (M_1^2 + M_2^2 + 2M_3^2) + c_1M_3 + 2c_2\lambda_0\lambda_3 + r_0\lambda_0. \quad (26)$$

After reduction (by eliminating the cyclic variable $\varphi + \psi$) the Hamiltonian can be represented as

$$H = \frac{1}{8}(L_1^2 + L_2^2 + 2L_3^2) - \frac{1}{4}(e + 2c_1)L_3 + r_0n_2 + 2c_2n_1n_2 + \frac{c^2}{8n_3^2}. \quad (27)$$
The integrability of the Hamiltonian (27) on the zero level set of the area integral was shown by H. M. Yehia [23]. In this case, the additional integral has the form

\[ F_3 = \left( L_1^2 - L_2^2 - c^2 \frac{n_1^2 - n_2^2}{n_3^2} + 8r_0n_2 \right)^2 + 
\]
\[ +4 \left( L_1L_2 - c^2 \frac{n_1n_2}{n_3^2} - 4r_0n_1 + 4c_2n_3^2 \right)^2 + 
\]
\[ +4(c + 2c_1)(L_3 - c - 2c_1) \left( L_1^2 + L_2^2 + c^2 \left( 1 + \frac{1}{n_3^2} \right) \right) - 
\]
\[ -16(c + 2c_1)n_3(c_2n_2L_1 + L_2(r_0 + c_2n_1)). \]

We present the second generalization of the Kovalevskaya case

\[ H = \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) + c_1(M_1\lambda_1 + M_2\lambda_2 + M_3\lambda_3) + c_2M_3 + r_0\lambda_0. \] (28)

After reduction on the zero level set of the area integral of the algebra e(3) the Hamiltonian (28) can be represented as

\[ H = \frac{1}{8}(L_1^2 + L_2^2 + 2L_3^2) + \frac{1}{4}(c + 2c_2)L_3 - \frac{1}{2}c_1(n_1L_3 - n_3L_1) + \frac{c^2}{8n_3^2} + r_0n_2. \] (29)

The family (29) was presented by H. M. Yehia [24], and the additional integral in this case has the form

\[ F = \left( L_1^2 - L_2^2 + 4c_1(L_1n_3 - L_3n_1) + 4c_1^2 + 8r_0n_2 - \frac{c^2(n_1^2 - n_2^2)}{n_3^2} \right)^2 + 
\]
\[ +4 \left( L_1L_2 + 2c_1(L_2n_3 - L_3n_2) - 4r_0n_1 - \frac{c^2n_1n_2}{n_3^2} \right)^2 + 
\]
\[ +4(c + 2c_2)(L_3 - c - 2c_2)(L_1^2 + L_2^2) + 16(c + 2c_2)(c_1n_2L_1L_2 - c_1n_1L_2^2 - 
\]
\[ -c_1^2L_3 - 2r_0n_3L_2) - \frac{4c^2(1 + 2c_2)}{n_3^2} (4c_1n_1 + c + 2c_2 - (1 + n_3^2)L_3). \]

To conclude, we consider a generalization of the quaternion case of Goryachev–Chaplygin

\[ H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) + c_1(M_1\lambda_1 + M_2\lambda_2 + 2M_3\lambda_3) + c_2M_3 + r_0\lambda_0. \] (30)

After reduction the Hamiltonian has the form

\[ H = \frac{1}{8}(L_1^2 + L_2^2 + 4L_3^2) - \frac{1}{4}(2c_2 + 3c)L_3 + \frac{1}{2}c_1(2n_1L_3 - n_3L_1) + \frac{c^2}{8n_3^2} - \frac{1}{2}c_1n_1 + r_0n_2. \]

The additional integral in this case was found in [19].
In this paper we have shown interrelations between different integrable systems resulting from reduction. These interrelations can be used in the inverse problem, that of recovering the dynamics. This procedure allows one to construct from an integrable system with two degrees of freedom an integrable three-degree-of-freedom system possessing additional symmetry. (In principle this procedure can also be applied to nonintegrable systems.) This issue is partially discussed in the book [4]. The issue of integrable generalizations of the Kovalevskaya case in which the potential is a superposition of terms linear and quadratic (linear in the direction cosines) in quaternions remains open.

The authors express their gratitude to P. E. Ryabov and Yu. N. Fedorov for useful discussions.

This research was supported by the Russian Scientific Foundation (project No 14-50-00005) at the Steklov Mathematical Institute of the Russian Academy of Sciences.

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