POINTWISE PROPERTIES OF MARTINGALES WITH VALUES IN BANACH FUNCTION SPACES

MARK VERAAR AND IVAN YAROSLAVTSEV

Abstract. In this paper we consider local martingales with values in a UMD Banach function space. We prove that such martingales have a version which is a martingale field. Moreover, a new Burkholder–Davis–Gundy type inequality is obtained.

1. Introduction

The discrete Burkholder–Davis–Gundy inequality (see [3, Theorem 3.2]) states that for any \( p \in (1, \infty) \) and martingales difference sequence \((d_j)_{j=1}^n\) in \( L^p(\Omega)\) one has

\[
\left\| \sum_{j=1}^n d_j \right\|_{L^p(\Omega)} \lesssim_p \left( \sum_{j=1}^n |d_j|^2 \right)^{1/2} \left\| \right\|_{L^p(\Omega)}.
\]

Moreover, there is the extension to continuous-time local martingales \( M \) (see [13, Theorem 26.12]) which states that for every \( p \in [1, \infty) \),

\[
\left\| \sup_{t \in [0, \infty)} |M_t| \right\|_{L^p(\Omega)} \lesssim_p \left\| [M]_{\infty}^{1/2} \right\|_{L^p(\Omega)}.
\]

Here \( t \mapsto [M]_t \) denotes the quadratic variation process of \( M \).

In the case \( X \) is a UMD Banach function space the following variant of (1.1) holds (see [25, Theorem 3]): for any \( p \in (1, \infty) \) and martingales difference sequence \((d_j)_{j=1}^n\) in \( L^p(\Omega; X)\) one has

\[
\left\| \sum_{j=1}^n d_j \right\|_{L^p(\Omega; X)} \lesssim_p \left( \sum_{j=1}^n |d_j|^2 \right)^{1/2} \left\| \right\|_{L^p(\Omega; X)}.
\]

Moreover, the validity of the estimate also characterizes the UMD property.

It is a natural question whether (1.2) has a vector-valued analogue as well. The main result of this paper states that this is indeed the case:

**Theorem 1.1.** Let \( X \) be a UMD Banach function space over a \( \sigma \)-finite measure space \((S, \Sigma, \mu)\). Assume that \( N : \mathbb{R}_+ \times \Omega \times S \to \mathbb{R} \) is such that \( N|_{[0,t] \times \Omega \times S} \) is \( \mathcal{B}([0,t]) \otimes \mathcal{F}_t \otimes \Sigma \)-measurable for all \( t \geq 0 \) and such that for almost all \( s \in S \),

\[
2010 \text{ Mathematics Subject Classification.} \text{ Primary: 60G44; Secondary: 60B11, 60H05, 60G48.}

Key words and phrases. local martingale, quadratic variation, UMD Banach function spaces, Burkholder-Davis-Gundy inequalities, lattice maximal function.

The first named author is supported by the Vidi subsidy 639.032.427 of the Netherlands Organisation for Scientific Research (NWO).
$N(\cdot, \cdot, s)$ is a martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$ and $N(0, \cdot, s) = 0$. Then for all $p \in (1, \infty)$,

$$
\| \sup_{t \geq 0} |N(t, \cdot, \cdot)| \|_{L^p(\Omega; X)} \approx_{p, X} \sup_{t \geq 0} \| N(t, \cdot, \cdot) \|_{L^p(\Omega; X)} \approx_{p, X} \| [N]_{L^p(\Omega; X)}^{1/2} \|_{L^p(\Omega; X)}.
$$

where $[N]$ denotes the quadratic variation process of $N$.

By standard methods we can extend Theorem 1.1 to spaces $X$ which are isomorphic to a closed subspace of a Banach function space (e.g. Sobolev and Besov spaces, etc.)

The two-sided estimate (1.4) can for instance be used to obtain two-sided estimates for stochastic integrals for processes with values in infinite dimensions (see [20] and [26]). In particular, applying it with $N(t, \cdot, s) = \int_0^t \Phi(\cdot, s) \, dW$ implies the following maximal estimate for the stochastic integral

$$
\left\| s \mapsto \sup_{t \geq 0} \left\| \int_0^t \Phi(\cdot, s) \, dW \right\|_{L^p(\Omega; X)} \approx_{p, X} \sup_{t \geq 0} \left\| \int_0^t \Phi(\cdot, s) \, dW \right\|_{L^p(\Omega; X)} \left( \int_0^\infty \Phi^2(t, s) \, dt \right)^{1/2} \right\|_{L^p(\Omega; X)}
$$

where $W$ is a Brownian motion and $\Phi : \mathbb{R}^+ \times \Omega \times S \to \mathbb{R}$ is a progressively measurable process such that the right-hand side of (1.5) is finite. The second norm equivalence was obtained in [20]. The norm equivalence with the left-hand side is new in this generality. The case where $X$ is an $L^2$-space was recently obtained in [1] using different methods.

It is worth noticing that the second equivalence of (1.4) in the case of $X = L^q$ was obtained by Marinelli in [18] for some range of $1 < p, q < \infty$ by using an interpolation method.

The UMD property is necessary in Theorem 1.1 by necessity of the UMD property in (1.3) and the fact that any discrete martingale can be transformed to a continuous-time one. Also in the case of continuous martingales, the UMD property is necessary in Theorem 1.1. Indeed, applying (1.3) with $W$ replaced by an independent Brownian motion $\tilde{W}$ we obtain

$$
\left\| \int_0^\infty \Phi \, dW \right\|_{L^p(\Omega; X)} \approx_{p, X} \left\| \int_0^\infty \Phi \, d\tilde{W} \right\|_{L^p(\Omega; X)},
$$

for all predictable step processes $\Phi$. The latter holds implies that $X$ is a UMD Banach space (see [10, Theorem 1]).

In the special case that $X = \mathbb{R}$ the above reduces to (1.2). In the proof of Theorem 1.1 the UMD property is applied several times:

- The boundedness of the lattice maximal function (see [2, 3, 22]).
- The $X$-valued Meyer–Yoeurp decomposition of a martingale (see Lemma 2.1).
- The square-function estimate (1.3) (see [22]).

It remains open whether there exists a predictable expression for the right-hand side of (1.4). One would expect that one needs simply to replace $[N]$ by its predictable compensator, the predictable quadratic variation $\langle N \rangle$. Unfortunately, this does not hold true already in the scalar-valued case: if $M$ is a real-valued martingale, then

$$
\mathbb{E} |M|^p_t \lesssim_p \mathbb{E} (M^p_t)^{\frac{p}{2}}, \quad t \geq 0, \quad p < 2, \quad \mathbb{E} |M|^p_t \gtrsim_p \mathbb{E} (M^p_t)^{\frac{p}{2}}, \quad t \geq 0, \quad p > 2,
$$

respectively.
where both inequalities are known not to be sharp (see [3, p. 40], [19, p. 297], and [22]). The question of finding such a predictable right-hand side in (1.4) was answered only in the case $X = \mathbb{L}^q$ for $1 < q < \infty$ by Dirksen and the second author (see [7]). The key tool exploited there was the so-called Burkholder-Rosenthal inequalities, which are of the following form:

$$E\|M_N\|^p \approx_{p, X} \|(M_n)_{0 \leq n \leq N}\|_{p, X}^p,$$

where $(M_n)_{0 \leq n \leq N}$ is an $X$-valued martingale, $\|\cdot\|_{p, X}$ is a certain norm defined on the space of $X$-valued $L^p$-martingales which depends only on predictable moments of the corresponding martingale. Therefore using approach of [7] one can reduce the problem of continuous-time martingales to discrete-time martingales. However, the Burkholder-Rosenthal inequalities are explored only in the case $X = \mathbb{L}^q$.

Thanks to (1.2) the following natural question arises: can one generalize (1.4) to the case $p = 1$, i.e. whether

$$(1.6) \quad \|\sup_{t \geq 0} |N(t, \cdot, \cdot)|\|_{L^1(\Omega; X)} \approx_{p, X} \|N|^{1/2}\|_{L^1(\Omega; X)}$$

holds true? Unfortunately the outlined earlier techniques cannot be applied in the case $p = 1$. Moreover, the obtained estimates cannot be simply extrapolated to the case $p = 1$ since those contain the UMD$_p$ constant, which is known to have infinite limit as $p \to 1$. Therefore (1.6) remains an open problem. Note that in the case of a continuous martingale $M$ inequalities (1.4) can be extended to the case $p \in (0, 1]$ due to the classical Lenglart approach (see Corollary 4.4).

Acknowledgment. The authors would like to thank the referee for helpful comments.

2. Preliminaries

Throughout the paper any filtration satisfies the usual conditions (see [12, Definition 1.1.2 and 1.1.3]), unless the underlying martingale is continuous (then the corresponding filtration can be assumed general).

A Banach space $X$ is called a UMD space if for some (or equivalently, for all) $p \in (1, \infty)$ there exists a constant $\beta > 0$ such that for every $n \geq 1$, every martingale difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega; X)$, and every $\{-1,1\}$-valued sequence $(\varepsilon_j)_{j=1}^n$ we have

$$\left(\mathbb{E} \left[ \sum_{j=1}^n \varepsilon_j d_j \right] \right)^p \leq \beta \left( \mathbb{E} \left[ \sum_{j=1}^n d_j \right] \right)^p.$$

The above class of spaces was extensively studied by Burkholder (see [4]). UMD spaces are always reflexive. Examples of UMD space include the reflexive range of $L^q$-spaces, Besov spaces, Sobolev, and Musielak-Orlicz spaces. Example of spaces without the UMD property include all nonreflexive spaces, e.g. $L^1(0,1)$ and $C([0,1])$. For details on UMD Banach spaces we refer the reader to [11, 23, 25].

The following lemma follows from [27, Theorem 3.1].

**Lemma 2.1** (Meyer-Yoeurp decomposition). Let $X$ be a UMD space and $p \in (1, \infty)$. Let $M : \mathbb{R}_+ \times \Omega \to X$ be an $L^p$-martingale that takes values in some closed subspace $X_0$ of $X$. Then there exists a unique decomposition $M = M^d + M^c$, where $M^c$ is continuous, $M^d$ is purely discontinuous and starts at zero, and $M^d$ and $M^c$
are $L^p$-martingales with values in $X_0 \subseteq X$. Moreover, the following norm estimates hold for every $t \in [0, \infty)$,

\begin{equation}
\|M^n(t)\|_{L^p(\Omega;X)} \leq \beta_{p,X} \|M(t)\|_{L^p(\Omega;X)},
\end{equation}

\begin{equation}
\|M^n(t)\|_{L^p(\Omega;X)} \leq \beta_{p,X} \|M(t)\|_{L^p(\Omega;X)}.
\end{equation}

Furthermore, if $A^0_X$ and $A^c_X$ are the corresponding linear operators that map $M$ to $M^n$ and $M^c$ respectively, then

\[ A^0_X = A^0_R \otimes \text{Id}_X, \]

\[ A^c_X = A^c_R \otimes \text{Id}_X. \]

Recall that for a given measure space $(S, \Sigma, \mu)$, the linear space of all real-valued measurable functions is denoted by $L^0(S)$.

**Definition 2.2.** Let $(S, \Sigma, \mu)$ be a measure space. Let $n : L^0(S) \to [0, \infty]$ be a function which satisfies the following properties:

(i) $n(x) = 0$ if and only if $x = 0$,
(ii) for all $x, y \in L^0(S)$ and $\lambda \in \mathbb{R}$, $n(\lambda x) = |\lambda| n(x)$ and $n(x + y) \leq n(x) + n(y)$,
(iii) if $x \in L^0(S)$, $y \in L^0(S)$, and $|x| \leq |y|$, then $n(x) \leq n(y)$,
(iv) if $0 \leq x_n \uparrow x$ with $(x_n)_{n=1}^\infty$ a sequence in $L^0(S)$ and $x \in L^0(S)$, then $n(x) = \sup_{n \in \mathbb{N}} n(x_n)$.

Let $X$ denote the space of all $x \in L^0(S)$ for which $\|x\| := n(x) < \infty$. Then $X$ is called the normed function space associated to $n$. It is called a Banach function space when $(X, \|\cdot\|_X)$ is complete.

We refer the reader to [31, Chapter 15] for details on Banach function spaces.

**Remark 2.3.** Let $X$ be a Banach function space over a measure space $(S, \Sigma, \mu)$. Then $X$ is continuously embedded into $L^0(S)$ endowed with the topology of convergence in measure on sets of finite measure. Indeed, assume $x_n \to x$ in $X$ and let $A \subseteq X$ be of finite measure. We claim that $1_{AX}x_n \to 1_{AX}x$ in measure. For this it suffices to show that every subsequence of $(x_n)_{n=1}^\infty$ has a further subsequence which converges a.e. to $x$. Let $(x_{n_k})_{k=1}^\infty$ be a subsequence. Choose a subsequence $(1_{AX}x_{n_{t_1}})_{t_1 \geq 1}$ such that $\sum_{t=1}^{\infty} |y_t - x| < \infty$. Then by [31, Exercise 64.1] $\sum_{t=1}^{\infty} |y_t - x|$ converges in $X$. In particular, $\sum_{t=1}^{\infty} |y_t - x| < \infty$ a.e. Therefore, $y_t \to x$ a.e. as desired.

Given a Banach function space $X$ over a measure space $S$ and Banach space $E$, let $X(E)$ denote the space of all strongly measurable functions $f : S \to E$ with $\|f\|_{X(E)} := \|s \mapsto \|f(s)\|_E\|_X \in X$. The space $X(E)$ becomes a Banach space when equipped with the norm $\|f\|_{X(E)}$.

A Banach function space has the UMD property if and only if (1.3) holds for some (or equivalently, for all) $p \in (1, \infty)$ (see [23]). A broad class of Banach function spaces with UMD is given by the reflexive Lorentz–Zygmund spaces (see [6]) and the reflexive Musielak–Orlicz spaces (see [17]).

**Definition 2.4.** $N : \mathbb{R}^+ \times \Omega \times S \to \mathbb{R}$ is called a (continuous) (local) martingale field if $N|_{[0,t] \times \Omega \times S}$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t \otimes \Sigma$-measurable for all $t \geq 0$ and $N(\cdot, \cdot, s)$ is a (continuous) (local) martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$ for almost all $s \in S$. 
Let $X$ be a Banach space, $I \subset \mathbb{R}$ be a closed interval (perhaps, infinite). A function $f : I \to X$ is called c\`adl\`ag (an acronym for the French phrase “continue \`a droite, limite \`a gauche”) if $f$ is right continuous and has limits from the left-hand side. We define a Skorohod space $\mathcal{D}(I; X)$ as a linear space consisting of all c\`adl\`ag functions $f : I \to X$. We denote the linear space of all bounded c\`adl\`ag functions $f : I \to X$ by $\mathcal{D}_b(I; X)$.

**Lemma 3.1.** Let $X$ be a UMD Banach function space and let $p \in (1, \infty)$. Let $N$ be a c\`adl\`ag martingale field with values in a finite dimensional subspace of $X$. Then for all $T > 0$,

$$\left\| \sup_{t \in [0, T]} |N(t, \cdot)|_{L^p(\Omega; X)} \right\|_{L^p(\Omega)} \lesssim_{p, X} \sup_{t \in [0, T]} \left\| N(t) \right\|_{L^p(\Omega; X)}$$

whenever one of the expression is finite.

**Proof.** Clearly, the left-hand side dominates the right-hand side. Therefore, we can assume the right-hand side is finite and in this case we have

$$\left\| N(T) \right\|_{L^p(\Omega; X)} = \sup_{t \in [0, T]} \left\| N(t) \right\|_{L^p(\Omega; X)} < \infty.$$ 

Since $N$ takes values in a finite dimensional subspace it follows from Doob’s $L^p$-inequality (applied coordinatewise) that the left-hand side is finite.

Since $N$ is a c\`adl\`ag martingale field and by Definition 2.2(iv) we have that

$$\lim_{n \to \infty} \left\| \sup_{0 \leq j \leq n} |N(jT/n, \cdot)|_{L^p(\Omega; X)} \right\|_{L^p(\Omega)} = \left\| \sup_{t \in [0, T]} |N(t, \cdot)|_{L^p(\Omega; X)} \right\|_{L^p(\Omega)}.$$ 

Set $M_j = N_{jT/n}$ for $j \in \{0, \ldots, n\}$ and $M_j = M_n$ for $j > n$. It remains to prove

$$\left\| \sup_{0 \leq j \leq n} |M_j(\cdot)|_{L^p(\Omega; X)} \right\|_{L^p(\Omega)} \leq C_{p, X} \left\| M_n \right\|_{L^p(\Omega; X)}.$$ 

If $(M_j)_{j=0}^n$ is a Paley–Walsh martingale (see [11, Definition 3.1.8 and Proposition 3.1.10]), this estimate follows from the boundedness of the dyadic lattice maximal operator [25, pp. 199–200 and Theorem 3]. In the general case one can replace $\Omega$ by a divisible probability space and approximate $(M_j)$ by Paley-Walsh martingales in a similar way as in [11, Corollary 3.6.7].

□
Therefore, \( (NT > n, m) \) holds in the special case that \( \xi \) is a Cauchy sequence and hence converges to some \( N \) from the space \( L^p(\Omega; X(D_0([0, \infty))) \). Clearly, \( N(t, \cdot) = M(t) \) and (1.1) holds in the special case that \( M \) becomes constant after \( T > 0 \).

In the case \( M \) is general, for each \( T > 0 \) we can set \( M^T(t) = M(t \wedge T) \). Then for each \( T > 0 \) we obtain a martingale field \( N^T \) as required. Since \( N^{T_1} = N^{T_2} \) on \([0, T_1 \wedge T_2] \), we can define a martingale field \( N \) by setting \( N(t, \cdot) = N^T(t, \cdot) \) on \([0, T] \). Finally, we note that

\[
\lim_{T \to \infty} \sup_{t \geq 0} \|M^T(t)\|_{L^p(\Omega;X)} = \sup_{t \geq 0} \|M(t)\|_{L^p(\Omega;X)}.
\]

Moreover, by Definition 2.2 (iv) we have

\[
\lim_{T \to \infty} \|M^T(t, \cdot)\|_{L^p(\Omega;X)} = \|M(t, \cdot)\|_{L^p(\Omega;X)}.
\]

Therefore the general case of (3.1) follows by taking limits.

Now let \( M \) be continuous, and let \( (M_n)_{n \geq 1} \) be as before. By the same argument as in the first part of the proof we can assume that there exists \( T > 0 \) such that \( M_t = M_{t \wedge T} \) for all \( t \geq 0 \). By Lemma 3.1 there exists a unique decomposition \( M_n = M^c_n + M^d_n \) such that \( M^d_n \) is purely discontinuous and starts at zero and \( M^c_n \) has continuous paths a.s. Then by (2.1)

\[
\|M(T) - M^c_n(T)\|_{L^p(\Omega;X)} \leq \beta_p, X \|M(T) - M_n(T)\|_{L^p(\Omega;X)} \to 0.
\]

Since \( M^c_n \) takes values in a finite dimensional subspace of \( X \) we can define a martingale field \( N_n \) by \( N_n(t, \omega, s) = M^c_n(t, \omega)(s) \). Now by Lemma 3.1

\[
\|\sup_{0 \leq s \leq T} |N_n(t, \cdot) - N_m(t, \cdot)|\|_{L^p(\Omega;X)} \approx_p, X \|M^c_n(T, \cdot) - M_m(T, \cdot)|\|_{L^p(\Omega;X)} \to 0.
\]

Therefore, \( (N_n)_{n \geq 1} \) is a Cauchy sequence and hence converges to some \( N \) from the space \( L^p(\Omega; X(C_0([0, \infty))) \). Analogously to the first part of the proof, \( N(t, \cdot) = M(t) \) for all \( t \geq 0 \). \( \square \)
Remark 3.3. Note that due to the construction of $N$ we have that $\Delta M_t(s) = \Delta N(t, s)$ for any stopping time $\tau$ and almost any $s \in S$. Indeed, let $(M_n)_{n \geq 1}$ and $(N_n)_{n \geq 1}$ be as in the proof of Theorem 3.2. Then on the one hand

$$\|\Delta M_t - \Delta(M_n)_T\|_{L^p(\Omega; X)} \leq \| \sup_{0 \leq t \leq T} |M(t) - M_n(t)|\|_{L^p(\Omega)} \approx_p \|M(T) - M_n(T)\|_{L^p(\Omega; X)} \to 0, \ n \to \infty.$$ 

On the other hand

$$\|\Delta N_t - \Delta(N_n)_T\|_{L^p(\Omega; X)} \leq \| \sup_{0 \leq t \leq T} |N(t) - N_n(t)|\|_{L^p(\Omega; X)} \approx_{p, X} \|N(T) - N_n(T)\|_{L^p(\Omega; X)} \to 0, \ n \to \infty.$$ 

Since $\|M_n(t) - N_n(t, \cdot)\|_{L^p(\Omega; X)} = 0$ for all $n \geq 0$, we have that by the limiting argument $\|\Delta M_t - \Delta(N_n)\|_{L^p(\Omega; X)} = 0$, so the desired follows from Definition 2.2(i).

One could hope there is a more elementary approach to derive continuity of $N$ in the case $M$ is continuous: if the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is generated by $M$, then $M(s)$ is $\mathbb{F}$-adapted for a.e. $s \in S$, and one might expect that $M$ has a continuous version. Unfortunately, this is not true in general as follows from the next example.

Example 3.4. There exists a continuous martingale $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ generated by $M$ and all $\mathbb{F}$-null sets, and a purely discontinuous nonzero $\mathbb{F}$-martingale $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$. Let $W : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a Brownian motion, $L : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a Poisson process such that $W$ and $L$ are independent. Let $\mathbb{P} = (\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by $W$ and $L$. Let $\sigma$ be an $\mathbb{F}$-stopping time defined as follows

$$\sigma = \inf\{u \geq 0 : \Delta L_u \neq 0\}.$$ 

Let us define

$$M := \int_{[0, \sigma]} 1_{[0, t]} \, dW = W^\sigma.$$ 

Then $M$ is a martingale. Let $\tilde{\mathbb{F}} := (\tilde{\mathcal{F}}_t)_{t \geq 0}$ be generated by $M$. Note that $\tilde{\mathcal{F}}_t \subset \mathcal{F}_t$ for any $t \geq 0$. Define a random variable

$$\tau = \inf\{t \geq 0 : \exists u \in [0, t]\} \text{ such that } M \text{ is a constant on } [u, t]\}.$$ 

Then $\tau = \sigma$ a.s. Moreover, $\tau$ is a $\tilde{\mathbb{F}}$-stopping time since for each $u \geq 0$

$$\mathbb{P}\{\tau = u\} = \mathbb{P}\{\sigma = u\} = \mathbb{P}\{\Delta L_u \neq 1\} \leq \mathbb{P}\{\Delta L_u \neq 1\} = 0,$$

and hence

$$\{\tau \leq u\} = \{\tau < u\} \cup \{\tau = u\} \subset \tilde{\mathcal{F}}_u.$$ 

Therefore $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ defined by

$$N_t := 1_{[\tau, \infty)}(t) - t \wedge \tau \geq 0,$$

is an $\tilde{\mathbb{F}}$-martingale since it is $\tilde{\mathbb{F}}$-measurable and since $N_t = (L_t - t)^\sigma$ a.s. for each $t \geq 0$, hence for each $u \in [0, t]$

$$\mathbb{E}(N_t)_{\mathbb{F}_u} = \mathbb{E}(\mathbb{E}(N_t | \mathcal{F}_u))_{\mathbb{F}_u} = \mathbb{E}(\mathbb{E}((L_t - t)^\sigma | \mathcal{F}_u))_{\mathbb{F}_u} = (L_u - u)^\sigma = N_u$$

due to the fact that $t \mapsto L_t - t$ is an $\mathbb{F}$-measurable $\mathbb{F}$-martingale (see [13, Problem 1.3.4]). But $(N_t)_{t \geq 0}$ is not continuous since $(L_t)_{t \geq 0}$ is not continuous.
4. Main result

Theorem 1.1 will be a consequence of the following more general result.

**Theorem 4.1.** Let $X$ be a UMD Banach function space over a σ-finite measure space $(S, \Sigma, \mu)$ and let $p \in (1, \infty)$. Let $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be a local $L^p$-martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$ and assume $M(0, \cdot) = 0$. Then there exists a mapping $N : \mathbb{R}_+ \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that

1. For all $t \geq 0$ and a.a. $\omega \in \Omega$, $N(t, \omega, \cdot) = M(t, \omega)$,
2. $N$ is a local martingale field,
3. the following estimate holds

\[
\sup_{t \geq 0} \|N(t, \cdot, \cdot)\|_{L^p(\Omega; X)} \lesssim_{p,X} \sup_{t \geq 0} \|M(t, \cdot, \cdot)\|_{L^p(\Omega)} \lesssim_{p,X} \|N\|^{1/2}_{L^p(\Omega; X)}. 
\]

To prove Theorem 1.1 we first prove a completeness result.

**Proposition 4.2.** Let $X$ be a Banach function space over a σ-finite measure space $S$, $1 \leq p < \infty$. Let

\[
\text{MQ}^p(X) := \{ N : \mathbb{R}_+ \times \Omega \times S \rightarrow \mathbb{R} : N(0, \cdot, \cdot, s) = 0 \quad \forall s \in S, \text{ and } \|N\|_{\text{MQ}^p(X)} < \infty \},
\]

where $\|N\|_{\text{MQ}^p(X)} := \|\|N\|^{1/2}_{L^p(\Omega; X)}$. Then $(\text{MQ}^p(X), \| \cdot \|_{\text{MQ}^p(X)})$ is a Banach space. Moreover, if $N_n \rightarrow N$ in $\text{MQ}^p$, then there exists a subsequence $(N_{n_k})_{k \geq 1}$ such that pointwise a.e. in $S$, we have $N_{n_k} \rightarrow N$ in $L^1(\Omega; \mathcal{D}_b([0, \infty)))$.

**Proof.** Let us first check that $\text{MQ}^p(X)$ is a normed vector space. For this only the triangle inequality requires some comments. By the well-known estimate for local martingales $M, N$ (see [13, Theorem 26.6(iii)]) we have that a.s.

\[
[M + N]_t = [M]_t + 2[M, N]_t + [N]_t 
\leq [M]_t + 2[M]^{1/2}_t [N]^{1/2}_t + [N]_t = ([M]^{1/2}_t + [N]^{1/2}_t)^2,
\]

Therefore, $[M + N]^{1/2}_t \leq [M]^{1/2}_t + [N]^{1/2}_t$ a.s. for all $t \in [0, \infty]$.

Let $(N_k)_{k \geq 1}$ be such that $\sum_{k \geq 1} \|N_k\|_{\text{MQ}^p(X)} < \infty$. It suffices to show that $\sum_{k \geq 1} N_k$ converges in $\text{MQ}^p(X)$. Observe that by monotone convergence in $\Omega$ and Jensen’s inequality applied to $\| \cdot \|_X$ for any $n > m \geq 1$ we have

\[
\left\| \sum_{k=m+1}^n \mathbb{E}[N_k]^{1/2} \right\|_X = \left\| \sum_{k=m+1}^n \mathbb{E}[N_k]^{1/2} - \sum_{k=1}^{m} \mathbb{E}[N_k]^{1/2} \right\|_X 
\leq \left\| \mathbb{E} \sum_{k=m+1}^n [N_k]^{1/2}_\infty \right\|_X \leq \left\| \sum_{k=m+1}^n [N_k]^{1/2}_\infty \right\|_{L^1(\Omega; X)} 
\leq \sum_{k=m+1}^n \left\| [N_k]^{1/2} \right\|_{L^p(\Omega; X)} \rightarrow 0, \quad m, n \rightarrow \infty,
\]

where the latter holds due to the fact that $\sum_{k \geq 1} \|N_k\|^{1/2}_{L^p(\Omega; X)} < \infty$. Thus $\sum_{k=1}^n \mathbb{E}[N_k]^{1/2}$ converges in $X$ as $n \rightarrow \infty$, where the corresponding limit coincides
with its pointwise limit \( \sum_{k \geq 1} \mathbb{E}[N_k]^{1/2} \) by Remark 2.3. Therefore, since any element of \( X \) is finite a.s. by Definition 2.2, we can find \( S_0 \in \Sigma \) such that \( \mu(S_0) = 0 \) and pointwise in \( S_0 \), we have \( \sum_{k \geq 1} \mathbb{E}[N_k]^{1/2} < \infty \). Fix \( s \in S_0 \). In particular, we find that \( \sum_{k \geq 1}[N_k]^{1/2} \) converges in \( L^1(\Omega) \). Moreover, since by the scalar Burkholder-Davis-Gundy inequalities \( \mathbb{E} \sup_{t \geq 0} |N_k(t, \cdot, s)| \leq \mathbb{E}[N_k(s)]^{1/2} \), we also obtain that

\[
N(\cdot, s) := \sum_{k \geq 1} N_k(\cdot, s) \text{ converges in } L^1(\Omega; \mathcal{D}_b([0, \infty))).
\]

Let \( N(\cdot, s) = 0 \) for \( s \notin S_0 \). Then \( N \) defines a martingale field. Moreover, by the scalar Burkholder-Davis-Gundy inequalities

\[
\lim_{m \to \infty} \left[ \sum_{k=n}^{m} N_k(\cdot, s) \right]^{1/2}_\infty \leq \sum_{k=n}^{\infty} [N_k(\cdot, s)]^{1/2}_\infty.
\]

It remains to prove that \( N \in \text{MQ}^p(X) \) and \( \sum_{k \geq 1} N_k \) with convergence in \( \text{MQ}^p(X) \). Let \( \varepsilon > 0 \). Choose \( n \in \mathbb{N} \) such that \( \sum_{k \geq n+1} \mathbb{E}[N_k]_{\text{MQ}^p(X)} < \varepsilon \). It follows from (4.3) that \( \mathbb{E}\left[ \sum_{k \geq 1} \left[ |N_k|^{1/2}_\infty \right] \right] < \infty \), so \( \sum_{k \geq 1} [N_k]^{1/2}_\infty \) a.s. converges in \( X \). Now by (4.5), the triangle inequality and Fatou’s lemma, we obtain

\[
\left\| \left[ \sum_{k \geq n+1} N_k \right]^{1/2}_{L^p(\Omega; X)} \right\|_{L^p(\Omega; X)} \leq \sum_{k=n+1}^{\infty} \mathbb{E}[N_k]^{1/2}_\infty \leq \sum_{k=n+1}^{\infty} \left\| [N_k]^{1/2}_\infty \right\|_{L^p(\Omega; X)} \leq \liminf_{m \to \infty} \sum_{k=n+1}^{m} \left\| [N_k]^{1/2}_\infty \right\|_{L^p(\Omega; X)} < \varepsilon^p.
\]

Therefore, \( N \in \text{MQ}^p(X) \) and \( \|N - \sum_{k=1}^{n} N_k\|_{\text{MQ}^p(X)} < \varepsilon \).

For the proof of the final assertion assume that \( N_n \to N \) in \( \text{MQ}^p(X) \). Choose a subsequence \( (N_{n_k})_{k \geq 1} \) such that \( \|N_{n_k} - N\|_{\text{MQ}^p(X)} \leq 2^{-k} \). Then \( \sum_{k \geq 1} \|N_{n_k} - N\|_{\text{MQ}^p(X)} < \infty \) and hence by (4.3) we see that pointwise a.e. in \( S \), the series \( \sum_{k \geq 1} (N_{n_k} - N) \) converges in \( L^1(\Omega; \mathcal{D}_b([0, \infty))) \). Therefore, \( N_{n_k} \to N \) in \( L^1(\Omega; \mathcal{D}_b([0, \infty); X)) \) as required. \( \square \)

For the proof of Theorem 4.1 we will need the following lemma presented in [8, Théorème 2].

**Lemma 4.3.** Let \( 1 < p < \infty \), \( M : \mathbb{R}_+ \times \Omega \to \mathbb{R} \) be an \( L^p \)-martingales. Let \( T > 0 \). For each \( n \geq 1 \) define

\[
R_n := \sum_{k=1}^{n} \left| M_{T_k} - M_{T_{k-1}} \right|^2.
\]

Then \( R_n \) converges to \( |M|_T \) in \( L^{p/2} \).
Proof of Theorem 4.1. The existence of the local martingale field $N$ together with the first estimate in (4.1) follows from Theorem 3.2. It remains to prove

$$\| \sup_{t \geq 0} \| M(t, \cdot) \|_X \|_{L^p(\Omega)} \lesssim_p \| N \|_{L^p(\Omega; X)}^{1/2}.$$  

Due to Definition 2.2(iv it suffices to prove the above norm equivalence in the case $M$ and $N$ becomes constant after some fixed time $T$.

Step 1: The finite dimensional case. Assume that $M$ takes values in a finite dimensional subspace $Y$ of $X$ and that the right hand side of (4.6) is finite. Then we can write $N(t, s) = M(t)(s) = \sum_{j=1}^n M_j(t)x_j(s)$, where each $M_j$ is a scalar-valued martingale with $M_j(T) \in L^p(\Omega)$ and $x_1, \ldots, x_n \in X$ form a basis of $Y$.

Note that for any $c_1, \ldots, c_n \in L^p(\Omega)$ we have that

$$\left\| \sum_{j=1}^n c_jx_j \right\|_{L^p(\Omega; X)} \lesssim_{p, Y} \sum_{j=1}^n \| c_j \|_{L^p(\Omega)}.$$  

Fix $m \geq 1$. Then by (1.3) and Doob’s maximal inequality

$$\| \sup_{t \geq 0} \| M(t, \cdot) \|_X \|_{L^p(\Omega)} \lesssim_p \| M(T, \cdot) \|_{L^p(\Omega; X)}$$  

$$= \left\| \sum_{i=1}^m M_{T_i} - M_{T_{i+1}} \right\|_{L^p(\Omega; X)}$$  

$$\lesssim_{p, Y} \left\| \left( \sum_{i=1}^m \left| M_{T_i} - M_{T_{i+1}} \right|^2 \right)^{1/2} \right\|_{L^p(\Omega; X)},$$

and by (4.7) and Lemma 4.3, the right hand side of (4.8) converges to

$$\| [M] \|_{L^p(\Omega; X)}^{1/2} \lesssim_p \| N \|_{L^p(\Omega; X)}^{1/2}.$$  

Step 2: Reduction to the case where $M$ takes values in a finite dimensional subspace of $X$. Let $M(T) \in L^p(\Omega; X)$. Then we can find simple functions $(\xi_n)_{n \geq 1}$ in $L^p(\Omega; X)$ such that $\xi_n \to M(T)$. Let $M_n(t) = \mathbb{E}(\xi_n | F_t)$ for all $t \geq 0$ and $n \geq 1$, $(N_n)_{n \geq 1}$ be the corresponding martingale fields. Then each $M_n$ takes values in a finite dimensional subspace $X_n \subseteq X$, and hence by Step 1

$$\| \sup_{t \geq 0} \| M_n(t, \cdot) - M_m(t, \cdot) \|_X \|_{L^p(\Omega)} \lesssim_{p, X} \| N_n - N_m \|_{L^p(\Omega; X)}^{1/2}$$

for any $m, n \geq 1$. Therefore since $(\xi_n)_{n \geq 1}$ is Cauchy in $L^p(\Omega; X)$, $(N_n)_{n \geq 1}$ converges to some $N$ in $\text{MQ}^p(X)$ by the first part of Proposition 4.2.

Let us show that $N$ is the desired local martingale field. Fix $t \geq 0$. We need to show that $N(\cdot, t, \cdot) = M_t$ a.s. on $\Omega$. First notice that by the second part of Proposition 4.2 there exists a subsequence of $(N_n)_{n \geq 1}$ which we will denote by $(N_{n_k})_{k \geq 1}$ as well such that $N_{n_k}(\cdot, t, \sigma) \to N(\cdot, t, \sigma)$ in $L^1(\Omega)$ for a.e. $\sigma \in S$. On the other hand by Jensen’s inequality

$$\| \mathbb{E}[N_{n_k}(\cdot, t, \cdot) - M_t] \|_X \leq \| \mathbb{E}[M_n(t) - M(t)] \|_X \leq \mathbb{E}\| M_n(t) - M(t) \|_X \to 0, \quad n \to \infty.$$  

Hence $N_{n_k}(\cdot, t, \cdot) \to M_t$ in $X(L^1(\Omega))$, and thus by Remark 2.3 in $L^0(S; L^1(\Omega))$. Therefore we can find a subsequence of $(N_{n_k})_{k \geq 1}$ (which we will again denote by $(N_{n_k})_{k \geq 1}$) such that $N_{n_k}(\cdot, t, \sigma) \to M_t(\sigma)$ in $L^1(\Omega)$ for a.e. $\sigma \in S$ (here we use the fact that $\mu$ is $\sigma$-finite), so $N(\cdot, t, \cdot) = M_t$ a.s. on $\Omega \times S$, and consequently by Definition 2.2(iii), $N(\omega, t, \cdot) = M_t(\omega)$ for a.a. $\omega \in \Omega$. Thus (4.6) follows by letting $n \to \infty$. 
Step 3: Reduction to the case where the left-hand side of (4.6) is finite. Assume that the left-hand side of (4.6) is infinite, but the right-hand side is finite. Since $M$ is a local $L^p$-martingale we can find a sequence of stopping times $(\tau_n)_{n \geq 1}$ such that $\tau_n \uparrow \infty$ and $\|M^{\tau_n}_{T}\|_{L^p(\Omega; X)} < \infty$ for each $n \geq 1$. By the monotone convergence theorem and Definition (2.2) we have

$$
\|N\|_{L^p(\Omega; X)} = \lim_{n \to \infty} \|N^{\tau_n}\|_{L^p(\Omega; X)} = \lim_{n \to \infty} \sup_{0 \leq t \leq T} \|M^{\tau_n}_{t}\|_{L^p(\Omega; X)} = \lim_{n \to \infty} \sup_{0 \leq t \leq T} \|M^{\tau_n}_{t}\|_{L^p(\Omega)} = \infty
$$

and hence the right-hand side of (4.6) is infinite as well.

We use an extrapolation argument to extend part of Theorem 4.1 to $p \in (0, 1]$ in the continuous-path case.

**Corollary 4.4.** Let $X$ be a UMD Banach function space over a $\sigma$-finite measure space and let $p \in (0, \infty)$. Let $M$ be a continuous local martingale $M : \mathbb{R}_+ \times \Omega \to X$ with $M(0, \cdot) = 0$. Then there exists a continuous local martingale field $N : \mathbb{R}_+ \times \Omega \times S \to \mathbb{R}$ such that for a.a. $\omega \in \Omega$, all $t \geq 0$, and a.a. $s \in S$, $N(t, \omega, \cdot) = M(t, \omega)(s)$ and

$$
\|\sup_{t \geq 0} \|M(t, \cdot)\|_{L^p(\Omega)} \|_{L^p(\Omega; X)} \leq \|N\|_{L^p(\Omega; X)}.
$$

**Proof.** By a stopping time argument we can reduce to the case where $\|M(t, \omega)\|_{X}$ is uniformly bounded in $t \in \mathbb{R}_+$ and $\omega \in \Omega$ and $M$ becomes constant after a fixed time $T$. Now the existence of $N$ follows from Theorem 4.1 and it remains to prove (4.9) for $p \in (0, 1]$. For this we can use a classical argument due to Lenglart. Indeed, for both estimates we can apply [16] or [24, Proposition IV.4.7] to the continuous increasing processes $Y, Z : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ given by

$$
Y_u = \mathbb{E} \sup_{t \in [0, u]} \|M(t, \cdot)\|_{X},
$$

$$
Z_u = \|s \mapsto [N(\cdot, \cdot, s)]_u^{1/2}\|_{X},
$$

where $q \in (1, \infty)$ is a fixed number. Then by (4.1) for any bounded stopping time $\tau$, we have

$$
\mathbb{E} Y_u^q = \sup_{t \geq 0} \|M(t \wedge \tau, \cdot)\|_{X}^q \approx_{q, X} \mathbb{E} \|s \mapsto [N(\cdot \wedge \tau, \cdot, s)]_{\tau}^{1/2}\|_{X}^q = \mathbb{E} Z_u^q,
$$

where we used [13, Theorem 17.5] in ($*$). Now (4.9) for $p \in (0, q)$ follows from [16] or [24, Proposition IV.4.7].

As we saw in Theorem 3.2 continuity of $M$ implies pointwise continuity of the corresponding martingale field $N$. The following corollaries of Theorem 4.1 are devoted to proving the same type of assertions concerning pure discontinuity, quasi-left continuity, and having accessible jumps.

Let $\tau$ be a stopping time. Then $\tau$ is called predictable if there exists a sequence of stopping times $(\tau_n)_{n \geq 1}$ such that $\tau_n < \tau$ a.s. on $\{\tau > 0\}$ for each $n \geq 1$ and $\tau_n \not\nearrow \tau$ a.s. A càdlàg process $V : \mathbb{R}_+ \times \Omega \to X$ is called to have accessible
jumps if there exists a sequence of predictable stopping times \((\tau_n)_{n \geq 1}\) such that 
\[ \{t \in \mathbb{R}_+ : \Delta V \neq 0\} \subset \{\tau_1, \ldots, \tau_n, \ldots\} \text{ a.s.} \]

**Corollary 4.5.** Let \(X\) be a UMD function space over a measure space \((S, \Sigma, \mu)\), \(1 < p < \infty\), \(M : \mathbb{R}_+ \times \Omega \to X\) be a purely discontinuous \(L^p\)-martingale with accessible jumps. Let \(N\) be the corresponding martingale field. Then \(N(\cdot, s)\) is a purely discontinuous martingale with accessible jumps for a.e. \(s \in S\).

For the proof we will need the following lemma taken from [27, Subsection 5.3].

**Lemma 4.6.** Let \(X\) be a Banach space, \(1 \leq p < \infty\), \(M : \mathbb{R}_+ \times \Omega \to X\) be an \(L^p\)-martingale, \(\tau\) be a predictable stopping time. Then \((\Delta M_t \mathbf{1}_{[0,t]}(\tau))_{t \geq 0}\) is an \(L^p\)-martingale as well.

**Proof of Corollary 4.5.** Without loss of generality we can assume that there exists \(T \geq 0\) such that \(M_t = M_T\) for all \(t \geq T\), and that \(M_0 = 0\). Since \(M\) has accessible jumps, there exists a sequence of predictable stopping times \((\tau_n)_{n \geq 1}\) such that a.s.
\[ \{t \in \mathbb{R}_+ : \Delta M \neq 0\} \subset \{\tau_1, \ldots, \tau_n, \ldots\}. \]

For each \(m \geq 1\) define a process \(M^m : \mathbb{R}_+ \times \Omega \to X\) in the following way:
\[ M^m(t) := \sum_{n=1}^m \Delta M_{\tau_n} \mathbf{1}_{[0,t]}(\tau_n), \quad t \geq 0. \]

Note that \(M^m\) is a purely discontinuous \(L^p\)-martingale with accessible jumps by Lemma 4.6. Let \(N^m\) be the corresponding martingale field. Then \(N^m(\cdot, s)\) is a purely discontinuous martingale with accessible jumps for almost any \(s \in S\) due to Remark 3.3. Moreover, for any \(m \geq \ell \geq 1\) and any \(t \geq 0\) we have that a.s.
\[ [N^m(\cdot, s)]_t \geq [N^\ell(\cdot, s)]_t. \]

Define \( F : \mathbb{R}_+ \times \Omega \times S \to \mathbb{R}_+ \cup \{+\infty\}\) in the following way:
\[ F(t, \cdot, s) := \lim_{m \to \infty} [N^m(\cdot, s)]_t, \quad s \in S, t \geq 0. \]

Note that \(F(\cdot, \cdot, s)\) is a.s. finite for almost any \(s \in S\). Indeed, by Theorem 4.1 and [27, Theorem 4.2] we have that for any \(m \geq 1\)
\[ \| [N^m]^{1/2} \|_{L^p(\Omega; X)} \leq \beta_p X M^m(T, \cdot) \|_{L^p(\Omega; X)} \leq \beta_p X M(T, \cdot) \|_{L^p(\Omega; X)}, \]
so by Definition 2.2(iv), \(F(\cdot, \cdot, s)\) is a.s. finite for almost any \(s \in S\) and
\[ \| F^{1/2} \|_{L^p(\Omega; X)} = \lim_{m \to \infty} \| [N^m]^{1/2} \|_{L^p(\Omega; X)} \leq \beta_p X \limsup_{m \to \infty} \| M^m(T, \cdot) \|_{L^p(\Omega; X)} \leq \beta_p X \| M(T, \cdot) \|_{L^p(\Omega; X)}. \]

Moreover, for almost any \(s \in S\) we have that \(F(\cdot, \cdot, s)\) is pure jump and
\[ \{t \in \mathbb{R}_+ : \Delta F \neq 0\} \subset \{\tau_1, \ldots, \tau_n, \ldots\}. \]

Therefore to this end it suffices to show that \(F(s) = [N(s)]\) a.s. on \(\Omega\) for a.e. \(s \in S\). Note that by Definition 2.2(iv),
\[ \| (F - [N^m])^{1/2}(\infty) \|_{L^p(\Omega; X)} \to 0, \quad m \to \infty \]
so by Theorem 4.1 \((M^m(T))_{m \geq 1}\) is a Cauchy sequence in \(L^p(\Omega; X)\). Let \(\xi\) be its limit, \(M^0 : \mathbb{R}_+ \times \Omega \to X\) be a martingale such that \(M^0(t) = \mathbb{E}(\xi|\mathcal{F}_t)\) for all
$t \geq 0$. Then by Proposition 2.14 \cite{21} $M^0$ is purely discontinuous. Moreover, for any stopping time $\tau$ a.s.

$$\Delta M^0_t = \lim_{m \to \infty} \Delta M^m_t = \lim_{m \to \infty} \Delta M_t 1_{[\tau_1, \ldots, \tau_m]}(\tau) = \Delta M_\tau,$$

where the latter holds since the set $\{\tau_1, \ldots, \tau_n, \ldots\}$ exhausts the jump times of $M$. Therefore $M = M^0$ since both $M$ and $M^0$ are purely discontinuous with the same jumps, and hence $[N] = F$ (where $F(s) = [M^0(s)]$ by \eqref{4.10}). Consequently $N(\cdot, s)$ is purely discontinuous with accessible jumps for almost all $s \in S$. \hfill \Box

**Remark 4.7.** Note that the proof of Corollary \cite{4.5} also implies that $M^m_t \to M_t$ in $L^p(\Omega; X)$ for each $t \geq 0$.

A càdlàg process $V : \mathbb{R}_+ \times \Omega \to X$ is called quasi-left continuous if $\Delta V_\tau = 0$ a.s. for any predictable stopping time $\tau$.

**Corollary 4.8.** Let $X$ be a UMD function space over a measure space $(S, \Sigma, \mu)$, $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \to X$ be a purely discontinuous quasi-left continuous $L^p$-martingale. Let $N$ be the corresponding martingale field. Then $N(\cdot, s)$ is a purely discontinuous quasi-left continuous martingale for a.e. $s \in S$.

The proof will exploit the random measure theory. Let $(J, \mathcal{J})$ be a measurable space. Then a family $\mu = \{\mu(\omega; dt, dx), \omega \in \Omega\}$ of nonnegative measures on $(\mathbb{R}_+ \times J; \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{J})$ is called a random measure. A random measure $\mu$ is called integer-valued if it takes values in $\mathbb{N} \cup \{\infty\}$, i.e. for each $A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{J}$ one has that $\mu(A) \in \mathbb{N} \cup \{\infty\}$ a.s., and if $\mu(\{t\} \times J) \in \{0, 1\}$ a.s. for all $t \geq 0$.

Let $X$ be a Banach space, $\mu$ be a random measure, $F : \mathbb{R}_+ \times \Omega \times J \to X$ be such that $\int_{\mathbb{R}_+ \times \Omega \times J} \|F\| \, d\mu < \infty$ a.s. Then the integral process $((F * \mu)_t)_{t \geq 0}$ of the form

$$(F * \mu)_t := \int_{\mathbb{R}_+ \times \Omega \times J} F(s, \cdot, x) 1_{[0, t]}(s) \mu(\cdot; ds, dx), \quad t \geq 0,$$

is a.s. well-defined.

Any integer-valued optional $\mathcal{P} \otimes \mathcal{J}$-\sigma-finite random measure $\mu$ has a compensator: a unique predictable $\mathcal{P} \otimes \mathcal{J}$-\sigma-finite random measure $\nu$ such that $\mathbb{E}(W * \mu)_\infty = \mathbb{E}(W + \nu)_\infty$ for each $\mathcal{P} \otimes \mathcal{J}$-measurable real-valued nonnegative $W$ (see \cite{12, Theorem II.1.8}). For any optional $\mathcal{P} \otimes \mathcal{J}$-\sigma-finite measure $\mu$ we define the associated compensated random measure by $\mu_c = \mu - \nu$.

Recall that $\mathcal{P}$ denotes the predictable $\sigma$-algebra on $\mathbb{R}_+ \times \Omega$ (see \cite{13} for details). For each $\mathcal{P} \otimes \mathcal{J}$-strongly-measurable $F : \mathbb{R}_+ \times \Omega \times J \to X$ such that $\mathbb{E}(|F| * \mu)_\infty < \infty$ (or, equivalently, $\mathbb{E}(|F| * \nu)_\infty < \infty$, see the definition of a compensator above) we can define a process $F*\mu_c$ by $F*\mu - F*\nu$. Then this process is a purely discontinuous local martingale. We will omit here some technicalities for the convenience of the reader and refer the reader to \cite{12, Chapter II.1}, \cite{7, Subsection 5.4-5.5}, and \cite{14, 13, 21} for more details on random measures.

**Proof of Corollary 4.8** Without loss of generality we can assume that there exists $T \geq 0$ such that $M_t = M_T$ for all $t \geq T$, and that $M_0 = 0$. Let $\mu$ be a random measure defined on $\mathbb{R}_+ \times X$ in the following way

$$\mu(A \times B) = \sum_{i \geq 0} 1_A(t) 1_{B \setminus \{0\}}(\Delta M_t),$$

where
where \( A \subset \mathbb{R}_+ \) is a Borel set, and \( B \subset X \) is a ball. For each \( k, \ell \geq 1 \) we define a stopping time \( \tau_{k, \ell} \) as follows

\[
\tau_{k, \ell} = \inf\{ t \in \mathbb{R}_+ : \#\{u \in [0, t] : \| \Delta M_u \|_X \in [1/k, k]\} = \ell \}.
\]

Since \( M \) has càdlàg trajectories, \( \tau_{k, \ell} \) is a.s. well-defined and takes its values in \([0, \infty]\). Moreover, \( \tau_{k, \ell} \to \infty \) for each \( k \geq 1 \) a.s. as \( \ell \to \infty \), so we can find a subsequence \( (\tau_{k_n, \ell_n})_{n \geq 1} \) such that \( k_n \geq n \) for each \( n \geq 1 \) and \( \inf_{m \geq n} \tau_{k_m, \ell_m} \to \infty \) a.s. as \( n \to \infty \).

Remark 4.11. \( \square \)

Note that if a local martingale \( M : \mathbb{R}_+ \times \Omega \to X \) is continuous, purely discontinuous quasi-left continuous, or purely discontinuous quasi-left continuous martingale of a local martingale \( M \) has some canonical decomposition, then this decomposition is unique (see \([13, 27, 28, 30]\)).

Corollary 4.9. Let \( X \) be a Banach space. A local martingale \( M : \mathbb{R}_+ \times \Omega \to X \) is called to have the canonical decomposition if there exist local martingales \( M^c, M^q, M^a : \mathbb{R}_+ \times \Omega \to X \) such that \( M^c \) is continuous, \( M^q \) and \( M^a \) are purely discontinuous, \( M^q \) is quasi-left continuous, \( M^a \) has accessible jumps, \( M^c_0 = M^q_0 = 0 \), and \( M = M^c + M^q + M^a \). Existence of such a decomposition was first shown in the real-valued case by Yoeurp in \([30]\), and recently such an existence was obtained in the UMD space case (see \([27, 28]\)).

Remark 4.10. Let \( X \) be a UMD Banach function space, \( 1 < p < \infty \), \( M : \mathbb{R}_+ \times \Omega \to X \) be an \( L^p \)-martingale. Let \( N \) be the corresponding martingale field. Let \( M = M^c + M^q + M^a \) be the canonical decomposition, \( N^c, N^q, \) and \( N^a \) be the corresponding martingale fields. Then \( N(s) = N^c(s) + N^q(s) + N^a(s) \) is the canonical decomposition of \( N(s) \) for a.e. \( s \in S \). In particular, if \( M_0 = 0 \) a.s., then \( M \) is continuous, purely discontinuous quasi-left continuous, or purely discontinuous with accessible jumps if and only if \( N(s) \) is so for a.e. \( s \in S \).

Proof. The first part follows from Theorem 3.2 Corollary 4.5 and Corollary 4.8 and the fact that \( N(s) = N^c(s) + N^q(s) + N^a(s) \) is then a canonical decomposition of a local martingale \( N(s) \) which is unique due to Remark 4.9. Let us show the second part. One direction follows from Theorem 3.2 Corollary 4.5 and Corollary 4.8. For the other direction assume that \( N(s) \) is continuous for a.e. \( s \in S \). Let \( M = M^c + M^q + M^a \) be the canonical decomposition, \( N^c, N^q, \) and \( N^a \) be the corresponding martingale fields of \( M^c, M^q, \) and \( M^a \). Then by the first part of the theorem and the uniqueness of the canonical decomposition (see Remark 4.9) we have that for a.e. \( s \in S \), \( N^q(s) = N^a(s) = 0 \), so \( M^q = M^a = 0 \), and hence \( M \) is continuous. The proof for the case of pointwise purely discontinuous quasi-left continuous \( N \) or pointwise purely discontinuous \( N \) with accessible jumps is similar. \( \square \)

Remark 4.11. It remains open whether the first two-sided estimate in (4.1) can be extended to \( p = 1 \). Recently, in \([29]\) the second author has extended the second
two-sided estimate in (4.1) to arbitrary UMD Banach spaces and to $p \in [1, \infty)$. Here the quadratic variation has to be replaced by a generalized square function.

References

[1] M. Antoni. Regular Random Field Solutions for Stochastic Evolution Equations. PhD thesis, 2017.
[2] J. Bourgain. Extension of a result of Benedek, Calderón and Panzone. Ark. Mat., 22(1):91–95, 1984.
[3] D.L. Burkholder. Distribution function inequalities for martingales. Ann. Probability, 1:19–42, 1973.
[4] D.L. Burkholder. A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional. Ann. Probab., 9(6):997–1011, 1981.
[5] D.L. Burkholder. Martingales and singular integrals in Banach spaces. In Handbook of the geometry of Banach spaces, Vol. I, pages 233–269. North-Holland, Amsterdam, 2001.
[6] F. Cobos. Some spaces in which martingale difference sequences are unconditional. Bull. Polish Acad. Sci. Math., 34(11-12):695–703 (1987), 1986.
[7] S. Dirksen and I.S. Yaroslavtsev. $L^q$-valued Burkholder-Rosenthal inequalities and sharp estimates for stochastic integrals. [arXiv:1707.00109], 2017.
[8] C. Doléans. Variation quadratique des martingales continues à droite. Ann. Math. Statist, 40:284–289, 1969.
[9] J. García-Cuerva, R. Macías, and J. L. Torrea. The Hardy-Littlewood property of Banach lattices. Israel J. Math., 83(1-2):177–201, 1993.
[10] D.J.H. Garling. Brownian motion and UMD-spaces. In Probability and Banach spaces (Zaragoza, 1985), volume 1221 of Lecture Notes in Math., pages 36–49. Springer, Berlin, 1986.
[11] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. Analysis in Banach Spaces. Volume I: Martingales and Littlewood-Paley Theory, volume 63 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Springer, 2016.
[12] J. Jacod and A.N. Shiryaev. Limit theorems for stochastic processes, volume 288 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, second edition, 2003.
[13] O. Kallenberg. Foundations of modern probability. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.
[14] O. Kallenberg. Random measures, theory and applications, volume 77 of Probability Theory and Stochastic Modelling. Springer, Cham, 2017.
[15] I. Karatzas and S.E. Shreve. Brownian motion and stochastic calculus, volume 113 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1991.
[16] E. Lenglart. Relation de domination entre deux processus. Ann. Inst. H. Poincaré Sect. B (N.S.), 13(2):171–179, 1977.
[17] N. Lindemulder, M.C. Veraar, and I.S. Yaroslavtsev. The UMD property for Musielak–Orlicz spaces. In Positivity and Noncommutative Analysis. Festschrift in honour of Ben de Pagter on the occasion of his 65th birthday, Trends in Mathematics. Birkhäuser, 2019. [arXiv:1810.13362], to appear.
[18] C. Marinelli. On maximal inequalities for purely discontinuous $L_q$-valued martingales. [arXiv:1311.7120], 2013.
[19] C. Marinelli and M. Röckner. On maximal inequalities for purely discontinuous martingales in infinite dimensions. In Séminaire de Probabilités XLVI, volume 2123 of Lecture Notes in Math., pages 293–315. Springer, Cham, 2014.
[20] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. Stochastic integration in UMD Banach spaces. Ann. Probab., 35(4):1438–1478, 2007.
[21] A.A. Novikov. Discontinuous martingales. Teor. Verojatnost. i Primenen., 20:13–28, 1975.
[22] A. Osękowski. A note on the Burkholder-Rosenthal inequality. Bull. Pol. Acad. Sci. Math., 60(2):177–185, 2012.
[23] A. Osękowski. Sharp martingale and semimartingale inequalities, volume 72 of Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series). Birkhäuser/Springer Basel AG, Basel, 2012.
[24] D. Revuz and M. Yor. Continuous martingales and Brownian motion, volume 293 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, third edition, 1999.
[25] J.L. Rubio de Francia. Martingale and integral transforms of Banach space valued functions. In Probability and Banach spaces (Zaragoza, 1985), volume 1221 of Lecture Notes in Math., pages 195–222. Springer, Berlin, 1986.
[26] M.C. Veraar and I.S. Yaroslavtsev. Cylindrical continuous martingales and stochastic integration in infinite dimensions. Electron. J. Probab., 21:Paper No. 59, 53, 2016.
[27] I.S. Yaroslavtsev. Martingale decompositions and weak differential subordination in UMD Banach spaces. [arXiv:1706.01731], to appear in Bernoulli, 2017.
[28] I.S. Yaroslavtsev. On the martingale decompositions of Gundy, Meyer, and Yoeurp in infinite dimensions. [arXiv:1712.00401] to appear in Ann. Inst. Henri Poincaré Probab. Stat., 2017.
[29] I.S. Yaroslavtsev. Burkholder–Davis–Gundy inequalities in UMD Banach spaces. [arXiv:1807.05573], 2018.
[30] Ch. Yoeurp. Décompositions des martingales locales et formules exponentielles. pages 432–480. Lecture Notes in Math., Vol. 511, 1976.
[31] A.C. Zaanen. Integration. North-Holland Publishing Co., Amsterdam; Interscience Publishers John Wiley & Sons, Inc., New York, 1967. Completely revised edition of An introduction to the theory of integration.