A state-sum invariant of tangles in oriented surfaces

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Abstract

We exploit a recent insight of [2], which explains how to obtain invariants of links in space, or tangles in surfaces, from quantities invariant only under a restricted set of Reidemeister moves. The main idea involves modifying diagrams to simplify their faces. This will now be used to define new state-sum invariants based on assigning symbols to faces, in a new way that avoids undesirable simplifications in the relations. Any such invariant has several properties of interest, among them functorial ones roughly like those of a TQFT: pasting surfaces with compatible tangles along some of their boundary components corresponds to obtaining the value of a matrix-valued invariant by multiplying matrices for the individual pieces.

As a concrete illustration of the ideas, simple assumptions yield what we call the \(u\)-invariant. For closed tangles (links in surfaces) it takes values in \(\mathbb{Z}[u]\), where \(u\) is a primitive fifth root of unity. It is well-adapted for computations, surprisingly strong for something whose definition (with verification) is so easy, and has several interesting properties.

1 Introduction

While this work is mainly concerned with invariants of links, it has its inspiration in the 3-manifold invariants introduced in two seminal articles: that by Turaev and Viro [10], where symbols in \{0, \ldots, r-1\} are assigned to the 2-cells of a special spine of a 3-manifold, and that by Reshetikhin and Turaev [8], where such symbols are assigned to the faces and component knots of a projection to \(S^2\) of a framed link in \(S^3\) representing a 3-manifold by surgery instructions. A state is such an assignment, subject to certain rules. See also Lickorish [7] and Kauffman-Lins [5] for connections of the Reshetikhin-Turaev theory with the Temperley-Lieb algebra and the Jones polynomial. Our motivating idea was to generalize and simplify the above constructions via an abstract algebraic approach, in a way roughly similar to the treatment of ideal Turaev-Viro invariants by King [6], but with a more radical reduction of the machinery involved. Euler characteristics of faces were relevant in [10] and will reappear below in a related role.

What we now wish to present arose from that project. It concerns a way to produce invariants of tangles in oriented surfaces \(S\), where the value of each state of a link or tangle diagram \(T\) is the product of variables that record information about the state, at least on faces and around vertices. The value of a diagram is obtained by summing the values of all its states. As usual, the notion of diagrams equivalent under moves leads to state-sum invariants, in rings satisfying relations obtained from pairs of diagrams.

To handle a problem related to a non-local property of Reidemeister moves of type 2, we were led to re-examine the purely diagrammatic foundations of the theory of links and tangles. The conclusion, justified in our companion article [2], is that to produce invariants it is enough to require invariance under a restricted set of Reidemeister moves, provided the tangle diagrams are first adjusted so as to be fine, as defined in the next section.

The present article applies the main result of [2] in order to construct families of link and tangle invariants with a remarkable property, much like one for topological quantum field theories: calculations made locally, using tangles in the pieces of any partition of the surface, can be merged (by multiplying matrices), to produce the value of the invariant, without any reliance on global information. Such computations are highly parallelizable. In contrast, the Kauffman bracket seems to be inherently global, as its calculation relies on obtaining the number of circles produced by each state. However, some

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even more general invariants, such as those of Khovanov-Rozansky homology, are amenable to local calculations, albeit difficult ones. They use a canopolis formalism based on planar algebras, so only plane projections of links are contemplated. For details one can consult Webster [11] and the references therein.

A complete analysis of the invariants we consider was possible after an especially simple abstraction from the case where the number \( r \) of symbols is 2. Only two different non-trivial invariants arose. One is known: it is essentially the state-sum invariant studied by Kauffman in Part II, Section 5 of [4]. It computes nothing more than the absolute values of linking numbers of unoriented links (in surfaces, if desired), and will not be discussed further. The other, called the \( \nu \)-invariant, is new. For links, it takes values in \( \mathbb{Z}[\nu] \), where \( \nu \) is a primitive fifth root of unity. Its definition is so easy that invariance can be checked by hand. We thought of the \( \nu \)-invariant as a toy test case until, after generating tables, we saw that its power to discriminate is not much weaker than that of the Kauffman bracket (Jones polynomial). Although extensive tests did not reveal any pair of links distinguished by the \( \nu \)-invariant but not by the Jones polynomial, we were unable to find any relation between values given by the two invariants. The \( \nu \)-invariant is the more easily calculable one, not just due to the locality property mentioned above, but because its more restrictive rules mean that the number of admissible states of link diagrams with \( n \) crossings (\( n + 2 \) faces if planar) tend to be only a small fraction of \( 2^n \). A small table for knots, and related graphics, appear in the appendixes. The last section develops some theory for the general invariants.

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2 The basic topological objects

Objects and maps are assumed throughout to be piecewise linear. Let \( S \) be a compact oriented surface of genus \( g \) whose boundary \( \partial S \) has \( c \) components, with orientation induced from that of \( S \). The Euler characteristic \( \chi_S \) of \( S \), usually defined from counts made after cutting \( S \) into simple pieces, is \( 2 - 2g - c \). Boundary components are topological circles, but our preference is to call them holes, as \( S \) can be obtained from a \( g \)-torus by removing the interiors of \( c \) mutually disjoint closed disks. A tangle \( T \) in \( S \) is a subset of \( S \) consisting of a finite set of unoriented curves, where each intersection is transverse and endowed with under-over crossing information, such that each curve that is open (not closed) has endpoints in \( \partial S \), and is otherwise disjoint from \( \partial S \). An \( n \)-tangle is a tangle with exactly \( n \) of its curves open. A face of a tangle is a connected component of \( S \setminus T \). For convenience the exposition will focus on framed tangles, where instead of Reidemeister moves of type 1 one uses those of type 1′, also known as ribbon moves.

We will ignore other possible refinements such as those where link components are oriented or coloured in some way.

![Figure 1: Ribbon moves (Reidemeister type 1′).](image)

(2.1) Definition. A tangle \( T \) in \( S \) is fine if each face \( f \) is, topologically, an open disk or an annulus (or cylinder) whose boundary \( \partial f \) in \( S \) intersects \( \partial S \) in a connected set, which is a boundary component of \( S \) if \( f \) is an annulus.

A diagram is an isotopy class of tangles in a surface \( S \). We refer to tangles but implicitly work only up to isotopy. Tangles that are not already fine can be adjusted to make them so. Some choice is involved but, as explained in [2], this can be accounted for. However, for invariants of the kind we consider, the requirement of fineness is too stringent, as it is more convenient to work with tangles satisfying only the following mild condition.
**2.2 Definition.** A tangle $T$ in $S$ is well-placed if, for each of its faces $f$, the boundary $\partial f$ intersects $\partial S$ in a connected set.

For a link in $S^2$, the removal of a point in the interior of an edge produces a long link: an infinite version of a 1-tangle in the plane, having exactly two unbounded faces. It consists of a long knot and possibly other components that are links. Long links in more general surfaces will not be contemplated here. The invariants below do not exploit these topological distinctions, treating indifferently links in $S^2$ and links or long links in the plane. Our main concern is to avoid tangles that are not well-placed (adapting the definition when treating long links). In particular, the only Reidemeister moves admitted are those that take place between well-placed tangles. Whenever tangles arise in other constructions, they should be converted to well-placed tangles.

Diagrams in surfaces will often be decorated by adding further structure. By an order we mean a linear (total) order, unless otherwise stated. The orientation on $S$ induces a cyclic order on each hole of $S$. Suppose two mutually disjoint and possibly empty sets of holes are chosen, and each listed in a fixed order. These holes are called respectively the upper and lower boundaries of the diagram, or inputs and outputs. At each hole there is a cyclically ordered set, possibly empty, of endpoints of curves in the tangle. We force these orders to become linear by marking a starting point on each hole, avoiding the tangle. There is an obvious way to form a category, where two isotopy classes of tangles can be composed when outputs of the first are compatible with inputs of the second. In order to be able to fuse strands of tangles where a pair of holes merges, aligned via the marked points, the holes must have anti-isomorphic linear orders (each isomorphic to the reverse of the other).

One useful variant is the category whose morphisms consist of equivalence classes of well-placed tangles, with equivalence given via Reidemeister moves between diagrams. Such definitions ensure good control over the Euler characteristics of faces obtained when pasting together tangles in surfaces. Yet another category is obtained by pasting classes of long links.

**Figure 2:** Composition of tangles (not well-placed) in surfaces with holes.

More importantly, the pasting procedure can be reversed, say starting with a tangle in a surface $S$, not assumed to be connected, and a cut that cleaves $S$ into two specified pieces $S_1$ and $S_2$ by removing closed curves disjoint from each other and from pre-existing holes, having only transverse intersections with the tangle. Each curve in a cut is required to be a boundary component of both $S_1$ and $S_2$. A tangle $T$ in $S$ gives tangles in $S_1$ and $S_2$ that can be replaced by equivalent well-placed tangles $T_1$ and $T_2$.

One can also cleave surfaces repeatedly, producing (after adjustment) well-placed tangles $T_i$ in surfaces $S_i$, ordered linearly and compatibly. When pasted together they produce a tangle in $S$ that is clearly well-placed. A noteworthy class of examples is given by rational links in the plane, or in $S^2$. By removing closed curves that intersect the link in exactly four points, these can be cut into simple pieces. One could if desired work within an even more general setting for cutting and pasting, much like that for the planar algebras of Jones [3], but with surfaces that need not be planar.
In the context of fine tangles, a Reidemeister move of type 2 is admissible only when the two faces \( f \) and \( g \) are distinct, as shown in Fig. 3, and at least one of these has boundary disjoint from \( \partial S \). Such conditions are useful in practice even though they are non-local: unlike Reidemeister moves of type 1 or 3, these ones cannot be verified merely by examining the given parts of the diagrams. The four kinds of move shown in Fig. 4, which are clearly local, form an adequate substitute for admissible moves of type 2, as moves between fine tangle diagrams. This is proved in Theorem 2.2 of [2]. However, to justify results for the wider class of well-placed tangles, it will be necessary to examine invariance under more general moves of type 2.

Finally, to see a useful composition of moves, consider a fine tangle having only one pair of external connections (a 1-tangle), as illustrated in Fig. 5, where the letters in the faces can for now be ignored. It is not hard to see that, via sequences of admissible moves, \( T \) can be moved through a crossing of either type (i.e., regardless of how the strands cross), provided \( T \) lies within a topological disk. Non-planar tangles do not have this property.

3 Facial state-sum evaluations of tangles

The first ingredient in the present approach to constructing state-sum invariants is an arbitrary finite set whose elements we call symbols, or coloured dots. A state of a tangle diagram is a function that assigns a symbol to each face. Something more general is needed, so by a partial state we mean a relation that assigns zero or more symbols to each face or, informally, places coloured dots in some faces. These will be assigned values in some fixed ring \( \mathcal{R} \) (when the tangle is a link) or in a certain overring \( \mathcal{R}^\sqrt{\cdot} \), described below. A partial state is inconsistent, and is assigned value zero, if it has a face with dots of different colours. Multiple copies of dots (same colour, in the same face) have no effect on values. Any assignment of a values to states can be extended to an assignment on partial states. One could just sum over all states that extend a given partial state, but we find it more useful to follow the convention of not counting the contribution (defined below) from faces to which the partial state has already assigned a symbol.
The general approach adopted for assigning a value to each state on the faces of a tangle diagram is to assign the product of variables that record local information about the state in various parts of the diagram. In what follows we proceed very simply, with nothing more than a variable at each face and at each crossing, and a notion of forbidding states where certain symbols lie in adjacent faces. Later we will study the relations necessary to ensure invariance of values under the relevant Reidemeister moves.

For a fine tangle in $S$, each face $f$ is either an internal face (its boundary $\partial f$ is disjoint from $\partial S$) or a boundary face ($\partial f$ meets $\partial S$ in a connected set, part or all of the boundary of a unique hole of $S$). Internal faces are topological disks, while boundary faces can be disks or annuli. Associate with each substate. For simplicity, the next formula is given only in the case where $T$ for forbidden. Thus, for example, the list corresponding to an empty boundary consists of a single empty some canonical way, of all possibilities for $x$, unless the boundary of $f$ intersects $\partial S$ in part but not all of some component of $\partial S$. In that case, the previous assignment must be divided by $\sqrt{\tau_i}$. This is consistent with the original formulas for $f$ a disk (one boundary, $\chi_f = 1$) or an annulus ($\chi_f = 0$).

In theory it suffices to assign a value to each tangle only via prior conversion into a fine tangle, in situations where this does not depend on the choices made, but it is more efficient to work directly with the much wider class of well-placed tangles. One can derive formulas appropriate for the more general faces that appear, but we give these now and will verify them only later, in Theorem 6.2. Just as $S$ has an Euler characteristic $\chi_S$, so does each face $f$. In a state where $f$ has symbol $i$, the contribution at $f$ is now defined to be $x_{ij}^\tau$, unless the boundary of $f$ intersects $\partial S$ in part but not all of some component of $\partial S$. In that case, the previous assignment must be divided by $\sqrt{\tau_i}$. This is consistent with the original formulas for $f$ a disk (one boundary, $\chi_f = 1$) or an annulus ($\chi_f = 0$).

At each crossing, viewed so that the overpass is the northeast-southwest strand, and with adjacent faces endowed with symbols $i, j, k, l$, in anticlockwise order starting from the east, associate a so-called 4j-variable $x_{ijkl}$ with that crossing. Since tangles are assumed to be unoriented, symmetry forces identities $x_{kitj} = x_{ijkt}$.

At edges it would be natural to define 2j-variables $x_{ij}$ from pairs $\{i, j\}$ of symbols. It appears at first that, at least when $\partial S$ is empty, such variables could be subsumed into the 4j-variables at crossings by an argument using half-edges and square roots of the $x_{ij}$, but that idea fails to deal properly with the trivial unknot. Still, by more careful arguments roughly like those used later in Theorem 6.3 and via Assumption 4.1 below, one could reduce to the situation that each $x_{ij}$ has value 0 or 1. A more refined approach (among several viable possibilities) would be to introduce some kind of 6j-variables that record the six symbols in faces around the two endpoints of each edge, as in [6]. There the situation becomes richer but much more difficult to analyse algebraically. The position we adopt is to avoid all such variables, giving instead a list of the pairs $\{i, j\}$ of symbols such that any state having adjacent faces marked with $i$ and $j$ is assigned value 0. Such pairs and states are said to be forbidden and are implicitly excluded from lists. All 4j (crossing) variables that involve forbidden pairs can and should be set to zero.

Now consider a tangle $T$ in a surface decorated as above with upper and lower boundaries. Let $u$ and $\lambda$ be partial states that assign symbols to those faces of $T$ that meet the upper (resp. lower) boundary. Recall that no face can meet both boundaries. Let $\Upsilon(T)$ and $\Lambda(T)$ denote lists, ordered in some canonical way, of all possibilities for $u$ and $\lambda$, respectively, excluding cases that are inconsistent or forbidden. Thus, for example, the list corresponding to an empty boundary consists of a single empty substate. For simplicity, the next formula is given only in the case where $T$ is fine and $C$ consists of two symbols called white and black, with corresponding facial variables $x$ and $y$. The evaluation of $T$ is the matrix $[T]$ indexed by pairs of substates in $\Lambda(T) \times \Upsilon(T)$ whose $(\lambda, u)$-entry is the sum

$$\sum_{\sigma} \left\{ x^{w(\sigma)} y^{b(\sigma)} \sqrt{x^{w'(\sigma)}} \sqrt{y^{b'(\sigma)}} \xi_{\sigma} \mid \sigma \text{ is a state of } T \text{ that extends both } \lambda \text{ and } u \right\} \in \mathcal{R}^\sqrt{\tau},$$

where $w(\sigma)$ and $b(\sigma)$ are, respectively, the number of internal faces whose symbol in state $\sigma$ is white or black. Similarly, $w'(\sigma)$ and $b'(\sigma)$ count colours for those boundary faces that are topological disks, while $\xi_{\sigma}$ denotes the product of the associated 4j-variables. An analogous formula defines evaluations for well-placed tangles and more general sets of symbols, in a way tailored to make the following theorem
\section*{4 Remarks on computing general ring-valued invariants}

In many cases where the rules for state assignment permit few choices, corresponding to small values of \( r \) in state-sum invariants similar to ones mentioned above, work has been carried out using algorithmic methods involving Groebner bases, where the invariants are the normal forms of polynomials modulo the ideal of relations. Harder analyses were performed using Singular \cite{1}, while tables of state-sum invariants were generated by programs written in Mathematica \cite{12}. Algorithmic aspects will not be discussed here. However, before examining the first interesting example that emerged from our systematic studies of such invariants, we briefly highlight the role played by well-known fundamental results on rings and fields.

For ring-valued state-sum invariants in general, the most appropriate ring is a polynomial ring \( R \) over \( \mathbb{Z} \) in the relevant variables, modulo the smallest possible ideal \( I \) of relations that force the desired invariance. As \( R \) is Noetherian, \( I \) is a finite intersection of primary ideals \( I_i \). Jointly, the invariants from the \( I_i \) have the same power to discriminate links as does the general invariant, since \( R/I \) injects into the direct sum of the \( R/I_i \). Whenever \( I_i \) is prime, this gives an invariant with values in a field (the quotient field of \( R/I_i \)). For an arbitrary primary ideal \( I_i \), a similar idea yields an invariant that can be considered to lie in a ring whose quotient, modulo the nilradical (some power of which is zero), is a field. One could just use fields as, when nilpotents are factored out, the discriminating power of invariants (measured on some set of links) rarely weakens. Some loss has been observed only for a few invariants that were already weak. There is, however, a motive for considering nilpotent elements. One can at times produce invariants that are not uselessly weak and can be computed relatively quickly, by adding extra relations to force values of selected variables to be nilpotent. In any case, the observations above justify the following standing assumption on the rings \( R \) (and also \( R^{\sqrt{\cdot}} \)) in which the invariants take values.

\subsection*{(4.1) Assumption.} For some integer \( N \), the divisors of zero in \( R \) are the elements \( r \) with \( r^N = 0 \).

Whenever an ideal \( I_i \) yields a weak invariant (tested against small tables of knots and links), it is discarded. Then, unless \( I_i \) is too complicated to handle well, it is worth finding some way to present values of the invariant in a better way than as normal forms relative to some Groebner basis. When the ideal is prime, we prefer to express each value in the associated quotient field in a canonical way as a quotient of polynomials. This can in theory be done, as any field of characteristic 0 is a 1-generated finite extension of a purely transcendent subfield, where free generators can be chosen from among the original variables. More generally, primary ideals \( I_i \) can also be handled. Work carried out with Singular
produced, in selected cases, explicit formulas, allowing invariants to be written in convenient forms. Our interest is in invariants with fairly simple formulas and good discriminating power.

5 A simple example: the \( \nu \)-invariant

Recall the general setup for states and partial states, where coloured dots (symbols) are placed in faces. This section is devoted to one simple example of an invariant that uses only two symbols, called black and white, subject to one rule: states are forbidden to have adjacent faces receiving a black dot. More precisely, if a forbidden state somehow arises it is given value 0, just as for inconsistent partial states where some face contains dots of different colours. Our rule for forbidden states was not imposed arbitrarily, but emerged from a complete analysis of the case \( r = 2 \) (two symbols), where an ideal of relations was decomposed as an intersection of primary ideals.

We consider only unoriented tangles, leaving more complex cases for articles in preparation. Thus, given a state of a diagram, at each vertex (crossing) there are at most five possible configurations with a dot in each of the four (possibly not different) adjacent faces, with associated \( 4j \)-variables \( \{z, a, b, e, j\} \) as shown in Fig. 6. There is a symmetry under a \( \pi \)-rotation, so \( d = a \) and \( h = b \). Each letter corresponds to a binary number obtained by drawing the crossing with the overpass from northeast to southwest, and reading the dots anticlockwise starting from the east, using 0 for a white dot and 1 for a black dot.

![Figure 6: Scheme to associate variables to crossings](image)

In addition to these five \( 4j \)-variables we use two facial or \( 1j \)-variables: a \( x \) in each white face and a \( y \) in each black face, provided the faces are disks with boundary disjoint from \( \partial S \). As mentioned before, and proved in the general setting of Theorem 6.2 there are related formulas for other kinds of faces.

When using only the theory of [2], the value of a tangle diagram is calculated only after an adjustment that produces a fine tangle. To obtain an invariant of framed tangles one need only verify invariance under Reidemeister moves of types \( 1' \) (ribbon move), 2 (restricted) and 3. The restriction is that the number of faces must change by two under the move. In particular, moves that disconnect a diagram are not allowed.

Fig. 7 shows all states, excluding forbidden ones, that can be assigned to faces in part of a tangle where a Reidemeister move of type 2 is performed. In the digon, a dot which is half white and half black indicates a superposition of two states. Cases labelled 2 and 8 have rightmost diagrams with inconsistent substates, so relations are imposed to force the left sides to have value 0. As explained below, evaluations here must be performed in a way that departs from the usual convention for diagrams showing partial states. Differences produce the following set of six polynomial relations that must be satisfied in the ring:

\[
P_2 = \{ z^2x^2 + abxy - 1, \; abx^2 - 1, \; zbx + ajy, \; ejx^2 - 1, \; zax + bey, \; abxy + ejy^2 - 1 \}.
\]

The diagrams form parts of a larger whole where an admissible move takes place between two fine tangles on a surface. Three different faces on the left, one a fully visible digon, merge to form one face on the right. There may, however, be other coincidences among the four boundary faces in the leftmost diagrams, so possible contributions of the upper and lower faces to the evaluation should be ignored for the purposes of obtaining relations. One might expect a common factor of a face variable (\( x \) or \( y \)) in the relations obtained by forcing pairs of consistent state diagrams to have equal values, but in fact these do not appear when one of the faces that fuses is an annulus. By fineness at least one of the leftmost and rightmost faces is an internal face, and one sees from this that the relations already given are sufficient to handle all
cases. We will later see a proof in a general setting that face variables are invertible, so can be cancelled from relations.

Figure 7: Relations inducing invariance of tangle evaluation under admissible non-local moves of type 2.

(5.1) Lemma. Suppose the relations in $P_2$ hold for certain elements of a ring $R$. Whenever $T$ and $T'$ are fine tangles differing only by an admissible Reidemeister move of type 2, their evaluations in $R^\sqrt{\cdot}$ are equal. The same is true for partial states of these tangles, where regions assigned symbols are not involved in the move.

Proof. In state sums for the leftmost diagram, one groups states that agree, except possibly on the digon for the move, with the corresponding state (if it exists) in the rightmost diagram. After evaluating, say ignoring contributions from faces with preassigned symbols, the differences within each group are clearly multiples of polynomials in $P_2$.

Figure 8: Polynomial relations for invariance of tangle evaluation under Reidemeister move 3.
The Reidemeister move 3 is easy to handle, as non-local features are absent. Fig. 8 shows 18 pairs of tangles with a complete assignment of states, possibly forming parts of larger diagrams. Differences of values yield polynomial relations that clearly suffice to guarantee invariance. Cases 0, 9, 18, 36 give zero, while the others, up to sign, give a set \( \mathcal{P}_3 \) of five distinct relations:

\[
\begin{align*}
1, 2, 8, 16) & \quad a(z^2 x + bey - zbx) \\
10, 17) & \quad b^2 ex - za^2 x - be^2 y \\
4) & \quad z^2 bx + a^2 jy - zb^2 x \\
5, 20, 32, 34, 40) & \quad a(zbx + ejy - bjr)
\end{align*}
\]

(5.2) Lemma. Suppose the relations in \( \mathcal{P}_3 \) hold in \( \mathbb{R} \). Whenever \( T \) and \( T' \) are fine tangles differing by a Reidemeister move of type 3, their evaluations in \( \mathbb{R}^\circ \) are equal. A similar result holds for partial states that agree, each leaving unassigned the central region of the move.

The factors of \( a \) appearing in some relations of \( \mathcal{P}_3 \) can be cancelled using the second relation in \( \mathcal{P}_2 \), which implies that \( a \) is invertible. The ideal generated by \( \mathcal{P}_2 \cup \mathcal{P}_3 \) in the polynomial ring over \( \mathbb{Z} \) with the variables as free generators was analysed using Singular. It is the intersection of two prime ideals, each of which is invertible. The ideal generated by \( P_1 \cup P_2 \) is invertible. The ideal generated by \( P_1 \cup P_2 \) is invertible.

(5.3) Definition. \( z \rightarrow -u^2 - u^3, \ a \rightarrow -u^3, \ b = h \rightarrow -u^2, \ e \rightarrow u, \ j \rightarrow u^4, \ x \rightarrow 1, \ y \rightarrow u^2 + u^3. \)

It is easy to verify by hand that the \( u \)-assignment, where \( 1 + u + u^2 + u^3 + u^4 = 0 \), annihilates the polynomials in \( \mathcal{P}_2 \cup \mathcal{P}_3 \). Invariance under the ribbon move is obvious, so we have now defined an invariant \( T \rightarrow [T]_u \) of framed tangles in oriented surfaces, henceforth called the \( u \)-invariant. One checks easily that, under a framing change of +1, values multiply by \( u \). Invariance under all Reidemeister moves of type 1 could then be arranged in the usual way via the self-writhe \( sw(T) \) of a tangle diagram \( T \) (the sum of the writhes of all curves in \( T \)). Thus gives an invariant \( u^{-sw(T)}[T]_u \) of unframed tangles that we prefer to name, as it suffices to apply the \( u \)-invariant to 0-framed diagrams of tangles.

All values lie in an extension of \( \mathbb{Z}[u] \) by a square root of \( u^2 + u^3 \), but as our focus is on closed tangles (links \( L \)) we work only with \( \mathbb{Z}[u] \). Since \( 1 = -(u + u^2 + u^3 + u^4) \), each value can be expressed uniquely in the form \( au + bu^2 + cu^3 + du^4 \), where \( a, b, c, d \) are integers. We write this as \( [a, b, c, d] \). Its complex conjugate is \( [d, c, b, a] \). Mirror images are taken with respect to the surface, altering all crossings. In both the framed and unframed cases, the corresponding invariants are complex conjugates of each other, as can be seen from the way values in \( \mathbb{Z}[u] \) were assigned to the variables. Thus:

(5.4) Proposition. If links \( L \) and \( L^* \) are mirror images and \( [L]_u = [a, b, c, d] \), then \( [L^*]_u = [d, c, b, a] \).

Figure 9: A planar 1-tangle passes through crossings with no change in value.

Diagrams are now being used to stand for their evaluations by the original (framed) \( u \)-invariant. Recall our convention in the evaluation of partial states that dots already assigned to faces do not contribute facial factors in the state sums. A result based on moving planar tangles through crossings, proved in general in Theorem 6.3, and illustrated here as it applies to the \( u \)-invariant, is:

(5.5) Theorem. Let \( T \) be a long link in the plane, shown here as an infinite 1-tangle. Then
and $u = \in \sqrt{5} \mathbb{Z}[u]$.

**Proof.** The leftmost diagrams have the same value, which also appears in the state sum for $[T]_u$ with factors of $y$, 1, $y$, where $y = u^2 + u^3$. One checks that $(1 + 2y)^2 = 5$. □

We write $\{T\}_u$ to denote the leftmost values, so $\{T\}_u = [T]_u/(1 + 2y)$. The new invariant assigns 1 to the trivial unknot and is clearly multiplicative under connected sums, so is the preferred form for recording tables of values for links in the plane, usually adjusted to have self-writhe 0. The previous result has the following interesting reformulation.

**Corollary.** For links in the plane or in $S^2$ the value of the $u$-invariant on a partial state having exactly one dot, coloured black, does not depend on which face contains the dot.

**Proof.** It suffices to compare two such values where the relevant faces are neighbours, for convenience drawn as the infinite faces of a long link, and treated just above. □

This does not extend to surfaces of positive genus. The following example of a tangle in a torus has three faces and exactly two non-zero states with a black dot. These have different values.

$$\text{Figure 10: Example in the torus showing that the corollary does not generalize.}$$

6 Further properties of the general state-sum invariants

In what follows, we assume that the variables, regarded as elements of a ring $R$ or $R\sqrt{5}$, satisfy relations analogous to those studied above. Thus they give an invariant of fine tangles and even, by preparation, an invariant $[T]$ of tangles $T$.

To avoid trivialities, for each symbol $i$ assume there is some $j$ such that $\{i,j\}$ not forbidden. If this failed for some $i$, that symbol would be almost useless, as its only possible contribution could be for tangles disjoint from some component of the surface they lie in. We can now verify earlier claims about facial variables $x_i$. Note that products of these were often cancelled from relations, but here they should be left in place to justify that practice.

**Theorem.** Under the mild assumption just above, all facial variables $x_i$ are invertible in $R$. There are formulas $x_i^{-1} = \sum_k x_k x_{kij} x_{ji} x_{ijk}$, for suitable $j$.

$$\text{Figure 11: Eight diagram with state labels.}$$

**Proof.** Consider state diagrams for the non-local Reidemeister move of type 2, like those in Fig. 7 especially the diagram pair marked 2, but with general states. Whenever certain boundary regions in the leftmost diagram have different symbols, giving an inconsistent substate on the right, the value of
the leftmost diagram, given by a state sum, is zero in $R$. Thus, in the following diagram, if $l$ is fixed, as well as $i$ and $j$, and a sum is taken over $k$, the value obtained is zero unless $l = i$. Invariance under a 2E (eight) move implies a sum over $k$ and $l$ which yields a polynomial $p_{ij}$ satisfying

$$\sqrt{x_i}\sqrt{x_j} = \sqrt{x_i}x_ip_{ij}, \quad \text{where} \quad p_{ij} = \sum_k x_kx_{kij}x_{jik}.$$

The value on the left is not zero, as $\{i,j\}$ is not forbidden. Then $1 - x_ip_{ij}$ is a divisor of zero, which in turn implies that $\sqrt{x_i}$ is not nilpotent. By symmetry, nor is $\sqrt{x_j}$. By a standing assumption about the ring, both can be cancelled from the equation. Thus $x_i^{-1}$ has a polynomial expression.

Next we return to an alternate method that assigns values directly to states of well-placed tangles $T$. Extending the original valuation method, the contribution of a face $f$ with symbol $i$ is defined to be $x_i\chi_f$, or this divided by $\sqrt{x_i}$ when $\partial f$ contains part but not all of some boundary component of $S$.

(6.2) Theorem. The value assigned directly to a well-placed tangle $T$ is the same as that originally obtained only after preparing $T$ to obtain a fine tangle.

Proof. It suffices to show that the new method gives values invariant under the preparation process, which involves a sequence of Reidemeister moves of type 2 between well-placed tangles, ending with a fine tangle. Looking at any such move made backwards, we focus throughout on the two external faces that will fuse to the digon and create a new face. The other faces present in the move play a minor role, even if they are not distinct from the faces under consideration, and will be ignored.

First assume that the two faces under study are distinct, as will hold for example if they contain different symbols. Globally, the difference in values of diagrams before and after the move is a multiple of an analogous difference calculated using only parts involved in the move. This difference is in turn a multiple, up to powers of facial variables, of the difference from a move between tame tangles. Thus, as a consequence of the relations for tame tangles, this difference has value 0. Assuming both faces have the same symbol $i$, with contributions $x_i^{f_1}$ and $x_i^{f_2}$, the contribution of the merged face is then $x_i^{f_1}x_i^{f_2}/x_i$, and the invariance result is equivalent to the polynomial formula for $x_i^{-1}$ in Theorem 6.1.

Now suppose the two faces are, globally, the same face $f$, of genus $g_f$ and with $c_f$ boundary components. Consider the two strands of $\partial f$ involved in the move. There are two cases, illustrated roughly in the diagrams. In the second, parts of both sides of $f$ are visible.

1. The strands lie in the same component of $\partial f$, which then splits into two, with no change in the genus of the face.
2. The strands lie in different components of $\partial f$, which coalesce. This creates a new handle with a hole, increasing the genus of the face by 1.

In both cases the Euler characteristic $2 - 2g_f - c_f$ of the face decreases by 1, giving invariance via the polynomial formula for $x_i^{-1}$.

Figure 12: Cases where two faces that fuse coincide globally.

Some aspects related to moving tangles (especially curls) through crossings will now be discussed. The next result generalizes part of Theorem 5.5 and was used in its proof.

(6.3) Theorem. Let $T$ be a long knot or link in the plane, with specified infinite faces $f$ and $g$, and let $i,j,k,l$ be among the symbols. Let $[T]_{ij}$ denote the value of the substate which assigns $i$ to $f$ and $j$ to $g$,
Proof. The diagrams of Fig. 5 have the same value. One then cancels a crossing variable. □

For arbitrary symbols $i$ and $j$, let $c_{ij}$ denote the factor in $R$ calculated from the positive curl in the next diagram, and let $c'_{ij}$ denote the factor for the positive curl (not shown) obtained from it by applying a ribbon move. Explicitly, $c_{ij} = \sum_k x_k x_{jiki}$, while $c'_{ij} = c_{ji}$ since we do not orient tangles. Invariance under the ribbon move, for diagrams with symbols assigned to the boundary faces, is thus the assertion that $c_{ij} = c_{ji}$ always holds. Excluding forbidden pairs $\{i, j\}$ (for which $c_{ij} = 0$), $c_{ij}$ has an inverse in $R$, calculable from a negative curl. This is clear from the Whitney trick.

Theorem 6.3 will give certain relations of the form $c_{ij} = c_{ji}$. By composing such relations, always avoiding forbidden states, one expects to see that invariants of regular isotopy produced by these constructions will often need no further specialization in order to give invariants of framed tangles. Certainly this holds for the $u$-invariant, where the factor for positive curls is $u$. In general, however, invariants need not have a single factor measuring changes of writhe, and examples could well be of interest.

![Figure 13: The Whitney trick (realizable by admissible Reidemeister moves).](image)

Finally we show, for links in closed surfaces $S$, how little an invariant is affected by adding a relation $x_0 = 1$, where 0 denotes some fixed symbol. Nothing is lost by taking $R$ to be the usual polynomial ring modulo relations. Instead of adding the above relation, one can introduce new variables, shown with primes, that satisfy relations $x'_ix_0 = x_i$ (so $x'_0 = 1$) and $x'_{ijkl} = x_{ijkl}x_0$. All earlier relations for moves between fine tangles can be rewritten in terms of the new variables and take exactly the same form, as the $x_0$ always cancels completely. In the case of admissible moves of type 2, as shown in Fig. 8 this is because one diagram has exactly two faces and two vertices more than the other.

One obtains a new invariant of links on $S$ by giving the primed variables values in the obvious subring $R'$ of $R$, and can then identify $R$ with the ring of Laurent polynomials $R'[x_0, x_0^{-1}]$. Now observe that whenever the primed invariant assigns value $r' \in R'$ to a link $T$ in $S$, the corresponding unprimed invariant has value $x_0^n r'$, for some integer $n$. From the definitions of the primed variables, the exponent of $x_0$ is the number of faces of $T$ minus the number of vertices. But this is just the Euler characteristic $\chi_S$ of $S$, as every vertex of $T$ has degree 4. Nothing more than that can be lost by assuming $x_0 = 1$.

7 Appendix A: Values of the $u$-invariant for knots up to 10 crossings

We record the values of the $u$-invariant, in its normalized form $\{K\}_u$ where the unknot has value 1, for all prime knots $K$ with at most 10 crossings, choosing one in each pair of mirror images and adjusting the framing to be 0. Each value can be represented by a vector of four integers giving the coefficients of $u$, $u^2$, $u^3$ and $u^4$, which reverses under taking the mirror image. Here we present this data with the vectors given in condensed form as 4-letter words, using $A, \ldots, Z$ for 1, \ldots, 26, and $a, \ldots, z$ for $-1, \ldots, -26$, while 0 is 0. In addition, write $\alpha$ for -27, $\beta$ for -30, $\gamma$ for -33.

From the table, one sees for example that the $u$-invariant distinguishes $8_{20}$ from its mirror image. By these two knots, as well as the connected sum of a trefoil with its mirror image, are not distinguished by knot Floer homology. On the other hand, the $u$-invariant fails to distinguish pairs such as $5_1$ and its mirror image, and does not even detect the unknottedness of $8_{19}$, $9_1$ or $10_{152}$. 

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7.1 Up to 7 crossing knots

\[ 3_1 \text{ AAa} \quad 4_1 \text{ bbb} \quad 5_1 \text{ ABa} \quad 5_2 \text{ Aab} \quad 6_1 \text{ ad} \quad 6_2 \text{ BBa} \quad 6_3 \text{ bee} \]

\[ 7_1 \text{ AAa} \quad 7_2 \text{ ad} \quad 7_3 \text{ BEA} \quad 7_4 \text{ aed} \quad 7_5 \text{ DFEA} \quad 7_6 \text{ DBb} \quad 7_7 \text{ chf} \]

7.2 8 crossing knots

\[ 8_1 \text{ Aab} \quad 8_2 \text{ ACE} \quad 8_3 \text{ bee} \quad 8_4 \text{ DEB} \quad 8_5 \text{ AEFD} \quad 8_6 \text{ cca} \quad 8_7 \text{ CCa} \]

7.3 9 crossing knots

\[ 9_1 \text{ aaaa} \quad 9_2 \text{ bbb} \quad 9_3 \text{ baBB} \quad 9_4 \text{ DEB} \quad 9_5 \text{ bee} \quad 9_6 \text{ dca} \quad 9_7 \text{ EFbb} \]

7.4 10 crossing knots

\[ 10_1 \text{ AAa} \quad 10_2 \text{ bba} \quad 10_3 \text{ aed} \quad 10_4 \text{ BEA} \quad 10_5 \text{ AEFD} \quad 10_6 \text{ BGC} \]

8 Appendix B: Images of the $u$-invariant of knots up to 10 crossings

We plot the complex numbers \{\(K\}\}_u for all prime knots \(K\) up to 10 crossings evaluated at \(u = e^{\frac{2\pi i}{10}}\) and \(u = e^{\frac{2\pi i}{5}}\), or equivalently at the conjugates \(u = e^{\frac{8\pi i}{10}}\) and \(u = e^{\frac{8\pi i}{5}}\). Since all mirror pairs of knots are included, these graphics are symmetric relative to the real axis. Moreover we include values for all writhe

of the knots involved, thus giving the image a dihedral 10-fold symmetry. A curious unexpected fact is the great difference in scale corresponding to the two values of \(u\). When non-prime knots are included
images become a little denser, from new products of points in the original graphic. Other interesting graphics arise by taking certain products and quotients of the above invariants.

Figure 14: The image of the $u$-invariant of knots up to 10 crossings, computed at $u = e^{2\pi i}$.

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