NICHOLS ALGEBRAS AND QUANTUM PRINCIPAL BUNDLES

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Abstract. We introduce a general framework for associating to a homogeneous quantum principal bundle a Yetter–Drinfeld module structure on the cotangent space of the base calculus. The holomorphic and anti-holomorphic Heckenberger–Kolb calculi of the quantum Grassmannians are then presented in this framework. This allows us to express the calculi in terms of the corresponding Nichols algebras. The extension of this result to all irreducible quantum flag manifolds is then conjectured.

1. Introduction

Exterior and symmetric algebras play a fundamental and ubiquitous role in classical differential geometry. Their quantum counterparts, however, are much more poorly understood, as is their role in noncommutative differential geometry. The most developed framework we have for understanding exterior and symmetric algebras in the noncommutative setting is the theory of Nichols algebras, an important class of braided Hopf algebras. Nichols algebras first appeared in [47] as a tool for the constructing new examples of Hopf algebras. Later they would arise independently in a number of works, for example, the work of Woronowicz on differential calculi [63, 64] and the work of Majid on braided groups [44, 43]. Nichols algebras were subsequently used to give an abstract construction of quantised enveloping algebras [42, 45]. Most famously, Nichols algebras are basic invariants of pointed Hopf algebras, and are crucial in Andruskiewitsch and Schneider’s remarkable classification program for Hopf algebras [7, 8, 2]. Other notable applications include the work of Bazlov [13], where Nichols algebras were used to describe the cohomology

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rings of the flag manifold $G/B$ of a semisimple Lie group $G$. This was later generalised to the case of the (small) quantum cohomology ring for $G/B$ in \cite{36}. Nichols algebras have also seen applications in conformal field theory \cite{55,56,57} and more recently \cite{39}.

Woronowicz rediscovered Nichols algebras as part of his investigation of bicovariant differential calculi over Hopf algebras. He showed that the left invariant forms of a bicovariant calculus possess a Yetter–Drinfeld structure, and hence the structure of a braided vector space. He then took the associated Nichols algebra, endowed it with a commutator differential, producing an extension of the first-order calculus to a differential graded algebra. Such an approach leads to noncommutative analogues of curvature, Bianchi identities, and Lie derivatives \cite{9}. However, an obvious problem is that any bicovariant calculus over a quantum coordinate algebra $\mathcal{O}_q(G)$ will not have classical dimension, even for the simplest example of $\mathcal{O}_q(SU_2)$ \cite{54}. As a result applications, of Nichols algebras to the noncommutative geometry of quantum groups remain limited.

In contrast to the situation for Drinfeld–Jimbo quantum groups, the framework of covariant differential calculi has proved itself to be an ideal setting for investigating the noncommutative geometry of quantum flag manifolds. In particular, the seminal work of Heckenberger and Kolb has shown that the irreducible quantum flag manifolds admit an essentially unique covariant $q$-deformation of their classical de Rham complex \cite{29,30}. These differential calculi arguably constitute the most important family of noncommutative differential structures in the theory of quantum groups. In the special case of the $A$-series, the irreducible quantum flag manifolds are precisely the quantum Grassmannians, which properly contain the quantum projective spaces $\mathcal{O}_q(\mathbb{CP}^n)$. In particular, when $n = 1$, this includes the celebrated Podleś sphere $\mathcal{O}_q(\mathbb{CP}^1) \cong \mathcal{O}_q(S^2)$.

Quantum flag manifolds are constructed as subspaces of right coinvariant elements with respect to a quantum Levi subgroup $\mathcal{O}_q(L_S)$. As such they only admit a left $\mathcal{O}_q(G)$-coaction, meaning that for these quantum spaces one can only speak of left $\mathcal{O}_q(G)$-covariant calculi. This places the Heckenberger–Kolb calculi outside Woronowicz’s bicovariant framework. However, left covariance of the calculi gives their cotangent spaces the structure of an $\mathcal{O}_q(L_S)$-comodule. Since $\mathcal{O}_q(L_S)$ is a coquasitriangular Hopf algebra, its category of comodules is a braided monoidal category with braiding $\sigma_R$. Therefore any $\mathcal{O}_q(L_S)$-comodule $V$ will have an associated Nichols algebra $\mathfrak{B}(V, \sigma_R)$, $q$-deforming the classical exterior algebra. Unfortunately, in most cases the dimension of $\mathfrak{B}(V, \sigma_R)$ will not be the same as the dimension of the classical exterior algebra, for example, $\mathfrak{B}(V, \sigma_R)$ will often be infinite-dimensional. To address this non-classical behaviour, Berenstein and Zwicknagl introduced the novel notion of quantum exterior algebras \cite{15}. Their approach is set in the framework of coboundary categories (as opposed to braided monoidal categories) and cactus groups (as opposed to braid groups). This solves the problem of non-classical dimension for a distinguished family of $U_q(g)$-modules classified in \cite{66}. In particular, for
the irreducible quantum flag manifolds, Berenstein and Zwicknagl’s quantum exterior algebras have classical dimension. These quantum exterior algebras would later be used by Krähmer and Tucker-Simmons in their construction of noncommutative Dolbeault–Dirac operators over the irreducible quantum flag manifolds [38].

In this paper, we return to the more standard braid group approach, producing for the first time a Nichols algebra presentation of the Heckenberger–Kolb calculi $\Omega_q^1(\text{Gr}_{n,m})$ of the quantum Grassmannians. We introduce a general approach, based around the theory of quantum principal bundles, for producing a Yetter–Drinfeld braiding on the cotangent space of a covariant first-order calculus over a quantum homogeneous space. Indeed, our approach can be considered as a generalisation of Woronowicz’s work to the setting of quantum homogeneous spaces. This general approach is then applied to the quantum principal bundle description of $\Omega_q^1(\text{Gr}_{n,m})$ introduced in [18], which uses a differential calculus on $\mathcal{O}_q(SU_n)$ constructed from the coquasitriangular structure of $\mathcal{O}_q(SU_n)$, following the approach of [14, §5]. For recent advances, see [10, 17, 16] and references therein. The resulting Yetter–Drinfeld structure is shown to be both diagonal and of Hecke type for the special case of quantum projective space, but non-diagonal and of non-Hecke type for the quantum Grassmannians $\mathcal{O}_q(\text{Gr}_{n,m})$, with $m \notin \{1, n-1\}$.

Next, we investigate the associated Nichols algebras, starting with the degree two terms where the braiding gives an explicit description of the commutation relations of the calculi. The remaining terms of the Hilbert–Poincaré series for the associated Nichols algebra are calculated using Poincaré duality for Nichols algebras, the fact that the maximal prolongation $\Phi(\Omega^{0,\bullet})$ is a Frobenius algebra (as observed in [38]), and quantum Howe duality for the quantum Levi subalgebra of the quantum Grassmannians. The Hilbert–Poincaré series of the Nichols algebra and that of $\Phi(\Omega^{0,\bullet})$ coincide, meaning that the two graded algebras coincide. This constitutes the principal result of the paper, and confirms that the braiding produced is novel and distinct from the (co)quasitriangular braiding. Moreover, it establishes an important point of contact between Nichols algebras and noncommutative geometry.

This project leads to a number of new questions. The first and most obvious is how to extend our results to all the irreducible quantum flag manifolds. Since the quantum enveloping algebras of the other series are also quasitriangular, it seems reasonable to expect that we can adapt the construction of [18] to this more general setting. Moreover, the calculation of the Hilbert–Poincaré series is far easier for cases outside the $A$-series. As we show in §5 the fact that the relations of a Nichols algebra of Hecke type are generated in degree two, means that the Hilbert–Poincaré series can be simply concluded from the action of the braiding on degree two forms. A further problem is to define Lie derivatives, contraction operators, Maurer–Cartan forms and the corresponding Cartan calculus in the manner of [9]. Another challenging, but important, goal is to produce a Nichols algebra description of the whole Dolbeault double complex of the irreducible quantum flag manifolds.
1.1. **Summary of Results.** The paper is organised as follows: In §2 we recall necessary preliminaries about braided monoidal categories, Yetter–Drinfeld modules, Nichols algebras, and differential calculi over Hopf algebras and quantum homogeneous spaces, focusing on complex structures, connections, holomorphic structures, principal comodule algebras, and strong principal connections.

In §3 we establish a general quantum principal bundle framework for constructing a Yetter–Drinfeld module structure on the cotangent space of a covariant first-order differential calculus. Along the way we establish some novel categorical equivalences to clarify the underlying processes at work.

In §4 the basic definitions and results of Drinfeld–Jimbo quantum groups are recalled. We then present the definition of a quantum flag manifold, focusing on the special case of the quantum Grassmannians $O_q(Gr_{n,m})$ and their Heckenberger–Kolb calculi $\Omega^1_q(Gr_{n,m})$. We apply the general results of §3 to the anti-holomorphic part of $\Omega^1_q(Gr_{n,m})$ and prove the main result of the paper:

**Theorem 1.1.** For any quantum Grassmannian, the anti-holomorphic tangent space $V^{(0,1)}$ of its Heckenberger–Kolb calculus admits the structure of an $O_q(LS)$-Yetter–Drinfeld module, such that the associated Nichols algebra is isomorphic, as an $O_q(LS)$-comodule algebra, to the maximal prolongation $V^{(1,0)}$. An analogous result holds for the holomorphic tangent space $V^{(1,0)}$.

In §5 we conjecture the existence of a Yetter–Drinfeld structure for the cotangent spaces of all the irreducible quantum flag manifolds. We also discuss the conjectured isomorphism between the Nichols algebra and calculus for each simple Lie algebra series, using explicit representation-theoretic calculations.

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2. **Preliminaries**

2.1. **Hopf algebras and braided Hopf algebras.** Throughout this paper $A$ and $H$ will denote Hopf algebras, and all Hopf algebras are assumed to unital, with bijective antipode, and defined over the complex numbers. We denote the coproduct, counit, and antipode of a Hopf algebra by $\Delta$, $\varepsilon$, and $S$, respectively. Throughout we use Sweedler notation, and write $a^+ := a - \varepsilon(a)1$, for any $a \in A$, and $V^+ := V \cap \ker(\varepsilon)$, for $V$ a subspace of $A$. 

2.1.1. Braided Hopf algebras. A braiding on a monoidal category \( C \) is a natural isomorphism \( \sigma \) between functors \( - \otimes - \) and \( - \otimes \text{op} - \) such that the relevant hexagonal diagrams commute, see [27, §8.1] for details.

A braided monoidal category is a pair consisting of a monoidal category and a braiding.

Let \( C \) be a braided monoidal category with braiding \( \sigma \). To give the tensor product \( A \otimes B \) the structure of an associative algebra in \( C \), we define a multiplication by the formula
\[
m_{A \otimes B} := (m_A \otimes m_B) \circ (\text{id}_A \otimes \sigma_{B,A} \otimes \text{id}_B).
\] (1)

A braided bialgebra in \( C \) is an object \( A \) in \( C \) endowed with an associative algebra structure in \( C \) and a coalgebra structure in \( C \) such that its coproduct \( \Delta \) and counit \( \varepsilon \) are algebra morphisms with respect to the multiplication in \( A \otimes A \) given by (1).

A braided Hopf algebra in \( C \) is a bialgebra in \( C \) admitting an antipode which is a morphism in \( C \). For further details on braided Hopf algebras, we direct the reader to [27] and [11].

2.1.2. Yetter–Drinfeld modules. An important example of a braided monoidal category is the category of (right) Yetter–Drinfeld modules \( V \) over a Hopf algebra \( H \), which are those right \( H \)-modules \( V \) with action \( \triangleleft \), and a right \( H \)-comodule structure such that
\[
v_{(0)} \triangleleft h_{(1)} \otimes v_{(1)} h_{(2)} = (v \triangleleft h_{(2)})_{(0)} \otimes h_{(1)} (v \triangleleft h_{(2)})_{(1)}, \quad \text{for } h \in H, \ v \in V.
\] (2)

We denote the category of Yetter–Drinfeld modules, endowed with its obvious monoidal structure, by \( \text{YD}_H \). A braiding for the category is defined by
\[
\sigma : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto w_{(0)} \otimes v \triangleleft w_{(1)}, \quad \text{for } v \in V, \ w \in W.
\] (3)

Note that for any \( V \in \text{YD}_H \), the tensor algebra \( \mathcal{T}(V) \) is a braided Hopf algebra in \( \text{YD}_H \) with
\[
\Delta(v) := v \otimes 1 + 1 \otimes v, \quad S(v) := -v, \quad \varepsilon(v) := 0, \quad \text{for } v \in V.
\]

2.2. Nichols algebras. For a detailed introduction to Nichols algebras we refer the reader to the surveys [3, 4]. Let \( B_n \) denote the braid group on \( n \) strands, that is, the group generated by \( n-1 \) elements \( \beta_1, \ldots, \beta_{n-1} \) subject to the relations
\[
\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}, \quad 1 \leq i \leq n-2,
\]
\[
\beta_i \beta_j = \beta_j \beta_i, \quad 1 \leq i, j \leq n-2, \ |i-j| \geq 2.
\]

When \( V \in \text{YD}_H \) is finite-dimensional as a vector space, we obtain a representation of the braid group on \( n \) strands
\[
\rho_n : B_n \to GL(V^\otimes n),
\]
given by
\[
\rho_n(\beta_i) = \text{id} \otimes \cdots \otimes \text{id} \otimes \sigma \otimes \text{id} \otimes \cdots \otimes \text{id},
\]
where \( \sigma \) is acting on \( V \otimes V \) in position \( i \) and \( i+1 \).
There is a canonical surjective group homomorphism onto the symmetric group $S_n$, 
\[ \varphi_n : B_n \to S_n, \]
which maps $\beta_i$ to the simple transposition $\tau_i = (i, i+1)$. Let $\ell(g)$ denote the length of an element $g \in S_n$. The projection $\varphi_n$ admits a set-theoretic section, called the Matsumoto section 
\[ s_n : S_n \to B_n, \]
which is determined by $s_n(\tau_i) = \beta_i$ and $s_n(\tau_i \tau_{i+1}) = s_n(\tau_i)s_n(\tau_{i+1})$, for $1 \leq i \leq n$, and $s_n(gf) = s_n(g)s_n(f)$ if $\ell(gf) = \ell(g) + \ell(f)$, for $g, f \in S_n$. Note that $s_n$ is not a group homomorphism.

The braided symmetriser is given by the map 
\[ S_\sigma^n(V) := \sum_{g \in S_n} \rho_n(s_n(g)) : V^\otimes n \to V^\otimes n. \]

We denote 
\[ \ker S_\sigma^n(V) := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \ker S_\sigma^n(V). \]

**Definition 2.1.** The Nichols algebra of $V$ is the braided Hopf algebra in $\mathcal{YD}_H^H$ defined by 
\[ \mathfrak{B}(V, \sigma) := T(V)/\ker S_\sigma^n(V). \]

In what follows we will write $\mathfrak{B}(V)$ when the braiding on $V$ is clear.

Since $\ker S_\sigma^n(V)$ is a homogeneous ideal of $T(V)$, the Nichols algebra $\mathfrak{B}(V)$ has a unique $\mathbb{Z}_{\geq 0}$-grading 
\[ \mathfrak{B}(V) \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathfrak{B}_n(V), \]
where $\mathfrak{B}_n(V) := T^n(V)/\ker S_\sigma^n(V)$.

**2.3. Principal comodule algebras.** For a right $H$-comodule $V$ with structure map $\Delta_R$, we say that an element $v \in V$ is (right) coinvariant if $\Delta_R(v) = v \otimes 1$. We denote the subspace of all $H$-coinvariant elements by $V^{\text{coin}(H)}$, and call it the (right) coinvariant subspace of the coaction.

A right $H$-comodule algebra $(P, \Delta_R)$ is a right $H$-comodule which is also an algebra such that the comodule structure map $\Delta_R : P \to P \otimes H$ is an algebra map. We say that $P$ is an $H$-Hopf–Galois extension of $B := P^{\text{coin}(H)}$ if, for $m_P$ the multiplication of $P$, an isomorphism $P \otimes_B P \simeq P \otimes H$ is given by 
\[ \text{can} := (m_P \otimes \text{id}) \circ (\text{id} \otimes \Delta_R) : P \otimes_B P \to P \otimes H. \]

If the functor $P \otimes_B - : {_B\text{Mod}} \to \text{Vect}$, from the category of left $B$-modules to the category of vector spaces, preserves and reflects exact sequences, then we say that $P$ is faithfully flat as a right $B$-module. The definition of faithful flatness for $P$ as a left $B$-module is analogous.
Definition 2.2. A principal right $H$-comodule algebra is a right $H$-comodule algebra $(P, \Delta_R)$ such that $P$ is an $H$-Hopf–Galois extension of $B := P^{co(H)}$ and $P$ is faithfully flat as a right and left $B$-module.

2.4. Quantum homogeneous spaces. Let $A$ and $H$ be Hopf algebras, and let $\pi : A \to H$ be a surjective Hopf algebra map. A right $H$-coaction, giving $A$ the structure of a right $H$-comodule algebra, is given by

$$\Delta_R := (\text{id} \otimes \pi) \circ \Delta : A \to A \otimes H.$$ 

We call the coinvariant subspace $B := A^{co(H)}$ of such a coaction a quantum homogeneous space. In this paper we will exclusively consider quantum homogeneous spaces $B = A^{co(H)}$ for which $A$ is faithfully flat as a right $B$-module, as it allows us to use Takeuchi’s equivalence, see §2.5 below. (We note that similar results hold under much weaker assumptions, see [60].) An important fact is that the coproduct of $A$ restricts to a left $A$-coaction

$$\Delta_L : B \to A \otimes B, \quad b \mapsto b_{(1)} \otimes b_{(2)},$$

giving $B$ the structure of a left $A$-comodule algebra.

A strong bicovariant splitting map is a unital linear map $i : H \to A$ splitting the projection $\pi : A \to H$ such that

$$(i \otimes \text{id}) \circ \Delta = \Delta_R \circ i, \quad (\text{id} \otimes i) \circ \Delta = \Delta_L \circ i. \quad (4)$$

The existence of a bicovariant splitting map implies that the associated quantum homogeneous space gives a principal comodule algebra. For a more detailed discussion of bicovariant splitting maps see [45, §24] and [14, §5].

We say that a Hopf algebra $H$ is cosemisimple if its category of comodules $^H\text{Mod}$ is a semisimple category. Equivalently, $H$ is cosemisimple if it is the direct sum of its simple subcoalgebras. The following technical lemma follows, for example, from the proof of Lemma 3.6 in [24].

Lemma 2.3. For $H$ a cosemisimple Hopf algebra, every surjective Hopf algebra map $\pi : A \to H$ admits a strong bicovariant splitting map, and hence the corresponding right $H$-comodule algebra is a principal comodule algebra.

2.5. Takeuchi’s categorical equivalence. In this subsection we recall the form of Takeuchi’s equivalence [61], for a quantum homogeneous space $\pi : A \to H$, best suited to the paper.

For any quantum homogeneous space $B = A^{co(H)}$, we define $^B\text{Mod}_B$ to be the category whose objects are left $A$-comodules $\Delta_L : \mathcal{F} \to A \otimes \mathcal{F}$, endowed with a $B$-bimodule structure such that $\Delta_L(bfc) = \Delta_L(b)\Delta_L(f)\Delta_L(c)$, for all $f \in \mathcal{F}, b, c \in B$, and whose morphisms are left $A$-comodule, $B$-bimodule, maps.

Let $^H\text{Mod}_B$ denote the category with objects left $H$-comodules $\Delta_L : V \to H \otimes V$, endowed with a right $B$-module structure such that $\Delta_L(vb) = v_{(-1)}\pi(b_{(1)}) \otimes v_{(0)}b_{(2)}$, for all $v \in V, b \in B$, and whose morphisms are left $H$-comodule maps.
Consider the functor
\[ \Phi : A_B^\text{Mod} \to H^\text{Mod}, \quad F \mapsto F / B^+, \]
where the left \( H \)-comodule structure of \( \Phi(F) \) is given by \( \Delta_L[f] := \pi(f(-1)) \otimes [f(0)] \), with square brackets denoting the coset of an element in \( \Phi(F) \). In the other direction, we use the cotensor product \( \Box_H \) to define a functor
\[ \Psi : H^\text{Mod} \to A_B^\text{Mod}, \quad V \mapsto A \Box_H V, \]
where the left \( A \)-comodule structure of \( \Psi(V) \) is defined on the first tensor factor, the right \( B \)-module structure is the diagonal one, and if \( \gamma \) is a morphism in \( H^\text{Mod} \), then \( \Psi(\gamma) := \text{id} \otimes \gamma \).

An adjoint equivalence of categories between \( A_B^\text{Mod} \) and \( H^\text{Mod} \), which we call Takeuchi’s equivalence, is given by the functors \( \Phi \) and \( \Psi \), the unit natural isomorphism
\[ U^n : F \to \Psi \circ \Phi(F), \quad f \mapsto f(-1) \otimes [f(0)], \]
and the counit natural transformation
\[ C^n := (\varepsilon \otimes \text{id}) : \Phi \circ \Psi(V) \to V. \]
The \emph{dimension} \( \dim(F) \) of an object \( F \in A_B^\text{Mod} \) is the vector space dimension of \( \Phi(F) \).

An important point to note is that for any \( B = A^{\text{co}(H)} \) a quantum homogeneous space, Takeuchi’s equivalence implies an isomorphism of Hopf algebras \( \phi : H \to A/B^+A \), such that for \( \text{proj} : A \to A/B^+A \) the canonical surjection, \( \pi = \phi \circ \text{proj} \). This means that we necessarily have that \( \ker(\pi) = B^+A \).

Consider \( A_B^\text{Mod}_0 \) the full subcategory of \( A_B^\text{Mod} \) consisting of those objects \( F \) satisfying \( B^+F = FB^+ \). The corresponding full subcategory \( H^\text{Mod}_0 \) of \( H^\text{Mod} \) is given by objects with the trivial right \( B \)-action. The category \( A_B^\text{Mod}_0 \) comes equipped with a monoidal structure given by the tensor product \( \otimes_B \). Moreover, with respect to the obvious monoidal structure on \( H^\text{Mod}_0 \), Takeuchi’s equivalence is readily endowed with the structure of a monoidal equivalence (see [48, §4]).

Finally, we consider \( B^\text{mod}_0 \) the full subcategory of \( B^\text{Mod}_0 \) whose objects are finitely generated as left \( B \)-modules, and note that it is a monoidal subcategory of \( B^\text{Mod}_0 \). The corresponding full monoidal subcategory \( H^\text{mod}_0 \) of \( H^\text{Mod} \) is given by the finite-dimensional left \( H \)-comodules.

2.6. The fundamental theorem of two-sided Hopf modules. In this subsection we consider a special case of Takeuchi’s equivalence, namely the fundamental theorem of two-sided Hopf modules. (This equivalence was originally considered in [53, Theorem 5.7] using a parallel but equivalent formulation. See also [52].) For a Hopf algebra \( A \), the counit \( \varepsilon : A \to \mathbb{C} \) is clearly a surjective Hopf algebra map. The associated quantum homogeneous space is given by \( A = A^{\text{co}(\mathbb{C})} \). In this case, the category \( A_B^\text{Mod} \) specialises to \( A^\text{Mod}_A \), and the category \( H^\text{Mod} \) reduces to the
category of right $A$-modules $\text{Mod}_A$. For this special case, we find it useful to denote the functor $\Phi$ as

$$F : A\text{-Mod} \to \text{Mod}_A, \quad \mathcal{F} \mapsto \mathcal{F}/A^+\mathcal{F},$$

Moreover, since the cotensor product over $\mathbb{C}$ is just the usual tensor product $\otimes$, we see that the functor $\Psi$ reduces to $A \otimes - : \text{Mod}_A \to A\text{-Mod}_A, \quad V \mapsto A \otimes V.

Since faithful flatness is trivially satisfied in this case, we have the following consequence of Takeuchi’s equivalence: The fundamental theorem of two-sided Hopf modules states that an adjoint equivalence between the categories $A\text{-Mod}_A$ and $\text{Mod}_A$ is given by the functors $F$ and $A \otimes -$, and the unit natural isomorphism

$$U : F \to A \otimes F(F), \quad f \mapsto f(-1) \otimes [f(0)],$$

and the counit natural transformation

$$C : F(A \otimes V) \to V, \quad [a \otimes v] \mapsto \varepsilon(a)v.$$

2.7. Differential calculi. A differential calculus $\left(\Omega^\bullet \simeq \bigoplus_{k \in \mathbb{N}_0} \Omega^k, d\right)$ is a differential graded algebra (dg-algebra) which is generated in degree 0 as a dg-algebra, that is to say, it is generated as an algebra by the elements $a, db$, for $a, b \in \Omega^0$. We call an element $\omega \in \Omega^\bullet$ a form, and if $\omega \in \Omega^k$, for some $k \in \mathbb{N}$, then $\omega$ is said to be homogeneous of degree $|\omega| := k$. The product of two forms $\omega, \nu \in \Omega^\bullet$ is denoted by $\omega \wedge \nu$, unless one of the forms is of degree 0, whereupon the product is denoted by juxtaposition. For a given algebra $B$, a differential calculus over $B$ is a differential calculus such that $\Omega^0 = B$.

2.7.1. Universal differential calculi. A first-order differential calculus over an algebra $B$ is a pair $(\Omega^1(B), d)$, where $\Omega^1(B)$ is a $B$-bimodule and $d : B \to \Omega^1$ is a linear map for which the Leibniz rule holds

$$d(ab) = a(db) + (da)b, \quad \text{for } a, b \in B,$$

and for which $\Omega^1(B)$ is generated as a left $B$-module by those elements of the form $db$, for $b \in B$. The universal first-order differential calculus over $B$ is the pair $(\Omega^1_u(B), d_u)$, where $\Omega^1_u(B)$ is the kernel of the multiplication map $m_B : B \otimes B \to B$ endowed with the obvious $B$-bimodule structure, and $d_u$ is the map defined by

$$d_u : B \to \Omega^1_u(B), \quad b \mapsto 1 \otimes b - b \otimes 1.$$

By [62, Proposition 1.1], every first-order differential calculus over $B$ is of the form $(\Omega^1_u(B)/N, \text{proj} \circ d_u)$, where $N$ is a $B$-subbimodule of $\Omega^1_u(B)$, and we have denoted by $\text{proj} : \Omega^1_u(B) \to \Omega^1_u(B)/N$ the quotient map. This gives a bijective correspondence between calculi and subbimodules of $\Omega^1_u(B)$. Moreover, every first-order differential calculus admits an extension to its maximal prolongation differential calculus, which is to say, one from which any other extension can be obtained by quotienting, see for example [48, §2.5].
For a Hopf algebra and $A$ a left $A$-comodule algebra, we say that a first-order differential calculus $\Omega^1(B)$ over $B$ is left $A$-covariant if there exists a (necessarily unique) map $\Delta_L: \Omega^1(B) \to A \otimes \Omega^1(B)$ satisfying
\[
\Delta_L(bdb') = \Delta_L(b)(id \otimes d)\Delta_L(b'), \quad \text{for } b, b' \in B.
\]
Similarly one can define a right $A$-covariant first-order differential calculus over a right $A$-comodule algebra.

For the special case of $A$ considered as a left $A$-comodule algebra over itself, we note that every left $A$-covariant first-order differential calculus over $A$ is an object in the category $A^A\text{Mod}_A$.

2.7.2. Covariant differential calculi over homogeneous spaces. In the case that $B$ is a quantum homogeneous space of the form $B = A^{co(H)}$, a left $A$-covariant first-order differential calculus $(\Omega^1(B), d)$ is a natural object in $A^B\text{Mod}_B$.

Let us note that $B^+$ can be viewed as an object in $H\text{Mod}_B$. It is easy to see that the map
\[
\xi_B: \Phi(\Omega^1_u(B)) \to B^+, \quad [b_1 db_2] \mapsto \varepsilon(b_1)(b_2)^+, \quad \text{for } b_1, b_2 \in B
\]
is an isomorphism in $H\text{Mod}_B$. Throughout the paper we will tacitly identify these two objects.

As was shown by Hermisson in [34] (see also [46]), we can classify left $A$-covariant first-order differential calculi in terms of subobjects $I_B \subseteq B^+$ in $H\text{Mod}_B$. In particular, if $\Omega^1(B) = \Omega^1_u(B)/N$ then the corresponding subobject is given by $I_B = \xi_B(\Phi(N))$.

Moreover we have the following commutative diagram
\[
\begin{array}{ccc}
\Omega^1(B) & \xrightarrow{(id \otimes \xi_B) \circ \Upsilon} & A \Box_H B^+/I_B \\
\downarrow & & \downarrow \\
B^+ & \xrightarrow{(id \otimes [-]) \circ \Delta_L} & \end{array}
\]

where $[-]: B^+ \to B^+/I_B$ is the canonical projection.

2.7.3. Bicovariant differential calculi over Hopf algebras. For the special case of a trivial quantum homogeneous space, Hermisson’s classification reduces to Woronowicz’s celebrated theorem classifying left-covariant calculi over a Hopf algebra $A$ [63, Theorem 1.5]. Namely, left-covariant first-order differential calculi over $A$ are classified by right ideals of $A^+$.

A first-order differential calculus $\Omega^1(A) = \Omega^1_u(A)/N$ over $A$ is called bicovariant if it is both a right and a left $A$-covariant differential calculus. According to [63, Theorem 1.8], a left-covariant first-order differential calculus over $A$ is bicovariant if the corresponding right ideal $\xi_A(F(N_A))$ of $A^+$ is invariant with respect to the right adjoint $A$-coaction $Ad_R$ given by
\[
Ad_R a = a_{(2)} \otimes S(a_{(1)})a_{(3)}, \quad \text{for } a \in A.
\]
Moreover, when $\Omega(A)$ is bicovariant, $F(\Omega(A))$ has the structure of a Yetter–Drinfeld module over $A$, see [37, §13 and §14].

2.8. Quantum principal bundles. For a right $H$-comodule algebra $(P, \Delta_R)$ with $B := P^\text{co}(H)$, it can be shown that the extension $B \hookrightarrow P$ is Hopf–Galois if and only if the sequence

$$0 \rightarrow P\Omega^1_u(B)P \xrightarrow{\iota} \Omega^1_u(P) \xrightarrow{\text{ver}} P \otimes H^+ \rightarrow 0,$$

(7)

is exact, where $\Omega^1_u(B)$ is the restriction of $\Omega^1_u(P)$ to $B$, $\iota$ is the inclusion map, $\text{ver}$ is the restriction of $\text{can}$ to $\Omega^1_u(P)$. It is useful to note that an explicit presentation of the action of $\text{ver}$ is given by $\text{ver}(a'da) = a'a_{(1)} \otimes \pi(a_{(2)}^+)$. The following definition generalises this sequence to general calculi which are not necessarily universal [14, §5].

**Definition 2.4.** A quantum $H$-principal bundle is a triple $(P, \Delta_R, \Omega^1_u(P))$, where

(i) $(P, \Delta_R)$ is a right $H$-comodule algebra such that $P$ is a Hopf–Galois extension of $B := P^\text{co}(H)$,

(ii) $\Omega^1_u(P) \simeq \Omega^1_u(B)/N$ is a left $H$-covariant first-order differential calculus over $P$,

\[\text{ver}(N) = P \otimes I,\]

for $I$ some right ideal of $H^+$ satisfying $\text{Ad}_H I \subseteq I \otimes H$, where $\text{Ad}_H$ is the adjoint coaction of $H$.

Denote by $\Omega^1(B)$ the restriction of $\Omega^1(P)$ to $B$. **Definition 2.4** implies that

$$0 \rightarrow P\Omega^1(B)P \xrightarrow{\iota} \Omega^1_u(P) \xrightarrow{\text{ver}} P \otimes (H^+/I) \rightarrow 0$$

(8)

is a well-defined exact sequence.

3. Nichols algebras from left $A$-covariant right $H$-covariant calculus

Let $\Omega^1(B)$ be a left $A$-covariant first-order differential calculus over a quantum homogeneous space $B$. In this section we establish sufficient criteria for $\Omega^1(B)$ which allow one to define a (right) Yetter–Drinfeld module structure on the space of left-coinvariant forms $\Phi(\Omega^1_u(B))$.

The objects of the category $\mathcal{A}^A\text{Mod}_H^A$ are $A$-bimodules $\mathcal{F}$ equipped with an $(A, H)$-bicomodule structure $(\Delta_L, \Delta_R)$ such that

\[\Delta_L(afa') = \Delta(a)\Delta_L(f)\Delta(a'), \quad \Delta_R(afa') = \Delta(a)\Delta_R(f)\Delta(a').\]

When $H = A$, the category $\mathcal{A}^A\text{Mod}_H^A$ is known as the category of tetramodules.

The objects of the category $\mathcal{YD}^H_A$ of relative Yetter–Drinfeld modules (see [37, §4]) are right $A$-modules $V$ equipped with a right $H$-comodule structure $\Delta_R$ satisfying the compatibility condition

\[v(0) \triangleleft a_{(1)} \otimes v(1)\pi(a_{(2)}) = (v \triangleleft a_{(2)})_{(0)} \otimes \pi(a_{(1)})(v \triangleleft a_{(2)})_{(1)}, \quad \text{for all } v \in V, a \in A.\]
We now consider a generalisation of the well-known equivalence between tetramodules and Yetter–Drinfeld modules [53]. For any \( V \in \text{YD}_A \) we can endow \( A \otimes V \) with the structure of an object in \( \text{AMod}_A \) by taking the left \( A \)-Hopf module structure of the first tensor factor and the right \( A \)-module structure given by the action on the tensor product. Moreover, we endow \( A \otimes V \) with a right \( H \)-comodule structure \( \Delta_R: A \otimes V \to A \otimes V \otimes H \) as follows
\[
\Delta_R(a \otimes v) := a_{(1)} \otimes v_{(0)} \otimes \pi(a_{(2)})v_{(1)}, \quad \text{for } a \in A, \ v \in V.
\]
It is clear that
\[
\Delta_R(af) = \Delta_R(a)\Delta_R(f), \quad \text{for } a \in A, \ f \in A \otimes V,
\]
and that \( A \otimes V \) is an \((A, H)\)-bicomodule. Moreover, we see that
\[
\Delta_R((b \otimes v)a) = \Delta_R(ba_{(1)} \otimes v \cdot a_{(2)}) = b_{(1)}a_{(1)} \otimes (v \cdot a_{(3)}(0)) \otimes \pi(b_{(2)})\pi(a_{(2)})(v \cdot a_{(3)}(1)), \quad (9)
\]
\[
\Delta_R(b \otimes v)\Delta_R(a) = \left( b_{(1)} \otimes v_{(0)} \otimes \pi(b_{(2)})v_{(1)} \right) \left( a_{(1)} \otimes \pi(a_{(2)}) \right) = b_{(1)}a_{(1)} \otimes v_{(0)} \cdot a_{(2)} \otimes \pi(b_{(2)})v_{(1)}\pi(a_{(3)}). \quad (10)
\]
Thus \( A \otimes V \) is an object in \( \text{AMod}_A^H \) and \( A \otimes - \) defines a functor from \( \text{YD}_A^H \) to \( \text{AMod}_A^H \), which acts on morphisms in the obvious way.

Conversely, consider an \( F \in \text{AMod}_A^H \) and recall the functor \( F : \text{AMod}_A^H \to \text{HMod}_A \) introduced in §2.6. We define a right \( H \)-comodule structure on \( F(\mathcal{F}) \) by
\[
F(\mathcal{F}) \to F(\mathcal{F}) \otimes H, \quad [f] \mapsto [f_{(2)}] \otimes \pi(f_{(1)})\pi(S(f_{(3)})),
\]
and hence give \( F(\mathcal{F}) \) a Yetter–Drinfeld structure. This gives a functor \( F \) from \( \text{AMod}_A^H \) to \( \text{YD}_A^H \), where morphisms are defined by descending to the quotient.

Finally, we note that an adjoint equivalence between the two categories is given by the unit natural transformation
\[
\mathcal{F} \to A \otimes (F(\mathcal{F})), \quad f \mapsto f_{(1)} \otimes [f_{(2)}],
\]
and the counit natural transformation
\[
F(A \otimes \mathcal{F}) \to \mathcal{F}, \quad a \otimes [f] \mapsto \varepsilon(a)f.
\]

**Remark 3.1.** In general, the formula for the Yetter–Drinfeld braiding [53] is not well defined for objects in \( \text{YD}_A^H \). This difficulty can be avoided if \( \pi \) admits a bicovariant splitting map \( i: H \to A \), as we will see below.

### 3.1. Left \( A \)-covariant right \( H \)-covariant first-order differential calculi.

If \( (\Omega^1(A), d) \) is a left \( A \)-covariant right \( H \)-covariant first-order differential calculus over \( A \) then \( \Omega^1(A) \) is an object in \( \text{AMod}_A^H \). In particular, since \( (\Omega^1(A), d) \) is left \( A \)-covariant there is a corresponding ideal \( I_A \) of \( A^+ \). From [16], §2] the right \( H \)-covariance condition is equivalent to the condition \( \text{Ad}_\pi I_A \subseteq I_A \otimes H \), where
\[
\text{Ad}_\pi a := (\text{id} \otimes \pi) \circ \text{Ad}_R(a) = a_{(2)} \otimes \pi(S(a_{(1)})a_{(3)}), \quad \text{for } a \in A, \quad (11)
\]
is the right $H$-coaction induced by the right adjoint $A$-coaction $\tilde{A}$.

3.2. Restrictions of left $A$-covariant right $H$-covariant first-order differential calculi. Let $\Omega^1(A) \cong \Omega^1_u(A)/N_A$ be a left $A$-covariant right $H$-covariant first-order differential calculus over $A$ and $I_A = \xi_A(F(N_A))$ the corresponding $A$-coinvariant ideal of $A^+$. For a quantum homogeneous space $B = A^{co(H)}$, let $\Omega^1(B) := \operatorname{span}_C \{adb \mid a, b \in B\}$ be the restriction of $\Omega^1(A)$ to $B$ and let $I_B$ be the corresponding ideal of $B^+$. Then $I_B = I_A \cap B^+$, see [14, Theorem 5.77] for details.

The adjoint right $H$-coaction $\text{Ad}_\pi^r$, induced by (11), acts on $b \in B$ as
\[
\text{Ad}_\pi^r b = b(2) \otimes \pi(S(b(1))b(3)) = b(2) \otimes \pi(S(b(1))), \quad \text{for } b \in B.
\] (12)
We have the following commutative diagram in the category $^A\text{Mod}$
\[
\begin{array}{ccc}
\Omega^1(A) & \xrightarrow{(id \otimes \xi_A) \circ U} & A \otimes F(\Omega^1(A)) \\
\downarrow{\iota'} & & \downarrow{id \otimes \iota} \\
\Omega^1(B) & \xrightarrow{(id \otimes \xi_B) \circ U} & A \square_H \Phi(\Omega^1(B))
\end{array}
\] (13)
where $\iota$ and $\iota'$ are the evident inclusions. It is important to note that $\iota$ is an $H$-comodule map. In what follows we identity $\Phi(\Omega^1(B))$ with its image under $\iota$.

In this paper, we will only deal with quantum principal bundles over quantum homogeneous spaces. In this special case, we see that we have the following commutative diagram
\[
\begin{array}{ccc}
\Omega^1_u(A) & \xrightarrow{\text{ver}} & A \square_H H^+ \\
\downarrow{U} & & \\
A \otimes A^+ & \xrightarrow{id \otimes \pi} & A \square_H H^+.
\end{array}
\]
This means that for any subbimodule $N \subseteq \Omega^1_u(A)$, with corresponding ideal $I := U(N)$, it holds that
\[
\text{can}(N) = A \otimes \pi(I).
\]
From this we see that the requirement that $\text{ver}(N) = A \otimes I$ is automatically satisfied.

**Proposition 3.2** ([14, §2]). Let $\pi : A \to H$ be a surjective Hopf algebra map such that $A$ is an $H$-Hopf–Galois extension of the associated quantum homogeneous space $B := A^{co(H)}$, and $\Omega^1(A)$ is a left $A$-covariant right $H$-covariant differential calculus over $A$. Then $(A, \Delta_R, \Omega^1(A))$ is a quantum $H$-principal bundle.
3.3. Constructing a Yetter–Drinfeld $H$-module structure. In this section we establish a necessary list of conditions allowing us to associate a Yetter–Drinfeld module to a quantum $H$-principal bundle over a quantum homogeneous space. This is our principal tool for constructing a Nichols algebra presentation of the quantum Grassmannian Heckenberger–Kolb calculus in §4.1.

**Theorem 3.3.** Suppose $(A, \Delta_R, \Omega^1(A))$ is a quantum $H$-principal bundle over a quantum homogeneous space $B := A^{\text{co}(H)}$, and let $\Omega^1(B)$ be the restriction of $\Omega^1(A)$ to $B$. Moreover, assume that

(i) $i(\Phi(\Omega^1(B)))$ is a subobject of $F(\Omega^1(A))$ in $\text{Mod}_A$,

(ii) $\Omega^1(B)$ is a finite-dimensional object of $\text{dMod}_0$.

Then

(a) a right $H$-action on $\Phi(\Omega^1(B))$ is defined by

$$\triangleright : \Phi(\Omega^1(B)) \otimes H \to \Phi(\Omega^1(B)), \quad [b] \otimes h \mapsto [b] \triangleright_A i(h) = [b i(h)],$$

where $i$ is a bicovariant splitting of $\pi$, and $\triangleright_A$ is the right $A$-action on $i(\Phi(\Omega^1(B)))$,

(b) the triple $(\Phi(\Omega^1(B)), \triangleright, \text{Ad}_\pi)$, where $\text{Ad}_\pi$ is defined by (12), defines a Yetter–Drinfeld module over $H$,

(c) the Yetter–Drinfeld module structure is independent of the choice of bicovariant splitting map $i$.

**Proof.** For notational simplicity set $V := \Phi(\Omega^1(B))$. For the $H$-action (14) to be well defined it should hold that

$$v \triangleright_A \left(i(hh') - i(h)i(h')\right) = 0, \quad \text{for all } v \in V, h, h' \in H. \quad (15)$$

Now $\pi \left(i(hh') - i(h)i(h')\right) = 0$, and as discussed in §2.5 it holds that $\ker \pi = B^+A$. Therefore $i(hh') - i(h)i(h') \in B^+A$. Hence, since $B^+$ acts trivially on $V$, we must have that (15) holds and that $i$ defines an action of $H$ on $V$.

We now show that $\text{Ad}_\pi$ and $\triangleright$ define a Yetter–Drinfeld structure on $V$. Note first that for $b \in B^+$ and $h \in H$ we have

$$([b] \triangleright_A h^{(2)})^{(1)} \otimes h^{(1)}([b] \triangleright_A h^{(2)})^{(1)} = [b i(h^{(2)})]^{(1)} \otimes h^{(1)}(b i(h^{(2)}))^{(1)}$$

$$= [(b i(h^{(2)}))^{(2)}] \otimes h^{(1)} \pi \left(S \left(b i(h^{(2)})^{(1)}\right)b i(h^{(2)})^{(3)}\right)$$

$$= [b i(h^{(2)})^{(2)}] \otimes h^{(1)} S \left(\pi \left(i(h^{(2)})^{(1)}\right)\right) \pi \left(S(b^{(1)})b^{(3)}\right) \pi \left(i(h^{(2)})^{(3)}\right).$$

Since $i$ is a bicovariant splitting map we have that

$$\pi (i(h)^{(1)} \otimes i(h)^{(2)} \otimes \pi (i(h)^{(3)})) = h^{(1)} \otimes i(h^{(2)}) \otimes h^{(3)}.$$
Combining this with the previous calculation we see that

\[
([b] \triangleleft h_{(2)}(0) \otimes h_{(1)}([b] \triangleleft h_{(2)}(1)) = [b_{(2)}] 	riangleleft h_{(1)}(S(h_{(2)})\pi(S(b_{(1)})b_{(3)})h_{(4)})
\]

\[
= [b_{(2)}] \triangleleft h_{(1)} \otimes \pi\left(S(b_{(1)})b_{(3)}\right)h_{(2)}
\]

\[
= [b] \triangleleft h_{(1)} \otimes [b]_{(1)}h_{(2)},
\]

as required. It follows that the action and coaction satisfy the Yetter–Drinfeld condition (2).

Finally, we show that the Yetter–Drinfeld structure is independent of the choice of bicovariant splitting map. Let \( i' \) be a second splitting map and note that since

\[
\pi(i(h) - i'(h)) = 0,
\]

we must have \( \text{Im}(i - i') \subseteq \ker(\pi) = B^+A \). In particular, \( i(h) - i'(h) \in B^+A \) and

\[
\varepsilon(i(h) - i'(h)) = 0.
\]

Since we have assumed that \( \Omega^1(B) \) is an object in \( \mathcal{H}_B \text{Mod}_0 \), the right \( B \)-action on \( V \) is trivial. Therefore

\[
[b] \triangleleft_A i(h) - [b] \triangleleft_A i'(h) = [b] \triangleleft_A \varepsilon(i(h) - i'(h)) = 0,
\]

implying that the two actions are equal. \( \square \)

**Remark 3.4.** Recall from §2.5 that \( \Psi \) is a monoidal functor. Thus the morphism \( \Psi(\sigma) \) satisfies the Yang–Baxter equation for linear operators on \( \Omega^1(B) \otimes_B \Omega^1(B) \). Thus we can consider the corresponding Nichols algebra \( \mathcal{B}(\Omega^1) \) as a quotient of the tensor algebra \( T_B(\Omega^1(B)) \) of \( \Omega^1(B) \) over \( B \), see [64]. This means that \( \mathcal{B}(\Omega^1(B)) \cong \Psi(\mathcal{B}(V)) \), giving us a Nichols algebra description of the differential calculus.

**Remark 3.5.** If \( i: H \rightarrow A \) is a bicovariant splitting then \( A_A \text{Mod}_{A}^H \cong YD_A^H \) and so the corresponding Nichols algebra makes sense with respect to the braiding

\[
\sigma(v \otimes w) = w_{(0)} \otimes v(\triangleleft_A \circ i)(w_{(1)}).
\]

4. **The Quantum Grassmannian Heckenberger–Kolb Calculi**

In this section we apply Theorem 3.3 to the first-order parts of the Heckenberger–Kolb calculi of the quantum Grassmannians on their cotangent spaces. We then build on this construction to present the holomorphic and anti-holomorphic subcomplexes as Nichols algebras.

4.1. **Preliminaries on Drinfeld–Jimbo quantum groups.** In this section we recall basic material about Drinfeld–Jimbo quantised universal enveloping algebras [25, 35] and their representation theory. We refer the reader to [37, 20, 45] for further details.
4.1.1. Drinfeld–Jimbo quantised universal enveloping algebras. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra of rank $r$. We fix a Cartan subalgebra $\mathfrak{h}$ with corresponding root system $\Delta \subseteq \mathfrak{h}^*$. Let a choice of simple roots $\{\alpha_1, \ldots, \alpha_r\}$. Denote by $(\cdot, \cdot)$ the symmetric bilinear form induced on $\mathfrak{h}^*$ by the Killing form of $\mathfrak{g}$, normalised so that any shortest simple root $\alpha_i$ satisfies $(\alpha_i, \alpha_i) = 2$. Let $\{w_1, \ldots, w_r\}$ denote the corresponding set of fundamental weights of $\mathfrak{g}$. The Cartan matrix $A = (a_{ij})$ of $\mathfrak{g}$ is the $(r \times r)$-matrix defined by $a_{ij} := (\alpha'_i, \alpha_j)$, where $\alpha'_i := 2\alpha_i/(\alpha_i, \alpha_i)$. Let $q \in \mathbb{C}$ such that $q$ is not a root of unity and denote $q_i := q^{(\alpha_i, \alpha_i)/2}$. The quantised universal enveloping algebra generated by the elements $E_i, F_i, K_i,$ and $K_i^{-1}$, for $i = 1, \ldots, r$, subject to the relations

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \quad K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

along with the quantum Serre relations

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \left[ 1 - \frac{a_{ij}}{s} \right] q_i^{1-a_{ij}-s} E_i^{1-a_{ij}-s} E_j E_i^s = 0, \quad \text{for } i \neq j,$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \left[ 1 - \frac{a_{ij}}{s} \right] q_i^{1-a_{ij}-s} F_i^{1-a_{ij}-s} F_j F_i^s = 0, \quad \text{for } i \neq j,$$

where we have used the $q$-binomial coefficients defined as follows

$$[n]_q! := [n]_q[n-1]_q \cdots [2]_q[1]_q, \quad \text{where } [k]_q := \frac{q^k - q^{-k}}{q - q^{-1}},$$

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$ 

A Hopf algebra structure is defined on $U_q(\mathfrak{g})$ by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(K_i) = K_i^{-1},$$

$$\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_i) = 1.$$ 

Let $\mathcal{P}$ be the weight lattice of $\mathfrak{g}$, and $\mathcal{P}^+$ its set of dominant integral weights. We consider $\text{Rep}_1 U_q(\mathfrak{g})$, the full subcategory of the category of (left) $U_q(\mathfrak{g})$-modules, whose the objects are finite-dimensional $U_q(\mathfrak{g})$-modules having a weight decomposition $V = \bigoplus_{\mu \in \mathcal{P}} V(\mu)$. Recall that a vector $v \in V$ is called a weight vector of weight $\mu \in \mathfrak{h}^*$ if $K_i \triangleright v = q^{(\alpha_i, \mu)} v$ for all $i = 1, \ldots, r$. The category $\text{Rep}_1 U_q(\mathfrak{g})$ is a semisimple tensor category whose simple objects are irreducible modules $V_\lambda$ with highest weight $\lambda \in \mathcal{P}^+$. The character of $V_\lambda$ is given by the classical Weyl character
formula for the irreducible $\mathfrak{g}$-module $\hat{V}_\lambda$ with highest weight $\lambda$. In fact, the category $\text{Rep}_1 U_q(\mathfrak{g})$ is equivalent to the category $\mathcal{O}_f$ of finite-dimensional representations of $\mathfrak{g}$. We refer to [27, §5.8] and [37, §7] for further details.

4.2. Quantum coordinate algebras. In this subsection we recall some necessary material about quantised coordinate algebras, see [37, §6 and §7] and [51] for further details. Let $V$ be a finite-dimensional left $U_q(\mathfrak{g})$-module, $v \in V$, and $f \in V^*$, where $V^*$ is the $\mathbb{C}$-linear dual of $V$ endowed with its right $U_q(\mathfrak{g})$-module structure. An important point to note is that, with respect to the equivalence of left and right $U_q(\mathfrak{g})$-modules given by the invertible antipode, the left module corresponding to $V^*_\mu$ is isomorphic to $V_{-w_0(\mu)}$, where $w_0$ denotes the longest element in the Weyl group of $\mathfrak{g}$.

Consider the function $c^{V}_{f,v} : U_q(\mathfrak{g}) \to \mathbb{C}$ defined by $c^{V}_{f,v}(X) := f(X \triangleright v)$. The coordinate ring of $V$ is the subspace $C(V) := \text{span}_\mathbb{C} \{ c^{V}_{f,v} | v \in V, f \in V^* \} \subseteq U_q(\mathfrak{g})^\circ$, where $U_q(\mathfrak{g})^\circ$ denotes the Hopf dual of $U_q(\mathfrak{g})$. A $U_q(\mathfrak{g})$-bimodule structure on $C(V)$ is given by

\[
(Y \triangleright c^{V}_{f,v} \triangleleft Z)(X) := f \left( (ZXY) \triangleright v \right) = c^{V}_{f \triangleleft Z,Y \triangleright v}(X). \tag{17}
\]

It is easily checked that $C(V) \subseteq U_q(\mathfrak{g})^\circ$, and moreover that a Hopf subalgebra of $U_q(\mathfrak{g})^\circ$ is given by

\[
\mathcal{O}_q(G) := \bigoplus_{\mu \in P^+} C(V_\mu). \tag{18}
\]

We call $\mathcal{O}_q(G)$ the quantum coordinate algebra of $G$, where $G$ is the compact, connected, simply-connected, simple Lie group having $\mathfrak{g}$ as its complexified Lie algebra.

Let $H$ be a Hopf algebra and $V$ a right $H$-comodule. Suppose that $U$ is a Hopf algebra which is dually paired with $H$ via a pairing $\langle \cdot, \cdot \rangle : U \otimes H \to \mathbb{C}$. Recall that a right $U$-action on $V$ is defined by

\[
\triangleright : V \otimes U \to V, \quad v \otimes X \mapsto v(0) \langle S(X), v(1) \rangle. \tag{19}
\]

Similarly one may define a left action of $U$ on a left $H$-comodule.

4.2.1. Quantum exterior algebras. The category $\text{Rep}_1 U_q(\mathfrak{g})$ is a braided monoidal category where the braiding comes from the universal $R$-matrix $R \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ as defined in [25]. For any two objects $V, W \in \text{Rep}_1 U_q(\mathfrak{g})$, let

\[
\rho_V : U_q(\mathfrak{g}) \to \text{End}(V), \quad \rho_W : U_q(\mathfrak{g}) \to \text{End}(W),
\]

be the corresponding structure maps. The braiding $\sigma_{R \otimes V \otimes W} : V \otimes W \to W \otimes V$ induced by $R$ is defined by

\[
\sigma_{R \otimes V \otimes W} := \tau \circ (\rho_V \otimes \rho_W)(R), \tag{20}
\]

where $\tau: V \otimes W \to W \otimes V$ is the ordinary flip. Following [26], define the normalised braiding $\hat{\sigma}_{R,V \otimes W}: V \otimes W \to W \otimes V$ by

$$\hat{\sigma}_{R,V \otimes W} := \sqrt{\sigma_{R,W \otimes V}^{-1} \sigma_{R,V \otimes W}^{-1}}.$$

\section*{Remark 4.1.}
Note that formula (21) is not well defined for an arbitrary braided monoidal category. However, it is well defined for the quantised universal enveloping algebra $U_q(\mathfrak{g})$ since the universal $R$-matrix $R \in U_q(\mathfrak{g}) \hat{\otimes} U_q(\mathfrak{g})$ can be decomposed as

$$R = R_0 R_1 = R_1 R_0,$$

where $R_0$ is “the diagonal part” of $R$, and $R_1$ is unipotent. We refer to [26, §3] for further discussion.

As in §2.1.1 we write $\hat{\sigma}_R$ instead of $\hat{\sigma}_{R,V \otimes W}$. Since $\hat{\sigma}_R^2 = \text{id}$, $\hat{\sigma}_R$ is a symmetric commutativity constraint but it does not satisfy the Yang–Baxter equation in general, see [26, §3] and [15].

For any $V \in \text{Rep}_1(U_q(\mathfrak{g}))$, denote

$$S_q^2 V := \{ x \in V \otimes V | \hat{\sigma}_R(x) = x \}, \quad \Lambda_q^2 V := \{ x \in V \otimes V | \hat{\sigma}_R(x) = -x \}. \quad (22)$$

Note that $V \otimes V = S_q^2 V \oplus \Lambda_q^2 V$. Following [15], define the quantum exterior algebra $\Lambda_q(V)$ of $V$ to be

$$\Lambda_q(V) := \mathcal{T}(V) / \langle S_q^2 V \rangle,$$

where $\mathcal{T}(V)$ is the tensor algebra of $V$ and $\langle J \rangle$ denotes the two-sided ideal generated by $I \subset \mathcal{T}(V)$. A $U_q(\mathfrak{g})$-module $V$ is called flat if the Hilbert–Poincaré series of $\Lambda_q(V)$ is the same as for its classical counterpart $\Lambda(V)$.

\section*{Example 4.2.}
Fix two positive integers $n$ and $m$ such that $n > m$. Let $V$ be a $U_q(\mathfrak{gl}_m)$-module and $W$ a $U_q(\mathfrak{sl}_{n-m})$-module. Denote by $V \boxtimes W$ the tensor product $V \otimes W$ viewed as a $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-module. We have the following isomorphism of $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-modules

$$(V_{\omega_1} \boxtimes V_{\omega_1}^*) \otimes (V_{\omega_2} \boxtimes V_{\omega_2}^*) \simeq (V_{\omega_2} \boxtimes V_{\omega_2}^*) \oplus (V_{\omega_1} \boxtimes V_{\omega_1}^*) \oplus (V_{\omega_2} \boxtimes V_{\omega_2}^*).$$

For details see Table 5 in [49].

As was shown in [15, Proposition 2.33], we have

$$S_q^2(V_{\omega_1} \boxtimes V_{\omega_1}^*) = (V_{\omega_2} \boxtimes V_{\omega_2}^*) \oplus (V_{\omega_2} \boxtimes V_{\omega_2}^*).$$

Moreover, the quantum exterior algebra $\Lambda_q(V_{\omega_1} \boxtimes V_{\omega_1}^*)$ has the same Hilbert–Poincaré series as the exterior algebra $\Lambda(V_{\omega_1} \boxtimes V_{\omega_1}^*)$. Hence $V_{\omega_1} \boxtimes V_{\omega_1}^*$ is flat.

\subsection*{4.2. Quantum Howe duality.}
In [19, Theorem 4.2.2] (see also [65]), it was shown that as $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-modules the quantum exterior algebra of $V_{\omega_1} \boxtimes V_{\omega_1}^*$ decomposes as

$$\Lambda_q(V_{\omega_1} \boxtimes V_{\omega_1}^*) = \bigoplus_{\lambda} V_{\lambda} \boxtimes V_{\lambda^t}, \quad (23)$$

where $\lambda$ varies over all $(n - m)$-bounded partitions, and $\lambda^t$ is the transpose of $\lambda$. 

\section*{Acknowledgments}

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4.3. Preliminaries on quantum flag manifolds. Let \( \{\alpha_i\}_{i \in S} \) be a subset of simple roots. In what follows, by abuse of notation, we denote by \( S \) not only an index subset but also the corresponding subset of simple roots \( \{\alpha_i\}_{i \in S} \). Consider the Hopf subalgebra
\[
U_q(l_S) := \langle K_i, E_j, F_j \mid i = 1, \ldots, r; j \in S \rangle.
\]
The Hopf algebra embedding \( \iota_S : U_q(l_S) \hookrightarrow U_q(g) \) induces a dual Hopf algebra map \( \iota_S^* : U_q(g)^* \rightarrow U_q(l_S)^* \). By construction \( O_q(G) \subseteq U_q(g) \), so the restriction map
\[
\pi_S := \iota_S^*|_{O_q(G)} : O_q(G) \rightarrow U_q(l_S)^*,
\]
defines a Hopf subalgebra \( O_q(L_S) := \pi_S(O_q(G)) \subseteq U_q(l_S)^* \). The quantum flag manifold associated to \( S \) is the quantum homogeneous space associated to the surjective Hopf algebra map \( \pi_S : O_q(G) \rightarrow O_q(l_S) \), and is denoted by
\[
O_q(G/L_S) := O_q(G)_{\text{co}(O_q(L_S))}.
\]

Remark 4.3. Since \( O_q(l_S) \) is cosemisimple, it follows from Lemma 2.3 that the extension \( O_q(G/L_S) \hookrightarrow O_q(G) \) is a principal comodule algebra and \( \pi_S \) admits a bicovariant splitting map.

4.3.1. The Heckenberger–Kolb calculus. Let \( S \) be a subset a simple roots of \( g \). Following the classical case (see, for example, [12]) we say that the quantum flag manifold associated to \( S \) is of irreducible type if \( g/l_S \) is a direct sum of two dual irreducible \( l_S \)-modules. The following theorem summarises [29, Theorem 7.2] and [30, Propositions 3.6 and 3.7].

Theorem 4.4. For any irreducible quantum flag manifold \( O_q(G/L_S) \), there exist exactly two non-isomorphic, irreducible, left \( O_q(G) \)-covariant, finite-dimensional, first-order differential calculi
\[
\Omega_q^{(1,0)}(G/L_S), \quad \Omega_q^{(0,1)}(G/L_S) \in O_q(G/L_S)^{\text{Mod}_0},
\]
and the corresponding maximal prolongations \( \Omega_q^{(\bullet,0)}(G/L_S) \) and \( \Omega_q^{(0,\bullet)}(G/L_S) \) have classical dimension.

Remark 4.5. For the reader’s convenience, we recall some detail about the proof of Theorem 4.4. The left-covariant first-order differential calculi were first classified in [29], where it was shown that, up to isomorphism, there exist precisely two irreducible finite-dimensional left-covariant differential calculi over the irreducible quantum flag manifolds. This was achieved by classifying the equivalent notation of a quantum tangent space using the coradical filtration of the locally finite part of the dual coalgebra of \( O_q(G/L_S) \). The maximal prolongation of these first-order calculi to differential calculi was then explicitly described in [30] where it was shown, among many other things, that the calculi have classical dimension.
4.4. Quantum Grassmannians. Consider the A-series irreducible quantum flag manifolds, namely, the quantum Grassmannians. Let \( g = \mathfrak{s}l_n \) and \( m \) be an integer such that \( 1 \leq m < n \). Fix \( S = \{ \alpha_i \mid i \in \{1, \ldots, n-1\} \setminus \{m\} \} \) a subset of simple roots of \( g \). In this case \( l_S = \mathfrak{s}l_m \oplus \mathfrak{s}l_{n-m} \). The quantum flag manifold associated to \( S \) is called the quantum \( (n, m) \)-Grassmannian and denoted by \( \mathcal{O}_q(\text{Gr}_{n,m}) \). We denote by \( \pi := \pi_S : \mathcal{O}_q(SU_n) \to \mathcal{O}_q(U_m \times SU_{n-m}) \) the Hopf algebra map corresponding to \( S \) \((24)\).

In what follows we use the fact that the quantum coordinate algebra \( \mathcal{O}_q(SU_n) \) can be described in terms of matrix coefficients of the first fundamental representation \( V_{\varpi_1} \) of \( U_q(\mathfrak{s}l_n) \), recalling the Faddeev–Reshetikhin–Takhtadzhyan approach \([51]\).

4.5. Quantum principal bundles over the quantum Grassmannians. In this section we recall necessary facts about the first-order part of the Heckenberger–Kolb calculi over the quantum Grassmannians. These allow us to apply Theorem \([5,3]\) to construct the associated Yetter–Drinfeld modules. Throughout we denote by \( \Omega^1_q(\text{Gr}_{n,m}) := \Omega^{1,0}_q(\text{Gr}_{n,m}) \oplus \Omega^{0,1}_q(\text{Gr}_{n,m}) \) the direct sum of the pair of the first order part of the Heckenberger–Kolb calculi over \( \mathcal{O}_q(\text{Gr}_{n,m}) \).

Proposition 4.6 \([18, \text{Corollary 5.3 and Proposition 5.8}]\). Let \( \Omega^{1,b}_{bc}(SU_n) \) be the bicovariant first-order differential calculus over \( \mathcal{O}_q(SU_n) \) associated to the R-matrix of \( U_q(\mathfrak{s}l_n) \). For every \( m = 1, \ldots, n-1 \), there is an ideal \( N_m \) of \( \Omega^{1,b}_{bc}(SU_n) \) such that

1) the quotient calculus \( \Omega^1_m(SU_n) := \Omega^{1,0}_{bc}(SU_n)/N_m \) is a left \( \mathcal{O}_q(SU_n) \)-covariant right \( \mathcal{O}_q(U_m \times SU_{n-m}) \)-covariant first order differential calculus,

2) the restriction of \( \Omega^1_m(SU_n) \) to \( \mathcal{O}_q(\text{Gr}_{n,m}) \) is \( \Omega^1_q(\text{Gr}_{n,m}) \).

Set \( V^{(1,0)} := \Phi(\Omega^{(1,0)}(\text{Gr}_{n,m})) \) and \( V^{(0,1)} := \Phi(\Omega^{(0,1)}(\text{Gr}_{n,m})) \).

Corollary 4.7. The quantum principal bundle above determines \( \mathcal{O}_q(U_m \times SU_{n-m}) \)-Yetter–Drinfeld module structures for \( V^{(1,0)} \) and \( V^{(0,1)} \).

Proof. As was noted in Remark \([5,3]\) \( \pi \) admits a bicovariant splitting map. From Proposition \([1,0]\) it follows that the triple \( (\mathcal{O}_q(SU_n), \mathcal{O}_q(\text{Gr}_{n,m}), \Omega^1_m(SU_n)) \) defines a quantum principal bundle. By Theorem \([4,3]\) we have that \( \Omega^{(1,0)}_q(\text{Gr}_{n,m}) \) and \( \Omega^{(0,1)}_q(\text{Gr}_{n,m}) \) are in \( \mathcal{O}_q(G/L_S)\text{-Mod}_0 \). Moreover, \( \iota(V^{(1,0)}) \) and \( \iota(V^{(0,1)}) \) are right \( \mathcal{O}_q(SU_n) \)-submodules, see \([26]\) below. Thus the conditions of Theorem \([5,3]\) are satisfied. \( \square \)

As noted in Remark \([5,3]\) since \( \Omega^1_m(SU_n) \) is a left \( \mathcal{O}_q(SU_n) \)-covariant and right \( \mathcal{O}_q(U_m \times SU_{n-m}) \)-covariant first order differential calculus, it admits a braiding by \([16]\). Determining the properties of the corresponding Nichols algebras presents itself as an interesting direct of research.
4.6. The Yetter–Drinfeld braiding on $V^{(0,1)} \otimes V^{(0,1)}$. Recall from §3.2 that we have an embedding of right $H$-comodules

$$\iota: V^{(1,0)} \oplus V^{(0,1)} \rightarrow F(\Omega^1_m(SU_n)).$$

Let $M := \{1, \ldots, m\}$, and $\overline{M} := \{m+1, \ldots, n\}$. From [18] Lemma 5.7 it holds that

(i) the set $\{[u^i_j] | (i, j) \in \overline{M} \times M\}$ is a basis of $\iota(V^{(1,0)})$,

(ii) the set $\{[u^i_j] | (i, j) \in M \times \overline{M}\}$ is a basis of $\iota(V^{(0,1)})$.

In what follows, we will use the bases of $V^{(1,0)}$ and $V^{(0,1)}$ induced by the bases of $\iota(V^{(1,0)})$ and $\iota(V^{(0,1)})$ from the previous proposition. The $O_q(U_m \times SU_{n-m})$-Yetter–Drinfeld module structure on $V^{(0,1)}$ is explicitly described below in terms of these bases.

As was shown in [18] Proposition B.3, the $O_q(U_m \times SU_{n-m})$-action on $V^{(1,0)}$ and $V^{(0,1)}$ is given as follows. For $i \neq j$ and $(p, s) \in (M \times M) \cup (\overline{M} \times \overline{M})$ we have

$$[u^i_j] \cdot \pi(u^p_s) = q^{-2/n} q^{\delta_{p,i} + \delta_{p,j}} [u^i_j],$$

$$[u^i_j] \cdot \pi(u^p_s) = q^{-2/n} \left( \theta(p-s) \delta_{s,i} (q-q^{-1}) [u^p_s] + \theta(s-p) \delta_{p,j} (q-q^{-1}) [u^i_j] \right),$$

$$[u^i_j] \cdot \pi(S(u^p_s)) = q^{2/n} q^{-\delta_{p,i} - \delta_{p,j}} [u^i_j],$$

$$[u^i_j] \cdot \pi(S(u^p_s)) = q^{2/n} \left( \delta_{s,i} \theta(p-s) (q^{-1} - q) [u^p_s] + \delta_{p,j} \theta(s-p) q^{2(p-s)} (q^{-1} - q) [u^i_j] \right).$$

By [12], the right $O_q(U_m \times SU_{n-m})$-coactions on $V^{(1,0)}$ and $V^{(0,1)}$ are given by

$$Ad_x[u^i_j] = \sum_{(a,b) \in M \times M} [u^a_b] \otimes \pi(S(u^a_b)u^i_j), \quad \text{for } (i, j) \in M \times M,$$

$$Ad_x[u^i_j] = \sum_{(a,b) \in \overline{M} \times \overline{M}} [u^a_b] \otimes \pi(u^a_bS(u^i_j)), \quad \text{for } (i, j) \in \overline{M} \times M.$$  

(26)

(27)

Lemma 4.8. For $[u^i_j] \otimes [u^k_l] \in V^{(0,1)} \otimes V^{(0,1)}$, and $\sigma$ the Yetter–Drinfeld braiding of $V^{(0,1)}$, it holds that

$$\sigma([u^i_j] \otimes [u^k_l]) = q^{-\delta_{k,i} + \delta_{l,j}} [u^i_j] \otimes [u^k_l] + q^{-\delta_{k,i}} (q-q^{-1}) \theta(l-j) [u^k_l] \otimes [u^i_j] - q^{\delta_{l,j}} (q-q^{-1}) \theta(k-i) [u^i_j] \otimes [u^k_l].$$

(28)

Proof. Applying (27) to (3), we see that the non-zero terms are

$$\sigma([u^i_j] \otimes [u^k_l]) = [u^k_l] \otimes \left( [u^i_j] \cdot \pi(S(u^k_l)u^i_j) \right) + [u^i_j] \otimes \left( [u^k_l] \cdot \pi(S(u^i_j)u^i_j) \right)$$

$$+ [u^i_j] \otimes \left( [u^k_l] \cdot \pi(S(u^i_j)u^i_j) \right) + [u^i_j] \otimes \left( [u^k_l] \cdot \pi(S(u^i_j)u^i_j) \right).$$

A further application of (26) gives us (28). □
4.7. The spectrum of the Yetter–Drinfeld braiding on $V^{(0,1)} \otimes V^{(0,1)}$. Following [29], we convert the $O(U_m \times SU_{n-m})$-coaction on $V^{(0,1)}$ to a $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-action. A direct computation confirms that the highest weight vectors in the $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-module $V^{(0,1)} \otimes V^{(0,1)}$ are given by

$$
\begin{align*}
&v_1 := [u_n^1] \otimes [u_n^1], \\
v_2 := -q[u_n^1] \otimes [u_{n-1}^1] + [u_{n-1}^1] \otimes [u_n^1], \\
v_3 := -q[u_n^1] \otimes [u_n^2] + [u_n^2] \otimes [u_n^1], \\
v_4 := q^2[u_n^1] \otimes [u_{n-1}^2] + [u_{n-1}^2] \otimes [u_n^1] - q([u_{n-1}^1] \otimes [u_n^2] + [u_n^2] \otimes [u_{n-1}^1]).
\end{align*}
$$

(29)

Recall from [29] that $V^{(0,1)} \simeq V_{\varpi_2} \boxtimes V_{\varpi_1}^*$. In terms of the decomposition given in Example 4.2, we have that $v_1$ is the highest weight vector of $V_{\varpi_2} \boxtimes V_{\varpi_2}^*$, that $v_2$ is the highest weight vector of $V_{\varpi_2} \boxtimes V_{\varpi_2}^*$, that $v_3$ is the highest weight vector of $V_{\varpi_1} \boxtimes V_{\varpi_2}^*$, and that $v_4$ is the highest weight vector of $V_{\varpi_1} \boxtimes V_{\varpi_1}^*$.

**Lemma 4.9.** The braiding (28) acts on the highest weight vectors (29) as follows:

$$
\begin{align*}
\sigma(v_1) &= v_1, & \sigma(v_2) &= -q^{-2}v_2, & \sigma(v_3) &= -q^2v_3, & \sigma(v_4) &= v_4.
\end{align*}
$$

Since we are interested in the $q$-deformed exterior algebra, it is natural to consider the Nichols algebra given by $-\sigma$; see [64, §3].

**Corollary 4.10.** It holds that

$$
\ker(\mathfrak{S}_2^{\sigma}) = V_{\varpi_2} \boxtimes V_{\varpi_2}^* \oplus V_{\varpi_1} \boxtimes V_{\varpi_2}^* = S_q^2(V^{(0,1)}).
$$

4.8. A Nichols algebra presentation of $\Omega_q^{(\bullet,0)}(\text{Gr}_{n,m})$ and $\Omega_q^{(0,\bullet)}(\text{Gr}_{n,m})$. In this section we present the main result of the paper. Namely, we show that the holomorphic and anti-holomorphic quantum exterior algebras of the subcomplex $\Omega_q^{(\bullet,0)}(\text{Gr}_{n,m})$ and the subcomplex $\Omega_q^{(0,\bullet)}(\text{Gr}_{n,m})$ are Nichols algebras.

**Theorem 4.11.** There exist $O_q(U_m \times SU_{n-m})$-comodule algebra isomorphisms, or equivalently, $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-module algebra isomorphisms,

$$
\mathfrak{B}(V^{(1,0)}, -\sigma) \simeq \Phi(\Omega_q^{(\bullet,0)}(\text{Gr}_{n,m})), \quad \mathfrak{B}(V^{(0,1)}, -\sigma) \simeq \Phi(\Omega_q^{(0,\bullet)}(\text{Gr}_{n,m})),
$$

where $\sigma$ is the Yetter–Drinfeld braiding (28).

**Proof.** Let $V := \Phi(\Omega_q^{(0,1)}(\text{Gr}_{n,m}))$ and $N := \dim V = m(n-m)$. Recall from [29, §6] that, as $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-modules,

$$
V \simeq V_{\varpi_1} \boxtimes V_{\varpi_1}^* \quad \text{and} \quad \Phi(\Omega_q^{(0,\bullet)}(\text{Gr}_{n,m})) \simeq \Lambda_q(V) = T(V)/\langle S_q^2(V) \rangle.
$$

By definition, $\mathfrak{B}(V, -\sigma) = T(V)/\ker \mathfrak{S}^{-\sigma}$. By Corollary 4.10, we have $\ker \mathfrak{S}_2^{\sigma} = S_q^2(V)$. Therefore $\langle S_q^2(V) \rangle$ is a sub-ideal in $\ker \mathfrak{S}^{-\sigma}$, and hence there is an homogeneous ideal $I$ in $\Lambda_q(V)$ such that

$$
\mathfrak{B}(V, -\sigma) \simeq \Lambda_q(V)/I.
$$

(30)
In particular, it follows that $\mathcal{B}(V,-\sigma)$ is finite-dimensional. Let $p: \Lambda_q(V) \to \mathcal{B}(V,-\sigma)$ be the canonical projection. Since both $\langle S_q^2(V) \rangle$ and ker $\mathcal{S}^{-\sigma}$ are $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-modules and graded ideals in $\mathcal{T}(V)$, it follows that $p$ is a graded $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-module map.

Let $d \in \mathbb{Z}_{>0}$ be the largest integer such that $\mathcal{B}_d(V,-\sigma) \neq 0$ and $\mathcal{B}_{d+1}(V,-\sigma) = 0$. It follows from (30) that $d \leq N$. For a finite-dimensional Nichols algebra (and more generally, for a finite-dimensional graded Hopf algebra in $\mathcal{YD}^H_q$) Poincaré duality holds, which is to say, $\dim \mathcal{B}_{d-k}(V,-\sigma) = \dim \mathcal{B}_k(V,-\sigma)$, for $k = 0, \ldots, \lfloor d/2 \rfloor$, see [5, Proposition 3.2.2]. In particular, dim $\mathcal{B}_0(V,-\sigma) = \dim \mathcal{B}_d(V,-\sigma) = 1$.

The quantum version of Howe duality (23) gives us a decomposition of $\Lambda_q(V)$ into a sum of irreducible $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-modules. There are exactly two 1-dimensional $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-modules in this decomposition, namely, $\Lambda_q^0(V)$ and $\Lambda_q^N(V)$. Therefore $p(\Lambda_q^0(V)) = \mathcal{B}_0(V,-\sigma)$ and $p(\Lambda_q^N(V)) = \mathcal{B}_d(V,-\sigma)$, implying that $d = N$.

Denote by $\wedge$ the multiplication in $\Lambda_q(V)$ and by $\wedge_{\mathcal{B}}$ the multiplication in $\mathcal{B}(V,-\sigma)$. By [38, Proposition 4.11], we know that $\Lambda_q(V)$ is a Frobenius algebra. In particular there exists a nondegenerate bilinear form

$$(\cdot, \cdot): \Lambda_q(V) \otimes \Lambda_q(V) \to \mathbb{C},$$

and an element vol $\in \Lambda_q^N(V)$ such that if $(v, v^c) = 1$ for $v, v^c \in \Lambda_q(V)$ then $v \wedge v^c = \text{vol}$.

Assume that $v \in I$ and $v \neq 0$. Then

$$p(v \wedge v^c) = p(v) \wedge_{\mathcal{B}} p(v^c) = 0.$$

On the other hand, since vol $\in \Lambda_q^N(V)$,

$$p(v \wedge v^c) = p(\text{vol}) \neq 0,$$

a contradiction. Hence there is no such $v$, and $I = \langle 0 \rangle$ and $\Lambda_q(V) \simeq \mathcal{B}(V,-\sigma)$.

The proof that $\Phi(\Omega_q^{(1,0)}(\text{Gr}_{n,m}))$ is a Nichols algebra is analogous since $V^{(1,0)}$ and $V^{(0,1)}$ are dual $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m})$-modules. $\square$

**Corollary 4.12.** The calculi $\Omega_q^{(1,0)}(\text{Gr}_{n,m})$ and $\Omega_q^{(0,1)}(\text{Gr}_{n,m})$ are Nichols algebras.

5. THE GENERAL CASE OF THE IRREDUCIBLE QUANTUM FLAG MANIFOLDS

At present we do not have a quantum principal bundle description of the Heckenberger–Kolb calculi for irreducible quantum flag manifolds outside the A-series. However, we expect that such a description can be obtained in the same manner as for the quantum Grassmannians. Hence we conjecture that the main results of the previous section hold for all irreducible quantum flag manifolds.

**Conjecture 5.1** (Analogue of Corollary 4.17). Let $\mathcal{O}_q(G/L_S)$ be an irreducible quantum flag manifold and $\Omega_q^{(1,0)}(G/L_S)$, $\Omega_q^{(0,1)}(G/L_S)$ be the Heckenberger–Kolb first-order differential calculi. Then $\Phi(\Omega_q^{(1,0)}(G/L_S))$ and $\Phi(\Omega_q^{(0,1)}(G/L_S))$ are Yetter–Drinfeld modules over $\mathcal{O}_q(L_S)$. 
Conjecture 5.2 (Analogue of Theorem 4.11). For every irreducible quantum flag manifolds $O_q(G/L_S)$ the maximal prolongations of the pair of Heckenberger–Kolb first-order differential calculi are Nichols algebras.

In what follows we outline a strategy for proving Conjecture 5.2, under the assumption that Conjecture 5.1 is true. It is enough to calculate the eigenvalues of the braiding on the irreducible components of the second tensor power of $\Phi(\Omega_q^{(0,1)}(G/L_S))$, as discussed in Lemma 4.9. This is in contrast to the more involved proof of Theorem 4.11 for the quantum Grassmannians.

5.1. Some general results for the irreducible quantum flag manifolds.

5.1.1. The spectrum for $R$-matrices. Let $V_{\lambda}$ be the irreducible type-1 representation of $U_q(g)$ with the highest weight $\lambda$. Recall that by [37, §8.4.3, Corollary 23] the $R$-matrix braiding (20) acts on $V_{\mu}$, an irreducible component with highest weight $\mu$ in the decomposition of $V_{\lambda} \otimes V_{\lambda}$, as multiplication by the scalar
\[ \pm q^{-(2(\lambda,\lambda+2\rho)-(\mu,\mu+2\rho))/2}, \tag{31} \]
where $\rho$ is the half-sum of positive roots in $g$.

5.1.2. Quantum exterior algebras. Let $O_q(G/L_S)$ be an irreducible quantum flag manifold and let $\Phi(\Omega_q^{(0,1)}(G/L_S)$ and $\Omega_q^{(1,0)}(G/L_S)$ be the corresponding pair of Heckenberger–Kolb first-order calculi. In what follows we denote $V := \Phi(\Omega_q^{(0,1)}(G/L_S))$.

First recall that $V$ is an irreducible $U_q(I_S)$-module with highest weight $\lambda$, which is the same as in the classical case (see the list in [29, §6] and Table 3.2 on p. 27 in [12]). Second, $V$ is a flat $U_q(I_S)$-module (for the complete list of flat modules see [66]) and the quantum exterior algebra $\Lambda_q(V)$ is isomorphic to $\Phi(\Omega_q^{(\bullet,0)}(G/L_S))$. Third, all irreducible components in $V \otimes V$ have multiplicity 1, see [49, Table 5]. Hence together with the previous facts we can determine which sign occurs in (31).

We also present pictorial descriptions of the (quantum) irreducible flag manifolds. Note that we denote them by the same symbols as in [23, Table 1], but the numbering of nodes in Dynkin diagrams follows [49, Table 1]. In what follows, $\mathfrak{k}_S$ denotes the maximal semisimple ideal of $I_S$.

5.1.3. Hecke-type braidings. Let $V$ be a Yetter–Drinfeld module and $\sigma$ be the corresponding braiding. The braiding $\sigma$ is of Hecke type if $(\sigma - \lambda)(\sigma + 1) = 0$, for some non-zero scalar $\lambda$. If $\lambda$ is not a root of unity or if $\lambda = 1$, then the Nichols algebra is quadratic, see for example [11, Proposition 2.3]. For example, the braiding (28) for the quantum Grassmannians $O_q(Gr_{n,m})$ is of Hecke type only for the special case of the quantum projective spaces (when $m = 1$ or $m = n$). In this case $\Lambda^2_q(V)$ and $S^2_q(V)$ are irreducible $U_q(I_S)$-modules, see §5.1.5 for details.
5.1.4. Quantum Grassmannians $\mathcal{O}_q(\text{Gr}_{n,m})$. Consider the pictorial description of the (quantum) Levi subalgebra $\mathfrak{l}_S$ corresponding to the crossed node

where in addition the numbered nodes determine the highest weight of the adjoint representation of $\mathfrak{g}$. In this case have that

$$\mathfrak{g} = \mathfrak{sl}_n, \quad \mathfrak{l}_S = \mathfrak{gl}_m \oplus \mathfrak{st}_{m-n}, \quad \mathfrak{k}_S = \mathfrak{sl}_m \oplus \mathfrak{st}_{m-n}, \quad V = V_{\varpi_1} \otimes V_{\varpi_2}.$$

For $n > m > 1$, the decomposition (22) of $V \otimes V$ into quantum symmetric and antisymmetric parts with respect to diagonalised $R$-matrix braiding is as follows

$$S^2_q(V) \simeq (V_{2\varpi_1} \otimes V_{\varpi_2}^*) \oplus (V_{\varpi_2} \otimes V_{2\varpi_1}^*), \quad \Lambda^2_q(V) \simeq (V_{2\varpi_1} \otimes V_{2\varpi_1}^*) \oplus (V_{\varpi_2} \otimes V_{\varpi_2}^*).$$

Conjecture 5.1 is proved in Corollary 4.7 and Conjecture 5.2 is proved in Theorem 4.11.

5.1.5. Quantum projective spaces $\mathcal{O}_q(\mathbb{CP}^n)$. Consider the pictorial description of the (quantum) Levi subalgebra $\mathfrak{l}_S$ corresponding to the crossed node

where in addition the numbered nodes determine the highest weight of the adjoint representation of $\mathfrak{g}$. In this case recall have

$$\mathfrak{g} = \mathfrak{sl}_{n+1}, \quad \mathfrak{l}_S = \mathfrak{gl}_n, \quad \mathfrak{k}_S = \mathfrak{sl}_n, \quad V = V_{\varpi_1}.$$

For $n > 1$, the decomposition (22) of $V \otimes V$ into quantum symmetric and antisymmetric parts with respect to diagonalised $R$-matrix braiding is as follows

$$S^2_q(V) \simeq V_{2\varpi_1} \oplus V_0, \quad \Lambda^2_q(V) \simeq V_{\varpi_2}.$$

Conjecture 5.1 is proved in Corollary 4.7 and Conjecture 5.2 is proved in Theorem 4.11. Let us note that in this case the proof of Theorem 4.11 is trivial since the Yetter–Drinfeld braiding is of Hecke type. Moreover, the quasitriangular braiding gives the same Nichols algebra.

5.1.6. Odd quantum quadrics $\mathcal{O}_q(Q_{2n+1})$. Consider the pictorial description of the (quantum) Levi subalgebra $\mathfrak{l}_S$ corresponding to the crossed node

where in addition the numbered node determines the highest weight of the adjoint representation of $\mathfrak{g}$. In this case we have that

$$\mathfrak{g} = \mathfrak{o}_{2n+1}, \quad \mathfrak{l}_S = \mathfrak{o}_{2n-1} \oplus \mathfrak{gl}_1, \quad \mathfrak{k}_S = \mathfrak{o}_{2n-1}, \quad V = V_{\varpi_1}.$$

For $n > 2$, the decomposition (22) of $V \otimes V$ into quantum symmetric and antisymmetric parts with respect to diagonalised $R$-matrix braiding is as follows

$$S^2_q(V) \simeq V_{2\varpi_1} \oplus V_0, \quad \Lambda^2_q(V) \simeq V_{\varpi_2}.$$
Assume that Conjecture 5.1 is true and let $\sigma$ be the corresponding Yetter–Drinfeld braiding. If $\ker \mathfrak{S}_2^{-\sigma} \simeq S^2_q V$, then, since $\Lambda^2_q(V)$ is irreducible, the (rescaled) braiding $-\sigma$ has two eigenvalues $-1$ on $S^2_q V$ and $\lambda$ on $\Lambda^2_q(V)$. Therefore, the braiding $-\sigma$ is of Hecke type and the corresponding Nichols algebra is generated in degree 2 and isomorphic to $\Lambda_q V$. This implies Conjecture 5.2.

Denote by $e_\mu$ the eigenvalue of the $R$-matrix braiding of $U_q(\mathfrak{f}_S)$ on the irreducible component with the highest weight $\mu$ of $V \otimes V$. We have

\begin{align*}
  e_{2\omega_1} &= q, & e_0 &= q^{-2(n-1)}, & e_{\omega_2} &= -q^{-1}.
\end{align*}

5.1.7. Quantum Lagrangian Grassmannians $O_q(\mathbb{L}_n)$. Consider the pictorial description of the (quantum) Levi subalgebra $\mathfrak{l}_S$ corresponding to the crossed node

where in addition the numbered node determines the highest weight of the adjoint representation of $\mathfrak{g}$. In this case we have that

\begin{align*}
  \mathfrak{g} &= \mathfrak{sp}_{2n}, & \mathfrak{l}_S &= \mathfrak{gl}_n, & \mathfrak{f}_S &= \mathfrak{sl}_n, & V &= V_{2\omega_1}.
\end{align*}

For $n > 2$, the decomposition (22) of $V \otimes V$ into quantum symmetric and antisymmetric parts with respect to diagonalized $R$-matrix braiding is as follows

\begin{align*}
  S^2_q V &\simeq V_{4\omega_1} \oplus V_{2\omega_2}, & \Lambda^2_q V &\simeq V_{2\omega_1+\omega_2}.
\end{align*}

Assume that Conjecture 5.1 is true and let $\sigma$ be the corresponding Yetter–Drinfeld braiding. If $\ker \mathfrak{S}_2^{-\sigma} \simeq S^2_q V$, then, since $\Lambda^2_q(V)$ is irreducible, the (rescaled) braiding $-\sigma$ has two eigenvalues $-1$ on $S^2_q V$ and $\lambda$ on $\Lambda^2_q(V)$. Therefore, the braiding $-\sigma$ is of Hecke type and the corresponding Nichols algebra is generated in degree 2 and isomorphic to $\Lambda_q V$. This implies Conjecture 5.2.

Denote by $e_\mu$ the eigenvalue of the $R$-matrix braiding of $U_q(\mathfrak{f}_S)$ on the irreducible component with the highest weight $\mu$ of $V \otimes V$. We have

\begin{align*}
  e_{4\omega_1} &= q^n(n-1), & e_{2\omega_2} &= q^{-2} n(n+2), & e_{2\omega_1+\omega_2} &= -q^{-4} n(n-1).
\end{align*}

5.1.8. Even quantum quadrics $O_q(Q_{2n})$. Consider the pictorial description of the (quantum) Levi subalgebra $\mathfrak{l}_S$ corresponding to the crossed node

where in addition the numbered node determines the highest weight of the adjoint representation of $\mathfrak{g}$. In this case we have that

\begin{align*}
  \mathfrak{g} &= \mathfrak{o}_{2n}, & \mathfrak{l}_S &= \mathfrak{o}_{2(n-1)} \oplus \mathfrak{gl}_1, & \mathfrak{f}_S &= \mathfrak{o}_{2(n-1)}, & V &= V_{\omega_1}.
\end{align*}
For $n > 2$, the decomposition (22) of $V \otimes V$ into quantum symmetric and antisymmetric parts with respect to diagonalised $R$-matrix braiding is as follows

$$S^2_q V \simeq V_{2\varpi_1} \oplus V_0, \quad \Lambda^2_q V \simeq V_{\varpi_2}.$$  

Assume that Conjecture 5.1 is true and let $\sigma$ be the corresponding Yetter–Drinfeld braiding. If $\ker S^2_2 \sigma \simeq S^2_q V$, then, since $\Lambda^2_q(V)$ is irreducible, the (rescaled) braiding $-\sigma$ has two eigenvalues $-1$ on $S^2_q V$ and $\lambda$ on $\Lambda^2_q(V)$. Therefore, the braiding $-\sigma$ is of Hecke type and the corresponding Nichols algebra is generated in degree 2 and isomorphic to $\Lambda_q V$. This implies Conjecture 5.2.

Denote by $e_\mu$ the eigenvalue of the $R$-matrix braiding of $U_q(\mathfrak{k}_S)$ on the irreducible component with the highest weight $\mu$ of $V \otimes V$. We have

$$e_{2\varpi_2} = q, \quad e_0 = q^{-2n+3}, \quad e_{\varpi_2} = -q^{-1}.$$  

5.1.9. Quantum spinor varieties $O_q(S_{2n})$. Consider the pictorial description of the (quantum) Levi subalgebra $\mathfrak{l}_S$ corresponding to the crossed node

where in addition the numbered node determines the highest weight of the adjoint representation of $\mathfrak{g}$. In this case we have that

$$\mathfrak{g} = \mathfrak{o}_{2n}, \quad \mathfrak{l}_S = \mathfrak{gl}_n, \quad \mathfrak{k}_S = \mathfrak{sl}_n, \quad V = V_{\varpi_2}.$$  

For $n > 5$, the decomposition (22) of $V \otimes V$ into quantum symmetric and antisymmetric parts with respect to diagonalised $R$-matrix braiding is as follows

$$S^2_q V \simeq V_{2\varpi_2} \oplus V_{\varpi_4}, \quad \Lambda^2_q V \simeq V_{\varpi_3 + \varpi_1}.$$  

Assume that Conjecture 5.1 is true and let $\sigma$ be the corresponding Yetter–Drinfeld braiding. If $\ker S^2_2 \sigma \simeq S^2_q V$, then, since $\Lambda^2_q(V)$ is irreducible, the (rescaled) braiding $-\sigma$ has two eigenvalues $-1$ on $S^2_q V$ and $\lambda$ on $\Lambda^2_q(V)$. Therefore, the braiding $-\sigma$ is of Hecke type and the corresponding Nichols algebra is generated in degree 2 and isomorphic to $\Lambda_q V$. This implies Conjecture 5.2.

Denote by $e_\mu$ the eigenvalue of the $R$-matrix braiding of $U_q(\mathfrak{k}_S)$ on the irreducible component with the highest weight $\mu$ of $V \otimes V$. We have

$$e_{2\varpi_2} = q^{\frac{2}{n}(n-2)}, \quad e_{\varpi_4} = q^{-\frac{2}{n}(4n-5)}, \quad e_{\varpi_1 + \varpi_3} = -q^{-\frac{2}{n}}.$$  

5.1.10. Quantum Caley plane $O_q(\mathbb{P}^2)$. Consider the pictorial description of the (quantum) Levi subalgebra $\mathfrak{l}_S$ corresponding to the crossed node


where in addition the numbered node determines the highest weight of the adjoint representation of \( g \). In this case we have that
\[
    g = e_6, \quad l_S = o_{10} \oplus gl_1, \quad \mathfrak{k}_S = o_{10}, \quad V = V_{\varpi_5}.
\]

The decomposition (22) of \( V \otimes V \) into quantum symmetric and antisymmetric parts with respect to diagonalised \( R \)-matrix braiding is as follows
\[
    S_q^2V \simeq V_{2\varpi_5} \oplus V_{\varpi_1}, \quad \Lambda^2_qV \simeq V_{\varpi_3}.
\]

Assume that Conjecture 5.1 is true and let \( \sigma \) be the corresponding Yetter–Drinfeld braiding. If \( \ker S_2^{-\sigma} \simeq S_q^2V \), then, since \( \Lambda^2_q(V) \) is irreducible, the (rescaled) braiding \( -\sigma \) has two eigenvalues \(-1\) on \( S_q^2V \) and \( \lambda \) on \( \Lambda^2_q(V) \). Therefore, the braiding \( -\sigma \) is of Hecke type and the corresponding Nichols algebra is generated in degree 2 and isomorphic to \( \Lambda_qV \). This implies Conjecture 5.2.

Denote by \( e_\mu \) the eigenvalue of the \( R \)-matrix braiding of \( U_q(\mathfrak{k}_S) \) on the irreducible component with the highest weight \( \mu \) of \( V \otimes V \). We have
\[
    e_{2\varpi_5} = q^{25/3}, \quad e_{\varpi_1} = q^{-28/3}, \quad e_{\varpi_3} = -q^{-4/3}.
\]

5.1.11. Quantum Freudenthal variety \( O_q(F) \). Consider the pictorial description of the (quantum) Levi subalgebra \( l_S \) corresponding to the crossed node

\[\begin{array}{c}
    \circ \quad \circ \quad \circ \quad \circ \quad \times
\end{array}\]

where in addition the numbered node determines the highest weight of the adjoint representation of \( g \). In this case we have that
\[
    g = e_7, \quad l_S = e_6 \oplus gl_1, \quad \mathfrak{k}_S = e_6, \quad V = V_{\varpi_5}.
\]

The decomposition (22) of \( V \otimes V \) into quantum symmetric and antisymmetric parts with respect to diagonalised \( R \)-matrix braiding is as follows
\[
    S_q^2V \simeq V_{2\varpi_5} \oplus V_{\varpi_1}, \quad \Lambda^2_qV \simeq V_{\varpi_4}.
\]

Assume that Conjecture 5.1 is true and let \( \sigma \) be the corresponding Yetter–Drinfeld braiding. If \( \ker S_2^{-\sigma} \simeq S_q^2V \), then, since \( \Lambda^2_q(V) \) is irreducible, the (rescaled) braiding \( -\sigma \) has two eigenvalues \(-1\) on \( S_q^2V \) and \( \lambda \) on \( \Lambda^2_q(V) \). Therefore, the braiding \( -\sigma \) is of Hecke type and the corresponding Nichols algebra is generated in degree 2 and isomorphic to \( \Lambda_qV \). This implies Conjecture 5.2.

Denote by \( e_\mu \) the eigenvalue of the \( R \)-matrix braiding of \( U_q(\mathfrak{k}_S) \) on the irreducible component with the highest weight \( \mu \) of \( V \otimes V \). We have
\[
    e_{2\varpi_5} = q^{2/3}, \quad e_{\varpi_1} = q^{-28/3}, \quad e_{\varpi_4} = -q^{-4/3}.
\]
5.2. Nichols algebras and Weyl groupoids. We are thankful to S. Lentner who brought the following observation to our attention. As shown in [6], any Nichols algebra is controlled (but not completely determined) by a Weyl groupoid. Weyl groupoids are generalisations of Weyl groups motivated by Serganova’s work on generalised root systems of basic Lie superalgebras [58] and examples coming from Nichols algebras, see [33], cf. [59] and [28].

A crystallographic arrangement \( \mathcal{A} \) is a finite set of hyperplanes in \( X := \mathbb{R}^r \) which can be described as kernels of a given set of root vectors \( \alpha_1, \ldots, \alpha_k \in V^* \) satisfying certain properties (see [50, 21] for an explicit definition). Let \( Y \) be a subspace of \( X \). Then the restriction \( \mathcal{A}^Y \) is defined to be the set of all hyperplanes of the form \( Y \cap H \) for \( H \in \mathcal{A} \). In general, \( \mathcal{A}^Y \) is not a crystallographic arrangement. When \( Y \) is an intersection of a subset of hyperplanes of \( \mathcal{A} \), \( \mathcal{A}^Y \) is called a parabolic restriction. In this case \( \mathcal{A}^Y \) is again a crystallographic arrangement. As was shown in [21], we can associate a crystallographic arrangement to a given Weyl groupoid.

The set of simple roots \( I \) of the crystallographic arrangements of the Nichols algebra \( \mathfrak{B}(V) \) enumerates the irreducible Yetter–Drinfeld submodules \( V_i \) of \( V = \bigoplus_{i \in I} V_i \). Let \( J \subset I \) be a subset of simple roots for \( \mathfrak{B}(V) \) and denote by \( V_J := \bigoplus_{i \in J} V_i \) the corresponding Yetter–Drinfeld submodule of \( V \). Consider the associated subalgebra of coinvariant elements \( \mathfrak{B}(\tilde{V}) := \mathfrak{B}(V)^{\text{co}(\mathfrak{B}(M_J))} \). The main result of [22] is that the root system of \( \mathfrak{B}(\tilde{V}) \) is the parabolic restriction \( \mathcal{A}^Y \), where \( Y \) is the kernel of \( J \). In particular, consider the Nichols algebra \( \mathfrak{B}(V) \) associated to the Borel part \( \mathfrak{u}_q(\mathfrak{g})^+ \) of Lusztig’s small quantum group \( \mathfrak{u}_q(\mathfrak{g}) \) at a root of unity [41, 40]. The restricted Nichols algebra \( \mathfrak{B}(\tilde{V}) \) belongs to the category of Yetter–Drinfeld modules of \( \mathfrak{u}_q(\mathfrak{g}_J) \), where \( \mathfrak{g}_J \) is the Lie subalgebra of \( \mathfrak{g} \) generated by the simple roots from \( J \).

The Nichols algebras corresponding to the Heckenberger–Kolb calculi for quantum Grassmannians have similar relations to these Nichols algebras. Thus we expect to see analogous behaviour for the Nichols algebras of the Heckenberger–Kolb calculi for all irreducible flag manifolds, assuming that Conjecture 5.2 is correct.

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