A categorical action on quantized quiver varieties

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Abstract
In this paper, we describe a categorical action of any symmetric Kac–Moody algebra on a category of quantized coherent sheaves on Nakajima quiver varieties. By “quantized coherent sheaves,” we mean a category of sheaves of modules over a deformation quantization of the natural symplectic structure on quiver varieties. This action is a direct categorification of the geometric construction of universal enveloping algebras by Nakajima.

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Let $\mathfrak{g}$ be an arbitrary Kac–Moody algebra with symmetric Cartan matrix, and $\gamma$ its associated graph (note that any graph with no loops will appear this way). Nakajima showed that there exists a remarkable connection between the algebra $U(\mathfrak{g})$ and certain varieties, called quiver varieties, constructed directly from the graph $\gamma$. This construction takes the form of a map from $U(\mathfrak{g})$ to the Borel–Moore homology of a quiver analogue of the Steinberg variety [37].

Both the source and target of this map have natural categorifications:

- the algebra $U(\mathfrak{g})$ is categorified by a 2-category $\mathcal{U}$. Actually several variations on the theme of this category have been introduced by Rouquier [39], Khovanov-Lauda [26]; Brundan has more recently has shown that these approaches all give the same 2-category

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\begin{table}[h]
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\begin{tabular}{|c|c|}
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[9], so there is no need on our part to distinguish between them. A 2-functor from this
category into another 2-category is called a **categorical action** of \( g \) in this 2-category.

- the Borel-Moore homology of the “Steinberg” of a symplectic resolution \( \mathcal{M} \) (such as a
  quiver variety) is categorified by a certain category of sheaves on \( \mathcal{M} \times \mathcal{M} \). The structure
  sheaf of \( \mathcal{M} \) possesses a quantization, in the sense of [4], and the category of interest to us
  is that of bimodules over this quantization which satisfy a “Harish–Chandra” property,
  as described by Braden, Proudfoot and the author [8, §6.1-2]. Viewed correctly, these
  bimodules on the quiver varieties associated to a single highest weight \( \lambda \) can be organized
  into a 2-category, which we denote \( Q^\lambda \).

Thus, this previous work suggests how to categorify Nakajima’s map:

**Theorem A** For each highest weight \( \lambda \), there is a categorical representation of \( g \) in the 2-
category \( Q^\lambda \); taking “characteristic cycles” of these bimodules recovers the geometric action
of \( U(g) \) on the cohomology of quiver varieties by Nakajima [37].

Furthermore, the form of this functor is strongly suggested by Nakajima’s work; his map is
defined by sending the Chevalley generators of \( U(g) \) to particular correspondences, called
**Hecke correspondences**, which have natural moduli-theoretic significance. We “upgrade”
these correspondences to modules over deformation quantizations, and show that these satisfy
the categorical analogues of the Chevalley presentation. For the experts, we should note that
this will not work with arbitrary quantizations. The quantizations we wish to consider are
classified up to isomorphism by classes in \( H^2(\mathcal{M}; \mathbb{C}) \) called **periods** following [4]. The
 correspondences can only be quantized when the period satisfies an integrality condition, as
we’ll discuss in much more detail in Sect. 3.1.

We regard this theorem as very strong evidence of the naturality of the notion of a cate-
gorical \( g \)-action currently circulating in the literature. While defined diagrammatically in a
way that might outwardly seem arbitrary, in fact, its relations are hard-coded in the geometry
of quiver varieties.

This action of a 2-category is also quite useful in understanding categories of sheaves on
quiver varieties. In particular, we’ll use it to understand the category of **core modules** for
 certain “integral” quantizations. These are closely related to the category of finite dimen-
sional modules over a global quantization of the quiver variety. Work of Bezrukavnikov and
Losev [5] following up on this paper has described this category for more general quantiza-
tions, resolving a conjecture of Etingof on the number of finite dimensional modules over a
symplectic reflection algebra.

This theorem fits into a context of older results. Very close analogues of the functors
that appear in this representation have already been constructed in work of Zheng [50] and
Li [31,32]. However, these authors work in a slightly different context, which is based on
constructible sheaves rather than deformation quantizations. The Riemann-Hilbert corre-
spodence has already established a tie between constructible sheaves on a space \( X \), and
certain modules over a deformation quantization of \( T^*X \): the (twisted) differential operators
\( \mathcal{D}_X \) on \( X \). From this perspective, if there were a space \( Y \) of which a given Nakajima quiver
variety were the cotangent bundle (there almost never is) then sheaves of modules over the
quantized structure sheaf could be thought of as a replacement for the category of D-modules
on the hypothetical space \( Y \). As pointed out by Zheng [50, §2.2], his work was in a sense
intended to understand constructible sheaves with the same philosophy.

Rouquier [40, 5.10] showed that Zheng’s action can be strengthened to an action of the
2-category \( \mathcal{U} \); while it is not obvious that Rouquier’s category is the same as that from [16],
this was later proven by Brundan [9]. Rouquier’s result is extremely close to the first clause...
of Theorem A, but a host of annoying details rise up if one tries to derive one from the other: the result [40, 5.10] only establishes that the functors induce an action on a subcategory of Zheng’s category, though this proof could likely be extended; all the above work is on $Q_{\ell}$-sheaves on a variety over finite fields rather than over $\mathbb{C}$, etc. None of these issues are insuperable, but we felt the reader would be better served by an exposition which is more native to the world of deformation quantizations.

We are also motivated by analogous results that have appeared in the literature on coherent sheaves, for example in the work of Cautis, Licata and Kamnitzer [12–15]. Amongst other things, these results show that the categories of coherent sheaves on quiver varieties carry a version of a categorical action. In particular, these results have lead to interesting equivalences between derived categories of coherent sheaves. From our perspective, the action on sheaves over deformation quantizations is easier to work with, since one can use topological methods for D-modules, and seems to be the more basic object. In recent work, Cautis, Dodd and Kamnitzer [11] have made the connection between classical and quantum situations precise, showing that the action on coherent sheaves of quiver varieties in [14] is a classical limit of the action presented here in the $sl_2$ case; it seems likely to the author that their result can be extended to the general case using the results of this paper.

More generally, this action is but one aspect of close ties between the geometry of quiver varieties and the theory of categorical Lie algebra actions. It builds on work of Rouquier, Varagnolo and Vasserot [40,41] and is expanded further in further work of the author [48], which relates other categories of modules over these deformation quantizations to known categorical $g$-actions.

Another perspective on these deformation quantizations is that they provide a replacement for the Fukaya category of a complex symplectic variety. Such a connection is suggested by Kapustin and Witten [30, §11] from a physical perspective, and the work of Nadler and Zaslow [38] relating constructible sheaves and the Fukaya category of a cotangent bundle is also quite suggestive along these lines. In particular, it would be very interesting to find a categorical Lie algebra action in the 2-category of Lagrangian correspondences constructed by Wehrheim and Woodward [49]. Hopefully, instead of finding modules supported on the Hecke correspondences, one would simply consider them as objects in the Fukaya category.

Our main technical tool is a theorem of Rouquier [40, 4.13] which greatly reduces the number of relations which need to be checked in order to confirm that a candidate is a categorical action. This result is quite similar to earlier works of Chuang and Rouquier ([17, 5.27] & [39, 5.27]) and Cautis and Lauda [16, Th. 1.1], which likewise reduce the number of calculations needed, but which require stronger hypotheses. In particular, we can rely on calculations of Varagnolo and Vasserot from [41] for the most important check of relations between 2-morphisms; the other conditions either follow from general principles or are close analogues of results proven by Zheng and Li, with proofs that can be adapted.

**Notation.** We consider an oriented graph $\gamma$ without loops with vertex set $I$ and the associated Kac–Moody algebra $g$. Consider the weight lattice $Y(g)$ and root lattice $X(g)$, and the simple roots $\alpha_i$ and coroots $\alpha_i^\vee$. Let $c_{ij} = \alpha_j^\vee (\alpha_i)$ be the entries of the Cartan matrix.

Choose an orientation $\Omega$ on $\gamma$, let $\epsilon_{ij}$ denote the number of edges oriented from $i$ to $j$, and fix

$$Q_{ij}(u, v) = \begin{cases} (-1)^{\epsilon_{ij}} (u - v)^{c_{ij}} & i \neq j \\ 0 & i = j. \end{cases}$$
We let \( U_q(g) \) denote the deformed universal enveloping algebra of \( g \); that is, the associative \( \mathbb{C}(q) \)-algebra given by generators \( E_i, F_i, K_\xi \) for \( i \) and \( \xi \in Y(g) \), subject to the relations:

(i) \( K_0 = 1, K_\xi K_{\xi'} = K_{\xi + \xi'} \) for all \( \xi, \xi' \in Y(g) \),
(ii) \( K_\xi E_i = q^{\alpha_i(\xi)} E_i K_\xi \) for all \( \xi \in Y(g) \),
(iii) \( K_\xi F_i = q^{-\alpha_i(\xi)} F_i K_\xi \) for all \( \xi \in Y(g) \),
(iv) \( E_i F_j - F_j E_i = \delta_{ij} \tilde{K}_i - \tilde{K}_{-i} \), where \( \tilde{K}_{\pm i} = K_{\pm \alpha_i} \),
(v) For all \( i \neq j \)
\[
\sum_{a+b=-c_{ij}+1} (-1)^a E_i^{(a)} E_j E_i^{(b)} = 0 \quad \text{and} \quad \sum_{a+b=-c_{ij}+1} (-1)^a F_i^{(a)} F_j F_i^{(b)} = 0.
\]

As usual, we use \( F_i^{(a)} := F_i^a / [a]_q ! \) to denote the quantum divided power.

### 1 The 2-category \( \mathcal{U} \)

Our primary object of study is a 2-category categorifying the universal enveloping algebra; versions of this category have been considered by Rouquier [39], Khovanov and Lauda [26] and Cautis and Lauda [16]. Since recent work of Brundan [9] has shown that the different versions of this category have been considered by Rouquier [39], Khovanov and Lauda [26]. For simplicity of notation, if \( u_1 \ldots u_n \) is the composition of \( n \) 1-morphisms in a 2-category, we let \( x(\ell) \) for a 2-morphism \( x : u_\ell \to u_\ell' \) be the horizontal composition \( 1_{u_1} \otimes 1_{u_2} \otimes \cdots \otimes 1_{u_{\ell-1}} \otimes x \otimes 1_{u_{\ell+1}} \otimes \cdots \otimes 1_{u_n} \), and similarly with \( x(\ell, \ell+1) \) for \( x : u_\ell u_{\ell+1} \to u_\ell' u_{\ell+1} \). Let \( \lambda' = \alpha_i' (\lambda) \).

**Definition 1.1** \( \mathcal{U}' \) is the 2-category with:

- objects given by the weight lattice;
- 1-morphisms freely generated under composition and direct sum by adjoint 1-morphisms \( \mathcal{F}_i \) and \( \mathcal{E}_i \) for \( i \in I \);
- 2-morphisms
\[
\begin{align*}
& \iota : \mathbb{1}_\lambda \to \mathcal{E}_i \mathcal{F}_i \mathbb{1}_\lambda \quad \epsilon : \mathcal{F}_i \mathcal{E}_i \mathbb{1}_\lambda \to \mathbb{1}_\lambda \\
& y_i : \mathcal{F}_i \to \mathcal{F}_i \\
& \psi_{ij} : \mathcal{F}_i \mathcal{F}_j \to \mathcal{F}_j \mathcal{F}_i \\
& \xi_{i,j,\lambda} : \mathcal{E}_i \mathcal{F}_j \mathbb{1}_\lambda \oplus \mathbb{1}_\lambda \oplus \mathbb{1}_\lambda \to \mathcal{F}_j \mathcal{E}_i \mathbb{1}_\lambda \oplus \mathbb{1}_\lambda \oplus \mathbb{1}_\lambda \max(0, -\lambda').
\end{align*}
\]

These 2-morphisms are subject to the relations that \( \iota, \epsilon \) make \( (\mathcal{E}_i, \mathcal{F}_i) \) into an adjoint pair, and furthermore:
\[
\begin{align*}
\psi_{ij} y_i^{(1)} &= y_j^{(2)} \psi_{ij} + \delta_{ij} \\
y_j^{(1)} \psi_{ij} &= \psi_{ij} y_j^{(2)} + \delta_{ij} \\
\psi_{ji} \psi_{ij} &= Q_{ij} \left( y_i^{(1)}, y_j^{(2)} \right) \\
\psi_{jk} \psi_{ik} \psi_{ij} &= \psi_{ij} \psi_{ik} \psi_{jk}^{(1,2)} + \delta_{ik} \frac{Q_{ij} \left( y_k^{(1)}, y_j^{(2)} \right) - Q_{ij} \left( y_k^{(3)}, y_j^{(2)} \right)}{y_i^{(1)} - y_k^{(3)}}.
\end{align*}
\]

Furthermore, let \( \sigma_{i,j,\lambda} : \mathcal{F}_j \mathcal{E}_i \mathbb{1}_\lambda \to \mathcal{E}_i \mathcal{F}_j \mathbb{1}_\lambda \) be given by
\[
\sigma_{i,j,\lambda} = (1_{\mathcal{E}_i} \mathcal{F}_j \otimes \epsilon)(1_{\mathcal{E}_i} \otimes \psi_{ij} \otimes 1_{\mathcal{E}_j})(\iota \otimes 1_{\mathcal{F}_j} \epsilon).
\]
We also have the relation

\[ \xi_{i,j,\lambda}^{-1} = \begin{cases} 
\sigma_{i,j,\lambda} & \text{if } i \neq j \\
\epsilon & i = j, \lambda^i > 0 \\
\epsilon \cdot y_i^{(1)} & i = j, \lambda^i < 0 \\
\sigma_{i,j,\lambda} \cdot t \cdot y_i^{(1)} \cdots t \cdot (y_i^{(1)})^{-\lambda^i-1} & i = j, \lambda^i \leq 0 
\end{cases} \]

The 2-morphism spaces in this 2-category are graded with

\[ \deg \psi_{ij} = \begin{cases} 
-2 & i = j \\
c_{ij} & i \neq j 
\end{cases} \quad \deg y_i = 2 \]

\[ \deg \epsilon = \langle \lambda, \alpha_i \rangle + 1 \quad \deg t = -\langle \lambda, \alpha_i \rangle + 1 \]

with \( i \) and \( \lambda \) as in (1.1a).

As in [26], we let \( \mathcal{U} \) be the 2-category whose objects are the same, whose 1-morphisms are sums of formal grading shifts of 1-morphisms in \( \mathcal{U}^e \) and whose 2-morphisms are degree 0 morphisms in \( \mathcal{U}^e \); that is, for 1-morphisms \( u, v \), we have \( \text{Hom}_\mathcal{U}(u(m), v(n)) \) is the space of morphisms of degree \( n - m \) in \( \text{Hom}_{\mathcal{U}^e}(u, v) \). Let \( \bar{\mathcal{U}} \) denote the 2-category obtained from \( \mathcal{U} \) by replacing every \( \text{Hom} \)-category by its idempotent completion; we note that since every object in \( \mathcal{U} \) has a finite-dimensional endomorphism algebra, every \( \text{Hom} \)-category in \( \bar{\mathcal{U}} \) satisfies the Krull-Schmidt property.

**Definition 1.2** We call a category **graded** if it is equipped with a grading shift equivalence \( A \mapsto A(1) \); triangulated categories are an example of such a category with homological shift, as are modules over a graded ring, equipped with degree 0 homomorphisms, and the usual grading shift subtracting 1 from the degree of each element.

Any graded category has a **degrading** given by the category with the same objects where morphisms are given by summing morphisms of all degrees.

Note that \( \mathcal{U} \) is a graded category and \( \mathcal{U}^e \) its degrading. We prefer this terminology (as opposed to thinking of the degrading category as the basic object) since it behaves better under the usual constructions on abelian categories. For example, the graded Grothendieck group \( K_q \) is the usual Grothendieck group of a graded category, thought of as a \( \mathbb{Z}[q, q^{-1}] \) module via the rule \( q[M] = [M(1)] \).

The 2-category \( \bar{\mathcal{U}} \) is a categorification of the universal enveloping algebra in the sense that:

**Theorem 1.3** ([43, 4.10]) The graded Grothendieck group of \( \bar{\mathcal{U}} \) is isomorphic to \( \bar{U}_q^\mathbb{Z} \), Lusztig’s integral modified quantum universal enveloping algebra.

The algebra \( \bar{U}_q \) can be thought of as \( U_q(\mathfrak{g}) \) with additional idempotents \( 1_\lambda \) for integral weights \( \lambda \) which satisfy the relations of projection to the \( \lambda \)-weight space. The integral form \( \bar{U}_q^\mathbb{Z} \) is generated over \( \mathbb{Z}[q, q^{-1}] \) by \( 1_\lambda, E_i^{(a)} 1_\lambda, F_i^{(b)} 1_\lambda \) for all \( \lambda \). The map from the Grothendieck group sends the class of the 1-morphism \([E_i : \lambda \rightarrow \lambda + \alpha_i]\) to \( E_i 1_\lambda \), and similarly for \( F_i \). This theorem was first conjectured by Khovanov and Lauda [26] and proven by them in the special case of \( \mathfrak{sl}_n \).
2 Quiver varieties

Definition 2.1 For each orientation $\Omega$ of $\gamma$ (thought of as a subset of the edges of the oriented double), a **representation of** $(\gamma, \Omega)$ **with shadows** is

- a pair of finite dimensional $\mathbb{C}$-vector spaces $V = \oplus_{i \in I} V_i$ and $W = \oplus_{i \in I} W_i$, graded by the vertices of $\gamma$, and
- a map $x_e : V_{\omega(e)} \to V_{\alpha(e)}$ for each oriented edge (as usual, $\alpha$ and $\omega$ denotes the head and tail of an oriented edge), and
- a map $z : V \to W$ that preserves grading.

We let $w, v \in \mathbb{Z}_{\geq 0}$ denote $I$-tuples of non-negative integers. Following the notation of Ginzburg [23, §3.1], we let the **Crawley-Boevey quiver** $\gamma^w$ be $\gamma$ with an additional vertex $\infty$ and $w_i$ new edges from $i$ to $\infty$; we let $\Omega^w$ be the oriented edge set of this quiver. As noted in [23, §3.1], there is a canonical bijection between representations of $\gamma^w$ with $V_\infty \cong \mathbb{C}$ and a representations of $(\gamma, \Omega)$ with shadows with the same $V_i$ and $W_i = \mathbb{C}^{w_i}$; this bijection is induced by matching the columns of the matrix $z$ with the maps $V_i \to \mathbb{C}$ attached to the edges $i \to \infty$.

For now, we fix an orientation $\Omega$, though we will sometimes wish to consider the collection of all orientations. With this choice, we have the **universal** $(w, v)$-dimensional representation

$$E_{v, w} = \bigoplus_{i \to j} \text{Hom}(C^{v_i}, C^{v_j}) \oplus \bigoplus_i \text{Hom}(C^{v_i}, C^{w_i}).$$

In moduli terms, this is the moduli space of actions of the quiver (in the sense above) on the vector spaces $C^{v_i}, C^{w_i}$, with their chosen bases considered as additional structure.

If we wish to consider the moduli space of representations where $V$ has fixed graded dimension (rather than of actions on a fixed vector space), we should quotient by the group of isomorphisms of quiver representations: $G_v = \prod_i \text{GL}(C^{v_i})$ acting by pre- and post-composition. The result is the **moduli stack of** $v$-dimensional representations shadowed by $\mathbb{C}^w$, which we can define as the stack quotient

$$X_{v, w} = E_{v, w} / G_v.$$

This is not a scheme in the usual sense, but rather an Artin stack. We will always consider this space as having the classical topology. This means that a smooth map of a complex manifold $Y$ to $X_{v, w}$ is the same as a principal $G_v$-bundle $\tilde{Y}$ over $Y$, and a smooth $G_v$-equivariant map $\tilde{Y} \to E_{v, w}$.

Any structure like a coherent sheaf or D-module on $X_{v, w}$ can be defined as an assignment for each smooth map $f : Y \to E_{v, w} / G_v$ of a coherent sheaf or D-module on $Y$ which is natural under pullback. An example would be the structure sheaf $\mathcal{O}_{X_{v, w}}$. Since we will only be interested in the categories of sheaves on this stack, we do not need the full machinery of Artin stacks, and could consider instead the equivariant derived category of $E_{v, w}$ as in the book of Bernstein and Lunts [6].

By convention, if $w_i = a_i(\lambda)$ and $\xi = \lambda - \sum v_i \alpha_i$, then $X_{\xi}^\lambda = X_{v, w}^\lambda$ (if the difference $\lambda - \xi$ is not in the positive cone of the root lattice, then this is by definition empty) $G_{\xi} = G_v$, and $X_{\xi}^\lambda = \cup_{\xi} X_{\lambda}^\lambda$.

As mentioned in the introduction, our construction is inspired by the work of Li [31] and that of Zheng [50]. Li defines a 2-category built from perverse sheaves on double framed quiver varieties.
Definition 2.2 Li’s 2-category is defined as follows:

- 0-morphisms are dimension vectors for the quiver $\gamma$.
- 1-morphisms between $d$ and $d'$ are objects in the localized derived category which Li denotes by $D^-(E_{-\Omega}(k^\lambda, k^d, k^{d'}))$, with product given by the convolution product of [31, (16)].
- 2-morphisms are morphisms in the category described above.

For certain technical purposes, it is much more convenient for us to use a different 2-category built using quantizations. Let $\det: G_\xi \to \mathbb{C}$ be the product of the determinant characters on each of the individual factors $G_{v_i}$. Let

$$\mathfrak{M}_\xi^\lambda = T^* E_\xi^\lambda \# \det G_\xi = \mu^{-1}(0)^s / G_\xi$$

be the Nakajima quiver variety attached to $\lambda$ and $\xi$; this is a smooth, quasi-projective variety which arises through geometric invariant theory as an open subset of the cotangent bundle of $X_\xi^\lambda$. See [36,37] for a more detailed discussion of the geometry of these varieties.

Any point in $T^* E_\xi^\lambda$ can be thought of as a representation of the doubled quiver $\tilde{\gamma}^\omega$ of $\gamma^\omega$. The subset $\mu^{-1}(0)$ can be thought of as parameterizing representations that descend to a certain quotient of the doubled path algebra called the preprojective algebra $\Pi(\gamma^\omega)$. If we let $e \mapsto e^*$ be the edge reversing map on the edge set of the doubled quiver $\tilde{\gamma}^\omega$, then the preprojective algebra is the quotient of the path algebra of $\tilde{\gamma}^\omega$ by the 2-sided ideal generated by the element $\sum_{\sigma \in \omega} e^* e - e e^*$. A particularly important result for us is a description of the stable locus in terms of representation theory:

Lemma 2.3 ([36, 3.5]) The subvariety $\mu^{-1}(0)^s$ is the subset whose associated $\Pi(\gamma^\omega)$-representation has no non-trivial subrepresentation killed by the shadow map $z$.

Recall that a quantization of the variety $\mathfrak{M}_\xi^\lambda$ as defined in [4] or [8, §3] is an $h$-adically complete and separated sheaf $A'_\xi$ of flat $\mathbb{C}[[h]]$-algebras $A'_\xi / h A'_\xi \cong \mathcal{O}_{\mathfrak{M}_\xi^\lambda}$ with the structure sheaf $\mathcal{O}_{\mathfrak{M}_\xi^\lambda}$ such that the induced Poisson structure on $\mathcal{O}_{\mathfrak{M}_\xi^\lambda}$ matches the standard holomorphic symplectic structure on a quiver variety (induced from the cotangent bundle $T^* E_\xi^\lambda$). For such a quantization, we let $A_\xi := A'_\xi [h^{-1}] = A'_\xi \otimes \mathbb{C}[[h]] \subset (h)$.

One method of constructing such quantizations is quantum Hamiltonian reduction. This operation was introduced in an algebraic context by Crawley-Boevey, Etingof and Ginzburg [10] (though in many contexts it appeared even earlier), and in the geometric form of interest to us in [27, 2.8(i)]. Throughout we’ll follow the conventions of [8] for Hamiltonian reduction of quantizations and refer the reader to constructions there (even those which have appeared in older papers) in the interest of consistency. The reader can refer to [8, §3.4 & 5.4] for more details on these constructions. We let $\mathcal{R}'$ be the sheaf of microlocal differential operators on $T^* E_\xi^\lambda$, that is, the Rees algebra for the usual order filtration of the sheaf $\mathcal{D} E_\xi^\lambda$ of differential operators sheafified over $T^* E_\xi^\lambda$. This algebra is naturally a quantization of $T^* E_\xi^\lambda$ (see [8, §4.1]). We let $\mathcal{R} = \mathcal{R}'[h^{-1}]$; this is a sheaf on $T^* E_\xi^\lambda$ such that taking pushforward under $\pi: T^* E_\xi^\lambda \to E_\xi^\lambda$ we obtain $\pi_* \mathcal{R} \cong \mathcal{D} E_\xi^\lambda((h))$. In particular,

$$\Gamma(T^* E_\xi^\lambda; \mathcal{R}) \cong \Gamma(E_\xi^\lambda; \mathcal{D} E_\xi^\lambda)((h)).$$

Differentiating the action of $G_\xi$ on $E_\xi^\lambda$ induces a Lie algebra map from $\mathfrak{g}$ to vector fields on $E_\xi^\lambda$, and thus a non-commutative moment map $m: U(\mathfrak{g}_\xi) \to \mathcal{R}$ with $\mathfrak{g}_\xi = \text{Lie}(G_\xi)$. Let

\[ Springer \]
\( \mathcal{R}' \), \( \mathcal{R} \mathcal{S} \) be the pullback of these sheaves to the stable locus \( \mathcal{S} \). As in [8, §3.4], we let
\[
\mathcal{E} = \mathcal{R}/\mathcal{R}m(g_\xi) \quad \mathcal{E} \mathcal{S} = \mathcal{R} \mathcal{S}/\mathcal{R} \mathcal{S}m(g_\xi).
\]
If let \( p^0 : \mu^{-1}(0) \to T^*X_\lambda^\xi \cong \mu^{-1}(0)/G_\xi \) be the quotient map, then \( p^0 \mathcal{E} \text{End}_{\mathcal{R}}(\mathcal{E}) \) is the microlocal differential operators on \( T^*X_\lambda^\xi \). We can define \( \mathcal{D}_{X_\lambda^\xi} \) by applying the same construction to \( \mathcal{D}_{E_\lambda^\xi}/\mathcal{D}_{E_\lambda^\xi}m(g_\xi) \). Since \( E_\lambda^\xi \to X_\lambda^\xi \) is a principally \( G_\xi \)-equivariant \( \mathcal{D}_{E_\lambda^\xi} \)-module, the category of \( \mathcal{D}_{X_\lambda^\xi} \)-modules is naturally supported on \( \mathcal{D}_{E_\lambda^\xi} \)-modules, that is, to \( G_\xi \)-equivariant \( \mathcal{D} \)-modules where the action agrees with the action of \( m(g_\xi) \) by left multiplication. This is probably a more familiar context for most readers.

One particularly important example of a \( \mathcal{D} \)-module on \( X_\lambda^\xi \) is provided by \( G_\xi \)-varieties \( Y \) equipped with an equivariant map \( p : Y \to E_\lambda^\xi \). The structure sheaf \( \mathcal{O}_Y \) is naturally a strongly equivariant left \( \mathcal{D}_Y \)-module, so its pushforward \( p_* \mathcal{O}_Y \) is a strongly equivariant \( \mathcal{D}_{E_\lambda^\xi} \)-module. The corresponding \( \mathcal{D}_{X_\lambda^\xi} \)-module is the pushforward of the structure sheaf \( \mathcal{O}_{Y/G} \) via the induced map \( Y/G \to X_\lambda^\xi \).

Now, we wish to incorporate the choice of stability condition. Consider the endomorphism sheaf \( \mathcal{E} \text{End}_{\mathcal{R} \mathcal{S}}(\mathcal{E} \mathcal{S}) \), which is naturally supported on \( \mu^{-1}(0)^\xi \). Let \( \mu^{-1}(0)^\xi \to M_\lambda^\xi \) be the quotient map.

**Definition 2.4** We let \( A_\xi := p_* \mathcal{E} \text{End}_{\mathcal{R} \mathcal{S}}(\mathcal{E} \mathcal{S}) \), the pushforward sheaf on \( M_\lambda^\xi \).

We actually have that \( A_\xi = A_\xi'[h^{-1}] \) for a quantization \( A_\xi' \) obtained from \( \mathcal{R}' \) by a similar reduction procedure as in [8, §3.4]. The quantizations of \( M_\lambda^\xi \) can be classified by a cohomological invariant called its period. This is a class in \( hH^2(M_\lambda^\xi; \mathbb{C}) \) and any such class can be realized by a quantization since \( H^2(M_\lambda^\xi; \mathbb{C}) = H^{1,1}(M_\lambda^\xi; \mathbb{C}) \).

On \( X_\lambda^\xi \), we have a tautological vector bundle \( \mathcal{Y}_\lambda \) whose fiber over a representation is \( V_i \), the part of that representation at node \( i \); let \( \mathcal{L}_i = \text{det}(\mathcal{Y}_i) \). By [8, 6.4], the period of \( A_\xi' \) is
\[
\frac{1}{2} \sum_{i \in I} \left( w_i + \sum_{j \neq i} v_{ij} - \sum_{i \neq j} v_{ij} \right) c_1(\mathcal{L}_i) h.
\]
Note that this period depends on the choice of orientation of \( \gamma \), but its class modulo \( hH^2(M_\lambda^\xi; \mathbb{Z}) \) does not. Also, this is not always an integral class; this is a generalization of the fact that differential operators, thought of as a quantization of a cotangent bundle, do not always have integral period (as [8, 3.10] shows). If, as suggested in the introduction, we think of the quiver variety as the cotangent bundle of a hypothetical space \( Y \), this would be the algebra of untwisted differential operators on \( Y \), and the quantization with period 0 would be the differential operators in the square root of the canonical bundle of \( Y \).

The quiver varieties carry a natural \( \mathbb{C}^* \)-action inherited from the action on \( T^*E_{v,w} \) scaling the cotangent fibers. The sheaf of algebras \( A_\xi \) carries an equivariant structure over \( \mathbb{C}^* \) (see [33, 2.3.3]). We let \( A_\xi \)-mod denote the category of \( \mathbb{C}^* \)-equivariant good modules over \( A_\xi \) (as defined in [8, §4]).

The Hamiltonian reduction realization of \( A_\xi \) gives us a functor \( \tau : \mathcal{D}_{X_\lambda^\xi} \)-mod \to \( A_\xi \)-mod (called the “Kirwan functor” in [8, §5.4], where this functor is studied extensively) from \( \mathcal{D} \)-modules on \( X_\lambda^\xi \) to \( A_\xi \)-modules. This functor proceeds by replacing a \( \mathcal{D} \)-module \( \mathcal{M} \) by its microlocalization \( \mu \mathcal{M} := \mathcal{R} \otimes_{\pi^{-1}} \mathcal{D}_{X_\lambda^\xi} \pi^{-1} \mathcal{M} \), which is a sheaf on \( T^*X_\lambda^\xi \cong \mu^{-1}(0)/G_\xi \), and then restricting to \( M_\lambda^\xi \subset T^*X_\lambda^\xi \). That is:

**Definition 2.5** The Kirwan functor is the restriction \( \tau(\mathcal{M}) = (\mu \mathcal{M})|_{M_\lambda^\xi} \).
While using stack language is elegant, one can also describe this in terms of the associated \( G_\xi \)-equivariant \( \mathcal{D} \)-module \( \mathcal{M}' \) on \( E_\xi^2 \); this has microlocalization \( \mu \mathcal{M}' \) supported on \( \mu^{-1}(0) \) by equivarience. We restrict this to the stable locus, and take the invariant pushforward

\[
\tau(\mathcal{M}) = p_* \text{Hom}_{\mathcal{R}\mathcal{S}}(\xi_\mathcal{S}, \mu \mathcal{M}'|_\mathcal{S}) = p_* (\mu \mathcal{M}'|_\mathcal{S}/m(g_\xi)\mu \mathcal{M}'|_\mathcal{S}),
\]

with its natural \( \mathcal{A}_\xi \)-action. Since on \( \mu^{-1}(0)^c \), the \( G_\xi \) action is free, we always stay within the world of varieties. This same functor appears in [27, 2.8] in the analytic context.

We’ll also be interested in the Kirwan functor for bimodules; by a bimodule over a pair of sheaves of algebras \( \mathcal{A} \) over \( X \) and \( \mathcal{B} \) over \( Y \), we mean of a sheaf of \( \mathcal{A} \otimes \mathcal{B}\text{op} \) algebras on \( X \times Y \). We can extend the Kirwan functor to the category of \( \mathcal{D}_{X_\xi^2} \cdot \mathcal{D}_{X_\xi^2}^{\prime} \)-bimodules, with the image being \( \mathcal{A}_\xi \cdot \mathcal{A}_\xi' \) bimodules in the sense discussed above. The only change we need to make is that we need to replace \( m \) by \( -m \) to obtain a quantum moment map into \( \mathcal{D}_{X_\xi^2}^{\prime} \).

There is an important compatibility of this functor with the change of orientation of the quiver \( \gamma \). We let \( \gamma' \) with \( \gamma \) with a single edge \( i \to j \) reversed to an edge \( j \to i \); we let \( \mathcal{D}_1 = \mathcal{D}_{E_\xi^2} \) for the quiver \( \gamma \), and \( \mathcal{D}_2 \) the same sheaf considered for the quiver \( \gamma' \). The global sections of these sheaves are isomorphic via a partial Fourier transform map, and this gives an equivalence of categories \( F_{\gamma, \gamma'}: \mathcal{D}_1\text{-mod} \to \mathcal{D}_2\text{-mod} \). Performing the reduction \( \tau \) as we’ve defined it on these two different categories gives two different quantizations \( A_1, A_2 \) of \( M_\xi^2 \). However, by [8, 5.2], there is a \( A_2 - A_1 \)-bimodule \( \mathcal{T}_1 \) such that tensor product with this bimodule gives an equivalence of categories; this is a quantization of the line bundle \( \mathcal{L}_i \otimes \mathcal{L}_j^{-\nu_i} \). By [8, 5.4], it follows that:

**Proposition 2.6** We have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}_1\text{-mod} & \xrightarrow{F_{\gamma, \gamma'}} & \mathcal{D}_2\text{-mod} \\
\downarrow \tau & & \downarrow \tau \\
A_1\text{-mod} & \xrightarrow{2 \mathcal{T}_1 \otimes -} & A_2\text{-mod}
\end{array}
\]

Perhaps the most important property for us is that:

**Proposition 2.7** ([8, 5.17], [35, 1.1]) The functor \( \tau \) admits left and right adjoints

\[
\tau_*: A_\xi\text{-mod} \to \mathcal{D}_{X_\xi^2}\text{-mod} \quad \tau^*: A_\xi\text{-mod} \to \mathcal{D}_{X_\xi^2}^{\prime}\text{-mod},
\]

such that \( \tau \circ \tau_* \cong \tau \circ \tau^* \cong \text{id} \).

Recall that \( \xi = \lambda - \sum_{i \in \ell} v_i \alpha_i \), and we’ll use \( (\xi', \nu') \) and \( (\xi_1, v_1) \), etc. for pairs of weights and dimension vectors connected by the same formula. In a product \( M_{\xi_1}^\lambda \times M_{\xi_2}^\lambda \times M_{\xi_3}^\lambda \), we’ll let \( p_{ij}: M_{\xi_1}^\lambda \times M_{\xi_2}^\lambda \times M_{\xi_3}^\lambda \to M_{\xi_i}^\lambda \times M_{\xi_j}^\lambda \) be the projection onto the \( i \)th and \( j \)th factors. We define the convolution of a complex \( \mathcal{H}_1 \) of \( A_{\xi_1} \otimes A_{\xi_2}^{\text{op}} \)-bimodules with a complex \( \mathcal{H}_2 \) over \( A_{\xi_2} \otimes A_{\xi_3}^{\text{op}} \) by

\[
\mathcal{H}_1 \star \mathcal{H}_2 := (p_{13})_* (p_{12}^* \mathcal{H}_1 \otimes A_{\xi_2} \otimes p_{23}^* \mathcal{H}_2) [-\dim(M_{\xi_1}^\lambda \times M_{\xi_3}^\lambda)].
\]

(2.1)

Note this sends pairs of bounded complexes to bounded complexes since the same is true of coherent sheaves on smooth varieties, and similarly with bounded above/below, etc.

\[\square\] Springer
Definition 2.8 We let $\Omega^\lambda$ be the 2-category where

- 0-morphisms are dimension vectors for the quiver $\gamma$,
- 1-morphisms $v \to v'$ given by the bounded-below derived category $D^+(A_{\xi} \boxtimes A_{\xi}'^{{\text{op}}})$ of complexes of modules over $A_{\xi} \boxtimes A_{\xi}'^{{\text{op}}}$. Composition of 1-morphisms $\mathcal{H}_1: v_1 \to v_2$ and $\mathcal{H}_2: v_2 \to v_3$ (that is, $\mathcal{H}_1$ is a complex of $A_{\xi_1} \boxtimes A_{\xi_2}^{{\text{op}}}$-modules considered up to quasi-isomorphism, etc.) is given by convolution $\mathcal{H}_1 \star \mathcal{H}_2$.
- 2-morphisms are morphisms in the category $D^+(A_{\xi} \boxtimes A_{\xi}'^{{\text{op}}})$; we consider this as a graded category (in the sense of Definition 1.2) with the homological shift giving the grading shift functor.

This 2-category receives a natural 2-functor from the analytic version of Li’s 2-category; the (classical topology) derived category of $X_{\lambda}^\xi$ has a functor to the derived category of $D_{X_{\lambda}^\xi}$-modules given by the Riemann-Hilbert correspondence, and the functor $r$ kills the necessary subcategories to induce a 2-functor from the localization. Li’s definition [31, §4.8] is not obviously the same as convolution as defined in (2.1), but their match can be confirmed using Lemma 4.4.

3 Hecke correspondences and categorical actions

We let $X_{\xi;\nu}^\lambda$ denote the moduli stack of short exact sequences (“Hecke correspondences”) where the subobject belongs in $X_{\xi}^\lambda$, the total object in $X_{\xi-\nu}^\lambda$, and the quotient in $X_{\lambda-\nu}^0$.

This moduli stack is naturally equipped with projections

$$
\begin{align*}
\pi_1 &: X_{\xi}^\lambda \to X_{\xi;\nu}^\lambda \\
\pi_2 &: X_{\xi-\nu}^\lambda \to X_{\xi;\nu}^\lambda \\
\pi_3 &: X_{\lambda-\nu}^0 \to X_{\xi;\nu}^\lambda
\end{align*}
$$

which we can think of more abstractly as taking the subobject, total object and quotient, respectively.

If $\lambda - \xi = \sum v'_i \alpha_i$ and $\nu = \sum v''_i \alpha_i$, then we can also think of this space more concretely. We let

$$
E_{\xi;\nu}^\lambda \cong \bigoplus_{i \to j} \text{Hom}(C^{v'_i}, C^{v'_j}) \oplus \text{Hom}(C^{v''_i}, C^{v'_j}) \oplus \text{Hom}(C^{v''_i}, C^{v''_j}) \oplus \bigoplus_i \text{Hom}(C^{v'_i}, C^{v''_i})
$$

(3.1)

and let $P_{\xi;\nu}$ be the parabolic in $G_\psi$ which preserves $C^{v'_i}$ inside of $C^{v_i} = C^{v'_i} \oplus C^{v''_i}$. We can alternatively define $X_{\xi;\nu}^\lambda := E_{\xi;\nu}^\lambda / P_{\xi;\nu}$. The projection maps are also easily understood from this perspective.

- The map $\pi_2$ is induced by the inclusion $E_{\xi;\nu}^\lambda \hookrightarrow E_{\xi-\nu}^\lambda$ induced by the isomorphism $C^{v'_i} = C^{v''_i} \oplus C^{v''_i}$.
- The map $\pi_1$ is induced by the map $E_{\xi;\nu}^\lambda \to E_{\lambda-\nu}^\xi$ restricting each map to the subspace $C^{v'_i}$ over each node. That is, we apply $\pi_2$, and then we precompose the resulting map with the inclusion $C^{v'_i} \hookrightarrow C^{v_i}$.

[Springer]
• The map $\pi_3$ is induced by the map $E^\lambda_{\xi} \to E^0_{-v}$ which takes the induced map on the spaces $\mathbb{C}^v_{ij}$. In the decomposition of (3.1), this corresponds to the projection to the summand $\text{Hom}(\mathbb{C}^v_{ij}, \mathbb{C}^{v''})$ for each arrow $i \to j$.

These maps are compatible with group homomorphisms from $P_{\xi, v}$ to $G_\nu$, $G_v$, $G_{v''}$, and thus induce maps of the appropriate quotient stacks.

We’ll want to consider the singular support of a D-module on $X^\lambda_{\xi}$, which is a subvariety of $\mu^{-1}(0)/G_\xi$. Since the dual of $\text{Hom}(V, V')$ is $\text{Hom}(V', V)$, we can think of an element of $T^*X^\lambda_{\xi}$ as a representation of the doubled Crawley-Boevey quiver; the moment map condition makes this into a representation of the preprojective algebra of this quiver, so $\mu^{-1}(0)/G_\xi$ can be interpreted as the moduli space of representations of the preprojective algebra.

We also have a moduli space of short exact sequences of modules over the preprojective algebra; we won’t be interested in this as a scheme, but we’ll want to consider the locus $\widehat{\mathcal{P}}_i \subset T^*X^\lambda_{\xi} \times T^*X^\lambda_{\xi-a_i}$ of pairs that appear as the submodule and total space in a short exact sequence, that is, the point in the left factor corresponds to a subrepresentation of the right factor. The intersection of $\widehat{\mathcal{P}}_i$ with the stable locus is precisely Nakajima’s Hecke correspondence $\mathcal{P}_i$ [37, Def. 5.6].

Let $\omega_{X^\lambda_{\xi}}$ be the canonical sheaf of $X^\lambda_{\xi}$. As discussed in [2, §4], the tensor product $\omega_X \otimes \mathcal{O}_X M$ of this sheaf with a left D-module $M$ on any smooth space $X$ is naturally a right D-module on $X$ (and this functor defines an equivalence). For each $\lambda, \xi, i$, we let

$$\tilde{\mathcal{F}}_i = \omega_{X^\lambda_{\xi-a_i}} \otimes \mathcal{O}_{X^\lambda_{\xi-a_i}} (\pi_1 \times \pi_2)_* \mathcal{O}_{X^\lambda_{\xi-a_i}}$$

(3.2)

$$\tilde{\mathcal{E}}_i = \omega_{X^\lambda_{\xi}} \otimes \mathcal{O}_{X^\lambda_{\xi}} (\pi_2 \times \pi_1)_* \mathcal{O}_{X^\lambda_{\xi-a_i}}$$

(3.3)

This defines a sheaf $\tilde{\mathcal{F}}_i$ on $X^\lambda_{\xi-a_i} \times X^\lambda_{\xi}$ which is a module over $\mathcal{D}_{X^\lambda_{\xi-a_i}} \otimes \mathcal{D}_{X^\lambda_{\xi}}^\text{op}$, and $\tilde{\mathcal{E}}_i$ which is on the same space, but with the roles of the factors reversed. For purposes of understanding the decategorification of this construction, we’ll be interested in the characteristic cycle of this sheaf.

**Lemma 3.1** The singular support of $\tilde{\mathcal{E}}_i$, $\tilde{\mathcal{F}}_i$ lies in the locus $\widehat{\mathcal{P}}_i$.

**Proof** This is essentially the same calculation as [34, §13], and we will use the same observation on characteristic cycles of pushforwards. For $\tilde{\mathcal{E}}_i$, this shows that its singular support lies in the locus $L$ of elements $(x, \xi) \in T^*(X^\lambda_{\xi} \times X^\lambda_{\xi-a_i})$ where $x = \pi_1 \times \pi_2(y)$ for $y \in X^\lambda_{\xi-a_i}$, and $\xi$ is perpendicular to the image of the tangent space at $y$. However, we must be careful about signs here; taking right D-modules means that we must negate this pairing on the second factor, since we use the opposite symplectic form.

Fix $y \in E^\lambda_{\xi-a_i}$; the image of $y$ under $\pi_1 \times \pi_2$ is simply considering the whole space and the submodule as separate quiver representations. An element of the cotangent space at the point can be thought of as two representations of the doubled quiver, one with underlying vector space $\mathbb{C}^v_{ij}$ and one with underlying space $\mathbb{C}^{vi}$. We claim that lying in the locus $L$ defined above is exactly the condition that the obvious inclusion of vector spaces is a map of representations of the doubled quiver. It’s enough to check this when the quiver has two vertices. For simplicity of notation, we’ll assume there is a single edge $e: i \to j$. Let $Q$ be the image of the map

$$\text{Hom}(\mathbb{C}^v_{ij}, \mathbb{C}^{vi}) \oplus \text{Hom}(\mathbb{C}, \mathbb{C}^{v'}) \hookrightarrow \text{Hom}(\mathbb{C}^v_{ij}, \mathbb{C}^{v'}) \oplus \text{Hom}(\mathbb{C}^v_{ij}, \mathbb{C}^{v'})$$

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sending \((a, b) \mapsto (a, a \oplus b)\). We’ll consider \(\text{Hom}(C^{v'_j}, C^{v'_i}) \oplus \text{Hom}(C^{v'_i}, C^{v_i})\) as the dual space to the target above, using the trace pairing on the first factor and its negative on the second. Given \((c, d)\) with \(c, d \in \text{Hom}(C^{v'_j}, C^{v'_i})\), this pair lies in \(Q^\perp\) if \(\text{Tr}(ac - ad') = 0\) for all \(a, b\). Of course, this holds if and only if \(c = d'\). Thus, \(Q^\perp\) contains the image of
\[
\text{Hom}(C^{v'_j}, C^{v'_i}) \hookrightarrow \text{Hom}(C^{v'_i}, C^{v_i}) \oplus \text{Hom}(C^{v'_j}, C^{v'_i}).
\]
On the other hand the codimension of this space is \(v_iv_j = v'_iv'_j + v'_j\), which is the dimension of \(Q\). The case where we have \(e: j \to i\) follows by switching the roles of \(Q\) and \(Q^\perp\). If there are multiple edges, we just take the direct sum of the appropriate number of the single edge cases.

As discussed before, we can apply the Kirwan functor to this bimodule and define
\[
\mathcal{F}_i = \mathcal{r} (\hat{\mathcal{F}}_i) \quad \mathcal{E}_i = \mathcal{r} (\hat{\mathcal{E}}_i).
\]

The sheaf \(\mathcal{F}_i\) is a module over \(A_{\xi} \boxtimes A_{\xi}^{\text{op}}\) and \(\mathcal{E}_i\) is a module over \(A_{\xi - a_i} \boxtimes A_{\xi}^{\text{op}}\). That is, by definition, these are 1-morphisms in \(\mathcal{Q}^\lambda\) between the appropriate dimension vectors. They are the images under the Riemann-Hilbert correspondence of the similarly named objects in Li’s development of the theory.

Our underlying philosophy is that these are categorified versions of Nakajima’s Hecke correspondences. One version of this follows immediately from Lemma 3.1:

**Corollary 3.2** The support of \(\mathcal{F}_i, \mathcal{E}_i\) lies in the Hecke correspondence \(\mathcal{P}_i\).

We now proceed to our principal result:

**Theorem 3.3** We have a 2-functor of graded categories \(\mathcal{G}_\lambda: \mathcal{U} \to \mathcal{Q}^\lambda\) sending \(\mathcal{E}_i \mapsto \mathcal{E}_i\) and \(\mathcal{F}_i \mapsto \mathcal{F}_i\).

For now, we postpone the proof of this theorem to Sect. 4, and instead discuss its variations and consequences in a bit more detail.

### 3.1 The non-integral case

The existence of the objects \(\mathcal{E}_i\) and \(\mathcal{F}_i\) depends very strongly on the fact that we use untwisted D-modules here. Choose \(a_i, b_i \in \mathbb{C}\) for each \(i \in I\). Consider the twists
\[
\chi_1 = \sum_{j \in I} a_i c_1 (L_j) \in H^2(X_{\xi}^\lambda) \quad \chi_2 = \sum_{j \in I} b_i c_1 (L'_j) \in H^2(X_{\xi - a_i}^\lambda)
\]
on the respective varieties, where we use \(L'_j\) to denote the tautological bundles on \(X_{\xi - a_i}^\lambda\). We let \(\mathcal{D}_X^\lambda (\chi_1)\) be the twisted differential operators corresponding to this twist; using descent again, the category of \(\mathcal{D}_X^\lambda (\chi_1)\)-modules can be identified via pullback with \(\mathcal{D}_{\xi} (\chi_2)\)-modules which are strongly equivariant for a different choice of quantum moment map: \(m_X(X) = m(X) - \chi(X)\).

**Proposition 3.4** There exists a line bundle \(L\) on \(X_{\xi - a_i}^\lambda\), such that
\[
\hat{\mathcal{F}}_i^L = \omega_{X_{\xi - a_i}^\lambda} \boxtimes \mathcal{O}_{X_{\xi - a_i}^\lambda} (\pi_1 \times \pi_2)_* L
\]
and
\[
\hat{\mathcal{E}}_i^L = \omega_{X_{\xi}^\lambda} \boxtimes \mathcal{O}_{X_{\xi}^\lambda} (\pi_2 \times \pi_1)_* L^{-1}
\]
are bimodules over \(\mathcal{D}_{X_{\xi}^\lambda} (\chi_1)\) and \(\mathcal{D}_{X_{\xi}^\lambda} (\chi_2)\) if and only if \(a_i, b_i, a_j - b_j \in \mathbb{Z}\) for all \(j \in I\).
A categorical action on quantized quiver varieties

Proof First we note that $X^λ_ξ$, $X^λ_ξ$ and $X^λ_ξ$ are each the quotient of an affine space by an affine algebraic group. These groups are $G_ξ$ and $G_ξ$ and a maximal parabolic in the latter, respectively. Thus the Picard groups of these spaces are naturally identified with the character group of the group in question. In practice, this means that

- $c_1(\mathcal{L}_j)$ is a basis of $H^2(X^λ_ξ; \mathbb{Z}) \cong H^2_G(E^λ_ξ; \mathbb{Z})$,
- $c_1(\mathcal{L}_j′)$ is a basis of $H^2(X^λ_ξ; \mathbb{Z}) \cong H^2_G(E^λ_ξ; \mathbb{Z})$ and
- $c_1(\pi^*_1 \mathcal{L}_j) \cup c_1(\pi^*_2 \mathcal{L}_j′)$ for $H^2(X^λ_ξ; \mathbb{Z}) \cong H^2_G(E^λ_ξ; \mathbb{Z})$.

A line bundle $\mathcal{L}$ on a smooth variety whose Picard group matches $H^2(-; \mathbb{Z})$ is a module over twisted D-modules for a single choice of twist: that given by its first Chern class. Thus, in order to have the desired left and right twisted D-module structure, we must have that the Chern class of $\pi^*_1 \omega_{X^λ_ξ} \otimes \mathcal{L}$ on $X^λ_ξ$ matches the pullback of the twist of $\mathcal{D}_{X^λ_ξ}(\chi_1) \boxtimes (\mathcal{D}_{X^λ_ξ}(\chi_2))^{op}$. This holds if and only if $c_1(\mathcal{L}) = \pi^*_1 \chi_1 - \pi^*_2 \chi_2$; by the identification of the Picard group with homology, such an $\mathcal{L}$ exists if and only if $\pi^*_1 \chi_1 - \pi^*_2 \chi_2 \in H^2(X^λ_ξ; \mathbb{Z})$.

For $j \neq i$, we have that $\pi^*_1 c_1(\mathcal{L}_j) = \pi^*_2 c_1(\mathcal{L}_j′)$, but for $i$, these are independent classes. Thus, we have that

$$\pi^*_1 \chi_1 - \pi^*_2 \chi_2 = a_i \pi^*_1 c_1(\mathcal{L}_i) - b_i \pi^*_2 c_1(\mathcal{L}_i′) + \sum_{j \neq i} (a_j - b_j) \pi^*_i c_1(\mathcal{L}_j),$$

which is integral if and only if $a_i, b_i, a_j - b_j \in \mathbb{Z}$.

Thus more generally, using Proposition 3.4, we can define such an action where we choose any quantization corresponding to differential operators in a line bundle on each $X^λ_ξ$, not just the particular one we have fixed. If we instead choose a not necessarily integral twist $\chi = \sum a_i c_1(\mathcal{L}_i)$, we only know at the moment how to construct a categorical action of the smaller Lie algebra generated by the simple root spaces where $a_i$ is integral.

This observation is particularly interesting in the case where $g$ is affine, $λ$ is the basic fundamental weight and $ξ = nδ$. In this case, the $C^*$-invariant section algebra $\Gamma(\mathfrak{g}_ξ; A_ξ)^{C^*}$ is a spherical symplectic reflection algebra for wreath product complex reflection group $S_n \wr H$, $H$ matches $g$ under the Mackay correspondence by [19,24,33]. This phenomenon of functors associated to roots appearing when particular functions on the parameter space are integral is quite suggestive in connection with Etingof’s conjecture relating finite dimensional modules for these symplectic reflection algebras to affine Lie algebras [21]. In fact, since the first version of this paper appeared as a preprint, Bezrukavnikov and Losev [5] have proven this theorem with these functors appearing as a tool in the proof.

One further technique we will need is to relate different orientations of $γ$. If we change the orientation of one of the edges of $γ$, then we can use a partial Fourier transform to relate the categories of $\mathcal{D}$-modules, as in Proposition 2.6.

Proposition 3.5 Given two orientations of the quiver $Γ$, the equivalence of Proposition 2.6 intertwines the bimodules $\mathcal{F}_i$, $\mathcal{F}_i$ constructed using these orientations.

Proof By Proposition 2.6, it’s enough to check this for $\mathcal{F}_i$; of course, to confirm this for a bimodule, it is enough to confirm it for the functor induced by tensor product with that bimodule. For $\mathcal{F}_i$, this is the functor of convolution with the constant sheaf on $X^0_ξ$. This commutes with Fourier transform by [34, Th. 5.4]; note that we have different conventions for homological shift in convolution from Lusztig (ours are chosen so that Verdier self-dual objects are preserved), so we do not need a homological shift as in the statement of [34, Th. 5.4]. This can be seen from the fact that Fourier transform commutes with Verdier duality.
3.2 Harish–Chandra and core modules

The 2-functor $\mathcal{G}_\lambda$ actually lands in a much smaller subcategory of $\Omega^\lambda$. In the product $\mathcal{M}_{\xi}^\lambda \times \mathcal{M}_{\xi'}^\lambda$, we still have a notion of “diagonal.” As discussed in Nakajima [37, 3.iii], there is a natural projective morphism $\mathcal{M}_{\xi}^\lambda \to \mathcal{M}_0(v, w)$, where the latter space is the categorical quotient $\text{Spec}\mathbb{C}[\mu^{-1}(0)]^{G_\xi}$; in many but not all cases, this is the affinization map of $\mathcal{M}_{\xi}^\lambda$. We let $\mathcal{M}_{\xi}^\lambda$ be the image of this map; this is the spectrum of the ring $\mathbb{C}[\mu^{-1}(0)]^{G_\xi}$ modulo the ideal of functions that vanish on the stable locus $\mu^{-1}(0)^s$.

By [37, 3.27], $\mathcal{M}_0(v, w)$ can be identified with the moduli space of semi-simple representations of the pre-projective algebra of the Crawley-Boevey quiver corresponding to $w$ with dimension vector given by $v$ and 1 on the new vertex, so $\mathcal{M}_{\xi}^\lambda$ is a subvariety of this moduli space. As noted in Nakajima, we have a natural inclusion $\mathcal{M}_0(v', w) \hookrightarrow \mathcal{M}_0(v, w)$ induced by direct sum with trivial representations (those with all quiver maps being 0) when $v'_i \leq v_i$ for all $i$. In terms of functions, the global functions on $\mathcal{M}_0(v, w)$ are generated by trace of the composition of the maps along an oriented cycle in the doubled Crawley-Boevey quiver of $\gamma$, and any relation between these functions that holds for all representations with dimension vector $v$ also holds for those of dimension vector $v'$.

We say a pair of such representations lies in the stable diagonal if they are identified under a suitable pair of such inclusions, that is, if they become isomorphic after the addition of trivial representations. Similarly, in terms of functions, we can define the stable diagonal to be the pairs in $\mathcal{M}_{\xi}^\lambda \times \mathcal{M}_{\xi'}^\lambda$ where the traces agree for any oriented cycle in the doubled Crawley-Boevey quiver.

Recall that $\mathcal{M}^\lambda = \bigsqcup_{\xi} \mathcal{M}_{\xi}^\lambda$. This is a (possibly infinite) union of smooth algebraic varieties. Following Nakajima [37, (7.1)], we let $Z$ denote the preimage of the stable diagonal in $\mathcal{M}^\lambda \times \mathcal{M}^\lambda$; this can also be thought of as the points where all the traces along oriented cycles coincide. In [8, §6.1], Braden, Proudfoot and the author define a 2-subcategory $\mathbf{HC}^8(\lambda)$ of good sheaves of $A^\xi \boxtimes A^\xi'$-modules called Harish–Chandra bimodules. This is the category of modules $\mathcal{M}$ such that:

1. the support of $\mathcal{M}$ is contained in $Z$.
2. there is an $A^\xi \boxtimes A^\xi'$-lattice $\mathcal{M}' \subset \mathcal{M}$ such that any global function vanishing on the stable diagonal $Z$ kills the coherent sheaf $\mathcal{M}'/h\mathcal{M}'$. This is a condition which should be thought of as an analogue of regularity of D-modules.

**Proposition 3.6** The image of $\mathcal{G}_\lambda$ lies in the 2-category $\mathbf{HC}^8(\lambda)$.

**Proof** As Corollary 3.2 shows, $\mathcal{E}_i$ and $\mathcal{F}_i$ have support that lies in $Z$. Recall that the regularity of a D-module on $X/G$ is equivalent to the regularity of its pullback to $X$, is the case for any principal bundle. Since $\mathcal{E}_i$ and $\mathcal{F}_i$ are the pushforward of regular D-modules by a proper map, by [25, Thm. 6.1.11] they are themselves regular. Thus, they possess good filtrations whose associated graded has radical annihilator; that is, they are killed by any function vanishing on $\mathcal{P}_i$. Taking the Rees module for this good filtration induces a lattice in the microlocalization of these D-modules, which modulo $h$ gives this associated graded.

Of course, a coherent sheaf possessing radical annihilator is preserved under pullback to an open subset, so $\mathcal{E}_i$ and $\mathcal{F}_i$ have lattices with the desired property (2) above.

Finally, we note that $\mathbf{HC}^8(\lambda)$ is closed under convolution. The property (1) is simply a consequence of the fact that $Z$ itself is closed under set-theoretic convolution.

For (2), the proof is identical to [8, Prop 6.7], so we will give a somewhat abbreviated explanation. Note that the pullback of the ideal sheaf $\mathcal{J}(Z)$ by $p_1^*\mathcal{P}_i$ is generated by the difference between the trace of an oriented cycles in the double Crawley-Boevey quiver on
the representations in the first and third factors. Using the difference with the same cycle on the second factor, we see that this lies in $p_{12}^* \mathcal{J}(Z) + p_{23}^* \mathcal{J}(Z)$. Using locally free resolutions as in [8, Prop 6.7], this shows that the cohomology of $p_{12}^* \mathcal{H}^L_1 \otimes \mathcal{A}_{\xi_2} \cdot p_{23}^* \mathcal{H}_2$ has a lattice which is killed by $p_{13}^* \mathcal{J}(Z)$ for $\mathcal{H}_1$, $\mathcal{H}_2$ Harish–Chandra bimodules. Pushing forward, we see the same is true for $\mathcal{J}(Z)$ acting on $\mathcal{H}_1 \ast \mathcal{H}_2$. □

This draws an analogy between the categorical action $G_{\lambda}$ and the action of the monoidal category of Harish–Chandra bimodules (in the classical sense) on various categories of representations of $\mathfrak{g}$. The latter is a categorification of the Hecke algebra, which has

- its original representation-theoretic description,
- a geometric one via the localization theorem of Beilinson and Bernstein [1], and
- a diagrammatic description in the guise of Soergel bimodules given by the work of Elias and Khovanov in type A [20] and work of Elias and Williamson in general [22].

The 2-category $\mathcal{U}$ was first defined in a purely diagrammatic manner, so it is striking evidence of its naturality (at least to the author) to see it arise in a geometric context as well.

This allows to conclude that $G_{\lambda}$ preserves any subcategory closed under convolution with Harish–Chandra bimodules. Consider a system of subvarieties $J_\xi \subset M^{(1)}_\xi$ which is closed under convolution with $Z$. Since the support of the convolution of two modules is contained in the convolution of their supports, the category of modules supported on $J_\xi$ is closed under this categorical action. Examples include:

- the cores $L^{(\lambda)}_\xi$ of the varieties $M^{(1)}_\xi$; that is, the subvariety of representations which are nilpotent as representations of the preprojective algebra $\Pi(\mathfrak{g}^w)$. Alternatively, the core $L^{(\lambda)}_\xi$ is the preimage of the unique fixed point of the conic $\mathbb{C}^*$-action on $\mathcal{Y}^{(\lambda)}_\xi$.

A sheaf of $\mathcal{A}_\xi$-modules is called a core module if it is supported on the core. Let $C^{(\lambda)}_\xi$ be the abelian category of core modules on $M^{(1)}_\xi$ for our fixed quantization $\mathcal{A}_\xi$ and $C^{(\lambda)} := \oplus_\xi C^{(\lambda)}_\xi$. Similarly, we let $D^b_{\mathcal{A}}(C^{(\lambda)}_\xi)$ be the subcategory of the bounded derived category $D^b(\mathcal{A}_\xi$-mod) given by complexes with cohomology in $C^{(\lambda)}_\xi$, and $D^b_{\mathcal{A}}(C^{(\lambda)})$ the sum of these categories.

- the points attracted to the core under a Hamiltonian $\mathbb{C}^*$-action. The modules supported on these subvarieties (subject to a regularity condition like $HC_{\mathfrak{g}}$) are an analogue of category $O$ and are studied in much greater detail by Braden, Licata, Proudfoot and the author in [7].

**Corollary 3.7** The sum $D^b_{\mathcal{A}}(C^{(\lambda)})$ carries a categorical $\mathfrak{g}$-action. □

Thus, we can easily construct examples of core modules by applying categorification functors to simple examples. Consider the sum

$$H_\xi := \bigoplus_i \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_n} \mathcal{A}^{(\lambda)}_{\xi}$$

(3.5)

for $i = (i_1, \ldots, i_n)$ ranging over the set of all sequences such that $\xi + \alpha_{i_1} + \cdots + \alpha_{i_n} = \lambda$.

**Lemma 3.8** The object $H_\xi$ is a sum of shifts of simple core modules.

**Proof** Consider the complex of $\mathcal{D}$-modules $\tilde{H}_\xi \cong \oplus_i \tilde{\mathcal{F}}_{i_1} \ast \cdots \ast \tilde{\mathcal{F}}_{i_n} \ast \mathcal{O}^{(\lambda)}_{\xi}$ (where $\mathcal{O}^{(\lambda)}_{\xi}$ is the structure sheaf of the point $E^{(\lambda)}_\xi$). This complex of $\mathcal{D}$-modules is quasi-isomorphic to a sum of shifts of simple $\mathcal{D}$-modules by the Decomposition Theorem. The reader may be concerned by the fact that we are working on a stack, but of course, we can consider the pullback of
By work of Nakajima [37], dim \(V\) is also a sum of shifts of simple objects. Thus, \(H_\xi \cong \tau(H_\xi)\) is also a sum of shifts of simple objects. □

Let \(V_\xi^\circ\) be the abelian subcategory of \(C_\xi^\circ\) generated by the simple modules that appear as summands of shifts of \(H_\xi\). As before, we let \(D^b_\lambda(V_\xi^\circ)\) be the subcategory of \(D^b(A_\xi\text{-mod})\) with cohomology in \(V_\xi^\circ\). Note that this subcategory also inherits a categorical \(g\)-action.

One of the powerful aspects of categorical actions is that they constrain the structure of a category. In particular, categorifications of simple representations are essentially unique, by work of Rouquier [39]. Attached to the weights \(\lambda\) and \(\xi\), there is an algebra \(R_\xi^\circ\), the cyclotomic KLR algebra, such that the projective modules over this algebra categorify the simple representation \(V_\lambda\). We’ll also use the deformed version of this quotient \(\check{R}_\xi^\circ\) as defined in [47, Def. 3.24] By this uniqueness result, we have the following:

**Theorem 3.9** \(\text{Ext}^\bullet(H_\xi, H_\xi) \cong R_\xi^\circ\).

In fact in [48, 4.7 & 5.1], we show that the dg-algebra \(\text{Ext}^\bullet(H_\xi, H_\xi)\) is formal, so Morita theory for dg-categories will imply that \(\text{Ext}^\bullet(H_\xi, H_\xi)\) induces an equivalence of dg-categories \(R_\xi^\circ\text{-dgmod} \cong D^b_\lambda(V_\xi^\circ)\).

**Proof** By [47, Cor. 3.26], we have an isomorphism of the deformed cyclotomic quotient \(\check{R}_\xi^\circ \otimes \check{R}_\xi^\circ \text{Ext}^\bullet(H_\lambda, H_\lambda) \rightarrow \text{Ext}^\bullet(H_\xi, H_\xi)\). Since \(H_\lambda = A_\lambda^\circ\), this tensor product is just \(R_\xi^\circ\), and we have the desired isomorphism \(\text{Ext}^\bullet(H_\xi) \cong R_\xi^\circ\).

Thus the non-isomorphic simple summands of \(H_\xi\) are in bijection with indecomposable projectives of \(R_\xi^\circ\). We know from [47, Prop. 3.21] that the K-group of \(R_\xi^\circ\) has dimension given by the weight multiplicity of \(\xi\) in \(V_\lambda\), so this is the number of non-isomorphic simple summands of \(H_\xi\).

**Remark 3.10** Recent unpublished work of Baranovsky and Ginzburg [3] shows that number of simples in \(C_\xi^\circ\) is less than or equal to the dimension of cohomology group \(H^{\text{mid}}(M_\xi^\circ; \mathbb{C})\).

By work of Nakajima [37], \(\dim H^{\text{mid}}(M_\xi^\circ; \mathbb{C}) = \dim(V_\lambda)_\xi\), the weight multiplicity of \(\xi\) in the simple \(g\)-representation \(V_\lambda\) with highest weight \(\lambda\).

The simples in \(V_\xi^\circ\) are in bijection with isomorphism classes of primitive idempotent endomorphisms of \(H_\xi\), that is, with indecomposable projectives over \(R_\xi^\circ\). By [47, Prop. 3.21], the number of these is also \(\dim(V_\lambda)_\xi\). Thus, assuming this unpublished work, we see that \(C_\xi^\circ = V_\xi^\circ\).

In finite and affine type A, the desired injectivity follows from the same injectivity for a larger category \(\mathcal{O}\) containing \(C_\xi^\circ\); this follows from [7, 6.5].

Let \(K(C_\xi^\circ)\) and \(K(V_\xi^\circ)\) be the Grothendieck groups of these abelian categories (which Remark 3.10 would show are isomorphic).

**Theorem 3.11** There is an isomorphism of \(g\)-representations \(K(V_\xi^\circ) \cong V_\lambda\).

We should emphasize that this is only true in the case where the quantization is integral (it corresponds to D-modules on an honest line bundle). There is always a map \(K(C_\xi^\circ) \rightarrow V_\lambda\) given by characteristic cycles (see [8, 6.2] or [29]), but outside of the integral case, it seems to never to be surjective. Recent work of Bezrukavnikov and Losev [5] has calculated the structure of this Grothendieck group in non-integral cases for finite type, and certain especially important affine cases.
The space $K(V^\lambda)$ is a $g$-representation, which has a weight decomposition by the definition of a categorical $g$-action. Theorem 3.9 and [47, 3.21] show that the weight multiplicities of this representation are no more than those of the simple $g$-module $V_\lambda$.

On the other hand $A_\lambda \cong \mathbb{C}$, thought of as a sheaf on a point. Thus, $V^\lambda_\lambda$ is equivalent to the category of $\mathbb{C}$-vector spaces, so $K(V^\lambda)$ has weight multiplicity 1 for $\lambda$. By our bound by $\dim(V^\lambda_\xi)$, the weight $\lambda$ is maximal among weights with non-zero multiplicity, since any higher weight corresponds to an empty quiver variety. Thus all vectors of weight $\lambda$ in $K(V^\lambda)$ are highest weight vectors.

This shows that $V_\lambda$ occurs as a composition factor with multiplicity 1, and by the bound on weight multiplicity, there can be no others. □

Note that Theorem 3.9 matches the indecomposable projective modules over $R^\xi_\xi$ to the simple modules in $C^\lambda_\xi$ (since both correspond to primitive idempotents). Since the main result of [41] show that these indecomposables are mapped to Lusztig’s canonical basis in the isomorphism $K^0(R^\lambda) \cong V_\lambda$, we have that:

**Corollary 3.12** The isomorphism $K^0(V^\lambda) \cong V_\lambda$ matches the classes of simples and Lusztig’s canonical basis. □

In particular, this shows that cyclotomic KLR algebras and Lusztig’s canonical basis for a simple have a natural geometric origin based on quiver varieties. Core modules may not seem like a familiar object to most readers, but they are closely linked to finite dimensional modules over the section algebra $A_\xi = \Gamma(\mathcal{M}^\xi_\xi; A_\xi)^{\mathbb{C}^*}$. It follows from [8, Th. B.1] that:

**Theorem 3.13** Every core module $\mathcal{M}$ has a finite dimensional space of $\mathbb{C}^*$-invariant sections $\Gamma(\mathcal{M}^\xi_\xi; \mathcal{M})^{\mathbb{C}^*}$. Furthermore, there exist choices of integral period such that $\Gamma(\mathcal{M}^\xi_\xi; -)^{\mathbb{C}^*}$ is an equivalence of categories between core modules and finite dimensional $A_\xi$-modules. □

Even when this sections functor fails to be an equivalence, it is often a derived equivalence. The paper [8] contains a much more detailed discussion of when localization and derived localization hold.

While it does not follow from such a simple uniqueness argument, one can generalize Theorem 3.9 to one connecting category $\mathcal{O}$’s to the weighted KLR algebras introduced in [44]. This is proven in [48, Th. A].

### 3.3 Canonical bases

Another canonical basis worth considering is that for the modified quantum universal enveloping algebra $\hat{U}$. By [45, Th. A], this canonical basis coincides with the classes of the indecomposable 1-morphisms in $\mathcal{U}$ when $\Gamma$ is an ADE Dynkin diagram.

In this section, we’ll compare those 1-morphisms with the simple objects in $\mathcal{Q}$.

**Lemma 3.14** For every simple $\mathcal{D}_{X^\alpha_\xi} \boxtimes \mathcal{D}^\text{op}_{X^\beta_\xi'}$-module $L$, the reduction $\tau(L)$ is simple, and every simple object in the heart of $\mathcal{Q}$ is a reduction of such a simple module.

**Proof** Consider a simple $K$ in $\mathcal{Q}$. Since $K = \tau(\tau(K))$ and the functor $\tau$ is exact, there must exactly one composition factor $L$ in $\tau(K)$ such that $\tau(L) = K$, and all others must be killed.

On the other hand, if $L$ is simple and $\tau(L) \neq 0$, then we have a non-zero map $K \rightarrow \tau(L)$ for some $K$, and thus a map $\tau(K) \rightarrow L$. Thus, $L$ must be the unique composition factor of $\tau(K)$ not killed by $\tau$, and so $\tau(L) = K$ and is thus simple. □
Fix a vertex $i$ (which we assume to be a source). It will frequently be useful to consider a variety intermediate to imposing all stability conditions and none of them:

**Definition 3.15** We let $\hat{X}^\lambda_\xi$ be the open locus in $X^\lambda_\xi$ where the sum

$$x_{out} : V_i \rightarrow W_i \oplus \bigoplus_{\alpha(e)=i} V_{\omega(e)}$$

of the maps along edges pointing out from $i$ is injective. We let $\hat{X}^\lambda_\xi_{\pm;\alpha;\alpha_i}$ be the restriction of the correspondence $X^\lambda_\xi_{\pm;\alpha;\alpha_i}$ to the same locus.

We have $M^\lambda_\xi \subset T^*\hat{X}^\lambda_\xi = (\mu^{-1}(0) \cap \hat{E}^\lambda_\xi)/G_\xi$, since any point where $x_{out}$ is not injective is destabilized by a subrepresentation on $i$ given by its kernel. We can define a bimodule $\hat{E}_i$ as the pullback of $\tilde{E}_i$ to the hatted varieties. We thus can factor the Kirwan functor $r$ through a similar functor $\hat{r}$:

$$\mathcal{D}\hat{X}^\lambda_\xi \text{-mod} \rightarrow \mathcal{A}_\xi \text{-mod}. $$

We can extend this functor to bimodules as before, and we have $\hat{r}(\hat{E}_i) = E_i$, $\hat{r}(\hat{F}_i) = F_i$.

**Lemma 3.16** The object $G_\lambda(P)$ for $P : \xi \rightarrow \xi'$ a 1-morphism in $\mathcal{U}$ is isomorphic to $r(M)$ for $M$ a sum of shifts of simple regular holonomic $\mathcal{D}\hat{X}^\lambda_\xi \boxtimes \mathcal{D}\hat{X}^\lambda_{\xi'}$-modules and is thus a sum of shifts of simple $\mathcal{A}_{\xi'} \boxtimes \mathcal{A}_\xi$-modules. If $\tilde{\psi}(P) = P$, then the $D$-module $M$ can be taken to be self-dual.

We should note that this is an analogue of Conjecture 4.13 in [31] in our situation.

**Proof** Given a sequence $i$, we let $E_i = E_{i_1} \cdots E_{i_n}$, letting $E_{-i} = F_i$. In order to show both of these statements, we need only show that for any sequence $i$ (including both positive and negative simple roots), the complex $G_\lambda(E_i)$ is the reduction of a self-dual sum of shifts of regular holonomic $D$-modules. We induct on the length of $i$.

Consider the identity 1-morphism, the sheaf $\mathcal{A}_\xi$ with the diagonal bimodule structure. We claim that this is a simple object. Any non-zero subobject must be supported on a Lagrangian subvariety of the diagonal. This is open since it is Lagrangian and closed since it is the support of a coherent sheaf. Since the diagonal is connected, its support must be the whole diagonal. When we consider the completion of this bimodule near a point of its support, the result is the diagonal bimodule over the Weyl algebra (corresponding to formal Darboux coordinates around that point). The simplicity of this bimodule over the Weyl algebra, and the fact that the inclusion of a subobject is non-zero on stalks at every point in the support, show that this subobject is $\mathcal{A}_\xi$ itself. It is the analogue of this point which is actually quite difficult in Li’s category, and thus is an obstruction to using the techniques described here in that situation.

We have the diagram of maps

$$\begin{array}{ccc}
\hat{X}^\lambda_\xi_{-;\alpha;\alpha_i} & \xrightarrow{f_1} & \hat{X}^\lambda_\xi_{-;\alpha;\alpha_i} \\
\downarrow{t} & & \downarrow{t} \\
X^\lambda_\xi_{-;\alpha_i} & & X^\lambda_\xi_{-;\alpha_i} \\
\hat{X}^\lambda_\xi_{+;\alpha;\alpha_i} & \xrightarrow{f_2} & \hat{X}^\lambda_\xi_{+;\alpha;\alpha_i} \\
\downarrow{e_1} & & \downarrow{e_2} \\
X^\lambda_\xi_{+;\alpha_i} & & X^\lambda_\xi_{+;\alpha_i} \\
\end{array}$$

(3.6)
where $f_k, e_k$ is the restriction of $\pi_k$ to the hatted locus, and $t_\ast$ is the appropriate inclusion. We note that the maps $f_k, e_k$ are proper and smooth and $t_\ast$ an open inclusion. We let

$$f_{1}^! := f_1^! [\dim \widehat{X}_\xi - \alpha_i - \dim \widehat{X}_\xi - \alpha_i] = f_2^! [\dim \widehat{X}_\xi - \alpha_i - \dim \widehat{X}_\xi - \alpha_i]$$

denote the unique shift of the pullback functor with commutes with Verdier duality, and similarly for $f_{2}^!, e_1^!, e_2^!$.

Consider any simple D-module $L$ on $X^\lambda_\xi$. Rewriting convolution with $\widehat{\mathcal{F}}_i, \delta_i$ in terms of the diagram (3.6), we have that

$$\tau(L\star \delta_i) = \tau((t_+)\ast (e_2)\ast e_1^! t^! L) \quad \tau(L\star \widehat{\mathcal{F}}_i) = \tau((t_-)\ast (f_1)\ast f_2^! t^! L).$$

Furthermore, applying Lemma 4.4, we have that

$$\tau(L)\star \delta_i = \tau((t_+)\ast (e_2)\ast e_1^! t^! L) \quad \tau(L)\star \widehat{\mathcal{F}}_i = \tau((t_-)\ast (f_1)\ast f_2^! t^! L).$$

The D-modules $e_1^! t^! L$ and $f_2^! t^! L$ are simple, since pullback by an open inclusion, and by a smooth map preserve simplicity. Thus the Decomposition Theorem and the properness of $f_1$ and $e_2$, the pushforwards $(e_2)\ast e_1^! t^! L$ and $(f_1)\ast f_2^! t^! L$ are a sum of shifts of simple D-modules on $\widehat{X}_\xi - \alpha_i$. Note that for any D-module $N$ on $\widehat{X}_\xi - \alpha_i$, we have a canonical map from the $(\pm)\ast N \rightarrow (\pm)\ast N$ where $(\pm)\ast N$ is the intermediate extension; the cokernel of this map is supported on $\widehat{X}_\xi - \alpha_i \setminus \widehat{X}_\xi - \alpha_i$. Since all D-modules supported on $\widehat{X}_\xi - \alpha_i \setminus \widehat{X}_\xi - \alpha_i$ are killed by $\tau$, we have $\tau((\pm)\ast N) \cong \tau((\pm)\ast N)$, with the isomorphism induced by this map.

Thus, we can replace $(\pm)\ast$ with the intermediate extension $(\pm)\ast$ in (3.7):

$$\tau(L)\star \delta_i = \tau((t_+)\ast (e_2)\ast e_1^! t^! L) \quad \tau(L)\star \widehat{\mathcal{F}}_i = \tau((t_-)\ast (f_1)\ast f_2^! t^! L).$$

Since intermediate extension preserves simplicity, we see that $\tau(L)\star \delta_i$ is the image under $\tau$ of a sum of shifts of simple D-modules by Lemma 3.14. By the inductive assumption, $\mathcal{G}_\lambda(\mathcal{E}_i) = \tau(M)$ for some object $M$ which is a self-dual sum of shifts of regular holonomic D-modules. Thus $\mathcal{G}_\lambda(\mathcal{E}_i \mathcal{E}_i)$ is also a reduction of a self-dual sum of shifts of regular holonomic D-modules (either $(t_-)\ast (f_1)\ast f_2^! t^! M$ or $(t_+)\ast (e_2)\ast e_1^! t^! M$) since the operations $(t_-)\ast, t^!, (f_1)\ast, f_2^!$, $(e_2)\ast, e_1^!$ all commute with duality (since $f_1, e_1$ are proper and smooth).

\begin{lemma}
If $\Gamma$ is an ADE Dynkin diagram, then the 2-functor $\mathcal{G}_\lambda$ is full on 2-morphisms, that is for any 1-morphisms $u, v: \mu \rightarrow v$, the map

$$\text{Hom}_\mathcal{U}(u, v) \rightarrow \text{Hom}_\mathcal{U}(\mathcal{G}_\lambda(u), \mathcal{G}_\lambda(v))$$

is surjective.
\end{lemma}

\begin{proof}
We induct downward on the usual order on the weight lattice generated by $\mu - \alpha_i < \mu$.

We have that $\mathfrak{M}_\lambda$ is a point, so the only non-trivial 1-morphism is the identity, and its endomorphisms are just the scalars. In this case, fullness is clear. This establishes the base case.

Assume that we know the result for 1-morphisms $\mu' \rightarrow v'$ where either $\mu' > \mu$ or $\mu' > v$. Assume that $u$ and $v$ are indecomposable. Recall that $\mathcal{U}$ has a “triangular decomposition” into two subcategories $\mathcal{U}^+$ and $\mathcal{U}^-$ generated by the $\mathcal{E}_i$’s and $\mathcal{F}_i$’s respectively. We now prove two smaller claims:

1. If $v$ is not in the image of $\mathcal{U}^-$, then $\text{Hom}_\mathcal{U}(u, v) \rightarrow \text{Hom}_\mathcal{U}(\mathcal{G}_\lambda(u), \mathcal{G}_\lambda(v))$.
2. If $u$ is not in the image of $\mathcal{U}^+$, then $\text{Hom}_\mathcal{U}(u, v) \rightarrow \text{Hom}_\mathcal{U}(\mathcal{G}_\lambda(u), \mathcal{G}_\lambda(v))$.
\end{proof}
Let us first consider (1). If \( v \) is not in the image of \( \mathcal{U}^- \) then by [45, 5.12], we have that \( v \) is a summand of \( \mathcal{E}_i v' \) for some 1-morphism \( v' : \mu + \alpha_i \to v \); let \( e : \mathcal{E}_i v' \to \mathcal{E}_i v'' \) by an idempotent whose image is \( v \), and \( v'' \) be the image of \( 1 - e \), that is the complementary summand. By assumption, we have a surjection

\[
\text{Hom}_{\mathcal{U}}(u, \mathcal{E}_i v') \cong \text{Hom}_{\mathcal{U}}(\mathcal{F}_i u, v') \to \text{Hom}_{\mathcal{Q}_\lambda}(\mathcal{G}_\lambda(\mathcal{F}_i u), \mathcal{G}_\lambda(v')) \cong \text{Hom}_{\mathcal{Q}_\lambda}(\mathcal{G}_\lambda(u), \mathcal{G}_\lambda(\mathcal{E}_i v')).
\]

With we compose this map with the idempotent \( \mathcal{G}_\lambda e \), then we obtain a surjection \( \text{Hom}_{\mathcal{U}}(u, \mathcal{E}_i v') \to \text{Hom}_{\mathcal{Q}_\lambda}(\mathcal{G}_\lambda(u), \mathcal{G}_\lambda(v')) \), which kills \( \text{Hom}_{\mathcal{U}}(u, v'') \); thus, the induced map \( \text{Hom}_{\mathcal{U}}(u, v) \to \text{Hom}_{\mathcal{Q}_\lambda}(\mathcal{G}_\lambda(u), \mathcal{G}_\lambda(v')) \) is surjective as desired. Claim (2) follows by a symmetric argument.

Thus, it remains to establish \( \text{Hom}_{\mathcal{U}}(u, v) \to \text{Hom}_{\mathcal{Q}_\lambda}(\mathcal{G}_\lambda(u), \mathcal{G}_\lambda(v)) \) for \( u \) in the image of \( \mathcal{U}^- \) and \( v \) in the image of \( \mathcal{U}^+ \). For reasons of weight, the target can only be non-zero if \( \mu = v \) and \( u = v = 1_\mu \). Thus, we must prove that

\[
\text{Hom}_{\mathcal{U}}(1, 1) \to \text{Hom}_{\mathcal{Q}_\lambda}(1, 1) \cong H^n(\mathcal{M}_\mu^\lambda).
\]

This surjectivity is a consequence of the “algebraic Kirwan surjectivity” discussed in [42], which only holds in the generality we need in the ADE case. Combining the surjectivity [42, Prop. 1.6] to the center of the cyclotomic quotient and the isomorphism [42, Prop. 3.6] of the center to the cohomology of the quiver variety, the map of (3.8) must be surjective. □

By a standard argument (see, for example, [46, Lemma 10]), this shows that the functor \( \mathcal{G}_\lambda \) sends each indecomposable 1-morphism to an indecomposable bimodule.

Combining Lemma 3.16 with the observation that indecomposability is preserved under the functor \( \mathcal{G}_\lambda \), we see that:

**Corollary 3.18** If \( \Gamma \) is an ADE Dynkin diagram, then for each indecomposable 1-morphism \( P \) in \( \mathcal{U} \), the sheaf \( \mathcal{G}_\lambda(P) \) is simple.

Let \( \mathcal{Q}_\lambda^\lambda \) denote the image of \( \mathcal{G}_\lambda \) as a functor between graded additive categories, where as before, the grading on \( \mathcal{Q}_\lambda \) arises from the homological grading; this is a full 2-subcategory of \( \mathcal{Q}_\lambda \), which is closed under convolution (but not under extensions). This is a mixed humorous category in the sense of [45, 1.11] by applying [45, 1.20] with \( \mathcal{F} \) given by the dg-subcategory \( \mathcal{Q}_\lambda \) generates equipped with the usual \( t \)-structure. The graded Grothendieck group \( K_q(\mathcal{Q}_\lambda^\lambda) \) is, as discussed before, the free abelian group spanned by indecomposable objects in this category, made into a \( \mathbb{Z}[q, q^{-1}] \) module by grading shift. Thus, if we let the canonical basis of \( K_q(\mathcal{Q}_\lambda^\lambda) \) be the classes of the simple modules, then [45, 1.15] implies that these are also canonical bases in the algebraic sense of bar-invariant almost-orthogonal vectors.

Convoluted also endows the graded Grothendieck group \( K_q(\mathcal{Q}_\lambda^\lambda) \) with an algebra structure, with an induced algebra map \( K_q(\mathcal{G}_\lambda) : K_q(\mathcal{U}) \to K_q(\mathcal{Q}_\lambda^\lambda) \). Finally, [45, 1.17] shows that:

**Proposition 3.19** If \( \Gamma \) is an ADE Dynkin diagram, each canonical basis vector in \( K_q(\mathcal{Q}_\lambda^\lambda) \) is the image of a unique canonical basis vector in \( \mathcal{U} \), and any other canonical basis vector in \( \mathcal{U} \) is killed by \( K_q(\mathcal{G}_\lambda) \).

### 3.4 Decategorification

Finally, we turn to understanding how this action decategorifies. As defined in [8, §6.2], based on work of Kashiwara and Schapira [29], we have a map CC from the \( K \)-group of sheaves
supported on $Z$ to $H_{\text{top}}^{BM}(Z)$ which intertwines convolution of sheaves with convolution of Borel-Moore classes. While this map has a general definition in terms of Hochschild homology (which can be used for any sheaf of algebras), in this case, there is a very concrete way of understanding it. The group $H_{\text{top}}^{BM}(Z)$ is freely spanned by the top dimensional components of $Z$; in fact, the semi-smallness of the map $\mathcal{M}_\xi^\lambda \to \mathcal{N}_\xi^\lambda$ shows that all components of this variety are top dimensional. Thus, describing a cycle in $H_{\text{top}}^{BM}(Z)$ just requires describing the coefficient we take it with. Let $\mathcal{M}$ be an object in $\Omega^\lambda$ supported on $Z$. By [29, 7.3.5], the characteristic cycle

$$CC(\mathcal{M}) = \sum_Y a_Y [Y]$$

(3.9)

is just a sum over with the components $Y$ of $Z$ with coefficients given by the dimension $a_Y$ of the fiber (that is, the tensor product of the stalk with the residue field) of the classical limit $\mathcal{M}'/h\mathcal{M}'$ (for $\mathcal{M}'$ an $A^\xi_\lambda$-lattice) at a generic point of the component. Put slightly differently, the completion of $A_\lambda \boxtimes A^\operatorname{op}_\lambda$ at a general point of the component is a Weyl algebra, and the completion of $\mathcal{M}$ is the direct sum of $a_Y$ copies of the completion of functions on $Y$ as a holonomic module over this Weyl algebra.

Composing the map induced on Grothendieck groups defined by $G_\lambda$ with $CC$, we obtain a homomorphism $C : K(\mathcal{U}) \to H_{\text{top}}^{BM}(Z)$.

**Proposition 3.20** We have a commutative diagram

$$
\begin{array}{ccc}
K(\mathcal{U}) & \overset{\sim}{\rightarrow} & \mathcal{U}(\mathfrak{g}) \\
\downarrow C & & \downarrow N \\
H_{\text{top}}^{BM}(Z) & & H_{\text{top}}^{BM}(Z)
\end{array}
$$

(3.10)

where $N : \mathcal{U}(\mathfrak{g}) \to H_{\text{top}}^{BM}(Z)$ is the map defined by Nakajima in [37].

**Proof** First, let us consider $CC(\hat{\mathcal{E}}_i)$. This is slightly more technical than it should be, since the diagonal map of $X^\lambda_\xi;\alpha \to X^\lambda_\xi \times X^\lambda_\xi;\alpha_i$ is not closed, and thus not proper. However, all the difficulty in this point is wrapped up in the fact that the diagonal in $X^\lambda_\xi$ itself is not proper.

We remove one source of such problems by considering the hatted varieties (requiring that the map out of $V_i$ is injective), and the other by absorbing it into the fact that $X^\lambda_\xi$ has non-proper diagonal.

That is, we consider the diagram

$$
\begin{array}{ccc}
\hat{X}^\lambda_\xi \times \hat{X}^\lambda_\xi;\alpha_i & & \hat{X}^\lambda_\xi \times \hat{X}^\lambda_\xi;\alpha_i \\
\downarrow \text{id} \times \pi_1 & & \downarrow \text{id} \times \pi_2 \\
\hat{X}^\lambda_\xi \times \hat{X}^\lambda_\xi & & \hat{X}^\lambda_\xi \times \hat{X}^\lambda_\xi;\alpha_i
\end{array}
$$

(3.11)

As before, let $(\text{id} \times \pi_1)^{!*}$ be the shift of $(\text{id} \times \pi_1)^{*}$ and $(\text{id} \times \pi_1)^!$ which commutes with Verdier duality. The sheaf $\mathcal{E}_i = (\text{id} \times \pi_2)_* (\text{id} \times \pi_1)^{!*} \mathcal{D}_{X^\lambda_\xi}$ can be constructed by starting with the diagonal sheaf on $\mathcal{D}_{X^\lambda_\xi}$ on as a bimodule on $\hat{X}^\lambda_\xi \times \hat{X}^\lambda_\xi$ and then pull-push on this diagram. This is a more congenial setup, since both maps are proper and smooth, making their action on
characteristic cycles straightforward to compute (see, for example, [28, 9.4.2]) by pull-push on the diagram below:

\[
\begin{array}{c}
T^*\hat{X}_\xi^\lambda \times \hat{Y}_\xi^\lambda;\alpha_i \\
\downarrow \text{id} \times \pi_1 \\
T^*\hat{X}_\xi^\lambda \times T^*\hat{X}_\xi^\lambda \\
\downarrow \text{id} \times \pi_2 \\
T^*\hat{X}_\xi^\lambda \times T^*\hat{X}_\xi^\lambda - \alpha_i \\
\end{array}
\]

(3.12)

where \(\hat{Y}_\xi^\lambda;\alpha_i\) is the pullback of the cotangent bundle \(T^*\hat{X}_\xi^\lambda\); we can think of this as the space of short exact sequences of representations of the preprojective algebra with appropriate dimension vectors such that \(x_{out}\) is injective in the total representation.

For any stack \(Y\), let \(\Delta_Y \subset Y \times Y\) be the diagonal. Note that in \(\hat{X}_\xi^\lambda \times \hat{X}_\xi^\lambda\), we have that the diagonal \(\Delta_{M^\lambda_\xi}\) is a locally closed Lagrangian subvariety of \(\hat{X}_\xi^\lambda \times \hat{X}_\xi^\lambda\). Since the characteristic cycle of \(A^\lambda_\xi\) is \([\Delta_{M^\lambda_\xi}]\), the characteristic cycle of \(D_{\hat{X}_\xi^\lambda}\) contains \([\Delta_{M^\lambda_\xi}]\) with multiplicity 1, and possibly some other unknown components which lie in the complement of \(M^\lambda_\xi \times M^\lambda_\xi\).

Convolving by the diagram (3.12) above, we obtain that the characteristic cycle of \(\hat{E}_i\) is the sum of \([\bar{P}_i]\) with multiplicity 1 plus components that lie in the complement of \(M^\lambda_\xi \times M^\lambda_\xi - \alpha_i\).

Thus the characteristic cycle of \(\hat{E}_i\) is precisely \([\bar{P}_i]\) with multiplicity 1, since all other components of the characteristic cycle of \(\hat{E}_i\) lie in the complement of \(M^\lambda_\xi \times M^\lambda_\xi - \alpha_i\). By a symmetric argument, the support variety of \(\hat{F}_i\) is the variety obtained from this one by reversing factors. Thus, we have that

\[
[E_i] \mapsto [\rho_i] \mapsto [\bar{P}_i] \\
[F_i] \mapsto [\bar{F}_i] \mapsto [\omega(\bar{P}_i)].
\]

By [37, 9.4], the homomorphism \(N: \hat{U} \rightarrow H^{BM}_{top}(Z)\) is the unique one with this property.

Kashiwara and Schapira have shown that there is a compatibility between convolution of bimodules and convolution of Borel-Moore cycles [29, 6.5.4]. The application of these results in our context is discussed in [8, 6.15-16]. These results show that:

**Corollary 3.21** The diagram (3.10) can be extended to a commutative diagram including the natural actions of \(K(\mathcal{U})\) on \(K(\mathcal{L}^\lambda)\) induced by \(G_\lambda\), Nakajima’s action of \(H^{BM}_{top}(Z)\) on \(H^{BM}_{top}(L^\lambda)\), and the usual action of \(U(\mathfrak{g})\) on \(V_\lambda\):

\[
\begin{array}{c}
K(\mathcal{U}) \xrightarrow{C} H^{BM}_{top}(Z) \xleftarrow{N} \hat{U}(\mathfrak{g}) \\
K(\mathcal{L}^\lambda) \xrightarrow{CC} H^{BM}_{top}(L^\lambda) \xleftarrow{\sim} V_\lambda
\end{array}
\]

\(\square\)
4 The Proof of Theorem 3.3

Now we proceed to the proof of Theorem 3.3 through a series of lemmata. Our general strategy is to use [40, 4.13]; this requires us to define a number of morphisms between convolutions of the sheaves $\mathcal{E}_i$ and $\mathcal{F}_i$, and to check relations between these morphisms. It is hard to do this directly in the 2-category $\mathcal{O}$, and thus more practical to do so using $D$-modules. Lemmata 4.1–4.4 show a number of compatibilities between $D$-modules. Lemma 4.5 uses this to confirm the biadjunction of $\mathcal{O}$, and the category of $D$-modules. Lemma 4.5 uses this to confirm the biadjunction of $\mathcal{O}$, which simplifies the confirmation of the desired relations. Following this, we will complete the proof.

**Lemma 4.1** We have a natural isomorphism in the derived category $M \boxtimes tN \cong \tau(t_M \boxtimes N)$ for any complex of $A_{\xi} \boxtimes A_{\xi}'$-modules $M$ and any complex of $D_{X_{\xi}} \boxtimes D_{X_{\xi}'}$-modules $N$.

**Proof** Assume for now that $M$ is an honest module over $A_{\xi} \boxtimes A_{\xi}'$. For a fixed anti-ample line bundle on $\mathcal{M}_{\xi}'$ and on $\mathcal{M}_{\xi}$, each power of the tensor product of these line bundles has a canonical quantization as an $A_{\xi} \boxtimes A_{\xi}'$-module by [8, Prop. 5.2]. Just as any coherent sheaf is a quotient of sums of these powers (since its twist with a sufficiently negative power is generated by global sections), the module $M$ is a quotient of a direct sum of quantizations of powers of this line bundle. This shows that an arbitrary complex $M$ in $D^b(A_{\xi} \boxtimes A_{\xi}')$ can be replaced by a quasi-isomorphic representative where each term of the complex of $M$ is of the form $M_1 \boxtimes M_2$ (and in fact, $M_i$ is a sum of quantizations of powers of our anti-ample line bundle).

Similarly, each term of $N$ can be assumed to be of the form $(D_{X_{\xi}} \boxtimes V') \boxtimes (D_{X_{\xi}'} \boxtimes V'')$ for $V', V''$ finite dimensional representations of $G_{\xi}$ and $G_{\xi}'$. Let

$$\mathcal{V}' = \tau(D_{X_{\xi}} \boxtimes V') \quad \mathcal{V}' = \tau(D_{X_{\xi}'} \boxtimes V'').$$

These sheaves are locally free, so $(\mathcal{V}')^* = \mathcal{H}om_{A_{\xi}'}(\mathcal{V}', A_{\xi})$ is again locally free as a right module.

We need only check that we have a natural isomorphism when $M$ and $N$ are one of these individual terms. In this case,

$$M \boxtimes tN \cong M_1 \boxtimes \Gamma(\mathcal{M}_{\xi}'; M_1 \boxtimes A_{\xi'} \mathcal{V}') \boxtimes \mathcal{V}'' \quad \tau(t_M \boxtimes N) \cong M_1 \boxtimes \Gamma(X_{\xi}'; (t_M M_1) \boxtimes V') \boxtimes \mathcal{V}'' \quad \text{where here we view the sections a vector space (i.e. a sheaf on a point).}$$

The result then follows from the isomorphism

$$\Gamma(\mathcal{M}_{\xi}'; M_2 \boxtimes A_{\xi'} \mathcal{V}') \cong \mathcal{H}om_{A_{\xi}'}((\mathcal{V}')^*, M_2),$$

$$\cong \mathcal{H}om_{D_{X_{\xi}'}^{\text{op}}}(D_{X_{\xi}'}^{\text{op}} \boxtimes (V')^*, t_A M_2) \cong \Gamma(X_{\xi}'; t_A M_2 \boxtimes V').$$

\[ \square \]

**Lemma 4.2** If $\mathcal{M}$ is a $D_{X_{\xi}}$-module whose singular support $\mu \text{ supp}(\mathcal{M})$ is contained in the unstable locus, then $\mu \text{ supp}(\mathcal{M} \boxtimes \mathcal{E}_i)$ is also contained in the unstable locus. That is, if $\tau(\mathcal{M}) = 0$, then $\tau(\mathcal{M} \boxtimes \mathcal{E}_i) = 0$.

**Proof** Note that $\tau(\mathcal{M}) = 0$ if and only if every element of the singular support of $\mathcal{M}$ lies in the unstable locus. That is, as a representation of the doubled quiver, it is has a destabilizing subrepresentation.
By Lemma 3.1, we have that $\mu \supp(\mathcal{E})$ is contained in $\mathcal{K}_i$. Applying the result on singular support of pushforward and pullback from [2, 9a & b], the singular support $\mu \supp(\mathcal{M} \ast \mathcal{E}_i)$ is contained in the set

$$\{(x, \varphi) \in T^* X_{\xi - \alpha_i}^\lambda ((x', \varphi'; x, \varphi) \in \mathcal{K}_i \text{ for some } (x', \varphi') \in \mu \supp(\mathcal{M})\}.$$  

As representations of the doubled quiver, this means that $(x', \varphi')$ must correspond to a submodule of $(x, \varphi)$. Thus, if all elements of $\mu \supp(\mathcal{M})$ contain a destabilizing subrepresentation, the same must be true of $\tau(\mathcal{M} \ast \mathcal{E}_i) = 0$. This shows that all points in this singular support must be unstable, so $\tau(\mathcal{M} \ast \mathcal{E}_i) = 0$. \hfill \Box

**Lemma 4.3**  

$\tau(\tilde{\mathcal{E}}_i \ast \cdots \ast \tilde{\mathcal{E}}_n) \cong \mathcal{E}_i \ast \cdots \ast \mathcal{E}_n$

**Proof**  

We induct on $n$; when $n = 1$, this is true by definition.

By the inductive hypothesis, $\tau(\tilde{\mathcal{E}}_i \ast \cdots \ast \tilde{\mathcal{E}}_{i-1}) \cong \mathcal{E}_i \ast \cdots \ast \mathcal{E}_{i-1}$. Thus, we have maps

$$\tau(\tilde{\mathcal{E}}_i \ast \cdots \ast \tilde{\mathcal{E}}_{i-1}) \to \tilde{\mathcal{E}}_i \ast \cdots \ast \tilde{\mathcal{E}}_{i-1} \to C(a)$$

which induce isomorphisms after applying $\tau$. Thus $C(a)$, the cone of the left-hand morphism above, has cohomology singularly supported on the unstable locus. By definition, we have an exact triangle

$$\tau(\tilde{\mathcal{E}}_i \ast \cdots \ast \tilde{\mathcal{E}}_{i-1}) \to \tilde{\mathcal{E}}_i \ast \cdots \ast \tilde{\mathcal{E}}_{i-1} \to C(a) \quad (4.1)$$

Thus, applying the triangulated functor $-\ast \mathcal{E}_n$ to the equation (4.1), we have an exact triangle

$$\tau(\mathcal{E}_i \ast \cdots \ast \mathcal{E}_{i-1} \ast \mathcal{E}_i) \to \mathcal{E}_i \ast \cdots \ast \mathcal{E}_{i-1} \to C(a) \ast \mathcal{E}_i \quad (4.2)$$

If we apply Lemma 4.1 with $M = \mathcal{E}_i \ast \cdots \ast \mathcal{E}_{i-1}$ and $N = \mathcal{E}_n$, we arrive at

$$\mathcal{E}_i \ast \cdots \ast \mathcal{E}_n \cong \tau(\mathcal{E}_i \ast \cdots \ast \mathcal{E}_{i-1} \ast \mathcal{E}_i). \quad (4.3)$$

Combining equations (4.2–4.3), we arrive at the desired isomorphism

$$\mathcal{E}_i \ast \cdots \ast \mathcal{E}_n \cong \tau(\mathcal{E}_i \ast \cdots \ast \mathcal{E}_{i-1} \ast \mathcal{E}_i).$$

\hfill \Box

**Lemma 4.4**  

*For any $D_{\lambda\xi}$-module $\mathcal{M}$, we have that:*

$$\hat{\tau}\mathcal{M} \ast \mathcal{F}_i \cong \hat{\tau}(\mathcal{M} \ast \hat{\mathcal{F}}_i) \quad \hat{\tau}\mathcal{E}_i \cong \hat{\tau}(\mathcal{E}_i)$$

**Proof**  

We let $\hat{\mathcal{E}}_i = T^* \tilde{X}_{\xi - \alpha_i}^\lambda \setminus \mathcal{M}_i^\lambda$ and

$$\hat{\mathcal{F}}_i : M_{\lambda, \mu}^\alpha \times M_{\lambda, \mu}^\alpha_{\xi - \alpha_i} \to T^* \tilde{X}_{\xi - \alpha_i}^\lambda \times M_{\lambda, \mu}^\lambda \quad \hat{\mathcal{E}}_i : \hat{\mathcal{E}}_i \to T^* \tilde{X}_{\xi - \alpha_i}^\lambda \times M_{\lambda, \mu}^\lambda$$

be the inclusion of the loci where the the first coordinate is (un)stable. By definition,

$$\hat{\tau}\mathcal{M} \ast \mathcal{F}_i \cong (\pi_{\alpha})_* \hat{\tau}(\pi_{\alpha}^* \mu \mathcal{M} \otimes \mu \mathcal{F}_i) \quad \hat{\tau}\mathcal{E}_i \cong (\pi_{\alpha})_* (\pi_{\alpha}^* \mu \mathcal{E}_i)$$

Thus, by the usual recollement, these will be isomorphic via the natural map if and only if $(\pi_{\alpha})_* \hat{\tau}\mathcal{F}_i \cong (\pi_{\alpha})_* (\pi_{\alpha}^* \mu \mathcal{M} \otimes \mu \mathcal{F}_i) = 0$; note here $\hat{\tau}\mathcal{F}_i$ is pullback with compact support (that is,
its sections on an open set $U \cap \text{image}(\hat{J})$ are those on the larger space $U$ supported on $U \cap \text{image}(\hat{J})$ and since $\hat{J}$ is a closed map, $\hat{J}_i$ is just usual pushforward. Thus, it suffices to show that $Y := \hat{J}_i((\pi_1^+ \mu \mathcal{M} \otimes \mu \hat{F}_i) = 0$. Any point which lies in the support of this $Y$ must have the following properties: its two coordinates correspond to framed modules $S_1, S_2$ over the preprojective algebra with an inclusion $S_2 \hookrightarrow S_1$, such that $S_2$ is stable, and $S_1$ has a destabilizing subrepresentation $Z_1 \subset S_1$. Furthermore, this destabilizing subrepresentation cannot lie solely on the vertex $i$. However, the cokernel $S_1/S_2$ is only supported on $i$, so $Z_1$ cannot inject into this quotient. The intersection $Z_1 \cap S_2$ must thus be non-trivial, providing a destabilizing subrepresentation of $S_2$. Thus, we have arrived at a contradiction, and this sheaf must have empty support, and thus be 0. The second equation follows by a similar argument $T^*\hat{X}^\lambda_1 \times \mathcal{M}^\lambda_{\xi+\alpha};$, or alternately, from Lemma 4.2. \hfill \Box

**Lemma 4.5** The left and right adjoint of $\mathcal{F}_i \ast -$ are the convolution functors

$$\mathcal{E}_i[1/2(m^\lambda_1 - m^\lambda_2-\alpha_i)] \ast -$$

and

$$\mathcal{E}_i[1/2(m^\lambda_2-\alpha_i - m^\lambda_1)] \ast - .$$

**Proof** As in [29, (2.3.18)], we can take the dual

$$\mathbb{D}\mathcal{E}_i \cong \mathcal{R}\text{Hom}_{\mathcal{A}_\xi \boxtimes \mathcal{A}_{\xi-\alpha}} (\mathcal{E}_i, \mathcal{A}_\xi \boxtimes \mathcal{A}_{\xi-\alpha}^{\text{op}}).$$

This naturally an $\mathcal{A}_{\xi-\alpha}$- $\mathcal{A}_\xi$-bimodule. Note that $\mathbb{D}\mathcal{A}_\xi = \mathcal{A}_\xi[-m^\lambda_1]$, since the same equality holds canonically for the Weyl algebra (i.e. the isomorphism is compatible with automorphisms of the Weyl algebra). Thus, $\hat{\tau}(\mathbb{D}\mathcal{D}_{\hat{X}^\lambda_1}) = \mathcal{A}_\xi[-m^\lambda_1]$. Applying [25, Thm. 2.7.1 & 2.7.2] to the diagram (3.11), we find that

$$\mathbb{D}\mathcal{E}_i \cong \hat{\mathcal{F}}_i \ast (\mathbb{D}\mathcal{D}_{\hat{X}^\lambda_1})[1/2(m^\lambda_1 - m^\lambda_2-\alpha_i)];$$

note that we have a different convention for homological shifts in taking dual than [25]. By Lemma 4.4, we thus have

$$\mathbb{D}\mathcal{E}_i[1/2(m^\lambda_1 + m^\lambda_2-\alpha_i)] \cong \mathcal{F}_i.$$ 

The smoothness of $\mathbb{P}_i$ and [29, 2.3.15] show that the left hand side of this equation must be an honest $\mathcal{A}_{\xi-\alpha}$- $\mathcal{A}_\xi$-bimodule, as our convention for homological shifts dictates, and that is must have support equal to $\omega(\mathbb{P}_i)$. By [29, 6.2.4], this homological shift is just convolution with the square root of the dualizing sheaf for $\mathcal{A}_{\xi-\alpha} \boxtimes \mathcal{A}_{\xi-\alpha}^{\text{op}}$. We denote the dualizing sheaf for $\mathcal{A}_\xi$ by $\omega_\xi$. By symmetry, we have that $\mathbb{D}\mathcal{F}_i[-1/2(m^\lambda_1 + m^\lambda_2-\alpha_i)] \cong \mathcal{E}_i .$

For any $\mathcal{A}_\xi$-module $\mathcal{M}$ and $\mathcal{A}_{\xi-\alpha}$-module $\mathcal{N}$, we have that

$$\mathcal{R}\text{Hom}_{\mathcal{A}_\xi} (\mathcal{M}, \mathcal{F}_i \ast \mathcal{N}) \cong (\mathbb{D}\mathcal{M} \ast (\mathcal{F}_i \ast \mathcal{N}) \cong (\mathbb{D}\mathcal{M} \ast \mathcal{N}) \ast \mathcal{F}_i \ast \mathcal{N}) \cong (\mathbb{D}\mathcal{E}_i \ast \mathcal{M}) \ast \mathcal{N} [1/2(m^\lambda_2-\alpha_i - m^\lambda_1)]$$

and

$$\mathcal{R}\text{Hom}_{\mathcal{A}_\xi} (\mathcal{E}_i \ast \mathcal{M}, \mathcal{F}_i \ast \mathcal{N}) \cong (\mathbb{D}\mathcal{E}_i \ast \mathcal{M}) \ast \mathcal{N} [1/2(m^\lambda_2-\alpha_i - m^\lambda_1)].$$

By symmetry, this gives the biadjunction. \hfill \Box

**Proof of Theorem 3.3** We wish to check the conditions of [40, 4.13]. We check these conditions below in the same order they are presented in [40]
The functors $\mathcal{E}_i \leftarrow$ and $\mathcal{F}_i \leftarrow$ are biadjoint up to shift: This follows from Lemma 4.5.

The sheaves $\bigoplus_i \mathcal{E}_i \cdots \mathcal{E}_i$ carry an action of the KLR algebra for the polynomials $Q$ we have specified: As observed by Lusztig [34, 3.2(b)], the sheaf $\mathcal{E}_i \cdots \mathcal{E}_i$ can be seen as a pushforward from a moduli space of quiver representations with compatible flags, denoted by $\hat{F}_y$ in [41, §1.1], for the sequence of simple roots $y = (\alpha_i, \ldots, \alpha_n)$. For us, this is the pushforward of the structure sheaf as a D-module; for Lusztig is the constructible sheaf obtained by applying the solutions functor to this D-module (that is, its image under Riemann-Hilbert). In particular, the solution sheaf (i.e., the image under the Riemann-Hilbert correspondence) of $\mathcal{E}_i \cdots \mathcal{E}_i$ is precisely the perverse sheaf that Varagnolo and Vasserot denote by $L_i$ in [41, (1.5)].

Since Riemann-Hilbert is an equivalence of categories, [41, 3.5] and [40, 5.7] (independently) show that

$$\text{Ext}^i \left( \bigoplus_i \mathcal{E}_i \cdots \mathcal{E}_i \right) \cong R$$

for the KLR algebra $R = \bigoplus R_y$. It follows that the image of these sheaves under any functor, in particular, by Lemma 4.3, the sheaf $\tau(\mathcal{E}_i \cdots \mathcal{E}_i) \cong \mathcal{E}_i \cdots \mathcal{E}_i$ carries this action.

The functors $\mathcal{E}_i \leftarrow$ and $\mathcal{F}_i \leftarrow$ are locally nilpotent. This follows from that fact that for fixed $\xi$, there are only finitely many integers such that $\mathcal{M}_k^{\xi + i\alpha_i}$ is non-empty.

The final condition is that for each $i$, we have

$$\mathcal{E}_i \mathcal{F}_i + A_{\xi}[\langle \alpha_i, \xi \rangle + 1] \oplus A_{\xi}[\langle \alpha_i, \xi \rangle + 3] + \cdots \oplus A_{\xi}[\langle \alpha_i, \xi \rangle - 1]$$

$$\cong \mathcal{F}_i \mathcal{E}_i \quad \text{if} \quad \langle \alpha_i, \xi \rangle \leq 0 \quad (4.5a)$$

$$\mathcal{F}_i \mathcal{E}_i + A_{\xi}[\langle \alpha_i, \xi \rangle - 1] \oplus A_{\xi}[\langle \alpha_i, \xi \rangle + 3] + \cdots \oplus A_{\xi}[\langle \alpha_i, \xi \rangle - 1]$$

$$\cong \mathcal{E}_i \mathcal{F}_i \quad \text{if} \quad \langle \alpha_i, \xi \rangle \geq 0. \quad (4.5b)$$

Once this is proven, the result will follow. These isomorphisms are equivalent to the existence of certain isomorphisms of constructible sheaves by the Riemann-Hilbert correspondence. That these isomorphisms exist is shown in the characteristic existence of certain isomorphisms of constructible sheaves by the Riemann-Hilbert correspondence of this paper, another version of this argument was given in the context of D-modules we give arguments in the deformation quantization setting for the same facts. After the appearance of this paper, another version of this argument was given in the context of D-modules on Grassmannians in [11, §6.2].

Applying Fourier transform as necessary, we can assume that $i$ is a source; Proposition 3.5 shows that this leaves the action unchanged. We consider the diagram (3.6) of maps again. Recall that the maps $e_i$ and $f_i$ are both proper and smooth; their fibers are projective spaces. We let $\hat{\mathcal{F}}_i$ denote the restriction of $\mathcal{F}_i$ to the locus $X_{\varepsilon + i\alpha_i} \times \hat{X}_{\varepsilon}$, and similarly for $\hat{\mathcal{E}}_i$.

We have an isomorphism of $\hat{\mathcal{E}}_i \hat{\mathcal{F}}_i$ with the pushforward by $\hat{f}_2 := f_2 \times f_2$ of the structure sheaf of $\hat{X}_{\varepsilon + i\alpha_i} \times \hat{X}_{\varepsilon - i\alpha_i}$, $\hat{X}_{\varepsilon \alpha_i}$ tensored with the canonical sheaf of the right factor. The sheaf $\hat{\mathcal{F}}_i \hat{\mathcal{E}}_i$ is derived in the same way from $e_1 := e_1 \times e_1$.

Now, we turn to showing that

$$\hat{\mathcal{E}}_i \hat{\mathcal{F}}_i + D_{\varepsilon + i\alpha_i}[\langle \alpha_i, \xi \rangle + 1] + D_{\varepsilon + i\alpha_i}[\langle \alpha_i, \xi \rangle + 3] + \cdots + D_{\varepsilon + i\alpha_i}[\langle \alpha_i, \xi \rangle - 1]$$

$$\cong \hat{\mathcal{F}}_i \hat{\mathcal{E}}_i \quad \text{if} \quad \langle \alpha_i, \xi \rangle \leq 0 \quad (4.6a)$$

$$\hat{\mathcal{F}}_i \hat{\mathcal{E}}_i + D_{\varepsilon + i\alpha_i}[\langle \alpha_i, \xi \rangle - 1] + D_{\varepsilon + i\alpha_i}[\langle \alpha_i, \xi \rangle + 3] + \cdots + D_{\varepsilon + i\alpha_i}[\langle \alpha_i, \xi \rangle - 1]$$

$$\cong \hat{\mathcal{E}}_i \hat{\mathcal{F}}_i \quad \text{if} \quad \langle \alpha_i, \xi \rangle \geq 0. \quad (4.6b)$$
where $\Delta$ denotes the diagonal in $\hat{X}_\xi^\lambda \times \hat{X}_\xi^\lambda$. The analogous calculation for $\ell$-adic sheaves is done by Li in [32, 1.13]; specifically, his equations [32, (19-20)] compute the two sides of the preceding displayed equations and show that they agree.

Since his proof is not especially difficult, let us give an account for the reader. A point in the space $\hat{X}_\xi^\lambda: a_i \times \hat{X}_\xi^\lambda: a_i$ consists of a representation such that $x_{\text{out}}$ is injective with image $U_1$, together with a choice of subspaces $U_{12}, U_2$ with

\[
\begin{array}{ccc}
U_1 & \xrightarrow{W_i} & \oplus_{\alpha(e)=i} V_{\omega(e)} \\
& \searrow & \\
& \downarrow & \\
& U_2 & \\
\end{array}
\]

with $\dim U_{12} = \dim U_1 + 1 = \dim U_2 + 1$, and the space $\hat{X}_\xi^\lambda: a_i \times \hat{X}_\xi^\lambda: a_i$ consists of representations such that $x_{\text{out}}$ is injective with image $U_1$, together with a choice of subspaces $U_{21}, U_2$ with

\[
\begin{array}{ccc}
U_1 & \xrightarrow{W_i} & \oplus_{\alpha(e)=i} V_{\omega(e)} \\
& \searrow & \\
& \downarrow & \\
& \text{U}_2 & \\
\end{array}
\]

with $\dim U_{12} = \dim U_1 - 1 = \dim U_2 - 1$. The maps $f_2: \hat{X}_\xi^\lambda: a_i \times \hat{X}_\xi^\lambda: a_i \to \hat{X}_\xi^\lambda \times \hat{X}_\xi^\lambda$ and $\varepsilon_1: \hat{X}_\xi^\lambda: a_i \times \hat{X}_\xi^\lambda: a_i \to \hat{X}_\xi^\lambda \times \hat{X}_\xi^\lambda$ forget the space $U_{12}$ and $U_{21}$ respectively. Thus, these maps have image given by the set $H_\xi^\lambda$ of representations (both with the maps $x_{\text{out}}$ injective at $i$) which are the same away from $i$, and where the images $U_1, U_2$ of $x_{\text{out}}$ at $i$ have $\dim(U_1 \cap U_2) \geq \dim(U_1) - 1$. Both $f_2$ and $\varepsilon_1$ induce an isomorphism on the locus in $H_\xi^\lambda$ where the subspaces $U_1$ and $U_2$ differ, since we must have $U_{12} = U_1 + U_2, U_{21} = U_1 \cap U_2$.

The map $\varepsilon_1$ has fiber over the diagonal given by $\mathbb{P}^{v_i - 1}$, since the fiber consists of all the ways of choosing a hyperplane $U_{21}$ in $V_i \cong U_1 = U_2$. The map $f_2$ has fiber $\mathbb{P}^{\langle \alpha_i, \xi \rangle + v_i - 1}$, since the fiber consists of the lines $U_{12}/U_1$ in the cokernel of $x_{\text{out}}$. Finally, the diagonal has codimension $\langle \alpha_i, \xi \rangle + 2v_i - 1$ inside $H_\xi^\lambda$. Thus, the map $\varepsilon_1$ is small if $\langle \alpha_i, \xi \rangle \geq 0$, and the map $f_2$ is small if $\langle \alpha_i, \xi \rangle \leq 0$. Let’s restrict to the former case for simplicity.

In this case, $\overline{\mathcal{F}}_i \star \overline{\mathcal{F}}_j$ is an irreducible D-module $Q$, the unique one on $H_\xi^\lambda$ which extends the pullback of the canonical sheaf by the first projection, since it is the pushforward by a small resolution of singularities. On the other hand, $\overline{\mathcal{F}}_j \star \overline{\mathcal{F}}_i$ is the pushforward of a resolution of singularities which is not necessarily small, and is thus of the form $Q \oplus Q'$ where $Q'$ is a sum of shifts of semi-simple D-modules supported on the diagonal by the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber (see, for example, [18, 1.6.1]).

Note that $\hat{X}_\xi^\lambda$ is the quotient of an affine bundle over a Grassmannian by a connected algebraic group, and thus simply connected, that is, it has no nontrivial local systems. This stack is homotopic to its Borel space, which is an affine bundle over the classifying space of a product of $GL_n$’s. Therefore the pullback of $\overline{\mathcal{F}}_i \star \overline{\mathcal{F}}_j$ to the diagonal is the pushforward by a proper algebraic fiber bundle to a simply connected space, and thus a sum of shifts of the structure sheaf. When we use Kashiwara’s theorem to think of this as a D-module on
$\tilde{X}_{\xi}^\lambda \times \tilde{X}_{\xi}^\lambda$, we obtain a sum of shifts of $\mathcal{D}_\Delta$. Furthermore, since the fiber is $p(\langle \alpha_i, \xi \rangle + v_i - 1)$, we know that it is the sum $\mathcal{D}_\Delta(\langle \alpha_i, \xi \rangle + v_i - 1) \oplus \cdots \oplus \mathcal{D}_\Delta(-\langle \alpha_i, \xi \rangle + 1)$. On the other hand, the pullback of $Q$ is $\mathcal{D}_\Delta(\langle \alpha_i, \xi \rangle + v_i - 1) \oplus \cdots \oplus \mathcal{D}_\Delta(-\langle \alpha_i, \xi \rangle + 3) \oplus \cdots \oplus \mathcal{D}_\Delta(\langle \alpha_i, \xi \rangle - 1)$ by the same argument. This is only possible if

$$Q' \cong \mathcal{D}_\Delta(-\langle \alpha_i, \xi \rangle + 1) \oplus \mathcal{D}_\Delta(-\langle \alpha_i, \xi \rangle + 3) \oplus \cdots \oplus \mathcal{D}_\Delta(\langle \alpha_i, \xi \rangle - 1).$$

This shows that (4.6b) holds. If we instead consider the case where $\langle \alpha_i, \xi \rangle \leq 0$, we can show (4.6a) by applying the same argument, switching the roles of the two sheaves.

Thus, applying (4.4) to the equations (4.6a–4.6b), we arrive at the desired isomorphisms (4.5a–4.5b). Thus, by [40, 4.13], we have a 2-functor from $\mathcal{U}$ to $\Omega^\lambda$ as desired. □

The conclusions of Theorem 3.3 and Proposition 3.20 now together complete the proof of Theorem A.

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