Expansion of Rényi entropy
for free scalar fields

J.S. Dowker\textsuperscript{1}

Theory Group,
School of Physics and Astronomy,
The University of Manchester,
Manchester, England

An expression for the effective action of a conformal scalar on odd spheres allows a relatively simple computation of the expansion coefficients of the Rényi entropy for any odd dimension, $d$. Explicit values are listed for $d = 3, 5$ and $7$. The alternative method, using a mapping to a flat conical manifold, is also employed, again for any odd dimension, and some mathematical details are presented on the computation of certain integrals.

\textsuperscript{1}dowker@man.ac.uk
1. Introduction.

There has been recent interest, Perlmutter, [1], Lee et al, [2], and Hung, Myers and Smolkin, [3], in the expansion of the Rényi entropy across a spherical surface, $S^{d-2}$, in a $d$–dimensional (Euclideanised) space–time for a conformal field theory, CFT.

The definition, on the basis of the replica method, of the Rényi entropy is,

$$S_n = \frac{nW(1) - W(1/n)}{1 - n},$$

where $W(q)$ is the effective action on the $d$–dimensional space–time deformed by a conical singularity of angle $2\pi/q$. I set $n = 1/q$.

The entanglement entropy across $S^{d-2}$, is, Callan and Wilczek [4],

$$S_E = -(1 + q\partial_q) W(q) \bigg|_{q \to 1} = \lim_{n \to 1} S_n,$$

and the expansion in question is that of $S_n$ about $n = 1$.

By conformal transformation to the hyperbolic cylinder, $S^1 \times \mathbb{H}^{d-1}$ (the thermalised open Einstein universe) the expansion coefficients are, to a factor, just the thermal averages of products of the CFT Hamiltonian at the special temperature that eliminates the conical singularity. The thermal state is then the conventional (Euclidean) vacuum state relevant for $\mathbb{R}^d$, cf [5]. In order to make the quantities meaningful on a non–compact manifold, it is necessary to introduce, ad hoc, the regularised volume of hyperbolic space.

For a general CFT, Perlmutter, [1], gives an expression for the first derivative, $\partial S_n/\partial n|_{n=1}$, in terms of the strength of the energy–momentum two–point function in $\mathbb{R}^d$ (the central charge) and the regularised volume of $\mathbb{H}^{d-1}$. He then tests this formula in various CFTs, including those for free fields which simple case is my sole concern here. I work in arbitrary, odd dimensions (the hardest case in the hyperbolic cylinder approach) and produce a general formula for conformal scalars. Such is the limited aim of this calculational note.

---

2 Beware that the $q$ in the present paper is the inverse of that in [1] and [2].
2. The basic formulae

Rather than the hyperbolic cylinder, I prefer to use the periodic $d$–lune of angle $2\pi/q$ which is a slice of the $d$–sphere with edges identified. This is a subdivision of the sphere, in contrast to the covering used in [6].

On the lune, an integral for the effective action, $W$, which it is easy to differentiate, has been given in [7] for all odd dimensions. This is,

$$W(d, q) = -\frac{1}{2}\zeta'(0)$$

$$= -\frac{1}{2d} \int_{-\infty}^{\infty} d\tau \frac{\coth q\tau/2 \cosh \tau/2}{\tau \sinh^{d-1} \tau/2} ,$$

where $y < 2\pi/q$. $W$ is often referred to as the free energy.

My previous method, [7], was just to numerically integrate and some graphs as $q$ varies were obtained this way. Now I wish to produce a closed expression in terms of the Riemann $\zeta$–function which seems to be the most popular type of representation. This is done by a residue computation of (3), pushing the contour to $y = \infty$.

It is easier to use $q$ rather than $n$ as the relevant parameter. The corresponding derivatives are simply related. I therefore wish to evaluate the derivatives $W^{(p)}_d \equiv (d/dq)^p W(d, q)|_{q=1}$ and so require,

$$\left. \frac{d^p}{dq^p} \coth(q\tau/2) \right|_{q=1} = \left. t^p \left( \frac{d^p}{dt^p} \coth t \right) \right|_{t=\tau/2} .$$

I treat odd and even $p$ separately because one then has the series,

$$\frac{d^p}{dt^p} \coth t = \sum_{r=1}^{(p+1)/2} A_r^p \cosech^{2r} t , \quad p \text{ odd}$$

$$= \coth t \sum_{r=1}^{p/2} A_r^p \cosech^{2r} t , \quad p \text{ even} .$$

I will consider the $A$ constants as known.

$W^{(p)}_d$ then takes the form of a sum, over $r$, of the terms,

$$-\frac{A_r^p}{2d} \int_{-\infty}^{\infty} dt \frac{t^{p-1} \cosh t}{\sinh^{d-1+2r} t} , \quad p \text{ even}$$

$$= -\frac{A_r^p}{2d} \frac{p-1}{d+2r-2} \int_{-\infty}^{\infty} dt \frac{t^{p-2} 1}{\sinh^{d-2+2r} t} , \quad p \text{ odd}$$

$$-\frac{A_r^p}{2d} \int_{-\infty}^{\infty} dt \frac{t^{p-1} 1 + \sinh^2 t}{\sinh^{d+2r} t} , \quad p \text{ even} .$$
One sees that the basic integral required is,

\[ L(m, l) \equiv \int_{-\infty}^{\infty} \frac{t^l}{\sinh^m t}, \quad l - m \text{ even} \]

\[ = i (-1)^{(l-m)/2} \int_{y/2-i\infty}^{y/2+i\infty} dz \frac{z^l}{\sin^m z}, \]

and \( W(p) \) is given by

\[ W_d^{(p)} = -\frac{p-1}{2d} \sum_{r=1}^{(p+1)/2} \frac{A_r^p}{d+2r-2} L(d+2r-2, p-2), \quad p \text{ odd} \]

\[ = -\frac{1}{2d} \sum_{r=1}^{p/2} A_r^p \left( L(d+2r, p-1) + L(d+2r-2, p-1) \right), \quad p \text{ even}. \]

\[ L(m, n) \] is easily evaluated by residues on pushing the contour to \( y = \infty. \)\(^3\) For this purpose the series,

\[ (z \csc z)^m = \sum_{k \geq 0} (-1)^k \frac{D^{(m)}_{2k}}{(2k)!} z^{2k}, \quad |z| < \pi, \]

in terms of the Nörlund \( D \)-numbers, is handy.

We have

\[ L(m, l) = (-1)^{(l-m)/2} 2\pi \sum_{n=1}^{\infty} \text{Res} \frac{z^l}{\sin^m z} \bigg|_{z=n\pi}, \]

with,

\[ \text{Res} \frac{z^l}{\sin^m z} \bigg|_{z=n\pi} = (-1)^{nd} \text{Co}_{l-1}(z+n\pi)^l \]

\[ = (-1)^n \text{Co}_{l-1} \sum_{j=0}^{l} \binom{l}{j} (n\pi)^{l-j} \sum_{k \geq 0} (-1)^k \frac{D^{(m)}_{2k}}{(2k)!} z^{2k-m} \]

\[ = (-1)^n \sum_{k \geq 0} \binom{l}{m-2k-1} (n\pi)^{l-m+2k+1} (-1)^k \frac{D^{(m)}_{2k}}{(2k)!}, \]

using the fact that \( d \) is odd.

\(^3\) The sum over residues diverges as \( y \to \infty \) and a continuation in \( l \) is required. I will not formalise this and take it as understood. Another method, given in [7], is to expand the powers of cosech as a (finite) series of derivatives of cosech and then integrate by parts. This yields the Dirichlet eta function directly using a known representation, [8].
Hence

\[ L(m, l) = (-1)^{(l-m)/2} 2\pi \sum_{k \geq 0} (-1)^k \left( \frac{l}{m-2k-1} \right) \frac{D_{2k}^{(m)}}{(2k)!} \sum_{n=1}^{\infty} (-1)^n (n\pi)^{l-m+2k+1} \]

\[ = -(-1)^{(l-m)/2} 2\pi \sum_{k \geq 0} (-1)^k \left( \frac{l}{m-2k-1} \right) \frac{D_{2k}^{(m)}}{(2k)!} \frac{\eta(m-l-2k-1)}{\pi^{m-l-2k-1}}, \]

(7)

where \( \eta \) is the Dirichlet eta function. The expression (7) has still to be inserted into (5) to give the effective action. The expressions are easily computed symbolically. I do not give examples just now but pass on to the entropy.

3. The entropy

By expanding the Rényi entropy, (1), about \( q = 1 \), it is easy to show that,

\[ S^{(p)} = W^{(p)} + \frac{1}{p+1} W^{(p+1)}, \]

where quantities without an argument are evaluated at \( q = 1 \).

In order to compare with the results in [1] and [2], the derivatives with respect to \( n = 1/q \) are required. I denote these generally by \( S^{(p)} \) and also by dashes.

Then, for any function, \( F \), of \( q \),

\[ F^{(p)}(q) = \left( -q^2 \frac{\partial}{\partial q} \right)^p F(q). \]

In particular, at \( q = 1 \),

\[ F'' = 2 F^{(1)} + F^{(2)} \]
\[ F''' = -6 F^{(1)} - 6 F^{(2)} - F^{(3)}. \]

Hence,

\[ S' \equiv S^{(1)} = \frac{1}{2} W^{(2)} \]
\[ S'' \equiv S^{(2)} = 2 W^{(2)} + \frac{1}{3} W^{(3)} \]
\[ S''' \equiv S^{(3)} = -9 W^{(2)} - 3 W^{(3)} - \frac{1}{4} W^{(4)}, \]

and so on. A general formula can be found.

The first derivative at \( q = 1 \), \( W^{(1)} \), vanishes as was also shown in [9] and [7]. It corresponds to the vanishing of the (local) Casimir energy in the open Einstein
universe, which follows either by direct computation from the Green function or by conformal transformation (in odd dimensions).

Algebraic computation gives the first derivative, \( S' \), for a real scalar as \( \left[ \frac{\pi^2}{128}, -\frac{\pi^2}{1024}, \frac{5\pi^2}{32768} \right] \) for \( d = 3, 5 \) and 7 respectively. I tabulate the higher derivatives,

\[
\begin{array}{cccc}
d & S'' & S''' & S'''' \\
3 & -\frac{\pi^2}{49} & \approx -0.219 & -\frac{\pi^4-32\pi^2}{256} \approx 0.853 & \frac{4\pi^4-84\pi^2}{105} \approx -4.185 \\
5 & \frac{\pi^2}{280} & \approx 0.035 & \frac{21\pi^4-640\pi^2}{245676} \approx -0.174 & -\frac{33\pi^4-665\pi^2}{3150} \approx 1.063 \\
7 & -\frac{5\pi^2}{8064} & \approx -0.006 & -\frac{825\pi^4-24896\pi^2}{4915200} \approx 0.034 & \frac{1275\pi^4-25487\pi^2}{534400} \approx -0.230.
\end{array}
\]

The values for \( S' \) and \( S'' \) in three dimensions agree with those in [2] which were calculated using a different representation for the effective action (free energy) computed in [6] on the hyperbolic cylinder.

The scalar results in [3], as noted there, are subject to a discrepancy which has been independently addressed in [2] and associated with the non-compactness of the hyperbolic cylinder.

4. The conical method

As a further check, a direct calculation using the mapping to the hyperbolic cylinder and thence to a flat conical manifold has been outlined in [2], the required vacuum average of the energy momentum tensor being taken from some existing cosmic string results. These were only for \( d = 3 \) and \( d = 4 \). In this section I follow the same route, for scalar fields, but in any (odd) dimension. For this I employ the calculations of [10] where expressions for the energy density were also derived for a cosmic string space–time. By transcription of coordinates and Euclideanisation, this is related to the quantity \( \langle T^{\text{conf}}_{\tau\tau} \rangle \) of [2] by

\[
\langle T^{\text{conf}}_{\tau\tau} \rangle = r^2 (d - 1) \langle T_{zz} \rangle ,
\]

using conformal transformation and tracelessness. I will work with \( \langle T_{zz} \rangle \), which was denoted by \( \langle T_{00} \rangle \) in [10] \(^4\).

\(^4\) Note that \( d \) in the present paper equals \( d + 1 \) in [10]
My coordinates are defined by the metric

\[ ds^2 = dr^2 + r^2 d\phi^2 + dz.dz \]

the last term being for a flat space of codimension 2 and the first two for a cone.

One can proceed more easily for even \( d \) but here I discuss odd \( d \), the harder option.

Using the representation of \( \langle T_{xx} \rangle \) as the coincidence limit of a differential operator acting upon a Green function, it is shown in [10] that

\[
\langle T_{zz} \rangle = \frac{1}{\pi (4\pi r^2)^{d/2}} q \left( W_d(q) - \frac{d - 2}{d - 1} W_{d-2}(q) \right)
\]

\[ \equiv \frac{1}{\pi (4\pi r^2)^{d/2}} Y(q), \tag{9} \]

which defines \( Y(q) \) and where \( W_d \), originally a contour integral, manipulates into,

\[
W_d(q) = \int_{0}^{\infty} \frac{d\tau}{\cosh^d \tau/2} \frac{\sin(\pi q)}{\cosh q \tau - \cos q \pi}, \tag{10} \]

where \( d \) is odd and \( q \leq 1 \) which range corresponds to a conical angular excess. The values \( q = 1/n, \ n \in \mathbb{N} \), give a multi–sheeted integral covering of the plane. At \( q = 1 \), \( W_d(q) \) is obviously, and correctly, zero.

I have altered the notation slightly and have set to zero the \( U(1) \) flux through the cone ‘axis’ (the codimension 2 manifold).

The conformal transformation from the hyperbolic cylinder shows, [2], that the derivatives (with respect to \( n = 1/q \)) of the entropy at \( q = 1 \) are, this time, given by

\[
\partial_n^p S \bigg|_{q=1} = \frac{(-1)^{|d/2|}(d - 1)}{2^{d-1}(p + 1)} \partial_q^p Y(q) \bigg|_{q=1}, \tag{11} \]

where (8) and (9) have been employed. This formula incorporates the regularised volume of hyperbolic space.

If \( q \geq 1 \) a representation different to (10) holds but agrees around \( q = 1 \). Hence, as a first step, I can proceed to differentiate (10) with respect to \( q \) and set \( q = 1 \), as before.

Symbolic manipulation quickly yields the examples for the \( q \)–derivatives,
\[ W_d^{(p)} \equiv \partial_q^p W_d(q) \big|_{q=1}, \]

| \(d\) | \(W_d^{(1)}\) | \(W_d^{(2)}\) | \(W_d^{(3)}\) | \(W_d^{(4)}\) |
|-----|-----|-----|-----|-----|
| 3   | \(\frac{3\pi^2}{32}\) | \(-\frac{3\pi^2}{40}\) | \(-\frac{15\pi^4 - 236\pi^2}{320}\) | \(\frac{33\pi^4 - 476\pi^2}{140}\) |
| 5   | \(\frac{5\pi^2}{64}\) | \(-\frac{5\pi^2}{112}\) | \(-\frac{525\pi^4 - 7792\pi^2}{10754}\) | \(\frac{375\pi^4 - 5360\pi^2}{2016}\) |
| 7   | \(\frac{35\pi^2}{512}\) | \(-\frac{35\pi^2}{1152}\) | \(-\frac{3675\pi^4 - 53764\pi^2}{76800}\) | \(\frac{69825\pi^4 - 1000516\pi^2}{475200}\) |
| 9   | \(\frac{63\pi^2}{1024}\) | \(-\frac{63\pi^2}{2816}\) | \(-\frac{363825\pi^4 - 5300936\pi^2}{7884800}\) | \(\frac{760725\pi^4 - 10939448\pi^2}{6406400}\) |

and so on. Then, from (11) and the definition of \(Y(q)\) in (9), one reproduces the values for the entropy \(n\)-derivatives given in section 3.

4. Comments

The even dimensional case leads to the standard conformal anomaly log prefactor. For the factored sphere the calculation has been performed before giving generalised Bernoulli polynomials and it is obvious that the conical method would yield the same answer using [10].

The method of evaluating the basic integral, (4), is, perhaps, overly complicated. A more direct one involves a conversion to real form by choosing \(y = \pi\).

The unsophisticated treatment of the conical singularity has produced the preferred values of the Rényi expansion coefficients.

The residue approach can be applied in the covering case, \(q = 1/n, n \in \mathbb{N}\).

It is not difficult to retain the U(1) flux giving charged Rényi entropies. This will be the subject of a further communication.

The extension to spin–half would be desirable.

Consult also Aros, Bugini and Diaz, [11], for recent, holographic work on Rényi entropies in field theory.

Appendix. Some integrals

As might be expected, one encounters, in the details of the conical approach, quantities similar to those in the earlier method of section 2. The computation
of the derivatives of the integral (10) ultimately devolves upon the evaluation of a series of integrals of the form, (cf (4)),

$$J(\sigma, \mu) \equiv \int_0^{\infty} d\xi \frac{\xi^{2\sigma}}{\cosh^{2\mu+1} \xi}. \quad (12)$$

Some algebraic integrators balk at values of \( \mu \) larger than 3. If so, then it is necessary to find \( J(\sigma, \mu) \) by hand. In this appendix I discuss this technical problem, which has an independent interest. I will present two, related methods.

A direct method of integration is to expand the \( \text{sech}^{2\mu+1} \xi \) in powers of \( e^{-2\xi} \) by the binomial theorem and then use standard moments of \( \xi^{2\sigma} \) against the exponentials. The resulting expression can be grouped according to the powers of \( \xi \), ranging from \( 2\sigma + 1 \) down to zero, the coefficients of which equal \( e^{-(2\mu+1)\xi} \) times the hypergeometric series, \( _2F_1, _3F_4, \ldots, _{2\sigma+1}F_{2\sigma+2} \), with arguments irrelevant for now.

Applying the limits, only the last term survives and the hypergeometric also simplifies giving the series,

$$J(\sigma, \mu) = 2^{2\mu+1} \frac{(2\sigma)!}{(2\mu)!} \sum_{j=0}^{\infty} (-1)^j \frac{(2\mu+j)(2\mu+j-1) \ldots (j+1)}{(2\mu+2j+1)^{2\sigma+1}}. \quad (13)$$

In traditional fashion, the numerator can be rewritten as a polynomial in \((2\mu+1+2j)^2\) and (13) becomes

$$J(\sigma, \mu) = 2 \frac{(2\sigma)!}{(2\mu)!} \sum_{\rho=0}^{\mu} (-1)^{\mu-\rho} \mathcal{G}_\rho \sum_{j=0}^{\infty} (-1)^j \frac{1}{(2\mu+2j+1)^{2\sigma+1-2\rho}}$$

where the coefficients, \( \mathcal{G}_\rho^\mu \), come from the expansion,

$$P(x^2) \equiv (x^2 - 1^2)(x^2 - 3^2)(x^2 - 5^2) \ldots (x^2 - (2\mu - 1)^2) = \sum_{\rho=0}^{\mu} (-1)^{\mu-\rho} \mathcal{G}_\rho^\mu x^{2\rho}. $$

They are related to the central factorial coefficients and, in terms of the differentials of nothing, [12],

$$\mathcal{G}_\rho^\mu = (-4)^{\mu-\rho} \frac{D^{2\rho+1} 0^{[2\mu-1]}}{(2\rho+1)!}. $$

A table of some of these integer valued (positive) coefficients was early given by Thiele, [13], p.36, but they are readily calculated from the recursion, [14],

$$\mathcal{G}_\rho^r = (2r - 1)^2 \mathcal{G}_\rho^{r-1} + \mathcal{G}_{\rho-1}^{r-1}, \quad (14)$$
equivalent to those in Steffensen, [12], and in [15] and [16].

Furthermore, because of zeros in the numerator, the lower \( j \)-summation limit can be adjusted to give,

\[
J(\sigma, \mu) = 2 \frac{(2\sigma)!}{(2\mu)!} \sum_{\rho=0}^{\mu} (-1)^{\mu-\rho} G_\mu^\rho \beta(2\sigma + 1 - 2\rho),
\]

where Dirichlet’s \( \beta \)-function is defined by,

\[
\beta(s) \equiv \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^s}.
\]

This function vanishes at negative odd integers and at positive ones involves an Euler number so that

\[
J(\sigma, \mu) = 2 \frac{(2\sigma)!}{(2\mu)!} \sum_{\rho=0}^{\min(\sigma,\mu)} (-1)^{\mu-\rho} G_\mu^\rho \beta(2\sigma + 1 - 2\rho)
= (2\sigma)! \frac{\min(\sigma,\mu)}{(2\mu)!} \sum_{\rho=0}^{\min(\sigma,\mu)} (-1)^{\mu-\rho} G_\mu^\rho (-1)^{\sigma-\rho} \frac{E_2(\sigma-\rho)}{(2(\sigma-\rho))!} \left( \frac{\pi}{2} \right)^{2\sigma+1-2\rho}
= (-1)^{\mu-\sigma} \frac{(2\sigma)!}{(2\mu)!} \sum_{\rho=0}^{\min(\sigma,\mu)} G_\mu^\rho \frac{E_2(\sigma-\rho)}{(2(\sigma-\rho))!} \left( \frac{\pi}{2} \right)^{2\sigma+1-2\rho},
\]

on using the known values of the \( \beta \)-function in terms of Euler numbers. This is the final answer. The upper limit can be replaced by just \( \mu \).

I now show that the same result can be obtained in an equivalent but differently organised manner by applying partial integration.

This time the \( G_\mu^\rho \) coefficients arise from the old, recursion–derived relation,

\[
\sech^{2\mu+1} \xi = P(D^2) \sech \xi
= \left( \frac{(-1)^\mu}{(2\mu)!} \sum_{\rho=0}^{\mu} (-1)^\rho G_\mu^\rho \frac{d^{2\rho}}{d\xi^{2\rho}} \right) \sech \xi,
\]

for odd powers in terms of even derivatives.

Next we evaluate the integral

\[
I(\sigma, \rho) \equiv \int_0^\infty d\xi \xi^{2\sigma} \frac{d^{2\rho}}{d\xi^{2\rho}} \sech \xi.
\]
by partial integration. For $\sigma < \rho$,
\[
I(\sigma, \rho) = (2\sigma)! \int_0^\infty d\xi \left( \frac{d}{d\xi} \right)^{2\rho - 2\sigma} \text{sech} \xi \\
= (2\sigma)! \left. \left( \frac{d^{2\rho - 2\sigma - 1}}{d\xi^{2\rho - 2\sigma - 1}} \text{sech} \xi \right) \right|_0^\infty \\
= 0
\]
which can be seen from the oddness of the function.

Next, for $\sigma \geq \rho$,
\[
I(\sigma, \rho) = \frac{(2\sigma)!}{(2\sigma - 2\rho)!} \int_0^\infty d\xi \xi^{2\rho - 2\sigma} \text{sech} \xi \\
= \frac{(2\sigma)!}{(2\sigma - 2\rho)!} (-1)^{\sigma - \rho} \left( \frac{\pi}{2} \right)^{2\rho + 1} E_{2\rho - 2\rho}
\]
in terms of Euler numbers, $E_n$. This is zero for $\sigma < \rho$ and so can be taken as generally true.

Combining (12), (17) and (18) yields,
\[
J(\sigma, \mu) = \frac{(-1)^{\mu}}{(2\mu)!} \sum_{\rho=0}^{\mu} (-1)^{\rho} G^\mu_\rho \cdot I(\sigma, \rho) \\
= \frac{(2\sigma)!}{(2\mu)!} (-1)^{\mu - \sigma} \sum_{\rho=0}^{\mu} \left( \frac{\pi}{2} \right)^{2\rho + 1} G^\mu_\rho \frac{E_{2\rho - 2\rho}}{(2\sigma - 2\rho)!},
\]
which is the same as (16). The values of the $\beta$ function are not required in this simpler approach and could, therefore, be derived.

The equations are now in a form amenable to machine algebra with no integration required.

References.
1. Perlmutter,E. A universal feature of CFT Rényi entropy ArXiv:1308.1083
2. Lee,J., Lewkowicz,A., Perlmutter,E. and Safdi,B.R. Rényi entropy. stationarity and entanglement of the conformal scalar ArXiv:1407.7816.
3. Hung,L.Y., Myers,R.C. and Smolkin,M. Twist operators in higher dimensions ArXiv:1407.6429.
4. Callan,C.G. and Wilczek,F. Phys. Letts. B333 (1994) 55.
5. Candelas,P. and Dowker,J.S. Phys. Rev. D19 (1979) 2902.
6. Klebanov, I.R., Pufu, S.S., Sachdev, S. and Safdi, B.R. *JHEP* 1204 (2012) 074.
7. Dowker, J.S. *J. Phys. A: Math. Theor.* 46 (2013) 2254.
8. Milgram, M.S., Journ. Maths. (Hindawi) 2013 (2013) 181724.
9. Dowker, J.S. *Entanglement entropy on odd spheres*. ArXiv:1012.1548.
10. Dowker, J.S. *Phys. Rev.* D36 (1987) 3095.
11. Aros, R., Bugini, F. and Diaz, D.E. *On the Renyi entropy for free conformal fields: holographic and q–analog recipes*. ArXiv:1408.1931.
12. Steffensen, J.F. *Interpolation*, (Williams and Wilkins, Baltimore, 1927).
13. Thiele, T.N. *Interpolationsrechnung* (Teubner, Leipzig, 1909).
14. Dowker, J.S. *Central Differences, Euler numbers and symbolic methods* ArXiv: 1305.0500.
15. Riordan, J. *Combinatorial Identities* (Wiley, New York, 1968).
16. Butzer, P.L., Schmidt, M., Stark, E.L. and Vogt, I. *Numer. Funct. Anal. Optim.* 10 (1989) 419.