EQUIDISTRIBUTION OF FEKETE POINTS ON COMPLEX MANIFOLDS

ROBERT BERMAN, SÉBASTIEN BOUCKSOM

Abstract. We prove the several variable version of a classical equidistribution theorem for Fekete points of a compact subset of the complex plane, which settles a well-known conjecture in pluripotential theory. The result is obtained as a special case of a general equidistribution theorem for Fekete points in the setting of a given holomorphic line bundle over a compact complex manifold. The proof builds on our recent work "Capacities and weighted volumes for line bundles".

1. Introduction

A classical potential theoretic invariant of a compact set $E$ in the complex plane $\mathbb{C}$ is given by its transfinite diameter:

$$d_{\infty}(E) := \lim_{k \to \infty} \left( \sup_{z_0, \ldots, z_k \in E} \prod_{0 \leq i < j \leq k} |z_i - z_j| \right)^{2/k(k-1)}. \quad (1.1)$$

i.e. the asymptotic geometric mean distance of points in $E$. A configuration $z^{(k)} \in E^{N_k}$ of points achieving the supremum for a fixed $k$ is called a $k$-Fekete configuration for $E$ (in classical terminology the corresponding points $(z_i^{(k)})$, for $k$ fixed, are called Fekete points). A basic classical theorem (see [9] for a modern reference and [6] for the relation to Hermitian random matrices) asserts that Fekete configurations equidistribute on the equilibrium measure $\mu_E$ of $E$, i.e.

$$\frac{1}{k+1} \sum_{i=1}^{N_k} \delta_{z^{(k)}} \to \mu_E$$

as $k \to \infty$. In the logarithmic pluripotential theory in $\mathbb{C}^n$ a higher dimensional version of the transfinite diameter was introduced by Leja in 1959

$$d_{\infty}(E) := \limsup_{k \to \infty} \left( \sup_{z_1, \ldots, z_{N_k} \in E} |\Delta(z_1, \ldots, z_{N_k})| \right)^{(n+1)!/nk^{n+1}}, \quad (1.2)$$

where $\Delta(z_1, \ldots, z_{N_k})$ is the following higher-dimensional Vandermonde determinant:

$$\Delta(z_1, \ldots, z_{N_k}) := \det(e^{\alpha}(z_j))_{|\alpha| \leq k, 1 \leq j \leq N_k},$$

where $e^{\alpha}(x) = x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \alpha \in \mathbb{N}^n$ denote the monomials and $N_k$ is the number of monomials of degree at most $k$ (so that $N_k = k^n/n! + O(k^{n-1})$). It was
shown by Zaharjuta [14] that the limsup in formula (1.2) is actually a true limit, thereby answering a conjecture by Leja. However, the corresponding convergence of higher-dimensional “Fekete configurations” (towards the pluripotential theoretic equilibrium measure of $E$) has remained an open problem (see the survey [8] by Levenberg on approximation theory in $\mathbb{C}^n$, where it is pointed out (p. 120) that “to this date, nothing is known if $n > 1$”).

There is also a weighted version of the previous setting where the set $E$ is replaced by the weighted set $(E, \phi)$, with $\phi$ a continuous function on the compact set $E$ (see the appendix by Bloom in [9]). The weighted transfinite diameter $d_\infty(E, \phi)$ is then obtained by replacing $|\Delta(z_1, \ldots, z_N)|$ in (1.2) by its weighted counterpart

$$|\Delta(z_1, \ldots, z_N)|e^{-k(\phi(z_1)+\ldots+\phi(z_N))}.$$

Even more generally, there is a variant of this weighted setting where $E$ may be assumed to be an unbounded set in $\mathbb{C}^n$ provided that $\phi$ growths sufficiently fast at infinity (cf. the appendix by Bloom in [9]).

The aim of this note is to prove a generalized version of the conjecture referred to above in the more general setting of a big line bundle $L$ over a complex manifold $X$ (recall that from an analytic perspective $L$ is big iff it admits a singular Hermitian metric with strictly positive curvature current). In this setting the role of $e^{-\phi}$ is played by a Hermitian metric on $L$ (we will call the additive object $\phi$ a weight - see [3] for further notation). A weighted subset $(E, \phi)$ will thus consist of a subset $E$ of $X$ and a continuous weight on $L$. The “classical setting” above corresponds to the hyperplane line bundle $\mathcal{O}(1)$ over the complex projective space $\mathbb{P}^n$ with $E$ a compact set in the affine piece $\mathbb{C}^n$ and the space $H^0(kL)$ of global holomorphic sections with values in $kL$ (the $k$-th tensor power of $L$, written in additive notation) may in this case be identified with the space of all polynomials on $\mathbb{C}^n$ of total degree at most $k$.

In order to state the theorem we first recall some further notation, mostly taken from [3]. Fix a compact subset $E$ of $X$ which is not locally pluripolar and a continuous weight $\phi$ on the line bundle $L \to X$. The equilibrium weight of $(E, \phi)$ is defined by

$$\phi_E = \sup \{ \varphi, \varphi \text{ psh weight on } L, \varphi \leq \phi \text{ on } E \},$$

where a weight $\varphi$ on $L$ is called psh if its curvature $dd^c \varphi$ is a positive current. In the $\mathbb{C}^n$-case above, $\phi_E$ is usually referred to as the weighted Siciak extremal function of $(E, \phi)$ (cf. e.g. the appendix by Bloom in [9]). The equilibrium measure of the weighted set $(E, \phi)$ is then defined as the Monge-Ampère measure

$$\mu_{(E, \phi)} := MA(\phi^*_E),$$

of the psh weight with minimal singularities $\phi^*_E$ (the usc envelope of $\phi_E$), that is the trivial extension to $X$ of the Bedford-Taylor wedge-product $(dd^c \phi^*_E)^n$ computed on a Zariski open subset where $\phi^*_E$ is locally bounded (see [3]).

Next let $(s_1, \ldots, s_N)$ be a basis of $H^0(L)$ and consider the following Vandermonde-type determinant

$$\det(s)(x_1, \ldots, x_N) := \det(s_i(x_j))_{i,j}$$
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which is a holomorphic section of the pulled-back line bundle $L^\otimes N$ over the $N$-fold product $X^N$.

A configuration of points $P = (x_1, ..., x_N) \in E^N$ is called a Fekete configuration for the weighted subset $(E, \phi)$ if it realizes the supremum on $E^N$ of the pointwise length $|\det(s)|_\phi$ of $\det(s)$ with respect to the metric induced by $\phi$. It is a basic observation that the definition of a Fekete configuration is actually independent of the choice of the basis $(s_i)$. Indeed, since the top exterior power of the vector space $H^0(L)$ is one-dimensional, replacing $s_i$ by $s'_i$ gives $\det(s') = c \det(s')$, where $c$ is a non-zero complex number.

As a last piece of notation, given a configuration $P = (x_1, ..., x_N) \in X^N$, we will denote by $\delta_P := 1/N \sum_{i=1}^{N} \delta x_i$ the associated probability measure.

**Theorem 1.1.** Let $L \to X$ be a big line bundle over a compact complex manifold. Assume that $P_k \in X^{N_k}$ is a Fekete configuration for $(E, k\phi)$ for each $k$ large enough. Then this sequence of configurations equidistributes towards the equilibrium measure of $(E, \phi)$, i.e.

$$\lim_{k \to \infty} \delta_{P_k} = M^{-1} \mu_{(E, \phi)}$$

weakly as measures on $X$, where $M$ is the total mass of $\mu_{(E, \phi)}$.

Here $N_k := h^0(kL) \simeq \text{vol}(L)k^n/n!$, where $\text{vol}(L) > 0$ is the volume of the big line bundle $L$, which is also equal to the total mass $M$ of $\mu_{(E, \phi)}$ (see [3]).

In the one dimensional case, i.e. when $X$ is a complex curve and if $(E, \phi)$ is taken as $(E, 0)$ where $E$ is a compact set contained in an affine piece of $X$ where $L$ has been trivialized, the theorem was obtained, using completely different methods, by Bloom and Levenberg [5] (see also [7] for the case when $X$ has genus zero and for a discussion of related interpolation problems in $\mathbb{R}^n$).

The proof of Theorem 1.1 builds on our previous work [3], where it was shown that (minus the logarithm of) the transfinite diameter of $(E, \phi)$, considered as a functional on the affine space of all continuous weights on $L$ (for $E$ fixed) is Fréchet differentiable: its differential at the weight $\phi$ may be represented by the corresponding equilibrium measure $\mu_{(E, \phi)}$. A similar argument was used very recently by the first author and Witt Nyström in [4] to obtain very general convergence results for “Bergman measures” (corresponding to Christoffel-Darboux functions in classical terminology). In fact, a unified treatment of these two convergence results can be given, which also includes the equidistribution of generic points of “small height” on an arithmetic variety obtained in [3] (which in turn generalizes Yuan’s arithmetic equidistribution theorem [10]). This will be further investigated elsewhere.

**Remark 1.2.** In the case that $L$ is ample the differentiability property referred to above can also be deduced from the Bergman kernel asymptotics for smooth weights in [2] as explained in section 1.4 in [3] (see also remark 11.2 in [3]). For
an essentially elementary proof of these asymptotics in the weighted case in $\mathbb{C}^n$ see [1].

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1.1. **Proof of Theorem 1.1.** Given a basis $s = (s_1, ..., s_N)$ of $H^0(L)$, set

$$D_\phi := \log |\det(s)|_{\phi},$$

so that $D_\phi$ achieves its maximum on $E^N$ exactly at Fekete configurations of $(E, \phi)$ by definition. As noticed above, $D_\phi$ only depends on the choice of the basis $s$ up to an additive constant. If $P = (x_1, ..., x_N) \in X^N$ is a given configuration, the more explicit formula

$$D_\phi(P) = \log |\det(s_i(x_j)) - (\phi(x_1) + ... + \phi(x_N))|$$

clearly shows that $\phi \mapsto D_\phi(P)$ is an affine function, with linear part given by integration against $-N\delta_P$.

In order to normalize $D_\phi$, we fix an auxiliary (smooth positive) volume form $\mu$ on $X$ and smooth metric $e^{-\psi}$ on $L$ and take $(s_1, ..., s_N)$ to be an orthonormal basis of $H^0(L)$ with respect to the corresponding $L^2$ scalar product. It is easily seen (cf. [3]) that $D_\phi$ becomes independent of the choice of such an orthonormal basis. The main result of [3] implies that

$$\lim_{k \to \infty} \frac{(n+1)!}{k^{n+1}} \sup_{E_{N_k}} D_{k\phi} = \mathcal{E}(\psi_X, \phi^*_E)$$

where $\psi_X$ is the equilibrium weight of $(X, \psi)$ and $\mathcal{E}$ denotes the Aubin-Yau energy, whose precise formula doesn’t matter here. The main point for what follows is that

$$\phi \mapsto \mathcal{E}(\psi_X, \phi^*_E)$$

is Fréchet differentiable on the space of all continuous weights on $L$, with derivative at $\phi$ in the tangent direction $\nu \in C^0(X)$ given by

$$(n+1) \int_X \nu \mu(E, \phi).$$

This is indeed the content of Theorem 5.7 of [3]. Now let $P_k \in E^{N_k}$ be a sequence of configurations, and set

$$F_k(\phi) := \frac{1}{kN_k} D_{k\phi}(P_k)$$

and

$$G(\phi) := \frac{1}{(n+1)M} \mathcal{E}(\phi^*_E, \psi_X).$$

We thus see that

$$\liminf_{k \to \infty} F_k(\phi) \geq G(\phi),$$
and furthermore
\[ \lim_{k \to \infty} F_k(\phi) = G(\phi) \]
if \( P_k \in E^{N_k} \) is a Fekete configuration for \((E,k\phi)\), as follows from (1.5) and the fact that \( N_k \simeq M k^n / n! \).

As noticed above, the functional \( F_k \) is affine, with linear part given by integration against \( \delta_{P_k} \). On the other hand the differentiability property of the energy writes
\[
\frac{d}{dt}_{t=0} G(\phi + tv) = M^{-1} \int_X v \mu(E,\phi).
\]

The proof of Theorem 1.1 is thus concluded by the following elementary result applied to \( f_k(t) := F_k(\phi + tv) \) and \( g(t) := G(\phi + tv) \), taking \( P_k \in E^{N_k} \) to be a Fekete configuration for \((E,k\phi)\) as in the Theorem.

**Lemma 1.3.** Let \( f_k \) be a sequence of concave functions on \( \mathbb{R} \) and let \( g \) be a function on \( \mathbb{R} \) such that
- \( \lim \inf_{k \to \infty} f_k \geq g \).
- \( \lim_{k \to \infty} f_k(0) = g(0) \).

If the \( f_k \) and \( g \) are differentiable at 0, then
\[
\lim_{k \to \infty} f'_k(0) = g'(0).
\]

**Proof.** Since \( f_k \) is concave, we have
\[
f_k(0) + f'_k(0)t \geq f_k(t)
\]
and it follows that
\[
\lim \inf_{k \to \infty} tf'_k(0) \geq g(t) - g(0).
\]
The result now follows by first letting \( t > 0 \) and then \( t < 0 \) tend to 0. \( \square \)

The same lemma underlies the proof of Yuan’s equidistribution theorem given in [3]. It is in fact inspired by the variational principle in the original proof by Szpiro-Ullmo-Zhang. The case of concave functions \( f_k \) pertains to the situation considered in [4].

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Université Grenoble I, Institut Fourier, Saint-Martin d’Hères, France  
E-mail address: robertb@math.chalmers.se

CNRS-Université Paris 7, Institut de Mathématiques, F-75251 Paris Cedex 05, France  
E-mail address: boucksom@math.jussieu.fr