Note on equivalence of cutpoint languages recognized by measure many quantum finite automata*

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Abstract

This note revisits the equivalence of languages recognized by measure many one way quantum finite automata with non/strict cutpoint. The main contributions are as follows:

(1) We provide an additional proof of the undecidability of non/strict emptiness of measure many one way quantum finite automata;

(2) By the undecidability of non/strict emptiness of measure many one way quantum finite automata, we show that the equivalence of languages recognized by measure many one way quantum finite automata with non/strict cutpoint is undecidable, implying the undecidability of containment problem of measure many one way quantum finite automata

Keywords: Measure many one way quantum finite automata, Equivalence of cut-point language, Emptiness, Containment, Undecidability

1 Introduction

Some decision problems are important and interesting in the mathematical theory of automata, for example, the emptiness problem, the equivalence problem, the minimization problem, which have received sufficient attention, see [1] by Rabin et al., [26] by Hopcroft et al., [24] by Paz and [25] by Eilenberg, and [27] by Sénizergues.

The quantum generalization of classical finite automata, whose definitions were first appeared in [23] by Knodacs et al. and [29] by Moore et al.,—quantum finite automata—employ quantum mechanism to control their behaviors. Many interesting properties about quantum finite automata have been discovered [10, 9, 8, 6, 4, 3, 21, 2, 14, 5, 7]. For instance, Yakaryılmaz et al. have shown that measure many one way quantum finite automata recognize all and only the stochastic languages in the unbounded error setting [3, 4]; And, Hirvensalo, who also introduced in [14] a model for finite automata with an open quantum evolution, whose basic properties are studied in the same article, has examined various aspects of quantum finite automata in [16]; Brodsky et al. have obtained some characterizations of one way quantum finite automata in [9]; It has been observed in [10] by Ambainis et al. that one way quantum finite automata can be very space-efficient.

In this note, we revisit equivalence of languages recognized by measure many one way quantum finite automata with non/strict cutpoint. This problem was first considered in [18] but without proof. In fact, the statement in [18] is incorrect and has already been found.† We have to say that this note benefits, in order of time, very much from Dr. Yakaryılmaz [17] and a reader of a journal.

The techniques used to deal with the equivalence of non/strict languages recognized by quantum finite automata are undecidability of non/strict emptiness of measure many one way quantum

*The objective is to discard a incorrect corollary, i.e. Corollary 3 in [18], and to suggest a corrected one.
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‡Originally, this problem was a modification of THEOREM 8.4 in [25] by Eilenberg (See, p. 146 of [25]), but without careful examining the difference between two definitions, which is the author’s mishandling. And, before [18] has been formally published, the author had deleted it in the updated version [19].
finite automata and constructing of special measure many one way quantum finite automata. It has been observed in [22, 21] that both nonstrict and strict emptiness of measure many one way quantum finite automata are shown to be undecidable in [20]. Here, we provide an additional proof of them which is due to Yakaryilmaz [17] and the author. Afterwards, we use these results to show, together with extra tool, that the problems of equivalence of languages recognized by measure many one way quantum finite automata with non/strict cutpoint are undecidable. And, some obvious consequence of the above results have been also summarized.

The rest of the note is structured in the following way. In Section 2, some basic notations and concepts are reviewed; The additional proof of undecidability of non/strict emptiness of measure many one way quantum finite automata are shown to be undecidable in Section 3; Section 4 contains our main purpose, where we show that the equivalence of languages recognized by measure many one way quantum finite automata with non/strict cutpoint is undecidable. We draw some conclusions in Section 5.

2 Preliminaries

For any finite set $S$, $|S|$ denotes the cardinality of $S$. Throughout this paper, $\Sigma$ denotes the non-empty finite alphabet, $\Sigma^*$ denotes the set of all finite words (including empty word $\epsilon$) over $\Sigma$, and $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$. Let $w$ be a word in $\Sigma^*$, then $|w|$ will denote the length of $w$. For example, let $\Sigma = \{0, 1\}$, then $|\epsilon| = 0$ and $|001101| = 6$.

We refer the reader to [13, 11, 12] for more quantum theories. In quantum theory, the unit-length column (resp. row) vector in a fixed (finite dimensional) Hilbert Space $\mathcal{H}$ will be denoted as $|\varphi\rangle$ (resp. $\langle\varphi|$). $|\varphi\rangle$ and $\langle\varphi|$ are satisfied the relation $|\varphi\rangle^\dagger = \langle\varphi|$ where $^\dagger$ denotes the conjugate-transpose of complex matrices. The inner product of two vectors $|\varphi\rangle$ and $|\psi\rangle$ is denoted as $\langle\varphi|\psi\rangle$. The length of the vector $|\varphi\rangle$, denoted by $||\varphi||$, is defined to be $||\varphi|| = \sqrt{\langle\varphi|\varphi\rangle}$.

2.1 Definitions

In the following, we recall the formal definitions of probabilistic automata (p.a.), measure once one way quantum finite automata (MOQFAs) and measure many one way quantum finite automata (MMQFAs).

**Definition 2.1.** A probabilistic automaton (p.a.) over an alphabet $\Sigma$ is a triplet $(x, \{M_a|a \in \Sigma\}, y)$, where $y \in \mathbb{R}^n$ is the initial probability distribution, each $M_a \in \mathbb{R}^{n \times n}$ is a Markov matrix, and $y \in \mathbb{R}^n$ is the final state vector whose $i$th coordinate is 1, if the $i$th state is final, and 0 otherwise. The probability accepting word $\omega \in \Sigma^*$ for $A$ is described by

$$P_A(\omega) = x^T M_a_1 \cdots M_a_k y.$$ 

**Definition 2.2.** An measure once one way quantum finite automaton (MOQFA) $A$ is a tuple $(Q, \Sigma, |\pi\rangle; \{U_\sigma\}_{\sigma \in \Sigma}, P_a)$ where $Q = \{q_1, \cdots, q_Q\}$ is the basic state set, $\Sigma$ the input alphabet, $|\pi\rangle$ the initial state vector with $||\pi|| = 1$, $U_\sigma$ an unitary matrix of dimension $|Q|$. The probability accepting word $\omega = x_1 \cdots x_n \in \Sigma^*$ is given by

$$P_A(\omega) = \|P_a U_{x_n} \cdots U_{x_1} |\pi\rangle\|^2$$

**Definition 2.3** (Modification of [23] by Knodacs et al.). An MMQA is an element of subset of all 2qfa’s, which is given by the tuple

$$A = (Q, \Sigma, \{U_\sigma\}_{\sigma \in \Sigma \cup \{\ell, \epsilon\}, q_0, Q_{acc}, Q_{rej}})$$

where $Q_0$ is the basic state set of $A$, $\Sigma$ the input alphabet, $\delta$ the transition function, $q_0$ the initial state, and $Q_{acc}$ and $Q_{rej}$ the accepting set and rejecting set, respectively. The probability accepting any word $\omega = x_1 \cdots x_n \in \Sigma^*$ is as follows

$$P_A(\omega) = \sum_{k=0}^{n+1} \|P_a U_{x_k} \left( \prod_{j=k-1}^0 (P_g U_{x_j}) \right) |q_0\|^2$$

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where $x_0 = 'L'$, $x_{n+1} = '$'$ and

$$P_a = \sum_{q \in Q_{acc}} |q\rangle \langle q|, \quad P_g = \sum_{q \in Q \setminus (Q_{acc} \cup Q_{rej})} |q\rangle \langle q|.$$  

The formal product $\prod_{i=0}^{n} M_i$ is defined to be $M_k \cdots M_0$.

Remark 1. This definition is equivalent to the one given in [18] which has no left end-marked ‘L’ due to a result in [9] by Brodsky et al. (Cf. Theorem A.2. in [9]). For convenience to prove Proposition 3.1, we adopt the above definition. In what follows, we often replace the initial state $q_0$ with an unit vector $|q_0\rangle$, which is equivalent.

2.2 Statement of problems

Now we proceed to define the languages recognized by $A$--a probabilistic automaton, or a measure over many one way quantum finite automata over the input alphabet $\Sigma$--with non/strict cutpoint $\lambda$. Then, identical with [16], we define

$$L_{> \lambda}(A) \triangleq \{ \omega \in \Sigma^* | \mathcal{P}_A(\omega) > \lambda \}$$  

and

$$L_{\geq \lambda}(A) \triangleq \{ \omega \in \Sigma^* | \mathcal{P}_A(\omega) \geq \lambda \}$$

where $\mathcal{P}_A(\omega)$ denote the probability of $A$ accepting the word $\omega$ and $0 < \lambda \leq 1$.

We call the language of $L_{\geq \lambda}(A)$ a nonstrict cutpoint language recognized by $A$, and $L_{> \lambda}(A)$ a strict cutpoint language recognized by $A$.

The non/strict emptiness problem of $A$ is to ask whether there is algorithm deciding $L_{\geq \lambda}(A) = \emptyset$? $L_{> \lambda}(A) = \emptyset$?. And, given two automata $A_1$ and $A_2$ with the same type, the equivalence of non/strict cutpoint language is to ask whether there exists algorithm deciding $L_{> \lambda}(A_1) = L_{> \lambda}(A_2)$? with $\succ$ is $\geq, >$. In similarity to [28], we define the non/strctic containment problem as to ask whether there exists algorithm deciding $L_{> \lambda_1}(A_1) \succ L_{> \lambda_2}(A_2)$ where $\succ \in \{ >, \geq \}$ and $\succ \in \{ \subseteq, \subset \}$.

2.3 Basic facts

We now list some basic facts which are useful in the sequel.

Lemma 2.1. ([24], see also [15]) Given a p.a. $A$, both the nonstrict and strict emptiness of it are undecidable.

Lemma 2.2. [22, Corollary 2.4] For any rational $\lambda$, $0 < \lambda \leq 1$, there is no algorithm that decides if a given quantum automata has a word $\omega$ for which $\mathcal{P}_A(\omega) \geq \lambda$.

3 Undecidability of emptiness problem of MMQFAs

We first show an auxiliary lemma, which is useful as we can see in the sequel.

Proposition 3.1. Let $\mathcal{M} = (\mathcal{Q}, \Sigma, \{ U_{\sigma} \}_{\sigma \in \Sigma}, |\pi\rangle, P_a)$ be an arbitrary MOQFA. Then there exists an MMQFA $\mathcal{A} = (\mathcal{Q}', \Sigma, \{ U_{\sigma}' \}_{\sigma \in \Sigma}, \{ q_0' \}, Q_{acc}', Q_{rej}')$ such that

$$\mathcal{P}_A(\omega) = \mathcal{P}_M(\omega)$$

for all $\omega \in \Sigma^*$.

Both $L_{> \lambda}(A_1) \subseteq L_{> \lambda}(A_2)$ and $L_{\geq \lambda}(A_1) \subseteq L_{\geq \lambda}(A_2)$ are called nonstrict containment. Similarly, both $L_{> \lambda}(A_1) \subset L_{> \lambda}(A_2)$ and $L_{\geq \lambda}(A_1) \subset L_{\geq \lambda}(A_2)$ are called strict containment.
**Proof.** Suppose that $n = |Q|$. Let

$$U'_\sigma = \begin{pmatrix} I_n & 0 & 0 \\ 0 & U_\sigma & 0 \\ 0 & 0 & I_n \end{pmatrix}$$

for any $\sigma \in \Sigma$ and $U'_E = U'_S = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_n \end{pmatrix}$.

Further, let

$$P'_a = \begin{pmatrix} P_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad P'_g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $P'_r = I_{3n} - (P'_a + P'_g)$, where $|q_0'\rangle = (\langle \pi |, 0, 0)^\dagger$, $Q'_{acc}$ and $Q'_{acc}$ is defined to be satisfied the conditions $P'_a = \sum_{q \in Q'_{acc}} |q\rangle\langle q|$ and $P'_r = \sum_{q \in Q'_{acc}} |q\rangle\langle q|$ and where $P'_r = \text{diag}(I_n, I_n, I_n) - P'_a - P'_g$.

Then we have

$$\mathcal{P}_A(x_1 x_2 \cdots x_n) = \sum_{k=0}^{n+1} \left\| P'_a U'_x \left( \prod_{i=k-1}^{0} (P'_g U'_x) \right) |q_0'\rangle \right\|^2$$

where $x_0 = \mathcal{L}$ and $x_{n+1} = \mathcal{S}$.

Note that

$$P'_a U'_\sigma = \begin{pmatrix} P_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & U_\sigma & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} P_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$P'_g U'_\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & U_\sigma & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & U_\sigma & 0 \\ 0 & 0 & I \end{pmatrix}$$

Thus, for $k \neq n + 1$ we see that

$$\left\| P'_a U'_\sigma \left( \prod_{i=k-1}^{0} (P'_g U'_x) \right) |\pi\rangle \right\|^2 = \left\| 0 \right\|^2 = 0, \quad k = 0; \tag{1}$$

$$\left\| P'_a U'_\sigma \left( \prod_{i=k-1}^{0} (P'_g U'_x) \right) |q_0'\rangle \right\|^2 = \left\| 0 \right\|^2 = 0, \quad 0 < k \leq n.$$

For $k = n + 1$, we get

$$\left\| P'_a U'_\sigma \left( \prod_{i=k-1}^{0} (P'_g U'_x) \right) |\pi\rangle \right\|^2 = \left\| P'_a U_{x_1} \cdots U_{x_n} |\pi\rangle \right\|^2 = \mathcal{P}_M(x_1 \cdots x_n) \tag{2}$$

From Eq. (1) and Eq. (2) we have

$$\mathcal{P}_A(x_1 \cdots x_n) = \sum_{k=0}^{n+1} \left\| P'_a U'_\sigma \left( \prod_{i=k-1}^{0} (P'_g U'_x) \right) |q_0'\rangle \right\|^2$$

$$= 0 + \cdots + 0 + \left\| P'_a U_{x_1} \cdots U_{x_n} |\pi\rangle \right\|^2$$

$$= \mathcal{P}_M(x_1 \cdots x_n). \quad \square$$

**Remark 2.** The proof of the above proposition essentially utilizes the idea of proof of Theorem 4 in [18]: constructing a special MMQFA $A$ such that the word function $\mathcal{P}_A(x_1, \cdots, x_n)$ is non-cumulative.
3.1 Non-strict emptiness

In this section, we provide an additional proof about the undecidability of emptiness of MMQFA. See Ref. [20] for original proof due to E. Jeandel.

In fact, by Lemma 2.2 in Section 2 and Proposition 3.1 developed in previous section, the following corollary is very clear.

**Corollary 3.1.** Let $A$ be an MMQFA over $\Sigma$. Then, for any rational $\lambda$, $0 < \lambda \leq 1$, there is no algorithm that decides if $A$ has a word $\omega$ for which $P_A(\omega) \geq \lambda$.

**Remark 3.** Also, this result can be derived from the undecidability of emptiness of p.a. and the result of [4], as observed in [17] by Dr. Abuzer.

3.2 Strict emptiness

The idea to the undecidability of strict emptiness of MMQFA is due to Yakaryilmaz [17]. We first list the following

**Lemma 3.1.** ([4, 3]) Any language recognized with cutpoint(or nonstrict cutpoint) $\frac{1}{2}$ RT-PFA (i.e. probabilistic finite automata) with $n$ internal states can be recognized with cutpoint (or nonstrict cutpoint) $\frac{1}{2}$ by a RT-KWQFA (i.e. measure many one way quantum finite automata) with $O(n)$ internal states.

By Lemma 3.1 and Lemma 2.1 in Section 2, the following is derived

**Corollary 3.2.** Let $A$ be an MMQFA over $\Sigma$. Then, for any rational $\lambda$, $0 < \lambda \leq 1$, it is undecidable that if $A$ has a word $\omega$ for which $P_A(\omega) > \lambda$.

4 Equivalence of non/strict cutpoint languages

This section, in which we will correct Corollary 3 in [18], is our main purpose, see the introduction in Section 1.

4.1 The nonstrict case

By Proposition 3.1, we only have to show the equivalence of languages recognized by MOQFAs with nonstrict cutpoint is undecidable.

By Lemma 2.2, it is undecidable whether or not $L_{=\lambda}(A) = \Sigma^*$ where $A$ is an MOQFA.

Then we construct a MOQFA

$$M = (\{q_1, q_2\}, \{I\}_{\sigma \in \Sigma}, P_a = \text{diag}(1, 0, 0), |\pi\rangle = (\sqrt{\lambda}, \sqrt{1 - \lambda}, 0)^T)$$

with $P_M(\omega) = \lambda$ hence $L_{\geq \lambda}(A) = \Sigma^*$. This implies

**Proposition 4.1.** Give two MOQFAs $A_i$, $i = 1, 2$, it is undecidable that whether $L_{\geq \lambda}(A_1) = L_{\geq \lambda}(A_2)$.

**Proof.** The proof is similar to that of Theorem 8.12 in [26] (Cf. p. 203). Obviously, an algorithm for deciding whether $L_{\geq \lambda}(A_1) = L_{\geq \lambda}(A_2)$, will lead to an algorithm for deciding whether $L_{\geq \lambda}(A) = \Sigma^*$. □

4.2 The strict case

We first present the following useful

**Example.** Let $0 < \lambda \leq 1$ and $c$: $0 < c < \lambda$. Let $A = (Q, \{I\}_{\sigma \in \Sigma}, P_a, |q_1\rangle)$ be an MOQFA where

$$Q = \{q_1, q_2, q_3\}, \quad P_a = \text{diag}(1, 0, 0) \quad \text{and} \quad |q_1\rangle = (\sqrt{\lambda - c}, \sqrt{1 - \lambda + c}, 0)^T$$
Then, it is clear that for all \( \omega \in \Sigma^* \)

\[
P_A(\omega) = \|P_aU_\omega|q_1\rangle\|_2^2 = \lambda - c < \lambda
\]

Thus, \( L_{>\lambda}(A) = \{ \omega \in \Sigma^* | P_A(\omega) > \lambda \} = \emptyset \).

**Remark 4.** By Proposition 3.1, we have an MMQFA over \( \Sigma \), say \( A' \), such that for all \( \omega \in \Sigma^* \),

\[
P_{A'}(\omega) = P_A(\omega)
\]

which means that \( \{ \omega \in \Sigma^* | P_{A'}(\omega) > \lambda \} = \emptyset \).

The approach to the following proposition attributes to E. Jeandel [17].

**Proposition 4.2.** Let \( A_i, i = 1, 2 \), be two MMQFAs over \( \Sigma \). Then it is undecidable that \( L_{>\lambda}(A_1) = L_{>\lambda}(A_2) \).

**Proof.** Let \( M \) be arbitrary an MMQFA over \( \Sigma \). Assume it is decidable that \( L_{>\lambda}(A_1) = L_{>\lambda}(A_2) \), then an algorithm for deciding whether \( \{ \omega \in \Sigma^* | P_M(\omega) > \lambda \} = \emptyset \) would follows because we can first construct an MMQFA \( M' \) such that \( L_{>\lambda}(M') = \emptyset \) due to Remark 4, and then decide whether or not \( L_{>\lambda}(M) = L_{>\lambda}(M') \). Contradict to Corollary 3.2. \( \square \)

**Remark 5.** In fact, the method coping with Proposition 4.2 is also applicable to the nonstrict case.

By Proposition 4.1 and Proposition 4.2, we can correct Corollary 3 in [18] as follows

**Corollary 4.1.** Given two MMQAs (or EQFAs\(^3\)) \( A_1 \) and \( A_2 \) over \( \Sigma \), it is undecidable whether or not \( L(A_1) = L(A_2) \) (i.e. both \( L_{>\lambda}(A_1) = L_{>\lambda}(A_2) \) and \( L_{>\lambda}(A_1) = L_{>\lambda}(A_2) \)). \( \square \)

The above corollary obviously implies

**Corollary 4.2.** Given two MMQAs (or EQFAs) \( A_1 \) and \( A_2 \) over \( \Sigma \), it is undecidable whether or not \( L_{>\lambda}(A_1) \subseteq L_{>\lambda}(A_2) \) where \( \gg \in \{>, \geq\} \). \( \square \)

Further, the technique of proving Proposition 4.2 is applicable to the following

**Corollary 4.3.** Given two MMQAs (or EQFAs) \( A_1 \) and \( A_2 \) over \( \Sigma \), it is undecidable whether or not \( L_{>\lambda}(A_1) \subset L_{>\lambda}(A_2) \) where \( \gg \in \{>, \geq\} \). \( \square \)

### 5 Conclusion

In this note, we have revisited the emptiness and equivalence of languages of measure many one way quantum finite automata. Specially, we have provided an additional proof of undecidability of the emptiness of measure many quantum finite automata; And then we show that both the equivalence of languages recognized by measure many one way quantum finite automata with non/strict cutpoint are undecidable. These results further imply that the containment problem of measure many quantum finite automata are undecidable. It is unknown, to the author's knowledge, whether it is decidable that the non/strict cutpoint language recognized by MMQFA is finite? In other words, whether there is a “pumping lemma” for MMQFAs? Note that MOQFAs accepts only subset of regular languages [9, 7], which have a pumping lemma.

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\(^3\) MMQFAs can be see as a special kind of EQFAs, see definition in [8] by Nayka.
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