A Polynomial Algorithm for Some Relaxed Subset Sum Problems with Real Numbers

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In this paper we study the subset sum problem with real numbers. Starting from the given problem, we formulate a quadratic maximization problem over a polytope, \( P \), which is eventually written as a distance maximization to a fixed point over the polytope. Next, starting from the obtained polytope, we construct an intersection of balls which includes the polytope and show that in case the subset sum problem has a solution we can find it by maximizing the distance to the fixed point over the intersection of balls. That is, we show that the points which maximize the distance to the fixed point over the polytope are the same points which maximize the distance to the fixed point over the constructed intersection of balls. For the latter problem we give an original result which allows the characterization of the optimum points as follows: with the centers of the balls and the said fixed point we form a function whom sublevel sets are polytopes. As such we obtain a uni-parameter, family of polytopes. We show that by increasing this parameter from 0 to a finite maximum value (which is computed through a convex optimization problem), the polytopes in the family evolve from initially containing the intersection of balls towards three possible outcomes: a) included in the interior of the intersection of balls b) included in the interior of the complementary of the intersection of balls c) the border of the intersection of balls and the polytope of maximum parameter share at least a point. Then we show that the maximum distance to the fixed point over the intersection of balls is given by a) the smallest parameter for which the polytopes enter the intersection of balls, b) the largest parameter for which the polytopes still share points with the intersection of balls c) the maximum value of the parameter.

Considering that the points which maximize the distance to the fixed point over the intersection of balls and those which maximize the distance over the polytope \( P \) are the same, if the subset sum has a solution, we show that the maximum distance is actually given by a) the smallest parameter for which the family polytopes enter the polytope \( P \), b) the largest parameter for which the family polytopes still share points with the polytope \( P \) c) the maximum value of the parameter.

Additional Key Words and Phrases: computational geometry, subset sum problem, quadratic optimization, 0-1 integer programming

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1 INTRODUCTION

This paper presents an approach to the subset sum problem. The subset sum problem is known to be NP-complete hence enjoys everything desired and undesired which characterizes the NP-complete class of problems: it would be nice to have a solution to it but getting it is hard! The algorithm we present here defines a relaxed problem and allows one to obtain a solution for the relaxed problem in polynomial time, if the instance happens to enjoy some requirement that we need for our result. We also give an heuristic algorithm which can be applied to any instance.

The results in this paper are highly original and the paper can be considered elementary and complete. Therefore we have many proofs for what one would correctly classify as trivial lemmas, but these are essential for the completeness of the statements we make in the main results. As such the presentation of the results is somewhat awkward since if we follow the traditional way of presenting results where all the supporting results are presented first, the reader would have to go through many pages with dense and seemingly uninteresting lemmas and remarks. We have learned this from the reviewers comments from several publication outlets now. Hence we have decided to have a later section where all the supporting results are being presented and have an early section where the main results are stated and proven with reference to the results from the supporting results section. The structure of the paper is the following: the first section, besides this text, contains several subsections which ought to align the reader with the author in subject matters like the notations used, the definition of the problems this paper is about, some approaches from the literature, the proposed approach and finally a subsection about how the main results of this paper can be used to solve one of the problems defined in Problem definition subsection.

The second section of the paper, upon giving some necessary definitions and geometric constructions, states the main theorem and gives a proof for it. The proof is not very long as it only has about three pages and is also divided in three steps, where the first step is the most important.

The next section is the largest as it presents the results and their proofs which stand behind the affirmations made in the proof of the main results. This sections is also divided in subsections to offer a little structure in what otherwise would be a pool of various mathematical statements. Also, in this section we give what can be considered the backbone of the paper, the one results which enable the paper practically, and is also the basis of a naive algorithm presented in the following section. This result found itself in this section due to the fact that the independent proof-readers of the manuscript found it to be not interesting in itself.

The next section gives a very short naive algorithm, which we will have to call heuristic at this point, since being somewhat out of the scope of this current publication, no proofs are given specifically for it. The naive algorithm it is just a broadening of the main results to allow the treatment of the instances of the subset sum
problem which do not meet the requirement for the main results. This algorithm can be the basis of a possible future work related to this subject.

The conclusions section follow after the naive algorithm section.

1.1 Notations

For $n, m \in \mathbb{N}$ and $\mathcal{D} \subseteq \mathbb{R}$ we denote by

$$\mathcal{D}^{n \times m} = \left\{ \begin{bmatrix} a_{1,1} & \ldots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,m} \end{bmatrix} : a_{i,j} \in \mathcal{D}, \; 1 \leq i \leq n, 1 \leq j \leq m \right\} \quad (1)$$

We denote by $I_n$ the unit matrix in $\mathbb{R}^{n \times n}$, with $1_{n \times m}$ the matrix of appropriate size where each entry is 1, and with $0_{n \times m}$ the matrix of appropriate size where each entry is 0.

For $x \in \mathbb{R}^{n \times 1}$ and $r > 0$, we denote by $B(x, r)$ the open ball centered in $x$ and of radius $r$, and with $\bar{B}(x, r)$ the closed ball centered at $x$ and of radius $r$, i.e

$$B(x, r) = \{ y \in \mathbb{R}^{n \times 1} : \| y - x \| < r \} \quad \bar{B}(x, r) = \{ y \in \mathbb{R}^{n \times 1} : \| y - x \| \leq r \} \quad (2)$$

For $\mathcal{D} \subseteq \mathbb{R}^{n \times 1}$ we denote by $\partial \mathcal{D}$ the boundary of $\mathcal{D}$.

1.2 Problem definition

Inspired from the classical subset sum problem we define the following problems:

**Definition 1.1 (RSSP).** Let us consider the real subset sum problem (RSSP): given $S \in \mathbb{R}^{n \times 1}$, we ask if exists $x \in \{0, 1\}^{n \times 1} \setminus 0_{n \times 1}$ such that $S^T \cdot x = 0$.

We call a solution of the RSSP problem a point which meets the requirements. We say that the RSSP problem has a solution if such a point exists.

We relax the above problem as follows:

**Definition 1.2 ($\epsilon$RRSSP).** For a given $\epsilon > 0$, we define the epsilon relaxed real subset sum problem ($\epsilon$RRSSP) as follows: given $S \in \mathbb{R}^{n \times 1}$, we ask if exists $x \in ([0, 2 \cdot \epsilon] \cup [1 - 2 \cdot \epsilon, 1])^{n \times 1} \setminus [0, 2 \cdot \epsilon]^{n \times 1}$ such that $S^T \cdot x \in [-\epsilon, 0]$.

We call a solution of the $\epsilon$RRSSP problem a point which meets the requirements. We say that the $\epsilon$RRSSP problem has a solution if such a point exists.

1.3 Approaches from literature

We will present some algorithms for the literature for solving the Subset Sum problem where the numbers are positive integers. These FPTAS use a very different approach to our work. One of them, is presented here, [1].

This algorithm first sorts the numbers in the list that we will call $S$ (i.e what we defined as a vector and left unsorted, this will now become a list, which can be sorted). Then it defines merging of two such ordered lists. Finally they provide a solution to the exact Subset Sum problem in terms of merging of lists. This proves to be an exponential algorithm. In order to improve on this, they define a trimming operation: remove an item in the list if it can be approximated (a proper definition of approximation is given) by another element. Then applying the previous algorithm with merging of lists together with trimming they provide a FPTAS to the subset sum problem. For more information see [1].

Another classical approach from the literature uses the tools from dynamic programming as follows: the list of integers is ordered and for simplicity of presentation it is assumed to be composed out of positive integers. Let $M$ be the sum of all the integers in the list. It is obvious that is not possible to have a subset which adds up to a
negative number, zero or a positive number (strictly) greater than $M$. It is also obvious that we have a subset which adds up to $M$. Let us ask if there is a subset of $S$ which adds up to $1 \leq T \leq M - 1$. Applying the dynamic programming based algorithm we (parasitically) obtain an answer to all the possible values of $T$. The algorithm works as follows: form a table with as many rows as numbers are in $S$ and as many columns as numbers are between 1 and $M - 1$ i.e $M - 1$. Assuming that the numbers in $S$ are ordered, proceed as follows to fill the table. The table will be filled with the logical values $1 - \text{true or } 0 - \text{false}$. At the end, if the column corresponding to $T$ has a 1 then there is a subset of $S$ which adds up to $T$ otherwise there is none.

So the table is filled as follows: the first line is filled with zeros, except for the column equal to the first number in $S$, where we put 1. All the other rows are filled with the values of the previous row until we meet the column equal to the number in $S$ the row corresponds to, where we put 1. Continuing on the same row, we fill each remaining column with the value from the previous row on the column obtained by subtracting from the current one the value of the number in $S$ corresponding to the row. For instance, assume that $S = \{x_1, \ldots, x_N\}$. Then we will have $N$ lines. Let’s denote the table by $A$. The line $k$ is filled as follows: $A(k, j) = A(k - 1, j)$ for all $1 \leq j < x_k$. Then $A(k, x_k) = 1$ and $A(k, x_k + p) = A(k - 1, p)$ for all $1 \leq p \leq M - 1 - x_k$. It can be proven that this approach gives the correct results. For more advanced algorithm using Dynamic Programming related tools see [2], [3], [4], [5].

Finally, another approach is based on writing an associated quadratic optimization problem to the subset sum problem, see [6]. Then a quadratic function is ought to be maximized over a convex set. We therefore mention here that such algorithms (maximization of quadratic functions over convex domains) are able to provide an approximate answer to the subset sum problem. Some references in this direction are: [8], [9], [10], and [11].

Overall a good presentation of the state of the art approaches can be found here [26].

1.4 Our proposed approach

For some $\gamma > 0$ we formulate the well known optimization problem associated to the subset sum problem:

$$\max \sum_{k=1}^{n} x_k \cdot (x_k - 1) + \gamma \cdot \sum_{k=1}^{n} x_k \cdot s_k \quad \text{s.t} \quad \begin{cases} s^T \cdot x \leq 0 \\ 0 \leq x_k \leq 1 \\ 1^T_{nx1} \cdot x - \frac{1}{2} \geq 0 \end{cases}$$

where $x = [x_1 \ldots x_n]^T \in \mathbb{R}^{nx1}$ and $S = [s_1 \ldots s_n]^T \in \mathbb{R}^{nx1}$. Please note that due to the last constraint, the origin $0_{nx1}$ does not belong to the search space. We assume throughout the paper that the feasible set has a non-empty interior.

As presented here [6] note that upon solving (3) the answer to the problem is positive if and only if the maximum is zero. Indeed, on the search space, the objective function is less than or equal to zero and reaches its maximum value of zero on $x^* \in \mathbb{R}^{nx1}$ if and only if $x^* \in \{0, 1\}^{nx1}$ and $S^T \cdot x^* = 0$.

We write the function in (3) as follows

$$\sum_{k=1}^{n} x_k \cdot (x_k - 1) + \gamma \cdot \sum_{k=1}^{n} x_k \cdot s_k = x^T \cdot x + (\gamma \cdot S - 1^T_{nx1})^T \cdot x$$

For a given $\frac{1}{2} \geq \epsilon > 0$, being able to assert the existence of an $x$ in the feasible set such that

$$-\epsilon \leq x^T \cdot (x - 1_{nx1}) + \gamma \cdot S^T \cdot x \leq 0$$

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means that we can assert the existence of an \( x \) such that
\[
-\epsilon \leq x_k \cdot (x_k - 1) \leq 0 \quad -\epsilon \leq y \cdot S^T \cdot x \leq 0
\]
which finally means \( x_k \in \left( \frac{1 + \sqrt{1 - 4 \cdot \delta}}{2} \right) \), for all \( k \in \{1, \ldots, n\} \). Please note that for \( \epsilon \to 0 \) we get \( x_k \in \{0, 1\} \), and for \( \gamma \geq 1 \) we get \( S^T \cdot x \to 0 \).

**Remark 1.** Therefore, in order to solve the \( \epsilon \)RRSSP problem, for a fixed \( \frac{1}{4} > \epsilon > 0 \), it is sufficient to assert if exists \( x \) in the feasible set, such that (5) holds for some \( \gamma \geq 1 \). Indeed, as shown above, since \( 0 \leq \delta \leq \epsilon \), this means that
\[
0 \leq x_k = \frac{1 - \sqrt{1 - 4 \cdot \delta}}{2} \leq \frac{1 - \sqrt{1 - 4 \cdot \epsilon}}{2} \leq 2 \cdot \epsilon
\]
or
\[
1 \geq x_k = \frac{1 + \sqrt{1 - 4 \cdot \delta}}{2} \geq \frac{1 + \sqrt{1 - 4 \cdot \epsilon}}{2} \geq 1 - 2 \cdot \epsilon
\]

We can rewrite the objective function in (3) as follows:
\[
x^T \cdot x + 2 \cdot x^T \cdot y \cdot S - 1_{nx1} \cdot \frac{1}{2} \|y \cdot S - 1_{nx1}\|^2 = \|x - \frac{1}{2} (1_{nx1} - y \cdot S)\|^2 - \frac{1}{2} \|y \cdot S - 1_{nx1}\|^2
\]
It is obvious that we can solve (3) for a given \( \gamma \) if we solve:
\[
\max \left\{ \|x - \frac{1}{2} \left( 1_{nx1} - \frac{\beta}{\|S\|} \cdot S \right) \|^2 \right\}
\text{s.t.} \quad 0 \leq x_k \leq 1, \quad 1_{nx1}^T \cdot x - \frac{1}{2} \geq 0
\]
for \( \beta = \|S\| \cdot y \). We take \( \beta \geq \|S\| \). In the following we focus on solving the problem (10).

Having a closer look at (10) let’s denote
\[
C_\beta = \frac{1}{2} \left( 1_{nx1} - \frac{\beta}{\|S\|} \cdot S \right) \quad \mathcal{P} = \{ x \in \mathbb{R}^{nx1} \mid S^T \cdot x \leq 0, 0 \leq e_k^T \cdot x \leq 1, 1_{nx1}^T \cdot x - \frac{1}{2} \geq 0 \}
\]
where \( e_k \) is the \( k \)’th column of the unit matrix in \( \mathbb{R}^{nxn} \). Then (10) becomes
\[
\max_{x \in \mathcal{P}} \|x - C_\beta\|^2
\]
i.e we have to find the furthest point in the polytope \( \mathcal{P} \subseteq \mathbb{R}^{nx1} \) to the fixed point \( C_\beta \in \mathbb{R}^{nx1} \) with \( \beta > \|S\| \).

**Lemma 1.3.** Let
\[
x_\beta^* \in \arg \max_{x \in \mathcal{P}} \|x - C_\beta\|^2
\]
and assume that for a fixed \( \frac{1}{4} \geq \gamma > 0 \) we are able to find \( \hat{x}_\beta^* \) in the feasible set, such that
\[
\left| \|x_\beta^* - C_\beta\|^2 - \|\hat{x}_\beta^* - C_\beta\|^2 \right| \leq \epsilon
\]
Then
If \( \|\hat{x}_\beta^\star - C\beta\| \geq \epsilon \) we have that \( \hat{x}_\beta^\star \) is a solution to the \( \epsilon \text{RSSP} \).

If \( \|\hat{x}_\beta^\star - C\beta\| < \epsilon \) we have that the RSSP does not have a solution.

**Proof.** Indeed

(1) since \( \hat{x}_\beta^\star \) is in the feasible set, it follows that from (9) and (11)

\[
0 > \|\hat{x}_\beta^\star - C\beta\|^2 - \|C\beta\|^2 \geq -\epsilon
\]

hence \( \hat{x}_\beta^\star \) is a solution to the \( \epsilon \text{RSSP} \) problem.

(2) we have the following:

\[
\|\hat{x}_\beta^\star - C\beta\|^2 - \|C\beta\|^2 = \|\hat{x}_\beta^\star - C\beta\|^2 - \|\hat{x}_\beta^\star - C\beta\|^2 + \|\hat{x}_\beta^\star - C\beta\|^2 - \|C\beta\|^2 \leq -\epsilon
\]

but since \( \|\hat{x}_\beta^\star - C\beta\|^2 - \|x_\beta^\star - C\beta\|^2 \geq -\frac{\epsilon}{2} \) follows

\[
-\frac{\epsilon}{2} + \|x_\beta^\star - C\beta\|^2 - \|C\beta\|^2 \leq -\epsilon \Rightarrow \|x_\beta^\star - C\beta\|^2 - \|C\beta\|^2 \leq -\frac{\epsilon}{2}
\]

which means that the maximum of (3) is strictly less than zero, hence RSSP does not have a solution.

\[\square\]

**Remark 2.** According to the above lemma, all we have to do is to find \( \hat{x}_\beta^\star \). However, this is not trivial at all, since the problem (13) requires the maximization of the distance to a fixed point over a polytope. This is a quadratic maximization problem over a convex domain and it is NP-hard in general.

### 1.5 What to expect from this paper?

In this paper we give an algorithm which stops for any input problem after a number of steps in \( O(poly(n,L,\frac{1}{\epsilon})) \) and requires a number of bits in \( O(poly(L)) \) where \( L \) is the number of bits used to store the inputs involved in the description of the problem.

Our algorithm will searches for \( \hat{x}_\beta^\star \).

Besides the guarantees mentioned above: stops in finite poly time and uses poly memory for any input instance, our algorithm has the following additional guarantees for any fixed \( \frac{1}{4} \geq \epsilon > 0 \):

(1) upon stopping it always provides a point \( \hat{x}_\beta^\star \in \mathbb{R}^{n \times 1} \) in the feasible set.

(2) for \( S \in \mathbb{R}^{n \times 1} \) which meets an easily apriori computable condition (more details later), if the RSSP has a solution, then the equation (14) for the output point \( \hat{x}_\beta^\star \) is guaranteed.

So, given \( S \in \mathbb{R}^{n \times 1} \), pick a desired \( 0 < \epsilon \leq \frac{1}{4} \). Assuming that \( S \) meets the condition (to run the algorithm with full theoretical guarantees), obtain \( \hat{x}_\beta^\star \) in poly time and memory usage. From the guarantees of the algorithm and Lemma 1.3 two options might happen:

(1) The point \( \hat{x}_\beta^\star \) returned by the algorithm is a solution to \( \epsilon \text{RSSP} \).

(2) If the point \( \hat{x}_\beta^\star \) returned by the algorithm is NOT a solution to \( \epsilon \text{RSSP} \). Then the RSSP cannot have a solution. Indeed, in order to prove this, assume that the RSSP has a solution. In such a case (14) is guaranteed by the algorithm and from Lemma (1.3) follows that then the RSSP does not have a solution, which is a contradiction.

Therefore, using this algorithm, for the given conditions, we either find a solution to \( \epsilon \text{RSSP} \) either prove that RSSP does not have a solution.
2 MAIN RESULTS

We begin with a short presentation of the necessary objects to define the property $S$ is supposed to meet. This is needed for the enunciation of the papers main theorem. Let $\rho > \beta \geq \max \{\|S\|, \sqrt{n}\}$ and we construct the following points for $k \in \{1, \ldots, n\}$:

\[
C_{k \pm} = \frac{1}{2} \cdot 1_{n \times 1} \pm \rho \cdot e_k, \quad C_h = \frac{1}{2} \cdot 1_{n \times 1} + \rho \cdot \frac{1_{n \times 1}}{\sqrt{n}}, \quad C_s = \frac{1}{2} - 1_{n \times 1} - \rho \cdot \frac{S}{\|S\|}, \quad C_\beta = \frac{1}{2} \cdot 1_{n \times 1} - \beta \cdot \frac{S}{\|S\|}
\]

Next let

\[
\begin{align*}
    r_{k \pm}^2 &= \frac{n - 1}{4} + \left(\rho + \frac{1}{2}\right)^2, \\
    r_h^2 &= \frac{n - 1}{4} - \left(\frac{\sqrt{n} - 1}{2} \cdot n\right)^2 + \left(\rho + \frac{\sqrt{n} - 1}{2} \cdot n\right)^2, \\
    r_s^2 &= \frac{n - 1}{4} - \left(\frac{S^T \cdot 1_{n \times 1}}{\|S\|}\right)^2 + \left(\rho - \frac{1}{2} \cdot \frac{S^T \cdot 1_{n \times 1}}{\|S\|}\right)^2
\end{align*}
\]

**Remark 3.** Please note that $\partial B(C_{k \pm}, r_{k \pm}), \partial B(C_h, r_h)$ and $\partial B(C_s, r_s)$ leave the same imprint on $\partial B \left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$ as the hyperplanes forming $P$ individually. That is, for instance, $\partial B(C_h, r_h) \cap \partial B \left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right) = \partial B \left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right) \cap \{x|1_{n \times 1} \cdot x - \frac{1}{2} = 0\}$

Let $\odot$ denote a symbol in the set $\{k \pm, h, s\}$ for all $k \in \{1, \ldots, n\}$. Then we define the intersection

\[
Q_\rho = \bigcap_{\odot \in \{k \pm, h, s\}} \hat{B}(C_\odot) = \bigcap_{\odot \in \{k \pm, h, s\}} \hat{B}(C_\odot, r_\odot)
\]

With the above, we define the following functions:

\[
h_\rho(x) = \max_{\odot \in \{k \pm, h, s\}} \|x - C_\odot\|^2 - r_\odot^2, \quad f_\beta(x) = \|x - C_\beta\|^2
\]

then $Q_\rho = \{x|h_\rho(x) \leq 0\}$. We have:

\[
(h_\rho - f_\beta)(x) = -\|x - C_\beta\|^2 + \max_{\odot \in \{k \pm, h, s\}} \|x - C_\odot\|^2 - r_\odot^2
\]

\[
= \max_{\odot \in \{k \pm, h, s\}} 2 \cdot \left(C_\beta - C_\odot\right) \cdot X + \|C_\odot\|^2 - \|C_\beta\|^2 - r_\odot^2
\]

Let us consider the following convex optimization problem:

\[
\mathcal{H}_{\rho, \beta} = \arg\min_{h_\rho(x) \leq 1} (h_\rho - f_\beta)(x)
\]

We choose the search space $\{x|h_\rho(x) \leq 1\}$ just to have a compact, larger set than $Q_\rho = \{x|h_\rho(x) \leq 0\}$. As such we also assure the fact that the result is finite.

We give the following definition:

**Definition 2.1 (Property $A(S, \epsilon)$).** We say that $S \in \mathbb{R}^{n \times 1}$ and $\frac{1}{4} \geq \epsilon > 0$ have the property $A(S, \epsilon)$ if exists $2 \cdot \max \{\|S\|, \sqrt{n}\} \geq \rho > \beta \geq \max \{\|S\|, \sqrt{n}\}$ such that

\[
\bigcup_{x \in \mathcal{H}_{\rho, \beta}} \mathcal{B}(x, \epsilon) \subseteq P \quad \text{or} \quad \bigcup_{x \in \mathcal{H}_{\rho, \beta}} \mathcal{B}(x, \epsilon) \subseteq \mathbb{R}^{n \times 1} \setminus Q_\rho
\]

**Remark 4.** Please note that given $S, \rho, \beta, \epsilon$ testing (24) in $A(S, \epsilon)$ is a convex problem. Indeed, $\mathcal{H}_{\rho, \beta}$ is a convex set. Testing if the polytope $P$ contains a convex set is a convex problem (since it boils down to $2 \cdot n + 2$ maximizations...
of linear objective functions over a convex set) and testing if $\mathcal{H}^\ast_{p,\beta} \subseteq \mathbb{R}^{n\times 1} \setminus Q_p$ is again a convex problem since it summarizes to asserting if $\mathcal{H}^\ast_{p,\beta} \cap Q_p = \emptyset$, i.e. asserting if the intersection of two convex sets is empty or not.

The main result of this paper is the following:

**Theorem 2.2.** For $S \in \mathbb{R}^{n\times 1}$ and $0 < \varepsilon \leq \frac{1}{4}$ such that $\exists \mathcal{B}(x, \varepsilon) \subseteq \mathcal{P}$ and having the property $A(S, \varepsilon)$, exists an algorithm with time complexity in $O \left( \text{poly} \left( n, L, \log \left( \frac{\max\{||S||, \sqrt{n}\}}{\varepsilon} \right) \right) \right)$ which always returns a point $x^\ast_{\beta}$ in the feasible set $\mathcal{P}$ for which, under the assumption that RSP has a solution, it is guaranteed that

$$||x^\ast_{\beta} - C\beta||^2 - ||\hat{x}^\ast_{\beta} - C\beta||^2 \leq \frac{\varepsilon}{2}$$  \hspace{1cm} (25)

where $L$ is the coding length of the input.

**Proof.** The proof will have three steps. For what is intended to be a good presentation of a fairly complicated result, we opted to put all the supporting results in a later section of the paper and use them in this main theorem by referencing them. Otherwise, there would be several pages of seemingly trivial or unrelated mathematical statements, before the main results.

Let $2 \cdot \max\{||S||, \sqrt{n}\} \geq \rho > \beta \geq \max\{||S||, \sqrt{n}\}$ be given by the property $A(S, \varepsilon)$. 

**Step 1** Consider the problems:

$$\mathcal{Y}^\ast_{p,\beta} = \arg\max_{y \in Q_p} ||y - C\beta||^2 \quad \mathcal{X}^\ast_{\beta} = \arg\max_{x \in \mathcal{P}} ||x - C\beta||^2$$  \hspace{1cm} (26)

For $Q_p$ defined above, we conclude from Lemma 3.3 and Lemma 3.4 that

$$\mathcal{P} \subseteq Q_p \quad \mathcal{Y}^\ast_{p,\beta} = \mathcal{X}^\ast_{\beta}$$  \hspace{1cm} (27)

In order to find $y^\ast_{p,\beta}$ we form the family of polytopes:

$$\mathcal{P}_{p,\beta,R^2} = \{x|h_p(x) - f\beta(x) \leq -R^2\}$$  \hspace{1cm} (28)

and define

$$-\infty < -R^2_{p,\beta} = (h_p - f\beta)(z^\ast_{p,\beta})$$  \hspace{1cm} (29)

where $z^\ast_{p,\beta} \in \mathcal{H}^\ast_{p,\beta}$ from (23). Observe that

1. $Q_p \subseteq \mathcal{P}_{p,\beta,0}$. Indeed, for all $x \in Q_p$ one has $h_p(x) \leq 0$ and since $f\beta(x) \geq 0$ easily follows that $x \in \mathcal{P}_{p,\beta,0}$.
2. Since $S$ has the property $A(S, \varepsilon)$ and $\mathcal{P}_{p,\beta,R^2_{p,\beta}} \cap \{x|h_p(x) \leq 1\} = \mathcal{H}^\ast_{p,\beta}$ follows that one of the following two is true, see Remark 6

(a) $\mathcal{P}_{p,\beta,R^2_{p,\beta}} \subseteq \bigcup_{x \in \mathcal{P}_{p,\beta,R^2_{p,\beta}}} \mathcal{B}(x, \varepsilon) \subseteq \mathcal{P}$.
(b) $\mathcal{P}_{p,\beta,R^2_{p,\beta}} \subseteq \bigcup_{x \in \mathcal{P}_{p,\beta,R^2_{p,\beta}}} \mathcal{B}(x, \varepsilon) \subseteq \mathbb{R}^{n\times 1} \setminus Q_p$

Let us denote

$$R^2_{p,\beta} = \max_{x \in Q_p} ||y - C\beta||$$  \hspace{1cm} (30)

then using Theorem 3.6 we obtain

1. if $\mathcal{P}_{p,\beta,R^2_{p,\beta}} \subseteq \mathcal{P} \subseteq Q_p$ then

$$R^2_{p,\beta} = \min\{R > 0|\mathcal{P}_{p,\beta,R^2_{p,\beta}} \subseteq Q_p\}$$  \hspace{1cm} (31)
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(2) if \( \mathcal{P}_{\rho,\beta,R^2} \subseteq \mathbb{R}^{n \times 1} \setminus Q_\rho \), then

\[
R^*_{\rho,\beta} = \max \{ R > 0 | \mathcal{P}_{\rho,\beta,R} \cap Q_\rho \neq \emptyset \} \quad y^*_{\rho,\beta} \in \mathcal{P}_{\rho,\beta,\{R^*_{\rho,\beta}\}^2} \cap Q_\rho
\]  

(32)

and \( Y^*_{\rho,\beta} \subseteq \mathcal{P}_{\rho,\beta,\{R^*_{\rho,\beta}\}^2} \).

While for the second item the computation of \( R^*_{\rho,\beta} \) and \( y^*_{\rho,\beta} \) is a convex optimization problem (and can be easily done using bisection after \( R \) in the interval \([0,R_{\rho,\beta}]\)), for the first item the problem is hard because asserting \( \mathcal{P}_{\rho,\beta,R^2} \subseteq Q_\rho \) is hard in general.

For now we focus on the first case: i.e if \( \mathcal{P}_{\rho,\beta,R^2} \subseteq \mathbb{R}^{n \times 1} \setminus Q_\rho \). Define the quantities

\[
\hat{R}^*_{\rho,\beta} = \min \{ R > 0 | \mathcal{P}_{\rho,\beta,R^2} \subseteq \mathcal{P} \} \quad \hat{y}^*_{\rho,\beta} \in \mathcal{P}_{\rho,\beta,\{\hat{R}^*_{\rho,\beta}\}^2} \setminus \text{int} (\mathcal{P})
\]

(33)
i.e we exchanged \( Q_\rho \) with \( \mathcal{P} \) in (31).

Finding \( \hat{R}^*_{\rho,\beta} \) is a convex optimization problem and can be done, again as above, by using bisection after \( R \) on the interval \([0,\hat{R}^*_{\rho,\beta}]\) since asserting \( \mathcal{P}_{\rho,\beta,R^2} \subseteq \mathcal{P} \) resumes to solving \( 2 \cdot n + 2 \) linear programs. The point \( \hat{y}^*_{\rho,\beta} \) is found in the process, as being the maximizer of a linear function over \( \mathcal{P} \). This is basically testing polytope containment in another polytope.

Next we prove that if the RSSP has a solution then \( R^*_{\rho,\beta} = \hat{R}^*_{\rho,\beta} \). This is done by showing that

\[
\mathcal{P}_{\rho,\beta,\{R^*_{\rho,\beta}\}^2} = \mathcal{P}_{\rho,\beta,\{\hat{R}^*_{\rho,\beta}\}^2}
\]

(34)

Since \( \mathcal{P} \subseteq Q_\rho \) it is sufficient to show that

\[
\mathcal{P}_{\rho,\beta,\{R^*_{\rho,\beta}\}^2} \subseteq \mathcal{P}
\]

(35)

Assume that exists \( \delta > 0 \) and \( x_0 \in \mathbb{R}^{n \times 1} \) such that \( B(x_0,\delta) \subseteq \mathcal{P}_{\rho,\beta,\{R^*_{\rho,\beta}\}^2} \setminus \mathcal{P} \). From Lemma 3.5 exists \( \alpha > 1 \) such that

\[
\mathcal{P} \subseteq Q_{\alpha,\rho} \subseteq \bigcup_{x \in \mathcal{P}} B \left( x, \frac{\delta}{2} \right)
\]

(36)
hence \( x_0 \notin Q_{\alpha,\rho} \). Define

\[
T^*_{\alpha,\rho,\beta} = \min \{ T > 0 | \mathcal{P}_{\alpha,\rho,\beta,T} \subseteq Q_{\alpha,\rho} \}
\]

(37)

and we will show that

\[
\mathcal{P}_{\alpha,\rho,\beta,\{T^*_{\alpha,\rho,\beta}\}^2} = \mathcal{P}_{\rho,\beta,\{R^*_{\rho,\beta}\}^2}
\]

(38)

From here the contradiction follows because from \( x_0 \notin Q_{\alpha,\rho} \) follows \( x_0 \notin \mathcal{P}_{\alpha,\rho,\beta,\{T^*_{\alpha,\rho,\beta}\}^2} \) hence \( x_0 \notin \mathcal{P}_{\rho,\beta,\{R^*_{\rho,\beta}\}^2} \).

Finally, in order to prove (38) we define

\[
Y^*_{\alpha,\rho,\beta} = \arg \max_{y \in Q_{\alpha,\rho}} \| y - C_{\alpha,\beta} \|^2
\]

(39)
and use Lemma 3.3 and Lemma 3.4 for \( Q_{\alpha, \beta} \) to obtain that

\[
\begin{align*}
\mathcal{P} & \subseteq Q_{\alpha, \beta} \\
\mathcal{Y}^*_{\alpha, \rho, \alpha, \beta} & = \mathcal{X}^*_{\beta} \quad \text{if the RSSP has a solution}
\end{align*}
\] (40)

Using Theorem 3.6 we obtain that \( \mathcal{Y}^*_{\alpha, \rho, \alpha, \beta} \subseteq \mathcal{P}_{\alpha, \rho, \alpha, \beta}(\mathcal{T}^*_{\alpha, \rho, \alpha, \beta})^2 \). Furthermore, from Lemma 3.7 we obtain that

\[
\exists \hat{\mathcal{T}}_{\alpha, \rho, \alpha, \beta} \text{ such that }
\]

\[
\begin{align*}
\mathcal{P}_{\alpha, \rho, \alpha, \beta}(\hat{\mathcal{T}}_{\alpha, \rho, \alpha, \beta})^2 & = \mathcal{P}_{\rho, \beta}(\mathcal{T}^*_{\rho, \beta})^2
\end{align*}
\] (41)

And lastly from Remark 7 we get

(1) \( Q_{\alpha, \rho} \subseteq Q_{\rho} \) for all \( \alpha \geq 1 \) i.e the set \( Q_{\rho} \) shrinks as \( \rho \) increases and \( Q_{\alpha, \rho} \cap \partial Q_{\rho} \subseteq \{0, 1\}^{n+1} \)

(2) \( \mathcal{P}_{\alpha, \rho, \alpha, \beta, \alpha, \beta} \subseteq \mathcal{P}_{\alpha, \rho, \alpha, \beta, \alpha, \beta, \alpha, \beta} \) for \( T_1 < T_2 \) i.e. the set \( \mathcal{P} \) shrinks as \( T \) increases and \( \mathcal{P}_{\alpha, \rho, \alpha, \beta, T_2} \cap \partial \mathcal{P}_{\alpha, \rho, \alpha, \beta, T_2} = \emptyset \)

Now assume that the RSSP has a solution. Then

\[
\mathcal{Y}^*_{\rho, \beta} = \mathcal{Y}^*_{\rho, \alpha, \rho, \alpha, \beta} = \mathcal{X}^*_{\beta} = \{x \in \{0, 1\}^{n+1}| S^T \cdot x = 0\}
\] (42)

and it is easy to show that

\[
\begin{align*}
\mathcal{Y}^*_{\alpha, \rho, \alpha, \beta} & \subseteq \partial \mathcal{P}_{\alpha, \rho, \alpha, \beta}(\mathcal{T}^*_{\alpha, \rho, \alpha, \beta})^2 \\
\mathcal{Y}^*_{\rho, \beta} & \subseteq \partial \mathcal{P}_{\rho, \beta}(\mathcal{T}^*_{\rho, \beta})^2
\end{align*}
\] (43)

hence

\[
\mathcal{X}^*_{\beta} \subseteq \partial \mathcal{P}_{\alpha, \rho, \alpha, \beta}(\hat{\mathcal{T}}_{\alpha, \rho, \alpha, \beta})^2 \cap \partial \mathcal{P}_{\rho, \beta}(\mathcal{T}^*_{\rho, \beta})^2
\] (44)

But, if \( \mathcal{T}^*_{\alpha, \rho, \alpha, \beta} > \hat{\mathcal{T}}_{\alpha, \rho, \alpha, \beta} \) then \( \mathcal{P}_{\alpha, \rho, \alpha, \beta}(\mathcal{T}^*_{\alpha, \rho, \alpha, \beta})^2 \cap \partial \mathcal{P}_{\alpha, \rho, \alpha, \beta}(\hat{\mathcal{T}}_{\alpha, \rho, \alpha, \beta})^2 = \emptyset \). This is a contradiction with (41), (43) and (44). Similarly, if \( \mathcal{T}^*_{\alpha, \rho, \alpha, \beta} < \hat{\mathcal{T}}_{\alpha, \rho, \alpha, \beta} \) then \( \mathcal{P}_{\alpha, \rho, \alpha, \beta}(\hat{\mathcal{T}}_{\alpha, \rho, \alpha, \beta})^2 \cap \partial \mathcal{P}_{\alpha, \rho, \alpha, \beta}(\mathcal{T}^*_{\alpha, \rho, \alpha, \beta})^2 = \emptyset \) hence we contradict (41), (43) and (44) therefore \( \hat{\mathcal{T}}_{\alpha, \rho, \alpha, \beta} = \mathcal{T}^*_{\alpha, \rho, \alpha, \beta} \).

Finally, if the RSSP has a solution, then similar with the above, we can prove for \( R^*_{\rho, \beta} \) in (32) that

\[
\begin{align*}
R^*_{\rho, \beta} & = \max\{R > 0 | \mathcal{P}_{\rho, \beta, \mathcal{R} \cap \mathcal{P} \neq \emptyset}\} \\
y^*_{\rho, \beta} & \in \mathcal{P}_{\rho, \beta}(\mathcal{P}^*_{\rho, \beta})^2 \cap \mathcal{P}
\end{align*}
\] (45)

**Step 2**

Using either (33) or (45), according to what cases \( S \) meets in the property \( A(S, \epsilon) \), we are now able to compute \( \hat{y}_{\rho, \beta} \in \mathcal{P} \) such that \( \|y^*_{\rho, \beta} - C_\beta\| < \frac{\epsilon}{16 \cdot \max\{|S|, \sqrt{n}\}} \). This can be done regardless of whether RSSP has a solution or not. If the RSSP has a solution we proved that \( y^*_{\rho, \beta} \) given by (33) or (45) is a solution to the RSSP. Then let us evaluate

\[
\begin{align*}
\|y^*_{\rho, \beta} - C_\beta\| - \|\hat{y}_{\rho, \beta} - C_\beta\| &= \|y^*_{\rho, \beta} - C_\beta\| - \|\hat{y}_{\rho, \beta} - y^*_{\rho, \beta} + y^*_{\rho, \beta} - C_\beta\| \\
&= \|y^*_{\rho, \beta} - C_\beta\| - \|y^*_{\rho, \beta} - \hat{y}_{\rho, \beta} + \hat{y}_{\rho, \beta} - C_\beta\| \\
&\leq \|y^*_{\rho, \beta} - \hat{y}_{\rho, \beta}\| + \|\hat{y}_{\rho, \beta} - y^*_{\rho, \beta}\| \cdot \|y^*_{\rho, \beta} - C_\beta\|
\end{align*}
\] (46)

It is easy to see that \( \|y^*_{\rho, \beta} - C_\beta\| \leq \frac{\beta}{2} + \frac{\sqrt{n}}{2} \leq 2 \cdot \max\{|S|, \sqrt{n}\} \). Let \( M = \max\{|S|, \sqrt{n}\} \). It follows that

\[
\|y^*_{\rho, \beta} - C_\beta\| - \|\hat{y}_{\rho, \beta} - C_\beta\| \leq \left(\frac{\epsilon}{16 \cdot M}\right)^2 + 2 \cdot \frac{\epsilon}{16 \cdot M} \cdot 2 \cdot M \leq \frac{\epsilon}{2}
\] (47)
We output \( \hat{x}_\beta^* \leftarrow \hat{y}_{\beta, \beta} \in \mathcal{P} \).

**Step 3**

We need to compute \( \hat{y}_{\beta, \beta} \) such that \( \|\hat{y}_{\beta, \beta} - y_{\beta, \beta}^*\| \leq \frac{\epsilon}{16M^2} \) where \( M = \max\{\|S\|, \sqrt{n}\} \). For the computation of \( \hat{y}_{\beta, \beta} \) we use linear programs. For instance, in order to decide if \( \mathcal{P} \) we only prove that

\[
d_k = \max A_k^T \cdot x \quad \text{s.t} \quad x \in \mathcal{P}_{\beta, \beta, R}^k
\]

where \( \mathcal{P} = \{x | A_k^T \cdot x + b_k \leq 0\} \) We have that \( \mathcal{P}_{\beta, \beta, R}^k \subseteq \mathcal{P} \) if \( d_k \leq b_k \) for all \( k \). In order to solve (48) we recall from (22) that \( \mathcal{P}_{\beta, \beta, R}^k \) is given by

\[
2 \cdot (C_b - C_a)^T \cdot x + ||C_a||^2 - ||C_b||^2 - r^2 + R^2 \leq 0 \quad \forall \phi \in \{k \pm h, s\}
\]

Let \( L \) be the coding length of the input \( S \) then the LP coefficients coding length is also in \( O(\text{poly}(L)) \). We have to solve these LP with \( \frac{\epsilon}{16M^2} \) precision. It is know that this can be done in \( O(\text{poly}(n, L, \log (\frac{16M}{\epsilon})) \) steps.

\[ \square \]

## 3 Supporting Results

In this section we give with proofs the mathematical statements used in the main theorem as well as some other results needed.

### 3.1 Basic geometry results

The following is a lemma concerning two \( n \) disks with different radii and a hyperplane. The centers of the \( n \)-disks are on the director vector of the hyperplane and the \( n \)-disks intersect on the hyperplane.

**Lemma 3.1.** Let us consider the following \( n \)-disks

\[
\mathcal{D}_1 = \{x \in \mathbb{R}^{n+1} | -q_1 \cdot e_1 - x \leq r_1\}
\]

\[
\mathcal{D}_2 = \{x \in \mathbb{R}^{n+1} | -q_2 \cdot e_1 - x \leq r_2\}
\]

with \( q_1 > q_2 > 0 \) and \( r_1 > r_2 \geq 0 \) such that exists

\[
0 < a^2 = r_1^2 - q_1^2 = r_2^2 - q_2^2
\]

that is the \( n \)-disks share a common \( n-1 \) sphere i.e \( \{x | e_1^T \cdot x = 0, ||x|| = a\} \). Let \( \mathcal{H} = \{x \in \mathbb{R}^{n+1} | e_1^T \cdot x \leq 0\} \) and \( \mathcal{G} = \{x \in \mathbb{R}^{n+1} | e_1^T \cdot x \geq 0\} = \mathbb{R}^{n+1} \setminus \text{int}(H) \). Then the following inclusions are true

\[ \mathcal{H} \cap \mathcal{D}_2 \subseteq \mathcal{H} \cap \mathcal{D}_1 \]

\[ \mathcal{G} \cap \mathcal{D}_1 \subseteq \mathcal{G} \cap \mathcal{D}_2 \]

**Proof.** Let \( x = q \cdot e_1 + v \) with \( e_1^T \cdot v = 0 \) and assume that \( x \in \mathcal{H} \cap \mathcal{D}_2 \), i.e \( q \leq 0 \) and \( ||q \cdot e_1 + v + (q \cdot e_1)\| \leq r_2^2 \). It follows that \( (q + q_1)^2 + ||v||^2 \leq r_2^2 \). We want to check if \( x \in \mathcal{D}_1 \), i.e \( ||q \cdot e_1 + v + q_1 \cdot e_1||^2 \leq r_1^2 \) i.e \( (q + q_1)^2 + ||v||^2 \leq r_1^2 \). However since

\[
(q + q_1)^2 + ||v||^2 \leq (q + q_1)^2 + \frac{r_2^2}{r_1^2}
\]

we only prove that

\[
(q + q_1)^2 + \frac{r_2^2}{r_1^2} - (q + q_2)^2 \leq r_1^2 \iff (q + q_1)^2 - (q + q_2)^2 \leq r_1^2 - r_2^2
\]

\[ \square \]
(q₁ - q₂) · (2 · q + (q₁ + q₂)) = 2 · (q₁ - q₂) · q + q₁² - q₂² ≤ r₁² - r₂² \quad (56)

But from (51) one has r₁² - r₂² = q₁² - q₂² hence (56) is equivalent to

2 · (q₁ - q₂) · q ≤ 0 \quad (57)

But since |q₁| > |q₂| and q₁ > 0 follows that q₁ - q₂ > 0. Finally because q ≤ 0 (57) is true and so is the claim (52). The claim in (53) is easily proved in a similar fashion. \quad \Box

Next we a lemma concerning three \( n \) \(-\) disks with different radii and a hyperplane. The centers of the \( n \)-disks are on the director vector of the hyperplane and the \( n \)-disks intersect on the hyperplane.

**Lemma 3.2.** Let us consider the following \( n \)-disks

\[
\begin{align*}
D₁ &= \{ x ∈ \mathbb{R}^{n×1} | ||−q₁ · e₁ − x|| ≤ r₁ \} \\
D₂ &= \{ x ∈ \mathbb{R}^{n×1} | ||−q₂ · e₁ − x|| ≤ r₂ \} \\
D₃ &= \{ x ∈ \mathbb{R}^{n×1} | ||−q₃ · e₁ − x|| ≤ r₃ \}
\end{align*}
\]

with \( q₁, q₂ > 0 \) and \( r₁ > r₂ > r₃ ≥ 0 \) such that exists

\[
0 < a² = r₁² - q₁² = r₂² - q₂² = r₃² - q₃² \quad (59)
\]

that is the \( n \)-disks share a common \( n \)-sphere \( i.e \{ x|e₁^T · x = 0, ||x|| = a \} \). Let \( \mathcal{H} = \{ x ∈ \mathbb{R}^{n×1}|e₁^T · x ≤ 0 \} \) and \( \mathcal{G} = \{ x ∈ \mathbb{R}^{n×1}|e₁^T · x ≥ 0 \} = \mathbb{R}^{n×1} \setminus \text{int}(\mathcal{H}) \). Then the following inclusions are true

\[
\begin{align*}
\mathcal{H} \cap D₃ &⊆ D₁ \cap D₃ \subseteq D₂ \cap D₃ \subseteq D₁ \cap D₂ \\
\mathcal{G} \cap D₁ &⊆ \mathcal{G} \cap D₂ ⊆ \mathcal{G} \cap D₃
\end{align*}
\]

\quad (60)

\quad (61)

**Proof.** For (61) once can successively apply Lemma 3.1 claim (53) to obtain the desired result. For (60) the last inclusion is obvious. For the first inclusion, let \( x ∈ \mathcal{H} \cap D₃ \). Then we already have from Lemma 3.1 that \( x ∈ \mathcal{H} \cap D₁ \cap D₃ \subseteq D₁ \cap D₃ \). For the second inclusion we prove the following:

\[
\begin{align*}
D₁ \cap D₃ \cap \mathcal{H} &⊆ D₂ \cap D₃ \cap \mathcal{H} \quad \text{and} \quad D₁ \cap D₃ \cap \mathcal{G} \subseteq D₂ \cap D₃ \cap \mathcal{G}
\end{align*}
\]

\quad (62)

Indeed, in the above equation for the first inclusion let \( x ∈ D₁ \cap D₃ \cap \mathcal{H} \). We have from Lemma 3.1 that \( D₃ \cap D₃ \cap \mathcal{H} \subseteq D₂ \cap D₃ \cap \mathcal{H} \) hence (since \( x ∈ D₃ \))

\[
x ∈ D₁ \cap D₃ \cap \mathcal{H} \cap D₃ \subseteq D₂ \cap D₃ \cap \mathcal{H}
\]

\quad (63)

Finally, let \( x ∈ D₁ \cap D₃ \cap \mathcal{G} \). Applying Lemma 3.1 we obtain that \( D₁ \cap \mathcal{G} \subseteq D₂ \cap \mathcal{G} \). It follows

\[
x ∈ D₃ \cap D₂ \cap \mathcal{G}
\]

\quad (64)

\quad \Box
3.2 Geometric properties of $\mathcal{Q}_\rho$

We can now give the result which is being used in the main results:

**Lemma 3.3.** For $\rho > \beta \geq \max \{||S||, \sqrt{n}\}$ and $\mathcal{Q}_\rho$ defined at (20) we have

$$\mathcal{P} \subseteq \mathcal{Q}_\rho \subseteq \mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$$

(65)

**Proof.** Recall the fact that $\rho > \beta \geq \max \{||S||, \sqrt{n}\}$ and from (19) we can see that $r_\circ \geq \frac{\sqrt{n}}{2}$ for all $\circ \in \{k, h, s\}$. Indeed, for instance

$$r_\circ^2 = \frac{n}{4} + \rho \cdot \left(\rho - \frac{S^T \cdot 1_{n \times 1}}{||S||}\right) \geq \frac{n}{4} \iff \rho \geq \frac{S^T \cdot 1_{n \times 1}}{||S||}$$

(66)

which happens since $S^T \cdot 1_{n \times 1} \leq ||S|| \cdot \sqrt{n} \leq ||S|| \cdot \max\{||S||, \sqrt{n}\} \leq ||S|| \cdot \rho$. Furthermore, we can also see that $C_T^T \cdot S = \frac{1}{2} \cdot S^T \cdot 1_{n \times 1} - \rho \cdot ||S|| < 0$ and therefore every ball center is in the corresponding halfspace defining the polytope $\mathcal{P}$. In order to prove $\mathcal{P} \subseteq \mathcal{Q}_\rho$, we first focus on

$$\mathcal{P} \subseteq \mathcal{Q}_\rho$$

(67)

Indeed, since we know that $\mathcal{P} \subseteq \mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$ in order to prove that $\mathcal{P} \subseteq \mathcal{Q}_\rho$ is enough to prove that any half space composing $\mathcal{P}$ intersected with $\mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$ is included in an n-disk composing $\mathcal{Q}_\rho$ intersected with $\mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$ W.l.o.g take in the above Lemma 3.2 $\mathcal{H} = \{x|S^T \cdot x \leq 0\}$, $\mathcal{D}_3 = \mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$ and $\mathcal{D}_1 = \mathcal{B}(C, r_\circ)$ to obtain $\mathcal{H} \cap \mathcal{D}_3 \subseteq \mathcal{D}_1 \cap \mathcal{D}_3$

Next we prove that

$$\mathcal{Q}_\rho \subseteq \mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$$

(68)

Indeed

$$\mathcal{Q}_\rho \subseteq \bigcap_{k=1}^{n} \left(\mathcal{B}(C_k^+, r_{k+}) \cap \mathcal{B}(C_k^-, r_{k-})\right) = \hat{\mathcal{Q}}_\rho$$

(69)

W.l.o.g let us analyze one of the intersecting balls:

$$\hat{\mathcal{B}}(C_k^+, r_{k+}) \subseteq \left\{x|e_k^T \cdot x \geq 0\right\} \cup \left(\left\{x|e_k^T \cdot x \leq 0\right\} \cap \mathcal{B}(C_k^+, r_{k+})\right)$$

(70)

It can be proven using the above Lemma 3.2 and the construction of the ball $\hat{\mathcal{B}}(C_k^+, r_{k+})$ that

$$\left(\left\{x|e_k^T \cdot x \leq 0\right\} \cap \mathcal{B}(C_k^+, r_{k+})\right) \subseteq \left(\left\{x|e_k^T \cdot x \leq 0\right\} \cap \mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)\right) \subseteq \mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$$

(71)

hence we obtained

$$\hat{\mathcal{B}}(C_k^+, r_{k+}) \subseteq \left\{x|e_k^T \cdot x \geq 0\right\} \cup \mathcal{B}\left(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2}\right)$$

(72)
Finally considering the following property regarding a finite intersection of sets for $A_k, B \subseteq \mathbb{R}^{n \times 1}$

$$\bigcap A_k \subseteq B \Rightarrow \bigcap (A_k \cup B) \subseteq B$$  \hspace{1cm} (73)

Therefore, let $\bigcap A_k$ be the unit hypercube and $B = \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)$ hence since the unit hypercube $\bigcap_{k=1}^{n} \{ x | e_k^T \cdot x \geq 0 \} \cap \{ x | e_k^T \cdot x \leq 1 \}$) $\subseteq \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)$ one obtains $\hat{Q}_\rho \subseteq \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)$.

**Lemma 3.4.** For $\rho > \beta \geq \max \{ ||S||, \sqrt{n} \}$, $Q_\rho$ defined at (20) and

$$y_{p,\beta}^* \in \mathcal{Y}_{p,\beta}^* = \arg\max_{y \in Q_\rho} ||y - C_\beta||^2$$

we have

1. $\left\| y_{p,\beta}^* - \frac{1}{2} \cdot 1_{n \times 1} \right\| = \frac{\sqrt{n}}{2}$ if $y_{p,\beta} \in \{0, 1\}^{n \times 1}$
2. If exists $0_{n \times 1} \neq y_1 \in \{0, 1\}^{n \times 1}$ with $S^T \cdot y_1 = 0$ then $y_1 \in \mathcal{Y}_{p,\beta}^*$
3. If exists $0_{n \times 1} \neq y_1 \in \{0, 1\}^{n \times 1}$ with $S^T \cdot y_1 = 0$ then for all $y_{p,\beta}^* \in \mathcal{Y}_{p,\beta}^*$ one has $y_{p,\beta} \in \{0, 1\}^{n \times 1}$ and $S^T \cdot y_{p,\beta}^* = 0$

**Proof.** From Lemma 3.3 follows $Q_\rho \subseteq \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)$. Then:

1. For this claim is easy to verify the reverse implication. We focus on the direct one. Let $\left\| y_{p,\beta}^* - \frac{1}{2} \cdot 1_{n \times 1} \right\| = \frac{\sqrt{n}}{2}$. Recall $\hat{Q}_\rho$ from (69).

   Since $y_{p,\beta}^* \in Q_\rho \cap \partial \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right) \subseteq \hat{Q}_\rho \cap \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)$

   follows that $x_{p,\beta}^* \in \{0, 1\}^{n \times 1}$ since it can be proven that, for $\rho, \beta$ as required in the hypothesis, one has $\hat{Q}_\rho \cap \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right) \subseteq \{0, 1\}^{n \times 1}$. From (69) we have that $\hat{Q}_\rho$ is basically an approximation of the unit hypercube and we will not make here further efforts to prove that it only shares its corners with the ball centered in $\frac{1}{2} \cdot 1_{n \times 1}$.

2. Let $y_1 \neq 0_{n \times 1}, y_1 \in \{0, 1\}^{n \times 1}$ with $S^T \cdot y_1 = 0$ then by construction we get $y_1 \in Q_\rho \cap \{ y | S^T \cdot y = 0 \}$. Hence $\left\| y_1 - \frac{1}{2} \cdot 1_{n \times 1} \right\| = \frac{\sqrt{n}}{2}$ and $\| C_\beta - y_1 \| = r$. We shall prove that $\| y_{p,\beta}^* - C_\beta \| \leq \| y_1 - C_\beta \|$. We prove this by showing that $\| y - C_\beta \| \leq \| y_1 - C_\beta \|$ for all $y \in Q_\rho$. This would imply that $y_1 \in \mathcal{Y}_{p,\beta}^*$. Please note that the points $\frac{1}{2} \cdot 1_{n \times 1}, C_\beta, C_s$ are on the same axis, therefore, we can use the Lemma 3.2 as follows.

Since $\rho > \beta > \frac{\sqrt{n}}{2}$ we can choose in Lemma 3.2 $D_1 = \bar{B}(C_s, r_s), D_3 = \bar{B}(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2})$ and $D_2 = \bar{B}(C_\beta, \| C_\beta - y_1 \|)$ to assert that $D_1 \cap D_3 \subseteq D_2$. Furthermore, because $Q_\rho \subseteq D_3 \cap D_1$, i.e $Q_\rho \subseteq \bar{B}(C_s, r_s) \cap \bar{B}(\frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2})$ follows that for all $y \in Q_\rho$ one has $y \in D_2$ hence $\| y - C_\beta \| \leq \| C_\beta - y_1 \|$. This is known from the previous point that $\| C_\beta - y_{p,\beta}^* \| = \| C_\beta - y_1 \|$. Then it follows that $\| C_\beta - x_{p,\beta}^* \| = r_s$ and $\| \frac{1}{2} \cdot 1_{n \times 1} - y_{p,\beta}^* \| = \frac{\sqrt{n}}{2}$. Indeed, assume that $\| C_s - y_{p,\beta}^* \| < r_s$ and/or $\| y_{p,\beta}^* - \frac{1}{2} \cdot 1_{n \times 1} \| < \frac{\sqrt{n}}{2}$. Then by the use of Lemma 3.2, similarly to previous point, one obtains $\| C_\beta - y_{p,\beta}^* \| < \| C_\beta - y_1 \|$, which is false. Since $\| y_{p,\beta}^* - \frac{1}{2} \cdot 1_{n \times 1} \| = \frac{\sqrt{n}}{2}$ using the second point in the lemma, we get that $y_{p,\beta} \in \{0, 1\}^{n \times 1}$. Finally because $\| y_{p,\beta}^* - \frac{1}{2} \cdot 1_{n \times 1} \| = \frac{\sqrt{n}}{2}$ and $\| C_s - y_{p,\beta}^* \| = r_s$ we can prove through a simple calculation that $S^T \cdot x_{p,\beta}^* = 0$. 

$\square$
The following is a result which shows that $Q_\rho$ can approximate $P$ arbitrarily well.

**Lemma 3.5.** Given $\delta > 0$ fixed and $\rho > \beta \geq \max\{\|S\|, \sqrt{n}\}$ if $\exists B(x, \delta) \subseteq P$ then exists $\rho_\delta > 0$ such that for all $\rho \geq \rho_\delta$ one has

$$P \subseteq Q_\rho \subseteq \bigcup_{x \in P} B(x, \delta) \quad (76)$$

**Proof.** The first inclusion is true from Lemma 3.3. For the second inclusion one has the following. From Figure 1 one has

$$x = r_1 - q_1 = \sqrt{q_1^2 + a^2} - q_1 = \frac{a^2}{q_1 + \sqrt{q_1^2 + a^2}} \quad (77)$$

For a fixed $a$, letting $q_1$ be large enough one can see that $x \leq \delta$ will eventually occur. Therefore for any point $u$ in the intersection of the large disk with the right half-space exists a point $v$ in the intersection of the hyper-plane with the small disk such that $u \in B(v, \delta)$. Consider w.l.o.g the set $H_s = \{x | S^T \cdot x \leq 0\}$ i.e a facet of $P$. Since from

![Plane problem](image)

**Fig. 1. Plane problem**

Lemma 3.3 one has $Q_\rho \subseteq \bar{B}(\frac{1}{2} \cdot 1_n x_1, \frac{\sqrt{n}}{2})$ we want to prove that

$$\bar{B}(C_s, r_s) \cap \bar{B}(\frac{1}{2} \cdot 1_n x_1, \frac{\sqrt{n}}{2}) \subseteq \bigcup_{x \in \mathcal{H}_s \cap \bar{B}(\frac{1}{2} \cdot 1_n x_1, \frac{\sqrt{n}}{2})} B(x, \delta) \quad (78)$$

Regarding the above equation, it is easy to see that

$$\mathcal{H}_s \cap \bar{B}(C_s, r_s) \cap \bar{B}(\frac{1}{2} \cdot 1_n x_1, \frac{\sqrt{n}}{2}) \subseteq \mathcal{H}_s \cap \bar{B}(\frac{1}{2} \cdot 1_n x_1, \frac{\sqrt{n}}{2}) \quad (79)$$
Therefore we will now focus only on the elements of
\[
\mathcal{G}_s \cap \bar{B}(C_s, r_s) \cap \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)
\]
(80)

where \( \mathcal{G}_s = \{ x \mid S^T \cdot x \geq 0 \} \). For this in Figure 1, take the large disk \( \bar{B}(C_s, r_s) \) and the small disk \( \bar{B}(P, r_s) \) where \( P \) is the projection of \( C_s \) on the hyperplane \( \{ x \mid S^T \cdot x = 0 \} \) and \( r_s^2 = r_s^2 - \| C_s - P \|^2 \). Here choose from the figure \( q_1 \) and \( a \) as \( q_1 = \| C_s - P \| \) and \( a = r_s \). Then, as shown, given \( \delta > 0 \) exists \( q_\delta \) such that for any \( q_1 \geq q_\delta \) one has
\[
\bar{B}(C_s, r_s) \cap \mathcal{G}_s \subseteq \bigcup_{x \in \mathcal{H}_s \cap \mathcal{G}_s \cap \bar{B}(P, r_s)} B(x, \delta)
\]
(81)

However, please note that due to construction one has
\[
\mathcal{H}_s \cap \mathcal{G}_s \cap \bar{B}(P, r_s) = \mathcal{H}_s \cap \mathcal{G}_s \cap \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{1 \sqrt{n}}{2} \right)
\]
\[
\subseteq \mathcal{H}_s \cap \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{1 \sqrt{n}}{2} \right)
\]
(82)

hence from (81) and (82) it is obtained:
\[
\mathcal{G}_s \cap \bar{B}(C_s, r_s) \cap \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right) \subseteq \mathcal{G}_s \cap \bar{B}(C_s, r_s) \subseteq
\]
\[
\subseteq \bigcup_{x \in \mathcal{H}_s \cap \mathcal{G}_s \cap \bar{B}(C_s, r_s)} B(x, \delta)
\]
\[
\subseteq \bigcup_{x \in \mathcal{H}_s \cap \mathcal{G}_s \cap \bar{B} \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{2} \right)} B(x, \delta)
\]
(83)

Finally, since \( \| P_s - \frac{1}{2} \cdot 1_{n \times 1} \| \) is fixed, form the existence of \( q_\delta \) the existence of \( \rho_\delta \) easily follows. Repeat the same reasoning for the other half-spaces forming \( \mathcal{P} \).

\[\square\]

3.3 Maximizing the distance to a point over an intersection of balls

We begin this subsection with the following remark:

**Remark 5.** For all \( \rho \geq \frac{\sqrt{n}}{2} \) exist \( \epsilon_\rho > 0 \) such that for all \( 0 < \epsilon \leq \epsilon_\rho \) one has
\[
\bigcup_{x \in Q_\rho} B(x, \epsilon) \subseteq \{ x \mid \rho_\rho(x) \leq 1 \}
\]
(84)

Indeed let \( y \in \bigcup_{x \in Q_\rho} B(x, \epsilon) \) then exists \( x \in Q_\rho \) such that \( \| y - x \| \leq \epsilon \). Then for all \( \rho \in \{ k \pm, h, s \} \) one has
\[
\| y - C_\rho \|^2 = \| y - x + x - C_\rho \|^2 = \| y - x \|^2 + 2 \cdot (y - x)^T \cdot (x - C_\rho) + \| x - C_\rho \|^2 \leq \epsilon^2 + 2 \cdot \epsilon \cdot r_\rho + r_\rho^2
\]
(85)

letting \( \epsilon_\rho > 0 \) such that \( \epsilon_\rho^2 + 2 \cdot \epsilon_\rho \cdot r_\rho \leq 1 \) then \( \epsilon_\rho = \min \{ \epsilon_\rho, \rho \} \in \{ k \pm, h, s \} \) proves the statement.

Assume that \( \rho \geq \beta \geq \max \{ \| S \|, \sqrt{n} \} \). Please recall from (23) and (28):
\[
\mathcal{H}_{\rho, \beta} = \arg \min_{\rho_\rho(x) \leq 1} (\rho_\rho - f_\beta)(x)
\]
(86)

and
\[ \mathcal{P}_{p,\beta,R^2} = \left\{ x \in \mathbb{R}^{n \times 1} \left| (h_p - f)(x) \leq -R^2 \right. \right\} \]

for some \(0 \leq R \leq R_{p,\beta}\) where \(R_{p,\beta}\) is given by (29). One can see \(\mathcal{P}_{p,\beta,R^2}\) as a parameterized family of sub-level sets (of \(h_p - f\beta\)). It is easy to see that

\[ \mathcal{P}_{p,\beta,R^2_{p,\beta}} \cap \{x|h_p \leq 1\} = \left\{ x \left| (h_p - f\beta)(x) \leq -R^2_{p,\beta} \right. \right\} \cap \{x|h_p \leq 1\} = \mathcal{H}^*_{p,\beta} \]  

**Remark 6.** Since \(S\) has the property \(A(S, \epsilon)\) and \(\mathcal{P}_{p,\beta,R^2_{p,\beta}} \cap \{x|h_p(x) \leq 1\} = \mathcal{H}^*_{p,\beta}\) follows that one of the following two is true:

\(1\)

\[ \bigcup_{x \in \mathcal{P}_{p,\beta,R^2_{p,\beta}}} \mathcal{B}(x, \epsilon) \subseteq \mathcal{P} \]  

Indeed, if \(\bigcup_{x \in \mathcal{H}^*_{p,\beta}} \mathcal{B}(x, \epsilon) \subseteq \mathcal{P}\) one can prove that \(\mathcal{P}_{p,\beta,R^2_{p,\beta}} \subseteq \{x|h_p(x) \leq 1\}\) hence \(\mathcal{P}_{p,\beta,R^2_{p,\beta}} = \mathcal{H}^*_{p,\beta}\). In order to do this, assume that \(\exists x_1 \in \mathcal{P}_{p,\beta,R^2_{p,\beta}} \setminus \{x|h_p(x) \leq 1\}\) and \(x_2 \in \mathcal{H}^*_{p,\beta}\). It follows that exists \(t^* \in (0,1)\) and \(x_3 = t^* \cdot x_1 + (1 - t^*) \cdot x_2\) such that \(h_p(x_3) = 1\) and \(x_3 \in \mathcal{P}_{p,\beta,R^2_{p,\beta}}\). Therefore \(x_3 \in \mathcal{H}^*_{p,\beta} \subseteq \mathcal{P}\). This is a contradiction with (84).

\(2\)

\[ \bigcup_{x \in \mathcal{P}_{p,\beta,R^2_{p,\beta}}} \mathcal{B}(x, \epsilon) \subseteq \mathbb{R}^{n \times 1} \setminus Q_p \]  

We know that \(\bigcup_{x \in \mathcal{H}^*_{p,\beta}} \mathcal{B}(x, \epsilon) \subseteq \mathbb{R}^{n \times 1} \setminus Q_p\). Let \(x \in \bigcup_{x \in \mathcal{P}_{p,\beta,R^2_{p,\beta}}} \mathcal{B}(x, \epsilon)\). Assume

\[ x \in Q_p \cap \bigcup_{x \in \mathcal{P}_{p,\beta,R^2_{p,\beta}}} \mathcal{B}(x, \epsilon) \subseteq \{x|h_p(x) \leq 1\} \cap \bigcup_{x \in \mathcal{P}_{p,\beta,R^2_{p,\beta}}} \mathcal{B}(x, \epsilon) \]  

\[ \subseteq \bigcup_{x \in \mathcal{P}_{p,\beta,R^2_{p,\beta}} \cap \{x|h_p(x) \leq 1\}} \mathcal{B}(x, \epsilon) = \bigcup_{x \in \mathcal{P}_{p,\beta,R^2_{p,\beta}} \cap \{x|h_p(x) \leq 1\}} \mathcal{B}(x, \epsilon) \subseteq \mathbb{R}^{n \times 1} \setminus Q_p \]  

which is a contradiction.

Before advancing to the backbone theorem of this paper we give the following remark:

**Remark 7.** Please note that

1. \(Q_{\alpha,p} \subseteq Q_p\) for all \(\alpha \geq 1\) i.e the set \(Q_p\) shrinks as \(p\) increases and \(Q_{\alpha,p} \cap \partial Q_p \subseteq \{0, 1\}^{n \times 1}\)
2. \(\mathcal{P}_{\alpha,p,\beta,T_{1}} \subseteq \mathcal{P}_{\alpha,p,\beta,T_{2}}\) for \(T_1 < T_2\) i.e the set \(\mathcal{P}_{\cdot,\cdot,\cdot}\) shrinks as \(T\) increases and \(\mathcal{P}_{\alpha,p,\beta,T_{1}} \cap \partial \mathcal{P}_{\alpha,p,\beta,T_{2}} = \emptyset\)

We let the proof of these to the reader.

The following is the fundamental result of the paper and the one which enables the main results presented in the corresponding section.

**Theorem 3.6.** For \(\rho \geq \beta \geq \max\{\|\mathcal{S}\|, \sqrt{n}\}\), consider the problem:

\[ R^*_{\rho,\beta} = \max_{y \in Q_p} \|y - C\rho\|^2 \]  

\[ (92) \]
where $Q_\rho$ is given by (20) has a nonempty interior and $C_\beta$ is given by (11). For $H^*_p,\beta$ given by (86) we have the following complete alternatives:

1. $H^*_p,\beta \subseteq \text{int} \{Q_\rho\}$ In such a case

\[ R^*_{p,\beta} = \min \{ R > 0 | \mathcal{P}_{p,\beta,R} \subseteq Q_\rho \} \]  \tag{93}

2. $H^*_p,\beta \subseteq \text{int} \left( \mathbb{R}^{n+1} \setminus Q_\rho \right)$ In such a case

\[ R^*_{p,\beta} = \max \{ R > 0 | \mathcal{P}_{p,\beta,R} \cap Q_\rho \neq \emptyset \} \]  \tag{94}

3. $H^*_p,\beta \cap \partial Q_\rho \neq \emptyset$ In such a case

\[ R^*_{p,\beta} = R_{p,\beta} \]  \tag{95}

with $R_{p,\beta}$ is given by (29) and $\mathcal{P}_{p,\beta,R}$ is given by (87). In each case a maximizer is found on $\partial Q_\rho \cap \mathcal{P}_{p,\beta,R\ast}$

**Proof.** Since for $R = 0$ we can easily see that $Q_\rho \subseteq \mathcal{P}_{p,\beta,R}$ therefore indeed the items in the enumerate cover all the possible outcomes if $R$ is increased from 0 to $R_{p,\beta}$ and also the existence of the $R^*_{p,\beta}$ as defined for each case.

The problem (92) is maximizing a convex function over a convex domain, hence the maximizer will be on the boundary. We give the proof for the following cases:

1. $H^*_p,\beta \subseteq \text{int} \{Q_\rho\}$ i.e $h_p(x^*_{p,\beta}) < 0$ for all $x^*_{p,\beta} \in H^*_p,\beta$: For some $0 < R < R^*_{p,\beta}$ we shall prove that

\[ \{ x | h_p(x) = 0 \} \setminus B(C_\beta, R) \subseteq \mathcal{P}_{p,\beta,R^*} \]  \tag{96}

which means that the points on the boundary of $Q_\rho$ whom distance to $C_\beta$ is greater than $R$, are inside $\mathcal{P}_{p,\beta,R^*}$. Indeed, this is easy to verify: let $h_p(x) = 0$ and $\|x - C_\beta\| > R$ then

\[ h_p(x) + R^2 - \|x - C_\beta\|^2 \leq 0 \iff (h_p - f_\beta)(x) \leq -R^2 \iff x \in \mathcal{P}_{p,\beta,R^*} \]  \tag{97}

Since $\mathcal{P}_{p,\beta,R^*} \cap \partial Q_\rho = 0$ for all $R > R^*_{p,\beta}$ (by the definition of $R^*_{p,\beta}$ and Remark 7), follows that $\not\exists x_2$ with $h_p(x_2) = 0$, i.e, on the boundary of $Q_\rho$, and $\|x_2 - C_\beta\| > R^*_{p,\beta}$. Indeed, assuming that $\exists \|x_2 - C_\beta\| = R_2 > R^*_{p,\beta}$ follows that $x_2 \in \mathcal{P}_{p,\beta,R^*} \setminus \partial Q_\rho$ since $h_p(x_2) - \|x_2 - C_\beta\|^2 = -R_2^2 < \left( R^*_{p,\beta} \right)^2$. Also, since $h_p(x_2) = 0$ follows that $x_2 \in \partial Q_\rho$ and a contradiction arises with the definition of $R^*_{p,\beta}$ and Remark 7 because now $x_2 \in \mathcal{P}_{p,\beta,R^*} \cap \partial Q_\rho$ and $R_2 > R^*_{p,\beta}$.

2. $H^*_p,\beta \subseteq \text{int} \left( \mathbb{R}^{n+1} \setminus Q_\rho \right)$ then $h_p(x^*_{p,\beta}) > 0$ for all $x^*_{p,\beta} \in H^*_p,\beta$. Assume that $\exists x_2 \in \partial Q_\rho$ such that $\|x_2 - C_\beta\| = R > R^*_{p,\beta}$. Then

\[ h_p(x_2) - \|x_2 - C_\beta\|^2 = -R^2 \Rightarrow x_2 \in \mathcal{P}_{p,\beta,R^*} \cap Q_\rho \]  \tag{98}

this is a contradiction with the definition of $R^*_{p,\beta}$.

3. $H^*_p,\beta \cap Q_\rho \neq \emptyset$ for all $R \in [0,R_{p,\beta}]$ Assume that $\exists x_2$ with $h_p(x_2) = 0$ and $\|x_2 - C_\beta\| = R > R_{p,\beta}$ then

\[ h_p(x_2) - \|x_2 - C_\beta\|^2 = -R^2 < -R_{p,\beta}^2 \]  \tag{99}

this is a contradiction with the definition of $R_{p,\beta}$ since $x_2 \in \{ x | h_p(x) \leq 1 \}$
3.4 Congruent polytopes

The following lemma shows an analysis of what happens by increasing $\rho$ and $\beta$ by the same factor.

**Lemma 3.7.** For some $\rho > \beta > \max \{||S||, \sqrt{n}\}$ consider $Q_{\beta}$ given by (20) and $C_{\beta}$ given by (11). For $\alpha \geq 1$ form $P_{\rho, \beta, R^2}$ and $P_{\rho, \alpha, \beta, T^2}$ for $R, T > 0$ as given by (87) i.e

\[
\begin{align*}
P_{\rho, \beta, R^2} &= \{ x | h_\rho(x) - \|x - C_\beta\|^2 \leq -R^2 \} \\
P_{\rho, \alpha, \beta, T^2} &= \{ x | h_{\alpha, \beta}(x) - \|x - C_{\alpha, \beta}\|^2 \leq -T^2 \}
\end{align*}
\]

with $C_\beta = \frac{1}{2} \cdot 1_{n \times 1} - \frac{\beta}{2} \cdot \frac{S}{||S||}$ and $C_{\alpha, \beta} = \frac{1}{2} \cdot 1_{n \times 1} - \frac{\alpha \beta}{2} \cdot \frac{S}{||S||}$ and

\[
\begin{align*}
h_\rho(x) &= \max_{\phi \in \{ \pm, \pm \}} \|x - C_{\phi, \rho}\|^2 - \rho^2 \\
h_{\alpha, \beta}(x) &= \max_{\phi \in \{ \pm, \pm \}} \|x - C_{\phi, \alpha, \beta}\|^2 - \rho^2
\end{align*}
\]

where

\[
\begin{align*}
C_{k, \rho} &= \frac{1}{2} \cdot 1_{n \times 1} + \rho \cdot e_k \\
C_{h, \rho} &= \frac{1}{2} \cdot 1_{n \times 1} + \rho \cdot \frac{1_{n \times 1}}{||1_{n \times 1}||} \\
C_{s, \rho} &= \frac{1}{2} \cdot 1_{n \times 1} - \rho \cdot \frac{S}{||S||}
\end{align*}
\]

\[
\begin{align*}
C_{k, \alpha, \beta} &= \frac{1}{2} \cdot 1_{n \times 1} + \alpha \cdot \rho \cdot e_k \\
C_{h, \alpha, \beta} &= \frac{1}{2} \cdot 1_{n \times 1} + \alpha \cdot \rho \cdot \frac{1_{n \times 1}}{||1_{n \times 1}||} \\
C_{s, \alpha, \beta} &= \frac{1}{2} \cdot 1_{n \times 1} - \alpha \cdot \rho \cdot \frac{S}{||S||}
\end{align*}
\]

and for $\phi \in \{ k, s, h \}$ $r_{\phi, \rho}$ and $r_{\phi, \alpha, \beta}$ are computed such that the ball $B(C_{\phi, \rho}, r_{\phi, \rho})$ and $B(C_{\phi, \alpha, \beta}, r_{\phi, \alpha, \beta})$ leave the same imprint (ie. have the same intersection) on the boundary of $B \left( \frac{1}{2} \cdot 1_{n \times 1}, \frac{\sqrt{n}}{T} \right)$ as the hyperplanes defining $P$ with $P$ given by (11). Let $P_\rho$ be the projection of $C_{\phi, \rho}$ on the corresponding facet of the polytope $P$. Please note that due to their construction $C_{\phi, \rho}$ and $C_{\phi, \alpha, \beta}$ have the same projection. Let

\[
\begin{align*}
\tilde{r}_{\rho}^2 &= \frac{n}{4} - \frac{1}{2} \cdot 1_{n \times 1} - P_\rho \|^2 \\
r_{\rho, \rho}^2 &= \|C_{\phi, \rho} - P_\rho\|^2 + \tilde{r}_{\rho}^2 \\
r_{\phi, \alpha, \beta}^2 &= \|C_{\phi, \alpha, \beta} - P_\rho\|^2 + \tilde{r}_{\rho}^2
\end{align*}
\]

Then for all $R > 0$ exists $T > 0$ such that

\[P_{\rho, \beta, R^2} = P_{\rho, \alpha, \beta, T^2}\]

**Proof.** To prove the above statement, we consider the half-spaces forming each polytope. Let $T, R > 0$ and $\alpha \geq 1$, then

\[
\begin{align*}
\|x - C_{\phi, \rho}\|^2 - r_{\phi, \rho}^2 - \|x - C_{\rho}\|^2 + R^2 &\leq 0 \\
\|x - C_{\phi, \alpha, \beta}\|^2 - r_{\phi, \alpha, \beta}^2 - \|x - C_{\alpha, \beta}\|^2 + T^2 &\leq 0
\end{align*}
\]

We want to show that for $R$ exists $T$ such that the two inequalities are the same for all $x \in \mathbb{R}^{n \times 1}$.

We write $C_{\phi, \rho} = \frac{1}{2} \cdot 1_{n \times 1} + \rho \cdot V_\rho$ and $C_{\phi, \alpha, \beta} = \frac{1}{2} \cdot 1_{n \times 1} + \alpha \cdot \rho \cdot V_\rho$ with $\|V_\rho\| = 1$. Then we get for the first inequality

\[ (C_{\beta} - C_{\phi, \rho})^T \cdot (2 \cdot x - C_{\phi, \rho} - C_{\beta}) + R^2 - r_{\phi, \rho}^2 \leq 0 \]

which is

\[ \left( -\frac{\beta}{2} \cdot \frac{S}{||S||} - \rho \cdot V_\rho \right)^T \cdot \left( 2 \cdot x - 1_{n \times 1} + \frac{\beta}{2} \cdot \frac{S}{||S||} - \rho \cdot V_\rho \right) + R^2 - r_{\phi, \rho}^2 \leq 0 \]

For the second inequality we get, in a similar way:

\[ \left( -\frac{\alpha \cdot \beta}{2} \cdot \frac{S}{||S||} - \alpha \cdot \rho \cdot V_\rho \right)^T \cdot \left( 2 \cdot x - 1_{n \times 1} + \alpha \cdot \frac{\beta}{2} \cdot \frac{S}{||S||} - \alpha \cdot \rho \cdot V_\rho \right) + T^2 - r_{\phi, \alpha, \beta}^2 \leq 0 \]
We want to show that exists with

Using (102) and the fact that by multiplying (107) with $20$, Vol. 1, No. 1, Article . Publication date: July 2022.

\[
(\frac{\beta}{2} \cdot \frac{S}{||S||} - \rho \cdot V_o) \cdot (2 \cdot x - 1_{nx1}) + (\frac{\beta}{2} \cdot \frac{S}{||S||} - \rho \cdot V_o) \cdot (\frac{\beta}{2} \cdot \frac{S}{||S||} - \rho \cdot V_o) + R^2 - r^2_{o,\rho} \leq 0 \tag{109}
\]

and (108) as

\[
(-\frac{\alpha \beta}{2} \cdot \frac{S}{||S||} - \alpha \rho \cdot V_o) \cdot (2 \cdot x - 1_{nx1}) + (-\frac{\alpha \beta}{2} \cdot \frac{S}{||S||} - \alpha \cdot \rho \cdot V_o) \cdot (\frac{\alpha \beta}{2} \cdot \frac{S}{||S||} - \alpha \rho \cdot V_o) + T^2 - r^2_{o,\alpha \rho} \leq 0 \tag{110}
\]

since the first term in second equations is already obtained by multiplying the first term in the first equation with $\alpha$, we focus on the rest of the term in both equations. These terms are for the first and second equation

\[
G_1 = -\left(\left(\frac{\beta}{2}\right)^2 - \rho^2\right) + R^2 - r^2_{o,\rho} \quad G_2 = -\left(\left(\frac{\alpha \cdot \beta}{2}\right)^2 - (\alpha \cdot \rho)^2\right) + T^2 - r^2_{o,\alpha \rho} \tag{111}
\]

Let $\theta_o = \alpha \cdot G_1 - G_2$ and we search for $T$ not depending on $\circ \in \{k, h, s\}$ such that $\theta_o = 0$. We obtain

\[
\theta_o = \alpha \cdot \left(\left(-\left(\frac{\beta}{2}\right)^2 - \rho^2\right) + R^2 - r^2_{o,\rho}\right) - \left(-\left(\frac{\alpha \cdot \beta}{2}\right)^2 - (\alpha \cdot \rho)^2\right) + T^2 - r^2_{o,\alpha \rho}
\]

\[
= \alpha \cdot R^2 - T^2 - \alpha \cdot r^2_{o,\rho} + r^2_{o,\alpha \rho} + (\alpha - \alpha^2) \cdot \rho^2 + (\alpha^2 - \alpha) \cdot \frac{\beta^2}{4} \tag{112}
\]

Let us evaluate using (103)

\[
-\alpha \cdot r^2_{o,\rho} + r^2_{o,\alpha \rho} = -\alpha \cdot \left(\left\|C_{o,\rho} - P_o\right\|^2 + \frac{n}{4} - \frac{1}{2} \cdot 1_{nx1} - P_o\right) + \left(\left\|C_{o,\alpha \rho} - P_o\right\|^2 + \frac{n}{4} - \frac{1}{2} \cdot 1_{nx1} - P_o\right)
\]

\[
= \frac{n}{4} \cdot (1 - \alpha) \cdot \alpha \cdot \left(\left\|C_{o,\rho} - P_o\right\|^2 - \frac{1}{2} \cdot 1_{nx1} - P_o\right) + \left(\left\|C_{o,\alpha \rho} - P_o\right\|^2 - \frac{1}{2} \cdot 1_{nx1} - P_o\right) \tag{113}
\]

we focus on the last two terms

\[
-\alpha \cdot \left(\left\|C_{o,\rho} - P_o\right\|^2 - \frac{1}{2} \cdot 1_{nx1} - P_o\right) + \left(\left\|C_{o,\alpha \rho} - P_o\right\|^2 - \frac{1}{2} \cdot 1_{nx1} - P_o\right)
\]

\[
= -\alpha \cdot \left(C_{o,\rho} - \frac{1}{2} \cdot 1_{nx1}\right)^T \cdot \left(C_{o,\rho} - \frac{1}{2} \cdot 1_{nx1} - 2 \cdot P_o\right) + \left(C_{o,\alpha \rho} - \frac{1}{2} \cdot 1_{nx1}\right)^T \cdot \left(C_{o,\alpha \rho} + \frac{1}{2} \cdot 1_{nx1} - 2 \cdot P_o\right) \tag{114}
\]

Using (102) and the fact that $C_{o,\rho} = \frac{1}{2} \cdot 1_{nx1} + \rho \cdot V_o$ we get

\[
C_{o,\rho} - \frac{1}{2} \cdot 1_{nx1} = \rho \cdot V_o \quad C_{o,\alpha \rho} - \frac{1}{2} \cdot 1_{nx1} = \alpha \cdot \rho \cdot V_o \quad C_{o,\rho} + \frac{1}{2} \cdot 1_{nx1} - 2 \cdot P_o = 1_{nx1} + \rho \cdot V_o - 2 \cdot P_o
\]

\[
C_{o,\alpha \rho} + \frac{1}{2} \cdot 1_{nx1} - 2 \cdot P_o = 1_{nx1} + \alpha \cdot \rho \cdot V_o - 2 \cdot P_o
\]

hence (114) becomes

\[
-\alpha \cdot \rho \cdot V_o^T \cdot (1_{nx1} + \rho \cdot V_o - 2 \cdot P_o) + \alpha \cdot \rho \cdot V_o^T \cdot (1_{nx1} + \alpha \cdot \rho \cdot V_o - 2 \cdot P_o) = -\alpha \cdot \rho^2 + \alpha^2 \cdot \rho^2 = \alpha \cdot (\alpha - 1) \cdot \rho^2 \tag{116}
\]
then
\[-\alpha \cdot r_{\alpha,p}^2 + r_{\alpha,\rho}^2 = \frac{n}{4} \cdot (1 - \alpha) + \alpha \cdot (\alpha - 1) \cdot \rho^2\]  
(117)

and finally from (112)
\[\theta_0 = \alpha \cdot R^2 - T^2 + \frac{n}{4} \cdot (1 - \alpha) + \alpha \cdot (\alpha - 1) \cdot \rho^2 + \alpha \cdot (1 - \alpha) \cdot \rho^2 + \alpha \cdot (\alpha - 1) \cdot \frac{\beta^2}{4} \]
\[= \alpha \cdot R^2 - (\alpha - 1) \cdot \frac{n}{4} + \alpha \cdot (\alpha - 1) \cdot \frac{\beta^2}{4} - T^2\]  
(118)

In order to have \(\theta_0 = 0\) just take \(T^2 = \alpha \cdot R^2 - (\alpha - 1) \cdot \frac{n}{4} + \alpha \cdot (\alpha - 1) \cdot \frac{\beta^2}{4}\) which does not depend on \(\diamond \in \{k\pm, h, s\}\)

4 A NAIVE ALGORITHM FOR THE RSSP PROBLEM

Although the main results of this paper concern a rigorous solution for the \(\varepsilon\)RSSP problem under the hypothesis that the given \(S\) from the RSSP problem has the property \(A(S, \varepsilon)\) for a desired \(\frac{1}{4} \geq \varepsilon > 0\), we give here a "naive" algorithm which solves the RSSP problem with no restrictions. We call it naive, because we assume here that the convex optimization problems can be solved exactly. In this conditions the RSSP problem can be solved exactly as well in a finite sequence of convex optimization problems. This algorithm captures our proposed approach towards the RSSP problem.

5 CONCLUSION AND FUTURE WORK

We provide a solution to a well known and studied problem in a more general case: the subset problem for real numbers. Our approach is to solve a classic optimization problem associated to the subset sum problem, i.e maximization of a quadratic function over a polytope. We rewrite this problem as a maximization of the distance to a fixed point over the polytope, and show that the subset sum problem has a solution if and only if the maximum distance has a certain easily computable value. Finally, for some instances of the subset sum problem, we give a polynomial algorithm which delivers the correct maximum, up to an arbitrary precision in case the subset problem has a solution. We also show how this result can be used to decide what we define to be the \(\varepsilon\) relaxed subset sum problem.

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Algorithm 1 A naive algorithm for the RSSP problem

Input $S \in \mathbb{R}^{n \times 1}$

Require: $\text{int}(\mathcal{P}) \neq \emptyset$ where $\mathcal{P}$ is given by (11)

Choose $\rho > \beta > \max\{\|S\|, \sqrt{n}\}$,

Form $C_\beta$ and $C_\circ$ for $\circ \in \{k \pm h, s\}$ as given by (18)

Form the functions $h_\rho(x)$ and $f_\beta(x)$ as given by (21)

Define for $R > 0$ the polytope

$$\mathcal{P}_{\rho,\beta,R^2} = \{x | h_\rho(x) - f_\beta(x) \leq -R^2\}$$

Compute

$$\mathcal{H}^*_{\rho,\beta} = \arg\min_{h_\rho(x) \leq 1} h_\rho(x) - f_\beta(x)$$

if $\mathcal{H}^*_{\rho,\beta} \subseteq \text{int}(\mathcal{P})$ then

$$R^* \leftarrow \min\{R > 0 | \mathcal{P}_{\rho,\beta,R^2} \subseteq \mathcal{P}\}$$

else

if $\mathcal{H}^*_{\rho,\beta} \subseteq \text{int}(\mathcal{R}^{n \times 1} \setminus \mathcal{P})$ then

$$R^* \leftarrow \max\{R > 0 | \mathcal{P}_{\rho,\beta,R^2} \cap \mathcal{P} \neq \emptyset\}$$

else

if $\mathcal{H}^*_{\rho,\beta} \cap \partial \mathcal{P} \neq \emptyset$ then

$$R^* \leftarrow \|x^* - C_\beta\| \quad \forall x^* \in \mathcal{H}^*_{\rho,\beta} \cap \partial \mathcal{P}$$

end if

end if

end if

if $R^* = \|C_\beta\|$ then

The RSSP problem does have a solution

else

The RSSP problem does NOT have a solution

end if

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