Modified MIT Bag Models:
Thermodynamical consistency, stability windows and symmetry group

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In this work we study different variations of the MIT bag model. We start with the so called non-ideal bag model and discuss it in detail. Then we implement a vector interaction in the MIT bag model that simulates a meson exchange interaction and fix the quark-meson coupling constants via symmetry group theory. At the end we propose an original model, inspired by the Boguta-Bodmer models, which allows us to control the repulsion interaction at high densities. For each version of the model we obtain a stability window as predicted by the Bodmer-Witten conjecture and discuss its thermodynamical consistency.

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I. INTRODUCTION

According to the quantum chromodynamics (QCD) phase diagram, deconfined quark matter existed in the early universe when the temperature was very high. The region, where it is foreseen is called quark-gluon plasma (QGP). At zero or low temperatures, however, deconfined quark matter may be present in the core of massive compact objects like neutron stars.

Bodmer and Witten proposed that the interior of those stars may consist of strange matter (SM), which in turn is composed of deconfined up, down and strange quarks, besides the leptons necessary to ensure charge neutrality and β-equilibrium [1, 2]. In addition to that, the Bodmer-Witten conjecture states that the SM may be the true ground state of all matter, which means that, as soon as the core of the star converts to the quark phase, the entire star converts and what is left is called a strange star [3].

There have been some models used to approach the SM hypothesis, the first of them being the original MIT bag model [4]. A modified version of this original model, called a non-ideal bag model [5, 6] has also been used to test this hypothesis. Furthermore the quark-mass density dependent (QMDD) [7] and the Nanbu-Jona-Lasinio (NJL) [8–12] were used to test this hypothesis. These non-linear terms mimic the Dirac sea contribution, which is absent in mean field approximation [20]. Finally the strength of the interaction is evaluated. Our task in then successfully accomplished.

II. ORIGINAL MIT BAG MODEL

The MIT bag model considers that each baryon is composed of three non-interacting quarks inside a bag. The bag, in its turn, corresponds to an infinity potential that confines the quarks. In this simple model the quarks are free inside the bag and are forbidden to reach its exterior. All the information about the strong force relies on the bag, which mimics the vacuum pressure. The MIT Lagrangian reads [4]:

\[ \mathcal{L} = \sum_{u,d,s} \left\{ \bar{\psi}_q [i\gamma^\mu \partial_\mu - m_q] \psi_q - B \right\} \Theta(\bar{\psi}_q \psi_q), \]  

where \( m_q \) is the \( q \) quark mass running from \( u, d \) and \( s \), \( \bar{\psi}_q \) is the Dirac quark field, \( B \) is the constant vacuum pressure and \( \Theta(\bar{\psi}_q \psi_q) \) is the Heaviside step function to assure that the quarks exist only confined to the bag.

Applying the Euler-Lagrange equations for the quarks, we have:

\[ [i\gamma^\mu \partial_\mu - m_q] \psi_q = 0, \]  

which gives us an energy eigenvalue for the quark \( q \):
Quarks are fermions with spin 1/2. Hence, the number density and the energy density of the quark matter can be obtained via Fermi-Dirac distribution [21]. At zero temperature approximation the Fermi-Dirac distribution becomes the Heaviside step function, and the energy eigenvalue matches the chemical potential, \( E_q = \mu_q \). Therefore the quark number density is:

\[
n_q = 2N_c \int_0^{k_f} \frac{d^3k}{(2\pi)^3} \Theta(\mu_q - E_q), = N_c \frac{k_f^3}{3\pi^2} \quad (4)
\]

while the energy density is:

\[
\epsilon_q = N_c \frac{k_f^3}{3\pi^2} \int_0^{k_f} E_q k^2 dk, \quad (5)
\]

where \( N_c \) is the number of colors. On other hand the Bag contribution to the energy density is easily obtained through the Hamiltonian: \( \mathcal{H} = -\langle \mathcal{L} \rangle = B \). Then, the total energy density is the sum over the three lightest quarks plus the Bag contribution. The pressure can be obtained via thermodynamic relations:

\[
\epsilon = \sum_q \epsilon_q + B \quad \text{and} \quad p = \sum_q n_q \mu_q - \epsilon = -\Omega \quad (6)
\]

where \( n \) is the number density, \( \mu \) is the chemical potential and \( \Omega \) is the thermodynamic potential. It is also useful to write down the pressure (as well the thermodynamic potential) as the derivative of the energy with respect to the volume.

\[
p_q = -\Omega = -\left( \frac{\partial E_q}{\partial V} \right)_T = \frac{N_c}{3\pi^2} \int_0^{k_f} k^4 dk \frac{E_q}{E_q}, \quad (7)
\]

and the total pressure is the sum of the pressure of each quark flavor plus the contribution of the bag from the Lagrangian: \( p = \langle \mathcal{L} \rangle \). Therefore we obtain:

\[
p = \sum_i p_i - B. \quad (8)
\]

Now, the Bag constant is not totally arbitrary. In fact \( B \) needs to assure that the two-flavored quark matter is unstable and it is not the ground state of the hadrons, i.e. the energy per baryon is higher than 930 MeV at zero pressure, otherwise the protons and neutrons would decay into u and d quarks. On the other hand, if the Bodmer-Witten conjecture [1, 2], that states that the true ground state of the hadronic matter is not the baryons, but strange matter consisting of \( \mu_u = \mu_d = \mu_s \), is true, the three-flavored quark matter needs to be stable (energy per baryon lower than 930 MeV), while the two-flavored quark matter is unstable. Therefore, the bag pressure \( B \) can only assume a range of values, known as the stability window [1, 2, 14]. These values depend on the quark masses. In this work, we assume that the \( u \) and \( d \) masses are 4 MeV. While in the past the mass of the \( s \) quark was very ambiguous, today it is known to be around 95 MeV [22]. With these values, we display the range of \( B \) for the stability window in Table I. Slightly different values are obtained if different quark masses are chosen, as seen in [14].

| MIT  | Min. \( B^{1/4} \) | Max. \( B^{1/4} \) |
|------|------------------|------------------|
|      | 148 MeV          | 159 MeV          |

TABLE I. Stability window for for MIT bag model.

III. NON IDEAL BAG MODEL

The non ideal bag model, presented in ref. 3, 8 is an empirical correction at the thermodynamical potential - \( \Omega \) - (or direct in the pressure as in ref. 22) to match QCD correction of \( O(\alpha_s) \). In this model, the deviation of the original MIT bag model comes from an adimensional parameter \( a_q \), where \( a_q = (1-2\alpha_s/\pi) \), included in an \( ad \) \( hoc \) way to the thermodynamical potential to reproduce some results coming from lattice QCD. When \( a_q = 1 \), we the original MIT bag model is recovered. For massless quarks the thermodynamical potential reads [3, 8, 23]:

\[
\Omega = -a_q \frac{N_f}{4\pi^2} u^4 + B_{eff}, \quad (9)
\]

where, \( \mu \) is the chemical potential of massless quarks, \( N_f \) is the number of massless quarks, and \( B_{eff} \) is an effective bag. With this model, the energy and the pressure for each of the massless quarks are given by [8]:

\[
p = a_q \frac{1}{4\pi^2} u^4 - B_{eff}, \quad
\]

\[
\epsilon = a_q \frac{3}{4\pi^2} u^4 + B_{eff}, \quad \epsilon = 3p + 4B_{eff}. \quad (10)
\]

Although this model does not modify the equation of state \( (p(\epsilon)) \), it was proposed in order that the parameter \( a_q \) could change the relation between the pressure and the chemical potential - \( p(\mu) \). We discuss here some subtleties of this model.

First, as this model, in principle, is not constructed at the Lagrangian level, it is hard to understand what kind of interaction could produce such behavior. The second is a little more serious. This model can present thermodynamic inconsistency if we do not proceed carefully.
one tries to compute the chemical potential by thermodynamical relations, for each flavor, for massless quarks ($\mu = k_f$) with $n$ given by eq. (4) and $N_c = 3$, it results in

$$\mu = \frac{\partial \epsilon}{\partial n} = a_4 \mu,$$

hence, this is true only if $a_4 = 1$. A similar result appears when we try to compute the pressure for the thermodynamical relation: $p = n\mu - \epsilon$:

$$p = \frac{3\mu^3}{3\pi^2} + \mu - \left( \frac{3a_2}{4\pi^2} a_4^2 + B_{eff} \right) = \frac{\mu^4(12 - 9a_4)}{12\pi^2} - B_{eff}. \tag{12}$$

Therefore, if we put $a_4 = 1$, we recover the result of eq. (7) for massless quarks. However, if $a_4 \neq 1$ we do not recover (9) because $12 - 9a_4 \neq a_4(12 - 9)$. The only way to recover the thermodynamic consistancy is assume from the start that $\mu = a_4 \mu$. However, when doing that, $p(\mu)$ is not affected by the parameter $a_4$ anymore, and the whole proposal is lost.

The authors of ref. [13] proposed that the number density $n_q$ has also a correction:

$$n_q = a_4 \left( N_c \frac{k_f^2}{3\pi^2} \right) = a_4 n_q. \tag{13}$$

In this case, we fully recover the thermodynamic consistancy of the model. More than that. Assuming only eq. (13), all eqs. (10) come as consequence for massless quarks. Therefore, the non ideal bag model is a modification in the quark number, instead of a correction in the MIT model itself. On other hand, the price we have to pay is that this modification violates the Fermi-Dirac distribution [21]. This model violates the Fermi-Dirac statistics! And this fact needs to be clear in mind.

To summarize, the non ideal bag model can be faced as an empirical correction of QCD, but the reader needs to notice that it has thermodynamic issues. With this in mind we construct a stability window for the non-ideal bag model and use massive quarks, with the same masses as used previously, in the original MIT model. Therefore, for each flavor, we have the following EoS:

$$\epsilon_q = a_4 \frac{N_c}{3\pi^2} \int_0^{k_f} E_q k^2 dk, \tag{14}$$

$$p_q = a_4 \frac{N_c}{3\pi^2} \int_0^{k_f} \frac{k^4 dk}{E_q},$$

$$n_q = a_4 N_c \frac{k_f^3}{3\pi^2} \tag{15}$$

and

$$\epsilon = \sum \epsilon_q + B_{eff} \quad \text{and} \quad p = \sum p_q - B_{eff}. \tag{16}$$

Now let’s redefine $a_4$ as $a_4 = 1 - c$ as made in ref. [6]. In this case, if $c = 0$ we recover the original MIT bag model. As we increase $c$ we deviate from the traditional model. We construct a stability window for values varying from 0 to 0.3, which coincides with the values used in ref. [6]. The stability window is displayed in fig. 1 and the corresponding values are presented in Tab. III

![Stability window for the non ideal bag model.](image)

**FIG. 1.** (Color online) Stability window for the non ideal bag model. If $c = 0$ we recover the original MIT bag model.

| $c$ | Min. $B^{1/4}$ | Max. $B^{1/4}$ |
|-----|----------------|----------------|
| 0.0 | 148 MeV        | 159 MeV        |
| 0.1 | 144 MeV        | 155 MeV        |
| 0.2 | 140 MeV        | 150 MeV        |
| 0.3 | 135 MeV        | 145 MeV        |

**TABLE II.** Stability windows for the non ideal bag model.

As we can see as we increase the value of $c$, we displace the stability window almost the same range. Increasing $c$ from 0 to 0.3 displaces the minimum value of $B^{1/4}$ from 148 MeV to 135 MeV. On other hand the range is almost the same, 11 MeV for $c = 0$ to 10 MeV for $c = 0.3$.

It is worth keeping in mind that the regions outside the stability window on the left and on the right have very different meaning and consequences. The $B^{1/4}$ cannot be lower than the minimum value presented in Tab. III because this would imply that two-flavor quark matter were stable and our known universe composed of protons and neutron would no longer exist. But $B^{1/4}$ can be higher than the maximum value. This implies that the strange quark matter is not the ground state of the matter. In this case deconfined quark matter can only be present in the core of massive hybrid stars, instead of forming quark stars.

With the stability window we can now study what the maximum mass of a stable strange star is. To accomplish this, we need to construct a neutral, beta-stable quark matter. We add leptons as free Fermi gas and impose chemical equilibrium:
\[ \mathcal{L}_{\text{EFP}} = \sum_i \bar{\psi}_i i \gamma^\mu \partial_\mu - m_i \psi_i, \]  

where the sum runs over the two lightest leptons (e and \( \mu \)), and:

\[
\begin{align*}
\mu_s &= \mu_d = \mu_u + \mu_e \quad \text{and} \quad \mu_e = \mu_\mu, \\
n_s + n_\mu &= \frac{1}{3} (2n_u - n_d - n_s).
\end{align*}
\] 

As the \( u \) and \( d \) quark masses are the same, to avoid saturating the figure, (or display several ones) we plot Fig. 2. The strangeness fraction instead of the individual particle population, for the values of \( c \) presented in Tab. II. The strangeness fraction is defined as:

\[ f_s = \frac{n_s}{\sum n_f} = \frac{1}{3} \frac{n_s}{n}. \] 

where \( n \) is the total baryon number density, \( n_s \) is the strange quark number density and the sum runs over the three quark flavors. As we can see, when we increase the value of \( c \) we also increase the strangeness fraction at a fixed density. This was expected once this non ideal bag model modifies the quark number, but not the chemical potential of the particles.

We display in Fig. 3 the EoS and the TOV solution for the minimum and maximum allowed values that produce stable strange quark matter. As we increase \( c \) we are able to use lower values of the Bag. Although we use massive quarks in the EoS, the masses are very low. This makes an almost linear EoS, as in the case of Eq. (10) for all values of \( c \). The EoS differ from each other only by a displacement proportional to the Bag value. Also, lower values of the Bag produce more massive quark stars, as pointed in ref. [2, 6]. Using \( c = 0.3 \) we are able to produce a 2.21 \( M_\odot \) stable strange star, while the original MIT bag model has a maximum mass of only 1.85 \( M_\odot \). Increasing the Bag value, the maximum strange quark star mass decreases, and lies outside the constraint imposed by the MSP J070+6620 [25], whose mass is \( 2.14^{+0.10}_{-0.09} \) \( M_\odot \) at 68% credibility interval (light blue in Fig. 3) and \( 2.14^{+0.20}_{-0.18} \) \( M_\odot \) at 95% credibility interval (light yellow in Fig. 3). Increasing the Bag constant to values beyond the stability window produces unstable strange matter that can be still present in the core of massive hybrid stars. A curious feature is the fact that while the central density decreases with the increase of \( c \), the strangeness fraction \( f_s \) remains almost the same.

Nowadays, an important discussion issue is the radii of canonical stars, \( M = 1.4 M_\odot \). The main results of our model are shown in Tab. III. In the last decade, several studies point towards a radius between 10 and 14 km. [12, 26–28]. However, here we pay special attention to a recent study based on the binary neutron-star merger GW170817. The authors [29] conclude that the canonical star radius cannot exceed 11.9 km. This result together with the existence of MSP J070+6620 puts strong constraints in the EoS of dense matter. As we can see, for \( c = 0.2 \) and \( c = 0.3 \) we fulfill both constraints.

To summarize this section, we state that we are able to produce massive stable strange stars for the modified MIT bag model. However we have to keep in mind that this model violates the Fermi-Dirac statistics.

| \( c \) | \( M/M_\odot = \text{B}_{(\text{Min})} \) R (km) | \( \varepsilon_c \) (MeV/fm\(^3\)) | \( f_s \) | \( R_{1.4} \) | \( M/M_\odot = \text{B}_{(\text{Max})} \) R (km) | \( \varepsilon_c \) (MeV/fm\(^3\)) | \( f_s \) | \( R_{1.4} \) |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.85 | 10.17 | 1286 | 0.322 | 10.38 | 1.61 | 8.93 | 1726 | 0.323 | 9.20 |
| 0.1 | 1.95 | 10.72 | 1158 | 0.322 | 10.81 | 1.70 | 9.31 | 1540 | 0.323 | 9.64 |
| 0.2 | 2.07 | 11.37 | 1038 | 0.321 | 11.30 | 1.81 | 9.94 | 1359 | 0.323 | 10.17 |
| 0.3 | 2.21 | 12.18 | 888 | 0.321 | 11.89 | 1.94 | 10.61 | 1205 | 0.322 | 10.74 |

TABLE III. Quark stars main properties for different values of \( c \). \( R_{1.4} \) is given in km.

IV. VECTOR MIT BAG MODEL

One way to introduce an interaction among the quarks in the MIT model is by coupling the quarks to a field. We next use a vector field that produces a repulsion between
the quarks. The inclusion of vector channels in the MIT bag model is not new, as can be seen in ref. [15–17]. Following ref. [16, 17], we introduce the Lagrangian that becomes:

$$L = \sum_{u,d,s} \left\{ \bar{\psi}_q \left[ \gamma^\mu \left( i \partial_\mu - g_{qqV} V_\mu \right) - m_q \right] \psi_q - B \right\} \Theta (\bar{\psi}_q \psi_q),$$

where the quark interaction is mediated by the vector channel $V_\mu$ analogous to the $\omega$ meson in QHD [30]. Indeed, in this work we consider that the vector channel is the $\omega$ meson itself. Unfortunately, in the previous papers, the authors missed the mass term of the vector channel. As we expose below, in mean field approximation, the vector channel becomes zero if the mass is zero. Therefore, we introduce the mass term of the $V_\mu$ ($\omega_\mu$) field as:

$$L_V = \frac{1}{2} m_{V_\mu}^2 V_\mu V^\mu.$$ (21)

Now, assuming mean field approximation (MFA) ($V^\mu \rightarrow \langle V \rangle \rightarrow \delta_{0,\mu} V^0$), we obtain the eigenvalue for the energy of the quarks and the equation of motion for the $V$ field respectively:

$$E_q = \mu = \sqrt{m_q^2 + k^2 + g_{qqV} V^0},$$

$$m_{V_0}^2 = \sum_{u,d,s} g_{qqV} \langle \bar{\psi}_q \gamma^0 \psi_q \rangle,$$ (22)

where the term $\langle \bar{\psi}_q \gamma^0 \psi_q \rangle$ can be recognized as the number density $n_q$ for each $q$ quark. It is clear from the expression above that if we do not take into account the mass of vector channel, the vector field itself needs to be zero. The energy density for the quarks is then:

$$\epsilon_q = \frac{N_c}{\pi^2} \int_0^{k_f} E_q k^2 dk.$$ (23)

We next need to compute the influence of the massive $\omega$ particle on the EoS. In MFA we have: $\epsilon = -\langle \mathcal{L} \rangle$. So, the total energy density reads:

$$\epsilon = \sum \epsilon_q + B - \frac{1}{2} m_{V_0}^2 V_0^2,$$ (24)

the last term of eq. (24) being absent in ref. [15–17]. Moreover, the bag value is not independent of the vector
field $V_0$, which ultimately depends on the strength of the coupling constant. The pressure is obtained via thermodynamical relation, $p = n\mu - \epsilon$, to guarantee thermodynamical consistency given in eq. (6).

To construct a new stability window, we have to fix the coupling constant $g$, as well the mass of the vector field. We consider that the vector channel is the physical $\omega$ meson, as in QHD models [31]. In relation to the coupling constant $g$, we have two concerns. The first one refers to its absolute value. There are very few studies trying to constrain its value, and the uncertainty is yet very high [31, 32]. Most of the models just consider it as a free parameter [33, 34]. Our second concern is about the relative strength of the $g$ constant for different quarks. In the literature, the $g$’s are universal, assuming the same value for all quark flavors. [15, 17, 31, 38]. Here we follow a new path. Instead of an universal coupling, we use symmetry group to fix the relative quark-vector field interaction. We obtain the relation:

$$g_{ssV} = \frac{2}{5} g_{uuV} = \frac{2}{5} g_{ddV}.\quad (25)$$

All the calculations are detailed in the appendix. The use of symmetry group to fix coupling constants is very common when we are dealing with baryons [37–45], but, as far as we known, it is an original approach in the quark sector.

In the following, we redefine $(g_{uuV}/mV)^2 = G_V$ and define $X_V$ as the ratio between $g_{ssV}$ to $g_{uuV}$:

$$X_V = \frac{g_{ssV}}{g_{uuV}}.\quad (26)$$

![FIG. 4. (Color online) Stability window obtained with the MIT bag model with vector interaction and different values of $X_V$.](image)

We construct a stability window for $G_V$ varying from 0 to 0.3 fm$^2$ for $X_V = 1$, which is usually found in the literature, and for $X_V = 0.4$, which is the value predicted by symmetry group. The results are presented in Fig. IV as well as in Tab. IV.

As we increase the value of $G_V$, the stability window is displaced to lower values. More than that, the stronger the vector field, the narrower the stability window. The same is true for different values of $X_V$. As $X_V = 1.0$ produces a stronger repulsion between the quarks, it yields a narrower stability window. As expected, the minimum value of the stability window is independent of $X_V$, and the differences increase as we increase $G_V$.

We next study stable strange stars within the vector MIT context. As we seek for massive strange stars, we only display in the figures the results for the minimum allowed Bag value. Nevertheless the star masses for the maximum values of the Bag are presented in the text as well as in Tab. IV.

We plot in Fig. V the strangeness fraction for different values of $G_V$ and the two different values of $X_V$ previously justified: an universal coupling $X_V = 1.0$ and $X_V = 0.4$. The strangeness fraction is independent of the Bag. As expected, in the universal coupling, the strangeness fraction is independent of the strength of the coupling of the quarks with the vector field. This can be easily seen from Eq. (22). As all $g_{qqV}$ have the same values, the chemical potential of the quarks are shifted by the same amount. This behaviour is similar to the Nambu Jona-Lasinio model for quarks, where the vector field does not affect the particle population either [33, 37, 38]. However, if the couplings are not the same, the $s$ quark, which has the lower value of the coupling constant, has also the lower shift in the chemical potential. As we increase the $G_V$, the difference in the shift becomes sharper. This cause, for instance, the strangeness fraction at $n = 1.0$ fm$^{-3}$ to grow from 0.32 for $G_V = 0.0$ fm$^2$ to 0.415 for $G_V = 0.3$ fm$^2$.

![FIG. 5. (Color online) The strangeness fraction in the universal and vector MIT context. As we increase the value of $G_V$, the strangeness fraction is displaced to lower values. More than that, the stronger the vector field, the narrower the stability window. The same is true for different values of $X_V$. As $X_V = 1.0$ produces a stronger repulsion between the quarks, it yields a narrower stability window. As expected, the minimum value of the stability window is independent of $X_V$, and the differences increase as we increase $G_V$.](image)

| $G_V$ (fm$^2$) | $X_V$ | Min. $B^{1/4}$ | Max. $B^{1/4}$ |
|---------------|-------|----------------|----------------|
| 0.0           | -     | 148 MeV        | 159 MeV        |
| 0.1           | 1.0   | 144 MeV        | 154 MeV        |
| 0.1           | 0.4   | 144 MeV        | 155 MeV        |
| 0.2           | 1.0   | 141 MeV        | 150 MeV        |
| 0.2           | 0.4   | 141 MeV        | 152 MeV        |
| 0.3           | 1.0   | 139 MeV        | 146 MeV        |
| 0.3           | 0.4   | 139 MeV        | 150 MeV        |

TABLE IV. Stability windows obtained with the vector MIT bag model.
properties of stable strange stars for the minimum allowed value of the Bag in the stability window. In this case, we have a positive feedback between the vector channel and the stability window. As we increase the parameter $G_V$ we stiffen the EoS. Nevertheless, this also reduces the minimum value of the Bag in the stability window, which in turn, also stiffens the EoS. The higher the $G_V$ value, the stiffer the EoS. Additionally, the higher the $G_V$, the lower is the minimum allowed value of the Bag. The lower the value of the Bag, the stiffer the EoS again [2]. As result of this combined effect we are able to produce very massive stable quark stars, fully compatible with the MSP J0740+6610 [23]. Our maximum mass can reach 2.61 $M_\odot$ for $X_V = 1.0$. Also, assuming an universal coupling, $X_V = 1.0$, the strangeness fraction at the center of the quark stars decreases as we increase $G_V$ but increases if we assume $X_V = 0.4$.

Unlike the minimum Bag value, the maximum allowed Bag value does depend on $X_V$ as can be seen in Fig. 4. As $X_V = 0.4$ produces smaller repulsion, the positive feedback plays its role again. A small repulsion produces a softer EoS, which produces a higher value of the maximum allowed Bag value, which in turn also softens the EoS. As a consequence, the maximum quark star mass for $G_V = 0.3$ fm$^2$ can vary from 2.14 $M_\odot$ to 2.61 $M_\odot$, a difference of 0.47 $M_\odot$, i.e. 22%. We emphasize that our radii are in agreement with ref. [20], except when we assume $B^{1/4} = 139$ MeV, $X_V = 1.0$, and $G_V = 0.3$ fm$^2$. This indicates that the bag value is too low, or/and $X_V = 0.4$ is a better approach to dense quark matter. Anyway, it is clear from Tab. V that we are able to produce stable strange stars with masses above 2.4 $M_\odot$ that fulfill all astrophysical constrains.

In this section we are able to construct very massive quark stars, and unlike the last section, all models here are thermodynamically consistent and none of them violate the Fermi-Dirac distribution. Another point worth mentioning: in our work the $G_V$ values are around ten times smaller than the one used in ref. [16, 17] where $G_V = 2.2$ fm$^2$. This is probably because the authors did not include the mass term of the vector field. Only for a comparison, if we remove the mass term, eq. (21) from our final Lagrangian, the maximum mass drops from 2.61 $M_\odot$ to only 2.08 $M_\odot$, a reduction of 0.53 $M_\odot$!

### V. SELF-INTERACTING VECTOR FIELD

The vector channel in mean field approximation takes into account only the valence quarks. This scenario is called "no sea approximation", once the Dirac sea of quarks is completely ignored [20]. As the vector field is borrowed directly from quantum hadrodynamics (QHD), the vector MIT bag model also becomes renormalizable [30]. However instead of transforming the mean field approximation (MFA) into a more complex relativistic Hartree or Hartree-Fock approximation, we can take the Dirac sea into account throughout modifications on the
effective Lagrangian as made in ref. [20]. Here, we introduce a quartic contribution for the vector field: $(V_\mu V^\mu)^2$ as a correction for the EoS at high density which will mimic the Dirac sea contribution.

When we add the vector channel in the MIT Lagrangian, it creates a repulsion term in the quark-quark interaction and, as result, the pressure (as well as the chemical potential and the energy density) increases. The stiffening of the EoS grows linear with the density. The quartic vector field makes the EoS more malleable. The introduction of self-interacting fields is not new in the relativistic models. Boguta and Bodmer [18] introduced self-interaction in the scalar sector to correct the compressibility of the symmetric nuclear matter. The same Bodmer also introduced quartic interaction in the vector sector [19] and others [49, 50] used quartic terms in order to correct the behaviour of nuclear matter at densities above $2n_0$. Now we introduce a self-interacting vector field in the vector MIT bag model as:

$$U(V^\mu) = b_4 \frac{(g^2 V_\mu V^\mu)^2}{4},$$

(27)

where $g = g_{uuV}$, and $b_4$ is a dimensionless parameter [20]. The self interactions of the vector field allow us to construct either a softer or a stiffer EoS when compared with the linear case. It also plays a crucial role in the relation between pressure and chemical potential $-\rho(\mu)$. This relation is important in hadron-quark phase transitions [3, 6, 12, 37]. As in the hadronic case [49], we do not expect any significant modification in the EoS for densities below $2n_0$. Using mean field approximation and solving the Euler-Lagrange equations of motion we obtain the following eigenvalues for the quarks and $V$ field respectively:

$$E_q = \mu = \sqrt{m_q^2 + k^2 + g_{qqV}V_0},$$

$$gV_0 + \left(\frac{g}{m_v}\right)^2 \left(b_4(gV_0)^2\right) = \left(\frac{g}{m_v}\right) \sum_{u,d,s}\frac{g_{qqV}}{m_v} n_q.$$

(28)

Again, this result is very similar to the Boguta and Bodmer terms in the scalar and vector fields in hadronic models [18, 19]. In order to produce only small deviations we impose that $|b_4| \leq 1$. Due to the quartic nature of the non linear term, the higher the density, the higher the deviation from the linear one. Also, as can be seen

FIG. 6. (Color online) EoS (top) and mass-radius relation (bottom) for $X_V = 1.0$ (left) and the $X_V = 0.4$ (right) with the minimum Bag value that produces stable strange star as a function of the parameter $G_V$. 
from Eq. (27), the quartic term has also a quartic dependence on the $g$ constant. So, the higher the value of $G_V$, the higher the influence of the quartic term. The advantage of using non linear terms is that we can modify the EoS at high density while keeping the stability window unaffected.

We are now in the position to construct an EoS in MFA, considering the Fermi-Dirac distribution to the quarks and the Hamiltonian to the vector field and the bag, $\mathcal{H} = -\langle \mathcal{L} \rangle$. We obtain:

$$
\epsilon_q = \frac{N_c}{\pi^2} \int_0^{k_f} E_q k^2 dk. \quad (29)
$$

$$
\epsilon = \sum \epsilon_q + B - \frac{1}{2} m_V^2 V^2_0 - U(V_0). \quad (30)
$$

The pressure is obtained via thermodynamical relation, $p = n \mu - \epsilon$, to guarantee thermodynamical consistency.

In Fig. 7 we display a 3-D stability window. For each value of $G_V$ we vary the dimensionless parameter $b_4$ from -0.4 to 1.0. As pointed out, the advantage of the self-interacting is to preserve the original stability window. This can be seen in this 3-D graphic. Each color wall is a stability window for a fixed $G_V$. The results are the same as the ones shown in Tab. IV being independent of $b_4$.

As can be seen from Eq. (27), the influence of the self-interaction does not only depend on $b_4$ but also on $g$, as presented in ref. [1] for the scalar meson. Hence, the higher the value of $G_V$, the stronger the influence of the quartic term. So we display here only the results for $G_V = 0.3 \text{ fm}^2$. As for lower values, the influence is significantly lower. We can also increase the strength of $G_V$ or $b_4$ but they would not modify the qualitative aspects and therefore, are beyond the scope of this work. We start by studying the influence of the non-linear term in the strangeness fraction $f_s$. If $b_4$ is negative, then the vector field $V^0$ increases with density when compared with the pure linear case. On the other hand, if $b_4$ is positive, then $V^0$ is lower when compared with the linear case. This effect is also reflected in the strangeness fraction if $X_V \neq 1.0$. As already pointed out in the last section, if $X_V = 1.0$ then the strangeness fractions is independent of the vector field. Due to these considerations, here we only discuss the strangeness fraction for $G_V = 0.3 \text{ fm}^2$ and $X_V = 0.4$, as predicted by symmetry group. The results are displayed in Fig. 8.

As can be seen, the strangeness fraction is the same up to densities about $0.4 \text{ fm}^{-3}$. From this point on, the results begin to apart from the linear case. For negative $b_4$ (-0.4), we have an increase of the vector field, which makes the differences of the coupling constant more evident, increasing the fraction of $s$ quarks. On the other hand, a positive $b_4$ (1.0) reduces the vector field as well as the strangeness fraction when compared with the linear case. For instance, fixing $n = 1.0 \text{ fm}^{-3}$ we have $f_s$ equal to 0.415, 0.421, and 0.406 for $b_4$ equal to 0.0, -0.4 and +1.0 respectively. We now discuss the effect of the self-interacting vector field on the EoS and on macroscopic properties of strange stars. The results for the minimal allowed bag value for $G_V = 0.3 \text{ fm}^2$ are displayed in Fig. 9.

As expected, the EoS is almost the same until the energy density reaches values around 400-500 MeV/fm$^3$. Then, as the strangeness fraction behavior, it starts to deviate from the linear case. We also see that the influence of the quartic term is higher for $X_V = 1.0$ as it produces a higher $V^0$ value. The softening of the EoS for a positive value of $b_4$ is analogue to the hadronic case, as shown in ref. [19, 51]. The results of the EoS are reflected in the mass-radius relation. The maximum quark star mass is higher for $b_4 = -0.4$ and lower for $b_4 = 1.0$. Also, from Fig. 9 and Tab. VI we can see that the influence of the quartic term is larger for the maximum bag value when compared with its minimum value. The influence is also stronger for $X_V = 1.0$ when compared with $X_V = 0.4$. For instance, the maximum star mass varies 0.08$M_\odot$ for $X_V = 1.0$ for the minimum bag value and 0.10$M_\odot$ for the maximum bag value. For $X_V = 0.4$, the mass variation is 0.04$M_\odot$ for the minimum bag value and 0.06$M_\odot$ for its maximum. The variation in the radius are not significant. Also, for $X_V = 0.4$ the strangeness fraction at the center of the maximum mass star increases if $b_4$ is negative and decreases if $b_4$ is positive.

We finish this section indicating that although for the specific case that $B^{1/4} = 139$ MeV, $G_V = 0.3 \text{ fm}^2$ and $X_V = 1.0$ the radii of canonical stars are in disagreement with ref. [29], in all other cases we are able to produce stable strange stars in agreement with this study. Indeed, even a 2.44$M_\odot$ star is obtained.

VI. CONCLUSIONS

In this work we revisited the MIT bag model, as well as some of its modified versions found in the literature [2, 6, 15, 17]. We start from the Lagrangian density and obtain the stability window for the original model.

Then we revisit the so called non-ideal bag model, obtain the stability windows for some values of the parameter $c$. We show that from the Lagrangian point of view that the model presents some thermodynamical inconsistencies if we preserve the Fermi-Dirac distribution for the quarks. However, if we modify the Fermi-Dirac distribu-
FIG. 7. (Color online) 3-D stability wall for $X_V = 1.0$ (left) and $X_V = 0.4$ (right). Each color wall represent a value of $G_V$. As expected the results are independent of $b$.

| $b$ | $X_V$ | $M/M_{\odot} - B_{(\alpha \gamma)}$ | $R$ (km) | $\epsilon_c$ (MeV/fm$^3$) | $f_s$ | $R_{1.4}$ | $M/M_{\odot} - B_{(\alpha \gamma)}$ | $R$ (km) | $\epsilon_c$ (MeV/fm$^3$) | $f_s$ | $R_{1.4}$ |
|-----|-------|-----------------------------------|---------|----------------|-------|----------|-----------------------------------|---------|----------------|-------|----------|
| 0.0 | 1.0   | 2.61                              | 12.97   | 795           | 0.317 | 12.08    | 2.40                              | 11.85   | 978           | 0.319 | 11.34    |
| -0.4| 1.0   | 2.65                              | 13.02   | 797           | 0.317 | 12.13    | 2.44                              | 11.85   | 986           | 0.319 | 11.36    |
| +1.0| 1.0   | 2.57                              | 12.96   | 791           | 0.317 | 12.08    | 2.34                              | 11.73   | 994           | 0.319 | 11.27    |
| 0.0 | 0.4   | 2.41                              | 12.33   | 893           | 0.402 | 11.81    | 2.14                              | 10.86   | 1154          | 0.413 | 10.86    |
| -0.4| 0.4   | 2.43                              | 12.45   | 903           | 0.407 | 11.84    | 2.16                              | 10.88   | 1188          | 0.417 | 10.88    |
| +1.0| 0.4   | 2.39                              | 12.35   | 869           | 0.396 | 11.80    | 2.10                              | 10.77   | 1185          | 0.406 | 10.69    |

TABLE VI. Quark star main properties for different values of $b$ and $X_V$.

FIG. 8. (Color online) Strangeness fraction obtained with $G_V = 0.3$ fm$^2$ and $X_V = 0.4$ for different choices of $b$.

We introduce a vector field in the Lagrangian density. We first correct this introduction, as originally proposed in ref. [15–17], by taking into account the mass term of the vector field. With the corrected Lagrangian we reduce the value of $G_V$ from 2.2 fm$^2$ to values ten times smaller. Moreover, besides the traditional universal coupling for all three quarks, with the help of symmetry group arguments, we calculate a new coupling constant for the $s$-quark mass, which is 40% of the $u$ and $d$ quark couplings.

We construct a stability window for this model and we are able to reproduce a star with a mass of $2.44M_{\odot}$ that fulfils all the astrophysical constraints while keeping the thermodynamical consistence.

At the end we propose a modification of the linear vector field, inspired in the models of QHD [18, 19, 49, 50]. We include a quartic term in the vector channel to mimic the contribution of the Dirac quark sea, which is absent in MFA. This quartic term allows us to modify the EoS while keeping the stability window unaltered. We see that a positive quartic term contribution causes a softening of the EoS at high densities, While a negative one causes its stiffening. Ultimately a stable strange star with mass of $2.44M_{\odot}$ that fulfils the above mentioned astrophysical constraints is obtained.
Appendix A: SU(3) and SU(6) symmetry group

To fix the quark coupling constant to the vector channel we use the hybrid group SU(6), which is invariant under both SU(3) flavor symmetry and SU(2) spin symmetry. We start from the Yukawa Lagrangian [41]:

\[ \mathcal{L} = -g(\bar{\psi}_q \psi_q)M, \]

(A1)

where \( \psi_q \) is the quark Dirac field, and \( M \) is the field of an arbitrary meson. This Lagrangian belongs to the irreducible representation IR\{1\}, an unitary singlet. The up, down, and strange quarks belong to the IR\{3\}, while the anti-quarks belong to the IR\{3*\} = D(0,1) [41]. The meson field can belong either to IR\{8\} or to IR\{1\}. The direct product of \{3\} \( \otimes \) \{3*\} = \{8\} \( \oplus \) \{1\} [46]. So, to preserve the IR\{1\} of the Lagrangian, the \( (\bar{\psi}_q \psi_q) \) needs to belong to the IR\{8\} if the meson \( M \) belongs to IR\{8\}, or belong to IR\{1\} if the meson belongs to IR\{1\}. The coupling constant for each quark can be written as [39]:

\[ \mathcal{L} = -g_0(\bar{\psi}_q \psi_q)M, \]

(A2)

for the mesons belonging to IR\{8\}. If the mesons belong to IR\{1\} we have:

\[ \mathcal{L} = -g_1(\bar{\psi}_q \psi_q)M, \]

(A3)

where the \( \mathcal{C} \) is the SU(3) Clebsch-Gordan (CG) coefficient [41]. We calculate the CG with the help of the algorithm presented in ref. [47] and the tables presented in ref. [48].

As the quarks are in IR\{3\} and the anti-quarks in IR\{3*\}; in \{3\} \( \otimes \) \{3*\} we have only one IR\{8\} as result, in opposition to the baryon case, where the baryons and anti-baryons are in IR\{8\}. In \{8\} \( \otimes \) \{8\} there are two possible \{8\}, typically called symmetric and anti-symmetric ones. This also implies that the \( \alpha_0 \) defined in ref. [41] is always equal to 1 in our case. The coupling constants reads:

\[
\begin{align*}
g_{uus\omega} &= \left( \frac{1}{\sqrt{6}} \right) \times \left( \frac{1}{\sqrt{3}} \right) = g_u \frac{1}{\sqrt{18}}, \\
g_{dds\omega} &= \left( -\frac{1}{\sqrt{6}} \right) \times \left( -\frac{1}{\sqrt{3}} \right) = g_d \frac{1}{\sqrt{18}}, \\
g_{sss\omega} &= \left( -\frac{2}{\sqrt{6}} \right) \times \left( \frac{2}{\sqrt{3}} \right) = -g_s \frac{2}{\sqrt{48}}, \\
g_{uud\phi_1} &= g_{dd\phi_1} = g_{ss\phi_1} = g_{1}.
\end{align*}
\]
The coupling of the strange quark is twice the value of the non-strange ones, and has the opposite sign of the $u-u$ and $d-d$ coupling. This result is exactly the same as the $\Sigma-\Sigma$ when compared with the N-N one (the reader can consult the table in ref. [48]). Nevertheless, as happens in the baryon octet case, this weird value will be washed out when we impose the mix of the singlet and octet states.

In nature, the observed $\omega$ and $\phi$ meson are not the theoretical $\omega_8$ and $\phi_1$ but a mixture of them [39, 40]. So, the coupling constant of the real vector mesons with the quarks now reads:

$$ g_{uu\omega} = g_{dd\omega} = g_1 \cos \theta + g_8 \frac{1}{\sqrt{48}} \sin \theta, $$

$$ g_{ss\omega} = g_1 \cos \theta - g_8 \frac{2}{\sqrt{48}} \sin \theta. \quad (A5) $$

Now, to eliminated the last of the free parameters, we impose SU(6) symmetry group, which give us an ideal mixing angle, $(\theta = 35.264)$, and $g_8 = \sqrt{6}g_1$ [39, 40]. We finally obtain:

$$ g_{ss\omega} = \frac{2}{5}g_{uu\omega} = \frac{2}{5}g_{dd\omega}. \quad (A6) $$

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