Characterization of Hardening Duffing Oscillator based on a Tensioned Wire System

M A Rahim¹, M S Z Azalan¹ and M N Arib²
¹Faculty of Electrical Engineering Technology, Universiti Malaysia Perlis
²Faculty of Mechanical Engineering Technology, Universiti Malaysia Perlis

Abstract—In this paper, the characterization of mechanical system that behaves as a hardening Duffing oscillator is presented. This mechanical system comprises a mass attached to a tensioned wire which exhibits a hardening stiffness behavior when the displacement of the mass is large. Firstly, the equation of motion of the system is derived to provide the relationship between the applied static force and the resulting displacement. Then, the effect of initial tension, and number of the wires on the force-displacement relationship are analyzed. It has been found that a higher tension will produce higher linear stiffness, whilst having a negligible effect on cubic stiffness. Moreover, the nonlinearity is less sensitive for small inequality between the length of wire on the left and right side of the mass. The results presented herein provide an insight of the system behavior for its application as a vibration isolator.

1. INTRODUCTION

Duffing oscillator is a well-known second order differential equation that has a linear and a nonlinear cubic stiffness terms [1]. It is a useful model for the nonlinear behavior of many engineering applications after its discovery by Georg Duffing. Examples of physical system that can be described as a Duffing oscillator are such as pendulum, isolators, beams and nonlinear electronic circuits [2-5].

In recent years, there has been a tremendous amount of work done on the Duffing oscillator. It is known that, the oscillator system can exhibit either a hardening or a softening characteristic, depending on the nature of the physical system being studied. In reference [6,7], extensive results have been reported on the response of Duffing oscillator with a hardening spring. Meanwhile, in [8,9], the behavior of Duffing oscillator with softening characteristic have been examined. On the other hand, the chaotic behavior of a Duffing oscillator system due to a certain system and external forcing parameter values have been investigated in the following studies [10,11].

The aim of this work is to investigate the character of a mechanical system that mimics the hardening Duffing oscillator. Although many researchers have described several physical systems that behave as Duffing oscillator [2-5], the effect of the system parameter values for a tensioned wire system have not been properly demonstrated. In addition, the research to date has tended to focus more on the studies of its solution methods [12] and various control strategies [13] for the nonlinear system.

The organization of this paper is as follows: In section 2, the proposed mechanical model of the hardening oscillator is presented, and its equation of motion is derived. Section 3 describes the effects of its system parameters, namely as the wire initial tension value, number of wires, and wire length ratio. Finally, conclusions are presented in Section 4.
2. MECHANICAL MODEL OF THE HARDENING DUFFING OSCILLATOR

The proposed mechanical system that behaves as hardening Duffing oscillator is illustrated in Figure 1. It comprises an isolated mass \( m \) which is attached to a stretched wire of initial length, \( L \) and tension, \( T \) where \( a \) and \( b \) are the length of the wire on the left and right side of the mass respectively.

![Figure 1. Model of a Duffing oscillator comprising a mass suspended on tensioned wires.](image)

When the mass, \( m \) moves in the \( x \) direction, the corresponding strain in the left, \( \varepsilon_a \) and right, \( \varepsilon_b \) wire can be expressed as

\[
\varepsilon_a = \frac{\sqrt{a^2 + x^2} - a}{a} \\
\varepsilon_b = \frac{\sqrt{b^2 + x^2} - b}{b}
\]  

(1)

As a consequence, the restoring force in left, \( F_a \) and right, \( F_b \) wire can be determined as

\[
F_a = (T + AE\varepsilon_a) \\
F_b = (T + AE\varepsilon_b)
\]  

(2)

where \( A \) and \( E \) are the cross-section area and the modulus of elasticity of the wire respectively.

By assuming no damping in the system, the resulting force on the mass \( m \) in the \( x \) direction is closely related to angular deflection due to forces on the left, \( F_a \) and right, \( F_b \) wires. This resulting force, \( F_r \) can be written as

\[
F_r = F_a \sin \theta_a + F_b \sin \theta_b
\]  

(3)

where the angular deflections on the left, \( \theta_a \) and right, \( \theta_b \) wire are given by

\[
\sin \theta_a = \frac{x}{\sqrt{a^2 + x^2}} \\
\sin \theta_b = \frac{x}{\sqrt{b^2 + x^2}}
\]  

(4)

By simplifying (1) and (4) using Taylors series expansion to third order with assumption \( x/a1 \) and \( x/b1 \), and substitute them into (3) will yield

\[
F_r = \frac{T L}{4ab} x + (AE - T) \left( \frac{b^3 + a^3}{2(ab)^2} \right) x^3
\]  

(5)

which is the relationship between the applied static force and the resulting displacement. The derivation of (5) is obtained by neglecting the terms higher than third order.

It is clear that (5) has form of Duffing equation, where the linear and cubic stiffness coefficients are given by \( k_1 = \frac{T L}{ab} \) and \( k_3 = (AE - T) \left( \frac{b^3 + a^3}{2(ab)^2} \right) \) respectively.
Note that, $\alpha = (k_3/k_1)Y^2$ is a factor that determines the degree of nonlinearity of the system [14], where $Y$ is the magnitude of base excitation. Therefore, the nonlinearity $\alpha$ in this system takes into account the magnitude of the base excitation $Y$, the linear stiffness $k_1$ and the cubic stiffness $k_3$. It is clear that the system defaults to a linear system if $\alpha = 0$.

3. EFFECTS OF SYSTEM PARAMETERS

In this section, the effects of system parameters, which are the value of the initial tension, and number of the wires are investigated. Both of the result from the formula solution which is from (5) and the exact solution (without simplification of (1) and (4)) of the force-displacement relation are compared. This is followed by analyzing the effect of wire length ratio on the nonlinearity of the system.

3.1 The Effect of Initial Tension

It is noticeable from (5) that the tension, $T$ will affect both the $k_1$ and $k_3$ terms. The value of $AE-T$ in the $k_3$ term has significant influence in the force-displacement relationship of the system. In order to observe this effect, the length for each wire is set to be equal ($a = b = 0.06$ m), and $AE$ is assume to be 1000 N. Dotted line in Figure 2 demonstrates that the system behaves as hardening stiffness for $T=10$ N, since $k_3$ is positive ($AE-T > 0$) when displacement of the mass is large. However, for a small mass displacement ($x < 0.005$ m), the system shows an approximately linear characteristic in force-displacement relationship. However, the exact solution (dashed line) demonstrates that its behavior is more hardening than the formula solution.

![Figure 2. Force-displacement relation of Duffing oscillator model for two different tension values. Dashed line (exact solution with T=10N), dotted line (formula solution with T=10N) and solid line (T=1000N).](image)

On the other hand, when the initial tension is equal to the value of $AE$ ($T=AE$), both of the formula and exact solution result are linear and exactly same. The solid line plot in Figure 2 indicates that the result of the formula and exact solution are overlapping each other for $T=1000$ N. It is obvious from (5), that the cubic stiffness term $k_3$ become zero in this case, which indicates that the system becomes fully linear system.

The softening stiffness occurs when the cubic stiffness $k_3$ becomes negative. In this system, the initial tension $T$ must be higher than $AE$. It is interesting to note that, for a small displacement the force-displacement relation of the system is still approximately linear as shown in Figure 3, even though $T>AE$. Meanwhile, the comparison between the formula (dotted) and exact solution (dashed line) exhibits a slightly difference in force-displacement relation as the displacement of the mass increases.
3.2 The Effect of Number of Wires

Generally, the resulting force on the mass is determined by the number of wires which is effectively connected to the mass. For example, the equation (5) is based on the two resulting forces from a pair of wires connected to the mass. Therefore, the resulting forces on the mass will increase by the number of wires which are effectively connected to the mass. By considering the wire connected to the mass is in pair, the general equation of force-displacement relationship can be written as

\[ F = n \left( \frac{TL}{ab} \right) x + n \left( AE - T \right) \left( \frac{b^3 + a^3}{2(ab)^3} \right) x^3 \]  

(6)

where \( n \) is the number of wire pair.

Therefore, it can be said that number of wire pair introduces scaling factor in the term of \( k_1 \) and \( k_3 \) as written in (6). As consequence, the system will be stiffer with the increment of wire pair connected to the mass. It also noticeable that, by considering a fixed magnitude of base excitation, \( Y \) the nonlinearity of the system will be similar even though with the increment of wire pair. This is because the scaling factor for both \( k_1 \) and \( k_3 \) terms in the nonlinearity \( \alpha = (k_3 / k_1) Y^2 \) will be cancelled out.

3.3 The Effect of Wire Length Ratio on the System Nonlinearity

It is worth to mention that, that the ratio between the wire lengths yields the sensitivity of the position of mass along the constant length of wire in resulting the linear \( k_1 \) and cubic stiffness \( k_3 \). In order to analyze this, the linear \( k_1 \) and cubic stiffness \( k_3 \) term from (5) is simplified into ratio of \( b/a \) as the following terms

\[ k_1 = \frac{TL}{ab} = \left( \frac{TL}{a^2} \right) \left( \frac{1}{b/a} \right) \]  

(7)

\[ k_3 = (AE - T) \left( \frac{b^3 + a^3}{2(ab)^3} \right) \left( \frac{AE - T}{2a^3} \right) \left( 1 + \frac{1}{(b/a)} \right) \]  

(8)

By assuming the tension \( T=10N \) and \( AE=1000N \), it is noticeable that both of the linear \( k_1 \) and cubic stiffness \( k_3 \) are increases symmetrically and have minimum value at ratio \( b/a=1 \) as the mass displaced from the center position along the length of the wire. The effect of the change in the linear \( k_1 \) and cubic stiffness \( k_3 \) are illustrated in Figure 4 and Figure 5 respectively. In both of the plots, the total length of wire is fixed at 0.12 m, while the change of wire length ratio is 0.1<b/a<10. It is worth to be highlighted that for 10% deviation of the mass from the center length of the wire (a=b), \( k_1 \) increases to 1.02%, while \( k_3 \) increases to 6.15%. It seems that \( k_3 \) is more sensitive than \( k_1 \). Nevertheless, it still can be said that the changes of \( k_1 \) and \( k_3 \) are not very sensitive for small changes of inequality between the length of \( a \) and \( b \).
Figure 4. Relationship of linear stiffness, $k_1$ and wire length ratio (b/a) with T=10N and AE=1000N.

Figure 5. Relationship of cubic stiffness, $k_3$ and wire length ratio (b/a) with T=10N and AE=1000N.

Figure 6. Relationship of nonlinearity with wire length ratio (b/a) with T=10N and AE=1000N.
As mentioned earlier in the section II, the nonlinearity of a system is determined by three values: linear stiffness $k_l$, cubic stiffness $k_3$ and the magnitude of base excitation $Y$. By using a fixed magnitude of base excitation ($Y=1\times10^4$ m), the nonlinearity characteristic of the system can be observed as the ratio between the wire changes. By applying initial tension $T=10$ N, Figure 6 demonstrates that the nonlinearity increases symmetrically and have minimum value at ratio $b/a=1$ as the mass displaced from the centre position along the length of the wire. It also can be said that, the nonlinearity is less sensitive for small inequality between the length of $a$ and $b$. This effect is due to change of ratio $k/k_l$ in the nonlinearity coefficient as the ratio of $b/a$ changes.

4. CONCLUSIONS

In this paper, a mechanical system consists of a mass suspended on tensioned wires which behaves as a hardening Duffing oscillator is presented. The effects of tension, and number of wires on the system behavior have been investigated. It has been found that a higher tension will produce higher linear stiffness, whilst having a negligible effect on cubic stiffness. Therefore, the system with strong nonlinearity could be acquired by having low tensioned wire as the mass largely displaced. Meanwhile, the increment number of wires introduce scaling factor in term of linear and cubic stiffness. Moreover, the nonlinearity is less sensitive for small inequality between the length of wire on the left and right side of the mass. In summary, the results and discussion in the paper have given a comprehensive understand on characteristic of mechanical system with a mass attached to a tensioned wire.

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