Gauge unification in noncommutative geometry

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Abstract – Gauge unification is widely considered to be a desirable feature for extensions of the standard model. Unfortunately the standard model itself does not exhibit a unification of its running gauge couplings but a unification is required by grand unified theories as well as the noncommutative version of the standard model (Connes A., Commun. Math. Phys., 182 (1996) 155; Chamseddine A. and Connes A., Commun. Math. Phys., 186 (1996) 731). We will consider here the extension of the noncommutative standard model by vector doublets as proposed in Squellari R. and Stephan C., J. Phys. A, 40 (2007) 10685. We will analyse greater in detail two consequences of this modification which had not been taken into account in the above-mentioned paper by Squellari and Stephan: 1) the altered relations of the coupling constants at unification energy compared to the well-known relation from grand unified theories and 2) the ability of the extended model to almost fulfil these relations at \( E \sim 10^{13} \text{ GeV} \).

It is generally believed that the standard model and the big desert are not the final theory describing the particle content of our universe. A hint for an underlying, more profound structure is the observation that the running gauge couplings almost converge, missing each other by roughly five orders of magnitude between \( \sim 10^{12} \text{ GeV} \) and \( \sim 10^{17} \text{ GeV} \). Grand unified theories require an exact convergence, but since the standard model cannot provide for this, extensions have to be considered. One of the most popular extensions is certainly supersymmetry which enlarges the particle content of the standard model roughly by a factor of two, introducing supersymmetric partners. Due to cancellations in renormalisation, this extension leads to an exact convergence of the gauge couplings. The price which has to be paid is a multitude of hitherto unobserved particles which should although be detectable at the LHC.

A different approach to the standard model is noncommutative geometry \cite{1} which, through the spectral action, also requires gauge unification \cite{3,2}. Here again the pure standard model cannot meet the conditions of the gauge couplings. The conditions of the gauge couplings coming from noncommutative geometry coincide for the standard model with the classical ones obtained from grand unified theories. In noncommutative geometry this unification is not thought of as having its origin in the breaking of a simple unifying group like \( SU(5) \) or \( SO(10) \) but as a modification of space-time itself.

Recently, extensions of the standard model within the framework of noncommutative geometry have been discovered \cite{4–6}. At least one of these extensions, the AC-model, has even a viable dark matter candidate \cite{7} and is compatible with high-precision measurements in particle physics \cite{8}.

In this publication we will examine the extension presented in \cite{5}, investigating its ability to cure the unification problem that occurs when the minimal central extension is employed. Here the particle content of the standard model is enlarged by fermions coupling vectorially to the electro-weak \( U(1)_Y \times SU(2)_w \) subgroup. For convenience we will call them vector doublets. A most interesting fact of these extensions is that the conditions of the gauge coupling unification get modified. If the mass of these vector doublets is taken to be of unification scale, \( \sim 10^{13} \text{ GeV} \), the altered unification conditions are almost fulfilled.

For comparison we will also consider the effect of these fermions on the classical unification conditions from grand unified theories. The purpose is to show that a simple modification can achieve a drastic amelioration since the vector doublets alter the running of the gauge couplings sufficiently to obtain a perfect convergence. This model

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does of course not originate from a unified theory but is ad hoc. So if one desires convergence of the gauge couplings this can also be achieved by vector doublets.

**Vector doublets.** – In noncommutative geometry the gauge group $G$ is extracted from the spectral triple either via the unimodularity condition [3,9] or via centrally extending the lift of the automorphism group of the associated algebra [10]. The two approaches coincide for a minimal central extension [10].

There are other constraints, on the fermionic representations, coming from the axioms of the spectral triple. They are conveniently captured in Krajewski diagrams which classify all possible finite-dimensional spectral triples [11,12].

The model considered here is an extension of the standard model by a set of fermions which couple vectorially to the $U(1)_Y \times SU(2)_W$ subgroup of the standard model. They are iso-spin doublets, i.e. they live in the fundamental representation of $SU(2)_W$, they have hypercharge $Y = -1/2$, and they are colour singlets. Since the particles couple vectorially to the gauge group, their masses are gauge invariant.

The charges follow immediately from the central charges of the standard model and the requirement that the resulting lift should be minimal [9]. After symmetry breaking one component of the vector doublet acquires a unit electric charge while the other becomes electrically neutral. This results in a slight mass difference of $\sim 350$ MeV due to radiative corrections, where the neutral particle is lighter than its charged partner.

A thorough presentation of this model containing the details of the construction of the spectral triple, the lift of the automorphisms, the Lagrangian and possible mass assignments, which could give viable dark-matter candidates, can be found in [5]. Extensions of the standard model within the noncommutative framework are rare and only a few viable ones are known [4-6]. Therefore the vector doublet model is far from ad hoc and its properties are quite remarkable. We will concentrate here on the ability of the model to achieve unification of the $U(1)_Y$-,$SU(2)_W$- and $SU(3)_c$-gauge couplings which has not been investigated in [5].

For this model all the axioms of noncommutative geometry [1] are fulfilled. Majorana neutrinos may be introduced at the expense of altering the orientability axiom [13]. Note also that this model is free of gauge anomalies and mixed gauge and gravitational anomalies for any number of vector doublets. This includes Witten’s $SU(2)$ anomaly. It is also interesting that this model resembles the Connes-Lott model [14] regarding the four summands in the algebra.

**The constraints on the gauge couplings.** – The spectral action is defined as the number of eigenvalues of the Dirac operator up to a cut-off $E$. As input one has this cut-off, the parameters of the inner Dirac operator, i.e. fermion masses and mixing angles and three positive parameters for the cut-off function.

As an output one obtains the Yang-Mills-Higgs action, in case of the spectral triple of the standard model it is exactly the desired standard model action [2,3], and additional constraints on the dimensionless couplings. For the standard model with three generations this implies the following relation for the gauge couplings at the cut-off $E$:

$$5g_1^2(E) = 3g_2^2(E) = 3g_3^2(E),$$

where $g_1$ is the $U(1)_Y$ coupling, $g_2$ the $SU(2)_W$ coupling and $g_3$ the $SU(3)_c$ coupling. These relations coincide with the unification conditions of grand unified theories. It is well known that this constraint cannot be met at any unification scale $E$ within the standard model alone and therefore extensions of the standard model have to be considered.

If we extend the standard model by vector doublets within the noncommutative framework, the spectral action produces a slightly different constraint for the gauge couplings at the cut-off [5]

$$\left(5 + \frac{N_v}{2}\right) g_1^2(E) = \left(3 + \frac{N_v}{2}\right) g_2^2(E) = 3g_3^2(E),$$

where $N_v$ denotes the number of vector doublets. This is quite remarkable since we have for the first time a deviation from the classical unification condition (1). It is not too surprising that additional particles change the constraint (1) since in noncommutative geometry adding new particles means changing the spectral triple and therefore the geometry itself.

Note that the constraints (1) and (2) depend only on the particle content of the model in consideration and not on the mass of the particles. Furthermore the difference of the constraints (1) and (2) underlines the fact that classical grand unified theories and the noncommutative approach are not necessarily compatible.

The strategy is now the following. Since the mass of the vector doublets is gauge invariant, it can be chosen freely. We will choose it in such a way that the running couplings of the standard model plus vector doublets meet condition (2), at a given energy scale $E$ which is then identified with the cut-off scale, as exactly as possible. Furthermore we will repeat this analysis for the classical condition (1).

Adding vector doublets changes of course the $\beta$-functions for the gauge couplings needed to evolve the constraints (1) and (2). We restrict ourselves to the one-loop $\beta$-functions. We set: $t := \ln(E/m_Z)$, $d\gamma/dt := \beta_\gamma$, $\kappa := (4\pi)^{-2}$ and we will neglect all fermion masses below the top mass and also neglect threshold effects.

By the Appelquist-Carazzone decoupling theorem [15] we distinguish two energy domains: $E > m_1$, and $E < m_i$, where $m_i$ is the mass of the $i$-th generation of vector doublets. At high energies, $E > m_1$, the $\beta$-functions are
for the standard model with three generations plus $N_\nu$ vector doublets [16,17]:

$$\beta_i = s_b g_i^3, \quad b_i = \left(\frac{41}{6} + \frac{2}{3} N_\nu, -\frac{19}{6} + \frac{2}{3} N_\nu, -7\right).$$

(3)

At low energies, $E < m_{\min}$, the $\beta$-functions are the same with $N_\nu$ put to zero. We suppose that all couplings (other than $g_\nu$ and $k$) are continuous at $E = m_\nu$, no threshold effects. The three gauge couplings have identical evolutions in both energy domains:

$$g_i(t) = g_0 \sqrt{1 - 2 s_b g_{10}^2 t}.$$  

(4)

The initial conditions are taken from experiment [18]:

$g_{10} = 0.3575$, $g_{20} = 0.6514$, $g_{30} = 1.221$.

We see that the effect of the vector doublets in noncommutative geometry is twofold. On the one hand, it alters directly the unification condition at the cut-off scale from (1) or (2). On the other hand, the $\beta$-functions of the gauge couplings (3) get modified since the vector doublets are charged under the electro-weak subgroup of the standard model gauge group.

We will therefore also investigate whether the unification condition (1) can be met by introducing vector doublets. This model is mainly for comparison and should be considered as an $ad$ $hoc$ extension of the standard model with the extra requirement of (1) at a cut-off scale.

To measure whether the evolution of the initial conditions for the coupling constants $g_{10} = 0.3575$, $g_{20} = 0.6514$, $g_{30} = 1.221$ with the $\beta$-functions (3) allows to fulfill either of the conditions (1) to (2), we introduce the quantity $\Delta$ to measure the possible defect,

$$\Delta := \log_{10} \left( \frac{E_{\max}}{E_{\min}} \right).$$

(5)

$E_{\max}$ and $E_{\min}$ are defined as follows: it is clear that the running couplings, together with each equality in the conditions (1) and (2), define the edges of a triangle in the plane of the coupling constants and the energy. $E_{\max}$ ($E_{\min}$) is now the maximal (minimal) energy of two running couplings to meet one equality in (1) or (2). By definition $\Delta \geq 0$. We say that a model achieves gauge unification under a given condition, such as (1) or (2), if $\Delta = 0$, i.e. $E_{\min} = E_{\max}$.

For the standard model with condition (1) it is well known that $g_1^2(E_{\min}) = 5/3 g_1^2(E_{\max})$ at $E_{\min} \sim 10^{13}$ GeV and $g_2^2(E_{\max}) = g_3^2(E_{\max})$ at $E_{\max} \sim 10^{17}$ GeV while $E_x$ for the equality $g_1^2(E_x) = 5/3 g_1^2(E_x)$ lies in between. Therefore we have $\Delta_{SM} \sim 4$ which tells us that unification is missed by roughly four orders of magnitude.

Let us first consider the noncommutative model, i.e. the standard model plus $N_\nu$ generations of vector doublets, with its constraint (2). To illustrate the mass dependence of the defect, we plot the $\Delta_{N_\nu}$ against the logarithm of the highest mass term (in units of GeV) of the $N_\nu$ vector doublets.

The three equations from constraint (2) are $(5 + N_\nu/2) g_1^2(E_1) = (3 + N_\nu/2) g_2^2(E_1)$, $(5 + N_\nu/2) g_3^2(E_2) = 3 g_3^2(E_2)$ and $(3 + N_\nu/2) g_3^2(E_3) = 3 g_3^2(E_3)$. For $N_\nu = 1, 2$ and 3 we always find that $E_{\min} = E_3$ and $E_{\max} = E_1$ during the evolution of the couplings.

In fig. 1 we assumed that the masses of the vector doublets are equal in all $N_\nu$ generations. The evolution of $\Delta_1 (N_\nu = 1)$ is depicted by the boxed line, the case $\Delta_2 (N_\nu = 2)$ by the crossed line and $\Delta_3 (N_\nu = 3)$ by the circled line. The masses of the vector doublets run up to $E_{\max}$, since above $E_{\max}$ the vector doublets cease to have an influence on the $\beta$-functions due to the Appelquist-Carazzone decoupling theorem.

In fig. 2 we assumed that the masses of the vector doublets for $N_\nu = 2, 3$ differ. We keep the light generation(s) fixed at $m_1 (m_2) = 10^5$ GeV and let only the one mass run up to $E_{\max}$. Again the evolution of the case $\Delta_2 (N_\nu = 2)$ is given by the crossed line and $\Delta_3 (N_\nu = 3)$ by the circled line. As a reference $\Delta_1 (N_\nu = 1)$ is also depicted by the boxed line.

We see immediately that only the model with one generation comes at least close to the required $\Delta = 0$ for unification with respect to constraint (2). One finds that $\Delta_1 (m_1 = E_{\max}) = 1.0$ with $E_{\max} = 8.8 \times 10^{13}$ GeV.

Fig. 1: Boxed lines correspond to $N_\nu = 2$ and circled lines to $N_\nu = 3$. The case of equal masses in all generations is shown.

Fig. 2: The case $m_{\max} = m_3$, $m_2 = m_1 = 10^5$ GeV for $N_\nu = 3$ and $m_{\max} = m_2$, $m_1 = 10^5$ GeV for $N_\nu = 2$ is shown. Same symbols as fig. 1.
generations unification, $\Delta = 0$ can be achieved with $m_1 = m_2 = 1.2 \times 10^4 \text{ GeV}$ and $E_{\text{min}} = E_{\text{max}} = 5 \times 10^{13} \text{ GeV}$. For three generations we find $\Delta_3 = 0$ for $m_1 = m_2 = m_3 = 2.0 \times 10^7 \text{ GeV}$ also with $E_{\text{min}} = E_{\text{max}} = 5 \times 10^{13} \text{ GeV}$. For $N_\nu = 1$ $m_1$ has to be well below 10 GeV and is therefore excluded experimentally.

In fig. 4 we have repeated the analysis as for fig. 3 but with mass differences. For $N_\nu = 2$ with $m_2 = 10^3 \times m_1$ we find $\Delta_2 = 0$ for $m_2 = 4.0 \times 10^6 \text{ GeV}$ with $E_{\text{min}} = E_{\text{max}} = 6.7 \times 10^{13} \text{ GeV}$. For $N_\nu = 3$ with $m_3 = 10^2 \times m_2 = 10^5 \times m_1$ we find $\Delta_3 = 0$ for $m_3 = 4.0 \times 10^9 \text{ GeV}$ with $E_{\text{min}} = E_{\text{max}} = 6.4 \times 10^{13} \text{ GeV}$. Again the case $N_\nu = 1$ is shown as a reference.

Conclusions. – Noncommutative geometry as well as grand unified theories impose constraints on the gauge couplings of Yang-Mills-Higgs models. They are assumed to be valid at a certain energy scale $E$, the unification scale. For the standard model these constraints coincide in the noncommutative setting and in the grand unified setting. But, since these conditions cannot be fulfilled when taking into account only the standard model particle content, one assumes that the big desert has to be populated.

We analysed here an extension of the standard model by vector doublets within noncommutative geometry [5]. This extension exhibits two main features:

- Adding the vector doublets changes the constraint that the gauge couplings have to fulfil at unification scale.
- One generation of vector doublets ($N_\nu = 1$) allows almost for gauge unification at $E \sim 10^{13} \text{ GeV}$ with respect to the new set of constraints (2).

The masses of the vector doublet is in the case of the modified constraints of the order of the unification scale.

For comparison we also considered the standard model with vector doublets with relation (1). Unification at $E \sim 10^{13} \text{ GeV}$ can be achieved by adding one, two or three generations of vector doublets ($N_\nu = 1, 2, 3$). Their masses range from $1.2 \times 10^4 \text{ GeV}$ to $4.0 \times 10^9 \text{ GeV}$, depending on the number of doublets added and the mass splitting. One generation would result in a mass of ca. 10 GeV which is excluded experimentally.

It is certainly possible to build more baroque models from the extensions proposed in [4-6] and [5] which also allow for gauge unification. But the model examined here has certainly the appeal of being very minimal.

The main aim of the vector doublet extension within the setting of noncommutative geometry was to fulfil the unification constraints which are imposed by the spectral action at the cut-off scale. Since in this case the mass of the vector doublet is of order $10^{13} \text{ GeV}$, it will certainly not be produced by any conceivable particle physics experiment. On the other hand, the lighter neutral component in the vector doublet could be a viable dark-matter candidate [5].

In the classical case the vector doublets are much lighter and could perhaps appear in particle accelerators like the LHC and also in experiments searching for dark matter.

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