RECOVERING A POTENTIAL FROM CAUCHY DATA
VIA COMPLEX GEOMETRICAL OPTICS SOLUTIONS

HOAI-MINH NGUYEN AND DANIEL SPIRN

Abstract. We revisit the problem of recovering a potential \( q \) in a domain in \( \mathbb{R}^d \) for \( d \geq 3 \) from the Dirichlet to Neumann map. This problem is related to the inverse conductivity problem of Calderón via the Liouville transformation. Using the method of complex geometrical optics solutions, along with some new averaging and iterative techniques, we find a more direct approach for establishing uniqueness for the potential. As a consequence we give new proofs of uniqueness for the Calderón problem for the class of \( W^{1,\infty} \) conductivities under a slight additional assumption, established by Haberman and Tataru [11], the class of \( W^{2-d/2} \) conductivities established by Nachman and Lavine [17], and a new result for the class of \( W^{s,3/s} (\not\in \, W^{2,3/2}) \) conductivities with \( 3/2 < s < 2 \) in three dimensions.

1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) \((d \geq 3)\) with \( C^1 \) boundary and let \( q \in L^{d/2}(\Omega) \), an assumption that will be weaken later. We consider the Dirichlet to Neumann map \( \Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega) \) given by

\[
\Lambda_q(f) = g,
\]

where

\[
g = \frac{\partial u}{\partial \eta} |_{\partial\Omega} ,
\]

and \( u \in H^1(\Omega) \) is the unique solution to the system

\[
\begin{cases}
\Delta u - qu = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial\Omega.
\end{cases}
\]

Here and in what follows \( \eta \) denotes a unit normal vector directed into the exterior of \( \Omega \). We assume here that 0 is not a Dirichlet eigenvalue for this problem; this implies \( \Lambda_q \) is well-defined (this assumption is not essential and is discussed later, see Remark 1). In this paper, we are interested in the injectivity of \( \Lambda_q \) for \( d \geq 3 \). This problem has a connection to the inverse conductivity problem posed by Calderón in [1]. In [3] Calderón asked whether one can determine \( \gamma \in L^\infty(\Omega) \) with \( \text{essinf}_\Omega \gamma > 0 \) from its Dirichlet to Neumann map \( \text{DtN}_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega) \) given by

\[
\text{DtN}_\gamma(f) = \gamma \frac{\partial u}{\partial \eta},
\]

where \( u \in H^1(\Omega) \) is the unique solution to the equation

\[
\text{div}(\gamma \nabla u) = 0 \text{ in } \Omega \text{ and } u = f \text{ on } \partial\Omega.
\]
In the same paper, Calderón proved the injectivity of the derivative of the map \( \gamma \rightarrow DtN_\gamma \) at \( \gamma = \text{constant} \). Kohn and Vogelius [15, 16] showed that if \( \partial \Omega \) is \( C^\infty \) then \( \Lambda_q \) determines \( q \) and all its derivatives on \( \partial \Omega \) and then used this to prove uniqueness for the class of piecewise analytic coefficients. Sylvester and Uhlmann [22] proved that \( \Lambda_q \) uniquely determines \( q \) if \( q \in C^\infty \); their method also gave the injectivity of \( \Lambda_q \) for \( q \in L^\infty \) (see also [20]). In [22], they introduced the concept of complex geometrical optics (CGO) solutions which plays an important role in establishing the uniqueness for inverse problems for \( d \geq 3 \). In one direction, the \( L^\infty \) uniqueness result was improved by Kenig and Jerison in [7] where they obtained the injectivity of \( \Lambda_q \) for \( q \in L^p \) for any \( p > d/2 \); in the same paper Chanillo established the injectivity of \( \Lambda_q \) for \( q \in L^{d/2} \) with small norm. Lavine and Nachman announced in [17] that the injectivity of \( \Lambda_q \) holds for \( q \in L^{d/2} \). Recently, this result has been extend by Ferreira, Kenig, and Salo in [10] for compact Riemannian manifolds with boundary which are conformally embedded in a product of the Euclidean line and a simple manifold. Their technique is based on Carleman estimates. In another direction, the injectivity of \( \Lambda_q \) was established for \( q \) which belongs to the Besov spaces \( B^{-s}_{\infty,2} \) \((0 < s < 1/2), q \in B^{-1/2}_{\infty,2}, \) and for \( q \) which belongs to the Sobolev spaces \( W^{-1/2, s} \) \((s > 2d)\) by Brown in [4], Päivärinta, Panchenko, and Uhlmann in [19], and Brown and Torres in [3], respectively. Recently, Haberman and Tataru in [11] established the injectivity of \( DtN_\gamma \) (Calderon’s problem) for \( \gamma \in C^1(\Omega) \) or \( \gamma \in W^{1,\infty}(\Omega) \) with a smallness assumption (see (1.9) below). The corresponding uniqueness result for \( \Lambda_q \) would hold for \( q \in W^{-1,\infty} \) with some kind of smallness assumption; however, obtaining this conclusion from their approach is not clear to us. The approach in [4, 19, 3] is via CGO solutions (see the introduction of [11] for an interesting account). The approach due to Haberman and Tataru is also via CGO solutions; on the other hand, the novelty in their approach stems from their use of weighted spaces and averaging arguments. These arguments will be discussed in detail later. Some refinements for piecewise smooth potentials \( q \) can be found in references therein (see also [12]). We note that the result of Lavine and Nachman is not a consequence of the one of Haberman and Tataru and vice versa since \( L^{d/2}(\Omega) \notin W^{-1,\infty}(\Omega) \) and \( W^{-1,\infty}(\Omega) \notin L^{d/2}(\Omega) \). In dimension 2, the injectivity of \( \Lambda_q \) was established by Astala and Päivärinta in [1]. Previous contributions in the 2d case can be found in [21, 5] and references therein.

The standard method to establish uniqueness for the Calderón problem is to prove the injectivity of \( \Lambda_q \). This can be done by Liouville’s transform and using the fact that one can recover the boundary data from the Dirichlet to Neumann map since

\begin{equation}
\Delta v - qv = 0 \text{ in } \Omega
\end{equation}

if and only if

\begin{equation}
\text{div}(\gamma \nabla u) = 0 \text{ in } \Omega,
\end{equation}

where \( u = \gamma^{1/2}v \) and \( q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \). It is known that (see e.g. [13, (5.0.4)]) if

\[ \Lambda_{q_1} = \Lambda_{q_2}, \]

then

\begin{equation}
\int_\Omega (q_1 - q_2)v_1 v_2 = 0
\end{equation}

for any \( v_i \in H^1(\Omega) \) \((i = 1, 2)\) a solution of the equation

\[ \Delta v_i - q_i v_i = 0 \text{ in } \Omega. \]
The crucial idea of Sylvester and Uhlmann in [22] is to find a (large) class of solutions of the equation
\[ \Delta u - qu = 0 \text{ in } \mathbb{R}^d \]
of the form
\[ v = (1 + w)e^{x \cdot \xi / 2} \text{ in } \mathbb{R}^d, \]
where \( \xi \in \mathbb{C}^d \) with \( \xi \cdot \xi = 0 \) and \( |\xi| \) is large. Since \( \xi \cdot \xi = 0 \), it follows that
\[ (1.3) \quad \Delta w + \xi \cdot \nabla w - qw = q \text{ in } \mathbb{R}^d; \]
here one extends \( q \) appropriately on \( \mathbb{R}^d \) and denotes the extension also by \( q \). Their key observation is
\[ (1.4) \quad \lim_{|\xi| \to 0} \|w\|_{H^1(B_r)} = 0 \text{ for } r > 0, \]
which is a consequence of the following fundamental estimate established in [22]:
\[ (1.5) \quad \|W\|_{H^1(B_r)} \leq \frac{C_r}{|\xi|} \|f\|_{H^1} \quad \forall \ r > 0, \]
if \( f \) has compact support. Here \( W \) is the solution to the equation
\[ (1.6) \quad \Delta W + \xi \cdot \nabla W = f \text{ in } \mathbb{R}^d. \]
By appropriate choices of \( \xi_1 \) and \( \xi_2 \) for the associated \( v_1 \) and \( v_2 \) with \( \xi_1 + \xi_2 = 2k \), a constant vector in \( \mathbb{C}^d \), then using (1.2) and (1.4), they show that
\[ \int_{\Omega} (q_1 - q_2)e^{k \cdot x} = 0 \text{ for all } k \in \mathbb{C}^d. \]
This in turn implies
\[ q_1 = q_2. \]

In [3, 4], the authors improved this estimate for solutions to (1.6) in a Besov spaces where \( f \) has \(-1/2\) derivatives. The proof in [19] is based on a different way of constructing CGO solutions.

We next discuss the approach due to Haberman and Tataru in [11]. The key point in [11] is to consider solutions to (1.6) in \( X^{1/2}_{\xi} \) with \( f \in X^{-1/2}_{\xi} \), where
\[ \|f\|_{X^{s}_{\xi}} := \|\|k|^2 + k \cdot \xi \|^{s}_{L^2} \text{ for } s \in \mathbb{R}. \]
These special function spaces have roots from the work of Bourgain in [2]. Their key estimates involves various quantities related to \( L^2 \)-norm of a function by its norm in \( X^{s}_{\xi} \) with \( s = -1/2 \) or \( 1/2 \). This is given in [11, Lemma 2.2]. Another ingredient in their proof is some average estimate for solutions to (1.6), [11, Lemma 3.1].

We now turn to the method of Kenig and Jerison in [7]. Following the approach of [22], they substitute a generalized Sobolev inequality, due to Kenig, Ruiz, and Sogge in [14], for (1.5) to obtain an estimate of the type (1.4). This Sobolev inequality for \( W \), a solution to (1.6), is of the form
\[ \|W\|_{L^p} \leq C \|f\|_{L^{p'}} \]
if \( 1 < p < +\infty \) and \( 1 < p' := pd/(d + 2) < +\infty \). In [7] the requirement \( p > d/2 \) is used to showed that
\[ \|W\|_{L^q} \leq C |\xi|^{-\alpha} \|U\|_{L^q}, \]
where \( \alpha = 2 - n/p \) and \( (q - 2)/q = 1/p \), and \( W \) is the solution to equation (1.7) below. This estimate was used in their iteration process to obtain solutions to (1.3).
The approach used by Ferreira, Kenig, and Salo in [10] is quite different. Their construction of CGO solutions is based on a limiting Carleman’s estimate originating in the work of [9].

The goal of the paper is to introduce an approach, rooted in the work of Sylvester and Uhlmann [22], to prove the following results:

i) \( \Lambda_q \) uniquely determines \( q \) if \( q = \text{div} \, g_1 + g_2 \) where \( \inf_{\phi \in [C(\hat{\Omega})]} \| g_1 - \phi \|_{L^\infty} \) is small, \( g_1 \in L^\infty(\Omega) \cap C^0(\overline{\Omega}_\delta) \) for some \( \delta > 0 \), \( \hat{g}_1 \in L^p \) for some \( p < 2 \), and \( g_2 \in L^d \). Here \( \overline{\Omega}_\delta = \{ \text{dist}(x, \partial \Omega) < \delta \} \cap \Omega \).

ii) \( \Lambda_q \) uniquely determines \( q \) for \( q \in L^{d/2} \).

iii) \( \Lambda_q \) uniquely determines \( q \) if \( q = \text{div} \, g_1 + g_2 \) where \( g_1 \in W^{t,3/(t+1)}(\Omega) \) for some \( t > 1/2 \) and \( g_2 \in L^{3/2}(\Omega) \).

We note here that i) and iii) are new to our knowledge; i) is a little more general than what one would expect from Haberman and Tataru’s result; ii) is Lavine and Nachman’s result. As a consequence, we give a new proof for Haberman and Tataru’s result under a mild additional assumption, Lavine and Nachman’s result, and prove the uniqueness of Calderón’s problem for the class of \( W^{s,3/2} \) (for some \( s > 3/2 \)) conductivities in \( 3d \); this last result is new to our knowledge.

Our approach is based on the construction of solutions to (1.3) and the corresponding estimates for them. This will be done in two steps. In the first step we establish appropriate estimates for solutions to the equation

\[
\Delta W + \xi \cdot \nabla W = qU \quad \text{in} \quad \mathbb{R}^d;
\]

which is the key for the iteration process to obtain a solution to (1.3) and an estimate for this solution. In the second step, some average arguments in the spirit in [11] will be used. Let us describe the ideas of the proof of each conclusion in more detail. Without loss of generality one may assume that \( \text{supp} \, q \subset B_1 \). Here and in what follows \( B_r(a) \) denotes the ball centered at \( a \) of radius \( r \), and \( B_r \) denotes \( B_r(0) \).

Concerning i), our new key estimate for solutions to (1.7) is

\[
\| \nabla W \|_{L^2(B_r)} + |\xi| \cdot \| W \|_{L^2(B_r)} \\
\leq \frac{C_r}{\| g_1 \|_{L^\infty} + \| g_2 \|_{L^d}} \left( \| \nabla U \|_{L^2(B_1)} + |\xi| \cdot \| U \|_{L^2(B_1)} \right),
\]

if \( q = \text{div} \, g_1 + g_2 \) and \( \text{supp} \, g_1, \text{supp} \, g_2 \subset B_1 \), see Lemma [2]. The proof of this inequality is based on an estimate for solutions to (1.6) in which \( f \in H^{-1} \) in the spirit of (1.5) and is presented in Lemma [1]. The proof of Lemma [1] is quite elementary and different from the proof in [22]. After this, we employ some average estimates, as in [11]. We remark that we will need \( g_1 \in C^0(\overline{\Omega}_\delta) \) to ensure the existence of a trace when turning the elliptic PDE (1.1) into the integral (1.2) since \textit{a priori} the trace does not make sense for such rough data.

Concerning ii), we split \( q \) into \( f + g \) where \( f \) is smooth and \( \| g \|_{L^{d/2}} \) is small and, we are able to conclude that

\[
\lim_{|\xi| \to \infty} \frac{\| W \|_{H^1(B_r)}}{\| U \|_{H^1(B_r)}} = 0
\]

by using (2.4) and (2.5). Statement ii) follows by the standard approach.

Concerning iii), our key ingredients are: the following estimate for solutions to (1.7)

\[
\| W \|_{H^1(B_r)} \leq E(q, \xi) \| U \|_{H^1}
\]
for some $E(q, \xi)$, and the observation that, roughly speaking, if $q \in H^{-1/2}$ with compact support then $E(q, \xi) \to 0$ as $\xi \to \infty$ for a large set of $\xi$’s. At this point we both average as in [11] and also average $E(q, \xi)$; the estimate for solutions of (1.7) depends on the direction of $\xi$ and $q$.

We state these results explicitly. Concerning i), we have

Theorem 1. Let $d \geq 3$, $\Omega$ be a smooth bounded subset of $\mathbb{R}^d$. Let $g_1, h_1 \in L^\infty(\Omega) \cap C^0(\Omega_\delta)$ for any $\delta > 0$, $g_2, h_2 \in L^d(\Omega)$ be such that $\|F(1_{\Omega} g_1)\|_{L^p} + \|F(1_{\Omega} h_1)\|_{L^p} < \infty$ for some $1 < p < 2$. Set

$q_1 = \text{div } g_1 + g_2 \quad \text{and} \quad q_2 = \text{div } h_1 + h_2.$

Assume that

$a_{q_1} = a_{q_2},$

then there exists a positive constant $c$ such that if

$\inf_{\phi \in C(\bar{\Omega})} \| g_1 - \phi \|_{L^\infty} + \inf_{\phi \in C(\bar{\Omega})} \| h_1 - \phi \|_{L^\infty} \leq c.$

then

$q_1 = q_2.$

As a consequence, we obtain Haberman and Tataru’s result under the mild additional assumption (1.8).

Corollary 1 (Haberman-Tataru). Let $d \geq 3$, $\Omega$ be a smooth bounded subset of $\mathbb{R}^d$, $\gamma_1, \gamma_2 \in W^{1,\infty}(\Omega) \cap C^1(\Omega_\delta)$ for some $\delta > 0$ be such that

$1/\lambda \leq \gamma_1(x), \gamma_2(x) \leq \lambda$ for a.e. $x \in \Omega$,

for some $\lambda > 0$ and

(1.8) $F(1_{\Omega} \nabla \ln \gamma_1) \in L^p$ for some $1 < p < 2$.

Assume that

$DtN_\gamma_1 = DtN_\gamma_2,$

then there exists a positive constant $c$ such that if

(1.9) $\inf_{\phi \in [C(\bar{\Omega})]^d} \| \nabla \ln \gamma_1 - \phi \|_{L^\infty} + \inf_{\phi \in [C(\bar{\Omega})]^d} \| \nabla \ln \gamma_2 - \phi \|_{L^\infty} \leq c,$

then

$\gamma_1 = \gamma_2.$

It is clear that (1.8) holds for $p = 2$ since $g_1, h_1 \in L^\infty(\Omega)$. Corollary 1 is slightly weaker than Haberman and Tataru’s result; in their result assumption (1.8) is not required.

Concerning ii), we have

Theorem 2 (Lavine-Nachman). Let $d \geq 3$, $\Omega$ be a smooth bounded subset of $\mathbb{R}^d$. Let $q_1, q_2 \in L^{d/2}(\Omega)$. Assume that

$a_{q_1} = a_{q_2},$

then

$q_1 = q_2.$

As a consequence of Theorem 2 one obtains

\footnote{Here $1_\Omega$ denotes the characteristic function of $\Omega$ and $F$ denotes the Fourier transform. This condition arises as a hypothesis for our averaging estimates in Lemma 4.}
Corollary 2 (Lavine-Nachman). Let $d \geq 3$, $\Omega$ be a smooth bounded subset of $\mathbb{R}^d$, $\gamma_1, \gamma_2 \in W^{2,d/2}(\Omega)$ be such that
\[ 1/\lambda \leq \gamma_1(x), \gamma_2(x) \leq \lambda \text{ for a.e. } x \in \Omega, \]
for some $\lambda > 0$. Assume that
\[ DtN_{\gamma_1} = DtN_{\gamma_2}, \]
then
\[ \gamma_1 = \gamma_2. \]

Concerning iii), we have

Theorem 3. Let $\Omega$ be a smooth bounded subset of $\mathbb{R}^3$, $g_1, h_1 \in W^{t,3/(t+1)}(\Omega)$ for some $t > 1/2$, $g_2, h_2 \in L^{3/2}(\Omega)$. Set
\[ q_1 = \text{div} \, g_1 + g_2 \text{ and } q_2 = \text{div} \, h_1 + h_2. \]
Assume that
\[ \Lambda_{q_1} = \Lambda_{q_2}, \]
then
\[ q_1 = q_2. \]

Here is a consequence of Theorem 3.

Corollary 3. Let $\Omega$ be a smooth bounded subset of $\mathbb{R}^3$, $\gamma_1, \gamma_2 \in W^{s,3/s}(\Omega)$ for some $s > 3/2$ be such that
\[ 1/\lambda \leq \gamma_1(x), \gamma_2(x) \leq \lambda \text{ for a.e. } x \in \Omega, \]
for some $\lambda > 0$. Assume that
\[ DtN_{\gamma_1} = DtN_{\gamma_2}, \]
then
\[ \gamma_1 = \gamma_2. \]

Remark 1. In Theorems 1, 2, and 3, 0 is assumed not a Dirichlet eigenvalue for the potential problems. Then the fact that $\Lambda_{q_1} = \Lambda_{q_2}$ implies $q_1 = q_2$. In fact this assumption can be weaken as follows. Assume that
\[ \frac{\partial u_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta}, \]
for any $u_1, u_2 \in H^1(\Omega)$ such that
\[ \Delta u_i - q_i u_i = 0 \text{ in } \Omega \text{ for } i = 1, 2, \text{ and } u_1 = u_2 \text{ on } \partial \Omega. \]
Then $q_1 = q_2$ under the same conditions on $q_i$, $i = 1, 2$. In fact, we prove Theorems 1, 2, and 3 under this weaker assumption.

The paper is organized as follows. In Section 2 we establish new estimates for CGO solutions in the spirit of Sylvester and Uhlmann. In Section 3 we establish Theorem 1 and Corollary 1. This is established by generating CGO solutions via a direct iteration method and averaging methods. We then turn to the proof of Theorem 2 and Corollary 2 in Section 4. Section 5 handles the proof of Theorem 3 and Corollary 3. Finally, in Appendix A we provide a few results on averaging of the kernel $K_\varepsilon(x)$ to (1.7) that are used crucially in our CGO arguments, and in Appendix B we establish that $\gamma_1 = \gamma_2$ on $\partial \Omega$ if $DtN_{\gamma_1} = DtN_{\gamma_2}$ and $\gamma_1, \gamma_2$ belong only $W^{1,1}(\partial \Omega)$. This result which will be used in analyzing the Dirichlet to Neumann operator.
2. New estimates for CGO solutions in the spirit of Sylvester and Uhlmann

In this section, we recall and extend the fundamental estimates due to Sylvester and Uhlmann in [22] concerning solutions of the equation
\[
\Delta u + \xi \cdot \nabla u = f
\]
where \( \xi \in C^d \) and \( \xi \cdot \xi = 0 \).

Given \( \xi \in C^d \) with \( |\xi| > 2 \) and \( \xi \cdot \xi = 0 \), define
\[
\hat{K}_\xi(k) = \frac{1}{-|k|^2 + i \xi \cdot k} \quad \text{for } k \in \mathbb{R}^d.
\]
Then for \( f \in H^{-1}(\mathbb{R}^d) \) with compact support, \( \hat{K}_\xi f \) is a solution to the equation
\[
\Delta u + \xi \cdot \nabla u = f \quad \text{in } \mathbb{R}^d,
\]
and
\[
\hat{K}_\xi \ast f = \hat{K}_\xi \cdot \hat{f} \in L^1 + L^2.
\]
We recall the following fundamental results due to Sylvester and Uhlmann in [22].

**Proposition 1** (Sylvester-Uhlmann). Let \(-1 < \delta < 0, \xi \in C^d \) with \( |\xi| > 2 \) and \( \xi \cdot \xi = 0 \), and let \( f \in L^2_{\text{loc}}(\mathbb{R}^d) \). Then
\[
\|K_\xi \ast f\|_{H^k_\delta} \leq \frac{C}{|\xi|} \|f\|_{H^{k+\delta}_1} \quad \text{for } k \geq 0,
\]
\[
\|K_\xi \ast f\|_{H^{k+1}_\delta} \leq C \|f\|_{H^{k+\delta}_1} \quad \text{for } k \geq 0,
\]
for some positive constant \( C \) independent of \( \xi \) and \( f \).

These estimates play an important role in their proof of the uniqueness of smooth potentials [22] and in the proofs of the improvements in [1, 19, 10].

We will extend the above results to negative derivatives and to the case with two derivative difference, which are crucial for the proof of Theorem [1]. Our proof for negative derivatives and the two derivative difference is rather elementary, and the same proof also gives the following estimates. Let \( f \in L^2(\mathbb{R}^d) \) with compact support. We have
\[
\|K_\xi \ast f\|_{H^k(B_r)} \leq \frac{C_r}{|\xi|} \|f\|_{H^k} \quad \text{for } k \geq 0,
\]
and
\[
\|K_\xi \ast f\|_{H^{k+1}(B_r)} \leq C_r \|f\|_{H^k} \quad \text{for } k \geq 0
\]
for some positive constant \( C \) independent of \( \xi \) and \( f \). These estimates are slightly weaker than the original ones of Sylvester and Uhlmann in [22] and [23]; however, they are sufficient for establishing the uniqueness of smooth potential in [22]. Here is the extension:
Lemma 1. Let $R > 0$, $\xi \in \mathbb{C}^d$ with $|\xi| > 2$ and $\xi \cdot \xi = 0$, and let $f \in H^{-1}(\mathbb{R}^d)$ with $\text{supp} \, f \subset B_R$. Then
\begin{equation}
\|K_\xi * f\|_{L^2(B_r)} \leq C_r \|f\|_{H^{-1}}
\end{equation}
and
\begin{equation}
\|K_\xi * f\|_{H^{k+1}(B_r)} \leq C_r |\xi| \cdot \|f\|_{H^{k}}, \quad \text{for } k \geq 0,
\end{equation}
for some $C_r$ which depends on $r$ and $R$ but is independent of $\xi$ and $f$.

Proof. We will prove (2.6); the proof of (2.7) as well as (2.4) and (2.5) follow similarly. Set
\[ \Gamma_\xi := \{ k \in \mathbb{R}^d; -|k|^2 + i\xi \cdot k = 0 \}. \]

It is clear that
\begin{equation}
|\hat{K}_\xi(k)| \leq \frac{C}{|\xi| \text{dist}(k, \Gamma_\xi)} \quad \text{if } |k| \leq 2|\xi|, \quad \text{and } |\hat{K}_\xi(k)| \leq \frac{C}{|k|^2} \quad \text{if } |k| \geq 2|\xi|,
\end{equation}
(see e.g. the proof of Lemma 2.2 of [11]). In this proof, $C$ denotes a positive constant independent of $\xi$ and $f$. Define $K_{1,\xi}$ and $K_{2,\xi}$ as follows
\begin{equation}
\hat{K}_{1,\xi}(k) = \begin{cases} 
\hat{K}_\xi(k) & \text{if dist}(k, \Gamma_\xi) \geq 1, \\
0 & \text{otherwise},
\end{cases}
\end{equation}
and
\begin{equation}
\hat{K}_{2,\xi}(k) = \hat{K}_\xi(k) - \hat{K}_{1,\xi}(k),
\end{equation}
and so
\begin{equation}
\|K_\xi * f\|_{L^2(B_r)} \leq \|K_{1,\xi} * f\|_{L^2(B_r)} + \|K_{2,\xi} * f\|_{L^2(B_r)}.
\end{equation}

Using Plancherel's theorem, we derive from (2.8) and (2.9) that
\begin{equation}
\|K_{1,\xi} * f\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{H^{-1}}.
\end{equation}

Fix
\begin{equation}
\varphi \in C_0^{\infty}(\mathbb{R}^d) \text{ with } \varphi = 1 \text{ in } B_{R+r}.
\end{equation}
Since $\text{supp} \, f \subset B_R$, it follows that $f = \varphi f$; hence
\[ \hat{f} = \hat{\varphi} * \hat{f}. \]

Define
\begin{equation}
\tilde{f}(k) = \sup_{\eta \in B_4(k)} |\hat{f}(\eta)|
\end{equation}
and
\begin{equation}
\tilde{\varphi}(k) = \sup_{\eta \in B_4(k)} |\hat{\varphi}(\eta)|.
\end{equation}

Since
\[ |\hat{f}| * |\hat{\varphi}|(\eta) = \int_{\mathbb{R}^d} |\hat{f}(\zeta)||\hat{\varphi}(\eta - \zeta)| \, d\zeta, \]
it follows from the definition of $\tilde{f}$ (2.14) and $\tilde{\varphi}$ (2.15) that
\begin{equation}
\tilde{f} \leq |\hat{f}| * \tilde{\varphi}.
\end{equation}
From the choice of $\varphi$ (2.13), we have
\begin{equation}
\|K_{2,\xi} \ast f\|_{L^2(B_r)}^2 \leq \|\varphi(K_{2,\xi} \ast f)\|_{L^2(\mathbb{R}^d)}^2
\end{equation}
(2.17)
\[ \leq \int_{\mathbb{R}^d} \left| \int_{\text{dist}(\eta, \Gamma_\xi) \leq 1} |\hat{\varphi}(k - \eta)| \cdot |\hat{K}_\xi(\eta)| \cdot |\hat{f}(\eta)| \ d\eta \right|^2 \ d k. \]

Using the fact that
\begin{equation}
\int_{|x| \leq 1} \frac{1}{|x_1| + |x_2|} \ dx_1 \ dx_2 < +\infty,
\end{equation}
(2.18)
we obtain
\begin{equation}
\int_{\text{dist}(\eta, \Gamma_\xi) \leq 1} |\hat{\varphi}(k - \eta)| \cdot |\hat{K}_\xi(\eta)| \cdot |\hat{f}(\eta)| \ d\eta \leq \frac{C}{|\xi|} \int_{\text{dist}(\eta, \Gamma_\xi) \leq 1} \hat{\varphi}(k - \eta) \hat{f}(\eta) \ d\eta.
\end{equation}
(2.19)

In fact, for $|\xi| > 2$, there exists $0 < r \leq 1$ (independent of $\xi$) such that for $\eta$ with $\text{dist}(\eta, \Gamma_\xi) \leq r$, there exists an unique pair $(\eta_1, \eta_2) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $\eta_1 \in \Gamma_\xi$, $\eta_2 \perp T_{\Gamma_\xi}(\eta_1)$, the tangent plane of $\Gamma_\xi$ at $\eta_1$, such that $|\eta_2| \leq r$ and $\eta_1 + \eta_2 = \eta$. Then
\begin{equation}
\int_{\text{dist}(\eta, \Gamma_\xi) \leq r} |\hat{\varphi}(k - \eta)| \cdot |\hat{K}_\xi(\eta)| \cdot |\hat{f}(\eta)| \ d\eta \leq C \int_{\eta_1 \in \Gamma_\xi} \int_{|\eta_2| \leq r: \eta_2 \perp T_{\Gamma_\xi}(\eta_1)} |\hat{\varphi}(k - \eta_1 - \eta_2)| \cdot |\hat{K}_\xi(\eta_1 + \eta_2)| \cdot |\hat{f}(\eta_1 + \eta_2)| \ d\eta_2 \ d\eta_1.
\end{equation}
(2.20)

Since
\begin{equation}
\int_{\eta_1 \in \Gamma_\xi} \int_{|\eta_2| \leq r: \eta_2 \perp T_{\Gamma_\xi}(\eta_1)} |\hat{\varphi}(k - \eta_1 - \eta_2)| \cdot |\hat{K}_\xi(\eta_1 + \eta_2)| \cdot |\hat{f}(\eta_1 + \eta_2)| \ d\eta_2 \ d\eta_1
\end{equation}
(2.21)
and, by (2.18),
\[ \int_{|\eta_2| \leq r: \eta_2 \perp T_{\Gamma_\xi}(\eta_1)} |\hat{K}_\xi(\eta_1 + \eta_2)| \ d\eta_2 \leq \frac{C}{|\xi|}, \]
(2.22)
it follows that
\begin{equation}
\int_{\text{dist}(\eta, \Gamma_\xi) \leq r} |\hat{\varphi}(k - \eta)| \cdot |\hat{K}_\xi(\eta)| \cdot |\hat{f}(\eta)| \ d\eta \leq \frac{C}{|\xi|} \int_{\eta_1 \in \Gamma_\xi} \sup_{|\eta_2| \leq r} |\hat{\varphi}(k - \eta_1 - \eta_2)| \sup_{|\eta_2| \leq r} |\hat{f}(\eta_1 + \eta_2)| \ d\eta_1.
\end{equation}
(2.23)

On the other hand, by the definition of $\tilde{f}$ and $\tilde{\varphi}$,
\begin{equation}
\int_{\eta_1 \in \Gamma_\xi} \sup_{|\eta_2| \leq r} |\hat{\varphi}(k - \eta_1 - \eta_2)| \sup_{|\eta_2| \leq r} |\hat{f}(\eta_1 + \eta_2)| \ d\eta_1 \leq C \int_{\text{dist}(\eta, \Gamma_\xi) \leq 1} \tilde{\varphi}(k - \eta) \tilde{f}(\eta) \ d\eta.
\end{equation}
(2.24)
A combination of (2.21) and (2.22) yields (2.19).

Applying Hölder’s inequality, we derive from (2.17) and (2.19) that
\begin{equation}
\|K_{2,\xi} \ast f\|_{L^2(B_r)}^2 \leq \frac{C}{|\xi|^2} \int_{\text{dist}(\eta, \Gamma_\xi) \leq 1} |\tilde{f}(\eta)|^2 \ d\eta.
\end{equation}
(2.25)
We now estimate the RHS of (2.23). Applying Hölder’s inequality, from (2.16) and the
fact that $\tilde{\varphi} \in L^1$, we have
\begin{equation}
\int_{\text{dist}(\eta, \Gamma_\varepsilon) \leq 1} |\tilde{f}(\eta)|^2 \, d\eta \leq C \int \tilde{\varphi}(\eta - k) |\tilde{f}(k)|^2 \, dk \, d\eta.
\end{equation}
Using Fubini’s theorem, we derive from (2.24) that
\begin{equation}
\int_{\text{dist}(\eta, \Gamma_\varepsilon) \leq 1} |\tilde{f}(\eta)|^2 \, d\eta \leq C \int |\tilde{f}(k)|^2 \int_{\text{dist}(\eta, \Gamma_\varepsilon) \leq 1} \tilde{\varphi}(\eta - k) \, d\eta \, dk.
\end{equation}
Since $\tilde{\varphi} \in S$, the Schwartz class, it follows that
\begin{equation}
\int |\tilde{f}(k)|^2 \int_{\text{dist}(\eta, \Gamma_\varepsilon) \leq 1} \tilde{\varphi}(\eta - k) \, d\eta \, dk \leq C \left( \int_{|k| \leq 2|\xi|} |\tilde{f}(k)|^2 \, dk + \int_{|k| > 2|\xi|} \frac{|\tilde{f}(k)|^2}{|k|^2} \, dk \right).
\end{equation}
From (2.25) and (2.26), we obtain
\begin{equation}
\frac{1}{|\xi|^2} \int_{\text{dist}(\eta, \Gamma_\varepsilon) \leq 1} |\tilde{f}(\eta)|^2 \, d\eta \leq C \|f\|_H^{-1}.
\end{equation}
Combining (2.23) and (2.27) yields,
\begin{equation}
\|K_{2\xi} \ast f\|_{L^2(B_r)}^2 \leq C \|f\|_H^{-1},
\end{equation}
and the conclusion now follows from (2.11), (2.12), and (2.28). \qed

3. Proof of Theorem 1 and Corollary 1

In this section, we prove Theorem 1 and Corollary 1. The proof of Theorem 1 contains
two main ingredients. The first one is a new useful inequality (Lemma 2) and its variant
(Lemma 3) to solutions to (2.1) whose the proof is based on estimates presented in Section 2. The second one is an averaging estimate (Lemma 4) with respect to $\xi$ for $K_\xi \ast q$
which has root in [11] and is presented in Appendix A.

3.1. Some useful lemmas. The following lemma is a new observation and plays an
important role in our analysis. Its proof is quite elementary, and can be seen as the replacement of [11] Lemma 2.3.

**Lemma 2.** Let $d \geq 3$, $\xi \in \mathbb{C}^d$ ($|\xi| > 2$) with $\xi \cdot \varepsilon = 0$, $g_1 \in [L^\infty(\mathbb{R}^d)]^d$, $g_2 \in L^d(\mathbb{R}^d)$ and $v \in H^1(\mathbb{R}^d)$ be such that supp $g_1$, supp $g_2 \subset B_1$. Set
\[ q = \text{div} \, g_1 + g_2 \]
and define
\[ u = K_\xi \ast (qv). \]
We have
\begin{equation}
\|\nabla u\|_{L^2(B_r)} + |\xi| \cdot \|u\|_{L^2(B_r)} \\
\leq C_r \left( \|g_1\|_{L^\infty} + \|g_2\|_{L^d} \right) \left( \|\nabla v\|_{L^2} + |\xi| \cdot \|v\|_{L^2} \right),
\end{equation}
for some positive constant $C_r$ independent of $\xi$, $g_1$, $g_2$, and $v$.

**Proof.** We have
\begin{equation}
qv = \text{div}(vg_1) - g_1 \cdot \nabla v + g_2 \varepsilon \text{ in } \mathbb{R}^d.
\end{equation}
Applying (2.25) with $k = 1$ and (2.7) with $k = 0$, we have
\[ \|\nabla u\|_{L^2(B_r)} \leq C_r \left( |\xi| \cdot \|\text{div}(vg_1)\|_{H^{-1}} + \|g_1 \cdot \nabla v\|_{L^2} + \|g_2 v\|_{L^2} \right). \]
which implies
\[ \| \nabla u \|_{L^2(B_r)} \leq C_r \left( |\xi| \cdot \| v g_1 \|_{L^2} + \| g_1 \cdot \nabla v \|_{L^2} + \| g_2 v \|_{L^2} \right). \]

It follows that
\[ \| \nabla u \|_{L^2(B_r)} \leq C_r \left( \| g_1 \|_{L^\infty} + \| g_2 \|_{L^d} \right) \left( |\xi| \cdot \| v \|_{L^2} + \| \nabla v \|_{L^2} \right). \]  

Similarly, using (3.2) and applying (2.4) with \( k = 0 \), and (2.6), we obtain
\[ |\xi| \cdot \| u \|_{L^2(B_r)} \leq C_r \left( |\xi| \cdot \| (g_1 v) \|_{H^{-1}} + \| g_1 \nabla v \|_{L^2} + \| g_2 v \|_{L^2} \right), \]
which implies
\[ |\xi| \cdot \| u \|_{L^2(B_r)} \leq C_r \left( |\xi| \cdot \| g_1 v \|_{L^2} + \| g_1 \nabla v \|_{L^2} + \| g_2 v \|_{L^2} \right). \]

It follows that
\[ |\xi| \cdot \| u \|_{L^2(B_r)} \leq C_r \left( \| g_1 \|_{L^\infty} + \| g_2 \|_{L^d} \right) \left( |\xi| \cdot \| v \|_{L^2} + \| \nabla v \|_{L^2} \right). \]  

A combination of (3.3) and (3.4) yields (3.1).

When \( g_1 \) and \( g_2 \) are smooth, we can improve the conclusion in Lemma 2 as follows.

**Lemma 3.** Let \( d \geq 3 \), \( \xi \in \mathbb{C}^d \) (\(|\xi| > 2\)) with \( \xi \cdot \xi = 0 \), \( g_1 \in [C^2(\mathbb{R}^d)]^d \), \( g_2 \in C^1(\mathbb{R}^d) \) with \( \text{supp} \, g_1, \text{supp} \, g_2 \subset B_1 \), and let \( v \in H^1(\mathbb{R}^d) \). Set
\[ q = \text{div} \, g_1 + g_2 \]
and define
\[ u = K_\xi * (qv). \]

We have, for \( r > 0 \),
\[ \| \nabla u \|_{L^2(B_r)} + |\xi| \cdot \| u \|_{L^2(B_r)} \leq C_r \left( \| g_1 \|_{C^2} + \| g_2 \|_{C^1} \right) \left( \| v \|_{L^2} + |\xi| \cdot \| v \|_{L^2} \right). \]  

Here \( C_r \) is a positive constant depending only on \( r \) and \( d \).

**Proof.** Applying (2.4) with \( k = 1 \), we have
\[ \| \nabla u \|_{L^2(B_r)} \leq \frac{C_r}{|\xi|} \left( \| v \text{div} \, g_1 \|_{H^1} + \| v g_2 \|_{H^1} \right) \]
\[ \leq \frac{C_r}{|\xi|} \| v \|_{H^1} \left( \| g_1 \|_{C^2} + \| g_2 \|_{C^1} \right). \]

Similarly,
\[ \| u \|_{L^2(B_r)} \leq \frac{C_r}{|\xi|} \left( \| v \text{div} \, g_1 \|_{L^2} + \| v g_2 \|_{L^2} \right) \]
\[ \leq \frac{C_r}{|\xi|} \| v \|_{L^2} \left( \| g_1 \|_{C^1} + \| g_2 \|_{C^0} \right). \]

A combination of (3.6) and (3.7) yields (3.5).
3.2. Construction of CGO solutions. We begin this section with an estimate for the solution of the equation
\[ \Delta u + \xi \cdot \nabla u - qu = q \text{ in } \mathbb{R}^d. \]

**Proposition 2.** Let \( \xi \in \mathbb{C}^d \) \( (|\xi| > 2) \) with \( \xi \cdot \xi = 0 \), \( g_1 \in [L^\infty(\mathbb{R}^d)]^d \), \( g_2 \in L^d(\mathbb{R}^d) \) with \( \text{supp } g_1, \text{supp } g_2 \subset B_1 \). Set \( q = \text{div } g_1 + g_2 \). Then there exists a positive constant \( c \) such that if

\[
\inf_{\phi \in [C(\mathbb{R}^d)]^d, \text{supp } \phi \subset B_1} \|g_1 - \phi\|_{L^\infty} \leq c,
\]

then there exists \( u \in H^1_{\text{loc}}(\mathbb{R}^d) \) such that

\[
u = K_\xi * (q + qu)
\]

and

\[
\|\nabla (u - u_1)\|_{L^2(B_r)} + |\xi| \cdot \|u - u_1\|_{L^2(B_r)} \leq C_r \left( \|\nabla u_1\|_{L^2(B_2)} + |\xi| \cdot \|u_1\|_{L^2(B_2)} \right) \quad \forall r > 0,
\]

for \( |\xi| \) large enough\(^2\). Here \( u_1 \) is given by

\[
u_1 = K_\xi * q.
\]

**Proof.** Let \( g_{i,j}, 1 \leq i, j \leq 2 \), such that

\[
g_{1,1} + g_{1,2} = g_1 \quad \text{and} \quad g_{2,1} + g_{2,2} = g_2.
\]

\( g_{1,2}, g_{2,2} \) are smooth with compact support in \( B_1 \),

\[
\|g_{1,1}\|_{L^\infty} + \|g_{2,1}\|_{L^d} \leq 2c,
\]

Set

\[
q = q_1 + q_2,
\]

where

\[
q_1 = \text{div } g_{1,1} + g_{2,1}
\]

and

\[
q_2 = \text{div } g_{1,2} + g_{2,2}.
\]

Let \( u_0 = 0 \) and consider the following iteration process:

\[
u_n = K_\xi * (q + qu_{n-1}) \quad \text{for } n \geq 1,
\]

which implies

\[
\Delta u_n + \xi \cdot \nabla u_n = q + qu_{n-1} \text{ in } \mathbb{R}^d, \text{ for } n \geq 1.
\]

Define

\[
u_{1,n} = K_\xi * (q_1 + q_1u_{n-1}) \quad \text{and} \quad \nu_{2,n} = K_\xi * (q_2 + q_2u_{n-1});
\]

this yields,

\[
\Delta u_{1,n} + \xi \cdot \nabla u_{1,n} = q_1 + q_1u_{n-1} \text{ in } \mathbb{R}^d
\]

and

\[
\Delta u_{2,n} + \xi \cdot \nabla u_{2,n} = q_2 + q_2u_{n-1} \text{ in } \mathbb{R}^d.
\]

We have

\[
u_n = u_{1,n} + u_{2,n}.
\]

Set

\[
w_{n+1} = u_{n+1} - u_n, \quad w_{1,n+1} = u_{1,n+1} - u_{1,n}, \quad w_{2,n+1} = u_{2,n+1} - u_{2,n}.
\]

\(^2\)The largeness of \( \xi \) depends only on \( g_1 \) and \( g_2 \).
It follows from Lemma 2 that
\begin{equation}
\| \nabla w_{1,n+1} \|_{L^2(B_r)} + |\xi| \cdot \| w_{1,n+1} \|_{L^2(B_r)} \leq C_r \left( \| g_{1,1} \|_{L^\infty} + \| g_{2,1} \|_{L^d} \right) \left( \| \nabla w_n \|_{L^2(B_2)} + |\xi| \cdot \| w_n \|_{L^2(B_2)} \right),
\end{equation}
and from Lemma 3 that
\begin{equation}
\| \nabla w_{2,n+1} \|_{L^2(B_r)} + |\xi| \cdot \| w_{2,n+1} \|_{L^2(B_r)} \leq \frac{C_r}{|\xi|} \left( \| g_{1,1} \|_{C^2} + \| g_{2,1} \|_{C^1} \right) \left( \| \nabla w_n \|_{L^2(B_2)} + |\xi| \cdot \| w_n \|_{L^2(B_2)} \right).
\end{equation}
A combination of (3.10), (3.11), and (3.12) yields
\begin{equation}
\| \nabla w_{n+1} \|_{L^2(B_r)} + |\xi| \cdot \| w_{n+1} \|_{L^2(B_r)} \leq C_r \left( \| g_{1,1} \|_{L^\infty} + \| g_{2,1} \|_{L^d} \right) + \frac{1}{|\xi|} \left( \| g_{1,2} \|_{C^2} + \| g_{2,2} \|_{C^1} \right) \left( \| \nabla w_n \|_{L^2(B_2)} + |\xi| \cdot \| w_n \|_{L^2(B_2)} \right).
\end{equation}
Thus, if $|\xi|$ is large enough, then
\begin{equation}
C_2 \left( \| g_{1,1} \|_{L^\infty} + \| g_{2,1} \|_{L^d} \right) + \frac{1}{|\xi|} \left( \| g_{1,2} \|_{C^2} + \| g_{2,2} \|_{C^1} \right) \leq 3/4.
\end{equation}
Hence, by standard fixed point arguments, it follows that
\begin{equation}
u_n \to u \text{ in } H^1(B_2).
\end{equation}
This implies, by (3.13),
\begin{equation}
u_n \to u \text{ in } H^1(B_r) \quad \text{for all } r > 0,
\end{equation}
and by (3.9)
\begin{equation}
u = K_\xi * (q + qu).
\end{equation}
We derive from (3.13) that
\begin{equation}
\| \nabla (u - u_1) \|_{L^2(B_2)} + |\xi| \cdot \| u - u_1 \|_{L^2(B_2)} \leq C \left( \| \nabla u_1 \|_{L^2(B_2)} + |\xi| \cdot \| u_1 \|_{L^2(B_2)} \right).
\end{equation}
Statement (3.8) now follows from (3.13), and the proof is complete.

To obtain some appropriate estimate for $u_1$ in Proposition 2, we use an averaging argument in the spirit of Haberman and Tataru in [11]. More precisely, we have the following lemma whose proof is quite lengthy and given in the appendix.

**Lemma 4.** Let $d \geq 3$, $s > 2$, $k \in \mathbb{R}^d$ with $|k| \geq 2$, $1 \leq p < 2$, and $R > 10$. We have
\begin{equation}
\frac{1}{R} \int_{R/2}^{2R} \int_{\sigma_1 \in S^{d-1}} \int_{\sigma_2 \in S^{d-1}} \left| \hat{K}_{s_2 - is_1}(k) \right|^p d\sigma_2 d\sigma_1 ds \leq C \min \left\{ \frac{1}{R^p |k|^p}, \frac{1}{|k|^{2p}} \right\},
\end{equation}
and
\begin{equation}
\frac{1}{R} \int_{R/2}^{2R} \int_{\sigma_1 \in S^{d-1}} \int_{\sigma_2 \in S^{d-1}} \int_{\sigma_3 \in S^{d-1}, \sigma_2} \left| \hat{K}_{s_2 + \sqrt{1 + s_1^2}}^{s_3} \right|^p d\sigma_3 d\sigma_2 d\sigma_1 ds \leq C \min \left\{ \frac{1}{R^p |k|^p}, \frac{1}{|k|^{2p}} \right\}.
\end{equation}
for some positive constant $C$ depending only on $d$ and $p$. Here

\begin{equation}
S_{\sigma_1}^{d-1} := \{ \sigma \in S^{d-1}; \sigma \cdot \sigma_1 = 0 \}
\end{equation}

and

\begin{equation}
S_{\sigma_1, \sigma_2}^{d-1} := \{ \sigma \in S^{d-1}; \sigma \cdot \sigma_1 = 0 \text{ and } \sigma \cdot \sigma_2 = 0 \}.
\end{equation}

**Remark 2.** Let $\sigma_1 \in S^{d-1}$, $\sigma_2 \in S_{\sigma_1}^{d-1}$, and $\sigma_3 \in S_{\sigma_1, \sigma_2}^{d-1}$. Set

$$
\xi_1 = s \sigma_2 - i s \sigma_1 \quad \text{and} \quad \xi_2 = -\frac{s^2 \sigma_2}{\sqrt{1 + s^2}} + \frac{s \sigma_3}{\sqrt{1 + s^2}} + i s \sigma_1,
$$

then

$$
\xi_1 \cdot \xi_1 = \xi_2 \cdot \xi_2 = 0 \quad \text{and} \quad \xi_1 + \xi_2 = \left(s - \frac{s^2}{\sqrt{1 + s^2}}\right) \sigma_2 + \frac{s \sigma_3}{\sqrt{1 + s^2}} \to \sigma_3
$$

uniformly with respect to $\sigma_1$ and $\sigma_2$ as $s \to \infty$.

Using Proposition 2 and Lemma 1, we obtain the following result:

**Proposition 3.** Let $d \geq 3$, $g_1, h_1 \in [L^\infty(\mathbb{R}^d)]^d$, $g_2, h_2 \in L^d(\mathbb{R}^d)$ with supports in $B_1$ be such that $\hat{g}_1, \hat{h}_1 \in L^p(\mathbb{R}^d)$ for some $1 < p < 2$. Assume that

$$
\inf_{\phi \in [C(\Omega)]^d} \|g_1 - \phi\|_\infty + \inf_{\phi \in [C(\Omega)]^d} \|h_1 - \phi\|_\infty \leq c,
$$

where $c$ is the constant in Proposition 2 or $g_1, h_1$ are continuous. Set

$$
q_1 = \text{div} \ g_1 + g_2 \quad \text{and} \quad q_2 = \text{div} \ h_1 + h_2.
$$

Then for any $0 < \varepsilon < 1$, $n > 2$, and $\sigma \in S^{d-1}$, there exist $\sigma_{1, \varepsilon}, \sigma_{2, \varepsilon}, \sigma_{3, \varepsilon} \in S^{d-1}$, $s_\varepsilon \in (n, 4n)$, and $u_\varepsilon, v_\varepsilon \in H^1_{\text{loc}}(\mathbb{R}^d)$ such that

\begin{equation}
\sigma_{1, \varepsilon} \cdot \sigma_{2, \varepsilon} = \sigma_{1, \varepsilon} \cdot \sigma_{3, \varepsilon} = \sigma_{2, \varepsilon} \cdot \sigma_{3, \varepsilon} = 0,
\end{equation}

\begin{equation}
|\sigma_{3, \varepsilon} - \sigma| \leq \varepsilon,
\end{equation}

\begin{equation}
\epsilon_\varepsilon = K_{\varepsilon_{1, \varepsilon}} * (q_1 + q_1 u_\varepsilon), \quad v_\varepsilon = K_{\varepsilon_{2, \varepsilon}} * (q_2 + q_2 v_\varepsilon),
\end{equation}

\begin{equation}
\|\nabla u_\varepsilon\|_{L^2(B_r)} + s_\varepsilon \|u_\varepsilon\|_{L^2(B_r)} \leq C_r / \varepsilon^{3d},
\end{equation}

and

\begin{equation}
\|\nabla v_\varepsilon\|_{L^2(B_r)} + s_\varepsilon \|v_\varepsilon\|_{L^2(B_r)} \leq C_r / \varepsilon^{3d},
\end{equation}

for some $C_r > 0$ independent of $\varepsilon$, $s$, and $\sigma$. Here

\begin{equation}
\xi_{1, \varepsilon} = s_\varepsilon \sigma_{2, \varepsilon} - i s_\varepsilon \sigma_{1, \varepsilon} \quad \text{and} \quad \xi_{2, \varepsilon} = -\frac{s_\varepsilon^2 \sigma_{2, \varepsilon}}{\sqrt{1 + s_\varepsilon^2}} + \frac{s_\varepsilon \sigma_{3, \varepsilon}}{\sqrt{1 + s_\varepsilon^2}} + i s_\varepsilon \sigma_{1, \varepsilon}.
\end{equation}

**Proof.** Applying Lemma 1, we have

\[
\frac{1}{n} \int_{\mathbb{R}^d} \int_{\sigma_1 \in S^{d-1}} \int_{\sigma_2 \in S_{\sigma_1}^{d-1}} \left| \hat{K}_{s_\sigma_2 - i s_\sigma_1}(k) \right|^p \, ds_3 \, ds_2 \, ds_1 \, ds \leq C \min \left\{ \frac{1}{n^p |k|^p}, \frac{1}{|k|^{2p}} \right\}.
\]
This implies (3.18) and (3.19) hold for some \( s, \varepsilon \in (n, 4n) \) and \( \sigma_{1, \varepsilon}, \sigma_{2, \varepsilon}, \sigma_{3, \varepsilon} \in S^{d-1} \), and

\[
(3.23) \quad \int_{\mathbb{R}^d} \left( |\hat{K}_{\xi_1, \varepsilon}(k)|^p |\hat{q}_1(k)|^p + |\hat{K}_{\xi_2, \varepsilon}(k)|^p |\hat{q}_2(k)|^p \right) (|k|^p + n^p) \, dk 
\leq \frac{C}{\varepsilon^{3d}} \int_{\mathbb{R}^d} \frac{1}{|k|^p} (|\hat{q}_1(k)|^p + |\hat{q}_2(k)|^p) \, dk \leq \frac{C}{\varepsilon^{3d}},
\]

where \( \xi_{1, \varepsilon} \) and \( \xi_{2, \varepsilon} \) are given by (3.22).

Define

\[ u_{1, \varepsilon} = K_{\xi_{1, \varepsilon}} * q_1 \text{ and } u_{2, \varepsilon} = K_{\xi_{2, \varepsilon}} * q_2. \]

It follows from (3.23) that

\[
\int_{\mathbb{R}^d} \left( |\hat{u}_{1, \varepsilon}|^p + |\hat{u}_{2, \varepsilon}|^p + n^p |\hat{u}_{1, \varepsilon}|^p + n^p |\hat{u}_{1, \varepsilon}|^p \right) \, dk \leq C/\varepsilon^{3d}.
\]

Applying Young’s inequality, we obtain

\[
\| \nabla u_{1, \varepsilon} \|_{L^2(B_r)} + \| \nabla u_{1, \varepsilon} \|_{L^2(B_r)} + n\| u_{1, \varepsilon} \|_{L^2(B_r)} + n\| u_{1, \varepsilon} \|_{L^2(B_r)} \leq C_r/\varepsilon^{3d/p},
\]

for all \( r > 0 \). The conclusion now follows from Proposition 2 \( \square \).

### 3.3. Proof of Theorem 1

Without loss of generality one may assume that \( \Omega \subset B_{1/2} \). Let \( g_{i,j}, 1 \leq i, j \leq 2 \), such that

\[
g_{1,1} + g_{1,2} = g_1 \text{ and } h_{1,1} + h_{1,2} = h_1.
\]

\( g_{1,1}, h_{1,1} \) are smooth with compact support in \( \Omega \),

\[
\|g_{1,2}\|_{L^\infty} + \|h_{1,2}\|_{L^\infty} \leq 2c,
\]

Extend \( g_{1,1} \) and \( h_{1,1} \) smoothly in \( \mathbb{R}^d \setminus \Omega \) with compact support in \( B_1 \) and denote these extension by \( G_{1,1} \) and \( H_{1,1} \). Extend \( g_{1,2}, h_{1,2}, g_2, h_2 \) by 0 outside \( \Omega \) and denote these extensions by \( G_{1,2}, H_{1,2}, G_2, H_2 \). Define

\[
G_1 = G_{1,1} + G_{1,2} \quad \text{and} \quad H_1 = H_{1,1} + H_{1,2}.
\]

Extend \( q_1 \) and \( q_2 \) in \( \mathbb{R}^d \) by \( \text{div} \, G_1 + G_2 \) and \( \text{div} \, H_1 + H_2 \) and still denote these extension by \( q_1 \) and \( q_2 \). Then \( q_1 \) and \( q_2 \) satisfy the assumptions of Proposition \( 3 \) since \( F(1_{\Omega}) \in L^r(\mathbb{R}^d) \) for \( r > 2n/(n+1) \) (see [18] Theorem 1). We claim that there exist \( \sigma \in S^{d-1} \) and \( u_{1,n}, u_{2,n} \in H^1_{\text{loc}}(\mathbb{R}^d) \) such that

\[
|\sigma - \sigma_0| \leq \varepsilon,
\]

\[
(3.24) \quad \Delta u_{i,n} + \xi_{i,n} \cdot \nabla u_{i,n} - q_i u_{i,n} = q_i \text{ in } \mathbb{R}^d \quad \text{for } i = 1, 2,
\]

\[
(3.25) \quad u_{i,n} \to 0 \text{ weakly in } H^1(B_2) \quad \text{for } i = 1, 2,
\]

for some \( \xi_{1,n}, \xi_{2,n} \in \mathbb{C}^d \) with \( \xi_{i,n} \cdot \xi_{i,n} = 0, |\xi_{1,n} + \xi_{2,n} - \sigma| \to 0 \) and \( |\xi_{i,n}| \to \infty \) as \( n \to \infty \).

Indeed, applying Proposition \( 3 \) there exist \( u_{i,n} \in H^1_{\text{loc}}(\mathbb{R}^d) \) \((i = 1, 2)\) such that

\[
u_{i,n} = K_{\xi_{i,n}} * (q_i + q_i u_{i,n}).
\]

Moreover,

\[
\| \nabla u_{i,n} \|_{L^2(B_r)} + n\| u_{i,n} \|_{L^2(B_r)} \leq C_r/\varepsilon^{3d/p},
\]

for some \( C_r > 0 \) which depends only on \( d, g_i, h_i \) \((i = 1, 2)\), and \( r \). Here

\[
\xi_{1,n} = s_n(\sigma_{2,n} - i s_n \sigma_{2,n})
\]

and

\[
\xi_{2,n} = s_n(-s_n \sigma_{2,n}/\sqrt{1 + s_n^2} + \sigma_{3,n}/\sqrt{1 + s_n^2}) + i s_n \sigma_{1,n}.
\]
for some $s_n \in (n, 4n)$, and $\sigma_{1,n}, \sigma_{2,n}, \sigma_{3,n}$ such that

$$\sigma_{1,n} \cdot \sigma_{2,n} = \sigma_{1,n} \cdot \sigma_{3,n} = \sigma_{2,n} \cdot \sigma_{3,n} = 0,$$

$$|\sigma_{3,n} - \sigma_0| \leq \varepsilon.$$  

Without loss of generality one might assume that $\sigma_{3,n} \rightarrow \sigma$ for some $\sigma \in S^{d-1}$. Then

$$\xi_{1,n} + \xi_{2,n} = \left(s_n - \frac{s_n^2}{\sqrt{1 + s_n^2}}\right)\sigma_{2,n} + \frac{s_n \sigma_{3,n}}{\sqrt{1 + s_n^2}} \rightarrow \sigma,$$

and the claim is proved.

We now can apply the complex geometric optics approach introduced by Sylvester and Uhlmann in [22]. Define, for $i = 1, 2$,

$$v_{i,n} = (1 + u_{i,n})e^{\xi_{i,n} \cdot x/2}.$$  

Since $u_{i,n}$ satisfies (3.25), it follows that (see for example [22])

$$\Delta v_{i,n} - q_i u_{i,n} = 0 \text{ in } \mathbb{R}^3 \text{ for } i = 1, 2.$$  

We derive from (1.2) that

$$\int_{B_2} (q_1 - q_2)(1 + u_{1,n})(1 + u_{2,n})e^{\sigma_n \cdot x/2} = 0,$$

where

$$\sigma_n = \xi_{1,n} + \xi_{2,n} \rightarrow \sigma \text{ as } n \rightarrow \infty.$$  

A combination of (3.25), (3.27), and (3.28) yields

$$\int_{B_2} (q_1 - q_2)e^{\sigma \cdot x/2} = 0.$$  

Since $\sigma_0 \in S^{d-1}$ and $\varepsilon > 0$ are arbitrary, it follows that

$$\int_{B_2} (q_1 - q_2)e^{\sigma_0 \cdot x/2} = 0 \text{ for all } \sigma_0 \in S^{d-1}.$$  

This implies

$$q_1 = q_2,$$

and the proof is complete.  

3.4. Proof of Corollary 1. Let $u_i \in H^1(\Omega)$ ($i = 1, 2$) be a solution to the equation

$$\text{div}(\gamma_i \nabla u_i) = 0 \text{ in } \Omega.$$  

Define

$$v_i = \gamma_i^{1/2} u_i \text{ in } \Omega.$$  

Then $v_i \in H^1(\Omega)$ is a solution to the equation

$$\Delta v_i - q_i v_i = 0 \text{ in } \Omega,$$

where

$$q_i = \frac{\Delta \gamma_i^{1/2}}{\gamma_i^{1/2}} = \Delta t_i - |\nabla t_i|^2 \text{ in } \Omega.$$  

Here $t_i$ ($i = 1, 2$) is given by

$$t_i = \ln \gamma_i^{1/2} \text{ in } \Omega.$$
Since $DtN_{\gamma_1} = DtN_{\gamma_2}$, it follows that
\[ \int_{\Omega} (q_1 - q_2)v_1v_2 = 0, \]
f for all solutions $v_i$ ($i = 1, 2$) to the equation
\[ \Delta v_i - q_i v_i = 0 \text{ in } \Omega. \]

Set
\[ g_i = \nabla t_i \text{ and } h_i = -t_i^2, \]
then $g_i$ and $h_i$ satisfy the assumptions of Theorem 1. Applying Theorem 1, we have
\[ q_1 = q_2 \text{ in } \Omega. \]
This implies,
\[ \Delta(t_1 - t_2) = |\nabla t_1|^2 - |\nabla t_2|^2 \in L^2(\Omega). \]

Hence
\[ (3.30) \quad \partial_{\eta} t_1 = \partial_{\eta} t_2 \text{ on } \partial \Omega, \]

We also have, by Proposition [31]
\[ (3.31) \quad t_1 = t_2 \text{ on } \partial \Omega, \]

We derive from (3.29) and the definition of $q_i$ that
\[ \Delta(t_1 - t_2) - \nabla t \cdot \nabla(t_1 - t_2) = 0 \text{ in } \Omega, \]
where $t = t_1 + t_2 \in W^{1,\infty}(\Omega)$. This implies $t_1 = t_2$ by (3.30), (3.31), and the unique continuation principle. Therefore, the conclusion follows. \\[ \square \]

4. Proof of Theorem 2 and Corollary 2

4.1. Construction of CGO solutions. We begin this section with an estimate for the solution to the equation
\[ \Delta u + \xi \cdot \nabla u + qu = q \text{ in } \mathbb{R}^d, \]
for $q \in L^{d/2}(\mathbb{R}^d)$. This estimate will play an important role in the proof of Lavine and Nachmann’s uniqueness result.

Proposition 4. Let $d \geq 3$, $\xi \in \mathbb{C}^d$ ($|\xi| > 2$) with $\xi \cdot \xi = 0$, and $q \in L^{d/2}(\mathbb{R}^d)$ with $\text{supp } q \subset B_1$. Then there exists $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ such that
\[ u = K_{\xi} * (q - qu). \]

Moreover,
\[ \limsup_{|\xi| \to \infty} \frac{\|u - u_1\|_{H^1(B_r)}}{\|u_1\|_{H^1(B_1)}} = 0, \]
where $u_1 \in H^1_{\text{loc}}(\mathbb{R}^d)$ is given by
\[ u_1 = K_{\xi} * q. \]

Proof. Let $f$ and $h$ be such that
\[ q = f + h, \]
where
\[ f \text{ is smooth with support in } B_1, \text{ and } \|h\|_{L^{d/2}} \text{ is small}. \]

Let $u_0 = 0$ and consider the following iteration process:
\[ u_n = K_{\xi} * (q - qu_{n-1}) \text{ for } n \geq 1. \]
Define

\[ u_{1,n} = K_\xi \ast (f - fu_{n-1}) \]

and

\[ u_{2,n} = K_\xi \ast (h - hu_{n-1}) \]

Then

\[ u_n = u_{1,n} + u_{2,n}, \]

\[ u_{2,n+1} - u_{2,n} = -K_\xi \ast [h(u_n - u_{n-1})] \]

and

\[ u_{1,n+1} - u_{1,n} = -K_\xi \ast [f(u_n - u_{n-1})]. \]  

(4.1)

We have, by (2.5),

\[ \|u_{2,n+1} - u_{2,n}\|_{H^1(B_r)} \leq C \|h(u_n - u_{n-1})\|_{L^2}. \]

Applying Hölder’s inequality, we obtain

(4.2)  

\[ \|u_{2,n+1} - u_{2,n}\|_{H^1(B_r)} \leq C_r \|h\|_{L^{d/2}} \|u_n - u_{n-1}\|_{H^1(B_1)}. \]

On the other hand, we derive from (2.4) and (4.1) that

\[ \|u_{1,n+1} - u_{1,n}\|_{H^1(B_r)} \leq \frac{C_r}{|\xi|} \|f(u_n - u_{n-1})\|_{H^1(B_1)}, \]

which implies

(4.3)  

\[ \|u_{1,n+1} - u_{1,n}\|_{H^1(B_r)} \leq \frac{C_r}{|\xi|} \|f\|_{W^{1,\infty}} \|u_n - u_{n-1}\|_{H^1(B_1)}. \]

A combination of (4.2) and (4.3) yields

\[ \|u_{n+1} - u_n\|_{H^1(B_r)} \leq C_r \left( \|h\|_{L^{d/2}} + \frac{1}{|\xi|} \|f\|_{W^{1,\infty}} \right) \|u_n - u_{n-1}\|_{H^1(B_1)}. \]

This implies

(4.4)  

\[ \sum_{m=1}^{n} \|u_{m+1} - u_m\|_{H^1(B_r)} \leq c(r, f, h) \sum_{m=0}^{n-1} \|u_{m+1} - u_m\|_{H^1(B_1)}, \]

where \( c(r, f, h) = C_r \left( \|h\|_{L^{d/2}} + \frac{1}{|\xi|} \|f\|_{W^{1,\infty}} \right). \) Assuming \( c(2, f, h) < 1/2, \) we have

\[ \sum_{m=1}^{n} \|u_{m+1} - u_m\|_{H^1(B_2)} \leq 2c(r, f, h)\|u_1 - u_0\|_{H^1(B_1)} = 2c(2, f, h)\|u_1\|_{H^1(B_1)}. \]

An appropriate splitting of \( q \) into \( h \) and \( f \) yields that \( u_m \rightarrow u \) in \( H^1_{loc}(\mathbb{R}^d) \) for large \( |\xi| \) and moreover, by (4.4),

\[ \limsup_{|\xi| \rightarrow \infty} \frac{\|u - u_1\|_{H^1(B_r)}}{\|u_1\|_{H^1(B_1)}} = 0. \]

The proof is complete. \( \square \)
4.2. **Proof of Theorem 2.** The proof is standard after Proposition 4. For the convenience of the reader, we present the proof. Let \( s > 2 \) and \( \sigma_1, \sigma_2, \sigma_3 \in S^{d-1} \) be such that
\[
\sigma_1 \cdot \sigma_2 = \sigma_1 \cdot \sigma_3 = \sigma_2 \cdot \sigma_3 = 0.
\]
Set
\[
\xi_1 = s \sigma_2 - i s \sigma_1 \
\xi_2 = s(-s \sigma_2 / \sqrt{1 + s^2} + \sigma_3 / \sqrt{1 + s^2}) + i s \sigma_1.
\]
Define, for \( i = 1, 2 \),
\[
v_i = (1 + u_i) e^{\xi_i \cdot \mathbf{x}} / 2.
\]
Since \( u_{i,n} \) satisfies (3.25), it follows that (see, e.g., [22])
\[
\Delta v_{i,n} + q_i u_{i,n} = 0 \text{ in } \mathbb{R}^d \text{ for } i = 1, 2.
\]
We derive from (1.2) that
\[
\int_{B_2} (q_1 - q_2)(1 + u_{1,n})(1 + u_{2,n}) e^{\sigma s \cdot \mathbf{x}} / 2 = 0,
\]
where
\[
\sigma_s = \xi_{1,n} + \xi_{2,n} \rightarrow \sigma_3 \text{ as } s \rightarrow \infty.
\]
A combination of (3.25), (4.5), and (4.6) yields
\[
\int_{B_2} (q_1 - q_2) e^{\sigma s \cdot \mathbf{x}} / 2 = 0.
\]
Since \( \sigma_3 \in S^{d-1} \) is arbitrary, it follows that
\[
q_1 = q_2,
\]
and the proof is complete.

4.3. **Proof of Corollary 2.** The proof is similar to the one of Corollary 1. The details are left to the reader.

5. **Uniqueness of Calderon's problem for conductivities of class \( W^{s,3/s} \) for \( s > 3/2 \) in 3d**

5.1. **Construction of CGO solutions.** We begin this section with

**Lemma 5.** Let \( \xi \in \mathbb{C}^3 \) with \( |\xi| > 2 \) and \( \xi \cdot \xi = 0 \), \( v \in H^1_{\text{loc}}(\mathbb{R}^3) \) and \( q \in H^{-1/2}(\mathbb{R}^3) \) with \( \text{supp } q \subset B_1 \). Define
\[
u = K_\xi * (qv).
\]
We have
\[
\|\nu\|_{H^1(B_r)}^2 \leq C_r \|v\|_{H^1(B_2)}^2 \cdot E(q, \xi),
\]
where
\[
E(q, \xi) = \int_{\mathbb{R}^3} |\tilde{q}(\eta)|^2 \int_{|\xi| \geq \text{dist}(k, \Gamma_\xi) \geq |k| / |\xi|} \frac{|k|^2 |\hat{K}_\xi(k)|^2}{|k - \eta|^2} dk d\eta
\]
\[
+ \int_{\mathbb{R}^3} |\tilde{q}(\gamma)|^2 \int_{\text{dist}(\gamma, \Gamma_\xi) \leq |\eta| / |\xi|} \frac{1}{|\eta - \gamma|^2} d\eta d\gamma + \|q\|_{H^{-1/2}}^2,
\]
where
\[
\tilde{q}(k) := \sup_{\eta \in B_4(k)} |\hat{q}(\eta)|.
\]
Proof. Without loss of generality, one may assume that supp\( v \subset B_2 \) and \( r > 1 \). Set
\( f = qv, \)
then
\( u = K_\xi * f. \)

Applying (2.6), we have
\( \|u\|_{L^2(B_r)} \leq C_r \|f\|_{H^{-1}}. \)

On the other hand, we have
\( \|f\|_{H^{-1}} \leq \int_{\mathbb{R}^3} \frac{1}{|k|^2 + 1} \left| \int_{\mathbb{R}^3} |\hat{q}(k - \eta)||\hat{v}(\eta)| \, d\eta \right|^2 \, dk. \)

Since
\( \int_{\mathbb{R}^3} |\hat{q}(k - \eta)||\hat{v}(\eta)| \, d\eta = \int_{|\eta| \leq |k|/2} |\hat{q}(k - \eta)||\hat{v}(\eta)| \, d\eta + \int_{|\eta| \geq |k|/2} |\hat{q}(k - \eta)||\hat{v}(\eta)| \, d\eta, \)
it follows that
\begin{align*}
(5.4) \quad &\frac{1}{2} \int_{\mathbb{R}^3} \frac{1}{|k|^2 + 1} \left| \int_{\mathbb{R}^3} |\hat{q}(k - \eta)||\hat{v}(\eta)| \, d\eta \right|^2 \, dk \\
&\leq \int_{\mathbb{R}^3} \int_{|\eta| \leq |k|/2} \frac{|\hat{q}(k - \eta)|^2}{(k^2 + 1)(|\eta|^2 + 1)} \, d\eta \int_{|\eta| \leq |k|/2} |\hat{v}(\eta)|^2 (|\eta|^2 + 1) \, d\eta \, dk \\
&\quad + \int_{\mathbb{R}^3} \int_{|\eta| \geq |k|/2} \frac{|\hat{q}(k - \eta)|^2}{(|k - \eta|^2 + 1)^{1/2}} \, d\eta \int_{|\eta| \geq |k|/2} \frac{|\hat{v}(\eta)|^2}{|k|^2 + 1} \, d\eta \, dk.
\end{align*}

We have, since \( |k - \eta| \leq |k|/2 \) implies \( 2|\eta| \geq |k| \geq 2|\eta|/3, \)
\begin{align*}
(5.5) \quad &\int_{\mathbb{R}^3} \int_{|\eta| \leq |k|/2} \frac{|\hat{q}(k - \eta)|^2}{(k^2 + 1)(|\eta|^2 + 1)} \, d\eta \, dk = \int_{\mathbb{R}^3} \int_{|k - \eta| \leq |k|/2} \frac{|\hat{q}(\eta)|^2}{(k^2 + 1)(|k - \eta|^2 + 1)} \, dk \, d\eta \\
&\leq \int_{\mathbb{R}^3} \frac{|\hat{q}(\eta)|^2}{(1 + |\eta|^2)^{1/2}} \int_{2|\eta| \geq |k|/2} \frac{1}{(|k - \eta|^2 + 1)(1 + |\eta|^2)^{1/2}} \leq C\|q\|_{H^{-1/2}}^2
\end{align*}
and
\begin{align*}
(5.6) \quad &\int_{\mathbb{R}^3} \int_{|\eta| \geq |k|/2} \frac{|\hat{v}(\eta)|^2 (|k - \eta|^2 + 1)^{1/2}}{|k|^2 + 1} \, d\eta \, dk \leq C\|v\|_{H^1}^2.
\end{align*}

Using (5.3), (5.4), (5.5), and (5.6), we derive from (5.3) that
\( \|f\|_{H^{-1}} \leq C\|v\|_{H^1} \|q\|_{H^{-1/2}}. \)
A combination of (5.2) and (5.7) yields
\( \|u\|_{L^2(B_r)} \leq C_r \|v\|_{H^1} \|q\|_{H^{-1/2}}. \)

It remains to prove
\( \|\nabla u\|_{L^2(B_r)} \leq C_r \|v\|_{H^1} \cdot E(q, \xi). \)

Set
\( \Gamma_\xi := \{ k \in \mathbb{R}^3; -|k|^2 + i\xi \cdot k = 0 \}. \)

Define \( K_1, K_2, \) and \( K_3, \) as follows
\( \hat{K}_1(k) = \begin{cases} 
\hat{K}(k) & \text{if } 4|\xi| \geq \text{dist}(k, \Gamma_\xi) > |k|/|\xi|, \\
0 & \text{otherwise},
\end{cases} \)

Define \( K_1, K_2, \) and \( K_3, \) as follows
\( \hat{K}_1(k) = \begin{cases} 
\hat{K}(k) & \text{if } 4|\xi| \geq \text{dist}(k, \Gamma_\xi) > |k|/|\xi|, \\
0 & \text{otherwise},
\end{cases} \)
\[
\tilde{K}_{2,\xi}(k) = \begin{cases} 
\tilde{K}_\xi(k) & \text{if dist}(k, \Gamma_\xi) \leq |k|/|\xi|, \\
0 & \text{otherwise,}
\end{cases}
\]

and
\[
\tilde{K}_{3,\xi}(k) = \begin{cases} 
\tilde{K}_\xi(k) & \text{if dist}(k, \Gamma_\xi) > 4|\xi|, \\
0 & \text{otherwise.}
\end{cases}
\]

Then
\[
\|\nabla(K_\xi * f)\|_{L^2(B_r)} \leq \|\nabla(K_{1,\xi} * f)\|_{L^2(B_r)} + \|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)} + \|\nabla(K_{3,\xi} * f)\|_{L^2(B_r)}.
\]

Since
\[
|\tilde{K}_\xi(k)| \leq \frac{1}{|k|^2} \text{ for } \text{dist}(k, \Gamma_\xi) \geq 4|\xi|,
\]

it follows that
\[
\|\nabla(K_{3,\xi} * f)\|_{L^2(B_r)} \leq \|\nabla(K_{3,\xi} * f)\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{H^{-1}}.
\]

A combination of (5.7) and (5.10) yields
\[
\|\nabla(K_{3,\xi} * f)\|_{L^2(B_r)} \leq C\|v\|_{H^1} \|q\|_{H^{-1/2}}.
\]

We next estimate the first two terms in the RHS of (5.9). We start with \(\|\nabla(K_{1,\xi} * f)\|_{L^2(B_r)}\). Since
\[
\|\nabla(K_{1,\xi} * f)\|_{L^2(B_r)}^2 \leq \|\nabla(K_{1,\xi} * f)\|_{L^2(\mathbb{R}^3)}^2,
\]

it follows from Plancherel’s theorem that
\[
\|\nabla(K_{1,\xi} * f)\|_{L^2(B_r)}^2 \leq C \int_{a \geq |K_\xi(k)| \geq k/s} |\hat{f}(k)|^2 |k|^2 |\tilde{K}_\xi(k)|^2 \, dk.
\]

From (5.11), we have
\[
\hat{f}(k) = \int_{\mathbb{R}^3} \hat{q}(\eta) \hat{v}(k-\eta) \, d\eta.
\]

Applying Hölder’s inequality, we obtain
\[
|\hat{f}(k)|^2 \leq \int_{\mathbb{R}^3} |\hat{q}(\eta)|^2 \frac{dk}{|k-\eta|^2} \int_{\mathbb{R}^3} |(k-\eta)\hat{v}(k-\eta)|^2 \, d\eta.
\]

A combination of (5.12) and (5.13) yields
\[
\int_{4|\xi| \geq |K_\xi(k)| \geq k/|\xi|} |\hat{f}(k)|^2 |k|^2 |\tilde{K}_\xi(k)|^2 \, dk \leq C\|\nabla v\|_{L^2}^2 \int_{\mathbb{R}^3} |\hat{q}(\eta)|^2 \int_{4|\xi| \geq |K_\xi(k)| \geq k/|\xi|} \frac{|k|^2 |K_\tilde{\xi}(k)|^2}{|k-\eta|^2} \, dk \, d\eta.
\]

We next estimate \(\|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}\). Fix
\[
\varphi \in C_0^\infty(\mathbb{R}^3) \text{ with } \varphi = 1 \text{ in } B_{2r}.
\]

Define
\[
\tilde{f}(k) = \sup_{\eta \in B_4(k)} |\hat{f}(\eta)|,
\]

and
\[
\tilde{\varphi}(k) = \sup_{\eta \in B_4(k)} |\hat{\varphi}(\eta)|.
\]
Since
\[
|\hat{f}| * |\hat{\varphi}|(\eta) = \int_{\mathbb{R}^d} |\hat{f}(\zeta)||\hat{\varphi}(\eta - \zeta)| \, d\zeta,
\]
and \(f = f \varphi\), it follows from the definition of \(\tilde{f}\) \((5.15)\) and \(\tilde{\varphi}\) \((5.16)\) that
\[
\tilde{f} \leq |\hat{f}| * \tilde{\varphi}.
\]
Since
\[
\|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}^2 \leq \|\nabla(\varphi \cdot K_{2,\xi} * f)\|_{L^2(\mathbb{R}^3)}^2,
\]
it follows that
\[
(5.17) \quad \|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}^2 \leq \int_{\mathbb{R}^d} |k|^2 \left| \int_{\text{dist}(\eta, \Gamma_{\xi}) \leq |\eta|/|\xi|} |\hat{\varphi}(k - \eta)| \cdot |\hat{K}_\xi(\eta)| \cdot |\hat{f}(\eta)| \, d\eta \right|^2 \, dk.
\]
Using the fact that \(\hat{K}_\xi(\eta) \leq C/(|\xi| \text{dist}(\eta, \Gamma_{\xi}))\) for \(|\eta| \leq 2|\xi|\) and
\[
\int_{|x| \leq 1} \frac{1}{|x_1| + |x_2|} \, dx < +\infty,
\]
as in \((2.19)\), we obtain
\[
(5.18) \quad \int_{\text{dist}(\eta, \Gamma_{\xi}) \leq 1} |\hat{\varphi}(k - \eta)| \cdot |\hat{K}_\xi(\eta)| \cdot |\hat{f}(\eta)| \, d\eta \leq \frac{C}{|\xi|} \int_{\text{dist}(\eta, \Gamma_{\xi}) \leq |\eta|/|\xi|} \hat{\varphi}(k - \eta) \cdot \hat{f}(\eta) \, d\eta.
\]
Applying Hölder’s inequality, we derive from \((5.17)\) and \((5.18)\) that
\[
(5.19) \quad \|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}^2 \leq \frac{C}{|\xi|^2} \int_{\mathbb{R}^3} \int_{\text{dist}(\eta, \Gamma_{\xi}) \leq |\eta|/|\xi|} |k|^2 |\hat{\varphi}(k - \eta)| |\hat{f}(\eta)|^2 \, d\eta \, dk.
\]
Since \(|k|^2 \leq C(|k - \eta|^2 + |\eta|^2)\) and \(\bar{\varphi}\) decays fast at infinity, it follows from \((5.16)\) and \((5.19)\) that
\[
(5.20) \quad \|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}^2 \leq C \int_{\text{dist}(\eta, \Gamma_{\xi}) \leq |\eta|/|\xi|} |\hat{f}(\eta)|^2 \, d\eta.
\]
Since
\[
\hat{f}(\eta) \leq \hat{q} * |\hat{v}|(\eta),
\]
it follows that
\[
\int_{\text{dist}(\eta, \Gamma_{\xi}) \leq |\eta|/|\xi|} |\hat{f}(\eta)|^2 \, d\eta \leq \int_{\text{dist}(\eta, \Gamma_{\xi}) \leq |\eta|/|\xi|} \left| \int_{\mathbb{R}^3} |\hat{q}(\gamma)| \left| \eta - \gamma \right| |\hat{v}(\eta - \gamma)| \, d\gamma \right|^2 \, d\eta.
\]
Using Hölder’s inequality, we obtain
\[
(5.21) \quad \int_{\text{dist}(\eta, \Gamma_{\xi}) \leq |\eta|/|\xi|} |\hat{f}(\eta)|^2 \, d\eta \leq C \|\nabla v\|_{L^2(\mathbb{R}^3)}^2 \int_{\mathbb{R}^3} |\hat{q}(\gamma)|^2 \int_{\text{dist}(\eta, \Gamma_{\xi}) \leq |\eta|/|\xi|} \frac{1}{|\eta - \gamma|^2} \, d\eta \, d\gamma.
\]
A combination of \((5.20)\) and \((5.21)\) yields
\[
(5.22) \quad \|\nabla(K_{2,\xi} * f)\|_{L^2(B_r)}^2 \leq C \|\nabla v\|_{L^2(\mathbb{R}^3)}^2 \int_{\mathbb{R}^3} |\hat{q}(\gamma)|^2 \int_{\text{dist}(\eta, \Gamma_{\xi}) \leq |\eta|/|\xi|} \frac{1}{|\eta - \gamma|^2} \, dq \, d\gamma.
\]
We derive from \((5.11)\), \((5.14)\), and \((5.22)\) that \((5.8)\) holds. The proof is complete. \(\square\)

To use Lemma 5, we need to choose \(\xi\) such that \(E(q, \xi)\) remains bounded. This can be done using the following average estimate for \(E(q, \xi)\) whose proof is in the spirit of the one of Lemma 4 and is presented in the appendix.
Lemma 6. Let $d = 3$ and $R > 10$. We have
\[
\frac{1}{R} \int_{R/2}^{2R} \int_{S^{d-1}} \int_{S_{\sigma_1}^2} \int_{S_{\sigma_2}^2} E(q, s\sigma_2 - is\sigma_1) \, d\sigma_2 \, d\sigma_1 \, ds \leq C \int_{\mathbb{R}^3} |\hat{q}(\eta)|^2 \min \left\{ \frac{\ln R}{R}, \frac{R \ln R}{|\eta|^2} \right\} \, d\eta
\]
and
\[
\frac{1}{R} \int_{R/2}^{2R} \int_{\sigma_1 \in S_2^2} \int_{\sigma_2 \in S_{\sigma_1}^2} \int_{\sigma_3 \in S_{\sigma_2}^2} E \left( q, \frac{s^2 \sigma_2}{\sqrt{1 + s^2}} + \frac{s \sigma_3}{\sqrt{1 + s^2}} - is\sigma_1 \right) \, d\sigma_3 \, d\sigma_2 \, d\sigma_1 \, ds \leq C \int_{\mathbb{R}^3} |\hat{q}(\eta)|^2 \min \left\{ \frac{\ln R}{R}, \frac{R \ln R}{|\eta|^2} \right\} \, d\eta.
\]

We recall that, by (3.16) and (3.17),
\[
S_{\sigma_1}^2 := \{ \sigma \in S_2^2; \sigma \cdot \sigma_1 = 0 \}
\]
and
\[
S_{\sigma_1, \sigma_2}^2 := \{ \sigma \in S_2^2; \sigma \cdot \sigma_1 = 0 \text{ and } \sigma \cdot \sigma_2 = 0 \}.
\]

We will show that the RHS of (5.23) will behave like $\|q\|_{H^{-1/2}}$ for appropriate choice of $s$. For this end, we need the following lemma.

Lemma 7. Let $(a_n)$ be a non-negative sequence. Define
\[
b_n = \sum_{l=1}^{n} 2^{l-n} a_l.
\]
Assume that $S = \sum_{n=1}^{\infty} a_n < +\infty$, then
\[
\liminf_{n \to \infty} nb_n = 0.
\]

Proof. The conclusion is a consequence of the following facts:
\[
\sum_{n=1}^{\infty} b_n \leq c \sum_{n=1}^{\infty} a_n < +\infty
\]
for some positive constant $c$, and
\[
\liminf_{n \to \infty} nb_n = 0,
\]
if
\[
\sum_{n=1}^{\infty} b_n < +\infty.
\]

Applying Lemmas 5, 6, and 7 we can obtain the following result which is a variant of Propositions 2 and 3 in this setting.

Proposition 5. Let $q_1, q_2 \in H^{-1/2}(\mathbb{R}^3)$ with support in $B_1$, and $\sigma_1, \sigma_2, \sigma_3 \in S^2$ be such that
\[
\sigma_1 \cdot \sigma_2 = \sigma_1 \cdot \sigma_3 = \sigma_2 \cdot \sigma_3 = 0.
\]
For any $\varepsilon > 0$, there exist a sequence $s_n \to \infty$, $\sigma_{1,\varepsilon}, \sigma_{2,\varepsilon}, \sigma_{3,\varepsilon} \in S^2$ and $u_{i,n} \in H^1_{loc}(\mathbb{R}^3)$ such that
\[
\sigma_{1,n} \cdot \sigma_{2,n} = \sigma_{1,n} \cdot \sigma_{3,n} = \sigma_{2,n} \cdot \sigma_{3,n} = 0,
\]
\[
|\sigma_{j,n} - \sigma_i| \leq \varepsilon \text{ for } j = 1, 2, 3,
\]

\[
\|q_1 \cdot u_{1,n} + q_2 \cdot u_{2,n} + \sigma_1 \cdot \sigma_2 \cdot u_{3,n} - \sigma_1 \cdot \sigma_3 \cdot u_{3,n} - \sigma_2 \cdot \sigma_3 \cdot u_{3,n} \|_{H^{-1/2}} < \varepsilon.
\]
and
\[ u_{i,n} = K\xi_{i,n} * (q_i - q_i u_{i,n}) \text{ for } i = 1, 2. \]

Here
\[ \xi_{1,n} = s_n \sigma_{2,n} - i s\sigma_{1,n} \quad \text{and} \quad \xi_{2,n} = -\frac{s_n^2 \sigma_{2,n}}{1 + s_n^2} + \frac{s_n \sigma_{3,n}}{1 + s_n^2} + is_n \sigma_{1,n}. \]

Moreover,
\[ \liminf_{n \to \infty} \|u_{i,n}\|_{H^1(B_r)} \leq C_r \|u_{i,1}\|_{H^1(B_1)} \quad \text{for } i = 1, 2. \]

**Proof.** For \( \varepsilon > 0 \), let \( q_{i,1} \in C^\infty(\mathbb{R}^3) \) and \( q_{i,2} \in C^\infty(\mathbb{R}^3) \) with supports in \( B_1 \) be such that
\[ q_{i,1} + q_{i,2} = q_i \]
and
\[ \|q_{i,2}\|_{H^{-1/2}} \leq \varepsilon. \]

Define
\[ a_{i,n} = \int_{2^n \leq |k| \leq 2^{n+1}} \frac{|\hat{q}_{i,2}(k)|^2}{|k|} dk, \]
then it is clear that
\[ \sum_{n=1}^{\infty} a_{i,n} \leq \|q_{i,2}\|_{H^{-1/2}}^2. \]

Define
\[ b_{i,n} = \sum_{l=1}^{n} 2^{l-n} a_{i,l} \sim \int_{2^l \leq |k| \leq 2^{l+1}} \frac{|\hat{q}_{i,2}(k)|^2}{2^{n+1}} dk. \]

By Lemma 7 there exists \( n_k \to \infty \) such that
\[ n_k b_{1,n_k} + n_k b_{2,n_k} \leq c \left( \|q_{i,1}\|_{H^{-1/2}}^2 + \|q_{i,2}\|_{H^{-1/2}}^2 \right). \]

Applying Lemma 6 there exist \( \sigma_{1,k}, \sigma_{2,k}, \sigma_{3,k} \in S^2 \) such that
\[ \sigma_{1,k} \cdot \sigma_{2,k} = \sigma_{1,n} \cdot \sigma_{3,k} = \sigma_{2,n} \cdot \sigma_{3,k} = 0, \]
\[ |\sigma_{j,k} - \sigma_i| \leq \varepsilon \text{ for } j = 1, 2, 3, \]
and
\[ (5.25) \quad E(q_{i,1}, \xi_{1,k}) + E(q_{i,2}, \xi_{2,k}) \leq C \varepsilon^2 \text{ for large } k. \]

Here
\[ \xi_{1,k} = s_k \sigma_{2,k} - i s_k \sigma_{1,k} \quad \text{and} \quad \xi_{2,k} = -\frac{s_k^2 \sigma_{2,k}}{1 + s_k^2} + \frac{s_k \sigma_{3,k}}{1 + s_k^2} + is_k \sigma_{1,k}. \]

Let \( u_{i,0} = 0 \) and consider the following iteration process:
\[ u_{i,n} = K\xi_k * (q_i - q_i u_{i,n-1}) \text{ for } n \geq 1; i = 1, 2. \]

Then, for \( n \geq 1 \) and \( i = 1, 2 \),
\[ u_{i,n+1} - u_{i,n} = -K\xi_k * (q_i u_{i,n-1}) = -K\xi_k * (q_{i,1} u_{i,n-1}) - K\xi_k * (q_{i,2} u_{i,n-1}). \]

Applying Lemma 5 and (2.4), we have
\[ \|u_{i,n+1} - u_{i,n}\|_{H^1(B_r)} \leq C_r \left( E(q_i, \xi_{i,k})^{1/2} + \frac{1}{|\xi_{i,k}|} \right) \|u_{i,n+1} - u_{i,n}\|_{H^1(B_1)}. \]

This implies
\[ (5.26) \quad \sum_{m=1}^{n} \|u_{i,m+1} - u_{i,m}\|_{H^1(B_r)} \leq c(r, f, h) \sum_{m=0}^{n-1} \|u_{i,m+1} - u_{i,m}\|_{H^1(B_1)}. \]
where \( c(r, f, h) = C_r \left( E(q_i, \xi, k)^{1/2} + \frac{1}{|k|} \right) \). Assuming \( c(2, f, h) < 1/2 \), we have

\[
\sum_{m=1}^{n} \| u_{i,m+1} - u_{i,m} \|_{H^1(B_2)} \leq 2c(r, f, h)\| u_{i,1} - u_{i,0} \|_{H^1(B_1)} = 2c(2, f, h)\| u_{i,1} \|_{H^1(B_1)}.
\]

Thus there exist \( u_{i,n} \in H^1_{\text{loc}}(\mathbb{R}^3) \) such that

\[
u_{i,n} = K_{i,\xi, k} \ast (q_i - q_i u_{i,n}),
\]

and, by (5.2),

\[
\| u_{i,n} \|_{H^1(B_r)} \leq C_r \varepsilon \| u_{i,1} \|_{H^1(B_1)}.
\]

The conclusion follows. \( \square \)

5.2. **Proof of Theorem** 3. Theorem 3 is a consequence of Proposition 5. The proof is standard and the details are left to the reader. We note that the condition \( t > 1/2 \) ensures the existence of the trace of \( g_1 \) on the boundary. \( \square \)

5.3. **Proof of Corollary** 3. The proof is similar to the one of Corollary 1. The details are left to the reader. \( \square \)

### Appendix A. Some averaging estimates

A.1. **Proof of Lemma** 4. It is clear that

\[
|\hat{K}_{s\sigma_2 - i\sigma_1}(k)|^p \leq \frac{C_p}{|k|^{2p}} \quad \text{for } |k| > 2s.
\]

Hence to obtain (3.14), it suffices to prove that

\[
\int_{\sigma_1 \in \mathbb{S}^{d-1}} \int_{\sigma_2 \in \mathbb{S}_s} |\hat{K}_{s\sigma_2 - i\sigma_1}(k)|^p \, d\sigma_2 \, d\sigma_1 \leq \frac{C_p}{|k|^{2sp}} \quad \text{for } |k| \leq 2s.
\]

Without loss of generality one may assume that \( k = te_1(\xi_1 = (1, 0, \cdots, 0)) \). Set

\[
\xi = s(\sigma_2 - i\sigma_1).
\]

We have

\[
\frac{1}{|k|^2 + ik \cdot \xi|^p} = \frac{1}{|t^2 + ist e_1 \cdot \sigma_1 + st e_1 \cdot \sigma_2|^p} \sim \frac{1}{|t^2 - st \sigma_1 \cdot e_1|^p + (st)^p |\sigma_2 \cdot e_1|^p}.
\]

Let \( \theta_1 \) be the angle between \( \sigma_1 \) and \( e_1 \) and let \( \theta_2 \) be the angle between \( \sigma_2 \) and \( v \) where \( v = e_1 - (e_1 \cdot \sigma_1) \sigma_1 = e_1 - \cos \theta_1 \sigma_1 \). Note that \( v \in \text{span}\{\sigma_1, e_1\} \), \( v \) is orthogonal to \( \sigma_1 \), and \(|v| = |\sin(\theta_1)|\). Using the spherical area element, we have

\[
C_p \int_{\sigma_1 \in \mathbb{S}^{d-1}} \int_{\sigma_2 \in \mathbb{S}_s} \frac{1}{|t^2 - st \sigma_1 \cdot e_1|^p + (st)^p |\sigma_2 \cdot e_1|^p} \, d\sigma_2 \, d\sigma_1 \\
\leq \int_0^{\pi/2} \int_0^\pi \frac{\theta_1^{d-2} \theta_2^{d-3}}{|t^2 - st \cos \theta_1|^p + (st)^p |\sin \theta_1 \cos \theta_2|^p} \, d\theta_2 \, d\theta_1 \\
+ \int_0^{\pi/2} \int_0^\pi \frac{(\pi - \theta_1)^{d-2} \theta_2^{d-3}}{|t^2 - st \cos \theta_1|^p + (st)^p |\sin \theta_1 \cos \theta_2|^p} \, d\theta_2 \, d\theta_1.
\]
Here we use $|\sigma_2 \cdot e_1| = |\sigma_2 \cdot v| = |\sin \theta_1 \cos \theta_2|$. It follows that
\[
\int_{\sigma_1 \in S^{d-1}} \int_{\sigma_2 \in S^{d-1}} \frac{1}{|k|^2 + i k \cdot \xi} d\sigma_2 d\sigma_1 \leq C_p \frac{(st)^p}{(st)^{1\frac{p}{4}}} \int_0^{\pi/2} \int_0^\pi \frac{\theta_1^{d-2} \theta_2^{d-3}}{|\frac{s}{t} - \cos \theta_1|^p + |\sin \theta_1 \cos \theta_2|^p} d\theta_2 d\theta_1.
\]
Fix $0 < \delta < 2 - p$, and consider the case $t \leq s$. Then
\[
\int_0^{\pi/2} \int_0^\pi \frac{\theta_1^{d-2} \theta_2^{d-3}}{|\frac{s}{t} - \cos \theta_1|^p + |\sin \theta_1 \cos \theta_2|^p} d\theta_2 d\theta_1 \leq C_p \int_0^{\pi/2} \int_0^\pi \frac{\theta_1^{d-2} \theta_2^{d-3}}{|\frac{s}{t} - \cos \theta_1|^{1-\delta} |\sin \theta_1 \cos \theta_2|^{p-1+\delta}} d\theta_2 d\theta_1.
\]
This implies
\[
A \text{ computation yields (A1)}
\]
\[
\int_0^{\pi/2} \int_0^\pi \frac{\theta_1^{d-2} \theta_2^{d-3}}{|\frac{s}{t} - \cos \theta_1|^{1-\delta} |\sin \theta_1 \cos \theta_2|^{p-1+\delta}} d\theta_2 d\theta_1 \leq C_p \int_0^{\pi/2} \frac{\theta_1^{d-1-p-\delta}}{|\frac{s}{t} - \cos \theta_1|^{1-\delta}} d\theta_1.
\]
On the other hand, let $\theta_0, \alpha_0$ be such that $\cos \theta_0 = \frac{s}{t}$ and $|\cos \theta_0 - \cos (\alpha + \theta_0)| \leq \frac{1}{2}$ for all $|\alpha| \leq \alpha_0$. We have, since $d - 1 - p - \delta \geq 2 - p - \delta > 0$,
\[
\text{(A2)} \quad C_p \int_0^{\pi/2} \frac{\theta_1^{d-1-p-\delta}}{|\frac{s}{t} - \cos \theta_1|^{1-\delta}} d\theta_1 
\leq \int_{|\theta - \theta_0| \leq \alpha_0} \frac{1}{|\cos \theta_0 - \cos \theta|^{1-\delta}} d\theta + \int_{|\theta - \theta_0| \leq \alpha_0} \frac{1}{|\cos \theta_0 - \cos \theta|^{1-\delta}} d\theta.
\]
We have
\[
\int_{|\theta - \theta_0| \leq \alpha_0} \frac{1}{|\cos \theta_0 - \cos \theta|^{1-\delta}} d\theta \leq \int_{|\theta - \theta_0| \leq \alpha_0} \frac{C_p}{|\sin ((\theta_0 + \theta)/2)|^{1-\delta} |\theta - \theta_0|^{1-\delta}} d\theta
\]
\[
\text{(A3)} \quad \leq \int_{|\theta - \theta_0| \leq \alpha_0} \frac{C_p}{|\sin \theta_0|^{1-\delta} |\theta - \theta_0|^{1-\delta}} d\theta \leq \frac{C_p}{(1 - \frac{1}{2}) \frac{1}{2}}
\]
and
\[
\int_{|\theta - \theta_0| \leq \alpha_0} \frac{1}{|\cos \theta_0 - \cos \theta|^{1-\delta}} d\theta \leq C_p.
\]
A combination of (A1), (A2), (A3), and (A4) yields
\[
\text{(A5)} \quad \int_0^{\pi/2} \int_0^\pi \frac{\theta_1^{d-2} \theta_2^{d-3}}{|s^2 - st \cos \theta_1|^p + |s^2 - st \cos \theta_2|^p} d\theta_2 d\theta_1 \leq \frac{C_p}{(st)^p (1 - \frac{1}{2}) \frac{1}{2}} + \frac{C_p}{(st)^p}.
\]
For \( s < t \leq 2s \), we have
\[
\int_0^{\pi/2} \frac{\theta_{d-4}^{t-1-p-\delta}}{|t^2 - \cos \theta_1|^{1-\delta}} d\theta_1 \leq C.
\]
Hence we also obtain (A5) in this case. Averaging (A5) in \( s \) yields bound (3.14).

We now establish (3.15). Define \( v_1 = e_1 - (e_1 \cdot \sigma_1)\sigma_1 - (e_1 \cdot \sigma_2)\sigma_2 = v - (v \cdot \sigma_2)\sigma_2 \) and let \( \theta_3 \) be the angle between \( \sigma_3 \) and \( v_1 \). We have, since \( \sigma_3 \cdot e_1 = \sigma_3 \cdot v_1 = |v_1|\cos \theta_3 \),
\[
\int_{\sigma_1 \in S^{d-1}} \int_{\sigma_2 \in S^{d-1}} \int_{\sigma_3 \in S^{d-1}} \frac{K}{1+\frac{s}{2}} \frac{s}{1+|s|} (k) \left| p \right| d\sigma_3 d\sigma_2 d\sigma_1
\]
\[
\leq C_p \int_0^{3\pi/4} \int_0^{\pi} |t^2 - st \cos \theta_1|^p (st)^p |\sin \theta_1| \cos \theta_2 - |v_1| \cos \theta_3/s|^p d\theta_3 d\theta_2 d\theta_1.
\]
Here \( \int_0^{\pi} f(\theta_3) \theta_3^{-d-4} d\theta_3 := f(\pi) + f(0) \) if \( d = 3 \). We will only consider the case \( d \geq 4 \), the case \( d = 3 \) follows similarly. We have
\[
\int_0^{\pi} \theta_3^{-d-4} d\theta_3 \leq C_p \int_0^{\pi} |t^2 - st \cos \theta_1|^p (st)^p |\sin \theta_1| \cos \theta_2 - |v_1| \cos \theta_3/s|^p \leq \frac{C_p}{t^2 - st \cos \theta_1} |v_1|.
\]
Since
\[
|v_1|^2 = |v|^2 - |v \cdot \sigma_2|^2 = \sin^2 \theta_1 \sin^2 \theta_2,
\]
\[
\int_0^{\pi} \theta_3^{-d-4} d\theta_3 \leq C_p \int_0^{\pi} \frac{1}{t^2 - st \cos \theta_1} |v_1| \sin \theta_1 \sin \theta_2 \sin \theta_3.
\]
This implies
\[
\int_0^{3\pi/4} \int_0^{\pi} |t^2 - st \cos \theta_1|^p (st)^p |\sin \theta_1| \cos \theta_2 - |v_1| \cos \theta_3/s|^p d\theta_3 d\theta_2 d\theta_1 \leq C_p \int_0^{3\pi/4} \int_0^{\pi} |t^2 - st \cos \theta_1|^p |v_1| d\theta_3 d\theta_2 d\theta_1.
\]
We have, since \( d \geq 4 \),
\[
\int_0^{3\pi/4} \int_0^{\pi} |t^2 - st \cos \theta_1|^{p-1} |\sin \theta_1 \sin \theta_2| d\theta_2 d\theta_1 \leq \int_0^{3\pi/4} \int_0^{\pi} \frac{1}{t^2 - st \cos \theta_1} |v_1| d\theta_3 d\theta_2 d\theta_1 \leq C_p \frac{1}{t^{p+1} s^{p-1}}.
\]
We obtain the conclusion. \( \square \)

A.2. Proof of Lemma 6. We first claim that, for \( k \in \mathbb{R}^3 \) with \( |k| \geq 2 \),
\[
(A6) \quad \frac{1}{R} \int_{\frac{2R}{R/2}}^{2R} \int_{\sigma_1 \in S^2} \int_{\sigma_2 \in S^2} |\Gamma_{k\sigma_2-\imath \sigma_1}(k)|^2 d\sigma_2 d\sigma_1 ds \leq C \min \left\{ \frac{\ln R}{R^2|k|^2}, \frac{1}{|k|^4} \right\}.
\]
Here \( \xi = \xi(s, \sigma_1, \sigma_2) = s\sigma_2 - \imath \sigma_1 \) and
\[
\Gamma_\xi := \{ k \in \mathbb{R}^3; -|k|^2 + i\xi \cdot k = 0 \}.
\]
Indeed, since
\[
|\Gamma_{k\sigma_2-\imath \sigma_1}(k)|^2 \leq \frac{C}{|k|^4} \text{ for } |k| > 2s,
\]
it suffices to prove that
\begin{align}
(A7) \quad \frac{1}{R} \int_{\sqrt{2}}^{2R} \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} |\hat{K}_{\sigma_2 - is\sigma_1}(k)|^2 \, d\sigma_2 \, d\sigma_1 \, ds \leq C \ln R \quad \text{for } |k| \leq 2s.
\end{align}

Without loss of generality, one may assume that \( k = te_1 = (t, 0, 0) \). As in the proof of Lemma \([4]\) we have
\begin{align}
(A8) \quad \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} \frac{1}{|k|^2 + (t/s)^2} \, d\sigma_2 \, d\sigma_1 \\
\leq \frac{C}{(st)^2} \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} \frac{\theta_1}{|\frac{t}{s} - \cos \theta_1|^2 + \sin \theta_1 \cos \theta_2^2 + s^{-4}} \, d\theta_2 \, d\theta_1.
\end{align}

A computation yields
\begin{align}
(A9) \quad \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} \frac{\theta_1}{|\frac{t}{s} - \cos \theta_1|^2 + \sin \theta_1 \cos \theta_2^2 + s^{-4}} \, d\theta_2 \, d\theta_1 \leq C \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} \frac{1}{|\frac{t}{s} - \cos \theta_1| + s^{-2}} \, d\theta_1.
\end{align}

and
\begin{align}
(A10) \quad \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} \frac{1}{|\frac{t}{s} - \cos \theta_1| + s^{-2}} \, d\theta_1 \leq C \ln s.
\end{align}

A combination of \((A8), (A9),\) and \((A10)\) yields \((A7)\); hence \((A6)\) is established.

In the rest, we only give the proof of \((5.23)\). The proof of \((5.24)\) follows similarly. Applying \((A6)\), we have
\begin{align}
\frac{1}{R} \int_{\sqrt{2}}^{2R} \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} |\hat{q}(\eta)|^2 \, \int_{4|\xi| \geq \text{dist}(k, \Gamma) \geq |k|/|\xi|} \frac{|k|^2 |\hat{K}_{\sigma_2 - is\sigma_1}(k)|^2}{|k - \eta|^2} \, dk \, d\eta \, d\sigma_2 \, d\sigma_1 \, ds \\
\leq C \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} \frac{\ln R}{|k - \eta|^2} \, dk \, d\eta.
\end{align}

Since
\begin{align}
\int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} \frac{\ln R}{|k - \eta|^2} \, dk \, d\eta \leq C \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} \frac{|\hat{q}(\eta)|^2}{|\eta|^2} \, \min\left\{ \frac{\ln R}{R}, \frac{R \ln R}{|\eta|^2} \right\} \, d\eta,
\end{align}

it follows that
\begin{align}
(A11) \quad \frac{1}{R} \int_{\sqrt{2}}^{2R} \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} |\hat{q}(\eta)|^2 \, \int_{4|\xi| \geq \text{dist}(k, \Gamma) \geq |k|/|\xi|} \frac{|k|^2 |\hat{K}_{\sigma_2 - is\sigma_1}(k)|^2}{|k - \eta|^2} \, dk \, d\eta \, d\sigma_2 \, d\sigma_1 \, ds \\
\leq C \int_{\Omega_{\delta_1}} \int_{\Omega_{\delta_2}} \int_{R/2}^{R} \frac{|\hat{q}(\eta)|^2}{|\eta|^2} \, \min\left\{ \frac{\ln R}{R}, \frac{R \ln R}{|\eta|^2} \right\} \, d\eta.
\end{align}

Define
\begin{align}
\bar{q}(k) := \sup_{\eta \in B_4(k)} |\hat{q}(\eta)|.
\end{align}
We have
\[ (A12) \quad \frac{1}{R} \int_{R/2}^{2R} \int_{\mathbb{S}^2} \left( \int_{\mathbb{R}^3} |\hat{q}(\gamma)|^2 \int_{\text{dist}(\eta, \Gamma_{\sigma_2 - i\sigma_1}) \leq |\eta|/|\xi|} \frac{1}{|\eta - \gamma|^2} \ d\eta \ d\gamma \right) \ d\sigma_2 \ d\sigma_1 \ ds \\
\leq C \int_{\mathbb{R}^3} |\hat{q}(\gamma)|^2 \min \left\{ \frac{R}{|\gamma|^2}, \frac{1}{R} \right\} \ d\gamma. \]

Fix \( q \in C^\infty_c(\mathbb{R}^3) \) such that \( \varphi = 1 \) in \( B_1 \) and \( \text{supp} \varphi \subset B_2 \) and define
\[ \tilde{\varphi}(k) = \sup_{\eta \in B_4(k)} |\hat{\varphi}(\eta)|. \]

Using the fact that
\[ (A13) \quad |\hat{q}| \leq \tilde{\varphi} \ast |\hat{q}|, \]
and applying Hölder’s inequality, we have
\[ \int_{\mathbb{R}^3} |\hat{q}(\gamma)|^2 \min \left\{ \frac{R}{|\gamma|^2}, \frac{1}{R} \right\} \ d\gamma \leq C \int_{\mathbb{R}^3} |\hat{q}(\beta)|^2 \int_{\mathbb{R}^3} \tilde{\varphi}(\gamma - \beta) \min \left\{ \frac{R}{|\gamma|^2}, \frac{1}{R} \right\} \ d\gamma \ d\beta. \]

It follows from (A13) that
\[ (A14) \quad \int_{\mathbb{R}^3} |\hat{q}(\gamma)|^2 \min \left\{ \frac{R}{|\gamma|^2}, \frac{1}{R} \right\} \ d\gamma \leq C \int_{\mathbb{R}^3} |\hat{q}(\beta)|^2 \min \left\{ \frac{R}{|\beta|^2}, \frac{1}{R} \right\} \ d\beta. \]

A combination of (A12) and (A14) yields
\[ (A15) \quad \frac{1}{R} \int_{R/2}^{2R} \int_{\mathbb{S}^2} \left( \int_{\mathbb{R}^3} |\hat{q}(\gamma)|^2 \int_{\text{dist}(\eta, \Gamma_{\sigma_2 - i\sigma_1}) \leq |\eta|/|\xi|} \frac{1}{|\eta - \gamma|^2} \ d\eta \ d\gamma \right) \ d\sigma_2 \ d\sigma_1 \ ds \\
\leq C \int_{\mathbb{R}^3} |\hat{q}(\beta)|^2 \min \left\{ \frac{R}{|\beta|^2}, \frac{1}{R} \right\} \ d\beta. \]

We derive (5.23) from (A11), and (A15).

\[ \square \]

\section*{Appendix B. Boundary determination}

In this appendix, we prove the following result

\textbf{Proposition B1.} \textit{Let \( d \geq 2 \), \( \Omega \) be an open subset of \( \mathbb{R}^d \) of class \( C^1 \), and \( \gamma_1, \gamma_2 \in W^{1,1}(\Omega) \). Assume \( \text{DtN}_{\gamma_1} = \text{DtN}_{\gamma_2} \), then we have}
\[ \gamma_1 = \gamma_2 \text{ on } \partial \Omega. \]

\textbf{Proof.} We give the proof in the case \( d \geq 3 \). The proof in the \( 2d \) case follows similarly. We prove this result by contradiction. Assume that the conclusion is not true. Hence there exists some \( z \) on \( \partial \Omega \) such that
\[ (B1) \quad \gamma_1(z) \neq \gamma_2(z) \]
\[ (B2) \quad \lim_{r \to 0} \int_{B(z,r) \cap \Omega} |\gamma_1(x) - \gamma_1(z)| = 0, \]
and
\[ (B3) \quad \lim_{r \to 0} \int_{B(z,r) \cap \Omega} |\gamma_2(x) - \gamma_2(z)| = 0. \]
These last two statement following from the fact that for $\mathcal{H}^{d-2}$ a.e. $y \in \partial \Omega$, we have (see e.g. [8, Theorem 2 on page 181])
\[
\lim_{r \to 0} \int_{B(y,r) \cap \Omega} |\gamma_1(x) - \gamma_1(y)| = 0,
\]
and
\[
\lim_{r \to 0} \int_{B(y,r) \cap \Omega} |\gamma_2(x) - \gamma_2(y)| = 0.
\]
Let $z_n$ be a sequence in $\mathbb{R}^d \setminus \Omega$ such that
\[\text{dist}(z_n, \Omega) = |z_n - z| \quad \text{and} \quad \lim_{n \to \infty} |z_n - z| = 0.
\]
For $n \in \mathbb{N}$, define
\[
v_n = \frac{1}{|x - z_n|^{d-2}},
\]
and let $u_{j,n} \in H^1(\Omega)$ ($j = 1, 2$) be the unique solution to the system
\[
\begin{align*}
\text{div}(\gamma_j \nabla u_{j,n}) &= 0 \quad \text{in } \Omega, \\
u_{j,n} &= v_n \quad \text{on } \partial \Omega.
\end{align*}
\]
Define
\[
w_{j,n} = u_{j,n} - v_n.
\]
It is clear that
\[
(B4) \quad \Delta v_n = 0 \quad \text{in } \Omega.
\]
We also have
\[
-\text{div}(\gamma_j \nabla w_{j,n}) = -\text{div}(\gamma_j \nabla u_{j,n}) - \text{div}(\gamma_j \nabla v_n) = -\text{div}([\gamma_j - \gamma_j(z)] \nabla v_n)
\]
where in the last identity, we used (B4). This implies
\[
\int_{\Omega} \gamma_j \nabla w_{j,n} = \int_{\Omega} [\gamma_j - \gamma_j(z)] \nabla v_n \nabla w_{j,n}.
\]
It follows from (B2) and (B3) that
\[
\|\nabla w_{j,n}\|_{L^2} \leq \|\gamma_j - \gamma_j(z)\|_{L^2} v_n \|\nabla w_{j,n}\|_{L^2} = \frac{o(1)}{|z - z_n|^{(d-2)/2}}.
\]
Here and in the following we let $o(1)$ denote a quantity going to 0 as $n \to \infty$; hence,
\[
\nabla w_{j,n} = \nabla v_n + \frac{g}{|z - z_n|^{(d-2)/2}},
\]
where $\|g\|_{L^2} \to 0$ as $n \to \infty$. On the other hand,
\[
\int_{\Omega} (\gamma_1 - \gamma_2) \nabla w_{1,n} \nabla w_{2,n} = 0
\]
which implies
\[
[\gamma_1(z) - \gamma_2(z)] \frac{1}{|z - z_n|^{d-2}} = o(1) \frac{1}{|z - z_n|^{d-2}}.
\]
Hence
\[
\gamma_1(z) = \gamma_2(z).
\]
This contradicts (B1), and the conclusion follows.

Acknowledgements
Hoai-Minh Nguyen was supported in part by NSF grant DMS-1201370 and by the Alfred P. Sloan Foundation. Daniel Spirn was supported in part by NSF grant DMS-0955687. We would like to thank Jean-Pierre Puel for pointing out an error in the proof of Theorem 2 in an earlier version. The first author thanks Gunther Uhlmann for interesting discussions on the subject.

References

[1] K. Astala and L. Päivärinta, Calderón’s inverse conductivity problem in the plane, Ann. of Math. 163 (2006), 265–299.
[2] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, Geometric And Functional Analysis 3 (1993), 107–156.
[3] R. M. Brown and R. H. Torres, Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in $L^p$, $p > 2n$. J. Fourier Anal. Appl. 9 (2003), 563–574.
[4] R. Brown, Global Uniqueness in the Impedance-Imaging Problem for Less Regular Conductivities, SIAM J. Math. Anal. 27 (1996), 1049–1056.
[5] R. Brown and G. Uhlmann, Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions, Comm. Partial Differential Equations 22 (1997), 1009–1027.
[6] A. P. Calderón, On an inverse boundary value problem, Seminar on Numerical Analysis and its Applications to Continuum Physics (1980), 65–73, Soc. Brasil. Mat., Rio de Janeiro.
[7] S. Chanillo, A Problem in Electrical Prospection and a n-Dimensional Borg-Levinson Theorem, Proceedings of the American Mathematical Society 108 (1990), 761–767.
[8] L. C. Evans and R. F. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
[9] D. S. Ferreira and C. Kenig and M. Salo and G. Uhlmann, Limiting Carleman Weights and Anisotropic Inverse Problems, Inventiones Math, 178 (2009), 119–171.
[10] D. S. Ferreira and C. Kenig and M. Salo Determining an unbounded potential from Cauchy data in admissible geometries Comm. Partial Differential Equations 38 (2013), 50–68.
[11] B. Haberman and D. Tataru, Uniqueness in Calderón’s problem with Lipschitz conductivities, Duke Math. J. 162 (2013), 497–516.
[12] V. Isakov, On uniqueness of recovery of a discontinuous conductivity coefficient, Comm. Pure Appl. Math. 41 (1988), 865–877.
[13] V. Isakov, Inverse problems for partial differential equations, Applied Mathematical Sciences, 127, Springer-Verlag, New York, Second Edition, 2006.
[14] C. E. Kenig and A. Ruiz and C. D. Sogge, Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators Duke Math. J. 55 (1987), 329–347.
[15] R. Kohn and M. Vogelius, Determining conductivity by boundary measurements, Comm. Pure Appl. Math. 37 (1984), 289–298.
[16] R. Kohn and M. Vogelius, Determining conductivity by boundary measurements. II. Interior results, Comm. Pure Appl. Math. 38 (1985), 643–667.
[17] R. Lavine and A. Nachman, Inverse scattering at fixed energy, Proceedings of the Xth Congress on Mathematical Physics, L. Schmdgen (Ed.), Leipzig, Germany, 1991, 434–441, Springer-Verlag.
[18] V. V. Lebelev, On the Fourier transform of the characteristic functions of domains with $C^1$ boundary, Func. Anal. Appl., 47 (2013) 27–37.
[19] A. Panchenko L. Päivärinta and G. Uhlmann, Complex geometrical optics solutions for Lipschitz conductivities, Revista Matematica Iberoamericana 1 (2003), 57–72.
[20] A. I. Nachman, Reconstructions from boundary measurements, Comm. Partial Differential Equations 128 (1988), 531–576.
[21] A. I. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, Ann. of Math. 143 (1996), 71–96.
[22] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. of Math. 125 (1987), 153–169.