EQUIVARIANT $D$-MODULES

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Dedicated to Professor Takeshi Kotake
on his sixtieth birthday

Abstract. The first part of these notes is devoted to an introduction to algebraic
$D$-modules. Several basic notions as in [2], [11] are introduced. In the second part,
$D$-modules with group action are treated. Several important examples in this situ-
ation are discussed in details. Particularly, the Harish-Chandra systems for group
characters and the Gelfand generalized hypergeometric systems are our main topics.

I. $D$-modules, an introduction

1. Systems of linear partial differential equations

Let $U$ be a complex domain in the $n$-dimensional complex affine space $\mathbb{C}^n$ and
$D(U)$ the ring of partial differential operators on $U$ with holomorphic coefficients.
Consider a system of linear partial differential equations

$$P_i u = 0 \quad (1 \leq i \leq m)$$

for $P_i \in D(U)$.

Let $F$ be a suitable function space on $U$ stable by the action of $D(U)$, e.g., $\mathcal{O}(U)$
the space of holomorphic functions, $C^\infty(U)$ that of $C^\infty$ functions or $\mathcal{D}'(U)$ that
of Schwarz distributions. If $\phi \in F$ is a solution to the above system of equations
($P_i \phi = 0 \ (1 \leq i \leq m)$), then the map

$$\hat{\phi} : D(U) \ni Q \mapsto Q\phi \in F$$

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is a left \( D(U) \)-linear by definition and \( \ker \hat{\phi} \) contains the \( P_i \)'s (\( 1 \leq i \leq m \)). Thus the \( D(U) \)-homomorphism \( \hat{\phi} \) factorizes to the \( D(U) \)-homomorphism

\[
\hat{\phi} : D(U)/I \longrightarrow F \quad (Q \mod I \mapsto Q\phi)
\]

where \( I = \sum_{i=1}^{m} D(U)P_i \) is the left ideal of the ring \( D(U) \) generated by the \( P_i \)'s.

Thus if we denote by \( M \) the left \( D(U) \)-module \( D(U)/I \), the space of solutions to the system in \( F \) is identified with the space of left \( D(U) \)-module homomorphisms

\[
\text{Hom}_{D(U)}(M, F)
\]

by the correspondence \( \phi \leftrightarrow \hat{\phi} \).

There are several reasons why we consider such algebraic objects, \( D \)-modules. First of all, an interpretation of solution spaces as \( \text{Hom}_{D(U)} \) prolongs naturally to use of homological algebra, which benefits us much enough. Secondly, as will be noted later, one of the basic invariants, the characteristic variety of a system can be correctly defined only when we consider the ideal generated by the \( P_i \)'s (a fixed set of generators is not enough for the definition).

2. Algebraic differential operators

Since all substantial examples in these notes are algebraic \( D \)-modules, we begin with basic notions on algebraic differential operators.

Simplest but important examples are linear differential operators with polynomial coefficients. The ring of differential operators with polynomial coefficients on the \( n \)-dimensional complex affine space \( \mathbb{C}^n \), denoted by \( D(\mathbb{C}^n) \), is called the Weyl algebra. The Weyl algebra \( D(\mathbb{C}^n) \) is a \( \mathbb{C} \)-algebra generated by \( x_i, \partial_i = \frac{\partial}{\partial x_i} \) (\( 1 \leq i \leq n \)) with Heisenberg commutator relations

\[
[\partial_i, x_j] = \delta_{ij}, \quad [x_i, x_j] = [\partial_i, \partial_j] = 0.
\]

Even on general smooth algebraic varieties, the situation does not differ much from the above. Let \( X \) be a smooth affine algebraic variety over \( \mathbb{C} \). This means the following. Let \( A \) be a commutative algebra finitely generated over \( \mathbb{C} \) with no nilpotent elements. The smoothness means that \( \dim_\mathbb{C} m/m^2 \) is constant (\( = \dim X \)) for every maximal ideal \( m \) of \( A \). The space \( X \) is identified with \( \text{Hom}_{\mathbb{C}_{-\text{alg}}}(A, \mathbb{C}) \), the set of all \( \mathbb{C} \)-algebra homomorphisms, which is also identified with \( \text{Spec} A \), the set of all maximal ideals of \( A \) by Hilbert’s Nullstellensatz \( (x \leftrightarrow \ker x = m_x (x \in \text{Hom}_{\mathbb{C}_{-\text{alg}}}(A, \mathbb{C})) \). The \( \mathbb{C} \)-algebra \( A \) is then denoted by \( \mathbb{C}[X] \) and called the algebra of regular functions on \( X \) \( (f(x) = x(f) \) for \( f \in \mathbb{C}[X], x \in \text{Hom}_{\mathbb{C}_{-\text{alg}}}(\mathbb{C}[X], \mathbb{C}) \)). The family of subsets \( X_f = \{ x \in X | f(x) \neq 0 \} \) \( (f \in \mathbb{C}[X]) \) forms a basis of open sets in \( X \) (the Zariski topology of \( X \)). Note that \( \mathbb{C}[X_f] = \mathbb{C}[X]_f = \mathbb{C}[X][f^{-1}] \) is the algebra of regular functions of an open affine subvariety \( X_f \) of \( X \).

The correspondence

\[
X_f \mapsto \mathbb{C}[X_f]
\]
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gives rise to the structure sheaf \( \mathcal{O}_X \) of \( X \) as a local ringed space \( (\mathcal{O}_X(X_f) = \Gamma(X_f, \mathcal{O}_X) = \mathbb{C}[X_f]) \). The stalk \( \mathcal{O}_{X,x} \) of \( \mathcal{O}_X \) at \( x \in X \) is the localization of \( \mathbb{C}[X] \) at the maximal ideal \( m_x \in \text{Specm} \mathbb{C}[X] \) (\( \lim_{\xrightarrow{\to}} \mathbb{C}[X_f] = \mathcal{O}_{X,m_x} \)).

In general, a smooth algebraic variety is defined to be a local ringed space \( (X, \mathcal{O}_X) \) such that every \( x \in X \) has an open neighborhood \( U \) such that \( (U, \mathcal{O}_X|_U) \) is isomorphic to a smooth affine variety as local ringed spaces as above. (Usually one adopts a further assumption, i.e., separability of the Zariski topology, which means that the diagonal map \( X \xrightarrow{\Delta} X \times X \) \( (\Delta(x) = (x, x)) \) is a closed immersion.)

Linear differential operators are defined as follows in algebraic geometry.

**Definition.** A \( \mathbb{C} \)-linear sheaf endomorphism \( P \in \text{End}_\mathbb{C} \mathcal{O}_X \) is called a linear differential operator of order not greater than \( m \) if

\[
(\text{ad} \mathcal{O}_X)^{m+1} P = 0.
\]

More precisely, for every open \( U \subset X \), \( P \) is a collection of \( \mathbb{C} \)-linear maps \( P_U \in \text{End}_\mathbb{C} \mathcal{O}_X(U) \) compatible with all sheaf restriction data \( \mathcal{O}_X(U) \to \mathcal{O}_X(V) \) \( (V \subset U) \) satisfying

\[
[f_0, [f_1, \cdots, [f_m, P_U] \cdots]] = 0 \quad \text{for every} \quad f_0, f_1, \cdots, f_m \in \mathcal{O}_X(U).
\]

By definition, if \( X \) is affine, a linear differential operator \( P \) of order not greater than \( m \) is seen to be a \( \mathbb{C} \)-linear endomorphism \( P \in \text{End}_\mathbb{C} \mathbb{C}[X] \) such that \( (\text{ad} \mathbb{C}[X])^{m+1} P = 0 \).

Denote by \( F_mD(X) \) the set of all linear differential operators on \( X \) of order not greater than \( m \). Clearly

\[
F_mD(X) \subset F_{m+1}D(X) \quad (m \geq 0)
\]

and it is easily seen that \( F_mD(X) F_lD(X) \subset F_{m+l}D(X) \). Thus the set of all linear differential operators on \( X \) forms a \( \mathbb{C} \)-algebra

\[
D(X) = \bigcup_{m=0}^{\infty} F_mD(X)
\]

with filtration \( F \). Note also that \( F_0D(X) = \mathcal{O}_X(X) \) by the correspondence \( P \mapsto P(1) \).

The sheaf \( D_X \) of algebras of linear differential operators on \( X \) is defined by the functor

\[
D_X : U \mapsto D(U) \quad \text{for every open} \quad U \subset X
\]

with obvious restriction maps. Thus \( D_X(U) = D(U) = \bigcup_{m=0}^{\infty} F_mD(U) \). The sheaf \( D_X \) also has the increasing filtration \( F \) by orders \( (F_mD_X)(U) = F_mD(U) \quad (m \geq 0) \).

The following lemma guarantees calculation in the algebraic case similar to the complex analytic case.
Lemma. In a smooth n-dimensional algebraic variety $X$, every point $p \in X$ has an affine open neighborhood $U$ with vector fields $\partial_i$ and functions $x_i$ ($1 \leq i \leq n$) on $U$ satisfying

\[ [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = \delta_{ij} \]

\[ F_m D_X(U) = \bigoplus_{|\alpha| \leq m} \mathcal{O}_X(U) \partial^\alpha \]

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index ($|\alpha| = \sum_{i=1}^n \alpha_i$) and

\[ \partial^\alpha = \prod_{i=1}^n \partial_i^{\alpha_i}. \]

Proof. Take $(x_i)$ to be a regular system at $p$, i.e., $\{x_i\}$ generates the maximal ideal of $\mathcal{O}_{X,p}$ and the differentials $dx_i$ are linearly independent at $p$. There then exists an open $U$ such that $f : U \rightarrow \mathbb{C}^n$ ($f(q) = (x_1(q), \cdots, x_n(q))$)

is an etale map. The standard vector fields $\frac{\partial}{\partial z_i}$ on $\mathbb{C}^n$ lift uniquely to $\partial_i$ on $U$ ($df(\partial_i) = \frac{\partial}{\partial z_i}$) and $\{x_i, \partial_i\}$ satisfies the requirement. In fact, for $P \in F_m D_X(U)$ and $\alpha$ such that $|\alpha| = m$, put

\[ a_\alpha(x) = (-1)^m (\alpha!)^{-1} (\text{ad} x_1)^{\alpha_1} \cdots (\text{ad} x_n)^{\alpha_n} P \]

where $\alpha! = \alpha_1! \cdots \alpha_n!$. Then $a_\alpha(x)$ is of order 0 and hence $a_\alpha(x) \in \mathcal{O}_X(U)$. It is easily seen that $P - \sum_{|\alpha| = m} a_\alpha(x) \partial^\alpha$ is of order less than $m$. By induction, the lemma has been proved. q.e.d.

Remark. Let $X_{an}$ be the underlying complex manifold of a smooth algebraic variety $X$ and $i : X_{an} \rightarrow X$ the natural morphism of local ringed spaces ($i^{-1} \mathcal{O}_X \rightarrow \mathcal{O}_{X_{an}}$ is the identification of regular functions on $X$ with holomorphic functions on $X_{an}$). Thus the sheaf $D_{X_{an}}$ of linear differential operators with holomorphic coefficients is regarded as $\mathcal{O}_{X_{an}} \otimes_i \mathcal{O}_X$, $i^{-1} D_X$. For a small open $U$ in $X_{an}$ (in the classical topology) the above choice of coordinates $\{x_i, \partial_i\}$ is a standard one in $D_{X_{an}}(U)$.

3. Filtrations of $D$-modules

3.1 Symbols.

The sheaf $D_X$ of algebras of linear differential operators has the increasing filtration $F$ by orders, i.e., for an open $U$ in $X$,

\[ (F_m D_X)(U) = \{ P \in D_X(U) \mid \text{ord } P \leq m \}. \]

(Almost tautologically, $\text{ord } P = m$ if and only if $P \in F_m D_X \setminus F_{m-1} D_X$.) Recall the following properties:
1) \( F_mD_X \subset F_{m+1}D_X \),
2) \( F_mD_X F_lD_X = F_{m+l}D_X \),
3) \( F_mD_X \) is \( \mathcal{O}_X \)-coherent,
4) \( D_X = \bigcup_{m=0}^{\infty} F_mD_X \).

Let \( \text{gr}D_X \) be the gradation of this algebra \( D_X \) by the order filtration \( F \),

\[
\text{gr}D_X = \bigoplus_{m=0}^{\infty} \text{gr}_mD_X
\]

where \( \text{gr}_mD_X = F_mD_X/F_{m-1}D_X \). Note that for an affine open \( U \),

\[
(\text{gr}_mD_X)(U) = F_mD_X(U)/F_{m-1}D_X(U).
\]

The graded algebra \( \text{gr}D_X \) is a commutative \( \mathcal{O}_X \)-algebra since

\[
\text{ord}[P, Q] \leq \text{ord}( PQ ) - 1, \quad (P, Q \in D_X).
\]

In the choice of local coordinates \( \{x_i, \partial_i\} \) in Lemma in 2, the projection

\[
F_mD_X(U) \longrightarrow \text{gr}_mD_X(U)
\]

is realized as the symbol map

\[
P = \sum_{|\alpha|\leq m} a_\alpha \partial^{\alpha} \longmapsto \sum_{|\alpha|=m} a_\alpha \xi^{\alpha} = \sigma_m(P)
\]

where \( \xi = (\xi_1, \cdots, \xi_n) \) is the linear coordinate system corresponding to \((x_i)\) on the
cotangent bundle \( T^*U \). Thus the symbol \( \sigma_m(P) \) is regarded as an element of the
polynomial algebra \( \mathcal{O}_X(U)[\xi_1, \cdots, \xi_n] \) over \( \mathcal{O}_X(U) \). It is a standard fact that this
symbol map \( \sigma \) is independent of the choice of coordinates and it gives rise to the
following global identification of the graded algebras:

\[
\text{gr}D_X \xrightarrow{\sim} \pi_*\mathcal{O}_{T^*X}
\]

where \( \pi : T^*X \to X \) is the cotangent bundle of \( X \) and \( \pi_* \) is the operation of a
direct image sheaf \( ((\pi_*\mathcal{O}_{T^*X})(U)) = \Gamma(\pi^{-1}(U), \mathcal{O}_{T^*X}) = \mathcal{O}_X(U)[\xi_1, \cdots, \xi_n] \).

3.2 Good filtrations.

A left \( D_X \)-module \( M \) simply means a sheaf of left \( D_X \)-modules: a sheaf \( M \) on
\( X \) such that for every open \( U \) in \( X \), \( M(U) \) is a left \( D_X(U) \)-module compatible with
restriction data. A right \( D_X \)-module is similarly defined. Since \( \mathcal{O}_X \) is a subalgebra
of \( D_X \), a \( D_X \)-module has the natural structure as an \( \mathcal{O}_X \)-module. On an algebraic
variety \( X \), we usually consider \( D_X \)-modules which are \( \mathcal{O}_X \)-\textit{quasi-coherent} in order
to pursue smooth manipulation in algebraic geometry. (An \( \mathcal{O}_X \)-module \( F \) is called
\textit{quasi-coherent} if \( F|_U \) is isomorphic to the sheaf made by localization of the \( \mathcal{O}_X \-
module \( F(U) \) on every affine open \( U \).)

However, we retain that for deeper analysis of solutions to equations, one often
needs non-\textit{quasi-coherent} \( \mathcal{O}_X \)-modules. In particular, for analysis on the complex
manifolds $X_{an}$, these are sometimes essential, e.g., a sheaf of distributions and/or hyperfunctions etc.,...

At any rate, usual systems of linear partial differential equations correspond to more restricted $D_X$-modules, $D_X$-coherent modules. Here we take the definition of the $D_X$-coherency as that of the local finite presentation of $D_X$-modules, i.e., we call a $D_X$-module $M$ on $X$ $D_X$-coherent if every point of $X$ has a neighborhood $U$ with an exact sequence

$$D^m_U \longrightarrow D^l_U \longrightarrow M|_U \longrightarrow 0$$

where $D_U$ is the sheaf of algebras of linear differential operators on $U$ considered as a $D_U$-module.

As is earlier introduced, the category of coherent $D_X$-modules corresponds to that of systems of linear partial differential equations on $X$. Note that on an affine open $U$, $M|_U$ is the localization of $M(U)$ and hence the above condition corresponds to the exact sequence of $D(U)$-modules:

$$D(U)^m \overset{\Phi}{\longrightarrow} D(U)^l \overset{\Psi}{\longrightarrow} M(U) \longrightarrow 0$$

**Remark.** The above coherent $D$-module corresponds to the system of linear partial differential equations

$$\sum_{j=1}^l P_{ij} u_j = 0 \quad (1 \leq i \leq m),$$

by the maps

$$\Phi(Q_1, \cdots, Q_m) = (Q_1, \cdots, Q_m)(P_{ij}),$$

$$\Psi(R_1, \cdots, R_l) = \sum_{j=1}^l R_j u_j$$

where $(P_{ij})$ is an $m \times l$-matrix of entries in $D(U)$.

In order to define the characteristic variety of a $D$-module, we introduce the notion of good filtrations matching the order filtration of $D_X$.

For a coherent $D_X$-module $M$, let $F_m M \subset M (m \in \mathbb{Z})$ be an increasing filtration by coherent $\mathcal{O}_X$-submodules (i.e., $\mathcal{O}_X$-submodules of locally finite presentation) such that

1) $F_m M = 0$ \quad ($m \ll 0$),

2) $M = \bigcup_{m \in \mathbb{Z}} F_m M$,

3) $F_l D_X F_m M \subset F_{l+m} M$ \quad ($l, m \in \mathbb{Z}$).

Then by 1), 2), 3), the gradation of $M$ by $F$

$$\text{gr}^F M = \bigoplus_{m \in \mathbb{Z}} F_m M / F_{m-1} M$$

is a graded $\text{gr} D_X$-module.

**Definition.** A filtration $F$ of a coherent $D_X$-module $M$ is called good if

$$F_l D_X F_m M = F_{l+m} M \quad \text{for} \ m \ \text{large enough and all} \ l \geq 0.$$  

Here the left hand side is the $\mathcal{O}_X$-submodule generated by the multiplication of $F_m M$ by $F_l D_X$.

The following is then rather easily proved ([11, II, Prop.1.2.3]).
Proposition. A filtration $F$ is good if and only if the graded module $\text{gr}^F M$ is a coherent $\pi_* \mathcal{O}_{T^*X}$-algebra.

By definition, a coherent $D_X$-module $M$ is locally finitely generated, i.e., on an open $U$

$$M|_U = \sum_{i=1}^l D_U u_i \quad (u_i \in M(U)).$$

Define $F^m M|_U = \sum_{i=1}^l F^m D_U u_i \quad (m \geq 0)$. Then $F^m$ is a good filtration of $M|_U$. Thus any coherent $D_X$-module locally has a good filtration. In the algebraic case, any coherent $D_X$-module globally has a good filtration thanks to the quasi-compactness of the Zariski topology (see [11, II,1.2]).

Example. Let $M = D_X u$ ($D_X$-cyclic by a section $u \in M(X)$). $F^m M = F^m D_X u$ then gives a good filtration of $M$. Put

$$I = \text{Ann} u = \{P \in D_X \mid Pu = 0\}$$

the annihilator of $u$ (thus $M \simeq D_X/I$). Then

$$\text{gr}^F M \simeq \text{gr} D_X / \text{gr} I$$

where

$$\text{gr} I = \bigoplus_{m \leq 0} F^m I / F^{m-1} I \subset \pi_* \mathcal{O}_{T^*X}, \quad (F^m I = I \cap F^m D_X).$$

4. Characteristic varieties

4.1 Definition.

Let $M$ be a coherent $D_X$-module and $F$ its good filtration. Then the graded $\pi_* \mathcal{O}_{T^*X}$-module $\text{gr}^F M$ is coherent where $\pi: T^*X \to X$ is the cotangent bundle. Since the sheaf pull-back $\pi^{-1} \text{gr}^F M$ on $T^*X$ is $\pi^{-1} \pi_* \mathcal{O}_{T^*X}$-coherent, we have an $\mathcal{O}_{T^*X}$-coherent module

$$\pi^*(\text{gr}^F M) = \mathcal{O}_{T^*X} \otimes_{\pi^{-1} \pi_* \mathcal{O}_{T^*X}} \pi^{-1}(\text{gr}^F M)$$

on the cotangent bundle $T^*X$ (the algebra homomorphism $\pi^{-1} \pi_* \mathcal{O}_{T^*X} \to \mathcal{O}_{T^*X}$ is a natural restriction of functions). The characteristic variety $\text{ch} M$ of $M$ is then defined to be the support of the $\mathcal{O}_{T^*X}$-coherent module $\pi^*(\text{gr}^F M)$. By the coherency, the characteristic variety is an algebraic subvariety of $T^*X$ conic along the fibers (= cotangent spaces).

We shall look at it in a more naive way. Let $U$ be a small affine open set in $X$. Then

$$(\pi_* \mathcal{O}_{T^*X})(U) = \mathcal{O}_X(U) \otimes \mathbb{C}[\xi_1, \cdots, \xi_n]$$

where $(\xi_1, \cdots, \xi_n)$ is a coordinate system of the cotangent space (we assume $T^*U \simeq U \times \mathbb{C}^n$). For $\text{gr}^F M$, we have

$$(\text{gr}^F M)(U) = \bigoplus_{m \in \mathbb{Z}} F^m M(U) / F^{m-1} M(U),$$
which is a graded module over the graded algebra \( O_X(U) \otimes_{\mathbb{C}} \mathbb{C}[\xi_1, \cdots, \xi_n] \). The \( O_{T^*X} \)-module \( \pi^* (\text{gr}^F M) \) is simply the localization of the \( O_X(U) \otimes_{\mathbb{C}} \mathbb{C}[\xi_1, \cdots, \xi_n] \)-module \( (\text{gr}^F M)(U) \) on \( T^*U \simeq U \times \mathbb{C}^n \) and hence the characteristic variety \( \text{ch} M \) on \( T^*U \) is nothing but the zeroes (affine subvariety) of the annihilator ideal in the algebra \( O_X(U) \otimes_{\mathbb{C}} \mathbb{C}[\xi_1, \cdots, \xi_n] \).

\[ \text{Ann}_{O_X(U) \otimes_{\mathbb{C}} \mathbb{C}[\xi_1, \cdots, \xi_n]} (\text{gr}^F M)(U). \]

By the gradedness, this variety is conic along the fibers \( \mathbb{C}^n \).

**Example.** Let \( M = D_X u \) be as in Example in 3.2. Then \( \text{ch} M = V(\text{gr} I) \) the zeroes defined by the ideal \( \text{gr} I \subset \pi_* O_{T^*X} \). Notice that even if \( P_1, \cdots, P_m \) are generators of the ideal \( I \) in \( D_X \), the symbols \( \sigma(P_1), \cdots, \sigma(P_m) \) are not necessarily generators of \( \text{gr} I \). That is, the characteristic variety \( \text{ch} M \) is not exactly the zeroes of the symbols \( \sigma(P_i) \) (1 \( \leq i \leq m \)) (is contained in those). Here we see the importance of the concept of \( D_X \)-modules in defining the characteristic varieties of systems of linear partial differential equations. It is however known that there exist generators \( P_i \)'s in \( I \) such that the symbols \( \sigma(P_i) \)'s generate \( \text{gr} I \) (see [11, II.2]).

**Theorem.** The characteristic variety \( \text{ch} M \) is independent of the choice of good filtrations of a coherent \( D_X \)-module \( M \).

For the proof, see [11, II,Th.2.1]. By this theorem, the characteristic variety turns out to be a true invariant of a \( D \)-module. Also in the analytic case, since a coherent \( D \)-module locally has a good filtration, the characteristic variety can be defined globally by a similar theorem.

### 4.2 The fundamental theorem

As is well-known, the cotangent bundle \( T^*X \) has the canonical symplectic structure \( \omega \) which is expressed in local coordinates

\[ \omega = \sum_{i=1}^n d\xi_i \wedge dx_i \quad (n = \dim X). \]

The symplectic structure \( \omega \) defines the Poisson bracket in the space of functions on \( T^*X \) which is expressed in local coordinates

\[ \{ f, g \} = \sum_{i=1}^n \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial \xi_i} \frac{\partial f}{\partial x_i} \right) \quad (f, g \in O_{T^*X}). \]

A subvariety \( V \) in \( T^*X \) is said to be *involutive* if the defining ideal \( I(V) \subset O_{T^*X} \) is closed under the Poisson Bracket \( \{ , \} \). It is easily seen that if \( V \) is involutive, then the tangent space of a smooth point \( p \in V \) is involutive with respect to the symplectic form \( \omega \) (\( T_p V^\perp \subset T_p V \) where \( \perp \) denotes the orthogonal complement with respect to \( \omega \) in \( T_p(T^*X) \)) and hence every irreducible component of \( V \) has dimension not less than \( \dim X = \frac{1}{2} \dim T^*X \) (if \( V \neq \emptyset \)).

The following theorem is called the fundamental theorem of algebraic analysis, which is first proved in [23]. Later O. Gabber gave a purely algebraic proof for this theorem [5] but still difficult.
Theorem (Sato-Kawai-Kashiwara). The characteristic variety of a coherent \(D\)-module is involutive.

In particular, if \(M \neq 0\) (\(M = 0 \iff \text{ch } M = \emptyset\)), then

\[
\dim(\text{component of } \text{ch } M) \geq \dim X.
\]

Note that the weaker statement “\(\dim \text{ch } M \geq \dim X\)” is much more easily proved (in the algebraic case) by J. Bernstein (see [1], [11, II, Th.5.1]).

Now we shall ask the following question. What kinds of non-zero coherent \(D\)-modules are of “smallest size”? In a monogenic case \(M = D_X u\), those must correspond to the case when the annihilator ideals \(\text{Ann}_{D_X} u\) have “largest size” which means that the corresponding systems of linear partial differential equations are “maximally overdetermined”. Taking the gradation \(\text{gr Ann } u\), the characteristic variety \(\text{ch } M = V(\text{gr Ann } u)\) must have “smallest size” as possible but these are of dimension not less than \(\dim X\). Thus we attain the case of holonomic \(D\)-modules.

Definition. A coherent \(D_X\)-module \(M\) is called **holonomic** if \(\dim \text{ch } M = \dim X\) or \(M = 0\).

All substantial examples in these notes are holonomic.

5. Examples

5.1. Ordinary differential equations.

Let \(P(x, \partial) = \sum_{i=0}^{m} a_i(x) \partial^i\) be a non-zero linear operator on \(\mathbb{C}\) (\(a_i(x) \in \mathbb{C}[x], \partial = \frac{d}{dx}, a_m(x) \neq 0\)). The ordinary differential equation

\[
P(x, \partial) u = 0
\]

corresponds to the \(D_\mathbb{C}\)-module

\[
M = D_\mathbb{C} u = D_\mathbb{C} / I \quad (I = D_\mathbb{C} P(x, \partial)).
\]

In this case, \(\text{gr } I\) turns out to be a principal ideal generated by the symbol \(\sigma_m(P)(x, \xi) = a_m(x)\xi^m \in \mathbb{C}[x, \xi]\) and hence

\[
\text{ch } M = \{ (x, \xi) \in \mathbb{C} \times \mathbb{C} \mid a_m(x)\xi^m = 0 \}
\]

\[
= \mathbb{C} \times 0 \cup \bigcup_{a_m(x) = 0} x \times \mathbb{C}
\]

i.e., the union of the zero section and the fibers at the singular points \(a_m(x) = 0\) (\(T^*\mathbb{C} = \mathbb{C} \times \mathbb{C}\)). Hence \(\dim \text{ch } M = 1\) and \(M\) is holonomic.

5.2 Connections.

A \(D_X\)-module \(M\) is called a **connection** if \(M\) is a locally free \(\mathcal{O}_X\)-module of finite rank (i.e., vector bundle as an \(\mathcal{O}_X\)-module).
Let $M$ be a connection and $U$ open in $X$. Let $\Theta_X$ be the sheaf of vector fields on $X$. Write the action of a vector field $\theta \in \Theta_X(U)$ on a section $s \in M(U)$ as
\[
\nabla_\theta s = \theta s.
\]
Then by the definition of the $D_X(U)$-action, we have
i) $\nabla_{f\theta}s = f\nabla_\theta s$ ($f \in \mathcal{O}_X(U)$),
ii) $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta s$ ($[\theta, f] = \theta(f)$),
iii) $\nabla_{[\theta, \theta']s} = [\theta, \theta']s$.
The conditions i), ii) are the usual ones for a “connection” and iii) corresponds to the “integrability”. Thus a connection in our sense is an integrable connection.

Conversely when a vector bundle $M$ is given with $\Theta_X$-action through $\nabla$ satisfying i), ii), iii), then this action extends to the left $D_X$-action and $M$ acquires the structure of a $D_X$-module.

**Example.** Let $M = D_C u = D_C/D_C P$ be as in 5.1. Then $M|_U$ is a connection on the open set $\{x \in \mathbb{C} | a_m(x) \neq 0\}$. In fact, let $u_i = \partial^i u$ ($0 \leq i < m$). Then
\[
M|_U = D_Uu \simeq \sum_{i=0}^{m-1} \mathcal{O}_U u_i
\]
since $a_m(x)^{-1} \in \mathcal{O}(U)$.

Let $M$ be a connection on $X$. Define a filtration $F$ on $M$ by
\[
F_i M = 0 \quad (i \leq 0), \quad F_i M = M \quad (i \geq 1).
\]
Since $M$ is $\mathcal{O}_X$-coherent, this is clearly a good filtration. Since
\[
gr^F M = M \quad \text{(concentrated at degree 1)},
\]
gr$_1 D_X$ then acts as 0 on gr$_1^F M = M$. This means, for instance, that locally every linear coordinate $\xi_i$ on $T^*X$ belongs to Ann gr$^F M$, which implies
\[
\ch M = T^*_X X \quad (= \text{zero-section of } \pi : T^*X \to X).
\]
Actually we know:

**Theorem.** For a coherent $D_X$-module $M$, the following are equivalent.
1) $M$ is a connection.
2) $M$ is $\mathcal{O}_X$-coherent.
3) $\ch M = T^*_X X$.

For the proof, see [11, II, Prop. 2.3].

We close this section by citing an important classical theorem in the analytic case.

**Theorem (Frobenius).** Let $M$ be an analytic connection on a complex manifold $X$. Define the subsheaf of vector spaces in $M$:
\[
(DR M)(U) = \{s \in M(U) | \nabla_\theta s = 0 \text{ for any vector field } \theta \in \Theta_X(U) \}
\]
for every open $U$ in $X$. Then $\text{DR} M$ is a local system of rank equal to $\text{rank} M$ (i.e., for a small connected $U$, $(\text{DR} M)(U) \cong \mathbb{C}^r$ where $r = \text{rank} M (M|_U \cong O^r_U)$).

On a complex manifold $X$, this correspondence
\[
\{ \text{analytic connections on } X \} \xrightarrow{\text{DR}} \{ \text{local systems on } X \}
\]
gives rise to a categorical equivalence. The quasi-inverse of the functor $\text{DR}$ (the de Rham functor) is given by
\[
L \mapsto O_X \otimes_C L
\]
where the connection in the right-hand-side is defined by $\nabla_{\theta}(f \otimes s) = \theta(f) \otimes s \quad (\theta \in \Theta_X, f \in O_X, s \in L)$.

In the algebraic case, the above correspondence $\text{DR}$ causes much delicate problems. Now let $X$ be a smooth algebraic variety and $X_{\text{an}}$ the underlying complex manifold. For an algebraic connection $M$ on $X$, in turn, define the functor $\text{DR}$ by
\[
\text{DR} M = \text{DR} M_{\text{an}} \quad \text{where } M_{\text{an}} = O_{X_{\text{an}}} \otimes_{O_X} M.
\]
Thus $\text{DR}$ is a functor from the category of algebraic connections on $X$ into that of local systems on the complex manifold $X_{\text{an}}$. In this case, $\text{DR}$ does not give rise to a categorical equivalence but by restricting the class of algebraic connections, P. Deligne established a nice equivalence between these categories ([4]). For this, the notion of regularities is necessarily involved, and an algebraic connection is called regular if its (unique) meromorphic extension to a compactification $\overline{X}$ has only regular singularities at the boundary $\overline{X} \setminus X$. (For the precise definition, see [11, IV].)

**Theorem (Deligne).** The functor $\text{DR}$ gives a categorical equivalence between the category of regular connections on an algebraic variety $X$ and that of local systems on the complex manifold $X_{\text{an}}$.

This correspondence $\text{DR}$ is intensively generalized to $D$-modules and plays a substantial role in the Riemann-Hilbert correspondence for regular holonomic $D$-modules (M. Kashiwara and Z. Mebkhout, see [16], [20], [11, V]).

**Example.** For a fixed $\lambda \in \mathbb{C}$, let $O_{\lambda}$ be the Euler system
\[
O_{\lambda} = D_{\mathbb{C}}u = D_{\mathbb{C}}/D_{\mathbb{C}}(x\partial - \lambda) \quad \text{on } \mathbb{C} \,(\partial = \frac{d}{dx}).
\]
Then $O_{\lambda}|_{\mathbb{C}^\times} \,(\mathbb{C}^\times = \mathbb{C} \setminus \{0\})$ is an algebraic connection and the analytic solution sheaf $S_{\lambda}$ is a local system of rank one generated by the multi-valued holomorphic function $x^\lambda$ on $\mathbb{C}^\times$.

On the other hand, $\text{DR}(O_{\lambda}|_{\mathbb{C}^\times}) = \mathbb{C}x^{-\lambda} u \subset O_{\lambda}|_{\mathbb{C}^\times}$. Since
\[
x\partial(x^{-\lambda} u) = -\lambda x^{-\lambda} u + x^{-\lambda}(x\partial u) = 0
\]
by $x\partial u = \lambda u$. In this sense, $S_{\lambda}$ and $\text{DR}(O_{\lambda}|_{\mathbb{C}^\times})$ are local systems dual to each other. Further we see, even for an algebraic $D_{\mathbb{C}}$-module $O_{\lambda}$ as above, this correspondence makes sense only in the analytic category (we need such analytic functions like $x^\lambda$).
6. Basic functors

6.1 Inverse images.

Let $X \to Y$ be a morphism of smooth algebraic varieties. For a $D_Y$-module $M$, let

$$f^* M = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M$$

be the inverse image as an $\mathcal{O}$-module. Then $f^* M$ turns out to be a $D_X$-module by the chain rule. That is, if $\{y_i, \partial_i\}$ is a local coordinate system on $Y$, vector fields $\theta \in \Theta_X$ act on $f^* M$ by

$$\theta(\phi \otimes u) = \theta(\phi) \otimes u + \phi \sum_{i=1}^n \theta(y_i \circ f) \otimes \partial_i u \quad (\phi \in \mathcal{O}_X, u \in M)$$

and these actions extend to the $D_X$-action.

Even if $M$ is $D_Y$-coherent, $f^* M$ is not necessarily $D_X$-coherent. For example, on $Y = \mathbb{C}^2$ consider an equation $\partial_y u = 0$ ($y$ is the second coordinate).

The corresponding $D_Y$-module $M \simeq D_Y / \partial_y \simeq \mathcal{D}_\mathbb{C} \boxtimes \mathcal{O}_\mathbb{C}$ has the inverse image

$$i^* M \simeq \bigoplus_{j=0}^{\infty} \mathcal{D}_\mathbb{C} \otimes \mathcal{O}_\mathbb{C}$$

for an inclusion $i : \mathbb{C} \hookrightarrow \mathbb{C}^2 (i(y) = (0, y))$. In the above $\boxtimes$ denotes the outer tensor product (over $\mathbb{C}$) on the product variety. Thus, as an $\mathcal{D}_\mathbb{C}$-module, $i^* M \simeq \bigoplus_{j=0}^{\infty} \mathcal{O}_\mathbb{C}$ is not a finitely generated $\mathcal{D}_\mathbb{C}$-module.

However, it is known that if $M$ is holonomic then so is $f^* M$ (see [11, III]).

6.2 Direct images (integrations along fibers).

In contrast with the inverse images, the definition of the “direct image” of $D$-modules is rather complicated. We want a certain $D_Y$-module “$f_* M$” on $Y$ for $f : X \to Y$ and for a $D_X$-module $M$.

As an extreme example, consider a closed immersion $i : X \to X \times \mathbb{C} = Y$ ($i(x) = (x, 0)$).

For a $D_X$-module $M$ on $X$, let $i_* M$ be the direct image in the sheaf theory:

$$(i_* M)(U) = M(U \cap (X \times 0)).$$

This becomes an $\mathcal{O}_Y$-module as usual but not a $D_Y$-module. Since $D_Y \simeq D_X \boxtimes D_\mathbb{C}$ $\bigoplus_{j=0}^{\infty}$ $D_\mathbb{C} \boxtimes \mathcal{O}_\mathbb{C} \partial^j$ ($\partial = \frac{\partial}{\partial t}$, $t$ coordinate of $\mathbb{C}$), we want a $\partial$-action on some $i_* M$, an extension of $i_* M$. Considering the infinite sum

$$\bigoplus_{j=0}^{\infty} i_* M \otimes \partial^j \quad (i_* M \otimes \partial^j \simeq i_* M \text{ as } D_X\text{-modules})$$
with the $\partial$-action by $\partial(u \otimes \partial^j) = u \otimes \partial^{j+1}$ ($O_C$ -action by $\phi(t)u \otimes \partial^j = \phi(0)u \otimes \partial^j$), $i_* M$ extends to a $D_{X \times C}$-module.

In general, we have to be much more careful. So far we have only considered left $D$-modules since they correspond naturally to systems of linear partial differential equations. But in order to understand the “direct images”, it is more convenient to consider right $D$-modules. These simply correspond to the adjoint systems of usual ones.

For example, let $P^*$ be the adjoint operator of $P \in D_{C^n}$ ($\partial_i \mapsto -\partial_i$). Then $(PQ)^* = Q^* P^*$. If $M$ is a left $D_{C^n}$-module, by the action

$$u P = P^* u \quad (u \in M, P \in D_{C^n})$$

$M$ is regarded as a right $D_{C^n}$-module.

Globally in general, this procedure can be defined by using Lie derivatives on highest differential forms. Let $\Omega_Y$ be the sheaf of highest differential forms on $X$ (the canonical line bundle). The Lie derivative $L_\theta \omega$ for a vector field $\theta \in \Theta_X$ and $\omega \in \Omega_X$ is by definition

$$(L_\theta \omega)(\theta_1, \cdots, \theta_n) = \theta(\omega(\theta_1, \cdots, \theta_n)) - \sum_{i=1}^n \omega(\theta_1, \cdots, [\theta, \theta_i], \cdots, \theta_n)$$

where $\theta_i \in \Theta_X$. Then $L_\phi \omega = L_\theta(\phi \omega)$ for $\phi \in O_X$. Hence defining the $\Theta_X$-action on $\Omega_X$ by $\omega \theta = -L_\theta \omega$, $\Omega_X$ gains the right $D_X$-module structure.

For a left $D_X$-module $M$, $M \otimes_{O_X} \Omega_X$ then turns out to be a right $D_X$-module by

$$(u \otimes \omega) \theta = -(\theta u) \otimes \omega + u \otimes \omega \theta \quad (u \otimes \omega \in M \otimes \Omega_X, \theta \in \Theta_X).$$

In the above example of $D_{C^n}$-modules, we have fixed a global section $dx_1 \wedge \cdots \wedge dx_n$ of $\Omega_{C^n}$ and identify $M$ with $M \otimes_{O_{C^n}} \Omega_{C^n}$.

Now let $f : X \to Y$ be a morphism of smooth algebraic varieties. As is seen in 6.1, the inverse image $f^* D_Y$ (as an $O$-module) is a left $D_X$-module. Simultaneously it is also a right $f^{-1} D_Y$-module commuting with the left $D_X$-action. We write

$$D_{X \to Y} = f^* D_Y$$

as a double $(D_X, f^{-1} D_Y)$-module.

If $M$ is a right $D_X$-module on $X$, then $M \otimes_{D_X} D_{X \to Y}$ turns out to be a right $f^{-1} D_Y$-module (coming from the right action on $D_{X \to Y}$). Hence if we take the sheaf direct image

$$f_*(M \otimes_{D_X} D_{X \to Y}),$$

then it becomes a right $D_Y$-module on $Y$.

For a left $D_X$-module $M$, we apply the left-right correspondence (tensoring $\otimes \Omega_X$) and take the above procedure. Thus the final form becomes the following messy one

$$f_*((M \otimes_{O_X} \Omega_X) \otimes_{D_X} D_{X \to Y} \otimes f^{-1} O_Y, f^{-1}(\Omega_{Y}^{-1}))$$

a left $D_Y$-module (tensoring $\Omega_Y^{-1}$ converts the right $D_Y$-structure into the left one). Furthermore, the description of the left $D_Y$-action is highly complicated in general.

Exercise. Check the earlier example for $X \leftarrow X \times C$. 


But still the above definition is not a final one in general (if $X, Y$ are not affine). For the correct definition of the direct images of $D$-modules, we have to use a concept of derived categories and several operations in them. We are not going into details now, but only write down

$$f\bullet M = \int_f M = Rf_\ast ((M \otimes_{\mathcal{O}_X} \Omega_X))^L \otimes_{D_X} (D_{X \to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\Omega_Y))$$

(see [11, I]).

This functor works well in the “Riemann-Hilbert” correspondence for regular holonomic $D$-modules and corresponds simply to the sheaf direct image functor for the solution sheaves or the de Rham complexes.
II. Equivariant $D$-modules, examples

1. Definition

When a variety has a group action, it is often important to consider a class of systems of linear partial differential equations with equivariance property under the group action. Here we start with defining the group equivariance property matching the $D$-module operations.

Let $G$ be an algebraic group acting on a smooth algebraic variety $X$ and $\alpha : G \times X \to X$ the group action morphism ($\alpha(g, x) = gx$ $(g \in G, x \in X)$). Naively, for instance, a sheaf $F$ on $X$ is considered as $G$-equivariant if it is given a datum of sheaf morphisms $F_x \simeq F_{gx}$ “simultaneously” for $(g, x) \in G \times X$. More strictly, $F$ is said to be $G$-equivariant if there exists a morphism

$$\alpha^{-1}F \xrightarrow{\sim} \text{pr}_X^{-1}F$$

satisfying the associativity condition coming from the group multiplication of $G$ (cocycle condition). ($\alpha^{-1}$, $\text{pr}_X^{-1}$ are the inverse image functor in the sheaf theory and $\text{pr}_X : G \times X \to X$ is the projection onto $X$.)

We follow an analogous approach in $D$-module operations but extend the setup a little wider, i.e., attach a twisting datum on the group $G$. This extension contains much more examples, in particular, Sato’s relative invariants on prehomogeneous vector spaces and Gelfand’s generalized hypergeometric equations.

A twisted datum is a connection $L$ on $G$. We want to define a $D_X$-module $M$ to be “$L$-twistedly” $G$-equivariant if there exists a $D$-module homomorphism (with naturality condition)

$$\alpha^*M \simeq L \boxtimes M \quad \text{on} \ G \times X.$$ 

Here $\alpha^*M$ is the inverse image of the $D$-module $M$ and $L \boxtimes M$ is the outer tensor product over $\mathbb{C}$ on the product space $G \times X$ ($L \boxtimes M = \text{pr}_G^{-1}L \otimes_\mathbb{C} \text{pr}_X^{-1}M \simeq \text{pr}_G^*L \otimes_{\mathcal{O}_{G \times X}} \text{pr}_X^*M$).

But then in the diagram

$$\begin{array}{c}
G \times G \times X \xrightarrow{i_G \times \alpha} G \times X \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
Furthermore in this case $L$ should necessarily be of rank one (line bundle). In fact, consider the maps

$$G \times e \xleftarrow{i} G \times G \xrightarrow{\mu} G \quad (e \in G : \text{identity}).$$

Then $\mu^* L \simeq L \boxtimes L$. Applying $i^*$, we have

$$L \simeq i^* \mu^* L$$

by naturality. But then the right hand side $\simeq L \boxtimes e^* L$ where $e^* L$ is the geometric stalk of $L$ at $e$. Hence $e^* L \simeq \mathbb{C}$, i.e., $L$ is a line bundle over $G$.

**Example.** Let $\mathfrak{g} = \text{Lie } G$ be the Lie algebra of $G$. Fix a Lie algebra homomorphism $\lambda : \mathfrak{g} \to \mathbb{C}$ where $\mathbb{C}$ is considered as an abelian Lie algebra. For $\theta \in \mathfrak{g}$, let $L_\theta$ be the corresponding right invariant vector field on $G$, i.e., $(L_\theta f)(x) = \frac{d}{dt} f(e^{-t\theta} x)|_{t=0}$.

The system of linear partial differential equations

$$(L_\theta - \lambda(\theta))u = 0 \quad (\theta \in \mathfrak{g})$$

corresponds to the $D_G$-module

$$\mathcal{O}_\lambda = D_G/\sum_{\theta \in \mathfrak{g}} D_G(L_\theta - \lambda(\theta)).$$

Then $\mathcal{O}_\lambda \simeq \mathcal{O}_G u$ is a rank one connection with

$$\nabla_\theta (fu) = \theta(\theta) u + \lambda(\theta) f u \quad (\theta \in \mathfrak{g})$$

and $\mathcal{O}_\lambda$ is $\mathcal{O}_\lambda$-twistedly $G$-equivariant.

**Examples of Example.**

1) Let $G$ be a linear group and $\det : G \to \mathbb{C}^\times$. A multi-valued function $\det(x)^\lambda$ $(x \in G)$ generates a rank one connection

$$D_G \det(x)^\lambda = \mathcal{O}_G \det(x)^\lambda.$$

This is a simplest connection with regular singularities.

2) On the additive group $\mathbb{G}_a = \mathbb{C}$, a similar connection

$$D_{\mathbb{C} e^{\lambda x}} = D_{\mathbb{C}}(\frac{d}{dx} - \lambda)$$

has an irregular singularity at $x = \infty$.

Finally, we arrive at the definition of equivariant $D$-modules.

**Definition.** Let $X$ be a smooth algebraic variety with algebraic group action $\alpha : G \times X \to X$. Let $L$ be a rank one connection on $G$. A $D_X$-module $M$ is called $L$-twistedly $G$-equivariant if there exists a $D$-module isomorphism

$$\phi : \alpha^* M \xrightarrow{\sim} L \boxtimes M \quad \text{on } G \times X$$
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satisfying the following cocycle condition:

\[ \text{pr}_G^* \circ (1_G \times \alpha)^*(\phi) \simeq (\mu \times 1_X)^*(\phi) \]

with respect to the commutative diagram

\[
\begin{array}{ccc}
G \times G \times X & \xrightarrow{\mu \times 1_X} & G \times X \\
\downarrow^{1_G \times \alpha} & & \downarrow^{\alpha} \\
G \times X & \xrightarrow{\alpha} & X.
\end{array}
\]

Here \( L \) itself necessarily turns out to be an \( L \)-twisted \( G \)-equivariant \( D_G \)-module under the group multiplication \( \mu : G \times G \to G \).

When \( L = O_\lambda \) for a Lie algebra character \( \lambda : g \to \mathbb{C} \), we say \( \lambda \)-twistedly for \( O_\lambda \)-twistedly.

Remark. An \( O_G(\lambda = 0) \)-twistedly \( G \)-equivariant \( D \)-module is just a \( G \)-equivariant \( D \)-module in [2, VII, 12.10], [24].

2. Equivariant systems of linear partial differential equations

All equivariant \( D \)-modules in these notes have the following forms:

**Theorem.** Let \( G \) acts on \( X \) and \( \lambda \) be a Lie algebra character. Let \( I \) be a finitely generated \( G \)-stable left ideal of the algebra of linear differential operators \( D_X(X) \). Then

\[ M = D_X/(I + \sum_{\theta \in g} D_X(L_\theta - \lambda(\theta))) \]

is \( \lambda \)-twistedly \( G \)-equivariant \( (L_\theta \) is the vector field on \( X \) given by the \( G \)-action: \( (L_\theta f)(x) = \frac{d}{dt}f(e^{-t\theta}x)|_{t=0} \)).

**Proof.** Let \( O_\lambda = D_Gv_\lambda = O_Gv_\lambda \) \( (L_\theta_G v_\lambda = \lambda(\theta)v_\lambda) \). \( L_\theta_G \) is the right invariant vector field on \( G \) corresponding to \( \theta \) and \( u \) the generator of \( M \), \( M = D_X u \) \((I u = (L_\theta - \lambda(\theta))u = 0)\). Take \( v_\lambda \boxtimes u \) a generator of \( O_\lambda \boxtimes M \) and \( \tilde{u} = 1 \otimes u \in (\alpha^* M)(X) \).

First we see the actions of \( L_{\theta,G} \boxtimes 1 \) and \( 1 \boxtimes L_\theta \) on \( \alpha^* M \). Taking a local coordinate system \( \{x_i, \partial_i\} \) on \( X \), we have \( \xi \tilde{u} = \sum_i \xi(x_i \circ \alpha) \otimes \partial_i u \) for a vector field \( \xi \) on \( G \times X \).

But then, by the right invariance of \( L_{\theta,G} \),

\[ L_{\theta,G}(x_i \circ \alpha) = (L_{\theta} x_i) \circ \alpha. \]

Hence

\[
(L_{\theta,G} \boxtimes 1)\tilde{u} = \sum_i 1 \otimes (L_{\theta} x_i) \partial_i u \\
= 1 \otimes L_\theta u \\
= \lambda(\theta)(1 \otimes u).
\]

For \( 1 \boxtimes L_\theta \), on the slice \( g \times X \rightarrow X \),

\[ L_\theta(x_i \circ \alpha)|_{g \times X} = L_{g \theta} x_i \quad (g \in G). \]
Hence on \( X \simeq g \times X \hookrightarrow G \times X \),

\[
(1 \boxtimes L_{\theta}) \tilde{u}|_{g \times X} = \sum_{i} L_{\theta}(x_i \circ \alpha) \otimes \partial_i u|_{g \times X} \\
= \sum_{i} L_{g \theta}(x_i) \partial_i u \\
= L_{g \theta} u \\
= \lambda(g \theta) u \\
= \lambda(\theta) u
\]

where \( g \theta = \text{Ad}_g \theta \) is the adjoint action of \( g \) on \( g \) and the last equality comes from the \( G \)-invariance of \( \lambda \in g^* \) (Lie algebra character). This means that the section 

\[
(1 \boxtimes L_{\theta} - \lambda(\theta)) \tilde{u}
\]

takes a value zero on every slice \( g \times X \). Since this section is in an \( O_{G \times X} \)-coherent subsheaf \( F_1 D_{G \times X} \tilde{u} \),

\[
(1 \boxtimes L_{\theta} - \lambda(\theta)) \tilde{u} = 0
\]

by the Nakayama lemma.

Secondly for \( P \in I \), we also have

\[
(1 \boxtimes P) \tilde{u}|_{g \times X} = P^g \quad \text{on } g \times X
\]

by the same computation (\( P^g \) is the \( g \)-translate of \( P \) under the \( G \)-action on \( D_X(X) \)). Since \( I \) is \( G \)-stable, \( P^g \in I \) and hence

\[
(1 \boxtimes P) \tilde{u}|_{g \times X} = 0 \quad \text{for } g \in G.
\]

Hence by the same reason as above,

\[
(1 \boxtimes P) \tilde{u} = 0 \quad \text{for } P \in I.
\]

We have thus proved \( \text{Ann } v_{\lambda} \boxtimes u \subset \text{Ann } \tilde{u} \) and hence the natural \( D \)-module homomorphism

\[
\phi : \mathcal{O}_{\lambda} \boxtimes M \longrightarrow \alpha^* M.
\]

By computation similar to the above, for the filtration \( F_m M = (F_mD_X )u \ (m \geq 0) \),

\[
\phi_m : \mathcal{O}_X \boxtimes F_m M \longrightarrow \alpha^* F_m M
\]

is surjective. Again restricting \( \phi_m \) on the slice \( g \times X \),

\[
\phi_m|_{g \times X} : \mathbb{C} \boxtimes F_m M \longrightarrow (\alpha^* F_m M)_{g \times X} \longrightarrow F_m M.
\]

Thus again by the Nakayama lemma, \( \phi_m \) is an isomorphism and hence so is \( \phi \).

q.e.d.

3. The Harish-Chandra system for characters
Let $G$ be a connected reductive algebraic group over $\mathbb{C}$ and $Z$ be the center of the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g} = \text{Lie} G$. Fix an algebra homomorphism $\chi : Z \to \mathbb{C}$.

Consider the $D_G$-module $M_\chi = D_G u$ defined by

$$M_\chi : \begin{cases} (\partial_z - \chi(z)) u = 0 & (z \in Z) \\ (L_\theta + R_\theta) u = 0 & (\theta \in \mathfrak{g}) \end{cases}$$

where $\partial_z$ is the two-sided invariant differential operator on $G$ corresponding to $z \in Z$ and $L_\theta$ (resp. $R_\theta$) is the right (resp. left) invariant vector field corresponding to $\theta \in \mathfrak{g}$. (Sorry! The symbols $L, R$ may be converse to the usual ones. They follow the earlier notation $L_\theta = L_{\theta, G}$, $R_\theta = R_{\theta, G}$.) The vector field $L_\theta + R_\theta$ corresponds to the vector field arising from the inner action of $G$ on itself.

Since $Z$ is the subalgebra of $D_G(G)$ invariant under the two-sides actions of $G$ and $L_\theta + R_\theta$ corresponds to the earlier $L_\theta$ on $G$ by the inner action, $M_\chi$ is (0-twistedly) $G$-equivariant under the inner action.

A distribution character of an irreducible admissible representation of a real form $G_\mathbb{R}$ of $G$ is a solution to $M_\chi$ for some $\chi$ (infinitesimal character). In order to analyze the behaviors of characters, Harish-Chandra extensively investigated this system of partial differential equations [10] and we call it the Harish-Chandra system.

We shall see $M_\chi$ is a holonomic $D_G$-module. Since $G$ is an affine variety, the $D_G$-module $M_\chi$ is the localization of the module of global sections $M_\chi(G) = \Gamma(G, M_\chi)$.

Put $D(G) = D_G(G)$ and

$$I_\chi = \sum_{z \in Z} D(G)(\partial_z - \chi(z)) + \sum_{\theta \in \mathfrak{g}} D(G)(L_\theta + R_\theta).$$

Then

$$M_\chi(G) = D(G)/I_\chi.$$ 

The filtration $F$ of $M_\chi(G)$ arising from the order filtration of $D(G)$ is good and

$$\text{gr}^F M_\chi(G) = \text{gr} D(G)/\text{gr} I_\chi$$

as is seen in I, 3.

Now let $\mathfrak{g}$ be identified with the left invariant vector fields ($\theta \leftrightarrow R_\theta$). Then under this identification the cotangent bundle $T^*G$ is trivialized as $G \times \mathfrak{g}^*$ and

$$\text{gr} D(G) \simeq \mathbb{C}[G] \otimes_\mathbb{C} \mathbb{C}[\mathfrak{g}^*].$$

On the other hand, by the Poincaré-Birkhoff-Witt theorem, we have

$$\text{gr} U(\mathfrak{g}) \simeq S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$$

where $S(\mathfrak{g})$ is the symmetric algebra over $\mathfrak{g}$. Considering the $G$-action on $U(\mathfrak{g})$ and $S(\mathfrak{g})$ arising from the adjoint action, since the center $Z$ is the subalgebra of $G$-invariants in $U(\mathfrak{g})$, we have

$$\text{gr} Z \simeq S(\mathfrak{g})^G \simeq \mathbb{C}[\mathfrak{g}^*]^G$$
as the subalgebras of the above three algebras (complete reducibility of $G$). Hence the symbols of $\partial_z - \chi(z)$ correspond to the subset $\mathbb{C}[\mathfrak{g}^*]^G_+$ where $\mathbb{C}[\mathfrak{g}^*]^G_+ = \mathbb{C}[\mathfrak{g}^*]^G \cap \mathbb{C}[\mathfrak{g}^*]$ in $\mathbb{C}[\mathfrak{g}^*]$.

On the other hand, the symbol of a vector field $L_\theta + R_\theta$ at $(x, \xi) \in G \times \mathfrak{g}^*$ is written as

$$\sigma_1 (L_\theta + R_\theta)(x, \xi) = \langle -\text{Ad}(x)\theta + \theta, \xi \rangle$$

where $\langle , \rangle$ denotes the natural pairing on $\mathfrak{g}$ and $\mathfrak{g}^*$. Thus the ideal generated by $\mathfrak{g}^*_{rs} = \mathfrak{g}^*$ and $\xi - x\xi$ on $G \times \mathfrak{g}^*$ $(x\xi = \text{Ad}(x)\xi)$. Hence for the characteristic variety $\text{ch} \ M_\chi$ we have the inclusion relation

$$\text{ch} \ M_\chi \subset X = \{(x, \xi) \in G \times \mathfrak{g}^* \mid P(\xi) = 0 \ (P \in C[\mathfrak{g}^*]^G_+), \ x\xi = \xi \}.$$

But then by Kostant’s theorem [18], the ideal generated by $\mathbb{C}[\mathfrak{g}^*]^G_+$ in $\mathbb{C}[\mathfrak{g}^*]$ is the defining ideal of the nilpotent variety $N$ in $\mathfrak{g}^*$ (the set of nilpotent elements under the identification $\mathfrak{g} \simeq \mathfrak{g}^*$ by a non-degenerate invariant bilinear form). Thus we have

$$X = \{(x, \xi) \in G \times N \mid \xi = x\xi \}.$$

We now see an irreducible component of $X$ is of dimension equal to $\dim G$. By Dynkin-Kostant, the nilpotent variety $N$ splits into finitely many $G$-orbits under the coadjoint action:

$$N = \prod_{i=1}^r O_G(\xi_i).$$

If $q : X \to N$ is the projection onto the second factor, then

$$q^{-1}O_G(\xi_i) = \{(x, \xi) \in G \times O_G(\xi_i) \mid x \in Z_G(\xi_i)\}$$

is a fiber bundle over $O_G(\xi_i)$ with standard fiber $Z_G(\xi_i) = \{x \in G \mid x\xi_i = \xi_i\}$. Since $\dim O_G(\xi_i) = \dim G - \dim Z_G(\xi_i)$, $\dim q^{-1}O_G(\xi_i) = \dim G$ and an irreducible component of $X$ is one of the closures of $q^{-1}O_G(\xi_i)$.

Thus we have seen $\dim \text{ch} \ M_\chi \leq \dim G$ and hence $M_\chi$ is holonomic.

More precisely, the following is known:

**Theorem.**

$$\text{ch} \ M_\chi = X.$$

Furthermore the characteristic cycle of $M_\chi$ (for definition, see [11, II]) is given by the intersection cycles of $V$ and $G \times N$ where $V = \{(x, \xi) \in G \times \mathfrak{g}^* \mid x\xi = \xi \}$ is the commuting variety.

**Proof.** See [14] for a Lie algebra version and [17] for a group version.

Let $G_{rs}$ be the set of regular semisimple elements $s \in G$, i.e., semisimple and the centralizer $Z_G(s)$ of $s$ in $G$ has the dimension equal to rank $G$. Then for $s \in G_{rs}$, if $\xi \in N$ and $s\xi = \xi$, then $\xi = 0$. Thus $\text{ch} \ M_\chi|_{G_{rs}} \subset X \cap (G_{rs} \times \mathfrak{g}^*) = G_{rs} \times 0 = T^*_{G_{rs}}(G_{rs})$ and hence by I, 5.2, $M_\chi|_{G_{rs}}$ is a connection. There is a well-known formula by Harish-Chandra describing this connection on $G_{rs}$. We shall introduce this formula in the $D$-module language.
Fix a maximal torus $T$ in $G$ and let $T_{\text{reg}} = T \cap G_{rs}$ the set of regular elements in $T$. Put $G_{rs} = G/T \times T_{\text{reg}}$ and define the map

$$p : \widetilde{G}_{rs} \rightarrow G_{rs}$$

by $p(gT, t) = gTg^{-1}$. Let $W = N_G(T)/T$ be the Weyl group for $T$. Then $W$ acts on $G_{rs}$ by $(gT, t) \rightarrow (g\hat{w}T, w^{-1}tw)$ ($w = \hat{w}T \in N_G(T)$) and $p$ is a Galois covering with group $W$ ($\widetilde{G}_{rs}/W \simeq G_{rs}$).

We shall describe the inverse image $p^*(M_{\chi}|_{G_{rs}})$ of the connection $M_{\chi}|_{G_{rs}}$. Let $R_+$ be the set of positive roots and put the difference

$$\Delta = \prod_{\alpha \in R_+} (t^{\alpha/2} - t^{-\alpha/2}) \quad (t \in T).$$

($\Delta$ is possibly a two-valued function on $T$ and $T_{\text{reg}} = \{t|\Delta(t) \neq 0\}$.) Let $t = \text{Lie}T$ and $S(t)$ the symmetric algebra over $t$ which is identified both with invariant differential operators on $T$ and with the polynomial algebra on $t^*$. Denote by $S(t)^W$ the subalgebra of $W$-invariants in $S(t)$.

For $\lambda \in t^*$, consider the $D_{T_{\text{reg}}}$-module $M_{\lambda}^\text{rad}$ defined by

$$\Delta^{-1}(P(\partial) - P(\lambda + \rho))\Delta v = 0 \quad (P \in S(t)^W).$$

where $\rho \in t^*$ is the half sum of positive roots in $R_+$. ($P(\partial)$ is considered as an invariant differential operator and $P(\lambda + \rho)$ is the value of the polynomial $P \in \mathbb{C}[t^*]$ at $\lambda + \rho \in t^*$.) It is easily seen that $M_{\lambda}^\text{rad}$ is a connection on $T_{\text{reg}}$.

By the standard theory of Chevalley and Harish-Chandra, the set of algebra homomorphisms $\text{Hom}_{\mathbb{C}-\text{alg}}(Z, \mathbb{C})$ is identified with the $W$-quotient $t^*/W$ where the $W$-action on $t^*$ is defined by

$$w.\lambda = w(\lambda + \rho) - \rho \quad (w \in W, \lambda \in t^*).$$

Hence for an infinitesimal character $\chi \in \text{Hom}_{\mathbb{C}-\text{alg}}(Z, \mathbb{C})$, there exists $\lambda \in t^*$ uniquely up to the $W$-action.

**Theorem (Harish-Chandra).** Let $\chi_{\lambda}$ be the infinitesimal character corresponding to $\lambda \in t^*$ as above. Then we have a $D$-module isomorphism on $G \times T_{\text{reg}}$

$$p^*(M_{\chi_{\lambda}}|_{G_{rs}}) \simeq \mathcal{O}_{G/T} \boxtimes M_{\lambda}^\text{rad}.$$

It is rather easy to see that $M_{\lambda}^\text{rad}$ and hence $M_{\chi_{\lambda}}|_{G_{rs}}$ is a regular connection in the sense of Deligne (I, 5.2). In the $D$-module theory, there exists a unique minimal extension $(M_{\chi}|_{G_{rs}})^\sim$ of $M_{\chi}|_{G_{rs}}$ on $G$ (related to the intersection cohomology theory of Goresky-MacPherson) and this extension $(M_{\chi}|_{G_{rs}})^\sim$ is a regular $D_G$-module. Finally, we close this section by citing the following:

**Theorem ([14], [17]).**

$$M_{\chi} \simeq (M_{\chi}|_{G_{rs}})^\sim.$$

In particular, $M_{\chi}$ is a regular $D_G$-module.

For applications of these considerations to the representation theory, we refer to [12], [13], [14], [15].
4. \(D\)-modules defined by a linear action and its orbits

Let \(G\) be an algebraic group acting linearly on a vector space \(V\). The dual space \(V^*\) of \(V\) is then acted by \(G\) through the contragredient action. Fix a special reference point \(\xi \in V^*\) and let \(I_\xi = I(O_G(\xi)) \subset \mathbb{C}[V^*]\) be the ideal consisting of functions vanishing on \(O_G(\xi)\), the \(G\)-orbit of \(\xi\). \((I_\xi\) is the defining ideal of the orbit closure \(\overline{O_G(\xi)}\)). Under the identification \(\mathbb{C}[V^*] \cong S(V)\), \(I_\xi\) is a \(G\)-stable ideal in the algebra \(S(V)\) of linear differential operators with constant coefficients. Hence if we fix a Lie algebra character \(\lambda : g \rightarrow \mathbb{C}\), the following defines a \(\lambda\)-twistedly \(G\)-equivariant \(D_V\)-module \(M_{\lambda, \xi}\):

\[
M_{\lambda, \xi} : \begin{cases} 
(L_\theta - \lambda(\theta)) u = 0 & (\theta \in g) \\
\partial u = 0 & (\partial \in I_\xi),
\end{cases}
\]

**Example 1.** Let \(V\) be a prehomogeneous vector space under group \(G\) (i.e., \(V\) has a dense \(G\)-orbit). Let \(\chi : G \rightarrow \mathbb{C}^\times\) be an algebraic group homomorphism and choose \(\lambda \in \mathbb{C}\). For a Lie algebra homomorphism \(\lambda d\chi : g \rightarrow \mathbb{C}\), \(M_{\lambda} = D_V u\) defined by

\[
(L_\theta - \lambda d\chi(\theta)) u = 0 \quad (\theta \in g)
\]

is \(\lambda d\chi\)-twistedly \(G\)-equivariant. In many cases, \(M_{\lambda}\) is known to be regular holonomic and if \(f\) is a relative invariant for \(\chi\) \((f(gx) = \chi(g)f(x))\), then \(f^\lambda\) is a solution to \(M_{\lambda}\). The \(D_V\)-module \(M_{\lambda}\) is important for the study of the \(b\)-function of \(f\). For recent results in this field, refer to works of M. Muro and A. Gyoja ([21], [9]).

**Example 2.** The Lie algebra version of the Harish-Chandra system for invariant eigendistributions introduced in the previous section is simply an example of this kind. Let \(G\) be a connected reductive algebraic group over \(\mathbb{C}\) and \(g\) its Lie algebra as in the previous section 3. Let \(\xi\) be an regular element in \(g^*\) (which means that the centralizer \(Z_G(\xi)\) has the dimension equal to the rank of \(g = \dim T\)). Consider the \((0\)-twistedly\) \(G\)-equivariant \(D_g\)-module \(M_\xi = M_{0, \xi}\) corresponding to the regular orbit \(O_G(\xi)\) on the Lie algebra \(g\).

Now by [18], it is known that \(I_\xi = I(\overline{O_G(\xi)})\) is generated by \(P - P(\xi)\) \((P \in \mathbb{C}[g^*]^G)\). Hence \(M_\xi\) is defined by

\[
M_\xi : \begin{cases} 
(P(\partial) - P(\xi)) u = 0 & (P(\partial) \in S(g)^G) \\
L_\theta u = 0 & (\theta \in g),
\end{cases}
\]

where we apply the identification

\[
S(g)^G \cong \mathbb{C}[g^*]^G.
\]

The above \(D_g\)-module \(M_\xi\) is of course defined for arbitrary \(\xi \in g^*\) but even if \(\xi\) is not regular, it gives the same \(D\)-module by the following reason. If \(\xi = \xi_s + \xi_n\) is the Jordan decomposition of the element \(\xi\), then \(P(\xi) = P(\xi_s)\) for any \(P \in \mathbb{C}[g^*]^G\).
Hence if \( \xi_{\text{reg}} \) is a regular element such that the semisimple part of \( \xi_{\text{reg}} \) coincides with \( \xi_s \) (such regular elements are unique up to the \( G \)-action), then \( M_\xi = M_{\xi_{\text{reg}}} \).

This \( D_\mathfrak{g} \)-module is investigated in details in [14]. As in 3, it is not difficult to see that this is holonomic but in contrast with the group case in 3, \( M_\xi \) is regular (in the algebraic sense) only for a nilpotent \( \xi \), which also follows from the theorem in the next section.

**Example 3.** Let \( G = T \) be an algebraic torus \( (\cong \mathbb{C}^n) \) acting linearly on a vector space \( V \). Assume the action is faithful and \( T \) contains all homotheties on \( V \).

Consider the contragredient action of \( T \) on \( V^* \) and take a full diagonal subgroup \( D \subset \text{GL}(V^*) \) such that \( T \hookrightarrow D \). Take a reference point \( \mathbf{i} \in V^* \) such that its \( D \)-orbit \( D.\mathbf{i} = \text{O}_{D}(\mathbf{i}) \) is open in \( V^* \) \( (D \cong \mathbb{C}^N \text{ (dim } V = N)) \). Then for \( \lambda \in \mathfrak{t}^* \) \( (\mathfrak{t} = \text{Lie } T \cong \mathbb{C}^n) \),

\[
M_\lambda : \begin{cases} 
(L_\theta - \lambda(\theta)) u = 0 & (\theta \in \mathfrak{t}) \\
P u = 0 & (P \in I(\mathcal{O}_T(\mathbf{i})))
\end{cases}
\]
gives a \( \lambda \)-twistedly \( T \)-equivariant \( \mathcal{D}_V \)-module.

Gelfand [7] calls \( M_\lambda \) a generalized hypergeometric system of linear partial differential equations. In the final section, we shall look at this \( D \)-module in more details. In particular, this turns out to be a regular holonomic \( D_V \)-module.

Gelfand and his other collaborators also consider certain equivariant holonomic \( D \)-modules which arise from unipotent group actions and call them generalized Airy equations (irregular at \( \infty \)) [6].

5. Some regularities

In this section, we prove the regularity of a certain equivariant \( D \)-module, which is suggested by a conversation with M. Kashiwara whom we sincerely thank. Here we assume some standard knowledge of algebraic \( D \)-modules as in [1], [2], [11].

Let \( G \) be a connected algebraic group acting on a smooth algebraic variety \( X \). Let \( \lambda : \mathfrak{g} \rightarrow \mathbb{C} \) be a Lie algebra character and \( \mathcal{O}_\lambda \) the rank one connection on \( G \) defined by \( \lambda \) in 2. In case \( \lambda = 0 \), the following theorem is in [11, VII, 12.11].

**Theorem.** Let \( M \) be a \( \lambda \)-twistedly \( G \)-equivariant coherent \( \mathcal{D}_X \)-module whose support \( \text{Supp } M \) (support in the sheaf theory) consists of finitely many \( G \)-orbits in \( X \). If the twisting datum \( \mathcal{O}_\lambda \) is a regular connection on \( G \), then \( M \) is regular holonomic.

**Proof.** We use the induction on the number of \( G \)-orbits in \( \text{Supp } M \). First we see the case that the orbit number is 1. Let

\[
\begin{array}{c}
G \xrightarrow{\pi} \text{Supp } M = \mathcal{O}_{G}(\mathbf{o}) \xrightarrow{i} X
\end{array}
\]

(\( \mathbf{o} \in \text{Supp } M \), \( i \) is a closed immersion). Since by the Kashiwara lemma \( M \cong \int i^! M \) \( (i^! = \mathcal{H}^0 i! \) is the twisted inverse functor) and \( i^! M \) is again \( G \)-equivariant, it is enough to show that \( i^! M \) is regular holonomic. Thus we may assume that \( \text{Supp } M = X = \mathcal{O}_{G}(\mathbf{o}) \). Considering the maps

\[
\begin{array}{c}
G \xrightarrow{i} G \times \mathbb{C} \xrightarrow{1 \times p} G \times X \xrightarrow{\alpha} X,
\end{array}
\]
we have
\[ \pi = \alpha \circ (1 \times p) \circ \iota \]
where \( p : o \hookrightarrow X \) is the inclusion map. Hence
\[
\begin{align*}
\pi^* M & \simeq \iota^*(1 \times p)^* \alpha^* M \\
& \simeq \iota^*(1 \times p)^* (\mathcal{O}_\lambda \boxtimes M) \\
& \simeq \mathcal{O}_\lambda \boxtimes p^* M.
\end{align*}
\]
Since \( M \) is \( D \)-coherent, \( p^* M \) is a finite dimensional vector space over \( \mathbb{C} \). Therefore \( \pi^* M \) is a finite sum of a regular connection \( \mathcal{O}_\lambda \). Since \( \pi \) is a smooth map, \( M \) is thus regular holonomic.

Secondly, we are going into a general case. Take a \( G \)-stable open set \( U \) in \( X \) such that \( U \cap \text{Supp} \, M \) is the union of all maximal dimensional \( G \)-orbits in \( \text{Supp} \, M \). Let
\[
U \xrightarrow{j} X \xleftarrow{i} Y = X \setminus U
\]
be the corresponding open and closed immersions. Then we have the exact triangle
\[
R\Gamma_Y(M) \longrightarrow M \longrightarrow j_* j^* M \xrightarrow{+1} R\Gamma_Y(M)
\]
in the derived category of coherent \( D \)-modules. Note that all cohomology \( D \)-modules of the complexes \( R\Gamma_Y(M) \) and \( j_* j^* M \) are again \( \lambda \)-twistedly \( G \)-equivariant. \( \text{Supp} \, j^* M \) is a disjoint union of \( G \)-orbits of the same dimensions and hence by the case of orbit number = 1, \( j^* M \) is regular holonomic. Thus \( j_* j^* M \) is a regular holonomic complex of \( D \)-modules. On the other hand, the number of \( G \)-orbits in \( \text{Supp} \, R\Gamma_Y(M) \) is less than that of \( \text{Supp} \, M \) and hence by the induction \( R\Gamma_Y(M) \) is regular holonomic. Thus the remaining vertex \( M \) in the exact triangle is regular holonomic. q.e.d.

Now we consider the case discussed in Section 4. Let \( G \) act on a vector space \( V \) and take \( \zeta \in V^* \) and a Lie algebra character \( \lambda \). Denote by \( M_{\lambda,\zeta} \) the \( D(V) = D_V(V)(\text{Weyl algebra}) \)-module \( D(V) \, u \) defined by
\[
\begin{cases}
(L_\theta - \lambda(\theta)) u = 0 & (\theta \in \mathfrak{g}) \\
P(\theta) u = 0 & (P(\theta) \in I_\zeta)
\end{cases}
\]
for \( I_\zeta = I(\mathcal{O}_G(\zeta)) \).

In general, for a Weyl algebra module \( M \) on \( V \), let \( \hat{M} \) be its Fourier transform. That is, \( \hat{M} \) is a \( D(V^*) \)-module on the dual space \( V^* \) defined by the substitutions \( x_i \mapsto -\partial_{\xi_i}, \partial_{x_i} \mapsto \xi_i \) for the dual coordinate systems \( (x_i) \) in \( V \) and \( (\xi_i) \) in \( V^* \). The Fourier transform \( \hat{M}_{\lambda,\zeta} \) is then defined by
\[
\begin{cases}
(\hat{L}_\theta - \lambda(\theta)) \hat{u} = 0 & (\theta \in \mathfrak{g}) \\
\hat{P}(\xi) \hat{u} = 0 & (P \in I_\zeta)
\end{cases}
\]
where \( \hat{L}_\theta \) is the operator after the above Fourier substitutions and \( \hat{u} \) is the generator of \( \hat{M}_{\lambda,\zeta} \).
A Weyl algebra module $M$ is said to be homogeneous if the action of the Euler operator $\sum x_i \partial x_i$ is locally nilpotent on $M$. If $M = D(V)/I$ and if the ideal $I$ is generated by homogeneous elements in the gradation of $D(V)$ by $\deg \partial x_i = 1$ and $\deg x_i = -1$, then $M$ is homogeneous in this sense.

It is known that a homogeneous Weyl algebra module $M$ is regular holonomic if and only if its Fourier transform $\hat{M}$ is regular holonomic (see for example [3, Th.7.4], [19]).

In the above examples, if $L_\theta$ has homogeneous degree 0 and the ideal $I_\zeta$ is homogeneous ($\iff O_G(\zeta)$ is conic), then $M_{\lambda,\zeta}$ (and $\hat{M}_{\lambda,\zeta}$) is homogeneous. Thus we have the following as a corollary of Theorem.

**Corollary.** Assume that $O_\lambda$ is a regular connection (this is the case when $G$ is reductive). If $M_{\lambda,\zeta}$ is homogeneous and $O_G(\zeta)$ consists of finitely many $G$-orbits, then $M_{\lambda,\zeta}$ is a regular holonomic $D_V$-module.

**Examples.** In Example 2 in Section 4, let $\xi$ be regular nilpotent. Then $O_G(\xi)$ is the nilpotent variety $N$ which is conic and consists of finitely many orbits. Thus $M_\xi = M_0$ is regular holonomic.

Example 3 (generalized hypergeometric systems) is also in this category as will be seen in the next section.

### 6. The Gelfand generalized hypergeometric systems

#### 6.1 Description.

In this section, we shall look into more details of the equations introduced in Example 3 in Section 4. Let $T = C^\times n$ be an $n$-dimensional algebraic torus acting linearly on an $N$-dimensional vector space $V$. Choose a coordinate system $(z_j)$ in $V$ such that the $T$-action is diagonalized and denote by $\chi = (\chi_{ij})$ the integral $(n \times N)$ matrix expressing the linear representation

$$\chi : T = C^\times n \rightarrow C^\times N,$$

i.e.,

$$t.z_j = \prod_{i=1}^n t_i^{\chi_{ij}} z_j \quad (t = (t_1, \cdots, t_n) \in T, \, 1 \leq j \leq N).$$

(We denoted by $\chi$ also the homomorphism given by $\chi_j$.) We assume that $\chi$ is injective ($\iff \text{rank } \chi = n$) and that the image $\chi(T)$ contains all homotheties, which means that there exists an integral vector $c = (c_1, \cdots, c_n) \in \mathbb{Z}^n$ such that

$$c\chi = (1, \cdots, 1) \quad \text{(all entries are 1)} \quad (\star)$$

(homogeneity condition).

In the Lie algebra $t = \text{Lie } T$, we choose the basis $t_i \partial_{t_i}$ ($1 \leq i \leq n$). Then the action of $t_i \partial_{t_i}$ on $V = C^N$ is given by

$$\sum_{j=1}^N \chi_{ij} z_j \partial_{z_j} \quad \text{for } 1 \leq i \leq n.$$
Thus if we fix a Lie algebra character $\lambda \in t^* \cong \mathbb{C}^n$ ($\lambda(t_i \partial_{t_i}) = \lambda_i \in \mathbb{C}$ ($1 \leq i \leq n$)), then the equations $(L_\theta - \lambda(\theta)) u = 0$ ($\theta \in t$) correspond to

$$
\sum_{j=1}^N \chi_{ij} z_j \partial_{z_j} - \lambda_i u = 0 \quad (1 \leq i \leq n).
$$

Secondly in the dual space $V^* \cong \mathbb{C}^N$ (by the dual coordinate system $(\xi_j)$ to $(z_j)$), fix a reference point

$$\mathbf{1} = (1, \cdots, 1) \in \mathbb{C}^N \quad \text{(all entries are 1)}.
$$

The $T$-orbit of $\mathbf{1}$ is given by

$$O_T(\mathbf{1}) = \{(t^{x_1}, \cdots, t^{x_N}) \mid t \in T\}
$$

where $t^{x_j} = \prod_{i=1}^N t_i^{x_{ij}}$ ($1 \leq j \leq N$). Define the sublattice $L$ in $\mathbb{Z}^N$ by

$$L = \text{Ker} \chi = \{a = (a_1, \cdots, a_N) \in \mathbb{Z}^N \mid \sum_{j=1}^N \chi_{ij} a_j = 0 \ (1 \leq i \leq n)\}.
$$

Then we have

$$O_T(\mathbf{1}) = \{\xi = (\xi_j) \in \mathbb{C}^X \mid \xi^a = 1 \ (a \in L)\}
$$

where $\xi^a = \prod_{j=1}^N \xi_{ij}^{a_j}$. Thus the defining ideal $I(O_T(\mathbf{1}))$ in $\mathbb{C}[\xi_1, \cdots, \xi_N]$ is generated by

$$\bigwedge_a = \prod_{a_j > 0} \xi_{ij}^{a_j} - \prod_{a_j < 0} \xi_{ij}^{a_j} \quad (a \in L).
$$

Hence the generalized hypergeometric system $M_\lambda$ ($\lambda \in \mathbb{C}^N$) in Example 3 in 4 is realized by the following system of linear partial differential equations:

$$M_\lambda : \begin{cases}
(\theta_i - \lambda_i) u = 0 & (1 \leq i \leq n) \\
\square_a u = 0 & (a \in L = \text{Ker} \chi)
\end{cases}
$$

where $\theta_i = \sum_{j=1}^N \chi_{ij} z_j \partial_{z_j}$ and $\square_a = \prod_{a_j > 0} \partial_{z_j}^{a_j} - \prod_{a_j < 0} \partial_{z_j}^{a_j}$ for $a \in L$.

Note that the homogeneity condition ($\blacklozenge$) implies $\sum_{j=1}^N a_j = 0$ for $a \in L$ and hence the operator $\square_a$ is of homogeneous degree $\sum_{a_j > 0} a_j = \sum_{a_j < 0} |a_j|$. Also the vector fields $\theta_i$ is of homogeneous degree 0 by $\deg z_j = -1$, $\deg \partial_{z_j} = 1$. Hence $M_\lambda$ is a homogeneous Weyl algebra module in the sense of Section 5.

6.2 Characteristic variety.

Let $D = D(V) = \mathbb{C}[z_1, \cdots, z_N, \partial_{z_1}, \cdots, \partial_{z_N}]$ be the Weyl algebra over $\mathbb{C}^N$ and write $M_\lambda = D \cdot u$, the global section of the $M_\lambda$ in 6.1. Then $I_\lambda = \text{Ann}_D u$ is generated by $\theta_i - \lambda_i$ ($1 \leq i \leq n$) and $\square_a$ ($a \in L$). Taking the gradation by the good filtration $F$ on $M_\lambda$ given by the order filtration of $D$, we have

$$\text{gr}^F M_\lambda = \text{gr} D / \text{gr} I_\lambda$$
where \( \text{gr} \, D = \mathbb{C}[z_1, \cdots, z_N, \xi_1, \cdots, \xi_N] = \mathbb{C}[z, \xi] \). Since \( \text{gr} \, I_\lambda \) contains

\[
\bar{\theta}_i = \sum_{j=1}^{N} \chi_{ij}z_j \xi_j \quad (1 \leq i \leq n),
\]

\[
[\xi] = \prod_{a_j > 0} \xi_j^{a_j} - \prod_{a_j < 0} \xi_j^{[a_j]} \quad (a \in L),
\]

there exists a surjective homomorphism

\[ A \twoheadrightarrow \text{gr}^F M_\lambda \]

where \( A = \mathbb{C}[z, \xi]/(\bar{\theta}_i, [\xi] \mid 1 \leq i \leq n, a \in L) \). Hence for the characteristic variety of \( M_\lambda \)

\[ \text{ch} M_\lambda \subset \text{Supp} \, A = \{ (z, \xi) \in \mathbb{C}^{2N} \mid \bar{\theta}_i = 0, [\xi] = 0 \ (1 \leq i \leq n, a \in L) \}. \]

So far, we have regarded \( \mathbb{C}^{2N} \) as the cotangent bundle \( T^*(V) \) of \( V \) (\( (\xi_j) \) is the fiber coordinate), but this same space can also be regarded as the cotangent bundle \( T^*V^* \) of the dual vector space \( V^* \) (\( (z_j) \) is in turn the fiber coordinate). Now the orbit closure

\[ \overline{O_T}(1) = \{ \xi \in V^* \mid [\xi] = 0 \ (a \in L) \} \]

in \( V^* \) splits into finitely many \( T \)-orbits:

\[ \overline{O_T}(1) = \bigcap_{k=1}^{r} O_T(p_k). \]

Then the equations \( \bar{\theta}_i = 0 \) gives the conormal condition at \( \xi \in O_T(p_k) \) and hence we have the following.

**Lemma.**

\[ \text{Supp} \, A = \bigcap_{k=1}^{r} T^*_{O_T(p_k)}(V^*) \]

where \( T^*_X(V^*) \) denotes the conormal bundle of \( X \subset V^* \). In particular, every irreducible component of \( \text{ch} M_\lambda \) is the closure of the conormal bundle of a \( T \)-orbit \( O_T(p_k) \) and hence \( N \)-dimensional.

We thus have:

**Theorem.** \( M_\lambda \) is a regular holonomic \( D_V \)-module.

**Proof.** The regularity follows from Corollary in 4, since \( M_\lambda \) is homogeneous and \( \overline{O_T}(1) \) consists of finitely many \( T \)-orbits.

**Remark.** In [7], the authors claim the isomorphism \( A \simeq \text{gr}^F M_\lambda \). Then the characteristic cycle of \( M_\lambda \) coincides with that of the commutative algebra \( A \), which has a nice description coming from the combinatorics of polytopes defined by the integral column vectors in the matrix \( \chi \).

We understand this isomorphism only in case that the orbit closure \( \overline{O_T}(1) \) is a normal variety. Most important examples seem to satisfy this normality condition.
In particular, Mutsumi Saito has checked this normality condition for all systems arising from the symmetric pairs [22].

6.3 Classical cases.

Kinds of classical hypergeometric functions are defined on the quotient space (or its compactification) by the torus action, instead on $\mathbb{C}^N$. We shall look at this situation on the generic part $\mathbb{C}^\times \rightarrow \mathbb{C}^\times N$. The torus action defines the injective homomorphism of tori $\chi : T = \mathbb{C}^\times n \rightarrow \mathbb{C}^\times N$. Let

$$
eq \mathbb{C}^\times n \xrightarrow{\chi} \mathbb{C}^\times N \xrightarrow{\pi} \mathbb{C}^\times l \rightarrow e$$

be the exact sequence of tori ($\mathbb{C}^\times l$ is the quotient and $l = N - n$). The quotient map $\pi$ is also given by an integral ($N \times l$) matrix (denoted also by the same symbol $\pi$) if a coordinate system on $\mathbb{C}^\times l$ is fixed. We want a system of differential equations on $\mathbb{C}^\times l$ whose inverse image on $\mathbb{C}^\times N$ is related to Gelfand’s $M_\lambda|\mathbb{C}^\times n$.

We take a heuristic viewpoint. For a function $v$ on $\mathbb{C}^\times l$, set the function $u$ on $\mathbb{C}^\times N$ as $u(z) = z^\Lambda v(\pi(z))$ for some $\Lambda = (\Lambda_1, \cdots, \Lambda_N) \in \mathbb{C}^N$ where $z^\Lambda = \prod_{j=1}^N z_j^{\Lambda_j}$. Choose a coordinate system $x = (x_k)$ in $\mathbb{C}^\times l$ (1 $\leq k \leq l$) and let

$$x_k = \pi(z)_k = \prod_{j=1}^N z_j^{\pi_{jk}} \quad (1 \leq k \leq l)$$

(here $\pi = (\pi_{jk})$ is an integral $N \times l$ matrix). Put $\vartheta_{z_j} = z_j \partial_{z_j}$ and $\vartheta_{x_k} = x_k \partial_{x_k}$. Then

$$\partial_{z_j} u = z_j^{-1} z^\Lambda \left( \sum_{k=1}^l \pi_{jk} \vartheta_{x_k} + \Lambda_j \right) v \quad (1 \leq j \leq N)$$

and hence

$$\vartheta_{z_j} u = z^\Lambda \left( \sum_{k=1}^l \pi_{jk} \vartheta_{x_k} + \Lambda_j \right) v.$$ 

Thus, if $u$ satisfies

$$\left( \theta_i - \lambda_i \right) u = 0 \quad (1 \leq i \leq n),$$

then $v$ satisfies

$$z^\Lambda \left( \sum_{j,k} \chi_{ij} \pi_{jk} \vartheta_{x_k} + \sum_{j=0}^N \chi_{ij} \Lambda_j - \lambda_i \right) v = 0.$$ 

But then since $\chi \pi = 0$ as a multiplication of matrices (by the definition of $\pi$), in order to get a non-trivial solution for $v$ to the above equations, we have the linear matrix equation for $\Lambda \in \mathbb{C}^N$:

$$\chi \Lambda = \lambda$$

where the matrix $\chi$ and the vector $\lambda \in \mathbb{C}^n$ are given.

In order to examine the second equations $\Box_a u = 0$, we compute the following iterated differentiations:

$$\partial_{z_j}^m u = z_j^{-m} z^\Lambda (D_j + \Lambda_j - m + 1)_m v.$$
where $D_j = \sum_{k=1}^{l} \pi_{jk} \theta_{x_k}$ and

$$(T)_m = T(T+1)\cdots(T+m-1) \quad (m \geq 1, \; T: \text{indeterminate}).$$

Hence $\Box_a u = 0$ corresponds to

$$\{ \prod_{a_j > 0} (D_j + \Lambda_j - a_j + 1) a_j - z^a \prod_{a_j < 0} (D_j + \Lambda_j + a_j + 1) |a_j| \} v = 0 \quad (a \in L) \quad (\sharp).$$

Since $a \in L = \text{Ker} \chi = \text{Im} \pi$, if $\pi_k \in \mathbb{Z}^N$ is the $k$-th column vector of the $N \times l$ matrix $\pi$ ($1 \leq k \leq l$), then

$$a = \sum_{k=1}^{l} m_k \pi_k \quad \text{for some } m_k \in \mathbb{Z}.$$

Hence

$$z^a = \prod_{j=1}^{N} z_j^{a_j} = \prod_{k=1}^{l} x_k^{m_k} \quad (x_k = z^{\pi_k})$$

and the equations (\sharp) are defined on $\mathbb{C}^l$. We thus have the following:

**Proposition.** For $\lambda \in \mathbb{C}^n$, choose $\Lambda \in \mathbb{C}^N$ such that $\chi \Lambda = \lambda$. Let $N_\Lambda = D_{\ell \times l} v$ on $\mathbb{C}^l$ defined by the equations (\sharp) for $a \in L$. Then the $D$-module $O_\Lambda \otimes \pi^* N_\Lambda$ on $\mathbb{C}^l \times \mathbb{C}^N$ is a quotient $D$-module of $M_\lambda |_{\mathbb{C}^l \times \mathbb{C}^N}$. ($O_\Lambda = D_{\mathbb{C}^N} z^\Lambda$).

**Remark.** Assume $l = N - n = 1$ and let $\pi = (\pi_1, \cdots, \pi_N) \in \mathbb{Z}^N$. Then the equations (\sharp) are reduced to a single equation for this $\pi$:

$$\{ \prod_{\pi_j > 0} (\pi_j \theta_x + \Lambda_j - \pi_j + 1) \pi_j - x \prod_{\pi_j < 0} (\pi_j \theta_x + \Lambda_j + \pi_j + 1) |\pi_j| \} v = 0$$

This ordinary differential equation is of Fuchsian type (regular in our language) and has a solution of the “classical generalized” hypergeometric function of one variable $pF_{p-1}$ ($p = \sum_{\pi_j > 0} \pi_j$) for suitable parameters expressed by $\pi$ and $\Lambda$.

For the further references in several variables ($l > 1$), see [7], [8].
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