Fields topology and perturbation theory

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Abstract

The fields nonlinear modes quantization scheme is discussed. New form of the perturbation theory achieved by unitary mapping the quantum dynamics in the space $W_G$ of (action, angle)-type collective variables. It is shown why the transformed perturbation theory contributions may accumulated exactly on the boundary $\partial W_G$. Abilities of the developed formalism are illustrated by examples from quantum mechanics and field theory.

- Introduction

One may use for my talk another titles. For instance:
- Quantization of nonlinear modes (waves)
- Strong-coupling perturbation theory
- Examples of unitary transformation of path-integral variables
- Quantum theory and symplectic geometry
- Path-Integral solution of H-atom Problem
- Particles Creation in the Integrable Systems
- Symmetry breaking in the $O(4,2)$-invariant field theories
- Ghost-free quantization of Yang-Mills fields

But the role of topologies in the perturbation theory structure appears so new for me that I offer concentrate the attention just on this question. (Details of the formalism one can find in hep-th/9811160.)

We will consider the expansion of integral:

$$A(F) = \int Du e^{i S(u)} F(u)$$

in vicinity of real-time path

$$u_c : \frac{\delta S(u_c)}{\delta u_c} \equiv 0.$$ 

This demands knowledge of Green function $G$. But the equation:

$$(\partial_\mu^2 + v''(u_c))_{x} G(x, x'; u_c) = \delta(x - x').$$

is translationally noninvariant if $u_c = u_c(x)$ and therefore has not explicit solution. By this reason this ordinary (WKB) method has not a future.

I would like to demonstrate the strict strong-coupling perturbation theory, $u_c = O(1/g)$. Actually I will follow the idea that the substitution may considerably simplify calculations, helping avoid above problem.

- Unitarity condition
Let us calculate probability to find somewhere a particle with energy $E$

$$\rho_1(E) = \int dx_1 dx_2 A_1(x_1, x_2; E) A_1^*(x_1, x_2; E).$$

using the spectral representation of one-particle amplitude:

$$A_1(x_1, x_2; E) = \sum_n \frac{\Psi_n^*(x_2) \Psi_n(x_1)}{E - E_n + i\varepsilon}, \quad \varepsilon \to +0,$$

All unnecessary contributions with $E \neq E_n$ were canceled in $\rho_1(E)$:

$$\rho_1(E) = \sum_n \left| \frac{1}{E - E_n + i\varepsilon} \right|^2 = \frac{\pi}{\varepsilon} \sum_n \delta(E - E_n), \quad (1)$$

together with real part of propagator $1/(E - E_n + i\varepsilon)$.

I would like to exclude such unnecessary contributions (like $E \neq E_n$). This means inclusion of last equality in $(1)$ into formalism, i.e. transition from $A_1 A_1^*$ to absorption part $\text{Im} A_1$.

Then the unitarity (optical theorem: $2i\varepsilon|A|^2 = A_1 - A_1^*$) becomes sufficient and necessary condition: I will show, it defines the complete set of contributions in the physically acceptable domains.

The formalism based on the statements:

(A) If $A \sim e^{iS(u)}$ and $S(u)$ is the action, if $\rho \sim |A|^2$ is measurable, if the equality $2i\varepsilon|A|^2 = A - A^*$ is taken into account, then:

$$\rho(F) = e^{-iK(e)} \int DM(u,j) e^{-iU(u,e)} \tilde{F}(u,e) \equiv \hat{O}(u) \tilde{F}(u)$$

where expansion over differential operator

$$\hat{K}(e) = \frac{1}{2} \int dx \frac{\delta}{\delta j(x)} \frac{\delta}{\delta e(x)} \equiv \frac{1}{2} \int dx j(x) e(x)$$

generates the perturbation series, functional $U(u,e) = O(e^3)$ describes interactions, the measure $DM$ is Diracian ($\delta$-like):

$$DM = \prod_x du(x) \delta \left( \frac{\delta S(u)}{\delta u(x)} + j(x) \right).$$

At the very end one should take $j = e = 0$.

Note: the variational principle is derivable from above first quantum principles.

(B) If coordinate $u_c(\xi, \eta)$ and corresponding momentum $p_c(\xi, \eta)$ obey the equations

$$\{u_c, h_j\} = \frac{\partial H_j}{\partial p_c}, \quad \{p_c, h_j\} = -\frac{\partial H_j}{\partial u_c}, \text{ at arbitrary } j, \quad (2)$$

where $H_j(u, p) = \frac{1}{2} p^2 + v(u) - j u$ and $\{,\}$ is the Poisson bracket, if

$$h_j(\xi, \eta) = H_j(u_c, p_c), \quad h(\eta) \equiv h_0(\xi, \eta), \quad (3)$$
then:

(a) the transformed measure has the form:

$$DM(\xi, \eta) = \prod_t d\xi(t)\eta(t)\delta(\dot{\xi} - \partial h_j/\partial \eta)\delta(\dot{\eta} + \partial h_j/\partial \xi),$$

(it is $T \to W_G$ mapping) since, as follows from (2, 3)

(a') $(u_c, p_c)$ are the solutions of incident (classical) Hamiltonian equations:

$$\{u^i_c, u^k_c\} = \{p^i_c, p^k_c\} = 0, \quad \{u^i_c, p^k_c\} = \delta^{ik}$$

(b) $\dim W_G \leq \dim T$, where $T$ is the incident phase space (reduction of $T$).

(c) $\dim W_G$ may be even or odd (splitting: $W_G = T^*G \times R$).

(C) If the Green function $g(t - t')$ of equations

$$\dot{\xi} = \partial h_j/\partial \eta, \quad \dot{\eta} = -\partial h_j/\partial \xi,$$

$h_j(\xi, \eta) = H_j(u_c, p_c)$, have the form:

$$g(t - t') = \theta(t - t'), \quad g(0) = 1,$$

then the quantum corrections to semiclassical approximation are accumulated on the boundaries of $T^*G$:

$$\rho = \rho^{sc} + \int_{\partial T^*G} d\rho^\eta,$$

where $\rho^{sc}$ is the semiclassical contribution. The explicit form of quantum corrections term $d\rho^\eta$ will be given.

• The generalization of formalism on the field theory, where $u_c = u_c(\vec{x}; \xi, \eta)$, becomes evident noting (b) and considering space coordinates as the indexes of special cell.

By the same reason ((b) and (c)) the formalism allows to consider also the situation where $(\xi, \eta) = (\xi, \eta)(x, t)$. Last one incorporates the gauge freedom.

(D) The measure

$$DM(\xi, \eta) = \prod_t d\xi(t)d\eta(t)\delta(\dot{\xi} - \omega(\eta) - j \partial u_c/\partial \eta)\delta(\dot{\eta} + j \partial u_c/\partial \xi),$$

admits the cotangent foliation of quantum force $j$:

$$DM(\xi, \eta) = \prod_t d\xi(t)d\eta(t)\delta(\dot{\xi} - \omega(\eta) - j \dot{\xi})\delta(\dot{\eta} - j \dot{\eta})$$

$$\dot{K}(j\epsilon) = \frac{1}{2} \int dt \{\dot{\epsilon}_\xi \dot{\epsilon}_\xi + \dot{\epsilon}_\eta \dot{\epsilon}_\eta\}$$

and

$$e \to e_c = e_\xi \frac{\partial u_c}{\partial \eta} - e_\eta \frac{\partial u_c}{\partial \xi}$$
Cotangent foliation of quantum force \( j \) gives the completely Hamiltonian description. The foliation allows quantize all classical degrees of freedom independently. The auxiliary variable \( e_c \) is invariant of canonical transformations. The perturbation theory describes fluctuations of classical flow through elementary cell \( \delta u \wedge \delta p \) in the \( T^* G \) cotangent (sub)space. The foliation solves the technical problem of functional determinants calculation.

**Content**

- **Introduction into formalism**
  - The representation \( \rho(F) = \hat{O}(u)\hat{F}(u) \) is derived.
  - The mechanism of canonical and coordinate transformations is shown.
- **Description of the perturbation theory**
  - The main properties of new perturbation theory are shown using simplest quantum-mechanical example.
- **Theory of transformation**
  - The Coulomb problem is solved.
  - The \( \rho \) is calculated for sin-Gordon model in the space of solitons parameters. The consequences are discussed.
  - The quantum measure of the scalar \( O(4, 2) \)-invariant field theory in the \( W_O = O(4, 2)/O(4) \times O(2) \) factor space is derived. It is shown that the scale invariance is broken.
  - The measure of Yang-Mills theory in the \( W_O \times G \) space, where \( G \) is the gauge group, is derived.

**Definitions**

- **S-matrix theory**
  The \( n \)-into-\( m \) particles transition amplitudes is

  \[
  a_{nm}(q'; q) = \prod_{k=1}^{m} \hat{\phi}(q'_k) \prod_{k=1}^{n} \hat{\phi}^*(q_k) Z(\phi),
  \]

  where \( q'_k \) and \( q_k \) are the in- (out)-going particles momenta. The ‘hat’ symbol means:

  \[
  \hat{\phi}(q) \equiv \int dx e^{-iqx} \frac{\delta}{\delta \phi(x)} \equiv \int dx e^{-iqx} \hat{\phi}(x).
  \]

  The vacuum-in-vacuum transition amplitude in the auxiliary background field \( \phi \) is

  \[
  Z(\phi) = \int D\phi e^{iS_0(u) - iV(u + \phi)},
  \]

  where \( S_0 \) is the free part of action:

  \[
  S_0(u) = \frac{1}{2} \int_{C_+} dx ((\partial_\mu u)^2 - m^2 u^2)
  \]
and $V$ describes the interactions:

$$V(u) = \int_{C_+} dx v(u).$$

The time integrals in (4) are defined on Mills time contour:

$$C_+ : t \rightarrow t + i\varepsilon, \quad \varepsilon \rightarrow +0, \quad -\infty \leq t \leq +\infty.$$

- **S-matrix interpretation of statistics**

Let us calculate now the probability

$$\hat{\rho}_{nm}(P) = \frac{1}{n!m!} \int d\omega_m(q')d\omega_n(q)\delta(P - \sum_{k=1}^{m} q'_k)\delta(P - \sum_{k=1}^{n} q_k)|a_{nm}|^2,$$

Summation over all $n, m$ gives the generating functional:

$$\rho(\alpha, z) = e^{-N_+(\alpha, z; \hat{\phi}) - N_-(\alpha, z; \hat{\phi})}\rho_0(\hat{\phi}) \equiv e^{-N(\alpha, z; \hat{\phi})}\rho_0(\phi).$$

in the Fourier-Mellin representation.

The external particles number operator

$$N_\pm(\alpha, z; \hat{\phi}) \equiv \int d\omega_1(q)e^{-iq\alpha_\pm z_\pm(q)}\hat{\phi}_\pm(q)\hat{\phi}_\pm(q).$$

and

$$\rho_0(\phi) = Z(\phi_+)Z^*(-\phi_-).$$

describes the vacuum-into-vacuum transition.

**Comments**

- $\rho(\alpha, z)$ may be used for generation of cross sections helping Fourier-Mellin transformation.
- If (i) $\alpha_\pm = (i\beta_\pm, 0)$, where

$$\tilde{\beta}_\pm : E_\pm \equiv -\frac{\partial}{\partial \beta_\pm} \ln \rho(\beta_\pm),$$

and if (ii) the fluctuations near $\tilde{\beta}_\pm$ are Gaussian, then $\rho(\beta, z)$ is the grand partition function. Then $\beta_\pm$ is the inverse temperature in the initial (final) state and $z(q)$ is the activity.

- Including the black-body environment $\rho(\beta, z)$ is the generating functional of the real-time finite temperature field theory of Schwinger-Keldysh type. If there is not correlations on the time infinities, then $\rho(\beta, z)$ may be continued on the Matsubara imaginary-time contour.

- **S-matrix interpretation allows to extend the formalism on the nonequilibrium media considering $\beta_\pm = \beta_\pm(Y)$, where $Y$ is the 4-coordinate of measurement.**

**Unitary definition of measure**
• **Factorization property**

In the expression

$$\rho(\alpha, z) = e^{-N(\alpha, z; \hat{\phi})} \rho_0(\phi),$$

the ‘external’ properties, fixed by $\alpha, z$, and ‘internal’ ones, described by $\rho_0(\phi)$, are factorized: the operator $e^{-N(\alpha, z; \hat{\phi})}$, where $N = N_+ + N_-$ and

$$N_\pm(\alpha, z; \hat{\phi}) \equiv \int d\omega(q) e^{-i\alpha\pm z_\pm(q)} \phi_\pm(q) \phi^*_\pm(q),$$

maps interacting fields system,

$$\rho_0(\phi) = \int Du_+ Du_- e^{iS_0(u_+) - iU(u_+ + \phi_+)} e^{-iS_0(u_-) + iU(u_- - \phi_-)},$$

on the observable states.

This property reflects adiabaticity of quantum perturbations.

• **Dirac measure**

We will use the substitution:

$$u(x)_\pm = u(x) \pm \varphi(x),$$

with boundary conditions:

$$\int_{\sigma_\infty} dx \varphi(x) \partial^\mu u(x) = 0 : \varphi(x)|_{(x)\in\sigma_\infty} = 0$$

to establish the equality: $2iAA^* = A - A^*$. Expanding over $\varphi$:

$$\rho_0(\phi) = e^{-iK(j, \phi)} \int DM(u, j) e^{-iU(u, \phi)} e^{i2\Re \int_{C_+} dx \varphi(x)(j(x) - v'(u))}$$

where

$$DM(u) = \prod_x du(x) \delta(\partial^2 u + m^2 u + v'(u) - j),$$

$$K(j, \phi) = \frac{1}{2} \Re \int_{C_+} dx j(x) \varphi^*(x),$$

$$U(u, \varphi) = V(u + \varphi) - V(u - \varphi) - 2\Re \int_{C_+} dx \varphi(x)v'(u) = O(\varphi^3).$$

**Note:** last exponent is linear over $u(x)$ because of $\delta$-likeness of measure $DM$.

• **Generating functional**

Substitution of $\rho_0$ gives (A):

$$\rho(\beta, z) = e^{-i\hat{K}} \int DM(u) e^{iS_0(u) - iU(u; \varphi)} e^{N(\beta, z; u)},$$

where

$$N(\beta, z; u) = n(\beta_+, z_+; u) + n^*(\beta_-, z_-; u)$$
and

\[ n(\beta, z; u) = \int d\omega(q)e^{-\beta(\varepsilon(q) + \mu(q))}|\Gamma(q, u)|^2, \]

where \( \mu(q) = \ln z(q)/\beta \) is the chemical potential and

\[ \Gamma(q, u) = \int_{C_+} du(x)\delta(\partial^2 u + m^2 u + \nu'(u) - j), \]

\[ DM(u) = \prod_x du(x)\delta(\partial^2 u + m^2 u + \nu'(u) - j), \]

\[ \hat{K}(j, \varphi) = \frac{1}{2} \Re \int_{C_+} dx j(x)\varphi(x), \]

\[ U(u, \varphi) = V(u + \varphi) - V(u - \varphi) - 2\Re \int_{C_+} dx \varphi(x)v'(u) = O(\varphi^3). \]

Note: if \( u_c(x) \) is a ‘good’ function, then \( \Gamma(q, u) = 0 \) since \( q^2 = m^2 \) by definition.

Comments

a. The functional \( \delta\)-function is defined as follows:

\[ \prod_x \delta(f_x(u)) = \int \prod_x \frac{de(x)}{\pi}e^{-2\Re \int_{C_+} dx e(x)f_x(u)} \]

So, considering the double integral \( AA^* \) we may introduce integration over two independent fields \( u \) and \( e \). Then, (i) integral over \( e \) gives the \( \delta\)-function and (ii) last one defines integral over \( u \). One can say: the real-time theories are ‘simple’.

It should be underlined that the measure \( DM \) was derived for real – time processes only.

b. Only strict solutions of equation

\[ \partial^2 u + \nu'(u) = 0, \quad (5) \]

should be taken into account.

c. \( \rho \) is described by the sum of all solutions of eq.(3), independently from theirs ‘nearness’ in the functional space.

d. \( \rho \) did not contain the interference terms from various topologically nonequivalent contributions. This displays the orthogonality of corresponding Hilbert spaces.

e. The measure \( DM \) and \( \hat{K} \) includes \( j(x) \) as the external source. Its fluctuation disturb the solutions of eq.(3).

f. If \( j \) is switched on adiabatically then the field disturbed by \( j(x) \) belongs to the same manifold (topology class) as the classical field \( u_c \).

g. (Selection rule) If \( V_{u_c} \) is the zero-modes volume occupied by given \( u_c \), then taking into account b, c and f one should leave the contribution with largest \( V_{u_c} \): if \( \dim V_{u_c} > \dim V_{u_c'} \), then \( u_c' \) contributions may be neglected with \( O(V_{u_c'}/V_{u_c}) \) accuracy.

Note: the imaginary-time (kink-like) contributions may be neglected iff theirs contribution are realized on zero \( (V_{u_c'}/V_{u_c} = 0) \) measure.
h. Our definition of $\rho$ restores the stationary phase methods perturbation theory in the vicinity of trivial extremum $u_c = 0$. The comparison of our and WKB perturbation theories is impossible for the case $u_c \neq 0$ since last one is unknown for this case.

The $i\varepsilon$-prescription should be used to avoid the singularities and for right definition of time analytical continuation to connect (if this is not in contradiction with topological principles) the real- and imaginary-time trajectories.

**Canonical transformation**

- **Introduction into the transformation theory**
  Let’s start consideration from $(1+0)$ field theory

$$A_1(x_1, x_2; E) = i \int_0^\infty dTe^{iET} \int \prod dx(t)dp(t)e^{iS_{C+}(x,p)},$$

assuming that $x(0) = x_1, x(T) = x_2$. Then

$$DM(x,p) = \delta(E - H_T) \prod_t dxdp\delta(\dot{x} - \frac{\partial H_j}{\partial p})\delta(p + \frac{\partial H_i}{\partial x}),$$

where

$$H_j = \frac{1}{2}p^2 + v(x) - jx, \quad H_T = H_{j=0}|_{t=T}.$$ 

Inserting

$$1 = \int D\theta Dh \prod_t \delta(h - \frac{1}{2}p^2 - v(x)) \times \delta(\theta - \int^x dx(2(h - v(x)))^{-1/2}).$$

If: $h_j(\theta, h) = H_j(x_c, p_c) = h - jx_c(\theta, h)$, then

$$DM(\theta, h) = \delta(E - h(T)) \prod_t d\theta dh\delta(\dot{\theta} - \frac{\partial h_j}{\partial h})\delta(\dot{h} + \frac{\partial h_i}{\partial \theta}),$$

- $j\partial x_c(\theta, h)/\partial h$ and $j\partial x_c(\theta, h)/\partial \theta$ in eqs.:

$$\dot{\theta} = \frac{\partial h_j}{\partial h} = 1 - j \frac{\partial x_c(\theta, h)}{\partial h}, \quad \dot{h} = \frac{\partial h_i}{\partial \theta} = j \frac{\partial x_c(\theta, h)}{\partial \theta}.$$ 

are the projections of $j$ on the axis of $W = (\theta, h)$ space. Using identity (at $j_b = e_b = 0$):

$$\prod \delta(a \pm bj) = e^{a^{-1}\int j_b e_b e^{-a}\int j_b e_b} \prod \delta(a - j_b)$$

($\alpha$ is arbitrary) one can complete the mapping:

$$DM = \delta(E - h_T) \prod_t d\theta dh\delta(\dot{\theta} - 1 - j\dot{\theta})\delta(\dot{h} - jh),$$

$$\hat{K}(j \cdot e) = \frac{1}{2} \Re \int_{C+} dt \{j\dot{\theta}e_\theta + j_h e_h\},$$
\[ e \to e_c = e_\theta \frac{\partial x_c(\theta, h)}{\partial h} - e_h \frac{\partial x_c(\theta, h)}{\partial \theta}. \]

Action of \( \exp\{-i\hat{K}(j \cdot e)\} \) gives:

\[ \rho(E) = \int_0^\infty dT \int DM : e^{-U(\hat{e}_c, x_c)} e^{iS_0(x_c)} :, \]

\[ \hat{e}_c = \{\hat{j} \wedge \hat{W}\} x_c, \ j = (j_\theta, j_h), \ W = (\theta, h). \]

This completes the (a) part of (B).

- **Zero modes**
  Noting that
  \[ \int \prod_t dX(t) \delta(\dot{X}) = \int dX(0) = V_X \]
  the measure
  \[ DM = \delta(E - h_T) \prod_t d\theta dh \delta(\dot{\theta} - 1 - j_\theta) \delta(\dot{h} - j_h) \sim d\theta(0) \]
  This is the translational zero modes measure.

- **Comments**
  - The zero modes differential measure was defined without Faddeev-Popov ansatz.
  - The Faddeev-Popov ghosts would not arise.
  
- The ghost-free quantization scheme may be shown for Yang-Mills field theories.
  - This removes (?) the problem of Gribov’s ambiguities.

- **Perturbation theory structure**

  \[ \rho(E) = \int_0^\infty dT \int DM : e^{-U(\hat{e}_c, x_c)} e^{iS_0(x_c)} :, \]

  \[ DM = \delta(E - h_T) \prod_t d\theta dh \delta(\dot{\theta} - 1 - j_\theta) \delta(\dot{h} - j_h), \]

Let \( g(t - t') \) be the Green function. The \( i\varepsilon \)-prescription gives:

\[ g(t - t') = \theta(t - t'), \ g(0) = 1, \ g(t - t') g(t' - t) = 0, \]

\[ g^2(t' - t) = g(t' - t), \ g(t' - t) + g(t - t') = 1. \]

\[ e^{-iU(x_c, \hat{e}_c)} = \prod_{n=1}^{\infty} \prod_{k=0}^{2n+1} e^{-iv_{k,n}(\hat{j}, x_c)}, \]

where

\[ V_{k,n}(\hat{j}, x_c) = \int_0^T dt (\dot{j}_\phi(t))^2 n^{-k+1}(\dot{j}_1(t))^k b_{k,n}(x_c). \]
The explicit form of $b_{k,n}(x_c)$ is not important.

\[ \hat{j}_X(t) = \int dt' g(t - t') \hat{X}(t'), \quad X = (\xi, \eta) \]

\[ \hat{j}_X(t_1)b_{k,n}(x_c(t_2)) = \Theta(t_1 - t_2)\partial b_{k,n}(x_c)/\partial X_0 \]

since $x_c = x_c(X(t) + X_0)$, or

\[ \hat{j}_X b_2 = \Theta_{12} \partial_X b_2 \]

since indices $(k, n)$ are not important.

$k = 0, m = 1$

\[ \hat{j}_1 b_1 = \Theta_{11} \partial_0 b_1 = \partial_0 b_1 \neq 0. \]

$k = 0, m = 2$

\[ \hat{j}_1 \hat{j}_2 b_1 b_2 = \Theta_{21} b_1^2 b_2 + b_1^1 b_2^1 + \Theta_{12} b_1 b_2^2, \]

($b_i^\alpha \equiv \partial^\alpha b_i$). Inserting $1 = \Theta_{12} + \Theta_{21}$:

\[ \hat{j}_1 \hat{j}_2 b_1 b_2 = \Theta_{21}(b_1^2 b_2 + b_1^1 b_2^1) + \Theta_{12}(b_1 b_2^2 + b_1^1 b_2^1) = \]

\[ = \partial_0(\Theta_{21} b_1 b_2 + \Theta_{12} b_1^1 b_2^1) \]

This important property of the perturbation theory is conserved in arbitrary order over $m$ and $k$.

This ends the statement (C):

\[ \rho = \rho^{sc} + \int_{\partial W_G} d\rho^q. \]

• General theory of transformation

If $J_i = J_i(x, p), i = 1, 2, ..., N$, are the first integrals in involution then the equations

\[ \dot{J} = -\frac{\partial H}{\partial Q}, \quad \dot{Q} = \frac{\partial H}{\partial J}, \]

\[ \eta = J(x, p), \quad \xi = Q(x, p) \]

(6)

solves mechanical problem (Liouville-Arnold).

Corresponding mapping:

\[ J : T \to W_G, \]

introduces integral manifold $J_\omega = J^{-1}(\omega)$ to which the classical phase flow belongs completely.

**Suggestion A:** If we know the classical phase flow $(x, p)_c$, then (i) one can restore $W_G$ without pointing out the canonical mapping (4); (ii) quantum dynamics representations in $(\xi, \eta) \in T^*G$ and $(x, p) \in T$ are isomorphic.

- This assumes following substitution:

\[ 1 = \frac{1}{\Delta} \int \prod_t d\xi d\eta \delta(u(t)x - u_c(\xi, \eta)x) \delta(p(t)x - p_c(\xi, \eta)x), \]

where $(u, p)_c$ obey the (B) conditions.
It was noted: \( \dim W_G \leq \dim T \); \( \dim W_G \) may be even or odd.

I wish to demonstrate the reduction:

\[
T^4 \rightarrow W^3_C = T^*G^2 \times R^1
\]

considering the **Coulomb problem:**

\[
U_T(r, e) = -S_0(r) + \int_0^T dt \left( \frac{1}{((r + e_r)^2 + r^2 e_\varphi^2)^{1/2}} - \frac{1}{((r - e_r)^2 + r^2 e_\varphi^2)^{1/2}} + \frac{2e_r}{r} \right)
\]

\[
e_r = e_m \frac{\partial r_c}{\partial \xi_1} - e_c \frac{\partial r_c}{\partial \eta_1}.
\]

\[
DM(\xi, \eta) = \delta(E - h(T)) \prod_i d^2 \xi_i d^2 \eta_i \delta(\dot{\xi}_1 - \omega_1 - j_{\xi_1})
\]

\[
\times \delta(\dot{\xi}_2 - \omega_2 - j_{\xi_2}) \delta(\dot{\eta}_1 - j_{\eta_1}) \delta(\dot{\eta}_2), \quad \omega_i = \partial h/\partial \eta_i.
\]

We have put \( j_{\xi_2} = e_{\xi_2} = 0 \) since \( r_c = r_c(\xi_1, \eta_1, \eta_2) \):

\[
\hat{K}(j, e) = \int_0^T dt (\hat{j}_\xi \hat{e}_{\xi_1} + \hat{j}_\eta \hat{e}_{\eta_1}).
\]

Using last \( \delta \)-functions:

\[
\rho(E) = \int_0^\infty dTe^{-i\hat{K}(j, e)} \int dM e^{-iU_T(r, e)} e^{iS_0},
\]

where

\[
dM = \frac{d\xi_1 d\eta_1}{\omega_2(E)}
\]

\[
r_c(t) = r_c(\eta_1 + \eta(t), \eta_2(E, T), \xi_1 + \omega_1(t) + \xi(t)).
\]

The integration range over \( \xi_1 \) and \( \eta_1 \) is as follows:

\[
\partial W_C : 0 \leq \xi_1 \leq 2\pi, \quad -\infty \leq \eta_1 \leq +\infty.
\]

Then,

\[
\rho(E) = \int_0^\infty dT \int dM \left\{ e^{iS_0(r_c)} + \frac{\partial}{\partial \xi_1} b_{\xi_1} + \frac{\partial}{\partial \eta_1} b_{\eta_1} \right\},
\]

and the mean value of quantum corrections in the \( \xi_1 \) direction are equal to zero:

\[
\int_0^{2\pi} d\xi_1 \frac{\partial}{\partial \xi_1} b_{\xi_1}(\xi_1, \eta_1) = 0
\]
since \( r_c \) is the closed trajectory independently from initial conditions.

In the \( \eta_1 \) direction the motion is classical:

\[
\int_{-\infty}^{+\infty} d\eta_1 \frac{\partial}{\partial \eta_1} b_{\eta_1}(\xi_1, \eta_1) = 0
\]

since (i) \( b_{\eta_1} \) is the series over \( 1/r_c^2 \) and (ii) \( r_c \to \infty \) when \( |\eta_1| \to \infty \).

This is the desired result:

\[
\rho(E) = \int_0^\infty dT \int dM e^{iS_0(r_c)}.
\]

Noting that

\[
S_0(r_c) = kS_1(E), \ k = \pm 1, \pm 2, ...
\]

where \( S_1(E) \) is the action over one classical period \( T_1 \):

\[
\frac{\partial S_1(E)}{\partial E} = T_1(E),
\]

and using the identity:

\[
\sum_{-\infty}^{+\infty} e^{inS_1(E)} = 2\pi \sum_{-\infty}^{+\infty} \delta(S_1(E) - 2\pi n),
\]

we find normalizing on zero-modes volume, that

\[
\rho(E) = \pi \sum_n \delta(E + 1/2n^2).
\]

**Suggestion B:** If \( v(u) \) interaction potential, if \( v_X(u_c) \) is the derivative of \( v(u_c) \) in the \( X = (\xi, \eta) \) direction, if \( \{v_X(u_c)\} \) is corresponding manifold, then the theory is exactly semiclassical iff

\[
\{v_X(u_c)\} \cap \partial_X T^*G = \emptyset, \ X(\xi, \eta) \in T^*G,
\]

and \( W_G = T^*G \times \mathbb{R}, \ T^*G \neq \emptyset \).

- The mapping

\[
T(\infty) \to W_{sG} = T^*G(2N), \ N = 1, 2, ...
\]

may be investigated considering \((1,1)\)-dimensional **sin-Gordon model**. We will calculate

\[
\rho_2(q) = e^{-i\hat{K}(q)} \int DM(u, j) e^{-iU(u, e)} e^{iS_0(u)} |\Gamma(q; u)|^2,
\]

The mapping \( J : \{u, p\}(x, t) \to \{\xi, \eta\}(t) \) gives, up to constant (infinite) coefficient:

\[
DM(u_c, p_c)_N = DM(\xi, \eta)
\]

\[
= \prod_t d\xi d\eta \delta(\xi - \frac{\partial h_j}{\partial \eta}) \delta(\eta + \frac{\partial h_j}{\partial \xi})
\]

\[
h(\eta) = \int dp \sigma(r) \sqrt{r^2 + m^2} + \sum_{i=1}^N h(\eta_i),
\]

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\[
\hat{K}(e_\xi, e_\eta; j_\xi, j_\eta) = \frac{1}{2} \text{Re} \int_{C_+} dt \{ \hat{j}_\xi(t) \cdot \hat{e}_\xi(t) + \hat{j}_\eta(t) \cdot \hat{e}_\eta(t) \}.
\]

\[
U(u_N; e_\xi, e_\eta) = -\frac{2m^2}{\lambda^2} \int dx dt \sin \lambda u_N (\sin \phi - \lambda e)
\]
with
\[
e(x, t) = e_\xi(t) \cdot \frac{\partial u_N(x; \xi, \eta)}{\partial \eta(t)} - e_\eta(t) \cdot \frac{\partial u_N(x; \xi, \eta)}{\partial \xi(t)}.
\]

One-soliton configuration (\(\beta = \frac{\lambda^2}{8}\)):
\[
u_s = -\frac{4}{\lambda} \arctan \{ \exp(m \cosh \beta \eta - \xi) \}
\]

Bounded mode:
\[
u_b = -\frac{4}{\lambda} \arctan \{ \frac{\beta \eta_2}{2} \frac{mx \sinh \frac{\beta \eta_2}{2}}{mx \cosh \frac{\beta \eta_2}{2}} - \xi_2 \}.
\]

Corresponding energies:
\[
h_s(\eta) = \frac{m}{\beta} \cosh \beta \eta, \quad h_b(\eta) = \frac{2m}{\beta} \cosh \frac{\beta \eta_1}{2} \sin \frac{\beta \eta_2}{2} \geq 0.
\]

Following to Suggestion B
\[
\rho_2(q) = 0.
\]

Indeed, \(v_X(u_c) = \sin \{ \lambda u_c \} \partial u_c / \partial X \) and
\[
\{ \frac{\partial u_c}{\partial X} \} \cap \partial_X T^* G = \emptyset
\]

- The mapping of scalar \(O(4, 2)\) field theory on the \(W_O(8) = T^*_G(4) \times R(5)\) space gives \(\rho_2(q) \neq 0\) since \(\{ \varphi \} \cap \inf \partial_X W_O \neq \emptyset\), where
\[
\varphi(x) = \left( \frac{4}{g \eta_1^2} \right)^{1/2} \left\{ \left( 1 + \frac{(x - x_0)^2}{\eta_1^2} \right)^2 + \left( 2 \frac{\eta_2 l_\mu(x - x_0)^\mu}{\eta_1} \right)^2 \right\}^{-1/2} = O(1/\sqrt{g}),
\]

where: \(l_\mu l^\mu = +1/\eta_2^2 \geq 0, \quad \vec{1}^2 = 1\). Note, other directions in the \(W_O\) space did not give contributions.

One should use:
\[
U(\varphi, e) = 2gR \int_{C_+} d^3 x dt \varphi(x) e^3(x)
\]
\[
DM(\xi, \eta) = d^3 x_0 d^3 \delta(\vec{1}^2 - 1) dt_0 \delta(\xi_1(0) - \xi_2(0)) \times \prod_t d^2 \xi(t) d^2 \eta(t) \delta^2(\dot{\xi} - \frac{\partial h_\xi}{\partial \eta}) \delta^2(\dot{\eta} + \frac{\partial h_\eta}{\partial \xi})
\]
\[
\hat{K}(j, e) = \frac{1}{2} \int dt \{ \hat{j}_\xi \cdot \hat{e}_\xi + \hat{j}_\eta \cdot \hat{e}_\eta \}
\]

Note, by definition, \(\rho_2(q) \sim \delta(q^2 - m^2)\), where \(m = h_r \neq 0\) is the renormalized energy of \(u_c\), \(h_r = h(u_c) + O(h), \ h(u_c) \sim 1/\eta_1 (!)\) by definition.
The mapping of Yang-Mills theory on $W_O \times G$ gives:

$$DM(\xi, \eta, \Lambda) = d^3x(0)d^3l\delta(l^2 - 1)\delta(\xi_{10} - \xi_{20})$$

$$\prod_{x,t,a} d^2\xi(t)d^2\eta(t)d\Lambda_a(x, t)\delta(\dot{\xi} - \omega - j_\xi)\delta(\dot{\eta} - j_\eta)$$

$$\hat{K}(j_\xi) = \frac{1}{2} \int dt \{ \hat{j}_\xi^a \cdot \hat{e}_\xi^a + \hat{j}_\eta^a \cdot \hat{e}_\eta^a \}$$

$$U(u, \bar{e}) = S_o(u) - \int d^4x \prod_a \left\{ \bar{e}_a \cdot \frac{\partial}{\partial u_a} \right\} v(u),$$

$$\bar{e}_a = e_\xi \frac{\partial u_a}{\partial \eta} - e_\eta \frac{\partial u_a}{\partial \xi}.$$

Note, measure is ghost fields free.