COMPLEXITY OF RANDOM SMOOTH FUNCTIONS OF MANY VARIABLES

ANTONIO AUFFINGER AND GÉRARD BEN AROUS

Abstract. We analyze the landscape of general smooth Gaussian functions on the sphere in dimension $N$, when $N$ is large. We give an explicit formula for the asymptotic complexity of the mean number of critical points of finite and diverging index at any level of energy and for the mean Euler characteristic of level sets. We then find two possible scenarios for the bottom energy landscape, one that has a layered structure of critical values and a strong correlation between indexes and critical values and another where even at energy levels below the limiting ground state energy the mean number of local minima is exponentially large. These two scenarios should correspond to the distinction between one-step replica symmetry breaking and full replica-symmetric breaking of the physics literature on spin glasses. In the former, we find a new way to derive the asymptotic complexity function as a function of the 1RSB Parisi functional.

1. Introduction

This work deals with the complexity of smooth functions of many variables. The questions addressed in this paper can be phrased as: What does a random Morse function look like on a high-dimensional manifold? How many critical values of given index, or below a given level? What can be said about the topology of its level sets? We study here general smooth Gaussian functions on the sphere in dimension $N$, when $N$ is large. We investigate the number of critical points of given index in level sets below a given value, as well as the topology of the level sets through their mean Euler characteristic. Our main result is that these functions have an exponentially large number of critical points of given index, and that the Euler characteristic of the level sets have a very interesting oscillatory behavior. Moreover we find an invariant to distinguish between two very different classes of complexity for these functions. These two classes should correspond to the distinction between one-step replica symmetry breaking and full replica-symmetric breaking of the physics literature on spin glasses. Indeed the general random Gaussian smooth functions on the sphere correspond exactly to the Hamiltonians of an important class of models of statistical physics of disordered media, i.e. mixed spherical spin-glasses. These mean-field models, as well as other spin glass models, are well-known to be very challenging to analyze. It is believed (see [CL04] and the references therein) that a subset of the spherical models that we study here share the same interesting static and dynamical behavior as the famous Sherrington-Kirkpatrick model at low temperature. As part of our study, we give further evidence for this claim and conjecture its domain of validity.

We start by calculating the averaged complexity of these functions i.e. the exponential rate function of the mean number of critical points of finite and diverging index at any level of energy. This initial computation uses the method developed in [ABC10], where

---

Date: November 8, 2011.
2000 Mathematics Subject Classification. 15A52,82D30,60G60.
Key words and phrases. Complexity of spin glasses, random matrices, Parisi formula, ground state energy.
this study was initiated for particular smooth covariance functions that appear as Hamiltonians of the pure spherical p-spin models.

In the general case, our first result is that the complexity of critical points of finite index can be decomposed into two pieces: only one of them present in the pure case (see Figure 1). This difference allows us to separate the models of Gaussian smooth functions on the sphere in two cases: one where the bottom landscape is qualitatively similar to the pure p-spin models and another which should correspond to the case of Full replica symmetry breaking where in particular even at energy levels below the limiting ground state energy, the mean number of local minima is exponentially large.

In the former case, that we call pure-like region, we prove a strong correlation between critical values and their indexes. There exist energy thresholds $-E_k$ such that below $-E_k$ with probability going to one it is only possible to find critical points of index less than $k$. In the latter case, called full mixture region this layered structure is not present and there is no difference between the complexity of critical points of finite index $k$ for any $k$ (not diverging with N). However, in both cases the complexity of critical points of any index is different from the pure case as we increase the level of energy. In particular, coexistence of local minima and local maxima is perfectly possible and the mean number of critical points of any finite index agree in a full neighborhood of energies around their "most typical" energy (see Theorem 2.1).

The understanding of the landscape of these Hamiltonians might prove useful for the study of both static and dynamical questions of these models. First, the layered structure described above may shed a light on the metastability of Langevin dynamics (in longer time scales than those studied in [BG97]). Second, it may provide an insight on the most important statics open question: to understand at any temperature the structure of the (random) Gibbs measure associated to these Hamiltonians. In this direction, a major breakthrough was done by Talagrand [Tal06a] based on the remarkable work of Guerra [Gue03] by computing the free energy at any temperature under a convexity assumption. The free energy is given as the infimum of the Parisi functional over the space of probability measures on $[0,1]$ (see (3.2)). Understanding the minimizer of this functional (uniqueness, for instance, is only known under certain conditions), called a Parisi measure, is also a major challenge. In the spherical pure p-spin it was shown in Proposition 2.2 of [Tal06a] that the model has a one step replica symmetry breaking (1-RSB) at low temperature, i.e the Parisi measure at low temperature is atomic with two atoms. For a mixed spin model, as far as we know, the structure of the Parisi measure remains an open question.

We also show that in the pure-like region, the 1-RSB picture is consistent with the complexity picture without any convexity assumption. Precisely, the complexity function of local minima can be characterized near its zero as a function of the Parisi Functional minimized over two-atomic measures and vice-versa. This is the content of Theorem 3.8. Furthermore, we show that concentration of the number of local minima implies 1-RSB at zero temperature for any mixture in the pure-like region. In the full mixture region, the 1-RSB Parisi functional does not describe the complexity function near its zero. In Theorem 3.1 we show that the ground state energy converges almost surely to a constant. In the full mixture region, this constant has positive complexity. This immediately implies that it is not possible to have concentration for the number of critical points around its mean and one may expect that the averaged complexity is strictly bigger than the quenched complexity (see the discussion after Theorem 3.8.)

Our picture is consistent and generalizes the one proposed by physicists. In [CL04], it is claimed that a $2 + p$ spherical spin glass model with $p \geq 4$, at low temperature is
either 1-RSB or its Parisi measure has an absolute continuous part (a Full RSB or a 1-Full RSB) depending on how much weight is assigned to the 2-spin model. The regions pure-like and full mixture seem to numerically agree and to extend (since we do not need the 2 spin component) the one proposed by [CL04]. We find this a remarkable fact: some mixtures of the spherical model are expected to have the same Gibbs structure as the Sherrington-Kirkpatrick Model on the hypercube. We conjecture that Full RSB holds in the full mixture region. Intuitively, since we prove that the average number of critical points at the ground state energy is exponentially large with $N$, the Gibbs measure at low temperature has plenty of candidates to sample from. However, our techniques are still worlds away to prove this fact. In particular we still do not know what is the typical overlap of two critical points.

Back to the topology of level sets, we show that the total number of critical points at a given level of energy is asymptotically equal to the number of critical points of a particular index (that depends on the level of energy). Loosely speaking, at lower levels, local minima dominate. In a certain threshold energy window $(-NE_{\infty}, NE_{\infty})$ critical points of diverging index give the main contribution to the total complexity. Above $NE_{\infty}$, the total complexity is equal to the complexity of local maxima.

This phenomena is related to the asymptotic mean Euler (or Euler-Poincaré) characteristic of the set of points below a certain level of energy. We show that in absolute value the mean Euler characteristic of these level sets is asymptotically equal to the number of total critical points at that level. It is therefore exponentially large for most energies. Moreover, we prove that is positive outside the window $(-NE_{\infty}, NE_{\infty})$ but oscillates $O(N)$ times from positive to negative (exponentially large) values inside $(-NE_{\infty}, NE_{\infty})$. We find this picture very interesting but quite hard to visualize.

The paper is organized as follows. In section 2, we define the model and we state the main results about the complexity of critical points. In section 3, we state a few relations between the structure of the Parisi measure, the global minima of the Hamiltonian and the complexity of critical points. We also define the regions pure-like and full-mixture. Next, in section 4, we state our results about the Euler characteristic of level sets. In section 5, we prove all Theorems about the complexity function. Their proofs follow the same strategy of [ABC10]. Namely, they will follow from an exact formula for the mean number of critical points of index $k$ that translates the problem to a Random Matrix Theory question. This formula is more envolving than the pure case since in a mixture the Hessian matrix gains an independent Gaussian component on the diagonal. This leads to different variational principle that we analyze. To end, in sections 6 and 7, we prove the results of section 3 while in section 8, we prove the results of section 4.

1.1. Acknowledgements. We want to underline our debt to Michel Ledoux's friendly help for the results of section 3. We also would like to thank Jiri Cerny for a careful reading of this manuscript and Yan Fyodorov for pointing out that the method used in this paper is similar to [Fyo04] and [FW07]. Both authors were partially supported by NSF Grant DMS 0806180. The first author was also partially supported by NSF grant DMS-0500923. The second author was also partially supported by NSF Grant OISE-0730136. We want to thank the hospitality from MSRI, IMPA, Université de Marseille and from the Université de Nice where a mini-course based on these results were given. A more pedagogical account of this subject that includes this work should appear in the MSRI publication series.
2. Complexity and Energy Landscape

The state space of the spherical spin-glass model is $S^{N-1}(\sqrt{N}) \subset \mathbb{R}^N$, the Euclidean sphere of radius $\sqrt{N}$. A configuration $\sigma$ is a vector of $\mathbb{R}^N$ satisfying the constraint

$$\frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 = 1. \quad (2.1)$$

The Hamiltonian of the pure $p$-spin model is the random function defined on $S^{N-1}(\sqrt{N})$ by

$$H_{N,p}(\sigma) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p} \sigma_{i_1} \ldots \sigma_{i_p}, \quad \sigma = (\sigma_1, \ldots, \sigma_N) \in S^{N-1}(\sqrt{N}), \quad (2.2)$$

where $J_{i_1,\ldots,i_p}$ are independent centered standard Gaussian random variables.

Equivalently, $H_{N,p}$ is the centered Gaussian process on the sphere $S^{N-1}(\sqrt{N})$ whose covariance is given by

$$\mathbb{E}[H_{N,p}(\sigma)H_{N,p}(\sigma')] = N^{1-p} \left( \sum_{i=1}^{N} \sigma_i \sigma'_i \right)^p = NR(\sigma, \sigma')^p, \quad (2.3)$$

where the normalised inner product $R(\sigma, \sigma') = \frac{1}{N} \langle \sigma, \sigma' \rangle = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \sigma'_i$ is usually called the overlap of the configurations $\sigma$ and $\sigma'$.

The study of the landscape of such Hamiltonians was done in [ABC10] via a Random Matrix Theory. In this paper, we will consider the analogous analysis for a mixed $p$-spin model, i.e. linear combinations, of these Hamiltonians. At a first sight, this question appears just as a simple generalization, however, as mentioned in the introduction we find a richer structure when considering mixed Hamiltonians instead of pure $p$-spins.

We now define a mixture of $p$-spins (or a mixed spin). Given a sequence $\beta = (\beta_p)_{p \in \mathbb{N}, p \geq 2}$ of positive real numbers such that

$$\sum_{p=2}^{\infty} 2^p \beta_p < \infty, \quad (2.4)$$

let

$$H_{N,\beta}(\sigma) = \sum_{p=2}^{\infty} \beta_p H_{N,p}(\sigma), \quad (2.5)$$

where for any pair of values $p \neq p'$ the Hamiltonians $H_{N,p}, H_{N,p'}$ are independent. Condition (2.4) is more than enough to guarantee that the above sum is a.s. finite and the Hamiltonian $H_{N,\beta}$ is a.s. smooth (see Theorem 11.3.1 of [AT07]).

In this case, we have that

$$\mathbb{E}[H_{N,\beta}(\sigma)H_{N,\beta}(\sigma')] = N \sum_{p=2}^{\infty} \beta_p^2 \left( R(\sigma, \sigma') \right)^p = N \nu(R(\sigma, \sigma')), \quad (2.6)$$

where

$$\nu(t) = \sum_{p=2}^{\infty} \beta_p^2 t^p. \quad (2.7)$$

A word of comment is needed here. By Schoenberg’s theorem [Sch42], if $\nu(R(\sigma, \sigma'))$ is a positive-definite function for all $N$ and all $\sigma, \sigma' \in S^{N-1}(\sqrt{N})$ then $\nu$ can be written as
a linear sum as in (2.7). This shows that we are exhausting all possible covariances given as (2.6).

From now on, we call the function \( \nu \) a mixture and we note that \( \nu \) is smooth with

\[
\nu'(1) \equiv \nu' \neq 0, \nu''(1) \equiv \nu'' > 0 \quad \nu(1) = \sum_{p=2}^{\infty} \beta_p^2 = 1.
\]

(2.8)

If we consider the random variable \( X \) that assign probability \( \beta_p^2 \) to the integer \( p \), then its probability measure is given by \( \mu_X = \sum \beta_p^2 \delta_p \) and

\[
\mathbb{E}X = \nu' \quad \text{and} \quad \alpha^2 \equiv \text{Var}X = \nu'' + \nu' - \nu'^2.
\]

(2.9)

A mixture is pure if and only if \( \alpha = 0 \). Furthermore, note that \( \nu'' \geq \nu' \) with equality only in the pure case with \( p = 2 \). The parameters \( \nu', \nu'' \) and \( \alpha^2 \) will be fundamental in our analysis.

We now introduce the complexity of spherical spin glasses as in [ABC10]. For any Borel set \( B \subset \mathbb{R} \) and any integer \( 0 \leq k < N \), we consider the (random) number \( \text{Crt}_{N,k}(B) \) of critical values of the Hamiltonian \( H_{N,\beta} \) in the set \( NB = \{Nx : x \in B\} \) with index equal to \( k \),

\[
\text{Crt}_{N,k}(B) = \sum_{\sigma : \nabla H_{N,\beta}(\sigma) = 0} 1\{H_{N,\beta}(\sigma) \in NB\} 1\{i(\nabla^2 H_{N,\beta}(\sigma)) = k\}.
\]

(2.10)

Here \( \nabla \), \( \nabla^2 \) are the gradient and the Hessian restricted to \( S^{N-1}(\sqrt{N}) \), and \( i(\nabla^2 H_{N,\beta}(\sigma)) \) is the number of negative eigenvalues of the Hessian \( \nabla^2 H_{N,\beta} \), called the index of the Hessian at \( \sigma \). We will also consider the total number \( \text{Crt}_{N}(B) \) of critical values of the Hamiltonian \( H_{N,\beta} \) in the set \( NB \) (whatever their index)

\[
\text{Crt}_{N}(B) = \sum_{\sigma : \nabla H_{N,\beta}(\sigma) = 0} 1\{H_{N,\beta}(\sigma) \in NB\}.
\]

(2.11)

Our first results will give exact and asymptotic formulas for the mean values \( \mathbb{E}\text{Crt}_{N,k}(B) \) and \( \mathbb{E}\text{Crt}_{N}(B) \), when \( N \to \infty \) and \( k, B \) and \( \nu \) are fixed. We will compute the limits of \( \frac{1}{N} \log \mathbb{E}\text{Crt}_{N,k}(B) \) and \( \frac{1}{N} \log \mathbb{E}\text{Crt}_{N}(B) \) as \( N \) tends to infinity.

2.1. Complexity functions for critical values of finite index.

Our first result is the existence and characterization of the asymptotic complexity of the mean number of critical points of index \( k \) in a certain level of energy.

**Theorem 2.1.** For any fixed integer \( k \geq 0 \), there exist a continuous function \( \theta_{k,\nu}(u) \), called the \( k \)-complexity function, explicitly given in (5.10), such that, for any open set \( B \subset \mathbb{R} \),

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}\text{Crt}_{N,k}(B) = \sup_{u \in B} \theta_{k,\nu}(u).
\]

(2.12)

We decided to postpone to section 5.2 the explicit expression of the \( k \)-complexity functions \( \theta_{k,\nu}(u) \). However, we describe some important properties of these functions (see Figure 3) in the proposition below. We first fix three important thresholds that depend on \( \nu \). Let

\[
E'_\infty = \frac{2\nu' \sqrt{\nu''}}{\nu' + \nu''}, \quad E_\infty = \frac{\nu'' - \nu' + \nu'^2}{\nu' \sqrt{\nu''}}
\]

(2.13)

and
Figure 1. \(k\)-complexity functions \(\theta_{k,\nu}(u)\) for \(-6 \leq u \leq -1\), \(k = 1, 2, 3, 5\) in the case where \(\nu\) is pure-like, i.e. \(\theta_{k,\nu}(-E_\infty) > 0\). The dashed line is the continuation of the parabola that describes \(\theta_{k,\nu}(u)\) in the interval \([-E_\infty, \infty)\) where they all agree.

\[
E_\infty^- = \frac{2\nu'\sqrt{\nu'} - \sqrt{4\nu''\nu'^2 - (\nu'' + \nu')(2(\nu'' - \nu' + \nu'^2) - (\nu'' + \nu' - \nu'^2)\log \frac{\nu''}{\nu'}}}{\nu' + \nu''}.
\]  (2.14)

Note that

\[
E_\infty^- \leq E'_\infty \leq E_\infty.
\]  (2.15)

Furthermore, \(E'_\infty = E_\infty\) if and only if \(E_\infty = E_\infty^-\) if and only if \(\nu'' + \nu' - \nu'^2 = 0\), that is, any equality in (2.15) implies a triple equality. It occurs if and only if the mixture is a pure \(p\)-spin (see (2.9)).

**Proposition 2.2.** For any mixture \(\nu\) and any \(k \geq 0\), the \(k\)-complexity functions \(\theta_{k,\nu}(u)\) satisfy the following:

1. \(\theta_{k,\nu}(u)\) is continuous on \(\mathbb{R}\) and differentiable on \(\mathbb{R} \setminus \{-E_\infty\}\).
2. \(\theta_{k,\nu}(u)\) is strictly increasing on \((-\infty, -E'_\infty)\) and strictly decreasing on \((-E'_\infty, \infty)\).
   
   Its unique maxima is independent of \(k\) and equal to
   
   \[
   \Sigma_\nu := \theta_{k,\nu}(-E'_\infty) = \frac{1}{2}\log \frac{\nu''}{\nu'} - \frac{\nu'' - \nu'}{\nu'' + \nu'} > 0.
   \]  (2.16)
3. \(\theta_{k,\nu}(u)\) has exactly two distinct zeros. The largest zero is given by \(-E_\infty^-\) and therefore is independent of \(k\).
4. For any \(k, k' \in \mathbb{N}\) with \(k < k'\), \(\theta_{k,\nu}(u) < \theta_{k',\nu}(u)\) for all \(u \in (-\infty, -E_\infty)\).
5. For any \(k, k' \in \mathbb{N}\) with \(k < k'\), \(\theta_{k,\nu}(u) = \theta_{k',\nu}(u)\) for all \(u \in [-E_\infty, \infty)\).

Immediately from Theorem 2.1 and Proposition 2.2 we obtain:

**Corollary 2.3.** The mean total number of critical points of index \(k\) satisfies

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Crt}_{N,k}(\mathbb{R}) = \Sigma_\nu.
\]  (2.17)
Furthermore, if \( B = (-\infty, u) \) with \( u \leq -E'_\infty \) then
\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Crt}_{N,k}(-\infty, u) = \theta_{k,\nu}(u).
\] (2.18)

**Remark 1.** By symmetry, Theorem 2.1 also holds as stated for the random variables \( \text{Crt}_{N,N-l}(B) \), with \( l \geq 1 \) fixed if one replaces \( \theta_{k,\nu}(u) \) by \( \theta_{k,\nu}(-u) \).

We now use Theorem 2.1 and Proposition 2.2 to describe the bottom landscape of the mixed spin glass models. For any integer \( k \geq 0 \), we introduce \( E_k = E_k(\nu) > 0 \) as the unique solution in \((E_\infty, \infty)\) to (see Figure 3 again)
\[
\theta_{k,\nu}(-E_k(\nu)) = 0.
\] (2.19)

That is, \(-E_k(\nu)\) is the smallest zero of the \( k \)-complexity function. It is important to note that, by items (4) and (5) of Proposition 2.2 the sequence \((E_k(\nu))_{k \in \mathbb{N}}\) is non-increasing. Its structure is of extreme importance and will be further explored in the next section. At this point we have the following consequence of Theorem 2.1:

**Theorem 2.4.** For \( k \geq 0 \) and \( \varepsilon > 0 \), let \( A_{N,k}(\varepsilon) \) to be the event “there is a critical value of the Hamiltonian \( H_{N,\beta} \) below the level \(-NE_k(\nu) + \varepsilon\) and with index larger or equal to \( k \)” , that is
\[
A_{N,k}(\varepsilon) = \left\{ \sum_{i=k}^{\infty} \text{Crt}_{N,i}(-E_k(\nu) - \varepsilon) > 0 \right\}
\]
and \( B_{N,k}(\varepsilon) \) be the event “there is a critical value of index \( k \) of the Hamiltonian \( H_{N,\beta} \) above the level \(-NE_\infty - \varepsilon\)”, that is
\[
B_{N,k}(\varepsilon) = \{ \text{Crt}_{N,k}((-E_\infty + \varepsilon, \infty)) > 0 \}
\]
Then for all \( k \geq 0 \) and \( \varepsilon > 0 \),
\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(A_{N,k}(\varepsilon)) < 0 \quad \text{and} \quad \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(B_{N,k}(\varepsilon)) < 0.
\] (2.20)

Theorem 2.4 says that with overwhelming probability all critical values of the Hamiltonian \( H_{N,\beta} \) of index \( k \) are inside the interval \([-NE_k, -NE_\infty]\). A similar result was derived for the pure spin glass models in [ABC10]. However, in the pure case it was shown (Theorem 2.2 of [ABC10]) that the probability of finding a critical point of finite index above the level \(-NE_\infty\) is asymptotically of order \( \exp(-NC) \). Hence, in the mixture case not only the window of possible values for the Hamiltonian for a critical point of finite index has changed, but also the probability of being outside of that window is of order \( \exp(-NC) \).

### 2.2. Complexity function for critical values of diverging index and the total number of critical points.

We end this section by studying the number of critical points with diverging index and the total number of critical points (regardless of index). Let \( k = k(N) \) be a sequence of integers such that as \( N \) goes to infinity
\[
\frac{k(N)}{N} \to \gamma \in (0,1).
\] (2.21)

Let \( s_\gamma \in (-\sqrt{2}, \sqrt{2}) \) be defined as solution of
\[
\frac{1}{\pi} \int_{-\sqrt{2}}^{s_\gamma} \sqrt{2-x^2} \, dx = \gamma.
\] (2.22)
The first result in this subsection is the analogue of Theorem 2.1 for critical points of diverging index.

**Theorem 2.5.** For any sequence \( k(N) \) satisfying (2.21), as \( N \) goes to infinity

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Crt}_{N,k(N)}(B) = \sup_{y \in B} \left\{ \frac{1}{2} \log \frac{\nu''}{\nu'} + \frac{1}{2} \left( s^2 - \frac{2\nu''}{\alpha^2} \left(s - \frac{\nu' y}{\nu'' \gamma} \right)^2 - y^2 \right) \right\}
\]

\[
= \sup_{y \in B} \theta_{\gamma,\nu}(u).
\]

**Remark 2.** From Theorem 2.5 one can easily get analogues of Theorems 2.4 and Corollary 2.3 for the case of critical points with diverging index. Its statements are adapted rewrites of the respective results. We leave it to the reader.

We also provide the complexity for the expected total number of critical values at a level of energy. Our next result can be described as follows: the mean number of critical points at level \( u \) is asymptotically given by the mean number of local minima, local maxima or critical points of index \( k(N) \sim \gamma(u) N \) if \( u \leq -E'_\infty, u \geq E'_\infty, -E'_\infty \leq u \leq E'_\infty \), respectively.

Here, \( \gamma(u) \in (0,1) \) is such that \( s_{\gamma(u)} = \sqrt{2 \nu u} \), see (2.22).

Precisely, define

\[
\theta_{\nu}(u) = \begin{cases} 
\theta_{0,\nu}(u) & \text{if } u \leq -E'_\infty \\
\theta_{0,\nu}(-u) & \text{if } u \geq E'_\infty \\
\frac{1}{2} \left( \log \frac{\nu''}{\nu'} - \frac{\nu''}{\nu'^2} \nu' u^2 \right) & \text{otherwise}
\end{cases}
\]

\[
= \sup_{y \in (0,1)} \theta_{\gamma,\nu}(u) = \theta_{\gamma(u),\nu}(u), \quad \text{otherwise.}
\]

**Theorem 2.6.** The total number of critical points satisfies

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Crt}_N(B) = \sup_{u \in B} \theta_{\nu}(u) := \Theta_{\nu}(u).
\]

3. The Ground State Energy, Pure-like and Full Mixtures and 1-step Replica Symmetry Breaking.

The goal of this section is to establish relations between the structure of the Parisi measure, the global minima of \( H_{N,\beta} \), and the results obtained for the asymptotic complexity in last section for different classes of mixed spin glass models. We start by recalling known results about the free energy at positive temperature, more precisely the Parisi formula as proved by Talagrand [Tal06a].

3.1. The Parisi Functional. The partition function of the \( p \)-spin spin glass is given by

\[
Z_{N,\nu}(\beta) = \int_{S^{N-1}(\sqrt{N})} e^{-\beta H_{N,\beta}(\sigma)} \Lambda_N(\mathrm{d}\sigma),
\]

where \( \Lambda_N \) is the normalized surface probability measure on the sphere \( S^{N-1}(\sqrt{N}) \). Let \( M[0,1] \) the space of probability measures on \( [0,1] \). By Theorem 1.1 of [Tal06a], if \( \nu \) is convex, the following limit holds almost surely,

\[
F_\infty(\beta) := \lim_{N \to \infty} \frac{1}{N} \log Z_{N,\nu}(\beta) = \inf_{\rho \in M[0,1]} F_{\nu}(\beta, \rho)
\]
A formula for $F_\nu(\beta, \rho)$ is given in (1.11) of [Tal06a] and we reproduce it now. Given a probability measure $\rho$ on $[0, 1]$ consider its distribution function $x_\rho: [0, 1] \rightarrow [0, 1]$. Write for $q \in [0, 1]$:

$$\hat{x}(q) = \int_q^1 x_\rho(s) \, ds.$$  \hfill (3.3)

Assuming that $x(\hat{q}) = 1$ for some $\hat{q} < 1$, then

$$F_\nu(\beta, \rho) = \frac{1}{2} \left( \beta^2 \int_0^1 x_\rho(q) \nu'(q) \, dq + \int_0^{\hat{q}} \frac{d}{d\hat{q}} \hat{x}(q) + \log(1 - \hat{q}) \right).$$  \hfill (3.4)

If $\hat{q} = 1$, we set $F_\nu(\beta, \rho) = \infty$. A measure that minimizes the right side of (3.2) is called a Parisi Measure. It is believed that the nature (atomic, absolutely continuous) of the Parisi Measure is of extreme importance to understand the statics of the spin glass model [Tal06a, Tal06b].

It is a difficult task to handle the infinite dimensional variational principle in (3.2). However, in some cases, (3.2) can be simplified. Let $M_1[0, 1]$ be the space of atomic probability measures on $[0, 1]$ that have at most 2 atoms. Define

$$F_1(\beta) := \inf_{\rho \in M_1[0, 1]} F_\nu(\beta, \rho).$$

Clearly, $F_{\infty}(\beta) \leq F_1(\beta)$. When equality holds, that is when $F_1(\beta) = F_{\infty}(\beta)$, we say that the model has a 1 step replica symmetry breaking (1RSB) at inverse temperature $\beta$. In [Tal06a], it was shown that the pure $p$-spin glass model is 1-RSB for $\beta$ sufficient large if $p$ is even. It is an open question to determine mixtures and values of $\beta$ that 1RSB holds.

Let

$$G_{SN} = \frac{1}{N} \inf_{\sigma \in S^{N-1}(\sqrt{N})} H_{N, \beta}(\sigma)$$  \hfill (3.5)

be the normalized absolute minima of the Hamiltonian, i.e., the energy of its Ground State. A straight forward exercise shows that, for any $\epsilon > 0$, with probability approaching one as $N$ goes to infinity,

$$-E_0(\nu) - \epsilon \leq G_{SN} \leq \lim_{\beta \rightarrow \infty} -\frac{1}{\beta} F_{\infty}(\beta) + \epsilon.$$  \hfill (3.6)

The main question we investigate in this section is whether the lower and upper bounds given on (3.6) are optimal, that is, whether is possible to identify the limit ground state energy using the partition function and the asymptotic complexity. The question to find the Ground State Energy is one of the foundational and most relevant questions in the study of spin glass system among probabilists [Tal03, Chapter 1]. The left and right sides of (3.6) are quantities that come from different computations, complexity and free energy, respectively. So, a priori, there is no reason to expect that these bounds match. We start our analysis by the following fact:

The upper bound in (3.6) is optimal in the sense that

**Theorem 3.1.** For any convex covariance function $\nu$:

1. The following limit exists

$$\lim_{\beta \rightarrow \infty} \frac{1}{\beta} F_{\infty}(\beta) \equiv f_{\infty} \in [0, \infty).$$  \hfill (3.7)

2. The ground-state energy $G_{SN}$ converges almost surely to $-f_{\infty}$. 
We turn to ask the same question about the lower bound. Our approach is simply to try to prove that \( f_\infty = E_0(\nu) \) directly\(^1\). The problem is that handling \( f_\infty \) is not an easy task since it comes from a rather complicated object, the infimum of the Parisi functional over the space of probability measures on \([0, 1]\). Instead our approach is to compare \( E_0(\nu) \) to the analogous constant as if the model was 1 RSB at low temperature. We will show in Lemma 7.1 that for any mixture the following limit exists

\[
    f_1 := \lim_{\beta \to \infty} \frac{1}{\beta} F_1(\beta) = \inf_{(a,b) \in \mathbb{R}^2} \left\{ \frac{1}{2} \left( b + \nu' a + \frac{1}{b} (\log \frac{a + b}{a}) \right) \right\}, \tag{3.8}
\]

where \( \epsilon \) is a positive constant depending on \( \nu \). When \( f_1 = f_\infty \) we say that the model is 1-RSB at zero temperature. Surprisingly, the comparison between \( f_1 \) and \( E_0 \) will heavily depend on the structure of the mixture \( \nu \) and on the bottom landscape of \( H_{N,\beta} \).

### 3.2. Pure-like mixtures and full mixtures.

In this subsection we relate the Parisi Functional to the asymptotic complexity of spin glasses and derive more precise information about the landscape of \( H_{N,\beta} \). We first identify the regions of mixtures mentioned in the introduction. We refer the reader to Figure 3.

Let

\[
    G(\nu', \nu'') := \log \frac{\nu''}{\nu'} - \frac{(\nu'' - \nu')(\nu'' - \nu' + \nu'^2)}{\nu'^2} = \theta_{0,\nu}(-E_\infty). \tag{3.9}
\]

**Definition 3.2.** A mixture \( \nu \) is called a pure-like mixture if and only if \( G(\nu', \nu'') > 0 \). If \( G(\nu', \nu'') < 0 \), \( \nu \) is called a full mixture. When \( G(\nu', \nu'') = 0 \), \( \nu \) is called critical.

**Example 1.** One can easily verify that all pure \( p \)-spins, \( \nu(x) = x^p \), \( p \geq 3 \) are pure-like while the spherical SK model, \( p = 2 \), is critical. A picture of these regions is given in Figure 3.

**Example 2.** Consider the case

\[
    \nu(t) = \mu t^2 + (1 - \mu)t \tag{3.10}
\]

where \( \mu \in [0, 1] \). Then, if \( p > 3 \) then it is possible to show that there exists a \( 0 < \mu_c(p) < 1 \) such that \( \nu \) is pure-like if and only if \( \mu \leq \mu_c(p) \). \( \mu_c(p) \) is given as the unique zero in \((0, 1)\) of

\[
    -\frac{(p^2 - 2p)(1 - \mu)(2p^2 - p) - 3(p^2 - 2p)\mu + (p - 2)^2 \mu^2}{2((p^2 - p)(1 - \mu) + 2\mu(p + 2\mu - p\mu))^2} + \frac{1}{2} \log \left[ 1 + p - \frac{2p}{p + 2\mu - p\mu} \right]
\]

see Figure 2. Remarkably, \( p = 3 \) in (3.10) is the only case where the mixture is a pure-like mixture for all values of \( t \).

Our first statement concerning pure-like mixtures is the following result about the bottom landscape. Let

\[
    E^+_k = \frac{2\nu'\sqrt{\nu'} + \sqrt{4\nu''\nu'^2 - (\nu'' + \nu')(2(\nu'' + \nu' + \nu'^2) - (\nu'' + \nu' - \nu'^2) \log \frac{\nu''}{\nu'}}}{\nu'' + \nu'}.
\]

It follows directly from the definition of pure-like and (2.19) that:

**Proposition 3.3.** If \( \nu \) is a pure-like mixture then the sequence \( E_k(\nu) \) is strictly decreasing and \( E_k(\nu) \) converges to \( E^+_\infty \) as \( k \) goes to infinity.

---

\(^1\) One could argue that if we prove a concentration result for the number of local minima then \( E_0(\nu) = f_\infty = \lim G_{SN} \) a.s.. Unfortunately, we still do not know how to control the second moment of \( \text{Crt}_{0,\nu} \), since we could not derive a manageable formula like (5.2) nor could we remove the expectation in Theorem 2.1.
Figure 2. Function $G(\nu', \nu'')$ in the case $\nu = \mu t^2 + (1 - \mu) t^{10}$.

Last Theorem combined with Theorem 2.4 says if the mixture $\nu$ is pure-like then the landscape of $\nu$ at low levels of energy is similar to the pure case. In particular, the same interesting layered structure for the lowest critical values of the Hamiltonian $H_{N, \beta}$ holds. Namely, the lowest critical values above the ground state energy are (with an overwhelming probability) only local minima, this being true up to the value $-NE_1(\nu)$, and that in a layer above, $(-NE_1(\nu), -NE_2(\nu))$, one finds only critical values with index 0 (local minima) or saddle point with index 1, and above this layer one finds only critical values with index 0, 1 or 2, etc.

Using the fact that $f_1$ comes from an easier variational principle, Lemma 5.3 of [ABC10] shows that miraculously in the pure case with $\nu(x) = x^p, p$ even, (3.6) is optimal as indeed we have $E_0(\nu) = f_\infty = f_1$. This result extends as:

**Theorem 3.4.** If $\nu$ is pure-like or critical then $f_1 = E_0(\nu)$.

Combining Theorems 3.1 and 3.8 we have the following:

**Corollary 3.5.** In the case of $\nu$ pure-like or critical, concentration of $\text{Crt}_{N,0}(-\infty, u)$ around its mean implies 1-RSB at zero temperature.

A word of comment is needed here. It is reasonable (although we do not have a proof at the moment) to believe that $\frac{1}{N} \log \text{Crt}_{N,0}(-\infty, u)$ concetrates around its mean. However, in Theorem 2.1 we study an "averaged" complexity instead of the possible smaller "quenched" complexity:

$$\lim_{N \to \infty} \frac{1}{N} E \log \text{Crt}_{N,0}(-\infty, u).$$

(3.11)

It is not clear if (and when) both quantities agree. We conjecture that in the pure-like region "quenched" is equal to "averaged" and indeed we have 1-RSB at zero temperature. We also believe that when $\nu$ is a full mixture the averaged complexity is indeed larger than the quenched complexity. These conjectures are supported by physicists [CL04] and by the following result that tells us that the complexity of minima can be constructed from the 1 RSB Parisi functional at zero temperature and vice-versa.

For $b \in (0, \infty)$ define

$$f_1(b) = \inf_{a \in (0, \infty)} \left\{ \frac{1}{2} \left( b + \nu' a + \frac{1}{b} (\log \frac{a + b}{a}) \right) \right\}.$$

(3.12)
and set
\[ c_\nu = \frac{\nu' - 2}{\sqrt{\nu'\nu' - 1}}, \quad g_1(x) = \begin{cases} -xf_1(x), & x > c_\nu \\ -c_\nu f_1(c_\nu), & x \leq c_\nu. \end{cases} \] (3.13)

**Theorem 3.6.** If \( \nu \) is pure like then for all \( u < -E_\infty \)
\[ \theta_{0,\nu}(u) = \min_{b \in [c_\nu, \infty)} (ub - bf_1(b)). \] (3.14)

Moreover, \( g_1(x) \) is a convex function, strictly convex in \((c_\nu, \infty)\) and if we set for \( u > E_\infty \),
\[ \psi(u) = -\theta_{0,\nu}(-u) \text{ then } \psi \text{ is the Legendre-Fenchel conjugate of } g_1(x): \]
\[ \psi(u) = \max_{x \in \mathbb{R}} (ux - g_1(x)). \] (3.15)

**Remark 3.** In fact, such duality is the first sign at zero temperature of an apparently deeper connection predicted by physicists [CGG99, CGPM05] between the Parisi Functional at finite temperature and the TAP complexity (see [ABC10], section 6 for a definition of TAP equations). We plan to explore this connection in the future.

The above connection does not hold if \( \nu \) is a full mixture. We end this section by analyzing this case.

If \( \nu \) is either critical or a full mixture it follows from Theorem 2.1 that for any \( k \) finite
the mean number of critical points of index \( k \) are asymptotically equal at any possible level of energy. In particular,

**Corollary 3.7.** If \( \nu \) is either critical or a full mixture then for any \( k, k' \in \mathbb{N} \)
\[ E_k(\nu) = E_{k'}(\nu) = E_0(\nu). \] (3.16)

Furthermore, for any \( k \in \mathbb{N} \) the probability of finding a critical value of index \( k \) below
the level \(-N(E_0(\nu) + \varepsilon)\) is exponentially small in \( N \).

**Theorem 3.8.** If \( \nu \) is full mixture, not necessarily convex, then \( f_1 < E_0(\nu) \).

If \( \nu \) is a full mixture, last theorem combined with Theorem 3.1 immediately implies that \(-E_0(\nu) < f_1 \leq GS_N \) for \( N \) large enough with probability one. Hence, we can not remove the expectation from Theorem 2.1. Moreover,

**Corollary 3.9.** If \( \nu \) is a full mixture, then for any \( u \in (-E_0(\nu), -f_\infty) \) the probability of having a critical value below u goes to zero while the mean number of local minima is exponentially large in \( N \). Namely for such \( u \) there exist constants \( 0 < C_1 < C_2 \) such that for \( N \) sufficiently large
\[ \mathbb{E} \text{ Crt}_{N,0}(-\infty, u) \geq e^{NC_1}, \quad \text{and} \quad \mathbb{P}\left( \text{Crt}_N(-\infty, u) > e^{NC_1} \right) \leq e^{-NC_2}. \] (3.17)

### 4. Euler characteristic of Level Sets

In this section, we investigate the landscape of the Hamiltonian \( H_{N,\beta} \) by analyzing
the mean Euler characteristic of level sets as \( N \) goes to infinity. In order to state our results we need further notation. The Hermite functions \( \phi_j, j \in \mathbb{N} \), are defined by
\[ \phi_j(x) = (2^j j! \sqrt{\pi})^{-1/2} h_j(x) e^{-x^2 \over 2}, \] (4.1)
Figure 3. Graph of $\nu' \times \nu''$. In blue, the level set $G(\nu', \nu'') = 0$ i.e. the case where $\nu$ is critical. Dotted lines are the possible values of $(\nu', \nu'')$ for the mixtures $2+6, 2+10$ and $4+30$. The gray region is outside the domain of possible values for $(\nu', \nu'')$.

where $h_j, j \in \mathbb{N}$ are Hermite polynomials,

$$h_j(x) = e^{x^2} \left(-\frac{d}{dx}\right)^j e^{-x^2}. \quad (4.2)$$

In particular, $h_0(x) = 1, h_1(x) = 2x, h_2(x) = 4x^2 - 2x$. The Hermite functions are orthonormal functions in $\mathbb{R}$ with respect to Lebesgue measure.

We denote by $\chi(A_u)$ the Euler characteristic of a level set

$$A_u := \{ \sigma \in S^{N-1}(\sqrt{N}) : \mathcal{H}_{N,\beta}(\sigma) \leq Nu \}.$$

$\chi(\cdot)$ is a topological invariant, integer valued function that is defined for any CW-complex as the alternating sum of Betti’s numbers $[War71]$. It is a functional that is invariant under homotopies and satisfies

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B), \quad \chi(\mathbb{B}) = 1 \quad \text{and} \quad \chi(S_N) = 1 + (-1)^{N-1} \quad (4.3)$$

where $\mathbb{B}$ denotes a $N$-dimensional unit ball, $S_N$ the $N$-dimensional unit sphere and $A$, $B$ are CW-complexes. $\chi(\cdot)$ roughly measures the number of connected components and its number of attached cylindrical holes and handles. Since we are only interested on
Euler characteristics of level sets of almost surely Morse functions, we use the equivalent definition that follows from Morse’s theorem (see [AT07] Theorem 9.3.2):
\[ \chi(A_u) := \sum_{k=0}^{N-1} (-1)^k \text{Crt}_k(A_u). \]

The strategy of using Rice’s formula to compute Euler characteristics of level sets was developed in [AT07 TTA05 TA03] and also explored in [AW09]. In fact, in a similar fashion, we prove the following proposition:

**Proposition 4.1.**
\[ \mathbb{E} \chi(A_u) = (-1)^{N-1} \left( \frac{\nu''}{\nu'} \right)^{\frac{N-1}{2}} 2^{-(N-1)N} \int_{-\infty}^{\infty} \int_{-\infty}^{u} h_{N-1} \left( \frac{\sqrt{N} (\nu' x - \alpha y)}{\sqrt{2 \nu''}} \right) e^{-\frac{N}{2} (x^2 + y^2)} \, dx \, dy. \]
(4.4)

Our main result in this section is the asymptotic formula for \( \mathbb{E} \chi(A_u) \) and its relation to the asymptotic complexity of the total number of critical points (see (2.24)).

**Theorem 4.2.** The mean Euler-Poincaré characteristic \( \mathbb{E} \chi(A_u) \) satisfies the following:
1. If \( u \leq -E'_\infty \),
   \[ \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \chi(A_u) = \Theta(u). \]
   (4.5)
2. If \( -E'_\infty < u \leq 0 \), with \( u = -E'_\infty \cos \omega, \omega \in (0, \pi) \)
   \[ \mathbb{E} \chi(A_u) = (-1)^{N-1} \frac{c(N, \nu)}{2^{\frac{1}{4}} \pi^2 N^{\frac{1}{2}}} \frac{e^{N \Theta(u)}}{f(\omega)(\sin \omega)^{1/2}} \sin \left[ N \tau(\omega) + \rho(\omega) \right] (1 + O(N^{-1})). \]
   (4.6)
   where
   \[ \tau(\omega) = \frac{1}{2} \left( \sin 2\omega - 2\omega \right), \quad \rho(\omega) = -\frac{1}{2} \tau(\omega) + \frac{3\pi}{4} + \alpha(\omega), \]
   \( c(N, \nu) \) is given in (8.22) and \( f(\omega), \alpha(\omega) \) are given in (8.23).
3. If \( u > 0 \) we have \( \mathbb{E} \chi(A_u) = \mathbb{E} \chi(A_{-u}) \) for \( N \) even and \( \mathbb{E} \chi(A_u) = 2 - \mathbb{E} \chi(A_{-u}) \) for \( N \) odd.

Let us describe in words the landscape picture emerging from Theorem 4.2. Roughly speaking, Theorem 4.2 says that the mean Euler Characteristic of \( A_u \) is in absolute value asymptotically equal to the total number of critical points at level \( Nu \) if \( u < E_0 \). This picture is fairly intuitive and easy to explain in the bottom of the landscape. As we increase the energy level \( u \) from negative infinity to \( -E'_\infty \), the level set \( A_u \) is "essentially" a union of disjoint simply connected neighborhoods of local minima. Since these are exponentially large and dominate the total number of critical points, the mean Euler characteristic is positive and of same size. As we cross the level \( -E'_\infty \), local minima cease to dominate. The total number of critical points and the Euler characteristic (in absolute value) is given by the critical values of dominant divergent index. The landscape is then hard to visualize. By increasing a tiny amount of energy it oscillates from a large positive to a large negative Euler characteristic (and vice versa). This oscillation continues up to level \( E'_\infty \). It would be of interest to find a simple and intuitive geometric reason for this large oscillation. By symmetry above \( E'_\infty \) we have "essentially" covered the whole sphere minus an exponentially large number of disjoint simply connected sets.
**Remark 4.** The above Theorem also holds as stated in the pure $p$-spin case. Only the complexity function $\Theta_\nu(u)$ needs to be replaced by its analogue given in Theorem 2.8 of [ABC10] (see also Remark 5 below).

Proposition 4.1 and Theorem 4.2 are proven in Section 8.

5. Complexity of critical points

5.1. Main Identity. In this section, we introduce the main identity that relates the mean number of critical points of index $k$ with the $k$-th smallest eigenvalue of the Gaussian Orthogonal Ensemble. This identity, given in Proposition 5.1, is the analogous of Theorem 2.1 of [ABC10] and it is the first step of the proofs of Theorems 2.1, 2.4, 2.6 and 3.3.

We fix our notation for the Gaussian Orthogonal Ensemble (GOE). The GOE is a probability measure on the space of real symmetric matrices. Namely, it is the probability distribution of the Gaussian Orthogonal Ensemble (GOE) ensemble of size $N \times N$, whose entries $(M_{ij}, i \leq j)$ are independent centered Gaussian random variables with variance

$$EM_{ij}^2 = \frac{1 + \delta_{ij}}{2N}. \quad (5.1)$$

We will denote by $E_{GOE}^N$ the expectation under the GOE ensemble of size $N \times N$.

Let $\lambda_1^N \leq \lambda_2^N \leq \cdots \leq \lambda_N$ be the ordered eigenvalues of $M^N$.

**Proposition 5.1.** The following identity holds for all $N$, $\nu$, $k \in \{0, \ldots, N-1\}$, and for all Borel sets $B \subset \mathbb{R}$,

$$\mathbb{E}[\text{Crt}_{N,k}(B)] = C(N, \nu', \nu'') \int_B E_{GOE}^N \left[ \exp \left\{ \frac{N}{2} \left( (\lambda^N_k)^2 - y^2 - \frac{2\nu''}{\alpha^2} (\lambda^N_k - \frac{\nu' y}{(2\nu'' \alpha^2)^{1/2}})^2 \right) \right\} \right] dy,$$  

where $C(N, \nu', \nu'') = 2 \sqrt{\frac{2\nu'' N}{\alpha^2}} \left( \frac{\nu''}{\alpha^2} \right)^{1/2}.$

**Proof.** Proof of Proposition 5.1 is a rewrite of the proof of Theorem 2.1 of [ABC10] with one subtle difference: the law of the Hessian in the mixed case gains an independent Gaussian component on its diagonal. In this proof, we use $H$ to denote $H_{N,\beta}$.

The hypothesis on $\nu$ allows us to apply Rice’s Formula, in the form of Lemma 3.1 of [ABC10]. It says that using $d\sigma$ to denote the usual surface measure on $S^{N-1}(\sqrt{N})$,

$$\mathbb{E}[\text{Crt}_{N,k}(B)] = \int_{S^{N-1}(\sqrt{N})} \mathbb{E}[\left| \det \nabla^2 H(\sigma) \right| 1\{H(\sigma) \in NB, i(\nabla^2 H(\sigma)) = k\} \left| \nabla H(\sigma) = 0 \right\}] \times \phi_\sigma(0) d\sigma \quad (5.3)$$

where $\phi_\sigma$ is the density of the gradient vector of $H$.

Now, since $H$ is invariant under rotations, to compute the above expectation it is enough to study the joint distribution of $(H, \nabla H, \nabla^2 H)$ at the north pole $n$. We fix a orthogonal base for the tangent plane at the north pole, and we consider $\nabla H(n), \nabla^2 H(n)$ with respect to that base. Denoting subscript by a derivative according to a orthonormal basis in $T_n S^{N-1}(\sqrt{N})$ we have that

**Lemma 5.2.** For all $1 \leq i \leq j \leq N - 1$,

$$\mathbb{E}[H(n)^2] = N, \quad \mathbb{E} [H(n)H_i(n)] = \mathbb{E} [H_i(n)H_j(n)] = 0,$$

$$\mathbb{E} [H(n)H_{ij}(n)] = -\nu' \delta_{ij}, \quad \mathbb{E} [H_i(n)H_j(n)] = \nu' \delta_{ij},$$

$$\mathbb{E} [H_i(n)H_j(n)] = \nu' \delta_{ij},$$

$$\mathbb{E} [H_i(n)H_j(n)] = \nu' \delta_{ij},$$

$$\mathbb{E} [H_i(n)H_j(n)] = \nu' \delta_{ij},$$

$$\mathbb{E} [H_i(n)H_j(n)] = \nu' \delta_{ij},$$

$$\mathbb{E} [H_i(n)H_j(n)] = \nu' \delta_{ij},$$

$$\mathbb{E} [H_i(n)H_j(n)] = \nu' \delta_{ij},$$

$$\mathbb{E} [H_i(n)H_j(n)] = \nu' \delta_{ij},$$

$$\mathbb{E} [H_i(n)H_j(n)] = \nu' \delta_{ij},$$

$$\mathbb{E} [H_i(n)H_j(n)] = \nu' \delta_{ij},$$
and
\[ \mathbb{E}[H_{ij}(n)H_{kl}(n)] = \frac{1}{N} [\nu''(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (\nu'' + \nu')\delta_{ij}\delta_{kl}] . \]

Furthermore, under the conditional distribution \( \mathbb{P}[\cdot|H(n) = x] \) the random variables \( H_{ij}(n) \) are Gaussian variables with
\[ \mathbb{E}[H_{ij}(n)] = -\frac{x}{N}\nu'\delta_{ij} \]
and
\[ \mathbb{E}[H_{ij}(n)H_{kl}(n)] = \frac{1}{N} [\nu''(1 + \delta_{ij})\delta_{ik}\delta_{jl} + \alpha^2\delta_{ij}\delta_{kl}] . \]
i.e., if \( M^{N-1} \) is distributed as a \( (N-1) \times (N-1) \) GOE matrix
\[ \mathbb{E}[\nabla^2 H|H(n)] = \frac{(N-1)^2}{N} M^{N-1} + \frac{1}{\sqrt{N}} \left( \alpha Z - \frac{1}{\sqrt{N}} \nu' H(n) \right) I \]
where \( Z \) is an independent standard Gaussian.

Last Lemma implies that \([5.3]\) can be rewritten as
\begin{align*}
\mathbb{E} \text{Crt}_{N,k}(B) \\
= \omega_N \mathbb{E} \left[ \left| \det \left( (\frac{N-1}{N} - 2\nu'')^N M^{N-1} + \frac{1}{\sqrt{N}} (\alpha Z - \nu' H(n)) I \right) \right| \right] \\
\times 1 \left\{ \left| \left( (\frac{N-1}{N} - 2\nu'')^N M^{N-1} + (\alpha \frac{Z}{\sqrt{N}} - \nu' H(n)) I \right) \right| = k \right\} \mathbb{E} \left[ I \{ H(n) \in NB \} | H(n) \right] \phi_n(n),
\end{align*}
(5.4)
where \( \omega_N \), the volume of the sphere \( S^{N-1}((\sqrt{N}) \cdot \nu) \), and \( \phi_n(n) \) are given by
\[ \omega_N = (\sqrt{N})^{N-1} \frac{2\pi^{N/2}}{\Gamma(N/2)} , \quad \phi_n(n) = (2\pi\nu')^{-(N-1)/2} . \quad (5.5) \]

Since we can assume \( \alpha \neq 0 \) (the case \( \alpha = 0 \), i.e. the pure p-spin was treated in \([ABC10]\)), we can rewrite the conditional expectation in \((5.4)\) as
\[ \frac{\sqrt{N}}{\sqrt{2\pi}} \left[ \frac{N-1}{N} - 2\nu'' \right]^{N-1} e^{-\frac{N^2}{2}} \mathbb{E} \left| \det \left( M^{N-1} - X(y) \right) I \right| 1 \left\{ \left| \left( M^{N-1} - X(y) \right) I \right| = k \right\} dy \]
(5.6)
where \( X(y) \) is a Gaussian random variable with mean \( m = \frac{\sqrt{N\nu y}}{(2\nu'' (N-1))^{1/2}} \) and variance \( t^2 = \frac{\nu'^2}{2\nu'' (N-1)} \). Hence, we can apply Lemma 3.3 of \([ABC10]\) with \( G = \mathbb{R} \) to get that \((5.6)\) is equal to
\[ \frac{\Gamma(\frac{N}{2})(\frac{N-1}{N})^{-\frac{N}{2}}}{\sqrt{\pi t^2}} \int_B \mathbb{E}_{\text{GOE}} \left[ \exp \left\{ \frac{N}{2} \left( (\lambda_N^N-y)^2 - 2\nu''(\lambda_N^N - \frac{\nu y}{(2\nu'')}^2) \right) \right\} \right] dy \]
(5.7)
Putting \((5.4), (5.5)\) and \((5.7)\) together we end the proof of Proposition 5.1. \( \square \)

5.2. Proof of Theorems 2.1, 2.4, 2.5 and 2.6
5.2.1. **Proving Theorem 2.1 and Proposition 2.2** In this subsection, we will compute the logarithm asymptotics of the left side of (5.2).

Let $F: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$F(\lambda, y) = \frac{1}{2} \left( -\frac{\nu'' + \nu'}{\nu'' + \nu' - \nu'' y^2} + \frac{2\sqrt{2} \sqrt{\nu''} y}{\nu'' + \nu' - \nu'' y^2} \lambda y - \frac{\nu'' - \nu' + \nu'^2}{\nu'' + \nu' - \nu'' y^2} \lambda^2 \right). \quad (5.8)$$

Note that $F(\lambda, y) = -ay^2 + by\lambda - c\lambda^2$ for some constants $a, b, c > 0$. Let

$$I_1(x) = \int_{\sqrt{2}}^{x} \sqrt{z^2 - 2} \, dz = \frac{1}{2} \left( x\sqrt{x^2 - 2} + \log [2] - 2 \log \left( x + \sqrt{x^2 - 2} \right) \right). \quad (5.9)$$

For any $k, \nu \in \mathbb{N}$ fixed, let

$$\theta_{k, \nu}(u) = \begin{cases} 
\frac{1}{2} \log \frac{\nu''}{\nu'} + F(-\sqrt{2}, u), & \text{if } -E_\infty \leq u, \\
\frac{1}{2} \log \frac{\nu''}{\nu'} + F(\lambda_k^*[u], u) - (k + 1)I_1(|\lambda_k^*[u]|), & \text{if } u \leq -E_\infty
\end{cases} \quad (5.10)$$

where $\lambda_k^*[u] \leq -\sqrt{2}$ is given by

$$\Psi(\lambda_k^*[u]) = 0, \quad \Psi(x) = \frac{2\nu'\sqrt{2\nu''}}{\alpha^2} u x - \frac{\nu'' - \nu' + \nu'^2}{\alpha^2} x^2 - 2(k + 1)I_1(|x|),$$

where $\lambda_k^*[u]$ is a solution on $(-\infty, -\sqrt{2}]$ of

$$\frac{\nu'}{\alpha^2} u x - \frac{\nu'' - \nu' + \nu'^2}{\alpha^2} \lambda_k^*[u] + (k + 1)\sqrt{(\lambda_k^*[u])^2 - 2} = 0. \quad (5.11)$$

Our goal in this section is to prove that $\theta_{k, \nu}$ is the $k$-complexity function. When $k = 0$ the formula for $\theta_{0, \nu}$ simplifies as follows.

**Proposition 5.3.** For all $u \in \mathbb{R}$,

$$\theta_{0, \nu}(u) = \begin{cases} 
\frac{1}{2} \left( \log[\nu''] - \frac{4(\nu'' + \nu'^2)}{\nu'^2 + \nu''} - 2 \frac{(\nu' + \nu'^2)}{\nu'^2 + \nu''} \right), & \text{if } -E_\infty \leq u, \\
\frac{1}{2} \log[u'] - 1 - \frac{4\nu'}{4(\nu'' - 1)} - I_1(-\frac{\nu'}{\sqrt{2}\nu'\nu'' - 1}), & \text{if } u \leq -E_\infty.
\end{cases} \quad (5.12)$$

**Remark 5.** It is possible to recover all complexity functions of the pure case by taking $\alpha$ to zero (i.e. recover the first results of [ABC10]). In particular, if $\alpha = 0$, $E_{\infty} = E_\infty$ and we do not have the intermediate regions where the $k$-complexity functions are equal for different $k$ and non-constant.

We postpone the proof of Proposition 5.3 to the end of this subsection since we will need another characterization of $\theta_{k, \nu}$.

**Proof of Theorem 2.1.** To prove Theorem 2.1 it suffices to show that $\theta_{k, \nu}(u)$ is the logarithm asymptotic limit of the left hand side of (5.2).

First, note that we can rewrite (5.2) as

$$C_N \mathbb{E}e^{-NA(\lambda_k^N, Y_N)} \mathbf{1} \{ Y_N \in B \}. \quad (5.13)$$

where $Y_N$ is a Gaussian random variable of mean zero and variance $N$ independent of $\lambda_k^N$, $\mathbb{E}$ is the expectation with respect to GOE and $Y_N$ and

$$\lim_{N \to \infty} \frac{1}{N} \log C_N = \frac{1}{2} \log \frac{\nu''}{\nu'}, \quad \Lambda(\lambda, y) = F(\lambda, y) + \frac{y^2}{2} = \frac{1}{2} \left( \lambda^2 - \frac{2\nu''}{\alpha^2} \left( \lambda - \frac{\nu'}{2\nu''} \right)^2 \right). \quad (5.14)$$
By the independence of $Y_N$ and $\lambda_N^k$ and Theorem A.1 of [ABC10], the sequence of random variables $(\lambda_N^k, Y_N)$ satisfies a large deviation principle of speed $N$ and rate function

$$I_k(\lambda, x) = \begin{cases} \frac{x^2}{2} + (k + 1)I_1(|\lambda|), & \text{if } \lambda \leq -\sqrt{2}, \\ \infty, & \text{otherwise.} \end{cases}$$

Therefore, in view of (5.13) and (5.14), we can apply Laplace-Varadhan Lemma (see e.g. [DZ98], Theorem 4.3.1 and Exercise 4.3.11) and get that

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Crt}_{N,k}(B) = \frac{1}{2} \left[ \log \frac{\nu''}{\nu'} + \max_{x \in B, \lambda \leq -\sqrt{2}} \left\{ \lambda^2 - \frac{1}{\alpha^2} (\nu' \sqrt{2\nu'' \lambda})^2 - 2I_k(\lambda, x) \right\} \right].$$

We will now analyse the above variational principle. We start by the case of $B = (-\infty, u)$. We want to find

$$\max_{x \leq u, \lambda \leq -\sqrt{2}} \left\{ -x^2 + \lambda^2 - \frac{1}{\alpha^2} (\nu' \sqrt{2\nu'' \lambda})^2 - 2(k + 1)I_1(|\lambda|) \right\}. \quad (5.15)$$

Case $u \geq -E'_{\infty}$: If $u \geq -E'_{\infty}$ then we maximize (5.15) in $x$ first. The maximum is obtained at $x = x_\lambda := \frac{\nu' \sqrt{2\nu'' \lambda}}{\nu' + \nu''} \lambda \leq u$. Plugging $x_\lambda$ back in (5.15), we get an increasing function in $\lambda$, since $I_1(|\lambda|)$ is itself decreasing. Thus the maxima is realized at

$$x = x_\lambda, \quad \lambda = -\sqrt{2}.$$  

This together with (5.2.1) proves Theorem 2.1 in the case $B = (-\infty, u)$ with $-E'_{\infty} \leq u$.

Case $u \leq -E'_{\infty}$: In the case $u \leq -E'_{\infty}$, $x_\lambda \leq u$ if and only if $\lambda \leq \frac{\sqrt{2u}}{E'_{\infty}}$. Therefore if $x^*$ maximizes (5.15) then

$$x^* = x_\lambda \Leftrightarrow \lambda \leq \frac{\sqrt{2u}}{E'_{\infty}} \quad \text{and} \quad x^* = u \Leftrightarrow \frac{\sqrt{2u}}{E'_{\infty}} \leq \lambda \leq -\sqrt{2}. \quad (5.16)$$

If we plug in the correspondent values of $x$ in each region we note that in the first case our function is again increasing in $\lambda$. Furthermore, since at $\lambda = \frac{\sqrt{2u}}{E'_{\infty}}$, $x_\lambda = u$, we are led to the following variational principle valid in both cases of (5.16)

$$\max_{\frac{\sqrt{2u}}{E'_{\infty}} \leq \lambda \leq -\sqrt{2}} \left\{ -u^2 + \lambda^2 - \frac{1}{\alpha^2} (\nu' \sqrt{2\nu'' \lambda})^2 - 2(k + 1)I_1(|\lambda|) \right\} =$$

$$= -(1 + \frac{\nu'^2}{\alpha^2})u^2 + \max_{\frac{\sqrt{2u}}{E'_{\infty}} \leq \lambda \leq -\sqrt{2}} \left\{ \frac{2\nu' \sqrt{2\nu'' \lambda}}{\alpha^2} u_{\lambda} - \frac{\nu'' - \nu' + \nu^2}{\alpha^2} \lambda^2 - 2(k + 1)I_1(|\lambda|) \right\}$$

$$= -(1 + \frac{\nu'^2}{\alpha^2})u^2 + \max_{\frac{\sqrt{2u}}{E'_{\infty}} \leq \lambda \leq -\sqrt{2}} \Psi(\lambda) = \max_{\frac{\sqrt{2u}}{E'_{\infty}} \leq \lambda \leq -\sqrt{2}} \Gamma(\lambda). \quad (5.17)$$

Note that $\Psi(\lambda)$ is a parabola $a\lambda^2 + b\lambda, a < 0$ plus an increasing function. The critical point of the parabola is given by

$$\lambda_c = \frac{\nu' \sqrt{2\nu'' \lambda}}{\nu'' - \nu' + \nu^2} \geq -\sqrt{2} \iff u \geq -E'_{\infty}. \quad (5.18)$$

Therefore if $u \geq -E'_{\infty}$, $\Psi$ is an increasing function in $\lambda$, so its maximum is attained at $\lambda = -\sqrt{2}$. This proves the Theorem in the region $-E_{\infty} \leq u \leq -E'_{\infty}$.
If \( u < -E_{\infty} \), equation (5.18) and the facts that \( \Psi'(-\sqrt{2}) < 0 \) and \( \Psi'(\lambda_c) > 0 \) imply that the maximum is taken in the interior of the interval \([\lambda_c, -\sqrt{2}]\) at \( \lambda^*_k[u] \). This ends the proof of the Theorem in the case \( B = (-\infty, u) \).

Now, it is easy to extend it to any open set \( B \). Let \( u^* \) be the point that realizes the sup\(_{\{u \in B\}} \theta_{k,u}(u) \). From the continuity and uniqueness of a local maxima of \( \theta_{k,u} \), it is clear that either \( u^* = -E'_{\infty} \) or \( u^* \) is in the boundary of \( B \). Assume without loss of generality that there exists an increasing sequence \( u_n \) in \( B \) approaching \( u^* \). Since \( B \) is open, there exist \( \epsilon_n > 0 \) such that

\[
E(Crt_{N,k}(-\infty, u_n) - Crt_{N,k}(-\infty, u_n - \epsilon_n)) = E Crt_{N,k}(u_n - \epsilon_n, u_n) \leq E Crt_{N,k}(B) \leq E Crt_{N,k}(-\infty, u^*).
\]

But since \( \theta_{k,u} \) is continuous and increasing for \( u \leq -E'_{\infty} \) last equation implies

\[
\theta_{k,u}(u_n) \leq \lim_{N \to \infty} \frac{1}{N} \log E Crt_{N,k}(B) \leq \theta_{k,u}(u^*),
\]

for all \( n \), which proves Theorem 2.1 for any \( B \) open.

It remains to prove Proposition 5.3. We first need the following miraculous Lemma.

**Lemma 5.4.** For all \( u < -E_{\infty} \),

\[
\frac{\partial}{\partial \nu''} \theta_{0,u}(u) = 0.
\]

**Proof.** The proof relies on how we derived \( \theta_{0,u}(u) \). When \( u < -E_{\infty} \), \( \theta_{0,u}(u) \) is the maximum over \( \lambda \) of a functional \( \Gamma \) (that depends on \( \nu'' \)) given in (5.17). Its maximizer \( \lambda^*(u) \) is the smallest root of a second degree polynomial that can be derived from (5.11). This second degree equation is given by \( A + B \lambda + C \lambda^2 = 0 \) where

\[
A = 2 + \frac{2u^2\nu''\nu''}{(\nu'' - \nu'' + \nu'')^2},
B = -\frac{2\sqrt{2u}\nu'\nu''((-1 + \nu')\nu' + \nu'')}{(\nu'' - \nu'' + \nu'')^2},
C = \frac{2((-1 + \nu')^2\nu'' + \nu''}{(\nu'' - \nu'' + \nu'')^2}.
\]

Now chain rule and the fact that \( \lambda^*(u) \) is a maximizer imply that \( \frac{\partial}{\partial \nu''} \theta_{0,u}(u) = 0 \) if and only if \( \frac{\partial}{\partial \nu''} (\Gamma'(\lambda^*(u))) = 0 \) if and only if \( \left( \frac{\partial}{\partial \nu''} \Gamma \right)(\lambda^*(u)) = 0 \). The last condition can be written down as a second degree equation of the form

\[
\frac{1}{2\nu''} + \frac{1}{2} \left( \frac{u^2(-\nu'' - \nu''))}{(\nu'' - \nu'' + \nu'')^2} - \frac{u^2}{(\nu'' - \nu'' + \nu'')^2} - \frac{2\sqrt{2u}\nu'\nu''}{(\nu'' - \nu'' + \nu'')^2} + \frac{2\sqrt{2u}\nu'\nu''}{(\nu'' - \nu'' + \nu'')^2} + \frac{\lambda^2}{\nu'' - \nu'' + \nu''} + \frac{(-\nu' + \nu' + \nu'')^2}{\nu'' - \nu'' + \nu''} \right) = 0.
\]

(5.20)

Comparing the coefficients of (5.20) with (5.19) one sees that their ratios are constant equal to \( \frac{1}{4\nu''} \). This immediately implies that they share the same roots. So \( \lambda^*(u) \) indeed satisfies \( \left( \frac{\partial}{\partial \nu''} \Gamma \right)(\lambda^*(u)) = 0 \) and the lemma is proven. \( \square \)
Proof of Proposition 5.3.} From Lemma 5.4 we know that for $u < -E_{\infty}^-$, $\theta_{k,\nu}$ does not depend on $\nu''$. By choosing $\nu'' = \nu'^2 - \nu' + \epsilon$ and taking $\epsilon$ to zero we get the desired result. Indeed, when $\epsilon$ goes to zero

$$\lambda^*(u) \to \frac{uv'}{\sqrt{2(\nu' - 1)^2}}, \quad F(\lambda^*(u), u) \to \frac{-u^2(\nu' - 2)}{4(\nu' - 1)}.$$

\[\square\]

5.2.2. Proof of Theorem 2.4. We want to prove that there are no critical values of index $k$ of the Hamiltonian above $-\nu = -N(E_{\infty}^- - \epsilon)$. The function $\theta_{k,\nu}$ is strictly decreasing on $(-E_{\infty}^-, \infty)$. Using Theorem 2.1, we have

$$\mathbb{E}[\text{Crt}_{N,k}(-E_{\infty}^- + \epsilon, \infty)] \leq \exp \{ N\theta_{k,\nu}(-E_{\infty}^- + \epsilon) + o(N) \}.$$ The constant $-E_{\infty}^-$ is defined by $\theta_{k,\nu}(-E_{\infty}^-) = 0$ for all $k$. Therefore, $\theta_{k,\nu}(-E_{\infty}^- + \epsilon) = c(k, \nu, \epsilon) < 0$. An application of Markov’s inequality as

$$\mathbb{P}\left( B_{N,k}(\epsilon) \right) \leq \mathbb{E}[\text{Crt}_{N,k}(-E_{\infty}^- + \epsilon, \infty)] \leq e^{-Nc(k,\nu,\epsilon)}$$

proves Theorem 2.4 for the event $B_{N,k}(\epsilon)$. The proof for the event $A_{N,k}(\epsilon)$ is analogous.

5.2.3. Proof of Theorem 2.5. The proof of Theorem 2.5 follows the same steps as the proof of Theorem 2.1. First by Lemma 3.5 of [ABC10], for any $\epsilon > 0$, there exists a constant $c = c(\gamma, \epsilon) > 0$ such that

$$\mathbb{P}(|\lambda_k^N - s_\gamma| > \epsilon) \leq e^{-cN^2}.$$ Therefore if we use Proposition 5.1 (5.14) and the above statement we have that for any $\epsilon > 0, \delta > 0$ there exists constants $c = c(\epsilon), d = d(\epsilon)$ such that for $N$ large enough

$$\mathbb{E}[\text{Crt}_{N,k}(B) \leq C_N \int_B e^{N\frac{N}{2}(F(\lambda_k^N, y))} \mathbf{1}\{\lambda_k^N \in (s_\gamma - \epsilon, s_\gamma + \epsilon)\} + e^{dN} e^{-cN^2}$$

$$\leq C_N \int_B e^{N\sup_{\lambda \in (s_\gamma - \epsilon, s_\gamma + \epsilon)}\{F(\lambda, y)\}} dy + e^{dN} e^{-cN^2}$$

$$\leq C_N e^{N\sup_{\lambda \in (s_\gamma - \epsilon, s_\gamma + \epsilon), y \in B}\{F(\lambda, y)\}} (1 + \delta) + e^{dN} e^{-cN^2}.$$ On the other hand we have the lower bound

$$\mathbb{E}[\text{Crt}_{N,k}(B) \geq C_N \int_B e^{N\frac{N}{2}(F(\lambda_k^N, y))} \mathbf{1}\{\lambda_k^N \in (s_\gamma - \epsilon, s_\gamma + \epsilon)\}$$

$$\geq C_N \int_B e^{N\inf_{\lambda \in (s_\gamma - \epsilon, s_\gamma + \epsilon)}\{F(\lambda, y)\}} dy$$

$$\geq C_N e^{N\inf_{\lambda \in (s_\gamma - \epsilon, s_\gamma + \epsilon)}\{\sup_{y \in B}\{F(\lambda, y)\}\}} (1 - \delta).$$ Taking $\frac{1}{N} \log$ on both bounds and taking $\epsilon$ to zero afterwards, we see that

$$\frac{1}{N} \log \mathbb{E}[\text{Crt}_{N,k}(B)] = \sup_{y \in B}\{F(s_\gamma, y)\}.$$
5.2.4. Proof of Theorem 2.6. We now prove the asymptotic limit of the mean number of critical points at some level of energy.

Since the total number of critical points is greater than the number of critical points of index \( k(N) \) with \( k(N) \) satisfying (2.21) for \( \gamma \in [0, 1] \) we clearly have the lower bound

\[
\sup_{\gamma \in [0, 1]} \sup_{u \in B} \theta_{\gamma, \nu}(u) \leq \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \text{Crt}_N(B). \tag{5.21}
\]

For \( u \leq -E'_\infty \), taking \( \gamma = 0 \) (i.e. considering the complexity of local minima) we get the right hand side of (2.24). For \( u \in (-E'_\infty, E'_\infty) \) the supremum on \( \gamma \) of \( \theta_{\gamma, \nu}(u) \) is attained at \( \gamma \in (0, 1) \) such that \( s_\gamma = \frac{\sqrt{2} u}{E'_\infty} \), plugging it back on the left hand side of (5.21), we get the right hand side of (2.24). Last, for \( u \geq E_\infty \), one just need to take the complexity of local maxima. This is enough to prove a lower bound.

To show a matching upper bound, we proceed as follows. A sum over \( k \) in Proposition 5.1, gives us that

\[
\mathbb{E}[\text{Crt}_N(B)] = 2N \sqrt{\frac{2}{\nu'}} \left( \frac{\nu''}{\nu'} \right)^{\frac{N}{2}} \int_B \mathbb{E}_{\text{GOE}}^N \int \exp \left\{ NF(z, y) \right\} dy L_N(dz).
\]

and \( L_N \) is the empirical spectral measure of the GOE matrix. The constant in front the integral gives a constant term \( C_\nu \) after the \( \frac{1}{N} \log \) limit. Furthermore,

\[
\int_B \mathbb{E}_{\text{GOE}}^N \int \exp \left\{ NF(z, y) \right\} dy L_N(dz) \leq N \int_B \sup_{z \in \mathbb{R}} \exp \left\{ NF(z, y) \right\} dy \leq N \int_B e^{-\frac{N}{2} \nu'' \nu' y^2} dy. \tag{5.22}
\]

So if \( B \cap (-E'_\infty, E'_\infty) \neq \emptyset \) this matches the right hand side of (2.24). If \( B \subseteq (-\infty, -E'_\infty) \) then we can estimate (5.22) with

\[
N \int_B \mathbb{E}_{\text{GOE}}^N \int \exp \left\{ NF(\lambda_0, y) \right\}.
\]

Applying log, dividing by \( N \) and taking limits we get Theorem 2.6 from Theorem 2.1.

6. Partition Function

In this section we prove Theorem 3.1 and we derive a formula for the 1-RSB solution at zero temperature that will be useful in the next section. We start by the proof of Theorem 3.1.

Proof of (a): By Holder’s inequality the function \( \frac{1}{N} \mathbb{E} \log Z_{N, \nu}(\beta) \) is convex in \( \beta \), therefore its limit \( F_\infty(\beta) \) is also convex. From (3.6),

\[
0 \leq \liminf_{\beta \to \infty} \frac{1}{\beta} F_\infty(\beta) \leq \limsup_{\beta \to \infty} \frac{1}{\beta} F_\infty(\beta) \leq E_0. \tag{6.1}
\]

So \( F(\beta) \) is convex, positive and grows at most linearly. This easily implies that

\[
\lim_{\beta \to \infty} \frac{1}{\beta} F_\infty(\beta) = \sup_{\beta} \frac{1}{\beta} F_\infty(\beta) \in [0, \infty). \tag{6.2}
\]
To prove item (b), we will need to introduce some notation and the proposition below.

Let \( \sigma^* \) be a point on the sphere such that \( H_{N,\beta}(\sigma^*) = \text{NGS}_N \) and let \( d \) denote the geodesic distance on the sphere. For \( \rho, \alpha, K > 0 \), let

\[
B_{N,\beta} \equiv \left\{ \sigma \in S_{N-1}(\sqrt{N}) : d(\sigma, \sigma^*) < \rho \right\}
\]

and \( A_{\epsilon,\alpha,K}(N) \), be the event

\[
A_{\epsilon,\alpha,K}(N) \equiv \left\{ \sup_{\sigma \in B_{N,\sqrt{N}}(\epsilon)} |H_{N,\beta}(\sigma) - \text{NGS}_N| \leq K N e^\alpha \right\}.
\]

(6.3)

### Lemma 6.1.
For any \( 0 < \alpha < 1 \) there exist constants \( K, K_1 > 0 \) so that for all \( \epsilon > 0 \) and all \( N \) sufficiently large

\[
\mathbb{P}(A_{\epsilon,\alpha,K}(N)^c) < 2e^{-K_1 N}.
\]

(6.4)

Note that this bound is independent of \( \epsilon \).

**Proof.**

Clearly,

\[
A_{\epsilon,K}(N) \supseteq \hat{A}_{\alpha,K}(N) \equiv \left\{ \|H_{N,\beta}\|_\alpha \leq KN^{1-\frac{\alpha}{2}} \right\}
\]

where

\[
\|H_{N,\beta}\|_\alpha = \sup_{\sigma,\sigma'} \frac{|H_{N,\beta}(\sigma) - H_{N,\beta}(\sigma')|}{d(\sigma, \sigma')^\alpha}.
\]

(6.5)

Now consider the centered Gaussian process \( X_\alpha \) field on \( S_{N-1}(\sqrt{N}) \times S_{N-1}(\sqrt{N}) \) given by

\[
X_\alpha(\sigma, \sigma') = \left\{ \frac{H_{N,\beta}(\sigma) - H_{N,\beta}(\sigma')}{d(\sigma, \sigma')^\alpha}, \quad \text{if} \quad d(\sigma, \sigma') > 0 \right\}
\]

(6.6)

Since the Gaussian field \( H_{N,\beta} \) is \( C^1 \) almost surely, then

\[
\mathbb{P}(\hat{A}_{\alpha,K}(N)^c) = \mathbb{P}\left( \sup_{\sigma,\sigma'} |X_\alpha(\sigma, \sigma')| > KN^{1-\frac{\alpha}{2}} \right).
\]

(6.7)

But now a simple computation yields for \( \sigma \neq \sigma' \),

\[
\mathbb{E}X_\alpha^2(\sigma, \sigma') = \frac{2N}{d(\sigma_1, \sigma'_1)^{2\alpha}} \left[ 1 - \nu \left( \frac{1}{N} \langle \sigma, \sigma' \rangle \right) \right] = \frac{2N}{(\sqrt{N} \theta)^{2\alpha}} \left[ 1 - \nu(\cos \theta) \right].
\]

(6.8)

where \( \theta \) is the angle between \( \sigma, \sigma' \) in \( \mathbb{R}^N \).

Therefore by the boundedness of \( \nu'(x) \) in \([-1,1]\) there exists a constant \( C \) independent of \( N \) such that (if \( \alpha < 1/2 \) or \( \alpha < 1 \) - using the boundedness of \( \nu'(x) \) and \( \nu''(x) \))

\[
\sup_{(\sigma, \sigma')} \mathbb{E}X_\alpha^2(\sigma, \sigma') \leq CN^{1-\alpha}.\]

(6.9)

Now, by Borell’s inequality, (see page 50 and 51 of [AT07], where we take \( u = KN^{1-\frac{\alpha}{2}}, \sigma_T \leq CN^{1-\alpha} \) for all \( \delta, \delta \), if \( N, K \) is large enough

\[
\mathbb{P}\left( \sup_{(\sigma, \sigma')} X_\alpha(\sigma, \sigma') > KN^{1-\frac{\alpha}{2}} \right) \leq e^{\delta KN^{1-\frac{\alpha}{2}} e^{-K N^{2(1-\frac{\alpha}{2})}} e^{-\frac{K^2 N^{2(1-\alpha)}}{2CN^{1-\alpha}}} \leq e^{-\frac{K^2 N}{4C}}.
\]

(6.10)

Taking \( K_1 = K^2/4C \) in the last equation, using (6.7) and symmetry of \( X_\alpha \) the lemma is proven. \( \square \)
Now we can start the

**Proof of (b):** We will show that for any \( \delta > 0 \) there exists \( \epsilon(\delta) \) so that if \( N \) is large enough

\[
P\left( |GS_N + f_\infty| > \delta \right) \leq P\left( A_{\epsilon(\delta),\alpha,K}(N)^c \right). \tag{6.11}
\]

The proof of (b) will then follow from (6.11) and Borel-Cantelli’s Lemma since for all \( \delta > 0 \) by Lemma 6.1

\[
\sum_{N=1}^{\infty} P\left( |GS_N + f_\infty| > \delta \right) < \infty. \tag{6.12}
\]

We will prove (6.11) by showing that for any \( \delta > 0 \) if \( N \) is large enough

\[A_{\epsilon,\alpha,K}(N) \subset \{ |GS_N + f_\infty| < \delta \}.
\]

On \( A_{\epsilon,\alpha,K}(N) \),

\[
Z_{N,\nu}(\beta) = \int_{S^{N-1}(\sqrt{N})} e^{-\beta H_{N,\beta}(\sigma)} \Lambda_N(d\sigma) \geq e^{-\beta NGS_N - K\beta N^\alpha} \Lambda_N(B_{N,\sqrt{N}e}). \tag{6.13}
\]

Recall that \( \Lambda_N(d\sigma) \) is the surface measure of \( S_N(\sqrt{N}) \) normalized to be a probability measure. We trivially have the bound

\[
\frac{1}{N} \log Z_{N,\nu}(\beta) \leq -\beta GS_N. \tag{6.14}
\]

Combining (6.13) and (6.14) we then have on \( A_{\epsilon,\alpha,K}(N) \),

\[- \frac{1}{N\beta} \log Z_{N,\nu}(\beta) - Ke^\alpha + \frac{1}{N\beta} \log \Lambda_N(B_{N,\sqrt{N}e}) \leq GS_N \leq - \frac{1}{N\beta} \log Z_{N,\nu}(\beta). \tag{6.15}
\]

Note that using spherical coordinates and the inequality \( \frac{2\theta}{\pi} \leq \sin \theta \) for \( \theta \leq \frac{\pi}{2} \), we have for \( \epsilon < \pi/2 \),

\[
\Lambda_N(B_{N,\sqrt{N}e}) = \left( \int_0^\epsilon \sin^{N-2}(\phi) \ d\phi \left( \int_0^\pi \sin^{N-2}(\phi) \ d\phi \right)^{-1}\right)^{-1} \geq \left( \frac{2\epsilon}{\pi} \right)^{N-1} \frac{1}{\pi(N-1)}. \tag{6.16}
\]

So on \( A_{\epsilon,\alpha,K}(N) \), for some constant \( C > 0 \)

\[- \frac{1}{N\beta} \log Z_{N,\nu}(\beta) - Ke^\alpha + C\epsilon \leq GS_N \leq - \frac{1}{N\beta} \log Z_{N,\nu}(\beta). \tag{6.17}
\]

Therefore by (3.2), for any \( \delta_1 > 0 \) one can take \( N \) large enough so that,

\[- \frac{F_\infty(\beta)}{\beta} - Ke^\alpha + C\epsilon - \frac{\delta_1}{\beta} \leq GS_N \leq - \frac{F_\infty(\beta)}{\beta} + \frac{\delta_1}{\beta}. \tag{6.18}
\]

By taking \( \beta \) large enough, part (a) of this Theorem and by choosing \( \epsilon \) sufficiently small, (6.11) is proven.  \( \square \)

7. **Proofs of Theorems 3.4, 3.6 and 3.8**

In this section, we prove Theorems 3.4, 3.6 and 3.8.
7.1. Calculating $f_1$. We now prove equation (3.8).

**Lemma 7.1.**  
(1) There exists $\epsilon, M > 0$ such that the following limit holds

$$f_1 := \lim_{\beta \to 0} \frac{1}{\beta} F_1(\beta) = \inf_{a,b \in [\epsilon, M]} \frac{1}{2} \left( b + \nu' a + \frac{1}{b} (\log \frac{a + b}{a}) \right). \quad (7.1)$$

(2) $f_1$ depends (continuously) only on the first derivative $\nu'$.

**Remark 6.** It is remarkable that while the $k$-complexity function depends on the first two derivatives at 1 of the covariance function $\nu$ and $f_1$ depends only on the first derivative $\nu'$ and $E_0(\nu) = f_1$ for any pure-like mixture.

**Proof.** First, taking $\mu = m\delta_r + (1 - m)\delta_q$, we can write the 1RSB Free Energy $F_1(\beta)$ via the Crisanti-Sommers representation (see (1.4) of [TP07]) as

$$F_1(\beta) = \inf_{m,r \in [0,1]} \left\{ \frac{r}{m(q - r) + 1 - q} + \frac{1}{m} \left( \log(m(q - r) + 1 - q) - \log(1 - q) \right) + \log(1 - q) + \nu(q) \right\}.$$

It is easy to show that the infimum above is attained at $r = 0$. Therefore,

$$F_1(\beta) = \inf_{m,q \in [0,1]} \left\{ \beta^2(1 - (1 - m)\nu(q)) + \log(1 - q) - \frac{1}{m} \log \left( \frac{1 - q}{1 - q + mq} \right) \right\}. \quad (7.2)$$

The conditions to be a critical points are

$$\beta^2 \nu'(q) = \frac{q}{(1 - q)(1 - q + qm)}$$

$$\log \left( \frac{1 - q + qm}{1 - q} \right) = \nu(q)m^2\beta^2 + \frac{mq}{1 - q(1 - m)}. \quad (7.3)$$

Let $(q^*, m^*) = (q^*, m^*)(\beta)$ be a solution of (7.3). First, from the first equation we deduce that either $q^*$ goes to 0, 1 or $1 - q^*(1 - m^*)$ goes to zero as $\beta$ goes to infinity. Analogously to the pure case, the case $q^*$ approaching zero is excluded via Lemma 3 of [TP07]. Since $1 - q^*(1 - m^*) \to 0$ implies $q^* \to 1$, it is then a fact that $q^* \to 1$. Looking at the second equation we see that $m^*$ has to go to zero. Indeed, if it is not the case $1 - q^* \sim \beta^{-2}$ implying $\log \beta^2 \sim \beta^2$ which is not possible.

If we put $A = 1 - q^*$, $B = m^*$, then $A, B$ go to zero as $\beta$ goes to infinity and conditions (7.3) become

$$\beta^2 \nu'(1) \sim \frac{1}{A^2 + AB}, \quad \log \left[ 1 + \frac{B}{A} \right] \sim B^2 \beta^2 + \left( 1 + \frac{B}{A} \right)^{-1}. \quad (7.4)$$

We will argue that $B \sim A$, i.e. $\lim_{\beta \to \infty} \frac{B}{A} = l \in (0, \infty)$. Suppose that $A >> B$. The first condition in (7.4) implies $B << \beta^{-2}$. But now the RHS in the second condition goes to 1 while the LHS goes to 0. Next, suppose that $B >> A$. First condition in (7.4) implies $\frac{1}{AB} \sim \beta^2$ while the second implies $\log(1 + \frac{B}{A}) \sim \frac{B}{A}$ which is a contradiction since $\frac{B}{A} \to \infty$. Therefore, not only we have $B \sim A$ but we can also write

$$A = a(\beta)a^*\beta^{-1}, \quad B = b(\beta)b^*\beta^{-1} \quad (7.5)$$

where $a(\beta), b(\beta)$ are functions that converge to 1 as $\beta$ goes to infinity and $a^*, b^* \in (0, \infty)$. Furthermore, $a^*, b^*$ satisfy the following relations:

$$\nu'(1) = \frac{1}{a^*(a^* + b^*)}, \quad \log \left[ 1 + \frac{b^*}{a^*} \right] = (b^*)^2 + \left( \frac{b^*}{a^*} \right)^{-1}. \quad (7.6)$$
To get the statement just note that replacing \( m = b \beta^{-1} \) and \( q = 1 - a \beta^{-1} \) in (7.2), we get

\[
\frac{1}{\beta} F_1(\beta) = \inf_{a,b \in [0,\beta]^2} P(a, b, \beta)
\]

where

\[
P(\beta, a, b) = \frac{1}{2} \left\{ \beta(1 + (b \beta^{-1} - 1)\nu[1 - a \beta^{-1}]) + (\beta - b^{-1}) \log (1 - (1 - a \beta^{-1})) + b^{-1} \log [1 - (1 - a \beta^{-1})(1 - b \beta^{-1})] \right\}.
\]

Clearly, for any \( a, b > 0 \) the function \( P(\beta, a, d) \) converges pointwise as \( \beta \) goes to infinity to \( \frac{1}{2} \left(b + \nu a + \frac{1}{b} \log \frac{a+b}{a}\right) \). Since we know that the location of the minima of \( \frac{1}{\beta} F_1(\beta) \) converges to \((a^*, b^*) \in (0, \infty)^2\), this is enough to guarantee the convergence stated in Lemma 7.1.

By solving for the critical points of (7.1), i.e. using equations (7.6) we can get an expression for \( f_1 \) in terms of \( \nu' \). Namely,

\[
f_1 = \frac{1}{2} \left( \frac{\nu' y^2 - 1}{\nu' y} + 1 + \frac{\nu' y}{\nu' y^2 - 1} \log(\nu' y^2) \right) = y + \frac{\nu' - 1}{\nu' y'}
\]

where \( y = y(\nu') \) is given by the unique solution of

\[
\left( \frac{\nu' y^2 - 1}{\nu' y} \right)^2 + \frac{\nu' y^2 - 1}{\nu' y} = y \log(\nu' y^2), \quad y > \nu'^{-1/2}.
\]

In other words, \( y = \sqrt[\nu']{a} \) where \( a \) is the unique solution of

\[
a \log[a] - a + 1 - \frac{(a - 1)^2}{\nu'} = 0, \quad a > 1.
\]

This ends the proof of Lemma 7.1. \( \square \)

7.2. Proof of Proposition 3.3.

Proof. If \( \nu \) is pure-like then \( \theta_{k,\nu}(-E_\infty) > 0 \). Since \( \theta_{k,\nu}(u) \) converges to negative infinity as \( u \) goes to negative infinity, \( E_k(\nu) \) are well-defined. Furthermore, as \( k \) goes to infinity, \( \lambda_k^*(u) \) converges to \(-\sqrt{2}\) for any \( u \leq -E_\infty \), implying that \( \theta_{k,\nu}(u) \) converges to \( F(-\sqrt{2}, u) \) pointwise. Therefore, taking \( u \) in a small neighborhood of \( E_\infty^- \) and using the fact that \( \theta_{k,\nu} \) are increasing in that neighborhood we see that the zero of \( \theta_{k,\nu} \) has to converge to the zero of \( F(-\sqrt{2}, u) \). Namely \( E_k(\nu) \) converges to \( E_\infty^+ \). \( \square \)

7.3. Proofs of Theorems 3.4, 3.6 and 3.8. We start by the case when \( \nu \) is critical, i.e. by the case when \( G(\nu', \nu'') = 0 \).

Proposition 7.2. A mixture \( \nu \) is critical if and only if

\[
f_1 = E_\infty = E_{0,\nu} = \frac{\nu'' - \nu' + \nu'^2}{\nu' \sqrt{\nu'^2}} \tag{7.7}
\]

Proof. If \( \nu \) is critical then \( y = \sqrt[\nu']{a} \) is the unique solution of (7.1) with \( y > \frac{1}{\nu'} \). Indeed,

\[
1 - \frac{\nu''}{\nu'} + \frac{(-\nu' + \nu'') (-\nu' + \nu'^2 + \nu'')}{\nu'^3} = \frac{(-1 + \nu'')^2}{\nu'} = 0.
\]
Plugging back the value of \( y \) in (7.1) we get \( f_1 \). On the other hand, if \( f_1 = \frac{\nu^4 - \nu^2 + \nu^2}{\nu^2 \nu^2} \) then one solve equation (7.1) in \( y \) to see that the only positive solution is \( y = \frac{\nu^2}{\nu^2} \). By the definition of \( y \) in (7.1) this immediately implies that \( \nu \) is critical. And trivially, \( \nu \) critical is precisely the case where \( E_\infty = E_{0,\nu} \).

Now we analyse the case where \( \nu \) is critical or a full mixture, i.e. the case where \( G(\nu', \nu'') \leq 0 \). In this case, the zero of the complexity function can be explicitly computed and is given by:

\[-E_{0,\nu} = -E_\infty^+,\]

where \( E_\infty^+ \) was defined in (2.14). Note that \( E_{0,\nu} \) is a function of \( \nu' \) and \( \nu'' \).

**Proposition 7.3.** If \( G(\nu', \nu'') \leq 0 \) then

\[
\frac{\partial}{\partial \nu''} E_{0,\nu} = 0 \text{ if and only if } G(\nu', \nu'') = 0.
\]

**Proof.** Let

\[
A(\nu', \nu'') = \sqrt{(\nu'' - \nu^2 + \nu') \left( (\nu' + \nu'') \log \left[ \frac{\nu''}{\nu'} \right] - 2(\nu'' - \nu') \right)}.
\]

Calculating the derivative \( \frac{\partial}{\partial \nu''} E_{0,\nu} \) one gets

\[
\left( \nu'^2 \nu'' (\nu' + \nu'') \log \left[ \frac{\nu''}{\nu'} \right] + (\nu'' - \nu') \left( \nu^3 + \nu'^2 - \nu^2 (1 + 3 \nu'') - 2 \nu' \sqrt{\nu'} A(\nu', \nu'') \right) \right. \times \left. \left( 2 \nu'' (\nu' + \nu'')^2 A(\nu', \nu'') \right)^{-1} \right.
\]

\[
(7.8)
\]

Sufficiency comes from a simplification of the above formula. To get necessity we solve a second degree equation on the variable \( M = \log \left[ \frac{\nu''}{\nu'} \right] \) to see that this second degree equation has a unique zero given by

\[
\frac{\nu'^2 - \nu'^3 - 2 \nu' \nu'' + \nu'^2 \nu'' + \nu'^2}{\nu'^2 \nu''}.
\]

This is precisely \( G(\nu', \nu'') = 0 \).

With the above propositions we now prove Theorem 3.8.

**Proof of Theorem 3.8.** If \( \nu \) is critical Theorem 3.8 is Proposition 7.2. Now suppose that \( \nu \) is pure-like. By Lemma 7.1 and (5.10), both \( f_1(\nu) \) and \( E_0(\nu) \) are independent of \( \nu'' \). Consider then another mixture \( \mu \) such that \( \mu' = \nu' \) and \( \mu \) satisfies \( G(\mu', \mu'') = 0 \). Since \( G \) is continuous on its domain, we have

\[
f_1(\nu) = f_1(\mu) = E_0(\mu) = E_0(\nu).
\]

**Proof of Theorem 3.6.** The proof is just a simple but major simplification. Note that by solving (3.12) in \( a \) we can rewrite (3.12) as

\[
\frac{1}{4} \left( -b(-2 + v) - \sqrt{2 + b^2 v^2} \right) + \frac{2 \log \left[ \frac{1}{2} \left( 2 + b^2 v' - b \sqrt{4 + b^2 v'} \right) \right]}{b}.
\]
This implies that a critical point $b^*(u)$ of $M(u, b) = ub - bf_1(b)$ is given by
\[
b^*_\pm(u) = \frac{-u(\nu' - 2) \pm \sqrt{\nu' \sqrt{4 - 4\nu' + u^2\nu'} \nu'}{2(\nu' - 1)}.
\]

Here, we choose $b^*_+(u)$ since $b^*_-(u)$ is negative and a local maxima. Furthermore, since
\[
(\frac{\partial}{\partial b} M)(u, c_\nu) = u + 2\sqrt{\frac{\nu' - 1}{\nu'}} < 0,
\]
b^*_+(u) is a global minima in $[c(\nu), \infty)$. So, replacing $b^*_+(u)$ in (3.12) and setting $z = -u(\nu' - 2) + \sqrt{\nu' \sqrt{4\nu' + 4 - 4\nu' + 4}} = -u(\nu' - 2) + \sqrt{\nu' \sqrt{4 - 4\nu' + u^2\nu'}} = -u(\nu' - 2) + \sqrt{\nu' \sqrt{4 - 4\nu' + u^2\nu'}}$
we get the function
\[
\frac{zu}{2(\nu' - 1)} + \frac{z}{2(\nu' - 1)} \left(-\frac{(\nu' - 2)z}{2(\nu' - 1)} + \sqrt{\frac{\nu'z^2}{4(\nu' - 1)^2} + 4}\right) + \frac{1}{2} \log \left[\frac{1}{2} \left(2 + \frac{\nu'z^2}{4(\nu' - 1)^2} + \frac{\nu'z}{2(\nu' - 1)} \sqrt{4 + \frac{\nu'z^2}{4(\nu' - 1)^2}}\right)\right] (7.9)
\]
Comparing term by term the logarithm, polynomial and fractional terms of (5.10) and (7.9) one gets the first part of the Theorem. The proof of the second part follows from Fenchel’s duality theorem (see [Roc70, Chapter 1]). □

**Proof of Theorem 3.8.** First, note that for a fixed $\nu'$, there exists $\nu'' c$ such that the condition that $\nu$ is a full mixture can be written as $\nu'' > \nu'' c$. From Proposition 7.3 we know that $\frac{\partial}{\partial \nu''} E_{0, \nu}$ does not change sign as we change $\nu''$. Taking $\nu''$ to infinity in (7.8) we see that $\frac{\partial}{\partial \nu''} E_{0, \nu} \geq 0$, meaning that $E_{0, \nu}$ is increasing in $\nu''$. Since at $\nu'' c$, $E_{0, \nu} = f_1$, we get that for any $\nu$ such that $G(\nu', \nu'') < 0$, $E_{0, \nu} > f_1$ proving the Theorem. □

8. PROOF OF PROPOSITION 4.1 AND THEOREM 4.2

In this section we prove Proposition 4.1 and Theorem 4.2.

**Proof of Proposition 4.1.** We start from the following identity:
\[
\mathbb{E} \chi(A_u) = \sum_{k=0}^{N-1} (-1)^k \text{Crt}_k(A(u)) = \sum_{k=0}^{N-1} (-1)^k \mathbb{E} \left( \left| \det \nabla^2 H_N(\sigma) \right| 1_{\{i(\nabla^2 H_N(\sigma)) = k\}} 1_{\{H_N(\sigma) \leq Nu\}} \right) \nabla H_N(\sigma) = 0
\]
\times \phi_{\sigma H_N(0)} d\sigma = (2\nu'\pi)^{-\frac{N-1}{2}} |S^{N-1}(\sqrt{N})| \frac{1}{\sqrt{2\pi N}} \times
\]
\times \sum_{k=0}^{N-1} \int_{-\infty}^{\infty} \mathbb{E} \left( (-1)^k | \det \nabla^2 H_N(\sigma) | 1_{\{i(\nabla^2 H_N(\sigma)) = k\}} H_N(\sigma) = x \right) e^{-\frac{1}{2} \sigma^2} d\sigma
\times \int_{-\infty}^{\infty} \mathbb{E} \left( \det \nabla^2 H_N(\sigma) \nabla H_N(\sigma) = x \right) e^{-\frac{1}{2} \sigma^2} d\sigma.
\]
\[
= (2\nu'\pi)^{-\frac{N-1}{2}} \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} N^{\frac{N-1}{2}} \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{\infty} \mathbb{E} \left( \det \nabla^2 H_N(\sigma) \nabla H_N(\sigma) = x \right) e^{-\frac{1}{2} \sigma^2} d\sigma
\times \int_{-\infty}^{\infty} \mathbb{E} \left( \det \nabla^2 H_N(\sigma) H_N(\sigma) = N x \right) e^{-\frac{N}{2} x^2} d\sigma.
\]
\[
= \nu'^{-\frac{N-1}{2}} \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} N^{\frac{N}{2}} \int_{-\infty}^{\infty} \mathbb{E} \left( \det \nabla^2 H_N(\sigma) H_N(\sigma) = N x \right) e^{-\frac{N}{2} x^2} d\sigma.
\]
Lemma 8.1. If $M_N$ is a $N \times N$ GOE with variance $\mathbb{E} M_{ij}^2 = \frac{1 + \delta_{ij}}{2N}$ then for any $x \in \mathbb{R}$
$$\mathbb{E} \det(M_N - xI) = 2^{-N} N^{-\frac{N}{2}} (-1)^N h_N(\sqrt{N}x)$$
where $h_N(x)$ is given in (4.2).

Proof. The proof, a straight-forward linear algebra exercise, can be found as Corollary 11.6.3 in [AT07]. □

Now by Lemma 5.2:
$$\mathbb{E} \chi(A_u) = \nu^{-\frac{N-1}{2}} 2^{-\frac{N+2}{2}} \frac{N^\frac{N}{2}}{\sqrt{2\pi}} \times$$
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E} \left[ \det \left( \frac{N-1}{N} 2\nu'' \right)^{1/2} M^{N-1} + (\alpha y - \nu' x)I \right] e^{-\frac{N-1}{2} x^2} e^{-\frac{N-1}{2} y^2} dx dy. \quad (8.1)$$
The double integral becomes:
$$\left( \frac{N-1}{N} 2\nu'' \right)^{1/2} \frac{N}{\sqrt{2\nu'}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E} \left[ \frac{1}{\nu''} M^{N-1} + (\alpha y - \nu' x)I \right] e^{-\frac{N}{2} x^2} e^{-\frac{N}{2} y^2} dx dy,$$
which by Lemma 8.1 can be rewritten as
$$(-1)^{N-1} \left( \frac{\nu''}{2N} \right)^{1/2} \frac{N}{\sqrt{2\nu'}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{N-1} \left( \frac{\sqrt{N}(\nu' x - \alpha y)}{\sqrt{2\nu'}} \right) e^{-\frac{N}{2} x^2} e^{-\frac{N}{2} y^2} dx dy. \quad (8.2)$$
Combining (8.1) and (8.2) we get Proposition 4.1. □

We will need the following Lemma to prove Theorem 4.2.

Lemma 8.2. Let $a$, $b$ be constants such that $a > 1/2$ and $b \geq 0$. Set
$$I_N(M) = \int_{-\infty}^{\infty} \phi_{N-1}(\sqrt{N}x)e^{ax^2 + bx} dx.$$
As $N$ goes to infinity:
1. If $\sqrt{2} \leq M$ then $I_N(M) = O(e^{-N(aM^2 + bM + I_1(M))}).$
2. If $-\sqrt{2} < M < \sqrt{2}$ and if we set $M = \sqrt{2} \cos \omega$ with $\epsilon < \omega < \pi - \epsilon$ then $I_N(M)$ is
equal to
$$\frac{2^{1/4}}{\pi^{1/2} N^{1/2}} e^{-N(aM^2 + bM)} \frac{1}{2|m'(2\epsilon(M))|} \sin \left( \frac{N}{2} - \frac{1}{4} \right) \left( \sin 2\omega - 2\omega \right) + \frac{3\pi}{4} + \alpha(M) \right) (1 + O(N^{-1})).$$
3. If $M \leq -\sqrt{2}$ then $I_N(M) = \frac{c}{N^{1/2}} e^{-N\lambda(a,b,M)}$ where $\lambda(a,b,M)$ is the minimum of
$ax^2 + bx + I_1(-x)$ in $[M, -\sqrt{2}]$ and $c$ is a positive constant that depends on $a$, $b$
and $M$.

A few comments before the proof of the above lemma. First, under the assumption
that $a > 1/2$ and $b > 0$ the major contribution to the integral in part (2) comes from
a small neighborhood of $M$, instead of the minimum of $ax^2 + bx$. This is due to
rapid oscillations of $\phi_{N-1}$ inside the ”bulk” $(-\sqrt{2}, \sqrt{2})$. Second, in part (3),
the condition that the minimizer of $ax^2 + bx + I_1(-x)$ lies inside $[M, -\sqrt{2}]$ is similar
to the condition on (5.11). It will lead to the asymptotic Euler characteristic in the region $u < -E'_\infty$. t The
Lemma 8.3 (Plancherel - Rotach). There exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$ the following asymptotics hold uniformly in each region:

1. If $x < -\sqrt{2} - \delta$,
\[
\phi_{N-1}(\sqrt{N}x) = (-1)^{N-1}e^{-N(I_1(-x))} e^{-\sqrt{N}h(x)(1 + O(N^{-1}))}.
\]

2. If $-\sqrt{2} - \delta < x < -\sqrt{2} + \delta$,
\[
\phi_{N-1}(\sqrt{N}x) = \frac{(-1)^{N-1}}{(2N)^{1/4}} \left\{ \frac{|x - \sqrt{2}|^{1/4} |3N/2|I_1(-x)|^{1/2} \sin \left( \frac{N}{2} + \frac{1}{4} \right) (\sin 2\omega - 2\omega) + \frac{3\pi}{4} \right\} (1 + O(N^{-1})).
\]

where $\sin \left( \frac{N}{2} + \frac{1}{4} \right)$.

3. If $-\sqrt{2} + \delta < x < \sqrt{2} - \delta$ and if we set $x = \sqrt{2}\cos \omega$ with $\epsilon < \omega < \pi - \epsilon$ then
\[
\phi_{N-1}(\sqrt{N}x) = \frac{e^{-N(I_1(x))}}{\sqrt{4\pi\sqrt{2N}}} h(x)(1 + O(N^{-1})).
\]

Proof of Lemma 8.2: Part (1): We can use the uniform asymptotics given by the exponential region (4) in Lemma 8.3. Precisely, by hypothesis, the function $K(x) := ax^2 + bx + I_1(x)$ is increasing in $[M, \infty)$ and by Laplace’s method:
\[
I_N(M) = \int_M^\infty \frac{e^{-N(ax^2+bx+I_1(x))}}{\sqrt{4\pi\sqrt{2N}}} h(x)(1 + O(N^{-1}))dx
\]
\[
= \frac{e^{-NK(M)}}{N|K'(M)|\sqrt{4\pi\sqrt{2N}}} h(M)(1 + O(N^{-1})).
\]

Part (2): Choose $\delta < \delta_0$ such that $-\sqrt{2} < M < \sqrt{2} - \delta$. We split the integral $I_N(M)$ into three parts
\[
I_N(M) = \left( \int_M^{\sqrt{2} - \delta} + \int_{\sqrt{2} - \delta}^{\sqrt{2} + \delta} + \int_{\sqrt{2} + \delta}^\infty \right) = I_1(M) + I_2 + I_3.
\]

We will show that the main contribution in this case comes from $I_1(M)$. As in part (1), it is easy to see that
\[
I_3 = O(e^{-NK(\sqrt{2})}).
\]
Next since $|x^{1/4} \text{Ai}(x)|$ and $|x^{-1/4} \text{Ai}'(x)|$ are bounded functions on $\mathbb{R}$ a change of variables $z = I_1(-x)$ when using Part (2) of Lemma 8.3 immediately implies that for any $\epsilon > 0$:

$$I_2 = O(e^{-N(a(\sqrt{2} - \delta)^2 + b(\sqrt{2} - \delta) + \epsilon}). \quad (8.5)$$

Now we estimate $I_1(M)$. Using the uniform asymptotics of $\phi_{N-1}$ we need to evaluate

$$\frac{2^{1/4}}{\pi^{1/2} N^{1/4}} \int_M \frac{e^{-N(ax^2 + bx)}}{(\sin \omega)^{1/4}} \sin \left( \left( \frac{N}{2} - \frac{1}{4} \right) (\sin 2\omega - 2\omega) + \frac{3\pi}{4} \right) dx. \quad (8.6)$$

Performing the change of variables $x = \sqrt{2} \cos \omega$, $0 < \omega < \pi$ the integral above becomes (for some different $\delta > 0$)

$$\sqrt{2} \int_{\iota(M)}^{\pi - \delta} e^{-N(2a \cos^2 \omega + \sqrt{2} b \cos \omega)} (\sin \omega)^{1/4} \sin \left( \left( \frac{N}{2} - \frac{1}{4} \right) (\sin 2\omega - 2\omega) + \frac{3\pi}{4} \right) d\omega \quad (8.7)$$

for $\iota(M) = \arccos(2^{1/2} M)$. We now rewrite $\cos^2 \omega = \frac{1 + \cos 2\omega}{2}$ and use the substitution $2\omega = z$ to obtain the integral

$$\frac{1}{\sqrt{2}} \int_{2\iota(M)}^{2\pi - 2\delta} e^{-N(a + a \cos z + \frac{b}{\sqrt{2}} \cos \frac{z}{2})} (\sin \frac{z}{2})^{1/4} \sin \left( \left( \frac{N}{2} - \frac{1}{4} \right) (\sin z - z) + \frac{3\pi}{4} \right) dz. \quad (8.8)$$

Last, we write

$$\sin \left( \left( \frac{N}{2} - \frac{1}{4} \right) (\sin z - z) + \frac{3\pi}{4} \right) = \frac{1}{2i} \left[ e^{i(\frac{N}{2} - \frac{1}{4})z} e^{if_1(z)} - e^{-i(\frac{N}{2} - \frac{1}{4})z} e^{-if_1(z)} \right], \quad (8.9)$$

where $f_1(z) = -\frac{1}{4}(\sin z - z) + \frac{3\pi}{4}$.

Therefore, we just need to evaluate the asymptotics of

$$\int_{2\iota(M)}^{2\pi - 2\delta} e^{-Nm(z)} j(z) dz, \quad \int_{2\iota(M)}^{2\pi - 2\delta} e^{-Nn(z)} k(z) dz \quad (8.10)$$

where $m$ and $n$ are entire functions given by

$$m(z) = a + a \cos z + \frac{b}{\sqrt{2}} \cos \frac{z}{2} - \frac{i}{2} (\sin z - z) \quad (8.11)$$

$$n(x) = a + a \cos z + \frac{b}{\sqrt{2}} \cos \frac{z}{2} + \frac{i}{2} (\sin z - z) \quad (8.12)$$

and $j(z) = \sin(\frac{z}{2})^{1/4} e^{if_1(z)}$, $k(z) = \sin(\frac{z}{2})^{1/4} e^{-if_1(z)}$.

We will change our contour of integration and apply Laplace’s Integral in the appropriate integrals. Notice that the steepest descent paths are given by the equations

$$\Im(m(z)) = \sin x \left( a \sinh y + \frac{\cosh y}{2} \right) + \frac{b}{\sqrt{2}} \sin \frac{x}{2} \sin \frac{y}{2} - \frac{x}{2} = \text{constant} \quad (8.13)$$

$$\Im(n(z)) = \sin x \left( a \sinh y - \frac{\cosh y}{2} \right) + \frac{b}{\sqrt{2}} \sin \frac{x}{2} \sin \frac{y}{2} + \frac{x}{2} = \text{constant}. \quad (8.14)$$

The phase diagram for the steepest paths of $m$ is described as follows. First all lines $x = 2k\pi$, $k \in \mathbb{N}$ are steepest paths. Second, for every $t \in (0, 2\pi)$ the steepest path that
passes through $t$ goes from $0 - i\infty$ to $\pi + i\infty$ if $b > 0$ and from $\pi - i\infty$ to $\pi + i\infty$ if $b = 0$. The real part of $m(z)$ is given by

$$
\Re(m(z)) = \cos x \left( a \cosh y + \frac{1}{2} \sinh y \right) + a + \frac{b}{2} \cos \frac{x}{2} \cosh y - \frac{y}{2}, \quad (8.15)
$$

$$
\Re(n(z)) = \cos x \left( a \cosh y - \frac{1}{2} \sinh y \right) + a + \frac{b}{2} \cos \frac{x}{2} \cosh y + \frac{y}{2}, \quad (8.16)
$$

If we integrate $m(z)$ between two points $\alpha, \beta \in (0, 2\pi)$, we can deform our contour to be equal to the two steepest paths that connect $\alpha$ and $\beta$ to $z = 0 - i\infty$. Precisely, we deform our contour into three pieces: we first follow the steepest descent path from $\alpha$ to a point with imaginary part $y_0 < 0$, |$y_0$| large. From there we go along the straight line $y = y_0$ until we reach the steepest path that passes through $\beta$, $\gamma$, and then we integrate on this steepest path back to $\beta$. From (8.15) we see that if we choose |$y_0$| large enough, every point in the straight segment $y = y_0$ that we cross has real part $x$ sufficiently close to $0$ so $\cos x > 0$. This together with $a > 1/2$ implies that $\Re(m(z))$ diverges to infinity as $y$ goes to negative infinity. The trivial bound

$$
|\int_{\gamma} e^{-Nm(z)} j(z) dz| \leq \int_{\gamma} e^{-N\Re(m(z))} dz \sup_{z \in \gamma} |j(z)| \quad (8.17)
$$

combined with the bounded length of $\gamma$ show that the contribution of this part can be made as small as we want by choosing $y_0$ large enough.

In the two remaining paths the imaginary part of $m$ is constant and therefore we can apply Laplace’s method to get the asymptotic behavior. Since we assumed that $M < \sqrt{2}$ the contribution at $2\pi - 2\delta$ is negligible compared to the one at $2\delta(M)$. Indeed, by formula (7.2.11) of [BH91],

$$
\int_{2\delta(M)}^{2\pi - 2\delta} e^{-Nm(z)} j(z) dz = \frac{e^{-Nm(2\delta(M)) + i(\pi - \alpha(M))} j(2\delta(M))}{N|m'(2\delta(M))|} (1 + O(N^{-1})), \quad (8.18)
$$

where $\alpha(M)$ is the angle of the steepest descent path of $m$ at $z = 2\delta(M)$:

$$
\alpha(M) = \arctan \left( \frac{1 - a \cos z}{2a \sin z + \frac{b \sin(x/2)}{\sqrt{2}} \right). \quad (8.19)
$$

The above argument adapted to the function $n$ implies

$$
\int_{2\delta(M)}^{2\pi - 2\delta} e^{-Nn(z)} k(z) dz = \frac{e^{-Nn(2\delta(M)) + i(\pi - \alpha(M))} k(2\delta(M))}{N|m'(2\delta(M))|} (1 + O(N^{-1})). \quad (8.20)
$$

Noting that for any $x \in (0, 2\pi)$ |$n'(x)$| = |$m'(x)$|, we can combine (8.9), (8.18) and (8.20) to recover that $I_1(M)$ is asymptotic equivalent to

$$
\frac{2^{1/4}}{\pi^{1/2} N^{1/2}} \frac{e^{-N(aM^2 + bM)}}{2|m'(2\delta(M))|(|\sin \omega|)^{1/2}} \sin \left[ \left( \frac{N}{2} - \frac{1}{4} \right)(\sin 2\omega - 2\omega) + \frac{3\pi}{4} + \alpha(M) \right] (1 + O(N^{-1})). \quad (8.21)
$$

This ends the proof of Part (2) of Lemma. The proof of part (3) follows from the proof of part (2) and Laplace’s method as in part (1) applied to the integral

$$
\int_{M}^{-\sqrt{2} - \delta} e^{ax^2 + bx + I_1(-x)} h(x) dx = O(e^{-N\lambda(M,a,b)}).
$$

We leave the details to the reader. □
We now turn to the proof of Theorem 4.2.

Proof of Theorem 4.2. We can rewrite (4.4) as

\[
\mathbb{E}(A_u) = (-1)^{N-1} \left( \frac{\nu''}{\nu'} \right)^{N-1} \frac{N-1}{2} c(N, \nu) \\
\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{N-1}(\sqrt{N}(\nu' - \alpha y)) e^{-N\nu''(x^2 + y^2)} e^{\frac{\nu}{2}(\nu' \alpha y - \alpha^2 y^2)} \, dx \, dy
\]

where

\[
c(N, \nu) = 2\nu''([N - 1]!\sqrt{\pi})^{1/2} 2^{-\frac{N-1}{2}} \frac{N}{\sqrt{\pi} \Gamma(N)}.
\]  (8.22)

For the case \(\alpha \neq 0\), we can change variables \(z = \nu' x - \alpha y, \ w = \alpha x + \nu' y\) to get

\[
x = \left( \nu' + \alpha w \right) \left( \frac{1}{\alpha^2 + \nu'^2} \right), \quad y = \left( \nu' w - \alpha z \right) \left( \frac{1}{\alpha^2 + \nu'^2} \right),
\]

and the above double integral becomes (using \(\alpha^2 = \nu'' + \nu' - \nu'^2\)):

\[
\frac{1}{\nu'' + \nu'} \int \int_{\nu' z + \alpha w \leq (\nu'' + \nu')} \phi_{N-1}(\sqrt{N}z)e^{-\frac{N\nu''(z^2 + w^2)}{\nu'' + \nu'}} e^{\frac{\nu}{2} z^2} \, dz \, dw.
\]

So we have to evaluate the asymptotic behavior of the following integral:

\[
J = \int_{-\infty}^{\infty} \phi_{N-1}(\sqrt{N}z)e^{-\frac{N\nu''(z^2 + w^2)}{\nu'' + \nu'}} e^{\frac{\nu}{2} z^2} \, dz \, dw.
\]

We write the outside integral \(\int_{-\infty}^{\infty} \, dz\) as \(\int_{-\infty}^{M} + \int_{M}^{\infty}\) with \(M = \frac{(\nu' + \nu'')u}{\sqrt{2\nu'^2}}\). The inside integral is just a Gaussian integral and therefore after a straight-forward computation the problem amounts to compute the asymptotics of the three following one-dimensional integrals:

\[
J_1 = \int_{-\infty}^{\infty} \phi_{N-1}(\sqrt{N}z)e^{-\frac{N\nu''(z^2 + w^2)}{\nu'' + \nu'}} e^{\frac{\nu}{2} z^2} \, dz
\]

\[
J_2 = \int_{-\infty}^{\infty} \phi_{N-1}(\sqrt{N}z)e^{-\frac{N\nu''(z^2 + w^2)}{\nu'' + \nu'}} e^{\frac{\nu}{2} z^2} \, dz
\]

as \(J = (J_1 + J_2)(1 + O(N^{-1/2}))\) if \(N\) is even and \(J = (J_1 - J_2)(1 + O(N^{-1/2}))\) if \(N\) is odd. Take \(u \leq 0\). We use Lemma 8.2 in both cases. Note that by (2.9), \(a = \frac{\nu'^2 + \nu'' - \nu'}{2(\nu'' + \nu' - \nu'^2)} > \frac{1}{2}\) and \(b = -\frac{\sqrt{2\nu'^2}u}{\nu'' + \nu' - \nu'^2} \geq 0\).

Now the condition \(M \leq -\sqrt{2} (M > -\sqrt{2})\) is exactly the condition \(u \leq -E'_\infty\) \((u > -E'_\infty)\). Applying the appropriate cases of Lemma 8.2 we see that the integral \(J_2\) is negligible compared to \(J_1\). A comparison with (2.23) and (5.11) gives the proof of part (1) and part (2) of the Theorem with \(a\) and \(b\) as above,

\[
\alpha(w) = \arctan \left( \frac{1 - a \cos \omega}{2a \sin \omega + \frac{b \sin(\omega/2)}{\sqrt{2}}} \right) \quad \text{and} \quad f(\omega) = \left( |m'(2\omega)| \sin^{1/2} \omega \right)^{-1}
\]  (8.23)

where \(m\) is given in (8.11). Part (3) follows from symmetry of the Hamiltonian and (4.3). \(\square\)
References

[ABC10] A. Auffinger, G. Ben Arous, and J. Cerny, *Complexity of spin glasses and random matrices*, Submitted. Available online at arxiv[0310.1129] (2010).

[AT07] R. J. Adler and J. E. Taylor, *Random fields and geometry*, Springer Science, 2007.

[AW09] J. M. Azais and M. Wschebor, *Level sets and extrema of random processes and fields*, John Wiley & Son, 2009.

[BG97] G. Ben Arous and A. Guionnet, *Large deviations for wigner’s law and voiculescu’s non-commutative entropy*, Probab. Theory Related Fields 108 (1997), no. 4, 517–542.

[BH91] N. Bleistein and R. A. Handelsman, *Asymptotic expansions of integrals*, Dover Publications, Inc., New York, 1991.

[C CGG99] A. Cavagna, J. Garrahan, and I. Giardina, *Quenched complexity of the mean-field p-spin spherical model with external magnetic field*, Journal of Physics A: Mathematical and General 32 (1999), no. 5, 711.

[CGPM05] A. Cavagna, I. Giardina, G. Parisi, and M. Mezard, *On the formal equivalence of the tap and thermodynamic methods in the sk model*, J. Phys. A: Math. Gen. 36 (2005), 1175.

[CL04] A. Crisanti and L. Leuzzi, *Spherical 2+p spin-glass model: An exactly solvable model for glass to spin-glass transition*, Phys. Rev. Lett. 93 (2004), no. 21, 21203–21207.

[DZ98] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, Springer Science, 1998.

[FW07] Y. V. Fyodorov and I. Williams, *Replica symmetry breaking condition exposed by random matrix calculation of landscape complexity*, J. Stat. Phys. 129 (2007), no. 5–6, 1081–1116.

[Fyo04] Yan V. Fyodorov, *Complexity of random energy landscapes, glass transition and absolute value of spectral determinant of random matrices*, Physical Rev. Let. 92 (2004), 240601.

[Gue03] F. Guerra, *Broken replica symmetry bounds in the mean field spin glass model*, Commun. Math. Phys. 233 (2003), 1–12.

[PR29] M. Plancherel and W. Rotach, *Sur les valeurs asymptotiques des polynomes de hermite*, Comm. Math. Helv. 1 (1929), 227–254.

[Roc70] R. Tyrrell Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. MR 0274683 (43 #445)

[Sch42] I. J. Schoenberg, *Positive definite functions on spheres*, Duke Math. J. 9 (1942), 96–108.

[TAL03] Jonathan E. Taylor and Robert J. Adler, *Euler characteristics for Gaussian fields on manifolds*, Ann. Probab. 31 (2003), no. 2, 533–563. MR 1964940 (2005g:58072)

[Tal03] M. Talagrand, *Spin glasses: A challenge for mathematicians*, Springer, 2003.

[Tal06a] Talagrand, *Spin glasses: A challenge for mathematicians*, Springer, 2003.

[Tal06b] M. Talagrand and D. Panchenko, *On the overlap in the multiple spherical sk models*, The Annals of Probability 35 (2007), no. 6, 2321–2355.

[T TA05] Jonathan Taylor, Akimichi Takemura, and Robert J. Adler, *Validity of the expected Euler characteristic heuristic*, Ann. Probab. 33 (2005), no. 4, 1362–1396. MR 2150192 (2006b:60073)

[War71] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Scott, Foresman and Co., Glenview, Ill.-London, 1971. MR 0295244 (45 #3142)

A. AUFFINGER, UNIVERSITY OF CHICAGO, 5734 S. UNIVERSITY AVENUE, CHICAGO, IL 60637, USA

E-mail address: auffing@math.uchicago.edu

G. BEN AROUS, COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, 251 MERCER STREET, NEW YORK, NY 10012, USA

E-mail address: benarous@cims.nyu.edu