Representations of the LL BFKL Kernel and the Bootstrap

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Different forms of the BFKL kernel both in coordinate and momentum representations may appear as a result of different gauge choices and/or inner scalar products of the color singlet states. We study a spectral representation of the BFKL kernel not defined on the Möbius space of functions but on a deformation of it, which provides the usual bootstrap property due to gluon reggeization. In this space the corresponding symmetry is made explicit introducing a deformed realization of the \( sl(2, \mathbb{C}) \) algebra.

Introduction

Small \( x \) physics in perturbative QCD has been long investigated, after the pioneering work in the LL approximation devoted to the study of the asymptotical behavior in the high energy limit of the physical cross sections\(^\text{[1]}\). The kinematics of such processes is related to the Regge limit wherein one observes the high energy factorization of the scattering amplitudes in terms of external particle impact factors and a Green’s function, which exponentiates the BFKL kernel, describing the evolution in rapidity of the states in the t-channel.

The leading energy contribution to a perturbative cross section is typically associated to the exchange of the BFKL Pomeron, which in the lowest approximation is realized as the exchange of a pair of two interacting reggeized gluons in the color singlet state. The color singlet impact factors by gauge invariance vanish as one of the two reggeized gluons carries a zero transverse momentum so that the BFKL kernel (Green’s function) is not unique: it depends for example on the gauge choices, and therefore one may choose different space of functions for the particle impact factors and the eigenstates which diagonalize the BFKL kernel dynamics.

Not less important is the color octet exchange channel: the kernel leads to an equation which has a bootstrap solution, a fundamental consistency property derived from the s-channel unitarity and corresponding to the reggeization of the gluon. This property can be formulated also in the NLL approximation\(^\text{[2]}\). Different bootstraps conditions can be defined but the most important is again the one directly related to the gluon reggeization (in the strong form), telling that all the colored impact factors are proportional to the gluon eigenstate of the octet kernel\(^\text{[2]}\).

\(^{[1]}\) see Fadin’s contribution at EDS05

\(^{[2]}\)
In the LL approximation the impact factors of the colored external particles are simply functions only depending on the total transverse momenta exchanged in the t-channel. A manifestation of this bootstrap property also takes place for the color singlet state of two gluons and inside the coupling of three or more gluons to colorless particle impact factors. In particular, it plays an important role in the Odderon solution \[^3\] which appears as a bound state of three reggeized gluons and in generalizations of it \[^4\].

After the construction of the BFKL kernel from a Feynman diagram analysis one faces two cases. For an amplitude related to the exchange of two reggeized gluons in the singlet state, due to the gauge invariance, one may move to the so called Möbius space of functions \[^5\],\[^6\], then the kernel (which acts on amputated functions) presents remarkable properties such as holomorphic factorization and Möbius symmetry. The latter is exploited in coordinate representation and the BFKL kernel can be easily diagonalized on the conformal eigenstate basis associated to the famous Polyakov ansatz.

On the other hand one is not allowed to do this when the two interacting reggeized gluons are in an octet state. The relevant kernel simply acts on a different space of functions. In the latter the singlet kernel inherits a property from bootstrap but the Möbius symmetry is no more so evident. It has been shown \[^7\] that the Möbius symmetry \(sl(2,C)\) is still a property of the singlet kernel in such a space (of non amputated functions) and that its realization is deformed. According to this one may still write a spectral, but deformed, representation of the BFKL kernel, which can be also useful in the study of coupling the BFKL kernel directly to quarks \[^8\],\[^9\].

**LL BFKL kernels, bootstrap and deformation**

In the following we shall illustrate in few steps the main result leaving out much of the details which can be found elsewhere \[^1\].

Let us start from the BFKL kernel \[^1\],\[^10\] operator which generates the Green’s function appearing in the high energy factorization of an amplitude:

\[
K^{(R)}_2 = -(\omega_1 + \omega_2) - \lambda_R V_{12},
\]

where \(R\) labels the colour representation of the two gluon state and in the singlet and octet channel one has respectively \(\lambda_1 = N_c\) and \(\lambda_8 = N_c/2\). The \(\omega\)’s are trajectories of the reggeized gluons (virtual contributions) and \(\lambda_R V_{12}\) are the real corrections due to the gluon emission strongly ordered in rapidity (Multi Regge Kinematics). The BFKL operator \(K_2\) is initially defined in momentum space and we do not need the explicit expressions here. In the octet channel the bootstrap condition reads:

\[
\bar{K}^{(8)}_{12} \otimes 1 = -\omega(q),
\]

where the barred operator is chosen to be amputated on the left and \(\otimes\) makes explicit the integration in momentum space. This relation implies on the singlet case the relation

\[
\bar{K}^{(1)}_{12} \otimes 1 = -2\omega(q) + \omega(k_1) + \omega(k_2) = \frac{1}{2} \bar{\alpha}_s \log \left( \frac{q^4}{k_1^2 k_2^2} \right),
\]

with \(\bar{\alpha}_s = \alpha_s N_c/\pi\), which is infrared finite in 4 dimensions.

In the singlet case because of gauge invariance an impact factor in momentum space is such that \(\Phi(k_1, k_2) \to 0\) as \(k_i \to 0\) and therefore we can add arbitrary terms, proportional to \(\delta^{(2)}(k_i)\), to the gluon propagators and to the BFKL kernel, since they do not alter the amplitude. In coordinate space this corresponds to the physical equivalence relation

\[
f(\rho_1, \rho_2) \sim \tilde{f}(\rho_1, \rho_2) = f(\rho_1, \rho_2) + f^{(1)}(\rho_1) + f^{(2)}(\rho_2),
\]

where

\[
\tilde{f}(\rho_1, \rho_2) = f(\rho_1, \rho_2) + f^{(1)}(\rho_1) + f^{(2)}(\rho_2).
\]
which permits to choose to work in the space of Möbius functions $f(\rho_1, \rho_2)$, such that $f(\rho, \rho) = 0$.

In such a space the operator $H_{12} = K_{12}^{(1)} 2/\alpha_s$ is Möbius invariant and can be written in the separable form

$$H_{12} = h_{12} + h_{12}^1, \quad h_{12} = \sum_{r=1}^2 \left( \ln p_r + \frac{1}{p_r} \ln(p_{12}) p_r - \Psi(1) \right), \tag{5}$$

where one may verify the invariance under action of the generators:

$$M^3_r = \rho_r \partial_r, \quad M^+ \partial_r, \quad M^- \partial_r = -\rho^2 \partial_r. \tag{6}$$

For two reggeized gluons one has $M^k = \sum_{r=1}^2 M^k_r$ and the Casimir operator is defined by $M^2 = |\vec{M}|^2 = -\rho^2 \partial_1 \partial_2$ and shares the eigenstates with the BFKL kernel of eq. (5):

$$E_{\rho,\rho}(\rho_{10}, \rho_{20}) \equiv \langle \rho|h \rangle = \left( \frac{\rho_{12}}{\rho_{10} \rho_{20}} \right)^\rho \left( \frac{\rho_{12}^*}{\rho_{10}^* \rho_{20}^*} \right)^\rho. \tag{7}$$

The BFKL kernel on the Möbius space can also be written in the dipole picture form

$$H_{12} f_\omega(\rho_1, \rho_2) = \int \frac{d^2 \rho_3}{\pi} \frac{|\rho_{12}|^2}{|\rho_{13}|^2 |\rho_{23}|^2} (f_\omega(\rho_1, \rho_2) - f_\omega(\rho_1, \rho_3) - f_\omega(\rho_2, \rho_3)). \tag{8}$$

The $sl(2, C)$ (Möbius) symmetry allows us to write a spectral representation for the kernel

$$\langle \rho|\tilde{K}_{12}^{(1)}|\rho' \rangle = \int d^2 \rho_0 \sum_h \frac{N_h}{|\rho_{12}|^4} \langle \rho|h \rangle \chi_h \langle h|\rho' \rangle. \tag{9}$$

This representation is not compatible with the eq. (3), since the state $|h\rangle$ is orthogonal to any octet impact factor (depending in momentum representation only on the total momentum $q$).

On noting that the Fourier transform of the Pomeron eigenstates, $\langle k|h \rangle = \langle k|h^A \rangle + \langle k|h^\delta \rangle$ is decomposed in the sum of two terms, the first meromorphic in the momenta and the second with $\delta(k_1)$ singularities, one can trace the appearance of the latter because of the gauge choice, made to work on the Möbius space of functions. Infact the original kernels in eq. (1) are meromorphic in the momenta.

It is therefore natural to define a completeness relation such that

$$\langle k|\tilde{K}_{12}^{(1)}|k' \rangle = \sum_h \tilde{N}_h \langle k|h^A \rangle \chi_h \langle h^A|k' \rangle, \tag{10}$$

which is associated to the scalar product (corresponding to the Analytic Feynman (AF) space)

$$\langle f|g \rangle \equiv \int d\mu(k) f^*(k_1, k_2) g(k_1, k_2), \quad d\mu(k) = k_1^2 k_2^2 \delta^2(q - k_1 - k_2) d^2 k_1 d^2 k_2. \tag{11}$$

This representation of the kernel gives the same result of the Möbius one when acting on colorless impact factors but it can also act on colored impact factors and in particular it satisfies the relation in eq. (3). In order to prove this fact we project the above relation on the spectral basis,

$$\epsilon_h (\langle h^A|P_\lambda \rangle)_{\lambda=0} = \frac{1}{2} (h^A) \frac{d}{d\lambda} (\langle P_\lambda \rangle)_{\lambda=0}, \quad \langle k|P_\lambda \rangle \equiv \frac{1}{k_1^2 k_2^2} \left( \frac{q^4}{k_1^4 k_2^4} \right)^\lambda, \tag{12}$$

where $\epsilon_h$ is the scaled BFKL kernel eigenvalue. This new relation can be verified by direct integration in momentum space.
It is interesting to understand better the relation between the two spaces under consideration (with the particular choice made above for the scalar product in the AF space) and which are spanned by the two basis $|h\rangle (M)$ and $|h^A\rangle (AF)$. Let us write the mapping between the two spaces in coordinate representation:

$$\Phi^{-1} : M \rightarrow AF, \quad E^A_h(\rho_{10}, \rho_{20}) = E^M_h(\rho_{10}, \rho_{20}) - \lim_{\rho_1 \to \infty} E^M_h(\rho_{10}, \rho_{20}) - \lim_{\rho_2 \to \infty} E^M_h(\rho_{10}, \rho_{20})$$

$$= E^M_h(\rho_{10}, \rho_{20}) - 2^{h-h^A} \left( E^M_h(\rho_{10}, -\rho_{10}) + E^M_h(-\rho_{20}, \rho_{20}) \right), \quad (13)$$

$$\Phi : AF \rightarrow M ; \quad E^M_h(\rho_{10}, \rho_{20}) = E^A_h(\rho_{10}, \rho_{20}) + \frac{1}{2^{h^A-h}} \left( E^A_h(\rho_{10}, -\rho_{10}) + E^A_h(-\rho_{20}, \rho_{20}) \right).$$

This mapping is therefore related to a special shift, defined by its projection on any basis vector. One can easily check that

$$\Phi \Phi^{-1} \equiv I_M, \quad \Phi^{-1} \Phi \equiv I_{AF} \quad (14)$$

so that one can really define, by means of this 1-1 mapping, the action of an operator given on one space also on the other space. In particular, by this similarity transformation we can define the action of the M"{o}bius group generators on the AF space:

$$\hat{M}_p^{AF} = \Phi^{-1} \hat{M}_p \Phi. \quad (15)$$

An explicit realization of these operators can be found\(^{[7]}\) and it turns out to be simpler to derive it from consideration in the momentum representation. Only the $M^-$ generator has a form different in the M and in the AF spaces and consequently also the Casimir is different. Let us note that with other choicea of the scalar product also the consideration changes. For example considering functions amputated implies that no $\delta$-like behavior is present and one has meromorphic functions from the beginning.

One interesting general fact which can be inferred is that the freedom in the gauge and scalar product choices can make a symmetry more or less evident and realized in different ways on the associated different space of functions.

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