SYNCHRONIZATION IN COUPLED STOCHASTIC SINE-GORDON WAVE MODEL

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Dedicated to Björn Schmalfuß on his 60th birthday

Abstract. The asymptotic synchronization at the level of global random attractors is investigated for a class of coupled stochastic second order in time evolution equations. The main focus is on sine-Gordon type models perturbed by additive white noise. The model describes distributed Josephson junctions. The analysis makes extensive use of the method of quasi-stability.

1. Introduction. Our main object of interest is the following model of \( N \) coupled sine-Gordon equations of the form

\[
\begin{align*}
\frac{d^2 u_1}{dt^2} - \nu \Delta u_1 + \gamma u_1' + \kappa (u_1' - u_2') + \lambda_1 \sin(u_1 + \alpha_1) &= f_1(x) + \dot{W}_1, \\
\frac{d^2 u_j}{dt^2} - \nu \Delta u_j + \gamma u_j' - \kappa (u_{j+1}' - 2u_j' + u_{j-1}') + \lambda_j \sin(u_j + \alpha_j) &= f_j(x) + \dot{W}_j, \\
\frac{d^2 u_N}{dt^2} - \nu \Delta u_N + \gamma u_N' + \kappa (u_N' - u_{N-1}') + \lambda_N \sin(u_N + \alpha_N) &= f_N(x) + \dot{W}_N,
\end{align*}
\]

on a smooth bounded domain \( \mathcal{O} \subset \mathbb{R}^d \) with the Neumann boundary conditions

\[
\frac{\partial u_j}{\partial n} \bigg|_{\partial \mathcal{O}} = 0, \quad j = 1, \ldots, N,
\]

and initial data

\[
u^j(0) = u_0^j, \quad u_1^j(0) = u_1^j, \quad j = 1, \ldots, N.
\]

Here \( \nu, \gamma, \kappa, \lambda, \alpha \) are real parameters, \( \dot{W}_j \) are trace-class white noise processes and \( f_j \) are given functions.

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As a particular case we can consider the model of two coupled equations:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial u}{\partial t} - \Delta u + \varepsilon (u - v) + \lambda_1 \sin u &= f_1(x) + \dot{W}_1, \\
\frac{\partial^2 v}{\partial t^2} + \gamma \frac{\partial v}{\partial t} - \Delta v + \varepsilon (v - u) + \lambda_2 \sin v &= f_2(x) + \dot{W}_2,
\end{align*}
\]

in a smooth domain \( \mathcal{O} \subset \mathbb{R}^d \) with the Neumann boundary conditions

\[
\frac{\partial u}{\partial n} \Big|_{\partial \mathcal{O}} = 0, \quad \frac{\partial v}{\partial n} \Big|_{\partial \mathcal{O}} = 0.
\]

Our goal is to study the long-time dynamics of the random dynamical system generated by the system (1). Under certain hypotheses we first prove the existence of a global random attractor and study its dependence on the interaction parameter \( \varepsilon \). Then we apply these results to analyse various synchronization phenomena, which we understand at the level of global attractors, i.e., in the synchronized regime the attractor of coupled system becomes “diagonal” in some sense. We also discuss briefly the possibility of synchronization in infinite-dimensional systems by means of finite-dimensional interaction operators.

We note that recently the subject of synchronization of coupled (identical or not) systems has received considerable attention. There are now quite a few monographs \([3, 28, 30, 34, 41, 43]\) in this field, which contain extensive lists of references. These sources deal mainly with finite-dimensional systems. For infinite dimensional systems the synchronization problem has been studied in \([4, 7, 8, 25, 38]\) for coupled parabolic systems, while synchronization in Berger plates was considered in \([31, 32, 33]\). General (deterministic) infinite-dimensional second order in time models were studied in \([14]\) (for the deterministic second order ODE see also \([1]\) and \([25]\)), while master-slave synchronization of coupled parabolic-hyperbolic PDE systems was considered in \([12, 13]\) (see also \([20]\) for the stochastic case). We also refer to the survey in the recent monograph \([16]\). To the best of our knowledge the synchronization in coupled second order stochastic PDE systems was not considered before.

As an important technical tool we make extensive use of the quasi-stability method that was developed in \([17]\) (see also also \([18, \text{Chapter 8}]\) and \([15]\)). This is essentially an asymptotic compactness method, which involves the contraction of pairs of solutions modulo a compact term. The quasi-stability estimates lead to appropriate uniform bounds for the attractors which are important for asymptotic synchronization. In the stochastic case this approach was applied earlier in \([21]\) in the study of a stochastic fluid-structure interaction model.

We also note that the different aspects of pullback random dynamics in a single sine-Gordon equations

\[
\frac{\partial^2 u}{\partial t^2} + \gamma \frac{\partial u}{\partial t} - \Delta u + \lambda \sin u = f(x) + \dot{W},
\]

with different types of boundary conditions were considered in \([22, 23, 24, 39]\). The papers \([22, 23, 24]\) deal with a simpler case of Dirichlet boundary conditions while Neumann boundary conditions are considered in \([39]\). However, in all these cases the stochastic noise is finite-dimensional and smooth in its spatial variables. Synchronized random dynamics in the finite dimensional case, e.g., SDE version of (1), were considered in \([5, 40]\).

To the best of our knowledge interacting systems like (1) have not been considered before even in the deterministic case. An additional difficulty which impacts on description of random dynamics is degeneracy of the linear part of the problem due to the Neumann boundary conditions.
The paper is organized as follows. The next Section 2 is devoted to preliminary considerations. There we describe our abstract model, formulate main hypotheses and provide well-posedness result for rather general situation. Our main results on attractors are presented in Section 3 and those on synchronization in Section 4. Finally, in Section 5 we briefly discuss other possible applications.

2. Preliminaries. First, we describe the problem and state our basic notations and hypotheses. We also discuss the basic notions from the theory of dissipative random dynamical systems (RDS) and then provide a result on the generation of an RDS by the system of equations (1).

2.1. Abstract model and main hypotheses. The model in (1) is a particular case of the following system of equations in a Hilbert space $H$:

\begin{align}
\begin{cases}
\frac{\partial u_1}{\partial t} + \nu A u_1 + D u_1 + \varepsilon K(u^1 - u^2) + B_1(u^1) = \dot{W}_1, \\
\frac{\partial u_2}{\partial t} + \nu A u_2 + D u_2 - \varepsilon K(u^{j+1} - 2u^j + u^{j-1}) + B_j(u^j) = \dot{W}_j, \quad j = 2, \ldots, N - 1, \\
\frac{\partial u_N}{\partial t} + \nu A u_N + D u_N + \varepsilon K(u^N - u^{N-1}) + B_N(u^N) = \dot{W}_N,
\end{cases}
\end{align}

equipped with the initial data

\begin{align}
u^j(0) = u^j_0, \quad u^j(0) = u^j_1, \quad j = 1, \ldots, N.
\end{align}

To obtain (1) from (2) we need to set $H = L_2(\Omega)$, $A = -\Delta$ with the domain

\[ D(A) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial n} |_{\partial \Omega} = 0 \right\}, \]

and also to take $D = \gamma I$, $K = I$ and $B_i(u) = \lambda_i \sin(u + \alpha_i)$.

We impose the following hypotheses.

Assumption 2.1. (i) $A$ is a self-adjoint nonnegative operator densely defined on a domain $D(A)$ in a separable Hilbert space $H$. We assume that $\dim \ker A = 1$ and the resolvent of $A$ is compact in $H$, which implies that there is an orthonormal basis $\{e_k\}_{k=1}^\infty$ in $H$ consisting of the eigenvectors of the operator $A$:

\[ Ae_k = \lambda_k e_k, \quad 0 \equiv \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lim_{k \to \infty} \lambda_k = \infty. \]

We denote by $\| \cdot \|$ and $\langle \cdot , \cdot \rangle$ the norm and the scalar product in $H$ respectively. We also denote by $H^s$ (with $s > 0$) the domain $D(A^s)$ equipped with the graph norm $\| \cdot \|_s = \|(I + A)^s \cdot \|$. $H^{-s}$ denotes the completion of $H$ with respect to the norm $\| \cdot \|_{-s} = \|(I + A)^{-s} \cdot \|$. The symbol $\langle \cdot , \cdot \rangle$ denotes not only the scalar product but also the duality between $H^s$ and $H^{-s}$. Below we also use the notation $H^s = H^s \times \cdots \times H^s = [H^s]^N$. The norm in $H^s$ is denoted by the same symbol as in $H^s$.

(ii) The damping operator $D$ is a linear positive self-adjoint operator and $D : H^{1/2} \rightarrow H^{-1/2}$ is a bounded mapping. In particular, there exist $c_1, c_2 > 0$ such that

\[ c_1 \|u\|^2 \leq (Du, u) \leq c_2 \|A^{1/2}u\|^2, \quad u \in H^{1/2}. \]

(iii) The interaction operator $K$ is a linear positive self-adjoint operator and $K : H^{1/2} \rightarrow H$. 
The nonlinear operators $B_i : H^{1/2} \to H$ are bounded and (globally) Lipschitz, i.e., there exist constants $M_B$ and $L_B$ such that
\[
\|B_i(u)\| \leq M_B, \quad \|B_i(u) - B_i(v)\| \leq L_B\|u - v\|^{1/2}, \quad i = 1, \ldots, N
\]
for all $u, v \in H^{1/2}$. In addition, we assume that $B_i$ are $d$-periodic in the direction $e_0$ (recall that $e_0 = 0$ and $\|e\| = 1$):
\[
\forall u \in H^{1/2} : \quad B_i(u + de_0) = B_i(u), \quad n = 0, \pm 1, \pm 2, \ldots
\]
(iv) The nonlinear operators $B_i : H^{1/2} \to H$ are bounded and (globally) Lipschitz, i.e., there exist constants $M_B$ and $L_B$ such that
\[
\|B_i(u)\| \leq M_B, \quad \|B_i(u) - B_i(v)\| \leq L_B\|u - v\|^{1/2}, \quad i = 1, \ldots, N
\]
for all $u, v \in H^{1/2}$. In addition, we assume that $B_i$ are $d$-periodic in the direction $e_0$ (recall that $e_0 = 0$ and $\|e\| = 1$):
\[
\forall u \in H^{1/2} : \quad B_i(u + de_0) = B_i(u), \quad n = 0, \pm 1, \pm 2, \ldots
\]
(v) $W_i$ is a continuous trace-class Brownian motion on $H$ with covariance operator $K_i$, i.e., with $\text{tr}_H K_i < \infty$, $i = 1, \ldots, N$. We do not assume that the $W_i$ are independent.

The system in (2) can be written as a single equation
\[
U_t + \nu AU + \partial U_t + \kappa KU + B(U) = \dot{W}, \quad U(0) = U_0, \quad U_0(0) = U_1, \quad (3)
\]
with $A = \text{diag} (1, \ldots, 1) \cdot A$, $D = \text{diag} (1, \ldots, 1) D$ and
\[
K = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 \\
0 & -1 & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}, \quad (4)
\]
as well as $B(U) = (B_1(u^1), \ldots, B_N(u^N))$, $U = (u^1, \ldots, u^N)$, and $W = (W_1, \ldots, W_N)$.}

2.2. **Random dynamical systems.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a measurable mapping
\[
\theta : (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}) \to (\Omega, \mathcal{F})
\]
satisfying the flow property:
\[
\theta(0, \cdot) = \text{id}_\Omega, \quad \theta(t, \cdot) \circ \theta(\tau, \cdot) = \theta(t + \tau, \cdot)
\]
and, for simplicity, write $\theta(t, \omega) = \theta_t \omega$. We assume that $\theta$ preserves the measure $\mathbb{P}$, i.e., $\theta_\omega \mathbb{P} = \mathbb{P}$ for $t \in \mathbb{R}$. Then $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system.

Let $H$ be a separable Banach space. A measurable mapping $\phi : (\mathbb{R}^+ \times \Omega \times H, \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(H)) \to (H, \mathcal{B}(H))$ satisfying the cocycle property
\[
\phi(t + \tau, \omega, u) = \phi(t, \theta_t \omega, \phi(\tau, \omega, u)) \quad \text{for } t, \tau \geq 0, \quad u \in H, \quad \omega \in \Omega,
\]
\[
\phi(0, \omega, \cdot) = \text{id}_H \quad \text{for } \omega \in \Omega, \quad t, \tau \in \mathbb{R}
\]
is called a random dynamical system (RDS). This RDS is called continuous if the mapping $u \mapsto \phi(t, \omega, u)$ is continuous for every $t > 0$ and $\omega \in \Omega$.

A mapping $\Omega \ni \omega \mapsto C(\omega) \neq \emptyset$ with closed values is called a random set (in $H$) if the mapping
\[
\omega \ni \Omega \ni \inf_{x \in C(\omega)} \|x - y\|_H
\]
is a random variable for any $y \in H$, see Castaing & Valadier [9], Chapter III.

A random variable $X \geq 0$ is called tempered if
\[
\lim_{t \to \pm \infty} \frac{\log^+ X(\theta_t \omega)}{|t|} = 0.
\]

\footnote{In the case of the coupled sine-Gordon system in (1) we have $e_0 = (\text{Vol}(\mathcal{O}))^{-1/2}$ and $d = 2\pi(\text{Vol}(\mathcal{O}))^{1/2}$.}
This condition can be written (see [2]) in the form
\[ \forall \beta > 0, \forall \omega \in \Omega : \sup_{t \in \mathbb{R}} \{ X(\theta_{t}\omega)e^{\beta|t|} \} < +\infty. \]

A random set \( C \) is called tempered if the random variable
\[ \omega \mapsto \sup_{y \in C(\omega)} \| y \| < \infty \]
is tempered. We denote the family of all tempered random sets by \( \mathcal{D} \).

**Definition 2.2.** A random set \( B \in \mathcal{D} \) is called pullback absorbing for the random dynamical system \( \phi \) with respect to \( \mathcal{D} \) if for every \( D \in \mathcal{D} \) and \( \omega \in \Omega \) there exists a \( T = T_{D}(\omega) \geq 0 \) such that
\[ \phi(s, \theta_{-s}\omega, D(\theta_{-s}\omega)) \subset B(\omega), \ s \geq T. \]

A random set \( B \) is called positively invariant if
\[ \phi(t, \omega, B(\theta_{-t}\omega)) \subset B(\omega), \ \text{for all} \ \omega \in \Omega, \ t \geq 0. \]

A random set \( C \in \mathcal{D} \) is called pullback attracting for the random dynamical system \( \phi \) with respect to \( \mathcal{D} \) if
\[ \lim_{t \to +\infty} \text{dist}_{H}(\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), C(\omega)) = 0, \ \text{for all} \ D \in \mathcal{D}, \]
where
\[ \text{dist}_{H}(X, Y) = \sup_{x \in X} \inf_{y \in Y} \| y - x \|. \]

A RDS \( \phi \) is said to be asymptotically compact in \( \mathcal{D} \) if there exists a compact pullback attracting random set.

The main objects of this article are random attractors given by

**Definition 2.3.** A random set \( \mathfrak{A} \in \mathcal{D} \) is called a random attractor if \( \mathfrak{A}(\omega) \) is compact for \( \omega \in \Omega \), pullback attracting in the sense of Definition 2.2 and satisfies the invariance property:
\[ \phi(t, \omega, \mathfrak{A}(\omega)) = \mathfrak{A}(\theta_{t}\omega), \ \text{for} \ t \geq 0 \ \text{and} \ \omega \in \Omega. \]

The existence of a random attractor is ensured by the conditions of the next theorem.

**Theorem 2.4.** Let \( \phi \) be a continuous asymptotically compact RDS with a pullback attracting compact set \( B \) in \( \mathcal{D} \). Then there exists a random attractor \( \mathfrak{A} \) of \( \phi \) which belongs to \( B \) with
\[ \mathfrak{A}(\omega) = \bigcap_{n \in \mathbb{N}} \phi(nT, \theta_{-nT}\omega, B(\theta_{-nT}\omega)), \ \forall T > 0. \]

For the proof see, for instance, Theorem 1.8.1 in [11].

2.3. **Ornstein-Uhlenbeck processes generated by wave equations.** Consider the following linear SPDE
\[ u_{tt} + \nu Au + u + Du_{t} = \dot{W}_{i}, \] (5)
where \( \dot{W}_{i} \) is a white noise on \( H \) with covariance operator \( K_{i} \) of trace class: \( \text{tr}_{H} K_{i} < \infty \), \( i = 1, \ldots, N \), as in Assumption 2.1 (v).

More precisely, let \( \dot{W}_{i} \) be a continuous two-sided trace-class Brownian motion on \( H \) with covariance \( K_{i} \) with respect to a metric dynamical system \( (\Omega, \mathcal{F}, P, \theta) \).
Therefore, we can consider the Ornstein-Uhlenbeck process \( t \) for \( W \), the natural filtration of \( W \), is a H"{o}lder–continuous version of this Brownian motion with time set \( \mathbb{R} \). According to the properties of \( K \), we have

\[
S_t = \exp \left( \int_{-\infty}^t \sigma \, dW_s \right)
\]

where

\[
\sigma = \begin{pmatrix}
\sigma_1 & \cdots & \sigma_N
\end{pmatrix}
\]

\( \sigma \) is a tempered random variable. Let \( (F_t)_{t \in \mathbb{R}} \) be the continuous semigroup \( S_t \) in \( \mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_N \), which we will also denote by \( W \).

Lemma 2.5. There exists a (perfect) random variable \( \Lambda : \Omega \to \mathcal{H} \) such that

\[
t \mapsto \Lambda_t(\theta_t \omega)
\]

is a H"{o}lder–continuous version of \( (6) \). In particular, this mapping is continuous and \( ||\Lambda_t||_{\mathcal{H}} \) is a tempered random variable.

We will also use the notation \( \eta = (\eta^1, \ldots, \eta^N) \). This variable \( \eta \) solves the equation

\[
\eta_{tt} + \nu \mathcal{A} \eta + \mathcal{D} \eta = W_t \text{ in } \tilde{H} = H \times \cdots \times H.
\]

Moreover we have

\[
\Lambda(t) = \left( \eta(t) \right) = \int_{-\infty}^t \tilde{S}_{t-s} \, d \left( \begin{pmatrix} 0 \\ W(s, \omega) \end{pmatrix} \right), \quad t \in \mathbb{R},
\]

in the space \( \tilde{H} = \tilde{H}_1^{1/2} \times \tilde{H} \), where \( \tilde{S}_t = \text{diag}(1, \ldots, 1) \) is a strongly continuous exponentially stable semigroup in \( \mathcal{H} \) generated by the equation

\[
V_{tt} + \nu \mathcal{A} V + V + \mathcal{D} V_t = 0.
\]

2.4. Random evolution equation. Introducing new variables \( v^i = u^i - \eta^i(\theta_t \omega) \) in (3), we obtain the random evolution equation

\[
V_{tt} + \nu \mathcal{A} V + \mathcal{D} V_t + \alpha \mathcal{K} V + \mathcal{B}(V + \eta) = \xi \equiv \eta - \alpha \mathcal{K} \eta
\]

for \( V = (v^1, \ldots, v^N) \), with the initial data

\[
V(0) = V_0 \equiv U_0 - \eta(0), \quad V_t(0) = V_1 \equiv U_1 - \eta_t(0).
\]

Below we deal with mild solutions of problem (8) on an interval \([0, T]\) which we define as a pair

\[
(V, \nu) \in C([0, T]; \tilde{H}_1^{1/2} \times \tilde{H}),
\]

satisfying the integral relation

\[
\left( \begin{array}{c}
V(t) \\
\nu(t)
\end{array} \right) = \tilde{S}_t \left( \begin{array}{c}
V_0 \\
\nu_1
\end{array} \right) + \int_0^t \tilde{S}_{t-s} \left( \begin{array}{c}
0 \\
G(V(s))
\end{array} \right) ds, \quad t \in [0, T],
\]

where \( G(V) = V - \alpha \mathcal{K} V - \mathcal{B}(V + \eta) + \xi \).
Theorem 2.6. Let $T > 0$ be arbitrary. Under the Assumption 2.1 for every $(V_0, V_1) \in \mathcal{H}^{1/2} \times \mathcal{H}$ there exists an unique mild solution of (8). This solution possesses the property $D^{1/2}u_t \in L_2(0, T; \mathcal{H})$ and satisfies the energy relation
\[
\mathcal{E}(V(t), V_i(t)) + \int_s^t (\mathcal{D}V_i(\tau), V_i(\tau)) d\tau + \int_s^t (\mathcal{B}(V(\tau) + \eta(\tau)) - \xi(\tau), V_i(\tau)) d\tau = \mathcal{E}(V(s), V_i(s)),
\]
where the (quadratic) energy $\mathcal{E}$ is defined by
\[
\mathcal{E}(V_0, V_1) = \sum_{i=1}^N E(v_i^0, v_i^1) + \mathcal{E}_{\text{int}}(V_0),
\]
with $V_0 = (v_0^1, \ldots, v_0^N)$, $V_1 = (v_1^1, \ldots, v_1^N)$. Here
\[
E(u_0, u_1) = \frac{1}{2} \left( \|u_1\|^2 + \nu \|A^{1/2}u_0\|^2 \right)
\]
and
\[
\mathcal{E}_{\text{int}}(V_0) = \frac{\kappa}{2} (\mathcal{K}V_0, V_0) = \frac{\kappa}{2} \sum_{j=1}^{N-1} \|K^{1/2}(v_0^{j+1} - v_0^j)\|^2.
\]

In addition, the following estimate holds:
\[
\mathcal{E}(V(t), V_i(t)) + \frac{1}{2} \int_s^t (\mathcal{D}V_i(\tau), V_i(\tau)) d\tau \leq \mathcal{E}(V(s), V_i(s)) + b \int_s^t (1 + \|\xi(\tau)\|^2) d\tau,
\]
where $b$ is a deterministic constant independent of $\kappa$.

Moreover, if $V^1$ and $V^2$ are solutions with different initial data and $Z = V^1 - V^2$, then
\[
\|Z_i(t)\|^2 + \nu \|Z(t)\|_{1/2}^2 + \kappa \|K^{1/2}Z(t)\|^2 \\
\leq \left( \|Z_i(s)\|^2 + \nu \|Z(s)\|_{1/2}^2 + \kappa \|K^{1/2}Z(s)\|^2 \right)e^{a(t-s)}
\]
for all $t > s \geq 0$, where $a$ is a deterministic constant independent of $\kappa$.

Proof. Since the nonlinearity is bounded and globally Lipschitz we can apply the standard deterministic arguments, see, e.g., [35]. The energy relation in (10) follows from (8) by multiplication by $V_i$. This formal procedure can be justified by considering finite-dimensional approximations of (8). To prove (11) we note that
\[
|<(\mathcal{B}(V + \eta) - \xi, V_i)| \leq (M_B + \|\xi\|)\|V_i\| \leq \sqrt{2} \max\{1, M_B\} \sqrt{1 + \|\xi\|^2 \|V_i\|} \\
\leq \frac{[\max\{1, M_B\}]^2}{2\epsilon} (1 + \|\xi\|^2) + \epsilon \|V_i\|^2,
\]
for every $\epsilon > 0$. Thus choosing $\epsilon$ small enough and using Assumption 2.1(ii) we obtain (11).

To prove (12) we note that $Z$ solves the problem
\[
Z_{tt} + \nu(A + I)Z + DZ_t + \kappa KZ = F,
\]
where $F = \mathcal{B}(V_2 + \eta) - \mathcal{B}(V_1 + \eta) + \nu Z$. We obviously have that
\[
|(F, Z)| \leq (1 + L_B)\|Z\|_{1/2}\|Z_t\| \leq \frac{1}{2}(DZ_t, Z_t) + c\|Z\|_{1/2}^2.
\]
Therefore from the energy relation for (13) we have that the function
\[
\Psi(t) = \|Z(t)\|^2 + \nu \|Z(t)\|_{1/2}^2 + \kappa \|K^{1/2}Z(t)\|^2
\]
satisfies the inequality
\[
\Psi(t) \leq \Psi(s) + c \int_s^t \Psi(\tau) d\tau, \; t > s.
\]
This implies (12). \qed

**Remark 2.7.** When the nonlinearity is a potential operator there is another form of the energy relation. Let \( B_i(u) \) be the Fréchet derivative of a functional \( \Pi_i(u) \) on \( H^{1/2} \), i.e., \( B_i(u) = \Pi_i'(u) \), which means that
\[
\lim_{\|v\|_{1/2} \to 0} \frac{1}{\|v\|_{1/2}} \left[ \Pi_i(u + v) - \Pi_i(u) - (\Pi_i'(u), v) \right] = 0
\]
and define \( \tilde{\mathcal{E}}(V, V_t) = \mathcal{E}(V, V_t) + \sum_{j=1}^N \Pi_j(v^j + \eta^j) \). Then
\[
\tilde{\mathcal{E}}(V(t), V_t(t)) + \int_s^t (D\mathcal{V}_i(\tau), V_i(\tau)) d\tau
- \int_s^t \left[ (\mathcal{B}(V(\tau) + \eta(\tau)), \eta(\tau)) + (\xi(\tau), V(\tau)) \right] d\tau
= \tilde{\mathcal{E}}(V(s), V_t(s)).
\]
This energy relation involves both components of the Ornstein-Uhlenbeck process \( \Lambda \) in (6).

By Theorem 2.6 problem (8) generates a RDS \((\mathcal{H}, S_t)\) in the space
\[
\mathcal{H} = H^{1/2} \times \tilde{H} \equiv [H^{1/2}]^N \times H^N
\]
with the cocycle operator defined by the relation
\[
\phi_t(V_0, V_1) = (V(t), V_t(t)),
\]
where \( V(t) \) is a solution to problem (8). The cocycle \( \phi_t \) possesses the following symmetry property
\[
\phi_t(V_0 + d\sqrt{N}m\psi_0, V_1) = \phi_t(V_0, V_1) + (dm\sqrt{N}\psi_0, 0), \; m = \pm 1, \pm 2, \ldots, \quad (14)
\]
where \( \psi_0 = (e_0, \ldots, e_0)/\sqrt{N} \), and \( e_0 \) is the normalized eigenvector of \( A \) with the zero eigenvalue.

We also note that using the cocycle \( \phi_t \) we can define the cocycle \( \Phi_t \) in the same space \( \mathcal{H} \) by the formula
\[
\Phi_t \left( \begin{array}{c} U_0 \\ U_1 \end{array} \right) = \Lambda(\theta_t \omega) + \phi_t \left[ \begin{array}{c} U_0 \\ U_1 \end{array} \right] - \Lambda(\omega),
\]
where \( \Lambda(\theta_t \omega) = \Lambda(t) \) is given by (7). One can see that this cocycle is generated by mild solutions of stochastic PDE in (3) in the sense that \( \Phi_t(U_0, U_1) = (U(t), U_t(t)) \) almost surely, where \( (U(t), U_t(t)) \) satisfies the integral equation
\[
\begin{align*}
\begin{pmatrix} U(t) \\ U_t(t) \end{pmatrix} &= \tilde{S}_t \left( \begin{array}{c} U_0 \\ U_1 \end{array} \right) \\
+ \int_0^t \tilde{S}_{t-s} \left( \begin{array}{c} 0 \\ U(s) - \kappa K U(s) - B(U(s)) \end{array} \right) ds, + \int_0^t \tilde{S}_{t-s} d \left( \begin{array}{c} 0 \\ W(s, \omega) \end{array} \right)
\end{align*}
\]
This follows from (9) by the standard (stochastic) integration by parts (see [36]). Thus all results presented in this article can be reformulated for the cocycle $\Phi$ and SPDE (3). However, for the sake of transparency and some simplifications we prefer to deal with the cocycle $\phi$ generated by the random PDE (8).

In the same way as in [42] for a single sine-Gordon model the symmetry in (14) allows us to define the dynamics in the corresponding factor-space. We first note that every element $V$ from $\bar{H}^{1/2}$ can written in the form

$$V = v + \psi \equiv [V - (V, \psi_0)\psi_0] + (V, \psi_0)\psi_0 \equiv Q_{\psi_0}V + P_{\psi_0}V,$$

where $v \in \bar{H}^{1/2} = \{v \in H^{1/2} : (v, \psi_0) = 0\}$ and $\psi \in \mathcal{L}_{\psi_0} \equiv \{c\psi_0 : c \in \mathbb{R}\}$. Therefore, using the symmetry above, we can correctly define evolution $\phi$ in the space

$$\tilde{H} = \bar{H}^{1/2} \times \{L_{\psi_0}/d\sqrt{N}, \cdot \psi_0\} \times \bar{H} \cong \bar{H}^{1/2} \times \mathbb{T}^1 \times \bar{H},$$

where $\mathbb{T}^1 = \mathbb{R}/d\sqrt{N} \cong [0, d\sqrt{N})$ is the 1D torus. To do this we take $(v_0, c_0, v_1)$ from $\bar{H}^{1/2} \times \mathbb{T}^1 \times \bar{H}$. Then we solve (8) with the initial data $V_0 = v_0 + a_0\psi_0$ and $V_1 = v_1$, where $a_0$ is a representative for the factor-element $c_0 \in \mathbb{T}^1$. Finally, using solution $V(t)$, we define the cocycle operator $\tilde{\phi}$

$$\tilde{\phi}_t(v_0, c_0, v_1) = (V(t) - (V(t), \psi_0)\psi_0, \{(V(t), \psi_0)\}, V(t)) = (Q_{\psi_0}V(t), \{(V(t), \psi_0)\}, V(t)), \quad (15)$$

where $\{(V(t), \psi_0)\} = (V(t), \psi_0)\psi_0 \mod d\sqrt{N}$ is the element in $\mathbb{T}^1$ generated by $(V(t), \psi_0)$, i.e.

$$\{(V(t), \psi_0)\} = \{(V(t), \psi_0) + md\sqrt{N} : m = \pm 1, \pm 2, \ldots\}.$$

We note that $\tilde{H}$ is not a vector space, but it is Polish (complete separable metric space). The metric in $\tilde{H}$ can be defined by the relation

$$\text{dist}_{\tilde{H}}(Y, Y^*) = \|v_0 - v_0^*\|_{1/2} + \|v_1 - v_1^*\| + \text{dist}_{\mathbb{T}^1}(c_0, c_0^*), \quad (16)$$

where $Y = (v_0, c_0, v_1)$ and $Y^* = (v_0^*, c_0^*, v_1^*)$ are elements from $\tilde{H} = \bar{H}^{1/2} \times \mathbb{T}^1 \times \bar{H}$. The distance on the torus $\mathbb{T}^1$ can be defined as follows

$$\text{dist}_{\mathbb{T}^1}(c_0, c_0^*) = \min \{\|a_0 - a_0^*\| : a_0 \in c_0, a_0^* \in c_0^*\}.$$

3. Global random attractors. In this section we prove the existence of a global random attractor for the RDS $(\tilde{H}, \tilde{\phi})$ and study its properties.

3.1. Dissipativity. For the synchronization phenomena it is important to have effective bounds for the attractor. Below we use an approach based Lyapunov type functions.

We first introduce two orthogonal projectors $P$ and $Q$:

$$PU \equiv P \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix} = MU \cdot \mathbb{I} \text{ and } Q = I - P \text{ with } MU = \frac{1}{N} \sum_{i=1}^{N} u^i \text{ and } I = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad (17)$$

in the space $\bar{H}$. We note that the (partially) average operator $M$ maps $\bar{H}$ into $\bar{H}$.

To prove dissipativity of the RDS $(\tilde{\phi}, \theta)$ we use a modification of the standard (deterministic) method (see, e.g., [10, 42]) based on Lyapunov type functions. This
modification is necessary because the operators \( A \) and \( K \) are degenerate. This leads to a double splitting procedure. We first use \( Q \) and \( P \) components of a solution and then split the \( P \)-component using complete averaging type operator \( P_0 \) (which is orthoprojector on \( c_0 \) in \( H \)).

Thus using the projectors \( Q \) and \( P \) we can split the equation in (8) as follows

\[
v_{tt} + \nu Av + Dv_t + \varepsilon K v + QB(v + uI + \eta) = Q \xi \text{ in } \tilde{H} \equiv HQ,
\]

\[
u_{tt} + \nu Au + Du_t + MB(v + uI + \eta) = M \xi \text{ in } H.
\]

We use here the facts that \( PK = KP = 0 \) and \( PD = DP \). The pair \((v, u)\) solve (18), if and only if \( V = v + uI \) solves (8). We also note that

\[
\tilde{H} = QHQ = \left\{ U \equiv (u^1, \ldots, u^N) \in H \times \ldots \times H : \sum_{i=1}^{N} u^i = 0 \right\}.
\]

**Lemma 3.1.** The operator \( K \) is strictly positive on \( \tilde{H} \). More precisely, there exist \( c_0, k_0 > 0 \) such that

\[
(Kv, v) = \sum_{i=1}^{N-1} \|K^{1/2}(v^{i+1} - v^i)\|^2 \geq c_0 \sum_{i=1}^{N} \|K^{1/2}v^i\|^2 \geq c_0 k_0\|v\|^2
\]

for every \( v = (v^1, \ldots, v^N) \in \tilde{H} \).

**Proof.** We have

\[
(Kv, v) = \sum_{i=1}^{N-1} \|K^{1/2}(v^{i+1} - v^i)\|^2 = \sum_{m=0}^{\infty} \sum_{i=1}^{N-1} \|(K^{1/2}v^{i+1}, e_m) - (K^{1/2}v^i, e_m)\|^2.
\]

For every \( v \in \tilde{H} \) we obviously have \( \sum_{i=1}^{N} (K^{1/2}v^i, e_m) = 0 \) for every \( m = 0, 1, \ldots \). Therefore an application of a standard scalar result for finite Jacobi matrices gives the estimate in (20) with \( c_0 = 4 \sin^2 \frac{\pi}{N} \). The constant \( k_0 \) is determined by the upper lower bound of the spectrum of \( K \).

**Lemma 3.2.** Let \( v \) satisfy (18a) and define

\[
E(t) := E(v; v_t) = \frac{1}{2} \left( \|v_t(t)\|^2 + \nu\|A^{1/2}v(t)\|^2 \right).
\]

Then

\[
E(t) + \varepsilon\|K^{1/2}v(t)\|^2 \leq c_1 \left( E(s) + \varepsilon\|K^{1/2}v(s)\|^2 \right) e^{-\gamma(t-s)} + c_2 \int_s^t (1 + \|Q\xi(\tau)\|^2) e^{-\gamma(t-\tau)} d\tau,
\]

where the positive constants \( c_i \) and \( \gamma \) do not depend on \( \varepsilon \geq \varepsilon_* \).

**Proof.** We consider the functional \( \Psi_1(t) = E(t) + \Phi_1(t) \), where \( \Phi_1(t) = \rho(v, v_t) + \frac{\varepsilon}{2}(Kv, v) \). Here \( \rho \) is a positive constant, which will be chosen later. Using Lemma 3.1 one can see that there exist \( 0 < \rho_0 < 1 \) and \( \beta_i > 0 \) independent of \( \varepsilon \geq \varepsilon_* \) such that

\[
\beta_0 \left[ E(t) + \varepsilon\|K^{1/2}v(t)\|^2 \right] \leq \Psi_1(t) \leq \beta_2 \left[ E(t) + \varepsilon\|K^{1/2}v(t)\|^2 \right]
\]

for all \( \rho \in (0, \rho_0) \). Using the energy relation

\[
E(t) + \frac{\varepsilon}{2}\|K^{1/2}v(t)\|^2 = E(s) + \frac{\varepsilon}{2}\|K^{1/2}v(s)\|^2 - \int_s^t (Dv_t + B(v + uI + \eta) - \xi, v_t) d\tau
\]
for the equation in (18a) we calculate the derivative

$$
\frac{d\Psi_1}{dt} = -(Dv_t, v_t) + (-QB + Q\xi, v_t) + \rho\left[\|v_t\|^2 - (Dv_t, v) - (Av, v) - \kappa(Kv, v) - (QB - Q\xi, v)\right].
$$

Since $D = \text{diag}(1, \ldots, 1)D$ is bounded from $H^{1/2}$ into $H^{-1/2}$, we obtain

$$
|\langle Dv_t, v \rangle| \leq \varepsilon|\langle Dv, v \rangle| + \varepsilon^{-1}|\langle Dv_t, v \rangle| \leq c_1\varepsilon \left[\|A^{1/2}v\|^2 + \|v\|^2\right] + c_2\varepsilon^{-1}|\langle Dv_t, v \rangle|
$$

for all $\varepsilon > 0$. Thus, using Lemma 3.1 and the lower bound in Assumption 2.1(ii), we obtain

$$
\frac{d\Psi_1}{dt} \leq -[1 - c_1\varepsilon - \rho\varepsilon_2(1 + \varepsilon^{-1})] \langle Dv_t, v_t \rangle + C\varepsilon(M_B + \|Q\xi\|^2)
$$

$$
- \rho\left[(1 - c_3\varepsilon)E(t) + (\kappa - c_4\varepsilon)(Kv, v)\right] + \rho C\varepsilon(M_B + \|Q\xi\|^2).
$$

After an appropriate choice of $\varepsilon$ and $\rho$ we find that there exists a $b_1 > 0$ independent of $\kappa$, such that

$$
\frac{d\Psi_1}{dt} + b_1\left[E(t) + \kappa(Kv, v)\right] \leq c(1 + \|Q\xi\|^2).
$$

Moreover, it follows from (22) and (23) that

$$
\Psi_1(t) \leq \Psi_1(s) e^{-\gamma(t-s)} + c \int_s^t (1 + \|Q\xi(t)\|^2)e^{-\gamma(t-\tau)}d\tau, \quad t \geq s,
$$

for some $\gamma, c > 0$. This implies the dissipativity property in (21).

**Lemma 3.3.** Let $u(t)$ satisfy (18b) and $P_0$ be the orthoprojector in $H$ on the eigenfunction $e_0$ with $Q_0 = I - P_0$. Let

$$
\tilde{E}(t) = \|u_t(t)\|^2 + \nu\|A^{1/2}Q_0u(t)\|^2.
$$

Then, the dissipativity estimate

$$
\tilde{E}(t) \leq c_1\tilde{E}(s)e^{-\gamma(t-s)} + c_2\int_s^t (1 + \|Q_0M\xi(t)\|^2)e^{-\gamma(t-\tau)}d\tau
$$

holds, where the positive constants $c_1$ and $\gamma$ do not depend on $\kappa \geq 0$.

We note that in contrast with Lemma 3.2 the constants in dissipativity relation (25) are uniform in all nonnegative $\kappa$. We have this due to the facts that (i) equation (18b) does not contain $\kappa$ in the explicit form and (ii) the nonlinearity in (18b) which depends on $\kappa$ in the implicit form admits bounds independent of $\kappa \geq 0$.

**Proof.** The argument is almost the same as in the previous lemma. We note in particular that the operator $A$ is not degenerate in the subspace $Q_0H$. This implies that

$$
\|A^{1/2}y\|^2 \leq \|y\|^2, \quad y \in Q_0H.
$$

We consider the functional $\Psi_2(t) = \tilde{E}(t) + \rho(Q_0u, u_t)$, where, as above, $\rho$ is a positive constant, which will be chosen later. One can see that there exist $0 < \rho_0 < 1$ and $\beta_i > 0$ independent of $\kappa$ such that

$$
\beta_0\tilde{E}(t) \leq \Psi_2(t) \leq \beta_2\tilde{E}(t)
$$

(26)
for all $\rho \in (0, \rho_0]$. Using the corresponding energy relation we calculate the derivative
\[
\frac{d\Psi_2}{dt} = -(Du_t, u_t) + (-Q_0MB + Q_0M\xi, u_t) + \rho[||Q_0u_t||^2 - (Du_t, Q_0u) - (AQ_0u, Q_0u) - (Q_0MB - Q_0M\xi, Q_0u)].
\]
As above,
\[
|(Du_t, Q_0u)| \leq \varepsilon(DQ_0u, Q_0u) + c\varepsilon^{-1}(Du_t, u_t) \leq c_1\varepsilon||A^{1/2}Q_0u||^2 + c_2\varepsilon^{-1}(Du_t, u_t)
\]
for all $\varepsilon > 0$. Thus, after an appropriate choice of $\varepsilon$ and $\rho$, we find that there exist $b_i > 0$ independent of $\varepsilon$ such that
\[
\frac{d\Psi_2}{dt} \leq -b_1(Du_t, u_t) - b_2||A^{1/2}Q_0u||^2 + b_3(1 + ||Q_0M\xi||^2).
\]
Using (26) we obtain that
\[
\Psi_2(t) \leq \Psi_2(s)e^{-\gamma(t-s)} + c\int_s^t (1 + ||Q_0M\xi||^2)e^{-\gamma(t-\tau)}d\tau, \ t \geq s, \quad (27)
\]
for some $\gamma, c > 0$. This implies the dissipativity property in (25). \qed

Now we are in position to state and prove a result on dissipativity of the RDS $(\tilde{\phi}, \theta)$.

**Theorem 3.4.** Let Assumption 2.1 hold. Then for every $\varkappa > 0$ the RDS $(\tilde{\phi}, \theta)$ is dissipative in $\mathcal{H}$, so there exists a tempered random variable $R_{\varkappa}(\omega)$ such that, for every tempered set $D(\omega)$ in $\mathcal{H}$,
\[
\tilde{\phi}_t(\theta^{-t}\omega)D(\theta^{-t}\omega) \subset \mathcal{B}_{\varkappa}(\omega)
\]
eq \{Y = (v_0, c_0, v_1) \in \tilde{H} \times T^1 \times \tilde{H} : \ ||v_0||_{\varkappa}^2 + ||v_1||^2 \leq R_{\varkappa}^2(\omega)\}

for all $t \geq T_B(\omega)$. Moreover, there exists a forward invariant pullback absorbing random set $\mathcal{B}_{\varkappa}^0(\omega)$ inside $\mathcal{B}_{\varkappa}(\omega)$.

If $Q\eta = 0$ (i.e., if the $W_i$ are identical), then the size $R_{\varkappa}(\omega)$ of the absorbing ball is independent of $\varkappa > \varkappa_0 > 0$. More precisely, there exists a tempered random variable $R(\omega)$ independent of $\varkappa$ such that the set
\[
\mathcal{B}_{\varkappa}^*(\omega)
\]
eq \{Y = (v_0, c_0, v_1) \in \tilde{H} \times T^1 \times \tilde{H} : \ ||v_0||_{\varkappa}^2 + ||v_1||^2 \leq R^2(\omega)\}

is pullback absorbing.

**Proof.** Let $\mathcal{R} : \tilde{H} \rightarrow \tilde{H}$ be defined by $\mathcal{R}(v_0, c_0, v_1) = (v_0, v_1)$. Let
\[
||(v_0, v_1)||_{\varkappa}^2 = ||v_0||^2 + ||A^{1/2}Q_0v_0||^2 + ||A^{1/2}Q_0Mv_0||^2 + \varkappa||K^{1/2}v_0||^2.
\]
Then Lemmas 3.2 and 3.3 yield
\[
|\mathcal{R}\tilde{\phi}_t(\omega)Y(\omega)||_{\varkappa}^2 \leq c_1||\mathcal{R}Y(\omega)||_{\varkappa}^2e^{-\gamma t} + c_2\int_0^t q(\theta_\tau\omega)e^{-\gamma(t-\tau)}d\tau,
\]
where
\[
Y \in \tilde{H} \text{ and } q(\omega) = 1 + ||Q_\omega(\omega)||^2 + ||Q_0M\xi(\omega)||^2.
\]
If \( Y \) belongs to a tempered bounded set, then, after change of integration variable, we obtain
\[
|\mathcal{R}_0(\theta_{-t}\omega)Y(\theta_{-t}\omega)|^2 \leq C(\theta_{-t}\omega)e^{-\gamma t} + c_2 \int_0^t q(\theta_{-\tau}\omega)e^{-\gamma \tau}d\tau,
\]
where \( C(\omega) \) is a tempered random variable. This implies the first statement of Theorem 3.4 with
\[
R^2(\omega) = c_2 \int_0^{+\infty} \left(1 + \|Q\xi(\theta_{-\tau}\omega)\|^2 + \|Q_0M\xi(\theta_{-\tau}\omega)\|^2\right)e^{-\gamma \tau}d\tau.
\]
In the case when \( Q\eta = 0 \) we have \( Q\xi = 0 \) and \( M\xi = M\eta \). Thus \( R(\omega) \) does not depend on \( \kappa \) in this case.

The existence of a forward invariant pullback absorbing subset \( \mathcal{B}_0^\omega(\omega) \) in \( \mathcal{B}_\omega(\omega) \) follows from (24) and (27). This set has the form
\[
\mathcal{B}_0^\omega(\omega) = \{(v_0, c_0, v_1) : \|v_0; v_1\|^2 + \rho[(Qv_0, v_1) + (Q_0Mv_0, Mv_1)] \leq cR^2(\omega)\},
\]
where \( \rho > 0 \) is some small parameter, \( c > 0 \) is a deterministic constant, and \( R(\omega) \) is given by (28).

### 3.2. Quasi-stability

The reader is referred to the monograph [15] for details about quasi-stability and its applications in the deterministic setting. The paper [21] uses the method for random systems.

**Proposition 3.5.** Let Assumption 2.1 hold and, in addition, assume that the \( B_1(u) \) is subcritical, \(^2\) i.e.,
\[
\exists \sigma_0 < 1/2 : \|B_1(u_1) - B_1(u_2)\| \leq L_B\|(I + A)\sigma_0(u_1 - u_2)\|, \quad \forall u_1, u_2 \in H^{1/2}.
\]
Let \( V^1(t) \) and \( V^2(t) \) be two solutions of (8) with (different) initial data \( (V_0^1, V_1^1) \) and \( (V_0^2, V_1^2) \). Let \( Z = V^1 - V^2 \). Then there exist \( C, \gamma > 0 \) such that
\[
E_Z(t) \leq C E_Z(0)e^{-\gamma t} + C \int_0^t e^{-\gamma(t-\tau)}\|Z(\tau)\|^2d\tau, \quad \forall t > 0,
\]
where
\[
E_Z(t) = \frac{1}{2} \left(\|Z(t)\|^2 + \nu\|A^{1/2}Z(t)\|^2 + \nu\|Z(t)\|^2 + \kappa\|K^{1/2}Z(t)\|^2\right).
\]
The constants \( C \) and \( \gamma \) are deterministic and independent of \( \kappa \).

**Proof.** We use the same line of argument as in the deterministic case (see [17] and also [15]), which involves Lemma 3.23 from [17]. This argument uses the fact that \( A \) is not degenerate. Therefore it is convenient to redefine the main part in the equation as \( \nu A \mapsto \nu(A + I) \) and introduce the modified nonlinearity \( B(V + \eta) \mapsto B(V + \eta) - \nu V \). Then we obtain the following relation (which follows from Lemma 3.23 in [17]):
\[
TE_Z(T) + \int_0^T E_Z(t)dt \leq c \left\{ \int_0^T \|DZ_t, Z_t\|dt + \int_0^T \|(DZ_t, Z)\|dt + \Psi_T(V^1, V^2) \right\}
\]

\(^2\)In the case of the sine-Gordon model (1) we have \( \sigma_0 = 0 \).
for every $T \geq T_0 \geq 1$, where $c > 0$ does not depend on $\alpha$ or $\varkappa, T$, and

$$
\Psi_T(V^1, V^2)
= \left| \int_0^T (G(\tau), Z_t(\tau))d\tau \right| + \left| \int_0^T \int_t^T (G(t), Z(t))dt \right|
$$

with

$$
G(t) = B(V^1(t) + \eta) - B(V^2(t) + \eta) - \nu Z(t).
$$

As above

$$
|\langle DZ_t, Z \rangle| \leq C_\varepsilon(\langle DZ_t, Z_t \rangle) + \varepsilon E_Z(t)
$$

for every $\varepsilon > 0$. Thus, choosing $\varepsilon$ in an appropriate way, we obtain

$$
TE_Z(T) + \int_0^T E_Z(t)dt \leq c_0 \int_0^T \langle DZ_t, Z_t \rangle dt + c_0 \Psi_T(V^1, V^2).
$$

Using the subcritical hypothesis in (29) we can show that

$$
\Psi_T(V^1, V^2) \leq \varepsilon \int_0^T E_Z(\tau)d\tau + b(\varepsilon, T) \int_0^T \|Z(\tau)\|^2d\tau
$$

for every $\varepsilon > 0$. The difference $Z(t)$ solves the problem

$$
Z_{tt} + \nu(\mathcal{A} + I)Z + DZ_t + \varkappa KZ + G(t) = 0.
$$

Therefore applying the energy relation for this equation we obtain

$$
\int_0^T \langle DZ_t, Z_t \rangle dt \leq E_Z(0) - E_Z(T) + \Psi_T(V^1, V^2).
$$

Thus, after appropriate choice of $\varepsilon$ and $T$, we arrive at the relation

$$
E_Z(T) \leq qE_Z(0) + b(T) \int_0^T \|Z(\tau)\|^2d\tau,
$$

where $q < 1$. This inequality allows us to apply the same procedure as in [17, p.100] to obtain (30).

By the definition (15) of the cocycle $\tilde{\phi}$ from (16) we have that

$$
dist_{\tilde{H}}(\tilde{\phi}_t(\omega)Y, \tilde{\phi}_t(\omega)Y^*) = C(||V(t) - V^*(t)||_{1/2} + ||V^*_t(t) - V^*_t(t)||),
$$

where $V(t)$ and $V^*(t)$ are solutions to (8) which correspond to the cocycle $\tilde{\phi}_t$ applied to the initial data $Y = (v_0, c_0, v_1)$ and $Y^* = (v_0', c_0', v_1')$ respectively. On the other hand

$$
\|V(t) - V^*(t)\| \leq \|Q_{\psi_0}V(t) - Q_{\psi_0}V^*(t)\| + \|(V(t), \psi_0) - (V^*(t), \psi_0)\|
$$

and

$$
|(V(t), \psi_0) - (V^*(t), \psi_0)| \leq |a_0 - a_0'\| + \int_0^t ||V_t(\tau) - V^*_t(\tau), \psi_0)\|d\tau,
$$

where $a_0 = (V(t), \psi_0)$ and $a_0' = (V^*(t), \psi_0)$. Thus, if we choose the representatives $a_0$ and $a_0'$ of the factor-elements $c_0$ and $c_0'$ such that

$$
|a_0 - a_0'| = \text{dist}_{\tilde{H}}(c_0, c_0'),
$$

then it follows from Proposition 3.5 that

$$
\text{dist}_{\tilde{H}}(\tilde{\phi}_t(\omega)Y, \tilde{\phi}_t(\omega)Y^*) \leq C\text{dist}_{\tilde{H}}(Y, Y^*) e^{-\beta t} + Cq^t(Y, Y^*)
$$

(31)
for every \( Y = (v_0, c_0, v_1) \) and \( Y^* = (v_0^*, c_0^*, v_1^*) \) from \( \mathcal{H} \), where \( C, \beta > 0 \) are deterministic constants and

\[
g^t(Y, Y^*) = \text{dist}_{\mathcal{T}}(c_0, c_0^*) + \left[ \int_0^t \left( e^{-\beta(t-\tau)} \|Q_{\psi_0}[V(\tau) - V^*(\tau)]\|^2 + |(V_t(\tau) - V_t^*(\tau), \psi_0)|^2 \right) d\tau \right]^{1/2}.
\]

Theorem 3.6. Let Assumption 2.1 and the property in (29) hold. Then for any \( \varepsilon > 0 \) the RDS \( (\tilde{\phi}, \theta) \) on the space \( \mathcal{H} \) has a compact pullback attractor \( \mathfrak{A}^\varepsilon \).

Proof. We apply the quasi-stability method in the form presented in [21] for random systems. However, it is worth to mention that in contrast with [21] the state space is metric (not Banach) in the case considered here. This gives us additional difficulties in the realization of the quasi-stability method. First of all we have to apply the Ceron-Lopes idea developed for metric spaces (see, e.g., Proposition 2.2.21 in [15]) to prove the existence of a pullback attractor.

By Theorem 3.4 the system \( (\tilde{\phi}, \theta) \) is dissipative with forward invariant absorbing set \( \mathcal{B}_0^\varepsilon \). Therefore to prove the existence of a compact pullback attractor we need to prove that this RDS is asymptotically compact. For this we show that this RDS is asymptotically compact. For this we show that

\[
K(\omega) = \bigcap_{t \geq 0} \overline{\tilde{\phi}(t, \theta_{-t} \omega, \mathcal{B}_0^\varepsilon(\theta_{-t} \omega))}
\]

is a nonempty pullback attracting compact set.

First we note that the pseudo-metric \( g^t \) given by (32) is compact on \( \mathcal{B}_0^\varepsilon(\omega) \) (this can be shown by the contradiction argument in the same way as in [21]). Therefore for every \( \varepsilon > 0 \) there exists a covering of \( \mathcal{B}_0^\varepsilon(\omega) \) by open \( \varepsilon \)-balls \( K_1, \ldots, K_m \) in the \( \rho^T \)-pseudo-metric.

Let \( \{C_j\} \) be a covering of \( \mathcal{B}_0^\varepsilon(\omega) \) with the closed sets of the diameter \( \text{diam}_{\mathcal{H}}(C_j) \) less than \( \alpha(\mathcal{B}_0^\varepsilon(\omega)) + \varepsilon \), where \( \alpha(B) \) denotes the Kuratowski \( \alpha \)-measure of non-compactness of \( B \) (see, e.g., [15, p.54]). It follows from the quasi-stability estimate in (31) that

\[
\text{diam}_{\mathcal{H}}(\tilde{\phi}_t(\omega)[K_i \cap C_j]) \leq C e^{-\beta t} \text{diam}_{\mathcal{H}}(\mathcal{B}_0^\varepsilon(\omega)) + 2C\varepsilon.
\]

Therefore,

\[
\alpha(\tilde{\phi}_t(\omega)\mathcal{B}_0^\varepsilon(\omega))) \leq C e^{-\beta t} \text{diam}_{\mathcal{H}}(\mathcal{B}_0^\varepsilon(\omega)).
\]

Thus substituting \( \omega \) by \( \theta_{-t} \omega \) and using the temperedness of \( \text{diam}_{\mathcal{H}}(\mathcal{B}_0^\varepsilon(\omega)) \) we conclude that

\[
\alpha(\tilde{\phi}_t(\theta_{-t} \omega)\mathcal{B}_0^\varepsilon(\theta_{-t} \omega))) \to 0 \text{ as } t \to +\infty, \forall \omega \in \Omega.
\]

Therefore \( K(\omega) \) is a nonempty compact set. By the contradiction argument it is easy to see that this set pullback attracts every tempered set. Thus we can apply Theorem 2.4.

Remark 3.7 (On the dimension of the pullback attractor). We do not claim the attractor given by Theorem 3.6 has a finite dimension. The main reason is that the phase space for the case considered is not Banach and thus we cannot apply the same stochastic version of the standard quasi-stability argument as in [21]. Although there is a version of the quasi-stability approach for complete metric spaces (see [17, Theorem 2.14]), its application requires appropriate estimates for local \( (\varepsilon, \rho) \)-capacities. These estimates can be obtained, but with random constants for which we cannot control their temperedness. On the other hand we can easily
prove the existence of finite-dimensional pullback attractor for the case when instead of the operator $A$ we consider $A + cf$ with a positive (even small) constant $c$. With this modification the main linear part of the model is not degenerate and we can apply the ideas developed in [21] to obtain the result in the space $H^{1/2} \times \overline{H}$.

4. Synchronization. We will consider various kinds of synchronization of the system of equations (8) as the parameter $\kappa$ changes.

4.1. Upper semi-continuity and synchronization. For the original system of equations (8) we can prove that the attractors $\mathcal{A}_\kappa$ are upper semicontinuous at every point $\kappa_0 > 0$, i.e.,

$$\lim_{n \to \infty} \left[ \sup \{ \text{dist}_H(y, \mathcal{A}_{\kappa_0}) : y \in \mathcal{A}_{\kappa_n} \} \right] = 0$$

(33)

for every sequence $\{\kappa_n\}$ such that $\kappa^n \to \kappa_0 > 0$ as $n \to \infty$.

For this we can apply the methods developed in [26, 27] and extended to the random case in [6, 37]. According to [37] the proof of (33) requires two ingredients (a) the existence of an attracting compact set locally independent of $\kappa$ and (b) the convergence of cocycles in some uniform way. The latter property is simple in our situation. Indeed, let $Y^\kappa = (V^\kappa_0, V^\kappa_1)$ and $Y = (V_0, V_1)$. In addition, let $V^\kappa$ and $V^\kappa_1$ be solutions to (8) with these initial data and parameters $\kappa$ and $\kappa_1$, respectively. One can see that $Z(t) = V^\kappa(t) - V^\kappa_1(t)$ satisfies the equation

$$Z_{tt} + \nu AZ + DZ_t + \kappa_1 KZ = F,$$

where

$$F = -(\kappa - \kappa_1)KV^\kappa - B(V^\kappa + \eta) + B(V^\kappa_1 + \eta).$$

We obviously have

$$\|F\| \leq c_1|\kappa_1 - \kappa_1|\|I + A\|^{1/2}V^\kappa_1 + c_2\|(I + A)^{1/2}Z\|.$$

Using (11) we can conclude

$$\|I + A\|^{1/2}V^\kappa_1 \leq C_{\alpha, \beta} \left[ \|Y^\kappa\| + \int_0^t \left(1 + \|\xi(\tau)\|\right) d\tau \right],$$

for every $\kappa \in [\alpha, \beta]$, where $0 < \alpha < \beta < \infty$ are arbitrary. Then standard energy type calculations give the estimate

$$\|V^\kappa(t) - V^\kappa_1(t)\|_H \leq C_T \left( |\kappa_1 - \kappa_1| \left[ \|Y^\kappa\| + \int_0^T \left(1 + \|\xi(\tau)\|\right) d\tau \right] + \|Y^\kappa - Y\|_H \right), \quad t \in [0, T].$$

Thus we have uniform continuity of the cocycle at every point $\kappa_0 > 0$. For the existence of an (locally uniform) attracting compact set, we need to impose stronger hypotheses on the system. For instance, in the basic model we need additional smoothness of the noises. We need this to justify an application of multipliers $A^\delta V_1$ and $A^\delta V$ with a positive $\delta$ in (8).

We can also consider the case when $\kappa \to +\infty$, which can be important in the asymptotic synchronization phenomena. To guarantee a uniform (in $\kappa$) bound of the absorbing ball we need to assume that all noises are identical (i.e., $Q\eta = 0$). Then, for every element $Y(\omega)$ from the attractor to (8) we have

$$\|Q\tilde{\phi}_1(\omega)Y(\omega)\| \leq \frac{R(\delta(\omega))}{\kappa} \to 0, \quad \kappa \to \infty.$$
By interpolation
\[ \|(1 + A)^{1/2 - \varepsilon}Q\hat{\phi}_\nu(\omega)Y(\omega)\| \to 0, \ \kappa \to \infty. \]

These observations show that the following limiting problem arises:
\[ p_{tt} + \nu Ap + \nabla p + (p + \eta_0) - \eta_0 = 0, \ w(0) = w_0, \ w_t(0) = w_1, \quad (34) \]
where \( B(v) = N^{-1} \sum_{i=1}^{N} B_i(v) \). Using the same argument as above one can show that problem (34) generates a RDS in the space \( Q_0 H^{1/2} \times \tilde{T}_0^1 \times H \), which possesses a compact global pullback attractor \( \tilde{A} \). Here \( \tilde{T}_0^1 = \mathbb{R}/d\mathbb{Z} \) is the corresponding factor-space.

Under additional hypotheses on the system we can also show that the attractors \( \tilde{A}_\kappa \) given in Theorem 3.6 converge in some sense to the set
\[ \tilde{A} = \{ (y, \ldots, y) : y \in A \}, \]
where \( A \) is the pullback attractor for the RDS generated by (34). This means that the attractor \( \tilde{A}_\kappa \) becomes “diagonal” in the limit of large intensity parameter \( \kappa \). Thus the components of the system become synchronized in this limit at the level of global (pullback) attractors. We do not provide full details here because our primary goal is synchronization for fixed \( \kappa \).

4.2. Synchronization for finite values of the interaction parameter. Now we consider the case of identical interacting subsystems, i.e., we assume that
\[ \eta_j \equiv \eta_0, \ B_j(w) \equiv B(w), \ j = 1, 2, \ldots. \quad (35) \]
In this case we observe asymptotic synchronization for finite values of \( \kappa \).

We start with the following auxiliary estimate.

**Lemma 4.1.** \( \|QB(U)\|^2 \leq 4L_B^2\|QU\|^2_{\sigma_0}. \)

**Proof.** One can see that
\[ [QB(U)]_j = \frac{1}{N} \sum_{i=1}^{N} [B(u_j) - B(u_i)]. \]
Thus, due to the structure of the projector \( P \),
\[ \|QB(U)\| = \frac{L_B}{N} \sum_{i=1}^{N} \|u_j - u_i\|_{\sigma_0} \leq \frac{L_B}{N} \sum_{i=1}^{N} (\|QU_j\|_{\sigma_0} + \|QU_i\|_{\sigma_0}). \]
This implies the conclusion. \( \square \)

Now we are in position to state the result on exponentially fast synchronization for finite \( \kappa \).

**Theorem 4.2.** Let Assumption 2.1, the property in (29) and also relations (35) hold. Assume that \( \kappa \) is positive, to say, \( \kappa \geq \kappa_* > 0 \). Let
\[ s_\kappa = \inf \left\{ \nu(Aw, w) + \kappa(Kw, w) : w = (w_1, \ldots, w_N) \in \tilde{H}^{1/2}, \ \sum_{j=1}^{N} w_j = 0, \ |w| = 1 \right\}. \]
Then there exist $s_\ast > 0$ and $\gamma > 0$ such that under the condition
\[ s_{\kappa} \geq s_\ast, \]
\[ \| \mathcal{Q} V(t) \|_{2}^{2} + \| (I + A)^{1/2} \mathcal{Q} V(t) \|_{2}^{2} + \kappa \| \mathcal{Q} V(t) \|_{2}^{2} \]
\[ \leq C(1 + \kappa) e^{-\gamma t} \left[ \| \mathcal{Q} V(t) \|_{2}^{2} + \| (I + A)^{1/2} \mathcal{Q} V(t) \|_{2}^{2} \right], \quad \forall \omega \in \Omega, \]
for every solution $V(t)$ to (8). The projector $\mathcal{Q}$ is defined in (17). In this case $\mathcal{X}^{\kappa} \equiv \mathcal{Q}$ for all $\kappa$ such that $s_{\kappa} \geq s_\ast$.

We note that (36) implies that
\[ \lim_{t \to \infty} e^{\gamma t} \left[ \| \mathcal{Q} V(t) \|_{2}^{2} + \| (I + A)^{1/2} \mathcal{Q} V(t) \|_{2}^{2} \right] = 0, \quad \forall \omega \in \Omega, \quad \forall \gamma < \gamma, \]
for every solution $V(t) = (v^{1}(t), \ldots, v^{N}(t))$ of (8). Since
\[ \sum_{i,j=1}^{N} \| (I + A)^{1/2} (v^{j} - v^{i}) \| \leq \sum_{i,j=1}^{N} \left( \| (I + A)^{1/2} [Q \mathcal{V}] \| + \| (I + A)^{1/2} [Q \mathcal{V}] \| \right) \]
\[ \leq C_{N} \| (I + A)^{1/2} \mathcal{Q} V(t) \|, \]
the limit (37) can be rewritten as
\[ \lim_{t \to \infty} e^{\gamma t} \sum_{i,j=1}^{N} \left[ \| v^{i}(t) - v^{i}(t) \|^{2} + \| (I + A)^{1/2} (v^{j}(t) - v^{i}(t)) \|^{2} \right] = 0 \]
for all $\omega \in \Omega$. Thus we observe exponential asymptotic synchronization.

**Proof.** In the case considered $Q \xi = 0$ and thus $Z = Q \mathcal{V}$ satisfies the equation
\[ Z_{tt} + \nu A Z + D Z_{t} + \kappa K Z + Q \mathcal{B}(V + \eta) = 0, \quad Z(0) = Z_{0}, \quad Z_{t}(0) = Z_{1}, \]
where $Z_{0} = QU_{0}$ and $Z_{1} = QU_{1}$.

We consider a Lyapunov type function of the form
\[ \Psi(t) = E(t) + \Phi(t), \]
where
\[ E(t) = \frac{1}{2} \left( \| Z(t) \|_{2}^{2} + \nu \| A^{1/2} Z(t) \|_{2}^{2} \right), \quad \Phi(t) = \rho(Z, Z_{t}) + \frac{\kappa}{2}(KZ, Z), \]
for a positive constant $\rho$, which will be chosen later.

It follows from Lemma 3.1 that
\[ |(Z, Z_{t})| \leq \frac{1}{2} \| Z_{t} \|_{2}^{2} + c_{0} \left( \| A^{1/2} Z \|_{2}^{2} + \kappa \| KZ, Z \| \right). \]

Therefore there exist $0 < \rho_{0} < 1$ and $\beta_{i} > 0$, independent of $\kappa \geq \kappa_{\ast}$ such that
\[ \beta_{0} [E_{\ast}(t) + (\kappa - \kappa_{\ast}/2) \| K^{1/2} Z(t) \|_{2}^{2}] \leq \Psi \leq \beta_{2} E_{\ast}(t) + \kappa \| K^{1/2} Z(t) \|_{2}^{2} \]
for all $\kappa \geq \kappa_{\ast}$ and $\rho \in (0, \rho_{0})$, where
\[ E_{\ast}(t) = \| Z(t) \|_{2}^{2} + \| A^{1/2} Z(t) \|_{2}^{2} + \| Z(t) \|_{2}^{2}. \]

For strong solutions we can calculate the derivative
\[ \frac{d \Psi}{dt} = - (D Z_{t}, Z_{t}) - (Q \mathcal{B}(V + \eta), Z_{t}) \]
\[ + \rho \| Z_{t} \|_{2}^{2} - (D Z_{t}, Z) - \nu (A Z, Z) - \kappa (KZ, Z) - (Q \mathcal{B}(V + \eta), Z). \]

Since $D$ is bounded from $H^{1/2}$ into $H$, we obtain
\[ |(D Z_{t}, Z)| \leq \varepsilon \left[ \| A^{1/2} Z \|_{2}^{2} + \| Z \|_{2}^{2} \right] + C \varepsilon^{-1} \| Z_{t} \|_{2}^{2}, \quad \forall \varepsilon > 0. \]

---

3One can see from Lemma 3.1 that $s_{\kappa} \geq c_{0} \kappa \cdot \inf \text{spec}(K)$. Thus, if $K$ is not degenerate, then $s_{\kappa} \to +\infty$ as $\kappa \to +\infty$. 
Thus there exist \( b_1 > 0 \), which are independent of \( \varkappa \), such that
\[
\frac{d\Psi}{dt} \leq - \left( [\mathcal{D}Z_t, Z_t] - b_1 \rho \| Z_t \|^2 \right) + C \| Z_t \| \| Z \|_{\sigma_0} \\
- b_2 \rho \left( E_\varepsilon(t) + (\varkappa - \varkappa_{\varepsilon}/2) \| \mathcal{K}^{1/2} Z \|^2 \right) + \rho \varepsilon \| Z(t) \|^2.
\]
Choosing \( \rho > 0 \) small enough and using the inequality
\[
\| Z_t \| \| Z \|_{\sigma_0} \leq \varepsilon \left( \| Z_t \|^2 + \| \mathcal{A}^{1/2} Z \|^2 \right) + C_\varepsilon \| Z \|^2, \quad \forall \varepsilon > 0,
\]
gives
\[
\frac{d\Psi}{dt} + \gamma \Psi(t) \leq - c_1 \left[ \nu \| \mathcal{A}^{1/2} Z \|^2 + (\varkappa - \varkappa_{\varepsilon}/2) \| \mathcal{K}^{1/2} Z \|^2 \right] + c_2 \| Z \|^2
\]
for some \( \gamma, c_1, c_2 > 0 \). Then, taking \( s_{\varkappa} \) large enough, we obtain
\[
\frac{d\Psi}{dt} + \gamma \Psi(t) \leq 0
\]
for some \( \gamma, C > 0 \). This implies (36). The equality \( \mathfrak{K} \varepsilon \equiv \tilde{\mathfrak{K}} \) follows from (36).

4.3. Synchronization by means of finite-dimensional coupling. One can see from the calculations made in Lemma 3.1 that
\[
\left( \mathcal{K} w, w \right) \geq c_0 \sum_{i=1}^{N} \| \mathcal{K}^{1/2} v_i \|^2, \quad \text{for all } w \in \mathcal{H},
\]
where \( \mathcal{K} \) is given by (4) with a nonnegative operator \( \mathcal{K} \) and \( \mathcal{H} \) is defined by (19). This implies that the parameter \( s_{\varkappa} \) can be estimated from below as follows:
\[
s_{\varkappa} \geq \inf \left\{ \sum_{j=1}^{N} \left[ \nu \| \mathcal{A}^{1/2} w' \|^2 + \varkappa c_0 \| \mathcal{K}^{1/2} w' \|^2 \right] : w_j \in \mathcal{H}^{1/2}, \sum_{j=1}^{N} w_j = 0, \sum_{j=1}^{N} \| w' \|^2 = \frac{1}{2} \right\}
\]
with some \( c_0 > 0 \). This means that it is not necessary to assume non-degeneracy of the operator \( \mathcal{K} \) to guarantee large \( s_{\varkappa} \). For instance, if \( \mathcal{K} = P_N \) is the orthoprojector onto \( \text{Span}\{ e_k : k = 0, 1, 2, \ldots, N \} \), then
\[
\nu \| \mathcal{A}^{1/2} w \|^2 + \varkappa b_0 \| \mathcal{K}^{1/2} w \|^2 \geq \sum_{k=0}^{N} (\nu \lambda_k + b_0 \varkappa) \| (w, e_k) \|^2 + \nu \sum_{k=N+1}^{\infty} \lambda_k \| (w, e_k) \|^2
\]
\[
\geq b_0 \varkappa \sum_{k=1}^{N} \| (w, e_k) \|^2 + \nu \lambda_{N+1} \sum_{k=N+1}^{\infty} \| (w, e_k) \|^2 \geq \min \{ b_0 \varkappa, \nu \lambda_{N+1} \} \| w \|^2.
\]
Thus, if \( b_0 \varkappa \geq \nu \lambda_{N+1} \), then we can guarantee a large value of \( s_{\varkappa} \) by an appropriate choice of \( N \).

This allows a generalization based on the assumption that \( \mathcal{K} \) is a “good” approximation (in some sense) for a strictly positive operator. In particular, we can use the theory of determining functionals (see [10] and the references therein) to obtain localized (in some sense) finite-dimensional forms of the interaction operator \( \mathcal{K} \). For instance, we can use interpolation operators related with a finite family \( \mathcal{L} \) of linear continuous functionals \( \{ l_j : j = 1, \ldots, N \} \) on \( \mathcal{H}^{1/2} \) given by
\[
K v = \sum_{j=1}^{N} l_j(v) \psi_j, \quad \forall v \in \mathcal{H}^{1/2},
\]
where \( \{ \psi_j \} \) is an appropriate finite set of elements from \( \mathcal{H}^{1/2} \). For details in the deterministic case see [14] and the references therein.
5. Application. As an application of the results presented above we can consider models of plates with coupling via elastic (Hooke type) links. Namely, we consider the following SPDEs

\[
\begin{align*}
    u_{tt} + \gamma u_t + \Delta^2 u + \kappa K(u - v) + \varphi_1(u) &= f_1 + \dot{W}_1 \text{ in } \mathcal{O} \subset \mathbb{R}^2, \\
    v_{tt} + \gamma v_t + \Delta^2 v + \kappa K(v - u) + \varphi_2(u) &= f_2 + \dot{W}_2 \text{ in } \mathcal{O} \subset \mathbb{R}^2,
\end{align*}
\]

with free type boundary conditions (see, e.g., [18] for discussion of these conditions).

As a particular case of the model above we can consider several versions of damped sine-Gordon equations. These are used to model the dynamics of Josephson junctions driven by a source of current (see, e.g., [42] for comments and references). For instance, we can consider the system

\[
\begin{align*}
    u_{tt} + \gamma u_t - \Delta u + \kappa(u - v) + \lambda \sin u &= f(x) + \dot{W}, \\
    v_{tt} + \gamma v_t - \Delta v + \kappa(v - u) + \lambda \sin v &= f(x) + \dot{W},
\end{align*}
\]

in a smooth domain \( \mathcal{O} \subset \mathbb{R}^d \) and equipped with the Neumann boundary conditions

\[
\frac{\partial u}{\partial n} \bigg|_{\partial \mathcal{O}} = 0, \quad \frac{\partial v}{\partial n} \bigg|_{\partial \mathcal{O}} = 0.
\]

(38c)

It is convenient to introduce new variables

\[
\begin{align*}
    w &= \frac{u - v}{2} \quad \text{and} \quad z = \frac{u + v}{2},
\end{align*}
\]

(39)

in which problem (38) can be rewritten in the form

\[
\begin{align*}
    w_{tt} + \gamma w_t - \Delta w + 2\kappa w + \lambda \sin w \cos z &= 0, \\
    z_{tt} + \gamma z_t - \Delta z + \lambda \cos w \sin z &= f(x) + \dot{W}, \\
    \frac{\partial w}{\partial n} \bigg|_{\partial \mathcal{O}} &= 0, \quad \frac{\partial z}{\partial n} \bigg|_{\partial \mathcal{O}} = 0.
\end{align*}
\]

(40a)

(40b)

(40c)

The main linear part in (40a) is not degenerate when \( \kappa > 0 \). Therefore the same calculations as in the proof of Theorem 4.2 shows that there exists \( \kappa_* \) such that

\[
\exists \eta > 0 : \|w(t)\|^2_{H^1(\mathcal{O})} + \|w_t(t)\|^2_{L^2(\mathcal{O})} \leq C_B e^{-\eta t}, \quad t > 0,
\]

when \( \kappa \geq \kappa_* \) for all initial data from a bounded set \( B \) in \( H^1(\mathcal{O}) \times L^2(\mathcal{O}) \). This means that every trajectory is asymptotically synchronized.

Moreover, it follows from the reduction principle (see [15, Section 2.3.3]) that the limiting (synchronized) dynamics is determined by the single equation

\[
    z_{tt} + \gamma z_t - \Delta z + \lambda \sin z = f(x) + \dot{W}, \quad \frac{\partial z}{\partial n} \bigg|_{\partial \mathcal{O}} = 0.
\]

The long-time dynamics of this equation is described in [22, 23, 40] for some special types of the noise.

Another coupled sine-Gordon systems of an interest is

\[
\begin{align*}
    u_{tt} + \gamma u_t - \Delta u + \lambda \sin (u - v) &= f_1(x) + \dot{W}, \\
    v_{tt} + \gamma v_t - \Delta v + \lambda \sin (v - u) &= f_2(x) + \dot{W},
\end{align*}
\]

(41a)

(41b)

(41c)

\[\quad \frac{\partial u}{\partial n} \bigg|_{\partial \mathcal{O}} = 0, \quad \frac{\partial v}{\partial n} \bigg|_{\partial \mathcal{O}} = 0.\]

4For simplicity we discuss a symmetric coupling of identical systems only.
Formally, this model is out of the scope the theory developed above. However, using the ideas presented there we can answer some questions about its synchronized regimes.\footnote{In the deterministic case the synchronization phenomena in (41) was studied in \cite{14}.} In particular with the variables $w$ and $z$ given by (39) we obtain the equations
\begin{align*}
w_{tt} + \gamma w_t - \Delta w + \lambda \sin 2w &= g(x), \\
z_{tt} + \gamma z_t - \Delta z &= h(x) + \dot{W}, \end{align*}
where
\begin{align*}
\frac{\partial w}{\partial n} \bigg|_{\partial \Omega} = 0, \quad \frac{\partial z}{\partial n} \bigg|_{\partial \Omega} = 0,
\end{align*}
and
\begin{align*}
g(x) &= \frac{1}{2} (f_1(x) - f_2(x)), \\
h(x) &= \frac{1}{2} (f_1(x) + f_2(x)).
\end{align*}
Thus the difference of two solutions converges to a deterministic attractor, while their average converges (in pullback sense) to a Ornstein-Uhlenbeck random variable.

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