Results on Resistance Distance and Kirchhoff Index of Graphs With Generalized Pockets

Qun Liu1* and Jiaqi Li2

1School of Mathematics and Statistics, Hexi University, Zhangye, China, 2Institute of Intelligent Information, Hexi University, Zhangye, China

F, Hμ are considered simple connected graphs on n and m + 1 vertices, and v is a specified vertex of Hμ and u1, u2, . . . uk ∈ F. The graph G = G|F, u1, . . . , uk, Hμ] is called a graph with k pockets, obtained by taking one copy of F and k copies of Hμ and then attaching the ith copy of Hμ to the vertex ui, i = 1, . . . , k, at the vertex v of Hμ. In this article, the closed-form formulas of the resistance distance and the Kirchhoff index of G = G|F, u1, . . . , uk, Hμ] are obtained in terms of the resistance distance and Kirchhoff index F and Hμ.

Keywords: resistance distance, Kirchhoff index, generalized inverse, Schur complement, generalized pockets

1 INTRODUCTION

All graphs considered in this article are simple and undirected. The resistance distance between vertices u and v of G was defined by Klein and Randić [1] to be the effective resistance between nodes u and v as computed with Ohm’s law when all the edges of G are considered to be unit resistors. The Kirchhoff index Kf(G) was defined in Ref. 1 as Kf(G) = ∑uv r(uv)G, where r(uv)G denotes the resistance distance between u and v in G. Resistance distance are, in fact, intrinsic to the graph, with some nice purely mathematical interpretations and other interpretations. The Kirchhoff index was introduced in chemistry as a better alternative to other parameters used for discriminating different molecules with similar shapes and structures [1]. The resistance distance and Kirchhoff index have attracted extensive attention due to their wide applications in physics, chemistry, and other fields. Until now, many results on the resistance distance and Kirchhoff index are obtained. The references in [2–5] can be referred to know more. However, the resistance distance and Kirchhoff index of the graph is, in general, a difficult thing from the computational point of view. The bigger the graph, the more difficult it is to compute the resistance distance and Kirchhoff index; so a common strategy is to consider a complex graph as a composite graph and to find relations between the resistance distance and Kirchhoff index of the original graphs. Let G = (V(G), E(G)) be a graph with the vertex set V(G) and edge set E(G). Let dvi be the degree of vertex i in G and D = diag {d1, d2, . . . |V(G)|} the diagonal matrix with all vertex degrees of G as its diagonal entries. For graph G, let AG and BG denote the adjacency matrix and vertex-edge incidence matrix of G, respectively. The matrix LG = D − AG is called the Laplacian matrix of G, where D is the diagonal matrix of vertex degrees of G. We use μ1(G) ≥ μ2(G) ≥ . . . ≥ μn(G) = 0 to denote the spectrum of LG. For other undefined notations and terminology from graph theory, the readers may refer to Ref. 6 and the references therein [7–23]. The computation of the resistance distance between two nodes in a resistor network is a classical problem in electric theory and graph theory. For certain families of graphs, it is possible to identify a graph by looking at the resistance distance and Kirchhoff index. More generally, this is not possible. In some cases, the resistance distance and Kirchhoff index of a relatively larger graph can be described in terms of the resistance distance and Kirchhoff index of some smaller (and simpler) graphs using some simple graph operations. There are results that discuss the resistance distance and Kirchhoff
and $H - v = C_4$ are considered. Taking $l = 1, 2$ and $3$, we obtained graphs $G_1 = G_1 [F; H_v, 1]$, $G_2 = G_2 [F; H_v, 2]$, and $G_3 = G_3 [F; H_v, 3]$, respectively. Figure 1 is referred. In this case, we described the resistance distance and Kirchhoff index of $G = G [F; H_v, l]$ in terms of the resistance distance and Kirchhoff index of $F$ and $H_v$. The results are contained in Section 3 of this article. Furthermore, when $F = F_1 \lor F_2$, $F_1$ is the subgraph of $F$ induced by the vertices $u_1, u_2, \ldots, u_k$ and $F_2$ is the subgraph of $F$ induced by the vertices $u_{k+1}, u_{k+2}, \ldots, u_n$. The considered three graphs $G_2, G_3$, and $G_4$ are shown in Figure 2, obtained from the two graphs $F = K_4$ and $H_v$ such that $H_v \setminus \{v\} = K_3$. It is observed that $F = K_1 \lor K_3, G_2, G_3$, and $G_4$ are graphs with 2, 3, and 4 pockets, respectively. Figure 2 can be referred. In this case, we described the resistance distance and Kirchhoff index of $G[F, u_1, u_2, \ldots, u_k; H_v, l]$ in terms of the resistance distance and Kirchhoff index of $F$ and $H_v$. These results are contained in Section 4.

### 2 Preliminaries

The [1]-inverse of $M$ is a matrix $X$ such that $MXM = M$. If $M$ is singular, then it has infinite [1]-inverse [16]. For a square matrix $M$, the group inverse of $M$, denoted by $M^g$, is the unique matrix $X$ such that $MXM = M, XMX = X$, and $MX = XM$. It is known that $M^g$ exists if and only if $\text{rank}(M) = \text{rank}(M^g)$ [16, 17]. If $M$ is really symmetric, then $M^g$ exists, and $M^g$ is a symmetric [1]-inverse of $M$. Actually, $M^g$ is equal to the Moore–Penrose inverse of $M$ since $M$ is symmetric [17].

It is known that the resistance distance in a connected graph $G$ can be obtained from any [1]-inverse of $G$ [13]. We used $M^{(1)}$ to denote any [1]-inverse of a matrix $M$, and $(M)^{(1)}_{uv}$ denotes the $(u, v)$-entry of $M$.

**Lemma 2.1.** [17]: Let $G$ be a connected graph, then

$$r_{uv}(G) = (L^{(1)\ast}_{G})_{uv} + (L^{(1)}_{G})_{uv} - (L^{(1)}_{G})_{uv} - (L^{(1)}_{G})_{uv} = (L^{(1)\ast}_{G})_{uv} + (L^{(1)}_{G})_{uv} - 2(I^{(1)}_{G})_{uv}.$$  

Let $I_n$ denote the column vector of dimension $n$ with all the entries equal to one. We often use $I_1$ to denote all-ones column vector if the dimension can be read from the context.

**Lemma 2.2.** [14]: For any graph, we have $(L^2_{G})_{1} = 0$.

**Lemma 2.3.** [18]: Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a nonsingular matrix. If $A$ and $D$ are nonsingular, then

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} = \begin{pmatrix} (A - BD^{-1}C)^{-1}A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} \end{pmatrix},$$

where $S = D - CA^{-1}B$. 

---

**Figure 1**: $[F; H_v, l]$ for different $l$.

**Figure 2**: Graphs having different numbers of pockets.
Lemma 2.4. [15]: Let \( L \) be the Laplacian matrix of a graph of order \( n \). For any \( a > 0 \), we have

\[
(L + aI_n - \frac{a}{n}I_{soc})^# = (L + aI)^{-1} - \frac{1}{an}I_{soc}.
\]

Lemma 2.5. [5]: Let \( G \) be a connected graph on \( n \) vertices, then

\[
Kf(G) = ntr(L^{(1)}_G) - I^2 L^{(1)}_G = ntr(L^#).
\]

Lemma 2.6. [19]: Let

\[
L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}
\]

be the Laplacian matrix of a connected graph. If \( D \) is nonsingular, then

\[
X = \begin{pmatrix} H^# & -H^#BD^{-1} \\ -D^{-1}B^TH^# & D^{-1} + D^{-1}B^TH^#BD^{-1} \end{pmatrix}
\]

is a symmetric \([1]\)-inverse of \( L \), where \( H = A - BD^{-1}B^T \).

3 THE RESISTANCE DISTANCE AND KIRCHHOFF INDEX OF \([F; H_v, L]s\)

Let \( F \) be a connected graph with the vertex set \([u_1, u_2, \ldots, u_m]s\). Let \( H_v \) be a connected graph on \( m + 1 \) vertices with a specified vertex \( v \) and \( V(H_v) = \{v_1, v_2, \ldots, v_m, v\} \) and \( G = G[F; H_v, l]s\). It is noted that \( G \) has \( n(m + 1) \) vertices. Let \( deg(v) = l \), \( 1 \leq l \leq m \). With loss of generality, it is assumed that \( N(v) = \{v_1, v_2, \ldots, v_l\} \). Let \( H_1 \) be the subgraph of \( H_v \) induced by the vertices in \([v_1, v_2, \ldots, v_l]s\) and \( H_2 \) be the subgraph of \( H_v \) induced by the vertices \([v_1, v_2, \ldots, v_m]s\). It is supposed that \( H_v = H_1 \cup (H_2 \cup \{v\}) \). In this section, we focused on determining the resistance distance and Kirchhoff index of \([F; H_v, l]s\) in terms of the resistance distance and Kirchhoff index of \( F \), \( H_1 \) and \( H_2 \).

Theorem 3.1. Let \( G = G[F; H_v, l]s\) be the graph, as described previously. It is supposed that \( H_v = H_1 \cup (H_2 \cup \{v\}) \). Let the Laplacian spectrum of \( H_1 \) and \( H_2 \) be \( \sigma(H_1) = (0 = \mu_1, \mu_2, \ldots, \mu_l) \) and \( \sigma(H_2) = (0 = v_1, v_2, \ldots, v_m) \). Then, \( G = G[F; H_v, l]s\) has the resistance distance and Kirchhoff index as follows:

(i) For any \( i, j \in V(F) \), we obtained

\[
r_{ij}(G[F; H_v, l]) = (L^#(F))_{ij} + (L^#(F))_{jj} - 2L^#(F)_{ij} = r_{ij}(F).
\]

(ii) For any \( i \in V(F) \) and \( j \in V(H_1) \), we obtained

\[
r_{ij}(G[F; H_v, l]) = (L^#(F))_{ij} + \left[\left(L(H_1) + (m - l + 1)I_j - \frac{l}{m - l + 1}I_{soc}\right)^{-1} \otimes I_n^+,\right.
\]

\[
\left.\left(L_1 \otimes I_n\right)L^#(F)(I_j^T \otimes I_n) - 2L^#(F)(I_j^T \otimes I_n)\right]_{ij}.
\]

(iii) For any \( i \in V(F) \) and \( j \in V(H_2) \), we obtained

\[
r_{ij}(G[F; H_v, l]) = (L^#(F))_{ij} + \left[\left(L(H_1) + (m - l + 1)I_i - \frac{l}{m - l + 1}I_{soc}\right)^{-1} \otimes I_n^+,\right.
\]

\[
\left.\left(1_l \otimes I_n\right)L^#(F)(I_i^T \otimes I_n) - 2L^#(F)(I_i^T \otimes I_n)\right]_{ij}.
\]

(iv) For any \( i \in V(H_1) \) and \( j \in V(H_2) \), we obtained

\[
r_{ij}(G[F; H_v, l]) = \left[\left(L(H_1) + (m - l + 1)I_i - \frac{l}{m - l + 1}I_{soc}\right)^{-1} \otimes I_n^+\right]
\]

\[
2L^#(F)_{ij}.
\]

(v) For any \( i \in V(H_2) \) and \( j \in V(H_2) \), we obtained

\[
r_{ij}(G[F; H_v, l]) = \left[\left(L(H_1) + (m - l + 1)I_i - \frac{l}{m - l + 1}I_{soc}\right)^{-1} \otimes I_n^+\right]
\]

\[
2L^#(F)_{ij}.
\]

(vi) Let

\[
Kf(G[F; H_v, l]) = n(m + 1) \left(\frac{1}{m - l + 1} + \frac{1}{l} + f^2\right)
\]

Proof: Let \( v_j^j \) denote the \( j \)-th vertex of \( H_1 \) in the \( i \)-th copy of \( H_2 \) in \( G \). For \( i = 1, 2, \ldots, n; \) \( j = 1, 2, \ldots, m \), and let \( V_j(H_v) = \{v_j^1, v_j^2, \ldots, v_j^m\} \). Then, \( V(F) \cup (\cup_{j=1}^m V_j(H_v)) \) is a partition of \( V(G) \). Using this partition, the Laplacian matrix of \( G = G[F; H_v, l]s\) can be expressed as

\[
L(G[F; H_v, l]) = \begin{pmatrix} L(F) + I_n & -I^n_j \otimes I_n & 0 \\ -I^n_i \otimes I_n & (L(H_1) + (m - l + 1)I) \otimes I_n & -I_j(\mu_m) \otimes I_n \\ 0 & -I_j(\mu_m) \otimes I_n & (L(H_2) + I_m) \otimes I_n \end{pmatrix}.
\]

We began with the computation of \([1]\)-inverse of the Laplacian matrix \( L(G) \) of \( G = G[F; H_v, l]s\). Let \( A = L(F) + I_m \) and \( B = (I^n_j \otimes I_n) \), \( B^T = (I^n_i \otimes I_n) \), \( D = (L(H_1) + (m - l + 1)I) \otimes I_n \), \( -I_j(\mu_m) \otimes I_n \), \( (L(H_2) + I_m) \otimes I_n \). First, we computed the \( D^{-1} \). By Lemma 2.3, we obtained

\[
(A_1 - B_1 D_1^{-1} C_1) = \left(\left(L(H_1) + (m - l + 1)I\right) \otimes I_n - \left(-I_j(\mu_m) \otimes I_n\right)\right)
\]

\[
\left(L(H_2) + I_m\right) \otimes I_n \}

\[
\left(L(H_1) + (m - l + 1)I\right) \otimes I_n - \left(-I_j(\mu_m) \otimes I_n\right)\right)
\]

\[
\left(L(H_2) + I_m\right) \otimes I_n \}

\[
\left(L(H_1) + (m - l + 1)I\right) \otimes I_n - \left(-I_j(\mu_m) \otimes I_n\right)\right)
\]

\[
\left(L(H_2) + I_m\right) \otimes I_n \}

\[
\left(L(H_1) + (m - l + 1)I\right) \otimes I_n - \left(-I_j(\mu_m) \otimes I_n\right)\right).
\]

so

\[
(A_1 - B_1 D_1^{-1} C_1)^{-1} = \left[L(H_1) + (m - l + 1)I \otimes I_n - \left(-I_j(\mu_m) \otimes I_n\right)\right]
\]

\[
\left(L(H_2) + I_m\right) \otimes I_n \}

\[
\left(L(H_1) + (m - l + 1)I \otimes I_n - \left(-I_j(\mu_m) \otimes I_n\right)\right).
\]
By Lemma 2.3, we obtained

\[
S^1 = \{D_1 - C_{1j}A_{1j}^{-1}\}^{-1} = \{L(H_1) + (m-l+1)I_{l}\}^{-1} \otimes I_n + (l_{m-\otimes I_n} \otimes I_n)\{L(H_1) + (m-l+1)I_{l}\}^{-1} \otimes I_n = \{L(H_1) + L_{m-\otimes I_n} + (m-l+1)I_{l}\}^{-1} \otimes I_n \\
= \{L(H_1) + H_{m-\otimes I_n} = (m-l+1)I_{l}\}^{-1} \otimes I_n = L_{m-\otimes I_n} = (L_{m-\otimes I_n} + (m-l+1)I_{l})^{-1} \otimes I_n \\
= \bigg( L(H_1) + H_{m-\otimes I_n} - \frac{1}{m-l+1} \bigg(I_{m-\otimes I_n}\bigg) \bigg)^{-1} \otimes I_n.
\]

By Lemma 2.3, we obtained

\[
-A^{-1}_1B_1S^{-1} = \bigg( L(H_1) + (m-l+1)I_{l}\bigg)^{-1} \otimes I_n \bigg( I_{m-\otimes I_n} \otimes I_n \bigg) \bigg( L(H_1) + (m-l+1)I_{l}\bigg)^{-1} \otimes I_n = \bigg( L(H_1) + H_{m-\otimes I_n} - \frac{1}{m-l+1} \bigg(I_{m-\otimes I_n}\bigg) \bigg)^{-1} \otimes I_n.
\]

Similarly, \( S^{-1}C_1A_1^{-1} = \big( -A^{-1}_1B_1S^{-1} \big)^T = \frac{1}{I_{m-\otimes I_n}} \otimes I_n. \) So

\[
D^{-1} = \begin{pmatrix}
\frac{1}{I_{m-\otimes I_n}} \otimes I_n & P_1 & Q_1
\end{pmatrix},
\]

where \( P_1 = \{(L(H_1) + (m-l+1)I_{l})^{-1} \otimes I_n, \quad Q_1 = \{(L(H_1) + H_{m-\otimes I_n} - \frac{1}{m-l+1} \bigg(I_{m-\otimes I_n}\bigg) \}^{-1} \otimes I_n. \) Now, we computed the \([1\otimes I]\)-inverse of \( G[F; H_{m-\otimes I_n}], \) By Lemma 2.6, we obtained

\[
H = A - BDT^{-1}B^T = L(F) + H_{m-\otimes I_n} - L_{m-\otimes I_n} \otimes I_n \left( \begin{array}{ll}
P_1 & Q_1 \\
Q_1 & I_{m-\otimes I_n}
\end{array} \right)
\]

so \( H^T = L^T(F). \) According to Lemma 2.6, we calculated \(-H^TBD^{-1} \) and \(-D^{-1}B^TH^T. \)

\[
-H^T BD^{-1} = L^T(F) \left( \begin{array}{ll}
I_{m-\otimes I_n} & \frac{1}{I_{m-\otimes I_n}} \otimes I_n
\end{array} \right)
\]

\[
L \left( L(F) \left( \begin{array}{ll}
I_{m-\otimes I_n} & \frac{1}{I_{m-\otimes I_n}} \otimes I_n
\end{array} \right) Q_1
\right)
\]

\[
\left( \begin{array}{ll}
1 & \frac{1}{I_{m-\otimes I_n}} \otimes I_n
\end{array} \right)
\]

and

\[
-D^{-1}B^T H^T = L^T(F) \left( \begin{array}{ll}
1 & \frac{1}{I_{m-\otimes I_n}} \otimes I_n
\end{array} \right),
\]

We are ready to compute the \( D^{-1}B^T H^T BD^{-1}. \)

\[
D^{-1}B^T H^T BD^{-1} = \left( \begin{array}{ll}
1 & \frac{1}{I_{m-\otimes I_n}} \otimes I_n
\end{array} \right) L^T(F) \left( \begin{array}{ll}
1 & \frac{1}{I_{m-\otimes I_n}} \otimes I_n
\end{array} \right).
\]

Let \( P = \{(L(H_1) + (m-l+1)I_{l} - m-l+1)I_{l}\}^{-1} \otimes I_n, \) \( Q = \{(L(H_1) + H_{m-\otimes I_n} - \frac{1}{m-l+1} \bigg(I_{m-\otimes I_n}\bigg) \}^{-1} \otimes I_n, \) and \( M = \frac{1}{I_{m-\otimes I_n}} \otimes I_n \).

\[
N = \begin{pmatrix}
(L^#(F) & L^#(F)(1^T_l \otimes I_n) & L^#(F)(1^T_{m-l} \otimes I_n) \\
(1 \otimes I_n)L^#(F) & P_1 & M \\
(1 \otimes I_n)L^#(F) & M^T & Q_1
\end{pmatrix}.
\]

is a symmetric \([1\otimes I]\)-inverse of \( G[F; H_{m-\otimes I_n}], \) where \( P_1 = P^{-1} + (1 \otimes I_n)L^#(F)(1^T_l \otimes I_n) \) and \( Q_1 = Q^{-1} + (1 \otimes I_n)L^#(F)(1^T_{m-l} \otimes I_n). \) For any \( i, j \in V(F), \) by Lemma 2.1 and Eq. 1, we obtained

\[
\begin{align*}
\begin{align*}
&\begin{align*}
\begin{pmatrix}
L^#(F) & L^#(F)(1^T_l \otimes I_n) & L^#(F)(1^T_{m-l} \otimes I_n) \\
(1 \otimes I_n) & P_1 & M \\
(1 \otimes I_n) & M^T & Q_1
\end{pmatrix}
\end{align*}

\end{align*}
\end{align*}
\]
\[ \text{tr}\left(\left( L(H_2) + lI_{m-1} - \frac{l}{m-l+1} J_{(m-l)} \right) \otimes I_n \right) = n \sum_{l=2}^{m} \frac{1}{\nu_l(H_2) + l} + \frac{n(m-l+1)}{l}. \]

Similarly,
\[ \text{tr}\left(\left( L(H_1) + (m-l+1)I_l - \frac{m-l}{l} J_{(m-l)} \right) \otimes I_n \right) = n \sum_{l=2}^{m} \frac{1}{\mu_l(H_1) + (m-l+1) + n}. \]

It is easily obtained
\[ \text{tr}\left((1 \otimes I_n) L^#(F)(1^T \otimes I_n) + \text{tr}\left((1_{m-1} \otimes I_n) L^#(F)(1^T_{m-1} \otimes I_n) \right) \right. \]
\[ = \text{tr}\left(J_{bd} \otimes L^#(F) + \text{tr}\left((1_{(m-l)\times(m-l)} \otimes L^#(F) \right) \right) \]
\[ = l \text{tr}\left(L^#(F) \right) + (m-l) \text{tr}\left(L^#(F) \right) = mtr(L^#(F)). \]

Let \( P = (L(H_1) + (m-l+1)I_l - \frac{m-l}{l} J_{bd}) \otimes I_n, \)
\[ \text{tr}\left(P^{-1} L(H_1) (m-l+1)I_l - \frac{m-l}{l} J_{bd} \right) = I_l. \]

Let \( Q = (L(H_2) + lI_{m-1} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)}) \otimes I_n, \)
\[ \text{tr}\left(Q^{-1} P^{-1} \right) = (1_{m-1}^T I_{m-1} \cdots 1_{m-1}^T) \left( \begin{array}{rrrr} Q^{-1} 0 & 0 & \cdots & 0 \\ 0 & Q^{-1} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & Q^{-1} \end{array} \right) \]
\[ = (m-l)^{m-l+1} \left( L(H_2) + lI_{m-1} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)} \right) = (m-l)^{m-l+1}. \]

Similarly, \( 1^T (1 \otimes I_n) L^#(F)(1^T \otimes I_n) I_1 = 0, 1^T (1_{m-1} \otimes I_n) L^#(F)(1^T_{m-1} \otimes I_n) I_1 = 0. \)

Plugging Eqs 2-6 and the aforementioned equations into \( K(G [F; H_v, l]), \)
we obtained the required result in (vi).

### 4 Resistance Distance and Kirchhoff Index of \( G [F, U_1, U_2, \ldots , U_K; H_v, l] \)

In this section, we considered the case when \( F = F_1 \cup F_2, \) where \( F_1 \) is the subgraph of \( F \) induced by the vertices \( u_1, u_2, \ldots , u_k \) and \( F_2 \) is the subgraph of \( F \) induced by the vertices \( u_{k+1}, u_{k+2}, \ldots , u_n. \)
In this case, we indicated the explicit formulation of the resistance distance and Kirchhoff index of \( G = G[F, u_1, u_2, \ldots , u_K; H_v, l] \) in terms of the resistance distance and Kirchhoff index of \( G \) and \( H_v. \)

**Theorem 4.1.** Let \( G = G[F, u_1, u_2, \ldots , u_K; H_v, l] \) be the graph, as described previously. Let \( \sigma(F_1) = (0 = \alpha_1, \alpha_2, \ldots , \alpha_k), \sigma(F_2) = (0 = \beta_1, \beta_2, \ldots , \beta_{n-k}), \sigma(H_v) = (0 = \mu_1, \mu_2, \ldots , \mu_l), \) and \( \sigma(H_2) = (0 = \nu_1, \nu_2, \ldots , \nu_{m-l}). \) Then, \( G \) has the resistance distance and Kirchhoff index as follows:

(i) For any \( i, j \in V(F_1), \)
\[ r_{ij}(G) = \left( (L(F_1) + (n-k)I_{n-k})^{-1} - \frac{n-k}{k} \right)_{ij} + \left( (L(F_1) + (n-k)I_{n-k})^{-1} - \frac{n-k}{k} \right)_{ij}. \]

(ii) For any \( i, j \in V(F_2), \)
\[ r_{ij}(G) = (L(F_2) + kI_{n-k})_{ij} + (L(F_2) + kI_{n-k})_{ij}. \]

(iii) For any \( i, j \in V(H_v), \)
\[ r_{ij}(G) = \left( (L(H_v) + (m-l+1)I_l - \frac{m-l}{l} J_{bd}) \otimes I_k \right)_{ij} + \left( (L(H_v) + (m-l+1)I_l - \frac{m-l}{l} J_{bd}) \otimes I_k \right)_{ij}. \]

(iv) For any \( i, j \in V(H_2), \)
\[ r_{ij}(G) = \left( (L(H_2) + lI_{m-1} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)}) \otimes I_n \right)_{ij} + \left( (L(H_2) + lI_{m-1} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)}) \otimes I_n \right)_{ij}. \]

(v) For any \( i \in V(F) \) and \( j \in V(H_2), \)
\[ r_{ij}(G) = \left( (L(H_2) + lI_{m-1} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)}) \otimes I_n \right)_{ij} + \left( (L(H_2) + lI_{m-1} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)}) \otimes I_n \right)_{ij}. \]
\( r_{ij}(G) = (L^\#(F))_{ij} \)
\[ + \left[ \left( L(H_i) + (m - l + 1)I_i - \frac{m - l}{l}J_{1l} \right) \right]^{-1} \otimes I_n \]
\[ - 2L^\#(F)_{ij} \].

(vi) For any \( i \in V(F) \) and \( j \in V(H_2) \), we obtained
\[ r_{ij}(G) = (L^\#(F))_{ij} + \left[ \left( L(H_i) + I_{m-l} - \frac{1}{m-l+1}J_{1(m-l+1)} \right) \right]^{-1} \otimes I_n \]
\[ - 2L^\#(F)_{ij} \].

(vii) For any \( i \in V(H_2) \) and \( j \in V(H_2) \), we obtained
\[ r_{ij}(G) = \left[ \left( L(H_i) + (m - l + 1)I_i - \frac{m - l}{l}J_{1l} \right) \right]^{-1} \otimes I_n \]
\[ + \left[ \left( L(H_2) + I_{m-l} - \frac{1}{m-l+1}J_{1(m-l+1)} \right) \right]^{-1} \otimes I_n \]
\[ - 2 \left[ \left( L(H_2) + (m - l + 1)I_i - \frac{m - l}{l}J_{1l} \right) \right]^{-1} \otimes I_n \].

(viii) For any \( i \in V(H_2) \) and \( j \in V(H_2) \), we obtained
\[ r_{ij}(G) = \left[ \left( L(H_i) + I_{m-l} - \frac{1}{m-l+1}J_{1(m-l+1)} \right) \right]^{-1} \otimes I_n \]
\[ + \left[ \left( L(H_2) + (m - l + 1)I_i - \frac{m - l}{l}J_{1l} \right) \right]^{-1} \otimes I_n \]
\[ - 2 \left[ \left( L(H_2) + I_{m-l} - \frac{1}{m-l+1}J_{1(m-l+1)} \right) \right]^{-1} \otimes I_n \].

(ix) Let
\[ K_f(G) = (n + mk) \left[ \sum_{\alpha = 1}^{(1)} \left( \frac{1}{n-k} \right) - \frac{1}{(n-k)} \right] + \sum_{\beta = 1}^{(1)} \frac{1}{\nabla^\beta} + \sum_{\gamma = 1}^{(1)} \frac{1}{\nabla^\gamma + 1} \left( \frac{l(2m+2l+1)}{m+l+1} \right) \]
\[ + \frac{k}{l} \left( \frac{l(m-l)}{l} \right) \left( \frac{l(m-l+1)}{l} \right) \]
\[ - \left( \frac{l(m-l+1)}{l} \right) \left( \frac{l(m-l+1)}{l} \right) \]
\[ + 2k \left( m-l \right) \]
\[ + \frac{k}{l} \left( \frac{l(m-l)}{l} \right) \left( \frac{l(m-l+1)}{l} \right) \]
\[ + \frac{k}{l} \left( \frac{l(m-l)}{l} \right) \left( \frac{l(m-l+1)}{l} \right) \]

Proof: Let \( v_j \) denote the \( j \)-th vertex of \( H \) in the \( 1 \)-th copy of \( H^v \) in \( G \), for \( i = 1, 2, \ldots, k, j = 1, 2, \ldots, m \), and let \( V_f(H_i) = \{ v_j^1, v_j^2, \ldots, v_j^m \} \). Then, \( V_f(F_1) \cup V_f(F_2) \cup \ldots \cup (\cup_{j=1}^{m} V_f(H_i)) \) is a partition of the vertex set of \( G = G[F, u_1, u_2, \ldots, u_k; H_m, I] \). Using this partition, the Laplacian matrix of \( G \) can be expressed as
\[ L(G) = \begin{pmatrix} L_1 & -J_{k \times (m-k)} & -I_l^T \otimes I_k & 0 & 0 \\ -J_{(m-k) \times k} & L_2 & 0 & 0 & 0 \\ -I_l \otimes I_k & 0 & L_3 & -J_{1 \times (m-l)} \otimes I_k & 0 \\ 0 & 0 & -J_{(m-l) \times k} \otimes I_k & L_4 & -J_{1 \times (m-l)} \otimes I_k \end{pmatrix} \]
where \( L_1 = (L(F_1) + (n - k + l)I_2), \)
\( L_2 = (L(F_2) + kI_{m-k}), \)
\( L_3 = (L(H_i) + (m - l + 1)I_2) \otimes I_k, \)
\( L_4 = (L(H_2) + I_{m-l} \otimes I_k). \)

Let \( A = L_1, \)
\( B = (-J_{k \times (m-k)} - I_l^T \otimes I_k 0), \)
\( B^T = (-J_{(m-k) \times k} - I_l \otimes I_k 0), \)
\( D = \begin{pmatrix} L_2 & 0 & 0 \\ 0 & L_3 & -J_{1 \times (m-l)} \otimes I_k \\ 0 & -J_{(m-l) \times k} \otimes I_k & L_4 \end{pmatrix}. \)

First, we computed
\[ D_1^{-1} = \left( -J_{(m-l) \times k} \otimes I_k - J_{1 \times (m-l)} \otimes I_k \right)^{-1}. \]
By Lemma 2.3, we obtained
\[ A_1 = B_1D_1^{-1}C_1 = (L(H_i) + (m - l + 1)I_2) \otimes I_k - \left( -J_{1 \times (m-l)} \otimes I_k \right) \]
\[ = (L(H_i) + (m - l + 1)I_2) \otimes I_k - \left( -J_{1 \times (m-l)} \otimes I_k \right) \]
\[ = (L(H_2) + (m - l + 1)I_2) \otimes I_k. \]

so \( (A_1 - B_1D_1^{-1}C_1)^{-1} = (L(H_i) + (m - l + 1)I_2 - J_{1 \times (m-l)} \otimes I_k)^{-1} \).
By Lemma 2.3, we obtained
\[ S_1 = (G - C_1A_1^{-1}B_1)^{-1} \]
\[ = (L(H_i) + I_{m-l} \otimes I_k) \]
\[ = (L(H_2) + I_{m-l} \otimes I_k) \]
\[ = (L(H_2) + I_{m-l} \otimes I_k)(L(H_2) + (m - l + 1)I_2) \]
\[ = (L(H_2) + I_{m-l} \otimes I_k). \]

By Lemma 2.3, we obtained
\[ A_1^{-1}B_1S_1^{-1} = -\left[ (L(H_i) + (m - l + 1)I_2) \right] \]
\[ \otimes I_k \]
\[ = \frac{1}{1 - t} \]
\[ \otimes I_k. \]

Similarly, \(-S_1^{-1}C_1A_1^{-1} = \left( -A_1^{-1}B_1S_1^{-1} \right)^T = \frac{1}{1 - t} \otimes I_k. \)
So
\[ D_1^{-1} = \begin{pmatrix} P_1 \frac{1}{1 - t} \otimes I_k \\ Q_1 \end{pmatrix}, \]
where \( P_1 = [L(H_2) + (m - l + 1)I_2 - J_{1 \times (m-l)} \otimes I_k], \)
\( Q_1 = [L(H_2) + I_{m-l} - J_{1 \times (m-l)} \otimes I_k]. \)

Now, we computed the \([1]^{-1}\) inverse of \( G[F, u_1, u_2, \ldots, u_k; H_m, I] \). Let \( P = [L(H_2) + (m - l + 1)I_2 - J_{1 \times (m-l)} \otimes I_k], \)
\( Q = [L(H_2) + I_{m-l} - J_{1 \times (m-l)} \otimes I_k]. \)

so \( H^\# = (L(F_1) + (n - k)I_k - \frac{m - l}{l})^\#_k. \)

By Lemma 2.4, we obtained \( H^\# = (L(F_1) + (n - k)I_k - \frac{m - l}{l})^\#_k. \)
According to Lemma 2.6, we calculated $-H^T BD^{-1}$ and $-D^{-1} B^T H^T$.

$$-H^T BD^{-1} = -H^T \left( -I_{k \times (n-k)} - I_k^T \otimes I_k \right) \left( L(F_2) + k I_{n-k} \right)^{-1} \left( \begin{array}{ccc} 0 & 0 & P^{-1} \frac{1}{1} T_{(m-l) \otimes I_k} Q^{-1} \\
0 & 0 & H^#(1_1^T \otimes I_k) \end{array} \right)$$

and

$$-D^{-1} B^T H^# = \left( \frac{1}{k} H^# I_{k \times (n-k)} H^#(1_1^T \otimes I_k) \right) \left( \begin{array}{c} 1 \otimes I_k \end{array} \right)$$

We are ready to compute $D^{-1} B^T H^T B^{-1}$. Let $M = 1_{m}^T \otimes I_k$ and $N = 1_2^T \otimes I_k$. Based on Lemma 2.6, the following matrix

$$T = \left( \begin{array}{cccc} H^# & H^# I_{k \times (n-k)} & H^# N & H^# M \\
\frac{1}{k} H^# I_{k \times (n-k)} & 0 & 0 & M^T H^# I_{k \times (n-k)} \\
N^T H^# & 0 & P^{-1} + N^T H^# N & N^T H^# M + \frac{1}{k} T_{I_k} \\
M^T H^# & 0 & M^T H^# N + \frac{1}{k} I_k \otimes Q^{-1} + M^T H^# M \end{array} \right).$$

is a symmetric $[1]$-inverse of $G = G[F, u_1, u_2, \ldots, u_k; H, l]$, where

and $Q = \left( \begin{array}{c} (L(H_1)) + (m - l + 1) I_{l} - \frac{1}{l} N_{I_{l}} \otimes I_k \end{array} \right).$

For any $i, j \in V(F_1)$, by Lemma 2.1 and Eq. 7, we obtained

$$r_{ij}(G) = \left( \left( L(H_1) + (m - l + 1) I_{l} - \frac{1}{l} N_{I_{l}} \otimes I_k \right) \right)_{ij}^{-1}$$

as stated in (i).

For any $i, j \in V(F_2)$, by Lemma 2.1 and Eq. 7, we obtained

$$r_{ij}(G) = \left( \left( L(F_2) + k I_{n-k} \right) \right)_{ij}^{-1} \left( \left( L(F_2) + k I_{n-k} \right) \right)_{ij}^{-1} - 2 \left( L(F_2) + k I_{n-k} \right)_{ij}^{-1},$$

as stated in (ii).

For any $i, j \in V(H_1)$, by Lemma 2.1 and Eq. 7, we obtained

$$r_{ij}(G) = \left( \left( \left( L(H_1) + (m - l + 1) I_{l} - \frac{1}{l} N_{I_{l}} \otimes I_k \right) \right) \right)_{ij}^{-1}$$

and

as stated in (iii).

For any $i, j \in V(H_2)$, by Lemma 2.1 and Eq. 7, we obtained

$$r_{ij}(G) = \left( \left( L(H_1) + (m - l + 1) I_{l} - \frac{1}{l} N_{I_{l}} \otimes I_k \right) \right)_{ij}^{-1} - 2 \left( L(H_1) + (m - l + 1) I_{l} - \frac{1}{l} N_{I_{l}} \otimes I_k \right)_{ij}^{-1},$$

as stated in (iv).

For any $i, j \in V(H_1)$, by Lemma 2.1 and Eq. 7, we obtained

$$r_{ij}(G) = \left( \left( L(H_1) + (m - l + 1) I_{l} - \frac{1}{l} N_{I_{l}} \otimes I_k \right) \right)_{ij}^{-1}$$

and

as stated in (v).

For any $i, j \in V(H_2)$, by Lemma 2.1 and Eq. 7, we obtained

$$r_{ij}(G) = \left( \left( L(H_1) + (m - l + 1) I_{l} - \frac{1}{l} N_{I_{l}} \otimes I_k \right) \right)_{ij}^{-1} - 2 \left( L(H_1) + (m - l + 1) I_{l} - \frac{1}{l} N_{I_{l}} \otimes I_k \right)_{ij}^{-1},$$

as stated in (vi).

For any $i, j \in V(H_1)$ and $j \in V(H_2)$, by Lemma 2.1 and Eq. 7, we obtained

$$r_{ij}(G) = \left( \left( L(H_1) + (m - l + 1) I_{l} - \frac{1}{l} N_{I_{l}} \otimes I_k \right) \right)_{ij}^{-1}$$

and

as stated in (vii).

For any $i, j \in V(H_2)$ and $j \in V(H_1)$, by Lemma 2.1 and Eq. 7, we obtained

$$r_{ij}(G) = \left( \left( L(H_1) + (m - l + 1) I_{l} - \frac{1}{l} N_{I_{l}} \otimes I_k \right) \right)_{ij}^{-1} - 2 \left( L(H_1) + (m - l + 1) I_{l} - \frac{1}{l} N_{I_{l}} \otimes I_k \right)_{ij}^{-1},$$

as stated in (viii).
as stated in (viii).

Now, we computed the Kirchhoff index of $G[F, u_1, u_2, \ldots, u_k; H, l]$ as $Kf(G[F, u_1, u_2, \ldots, u_k; H, l])$

$$= (n + mk)tr(T) - 1^{-1}T1$$

$$= (n + mk)\left(\begin{array}{c} (L(F_1) + (n - k)I_k)^{-1} - \frac{1}{k(n - k)J_{k,k}} \\
+ ktr\left(L(H_1) + (m - l + 1)I_l - \frac{m - l - I_{J_{l,l}}}{I_{J_{l,l}}}\right)^{-1} \\
+ ltr\left(I_{J_{l,l}} \otimes I_k\right) \right) = 1^{-1}T1.$$  

It is noted that the eigenvalues of $(L(F_1) + (n - k)I_k)$ are $\alpha_1 + (n - k), \alpha_2 + (n - k), \ldots, \alpha_k + (n - k)$. Then,

$$tr\left(L(F_1) + (n - k)I_k\right)^{-1} = \sum_{i=1}^{n-k} \frac{1}{\alpha_i + (n - k) - k}.$$  

Similarly, $tr\left((L(F_2) + kI_{n-k})^{-1}\right) = \sum_{i=1}^{n-k} \frac{1}{\beta_i + (n - k) - k}$. It is noted that the eigenvalues of $L(H_1) + (m - l + 1)I_l$ are $0 + (m - l + 1), \mu_2(H_1) + (m - l + 1), \ldots, \mu_n(H_1) + (m - l + 1)$ and the eigenvalues of $J_{(m-1)x(m-1)}$ are $(m - l), 0^{m-l-1}$. Then,

$$tr\left(L(H_1) + (m - l + 1)I_l - \frac{m - l - I_{J_{l,l}}}{I_{J_{l,l}}}\right)^{-1} = k\sum_{i=2}^{m-l} \frac{1}{\mu_i + (m - l + 1) + k}.$$  

Similarly,

$$tr\left((L(H_2) + kI_{n-k})^{-1}\right)^{-1} = k\sum_{i=1}^{n-k} \frac{1}{\beta_i + (n - k) + k}.$$  

It is easily obtained that $tr(J_{l,x(l)} \otimes I_k) = l_k$, $tr(J_{l,x(l)} \otimes I_k) = (m - l)k$ and $tr(N^TH^N) + tr(M^TH^M) = tr(J_{l,x} \otimes H^T) + tr(J_{l,x} \otimes H^T) = tr(H^T) + tr(H^T) = mtr(H^T)$. Since $1_k^T H^T = 1_k$, $tr(H^T) - \frac{1}{k(n-k)}J_{k,k} = \frac{1}{n-k} - \frac{1}{k(n-k)}1_k^T 0$, then

$$1^{-1}N1 = 1^{-1}(L(F_2) + kI_{n-k})^1 + 1^{-1}P^{-1} + 1^{-1}Q^{-1}1$$

$$+ 1^{-1}N^TH^N1 + 1^{-1}N^TH^M1 + 1^{-1}M^TH^N1 + 1^{-1}M^TH^M1$$

By the process of Theorem 4.1, we obtained

$$1^{-1}P^{-1} = I, 1^{-1}Q^{-1}1 = (m - l)^{-1}.$$  

$$1^{-1}(N^TH^N)1 = I_k^T (I_k \otimes I_k)H^#(I_k \otimes I_k)1_{lk}$$

$$= \left(\begin{array}{c} I_k^T I_k \cdots I_k \\
I_k \cdots I_k \\
I_k \cdots I_k \\
I_k \end{array}\right) H^#$$

Similarly, $1^{-1}(M^TH^M)1 = 0, 1^{-1}N^TH^M1 = 0$, and $1^{-1}M^TH^N1 = 0$.

$$1^{-1}(J_{l,x(l)} \otimes I_k) = \left(\begin{array}{c} I_k^T I_k \cdots I_k \\
I_k \cdots I_k \\
I_k \cdots I_k \\
I_k \end{array}\right)$$

similarly, $1^{-1}(J_{l,x(l)} \otimes I_k) = lk(m - l)$. Applying the aforementioned equations into $Kf(G[F, u_1, u_2, \ldots, u_k; H, l])$, we obtained the required result in (ix).

**DATA AVAILABILITY STATEMENT**

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

**AUTHOR CONTRIBUTIONS**

All the authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

**FUNDING**

This work was supported by the National Natural Science Foundation of China (no. 61963013), the Science and Technology Plan of Gansu Province(18JR33RG06), and the Research and Innovation Fund Project of President of Hexi University(XZZD2018003).

2. Huang S, Zhou J, Bu C. Some Results on Kirchhoff index and Degree-Kirchhoff index. MATCH Commun Math Comput Chem (2016) 75: 207–22.

3. Cao J, Liu J, Wang S. Resistance Distances in corona and Neighborhood corona Networks Based on Laplacian Generalized Inverse

**REFERENCES**

1. Klein DJ, Randić M. Acute Resistance Distance. J Math Chem (1993) 12:81–95. doi:10.1007/bf01164627
Approach. J Algebra Appl (2019) 18(3):1950053. doi:10.1142/s0219498819500533
4. Liu J-B, Pan X-F, Yu L, Li D. Complete Characterization of Bicyclic Graphs with Minimal Kirchhoff index. Discrete Appl Maths (2016) 200:95–107. doi:10.1016/j.dam.2015.07.001
5. Sun L, Wang W, Zhou J, Bu C. Some Results on Resistance Distances and Resistance Matrices. Linear and Multilinear Algebra (2015) 63(3):523–33. doi:10.1080/03081087.2013.877011
6. Bapat RB. Graphs and Matrices. London/New Delhi: Springer/Hindustan Book Agency (2010). Universitext.
7. Chen H, Zhang F. Resistance Distance and the Normalized Laplacian Spectrum. Discrete Appl Maths (2007) 155:654–61. doi:10.1016/j.dam.2006.09.008
8. Xiao W, Gutman I. Resistance Distance and Laplacian Spectrum. Theor Chem Acc Theor Comput Model (Theoretica Chim Acta) (2003) 110:284–9. doi:10.1007/s00214-003-0460-4
9. Yang Y, Klein DJ. A Recursion Formula for Resistance Distances and its Applications. Discrete Appl Maths (2013) 161:2702–15. doi:10.1016/j.dam.2012.07.015
10. Yang Y, Klein DJ. Resistance Distance-Based Graph Invariants of Subdivisions and Triangulations of Graphs. Discrete Appl Maths (2015) 181:260–74. doi:10.1016/j.dam.2014.08.039
11. Barik S. On the Laplacian Spectra of Graphs with Pockets. Linear and Multilinear Algebra (2008) 56:481–90. doi:10.1080/03081080600906463
12. Barik S, Sahoo G. Some Results on the Laplacian Spectra of Graphs with Pockets. Electron Notes Discrete Maths (2017) 63:219–28. doi:10.1016/j.endm.2017.11.017
13. Bapat RB, Gupta S. Resistance Distance in Wheels and Fans. Indian J Pure Appl Math (2010) 41:1–13. doi:10.1007/s13226-010-0004-2
14. Bu C, Yan B, Zhou X, Zhou J. Resistance Distance in Subdivision-Vertex Join and Subdivision-Edge Join of Graphs. Linear Algebra its Appl (2014) 458:454–62. doi:10.1016/j.laa.2014.06.018
15. Liu X, Zhou J, Bu C, Resistance Distance and Kirchhoff index of R-Vertex Join and R-Edge Join of Two Graphs. Discrete Appl Maths (2015) 187:130–9. doi:10.1016/j.dam.2015.02.021
16. Ben-Israel A, Greville TNE. Generalized Inverses: Theory and Applications. 2nd ed. New York: Springer (2003).