The number of perfect matchings, and the sandwich conjectures, of random regular graphs

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Abstract

We prove that the number of perfect matchings in $G(n, d)$ is asymptotically normal when $n$ is even, $d \to \infty$ as $n \to \infty$, and $d = O(n^{1/7}/\log^2 n)$. This is the first distributional result of spanning subgraphs of $G(n, d)$ when $d \to \infty$.

Moreover, we prove that $G(n, d-1)$ and $G(n, d)$ can be coupled so that $G(n, d-1)$ is a subgraph of $G(n, d)$ with high probability when $d \to \infty$ and $d = o(n^{1/3})$. Further, if $d = \Omega(\log^7 n)$, $d = O(n^{1/7}/\log^2 n)$, and $d \leq d' \leq n - 1$ then $G(n, d)$ and $G(n, d')$ can be coupled so that asymptotically almost surely (a.a.s.) $G(n, d)$ is a subgraph of $G(n, d')$.

1 Introduction

We address two problems in this paper regarding $G(n, d)$, the random $d$-regular graph: the limiting distribution of the number of perfect matchings in $G(n, d)$, and the sandwich conjectures of $G(n, d)$.

1.1 The number of perfect matchings

The study of subgraphs lies in the centre of random graph theory. The commonly studied examples include spanning subgraphs such as perfect matchings, Hamilton cycles, spanning trees, $H$-factors where $H$ has a fixed size, as well as smaller subgraphs such as independent sets, cycles, and in general subgraphs isomorphic to some given $H$ of fixed size. Let $Z_H$ denote the number of subgraphs isomorphic to $H$. The phase transition of positive $Z_H$ and the distribution of $Z_H$ have been well studied in $G(n, p)$ and in $G(n, m)$ for both small and large $H$. It is interesting that $Z_H$ has different types of distributions for small and large $H$. If $H$ has fixed size and is balanced, then $Z_H$ is asymptotically normally distributed in $G(n, p)$ and $G(n, m)$ when $\mathbb{E}Z_H \to \infty$ \cite{21}. The distribution of $Z_H$ for $H$ with linear size becomes complicated. For $p \gg n^{-1/2}$ and $p$ not too close to 1, the numbers of perfect matchings, Hamilton cycles, and spanning trees are log-normally distributed in $G(n, p)$, but are normally distributed in $G(n, m)$ when $m \gg n^{3/2}$ \cite{11}. For $m = \Theta(n^{3/2})$, these random variables also become log-normally distributed in $G(n, m)$. It is not known if they remain log-normally
distributed in $G(n,p)$ when $p = O(n^{-1/2})$ and in $G(n,m)$ when $m = o(n^{3/2})$, although it is conjectured so [11]. The distributional phase transition of $Z_H$ when the size of $H$ grows from constant to linear size has been studied in [7], where $H$ is the number of $\ell$-matchings (i.e. matchings of size $\ell$). Its distribution in $G(n,p)$ changes from normal to log-normal at the critical value $\ell = \ell(p) \approx n^{1/3}$. Such distributional phase transition is also observed in $G(n,d)$ when $d$ is a fixed constant. It is well known that the distributions of short cycles in $G(n,d)$ are asymptotically Poisson [11, 22], whereas the distribution of the number of large subgraphs such as perfect matchings and Hamilton cycles in $G(n,d)$ is of an unusual type [13] as follows. Suppose that $Z$ is the number of perfect matchings (or the number of Hamilton cycles) in $G(n,d)$. Then the limiting distribution of the distribution of the logarithm of $Z/\mathbb{E}Z$ is an infinite linear combination of independent Poisson variables. More precisely,

$$\frac{Z}{\mathbb{E}Z} \rightarrow \prod_{i=1}^{\infty} (1 + \delta_i) e^{-\lambda_i \delta_i}, \quad \text{as } n \rightarrow \infty,$$

(1)

where $X_1, X_2, \ldots$ are independent Poisson variables with mean $\lambda_1, \lambda_2, \ldots$, and $\delta_1, \delta_2, \ldots$ are real numbers whose values depend on which subgraphs (i.e. perfect matchings or Hamilton cycles) $Z$ counts. The distribution of $Z$ is determined by using the small subgraph conditioning method, originally developed by Robinson and Wormald [19, 20] to prove Hamiltonicity of $G(n,d)$. The argument is then tuned to produce the distribution result of $Z$ by Janson [13]. Recently, Greenhill, Isaev and Liang [10] proved that the number of spanning trees in $G(n,d)$ has the same type of distribution as [1]. On the other hand, Garmo [9] studied the distributional phase transition of the number of $\ell$-cycles in $G(n,d)$ as $\ell$ grows from constant to linear in $n$. Its limiting distribution changes from a linear combination of independent Poisson variables to the exponential of that form, and the critical phase transition occurs when $\ell$ becomes linear in $n$.

It is natural to ask, in the case $d \rightarrow \infty$, whether the distribution type of these subgraphs (e.g. perfect matchings, Hamilton cycles, spanning trees) are the same as, or analogous to, that for constant $d$, and whether the distributional phase transitions occur when the size of the subgraphs (e.g. $\ell$-matchings and $\ell$-cycles) grows from constant to linear in $n$, as for constant $d$. We give a negative answer to this question. There have been few distributional results that are known for the number of subgraphs of $G(n,d)$ when $d \rightarrow \infty$, even for small subgraphs. The limiting distribution of the number of $\ell$-cycles was extended from constant $d$ and $\ell$ to those such that $(d-1)^{2\ell-1} = o(n)$ by McKay, Wormald and Wysocka [18]. Z. Gao and Wormald determined the limiting distributions of strictly balanced graphs of fixed sizes for $d$ that grows sufficiently slowly with $n$ [5]. There has been no result on the distribution of the number of subgraphs whose size is beyond $\log n$ when $d \rightarrow \infty$. In particular, the analysis for the number of perfect matchings, Hamilton cycles, and the spanning trees when $d = O(1)$, based on the configuration model, can no longer apply to obtain the distributional result when $d \rightarrow \infty$.

One may expect that the number of large subgraphs such as perfect matchings or Hamilton cycles would be of log-normal type in $G(n,d)$ as $d \rightarrow \infty$, which can be viewed as an analog of [1]. One may also expect that the number of $\ell$-matchings may exhibit a distributional phase transition as $\ell$ grows from constant to linear in $n$, as what happens for the $\ell$-matchings in $G(n,p)$ and for the $\ell$-cycles in $G(n,d)$ for constant $d$. In contrast with the
intuition, we show in this paper that the number of perfect matchings is asymptotically normally distributed in \( G(n,d) \) when \( d \to \infty \) as \( n \to \infty \) and \( d = O(n^{1/7}/\log^2 n) \). The power of the logarithmic term is not optimised.

**Theorem 1.** Let \( Y \) denote the number of perfect matchings in \( G(n,d) \) where \( n \) is even. Then \( Y \) is asymptotically normally distributed if \( d \to \infty \) as \( n \to \infty \) and \( d = O(n^{1/7}/\log^2 n) \). More formally,

\[
\frac{Y - \mathbb{E}Y}{\mathbb{E}Y/\sqrt{6d^3}} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } n \to \infty.
\]

This result suggests that there is likely no distributional phase transition on the number of \( \ell \)-matchings as \( \ell \) grows. The condition \( d = O(n^{1/7}/\log^2 n) \) in the result is imposed only for technical reasons and we believe that the same distribution holds for all \( d \to \infty \) until \( d \) is too close to \( n-1 \).

To our knowledge this is the first result on the limiting distribution of the number of spanning subgraphs in \( G(n,d) \) when \( d \to \infty \). The main contribution of Theorem 1 is the discovery of the distribution type of the number of perfect matchings, and we believe that this phenomenon is ubiquitous among other spanning subgraphs such as the number of Hamilton cycles. For future research, it is interesting to determine the limiting distributions of the number of \( \ell \)-matchings and \( \ell \)-cycles in \( G(n,d) \) for all \( \ell \).

**Conjecture 2.** The numbers of perfect matchings, Hamilton cycles, spanning trees, and \( k \)-factors where \( k \leq d-1 \), are all asymptotically normally distributed in \( G(n,d) \) for all \( d \) where \( dn \) is even and \( \min\{d,n-d\} \to \infty \) as \( n \to \infty \) (where \( n \) is even in the case of perfect matchings, and \( kn \) is even in the case of \( k \)-factors).

**Conjecture 3.** Suppose \( \min\{d,n-d\} \to \infty \) as \( n \to \infty \). The number of \( \ell \)-cycles in \( G(n,d) \) is asymptotically normal for all \( 3 \leq \ell \leq n \). The number of \( \ell \)-matchings of \( G(n,d) \) is asymptotically normal for all \( 3 \leq \ell \leq n/2 \).

**Remark 4.** The condition \( n - d \to \infty \) in the above conjectures are likely not necessary. Indeed, when \( n - d = o(n^{1/3}) \) and \( k = o(n^{1/3}) \) the asymptotic number of \( k \)-factors of a \( d \)-regular graph \( G \) is independent of \( G \) and can be obtained by Theorem 1 in Section 3.

### 1.2 The sandwich conjectures of \( G(n,d) \)

Analysis in \( G(n,d) \) is highly nontrivial, especially when \( d \to \infty \). Kim and Vu initiated the study of approximating \( G(n,d) \) by \( G(n,p) \), known as the sandwich conjecture [14]. Since then the conjecture has been worked on by several groups [2, 5, 15, 3], and the conjecture is close to being fully resolved. Along the line of the research there has been new conjectures that are proposed, one of which is stated as follows [5, Conjecture 1.2].

**Conjecture 5.** Let \( 0 \leq d_1 \leq d_2 \leq n-1 \) be integers, other than \((d_1,d_2) = (1,2)\) or \((d_1,d_2) = (n-3,n-2)\). Assume that \( d_1n \) and \( d_2n \) are both even. Then there exists a coupling \((G_1, G_2)\) such that \( G_1 \sim G(n,d_1), G_2 \sim G(n,d_2) \) and \( P(G_1 \subseteq G_2) = 1 - o(1) \).
The conjecture is only known to be true for \((d_1, d_2)\) where \(d_1 = 1\) and \(3 \leq d_2 \leq n - 1\), as well as for \((d_1, d_2)\) where \(d_2 - d_1\) is larger than some function of \(d_1\) (see [5, Corollary 1.7] for the precise statement). When \(d_1\) and \(d_2\) are both fixed constants and \((d_1, d_2) \neq (1, 2)\), it is known that \(G(n, d_1)\) is contiguous to the union of two independent copies of \(G(n, d_1)\) and \(G(n, d_2 - d_1)\) conditional on \(G(n, d_1)\) and \(G(n, d_2 - d_1)\) being disjoint (see more contiguity results in [23, Section 4]). However, contiguity does not imply a coupling as in the conjecture. In this paper we prove Conjecture 5 for a certain range of \(d_1\).

**Theorem 6.** Conjecture 5 holds for all integers \(d_1 \leq d_2 \leq n - 1\) where \(d_1 = \Omega(\log^7 n)\) and \(d_1 = O(n^{1/7}/\log^2 n)\) if \(n\) is even.

Theorem 6 follows as a corollary of [3, Theorem 2] and the following theorem that simultaneously couple a sequence of random regular graphs.

**Theorem 7.** Suppose \(d \to \infty\) and \(d = O(n^{1/7}/\log^2 n)\). For any \(\epsilon_n = o(1)\), there is a multiple coupling \((G_d, G_{d+1}, \ldots, G_{[(1+\epsilon_n)d]}\) such that marginally \(G_i \sim G(n, i)\) for all \(d \leq i \leq [(1+\epsilon_n)d]\) and jointly with probability \(1-o(1)\), \(G_d \subseteq G_{d+1} \subseteq \cdots \subseteq G_{[(1+\epsilon_n)d]}\) simultaneously.

If we only consider \((d_1, d_2)\) where \(d_2 = d_1 + 1\) then we have the following coupling theorem which holds for a much larger range of \(d\).

**Theorem 8.** Suppose \(d \to \infty\) and \(d = o(n^{1/3})\). There is a coupling \((G_d, G_{d+1})\) where marginally \(G_d \sim G(n, d)\) and \(G_{d+1} \sim G(n, d + 1)\), and jointly with probability \(1 - o(1)\), \(G_d \subseteq G_{d+1}\).

Theorem 8 follows as a corollary of a more general version (Theorem 18) which we state in Section 4. Theorem 8 indeed holds for all \(d \to \infty\) and \(d = o(n^{1/2})\). However for a simpler proof we did not pursue that. See Remark 15 for more explanations.

## 2 Proof of Theorem 1

Recall that \(Y\) denotes the number of perfect matchings in \(G(n, d)\). Throughout the paper we assume that \(n\) is even. Let \(X\) denote the number of triangles in \(G(n, d)\). We will approximate \(Y\) by a linear function of \(X\) using linear regression, and then study the distribution of \(Y\) by the distribution of \(X\). This method is known as orthogonal decomposition and projection, developed by Janson [12]. Originally this method is developed to determine the limiting distribution of the number of (large) subgraphs in \(G(n, p)\), and Janson also applied the method to determine the distributions of the numbers of spanning trees, perfect matchings and Hamilton cycles in \(G(n, m)\) [11]. We are not aware of its applications in \(G(n, d)\) before.

More specifically, we approximate \(Y\) by \(Y^* = aX + b\) where \(a = \text{Cov}(X, Y)/\text{Var}X\) and \(b = \mathbb{E}Y - a\mathbb{E}X\). The values of \(a\) and \(b\) are chosen so that \(\mathbb{E}(Y - Y^*) = 0\) and \(\mathbb{E}((Y - Y^*)^2)\) is minimised. We prove that \(\mathbb{E}((Y - Y^*)^2)\) is sufficiently small and thus the distribution of \(Y\) is determined by the distribution of \(aX + b\). Since \(X\) is asymptotically normally distributed, so is \(Y\). The expectation \(\mathbb{E}X\), the variance \(\text{Var}X\) and the limiting distribution of \(X\) have been studied in [1, Theorems 8 and 10], which we state below.
Theorem 9. Suppose \( d = o(n^{2/5}) \). Then,

\[
\begin{align*}
\mathbb{E}X &= \frac{(d-1)^3}{6} (1 + O(1/n)), \quad \text{Var}X \sim \mathbb{E}X, \\
\frac{X - \mathbb{E}X}{\sqrt{\mathbb{E}X}} &\xrightarrow{d} \mathcal{N}(0,1), \, \text{as } n \to \infty.
\end{align*}
\]

Next, we calculate the expectation \( \mathbb{E}Y \), the second moment \( \mathbb{E}Y^2 \) and the covariance \( \text{Cov}(X,Y) \), which allow us to estimate \( a \) and \( b \) and to bound \( \mathbb{E}((Y - Y^*)^2) \).

Theorem 10.

\[
\begin{align*}
\mathbb{E}Y &= \frac{n!}{(n/2)!2^{n/2}} \left( \frac{e}{n} \right)^{n/2} \left( \frac{d-1}{d} \right)^{(\frac{d-1}{d})n} d^{\frac{n}{2}} \exp \left( \frac{1}{4} + O \left( \frac{d^3}{n} \right) \right), \quad (2) \\
\mathbb{E}Y^2 &= \left( 1 + \frac{1}{6d^3} + O \left( d^{-4} + \frac{d^3}{n} + \sqrt{\frac{d}{n}} \log^6 n \right) \right) (\mathbb{E}Y)^2 \quad (3) \\
\text{Cov}(X,Y) &= \left( -\frac{1}{d^3} + O \left( d^{-4} + \frac{d}{n} \right) \right) \mathbb{E}X \mathbb{E}Y. \quad (4)
\end{align*}
\]

Proof of Theorem 7. Recall the definition of \( a \) and we make the following claim.

Claim 11. \( \mathbb{E}((Y - Y^*)^2) = o(a^2 \text{Var}X) \).

Since

\[
Y = Y^* + (Y - Y^*) = a(X - \mathbb{E}X) + a\mathbb{E}X + b + (Y - Y^*),
\]

it follows then that

\[
\frac{X - \mathbb{E}X}{\sqrt{\mathbb{E}X}} = \frac{Y - a\mathbb{E}X - b + (Y - Y^*)}{a\sqrt{\mathbb{E}X}}.
\]

By Claim 11, Theorem 9 and Markov’s inequality, asymptotically almost surely (a.a.s.)

\[
|Y - Y^*| = o(a\sqrt{\text{Var}X}) = o(a\sqrt{\mathbb{E}X}). \quad (5)
\]

Thus,

\[
\frac{Y - a\mathbb{E}X - b}{a\sqrt{\mathbb{E}X}} = \frac{X - \mathbb{E}X}{\sqrt{\mathbb{E}X}} - \frac{Y - Y^*}{a\sqrt{\mathbb{E}X}},
\]

where the left hand side above is

\[
\frac{Y - \mathbb{E}Y}{a\sqrt{\mathbb{E}X}} \sim \frac{Y - \mathbb{E}Y}{\mathbb{E}Y/\sqrt{6d^3}},
\]

and the right hand side is a random variable whose limiting distribution is \( \mathcal{N}(0,1) \) by (5) and Theorem 9. Consequently,

\[
\frac{Y - \mathbb{E}Y}{\mathbb{E}Y/\sqrt{6d^3}} \xrightarrow{d} \mathcal{N}(0,1), \, \text{as } n \to \infty. \quad \blacksquare
\]
Proof of Claim 11. By the definitions of $a$ and $b$,
\[
\mathbb{E}((Y - Y^*)^2) = \mathbb{E}Y^2 - 2\mathbb{E}Y(aX + b) + \mathbb{E}(aX + b)^2
\]
\[
= \mathbb{E}Y^2 - 2a\mathbb{E}XY - 2(\mathbb{E}Y - a\mathbb{E}X)\mathbb{E}Y + a^2\mathbb{E}X^2 + 2a(\mathbb{E}Y - a\mathbb{E}X)\mathbb{E}X + (\mathbb{E})^2
\]
\[
- 2a\mathbb{E}XY + a^2(\mathbb{E}X)^2
\]
\[
= (\mathbb{E}Y^2 - (\mathbb{E}Y)^2) - 2a(\mathbb{E}XY - \mathbb{E}X\mathbb{E}Y) + a^2(\mathbb{E}X^2 - (\mathbb{E}X)^2)
\]
\[
= \text{Var}Y - 2a\text{Cov}(X,Y) + a^2\text{Var}(X)
\]
\[
= \text{Var}Y - \frac{\text{Cov}(X,Y)^2}{\text{Var}X}.
\]
By the definition of $a$, $a^2\text{Var}X = \text{Cov}(X,Y)^2/\text{Var}X$. Hence it is sufficient to verify that
\[
\text{Var}Y - \frac{\text{Cov}(X,Y)^2}{\text{Var}X} \ll \frac{\text{Cov}(X,Y)^2}{\text{Var}X}.
\]
It is thus sufficient to prove that
\[
\text{Var}Y \sim \frac{\text{Cov}(X,Y)^2}{\text{Var}X}. \quad (6)
\]
By Theorem 10,
\[
\text{Var}Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \left(\frac{1}{6d^3} + O\left(d^{-4} + \frac{d^3}{n} + \sqrt{\frac{d}{n}}\log^6 n\right)\right)(\mathbb{E}Y)^2,
\]
and by Theorems 10 and 9
\[
\frac{\text{Cov}(X,Y)^2}{\text{Var}X} \sim \left(-\frac{1}{d^3} + O(d^{-4} + d/n)\right)^2 (\mathbb{E}X\mathbb{E}Y)^2 \sim \frac{1}{6d^3}(\mathbb{E}Y)^2.
\]
Now (6) follows, since $d \to \infty$ and $d = O(n^{1/7}/\log^2 n)$ and thus $\text{Var}Y \sim (\mathbb{E}Y)^2/6d^3$. \qed

3 Proof of Theorem 10

We will use the tools from [6 Theorem 1] and [17 Theorem 4.6] to estimate $\mathbb{E}Y$, $\mathbb{E}Y^2$ and $\text{Cov}(X,Y)$.

3.1 Edge and subgraph probabilities in $G(n,d)$

Let $H$ be a graph on $[n]$ and let $d^H = (d_1^H, \ldots, d_n^H)$ denote the degree sequence of $H$. Suppose that $d_i^H \leq d$ for every $1 \leq i \leq n$. Let $H^+$ denote the event that $H$ is a subgraph of $G(n,d)$. Let $|H|$ denote the number of edges in $H$. The following result is a special case of [6 Theorem 1] for the conditional edge probability $\mathbb{P}(uv \in G(n,d) \mid H^+)$. 

Theorem 12. Suppose $d = o(n)$ and suppose that $H$ is a graph on $[n]$ such that $d_i^H \leq d$ for every $1 \leq i \leq n$ and $dn - 2|H| = \Omega(dn)$.

\[
\mathbb{P}(uv \in G(n,d) \mid H^+) = \left(1 + O\left(\frac{d}{n}\right)\right) \frac{(d - d_u^H)(d - d_v^H)}{M - 2|H|}.
\]
We will apply the following enumeration result of McKay [17, Theorem 4.6] to estimate the probability of a (large) subgraph of $\mathcal{G}(n, d)$.

**Theorem 13.** Let $g = (g_1, \ldots, g_n)$ be a sequence of non-negative integers. Let $m = m(g) = \|g\|_1/2$. Let $X$ be a simple graph on $[n]$ with degree sequence $x$. Let $\Delta(g)$ and $\Delta(x)$ denote the maximum components of $g$ and $x$ respectively. Suppose $\Delta(g) \geq 1$, $\Delta = o(m)$ where $\hat{\Delta}(g) = \Delta(g)^2 + \Delta(g)\Delta(x)$. Define

$$\lambda = \lambda(g) = \frac{1}{4m(g)} \sum_{j=1}^n (g_j)2, \quad \mu = \mu(g, X) = \frac{1}{2m(g)} \sum_{ij \in X} g_ig_j.$$ 

Let $N(g, X)$ denote the number of simple graphs with degree sequence $g$ and with no edge in common with $X$. Then,

$$N(g, X) = \frac{(2m)!}{m!2^m \prod_{i=1}^n g_i!} \exp \left( -\lambda(g) - \frac{(g_i)2}{2} - \mu(g, X) + O(\hat{\Delta}(g)^2/m(g)) \right).$$

**Corollary 14.** Let $0 \leq k \leq n/2$ be an integer. Let $H$ be a graph on $[n]$ containing $k$ isolated edges and a collection of disjoint cycles spanning the remaining $n - 2k$ vertices. Then, with $\alpha = 2k/n$,

$$\mathbb{P}(H \subseteq \mathcal{G}(n, d)) = \frac{(d-2)n + 2k)!d^{2n-k}d^n(d-1)^{n-2k}}{(d-2)n + k)!((d)n)!} \exp \left( \frac{\phi(d, \alpha)}{d^3/n} \right),$$

where

$$\phi(d, \alpha) = \frac{4(d-2)^2 - (d^2 - 5)\alpha^2 - (2d^2 - 14d + 20)\alpha}{4(d^2 + \alpha^2)}.$$  

(7)

**Remark 15.** Theorem 12 can be deduced from earlier work than [6], e.g. by McKay [16]. We cite [6, Theorem 1] as it is written in form of conditional edge probabilities, which is what we need in this paper.

A stronger version of Theorem 12 is available in [4, Theorem 6] which estimates the conditional edge probabilities up to a relative error $d^2/n^2$ instead of $d/n$. Using that result, we can deduce Corollary 14 with a smaller error $O(d^3/n)$ than $O(d^3/n)$. This will result in an improvement in the range of $d$ in several of theorems in the paper, e.g. in Theorems 3 [10] and [18]. However, applying [4, Theorem 6] involves more intensive calculations, and for a simpler proof we deduce Corollary 14 from Theorem 13 instead.

**Proof of Corollary 14.** Let $d^H$ denote the degree sequence of $H$ and let $g = d - d^H$ where $d = (d, \ldots, d)$. Then, $g$ has exactly $2k$ components of value $d - 1$ and $n - 2k$ components of value $d - 2$. Hence,

$$2m(g) = (d - 2)n + 2k,$$

$$\lambda(g) = \frac{1}{2((d - 2)n + 2k)} \left( (d - 1)(d - 2) \cdot 2k + (d - 2)(d - 3)(n - 2k) \right),$$

$$\mu(g, H) = \frac{1}{(d - 2)n + 2k} \left( (d - 1)^2 \cdot k + (d - 2)^2(n - 2k) \right).$$
and
\[2m(d) = dn, \quad \lambda(d) = \frac{1}{2dn} (d(d - 1)n), \quad \mu(d, \emptyset) = 0.\]

Moreover,
\[\hat{\Delta}(g) \cdot \hat{\Delta}(d) = O(d^2) \quad \text{and} \quad m(g), m(d) = \Omega(dn).\]

Thus,
\[\mathbb{P}(H \subseteq G(n, d)) = \frac{N(g, H)}{N(d, \emptyset)} = \frac{((d - 2)n + 2k)!/ \left(\frac{(d-2)n+2k}{2}(d-2)n+2k\right)}{(dn)!/ \left(\frac{dn}{2} \cdot \frac{dn}{2}\right)} \cdot d^n(d - 1)^{n - 2k} \exp(\phi(d, \alpha) + O(d^3/n)),\]

where
\[\phi(d, \alpha) = -\lambda(g) - \lambda(g)^2 - \mu(g, H) + \lambda(d) + \lambda(d)^2 + \mu(d, \emptyset) = \frac{4(d - 2)^2 - (d^2 - 5)\alpha^2 - (2d^2 - 14d + 20)\alpha}{4(d - 2 + \alpha)}.\]

See Maple calculations of (9) in the Appendix. Now the corollary follows by applying the Stirling formula to the factorials in (8). The relative error \(O(1/dn)\) in the Stirling formula is absorbed by \(O(d^3/n)\).

### 3.2 Cov(X, Y)

Fix a perfect matching \(H\) of \(K_n\). There are \(n/2 \cdot (n - 2)\) ways to choose a triangle \(T\) such that \(|H \cap T| = 1\). By Theorem 12, the conditional probability of \(T\) given \(H\) is \((d - 1)^2(d - 1)/2/(M - n)^2(1 + O(d/n))\). There are \(\binom{n}{3} - (n/2 \cdot (n - 2)) = (1 + O(1/n))n^3/6\) ways to choose \(T\) such that \(H \cap T = \emptyset\). The conditional probability of \(T\) given \(H\) is \((d - 1)^3/2/(M - n)^2(1 + O(d/n))\). Hence,
\[\mathbb{E}XY = \sum_H \mathbb{P}(H^+ \cap T) \sum_T \mathbb{P}(T | H^+),\]

where the first summation is over all perfect matchings in \(K_n\) and the second summation is over all triangles in \(K_n\). By the discussions above, \(\sum_T \mathbb{P}(T | H^+\) are the same for every perfect matching \(H\). Noting that \(\sum_H \mathbb{P}(H^+) = \mathbb{E}Y\), we have
\[\mathbb{E}XY = \mathbb{E}Y \left(\frac{n^2(d - 1)^3(d - 2)}{2(M - n)^2} + \frac{n^3}{6} \cdot \frac{(d - 1)^3(d - 2)^3}{(M - n)^3}(1 + O(d/n))\right)\]
\[= \mathbb{E}Y \left(\frac{d^2}{2} (1 - x)(1 - 2x) + \frac{d^3}{6} (1 - 3x)(1 - 2x)^3\right) (1 + O(d/n))\]
\[= \left(1 - 1/d^3 + O(1/d^4 + d/n)\right) \mathbb{E}X \mathbb{E}Y,\]

where \(x = 1/d\) in the second equation above, and the last equation above is obtained by taking the product of \(\frac{d^2}{2} (1 - x)(1 - 2x) + \frac{d^3}{6} (1 - 3x)(1 - 2x)^3\) and \((\mathbb{E}X)^{-1} = (1 + O(1/n))\frac{6}{d^4(1-x)^3}\) from Theorem 9 and then taking the Taylor expansion of the product at \(x = 0\). We include the Maple expansion formulae in the Appendix.
3.3 \( \mathbb{E}Y \)

Let \( H \) be a perfect matching of \( K_n \). By Corollary \[14\] with \( k = n/2 \) (i.e. \( \alpha = 1 \)), we have \( \phi(d, \alpha) = 1/4 \) and thus,

\[
\mathbb{P}(H \subseteq \mathcal{G}(n, d)) = (1 + O(d^3/n))\rho_1(n, d).
\]

where

\[
\rho_1(n, d) = \left(\frac{e}{n}\right)^{n/2} \left(\frac{d}{d - 1}\right)^{(n-1)/2} \frac{dn}{d^2} \exp\left(\frac{1}{4}\right)
\] (11)

Hence,

\[
\mathbb{E}Y = \frac{n!}{(n/2)!2^{n/2}}\rho_1(n, d).
\] (12)

3.4 \( \mathbb{E}Y^2 \)

Let \( 0 \leq k \leq n/2 \) be an integer. Fix two perfect matchings \( H_1 \) and \( H_2 \) of \( K_n \) such that \( |H_1 \cap H_2| = k \). Let \( \alpha = \alpha(k) = 2k/n \). Then, by Corollary \[14\],

\[
\mathbb{P}(H_1 \cup H_2 \subseteq \mathcal{G}(n, d)) = (1 + O(d^3/n))\rho_2(n, d, \alpha)
\]

where

\[
\rho_2(n, d, \alpha) = \frac{((d - 2)n + 2k)!dn^{dn-k}dn(d - 1)^{n-k}}{(d-2n)^{n-k}} \exp(\phi(d, \alpha))
\] (13)

\[
= \left(\frac{e}{n}\right)^{(1-\alpha)n} \left(\frac{d - 2 + \alpha}{d}\right)^{(n-1)/(2\alpha)} d^{2n} (d - 1)^{(1-\alpha)n} \exp(\phi(d, \alpha) + O(1/dn)),
\] (14)

with \( \phi(d, \alpha) \) defined in (7). Next we count pairs \((H_1, H_2)\) of perfect matchings of \( K_n \) such that \( |H_1 \cap H_2| = k \).

Lemma 16. The number of pairs \((H_1, H_2)\) of perfect matchings of \( K_n \) such that \( |H_1 \cap H_2| = k \) is

\[
(1 + O((n - 2k)^{-1})) \frac{n!}{2^{k}k!(\pi(n - 2k)/2)\sqrt{n}}.
\]

Proof. The exponential generating function for alternating cycles of length at least 4 is

\[
F(z) = \sum_{n=2}^{\infty} \frac{(2n)!}{2 \cdot 2n} \cdot \frac{z^{2n}}{(2n)!} = -\frac{1}{2} (\log(1 - x^2) + x^2).
\]

Thus, the number of pairs \((H_1, H_2)\) of disjoint perfect matchings of \( K_{2m} \) is

\[
(2m)! \cdot [z^{2m}] e^{F(z)} = (2m)! \cdot [z^{2m}] \frac{e^{-z^2/2}}{\sqrt{1-z^2}} = \frac{(2m)!}{\sqrt{\pi}m} (1 + O(m^{-1})).
\]
Thus, the number of pairs \((H_1, H_2)\) of perfect matchings of \(K_n\) such that \(|H_1 \cap H_2| = k\) is
\[
\binom{n}{2k} \cdot \frac{(2k)!}{2^{k!}} \cdot \frac{(n - 2k)!}{\sqrt{e\pi (n - 2k)/2}} (1 + O((n - 2k)^{-1})) = (1 + O((n - 2k)^{-1})) \frac{n!}{2^{k!} \sqrt{e\pi (n - 2k)/2}}. \]

By Lemma 16
\[
\mathbb{E} Y^2 = \sum_{k=0}^{n/2 - 1} (1 + O((n - 2k)^{-1} + d^2/n)) \frac{n!}{2^{k!} \sqrt{e\pi (n - 2k)/2}} \rho_2(n, d, \alpha(k)) + \mathbb{E} Y. \tag{15}
\]

Next, we show that the main contribution to \(\mathbb{E} Y^2\) are from \(k\) near some specific value. The proof of the lemma is postponed till Section 3.4.2.

**Lemma 17.** Let
\[
\bar{\alpha} = \frac{1}{d}, \quad \bar{k} = \lfloor \bar{\alpha} n / 2 \rfloor, \quad \bar{\delta} = \frac{2d}{n} \frac{d(d - 2)}{(d - 1)^2}.
\]
Then,
\[
\mathbb{E} Y^2 = (1 + O(d^3/n)) \frac{2}{\sqrt{e\bar{\delta} \bar{k}!} 2^{\bar{k}}} \sqrt{n - 2\bar{k}} \rho_2(n, d, \bar{\alpha}) + \mathbb{E} Y.
\]

**3.4.1 Comparing \(\mathbb{E} Y^2\) with \((\mathbb{E} Y)^2\)**

We complete the proof of Theorem 10 by verifying that
\[
\mathbb{E} Y^2 = \left(1 + \frac{1}{6d^3} + O(\xi)\right) (\mathbb{E} Y)^2, \tag{16}
\]
where \(\xi = d^{-4} + \frac{d^3}{n} + \sqrt{\frac{d}{n}} \log^6 n\). By (12) and Lemma 17
\[
(\mathbb{E} Y)^2 = (1 + O(\xi)) 2 \left(\frac{n}{e}\right)^n \rho_1(n, d)^2,
\]
and
\[
\mathbb{E} Y^2 = (1 + O(\xi)) \frac{2}{\sqrt{e\bar{\delta}}} \sqrt{\frac{n}{\bar{k}}} \left(\frac{n}{e}\right)^n \rho_2(n, d, \bar{\alpha}) \rho_1(n, d)^2.
\]
Hence,
\[
\frac{\mathbb{E} Y^2}{(\mathbb{E} Y)^2} = (1 + O(\xi)) \sqrt{\frac{n}{e\bar{\delta}k(n - 2k)}} \frac{\rho_2(n, d, \bar{\alpha})}{(2k/e)^k \rho_1(n, d)^2}
\]
With straightforward but tedious calculations (see Appendix for more details)
\[
\sqrt{\frac{n}{e\bar{\delta}k(n - 2k)} (2k/e)^k \rho_1(n, d)^2} = \left(1 + \frac{1}{6d^3} + O(\xi)\right), \tag{17}
\]
and now (16) follows. □
3.4.2 Proof of Lemma 17

Recall from (18) that

\[ \mathbb{E} Y^2 = \sum_{k=0}^{n/2-1} \left( 1 + O((n - 2k)^{-1} + d^2/n) \right) \frac{n!}{2^k k! \sqrt{\pi(n - 2k)/2}} \rho_2(n, d, \alpha(k)) + \mathbb{E} Y \]

\[ = \sum_{k=0}^{n/2-1} \left( 1 + O((n - 2k)^{-1} + d^2/n) \right) \sqrt{\frac{2}{e\pi n}} n! \varphi(k) + \mathbb{E} Y, \tag{18} \]

where

\[ \varphi(k) = \frac{\rho_2(n, d, \alpha(k))}{2^k k! \sqrt{n - 2k}}. \]

The proof of Lemma 17 is standard. We prove that the summand in (18) is maximised at \( \bar{k} \). Then we approximate the summation around \( \bar{k} \) by an integral of a function of form \( e^{-x^2} \).

The contributions to (18) from \( k \) far away from \( \bar{k} \) is negligible.

It is easy to see then that \( \exp(\phi(d, \alpha(k) - \phi(d, \alpha(k-1)))) = \exp(O(1/n)) \). Hence, by (13),

\[ \frac{\varphi(k)}{\varphi(k-1)} = \frac{(d - 2)n + 2k}{2(d - 1)^2 k} \left( 1 + O \left( \frac{1}{n} \right) \right) \quad \text{for all } k. \tag{19} \]

The maximum is achieved at \( k = \bar{k} \). Moreover,

\[ \frac{\varphi(\bar{k} + j)}{\varphi(\bar{k})} = \left( 1 + O \left( \frac{j}{n} \right) \right) \prod_{i=1}^{j} \left( 1 + i \left( \frac{2}{(d - 2)n + 2\bar{k}} - \frac{1}{\bar{k}} \right) + O \left( \frac{j^2}{\bar{k}^2} \right) \right) \]

\[ = \exp \left( -\bar{\delta} j + O \left( \frac{j^3}{\bar{k}^2} + \frac{j}{n} \right) \right) \]

\[ = \exp \left( -\frac{\bar{\delta}}{2} j^2 + O \left( \bar{\delta} j \right) \right), \tag{20} \]

recalling that \( \bar{\alpha} = 1/d, \) \( \bar{k} = [\bar{\alpha} \cdot n/2] \) and

\[ \bar{\delta} = \frac{2d}{n} \frac{d(d - 2)}{(d - 1)^2} = \left( \frac{1}{\bar{k}} - \frac{2}{(d - 2)n + 2\bar{k}} \right) \left( 1 + O \left( \frac{1}{\bar{k}} \right) \right) \]

\[ = \left( \frac{1}{\bar{k}} - \frac{2}{(d - 2)n + 2\bar{k}} \right) \left( 1 + O \left( \frac{1}{n} \right) \right) \].

It follows then that

\[ \sum_{k - \delta^{-1/2} \log^2(1/\delta) \leq k \leq k + \delta^{-1/2} \log^2(1/\delta)} \varphi(k) = (1 + O(\xi)) \varphi(\bar{k}) \sqrt{\frac{2}{\bar{\delta}}} \int_{-\infty}^{+\infty} e^{-x^2} dx = (1 + O(\xi)) \sqrt{\frac{2\pi}{\bar{\delta}}} \varphi(\bar{k}), \]
where
\[ \xi = \bar{\delta}^{1/2} \log^2(1/\bar{\delta}) + \frac{\bar{\delta}^{-3/2} \log^6(1/\bar{\delta})}{k^2} + \bar{\delta}^{1/2} = O \left( \sqrt{\frac{d}{n} \log^6 n} \right). \]

Note that the first two terms in \( \xi \) come from the accumulative error \( O(\bar{\delta} j + j^3 / \bar{k}^2) \) in [20], and the last term comes from approximating the sum of \( \exp(-\bar{\delta} j^2 / 2) \) by an integral. The contributions to (18) from \( k \) where \( |k - \bar{k}| > \bar{\delta}^{-1/2} \log^2(1/\bar{\delta}) \) is smaller than \( n^{-1} \) as a relative error by (19) and standard calculations on summing a geometrically bounded series. So

\[ \mathbb{E}Y^2 = \left( 1 + O \left( \sqrt{\frac{d}{n} \log^6 n + \frac{d^2}{n}} \right) \right) \frac{2 n! \rho_2(n, d, \alpha)}{\sqrt{c_3} k! 2^k \sqrt{n - 2k}}, \]

### 4 Proofs of Theorems 6 – 8

#### 4.1 Proof of Theorem 6

We prove Theorem 6 assuming Theorem 7. Let \( d = \Omega(\log^7 n) \). By [3, Theorem 2], there exists \( \delta_n = o(1) \) such that \( \mathcal{G}(n, d_1) \) can be coupled with \( \mathcal{G}(n, (1 + \delta_n)d_1/n) \) such that a.a.s. \( \mathcal{G}(n, d_1) \subseteq \mathcal{G}(n, (1 + \delta_n)d_1/n) \). Suppose \( d_2 \geq (1 + 2\delta_n)d_1 \). Then, again by [3, Theorem 2], a.a.s. \( \mathcal{G}(n, d_2) \) can be coupled with \( \mathcal{G}(n, (1 + \delta_n)d_1/n) \) such that a.a.s. \( \mathcal{G}(n, (1 + \delta_n)d_1/n) \subseteq \mathcal{G}(n, d_2) \). It follows now that Conjecture 5 holds for any \( (d_1, d_2) \) where \( d_1 = \Omega(\log^7 n) \) and \( d_2 - d_1 \geq 2\delta_n d_1 \). Now suppose \( d_1 + 1 \leq d_2 < (1 + 2\delta_n)d_1 \) and \( d_1 = O(n^{1/7} / \log^2 n) \). Then, there is a coupling where a.a.s. \( \mathcal{G}(n, d_1) \subseteq \mathcal{G}(n, d_2) \) by Theorem 7. Now Theorem 6 follows.

#### 4.2 Couple \( \mathcal{G}(n, d) \) and \( \mathcal{G}(n, d + 1) \)

Throughout this section we assume \( d \to \infty \) and \( d = o(n^{1/3}) \). Our goal is to couple \( \mathcal{G}(n, d) \) with \( \mathcal{G}(n, d + 1) \) so that \( \mathcal{G}(n, d) \subseteq \mathcal{G}(n, d + 1) \) with sufficiently high probability. In the next section, we “stitch” a sequence of such couplings together to obtain a simultaneous coupling as in Theorem 7.

Given \( \alpha = \alpha_n = o(1) \), define \( \eta = \eta(\alpha) \) where
\[
\eta(\alpha) = 2\alpha + \frac{1}{d^3 \alpha^2} + \frac{C d^3}{n \alpha^2} + \frac{C' \sqrt{d / n} \log^6 n}{\alpha^2}, \quad \text{where} \quad C' > 0 \text{ is a sufficiently large constant.}
\]

We prove the following stronger version of Theorem 8.

**Theorem 18.** Assume \( \alpha = o(1) \) is such that \( \eta(\alpha) = o(1) \). There is a coupling \((G_d, G_{d+1})\) where marginally \( G_d \sim \mathcal{G}(n, d) \) and \( G_{d+1} \sim \mathcal{G}(n, d + 1) \), and jointly with probability \( 1 - O(\eta) \), \( G_d \subseteq G_{d+1} \).

#### 4.2.1 The coupling procedure

For \( G \in \mathcal{G}(n, d + 1) \) let \( Y(G) \) denote the number of perfect matchings of \( G \). For \( G \in \mathcal{G}(n, d) \) let \( Z(G) \) denote the number of perfect matchings in \( K_n \setminus G \). We say \( G \) and \( G' \) are related, denoted by \( G \sim G' \), for \( G \in \mathcal{G}(n, d) \) and \( G' \in \mathcal{G}(n, d + 1) \), if \( G \subseteq G' \). We can represent this
relation using an auxiliary directed bipartite graph $X$ where $V(X) = G(n, d) \cup G(n, d + 1)$, and $(G, G')$ is an arc if $G \subseteq G'$. Thus, a $d$-regular graph $G$ in $X$ has out-degree $Z(G)$, and a $(d + 1)$-regular graph $G'$ in $X$ has in-degree $Y(G')$.

Let $\alpha = \alpha_n = o(1)$. By Theorem 10 and Chebyshev’s inequality,

$$\mathbb{P}_{G(n,d+1)}(|Y - EY| \geq \alpha EY) \leq \frac{\text{Var}Y}{\alpha^2(EY)^2} = \frac{1}{6d^3\alpha^2} + O\left(\frac{1}{d^4\alpha^2} + \frac{d^3}{n\alpha^2} + \frac{\sqrt{d/n\log^6 n}}{\alpha^2}\right).$$

By Theorem 13 (with $\mathbf{g}$ being the all one vector and $X$ being a $d$-regular graph) and Theorem 10, there exists a sufficiently large constant $C > 0$ such that

$$|Z(G) - Z^*| \leq \frac{Cd^2}{n} \cdot Z^*, \text{ for all } d\text{-regular graph } G,$$

$$|EY(G) - Y^*| \leq \frac{Cd^3}{n} \cdot Y^*, \text{ for } G \sim G(n, d + 1),$$

where

$$Z^* = \frac{n!e^{-d/2}}{(n/2)!2^{n/2}}, \quad Y^* = \frac{n!e^{1/4}}{(n/2)!2^{n/2}} \left(\frac{e}{n}\right)^{n/2} \left(\frac{d - 1}{d}\right)^{(d-1)n/2} d^{\frac{d^2}{2}}.$$

Let

$$Y' = (1 - \alpha - Cd^3/n) Y^*$$

$$Z = \left(1 + \frac{Cd^2}{n}\right) Z^*$$

Define

$$B = \{G \in G(n,d+1) : Y(G) < Y\} \quad \text{ and } \quad B' = \{G \in G(n,d+1) : Y(G) > (1 + \alpha + Cd^3/n) Y^*\}.$$  \hspace{1cm} (26)

By (21) and (23),

$$\mathbb{P}(B \cup B') = \frac{1}{6d^3\alpha^2} + O\left(\frac{1}{d^4\alpha^2} + \frac{d^3}{n\alpha^2} + \frac{\sqrt{d/n\log^6 n}}{\alpha^2}\right).$$  \hspace{1cm} (28)

Let $\hat{D}$ and $\hat{D}'$ denote the total in-degrees of $\mathcal{G}(n,d+1)$ and $\mathcal{G}(n,d+1) \setminus B$ respectively in $X$. That is,

$$D = |\{(G, G') \in \mathcal{G}(n,d) \times \mathcal{G}(n,d+1) : G \sim G'\}|$$

$$\hat{D} = |\{(G, G') \in \mathcal{G}(n,d) \times (\mathcal{G}(n,d+1) \setminus B) : G \sim G'\}|.$$  \hspace{1cm} (29)

Further, let $d^-(B)$ and $d^-(B')$ denote the total in-degrees of $B$ and $B'$ respectively in $X$. We prove the following bounds on $d^-(B)$ and $d^-(B')$. Recall that

$$\eta(\alpha) = 2\alpha + \frac{1}{d^3\alpha^2} + \frac{C'd^3}{n\alpha^2} + \frac{C' \sqrt{d/n\log^6 n}}{\alpha^2},$$

where $C' > 0$ is a sufficiently large constant.
Lemma 19. Assume $\alpha = o(1)$ is such that $\eta(\alpha) = o(1)$. Then, $d^-(B) + d^-(B') \leq \eta D$.

Proof. The number of edges in $X$ is $D = |\mathcal{G}(n, d + 1)| \mathbb{E} Y$. This can be rewritten as

$$d^-(B) + d^-(B') + |\mathcal{G}(n, d + 1) \setminus (B \cup B')| \cdot \mathbb{E} Y (1 + \xi),$$

where $|\xi| \leq \alpha + O(d^3/n)$ since $|Y(G)/\mathbb{E} Y - 1| \leq \alpha + O(d^3/n)$ for all $G \notin B \cup B'$ by the definition of $B$ and $B'$. By (28), the above is equal to

$$d^-(B) + d^-(B') + |\mathcal{G}(n, d + 1)| (1 - \xi') \cdot \mathbb{E} Y (1 + \xi)$$

where

$$0 \leq \xi' \leq \frac{1}{6d^3n^2} + O\left(\frac{1}{d^4n^2} + \frac{d^3}{n\alpha^2} + \frac{\sqrt{d/n \log^6 n}}{\alpha^2}\right).$$

Thus,

$$|\mathcal{G}(n, d + 1)| \mathbb{E} Y = d^-(B) + d^-(B') + |\mathcal{G}(n, d + 1)| \cdot \mathbb{E} Y (1 - \xi' + \xi - \xi' \xi),$$

which implies that

$$d^-(B) + d^-(B') = |\mathcal{G}(n, d + 1)| \cdot \mathbb{E} Y (\xi' - \xi + \xi' \xi) < \eta D,$$

by the definition of $\eta$. ■

Finally we are ready to define the coupling $(G_d, G_{d+1})$.

- Let $G_d$ be a uniformly random graph in $\mathcal{G}(n, d)$ and let $G'$ be the graph obtained from $G_d$ by adding a uniformly random perfect matching of $K_n \setminus G_d$. Let $H$ be a uniformly random graph in $\mathcal{G}(n, d + 1)$ independent of $G'$.

- If $G' \in B$ then let $G_{d+1} = G'$ with probability $(1 - \eta) \frac{Z(G)}{Z}$ and let $G_{d+1} = H$ with the remaining probability.

- If $G' \in \mathcal{G}(n, d + 1) \setminus B$, then

$$G_{d+1} = \begin{cases} 
G' & \text{with probability } (1 - \eta) \frac{Z(G)}{Z} \cdot \frac{Y(G')}{Y(G)} \\
G'' & \text{with probability } (1 - \eta) \frac{Z(G)}{Z} \cdot \frac{Y(G')}{D} \text{ for every } G'' \in B \\
H & \text{with the remaining probability.}
\end{cases}$$

The following lemma justifies that the coupling procedure is well defined.

Lemma 20. \hfill

$$\frac{Z(G)}{Z} \geq 1 - 3C\frac{d^2}{n}, \quad \text{for every } d\text{-regular graph } G,$$

$$\frac{Y}{Y(G')} \geq 1 - 3\alpha - 3C\frac{d^3}{n}, \quad \text{for every } (d + 1)\text{-regular graph } G \in \mathcal{G}(n, d + 1) \setminus (B \cup B'),$$

and for every $(G, G') \in \mathcal{G}(n, d) \times (\mathcal{G}(n, d + 1) \setminus B)$ where $G \sim G'$,

$$(1 - \eta) \frac{Z(G)}{Z} \cdot \frac{Y}{Y(G')} + \sum_{G'' \in B} (1 - \eta) \frac{Z(G)}{Z} \frac{(Y - Y(G''))}{D} \leq 1.$$
Proof. The first inequality in the lemma follows by (22) and (25). The second inequality in the lemma follows by (24), (26) and (27). For the last inequality, note that
\[
\frac{Z(G)}{Z}, \frac{Y}{Y(G')} \leq 1
\]
always and thus it is sufficient to show that
\[
\sum_{G'' \in B} \left( \frac{Y - Y(G'')}{\hat{D}} \right) \leq \eta.
\]
By (21),
\[
|B| \leq \left( \frac{1}{6d^3\alpha^2} + O \left( \frac{1}{d^4\alpha^2} + \frac{d^3}{n\alpha^2} + \frac{\sqrt{d/n}\log^6 n}{\alpha^2} \right) \right) |G(n,d + 1)|.
\]
By Lemma 19,
\[
\hat{D} = (1 + O(\eta))D = (1 + O(\eta))|G(n,d + 1)| |\mathbb{Y}| = (1 + O(\eta))|G(n,d + 1)| |\mathbb{Y}|
\]
Thus,
\[
\sum_{G'' \in B} (Y - Y(G'')) \leq |B| \leq \left( \frac{1}{6d^3\alpha^2} + O \left( \frac{1}{d^4\alpha^2} + \frac{d^3}{n\alpha^2} + \frac{\sqrt{d/n}\log^6 n}{\alpha^2} \right) \right) |G(n,d + 1)| |\mathbb{Y}| \leq \eta \hat{D},
\]
by the definition of \( \eta \). Thus, the last inequality of the lemma follows. \( \blacksquare \)

4.2.2 Proof of Theorem 18

By the construction, \( G_d \) is obviously distributed as \( G(n,d) \) marginally. We prove that the marginal distribution of \( G_{d+1} \) is \( G(n,d+1) \). Let \( G' \) be a \((d+1)\)-regular graph. Let
\[
\sigma_d = \frac{1}{|G(n,d)|}, \quad \sigma_{d+1} = \frac{1}{|G(n,d + 1)|}.
\]
If \( \hat{G} \in G(n,d + 1) \setminus B \) then
\[
\mathbb{P}(G_{d+1} = \hat{G}) = \sum_{G:G \sim G} \frac{\sigma_d}{Z(G)} \cdot (1 - \eta) \frac{Z(G)}{Z} \cdot \frac{Y}{Y(G)} + \varphi,
\]
where
\[
\varphi = \sum_{(G,G'): G'' \in B} \frac{\sigma_d}{Z(G)} \left( 1 - (1 - \eta) \frac{Z(G)}{Z} \right) \sigma_{d+1}
\]
\[
+ \sum_{(G,G'): G'' \not\in B} \frac{\sigma_d}{Z(G)} \left( 1 - (1 - \eta) \frac{Z(G)}{Z} \cdot \frac{Y}{Y(G')} - \sum_{G'' \in B} (1 - \eta) \frac{Z(G) (Y - Y(G''))}{\hat{D}} \right) \sigma_{d+1}.
\]

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In the first summation in (31), \( \sigma_d/Z(G) \) is the probability that \( G_d = G \) and \( G' = \hat{G} \). Conditioning on that, \( (1 - \eta) \frac{Z(G)}{Z} \) is the probability that \( G_{d+1} = G' \). Thus this summation gives the contribution to \( \mathbb{P}(G_{d+1} = \hat{G}) \) from the case that \( G' = \hat{G} \) and \( G_{d+1} \) is set \( G' \). Similarly, it is easy to see that \( \varphi \) is the probability that \( G_{d+1} \) is set as a uniformly random graph in \( \mathcal{G}(n, d+1) \), no matter \( G' \) is in \( \mathcal{B} \) or not. Note that the value of \( \varphi \) is independent of \( \hat{G} \). Hence, by noting that \( Y(\hat{G}) = |\{G : G \sim \hat{G}\}| \), we obtain

\[
\mathbb{P}(G_{d+1} = \hat{G}) = (1 - \eta)\sigma_d \frac{Y}{Z} + \varphi,
\]

which is independent of \( \hat{G} \) for all \( \hat{G} \in \mathcal{G}(n, d+1) \setminus \mathcal{B} \).

Next, suppose \( \hat{G} \in \mathcal{B} \). Then,

\[
\mathbb{P}(G_{d+1} = \hat{G}) = \sum_{G:G \sim \hat{G}} \frac{\sigma_d}{Z(G)} (1 - \eta) \frac{Z(G)}{Z} + \sum_{(G,G') \in \mathcal{B}: G' \notin \mathcal{B}} \frac{\sigma_d}{Z(G)} (1 - \eta) \frac{Z(G)}{Z} \frac{Y - Y(\hat{G})}{D} + \varphi,
\]

where the second summation above is from the case where \( G_d = G, G' \notin \mathcal{B} \) and \( G_{d+1} \) is set to be \( \hat{G} \) which occurs with probability \( (1 - \eta) \frac{Z(G)}{Z} \frac{Y - Y(\hat{G})}{D} \), given \( (G, G') \). The first summation above gives \( (1 - \eta)\sigma_d Y(\hat{G})/Z \). The second summation above gives \( (1 - \eta)\sigma_d Y - Y(\hat{G}))/Z \) by (30). Hence,

\[
\mathbb{P}(G_{d+1} = \hat{G}) = (1 - \eta)\sigma_d \frac{Y}{Z} + \varphi,
\]

which is independent of \( \hat{G} \) for all \( \hat{G} \in \mathcal{B} \), and is the same for all \( \hat{G} \in \mathcal{G}(n, d+1) \setminus \mathcal{B} \). This confirms that the marginal distribution of \( G_{d+1} \) is uniform in \( \mathcal{G}(n, d+1) \).

Finally, we prove that \( G_d \subseteq G_{d+1} \) with probability \( 1 - O(\eta) \). Note that \( G_d \subseteq G_{d+1} \) if \( G_{d+1} = G' \) in the construction. Thus, it is sufficient to show that the probability that \( G_{d+1} = H \) or \( G_{d+1} = G'' \) for some \( G'' \in \mathcal{B} \) in the case \( G' \notin \mathcal{B} \) is \( O(\eta) \).

Suppose \( G' \in \mathcal{B} \).

\[
\mathbb{P}(G_{d+1} = H) = \sum_{G \in \mathcal{G}(n, d)} \sum_{G', G' \sim G} \frac{\sigma_d}{Z(G)} (1 - (1 - \eta) \frac{Z(G)}{Z}) = \sum_{G \in \mathcal{G}(n, d)} \sigma_d \sum_{G' : G' \sim G} \frac{1}{Z(G)} = O(\eta).
\]

Suppose \( G' \notin \mathcal{B} \).

\[
\mathbb{P}\left(G_{d+1} = H \text{ or } G_{d+1} = G'' \text{ for some } G'' \in \mathcal{B}\right) = \sum_{G \in \mathcal{G}(n, d)} \sum_{G', G' \sim G} \frac{\sigma_d}{Z(G)} (1 - (1 - \eta) \frac{Z(G)}{Z} \frac{Y}{Y(G')}) \leq \sum_{G \in \mathcal{G}(n, d)} \sum_{G', G' \sim G} \frac{\sigma_d}{Z(G)} O(\eta) + \sum_{G \in \mathcal{G}(n, d)} \sum_{G', G' \sim G} \frac{\sigma_d}{Z(G)}.
\]
as for every $G \notin B \cup B'$, $1 - (1 - \eta)\frac{Z(G)}{Z} \cdot \frac{Y}{\sqrt{|G'|}} = O(\eta)$ by Lemma \ref{lem:extent_bound} and for $G \in B \cup B'$ we use the trivial upper bound $1 - (1 - \eta)\frac{Z(G)}{Z} \cdot \frac{Y}{\sqrt{|G'|}} \leq 1$. Since $|\{G' : G' \sim G, G' \notin B \cup B'\}| \leq |\{G' : G' \sim G\}| = Z(G)$, and $\sigma_d \cdot |G(n, d)| = 1$, the first double summation above is $O(\eta)$. The second double summation above is equal to

$$(1 + O(d^2/n)) \frac{\sigma_d}{Z} |\{G, G' \in G(n, d) \times B' : G \sim G'\}| = (1 + O(d^2/n)) \frac{\sigma_d}{Z} \eta D \quad \text{(by Lemma \ref{lem:density_bound})},$$

$$= (1 + O(d^2/n)) \frac{\sigma_d}{Z} \eta |G(n, d)| \mathbb{E}Z = O(\eta),$$

where the last equality above holds because $Z^* \sim \mathbb{E}Z$ and $\sigma_d |G(n, d)| = 1$.  

\subsection*{4.3 Proof of Theorem \ref{thm:finite_expansion}}

Suppose $d \to \infty$ and $d = o(n^{1/3})$. Then there exists $\alpha = o(1)$ such that $\eta(\alpha) = o(1)$. Theorem \ref{thm:finite_expansion} follows by Theorem \ref{thm:finite_expansion_bound} with such a choice of $\alpha$.  

\subsection*{4.4 Proof of Theorem \ref{thm:finite_expansion_bound}}

Suppose $d \to \infty$ and $d = O(n^{1/7}/\log^2 n)$. Let $\alpha = 1/d$. It follows now that $\eta = O(1/d)$. We prove that for each $1 \leq j \leq \lfloor \epsilon_n d \rfloor$, there is a constant $C > 0$, and a coupling $(G_d, \ldots, G_{d+j})$ where $G_i \sim G(n, i)$ for every $d \leq i \leq d + j$ and with probability at least $1 - Cj/d$, $G_d \subseteq G_{d+1} \subseteq \cdots \subseteq G_{d+j}$. Then Theorem \ref{thm:finite_expansion_bound} follows by taking $j = \lfloor \epsilon_n d \rfloor$.

We prove by induction on $j$. The base case $j = 1$ follows directly by Theorem \ref{thm:finite_expansion_bound} with our choice of $\alpha$. Suppose the statement holds for some $1 \leq j < \lfloor \epsilon_n d \rfloor$. Let $\pi_j$ be the joint probability distribution of such a coupling $(G_d, \ldots, G_{d+j})$. Again, by Theorem \ref{thm:finite_expansion_bound} there is a coupling $(G_{d+j}, G_{d+j+1})$ where $G_{d+j} \sim G(n, d + j)$, $G_{d+j+1} \sim G(n, d + j + 1)$ and with probability at least $1 - C/d$, $G_{d+j} \subseteq G_{d+j+1}$. Let $\pi$ denote the joint probability of this coupling $(G_d, G_{d+j+1})$. We construct a coupling $(G_d, \ldots, G_{d+j+1})$ by first sample $(G_d, \ldots, G_{d+j})$ according to the distribution $\pi_j$, and then sample $G_{d+j+1}$ according to the conditional probability $\pi(G_{d+j+1} | G_{d+j})$. The resulting coupling satisfies the required marginal distribution conditions. Moreover, the probability that either $G_d \subseteq \cdots \subseteq G_{d+j}$ fails or $G_{d+j} \subseteq G_{d+j+1}$ fails is at most $Cj/d + C/d = C(j + 1)/d$ by the union bound. The assertion follows by induction.

\section{Future research}

As mentioned in Remark \ref{rem:future_research}, the error $d^3/n$ in Corollary \ref{cor:finite_expansion} can be improved to $d^3/n$ if we apply \cite[Theorem 6]{finite_expansion} and go through more involved calculations. Another approach is to improve the error in Theorem \ref{thm:finite_expansion_bound}, which is of independent interest and can lead to improvements of many other existing results on subgraphs of $G(n, d)$.

We solved Conjecture \ref{conj:finite_expansion} for a certain range of $d_1$, by simultaneously coupling a sequence of random regular graphs. This is certainly not necessary, and is the cause of the restrictions.
on $d_1$ in Theorem 6. A more direct approach would be to prove concentration of the number of $k$-factors in $G(n,d)$. This would significantly relax the restrictions on $d_1$, and itself has independent interest.

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**Appendix**

- Justify (10)

\[ f(d) = \frac{d^2}{2} \left( 1 - \frac{1}{d} \right) \left( 1 - \frac{2}{d} \right) + \frac{d^3}{6} \left( 1 - \frac{2}{d} \right)^3 \]

simplify(f(d))

\[ \frac{(d-2)(d^2 - d + 1)}{(d-1)^3} \]

\[ \text{taylor} \left( \frac{(1-2x)(1-x+x^2)}{(1-x)^3}, x=0,4 \right) \]

\[ 1 - x^3 + O(x^4) \]

- Justify (9)
Justify (17)

We verify

\[ \frac{n}{\sqrt{e^\delta(\bar{\alpha}n/2)(n - \bar{\alpha}n)}} \rho_2(n, d, \bar{\alpha}) = \left(1 + \frac{1}{6d^5} + O(\xi)\right) \rho_1(n, d)^2(\bar{\alpha}n/e)^{\bar{\alpha}n/2}. \]  

(32)

Recall that \( \xi = d^{-4} + \frac{d^2}{n} + \sqrt{\frac{d}{n}} \log^6 n \) and \( \bar{\alpha} = 1/d \) and

\[ \rho_1(n, d) = \left(\frac{c}{n}\right)^{n/2} \left(\frac{d - 1}{d}\right)^{(d-1)n} \frac{d^{\frac{n}{2}}}{d^{\frac{n}{2}}} \exp\left(\frac{1}{4}\right), \]

and

\[ \rho_2(n, d, \bar{\alpha}) = \left(\frac{c}{n}\right)^{(1-\frac{2}{n})n} \left(\frac{d - 2 + \alpha}{d}\right)^{(d-2+\alpha)n} d^{\frac{n}{2}} (d - 1)^{(1-\alpha)n} \exp\left(\phi(d, \bar{\alpha}) + O(n^{-1})\right). \]

It is easy to check that all exponential terms cancel exactly from both sides of (32). By Corollary 14 with \( \alpha = \bar{\alpha} \) (See Maple calculations and expansions below),

\[ \phi(d, \bar{\alpha}) = \frac{4d^2 - 10d + 5}{4(d - 1)^2} = 1 - \frac{1}{2d} - \frac{3}{4d^2} - \frac{1}{d^3} + O(d^{-4}). \]

The polynomially bounded term on the left hand side of (32) is

\[ \sqrt{\frac{n}{e^\delta(\bar{\alpha}n/2)(n - \bar{\alpha}n)}} \exp(\phi(d, \bar{\alpha})) = \sqrt{\frac{d - 1}{e(d - 2)}} \exp\left(1 - \frac{1}{2d} - \frac{3}{4d^2} - \frac{1}{d^3} + O(d^{-4})\right) \]

\[ = \exp(-1/2) \exp\left(\frac{1}{2d} + \frac{3}{4d^2} + \frac{7}{6d^3} + O(d^{-4})\right) \exp\left(1 - \frac{1}{2d} - \frac{3}{4d^2} - \frac{1}{d^3} + O(d^{-4})\right) \]

\[ = \exp\left(\frac{1}{2} + \frac{1}{6d^3} + O(\xi)\right). \]
See Maple expansion of $\sqrt{(d - 1)/(d - 2)}$ below where $x = 1/d$:

\[
\text{taylor}\left[\ln\left(1 + \frac{x}{1 - 2x}\right)^{\frac{1}{2}}, x = 0.4\right]
\]

\[
\frac{1}{2} x + \frac{3}{4} x^2 + \frac{7}{6} x^3 + O(x^4)
\]

The polynomially bounded term on the right hand side of (32) is

\[
\left(1 + \frac{1}{6d^3} + O(\xi)\right) \exp(1/2) = \exp\left(\frac{1}{2} + \frac{1}{6d^3} + O(\xi)\right).
\]