SOME RESULTS ON $r$-TRUNCATED DEGENERATE POISSON RANDOM VARIABLES

TAEKYUN KIM, DAE SAN KIM, SI-HYEON LEE, SEONG-HO PARK, AND LEE-CHAE JANG

Abstract. The zero-truncated Poisson distributions are certain discrete probability distributions whose supports are the set of positive integers, which are also known as the conditional Poisson distributions or the positive Poisson distributions. Recently, as a natural extension of those distributions, Kim-Kim studied the zero-truncated degenerate Poisson distributions. In this paper, we introduce the $r$-truncated degenerate Poisson random variable with parameter $\alpha > 0$, whose probability mass function is given by $p_{\lambda,r}(i) = \frac{(1)_{i,r}}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha) i!} \alpha^i (i = r+1, r+2, r+3, \ldots)$, and investigate various properties of this random variable.

1. Introduction

It is well known that a random variable $X$, taking on one of the values $0, 1, 2, \ldots$, is said to be the Poisson random variable with parameter $\alpha > 0$, if the probability mass function of $X$ is given by

$$p(i) = P\{X = i\} = e^{-\alpha} \frac{\alpha^i}{i!}, \quad (i = 0, 1, 2, \ldots),$$

(see [11, 13]).

We note that a Poisson random variable indicates how many events occurred in a given period of time. Further, a random variable $X_r$, taking on one of the values $r+1, r+2, \ldots$, is called the $r$-truncated Poisson random variable with parameter $\alpha > 0$, if the probability mass function of $X_r$ is given by

$$p_r(i) = P\{X_r = i\} = \frac{1}{e^{\alpha} - \sum_{l=0}^{r} \frac{\alpha^l}{l!}},$$

where $i = r+1, r+2, r+3, \ldots$, for $r \geq 0$. In the special case of $r = 0$, we obtain the zero-truncated Poisson distribution.

On the other hand, the degenerate Poisson random variables with parameter $\alpha > 0$, whose probability mass function is given by (see (1))

$$p_{\lambda}(i) = e^{-\frac{1}{\lambda}} \frac{\lambda^i}{i!} (1)_{i,\lambda} \quad (i = 0, 1, 2, \ldots),$$

were studied by Kim-Kim-Jang-Kim in [10]. In addition, the zero-truncated degenerate Poisson random variables with parameter $\alpha > 0$, whose probability mass function is given by

$$p_{\lambda,0}(i) = \frac{1}{e_{\lambda}(\alpha) - 1} \frac{\alpha^i}{i!} (1)_{i,\lambda} \quad (i = 1, 2, 3, \ldots),$$

were studied in [9].

The aim of this paper is to generalize the results on the zero-truncated degenerate Poisson distribution in [9] to the case of the $r$-truncated degenerate Poisson distribution as a natural extension.

2010 Mathematics Subject Classification. 11B73; 60G50.

Key words and phrases. $r$-truncated degenerate Poisson random variables; $r$-truncated degenerate Stirling numbers of the second kind.
of the $r$-truncated Poisson distribution. We will obtain, among other things, its expectation, its variance, its $n$-th moment, and its cumulative distribution function.

One motivation for this research is its potential applications to the coronavirus pandemic. It has been spreading unpredictably around the world, terrifying many people. Although several vaccines for the coronavirus have been developed and many people are getting vaccinated, they have many unexpected side effects as well. We would like to predict stability of the coronavirus vaccines after $r$-days the vaccines were shot. For this purpose, we study the $r$-truncated degenerate Poisson random variables (see (6)), which has the ‘degenerate factor’ $\lambda$ reflecting abnormal situations. Indeed, we think that Theorem 5 is useful in predicting the probability of the coronavirus vaccines becoming stable after the $r$-days of getting vaccinated. Another motivation is applications of various probabilistic methods in studying special numbers and polynomials arising from combinatorics and number theory. For this, we let the reader refer to [5-10,14 and the references therein]. For the rest of this section, we recall the necessary facts that are needed throughout this paper.

For any $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

\begin{equation}
\left(1 + \lambda t\right)^{x} = \sum_{k=0}^{\infty} \left(x\right)_{k,\lambda \mathbf{k}} \frac{t^{k}}{k!}, \quad \text{see } [1-3,5-10,14],
\end{equation}

where $(x)_{0,\lambda} = 1, (x)_{n,\lambda} = x(x-\lambda) \cdots (x-(n-1)\lambda), (n \geq 1)$.

In particular, for $x = 1$, we denote $e_{\lambda}(t)$ by $e_{\lambda}(t)$.

In addition, it is convenient to introduce $e_{\lambda,r}(t)$, for any integer $r \geq 0$, given by

\begin{equation}
e_{\lambda,r}(t) = \sum_{k=0}^{r} \left(1\right)_{k,\lambda \mathbf{k}} \frac{t^{k}}{k!}.
\end{equation}

The Bell polynomials are given by

\begin{equation}
\left(x^{(r)}\right)^{-1} = \sum_{n=0}^{\infty} \text{Bel}_{n}(x) \frac{t^{n}}{n!}, \quad \text{see } [4,12]).
\end{equation}

In light of (3), the degenerate Bell polynomials are defined in [9] by

\begin{equation}
e_{\lambda,r}^{-1}(x)e_{\lambda}(xe' = \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}(x) \frac{t^{n}}{n!}.
\end{equation}

In particular, for $x = 1$, $\text{Bel}_{n,\lambda} = \text{Bel}_{n,\lambda}(1), (n \geq 0)$, are called the degenerate Bell numbers. Note that

\begin{equation}
\lim_{\lambda \to 1} \text{Bel}_{n,\lambda}(x) = \text{Bel}_{n}(x), \quad (n \geq 0).
\end{equation}

Let $X$ be a Poisson random variable with parameter $\alpha > 0$. Then the moment of $X$ is given by

\begin{equation}
E[X^{n}] = \text{Bel}_{n}(\alpha), \quad (n \geq 0), \quad \text{see } [11,13]).
\end{equation}

Let $Y$ be a discrete random variable taking values in the nonnegative integers. Then the probability generating function of $Y$ is given by

\begin{equation}
F(t) = E[t^{Y}] = \sum_{i=0}^{\infty} p(i)t^{i}, \quad \text{see } [11,13]),
\end{equation}

where $p(i) = P\{Y = i\}$ is the probability mass function of $Y$.

The degenerate Stirling numbers of the second kind $S_{2,\lambda}(n,k)$ are defined by

\begin{equation}
\left(x\right)_{n,\lambda} = \sum_{i=0}^{n} S_{2,\lambda}(n,i)\left(x\right)_{i}, \quad (n \geq 0), \quad \text{see } [5]),
\end{equation}
or equivalently by

\[
\frac{1}{k!} \left( e^{\lambda} - 1 \right)^k = \sum_{n=k}^{\infty} S_{2,\lambda} (n, k) \frac{r^n}{n!}.
\]

Here \((x)_0 = 1, \ (x)_n = x(x-1)(x-2) \cdots (x-n+1), \ (n \geq 1)\).

Now, we consider the \(r\)-truncated degenerate Stirling numbers of the second kind \(S_{2,\lambda}^{[r]} (n, kr + k)\) given by

\[
\frac{1}{k!} \left( e^{\lambda} (t) - e^{\lambda, r} (t) \right)^k = \sum_{n=kr+k}^{\infty} S_{2,\lambda}^{[r]} (n, kr + k) \frac{r^n}{n!}.
\]

Note that

\[
S_{2,\lambda}^{[0]} (n, k) = S_{2,\lambda} (n, k), \quad \text{(see [5]).}
\]

2. \(r\)-TRUNCATED DEGENERATE POISSON RANDOM VARIABLES

The random variable \(X_{\lambda, r}\) is called the \(r\)-truncated degenerate Poisson random variable with parameter \(\alpha > 0\), if the probability mass function of \(X_{\lambda, r}\) is given by

\[
p_{\lambda, r}(k) = \frac{(1)_{k, \lambda}}{e^{\lambda} (\alpha) - e^{\lambda, r} (\alpha)} \frac{\alpha^k}{k!},
\]

where \(k = r + 1, r + 2, \ldots, \text{ with } r \geq 0\).

Here we must observe that

\[
\sum_{k=r+1}^{\infty} p_{\lambda, r}(k) = \frac{1}{e^{\lambda} (\alpha) - e^{\lambda, r} (\alpha)} \sum_{k=r+1}^{\infty} \frac{\alpha^k}{k!} (1)_{k, \lambda}
\]

\[
= \frac{1}{e^{\lambda} (\alpha) - e^{\lambda, r} (\alpha)} (e^{\lambda} (\alpha) - e^{\lambda, r} (\alpha))
\]

\[
= 1.
\]

In addition, we note that

\[
\lim_{\lambda \to 0} p_{\lambda, r}(k) = \frac{1}{e^{\alpha} - \sum_{l=0}^{r} \frac{\alpha^l}{l!}} \frac{\alpha^k}{k!}
\]

is the probability mass function of the \(r\)-truncated Poisson random variable with parameter \(\alpha > 0\).
Let us assume that $X_{\lambda,r}$ is the $r$-truncated degenerate Poisson random variable with parameter $\alpha > 0$. Then we note that the expectation of $X_{\lambda,r}$ is given by

\begin{align*}
E[X_{\lambda,r}] &= \sum_{n=r+1}^{\infty} n p_{\lambda,r}(n) = \frac{1}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \sum_{n=r+1}^{\infty} \frac{\alpha^n}{n!} (1)_{n,\lambda} \\
&= \frac{\alpha}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \sum_{n=r+1}^{\infty} \frac{\alpha^n}{n!} (1)_{n+1,\lambda} \\
&= \frac{\alpha}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \left( \sum_{n=r+1}^{\infty} \frac{\alpha^n}{n!} (1)_{n+1,\lambda} + \frac{\alpha^r}{r!} (1)_{r+1,\lambda} \right) \\
&= \frac{\alpha}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \left( \sum_{n=r+1}^{\infty} \frac{\alpha^n}{n!} (1)_{n,\lambda}(1-n\lambda) + \frac{\alpha^r}{r!} (1)_{r+1,\lambda} \right) \\
&= \alpha - \frac{\alpha \lambda}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \sum_{n=r+1}^{\infty} \frac{\alpha^n}{n!} (1)_{n,\lambda} + \frac{\alpha}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \frac{\alpha^r}{r!} (1)_{r+1,\lambda} \\
&= \alpha - \frac{\alpha \lambda}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} E[X_{\lambda,r}] + \frac{\alpha}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \frac{\alpha^r}{r!} (1)_{r+1,\lambda}.
\end{align*}

From (7), we obtain the following theorem.

**Theorem 1.** Let $X_{\lambda,r}$ be the $r$-truncated degenerate Poisson random variable with parameter $\alpha > 0$. Then, for $\lambda \neq -\frac{1}{\alpha}$, the expectation of $X_{\lambda,r}$ is given by

\begin{align*}
E[X_{\lambda,r}] &= \frac{1}{1 + \alpha \lambda} \left[ \alpha + \frac{\alpha^{r+1}}{r!} \frac{(1)_{r+1,\lambda}}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \right].
\end{align*}

Now, we observe that

\begin{align*}
E[X_{\lambda,r}^2] &= \sum_{n=k+1}^{\infty} n^2 p_{\lambda,r}(n) = \frac{1}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \sum_{n=k+1}^{\infty} \frac{n^2 \alpha^n}{n!} (1)_{n,\lambda} \\
&= \frac{1}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \left( \sum_{n=k+1}^{\infty} \frac{n^2 \alpha^n}{(n-2)!} (1)_{n,\lambda} + \sum_{n=k+1}^{\infty} \frac{n \alpha^n}{(n-1)!} (1)_{n,\lambda} \right) \\
&= \frac{1}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \left( \sum_{n=k+1}^{\infty} \frac{n \alpha^n}{(n-1)!} (1)_{n,\lambda} + \sum_{n=k+1}^{\infty} \frac{\alpha^n}{(n-1)!} (1)_{n,\lambda} \right) \\
&+ \frac{1}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \frac{\alpha^{r+1}}{(r-1)!} (1)_{r+1,\lambda} \\
&= \frac{1}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \left( \alpha \sum_{n=k+1}^{\infty} \frac{\alpha^n}{(n-1)!} (1)_{n,\lambda} + \sum_{n=k+1}^{\infty} \frac{n \alpha^n}{n!} (1)_{n,\lambda} \right) \\
&+ \frac{1}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \frac{\alpha^{r+1}}{r!} (1)_{r+1,\lambda} \\
&= \frac{\alpha}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \sum_{n=r+1}^{\infty} \frac{n \alpha^n}{n!} (1)_{n,\lambda}(1-n\lambda) + E[X_{\lambda,r}] \right) \\
&+ \frac{r \alpha^{r+1}}{r!} (1)_{r+1,\lambda} \\
&+ \frac{\alpha}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)} \frac{\alpha^r}{r!} (1)_{r+1,\lambda} \\
&= \alpha E[X_{\lambda,r}] - \lambda \alpha E[X_{\lambda,r}^2] + E[X_{\lambda,r}] + \frac{\alpha^{r+1}}{r!} \frac{r(1)_{r+1,\lambda}}{e_{\lambda}(\alpha) - e_{\lambda,r}(\alpha)}.$


From (8), we note that

\[ E[X^2_{\lambda,r}] = \frac{\alpha + 1}{1 + \lambda \alpha} E[X_{\lambda,r}] + \frac{1}{1 + \lambda \alpha} \sum_{r=1}^{n} r(1)_{r+1,\lambda} \frac{\alpha^{r+1}}{r!}, \]

where \( \lambda \neq -\frac{1}{\alpha} \).

For \( \lambda \neq -\frac{1}{\alpha} \), we note that the variance of \( X_{\lambda,r} \) is given by

\[ \text{Var}(X_{\lambda,r}) = [E[X^2_{\lambda,r}] - (E[X_{\lambda,r}])^2] \]

\[ = \frac{1}{(1 + \lambda \alpha)^2} \left( \alpha + \frac{\alpha^{r+1}}{r!} (1)_{r+1,\lambda} \frac{(1)_{r+1,\lambda}}{r!} \frac{\alpha^{r+1}}{r!} \right) \left( 1 - \frac{\alpha^{r+1}}{r!} (1)_{r+1,\lambda} \frac{\alpha^{r+1}}{r!} \right) \]

Therefore, by (10), we obtain the following theorem.

**Theorem 2.** Let \( X_{\lambda, r} \) be the \( r \)-truncated degenerate Poisson random variable with parameter \( \alpha > 0 \).

For \( \lambda \neq -\frac{1}{\alpha} \), we have

\[ \text{Var}(X_{\lambda,r}) = \frac{1}{(1 + \lambda \alpha)^2} \left( \alpha + \frac{\alpha^{r+1}}{r!} (1)_{r+1,\lambda} \frac{(1)_{r+1,\lambda}}{r!} \frac{\alpha^{r+1}}{r!} \right) \left( 1 - \frac{\alpha^{r+1}}{r!} (1)_{r+1,\lambda} \frac{\alpha^{r+1}}{r!} \right) \]

Let us consider the generating function of the moments of the \( r \)-truncated degenerate Poisson random variable with parameter \( \alpha > 0 \). Then, from (4), we have

\[ \sum_{n=0}^{\infty} E[X^n_{\lambda,r}] \frac{t^n}{n!} = E[e^{X_{\lambda,r}}} = \sum_{m=0}^{\infty} e^{\alpha m} p_{\lambda,r}(m) \]

\[ = \frac{1}{e_{\lambda}(\alpha) - e_{\lambda, r}(\alpha)} \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} (1)_{m, \lambda} e^{\alpha m} \]

\[ = \frac{e_{\lambda}(\alpha) - e_{\lambda, r}(\alpha)}{e_{\lambda}(\alpha) - e_{\lambda, r}(\alpha)} \left( e_{\lambda}(\alpha e^\alpha) - \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} (1)_{m, \lambda} e^{\alpha m} \right) \]

\[ = \frac{e_{\lambda}(\alpha)}{e_{\lambda}(\alpha) - e_{\lambda, r}(\alpha)} \left( \sum_{n=0}^{\infty} \frac{\alpha^m}{m!} (1)_{m, \lambda} \frac{n^m}{n!} \right) \]

\[ = \frac{e_{\lambda}(\alpha)}{e_{\lambda}(\alpha) - e_{\lambda, r}(\alpha)} \left( \sum_{n=0}^{\infty} \frac{\alpha^m}{m!} (1)_{m, \lambda} \frac{n^m}{n!} \right) \]
Thus, by (11), we get the next theorem.

**Theorem 3.** Let $X_{\lambda,r}$ be the $r$-truncated degenerate Poisson random variable with parameter $\alpha > 0$. For $n \geq 0$, we have

$$E[X_{\lambda,r}^n] = \frac{e_\lambda(\alpha)}{e_\lambda(\alpha) - e_{\lambda,r}(\alpha)} \left( \text{Bel}_{n,\lambda}(\alpha) - \sum_{m=0}^{r} \frac{\alpha^m}{m!} (1)_{m,\lambda} m^n \right).$$

For $x \geq r + 1$, we note that the cumulative distribution function is given by

$$F_{X_{\lambda,r}}(x) = P\{X_{\lambda,r} \leq x\} = \sum_{k=r+1}^{[x]} p_{\lambda,r}(k)$$

$$= \frac{1}{e_\lambda(\alpha) - e_{\lambda,r}(\alpha)} \sum_{k=r+1}^{[x]} \frac{\alpha^k}{k!} (1)_{k,\lambda}$$

$$= \frac{1}{e_\lambda(\alpha) - e_{\lambda,r}(\alpha)} \left( \sum_{k=0}^{[x]} \frac{\alpha^k}{k!} (1)_{k,\lambda} - \sum_{k=0}^{r} \frac{\alpha^k}{k!} (1)_{k,\lambda} \right).$$

From (12) and (2), we can derive the following equation.

$$F_{X_{\lambda,r}}(x) = \frac{1}{e_\lambda(\alpha) - e_{\lambda,r}(\alpha)} (e_{\lambda,\alpha}(\alpha) - e_{\lambda,r}(\alpha)).$$

**Theorem 4.** Assume that $X_{\lambda,r}$ is the $r$-truncated degenerate Poisson random variable with parameter $\alpha > 0$. For $x \geq r + 1$, the cumulative distribution function of $X_{\lambda,r}$ is given by

$$F_{X_{\lambda,r}}(x) = \frac{1}{e_\lambda(\alpha) - e_{\lambda,r}(\alpha)} (e_{\lambda,\alpha}(\alpha) - e_{\lambda,r}(\alpha)).$$

Let us assume that $X_{\lambda,r}^{(1)}, X_{\lambda,r}^{(2)}, \ldots, X_{\lambda,r}^{(k)}$ are identically independent $r$-truncated degenerate Poisson random variables with parameter $\alpha > 0$, and let

$$X_{\lambda,r} = \sum_{i=1}^{k} X_{\lambda,r}^{(i)}, \quad (k \in \mathbb{N}).$$

From the probability generating function of random variable, we note that

$$E[t^{X_{\lambda,r}}] = \sum_{n=r+1}^{\infty} t^n P[X_{\lambda,r} = n]$$

$$= \frac{1}{e_\lambda(\alpha) - e_{\lambda,r}(\alpha)} \sum_{n=r+1}^{\infty} \frac{\alpha^n}{n!} (1)_{n,\lambda} t^n$$

$$= \frac{e_\lambda(\alpha t) - e_{\lambda,r}(\alpha t)}{e_\lambda(\alpha) - e_{\lambda,r}(\alpha)}.$$

By (14), we get

$$E[t^{X_{\lambda,r}}] = \prod_{i=1}^{k} E[t^{X_{\lambda,r}^{(i)}}]$$

$$= \left( \frac{1}{e_\lambda(\alpha) - e_{\lambda,r}(\alpha)} \right)^k \frac{k!}{k!} (e_{\lambda}(\alpha t) - e_{\lambda,r}(\alpha t))^k$$

$$= \left( \frac{k!}{e_\lambda(\alpha) - e_{\lambda,r}(\alpha)} \right)^k \sum_{n=kr+k}^{\infty} S_{n,\lambda}(n, kr + k) \frac{\alpha^n r^n}{n!}.$$
On the other hand,

\[
E[t^{X_{\lambda, r}}] = E[t^{X_{\lambda, r}^{(1)} + X_{\lambda, r}^{(2)} + \cdots + X_{\lambda, r}^{(k)}}] = \sum_{n=kr+r}^{\infty} e^{\alpha} \frac{\alpha^n}{n!} P_X(n) \]

Therefore, by (15) and (16), we obtain the following theorem.

**Theorem 5.** Let \(X_{\lambda, r}^{(1)}, X_{\lambda, r}^{(2)}, \ldots, X_{\lambda, r}^{(k)}\) be identically independent \(r\)-truncated degenerate Poisson random variables with parameter \(\alpha > 0\), and let \(X_{\lambda, r} = \sum_{i=1}^{k} X_{\lambda, r}^{(i)}\). Then the probability for \(X_{\lambda, r}\) is given by

\[
P[X_{\lambda, r} = n] = \begin{cases} 
\frac{k!}{(e^{\alpha} - e^{\alpha, r}(\alpha))^k} e^{\alpha} \frac{\alpha^r}{r!} P_X(n, kr + k), & \text{if } n \geq kr + r, \\
0, & \text{otherwise.}
\end{cases}
\]

3. Conclusion

We generalized the results on the zero-truncated degenerate Poisson distribution to the case of the \(r\)-truncated degenerate Poisson distribution, with its potential applications to the coronavirus pandemic and applications of probabilistic methods to the study of some special numbers and polynomials in mind.

Let \(X_{\lambda, r}\) be the \(r\)-truncated degenerate Poisson random variable with parameter \(\alpha\). Then, for the random variable \(X_{\lambda, r}\), we derived its expectation, its variance, its \(n\)-th moment, and its cumulative distribution function. In addition, we obtained two different expressions for the probability generating function of a finite sum of independent \(r\)-truncated degenerate Poisson random variables with equal parameters.

As one of our future projects, we would like to continue this line of research, namely to explore applications of various methods of probability theory to science, engineering and social science, and to the study of some special polynomials and numbers.

**Acknowledgements:** Not applicable.

**Funding:** Not applicable.

**Availability of data and material:** Not applicable.

**Competing interests:** The authors declare no conflict of interest.

**Author Contributions:** D.S.K. and T.K. wrote the paper; L.-C.J. and S.-H. L. and S.-H. P. checked the results of the paper and typed the paper. All authors have read and agreed to the published version of the manuscript.

**References**

[1] Carlitz, L. Degenerate Stirling, Bernoulli and Eulerian numbers, Utilitas Math. 15 (1979), 51–88.

[2] Carlitz, L. A degenerate Staudt-Clausen theorem, Arch. Math. (Basel) 7 (1956), 28–33.

[3] Lewis, P. A. W. Some results on tests for Poisson processes, Biometrika 52 (1965), 67–77.

[4] Gun, D.; Simsek, Y. Combinatorial sums involving Stirling, Fubini, Bernoulli numbers and approximate values of Catalan numbers, Adv. Stud. Contemp. Math. (Kyungshang) 30 (2020), no. 4, 503–513.

[5] Kim, D. S.; Kim, T. A Note on a New Type of Degenerate Bernoulli Numbers, Russ. J. Math. Phys. 27 (2020), no. 2, 227—235.
Two variable degenerate Bell polynomials associated with Poisson degenerate central moments, Proc. Jangjeon Math. Soc. 23 (2020), no. 4, 587–596.

A note on truncated degenerate exponential polynomials, Proc. Jangjeon Math. Soc. 24 (2021), no. 1, 63–76.

Note on extended Lah-Bell polynomials and degenerate extended Lah-Bell polynomials, Adv. Stud. Contemp. Math. (Kyungshang) 30 (2020), no. 4, 547–558.

Degenerate Zero-Truncated Poisson Random Variables, Russ. J. Math. Phys. 28 (2021), no. 1, 66–72.

A note on discrete degenerate random variables, Proc. Jangjeon Math. Soc. 23 (2020), no. 1, 125–135.

Probability and random processes for Electronical Engineering, 3’rd ed., Addition-Wesley, Massachusetts.

The umbral calculus, Pure and Applied Mathematics, 111. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.

Introduction to probability models, 12th edition, Academic Press, London, 2019.

New construction of type 2 degenerate central Fubini polynomials with their certain properties, Adv. Difference Equ. 2020, 2020:587.

DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA
Email address: tkim@kw.ac.kr

DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, REPUBLIC OF KOREA
Email address: dskim@sogang.ac.kr*

DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA
Email address: ugug11@naver.com

DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA
Email address: abcd2938471@kw.ac.kr

GRADUATE SCHOOL OF EDUCATION, KONKUK UNIVERSITY, SEOUL 143-701, REPUBLIC OF KOREA
Email address: ljang@konkuk.ac.kr