Note on universal algorithms for learning theory

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Abstract

We propose the general way of study the universal estimator for the regression problem in learning theory considered in [1] and [2]. This new approach allows us, for example, to improve the results from [1].

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1 Introduction

I am recalling this paper since it is somehow important. We wrote this paper to improve one of the results from [1]. It was written in [1] that “It has been communicated to us by Lucien Birgé that one can derive from one of his forthcoming papers (Birgé, 2004) [published in (2006)] that for any class Θ satisfying (5), (namely Θ ⊂ L^2(X, ρ_X) with a condition on the entropy number which is similar to the assumption (10)), there is an estimator f_z satisfying

\[ E\|f_\theta - f_z\|^2 = O\left(\frac{1}{m}\right)^{\frac{2s}{1+2s}}. \]

whenever f_\theta ∈ Θ”. Our paper shows how to construct an estimator with straightforward reasoning and I can not find the mentioned proof so I recall the proof of the result.
S. Cucker and S. Smale in their paper [4] determined the scope of the learning theory. We would like to present a general approach which corresponds to the papers [1] and [2]. The problem is the following.

Let $X = [0, 1]^d$ and $Y = [-A, A]$. On a product space $Z = X \times Y$ it is unknown probability Borel measure $\rho$. We shall assume that the marginal probability measure $\rho_X(S) = \rho(S \times Y)$ on $X$ is a Borel measure. We have

$$d\rho(x, y) = d\rho(y|x)d\rho_X(x).$$

We are given the data $z \subset Z$ of $m$ independent random observation $z_j = (x_j, y_j), j = 1, 2, \ldots, m$ identically distributed according to $\rho$. We are interested in estimating the **regression function**

$$f_\rho(x) := \int_Y yd\rho(y|x)$$

in $L^2(X, \rho_X)$ norm which will be denoted by $\| \cdot \|$.

To do it let $\mathbf{M} = \{M_v\}_{v \in T}$ denote any family of measurable functions on $X$ such that for all $v \in T$

$$0 \leq M_v(x) \leq 1, \quad x \in X \tag{1}$$

and

$$\sum_{v \in T} M_v(x) = 1, \quad x \in X. \tag{2}$$

One of examples of this kind of family $\mathbf{M}$ is the family $\{\chi_I\}_{I \in T}$, where $\chi_I$ denotes the indicator function of $I$ and $\{I : I \in T\}$ is any partition of the set $X$ (in [1] the sets $I$ are dyadic cubes). Another example we get if we conside the triangulation of $X$ with the vertices $\{v\}_{v \in T}$.

To define piecewise linear and continuous function corresponding to every vertex $v \in T$ it is sufficient to define such function on vertices. We define basis

$$M_v(w) = \begin{cases} 1 & \text{for vertices } w = v \\ 0 & \text{for } w \neq v. \end{cases}$$

It is not hard to check that family $\{M_v\}_{v \in T}$ satisfies (1) and (2).

Now for a given family $\mathbf{M}$ we define the operator

$$Q_M f(x) = \sum_{v \in T} c_v(f) M_v(x),$$

where $c_v(f)$
where
\[ c_v(f) = \frac{\alpha_v(f)}{q_v}, \quad \alpha_v(f) = \int_X f M_v d\rho_X, \quad q_v = \int_X M_v d\rho_X \]
and the estimator
\[ f_z(x) = \sum_{v \in T} c_v(z) M_v(x), \]
where
\[ c_v(z) = \frac{\alpha_v(z)}{q_v(z)}, \quad \alpha_v(z) = \frac{1}{m} \sum_{j=1}^m y_j M_v(x_j), \quad q_v(z) = \frac{1}{m} \sum_{j=1}^m M_v(x_j). \]
If \( q_v = 0 \) then we define \( c_v = 0 \) and if \( q_v(z) = 0 \) then we put \( c_v(z) = 0 \). Note also that \( E\alpha_v(z) = \alpha_v \) (here and subsequently \( \alpha_v := \alpha_v(f_{\theta}) \), \( c_v := c_v(f_{\theta}) \)) and \( Eq_v(z) = q_v \). Moreover
\[ \text{Var}(y M_v(x)) \leq \int_Z y^2 M_v^2(x) d\rho(x, y) \leq A^2 \int_X M_v^2(x) d\rho_X(x), \]

hence
\[ \text{Var}(y M_v(x)) \leq A^2 \int_X M_v(x) d\rho_X(x) = A^2 q_v, \quad (3) \]
\[ \text{Var}(M_v(x)) \leq E(M_v(x))^2 \leq E(M_v(x)) = q_v. \quad (4) \]
Therefore by Bernstein’s inequality, see for instance [1] we have for any \( \epsilon > 0 \)
\[ \text{Prob}\{ |\alpha_v - \alpha_v(z)| \geq \epsilon \} \leq 2e^{-\frac{3m^2}{6m^2 q_v + 4\epsilon^2}}, \quad (5) \]
\[ \text{Prob}\{ |q_v - q_v(z)| \geq \epsilon \} \leq 2e^{-\frac{3m^2}{6q_v + 4\epsilon^2}}. \quad (6) \]

The main result of this paper is

**Theorem 1.1** For any family \( M \)
\[ E\|Q_M f_{\theta} - f_z\|^2 = O\left(\frac{N}{m}\right), \]
where \( N = |T| \).
The new idea of the proof presented below allows us to improve the result from [1] (in Corollary 2.2 [1] the above expectation is bounded by $O(N/m \cdot \log N)$).

**Proof.** By (1), (2) and the convexity of the square functions we have

$$E\|QMf_\nu - f_x\|^2 \leq \int_X \sum_{\nu \in T} E|c_\nu - c_\nu(z)|^2 M_\nu(x) d\rho_X(x)$$

$$= \sum_{\nu \in T} E|c_\nu - c_\nu(z)|^2 \rho_\nu.$$

Note that if $\rho_\nu = 0$ then $E\rho_\nu(z) = 0$ hence $\rho_\nu(z) = 0 \rho_\nu$ a.e. Consequently

$$E\|QMf_\nu - f_x\|^2 \leq \sum_{\nu \in T, \rho_\nu > 0} E|c_\nu - c_\nu(z)|^2 \rho_\nu.$$

Let us fix $\nu$ such that $\rho_\nu > 0$. We can write

$$E|c_\nu - c_\nu(z)|^2 = \int_{\rho_\nu(z) > 0} |c_\nu - c_\nu(z)|^2 + \int_{\rho_\nu(z) = 0} |c_\nu|^2.$$

Note that if $\rho_\nu(z) = 0$ $\rho_\nu$ a.e. then for all $j M_\nu(x_j) = 0$, hence $\alpha_\nu(z) = 0 \rho_\nu$ a.e. Thus

$$E|c_\nu - c_\nu(z)|^2 = \int_{\rho_\nu(z) > 0} |c_\nu - c_\nu(z)|^2 + \int_{\rho_\nu(z) = 0} \frac{\alpha_\nu - \alpha_\nu(z)}{\rho_\nu} |c_\nu|^2.$$

For $b \neq 0$ and $t \neq 0$ we use the simple inequality

$$\left| \frac{a}{b} - \frac{s}{t} \right| \leq \frac{1}{|b|} |a - s| + \frac{|s|}{|bt|} |t - b| \quad (7)$$

to get

$$\left| \frac{a}{b} - \frac{s}{t} \right|^2 \leq 2 \frac{|a - s|^2}{b^2} + 2 \frac{1}{b^2 t^2} |t - b|^2, \quad (8)$$

which gives in particular that

$$\left| \frac{a_\nu}{\rho_\nu} - \frac{a_\nu(z)}{\rho_\nu(z)} \right|^2 \leq 2 \left| \frac{a_\nu - a_\nu(z)}{\rho_\nu^2} \right|^2 + 2 \left( \frac{a_\nu(z)}{\rho_\nu(z)} \right)^2 \left| \frac{\rho_\nu - \rho_\nu(z)}{\rho_\nu^2} \right|^2.$$

For $\rho_\nu(z) > 0$ we have

$$\frac{\alpha_\nu(z)^2}{\rho_\nu(z)^2} \leq A^2.$$
thus

\[ E|c_v - c_v(z)|^2 \leq \frac{3}{mg_v^2} \text{Var}(yM_v(x)) + \frac{2A^2}{mg_v^2} \text{Var}(M_v(x)). \]

Consequently

\[ E\|Q_T f_\theta - f_z\|^2 \leq C \sum_{v \in T} \frac{1}{mg_v^2} (\text{Var}(yM_v(x)) + \text{Var}(M_v(x))) \varphi_v. \]

By (3) and (4) we get

\[ E\|Q_T f_\theta - f_z\|^2 \leq O\left( \sum_{v \in T} \frac{1}{m} \right) = O\left( \frac{N}{m} \right) \]

and this finishes the proof.

Let us note that if we take \( N = m^{\frac{1}{1+2s}} \) for fixed \( s > 0 \) then

\[ E\|Q_M f_\theta - f_z\|^2 = O\left( \frac{1}{m} \right)^{\frac{2s}{1+2s}}. \] (9)

To unify approach to linear and nonlinear approach in estimation let us introduce the sets \( A^s \) similar to definition given in [1]. We have that \( f \in A^s \), \( 0 < s \) (in fact it makes sense to consider \( 0 < s \leq 2 \)) if \( f \in L^2(\varphi_X) \) and there is \( C \) such that for all \( N \) there is a family \( M = \{M_v\}_{v \in T} \) with properties (1) and (2) such that \( N = |T| \) and

\[ \| f - Q_M f \| \leq CN^{-s}. \] (10)

By Theorem 1.2, (9) and (10) and since

\[ E\|f_\theta - f_z\|^2 \leq 2E\|f_\theta - Q_M f_\theta\|^2 + 2E\|Q_M f_\theta - f_z\|^2 \]

we get the optimal rate of estimation (see [5]). This approach improves the rate of estimation in [(1)].

**Theorem 1.2** Let \( f_\theta \in A^s \) and let \( M \) be the family from definition of space \( A^s \) such that \( N = |T| = [m^{\frac{1}{1+2s}}] \). Then

\[ E\|f_\theta - f_z\|^2 = O\left( \frac{1}{m} \right)^{\frac{2s}{1+2s}}. \]
Finally, we will show the general version of the Theorem 2.1 in [1]. Our proof is very analogous but partially simplified, so we present it for the sake of completeness. We improve the constant in estimation.

**Theorem 1.3** For any family \( M \) and any \( \eta > 0 \)

\[
Prob\{\|Q_M f_\theta - f_x\| > \eta\} \leq 4Ne^{-\frac{c\eta^2}{N}},
\]

where \( N := |T| \) and \( c \) depends only on \( A \).

**Proof.** By the convexity of the square function we have that

\[
\|Q_M f_\theta - f_x\|^2 \leq \int_X \sum_{v \in T} |c_v - c_v(z)|^2 M_v(x) d\mu_X(x) = \sum_{v \in T} |c_v - c_v(z)|^2 \varrho_v.
\]

This gives

\[
Prob\{\|Q_M f_\theta - f_x\| > \eta\} \leq Prob\{\sum_{v \in T} |c_v - c_v(z)|^2 \varrho_v > \eta^2\}
\]

\[
\leq \sum_{v \in T} Prob\{|c_v - c_v(z)| > \frac{\eta}{\sqrt{N} \varrho_v}\}.
\]

Let us note that

\[
Prob\{|c_v - c_v(z)| > \frac{\eta}{\sqrt{N} \varrho_v}\} = 0
\]

provided \( \varrho_v \leq \frac{\eta^2}{24A^2N} \). To see this it is enough to transform this assumption to the form \( \frac{\eta}{\sqrt{N} \varrho_v} \geq 2A \) and recall that \( |c_v| \) and \( |c_v(z)| \) are less than \( A \).

Therefore we can write

\[
Prob\{\|Q_M f_\theta - f_x\| > \eta\} \leq \sum_{v: \varrho_v > \frac{\eta^2}{4A^2N}} Prob\{|c_v - c_v(z)| > \frac{\eta}{\sqrt{N} \varrho_v}\}.
\]

To estimate the last sum let us note that if

\[
|\alpha_v(z) - \alpha_v| \leq \frac{\varrho_v \eta}{4\sqrt{N} \varrho_v}
\]

and

\[
|\varrho_v(z) - \varrho_v| \leq \frac{\varrho_v \eta}{4A \sqrt{N} \varrho_v}
\]
then (we know that \( \varrho_v > \frac{\eta^2}{4A^2N} \))

\[
|\varrho_v(z) - \varrho_v| \leq \frac{\varrho_v \eta}{4A \sqrt{N} \frac{\eta^2}{4A^2N}} = \frac{1}{2} \varrho_v
\]

(this gives in particular that \( |\varrho_v(z)| \geq \frac{1}{2} \varrho_v \)) and using (7) we get

\[
|c_v(z) - c_v| = \left| \frac{\alpha_v(z)}{\varrho_v(z)} - \frac{\alpha_v}{\varrho_v} \right|
\leq \frac{1}{2} \frac{\varrho_v |\alpha_v(z) - \alpha_v|}{|\varrho_v(z)|} + \frac{|\alpha_v|}{|\varrho_v(z)|} |\varrho_v(z) - \varrho_v|
\leq \frac{1}{2} \frac{\varrho_v \eta}{4N \varrho_v} + \frac{A}{2} \frac{\varrho_v \eta}{4A \sqrt{N} \varrho_v} = \frac{\eta}{\sqrt{N} \varrho_v}.
\]

Therefore

\[
\text{Prob}\left\{ |c_v - c_v(z)| > \frac{\eta}{\sqrt{N} \varrho_v} \right\}
\leq \text{Prob}\left\{ |\alpha_v(z) - \alpha_v| > \frac{\varrho_v \eta}{4 \sqrt{N} \varrho_v} \right\} + \text{Prob}\left\{ |\varrho_v(z) - \varrho_v| > \frac{\varrho_v \eta}{4A \sqrt{N} \varrho_v} \right\}.
\]

If we first use (5), (6) and then the fact that \( \frac{\eta}{\sqrt{N} \varrho_v} \leq 2A \) we get finally

\[
\text{Prob}\{\|Q_M f_\varrho - f_z\| > \eta\} \leq \sum_{v: \varrho_v > \frac{\eta^2}{4A^2N}} \left( 2e^{-\frac{3\eta^2}{16N(6A^2 + 4A \sqrt{N} \varrho_v)}} + 2e^{-\frac{3\varrho_v \eta^2}{16A^2N(6 + 2A \sqrt{N} \varrho_v)}} \right)
\leq \sum_{v: \varrho_v > \frac{\eta^2}{4A^2N}} 2 \left( e^{-\frac{3\eta^2}{16N A^2}} + e^{-\frac{3\eta^2}{16A^2N}} \right) \leq 4Ne^{-\frac{3\eta^2}{128A^2}}.
\]

which complete the proof of (11) with \( c = \frac{3}{128A^2} \).

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