RECOVERY OF A TIME-DEPENDENT HERMITIAN CONNECTION AND
POTENTIAL APPEARING IN THE DYNAMIC SCHRÖDINGER EQUATION

ALEXANDER TETLOW

Abstract. We consider, on a trivial vector bundle over a Riemannian manifold with boundary, the inverse problem of uniquely recovering time- and space-dependent coefficients of the dynamic, vector-valued Schrödinger equation from the knowledge of the Dirichlet-to-Neumann map. We show that the D-to-N map uniquely determines both the connection form and the potential appearing in the Schrödinger equation, under the assumption that the manifold is either a) two-dimensional and simple, or b) of higher dimension with strictly convex boundary and admits a smooth, strictly convex function.

1. Introduction
1.1. Statement of the Problem. Let $T > 0$ be fixed, and let $(M, g)$ be a connected, compact, smooth Riemannian manifold of dimension $m \geq 2$ with boundary $\partial M$. In what follows, we shall additionally assume that $(M, g)$ is non-trapping. Consider a trivial Hermitian vector bundle $E = M \times \mathbb{C}^n$ equipped with the Hermitian inner product $\langle \cdot, \cdot \rangle_E$.

We say that a connection $\nabla : C^\infty(M; E) \to C^\infty(M; E \otimes T^*M)$ is compatible with the Hermitian structure of $E$ if for any sections $u, v \in C^\infty(M; E)$ it holds that

$$d \langle u, v \rangle_E = \langle \nabla u, v \rangle_E + \langle u, \nabla v \rangle_E,$$

where both sides of the above are regarded as sections of the cotangent bundle.

Such a connection has the form $\nabla = d + A$, where $A = A_i dx^i$ and each $A_i(x)$ is given by an $n \times n$ skew-Hermitian matrix. In what follows, we allow the connection form $A$ to also depend smoothly on time, and write $\nabla^A : C^\infty((0, T) \times M; E) \to C^\infty((0, T) \times M; E \otimes T^*M)$ for the time-dependent connection corresponding to the connection form $A$. In other words, $\nabla^A(t)$ is a connection on $C^\infty(M; E)$ for each $t \in [0, T]$, and each $\nabla^A(t)$ is compatible with the Hermitian metric on $E$.

We can define a natural $L^2$-inner product on $C^\infty((0, T) \times M; E)$ via

$$\langle u, v \rangle_{L^2((0, T) \times M; E)} = \int_0^T \int_M \langle u, v \rangle_E dV dt,$$

where $dV$ denotes the usual Riemannian volume measure of $(M, g)$. We can similarly define a natural $L^2$-inner product on $C^\infty((0, T) \times M; E \otimes T^*M)$. For $E$-valued 1-forms $\alpha = \alpha_j dx^j$ and $\beta = \beta_j dx^j$, we set

$$\langle \alpha, \beta \rangle_{L^2((0, T) \times M; E \otimes T^*M)} = \int_0^T \int_M g^{ij} \langle \alpha_i, \beta_j \rangle_E dV dt,$$

where $g^{ij}$ denotes the inverse of the metric tensor.

We let $(\nabla^A)^*$ denote the adjoint of $\nabla^A$ with respect to the above inner products. We can then define the connection Laplacian $\Delta_A = - (\nabla^A)^* \nabla^A$, which corresponds to the connection form $A$.

We can compute local expressions for $(\nabla^A)^*$ and $\Delta_A$. Consider a section $u \in C^\infty(M; E)$ and an $E$-valued 1-form $\beta = \beta_j dx^j$ supported on a local trivialisation. Since $A$ is skew-Hermitian, it holds that

$$\langle A u, \beta \rangle_{L^2((0, T) \times M; E \otimes T^*M)} = \int_0^T \int_M g^{ij} \langle A_i u, \beta_j \rangle_E dV dt = - \int_0^T \int_M \langle u, g^{ij} A_i \beta_j \rangle_E dV dt.$$
Letting \((A, \beta)_g = g^{ij} A_i \beta_j\), we see that \((\nabla^A)^* = d^* - (A, \cdot)_g\). Therefore, we have
\[
\Delta_A u = -d^* du - d^*(Au) + (A, du)_g + (A, Au)_g.
\]
Recall that \(-d^* \alpha = -|g|^{-\frac{1}{2}} \partial_t(|g|^{\frac{1}{2}} g^{ij} \alpha_{ij})\), hence \(d^*(Au) = (d^* A) u - (A, du)_g\). Thus, we conclude that
\[
(1.2) \quad \Delta_A u = -d^* du + 2(A, du)_g - (d^* A) u + (A, Au)_g.
\]

Lastly, we say that a section \(V \in C^\infty((0, T) \times M; \mathbb{C}^{n \times n})\) is a potential if \(V\) is symmetric or, equivalently, for any sections \(u, v \in C^\infty((0, T) \times M; E)\) it holds that
\[
(V u, v)_E = (u, V v)_E.
\]

Let \(\Delta_A\) and \(V\) be as above and consider the following initial and boundary value problem for sections \(u \in C^\infty((0, T) \times M; E)\).
\[
i\partial_t u(t, x) + \Delta_A(t) u(t, x) + V(t, x) u(t, x) = 0 \text{ in } (0, T) \times M,
\]
\[
\begin{align*}
    u(t, x) &= f \text{ on } (0, T) \times \partial M, \\
    u(0, x) &= 0 \text{ in } M,
\end{align*}
\]
where the inhomogeneous Dirichlet data is given by \(f \in C^\infty((0, T) \times \partial M; E)\) satisfying \(f|_{t=0} = \partial_t f|_{t=0} = 0\).

We can then define the associated Dirichlet-to-Neumann map via
\[
\Lambda_{A, V} f = \nabla^A_{\nu} u \big|_{\partial M},
\]
where \(\nu\) denotes the outward pointing unit normal vector field on \(\partial M\).

There is a natural gauge group associated with the equation above. Let \(G : M \to U(n)\) be a smooth map such that \(G|_{\partial M} = 1\), and choose \(A_2 = G^{-1} A_1 G + G^{-1} \partial_t G\) and \(V_2 = G^{-1} V_1 G + iG^{-1} \partial_t G\). It then holds that \(\nabla A_2 = G^{-1} \nabla A_1 G\), and hence that \(\Delta A_2 = G^{-1} \Delta A_1 G\). We observe that if \(u\) solves \((1.3)\) with \(A = A_2\) and \(V = V_2\), then \(Gu\) solves the equation with \(A = A_1\) and \(V = V_1\), since
\[
(i\partial_t + \Delta A_1 + V_1) Gu = G(i\partial_t + \Delta A_2 + V_2) u = 0.
\]
Furthermore, we observe that when the pairs \((A_1, V_1)\) and \((A_2, V_2)\) are as above, it holds that \(\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}\). Therefore, we can only hope to recover the pair \((A, V)\) up to a gauge transform. The aim of the present work is to establish unique recovery of the connection form and potential from the knowledge of the Dirichlet-to-Neumann map, modulo gauge invariance.

1.2. History of the Problem. Literature dealing with the recovery of space- and time-dependent potentials of the dynamic Schrödinger equation is limited, even in the scalar case. For Euclidean domains, it was shown in [12] that the time-dependent electromagnetic potentials are uniquely determined by the Dirichlet-to-Neumann map. Logarithmic-stable determination was shown for the electric potential in [10], and this result was extended to the full electromagnetic potential in [8], provided that the time-independent part of the magnetic potential is sufficiently small. Indeed, it was only recently shown in [15] that time-dependent electromagnetic potentials in a Euclidean domain can be Hölder-stably recovered from the knowledge of the D-to-N map. We also mention here the recent work of [3], which establishes logarithmic and double-logarithmic stability estimates for the same problem with partial data.

In the Riemannian setting, [4] and [5] establish, respectively, Hölder-stable recovery of a time-independent magnetic and electric potential of the dynamic Schrödinger equation on a simple manifold. These results were extended to simultaneous recovery of both electromagnetic potentials in [2]. In the case of time-dependent potentials in the Riemannian context, the only result is that of [16], establishing, on a simple manifold, the Hölder-stable recovery of both potentials from the knowledge of the Dirichlet-to-Neumann map.

In the case of the vector-valued dynamic Schrödinger equation, there are, to the best of the author’s knowledge, no results establishing unique recovery even for time-independent coefficients. However, such results do exist for the related case of the stationary Schrödinger equation. In particular, for the stationary Schrödinger equation on a trivial vector bundle over a Euclidean domain, [13] establishes unique recovery of a connection form and potential from the knowledge of the Dirichlet-to-Neumann map. Additionally, for
the stationary Schrödinger equation on a Hermitian vector bundle over a two-dimensional Riemann surface, 
[1] uniquely recovers the coefficients from Cauchy data at the boundary. Let us also mention the results of [6],
where it is conjectured that the Dirichlet-to-Neumann maps for two connection Laplacians coincide in the
case of the stationary Schrödinger equation if and only if the associated connection forms are gauge
equivalent. The present work solves this conjecture in the non-stationary case. More precisely, we show that
the Dirichlet-to-Neumann map uniquely determines, up to gauge invariance, the space- and time-dependent
connection form and potential appearing in the dynamic Schrödinger equation on a trivial vector bundle
over a Riemannian manifold, provided that the manifold in question satisfies certain geometric conditions.

Finally, let us mention the works [7], [17], and [9], where inverse problems for partial differential equations
involving connections are considered. In particular, [7] establishes unique recovery of a time-independent
unitary Yang-Mills connection on both smooth Hermitian line bundles and analytic bundles of higher rank,
as well as recovering the bundle structure. Similarly, in the case of the wave equation, [17] uses techniques
from the boundary control method to reconstruct a Riemannian manifold and Hermitian vector bundle with
a time-independent compatible connection from the knowledge of the associated hyperbolic Dirichlet-to-
Neumann map. Lastly, [9] considers the inverse problem of recovering a time-independent connection from
a cubic wave equation on a Hermitian vector bundle over the Minkowski space \( \mathbb{R}^{1+3} \).

1.3. Geodesics and Parallel Transport. Let us assume that \((M, g)\) is non-trapping, which is to say that
every geodesic in \( M \) reaches the boundary in finite time. We now take a moment to recall certain key facts
relating to the geodesics in \( M \).

Given \( x \in M \) and \( \theta \in T_x M \), we denote by \( \gamma_{x, \theta} \) the geodesic with initial point \( x \) and initial direction \( \theta \).
We define the sphere bundle of \( M \) via
\[
SM = \{ (x, \theta) \in TM : |\theta|_g = 1 \}.
\]
Likewise, we define the submanifold of inner vectors \( \partial_+ SM \) via
\[
\partial_+ SM = \{ (x, \theta) \in SM : x \in \partial M, \langle \theta, \nu(x) \rangle_{g(x)} < 0 \},
\]
where \( \nu(x) \) is the outward pointing unit normal vector at \( x \in \partial M \), and we define the submanifold of outer
vectors \( \partial_- SM \) via
\[
\partial_- SM = \{ (x, \theta) \in SM : x \in \partial M, \langle \theta, \nu(x) \rangle_{g(x)} > 0 \}.
\]
Then, for \( \gamma_{x, \theta} \) such that \( (x, \theta) \in \partial_+ SM \), we can define the exit-time \( \rho_+(x, \theta) \) of the geodesic in \( M \) by
\[
\rho_+(x, \theta) = \min\{ s > 0 : \gamma_{x, \theta}(s) \in \partial M \}.
\]
Given the above, we further recall the parallel transport equations associated to a connection \( \nabla^A \). For
any geodesic \( \gamma_{x, \theta} \) with \( (x, \theta) \in \partial_+ SM \), and any initial vector \( w \in E_x \), we consider the parallel transport
equation along \( \gamma_{x, \theta} \), given by
\[
\begin{align*}
\left[ \partial_r + A \gamma_{x, \theta}'(r) \right] W &= 0 \\
W(0) &= w.
\end{align*}
\]
The transport of \( w \in E_x \) along \( \gamma_{x, \theta} \) is thus given by \( W(r) \).

It is frequently helpful to consider the fundamental matrix solution \( U_A : [0, \rho_+(x, \theta)] \to U(n) \) of the
parallel transport equation:
\[
\begin{align*}
\left[ \partial_r + A \gamma_{x, \theta}'(r) \right] U_A &= 0 \\
U_A(0) &= \text{Id}.
\end{align*}
\]

It is clear from the above that the transport of \( w \in E_x \) along \( \gamma_{x, \theta} \) is given by \( W(r) = U_A(r) \cdot w \).

Given \( (x, \theta) \in \partial_+ SM \), we define the scattering data for the connection as the map \( C_A : \partial_+ SM \to U(n) \)
given by
\[
C_A(x, \theta) := U_A(\rho_+(x, \theta)).
\]
1.4. Main Results.

Theorem 1. Suppose that for \( j = 1, 2 \), we have connection forms \( A_j \in C^\infty((0, T) \times M; \mathbb{C}^{n \times n} \otimes T^* M) \), and potentials \( V_j \in C^\infty((0, T) \times M; \mathbb{C}^{n \times n}) \). Suppose further that \( A_1 = A_2 \) on \( \partial M \). Then \( \Lambda_{A_1, V_1} = \Lambda_{A_2, V_2} \) implies that \( C_{A_1} = C_{A_2} \).

Theorem 2. Assume the conditions of Theorem 1 hold, and assume further that \( M \) is either 2-dimensional and simple, or has dimension \( m \geq 3 \) and satisfies the foliation condition of [20] and \( \partial M \) is convex. Then \( (A_1, V_1) \) is gauge equivalent to \( (A_2, V_2) \).

These results are, as far as the author is aware, the first dealing with the recovery of space- and time-dependent coefficients for the dynamic Schrödinger equation on a vector bundle. In fact, the above results are the first showing recovery of time-dependent coefficients of a linear second-order partial differential equation with variable coefficients of order two, in the vector-valued case. The proof of these results relies on the construction of Gaussian beam solutions which allow recovery of the scattering data corresponding to the connection form. This data is then used to recover the connection form and potential of the Schrödinger equation via the inversion of attenuated ray transforms with specific matrix-weights corresponding to the connection forms and potentials we wish to recover. This last step relies on the results of [19] and [20], which guarantee that the appropriate attenuated ray-transform is invertible when the base manifold is either i) two-dimensional and simple or ii) of higher dimension with strictly convex boundary and admits a smooth strictly convex function.

Here follows an outline of the present work. In section 2, we give some regularity results for the forward problem and for the Neumann trace. In section 3 we construct special Gaussian beam solutions for the Schrödinger equation. The proofs of Theorems 1 and 2 are given in sections 4 and 5 respectively.

2. The Forward Problem

The unique solvability of (1.3) can be established in a similar manner to the scalar valued case. For the reader’s convenience, we outline the necessary energy estimates here. Existence and uniqueness can then be proven using, for example, the Galerkin approach (see e.g. [18, Section 3, Theorem 10.1] or [15, Theorem 2.3]). Thus, for \( F \in C^\infty((0, T) \times M; E) \), we consider the solution of the following source problem for the Schrödinger equation:

\[
(i \partial_t + \Delta_A + V)u = F(t, x) \quad \text{in} \quad (0, T) \times M,
\]

\[
\begin{align*}
    u(t, x) &= 0 \quad \text{on} \quad (0, T) \times \partial M, \\
    u(0, x) &= 0 \quad \text{in} \quad M.
\end{align*}
\]

By taking the inner product of (2.1) with \( u \) and integrating by parts, we deduce that

\[
\int_M \langle \partial_t u, u \rangle_E dV_g - \int_M g^{ij} \langle \nabla^A u, \nabla^A u \rangle_E dV_g + \int_M \langle Vu, u \rangle_E dV_g = \int_M \langle F, u \rangle_E dV_g.
\]

Taking the imaginary part of (2.1) yields

\[
\frac{d}{dt} \left( \|u(t)\|^2_{L^2(M; E)} \right) \leq C \left( \|F(t, \cdot)\|_{L^2(M; E)} \|u(t)\|_{L^2(M; E)} + \|u(t)\|^2_{L^2(M; E)} \right).
\]

Then Grönwall’s inequality tells us that

\[
\|u\|_{L^\infty(0, T; L^2(M; E))} \leq C \|F\|_{L^2((0, T) \times M; E)}.
\]

On the other hand, taking the inner product of (2.1) with \( \partial_t u \), we can integrate by parts to deduce that

\[
\int_M \langle \partial_t u, \partial_t u \rangle_E dV_g - \int_M g^{ij} \langle \nabla^A u, \nabla^A \partial_t u \rangle_E dV_g + \int_M \langle Vu, \partial_t u \rangle_E dV_g = \int_M \langle F, \partial_t u \rangle_E dV_g.
\]

Then, by setting

\[
\alpha(t; u, v) = \int_M g^{ij} \langle \nabla^A u, \nabla^A v \rangle_E dV_g + \int_M \langle Vu, v \rangle_E dV_g
\]
and
\[ \alpha'(t; u, u) = \int_M g^{ij} \langle (\partial_t A)u, \nabla^A u \rangle_E \, dV_g + \int_M g^{ij} \langle \nabla^A u, (\partial_t A)u \rangle_E \, dV_g + \int_M \langle (\partial_t V)u, u \rangle_E \, dV_g, \]
we can take the real part of (2.4) to conclude that
\[ \frac{d}{ds} \alpha(s; u, u) = \alpha'(s; u, u) - 2 \, \text{Re} \int_M \langle F(s, \cdot), \partial_t u(s) \rangle_E \, dV_g. \]
By integrating, we can rewrite the above as
\[ \alpha(t; u, u) = \int_0^t \alpha'(s; u, u) \, ds - 2 \int_0^t \langle F(s, \cdot), \partial_t u(s) \rangle_{L^2(M; E)} \, ds, \]
and since \( \int_0^t \langle F(s, \cdot), \partial_t u(s) \rangle_{L^2(M; E)} \, ds = \langle F(s, \cdot), u(s) \rangle_{L^2(M; E)} - \int_0^t \langle \partial_t F(s, \cdot), u(s) \rangle_{L^2(M; E)} \, ds \), we can rewrite the identity (2.5) in the form
\[ \alpha(t; u, u) \leq \int_0^t \alpha'(s; u, u) \, ds + C \| F \|_{H^1,0((0, T) \times M; E)} \| u \|_{L^2((0, T) \times M; E)}. \]
Further, Hölder’s inequality tells us that
\[ \| \nabla^A u(t) \|_{L^2(M; E \otimes T^* M)}^2 \geq \frac{1}{2} \| \nabla u(t) \|_{L^2(M; E \otimes T^* M)}^2 - 2 \| A(t) \|_{L^\infty(M; \mathbb{C}^{n \times n} \otimes T^* M)} \| u(t) \|_{L^2(M; E)}^2, \]
whence
\[ \alpha(t; u, u) + \lambda \| u(t) \|_{L^2(M; E)}^2 \geq \frac{1}{2} \| u(t) \|_{H^1(M; E)}^2, \]
for \( \lambda = \frac{1}{2} + \| V(t) \|_{L^\infty(M; \mathbb{C}^{n \times n})}^2 + 2 \| A(t) \|_{L^\infty(M; \mathbb{C}^{n \times n} \otimes T^* M)}^2 \). Then, combining the above with identity (2.6) and the definition of \( \alpha'(t; u, v) \), we may deduce that
\[ \| u(t) \|_{H^1(M; E)} \leq C \left( \int_0^t \| u(t) \|_{H^1(M; E)} \, ds + \| F \|_{H^1,0((0, T) \times M; E)} \| u \|_{L^2((0, T) \times M; E)} \right). \]
Then an application of Grönwall’s inequality tells us that
\[ \| u \|_{L^\infty(0, T; H^1(M; E))} \leq C \| F \|_{H^1,0((0, T) \times M; E)}. \]
For the next estimate, we begin by applying \( \partial_t \) to (2.1). Using the expression (1.2) for the connection Laplacian, we deduce that
\[ (i \partial_t + \Delta_A + V) \partial_t u = \partial_t F - 2(\partial_t A, du) + (\partial_t d^* A)u - (\partial_t A, Au) - (A, (\partial_t A)u) - (\partial_t V)u. \]
We now apply the estimate (2.3) to \( \partial_t u \), replacing \( F \) appearing in (2.3) by the right-hand side of (2.9). We deduce, therefore, that
\[ \| \partial_t u \|_{L^\infty(0, T; L^2(M; E))} \leq C \| \partial_t F - 2(\partial_t A, du) + (\partial_t d^* A)u - (\partial_t A, Au) - (A, (\partial_t A)u) - (\partial_t V)u \|_{L^2((0, T) \times M; E)}. \]
Using the estimates (2.3) and (2.8), on the right-hand side of the above, we observe that
\[ \| \partial_t u \|_{L^\infty(0, T; L^2(M; E))} \leq C \| F \|_{H^1,0((0, T) \times M; E)}. \]
Lastly, we rearrange (2.1) to obtain
\[ (\Delta_A + V)u = F - i \partial_t u \text{ in } (0, T) \times M \]
\[ u = 0 \text{ on } (0, T) \times \partial M. \]
Then the bounds (2.3), (2.8) and (2.10) immediately imply the desired energy estimate
\[ \| u \|_{H^1,2((0, T) \times M; E)} \leq C \| F \|_{H^1,0((0, T) \times M; E)}. \]
We now turn to establishing a bound for the Dirichlet-to-Neumann map. Recall the initial and boundary value problem (1.3):
\[ i \partial_t u(t, x) + \Delta_A(t)u(t, x) + V(t, x)u(t, x) = 0 \text{ in } (0, T) \times M, \]
\[ u(t, x) = f \text{ on } (0, T) \times \partial M, \]
\[ u(0, x) = 0 \text{ in } M, \]
where the inhomogeneous Dirichlet data is given by \( f \in C^\infty((0, T) \times \partial M; E) \) satisfying \( f|_{t=0} = \partial_t f|_{t=0} = 0 \).

Note that we can find \( \Phi \in C^\infty((0, T) \times M; E) \) such that
\[
\Phi(0, \cdot) = \partial_t \Phi(0, \cdot) = 0 \text{ in } M \text{ and } \Phi = f \text{ on } \partial M,
\]
and
\[
\|\Phi\|_{H^{3,2}((0, T) \times M; E)} \leq C \|f\|_{H^{3,2}((0, T) \times M; E)},
\]
for some \( C > 0 \), depending only on \( M \) and \( T \). See [18, Chapter 4, Section 2] for a proof of this fact in the scalar case. The proof for vectors is analogous and, therefore, omitted. From the above, it holds that
\[
F := -(i\partial_t + \Delta_A + V)\Phi
\]
satisfies \( F(0, \cdot) = 0 \) in \( M \). Then, letting \( v \) be the solution of (2.1) corresponding to the source term \( F \) defined in (2.14), we see that \( u = \Phi + v \) is a solution to (1.3). Then, it follows by (2.12) that
\[
\|u\|_{H^{1,2}((0, T) \times M; E)} \leq C \|f\|_{H^{3,2}((0, T) \times M; E)},
\]
and applying this estimate with \( f = 0 \) implies that such a solution \( u \) is unique. Finally, we observe that
\[
\|\Lambda_{A,v} f\| \leq \|u\|_{H^{1,2}((0, T) \times M; E)} \leq C \|\Phi\|_{H^{3,2}((0, T) \times M; E)} \leq C \|f\|_{H^{3,2}((0, T) \times M; E)},
\]
and so \( \Lambda_{A,v} \) is bounded from \( \{f \in H^{3,2}((0, T) \times M; E) : f|_{t=0} = 0\} \) into \( L^2((0, T) \times M; E) \).

3. Construction of Gaussian Beam Solutions

In this section, we shall construct Gaussian beam solutions to the Schrödinger equation which concentrate along geodesics in the high frequency limit.

Let \((M, g)\) be a closed manifold. Recall that for a geodesic segment \( \gamma : (a, b) \to M \) with no closed loops, there exist only finitely many values of \( r \in (a, b) \) for which \( \gamma \) self-intersects at \( \gamma(r) \). We begin by recording the following system of Fermi coordinates near a geodesic, which we shall later use to construct our Gaussian beam solutions.

**Lemma 1.** Let \((M, g)\) be a compact \( m \)-dimensional manifold without boundary, \( m \geq 2 \), and assume that \( \gamma : (a, b) \to M \) is a unit-speed geodesic with no closed loops. Given a closed sub-interval \([a_0, b_0]\) of \((a, b)\) such that \( \gamma|_{[a_0, b_0]} \) self-intersects only at \( \gamma(r_j) \) with \( a_0 < r_1 < \cdots < r_k < b_0 \), and setting \( r_0 = a_0, r_{N+1} = b_0 \), there exists an open cover \( \{U_j, \phi_j\}_{j=0}^{k+1} \) of \( \gamma([a_0, b_0]) \) consisting of coordinate neighbourhoods with the following properties:

- \( \phi_j(U_j) = I_j \times B \), where \( I_j \) are open intervals and \( B = B(0, \delta') \) is an open ball in \( \mathbb{R}^{m-1} \), where \( \delta' \) can be taken arbitrarily small.
- \( \phi_j(\gamma(r)) = (r, 0) \) for \( r \in I_j \)
- \( r_j \) only belongs to \( I_j \) and \( \overline{I_j} \cap \overline{I_k} = \emptyset \), unless \( |j - k| \leq 1 \)
- \( \phi_j = \phi_k \text{ on } \phi_j^{-1}(I_j \cap I_k) \times B \)

Furthermore, the metric in these coordinates satisfies \( g^{jk}|_{\gamma(r)} = \delta^{jk} \) and \( \partial_t g^{jk}|_{\gamma(r)} = 0 \).

**Proof.** See e.g. [11, Lemma 3.5] for details. \( \square \)

We now turn to the construction of the Gaussian beam solutions. We consider here a non-tangential unit-speed geodesic \( \gamma : [0, L] \to M \). That is, \( \gamma'(0) \) and \( \gamma'(L) \) are both non-tangential to \( \partial M \), and \( \gamma(r) \in M^{int} \) for \( 0 < r < L \). Note that this implies that the geodesic \( \gamma \) is not a closed loop in \( M \).

We may then embed \((M, g)\) in some slightly larger closed manifold \( M_1 \), and extend \( \gamma \) to \( M_1 \) as a unit-speed geodesic \( \gamma : [-\varepsilon, L + \varepsilon] \to M_1 \). Our aim is to construct a Gaussian beam solution near \( \gamma([0, L]) \). We fix a point \( x_0 \) on \( \gamma \), and apply Lemma 1 with \( M = M_1 \), and \( a_0 < 0 \) and \( b_0 > L \) chosen so that \( \gamma(a_0) \) and \( \gamma(b_0) \) are in the interior of \( M_1 \setminus M \). This gives us a system of coordinates \((r, y)\) around \( x_0 = (r_0, 0) \), defined in a set \( U = \{(r, y) : |r - r_0| < \delta, |y| < \delta'\} \) such that the geodesic near \( x_0 \) is given by \( \Gamma = \{(r, 0) : |r - r_0| < \delta\} \).
Our aim is to construct an approximate solution \( v \) of the Schrödinger equation (1.3), having the form
\[
v = e^{i s \langle \Psi(r, y), a(s, t, r, y) \rangle} \Psi(s, t, r, y),
\]
where \( s > 1 \) is a large real parameter, and \( \Psi \in C^\infty(M; \mathbb{C}) \), \( a \in C^\infty(M; E) \) are given near \( \Gamma \) with \( a \) supported in \( \{|y| < \delta'/2\} \). For convenience, we shall suppress the dependence on \( s \) of the amplitude function \( a \).

We first compute the Schrödinger operator applied to \( v \):
\[
(i \partial_t + \Delta_A + V)v = e^{i s \langle \Psi, a \rangle} (i \partial_t + \Delta_A + V)a + s^2 e^{i s \langle \Psi, a \rangle} \left( 1 - \langle d\Psi, d\Psi \rangle_G \right) a
+ 2 is e^{i s \langle \Psi, a \rangle} \left( \langle d\Psi, \nabla^A a \rangle_G + \frac{1}{2} \langle \Delta_g \Psi, a \rangle \right).
\]

In light of the above, we begin by seeking \( \Psi \) so that
\[
\langle d\Psi, d\Psi \rangle_G - 1 = 0 \text{ to } N \text{th order on } \Gamma.
\]
In fact, we seek \( \Psi \) of the form \( \Psi = \sum_{j=0}^N \Psi_j \), where
\[
\Psi_j(r, y) = \sum_{|\alpha| = j} \frac{\Psi_{j,\alpha}(r)}{\alpha!} y^\alpha.
\]
Let us also write the metric in the form \( g_{jk} = \sum_{i=0}^N g_{jk}^i + r_{N+1}^j \), where
\[
g_{jk}^i(r, y) = \sum_{|\beta| = i} \frac{g_{jk}^i(r)}{\beta!} y^\beta, \quad r_{N+1}^j = O(|y|^{N+1}).
\]

By the properties of the Fermi coordinates, we observe that \( g_{jk}^0 = \delta^{jk} \) and \( g_{jk}^1 = 0 \). Thus, we can immediately choose \( \Psi_0(r) = r \) and \( \Psi_1(r, y) = 0 \). Then, for \( j, k = 1...m \) and \( \alpha, \beta = 2...m \), we have
\[
g_{jk} \partial_j \Psi \partial_k \Psi - 1 = (1 + g_{21}^1 + \cdots)(1 + \partial_r \Psi_2 + \cdots)(1 + \partial_r \Psi_2 + \cdots)
+ 2(g_{10}^1 + \cdots)(1 + \partial_r \Psi_2 + \cdots)(\partial_{y\alpha} \Psi_2 + \cdots)
+ (\delta^{\alpha\beta} + g_{22}^1 + \cdots)(\partial_{y\alpha} \Psi_2 + \partial_{y\alpha} \Psi_3 + \cdots)(\partial_{y\beta} \Psi_2 + \partial_{y\beta} \Psi_3 + \cdots)
= [2\partial_r \Psi_2 + \nabla_y \Psi_2 \cdot \nabla_y \Psi_2 + g_{11}^1] + \sum_{p=3}^N \left[ 2\partial_r \Psi_p + \nabla_y \Psi_2 \cdot \nabla_y \Psi_p + \sum_{l=0}^p g_{11}^l \sum_{j+k=p-l, 0 \leq j < k < l} \partial_r \Psi_j \partial_r \Psi_k \right.
+ \sum_{l=2}^p 2 g_{11}^l \sum_{j+k=p-1-l, 2 \leq j < k < p} \partial_{y\alpha} \Psi_j \partial_{y\beta} \Psi_k + \sum_{j+k=p+2-l} \partial_{y\alpha} \Psi_j \partial_{y\beta} \Psi_k \bigg] + O(|y|^{N+1}).
\]
In the last equality, we have chosen to collect the terms into homogeneous polynomials in \( y \) (so that the first term is the second degree part of the right-hand side, and the rest are the parts of degree \( p = 3, \ldots, N \)). We first choose \( \Psi_2 \) such that the second-degree term \([2\partial_r \Psi_2 + \nabla_y \Psi_2 \cdot \nabla_y \Psi_2 + g_{11}^1]\) vanishes.

To this end, we choose \( \Psi_2(r, y) = \frac{1}{2} H(r) y \cdot y \), where \( H \) is a smooth, symmetric, complex matrix solving the matrix Riccati equation
\[
H'(r) + H^2(r) = F(r),
\]
and \( F(r) \) is the symmetric matrix such that \( g_{11}^2(r, y) = -F(r) y \cdot y \). If we impose some initial condition \( H(r_0) = H_0 \) on this equation, where \( H_0 \) is chosen to be a complex symmetric matrix with \( \text{Im}(H_0) \) positive definite, then [14, Lemma 2.56] implies that the matrix Riccati equation above has a unique smooth symmetric solution \( H(r) \), for which \( \text{Im}(H(r)) \) is positive definite.
We now choose \( \Psi_3 \) so that the term corresponding to \( p = 3 \) in the right-hand side of (3.3) vanishes. We obtain the equation

\[
2\partial_y \Psi_3 + 2\nabla_y \Psi_2 \cdot \nabla_y \Psi_3 = F(r, y),
\]

where \( F \) is a third-order polynomial in \( y \) which only depends on \( \Psi_2 \) and \( g \). This gives us a linear system of first-order ODEs for the Taylor coefficients \( \Psi_{3,a}(r) \), which can be solved uniquely if we prescribe some initial conditions at \( r_0 \). We may, then, repeat this argument in order to obtain \( \Psi_4, \ldots, \Psi_N \) by solving ODEs on \( \Gamma \), given initial conditions at \( r_0 \).

Thus, we have \( \Psi(r, y) = r + \frac{1}{2} H(r)y \cdot y + \hat{\Psi} \), where \( \hat{\Psi} = O(|y|^3) \). We now turn to finding the amplitude \( a \) such that, up to a small error, we have

\[
s\left( \langle d\Psi, \nabla^A a \rangle_g + \frac{1}{2}(\Delta_g a) \right) - \frac{i}{2}(i\partial_t + \Delta_A + V)a = 0 \quad \text{to Nth order on } \Gamma.
\]

We choose \( a \) of the form

\[
a = s^{m-1}(a_0 + s^{-1}a_1 + \cdots + s^{-N}a_N)\chi(y/\delta'),
\]

where \( \chi \) is a smooth function such that \( \chi = 1 \) for \( |y| \leq 1/4 \) and \( \chi = 0 \) for \( |y| \geq 1/2 \). Letting \( \eta = \Delta_g \Psi \), it is enough to find \( a_j \) such that

\[
\langle d\Psi, \nabla^A a_0 \rangle_g + \frac{1}{2}\eta a_0 = 0 \quad \text{to Nth order on } \Gamma
\]

\[
\langle d\Psi, \nabla^A a_1 \rangle_g + \frac{1}{2}\eta a_1 - \frac{i}{2}(i\partial_t + \Delta_A + V)a_0 = 0 \quad \text{to Nth order on } \Gamma
\]

\[
\vdots
\]

\[
\langle d\Psi, \nabla^A a_N \rangle_g + \frac{1}{2}\eta a_N - \frac{i}{2}(i\partial_t + \Delta_A + V)a_{N-1} = 0 \quad \text{to Nth order on } \Gamma.
\]

By putting \( \eta = \eta_0 + \cdots + \eta_N \) and \( a_0 = a_00 + \cdots + a_0N \), where \( \eta_j, a_0j \) are polynomials of order \( j \) in \( y \), and letting \( A = A_1 dr + A_2 dy^a \), we can rewrite the transport equation for \( a_0 \) above in the form

\[
\left( 1 + g_2^{11} + \cdots \right) \left( 1 + \partial_r \Psi_2 + \cdots \right) \left( (\partial_r + A_1)a_{00} + (\partial_r + A_1)a_{01} + \cdots \right)
\]

\[
+ \left( g_2^{1\alpha} + \cdots \right) \left( 1 + \partial_r \Psi_2 + \cdots \right) \left( (\partial_r^\alpha + A_\alpha)a_{00} + (\partial_r^\alpha + A_\alpha)a_{01} + \cdots \right)
\]

\[
+ \left( g_2^{\alpha\beta} + \cdots \right) \left( 1 + \partial_r^\alpha \Psi_2 + \cdots \right) \left( \partial_r^\beta + A_\beta a_{00} + \partial_r^\beta + A_\beta a_{01} + \cdots \right)
\]

\[
+ \left( \delta^{\alpha\beta} + g_2^{\alpha\beta} + \cdots \right) \left( \partial_r^\beta + \partial_r^\gamma \Psi_3 + \cdots \right) \left( (\partial_r^\beta + A_\beta)a_{00} + (\partial_r^\beta + A_\beta)a_{01} + \cdots \right)
\]

\[
= \partial_r a_0 + A_1 a_0 + \frac{1}{2}\eta a_0
\]

\[
+ \partial_r a_0 + A_1 a_0 + \nabla_y \Psi_2 \cdot \nabla y a_0 + \frac{1}{2}\eta a_0 + \frac{1}{2}\eta a_0
\]

\[
+ \cdots
\]

We wish to find \( a_0 \) such that the first line on the right-hand side vanishes. To this end, we note that \( \eta_0(r) = \Delta_g \Psi(r, 0) = g^{\alpha\beta} \partial_r^\gamma \left[ H_{\alpha\beta} (y^\gamma) \right] = \text{tr} H(r) \). We therefore choose \( a_0 \) such that

\[
\partial_r a_0 + A_\gamma(r)a_0 + \frac{1}{2}(\text{tr } H(r))a_0 = 0.
\]

This equation has the solution

\[
a_0 = c_0 \tilde{\chi}(t) e^{-\frac{1}{R} \int_{r_0}^{r} \text{tr } H(f) df} \cdot U_A w,
\]

where \( \tilde{\chi} \in C_\infty^0([r, T - \tau]) \) satisfies \( \tilde{\chi} = 1 \) on \([2\tau, T - 2\tau]\), \( 0 \leq \tilde{\chi} \leq 1 \) and \( \|\tilde{\chi}\|_{W^{k, \infty}(\mathbb{R})} \leq C_k \tau^{-k} \) with \( C_k \) independent of \( \tau \), \( w \) is some arbitrary initial vector, and \( c_0 = a_{00}(r_0) \). Since it will simplify later calculations,
we choose the value
\[
(3.6) \quad c_0 = \frac{\sqrt[\scriptstyle \pi]\text{det} \text{Im}(H(r_0))}{\sqrt[\scriptstyle \pi\text{m-1}]{\int_{r_{m-1}} e^{-|y|^2} \, dy}} = \frac{\sqrt[\scriptstyle \pi\text{m-1}]{\text{det} \text{Im}(H(r_0))}}{\sqrt[\scriptstyle \pi]{\int_{r_{m-1}} e^{-|y|^2} \, dy}}.
\]

We can then obtain the rest of \( a_0, \ldots, a_N \) by solving linear first-order ODEs. The sections \( a_1, \ldots, a_N \) may then be determined in much the same manner as \( a_0 \), so that the transport equations in (3.5) are satisfied to \( N \text{th} \) degree on \( \Gamma \).

Thus, we have constructed a function \( v = e^{i\pi(\Psi - st)}a \) in \( U \) such that:
\[
\Psi(r, y) = r + \frac{1}{2} H(r)y \cdot y + \Psi, \quad \text{where} \quad \Psi = O(|y|^3),
\]
\[
a(r, y) = s^{\frac{m-1}{2}} (a_0 + s^{-1}a_{-1} + \cdots + s^{-N}a_{-N}) \chi(y/\delta'),
\]
\[
a_0(r, 0) = c_0 e^{-\frac{1}{2}\int_0^r tr H(t) dt} UAw.
\]

We now turn to establishing some norm estimates for \( v \) in \( U \), for the choice \( N = 5 \). Note that
\[
|e^{i\pi(\Psi - st)}| = e^{-\frac{1}{2} \text{Im} H(r)y \cdot y} e^{\Psi}.
\]

Note also that \( \text{Im}(H(r)) \geq c |y|^2 \) for \((r, y) \in U\), where the constant \( c > 0 \) depends on \( H_0 \) and the value of \( \delta \) appearing in the definition of \( U \). From this fact, together with the transport equations (3.5) and the definition of \( \chi \), we can deduce that for \( r \) in a compact interval, possibly after decreasing \( \delta' \), we have
\[
|v(t, r, y)| \leq CS^{m-1} e^{-\frac{1}{2} c s |y|^2} \chi(y/\delta'), \quad s \gg 1.
\]

As a result, we have that for \( s \gg 1 \),
\[
(3.7) \quad \|v\|_{L^2((0, T) \times U; E)} \leq C \left\| S^{m-1} e^{-\frac{1}{2} c s |y|^2} \right\|_{L^2((0, T) \times U; E)} = O(1).
\]

Similarly, we can deduce that for \( s \gg 1 \),
\[
(3.8) \quad \|v\|_{H^k((0, T) \times U; E)} \leq CS^{2k} \left\| S^{m-1} e^{-\frac{1}{2} c s |y|^2} \right\|_{L^2((0, T) \times U; E)} = O(s^{2k}).
\]

Further, from (3.1) and (3.5), we deduce that
\[
(3.9) \quad \|i\partial_t + \Delta_A + V\|v\|_{H^{1,0}((0, T) \times U; E)} \lesssim \left\| S^{m-1} e^{-\frac{1}{2} c s |y|^2} \left( s^2 |y|^6 + s^{-1} \right) \right\|_{L^2((0, T) \times U; E)} = O(s^{-1}).
\]

In a similar manner, we derive the estimate
\[
(3.10) \quad \|i\partial_t + \Delta_A + V\|v\|_{H^{1/0}((0, T) \times U; E)} \lesssim \left\| S^{m-1} e^{-\frac{1}{2} c s |y|^2} \left( s^2 |y|^6 + s^{-5} \right) \right\|_{L^2((0, T) \times U; E)} s^2 = O(s).
\]

Recall the definition of the sets \( U_j \) from Lemma 1. Fix \( \delta' \) and choose an open cover \( U_0, \ldots, U_K \) of \( \gamma([a_0, b_0]) \), with each \( U_j \) corresponding to an interval \( I_j \), as in Lemma 1. We first find a function \( v^{(0)} = e^{i\pi(\Psi^{(0)} - st)}a^{(0)} \) in \( U_0 \) following the method above, with some fixed initial conditions at \( r_0 \) for the ODEs that determine \( \Psi^{(0)} \) and \( a^{(0)} \). We continue by choosing some \( \tilde{r}_1 \) so that \( \gamma(\tilde{r}_1) \in U_0 \cap U_1 \), and construct \( v^{(1)} = e^{i\pi(\Psi^{(1)} - st)}a^{(1)} \) in \( U_1 \) again by following the above method, choosing our initial conditions for \( \Psi^{(1)} \), \( a^{(1)} \) at \( \tilde{r}_1 \) to be, respectively, the values of \( \Psi^{(0)} \) and \( a^{(0)} \) at \( \tilde{r}_1 \). In this manner, we can proceed to determine \( v^{(K)} \). We choose a partition of unity \( \{\rho_j(r)\} \) for \([a_0, b_0]\) corresponding to the family of intervals \( \{I_j\} \), and let \( \tilde{\rho}_j(r, y) = \rho_j(r) \) in \( U_j \). We can then define
\[
v = \sum_{j=0}^K \tilde{\rho}_j v^{(j)}.
\]

Since the ODEs for the phases and amplitudes have the same initial value in \( U_j \) as in \( U_{j+1} \), we can deduce that \( v^{(j)} = v^{(j+1)} \) in \( U_j \cap U_{j+1} \). Therefore, we conclude that the \( L^2 \)-bounds (3.7)-(3.8) for \( v \) and (3.9)-(3.10) for \((i\partial_t + \Delta_A + V)v\) follow with \( U = M \) from the corresponding bounds in \( U_j \) for each \( v^{(j)} \).

Before we proceed, we note also the following partition, which shall be useful later. Suppose that \( p_1, \ldots, p_K \) are the distinct points where the geodesic self-intersects, \( 0 < r_1 < \cdots < r_K < L \) are the
times where the geodesic self-intersects, and $V_1, \ldots, V_K$, are small balls centered at $p_j$. By choosing $\delta'$ small enough we have a cover

\begin{equation}
\text{supp}(v) \cap M \subset \left( \bigcup_{j=1}^{K'} V_j \right) \cup \left( \bigcup_{k=1}^\infty W_k \right)
\end{equation}

such that

\[ v|_{V_j} = \sum_{\gamma(t_i)=p_j} v^{(i)} \]

and in each $W_k$ there is $l_k$ so that

\[ v|_{W_k} = v^{(l_k)}. \]

4. Determination of the Scattering Data

For $j = 1, 2$, we construct Gaussian beam solutions $u_j$ of the Schrödinger equations

\begin{equation}
(i \partial_t + \Delta_{A_j} + V_j) u_j(t, x) = 0 \quad \text{in} \ (0, T) \times M,
\end{equation}

\[ u_1(0, \cdot) = u_2(T, \cdot) = 0 \quad \text{in} \ M. \]

To this end, we fix some geodesic $\gamma_{x, \theta}$ in $M_1$ for $x \in \partial M$, and choose a system of Fermi coordinates along $\gamma_{x, \theta}$ as in Lemma 1. Using the work of the previous section, we construct approximate solutions $v_j$ of the form $e^{is(\Psi - \nu t)}(A_j)$. We can turn these $v_j$ into exact solutions $u_j = v_j + R_j$ by solving

\begin{align*}
(i \partial_t + \Delta_{A_j} + V_j) R_j &= -(i \partial_t + \Delta_{A_j} + V_j) v_j \quad \text{in} \ (0, T) \times M, \\
R_j &= 0 \quad \text{on} \ (0, T) \times \partial M, \\
R_j(0, \cdot) &= R_j(T, \cdot) = 0 \quad \text{in} \ M.
\end{align*}

Note that for $s \gg 1$, (3.9) yields

\begin{equation}
\|R_j\|_{L^2((0, T) \times M; E)} \leq C \|\langle i \partial_t + \Delta_{A_j} + V_j \rangle v_j\|_{L^2((0, T) \times M; E)} = O(s^{-1}),
\end{equation}

whereas (3.10) and the energy estimate (2.12) yields

\[ \|R_j\|_{L^2((0, T) \times M; E)} \leq C \langle\|i \partial_t + \Delta_{A_j} + V_j\| v_j\rangle_{L^2((0, T) \times M; E)} = O(s). \]

Interpolating between (4.2) and the above, we conclude that for $s \gg 1$ we have

\begin{equation}
\|R_j\|_{L^2((0, T) \times M; E)} \leq C \|\langle i \partial_t + \Delta_{A_j} + V_j \rangle v_j\|_{L^2((0, T) \times M; E)} = O(1).
\end{equation}

We then set $\phi_j = u_j$ on $(0, T) \times \partial M$ and consider $\omega \in H^{1, 2}((0, T) \times M; E)$ the solution of

\begin{align*}
(i \partial_t + \Delta_{A_2(t)} + V_2(t, x)) \omega(t, x) &= 0 \quad \text{in} \ (0, T) \times M, \\
\omega(t, x) &= \phi_1 \quad \text{on} \ (0, T) \times \partial M, \\
\omega(0, \cdot) &= 0 \quad \text{in} \ M.
\end{align*}

We observe that the difference $\omega - u_1$ solves the following Schrödinger equation:

\begin{align*}
(i \partial_t + \Delta_{A_2} + V_2)(\omega - u_1) &= 2(A_1 - A_2, du_1)_g + Qu_1 \quad \text{in} \ (0, T) \times M, \\
\omega - u_1 &= 0 \quad \text{on} \ (0, T) \times \partial M, \\
\omega(0, x) - u_1(0, x) &= 0 \quad \text{in} \ M,
\end{align*}

where $Qu_1 = (V_1 - V_2)u_1 + (A_1, A_1 u_1)_g - (A_2, A_2 u_1)_g - (d^* A_1) u_1 + (d^* A_2) u_1$.

Taking the Hermitian inner product of the above equation with $u_2$, we deduce that

\begin{equation}
\int_0^T \int_M \langle 2(A_1 - A_2, du_1)_g + Qu_1, u_2 \rangle_E \ dV_g \ dt = \int_0^T \int_{\partial M} \langle \partial_\nu (\omega - u_1), u_2 \rangle_E \ d\sigma_g \ dt.
\end{equation}

The RHS of the above is bounded by

\[ \int_0^T \int_{\partial M} \langle \partial_\nu (\omega - u_1), u_2 \rangle_E \ d\sigma_g \ dt \leq C \| (A_{A_1, V_1} - A_{A_2, V_2}) \phi_1 \|_{L^2((0, T) \times \partial M; E)} \| \phi_2 \|_{L^2((0, T) \times \partial M; E)} \].
which together with (3.8) and the trace theorem implies

\[
(4.7) \quad \left| \int_0^T \int_M \langle \partial_t (\omega - u_1), u_2 \rangle_E \, d\sigma_g dt \right| \leq C \left\| \left( \Lambda_{A_1} V_1 - \Lambda_{A_2} V_2 \right) \right\|_{L^2(0,T)} \left\| \phi_1 \right\|_{L^2(0,T) \times M; E} \left\| \phi_2 \right\|_{L^2(0,T) \times \partial M; E}
\leq C \left\| \Lambda_{A_1} V_1 - \Lambda_{A_2} V_2 \right\| s^8.
\]

On the other hand, since \( du_1 = i s (d\Psi) u_1 + e^{i s (\Psi - g \sigma)} da^{(A_1)} + dr_1 \), the LHS of (4.6) can be written as

\[
\int_0^T \int_M \langle (A_1 - A_2, du_1), (d\Psi) u_1, u_2 \rangle_E \, dV_g dt = i s \int_{(0,T) \times M} \langle (A_1 - A_2, (d\Psi) u_1), u_2 \rangle_E \, dV_g dt
+ \int_{(0,T) \times M} \langle (A_1 - A_2, e^{i s (\Psi - g \sigma)} da^{(A_1)} + dr_1), u_2 \rangle_E \, dV_g dt.
\]

We can divide the above by \( s \) and use the bounds (4.3) and (3.7) to deduce that

\[
(4.8) \quad \left| \int_{(0,T) \times M} \langle (A_1 - A_2, (d\Psi) u_1), u_2 \rangle_E \, dV_g dt \right| \leq s^{-1} \int_{(0,T) \times M} \langle (A_1 - A_2, du_1), u_2 \rangle_E \, dV_g dt + O(s^{-1}).
\]

By combining (4.6), (4.7), and (4.8), we conclude that

\[
\left| \int_{(0,T) \times M} \langle (A_1 - A_2), (d\Psi) u_1, u_2 \rangle_E \, dV_g dt \right| \leq C \left( \left\| \Lambda_{A_1} V_1 - \Lambda_{A_2} V_2 \right\| s^7 + s^{-1} \right).
\]

Thus, if \( \Lambda_{A_1} V_1 = \Lambda_{A_2} V_2 \), we can let \( s \to \infty \) in the RHS above to conclude that

\[
(4.9) \quad \lim_{s \to \infty} \int_{(0,T) \times M} \langle (A_1 - A_2), (d\Psi) u_1, u_2 \rangle_E \, dV_g dt = 0.
\]

We now make use of the following Lemma:

**Lemma 2.**

\[
\lim_{s \to \infty} \int_{(0,T) \times M} \langle (A_1 - A_2, (d\Psi) u_1), u_2 \rangle_E \, dV_g dt = \int_0^T \chi^2 \int_0^{\rho_+(x,\theta)} \langle (A_1 \gamma'_{x,\theta} - A_2 \gamma'_{x,\theta}) U_A, w_1, U_A w_2 \rangle_E \, dr \, dt.
\]

Before proving the above result, we first conclude the proof of Theorem 1. Since \( \tau \in (0,T/4) \) in the definition of \( \tilde{\chi} \) is arbitrary, we deduce that

\[
(4.10) \quad \int_0^{\rho_+(y,\theta)} \langle (A_1 \gamma'_{x,\theta} - A_2 \gamma'_{x,\theta}) U_A, w_1, U_A w_2 \rangle_E \, dr = 0.
\]

Note that the scattering data \( C_{A_i} \) takes values in \( U(n) \), so that we can define \( C_{A_i}^{-1} \) as the matrix inverse of \( C_{A_i} \). Making use of (1.1), (1.5), and (1.6), it can be shown that

\[
(4.11) \quad \langle (C_{A_2}^{-1} C_{A_1} - \text{Id}) w_1, w_2 \rangle_E = \int_0^{\rho_+(y,\theta)} \partial_t \langle U_A, w_1, U_A w_2 \rangle_E \, dr
= \int_0^{\rho_+(y,\theta)} \langle \nabla_{\theta} U_A, w_1, U_A w_2 \rangle_E \, dr + \int_0^{\rho_+(y,\theta)} \langle U_A, w_1, \nabla_{\theta} U_A w_2 \rangle_E \, dr
= \int_0^{\rho_+(y,\theta)} \langle (A_2 \gamma'_{x,\theta} - A_1 \gamma'_{x,\theta}) U_A, w_1, U_A w_2 \rangle_E \, dr.
\]

However, (4.10) and (4.11) imply that \( \langle (C_{A_2}^{-1} C_{A_1} - \text{Id}) w_1, w_2 \rangle_E = 0 \). Since \( w_1, w_2 \) were arbitrary, we conclude that \( C_{A_2}^{-1} C_{A_1} = \text{Id} \), whence \( C_{A_1} = C_{A_2} \) and the proof of Theorem 1 is complete. It remains for us to provide a proof of Lemma 2.
we denote by $x, v$. We will often suppress the dependence on $(V_j \cap M)$ or $(W_k \cap M)$. We shall first prove the latter case. Recall that $a^{(A_j)} = s^\frac{m-1}{2}(a_0 + O(s^{-1})\chi(y/\delta'))$. We can observe that

$$\lim_{s \to \infty} \int_{(0,T) \times M} \left( (A_1 - A_2), (d\Psi) u_1 \right)_g, u_2 \right)_E dV_g dt = \lim_{s \to \infty} \int_{0}^{T} \int_{\mathbb{R}^{m-1}} e^{-s\text{Im}(H(r))} |y| e^{sO(|y|^2)} s^{-\frac{m+1}{2}} \chi^2(y/\delta') \times$$

$$\left[ \left( (A_1(t, r, y) - A_2(t, r, y)), d\Psi(t, r, y) a_0^{(A_j)}(t, r, y) \right)_g, a_0^{(A_j)}(t, r, y) \right]_E + O(s^{-1}) \left| g(r, 0) \right|^2 dy dt.$$

We make the substitution $y \mapsto s^{-\frac{1}{2}}y$ above, and recall that $d\Psi|_{r=0} = \partial_r = \gamma'_{x, \theta}$. Then, since $\text{Im} H$ is positive definite and $\delta'$ is small, we note that the exponential term involving $\text{Im} H$ dominates the others. Thus, we can conclude that the RHS of the above is given by

$$\int_{0}^{T} \int_{\mathbb{R}^{m-1}} e^{-|y|^2} dy \int_{\mathbb{R}^{m-1}} e^{-s\text{Im}(H(r))} |y|^2 \left( (A_1(t, r, y) - A_2(t, r, y)), d\Psi(t, r, y) a_0^{(A_j)}(t, r, y) \right)_g, a_0^{(A_j)}(t, r, y) \right]_E |g(r, 0)|^2 dy dt.$$

Evaluating the integral in $y$ and using that $|g(r, 0)|^2 = 1$, we rewrite the above integral in the form

$$\int_{0}^{T} \chi^2(t) \int_{\mathbb{R}^{m-1}} e^{-|y|^2} dy \int_{\mathbb{R}^{m-1}} e^{-s\text{Im}(H(r))} \left( (A_1(t, r, y) - A_2(t, r, y)), d\Psi(t, r, y) a_0^{(A_j)}(t, r, y) \right)_g, a_0^{(A_j)}(t, r, y) \right]_E |g(r, 0)|^2 dy dt.$$

We now use the fact that, according to [14, Lemma 2.58], matrices which solve the Riccati equation (3.4) have the property

$$\text{det } \text{Im}(H(r)) = \text{det } \text{Im}(H(r_0)) e^{-2\int_{r_0}^{r} \text{tr } \text{Re}(H(s)) ds}.$$

This fact, together with our choice of constant $c_0$ in (3.6) is sufficient to conclude that

$$\lim_{s \to \infty} \int_{(0,T) \times M} \left( (A_1 - A_2), (d\Psi) u_1 \right)_g, u_2 \right)_E dV_g dt = \int_{0}^{T} \int_{\mathbb{R}^{m-1}} e^{-s\text{Im}(H(r))} \left( (A_1(t, r, y) - A_2(t, r, y)), U_A w_1, U_A w_2 \right)_E dy dt.$$

when $A_1 - A_2$ is compactly supported in $W_k \cap M$. For $A_1 - A_2$ compactly supported in $V_j \cap M$, we can write $v = \sum_{\gamma_{(t)} = p_j} v^{(l)}$. Thus, the limit has the form

$$\lim_{s \to \infty} \int_{(0,T) \times M} \left( (A_1 - A_2), (d\Psi) u_1 \right)_g, u_2 \right)_E dV_g dt = \sum_{\gamma_{(t)} = p_j} \int_{(0,T) \times M} \left( (A_1 - A_2), (d\Psi) v^{(l)} \right)_g, v^{(l)} \right)_E dV_g dt.$$

We observe that the first sum converges to the required limit by the same computations used to prove the limit in $W_k \cap M$, and the second sum vanishes via stationary phase arguments as in the proof of [11, Proposition 3.1], thus completing the proof of Lemma 2. \qed

5. PROOF OF GAUGE EQUIVALENCE

In what follows, we denote by $\varphi_r(x, v)$ the geodesic flow given by $\varphi_r(x, v) = (\gamma_{x, v}(r), \gamma'_{x, v}(r)) \in TM$. We will often suppress the dependence on $(x, v)$ and simply write $\varphi_r$ for the geodesic flow. Additionally, we denote by $X$ the geodesic vector field on $M$, which satisfies $\partial_r (F \circ \varphi_r) = X(F) \circ \varphi_r$, for any function $F : SM \to \mathbb{C}^n$.

For a function $F : SM \to \mathbb{C}^n$ and a connection 1-form $B : TM \to \mathbb{C}^{n \times n}$, we can consider the transport equation:

$$Xw + Bw = -F \text{ in } SM,$$

$$w|_{\partial_- SM} = 0,$$
where $B$ acts on $w : SM \to \mathbb{C}^n$ by multiplication at each point. In the present work, we shall consider only the cases $F(x, \theta) = f(x)$ or $F(x, \theta) = \alpha_j(x)\theta^j$, where $f, \alpha_j : M \to \mathbb{C}^n$. Using the transport equation (5.1), we can define the attenuated ray transform of $F$ with attenuation due to $B$ (see e.g. [20, Section 1]), via:

\begin{equation}
I_B F = w|_{\partial_+SM}.
\end{equation}

We can now proceed to the proof of Theorem 2 below.

**Proof of Theorem 2.** In order to prove Theorem 2, we further assume that $M$ is either i) 2-dimensional and simple, or ii) of dimension $m \geq 3$ with strictly convex boundary and admits the existence of a smooth strictly convex function $\phi : M \to \mathbb{R}$. We note that this second condition is conjectured to be true for all simple manifolds (amongst others, see e.g. [20, Section 2] for discussion), although the question is still open at present.

Consider the candidate gauge $G(t) = U_{A_1(t)}(U_{A_2(t)})^{-1}$. Note that $G(t)$ is unitary, $G(t) : SM \to U(n)$, and further $G$ is smooth by construction, since the $U_{A_j}$ are smooth in both space and time. Further, since $C_{A_1} = C_{A_2}$, it holds that $G(t)|_{\partial_+SM} = Id$ and additionally that $G(t)|_{\partial_-SM} = Id$, since $U_{A_j} = Id$ on $\partial_-SM$. We can observe that

\begin{equation}
XG + A_1G - GA_2 = 0
\end{equation}

(5.3)

$G|_{\partial SM} = Id$.

It remains to show that $G(t)$ depends only on the base-point $x \in M$. Note that (5.3) is equivalent to

\begin{align}
X(G - \text{Id}) + A_1(G - \text{Id}) - (G - \text{Id})A_2 &= A_2 - A_1 \\
(G - \text{Id})|_{\partial SM} &= 0.
\end{align}

(5.4)

We can henceforth fix some $t \in (0, T)$ and define a new connection form $B$ via $B(W) = A_1W - WA_2$ for $W \in \mathcal{C}^\infty(SM; \mathbb{C}^{n \times n})$. Then (5.4) becomes

\begin{align}
X(G - \text{Id}) + B(G - \text{Id}) &= A_2 - A_1 \\
(G - \text{Id})|_{\partial SM} &= 0.
\end{align}

(5.5)

We can interpret the above equation as a ray transform (see e.g. [20, Section 1]), as $I_B(A_1 - A_2) = 0$, where $I_B$ denotes the ray transform with attenuation due to the connection form $B$.

We now apply either [19, Theorem 1.3] if $M$ is 2-dimensional and simple, or [20, Theorem 1.6] if $M$ is of dimension $m \geq 3$ with strictly convex boundary and admits a smooth strictly convex function. Thus, we conclude that $A_2(t) - A_1(t) = \nabla B(t)p(t)$ for some $p(t) : M \to \mathbb{C}^{n \times n}$, and (5.5) then implies that $G(t) = \text{Id} + p(t)$. Hence, we have shown that $G(t)$ depends only on the base-point $x \in M$. Therefore, $G$ satisfies all the necessary conditions to be a gauge, and it follows from (5.3) that $A_1$ is gauge-equivalent to $A_2$ via the gauge-transform $A_2 = G^{-1}dG + G^{-1}A_1G$, as required.

We now turn to showing unique determination of the potential. Note that since $A_2 = G^{-1}dG + G^{-1}A_1G$, it remains only to show that $V_2 = G^{-1}V_1G + iG^{-1}\partial_t G$. We define $V_3 = G^{-1}V_1G + iG^{-1}\partial_t G$. By gauge invariance, it holds that $\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}$. Thus, by the assumption $\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}$ it follows that

\begin{equation}
\Lambda_{A_2, V_2} = \Lambda_{A_2, V_2}.
\end{equation}

(5.6)

It remains to show that condition (5.6) implies that $V_2 = V_3$. Note that we can take $A_1 = A_2 = A$ in (4.6), and argue as in the derivation of (4.10) to deduce that for $V = V_3 - V_2$ and each $t \in (0, T)$, $y \in \partial M$ and $\theta \in \partial_+S_yM$ we have

\[ \int_0^{\rho_+(y, \theta)} (VU_A w_1, U_A w_2)_{E} \circ \varphi_r(y, \theta)dr = 0, \]

where $\varphi_r(y, \theta)$ is the geodesic flow defined by $\varphi_r(y, \theta) = (\gamma_{y, \theta}(r), \gamma'_{y, \theta}(r))$. Then, by linearity, we conclude that

\begin{equation}
\int_0^{\rho_+(y, \theta)} U_A^{-1}VU_A \circ \varphi_r(y, \theta)dr = 0.
\end{equation}

(5.7)
In order to finish the proof, we wish to interpret (5.7) as an attenuated ray transform.

For \( W \in C^\infty(SM; \mathbb{C}^N) \), we can define the map \( BW = AW - WA = [A, W] \). Since \( A(x, v) = A_j(x)v^j \), we can observe that \( BW(x, v) = [A_j(x), W]v^j \). Note that we can regard \( V \) as a function on \( M \) taking values in \( \mathbb{C}^N \), where \( N = n^2 \). Thus, we can regard the map \( B \) as a smooth \( \mathbb{C}^{N \times N} \)-valued 1-form.

Recall that the inner product \( \langle \cdot, \cdot \rangle_E \) on the trivial bundle \( M \times \mathbb{C}^N \) induces an inner product on the endomorphism bundle \( \text{End}(E) = M \times \mathbb{C}^{N \times N} \) via \( \langle X, Y \rangle_{\text{End}(E)} = \text{tr}(X^*Y) \). Note that we can regard the 1-form \( B \) as a connection form on \( \text{End}(E) \). Further, by the cyclic property of trace, we observe that

\[
\langle BX, Y \rangle_{\text{End}(E)} = \text{tr}(-X^*AY + AX^*Y) = \text{tr}(-X^*AY + X^*YA) = \langle X, -BY \rangle_{\text{End}(E)},
\]

and that \( B \) is, therefore, a unitary connection on the endomorphism bundle.

Letting \( X \) once again denote the geodesic vector field, we consider \( \omega \) the solution of the transport equation

\[
X \omega + B \omega = -V \quad \text{in } SM, \quad \omega = 0 \quad \text{on } \partial_-SM.
\]

Using (5.8), we now observe that

\[
X(U_A^{-1}\omega U_A) = U_A^{-1}A\omega U_A + U_A^{-1}(-V - B\omega)U_A - U_A^{-1}\omega AU_A = -U_A^{-1}VU_A.
\]

The above then implies that \( \partial_1[(U_A^{-1}\omega U_A) \circ \varphi_r(y, \theta)] = -U_A^{-1}VU_A \circ \varphi_r(y, \theta) \), using the definitions of the geodesic vector field and geodesic flow. We can integrate this last expression to obtain

\[
- \int_0^{\rho_+(y, \theta)} U_A^{-1}VU_A \circ \varphi_r(y, \theta) dr = (U_A^{-1}\omega U_A) \circ \varphi_{\rho_+(y, \theta)} - (U_A^{-1}\omega U_A) \circ \varphi_0.
\]

Note that \( \phi_{\rho_+(y, \theta)} \in \partial_-SM \) and \( \phi_0 \in \partial_+SM \). By recalling that that \( U_A = \text{Id} \) on \( \partial_+SM \) and \( \omega = 0 \) on \( \partial_-SM \), we observe that the left-hand side of (5.9) is just \( -\omega|_{\partial_+SM} \).

Therefore, (5.7) and (5.8) tell us that

\[
I_B V = \omega|_{\partial_+SM} = \int_0^{\rho_+(y, \theta)} U_A^{-1}VU_A \circ \varphi_r(y, \theta) dr = 0.
\]

Hence, if \( M \) is 2-dimensional and simple, we can apply [19, Theorem 1.3] to conclude that the above implies \( V = 0 \). On the other hand, if \( M \) is of dimension \( m \geq 3 \) with strictly convex boundary and admits a smooth strictly convex function, we instead apply the result of [20, Theorem 1.6] to conclude that \( V = 0 \). Thus, it holds that \( V_2 = G^{-1}V_1G + iG^{-1}\partial_tG \), and hence \((A_1, V_1)\) is gauge-equivalent to \((A_2, V_2)\), as required. \( \square \)

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