Symmetric (not Complete Intersection) Numerical Semigroups Generated by Six Elements

Leonid G. Fel

Department of Civil Engineering, Technion, Haifa 32000, Israel
e-mail: lfel@technion.ac.il

Abstract

We consider symmetric (not complete intersection) numerical semigroups $S_6$, generated by a set of six positive integers $\{d_1, \ldots, d_6\}$, $\gcd(d_1, \ldots, d_6) = 1$, and derive inequalities for degrees of syzygies of such semigroups and find the lower bound for their Frobenius numbers. We show that this bound may be strengthened if $S_6$ satisfies the Watanabe lemma.

Keywords: symmetric (not complete intersection) semigroups, Betti’s numbers, Frobenius number

2010 Mathematics Subject Classification: Primary – 20M14, Secondary – 11P81.

1 Symmetric numerical semigroups generated by six integers

Let a numerical semigroup $S_m = \langle d_1, \ldots, d_m \rangle$ be generated by a set of positive integers $\{d_1, \ldots, d_m\}$, $d_1 < \ldots < d_m$, such that $\gcd(d_1, \ldots, d_m) = 1$, where $d_1$ and $m$ denote multiplicity and embedding dimension ($edim$) of $S_m$. There exist $m - 1$ polynomial identities [5] for degrees of syzygies associated with semigroup ring $k[S_m]$. They are a source of various relations for semigroups of different nature. In the case of complete intersection (CI) semigroups such relation for degrees of the 1st syzygy was found in [5], Corollary 1. The next nontrivial case exhibits a symmetric (not CI) semigroup generated by $m \geq 4$ integers. In [4] and [6], such semigroups with $m = 4$ and $m = 5$ were studied and the lower bound for the Frobenius numbers $F(S_4)$ and $F(S_5)$ were found. In the present paper we deal with more difficult case of symmetric (not CI) semigroups $S_6$.

Consider a symmetric numerical semigroup $S_6$, which is not CI and generated by six positive integers. Its Hilbert series $H(S_6; t)$ with independent Betti’s numbers $\beta_1, \beta_2$ reads:

$$H(S_6; t) = \frac{Q_6(t)}{\prod_{i=1}^{6} (1 - t^{d_i})},$$

$$Q_6(t) = 1 - \sum_{j=1}^{\beta_1} t^{x_j} + \sum_{j=1}^{\beta_2} t^{y_j} - \sum_{j=1}^{\beta_2} t^{g - y_j} + \sum_{j=1}^{\beta_1} t^{g - x_j} - t^g;$$

$$x_j, y_j, g \in \mathbb{Z}_>, \quad 2d_1 \leq x_j, y_j < g.$$

The Frobenius number $F(S_6)$ of numerical semigroup $S_6$ is related to the largest degree $g$ as follows:

$$F(S_6) = g - \sigma_1, \quad \sigma_1 = \sum_{j=1}^{6} d_j.$$

There are two constraints more, $\beta_1 > 5$ and $d_1 > 6$. The inequality $\beta_1 > 5$ holds since $S_6$ is not CI, and the condition $d_1 > 6$ is necessary since a semigroup $\langle m, d_2, \ldots, d_m \rangle$ is never symmetric [3].
## Polynomial identities for degrees of syzygies

Polynomial identities for degrees of syzygies for numerical semigroups were derived in [5], Thm 1. In the case of symmetric (not CI) semigroup $S_6$, they read:

\begin{align*}
\sum_{j=1}^{\beta_1} x_j^r - \sum_{j=1}^{\beta_2} y_j^r + \sum_{j=1}^{\beta_2} (g - y_j)^r - \sum_{j=1}^{\beta_1} (g - x_j)^r + g^r = 0, & \quad r \leq 4, \\
\sum_{j=1}^{\beta_1} x_j^5 - \sum_{j=1}^{\beta_2} y_j^5 + \sum_{j=1}^{\beta_2} (g - y_j)^5 - \sum_{j=1}^{\beta_1} (g - x_j)^5 + g^5 = 120\pi_6, & \quad \pi_6 = \prod_{j=1}^{6} d_j.
\end{align*}

Only three of five identities in (2) are not trivial, these are for $r = 1, 3, 5$:

\begin{align*}
B_6g + \sum_{j=1}^{\beta_1} x_j = \sum_{j=1}^{\beta_2} y_j, & \quad B_6 = \frac{\beta_2 - \beta_1 + 1}{2}, \\
B_6g^3 + \sum_{j=1}^{\beta_1} x_j^2 (3g - 2x_j) = \sum_{j=1}^{\beta_2} y_j^2 (3g - 2y_j), & \quad (4) \\
B_6g^5 + \sum_{j=1}^{\beta_1} x_j^3 (10g^2 - 15gx_j + 6x_j^2) - 360\pi_6 = \sum_{j=1}^{\beta_2} y_j^3 (10g^2 - 15gy_j + 6y_j^2). & \quad (5)
\end{align*}

where $B_6$ is defined according to the expression for an arbitrary symmetric semigroup $S_m$ in [5], Formulas (5.7, 5.9). The sign of $B_6$ is strongly related to the famous Stanley Conjecture 4b [10] on the unimodal sequence of Betti’s numbers in the 1-dim local Gorenstein rings $k[[S_m]]$. We give its simple proof in the case $edim = 6$.

**Lemma 1.** Let a symmetric (not CI) semigroup $S_6$ be given with the Hilbert series $H(S_6; z)$ in accordance with (1). Then

\begin{equation}
\beta_2 \geq \beta_1 + 1. 
\end{equation}

**Proof.** According to the identity (3) and constraints on degrees $x_j$ of the 1st syzygies (1) we have,

\begin{equation}
\sum_{j=1}^{\beta_2} y_j < B_6g + \beta_1 g = \frac{\beta_2 + \beta_1 + 1}{2} g.
\end{equation}

On the other hand, there holds another constraint on degrees $y_j$ of the 2nd syzygies,

\begin{equation}
\sum_{j=1}^{\beta_2} y_j < \beta_2 g.
\end{equation}

Inequality (8) holds always, while inequality (7) is not valid for every set $\{x_1, \ldots, x_{\beta_1}\}$, but only when (3) holds. In order to make the both inequalities consistent, we have to find a relation between $\beta_1$ and $\beta_2$ where both inequalities (7) and (8) are satisfied, even if (7) is stronger than (8). To provide these inequalities to be correct, it is enough to require $(\beta_2 + \beta_1 + 1)/2 \leq \beta_2$, that leads to (6). 

Another constraint for Betti’s numbers $\beta_j$ follows from the general inequality for the sum of $\beta_j$ in the case of non-symmetric semigroups [4]. Formula (1.9),

\begin{equation}
\sum_{j=0}^{m-1} \beta_j \leq d_1 2^{m-1} - 2(m - 1), \quad \beta_0 = 1.
\end{equation}
Applying the duality relation for Betti’s numbers, \( \beta_j = \beta_{m-j-1}, \beta_{m-1} = 1 \), in symmetric semigroups \( S_6 \) to inequality (9) and combining it with Lemma 1, we obtain
\[
\beta_1 < 2(4d_1 - 1).
\] (10)

To study polynomial identities (3,4,5) and their consequences, start with observation, which follows by numerical calculations for two real functions \( R_1(z), R_2(z) \) and is presented in Figure 1.

\[
R_1(z) \geq A_\star R_2(z), \quad 0 \leq z \leq 1,
\]
where
\[
R_1(z) = z^2\sqrt{10 - 15z + 6z^2}, \quad R_2(z) = z^2(3 - 2z), \quad A_\star = 0.9682.
\]

The constant \( A_\star \) is chosen by requirement of the existence of such a coordinate \( z_\star \in [0, 1] \) providing two equalities,
\[
R_1(z_\star) = A_\star R_2(z_\star), \quad R_1'(z_\star) = A_\star R_2'(z_\star), \quad z_\star \simeq 0.8333,
\]
where \( R_1'(z_\star) = \frac{dR_1(z)}{dz} \big|_{z=z_\star} \).

Substituting \( z = y_j / g, \ 0 < z < 1 \), into inequality (11) and making summation over \( 1 \leq j \leq \beta_2 \), we get
\[
A_\star \sum_{j=1}^{\beta_2} y_j^3(3g - 2y_j) < \sum_{j=1}^{\beta_2} y_j^2\sqrt{10g^2 - 15gy_j + 6y_j^2}.
\] (12)

Applying the Cauchy-Schwarz inequality \( \left( \sum_{j=1}^{N} a_j b_j \right)^2 \leq \left( \sum_{j=1}^{N} a_j^2 \right) \left( \sum_{j=1}^{N} b_j^2 \right) \) to the right-hand side of inequality (12), we obtain
\[
\left( \sum_{j=1}^{\beta_2} y_j^{3/2} \sqrt{10g^2 - 15gy_j + 6y_j^2} y_j^{1/2} \right)^2 \leq \sum_{j=1}^{\beta_2} y_j^3(10g^2 - 15gy_j + 6y_j^2) \sum_{j=1}^{\beta_2} y_j.
\] (13)

Combining (12) and (13), we arrive at inequality
\[
A_\star^2 \left( \sum_{j=1}^{\beta_2} y_j^2(3g - 2y_j) \right)^2 < \sum_{j=1}^{\beta_2} y_j^3(10g^2 - 15gy_j + 6y_j^2) \sum_{j=1}^{\beta_2} y_j.
\] (14)
Denote by \( X_k \) the \( k \)-th power symmetric polynomial \( X_k(x_1, \ldots, x_{\beta}) = \sum_{j=1}^{\beta_1} x_j^k \), \( x_j < g \), and substitute identities (3,4,5) into inequality (14),
\[
A^2 \left( B_6 g^3 + 3gX_2 - 2X_3 \right)^2 < \left( B_6 g^5 - 360\pi_6 + 10g^2 X_3 - 15gX_4 + 6X_5 \right) \left( B_6 g + X_1 \right). \tag{15}
\]

On the other hand, similarly to inequalities (12,13,14), let us establish another set of inequalities for \( X_k \) by replacing \( y_j \to x_j \). We write the last of them, which is similar to (14),
\[
A^2 \left( \sum_{j=1}^{\beta_1} x_j^2 (3g - 2x_j) \right)^2 < \sum_{j=1}^{\beta_1} x_j^3 (10g^2 - 15gx_j + 6x_j^2) \sum_{j=1}^{\beta_1} x_j, \tag{16}
\]
and present (16) in terms of \( X_k \),
\[
A^2 (3gX_2 - 2X_3)^2 < (10g^2 X_3 - 15gX_4 + 6X_5) X_1. \tag{17}
\]

Represent the both inequalities (15) and (17) as follows:
\[
360\pi_6 - B_6 g^5 + A^2 \frac{\left( B_6 g^3 + 3gX_2 - 2X_3 \right)^2}{B_6 g + X_1} < 10g^2 X_3 - 15gX_4 + 6X_5, \tag{18}
\]
\[
\frac{A^2 (3gX_2 - 2X_3)^2}{X_1} < 10g^2 X_3 - 15gX_4 + 6X_5. \tag{19}
\]

Inequality (19) holds always, while inequality (18) is not valid for every set \( \{x_1, \ldots, x_{\beta_1}, g\} \). In order to make the both inequalities consistent, we have to find a range of \( g \) where both inequalities (18) and (19) are satisfied. To provide these inequalities to be correct, it is enough to require that inequality (19) implies inequality (18), i.e.,
\[
\frac{360\pi_6 - B_6 g^5}{A^2} + \frac{\left( B_6 g^3 + 3gX_2 - 2X_3 \right)^2}{B_6 g + X_1} < \frac{(3gX_2 - 2X_3)^2}{X_1}. \tag{20}
\]

Simplifying the above expressions, we present the last inequality (20) as follows:
\[
CX_1(X_1 + B_6 g) < (3gX_2 - 2X_3 - g^2 X_1)^2, \quad C = \frac{360\pi_6 - \alpha B_6 g^5}{A^2 B_6 g}, \tag{21}
\]

where \( \alpha = 1 - A^2 \simeq 0.06259 \) and \( B_6 \geq 1 \) due to Lemma [1]. An inequality (21) holds always if its left-hand side is negative, i.e., \( C < 0 \), that results in the following constraint,
\[
g > q_6, \quad q_6 = \sqrt[3]{\frac{360}{\alpha B_6}} \sqrt[3]{\pi_6}, \quad \text{where} \quad \sqrt[3]{\frac{360}{\alpha}} \simeq 5.649. \tag{22}
\]

The lower bound \( q_6 \) in (22) provides a sufficient condition to satisfy the inequality (21). In fact, a necessary condition has to produce another bound \( g_6 < q_6 \).

3 The lower bound for the Frobenius numbers of semigroups \( S_0 \)

An actual lower bound of \( g \) precedes that, given in (22), since the inequality (21) may be satisfied for a sufficiently small \( C > 0 \). To find it, we introduce another kind of symmetric polynomials \( \mathcal{X}_k \):
\[
\mathcal{X}_k = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \ldots x_{i_k}, \quad \mathcal{X}_0 = 1, \quad \mathcal{X}_1 = \sum_{i=1}^{\beta_1} x_i, \quad \mathcal{X}_2 = \sum_{i<j} x_i x_j, \quad \mathcal{X}_3 = \sum_{i<j<r} x_i x_j x_r, \ldots, \quad \mathcal{X}_{\beta_1} = \prod_{i=1}^{\beta_1} x_i.
\]
which are related to polynomials $X_k$ by the Newton recursion identities,

$$mX_m = \sum_{k=1}^{m} (-1)^{k-1} X_k X_{m-k}, \quad \text{i.e.,}$$

$$X_1 = X_1, \quad X_2 = X_1^2 - 2X_2, \quad X_3 = X_1^3 - 3X_2X_1 + 3X_3, \quad \ldots.$$ \tag{23}

Recall the Newton-Maclaurin inequalities \cite{7} for polynomials $X_k$,

$$\frac{X_1}{\beta_1} \geq \left( \frac{X_2}{\beta_1^2} \right)^{\frac{1}{2}} \geq \left( \frac{X_3}{\beta_1^3} \right)^{\frac{1}{3}} \geq \ldots \geq \sqrt[\beta_1]{\sqrt[\beta_1]{\ldots}}.$$ \tag{24}

Consider the master inequality \cite{21} in the following form

$$CX_1(X_1 + B_6 g) < 9g^2 X_2^2 + 4X_3^2 + 4g X_1^4 + 2g^2 X_1 X_3 - 12gX_2X_3 - 6g^2 X_1 X_2,$$ \tag{25}

and substitute Newton’s identities \cite{23} into \cite{25},

$$X_1 P(X_1, X_2, X_3) < X_1 Q_1(X_1) + X_2 Q_2(X_1, X_2) + X_3 Q_3(X_1, X_2, X_3),$$ \tag{26}

where

$$P(X_1, X_2, X_3) = 4g^2 X_1 X_2 + 2X_3^2 X_2 + 6X_2 X_3 + 6g X_2^2 + 3g X_1 X_3,$$

$$Q_1(X_1) = \frac{1}{3} X_1^5 - g X_1^4 + \frac{13}{12} g^2 X_1^3 - \frac{1}{2} g^3 X_1^2 + \frac{g^4 - C}{12} X_1 - \frac{C}{12} X_1B_6 g,$$

$$Q_2(X_1, X_2) = 3g^2 X_2 + 3X_2 X_1^2 + 5g X_1^3 + g^3 X_1,$$

$$Q_3(X_1, X_2, X_3) = 3X_3 + 2X_3^2 + g^2 X_1 + 6g X_2.$$  

Applying inequalities \cite{24} to $Q_2(X_1, X_2)$ and $Q_3(X_1, X_2, X_3)$, we obtain

$$Q_2(X_1, X_2) < X_1 Q_{21}(X_1), \quad Q_{21}(X_1) = 3 \frac{X_1}{\beta_1^2} \left( \frac{\beta_1}{2} \right) \left( g^2 + X_1^2 \right) + 5g X_1^2 + g^3,$$

$$Q_3(X_1, X_2, X_3) < X_1 Q_{31}(X_1), \quad Q_{31}(X_1) = 3 \frac{X_1^2}{\beta_1^3} \left( \frac{\beta_1}{3} \right) + 2X_1^2 + g^2 + 6g \frac{X_1}{\beta_1^2} \left( \frac{\beta_1}{2} \right).$$ \tag{27}

Substituting inequalities \cite{27} into \cite{26} and applying again \cite{24}, we obtain

$$P(X_1, X_2, X_3) < Q_1(X_1) + \frac{X_1^2}{\beta_1^2} \left( \frac{\beta_1}{2} \right) Q_{21}(X_1) + \frac{X_1^3}{\beta_1^3} \left( \frac{\beta_1}{3} \right) Q_{31}(X_1).$$ \tag{28}

Represent the right-hand side of inequality \cite{28} as a polynomial $E(X_1)$ of the 5th order in $X_1$,

$$E(X_1) = \sum_{k=0}^{5} E_k g^{5-k} X_1^k, \quad \text{where}$$

$$E_0 = -\frac{B_6 C g^{-4}}{12}, \quad E_1 = \frac{1 - C g^{-4}}{12}, \quad E_2 = \frac{1}{\beta_1^2} \left( \frac{\beta_1}{2} \right) - \frac{1}{2} = -\frac{1}{2\beta_1},$$

$$E_3 = \frac{3}{\beta_1^2} \left( \frac{\beta_1}{2} \right)^2 + \frac{1}{\beta_1^3} \left( \frac{\beta_1}{3} \right) + \frac{13}{12}, \quad E_4 = \frac{5}{\beta_1^2} \left( \frac{\beta_1}{2} \right) + \frac{6}{\beta_1^3} \left( \frac{\beta_1}{3} \right) \left( \frac{\beta_1}{3} \right) - 1,$$

$$E_5 = \frac{3}{\beta_1^2} \left( \frac{\beta_1}{2} \right)^2 + \frac{3}{\beta_1^3} \left( \frac{\beta_1}{3} \right)^2 + \frac{2}{\beta_1^3} \left( \frac{\beta_1}{3} \right) + \frac{1}{3}.$$
Thus, the master inequality (21) reads:

\[ P(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3) < E(\mathcal{X}_1). \] (30)

On the other hand, applying (24) to the polynomial \( P(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3) \), we have another inequality,

\[ P(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3) < J(\mathcal{X}_1), \quad J(\mathcal{X}_1) = \sum_{k=3}^{5} J_k g^{5-k} \mathcal{X}_1^k, \] (31)

\[ J_5 = \frac{2}{\beta_1^2} \left( \frac{\beta_1}{2} \right) \left[ 1 + \frac{3}{\beta_1^3} \left( \frac{\beta_1}{3} \right) \right], \quad J_4 = \frac{6}{\beta_1^2} \left( \frac{\beta_1}{2} \right)^2 + \frac{3}{\beta_1^3} \left( \frac{\beta_1}{3} \right), \quad J_3 = \frac{4}{\beta_1^2} \left( \frac{\beta_1}{2} \right). \]

Inequality (31) holds always, while inequality (30) is not valid for every set \( \{x_1, \ldots, x_{\beta_1}, g\} \). In order to make both inequalities consistent, we have to find a range for \( g \) where both inequalities (30) and (31) are satisfied. To provide both inequalities to be correct, it is enough to require that (31) implies (30),

\[ E(\mathcal{X}_1) > J(\mathcal{X}_1), \quad \text{or} \]

\[ (E_5 - J_5)\mathcal{X}_1^5 + (E_4 - J_4)g\mathcal{X}_1^4 + (E_3 - J_3)g^2\mathcal{X}_1^3 + E_2g^3\mathcal{X}_1^2 + E_1g^4\mathcal{X}_1 + E_0g^5 > 0, \] (32)

\[ E_5 - J_5 = \frac{3}{\beta_1^2} \left( \frac{\beta_1}{2} \right)^2 + \frac{3}{\beta_1^3} \left( \frac{\beta_1}{3} \right)^2 + \frac{2}{\beta_1^2} \left( \frac{\beta_1}{3} \right) + \frac{1}{\beta_1^3} - \frac{2}{\beta_1^2} \left( \frac{\beta_1}{2} \right) \left[ 1 + \frac{3}{\beta_1^3} \left( \frac{\beta_1}{3} \right) \right] = \frac{1}{3\beta_1^4}, \]

\[ E_4 - J_4 = \frac{5}{\beta_1^2} \left( \frac{\beta_1}{2} \right)^2 + \frac{6}{\beta_1^3} \left( \frac{\beta_1}{2} \right) \left( \frac{\beta_1}{3} \right) - 1 - \frac{6}{\beta_1^4} \left( \frac{\beta_1}{2} \right)^2 - \frac{3}{\beta_1^2} \left( \frac{\beta_1}{3} \right) = - \frac{1}{\beta_1^4}, \]

\[ E_3 - J_3 = \frac{3}{\beta_1^2} \left( \frac{\beta_1}{2} \right)^2 + \frac{1}{\beta_1^3} \left( \frac{\beta_1}{3} \right) + \frac{13}{12} - \frac{4}{\beta_1^2} \left( \frac{\beta_1}{2} \right)^2 = \frac{13}{12\beta_1^4}. \] (33)

Substituting expressions \( E_k - J_k, k = 3, 4, 5 \) from (33) and \( E_0, E_1, E_2 \) from (29) into (32), we obtain

\[ \frac{C}{g^4} < G(b, u), \quad G(b, u) = \frac{u}{u + b}(1 - u)^2(1 - 2u)^2, \quad u = \frac{\mathcal{X}_1}{\beta_1 g}, \quad b = \frac{B_6}{\beta_1}. \] (34)

The function \( G(b, u) \) is continuous (see Figure 2) and attains its global maximal value \( G(b, u_m) \) at

![Figure 2: Plot of the functions G(b, u) with different b: (in brown) b = 1.75, u_m = 0.125; (in blue) b = 0.85, u_m = 0.117; (in red) b = 0.5, u_m = 0.112; (in black) b = 0.35, u_m = 0.107.](image)

\( u_m(b) \in (0, 1/2) \), where \( u_m = u_m(b) \) is a smaller positive root of cubic equation,

\[ 8u_m^3 + 2(5b - 3)u_m^2 - 9bu_m + b = 0, \]
with asymptotic behavior of \( u_m(b) \) and \( G(b, u_m) \) (see Figure 3),

\[
\begin{align*}
\lim_{b \to 0} \sqrt{\frac{b}{6}} & \quad \lim_{b \to \infty} \frac{v_1 - v_2}{b} = \frac{9 - \sqrt{41}}{20} \approx 0.1298, \quad v_1 = \frac{9 - \sqrt{41}}{20} \approx 0.013, \\
G(b, u_m) & \xrightarrow{b \to 0} 1, \quad G(b, u_m) \xrightarrow{b \to \infty} \frac{w}{b}, \quad w = \frac{411 + 41\sqrt{41}}{12500} \approx 0.05388.
\end{align*}
\]

(35)

**Theorem 1.** Let a symmetric (not CI) semigroup \( S_6 \) be given with its Hilbert series \( H(S_6; z) \) in accordance with (1). Then the following inequality holds:

\[
g > g_6, \quad g_6 = \lambda_6 \sqrt{\pi}, \quad \lambda_6 = \sqrt[5]{\frac{360}{B_6 K(b, A_4)}}, \quad K(b, A_4) = \alpha + A_4 G(b, u_m). \quad (36)
\]

**Proof.** Substitute into (34) the expression for \( C \), given in (21), and arrive at inequality

\[
\frac{360\pi - \alpha B_6 g^5}{A_4^2 B_6 g^5} < G(b, u_m),
\]

which gives rise to the lower bound \( g_6 \) in (36).

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![Figure 3: Plot of the functions (a) \( u_m(b) \) and (b) \( G(b, u_m) \) in a wide range of \( b \).](image)

Formula for \( \lambda_6 \) in (36) shows a strong dependence on \( B_6 \), even the last is implicitly included into \( G(b, u_m) \) by a slowly growing function \( u_m(b) \) when \( b > 1 \). Such dependence \( \lambda_6(B_6) \) may lead to a very small values of \( \lambda_6 \) if \( B_6 \) is not bounded from above, but \( b \) is fixed, and results in an asymptotic decrease of the bound, \( g_6 \xrightarrow{B_6 \to \infty} 0 \). The last limit poses a question: does formula (36) for \( g_6 \) contradict the known lower bound [9] for the Frobenius number in the 6-generated numerical semigroups of the arbitrary nature, i.e., not assuming their symmetricity. If the answer is affirmative then it arises another question: what should be required in order to avoid such contradiction. We address both questions in the next section in a slightly different form: are there any constraints on Betti’s numbers.

4 **Are there any constraints on Betti’s numbers of symmetric (not CI) semigroups \( S_6 \)**

Denote by \( \tilde{g}_6 \) and \( g_6 \) the lower bounds of the largest degree of syzygies for non-symmetric [9] and symmetric CI [5] semigroups generated by six integers, respectively. Compare \( g_6 \) with \( \tilde{g}_6 \) and \( g_6 \) and require that the following double inequality hold:

\[
\tilde{g}_6 < g_6 < g_6, \quad g_6 = \sqrt[5]{120\sqrt{\pi}}, \quad \tilde{g}_6 = 5\sqrt[5]{\pi}. \quad (37)
\]
Substituting the expression for \( g_6 \) from (36) into (37), we obtain
\[
\frac{72}{625} K(b, A_*) < B_0 < \frac{3}{K(b, A_*)}, \quad \frac{72}{625} = 0.1152, \quad K(b, A_*) \xrightarrow{b \to 0} 1, \quad K(b, A_*) \xrightarrow{b \to \infty} \alpha, \quad (38)
\]
where the two limits follow by (35,36). The double inequality (38) determines the upper and lower bounds for varying \( B_0 \) in the plane \((b, B_0)\) as monotonic functions (see Figure 4b) with asymptotic behavior,
\[
\text{Upp. bound : } B_0 \xrightarrow{b \to 0} 3, \quad B_0 \xrightarrow{b \to \infty} 47.92; \quad \text{Low. Bound : } B_0 \xrightarrow{b \to 0} 0.1152, \quad B_0 \xrightarrow{b \to \infty} 1.84. \quad (39)
\]
According to Lemma 1, the lower bound in (38) may be chosen as \( B_0 = 1 \).

Find the constraints on Betti’s numbers. For this purpose, the inequality (38) has to be replaced by
\[
1 < \beta_2 - \beta_1 < \frac{6}{K(b, A_*)} - 1, \quad \beta_2 - \beta_1 \xrightarrow{b \to 0} 94.84, \quad \beta_2 - \beta_1 \xrightarrow{b \to \infty} 5, \quad (40)
\]
and the plot in Figure 4b has to be transformed by rescaling the coordinates \((b, B_0)\) with inversion, \( b \to \beta_1 = B_0/b \), and shift, \( B_0 \to \beta_2 - \beta_1 = 2B_0 - 1 \) (see Figure 4b). Following sections 1 and 2 the constraints (38) have to be supplemented by another double inequality \( 5 < \beta_1 < 2(4d_1 - 1) \).

![Figure 4: Plots of the lower (blue) and upper (red) bounds in the planes (a) \((b, B_0)\) and (b) \((\beta_1, \beta_2 - \beta_1)\).](image)

The double inequality (40) manifests a phenomenon, which does not exist in symmetric (not CI) semigroups \( S_m \), generated by four [4] and five [6] integers, where inequalities \( \tilde{g}_m < g_m < \overline{g}_m \), are always satisfied and independent of Betti’s numbers \( \beta_1 = 5 \) for \( S_4 \) and \( \beta_1 = \beta \) for \( S_5 \):
\[
\tilde{g}_m < \lambda_4 \rho_m - \sqrt[3]{\pi_m} < \overline{g}_m, \quad \left\{ \begin{array}{l}
\lambda_4 = \frac{3}{25}, \\
\lambda_5 = \frac{\sqrt{192(\beta - 1)/\beta}}{\sqrt{\pi_m}}.
\end{array} \right. \quad (41)
\]
Note, that constraints (40) do not contradict Bresinsky’s theorem [2] on the arbitrary large finite value of \( \beta_1 \) for generic semigroup \( S_m, m \geq 4 \). Below, we put forward some considerations about validity of (40) for Betti’s numbers \( \beta_1, \beta_2 \) of symmetric (not CI) semigroup \( S_6 \).

The double inequality (40) has arisen by comparison of \( g_6 \) with two other bounds \( \tilde{g}_6 \) and \( \overline{g}_6 \) and, strictly speaking, a validity of (40) is dependent on how small is a discrepancy \( \delta R(z) \) in Figure 1. If \( \delta R(z) \) is not small enough and its neglecting in (11) is a far too rude approximation, then there may exist symmetric (not CI) semigroups \( S_6 \) with Betti’s numbers \( \beta_1, \beta_2 \), where (40) is broken. Such violation should indicate a necessity to improve the lower bound \( g_6 \) in (36) to restore the relationship \( \tilde{g}_6 < g_6 < \overline{g}_6 \). Note, that such improvement is very hard to provide even by replacing \( A_s \to A \) in inequality (12), where \( A_s < A < 1 \), and still preserving (12) with a new \( A \). Such replacement leads
In [6], we introduced a notion of the symmetric (not CI) semigroups $S_6$, where the double inequality (40) is broken, then there arises a much more deep question: why do the constraints on Betti’s numbers exist. This problem is strongly related to the structure of minimal relations of the first and second syzygies in the minimal free resolution for the 1–dim Gorenstein (not CI) ring $k[S_6]$ and has to be addressed in a separate paper.

5 Symmetric (not CI) semigroups $S_6$ with the $W$ and $W^2$ properties

In [6], we introduced a notion of the $W$ property for the $m$-generated symmetric (not CI) semigroups $S_m$ satisfying Watanabe’s Lemma [11]. We recall this Lemma together with the definition of the $W$ property and two other statements relevant in this section.

Lemma 2. ([11]). Let a semigroup $S_{m-1} = \langle \delta_1, \ldots, \delta_{m-1} \rangle$ be given and $a \in \mathbb{Z}$, $a > 1$, such that \(\gcd(a, d_m) = 1\), $d_m \in S_{m-1} \setminus \{\delta_1, \ldots, \delta_{m-1}\}$. Consider a semigroup $S_m = \langle a\delta_1, \ldots, a\delta_{m-1}, d_m \rangle$ and denote it by $S_m = \langle aS_{m-1}, d_m \rangle$. Then $S_m$ is symmetric if and only if $S_{m-1}$ is symmetric, and $S_m$ is symmetric CI if and only if $S_{m-1}$ is symmetric CI.

Corollary 1. ([6]). Let a semigroup $S_{m-1} = \langle \delta_1, \ldots, \delta_{m-1} \rangle$ be given and $a \in \mathbb{Z}$, $a > 1$, such that \(\gcd(a, d_m) = 1\), $d_m \in S_{m-1} \setminus \{\delta_1, \ldots, \delta_{m-1}\}$. Consider a semigroup $S_m = \langle aS_{m-1}, d_m \rangle$. Then $S_m$ is symmetric (not CI) if and only if $S_{m-1}$ is symmetric (not CI).

Definition 1. ([6]). A symmetric (not CI) semigroup $S_m$ has the property $W$ if there exists another symmetric (not CI) semigroup $S_{m-1}$ giving rise to $S_m$ by the construction, described in Corollary 1.

Theorem 2. ([6]). A minimal $\text{edim}$ of symmetric (not CI) semigroup $S_m$ with the property $W$ is $m = 5$.

In this section we study the symmetric (not CI) semigroups $S_6$ satisfying Watanabe’s Lemma [11]. To distinguish such semigroups from the rest of symmetric (not CI) semigroups $S_6$ without the property $W$ we denote them by $\mathcal{W}_6$.

Lemma 3. Let two symmetric (not CI) semigroups $\mathcal{W}_6 = \langle aS_5, d_6 \rangle$ and $S_5 = \langle q_1, \ldots, q_5 \rangle$ be given and $\gcd(a, d_6) = 1$, $d_6 \in S_5 \setminus \{q_1, \ldots, q_5\}$. Let the lower bound $F_{6w}$ of the Frobenius number $F(\mathcal{W}_6)$ of the semigroup $\mathcal{W}_6$ be represented as, $F_{6w} = g_{6w} - \left(a \sum_{j=1}^{5} q_j + d_6 \right)$. Then

$$g_{6w} = a \left(\lambda_5 \sqrt[4]{\pi_5(q)} + d_6\right), \quad \pi_5(q) = \prod_{j=1}^{5} q_j,$$  (42)

where $\lambda_5$ is defined in (41).

Proof. Consider a symmetric (not CI) numerical semigroup $S_5$ generated by five integers (without the $W$ property), and apply the recent result [6] on the lower bound $F_5$ of its Frobenius number, $F(S_5)$,

$$F(S_5) \geq F_5, \quad F_5 = h_5 - \sum_{j=1}^{5} q_j, \quad h_5 = \lambda_5 \sqrt[4]{\pi_5(q)}.$$  (43)

The following relationship between the Frobenius numbers $F(\mathcal{W}_6)$ and $F(S_5)$ was derived in [1]:

$$F(\mathcal{W}_6) = aF(S_5) + (a - 1)d_6.$$  (44)
Substituting $F(W_6) = g - \left( a \sum_{j=1}^{5} q_j + d_6 \right)$ and the representation (43) for $F(S_5)$ into (44), we obtain

$$g - a \sum_{j=1}^{5} q_j - d_6 = ah_5 - a \sum_{j=1}^{5} q_j + (a - 1)d_6 \quad \Rightarrow \quad g = a(h_5 + d_6).$$

Comparing the last equality in (45) with the lower bound of $h_5$ in (43), we arrive at (42).

Following Corollary 1, let us apply the construction of a symmetric (not CI) semigroup $S_n$ to a symmetric (not CI) semigroup $S_{m-1}$, which already has such property.

**Definition 2.** A symmetric (not CI) semigroup $S_n$ has the property $W^2$ if there exist two symmetric (not CI) semigroups $S_{m-1} = \langle q_1, \ldots, q_{m-1} \rangle$ and $S_{m-2} = \langle p_1, \ldots, p_{m-2} \rangle$ giving rise to $S_n$ by the construction, described in Corollary 1.

$$S_n = \langle a_1S_{m-1}, d_m \rangle, \quad d_m \in S_{m-1} \setminus \{ q_1, \ldots, q_{m-1} \}, \quad \gcd(a_1, d_m) = 1,$$

$$S_{m-1} = \langle a_2S_{m-2}, q_{m-1} \rangle, \quad q_{m-1} \in S_{m-2} \setminus \{ p_1, \ldots, p_{m-2} \}, \quad \gcd(a_2, q_{m-1}) = 1.$$

**Theorem 3.** A minimal edim of symmetric (not CI) semigroup $S_n$ with the property $W^2$ is $m = 6$.

**Proof.** This statement follows if we combine Definition 2 and Theorem 2.

In this section we denote the symmetric (not CI) semigroups $S_6$ with the property $W^2$ by $W_6^2$.

**Lemma 4.** Let three symmetric (not CI) semigroups $W_6^2 = \langle a_1W_5, d_6 \rangle$, $W_5 = \langle a_2S_4, q_5 \rangle$, and $S_4 = \langle p_1, \ldots, p_4 \rangle$, where $q_j = a_2p_j$, $1 \leq j \leq 4$, be given in such a way that

$$d_6 \in W_5 \setminus \{ q_1, \ldots, q_5 \}, \quad q_5 \in S_4 \setminus \{ p_1, \ldots, p_4 \}, \quad \gcd(a_1, d_6) = \gcd(a_2, q_5) = 1.$$

Let the lower bound $F_{6^2W_5}$ of the Frobenius number $F(W_6^2)$ of the semigroup $W_6^2$ be represented as,

$$F_{6^2W_5} = g_{6W^2} - \left( a_1a_2 \sum_{j=1}^{4} p_j + a_1q_5 + d_6 \right).$$

Then

$$g_{6W^2} = a_1 \left[ a_2 \left( \lambda_4 \sqrt[4]{\pi_4(p)} + q_5 \right) + d_6 \right], \quad \pi_4(p) = \prod_{j=1}^{4} p_j,$$

where $\lambda_4$ is defined in (44).

**Proof.** By Lemma 2 in [1], the lower bound $F_{5W}$ of its Frobenius number $F(W_5)$ of the symmetric (not CI) semigroup $W_5$ reads:

$$F_{5W} = g_{5W} - \left( a_2 \sum_{j=1}^{4} p_j + q_5 \right), \quad g_{5W} = a_2 \left( \lambda_4 \sqrt[4]{\pi_4(p)} + q_5 \right).$$

Consider a symmetric (not CI) semigroup $W_6^2$, generated by six integers, and make use of a relationship between the Frobenius numbers $F(W_6^2)$ and $F(W_5)$ derived in [1]:

$$F(W_6^2) = a_1F(W_5) + (a_1 - 1)d_6.$$  \hspace{1cm} (48)

Substituting $F(W_6^2) = g_{6W^2} - \left( a_1a_2 \sum_{j=1}^{4} p_j + a_1q_5 + d_6 \right)$ and the representation (47) for $F(S_5)$ into (48), we obtain

$$g_{6W^2} - a_1a_2 \sum_{j=1}^{4} p_j - a_1q_5 - d_6 = a_1 \left[ g_{5W} - \left( a_2 \sum_{j=1}^{4} p_j + q_5 \right) \right] + (a_1 - 1)d_6.$$  \hspace{1cm} (49)

Simplifying the last equality (49), we arrive at (46).
Among the subsets \( \{ W_0^2 \}, \{ W_6 \} \) and the entire set \( \{ S_6 \} \) of symmetric (not CI) semigroups, generated by six integers, the following containment holds:

\[
\{ W_0^2 \} \subset \{ W_6 \} \subset \{ S_6 \}.
\]

Below we present twelve symmetric (not CI) semigroup generated by six integers: \( V_1, V_2, V_3, V_4 \) – without the \( W \) property, \( V_5, V_6, V_7, V_8 \) – with the \( W \) property, \( V_9, V_{10}, V_{11}, V_{12} \) – with the \( W^2 \) property.

\[
\begin{align*}
V_1 &= \langle 7, 9, 11, 12, 13, 15 \rangle, & V_3 &= \langle 12, 20, 28, 30, 38, 41 \rangle, & V_9 &= \langle 30, 33, 36, 37, 42, 48 \rangle, \\
V_2 &= \langle 7, 9, 10, 11, 12, 13 \rangle, & V_6 &= \langle 12, 20, 28, 30, 38, 46, 47 \rangle, & V_{10} &= \langle 42, 45, 48, 54, 59, 78 \rangle, \\
V_4 &= \langle 12, 13, 14, 15, 17, 19 \rangle, & V_7 &= \langle 14, 24, 26, 36, 46, 49 \rangle, & V_{11} &= \langle 40, 42, 48, 54, 71, 78 \rangle, \\
V_5 &= \langle 12, 13, 14, 15, 18, 19 \rangle, & V_8 &= \langle 38, 46, 58, 62, 74, 79 \rangle, & V_{12} &= \langle 46, 48, 75, 78, 90, 102 \rangle.
\end{align*}
\]

We give a comparative Table 1 for the largest degree \( g \) of syzygies and its lower bounds \( g_6, g_{6w}, g_{6w^2} \) and \( \bar{g}_6 \), calculated by formula (37).

| \( S_6 \) | \( V_1 \) | \( V_2 \) | \( V_3 \) | \( V_4 \) | \( V_5 \) | \( V_6 \) | \( V_7 \) | \( V_8 \) | \( V_9 \) | \( V_{10} \) | \( V_{11} \) | \( V_{12} \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \beta_1 \) | 13 | 14 | 10 | 10 | 8 | 9 | 10 | 14 | 7 | 7 | 7 | 7 |
| \( \beta_2 \) | 31 | 35 | 19 | 22 | 19 | 18 | 23 | 37 | 16 | 16 | 16 | 16 |
| \( B_6 \) | 9.5 | 11 | 5 | 6.5 | 6 | 5 | 7 | 12 | 5 | 5 | 5 | 5 |
| \( g \) | 84 | 77 | 125 | 126 | 256 | 292 | 302 | 638 | 387 | 603 | 598 | 816 |
| \( g_{6w^2} \) | – | – | – | – | – | – | – | – | 385.6 | 595.3 | 590.3 | 811.2 |
| \( g_{6w} \) | – | – | – | – | 240.4 | 271.2 | 286 | 609.2 | 359.8 | 554.8 | 548 | 746.9 |
| \( g_6 \) | 55 | 49.6 | 88 | 86.5 | 173.3 | 196 | 196.6 | 395.4 | 274.4 | 420.8 | 426.6 | 586.6 |
| \( \bar{g}_6 \) | 45.5 | 42 | 66.2 | 66.9 | 130.4 | 146 | 153.4 | 338 | 199.9 | 306.5 | 310.7 | 427.2 |

Table 1. The largest degree \( g \) of syzygies for symmetric (not CI) semigroups \( S_6 \) with different Betti’s numbers \( \beta_1, \beta_2 \) and its lower bounds \( g_6, g_{6w}, g_{6w^2}, \bar{g}_6 \).

For symmetric (not CI) semigroups \( W_0^2 \), presented in Table 1, there following inequalities hold:

\[
g > g_{6w^2} > g_{6w} > g_6 > \bar{g}_6.
\]

(50)

For the rest of symmetric (not CI) semigroups \( W_6 \) and \( S_6 \) the bounds \( g_{6w^2} \) and \( g_{6w} \) are skipped in inequalities (50) depending on the existence (or absence) of the \( W \) property in these semigroups. It is easy to verify that the Betti numbers of all semigroups from Table 1 satisfy the constraints (40).

Acknowledgement

The research was partly supported by the Kamea Fellowship.

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