Multiple phase transitions of the susceptible-infected-susceptible epidemic model on complex networks

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We show that the susceptible-infected-susceptible (SIS) epidemic dynamics running on the top of networks with a power law degree distribution can exhibit multiple phase transitions. Three main transitions involving different mechanisms responsible by sustaining the epidemics are identified: A short-term epidemics concentrated around the most connected vertex; a long-term (asymptotically stable) localized epidemics with a vanishing threshold; and an endemic phase occurring at a finite threshold. The different transitions are suited through different mean-field approaches. We finally show that the multiple transitions are due to the activations of different domains of the network that are observed in rapid (singular) variations of both stationary density of infected vertices and the participation ratio against the infection rate.

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The existence/absence of finite epidemic thresholds involving an endemic phase of the susceptible-infected-susceptible (SIS) model on the top of scale-free (SF) networks with a degree distribution $P(k) \sim k^{-\gamma}$, where $\gamma$ is the degree exponent, has been target of a recent and intense investigation [1–6]. In the SIS model, individuals in a population can be only in one of two states: infected or susceptible. Infected individuals become spontaneously healthy at rate 1 (this choice fixes the time scale), while the susceptible ones are infected at rate $\lambda n_i$, where $n_i$ is the number of infected contacts of a vertex $i$. Distinct theoretical approaches were devised at the aim of determining an epidemic threshold $\lambda_c$ of the SIS model, that separates an absorbing disease-free phase from an active phase where epidemics can last forever. The quenched mean-field (QMF) theory [7] explicitly includes the entire structure of the network through its adjacency matrix in a perturbative analysis of the dynamical equations for each vertex while the heterogeneous mean-field (HMF) theory [8, 9] performs a coarse-graining of the network structure grouping vertices accordingly their degree. The HMF theory predicts a vanishing threshold for the SIS model running on the top of SF networks for range $2 < \gamma < 3$ while a finite threshold is expected for degree exponent $\gamma > 3$. Conversely, the QMF theory states a threshold inversely proportional to the largest eigenvalue of the adjacency matrix, which diverges with size for networks with power law degree distributions, and the threshold vanishes for any value of $\gamma$. [1–10].

Recently, Boguná et al. [1] proposed a semi-analytical approach taking into account a long-range reinfection mechanism and again found a vanishing epidemic threshold for $\gamma > 3$ that decays with the network size much slower than in the QMF theory. They compared their theoretical predictions with simulations starting from a single infected vertex and a diverging epidemic lifespan was used as a criterion to determine the thresholds. Lee et al. [6], however, criticize these finds claiming that long-range reinfections are not sufficiently strong to guarantee an endemic phase with a finite fraction of infected vertices. Lee et al. [5] proposed that for a range $\lambda_{c_{QMF}} < \lambda < \lambda_c$ with a nonzero $\lambda_c$, the outliers (hubs) in a random network become infected but their activities are restricted to their neighborhoods. High-degree vertices produce independently active domains only when they are not directly connected. The sizes of these local domains increase for increasing $\lambda$ leading to the overlap among them and, finally, to an endemic phase for $\lambda > \lambda_c$. However, on networks where hubs are directly connected, it is possible to sustain an endemic state even in the limit $\lambda \to 0$ due to the mutual reinfection of connected hubs.

In this paper, we show that the SIS dynamics running on the top of SF networks with $\gamma > 3$ can exhibit multiple phase transitions and, consequently, more than one threshold whenever a set of outliers forming gaps in the degree distribution is considered. The vanishing threshold predicted by the lifespan method [4] is also found in our analysis, but we show that it usually represents a transition to a localized epidemics with an long-term activity. On other hand, a vanishing threshold agreeing with the QMF theory is also observed, but it represents a short-term epidemics highly localized in the neighborhood of the most connected vertex. Finally, our numerical results strongly indicate a transition to the endemic state occurring at a finite threshold.

The simulations were performed with the quasistationary (QS) method [11, 12] that, to our knowledge, is the...
most robust approach to deal with absorbing states in finite-size systems. The method consists in using QS states obtained from the simulations as initial conditions every time the dynamics gets trapped into an absorbing state. It is important to notice that the QS method becomes expendable for a large part of our simulations since the system never visits the absorbing state for the considered simulation times.

We used different criteria to determine the thresholds, relied on the fluctuations or singularities of the order parameters in analogy with critical phenomena [13, 14]. We define the susceptibility as $$\chi = N((\rho^2) - \langle \rho^2 \rangle)/\rho$$ [15] that does exhibit a pronounced divergence at the transition point for SIS [3, 5, 10] and contact process [16, 17] models on networks. Ref. [4] misleadingly claims that this susceptibility method is unreliable for networks with degree exponents $$\gamma > 3$$. In order to address this issue, we implemented both QS and lifespan methods. We performed simulations of the SIS model on networks generated by the uncorrelated configuration (UCM) model [18] with $$\gamma = 3.50$$, minimum degree $$k_0 = 3$$, and upper cutoff $$k_{\text{max}} = \langle k_{\text{max}} \rangle$$ [19] in analogy with Refs. [11, 12, 10]. The QS simulation parameters are similar to those used in Ref. [10]. The lifespan method was implemented exactly as in Ref. [1] (see Ref. [20]). Figures 1(a) and (b) show the lifespan and susceptibility against infection rate for networks of different sizes. The peak positions $$\lambda_p$$ against network size are compared in Fig. 1(c). As can be clearly seen, the right susceptibility peaks are very close to the lifespan ones showing that the susceptibility method is able to capture the same transitions as the lifespan method does but going beyond as discussed below. Moreover, the lifespan is also easily obtained in the QS method [11, 12]. We have checked that the lifespans obtained in the QS method and those of Ref. [4] diverge at the same threshold.

Two-peaks in the susceptibility were firstly reported in Ref. [2] but there was not realized that the right peak represents a phase transition to a long-term epidemics. However, the susceptibility curves can exhibit much more complex behaviors with multiple peaks for values of $$\lambda$$ larger than those shown in both Fig. 1 and in Refs. [3, 10] depending on the network realization and these behaviors become very frequent for large random SF networks. From now on, the constraint $$k_{\text{max}} = \langle k_{\text{max}} \rangle$$ is dropped out in the rest of this paper. Figure 2(a) shows a typical susceptibility curve exhibiting such a complex behavior for a network generated without constraints in the upper cutoff, whose degree distribution is shown in Fig. 2(d). Multiple peaks appear whenever the network has outliers forming gaps in the degree distribution. Notice that these multiple peaks are not detected by the lifespan method [11]. The role played by outliers is elicited by their immunizations [21] as illustrated in Fig. 2. For instance, the immunization of three most connected vertices is enough to destroy two peaks and to enhance others in the network used in Fig. 2.

The stationary density varies abruptly close to the thresholds, which is hallmark by sharp peaks in the logarithmic derivative, as seen in Figs. 2(b) and (c).

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In order to obtain a physical explanation for the multiple peaks, we analyzed the participation ratio (PR) de-
defined as
\[ \Phi = \frac{1}{N} \left( \sum_i \rho_i \right)^2, \]
where \( \rho_i \) is the probability that the vertex \( i \) is infected in the stationary state. The inverse of the participation ratio is a standard measure for localization/delocalization of states in condensed matter \[22\] and has been applied to statistical physics problems \[23\] including epidemic spreading on networks \[2, 24\]. The limiting cases of totally delocalized (\( \rho_i = \rho \forall i \)) and localized (\( \rho_i = \rho \delta_{i,0} \) where 0 is vertex where localization occurs) states are \( \Phi = 1 \) and \( \Phi = 1/N \), respectively. The PR as a function of the infection rate is shown in Fig. 3. The PR also varies abruptly at the susceptibility peak positions (inset of Fig. 3). The PR is an estimate of the fraction of vertices that effectively contributes to sustaining activity. The multiple transitions are due to rapid delocalization processes of the epidemics as \( \lambda \) increases. The onset of peaks will stop when the PR corresponds to a finite fraction of the network in the thermodynamical limit and no abrupt variations with \( \lambda \) are allowed, constituting an authentic endemic state. The PR against network size for fixed distance to either lifespan and rightmost thresholds is shown in Fig. 4. As one can see, in this range the PR decays very consistently as \( N^{-1} \). Analogous results are obtained for \( \bar{\rho} \) vs \( N \) curves. These decays are evidences of epidemics localization. On other hand, the constant dependence on \( N \) observed above the rightmost thresholds represents an endemic phase \[25\].

![FIG. 3. PR as a function of the infection rate for the same networks and immunization strategies shown in Fig. 2. Inset shows the logarithmic derivative of the PR. Symbols as in Fig. 2.](image)

![FIG. 4. PR against size for a fixed distance to either lifespan and rightmost thresholds for SF networks. The distance \( \lambda - \lambda_p = 0.012 \) was used.](image)

The case \( \gamma > 3 \) encompasses three competing mean-field theories. The leftmost peak represents a transition to a short-term epidemics highly concentrated at the most connected vertex and its neighborhood. The threshold dependence on size is very well described by QMF theories \[1, 3, 10\]. A transition to a long-term epidemics is observed for a threshold described by the theory of Ref. \[4\] but it is not endemic since PR decays as network size increases quite above the threshold. The lifespan threshold vanishes as network size increases much slower than the leftmost peak. In particular, decays \( \lambda_p \sim N^{-a} \) where \( a \approx 0.13 \) and \( a \approx 0.08 \) were found for \( \gamma = 3.5 \) and 4.0, respectively. Finally, at least in our numerical accuracy, a transition to an endemic phase is observed at a finite threshold exactly as formerly predicted by the HMF theory \[8, 9\].

![FIG. 5. Thresholds for SIS dynamics on SF networks with degree exponents \( \gamma = 3.5 \) (left) and \( \gamma = 4.0 \) (right). Averages were done over 3 network realizations. The results predicted by pair QMF \[10\] and pair HMF \[17\] theories are shown as dashed and doted lines, respectively. Solid lines are power law regressions.](image)

Our results do not point the correct neither the best theory for SIS dynamics on networks. Actually, we show that there are complementary theories able to describe distinct transitions that may appear depending on the network structure. Moreover, it is important to stress...
that the transitions to non-endemic phases are not negligible since they become long-term and, in the case of the lifespan threshold, an outbreak will eventually reach a finite fraction of the network in the thermodynamical limit with a non negligible chance [4]. This peculiar result is unthinkable for other substrates than complex networks sharing the small-world and scale-free properties. Actually, it is well known that some computer viruses can survive for long periods of time (years) in a very low density (below $10^{-4}$) [26] exemplifying the importance of metastable non-endemic states. Very recently, it was found a double phase transition for bond percolation on SF networks with high clustering [27], where transitions were hallmarked by two peaks in the susceptibility in analogy with our results for epidemics. However, the transitions we identified are from a completely different origin involving dynamical processes on low-clustering substrates.

It is evident that the outliers and the shape of the degree distribution play a central role on sustaining epidemic activity on the hole network. In order to highlight the effects of the outliers, we performed simulations on networks with a hard cutoff. Networks without an rigid upper bound in the degree distribution have a highly fluctuating natural cutoff with an average value $\langle k_{\text{max}} \rangle \simeq k_0N^{1/(\gamma - 1)}$ [28]. We introduce a hard cutoff as $k_{\text{max}} = k_0N^{0.75/(\gamma - 1)}$, which suppresses emergence of outliers in networks (inset of Fig. 4). Top panel of Fig. 4 shows the stationary density for hard and natural cutoffs. The infectiousness for $\lambda < \lambda_p$ is highly reduced in the absence of outliers. Also, the hard cutoff thresholds are quite close to those given by the rightmost peaks obtained for the natural cutoff, as shown in Fig. 4. One can also see that that the densities get closer above the threshold of $\lambda_p \approx 0.18$ for both cutoffs in agreement with qualitative HMF theories where the thresholds for $\gamma > 3$ are asymptotically independent of $k_0$ diverges [8, 17].

Our finds are partially consistent with the conjecture proposed by Lee et al. [5]. The finite thresholds of the endemic phase observed for the natural cutoff are due to an isolated feature of the outliers that generates almost independent domains of activity. Moreover, since the lifespans of independent domains grow exponentially with their sizes, long-term states are sustainable even in non-endemic phases, explaining the intermediary phase transitions happening for $\lambda_{c}\text{endemic} < \lambda < \lambda_{c}\text{endemic}$. In the case of a hard cutoff, outliers and, consequently, independent domains of the activity are absent. So, according to Ref. [5], the threshold could be null for $N \to \infty$. Indeed, a extremely slow vanishing (logarithmic) for a hard cutoff cannot be discarded from our numerical data shown in Fig. 5. Our results also do not rule out the mean-field analysis of Ref. [2]. The intermediary transitions can be associated to distinct localized eigenvectors that are centered on the outliers while the endemic threshold involves a delocalized eigenvector with a finite eigenvalue.

In conclusion, we thoroughly simulated the dynamics of the SIS epidemic model on complex networks with power law degree distributions with exponent $\gamma = 4.0$ using both natural and hard cutoffs. Inset shows the tail of the degree distributions. Bottom: Threshold against system size for hard (open symbols) and natural (filled symbols) cutoffs. Averages were done over three networks and error bars for hard cutoff are smaller than symbols. Lines are logarithmic regressions.

FIG. 6. Top: QS density against infection rate for a network with $N = 10^8$ vertices and degree exponent $\gamma = 4.0$ using both natural and hard cutoffs. Inset shows the tail of the degree distributions. Bottom: Threshold against system size for hard (open symbols) and natural (filled symbols) cutoffs. Averages were done over three networks and error bars for hard cutoff are smaller than symbols. Lines are logarithmic regressions.

In conclusion, we thoroughly simulated the dynamics of the SIS epidemic model on complex networks with power law degree distributions with exponent $\gamma = 4.0$, for which controversial theories discussing the existence of not of a finite epidemic threshold for an endemic phase have recently been proposed [1, 2, 4–6]. We show that the SIS dynamics can indeed exhibit several phase transitions associated to different epidemiological scenarios and the one associated to the transition to an endemic state, in which a finite fraction of network is infected, occurs at a finite threshold as formerly and now surprisingly foreseen by the HMF theory [8]. We show that the threshold obtained in a recent semi-analytical mean-field theory [4] represents a transition to a long-term epidemics, characterized by a diverging lifespan, but it is not an endemic state. Our simulations also confirms the existence of a transition to a highly localized and short-term epidemic state with threshold well described by the QMF theory. The multiple phase transitions are associated to gaps in the degree distribution, which also implies in gaps in the eigenvalue spectrum of the adjacency matrix. Our numerical results call for general theoretical approaches to concomitantly describe the multiple phase transitions. This work was partially supported by the Brazilian agencies CAPES, CNPq and FAPEMIG. Authors thank profitable and intense discussions with R. Pastor-Satorras and C. Castellano.
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