Convexification of Restricted Dirichlet-to-Neumann Map

Michael V. Klibanov
Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA
mklibanv@uncc.edu

Abstract

By our definition, “restricted Dirichlet-to-Neumann map” (DN) means that the Dirichlet and Neumann boundary data for a Coefficient Inverse Problem (CIP) are generated by a point source running along an interval of a straight line. On the other hand, the conventional DN data can be generated, at least sometimes, by a point source running along a hypersurface. CIPs with the restricted DN data are non-overdetermined in the $n$-D case with $n \geq 2$. We develop, in a unified way, a general and a radically new numerical concept for CIPs with restricted DN data for a broad class of PDEs of the second order, such as, e.g. elliptic, parabolic and hyperbolic ones. Namely, using Carleman Weight Functions, we construct globally convergent numerical methods. Hölder stability and uniqueness are also proved. The price we pay for these features is a well acceptable one in the Numerical Analysis: we truncate a certain Fourier-like series with respect to some functions depending only on the position of that point source. At least three applications are: imaging of land mines, crosswell imaging and electrical impedance tomography.

Key Words: restricted Dirichlet-to-Neumann data, convexification, global strict convexity, Carleman Weight Functions

2010 Mathematics Subject Classification: 35R30.

As of August 6, 2017, the corresponding paper is available online of Journal of Inverse and Ill-Posed Problems, DOI: 10.1515/jiip-2017-0067, https://www.degruyter.com/printahead/j/jiip

1 Introduction

The conventional Dirichlet-to-Neumann map (DN) data for a Coefficient Inverse Problem (CIP) can be generated, at least sometimes, by the point source running along a hypersurface, see pages 10-14 in [15] for DN and [10] for the Neumann-to-Dirichlet map data. We define “restricted DN data” for a CIP as the ones in which Dirichlet and Neumann boundary data are generated by a point source running along an interval of a straight line. These data are non-overdetermined in the $n$--D case with $n \geq 2$.

We present in this paper a general and a radically new concept of constructing globally convergent numerical methods for CIPs with restricted DN data. This concept also covers
both Hölder stability and uniqueness results for the CIPs we consider. Our construction is independent on a specific PDE operator: it is the same for those PDEs of the second order, which admit Carleman estimates. In particular, it works for three main types of PDEs of the second order: elliptic, parabolic and hyperbolic ones. The Dirichlet and Neumann data in elliptic and parabolic cases can be given on a part of the boundary.

The price we pay for our concept is a well acceptable one in the Numerical Analysis: we truncate a Fourier-like series with respect to a certain orthonormal basis in the $L_2$ space of functions depending only on the position of that point source. Next, to find spatially dependent coefficients of that truncated series, we construct a weighted globally strictly convex Tikhonov-like functional with the Carleman Weight Function (CWF) in it. This is the function, which is involved in the Carleman estimate for the corresponding PDE operator. Also, we establish the global convergence of the gradient projection method to the exact solution under the natural condition that the noise in the data tends to zero. As some applications, we mention detection and identification of land mines, crosswell imaging and electrical impedance tomography.

The construction of weighted strictly convex Tikhonov-like functionals with CWFs in them was started by the author in 1997 [14] with the recently renewed interest in [4, 17, 19]. However, all these works consider only CIPs with a single measurement data, as opposed to many measurements of the current paper. In [18] this technique was applied, for the first time, to ill-posed Cauchy problems for a class of quasilinear PDEs of the second order. The idea of [18] was further explored in [2]. Numerical results can be found in [2, 17].

As to the DN data, a very substantial number of works have been published. Since this paper is not a survey of DN, we refer to only a very few of them, for brevity, and the reader can find other references in these publications. Global uniqueness theorems for the elliptic case, i.e. for the Calderon problem, were obtained in [24, 27, 29]. Some reconstruction procedures can be found in [22, 24, 27, 28]. In the reconstruction procedure of [28], a certain infinite matrix is truncated, which is philosophically close to our truncation of that Fourier-like series. We refer to [1, 9, 11] for numerical studies of DN. In [5] and [12] reconstruction procedures for DN for hyperbolic PDEs were developed, and they were computationally tested in [6] and [12, 13] respectively.

We point out that since our goal here is to present a new numerical concept, for brevity, we are not concerned below with some issues related to solutions of forward problems, since they can be discussed in later publications. These issues are: the minimal smoothness assumptions, existence and uniqueness of the solutions of the forward problems under considerations, the positivity of those solutions and also the continuous differentiability of those solutions with respect to the position of the point source, see Conditions 1-3 in section 2.2. We just assume below that these properties hold.

In sections 2-4 we present our concept for the case of a general PDE of the second order, for which a Carleman estimate is valid. Next, we specify this concept for elliptic, parabolic and hyperbolic PDEs in sections 5, 6 and 7 respectively. In particular, we outline in section 5 applications to detection and identification of land mines, crosswell imaging and electrical impedance tomography. Finally, we present in section 8 some thoughts about numerical studies.
2 A CIP With the Restricted DN Data

2.1 The Carleman estimate

Below all functions are real valued, unless stated otherwise. The material of section 2.1 is a somewhat modified material of section 2.1.2 of [15]. Below \( x = (x_1, ..., x_n) \in \mathbb{R}^n \).

Also, below \( \alpha = (\alpha_1, ..., \alpha_n) \) is the multiindex with integer coordinates \( \alpha_i \geq 0 \) and with \( |\alpha| = \alpha_1 + ... + \alpha_n \). Consider a general Partial Differential Operator of the second order

\[
A(x, u) = \sum_{|\alpha| \leq 2} a_{\alpha}(x) D_x^\alpha u = A_0(x, u) + A_1(x, u), x \in \mathbb{R}^n, \tag{2.1}
\]

\[
A_0(x, u) = \sum_{|\alpha| = 2} a_{\alpha}(x) D_x^\alpha u, \ A_1(x, u) = \sum_{|\alpha| = 1} a_{\alpha}(x) D_x^\alpha u + a_0(x) u. \tag{2.2}
\]

Thus, \( A_0(x, u) \) is the principal part of the operator \( A(x, u) \) and the operator \( A_1(x, u) \) contains lower order terms. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with a piecewise smooth boundary. Let \( Z > 0 \) be a given number. We assume that coefficients

\[
a_{\alpha}(x) = \hat{a}_{\alpha} = \text{const. for } x \notin \Omega \text{ and for all } \alpha \text{ with } |\alpha| \leq 2, \tag{2.3}
\]

\[
a_{\alpha} \in C^1(\mathbb{R}^n) \text{ for } |\alpha| \leq 2, \tag{2.4}
\]

\[
\|a_{\alpha}\|_{C^1(\overline{\Omega})} \leq Z \text{ for } |\alpha| \leq 1. \tag{2.5}
\]

Let \( \Gamma \in C^2, \Gamma \subseteq \partial \Omega \) be a part of the boundary of the domain \( \Omega \). We assume that any part of \( \Gamma \) is not a characteristic surface of the operator \( A_0(x, u) \). Let the function \( \xi \in C^\infty(\overline{\Omega}) \) and \( |\nabla \xi| \neq 0 \) in \( \overline{\Omega} \). For a number \( d > 0 \) denote

\[
\xi_d = \{ x \in \Omega : \xi(x) = d \}, \Omega_d = \{ x \in \Omega : \xi(x) > d \}. \tag{2.6}
\]

We assume below that \( \Omega_d \neq \emptyset \) and that \( (\overline{\Omega}_d \cap \partial \Omega) = \Gamma_d \subseteq \Gamma \). Hence,

\[
\Gamma_d = \{ x \in \Gamma : \xi(x) > d \}. \tag{2.7}
\]

Hence, the boundary of the domain \( \Omega_d \) consists of two parts,

\[
\partial \Omega_d = \partial_1 \Omega_d \cup \partial_3 \Omega_d, \ \partial_1 \Omega_d = \xi_d, \partial_2 \Omega_d = \Gamma_d. \tag{2.8}
\]

We assume below that \( \partial \Omega_d \) is piecewise smooth. Below \( C_1 = C_1(A_0, \Omega_d) > 0 \) denotes different constants depending only on the operator \( A_0 \) and the domain \( \Omega \). Let \( \lambda > 1 \) be a large parameter. Consider the function \( \varphi_\lambda(x) \),

\[
\varphi_\lambda(x) = \exp(\lambda \xi(x)). \tag{2.9}
\]

It follows from (2.6)-(2.8) that

\[
\min_{\overline{\Omega}_d} \varphi_\lambda(x) = \varphi_\lambda(x) |_{\xi_d} = e^{\lambda d}, \tag{2.10}
\]

\[
m = \max_{\overline{\Omega}_d} \xi(x) \Rightarrow \max_{\overline{\Omega}_d} \varphi_\lambda(x) = e^{\lambda m}. \tag{2.11}
\]

**Definition 2.1.** We say that the operator \( A_0 \) with its coefficients \( a_{\alpha}(x) \) satisfying conditions (2.2), (2.3) admits the pointwise Carleman estimate in the domain \( \Omega_d \) with
the CWF \( \varphi_\lambda (x) \) if there exist constants \( \lambda_0 = \lambda_0 (A_0, \Omega_d) > 1 \), \( C_1 = C_1 (A_0, \Omega_d) > 0 \), depending only on listed parameters, such that the following estimates hold

\[
(A_0 u)^2 \varphi_\lambda^2 (x) \geq C_1 \lambda (\nabla u)^2 \varphi_\lambda^2 (x) + C_1 \lambda^2 u^2 \varphi_\lambda^2 (x) + \text{div} \, U, \quad (2.12)
\]

\[
|U (x)| \leq C_1 \lambda^3 \left[ (\nabla u)^2 + u^2 \right] \varphi_\lambda^2 (x), \quad \forall \lambda \geq \lambda_0, \forall x \in \Omega_d, \forall u \in C^2 (\Omega_d). \quad (2.13)
\]

\[
\forall \lambda \geq \lambda_0, \forall x \in \Omega_d, \forall u \in C^2 (\Omega_d). \quad (2.14)
\]

2.2 Statement of the problem

Denote \( \pi = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \). Below \( \pi^0 \in \mathbb{R}^{n-1} \) is a fixed point of \( \mathbb{R}^{n-1} \) and \( x_0 \in [0, 1] \) is a varying parameter. Consider an interval \( I \) of a straight line such that

\[
I = \{ x = (x_0, \pi^0) : x_0 \in (0, 1) \}, \quad (2.15)
\]

\[
I \cap \overline{\Omega} = \emptyset. \quad (2.16)
\]

Consider the following equation

\[
A (u) = -\delta \left( x_1 - x_0, \pi - \pi^0 \right), \quad x \in \mathbb{R}^n, \forall x_0 \in [0, 1], \quad (2.17)
\]

where \( u = u(x, x_0) \) is a distribution with respect to \( x \). Since we do not impose any condition at the infinity on the distribution \( u \), equation (2.17) might have many solutions or even none. Suppose that it has a solution, which we still denote as \( u(x, x_0) \). We assume that the following conditions are valid for this solution:

**Condition 1.** For each \( x_0 \in [0, 1] \) the function \( u(x, x_0) \in C^2 (\Omega) \).

**Condition 2.** For each \( x \in \overline{\Omega} \), the functions \( D^a_x u (x, x_0) \), are differentiable with respect to \( x_0 \in (0, 1) \) and functions \( \partial_{x_0} D^a_x u (x, x_0) \in C (\overline{\Omega} \times [0, 1]) \) for \( k = 0, 1; |\alpha| \leq 2 \).

**Condition 3.** \( u(x, x_0) \geq \beta = \text{const.} > 0, \forall (x, x_0) \in \overline{\Omega} \times [0, 1] \), see Remark 2.1.

**Condition 4.** The following Dirichlet and Neumann boundary conditions hold for the function \( u(x, x_0) \):

\[
u (x, x_0) \mid_{x \in \Gamma, x_0 \in [0, 1]} = g_0 (x, x_0), \quad \partial_n u(x, x_0) \mid_{x \in \Gamma, x_0 \in [0, 1]} = g_1 (x, x_0), \quad (2.18)
\]

where \( g_0 (x, x_0) \) and \( g_1 (x, x_0) \) are two given functions of \( (x, x_0) \in \Gamma \times [0, 1] \).

We call the Dirichlet and Neumann boundary data (2.18) “restricted DN data”.

**Coefficient Inverse Problem 1 (CIP1).** Suppose that for each value \( x_0 \in [0, 1] \) of the parameter \( x_0 \) there exists a distribution \( u(x, x_0) \) satisfying equation (2.17) and Conditions 1-4. Determine the coefficient \( a_0(x) \) in (2.2) from functions \( g_0 (x, x_0) \) and \( g_1 (x, x_0) \) in (2.18).

**Remark 2.1.** Thus, (2.17) and (2.18) mean that the source \( (x_0, \pi^0) \) runs along the interval \( I \). In the cases of elliptic and parabolic PDEs Condition 3 can often be established via the maximum principle [7, 8].

Sometimes it is hard to prove the validity of Conditions 1-3 in the case when the fundamental solution (2.17) of the operator \( A \) is considered. Hence, we formulate now the second CIP with restricted DN data. Let \( \varepsilon > 0 \) be a sufficiently small number. Let the functions \( f \in C^\infty (\mathbb{R}) \) and \( \chi (\pi) \in C^\infty (\mathbb{R}^{n-1}) \) be such that \( f (0) \chi (0) \neq 0 \) and also \( f (z) = 0 \) for \( |z| > \varepsilon \) as well as \( \chi (y) = 0 \) for \( y \in \{|y| > \varepsilon \} \). Let \( I_\varepsilon = \{ x \in \mathbb{R}^n : \text{dist} \, (x, I) < \varepsilon \} \), where \( \text{dist} \, (x, I) \) is the Hausdorff distance between the point \( x \) and the interval \( I \). Let \( G \subset \)}
Let $\mathbb{R}^n$ be a bounded domain with its boundary $\partial G \in C^1$ and such that $\Omega \subset G$, $\partial \Omega \cap \partial G = \emptyset$. We assume that $I_\varepsilon \subset (G \setminus \overline{\Omega})$.

We now replace (2.17) with

$$A(\tilde{u}) = f(x_1 - x_0) \chi(x - x^0), \forall x_0 \in [0, 1],$$

(2.19)

$$\tilde{u} \mid_{x \in \partial G} = 0, \forall x_0 \in [0, 1].$$

(2.20)

**Coefficient Inverse Problem 2 (CIP2).** Assume that the function $\tilde{u}(x, x_0)$ satisfies Conditions 1-4, (2.19) and (2.20). Determine the coefficient $a_0(x)$ in (2.3) from functions $g_0(x, x_0)$ and $g_1(x, x_0)$ in (2.18).

Both CIP1 and CIP2 are non overdetermined. Indeed the number $n$ of free variables in the data (2.18) coincides with the number of free variables in the unknown coefficient.

Since our method of the numerical solution of CIP2 is exactly the same as the one of CIP1, we consider below CIP1 in most cases.

### 2.3 A special orthonormal basis in $L_2(0, 1)$

We need to construct such an orthonormal basis in the space $L_2(0, 1)$ of functions depending on $x_0$ that the first derivative with respect to $x_0$ of any element of this basis is not identically zero. In addition, this derivative should be a linear combination of a finite number of elements of this basis. Neither the basis of trigonometric functions nor the basis of standard orthonormal polynomials are suitable for this goal. Therefore, we construct a new basis. Our basis is similar with Laguerre functions, which, however, form an orthonormal basis in $L_2(0, \infty)$ rather than in $L_2(0, 1)$.

For $x_0 \in (0, 1)$, consider the set of functions $\{x_0^ke^{x_0}\}_{k=0}^{\infty}$. Clearly these functions are linearly independent and form a complete set in $L_2(0, 1)$. We apply the classical Gram-Schmidt Orthonormalization procedure to this set. We start from $e^{x_0}$. Then we take $x_0e^{x_0}$, then $x_0^2e^{x_0}$, etc. As a result, we obtain an orthonormal basis in $L_2(0, 1)$, which consists of functions $\{P_m(x_0)e^{x_0}\}_{m=0}^{\infty} = \{\psi_m(x_0)\}_{m=0}^{\infty}$, where $P_m(x_0)$ is a polynomial of the degree $m$. Denote $[,]$ the scalar product in $L_2(0, 1)$. Let $Q_s(x_0)$ be an arbitrary polynomial of the degree $s \geq 0$. By the construction of functions $\psi_m(x_0)$, there exists

$$Q_s(x_0) = \sum_{j=0}^{s} b_j(Q_s) P_j(x_0).$$

(2.21)

**Remark 2.2.** In the computational practice, one can use a symbolic computations software, such as, e.g. Mathematica or Maple to figure out a reasonable number of functions $\psi_m(x_0)$.

**Theorem 2.1.** We have

$$a_{mk} = [\psi'_k, \psi_m] = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k < m. \end{cases}$$

(2.22)

Let $N > 1$ be an integer. Consider the $N \times N$ matrix $M_N = (a_{mk})_{(k,m) = (0,0)}^{(N-1,N-1)}$. Then (2.22) implies that $\det(M_N) = 1$, which means that there exists the inverse matrix $M_N^{-1}$.
Carleman Weight Functions

Proof. We have \( \psi'_k(x_0) = P_k(x_0)e^{x_0} + P'_k(x_0)e^{x_0} = \psi_k(x_0) + P'_k(x_0)e^{x_0} \). Since the degree of the polynomial \( P'_k(x_0) \) is less than \( k \), then (2.21) implies that the function \( P_k(x_0)e^{x_0} \) is a linear combination of functions \( \psi_j(x_0) \) with \( j \leq k - 1 \). Hence,

\[
\psi'_k(x_0) = \psi_k(x_0) + \sum_{j=0}^{k-1} b_{jk}\psi_j(x_0).
\]

First, let \( m = k \). Since \([\psi_j, \psi_m] = 0 \) for \( j < m \), then (2.23) implies that \([\psi'_m(x_0), \psi_m(x_0)] = 1 \). Consider now the case \( m > k \). Then we obtain similarly from (2.22) that \([\psi'_k(x_0), \psi_m(x_0)] = 0 \). Thus, (2.23) is established. \( \square \)

3 An Ill-Posed Problem for a Coupled System of Quasilinear PDEs

If we say below that a certain vector function belongs to a functional space, then this means that each component of this function belongs to this space. The norm of that vector function in that space is defined as the square root of the sum of squares of norms of its components.

It follows from Condition 3 of section 2.2 that we can consider the function \( v(x, x_0) = \ln u(x, x_0) \) for \( x \in \overline{\Omega} \). Substituting \( u = e^v \) in (2.17) for \( x \in \Omega \) and using (2.11)-(2.5), (2.15), (2.16) and (2.18), we obtain

\[
A_0(x, v) + F_1(x, \nabla v) = -a_0(x), \ x \in \Omega, x_0 \in [0, 1],
\]

(3.1)

\[
v(x, x_0) |_{x \in \Gamma, x_0 \in [0,1]} = \tilde{g}_0(x, x_0), \ \partial_n v(x, x_0) |_{x \in \Gamma, x_0 \in [0,1]} = \tilde{g}_1(x, x_0),
\]

(3.2)

where the function \( F_1 \in C^1(\mathbb{R}^{2n}) \), and it is quadratic with respect to derivatives \( \partial x v \). Denote \( \psi_{x_0}(x_0) = \partial_{x_0} v(x, x_0) \). Differentiate both sides of (3.1) with respect to \( x_0 \). Since \( \partial_{x_0} (a_0(x)) \equiv 0 \), then, using (3.2), we obtain

\[
A_0(x, v_{x_0}) + F_2(x, \nabla v, \nabla v_{x_0}) = 0, \ x \in \Omega, x_0 \in [0, 1],
\]

(3.3)

\[
v_{x_0}(x, x_0) |_{x \in \Gamma, x_0 \in [0,1]} = \partial_{x_0} \tilde{g}_0(x, x_0), \ \partial_n v_{x_0}(x, x_0) |_{x \in \Gamma, x_0 \in [0,1]} = \partial_{x_0} \tilde{g}_1(x, x_0),
\]

(3.4)

where the function \( F_2 \in C^1(\mathbb{R}^{3n}) \) is quadratic with respect to derivatives \( \partial x v, \partial x v_{x_0} \).

It follows from Conditions 1-3 of section 2.2 that, for each \( x \in \overline{\Omega} \), the function \( v(x, x_0) \) can be represented as a Fourier-like series with respect to the orthonormal basis \( \{\psi_m(x_0)\}_{m=0}^\infty \). Coefficients of this series depend on \( x \). We, however, assume that the function \( v(x, x_0) \) can be represented as a truncated series,

\[
v(x, x_0) = \sum_{k=0}^{N-1} v_k(x) \psi_k(x_0), \forall x \in \overline{\Omega}, \forall x_0 \in [0, 1],
\]

(3.5)

where coefficients \( v_k(x) \in C^2(\overline{\Omega}) \) and \( N \geq 2 \) is an integer of ones choice. Substituting (3.5) in (3.3), we obtain

\[
\sum_{k=0}^{N-1} A_0(x, v_k) \psi'_k(x_0) + F_2 \left( x, \sum_{m=0}^{N-1} \nabla v_m(x) \psi_m(x_0), \sum_{k=0}^{N-1} \nabla v_k(x) \psi'_k(x_0) \right) = 0,
\]

(3.6)
where \( x \in \Omega, x_0 \in [0, 1] \). Let the integer \( m \in [0, N - 1] \). Multiply both sides of (3.6) by the function \( \psi_m (x_0) \) and integrate with respect to \( x_0 \in (0, 1) \). We obtain

\[
\sum_{k=0}^{N-1} A_0 (x, v_k) [\psi_k (x_0), \psi_m (x_0)] = - \left[ F_2 \left( x, \sum_{k=0}^{N-1} \nabla v_k (x) \psi_k (x_0), \sum_{k=0}^{N-1} \nabla v_k (x) \psi_k (x_0) \right), \psi_m (x_0) \right],
\]

where \( x \in \Omega, m \in [0, N - 1] \). Denote

\[ V (x) = (v_0 (x), ..., v_{N-1} (x))^T, A_0 (x, V) = (A_0 (x, v_0), ..., A_0 (x, v_{N-1}))^T. \]

Also, let \( F (x, \nabla V) = (F_{2,0} (x, \nabla V), ..., F_{2,N-1} (x, \nabla V))^T \) be the vector of right hand sides of equations (3.7). Then (3.7) can be rewritten as

\[ M_N A_0 (x, V) = F (x, \nabla V), \]

where \( M_N \) is the matrix of Theorem 2.1. Applying Theorem 2.1 to (3.9) and denoting \( P (x, \nabla V) = M_N^{-1} F (x, \nabla V) \), we obtain

\[ A_0 (x, V) - P (x, \nabla V) = 0, \]

where vector functions \( p_0 (x) \) and \( p_1 (x) \) are obtained from functions \( \partial_{x_0} \tilde{g}_0 (x, x_0) \) and \( \partial_{x_0} \tilde{g}_1 (x, x_0) \) of (3.4) in an obviously similar manner, the \( N \)-Dimensional vector function \( P \in C^1 (\mathbb{R}^n) \), \( s_1 = n (N + 1) \), and each component of \( P \) is a quadratic function of the first derivatives \( \partial_{x_k} v_i (x) \), where \( k = 1, ..., n \) and \( i = 0, ..., N - 1 \).

Equalities (3.10), (3.11) form an ill-posed problem for the coupled system of quasilinear equations. A similar problem was considered in [2, 13] for the case of a single quasilinear PDE. Thus, we proceed below similarly with [2, 13]. It follows from (3.1), (3.5) and (3.8) that, given the vector function \( V (x) \), we can find the unknown coefficient \( a_0 (x) \). However, since only \( u \) and \( \nabla u \) are involved in the Carleman estimate (2.12), while the second derivatives \( u_{x_k x_j} \) are not involved, we formulate all theorems below in terms of the vector functions \( V, \nabla V \) rather than in terms of the unknown coefficient \( a_0 (x) \). At the same time, it is well known that in the case of parabolic and elliptic operators (unlike hyperbolic ones) derivatives involved in their principal parts can be incorporated in corresponding Carleman estimates, see, e.g. theorem 2.5 in [16]. Hence, the Hölder stability result of Theorem 3.1 as well as the global convergence result (Theorem 4.4 below) can be reformulated in terms of \( a_0 (x) \) in these cases. We are not doing this here for brevity.

**Theorem 3.1** (Hölder stability and uniqueness). *Suppose that there exist two vector functions \( V^{(1)}, V^{(2)} \in C^2 (\bar{\Omega}) \) satisfying equation (3.10) and with two pairs of boundary conditions (3.11), \( V^{(i)} \big|_\Gamma = p_0^{(i)} (x) \) and \( \partial_n V^{(i)} \big|_\Gamma = p_1^{(i)} (x) \), \( i = 1, 2 \). Let \( K > 0 \) be such a number that \( \| V^{(i)} \|_{C^1 (\bar{\Omega})} \leq K \). Let \( Z > 0 \) be the number defined in (2.5). Let \( \sigma \in (0, 1) \) be the level of the error in the data (3.11), i.e.

\[
\left\| p_0^{(1)} - p_0^{(2)} \right\|_{H^1 (\Gamma)} \leq \sigma, \left\| p_1^{(1)} - p_1^{(2)} \right\|_{L_2 (\Gamma)} \leq \sigma.
\]
Carleman Weight Functions

Choose a number $c > 0$ such that $\Omega_{d+c} \neq \emptyset$. Then there exists a sufficiently small constant $\sigma_0 = \sigma_0(\Omega, K, Z, \xi, m, c) \in (0, 1)$ and a constant $C_2 = C_2(\Omega, K, Z, \xi, m, c) > 0$, both depending only on listed parameters, such that for all $\sigma \in (0, \sigma_0)$ the following Hölder stability estimate is valid

$$\|V^{(1)} - V^{(2)}\|_{H^1(\Omega_{d+c})} \leq C_2 \left( 1 + \|V^{(1)} - V^{(2)}\|_{H^2(\Omega)} \right) \sigma^\rho, \rho = c/(m + c). \quad (3.13)$$

In particular, if $p_0^{(1)} = p_0^{(2)}$ and $p_1^{(1)} = p_1^{(2)}$, i.e. if $\sigma = 0$, then $V^{(1)}(x) = V^{(2)}(x)$ in $\Omega_d$, which means that uniqueness of the problem (3.10), (3.11) holds in the domain $\Omega_d$. Since $\Omega = \Omega_d \neq \emptyset$,

**Proof.** Uniqueness in the domain $\Omega_d$ follows from (3.13) immediately. In this proof, $C_2 = C_2(\Omega, K, Z, \xi, m, c) > 0$ denotes different positive constants. Consider the set of vector functions $Y = Y(K) = \left\{ V \in C^2(\overline{\Omega}) : \|V\|_{C^1(\overline{\Omega})} \leq K \right\}$. Denote $\tilde{V}(x) = V^{(1)}(x) - V^{(2)}(x)$. Then $\tilde{V}(x) = (\tilde{v}_0(x), ..., \tilde{v}_{N-1}(x))^T$. Since each component of the vector function $P(x, \nabla V)$ is a quadratic function with respect to the first derivatives $\partial_i v_i(x)$, then

$$P\left(x, \nabla V^{(1)}\right) - P\left(x, \nabla V^{(2)}\right) = \tilde{P}\left(x, \nabla V^{(1)}, \nabla V^{(2)}\right) \nabla \tilde{V}(x), \quad (3.14)$$

where the matrix $\tilde{P}\left(x, \nabla V^{(1)}, \nabla V^{(2)}\right)$ is such that

$$\max_{V^{(1)}, V^{(2)}} \left\| \tilde{P} \left(x, \nabla V^{(1)}, \nabla V^{(2)}\right) \right\|_{C(\overline{\Omega})} \leq C_2. \quad (3.15)$$

We obtain from (3.10), (3.11), (3.14) and (3.15)

$$|A_0(x, \tilde{v}_i)| \leq C_2 |\nabla \tilde{V}(x)|, \forall x \in \Omega, i = 0, ..., N - 1, \quad (3.16)$$

$$|\tilde{v}_i| \leq \tilde{p}_{0,i}(x), \partial_i \tilde{v}_i | \leq \tilde{p}_{1,i}(x), i = 0, ..., N - 1, \quad (3.17)$$

where $\tilde{p}_{k,i}(x) = (p_{k,i}^{(1)} - p_{k,i}^{(2)})(x), k = 0, 1$. Square both sides of (3.16), sum up with respect to $i = 0, ..., N - 1$, multiply by the function $\varphi_\lambda^2(x)$ defined in (2.9), integrate over the domain $\Omega_d$, and then apply (2.8), (2.12), (2.14) as well as the Gauss’ formula. Also, use (2.10), (2.11) and (3.12). Since $\|\tilde{V}\|_{L^2(\xi_d)}, \|\nabla \tilde{V}\|_{L^2(\xi_d)} \leq C_1 \|\tilde{V}\|_{H^2(\Omega)}$, then we obtain for $\lambda \geq \lambda_0$

$$C_1 \lambda \int_{\Omega_d} \left| \nabla \tilde{V}(x) \right|^2 \varphi_\lambda^2 dx + C_1 \lambda^3 \int_{\Omega_d} \left| \tilde{V}(x) \right|^2 \varphi_\lambda^2 dx \quad (3.18)$$

$$\leq C_1 \lambda^3 e^{2\lambda m} \sigma^2 + C_1 \lambda^3 e^{2\lambda d} \left\| \tilde{V} \right\|^2_{H^2(\Omega)} + C_2 \int_{\Omega_d} \left| \nabla \tilde{V}(x) \right|^2 \varphi_\lambda^2 dx.$$

Choose $\lambda_1 = \lambda_1(\Omega, K, Z, c) > \max(\lambda_0, 1)$ so large that $C_2 < C_1 \lambda_1/2$. Then we obtain from (3.13) for $\lambda \geq \lambda_1$

$$\lambda \int_{\Omega_d} \left| \nabla \tilde{V}(x) \right|^2 \varphi_\lambda^2 dx + \lambda^3 \int_{\Omega_d} \left| \tilde{V}(x) \right|^2 \varphi_\lambda^2 dx \leq C_1 \lambda^3 e^{2\lambda m} \sigma^2 + C_1 \lambda^3 e^{2\lambda d} \left\| \tilde{V} \right\|^2_{H^2(\Omega)}. \quad (3.19)$$

Since $\Omega_{d+c} \subset \Omega_d$, $\Omega_{d+c} \neq \emptyset$ and also since

$$\varphi_\lambda^2(x) > e^{2\lambda(d+c)} \text{ for } x \in \Omega_{d+c}, \quad (3.20)$$

8
we obtain from (3.19)

\[
\| \tilde{V} \|_{H^1(\Omega_{d+c})}^2 \leq C_2 e^{2\lambda_m \sigma^2} + C_2 e^{-2\lambda_c} \| \tilde{V} \|_{H^2(\Omega)}^2, \forall \lambda \geq \lambda_1.
\] (3.21)

Choose \( \lambda = \lambda(\sigma, m, c) \) such that \( e^{2\lambda_m \sigma^2} = e^{-2\lambda_c} \). Hence, \( \lambda = \ln \sigma^{-1/(m+c)} \). We assume that the number \( \sigma_0 \) is so small that \( \ln \sigma_0^{-1/(m+c)} > \lambda_1 \). Hence, by (3.21) for \( \sigma \in (0, \sigma_0) \)

\[
\| \tilde{V} \|_{H^1(\Omega_{d+c})}^2 \leq C_2 \left( 1 + \| \tilde{V} \|_{H^2(\Omega)}^2 \right) \sigma^{2\rho}, \quad \rho = c/(m + c). \quad \Box
\] (3.22)

4 Convexification

4.1 Weighted Tikhonov-like functional

Assume that there exists a vector function \( p \in C^2(\Omega) \) such that

\[
p \big|_{\Gamma} = p_0(x), \partial_n p \big|_{\Gamma} = p_1(x),
\] (4.1)

where functions \( p_0, p_1 \) are defined in (3.11). Consider the vector function \( W(x) \),

\[
W(x) = (w_0, w_1, ..., w_{N-1})^T(x) = V(x) - p(x).
\] (4.2)

Then the problem (3.10), (3.11) becomes

\[
L(x, p, W) = A_0 W - Q(x, \nabla p, \nabla W) + A_0 p = 0,
\] (4.3)

\[
W \big|_{\Gamma} = \partial_n W \big|_{\Gamma} = 0,
\] (4.4)

where the \( N \)-Dimensional vector function \( Q \in C^1(\mathbb{R}^{n_s}) \), \( s_2 = n(2N + 1) \) and each component of \( Q \) is a quadratic function with respect to first derivatives \( \partial_{x_k} w_i(x), \partial_{x_k} p_i(x) \), where \( k = 1, ..., n \) and \( i = 0, ..., N - 1 \).

Let \( s = \lceil n/2 \rceil + 2 \), where \( \lceil n/2 \rceil \) is the largest integer, which does not exceed \( n/2 \). Consider the space \( H^s(\Omega) \). By the embedding theorem \( H^s(\Omega) \subset C^1(\overline{\Omega}) \) and with a generic constant \( C > 0 \)

\[
\| f \|_{C^1(\overline{\Omega})} \leq C \| f \|_{H^s(\Omega)}, \forall f \in H^s(\Omega).
\] (4.5)

Introduce the space \( H^s_{0,\Gamma}(\Omega) \) of \( N \)-Dimensional vector functions \( W(x) \) as

\[
H^s_{0,\Gamma}(\Omega) = \{ W \in H^s(\Omega) : W \big|_{\Gamma} = \partial_n W \big|_{\Gamma} = 0 \}.
\]

Let \( R > 0 \) be an arbitrary number. Denote

\[
B(R) = \left\{ W \in H^s_{0,\Gamma}(\Omega) : \| W \|_{H^s(\Omega)} < R \right\}.
\]

As in Theorem 3.1, choose a number \( c > 0 \) such that \( \Omega_{d+c} \neq \emptyset \). Obviously \( \Omega_{d+c} \subset \Omega_d \). To solve the problem (4.3), (4.4) numerically, consider the following weighted Tikhonov-like functional with the CWF \( \varphi^2_\lambda(x) \) in it:

\[
J_{\lambda, \gamma}(W) = e^{-2\lambda(d+c)} \int_{\Omega} \| L(x, p, W) \|^2 \varphi^2_\lambda(x) \, dx + \gamma \| W \|_{H^s(\Omega)}^2,
\] (4.6)
where $\gamma > 0$ is the regularization parameter and the multiplier $e^{-2\lambda(d+c)}$ is introduced here in order to balance first and second terms in the right hand side of (4.6).

**Minimization Problem.** Minimize the functional $J_{\lambda,\gamma}(W)$ on the closed ball $B(R)$.

The second term in the right hand side of (4.6) is taken in the norm of the space $H^s(\Omega)$ in order to make sure that the iterative terms of the gradient projection method applied to the functional $J_{\lambda,\gamma}(W)$ belong to the space $C^1(\overline{\Omega})$, see (4.5).

**Theorem 4.1.** The functional $J_{\lambda,\gamma}(W)$ has the Frechét derivative $J'_{\lambda,\gamma}(W)$ at every point $W \in H^s_{0,\Gamma}(\Omega)$. This derivative satisfies the Lipschitz condition in $B(R)$, i.e. there exists a constant $\operatorname{Lip} = \operatorname{Lip}(\lambda, \gamma, Z, R) > 0$ depending only on listed parameters such that for all $\lambda, \gamma > 0$

$$
\|J'_{\lambda,\gamma}(W_1) - J'_{\lambda,\gamma}(W_2)\|_{H^s(\Omega)} \leq \operatorname{Lip}\|W_1 - W_2\|_{H^s(\Omega)}, \ \forall W_1, W_2 \in B(R).
$$

**Theorem 4.2** (global strict convexity). Choose a number $D > 0$ such that $\|p\|_{C^2(\overline{\Omega})} \leq D$. There exists a sufficiently large number $\lambda_2 = \lambda_2(\Omega, R, Z, D, d, \xi)$ such that for all $\lambda \geq \lambda_2$ and for $\gamma \in [e^{-\lambda c}, 1)$ the functional $J_{\lambda,\gamma}(W)$ is strictly convex on $B(R)$, i.e.

$$
J_{\lambda,\gamma}(W_2) - J_{\lambda,\gamma}(W_1) - J'_{\lambda,\gamma}(W_1)(W_2 - W_1) \geq C_1\|W_2 - W_1\|^2_{H^1(\Omega_{d+c})} + \frac{\gamma}{2}\|W_2 - W_1\|^2_{H^s(\Omega)}, \ \forall W_1, W_2 \in B(R).
$$

**Remark 4.1.** Since the regularization parameter $\gamma \in (e^{-\lambda c}, 1)$, then this allows values of $\gamma$ to be small. Also, the presence of the first term in the right hand side of (4.7) indicates that the stable reconstruction should be expected in the subdomain $\Omega_{d+c}$ rather than in the whole domain $\Omega$. Theorem 4.4 confirms the latter.

Let $P_{B(R)} : H^s_{0,\Gamma}(\Omega) \rightarrow B(R)$ be the projection operator of the Hilbert space $H^s_{0,\Gamma}(\Omega)$ on the closed ball $B(R)$. Let $\varsigma \in (0, 1)$ be a number which we will choose later. Let $W_0 \in B(R)$ be an arbitrary point. The gradient projection method of the minimization of the functional $J_{\lambda,\gamma}(W)$ on the set $B(R)$ is defined by the following sequence:

$$
W_n = P_{B(R)}(W_{n-1} - \varsigma J'_{\lambda,\gamma}(W_{n-1})); \ n = 1, 2, ...
$$

**Theorem 4.3.** Let $\lambda_2 = \lambda_2(\Omega, R, Z, D, c, d, \xi)$ be the number introduced in Theorem 4.2. Fix a number $\lambda \geq \lambda_2$ and let the regularization parameter $\gamma \in [e^{-\lambda c}, 1)$. Then there exists unique minimizer $W_{\min} \in B(R)$ of the functional $J_{\lambda,\gamma}(W)$ on the set $B(R)$. Furthermore, there exists a sufficiently small number $\varsigma_0 = \varsigma_0(\Omega, R, Z, D, c, d, \xi, \lambda) \in (0, 1)$ depending only on listed parameters such that for every $\varsigma \in (0, \varsigma_0)$ there exists a number $q = q(\varsigma) \in (0, 1)$ such that the sequence (4.8) converges to $W_{\min}$,

$$
\|W_n - W_{\min}\|_{H^s(\Omega)} \leq q^n\|W_0 - W_{\min}\|_{H^s(\Omega)}, \ n = 1, 2, ...
$$

Consider now the question of the convergence of the sequence (4.8) to the exact solution $W^*$ of the problem (4.3).

**Theorem 4.4.** Assume that there exists exact solution $W^* \in B(R)$ of the problem (4.3), (4.4) with the exact data $p^* \in C^2(\overline{\Omega})$. Let $p \in C^2(\overline{\Omega})$ be the noisy data. Assume that $\|p - p^*\|_{C^2(\overline{\Omega})} \leq \sigma$, where $\sigma \in (0, 1)$ is the level of the error in the data. Also, assume that the $C^2(\overline{\Omega})$-norm of the exact data $p^*$ is bounded by an a priori given constant $M^*$, i.e. $\|p^*\|_{C^2(\overline{\Omega})} \leq M^*$ (then $\|p\|_{C^2(\overline{\Omega})} \leq M^* + 1$). Let $\lambda_2 = \lambda_2(\Omega, R, Z, D, c, d, \xi)$ be the
number of Theorem 4.2. Then there exists a number \( \lambda_3 = \lambda_3(\Omega, R, Z, D, c, d, \xi, M^*) > \lambda_2 \), a sufficiently small number \( \sigma_1 = \sigma_1(\Omega, R, Z, D, c, d, \xi, M^*) \in (0, 1) \) and a number \( \theta = c/(8n) \in (0, 1) \) such that if \( \ln \sigma_1^{-29/c} > \lambda_3 \), then if for any \( \sigma \in (0, \sigma_1) \) one chooses \( \lambda = \ln \sigma^{-29/c} \) and \( \gamma = e^{-\lambda c} = \sigma^{29} \), then the following convergence estimate holds for the sequence (4.8):

\[
\|W^* - W_n\|_{H^1(\Omega_{d+c})} \leq C_4 \sigma^\theta + q^n \|W_0 - W_{\min}\|_{H^1(\Omega)}, \quad n = 1, 2, \ldots, \tag{4.9}
\]

where the number \( q = q(\varsigma) \in (0, 1) \) and \( \varsigma = (0, \varsigma_1) \), where \( \varsigma_1 = \varsigma_1(\Omega, R, Z, D, c, d, \xi, M^*) \in (0, 1) \) is a sufficiently small number. In (4.3) \( C_4 = C_4(\Omega, R, Z, D, c, d, \xi, M^*) = \text{const.} > 0 \). All numbers here depend only on listed parameters.

Theorem 4.2 is the central one among theorems 4.1-4.4. Thus, we prove Theorem 4.2 below. A similar theorem 3.2 was proven in [2] for the case of a single quasilinear PDE, as opposed to the coupled system (4.3) of quasilinear PDEs. As to the rest of theorems of this section, we omit their proofs referring the reader to proofs of similar theorems in [2] for the case of a single quasilinear PDE.

**Remark 4.2.** Theorem 4.2 means the convexification. Unlike (4.9), in the case of non-convex functionals there is no guarantee that a gradient-like method converges to the exact solution starting from an arbitrary point. Since the starting point \( W_0 \in B(R) \) of the iterative process (4.8) is an arbitrary one and since smallness restrictions on the radius \( R \) are not imposed, then convergence estimate (4.9) means the global convergence in the space \( H^1(\Omega_{d+c}) \).

**Proof of Theorem 4.2.** In this proof, \( C_3 = C_3(\Omega, R, Z, D, c, d, \xi) > 0 \) denotes different constants depending only on listed parameters. Let \( W_1, W_2 \in B(R) \) be two arbitrary functions. Denote \( W_2 - W_1 = h = (h_0(x), \ldots, h_{N-1}(x))^T \). Since each component of the vector function \( Q(x, \nabla p, \nabla W) \) in (4.3) is a quadratic function with respect to first derivatives \( \partial_{x_k} w_i(x), \partial_{x_k} p_i(x) \), we have

\[
Q(x, \nabla p, \nabla W_1 + \nabla h) = Q(x, \nabla p, \nabla W_1) + Q^{(1)}(x, \nabla p, \partial_{x_k} W_1, \nabla h) + Q^{(2)}(x, \nabla p, \nabla W_1, \nabla h). \tag{4.10}
\]

Here, each component of the vector function \( Q^{(1)} \) is linear with respect to derivatives \( \partial_{x_k} h_i \) and each component of the vector function \( Q^{(2)} \) contains only quadratic terms \( (\partial_{x_k} h_i) \cdot (\partial_{x_j} h_j) \). Hence, the following estimates hold for all \( x \in \Omega \):

\[
|Q^{(1)}(x, \nabla p, \nabla W_1, \nabla h)| \leq C_3 |\nabla h|, \quad |Q^{(2)}(x, \nabla p, \nabla W_1, \nabla h)| \leq C_3 |\nabla h|^2. \tag{4.11}
\]

By (4.3) and (4.10)

\[
[L(x, p, W_1 + h)]^2 - [L(x, p, W_1 + h)]^2 = \text{Lin}(x, p, h) + (A_0(h))^2 + 2A_0(h) \left[ Q^{(1)}(x, \nabla p, \nabla W_1, \nabla h) + Q^{(2)}(x, \nabla p, \nabla W_1, \nabla h) \right] + 
\left[ Q^{(1)}(x, \nabla p, \nabla W_1, \nabla h) + Q^{(2)}(x, \nabla p, \nabla W_1, \nabla h) \right]^2,
\]

where the functional \( \text{Lin}(x, p, h) \) depends linearly on \( h \). Combining (4.12) with the Cauchy-Schwarz inequality and as well as with (4.11), we obtain

\[
[L(x, p, W_1 + h)]^2 - [L(x, p, W_1 + h)]^2 - \text{Lin}(x, p, h) \geq \frac{1}{2} (A_0(h))^2 - C_3 (\nabla h)^2. \tag{4.13}
\]
Let \( \{,\} \) be the scalar product in the space of such real valued \( N \)-dimensional vector functions whose components belong to \( H^s (\Omega) \). Then (4.6) and (4.13) imply that

\[
J_{\lambda, \gamma} (W_1 + h) - J_{\lambda, \gamma} (W_1) - e^{-2\lambda(d+c)} \int_\Omega \text{Lin} (x, p, h) \varphi_\lambda^2 dx - 2\gamma \{ W, h \}
\]

\[
\geq \frac{1}{2} \int_\Omega (A_0 (h))^2 \varphi_\lambda^2 dx - C_3 \int_\Omega (\nabla h)^2 \varphi_\lambda^2 dx + \gamma \| h \|^{2} _{H^s(\Omega)}. \tag{4.14}
\]

It easily follows from the proof of theorem 3.1 of [2], which is a close analog of Theorem 4.1, that

\[
J'_{\lambda, \gamma} (W_1) (h) = e^{-2\lambda(d+c)} \int_\Omega \text{Lin} (x, p, h) \varphi_\lambda^2 dx + 2\gamma \{ W, h \}. \tag{4.15}
\]

Applying the Carleman estimate (2.12)-(2.14) to the right hand side of (4.14), we obtain

\[
e^{-2\lambda(d+c)} \int_\Omega (A_0 (h))^2 \varphi_\lambda^2 dx - C_3 e^{-2\lambda(d+c)} \int_\Omega (\nabla h)^2 \varphi_\lambda^2 dx + \gamma \| h \|^{2} _{H^s(\Omega)} \geq
\]

\[
e^{-2\lambda(d+c)} \int_\Omega (A_0 (h))^2 \varphi_\lambda^2 dx - C_3 e^{-2\lambda(d+c)} \int_\Omega (\nabla h)^2 \varphi_\lambda^2 dx + \gamma \| h \|^{2} _{H^s(\Omega)} \geq
\]

\[
\geq C_1 e^{-2\lambda(d+c)} \int_\Omega (\nabla h)^2 \varphi_\lambda^2 dx + C_1 e^{-2\lambda(d+c)} \int_\Omega (h^2 \varphi_\lambda^2 dx - C_3 e^{-2\lambda(d+c)} \int_\Omega (\nabla h)^2 \varphi_\lambda^2 dx
\]

\[
- C_3 e^{-2\lambda(d+c)} \int_\Omega (\nabla h)^2 \varphi_\lambda^2 dx - C_1 \lambda^2 e^{-2\lambda(c)} \int_{\xi_d} ((\nabla h)^2 + h^2) dS + \gamma \| h \|^{2} _{H^s(\Omega)}. \tag{4.16}
\]

Choose \( \lambda_2 = \lambda_2 (\Omega, R, Z, D, d, \xi) \geq \lambda_0 \) so large that \( C_1 \lambda_2 / 2 > C_3 \). Also, observe that

\[
\varphi_\lambda^2 (x) \leq e^{2\lambda d}, \forall x \in \Omega \setminus \Omega_d \text{ and } \| \nabla h \|^{2} _{L^2(\xi_d)} + \| h \|^{2} _{L^2(\xi_d)} \leq C_3 \| h \|^{2} _{H^s(\Omega)}.
\]

Hence, taking into account (3.20), we obtain from (4.16)

\[
e^{-2\lambda(d+c)} \int_\Omega (A_0 (h))^2 \varphi_\lambda^2 dx - C_3 e^{-2\lambda(d+c)} \int_\Omega (\nabla h)^2 \varphi_\lambda^2 dx + \gamma \| h \|^{2} _{H^s(\Omega)} \geq
\]

\[
C_1 \| h \|^{2} _{H^s(\Omega_d)} + ( \gamma - C_3 e^{-2\lambda(c)} ) \| h \|^{2} _{H^s(\Omega)}, \forall \lambda \geq \lambda_2.
\]

Since \( \gamma \in [e^{-\lambda c}, 1], \) then (4.14), (4.15) and (4.17) imply (4.17). \( \square \)
4.2 Numerical scheme

The numerical scheme for the above technique is as follows:

**Step 1.** Using symbolic computations for the Gram-Schmidt Orthonormalization procedure in $L^2(0,1)$, obtain functions \( \{ \psi_m(x_0) \}_{m=0}^{N-1}, x_0 \in (0,1) \) from functions \( \{ x_m e^{x_0} \}_{m=0}^{N-1} \) for a reasonable integer \( N \geq 2 \). Alternatively, if \( x_0 \in (0,\infty) \), then use Laguerre functions.

**Step 2.** Sequentially obtain problems (3.3), (3.4), then (3.10), (3.11) and then (4.3), (4.4) for the specific operator \( A \).

**Step 3.** Minimize the functional (4.6) on the set \( B(R) \) using the gradient projection method.

**Step 4.** Let \( W_{\min}(x) \) be the minimizer of the functional (4.6) on the set \( B(R) \) (Theorem 4.3). Set \( V_{\min}(x) = W_{\min}(x) + p(x) \). Next, use the first formula (3.8), then use (3.5) and finally use (3.1).

5 Elliptic Equation

The goal of sections 5-8 is to provide some specific examples of CIPs for which the above technique works. We point out that a variety of other examples are possible.

5.1 The general case

In this case conditions (2.1), (2.2) are specified as

\[
Au = \sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i x_j} + \sum_{j=1}^{n} b_j(x) u_{x_j} + a_0(x) u, \quad x \in \mathbb{R}^n, \quad (5.1)
\]

\[
A_0 u = \sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i x_j}, \quad (5.2)
\]

where \( a_{i,j}(x) = a_{j,i}(x), \forall i, j \). We assume that obvious analogs of conditions (2.3)-(2.5) are valid. Also, we assume that there exist two constants \( \mu_1, \mu_2 > 0, \mu_1 \leq \mu_2 \) such that

\[
\mu_1 |\eta|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x) \eta_i \eta_j \leq \mu_2 |\eta|^2, \quad \forall x \in \mathbb{R}^n, \forall \eta = (\eta_1, ..., \eta_n) \in \mathbb{R}^n. \quad (5.3)
\]

Let the domain \( \Omega \subset \{ x_n > 0 \} \) and \( \overline{x} = (0, ..., 0, -1) \). Let \( I \) be the interval defined in (2.15). Hence, (2.16) is valid. Choose a number \( \omega > 1 \) and assume that

\[
\Gamma = \left\{ x_n = 0, (x_1 - 1/2)^2 / \omega^2 + \sum_{k=2}^{n-1} x_k^2 < \frac{1}{4} \right\} \subset \partial \Omega. \quad (5.4)
\]

Hence, \( \{ x_1 \in (0, 1), \overline{x} = 0 \} \subset \left( \Gamma \cap \{ x_2 = ... = x_{n-1} = 0 \} \right) \), which means that the interval \( I \), over which the point source is running, is “observable” from the hypersurface \( \Gamma \), where our Dirichlet and Neumann data are given, see Condition 4 in section 2.2. This “observability” seems to be important for successful numerical studies.
Suppose that a distribution $u ( x, x_0 )$ satisfies equation (2.17) and Conditions 1-4 of section 2.2. Define the functions $\xi ( x )$ and the CWF $\varphi_\lambda ( x )$ as

$$\xi ( x ) = \left[ x_n + ( x_1 - 1/2 )^2 / \omega^2 + \sum_{k=2}^{n-1} x_k^2 + \frac{1}{4} \right]^{-\nu}, \quad \varphi_\lambda ( x ) = \exp ( \lambda \xi ( x ) ), \tag{5.5}$$

where $\nu = \nu ( \omega ) > 1$ is a parameter depending on $\omega$. Assume that $\{ \xi ( x ) > 2^\nu, x_n > 0 \} = \Omega_{2^\nu} \subseteq \Omega$. Then by (2.7) (5.3) and (5.5) $\Gamma = \Gamma_{2^\nu}$. It follows from results of §1 of Chapter 4 of [23] that a direct analog of the Carleman estimate of (2.12)-(2.14) is valid for the operator $A_0$ in (5.2) with conditions (2.4) and (5.3) and with the CWF (5.5), as long as the parameter $\nu$ is sufficiently large.

Therefore, the above construction works in this case. The unknown coefficient $a_0 ( x )$ can be Hölder-stable reconstructed numerically by the above method in the domain $\Omega_{2^\nu+c}, c > 0$, as long as $\Omega_{2^\nu+c} \neq \emptyset$. On the other hand, uniqueness is guaranteed in the entire domain $\Omega$. This can be proven similarly with the conventional proof of the uniqueness of the problem of the continuation of solutions of elliptic equations.

### 5.2 Helmholtz equation

We now specify the discussion of section 6.1 for the case of the Helmholtz equation, since it is interesting for many applications. Let $x \in \mathbb{R}^3$. Let the function $c ( x )$ be smooth in $\mathbb{R}^3$ and be such that $c ( x ) \geq 1$ in $\mathbb{R}^3$ and $c ( x ) = 1$ for $x \in \mathbb{R}^3 \setminus \Omega$. The Helmholtz equation for the function $u ( x, k )$ with the radiation condition is and with the source located at $( x_0, x_0^0 ) = ( x_0, 0, -1 ) = y ( x_0 )$ is

$$\Delta u + k^2 c ( x ) u = -\delta ( x_1 - x_0 ) \delta ( x_2, x_3 + 1 ), \tag{5.6}$$

$$\lim_{r \to \infty} [ r ( \partial_r u - i k u ) ] = 0, r = | x |, \tag{5.7}$$

where $k$ is the wavenumber. In applications to scattering of electromagnetic waves, $c ( x )$ is a spatially distributed dielectric constant. Using a comparison with the solution of the Maxwell’s equations, it was demonstrated numerically in [3] that a simplified model of propagation of electric wave field, based on (5.6), (5.7), can be used instead of the Maxwell’s equations. This conclusion was confirmed by many accurate imaging results of the author with coauthors, using experimentally measured microwave data, see [25][26] for the frequency domain data and references cited there for the time domain data.

Let $a$ and $B$ be two numbers, where $a > 1, B > 0$. We set

$$\Omega = \{ x : -a < x_1, x_2 < a, x_3 \in ( 0, B ) \}. \tag{5.8}$$

We consider two Coefficient Inverse Scattering Problems (CISPs): one with backscattering data and another one with transmitted data.

**Coefficient Inverse Scattering Problem 1** (CISP1, backscattering data). Let $\Gamma_b = \{ x : -a < x_1, x_2 < a, x_3 = 0 \}$. Determine the unknown coefficient $c ( x )$ for $x \in \Omega$, assuming that the following two functions $g_0 ( x_1, x_2, x_0 ) , g_1 ( x_1, x_2, x_0 )$ are known for a fixed value of the wavenumber $k = k_0$:

$$u ( x, k_0 ) \mid_{\Gamma_b} = g_0 ( x_1, x_2, x_0 ) , \partial_n u ( x, k_0 ) \mid_{\Gamma_b} = g_1 ( x_1, x_2, x_0 ) , \forall x_0 \in [ 0, 1 ]. \tag{5.9}$$
Coefficient Inverse Scattering Problem 2 (CISP2, transmitted data). Let \( \Gamma_{tr} = \{ x : -a < x_1, x_2 < a, x_3 = B \} \). Determine the unknown coefficient \( c(x) \) for \( x \in \Omega \), assuming that the following two functions \( g_0(x_1, x_2, x_0), g_1(x_1, x_2, x_0) \) are known for a fixed value of the wavenumber \( k = k_0 \):

\[
u(x, k_0) |_{\Gamma_{tr}} = g_0(x_1, x_2, x_0), \quad \partial_n u(x, k_0) |_{\Gamma_{tr}} = g_1(x_1, x_2, x_0), \quad \forall x_0 \in [0, 1]. \tag{5.10}\]

CISP1 has direct applications in imaging of land mines and improvised explosive devices (IEDs) \([25, 26]\). As to the CISP2, it has direct applications in crosswell imaging, see Figures 1a),b).

![Figure 1: Schematic measurement diagrams for two applications. a) Imaging of land mines and improvised explosive devices (CISP1). b) Crosswell imaging (CISP2).](image)

As to the first application, the author has many publications on this subject, in which values of dielectric constants and locations of objects mimicking land mines and IEDs were accurately imaged, including many cases of real data, see, e.g. \([25, 26]\) and references cited there. In these references a globally convergent numerical method was applied. However, a single location of the source and an interval of wavenumbers were used in these publications, unlike the current case of the restricted DN data and a single wavenumber.

Remark 5.1. The question “How do we know both functions \( g_0 \) and \( g_1 \) in (5.9), (5.10) if only the function \( u \) is usually actually measured in experiments?” is addressed in \([25, 26]\) via the so-called “data propagation procedure”, see a detailed description in \([26]\). In the case of backscattering data, the procedure “moves” the data from the plane \( \{ x_3 = -z = \text{const.} < 0 \} \), where the data are originally collected, to the surface \( \Gamma_b \subset \{ x_3 = 0 \} \), which is closer to the targets to be imaged. A similar procedure can be arranged for transmitted data in the case of CISP2.

It was established in \([20]\) that, under certain conditions imposed on the coefficient \( c(x) \), the following asymptotic expansion of the function \( u(x, x_0, k) \) takes place

\[
u(x, x_0, k) = A(x, y) e^{ik\tau(x, y)} (1 + h(x, y(x_0), k)), \quad k \to \infty, \forall x \in \Omega, \forall x_0 \in [0, 1], \tag{5.11}\]

where the function \( h(x, y(x_0), k) \) is such that \( |h(x, y(x_0), k)| \leq C_5/k, \forall x \in \Omega, \) where \( C_5 > 0 \) is a certain constant independent on \( x, k \). In (5.11) \( A(x, y(x_0)) > 0, \forall x \in \Omega, \forall x_0 \in \)
respectively. Let the function \( \sigma_k \) for reasonable values of \( k \). This indicates that the technique of the current paper can actually work for the case of Helmholtz equation. Furthermore, \( \log u (x, x_0, k) \) can be uniquely defined as:

\[
\log u (x, x_0, k) = ik \tau (x, y (x_0)) + \ln A (x, y (x_0)) + \log (1 + h (x, y (x_0), k))
\]

\[
:= ik \tau (x, y (x_0)) + \ln A (x, y (x_0)) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} h^n (x, y (x_0), k), \quad x \in \Omega.
\]

In the case of CISP1 the CWF (5.1) can be chosen. In the case of CISP2 an obvious modification of (5.5) can be chosen. We conclude, therefore, that the above technique is applicable to both CISP1 and CISP2, as long as \( k_0 \) is sufficiently large.

**Remark 5.2.** Note that even though the theory of the method of [23, 26] also works only for sufficiently large values of \( k \), successful numerical studies for real data were conducted in [23, 26] for quite reasonable values of \( k \): \( k \in [6.25, 6.7] \) in [23] and \( k \in [5.31, 5.69] \) in [26], see [25, 26] for explanations of these choices from the standpoint of Physics. This indicates that the technique of the current paper can actually work for CISP1 and CISP2 for reasonable values of \( k_0 \).

### 5.3 Electrical impedance tomography (EIT)

In this case we consider CIP2 of section 2.2. Below for any \( \alpha \in (0, 1) \) and any integer \( k \geq 0 \) let \( C^{k+\alpha} \) and \( C^{2k+\alpha, k+\alpha/2} \) be the Hölder spaces for elliptic and parabolic equations respectively. Let the function \( \sigma \in C^{2+\alpha} (\overline{G}) \), \( \sigma (x) \geq \sigma_0 = \text{const.} > 0, \forall x \in G \) and \( \sigma (x) = 1 \) in \( G \setminus \Omega \). In addition, let \( \partial G \subset C^{2+\alpha} \). The boundary value problem for EIT is

\[
\text{div} (\sigma (x) \nabla u) = -f (x_1 - x_0) \chi (\overline{G} - \overline{G}), \forall x_0 \in [0, 1],
\]

\[
u |_{x \in \partial G} = 0, \forall x_0 \in [0, 1].
\]

Introducing the well known change of variables \( v = \sqrt{\sigma} u \), we obtain

\[
\Delta v + a_0 (x) v = -f (x_1 - x_0) \chi (\overline{G} - \overline{G}), \forall x_0 \in [0, 1],
\]

\[
v |_{x \in \partial G} = 0, \forall x_0 \in [0, 1],
\]

where \( a_0 (x) = \Delta (\sqrt{\sigma (x)}) / \sqrt{\sigma (x)} \). Recall that \( f (0) \chi (0) \neq 0 \). Assume that \( f (z), \chi (\overline{G}) \geq 0, \forall z \in \mathbb{R}, \forall \overline{G} \in \mathbb{R}^{n-1} \) and that \( a_0 (x) \leq 0 \) in \( \Omega \). Then theorem 6.14 of [8] guarantees that there exists unique solution \( v \in C^{2+\alpha} (\overline{G}) \) of the problem (5.12), (5.13), for every \( x_0 \in [0, 1] \). Next, the strong maximum principle of theorem 3.5 of [8] ensures that the function \( v (x, x_0) \) satisfies Condition 3 of section 2.2. Thus, it follows from results of section 5.1 that the above technique is applicable to EIT.
6 Parabolic Equation

Let $T > 1$ be an arbitrary number. Denote $D_T^{n+1} = \mathbb{R}^n \times (0, T)$. Consider the parabolic operator in $D_T^{n+1}$

\begin{align*}
Au &= u_t - \sum_{i,j=1}^{n} a_{i,j}(x, t) u_{x_i x_j} - \sum_{j=1}^{n} b_{j}(x, t) u_{x_j} + a_0(x, t) u, \quad (6.1) \\
A_0 u &= u_t - \sum_{i,j=1}^{n} a_{i,j}(x, t) u_{x_i x_j}, \quad (6.2)
\end{align*}

where $a_0(x, t)$ is the unknown coefficient and $a_{i,j}(x, t) = a_{j,i}(x, t), \forall i, j$. We assume that all coefficients of the operator $(6.1)$ belong to $C^1(D_T^{n+1})$ and also that the obvious analog of the ellipticity condition $(5.3)$ holds.

We take the same $\overline{\mathbb{R}}^d = (0, ..., 0, -1)$ as in section 5.1 and let the domain $\Omega_1 \subset \{x_n > 0\}$. We assume that $Au = u_t - \Delta u$ for $x \in \mathbb{R}^n \setminus \Omega_1$. In the case of the parabolic equation the point source runs over $I_1 = I \times \{t = 0\}$, where the interval $I$ is defined in (2.15). Consider the fundamental solution $u(x, t, x_0)$ of the operator $A$,

\begin{align*}
Au = \delta(x_1 - x_0, \overline{x} - x_0), (x, t) \in D_T^{n+1}, \quad (6.3) \\
\frac{\partial u}{\partial \nu}(x_0) = 0, \forall x_0 \in [0, 1]. \quad (6.4)
\end{align*}

It is well known that there exists unique solution $u(x, t, x_0) \in C^{2+\alpha,1+\alpha/2}(\overline{D_T^{n+1}} \setminus I_{1\varepsilon}), \forall \varepsilon > 0, \forall x_0 \in [0, 1]$ of the problem $(6.3)$, $(6.4)$, see chapter 4 of [21]. Here $I_{1\varepsilon} \subset D_T^{n+1}$ is defined similarly with $I_\varepsilon$ in section 2.2. Furthermore, $u(x, t, x_0) > 0$ for $t > 0$, see theorem 11 in chapter 2 of [7]. Hence (see Condition 3 in section 2.2), we define $\Omega = \Omega_1 \times (\zeta, T)$, where $\zeta \in (0, (T - 1)/2)$ is a sufficiently small number.

We define the hypersurface $\Gamma$ and the function $\xi(x, t)$ similarly with (5.4) and (5.5). Choose a number $\omega > 1$ and set

\[ \Gamma_0 = \left\{ x \in \mathbb{R}^n : x_n = 0, (x_1 - 1/2)^2 / \omega^2 + \sum_{k=2}^{n-1} x_k^2 < 1/4 \right\}, \Gamma = \Gamma_0 \times (\zeta, T). \]

\[ \xi(x, t) = \left[ x_n + (x_1 - 1/2)^2 / \omega^2 + \sum_{k=2}^{n-1} x_k^2 + (t - T/2)^2 + 1/4 \right]^{-\nu}. \quad (6.5) \]

and $\varphi_\lambda(x, t) = \exp(\lambda \xi(x, t))$, where $\nu = \nu(\omega) > 1$ is a parameter depending on $\omega$. We assume that $\Gamma_0 \subset \partial \Omega_1$. Hence, $\Gamma \subset \partial \Omega$. In addition, we assume that $\{ \xi(x, t) > 2^\nu, x_n > 0 \} = \Omega_{2^\nu} \subset \Omega$. Note that

\[ \Gamma_{2^\nu} = \left\{ x_n = 0, (x_1 - 1/2)^2 / \omega^2 + \sum_{k=2}^{n-1} x_k^2 + (t - T/2)^2 < 1/4 \right\} \subset \Gamma. \]

Let the restricted DN data be given on $\Gamma$,

\[ u(x, t, x_0) = g_0(x, t, x_0), \partial_n u(x, t, x_0) = g_1(x, t, x_0), \forall (x, t, x_0) \in \Gamma \times [0, 1]. \quad (6.6) \]
Similarly with section 5.1, results of §1 of Chapter 4 of [23] imply that a direct analog of the Carleman estimate of (2.12)-(2.14) is valid for the operator $A_0$ in (6.2), as long as the parameter $\nu$ is sufficiently large. Hence, the above construction works in this case. The unknown coefficient $a_0(x, t)$ can be Hölder-stable reconstructed numerically by the above method in the domain $\Omega_{2\nu+c}$ for any $c > 0$ such that $\Omega_{2\nu+c} \neq \emptyset$. As to the uniqueness, the entire domain $\Omega_1 \times (0, T)$ works: similarly with the end of section 5.1.

7 Hyperbolic equation

In this case we consider CIP2 of section 2.2. Let the domains $\Omega_k \subset \mathbb{R}^3$ be defined as $\Omega_k = \{|x| < k\}, k = 1, 2, 3$. Let $I_\varepsilon$ be the set defined in section 2.2. We assume that

$$I_\varepsilon \subset (\Omega_3 \setminus \Omega_2). \quad (7.1)$$

Let the function $a(x, t) \in C\left(D_\varepsilon^4\right)$ be such that

$$a(x, t) \geq 0 \text{ in } D_\varepsilon^4, \ a(x, t) = 0 \text{ for } x \in \mathbb{R}^3 \setminus \Omega_2. \quad (7.2)$$

Let $f(z), z \in \mathbb{R}$ and $\chi(\overline{x}), \overline{x} \in \mathbb{R}^{n-1}$ be functions defined in section 2.2. Recall that $f(0) \chi(0) \neq 0$. We assume that

$$f(z) \geq 0, \forall z \in \mathbb{R} \text{ and } \chi(\overline{x}) \geq 0, \forall \overline{x} \in \mathbb{R}^{n-1}. \quad (7.3)$$

Consider the following Cauchy problem for the function $u(x, t, x_0)$

$$u_{tt} = \Delta u + a(x, t) u + f(x_1 - x_0) \chi(\overline{x} - \overline{t}^0), (x, t) \in D_\varepsilon^4, \quad (7.4)$$

$$u(x, 0, x_0) = u_t(x, 0, x_0) = 0, \quad (7.5)$$

where $x_0 \in [0, 1]$ is a parameter. Then the problem (1.1), (1.3) is equivalent with:

$$u(x, t, x_0) = \int_{|x-\eta|<t} \frac{f(\eta_1 - x_0) \chi(\overline{\eta} - \overline{t}^0)}{4\pi|x-\eta|} d\eta + \int_{|x-\eta|<t} \frac{(au)(\eta, t - |x-\eta|)}{4\pi|x-\eta|} d\eta. \quad (7.6)$$

One can prove (see [23] for a similar result) that the integral equation (7.6) can be rewritten as Volterra integral equation, whose solution can be represented as a series, which converges absolutely and uniformly in any subdomain $(G \times (0, T)) \subset D_\varepsilon^4$ and for any $x_0 \in [0, 1]$, where $G \subset \mathbb{R}^3$ is an arbitrary bounded domain. This series is

$$u = \sum_{n=0}^{\infty} u_n, u_0 = \int_{|x-\eta|<t} \frac{f(\eta_1 - x_0) \chi(\overline{\eta} - \overline{t}^0)}{4\pi|x-\eta|} d\eta, \quad (7.7)$$

$$u_n = \int_{|x-\eta|<t} \frac{(au_{n-1})(\eta, t - |x-\eta|)}{4\pi|x-\eta|} d\eta, n \geq 1. \quad (7.8)$$

We now prove that

$$u(x, t, x_0) \geq C_5 T, \forall x \in \Omega_1, \forall x_0 \in [0, 1], \forall t \in (T/4, T), \forall T > 20, \quad (7.9)$$

Carleman Weight Functions
where the constant $C_5 = C_5(I_e, f, \chi) > 0$ depends only on listed parameters and is independent on $T$.

Indeed, let $x \in \Omega_1$ and $\eta \in \Omega_3$ be two arbitrary points. Let $t \in (T/4, T)$ and $T > 20$. Then

$$|x - \eta| \leq |x| + |\eta| < 4 < t.$$  

(7.10)

Since by (7.3) $f(0) \chi(0) > 0$, then (7.9) follows from (7.1)-(7.3), (7.7), (7.8) and (7.10).

We now set $\Omega = \Omega_1 \times (T/4, T)$, where $T > 20$. Next, let

$\xi(x, t) = |x|^2 - q^2 (t - T/2)^2, \varphi_\lambda(x, t) = \exp(\lambda \xi(x, t))$.  

(7.11)

Choose any number $d \in (0, 1)$. Next, choose $q \in (4\sqrt{1 - d}/T, 1)$. Let $\Omega_d = \{(x,t) : x \in \Omega_1, \xi(x,t) > d\}$.

Then

$$\Omega_d \subset \Omega, \Omega_d \cap \{t = T/4\} = \Omega_d \cap \{t = T\} = \emptyset.$$  

(7.12)

Hence, we define $\Gamma$ and $\Gamma_d$ as

$$\Gamma = \{(x,t) : |x| = 1, t \in (T/4, T)\}, \Gamma_d = \{(x,t) : |x| = 1, \xi(x,t) > d\}.$$  

(7.13)

It follows from (7.11)-(7.13) that $\Gamma_d \subset \Gamma$.

Similarly with (2.18) we define the CIP in this case as the problem of determining the unknown coefficient $a(x,t) \in C(D^{\frac{3}{4}}T)$ satisfying conditions (7.2), given functions $g_0(x,t,x_0), g_1(x,t,x_0)$, where

$$u(x,t,x_0) = g_0(x,t,x_0), \partial_n u(x,t,x_0) = g_1(x,t,x_0), \forall (x,t,x_0) \in \Gamma \times [0, 1].$$

An analog of the Carleman estimate of (2.12)-(2.14) works for the operator $\partial^2_t - \Delta$ with the CWF $\varphi_\lambda(x,t)$ given in (7.11), see theorem 2.2.5 in [15]. Therefore, the above construction works for this CIP. The function $a(x,t)$ can be reconstructed numerically by the above method in $\Omega_{d+c}$ for any $c \in (0, 1 - d)$.

8 Some Numerical Considerations

We discuss in this section some practical ideas for the numerical implementation of the procedure of this paper. These ideas are generated by the numerical experience of the author in working with the convexification for a CIP with single measurement data [17] as well as for an ill-posed Cauchy problem for a quasilinear parabolic PDE [2].

First, even though the above theory is valid only for sufficiently large values of $\lambda$, in fact, $\lambda \in [1, 3]$ worked well in [2, 17]. Another observation is that it is more effective to work with such functions $\xi(x)$, which are simple and change rather slowly. However, the function $\xi(x)$ in (5.5) changes rapidly due to the presence of the parameter $\nu > 1$. On the other hand, it was heuristically established in [25, 26] that the stability of the numerical solution of an analog of CISP1 (section 5.2) can be improved if the Dirichlet and Neumann data at the backscattering side $\Gamma_b$ of $\Omega$ are complemented on the rest of $\partial \Omega$ by the Dirichlet data generated by the solution of the problem (5.6) for the case $c(x) \equiv 1$. At the same time, it was also observed in [25, 26] that this complement does influence the accuracy of the solution insignificantly.

One can prove that the CWF in the latter case can be chosen as
Carleman Weight Functions

REFERENCES

Carleman Weight Functions

\[ \varphi^{(1)}_\lambda (x) = \exp [\lambda (x_3 - B - b_1)^2], \]
where \( b_1 > 0 \) is any number. One can simplify even this choice via choosing another CWF as \( \varphi^{(2)}_\lambda (x) = \exp (-\lambda x_3) \) : this CWF works for the 1-D operator \( d^2/dx_3^2 \), see lemma 6.1 in [17]. Then, however, one needs to assume that all derivatives with respect to \( x_1 \) and \( x_2 \) are written in finite differences, unlike derivatives with respect to \( x_3 \). In this case, the parameter \( \lambda \) would depend on the grid step size. One can proceed similarly in the case of CISP2.

Assume now that the restricted DN data for the elliptic case are given on the sphere \( S = \{ |x| = 1 \} \) and that \( \Omega = \{ \kappa < |x| < 1 \} \), where \( \kappa = \text{const.} \in (0, 1) \). In addition, assume that \( A_0 = \Delta \). Writing the operator \( \Delta \) in spherical coordinates as \( \Delta_{\kappa, \varphi, \theta} \), where \( r \in (\kappa, 1), \varphi \in (0, 2\pi), \theta \in (0, \pi) \), one can prove an analog of the Carleman estimate (2.12)-(2.14) for \( (\sqrt{r}\Delta_{\kappa, \varphi, \theta} u - 2u_r/\sqrt{r})^2 e^{2\lambda r} \). However, when integrating the analog of (2.12) over \( \Omega \), one should replace the conventional \( r^2 \sin \theta dr d\varphi d\theta \) with \( \sin \theta dr d\varphi d\theta \). As to the term \( (-2u_r/\sqrt{r}) \), recall that Carleman estimates are independent on terms with derivatives, whose order is less than the order of derivatives in the principal part of a corresponding PDE operator, see page 39 in [15]. Hence, the function \( \varphi^{(3)}_\lambda (r) = e^{2\lambda r} \) might be an appropriate choice of the CWF in this case. The 2-D case is similar. It is worthy to repeat now that if the data are given rather far from the domain \( \Omega \), the data propagation procedure is recommended, see [26] for a detailed description.

Considerations about the CWF, which are similar with the ones above in this section, can be also brought in for the case of CIPs with restricted DN data for parabolic PDEs (section 6). Here is an example: Let \( n = 3 \) and let the domain \( \Omega_1 \) be the same as the domain \( \Omega \) in (5.3). Also, assume that the restricted DN data (6.6) are given at \( \partial \Omega_1 \cap \{ x_3 = 0 \} \times (\zeta, T) = \Psi \times (\zeta, T) \). In addition, assume that we have Dirichlet data at \( \partial \Omega_1 \setminus \Psi \times (\zeta, T) = \Phi \) and that in (6.2) the operator \( A_0 = \partial_x - \Delta \). In this case the CWF of section 6 can be simplified as \( \varphi^{(2)}_\lambda (x, t) = \exp [\lambda \left((x_3 - B - b_1)^2 - (t - T/2)^2\right)] \), where \( b_1 > 0 \) is any number.

In some applications the point source might run along a circle surrounding either the entire domain \( \Omega \) in the 2-D case or a 2-D cross-section of \( \Omega \) in the 3-D case. The above process can be modified as follows then: Let \( \{(r, \varphi, \theta) : r = 1, \varphi \in (0, 2\pi), \theta = \pi/2\} \) be that circle. Choose a small number \( \varepsilon \in (0, \pi) \). Next, in the process of sections 2-4, replace \( x_0 \in [0, 1] \) in (2.19) with \( \varphi_0 \in \varepsilon, 2\pi - \varepsilon \) and modify the orthonormal basis of section 2.3 accordingly.

Acknowledgments

This work was supported by US Army Research Laboratory and US Army Research Office grant W911NF-15-1-0233 and by the Office of Naval Research grant N00014-15-1-2330.

References

[1] G. S. Alberti, H. Ammari, B. Jin, J.-K. Seo and W. Zhang, The linearized inverse problem in multifrequency electrical impedance tomography, SIAM J. Imaging Sciences, 9, 1525-1551, 2016.

[2] A.B. Bakushinskii, M.V. Klibanov and N.A. Koshev, Carleman weight functions for a globally convergent numerical method for ill-posed Cauchy problems for some quasilinear PDEs, Nonlinear Analysis: Real World Applications, 34, 201-224, 2017.
[3] L. Beilina, Energy estimates and numerical verification of the stabilized domain decomposition finite element/finite difference approach for the Maxwell’s system in time domain, *Central European Journal of Mathematics*, 11, 702–733, 2013.

[4] L. Beilina and M.V. Klibanov, Globally strongly convex cost functional for a coefficient inverse problem, *Nonlinear Analysis: Real World Applications*, 22, 272-278, 2015.

[5] M.I. Belishev, Recent progress in the boundary control method, *Inverse Problems*, 23, R1-R67, 2007.

[6] M. I. Belishev, I. B. Ivanov, I. V. Kubyshkin and V. S. Semenov, Numerical testing in determination of sound speed from a part of boundary by the BC-method, *J. Inverse and Ill-Posed Problems*, 24, 159–180, 2016.

[7] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc., Englewood Cliffs, 1964.

[8] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York, 1983.

[9] B. Hararah and M.N. Minh, Enhancing residual-based techniques with shape reconstruction features in Electrical Impedance Tomography, *Inverse Problems*, 32, 125002, 2016.

[10] N. Hyvonen, P. Piironen and O. Seiskari, Point measurements for a Neumann-to-Dirichlet map and the Calderon problem in the plane, *SIAM J. Math. Analysis*, 44, 3526-3536, 2012.

[11] B. Jin, Y. Xu and J. Zou, A convergent adaptive finite element method for electrical impedance tomography, *IMA J. of Numerical Analysis*, published online, doi: 10.1093/imanum/drw045, 2016.

[12] S. I. Kabanikhin, A. D. Satybaev and M. A. Shishlenin, *Direct Methods of Solving Multidimensional Inverse Hyperbolic Problems*, VSP, Utrecht, 2004.

[13] S. I. Kabanikhin, K. K. Sabelfeld, N.S. Novikov and M. A. Shishlenin. Numerical solution of the multidimensional Gelfand-Levitan equation. *J. Inverse and Ill-Posed Problems*, 23, 439-450, 2015.

[14] M.V. Klibanov, Global convexity in a three-dimensional inverse acoustic problem, *SIAM J. Mathematical Analysis*, 28, 1371-1388, 1997.

[15] M.V. Klibanov and A. Timonov, *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications*, VSP, Utrecht, 2004.

[16] M.V. Klibanov, Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems, *J. Inverse and Ill-Posed Problems*, 21, 477-560, 2013.
REFERENCES

[17] M.V. Klibanov and N.T. Thành, Recovering of dielectric constants of explosives via a globally strictly convex cost functional, *SIAM J. Applied Mathematics*, 75, 518-537, 2015.

[18] M.V. Klibanov, Carleman weight functions for solving ill-posed Cauchy problems for quasilinear PDEs, *Inverse Problems*, 31, 125007, 2015.

[19] M.V. Klibanov and V.G. Kamburg, Globally strictly convex cost functional for an inverse parabolic problem, *Mathematical Methods in the Applied Sciences*, 39, 930-940, 2016.

[20] M.V. Klibanov and V.G. Romanov, Reconstruction procedures for two inverse scattering problems without the phase information, *SIAM J. Appl. Math.*, 76, 178-196, 2016.

[21] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, AMS, Providence, R.I, 1968.

[22] E.L. Lakshtanov, R.G. Novikov and B.R. Vainberg, A global Riemann-Hilbert problem for two-dimensional inverse scattering at fixed energy, *Rend. Istit. Mat. Univ. Trieste* 48, 21–47, 2016.

[23] M.M. Lavrentiev, V.G. Romanov and S.P. Shishatskii, *Ill-Posed Problems of Mathematical Physics and Analysis*, AMS, Providence, R.I., 1986.

[24] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, *Annals of Mathematics*, 143, 71-96, 1996.

[25] D.-L. Nguyen, M.V. Klibanov, L.H. Nguyen, A.E. Kolesov, M.A. Fiddy and H. Liu, Numerical solution of a coefficient inverse problem with multi-frequency experimental raw data by a globally convergent algorithm, *J. Computational Physics*, 345, 17-32, 2017.

[26] D.-L. Nguyen, L.H. Nguyen, M.V. Klibanov and M.A. Fiddy, Imaging of buried objects from multi-frequency experimental data using a globally convergent inversion method, *arxiv*: 1705.01219, 2017; accepted for publication in *J. Inverse and Ill-Posed Problems*.

[27] R.G. Novikov, A multidimensional inverse spectral problem for the equation $-\Delta \psi + (v(x) - Eu(x))\psi = 0$, *Funct. Anal. Appl.*, 22, 263–272, 1988.

[28] R.G. Novikov and M. Santacesaria, Monochromatic reconstruction algorithms for two dimensional multi-channel inverse problems, *International Mathematics Research Notices*, Oxford University Press, 1205-1229, 2013.

[29] J. Sylvester and G. Uhlmann, Global uniqueness theorem for an inverse boundary problem, *Annals of Mathematics*, 39, 91-112, 1987.