Cooperative parametric resonance of the spin one half system of the dense atomic gas

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The cooperative resonance for a spin one half system interacting with dc and ac magnetic field is considered. This interaction in the system collective regime can result in parametric resonance and rapid excitation of the excited spin state of the dense atomic gas. The phenomenon is studied using the density matrix approach. We discuss the implementation of this effect and possible applications of the quantum amplification by superradiant emission of radiation.

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I. INTRODUCTION

The paradigm of two-level systems in an electromagnetic field plays an extremely important role in two quite different branches of physics, quantum optics and magnetic resonance theory. Using mathematical analogy, one can transfer the idea of the effect developed in one branch to another branch and a similar effect to predict. Thus, we have an exciting opportunity and playground for the development of approaches using the atomic population in the excited state, Ωs = ℏωs/h, is the Rabi frequency of the radiation field interacting with the atomic ensemble, ρaa is the atomic population in the excited state, ρab is the atomic coherence, and Ωa is the cooperative frequency defined by

When two-level atoms are in the excited state (ρaa > 1/2), their transition to the ground state leads to the generation of a superradiant pulse [6].

If the atoms are close to the ground state (ρaa < 1/2), then Eq. (1) describes a harmonic oscillator. So far as the energy

\[ E = \left( \hbar \omega_{ab} N \rho_{aa} + \frac{|E_s|^2}{4\pi} \right) V_{cavity} \]

stored in the atoms and the laser field is conserved, the radiation oscillates at the cooperative frequency Ωa and the energy goes from the radiation to excite atoms and back.

The amplitude of this oscillation rapidly increases if the atomic population is modulated as \( \rho_{aa} = \rho_{aa}^0 + \delta \rho_{aa} \cos(\nu t) \) near the cooperative frequency \( \nu \approx \Omega_a \). In this case the equation for radiation becomes the Mathieu equation, and we have the parametric resonance leading to the growth of the oscillation field amplitude. The energy increases due to interaction of the atomic gas with the external modulation field, which leads to the population excitation and simultaneously increases the laser field.

The above simple consideration showed that the cooperative resonance is a promising tool for the development of new sources for the coherent radiation generation. However, in order to build more realistic approach in the optical range, we have to take into account the velocity distribution of atoms. The cooperative frequency is not well-defined in this case and to observe the cooperative effects, some additional conditions are required. At the same time, we can develop similar models in the RF (or microwave) range. The interaction of intense ultrashort pulses with atomic system can be studied with RF pulses as a model system [7]. We acknowledge also the ground-breaking experiments performed with RF radiation [8]. Such experiments in the RF region might furnish physical insight for the development of approaches using the cooperative resonances.
FIG. 1. Level structure of (a) spin-1/2 system, (b) hydrogen atom, and (c) Rb atom. The population is optically pumped to the state \( |d⟩ \), namely: (a) \(|-1/2⟩\), (b) \(|F = 0, M = 0⟩\), and (c) \(|F = 1, M = +1⟩\).

II. MODEL

In this section we consider a two-level system, spin-1/2, interacting with the circularly polarized magnetic field. Schematically this simple system is shown in Fig. 1a, and the basic equations are given in Appendix A.

In addition, we consider two examples of real two-level system in application to atoms, namely, hydrogen atom (see, Fig. 1b) and Rb atom (see, Fig. 1c).

One can optically pump all population to the particular state \( |d⟩ \) (marked in Fig. 1a-c by a bullet) and then apply the circular polarized RF field between two levels as shown in the same figure. The frequency of the RF transitions can be controlled by a dc magnetic field.

The experiment can be performed with RF (or, microwave) fields. Note that another possible realization of the present results involves experiments with atoms in Rydberg states, as in [9], or using a RF field resulting in magnetic dipole transitions between levels with the same \( F \) and different \( M \) in the magnetic field [8].

The spin Hamiltonian in the circular magnetic field \( B_x = B_s \cos(\nu t) \) and \( B_y = -B_s \sin(\nu t) \), is given by

\[
H = \mu_x B_x + \mu_y B_y = \frac{B_s}{2} [\mu_+ e^{-i\nu t} + \mu_- e^{i\nu t}],
\]

where \( \mu_\pm = (\mu_x \pm i \mu_y)/\sqrt{2} \) and \( \vec{\mu} = \mu_B (\vec{L} + 2 \vec{S}) \) is the magnetic moment.

Evolution of the state vector

\[
|\Psi⟩ = \sum_{FM} a_{FM} e^{-i\omega_F t} |FM⟩,
\]

is determined by the equation

\[
i\dot{a}_{FM} = \frac{B_0}{2\hbar} \sum_{F′M′} (⟨FM|μ_+|F′M′⟩ \exp[-i(\nu + \omega_{F′F} t)] + ⟨FM|μ_-|F′M′⟩ \exp[i(\nu - \omega_{F′F} t)]) a_{F′M′}.
\]

The elements of matrices \( ⟨FM|μ_±|F′M′⟩ \) and the explicit form of Eq. (6) can be found in Appendix B. To find the magnetization of the spins, we solve the set of Eqs. (6).

We can also use the following equations of motion for the density matrix \( \rho \) at the times much shorter than the relaxation times:

\[
\dot{\rho}_{aa} = 2 \Omega_s \text{Im} \rho_{ab},
\]

\[
\dot{\rho}_{ab} = i(1 - 2\rho_{aa}) \Omega_s.
\]

Now we turn our attention to the coupling of the induced magnetic polarization to the probe magnetic field \( B_s \) that is created by the resonant RF circuit (with high quality factor \( Q \)), which consists of the inductance \( L_s \) and a capacitor \( C_s \) (see, Fig. 2) and has a resonant frequency at the frequency of the atomic a-b transition, is to be excited by Rb atoms.

It is possible to consider a proof-of-principle experiment to demonstrate the mechanism of radiation generation. Using the Zeeman splitting of hyperfine magnetic sublevels, one can drive the system with a detuned RF field (see Fig. 2) the RF generator drives current \( I \)
The coil current \( I \) and inductance \( L \) is defined by the relation through the coil to create a magnetic field \( B \) generated from a resonant contour.

The configuration of the Zeeman sublevels driven by RF field \( \Omega \) created by RF generator, and the RF field \( \Omega_s \) generated in a resonant contour.

The oscillating magnetic dipole creates the electric field that in the case of \( kr \ll 1 \) is given by

\[
\vec{E} = -\frac{\mu_0 k^2}{4\pi} \vec{n} \times \vec{\mu} \frac{e^{ikr}}{r} \left( 1 + \frac{i}{kr} \right) \simeq -i \frac{\mu_0 k}{4\pi} \vec{n} \times \vec{\mu} \frac{1}{r^2},
\]

where \( \mu_0 \) is the permeability of vacuum, \( k = \omega/c \), \( \vec{\mu} \) is the oscillating magnetic moment,

\[
\vec{\mu} = \vec{\mu}_{ab} N \rho_{ab} e^{-i\omega t},
\]

\( N = NV_{\text{sample}} \) is the number of the atomic spins, which depends on the spin density \( N \) and the volume of the sample. For simplicity, let us consider the magnetic spins being in the center of a circle wire, and, then, the electromotive force generated in the electric circuit is given by

\[
V = \oint_{L_s} \vec{E} \cdot d\vec{l} = -i \frac{\mu_0 k}{4\pi} \oint_{L_s} \frac{\vec{n} \times \vec{\mu}}{r^2} \cdot d\vec{l} = -i \frac{\mu_0 k}{4\pi} \vec{n} \times \vec{\mu} \oint_{L_s} \frac{d\vec{l}}{r^2} = -i \frac{\mu_0 k}{4\pi} \vec{\mu} \cdot \vec{J},
\]

where

\[
\vec{J} = \oint_{L_s} \frac{d\vec{l} \times \vec{n}}{r^2}.
\]

The coil current \( I_s \) is then given by

\[
L_s \dot{I}_s + \frac{q_s}{C_s} = V,
\]

and

\[
\dot{I}_s + \omega_s^2 I_s = \frac{V}{L_s},
\]

where \( \omega_s = (L_s C_s)^{-1/2} \).

The current \( I_s \) and the magnetic field \( B_s \) created by the coil follow from the Biot-Savart law

\[
\vec{B}_s = \frac{\mu_0 I_s}{4\pi} \oint_{L_s} \frac{d\vec{l} \times \vec{r}}{r^3} = \frac{\mu_0 I_s}{4\pi} \vec{J}.
\]

One can write

\[
\ddot{B}_s + \omega_s^2 B_s = \frac{\mu_0}{\pi L_s} \dot{V} = \frac{\dot{V}}{2a_s L_s},
\]

or,

\[
\ddot{B}_s + \omega_s^2 B_s = -\frac{k}{L_s} \left( \frac{\mu_0}{4\pi} \right)^2 \left( \vec{\mu} \cdot \vec{J} \right) \vec{J}.
\]

Using the slowly varying amplitude approximation, \( B_s = B_s e^{-i\omega_s t} \), we obtain

\[
-2i\omega_s \dot{B}_s = -i \frac{k}{L_s} \left( \frac{\mu_0}{4\pi} \right)^2 \left( \vec{\mu} \cdot \vec{J} \right) \vec{J}.
\]

Introducing \( \Omega_s = \vec{\mu}_{ab} \cdot \vec{B}_s / \hbar \), one has

\[
\dot{\Omega}_s = -i \frac{\omega_s}{2\hbar c L_s} \left( \frac{\mu_0 \vec{\mu}_{ab} \cdot \vec{J}}{4\pi} \right)^2 \rho_{ab} = -i \Omega_s^2 \rho_{ab},
\]

where

\[
\Omega_s^2 = \frac{\omega_s}{2\hbar c L_s} \left( \frac{\mu_0 \vec{\mu}_{ab} \cdot \vec{J}}{4\pi} \right)^2 \rho_{ab} = \frac{\omega_s \mu_{ab}^2 N^2}{8\hbar c L_s a_s^2},
\]

and \( a_s \) is the radius of the coil.

### III. COOPERATIVE PARAMETRIC RESONANCE

Let us consider theoretically the interaction of strong coherent field effects on the population of two atomic spin states of a dense atomic gas interacting in a collective regime. It is an interesting way of generation of a laser field that is not based on population inversion, but rather on the cooperative interaction with the ensemble of two-level atoms. The key feature of the approach is that the laser field is generated together with the population in the excited state.

To demonstrate the cooperative parametric resonance, we write the Rabi driving frequency in the rotating wave approximation as

\[
\Omega = \Omega_0 \cos(\nu t).
\]
The obtained characteristic numbers are:

\[ A \]

Here \( \omega \) where \( \Omega \) are shown in Fig. 3.

To support the simplified calculations, we performed simulations of the set of Eq.(7) and Eq.(19). The results correspond to practically total population inversion.

The excitation occurs at the cooperative resonance \( (\nu \approx \Omega_a) \). As can be seen from Fig. 3, using a weak driving field \( \Omega_0 = 0.01\Omega_a \), it is possible to excite the spin magnetization at almost the same the level \( (\Omega_a \approx 0.4 \Omega_a) \) as it would occured with superradiant generation corresponding to practically total population inversion.

This results can be observed in the experimental setup similar to the one described in [11]. Using a longitudinal magnetic field \( B = 500 \) G, the splitting is \( \omega_{ab} \approx 318.3 \) MHz. Then, for the atomic density \( N = 10^{13} \text{ cm}^{-3} \), the cooperative frequency is \( \Omega_a = 10^3 \text{ s}^{-1} \), which is much larger than the ground state relaxation \( \gamma_0 = 3 \cdot 10^3 \text{ s}^{-1} \).
The obtained above results can probably be applied to the nuclear spin systems (see, e.g., Ref. [12]). It provides a new approach to the detection of nuclear magnetic resonance. Usually, NMR is detected by the measuring of additional losses in the coils at the nuclear. Meanwhile, the current approach allows one to induce the magnetic polarization the population difference is given by $\Delta N_{ab} = N_0 \Delta E/k_BT$ where the density is $N_0 \approx 10^{23} \text{ cm}^{-3}$ the cooperative frequency is $\Omega_a = 10^4 \text{ s}^{-1}$. Then, modulation at the frequency $\Delta = \Omega_a$ leads to the strong magnetization of the nuclear transition.

We should mention another interesting opportunity to find analogies (and therefore enrich our approach) using magnetic dynamics theory. The point is that the theory of parametric resonance of magnons, quanta of spin waves in magnetoordered systems is well-developed [13]. So far as the roles of nonlinear medium and resonator cavity are theoretically well clarified in this branch of physics, a similar structure of dynamic equations gives useful hints how to update our simple model introducing new important parameters.

IV. DISCUSSION

In the paper, we study the new way of generation of coherent field based on the cooperative resonance in the system of spins which we consider as a good way to implement a proof-of-principle experimental realization as well as probably new way of nuclear magnetic resonance detection.

This new approach is not based on the population inversion which is required for lasing, as it is well-known to implement lasing, population inversion usually is needed to overcome stimulated absorption [4]. Also the technique is not related to the concept of lasing without population inversion (LWI) [5] that appears as a result of coherent effects [5][13][18] in atomic or molecular media [19–21]. The LWI was demonstrated experimentally [22, 23] and it was even shown that lasing can exist without any inversion in any reservoir, even under thermodynamical equilibrium [24].

The physics of new way of generation is closely related to the cooperative response of quantum ensemble that are all practically in the ground state. Because of collective motion of the spin excitation at the cooperative frequency, the system undergoes the process of coherent excitation together with generation of coherent field. The physics is closely related to the so-called Dicke superradiance [5] which is usually related to the spontaneous emission in the excited medium. Usually, spontaneous emission is an incoherent process that leads to relaxation of excitation in media. But in sufficiently dense media, the spontaneous emission becomes a collective process as was shown by Dicke [4]. The cooperative behavior of $N$ atoms can speed up the relaxation processes, leading to a burst of radiation with an intensity proportional to the square of the number of atoms $N^2$. Recently, in a single photon superradiance [25][27], the collective Lamb shift was predicted and observed [28][29].

In conclusion, we have studied and demonstrated several cases when the resonance with the cooperative frequency creates the possibility to generate coherent radiation. In particular, we consider a gas of two-level atoms in the presence of either a CW or a pulsed coherent field that leads to substantial enhancement of the generated radiation under the condition that uses the cooperative frequency. A number of important applications of the generation of coherent radiation based on cooperative resonance phenomena that may lead to a significant progress in fundamental and applied physics.

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The magnetic field has a dc longitudinal component $B_z$ and the time-dependent transverse components $B_x$ and $B_y$.

The Hamiltonian for a spin-1/2 system in a magnetic field is defined by

$$\mathcal{H} = -\vec{\mu} \cdot \vec{B}, \quad (A1)$$

where

$$\vec{\mu} = \mu_B(\sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}). \quad (A2)$$

Here the basis of two states, spin-up $|+\rangle$ and spin-down $|−\rangle$, is described by Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A3)$$

The magnetic field has a dc longitudinal component $B_z$ and the time-dependent transverse components

$$\vec{B} = B_x \hat{x} + B_y \cos(\nu t) \hat{x} + \sin(\nu t) \hat{y}. \quad (A4)$$

The longitudinal magnetic field causes splitting of the spin-up and spindown states as

$$\mathcal{H}_0 = \mu_B B_z \sigma_z = \mu_0 B_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A5)$$

The frequency, that corresponds to the transition between levels is defined by

$$\omega_c = 2\mu_B B_z / \hbar. \quad (A6)$$

The transverse magnetic field causes coupling between these states

$$\mathcal{V} = \mu_B B_x e^{i\frac{\mu_0}{\hbar}[\sigma_x \cos(\nu t)]} + \sigma_y \sin(\nu t) e^{-i\frac{\mu_0}{\hbar}}. \quad (A7)$$

Interaction Hamiltonian is given by

$$\mathcal{V} = -\mu_B B_x e^{i\frac{\mu_0}{\hbar}} \begin{pmatrix} 0 & e^{-i\nu t} \\ e^{i\nu t} & 0 \end{pmatrix} e^{-i\frac{\mu_0}{\hbar}}. \quad (A8)$$

We see that the matrix elements,

$$\langle +|\mathcal{V}|−\rangle = -\mu_B B_x e^{-i(\nu-\omega_c)t}, \quad (A9)$$

$$\langle −|\mathcal{V}|+\rangle = -\mu_B B_x e^{i(\nu-\omega_c)t}, \quad (A10)$$

coincide with rotating wave approximation for the left-circularly polarized magnetic field. For the right-circularly polarized magnetic field the matrix elements are given by

$$\langle +|\mathcal{V}_{rcp}|−\rangle = -\mu_B B_x e^{-i(\nu+\omega_c)t}, \quad (A11)$$

$$\langle −|\mathcal{V}_{rcp}|+\rangle = -\mu_B B_x e^{i(\nu+\omega_c)t}. \quad (A12)$$

Note that either the right-circularly or left-circularly polarized magnetic field is able to couple these two states.

For the linearly polarized transverse magnetic field $\mathcal{B} = B_y \cos(\nu t)$,

$$\mathcal{B} = B_y \cos(\nu t), \quad (A13)$$

the matrix elements have the following form:

$$\langle +|\mathcal{V}|−\rangle = -\mu_B B_y [e^{-i(\nu-\omega_c)t} + e^{i(\nu+\omega_c)t}], \quad (A14)$$

$$\langle −|\mathcal{V}|+\rangle = -\mu_B B_y [e^{i(\nu-\omega_c)t} + e^{-i(\nu+\omega_c)t}]. \quad (A15)$$

**Appendix B: Matrix elements of $\langle FM|\mu_\pm|F'M'\rangle$**

For convenience to work with two-level H atom and Rb atom, here we derive the complete sets of equations.

1. **H atom**

The $^1\text{H}$ atom has a ground state split by a hyperfine interaction of electron and nuclear spins $(L = 0, S = 1/2, I = 1/2)$ in two levels with different total angular moment $F = L + S + I$, which has two values 0 and 1 as shown in Fig. 1b. The states are given by the elements of matrices $\langle FM|\mu_\pm|F'M'\rangle$, which can be calculated using
\[ |F = 1, M = 1\rangle = |S_{m_s} = -1/2\rangle |I_m_i = -1/2\rangle, \quad |F = 1, M = -1\rangle = |S_{m_s} = -1/2\rangle |I_m_i = -1/2\rangle, \]  
\[ |F = 1, M = 0\rangle = \frac{1}{\sqrt{2}} |S_{m_s} = -1/2\rangle |I_m_i = 1/2\rangle + \frac{1}{\sqrt{2}} |S_{m_s} = 1/2\rangle |I_m_i = -1/2\rangle, \]  
\[ |F = 0, M = 0\rangle = \frac{1}{\sqrt{2}} |S_{m_s} = -1/2\rangle |I_m_i = 1/2\rangle - \frac{1}{\sqrt{2}} |S_{m_s} = 1/2\rangle |I_m_i = -1/2\rangle. \]

The nuclear magnetic moment can be neglected, because it is much smaller than the electron magnetic moment. We calculate the elements of matrices as follows

\[ \langle S, 1/2 | \mu_+ | S, -1/2 \rangle = \langle S, 1/2 | \mu_- | S, -1/2 \rangle = \mu_S, \]  
where \( \mu_S = g_S \mu_B \), \( g_s = 2 \), and \( \mu_B \) is the Bohr’s magneton. For example,

\[ \langle F = 1, M = 1 | \mu_+ | F = 1, M = 0 \rangle = \langle S, 1/2 | I, 1/2 | \mu_+ \left( \frac{1}{\sqrt{2}} | S, -1/2 \rangle | I, 1/2 \rangle + \frac{1}{\sqrt{2}} | S, 1/2 \rangle | I, -1/2 \rangle \right) = \frac{\mu_S}{\sqrt{2}}. \]

Other elements can be calculated similarly:

\[ -\langle F = 0, M = 0 | \mu_+ | F = 1, M = -1 \rangle = \langle F = 0, M = 0 | \mu_+ | F = 1, M = 1 \rangle = \mu_S, \]  
\[ \langle F = 1, M = 0 | \mu_+ | F = 1, M = -1 \rangle = \langle F = 1, M = 0 | \mu_+ | F = 1, M = 1 \rangle = \mu_S. \]

Then, Eqs. (6) can be explicitly written as

\[ \frac{d}{dt} \begin{pmatrix} a_{1^{-1}} \\ a_{10} \\ a_{11} \\ a_{00} \end{pmatrix} = \Omega_0 \tilde{L} \begin{pmatrix} a_{1^{-1}} \\ a_{10} \\ a_{11} \\ a_{00} \end{pmatrix}, \]  
where \( \Omega_0 = \mu_B B_s / \hbar \) and \( \tilde{L} \) is the following matrix

\[ \begin{pmatrix} 0 & \Omega e^{-i\omega t} & 0 & \Omega e^{-i(\nu + \omega) t} \\ \Omega e^{-i\nu t} & 0 & \Omega e^{i\nu t} & 0 \\ 0 & \Omega e^{-i\omega t} & 0 & \Omega e^{i(\nu - \omega) t} \\ \Omega e^{i(\nu + \omega) t} & 0 & \Omega e^{-i(\nu - \omega) t} & 0 \end{pmatrix}. \]

This set of equations has been solved numerically.

### 2. Rb atom

The \(^{87}\text{Rb}\) atom has a ground state split by a hyperfine interaction of electron and nuclear spins (\( L = 0, \))

\[ |F = 1, M = -1\rangle = -\frac{1}{2} |S_{m_s} = -1/2\rangle |I_m_i = -1/2\rangle + \frac{\sqrt{3}}{2} |S_{m_s} = 1/2\rangle |I_m_i = -3/2\rangle, \]  
\[ |F = 1, M = 0\rangle = -\frac{1}{\sqrt{2}} |S_{m_s} = -1/2\rangle |I_m_i = 1/2\rangle + \frac{1}{\sqrt{2}} |S_{m_s} = 1/2\rangle |I_m_i = -1/2\rangle, \]  
\[ |F = 1, M = 1\rangle = -\frac{\sqrt{3}}{2} |S_{m_s} = -1/2\rangle |I_m_i = 3/2\rangle + \frac{1}{2} |S_{m_s} = 1/2\rangle |I_m_i = 1/2\rangle, \]  
\[ |F = 2, M = -2\rangle = |S_{m_s} = -1/2\rangle |I_m_i = -3/2\rangle, \]  
\[ |F = 2, M = -1\rangle = \frac{\sqrt{3}}{2} |S_{m_s} = -1/2\rangle |I_m_i = -1/2\rangle + \frac{1}{2} |S_{m_s} = 1/2\rangle |I_m_i = -3/2\rangle, \]  
\[ |F = 2, M = 0\rangle = \frac{1}{\sqrt{2}} |S_{m_s} = -1/2\rangle |I_m_i = 1/2\rangle + \frac{1}{\sqrt{2}} |S_{m_s} = 1/2\rangle |I_m_i = -1/2\rangle, \]  
\[ |F = 2, M = 1\rangle = \frac{1}{2} |S_{m_s} = -1/2\rangle |I_m_i = 3/2\rangle + \frac{\sqrt{3}}{2} |S_{m_s} = 1/2\rangle |I_m_i = 1/2\rangle, \]  
\[ |F = 2, M = 2\rangle = |S_{m_s} = 1/2\rangle |I_m_i = 3/2\rangle. \]
The nuclear magnetic moment can be neglected, because it is much smaller than the electron magnetic moment. We calculate all elements of matrixes as follows

\[ \langle S, 1/2 | \mu_+ | S, -1/2 \rangle = \langle S, 1/2 | \mu_- | S, -1/2 \rangle = \mu_S, \]  

(B17)

where \( \mu_S = g_S \mu_B \), \( g_s = 2 \), and \( \mu_B \) is the Bohr’s magneton. For example,

\[ \langle F = 2, M = 2 | \mu_+ | F = 1, M = 1 \rangle = \langle S, 1/2 | (I, 3/2 | \mu_+ \left( -\frac{\sqrt{3}}{2} | S, -1/2 \rangle | I, 3/2 \rangle + \frac{1}{2} | S, 1/2 \rangle | I, 1/2 \rangle \right) \rangle = -\frac{\sqrt{3}}{2} \mu_S. \]  

(B18)

Other elements can be calculated similarly

\[ \langle F = 2, M = 2 | \mu_+ | F = 2, M = 1 \rangle = \frac{\sqrt{3}}{2} \mu_S, \langle F = 2, M = 2 | \mu_- | F = 1, M = 1 \rangle = -\frac{\sqrt{3}}{2} \mu_S, \]  

(B19)

\[ \langle F = 2, M = 1 | \mu_+ | F = 2, M = 0 \rangle = \frac{\sqrt{6}}{4} \mu_S, \langle F = 2, M = 1 | \mu_- | F = 1, M = 0 \rangle = -\frac{\sqrt{6}}{4} \mu_S, \]  

(B20)

\[ \langle F = 2, M = 0 | \mu_+ | F = 2, M = -1 \rangle = \frac{\sqrt{6}}{4} \mu_S, \langle F = 2, M = 0 | \mu_- | F = 1, M = -1 \rangle = -\frac{\sqrt{6}}{4} \mu_S, \]  

(B21)

\[ \langle F = 2, M = -1 | \mu_+ | F = 2, M = -2 \rangle = \frac{\sqrt{2}}{4} \mu_S, \langle F = 1, M = 1 | \mu_+ | F = 1, M = 0 \rangle = -\frac{\sqrt{2}}{4} \mu_S, \]  

(B22)

\[ \langle F = 1, M = 0 | \mu_+ | F = 2, M = -1 \rangle = -\frac{\sqrt{2}}{4} \mu_S, \langle F = 1, M = 1 | \mu_+ | F = 2, M = 0 \rangle = \frac{\sqrt{2}}{4} \mu_S, \]  

(B23)

\[ \langle F = 1, M = 0 | \mu_+ | F = 2, M = -2 \rangle = -\frac{\sqrt{3}}{2} \mu_S. \]  

(B24)

Then, Eqs. (6) can be explicitly written as

\[
\frac{d}{dt} \begin{pmatrix}
a_{1-1} \\
a_{10} \\
a_{11} \\
a_{2-2} \\
a_{2-1} \\
a_{20} \\
a_{21} \\
a_{22}
\end{pmatrix} = \frac{\mu_B}{\hbar} \hat{L} \begin{pmatrix}
a_{1-1} \\
a_{10} \\
a_{11} \\
a_{2-2} \\
a_{2-1} \\
a_{20} \\
a_{21} \\
a_{22}
\end{pmatrix},
\]

(B25)

where \( \Omega_0 = \mu_B B_s / \hbar \) and \( \hat{L} \) is the following matrix

\[
\begin{pmatrix}
0 & -\frac{\sqrt{3}}{4} e^{i \nu t} & 0 & \frac{\sqrt{3}}{4} e^{-i (\nu + \omega_e) t} & 0 & -\frac{\sqrt{3}}{4} e^{i (\nu - \omega_e) t} & 0 & 0 \\
-\frac{\sqrt{3}}{4} e^{-i \nu t} & 0 & -\frac{\sqrt{3}}{4} e^{i \nu t} & 0 & \frac{\sqrt{3}}{4} e^{-i (\nu + \omega_e) t} & 0 & -\frac{\sqrt{3}}{4} e^{i (\nu - \omega_e) t} & 0 \\
\frac{\sqrt{3}}{2} e^{i (\nu + \omega_e) t} & 0 & 0 & 0 & \frac{\sqrt{3}}{2} e^{i (\nu + \omega_e) t} & 0 & \frac{\sqrt{3}}{2} e^{-i (\nu - \omega_e) t} & 0 \\
0 & \frac{\sqrt{3}}{4} e^{i (\nu + \omega_e) t} & 0 & \frac{\sqrt{3}}{4} e^{i (\nu + \omega_e) t} & 0 & \frac{\sqrt{3}}{4} e^{-i (\nu - \omega_e) t} & 0 & \frac{\sqrt{3}}{4} e^{i (\nu - \omega_e) t} \\
-\frac{\sqrt{3}}{2} e^{i (\nu + \omega_e) t} & 0 & \frac{\sqrt{3}}{4} e^{-i (\nu - \omega_e) t} & 0 & 0 & \frac{\sqrt{3}}{4} e^{-i (\nu + \omega_e) t} & 0 & \frac{\sqrt{3}}{4} e^{i (\nu + \omega_e) t} \\
0 & -\frac{\sqrt{3}}{4} e^{-i (\nu - \omega_e) t} & 0 & -\frac{\sqrt{3}}{4} e^{-i (\nu - \omega_e) t} & 0 & -\frac{\sqrt{3}}{4} e^{i (\nu + \omega_e) t} & 0 & -\frac{\sqrt{3}}{4} e^{i (\nu + \omega_e) t} \\
0 & 0 & -\frac{\sqrt{3}}{4} e^{-i (\nu - \omega_e) t} & 0 & 0 & \frac{\sqrt{3}}{4} e^{i (\nu + \omega_e) t} & 0 & \frac{\sqrt{3}}{4} e^{i (\nu - \omega_e) t} \\
0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{4} e^{i \nu t} & 0 & \frac{\sqrt{3}}{4} e^{i \nu t}
\end{pmatrix}
\]