ON PSEUDO-HERMITIAN MAGNETIC CURVES IN SASAKIAN MANIFOLDS

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Abstract. We define pseudo-Hermitian magnetic curves in Sasakian manifolds endowed with the Tanaka-Webster connection. After we have given a complete classification theorem, we shall construct parametrizations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$.

Keywords: magnetic curve; slant curve; Sasakian manifold; the Tanaka-Webster connection.

1. Introduction

The study of the motion of a charged particle in a constant and time-independent static magnetic field on a Riemannian surface is known as the Landau–Hall problem [16]. The main problem is to study the movement of a charged particle moving in the Euclidean plane $\mathbb{E}^2$. The solution of the Lorentz equation (called also the Newton equation) corresponds to the motion of the particle. The trajectory of a charged particle moving on a Riemannian manifold under the action of the magnetic field is a very interesting problem from a geometric point of view [16].

Let $(N, g)$ be a Riemannian manifold, and $F$ a closed 2-form, $\Phi$ the Lorentz force, which is a $(1, 1)$-type tensor field on $N$. $F$ is called a magnetic field if it is associated to $\Phi$ by the relation

$$F(X, Y) = g(\Phi X, Y),$$

where $X$ and $Y$ are vector fields on $N$ (see [1], [3] and [8]). Let $\nabla$ be the Riemannian connection on $N$ and consider a differentiable curve $\alpha : I \rightarrow N$, where $I$ denotes an open interval of $\mathbb{R}$. $\alpha$ is said to be a magnetic curve for the magnetic field $F$, if it is a solution of the Lorentz equation given by

$$\nabla_{\alpha'(t)}\alpha'(t) = \Phi(\alpha'(t)).$$

Received February 28, 2020; accepted April 04, 2020
2020 Mathematics Subject Classification. Primary 53C25; Secondary 53C40, 53A04

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From the definition of magnetic curves, it is straightforward to see that their speed is constant. Specifically, unit-speed magnetic curves are called normal magnetic curves [9].

In [9], Drut˘ă-Romaniuc, Inoguchi, Munteanu and Nistor studied magnetic curves in a Sasakian manifold. Magnetic curves in cosymplectic manifolds were studied in [10] by the same authors. In [13], 3-dimensional Berger spheres and their magnetic curves were considered by Inoguchi and Munteanu. Magnetic trajectories of an almost contact metric manifold were studied in [14], by Jleli, Munteanu and Nistor. The classification of all uniform magnetic trajectories of a charged particle moving on a surface under the action of a uniform magnetic field was obtained in [19], by Munteanu. Furthermore, normal magnetic curves in para-Kaehler manifolds were researched in [15], by Jleli and Munteanu. In [17], Munteanu and Nistor obtained the complete classification of unit-speed Killing magnetic curves in \( S^2 \times \mathbb{R} \). Moreover, in [18], they studied magnetic curves on \( S^{2n+1} \). 3-dimensional normal para-contact metric manifolds and their magnetic curves of a Killing vector field were investigated in [5], by Calvaruso, Munteanu and Perrone. In [20], the present authors studied slant curves in contact Riemannian 3-manifolds with pseudo-Hermitian proper mean curvature vector field and pseudo-Hermitian harmonic mean curvature vector field for the Tanaka-Webster connection in the tangent and normal bundles, respectively. The second author gave the parametric equations of all normal magnetic curves in the 3-dimensional Heisenberg group in [21]. Recently, the present authors have also considered slant magnetic curves in S-manifolds in [11].

These studies motivate us to investigate pseudo-Hermitian magnetic curves in (2n + 1)-dimensional Sasakian manifolds endowed with the Tanaka-Webster connection. In Section 2, we summarize the fundamental definitions and properties of Sasakian manifolds and the unique connection, namely the Tanaka-Webster connection. We give the main classification theorems for pseudo-Hermitian magnetic curves in Section 3. We show that a pseudo-Hermitian magnetic curve cannot have osculating order greater than 3. In the last section, after a brief information on \( \mathbb{R}^{2n+1}(-3) \), we obtain the parametric equations of pseudo-Hermitian magnetic curves in \( \mathbb{R}^{2n+1}(-3) \) endowed with the Tanaka-Webster connection.

2. Preliminaries

Let \( N \) be a \((2n + 1)\)-dimensional Riemannian manifold satisfying the following equations

\[
\begin{align*}
\phi^2(X) &= -X + \eta(X)\xi, & \eta(\xi) &= 1, & \phi(\xi) &= 0, & \eta \circ \phi &= 0, \\
g(X, \xi) &= \eta(X), & g(X, Y) &= g(\phi X, \phi Y) + \eta(X)\eta(Y),
\end{align*}
\]

for all vector fields \( X, Y \) on \( N \), where \( \phi \) is a \((1, 1)\)-type tensor field, \( \eta \) is a 1-form, \( \xi \) is a vector field and \( g \) is a Riemannian metric on \( N \). In this case, \((N, \phi, \xi, \eta, g)\) is said to be an almost contact metric manifold [2]. Moreover, if \( d\eta(X, Y) = \Phi(X, Y) \),
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where $\Phi(X, Y) = g(X, \phi Y)$ is the fundamental 2-form of the manifold, then $N$ is said to be a contact metric manifold [2].

Furthermore, if we denote the Nijenhuis torsion of $\phi$ by $[\phi, \phi]$, for all $X, Y \in \chi(N)$, the condition given by

$$[\phi, \phi](X, Y) = -2\phi g(X, Y)$$

is called the normality condition of the almost contact metric structure. An almost contact metric manifold turns into a Sasakian manifold if the normality condition is satisfied [2].

From Lie differentiation operator in the characteristic direction $\xi$, the operator $h$ is defined by

$$h = \frac{1}{2} L_\xi \phi.$$ 

It is directly found that the structural operator $h$ is symmetric. It also validates the equations below, where we denote the Levi-Civita connection by $\nabla$:

$$h\xi = 0, \quad h\phi = -\phi h, \quad \nabla \xi = -\phi X - \phi hX,$$

(see [2]).

If we denote the Tanaka-Webster connection on $N$ by $\tilde{\nabla}$ ([22], [24]), then we have

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)(\phi X + \phi hX) - g(\phi X + \phi hX, Y)\xi,$$

(2.4) for all vector fields $X, Y$ on $N$. By the use of equations (2.3), the Tanaka-Webster connection can be calculated as

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)(\phi X + \phi hX) - g(\phi X + \phi hX, Y)\xi.$$

The torsion of the Tanaka-Webster connection is

$$\tilde{T}(X, Y) = 2g(X, \phi Y)\xi + \eta(Y)\phi hX - \eta(X)\phi hY.$$

(2.5)

In a Sasakian manifold, from the fact that $h = 0$ (see [2]), the equations (2.4) and (2.5) can be rewritten as:

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)\phi X - g(\phi X, Y)\xi,$$

$$\tilde{T}(X, Y) = 2g(X, \phi Y)\xi.$$ 

(2.6)

The following proposition states why the Tanaka-Webster connection is unique:

**Proposition 2.1.** [23] The Tanaka-Webster connection on a contact Riemannian manifold $N = (N, \phi, \xi, \eta, g)$ is the unique linear connection satisfying the following four conditions:

(a) $\tilde{\nabla}\eta = 0$, $\tilde{\nabla}\xi = 0$;

(b) $\tilde{\nabla}g = 0$, $\tilde{\nabla}\phi = 0$;

(c) $\tilde{T}(X, Y) = -\eta([X, Y])\xi$, $\forall X, Y \in \mathcal{D}$;

(d) $\tilde{T}(\xi, \phi Y) = -\phi \tilde{T}(\xi, Y)$, $\forall Y \in \mathcal{D}$. 


3. Magnetic Curves with respect to the Tanaka-Webster Connection

Let \((N, \phi, \xi, \eta, g)\) be an \(n\)-dimensional Riemannian manifold and \(\alpha : I \to N\) a curve parametrized by arc-length. If there exists \(g\)-orthonormal vector fields \(E_1, E_2, \ldots, E_r\) along \(\alpha\) such that

\[
\begin{align*}
E_1 &= \alpha', \\
\hat{\nabla}_{E_1} E_1 &= \hat{k}_1 E_2, \\
\hat{\nabla}_{E_1} E_2 &= -\hat{k}_1 E_1 + \hat{k}_2 E_3, \\
\vdots & \\
\hat{\nabla}_{E_1} E_r &= -\hat{k}_{r-1} E_{r-1},
\end{align*}
\]

(3.1)

then \(\alpha\) is called a Frenet curve for \(\hat{\nabla}\) of osculating order \(r\), \((1 \leq r \leq n)\). Here \(\hat{k}_1, \ldots, \hat{k}_{r-1}\) are called pseudo-Hermitian curvature functions of \(\alpha\) and these functions are positive valued on \(I\). A geodesic for \(\hat{\nabla}\) (or pseudo-Hermitian geodesic) is a Frenet curve of osculating order 1 for \(\hat{\nabla}\). If \(r = 2\) and \(\hat{k}_1\) is a constant, then \(\alpha\) is called a pseudo-Hermitian circle. A pseudo-Hermitian helix of order \(r\) \((r \geq 3)\) is a Frenet curve for \(\hat{\nabla}\) of osculating order \(r\) with non-zero positive constant pseudo-Hermitian curvatures \(\hat{k}_1, \ldots, \hat{k}_{r-1}\). If we shortly state pseudo-Hermitian helix, we mean its osculating order is 3 [7].

Let \(N = (N^{2n+1}, \phi, \xi, \eta, g)\) be a Sasakian manifold endowed with the Tanaka-Webster connection \(\hat{\nabla}\). Let us denote the fundamental 2-form of \(N\) by \(\Omega\). Then, we have

\[
\Omega(X, Y) = g(X, \phi Y),
\]

(3.2)

(see [2]). From the fact that \(N\) is a Sasakian manifold, we have \(\Omega = d\eta\). Hence, \(d\Omega = 0\), i.e., it is closed. Thus, we can define a magnetic field \(F_q\) on \(N\) by

\[
F_q(X, Y) = q\Omega(X, Y),
\]

namely the contact magnetic field with strength \(q\), where \(X, Y \in \chi(N)\) and \(q \in \mathbb{R}\) [14]. We will assume that \(q \neq 0\) to avoid the absence of the strength of magnetic field (see [4] and [9]).

From (1.1) and (3.2), the Lorentz force \(\Phi\) associated to the contact magnetic field \(F_q\) can be written as

\[
\Phi = -q\phi.
\]

So the Lorentz equation (1.2) is

\[
\nabla_{E_1} E_1 = -q\phi E_1,
\]

(3.3)

where \(\alpha : I \to N\) is a curve with arc-length parameter, \(E_1 = \alpha'\) is the tangent vector field and \(\nabla\) is the Levi-Civita connection (see [9] and [14]). By the use of equations (2.6) and (3.3), we have

\[
\hat{\nabla}_{E_1} E_1 = \left[-q + 2\eta(E_1)\right]\phi E_1.
\]

(3.4)
Definition 3.1. Let $\alpha : I \to N$ be a unit-speed curve in a Sasakian manifold $N = (N^{2n+1}, \phi, \xi, \eta, g)$ endowed with the Tanaka-Webster connection $\hat{\nabla}$. Then it is called a normal magnetic curve with respect to the Tanaka-Webster connection $\hat{\nabla}$ (or shortly a pseudo-Hermitian magnetic curve) if it satisfies equation (3.4).

If $\eta(E_1) = \cos \theta$ is a constant, then $\alpha$ is called a slant curve [6]. From the definition of pseudo-Hermitian magnetic curves, we have the following direct result as in the Levi-Civita case:

**Proposition 3.1.** If $\alpha$ is a pseudo-Hermitian magnetic curve in a Sasakian manifold, then it is a slant curve.

**Proof.** Let $\alpha : I \to N$ be a pseudo-Hermitian magnetic curve. Then, we find

\[
\frac{d}{dt} g(E_1, \xi) = g(\hat{\nabla}_E_1 E_1, \xi) + g(E_1, \hat{\nabla}_{E_1} \xi) = g([-q + 2\eta(E_1)] \phi E_1, \xi) = 0.
\]

So we obtain

\[
\eta(E_1) = \cos \theta = \text{constant},
\]

which completes the proof. □

As a result, we can rewrite equation (3.4) as

\[
(3.5) \quad \hat{\nabla}_E_1 E_1 = (-q + 2 \cos \theta) \phi E_1,
\]

where $\theta$ is the contact angle of $\alpha$. Now, we can state the following theorem:

**Theorem 3.1.** Let $(N^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian manifold endowed with the Tanaka-Webster connection $\hat{\nabla}$. Then $\alpha : I \to N$ is a pseudo-Hermitian magnetic curve if and only if it belongs to the following list:

(a) pseudo-Hermitian non-Legendre slant geodesics (including pseudo-Hermitian geodesics as integral curves of $\xi$);

(b) pseudo-Hermitian Legendre circles with $\hat{k}_1 = |q|$ and having the Frenet frame field (for $\hat{\nabla}$)

\[
\{ E_1, -\text{sgn}(q) \phi E_1 \};
\]

(c) pseudo-Hermitian slant helices with

\[
\hat{k}_1 = |-q + 2 \cos \theta| \sin \theta, \quad \hat{k}_2 = |-q + 2 \cos \theta| \varepsilon \cos \theta
\]

and having the Frenet frame field (for $\hat{\nabla}$)

\[
\left\{ E_1, \frac{\delta}{\sin \theta} \phi E_1, \frac{\varepsilon}{\sin \theta} (\xi - \cos \theta E_1) \right\},
\]

where $\delta = \text{sgn}(-q + 2 \cos \theta)$, $\varepsilon = \text{sgn}(\cos \theta)$ and $\cos \theta \neq \frac{q}{2}$. 

Proof. Let us assume that \( \alpha : I \to N \) is a normal magnetic curve with respect to \( \hat{\nabla} \). Consequently, equation (3.5) must be validated. Let us assume \( \hat{k}_1 = 0 \). Hence, we have \( \cos \theta = \frac{q}{2} \) or \( \phi E_1 = 0 \). If \( \cos \theta = \frac{q}{2} \), then \( \alpha \) is a pseudo-Hermitian non-Legendre slant geodesic. Otherwise, \( \phi E_1 = 0 \) gives us \( E_1 = \pm \xi \). Thus, \( \alpha \) is a pseudo-Hermitian geodesic as an integral curve of \( \pm \xi \). So we have just proved that \( \alpha \) belongs to (a) from the list, if the osculating order \( r = 1 \). Now, let \( \hat{k}_1 \neq 0 \). From equation (3.5) and the Frenet equations for \( \hat{\nabla} \), we find

\[
\hat{\nabla}_{E_1} E_1 = \hat{k}_1 E_2 = (-q + 2 \cos \theta) \phi E_1.
\]

(3.6)

Since \( E_1 \) is unit, the equation (2.2) gives us

\[
g(\phi E_1, \phi E_1) = \sin^2 \theta.
\]

(3.7)

By the use of (3.6) and (3.7), we obtain

\[
\hat{k}_1 = | -q + 2 \cos \theta | \sin \theta,
\]

(3.8)

which is a constant. Let us denote \( \delta = \text{sgn}(-q + 2 \cos \theta) \). From (3.8), we can write

\[
\phi E_1 = \delta \sin \theta E_2.
\]

(3.9)

Let us assume \( \hat{k}_2 = 0 \), that is, \( r = 2 \). From the fact that \( \hat{k}_1 \) is a constant, \( \alpha \) is a pseudo-Hermitian circle. (3.9) gives us

\[
\eta(\phi E_1) = 0 = \delta \sin \theta \eta(E_2),
\]

which is equivalent to

\[
\eta(E_2) = 0.
\]

Differentiating this last equation with respect to \( \hat{\nabla} \), we obtain

\[
\hat{\nabla}_{E_1} \eta(E_2) = 0 = g\left( \hat{\nabla}_{E_1} E_2, \xi \right) + g\left( E_2, \hat{\nabla}_{E_1} \xi \right).
\]

Since \( \hat{\nabla} \xi = 0 \) and \( r = 2 \), we have

\[
g(-\hat{k}_1 E_1, \xi) = 0,
\]

that is, \( \eta(E_1) = 0 \). Hence, \( \alpha \) is Legendre and \( \cos \theta = 0 \). From equation (3.8), we get \( \hat{k}_1 = |q| \). In this case, we also obtain \( \delta = -\text{sgn}(q) \) and \( E_2 = -\text{sgn}(q) \phi E_1 \). We have proved that \( \alpha \) belongs to (b) from the list, if the osculating order \( r = 2 \). Now, let us assume \( \hat{k}_2 \neq 0 \). If we use \( \hat{\nabla} \phi = 0 \), we calculate

\[
\hat{\nabla}_{E_1} \phi E_1 = \hat{k}_1 \phi E_2.
\]

(3.10)

From (2.1) and (3.9), we find

\[
\phi^2 E_1 = -E_1 + \cos \theta \xi = \delta \sin \theta \phi E_2,
\]

(3.11)
which gives us
\[ \phi E_2 = \frac{\delta}{\sin \theta} (-E_1 + \cos \theta \xi). \]
So equation (3.10) becomes
\[ \hat{\nabla}_{E_1} \phi E_1 = \hat{k}_1 \frac{\delta}{\sin \theta} (-E_1 + \cos \theta \xi). \]

If we differentiate the equation (3.9) with respect to \( \hat{\nabla} \), we also have
\[ \hat{\nabla}_{E_1} \phi E_1 = \delta \sin \theta \hat{\nabla}_{E_1} E_2 = \delta \sin \theta \left( -\hat{k}_1 E_1 + \hat{k}_2 E_3 \right). \]

By the use of (3.12) and (3.13), we obtain
\[ \hat{k}_1 \cot \theta (\xi - \cos \theta E_1) = \hat{k}_2 \sin \theta E_3. \]
One can easily see that
\[ g(\xi - \cos \theta E_1, \xi - \cos \theta E_1) = \sin^2 \theta. \]
From (3.14), we calculate
\[ \hat{k}_2 = |-q + 2 \cos \theta| \varepsilon \cos \theta, \]
where we denote \( \varepsilon = \text{sgn}(\cos \theta) \). As a result, we get
\[ E_3 = \frac{\varepsilon}{\sin \theta} (\xi - \cos \theta E_1), \]
\[ E_2 = \frac{\delta}{\sin \theta} \phi E_1. \]
If we differentiate (3.15) with respect to \( \hat{\nabla} \), since \( \phi E_1 \parallel E_2 \), we find \( \hat{k}_3 = 0 \). So we have just completed the proof of (c). Considering the fact that \( \hat{k}_3 = 0 \), the Gram-Schmidt process ends. Thus, the list is complete.

Conversely, let \( \alpha : I \to N \) belong to the given list. It is easy to show that equation (3.5) is satisfied. Hence, \( \alpha \) is a pseudo-Hermitian magnetic curve. \( \square \)

A pseudo-Hermitian geodesic is said to be a pseudo-Hermitian \( \phi \)-curve if the set \( sp \{E_1, \phi E_1, \xi\} \) is \( \phi \)-invariant. A Frenet curve of osculating order \( r = 2 \) is said to be a pseudo-Hermitian \( \phi \)-curve if \( sp \{E_1, E_2, \xi\} \) is \( \phi \)-invariant. A Frenet curve of osculating order \( r \geq 3 \) is said to be a pseudo-Hermitian \( \phi \)-curve if \( sp \{E_1, E_2, \ldots, E_r\} \) is \( \phi \)-invariant.

**Theorem 3.2.** Let \( \alpha : I \to N \) be a pseudo-Hermitian \( \phi \)-helix of order \( r \leq 3 \), where \( N = (N^{2n+1}, \phi, \xi, \eta, g) \) is a Sasakian manifold endowed with the Tanaka-Webster connection \( \hat{\nabla} \). Then:
If \( \cos \theta = \pm 1 \), then it is an integral curve of \( \xi \), i.e. a pseudo-Hermitian geodesic and it is a pseudo-Hermitian magnetic curve for \( F_q \) for arbitrary \( q \);

(b) If \( \cos \theta \notin \{-1,0,1\} \) and \( \hat{k}_1 = 0 \), then it is a pseudo-Hermitian non-Legendre slant geodesic and it is a pseudo-Hermitian magnetic curve for \( F_{2\cos \theta} \);

(c) If \( \cos \theta = 0 \) and \( \hat{k}_1 \neq 0 \), i.e. \( \alpha \) is a Legendre \( \phi \)-curve, then it is a pseudo-Hermitian magnetic circle generated by \( F_{-\delta \hat{k}_1} \), where \( \delta = \text{sgn}(g(\phi E_1, E_2)) \);

(d) If \( \cos \theta = \frac{\varepsilon \hat{k}_2}{\sqrt{k_1^2 + k_2^2}} \) and \( \hat{k}_2 \neq 0 \), then it is a pseudo-Hermitian magnetic curve for \( F_{-\delta \sqrt{k_1^2 + k_2^2} + \frac{2\varepsilon}{\sqrt{k_1^2 + k_2^2}}} \), where \( \delta = \text{sgn}(g(\phi E_1, E_2)) \) and \( \varepsilon = \text{sgn}(\cos \theta) \).

(e) Except above cases, \( \alpha \) cannot be a pseudo-Hermitian magnetic curve for any \( F_q \).

Proof. Firstly, let us assume \( \cos \theta = \pm 1 \), that is, \( E_1 = \pm \xi \). As a result, we have

\[ \hat{\nabla}_{E_1} E_1 = 0, \phi E_1 = 0. \]

Hence, equation (3.5) is satisfied for arbitrary \( q \). This proves (a). Now, let us take \( \cos \theta \notin \{-1,0,1\} \) and \( \hat{k}_1 = 0 \). In this case, we obtain

\[ \hat{\nabla}_{E_1} E_1 = 0, \phi E_1 \neq 0. \]

So equation (3.5) is valid for \( q = 2\cos \theta \). The proof of (b) is over. Next, let us assume \( \cos \theta = 0 \) and \( \hat{k}_1 \neq 0 \). One can easily see that \( \alpha \) has the Frenet frame field (for \( \hat{\nabla} \))

\[ \{E_1, \delta \phi E_1\} \]

where \( \delta \) corresponds to the sign of \( g(\phi E_1, E_2) \). Consequently, we get

\[ \hat{\nabla}_{E_1} E_1 = \delta \hat{k}_1 \phi E_1, \]

that is, \( \alpha \) is a pseudo-Hermitian magnetic curve for \( q = -\delta \hat{k}_1 \). We have just proven (c). Finally, let \( \cos \theta = \frac{\varepsilon \hat{k}_2}{\sqrt{k_1^2 + k_2^2}} \) and \( \hat{k}_2 \neq 0 \). So \( \alpha \) has the Frenet frame field (for \( \hat{\nabla} \))

\[ \left\{ E_1, \frac{\delta}{\sin \theta} \phi E_1, \frac{\varepsilon}{\sin \theta} \left( \xi - \cos \theta E_1 \right) \right\}, \]

where \( \delta = \text{sgn}(g(\phi E_1, E_2)) \) and \( \varepsilon = \text{sgn}(\cos \theta) \). After calculations, it is easy to show that equation (3.5) is satisfied for \( q = -\delta \sqrt{k_1^2 + k_2^2} + \frac{2\varepsilon}{\sqrt{k_1^2 + k_2^2}} \). Hence, the proof of (d) is completed. Except above cases, from Theorem 3.1, \( \alpha \) cannot be a pseudo-Hermitian magnetic curve for any \( F_q \). \( \square \)
4. Parametrizations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$

In this section, our aim is to obtain parametrizations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$. To do this, we need to recall some notions from [2]. Let $N = \mathbb{R}^{2n+1}$. Let us denote the coordinate functions of $N$ with $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$. One may define a structure on $N$ by $\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y_i dx_i)$, which is a contact structure, since $\eta \wedge (d\eta)^n \neq 0$. This contact structure has the characteristic vector field $\xi = 2\frac{\partial}{\partial z}$. Let us also consider a $(1,1)$-type tensor field $\phi$ given by the matrix form as

$$\phi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}.$$ 

Finally, let us take the Riemannian metric on $N$ given by $g = \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^{n} ((dx_i)^2 + (dy_i)^2)$. It is known that $(N, \phi, \xi, \eta, g)$ is a Sasakian space form and its $\phi$-sectional curvature is $c = -3$. This special Sasakian space form is denoted by $\mathbb{R}^{2n+1}(-3)$ [2]. One can easily show that the vector fields

$$(4.1) \quad X_i = 2\frac{\partial}{\partial y_i}, \quad X_{n+i} = \phi X_i = 2(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}), \quad i = 1, n, \quad \xi = 2\frac{\partial}{\partial z}$$

are $g$-unit and $g$-orthogonal. Hence, they form a $g$-orthonormal basis [2]. Using this basis, the Levi-Civita connection of $\mathbb{R}^{2n+1}(-3)$ can be obtained as

$$\nabla X_i X_j = \nabla X_{m+i} X_{n+j} = 0, \quad \nabla X_i X_{m+j} = \delta_{ij} \xi, \quad \nabla X_{m+i} X_j = -\delta_{ij} \xi,$$

$$\nabla X_i \xi = \nabla \xi X_i = -X_{m+i}, \quad \nabla X_{m+i} \xi = \nabla \xi X_{m+i} = X_i,$$

(see [2]). As a result, the Tanaka-Webster connection of $\mathbb{R}^{2n+1}(-3)$ is

$$\tilde{\nabla} X_i X_j = \tilde{\nabla} X_{m+i} X_{n+j} = \tilde{\nabla} X_i X_{m+j} = \tilde{\nabla} X_{m+i} X_j = \tilde{\nabla} X_i \xi = \tilde{\nabla} \xi X_i = \tilde{\nabla} X_{m+i} \xi = \tilde{\nabla} \xi X_{m+i} = 0,$$

which was calculated in [12]. Now, we can investigate the parametric equations of pseudo-Hermitian magnetic curves in $\mathbb{R}^{2n+1}(-3)$ endowed with the Tanaka-Webster connection.

Let $N = \mathbb{R}^{2n+1}(-3)$ endowed with the Tanaka-Webster connection $\tilde{\nabla}$. Let $\alpha : I \subseteq \mathbb{R} \to N$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n}, \alpha_{2n+1})$ be a pseudo-Hermitian magnetic curve. Then, the tangential vector field of $\alpha$ can be written as

$$E_1 = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \alpha_{n+i} \frac{\partial}{\partial y_i} + \alpha_{2n+1} \frac{\partial}{\partial z}.$$

In terms of the $g$-orthonormal basis, $E_1$ is rewritten as

$$E_1 = \frac{1}{2} \left[ \sum_{i=1}^{n} \alpha_{n+i} X_i + \sum_{i=1}^{n} \alpha_i X_{n+i} + \left( \alpha_{2n+1} - \sum_{i=1}^{n} \alpha_i \alpha_{n+i} \right) \xi \right].$$
From Proposition 3.1, $\alpha$ is a slant curve. Hence, we have
\[ \eta(E_1) = \cos \theta = \text{constant}, \]
which is equivalent to
\[ \alpha'_{2n+1} = 2 \cos \theta + \sum_{i=1}^{n} \alpha_i' \alpha_{n+i}. \]  
(4.2)

From the fact that $\alpha$ is parametrized by arc-length, we also have
\[ g(E_1, E_1) = 1, \]
that is,
\[ \sum_{i=1}^{2n} (\alpha'_i)^2 = 4 \sin^2 \theta. \]  
(4.3)

Differentiating $E_1$ with respect to $\hat{\nabla}$, we obtain
\[ \hat{\nabla}_{E_1} E_1 = \frac{1}{2} \left( \sum_{i=1}^{n} \alpha''_{n+i} X_i + \sum_{i=1}^{n} \alpha''_i X_{n+i} \right). \]

We also easily find
\[ \phi_{E_1} = \frac{1}{2} \left( -\sum_{i=1}^{n} \alpha'_i X_i + \sum_{i=1}^{n} \alpha'_{n+i} X_{n+i} \right). \]

Since $\alpha$ is a pseudo-Hermitian magnetic curve, it must satisfy
\[ \hat{\nabla}_{E_1} E_1 = (q - 2 \cos \theta) \phi_{E_1}. \]

Then, we can write
\[ \frac{\alpha''_{n+1}}{-\alpha'_1} = \ldots = \frac{\alpha''_{2n}}{-\alpha'_n} = \frac{\alpha''_1}{\alpha''_{n+1}} = \ldots = \frac{\alpha''_n}{\alpha''_{2n}} = -\lambda, \]

where $\lambda = q - 2 \cos \theta$. From the last equations, we can select the pairs
\[ \frac{\alpha''_{n+1}}{-\alpha'_1} = \frac{\alpha''_1}{\alpha''_{n+1}}, \ldots, \frac{\alpha''_{2n}}{-\alpha'_n} = \frac{\alpha''_n}{\alpha''_{2n}}. \]  
(4.4)

Firstly, let $\lambda \neq 0$. Solving the ODEs, we have
\[ (\alpha_i')^2 + (\alpha_{i+1}')^2 = c_i^2, \quad i = 1, \ldots, n \]
for some arbitrary constants $c_i$ ($i = 1, \ldots, n$) such that
\[ \sum_{i=1}^{n} c_i^2 = 4 \sin^2 \theta. \]
So we have

\[ \alpha_i' = c_i \cos f_i, \quad \alpha_{n+i}' = c_i \sin f_i \]

for some differentiable functions \( f_i : I \to \mathbb{R} \) \((i = 1, \ldots, n)\). From (4.4), we get

\[ \frac{\alpha''_{n+i}}{-\alpha'_i} = -f'_i = -\lambda, \]

which gives us

\[ f_i = \lambda t + d_i \]

for some arbitrary constants \( d_i \) \((i = 1, \ldots, n)\). Here, \( t \) denotes the arc-length parameter. Then, we find

\[ \alpha_i' = c_i \cos (\lambda t + d_i), \quad \alpha_{n+1}' = c_i \sin (\lambda t + d_i). \]

Finally, we obtain

\[ \alpha_i = \frac{c_i}{\lambda} \sin (\lambda t + d_i) + h_i, \]

\[ \alpha_{n+i} = -\frac{c_i}{\lambda} \cos (\lambda t + d_i) + h_{n+i}, \]

\[ \alpha_{2n+1} = 2t \cos \theta + \sum_{i=1}^{n} \left\{ \frac{-c_i^2}{4\lambda^2} \left[ 2 (\lambda t + d_i) + \sin (2 (\lambda t + d_i)) \right] \right. \\
+ \left. \frac{c_i h_{n+i}}{\lambda} \sin (\lambda t + d_i) \right\} + h_{2n+1} \]

for some arbitrary constants \( h_i \) \((i = 1, \ldots, 2n+1)\).

Secondly, let \( \lambda = 0 \). In this case, \( q = 2 \cos \theta \) and \( \hat{k}_1 = 0 \). Hence, we have

\[ \nabla_{E_1} E_1 = \frac{1}{2} \left( \sum_{i=1}^{n} \alpha''_{n+i} X_i + \sum_{i=1}^{n} \alpha''_i X_{n+i} \right) = 0, \]

which gives us

\[ \alpha_i = c_i t + d_i, \quad i = 1, \ldots, 2n, \]

\[ \alpha_{2n+1} = 2t \cos \theta + \sum_{i=1}^{n} c_i \left( \frac{c_{n+i}}{2} t^2 + d_{n+i} t \right) + c_{2n+1}, \]

where \( c_i \) \((i = 1, 2, \ldots, 2n+1)\) and \( d_i \) \((i = 1, 2, \ldots, 2n)\) are arbitrary constants such that

\[ \sum_{i=1}^{2n} c_i^2 = 4 \sin^2 \theta. \]

To conclude, we can state the following theorem:
Theorem 4.1. The pseudo-Hermitian magnetic curves on $\mathbb{R}^{2n+1}(-3)$ endowed with the Tanaka-Webster connection have the parametric equations

$$\alpha : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^{2n+1}(-3), \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n}, \alpha_{2n+1}),$$

where $\alpha_i$ ($i = 1, \ldots, 2n+1$) satisfies either

(a) $$\alpha_i = \frac{c_i}{\lambda} \sin (\lambda t + d_i) + h_i,$$
$$\alpha_{n+i} = \frac{-c_i}{\lambda} \cos (\lambda t + d_i) + h_{n+i},$$

$$\alpha_{2n+1} = 2 \cos \theta t + \sum_{i=1}^{n} \left\{ \frac{-c_i^2}{4\lambda^2} \left[ 2 (\lambda t + d_i) + \sin (2 (\lambda t + d_i)) \right] + \frac{c_i h_{n+i}}{\lambda} \sin (\lambda t + d_i) \right\} + h_{2n+1},$$

where $\lambda = q - 2 \cos \theta \neq 0$, $c_i$, $d_i$ ($i = 1, \ldots, n$) and $h_i$ ($i = 1, \ldots, 2n+1$) are arbitrary constants such that

$$\sum_{i=1}^{n} c_i^2 = 4 \sin^2 \theta;$$

or

(b) $$\alpha_i = c_i t + d_i,$$

$$\alpha_{2n+1} = 2t \cos \theta + \sum_{i=1}^{n} c_i \left( \frac{c_{n+i} t^2}{2} + d_{n+i} t \right) + c_{2n+1},$$

where $q = 2 \cos \theta$ and $c_i$ ($i = 1, 2, \ldots, 2n+1$), $d_i$ ($i = 1, 2, \ldots, 2n$) are arbitrary constants such that

$$q^2 + \sum_{i=1}^{2n} c_i^2 = 4.$$
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