SUPER VUST THEOREM AND SCHUR-SERGEEV DUALITY FOR PRINCIPAL FINITE W-SUPERALGEBRAS

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Abstract. In this paper, we first formulate a super version of Vust theorem associated with a regular nilpotent element \(e \in \mathfrak{gl}(V)\). As an application of this theorem, we then obtain the Schur-Sergeev duality for principal finite \(W\)-superalgebras which is partially a super version of Brundan-Kleshchev’s higher level Schur-Weyl duality established in [8].

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0. Introduction

The purpose of the present paper is to prove a super Vust theorem, and then, as an application, to extend Brundan-Kleshchev’s higher Schur-Weyl duality to the super case. We successfully achieve a super version of Brundan-Kleshchev’s result when \(e\) is regular nilpotent (see Definition 1.1).

0.1. Let \(G = \text{GL}(V)\) and \(g = \mathfrak{gl}(V)\) be the general linear Lie group and its Lie algebra on \(V\) over \(\mathbb{C}\) with \(\dim_{\mathbb{C}} V = n\) respectively, and \(e\) a nilpotent element of \(g\). Set \(g_e = \{X \in g \mid \text{ad}_e(X) = 0\}\), the centralizer of \(e\) in \(g\). There is a natural action \(\phi\) of \(g_e\) on the tensor product \(V^\otimes d\). The Vust theorem generalizes the classical Schur-Weyl duality in the setup of \(g_e\) (see [26]). We briefly give an account of it below, in connection with the finite \(W\)-algebras and the degenerate cyclotomic affine Hecke algebras.

Recall that the degenerate affine Hecke algebra \(H_d\) is an associative algebra equal to a vector space of the tensor product \(\mathbb{C}[x_1, \ldots, x_d] \otimes \mathbb{C}S_d\) of a polynomial algebra and the group algebra of the symmetric group \(S_d\), and \(H_d\) is defined via generators \(\{x_i, s_j = (j, j+1) \in S_d \mid i = 1, \ldots, d; j = 1, \ldots, d-1\}\) and relations \(s_ix_j = x_js_i\) if \(j \neq i, i+1\), and \(s_ix_{i+1} = x_is_i + 1\).

On the other hand, associated with \(e\) there is a twisted truncated polynomial algebra \(\mathbb{C}_l[x_1, \ldots, x_d]\) and its action \(\psi\) on \(V^\otimes d\), where \(l\) is a positive integer determined by the Young diagram of the
adjoint orbit $G.e$, and $C_l[x_1, \ldots, x_d]$ is a quotient of the polynomial algebra $\mathbb{C}[x_1, \ldots, x_d]$ by the ideal generated by $x_i^2, i = 1, \ldots, d$. The action is twisted from the usual right action of $S_d$ on $V^\otimes d$, by letting $x_i$ be as the endomorphism $1^\otimes(i-1) \otimes e \otimes 1^\otimes(d-i)$. Then we can define an algebra $C_l[x_1, \ldots, x_d] \otimes S_d$ which has the ground space $C_l[x_1, \ldots, x_d] \otimes S_d$, and is subjected to the equations $s_i x_j = x_j s_i$ if $j \neq i, i + 1$, and $s_ix_{i+1} = x_is_i + 1$; and to the provision that the algebra $C_l[x_1, \ldots, x_d] \otimes S_d$ extends the subalgebras $C_l[x_1, \ldots, x_d]$ (identified with $C_l[x_1, \ldots, x_d] \otimes 1$) and $C\mathfrak{g}\mathfrak{sl}_d$ (identified with $1 \otimes S_d$), respectively. Then $V^\otimes d$ becomes a natural representation over $C_l[x_1, \ldots, x_d] \otimes S_d$. The classical Vust’s theorem says

\begin{equation}
\psi(C_l[x_1, \ldots, x_d] \otimes S_d) = \text{End}_{U(e)}(V^\otimes d).
\end{equation}

Then a generalized Schur-Weyl duality, associated with $e$ further says that $g_e$ and $C_l[x_1, \ldots, x_d] \otimes S_d$ are double-centralizers in $\text{End}_{\mathbb{C}}(V^\otimes d)$. This is to say, addition to (0.1), one has

\begin{equation}
\text{End}_{C_l[x_1, \ldots, x_d] \otimes S_d}(V^\otimes d) = \phi(U(g_e)).
\end{equation}

Starting from here, Brundan and Kleshchev go further, considering the filtered deformation of the above double-centralizers. Consider the finite $W$-algebra $W_\chi$ with $\chi$ being the linear dual of $e$ via the Killing form on $g$. According to [34], a well-known result says that $W_\chi$ is actually a filtered-deformation of $U(g_e)$. They obtained the following higher Schur-Weyl duality:

\begin{equation}
\text{End}_{H_d(A)}(V^\otimes d) = \Phi(W_\chi),
\end{equation}

\begin{equation}
\Psi(H_d(A)) = \text{End}_{V_\chi}(V^\otimes d).
\end{equation}

Here $H_d(A)$ is a degenerate cyclotomic Hecke algebra which is a quotient algebra of $H_d$.

0.2. Recall that Sergeev’s super-version of Schur-Weyl duality (called Schur-Sergeev duality in the present paper) describes the double centralizer property of general linear Lie superalgebras and symmetry groups happening on the tensor products of super spaces (see Theorem 1.2 precisely).

Consider $V = \mathbb{C}^{m|n}$, $g = \mathfrak{gl}(m|n)$ and the natural action of $g$ on $V^\otimes d$ is defined via

\begin{equation}
\rho(X)(v_1 \otimes v_2 \otimes \cdots \otimes v_d) = \sum_{i=1}^{d} (-1)^{|X|(|v_1|+|v_2|+\cdots+|v_{i-1}|)} v_1 \otimes v_2 \otimes \cdots \otimes X v_i \otimes \cdots \otimes v_d
\end{equation}

for any $\mathbb{Z}_2$-homogeneous $X \in \mathfrak{gl}(m|n)$ and $v_i \in V$ $(i = 1, \ldots, d)$ with an appointment $v_0 = 0$. Here and further $|w|$ always stands for the parity of $w \in W[|w|]$ when we talk about a superspace $W = W_0 \oplus W_1$.

0.2.1. Super Vust theorem. Let $e$ be any given regular nilpotent element in $g_0$. As to the twisted action $\pi_d$ of the symmetry group $S_d$ generated by transpositions $\{s_i = (i \ i+1) \mid i = 1, 2, \ldots, d-1\}$, the action of $s_i$ is defined as below:

\begin{equation}
\psi_d(s_i) : v_1 \otimes \cdots \otimes v_d \mapsto v_1 \otimes \cdots \otimes \tau(v_i \otimes v_{i+1}) \otimes \cdots \otimes v_d,
\end{equation}

where $\tau : V \otimes V \to V \otimes V$ sends $v \otimes w$ to $(-1)^{|w| |v|} w \otimes v$.

One of our main results is the following super Vust theorem.

**Theorem 0.1.** Keep the notations and assumptions as above. In particular, $g = \mathfrak{gl}(m|n)$, $e \in g_0$ is a regular nilpotent element. Let $A$ denote the algebra generated by $\pi_d(S_d)$ and $\rho(ge)$ in $\text{End}_{\mathbb{C}}(V^\otimes d)$. Then $\text{End}_{\rho(e)}(V^\otimes d) = A$.

The proof will be given in §2. An important ingredient there is the argument on the orbit of $e$ under the adjoint action of the general linear Lie supergroup (see Lemma 2.1).
0.2.2. For the other side of the desired double centralizers, a careful inspection and thorough argument involving parities shows that Brundan-Kleshchev’s original strategy in [8] can be carried out here. Roughly speaking, our strategy is that we first extend the Schur-Sergeev duality for \( \mathfrak{gl}(m|n) \) to the level of \( \mathfrak{gl}(m|n)_{e} \) for a regular nilpotent \( e \in \mathfrak{gl}(m|n)_{0} \) where the general Lie superalgebra \( \mathfrak{gl}(m|n) \) has \( \mathbb{Z}_{2} \)-superspace decomposition \( \mathfrak{gl}(m|n) = \mathfrak{gl}(m|n)_{0} \oplus \mathfrak{gl}(m|n)_{1} \) for \( \mathbb{Z}_{2} = \{0,1\} \), which says that \( U(\mathfrak{gl}(m|n)_{e}) \) and \( \mathbb{C}_{n}[x_{1},\ldots,x_{d}] \otimes \mathcal{C}_{d} \) (assume \( n \geq m \)) play the role of the double centralizers in \( \text{End}_{\mathbb{C}}(V^{\otimes d}) \). Here the algebra \( \mathbb{C}_{n}[x_{1},\ldots,x_{d}] \otimes \mathcal{C}_{d} \) is the same as defined previously. However, the action on \( V^{\otimes d} \) of the generators \( s_{i} \) has the super meaning (see [41]). Then we exploit Brundan-Kleshchev’s arguments in [7] to establish the Schur-Sergeev duality for the corresponding finite \( W \)-superalgebra. This is carried out for regular nilpotent elements in the present paper. We accomplish this by a couple of steps.

Owing to [3], associated with a regular nilpotent element \( e \in \mathfrak{gl}(m|n)_{0} \) the centralizer \( \mathfrak{gl}(m|n)_{e} \) is described clearly. Consequently, \( U(\mathfrak{g}_{e}) \) and its filtered deformation finite \( W \)-superalgebra \( W_{\chi} \) are well investigated in [3]. Those results are the start-point of our arguments.

With a super version of Vust theorem already established, we then develop some counterpart theory in the super case as some necessary preparations, i.e., the super version of Premet’s theorem on the gradation isomorphism of finite \( W \)-algebras.

0.2.3. Grading (in natural sense/Kazhdan sense) on \( \mathfrak{g}_{e} \) associated with \( e \). For any basic classical Lie superalgebra \( \mathfrak{g} \) over \( \mathbb{C} \) and any given nilpotent element \( e \in \mathfrak{g}_{0} \), fix an \( \mathfrak{sl}_{2} \)-triple \( h,f,e \in \mathfrak{g}_{0} \), and denote by \( \mathfrak{g}_{e} = \text{Ker}(\text{ad}e) \) in \( \mathfrak{g} \). The linear operator \( \text{ad}h \) defines a \( \mathbb{Z} \)-grading \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i) \) (natural grading). Define the Kazhdan degree on \( \mathfrak{g} \) by declaring \( x \in \mathfrak{g}(j) \) is \( (j+2) \). Recall that in [42] a finite \( W \)-superalgebra is introduced for any basic classical Lie superalgebra \( \mathfrak{g} \) over \( \mathbb{C} \) and any given nilpotent element \( e \in \mathfrak{g}_{0} \) (see Definition [11] for the definition of \( W_{\chi} \) corresponding to \( e \), where \( \chi = (e,-) \) is the linear dual of \( e \) associated with the non-degenerate bilinear form defined on \( \mathfrak{g}_{0} \), and the PBW structure theorem for the finite \( W \)-superalgebra \( W_{\chi} \) is presented. It is shown that the construction of finite \( W \)-superalgebras \( W_{\chi} \) can be divided into two cases, in virtue of the parity of the so-called “judging number” \( r \) which is equal to the dimension of \( \mathfrak{g}(-1)_{0} \) (see [42] Theorem 4.5). However, under the Kazhdan grading \( W_{\chi} \) is a filtered deformation of \( S(\mathfrak{g}_{e}) \) or \( S(\mathfrak{g}_{e}) \otimes \mathbb{C}[\Theta] \) with \( \mathbb{C}[\Theta] \) being an exterior algebra generated by an element \( \Theta \), depending on the parity of \( r \) we discussed above (see Theorem [13]).

Another important ingredient in our arguments is to prove an analogue of Premet’s result [32 Proposition 2.1], which says that when the grading associated with \( r \) is even, associated with the filtration of \( W_{\chi} \) arising from the natural grading on the above \( (\mathfrak{g},e) \) provided by \( \mathfrak{sl}(2) \)-triple \( \{e,h,f\} \), \( W_{\chi} \) is indeed a filtered deformation of \( U(\mathfrak{g}_{e}) \). In the following, the associated grading is denoted by \( \mathfrak{gr} \).

**Theorem 0.2.** (see Theorem [4.3]) The associated graded algebra \( \mathfrak{gr}W_{\chi} \) is isomorphic to \( U(\mathfrak{g}_{e}) \) if \( r \) is even.

It is notable that \( r \) is always an even number for any nilpotent element \( e \in \mathfrak{g}_{0} \) for \( \mathfrak{g} = \mathfrak{gl}(m|n) \) (see Remark [14]). In particular, the above theorem is valid to the case \( \mathfrak{g} = \mathfrak{gl}(m|n) \). However, for a general pair \( (\mathfrak{g},e) \) the judging number \( r \) is not necessarily even, then in the above statement \( W_{\chi} \) has to be replaced by a refined \( W \)-superalgebra \( W'_{\chi} \) (see Theorem [4.8]). The refined \( W \)-superalgebra is defined as

\[
W'_{\chi} := Q_{\chi}^{adm'},
\]

where \( m' = m \) if \( r \) is an even number, and \( m' \) is a subspace properly containing \( m \) when \( r \) is an odd number (see [31] for details). Clearly, \( W'_{\chi} \) is a subalgebra of \( W_{\chi} \). In fact, the PBW theorem of \( W'_{\chi} \) is of much the same as that of Theorem [1.2] with \( r \) being odd, just abandoning the related topics on the element \( \Theta_{t+q+1} \) (see [1.2] for details). The authors also showed that \( \mathfrak{gr}(W'_{\chi}) \cong S(\mathfrak{g}_{e}) \).
as $\mathbb{C}$-algebras under the Kazhdan grading in [44] Corollary 3.8]. We refer to [44] Theorem 3.7] for more details.

0.2.4. Filtered deformations and Schur-Sergeev duality for principal $W$-superalgebras. Turn back to the case $\mathfrak{g} = \mathfrak{gl}(m|n)$ which is a general linear Lie superalgebra on the superspace $V = \mathbb{C}^{m|n}$ with basis $\{v_1, \ldots, v_m \in V \} \cup \{v_1, \ldots, v_n \in V \}$, and $e = \sum_{i=1}^{m-1} e_{i,i+1} + \sum_{j=1}^{n-1} e_{j,j+1}$ for $I(m|n) = \{1, \ldots, m; 1, \ldots, n\}$ with parity $|i| = 0$, and $|j| = 1$ ($i = 1, \ldots, m$ and $j = 1, \ldots, n$). Associated with $e$, we define a pyramid diagram as in (3.1), where we make an assumption $m \leq n$ without loss of generality. Define a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ on $\mathfrak{g}$ with $\mathfrak{g}(0)$ being spanned by $e_{s,t}$ satisfying $i = \deg(e_{s,t}) := \text{col}(t) - \text{col}(s)$ where $\text{col}(t)$ and $\text{col}(s)$ respectively stand for the column coordinates of the positions where $t$ and $s$ lie in the pyramid diagram (3.1). Set $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$, $\mathfrak{h} = \mathfrak{g}(0)$ and $\mathfrak{m} = \bigoplus_{i < 0} \mathfrak{g}(i)$. By [3], $\mathfrak{g}_c$ is a graded subalgebra of $\mathfrak{p}$, precisely described there. Consider the natural representation $\rho$ of $\mathfrak{g} = \mathfrak{gl}(m|n)$ on $V = \mathbb{C}^{m|n}$, and the natural action of $\mathfrak{g}$ on $V^{\otimes \mathfrak{d}}$. Then one has $\mathfrak{g}_c$-representation $\phi_\mathfrak{d}$ on $V^{\otimes \mathfrak{d}}$, and the action $\psi_\mathfrak{d}$ of the symmetry group $\Phi_\mathfrak{d}$ on $V^{\otimes \mathfrak{d}}$ (see 0.2.1). Let $\psi_\mathfrak{d} : \mathbb{C}[x_1, \ldots, x_\mathfrak{d}] \otimes \mathbb{C} \Phi_\mathfrak{d} \rightarrow \text{End}_\mathbb{C}(V^{\otimes \mathfrak{d}})^\mathfrak{op}$ be the homomorphism arising from the right action of $\mathbb{C}[x_1, \ldots, x_\mathfrak{d}] \otimes \mathbb{C} \Phi_\mathfrak{d}$ on $V^{\otimes \mathfrak{d}}$.

**Theorem 0.3.** When an even element $e$ is a regular nilpotent element from $\mathfrak{g} = \mathfrak{gl}(m|n)$, the following double centralizer property holds:

$$
\phi_\mathfrak{d}(U(\mathfrak{g}_c)) = \text{End}_{\mathbb{C}[x_1, \ldots, x_\mathfrak{d}] \otimes \mathbb{C} \Phi_\mathfrak{d}}(V^{\otimes \mathfrak{d}}),
$$

$$
\text{End}_{U(\mathfrak{g}_c)}(V^{\otimes \mathfrak{d}})^\mathfrak{op} = \psi_\mathfrak{d}(\mathbb{C}[x_1, \ldots, x_\mathfrak{d}] \otimes \mathbb{C} \Phi_\mathfrak{d}).
$$

The complete proof of the above theorem will be left till 3.7.

0.3. Now we turn to the finite $W$-superalgebras. Thanks to [3], $W_\chi$ for a regular nilpotent $e \in \mathfrak{g}_0$ can be describe precisely, which is a subalgebra of $U(\mathfrak{p})$, isomorphic to $U(\mathfrak{h})$ as algebras via Miura transform (see Remark 5.2.3)). Under the filtration arising from the grading on $U(\mathfrak{p})$ as an enveloping algebra, $W_\chi$ is a filtered subalgebra of $U(\mathfrak{p})$, whose gradation is isomorphic to $U(\mathfrak{g}_c)$.

For any given $\mathfrak{c} = (c_1, \ldots, c_n) \in \mathbb{C}^n$, there is a one-dimensional $U(\mathfrak{p})$-module $\mathbb{C}_\mathfrak{c} = \mathbb{C}_{\mathfrak{c}}$ with the action for any $e_{i,j} \in \mathfrak{p}$ on the generator $1_\mathfrak{c}$ as $e_{i,j} |_\mathfrak{c} = \delta_{i,j} (-1)^{|i|} |_\mathfrak{c}$ for all $i \leq j$. There is a tensor representation $\Phi_\mathfrak{d,\mathfrak{c}}$ of $W_\chi$ on $V^{\otimes \mathfrak{d}}$ where $V^{\otimes \mathfrak{d}} = \mathbb{C}_\mathfrak{c} \otimes V^{\otimes \mathfrak{d}}$.

Let $\Lambda_\mathfrak{c} = \sum_{i=1}^{n} \Lambda_{c_i}$ be an element of the free abelian group generated by symbols $\{\Lambda_a \mid a \in \mathbb{C}\}$. The corresponding degenerate cyclotomic Hecke algebra $H_\mathfrak{d}(\Lambda_\mathfrak{c})$ is the quotient of the degenerate affine Hecke algebra $H_\mathfrak{d}$ (see Definition 1.3) by the two-sided ideal generated by $\prod_{i=1}^{n} (x - c_i)$. The operator

$$
\Omega = \sum_{i,j \in I(m|n)} (-1)^{|j|} e_{i,j} \otimes e_{j,i}
$$

on $\mathbb{C}_\mathfrak{c} \otimes V^{\otimes \mathfrak{d}}$ will play a crucial role in the arguments. By exploiting the arguments in 7 and 8 to the present case, we will show the right action of $H_\mathfrak{d}$ on $V^{\otimes \mathfrak{d}}$ via

$$
x_i := 1^{d-i} \otimes x \otimes 1^{i-1}, \quad s_j := 1^{d-j-1} \otimes s \otimes 1^{j-1},
$$

where the operators $x$ and $s$ are defined through $\Omega$ respectively (see 3.3 for details), and $1 \leq i \leq d$ for all $d \geq 1$. So we finally have a homomorphism

$$
\Psi_\mathfrak{d,\mathfrak{c}} : H_\mathfrak{d}(\Lambda_\mathfrak{c}) \rightarrow \text{End}_{\mathbb{C}_\mathfrak{c}}(V^{\otimes \mathfrak{d}}).
$$

Then we have the following homomorphisms:

$$
W_\chi \xrightarrow{\phi_\mathfrak{d,\mathfrak{c}}} \text{End}_{\mathbb{C}_\mathfrak{c}}(V^{\otimes \mathfrak{d}}) \xrightarrow{\Psi_\mathfrak{d,\mathfrak{c}}} H_\mathfrak{d}(\Lambda_\mathfrak{c}).
$$
Owing to Theorem 0.3 and the gradation property of $W_\chi$, the images of the above homomorphisms turn out to be double-centralizers in $\text{End}_{\mathbb{C}}(V_\chi \otimes d)$ (see Theorem 0.7). Taking a special value of $c = (0, 0, \ldots, 0) := 0$ and $\Lambda = \Lambda_0$, we finally have the following result:

**Theorem 0.4.** When an even element $e$ is a regular nilpotent element from $\mathfrak{g} = \mathfrak{gl}(m|n)$, the following double centralizer properties hold:

$$\Phi_d(W_\chi) = \text{End}_{H_d}(V_\chi \otimes d),$$

$$\text{End}_{W_\chi}(V_\chi \otimes d)^{\text{op}} = \Psi_d(H_d(A)).$$

The proof of the above theorem will be given in the concluding section §6.4. It is worthwhile mentioning that according to some anonymous referee’s statements, there is an alternative proof via some work on categorifications in [16] or [15].

### 0.4. The structure of this paper.

The paper is organized as follows. In §1 we introduce some preliminary material. §2 is devoted to the proof of the super Vust theorem (Theorem 0.1). In §3, in most of the time we specially take $e$ to be regular nilpotent. In order to prove the first equation in Theorem 0.3, we need to compute the centralizer of $G \subset \text{End}_{\mathbb{C}}(V_\chi \otimes d)$. For this, we will work with the distribution space of $M_e$ (where $M_e$ is the ground space of $\mathfrak{g}_e$), and identify $U(\mathfrak{g}_e)$ with $\text{Dist}(M_e)$. By analysing the comodule structure of $\mathbb{C}[M_e]$ on $V_\chi \otimes d$, we finally verify the first equation of Theorem 0.3. In §4, we show that $U(\mathfrak{g}_e)$ is actually a contraction of the corresponding finite W-superalgebra. In §5, we will analysis the tensor representations of $W_\chi$ on $V_\chi \otimes d$, by Skryabin’s equivalence. In the concluding section §6, we will introduce the degenerate affine Hecke algebras, introduce the filtered deformation of the double centralizer property, and finally accomplish the proof of Theorem 0.4.

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## 1. Preliminaries

Throughout the paper, we always assume $V = \mathbb{C}^{m|n}$ unless otherwise specified, which is a vector superspace with $V = V_0 \oplus V_1$, and the even part and odd part are isomorphic to $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. The general linear Lie supergroup $\text{GL}(V)$ and its Lie superalgebra $\mathfrak{gl}(V)$ are simply written as $\text{GL}(m|n)$ and $\mathfrak{gl}(m|n)$ respectively. We will list some basic material on them, for which the reader can refer to [10], [27] or [28] §6. Note that $\mathfrak{gl}(m|n) \cong \mathfrak{gl}(n|m)$ (see [12] Remark 1.6)). Throughout the paper, we always assume that $m \leq n$ without any loss of generality.

Generally, for a $\mathbb{C}$-vector superspace $M = M_0 \oplus M_1$, we denote by $|m| \in \mathbb{Z}_2$ the parity of a $\mathbb{Z}_2$-homogeneous vector $m \in M_{|m|}$. Throughout the paper, the terminology of ideas, subalgebras, modules etc. of a Lie superalgebra instead of superideals, subsuperalgebras, supermodules, etc. are adopted.

### 1.1. General linear Lie superalgebras and their nilpotent elements.

For $V = \mathbb{C}^{m|n}$, we have the $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$ with the even part $V_0 \cong \mathbb{C}^m$ and the odd part $V_1 \cong \mathbb{C}^n$. Choose ordered bases for $V_0$ and $V_1$ that combine to a homogeneous ordered basis of $V$. We will make it a convention to parameterize such a basis by the set $I(m|n) = \{\overline{1}, \ldots, \overline{m}, 1, \ldots, n\}$ with total order $\{\overline{1} < \cdots < \overline{m} < 1 < \cdots < n\}$ (where $I(m|n)$ will be simply written as $I$ afterwards). The elementary matrices are accordingly denoted by $e_{i,j}$ corresponding to $(i,j)$-entry $(i,j \in I(m|n))$, which stands for the matrix of all zeros in any entry with exception $(i,j)$-entry equal to 1. With respect to such an ordered basis, $\mathfrak{gl}(V)$ can be realized as $\mathfrak{gl}(m|n)$ consisting of $(m+n) \times (m+n)$ complex matrices of the block form $e = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c$ and $d$ are respectively $m \times m$, $m \times n$, $n \times m$ and $n \times n$ respectively.
of $C$ just $g$ and $n$ to the Hopf ideal $GL(m) \times GL(n)$.

The general linear Lie supergroup $GL(m|n)$ can be defined, via the setting-up of affine supergroup schemes, which is a functor sending a commutative $C$-superalgebra $A$ to the group $GL(m|n; A)$ of all invertible $(m+n) \times (m+n)$ matrices of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a$ and $d$ are $m \times m$ and $n \times n$ matrices, respectively, with entries in $A_0$; $b$ and $c$ are $m \times n$ and $n \times m$ matrices, respectively, with entries in $A_1$. It is well known (cf. for example, [17] or [27]) that $g$ is invertible if and only if both $a$ and $d$ are invertible. The Lie superalgebra of $GL(m|n)$ is just $gl(m|n)$ (cf. [17] or [27]).

Let $Mat(m|n)$ be the affine superscheme with $Mat(m|n; A)$ consisting of all $(m+n) \times (m+n)$ matrices of the form (1.1). Then the affine coordinate superalgebra $C[GL(m|n)]$ is the localization of $C[Mat(m|n)]$ at the function $\det : g \mapsto \det(a)\det(d)$. A closed subgroup of $GL(m|n)$ is an affine supergroup scheme with coordinate superalgebra that is a quotient of $C[GL(m|n)]$ by a Hopf ideal $I$. In particular, the underlying purely even group of $GL(m|n)$, denoted by $GL(m|n)_{ev}$, corresponds to the Hopf ideal $C[GL(m|n)]/C[GL(m|n)]I$. That is to say, $GL(m|n)_{ev}$ is completely determined by its coordinate superalgebra that is $C[GL(m|n)]/C[GL(m|n)]C[GL(m|n)]I$. It is well known that $GL(m|n)_{ev} \cong GL(m) \times GL(n)$ (cf. [17] or [27]).

1.1.1. Nilpotent orbits. From now on, we set $g = gl(m|n)$ and set $G$ to be the corresponding Lie supergroup $GL(m|n)$. For an (even) nilpotent element $e \in g_0$, and any of its conjugates $e' = ad(g)e$ for $g \in G_{ev}$, we have $g_e \cong g_{e'}$. Both of them share the same double centralizer property in $\text{End}_C(V^{\otimes d})$. So we can choose nilpotent matrices of Jordan standard forms, as typical representatives in their $G_{ev}$-adjoint orbits for the study of the double centralizer property we are concerned.

Recall that all nilpotent orbits under $G_{ev}$ in $g_0$ are parameterized by all partition $(\lambda, \mu)$ of $(m|n)$, which means that both sequences $\lambda$ and $\mu$ of positive integers with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ for a certain $s$, and $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$ for a certain $t$ satisfy the condition that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$, and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_t > 0$ along with $|\lambda| := \sum_i \lambda_i = m$ and $|\mu| := \sum_j \mu_j = n$. (The order of a partition may be defined in a reversed one, i.e., components go increasing.)

1.1.2. Regular nilpotent elements.

**Definition 1.1.** A nilpotent element $e \in g_0$ is said to be regular or principal if the adjoint orbit $G_{ev}.e$ has the largest dimension in all $G_{ev}$-orbits of the nilpotent cone consisting of all nilpotent elements in $g_0$.

Recall that all regular nilpotent elements in $g$ are in the same $G_{ev}$-orbit. We choose a standard representative $e$ of regular nilpotent elements as below (which is called a canonical regular nilpotent element):

$$e = \sum_{i=1}^{m-1} e_{i,i+1} + \sum_{j=1}^{n-1} e_{j,j+1}.$$

Here and further, $e_{i,j}$ with $i, j \in I := I(m|n)$ stands for the $(m+n) \times (m+n)$ matrix in $gl(m|n)$ with all entries equal to 0 except the $(i, j)$-entry equals 1.
1.2. **Schur-Sergeev duality.** For $V = \mathbb{C}^{|m|n}$ as a natural $\mathfrak{g}$-module, the tensor product module $V^{\otimes d}$ over $\mathfrak{g}$ (represented by $\phi_d$) is defined as in [34]. The symmetry group $\mathfrak{S}_d$ acts on $V^{\otimes d}$ defined as in [14].

**Theorem 1.2.** (Schur-Sergeev duality, see [34] or [12]) The images of $\phi_d$ and $\psi_d$ satisfy the double centralizer property, i.e.,

$$\phi_d(U(\mathfrak{g})) = \text{End}_{\mathfrak{S}_d}(V^{\otimes d}),$$
$$\text{End}_{U(\mathfrak{g})}(V^{\otimes d}) = \psi_d(\mathfrak{S}_d).$$

1.3. **Degenerate affine Hecke algebras in the super case.** Now we introduce the so-called degenerate affine Hecke algebra (dAHA for short) in the super case, which are actually the same as defined in [8]. However, the action on the tensor product of superspaces is different from that one from [8], where the former is introduced by Sergeev. Below we will take use the notation $H_\ell$ for such an algebra, instead of the usual one.

**Definition 1.3.** The degenerate affine Hecke algebra $H_\ell$ is an associative algebra with generators $x_i (i = 1, \ldots, \ell)$ and $s_i (i = 1, \ldots, \ell - 1)$, subjected to relations:

\[
s_i^2 = 1, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \quad s_is_j = s_js_i, \quad |i - j| > 1, \\
x_js_i = s_ix, \quad (j \neq i, i + 1), \\
x_{i+1}s_i - s_ix_i = 1, \\
x_ix_j = x_jx_i, \quad (i \neq j).
\]

2. **Super VUST theorem**

For a finite-dimensional vector space $V = \mathbb{C}^{|m|n}$ over $\mathbb{C}$ and $A \in \text{End}_\mathbb{C}(V)$ whose centralizer subgroup in $G = \text{GL}(m|n)$ is denoted by $G_A$, the classical Vust theorem says as introduced in §11.4 that on $V^{\otimes d}$, the action of $G_A$ and the twisted action of symmetry group $\mathfrak{S}_d$ presented as the algebra $\mathbb{C}[x_1, \ldots, x_d] \otimes \mathfrak{S}_d$ are double-centralizers in $\text{End}_\mathbb{C}(V^{\otimes d})$ (see [39] or [26]).

In this section, we will prove the super VUST theorem (Theorem 0.1), taking the same strategy as in the classical case (see [26] §2.6).

2.1. **Preparation for the proof of Theorem 0.1**

2.1.1. **Adjoint orbits.** Generally, for an affine algebraical supergroup $G$, the adjoint action $Ad$ of $G$ on $\mathfrak{g} := \text{Lie}(G)$ can be naturally defined (for example, see [10] §11.5 or [28] §6). Denote by $Ad$ as usual the adjoint action. Then $Ad$ is defined as the natural transformation $Ad : G \rightarrow \text{GL}(\mathfrak{g})$.

In particular, take $G := \text{GL}(m|n)$ and $\mathfrak{g} = \mathfrak{gl}(m|n)$ over $\mathbb{C}$. Let $e$ be any given nilpotent element with $e \in \mathfrak{g}_0$. Under the action $Ad$, $G.e$ can be regarded a super scheme of $\mathfrak{g}$ defined as

$$G.e(R) = \{Ad(g)(e) = geg^{-1} \mid g \in G(R)\}$$

for any commutative $\mathbb{C}$-superalgebra $R$.

Let $\mathcal{O}_X$ denote the structure sheaf on a super subscheme $X$ of $\mathfrak{g}$. A superscheme can be equivalently defined as a superspace $X = (X, \mathcal{O}_X)$ satisfying that $(X, \mathcal{O}_X)$ is an ordinary subscheme of $\mathfrak{g}_0$ and $(\mathcal{O}_X)_1$ is a quasi-coherent sheaf of $(\mathcal{O}_X)_0 := \mathcal{O}_X$ (see [10] §3.3). Set $\mathbb{C}[X] := \mathcal{O}_X(X)$. A general setting-up for adjoint orbits can be referred to [28] §6.2.

Recall that an affine superscheme $X$ is a closed subscheme of $\mathfrak{g}$ if $\mathbb{C}[X]$ is a $\mathbb{Z}_2$-graded quotient algebra of $\mathbb{C}[\mathfrak{g}]$. The orbit closure $\overline{G.e}$ can be defined as below. The orbit map $g \mapsto g.e := Ad(g)e$ gives a natural transformation of functors $\mu : G \to G.e \subset \mathfrak{g}$. By Yoneda’s lemma (see [10] Lemma
3.4.3), μ induces an algebra homomorphism μ* : C[G.e] → C[G], and G.e is the closed subscheme of g defined by the ideal ker μ*. By definition, there is an embedding

\[ C[G.e] \hookrightarrow C[G] \].

Note that G.ev = GL(m) × GL(n) is a connected algebraic group. So both G.ev,e and G.ev,e are irreducible varieties in g0. Denote by Frac(C[G(ev.e)]) (resp. Frac(C[G(ev.e)])) the fractional field of C[G(ev.e)] (resp. C[G(ev.e)]). Then we have Frac(C[G(ev.e)]) = Frac(C[G(ev.e)]) because G.ev,e is an open subset in G.ev,e (see [11] or [23]).

Now set Y = G.e and Z = G.e. Thanks to [28, Theorem 6.8], dim \( \bar{Z} \) = dim \( \bar{G} \) = dim \( \bar{G.e} \). By the above arguments, we have the following

\[ \text{(2.1)} \quad \text{Frac}(\mathbb{C}[Y]) = \text{Frac}(\mathbb{C}[G.e]) = \text{Frac}(\mathbb{C}[Z]). \]

2.1.2. Fractional algebras. We introduce the fractional superalgebras of C[Y] and of C[Z] for Y = G.e and Z = G.e respectively. Set

\[ \text{Frac}(\mathbb{C}[Y]) := \text{Frac}(\mathbb{C}[Y]) \otimes_{\mathbb{C}[Y]} \mathbb{C}[Y], \]

\[ \text{Frac}(\mathbb{C}[Z]) := \text{Frac}(\mathbb{C}[Z]) \otimes_{\mathbb{C}[Z]} \mathbb{C}[Z]. \]

Lemma 2.1. Let G = GL(m|n) and g = gl(m|n), and let e ∈ g0 be a nilpotent element. Then the following statements hold.

1. \( \text{Frac}(\mathbb{C}[G.e]) = \text{Frac}(\mathbb{C}[G.e]) \).
2. \( \mathbb{C}[G.e] = \mathbb{C}[G.e] \).

Proof. (1) Note that the superscheme G.e is of dimension \( \text{dim}(G.e, \text{dim}(\mathbb{C}[Y])) \) (see [28, Theorem 6.8]). And the superscheme G.e has the same dimension as the previous one. Keep the notations as in (2.2). Note that C[Z] is a subalgebra of C[Y]. By definition, C[Y] = C[i Y i], C[Z] = C[Z i]. And both C[Y] and C[Z] are finite-dimensional, with the same dimension. So C[Z] = C[Y]. From (2.1), the statement follows.

(2) According to the nilpotent orbit theory of general linear Lie algebras, the closure of the nilpotent G.ev,e is normal (see [26] or [23]), so \( \mathbb{C}[G(ev.e)] = \mathbb{C}[G(ev.e)] \). By definition, i G.e is an irreducible closed subset in g0. Now the superscheme G.e is of dimension \( \text{dim}(G.e, \text{dim}(\mathbb{C}[Y])) \), including a closed subset G.ev,e. It is deduced that i G.e := G.ev,e. So, C[Y] = C[Z] = C[G.ev,e] = C[G.ev,e]. It is already known above that C[G.e] = C[G.e]. Consequently, C[G.e] = C[G.e].

2.1.3. For a given e ∈ g0, consider the centralizer supergroup G.e of e in G, as a closed subgroup of G (see [10, §11.8], or see [22, §1.2.6] as an analogue of group schemes). The following fact is clear, as an analogue of the classical result (see [20, Proposition 10.6]).

Lemma 2.2. Keep the notations as in Lemma 2.1. Then \( \text{Lie}(G.e) = g.e \).

2.2. From now on till §2.4, we prove Theorem 1.11 First of all, by a direct computation it is easy to verify that

\[ \text{End}_{\rho(e)}(V^{\otimes d}) \supseteq \mathcal{A}. \]

We only need to prove the other direction. For the simplicity of the subsequent arguments, we will adopt some technique of formulations of super groups which could be familiar to some physicists (see [18]).
2.2.1. Associated with an $N$-dimensional vector space over $\mathbb{C}$, one has the Grassmannian algebra which is only dependent on the degree $N$ up to isomorphism, denoted by $\Lambda(N)$. Set $\Lambda = \varinjlim \Lambda(N)$, which becomes an infinite-dimensional commutative superalgebra over $\mathbb{C}$.

Let $V := V_{\mathbb{C}} \otimes_{\mathbb{C}} \Lambda$, $V^* = \text{Hom}_{\Lambda}(V, \Lambda)$. The general linear Lie supergroup on $V$ can be defined as:

$$GL(V) \cong \{ g \in \text{End}_{\Lambda}(V)_0 \mid g \text{ is invertible} \}.$$ 

Let $\widehat{\mathfrak{gl}(V)} = \mathfrak{gl}(V) \otimes_{\mathbb{C}} \Lambda$ which is regarded as a Lie superalgebra over $\Lambda$ in the sense that there is a $\Lambda$-bilinear Lie bracket on $\widehat{\mathfrak{gl}(V)}$, via $[X \otimes a, Y \otimes b] = [X, Y] \otimes (-1)^{|a||Y|} ab$ for $\mathbb{Z}_2$-homogeneous $X,Y \in \mathfrak{gl}(V)$ and $a,b \in \Lambda$. Then $\mathfrak{gl}(V) = (\mathfrak{gl}(V) \otimes_{\mathbb{C}} \Lambda)_0$ forms a Lie subalgebra of $\widehat{\mathfrak{gl}(V)}$ over $\Lambda_0$, which can be referred to as the Lie algebra of $GL(V)$. The action of $\mathfrak{gl}(V)$ on $V$ is exactly the restriction of the one of $\widehat{\mathfrak{gl}(V)}$ on $V$, the latter of which is clear by its definition.

In this new formulation, we have the following basic fact.

**Lemma 2.3.** ([24 Lemma 2.6]) The following hold:

1. For any $X \in \mathfrak{gl}(V)$, let $\exp(X) := \sum_{r=0}^{\infty} \frac{X^r}{r!}$. Then $\exp(X)$ is well defined as an automorphism of $V$ which lies in $GL(V)$. Hence, there is a mapping

$$\exp : \mathfrak{gl}(V) \to GL(V), X \mapsto \exp(X).$$

2. The image of $\mathfrak{gl}(V)$ under $\exp$ is $GL(V)$.

2.2.2. Let us investigate $\mathcal{B} := \text{End}_{\rho(G.e)}(V^\otimes d)$ (for simplicity we do not distinguish the notations of $\mathfrak{gl}(V)$-action and $GL(V)$-action on $V^\otimes d$), which is exactly $\text{End}_{\rho(G.e)}(V^\otimes d) \otimes \Lambda$ by Lemma 2.2. Note that $\mathcal{B}$ contains the subalgebra $\text{End}_{\Lambda}(V^\otimes d)_0^G$, the latter of which by Schur-Sergeev duality, is just the image $\pi_d(\mathfrak{g}_d)$ in $\text{End}_{\Lambda}(V^\otimes d)_0$ by base change. Then $\mathcal{B}$ contains $\pi_d(\mathfrak{g}_d)$ and $e$, thereby contains the subalgebra generated by them. We need to decide the exact generators of $\mathcal{B}$. For this, we first recall a general fact on the group action. Suppose $H$ is a group, and $X_i$, $i = 1,2$ are two $H$-sets. If two points $x_i \in X_i$, $i = 1,2$ satisfy the condition concerning the stabilizer subgroups $\text{Stab}_H(x_1) \subset \text{Stab}_H(x_2)$, then one can define $H$-equivariant map from $G.x_1 \subset X_1$ to $G.x_2 \subset X_2$ which sends $x_1$ onto $x_2$. We have the following basic observation.

**Lemma 2.4.** Suppose $H$ is a connected reductive algebraic group over $\mathbb{C}$, $e \in \text{Lie}(H)$ is a regular nilpotent element. Suppose $(\rho, M)$ is an $H$-module with $b \in \text{End}_{\mathbb{C}}(M)$ satisfying $\text{Stab}_H(e) \subset \text{Stab}_H(b)$. Then there exists an $H$-equivariant morphism $\varphi$ from $\text{Lie}(H)$ to $\text{End}_{\mathbb{C}}(M)$ such that $\varphi(e) = b$.

**Proof.** Denote $\mathcal{H} = \text{Lie}(H)$. By the arguments before the lemma, there is a $H$-equivariant morphism $\varphi$ from $G.e$ to $G.b$. Note that $e$ is a regular nilpotent, thereby $H.e$ is an open dense set in the nilpotent cone $\mathcal{N}(H)$ of all nilpotent elements in $\mathcal{H}$. Hence $\varphi$ can be extended and becomes an $H$-equivariant morphism from $\mathcal{N}(H)$ to $\text{End}_{\mathbb{C}}(M)$. On the other hand, classical Jordan-Chevalley decomposition theorem says that for any $h \in \mathcal{H}$ we have $h = h_s + h_n$ for $h_s \in \mathcal{H}$ semisimple and $h_n \in \mathcal{H}$ nilpotent, satisfying $[h_s,h_n] = 0$. The set of all semisimple elements in $\mathcal{H}$ are $H$-invariant. Hence $\varphi$ can be naturally extended to be the desired one from $H$ to $\text{End}_{\mathbb{C}}(M)$. \(\square\)

**Lemma 2.5.** For any $b \in \mathcal{B} = \text{End}_{\rho(G.e)}(V^\otimes d)$, there is a $G$-equivariant morphism

$$\Phi : \text{End}_{\Lambda}(V) \to \text{End}_{\Lambda}(V^\otimes d) = \text{End}_{\Lambda}(V) ^\otimes d$$

such that $\Phi(e) = b$. 
Proof. Note that $G$ acts on $\text{End}_{\Lambda}(V)$ and $\text{End}_{\Lambda}(V^{\otimes d})$ by conjugation respectively (as a realization in Lemma 2.3). From the assumption $b \in (\text{End}_{\Lambda}V^{\otimes d})^G$, it follows that $\text{Stab}_G(e) \subset \text{Stab}_G(b)$. What remains to do is to apply Lemma 2.3.

**Lemma 2.6.** Keep the notations and assumptions as in Lemma 2.5. Let $\mathcal{V} \triangleq \{ \psi : \text{End}_{\Lambda}(V) \to \text{End}_{\Lambda}(V)^{\otimes d} | \psi \circ g = g \circ \psi, \forall g \in G \}$. Then $\mathcal{V}$ is a $\Lambda$-module generated by the elements as follows:

$$X \mapsto \sigma \cdot (X^{h_1} \otimes X^{h_2} \otimes \ldots \otimes X^{h_d}),$$

where $h_i \in \mathbb{Z}_{\geq 0}$, and $\sigma \in \mathcal{S}_d$ gives rise to a permutation on the positions of tensor homogeneous parts.

**Proof.** Recall that $\mathcal{V} = \{ \psi : \text{End}_{\Lambda}(V) \to \text{End}_{\Lambda}(V)^{\otimes d} | \psi \circ g = g \circ \psi, \forall g \in G \}$, where $g \cdot (X_1 \otimes \ldots \otimes X_d) = \text{Ad}_g(X_1) \otimes \ldots \otimes \text{Ad}_g(X_d)$. For a given $\Phi \in \mathcal{V}$, we define a $\Lambda$-function $\bar{\Phi}$ on $\text{End}(V)^{\otimes (d+1)}$, via

$$\bar{\Phi} : X \otimes Y_1 \otimes \ldots \otimes Y_d \mapsto \text{str}(\Phi(X)(Y_1 \otimes \ldots \otimes Y_d)),$$

which is $G$-invariant, and $\Lambda$-linear in $Y_1, Y_2, \ldots, Y_d$. Denote by $\mathcal{R}$ the $\Lambda$-ring of all $G$-invariant functions on $\text{End}(V)^{\otimes (d+1)}$. By the non-degenerate property of the supertrace we see that the mapping $\Phi \to \bar{\Phi}$ is injective from $\mathcal{V}$ to $\mathcal{R}$.

On the other hand, all elements in $\mathcal{R}$ are of the form as follows

$$\sum a_X \text{str}(X^{q_1}Y_{i_1}X^{q_2}Y_{i_2} \ldots)\text{str}(X^{p_1}Y_{j_1}X^{p_2}Y_{j_2} \ldots)\ldots \text{str}(X^{s_1}Y_{t_1}X^{s_2}Y_{t_2} \ldots),$$

where $a_X \in \Lambda$, $q_i, p_j, s_k \in \mathbb{Z}_{\geq 0}$ and $\sigma = (i_1i_2 \ldots)(j_1j_2 \ldots)(t_1t_2 \ldots) \in \mathcal{S}_d$ in the expression is a product of some cycles not intersecting mutually. This can be ensured by the forthcoming Lemma 2.7(1). In addition, from Lemma 2.7(2)-(3) it follows that these elements are of the form

$$\text{str}(\Phi(X)(Y_1 \otimes Y_2 \otimes \ldots \otimes Y_d)),$$

where $\Phi(X)$ is $\mathcal{R}$-spanned by the mapping $X \mapsto \sigma \cdot (X^{h_1} \otimes X^{h_2} \otimes \ldots \otimes X^{h_d})$.

The injective property of the mapping sending $\Phi$ to $\bar{\Phi}$ entails that the second statement of the lemma. The proof is completed.

2.3. Note that there is an isomorphism

$$\pi : ((V^*)^{\otimes d} \otimes V^{\otimes d})^* \cong \text{End}_{\Lambda}(V^{\otimes d}) = (\text{End}_{\Lambda}(V))^{\otimes d}$$

determined by the non-degenerate bilinear form as what we define for $\Omega \in (\text{End}_{\Lambda}V^{\otimes d})$, $v_1 \otimes v_2 \otimes \ldots \otimes v_d \in V^{\otimes d}$ and $f_1 \otimes f_2 \otimes \ldots \otimes f_d \in (V^*)^{\otimes d}$ such that

$$\langle \Omega, f_1 \otimes f_2 \otimes \ldots \otimes f_d \otimes v_1 \otimes \ldots \otimes v_d \rangle$$

$$= \langle f_1 \otimes f_2 \otimes \ldots \otimes f_d, \Omega(v_1 \otimes \ldots \otimes v_d) \rangle.$$

Here and further, we use the notation $\langle \cdot, \cdot \rangle$ to stand for the valuation of the specific pair of linear duals.

Now we separately list some results as below, which were used previously in the proof of Lemma 2.6.

**Lemma 2.7.** (1) The $G$-invariants of $\Lambda$-function ring in $\text{End}_{\Lambda}(V)^q$ are generated by the supertrace functions at $(X_1, \ldots, X_q) \in \text{End}_{\Lambda}(V)^q$ (the space of $q$-tuples of elements of $\text{End}_{\Lambda}(V)$):

$$\text{str}(X_{i_1}X_{i_2} \cdots X_{i_j}), i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, q\}.$$
(2) Furthermore, the $G$-invariants in $\text{End}_A(V)^{\otimes d}$ are $\Lambda$-linear combinations of the following basic invariants:

$$\theta_\sigma : h_1 \otimes h_2 \otimes \cdots \otimes h_d \otimes v_1 \otimes \cdots \otimes v_d \mapsto (-1)^{J(h,v,\sigma)} \prod_j \langle h_{\sigma(j)}, v_j \rangle$$

where $\sigma \in \mathfrak{S}_d$, $J(h,v,\sigma) = \sum_{i=1}^d t_i$ with $t_i = |v_i| \sum_{j=1}^d |h_{\sigma(j)}|$. 

(3) Under the identification between $(V^*)^d \otimes V^d$ and $\text{End}_A(V)^{\otimes d}$, we have

$$\theta_\sigma(X_1 \otimes \cdots \otimes X_d) = \text{str}(X_1 \cdots X_i) \text{str}(X_{j_1} \cdots X_{j_k}) \cdots \text{str}(X_{k_1} \cdots X_{k_t})$$

where $X_1, X_2, \ldots, X_d \in \text{End}_A(V)$, and $\sigma = (i_1 i_2 \cdots i_r)(j_1 j_2 \cdots j_s)(k_1 k_2 \cdots k_t) \in \mathfrak{S}_d$ is an expression in product of cycles not intersecting mutually.

Proof. For (1) and (3), the proof is the same as in [33, Theorem 1.3], and we omit it here. In the following, we only deal with the second statement.

(2) By [24, Theorem 3.6] we know that $\theta_\sigma$ is $G$-invariant, which can be achieved by the same arguments in the proof of [33, Theorem 1.1]. According to Schur-Sergeev duality (see [34]), $\text{End}_G(V^{\otimes d})$ is generated by the twisted action of the generators $(i, i + 1) \in \mathfrak{S}_d$ as below

$$(i, i + 1) \cdot v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_d = (-1)^{|v_i||v_{i+1}|} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_d, \quad 1 \leq i \leq d - 1.$$ 

Recall that there is an isomorphism as presented previously

$$\pi : ((V^*)^{\otimes d} \otimes V^{\otimes d})^* \cong \text{End}(V^{\otimes d}) = (\text{End}(V))^{\otimes d}.$$ 

Hence, the generators of invariants in the Schur-Sergeev duality become $\theta_\sigma$ for $\sigma \in \mathfrak{S}_d$ which are defined via

$$\langle \theta_\sigma, h_1 \otimes h_2 \otimes \cdots \otimes h_d \otimes v_1 \otimes \cdots \otimes v_d \rangle = (-1)^{J(h,v,\sigma)} \prod_j \langle h_{\sigma(j)}, v_j \rangle.$$ 

\[ \square \]

2.4 Summation for the proof of Theorem 0.1. Let us turn back to the proof of Theorem 0.1. Let $B_C = \text{End}_{\rho(\mathfrak{g}_e)}(V^{\otimes d})$, a subalgebra of $\text{End}_C(V^{\otimes d})$. By our analysis at the beginning of §2.2, we need to decide the generators of $B_C$. One can easily reformulate the $C$-versions of Lemmas 2.5, 2.6 and 2.7 just replacing $A$ with $C$ in the original statements.

By Lemma 2.5 for any $b \in B_C$ there is a $G$-equivariant morphism $\Phi : \text{End}_C(V) \to \text{End}_C(V^{\otimes d}) = \text{End}_C(V)^{\otimes d}$ such that $\Phi(e) = b$. By Lemma 2.6 such a $\Phi$ is generated by some elements of the form: $X \mapsto \sigma \cdot (X^{h_1} \otimes X^{h_2} \otimes \cdots \otimes X^{h_d})$ for $\sigma \in \mathfrak{S}_d$. Hence the theorem follows.

3. DISTRIBUTION SUPER ALGEBRAS AND DOUBLE CENTRALIZERS

As a consequence of Theorem 0.1, the second equation of Theorem 0.3 has been already proved. In this section, we will accomplish the remaining proof of Theorem 0.3 by the same strategy as in §8.

Maintain the notations in §1. In particular, keep in mind that $V = C^m|_e$, $\mathfrak{g} = \mathfrak{gl}(V)$ and $e \in \mathfrak{g}_0$ is a regular nilpotent element. Recall that as in the introduction, we have the action $\psi_d$ of $C_n[x_1, \ldots, x_d] \otimes C\mathfrak{S}_d$ on $V^{\otimes d}$, and the action $\phi_d$ of $U(\mathfrak{g}_e)$ on $V^{\otimes d}$. In order to prove that the centralizer of $\psi_d(C_n[x_1, \ldots, x_d] \otimes C\mathfrak{S}_d)$ is exactly $\phi_d(U(\mathfrak{g}_e))$, we will realize the latter by the action of $C[M_e]^*_d$ on $V^{\otimes d}$, where $C[M_e]^*_d$ is the homogeneous dual space of polynomial degree $d$ of $C[M_e]$ (see §3.3). To carry this out, we will take use of the description of $\mathfrak{g}_e$ via “pyramid” expression technique, along with the description of $U(\mathfrak{g}_e)$ by the distribution algebra of $M_e$. 
3.1. **Pyramids and a pyramid description of $\mathfrak{g}_e$.** We take a standard regular nilpotent element $e := \sum_{i=1}^{m-1} e_{i,i+1} + \sum_{j=1}^{n-1} e_{j,j+1} \in \mathfrak{g}_0$. Accordingly, we have a pyramid associated with $(m,n)$. Roughly speaking, a pyramid is a graph consisting of boxes, with connected horizontal lines of boxes. Corresponding to $(m,n)$, we have the following pyramid

\[(3.1) \quad \Xi = \begin{array}{cccccc}
1 & \cdot & \cdot & \cdot & 12 \cdot & n \\
1 & 2 & 3 & \cdot & \cdot & \cdot
\end{array} \]

Let us define the coordinates for all $i$ and $j$ appearing in the pyramid $\Xi$. Define the coordinate of $i$ to be $(1,i)$ for $i = 1, \ldots, n$, and the coordinate of $j$ to be $(\bar{1},j)$ for $j = 1, \ldots, m$. Then we have the following set:

- $I := I(m|n) = \{1, \ldots, \bar{m}; 1, \ldots, n\}$,
- $J := \{(\ell, h) \in I \times I : \ell = 1 \text{ or } 1 - n - m \leq h \leq n \text{ if } (\ell, h) = (\bar{1},j) \text{ for } j \in \{1, \ldots, n\}\}$,
- $K := \{1, \bar{1}, 0 \}, \ldots, (\bar{1},1, m-1); (1,1, n-m-1), \ldots, (\bar{1},1, n-1); (1,\bar{1},0), \ldots, (1,\bar{1}, m-1); (1,1,0), \ldots, (1,1, n-1)\}.$

We define a map from $J$ to $K$ by

\[(3.2) \quad v : (i,j) \mapsto (\text{row}(i), \text{row}(j), \text{col}(j) - \text{col}(i))\]

which is bijective. There is a well-known explicit description of the centralizer of any nilpotent element in a general linear Lie algebra associated to a pyramid (see [2] Lemma 7.3 and [38, IV.1.6]). The same description is still true for our regular nilpotent $e \in \mathfrak{gl}(m|n)$ associated to $\Xi$ by the same discussion as in [3, Lemma 4.2].

**Lemma 3.1.** For $(i,j,r) \in K$, set

\[(3.3) \quad e_{i,j,r} = \sum_{\substack{h,k \in I \\text{row}(k) = i, \text{row}(k) = j \\text{col}(k) - \text{col}(h) = r}} e_{h,k}.\]

Then $\{e_{i,j,r} \mid (i,j,r) \in K\}$ constitute a basis of $\mathfrak{g}_e$, and all of them are $\mathbb{Z}_2$-homogeneous.

Alternatively, for $(i,j) \in J$ we set

\[(3.4) \quad \xi_{i,j} := e_{i,j; \text{col}(j) - \text{col}(i)}.\]

Then $\{\xi_{i,j} \mid (i,j) \in J\}$ constitute a basis of $\mathfrak{g}_e$.

**Proof.** Note that $e \in \mathfrak{g}_0$. The centralizer of $e$ in $\mathfrak{g} = \mathfrak{gl}(m|n)$ is the same as a vector space as the centralizer of $e$ viewed as an element of $\mathfrak{gl}(m+n)$. The lemma follows from [38, IV.1.6] or [2] Lemma 7.3. \qed

3.2. **Some special notations for tensor products.** We need some other notations.

3.2.1. Let $I^d$ and $J^d$ denote respectively, a collection of all $d$-tuples $i = (i_1, \ldots, i_d)$ and a collection of all pairs $(i,j)$ with $i = (i_1, \ldots, i_d), j = (j_1, \ldots, j_d)$ satisfying $(i_k, j_k) \in J$. Denote by $K^d$ a collection of all triples $(i,j,r)$ with $i = (i_1, \ldots, i_d), j = (j_1, \ldots, j_d), r = (r_1, \ldots, r_d)$ satisfying each $(i_k, j_k, r_k) \in K$.

As a generalization of (3.2), there is a bijective map

\[\Upsilon : J^d \longrightarrow K^d\]

defined as

\[(i,j) \mapsto (\text{row}(i), \text{row}(j), \text{col}(j) - \text{col}(i)).\]
where $\text{row}(i) = (\text{row}(i_1), \ldots, \text{row}(i_d))$, and $\text{col}(i) = (\text{col}(i_1), \ldots, \text{col}(i_d))$.

3.2.2. For $i = (i_1, \ldots, i_d) \in I^d$, we will set $\epsilon_i := (|i_1|, \ldots, |i_d|) \in \mathbb{Z}_2^d$. And set $|\epsilon_i| = |i_1| + \cdots + |i_d|$. Furthermore, following [5] we adopt some more notations as below. For $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_d)$, and $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{Z}_2^d$, set

$$\alpha(\epsilon, \delta) := \prod_{1 \leq s < t \leq d} (-1)^{\delta_s \epsilon_t}.$$

Then we have $(X_1 \otimes X_2 \otimes \cdots \otimes X_d)(w_1 \otimes \cdots \otimes w_d) = \alpha((|X_1|, \ldots, |X_d|), (|w_1|, \ldots, |w_d|))X_1(w_1) \otimes \cdots \otimes X_d(w_d)$ for $\mathbb{Z}_2$-homogeneous elements $X_i \in \text{End}_\mathbb{C}(V_{|X_i|})$, and $w_j \in V_{|w_j|}$, $i, j \in I$, which enables us to identify $\text{End}_\mathbb{C}(V^{\otimes d})$ with $\text{End}_\mathbb{C}(V^{\otimes d})$.

3.2.3. Note that any $\sigma \in S_d$ acts on the right on $I^d$ by permutation; this is to say, $i.\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(d)})$ for $i = (i_1, \ldots, i_d) \in I^d$. For $\epsilon \in \mathbb{Z}_2^d$ and $\sigma \in S_d$, following [5] again we set

$$\nu(\epsilon, \sigma) := \prod_{1 \leq s < t \leq d} (-1)^{\epsilon_s \epsilon_t}.$$

Arising from $\psi$ in (0.4), the action of $S_d$ on the right on $V^{\otimes d}$ becomes $(w_1 \otimes \cdots \otimes w_d).\sigma = \nu((|w_1|, \ldots, |w_d|), \sigma)w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(d)}$.

We can further define the action of $S_d$ on the right on $J^d$ diagonally, which means $(i, j).\sigma := (i.\sigma, j.\sigma)$ for any $\sigma \in S_d$. This conjugation is denoted by $(i, j)\overset{\sigma}{\curvearrowright}(h, k)$ for $(h, k) = (i, j).\sigma$. Similarly, $S_d$ acts diagonally on $K^d$. Choose the sets of orbit representatives $J^d/S_d$ and $K^d/S_d$. For $i \in I^d$, write $\text{row}(i) = (\text{row}(i_1), \ldots, \text{row}(i_d))$ and define $\text{col}(i)$ similarly. In the same sense, set $\text{col}(j) = (\text{col}(j_1), \ldots, \text{col}(j_d))$. Then the map $\Upsilon : J^d \to K^d$, $(i, j) \mapsto (\text{row}(i), \text{row}(j), \text{col}(j) - \text{col}(i))$ is $S_d$-equivariant.

We can write

$$(3.5) \quad \Upsilon(i, j) \overset{\sigma}{\curvearrowright}\Upsilon(s, t)$$

for $(\text{row}(i), \text{row}(j), \text{col}(j) - \text{col}(i)).\sigma = (\text{row}(s), \text{row}(t), \text{col}(t) - \text{col}(s))$.

3.3. The distribution superalgebra associated to $e$. Let $M_e$ be a superalgebra consisting of all matrices in $\text{Mat}(m|n; \mathbb{C})$ that commute with $e$. It is obvious that $M_e$ is equal to $\mathfrak{g}_e$ as vector spaces. There is a $\mathbb{Z}_2$-graded structure on the coordinate ring $\mathbb{C}[M_e]$. Hence

$$(3.6) \quad \mathbb{C}[M_e] \cong S((\mathfrak{g}_e^1 \otimes \Lambda((\mathfrak{g}_e^1)^*))).$$

This superalgebra is super-commutative, and linearly spanned by canonical monomial elements, which can be described as below (modulo the order)

$$(3.7) \quad \prod_{(i, j) \in J} x_{\text{row}(i), \text{row}(j); \text{col}(j) - \text{col}(i)}^{s_{i,j}}$$

where $s_{i,j} \in \mathbb{Z}_{\geq 0}$ if $|\text{row}(i)| + |\text{row}(j)| = \overline{0}$, and $s_{i,j} \in \{0, 1\}$ if $|\text{row}(i)| + |\text{row}(j)| = \overline{1}$, and the factors $\{x_{h,k;t} \in M_e^* \mid (h, k, t) \in K\}$ are the linear duals of $\{e_{i,j,r} \in M_e \mid (i, j, r) \in K\}$ as in Lemma 3.1 with $(e_{i,j,r}, x_{h,k;t}) = \delta_{(i,j,r),(h,k,t)}(-1)^{|i|+|j|}$.

The super-commutative superalgebra $\mathbb{C}[M_e]$ can be endowed a Hopf superalgebra structure via

$$\Delta(x_{i,j,r}) = \sum_{r_1+r_2=r; (i,h,r_1),(j,h,r_2) \in K} (-1)^{|(i)|+|h|+|j|+|l|} x_{i,h,r_1} \otimes x_{j,h,r_2}.$$
defined via \((xy)f = (x \bar{\circ} y)\Delta(f)\) for \(x, y \in \mathbb{C}[M_e]^*\) and \(f \in \mathbb{C}[M_e]\). Here and further the notation \(\bar{\circ}\) means the operation obeying in the superalgebra rule of signs as:
\[
(x \bar{\circ} y)(f \bar{\circ} g) = (-1)^{|g||f|}x(f)y(g).
\]
Furthermore, \(\text{Dist}(M_e)\) can be endowed with a Hopf algebra structure, which is isomorphic to \(U(\mathfrak{g}_e)\) as Hopf superalgebras (see \([8 \S 2.2]\) or \([35 \S 2.2]\)).

3.3.1. Set the degree of each polynomial generator to be 1. Then \(\mathbb{C}[M_e]\) is endowed with a graded algebra structure:
\[
\mathbb{C}[M_e] = \bigoplus_{i \geq 0} \mathbb{C}[M_e]_i.
\]

Moreover, each graded dual subspace \(\mathbb{C}[M_e]^*_d\) of the dual algebra \(\mathbb{C}[M_e]^*\) is finite dimensional.

Recall that \([27]\) provides a basis of the coordinate superalgebra \(\mathbb{C}[M_e]\). Then \(\mathbb{C}[M_e]^*_d\) has the corresponding dual basis.

Now we write a set of basis elements for \(\mathbb{C}[M_e]^*_d\) specifically. It follows from \([16,6]\) that the base of \(\mathbb{C}[M_e]^*_d\) is of the form \(x_{i,j,k} = x_{i,j,k_1} \cdots x_{i,j,k_d}\) for \((i,j,k) \in K^d / \mathcal{S}_d\). Now we introduce a set of basis elements \(\xi_{i,j,k} \mid (i,j,k) \in K^d / \mathcal{S}_d\) of \(\mathbb{C}[M_e]^*_d\) dual to \(\{x_{i,j,k} \mid (i,j,k) \in K^d / \mathcal{S}_d\}\). Note that the bijective map \(\Upsilon : J^d \rightarrow K^d\) satisfies \(\mathcal{S}_d\)-equivariance and the property \(\nu(\epsilon_i + \epsilon_j, \sigma) = \nu(\epsilon_{\text{row}(i)} + \epsilon_{\text{row}(j)}, \sigma)\) because \(\epsilon_i = \epsilon_{\text{row}(i)}\) for any \(i \in I^d\). Then for any \((i,j) \in J^d\) we have
\[
\langle \xi_{\Upsilon(i,j)}, x_{\Upsilon(i,j)} \rangle = \alpha(\epsilon_i + \epsilon_j, \epsilon_i + \epsilon_j)
\]
and
\[
\langle \xi_{\text{row}(i), \text{row}(j); \text{col}(j) - \text{col}(i)}, x_{s, t; \text{col}(t) - \text{col}(s)} \rangle = \begin{cases} 
\alpha(\epsilon_i + \epsilon_j, \epsilon_i + \epsilon_j)\nu(i + j, \sigma), & \text{if there exists } \sigma \in \mathcal{S}_d \text{ such that } \Upsilon(i,j) = \Upsilon(s,t), \\
0, & \text{otherwise.}
\end{cases}
\tag{3.8}
\]

3.4. By the above arguments, we can identify \(U(\mathfrak{g}_e)\) with \(\text{Dist}(M_e)\). Let us define an algebra homomorphism:
\[
\pi_d : U(\mathfrak{g}_e) \to \mathbb{C}[M_e]^*_d
\]
such that for all \(u \in U(\mathfrak{g}_e), x \in \mathbb{C}[M_e]^*_d\), we have \(\pi_d(u)(x) = \langle u, x \rangle\).

**Lemma 3.2.** \(\pi_d\) is surjective.

**Proof.** The arguments are the same as in the Lie algebra case (see \([8 \text{ Lemma 2.1}]\)). Suppose the contrary. Then we can find \(0 \neq x \in \mathbb{C}[M_e]^*_d\) such that \(\pi_d(u)(x) = 0\) for all \(u \in U(\mathfrak{g}_e)\). By identifying \(U(\mathfrak{g}_e)\) with \(\text{Dist}(M_e)\), we have \(u(x) = 0\) for each \(m \geq 1\) and all \(u \in \mathbb{C}[M_e]^*\) with \(u((\ker \varepsilon)^m) = 0\). Note that \(u((\ker \varepsilon)^m) = 0\) if and only if \(x \in (\ker \varepsilon)^m\) by definition. So \(x \in (\ker \varepsilon)^m\). But we know \(\bigcap_{m \geq 1}(\ker \varepsilon)^m = 0\), which is a contradiction. \(\square\)

3.5. **Comodule structure on \(V^\otimes d\).** Let us first observe that \(\mathfrak{g}_e\)-module \(V\) can be endowed with a natural right \(\mathbb{C}[M_e]\)-comodule structure as follows.

Denote by \(\rho\) the action of \(\mathfrak{g}\) on the natural module \(V\). Fix basis elements \(v_1, \ldots, v_m, v_1, \ldots, v_n\) of \(V\) such that \(v_1, \ldots, v_m \in V_0\) and \(v_1, \ldots, v_n \in V_1\). We can write
\[
\rho(x)v_i = \sum_{j=1}^m \rho_{ji}(x)v_j + \sum_{j=1}^n \rho_{ji}(x)v_j, \quad \rho_{ji} \in \mathfrak{g}^*.
\]
3.5.2. Let us return to the case (3.11), with the initial step (3.12). initiated from (3.9), we have that comodule structure on
\[
(3.13)
\]
where the requirement of the parameters in the sum means that
\[
s = (s_1, \ldots, s_d) \in \mathbb{N}^d
\]
and
\[
t = (t_1, \ldots, t_d) \in \mathbb{N}^d
\]
Consequently, the structure of comodule on \(V^{\otimes d}\) can be described arising from
\[
\Delta_1(v_i) = \sum_{j=1}^{m} v_j \otimes \rho_{ji} + \sum_{j=1}^{n} (-1)^{|j|+|i|} v_j \otimes \rho_{ji}.
\]

Turning to the \(g_n\)-module over \(V\), we first have

**Lemma 3.3.** The comodule structure on \(V\) can be given as follows:
\[
\Delta^{(1)} : V \to V \otimes \mathbb{C}[M]_c
\]
\[
v_i \mapsto \sum_{j \in I: (j,i) \in I} (-1)^{|j|+|i|} v_j \otimes x_{v(j,i)}.
\]

**Proof.** It follows by a straightforward computation. \(\square\)

**Remark 3.4.** In [6, §10] Brundan-Kleshchev gave some similar arguments for Lie superalgebras of type \(Q\).

3.5.1. Generally, for a given bi-superalgebra \(A\), and a (super) comodule \(C\) over \(A\) with defining map \(\Delta^{(1)} : C \to C \otimes A\) of the image \(\Delta^{(1)}(c) = \sum c_0 \otimes c_1\), using \(A\)-comodule structure on \(C\) one can define an \(A\)-comodule on \(C \otimes C\) as below (see [1] for the details):
\[
\Delta^{(2)}(c \otimes d) = \sum c_0 \otimes d_0 \otimes c_1 d_1, \quad \forall c, d \in C.
\]

Inductively, one can endow \(C^{\otimes d}\) with \(A\)-comodule structure for any positive integer \(d\), of which the defining map is
\[
\Delta^{(d)} : C^{\otimes d} \to C^{\otimes d} \otimes A.
\]

Furthermore, \(A^*\) is also a superalgebra. One can define a super \(A^*\)-module on \(C\) as below
\[
f.c = \sum \langle f, c_1 \rangle c_0, f \in A^*, c \in C.
\]

Inductively, one can endow \(C^{\otimes d}\) with \(A^*\)-module structure for any positive integer \(d\) arising from (3.11), with the initial step (3.12).

3.5.2. Let us return to the case \(\mathbb{C}[M]_c\). Recall that \(\mathbb{C}[M]_c\) is a bi-superalgebra, and there is (super) comodule structure on \(V^{\otimes d}\), with comodule map \(\Delta^{(d)} : \mathbb{C}[M]_c \to V^{\otimes d} \otimes \mathbb{C}[M]_c\), which is defined inductively from (3.13). To be precise, for any \(t = (t_1, \ldots, t_d) \in I^d\) and any monomial basis vector \(v_t := v_{t_1} \otimes v_{t_2} \otimes \cdots \otimes v_{t_d} \in V^{\otimes d}\), combining Lemma 3.3 along with (3.10)-(3.11), and the induction initiated from (3.13), we have that
\[
\Delta^{(d)}(v_t) = \sum_{s \in I^d \setminus \{t\}} (-1)^{|s_1|+|\epsilon_t|} \alpha(\epsilon_s, s_1 \otimes \epsilon_t) v_s \otimes x_{Y(s,t)}
\]
where the requirement of the parameters in the sum means that
\[
s = (s_1, s_2, \ldots, s_d) \in I^d
\]
and
\[
t = (t_1, \ldots, t_d) \in I^d
\]
Consequently, \(V^{\otimes d}\) becomes a \(\mathbb{C}[M]_c^*\)-module. This representation of \(\mathbb{C}[M]_c^*\) on \(V^{\otimes d}\) is denoted by \(\omega_d\) which will play a critical role in the sequel.
Lemma 3.6. for \((i, j) \in J^d\) (see [3.2]), set

\[
\Theta_{i,j} := \omega_d(\xi_{\text{row}(i),\text{row}(j);\text{col}(j) - \text{col}(i))}.
\]

Recall \(V\) has a basis \(\{v_i \mid i \in I\}\). Then \(V^{\otimes d}\) admits a basis \(\{v_i := v_i \otimes \cdots \otimes v_i \mid i = (i_1, \ldots, i_d) \in I^d\}\).

Recall in [3.4], we have defined \(\xi_{s,t} \) for \((s, t) \in J\). For \((s, t) \in J, i = 1, \ldots, d\) and \(s = (s_1, \ldots, s_d), t = (t_1, \ldots, t_d)\), we continue to define \(\xi_{s,t} := \xi_{s_1,t_1} \otimes \cdots \xi_{s_d,t_d} \in \text{End}_C(V)^{\otimes d} \cong \text{End}_C(V^{\otimes d})\). Then we can precisely formulate \(\Theta_{\delta_{s,t}}\) as below.

**Lemma 3.5.** Keep the notations as above and in [3.3]. Then

\[
\Theta_{i,j} = \alpha(\varepsilon_i + \varepsilon_j, \varepsilon_i + \varepsilon_j) \sum_{\sigma \in \mathcal{S}_d} \nu(i + j; \sigma)\xi_{\delta_{s,t}}.
\]

**Proof.** For any basis element \(v_t \in V^{\otimes d}\) \((t \in I^d)\), by (3.13) we have

\[
\Theta_{i,j}v_t = (1 \boxtimes \xi_{\text{row}(i),\text{row}(j);\text{col}(j) - \text{col}(i)})\Delta(\delta)(v_t)
= \sum_{s \in I^d} (-1)^{|s_t|(|\varepsilon_i| + |\varepsilon_j|) + |s|(|\varepsilon_i| + |\varepsilon_j|)} \alpha(\varepsilon_s + \varepsilon_t, \varepsilon_s)\xi_{\text{row}(i),\text{row}(j);\text{col}(j) - \text{col}(i), x_T(s, t)}v_s.
\]

Note that \(\langle \xi_{\text{row}(i),\text{row}(j);\text{col}(j) - \text{col}(i), x_T(s, t)} \rangle\) is nonzero if and only if there exists \(\sigma \in \mathcal{S}_d\) such that \(\Upsilon(i, j) = \Upsilon(s, t)\). In this case, by (3.8) we have \(\langle \xi_{\text{row}(i),\text{row}(j);\text{col}(j) - \text{col}(i), x_T(s, t)} \rangle = \alpha(\varepsilon_i + \varepsilon_j, \varepsilon_i + \varepsilon_j)\nu(i + j, \sigma)\). So

\[
\Theta_{i,j}v_t = \alpha(\varepsilon_i + \varepsilon_j, \varepsilon_i + \varepsilon_j) \sum_{\sigma \in \mathcal{S}_d} \alpha(\varepsilon_s + \varepsilon_t, \varepsilon_s)\nu(i + j, \sigma)v_s.
\]

Keep in mind that

\[
e_{h,k}v_t = \delta_{h,t}\alpha(\varepsilon_h + \varepsilon_k, \varepsilon_t)v_h
\]

for \(e_{h,k} := e_{h_1,k_1} \otimes \cdots \otimes e_{h_d,k_d}\) with \(h = (h_1, \ldots, h_d)\) and \(k = (k_1, \ldots, k_d)\). By a straightforward computation, we have

\[
\xi_{h,k}v_t = \delta_{h,t}\alpha(\varepsilon_h + \varepsilon_k, \varepsilon_t)v_h.
\]

This proves the formula for \(\Theta_{i,j}\). \(\square\)

**Lemma 3.6.** The representation \(\omega_d\) is faithful.

**Proof.** By Lemma 3.3 it follows that \(\omega_d\) maps the basis elements \(\xi_{\text{row}(i),\text{row}(j);\text{col}(j) - \text{col}(i)}\) of \(\mathbb{C}[M_d]^*\) to linearly independent elements.

Note that \(\text{End}_C(V^{\otimes d}) \cong \text{End}_C(V)^{\otimes d}\). Then we will identify both of them. For any \(E \in \text{End}_C(V^{\otimes d})\), we can write \(E = \sum_{(i,j) \in I^d \times I^d} a_{i,j}e_{i,j}\) for \(a_{i,j} \in \mathbb{C}\) and \(e_{i,j} := e_{i_1,j_1} \otimes \cdots \otimes e_{i_d,j_d}\) with \(i = (i_1, \ldots, i_d)\) and \(j = (j_1, \ldots, j_d)\). Recall that we have already defined the algebra \(\mathbb{C}_n[x_1, \ldots, x_d] \otimes \mathcal{S}_d\), and its action on \(V^{\otimes d}\) in the introduction.

**Lemma 3.7.** Keep the notations as above. Suppose \(E = \sum_{(i,j) \in I^d \times I^d} a_{i,j}e_{i,j} \in \text{End}_C(V^{\otimes d})\). If \(E\) commutes with the action of \(\mathbb{C}_n[x_1, \ldots, x_d] \otimes \mathcal{S}_d\), then those coefficients \(a_{i,j}\) satisfy the following axioms:

1. \(a_{i,j} = 0\) if there exits \(k \in \{1, \ldots, d\}\) such that \(\text{col}(i_k) > \text{col}(j_k)\). This means that the nonzero terms come from the parameters \((i, j) \in J^d\).
2. \(a_{i,j} = \nu(\varepsilon_i + \varepsilon_j, \sigma)a_{s_{i,j},\sigma}\) for all \(\sigma \in \mathcal{S}_d\).
3. \(a_{i,j} = a_{s,t}\) if \(\Upsilon(i, j) = \Upsilon(s, t)\).
(4) \( a_{ij} = \nu(\epsilon_1 + \epsilon_j, \sigma)a_{s,t} \) if there exists some \( \sigma \in \mathcal{S}_d \) such that \( \Upsilon(i,j) \overset{\sigma}{\to} \Upsilon(s,t) \).

Proof. (1) and (2) can be verified by straightforward computation. (4) can be deduced from (2) and (3) because \( \Upsilon \) is \( \mathcal{S}_d \)-equivariant. So we only need to check (3).

(3) Take \( z_k = 1 \otimes \cdots \otimes \hat{e} \otimes \cdots \otimes 1 \in \text{End}_C(V)^{\otimes d} \), of which all the components in the tensor product are the identity except the \( k \)-th component equal to \( e \). Then \( z_k \) is naturally an even element in \( \text{End}_C(V)^{\otimes d} \). From the assumption \( z_k \circ E = E \circ z_k \), we verify the statement in (3).

For this, we first adopt some notations. For \( \ell \in I \), we set

\[
e_{\ell-1, \ell} := \begin{cases} 0, & \text{if } \ell = 1, \text{ or } \bar{1}; \\ e_{j-1,j}, & \text{if } \ell = j \in \{2, \ldots, m\}; \\ e_{\bar{j}-\bar{1}, \bar{j}} & \text{if } \ell = \bar{j} \in \{2, \ldots, \bar{n}\}; \end{cases}
\]

Actually, we are implicitly using the notation \( \ell - i \) as above. In the same spirit, we can talk about \( \ell + i \). Furthermore, for any \( j = (j_1, \ldots, j_d) \in I^d \) and \( k \in \{1, \ldots, d\} \), we set

\[
e_{j-1,k,j} := e_{j_1,j_1} \otimes e_{j_2,j_2} \otimes \cdots \otimes e_{j_k-1,j_k-1} \otimes e_{j_k-\bar{1},j_k} \otimes e_{j_{k+1},j_{k+1}} \otimes \cdots \otimes e_{j_d,j_d},
\]

and

\[
v_{j-1,k} := \begin{cases} 0, & \text{if } j_k = 1 \text{ or } \bar{1}; \\ v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_{k-1}} \otimes v_{j_k-\bar{1}} \otimes v_{j_{k+1}} \otimes \cdots \otimes v_{j_d}, & \text{otherwise}. \end{cases}
\]

Using (3.15), we have \( z_k(v_j) = \alpha(\epsilon_{j_1} + \epsilon_j, \epsilon_j)v_{j-1,k} \). Note that \( \alpha(\epsilon_{j_1} + \epsilon_j, \epsilon_j) = 1 \), then we have further

\[
z_k(v_j) = v_{j-1,k}.
\]

Next, for any given \( v_j \in V^{\otimes d} \) the assumption assures that \( E \circ z_k(v_j) = z_k \circ E(v_j) \). By (3.17) we have

\[
E \circ z_k(v_j) = \sum_{s,t \in I^d} a_{s,t}e_{s,t}z_k(v_j)
= \sum_{s,t \in I^d} a_{s,t}e_{s,t}v_{j-1,k}
= \sum_{s \in I^d} a_{s,j-1,k} \alpha(\epsilon_s + \epsilon_{j-1,k}, \epsilon_{j-1,k})v_s.
\]

On the other hand, by (3.15) again we have

\[
z_k \circ E(v_j) = z_k(\sum_{s \in I^d} a_{s,j} \alpha(\epsilon_s + \epsilon_j, \epsilon_j)v_s)
= \sum_{s \in I^d} a_{s,j} \alpha(\epsilon_s + \epsilon_j, \epsilon_j)z_k(v_s)
= \sum_{s \in I^d} a_{s,j} \alpha(\epsilon_s + \epsilon_j, \epsilon_j)v_{s-1,k}.
\]

Comparing \( E \circ z_k(v_j) \) with \( z_k \circ E(v_j) \), we have

\[
a_{s,j-1,k} \alpha(\epsilon_s + \epsilon_{j-1,k}, \epsilon_{j-1,k}) = a_{s+\bar{i},j} \alpha(\epsilon_{s+\bar{i}} + \epsilon_j, \epsilon_j).
\]

Note that \( \alpha(\epsilon_{s+\bar{i}} + \epsilon_j, \epsilon_j) = \alpha(\epsilon_s + \epsilon_{j-1,k}, \epsilon_{j-1,k}) = \alpha(\epsilon_s + \epsilon_{j-1,k}, \epsilon_{j-1,k}) \). So we finally have \( a_{s,j-1,k} = a_{s+\bar{i},j} \) for any \( s,j \in I^d \) and \( k \in \{1, \ldots, d\} \) whenever \( s + \bar{i} \) and \( j - \bar{i} \) make sense. Iterating the process, we can obtain \( a_{s,j} = a_{1,t} \) if \( \Upsilon(s,j) = \Upsilon(i,t) \). This proves the statement in (3).

The proof is completed. \( \square \)
As an immediate consequence, we have the following result.

**Corollary 3.8.** If \( E \in \text{End}_C(V^{\otimes d}) \) commutes with the action of \( \mathbb{C}[x_1, \ldots, x_d] \otimes \mathfrak{S}_d \), then \( E \) is a linear combination of \( \Theta_{i,j} \) with \((i,j) \in J^d\) in \([3.17] \).

### 3.7. Double centralizer property and the proof of Theorem 0.3

Let us continue to accomplish the proof of double centralizer property. Keep in mind that the matrix action of \( \mathfrak{g}_e \) on the tensor space. By definition, \( \phi_d \) is precisely the composite map \( \omega_d \circ \pi_d \). By virtue of the arguments in the proof of Lemmas 3.2 and 3.6, the image \( \text{Im}\phi_d \) is isomorphic to \( \mathbb{C}[M]^{\mathfrak{g}_e} \). Let \( \psi_d : \mathbb{C}[x_1, \ldots, x_d] \otimes \mathfrak{S}_d \rightarrow \text{End}_C(V^{\otimes d})^{\mathfrak{g}_e} \) be the homomorphism arising from the right action of \( \mathbb{C}[x_1, \ldots, x_d] \otimes \mathfrak{S}_d \) on \( V^{\otimes d} \). Thus, we have the following maps

\[
U(\mathfrak{g}_e) \xrightarrow{\phi_d} \text{End}_C(V^{\otimes d}) \xrightarrow{\psi_d} \mathbb{C}[x_1, \ldots, x_d] \otimes \mathfrak{S}_d.
\]

The (natural) \( \mathbb{Z} \)-grading on \( \mathfrak{g} \) extends to a grading \( U(\mathfrak{g}) = \bigoplus_{r \in \mathbb{Z}} U(\mathfrak{g})_r \) on its universal enveloping algebra, and \( U(\mathfrak{g}_e) \) is a graded subalgebra. Consider \( V \) as a graded module by declaring that each \( v_i \) is of degree \((n - \text{col}(i))\). There are induced gradings on \( V^{\otimes d} \) and its endomorphism algebra \( \text{End}_C(V^{\otimes d}) \), so that the map \( \phi_d \) is then a homomorphism of graded algebras. Also define a grading on \( \mathbb{C}[x_1, \ldots, x_d] \otimes \mathfrak{S}_d \) by declaring that each \( x_i \) is of degree 1 and each \( x \in \mathfrak{S}_d \) is of degree 0. The map \( \psi_d \) is then a homomorphism of graded algebras too.

Now we are in the position to prove Theorem 0.3.

**Proof.** Recall that \( e \in \mathfrak{g}_e \) is regular nilpotent. The second equation is based on Theorem 0.3. Owing to Lemma 3.7 and its corollary, any element of \( \text{End}_{\mathbb{C}[x_1, \ldots, x_d] \otimes \mathfrak{S}_d} (V^{\otimes d}) \) is a linear combination of the elements \( \Theta_{i,j} \). These belong to the image of \( \phi_d \) by Lemma 3.2. Thus, the first equation follows. \( \square \)

## 4. \( U(\mathfrak{g}_e) \) as a contraction of finite \( W \)-superalgebra for any nilpotent \( e \)

In this section, we suppose that \( \mathfrak{g} \) is any basic classical Lie superalgebra over \( \mathbb{C} \) as defined in \([21, 22, 43 \text{ or } 44]\), and \( e \) is any nilpotent element in \( \mathfrak{g}_e \).

In \([31]\) Premet introduced and developed the theory of finite \( W \)-algebra \( U(\mathfrak{g}, e) \) (in the present paper, we instead use the notation \( W_\chi \) for \( \chi = (e, -) \), see \([22, 23]\)). Partial counterpart theory in the super case has been established in \([41, 42, 43 \text{ and } 44]\). In this section, we will continue to extend a result in \([32]\) where the author constructed a \( \mathbb{C}[t] \)-algebra \( \mathcal{H}_\chi \) which is also a free module over \( \mathbb{C}[t] \) satisfying \( \mathcal{H}_\chi / t\mathcal{H}_\chi \cong U(\mathfrak{g}_e) \).

### 4.1. Kazhdan filtration

First recall some results in \([42]\). For any given nilpotent element \( e \in \mathfrak{g}_e \), let \( e, f, h \) be an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \), then under the action of \( \text{ad} h \) we have \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i) \) (natural grading), where \( \mathfrak{g}(i) = \{ x \in \mathfrak{g} \mid [h, x] = ix \} \). Then the grading on \( \mathfrak{g} \) can be generalized to the ones on \( U(\mathfrak{g}) \) and its subalgebras under the action of \( \text{ad} h \). It follows from \([42] \) Proposition 2.1 that there exists an even non-degenerate supersymmetric invariant bilinear form \( \langle \cdot, \cdot \rangle \) such that \( \langle e, f \rangle = 1 \). For any \( x \in \mathfrak{g} \), define \( \chi(x) = \langle e, x \rangle \). Since the even part \( \mathfrak{g}_0 \) of \( \mathfrak{g} \) is a reductive Lie algebra, it is immediate that \( (\mathfrak{g}_0)_0 \cap (\mathfrak{g}_e)_0 \) is a Levi subalgebra of \( (\mathfrak{g}_e)_0 \). Pick a maximal toral subalgebra \( t_e \subseteq \mathfrak{g}(0)_0 \) of this Levi subalgebra, and a Cartan subalgebra \( t \) of \( \mathfrak{g} \) containing \( t_e \) and \( h \).

There exists a symplectic (resp. symmetric) bilinear form \( \langle \cdot, \cdot \rangle' \) on the \( \mathbb{Z}_2 \)-graded subspace \( \mathfrak{g}(-1)_{\overline{0}} \) (resp. \( \mathfrak{g}(-1)_{\overline{1}} \)) given by \( \langle x, y \rangle' := \langle e, [x, y] \rangle = \chi(x, y) \) for all \( x, y \in \mathfrak{g}(-1)_{\overline{0}} \) (resp. \( x, y \in \mathfrak{g}(-1)_{\overline{1}} \)). There exist bases \( \{ u_1, \ldots, u_{2s} \} \) of \( \mathfrak{g}(-1)_{\overline{0}} \) and \( \{ v_1, \ldots, v_r \} \) of \( \mathfrak{g}(-1)_{\overline{1}} \) such that \( \langle u_i, v_j \rangle = i^s \delta_{i+j,2s+1} \) for \( 1 \leq i, j \leq 2s \), and \( \langle v_i, v_j \rangle = \delta_{i+j,r+1} \) for \( 1 \leq i, j \leq r \).
Set $\mathfrak{m} := \bigoplus_{i \leq -2} \mathfrak{g}(i) \oplus \mathfrak{g}(-1)'$ with $\mathfrak{g}(-1)' = \mathfrak{g}(-1)_{0}' \oplus \mathfrak{g}(-1)_{1}'$, where $\mathfrak{g}(-1)_{0}'$ is the $\mathbb{C}$-span of $u_{s+1}, \ldots, u_{2s}$ and $\mathfrak{g}(-1)_{1}'$ is the $\mathbb{C}$-span of $v_{r+1}, \ldots, v_{r}$ (resp. $v_{r+3}, \ldots, v_{r}$) when $r := \dim \mathfrak{g}(-1)'$ is even (resp. odd), then $\chi$ vanishes on the derived subalgebra of $\mathfrak{m}$. Define

\begin{equation}
\mathfrak{p} := \bigoplus_{i \geq 0} \mathfrak{g}(i), \quad \mathfrak{m}' := \begin{cases} \mathfrak{m} & \text{if } r \text{ is even;} \\ \mathfrak{m} \oplus \mathbb{C}v_{r+1} & \text{if } r \text{ is odd.} \end{cases}
\end{equation}

As in [44] §2 we can choose a basis $x_{1}, \ldots, x_{1}, x_{r+1}, \ldots, x_{m} \in \mathfrak{p}_{0}, y_{1}, \ldots, y_{q}, y_{q+1}, \ldots, y_{n} \in \mathfrak{p}_{1}$ of the free $\mathfrak{g}$-module $\mathfrak{p}$ such that

(a) $x_{i} \in \mathfrak{g}(k_{i})_{0}, y_{j} \in \mathfrak{g}(k_{j}')_{1}$, where $k_{i}, k_{j}' \in \mathbb{Z}_{+}$ with $1 \leq i \leq m$ and $1 \leq j \leq n$;

(b) $x_{1}, \ldots, x_{i}$ is a basis of $(\mathfrak{g}_{0})_{i}$ and $y_{1}, \ldots, y_{q}$ is a basis of $(\mathfrak{g}_{e})_{i}$;

(c) $x_{1}, \ldots, x_{m} \in [f, \mathfrak{g}_{0}]$ and $y_{q+1}, \ldots, y_{n} \in [f, \mathfrak{g}_{1}]$.

Recall that a generalized Gelfand-Graev $\mathfrak{g}$-module associated with $\chi$ is defined by

\begin{equation}
Q_{\chi} := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_{\chi},
\end{equation}

where $\mathbb{C}_{\chi} = \mathbb{C}1_{\chi}$ is a one-dimensional $\mathfrak{m}$-module such that $x_{1} \chi = \chi(x)1_{\chi}$ for all $x \in \mathfrak{m}$. For $k \in \mathbb{Z}_{+}$, define

\begin{equation}
\mathbb{Z}_{i}^{+} := \{(i_{1}, \ldots, i_{k}) | i_{j} \in \mathbb{Z}_{+}\}, \quad \mathbb{Z}_{i}^{-} := \{(i_{1}, \ldots, i_{k}) | i_{j} \in \{0, 1\}\}
\end{equation}

with $1 \leq j \leq k$. For $i := (i_{1}, \ldots, i_{k})$ in $\mathbb{Z}_{i}^{+}$ or $\mathbb{Z}_{i}^{-}$, set $|i| = i_{1} + \cdots + i_{k}$. For any real number $a \in \mathbb{R}$, let $[a]$ denote the largest integer lower bound of $a$, and $\lceil a \rceil$ the least integer upper bound of $a$. Given $(a, b, c, d) \in \mathbb{Z}_{m}^{+} \times \mathbb{Z}_{n}^{+} \times \mathbb{Z}_{1}^{+} \times \mathbb{Z}_{1}^{+}$ (where $t := \lfloor \mathcal{F} \rceil = \left\lfloor \frac{\dim \mathfrak{g}(-1)_{1}'}{2} \right\rfloor$), let $x_{m}y_{n}u_{c}v_{d}$ denote the monomial $x_{1}a_{1}^{m}x_{m}^{m}y_{1}a_{1}^{m}y_{n}a_{1}^{m}u_{c}v_{c}^{1} \cdots u_{c}v_{d}^{t} v_{d}$ in $U(\mathfrak{g})$, denote by $wt(x_{m}y_{n}u_{c}v_{d}) = (\sum k_{i}a_{i}) + (\sum k_{i}'b_{i}) - |c| - |d|$ the weight of $x_{m}y_{n}u_{c}v_{d}$, and set

\begin{equation}
|(a, b, c, d)|_{c} := \sum_{i=1}^{m} a_{i}(k_{i} + 2) + \sum_{i=1}^{n} b_{i}(k_{i}' + 2) + \sum_{i=1}^{m} c_{i} + \sum_{i=1}^{t} d_{i}.
\end{equation}

**Definition 4.1.** Define the finite $W$-superalgebra over $\mathbb{C}$ by

\begin{equation}
W_{\chi} := (\text{End}_{\mathfrak{g}}Q_{\chi})^{\text{op}}
\end{equation}

where $(\text{End}_{\mathfrak{g}}Q_{\chi})^{\text{op}}$ denotes the opposite algebra of the endomorphism algebra of $\mathfrak{g}$-module $Q_{\chi}$.

Let $P_{\chi}$ denote the $\mathbb{Z}_{2}$-graded ideal of codimension one in $U(\mathfrak{m})$ generated by all $x - \chi(x)$ with $x \in \mathfrak{m}$. Set

\begin{equation}
P_{\chi} := U(\mathfrak{g})P_{\chi}.
\end{equation}

Then $Q_{\chi} \cong U(\mathfrak{g})/P_{\chi}$ as $\mathfrak{g}$-modules. The fixed point space $(U(\mathfrak{g})/P_{\chi})^{\text{ad}}$ carries a natural algebra structure given by

\begin{equation}
(x + P_{\chi}) \cdot (y + P_{\chi}) := (xy + P_{\chi})
\end{equation}

for all $x, y \in U(\mathfrak{g})$. By [44] Theorem 2.12, we have $(\text{End}_{\mathfrak{g}}Q_{\chi})^{\text{op}} \cong Q_{\chi}^{\text{ad}}$ as $\mathbb{C}$-algebras.

Let $w_{1}, \ldots, w_{c}$ be a basis of $\mathfrak{g}$ over $\mathbb{C}$. Let $U(\mathfrak{g}) = \bigcup_{i \in \mathbb{Z}_{+}} U^{i}(\mathfrak{g})$ be a filtration of $U(\mathfrak{g})$, where $U^{i}(\mathfrak{g})$ is the $\mathbb{C}$-span of all $w_{i_{1}} \cdots w_{i_{k}}$ with $w_{i_{1}} \in \mathfrak{g}(j_{1}), \ldots, w_{i_{k}} \in \mathfrak{g}(j_{k})$ and $(j_{1} + 2) + \cdots + (j_{k} + 2) \leq i$. This filtration is called Kazhdan filtration. The Kazhdan filtration on $Q_{\chi}$ is defined by $Q_{\chi}^{i} := \pi(U^{i}(\mathfrak{g}))$ with $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/P_{\chi}$ being the canonical homomorphism, which makes $Q_{\chi}$ into a filtered $U(\mathfrak{g})$-module. Then there is an induced Kazhdan filtration $W_{\chi}^{i}$ on the subspace $W_{\chi} = Q_{\chi}^{\text{ad}}$ of $Q_{\chi}$ such that $W_{\chi}^{j} = 0$ unless $j \geq 0$. We see that $W_{\chi}^{i}, W_{\chi}^{j} \subseteq W_{\chi}^{i+j}$ for any $i, j \in \mathbb{Z}_{+}$.
4.2. Recall that \( \{x_1, \ldots, x_l\} \) and \( \{y_1, \ldots, y_q\} \) are bases of \( \g^e_0 \) and \( \g^e_1 \), respectively. Set

\[
Y_i := \begin{cases} 
  x_i & \text{if } 1 \leq i \leq l; \\
  y_{i-l} & \text{if } l+1 \leq i \leq l+q; \\
  v_{i+1} & \text{if } i = l + q + 1 \text{ whenever } r \text{ is odd.}
\end{cases}
\]

By assumption it is immediate that \( Y_i \in \g_e \) for \( 1 \leq i \leq l + q \). The term \( Y_{l+q+1} \notin \g_e \) occurs only when \( r = \dim \g(-1) \) is odd. Assume that \( Y_i \) belongs to \( \g(m_i) \) for \( 1 \leq i \leq l + q + 1 \). In the sequent arguments, we further set

\[
q' := \begin{cases} 
  q & \text{if } r \text{ is even;} \\
  q + 1 & \text{if } r \text{ is odd.}
\end{cases}
\]

Let \( \tilde{\Theta}_i \) denote the image of \( \Theta_i \in W_\chi \) in the graded algebra \( \gr(W_\chi) \) under the Kazhdan grading.

In [42] Theorem 4.5, the authors introduced the following PBW theorem of finite \( W \)-superalgebra \( W_\chi \) as follows:

**Theorem 4.2.** (42) The following PBW structural statements for the finite \( W \)-superalgebra \( W_\chi \) hold, corresponding to the cases when \( r \) is even and when \( r \) is odd, respectively.

1. There exist homogeneous elements \( \Theta_1, \ldots, \Theta_l \in (W_\chi)_0 \) and \( \Theta_{l+1}, \ldots, \Theta_{l+q'} \in (W_\chi)_1 \) such that

\[
\Theta_k(1_\chi) = (Y_k + \sum_{|a| + |b| = m_k + 2, |c| + |d| \geq 2} \lambda^k_{a,b,c,d} x^ay^b c^d d^e) \\
+ \sum_{|a| + |b| + |c| + |d| < m_k + 2} \lambda^k_{a,b,c,d} x^ay^b c^d d^e \otimes 1_\chi
\]

for \( 1 \leq k \leq l + q \), where \( \lambda^k_{a,b,c,d} \in \mathbb{Q} \), and \( \lambda^k_{a,b,c,d} = 0 \) if \( a_{i+1} = \cdots = a_m = b_{q+1} = \cdots = b_n = c_1 = \cdots = c_s = d_1 = \cdots = d_q = 0 \).

Additionally set \( \Theta_{l+q+1}(1_\chi) = \frac{v_{q+1}}{2} \otimes 1_\chi \) when \( r \) is odd.

2. The monomials \( \Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_{l+1}} \cdots \Theta_{l+q'}^{b_{l+q'}} \) with \( a_i \in \mathbb{Z}_{+}, b_j \in \mathbb{Z}_{2} \) for \( 1 \leq i \leq l \) and \( 1 \leq j \leq q' \) form a basis of \( W_\chi \) over \( \mathbb{C} \).

3. For \( 1 \leq i \leq l + q' \), the elements \( \tilde{\Theta}_i = \Theta_i + W_\chi^{m_i+1} \in \gr(W_\chi) \) are algebraically independent and generate \( \gr(W_\chi) \). In particular, \( \gr(W_\chi) \) is a graded polynomial superalgebra with homogeneous generators of degrees \( m_1 + 2, \ldots, m_{l+q'} + 2 \).

4. For \( 1 \leq i, j \leq l + q' \), we have

\[
[\Theta_i, \Theta_j] \in W_\chi^{m_i + m_j + 2}.
\]

Moreover, if the elements \( Y_i, Y_j \in \g_e \) for \( 1 \leq i, j \leq l + q \) satisfy \( Y_i Y_j = \sum_{k=1}^{l+q} \alpha_{ij}^k Y_k \) in \( \g_e \), then

\[
[\Theta_i, \Theta_j] = \sum_{k=1}^{l+q} \alpha_{ij}^k \Theta_k + q_{ij}(\Theta_1, \ldots, \Theta_{l+q'}) \mod W_\chi^{m_i + m_j + 1},
\]

where \( q_{ij} \) is a super-polynomial in \( l + q' \) variables in \( \mathbb{Q} \) whose constant term and linear part are zero.

5. When \( r \) is odd, if one of \( i \) and \( j \) happens to be \( l + q + 1 \), we can find super-polynomials \( F_{l+j+1} \) and \( F_{l+q+1+j} \) in \( l + q + 1 \) invariants over \( \mathbb{Q} \) \( \{1 \leq i, j \leq l + q\} \) such that

\[
[\Theta_i, \Theta_{l+q+1}] = F_{i, l+q+1}(\Theta_1, \ldots, \Theta_{l+q+1}),
[\Theta_{l+q+1}, \Theta_j] = F_{l+q+1, j}(\Theta_1, \ldots, \Theta_{l+q+1}),
\]
where the Kazhdan degree for all the monomials in the \( \mathbb{Q} \)-span of \( F_{i,l+q+1} \)'s and \( F_{l+q+1,j} \)'s is less than \( m_i + 1 \) and \( m_j + 1 \), respectively.

Moreover, if \( i = j = l + q + 1 \), then
\[
[\Theta_{l+q+1}, \Theta_{l+q+1}] = \text{id.}
\]

By virtue of Theorem 4.2, we see that the Kazhdan degree of the monomial \( \Theta_i^{a_i} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_{l+1}} \cdots \Theta_{l+q'}^{b_{l+q'}} \) for the PBW basis of \( W_\chi \) is \( \sum_{i=1}^l a_i(m_i + 2) + \sum_{j=1}^{q'} b_j(m_{l+i} + 2) \), and we write
\[
\deg_e(\Theta_i^{a_i} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_{l+1}} \cdots \Theta_{l+q'}^{b_{l+q'}}) = \sum_{i=1}^l a_i(m_i + 2) + \sum_{i=1}^{q'} b_i(m_{l+i} + 2)
\]
for simplicity. Moreover, the corresponding graded algebra \( \text{gr}(W_\chi) = \sum_{k \geq -1} W_\chi^{k+1} / W_\chi^k \) (we set \( W_\chi^{-1} = \{0\} \) here) is a supercommutative algebra, i.e.,

**Theorem 4.3.** ([42, Theorem 4.5], [45, Theorem 1.7], [39, Corollary 3.9(2)]) For any basic classical Lie superalgebra \( \mathfrak{g} \), we have

1. \( \text{gr}(W_\chi) \cong S(\mathfrak{g}_e) \) when \( \dim \mathfrak{g}(-1)_\uparrow \) is even;
2. \( \text{gr}(W_\chi) \cong S(\mathfrak{g}_e) \otimes \mathbb{C}[\Theta] \) when \( \dim \mathfrak{g}(-1)_\uparrow \) is odd,

where \( S(\mathfrak{g}_e) \) denotes the supersymmetric algebra on \( \mathfrak{g}_e \), and \( \mathbb{C}[\Theta] \) is an external algebra generated by an element \( \Theta \).

**Remark 4.4.** (1) It is notable that for the case with \( \mathfrak{g} = \mathfrak{gl}(m|n) \), \( r = \dim \mathfrak{g}(-1)_\uparrow \) is always an even number (see [40, §3.2] for more details).

(2) In the case \( \mathfrak{g} = \mathfrak{gl}(m|n) \) with \( e \in \mathfrak{g}_0 \) being regular nilpotent, the first statement of Theorem 4.3 was formulated in [38, Theorem 4.1].

Moreover, in [41, Appendix I, Lemma I.1] Zeng showed that

**Lemma 4.5.** ([41]) Each generator \( \Theta_k \) of the finite \( W \)-superalgebra \( W_\chi \) can be chosen to be a weight vector for \( \mathfrak{t}_e \) of the same weight as \( Y_k \).

Consider the linear mapping
\[
\Theta : \mathfrak{g}_e \to W_\chi \quad x \mapsto \Theta_x
\]
such that \( \Theta_{Y_i} = \Theta_i \) for all \( i \). Thanks to Lemma 4.5, \( \Theta \) is an injective homomorphism of \( \mathfrak{t}_e \)-module. Although \( \Theta \) is not a Lie superalgebra homomorphism, in general, we have

**Lemma 4.6.** For \( Y_i, Y_j \in \mathfrak{g}_e \), we have
\[
[\Theta_{Y_i}, \Theta_{Y_j}] \equiv \Theta_{[Y_i, Y_j]} + q_{ij}(\Theta_1, \ldots, \Theta_{l+q'}) \quad (\text{mod } W_\chi^{m_i+m_j}),
\]
where \( q_{ij} \) is a super-polynomial in \( l + q' \) variables with initial form of total degree \( \geq 2 \).

**Proof.** For any \( x \in \mathfrak{g}(i) \), set \( \sigma(x) = (-1)^i x \). Since \( \sigma \) preserves \( I_\chi \) and \( m \), it acts on \( W_\chi \cong Q_\chi^{ad,m} \) as algebra automorphisms. For any monomial \( x^a y^b u^c v^d \otimes 1_\chi \) in \( Q_\chi \) with \( (a, b, c, d) \in \mathbb{Z}_+^m \times \mathbb{Z}_2^n \times \mathbb{Z}_+^2 \times \mathbb{Z}_2^2 \), note that
\[
\sigma(x^a y^b u^c v^d \otimes 1_\chi) = (-1)^{(a,b,c,d)} e x^a y^b u^c v^d \otimes 1_\chi.
\]
Now the lemma follows from Theorem 4.2(4), Lemma 4.5, and (4.11). \( \square \)
4.3. In [44] Definition 4.8, the authors introduced the refined $W$-superalgebra

$$W' := Q_{adm}'$$

which is a subalgebra of the finite $W$-superalgebra $W_\chi$. When $r$ is an even number, as $m' = m$ by definition, we have $W_\chi = W'_r$. However, the situation changes in the case when $r$ is an odd number. Since $m$ is a proper subalgebra of $m'$, it follows that $W'_r$ is a subalgebra of $W_\chi$. In fact, the PBW theorem of $W'_r$ is of much the same as that of Theorem 4.2 with $r$ being odd, just abandoning the related topics on the element $\Theta_{l+q+1}$. The authors also showed that $\text{gr}(W'_r) \cong S(g_e)$ as $C$-algebras under the Kazhdan grading in [44, Corollary 3.8]. We refer to [44, Theorem 3.7] for more details.

Now we are in a position to introduce the main results of this section.

**Theorem 4.7.** There exists an associative $C[t]$-superalgebra $H_\chi$ free as a module over $C[t]$ such that

$$H_\chi/(\lambda - t)H_\chi \cong \begin{cases} W'_r & \text{if } \lambda \neq 0; \\ U(\frak{g}_e) & \text{if } \lambda = 0. \end{cases}$$

In other words, the enveloping algebra $U(\frak{g}_e)$ is a contraction of $W'_r$.

**Proof.** Our proof is similar to that in [32, Proposition 2.1]. Consider the superalgebra $H(R) = R \otimes W'_\chi$ over the ring of Laurent polynomials $R = C[t, t^{-1}]$ obtained from $W'_\chi$ by extension of scalars, and identify $W'_r$ with the subspace $C \otimes W'_r$ of hyperspace $H(R)$. Set the elements in $R$ to be even. Define an invertible $R$-linear transformation $\pi$ on $H(R)$ by setting

$$(4.12) \quad \pi(\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q}^{b_q}) = t^{a_1m_1 + \cdots + a_lm_l + b_1m_{l+1} + \cdots + b_qm_{l+q}} \Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q}^{b_q}$$

and extending to $H(R)$ by $R$-linearity. We view $\pi$ as an isomorphism from $H(R)$ onto a new $R$-superalgebra $H(R, \pi)$ with underlying $R$-module $R \otimes W'_\chi$ and with associative product given by $(x \cdot y)_\pi := \pi^{-1}(\pi(x) \cdot \pi(y))$, for all $x, y \in R \otimes W'_\chi$. We denote by $H_\chi$ the free $C[t]$-submodule of $H(R, \pi)$ generated by $\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q}^{b_q}$, $\forall a_i \in \mathbb{Z}_+, b_j \in \mathbb{Z}_2$. It follows from (4.3) and Theorem 4.2 that

$$\deg_e(\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q}^{b_q}) = \sum_{i=1}^{l} a_i m_i + \sum_{i=1}^{q} b_i m_{l+i} + 2 \sum_{i=1}^{l} m_i + 2 \sum_{i=1}^{q} m_{l+i}.$$  

By virtue of (4.10) this yields

$$\langle \Theta_i, \Theta_j \rangle - \langle \Theta_i, \Theta_j \rangle_{\pi} = \pi^{-1}(t^{m_i + m_j} \langle \Theta_i, \Theta_j \rangle_{\pi}) \equiv \Theta_{[y_i, y_j]} \mod tH_\chi$$

(since the initial form of $q_{ij}$ has total degree $\geq 2$ and $\deg_e q_{ij}(\Theta_1, \ldots, \Theta_{l+q}) = m_i + m_j + 2$).

Using induction on the Kazhdan degree of $\Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_1} \cdots \Theta_{l+q}^{b_q}$ and the supercommutativity of $\text{gr}(W'_r)$, we have $(\Theta_i : H_\chi)_\pi \subseteq H_\chi$ for $1 \leq i \leq l + q$. So $H_\chi$ is a $C[t]$-subalgebra of $H(R, \pi)$.

If $\lambda \neq 0$, then the homomorphism $C[t] \to C$ taking $t$ to $\lambda$ extends to a homomorphism $R \to C$. The isomorphism $\pi^{-1}$ injects $(t - \lambda)H(R, \pi)$ onto $(t - \lambda)H(R)$. Because $H_\chi \cap (t - \lambda)H(R, \pi) = (t - \lambda)H_\chi$, $W'_\chi \cap (t - \lambda)H(R) = 0$, we have

$$H_\chi/(t - \lambda)H_\chi \cong H(R, \pi)/(t - \lambda)H(R, \pi) \cong H(R)/(t - \lambda)H(R) \cong W'_r,$$

by the theorem on isomorphism.

Now put $\tilde{H}_\chi := H_\chi/tH_\chi$, and identify the generators $\Theta_i = \Theta_{Y_i}$ of $H_\chi$ with their images in $\tilde{H}_\chi$. It is immediate from our earlier discussion that these images satisfy the relations $[\Theta_{Y_i}, \Theta_{Y_j}] = \Theta_{[Y_i, Y_j]}$ for $1 \leq i, j \leq l + q$. By the universality property of the enveloping superalgebra $U(\frak{g}_e)$, there is an
algebra homomorphism \( \phi: U(\mathfrak{g}_e) \rightarrow \mathcal{H}_X \) with \( \phi(Y_i) = \Theta_i \) for \( 1 \leq i \leq l + q \). Since \( \mathcal{H}_X \) is a free \( \mathbb{C}[t] \)-module, the monomials \( \Theta_1^{a_1} \cdots \Theta_l^{a_l} \Theta_{l+1}^{b_{l+1}} \cdots \Theta_{l+q}^{b_{l+q}} \) are linearly independent in \( \mathcal{H}_X \). As a consequence, \( \phi \) is an isomorphism. \( \square \)

**Theorem 4.8.** The graded algebra of the refined \( W \)-superalgebra is isomorphic to the universal enveloping superalgebra of \( \mathfrak{g}_e \) under the natural grading, i.e.,

\[
\text{gr}_z: \quad \text{gr}_z(W'_X) \rightarrow U(\mathfrak{g}_e) \\
\text{gr}_z(\Theta_i) \mapsto Y_i
\]

for \( 1 \leq i \leq l + q \).

**Proof.** Pick a homogeneous basis \( Y_1, \ldots, Y_{l+q} \) for \( \mathfrak{g}_e \) as in \( \S 1.2 \) so that the monomials \( \Theta_1^{a_1} \cdots \Theta_l^{a_l} \cdot \Theta_{l+1}^{b_{l+1}} \cdots \Theta_{l+q}^{b_{l+q}} \) with \( a_i \in \mathbb{Z}_+ \), \( b_j \in \mathbb{Z}_2 \) for \( 1 \leq i \leq l \) and \( 1 \leq j \leq q \) form a basis for \( W'_X \). By virtue of \( \S 4.3 \) and Lemma 4.6, one can easily conclude that

\[
\Theta_1^{a_1} \cdots \Theta_l^{a_l} \cdot \Theta_{l+1}^{b_{l+1}} \cdots \Theta_{l+q}^{b_{l+q}} = Y_1^{a_1} \cdots Y_l^{a_l} \cdot Y_{l+1}^{b_{l+1}} \cdots Y_{l+q}^{b_{l+q}} + (\dagger),
\]

where the first term on the right hand side of \( (4.13) \) lies in \( U(\mathfrak{p})(\sum_{i=1}^{l} a_i m_i + \sum_{j=1}^{q} b_j m_{l+i}) \) and the term \( (\dagger) \) lies in the sum of all strictly lower graded components. Hence the monomials \( Y_1^{a_1} \cdots Y_l^{a_l} \cdot Y_{l+1}^{b_{l+1}} \cdots Y_{l+q}^{b_{l+q}} \) constitute a homogeneous basis for \( \text{gr}_z(W'_X) \). The same monomials constitute a homogeneous basis for \( U(\mathfrak{g}_e) \) by the PBW theorem. \( \square \)

**Remark 4.9.** In the case \( \mathfrak{g} = \mathfrak{gl}(m|n) \) with \( e \in \mathfrak{g}_0 \) being a regular nilpotent element, the above theorem was formulated in \( \S 3 \) Remark 4.7, where \( U(\mathfrak{g}_e) \) can be realized precisely as a subalgebra of \( U(\mathfrak{p}) \) isomorphic to \( U(\mathfrak{h}) \) (see Remark 5.2(3) for more details).

5. **Deformed action on \( V^{\otimes d} \): as a module over finite \( W \)-superalgebras**

In this section we will discuss the filtered deformation of Vust’s double centralizer property. To achieve this we need another formation of finite \( W \)-superalgebras.

5.1. **Pyramids and finite \( W \)-superalgebras.** Now we turn to the case \( \mathfrak{g} = \mathfrak{gl}(m|n) \). Consider the regular nilpotent orbit with the standard representative element

\[
e = \sum_{i=1}^{m-1} e_{i,i+1} + \sum_{j=1}^{n-1} e_{j,j+1} \in \mathfrak{g}_0.
\]

We have a pyramid associated with the pair \( (m \leq n) \) as in \( \S 5.1 \). Then \( \mathfrak{g} = \sum_{r \in \mathbb{Z}} \mathfrak{g}(r) \) by declaring that \( e_{i,j} \) is of degree

\[
\deg(e_{i,j}) = \text{col}(j) - \text{col}(i).
\]

This is a good grading for \( e \), which means that \( e \in \mathfrak{g}(2) \) and the centralizer \( \mathfrak{g}_e \) of \( e \) in \( \mathfrak{g} \) belongs to \( \sum_{r \geq 0} \mathfrak{g}(r) \) (see \( \S 19 \) for more about good gradings). In this case, we take

\[
p := \sum_{r \geq 0} \mathfrak{g}(r) = \sum_{\text{col}(i) \leq \text{col}(j)} \mathbb{C} e_{i,j}; \quad \mathfrak{h} := \mathfrak{g}(0) = \sum_{\text{col}(i) = \text{col}(j)} \mathbb{C} e_{i,j}; \quad \mathfrak{m} := \sum_{r < 0} \mathfrak{g}(r) = \sum_{\text{col}(i) > \text{col}(j)} \mathbb{C} e_{i,j}.
\]

Note that we have \( U(\mathfrak{g}) = U(\mathfrak{p}) \oplus \mathfrak{I}_X \) by definition. Let \( \text{Pr}: U(\mathfrak{g}) \rightarrow U(\mathfrak{p}) \) denote the corresponding linear projection. Then we have

**Lemma 5.1.** (\( \S 3 \) 4 or \( \S 12 \) Remark 2.14) Keep the notations and assumptions as above. Then we can identify \( W_X \) with the following subalgebra in \( U(\mathfrak{p}) \):

\[
\{ u \in U(\mathfrak{p}) \mid \text{Pr}(x,u) = 0 \text{ for any } x \in \mathfrak{m} \}.
\]
Furthermore, with the above identification there is an isomorphism of \( \mathbb{C} \text{-algebras} \)

\[
\varphi : W_\chi \longrightarrow Q^{\text{odd}}_\chi \quad u \mapsto u(1 + I_\chi).
\]

**Remark 5.2.** (1) Up to isomorphisms, the finite \( W \)-superalgebra \( W_\chi \) is only dependent on the set \( \{m, n\} \) (see [3, Remark 4.7]).

(2) Here the definition is subjected to the condition “Pr (\([m, u]\) = 0” with left-multiplication by \( m \), which can be regarded a “left-handed” formulation. There is another so-called “right-handed” formulation. Both of definitions are equivalent because two finite \( W \)-superalgebras defined in different ways are isomorphic as algebras (see [3, Remark 4.7]).

(3) Furthermore, \( W_\chi \) can be precisely realized as a subalgebra of \( U(\mathfrak{p}) \) isomorphic to \( U(\mathfrak{h}) \) as below. Associated with Pyramid (3.1), one can define an automorphism \( \eta^0 \) of \( U(\mathfrak{p}) \) via the shift

\[
e_{i,j} \mapsto e_{i,j} - \delta_{i,j}d_i
\]

for each \( e_{i,j} \in \mathfrak{p} \), where

\[
d_i = \begin{cases} 
1 - (n - m), & \text{if } 1 \leq \text{col}(i) \leq m; \\
n - \text{col}(i), & \text{if } m < \text{col}(i) \leq n.
\end{cases}
\]

The Miura transform \( \mu^0 \) is defined as the composition of \( \eta^0|_{W_\chi} \) and the projection of \( U(\mathfrak{p}) \) onto \( U(\mathfrak{h}) \) arising from the canonical projection \( \mathfrak{p} \rightarrow \mathfrak{h} \). Then \( \mu^0 \) gives rise to an isomorphism between \( W_\chi \) onto \( U(\mathfrak{h}) \) (cf. [3]). However, we are not going to take \( \eta^0 \) into our arguments in the next section for simplicity.

5.2. From this section on, we will identify \( W_\chi \) with the subalgebra of \( U(\mathfrak{p}) \) as described in (5.3).

By [13] Proposition 3.7], there exists a one-dimension module over \( W_\chi \), which we denote by \( C_c \).

Under the identification of \( W_\chi \) in Lemma 5.1, we can extend the one-dimensional \( W_\chi \)-module \( C_c \) to a \( U(\mathfrak{p}) \)-module. In the following we will describe \( U(\mathfrak{p}) \)-module \( C_c \) precisely.

From now on, fix \( c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n \). Let us define a one-dimension \( \mathfrak{p} \)-module \( C_c = C_1c \)

by defining the action for any \( e_{i,j} \in \mathfrak{p} \) on the generator \( 1_c \) as

\[
e_{i,j}1_c = \delta_{i,j}(-1)^{|i|}c_{\text{col}(i)}.
\]

By a direct check, it does make sense. Furthermore, consider an automorphism

\[
\eta_c : U(\mathfrak{p}) \rightarrow U(\mathfrak{p}) \quad e_{i,j} \mapsto e_{i,j} + \delta_{i,j}(-1)^{|i|}c_{\text{col}(i)}
\]

for each \( e_{i,j} \in \mathfrak{p} \). Denote by \( V^{\otimes d}_c \) the graded \( U(\mathfrak{p}) \)-module which is equal to \( V^{\otimes d} \), as a graded vector space, and endowed with the action obtained by twisting the natural one by the automorphism \( \eta_c \).

This is to say, \( u \cdot v = \eta_c(u)v \) for \( u \in U(\mathfrak{p}) \) and \( v \in V^{\otimes d}_c \).

Naturally, we can identify \( \mathbb{C}_c \otimes V^{\otimes d} \) with \( V^{\otimes d}_c \) so that \( 1_c \otimes v = v \) for any \( v \in V^{\otimes d}_c \). The above \( \mathbb{C}_c \) and \( V^{\otimes d}_c \) can be naturally considered as \( W_\chi \)-modules by restriction.

It follows from Lemma 3.1 that \( V^{\otimes d}_c \) becomes a \( \mathfrak{g}_e \)-module by restriction. So we have the following tensor representation of \( U(\mathfrak{g}_e) \) with

\[
\phi_{d,c} : U(\mathfrak{g}_e) \rightarrow \text{End}_{\mathbb{C}}(V^{\otimes d}_c),
\]

which is the composition of \( \phi_d \) (see [3, 7]) and \( \eta_c \). The image of \( \phi_{d,c} \) is the same as the image of \( \phi_d \). So the Vust theorem is still true to the case with \( V^{\otimes d} \) being replaced by \( V^{\otimes d}_c \). Correspondingly, we have the following representation of \( W_\chi \) with

\[
\Phi_{d,c} : W_\chi \rightarrow \text{End}_{\mathbb{C}}(V^{\otimes d}_c).
\]
Recall that the grading on $V$ is defined by letting $v_i$ be $(n - \text{col}(i))$ (see §3.7). Then both $V_c \otimes^d$ and $\text{End}_c(V_c \otimes^d)$ also have gradations. Therefore, $\text{End}_c(V_c \otimes^d)$ can be considered as a filtered superalgebra via

$$
(5.7) \quad F_r \text{End}_c(V_c \otimes^d) = \bigoplus_{s \leq r} \text{End}_c(V_c \otimes^d)_s,
$$

and then the homomorphism $\Phi_{d,c}$ is a homomorphism of filtered superalgebras. Moreover, if we identify the associated graded algebra gr($\text{End}_c(V_c \otimes^d)$) with $\text{End}_c(V_c \otimes^d)$ in the obvious way, Theorem 4.8 shows that the associated graded map

$$
\text{gr} \Phi_{d,c} : \text{gr}(W_{\chi}) \rightarrow \text{End}_c(V_c \otimes^d)
$$
coinsides with the map $\phi_d$ in §3.7.

### 5.3. Whittaker functor and Skryabin’s equivalence.

In this subsection, we turn back to the case of $\mathfrak{g}$ being a general basic classical Lie superalgebra as in §4. We will recall the connection between the super module category of the finite $W$-superalgebra $W_{\chi}$ associated with $\mathfrak{g}$ and a nilpotent element in $\mathfrak{g}_0$ and the Whittaker super module category of $\mathfrak{g}$. As usual, we make an appointment that for each category, the homomorphisms between objects are even. In the sequent arguments, supermodules will be simply called modules, and the notations in §4 are maintained.

**Definition 5.3.** A $\mathfrak{g}$-module $L$ is called a Whittaker module, if for any $a \in \mathfrak{m}$, there exists $p > 0$ such that $(a - \chi(a))^p$ vanishes $L$. A vector $v \in L$ is called a Whittaker vector if $\forall a \in \mathfrak{m}$, we have $(a - \chi(a))v = 0$.

Let $\mathfrak{g} \cdot W \text{-mod}^\chi$ denote the category of all the finitely generated Whittaker $\mathfrak{g}$-module, and assume that all the homomorphisms are even as mentioned previously. Let $\text{Wh}(L) = \{v \in L \mid (a - \chi(a))v = 0, \forall a \in \mathfrak{m}\}$ be spanned by the Whittaker vectors in $L$. Recall that $W_{\chi} \cong (U(\mathfrak{g})/I_{\chi})^\text{ad}m$, and denote by $\bar{y} \in U(\mathfrak{g})/I_{\chi}$ the coset associated to $y \in U(\mathfrak{g})$. Then we have

**Theorem 5.4.** ([42, Theorem 2.16])

(i) Given a Whittaker $\mathfrak{g}$-module $L$ with an action map $\rho$, Wh($L$) is naturally a $W_{\chi}$-module by letting

$$
\bar{y}.v = \rho(y)v
$$

for $v \in \text{Wh}(L)$ and $\bar{y} \in U(\mathfrak{g})/I_{\chi}$.

(ii) For $M \in W_{\chi}$-mod, $Q_{\chi} \otimes_{W_{\chi}} M$ is a Whittaker $\mathfrak{g}$-module by letting

$$
y.(q \otimes v) = (y.q) \otimes v
$$

for $y \in U(\mathfrak{g})$ and $q \in Q_{\chi}$, $v \in M$.

Moreover, the following theorem shows that there exist category equivalences between $\mathfrak{g} \cdot W \text{-mod}^\chi$ and $W_{\chi}$-mod, which is the super version of Skryabin’s equivalence for finite $W$-algebras.

**Theorem 5.5.** ([42, Theorem 2.17]) The functor $Q_{\chi} \otimes_{W_{\chi}} - : W_{\chi}$-mod $\rightarrow \mathfrak{g} \cdot W \text{-mod}^\chi$ is a category equivalence, with $\text{Wh} : \mathfrak{g} \cdot W \text{-mod}^\chi \rightarrow W_{\chi}$-mod being its quasi-inverse.

### 5.4. Functor $- \otimes X$.

From now on, we will be concerned with $\mathfrak{g} = \mathfrak{gl}(m|n)$ and keep the same notations as in §3 and §4.

As an analogue of Lie algebra case (see [40, §8]), associated with a $\mathfrak{g}$-module $X$, in this subsection we will introduce a functor from the category of $W_{\chi}$-modules to itself with

$$
- \otimes X : W_{\chi}$-mod $\rightarrow W_{\chi}$-mod.

Set $1_{\chi}$ to be the image of 1 in $Q_{\chi}$ where $Q_{\chi}$ is identified with $U(\mathfrak{g})/I_{\chi}$, and define the dot action of $u \in U(\mathfrak{p})$ on $Q_{\chi}$ by $u \cdot u'1_{\chi} := \eta_c(u)u'1_{\chi}$ for all $u' \in U(\mathfrak{g})$. Let $\{y_1, \ldots, y_r\}$ and $\{y_{r+1}, \ldots, y_t\}$. 

be the bases of the even part and odd part of \( m \), respectively. Further assume that \( y_i \in \mathfrak{g}(-d_i) \) for \( 1 \leq i \leq \iota \). Then the elements \([y_i, e]\) are linearly independent, and \([y_i, e] \in \mathfrak{g}(2 - d_i)\). There exist elements \( z_1, \ldots, z_r \) and \( z_{r+1}, \ldots, z_i \in \mathfrak{p} \) such that \( z_i \in \mathfrak{g}(d_i - 2) \) with 

\[
(y_i, [z_j, e]) = ([y_i, z_j], e) = \delta_{i,j}.
\]

**Lemma 5.6.** The right \( W_\chi \)-module \( Q_\chi \) is free with basis 

\[
\{z_1^{i_1} \cdots z_r^{i_r} z_{r+1}^{\epsilon_{r+1}} \cdots z_i^{\epsilon_i} \cdot 1_\chi \mid i_1, \ldots, i_r \geq 0 \text{ and } \epsilon_{r+1}, \ldots, \epsilon_i \in \mathbb{Z}_2\}.
\]

**Proof.** We sketch a proof as in the Lie algebra case in [37]. Write 

\[
\Theta = \{i = (i_1, \ldots, i_r; \epsilon_{r+1}, \ldots, \epsilon_i) \mid i_j \geq 0, \epsilon_t \in \{0, 1\}\}.
\]

For \( i \in \Theta \), put 

\[
|i| = \sum_{i=r+1}^{r} i_j + \sum_{t=r}^{i} \epsilon_t, \quad \text{wt } i = \sum_{j=1}^{r} i_j d_j + \sum_{t=r+1}^{i} \epsilon_t d_t,
\]

\[
Z^i = z_1^{i_1} \cdots z_r^{i_r} z_{r+1}^{\epsilon_{r+1}} \cdots z_i^{\epsilon_i} \in U(p),
\]

\[
u_i = (y_1 - \chi(y_1))^{i_1} \cdots (y_r - \chi(y_r))^{i_r} \cdots (y_{r+1} - \chi(y_{r+1}))^{\epsilon_{r+1}} \cdots (y_i - \chi(y_i))^{\epsilon_i} \in U(m).
\]

Consider any linear order on \( \Theta \) subject to the condition: \( i < j \) whenever \( \text{wt } i < \text{wt } j \) or \( \text{wt } i = \text{wt } j \) and \( |i| > |j| \). If \( i \neq 0 \), let \( i' \) denote the predecessor of \( i \). Denote by \( I_i \) the linear span of \( u_i \) with \( j > i \). Now we identify \( Q_\chi \) and \( U(p) \) via the map \( \text{Pr} \). Let \( Q^i_\chi = \{u \in Q_\chi \mid I_i u = 0\} \). Multiplying by \( u_i \) defines a linear map 

\[
\tau_i : Q^i_\chi \to Q^0_\chi = Q^{1d}_\chi = W_\chi.
\]

Arguing as in [37], we can show that \( Q^i_\chi / Q^{i'}_\chi \cong Q^0_\chi = W_\chi \) as right \( W_\chi \)-modules, and the image of \( Z^i \) in \( Q^i_\chi / Q^{i'}_\chi \) is a set of free generators of that module. The lemma follows from the fact that \( Q_\chi = \bigcup Q^i_\chi \). \( \square \)

It follows from Lemma 5.6 that there is a unique right \( W_\chi \)-module homomorphism \( p : Q_\chi \to W_\chi \) with 

\[
p(z_1^{i_1} \cdots z_r^{i_r} z_{r+1}^{\epsilon_{r+1}} \cdots z_i^{\epsilon_i} \cdot 1_\chi) = \delta_{i_1,0} \cdots \delta_{i_\iota,0} \delta_{\epsilon_{r+1},0} \cdots \delta_{\epsilon_i,0}
\]

for all \( i_1, \ldots, i_r \geq 0, \epsilon_{r+1}, \ldots, \epsilon_i \in \mathbb{Z}_2 \). In particular, \( p(1_\chi) = 1 \). And \( p \) is an even map.

5.5. **Tensor identities.** Now let \( X \) be a finite-dimensional \( \mathfrak{g} \)-module with fixed basis \( w_1, \ldots, w_{r_0} \in X_\mathfrak{g}, w_{r_0+1}, \ldots, w_t \in X_\mathfrak{f} \). For any \( u \in U(\mathfrak{g}) \), define the coefficient functions \( c_{i,j} \in U(\mathfrak{g})^* \) by the equation 

\[
u_i \cdot w_j = \sum c_{i,j}(u) w_i.
\]

Given any \( W_\chi \)-mod \( M \), it is clear that the usual \( \mathfrak{g} \)-module tensor product \( M \otimes X \) also belongs to the category \( W_\chi \)-mod. With aid of the Skryabin’s equivalence, we define a functor of the category \( W_\chi \)-mod to itself as below: for a \( W_\chi \)-module \( M \), set \( (M \otimes -) \) by 

\[
(M \otimes -)(X) = \text{Wh}((Q_\chi \otimes W_\chi M) \otimes X),
\]

thereafter whose image will be directly denoted by \( M \otimes X \). This is an exact functor in \( W_\chi \)-mod. In the following, we introduce an important result which is a super version of [9, Theorem 8.1] which will be used later. For this, firstly we fix a right \( W_\chi \)-module homomorphism \( p : Q_\chi \to W_\chi \) with \( p(1_\chi) = 1 \) (see (5.8)). Note that \( p \) is an even map. By Skryabin’s equivalence (Theorem 5.5),
the same arguments as in [9] Theorem 8.1 entail that the restriction of the map \((Q_\cdot W_\cdot M) \otimes X \to M \otimes X\) sending \((u_1 \cdot m) \otimes w\) to \(p(u_1) m \otimes w\) defines a natural even super-space isomorphism 
\[
\chi_{M,X} : M \oplus X \to M \otimes X.
\]
When considering the regular \(W_\cdot\)-module \(M\) which is \(W_\cdot\) itself, the inverse image of \(1 \otimes w\) under the isomorphism \(\chi_{W_\cdot,X}\) can be written as

\[
\chi_{W_\cdot,X}^{-1}(1 \otimes w_j) = \sum_{i=1}^{t} (x_{ij} \cdot 1_\cdot 1) \otimes w_i \quad \text{for unique elements } x_{ij} \in U(p).
\]

Since the map \(\chi_{W_\cdot,X}\) is even, all \(x_{ij}\) are even elements. Furthermore, for \(x \in m\) we have

\[
(x - \chi(x)) \cdot \sum_{i=1}^{t} (x_{ij} \cdot 1_\cdot 1) \otimes w_i = \sum_{i=1}^{t} ([x, x_{ij}] \cdot 1_\cdot 1) \otimes w_i + \sum_{i=1}^{t} (x_{ij} \cdot 1_\cdot 1) \otimes x \cdot w_i.
\]

\[
(5.12) \quad \sum_{i=1}^{t} \left( \sum_{s=1}^{t} c_{i,s}(x)x_{s,j} \right) \cdot 1_\cdot 1 \otimes w_i.
\]

Note that \(\chi_{W_\cdot,X}^{-1}(1 \otimes w_j)\) lies in \(\text{Wh}(Q_\cdot \otimes C) \otimes X)\). By definition, (5.11) and (5.12) give rise to the following equation

\[
[x, \eta_c(x_{ij})] + \sum_{s=1}^{t} c_{i,s}(x)\eta_c(x_{s,j}) \in I_X.
\]

Taking the parities into account as above, by the same arguments as in the proof of [9] Theorem 8.1 we have the following theorem.

**Theorem 5.7.** Let \(M\) be any left \(W_\cdot\)-module, and \(X\) be a finite-dimensional \(g\)-module as above, with super dimension \(\dim X = (r_0|t - r_0)\). For the natural vector spaces isomorphism 
\[
\chi_{M,X} : M \oplus X \to M \otimes X,
\]
its inverse map of \(\chi_{M,X}\) can be described by mapping \(m \otimes w_j\) to \(\sum_{i=1}^{t} (x_{ij} \cdot 1_\cdot m) \otimes w_i\), where \((x_{ij})_{t \times t}\) is an invertible supermatrix with entries \(x_{ij} \in U(p)\) being uniquely determined by the properties:

(i) \(|x_{ij}| = \overline{0}\) and \(p(x_{ij} \cdot 1_\cdot) = \delta_{i,j}\).

(ii) \([x, \eta_c(x_{ij})] + \sum_{s=1}^{t} c_{i,s}(x)\eta_c(x_{s,j}) \in I_X\) for all \(x \in m\) for all \(i, j\).

**Proof.** With the arguments prior to the theorem, what remains are essentially to repeat verbatim the proof of [9] Theorem 8.1. Particularly, all \(x_{ij}\) are even because \(\chi \in g^\ast\) is even, and the isomorphism \(\chi_{M,X}\) is even. \(\square\)

Then we have the following corollary.

**Corollary 5.8.** Keep the notations as above. Additionally, suppose \(M\) is a \(p\)-module, and \(X\) is a finite-dimensional \(g\)-module as defined above. For any \(u \in U(p)\), \(m \in M\) and \(w \in X\), the restriction of the map 
\[
\hat{\mu}_{M,X} : (Q_\cdot W_\cdot M) \otimes X \to M \otimes X
\]
\[
(u \cdot 1_\cdot m) \otimes w \mapsto um \otimes w
\]
defines an isomorphism \(\mu_{M,X}\) of \(W_\cdot\)-modules by 
\[
\mu_{M,X} : M \oplus X \cong M \otimes X.
\]
Here, the $U(p)$-modules $M$ and $M \otimes X$ are regarded the left and right hand sides as $W_X$-modules by restriction. The inverse map sends $m \otimes w_k$ to $\sum_{i,j=1}^t (x_{i,j} \cdot 1 \otimes y_{j,km}) \otimes w_i$, where $(y_{i,j})_{t \times t}$ is the inverse matrix of $(x_{i,j})_{t \times t}$ defined in Theorem 5.7.

Proof. The proof is similar to that in [9, Corollary 8.2].

The composition of the functor $\oplus$ is similar to that in [9 (8.8)-(8.10)]. In particular, for a finite dimension $g$-module $Y$, there is a natural isomorphism:

\[ \alpha_{M,Y} : (M \oplus X) \oplus Y \cong M \oplus (X \otimes Y). \]

One can define $M \oplus Y$ and has a natural isomorphism $M \oplus Y \cong M \oplus Y$. In the following we will consider tensor products involving $\oplus$ powers.

5.6. From now on, we focus on $M = \mathbb{C}_c$ which is regarded as a $U(p)$-module. We can define the $(W_X,H_d)$-bimodule $\mathbb{C}_c \otimes V \otimes d$. Then by the arguments as in [8 §3.3], the induction starting from (5.13) gives rise to the following isomorphism of $W_X$-modules:

\[ \mu_{\mathbb{C}_c, V \otimes d} : \mathbb{C}_c \otimes V \otimes d \sim \mathbb{C}_c \otimes V \otimes d = V \otimes d. \]

Now we have the following corollary of Theorem 5.7 and Corollary 5.8 for the case $M = \mathbb{C}_c$ and $X = V$.

**Corollary 5.9.** Keep the notation $I := I(m|n)$. The following hold:

1. There exists an element $x_{r,t} := x_{r_1,t_1} \ldots x_{r_d,t_d} \in U(p)$ with $r = (r_1, \ldots, r_d), t = (t_1, \ldots, t_d) \in I^d$, satisfying

   - (i) $e_{i,j}, \eta_c(x_{r,t})] + \sum_{s \in I^d} \eta_c(x_{s,t}) \in I_X$ as long as $e_{i,j} \in \mathfrak{m}$, where the sum is over all $s \in I^d$ obtained from $r$ by replacing an entry equal to $i$ by $j$;

   - (ii) The action of $x_{r,t}$ on $\mathbb{C}_c$ is given by $x_{i,j}1_c = \delta_{i,j}1_c$;

2. Under the isomorphism $\mu_{\mathbb{C}_c, V \otimes d}$, the inverse element of $v_1 \in V \otimes d$ is $\sum_{i \in I^d} (\eta_c(x_{i,j})1 \otimes 1_c) \otimes v_1 \in \mathbb{C}_c \otimes V \otimes d$.

**Proof.** This corollary follows from Theorem 5.7 and Corollary 5.8 along with the computation as below.

Note that $e_{i,j}(v_1) = \delta_{j,t}v_1$, and we have $c_{r,s}(e_{i,j}) = \delta_{s,j}e_{r,t}$ for $i, j, r, s \in I = I(m|n)$. By Theorem 5.7, if $e_{i,j} \in \mathfrak{m}$, then

\[ [e_{i,j}, \eta_c(x_{r,t})] + \sum_{s \in I^d} c_{r,s}(e_{i,j}) \eta_c(x_{s,t}) = [e_{i,j}, \eta_c(x_{r,t})] + \delta_{r,t}x_{i,j} \in I_X. \]

Note that for $r' = (r_1, \ldots, r_{d-1}), t' = (t_1, \ldots, t_{d-1}) \in I^{d-1}$, we have

\[ [e_{i,j}, \eta_c(x_{r,t})] = [e_{i,j}, \eta_c(x_{r',t'})] \eta_c(x_{r_{d-1},t_d}) + (-1)^{|e_{i,j}|} x_{r',t'} [e_{i,j}, \eta_c(x_{r_{d-1},t_d})]. \]

Taking (5.14) into consideration, we have (1)-(i). The remaining statements follow the above arguments. \(\square\)

6. Degenerate affine Hecke algebras and their double centralizers with $W_X$

In this section, on the basis of §5 we exploit the arguments of [7] on finite $W$-algebras to the super case, and finally obtain the duality presented in Theorem 0.4.
6.1. The natural transformations of tensor functors and the dAHAs. Following the approach of Chuang-Rouquier [14 §7.4], Brundan-Kleshchev introduced the degenerate affine Hecke algebra into the study of tensor representations of finite $W$-algebras in [5] and [9]. Their arguments are in principle available to the super case. Let $\mathfrak{g} = \mathfrak{gl}(V)$ for $V = \mathbb{C}^{m|n}$. For a given $W_\chi$-module $M$ and a finite-dimensional $\mathfrak{g}$-module $V$, define a $W_\chi$-module endomorphism $\Omega : M \otimes V \to M \otimes V$ by

$$\Omega = \sum_{i,j \in I(m|n)} (-1)^{|i|}e_{i,j} \otimes e_{j,i}.$$

Recall $V = \mathbb{C}^{m|n}$ admits a standard basis $v_1, \ldots, v_m, v_1, \ldots, v_n$. Consider the functor $- \otimes V$ of the category $W_\chi$-mod. Define a natural transformation $x$ from the functor $- \otimes V$ to itself: for any $W_\chi$-module $M$, $x_M$ is an endomorphism of $(- \otimes V)(M) = M \otimes V$, defined by left multiplication by $\Omega$, precisely

$$(6.1) \quad \Omega((u1_\chi \otimes m) \otimes v) = \sum_{i,j \in I(m|n)} (-1)^{|i|+(|i|+|j|)(|u|+|v|)}(e_{i,j}u1_\chi \otimes m) \otimes e_{j,i}v$$

where the $\mathfrak{g}$-module $Q_\chi \otimes W_\chi M$ is regarded as the first tensor position, and $V$ as the second. Furthermore, keeping the map $\alpha_{M,V,Y}$ (see (5.13)) in mind we can define another natural transformation $s$ from the functor $(- \otimes V) \otimes V$ to itself as below. Take

$$s_M : (M \otimes V) \otimes V \to (M \otimes V) \otimes V$$

to be the composite map $\alpha^{-1}_{M,V,V} \circ \tilde{s}_M \circ \alpha_{M,V,V}$, where $\tilde{s}_M$ is the endomorphism of $M \otimes (V \otimes V) = \text{Wh}((Q_\chi \otimes W_\chi M) \otimes V \otimes V)$ arising from left multiplication by $\Omega^{1,3}$, which means

$$\Omega^{1,3}((u1_\chi \otimes m) \otimes v \otimes v') := \sum_{i,j \in I(m|n)} (-1)^{|i|+(|i|+|j|)|v|}(u1_\chi \otimes m) \otimes e_{i,j}v \otimes e_{j,i}v'.$$

By a direct check (see [14 Proposition 5.1]), the above can be reformulated as below

$$(6.2) \quad \Omega^{1,3}((u1_\chi \otimes m) \otimes v \otimes v') = (u1_\chi \otimes m) \otimes v' \otimes v.$$

More generally, for any $d \geq 1$ we can introduce the following natural endomorphisms of the functor $- \otimes V^\otimes d$.

$$(6.3) \quad x_i := 1^{d-i}x1^{i-1}, \quad s_j := 1^{d-j-1}s1^{j-1}$$

for $1 \leq i \leq d$ with $d \geq 1$ and $1 \leq j \leq d - 1$ for all $d \geq 2$. By the arguments parallel to [5 §3] and [9 §8], these induce a well-defined right action of the dAHA $H_d$ on $M \otimes V^\otimes d$ for any $W_\chi$-module $M$.

There is a more convenient way to describe them. To achieve this, we exploit the natural isomorphism

$$(6.4) \quad a_d : (- \otimes V)^d \simto - \otimes V^{\otimes d},$$

which is obtained by iterating the map in (4.13). For $1 \leq i \leq d$ and $1 \leq j \leq d - 1$, let $(\tilde{x}_i)_M$ and $(\tilde{s}_j)_M$ denote the endomorphisms of the image $(- \otimes V^{\otimes d})(M)$ defined by the left multiplication of the elements $\sum_{h=1}^i \Omega^{[h,i+1]}$ and $\Omega^{[j+1,j+2]}$, respectively. Here and thereafter, for $s > r$ set

$$\Omega^{[r,s]} = \sum_{i,j \in I(m|n)} (-1)^{|i|}1^{\otimes(r-1)} \otimes e_{i,j} \otimes 1^{\otimes(s-r-1)} \otimes e_{j,i} \otimes 1^{\otimes(d+1-s)}.$$ 

Then we have the following reformulation of $x_i$ and $s_j$:

$$(6.5) \quad x_i = a_d^{-1} \circ \tilde{x}_i \circ a_d, \quad s_j = a_d^{-1} \circ \tilde{s}_j \circ a_d.$$
As in the ordinary case dealt in [7] and [8], we have the following important property for those natural endomorphisms $x_i, s_j$ of the functor $- \otimes V^{\otimes d}$.

**Theorem 6.1.** The above $x_i$ and $s_j$ satisfy the axioms in Definition 1.3 for $H_d$.

**Proof.** In the super case, by (8.2) a relation crucially making potential difference is kept true as in the ordinary case [9] §4.4 and §8.3. By a direct check as in the ordinary case (see [7] §3.4] and [9] §8.3), all axioms in Definition 1.3 of $H_d$ are satisfied for the above $x_i$ and $s_j$. The theorem follows. 

By the above theorem, there is a natural representation of the degenerate affine Hecke Algebra $H_d$ on $M \oplus V^{\otimes d}$ for any $W_X$-module $M$.

**Remark 6.2.** In the case of the functor $- \otimes V^{\otimes d}$, the corresponding natural transformations $x_i$ and $s_j$ can be similarly defined. They also satisfy the axioms of $H_d$ (see [12] §5.1]).

6.2. Turn to the case $M = C_c$. In [5.6] there is already an isomorphism of $W_X$-modules:

$$C_c \oplus V^{\otimes d} \cong C_c \otimes V^{\otimes d} = V^{\otimes d}.$$ 

The $H_d$-module $C_c \otimes V^{\otimes d}$ can be lifted to $C_c \oplus V^{\otimes d}$, and then to $V^{\otimes d}$. This makes them into $(W_X, H_d)$-bimodules. Let us describe the actions of the generators of $H_d$ on $C_c \oplus V^{\otimes d}$ and $V^{\otimes d}$ explicitly. Make $x_i$ and $s_j$ act on

$$C_c \otimes V^{\otimes d} \subseteq (Q_X \otimes_{W_X} C_c) \otimes V^{\otimes d}$$

in the same way as the endomorphisms of these modules defined by the multiplication of $\sum_{h=1}^{i} \Omega_{[h,i+1]}$ and $\Omega_{[j+1,j+2]}$, respectively. Regard $Q_X \otimes_{W_X} C_c$ as the first tensor position and the copies of $V$ as positions $2, 3, \ldots, (d + 1)$.

We now describe the action of $H_d$ on $V^{\otimes d}$ in this special case. Each $s_j$ acts by permuting the $i$th and $(i + 1)$th tensor positions, arising from the morphism $\mu_{C_c, V^{\otimes d}} : C_c \otimes V^{\otimes d} \rightarrow V^{\otimes d}$ as described in Corollary 5.8. The action of each $x_s$ is described by the following lemma.

**Lemma 6.3.** Set $I = I(m|n)$ as before. For $i \in I^d$ with $1 \leq s \leq d$, the action of $x_s$ on $V^{\otimes d} \otimes V^{\otimes d}$ satisfies

$$v_{t}x_s = c_{col(i_s)}v_1 + (-1)^{|i_s|}v_{1-i_s} + \sum_{1 \leq t < s \atop col(i_t) \geq col(i_s)} (-1)^{|i_t|}v_{i(t s)} \sum_{s < t \leq d \atop col(i_t) < col(i_s)} (-1)^{|i_t|}v_{i(s t)},$$

where $i(t s)$ stands for the transposition interchanging $t$ and $s$ in $S_d$, and the meaning of $v_{1-i_s}$ is the same as in (5.10).

**Proof.** First note that $x_{j+1} = s_jx_j s_j + s_j$ for $j = 1, 2, \ldots, d - 1$. Carrying induction on $j$, we just need to check the situation with $j = 1$. Let $\mu$ denote the map $\mu_{C_c, V^{\otimes d}}$ for simplicity. For $i = (i_1, \ldots, i_d), j = (j_1, \ldots, j_d) \in I^d$, take $x_{i_1} \in U(p)$ as in Corollary 5.9. Recall that $\Omega^{[1,2]} =$
\[ \sum_{i,j \in I} (-1)^j e_{i,j} \otimes e_{j,i} \otimes 1^{\otimes d-1} \]. Keeping (6.1) in mind, by definition we then have
\[ v_1 x_1 = \mu \left( \Omega^{[1,2]} \left( \sum_{j \in I^d} (\eta_e(x_{j,1}) 1_\chi \otimes 1_c) \otimes v_j \right) \right) \]
\[ = \sum_{j,k \in I^d} (-1)^{|j| + |k|} \mu((e_{j,k,1} \eta_e(x_{j,1}) 1_\chi \otimes 1_c) \otimes v_k) \]
\[ = \sum_{j,k \in I^d} (-1)^{|j|} \mu((e_{j,k,1} \eta_e(x_{j,1}) 1_\chi \otimes 1_c) \otimes v_k) \]
where in the last summands, \( j, k \in I^d \) satisfy the condition that \( j_2 = k_2, \ldots, j_d = k_d \); and the last equation is due to the fact \( |x_{j,1}| = 0 \) (see Theorem 5.7(i)). In the following, we proceed the arguments in different cases.

(i) First assume that \( \text{col}(j_1) \leq \text{col}(k_1) \). According to Corollary 5.9(1)-(ii) and the definitions of \( \mu \) and \( C_c \), we have \( \mu((e_{j_1,k_1} \eta_e(x_{j,1}) 1_\chi \otimes 1) \otimes v_k) = 0 \) except \( i = j = k \), and in this exceptional case we have
\[ (-1)^{|j_1|} \mu((e_{j_1,k_1} \eta_e(x_{j,1}) 1_\chi \otimes 1_c) \otimes v_k) = (-1)^{|i|} \mu((e_{i_1,i_1} \eta_e(x_{i,1}) 1_\chi \otimes 1_c) \otimes v_i) \]
\[ = c_{\text{col}(i_1)} v_i. \]

(ii) Now consider the case with \( \text{col}(k_1) < \text{col}(j_1) \). With the identification of \( W_\chi \) in Lemma 5.3, \( e_{j_1,k_1} \) becomes an element of \( m \) (see (5.2)). Note that
\[ e_{j_1,k_1} \eta_e(x_{j,1}) 1_\chi = \eta_e(x_{j,1}) e_{j_1,k_1} 1_\chi + [e_{j_1,k_1}, \eta_e(x_{j,1})] 1_\chi. \]
So we have
\[ (-1)^{|j_1|} \mu((e_{j_1,k_1} \eta_e(x_{j,1}) 1_\chi \otimes 1_c) \otimes v_k) \]
\[ = (-1)^{|k_1|} \mu((\eta_e(x_{j,1}) e_{j_1,k_1} 1_\chi \otimes 1_c) \otimes v_k) + (-1)^{|k_1|} \mu(([e_{j_1,k_1}, \eta_e(x_{j,1})] 1_\chi \otimes 1_c) \otimes v_k). \]
In the above equation, for the first summand on the right hand side we know that \( \chi(e_{j_1,k_1}) \) is zero unless \( k_1 = j_1 - 1 \) because the corresponding nilpotent element \( e \) is regular of the form (5.1). In the case, Corollary 5.9(2) entails
\[ \sum_{j,k \in I^d} (-1)^{|k_1|} \mu((\eta_e(x_{j,1}) e_{j_1,k_1} 1_\chi \otimes 1_c) \otimes v_k) = (-1)^{|i_1|} v_{i_1-i_1} \]
where the meaning of \( v_{i_1-i_1} \) is the same as in (3.16).

For the second summand on the right side of (6.6), thanks to Corollary 5.9(1)-(i), by calculation we can conclude that
\[ \sum_{j,k \in I^d} (-1)^{|k_1|} \mu([e_{j_1,k_1}, \eta_e(x_{j,1})] 1_\chi \otimes 1_c) \otimes v_k) \]
\[ = \sum_{k \in I^d} (-1)^{|k_1|} \left( \sum_{s \in I^d} \mu((\eta_e(x_{s,i}) 1_\chi \otimes 1_c) \otimes v_k) \right) \]
where \( s \in I^d \) is obtained from \( j \) by replacing the entry equal to \( j_1 \) by \( k_1 \). Keeping in mind that \( k_t = j_t \) for \( t = 2, \ldots, d \), Corollary 5.8 and \( x_{s,i} 1_c = \delta_{s,i} 1_c \) (Corollary 5.9(1)-(ii)), we have that \( i \) is obtained from \( j \) by replacing an entry equal to \( j_1 \) by \( k_1 \), and all possible \( j \) are of the form \( (i_1, i_2, \ldots, i_{t-1}, i_1, i_{t+1}, \ldots, i_d) \) for \( t \in \{2, \ldots, d\} \) as long as \( \text{col}(i_1) > \text{col}(i_t) \). This implies that \( k_1 = i_1 \). Correspondingly, all possible \( k \) are of the following form
\[ (i_t, i_2, \ldots, i_{t-1}, i_1, i_{t+1}, \ldots, i_d) \text{ with } i_t < i_1. \]
By summing up, we have
\[
\sum_{j,k \in I^d} (-1)^{|k_1|} \mu([e_{j_1,k_1}, \eta_c(x_{j_1})] 1_\chi \otimes 1_c \otimes v_k) 
= - \sum_{1 < t \leq d, \text{col}(i_t) < \text{col}(i_1)} (-1)^{|i_t|} \mu((1_\chi \otimes 1_c) \otimes v_{i_t})
\]
(6.8)
\[
= - \sum_{1 < t \leq d, \text{col}(i_t) < \text{col}(i_1)} (-1)^{|i_t|} v_{i_t}(1) t.
\]
Summing up, we have
\[
v_{i_1 x_1} = c_{\text{col}(i_1)} v_{i_1} + (-1)^{|i_1|} v_{i_1 - i_1} - \sum_{1 < t \leq d, \text{col}(i_t) < \text{col}(i_1)} (-1)^{|i_t|} v_{i_t}(1) t.
\]

6.3. Degenerate cyclotomic Hecke algebra.

Lemma 6.4. Keep the notations as before, in particular \( \mathfrak{g} = \mathfrak{gl}(m|n) \) \( (m \leq n) \). For \( d \geq 2 \), the minimal polynomial of the endomorphism of \( V_c^{\otimes d} \) defined by the action of \( x_1 \) is \( \prod_{i=1}^n (x - c_i) \).

Proof. Keeping in mind (6.16) and (6.7), by (6.9) we have that the minimal polynomial of \( \mathfrak{gl}(m|n) \) divides \( \prod_{i=1}^n (x - c_i) \) for any \( d \geq 1 \). Recall the filtration of \( \text{End}_C(V_c^{\otimes d}) \) defined in (6.7). Thanks to Lemma 6.3 the endomorphism of \( V_c^{\otimes d} \) defined by \( x_1 \) belongs to \( \mathfrak{F}_1 \text{End}_C(V_c^{\otimes d}) \), and the associated graded endomorphism of \( V_c^{\otimes d} \) is equal to \( \varphi \circ e \otimes 1^{\otimes (d-1)} \) where \( \varphi \in \text{End}_C(V) \) is defined via \( \varphi(v) = (-1)^{|v|} v \) for any \( \mathbb{Z}_2 \)-homogeneous vector \( v \in V \). Since \( e \) is regular nilpotent with Jordan block of size \( n \), the corresponding minimal polynomial of \( (\varphi \circ e \otimes 1^{\otimes (d-1)}) \) is exactly equal to \( x^n \). This implies that the degree of minimal polynomial of the endomorphism defined by the action of \( x_1 \) cannot be of smaller than \( n \), completing the proof.

Let \( \Lambda_e = \sum_{i=1}^n \Lambda_{c_i} \), be an element of the free abelian group generated by symbols \( \{ \Lambda_{c} \mid a \in \mathbb{C} \} \). The corresponding degenerate cyclotomic Hecke algebra \( H_d(\Lambda) \) is the quotient of \( H_d \) by the two-sided ideal generated by \( \prod_{i=1}^n (x - c_i) \). In view of Theorem 6.1 and Lemma 6.4, the right action of \( H_d \) on \( V_c^{\otimes d} \) factors through the quotient \( H_d(\Lambda) \), and we can obtain a homomorphism
\[
\Psi_d : H_d(\Lambda) \to \text{End}_C(V_c^{\otimes d})^{\text{op}}.
\]
(6.10)

Define a filtration \( F_0 H_d(\Lambda) \subseteq F_1 H_d(\Lambda) \subseteq \cdots \) by declaring that \( F_r H_d(\Lambda) \) is the span of all \( x_1^{i_1} \cdots x_d^{i_d} w \) for \( i_1, \ldots, i_d \geq 0 \) and \( w \in \mathfrak{S}_d \) with \( i_1 + \cdots + i_d \leq r \). Then there is a well-defined surjective homomorphism of graded superalgebras
\[
\zeta_d : C[x_1, \ldots, x_d] \otimes C \mathfrak{S}_d \to \text{gr} H_d(\Lambda)
\]
such that \( x_i \mapsto \text{gr} x_i, s_j \mapsto \text{gr} s_j \) for each \( i, j \). By Lemma 6.3 we see that the map \( \Psi_d \) defined in (6.10) is a homomorphism of filtered algebras and
\[
(\text{gr} \Psi_d) \circ \zeta_d = \tilde{\psi}_d.
\]
Moreover, by the similar discussion as in [8, Lemma 3.5], we have

Lemma 6.5. the map \( \zeta_d : C[x_1, \ldots, x_d] \otimes C \mathfrak{S}_d \to \text{gr} H_d(\Lambda) \) is an isomorphism of graded superalgebras.
6.4. **Proof of Theorem 0.4:** a higher level Schur-Sergeev duality. The following lemma is a generalization of [8, Lemma 3.6].

**Lemma 6.6.** Let $\Phi : B \to A$ and $\Psi : C \to A$ be homomorphisms of filtered superalgebras such that $\Phi(B) \subseteq Z_A(\Psi(C))$, where $Z_A(\Psi(C))$ is the centralizer of $\Psi(C)$ in $A$. View the subsuperalgebras $\Phi(B)$, $\Psi(C)$ and $Z_A(\Psi(C))$ of $A$ as filtered superalgebras with filtrations induced by the one on $A$, so that the associated graded superalgebras are naturally subalgebras of $\text{gr } A$. Then

$$\text{(gr } \Phi)(\text{gr } B) \subseteq \text{gr } \Phi(B) \subseteq \text{gr } Z_A(\Psi(C)) \subseteq Z_{\text{gr } A}(\text{gr } \Psi(C)) \subseteq Z_{\text{gr } A}(\text{gr } \Psi(\text{gr } C)).$$

Now we consider the following maps:

$$W_\chi \Phi_{d,e} \text{End}_C(V_c^\otimes d) \Psi_d H_d(\Lambda_e).$$

Thanks to Theorem 4.8, we can identify the associated graded map $\text{gr } \Phi_{d,e}$ with the map $\phi_{d,e}$ in §5.2. Moreover, Lemma 6.5 and (6.11) enable us to identify $\text{gr } \Psi_d$ with $\bar{\psi}_d$. Taking $A = \text{End}_C(V_c^\otimes d)$ and $(B, C, \Phi, \Psi) = (W_\chi, H_d(\Lambda)^\text{op}, \Phi_d, \Psi_d)$ or $(H_d(\Lambda)^\text{op}, W_\chi, \Psi_d, \Phi_d)$, it follows from Theorem 0.3 that

$$(\text{gr } \Phi)(\text{gr } B) = Z_{\text{gr } A}(\text{gr } \Psi(\text{gr } C)).$$

By virtue of Lemma 6.6, we have the following theorem.

**Theorem 6.7.** Keep the notations as previously, in particular, $g = \text{gl}(m|n)$ ($m \leq n$), and $e \in g_0$ is a regular nilpotent element. Then the following double centralizer properties hold:

$$\Phi_{d,e}(W_\chi) = \text{End}_{H_d(\Lambda)}(V_c^\otimes d),$$

$$\text{End}_{W_\chi}(V_c^\otimes d)^\text{op} = \Psi_d(H_d(\Lambda_e)).$$

As the special case when $c = (0, \ldots, 0)$, Theorem 0.4 follows from the above general statement.

**Question 6.8.** (1) There is a natural question if the super Vust theorem is true for all nilpotent elements.

(2) Under the circumstance of the positive answer to (1), there is another question if the Schur-Sergeev duality for principal $W$-algebras in the present paper could be established for all finite $W$-superalgebras associated with $\text{gl}(m|n)$ and any nilpotent elements in $\text{gl}(m|n)^\ast$.

Related to the above question, there is a work [30] worth being mentioned where the author explicitly gave a superalgebra isomorphism between the finite $W$-superalgebra associated with an arbitrary nilpotent $e \in \text{gl}(m|n)^\ast$ and a quotient of a certain subalgebra of the super Yangian $Y(m|n)$. This result generalizes the main result of [7] for $\text{gl}(m|n)$.

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