MODULI SPACES OF SL(r)-BUNDLES 
ON SINGULAR IRREDUCIBLE CURVES

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INTRODUCTION

One of the problems in moduli theory, motivated by physics, is to study the degeneration of moduli spaces of semistable $G$-bundles on curves of genus $g \geq 2$. When a smooth curve $Y$ specializes to a stable curve $X$, one expects that the moduli space of semistable $G$-bundles on $Y$ specializes to a (nice) moduli space of generalized semistable $G$-torsors on $X$. It is well known ([Si]) that for any flat family $C \to S$ of stable curves there is a family $U(r,d)_S \to S$ of moduli spaces $U_C(r,d)$ of (s-equivalence classes of) semistable torsion free sheaves of rank $r$ and degree $d$ on curves $C_s$ ($s \in S$). If we fix a suitable representation $G \to \text{GL}(r)$, one would like to define a closed subscheme $U_X(G) \subset U_X(r,d)$ which should be a moduli space of suitable $G$-sheaves on $X$. Moreover, it should behave well under specialization, i.e. if a smooth curve $Y$ specializes to $X$, then the moduli space of $G$-bundles on $Y$ specializes to $U_X(G)$. By my knowledge, the problem is almost completely open except for special case like $G = \text{SO}(r)$ or $G = \text{Sp}(r)$ ([Fa1], [Fa2]), where one has a generalisation of $G$-torsors which extends the case $G = \text{GL}(r)$. It is open even for $G = \text{SL}(r)$ (See [Fa1], [Fa2] for the introduction).

In this paper, we will consider the case $G = \text{SL}(r)$ and $X$ being irreducible (the case of a reducible curve with one node was studied in [Su2]). For any projective curve $X$, we will use $U_X(r,d)$ to denote the moduli space of semistable torsion free sheaves of rank $r$ and degree $d$ on $X$. If $X_\eta$ is a smooth curve and $L_\eta$ is a line bundle of degree $d$ on $X_\eta$, we use $U_{X_\eta}(r,L_\eta)$ to denote the moduli space of semistable vector bundles of rank $r$ with fixed determinant $L_\eta$ on $X_\eta$, which is a closed subvariety of $U_{X_\eta}(r,d)$. It is known that when $X_\eta$ specializes to $X$ the moduli space $U_{X_\eta}(r,d)$ specializes to $U_X(r,d)$. It is natural to expect that if $L_\eta$ specializes to a torsion free sheaf $L$ on $X$ then $U_{X_\eta}(r,L_\eta)$ specializes to a closed subscheme $U_X(r,L) \subset U_X(r,d)$. It is important that we should look for an intrinsic $U_X(r,L)$ (i.e. independent of $X_\eta$) which should not be too bad and should represent a moduli problem.

Let $S = \text{Spec}(A)$ where $A$ is a discrete valuation ring, let $C \to S$ be a proper flat family of curves with closed fibre $C_0 \cong X$ and smooth generic fibre $C_\eta$. Then we have a $S$-flat scheme $U(r,d)_S \to S$ with generic fibre $U_{C_\eta}(r,d)$ and closed fibre being $U_X(r,d)$. For any line bundle $L_\eta$ of degree $d$ on $C_\eta$, there is a unique extension $L$.
on $C$ such that $L|_{C_0} := L$ is torsion free of degree $d$ (since $X$ is irreducible). Then $U_{C_0}(r, L_\eta) \subset U(r, d)_S$ is an irreducible, reduced, locally closed subscheme. Let

$$f : U(r, L)_S := \overline{U_{C_0}(r, L_\eta)} \subset U(r, d)_S \to S$$

be the Zariski closure of $U_{C_0}(r, L_\eta)$ in $U(r, d)_S$. Then $f : U(r, L)_S \to S$ is flat and projective, but there is no reason that its closed fibre $f^{-1}(0)$ (even its support $f^{-1}(0)_{\text{red}}$) is independent of the family $C \to S$ and $L_\eta$. However, there are conjectures ([NS]) that $f^{-1}(0)$ is intrinsic for irreducible curves $X$ with only one node. To state them, we introduce the notation for any stable irreducible curves. Let $X$ be an irreducible stable curve with $\delta$ nodes $\{x_1, \ldots, x_\delta\}$, and $L$ a torsion free sheaf of rank one and degree $d$ on $X$. A torsion free sheaf $F$ of rank $r$ and degree $d$ on $X$ is called with a determinant $L$ if there exists a morphism $(\wedge^r F) \to L$ which is an isomorphism outside the nodes of $X$. The subset $U_X(r, L) \subset U_X(r, d)$ consists of $s$-equivalence classes $[F] \in U_X(r, d)$ such that $[F]$ contains a sheaf with a fixed determinant $L$. Then D.S. Nagaraj and C.S. Seshadri made the following conjectures (See Conjecture (a) and (b) at page 136 of [NS]):

1. If $L$ is a line bundle on $X$ and $U_X(r, L)^0 \subset U_X(r, d)$ is the subset of locally free sheaves, then $U_X(r, L)$ is the closure of $U_X(r, L)^0$ in $U_X(r, d)$.
2. Let $L_\eta$ (resp. $L$) be a line bundle (resp. torsion free sheaf of rank one) of degree $d$ on smooth curve $Y$ (resp. $X$). Assume that $L_\eta$ specializes to $L$ as $Y$ specializes to $X$. Then $U_X(r, L)$ is the specialization of $U_Y(r, L_\eta)$.

We answer (1) completely. In fact, even if $L$ is not locally free (thus $U_X(r, L)$ contains no locally free sheaf), we prove that torsion free sheaves of type 1 (See Section 1) are dense in $U_X(r, L)$.

**Theorem 1.** Let $L$ be a torsion free sheaf of rank 1 and degree $d$. Define

$$U_X(r, L)^0 = \{ F \in U_X(r, L) \mid (\wedge^r F) \cong L \}$$

which coincides with the subset of locally free sheaves when $L$ is locally free. Then

1. $U_X(r, L)$ is the closure of $U_X(r, L)^0$. If $L$ is not locally free, $U_X(r, L)^0$ is the subset of torsion free sheaves of type 1.
2. There is a canonical scheme structure on $U_X(r, L)^0$, which is reduced when $L$ is locally free, such that when smooth curve $C_\eta$ specializes to $X$ and $L_\eta$ specializes to $L$ on $X$, the specialization $f^{-1}(0)$ of $U_{C_0}(r, L_\eta)$ contains a dense open subscheme which is isomorphic to $U_X(r, L)^0$. In particular,

$$f^{-1}(0)_{\text{red}} \cong U_X(r, L).$$

If the specialization $f^{-1}(0)$ has no embedded point, then our theorem also proved Conjecture (2). Unfortunately, $U_X(r, L)$ seems not represent a nice moduli functor, we can not say anything about the scheme structure of $U_X(r, L)$. To remedy this, we consider the specialization of $U_{C_0}(r, L_\eta)$ in the so called generalized Gieseker space $G(r, d)$ (See [NSe]). Let $X$ be an irreducible stable curve with only one node $p_0$ and $L$ be a line bundle of degree $d$ on $X$. Then, when $r \leq 3$, or $r = 4$ and the normalization $\tilde{X}$ is not hyperelliptic, we show that there is a Cohen-Macaulay closed subscheme $G(r, L) \subset G(r, d)$ of pure dimension $(r^2 - 1)(r - 1)$.
which represents a nice moduli functor (See Definition 3.2). Moreover, $G(r, L)$ satisfies the requirements in (2) for specializations. It is known ([NSe] that there is a canonical birational morphism $\theta : G(r, d) \to U_X(r, d)$. We prove in Lemma 3.5 that the set-theoretic image of $G(r, L)$ is $U_X(r, L)$. Thus we can endow $U_X(r, L)$ a scheme structure by the scheme-theoretic image of $G(r, L)$. Then we have

**Theorem 2.** Let $X$ be an irreducible curve of genus $g \geq 2$ with only one node $p_0$. Let $L$ be a line bundle of degree $d$ on $X$. Assume that $r \leq 3$, or $r = 4$ and the normalization of $X$ is not hyperelliptic. Then, when $(r, d) = 1$, we have

1. There is a Cohen-Macaulay projective scheme $G(r, L)$ of pure dimension $(r^2 - 1)(g - 1)$, which represents a moduli functor.
2. Let $C \to S$ be a proper family of curves over a discrete valuation ring, which has smooth generic fibre $C_0$ and closed fibre $C_0 \cong X$. If there is a line bundle $L$ on $C$ such that $L|_{C_0} \cong L$. Then there exists an irreducible, reduced, Cohen-Macaulay $S$-projective scheme $f : G(r, L)_S \to S$, which represents a moduli functor, such that $f^{-1}(0) \cong G(r, L)$, $f^{-1}(\eta) \cong U_{C_0}(r, L_\eta)$.
3. There exists a proper birational $S$-morphism $\theta : G(r, L)_S \to U(r, L)_S$ which induces a birational morphism $\theta : G(r, L) \to U_X(r, L)$.

Theorem 1 is proved in Section 1. In Section 2, we introduce the objects which are used to define Gieseker moduli space. Then Theorem 2 is proved in Section 3.

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### §1 Torsion-free sheaves with fixed determinant on irreducible curves

Let $X$ be a stable irreducible curve of genus $g$ with $\delta$ nodes $x_1, \ldots, x_\delta$. Any torsion free sheaf $\mathcal{F}$ of rank $r$ on $X$ can be written into (locally at $x_i$)

$$\mathcal{F} \otimes \mathcal{O}_{X, x_i} \cong \mathcal{O}_{X, x_i}^{\oplus a_i} \oplus m_{x_i}^{\oplus (r - a_i)}.$$

We call that $\mathcal{F}$ has type $r - a_i$ at $x_i$. Let $U_X(r, d)$ be the moduli space of $s$-equivalence classes of semistable torsion free sheaves of rank $r$ and degree $d$ on $X$. Inspired by [NS], we make the following definition.

**Definition 1.1.** Let $L$ be a torsion free sheaf of rank one and degree $d$ on $X$. A torsion free sheaf $\mathcal{F}$ of rank $r$ and degree $d$ on $X$ is called with a determinant $L$ if there exists a non-trivial morphism $\wedge^r \mathcal{F} \to L$ which is an isomorphism outside the nodes.

**Lemma 1.2.** For any exact sequence $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}_2 \to 0$ of torsion free sheaves with rank $r_1$, $r$, $r_2$ respectively, we have a morphism

$$(\wedge^{r_1} \mathcal{F}_1) \otimes (\wedge^{r_2} \mathcal{F}_2) \to \frac{\wedge^r \mathcal{F}}{\text{torsion}},$$

which is isomorphic outside the nodes. In particular, if a semistable sheaf $\mathcal{F}$ has a fixed determinant $L$, then the associated graded torsion free sheaf $\text{gr}(\mathcal{F})$ will also have the fixed determinant $L$. 


Proof. There is a morphism $\bigwedge^r \mathcal{F}_2 \to \mathcal{H}om(\bigwedge^r \mathcal{F}_1, \bigwedge^r \mathcal{F}/\text{torsion})$, which locally is defined as follows: For any $\omega \in \bigwedge^r \mathcal{F}_2$, choose a preimage $\tilde{\omega} \in \bigwedge^r \mathcal{F}$ with respect to $\bigwedge^r \beta$. Then the image of $\omega$ is defined to be the morphism

$$
\bigwedge^r \mathcal{F}_1 \to \bigwedge^r \mathcal{F}/\text{torsion},
$$

which takes any $f \in \bigwedge^r \mathcal{F}_1$ to the section $(\bigwedge^r \alpha)(f) \cdot \tilde{\omega} \in \bigwedge^r \mathcal{F}/\text{torsion}$, which does not depend on the choice of $\tilde{\omega}$ since the image of $\bigwedge^r \beta$ is a torsion sheaf. The morphism defined above is isomorphism outside the nodes (See Lemma 1.2 of [KW]). Thus we have the desired morphism

$$
(\bigwedge^r \mathcal{F}_1) \otimes (\bigwedge^r \mathcal{F}_2) \to (\bigwedge^r \mathcal{F}_1) \otimes \mathcal{H}om(\bigwedge^r \mathcal{F}_1, \bigwedge^r \mathcal{F}/\text{torsion}) \to \bigwedge^r \mathcal{F}/\text{torsion}.
$$

Definition 1.3. The subset $\mathcal{U}_X(r, L) \subset \mathcal{U}_X(r, d)$ and $\mathcal{U}_X(r, L)^0 \subset \mathcal{U}_X(r, L)$ are defined to be

$$
\mathcal{U}_X(r, L) = \left\{ \text{s-equivalence classes } [\mathcal{F}] \in \mathcal{U}_X(r, d) \text{ such that } \begin{cases} s \text{-equivalence classes } [\mathcal{F}] \in \mathcal{U}_X(r, L) & \text{contains a sheaf with a fixed determinant } L \\ \end{cases} \right\}
$$

$$
\mathcal{U}_X(r, L)^0 = \{ [\mathcal{F}] \in \mathcal{U}_X(r, L) | \bigwedge^r \mathcal{F} \cong L \}
$$

When $L$ is a line bundle, $\mathcal{U}_X(r, L)^0$ consists of locally free sheaves with the fixed determinant $L$. When $L$ is not a line bundle, $\mathcal{U}_X(r, L)^0$ consists of torsion free sheaves of type 1 at each node of $X$.

We first consider the case that $L$ is a line bundle and $X$ has only one node $p_0$. Let $\pi : \tilde{X} \to X$ be the normalization with $\pi^{-1}(p_0) = \{ p_1, p_2 \}$. The normalization $\phi : \mathcal{P} \to \mathcal{U}_X(r, d)$ was studied in [Su1], where $\mathcal{P}$ is the moduli spaces of semistable generalized parabolic bundles (GBP) of degree $d$ and rank $r$ on $\tilde{X}$. A GBP of degree $d$ and rank $r$ on $\tilde{X}$ is a pair $(E, Q)$ consisting of a vector bundle $E$ of degree $d$ and rank $r$ on $\tilde{X}$ and a $r$-dimensional quotient $E_{p_1} \oplus E_{p_2} \to Q$. There is a flat morphism (See Lemma 5.7 of [Su1])

$$
\text{Det} : \mathcal{P} \to J^d_{\tilde{X}}
$$

sending $(E, Q)$ to $\text{det}(E)$. Let $\tilde{L} = \pi^*(L)$ and $\mathcal{P}^L = \text{Det}^{-1}(\tilde{L})$. Then $\mathcal{P}^L$ is an irreducible projective variety (See the proof of Lemma 5.7 in [Su1]). Let $\mathcal{D}_i (i = 1, 2)$ be the divisor consisting of $(E, Q)$ such that $E_{p_i} \to Q$ is not an isomorphism (See [Su1] for details). Let $\mathcal{D}^L_i = \mathcal{D}_i \cap \mathcal{P}^L$.

Lemma 1.4. The set $\mathcal{U}_X(r, L)$ is contained in the image $\phi(\mathcal{P}^L)$. Moreover,

$$
\mathcal{U}_X(r, L) \setminus \mathcal{U}_X(r, L)^0 \subset \phi(\mathcal{D}^L_1 \cap \mathcal{D}^L_2).
$$

Proof. Let $F \in \mathcal{U}_X(r, L)$ with $F \otimes \hat{\mathcal{O}}_{p_0} \cong \hat{\mathcal{O}}^{\oplus a}_{p_0} \oplus m^{\oplus (r-a)}_{p_0}$. Let $\tilde{E} = \pi^* F/\text{torsion}$. Then, by local computations (See, for example, Remark 2.1, 2.6 of [NS]), we have

$$(1.1) \quad \phi_{|\mathcal{D}^L_i}(F) \cong \tilde{E} \otimes \hat{\mathcal{O}}_{p_0} \to 0$$
where \( \dim(\tilde{Q}) = a \) and the quotient \( \pi_*\tilde{E} \to p_0\tilde{Q} \) induces two surjective maps \( \tilde{E}_{p_i} \to \tilde{Q} \) \( i = 1, 2 \). Denote their kernel by \( K_i \), we have

\[
0 \to K_i \to \tilde{E}_{p_i} \to \tilde{Q} \to 0.
\]

On the other hand, for \( F \in \mathcal{U}_X(r, L) \), let \( Q \) be the cokernel of \( \wedge^r F \to L \), then

\[
0 \to \text{det}(\tilde{E}) \to \tilde{L} \to \pi^*Q \to 0
\]

where \( \pi^*Q = p_1V_1 \oplus p_2V_2 \) and \( n_1, n_2 \) is respectively the dimension of \( V_1, V_2 \). Thus

\[
\text{det}(\tilde{E}) = \tilde{L} \otimes \mathcal{O}_{\tilde{X}}(-n_1p_1 - n_2p_2)
\]

where \( n_i \geq 0 \) and \( n_1 + n_2 = r - a \).

Let \( h : \tilde{E} \to E \) be the Hecke modifications at \( p_1 \) and \( p_2 \) such that \( \ker(h_{p_i}) \subset K_i \) has dimension \( n_i \) for \( i = 1, 2 \). Then we have

\[
0 \to \tilde{E} \xrightarrow{h} E \to p_1\tilde{Q}_1 \oplus p_2\tilde{Q}_2 \to 0
\]

with \( \dim(\tilde{Q}_i) = n_i \). Thus

\[
\text{det}(E) = \text{det}(\tilde{E}) \otimes \mathcal{O}_{\tilde{X}}(n_1p_1 + n_2p_2) = \tilde{L} \quad \text{and} \quad \phi(E, Q) = F
\]

if we define \( Q \) by the exact sequence

\[
0 \to F \xrightarrow{(\pi_*h)^d} \pi_*E \to p_0Q \to 0.
\]

To describe the GPB \( (E, E_{p_1} \oplus E_{p_2} \xrightarrow{q} Q \to 0) \), note that (1.3) induces

\[
F_{p_0} \xrightarrow{d_{p_0}} \tilde{E}_{p_1} \oplus \tilde{E}_{p_2} \xrightarrow{h_{p_1} \oplus h_{p_2}} E_{p_1} \oplus E_{p_2} \xrightarrow{q} Q \to 0.
\]

Then \( d_{p_0}(F_{p_0}) \cap \tilde{E}_{p_i} = K_i \) by (1.1) and \( h_{p_i}(K_i) = \ker(q_i) \) by the exactness of (1.3), where \( q_i : E_{p_i} \to Q \) \( i = 1, 2 \) are projections induced by \( E_{p_1} \oplus E_{p_2} \xrightarrow{q} Q \to 0 \). Thus \( \dim(\ker(q_i)) = r - a - n_i \) by the construction of \( h \).

For any \( F \in \mathcal{U}_X(r, L) \setminus \mathcal{U}_X(r, L)^0 \), the cokernel \( Q \) of \( \wedge^r F \to L \) must be non-trivial. This implies that both \( V_1 \) and \( V_2 \) in \( \pi^*Q = p_1V_1 \oplus p_2V_2 \) are non-trivial since for any \( i = 1, 2 \), we have

\[
\text{Hom}_{\mathcal{O}_{\tilde{X}}}(p, V_i, p_i\mathbb{C}) = \text{Hom}_{\mathcal{O}_{\tilde{X}}}(\pi^*Q, p_i\mathbb{C}) = \text{Hom}_{\mathcal{O}_{\tilde{X}}}(Q, \pi_* (p_i\mathbb{C})) \neq 0.
\]

Thus their dimensions \( n_1 \) and \( n_2 \) must be positive and \( n_1 + n_2 = r - a \), which means that \( \ker(q_i) \neq 0 \) \( i = 1, 2 \) and the GPB \( (E, Q) \) must be in \( \mathcal{D}_1 \cap \mathcal{D}_2 \). Thus

\[
\mathcal{U}_X(r, L) \setminus \mathcal{U}_X(r, L)^0 \subset \phi(\mathcal{D}_1 \cap \mathcal{D}_2^\perp).
\]

**Remark 1.5.** This is also indicated in the following consideration. There is a \( \mathbb{P}^1 \)-bundle \( p : \mathbb{P} \to J^d_X \) and the normalization map \( \phi_1 : \mathbb{P} \to J^d_X \). The morphism \( \text{Det} : \mathcal{P} \to J^d_X \) can be lift to a rational morphism

\[
\tilde{\text{Det}} : \mathcal{P} \dashrightarrow \mathbb{P} \xrightarrow{\phi_1} J^d_X,
\]

which is well-defined on \( \mathcal{P} \setminus \mathcal{D}_1 \cap \mathcal{D}_2 \). When \( L \) is a line bundle, \( \tilde{\text{Det}}^{-1}(L) \) is disjoint with \( \mathcal{D}_1 \cup (\mathcal{D}_1 \cap \mathcal{D}_2) \).
Lemma 1.6. Let $\Lambda$ be a discrete valuation ring and $T = \text{Spec}(\Lambda)$. Then, for any $F \in \mathcal{U}_X(r, L)$, there is a $T$-flat sheaf $\mathcal{F}$ on $X \times T$ such that

1. $\mathcal{F}_t = \mathcal{F}|_{X \times \{t\}}$ is locally free for $t \neq 0$ and $\mathcal{F}_0 = F$,
2. $\Lambda^r(\mathcal{F}|_{X \times (T \setminus \{0\})}) = p_X^* L$.

In particular, $\mathcal{U}_X(r, L)^0$ is dense in $\mathcal{U}_X(r, L)$.

Proof. Let $(E, Q) \in \mathcal{P}^L$ be the GPB such that $\phi(E, Q) = F$ (Lemma 1.4). Then there exists a $T$-flat family of vector bundles $\mathcal{E}$ on $\tilde{X} \times T$ with $\text{det}(\mathcal{E}) = p_X^* \tilde{L}$, and a $T$-flat quotient

$$\mathcal{E}_{p_1} \oplus \mathcal{E}_{p_2} \twoheadrightarrow Q \rightarrow 0$$

such that $(\mathcal{E}_0, Q_0) = (\mathcal{E}, Q)|_{\tilde{X} \times \{0\}} = (E, Q)$. The quotient $\mathcal{E}_{p_1} \oplus \mathcal{E}_{p_2} \twoheadrightarrow Q \rightarrow 0$ is determined by the two projections $q_i : \mathcal{E}_{p_i} \rightarrow Q$ ($i = 1, 2$), which can be chosen to be isomorphisms for $t \neq 0$ since $\mathcal{P}^L$ is irreducible. The two maps $q_i$ are given by two matrices

$$
\begin{pmatrix}
  t^{a_1} & 0 & \cdots & 0 \\
  0 & t^{a_2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & t^{a_r}
\end{pmatrix},
\begin{pmatrix}
  t^{b_1} & 0 & \cdots & 0 \\
  0 & t^{b_2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & t^{b_r}
\end{pmatrix}
$$

where $0 \leq a_1 \leq a_2 \leq \cdots \leq a_r$ and $0 \leq b_1 \leq b_2 \leq \cdots \leq b_r$. When $t = 0$, they give the GPB $(E, Q)$. We recall that when $F$ is not locally free, the numbers $n_1$ and $n_2$ in the proof of Lemma 1.4 are positive. Thus the two projections $E_{p_i} \rightarrow Q$ are not isomorphism. Namely, there are $k_1, k_2$ such that $a_{k_1} > 0$, $b_{k_2} > 0$ but $a_j = 0$ ($j < k_1$) and $b_j = 0$ ($j < k_2$). It is clear now that we can change the positive numbers $a_{k_1}, \ldots, a_r, b_{k_2}, \ldots, b_r$ freely such that the resulted family $(\mathcal{E}, Q)$ has the property that $(\mathcal{E}_0, Q_0) = (E, Q)$. We modify the $T$-flat quotient $\mathcal{E}_{p_1} \oplus \mathcal{E}_{p_2} \twoheadrightarrow Q \rightarrow 0$ by choosing $a_{k_1}, \ldots, a_r, b_{k_2}, \ldots, b_r$ such that

$$\sum_{i=k_1}^{r} a_i - \sum_{i=k_2}^{r} b_i = 0.$$ 

Thus we get a $T$-flat sheaf $\mathcal{F}$ on $X \times T$ such that $\mathcal{F}_0 = F$. Moreover, on $T \setminus \{0\}$, $\mathcal{F}$ is obtained from $\mathcal{E}|_{\tilde{X} \times (T \setminus \{0\})}$ by identifying $\mathcal{E}_{p_1}$ and $\mathcal{E}_{p_2}$ through the isomorphism

$$q_1 \cdot q_2^{-1} : \mathcal{E}_{p_1} \rightarrow \mathcal{E}_{p_2}.$$ 

$\Lambda^r(\mathcal{F}|_{X \times (T \setminus \{0\})})$ is obtained from $\text{det}(\mathcal{E})|_{\tilde{X} \times (T \setminus \{0\})} = p_X^* \tilde{L}$ by identifying $\tilde{L}_{p_1} \otimes K(T)$ and $\tilde{L}_{p_2} \otimes K(T)$ through the isomorphism $\Lambda^r(q_1 \cdot q_2^{-1})$, where $K(T)$ denote the field of rational functions on $T$. By the choice of $a_{k_1}, \ldots, a_r, b_{k_2}, \ldots, b_r$, we know that $\Lambda^r(q_1 \cdot q_2^{-1})$ is the identity map. Thus

$$\Lambda^r(\mathcal{F}|_{X \times (T \setminus \{0\})}) = (p_X^* L)|_{X \times (T \setminus \{0\})}.$$
Lemma 1.7. For any stable irreducible curve \( X, \mathcal{U}_X(r, L)^0 \) is dense in \( \mathcal{U}_X(r, L) \).

Proof. Let \( \delta \) be the number of nodes of \( X \), we will prove the lemma by induction on \( \delta \). When \( \delta = 1 \), it is Lemma 1.6. Assume that the lemma is true for curves with \( \delta - 1 \) nodes. Then we show that for any \( F \in \mathcal{U}_X(r, L) \) there is a \( T \)-flat sheaf \( \mathcal{F} \) on \( X \times T \), where \( T = \text{Spec}(\Lambda) \) and \( \Lambda \) is a discrete valuation ring, such that

1. \( \mathcal{F}_t = \mathcal{F}|_{X \times \{t\}} \) is locally free for \( t \neq 0 \) and \( \mathcal{F}_0 = F \),
2. \( \wedge^r(\mathcal{F}|_{X \times (T \setminus \{0\})}) = p_X^* L \).

For \( F \in \mathcal{U}_X(r, L) \), we can assume that \( F \) is not locally free. Let \( p_0 \in X \) be a node at which \( F \) is not locally free. Let \( \pi : \tilde{X} \to X \) be the partial normalization at \( p_0 \) and \( \pi^{-1}(p_0) = \{p_1, p_2\} \). Let \( \tilde{L} = \pi^* L \) and \( \tilde{E} = \pi^* F/torsion \), then by the same arguments of Lemma 1.4

\[
0 \to F \xrightarrow{d} \pi_* \tilde{E} \to p_0 \tilde{Q} \to 0.
\]

Note that \( \wedge^r \tilde{E} = \pi^*(\wedge^r F)/(\text{torsion at } \{p_1, p_2\}) \) and the cokernel of \( \wedge^r \tilde{E} \to \tilde{L} \) at \( \{p_1, p_2\} \) is \( p_1 \mathbb{C}^{n_1} \oplus p_2 \mathbb{C}^{n_2} \), we have the morphism

\[
\wedge^r \tilde{E} \to \tilde{L} \otimes \mathcal{O}_{\tilde{X}}(-n_1 p_1 - n_2 p_2)
\]

which is an isomorphism outside the nodes of \( \tilde{X} \). As the same with proof of Lemma 1.4, we have the Hecke modification \( E \) of \( \tilde{E} \) at \( p_1 \) and \( p_2 \) such that

\[
0 \to \tilde{E} \xrightarrow{h} E \to p_1 \tilde{Q}_1 \oplus p_2 \tilde{Q}_2 \to 0
\]

with \( \text{dim}(\tilde{Q}_i) = n_i \). Thus \( \wedge^r E \cong (\wedge^r \tilde{E}) \otimes \mathcal{O}_{\tilde{X}}(n_1 p_1 + n_2 p_2) \to \tilde{L} \) and the generalized parabolic sheaf (GPS) \( (E, Q) \) defines \( E \) by the exact sequence

\[
0 \to F \xrightarrow{(\pi_* h \cdot d)} \pi_* E \to p_0 Q \to 0,
\]

where \( Q \) is defined by requiring above sequence exact. The two projections \( E_{p_i} \to Q \) \( (i = 1, 2) \) are not isomorphism, thus, by choosing suitable bases of \( E_{p_1} \) and \( Q \), they are given by matrices

\[
P_1 = \begin{pmatrix}
1 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \ldots & 0
\end{pmatrix},
\]

\[
P_2 = A \cdot \begin{pmatrix}
1 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \ldots & 0
\end{pmatrix} \cdot B
\]

where \( A, B \) are invertible \( r \times r \) matrices and \( \text{rank}(P_i) = r_i < r \) \((i = 1, 2)\). Since \( E \in \mathcal{U}_X(r, \tilde{L}) \), by the assumption, there is a \( T \)-flat sheaf \( \mathcal{E} \) on \( \tilde{X} \times T \) such that \( \mathcal{E}_0 := \mathcal{E}|_{\tilde{X} \times \{0\}} = E \) and \( \mathcal{E}|_{\tilde{X} \times (T \setminus \{0\})} \) locally free with determinant \( p_X^*(\tilde{L}) \). Define the morphisms \( q_i : \mathcal{E}_{p_i} := \mathcal{E}|_{\{p_i\} \times T} \to Q \otimes \mathcal{O}_T \) \((i = 1, 2)\) by using matrices

\[
Q_1 = \begin{pmatrix}
1 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & 0 & \ldots \\
0 & \ldots & 0 & t^{a_{r_1+1}} & \ldots \\
0 & \ldots & 0 & 0 & \ldots \\
\end{pmatrix},
\]

\[
Q_2 = A \cdot \begin{pmatrix}
1 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & 0 & \ldots \\
0 & \ldots & 0 & t^{b_{r_2+1}} & \ldots \\
0 & \ldots & 0 & 0 & \ldots \\
\end{pmatrix} \cdot B
\]
where $t$ is the local parameter of $\Lambda$, $a_{r_1+1}, \ldots, a_r, b_{r_2+1}, \ldots, b_r$ are positive integers satisfying $a_{r_1+1} + \cdots + a_r = b_{r_2+1} + \cdots + b_r$, and $c$ is any constant. Then these morphisms $q_i$ ($i = 1, 2$) define a family $(\mathcal{E}, Q \otimes \mathcal{O}_T)$ of GPS, which induces a $T$-flat sheaf $\mathcal{F}$ on $X \times T$ such that $\mathcal{F}_0 = F$ and $\mathcal{F}_t$ ($t \neq 0$) are locally free. The determinant $\det(\mathcal{F}|_{X \times T_0})$, where $T_0 = T \setminus \{0\}$, is defined by the sheaf $(\det(\mathcal{E}|_{\tilde{X} \times T_0}) = p_X^*(\tilde{L})$ through the isomorphism

$$\det(q_2^{-1} \cdot q_1) : (\det(\mathcal{E}|_{\tilde{X} \times T_0})|_{p_1}) = (\wedge^r \mathcal{E}_{p_1})|_{T_0} \to (\wedge^r \mathcal{E}_{p_2})|_{T_0} = (\det(\mathcal{E}|_{\tilde{X} \times T_0})|_{p_2},$$

which is a scale product by $\det(Q_2^{-1} \cdot Q_1) = \det(AB)^{-1} \cdot c$. Thus we can choose suitable constant $c$ such that $\det(\mathcal{F}|_{X \times T_0}) = p_X^*(L)$. We are done.

**Lemma 1.8.** When $L$ is not locally free, $U_X(r, L)^0$ consists of torsion free sheaves of type $1$ at each node of $X$, which is dense in $U_X(r, L)$.

**Proof.** The proof follows the same idea. For simplicity, we assume that $X$ has only one node $p_0$. Let $F$ be a torsion free sheaf of rank $r$ and degree $d$ on $X$ with type $t(F) \geq 1$ at $p_0$. Then

$$\deg(\wedge^r F/torsion) = d - t(F) + 1.$$ 

Thus $F \in U_X(r, L)^0$ if and only if $t(F) = 1$.

For any $F \in U_X(r, L)$ of type $t(F) > 1$, let $\widetilde{E} = \pi^* F/torsion$, then

$$0 \to F \xrightarrow{d} p_0^* \widetilde{E} \to p_0^* \widetilde{Q} \to 0$$

where $\dim(\widetilde{Q}) = r - t(F)$. Let $\tilde{L} = \pi^* L/torsion$, then $\deg(\tilde{L}) = d - 1$ and $L = \pi_\ast \tilde{L}$. The condition that $F \in U_X(r, L)$ implies that $\det(\widetilde{E}) = \tilde{L}(-n_1p_1 - n_2p_2)$ where $n_i \geq 0$ and $n_1 + n_2 = t(F) - 1$. As in the proof Lemma 1.4, let $h : \tilde{E} \to E$ be the Hecke modifications at $p_1$ and $p_2$ such that $\dim(\ker(h_{p_1})) = n_1 + 1$ and $\dim(\ker(h_{p_2})) = n_2$. Then we have $\det(E) = \det(\tilde{E}) \otimes \mathcal{O}_{\tilde{X}}((n_1 + 1)p_1 + n_2p_2) = \tilde{L}(p_1)$, and there is an GPB $(E, E_{p_1} \oplus E_{p_2} \xrightarrow{\phi} Q \to 0)$ such that $\phi(E, Q) = F$, where $q_i : E_{p_i} \to Q$ ($i = 1, 2$) satisfy $\dim(\ker(q_1)) = t(F) - n_1 - 1$ and $\dim(\ker(q_2)) = t(F) - n_2$. The two projections $q_i : E_{p_i} \to Q$ ($i = 1, 2$) are are given by matrices

$$P_1 = \begin{pmatrix} 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix}, \quad P_2 = A \cdot \begin{pmatrix} 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix} \cdot B$$

where $\operatorname{rank}(P_1) = r - t(F) + n_1 + 1$, $\operatorname{rank}(P_2) = r - t(F) + n_2$. Let $T = \operatorname{Spec}(\mathbb{C}[t])$ and $E = p_X^* E$. Choose deformations $P_i(t)$ of $P_i$ ($i = 1, 2$) as following

$$\begin{pmatrix} 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & \ldots & 0 \\ 0 & \ldots & t & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & t \end{pmatrix}, \quad A \cdot \begin{pmatrix} 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & \ldots & 0 \\ 0 & \ldots & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix} \cdot B$$
Thus \( F \) which induce a morphism \( \vartheta : A \times X \rightarrow N \) sheaf \( \) where the number of \( t \) in \( P_2(t) \) is \( t(F) - n_2 - 1 \). Then we get a family \( (E, Q \otimes \mathcal{O}_T) \) of GPB on \( \tilde{X} \times T \), which induces a \( T \)-flat sheaf \( F \) on \( X \times T \) such that \( F_0 = F \) and \( F_t \) \( (t \neq 0) \) are torsion free of type 1. To see that \( \wedge^r F_t \cong L \) \( (t \neq 0) \), we note that \( \text{det}(E) = p^*_X \tilde{L}(p_1) \) and \( L \) is determined by the GPB

\[
(\text{det}(E) = \tilde{L}(p_1), G \subset \text{det}(E)_{p_1} \oplus \text{det}(E)_{p_2})
\]

where \( G \) is the graph of zero map \( \text{det}(E)_{p_2} \rightarrow \text{det}(E)_{p_1} \). Thus we have a non-trivial morphism \( \wedge^r F_t \rightarrow L \), which must be an isomorphism when \( t \neq 0 \).

Next we will prove that \( \mathcal{U}_X(r, L) \) is the underlying scheme of specialization of the moduli spaces of semistable bundles with fixed determinant. This in particular implies that \( \mathcal{U}_X(r, L) \subset \mathcal{U}_X(r, d) \) is a closed subset. Let \( S = \text{Spec}(R) \) and \( R \) be a discrete valuation ring. Let \( \mathcal{X} \rightarrow S \) be a flat proper family of curves with smooth generic fibre and closed fibre \( \mathcal{X}_0 = X \). Let \( \mathcal{L} \) be a relative torsion free sheaf on \( \mathcal{X} \) of rank one and (relative) degree \( d \) such that \( \mathcal{L}|_{\mathcal{X}} = L \). It is well known that there exists a moduli scheme \( f : \mathcal{U}(r, d)_S \rightarrow S \) such that for any \( s \in S \) the fibre \( f^{-1}(s) \) is the moduli space of semistable torsion free sheaves of rank \( r \) and degree \( d \) on \( \mathcal{X}_s \) (where \( \mathcal{X}_s \) denote the fibre of \( \mathcal{X} \rightarrow S \) at \( s \)). Since \( \mathcal{X} \) is smooth over \( S^0 = S \setminus \{0\} \), there is a family \( \mathcal{U}(r, \mathcal{L}|_{S^0})_{S^0} \rightarrow S^0 \) of moduli spaces of semistable bundles with fixed determinant \( \mathcal{L}|_{\mathcal{X}_s} \) on \( \mathcal{X}_s \) \( (s \in S^0) \). We have

\[
\mathcal{U}(r, \mathcal{L}|_{S^0})_{S^0} \subset \mathcal{U}(r, d)_S.
\]

Let \( Z \) be the Zariski closure of \( \mathcal{U}(r, \mathcal{L}|_{S^0})_{S^0} \) inside \( \mathcal{U}(r, d)_S \). We get a flat family

\[
f : Z \rightarrow S
\]

of projective schemes. For any \( 0 \neq s \in S \), the fibre \( Z_s \) is the moduli space of semistable bundles on \( \mathcal{X}_s \) with fixed determinant \( \mathcal{L}|_{\mathcal{X}_s} \).

**Lemma 1.9.** The fibre \( Z_0 \) of \( f : Z \rightarrow S \) at \( s = 0 \) is contained in \( \mathcal{U}_X(r, L) \) as a set.

**Proof.** We can assume that for any \( [F] \in Z_0 \) there is a discrete valuation ring \( A \) and \( T = \text{Spec}(A) \rightarrow S \) such that there is a \( T \)-flat family of torsion free sheaves \( \mathcal{F} \) on \( \mathcal{X}_T = \mathcal{X} \times_S T \rightarrow T \), so that

\[
\wedge^r \mathcal{F}_\eta \cong \mathcal{L}_\eta, \quad \mathcal{F}|_{\mathcal{X}} \cong F.
\]

By Proposition 5.3 of [Se] and its proof (see [Se], it deals with one node curve, but generalization to our case is straightforward since its proof is completely local), there is a birational morphism \( \sigma : \Gamma \rightarrow \mathcal{X}_T \) and a vector bundle \( \mathcal{E} \) on \( \Gamma \) such that \( \sigma_* \mathcal{E} = F \). Moreover, the morphism \( \sigma \) is an isomorphism over \( \mathcal{X}_T \setminus \{x_1, \ldots, x_k\} \). Since \( (\wedge^r \mathcal{E})|_{\Gamma_\eta} \cong (\sigma^* \mathcal{L})^\vee |_{\Gamma_\eta} \), note that \( (\wedge^r \mathcal{E})^{-1} \otimes (\sigma^* \mathcal{L})^\vee \) is torsion free (thus \( T \)-flat), we can extend the isomorphism into a morphism \( \wedge^r \mathcal{E} \rightarrow (\sigma^* \mathcal{L})^\vee \). Since \( \sigma_* \) and \( \sigma^* \) are adjoint functors, \( \sigma_* \mathcal{O}_\Gamma = \mathcal{O}_{\mathcal{X}_T} \), we have \( \sigma_* ((\sigma^* \mathcal{N})^\vee) = \mathcal{N}^\vee \) for any coherent sheaf \( \mathcal{N} \). Then, by using \( \sigma^* (\mathcal{N}^\vee) = \sigma_* \sigma^* ((\sigma^* \mathcal{N})^\vee) \rightarrow (\sigma^* \mathcal{N})^\vee \), we have a canonical morphism \( \sigma_* ((\sigma^* \mathcal{N})^\vee) \rightarrow \mathcal{N}^\vee \). In particular, there is a canonical morphism

\[
\sigma_* ((\sigma^* \mathcal{L})^\vee) \rightarrow \mathcal{L}^\vee \cong \mathcal{L}
\]

which induce a morphism \( \vartheta : \wedge^r \mathcal{F} = \wedge^r (\sigma_* \mathcal{E}) \rightarrow \sigma_* \wedge^r \mathcal{E} \rightarrow \mathcal{L} \). Modified by some power of the maximal ideal of \( A \), we can assume the morphism \( \vartheta \) being nontrivial on \( X \), which means that \( \vartheta \) is an isomorphism on \( \mathcal{X}_T \setminus \{x_1, \ldots, x_k\} \) since \( X \) is irreducible. Thus \( [F] \in \mathcal{U}_X(r, L) \).
Theorem 1.10. \( U_X(r, L) \) is the closure of \( U_X(r, L)^0 \) in \( U_X(r, d) \). When smooth curve \( X_s \) specializes to \( X_0 = X \) and \( L_s \) specializes to \( L \), the moduli spaces \( U_X(r, L) \) of semistable bundles of rank \( r \) with fixed determinant \( L \) on \( X_s \) specializes to an irreducible scheme \( Z_0 \) with \( (Z_0)_{\text{red}} \cong U_X(r, L) \).

Proof. Let \( U(r, d)^0 \subset U(r, d)_S \) be the open subscheme of torsion free sheaves of type at most 1. Then there is a well-defined \( S \)-morphism (taking determinant \( \det(\bullet) = \bigwedge^r(\bullet) \))

\[
\det : U(r, d)^0_S \to U(1, d)_S.
\]

The given family of torsion free sheaves \( \mathcal{L} \) on \( X \) of rank one and degree \( d \) gives a \( S \)-point \([\mathcal{L}] \subset U(1, d)_S \). It is clear that

\[
Z^0 := (\det)^{-1}(\{\mathcal{L}\}) \subset Z
\]

and the fibre of \( f|_{Z^0} : Z^0 \to S \) at \( s = 0 \) is irreducible with support \( U_X(r, L)^0 \) (it is also reduced when \( L \) is a line bundle). Thus \( f^{-1}(0) = Z_0 \) contains the closure \( \overline{U_X(r, L)}^0 \) of \( U_X(r, L)^0 \) in \( U_X(r, d) \). On the other hand, by Lemma 1.9, Lemma 1.8 and Lemma 1.7, we have

\[
\overline{U_X(r, L)}^0 \subset (Z_0)_{\text{red}} \subset U_X(r, L) \subset \overline{U_X(r, L)}^0.
\]

Hence \( U_X(r, L) = \overline{U_X(r, L)}^0 = (Z_0)_{\text{red}} \). In particular, the fibre of \( f : Z \to S \) at \( s = 0 \) is irreducible.

§2 Stability and Gieseker functor

Let \( X \) be a stable curve with \( \delta \) nodes \( \{x_1, \ldots, x_\delta\} \). Any semistable curve with stable model \( X \) can be obtained from \( X \) by destablizing the nodes \( x_i \) with chains \( R_i \) (\( i = 1, \ldots, \delta \)) of projective lines. It will be denoted as \( X_\vec{n} \), where \( \vec{n} = (n_1, \ldots, n_\delta) \) and \( n_i \) is the length of \( R_i \) (See [NSe] for the example of \( \delta = 1 \)). Then \( X_\vec{n} \) are the curves which are semi-stably equivalent to \( X \), we use \( \pi : X_\vec{n} \to X \) to denote the canonical morphing contracting \( R_1, \ldots, R_\delta \) to \( x_1, \ldots, x_\delta \) respectively. A vector bundle \( E \) of rank \( r \) on a chain \( \bar{R} = \cup C_i \) of projective lines is called positive if \( a_{ij} \geq 0 \) in the decomposition \( E|_{C_i} = \bigoplus_{j=1}^r \mathcal{O}(a_{ij}) \) for all \( i \) and \( j \). A positive \( E \) is called strictly positive if for each \( C_i \) there is at least one \( a_{ij} > 0 \). \( E \) is called standard (resp. strictly standard) if it is positive (resp. strictly positive) and \( a_{ij} \leq 1 \) for all \( i \) and \( j \) (See [NSe], [Se]).

For any semistable curve \( X_\vec{n} = \cup X_\vec{n}^k \) of genus \( g \geq 2 \), let \( \omega_{X_\vec{n}} \) be its canonical bundle and

\[
\lambda_k = \frac{\deg(\omega_{X_\vec{n}}|_{X_\vec{n}^k})}{2g - 2},
\]

it is easy to see that \( \lambda_k = 0 \) if and only if the irreducible component \( X_\vec{n}^k \) is a component of the chains of projective lines.

Definition 2.1. A sheaf \( E \) of constant rank \( r \) on \( X_\vec{n} \) is called (semi)stable, if for every subsheaf \( F \subset E \), we have

\[
\chi(F) < (\leq) \frac{\chi(E)}{r} \cdot r(F) \quad \text{when} \ r(F) \neq 0, r,
\]

\[
\chi(F) \leq 0 \quad \text{when} \ r(F) = 0, \text{and} \ \chi(F) < \chi(E) \quad \text{when} \ r(F) = r, F \neq E,
\]

where, for any sheaf \( F \), the rank \( r(F) \) is defined to be \( \sum \lambda_k \cdot \text{rank}(F|_{X_\vec{n}^k}) \).

Let \( C = X_\vec{n} \) and \( C_0 = X_{(0,n_2,\ldots,n_\delta)} \) (namely, \( C_0 \) is obtained from \( C \) by contracting the chain \( R_0 = l \cup \cup_{i=2}^\delta \mathbb{P}^1 \) of projective lines \( \mathbb{P}^1 \)).
Lemma 2.2. Let $\pi : C \to C_0$ be the canonical morphism, let $E$ be a torsion free sheaf that is locally free on $R_1$. If $E|_{R_1}$ is positive and $\pi_*E$ is stable (semistable) on $C_0$, then $E$ is stable (semistable) on $C$. In particular, a vector bundle on $X_{\tilde{\pi}}$ is stable (semistable) if $E|_{R_i}$ ($1 \leq i \leq \delta$) are positive and $\pi_*E$ is stable (semistable) on $X$, where $\pi : X_{\tilde{\pi}} \to X$ is the canonical morphism contracting $R_1, \ldots, R_\delta$ to $x_1, \ldots, x_\delta$.

Proof. Let $C = \tilde{C}_0 \cup R_1$ and $\tilde{C}_0 \cap R_1 = \{p_1, p_2\}$, where $\pi : \tilde{C}_0 \to C_0$ is the partial normalization of $C_0$ at $x_1$. Let $\tilde{E} = E|_{\tilde{C}_0}$, $E = E|_{R_1}$. Then we have exact sequence

\begin{align*}
0 & \to E'(-p_1 - p_2) \to E \to \tilde{E} \to 0.
\end{align*}

If $E|_{R_1}$ is positive and $\pi_*E$ stable (semistable), then $\pi_*E'(-p_1 - p_2) = 0$. For any $E_1 \subset E$, consider the sequence (2.1), let $\tilde{E}_1 \subset \tilde{E}$ be the image of $E_1$ in $\tilde{E}$ and $K \subset E'(-p_1 - p_2)$ be the kernel of $E_1 \to \tilde{E}_1$, then we have

\begin{align*}
0 & \to \pi_*E_1 \to \pi_*\tilde{E}_1 \to R^1\pi_*K = x_1H^1(K),
\end{align*}

and $\chi(E_1) = \chi(\tilde{E}_1) + \chi(K) = \chi(\pi_* \tilde{E}_1) - h^1(K) \leq \chi(\pi_* E_1)$. Since $r(E_1) = r(\pi_* E_1)$,

\begin{align*}
\chi(E_1) - \frac{\chi(E)}{r}r(E_1) & \leq \chi(\pi_* E_1) - \frac{\chi(\pi_* E)}{r}r(\pi_* E_1).
\end{align*}

Thus we will be done if we can check that $\chi(E_1) < \chi(E)$ when $r(E_1) = r(E)$ and $E_1 \neq E$. In this case, the quotient $E_2 = E/E_1$ is torsion outside the chains $\{R_i\}$. If $E_2|_R = 0$, where $R = \cup R_i$, then $E_2$ is a nontrivial torsion and we are done. If $E_2|_R \neq 0$, then $\chi(E_2) \geq \chi(E_2|_R)$. Since $E|_R$ is positive and the surjective map

\begin{align*}
E|_R = \bigoplus_{j=1}^r L_j \to E_2|_R \to 0,
\end{align*}

we have $H^1(E_2|_R) = 0$ and there is at least one line bundle $L_j$ such that $L_j \hookrightarrow E_2|_R$ on a sub-chain. Thus $\chi(E_2) \geq \chi(E_2|_R) = h^0(E_2|_R) > 0$ and $\chi(E_1) < \chi(E)$.

Remark 2.3. It is easy to show that if $E$ is semistable on $X_{\tilde{\pi}}$, then $E$ is standard on the chains and $\pi_*E$ is torsion free. It is expected that (semi)stability of $E$ also implies the (semi)stability of $\pi_* E$.

Definition 2.4. Let $\mathcal{C} \to S$ be a flat family of stable curves of genus $g \geq 2$. The associated functor $\mathcal{G}_S$ (called the Gieseker functor) is defined as follows:

$$
\mathcal{G}_S : \{S \text{- schemes}\} \to \{\text{sets}\},
$$

where $\mathcal{G}_S(T) =$ set of closed subschemes $\Delta \subset \mathcal{C} \times_S T \times_S \text{Gr}(m, r)$ such that

1. the induced projection map $\Delta \to T \times_S \text{Gr}(m, r)$ over $T$ is a closed embedding over $T$. Let $\mathcal{E}$ denote the rank $r$ vector bundle on $\Delta$ which is induced by the tautological rank $r$ quotient bundle on $\text{Gr}(m, r)$.

2. the projection $\Delta \to T$ is a flat family of semistable curves and the the projection $\Delta \to \mathcal{C} \times_T T$ over $T$ is the canonical morphism $\pi : \Delta \to \mathcal{C} \times_S T$ contracting the chains of projective lines.

3. the vector bundles $\mathcal{E}_t = \mathcal{E}|_{\Delta_t}$, on $\Delta_t$ ($t \in T$) are of rank $r$ and degree $d = m + r(g - 1)$. The quotients $\mathcal{O}_{\Delta_t}^m \to \mathcal{E}_t$ induce isomorphisms

$$
H^0(\mathcal{O}_{\Delta_t}^m) \cong H^0(\mathcal{E}_t).
$$
Lemma 2.5 ([Gi],[NSe],[Se]). The functor $G$ is represented by a $PGL(m)$-stable open subscheme $Y \rightarrow S$ of the Hilbert scheme. The fibres $Y_s$ ($s \in S$) are reduced, and the singularities of $Y_s$ are products of normal crossings. A point $y \in Y_s$ is smooth if and only if the corresponding curve $\Delta_y$ is a stable curve, namely all chains in $\Delta_y$ are of length 0.

Let $Quot$ be the Quot-scheme of rank $r$ and degree $d$ quotiens of $O^m_C$ on $C \rightarrow S$ (we choose the canonical polarization on any flat family $C \rightarrow S$ of stable curves of genus $g \geq 2$). There is a universal quotient

$$O^m_{C \times_S Quot} \rightarrow F \rightarrow 0$$
on $C \times_S Quot \rightarrow Quot$. Let $R \subset Quot$ be the $PGL(m)$-stable open subscheme consisting of $q \in Quot$ such that the quotient map $O^m_{C \times_S \{q\}} \rightarrow F_q \rightarrow 0$ induces an isomorphism $H^0(O^m_{C \times_S \{q\}}) \cong H^0(F_q)$ (thus $H^1(F_q) = 0$). We can assume that $d$ is large enough so that all semistable torsion free sheaves of rank $r$ and degree $d$ on $C \rightarrow S$ can be realized as points of $R$. Let $R^s$ ($R^{ss}$) be the open set of stable (semistable) quotients, and let $W$ be the closure of $R^{ss}$ in $Quot$. Then there is an ample $PGL(m)$-line bundle $O_W(1)$ on $W$ such that $R^s$ (resp. $R^{ss}$) is precisely the set of GIT stable (resp. GIT semistable) points. Thus the moduli scheme $U(r,d) \rightarrow S$ is the GIT quotient of $R^{ss} \rightarrow S$.

Let $\Delta \subset C \times_S Y \times_S Gr(m,r)$ be the universal object of $G_S(Y)$, and

$$O^m_{\Delta} \rightarrow E \rightarrow 0$$

be the induced quotient on $\Delta$ by the universal quotient on Grassmannian over $Y$. Then there is a commutative diagram over $S$

$$\begin{array}{ccc}
\Delta & \xrightarrow{\pi} & C \times_S Y \\
\downarrow & & \downarrow \\
Y & \cong & Y
\end{array}$$

Lemma 2.6. If $S$ is a smooth scheme, then $\pi_*O_\Delta = O_{C \times_S Y}$ and there is a birational $S$-morphism

$$\theta : Y \rightarrow R$$

such that pullback of the universal quotient $O^m_{C \times_S R} \rightarrow F \rightarrow 0$ (by $id \times \theta$) is

$$O^m_{C \times_S Y} \rightarrow \pi_*E \rightarrow 0.$$

Proof. Similar with Proposition 6 and Proposition 9 of [NSe] (See also [Se]).

Lemma 2.7. Let $Y^s = \theta^{-1}(R^s)$ and $Y^0 = \theta^{-1}(R^{ss})$. Then

$$\theta : Y^s \rightarrow R^s, \quad \theta : Y^0 \rightarrow R^{ss}$$

are proper birational morphisms.

Proof. The proof in [NSe] and [Se] for irreducible one node curves is completely local. Thus can be generalized to general stable curves.
There is a $PGL(m)$-equivariant factorisation (See [NSe], [Se], [Sch])

$$
\begin{align*}
\mathcal{Y}^s & \xrightarrow{\iota} \mathcal{Y}^0 \xrightarrow{1} \mathcal{H} \\
\theta & \downarrow \quad \theta \downarrow \quad \lambda \downarrow \\
\mathcal{R}^s & \xrightarrow{1} \mathcal{R}^{ss} \xrightarrow{1} \mathcal{W}
\end{align*}
$$

and linearisation $\mathcal{O}_\mathcal{H}(1)$, where $\iota$ is open embedding. Let $L_a = \lambda^*(\mathcal{O}_\mathcal{Y}(a)) \otimes \mathcal{O}_\mathcal{H}(1)$. Then, for $a$ large enough, the set $\mathcal{H}(L_a)^{ss} (\mathcal{H}(L_a)^s)$ of GIT-semistable (stable) points satisfies: (i) $\mathcal{H}(L_a)^{ss} \subset \lambda^{-1}(\mathcal{R}^{ss})$, (ii) $\mathcal{H}(L_a)^s = \lambda^{-1}(\mathcal{R}^s)$. By Lemma 2.7, $\theta$ is proper, we have $\lambda^{-1}(\mathcal{R}^{ss}) = \mathcal{Y}^0$ and $\lambda^{-1}(\mathcal{R}^s) = \mathcal{Y}^s$. Thus

$$
\mathcal{H}(L_a)^s = \mathcal{Y}^s = \theta^{-1}(\mathcal{R}^s), \quad \mathcal{H}(L_a)^{ss} \subset \mathcal{Y}^0 = \theta^{-1}(\mathcal{R}^{ss}).
$$

**Notation 2.8.** $\mathcal{G}(r,d)_S = \mathcal{H}(L_a)^{ss}/PGL(m)$ is called (according to [NSe]) the generalized Gieseker semistable moduli space (or Gieseker space for simplicity). It is intrinsic by recent work [Sch].

Let $y = (\Delta_y, O^m_{\Delta_y} \rightarrow \mathcal{E}_y \rightarrow 0) \in \mathcal{Y}^0$. Obviously, for $y \in \mathcal{H}(L_a)^{ss} \setminus \mathcal{H}(L_a)^s$, we have to add extra conditions besides the semistability of $\pi_*\mathcal{E}_y$. Alexander Schmitt ([Sch]) recently figure out a sheaf theoretic condition $(H_3)$ (See Definition 2.2.10 in [Sch]) for $\pi_*\mathcal{E}_y$, which is a sufficient and necessary condition for $y \in \mathcal{H}(L_a)^{ss}$. The pair $(C, E)$ of a semistable curve $C$ with a vector bundle $E$ is called $H$-(semi)stable (See [Sch]) if $E$ is strictly positive on the chains of projective lines, and the direct image (on stable model of $C$) $\pi_*E$ is semistable satisfying the condition $(H_3)$.

**Theorem 2.9.** The projective $S$-scheme $\mathcal{G}(r,d)_S \rightarrow S$ universally corepresents the moduli functor $\mathcal{G}(r,d)^\sharp_S : \{S\text{-schemes}\} \rightarrow \{\text{sets}\}$,

$$
\mathcal{G}(r,d)^\sharp_S(T) = \begin{cases}
\text{Equivalence classes of pairs $(\Delta_T, \mathcal{E}_T)$, where $\Delta_T \rightarrow T$ is a flat family of semistable curves with stable model $C \times_S T \rightarrow T$ and $\mathcal{E}_T$ is an $T$-flat sheaf such that for any $t \in T$, $(\mathcal{E}_T)|_{\Delta_t}$ is $H$-(semi)stable vector bundle of rank $r$ and degree $d$.}
\end{cases}
$$

We call that $(\Delta_T, \mathcal{E}_T)$ is equivalent to $(\Delta'_T, \mathcal{E}'_T)$ if there is an $T$-automorphism $g : \Delta_T \rightarrow \Delta'_T$, which is identity outside the chains, such that $\mathcal{E}_T$ and $g^*\mathcal{E}'_T$ are fibrewisely isomorphic.

§3 A GIESEKER TYPE DEGENERATION FOR SMALL RANK

Let $\mathcal{C} \rightarrow S$ be a flat family of irreducible stable curves and $\mathcal{L}$ be a line bundle on $\mathcal{C}$ of relative degree $d$. We simply call the families in $\mathcal{G}(r,d)^\sharp_S(T)$, the families of semistable Gieseker bundles parametrized by $T$.

**Definition 3.1.** The subfunctor $\mathcal{G}_\mathcal{L} : \{S\text{-schemes}\} \rightarrow \{\text{sets}\}$ of $\mathcal{G}$ is defined to be

$$
\mathcal{G}_\mathcal{L}(T) = \begin{cases}
\Delta \in \mathcal{G}(T) \text{ such that for any } t \in T \text{ there is a morphism } \det(\mathcal{E}|_{\Delta_t}) \rightarrow \pi^*\mathcal{L}_t \text{ on } \Delta_t \text{ which is an isomorphism outside the chain of $\mathbb{P}^1$s}.
\end{cases}
$$
Definition 3.2. The moduli functor $G(r, L)^S_\mathfrak{d}$ of semistable Gieseker bundles with a fixed determinant is defined to be

$$G(r, L)^S_\mathfrak{d}(T) = \left\{ (\Delta_t, E_T) \in G(r, d)^S_\mathfrak{d}(T) \text{ such that for any } t \in T \right. $$

$$\text{there exists a morphism } det(E_T|_{\Delta_t}) \to \pi^*L_t \text{ on } \Delta_t \left. \right\}.$$  

When $S = Spec(\mathbb{C})$, the above defined functor is denoted by $G(r, L)^\sharp$.

Let $S = Spec(D)$ where $D$ is a discrete valuation ring. Let $C \to S$ be a family of curves with smooth generic fibre and closed fibre $C_0 = X$. Assume that $X$ is irreducible with only one node $p_0$. Then we have the following result that is similar with Lemma 1.19 of [Vi].

Lemma 3.3. When $r \leq 3$, or $r = 4$ but the normalization $\tilde{X}$ of $X$ is not hyperelliptic, the moduli functor $G(r, L)^S_\mathfrak{d}$ is a locally closed subfunctor of $G(r, d)^S_\mathfrak{d}$. More precisely, for any family $(\Delta_T, E_T) \in G(r, d)^S_\mathfrak{d}(T)$, there exists a locally closed subscheme $T' \subset T$ such that a morphism $T_1 \to T$ of schemes factors through $T_1 \to T' \to T$ if and only if

$$(\Delta_T \times_T T_1, pr_1^*E_T) \in G(r, L)^S_\mathfrak{d}(T_1).$$

Similarly, $G_L$ is a locally closed subfunctor of $G$.

Proof. Let $\pi : \Delta_T \to C \times_S T$ be the birational morphism contracting the chain of rational curves and $L_T$ be the pullback $\pi^*L$ to $\Delta_T$. Let $f : \Delta_T \to T$ be the family of semistable curves (thus $f_*(\mathcal{O}_{\Delta_T}) = \mathcal{O}_T$). Then the condition that defines the subfunctor is equivalent to the existence of a global section of $det(E_T|_{\Delta_t})^{-1} \otimes \pi^*L_t$ which is nonzero outside the chain $R_t \subset \Delta_t$ of $\mathbb{P}^1$s. There is a complex

$$(3.1)$$

$$K_T^\bullet : k_T^0 \xrightarrow{\delta_T} k_T^1$$

of locally free sheaves on $T$ such that for any base change $T_1 \to T$ the pullback of $K_T^\bullet$ to $T_1$ computes the direct image of $det(E_{T_1})^{-1} \otimes L_{T_1}$ (which equals to the kernel of $\delta_{T_1} : k_{T_1}^0 \to k_{T_1}^1$). There is a canonical closed subscheme of $T$ (defined locally by some minors of $\delta_T$) where $\delta_T$ is not injective. Replace $T$ by this closed subscheme, we assume that $f_*(det(E_T)^{-1} \otimes \pi^*L) \neq 0$. Let $U \subset T$ be the largest open subscheme such that for any $t \in U$

$$\dim(H^0(\Delta_t, det(E_T|_{\Delta_t})^{-1} \otimes \pi^*L_t)) = 1.$$ 

Let $Y \subset \Delta_U$ be the support of the cokernel of the map

$$f^*f_*(det(E_T)^{-1} \otimes \pi^*L) \to det(E_T)^{-1} \otimes \pi^*L.$$ 

Let $U_0 \subset U$ be the fibre of $U \to S$ at the closed point $0 \in S$. Then

$$\pi^{-1}(\{p_0\} \times U_0) \subset \Delta_U$$

consists of the chains of $\mathbb{P}^1$s. Note that $\pi(Y) \subset X \times U_0$, let $Y' \subset Y$ be the union of irreducible components $Y_i$ such that $p_1(\pi(Y_i)) \neq p_0$ where $p_1 : X \times U_0 \to X$ is the projection. Then we define that $T' = U \setminus f(Y')$. 


Let $T_1 \to T$ be a morphism. If it factors through $T_1 \to T' \to T$, it is clear

$$(\Delta_T \times_T T_1, pr_1^*\mathcal{E}_T) \in \mathcal{G}(r, \mathcal{L})^\sharp_T(T_1)$$

since $t \in T'$ if and only if dim($H^0(\Delta_t, \text{det}(\mathcal{E}_T|_{\Delta_t})^{-1} \otimes \pi^*\mathcal{L}_t)$) = 1 and

$$\mathcal{O}_{\Delta_t} \cong f^*f_*(\text{det}(\mathcal{E}_T)^{-1} \otimes \pi^*\mathcal{L}|_{\Delta_t}) \to \text{det}(\mathcal{E}_T|_{\Delta_t})^{-1} \otimes \pi^*\mathcal{L}_t$$

is an isomorphism outside the chain of $\mathbb{P}^1$s. On the other hand, if

$$(\Delta_T \times_T T_1, pr_1^*\mathcal{E}_T) \in \mathcal{G}(r, \mathcal{L})^\sharp_T(T_1),$$

then it factors firstly through the closed subscheme of $T$ where $H^0$ do not vanish. Then we have to show that the image of $T_1$ falls in the open set $U$, here we need the assumptions that $r \leq 3$, or $r = 4$ but $\tilde{X}$ is not hyperelliptic. To check it, let $t \in T_1$, then dim($H^0(\Delta_t, \text{det}(\mathcal{E}_T|_{\Delta_t})^{-1} \otimes \pi^*\mathcal{L}_t)$) = 1 when $\Delta_t$ has no chain of $\mathbb{P}^1$s. If $\Delta_t = \tilde{X} \cup R$ has a chain $R$, let $\{p_1, p_2\} = \tilde{X} \cap R$, then

$$H^0(\Delta_t, \text{det}(\mathcal{E}_T|_{\Delta_t})^{-1} \otimes \pi^*\mathcal{L}_t) = H^0(\tilde{X}, (\text{det}(\mathcal{E}_T|_{\Delta_t})^{-1} \otimes \pi^*\mathcal{L}_t)|_{\tilde{X}}(-p_1 - p_2)),$$

which has at most dimension 1 since deg($\text{det}(\mathcal{E}_T|_{\Delta_t})^{-1} \otimes \pi^*\mathcal{L}_t)|_{\tilde{X}}(-p_1 - p_2)$) $\leq 1$ when $r \leq 3$, or deg($\text{det}(\mathcal{E}_T|_{\Delta_t})^{-1} \otimes \pi^*\mathcal{L}_t)|_{\tilde{X}}(-p_1 - p_2)$ $\leq 2$ when $r = 4$ but $\tilde{X}$ is not hyperelliptic. Thus the morphism $T_1 \to T$ factors through $T_1 \to U$, then it factors through $T_1 \to T'$ by the definition of functor.

For simplicity, we assume that $r$ and $d$ are coprime $(r, d) = 1$. In this case, the functor $\mathcal{G}(r, d)_{/S}$ is representable by an irreducible Cohen-Macaulay $S$-scheme $\mathcal{G}(r, d)_S \to S$ (See [NSe]), whose fibres are reduced, irreducible projective schemes with at most normal crossing singularities. Moreover, there is a canonical proper birational $S$-morphism

$$(3.2) \quad \theta : \mathcal{G}(r, d)_S \to \mathcal{U}(r, d)_S,$$

where $\mathcal{U}(r, d)_S \to S$ is the family (associated to $\mathcal{C} \to S$) of moduli spaces of semistable torsion free sheaves with rank $r$ and degree $d$.

By the above Lemma 3.3, the functor $\mathcal{G}(r, \mathcal{L})^\sharp_{/S}$ is representable by a locally closed subscheme $\mathcal{G}(r, \mathcal{L})_S \subset \mathcal{G}(r, d)_S$ when $r \leq 3$, or $r = 4$ but $\tilde{X}$ is not hyperelliptic.

**Lemma 3.4.** $\mathcal{G}(r, \mathcal{L})_S \subset \mathcal{G}(r, d)_S$ is a closed subscheme of $\mathcal{G}(r, d)_S$. In fact, for the closed fibre $\mathcal{C}_0 = X$, we have

$$(3.3) \quad \mathcal{G}(r, \mathcal{L})^\sharp_{/S}(\{0\}) = \theta^{-1}(\mathcal{U}_X(r, \mathcal{L}_0)).$$

**Proof.** It is enough to prove (3.3). For any $(\Delta, E) \in \mathcal{G}(r, d)_{/S}(\{0\})$, let

$$\pi : \Delta \to X$$

be the morphism contracting the chain $R$ of $\mathbb{P}^1$s. Then, by definition of $\theta$,

$$\theta((\Delta, E)) = \pi_*(E) \in \mathcal{U}_X(r, d).$$
Note that $F$ has type of $t(F) = \deg(E|_R)$ (See [NSe]), then $\pi_*(\text{det}(E))$ has torsion of dimension $t(F) - 1$ supported at the node $p_0 = \pi(R)$. There is a natural morphism

$$\wedge^r F = \wedge^r (\pi_* E) \to \pi_*(\wedge^r E) = \pi_*(\text{det}(E)),$$

which is an isomorphism outside $p_0$. Thus we have an isomorphism

$$\wedge^r F/\text{torsion} \cong \pi_*(\text{det}(E))/\text{torsion}$$

since $\deg(\wedge^r F/\text{torsion}) = \deg(\pi_*(\text{det}(E))/\text{torsion}) = d - t(F) + 1$. By using this isomorphism, it is clear that

$$(\Delta, E) \in \mathcal{G}(r, \mathcal{L})_S^0(\{0\}) \iff \theta((\Delta, E)) \in \mathcal{U}_X(r, d).$$

$\mathcal{G}(r, \mathcal{L})_S$ is in fact a degeneracy loci of a map of vector bundles. To study it, we recall some standard results (See [FP] for example). Let $\varphi : F \to E$ be a morphism of vector bundles on a variety $M$ with $rk(F) = m$ and $rk(E) = n$. The closed subsets of $M$

$$D_r(\varphi) = \{ x \in M \mid rank(\varphi_x) \leq r \}$$

are the so called degeneracy locus of $\varphi$. We collect the results into

**Lemma 3.5.** The codimension of each irreducible component of $D_r(\varphi)$ is at most $(n - r)(m - r)$. If $M$ is Cohen-Macaulay and the codimension of each irreducible of $D_r(\varphi)$ equals to $(n - r)(m - r)$, then $D_r(\varphi)$ is Cohen-Macaulay.

In (3.1), $rk(K^0_T) - rk(K^0_U) = g - 1$ since $\text{det}(\mathcal{E}_T) \otimes \mathcal{L}_T$ has relative degree 0. Replace $T$ by an open set $U \subset \mathcal{G}(r, d)_S$, one sees that

$$\mathcal{G}(r, \mathcal{L})_S = D_{k_0}(\delta_U), \quad k_0 = rk(K^0_U) - 1.$$ 

In what follows, we will use $\text{Codim}(\bullet)$ to denote: codimension of each irreducible component of $\bullet$. Thus $\text{Codim}(\mathcal{G}(r, \mathcal{L})_S) \leq g$, and it is Cohen-Macaulay if

$$\text{Codim}(\mathcal{G}(r, \mathcal{L})_S) = g.$$ 

In particular, let $X$ be the singular fibre of $\mathcal{C} \to S$ and $L = \mathcal{L}|_X$. The closed fibre $G(r, d)$ of $\mathcal{G}(r, d)_S \to S$ is the so called generalized Gieseker moduli space (associated to $X$) of [NSe], which has normal crossing singularities. The closed fibre of $\mathcal{G}(r, \mathcal{L})_S \to S$, denoted by $G(r, L)$, is the degeneracy loci

$$D_{k_0}(\delta_{U_0}) \subset U_0 \subset G(r, d)$$

of $\delta_{U_0} : K^0_{U_0} \to K^1_{U_0}$, where $U_0$ is the closed fibre of $U \to S$. Thus

$$\text{Codim}(G(r, L)) \leq g$$

and $G(r, L)$ is Cohen-Macaulay if $\text{Codim}(G(r, L)) = g$. When $r \leq 3$, or $r = 4$ but $\widetilde{X}$ is not hyperelliptic, $G(r, L) \subset G(r, d)$ is a closed subscheme that represents a moduli functor (See Theorem 3.7 for definition).
Lemma 3.6. Codim(G(r, L)) = g. In particular, G(r, L)_S \subset G(r, d)_S is an irreducible, reduced, Cohen-Macaulay subscheme of codimension g.

Proof. Assume that Codim(G(r, L)) = g. Note that there is a unique irreducible component of G(r, L)_S with codimension g dominates S since C \to S has smooth generic fibre. Thus other irreducible components (if any) of G(r, d)_S will fall in G(r, L) and their codimension in G(r, d) are at most g - 1 since G(r, d)_S \to S is flat over S. This contradicts Codim(G(r, L)) = g. Hence G(r, L)_S \subset G(r, d)_S is an irreducible, Cohen-Macaulay subscheme of codimension g. It has to be reduced since it is Cohen-Macaulay and has a reduced open subscheme.

Now we prove that Codim(G(r, L)) = g in G(r, d). Let J^0_X be the Jacobian of line bundles of degree 0 on X. Consider a morphism

\( \phi : G(r, L) \times J^0_X \to G(r, d) \)

that sends any \( \{(\Delta, E), N\} \in G(r, L) \times J^0_X \) to \( (\Delta, E \otimes \pi^*N) \in G(r, d) \), where \( \pi : \Delta \to X \) is the morphism contracting the chain \( R \) of \( \mathbb{P}^1 \)'s. We claim that

\[ \dim \phi^{-1}((\Delta, E_0)) \leq 1, \quad \text{for any } (\Delta, E_0) \in G(r, d). \]

Let \( \sigma : J^0_X \to J^0_X \) be the morphism induced by pulling back line bundles on X to its normalization \( \tilde{X} \). The fibres of \( \sigma \) are of dimension 1. On the other hand, it is easy to see that the projection \( G(r, L) \times J^0_X \to J^0_X \) induces an injective morphism

\[ \rho : \phi^{-1}((\Delta, E_0)) \to J^0_X. \]

To prove the claim, it is enough to show that the image \( \text{Im}(\rho) \) falls in a finite number of fibres of \( \sigma \). Note that, for any \( \{(\Delta, E), N\} \in \phi^{-1}((\Delta, E_0)) \), we have

\[ \text{det}(E) \otimes \pi^*(N^\otimes r) = \text{det}(E_0) \]

on \( \Delta \). Recall that, by definition of \( G(r, L) \), there is a morphism \( \text{det}(E) \to \pi^*L \) which is an isomorphism outside the chain \( R \) of \( \mathbb{P}^1 \)'s. We have

\[ \text{det}(E)|_{\tilde{X}} = \pi^*L|_{\tilde{X}}(n_1p_1 - n_2p_2) = \tilde{L}(n_1p_1 - n_2p_2), \]

where \( \tilde{L} \) is the pullback of \( L \) to \( \tilde{X} \), \( n_1, n_2 \) are nonnegative integers such that

\[ n_1 + n_2 = \deg(E_0|_R) = t(F_0), \quad F_0 := \pi_*(E_0). \]

Thus \( \sigma \circ \rho((\{(\Delta, E), N\})) = \sigma(N) = \tilde{N} \in J^0_X \) falls in the set

\[ \{\tilde{N} \in J^0_X | \tilde{N}^\otimes r = \text{det}(E_0)|_{\tilde{X}} \otimes \tilde{L}^{-1}(n_1p_1 + n_2p_2)\}, \]

which is clearly a finite set. This proves that fibres of \( \phi \) are at most dimension 1.

There is a unique irreducible component \( G(r, L)^0 \) of \( G(r, L) \) containing \( \Delta \cong X \), which has codimension \( g \). For any other irreducible component (if any), say \( G(r, L)^+ \), all of \( \Delta s \) in \( G(r, L)^+ \) must have chain (with positive length) of \( \mathbb{P}^1 \)'s. Then the image \( \phi(G(r, L)^+ \times J^0_X) \) has to fall in a subvariety of \( G(r, d) \), which has codimension at least 1. Thus \( \dim(G(r, L)^+ \times J^0_X) \leq \dim G(r, d) \), that is,

\[ \text{Codim}(G(r, L)^+) \geq g. \]

By Lemma 3.5, \( G(r, L) \) is Cohen-Macaulay of pure codimension \( g \).
**Theorem 3.7.** Let $X$ be an irreducible curve of genus $g \geq 2$ with only one node $p_0$. Let $L$ be a line bundle of degree $d$ on $X$. Assume that $r \leq 3$, or $r = 4$ and the normalization of $X$ is not hyperelliptic. Then, when $(r, d) = 1$, we have

1. There is a Cohen-Macaulay projective scheme $G(r, L)$ of pure dimension $(r^2 - 1)(g - 1)$, which represents the moduli functor

$$G(r, L)^\sharp : (\mathbb{C} - \text{schemes}) \to (\text{sets})$$

which is defined in Definition 3.2.

2. Let $C \to S$ be a proper family of curves over a discrete valuation ring, which has smooth generic fibre $C_\eta$ and closed fibre $C_0 \cong X$. If there is a line bundle $\mathcal{L}$ on $C$ such that $\mathcal{L}|_{C_0} \cong L$. Then there exists an irreducible, reduced, Cohen-Macaulay $S$-projective scheme $f : G(r, \mathcal{L})_S \to S$ such that

$$f^{-1}(0) \cong G(r, L), \quad f^{-1}(\eta) \cong U_{C_\eta}(r, \mathcal{L}_\eta).$$

Moreover $G(r, \mathcal{L})_S$ represents the moduli functor $G(r, \mathcal{L})^\sharp_S$ in Definition 3.2.

3. There exists a proper birational $S$-morphism $\theta : G(r, \mathcal{L})_S \to U(r, \mathcal{L})_S$ which induces a birational morphism $\theta : G(r, L) \to U_X(r, L)$.

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