Abstract  We study the phase transition between G-instantons and D3-branes quantitatively. A G-instanton is a classical solution to the self-dual equation of the M/F-theory three-form tensor field C in the complex fourfold. This phase transition is dual to that between ‘vertical’ small instantons and 5-branes in the heterotic string. Using G as a background gauge flux, we may dynamically control the gauge symmetry breaking and connect between different vacua of F-theory. We may understand the amount of flux undergoing the phase transition and the resulting number of D3-branes in terms of group-theoretical quantities. We also discuss the resulting chirality change and preservation of anomaly freedom.

1  Introduction

String theory provides a framework for completion of the Standard Model. The global consistency conditions of string theory, principally arising from the one-loop vacuum-to-vacuum amplitude, are more fundamental than those of field theory. For instance, modular invariance of the closed-string holomorphic partition function imposes more restrictive conditions than anomaly cancellation alone [1–10]. Furthermore, string theory involves high-rank tensor fields which induce more general anomaly cancellation mechanisms such as the Green–Schwarz mechanism. These provide important guidelines for evading the swampland [11].

Although the consistency conditions allow for many different vacuum configurations, we have also seen that many of them are dynamically connected. In other words, spontaneous symmetry breaking relates different vacua and the corresponding moduli spaces turn out to be linked. Brane separation and recombination are good examples [12–20]. Dual to this process is a phase transition between small instantons and heterotic 5-branes [21]: by emitting and absorbing branes, the structure group of the instantons changes, and hence the gauge symmetry and matter spectrum can also change. Increasingly many phenomena have been unified under such dualities. For instance, recently it has been understood that twisted strings localized on orbifolds can take part in the transition [22].

Protected by supersymmetry and BPS conditions, these dynamical transitions are continuous and energy-cost free. In general, these mechanisms change the local chirality and reorganize the spectrum, while the overall chirality is preserved and counted by the global consistency conditions. This may explain the origin of chirality and the absence of anomalies in the Standard Model.

For every gauge symmetry breaking source, we may consider such a dynamical transition. It is then natural to ask whether there exists a similar dynamical transition in F-theory. In this work, we show that there is a phase transition between M2/D3-branes and G-instantons. A G-instanton is defined as a classical solution to the self-dual equation of the four-form field strength G on a four-complex-dimensional manifold. In M-theory, there is a three-form tensor field C with corresponding field strength

\[ G = dC . \]  

(1)

In what follows, we borrow this M-theoretical description and apply it in the T-dual theory of F-theory. In the F-theory context, the G-instanton may shrink to zero size and undergo a phase transition into a D3-brane. The total chirality is preserved along with the global consistency condition [9,10]

\[ \chi(Y) = \frac{1}{8\pi^2} \int_Y G \wedge G + n , \]  

(2)

where \( \chi(Y) \) is the Euler characteristic of the Calabi–Yau fourfold Y on which F-theory is compactified, and n is the number of D3-branes filling the noncompact dimensions. Each term in (2), especially n, must be nonnegative integer to ensure the absence of anti-branes and preserve to \( N = 2 \) or \( N = 1 \) supersymmetry in four dimensions.

\[ ^{1} \text{In the literature, sometimes G-instanton refers to an instanton with the structure group G [23].} \]

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We may write the condition (2) in a more democratic form, using $\chi(Y) = \int_Y c_4(Y)$, as

$$
d \ast dC = \frac{1}{24} c_4(Y) - \frac{1}{8\pi^2} G \wedge G - \sum_{a=1}^n \delta^{(8)}(y - y_a),
$$  

where the interpretation in terms of $d \ast dC$ is possible in eleven and twelve dimensions [24,25]. We see that the transition is natural because the delta function can be interpreted as either (i) a D3-brane source for the equation of motion for $C$ or (ii) a part of the quantized $G$-flux.

We first show that this phase transition is dual to that between small instantons and 5-branes in the heterotic string [21]. It is known [26] that the phase transition and emergence of a horizontal brane corresponds to blowing-up in the base of F-theory. However, this transition may also occur in the case of a vertical brane [23,27,28]. As such, it should be translated to a phase transition of $G$-instantons. Therefore we may prove that there is a transition between small $G$-instantons and D3-branes. We can understand their quantitative properties by group-theoretical invariants.

$G$-flux is an important source for gauge symmetry breaking as well as chirality. It is induced to the branes and hence to matter curves giving rise to a chiral spectrum in four dimensions [29,30]. An important application is to break the Grand Unification group down to the Standard Model group using a line bundle flux along the hypercharge direction, which can give different chirality for doublet and triplet Higgses [31–34]. To this end, we consider the toy example of an $SU(5)$ gauge group dynamically broken down to its subgroup by a $G$-instanton phase transition.

Although D3-branes do not directly contribute to anomaly cancellation in four dimensions [35], they can contribute indirectly by converting to a $G$-instanton. We demonstrate chirality change in our toy examples and verify that anomaly cancellation is preserved, both before and after the phase transition.

2 Gauge theory from duality

We first review the duality between F-theory and heterotic string theory. We establish quantitative relations between the F-theory geometry and the heterotic gauge field. Following this, we identify the background gauge bundle.

2.1 Gauge fields from the three-form field

We use the duality between F-theory on K3 and the heterotic string on $T^2$ to show the relation between the M-theory three-form $C$ and the gauge field $A$ of the heterotic string. We may expand the field strength $G$ along the harmonic $(1,1)$-forms of the K3 surface [36].

$$
\frac{G}{2\pi} = \sum_I F^I \wedge e_I, \quad e_I \in H^{1,1}(K3).
$$  

Via this expansion, a single Abelian field strength $G$ can give rise to the Cartan subalgebra of the Yang–Mills field strengths $F^I$.

Non-Abelian structure arises if the K3 develops a singularity by shrinking some cycle, which determines the unbroken set of simple gauge groups $g$. Then we have

$$
\frac{1}{2} \int_{K3} \frac{G}{2\pi} \wedge \frac{G}{2\pi} = \frac{1}{2} \sum_{I,J=1}^{20} \int_{K3} e_I \wedge e_J \wedge F^I \wedge F^J
$$

$$
= -\frac{1}{2} \sum_{i,j=1}^{8} A^i_{E_8} F_1^i \wedge F_1^j - \frac{1}{2} \sum_{i,j=1}^{8} A^i_{E_8} F_2^i \wedge F_2^j + \frac{1}{2} \sum_{a=1}^{2} \sum_{i,j} U^{ij} F_a^i \wedge F_a^j
$$  

where $U = (0 \, 1 \mid 1 \, 0)$. For each simple Lie algebra component $g$, the McKay correspondence holds between the intersection numbers and the symmetrized Cartan matrix of the algebra $g$,

$$
\frac{2}{(\alpha^i, \alpha^j)_g} A^i_{g} \equiv (\alpha^i_g)^{ij}_+ = - \int_{K3} e_i \wedge e_j.
$$  

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where the indices $i$, $j$ are appropriately arranged in (5) and $(\cdot, \cdot)$ is the inner product of root vectors $\alpha^{(i)}$ and/or coroot vectors $\alpha^{(i)\vee}$. Here, we use the Chevalley basis, which has the commutation relations [37]

\[
[H^i, H^j] = 0, \\
[H^i, E^j_\pm] = \pm A^{ij} E^j_\pm, \\
[E^i_+, E^j_-] = \delta_{ij} H^i, \\
(\text{ad}_{E^j_\pm})^{1-A^{ij}} E^j_\pm = 0
\]

where we have suppressed the algebra index. Each set of Cartan generators $H^i$ and ladder operators $E^j_\pm$ is related to a simple root $\alpha^{(i)}$, and we define $\text{ad}_A B \equiv [A, B]$. Also in the second and fourth lines the signs are correlated. These matrices are represented in the fundamental basis, which satisfies

\[
\text{Tr} T^a T^b = \frac{(\theta, \theta)}{2} \delta^{ab} \equiv \kappa^{ab}.
\]

This also defines the Killing form $\kappa^{ab}$ serving as the metric on the algebra space. Here, $\theta$ is the highest weight and we employ the convention $(\theta, \theta) = 2$.

Ultimately, we are dealing with the real Lie algebra: in the Chevalley basis, the diagonal element $-2 = A^{ii}$ corresponds to the normal form of the algebra $SL(n, \mathbb{R})$, for instance [37]. If the above $T^a$ generates $SL(n, \mathbb{R})$, the $SU(n)$ is generated by $i T^a$ having the diagonal element $-2 = -A^{ii}_{SU(n)}$. We may extend this convention to the exceptional groups.

The Killing form is proportional to other traces in different representations. In particular, for the adjoint representation, the normalization constant is the dual Coxeter number $h_g^\vee$,

\[
\text{Tr} T^a T^b = h_g^\vee (\theta, \theta) \delta^{ab} = 2h_g^\vee \kappa^{ab},
\]

where $\text{Tr}$ means the trace over the adjoint matrices. In the Chevalley basis, this decomposes into [37]

\[
\frac{1}{h_g^\vee} \text{Tr} H^i H^j = \frac{2}{(\alpha^i, \alpha^j)} A^{ij} = \kappa^{ij},
\]

\[
\frac{1}{h_g^\vee} \text{Tr} E^\alpha E^\beta = \frac{2}{(\alpha, \alpha)} \delta_{\alpha - \beta} = \kappa^{\alpha - \beta},
\]

\[
\frac{1}{h_g^\vee} \text{Tr} E^\alpha H^i = 0 = \kappa^{\alpha i}.
\]

Since the instanton number is the generalization of the second Chern class, which is an integer in the fundamental representation, we need the normalization

\[
\frac{1}{2h_g^\vee} \text{Tr} F \wedge F = \frac{1}{2h_g^\vee} \sum_{a,b=1}^{\dim \mathfrak{g}} \text{Tr} T^a T^b F^a \wedge F^b = \frac{1}{2h_g^\vee} \sum_{a,b=1}^{\dim \mathfrak{g}} \kappa^{ab} F^a \wedge F^b
\]

\[
= \frac{1}{2h_g^\vee} \sum_{i,j=1}^r \text{Tr} H^i H^j F^i \wedge F^j + \frac{1}{2h_g^\vee} \sum_{\alpha, \beta \in \Phi} \text{Tr} E^\alpha E^\beta F^\alpha \wedge F^\beta
\]

\[
= -\frac{1}{2} \sum_{i,j} A^{ij} F^i \wedge F^j - \frac{1}{2} \sum_{\alpha \in \Phi} F^\alpha \wedge F^{-\alpha}.
\]

Here $\Phi$ is the set of the root vectors and we used the Killing form (8)-(10). Since $E_\Phi$ and its subgroups embedded therein are self-dual, we fix $(\alpha, \alpha) = 2$ so that the normalization yields integer instanton numbers upon integration over a four-cycle. Thus, (5) contains the correct contents and normalization for the Cartan components.

It is known [38] that the off-diagonal parts describing $W$-bosons are sourced by the D3-branes wrapped on the two-cycles $e^l$ of (4) with the wrapping number forming the weight vectors in the Dynkin basis.

2.2 Background vector bundles

In F-theory, the gauge symmetry is configured by the singular elliptic fiber by choosing the elliptic Calabi–Yau fourfold $\pi : Y \to B'$. The setup is described by the Weierstrass equation,

\[
y^2 = x^3 + f x + g,
\]

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with \( f \) and \( g \) being sections of powers of the canonical bundle of \( B' \). Its discriminant locus \( D : \{ 4f^3 + 27g^2 = 0 \} \) determines the cycle supporting the worldvolume gauge theory. The unbroken gauge group \( h \) can be identified by the Kodaira–Tate algorithm [39]. We leave the details of the construction to review papers, e.g., [38].

We define the Euler characteristic of a singular Calabi–Yau as that of the resolved geometry. As seen in (5), with the K3 fibration of \( Y \), the unbroken gauge symmetry is \( E_8 \times E_8 \). The corresponding Calabi–Yau manifold \( Y_{\text{sing}} \) is also the most singular, with Euler characteristic [41–43]

\[
\chi(Y_{\text{sing}}) = \int_B (c_2 + 11c_1^2).
\]

Here, using the elliptic fibration structure and the vanishing of the first Chern class, we may express the Euler characteristic using Chern classes of the base \( B' \) of the elliptic fibration [9,44]. Then the duality between F-theory and the heterotic string requires that the base \( B' \) is a \( \mathbb{P}^1 \) fibration over a two-base \( B = B_1 \times B_2 \). Here and in what follows we understand Chern classes without an argument as those of \( B, c_k := c_k(B) \). It is known that the Euler characteristic does not depend on the method of resolution, that is, does not depend on the choice of flop.

A smoother singularity may be induced by deformation and we obtain a smaller unbroken gauge group. This is done by adding lower-order terms in the Weierstrass equation. This deformation will give an additional contribution to the Euler characteristic—it is known that this translates precisely to the second Chern class of the vector bundle on the heterotic side, i.e., the correction to the Euler characteristic,

\[
\chi(Y_{\text{tot}}) = \chi(Y_{\text{sing}}) + \Delta \chi(Y_g),
\]

where

\[
\Delta \chi(Y_g) = \int_B \tilde{e}(g),
\]

\[
\tilde{e}(g) \equiv h_g' \eta \left( n_g \eta - (\text{rk} g + 1) c_1 + \dim g c_1^2 \right).
\]

Here, \( \eta \) is a combination of \( c_1 \) and \( t \), which is to be determined shortly. Also \( n_g \) is the instanton number corresponding to the non-Higgsable cluster. We have \( n_g = 3, 4, 6, 8, 12 \) for \( SU(n), SO(n), E_6, E_7, E_8 \), respectively. Note that although it is described by group-theoretical quantities such as the dimension \( \dim g \), rank \( \text{rk} g \) and dual Coxeter number \( h_g' \), this contribution is purely geometric. We do not need to assume the duality to the heterotic string [41,42,45,47].

Nevertheless, we should have the same unbroken gauge group \( h \) on the heterotic side. In particular, we consider the dual \( E_8 \times E_8 \) heterotic string on an elliptic Calabi–Yau threefold \( X \) which is a fibration over the same two-base \( B, \pi_B : X \to B \). The unbroken gauge group \( h \) is the commutant to the structure group \( g \) of the vector bundles \( V_1 \) and \( V_2 \) in \( E_8 \times E_8 \). Each of them gives an additional contribution to the second Chern class,

\[
c_2(V) = \eta \sigma - \tilde{e}(g),
\]

where \( \sigma \) is the section of the base \( B \) in \( X \) and \( \tilde{e}(g) \) is defined in (16).

For example, if we wish to have an unbroken \( SU(5) \) gauge group, we need background gauge bundles \( V_1 \) and \( V_2 \) whose structure groups should be \( SU(5) \) and \( E_8 \), respectively, such that [47,48]

\[
c_2(V_1) = \eta_1 \sigma - 5c_1^2 - 5 \eta_1 (\eta_1 - 5c_1),
\]

\[
c_2(V_2) = \eta_2 \sigma - 310c_1^2 - 15 \eta_2 (\eta_2 - 9c_1).
\]

Meanwhile, the second Chern class for the tangent bundle of \( X \) is [47]

\[
c_2(X) = c_2 + 11c_1^2 + 12 \sigma c_1.
\]

Note that this is just the integrand of (13) plus \( 12 \sigma c_1 \). We may have a mismatch between them [47],

\[
c_2(X) - c_2(V_1) - c_2(V_2) \neq 0,
\]

apparently failing to satisfy the global consistency condition, namely the Bianchi identity for the Kalb–Ramond field strength \( H \),

\[
\frac{2}{\alpha'} dH = \frac{1}{2} \text{Tr} R \wedge R - \frac{1}{2h_{E_8}'} \text{Tr} F_1 \wedge F_1 - \frac{1}{2h_{E_8}'} \text{Tr} F_2 \wedge F_2
\]

\[
= c_2(X) - c_2(V_1) - c_2(V_2),
\]

\( ^2 \) We may also have additional \( U(1) \) factors and the total gauge group can be enhanced. Non-perturbatively enhanced gauge groups are also possible [26,40]. In this paper, we restrict our focus to the perturbative gauge group in ten dimensions.

\( ^3 \) In this paper, we use the same notation for a line bundle, its dual divisor and its first Chern class, without confusion.
where tr is the trace over the vector representation of the Lorentz group while Tr is taken over the adjoint representation of $E_8$. The dual Coxeter number of $E_8$ is $h_{E_8} = 30$. This mismatch should be accounted for by heterotic 5-branes, providing magnetic sources for the Kalb–Ramond field $B$.

2.3 $G$-flux

The gauge symmetry from the singular fiber can be broken further by $G$-flux (1). While there can be nontrivial flux that preserves the gauge symmetry, in any case it will induce a chiral spectrum in four dimensions. The gauge bosons of the broken symmetry acquire mass via the Stükelberg mechanism [32]. In addition, this background flux induces the D3-charge as in (2) and (3).

On the heterotic side, the dual of the $G$-flux provides an extra component of the vector bundle and thus also breaks the gauge group [23]. This is reflected in the second Chern class (17),

$$c_2(V) = \eta \sigma - e(g) + \frac{1}{8\pi^2} \int_{K3} G \wedge G, \tag{23}$$

where we understand the last integral is done by expansion as in (5). The relation between M-theory and $E_8 \times E_8$ heterotic string theory imposes the Freed–Witten quantization condition for the cohomology of $G$ [49],

$$\frac{G}{2\pi} + \frac{1}{2} c_2(Y) \in H^4(Y, \mathbb{Z}). \tag{24}$$

That is, if $\frac{1}{2} c_2(Y)$ is not integrally quantized, then $G$ should be also half-integrally quantized.

Supersymmetry restricts $G$ to be a primitive $(2,2)$-form. This flux should not break the Poincaré symmetry on the F-theory side, thus one of the legs of the four-form field $G$ should be along the fiber. This is expressed as [50–52]

$$G \cdot Z \cdot D_a = 0, \tag{25}$$

$$G \cdot D_a \cdot D_b = 0, \tag{26}$$

where $Z$ is the section of the elliptic fibration and $D_a, D_b$ are pullbacks of base divisors in $B'$. Here and in what follows, a dot product among divisors means intersection inside the Calabi–Yau manifold $Y$.

When the fiber becomes singular, giving rise to the gauge group $h$, we may blow-up these singularities to obtain exceptional divisors $E_i$. It turns out that these are $\mathbb{P}^1$ fibrations over the discriminant locus $D$, whose fibers are the above $e_i$, c.f. (4). Again, these obey a generalized McKay relation, that is, they are in one-to-one correspondence with the roots of the Lie algebra,

$$E_i \cdot E_j \cdot D_a \cdot D_b = -A_{ij} \delta^b_a, \tag{27}$$

where $A_{ij}$ is the Cartan matrix of $h$.

It is known [38] that the vertical divisor can be made of a bi-product of two $(1,1)$-forms. So we consider fluxes of the type

$$E_i \wedge D_a. \tag{28}$$

Here, we assume that there are no $U(1)$ gauge groups arising from rational sections. We may also construct $G$-flux dual to a matter curve having the form

$$E_i \wedge E_j, \tag{29}$$

which should also satisfy the conditions (25) and (26). Furthermore, a pair of $E_j$ divisors should be chosen which do not produce a nonzero value upon integration along the K3.

Usually we construct a $G$-flux preserving the whole gauge symmetry arising from the singularity,

$$G \cdot E_j \cdot D_a = 0, \quad a^j \in \Phi(h), \tag{30}$$

in order to obtain a chiral spectrum. However, we may also relax one or more of these conditions in order to introduce additional symmetry breaking $G$-flux.

2.4 Chirality

An important role of $G$-flux is to induce chirality in four dimensions.

For localized matter fields on the matter curve $\Sigma_R$, the local gauge symmetry on the discriminant locus is enhanced, and the branching of the adjoint of the enhanced gauge group determines the representation $R$ [30]. The curve is thus promoted to a matter surface, a $\mathbb{P}^1$ fibration over the matter curve [53]. To be precise, the fiber is a linear combination of $\mathbb{P}^1$s reflecting the weights of components of $R$. Thus, we have as many slices of matter surfaces as the dimension of $R$. We define the matter surface $\Sigma_R$ as that corresponding to the highest weight component of $R$. 

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A chiral fermion in six dimensions compactified on a smooth two-manifold, which is a matter curve $\Sigma_R$ in our case, gives vector-like fermions in four dimensions. However, the zero modes may become chiral if we turn on nontrivial magnetic flux on $\Sigma_R$. The $G$-flux induces magnetic flux on the matter curve, which is derived from that induced on the matter surface $S_R$ [30,45,52–55]

$$\chi(R) = \int_{S_R} G.$$ (31)

If the $G$-flux breaks the gauge symmetry given by the singularity, it gives rise to different chiralities in the different branched representations. The gauge symmetry is broken to $g \times h \times h'$, where fluxes $F'$ belong to the structure group $h'$. Accordingly, the representation also branches as $R \rightarrow (r, 1) + (1, r')$. The latter two differ by their respective roots, and the nontrivial intersection with the $G$-flux may induce different chiralities in each representation. In particular, if the $G$-flux is proportional to a Cartan element $E_I$, then the resulting chirality is proportional to the $U(1)$ charge along the $E_I$ direction.

In addition, the vector multiplet branches and its ‘off-diagonal’ components $R$ and $\bar{R}$ may become chiral. The chirality is determined by the Riemann–Roch–Hirzebruch index theorem [56],

$$\chi(R) \equiv n_R - \bar{n}_R = \int_B c_1(F) \wedge c_1(B).$$ (32)

It is defined as the number of zero modes for a complex representation $R$ minus that of $\bar{R}$.

### 3 Small $G$-instantons

Using heterotic/F-theory duality, we show that small $G$-instantons may shrink and become D3-branes via a phase transition. F-theory on a Calabi–Yau threefold is dual to heterotic string theory on a K3 manifold. This is extended in higher-dimensional Calabi–Yau manifolds by replacing the common $\mathbb{P}^1$ base with any rational manifold $B$. For each base divisor $C_a$ of $B$, we may define Poincaré-dual curves $C^b$ satisfying

$$\int_B C_a \wedge C^b = \delta^b_a.$$ (33)

Sections $\sigma C_a$ are curves in $X$, while pullbacks $\pi^a_b C^a$ are surfaces in $X$ providing Hodge duals of $\sigma C_a$, which arises from the relation (33) since

$$\delta^a_b = \int_X \sigma B \wedge \pi^a_b C^a \wedge \pi^a_b C_a = \int_X \sigma C_a \wedge \pi^a_b C_b,$$ (34)

where we have implicitly used the essential property of the fiber $E$ that

$$\int_X \sigma B \wedge E = 1.$$ (35)

#### 3.1 Small-instanton transition

There is a transition between small instantons and 5-branes [21]. If an instanton shrinks to zero size, the gauge symmetry is recovered and the instanton becomes a 5-brane. The zero size instanton is described by a delta function, which also describes a 5-brane source for the equation of motion of $*H$:

$$\frac{2}{\alpha'} dH = \frac{1}{2} \text{Tr } R \wedge R - \frac{1}{2h^*_{Es}} \text{Tr } F_1' \wedge F_1' - \frac{1}{2h^*_{Es}} \text{Tr } F_2' \wedge F_2' - \sum_{A=1}^{n} \delta^{(4)}(C_A).$$ (36)

Here, each delta function is nonvanishing along the locus of a curve $C_A$. We may think of the argument of the function as being the zeroes of the defining equation of $C_A$. Being protected by supersymmetry, the transition takes place without energy loss.

Depending on the direction of $F \wedge F$, different 5-branes may or may not wrap the elliptic fiber $E$. If a 5-brane wraps the elliptic fiber of the heterotic string, we refer to it as a vertical brane; if the brane is transverse to $E$, we call it a horizontal brane.

The properties of instantons differ depending on the type of string. For instance, the $SO(32)$ heterotic string theory is dual to the type I string, and the gauge group is constructed via D9-branes/O9-planes. The orientifold introduces a further projection on the gauge group described by D9 and D5-branes. Each D5-brane on top of O9-planes describes an $Sp(1) \simeq SU(2)$ gauge group, which is further enhanced to $Sp(k)$ for $k$ coincident branes. Therefore the gauge anomaly is constrained. In the Coulomb branch, we may detach the D5-branes, whose locations are described by adjoint-valued scalars in the vector multiplet.

Meanwhile, the heterotic 5-brane is dual to a D5-brane inside the D9-brane. It is well known that this D9-D5 system describes instantons [21]. Finite-size instantons correspond to flux sourced by D5-branes while shrinking instantons can become an unbounded state under a phase transition with no energy cost. This process is guided by open strings connecting D9 and D5-branes. The
development of vacuum expectation values in the corresponding low-energy fields is interpreted as the instantons growing in size and corresponds to points in moduli space along the Higgs branch.

In the $E_8 \times E_8$ heterotic string, a new interval opens up in the strong coupling limit [57, 58]. It is dual to M-theory compactified on this interval, at whose boundaries exist M9-branes, each describing one $E_8$ factor. The resulting 5-brane is an M5-brane and is emitted into the eleven-dimensional bulk. This corresponds to a tensor branch for which we have a new antisymmetric tensor field, whose scalar partner in the tensor multiplet describes the location of NS5-branes. From the point of view of the M9-branes, we cannot simultaneously observe the gauge group described by the 5-branes.

In F-theory, M9-branes and M5-branes are mapped to 7-branes and D3-branes, respectively. Depending on the particular $(p, q)$ charges of the branes, a stack of 7-branes can describe various gauge groups extending the aforementioned groups. The discussion for the D9-D5 system is also generalized: there is a shrinking instanton on the 7-branes which becomes a codimension-four brane, in this case a D3-brane. A heterotic 5-brane wrapping the elliptic fiber, or a vertical brane, is mapped to this D3-brane. The shrinking small-instanton behavior is again understood as the un-Higgsing of open strings stretched between the D3 and the 7-branes. When the D3-branes sit on top of the 7-branes, 3-7 strings describe the size of 3-branes inside the 7-branes. Roughly, the expectation value of the corresponding low-energy field parameterizes the size of the instanton, while in moduli space the value is along the Higgs branch. Finally, we remark that this scenario is further generalized when the adjoint-valued Higgs field describing the 7-brane configuration obtains non-diagonal expectation values. Such configurations are described by T-branes [59–61].

3.2 $G$-instanton

For a four-complex-dimensional manifold $Y$, we may consider the self-dual equation for the $G$-flux [36, 62],

$$\ast_Y G = G \quad (37)$$

We refer to the classical solution of this equation as a $G$-instanton. Using the representation $J = ig_{ab}dz^a \wedge d\bar{z}^b$ in the Kähler manifold, the primitivity condition $J \wedge G = 0$ implies

$$g^{ab}G_{abcd} = 0 \quad (38)$$

This is one of the necessary conditions for supersymmetry, the others being

$$G_{abcd} = G_{abcd} = 0 \quad (39)$$

In this work, we are interested in $G$-instantons which are dual to heterotic instantons. Expanding $G$ in terms of the exceptional divisors as in (4), the self-duality of $G$ in (37) translates into the (anti-)self-duality

$$\ast_B F = \pm F \quad (40)$$

in $B$, since $E_i$ are (anti-)self-dual in K3. The K3 manifold has signature $(19, 3)$, meaning that there are 19 self-dual and 3 anti-self-dual two forms $E_i$. In particular, all Cartan subalgebra elements of $E_8 \times E_8$ belong to the self-dual part. The self-dual equation of $F$ translates to the Hermitian Yang–Mills (HYM) equation in the heterotic string [62],

$$g^{ab}F_{ab} = 0 \quad (41)$$

along the Cartan directions. Meanwhile, the last equation of (39) becomes

$$F_{ab} = 0 \quad (42)$$

and states that the gauge field is holomorphic. Ten-dimensional supersymmetry imposes these conditions on a general gauge field—the instanton solution is a special case of this.

Since the heterotic string does not distinguish between the base $B$ and the fiber $E$ directions, we may consider an instanton orientated along a different subspace, with a self-duality of the form

$$\ast_C F = \pm F \quad (43)$$

where we may take an element $C_a \in H^2(B, \mathbb{Z})$ such that the pullback of its Poincaré dual is a four-cycle, $\pi_1^a C_a \in H_4(X, \mathbb{Z})$. The fiber $E$ is not accessible in the F-theory side, so this does not appear as a $G$-instanton.

3.3 Horizontal brane

There is a transition between small instantons and 5-branes, which corresponds to blowing-up in the base $B'$ of an elliptic fibration of $Y$ on the F-theory side. If the components of $F \wedge F$ lie on the elliptic fiber,

$$\frac{1}{2h' g} \text{Tr} F \wedge F|_{\pi_1^a C_a} = \frac{1}{2h' g} \text{Tr} F' \wedge F'|_{\pi_1^a C_a} = \sum_{\text{horizontal}} \delta(\sigma C_a) \quad (44)$$
then the corresponding instanton shrinks to a point and the resulting 5-brane becomes transverse to the elliptic fiber. This is known as a horizontal brane and wraps a four-cycle $\pi''_h C^4$. Recall that in our notation, $C_4$ is the section of a curve in the homology, and it is Poincaré dual to a four-form. From (34), it is natural to denote

$$\delta^{(4)}(\sigma C^a) = \sigma C^a.$$  

(45)

It is known [26] that the horizontal instantons are ‘conformal matter’ [63] localized at the intersections between discriminant locus components, when the base manifold is the $\mathbb{P}^1$ fiber bundle $\pi'': B' \to B$. The discriminant lies on the class $D$, given by adjunction as

$$D = 12c_1(B') = 12c_1 + 24r + 12t,$$

where $r$ is the section of the above fibration $\pi''$. Since we have two order ten singularities $\Pi^+\Pi^+$ for $E_8$ at $r$ and $r + t$, respectively, we may decompose the discriminant locus into irreducible components. The remaining one becomes a horizontal ‘instanton locus’ [26]

$$D_{\text{hor}} = D - 10r - 10(r + t) = 12c_1 + 4r + 2t.$$

That is, we have $E_8$ instantons when this surface intersects the discriminant locus component at $r = 0$ and $r = \infty$, respectively,

$$D_{\text{hor}}|_r = (12c_1 + 2r)r = 2(6c_1 - t)r, \quad D_{\text{hor}}|_{r+t} = 2(6c_1 + t)(r + t),$$

(46)

where we have used the property $r(r + t) = 0$. This hypersurface is six dimensional and mapped to a cycle on which heterotic 5-branes are wrapped [26,64]. The branes are horizontal. The extra factor 2 reflects the fact that the singularity is of Kodaira type II: the discriminant is the double zero of the corresponding coefficient of the Weierstrass equation, i.e., $\Delta \sim (g_{6c_1 + r}z^3)^2$ [26]. From this, we identify

$$\eta_1 = \pi''(\frac{1}{2}D_{\text{hor}}|_r) = 6c_1 - t, \quad \eta_2 = \pi''(\frac{1}{2}D_{\text{hor}}|_{r+t}) = 6c_1 + t,$$

(47)

where $\pi''(r) = 1$. Deformation of the singularity corresponds to the addition of more terms in $f$ and $g$ in (12), however the coefficient of $g_{6c_1 + r}z^3$ in $g$, and hence the number of embedded instantons, remains the same.

For the above choice (47), we do not need horizontal 5-branes because the coefficients of $\sigma$ in $c_2(X)$ and $c_2(V_1) + c_2(V_2)$ are the same, that is, all instantons are contained in $V_1$ and $V_2$. By a phase transition some of the instantons can be emitted as horizontal branes. The above process can be rephrased as

$$\eta \sigma = (6c_1 - t)\sigma = \frac{1}{2h_g} \text{Tr} F \wedge F \wedge F$$

$$= \frac{1}{2h'_g} \text{Tr} F' \wedge F' \wedge F' + C^a \sigma = \eta' \sigma + C^a \sigma.$$  

(48)

In this case, the duality is well known [26,41,65–67].

4 **Gauge symmetry breaking via $G$-instanton transitions**

Finally, we study the transition between $G$-instantons and D3-branes using the dual transition between heterotic instantons and vertical 5-branes. In the process, we break the gauge symmetry dynamically. A quantitative understanding of this process comes from group theory.

4.1 **Vertical brane**

In the previous section, we have seen that there is a transition between small instantons and horizontal 5-branes. From the viewpoint of the heterotic string, there is no difference between the horizontal and vertical directions. That is, there should also be small-instanton transitions along the base directions. The component of $F \wedge F|_B$ transverse to the elliptic fiber $E$ shrinks to a point and undergoes a phase transition,

$$\frac{1}{2h_g} \text{Tr} F \wedge F|_B = \frac{1}{2h'_g} \text{Tr} F' \wedge F'|_B + \sum_{\text{vertical}} \delta^{(4)}(E).$$  

(49)

The new vector bundle $V'$, whose related quantities are primed, belongs to a smaller structure group than the original bundle $V$. The resulting 5-brane is a vertical brane that wraps the fiber $E$. 

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From the orthonormality of the basis (33), a four-form becomes a Dirac delta function in the base $B$, characterizing the brane wrapping $E$. Its location in $B$ is a point at the intersection of two curves $C_a$ and $C^a$. Since only the homology class of this point matters, we loosely denote the delta function in (49) in the following sense:

$$\int_{M^4} d^4x \delta^{(4)}(E) = \int_{M^4} C_a \wedge C^a,$$

where $M^4$ is a real four-dimensional hyperplane. If the curve $C_a$ is self-dual like the self-dual field $F$,

$$C^a = *_B C^a = C_a,$$

then the above phase transition can be explained by a transition of the form

$$\frac{G}{2\pi} = \frac{G'}{2\pi} + P_a \wedge C_a.$$

Here, $P_a$ is a linear combination of exceptional divisors $E_i$ related to the simple roots $\alpha^{(i)} \in \Phi_8(h)$ of the unbroken gauge group $h$,

$$P_a = \sum_i c_a^i E_i, \quad c_a^i \in \mathbb{Z}.$$

The coefficients are integer in general and determined by the quantization condition (24). As long as we choose $E_i$ orthogonal to the structure group of $G$ such that

$$\forall c_a^i \neq 0, \quad \alpha^{(i)} \in \Phi_8(h) \implies G' \cdot E_i \cdot D_a = 0,$$

we have no cross-term in the expansion

$$\frac{1}{8\pi^2} G \wedge G = \frac{1}{8\pi^2} G' \wedge G' + \frac{1}{2} \pi^* C_a \wedge \pi^* C_a \wedge P_a \wedge P_a.$$

This $P_a$ has one-to-one correspondence with the weight vector $c_a = (c_a^i)$.

The delta function describes a heterotic 5-brane wrapped on $T^2$, where the wrapped volume has been factored out. It is mapped to a D3-brane on the F-theory side,

$$\delta^{(8)}(x - x_a) = \frac{1}{2} P_a \wedge P_a \wedge \pi^* C_a \wedge \pi^* C_a.$$

The normalization factor $1/2$ is set by the relation (5). Some part of the $G$-flux may be emitted and become D3-branes, described by the delta function in (3). During the transition the total D3-charge is conserved, so the relation (3) remains satisfied.

We can see that the number of heterotic 5-branes is always an integer. This corresponds to the diagonal elements of the Cartan matrix, which denote the Cartan subalgebra. Since we obtained $e_i$ and $E_i$ from the $H^2$ basis of the K3, for $E_8$ and its subgroup, the number of instantons is guaranteed to be an integer because the product

$$-\frac{1}{2} \int_{K3} P_a \wedge P_a = -\frac{1}{2} \int_{K3} \left( \sum_i (c_a^i)^2 E_i \wedge E_i + 2 \sum_{i<j} c_a^i c_a^j E_i \wedge E_j \right)$$

$$= \sum_i (c_a^i)^2 - \sum_{i<j} c_a^i c_a^j A_{ij}$$

is always an integer if $c_a^i$ are integrally quantized as in (53). Hence, we see that the instantonic $G$-flux is integrally quantized, so the number of D3-branes should be an integer. Thus, the quantitative aspect of the small-instanton transition is guided by the fact that the $E_8$ lattice is even and the $E_8$ algebra is encoded in the intersection structure of K3 through the expansion (5).

### 4.2 $G$-instantons breaking gauge symmetry

We wish to use the $G$-instanton transition, which comes from the phase transition of D3-branes, to explain dynamical transitions among vacua. We may break the gauge symmetry using $G$ by relaxing some of the conditions (54) [43,45,68]. Let $G^\text{h}$ the $h$-preserving flux consistent with the quantization condition. We have

$$G = G^\text{h} + G^\text{inst},$$

where

$$\frac{G^\text{inst}}{2\pi} = \sum_i a_i A_{(i)} \wedge F.$$
The two-form $\mathcal{F}$ satisfies the anti-self-duality condition in $B$,

$$^*_{B} \mathcal{F} = -\mathcal{F}. \quad (60)$$

Among the two sign possibilities in (40), we choose the minus sign in order to ensure that the resulting D3-brane energy density is positive definite. This is the same form as (52) because the first factor $\Lambda_{(i)}$ is expressed in terms of the exceptional divisors $E_i$, while the second factor corresponds to base divisors. One may verify that this $G_{\text{inst}}$ satisfies the conditions (25) and (26). The coefficient is fixed because both $G$ and $G^h$ satisfy the quantization condition (24).

Since the divisor $P_a$ of (53) is expanded in terms of the cycles $E_i$ which satisfy the McKay correspondence with the roots $\alpha^{(i)}$ of the Lie algebra, we can use this to control the symmetry breaking direction. Hence, we introduce fundamental weight divisors,

$$\Lambda_{(i)} = (A^{-1})^{ij} E_j, \quad (61)$$

using the inverse $(A^{-1})^{ij}$ of the Cartan matrix (6). By construction they satisfy the orthonormality condition

$$\Lambda_{(i)} \cdot E_j \cdot D_a \cdot D^b = -\delta^{ij}_{a b}. \quad (62)$$

This basis requires a stronger quantization condition for the coefficients $a_i$ (59) because the inverse Cartan matrix $A^{-1}$ has fractional entries, so a larger multiplicity is needed to ensure that all $E_i$ cycles are wrapped an integer number of times.

The induced D3-brane charge is given by

$$\frac{1}{8\pi^2} \int_B G \wedge G = \frac{1}{8\pi^2} (G^h + G_{\text{inst}}) \cdot (G^h + G_{\text{inst}}) = \frac{1}{8\pi^2} (G^h)^2 + \frac{1}{8\pi^2} (G_{\text{inst}})^2. \quad (63)$$

Owing to the relation (30), there is no cross-term, since

$$G^h \cdot G_{\text{inst}} = G^h \cdot \sum a_i \Lambda_{(i)} \cdot \mathcal{F} = 0, \quad (64)$$

where in the second equality we have used the fact that $\Lambda^{(i)}$ is a linear combination of exceptional divisors $E_i$. Hence, we see that this provides a good description of the phase transition between $G$-flux and D3-branes.

We now apply the transition between $G$-flux and D3-branes to gauge symmetry breaking. The part $\frac{1}{2}G_{\text{inst}} \cdot G_{\text{inst}}$ becomes a delta function and describes the location of the D3-branes after the phase transition. Explicitly, we have

$$\frac{1}{8\pi^2} (G_{\text{inst}})^2 = \frac{1}{2} \left( \sum a_i \Lambda_{(i)} \right)^2 \cdot \mathcal{F} \cdot \mathcal{F} = -\frac{1}{2} \sum_{i,j} a_i a_j (A^{-1})^{ij} \int_B \mathcal{F} \wedge \mathcal{F}. \quad (65)$$

We see that the number of D3-branes emitted is determined by elements of the inverse Cartan matrix of $h$.

We also wish to translate the above picture to the heterotic side. The instanton flux $G_{\text{inst}}$ is of the form (59) which is a simple wedge product of exceptional and base divisors, so it can simply be expanded in Chevalley basis. The $h$-preserving flux is more subtle: it corresponds to shift in the second Chern class of the gauge bundle $c_2(V_1)$. However, this ‘traceless’ component cannot be seen from the base geometry, as it is projected out [47]. Nevertheless, we can trace the contribution for the heterotic 5-branes.

The change of the heterotic flux is, using the reduction (11),

$$\frac{1}{2h^2} \int_B \text{Tr} F_{\text{inst}} \wedge F_{\text{inst}} = -\frac{1}{2} \sum_{i,j} a_i a_j (A^{-1})^{ij} \int_B (A^{-1})^{ij} \mathcal{F} \wedge \mathcal{F} \quad (66)$$

This is the number of heterotic 5-branes after the transition. Note that this agrees with direct calculation of the number of D3-branes in the F-theory side.

After emission of the flux, the $G$-flux can become ‘smaller,’ and thus the unbroken gauge group becomes smaller. Therefore, many vacua having different gauge groups and spectra can be connected. For example, starting from $SU(5)$ we may break the gauge group down to its subgroups. We may summarize this data as weight vectors in the Dynkin basis.
4.3 Example

As an example, we construct an $SU(5)$ surface with blowups and consider the phase transition that reduces the unbroken gauge group to $SU(4) \times U(1)$ or $SU(3) \times SU(2) \times U(1)$. Here, we take the former as an example, since the latter is well known.

The exceptional divisors $E_1, E_2, E_3, E_4$ satisfy the correspondence (27), whose relations are summarized by the Cartan matrix

$$A^{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

We adopt the resolution used in [45,70] where we take $E_2 \cdot E_4$ to be invariant.

The $SU(5)$-preserving flux is obtained from the solution

$$G^{SU(5)} \cdot E_i \cdot D_{a_i} = 0, \quad i = 1, 2, 3, 4, \quad a^{(i)} \in \Phi_4(SU(5)).$$

These conditions can be imposed on a $G$-flux ansatz satisfying the quantization conditions (24) and verticality conditions (25) and (26), namely

$$\frac{G}{2\pi} = \sum_{i=1}^{4} E_i \cdot (a_i c_1(B') + b_i B) + \frac{1}{2}(E_2 + E_3) \cdot c_1(B') + \left(p + \frac{1}{2}\right) E_2 \cdot E_4,$$

where $B$ is the section of the two-base in $Y$ and flux quantization restricts $a_i, b_i, p \in \mathbb{Z}$. Imposing $SU(5)$-invariance (67) leads to a unique $G$-flux [45,52],

$$G^{SU(5)} = \lambda \left(E_2 \cdot E_4 + \frac{1}{5}(2E_1 - E_2 + E_3 - 2E_4) \cdot c_1(B')\right) \equiv \lambda \, \hat{G}^{SU(5)}.$$

where the quantization condition on $\lambda$ can be expressed as

$$\lambda = -\frac{5}{2}(1 + 2a_i), \quad a_i \equiv a_3 - a_1 \in \mathbb{Z}.$$ (70)

If we relax (67) by requiring only three of the four conditions, we may introduce additional flux which spontaneously breaks $SU(5)$ to a subgroup, either $SU(3) \times SU(2)$ or $SU(4)$ depending on the breaking direction. Relaxing the $i$th condition of (67) and imposing the remaining constraints on the ansatz (68) yields expressions of the form

$$G^{(i)}_{\text{tot}} = G^{SU(5)} + G^{(i)};$$

$$G^{(i)}_{\text{tot}} = a^{(i)} \left(\Lambda^{(i)} \cdot c_1(B') + \frac{G^{SU(5)} + b^{(i)} \Lambda^{(i)} \cdot B}{2\pi}\right) + b^{(i)} \Lambda^{(i)} \cdot B,$$

where $a^{(i)}$ and $b^{(i)}$ are new parameters constructed from linear combinations of the $a_i$s and $b_i$s, respectively, while explicit calculation gives $a^{(i)} = (-2, 1, 4, 2)^t$. Here, we have also introduced the divisor related to fundamental weights as in (61), where the inverse Cartan matrix for $SU(5)$ is defined as

$$(A^{-1})^{ij} = \frac{1}{5} \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$ (72)

Flux quantization requires $a_i, b_i \in \mathbb{Z}$, which in turn fixes $b^{(i)} \in 5\mathbb{Z}$. This can be seen heuristically from the fact that a factor of 5 is required to cancel the 1/5 in $\Lambda^{(i)}$ to recover an integral cycle. However, the quantization of $a^{(i)}$ is more subtle due to the $SU(5)$-invariant contribution $G^{SU(5)}$. Quantization of the total flux in practice only requires it to be either an integer or odd half-integer. Only if we require each term in (71) to be quantized separately is $a^{(i)}$ further constrained to be a multiple of five. If this is the case, the resulting flux can be simplified to the form

$$G^{(i)}_{\text{tot}} = \frac{G^{SU(5)} + 5\Lambda^{(i)} \cdot (ac_1(B') + bB)}{2\pi},$$ (73)

$^4$ A $U(1)$ gauge group may also be preserved, unless its gauge boson acquires a Stückelberg mass. Its condition is topological: the dual divisor corresponding to $U(1)$ should be trivial in the base threefold [31,45]. Since it depends on the generous geometry of the Calabi–Yau fourfold in F-theory, this mechanism has no dual in the heterotic side. An explicit construction for such $G$-flux in native F-theory is given [69].
where \(a^{(i)} \equiv 5a, b^{(i)} \equiv 5b\), and \(\lambda'\) is defined such that

\[
a_{ij} = \frac{a_{ij}}{n^{(i)}}a
\]

(74)

for each \(i\). Note that while \(a\) may not always be an integer, the combination \(n^{(i)}a\) is always integral. Finally, if we require \(G^{(i)}\) to be self-dual, given that the exceptional divisors \(E_i\) are self-dual on the K3, the corresponding base divisor on \(B\) must be self-dual. However, this is generally not the case for \((\text{the pullback of})\ c_1(B')\), so for an instanton transition we may restrict

\[
a = 0.
\]

(75)

Thus, \(\lambda\) cannot change during an instanton transition.

Finally, note that the ansatz (68) may be too restrictive, since it uses a basis of fluxes specifically designed to cancel \(SU(5)\) breaking terms: in the conditions (67), double and triple intersections of the \(E_i\)s can always be reduced to terms proportional to \(c_1(B')\) or \(B\) only. Hence, if we relax any of the conditions (67), further contributions of the form \(c^{ai}E_i \cdot D_a\) may be allowed, with \(D_a\) the pullback of base divisors and \(c^{ai} \in \mathbb{Z}\). By the same arguments, and taking into account (75), the most general flux breaking \(SU(5)\) along the \(i\)th Cartan direction takes the form

\[
\frac{G^{(i)}}{2\pi} = 5\Lambda_{(i)} \wedge \sum a c^a C_a,
\]

where \(c^a \in \mathbb{Z}\) and we have expanded the discriminant locus \(B\) in the basis \(C_a\).

To construct a self-dual flux, we require that the base divisors \(C_a\) are self-dual. The \(C_a\) divisors form a self-dual orthonormal basis in the cohomology \(H^2(B, \mathbb{Z})\), i.e.,

\[
\int C^a \wedge C_b = -\delta^a_b,
\]

(77)

where \(C^a = -C_a\) for anti-self-dual divisors. If we consider a single \(C_a\) component, the resulting instanton contribution is then given by

\[
\frac{1}{8\pi^2} \int G^{(i)} \wedge G^{(i)} = \frac{25}{2} (A^{-1})^{ii} c^{ai} c^{ai}.
\]

(78)

For example, taking the breaking direction as the fourth direction,

\[
\frac{G^{(i)}}{2\pi} = 5\Lambda_{(4)} \wedge c^a D_a = (E_1 + 2E_2 + 3E_3 + 4E_4) \wedge c^a D_a,
\]

(79)

the instanton contribution for \(SU(4)\) invariance is

\[
\frac{1}{8\pi^2} \int G^{(4)} \wedge G^{(4)} = 10 c^{a} c^{a}.
\]

(80)

Similarly, if we choose \(i = 3\) in order to preserve \(SU(3) \times SU(2)\), the instanton becomes

\[
\frac{1}{8\pi^2} \int G^{(3)} \wedge G^{(3)} = 15 c^{a} c^{a}.
\]

(81)

Here, we see that 10 D3-branes are absorbed to break to \(SU(4)\), while 15 are needed to reach \(SU(3) \times SU(2)\).

We can generalize this result to the \(SU(n)\) case: \(G\)-flux of the form

\[
\frac{G^{(i)}}{2\pi} = n\Lambda_{(i)} \wedge \sum c^a C_a
\]

(82)

spontaneously breaks \(SU(n)\) along the \(i\)th Cartan direction. The minimal instanton number is then

\[
\frac{1}{8\pi^2} \int G^{(i)} \wedge G^{(i)} = \frac{n^2}{2} (A^{-1})^{ii} = n \sum_{j=1}^{n-1} (A^{-1})^{ij},
\]

(83)

where in the second equality we have used the group theory identity

\[
\sum_{j=1}^{n-1} (A^{-1})^{ij} = \frac{n}{2} (A^{-1})^{ii}.
\]

(84)

Therefore, the instanton number required to spontaneously break \(SU(n)\) in the \(i\)th Cartan direction is just the sum of the integer coefficients appearing in the \(i\)th row of the inverse Cartan matrix. For other gauge groups this is not the case, however the number of instantons will always be an integer multiple of the diagonal elements \((A^{-1})^{ii}\).
4.4 Anomaly freedom is preserved

The phase transition of D3-branes into G-flux gives rise to further gauge symmetry breaking. It is continuous deformation preserving the global consistency condition (3). Spontaneous symmetry breaking takes one anomaly-free vacuum into another. Thus, we expect that the process preserves anomaly freedom.

We verify this for the above SU(5) example. Four-dimensional gauge anomaly cancellation requires a special combination of matter fields. Although we have as many matter curves as the dimension of the representation, each one gives the same intersection number. Thus, we may choose the highest weight representations and construct the matter surfaces $\mathcal{S}_F$ and $\mathcal{S}_{10}$ as in the following table.

| Reprs. | Highest weight | Matter surfaces |
|--------|----------------|-----------------|
| 5      | $\mu_5 = [0, 0, 0, 1]$ | $\mathcal{S}_F \equiv E_2 \cdot (3c_1(B') + \sigma - E_1 - 2E_2 - 3E_3 - 2E_4) - (8c_1(B') - 5B) \cdot (E_1 + E_2)$ |
| 10     | $\mu_{10} = [0, 1, 0, 0]$ | $\mathcal{S}_{10} \equiv E_2 \cdot E_4 + (E_1 + E_2 + E_3) \cdot c_1(B')$ |

Bold refers to representations of non-Abelian gauge groups labelled by their dimension.

Accordingly, the chirality is

$$\chi(\bar{5}) = \int_{\mathcal{S}_F} G^{SU(5)}_\lambda = 5\lambda \eta \cdot (\eta - 5B),$$

and

$$\chi(10) = \int_{\mathcal{S}_{10}} G^{SU(5)}_\lambda = 5\lambda \eta \cdot (\eta - 5B).$$

Thus, we have an anomaly-free theory, since

$$-\chi(\bar{5}) + \chi(10) = 0.$$  \hspace{1cm} (87)

Now assume that a number of D3-branes become additional flux, $G^{(4)} = A^{(4)} \wedge \mathcal{F}$, as in (79). The gauge symmetry is broken to $SU(4) \times U(1)$ and the vector multiplet branches as

$$24 \rightarrow 15_1 + 4_{-5} + \bar{4}_5 + 1_0.$$  \hspace{1cm} (88)

While the diagonal elements remain as vector multiplets, the $4_{-5}$ component becomes chiral. Its multiplicity is, according to (32),

$$\chi(4_{-5}) = \int_{B} c_1 \wedge \mathcal{F},$$

(89)

where $c_1 \equiv c_1(B)$ as before. The matter fields also branch as

$$\bar{5} \rightarrow \bar{4}_1 + 1_{-4},$$

$$10 \rightarrow \bar{6}_{-2} + 4_3.$$

We have the following weight vectors and corresponding matter surfaces.

| Reprs. | Highest weight | Matter surfaces |
|--------|----------------|-----------------|
| $\bar{4}_1$ | $\mu_{\bar{4}}$ | $\mathcal{S}_F$ |
| $1_{-4}$ | $\mu_{1} - a^{(1)} - a^{(2)} - a^{(3)} - a^{(4)}$ | $\mathcal{S}_F + (E_1 + E_2 + E_3 + E_4) \cdot (8c_1(B') - 5B)$ |
| $\bar{6}_{-2}$ | $\mu_{\bar{6}}$ | $\mathcal{S}_{10}$ |
| $4_3$ | $\mu_{4} - a^{(2)} - a^{(3)} - a^{(4)}$ | $\mathcal{S}_{10} - (E_2 + E_3 + E_4) \cdot c_1(B')$ |

Bold refers to representations of non-Abelian gauge groups labelled by their dimension.

Note that the first term of each matter surface is that of the mother representation of the unified group, since all of them belong to the same highest weight module.

We obtain the chirality for the representation $\bar{4}_1$ as

$$\chi(\bar{4}_1) = \int_{\mathcal{S}_F} G^{(4)}_{\text{tot}} = \mathcal{S}_F \cdot G_{\lambda} + \mathcal{S}_F \cdot A^{(4)} \cdot \mathcal{F}$$

$$= \chi(\bar{5}) + \frac{1}{5} \int_{B} (8c_1(B') - 5B) \wedge \mathcal{F}$$

$$= \chi(\bar{5}) + \frac{1}{5} \int_{B} (8c_1 - 3t) \wedge \mathcal{F}. \hspace{1cm} (90)$$
Thus, with the contribution from the bulk fermion, the Wilson contribution is proportional to the $L_{\alpha}(\lambda)$ centroid, such that the additional term is proportional to the $U(1)$ charge. Likewise, we get the matter surfaces as follows.

\[
\chi(1_{-4}) = \chi(\bar{5}) + \frac{1}{5} \int_B (8c_1 - 3t) \wedge F - \frac{4}{5} \int_B (8c_1 - 3t) \wedge F,
\]

\[
= \chi(\bar{5}) - \frac{4}{5} \int_B (8c_1 - 3t) \wedge F, \tag{91}
\]

such that the additional term is proportional to the $U(1)$ charge. Likewise,

\[
\chi(6_{-2}) = \chi(10) - \frac{2}{5} \int_B c_1(B') \wedge F = \chi(10) - \frac{2}{5} \int_B (c_1 - t) \wedge F,
\]

\[
\chi(4_{3}) = \chi(10) + \frac{3}{5} \int_B (c_1 - t) \wedge F.
\]

Thus, with the contribution from the bulk fermion $4_{-5}$ in (89), we may verify that there is no anomaly in the $SU(4)$ vacuum. Since the representations $6$ and $1$ are real and do not take part in the anomaly, we obtain

\[
- \chi(\bar{4}) + \chi(4) + \chi(4_{-5}) = -\chi(\bar{5}) - \frac{1}{5} \int_B (8c_1 - 3t) \wedge F + \chi(10) + \frac{3}{5} \int_B (c_1 - t) \wedge F + \int_B c_1 \wedge F = 0,
\]

using the relation (87).

We may also consider $SU(5)$ breaking into $SU(3) \times SU(2) \times U(1)$ under the phase transition induced by $G^{(3)} = \Lambda^{(3)} \wedge F$. The $SU(5)$ representations branch as

\[
24 \rightarrow (8, 1)_0 + (1, 3)_0 + (1, 1)_0 + (3, 2)_{-5/6} + (\bar{3}, 2)_{5/6},
\]

\[
\bar{5} \rightarrow (\bar{3}, 1)_{1/3} + (1, 2)_{-1/2},
\]

\[
10 \rightarrow (3, 2)_{1/6} + (\bar{3}, 1)_{-2/3} + (1, 1)_1.
\]

The matter surfaces are as follows.

| Reprs. | Highest weight | Matter surfaces |
|-------|---------------|-----------------|
| $(\bar{3}, 1)_{1/3}$ | $\mu_{\bar{5}}$ | $\bar{5}$ |
| $(1, 2)_{-1/2}$ | $\mu_{\bar{5}} - \alpha^{(1)} - \alpha^{(2)} - \alpha^{(3)}$ | $\bar{5}$ + $(E_1 + E_2 + E_3) \cdot (8c_1(B') - 5B)$ |
| $(\bar{3}, 1)_{-2/3}$ | $\mu_{10}$ | $\bar{5}$ |
| $(3, 2)_{1/6}$ | $\mu_{10} - \alpha^{(1)} - \alpha^{(2)} - \alpha^{(3)}$ | $\bar{5}$ |
| $(1, 1)_1$ | $\mu_{10} - \alpha^{(1)} - 2\alpha^{(2)} - 2\alpha^{(3)} - \alpha^{(4)}$ | $\bar{5}$ + $(E_1 + E_2 + E_3) \cdot c_1(B')$ |

Bold refers to representations of non-Abelian gauge groups labelled by their dimension.

Thus, the chiralities are

\[
\chi((\bar{3}, 2)_{-5/6}) = \int_B c_1 \wedge F,
\]

\[
\chi((\bar{3}, 1)_{1/3}) = \chi(\bar{5}) + \frac{2}{5} \int_B (8c_1 - 3t) \wedge F,
\]

\[
\chi((1, 2)_{-1/2}) = \chi(\bar{5}) - \frac{3}{5} \int_B (8c_1 - 3t) \wedge F,
\]

\[
\chi((\bar{3}, 1)_{-2/3}) = \chi(10) - \frac{4}{5} \int_B (c_1 - t) \wedge F,
\]

\[
\chi((3, 2)_{1/6}) = \chi(10) + \frac{1}{5} \int_B (c_1 - t) \wedge F,
\]

\[
\chi((1, 1)_1) = \chi(10) + \frac{6}{5} \int_B (c_1 - t) \wedge F.
\]
The total $SU(3)$ anomaly is therefore

$$-\chi((\bar{3}, 1)_{1/3}) - \chi((\bar{3}, 1)_{-2/3}) + 2\chi((3, 2)_{1/6}) + 2\chi((3, 2)_{-5/6})$$

$$= -\chi(\bar{5}) - \frac{2}{5} \int_B (8c_1 - 3t) \wedge \mathcal{F} - \chi(10) + \frac{4}{5} \int_B (c_1 - t) \wedge \mathcal{F}$$

$$+ 2\chi(10) + 2 \cdot \frac{1}{5} \int_B (c_1 - t) \wedge \mathcal{F} + 2 \int_B c_1 \wedge \mathcal{F}$$

$$= 0. \quad (93)$$

Again, we find a theory with highly nontrivial anomaly cancellation.

4.5 Comments on the moduli space

Lastly, we make brief comments on the moduli space of small instantons. The dual F-theory description is useful in understanding the moduli space of instantons on the heterotic side. Vector bundles in the heterotic description are mapped to spectral covers and a spectral line bundle thereon, via a Fourier–Mukai transform $[31,54]$. The spectral covers are generalized to stacks of 7-branes supporting worldvolume gauge theories and their intersection structure determines the localized matter field content.

During the transition between small instantons and D3-branes, the 7-brane configuration on the F-theory side remains unchanged $[23,71]$. The D3-branes, dual to heterotic 5-branes, are pointlike and do not affect the configuration of the 7-branes at leading order. When they dissolve into the 7-branes on the discriminant locus, they become a background gauge bundle $M$ on the base $B$. This can be seen on the heterotic side, on which they become the vector bundle

$$V = V' \oplus \pi^* M, \quad (94)$$

now provided by the dissolved heterotic 5-branes $[23]$. In other words, the small instantons from the 5-branes are horizontal. Thus, the moduli space of the small instantons can be studied via the D3–7-brane system.

The study of T-branes teaches us that, in general, the instanton configuration is not clearly separated from that of the 7-branes $[59–61]$. The BPS condition arises again in the form of the HYM equation of the holomorphic vector bundle. In the presence of the adjoint Higgs field $\Phi$, the HYM equation (41) generalizes to

$$g^{\bar{a}b} F_{\bar{a}b} + \frac{i}{2} [\Phi^\dagger, \Phi] = 0, \quad (95)$$

with the covariantly holomorphic Higgs field

$$D_\xi \Phi = 0, \quad (96)$$

where $D_\xi$ is the anti-holomorphic covariant derivative with respect to the background gauge field.

If the 7-brane configuration parameterized by the Higgs field $\Phi$ is diagonal, it reduces to the usual HYM equation. However, if it is of T-brane type—that is, containing off-diagonal elements—then we cannot separate the instanton configuration from that of the 7-branes: the gauge field is no longer a general local pure gauge field, but is related to a covariantly holomorphic Higgs field through Eq. (95) $[59]$. The resulting moduli space of the instanton bundle is extended and unified with that of the Higgs bundle.

It is interesting to consider whether this backreaction of 7-branes is similar to Born–Infeltons (BIons), arising from the interaction between D$p$ and D$(p + 2)$ branes $[72]$. In the case of $SU(n)$ gauge theory on D7-branes, the next-to-leading order terms of the Dirac–Born–Infeld action give rise to a backreaction potential proportional to $[\Phi^\dagger, \Phi]^2$.

5 Discussion

In this work, we studied the transition between D3-branes and small $G$-instantons. This phenomenon is dual to the transition between (vertical) 5-branes and small instantons in the heterotic string. We can understand this phenomenon quantitatively in F-theory. The non-Abelian structure of $G$-flux is nontrivial and we may obtain selection rules for such transitions in terms of group theory. Furthermore, we have seen explicitly that, since the transition preserves the global consistency condition (3), it does not give rise to any anomaly.

One mysterious feature of this process is the role of D3-branes in the global consistency condition. Nevertheless, we may understand how D3-branes contribute indirectly to the chirality. Their conversion to $G$-flux may affect the chirality locally, but the global sum of chiralities measured by the anomaly remains zero.

Extrapolating this transition mechanism, we may infer that many different-looking vacua are in fact connected. Since the phase transition of small $G$-instantons can change the gauge symmetry breaking direction, we may apply the transition to a phenomenological model in which we arrive at the Standard Model gauge group.
A big question is whether all vacua are connected via this mechanism. As we have seen in Sect. 4.3, not all possible $G$-fluxes satisfying the consistency conditions can be obtained from the transition from D3-branes. In the $SU(5)$ example, there is an irreducible component that cannot take the form of (54). Moreover, there are selection rules for the transition because the non-Abelian structure of $G$-flux is nontrivial. It would be interesting to further understand this contribution and its relation to the allowed space of consistent vacua.

Another interesting application would be to introduce non-perturbative effects in the Kähler and superpotentials and stabilize the vacuum. In this case, the dynamics of vacuum transitions will again be guided by group theory. In particular, whether or not the Standard Model vacuum is favored is a crucial question.

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References

1. P. Candelas, G.T. Horowitz, A. Strominger, E. Witten, Vacuum configurations for superstrings. Nucl. Phys. B 258, 46 (1985). https://doi.org/10.1016/0550-3213(85)90602-9
2. L.J. Dixon, J.A. Harvey, C. Vafa, E. Witten, Strings on orbifolds. Nucl. Phys. B 261, 678 (1985). https://doi.org/10.1016/0550-3213(85)90593-0
3. L.J. Dixon, J.A. Harvey, C. Vafa, E. Witten, Strings on orbifolds. 2. Nucl. Phys. B 274, 285 (1986). https://doi.org/10.1016/0550-3213(86)90287-7
4. M. Bershadsky, A. Sagnotti, Phys. Rept. 371 (2002) 1; Erratum: [Phys. Rept. 376 (2003) no.6, 407]. https://doi.org/10.1016/S0370-1573(02)00273-9
5. 54. M. Schellekens, N.P. Warner, Phys. Lett. B 177, 317 (1986). https://doi.org/10.1016/0370-2693(86)90760-4
6. A. Klemm, B. Lian, S.S. Roan, S.T. Yau, Nucl. Phys. B 507, 445–474 (1997). https://doi.org/10.1016/S0550-3213(97)00563-4
7. 480. M. Cvetic, T.W. Grimm, T.W. Ha, A. Klemm, D. Klevers, Nucl. Phys. B 703 (1985). https://doi.org/10.1016/0550-3213(85)90593-0
8. 317. J. Fuchs, C. Schweigert, Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists
9. 219. M. Berthold, K. Schubert, Nucl. Phys. B 655, 219 (2003) no.6, 407 https://doi.org/10.1016/S0550-3213(02)00273-9
45. J. Marsano, S. Schafer-Nameki, JHEP 1111, 098 (2011). https://doi.org/10.1007/JHEP11(2011)098 [arXiv:1108.1794 [hep-th]]

46. R. Blumenhagen, T.W. Grimm, B. Jurke, T. Weigand, Nucl. Phys. B 829, 325 (2010). https://doi.org/10.1016/j.nuclphysb.2009.12.013 [arXiv:0908.1784 [hep-th]]

47. R. Friedman, J. Morgan, E. Witten, Commun. Math. Phys. 187, 679 (1997). https://doi.org/10.1007/s002200050154 [hep-th/9701162]

48. B. Andreas, G. Curio, Three-branes and five-branes in N = 1 dual string pairs. Nucl. Phys. B 417, 41 (1998). https://doi.org/10.1016/S0370-2693(97)01342-7

49. E. Witten, J. Geom. Phys. 22, 1 (1997). https://doi.org/10.1016/S0393-0440(96)00042-3 [hep-th/9609122]

50. T.W. Grimm, R. Savelli, Phys. Rev. D 85, 026003 (2012). https://doi.org/10.1103/PhysRevD.85.026003 [arXiv:1109.3191 [hep-th]]

51. T.W. Grimm, H. Hayashi, JHEP 03, 027 (2012). https://doi.org/10.1007/JHEP03(2012)027 [arXiv:1111.1232 [hep-th]]

52. T.W. Grimm, R. Savelli, Phys. Rev. D 88, 096002 (2013). https://doi.org/10.1103/PhysRevD.88.096002 [arXiv:1302.2862 [hep-th]]

53. S. Krause, C. Mayrhofer, T. Weigand, Nucl. Phys. B 858, 1 (2012). https://doi.org/10.1016/j.nuclphysb.2011.12.013 [arXiv:1111.1232 [hep-th]]

54. R. Donagi, M. Wijnholt, Adv. Theor. Math. Phys. 15(5), 1237–1317 (2011). https://doi.org/10.4310/ATMP.2011.v15.n5.a2 [arXiv:1101.0046 [hep-th]]

55. A.P. Braun, A. Collinucci, R. Valandro, Nucl. Phys. B 856, 129–179 (2012). https://doi.org/10.1016/j.nuclphysb.2011.10.034 [arXiv:1107.5337 [hep-th]]

56. C. Beasley, J.J. Heckman, C. Vafa, JHEP 0901, 058 (2009). https://doi.org/10.1088/1126-6708/2009/01/058 [arXiv:0802.3391 [hep-th]]

57. P. Horava, E. Witten, Eleven-dimensional supergravity on a manifold with boundary. Nucl. Phys. B 460, 506 (1996). https://doi.org/10.1016/0550-3213(96)00621-4 [arXiv:1107.5337 [hep-th]]

58. P. Horava, E. Witten, Heterotic and type I string dynamics from eleven-dimensions. Nucl. Phys. B 460, 506 (1996). https://doi.org/10.1016/0550-3213(96)00621-4 [arXiv:1107.5337 [hep-th]]