PDEs for the Gaussian Ensemble with External Source and the Pearcey Distribution

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Abstract

The present paper studies a Gaussian Hermitian random matrix ensemble with external source, given by a fixed diagonal matrix with two eigenvalues ±a. As a first result, the probability that the eigenvalues of the ensemble belong to an interval E satisfies a fourth-order PDE with quartic nonlinearity; the variables are the eigenvalue a and the boundary of E. This equation enables one to find a PDE for the Pearcey distribution. The latter describes the statistics of the eigenvalues near the closure of a gap, i.e., when the support of the equilibrium measure for large-size random matrices has a gap that can be made to close. The Gaussian Hermitian random matrix ensemble with external source, described above, has this feature. The Pearcey distribution is shown to satisfy a fourth-order PDE with cubic nonlinearity. This also gives the PDE for the transition probability of the Pearcey process, a limiting process associated with nonintersecting Brownian motions on \( \mathbb{R} \). © 2006 Wiley Periodicals, Inc.

Brézin and Hikami [10, 11, 12, 13] considered a random Gaussian Hermitian ensemble with external source,

\[
\frac{1}{Z} e^{-\frac{1}{2} \text{Tr}(M-A)^2} dM,
\]

where \( M \) is random and \( A \) is deterministic. Notice that this matrix ensemble, which also appeared in [17, 20, 22], ceases to be unitary invariant. The matrix \( A \) is chosen so that the support of the equilibrium measure has a gap when the size of the random matrices tends to infinity. Through a fine tuning of \( A \), the gap can be made to close at the origin. Brézin and Hikami propose a simple model having this feature, where the matrix \( A \) is diagonal, with two eigenvalues \( a \) and \(-a\) of equal multiplicity. Thus, upon letting the size of the matrix go to \( \infty \) and after appropriate rescaling, they discover a new critical distribution, specified by a kernel involving Pearcey integrals [23] and having universality properties.

P. Zinn-Justin [26, 27] established the determinantal form of the correlation functions for the eigenvalues of the finite model. Then extending classical connections between random matrix theory and nonintersecting random paths (see Dyson...
Grabiner [16], Johansson [18] and Aptekarev, Bleher, and Kuijlaars [6] gave a nonintersecting Brownian motion interpretation of this Gaussian ensemble with external source. In [6] the authors showed that multiple orthogonal polynomials are the right tools for studying this model and its limit (see [6, 7, 8, 9]).

The present paper studies the Gaussian Hermitian random matrix ensemble $\mathcal{H}_n$ with external source $A$, given by the diagonal matrix (set $n = k_1 + k_2$)

$$
A := \begin{pmatrix}
a & & \\
& \ddots & \\
& & a
\end{pmatrix} \oplus \begin{pmatrix}
O \\
& \ddots & \\
& & -a
\end{pmatrix} \oplus k_2
$$

and density

$$
\frac{1}{Z_n} e^{-\text{Tr}(\frac{1}{2}M^2 - AM)} dM.
$$

Given a disjoint union of intervals $E := \bigcup_{i=1}^{r} [b_{2i-1}, b_{2i}] \subset \mathbb{R}$, define the algebra of differential operators generated by

$$
B_k = \sum_{i=1}^{2r} b_{k+1}^{i+1} \frac{\partial}{\partial b_i}.
$$

Consider the following probability:

$$
\mathbb{P}_n(a; E) := \mathbb{P}(\text{all eigenvalues } \in E) = \frac{1}{Z_n} \int_{\mathcal{H}_n(E)} e^{-\text{Tr}(\frac{1}{2}M^2 - AM)} dM,
$$

where $\mathcal{H}_n(E)$ is the set of all Hermitian matrices with all eigenvalues in $E$. In [1], we have shown that for $A = 0$, the probability for this Gaussian Hermitian ensemble (GUE) satisfies a fourth-order PDE with quadratic nonlinearity (reducing to Painlevé IV in the case of one boundary point):

$$
(B_{-1}^4 + (4n + 6B_{-1}^2 \log \mathbb{P}_n)B_{-1}^2 + 3B_0^2 - 4B_1 + 6B_0) \log \mathbb{P}_n = 0.
$$

The first question discussed in this paper: Does the integral (0.4), with $A$ as in (0.1), satisfy a PDE? Indeed, we prove the following:

**Theorem 0.1** The log of the probability $\mathbb{P}_n(a; E)$ satisfies a fourth-order PDE in $a$ and in the endpoints $b_1, \ldots, b_{2r}$ of the set $E$ with quartic nonlinearity. The
The equation can be written in two different ways (in terms of the Wronskians \( \{f, g\}_X = gXf - fXg \)):

\[
0 = (F^+ B_{-1} G^- + F^- B_{-1} G^+)(F^-, F^+)_{B_{-1}} \\
- (F^+ G^- + F^- G^+) B_{-1}(F^-, F^+)_{B_{-1}}
\]

\[
= \det \left( \begin{array}{ccc} -G^+ & B_{-1} F^+ & -F^+ \\
G^- & B_{-1} F^- & -F^- \\
-B_{-1} G^+ & B_{-1}^2 F^+ & 0 & -F^+ \\
B_{-1} G^- & B_{-1}^2 F^- & 0 & -F^- \end{array} \right)
\]

(0.5)

where

\[
F^+ := 2 B_{-1} \left( \frac{\partial}{\partial a} - B_{-1} \right) \log \mathbb{P}_n - 4 k_1, \quad F^- = F^+ \frac{\partial}{\partial a} \bigg|_{a \rightarrow -a}
\]

\[
2 G^+ := \{H_1^+, F^+\}_{B_{-1}} - \{H_2^+, F^+\}_{\partial/\partial a}, \quad G^- = G^+ \frac{\partial}{\partial a} \bigg|_{a \rightarrow -a}
\]

with

\[
H_1^+ := \frac{\partial}{\partial a} \left( B_0 - a \frac{\partial}{\partial a} - a B_{-1} \right) \log \mathbb{P}_n + \left( B_0 B_{-1} + 4 \frac{\partial}{\partial a} \right) \log \mathbb{P}_n + 4 k_1 \left( a + k_2 \right)
\]

(0.6)

\[
H_2^+ := \frac{\partial}{\partial a} \left( B_0 - a \frac{\partial}{\partial a} - a B_{-1} \right) \log \mathbb{P}_n - \left( B_0 - 2 a B_{-1} - 2 \right) B_{-1} \log \mathbb{P}_n.
\]

**Remark.** The change of variables \( a \leftrightarrow -a, \ k_1 \leftrightarrow k_2 \), in the definition of \( F^- \) and \( G^- \) act at the level of the operators. In fact, later it will be clear that \( \mathbb{P}_n(a; E) \) is invariant under that change of variables.

**Remark.** The equation (0.5) can be checked by hand (or by computer) for small values of \( k_1 \) and \( k_2 \), (e.g., for \( k_1, k_2 = 1, 2, 3 \)). How to do this will be explained in the final remark of Section 4.

To prove Theorem 0.1, we provide a natural integrable deformation of (0.4) (Section 1). As is well-known, the probability for \( A = 0 \) relates to the standard Toda lattice and the one-component KP equation (see [1]), the spectrum of coupled random matrices to the 2-Toda lattice, and the two-component KP (see [2]), whereas we show that the model (0.4) relates to the three-component KP equation (Section 2). This deformation enjoys Virasoro constraints as well (Section 3), which together with the bilinear relations arising from 3-KP leads to the PDE of Theorem 0.1 (Section 4).

The second question concerns the Pearcey distribution, which we now explain. Following [18] and [6], consider \( n = 2k \) nonintersecting Brownian motions on \( \mathbb{R} \) (Dyson’s Brownian motions), all starting at the origin, such that the \( k \) left paths end up at \(-a\) and the \( k \) right paths end up at \(+a\) at time \( t = 1 \). As observed in [6], the
Karlin-McGregor formula [19] enables one to express the transition probability in terms of the Gaussian Hermitian random matrix probability $P(a; E)$ with external source, as in (0.4),

$$P_0^{\pm a}(\text{all } x_j(t) \in E)$$

$$= \lim_{\gamma_i \to 0, \delta_{k+1}, \delta_{2k} \to a} \int \frac{1}{Z_n} \det(p(t; \gamma_i, x_j))_{1 \leq i, j \leq n}$$

$$\cdot \det(p(1-t; x_{j'}, \delta_{j'}))_{1 \leq i', j' \leq n} \prod_{i} dx_i,$$

(0.7)  

$$= P_n(a \sqrt{\frac{2t}{1-t}}; E \sqrt{\frac{2}{t(1-t)}}),$$

where $p(t, x, y)$ is the Brownian transition probability

$$p(t, x, y) := \frac{1}{\sqrt{\pi t}} e^{-\frac{(y-x)^2}{t}}.$$

Let now the number $n = 2k$ of particles go to infinity, and let the points $a$ and $-a$ go to $\pm \infty$. This forces the left $k$ particles to $-\infty$ and the right $k$ particles to $+\infty$ at $t = 1$. Since the particles all leave from the origin at $t = 0$, it is natural to believe that for small times the equilibrium measure (mean density of particles) is supported by one interval, and for times close to 1, the equilibrium measure is supported by two intervals. With a precise scaling, $t = \frac{1}{2}$ is critical in the sense that for $t < \frac{1}{2}$, the equilibrium measure for the particles is supported by one interval, and for $t > \frac{1}{2}$, it is supported by two. The Pearcey process $P(s)$ is now defined as the motion of an infinite number of nonintersecting Brownian paths, just around time $t = \frac{1}{2}$, with the precise scaling (see [6]):

(0.9) $n = 2k = \frac{2}{z^4}, \quad \pm a = \pm \frac{1}{z^2}, \quad x_i \mapsto x_iz, \quad t \mapsto \frac{1}{2} + tz^2, \quad \text{for } z \to 0.$

Even though the pathwise interpretation of $P(t)$ is unclear and still deserves investigation, it is natural to define the following probability for $t \in \mathbb{R}$:

$$\mathbb{P}(P(t) \cap E = \emptyset) := \lim_{z \to 0} \mathbb{P}_0^{\pm 1/z^2} \left( \text{all } x_j \left( \frac{1}{2} + tz^2 \right) \notin zE, \ 1 \leq j \leq n \right) \bigg|_{n=2/z^4}.$$  

Brézin and Hikami [10, 11, 12, 13] for the Pearcey kernel and Tracy-Widom [24] for the extended kernels show that this limit exists and equals a Fredholm determinant:

$$\mathbb{P}(P(t) \cap E = \emptyset) = \det(I - K_t \chi_E),$$
where $K_t(x, y)$ is the Pearcey kernel, defined as follows:

$$K_t(x, y) := \frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - tp(x)q(y)}{x - y}$$

(0.10)

where (note $\omega = e^{i\pi/4}$)

$$p(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{u^4 + \omega^2}{2} - iux} du,$$

$$q(y) := \frac{1}{2\pi i} \int_{X} e^{\frac{u^4 - iu^2 + u}{4}} du = \text{Im} \left[ \frac{\omega}{\pi} \int_{0}^{\infty} \text{due} - \frac{u^4}{4} - it^2u^2 \left( e^{\omega uy} - e^{-\omega uy} \right) \right],$$

satisfy the differential equations

$$p''' - tp' - xp = 0 \quad \text{and} \quad q''' - tq' + yq = 0.$$ 

The contour $X$ is given by the ingoing rays from $\pm \infty e^{i\pi/4}$ to 0 and the outgoing rays from 0 to $\pm \infty e^{-i\pi/4}$, i.e., $X$ stands for the contour

$$
\begin{array}{c}
\nearrow \\
0 \\
\searrow
\end{array}
$$

The second result of this paper\footnote{Tracy and Widom show in [24] the existence of a large system of PDEs involving a large system of auxiliary variables for $Q$ also covering the case of the joint probabilities at different times.} is to give a PDE for the Pearcey distribution below in terms of the parameter $t$ appearing in the kernel (0.10). Since this Pearcey distribution with the parameter $t$ can also be interpreted as the transition probability for the Pearcey process, we prove the following:

**THEOREM 0.2** For compact $E = \bigcup_{i=1}^{r} [x_{2i-1}, x_{2i}]$ and $B_j = \sum_{i=1}^{2r} x_i^{j+1} \frac{\partial}{\partial x_i}$,

(0.11)

$$Q(t; x_1, \ldots, x_{2r}) = \log \mathbb{P}(P(t) \cap E = \emptyset) = \log \det(I - K_t \chi_E)$$

satisfies a fourth-order and third-degree PDE, which can be written as a single Wronskian:

(0.12)

$$\left\{ \frac{1}{2} \frac{\partial^3 Q}{\partial t^3} + (B_0 - 2) B_{-1}^2 Q + \frac{1}{16} \left\{ B_{-1} \frac{\partial Q}{\partial t}, B_{-1}^2 Q \right\}_{B_{-1}}, B_{-1}^2 \frac{\partial Q}{\partial t} \right\}_{B_{-1}} = 0.$$ 

The proof of this statement, based on taking a scaling limit on the PDE of Theorem 0.1, will be given in Section 5.

As is well-known, the transition probabilities for $n$-dimensional diffusions are solutions of second-order parabolic PDEs (heatlike equations) in $n$ variables. The Pearcey process is an infinite-dimensional diffusion and can be thought of as a “continuous Markov cloud”; it can be described by an infinite-dimensional Laplacian augmented with an infinite-dimensional first-order part containing the drift.
Although this may be conceptually pleasing, it is not clear how concrete information can be extracted from these infinite-dimensional diffusion equations! Here we propose for the Pearcey process a description in terms of a finite-dimensional PDE; the price one pays is the nonlinearity of the equation (0.12). We expect such a nonlinear equation to provide information, e.g., on large-time asymptotics. It therefore seems of great interest to study these “Markov clouds” and the finite-dimensional PDEs governing their transition probabilities.

Open Questions

1. Find “initial” or “final” conditions for equation (0.12), i.e., for \( t = -\infty \) or \( +\infty \).

2. How do the Markov property and, in particular, the Chapman-Kolmogorov equations for the transition probabilities reflect themselves in these equations? What are the PDEs for \( \mathbb{P}(\mathcal{P}(t_1) \cap E_1 = \emptyset, \ldots, \mathcal{P}(t_k) \cap E_k = \emptyset) \), involving several times \( t_1 < \cdots < t_k \).

3. Can one extract large-time asymptotics from the nonlinear equation (0.12)? For the Airy process \( A(t) \), we showed in [4] that the joint probability admits the asymptotic series

\[
\mathbb{P}
\left(\frac{A(t_1) \leq u}{A(t_2) \leq v}\right) = F_2(u)F_2(v) + \frac{F_2'(u)F_2'(v)}{t^2} + \frac{\Phi(u, v) + \Phi(v, u)}{t^4} + O\left(\frac{1}{t^6}\right),
\]

where \( F_2(u) \) is the Tracy-Widom distribution and where \( \Phi(u, v) \) is a polynomial of integrals involving the solution of the Painlevé II equation behaving at \( \infty \) like the Airy function. Can similar formulae be obtained for the Pearcey process?

4. Do the PDEs (0.5) and (0.12) in Theorems 0.1 and 0.2 contain some hidden Painlevé equation? Can one take the limit of equation (0.5) when \( a \to 0 \) and obtain the Painlevé IV equation?

1 An Integrable Deformation of Gaussian Random Ensemble with External Source

Consider an ensemble of \( n \times n \) Hermitian matrices with an external source given by a diagonal matrix

\[
A = \text{diag}(a_1, \ldots, a_n)
\]

and a general potential \( V(z) \) with density

\[
\mathbb{P}_n(M \in [M, M + dM]) = \frac{1}{Z_n} e^{-\text{Tr}(V(M) - AM)} dM.
\]

For the disjoint union of intervals \( E = \bigcup_{i=1}^{j} [b_{2i-1}, b_{2i}] \), the following probability can be transformed by the Harish-Chandra-Itzykson-Zuber formula, with \( D := \frac{1}{Z_2} \):
\[
\text{diag}(\lambda_1, \ldots, \lambda_n), \Delta_n(\lambda) := \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j) \text{ and all distinct } a_i:
\]

\[
\mathbb{P}_n(\text{spectrum } M \subset E) = \frac{1}{Z_n} \int_{H_n(E)} e^{-\text{Tr}(V(M) - AM)} dM
\]

\[
= \frac{1}{Z_n} \int_{E^n} \Delta_n^2(\lambda) \prod_{i=1}^n e^{-V(\lambda_i)} d\lambda_i \int_{U(n)} e^{\text{Tr} AU DU^{-1}} dU
\]

\[
= \frac{1}{Z_n} \int_{E^n} \Delta_n^2(\lambda) \prod_{i=1}^n e^{-V(\lambda_i)} d\lambda_i \frac{\det[e^{a_i\lambda_j}]_{1 \leq i, j \leq n}}{\Delta_n(\lambda) \Delta_n(a)}
\]

\[
= \frac{1}{Z_n} \int_{E^n} \Delta_n(\lambda) \det[e^{-V(\lambda_j) + a_i\lambda_j}]_{1 \leq i, j \leq n} \prod_{i=1}^n d\lambda_i
\]

(1.1)

with \(a_i \neq a_j\) and the Vandermonde \(\Delta_n(\lambda) = \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)\). The formula remains valid in the limit, when some \(a_i\)'s coincide, upon making differences of rows and dividing by the appropriate \((a_i - a_j)\)'s. In the following proposition, we consider a general situation, of which (1.1) with \(A = \text{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n)\) is a special case, by setting \(\varphi^+ = e^{az}\) and \(\varphi^- = e^{-az}\). Consider the Vandermonde determinant of the variables \(x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}\), namely,

\[
\Delta_n(x, y) := \Delta_n(x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}).
\]

Then we have the following:

**Proposition 1.1** Given an arbitrary potential \(V(z)\) and arbitrary functions \(\varphi^+(z)\) and \(\varphi^-(z)\), define \((n = k_1 + k_2)\)

\[
(\rho_1, \ldots, \rho_n) := e^{-V(z)}(\varphi^+(z), z\varphi^+(z), \ldots, z^{k_1-1}\varphi^+(z), \\
\varphi^-(z), z\varphi^-(z), \ldots, z^{k_2-1}\varphi^-(z))
\]

we have

\[
\frac{1}{n!} \int_{E^n} \Delta_n(z) \det(\rho_i(z_j))_{1 \leq i, j \leq n} \prod_{i} dz_i
\]

\[
= \frac{1}{k_1! k_2!} \int_{E^n} \Delta_n(x, y) \Delta_{k_1}(x) \Delta_{k_2}(y) \prod_{i=1}^{k_1} \varphi^+(x_i) e^{-V(x_i)} dx_i
\]

\[
\quad \cdot \prod_{i=1}^{k_2} \varphi^-(y_i) e^{-V(y_i)} dy_i
\]
\[
= \det \left( \begin{array}{c}
\int_{E}^{j+j-1} z^{(z)} e^{-V(z)} d\sigma_{1}^{1} \leq i \leq k_{1}^{1} \\
\int_{E}^{j+j-1} z^{(z)} e^{-V(z)} d\sigma_{2}^{1} \leq i \leq k_{2}^{1}
\end{array} \right)
\]

\text{Proof:} On the one hand, using
\[
\det(a_{ik})_{1 \leq i, k \leq n} \det(b_{ik})_{1 \leq i, k \leq n} = \sum_{\sigma \in S_{n}} \det(a_{i, \sigma(j)} b_{j, \sigma(j)})_{1 \leq i, j \leq n}
\]
and distributing the integration over the different columns, one computes
\[
\int_{E^{n}} \Delta_{n}(z) \det(\rho_{i}(z_{j}))_{1 \leq i, j \leq n} \prod_{j}^{n} dz_{i}
\]
\[
= \sum_{\sigma \in S_{n}} (-1)^{\sigma} \Delta_{n}(z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)}) \prod_{i=1}^{n} \rho_{i}(z_{\sigma(i)}) \prod_{j}^{n} dz_{\sigma(i)}
\]
\[
= \sum_{\sigma \in S_{n}} (-1)^{\sigma} \Delta_{n}(z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)}) \prod_{i=1}^{n} \rho_{i}(z_{\sigma(i)}) \prod_{j}^{n} dz_{i}
\]
\[
= \sum_{\sigma \in S_{n}} (-1)^{\sigma} \Delta_{n}(z_{1}, \ldots, z_{n}) \prod_{i=1}^{n} \rho_{i}(z_{i}) \prod_{j}^{n} dz_{i}
\]
\[
= \sum_{\sigma \in S_{n}} (-1)^{\sigma} \int_{E^{n}} \Delta_{n}(z_{1}, \ldots, z_{n}) \prod_{i=1}^{n} \rho_{i}(z_{i}) d\sigma_{1}^{1} \leq i \leq k_{1}^{1}
\]
\[
= \sum_{\sigma \in S_{n}} (-1)^{\sigma} \int_{E^{n}} \Delta_{n}(z_{1}, \ldots, z_{n}) \prod_{i=1}^{n} \rho_{i}(z_{i}) d\sigma_{2}^{1} \leq i \leq k_{2}^{1}
\]
\[
= n! \int_{E^{n}} \Delta_{n}(x, y) \prod_{i=1}^{k_{1}} \int_{E}^{j+j-1} z^{(z)} e^{-V(z)} d\sigma_{1}^{1} \leq i \leq k_{1}^{1} \prod_{i=1}^{k_{2}} \int_{E}^{j+j-1} z^{(z)} e^{-V(z)} d\sigma_{2}^{1} \leq i \leq k_{2}^{1}
\]
\[
\frac{n!}{k_1!k_2!} \int \Delta_n(x, y) \Delta_{k_1}(x) \Delta_{k_2}(y) \prod_{i=1}^{k_1} \varphi^+(x_i) e^{-V(x_i)} \, dx_i \prod_{i=1}^{k_2} \varphi^-(y_i) e^{-V(y_i)} \, dy_i,
\]

where \( \Delta_n(x, y) \) is defined in (1.2). In the last identity, one uses twice the following general identity for a skew-symmetric function \( F(x_1, \ldots, x_k) \) and a general measure \( \mu(dx) \):

\[
\int_{\mathbb{R}^k} F(x_1, \ldots, x_k) \Delta_k(x) \prod_{i=1}^k \mu(dx_i) = \int_{\mathbb{R}^k} \sum_{\sigma \in S_k} (-1)^\sigma \prod_{i=1}^k x_{\sigma(i)}^{i-1} \mu(dx_{\sigma(i)})
\]

This ends the proof of Proposition 1.1. \( \square \)

Add extra variables in the exponentials, one set for each Vandermonde determinant:

\[
t = (t_1, t_2, \ldots), \quad s = (s_1, s_2, \ldots), \quad u = (u_1, u_2, \ldots), \quad \text{and} \quad \beta.
\]

Then, setting \( n = k_1 + k_2 \),

\[
V(z) := \frac{z^2}{2} + \sum_{i=1}^{\infty} t_i z^i, \quad \varphi^+(z) = e^{az + \beta z^2 - \sum_{i=1}^{\infty} s_i z^i}, \quad \varphi^-(z) = e^{-az - \beta z^2 - \sum_{i=1}^{\infty} u_i z^i},
\]
Proposition 1.1 implies

\begin{equation}
\tau_{k_1,k_2}(t, s, u; \beta; E) := \det m_{k_1,k_2}(t, s, u; \beta; E)
\end{equation}

\[ = \frac{1}{k_1! k_2!} \int_E \Delta_n(x, y) \prod_{j=1}^{k_1} e^{\sum_{i=1}^{\infty} a_{ij}^2} \prod_{j=1}^{k_2} e^{\sum_{i=1}^{\infty} u_{ij}^2} \int
\nonumber
\end{equation}

where

\begin{equation}
m_{k_1,k_2}(t, s, u; \beta; E) := \begin{pmatrix}
\mu_{ij}^+(t, s; \beta, E) & \mu_{ij}^-(t, s; \beta, E) \\
\mu_{ij}^-(t, u; \beta, E) & \mu_{ij}^+(t, u; \beta, E)
\end{pmatrix},
\end{equation}

with

\begin{equation}
\mu_{ij}^+(t, s; \beta, E) = \int_E z^{i+j-1} e^{-\frac{z^2}{2} + \beta z} e^{\sum_{k=1}^{\infty} (u_k - s_k) z^k} dz
\end{equation}

\begin{equation}
\mu_{ij}^-(t, u; \beta, E) = \int_E z^{i+j-1} e^{-\frac{z^2}{2} - \beta z} e^{\sum_{k=1}^{\infty} (u_k - u_k) z^k} dz.
\end{equation}

In particular, by (1.3), the integral in (1.1) has a determinantal representation in terms of moments, and thus taking a limit for the probability, one finds

\begin{equation}
\lim_{a_1, \ldots, a_k \to -a} \frac{1}{Z_n n!} \int_E \Delta_n(\lambda) \det e^{-\lambda_i^2/2 + a_i \lambda_i} d\lambda_i
\end{equation}

\begin{equation}
= \frac{1}{Z_n} \det \begin{pmatrix}
\left( \int_E z^{i+j-1} e^{-\frac{z^2}{2} + a z} dz \right)_{0 \leq i+j \leq k_1+k_2-1}
\end{pmatrix}
\end{equation}

and so

\begin{equation}
P_n(\text{spec } M \subset E) = \frac{\tau_{k_1,k_2}(t, s, u; \beta; E)}{\tau_{k_1,k_2}(t, s, u; \beta; \mathbb{R})} \bigg|_{t=s=u=\beta=0}.
\end{equation}

Remark. The integral enjoys the obvious duality:

\begin{equation}
x \leftrightarrow y, \ k_1 \leftrightarrow k_2, \ t \leftrightarrow u, \ a \leftrightarrow -a, \ \beta \leftrightarrow -\beta.
\end{equation}
satisfy the bilinear identity

\[ (2.1) \]

where the integral is taken along a small contour about \( \lambda \). Bilinear identity (2.3) follows at once from identity (2.2), and E fixed, the wave matrices

\[ (\lambda; t, s, u) \]

and

\[ (\lambda; t, s, u) \]

with wave functions

\[ (2.1) \]

satisfy the bilinear identity

\[ (2.2) \]

where the integral is taken along a small contour about \( \infty \), for all integers \( k_1, k_2, \ell_1, \ell_2 \geq 0 \) and all \( t, s, u, t', s', u' \in \mathbb{C}^\infty \). It implies that the functions \( \tau_{k_1,k_2}(t, s, u) \) satisfy the bilinear identities

\[ (2.3) \]

The proof of this theorem (identity (2.2)) appears in [5] and is based on the observation that the shifted expressions in (2.1) are closely related to multiple orthogonal polynomials. Bilinear identity (2.3) follows at once from identity (2.2).

---

[2] For \( z \in \mathbb{C} \), define \([z] := (z, z^2/2, z^3/3, \ldots) \in \mathbb{C}^\infty\).

[3] Integrals (2.2) and (2.3) are contour integrals along a small circle about \( \infty \), with formal Laurent series as integrand.
using the explicit expressions (2.1) for the Ψ’s and computing the (1,1)-entry in the matrix multiplication (2.2). These identities define the three-component KP hierarchy, as described by Ueno and Takasaki [25].

The next point is to evaluate the integral (2.3) by computing the residue about λ = ∞. To do so efficiently, one needs to define the Hirota symbol between functions \( f = f(t_1, t_2, \ldots) \) and \( g = g(t_1, t_2, \ldots) \), given a polynomial \( p(t_1, t_2, \ldots) \), namely,

\[
p \left( \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots \right) f \circ g := p \left( \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots \right) f(t + y)g(t - y) \bigg|_{y=0}.
\]

This operation extends readily to the case where \( p(t_1, t_2, \ldots) \) is a Taylor series in \( t_1, t_2, \ldots \). We also need the elementary Schur polynomials \( s_\ell \), which are defined by \( e^{\sum_{i=0}^\infty z^i} := \sum_{i \geq 0} s_i(t) z^i \) for \( i \geq 0 \) and \( s_i(t) = 0 \) for \( i < 0 \); moreover, set

\[
s_\ell(\mathring{\partial}) := s_\ell \left( \frac{\partial}{\partial t_1}, \frac{\partial}{2 \partial t_2}, \frac{\partial}{3 \partial t_3}, \ldots \right).
\]

**Corollary 2.2** The bilinear identities (2.3) imply PDEs for \( \tau_{k_1k_2}(t, s, u) \), expressed in terms of Hirota’s symbol, for \( j = 1, 2, \ldots \).

\[
s_j(\mathring{\partial}) \tau_{k_1+1,k_2} \circ \tau_{k_1-1,k_2} = -\tau_{k_1k_2}^2 \frac{\partial^2}{\partial s_1 \partial t_{j+1}} \log \tau_{k_1k_2}
\]

(2.4)

\[
s_j(\mathring{\partial}) \tau_{k_1-1,k_2} \circ \tau_{k_1+1,k_2} = -\tau_{k_1k_2}^2 \frac{\partial^2}{\partial t_1 \partial s_{j+1}} \log \tau_{k_1k_2}
\]

(2.5)

yielding

\[
\frac{\partial^2 \log \tau_{k_1,k_2}}{\partial t_1 \partial s_1} = -\tau_{k_1+1,k_2} \tau_{k_1-1,k_2} \frac{\tau_{k_1,k_2}^2}{\tau_{k_1,k_2}}
\]

(2.6)

and

\[
\frac{\partial}{\partial t_1} \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} = \frac{\partial}{\partial s_1} \frac{\log \tau_{k_1,k_2}}{\tau_{k_1,k_2}},
\]

(2.7)

\[
-\frac{\partial}{\partial s_1} \log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}} = \frac{\partial}{\partial t_1} \frac{\log \tau_{k_1,k_2}}{\tau_{k_1,k_2}}.
\]

(2.8)

**Proof:** The methods used below are due to Date, Kashiwara, Jimbo, and Miwa [14]. As a first step, given two functions \( f(t) \) and \( g(t) \), with \( t = (t_1, t_2, \ldots), a = (a_1, a_2, \ldots) \), \( u = (u_1, u_2, \ldots) \in \mathbb{C}^\infty \), and \( \lambda \in \mathbb{C} \), check the following Taylor series:

\[
f(t + a + [\lambda^{-1}]g(t - a - [\lambda^{-1}]) = f(t + u + a + [\lambda^{-1}]g(t - u - a - [\lambda^{-1}]) \bigg|_{u=0}
\]
\[
\sum_{k=0}^{\infty} \sum_{l_1 \leq k_2, k} \sum_{l_1 + j_1 \leq k_2 + j_2} (-2a) s_{j_k} (\tilde{\partial}_k) e^{\frac{a}{2} + b_k \frac{2}{\pi k} + c \frac{2}{\pi k}} \tau_{k_1, k_2} \circ \tau_{l_1, k_2} = 0.
\]

Introducing the shifts
\[
\begin{align*}
t & \mapsto t - a, \quad t' \mapsto t + a, \\
 s & \mapsto s - b, \quad s' \mapsto s + b, \\
u & \mapsto u - c, \quad u' \mapsto u + c,
\end{align*}
\]
in the contour integrations (2.3), one finds by using the residue formula (2.9) that

\[
\sum_{j=0}^{\infty} \sum_{k_1 \leq k_2, k} \sum_{l_1 + j_1 \leq k_2 + j_2} (-2b) s_{j_k} (\tilde{\partial}_k) e^{\frac{a}{2} + b_k \frac{2}{\pi k} + c \frac{2}{\pi k}} \tau_{k_1, k_2} \circ \tau_{l_1, k_2} = 0.
\]

\[
\sum_{j=0}^{\infty} \sum_{k_1 \leq k_2, k} \sum_{l_1 + j_1 \leq k_2 + j_2} (-2c) s_{j_k} (\tilde{\partial}_k) e^{\frac{a}{2} + b_k \frac{2}{\pi k} + c \frac{2}{\pi k}} \tau_{k_1, k_2} \circ \tau_{l_1, k_2} = 0.
\]

This formula enables one to compute the following contour integral about \( \infty \) by the method of residues (with \( \alpha \in \mathbb{Z} \)):

\[
\oint \prod_{\infty} d\lambda \lambda^\alpha e^{-\sum_{i=0}^{\infty} (2a_i) \lambda^i} f(t + a + [\lambda^{-1}]) g(t - a - [\lambda^{-1}]) = \oint \prod_{\infty} d\lambda \lambda^\alpha \left( \sum_{i=0}^{\infty} s_i (-2a) \lambda^i \right) \left( \sum_{i=0}^{\infty} s_j (\tilde{\partial}) \lambda^{-j} \right) e^{\sum_{k=0}^{\infty} \frac{a_k}{2} \lambda^k} f \circ g
\]

\[
(2.9)
\]

Introducing the shifts
\[
\begin{align*}
t & \mapsto t - a, \quad t' \mapsto t + a, \\
 s & \mapsto s - b, \quad s' \mapsto s + b, \\
u & \mapsto u - c, \quad u' \mapsto u + c,
\end{align*}
\]
in the contour integrations (2.3), one finds by using the residue formula (2.9) that

\[
\sum_{j=0}^{\infty} \sum_{k_1 \leq k_2, k} \sum_{l_1 + j_1 \leq k_2 + j_2} (-2b) s_{j_k} (\tilde{\partial}_k) e^{\frac{a}{2} + b_k \frac{2}{\pi k} + c \frac{2}{\pi k}} \tau_{k_1, k_2} \circ \tau_{l_1, k_2} = 0.
\]

\[
\sum_{j=0}^{\infty} \sum_{k_1 \leq k_2, k} \sum_{l_1 + j_1 \leq k_2 + j_2} (-2c) s_{j_k} (\tilde{\partial}_k) e^{\frac{a}{2} + b_k \frac{2}{\pi k} + c \frac{2}{\pi k}} \tau_{k_1, k_2} \circ \tau_{l_1, k_2} = 0.
\]
Setting $\ell_1 = k_1 + 2$ and $\ell_2 = k_2$, this equation becomes

$$
\sum_{j=0}^{\infty} s_{j+1}(-2a) s_j(\tilde{\delta}) e^{\sum_{i} (a_i \frac{\partial}{\partial t_i} + b_i \frac{\partial}{\partial s_i} + c_i \frac{\partial}{\partial t_i s_i}) \tau_{k_1+k_2} \circ \tau_{k_1} \circ k_2} - 2b) s_j(\tilde{\delta}) e^{\sum_{i} (a_i \frac{\partial}{\partial t_i} + b_i \frac{\partial}{\partial s_i} + c_i \frac{\partial}{\partial t_i s_i}) \tau_{k_1} \circ k_1 \circ k_2} - \sum_{j=0}^{\infty} s_{j+1}(-2c) s_j(\tilde{\delta}) e^{\sum_{i} (a_i \frac{\partial}{\partial t_i} + b_i \frac{\partial}{\partial s_i} + c_i \frac{\partial}{\partial t_i s_i}) \tau_{k_1} \circ k_1 \circ k_2} = 0.
$$

The next step is to Taylor-expand expression (2.10) for small $a_j$, $b_j$, and $c_j$ and to concentrate on the purely linear term $a_{j+1}$, which is found in the first two integrals; to be specific, in $s_{j+1}(-2a) = -2a_{j+1} + \cdots$ in the first integral and in the exponential appearing in the second integral. Of course, its coefficient must vanish:

$$
a_{j+1}(-2s_j(\tilde{\delta}) \tau_{k_1+k_2} \circ \tau_{k_1} \circ k_2) + O(a_{j+1}^2) = 0,
$$

yielding equation (2.4) after setting $k_1 \to k_1 - 1$. Similarly, the coefficient of $b_{j+1}$ in equation (2.10) must vanish upon picking $\ell_1 = k_1$ and $\ell_2 = k_2$, yielding equation (2.5).

Specializing equation (2.4) to $j = 0$ and 1, respectively, yields (since $s_1(t) = t_1$ implies $s_1(\tilde{\delta}) = \frac{\partial}{\partial t_1}$; also $s_0 = 1$):

$$
\frac{\partial^2 \log \tau_{k_1+k_2}}{\partial t_1 \partial s_1} = -\frac{\tau_{k_1+k_2} \tau_{k_1-1,k_2}}{\tau_{k_1,k_2}^2}
$$

and

$$
\frac{\partial^2 \log \tau_{k_1+k_2}}{\partial s_1 \partial t_2} = -\frac{1}{\tau_{k_1,k_2}^2} \left[ \left( \frac{\partial}{\partial t_1} \tau_{k_1+k_2} \right) \tau_{k_1-1,k_2} - \tau_{k_1+k_2} \left( \frac{\partial}{\partial t_1} \tau_{k_1-1,k_2} \right) \right].
$$

Upon dividing the second equation by the first, we find equation (2.7) and similarly equation (2.8) follows from equation (2.5).  

3 Virasoro Constraints for the Integrable Deformations

Given the Heisenberg and Virasoro operators, for $m \geq -1$, $k \geq 0$,

$$
\mathcal{J}^{(1)}_{m,k}(t) = \frac{\partial}{\partial t_m} + (-m)t_{-m} + k\delta_{0,m},
$$

$$
\mathcal{J}^{(2)}_{m,k}(t) = \frac{1}{2} \left( \sum_{i+j=m} \frac{\partial^2}{\partial t_i \partial t_j} + 2 \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i + m} + \sum_{i+j=-m} i t_i j_t \right)
$$

$$
+ \left( k + \frac{m+1}{2} \right) \left( \frac{\partial}{\partial t_m} + (-m)t_{-m} \right) + \frac{k(k+1)}{2} \delta_{m,0}.
$$
we now state the following:

**Theorem 3.1** The integral \( \tau_{k_1, k_2} (t, s, u; \beta; E) \) as defined in (1.5) satisfies

\[
B_m \tau_{k_1, k_2} = \nabla^m \tau_{k_1, k_2} \quad \text{for } m \geq -1,
\]

where \( B_m \) and \( \nabla_m \) are differential operators:

\[
B_m = \sum_{i=1}^{2r} b_i^{m+1} \frac{\partial}{\partial b_i} \quad \text{for } E = \bigcup_{i=1}^{2r} [b_{2i-1}, b_{2i}] \subset \mathbb{R}
\]

and

\[
\nabla^k \boldsymbol{m} := \nabla^{k_1} (t) - (m + 1) \nabla^{k_2} (t) + \nabla^{k_1} (-s) + a \nabla^{k_2} (s) - (1 - 2\beta) \nabla^{k_2} (-s) + \nabla^{k_1} (-u) - a \nabla^{k_2} (-u) - (1 + 2\beta) \nabla^{k_2} (-u).
\]

We state the following lemmas:

**Lemma 3.2** (Adler and van Moerbeke [3]) Given

\[
\rho = e^{-V} \quad \text{with} \quad -\rho' \rho = V' = g_f = \sum_{i=0}^{\infty} \beta_i z_i \sum_{i=0}^{\infty} \alpha_i z_i,
\]

the integrand

\[
dI_n(x) := \Delta_n(x) \prod_{k=1}^{n}(e^{\sum_{i=1}^{\infty} t_i x_i} \rho(x_i) dx_k)
\]

satisfies the following variational formula:

\[
\frac{d}{de} dI_n(x_i \mapsto x_i + \varepsilon f (x_i) x_i^{m+1}) \bigg|_{\varepsilon=0} = \sum_{\ell=0}^{\infty} (\alpha_\ell \nabla^{(2)} m+\ell, n - \beta_\ell \nabla^{(1)} m+\ell, n) dI_n.
\]

The contribution coming from \( \prod_{i=1}^{n} dx_j \) is given by

\[
\sum_{\ell=0}^{\infty} \alpha_\ell (\ell + m + 1) \nabla^{(1)} m+\ell, n dI_n.
\]

**Lemma 3.3** Setting

\[
dI_n = \Delta_n (x, y) \prod_{j=1}^{k_1} e^{\sum_{i=1}^{\infty} u_i x_i^j} \prod_{j=1}^{k_2} e^{\sum_{i=1}^{\infty} u_i y_i^j} \cdot \left( \Delta_{k_1} (x) \prod_{j=1}^{k_1} e^{-\frac{1}{2} x_i^2 + ax_i + \beta x_i^2} e^{-\sum_{i=1}^{\infty} s_i x_i^j} dx_j \right) \cdot \left( \Delta_{k_2} (y) \prod_{j=1}^{k_2} e^{-\frac{1}{2} y_i^2 - ay_i - \beta y_i^2} e^{-\sum_{i=1}^{\infty} u_i y_i^j} dy_j \right),
\]
the following variational formula holds for \( m \geq -1 \):

\[
\frac{d}{d\varepsilon} dI_n \left( x_i \mapsto x_i + \varepsilon x_i^{m+1}, y_i \mapsto y_i + \varepsilon y_i^{m+1} \right) \bigg|_{\varepsilon=0} = \Psi_{m}^{k_1, k_2}(dI_n).
\]

**Proof:** The variational formula (3.4) is an immediate consequence of applying the variational formula (3.2) separately to the three factors of \( dI_n \) and in addition applying formula (3.3) to the first factor to account for the fact that

\[
\prod_{j=1}^{k_1} dx_j \prod_{j=1}^{k_2} dy_j
\]

is missing from the first factor.

**Proof of Theorem 3.1:** Formula (3.1) follows immediately from formula (3.4) by taking into account the variation of the boundary of \( E \) under the change of coordinates.

Using the identity, valid when acting on \( \tau_{k_1 k_2}(t, s, u; \beta; E) \),

\[
\frac{\partial}{\partial t_n} = -\frac{\partial}{\partial s_n} - \frac{\partial}{\partial u_n},
\]

one obtains by explicit computation for \( m \geq -1 \),

\[
\psi_{m}^{k_1, k_2} := \Psi_{m, k_1 + k_2}^{(2)}(t) - (m + 1)\Psi_{m, k_1 + k_2}^{(1)}(t) + \Psi_{m, k_1}^{(2)}(-s) + a\Psi_{m+1, k_1}^{(1)}(-s) - (1 - 2\beta)\Psi_{m+2, k_1}^{(1)}(-s)
\]

\[
+ \Psi_{m, k_2}^{(2)}(-u) - a\Psi_{m+1, k_2}^{(1)}(-u) - (1 + 2\beta)\Psi_{m+2, k_2}^{(1)}(-u)
\]

is

\[
= \frac{1}{2} \sum_{i+j=m} \left( \frac{\partial^2}{\partial t_i \partial t_j} + \frac{\partial^2}{\partial s_i \partial s_j} + \frac{\partial^2}{\partial u_i \partial u_j} \right)
\]

\[
+ \sum_{i \geq 1} \left( i_t \frac{\partial}{\partial t_{i+m}} + i_s \frac{\partial}{\partial s_{i+m}} + i_u \frac{\partial}{\partial u_{i+m}} \right)
\]

\[
+ (k_1 + k_2) \left( \frac{\partial}{\partial t_m} + (-m) t_m \right) - k_1 \left( \frac{\partial}{\partial s_m} + (-m) s_m \right)
\]

\[
- k_2 \left( \frac{\partial}{\partial u_m} + (-m) u_m \right) + (k_1^2 + k_1 k_2 + k_2^2) \delta_{m0}
\]

\[
+ a(k_1 - k_2) \delta_{m+1,0} + \frac{m(m+1)}{2} (t_m + s_m + u_m)
\]

\[
- \frac{\partial}{\partial t_{m+2}} + a \left( -\frac{\partial}{\partial s_{m+1}} + \frac{\partial}{\partial u_{m+1}} + (m+1)(s_{m-1} - u_{m-1}) \right)
\]

\[
+ 2\beta \left( \frac{\partial}{\partial u_{m+2}} - \frac{\partial}{\partial s_{m+2}} \right).
\]
The following identities, valid when acting on \( \tau_{s_1 s_2}(t, s; u; \beta; E) \), will also be used:

\[
\frac{\partial}{\partial s_1} = -\frac{1}{2} \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial a} \right), \quad \frac{\partial}{\partial s_2} = -\frac{1}{2} \left( \frac{\partial}{\partial t_2} + \frac{\partial}{\partial \beta} \right), \quad \frac{\partial}{\partial u_1} = -\frac{1}{2} \left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial a} \right), \quad \frac{\partial}{\partial u_2} = -\frac{1}{2} \left( \frac{\partial}{\partial t_2} - \frac{\partial}{\partial \beta} \right).
\]

**Corollary 3.4** The tau function \( \tau = \tau_{s_1 s_2}(t, s; u; \beta; E) \) satisfies the following differential identities, with \( B_m = \sum_{i=1}^{2r} k_i^{m+1} \frac{\partial}{\partial \nu_i} \):

\[
-B_{-1} \tau = \left( \frac{\partial}{\partial t_1} - 2\beta \frac{\partial}{\partial a} \right) \tau - \sum_{i \geq 2} \left( i \tau \frac{\partial}{\partial t_{i-1}} + is_i \frac{\partial}{\partial s_{i-1}} + iu_i \frac{\partial}{\partial u_{i-1}} \right) \tau + a(k_2 - k_1) \tau + (k_1 s_1 + k_2 u_1 - (k_1 + k_2) t_1) \tau, \tag{3.5}
\]

\[
\frac{1}{2} \left( B_{-1} - \frac{\partial}{\partial a} \right) \tau = \left( \frac{\partial}{\partial s_1} + \beta \frac{\partial}{\partial a} \right) \tau + \frac{1}{2} \left( \frac{\partial}{\partial s_1} + \frac{\partial}{\partial u_1} \right) \tau + \frac{1}{2} \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_{i-1}} + \frac{\partial}{\partial s_{i-1}} + \frac{\partial}{\partial u_{i-1}} \right) \tau,
\]

\[
-B_0 = \left( k_1^2 + k_2^2 + k_1 k_2 \right) \tau - 2\beta \frac{\partial \tau}{\partial \beta} - \sum_{i \geq 1} \left( i \tau \frac{\partial}{\partial t_i} + is_i \frac{\partial}{\partial s_i} + iu_i \frac{\partial}{\partial u_i} \right) \tau, \tag{3.6}
\]

\[
\frac{1}{2} \left( B_0 - a \frac{\partial}{\partial a} - \frac{\partial}{\partial \beta} \right) \tau = \frac{\partial \tau}{\partial s_2} + \frac{1}{2} \left( k_1^2 + k_2^2 + k_1 k_2 \right) \tau + \beta \frac{\partial \tau}{\partial \beta} + \frac{1}{2} \sum_{i \geq 1} \left( i \tau \frac{\partial}{\partial t_i} + is_i \frac{\partial}{\partial s_i} + iu_i \frac{\partial}{\partial u_i} \right) \tau.
\]

**Corollary 3.5** On the locus \( \mathcal{L} = \{ t = s = u = 0, \beta = 0 \} \), the function \( f = \log \tau_{s_1 s_2}(t, s; u; \beta; E) \) satisfies the following differential identities:

\[
\frac{\partial f}{\partial t_1} = -B_{-1} f + a(k_1 - k_2),
\]

\[
\frac{\partial f}{\partial s_1} = \frac{1}{2} \left( B_{-1} - \frac{\partial}{\partial a} \right) f + \frac{a}{2} (k_2 - k_1), \tag{3.6}
\]

\[
\frac{\partial f}{\partial t_2} = \left( -B_0 + a \frac{\partial}{\partial a} \right) f + k_1^2 + k_2^2 + k_1 k_2,
\]

\[
\frac{\partial f}{\partial s_2} = \frac{1}{2} \left( B_0 - a \frac{\partial}{\partial a} - \frac{\partial}{\partial \beta} \right) f - \frac{1}{2} (k_1^2 + k_2^2 + k_1 k_2).
\]
\[ 2 \frac{\partial^2 f}{\partial t_1 \partial s_1} = B_{-1} \left( \frac{\partial}{\partial a} - B_{-1} \right) f - 2k_1, \]

(3.7) \[ 2 \frac{\partial^2 f}{\partial t_1 \partial s_2} = \left( a \frac{\partial}{\partial a} + \frac{\partial}{\partial \beta} - B_0 + 1 \right) B_{-1} f - 2 \frac{\partial f}{\partial a} - 2a(k_1 - k_2), \]

\[ 2 \frac{\partial^2 f}{\partial s_2 \partial s_1} = \frac{\partial}{\partial a} \left( B_0 - a \frac{\partial}{\partial a} + aB_{-1} \right) f - B_{-1}(B_0 - 1) f - 2a(k_1 - k_2). \]

**Proof:** Upon dividing equations (3.5) by \( \tau \) and restricting to the locus \( \mathcal{L} \), equations (3.6) follow immediately. The essence of deriving (3.7) is that the Virasoro operators \( \mathcal{V}_n \) and the boundary operators \( \mathcal{B}_n \) commute. To derive, say, the first equation in the list (3.7), rewrite the two first equations of (3.5) as

\[ -B_{-1} f = \frac{\partial f}{\partial t_1} + a(k_2 - k_1) + L_1(f) + \ell_1, \]

\[ \frac{1}{2} \left( B_{-1} - \frac{\partial}{\partial a} \right) f = \frac{\partial f}{\partial s_1} + \frac{1}{2} a(k_1 - k_2) + L_2(f) + \ell_2, \]

where \( L_i \) are linear operators vanishing on \( \mathcal{L} \) and the \( \ell_i \) are functions vanishing on \( \mathcal{L} \). This yields

\[ (-B_{-1}) \frac{1}{2} \left( B_{-1} - \frac{\partial}{\partial a} \right) f \bigg|_{\mathcal{L}} \]

\[ = \left( \frac{\partial}{\partial s_1} + \beta \frac{\partial}{\partial a} \right) (-B_{-1} f) \bigg|_{\mathcal{L}} \]

\[ + \frac{1}{2} \sum_{i \geq 2} \left( it_i \frac{\partial}{\partial t_{i-1}} + is_i \frac{\partial}{\partial s_{i-1}} + iu_i \frac{\partial}{\partial u_{i-1}} \right) (-B_{-1} f) \bigg|_{\mathcal{L}} \]

\[ = \frac{\partial}{\partial s_1} (-B_{-1} f) \bigg|_{\mathcal{L}} \]

\[ = \frac{\partial}{\partial s_1} \left( \frac{\partial}{\partial t_1} - 2\beta \frac{\partial}{\partial a} \right) f + a(k_2 - k_1) \]

\[ + \sum_{i \geq 2} \left( it_i \frac{\partial}{\partial t_{i-1}} + is_i \frac{\partial}{\partial s_{i-1}} + iu_i \frac{\partial}{\partial u_{i-1}} \right) f \bigg|_{\mathcal{L}} \]

\[ = \frac{\partial^2}{\partial s_1 \partial t_1} f + k_1. \]

The other identities (3.7) can be obtained in a similar way. \( \square \)

**4 A PDE for the Gaussian Ensemble with External Source**

**Proof of Theorem 0.1:** First observe that, with \( n = k_1 + k_2 \),

\[ P_n(a; E) = \frac{1}{Z_n} \int_{\mathcal{H}_n(E)} e^{-\text{Tr}(\frac{1}{2} M^2 - AM)} dM = \frac{\tau_{k_1k_2}(t, s, u; \beta; E)}{\tau_{k_1k_2}(t, s, u; \beta; \mathbb{R})} \bigg|_{t=s=u=\beta=0}. \]
where

Thus we need to concentrate on

or, equivalently,

obtained, which yields

From these three equations, the expression \(\log(4.3)\)

and (4.2) and (4.3). Subsequently one eliminates \(\log(4.2)\)

subtracting the first equations in (4.2) and (4.3) and then the second equations in

\[ c_{k_1k_2} a^{k_1k_2} e^{(k_1+k_2)a^2/2} . \]

This is obtained from the representation (1.5) in terms of moments, which themselves are Gaussian integrals, as shown in Appendix A. From this formula, it follows that

\[
\log \tau_{k_1k_2}(t, s, u; \beta; \mathbb{R}) \bigg|_{t=s=u=\beta=0} = \frac{k_1 + k_2}{2} a^2 + k_1k_2 \log a + C_{k_1k_2},
\]

where \(c_{k_1k_2}\) and \(C_{k_1k_2}\) are constants depending on \(k_1\) and \(k_2\) only. It follows that

\[
(4.1) \quad \log \mathbb{P}_n(a; E) = \log \tau_{k_1k_2}(0, 0, 0; E) - \frac{k_1 + k_2}{2} a^2 - k_1k_2 \log a - C_{k_1k_2}.
\]

Thus we need to concentrate on \(\tau_{k_1k_2}(t, s, u; \beta; E)\), which, by Theorem 2.1, satisfies the bilinear identity (2.2) and thus the identities (2.7) and (2.8) of Corollary 2.2:

\[
\frac{\partial}{\partial t_1} \log \tau_{k_1 + 1, k_2} = \frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{k_1k_2},
\]

\[
\frac{\partial}{\partial s_1} \log \tau_{k_1 + 1, k_2} = -\frac{\partial^2}{\partial t_1 \partial s_2} \log \tau_{k_1k_2},
\]

whereas the first two Virasoro equations (3.6) yield, specializing to the locus \(\mathcal{L} = \{t = s = u = 0, \beta = 0\}\) and the indices \(k_1 \pm 1, k_2\),

\[
\frac{\partial}{\partial t_1} \log \tau_{k_1 + 1, k_2} = -B_{-1} \log \tau_{k_1 + 1, k_2} + 2a,
\]

\[
\frac{\partial}{\partial s_1} \log \tau_{k_1 + 1, k_2} = \frac{1}{2} \left( B_{-1} - \frac{\partial}{\partial a} \right) \log \tau_{k_1 + 1, k_2} - a.
\]

From these three equations, the expression \(\log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}}\) can be eliminated, by first subtracting the first equations in (4.2) and (4.3) and then the second equations in (4.2) and (4.3). Subsequently one eliminates \(\log \frac{\tau_{k_1+1,k_2}}{\tau_{k_1-1,k_2}}\) from the equations thus obtained, which yields

\[
\frac{1}{2} \left( B_{-1} - \frac{\partial}{\partial a} \right) \left( \frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{k_1k_2} - 2a \right) = B_{-1} \left( \frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{k_1k_2} - a \right)
\]

or, equivalently,
Using the Virasoro relations (3.7), one obtains along the locus \( \mathcal{L} = \{ t = s = u = 0, \beta = 0 \} \):

\[
(4.4) \quad \mathcal{B}_- \left( \frac{-\partial^2}{\partial t_2 \partial s_1} \log \tau_{k_1k_2} \right) - \frac{\partial}{\partial a} \left( \frac{-\partial^2}{\partial t_1 \partial s_1} \log \tau_{k_1k_2} \right) = 0.
\]

 Confirming (0.6). Notice that the expressions above do not contain partials in \( \beta \), except for the \( \beta \)-partial appearing in the second expression of (4.5). Putting expressions (4.5) into (4.4) yields

\[
(4.6) \quad \left\{ \mathcal{B}_- \frac{\partial}{\partial \beta} \log \tau_{k_1k_2} \bigg| \mathcal{L} \right\}_{\mathcal{B}_-} = \left\{ H^+_1, \frac{1}{2} F^+ \right\}_{\mathcal{B}_-} - \left\{ H^+_2, \frac{1}{2} F^+ \right\}_{\partial / \partial a} =: G^+.
\]
and by involution $a \mapsto -a$, $\beta \mapsto -\beta$, $k_1 \leftrightarrow k_2$,

\begin{equation}
\left\{ \mathcal{B}_{-1} \frac{\partial}{\partial \beta} \log \tau_{k_1 k_2} \bigg| \mathcal{L} , F^- \right\}_{\mathcal{B}_{-1}} = \left\{ H^-_1, \frac{1}{2} F^- \right\}_{\mathcal{B}_{-1}} - \left\{ H^-_2, \frac{1}{2} F^- \right\}_{-\partial/\partial a} =: G^-,
\end{equation}

where

$$F^- = F^+|_{a \rightarrow -a}, \quad H^-_i = H^+_i|_{a \rightarrow -a}.$$  

Remember that the change of variables $a \mapsto -a$, $\beta \mapsto -\beta$, and $k_1 \leftrightarrow k_2$ acts on the operators, since $\tau_{k_1 k_2}$ is invariant under this change; see (1.6). Equations (4.6) and (4.7) yield a linear system of equations in

$$\mathcal{B}_{-1} \frac{\partial}{\partial \beta} \log \tau_{k_1 k_2} \bigg| \mathcal{L} \quad \text{and} \quad \mathcal{B}^2_{-1} \frac{\partial}{\partial \beta} \log \tau_{k_1 k_2} \bigg| \mathcal{L},$$

with solution given by

$$\mathcal{B}_{-1} \frac{\partial}{\partial \beta} \log \tau_{k_1 k_2} \bigg| \mathcal{L} = \frac{G^- F^+ + G^+ F^-}{-F^- (\mathcal{B}_{-1} F^+) + F^+ (\mathcal{B}_{-1} F^-)},$$

$$\mathcal{B}^2_{-1} \frac{\partial}{\partial \beta} \log \tau_{k_1 k_2} \bigg| \mathcal{L} = \frac{G^- (\mathcal{B}_{-1} F^+) + G^+ (\mathcal{B}_{-1} F^-)}{-F^- (\mathcal{B}_{-1} F^+) + F^+ (\mathcal{B}_{-1} F^-)}.$$  

Subtracting the second equation from $\mathcal{B}_{-1}$ of the first equation yields the differential equation

\begin{equation}
(F^+ \mathcal{B}_{-1} G^- + F^- \mathcal{B}_{-1} G^+) (F^+ \mathcal{B}_{-1} F^- - F^- \mathcal{B}_{-1} F^+) - (F^+ G^- + F^- G^+) (F^+ \mathcal{B}^2_{-1} F^- - F^- \mathcal{B}^2_{-1} F^+) = 0,
\end{equation}

establishing the first equation (0.5) of Theorem 0.1.

To prove the second equation (0.5), set

$$X := \mathcal{B}_{-1} \frac{\partial}{\partial \beta} \log \tau_{k_1 k_2} \bigg| \mathcal{L};$$

then the system (4.6) and (4.7) and that same system acted upon by $\mathcal{B}_{-1}$ can be written

$$0 = -(X, F^+)_{\mathcal{B}_{-1}} + G^+,$$

$$0 = -(X, F^-)_{\mathcal{B}_{-1}} - G^-,$$

$$0 = \mathcal{B}_{-1} (X, F^+)_{\mathcal{B}_{-1}} + \mathcal{B}_{-1} G^+,$$

$$0 = \mathcal{B}_{-1} (X, F^-)_{\mathcal{B}_{-1}} - \mathcal{B}_{-1} G^-.$$

In matrix notation, this is

$$\begin{pmatrix}
G^+ & \mathcal{B}_{-1} F^+ & -F^+ & 0 \\
-G^- & \mathcal{B}_{-1} F^- & -F^- & 0 \\
\mathcal{B}^2_{-1} G^+ & \mathcal{B}^2_{-1} F^+ & 0 & -F^+ \\
-\mathcal{B}^2_{-1} G^- & \mathcal{B}^2_{-1} F^- & 0 & -F^-
\end{pmatrix}
\begin{pmatrix}
1 \\
X \\
\mathcal{B}_{-1} X \\
\mathcal{B}^2_{-1} X
\end{pmatrix}
= 0,$$
and thus the matrix must be singular, establishing the second formula of (0.5). This ends the proof of Theorem 0.1.

\[ \square \]

Remark. Equation (4.8) can be checked by hand for small values of \( k_1 \) and \( k_2 \), (e.g., for \( k_1 = 1, k_2 = 1, 2, 3 \)). Remembering

\[
\int \Delta_n(x, y) \left( \Delta_k(x) \prod_{j=1}^{k_1} e^{-\frac{x_j^2}{t} + ax_j} \right) \left( \Delta_k(y) \prod_{j=1}^{k_2} e^{-\frac{y_j^2}{t} + ay_j} \right) = \det \begin{pmatrix} \int_E z^{i+j-1} e^{-\frac{z^2}{t} + az} \ dz & 1 \leq i \leq k_1 \\ \int_E z^{i+j-1} e^{-\frac{z^2}{t} - az} \ dz & 1 \leq i \leq k_2 \end{pmatrix} \begin{pmatrix} 0 \leq j \leq k_1+k_2-1 \end{pmatrix},
\]

picking \( E = (-\infty, b) \), and setting

\[
\mu_{ij}^\pm = \int_{-\infty}^b z^{i+j-1} e^{-\frac{z^2}{t} \pm az} \ dz,
\]

one readily checks

\[
\frac{\partial}{\partial a} \mu_{ij}^\pm = \pm \mu_{i+1,j}^\pm,
\]

\[
\mathcal{B}_{-1} \mu_{ij}^\pm = \int_{-\infty}^b dz \frac{\partial}{\partial z}(z^{i+j-1} e^{-\frac{z^2}{t} \pm az}) = (i + j - 1) \mu_{i-1,j}^\pm - \mu_{i+1,j}^\pm \pm a \mu_{i,j}^\pm,
\]

\[
\mathcal{B}_0 \mu_{ij}^\pm = \int_{-\infty}^b dz \frac{\partial}{\partial z}(z^{i+j} e^{-\frac{z^2}{t} \pm az}) = (i + j) \mu_{i+1,j}^\pm - \mu_{i+2,j}^\pm \pm a \mu_{i,j}^\pm,
\]

from which differential equation (4.8) can be verified.

### 5 A PDE for the Pearcey Transition Probability

For nonintersecting Brownian motions \( x_j(t) \), we have from the Karlin-McGregor formula

\[
\mathbb{P}\left( \text{all } x_i(t) \in E, 1 \leq i \leq n \left| \begin{array}{c} x_i(0) = \gamma_i \\ x_i(1) = \delta_i \end{array} \right. \right) = \frac{1}{Z_n} \det(p(t; \gamma_i, x_j))_{1 \leq i, j \leq n} \det(p(1-t; \gamma_i', x_j'))_{1 \leq i', j' \leq n} \prod_{i=1}^n dx_i
\]

for the Brownian motion kernel

\[
p(t, x, y) := \frac{1}{\sqrt{\pi t}} e^{-\frac{(y-x)^2}{2t}}.
\]

Following Johansson [18], Aptekarev, Bleher, and Kuijlaars introduced in [6] a change of variables transforming the Brownian motion problem into the Gaussian
random ensemble with external source. For \( E := \bigcup_{i=1}^{r} [b_{2i-1}, b_{2i}] \), we have, using this change of variables,
\[
x_i = x_i' \sqrt{\frac{t(1-t)}{2}} \quad \text{and} \quad y_i = y_i' \sqrt{\frac{t(1-t)}{2}}
\]
in equality \(*\):
\[
\mathbb{P}^{\pm a} (\text{all } x_j(t) \in E) := \mathbb{P} \left( \text{all } x_j(t) \in E \middle| \begin{array}{l}
\text{k left paths end up at } -a \text{ at time } t = 1, \\
\text{k right paths end up at } +a \text{ at time } t = 1
\end{array} \right)
\]
\[
= \lim_{\delta_1, \ldots, \delta_{\frac{k}{2}} \rightarrow -a} \mathbb{E}_n \left[ \prod_{i=1}^{n} e^{-\frac{y_i'^2}{2} - \frac{2a y_i'}{e^{\sqrt{2t} y_i'}}} \right] 
\]
with \( \mathbb{P}_n \) of Theorem 0.1, using (1.1) and (1.3), with \( k = k_1 = k_2 \). Setting
\[
(5.1) \quad e^{g(t)} := \sqrt{\frac{2t}{1-t}}, \quad e^{h(t)} := \sqrt{\frac{2}{t(1-t)}},
\]
\[
(5.2) \quad \tilde{B}_k = \sum_{i} v_j^{k+1} \frac{\partial}{\partial v_i}, \quad B_k = \sum_{i} b_j^{k+1} \frac{\partial}{\partial b_i},
\]
we find
\[
\mathbb{P}_0^\pm(t; b_1, \ldots, b_{2r}) = \mathbb{P}_n(a e^{h(t)}; b_1 e^{h(t)}, \ldots, b_{2r} e^{h(t)})
\]
(5.3)
= \mathbb{P}_n(u; v_1, \ldots, v_{2r}) \bigr|_{u=a e^{h(t)}, \atop v=b e^{h(t)}}.

From Theorem 0.1, it follows that \( \mathbb{P}_n(u; v_1, \ldots, v_{2r}) \) satisfies the nonlinear equation (0.5), with \( a \) and all \( b_i \)'s replaced by \( u \) and \( v_i \), respectively.

In order to find the equation for \( \mathbb{P}_0^\pm(t; b_1, \ldots, b_{2r}) \), one needs to compute the partial derivatives in \( t_i \) and \( b_i \) in terms of partials in \( u \) and \( v_i \), appearing in equation (0.5), and use the relationship (5.3). To be precise, compute
\[
\left( \frac{\partial}{\partial t} \right)^i (B_0)^j(B_{-1})^\ell \mathbb{P}_0^\pm \quad \text{with} \quad i + j + \ell \leq 4 \quad \text{and} \quad i, j, \ell \geq 0.
\]
(5.4)
This yields a system of 34 linear equations in 34 unknowns,
\[
\left( \frac{\partial}{\partial u} \right)^i (\tilde{B}_0)^j(\tilde{B}_{-1})^\ell \mathbb{P}_n \quad \text{with} \quad i + j + \ell \leq 4 \quad \text{and} \quad i, j, \ell \geq 0,
\]
(5.5)
which one solves. Notice that one always writes \((\tilde{B}_0)^j(\tilde{B}_{-1})^\ell \mathbb{P}_n \) in that order, using the commutation relation \([\tilde{B}_{-1}, \tilde{B}_0] = \tilde{B}_{-1}\). For instance,
\[
(B_{-1})^j \mathbb{P}_0^\pm = e^{h(t)}(\tilde{B}_{-1})^j \mathbb{P}_n, \quad (B_0)^j \mathbb{P}_0^\pm = (\tilde{B}_0)^j \mathbb{P}_n, \quad j = 1, \ldots, 4,
\]
\[
\frac{\partial}{\partial t} \mathbb{P}_0^\pm = \left( g'(t)u \frac{\partial}{\partial u} + h'(t)\tilde{B}_0 \right) \mathbb{P}_n,
\]
\[
\frac{\partial^2}{\partial t^2} \mathbb{P}_0^\pm = \left( g'(t)u \frac{\partial}{\partial u} + h'(t)\tilde{B}_0 \right) \left( g'(t)u \frac{\partial}{\partial u} + h'(t)\tilde{B}_0 \right) \mathbb{P}_n
\]
\[
+ \left( g''(t)u \frac{\partial}{\partial u} + h''(t)\tilde{B}_0 \right) \mathbb{P}_n
\]
\[
:.
\]
The partials (5.5) thus obtained are now substituted into the fourth-order equation (0.5), with \( a \) and \( b_i \) replaced by \( u \) and \( v_i \), and thus the \( B_j \) by \( \tilde{B}_j \), yielding a new fourth-order equation involving the partials (5.4).

Now let the number of particles \( n \) go to infinity, together with the corresponding scaling (see [6, 24]), with \( s \in \mathbb{R} \),
\[
n = 2k = \frac{2}{\varepsilon^4}, \quad \pm a = \pm \frac{1}{\varepsilon^2}, \quad b_i = x_i z, \quad t = \frac{1}{2} + sz^2 \quad \text{for} \quad z \to 0.
\]
(5.6)
It is convenient to replace the \( \pm \) in (5.6) by the variable \( \varepsilon \), which one keeps in the computation as a variable. The scaling combined with the change of variables (5.3)
leads to the following expressions $u$ and $v_i$ in terms of $z$:

\[ 2k = \frac{2}{z^4}, \]

\[ u = a e^{g(t)} = a \sqrt{\frac{2t}{1-t}} = \epsilon \frac{\sqrt{2}}{z^2} \sqrt{\frac{1}{2} + sz^2}, \]

\[ v_i = b_i e^{h(t)} = b_i \sqrt{\frac{2}{t(1-t)}} = xi z \frac{\sqrt{2}}{\sqrt{1/4 - s^2 z^4}}. \]

So, the question now is to estimate

\[ \left\{ (F^+ \tilde{B}^-_1 G^- + F^- \tilde{B}^-_1 G^+)(F^+ \tilde{B}^-_1 F^- - F^- \tilde{B}^-_1 F^+) - (F^+ G^- + F^- G^+)(F^+ \tilde{B}^-_2 F^- - F^- \tilde{B}^-_2 F^+) \right\} \bigg|_{u \mapsto \frac{2}{z^4} \sqrt{\frac{1}{2} + sz^2}}^n \]

\[ v_i \mapsto xi \frac{\sqrt{2}}{\sqrt{1/4 - s^2 z^4}}, \]

\[ n \mapsto \frac{2}{z^4}. \]

For this, we need to compute the expressions $F^\pm, \tilde{B}^-_1 F^\pm, \tilde{B}^-_2 F^\pm, G^\pm$, and $\tilde{B}^-_1 G^\pm$ appearing in (5.8) in terms of

\[ Q_z(s; x_1, \ldots, x_{2r}) := \log \mathbb{P}_{2/z^4} \left( \frac{\epsilon \sqrt{2}}{z^2} \sqrt{\frac{1}{2} + sz^2}, x_1 \frac{z \sqrt{2}}{\sqrt{1/4 - s^2 z^4}}, \ldots, x_2, x_{2r} \frac{z \sqrt{2}}{\sqrt{1/4 - s^2 z^4}} \right) \]

\[ = Q(s; x_1, \ldots, x_{2r}) + O(z), \]

with

\[ Q(s; x_1, \ldots, x_{2r}) = \log \det(I - K_{1/K^o}) \]

in terms of the Pearcey kernel, as shown in [24]; for the Pearcey kernel, see (0.10). Before taking a limit of $Q_z(s; x_1, \ldots, x_{2r})$, one computes

\[ F^e = -4 \frac{4}{z^4} - \frac{1}{4z^2} B^2_{-1} Q_z + \frac{\epsilon}{4z} B_{-1} \frac{\partial Q_z}{\partial s} + O(z), \]

\[ \frac{1}{\sqrt{2}} \tilde{B}^-_1 F^e = -\frac{1}{16z^2} B^3_{-1} Q_z + \frac{\epsilon}{16z^2} B_{-1}^2 \frac{\partial Q_z}{\partial s} - \frac{\epsilon s}{8} B_{-1}^3 \frac{\partial Q_z}{\partial s} + O(z), \]

\[ \tilde{B}^-_2 F^e = -\frac{1}{32z^4} B^4_{-1} Q_z + \frac{\epsilon}{32z^3} B_{-1}^3 \frac{\partial Q_z}{\partial s} - \frac{\epsilon s}{16z} B_{-1}^4 \frac{\partial Q_z}{\partial s} + O(1), \]
\[ G^\varepsilon = \frac{3\varepsilon}{8z^9} B^{-1}_1 Q_z + \frac{\varepsilon s}{4z^7} B^3_1 Q_z \]
\[ - \frac{1}{128z^6} \left[ \left( B^{-1}_1 \frac{\partial Q_z}{\partial s} \right) \left( B^3_1 Q_z \right) + 32B_0B^2_1 Q_z \right. \]
\[ - \left( B^2_1 Q_z + 64sB^3_1 \frac{\partial Q_z}{\partial s} \right) - 64B^2_1 Q_z \]
\[ + 16 \left( \frac{\partial Q_z}{\partial s} \right)^3 \] 
+ \( O \left( \frac{1}{z^5} \right) \).

\[ \frac{1}{\sqrt{2}} \tilde{B}^{-1}_1 G^\varepsilon = \frac{3\varepsilon}{32z^{10}} B^4_1 Q_z + \frac{\varepsilon s}{16z^8} B^4_1 Q_z \]
\[ + \frac{1}{512z^7} \left[ - \left( B^{-1}_1 \frac{\partial Q_z}{\partial s} \right) \left( B^4_1 Q_z \right) - 32B_0B^3_1 Q_z \right. \]
\[ + \left( B^2_1 Q_z + 64sB^3_1 \frac{\partial Q_z}{\partial s} \right) + 32B^3_1 Q_z \]
\[ - 16B^{-1}_1 \frac{\partial Q_z}{\partial s} \] 
+ \( O \left( \frac{1}{z^6} \right) \).

The formulae needed to obtain the expansions above for \( G^\varepsilon \) and \( \tilde{B}^{-1}_1 G^\varepsilon \) are given in Appendix B. From the expressions above one readily deduces

\[ F^+ \tilde{B}^{-1}_1 G^- + F^- \tilde{B}^{-1}_1 G^+ = -\frac{\sqrt{2}}{64z^{11}} \left( 2(B^{-1}_1 \frac{\partial Q_z}{\partial s})(B^4_1 Q_z) \right. \]
\[ - 32(B_0 - 2s\frac{\partial Q_z}{\partial s} - 1)B^3_1 Q_z \]
\[ + \left( B^2_1 \frac{\partial Q_z}{\partial s} \right)(B^3_1 Q_z) - 16B^{-1}_1 \frac{\partial Q_z}{\partial s} + 1 \] 
+ \( O \left( \frac{1}{z^5} \right) \),

\[ F^+ \tilde{B}^{-1}_1 F^- - F^- \tilde{B}^{-1}_1 F^+ = \varepsilon \frac{\partial}{\partial s} B^2_1 Q_z \]
\[ \frac{\sqrt{2}}{2z^6} + O \left( \frac{1}{z^4} \right) \],

\[ F^+ G^- + F^- G^+ = -\frac{1}{16z^{10}} \left( 2(B^{-1}_1 \frac{\partial Q_z}{\partial s})(B^3_1 Q_z) \right. \]
\[ - 32(B_0 - 2s\frac{\partial Q_z}{\partial s} - 2)B^2_1 Q_z \]
\[ + \left( B^2_1 \frac{\partial Q_z}{\partial s} \right)(B^2_1 Q_z) - 16\frac{\partial Q_z}{\partial s} \] 
+ \( O \left( \frac{1}{z^8} \right) \),

\[ F^+ \tilde{B}^2_1 F^- - F^- \tilde{B}^2_1 F^+ = \varepsilon \frac{\partial}{\partial s} B^3_1 Q_z \]
\[ \frac{4z^7}{4z^3} + O \left( \frac{1}{z^5} \right) \].
Using these expressions, one easily deduces for small $z$,

$$
0 = \left\{ (F^+ \tilde{B}^{-1} G^+ + F^- \tilde{B}^{-1} G^+)(F^+ \tilde{B}^{-1} F^- - F^- \tilde{B}^{-1} F^+) \right\} \bigg|_{m \to \frac{1}{2z^2} \sqrt{V_{1/2-z^2}}} \bigg|_{v_i \to x_i \sqrt{1/4-z^2}^4} \bigg|_{\nu \to \frac{2}{z}}
$$

$$
= -\frac{\varepsilon}{2z^{17}} \left( \frac{B^2_{-1} \frac{\partial Q}{\partial s} + (B_0 - 2) B^2_{-1} Q_z}{B_{-1}} + \frac{1}{16} B_{-1} \frac{\partial Q}{\partial s} \left( B^3_{-1} Q_z, B^2_{-1} \frac{\partial Q}{\partial s} \right) \right) + O\left( \frac{1}{z^{15}} \right)
$$

$$
= -\frac{\varepsilon}{2z^{17}} \left( \text{the same expression for } Q_z(s; x_1, \ldots, x_{2r}) \right) + O\left( \frac{1}{z^{16}} \right),
$$

using (5.10) in the last equality. Taking the limit when $z \to 0$ yields equation (0.5) of Theorem 0.2.

**Appendix A: Evaluation of an Integral over the Full Range**

Setting

$$
\mu_{i+j-1}(\pm a) := \mu_{i+j}(t, s, u; \beta, \mathbb{R}) \bigg|_{t=s=u=\beta=0} = \int_{\mathbb{R}} z^{i+j-1} e^{-\frac{z^2}{2}} \frac{a}{z} d\tau,
$$

one computes:4

**Lemma A.1**

$$
\tau_{k_1 k_2}(t, s, u; \beta, \mathbb{R}) \bigg|_{t=s=u=\beta=0} = \det \left( \begin{array}{c} \mu_{i+j}(a) \big|_{0 \leq i < k_1 - 1} \\ \mu_{i+j}(-a) \big|_{0 \leq i < n-1} \end{array} \right) = c_{k_1 k_2} a^{k_1 k_2} e^{\frac{k_1 + k_2}{2} a^2}
$$

with

$$
c_{k_1 k_2} = (-2)^{k_1 k_2} (2\pi)^{k_1 + k_2} \prod_{j=0}^{k_1} j! \prod_{j=0}^{k_2} j! .
$$

**Proof:** By explicit integration, one computes

$$
\mu_0(a) = \sqrt{2\pi} e^{\frac{a^2}{2}} \quad \text{and} \quad \mu_i(\pm a) = \sqrt{2\pi} \left( \pm \frac{d}{da} \right)^i e^{\frac{a^2}{2}}.
$$

Define the Hermite polynomials (except for a minor change of variables)

$$
p_i(a) := e^{-\frac{a^2}{2}} \left( \frac{d}{da} \right)^i e^{\frac{a^2}{2}} = \left( \frac{d}{da} + a \right) p_{i-1}(a).
$$

4 Remember $n = k_1 + k_2$.  

The following holds:

\[ p_{2i}(a) = \text{even polynomial}, \quad p_{2i+1}(a) = \text{odd polynomial of } a, \]

which is used in equality \( \overset{**}{=} \) below, and

\[ p_{k+n}(a) = p_k^{(n)} + \beta_1(a)p_k^{(n-1)} + \beta_2(a)p_k^{(n-2)} + \cdots + \beta_n p_k, \]

where \( p_k^{(n)} := (\frac{d}{da})^n p_k \) and where \( \beta_i(a) \) are polynomials in \( a \), independent of \( k \); this feature is used in equality \( \overset{**}{=} \) below. Then we compute:

\[
\begin{align*}
\tau_{k_1k_2}(t, s, u; \beta, \mathbb{R})_{t=s=\beta=0} &= (\sqrt{2\pi})^n e^{\frac{nu^2}{2}} \det \left( \begin{pmatrix}
(p_{i+j})_{0 \leq i \leq k_1-1, 0 \leq j \leq n-1}
(-1)^{i+j} (p_{i+j})_{0 \leq i \leq k_2-1, 0 \leq j \leq n-1}
\end{pmatrix} \right) \\
&= (\sqrt{2\pi})^n (-1)^{\frac{k_1(k_1-1)}{2}} e^{\frac{nu^2}{2}} \det \left( \begin{pmatrix}
(p_{i+j})_{0 \leq i \leq k_1-1, 0 \leq j \leq n-1}
(-1)^j p_{i+j} & 0 \leq i \leq k_2-1, 0 \leq j \leq n-1
\end{pmatrix} \right) \\
&= (\sqrt{2\pi})^n (-1)^{\frac{k_1(k_1-1)}{2}} e^{\frac{nu^2}{2}} \det \left( \begin{pmatrix}
(p_{j}^{(i)})_{0 \leq i \leq k_1-1, 0 \leq j \leq n-1}
(-1)^j p_{j}^{(i)} & 0 \leq i \leq k_2-1, 0 \leq j \leq n-1
\end{pmatrix} \right) \\
&= \overset{**}{=} c_{k_1k_2} e^{\frac{nu^2}{2}} \det \left( \begin{pmatrix}
((a^{j-1})^{(i)})_{0 \leq i \leq k_1-1, 1 \leq j \leq n}
((-a)^{j-1})^{(i)} & 0 \leq i \leq k_2-1, 1 \leq j \leq n
\end{pmatrix} \right) \\
&= c_{k_1k_2} e^{\frac{nu^2}{2}} \det \left( \begin{pmatrix}
(\alpha_{ij} a^{j-1-i})_{1 \leq i \leq k_1, 1 \leq j \leq n}
(\alpha_{ij} a^{j-1-i+k_1})_{k_1+1 \leq i \leq n}
\end{pmatrix} \right) \\
&= c_{k_1k_2} e^{\frac{nu^2}{2}} \sum_{\sigma \in S_n} (-1)^\sigma \prod_{1 \leq i \leq k_1} \alpha_{i\sigma(i)} a^{\sigma(i)-i} \prod_{k_1+1 \leq i \leq n} \alpha_{i\sigma(i)} a^{\sigma(i)-i+k_1} \\
&= c_{k_1k_2} e^{\frac{nu^2}{2}} \sum_{\sigma \in S_n} (-1)^\sigma a^{\sum_{i=1}^n (\sigma(i)-i) (a_i)^{k_2}} \prod_{1 \leq i \leq n} \alpha_{i\sigma(i)} \\
&= c'_{k_1k_2} e^{\frac{(k_1+k_2)^2}{2\pi n^2}} a^{k_1k_2},
\end{align*}
\]
where the $\alpha_{ij}$ are coefficients, some of which vanish. Indeed, each of the blocks in the matrix above is upper-triangular. To evaluate $c_{k_1k_2}'$, observe, upon completing the squares in the exponentials and setting $x_j \mapsto x_j - a$ and $y_j \mapsto y_j + a$ in the integral,

$$
\tau_{k_1k_2}(t, s, u; \beta; \mathbb{R})\big|_{t=s=u=\beta=0} = \frac{1}{k_1!k_2!} \int_{\mathbb{R}^{k_1+k_2}} \Delta_{k_1+k_2}(x, y) \left( \Delta_{k_1}(x) \prod_{j=1}^{k_1} e^{-\frac{x_j^2}{2} - ax_j} \, dx_j \right)

\cdot \left( \Delta_{k_2}(y) \prod_{j=1}^{k_2} e^{-\frac{y_j^2}{2} - ayy} \, dy_j \right).
$$

This integral equals

$$
e^{(k_1+k_2)\frac{z^2}{2}} \left( -2a \right)^{k_1k_2} c_{k_1,0}c_{0,k_2} + \text{lower-order terms in } a
$$

$$
= e^{(k_1+k_2)\frac{z^2}{2}} \left( -2a \right)^{k_1k_2} (2\pi)^{k_1+k_2} \prod_{0}^{k_1-1} j! \prod_{0}^{k_2-1} j! + \text{lower-order terms in } a
$$

The result in the first part of this proof implies the absence of the lower terms and thus Lemma A.1 follows.

Appendix B: Some Asymptotics

In order to compute the asymptotics (5.9) for the expression $G^\epsilon$ and $B_{-1}G^\epsilon$, as defined in (0.6), one needs the following asymptotics:

$$F^\epsilon = -\frac{4}{z^4} - \frac{1}{4z^2} B_{-1}^2 Q_z + \frac{\epsilon}{4z} B_{-1} \frac{\partial Q_z}{\partial s} + O(z),$$

$$\frac{1}{\sqrt{2}} \tilde{B}_{-1} F^\epsilon = -\frac{1}{16z^3} B_{-1}^3 Q_z + \frac{\epsilon}{16z^2} B_{-1} \frac{\partial Q_z}{\partial s} - \frac{\epsilon s}{8} B_{-1}^2 \frac{\partial Q_z}{\partial s} + O(z),$$

$$\tilde{B}_2^2 F^\epsilon = -\frac{1}{32z^4} B_{-1}^4 Q_z + \frac{\epsilon}{32z^3} B_{-1}^3 \frac{\partial Q_z}{\partial s} - \frac{\epsilon s}{16z} B_{-1}^2 \frac{\partial Q_z}{\partial s} + O(1),$$

$$\frac{1}{\sqrt{2}} \frac{\partial}{\partial a} F^\epsilon = -\frac{\epsilon}{16z^2} B_{-1}^2 \frac{\partial Q_z}{\partial s} + O \left( \frac{1}{z} \right),$$

$$\frac{\partial}{\partial a} \tilde{B}_{-1} F^\epsilon = -\frac{\epsilon}{32z^3} B_{-1}^3 \frac{\partial Q_z}{\partial s} + O \left( \frac{1}{z^2} \right),$$

$$\frac{1}{\sqrt{2}} H_1^\epsilon = \frac{6\epsilon}{z^6} + \frac{4\epsilon s}{z^4} - \frac{1}{8z^3} B_{-1} \frac{\partial Q_z}{\partial s} + O \left( \frac{1}{z^7} \right).$$
\[
\begin{align*}
\mathcal{B}_{-1} H^\varepsilon_1 &= -\frac{1}{16z^4} B_{-1}^2 \frac{\partial Q_z}{\partial s} - \frac{\varepsilon}{16z^3} B_{-1} \frac{\partial^2 Q_z}{\partial s^2} + \frac{1}{8z^2} B_0 B_{-1}^2 Q_z \\
&+ O\left(\frac{1}{z}\right), \\
\frac{1}{\sqrt{2}} \mathcal{B}_{-1}^2 H^\varepsilon_1 &= -\frac{1}{64z^5} B_{-1}^3 \frac{\partial Q_z}{\partial s} - \frac{\varepsilon}{64z^4} B_{-1}^2 \frac{\partial^2 Q_z}{\partial s^2} + \frac{1}{32z^3} (B_0 + 1) B_{-1}^3 Q_z \\
&+ O\left(\frac{1}{z^2}\right), \\
\frac{1}{\sqrt{2}} H^\varepsilon_2 &= \frac{\varepsilon}{4z^4} B_{-1}^2 Q_z + O\left(\frac{1}{z^3}\right), \\
\frac{\partial}{\partial a} H^\varepsilon_2 &= \frac{1}{8z^4} B_{-1}^3 \frac{\partial Q_z}{\partial s} - \frac{1}{16z^3} B_{-1} \frac{\partial^2 Q_z}{\partial s^2} - \frac{1}{16z^2} \left(\frac{3}{8} \frac{\partial^3}{\partial s^3} - 4B_{-1}^2\right) Q_z \\
&+ O\left(\frac{1}{z}\right), \\
\tilde{B}_{-1} H^\varepsilon_2 &= \frac{\varepsilon}{8z^5} B_{-1}^3 Q_z + O\left(\frac{1}{z^4}\right), \\
\frac{1}{\sqrt{2}} \frac{\partial}{\partial a} \tilde{B}_{-1} H^\varepsilon_2 &= \frac{1}{32z^5} B_{-1}^3 \frac{\partial Q_z}{\partial s} - \frac{\varepsilon}{64z^4} B_{-1}^2 \frac{\partial^2 Q_z}{\partial s^2} \\
&- \frac{1}{64z^3} \left(\frac{3}{8} \frac{\partial^3}{\partial s^3} - 4B_{-1}^2\right) B_{-1} Q_z + O\left(\frac{1}{z^2}\right).
\end{align*}
\]

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