Direct Proof of Mirror Theorem of Projective Hypersurfaces up to degree 3 Rational Curves

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Abstract
In this paper, we directly derive generalized mirror transformation of projective hypersurfaces up to degree 3 genus 0 Gromov-Witten invariants by comparing Kontsevich localization formula with residue integral representation of the virtual structure constants. We can easily generalize our method for rational curves of arbitrary degree except for combinatorial complexities.

1 Introduction
In this paper, we prove generalized mirror transformation of genus 0 Gromov-Witten invariants of degree $k$ hypersurface in $CP^{N-1}$ (we denote it by $M^N_k$) up to degree 3 rational curves. For this purpose, we introduce here the virtual structure constants of $M^N_k$, that were first defined in our ancient work with A. Collino [2].

Definition 1 The virtual structure constants $\tilde{L}^{N,k,d}_{n}$ ($d \leq 3$, $\tilde{L}^{N,k,d}_{n} \neq 0$ only if $0 \leq n \leq N - 1 - (N - k)d$) are rational numbers defined by the initial condition and the recursive formula:

$$\sum_{n=0}^{k-1} \tilde{L}^{N,k,1}_{n} w^n = k \cdot \prod_{j=1}^{k-1} (jw + (k - j)), \quad (1.1)$$

$$\tilde{L}^{N,k,1}_{n} = \tilde{L}^{N+1,k,1}_{n}, \quad (1.2)$$

$$\tilde{L}^{N,k,2}_{n} = \frac{1}{2} \frac{\tilde{L}^{N+1,k,2}_{n-1}}{2} + \frac{1}{2} \frac{\tilde{L}^{N+1,k,2}_{n-2}}{2} + \frac{1}{2} \tilde{L}^{N+1,k,1}_{n} \cdot \tilde{L}^{N+1,k,1}_{n+(N-k)}, \quad (1.3)$$

$$\tilde{L}^{N,k,3}_{n} = \frac{2}{9} \tilde{L}^{N,k,3}_{n-3} + \frac{5}{9} \tilde{L}^{N,k,3}_{n-2} + \frac{2}{9} \tilde{L}^{N+1,k,3}_{n-3}$$

$$+ \frac{4}{9} \frac{\tilde{L}^{N+1,k,2}_{n-2} \cdot \tilde{L}^{N+1,k,1}_{n+2(N-k)}}{2} + \frac{1}{3} \tilde{L}^{N+1,k,2}_{n} \cdot \tilde{L}^{N+1,k,1}_{n+2(N-k)}$$

$$+ \frac{2}{9} \tilde{L}^{N+1,k,2}_{n} \cdot \tilde{L}^{N+1,k,1}_{n+2(N-k)} + \frac{1}{3} \tilde{L}^{N+1,k,2}_{n} \cdot \tilde{L}^{N+1,k,1}_{n+2(N-k)} + \frac{1}{3} \tilde{L}^{N+1,k,1}_{n} \cdot \tilde{L}^{N+1,k,1}_{n+2(N-k)} + \frac{1}{3} \tilde{L}^{N+1,k,1}_{n} \cdot \tilde{L}^{N+1,k,1}_{n+2(N-k)} + \frac{1}{3} \tilde{L}^{N+1,k,1}_{n} \cdot \tilde{L}^{N+1,k,1}_{n+2(N-k)} + \frac{1}{3} \tilde{L}^{N+1,k,1}_{n} \cdot \tilde{L}^{N+1,k,1}_{n+2(N-k)} + \frac{1}{3} \tilde{L}^{N+1,k,1}_{n} \cdot \tilde{L}^{N+1,k,1}_{n+2(N-k)}.$$
We can indeed prove (1.8) holds true for arbitrary residue integral representation of \( \tilde{L}_n^{N,k,d} \). In (1.8), \( h \) is hyperplane class of \( CP^{N-1} \), \( \bar{M}_{0,n}(CP^{N-1},d) \) represents moduli space of degree \( d \) stable maps from genus \( 0 \) stable curve to \( CP^{N-1} \) with \( n \) marked points, \( e_v : \bar{M}_{0,n}(CP^{N-1},d) \to CP^{N-1} \) is evaluation map of the \( i \)-th marked point and \( \tau : \bar{M}_{0,2}(CP^{N-1},d) \to \bar{M}_{0,2}(CP^{N-1},d) \) is forgetful map. Definition of the virtual structure constants for arbitrary degree \( d \) (\( \geq 1 \)) can be seen in [7]. In our previous paper [9], we conjectured a residue integral representation of \( \tilde{L}_n^{N,k,d} \), which can be interpreted as a result of localization computation on the moduli space of Gauged Sigma Model. In the following, we prepare some notations to describe the formula we conjectured. First, we define rational functions in \( u, v \) by,

\[
e(k, d; u, v) := \prod_{m=0}^{kd} \left( \frac{mu + (kd - m)v}{d} \right),
\]

\[
t(N, d; u, v) := \prod_{m=1}^{d-1} \left( \frac{mu + (d - m)v}{d} \right)^N.
\]

Next, we introduce ordered partition of positive integer \( d \):

**Definition 2** Let \( OP_d \) be the set of ordered partitions of positive integer \( d \):

\[
OP_d = \{ \sigma_d = (d_1, d_2, \cdots, d_{l(\sigma_d)}) \mid \sum_{j=1}^{l(\sigma_d)} d_j = d, \; d_j \in \mathbb{N} \}.
\]

From now on, we denote a ordered partition \( \sigma_d \) by \((d_1, d_2, \cdots, d_{l(\sigma_d)})\). In (1.7), we denote the length of the ordered partition \( \sigma_d \) by \( l(\sigma_d) \).

With these set up, the residue integral representation is given as follows:

\[
\frac{\tilde{L}_n^{N,k,d}}{d} = \frac{1}{k} \sum_{\sigma_d \in OP_d} \frac{1}{(2\pi i)^{l(\sigma_d)+1}} \prod_{j=1}^{l(\sigma_d)} \prod_{j=1}^{l(\sigma_d)-1} \int_C \frac{dx_{j+x_{j-1}}}{x_{j+x_{j-1}}} \sum_{j=0}^{l(\sigma_d)} \frac{1}{\prod_{j=1}^{l(\sigma_d)} \prod_{j=1}^{l(\sigma_d)-1} k x_j \left( \frac{x_{j+x_{j-1}}}{d_j} + \frac{x_{j+x_{j+1}}}{d_{j+1}} \right)} e(k; d_j; x_{j-1}, x_j) t(N; d_j; x_{j-1}, x_{j})
\]

In (1.8), \( \frac{1}{2\pi i} \int_C \) represents operation of taking residue at \( x_j = 0 \). We have to mention here that the residue integral in (1.8) severely depends on order of integration. To be more precise, we have to take the residues of \( x_j \)'s in descending (or ascending) order of subscript \( j \). In the appendix of this paper, we prove,

**Theorem 1** (1.8) holds true if \( d \leq 3 \).

We can indeed prove (1.8) holds true for arbitrary \( d \), but we write down proof for \( d \leq 3 \) cases in this paper, mainly for convenience of space. Full proof will appear elsewhere. If \( N \leq k \), the Gromov-Witten invariant \( \frac{1}{k} \langle O_{h^{N-2-n}}O_{h^{N-1}(N-k)d} \rangle_{0,d} \) and \( \tilde{L}_n^{N,k,d} \) are different. In this case, we can write the former by weighted homogeneous polynomial of \( \tilde{L}_m^{N,k,d'} \) (\( d' \leq d \)). This formula is the generalized mirror transformation in our sense. Main result of this paper is a proof of this transformation up to \( d = 3 \) case, that was conjectured in [8]:

**Theorem 2**

\[
\frac{1}{k} \langle O_{h^{N-2-n}}O_{h^{N-1}(N-k)} \rangle_{0,1} = \tilde{L}_n^{N,k,1} - \tilde{L}_1^{N,k,1}.
\]

\[
\frac{1}{k} \langle O_{h^{N-2-n}}O_{h^{N-1}2(N-k)} \rangle_{0,2} = \frac{1}{2} \left( \tilde{L}_n^{N,k,2} - \tilde{L}_1^{N,k,1} \left( 1+2(k-N) \right) - \tilde{L}_1^{N,k,1} \sum_{j=0}^{k-N} \left( \tilde{L}_n^{N,k,1} - \tilde{L}_1^{N,k,1} \left( 1+2(k-N-j) \right) \right) \right).
\]
where
\[
A_j := j + 1, \quad \text{if} \quad 0 \leq j \leq k - N, \quad A_j := 1 + 2(k - N) - j, \quad \text{if} \quad (k - N) \leq j \leq 2(k - N),
\]
\[
C_{1,1}^{N,k,3}(n) = \sum_{j=0}^{(k-N)-1} \sum_{m=0}^{j} \left( \sum_{n-m=0} \tilde{L}^{N,k,3}_{n-m} \tilde{L}^{N,k,1}_{n-2(k-N)+j-m} - \tilde{L}^{N,k,1}_{(k-N)+j+2} \left( \sum_{m=0}^{2(k-N)} \tilde{L}^{N,k,1}_{n-m} \right) \right)
\]
\[
+ \tilde{L}^{N,k,1}_{1+(k-N)} \left( \sum_{m=j+1}^{2(k-N)-j-1} \tilde{L}^{N,k,1}_{n-m} \right)
\]
\[
- \sum_{j=0}^{(k-N)-1} \sum_{m=0}^{j} \left( \sum_{n-m=0} \tilde{L}^{N,k,1}_{1+(k-N)+m} \tilde{L}^{N,k,1}_{(k-N)+j+m} - \tilde{L}^{N,k,1}_{1+(k-N)+j+2} \sum_{m=0}^{2(k-N)} \tilde{L}^{N,k,1}_{n-m} \right)
\]
\[
+ \tilde{L}^{N,k,1}_{1+(k-N)} \left( \sum_{m=j+1}^{2(k-N)-j-1} \tilde{L}^{N,k,1}_{1+(k-N)-m} \right).
\]

Of course, the above formulas can be derived by using known methods presented in various papers: [1, 5, 6, 11]. In these works, generalized mirror transformation is derived as the effect of coordinate change of the B-model deformation parameters into the A-model ones. We feel that this process is a little bit too sophisticated to capture geometrical image of generalized mirror transformation: change of the moduli space of Gauged Linear Sigma Model into the one of stable maps. In this paper, we present an elementary and direct proof of Theorem 2 by using the result of Kontsevich [10] and Theorem 1. Our strategy is the following. First, we write down explicit formula of \( \langle \mathcal{O}_{h^{N-2-n}} \mathcal{O}_{h^{N-1+3(N-k)}} \rangle_d \) that follows from localization computation of Kontsevich. This formula includes combinatorially complicated summations with characters of torus action \( \lambda_j, (j = 1, \cdots, N) \), but we can rewrite these summations into residue integrals of finite complex variables. This process is a generalization of the well-known computation on Bott residue theorem, that can be seen p.434-435 of [4]. After this operation, we take non-equivariant limit \( \lambda_j \rightarrow 0 \). Resulting formula is very close to our residue integral representation of \( \tilde{L}^{N,k,d}_n \) in (1.8). With this formula, what we need for proof of Theorem 2 is elementary combinatorial decomposition of rational functions in the integrands.

This paper is organized as follows. In Section 2, we explain the process to reduce combinatorial summations in Kontsevich’s localization formula to residue integrals in finite variables. Then we present residue integral representation of 2-point Gromov-Witten invariants. This representation can be directly compared with the r.h.s of (1.8) after taking non-equivariant limit \( \lambda_j \rightarrow 0 \). In Section 3, we prove Theorem 2 by using decomposition of rational functions in the integrands. Section 3 gives concluding remarks. In Appendix, we prove Theorem 1, that plays important role in the proof of Theorem 2.

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## 2 Reduction of Localization Formula to Residue Integral

We start from the Kontsevich’s localization formulas for 2-point genus 0 Gromov-Witten invariants of \( M^k_N \). For compact presentation of these formulas, we introduce several notations:

\[
w_a(u, v) := \frac{u^a - v^a}{u - v} = \sum_{p+q=a-1, p,q \geq 0} u^p v^q,
\]
\[ w_a(u, v, w) := \sum_{p+q+r=a-2, p, q, r \geq 0} u^p v^q w^r, \]  

(2.13)

and,

\[ E(k; d; i, j) := \prod_{m=0}^{kd} \left( \frac{m\lambda_i + (kd - m)\lambda_j}{d} \right), \]  

\[ V(N; i) := \prod_{j \neq i, 1 \leq j \leq N} (\lambda_j - \lambda_i), \]  

\[ T(N; d; i, j) := \prod_{k=1}^{N} \prod_{m=1}^{d-1} \left( \frac{m\lambda_i + (d - m)\lambda_j}{d} - \lambda_k \right). \]  

(2.14)

In (2.14), \( \lambda_j \) \((j = 1, \cdots, N)\) are characters of torus action on \( CP^{N-1} \):

\[ (X_1 : \cdots : X_N) \to (e^{\lambda_1 t}X_1 : \cdots : e^{\lambda_N t}X_N). \]  

(2.15)

Here, we also introduce an elementary equality that will be used later in this paper:

\[ w_a(x_1, x_2) + w_a(x_2, x_3) = (2x_2 - x_1 - x_3)w_a(x_1, x_2, x_3) + 2w_a(x_1, x_3). \]  

(2.16)

With these set-up’s, the localization formulas that represent \( \langle \mathcal{O}_{h^a} \mathcal{O}_{h^b} \rangle_{0, d} \) \((a = N - 2, \ b = n - 1 + (N - k)d)\) are described as follows:

**Fact 1** (Kontsevich)

\[ \langle \mathcal{O}_{h^a} \mathcal{O}_{h^b} \rangle_{0, 1} = -\frac{1}{2} \sum_{i \neq j} E(k; 1; i, j)(\lambda_i - \lambda_j)^2 \frac{V(N; i)V(N; j)}{V(N; i)V(N; j)} \cdot w_a(\lambda_i, \lambda_j)w_b(\lambda_i, \lambda_j), \]  

(2.17)

\[ \langle \mathcal{O}_{h^a} \mathcal{O}_{h^b} \rangle_{0, 2} = -\frac{1}{4} \sum_{i \neq j} E(k; 2; i, j)(\lambda_i - \lambda_j)^2 \frac{T(N; 2; i, j)V(N; i)V(N; j)}{T(N; i)V(N; j)V(N; i)V(N; j)} \cdot w_a(\lambda_i, \lambda_j)w_b(\lambda_i, \lambda_j) + \]  

\[ + \frac{1}{2} \sum_{i \neq j \neq l} E(k; 1; i, j)E(k; 1; j, l) \frac{1}{V(N; i)V(N; j)V(N; l)k\lambda_j} \cdot \frac{1}{\lambda_j - \lambda_i + \lambda_j - \lambda_i} \times \]  

\[ \times (w_a(\lambda_i, \lambda_j) + w_a(\lambda_i, \lambda_l))(w_b(\lambda_i, \lambda_j) + w_b(\lambda_i, \lambda_l)), \]  

(2.18)

\[ \langle \mathcal{O}_{h^a} \mathcal{O}_{h^b} \rangle_{0, 3} = \frac{1}{6} \sum_{i \neq j} E(k; 3; i, j)(\lambda_i - \lambda_j)^2 \frac{T(N; 3; i, j)V(N; i)V(N; j)}{T(N; i)V(N; j)V(N; i)V(N; j)} \cdot w_a(\lambda_i, \lambda_j)w_b(\lambda_i, \lambda_j) + \]  

\[ + \frac{1}{2} \sum_{i \neq j \neq l \neq m} \frac{E(k; 2; i, j)E(k; 1; j, l)}{T(N; 2; i, j)V(N; i)V(N; j)V(N; l)V(N; i)V(N; j)} \frac{1}{k\lambda_j} \cdot \frac{1}{\lambda_j - \lambda_i + \lambda_j - \lambda_i} \times \]  

\[ \times (w_a(\lambda_i, \lambda_j) + w_a(\lambda_i, \lambda_l))(w_b(\lambda_i, \lambda_j) + w_b(\lambda_i, \lambda_l)) - \]  

\[ - \frac{1}{2} \sum_{i \neq j \neq l \neq m} E(k; 1; i, j)E(k; 1; j, l)E(k; 1; l, m) \frac{1}{V(N; i)V(N; j)V(N; l)V(N; m)k\lambda_j} \times \]  

\[ \times (w_a(\lambda_i, \lambda_j) + w_a(\lambda_i, \lambda_l))(w_b(\lambda_i, \lambda_j) + w_b(\lambda_i, \lambda_l)) \times \]  

\[ \times (w_a(\lambda_i, \lambda_l) + w_a(\lambda_i, \lambda_m))(w_b(\lambda_i, \lambda_l) + w_b(\lambda_i, \lambda_m)) \times \]  

\[ \times (w_a(\lambda_i, \lambda_m) + w_a(\lambda_i, \lambda_m))(w_b(\lambda_i, \lambda_m) + w_b(\lambda_i, \lambda_m)). \]  

(2.19)
**Remark 1** In (2.17), (2.18) and (2.19), each summand corresponds to a tree graph that represents degeneration type of stable maps [10]. The r.h.s.’s of these equalities are invariant under variation of characters of torus action, but in (2.18) and (2.19) each summand indeed varies under variation of them.

Though some elementary simplification of complicated terms is operated, these formulas follow from the results in [10]. The above formulas includes many complicated summations, but we can rewrite these summations into residue integrals in finite complex variables as follows:

**Proposition 1**

\[
(\mathcal{O}_{h^s} \mathcal{O}_{h^s})_1 = -\frac{1}{2} (2\pi \sqrt{-1})^2 \int_{C_0} dx_2 \int_{C_0} dx_1 \frac{e(k,1;x_1,x_2)(x_1 - x_2)^2}{(x_1)^N(x_2)^N} \cdot w_a(x_1,x_2)w_b(x_1,x_2), \tag{2.20}
\]

\[
(\mathcal{O}_{h^s} \mathcal{O}_{h^s})_2 = -\frac{1}{4} (2\pi \sqrt{-1})^2 \int_{C_0} dx_2 \int_{C_0} dx_1 \frac{e(k,2;x_1,x_2)(x_1 - x_2)^2}{(x_1 + x_2)^N(x_1)^N(x_2)^N} \cdot w_a(x_1,x_2)w_b(x_1,x_2) + \frac{1}{2} (2\pi \sqrt{-1})^2 \int_{C_0} dx_2 \int_{C_0} dx_1 \frac{e(k,1;x_1,x_2)e(k,1;x_2,x_3)}{(x_1)^N(x_2)^N(x_3)^N} \cdot \frac{1}{x_2 - x_1 + x_2 - x_3} \times (w_a(x_1,x_2) + w_a(x_2,x_3))(w_b(x_1,x_2) + w_b(x_2,x_3)), \tag{2.21}
\]

\[
(\mathcal{O}_{h^s} \mathcal{O}_{h^s})_3 = -\frac{1}{6} (2\pi \sqrt{-1})^2 \int_{C_0} dx_2 \int_{C_0} dx_1 \frac{e(k,3;x_1,x_2)(x_1 - x_2)^2}{(x_1 + x_2 + x_3)^N(x_1)^N(x_2)^N(x_3)^N} \cdot w_a(x_1,x_2)w_b(x_1,x_2) + \frac{1}{4} (2\pi \sqrt{-1})^2 \int_{C_0} dx_2 \int_{C_0} dx_1 \frac{e(k,2;x_1,x_2)e(k,1;x_2,x_3)}{(x_1 + x_2 + x_3)^N(x_1)^N(x_2)^N(x_3)^N} \cdot \frac{1}{x_2 - x_1 + x_2 - x_3 + x_3 - x_1} \times (2w_a(x_1,x_2) + w_a(x_2,x_3))(2w_b(x_1,x_2) + w_b(x_2,x_3)) + \frac{1}{4} (2\pi \sqrt{-1})^2 \int_{C_0} dx_2 \int_{C_0} dx_1 \frac{e(k,1;x_1,x_2)e(k,2;x_2,x_3)}{(x_1 + x_2 + x_3)^N(x_1)^N(x_2)^N(x_3)^N} \cdot \frac{1}{x_2 - x_1 + x_2 - x_3} \times (w_a(x_1,x_2) + 2w_a(x_2,x_3))(w_b(x_1,x_2) + 2w_b(x_2,x_3)) - \frac{1}{2} (2\pi \sqrt{-1})^2 \int_{C_0} dx_2 \int_{C_0} dx_3 \int_{C_0} dx_2 \int_{C_0} dx_1 \frac{e(k,1;x_1,x_2)e(k,1;x_2,x_3)}{(x_1 + x_2 + x_3)^N(x_1)^N(x_2)^N(x_3)^N} \cdot \frac{1}{x_2 - x_1 + x_2 - x_3 + x_3 - x_1} \times (w_a(x_1,x_2) + w_a(x_2,x_3) + w_a(x_3,x_4))(w_b(x_1,x_2) + w_b(x_2,x_3) + w_b(x_3,x_4)) - \frac{1}{6} (2\pi \sqrt{-1})^2 \int_{C_0} dx_2 \int_{C_0} dx_3 \int_{C_0} dx_2 \int_{C_0} dx_1 \frac{e(k,1;x_1,x_2)e(k,1;x_2,x_3)e(k,1;x_1,x_4)}{(x_1 + x_2 + x_3)^N(x_1)^N(x_2)^N(x_3)^N(x_4)^N} \cdot \frac{1}{x_2 - x_1 + x_2 - x_3 + x_3 - x_1} \times (w_a(x_1,x_2) + w_a(x_1,x_3) + w_a(x_1,x_4))(w_b(x_1,x_2) + w_b(x_1,x_3) + w_b(x_1,x_4)). \tag{2.22}
\]

**proof** For convenience of space, we write down proof of $d = 1, 2$ cases. Proof of $d = 3$ case can be done in the same way as the proof below. We start from $d = 1$ case. By elementary residue theorem, we can rewrite the r.h.s. of (2.17) into the following residue integral:

\[
(\mathcal{O}_{h^s} \mathcal{O}_{h^s})_1 = -\frac{1}{2} (2\pi \sqrt{-1})^2 \int_{C_{(0,R)}} dx_2 \int_{C_{(0,R)}} dx_1 \frac{e(k,1;x_1,x_2)(x_1 - x_2)^2}{N_{j=1}((x_1 - \lambda_j)(x_2 - \lambda_j))} \cdot w_a(x_1,x_2)w_b(x_1,x_2). \tag{2.23}
\]

In (2.23), $C(0,a)$ denotes the circle with center at 0 and with radius $a$, and $R$ is a sufficiently large positive real number greater than max.$\{|\lambda_j| \mid j = 1, \cdots, N\}$. In $d = 2$ case, we also have similar equalities:

\[
-\frac{1}{4} (2\pi \sqrt{-1})^2 \int_{C_{(0,R)}} dx_2 \int_{C_{(0,R)}} dx_1 \frac{e(k,2;x_1,x_2)(x_1 - x_2)^2}{N_{j=1}((x_1 - \lambda_j)(x_2 - \lambda_j))} \cdot w_a(x_1,x_2)w_b(x_1,x_2) = -\frac{1}{4} \sum_{i \neq j} \frac{E(k,2;i,j)(\lambda_i - \lambda_j)^2}{N(2;i)V(N;i)V(N;j)} \cdot w_a(\lambda_i, \lambda_j)w_b(\lambda_i, \lambda_j) - 2\sum_{i \neq j} \frac{e(k,2;\lambda_i,2\lambda_j - \lambda_i)(\lambda_i - \lambda_j)^2}{N(2(i,j)(\lambda_i - \lambda_j))^2} \cdot w_a(\lambda_i, 2\lambda_j - \lambda_i)w_b(\lambda_i, 2\lambda_j - \lambda_i). \tag{2.24}
\]
and,

\[
\frac{1}{2} \frac{1}{(2\pi \sqrt{-1})^3} \int_{C(0,5R)} dx_3 \int_{C(0,5R)} dx_2 \int_{C(0,5R)} dx_1 \frac{e(k,1;x_1,x_2) e(k,1;x_2,x_3)}{kx_2 \prod_{j=1}^{N} ((x_1 - \lambda_j)(x_2 - \lambda_j)(x_3 - \lambda_j))} \times \\
\times \frac{1}{2x_2 - x_1 - x_3} (w_a(x_1, x_2) + w_a(x_2, x_3)) (w_b(x_1, x_2) + w_b(x_2, x_3)) = \\
= \frac{1}{2} \sum_{i \neq j, j \neq l} E(k,1;i,j) E(k,1;j,l) \frac{(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)}{2\lambda_j - \lambda_l - \lambda_i} \left( w_a(\lambda_i, \lambda_j) + w_a(\lambda_j, \lambda_i) \right) \left( w_b(\lambda_i, \lambda_j) + w_b(\lambda_j, \lambda_i) \right) + \\
+ \frac{1}{2} \sum_{i \neq j} E(k,1;i,j) E(k,1;i,j) \frac{(\lambda_j - \lambda_i)^2}{k\lambda_j^2} \left( w_a(\lambda_i, \lambda_j) + w_a(\lambda_j, \lambda_i) \right) \left( w_b(\lambda_i, \lambda_j) + w_b(\lambda_j, \lambda_i) \right) \times \\
\times (w_b(\lambda_i, \lambda_j) + w_b(\lambda_j, \lambda_i)).
\]  

(2.25)

On the other hand, we can easily see the following relations:

\[
e(k,2;\lambda_i,2\lambda_j - \lambda_i) = \frac{e(k,1;\lambda_i,\lambda_j) e(k,1;\lambda_j,2\lambda_j - \lambda_i)}{k\lambda_j},
\]

\[w_a(\lambda_i, \lambda_j) + w_a(\lambda_j, 2\lambda_j - \lambda_i) = 2w_a(\lambda_i, 2\lambda_j - \lambda_i).
\]  

(2.26)

The second equality follows from (2.16). With these relations, the second terms of the r.h.s.’s of (2.24) and (2.25) cancel. Then we obtain,

\[
\langle O_{R^2}, O_{R^2} \rangle_2 = \\
= -\frac{1}{4} \frac{1}{(2\pi \sqrt{-1})^3} \int_{C(0,5R)} dx_2 \int_{C(0,5R)} dx_1 \frac{e(k,2;x_1,x_2)(x_1 - x_2)^2}{\prod_{j=1}^{N} ((x_1 - \lambda_j)(x_2 - \lambda_j)(x_3 - \lambda_j))} \cdot w_a(x_1, x_2) w_b(x_1, x_2) + \\
+ \frac{1}{2} \frac{1}{(2\pi \sqrt{-1})^3} \int_{C(0,5R)} dx_3 \int_{C(0,5R)} dx_2 \int_{C(0,5R)} dx_1 \frac{e(k,1;x_1,x_2) e(k,1;x_2,x_3)}{kx_2 \prod_{j=1}^{N} ((x_1 - \lambda_j)(x_2 - \lambda_j)(x_3 - \lambda_j))} \times \\
\times \frac{1}{2x_2 - x_1 - x_3} (w_a(x_1, x_2) + w_a(x_2, x_3)) (w_b(x_1, x_2) + w_b(x_2, x_3)).
\]  

(2.27)

If we look at the r.h.s.’s of (2.23) and (2.27), we can easily see by coordinate change \(x_j = \frac{1}{z_j}\), that each summand is invariant under variation of \(\lambda_j’s\). Therefore, we can take non-equivariant limit \(\lambda_j \to 0\) (we also take \(R \to 0\) limit). This operation leads us to the equalities of the proposition.

Remark 2 In Remark 1, we noticed that each summand in (2.18) is not invariant under variation of characters. But as can be seen in (2.24) and (2.25), we can make it invariant by adding a suitable rational function of characters. These additional rational functions cancel out after adding up summands that correspond to tree graphs. The same mechanism also works in \(d = 3\) case.

3 Proof of Theorem 2

Before we go into proof of Theorem 2, we note an equality:

\[
\hat{L}^{N,k,d} \sim \tilde{L}^{N,k,d},
\]  

(3.28)

which naturally follows from Theorem 1.

Let us start from \(d = 1\) case. In this case, we apply a trivial equality:

\[
(x_1 - x_2)^2 w_a(x_1, x_2) w_b(x_1, x_2) = (x_1^2 - x_2^2)(x_2^b - x_1^b) = x_1^a x_2^b + x_1^b x_2^a - x_1^{a+b} - x_2^{a+b},
\]  

(3.29)

to the r.h.s of (2.20). Then the theorem follows from Theorem 1 and (3.28). In \(d = 2\) case, we apply (3.29) to the first summand of the r.h.s. of (2.21). To the second summand of it, we apply the following decomposition of the rational function in the integrand:

\[
(w_a(x_1, x_2) + w_a(x_2, x_3))(w_b(x_1, x_2) + w_b(x_2, x_3)) = \\
\]
\[
\frac{2}{2x_2 - x_1 - x_3} = \frac{((x_1)^{a} - (x_3)^{a})((x_3)^{b} - (x_1)^{b})}{2x_2 - x_1 - x_3} + (x_2 - x_1)w_a(x_1, x_2)w_b(x_1, x_2) + (x_2 - x_3)w_a(x_2, x_3)w_b(x_2, x_3). 
\]

(3.30)

Then the first summand in the r.h.s. of (2.21) and the first term in the decomposition (3.30) add up to,

\[
\frac{1}{2}k(\hat{L}_{n,k,d} - \hat{L}_{1+2(k-N)}),
\]

by using (3.28) and Theorem 1. The second and the third terms in the decomposition (3.30) result in,

\[
k\sum_{j=0}^{b-1} \hat{f}_{1-N,k} (\hat{L}_{1+N,k} - \hat{L}_{1+N,k+j+k-N}),
\]

(3.32) by using \((x_i - x_j)w_a(x_i, x_j) = x_i^a - x_j^a\), (3.28) and Theorem 1. But if we look back at \(a = N - 2 - n\), \(b = n - 1 - 2(k - N)\) and (3.28), (3.32) turns out to be,

\[
-k\sum_{j=0}^{k-N} \hat{f}_{1-N,k} (\hat{L}_{N,k} - \hat{L}_{1+N,k+N-j}).
\]

(3.33)

This completes the proof of \(d = 2\) case.

Now, we turn into \(d = 3\) case. In the same way as the \(d = 1, 2\) cases, we apply (3.29) to the first summand of the r.h.s. of (2.22). To the second and the third summands of it, we apply the decompositions:

\[
\frac{2}{2x_2 - x_1 - x_3} = \frac{2(2w_a(x_1, x_2) + w_a(x_2, x_3))(2w_b(x_1, x_2) + w_b(x_2, x_3))}{2x_2 - x_1 - x_3} =
\]

\[
= (x_2 - x_1)(x_2 - x_3)(2w_a(x_1, x_2) + w_a(x_2, x_3))(2w_b(x_1, x_2) + w_b(x_2, x_3)) =
\]

\[
= 2\left(\frac{((x_1)^{a} - (x_3)^{a})((x_3)^{b} - (x_1)^{b})}{2x_2 - x_1 - x_3} + 4(x_2 - x_1)w_a(x_1, x_2)w_b(x_1, x_2) + 2(x_2 - x_3)w_a(x_2, x_3)w_b(x_2, x_3)\right).
\]

(3.34)

and,

\[
\frac{2}{2x_2 - x_1 - x_3} = \frac{2w_a(x_2, x_3)(w_b(x_1, x_2) + 2w_b(x_2, x_3))}{2x_2 - x_1 - x_3} =
\]

\[
= (x_2 - x_1)(x_2 - x_3)(w_a(x_1, x_2) + 2w_a(x_2, x_3))(2w_b(x_1, x_2) + w_b(x_2, x_3)) =
\]

\[
= 2\left(\frac{((x_1)^{a} - (x_3)^{a})((x_3)^{b} - (x_1)^{b})}{2x_2 - x_1 - x_3} + 2(x_2 - x_1)w_a(x_1, x_2)w_b(x_1, x_2) + 4(x_2 - x_3)w_a(x_2, x_3)w_b(x_2, x_3)\right).
\]

(3.35)

Lastly, we apply the following decomposition to the fourth summand.

\[
- \frac{w_a(x_1, x_2) + w_a(x_2, x_3) + w_a(x_3, x_4)(w_b(x_1, x_2) + w_b(x_2, x_3) + w_b(x_3, x_4))}{x_2 - x_1} =
\]

\[
= (x_2 - x_1)(x_3 - x_4)(w_a(x_1, x_2) + w_a(x_2, x_3) + w_a(x_3, x_4))(w_b(x_1, x_2) + w_b(x_2, x_3) + w_b(x_3, x_4))
\]

\[
= \frac{((x_1)^{a} - (x_4)^{a})((x_4)^{b} - (x_1)^{b})}{r_1} +
\]

\[
+ \frac{2(x_3 - x_1)w_a(x_1, x_3)w_b(x_1, x_3) + (x_3 - x_4)w_a(x_3, x_4)w_b(x_3, x_4)}{r_1} +
\]

(7)
\[ + \frac{2(x_2 - x_4)w_a(x_2, x_4)w_b(x_2, x_4) + (x_2 - x_1)w_a(x_1, x_2)w_b(x_1, x_2)}{r_2} + \]
\[ + \frac{1}{2} \left( w_a(x_1, x_2)w_b(x_1, x_2) + w_a(x_3, x_4)w_b(x_3, x_4) + w_a(x_1, x_2)w_b(x_2, x_3) + w_a(x_2, x_3)w_b(x_1, x_2) + \right. \]
\[ + w_a(x_2, x_3)w_b(x_3, x_4) + w_a(x_3, x_4)w_b(x_2, x_3) + \]
\[ + (x_3 - x_1) \left( w_a(x_1, x_3)w_b(x_1, x_2, x_3) + w_a(x_1, x_2, x_3)w_b(x_1, x_3) + \frac{1}{2} r_1 w_a(x_1, x_2, x_3)w_b(x_1, x_2, x_3) \right) + \]
\[ + \left. (x_2 - x_4) \left( w_a(x_2, x_4)w_b(x_2, x_3, x_4) + w_a(x_2, x_3, x_4)w_b(x_2, x_4) + \frac{1}{2} r_2 w_a(x_2, x_3, x_4)w_b(x_2, x_3, x_4) \right) \right), \] (3.36)

where \( r_1 = 2x_2 - x_1 - x_3 \), \( r_2 = 2x_3 - x_2 - x_4 \). The first summand of the r.h.s. of (2.22), the first terms of (3.34) and (3.35), and the term with denominator \( r_1 r_2 \) in (3.36) add up to \( \frac{1}{k} k L_{n,k,3} - L_{1+3(k-N)} \) by (3.28) and Theorem 1. The second term of (3.34), the third term of (3.35) and

\[ \frac{2(x_3 - x_1)w_a(x_1, x_3)w_b(x_1, x_3)}{r_1}, \quad \frac{2(x_2 - x_4)w_a(x_2, x_4)w_b(x_2, x_4)}{r_2}, \] (3.37)

in (3.36) add up to,

\[ -k \sum_{j=0}^{k-N} \tilde{L}^{N,k,1}_{1+k-N}(\tilde{L}^{N,k,2}_{n-j} - \tilde{L}^{N,k,2}_{1+3(k-N)-j}). \] (3.38)

by Theorem 1 and the same reorganization used to derive (3.33). The third term of (3.34), the second term of (3.35) and

\[ \frac{(x_3 - x_4)w_a(x_3, x_4)w_b(x_3, x_4)}{r_1}, \quad \frac{(x_2 - x_1)w_a(x_1, x_2)w_b(x_1, x_2)}{r_2}, \] (3.39)

in (3.36) add up to,

\[ -k \cdot \frac{1}{2} \sum_{j=0}^{2(k-N)} \tilde{L}^{N,k,2}_{1+2(k-N)}(\tilde{L}^{N,k,1}_{n-j} - \tilde{L}^{N,k,1}_{1+3(k-N)-j}). \] (4.0)

in the same way as the previous argument. Remaining terms of the decomposition (3.36) and the fifth summand of the r.h.s. of (2.22) are reorganized as follows:

\[ k \tilde{L}^{N,k,1}_{1+k-N} \left( \frac{1}{2} \sum_{i=0}^{a-1} \sum_{j_0=0}^{b-1} \tilde{L}^{N,k,1}_{1+i+j+k-N} \tilde{L}^{N,k,1}_{2+2(k-N)} + \right. \]
\[ + \frac{1}{2} \sum_{i=0}^{a-1} \sum_{j_0=0}^{b-1} \tilde{L}^{N,k,1}_{1+i+k-N} \tilde{L}^{N,k,1}_{2+j+2(k-N)} + \frac{1}{2} \sum_{i=0}^{a-1} \sum_{j_0=0}^{b-1} \tilde{L}^{N,k,1}_{1+j+k-N} \tilde{L}^{N,k,1}_{2+i+2(k-N)} + \]
\[ + \frac{1}{2} \sum_{i=0}^{a-1} \sum_{j_0=0}^{b-1} \tilde{L}^{N,k,1}_{1+i+j+k-N} \tilde{L}^{N,k,1}_{2+i+j+k-N} + \]
\[ + \frac{1}{2} \sum_{i=0}^{a-2} \sum_{j_0=0}^{b} \tilde{L}^{N,k,1}_{1+i+j+k-N} \tilde{L}^{N,k,1}_{a+i+j+2(k-N)} + \tilde{L}^{N,k,1}_{2+i+j+k-N} \tilde{L}^{N,k,1}_{2+j+2(k-N)} + \]
\[ + \sum_{i=0}^{a-2} \sum_{j_0=0}^{b} \tilde{L}^{N,k,1}_{1+i+j+k-N} \tilde{L}^{N,k,1}_{a+i+j+2(k-N)} + \tilde{L}^{N,k,1}_{2+j+2(k-N)} + \tilde{L}^{N,k,1}_{2+i+j+k-N} \tilde{L}^{N,k,1}_{a+i+j+2(k-N)} \right) \]
+ \sum_{j=0}^{a-1} \left( \tilde{L}^{N,k,1}_{n,j-2(k-N)} - \tilde{L}^{N,k,1}_{n-j+(k-N)} \right) + \\
+ \frac{1}{2} \sum_{i=0}^{a-1} \sum_{j=0}^{k-N} \left( \tilde{L}^{N,k,1}_{n+i+(k-N)} - \tilde{L}^{N,k,1}_{n+j+(k-N)} \right) = \\
= k \tilde{L}^{N,k,1}_{1+k-N} \left( \sum_{i=0}^{(k-N)-1} \sum_{j=0}^{2(k-N)} \tilde{L}^{N,k,1}_{n+i+j} - \tilde{L}^{N,k,1}_{n+i+(k-N)} \right) + \\
+ \sum_{i=0}^{2(k-N)-1} \sum_{j=0}^{k-N} \left( \tilde{L}^{N,k,1}_{i+k-N} - \tilde{L}^{N,k,1}_{i+j+(k-N)} \right) = \\
= -k \tilde{L}^{N,k,1}_{1+k-N} C_{1,1}^{N,k,3}(n) + k \cdot \frac{3}{2} \left( \tilde{L}^{N,k,1}_{1+k-N} \right)^2 \sum_{j=0}^{2(k-N)} A_j \left( \tilde{L}^{N,k,1}_{n-j} - \tilde{L}^{N,k,1}_{n+2(k-N)-j} \right). \quad (3.41)

In this derivation, we only use the conditions:

\begin{align*}
a &= N - 2 - n, \quad b = n - 1 - 3(k - N), \quad \tilde{L}^{N,k,1}_n = \tilde{L}^{N,k,1}_{k-1-n}, \quad (3.42)
\end{align*}

but need careful treatment of summations. Anyway, the final formula completes the proof of Theorem 2.

4 Conclusion

Our motivation of this paper is to understand explicitly difference between the moduli space of Gauged Linear Sigma Model and the moduli space of stable maps. Unfortunately, this paper’s treatment is quite computational. It gives us an elementary and explicit proof of mirror theorem for rational curves of lower degrees but lacks geometrical vision. We can indeed extend this paper’s method to rational curves of higher degrees because we don’t use geometrical simplicity of moduli space of rational curves of lower degrees. As can be seen in [6] and [9], generalized mirror transformation of $d = 4,5$ rational curves has quite complicated structure. Therefore, we need combinatorial sophistication of our method. For this purpose, we had better search for geometrical meaning of decomposition of rational functions, such as (3.36).

One of the main features of this paper may be translation of combinatorial summations in Kontsevich’s localization formula into residue integrals, that enabled us to compare directly the Gromov-Witten invariants with the virtual structure constants. This translation can be applied to various examples. At least, we can use it to prove mirror theorem of $\mathcal{O}(1) \oplus \mathcal{O}(-3) \rightarrow \mathbb{P}^1$ [3]. We also think that we can apply it to prove mirror theorem at higher genus. Anyway, we have to pursue combinatorial sophistication of our method.
Appendix: Proof of Theorem 1

We prove Theorem 1 by showing that the r.h.s. of (1.8) satisfies the initial condition and the recursive formulas (1.1), (1.2), (1.3) and (1.4). For this purpose, we note here a relation between rational functions that appear in the residue integrals:

\[
\prod_{j=0}^{l(\sigma_d)} \frac{1}{(x_j)^N} \prod_{j=1}^{l(\sigma_d)-1} \frac{1}{k_{x_j}} \left( \frac{x_j-x_{j-1}}{d_j} + \frac{x_j-x_{j+1}}{d_{j+1}} \right) \prod_{j=1}^{l(\sigma_d)} \frac{e(k, d_j; x_{j-1}, x_j)}{l(N, d_j; x_{j-1}, x_j)} =
\]

\[
= \prod_{j=0}^{l(\sigma_d)} \frac{1}{(x_j)^{N+1}} \prod_{j=1}^{l(\sigma_d)-1} \frac{1}{k_{x_j}} \left( \frac{x_j-x_{j-1}}{d_j} + \frac{x_j-x_{j+1}}{d_{j+1}} \right) \prod_{j=1}^{l(\sigma_d)} \frac{e(k, d_j; x_{j-1}, x_j)}{l(N+1, d_j; x_{j-1}, x_j)} \times
\]

\[
\times (x_0 x_1 \cdots x_{l(\sigma_d)}) \prod_{j=1}^{l(\sigma_d)} \prod_{i=1}^{d_j-1} \left( \frac{ix_{j-1} + (d_j - i)x_j}{d_j} \right). \tag{A.43}
\]

In \(d = 1\) case, the r.h.s of (1.8) becomes,

\[
\frac{1}{k} \left( \frac{1}{2\sqrt{-1}} \right)^2 \oint_{C_0} dx_1 \oint_{C_0} dx_0 \frac{x_0^{N-2-n} x_1^{n-1+N-k} e(k, 1; x_0, x_1)}{x_0^{N+1} x_1^{N}} =
\]

\[
= k \left( \frac{1}{2\sqrt{-1}} \right)^2 \oint_{C_0} dx_1 \oint_{C_0} dx_0 \frac{\prod_{j=1}^{l(\sigma_d)-1} (jx_0 + (k-j)x_1)}{x_0^{n+1} x_1^{n}}. \tag{A.44}
\]

Hence (1.1) and (1.2) automatically hold true.

(A.43) tells us that the recursive formulas in \(d = 2, 3\) cases follow from adequate decomposition of,

\[(x_0 x_1 \cdots x_{l(\sigma_d)}) \prod_{j=1}^{l(\sigma_d)} \prod_{i=1}^{d_j-1} \left( \frac{ix_{j-1} + (d_j - i)x_j}{d_j} \right).\]

Explicitly, decompositions are given as follows:

**d=2 case**

\[
s_2 = (2) : \quad x_0 x_1 \frac{x_0 + x_1}{2},
\]

\[
s_2 = (1, 1) : \quad x_0 x_1 x_2 = x_0 x_2 \left( \frac{x_0 + x_2}{2} + \frac{1}{2} r x_0 x_2, \quad (r = 2x_1 - x_0 - x_2). \tag{A.45}\right.
\]

**d=3 case**

\[
s_3 = (3) : \quad x_0 x_1 \frac{2x_0 + x_1 x_0 + 2x_1}{3} = x_0 x_1 \left( \frac{2x_0}{9} + \frac{5}{9} x_0 x_1 + \frac{2}{9} x_1^2 \right),
\]

\[
s_3 = (2, 1) : \quad x_0 x_1 x_2 \frac{2x_0 + x_1}{2} = x_0 x_2 \left( \frac{2}{9} x_0^2 + \frac{5}{9} x_0 x_2 + \frac{2}{9} x_2^2 + 2 \frac{1}{3} x_0 + r \frac{4}{3} x_1 + \frac{2}{9} x_2 \right),
\]

\[
(r_1 = \frac{x_1 - x_0}{2} + x_1 - x_2),
\]

\[
s_3 = (1, 2) : \quad x_0 x_1 x_2 \frac{x_1 + x_2}{2} = x_0 x_2 \left( \frac{2}{9} x_0^2 + \frac{5}{9} x_0 x_2 + \frac{2}{9} x_2^2 + 2 \frac{1}{3} x_0 + \frac{2}{3} x_1 + \frac{2}{9} x_2 \right),
\]

\[
(r_2 = x_1 - x_0 + \frac{x_1 - x_2}{2}),
\]

\[
s_3 = (1, 1, 1) : \quad x_0 x_1 x_2 x_3 = x_0 x_3 \left( \frac{2}{9} x_0^2 + \frac{5}{9} x_0 x_3 + \frac{2}{9} x_3^2 + 2 \frac{1}{3} x_0 + \frac{1}{3} x_1 + \frac{4}{9} x_3 \right) +
\]

\[
+ 2 \frac{4}{9} x_0 x_1 + \frac{2}{3} x_2 + \frac{2}{9} x_3 + r_4 \frac{2}{3} x_1 + \frac{1}{3} x_3 r_4,
\]

\[
(r_3 = 2x_1 - x_0 - x_2, \quad r_4 = 2x_2 - x_1 - x_3). \tag{A.46}
\]

With these decompositions, the same argument on residue integrals as the one used in the proof of Theorem 2 leads us to the desired recursive formulas. We can prove the recursive formulas for higher degree by extending this kind of discussion.
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