Ergodic Theorems for Laminations and Foliations: Recent Results and Perspectives

Viêt-Anh Nguyên¹,²

Dedicated to My Beloved Father

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Abstract
This report discusses recent results as well as new perspectives in the ergodic theory for Riemann surface laminations, with an emphasis on singular holomorphic foliations by curves. The central notions of these developments are leafwise Poincaré metric, directed positive harmonic currents, multiplicative cocycles, and Lyapunov exponents. We deal with various ergodic theorems for such laminations: random and operator ergodic theorems, (geometric) Birkhoff ergodic theorems, Oseledec multiplicative ergodic theorem, and unique ergodicity theorems. Applications of these theorems are also given. In particular, we define and study the canonical Lyapunov exponents for a large family of singular holomorphic foliations on compact projective surfaces. Topological and algebro-geometric interpretations of these characteristic numbers are also treated. These results highlight the strong similarity as well as the fundamental differences between the ergodic theory of maps and that of Riemann surface laminations. Most of the results reported here are known. However, sufficient conditions for abstract heat diffusions to coincide with the leafwise heat diffusions (Section 5.2) are new ones.

Keywords  Riemann surface lamination · Singular holomorphic foliation · Leafwise Poincaré metric · Positive harmonic currents · Multiplicative cocycles · Ergodic theorems · Lyapunov exponents

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¹ Laboratoire de mathématiques Paul Painlevé, CNRS U.M.R. 8524, Université de Lille, 59655, Villeneuve d’Ascq Cedex, France
² Vietnam Institute for Advanced Study in Mathematics, 157 Chua Lang Street, Hanoi, Vietnam
1 Introduction

1.1 Prelude

The goal of these notes is to explain recent ergodic theorems for laminations by Riemann surfaces (without and with singularities), and particularly those for singular holomorphic foliations by curves. We make an emphasis on the analytic approach to the dynamical theory of laminations and foliations. This illustrates a prominent role of the theory of currents in the field.

There is a natural correspondence between the dynamics of Riemann surface laminations and those of iterations of continuous maps. More specifically in the meromorphic category, this correspondence becomes a connection between the dynamics of singular holomorphic foliations in dimension $k \geq 2$ and those of iterations of meromorphic maps in dimension $k - 1$. Ergodic theorems for measurable maps are by now well-understood, see, for instance, the monograph of Krengel [68]. Those for the subclass of all meromorphic maps have been studied intensively only during the last three decades, see the survey of Dinh-Sibony [41]. So a natural question arises whether one can obtain analogous ergodic theorems for Riemann surface laminations, and in particular, for the subclass of singular holomorphic foliations by curves. In this article, we try to answer this fundamental question by analyzing some known results and by proving some new ones. It is worth noting that the ergodic theory of laminations and foliations requires many new ideas and presents many difficulties. Traditional dynamical system techniques are based on a singly generated system, and these methods require the existence of invariant measures. For laminations and foliations, the dynamics are defined by the local actions of the holonomy maps, which provides a more complex system, and often precludes the existence of invariant measures. Therefore, the ergodic theory of laminations and foliations is rich in ideas and problems, where every insight opens new avenues of exploration.

In the next subsection, we will recall two basic ergodic theorems: one for measurable maps and the other for holomorphic maps. These theorems will serve as our starting models in order to look for ergodic theorems in the context of laminations-foliations. The last subsection outlines the organization of the paper. These notes may be considered as the continuation of our previous survey [79]. However, in the latter article, we are interested in the whole ergodic theory of Riemann surface laminations, which is clearly a broader topic. In the present work, we only specialize in ergodic theorems and related matters. So some fundamental topics such as the entropies etc. are not treated here. It should be noted that some progress has been made in this area since the publication of our previous survey [79]. Namely, Problem 4.7 (zero Lelong numbers), Problem 5.8 (unique ergodicity), and Problem 7.7 (negative Lyapunov exponent) therein have recently been solved in [81], [35] and [80] respectively. Moreover, we try to rewrite several parts of [79] in a somewhat more general context of (not necessarily hyperbolic) Riemann surface laminations with singularities. We hope that the ideas reviewed in these two surveys will be developed and expanded in the future. In writing the present article, we are inspired by the surveys and lecture notes of Deroin [28], Dinh-Sibony [46], Fornæss-Sibony [50], Ghys [56], Hurder [59], Zakeri [98], etc.

1.2 Two Ergodic Theorems in the Dynamics of Iterations of Maps

To state the first ergodic theorem, we need to introduce some notations and terminology. Let $f : X \rightarrow X$ be a measurable map on a probability measure space $(X, \mathcal{A}, \mu)$ and suppose $\varphi$ is a $\mu$-integrable function, i.e., $\varphi \in L^1(\mu)$. Then, we define the following averages:
• Time average (up to level \( n \in \mathbb{N} \)): This is defined as the average over iterations of \( f \) from 0 up to the \((n - 1)\)-iteration starting from some initial point \( x \in X \)

\[
\frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)).
\]

This sum can also be rewritten as \( \langle m_{x,n}^+, \varphi \rangle \), where \( m_{x,n}^+ \) is a probability measure which is the average of Dirac masses at forward orbit of \( x \) from time 0 up to time \( n - 1 \)

\[
m_{x,n}^+ := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}.
\]

Here, for \( a \in X \), let \( \delta_a \) denotes the Dirac mass at \( a \). The sign + in \( m_{x,n}^+ \) emphasizes that we are concerned with the forward orbit of \( x \).

• Space average

\[
\langle \mu, \varphi \rangle = \int_X \varphi \, d\mu.
\]

In general, the limit of time averages (if exists) as \( n \to \infty \) and space average may be different.

We say that \( \mu \) is ergodic with respect to \( f \) if for every element \( A \) of the \( \sigma \)-algebra \( \mathcal{A} \) with \( f^{-1}(A) = A \), then either \( \mu(A) = 0 \) or \( \mu(A) = 1 \). We say that \( \mu \) is invariant with respect to \( f \) if \( f_\ast \mu = \mu \), i.e., \( \mu(f^{-1}(A)) = \mu(A) \) for \( A \in \mathcal{A} \). The following theorem plays a fundamental role in the dynamics of maps-iterations.

**Theorem 1.1** (Birkhorff Ergodic Theorem [3, 68, 100]) If \( \mu \) is invariant and ergodic, and \( \varphi \) is as above, then

\[
\lim_{n \to \infty} \langle m_{x,n}^+, \varphi \rangle = \langle \mu, \varphi \rangle \quad \text{for } \mu\text{-almost everywhere } x \in X.
\]

In other words, as the time \( n \) tends to infinity, the limit of time averages is equal to the space average \( \mu \)-almost everywhere. In particular, \( m_{x,n}^+ \to \mu \) weakly as \( n \) tends to infinity.

There are many ergodic theorems for iterations of maps (see [68]).

Now, we turn to the statement of the unique ergodicity of surjective holomorphic maps defined on compact Kähler manifolds. Let \( X \) be a compact Kähler manifold of dimension \( k \) and \( \omega \) a Kähler form on \( X \) so normalized that \( \omega^k \) defines a probability measure on \( X \).

Let \( f : X \to X \) be a surjective holomorphic map. Let \( d_p(f) \) (or \( d_p \) if there is no possible confusion), \( 0 \leq p \leq k \), be the dynamical degree of order \( p \) of \( f \). This is a bi-meromorphic invariant which measures the norm growth of the operators \( (f^n)^\ast \) acting on the Hodge cohomology group \( H^{p,p}(X, \mathbb{C}) \) when \( n \) tends to infinity, that is,

\[
d_p = d_p(f) := \lim_{n \to \infty} \| (f^n)^\ast \|^\frac{1}{n}, \quad \text{where} \quad (f^n)^\ast : H^{p,p}(X, \mathbb{C}) \to H^{p,p}(X, \mathbb{C}).
\]

We always have \( d_0(f) = 1 \). The last dynamical degree \( d_k(f) \) is the topological degree of \( f \): it is equal to the number of points in a generic fiber of \( f \). We also denote it by \( d_t(f) \) or simply by \( d_t \).
We say that $f$ is with dominant topological degree if $d_t > d_p$ for every $0 \leq p \leq k - 1$. It is well-known that for such a map $f$, the following weak limit of probability measures

$$
\mu := \lim_{n \to \infty} \frac{1}{d^n_t} (f^n)^* \omega^k
$$

exists. The probability measure $\mu$ is called the equilibrium measure of $f$. It has no mass on proper analytic subsets of $X$, is totally invariant: $d_t^{-1} f^* (\mu) = f_* (\mu) = \mu$ and is exponentially mixing. The measure $\mu$ is also the unique invariant measure with maximal entropy $\log d_t$. We refer the reader to [39, 40, 58] for details, see also [97] for a recent result.

The set of preimages $f^{-n}(x)$ of $f^n$ consists of $d^n_t$ points counted with multiplicity (see, e.g., [37, Lemma 4.7]). So for $n \in \mathbb{N}$, the probability measure

$$
m_{x,n} := \frac{1}{d^n_t} (f^n)^* \delta_x = \frac{1}{d^n_t} \sum_{y \in f^{-n}(x)} \delta_y
$$

is the average of Dirac masses at backward orbit of $x$ up to past-time $n$. The sign $-$ in the notation $m_{x,n}$ emphasizes that we are concerned with the backward orbit of $x$. See Fig. 1.

The following theorem gives a equidistribution of preimages of points by $f^n$.

**Theorem 1.2** (Unique ergodicity for holomorphic maps) Let $f : X \to X$ be a surjective holomorphic map with dominant topological degree $d_t$ on a compact Kähler manifold $(X, \omega)$ and let $\mu$ be its equilibrium measure. Then, there is a (possibly empty) proper analytic set $E = E_f$ of $X$ such that we have

$$
m_{x,n} \to \mu \quad \text{as} \quad n \to \infty
$$

1In some references, such a map is said to be with large topological degree; we think the word “dominant” is more appropriate.
if and only if \( x \notin \mathcal{E} \). In fact, the exceptional set \( \mathcal{E} \) is characterized by the following two conditions: (1) \( f^{-1}(\mathcal{E}) \subseteq \mathcal{E} \); (2) any proper analytic subset of \( X \) satisfying (1) is contained in \( \mathcal{E} \). Moreover, we have \( \mathcal{E} = f^{-1}(\mathcal{E}) = f(\mathcal{E}) \).

The above result describes the dichotomous behavior of the equidistribution of preimages of points by \( f^n \). Namely, the typical case \( x \notin \mathcal{E} \) is characterized by the fact that \( m_{x,\mu} \) tends to the equilibrium measure \( \mu \). The complementary case (i.e., \( x \in \mathcal{E} \)) is non-typical since the exceptional set \( \mathcal{E} \) is small: it is a (possibly empty) proper analytic set. The above theorem was obtained for holomorphic endomorphisms of \( \mathbb{P}^k \) in [7, 8, 12, 38, 48]. For the case of dimension 1, see [10, 53, 69, 93]. A proof of this theorem even for the broader class of meromorphic maps is given in [36, Theorem 1.3]. The reader is invited to consult the survey [41] for a comprehensive treatment of the equidistribution of preimages by a holomorphic map in \( \mathbb{P}^k \).

When \( X = \mathbb{P}^k \), the space of all surjective holomorphic maps with a given algebraic degree \( d > 1 \) is canonically identified to a Zariski open set \( \Omega_{k,d} \) of some \( \mathbb{P}^N \). It is well-known that for \( f \in \Omega_{k,d} \), we have \( dk(f) = dk \), and hence \( f \) is with dominant topological degree. Moreover, we have \( \mathcal{E}_f = \emptyset \) for \( f \) belonging to a Zariski open subset of \( \Omega_{k,d} \). In particular, for a generic surjective holomorphic map \( f \) of a given degree \( d > 1 \) on \( \mathbb{P}^k \) of a given degree \( d > 1 \), we have that \( \mathcal{E}_f = \emptyset \).

### 1.3 Outline of the Article

In this article, we undertake the following two tasks. First, we survey various ergodic theorems in the context of laminations-foliations which follow the models of the above two ergodic Theorems 1.1 and 1.2. Second, in Section 5.2, we prove some new comparison results regarding the heat diffusions. Many concepts in the context of maps-iterations will find their analogues in the new context of laminations. Some have even more than one analogue. The work is organized as follows.

In Section 2, we will recall basic facts on Riemann surface laminations (without and with singularities), singular holomorphic foliations. Throughout the paper, unless otherwise stated, we will refer to all these objects as the common abridged name lamination. Moreover, (singular) (holomorphic) foliations mean (singular) (holomorphic) foliations by curves. The hyperbolicity and the leafwise Poincaré metric as well as the hyperbolic and parabolic parts of lamination will be introduced. The leafwise Poincaré metric is regarded as the hyperbolic time for the laminations. Consequently, we will develop the leafwise heat diffusions and define various concepts of harmonic measures for laminations. Next, we study companion notions for harmonic measures, namely, directed positive harmonic current for Riemann surface laminations and directed positive \( dd^c \)-closed current for holomorphically immersed Riemann surface laminations. The latter class of laminations contains the family of singular holomorphic foliations. We investigate the relationship \( \text{(positive quasi-) harmonic measures} \rightleftharpoons \text{directed (positive) harmonic currents} \). Historically, Garnett in [54] introduces the notion of harmonic measures and considers the diffusions of the heat equation in the Riemannian context. Her idea is further developed by Candel-Conlon [19] and Candel [17]. The notions of directed positive \( dd^c \)-closed currents on singular holomorphic foliations and on singular laminations living in a complex manifold are introduced in the articles of Berndtsson-Sibony [2] and Fornæss-Sibony [49, 50] respectively. We also collect from [32] basic facts about positive \( dd^c \)-closed currents on complex manifolds. The sample-path space and the holonomy of lamination are presented in this section. These
typical objects distinguish the laminations from the maps. A short digression to the isolated singularities for singular holomorphic foliations is given. Singular holomorphic foliations by Riemann surfaces in $\mathbb{P}^k (k > 1)$ provide a large family of examples where all the above notions apply. We will describe the properties of a generic holomorphic foliation in $\mathbb{P}^k$ with a given degree $d > 1$. The section is ended with a discussion on Sullivan’s dictionary. This is a kind of philosophical correspondence between the world of maps (or roughly speaking, discrete dynamics) and the world of laminations (or more generally, continuous dynamics).

The random ergodic theorem is presented in Section 3. The first part of this section introduces the Wiener measure for the sample-path space associated to a given point. These play the same role as the counting measures do for the orbit of a point in the context of map-iterations. The material for this section is mainly taken from [75].

In the first part of Section 4, we first will introduce a function $\eta$ which measures the ratio between the ambient metric and the leafwise Poincaré metric of a lamination. This function plays an important role in the study of laminations. We also introduce the class of Brody hyperbolic laminations. This class not only contains all compact laminations by hyperbolic Riemann surfaces, but also includes many interesting singular holomorphic foliations. We then state some recent results on the regularity of Brody hyperbolic laminations which arise from our joint-works with Dinh and Sibony in [33, 34]. The second part of Section 4 is devoted to the mass-distribution of positive $dd^c$-closed currents in both local and global settings. Understanding the mass-distribution is one of the main steps in establishing ergodic theorems for singular holomorphic foliations.

In Section 5, we introduce the abstract diffusions of the heat equation for two situations:

- For Riemann surface laminations (possibly with singularities) with respect to a harmonic measure
- For holomorphically immersed Riemann surface laminations (possibly with singularities) with respect to a (not necessarily directed) positive $dd^c$-closed current.

This approach allows us in [32] to extend the classical theory of Garnett [54] and Candel [17] to Riemann surface laminations (without or with singularities) or to singular holomorphic foliations with not necessarily bounded geometry. Our method is totally different from those of Garnett [54], Candel-Conlon [19], and Candel [17]. We give two versions of ergodic theorems for such currents: one associated with the abstract heat diffusions and one of geometric nature close to Birkhoff’s averaging on orbits of a dynamical system. Another consequence of this method is a sufficient condition for the abstract heat diffusions to coincide with the leafwise heat diffusions (see Theorem 5.17). This result and its consequences are new.

In Section 6, we present some unique ergodicity theorems for compact Riemann surface laminations without singularities and for singular holomorphic foliations. In the first subsection, we consider the case when the lamination is compact and transversally conformal. We state a unique ergodicity theorem due to Deroin-Kleptsyn [30] in this context. The second subsection is devoted to singular holomorphic foliations in $\mathbb{P}^2$. Then, the works of Fornæss-Sibony [51] and Dinh-Sibony [45] describe a dichotomous behavior of the unique ergodicity when the singular holomorphic foliation admits only hyperbolic singularities. So the panoramic picture in this special case is rather complete, at least when the singular holomorphic foliation admits only hyperbolic singularities. In the last subsection, we state our recent unique ergodicity theorems for Riemann surface laminations without singularities and for singular holomorphic foliations on compact Kähler surfaces. This result is obtained in collaboration with Dinh and Sibony [35]. Our results give a trichotomous behavior in these general settings.
In Section 7, we give a sketchy proof of our unique ergodicity theorems. We outline the theory of densities for a class of non $ddc$-closed currents developed in [35]. This theory is one of the main ingredients in our approach.

Section 8 is devoted to the Lyapunov–Oseledec theory for Riemann surface laminations (without or with singularities). Here, we deal with the (multiplicative) cocycles which are modelled on the holonomy cocycle of a foliation. The Oseledec multiplicative ergodic theorem for laminations is the main result of this theory. We apply this theorem to smooth compact laminations by hyperbolic Riemann surfaces and to compact singular holomorphic foliations. The material for this section is taken from our memoir [75].

Finally, Section 9 discusses some applications of the theory developed here. We define and study the canonical Lyapunov exponents for a large family of singular holomorphic foliations on compact projective surfaces. We also study the topological and algebro-geometric interpretations of these characteristic numbers. When the lamination in question is hyperbolic, smooth, and compact, we characterize geometrically the Lyapunov exponents of a smooth cocycle with respect to a harmonic measure. This section is based on our works in [76, 78, 80].

Several open problems develop in the course of the exposition.

**Main notation.** Throughout the paper,

- $\mathbb{R}^+$ (resp. $\mathbb{N}$) denotes $[0, \infty)$ (resp. $\{n \in \mathbb{Z} : n \geq 0\}$)
- $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$, $r\mathbb{D}$ denotes the disc of center 0 and of radius $r$, and $\mathbb{D}_R \subset \mathbb{D}$ is the disc of center 0 and of radius $R$ with respect to the Poincaré metric $g_P$ on $\mathbb{D}$, i.e., $\mathbb{D}_R = r\mathbb{D}$ with $R := \log((1 + r)/(1 - r))$. Recall that $g_P(\zeta) = \frac{2}{(1 - |\zeta|^2)^2} i d\zeta \wedge d\bar{\zeta}$ for $\zeta \in \mathbb{D}$.

Poincaré metric on a hyperbolic Riemann surface, in particular on $\mathbb{D}$ and on the hyperbolic leaves of a Riemann surface lamination, is given by a positive $(1, 1)$-form that we often denoted by the same symbol $g_P$. The associated distance is denoted by $\text{dist}_P$.

Given a Riemann surface lamination with singularities $\mathcal{F} = (X, \mathcal{L}, E)$, $E$ is the set of singularities, a leaf through a point $x \in X \setminus E$ is often denoted by $L_x$. Hyp($\mathcal{F}$) (resp. Par($\mathcal{F}$)) denotes the hyperbolic part (resp. the parabolic part) of $\mathcal{F}$.

Recall that $i := \sqrt{-1}$ and $d^c := \frac{i}{2\pi}(\partial - \bar{\partial})$ and $dd^c = \frac{i}{\pi} \partial \bar{\partial}$.

### 2 Basic Laminations and Foliations Concepts

#### 2.1 Riemann Surface Laminations and Singular Foliations

Let $X$ be a locally compact space. A **Riemann surface lamination** $(X, \mathcal{L})$ is the data of a (lamination) atlas $\mathcal{L}$ of $X$ with ( laminated) charts

$$
\Phi_p : \mathbb{U}_p \rightarrow \mathbb{B}_p \times T_p.
$$

Here, $T_p$ is a locally compact metric space, $\mathbb{B}_p$ is a domain in $\mathbb{C}$, $\mathbb{U}_p$ is an open set in $X$, and $\Phi_p$ is a homeomorphism, and all the changes of coordinates $\Phi_p \circ \Phi_q^{-1}$ are of the form

$$
x = (y, t) \mapsto x' = (y', t'), \quad y' = \Psi(y, t), \quad t' = \Lambda(t),
$$

where $\Psi$ and $\Lambda$ are continuous functions and $\Psi$ is holomorphic in $y$. See Fig. 2.
The open set $\mathcal{U}_p$ is called a flow box and the Riemann surface $\Phi_p^{-1}(t = c)$ in $\mathcal{U}_p$ with $c \in \mathbb{T}_p$ is a plaque. The property of the above coordinate changes insures that the plaques in different flow boxes are compatible in the intersection of the boxes. Two plaques are adjacent if they have non-empty intersection. A transversal in a flow box is a closed set of the box which intersects every plaque in one point. In particular, $\Phi_p^{-1}(\{x\} \times \mathbb{T}_p)$ is a transversal in $\mathcal{U}_p$ for any point $x \in \mathbb{B}_p$. For the sake of simplicity, we often identify $\mathbb{T}_p$ with $\Phi_p^{-1}(\{x\} \times \mathbb{T}_p)$ for some $x \in \mathbb{B}_p$, or even identify $\mathcal{U}_p$ with $\mathbb{B}_p \times \mathbb{T}_p$ via the map $\Phi_p$.

A leaf $L$ is a minimal connected subset of $X$ such that if $L$ intersects a plaque, it contains that plaque. So a leaf $L$ is a Riemann surface immersed in $X$ which is a union of plaques. For every point $x \in X$, denote by $L_x$ the leaf passing through $x$. A subset $M \subset X$ is called leafwise saturated if $x \in M$ implies $L_x \subset M$.

We say that a Riemann surface lamination $(X, \mathcal{L})$ is $\mathcal{C}^k$-smooth (resp. smooth) if each map $\Psi$ above is $\mathcal{C}^k$-smooth (resp. smooth) with respect to $y$, and its partial derivatives of any total order $\leq k$ (resp. any order) with respect to $y$ and $\bar{y}$ are jointly continuous with respect to $(y, t)$.

We are mostly interested in the case where the $\mathbb{T}_p$ are closed subsets of smooth real manifolds (resp. of some complex manifolds) and the functions $\Psi$ and $\Lambda$ are $\mathcal{C}^k$-smooth (resp. smooth, holomorphic) in all variables. In this case, we say that the lamination $(X, \mathcal{L})$ is $\mathcal{C}^k$-transversally smooth (resp. transversally smooth, transversally holomorphic). If, moreover, $X$ is compact, we can embed it in an $\mathbb{R}^N$ in order to use the distance induced by a Riemannian metric on $\mathbb{R}^N$.

We say that a transversally smooth Riemann surface lamination $(X, \mathcal{L})$ is a smooth foliation if $X$ is a manifold and all leaves of $\mathcal{L}$ are Riemann surfaces immersed in $X$. 

![Diagram of laminated charts $\Phi_p$ and $\Phi_q$ of a lamination](image-url)
We say that a Riemann surface lamination \((X, \mathcal{L})\) is a holomorphic foliation if \(X\) is a complex manifold (of dimension \(k\)) and there is an atlas \(\mathcal{L}\) of \(X\) with (foliated) charts \(\Phi_p : U_p \to \mathbb{B}_p \times \mathbb{T}_p\), where the \(\mathbb{T}_p\)'s are open sets of \(\mathbb{C}^{k-1}\) and all above maps \(\Psi\) and \(\Lambda\) are holomorphic.

We call Riemann surface lamination with singularities the data \(\mathcal{F} = (X, \mathcal{L}, E)\) where \(X\) is a locally compact space, \(E\) a closed subset of \(X\) such that \(X \setminus E = X\) and \((X \setminus E, \mathcal{L})\) is a Riemann surface lamination. The set \(E\) is the singularity set of the lamination.

We say that \(\mathcal{F} := (X, \mathcal{L}, E)\) is a singular foliation (resp. singular holomorphic foliation) if \(X\) is a manifold (resp. a complex manifold) and \(E \subset X\) is a closed subset such that \(X \setminus E = X\) and \((X \setminus E, \mathcal{L})\) is a smooth foliation (resp. a holomorphic foliation). \(E\) is said to be the set of singularities of the foliation \(\mathcal{F}\). We say that \(\mathcal{F}\) is compact if \(X\) is compact.

Many examples of abstract compact Riemann surface laminations are constructed in [18] and [56]. Suspensions of group actions give already a large class of laminations without singularities.

Remark 2.1 In the above definitions, if we allow \(\mathbb{B}_p\) to be a domain in \(\mathbb{R}^N\) (resp. in \(\mathbb{C}^N\)) for a fixed number \(N \in \mathbb{N}\), then we obtain \(N\)-real (resp. \(N\)-complex) dimensional laminations/foliations. For a comprehensive recent exposition on \(N\)-dimensional laminations/foliations, the reader is invited to consult the textbooks by Candel-Conlon [18, 19], by Walczak [99], by Cano-Cerveau-Déserti [20], etc.

2.2 Hyperbolicity and Leafwise Poincaré Metric

Let \(L\) be an arbitrary, not necessarily simply connected, Riemann surface. Riemann’s mapping theorem states that its universal covering surface \(\tilde{L}\), which is always simply connected, can be mapped conformally onto a domain of exactly one of the following types:

- The extended complex plane \(\mathbb{C} \cup \{\infty\} = \mathbb{P}^1\) (the case of a Riemann surface of elliptic type);
- The finite complex plane \(\mathbb{C}\) (is of parabolic type);
- Or the unit disc \(\mathbb{D}\) (is of hyperbolic type).

We say that \(L\) is uniformized by the corresponding domain of its type. Since the elliptic case differs from the others already from the topological point of view, the difficult problem of recognizing whether a given Riemann surface is of hyperbolic or parabolic type is still left. It is known that a closed Riemann surface of genus \(g\) for \(g = 0\) is of elliptic type, for \(g = 1\) it is of parabolic type, and for \(g > 1\) of hyperbolic type; therefore, the problem of types is mainly important for open Riemann surfaces.

Consider now a Riemann surface lamination with singularities \(\mathcal{F} = (X, \mathcal{L}, E)\).

Definition 2.2 A leaf \(L\) of \(\mathcal{F}\) is said to be hyperbolic if it is a hyperbolic Riemann surface, i.e., it is of hyperbolic type, i.e., it is uniformized by \(\mathbb{D}\). Otherwise (i.e., when \(L\) is either of parabolic type or of elliptic type), \(L\) is called parabolic.

\(\mathcal{F}\) is said to be hyperbolic if its leaves are all hyperbolic.

The hyperbolic part of the lamination \(\mathcal{F}\), denoted by \(\text{Hyp}(\mathcal{F})\), is the union of all hyperbolic leaves, whereas the union of all parabolic leaves, denoted by \(\text{Par}(\mathcal{F})\), is called the parabolic part (or equivalently, the non-hyperbolic part of \(\mathcal{F}\)). These are disjoint leafwise saturated measurable sets of \(X\) and \(\text{Hyp}(\mathcal{F}) \cup \text{Par}(\mathcal{F}) = X \setminus E\) (see [32, Proposition 3.1]).
Consider also a Hermitian metric on \( X \), i.e., Hermitian metrics on the leaves of \( L \) whose restriction to each flow box defines Hermitian metrics on the plaques that depend continuously on the plaques. It is not difficult to construct such a metric using a partition of unity. Observe that all the Hermitian metrics on \( X \) are locally equivalent. From now on, fix a Hermitian metric on \( X \).

For every \( x \in \text{Hyp}(\mathcal{F}) \), consider a universal covering map
\[
\phi_x : \mathbb{D} \to L_x \quad \text{such that } \phi_x(0) = x.
\] (2.1)
This map is uniquely defined by \( x \) up to a rotation on \( \mathbb{D} \). Then, by pushing forward the Poincaré metric \( g_P \) on \( \mathbb{D} \) via \( \phi_x \) (see Main notation), we obtain the so-called Poincaré metric on \( L_x \) which depends only on the leaf. The latter metric is given by a positive \((1, 1)\)-form on \( L_x \) that we also denote by \( g_P \) for the sake of simplicity. So \( g_P \) is a measurable \((1, 1)\)-form defined on \( \text{Hyp}(\mathcal{F}) \) [32, Proposition 3.1]. For a systematic exposition on the Poincaré metric and its generalizations, see the book by Kobayashi [67].

### 2.3 Leafwise Heat Diffusions and Harmonic Measures

Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a Riemann surface lamination with singularities. The leafwise Poincaré metric \( g_P \) induces the corresponding Laplacian \( \Delta_P \) on hyperbolic leaves (see formula (2.5) below for the case of the Poincaré disc \((\mathbb{D}, g_P)\) and formula (2.7) for the case of a hyperbolic leaf). For every point \( x \in \text{Hyp}(\mathcal{F}) \), consider the heat equation on \( L_x \)
\[
\frac{\partial p(x, y, t)}{\partial t} = \Delta_{P, y} p(x, y, t), \quad \lim_{t \to 0^+} p(x, y, t) = \delta_x(y), \quad y \in L_x, \ t \in \mathbb{R}^+. \tag{2.2}
\]
Here, \( \delta_x \) denotes the Dirac mass at \( x \), \( \Delta_{P, y} \) denotes the Laplacian \( \Delta_P \) with respect to the variable \( y \), and the limit is taken in the sense of distribution, that is,
\[
\lim_{t \to 0^+} \int_{L_x} p(x, y, t) f(y) g_P(y) = f(x)
\]
for every smooth function \( f \) compactly supported in \( L_x \).

The smallest positive solution of the above equation, denoted by \( p(x, y, t) \), is called the heat kernel. Such a solution exists because \((L_x, g_P)\) is complete and of bounded geometry (see, for example, [19, 22]). The heat kernel \( p(x, y, t) \) gives rise to a one parameter family \( \{D_t : t \geq 0\} \) of leafwise heat diffusion operators defined on bounded measurable functions on \( \text{Hyp}(\mathcal{F}) \) by
\[
D_t f(x) := \int_{L_x} p(x, y, t) f(y) g_P(y), \quad x \in \text{Hyp}(\mathcal{F}). \tag{2.3}
\]
This family is a semi-group, that is,
\[
D_0 = \text{id} \quad \text{and} \quad D_t 1 = 1 \quad \text{and} \quad D_{t+s} = D_t \circ D_s \quad \text{for } t, s \geq 0, \tag{2.4}
\]
where \( 1 \) denotes the function which is identically equal to 1.

We also denote by \( \Delta_P \) the Laplacian on the Poincaré disc \((\mathbb{D}, g_P)\), that is, for every function \( f \in \mathcal{C}^2(\mathbb{D}) \),
\[
(\Delta_P f) g_P = \pi dd^c f = i \partial \bar{\partial} f \quad \text{on } \mathbb{D}. \tag{2.5}
\]
Let \( \text{dist}_P \) denote the Poincaré distance on \((\mathbb{D}, g_P)\). For \( \zeta \in \mathbb{D} \) write \( \rho := \text{dist}_P(0, \zeta) \). So
\[
\rho := \log \frac{1 + |\zeta|}{1 - |\zeta|}.
\]
Recall a formula in Chavel [22, p. 246] for the heat kernel of the Poincaré disc \((\mathbb{D}, g_P)\):
\[
p_{\mathbb{D}}(0, \zeta, t) = \frac{\sqrt{2} e^{-t/4}}{(2\pi t)^{3/2}} \int_{\rho}^{\infty} \frac{se^{-\frac{s^2}{4\rho}}}{\sqrt{\cosh s - \cosh \rho}} ds.
\]
For every function \( f \in \mathcal{C}^1(\mathbb{D}) \), we also denote by \( |df|_P \) the length of the differential \( df \) with respect to \( g_P \), that is, \( |df|_P = |df| \cdot g_P^{-1/2} \) on \( \mathbb{D} \), where \( |df| \) denotes the Euclidean norm of \( df \).

Let \( x \in \text{Hyp}(\mathbb{F}) \). We often identify the fundamental group \( \pi_1(L_x) \) of \( L_x \) with the group of deck-transformations of the universal covering map \( \phi_x : \mathbb{D} \to L_x \) given in (2.1). It is well-known that \( \pi_1(L_x) \) is at most countable.

The Laplace operator \( \Delta_P \) on the leaf \((L_x, g_P|_{L_x})\) lifts, via \( \phi_x \), to \( \Delta_P \) on the Poincaré disc \((\mathbb{D}, g_P)\). More precisely, for \( x \in \text{Hyp}(\mathbb{F}) \) and for every \( \mathcal{C}^2 \)-smooth function \( f \) defined on \( L_x \),
\[
\Delta_P(f \circ \phi_x) = (\Delta_P f) \circ \phi_x \quad \text{on} \quad \mathbb{D}.
\]
The heat kernel \( p(x, y, t) \) for \((L_x, g_P)\) is related to \( p_{\mathbb{D}}(\tilde{x}, \tilde{y}, t) \) for \((\mathbb{D}, g_P)\) by
\[
p(x, y, t) = \sum_{\gamma \in \pi_1(L_x)} p_{\mathbb{D}}(\tilde{x}, \gamma \tilde{y}, t),
\]
where \( \tilde{x} \) (resp. \( \tilde{y} \)) is a preimage of \( x \) (resp. \( y \)) by the map \( \phi_x \). Moreover, \( p_{\mathbb{D}} \) is invariant under deck-transformations, that is,
\[
p_{\mathbb{D}}(\gamma \tilde{x}, \gamma \tilde{y}, t) = p_{\mathbb{D}}(\tilde{x}, \tilde{y}, t)
\]
for all \( \gamma \in \pi_1(L_x) \) and \( \tilde{x}, \tilde{y} \in \mathbb{D} \) and \( t \geq 0 \). As an immediate consequence of identity (2.8), we obtain the following relation between \( D_t \) given in (2.3) and the heat diffusions \( D_t \) on \( \mathbb{D} \). For \( x \in X \) and for every bounded measurable function \( f \) defined on \( L_x \), we have
\[
D_t(f \circ \phi_x) = (D_t f) \circ \phi_x \quad \text{on} \quad \mathbb{D} \quad \text{for all} \quad t \in \mathbb{R}^+.
\]
See [75, Proposition 2.7] for a proof.

### 2.4 Directed Differential Forms, Directed Positive Forms, Directed Currents, and Harmonic Measures

We recall now the notion of currents on a manifold. Let \( M \) be a real oriented manifold of dimension \( m \) (resp. a complex manifold of dimension \( k \)). We fix an atlas of \( M \) which is locally finite. Up to reducing slightly the charts, we can assume that the local coordinate system associated to each chart is defined on a neighborhood of the closure of this chart. For \( 0 \leq p \leq m \) (resp. for \( 0 \leq p, q \leq k \)) and \( l \in \mathbb{N} \), denote by \( \mathcal{D}_l^P(M) \) (resp. \( \mathcal{D}_l^{p,q}(M) \)) the space of \( p \)-forms (resp. \( (p, q) \)-forms) of class \( \mathcal{C}^l \) with compact support in \( M \), and \( \mathcal{D}_l^P(M) \) (resp. \( \mathcal{D}_l^{p,q}(M) \)) their intersection for \( l \in \mathbb{N} \). If \( \alpha \) is a \( p \)-form (resp. \( (p, q) \)-form) on \( M \), denote by \( \|\alpha\|_{q^l} \) the sum of the \( \mathcal{C}^l \)-norms of the coefficients of \( \alpha \) in the local coordinates. These norms induce a topology on \( \mathcal{D}_l^P(M) \) and \( \mathcal{D}_l^P(M) \) (resp. on \( \mathcal{D}_l^{p,q}(M) \) and \( \mathcal{D}_l^{p,q}(M) \)). In particular, a sequence \( \alpha_j \) converges to \( \alpha \) in \( \mathcal{D}_l^P(M) \) (resp. in \( \mathcal{D}_l^{p,q}(M) \)) if these forms are supported in a fixed compact set and if \( \|\alpha_j - \alpha\|_{q^l} \to 0 \) for every \( l \).

Let \( M \) be a real oriented manifold of dimension \( m \). A current of degree \( p \) (or equivalently, a current of dimension \( m - p \) on \( M \), or a \( p \)-current for short) is a continuous linear form \( T \)
on $\mathcal{D}^{m-p}(M)$ with values in $\mathbb{C}$. The value of $T$ on a test form $\alpha$ in $\mathcal{D}^{m-p}(M)$ is denoted by $\langle T, \alpha \rangle$ or $T(\alpha)$. The current $T$ is of order $\leq l$ if it can be extended to a continuous linear form on $\mathcal{D}^{m-p}(M)$. The order of $T$ is the minimal integer $l \geq 0$ satisfying this condition. It is not difficult to see that the restriction of $T$ to a relatively compact open set of $M$ is always of finite order. Define

$$
\|T\|_{l,K} := \sup \{ |\langle T, \alpha \rangle|, \alpha \in \mathcal{D}^{m-p}(M), \|\alpha\|_{\mathcal{Q}^l} \leq 1, \text{supp}(\alpha) \subset K \}
$$

for $l \in \mathbb{N}$ and $K$ a compact subset of $M$. This quantity may be infinite when the order of $T$ is larger than $l$.

Let $M$ be a complex manifold of dimension $k$. A $(p, q)$-current on $M$ (or equivalently, a current of bidimension $(p, q)$, or equivalently, a current of bidimension $(k - p, k - q)$) is a continuous linear form $T$ on $\mathcal{D}^{k-p,k-q}(M)$ with values in $\mathbb{C}$.

Consider now a Riemann surface lamination with singularities $\mathcal{F} = (X, \mathcal{L}, E)$. The notion of differential forms on manifolds can be extended to laminations (see Sullivan [92]). A (directed) $p$-form (resp. a (directed) $(p, q)$-form) on $\mathcal{F}$ can be seen on the flow box $U \simeq \mathbb{B} \times \mathbb{T}$ as a $p$-form (resp. $(p, q)$-form) on $\mathbb{B}$ depending on the parameter $t \in \mathbb{T}$. For $0 \leq p \leq 2$ (resp. $0 \leq p, q \leq 1$), denote by $\mathcal{D}_p^0(\mathcal{F})$ (resp. $\mathcal{D}_p^{0,q}(\mathcal{F})$) the space of $p$-forms (resp. $(p, q)$-form) $\alpha$ with compact support in $X \setminus E$ satisfying the following property: $\alpha$ restricted to each flow box $U \simeq \mathbb{B} \times \mathbb{T}$ is a $p$-form (resp. $(p, q)$-form) of class $\mathcal{C}^{1,1}$ on the plaques whose coefficients and all their derivatives up to order $l$ depend continuously on the plaque. The norm $\|\cdot\|_{\mathcal{Q}^l}$ on this space is defined as in the case of real manifold using a locally finite atlas of $\mathcal{F}$. We also define $\mathcal{D}_p^0(\mathcal{F})$ (resp. $\mathcal{D}_p^{0,q}(\mathcal{F})$) as the intersection of $\mathcal{D}_p^0(\mathcal{F})$ (resp. $\mathcal{D}_p^{0,q}(\mathcal{F})$) for $l \geq 0$. A (directed) current of degree $p$ (or equivalently, a (directed) current of bidimension $2 - p$) on $\mathcal{F}$ is a continuous linear form on the space $\mathcal{D}^{2-p}(\mathcal{F})$ with values in $\mathbb{C}$. A $p$-current is of order $\leq l$ if it can be extended to a linear continuous form on $\mathcal{D}_l^{2-p}(\mathcal{F})$. The restriction of a current to a relatively compact open set of $X \setminus E$ is always of finite order. The norm $\|\cdot\|_{-l,K}$ on currents is defined as in the case of manifolds. We often write for short $\mathcal{D}(\mathcal{F})$ instead of the space of functions $\mathcal{D}^0(\mathcal{F})$. A (directed) $(p + q)$-current is said to be of bidegree $(p, q)$ (or equivalently, of bidimension $(1 - p, 1 - q)$) if it vanishes on forms of bidegree $(1 - p', 1 - q')$ for $(p', q') \neq (p, q)$.

A form $\alpha \in \mathcal{D}^{1,1}(\mathcal{F})$ is said to be positive if its restriction to every plaque is a positive measure in the usual sense, that is, in every flow box $U \simeq \mathbb{B} \times \mathbb{T}$,

$$
\alpha(z, t) = a(z, t) idz \wedge d\bar{z} \quad \text{for} \quad (z, t) \in \mathbb{B} \times \mathbb{T},
$$

where $a$ is a non-negative-valued function.

**Definition 2.3** Let $\Delta_p$ be the Laplacian on $\mathcal{F}$, that is, the aggregate of the leafwise Laplacians $\{\Delta_{p,x}\}$, where $x \in \text{Hyp}(\mathcal{F})$ (see (2.5) and (2.10)). Let $\mu$ be a locally finite\(^2\) real-valued signed Borel measure on $X$ whose variation $|\mu|$ gives no mass to $\text{Par}(\mathcal{F}) \cup E$.

1. $\mu$ is said to be quasi-harmonic if

$$
\int_X \Delta_p f \, d\mu = 0
$$

for all functions $f \in \mathcal{D}(\mathcal{F})$.

---

\(^2\) A real-valued signed Borel measure $\mu$ on a topological space $X$ is said to be locally finite if every point $x \in X$ has a neighborhood $U_x$ such that $|\mu|(U_x)$ is finite, where $|\mu|$ is the variation of $\mu$. 

\[\mathcal{C}\] Springer
(2) $\mu$ is called very weakly harmonic (resp. weakly harmonic) if $\mu$ is finite positive and the following property is satisfied for $t = 1$ (resp. for all $t \in \mathbb{R}^+$):

$$\int_X D_t f d\mu = \int_X f d\mu$$

for all bounded measurable functions $f$ defined on $X$.

(3) $\mu$ is said to be harmonic if it is both weakly harmonic and quasi-harmonic.

## 2.5 Directed Positive Harmonic Currents vs Harmonic Measures

Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a Riemann surface lamination with singularities. For each chart $\Phi : U \to \mathbb{B} \times \mathbb{T}$, the complex structure on $\mathbb{B}$ induces a complex structure on the leaves of $X$. Therefore, the operator $\partial, \overline{\partial}, d$ and $d^c$ can be defined so that they act leafwise on forms as in the case of complex manifolds. Let $T$ be a directed $(p, q)$-current, then $\partial T$ and $\overline{\partial} T$ are defined as follows.

For $T$ is a $(0, q)$-current, then

$$\langle \partial T, \alpha \rangle := (-1)^{q+1} \langle T, \partial \alpha \rangle$$

for all test forms $\alpha \in \mathcal{D}^{0,1-q}(\mathcal{F})$.

If $T$ is a $(1, q)$-current, then $\partial T := 0$.

If $T$ is a $(p, 0)$-current, then

$$\langle \overline{\partial} T, \alpha \rangle := (-1)^{p+1} \langle T, \overline{\partial} \alpha \rangle$$

for all test forms $\alpha \in \mathcal{D}^{1-p,0}(\mathcal{F})$.

If $T$ is a $(p, 1)$-current, then $\overline{\partial} T := 0$.

So we get easily that

$$d := \partial + \overline{\partial} : \mathcal{D}^p(\mathcal{F}) \to \mathcal{D}^{p+1}(\mathcal{F})$$

for $0 \leq p \leq 2$ with $\mathcal{D}^3(\mathcal{F}) \equiv \{0\}$;

$$dd^c := \frac{i}{\pi} \overline{\partial} \partial: \mathcal{D}(\mathcal{F}) \to \mathcal{D}^{1,1}(\mathcal{F})$$.

### Definition 2.4 (Garnett [54], see also Sullivan [92])

Let $T$ be a directed current of bidimension $(1, 1)$ on $\mathcal{F}$.

- $T$ is said to be positive if $T(\alpha) \geq 0$ for all positive forms $\alpha \in \mathcal{D}^{1,1}(\mathcal{F})$.
- $T$ is said to be closed if $dT = 0$ in the weak sense (namely, $T(df) = 0$ for all directed forms $f \in \mathcal{D}(\mathcal{F})$).
- $T$ is said to be harmonic if $dd^c T = 0$ in the weak sense (namely, $T(dd^c f) = 0$ for all functions $f \in \mathcal{D}(\mathcal{F})$).

We have the following decomposition.

### Proposition 2.5 [32, Proposition 2.3] and [75, Proposition 2.10]

Let $T$ be a directed harmonic current on $\mathcal{F}$. Let $U \simeq \mathbb{B} \times \mathbb{T}$ be a flow box which is relatively compact in $X$. Then, there is a positive Radon measure $v$ on $\mathbb{T}$ and for $v$-almost every $t \in \mathbb{T}$, there is a harmonic function $h_t$ on $\mathbb{B}$ such that if $K$ is compact in $\mathbb{B}$, the integral $\int_K \|h_t\|_{L^1(K)} dv(t)$ is finite and

$$T(\alpha) = \int_{\mathbb{T}} \left( \int_{\mathbb{B}} h_t(y) \alpha(y, t) \right) dv(t)$$

for every form $\alpha \in \mathcal{D}^{1,1}(\mathcal{F})$ compactly supported on $U$. If moreover, $T$ is positive, then for $v$-almost every $t \in \mathbb{T}$, the harmonic function $h_t$ is positive on $\mathbb{B}$. If moreover, $T$ is closed, then for $v$-almost every $t \in \mathbb{T}$, the harmonic function $h_t$ is constant on $\mathbb{B}$. 
Definition 2.6 A directed positive harmonic current \( T \) on \( \mathcal{F} = (X, \mathcal{L}, E) \) is said to be \textit{diffuse} if for any decomposition of \( T \) in any flow box \( U \simeq \mathbb{B} \times \mathbb{T} \) as in Proposition 2.5, the measure \( \nu \) has no mass on each single point of the transversal \( T \).

Definition 2.7 Recall that a positive finite measure \( \mu \) on the \( \sigma \)-algebra of Borel sets in \( X \) with \( \mu(E) = 0 \) is said to be \textit{ergodic} if for every leafwise saturated Borel measurable set \( Z \subset X \), \( \mu(Z) \) is equal to either \( \mu(X) \) or 0.

A directed positive harmonic current \( T \) is said to be \textit{extremal} if the Poincaré mass of \( T \) with respect to Poincaré metric \( g_P \) on \( X \setminus E \), i.e.,

\[
\mu := T \wedge g_P \quad \text{on} \quad X \setminus (E \cup \text{Par}(\mathcal{F})) \quad \text{and} \quad \mu(E \cup \text{Par}(\mathcal{F})) = 0.
\]

We call \textit{Poincaré mass} of \( T \) the mass of \( T \) with respect to Poincaré metric \( g_P \) on \( X \setminus E \), i.e., the mass of the positive measure \( \mu = \Phi(T) \).

A priori, Poincaré mass may be infinite near the singular points. The following result, which is implicitly proved in [32], relates the notions of harmonic measures and directed positive harmonic currents (see also [78]).

Theorem 2.9 We keep the assumption and notation of Definition 2.8.

1. \( \Phi \) is a bijection from the convex cone of all directed positive harmonic currents \( T \) of \( \mathcal{F} \) giving no mass to \( \text{Par}(\mathcal{F}) \) onto the convex cone of all positive quasi-harmonic measures \( \mu \).
2. If \( T \) is a directed positive harmonic current such that \( T \) gives no mass to \( \text{Par}(\mathcal{F}) \) and that its Poincaré mass is finite, then \( \mu := \Phi(T) \) is an extremal positive quasi-harmonic measure iff \( \mu \) is ergodic iff \( T \) is extremal.
3. A positive quasi-harmonic measure is harmonic if and only if it is finite.

Assertions (1) and (2) follows from the definitions.

By definition, a harmonic measure is necessarily finite. Therefore, to complete the proof of assertion (3), we need to show that a finite positive quasi-harmonic measure is weakly harmonic.

For this purpose, we introduce the following operator \( A_R : L^\infty(\text{Hyp}(\mathcal{F})) \rightarrow L^\infty(\text{Hyp}(\mathcal{F})) \) given by

\[
A_R u(x) := \frac{1}{\mathcal{M}_R} \int_{\mathbb{D}_R} (\phi_x)^*(ug_P) \quad \text{for} \quad x \in \text{Hyp}(\mathcal{F}), \quad \text{where} \quad \mathcal{M}_R := \int_{\mathbb{D}_R} (\phi_x)^*(g_P).
\]

Note that \( \mathcal{M}_R \) is the Poincaré area of \( \mathbb{D}_R \) which is also the Poincaré area of \( \phi_x(\mathbb{D}_R) \) counted with multiplicity. It is immediate from the definition that the norm of \( A_R \) is equal to 1. Since \( \mu(E \cup \text{Par}(\mathcal{F})) = 0 \), we extends the domain of the definition of \( A_R \) in (2.12) in a natural way so that \( A_R : L^\infty(\mu) \rightarrow L^\infty(\mu) \) with norm 1.

The following result is needed.
Lemma 2.10 If \( \mu \) is a finite positive quasi-harmonic measure, then
\[
\int (A_R u) d\mu = \int u d\mu \quad \text{for} \quad u \in L^1(\mu).
\]

Taking for granted the above lemma, we arrive at the

End of the proof of Theorem 2.9 Let \( \mu \) be a finite positive quasi-harmonic measure. We only need to show that \( \mu \) is weakly harmonic. Fix an arbitrary function \( u \) in \( \mathcal{D}(\mathcal{F}) \) and a time \( t > 0 \). So it is sufficient to show that
\[
\int_X D_t u d\mu = \int_X u d\mu.
\]

For \( z \in \mathbb{D} \), write \( r = |z| \) and let \( R \) be defined by (5.30), that is, \( \mathbb{D}_R = r \mathbb{D} \). Write the Poincaré metric in \( \mathbb{D} \) as follows:
\[
dg_P(z) = d\sigma_R(z) dR, \tag{2.13}
\]
where \( d\sigma(z) \) is the Poincaré-length form on \( \partial \mathbb{D}_R \) which is the circle of center 0 and of Poincaré radius \( R \). It follows from (2.13) that the derivative of \( \mathcal{M}_R \) with respect to \( R \) is equal to
\[
\mathcal{M}'_R := \lim_{s \to 0} \frac{\mathcal{M}_{R+s} - \mathcal{M}_R}{s} = \int_{\partial \mathbb{D}_R} d\sigma_R(z).
\]
Since \( p_{\mathbb{D}}(0, \cdot, t) \) is a radial positive function smooth on \( \mathbb{D} \setminus \{0\} \) and it satisfies \( \int_{\mathbb{D}} p_{\mathbb{D}}(0, y, t) g_P(y) = 1 \) for every \( t > 0 \), we infer from the last line and (2.13) that
\[
\int_0^\infty p_{\mathbb{D}}(0, r_R, t) \mathcal{M}'_R dR = 1, \tag{2.14}
\]
where \( R \) and \( r_R \) are related by (5.30). On the one hand, since \( u \in \mathcal{D}(\mathcal{F}) \), we deduce from (2.13) that for \( x \in \text{Hyp}(\mathcal{F}) \), the derivative of \( (\mathcal{M}_R A_R)u(x) \) with respect to \( R \) is equal to
\[
(\mathcal{M}_R A_R)'u(x) := \lim_{s \to 0} \frac{\mathcal{M}_{R+s} A_{R+s}u(x) - \mathcal{M}_R A_R u(x)}{s} = \lim_{s \to 0} \frac{1}{s} \int_{\partial \mathbb{D}_R, \partial \mathbb{D}_R} \phi^*_s(u g_P)
\]
\[
= \int_{\partial \mathbb{D}_R} \phi^*_s(u \sigma_R).
\]
Moreover, by Lemma 2.10,
\[
\int_X (\mathcal{M}_R A_R)'u(x) d\mu(x) = \lim_{s \to 0} \frac{\int_X \mathcal{M}_{R+s} A_{R+s}u(x) d\mu(x) - \int_X \mathcal{M}_R A_R u(x) d\mu(x)}{s}
\]
\[
= \lim_{s \to 0} \frac{\mathcal{M}_{R+s} - \mathcal{M}_R}{s} \cdot \int_X u(x) d\mu(x)
\]
\[
= \mathcal{M}'_R \int_X u(x) d\mu(x).
\]
On the other hand, by (2.10), we have
\[
(D_t u)(x) := \int_{\mathbb{D}} p_{\mathbb{D}}(0, \cdot, t)(u \circ \phi_x) g_P \quad \text{for} \quad x \in \text{Hyp}(\mathcal{F}).
\]
Therefore, using (2.13) again and the last expression for \( (\mathcal{M}_R A_R)'u(x) \), we get that
\[
(D_t u)(x) = \int_0^\infty p_{\mathbb{D}}(0, r_R, t) \left( \int_{\partial \mathbb{D}_R} \phi^*_s(u \sigma_R) \right) dR = \int_0^\infty p_{\mathbb{D}}(0, r_R, t) \left( (\mathcal{M}_R A_R)'u(x) \right) dR.
\]
Integrating the last equalities with respect to \( d\mu \) and using Fubini theorem, we obtain

\[
\int_X (D_t u)(x) d\mu(x) = \int_0^\infty p_D(0, r, t) \left( \int_X (\mathcal{M}_R A_R)' u(x) d\mu(x) \right) dR.
\]

Using the last expression for the inner integral on the right hand side, it follows that

\[
\int_X (D_t u)(x) d\mu(x) = \left( \int_0^\infty p_D(0, r, t) \mathcal{M}_R dR \right) \left( \int_X u(x) d\mu(x) \right)
\]

By (2.14), the right hand side is equal to \( \int_X u(x) d\mu(x) \). Hence, \( \mu \) is a weakly harmonic measure. \(\square\)

In order to prove Lemma 2.10, we need some preparations. Consider now a flow box \( \Phi : U \to \mathbb{B} \times \mathbb{T} \) as above. Recall that for simplicity, we identify \( U \) with \( \mathbb{B} \times \mathbb{T} \) and \( T \) with the transversal \( \Phi^{-1}([z] \times \mathbb{T}) \) for some point \( z \in \mathbb{B} \). We have the following result.

**Lemma 2.11** [32, Proposition 3.2] Let \( \nu \) be a positive Radon measure on \( T \). Let \( T_1 \subset T \) be a measurable set such that \( \nu(T_1) > 0 \) and \( L_x \) is hyperbolic for any \( x \in T_1 \). Then, for every \( \epsilon > 0 \) there is a compact set \( T_2 \subset T_1 \) with \( \nu(T_2) > \nu(T_1) - \epsilon \) and a family of universal covering maps \( \phi_x : D \to L_x \) with \( \phi_x(0) = x \) and \( x \in T_2 \) that depends continuously on \( x \).

For \( x \in \text{Hyp}(\mathcal{F}) \) and \( R > 0 \), denote \( L_{x, R} := \phi_x(D_R) \), where we recall from (2.1) that \( \phi_x : D \to L_x \) is a universal covering map with \( \phi_x(0) = x \), and \( D_R \subset D \) is the disc of center 0 and of radius \( R \) (with respect to the Poincaré metric \( g_P \) on \( D \)). Since \( \phi_x \) is unique up to a rotation on \( D \), \( L_{x, R} \) is independent of the choice of \( \phi_x \). We will need the following result.

**Lemma 2.12** [32, Corollary 3.3] Let \( R > 0 \) be a positive constant. Then, under the hypothesis of Lemma 2.11, there is a countable family of compact sets \( \mathcal{S}_n \subset T_1 \), \( n \geq 1 \), with \( \nu(\mathcal{U}_n \mathcal{S}_n) = \nu(T_1) \) such that \( L_{x, R} \cap \mathcal{S}_n = \{x\} \) for every \( x \in \mathcal{S}_n \). Moreover, there are universal covering maps \( \phi_x : D \to L_x \) with \( \phi_x(0) = x \) which depend continuously on \( x \in \mathcal{S}_n \).

**Proof of Lemma 2.10** (see [32, Proposition 7.3]) We can assume that \( u \) is positive and using a partition of unity, we can also assume that \( u \) has support in a compact set of a flow box \( U \simeq \mathbb{B} \times \mathbb{T} \). Let \( T_1 \) be the set of \( T \cap \text{Hyp}(\mathcal{F}) \). We will use the decomposition of \( T \) and the notation as in Proposition 2.5. By hypothesis, we can assume that the measure \( \nu \) has total mass on \( T_1 \). Now, we apply Lemma 2.12 to \( \nu \) and \( 4\lambda R \) instead of \( R \) for a fixed constant \( \lambda \) large enough. Let \( \Sigma_n(R) \) denote the union of \( L_{x, R} \) for \( x \in \mathcal{S}_n \). Define by induction the function \( u_n \) as follows: \( u_1 \) is the restriction of \( u \) to \( \Sigma_1(R) \) and \( u_n \) is the restriction of \( u - u_1 - \cdots - u_{n-1} \) to \( \Sigma_n(R) \). We have \( u = u_n \). So, it is enough to prove the proposition for each \( u_n \).

We use now the properties of \( \mathcal{S}_n \) given in Lemma 2.12. The set \( \Sigma_n(4\lambda R) \) is a smooth lamination and the restriction \( T_n \) of \( T \) to \( \Sigma_n(4\lambda R) \) is a positive harmonic current. Observe that \( A_{Rn} \) vanishes outside \( \Sigma_n(\lambda R) \) and does not depend on the restriction of \( T \) to \( X \setminus (E \cup \text{Par}(\mathcal{F}) \cup \Sigma_n(2\lambda R)) \). Since there is a natural projection from \( \Sigma_n \) to the transversal \( \mathcal{S}_n \), the extremal positive harmonic currents on \( \mathcal{S}_n \) are supported by a leaf and defined by a harmonic function. Therefore, we can reduce the problem to the case where \( T = h[L_{x, 4\lambda R}] \) with \( x \in \mathcal{S}_n \) and \( h \) is positive harmonic on \( L_{x, 4\lambda R} \).
Define \( \hat{u} := u_n \circ \phi_x, \hat{h} := h \circ \phi_x \) and \( \hat{A}_R u := (A_R u) \circ \phi_x \). The function \( \hat{h} \) is harmonic on \( \mathbb{D}_{4\lambda R} \). Choose a measurable set \( \Theta \subset \mathbb{D}_{2\lambda R} \) such that \( \phi_x \) defines a bijection between \( \Theta \) and \( L_{x,2\lambda R} \). We first observe that

\[
\hat{A}_R u(0) := \frac{1}{M_R} \int_{\text{dist}_P(\xi,0)<R} \hat{u}(\xi) g_P(\xi).
\]

If \( \eta \) is a point in \( \mathbb{D} \) and \( \tau : \mathbb{D} \to \mathbb{D} \) is an automorphism such that \( \tau(0) = \eta \), then \( \phi_x \circ \tau \) is also a covering map of \( L_x \) but it sends 0 to \( \phi_x(\eta) \). We apply the above formula to this covering map. Since \( \tau \) preserves \( g_P \) and \( \text{dist}_P \), we obtain

\[
\hat{A}_R u(\eta) := \frac{1}{M_R} \int_{\text{dist}_P(\xi,\eta)<R} \hat{u}(\xi) g_P(\xi).
\]

Hence, we have to show the following identity

\[
\int_{\Theta} \left[ \frac{1}{M_R} \int_{\text{dist}_P(\xi,\eta)<R} \hat{u}(\xi) g_P(\xi) \right] \hat{h}(\eta) g_P(\eta) = \int_{\Theta} \hat{u}(\xi) \hat{h}(\xi) g_P(\xi).
\]

Let \( W \) denote the set of points \( (\xi, \eta) \in \mathbb{D}^2 \) such that \( \eta \in \Theta \) and \( \text{dist}_P(\xi, \eta) < R \). Let \( W' \) denote the symmetric of \( W \) with respect to the diagonal, i.e., the set of \( (\xi, \eta) \) such that \( \xi \in \Theta \) and \( \text{dist}_P(\xi, \eta) < R \). Since \( \hat{h} \) is harmonic, we have

\[
\hat{h}(\xi) = \frac{1}{M_R} \int_{\text{dist}_P(\xi,\eta)<R} \hat{h}(\eta) g_P(\eta).
\]

Therefore, our problem is to show that the integrals of \( \Phi := \hat{u}(\xi) \hat{h}(\eta) g_P(\xi) \wedge g_P(\eta) \) on \( W \) and \( W' \) are equal.

Consider the map \( \phi := (\phi_x, \phi_x) \) from \( \mathbb{D}^2 \) to \( L_x^2 \). The fundamental group \( \Gamma := \pi_1(L_x) \) can be identified with a group of automorphisms of \( \mathbb{D} \). Since, \( \Gamma^2 \) acts on \( \mathbb{D}^2 \) and preserves the form \( \Phi \), our problem is equivalent to showing that each fiber of \( \phi \) has the same number of points in \( W \) and in \( W' \). We only have to consider the fibers of points in \( L_{x,2\lambda R} \times L_{x,2\lambda R} \) since \( A_R u_n \) is supported on \( L_{x,\lambda R} \). Fix a point \( (\xi, \eta) \in \Theta' \) and consider the fiber \( F \) of \( \phi(\xi, \eta) \). By the definition of \( \Theta \), the numbers of points in \( F \cap W \) and \( F \cap W' \) are respectively equal to

\[
\# \{ \gamma \in \Gamma, \text{dist}_P(\gamma \cdot \xi, \eta) < R \} \quad \text{and} \quad \# \{ \gamma \in \Gamma, \text{dist}_P(\xi, \gamma \cdot \eta) < R \}.
\]

Since \( \Gamma \) preserves the Poincaré metric \( g_P \) on \( \mathbb{D} \), the first set is equal to

\[
\{ \gamma \in \Gamma, \text{dist}_P(\zeta, \gamma^{-1} \cdot \eta) < R \}
\]

It is now clear that the two numbers are equal. This completes the proof.

**Remark 2.13** Theorem 2.9 (3) gives a very effective necessary and sufficient criterion for a positive quasi-harmonic measure to be harmonic: the finiteness of the measure. This is a peculiar property of the leafwise Poincaré metric \( g_P \). For general families of leafwise metrics on Riemann surface laminations with singularities, or more generally, for \( N \)-real or complex dimensional laminations, it is an important question to determine when a positive finite quasi-harmonic measure is harmonic.

**Problem 2.14** For general families of leafwise metrics on Riemann surface laminations with singularities, or more generally, for \( N \)-real or complex dimensional laminations, find sufficient and effective conditions for a positive finite quasi-harmonic measure to be harmonic.
2.6 Positive $dd^c$-closed Currents on Complex Manifolds

Let $M$ be a complex manifold of dimension $k$. A $(p, p)$-form on $M$ is positive if it can be written at every point as a combination with positive coefficients of forms of type

$$i\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha}_p,$$

where the $\alpha_j$ are $(1, 0)$-forms. A $(p, p)$-current or a $(p, p)$-form $T$ on $M$ is weakly positive if $T \wedge \varphi$ is a positive measure for any smooth positive $(k-p, k-p)$-form $\varphi$. A $(p, p)$-current $T$ is positive if $T \wedge \varphi$ is a positive measure for any smooth weakly positive $(k-p, k-p)$-form $\varphi$. If $M$ is given with a Hermitian metric $\beta$ and $T$ is a positive $(p, p)$-current on $M$, $T \wedge \beta^{k-p}$ is a positive measure on $M$. The mass of $T \wedge \beta^{k-p}$ on a measurable set $A$ is denoted by $\|T\|_A$ and is called the mass of $T$ on $A$. The mass $\|T\|$ of $T$ is the total mass of $T \wedge \beta^{k-p}$ on $M$. A $(p, p)$-current $T$ on $M$ is strictly positive if we have locally $T \geq \epsilon\beta^p$, i.e., $T - \epsilon\beta^p$ is positive, for some constant $\epsilon > 0$. The definition does not depend on the choice of $\beta$.

A $(p, p)$-current on $M$ is closed if $dT = 0$ in the weak sense (namely, $T(d\alpha) = 0$ for every form $\alpha \in \mathcal{G}^{k-p,k-p-1}(M) \oplus \mathcal{G}^{k-p-1,k-p}(M)$). A $(p, p)$-current on $M$ is $dd^c$-closed if $dd^cT = 0$ in the weak sense (namely, $T(dd^c\alpha) = 0$ for every form $\alpha \in \mathcal{G}^{k-p-1,k-p-1}(M)$).

For every $r > 0$ let $\mathbb{B}_r$ denote the ball of center $0$ and of radius $r$ in $\mathbb{C}^k$. The following local property of positive $dd^c$-closed currents has been discovered by Skoda [91].

**Proposition 2.15** (Skoda [91]) Let $T$ be a positive $dd^c$-closed $(p, p)$-current in a ball $\mathbb{B}_{r_0}$. Define $\beta := dd^c\|z\|^2$ the standard Kähler form where $z$ is the canonical coordinates on $\mathbb{C}^p$. Then the function $r \mapsto \pi^{-(k-p)r-2(k-p)}\|T \wedge \beta^{k-p}\|_{\mathbb{B}_r}$ is increasing on $0 < r \leq r_0$. In particular, it is bounded on $(0, r_1]$ for any $0 < r_1 < r_0$.

**Definition 2.16** Under the hypothesis and notation of Proposition 2.15, the limit of the above function when $r \to 0$ is called the Lelong number of $T$ at $0$, and is denoted by $\nu(T, 0)$.

For a general positive $dd^c$-closed current $T$ defined on an open neighborhood of a point $x$ in a complex manifold $M$, we define the Lelong number of $T$ at $x$ similarly by using a local holomorphic coordinate system near $x$ which identifies $x$ to $0 \in \mathbb{C}^k$. This number is denoted by $\nu(T, x)$. By Siu [90] (for positive closed currents) and by Alessandrini-Bassanelli (for positive $dd^c$-closed currents), $\nu(T, x)$ is independent of the choice of a coordinate system. By Proposition 2.15, Lelong number always exists and is finite non-negative. The next simple result allows for extending positive $dd^c$-closed currents of bidimension $(1, 1)$ through isolated points.

**Proposition 2.17** (Dinh-Nguyen-Sibony [32, Lemma 2.5], Fornæss-Sibony-Wold [52, Lemma 17]) Let $T$ be a positive current of bidimension $(1, 1)$ with compact support on a complex manifold $M$. Assume that $dd^cT$ is a negative measure on $M \setminus E$ where $E$ is a finite set. Then $T$ is a positive $dd^c$-closed current on $M$.

Now we come to the notion of holomorphically immersed laminations.

**Definition 2.18** Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a Riemann surface laminations with singularities and let $M$ be a complex manifold. We say that $\mathcal{F}$ is holomorphically immersed in $M$ if $X$ is a closed subset of $M$ and the leaves of $(X \setminus E, \mathcal{L})$ are Riemann surfaces holomorphically immersed in $M$. 
If $\mathcal{F} := (X, \mathcal{L}, E)$ is a singular holomorphic foliation, then it is clearly a Riemann surface lamination with singularities which is holomorphically immersed in $X$.

**Remark 2.19** Let $j : X \hookrightarrow M$ be the canonical injection from $X$ into $M$. The condition in Definition 2.18 means the following properties (i)–(ii):

(i) $X$ is a closed subset of $M$ and $j$ is continuous;

(ii) The restriction $j_x$ of $j$ on each leaf $L_x$, with $x \in X \setminus E$, is a holomorphic immersion from $L_x$ into $M$.

In particular property (i) implies that the topology of the lamination $X$ coincides with the topology induced on the closed set $X$ from $M$.

The aggregate of the pull-back via $j_x$, with $x \in X \setminus E$, of each test form $\alpha \in \mathcal{D}^{1,1}(M \setminus E)$ defines a form in $\mathcal{D}^{1,1}(\mathcal{F})$ denoted by $j^* \alpha$. So we obtain a canonical map

$$j^* : \mathcal{D}^{1,1}(M \setminus E) \to \mathcal{D}^{1,1}(\mathcal{F}) \quad \text{given by} \quad \alpha \mapsto j^* \alpha.$$ 

We see easily that the image $I$ of $j^*$ is dense in $\mathcal{D}^{1,1}(\mathcal{F})$.

The original notions of directed positive $dd^c$-closed currents for singular holomorphic foliations (resp. for singular laminations which are holomorphically immersed in a complex manifold) with a small set of singularities were introduced by Berndtsson-Sibony [2] and Fornaess-Sibony [49, 50]) respectively. We give here another notion of directed positive $dd^c$-closed currents for singular Riemann surface laminations. Our notion coincides with the previous ones when the lamination is $C^2$-transversally smooth. The advantage of our notion is that it is relevant even when the set of singularities is not small.

**Definition 2.20** Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a Riemann surface lamination with singularities which is holomorphically immersed in a complex manifold $M$. A directed positive $dd^c$-closed current (resp. a directed positive closed current) on $\mathcal{F}$ is a positive $dd^c$-closed current $T$ (resp. a positive closed current $T$) of bidimension $(1, 1)$ on $M$ such that the following properties (i)–(ii)–(iii) are satisfied:

(i) The support of $T$ is contained in $X$;

(ii) $T$ does not give mass to $E$, i.e., the mass $\|T\|_E$ of $T$ on $E$ is zero;

(iii) $T$ is a directed positive harmonic current (resp. a directed positive closed current) on $\mathcal{F}$ in the sense of Definition 2.4.

Moreover, we say that $T$ is diffuse if it is diffuse in the sense of Definition 2.6 as a directed positive harmonic current (resp. a directed positive closed current) on $\mathcal{F}$.

**Remark 2.21** Property (iii) of Definition 2.20 means the following two properties (iii-a)–(iii-b):

(iii-a) $\langle T, \alpha \rangle = \langle T, \beta \rangle$ for $\alpha, \beta \in \mathcal{D}^{1,1}(M \setminus E)$ such that $j^* \alpha = j^* \beta$; so the current

$$\tilde{T} : \mathcal{I} \to \mathbb{C} \quad \text{given by} \quad \langle \tilde{T}, j^* \alpha \rangle := \langle T, \alpha \rangle \quad \text{for} \quad \alpha \in \mathcal{D}^{1,1}(M \setminus E)$$

is well-defined;

(iii-b) The current $\tilde{T}$ defined in (iii-a) can be uniquely extended from $\mathcal{I}$ to $\mathcal{D}^{1,1}(\mathcal{F})$ by continuity (as $\mathcal{I}$ is dense in $\mathcal{D}^{1,1}(\mathcal{F})$) to a current $\tilde{T}$ of order zero, and $\tilde{T}$ is a directed positive harmonic current (resp. directed positive closed current) on $\mathcal{F}$ in the sense of Definition 2.4.

Property (iii-b) holds automatically since $T$ is a positive $dd^c$-closed current (resp. positive closed current) on $M$. So property (iii) is equivalent to the single property (iii-a). If there is no confusion, we often denote $\tilde{T}$ and $\tilde{T}$ simply by $T$. 
Remark 2.22 In the original definitions of directed positive $dd^c$-closed current (resp. directed positive closed current) which have been introduced by Berndtsson-Sibony [2] and Fornæss-Sibony [49, 50]), these authors assume the following conditions (i)’–(ii)’–(iii)’ (see [50, Definition 7]):

(i)’ $T$ is a positive $dd^c$-closed current (resp. positive closed current) on $M$ with support in $X$;
(ii)’ $E$ is a small set in the sense that $\Lambda_2(E) = 0$, where $\Lambda_2$ denotes the two dimensional Hausdorff measure;
(iii)’ A decomposition of $T$ as in Proposition 2.5 is valid in any flow box outside the singularities.

In fact, the assumption (ii)’ ensures that $T$ does not give mass to $E$. The reader may consult the proof of Theorems 10 and 23 in [50] in order to see that our definition of directed positive closed currents (resp. directed positive $dd^c$-closed currents) is equivalent to theirs when the lamination $\mathcal{F}$ satisfies condition (ii)’ and $\mathcal{F}$ is $C^1$-transversally smooth (resp. $C^2$-transversally smooth).

It is worthy noting that these authors also introduce a notion of positive $dd^c$-closed currents weakly directed by a lamination in a complex manifold (see [50, Section 5]). Roughly speaking, this weaker notion means that the considered current is directed by a collection of continuous forms defining the considered lamination.

The existence of nonzero directed positive harmonic currents for compact nonsingular laminations was proved by Garnett [54]. The case of compact singular holomorphic foliations was proved by Berndtsson-Sibony [2, Theorem 1.4] under reasonable assumptions. The more general case of compact $C^2$-transversally smooth Riemann surface laminations with singularities was proved by Fornæss-Sibony [50, Theorem 23], see also Sibony [88] for the existence of positive $dd^c$-closed currents directed by a Pfaff system.

Recall that a subset $E$ of a complex manifold $M$ is said to be locally pluripolar if, for every $a \in M$, there is a plurisubharmonic function $u$ in some open neighborhood $U$ of $a$ in $M$ such that $\{u = -\infty\} \supset E \cap U$. Moreover, $E$ is said to be locally complete pluripolar if, for every $a \in M$, there is a plurisubharmonic function $u$ in some open neighborhood $U$ of $a$ in $M$ such that $\{u = -\infty\} = E \cap U$. Here is a synthesis of the above-mentioned existence results.

Theorem 2.23 Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a singular Riemann surface lamination which is holomorphically immersed in a complex manifold $M$. Assume moreover that $X$ is compact and we are in at least one of the following two situations:

(1) (Fornæss-Sibony [50, Theorem 23]) $\mathcal{F}$ is $C^2$-transversally smooth and $E$ is locally complete pluripolar in $M$;
(2) (Berndtsson-Sibony [2, Theorem 1.4]) $\mathcal{F}$ is a singular holomorphic foliation (so $M = X$) and $E$ is locally pluripolar in $M$.

Then there is a nonzero positive $dd^c$-closed current $T$ of bidimension $(1, 1)$ supported on $X$ such that the restriction of $T$ on $X \setminus E$ defines a directed positive harmonic current on $(X \setminus E, \mathcal{L})$. In particular, if there is no nonzero positive $dd^c$-closed current of bidimension $(1, 1)$ which gives mass to $E$ (e.g., if $\Lambda_2(E) = 0$, where $\Lambda_2$ denotes the two dimensional Hausdorff measure), then $T$ is a nonzero directed positive $dd^c$-closed current in the sense of Definition 2.20.

When a leaf $L_x$ is hyperbolic, an exhaustion-average on $L_x$ (using the Nevanlinna current $\tau_{x, R}$ given in formula (5.31)) was introduced by Fornæss-Sibony [49] (see also [50, Corollary 3]). It provides another construction of directed positive $dd^c$-closed currents.
By Burns-Sibony [13], if a singular holomorphic foliation on a compact complex manifold admits a parabolic leaf in the sense of the potential theory, then there is a nonzero positive closed current weakly directed by the foliation (see Remark 2.22 for the notion of weakly directed positive currents). If moreover, \( F \) satisfies assumption (2) of Theorem 2.23, then this current is also directed by the foliation in the sense of Definition 2.20. The reader is invited to consult Păun-Sibony [85] for a fruitful discussion on the link between value distribution theory and positive closed currents directed by singular holomorphic foliations.

The following notion will be needed later on.

**Definition 2.24** Let \( \mathcal{F} := (X, \mathcal{L}, E) \) be a Riemann surface lamination with singularities which is holomorphically immersed in a complex manifold \( M \). A pure 1-dimensional complex analytic set \( L \) in \( M \setminus E \) is called an invariant (analytic) curve of \( \mathcal{F} \) if \( L \) is itself a leaf of \( \mathcal{F} \). A pure 1-dimensional complex analytic set \( L \) in \( M \) is called an invariant closed (analytic) curve of \( \mathcal{F} \) if \( L \setminus E \) is itself a leaf of \( \mathcal{F} \).

If the current of integration \([L]\) associated to an invariant closed analytic curve \( L \) of \( \mathcal{F} \) (as it is a pure 1-dimensional complex analytic set) does not give mass to \( E \), then \([L]\) is a directed positive closed current which is clearly not diffuse (see Definitions 2.20 and 2.6).

By Fornæss-Sibony [50, Proposition 3], if \( X \) is compact and either \( \Lambda_1(E) = 0 \) or \( \Lambda_2(E) = 0 \) and \( E \) is locally complete pluripolar, then the closure \( \overline{L} \) in \( M \) of an invariant (analytic) curve \( L \) of \( \mathcal{F} \) is an invariant closed (analytic) curve of \( \mathcal{F} \). Here, \( \Lambda_k \) denotes the \( k \)-dimensional Hausdorff measure.

When \( M \) is an algebraic manifold, an invariant analytic curve is also called invariant algebraic curve.

### 2.7 Sample-Path Space and Shift-Transformations

Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a Riemann surface lamination with singularities. Let \( \Omega := \Omega(\mathcal{F}) \) be the space consisting of all continuous paths \( \omega : [0, \infty) \to X \) with image fully contained in a single leaf. This space is called the sample-path space associated to \( \mathcal{F} \). Observe that \( \Omega \) can be thought of as the set of all possible paths that a Brownian particle, located at \( \omega(0) \) at time \( t = 0 \), might follow as time progresses. For each \( x \in X \setminus E \), let \( \Omega_x = \Omega_x(\mathcal{F}) \) be the space of all continuous leafwise paths starting at \( x \) in \( X \setminus E \), that is,

\[
\Omega_x := \{ \omega \in \Omega : \omega(0) = x \}. \tag{2.15}
\]

Consider the shift-transformations \( \sigma_t : \Omega \to \Omega \) defined by

\[
\sigma_t(\omega)(s) := \omega(s + t), \quad \omega \in \Omega, \ s \in \mathbb{R}^+. \tag{2.16}
\]

### 2.8 Holonomy and Monodromy

Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a Riemann surface lamination with singularities and let \( L \) be a leaf of \( \mathcal{F} \). Fix two points \( x, y \in L \) and consider small transversals \( T_x, T_y \) to \( L \) at \( x, y \) respectively. If \( \mathcal{F} \) is a singular holomorphic foliation and \( X \) is a complex manifold of dimension \( k \), we can choose \( T_x, T_y \simeq \mathbb{D}^{k-1} \). Let \( y : [0, 1] \to L \) be a continuous path

---

3Here, a Riemann surface \( L \) is called *parabolic in the sense of the potential theory* (or equivalently, *parabolic in the sense of Ahlfors*), if bounded subharmonic functions on \( L \) are constant. If \( L \) is parabolic (see Definition 2.2) then it is parabolic in the sense of the potential theory. But the converse statement is in general not true.
with \( \gamma(0) = x \) and \( \gamma(1) = y \). For each \( z \in T_x \) near \( x \) one can travel on \( L_z \) over \( \gamma \) to reach \( T_y \) at some point \( z' \). More precisely, let \( \{(U_p, \Phi_p)\}_{0 \leq p \leq n-1} \) be laminated charts of \( \mathcal{F} \) and \( 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1 \) be a partition of \([0, 1]\) such that if \( U_p \cap U_q \neq \emptyset \) then \( U_p \cup U_q \) is contained in a laminated chart, and \( \gamma[t_p, t_{p+1}] \subset U_p \) for \( 0 \leq p \leq n-1 \). For each \( 1 \leq p \leq n-1 \), choose a transversal \( T_p \) to \( L \) at \( \gamma(t_p) \), and set \( T_0 := T_x \) and \( T_n := T_y \). Then for each \( z \in T_p \) sufficiently close to \( \gamma(t_p) \), the plaque of \( U_p \) passing through \( z \) meets \( T_{p+1} \) in a unique point \( f_p(z) \), and \( z \mapsto f_p(z) \) is homeomorphic with \( f_p(\gamma(t_p)) = \gamma(t_{p+1}) \). This map is smooth/holomorphic if \( \mathcal{F} \) is a transversally smooth lamination/singular holomorphic foliation. We see that the composition

\[
\text{hol}_\gamma := f_{n-1} \circ \cdots \circ f_0
\]

is well-defined for \( z \in T_x \) near \( x \), with \( \text{hol}_\gamma(x) = y \) (see Fig. 3).

**Definition 2.25** The map \( \text{hol}_\gamma \) is called the **holonomy** associated with \( \gamma \).

The following properties of the holonomy can be checked directly from the definition.

**Remark 2.26** \( \text{hol}_\gamma \) is independent of the chosen transversals \( T_p \), \( 1 \leq p \leq n - 1 \), and the laminated/foliated charts \( U_p \). Hence, \( T_x \), \( T_y \) and \( \gamma \) determine the germ of \( \text{hol}_\gamma \) at \( x \).

The germ of \( \text{hol}_\gamma \) at \( x \) depends only on the homotopy class of \( \gamma \) with fixed endpoints. More concretely, if \( \alpha : [0, 1] \rightarrow L \) is another continuous path with \( \alpha(0) = x \) and \( \alpha(1) = y \) which is homotopic to \( \gamma \) in \( L \), then the germ of \( \text{hol}_\alpha \) coincides with that of \( \text{hol}_\gamma \). Otherwise, if \( \alpha \) and \( \gamma \) are not homotopic in \( L \), the germ of \( \text{hol}_\alpha \) is, in general, different from that of \( \text{hol}_\gamma \), see Fig. 4.

If \( \gamma^{-1}(t) := \gamma(1-t) \), then \( \text{hol}_{\gamma^{-1}} = (\text{hol}_\gamma)^{-1} \). Consequently, \( \text{hol}_\gamma \) represents the germ of a local homeomorphism/diffeomorphism/biholomorphism.

Let \( T_x' \) and \( T_y' \) be other transversals to \( L \) at \( x \) and \( y \), respectively. Let \( h : T_x \rightarrow T_x' \) and \( \tilde{h} : T_y \rightarrow T_y' \) be projections along the plaques of \( \mathcal{F} \) in a neighborhood of \( x \) and \( y \), respectively. Then the holonomy \( \text{hol}_\gamma' : T_x' \rightarrow T_y' \) satisfies \( \text{hol}_\gamma' = \tilde{h} \circ \text{hol}_\gamma \circ h^{-1} \).

Consider the special case \( x = y \) and \( T_x = T_y \). We obtain a generalization of the Poincaré first return map. The holonomy map \( \text{hol}_\gamma \) is called the **monodromy map** of \( L \) associated with \( \gamma \). By the above remarks, the germ of \( \text{hol}_\gamma \) at \( x \) depends only on the homotopy class \([\gamma]\). We

Fig. 3 The holonomy \( \text{hol}_\gamma \) associated with a path \( \gamma \) going from \( x \) to \( y \)
see that \([\gamma] \mapsto \text{hol}_\gamma\) is a homomorphism from the first fundamental group \(\pi_1(L, p)\) into the group of germs of homeomorphisms/diffeomorphisms/biholomorphisms of \(T_x\) fixing \(x : [\gamma \circ \alpha] \mapsto \text{hol}_\gamma \circ \text{hol}_\alpha\).

### 2.9 Holomorphic Vector Fields and Singular Holomorphic Foliations

In order to review the local theory of singular holomorphic foliations we start with holomorphic vector fields.

**Definition 2.27** Let \(Z = \sum_{j=1}^{k} F_j(z) \frac{\partial}{\partial z_j}\) be a holomorphic vector field defined in a neighborhood \(U\) of \(0 \in \mathbb{C}^k\). Consider the holomorphic map \(F := (F_1, \ldots, F_k) : U \to \mathbb{C}^k\). We say that \(Z\) is

1. **Singular at 0** if \(F(0) = 0\).
2. **Generic linear** if it can be written as

\[
Z(z) = \sum_{j=1}^{k} \lambda_j z_j \frac{\partial}{\partial z_j},
\]

where \(\lambda_j\) are nonzero complex numbers. The \(k\) hyperplanes \([z_j = 0]\) for \(1 \leq j \leq k\) are said to be the *invariant hypersurfaces*.

3. **With non-degenerate singularity at 0** if \(Z\) is singular at 0 and the eigenvalues \(\lambda_1, \ldots, \lambda_k\) of the Jacobian matrix \(DF(0)\) are all nonzero. We say that the singularity is in the Poincaré domain if the convex hull in \(\mathbb{C}\) of \(\{\lambda_1, \ldots, \lambda_k\}\) does not contain the origin, it is in the Siegel domain otherwise.

4. **With weakly hyperbolic singularity at 0** if \(Z\) is singular at 0 and the eigenvalues \(\lambda_1, \ldots, \lambda_k\) of the Jacobian matrix \(DF(0)\) are all nonzero and there are some \(1 \leq j \neq l \leq k\) with \(\lambda_j/\lambda_l \notin \mathbb{R}\).
(5) With hyperbolic singularity at 0 if \( Z \) is singular at 0 and the eigenvalues \( \lambda_1, \ldots, \lambda_k \) of the Jacobian matrix \( DF(0) \) are all nonzero and \( \lambda_j/\lambda_i \notin \mathbb{R} \) for all \( 1 \leq j \neq i \leq k \).

The integral curves of \( Z \) define a singular holomorphic foliation on \( U \). The condition \( \lambda_j \neq 0 \) implies that the foliation has an isolated singularity at 0.

Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a singular holomorphic foliation such that \( E \) is an analytic subset of \( X \) with codim(\( E \)) \( \geq 2 \). Then \( \mathcal{F} \) is given locally by holomorphic vector fields and its leaves are locally, integral curves of these vector fields, and the set of non-removable singularities of \( \mathcal{F} \) coincide with the union of the zero sets of these vector fields. Here a point \( z \in E \) is called a removable singularity of \( \mathcal{F} \) if there is a singular holomorphic foliation \( \mathcal{F}' = (X, \mathcal{L}', E') \) such that \( E' \subset E \setminus \{z\} \) and that \( \mathcal{L}' = \mathcal{L}|_{X \setminus E} \). A point \( a \in E \) is said to be a non-removable singularity if it is not a removable singularity. So such a foliation \( \mathcal{F} \) is given by an open covering \( \{U_j\} \) of \( X \) and holomorphic vector fields \( v_j \in H^0(\cup U_j, Tan(X)) \) such that

\[
v_j = g_{j\ell} v_k \quad \text{on} \quad U_j \cap U_\ell
\]

for some non-vanishing holomorphic functions \( g_{j\ell} \in H^0(U_j \cap U_\ell, \Omega^*_X) \). Its leaves are locally integral curves, of these vector fields. The set of non-removable singularities of \( \mathcal{F} \) is precisely the union of the zero sets of these local vector fields.

We say that a singular point \( a \in E \) is linearizable (resp. weakly hyperbolic or hyperbolic) if there is a local holomorphic coordinate system of \( X \) near \( a \) on which the leaves of \( \mathcal{F} \) are integral curves of a generic linear vector field (resp. of a holomorphic vector field admitting 0 as a weakly hyperbolic singularity or a hyperbolic singularity). Clearly, \( a \) is an isolated point of \( E \). Moreover, if \( a \) is a hyperbolic singularity then it is clearly a weakly hyperbolic singularity. The converse is true only in dimension \( k = 2 \) (i.e., \( \dim X = 2 \)).

Now we focus on the dimension \( k = 2 \). If \( a \) is a hyperbolic singularity, then there is a local holomorphic coordinates system of \( X \) near \( a \) on which the leaves of \( \mathcal{F} \) are integral curves of a vector field \( Z(z_1, z_2) = \lambda_1 z_1 \frac{\partial}{\partial z_1} + \lambda_2 z_2 \frac{\partial}{\partial z_2} \), where \( \lambda_1, \lambda_2 \) are some nonzero complex numbers with \( \lambda_1/\lambda_2 \notin \mathbb{R} \). In particular, \( a \) is a linearizable singularity. Clearly, in dimension 2 a hyperbolic singularity is always in the Poincaré domain. The analytic curves (invariant hypersurfaces) \( \{z_1 = 0\} \) and \( \{z_2 = 0\} \) are called separatrices at \( a \).

The following result says roughly that for a hyperbolic singularity, the topological type of the foliation around this point is determined by the eigenvalues of its linear part.

**Theorem 2.28** (Chaperon [21]) Let \( Z \) be a germ of a holomorphic vector field in \((\mathbb{C}^k, 0)\). If 0 is a hyperbolic singularity, then \( Z \) is topologically linearizable. This means that there is a homeomorphism \( \Phi : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0) \) sending the foliation defined by \( Z \) to the foliation defined by the vector field \( Z_0 = \sum \lambda_j \frac{\partial}{\partial z_j} \) where the \( \lambda_j \) are the eigenvalues of \( DZ(0) \).

Now we discuss the biholomorphic type of a holomorphic foliation near a singularity. When we write the formal conjugation of a holomorphic vector field near a singularity, to its linear part, we have to divide by the quantities \( \langle \lambda, m \rangle = \sum_{j=1}^k \lambda_j m_j \). Here,

\[
\lambda := (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k, \quad m = (m_1, \ldots, m_k) \in \mathbb{N}^k, \quad |m| = \sum_{j=1}^k m_j \geq 2,
\]

and \( \langle \lambda, m \rangle \) denotes the inner product \( \sum_{j=1}^k \lambda_j m_j \). To prove the convergence we need that these quantities are nonzero and not too close to zero.
The resonances of $\lambda \in \mathbb{C}^k$ are defined by
$$\mathcal{R} := \left\{ (m, j) : m = (m_1, \ldots, m_k) \in \mathbb{N}^k, \quad |m| \geq 2, \langle \lambda, m \rangle - \lambda_j = 0 \right\}.$$ Notice that the set $\{ \langle \lambda, m \rangle - \lambda_j : |m| \geq 2 \}$ has zero as a limit point if and only if $\lambda$ belongs to the Siegel domain.

We are in the position to state some classical results of the local theory of singular holomorphic foliations near an isolated singular point in any dimension.

**Theorem 2.29** (Poincaré [1]) A germ of a singular holomorphic vector field in $(\mathbb{C}^k, 0)$ with a non resonant linear part (i.e., $\mathcal{R}$ is empty) such that $\lambda$ is in the Poincaré domain is holomorphically equivalent to its linear part.

To get linearization for $\lambda$ in the Siegel domain, the following fundamental result assumes the so called Brjuno condition: condition (B) [1, 7]. Define for $n \in \mathbb{N}$
$$\Omega(n) := \inf_{1 \leq j \leq k} \left\{ |\langle \lambda, m \rangle - \lambda_j| : m = (m_1, \ldots, m_k), |m| \leq 2^{n+1} \right\}.$$ Condition (B) is satisfied when $\mathcal{R}$ is empty and
$$\sum_{n \geq 1} \frac{\log (1/\Omega(n))}{2^n} < \infty.$$ In dimension 2, if $\lambda \in \mathbb{R}$ and $\lambda < 0$, then the Brjuno condition becomes $\sum_{n \geq 1} \frac{\log q_{n+1}/q_n}{q_n} < \infty$ where $p_n/q_n$ is the $n$th approximant of $-\lambda$.

**Theorem 2.30** (Brjuno [1, 9]) A germ of a singular holomorphic vector field in $(\mathbb{C}^k, 0)$ with non resonant linear part and which satisfies the Brjuno condition is holomorphically linearizable.

### 2.10 Examples

**Example 2.31** Suspension: The simplest examples of laminations are obtained by the process of suspension. Now we present these examples in our context [14]. Let $S$ be a compact Riemann surface of genus $g \geq 2$. Consider a homomorphism $h : \pi_1(S) \to \text{Aut}(\mathbb{P}^1)$. Let $\phi : \mathbb{D} \to S$ be a universal covering map. Note that $\pi_1(S)$ may be considered as a subgroup of $\text{Aut}(\mathbb{D})$. Thus $h$ induces a natural action $\tilde{h}$ on $\mathbb{D} \times \mathbb{P}^1$. More precisely:
$$\tilde{h} : \pi_1(S) \to \text{Aut}(\mathbb{D} \times \mathbb{P}^1),$$
$$\tilde{h}[\alpha](z, w) := ([\alpha] \cdot z, h[\alpha] \cdot w).$$ Observe that the action of $\tilde{h}$ on $\mathbb{D} \times \mathbb{P}$ is free and properly discontinuous. Therefore, we can consider the manifold $M = M_h := \mathbb{D} \times \mathbb{P}^1/\tilde{h}$ which is the quotient by the above action. The map $\phi : \mathbb{D} \to S$ induces a natural projection $\pi : M \to S$. Moreover, the trivial foliation on $\mathbb{D} \times \mathbb{P}^1$, with leaves $\mathbb{D} \times \{w\}$ induces a foliation $\mathcal{F}_h$ on $M$, whose leaves are coverings of $S$, hence hyperbolic Riemann surfaces. So to any representation $h$ of $\pi_1(S)$ into $\text{PSL}(2, \mathbb{C})$ corresponds a foliation. E. Ghys [56] shows that $M_h$ is a projective surface.

**Example 2.32** Levi-flats (or equivalently Cauchy-Riemann foliations): Let $M$ be a complex surface, $\text{Tan}(M)$ be its tangent bundle and $J$ be the endomorphism of $\text{Tan}(M)$ given by the complex structure (satisfying $J^2 = -\text{id}$). Given a real hypersurface $X$ in $M$, we define the Cauchy-Riemann distribution on $X$ as follows. To each point $x \in X$ we associate the
unique complex line contained in $\text{Tan}_x(X)$, i.e., the distribution $\text{Tan}(X) \cap J\text{Tan}(X)$. The hypersurface $X$ is called Levi-flat if the Cauchy-Riemann distribution is integrable in the sense of Frobenius. This means that through any point of $X$ passes a nonsingular holomorphic curve of $M$ that is completely contained in $X$. These curves correspond then to the leaves of a foliation on $X$, called the Cauchy-Riemann foliation or CR foliation. The condition that ensures a real hypersurface to be Levi-flat can be characterized by the vanishing of the Levi form of $X$.

Example 2.33 **Singular holomorphic foliations on $\mathbb{P}^k$ with $k \geq 2$**: Let $\pi: \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ denote the canonical projection. Let $\mathcal{F}$ be a singular holomorphic foliations on $\mathbb{P}^k$. It can be shown that $\pi^*\mathcal{F}$ is a singular foliation on $\mathbb{C}^{k+1}$ associated to a vector field $Z$ of the form

$$Z := \sum_{j=0}^{k} F_j(z) \frac{\partial}{\partial z_j},$$

where the $F_j$ are homogeneous polynomials of degree $d \geq 1$. We call $d$ the degree of the foliation. Here $\pi^*\mathcal{F}$ should be understood as a foliation such that each leaf is contained in the preimage of a leaf of $\mathcal{F}$ by $\pi$, and that the foliation is invariant under all dilatation maps $A_\lambda : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{C}^{k+1} \setminus \{0\}$ defined by $z \mapsto \lambda z$, with $\lambda \in \mathbb{C} \setminus \{0\}$.

In dimension $k = 2$, the number of tangencies of a generic line with a foliation is exactly its degree.

For $d \geq 2$, let $\mathcal{F}_d(\mathbb{P}^k)$ be the space of singular holomorphic foliations of degree $d$ in $\mathbb{P}^k$. Using the above form of $Z$, we can show that $\mathcal{F}_d(\mathbb{P}^k)$ can be canonically identified with a Zariski open subset of $\mathbb{P}^N$, where $N := (d + k + 1) \frac{(d+k-1)!}{(k-1)!} - 1$ (see [11]).

A point $x \in \mathbb{P}^k$ is a singularity of $\mathcal{F}$ if $F(x)$ is colinear with $x$, i.e., if $x$ is either an indeterminacy point or a fixed point of $f = [F_0 : \ldots : F_k]$ as a meromorphic map in $\mathbb{P}^k$. If $f$ is non holomorphic, then its indeterminacy set is analytic of codimension $\geq 2$, it can be of positive dimension when $k \geq 3$. Assume that $f$ is holomorphic. To count the fixed points we only need to apply the Bézout theorem to the equations $F_j(z) - t^{d-1}z_j = 0$ in $\mathbb{P}^{k+1}$ with homogeneous coordinates $[z : t]$ and observe that $[0 : \ldots : 0 : 1]$ is a solution. The number of fixed points counted with multiplicity is $d^{k+1} - 1$. So the singularity set, $\text{Sing}(\mathcal{F})$, of any holomorphic foliation $\mathcal{F} \in \mathcal{F}_d(\mathbb{P}^k)$ is always non-empty.

The next result describes some typical properties of a generic foliation $\mathcal{F} \in \mathcal{F}_d(\mathbb{P}^k)$.

**Theorem 2.34** Let $d, k > 1$.

1. (Jouanolou [63], Lins Neto-Soares [74]) There is a real Zariski dense open set $\mathcal{H}(d) \subset \mathcal{F}_d(\mathbb{P}^k)$ such that for every $\mathcal{F} \in \mathcal{H}(d)$, all the singularities of $\mathcal{F}$ are hyperbolic and $\mathcal{F}$ does not possess any invariant algebraic curve.
2. (Glutsyuk [57], Lins Neto [73]) If all the singularities of a foliation $\mathcal{F} \in \mathcal{F}_d(\mathbb{P}^k)$ are non-degenerate, then $\mathcal{F}$ is hyperbolic.
3. (Brunella [11]) If all the singularities of a foliation $\mathcal{F} \in \mathcal{F}_d(\mathbb{P}^k)$ are hyperbolic and $\mathcal{F}$ does not possess any invariant algebraic curve, then $\mathcal{F}$ admits no nontrivial directed positive closed current.

Moreover, Loray-Rebelo [70] constructed a non-empty open set $\mathcal{U}(d)$ of $\mathcal{F}_d(\mathbb{P}^k)$ such that every leaf of $\mathcal{F} \in \mathcal{U}(d)$ is dense. By Theorem 2.34, Theorem 2.23 applies to every generic foliation in $\mathbb{P}^k$ with a given degree $d > 1$. 
2.11 Sullivan’s Dictionary

Here is part of the correspondence between the world of maps and that of laminations/foliations.

| Notion                  | Maps                                      | Laminations/Foliations                        |
|------------------------|------------------------------------------|----------------------------------------------|
| Orbit of a point \(x \in X \setminus E\) | the sequence \(x, f(x), f^2(x), \ldots, f^n(x), \ldots\) (only one orbit) | Leaf \(L_x\) \n Leafwise continuous paths \(\Omega_x\) \n (infinitely many paths \(\omega \in \Omega_x\)) |
| Orbit of a point \(x \in X \setminus E\) | Dirac mass at \(x\)                      | Wiener measure \(W_x\) on \(\Omega_x\) at \(x \in \text{Hyp}(\mathcal{F})\) |
| Counting time          | linear time \(t \in \mathbb{N}\)        | hyperbolic \(t \in \mathbb{D}\)              |
| Nonsingular objects    | continuous map                           | Riemann surface lamination                   |
|                        | smooth map                               | smooth lamination                            |
| Singular holomorphic   | meromorphic map                          | singular holomorphic foliation               |
| objects                | indeterminacy point/set                 | singular point/set                           |
| Invariant dynamical    | invariant measures                       | directed posi. closed currents               |
| objects                | ...                                       | directed posi. harmonic currents             |
|                        | ...                                       | ...                                          |

By this dictionary, the notion of *orbit of a point* \(x\) has only one entry in the world of maps (that is, the unique sequence of points \(x, f(x), f^2(x), \ldots, f^n(x), \ldots\)). However, this notion has two entries in the world of laminations/foliations. The first one is the whole leaf \(L_x\) which is a Riemann surface. The second entry is the space \(\Omega_x\) of leafwise continuous paths \(\omega : \mathbb{R}^+ \to L_x\) starting from \(x\), i.e., \(\omega(0) = x\).

Similarly, the entry *invariant positive measures* in the world of maps possesses two entries in the world of laminations/foliations. The first one is the concept of directed positive closed currents, the second entry is the notion of directed positive harmonic currents. Although the first entry is very close to the original notion of invariant measures, it turns out that the second entry is more appropriate because laminations/foliations which have invariant measures are rather scarce. Moreover, directed positive harmonic currents provide a natural framework for a good theory to be developed.

3 Random and Operator Ergodic Theorems

In this section we follow the expositions given in [75, Sections 2.2, 2.4 and 2.5] and in [78, Subsection 2.4]. We are partly inspired by the constructions given in [17, 19]. The \(\sigma\)-algebra generated by a family \(\mathcal{F}\) of subsets of \(\Omega\) is, by definition, the smallest \(\sigma\)-algebra containing \(\mathcal{F}\).

3.1 Wiener Measures

Let \(\mathcal{F} = (X, \mathcal{L}, E)\) be a Riemann surface lamination with singularities endowed with the leafwise Poincaré metric \(g_P\). Recall from Section 2.7 that \(\Omega := \Omega(\mathcal{F})\) is the sample-path space associated to \(\mathcal{F}\) and that for each \(x \in X \setminus E\), \(\Omega_x = \Omega_x(\mathcal{F})\) denotes the space
of all continuous leafwise paths starting at \( x \) in \( X \setminus E \). Garnett developed in [54] a theory of leafwise Brownian motion in this context by constructing a \( \sigma \)-algebra \( (\Omega, \mathcal{A}) \) together with a family of Wiener measures (see also [17, 19]). Now recall briefly her construction.

A cylinder set (in \( \Omega \)) is a set of the form

\[
C = C([t_i, B_i] : 1 \leq i \leq m) := \{ \omega \in \Omega : \omega(t_i) \in B_i, \quad 1 \leq i \leq m \},
\]

where \( m \) is a positive integer and the \( B_i \) are Borel subsets of \( X \setminus E \), and \( 0 \leq t_1 < t_2 < \cdots < t_m \) is a set of increasing times. In other words, \( C \) consists of all paths \( \omega \in \Omega \) which can be found within \( B_i \) at time \( t_i \). For each point \( x \in \text{Hyp}(\mathcal{F}) \), let

\[
W_x(C) := (D_{t_1}(1_{B_1} D_{t_2 - t_1}(1_{B_2} \cdots 1_{B_{m-1}} D_{t_m - t_{m-1}}(1_{B_m}) \cdots))) (x),
\]

where \( C := C([t_i, B_i] : 1 \leq i \leq m) \) as above, \( 1_{B_i} \) is the characteristic function of \( B_i \) and \( D_t \) is the leafwise diffusion operator given by (2.3). Let \( \mathcal{A} = \mathcal{A}(\mathcal{F}) \) be the \( \sigma \)-algebra generated by all cylinder sets. It can be proved that \( W_x \) extends to a probability measure on \( (\Omega, \mathcal{A}) \).

In the monograph [75] we introduce another \( \sigma \)-algebra \( \mathcal{A} \) on \( \Omega \), which is bigger (finer) than \( \mathcal{A} \). In fact, \( \mathcal{A} \) takes into account the holonomy phenomenon, whereas \( \mathcal{A} \) does not so. Here is our construction in the present context. The covering lamination \( \mathcal{F} = (X, \mathcal{L}) \) of a Riemann surface lamination with singularities \( \mathcal{F} \) is, in some sense, its universal cover. More specifically, for every leaf \( L \) of \( \mathcal{F} \) (hyperbolic or parabolic) and every point \( x \in L \), let \( \pi_1(L, x) \) denotes the first fundamental group of all continuous closed paths \( \gamma : [0, 1] \to L \) based at \( x \), i.e., \( \gamma(0) = \gamma(1) = x \). Let \( [\gamma] \in \pi_1(L, x) \) be the class of a closed path \( \gamma \) based at \( x \). Then the pair \( (x, [\gamma]) \) represents a point of \( \tilde{X} \). Thus the set of points \( \tilde{X} \) of \( \mathcal{F} \) is well-defined. The leaf \( \tilde{L} \) passing through a given point \( (x, [\gamma]) \in \tilde{X} \), is by definition, the set

\[
\tilde{L} := \{(y, [\delta]) : y \in L_x, \ [\delta] \in \pi_1(L, y)\},
\]

which is the universal cover of \( L_x \). We put the following topological structure on \( \tilde{X} \) by describing a basis of open sets. Such a basis consists of all sets \( \mathcal{N}(U, \alpha) \). Here, \( U \) is an open subset of \( X \setminus E \) and \( \alpha : U \times [0, 1] \to X \setminus E \) is a continuous function such that

\[
\alpha_x := \alpha(x, \cdot) \text{ is a closed path in } L_x \text{ based at } x \text{ for each } x \in U,
\]

and

\[
\mathcal{N}(U, \alpha) := \{(x, [\alpha_x]) : x \in U\}.
\]

The projection \( \pi : \tilde{X} \to X \setminus E \) is defined by \( \pi(x, [\gamma]) := x \). It is clear that \( \pi \) is locally homeomorphic and is a leafwise map. By pulling-back the lamination atlas \( \mathcal{L} \) of \( \mathcal{F} \) via \( \pi \), we obtain a natural lamination atlas \( \tilde{\mathcal{L}} \) for the Riemann surface lamination \( \tilde{\mathcal{F}} \). Denote by \( \tilde{\Omega} \) the sample-path space \( \Omega(\tilde{\mathcal{F}}) \) associated with the Riemann surface lamination \( \tilde{\mathcal{F}} \). Similarly, by pulling-back the leafwise Poincaré metric \( g_p \) defined on \( \text{Hyp}(\mathcal{F}) \) via \( \pi \), we obtain a natural leafwise metric \( \pi^*g_p \) defined on the hyperbolic part \( \text{Hyp}(\mathcal{F}) \) of \( \tilde{\mathcal{F}} \).

Let \( x \in X \setminus E \) and \( \tilde{x} \) an arbitrary point in \( \pi^{-1}(x) \subset \tilde{X} \). Similarly as in (2.15), let \( \tilde{\Omega}_{\tilde{x}} = \Omega_{\tilde{x}}(\tilde{\mathcal{F}}) \) be the space of all paths in \( \tilde{\Omega} \) starting at \( \tilde{x} \). Every path \( \omega \in \Omega_{\tilde{x}} \) lifts uniquely to a path \( \tilde{\omega} \in \tilde{\Omega}_{\tilde{x}} \) in the sense that \( \pi \circ \tilde{\omega} = \omega \). In what follows this bijective lifting is denoted by \( \pi^{-1}_x : \Omega_x \to \tilde{\Omega}_{\tilde{x}} \). So \( \pi \circ (\pi^{-1}_x(\omega)) = \omega, \omega \in \Omega_x \).

**Definition 3.1** Let \( \mathcal{A} = \mathcal{A}(\mathcal{F}) \) be the \( \sigma \)-algebra generated by all sets of the following family

\[
\{ \pi \circ \tilde{A} : \text{cylinder set } \tilde{A} \text{ in } \tilde{\Omega} \},
\]

where \( \pi \circ \tilde{A} := \{ \pi \circ \tilde{\omega} : \tilde{\omega} \in \tilde{A} \} \).
Observe that \( \tilde{A} \subset A \) and that the equality holds if every leaf of the lamination is homeomorphic to the disc \( \mathbb{D} \).

Now we construct a family \( \{W_x\}_{x \in \text{Hyp}(\mathcal{F})} \) of probability Wiener measures on \((\Omega, \mathcal{A})\). Let \( x \in \text{Hyp}(\mathcal{F}) \) and \( C \) an element of \( \mathcal{A} \). Then we define the so-called Wiener measure \( W_x \) by the following formula

\[
W_x(C) := W_{\tilde{x}}(\pi^{-1}_{\tilde{x}} C), \tag{3.2}
\]

where \( \tilde{x} \) is an arbitrary point in \( \pi^{-1}(x) \), and

\[
\pi^{-1}_{\tilde{x}} C := \left\{ \pi^{-1}_{\tilde{x}} \omega : \omega \in C \cap \Omega_{\tilde{x}} \right\},
\]

and \( W_{\tilde{x}} \) is the probability measure on \((\tilde{\Omega}, \tilde{\mathcal{A}}(\tilde{\mathcal{F}}))\) which was defined by (3.1). Given a positive finite Borel measure \( \mu \) on \( X \) which gives no mass to \( \text{Par}(\mathcal{F}) \cup E \), consider the measure \( \tilde{\mu} \) on \((\Omega, \mathcal{A})\) defined by

\[
\tilde{\mu}(A) := \int_X \left( \int_{\omega \in A \cap \Omega_{\tilde{x}}} dW_{\tilde{x}} \right) d\mu(x), \quad A \in \mathcal{A}. \tag{3.3}
\]

The measure \( \tilde{\mu} \) is called the Wiener measure with initial distribution \( \mu \).

Here are some important properties of \( \tilde{\mu} \).

**Proposition 3.2** We keep the above hypotheses and notation.

(i) The value of \( W_x(C) \) defined in (3.2) is independent of the choice of \( \tilde{x} \). Moreover, \( W_x \) is a probability measure on \((\Omega, \mathcal{A})\).

(ii) \( \tilde{\mu} \) given in (3.3) is a positive finite measure on \((\Omega, \mathcal{A})\) and \( \tilde{\mu}(\Omega) = \mu(X) \).

**Proof** Assertion (i) has been proved in [75, Theorem 2.15]. Assertion (ii) has been established in [75, Theorem 2.16]. \( \square \)

### 3.2 Random and Operator Ergodic Theorems

Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a Riemann surface lamination with singularities. Recall from (2.16) the shift-transformations \( \sigma_t, t \in \mathbb{R}^+ \).

**Theorem 3.3** (i) If \( \mu \) is a very weakly harmonic measure (resp. weakly harmonic measure), then \( \tilde{\mu} \) is unit time-invariant (resp. time-invariant), that is,

\[
\int_{\Omega} F(\sigma_t(\omega)) d\tilde{\mu}(\omega) = \int_{\Omega} F(\omega) d\tilde{\mu}(\omega)
\]

for \( t = 1 \) (resp. for all \( t \in \mathbb{R}^+ \)) and \( F \in L^1(\Omega, \tilde{\mu}) \).

(ii) If \( \mu \) is a very weakly harmonic measure, then \( \mu \) is ergodic if and only if \( \tilde{\mu} \) is ergodic for \( \sigma_1 \). If moreover \( \mu \) is weakly harmonic and is ergodic, then \( \tilde{\mu} \) is ergodic for all \( \sigma_t \) with \( t \in \mathbb{R}^+ \setminus \{0\} \).

**Proof** Part (i) follows from [75, Theorem 2.20] where the general case of an \( N \)-real or complex dimensional lamination with a general leafwise metric was treated.

Part (ii) is a consequence of [75, Theorem 4.6]. \( \square \)

The following Operator Ergodic Theorem may be regarded as Akcoglu’s ergodic theorem (see [68, Theorem 2.6, p. 190]) in the context of laminations.
Theorem 3.4 Let \( \mu \) be a very weakly harmonic measure which is ergodic. Then the following properties hold.

(i) If \( D_1 f = f \) \( \mu \)-almost everywhere for some \( f \in L^1(X, \mu) \), then \( f = \text{const} \) \( \mu \)-almost everywhere.

(ii) For every \( f \in L^1(X, \mu) \), \( \frac{1}{n} \sum_{j=0}^{n-1} D_j f \) converges to \( \int_X f d\mu \) \( \mu \)-almost everywhere.

(iii) If \( \mu \) is weakly harmonic, then the following two properties hold for all \( t_0 > 0 \).

(i) If \( D_{t_0} f = f \) \( \mu \)-almost everywhere for some \( f \in L^1(X, \mu) \), then \( f = \text{const} \) \( \mu \)-almost everywhere.

(ii) For every \( f \in L^1(X, \mu) \), \( \frac{1}{n} \sum_{j=0}^{n-1} D_{t_j} f \) converges to \( \int_X f d\mu \) \( \mu \)-almost everywhere.

Proof It follows from [75, Theorem B.16] where the general case of an \( N \)-real or complex dimensional lamination with a general leafwise metric was investigated.

Problem 3.5 It seems of interest to find sufficient conditions to ensure that a very weakly harmonic measure (resp. a weakly harmonic measure) is harmonic.

4 Regularity of the Leafwise Poincaré Metric and Mass-distribution of Currents

4.1 Regularity of the Leafwise Poincaré Metric

Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a Riemann surface lamination with singularities. Let \( g_P \) be as usual the leafwise Poincaré metric for the lamination given in Section 2.2. Let \( g_X \) be a Hermitian metric on the leaves which is transversally continuous. We can construct such a metric on flow boxes and glue them using a partition of unity. When \( \mathcal{F} \) is holomorphically immersed in a complex manifold \( M \), we often fix an ambient Hermitian metric \( g_M \) on \( M \) and consider its restriction to the leaves. We denote by \( g_X \) the Hermitian metric on the leaves obtained in this way. Consider the function \( \eta : X \setminus E \to [0, \infty] \) given by

\[
\eta(x) = \sup \{ \| (D\phi)(0) \|, \phi : \mathbb{D} \to L_x \text{ holomorphic such that } \phi(0) = x \}. \quad (4.1)
\]

Here, and for the norm of the differential \( D\phi \) we use the Poincaré metric on \( \mathbb{D} \) and the Hermitian metric \( g_X \) restricted to \( L_x \). Using a map sending \( \mathbb{D} \) to a plaque, we see that the function \( \eta \) is locally bounded from below on \( X \setminus E \) by a strictly positive constant. Moreover, when \( \mathcal{F} \) is holomorphically immersed in a complex manifold, we can show that \( \eta \) is lower-semi continuous on \( X \setminus E \) (see [50, Theorem 20]). Note that \( \{ x \in X \setminus E : \eta(x) = \infty \} = \text{Par}(\mathcal{F}) \). When \( X \) is compact and \( \text{Par}(\mathcal{F}) = E = \emptyset \), the classical Brody lemma (see [67, p. 100]) implies that \( \eta \) is also bounded from above.

The extremal property of the Poincaré metric implies that, for \( x \in \text{Hyp}(\mathcal{F}) \),

\[
g_X = \eta^2 g_P, \quad \text{where } \eta(x) := \| D\phi_x(0) \| \quad (\text{see } (2.1)). \quad (4.2)
\]

Given a point \( x \in \text{Hyp}(\mathcal{F}) \) and a differentiable function \( f : L_x \to \mathbb{C} \), we define \( |df|_P : L_x \to \mathbb{R}^+ \) by

\[
|df|_P(y) := \eta(y)|df(y)| \quad \text{for } y \in L_x, \quad (4.3)
\]

where for the norm \( |df(y)| \) we use the Hermitian metric \( g_X \) on \( L_x \) and the Euclidean norm \( |\cdot| \) of \( \mathbb{C} \).

The continuity of the function \( \eta \) was studied by Candel, Ghys, Verjovsky (see [16, 56, 95]). The survey [50] establishes this result as a consequence of Royden’s lemma. Indeed
with his lemma, Royden proved the upper-semicontinuity of the infinitesimal Kobayashi metric in a Kobayashi hyperbolic manifold (see [67, p. 91 and p. 153]). The following theorem gives refinements of the previous results.

**Theorem 4.1** (Dinh-Nguyen-Sibony [33]) Let \( \mathcal{F} = (X, \mathcal{L}) \) be a transversally smooth compact lamination by hyperbolic Riemann surfaces. Then the Poincaré metric on the leaves is Hölder continuous, that is, the function \( \eta \) defined in (4.2) is Hölder continuous on \( X \). Moreover, the exponent of Hölder continuity can be estimated in geometric terms.

The main tool of the proof of Theorem 4.1 is to use Beltrami’s equation in order to compare universal covering maps of any leaf \( L_y \) near a given leaf \( L_x \). More precisely, for \( R > 0 \) let \( \mathbb{D}_R \) be the disc of center 0 with radius \( R \) with respect to the Poincaré metric on \( \mathbb{D} \) (see Main notation). We first construct a non-holomorphic parametrization \( \psi \) from \( \mathbb{D}_R \) to \( L_y \) which is close to a universal covering map \( \phi_x : \mathbb{D} \to L_x \) for each \( R \) large enough. Next, precise geometric estimates on \( \psi \) allow us to modify it, using Beltrami’s equation. We then obtain a holomorphic map that we can explicitly compare with a universal covering map \( \phi_y : \mathbb{D} \to L_y \).

Next, we investigate the regularity of the leafwise Poincaré metric \( g_\rho \) of a compact singular holomorphic foliation. Here an important difficulty emerges: a leaf of the foliation may visit singular flow boxes without any obvious rule. We introduce the following class of laminations.

**Definition 4.2** (Dinh-Nguyen-Sibony [34]) A hyperbolic Riemann surface lamination with singularities \( \mathcal{F} = (X, \mathcal{L}, E) \) with \( X \) compact is said to be *Brody hyperbolic* if there is a constant \( c_0 > 0 \) such that

$$
\| D\phi(0) \| \leq c_0
$$

for all holomorphic maps \( \phi \) from \( \mathbb{D} \) into a leaf, in other words, if the function \( \eta \) is uniformly bounded from above.

**Remark 4.3** It is clear that if the lamination is Brody hyperbolic then its leaves are hyperbolic in the sense of Kobayashi. Conversely, the Brody hyperbolicity is a consequence of the non-existence of holomorphic non-constant maps \( \mathbb{C} \to X \) such that out of \( E \) the image of \( \mathbb{C} \) is locally contained in leaves (see [50, Theorem 15]).

On the other hand, Lins Neto proved in [73] that for every holomorphic foliation of degree larger than 1 in \( \mathbb{P}^k \), with non-degenerate singularities (see Definition 2.27), there is a smooth metric with negative curvature on its tangent bundle (see also Glutsyuk [57]). Hence, these foliations are Brody hyperbolic. Consequently, holomorphic foliations in \( \mathbb{P}^k \) are generically Brody hyperbolic (see Theorem 2.34 (1)). The reader may find in [89] a nice discussion on the topology and the conformal structures of leaves of a singular holomorphic foliation which is Brody hyperbolic.

Denote by \( \log^* (\cdot) := 1 + |\log (\cdot)| \) a log-type function, and by \( \text{dist} \) the distance on \( X \) induced by the Hermitian metric \( g_X \). The following result is a counterpart of Theorem 4.1 in the context of singular holomorphic foliations.

**Theorem 4.4** (Dinh-Nguyen-Sibony [34]) Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a Brody hyperbolic singular holomorphic foliation on a Hermitian compact complex manifold \( X \). Assume that
the singular set $E$ is finite and that all points of $E$ are linearizable. Then, there are constants $c > 0$ and $0 < \alpha < 1$ such that

$$|\eta(x) - \eta(y)| \leq c \left( \frac{\max\{\log^*\text{dist}(x, E), \log^*\text{dist}(y, E)\}}{\log^*\text{dist}(x, y)} \right)^\alpha$$

for all $x, y$ in $X \setminus E$.

To prove this theorem, we analyze the behavior and get an explicit estimate on the modulus of continuity of the Poincaré metric on leaves. The following estimates are crucial in our method. They are also useful in other problems.

**Proposition 4.5** (Dinh-Nguyen-Sibony [34]) Under the hypotheses of Theorem 4.4, there exists a constant $c_1 > 1$ such that

$$c_1^{-1}s \log^*s \leq \eta(x) \leq c_1s \log^*s$$

for $x \in X \setminus E$ and $s := \text{dist}(x, E)$.

The Poincaré metric on the leaves of a hyperbolic foliation is a fundamental object which is extremely delicate to understand. As we see in Theorem 4.4, the regularity in the direction transverse to the foliation is quite weak. This is partly due to the presence of the singularities. We end the subsection with the following open question.

**Problem 4.6** Let $(X, \mathcal{L}, E)$ be a compact singular holomorphic foliation. Assume that every point $a \in E$ is a non-degenerate singularity. Study the regularity of the function $\eta$. In case $\dim(X) = 2$, we may investigate the problem where the singularities are not necessarily non-degenerate.

### 4.2 Mass-distribution of Undirected and Directed Positive $dd^c$-closed Currents

Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a Riemann surface lamination with singularities which is holomorphically immersed in a complex Hermitian manifold $(M, g_M)$. Let $T$ be a positive $dd^c$-closed current of bidimension $(1, 1)$ on $M$ whose support is in $X$. Here, $T$ may or may not be directed by $\mathcal{F}$. If $T$ is not necessarily directed by $\mathcal{F}$, we say that it is undirected. Clearly, the mass of $T$ with respect to the Hermitian metric $g_M$ (i.e., the mass of the positive measure $T \wedge g_M$) is locally finite on $X \setminus E$. If moreover, $T$ can be extended trivially through $E$ to a undirected (resp. directed) positive $dd^c$-closed current (see Definition 2.20), then its mass is locally finite on $X$. Consider the following concept (see Definition 2.8 for a similar notion in the context of directed positive harmonic currents).

**Definition 4.7** Consider the map $T \mapsto \Phi(T) := \mu$ which is defined by the following formula on the convex cone of all positive $dd^c$-closed currents $T$ of bidimension $(1, 1)$ on $M$ whose support is in $X$:

$$\mu := T \wedge g_P \quad \text{on} \quad X \setminus (E \cup \text{Par}(\mathcal{F})) \quad \text{and} \quad \mu(E \cup \text{Par}(\mathcal{F})) = 0.$$ 

We call Poincaré mass of $T$ the mass of $T$ with respect to Poincaré metric $g_P$ on $X \setminus E$, i.e., the mass of the positive measure $\mu = \Phi(T)$.

For many ergodic problems, the local finiteness mass of the Poincaré mass of $T$ is very important. The following proposition gives us a criterion for this local finiteness. It can be applied to generic foliations in $\mathbb{R}^k$ (see Theorem 2.34).
Proposition 4.8 (Dinh-Nguyen-Sibony [32]) Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a singular holomorphic foliation. If $a \in E$ is a linearizable singularity, then any positive $dd^c$-closed current on $X$ has locally finite Poincaré mass near $a$.

The proof of this result is based on the finiteness of the Lelong number of $T$ at $a$ (see Proposition 2.15). We have a more precise result when $T$ is directed and the singular point is weakly hyperbolic (see Definition 2.27).

Theorem 4.9 (Nguyen [77, 81]) Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a singular holomorphic foliation with $\dim X = k \geq 2$. If $a \in E$ is a linearizable singularity which is also a weakly hyperbolic singularity, then for any directed positive $dd^c$-closed current $T$ on $X$ which does not give mass to any of the $k$ invariant hypersurfaces at $a$, the Lelong number of $T$ at $a$ vanishes.

An immediate consequence of Theorem 4.9 is the following result on the Lelong numbers of a directed positive $dd^c$-closed current.

Corollary 4.10 (Nguyen [77, 81]) Let $\mathcal{F} = (X, \mathcal{F}, E)$ be a singular holomorphic foliation with $X$ a compact complex manifold. Assume that all the singularities are not only linearizable but also hyperbolic and that the foliation has no invariant analytic curve and that the foliation is Brody hyperbolic. Then for every positive $dd^c$-closed current $T$ directed by $\mathcal{F}$, the Lelong number of $T$ vanishes everywhere in $X$.

The above corollary can be applied to every generic foliation in $\mathbb{P}^k$ with a given degree $d > 1$ (see Theorem 2.34).

Remark 4.11 Theorem 4.9 answers positively Problem 4.7 raised in [79]. In fact, the special case $\dim X = 2$ of this theorem was proved in [77].

When the singularities are linearizable but not weakly hyperbolic, the study of Lelong numbers seems difficult (see Chen’s recent article [23] for a partial result).

We end the section with the following open question.

Problem 4.12 Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a compact singular holomorphic foliation and let $T$ be a directed positive $dd^c$-closed current for $\mathcal{F}$. Find sufficient conditions on the nature of the set of singularities $E$ to ensure that the Poincaré mass of $T$ is finite.

5 Heat Equation and Ergodic Theorems

Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a Riemann surface lamination with singularities. In collaboration with Dinh and Sibony [32], we introduce the heat equation relative to

- A harmonic measure $\mu$ of $\mathcal{F}$;
- A positive $dd^c$-closed current $T$ on a complex manifold $M$ in the case where $\mathcal{F}$ is holomorphically immersed in $M$, the current $T$ is not necessarily directed, but its support is assumed to be in $X$ and its Poincaré mass is assumed to be finite.

This permits us to construct the abstract heat diffusion with respect to various Laplacians that could be defined almost everywhere with respect to the harmonic measure/positive $dd^c$-closed current. In this section, we follow closely the exposition of [32]. Note however
that there are two differences. The first one is that in the present article we only consider laminations by Riemann surfaces and their leafwise Poincaré metric, whereas in [32] the case of $N$-dimensional laminations endowed with a general leafwise Riemannian metrics was studied. So this difference limits the scope of the present article. The second difference is that the Riemann surface laminations considered in this article may be not compact and their singularities may be neither isolated nor finite. This new situation leads us to introduce some spaces of test functions slightly more general than those given in [32].

Recall some classical results of functional analysis. The reader will find an exposition in Brezis [6]. A linear operator $A$ on a Hilbert space $L$ is called monotone if $\langle Au, u \rangle \geq 0$ for all $u$ in the domain $\text{Dom}(A)$ of $A$. Such an operator is maximal monotone if moreover for any $f \in L$ there is a $u \in \text{Dom}(A)$ such that $u + Au = f$. In this case, the domain of $A$ is always dense in $L$ and the graph of $A$ is closed.

A family $S(t) : L \to L$, $t \in \mathbb{R}_+$, is a semi-group of contractions if $S(t + t') = S(t) \circ S(t')$ and if $\|S(t)\| \leq 1$ for all $t, t' \geq 0$. We will apply the following theorem to our Laplacian operators. It says that any maximal monotone operator is the infinitesimal generator of a semi-group of contractions.

**Theorem 5.1** (Hille-Yosida) Let $A$ be a maximal monotone operator on a Hilbert space $L$. Then there is a semi-group of contractions $S(t) : L \to L$, $t \in \mathbb{R}_+$, such that for $u_0 \in \text{Dom}(A)$, $u(t, \cdot) = S(t)u_0$ is the unique function

$$u \in \mathcal{C}^1(\mathbb{R}_+, L) \cap \mathcal{C}(\mathbb{R}_+, \text{Dom}(A))$$

which satisfies

$$\frac{\partial u(t, \cdot)}{\partial t} + Au(t, \cdot) = 0 \quad \text{and} \quad u(0, \cdot) = u_0.$$ 

When $A$ is self-adjoint and $u_0 \in L$, then

$$u \in \mathcal{C}(\mathbb{R}_+, L) \cap \mathcal{C}^1(\mathbb{R}_+, L) \cap \mathcal{C}(\mathbb{R}_+, \text{Dom}(A)),$$

where $\mathbb{R}_+ := \mathbb{R}^+ \setminus \{0\} = (0, \infty)$, and we have the estimate

$$\left\| \frac{\partial u}{\partial t} \right\| \leq \frac{1}{t} \|u_0\| \quad \text{for} \ t > 0.$$

In order to check that our operators are maximal monotone, we will apply the following result.

**Theorem 5.2** (Lax-Milgram) Let $e(u, v)$ be a continuous bilinear form on a Hilbert space $H$. Assume that $e(u, u) \geq \|u\|_H^2$ for $u \in H$. Then for every $f$ in the dual $H^*$ of $H$ there is a unique $u \in H$ such that $e(u, v) = \langle f, v \rangle$ for $v \in H$.

### 5.1 Heat Equation on Riemann Surface Laminations with Singularities

Consider a Riemann surface lamination with singularities $\mathcal{F} = (X, \mathcal{L}, E)$ and a positive quasi-harmonic measure $\mu$.

In a flow box $U \simeq \mathbb{B} \times \mathbb{T}$, by Proposition 2.5, the current $T$ can be written as

$$T = \int h_a[\mathbb{B} \times \{a\}]dv(a), \quad (5.1)$$

where $h_a$ is a positive harmonic function on $\mathbb{B}$ and $v$ is a positive measure on the transversal $\mathbb{T}$.
By Theorem 2.9 (1), there is a unique directed positive harmonic current $T$ giving no mass to $\text{Par}(\mathcal{F})$ such that
\[
\mu = T \wedge g_P \quad \text{on} \quad \text{Hyp}(\mathcal{F}) \quad \text{and} \quad \mu = 0 \quad \text{on} \quad \text{Par}(\mathcal{F}) \cup E. \quad (5.2)
\]
It follows from (2.5) and (2.7) that the following identity holds for $u \in \mathcal{D}(\mathcal{F})$,
\[
\int_X (\Delta_P u) d\mu = \int_X i \hat{\partial} u \wedge T.
\]
In what follows, the differential operators $\nabla$, $\Delta_P$ and $\tilde{\Delta}_P$ are considered in $L := L^2(\mu)$.

We introduce the Hilbert space $H := H^1(\mu)$ as the completion of $\mathcal{D}(\mathcal{F})$ with respect to the norm
\[
\|u\|_H^2 := \int |u|^2 d\mu + \int |\nabla u|^2 d\mu. \quad (5.3)
\]
Recall that the gradient $\nabla$ is defined by
\[
\langle \nabla u, \xi \rangle_{g_P} = du(\xi) \quad (5.4)
\]
for all tangent vector $\xi$ along a leaf and for $u \in \mathcal{D}(\mathcal{F})$. In comparison with (4.3), we see that
\[
|\nabla u| = |d u{|_P}. \quad (5.5)
\]
We consider $\nabla$ as an operator in $L^2(\mu)$ and $H^1(\mu)$ is its domain of definition.

Define in a flow box $U \simeq B \times T$ as above the Laplace type operator
\[
\tilde{\Delta}_P u = \Delta_P u + \langle h^{-1}_a \nabla h_a, \nabla u \rangle_{g_P} = \Delta_P u + Fu,
\]
where $F$ is a vector field. The uniqueness of $h_a$ and $\mu$ implies that $F$ does not depend on the choice of the flow box. Therefore, $Fu$ and $\tilde{\Delta}_P u$ are defined globally $\mu$-almost everywhere when $u \in \mathcal{D}(\mathcal{F})$.

Remark 5.3 We say that a quasi-harmonic measure $\mu$ is invariant if for every flow box $U$ the functions $h_a$ given in (5.1) are constants $c(a)$ for $\nu$-almost every $a \in T$. If $\mu$ is invariant then the two laplacians $\Delta_P$ and $\tilde{\Delta}_P$ coincide.

Define for $u, v \in \mathcal{D}(\mathcal{F})$
\[
q(u, v) := -\int (\Delta_P u) v d\mu, \quad e(u, v) := q(u, v) + \int uv d\mu
\]
and
\[
\tilde{q}(u, v) := -\int (\tilde{\Delta}_P u) v d\mu = q(u, v) - \int (Fu) v d\mu, \quad \tilde{e}(u, v) := \tilde{q}(u, v) + \int uv d\mu.
\]
Note that these identities still hold for $v \in L^2(\mu)$ and $u$ in the domain of $\Delta_P$ and of $\tilde{\Delta}_P$ that we will define later.

The main properties of these bilinear forms are described in the following lemma. In particular, the lemma says that $\tilde{\Delta}_P$ is self-adjoint. This is the main advantage of $\tilde{\Delta}_P$ over $\Delta_P$.

Lemma 5.4 We have for $u, v \in \mathcal{D}(\mathcal{F})$,
\[
\tilde{q}(u, v) = \int \langle \nabla u, \nabla v \rangle_{g_P} d\mu = \int i \hat{\partial} u \wedge \hat{\partial} v \wedge T, \quad \int (\tilde{\Delta}_P u) v d\mu = \int u(\tilde{\Delta}_P v) d\mu.
\]
In particular, $\widetilde{q}(u, v)$ and $\widetilde{e}(u, v)$ are symmetric in $u, v$. Moreover,
\[
\int \Delta_P u d\mu = \int \Delta_P u d\mu = \int F u d\mu = 0 \quad \text{for} \quad u \in \mathscr{D}(\mathcal{F}).
\]

**Proof** Using a partition of unity, we can assume that $u$ and $v$ have compact support in a flow box as above. Using (5.1) it is then enough to consider the case where $T$ is supported by a plaque $\mathbb{B} \times \{a\}$ and given by a harmonic function $h_a$. Using (5.2), (5.4) and (2.5), we have that
\[
\widetilde{q}(u, v) = q(u, v) - \int_{\mathbb{B} \times \{a\}} (i \partial_\mathcal{F} u) v h_a - \int_{\mathbb{B} \times \{a\}} i v \partial_\mathcal{F} u \wedge \overline{h}_a = - \int_{\mathbb{B} \times \{a\}} (i \partial_\mathcal{F} u) v h_a - \int_{\mathbb{B} \times \{a\}} i v \partial_\mathcal{F} u \wedge \overline{h}_a,
\]
where the last equality holds because the last integral in the last line is equal to 0 by Stokes’ theorem. After expanding the differential expression $d (v h_a \partial_\mathcal{F} u)$ in the last term in the last line, and then simplifying the last line, we get that
\[
\widetilde{q}(u, v) = \int_{\mathbb{B} \times \{a\}} i h_a \partial_\mathcal{F} u \wedge \overline{v}.
\]
This proves the first identity of the lemma.

It follows from this identity that $\widetilde{q}$ and $\widetilde{e}$ are symmetric.

The second identity of the lemma ($\widetilde{\Delta}_P$ is self-adjoint) is an immediate consequence of the first one. Applying the second identity to the case where $v = 1$ on a neighborhood of the support of $u$, so $u \Delta v = 0$ and we obtain the other identities of the lemma.

Note that the lemma still holds for $u, v$ in the domain of $\Delta$ and $\widetilde{\Delta}$ that we will define later.

**Lemma 5.5** The bilinear forms $\widetilde{q}$ and $\widetilde{e}$ extend continuously to $H^1(\mu) \times H^1(\mu)$. If the measure $\mu$ is finite, then $q$ and $e$ also extend continuously to $H^1(\mu) \times H^1(\mu)$. Moreover, we have $q(u, u) = \widetilde{q}(u, u)$ and $e(u, u) = \widetilde{e}(u, u)$ for $u \in H^1(\mu)$.

**Proof** The first identity in Lemma 5.4 implies that $\widetilde{q}$ and $\widetilde{e}$ extend continuously to $H^1(\mu)$ and the identity is still valid for the extension of $\widetilde{q}$. In order to prove the same property for $q$ and $e$ when $\mu$ is of finite mass, it is enough to show that $q - \widetilde{q}$ is bounded on $H^1(\mu) \times H^1(\mu)$. For this we follow the proof of Lemma 5.30 below.

**Definition 5.6** Define the domain $\text{Dom}(\pm \Delta_P)$ of $\pm \Delta_P$ (resp. $\text{Dom}(\pm \widetilde{\Delta}_P)$ of $\pm \widetilde{\Delta}_P$) as the space of $u \in H^1(\mu)$ such that $q(u, \cdot)$ (resp. $\widetilde{q}(u, \cdot)$) extends to a linear continuous form on $L^2(\mu)$.

**Remark 5.7** Using Lemmas 5.4 and 5.5 we can show that when the measure $\mu$ is finite, $\text{Dom}(\pm \Delta_P) = \text{Dom}(\pm \widetilde{\Delta}_P)$ (see also Remark 5.31). Moreover, we can prove that $\text{Dom}(\pm \Delta_P)$ is equal to $H_P(\mu)$, where $H_P(\mu)$ is the completion of $\mathscr{D}(\mathcal{F})$ for the norm
\[
\|u\|_{H_P(\mu)} := \sqrt{\|u\|^2_{L^2(\mu)} + \|\Delta_P u\|^2_{L^2(\mu)}}.
\]
For more details see (5.7) and (5.8) in the proof of Proposition 5.8 below. It is clear that if $u \in \text{Dom}(-\Delta_P)$ then $\Delta_P u$ in the sense of distributions with respect to $\mathscr{D}(\mathcal{F})$ as test
functions, is in $L^2(\mu)$. This allows us to extend Lemma 5.4 to $u, v$ in $\text{Dom}(-\Delta_P)$, or more generally to $u \in \text{Dom}(-\Delta_P)$ and $v \in H^1(\mu)$.

The existence of abstract heat diffusions are given by the following proposition.

**Proposition 5.8** Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a Riemann surface lamination with singularities endowed with the leafwise Poincaré metric $g_P$. Let $\mu$ be a positive quasi-harmonic measure. Then the associated operator $-\tilde{\Delta}_P$ is maximal monotone on $L^2(\mu)$. If, moreover, the measure $\mu$ is finite, then the associated operator $-\Delta_P$ is also maximal monotone on $L^2(\mu)$. In particular, they are infinitesimal generators of semi-groups of contractions on $L^2(\mu)$ and their graphs are closed.

**Proof** The last assertion is the consequence of the first one, Theorem 5.1 and the properties of maximal monotone operators. So, we only have to prove the first assertion. By Lemma 5.4 we have

$$\langle -\tilde{\Delta}_P u, u \rangle_{g_P} = q(u, u) = \int i \partial u \wedge \bar{\partial} u \wedge T \geq 0. \quad (5.7)$$

This, combined with Lemma 5.5, yields that

$$\langle -\Delta_P u, u \rangle_{g_P} = q(u, u) = \tilde{q}(u, u) = \int i \partial u \wedge \bar{\partial} u \wedge T \geq 0. \quad (5.8)$$

By continuity, we can extend the inequalities to $u$ in $\text{Dom}(-\Delta_P) = \text{Dom}(-\tilde{\Delta}_P)$. So, $-\Delta_P$ and $-\tilde{\Delta}_P$ are monotone. Pick $u \in \mathcal{D}(\mathcal{F})$. By Lemmas 5.4 and 5.5, we have for $u \in H^1(\mu)$

$$\tilde{e}(u, u) \geq \|u\|_{H^1}^2 \quad \text{and} \quad e(u, u) \geq \|u\|_{H^1}^2. \quad (5.9)$$

By Theorem 5.2, for any $f \in L^2(\mu)$, there is $u \in H^1(\mu)$ such that

$$e(u, v) = \langle f, v \rangle_{L^2(\mu)} \quad \text{for} \quad v \in H^1(\mu).$$

So, $u$ is in $\text{Dom}(\Delta_P)$ and the last equation is equivalent to $u - \Delta_P u = f$. Hence, $-\Delta_P$ is maximal monotone. The case of $-\tilde{\Delta}_P$ is treated in the same way. Note that since $-\tilde{\Delta}_P$ is symmetric and maximal monotone, it is self-adjoint but $-\Delta_P$ is not symmetric. $\square$

When the measure $\mu$ is finite, we obtain the following ergodic theorem for abstract heat diffusions which is stronger than Proposition 5.8.

**Theorem 5.9** Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a Riemann surface lamination with singularities endowed with the leafwise Poincaré metric $g_P$. Let $\mu$ be a harmonic measure. Let $S(t)$, $t \in \mathbb{R}^+$, denote the semi-group of contractions associated with the operator $-\Delta$ or $-\Delta_P$ which is given by the Hille-Yosida theorem. Then the measure $\mu$ is $S(t)$-invariant and $S(t)$ is a positive contraction in $L^p(\mu)$ for all $1 \leq p \leq \infty$.

**Proof** We prove that $\mu$ is invariant, that is

$$\langle \mu, S(t)u_0 \rangle = \langle \mu, u_0 \rangle \quad \text{for} \quad u_0 \in \mathcal{D}(\mathcal{F}).$$

We will see later that this identity holds also for $u_0 \in L^1(\mu)$ because $S(t)$ is a contraction in $L^1(\mu)$ and $\mathcal{D}(\mathcal{F})$ is dense in $L^1(\mu)$. Define $u := S(t)u_0$ and

$$\eta(t) := \langle \mu, S(t)u_0 \rangle = \langle \mu, u(t, \cdot) \rangle \quad \text{for} \quad t \in \mathbb{R}^+. \qed$$
We deduce from Theorem 5.1 that $\eta$ is of class $C^1$ on $\mathbb{R}^+$ and that

$$\eta'(t) = \langle \mu', S'(t)u_0 \rangle = \langle \mu, Au(t, \cdot) \rangle,$$

where $A$ is the operator $-\Delta_p$ or $-\tilde{\Delta}_p$. By Lemma 5.4, the last integral vanishes. So, $\eta$ is constant and hence $\mu$ is invariant.

In order to prove the positivity of $S(t)$, it is enough to show the following maximum principle: if $u_0$ is a function in $\mathcal{D}(\mathcal{F})$ such that $u_0 \leq K$ for some constant $K$, then $u(t, x) \leq K$. To show the maximum principle we use a trick due to Stampacchia [6]. Fix a smooth bounded function $G : \mathbb{R} \to \mathbb{R}^+$ with bounded first derivative such that $G(t) = 0$ for $t \leq 0$ and $G'(t) > 0$ for $t > 0$. Put

$$H(s) := \int_0^s G(t)dt.$$ 

Consider the non-negative function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$\xi(t) := \int H(u(t, \cdot) - K)d\mu.$$ 

Here we make use of the assumption that $\mu$ is finite. By Theorem 5.1, $\xi$ is of class $C^1$. We want to show that it is identically zero. Define $v(t, x) := u(t, x) - K$. We have $Av(t, x) = Au(t, x)$. Using in particular that $G$ is bounded, we obtain

$$\xi'(t) = \int G(u(t, \cdot) - K) \frac{\partial u(t, \cdot)}{\partial t}d\mu = -\int G(u(t, \cdot) - K)Au(t, \cdot)d\mu.$$ 

When $A = -\Delta_p$, by Theorem 5.1 $v(t, \cdot) \in H^1(\mu)$, hence by Lemma 5.4, the last integral is equal to

$$-\int \langle \nabla G(v), \nabla v \rangle_{\mathcal{S}_p}d\mu = -\int G'(v)|\nabla v|^2d\mu \leq 0.$$ 

Thus, $\xi'(t) \leq 0$. This, combined with $\xi(0) = 0$ and $\xi(t) \geq 0$ for $t \in \mathbb{R}^+$, implies that $\xi = 0$. Hence $u(x, t) \leq K$.

When $A = -\Delta_p$, by Theorem 5.1 $v(t, \cdot) \in H^1(\mu)$, the considered integral is equal to

$$-\int G'(v)|\nabla v|^2d\mu + \int G(v)Fvd\mu = -\int G'(v)|\nabla v|^2d\mu + \int FH(v)d\mu.$$ 

By Lemma 5.4, the last integral vanishes. So, we also obtain that $\xi'(t) \leq 0$. This completes the proof of the maximum principle which implies the positivity of $S(t)$.

The positivity of $S(t)$ together with the invariance of $\mu$ imply that

$$\|S(t)u_0\|_{L^1(\mu)} \leq \|u_0\|_{L^1(\mu)} \text{ for } u_0 \in \mathcal{D}(\mathcal{F}).$$ 

It follows that $S(t)$ extends continuously to a positive contraction in $L^1(\mu)$ since $\mathcal{D}(\mathcal{F})$ is dense in $L^1(\mu)$. The uniqueness of the solution in Theorem 5.1 implies that $S(t)1 = 1$. This together with the positivity of $S(t)$ imply that $S(t)$ is a contraction in $L^\infty(\mu)$. Finally, the classical theory of interpolation between the Banach spaces $L^1(\mu)$ and $L^\infty(\mu)$ implies that $S(t)$ is a contraction in $L^p(\mu)$ for all $1 \leq p \leq \infty$ (see Triebel [94]).

An important consequence of Theorem 5.9 is the following ergodic theorem.
Theorem 5.10 Under the hypothesis of Theorem 5.9, for all \( u_0 \in L^p(\mu) \), \( 1 \leq p < \infty \), the average

\[
\frac{1}{R} \int_0^R S(t) u_0 dt
\]

converges pointwise \( \mu \)-almost everywhere and also in \( L^p(\mu) \) to an \( S(t) \)-invariant function \( u^*_0 \) when \( R \) goes to infinity. Moreover, \( u^*_0 \) is constant on the leaf \( L_a \) for \( \mu \)-almost every \( a \). If \( \mu \) is an ergodic harmonic measure, then \( u \) is constant \( \mu \)-almost everywhere.

Proof By Theorem 5.9, \( S(t) \) is a positive contraction in \( L^p(\mu) \) for all \( 1 \leq p \leq \infty \). Consequently, the pointwise \( \mu \)-almost everywhere convergence is a consequence of the ergodic theorem as in Dunford-Schwartz [47, Th. VIII.7.5]. We get a function \( u^*_0 \) which is \( S(t) \)-invariant. The \( L^p \) convergence follows from the \( L^p \) ergodic theorem of Von Neumann. For the rest of the proof, since \( S(t) \) is a contraction in \( L^p(\mu) \) for, it is enough to consider the case where \( u_0 \) is in \( D(F) \).

We first prove that \( u^*_0 \) is in the domain of \( A \) and

\[
Au^*_0 = 0. \tag{5.10}
\]

Define

\[
u_R := \frac{1}{R} \int_0^R S(t) u_0 dt.
\]

This function belongs to \( \text{Dom}(A) \). Since \( u_R \) converges to \( u^*_0 \) in \( L^2(\mu) \) and the graph of \( A \) is closed in \( L^2(\mu) \times L^2(\mu) \), it is enough to show that \( Au_R \to 0 \) in \( L^2(\mu) \). We have

\[
Au_R = \frac{1}{R} \int_0^R Au(t, \cdot) dt = -\frac{1}{R} \int_0^R \frac{\partial}{\partial t} u(t, \cdot) dt = \frac{1}{R} u_0 - \frac{1}{R} u(R, \cdot).
\]

Since \( S(t) \) is a contraction in \( L^2(\mu) \), the last expression tends to 0 in \( L^2(\mu) \). This proves (5.10).

By Lemmas 5.4 and 5.5, we deduce from equality (5.10) that

\[
\int |\nabla u_0|^2 d\mu = -\int (\Delta_P u_0) u_0 d\mu = \int (\Delta_P u_0) u_0 d\mu = 0.
\]

It follows that \( \nabla u_0 = 0 \) almost everywhere with respect to \( \mu \). Thus, \( u_0 \) is constant on the leaf \( L_a \) for \( \mu \)-almost every \( a \). When \( \mu \) is extremal, this property implies that \( u_0 \) is constant \( \mu \)-almost everywhere, since every measurable set of leaves has zero or full \( \mu \) measure.

We will need the following lemmas.

Lemma 5.11 Let \( \widehat{\mu} = \theta \mu \) be a (signed) quasi-harmonic measure, not necessarily positive, where \( \theta \) is a function in \( L^2(\mu) \) (see Definition 2.3). Let \( \widehat{\mu} = \widehat{\mu}^+ - \widehat{\mu}^- \) be the minimal decomposition of \( \widehat{\mu} \) as the difference of two positive measures. Then \( \widehat{\mu}^\pm \) are harmonic.

Proof We start with assertion (1). Let \( S(t) \) be the semi-group of contractions in \( L^1(\mu) \) associated with \( -\Delta_P \) as above. Define the action of \( S(t) \) on measures by

\[
\langle S(t) \widehat{\mu}, u_0 \rangle := \langle \widehat{\mu}, S(t) u_0 \rangle \quad \text{for} \quad u_0 \in L^2(\mu).
\]

Consider a function \( u_0 \in \mathcal{D}(\mathcal{F}) \) and define \( \eta(t) := \langle S(t) \widehat{\mu}, u_0 \rangle \). By Theorem 5.1, this is a \( C^1 \) function on \( \mathbb{R}^+ \). We have since \( \widehat{\mu} \) is quasi-harmonic and \( \theta \) is in \( L^2(\mu) \)

\[
\eta'(t) = \langle \widehat{\mu}, S'(t) u_0 \rangle = \langle \widehat{\mu}, -\Delta_P (S(t) u_0) \rangle = 0.
\]
To see the last equality, we can use a partition of unity and the local description of \( \hat{\mu} \) on a flow box. So, \( \eta \) is constant. It follows that \( S(t)\hat{\mu} = \hat{\mu} \). Since \( S(t) \) is a positive contraction, we deduce that \( S(t)\hat{\mu}^\pm = \hat{\mu}^\pm \). So, the functions \( \eta^\pm(t) := \langle \hat{\mu}^\pm, S(t)u_0 \rangle \) are constant. As above, we have
\[
\langle \hat{\mu}^\pm, \Delta_Pu_0 \rangle = -\langle \hat{\mu}^\pm, S'(0)u_0 \rangle = (\eta^\pm)'(0) = 0.
\]
Hence, \( \hat{\mu}^\pm \) are quasi-harmonic. Since they are finite positive, they are also harmonic by Theorem 2.9 (3).

Let \( \mu_1 \) and \( \mu_2 \) be two finite positive measures. Define \( \mu := \mu_1 + \mu_2 \) and \( \theta \) a function in \( L^1(\mu) \), \( 0 \leq \theta \leq 1 \), such that \( \mu_i = \theta_i \mu \). Define also \( \mu_1 \lor \mu_2 := \max\{\theta_1, \theta_2\} \mu \) and \( \mu_1 \land \mu_2 := \min\{\theta_1, \theta_2\} \mu \).

**Lemma 5.12** Let \( \mu_1 \) and \( \mu_2 \) be two harmonic measures. Then \( \mu_1 \lor \mu_2 \) and \( \mu_1 \land \mu_2 \) are also harmonic.

**Proof** We use the notation introduced just before the lemma. By Lemma 5.11, since the signed finite measure \( \mu_1 - \mu_2 \) is quasi-harmonic, \( \mu' := \max\{\theta_1 - \theta_2, 0\} \mu \) is harmonic. It follows that \( \mu_1 \lor \mu_2 = \mu' + \mu_2 \) and \( \mu_1 \land \mu_2 = \mu_1 - \mu' \) are harmonic.

We also obtain the following result (see Candel-Connon [19]).

**Corollary 5.13** Under the hypothesis of Theorem 5.9, the family \( \mathcal{H} \) of harmonic probability measures of \( \mathcal{F} \) is a non-empty compact and for any \( \mu \in \mathcal{H} \) there is a unique probability measure \( \nu \) on the set of extremal elements \( \mathcal{E} \) in \( \mathcal{H} \) such that \( \mu = \int_{\mathcal{E}} md\nu(m) \). Moreover, two different extremal harmonic probability measures are mutually singular.

**Proof** Under the hypothesis of Theorem 5.9, the family \( \mathcal{H} \) of harmonic probability measures of \( \mathcal{F} \) is a non-empty. Clearly, \( \mathcal{H} \) is compact. By Choquet’s representation theorem [24], we can decompose \( \mu \) into extremal measures as in the corollary. Using Lemma 5.12, the uniqueness of the decomposition is a consequence of the Choquet-Meyer theorem [24, p. 163].

The following result will be needed in Section 5.4.

**Proposition 5.14** Let \( \mu = \int_{m \in \mathcal{E}} md\nu(m) \) be as in Corollary 5.13. Then, the closures of \( \Delta_P(\mathcal{D}(\mathcal{F})) \) and of \( \Delta_P(\mathcal{D}(\mathcal{F})) \) in \( L^p(\mu) \), \( 1 \leq p \leq 2 \), are the space of functions \( u_0 \in L^p(\mu) \) such that \( \int u_0dm = 0 \) for \( \nu \)-almost every \( m \). In particular, if \( \mu \) is an ergodic harmonic probability measure, then this space is the hyperplane of \( L^p(\mu) \) defined by the equation \( \int u_0dm = 0 \).

**Proof** We only consider the case of \( \Delta_P \); the case of \( \Delta_P \) is treated in the same way. It is clear that \( \Delta_P(\mathcal{D}(\mathcal{F})) \) is a subset of the space of \( u_0 \in L^p(\mu) \) such that \( \int u_0dm = 0 \) for \( \nu \)-almost every \( m \) and the last space is closed in \( L^p(\mu) \). Consider a function \( \theta \in L^q(\mu) \), with \( 1/p + 1/q = 1 \), which is orthogonal to \( \Delta_P(\mathcal{D}(\mathcal{F})) \). So, \( \theta \mu \) is a quasi-harmonic signed measure. Since \( p \leq 2 \), we have \( \theta \in L^2(\mu) \). We have to show that \( \theta \) is constant with respect to \( \nu \)-almost every \( m \).

Consider the disintegration of \( \mu \) along the fibers of \( \theta \). There are a probability measure \( \nu' \) on \( \mathbb{R} \) and a probability measures \( \mu_c \) on \( \{\theta = c\} \) such that \( \mu = \int_{c \in \mathbb{R}} \mu_c d\nu'(c) \). By Lemma 5.12, for any \( c \in \mathbb{R} \), the measure \( \max\{\theta, c\} \mu \) is quasi-harmonic. Therefore, \( \mu_c \) is
harmonic for \( \nu' \)-almost every \( c \). If \( \nu_c \) is the probability measure on \( \mathcal{E} \) associated with \( \mu_c \) as in Corollary 5.13, we deduce from the uniqueness in this corollary that

\[
v = \int_{c \in \mathbb{R}} \nu_c \, dv'(c).
\]

Now, since \( \theta \) is constant \( \mu_c \)-almost everywhere, it is constant with respect to \( \nu_c \)-almost every \( m \). So

\[
\int_{c \in \mathbb{R}} \int_{m \in \mathcal{E}} \int_X \left| \theta - \int_X \theta \, dm \right| \, dv_c(m) \, dv'(c) = 0.
\]

We deduce from the last two equalities that \( \theta \) is equal to the constant \( \int_X \theta \, dm \) for \( \nu \)-almost every \( m \). This completes the proof.

The following result gives a version of the mixing property in our context. The classical case is due to Kaimanovich [65] who uses in particular the smoothness of the Brownian motion, see also Candel [17] who relies on a version of the zero-two law due to Ornstein and Sucheston [83].

**Theorem 5.15**  ([32, Theorem 5.12]) Under the hypothesis of Theorem 5.9, assume moreover that \( \mu \) is ergodic. If \( S(t) \) is associated to \( -\Delta_P \), then \( S(t)u_0 \) converge to \( \langle \mu, u_0 \rangle \) in \( L^p(\mu) \) when \( t \to \infty \) for \( u_0 \in L^p(\mu) \) with \( 1 \leq p < \infty \). In particular, \( S(t) \) is mixing, i.e.,

\[
\lim_{t \to \infty} \langle S(t)u_0, v_0 \rangle = \langle \mu, u_0 \rangle \langle \mu, v_0 \rangle \text{ for } u_0, v_0 \in L^2(\mu).
\]

**Remark 5.16** When the lamination \( \mathcal{F} \) is compact nonsingular hyperbolic (i.e., \( E = \emptyset \), \( X \) is compact and \( \text{Hyp}(\mathcal{F}) = X \)), several results in this subsection for \( \Delta_P \) can be deduced from Candel-Conlon [19] and Garnett [54]. But results on Riemann surface laminations with singularities and on \( \Delta_P \) are new.

### 5.2 When do the Abstract Diffusions Coincide with the Leafwise Diffusions?

As mentioned in the introduction, the results of this subsection are new. As observed in Section 5.1, the abstract heat diffusions associated with the operator \( A := -\Delta_P \) and the harmonic measure \( \mu \) enjoy many important ergodic properties. The only drawback is that these diffusions are not so concrete. On the other hand, the leafwise heat diffusions given in (2.3) have the advantage of being quite explicit because of concrete formulas (2.6)–(2.9). So the natural question arises whether these two heat diffusions coincide. The following result gives an effective criterion for this coincidence.

**Theorem 5.17** Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a Riemann surface lamination with singularities endowed with the leafwise Poincaré metric \( g_P \). Let \( g_X \) be a Hermitian metric on the leaves which is transversally continuous. Let \( \mu \) be a harmonic measure. Suppose that

(i) (local upper-boundedness of \( \eta \)) the function \( \eta \) given in (4.1) is locally bounded from above by strictly positive constants on \( X \setminus E \);

(ii) (membership test) if \( u \) is a measurable function on \( X \setminus E \) such that \( \|u\|_{L^\infty} < \infty \), \( \|du|_P\|_{L^\infty} < \infty \) and \( \|\Delta_P u\|_{L^\infty} < \infty \), then \( u \) belongs necessarily to \( H^1(\mu) \) (for the notation \( |du|_P \) see (4.3) and for \( H^1(\mu) \) see (5.3)).

Then the abstract heat diffusions associated to \( \mu \) coincide with the leafwise heat diffusions.
Let $S(t)$, $t \in \mathbb{R}_+$, denote the semi-group of contractions associated with the operator $-\Delta_P$ given by Proposition 5.8. Then the conclusion of Theorem 5.17 says that

$$S(t)u = D_t u \quad \text{for} \quad u \in \text{Dom}(-\Delta_P) \quad \text{and} \quad t \in \mathbb{R}^+,$$

where $D_t$ is the leafwise heat diffusion given in (2.3).

We start the proof with the following observation. Since $\mathscr{D}(\mathcal{F})$ is dense in $\text{Dom}(-\Delta_P) = H_P(\mu) \subset L^2(\mu)$ by Remark 5.7 and both diffusions are positive contractions, we only need to prove that for an arbitrary non-negative function $u_0 \in \mathscr{D}(\mathcal{F})$,

$$S(t)u_0 = D_t u_0 \quad \text{for} \quad t \in \mathbb{R}^+.$$  \hfill (5.11)

Fix such a function $u_0$. Since $\eta$ is locally bounded from above on $X \setminus E$, we infer that $\text{Par}(\mathcal{F}) = \emptyset$. By Theorem 5.1, there is a unique $\mathcal{C}^1$ function $U$ from $\mathbb{R}^+$ to $L^2(m)$ with values in $\text{Dom}(-\Delta_P)$ which satisfies

$$\frac{\partial U(t, \cdot)}{\partial t} - \Delta_P U(t, \cdot) = 0 \quad \text{and} \quad U(0, \cdot) = u_0.$$  \hfill (5.12)

Consider the function $u : \mathbb{R}^+ \times (X \setminus E) \to \mathbb{R}$ defined by

$$u(t, \cdot) := D_t u_0.$$  \hfill (5.13)

To complete the proof of (5.11) it suffices to show that

$$U = u.$$  \hfill (5.14)

We will prove (5.14) using the proof of the uniqueness in Theorem 5.1. The proof relies on three facts. The first fact is that the function $u(t, \cdot) : X \setminus E \to \mathbb{R}$ constructed in (5.12) is measurable and it satisfies

$$\|u(\cdot, \cdot)\|_p L^\infty < \infty, \quad \text{and for each} \quad t > 0, \quad \|d u(t, \cdot)\|_p L^\infty < \infty, \quad \|\Delta_P u(t, \cdot)\|_L^\infty < \infty.$$  \hfill (5.15)

The second fact is that

$$u \in \mathcal{C}^1(\mathbb{R}^+_\star, L^2(\mu)) \quad \text{and} \quad \frac{\partial u(t, \cdot)}{\partial t} - \Delta_P u(t, \cdot) = 0 \quad \text{for} \quad t \in \mathbb{R}^+_\star.$$  \hfill (5.16)

The third fact is the limit

$$\lim_{t \to 0} u(t, \cdot) = u_0 \quad \text{in} \quad L^2(\mu).$$  \hfill (5.17)

Taking for granted these three facts, we arrive at the

End of the proof of Theorem 5.17 Since the function $u : \mathbb{R}^+ \times (X \setminus E) \to \mathbb{R}$ enjoys the properties stated in the first fact (5.15), we infer from the membership test (ii) that $u(t, \cdot)$ belongs to $H^1(\mu)$ for each $t \in \mathbb{R}^+_\star$. So does $U(t, \cdot) - u(t, \cdot)$. Consequently, we infer from the second fact (5.16) that

$$\left\{ \frac{\partial}{\partial t} (U(t, \cdot) - u(t, \cdot)), U(t, \cdot) - u(t, \cdot) \right\} = (-\Delta_P (U(t, \cdot) - u(t, \cdot)), U(t, \cdot) - u(t, \cdot)) \quad \text{for} \quad t \in \mathbb{R}^+_\star.$$  

Fix a $t \in \mathbb{R}^+_\star$ and consider the function $\tilde{u} := U(t, \cdot) - u(t, \cdot) \in H^1(\mu)$. By (5.3), for every $\epsilon > 0$ there is $\tilde{u}_\epsilon \in \mathscr{D}(\mathcal{F})$ such that $\|\tilde{u} - \tilde{u}_\epsilon\|_{H^1(\mu)} < \epsilon$. On the one hand, we have $\tilde{u}_\epsilon \in H_P(\mu)$ (see (5.6)) and $\tilde{u} \in H^1(\mu)$. On the other hand, by (5.12) and (5.15)-(5.16) we get $\Delta_P \tilde{u} = \Delta_P U(t, \cdot) - \Delta_P u(t, \cdot) \in L^2(\mu)$. Consequently, by Lemma 5.4 (see Remark 5.7) and the inequality $\int i \partial \tilde{u} \wedge \overline{\partial} \tilde{u} \wedge T \geq 0$, we infer that

$$\tilde{q}(\tilde{u}, \tilde{u}_\epsilon) = \int i \partial \tilde{u} \wedge \overline{\partial} \tilde{u}_\epsilon \wedge T = \int i \partial \tilde{u} \wedge \overline{\partial} \tilde{u} \wedge T + O(\epsilon) \geq O(\epsilon).$$
Letting $\varepsilon$ tend to 0, the above inequality together with the estimate $\|\tilde{u} - \tilde{u}_v\|_{H^1(\mu)} < \varepsilon$ imply that $\tilde{q}(\tilde{u}, \tilde{u}) \geq 0$. Hence, by Lemma 5.5 we get that $q(\tilde{u}, \tilde{u}) = \tilde{q}(\tilde{u}, \tilde{u}) \geq 0$. So for $t \in \mathbb{R}^+$, 
\[
\left\{ \frac{\partial}{\partial t} (U(t, \cdot) - u(t, \cdot)), U(t, \cdot) - u(t, \cdot) \right\} = \langle -\Delta_p U(t, \cdot) - u(t, \cdot), U(t, \cdot) - u(t, \cdot) \rangle \leq 0.
\]
Hence,
\[
\frac{1}{2} \frac{\partial}{\partial t} \|U(t, \cdot) - u(t, \cdot)\|^2_{L^2(\mu)} = \left\{ \frac{\partial}{\partial t} (U(t, \cdot) - u(t, \cdot)), U(t, \cdot) - u(t, \cdot) \right\} \leq 0.
\]
So the function $\mathbb{R}^+ \ni t \mapsto \|U(t, \cdot) - u(t, \cdot)\|^2_{L^2(\mu)}$ is decreasing. Using the third fact (5.17) we see that $U(0, \cdot) - u(0, \cdot) = u_0 - u_0 = 0$ and that $\lim_{t \to 0} U(t, \cdot) - u(t, \cdot) = 0$ in $L^2(\mu)$. Hence, $U(t, \cdot) - u(t, \cdot) = 0$ for all $t \in \mathbb{R}^+$. The proof of (5.14) is thereby completed. \qed

To finish the proof of Theorem 5.17 we need to prove the three facts. To this end we do some simplifications. Using a countable partition of unity of $X \setminus E$, we may assume without loss of generality that the function $u_0$ is compactly supported in a given flow box $U \cong \mathbb{B} \times \mathbb{T}$, where $\mathbb{B}$ is simply the unit disc $\mathbb{D}$. The center of a plaque $\mathbb{B} \times a$ with $a \in \mathbb{T}$ is the point $0 \times a$, where $0$ is the center of $\mathbb{D}$. Fix a point $x_0 \in X \setminus E$ and a finite time $t_0 > 0$. Let $(\Omega_j)_{j \in J}$ be all connected components of $\phi^{-1}_{x_0}(U) \subset \mathbb{D}$. So the (eventually empty) index set $J$ (depending on $U$ and $x_0$) is at most countable. Since $\eta$ is locally bounded from above and below by strictly positive constants outside $E$, there is a positive constant $c_1 > 1$ such that $c_1^{-1} g_X \leq g_p \leq c_1 g_X$ on plaques of $U$. Observe that to travel from the center of a plaque of $U$ to the center of another plaque, we cover a distance $\geq c_3$ with respect to $g_X$, hence a distance $\geq c_1^{-1} c_2$ with respect to $g_p$. Consequently, there are constants $c_3, c_4 > 0$ (which depends only on $U$ and which does not depend on $x_0$) such that
\[
\text{diam}_p(\Omega_j) \leq c_3 \quad \text{and} \quad \text{dist}_p(\Omega_p, \Omega_q) \geq c_4 \quad \text{for} \quad p \neq q, \quad p, q \in J. \tag{5.18}
\]
Let $v : \mathbb{R}^+ \times \mathbb{D} \to \mathbb{R}$ be the function defined on by
\[
v(t, \zeta) = u(t, \phi_{x_0}(\zeta)) \quad \text{for} \quad t \in \mathbb{R}^+, \zeta \in \mathbb{D}. \tag{5.19}
\]
For $j \in J$ let
\[
u_j := u_0 \circ \phi_{x_0} \quad \text{on} \quad \Omega_j. \tag{5.20}
\]
Since $u_0$ is compactly supported in the flow box $U \cong \mathbb{B} \times \mathbb{T}$, it follows that $u_j \in \mathcal{D}(\Omega_j)$. Now $u_j$ can be regarded as a non-negative smooth function compactly supported on $\mathbb{D}$ by simply setting $u_j = 0$ outside $\Omega_j$. Set
\[
v_j(t, z) = (D_t u_j)(z) \quad \text{for} \quad z \in \mathbb{D}. \tag{5.21}
\]
So we have
\[
v = \sum_{j \in J} v_j. \tag{5.22}
\]
Since $u_j \in \mathcal{D}(\mathbb{D})$, we deduce from (2.2) that on $\mathbb{D}$
\[
\frac{\partial v_j(t, \cdot)}{\partial t} - \Delta_p v_j(t, \cdot) = 0 \quad \text{and} \quad v_j(0, \cdot) = u_j. \tag{5.23}
\]

**Lemma 5.18** For every $t_0 > 0$, there is a constant $c_5$ which depends only on $U$ and $t_0$ (so $c_5$ does not depend on $x_0 \in X \setminus E$) such that
\[
\sum_{j \in J, t \in [t_0/2, 2t_0]} \left( \sup_{(x, y, t) \in \mathbb{D}^2 : \text{dist}_p(x, 0) < 1} |\Phi(x, y, t)| \right) \leq c_5
\]
in the following cases:

1. \( \Phi(x, y, t) = p^D(x, y, t) \);
2. \( \Phi(x, y, t) = \frac{\partial p^D(x, y, t)}{\partial t} \);
3. \( \Phi(x, y, t) = (\Delta p)^x p^D(x, y, t) \);
4. \( \Phi(x, y, t) = |dx p^D(x, y, t)|^p \).

**Proof** Since the proof of Case (1) is similar and even simpler than that of Case (2), we treat directly Case (2).

Denote the Poincaré distance between \( x \) and \( y \) by \( \rho := \text{dist}_P(x, y) \). We deduce from formula (2.6) and identity (2.9) that

\[
p^D(x, y, t) = \sqrt{2} e^{-t/4} \int_\rho^\infty \frac{se^{-s^2/4}}{\sqrt{\cosh s - \cosh \rho}} \, ds \quad \text{for} \quad t \in \mathbb{R}^+.
\]

It can be checked that \( \sqrt{\cosh s - \cosh \rho} \gtrsim e^s/2 \) when \( s \geq \rho + 1 \) and \( \sqrt{\cosh s - \cosh \rho} \gtrsim e^s/2 \sqrt{s - \rho} \) for \( \rho < s \leq \rho + 1 \). Consequently,

\[
\frac{\partial p^D(x, y, t)}{\partial t} \lesssim e^{-t/4} (t^{-3/2} + t^{-5/2} + t^{-7/2}) \left( \int_{\rho+1}^\infty s^3 e^{-s^2/4} \, ds + \int_{\rho}^{\rho+1} s^3 e^{-s^2/4} \, ds \right).
\]

We have the elementary inequality for \( t \in [t_0/2, 2t_0] \) and \( s > 0 \),

\[
e^{-s^2/4} \leq C(N, t_0) e^{-Ns},
\]

where \( N = N(t_0) \) can be made arbitrarily large and \( c(N), t_0 \) is a constant which depends on \( N \) and \( t_0 \). Since both integrands on the right hand side of the estimate for \( \frac{\partial p^D(x, y, t)}{\partial t} \) contain the term \( e^{-s^2/4} \), we infer from the last line that for \( t \in [t_0/2, 2t_0] \),

\[
\frac{\partial p^D(x, y, t)}{\partial t} \leq c(N, t_0) e^{-N\rho}.
\]

So Case (2) will follow if one can show that

\[
\sum_{j \in J} \sup_{\text{dist}_P(x, 0) < 1, y \in \Omega_j} e^{-N\text{dist}_P(x, y)} < c_5.
\]

Using (5.18) we see that when \( N \) is chosen large enough with respect to \( c_3 \) and \( c_4 \), the above inequality holds. This completes the proof of Case (2).

Using the first identity of (2.2) and the symmetry \( p^D(x, y, t) = p^D(y, x, t) \) (see [22]), Case (3) is equivalent to Case (2).

To prove Case (4) the following elementary result is needed.

**Lemma 5.19** Let \( f \in C^2(2D) \). Then

\[
\sup_{D} \|df\| \leq c_3 \left( \sup_{2D} \|f\| + \sup_{2D} \|d^2f\| \right).
\]

Here we recall that \( rD \) denotes the disc with center 0 and radius \( r \).
Proof of Lemma 5.19 Let \( f \) be a function on \( 2\mathbb{D} \) such that \( \mu \):= \( dd^c f \) is a Radon measure of total finite mass \( \|\mu\| \). Let \( \mathcal{P}[\mu] \) be the function defined on \( 2\mathbb{D} \) by

\[
\mathcal{P}[\mu](z) := \frac{1}{\pi} \int_{\zeta \in 2\mathbb{D}} \log |z - \zeta| d\mu(\zeta), \quad z \in 2\mathbb{D}.
\]

Then we deduce from this integral formula that \( d\mathcal{P}[\mu] \in L^\infty(2\mathbb{D}) \). Moreover, \( u - \mathcal{P}[\mu] \) is a harmonic function on \( 2\mathbb{D} \) and \( \| f - \mathcal{P}[\mu] \|_{\varphi^1(2\mathbb{D})} \leq c' \| f - \mathcal{P}[\mu] \|_{L^1((3/2)\mathbb{D})} \leq c \left( \| \mu \| + \| f \|_{L^1(2\mathbb{D})} \right) \).

Here \( c, c' \) are constants independent of \( f \). This completes the proof of the lemma. \( \square \)

Using this lemma, Case (4) follows from combining Case (1) and Case (3). \( \square \)

Proof of the first fact (5.15) Since the norm of \( D_t \) on \( L^\infty(X) \) is 1, we get that

\[
\| u(\cdot, \cdot) \|_{L^\infty} \leq \| u_0 \|_{L^\infty}.
\]

This proves the first inequality in (5.15).

Fix \( t_0 > 0 \). We prove the following stronger inequality than the second inequality in (5.15) sup \( t \in [t_0/2, 2t_0] \| |d u(t, \cdot)|_p \|_{L^\infty} < \infty \). In fact, it is enough to show that

\[
\sup_{t \in [t_0/2, 2t_0], x \in \phi_{x_0}(\mathbb{D})} \| |d_x u(t, x)|_p \|_{L^\infty} < c,
\]

where \( c \) is a constant independent of \( x_0 \) and we use (5.30) for the notation \( \mathbb{D}_R \). Fix a point \( x_0 \in X \setminus E \). Using (5.20), (5.21), (5.22), we get for \( t \in [t_0/2, 2t_0] \) and \( x \in \mathbb{D}_1 \) that

\[
v(t, x) = \sum_{j \in J} v_j(t, x) = \sum_{j \in J} \int_{\Omega_j} p_{\mathbb{D}}(x, y, t) u_0(\phi_{x_0}(y)) d\mu(y).
\]

On the other hand, the first inequality of (5.18) implies that there is a constant \( c \) independent of \( j \in J \) such that \( \int_{\Omega_j} d\mu(y) < c \). Therefore, applying Lemma 5.18 to \( \Phi(x, y, t) = |d_x p_{\mathbb{D}}(x, y, t)|_p \) and using Lebesgue dominated convergence in order to differentiate the sum in (5.24), we get that

\[
|d_x v(t, x)|_p \leq c \|u_0\|_{L^\infty} \quad \text{for} \quad t \in [t_0/2, 2t_0] \quad \text{and} \quad x \in \mathbb{D}_1.
\]

This, combined with (5.13) and (5.19), implies the desired conclusion.

For the last inequality in (5.15) it is enough to show that

\[
\sup_{t \in [t_0/2, 2t_0], x \in \phi_{x_0}(\mathbb{D}_1)} \| \Delta_p u(t, x) \|_{L^\infty} < c,
\]

where \( c \) is a constant independent of \( x_0 \). We argue as above using this time Lemma 5.18 applied to \( \Phi(x, y, t) = (\Delta_p)_{x_0} p_{\mathbb{D}}(x, y, t) \). \( \square \)

Proof of the second fact (5.16) We use (5.22), (5.21), (5.20). We also apply Lebesgue’s dominated convergence and Lemma 5.18 for \( \Phi(x, y, t) = \frac{\partial p_{\mathbb{D}}(x, y, t)}{\partial t} \). Consequently, we infer from equality (5.23) that

\[
\frac{\partial v(t, x)}{\partial t} = \sum_{j \in J} \frac{\partial v_j(t, x)}{\partial t} = \sum_{j \in J} \Delta_p v_j(t, x) = \Delta_p \sum_{j \in J} v_j(t, x) = \Delta_p v(t, x)
\]

for \( t \in [t_0/2, 2t_0] \) and \( x \in \mathbb{D}_1 \). So

\[
\frac{\partial u(t, x)}{\partial t} = \Delta_p u(t, x).
\]

This, combined with (5.13) and (5.19), implies the desired conclusion. \( \square \)
To prove the third fact, we need a counterpart of Lemma 5.18 when the time \( t \) is small.

**Lemma 5.20** There is a constant \( c_6 \) which depends only on \( U \) and \( t_0 \) (so \( c_6 \) does not depend on \( x_0 \in X \setminus E \)) such that for every \( 0 < t < t_0 < 1 \), we have

\[
\sum_{j \in J} \sup_{(x,y) \in D^2 : \text{dist}_P(x,0) < 1, \text{dist}_P(y,0) > 2, y \in \Omega_j} p_D(x, y, t) < c_6 t_0.
\]

**Proof** Denote the Poincaré distance between \( x \) and \( y \) by \( \rho := \text{dist}_P(x, y) \). We deduce from \( \text{dist}_P(x,0) < 1 \) and \( \text{dist}_P(y,0) > 2 \) that \( \rho > 1 \). As in the proof of Lemma 5.18, using formula (2.6) and identity (2.9) as well as the estimates

\[
\sqrt{\cosh s - \cosh \rho} \gtrsim e^{s/2} \quad \text{when} \quad s \geq \rho + 1 \quad \text{and} \quad \sqrt{\cosh s - \cosh \rho} \gtrsim e^{s/2} \sqrt{s - \rho} \quad \text{for} \quad \rho < s \leq \rho + 1,
\]

we get that

\[
p_D(x, y, t) \lesssim e^{-t/4} t^{-3/2} \left( \int_{\rho+1}^{\infty} s e^{-\frac{s^2}{\pi}} ds + \int_{\rho}^{\rho+1} \frac{s e^{-\frac{s^2}{\pi}}}{\sqrt{s - \rho}} ds \right).
\]

We use the following elementary estimate: for \( 0 < t < t_0 \ll 1 \) and \( s > 1 \)

\[
e^{-t/4} t^{-3/2} e^{-\frac{s^2}{\pi}} \leq C(N, t_0) t^4 e^{-Ns},
\]

where \( N = N(t_0) \) can be made arbitrarily large when \( t_0 > 0 \) is sufficiently small, and \( c(N, t_0) \) is a constant which depends on \( N \) and \( t_0 \). Since both integrands on the right hand side of the estimate for \( p_D(x, y, t) \) contain the term \( e^{-\frac{s^2}{\pi}} \), we infer from the last line that for \( 0 < t < t_0 \ll 1 \) and \( \rho > 1 \) that

\[
p_D(x, y, t) \leq c(N, t_0) t_0 e^{-N\rho}.
\]

So the lemma follows from the inequality

\[
\sum_{j \in J} \sup_{(x,y) \in D^2 : \text{dist}_P(x,0) < 1, y \in \Omega_j} e^{-N \text{dist}_P(x,y)} < c_5,
\]

which has been established at the end of the proof of Lemma 5.18. \( \square \)

Now we complete the proof of Theorem 5.17.

**Proof of the third fact (5.17)** Let \( j_0 \in J \) be the (unique if exists) index such that \( \Omega_{j_0} \) contains 0. So the existence of such \( j_0 \) is equivalent to the condition that \( x_0 \in U \). We will see that if there is no such an index \( j_0 \) then the rest of the proof is trivially finished. Using Lemma 5.20, we see that

\[
\sum_{j \in J, j \neq j_0} |v_j(t, \zeta)| \leq C t_0 \quad \text{for} \quad t \in (0, t_0], \quad \zeta \in \mathbb{D}.
\]

On the other hand, applying the second equation in (2.2) to \( u_{j_0} \in \mathcal{D}(\Omega_{j_0}) \) yields that \( \lim_{t \to 0} v_{j_0}(t, \cdot) = u_{j_0} \) on \( \mathbb{D} \). Combining this with the previous estimate, and using (5.22) and (5.20), we see that \( \lim_{t \to 0} v(t, \cdot) = u_0 \circ \phi_{x_0} \) on \( \mathbb{D} \). So by (5.19) and (5.13), \( \lim_{t \to 0} u(t, \cdot) = u_0 \) everywhere on \( X \setminus E \). By the first fact (5.15), \( u(t, \cdot) \) is uniformly bounded. Therefore, by Lebesgue dominated convergence we get the conclusion (5.17) of the third fact. \( \square \)

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Corollary 5.21 Let $\mathcal{F} = (X, \mathcal{L})$ be a compact hyperbolic Riemann surface lamination (without singularities). Let $\mu$ be a harmonic measure. Then the abstract heat diffusions associated to $\mu$ coincide with the leafwise heat diffusions.

Proof Observe that the function $\eta$ is uniformly bounded from above and below by strictly positive constants. So condition (i) is satisfied. To prove the membership test (ii), pick a sequence $u_n \in \mathcal{D}(\mathcal{F})$ of approximants of $u$ such that $\|u_n - u\|_{H^1(\mu)} \to 0$ as $n$ tends to infinity. The interested reader may also see the proof of Proposition 5.22 below for more details. This proves (ii).

Applying now Theorem 5.17 gives the result. \qed

Proposition 5.22 Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a singular holomorphic foliation (not necessarily compact) such that $E$ is a finite set. Assume that the complex manifold $X$ is endowed with a Hermitian metric $g_X$ and the function $\eta$ given in (4.1) is locally bounded from above by strictly positive constants on $X \setminus E$ and that there is a constant $c > 0$ such that $\eta(x) \geq c \text{dist}(x, E)$ for $x \in X$. Here dist is the distance associated to the Hermitian metric on $X$.

Let $T$ be a directed positive harmonic current such that the Lelong number $\nu(T, x)$ for $x \in E$. Let $\mu$ be a harmonic measure. Then the abstract heat diffusions associated to $\mu$ coincides with the leafwise heat diffusions.

Proof We only need to check the membership test (ii) in Theorem 5.17.

Recall that for $r > 0$, $B_r$ denotes the ball in $\mathbb{C}^k$ centered at 0 with radius $r$. We often denote the unit ball by $B$, i.e., $B = B_1$. Fix a non-negative smooth function $\chi : \mathbb{C}^k \to [0, 1]$ such that $\chi(z) = 1$ for $|z| \leq 1/2$ and $\chi(z) = 0$ for $|z| \geq 1$. Let $\epsilon_0 := \int_{\mathbb{C}^k} \chi(z) d\text{vol}(z)$, where $\text{vol}(z)$ denotes the Lebesgue measure of $\mathbb{C}^k$. For $0 < \epsilon < 1$ consider the function $\chi_\epsilon(z) := \epsilon_0^{-1} \epsilon^{-2k} \chi(z/\epsilon)$, $z \in \mathbb{C}^k$. So $\int \chi_\epsilon(z) d\text{vol}(z) = 1$.

Consider a countable or finite cover $\mathcal{U} = (U_p)_{p \in I}$ of $X$ (of dimension $k$) by open sets such that
- For each $p \in I$, $U_p$ is the unit ball $B$ of $\mathbb{C}^k$ in a suitable local coordinate system;
- For each $p \in I \setminus E$, $U_p$ is a flow box and in a foliated chart $U_p \simeq D \times T_p$, with $D$ as usual the unit disc and $T_p$ an open set in $\mathbb{C}^{k-1}$, and $2U_p \cap E = \emptyset$, where $2U_p$ is $B_2$ in the above local coordinate system.
- For each $a \in E$, $U_a \cap E = \{a\}$.

Let $u$ be a measurable function on $X \setminus E$ such that
$$
\|u\|_{L^\infty} < \infty, \quad \|du\|_{L^\infty} < \infty, \quad \|\Delta pu\|_{L^\infty} < \infty. \quad (5.25)
$$

We need to prove that $u$ belongs to $H^1(\mu)$. Since $\eta$ is locally bounded from above by strictly positive constants on $X \setminus E$. It is uniformly bounded from above and below by strictly positive constants on $\bigcup_{p \in I \setminus E} 2U_p$. Consequently, using a partition of unity subordinate to $\mathcal{U}$ and (5.25) we may assume that $u$ is compactly supported in a single $U_p$. There are two cases to consider.

Case $p \notin E$. 

\[\text{ Springer}\]
So $u$ is compactly supported in a flow box $U_p$. For $n \geq 1$ consider the convolution $u_n := u \star \chi_{1/n}$, where

$$(u \star \chi_{1/n})(z) := \int_{C^k} u(z - w) \chi_{1/n}(w) d\text{vol}(w) \quad \text{for} \quad z \in C^k.$$ 

Observe that $u_n \in D(F)$. Since $\eta$ is uniformly bounded from above and below by strictly positive constants on $2U_p$, we infer from (5.25) that $\|u_n - u\|_{H^1(\mu)} \to 0$ as $n$ tends to infinity. Hence, $u$ belongs to $H^1(\mu)$.

**Case $p \in E$.**

Let $a$ be the unique point $E \cap U_p$. Using the local coordinate system associated to $U_p$ we may assume that $a = 0$ and $U_p = B$. So $u$ is compactly supported in $B$. For $0 < \epsilon < 1$ consider the function $v_\epsilon : C^k \to \mathbb{R}$ defined by

$$v_\epsilon(z) := (1 - \chi(\epsilon^{-1}z))u(z) \quad \text{for} \quad z \in C^k.$$ 

Observe that $v_\epsilon = u$ outside $B_\epsilon$, $v_\epsilon = 0$ on $B_\epsilon/2$, and $|u - v_\epsilon| \leq |u|$ everywhere. Moreover, it follows from (5.25) that $v_\epsilon$ also satisfies (5.25) for every $0 < \epsilon < 1$. Since the support of $v_\epsilon$ does not meet $E$, we can argue as in Step 1 to prove that $v_\epsilon \in H^1(\mu)$ for every $0 < \epsilon < 1$. So the proof of the membership $u \in H^1(\mu)$ will follow if one can show that

$$\lim_{\epsilon \to 0} \|u - v_\epsilon\|_{H^1(\mu)} = 0. \quad (5.26)$$

On the one hand, since $\int_X |u|^2 d\mu < \infty$ and $\mu(\{a\}) = 0$, we get

$$\int_X |u - v_\epsilon|^2 d\mu = \int_{B_\epsilon} |u - v_\epsilon|^2 d\mu \leq \int_{B_\epsilon} |u|^2 d\mu \to 0 \quad \text{as} \quad \epsilon \to 0.$$ 

On the other hand, we have that

$$\int_X |\nabla(u - v_\epsilon)|^2 d\mu = \int_X i\partial(u - v_\epsilon) \wedge \overline{\partial}(u - v_\epsilon) \wedge T = \int_{B_\epsilon} i\partial \left( u(z) \chi(\epsilon^{-1}z) \right) \wedge \overline{\partial} \left( u(z) \chi(e^{-1}z) \right) \wedge T(z).$$ 

The integral on the right hand side is dominated by a constant times

$$\int_{B_\epsilon} i\partial u \wedge \overline{\partial} u \wedge T + \epsilon^{-2} \int_{B_\epsilon \setminus B_\epsilon/2} |u|^2 T \wedge \beta + \epsilon^{-1} \int_{B_\epsilon \setminus B_\epsilon/2} |du| \cdot T \wedge \beta =: I_1 + I_2 + I_3.$$ 

Here $\beta := dd^c \|z\|^2$ (see the notation in Proposition 2.15). By (5.25), $\|u\|_{L^\infty} < \infty$ and $\|\nabla u\|_{L^\infty} < \infty$. Moreover, $\mu(\{a\}) = 0$. So

$$I_1 = \int_{B_\epsilon} |\nabla u|^2 d\mu \lesssim \int_{B_\epsilon} d\mu \to 0 \quad \text{as} \quad \epsilon \to 0.$$ 

On the other hand, by Proposition 2.15 and Definition 2.16, we have

$$I_2 \lesssim \epsilon^{-2} \int_{B_\epsilon \setminus B_\epsilon/2} T \wedge \beta \to 0 \quad \text{as} \quad \epsilon \to 0,$$

because by the hypothesis $v(T, a) = 0$. By (4.3) and (5.5), we get that

$$|du(z)| = \frac{|du(z)|_p}{\eta(z)} = \frac{|\nabla u(z)|}{\eta(z)} \quad \text{for} \quad z \in B.$$
By the hypothesis, $\eta(z) \gtrsim c\epsilon$ for $z \in \mathbb{B}_\epsilon \setminus \mathbb{B}_{\epsilon/2}$. Therefore, we argue as in estimating $I_2$ that for $z \in \mathbb{B}_\epsilon \setminus \mathbb{B}_{\epsilon/2}$,

$$I_3 \lesssim \epsilon^{-2} \int_{\mathbb{B}_\epsilon} T \wedge \beta \to 0 \quad \text{as} \quad \epsilon \to 0.$$  

The proof of (5.26) is thereby completed. This finishes the proof of the proposition.

**Corollary 5.23** If $\mathcal{F} = (X, \mathcal{L}, E)$ is a Brody hyperbolic compact singular holomorphic foliation. Suppose that all the singularities are linearizable as well as weakly hyperbolic. Then for every positive quasi-harmonic measure, the abstract heat diffusions coincide with the leafwise heat diffusions. In particular, the abstract heat diffusions are unique, i.e., they are independent of the quasi-harmonic measures.

**Proof** Since the foliation $\mathcal{F}$ is compact with only linearizable singularities. By Proposition 4.8, the mass of every positive quasi-harmonic measure $\mu$ is finite. Therefore, by Theorem 2.9 (3), every positive quasi-harmonic measure $\mu$ is harmonic. Let $T$ be the directed positive harmonic current given by (5.2). So $T$ is positive $dd^c$-closed current of bidimension $(1,1)$ on $X$. Since all the singularities are linearizable as well as weakly hyperbolic, it follows from Theorem 4.9 that $\nu(T,a) = 0$ for $a \in E$. Moreover, as all the singularities are linearizable, we know by Proposition 4.5 that $\eta(x) \approx \text{dist}(x,E) \log^* \text{dist}(x,E) \gtrsim \text{dist}(x,E)$. Hence, the result follows from Proposition 5.22.

**Problem 5.24** Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a singular holomorphic foliation, where $E$ is a subvariety of $X$ with $\text{codim}_X(E) \geq 2$. Let $T$ be a directed positive harmonic current giving no mass to $\text{Par}(\mathcal{F}) \cup E$. Find sufficient conditions for $T$ and $E$ so that the measure $\Phi(T)$ given by (2.11) is finite (that is, by Theorem 2.9 (3), $\Phi(T)$ is harmonic). Moreover, when the measure $\Phi(T)$ is harmonic, find sufficient conditions for $T$ and $E$ so that the abstract heat diffusions coincide with the leafwise heat diffusions.

**Problem 5.25** In this subsection we have studied the problem of unique heat diffusions for the completion of the space $\mathcal{D}(\mathcal{F})$ with respect to the norm $\| \cdot \|_{H^1(\mu)}$. It seems to be of interest to study the problem for other spaces of test functions with respect to other norms.

### 5.3 Heat Equation on Holomorphically Immersed Riemann Surface Laminations

Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a Riemann surface laminations with singularities which is holomorphically immersed in a complex manifold $M$ (see Definition 2.18). For simplicity, fix a Hermitian form $g_M$ on $M$. Let $T$ be a positive $dd^c$-closed current of bidimension $(1,1)$ in $M$, $T$ is not necessarily directed by $\mathcal{F}$. So, $T \wedge g_M$ is a positive measure in $M$. We assume that there is a $(1,0)$-form $\tau$ defined almost everywhere with respect to $T \wedge g_M$ such that $\partial T = \tau \wedge T$. In this context, the Hörmander $L^2$-estimates are proved in [2] for the $\overline{\partial}$-equation induced on $T$.

We also assume that $T \wedge g_M$ is absolutely continuous with respect to $T \wedge g_P$. In particular, this implies implicitly that $T$ is supported in $X$ and $T$ does not give mass to $\text{Par}(\mathcal{F}) \cup E$. This condition does not depend on the choice of $g_M$ and allows us to define the operators $\nabla^\partial_P$, $\nabla^\tau_P$ and $\nabla_P$ on $u \in \mathcal{D}(M)$ by

$$(\nabla^\partial_P u) T \wedge g_P := i \bar{\partial} u \wedge \tau \wedge T = i \partial (u \bar{\partial} T), \quad (\nabla^\tau_P u) T \wedge g_P := -i \partial u \wedge \tau \wedge T = -i \partial (u \partial T).$$
and
\[ \nabla := \nabla^\partial + \nabla^\bar{\partial}. \]
Define also the operators \( \Delta_\rho \) and \( \tilde{\Delta}_\rho \) on \( u \in \mathcal{D}(M) \) by
\[ (\Delta_\rho u)T \wedge g_P := i\partial \bar{\partial} u \wedge T \quad \text{and} \quad \tilde{\Delta}_\rho := \Delta_\rho + \frac{1}{2} \nabla. \] (5.27)
Here, for the definition of \( \Delta_\rho \) in the first equation of (5.27) we have used the assumption that \( T \wedge g_M \) is absolutely continuous with respect to \( T \wedge g_P \). We will extend the definition of \( \Delta_\rho \) to larger spaces, suitable for developing \( L^2 \)-techniques. To this end, consider the measure \( \mu \) on \( X \) defined by
\[ \mu := T \wedge g_P \text{ on } \text{Hyp}(\mathcal{F}) \quad \text{and} \quad \mu := 0 \text{ on } \text{Par}(\mathcal{F}) \cup E. \] (5.28)
Finally, we assume that the measure \( \mu \) is finite, or equivalently, the Poincaré mass of \( T \) is finite (see Definition 4.7). Consider the Hilbert space \( L = L^2(T) := L^2(\mu) \). We also introduce the Hilbert space \( H = H^1(T) \subset L^2(T) \) associated with \( T \) as the completion of \( \mathcal{D}(M) \) with respect to the Dirichlet norm
\[ \|u\|_{H^1(T)}^2 := \int |u|^2 T \wedge g_P + i \int \partial u \wedge \bar{\partial} u \wedge T. \]
Since \( \mathcal{D}(M \setminus E) \subset \mathcal{D}(M) \), we infer from the definition of \( H^1(T) \) and (5.28) and (5.3) that if \( T \) is directed by \( \mathcal{F} \) then \( H^1(\mu) \subset H^1(T) \).

Observe the operators \( \nabla^\partial, \nabla^\bar{\partial} \) and \( \nabla \) are defined on \( H^1(T) \) with values in \( L^1(T) \).
Define also for \( u,v \in \mathcal{D}(M) \) (for simplicity, we only consider real-valued functions)
\[ q(u,v) := -\int (\Delta_\rho u)vT \wedge g_P, \quad e(u,v) := q(u,v) + \int uvT \wedge g_P \]
and
\[ \tilde{q}(u,v) := -\int (\tilde{\Delta}_\rho u)vT \wedge g_P, \quad \tilde{e}(u,v) := \tilde{q}(u,v) + \int uvT \wedge g_P. \]
We will define later the domain of \( \Delta_\rho \) and \( \tilde{\Delta}_\rho \) which allows us to extend these identities to more general \( u \) and \( v \). The following lemma also holds for \( u, v \) in the domain of \( \Delta_\rho \) and \( \tilde{\Delta}_\rho \).

**Lemma 5.26** We have for \( u,v \in \mathcal{D}(M) \),
\[ \tilde{q}(u,v) = \Re \int i\partial \bar{u} \wedge \bar{v} \wedge T \quad \text{and} \quad \int (\tilde{\Delta}_\rho u)vT \wedge g_P = \int u(\tilde{\Delta}_\rho v)T \wedge g_P. \]
In particular, \( \tilde{q}(u,v) \) and \( \tilde{e}(u,v) \) are symmetric in \( u,v \) and
\[ \int (\tilde{\Delta}_\rho u)T \wedge g_P = \int (\Delta_\rho u)T \wedge g_P = \int (\nabla_\rho u)T \wedge g_P = 0 \quad \text{for } u \in \mathcal{D}(M). \]

**Proof** Since \( T \) is \( dd^c \)-closed, the integral of \( i\partial \bar{\partial}(u^2) \wedge T \) vanishes. We deduce using Stokes’ formula that
\[ \tilde{q}(u,v) = -\int (\Delta_\rho u + \frac{1}{2} \nabla_\rho u) \wedge g_P = -\int i\partial \bar{\partial} u \wedge vT - \Re \int i\partial u \wedge \bar{\partial} \wedge vT \]
\[ = -\Re \int i\partial \bar{\partial} u \wedge vT - \Re \int i\partial u \wedge \bar{\partial} \wedge vT \]
\[ = \Re \int i\partial u \wedge [\bar{\partial}(vT) - v\bar{\partial}T] = \Re \int i\partial u \wedge \bar{\partial} v \wedge T. \]
This gives the first identity in the lemma. We also have since $T$ is $dd^c$-closed
\[ \int (\nabla_p u) T \wedge g_P = 2\text{Re} \int i \partial u \wedge \bar{\partial} T = 2\text{Re} \int -i u \partial \bar{\partial} T = 0. \]
The other assertions are obtained as in Lemma 5.4. \qed

**Definition 5.27** Let $T$ be a positive $dd^c$-closed bidimension $(1, 1)$-current on $M$ such that there is a $(1, 0)$-form $\tau$ defined almost everywhere with respect to $T \wedge g_M$ such that $\partial T = \tau \wedge T$. Then $T$ is called **Poincaré-regular** if there is a constant $c > 0$ such that $i \tau \wedge \bar{\tau} \wedge T \leq c \cdot T \wedge g_P$.

The following result gives a typical example of Poincaré-regularity.

**Proposition 5.28** Let $T$ be a positive $dd^c$-closed current directed by $F$. Assume that $T$ does not give mass to $\text{Par}(F)$. Then
1. $T \wedge g_M$ is absolutely continuous with respect to $\mu$ defined by (5.28).
2. $T$ is Poincaré-regular.
3. Let $G$ be a singular holomorphic foliation on $M$ such that the restriction of $G$ on $X \setminus E$ induces $F$. If all points of $E$ are linearizable singularities of $G$, then the Poincaré mass of $T$ is finite.

**Proof** In the decomposition (5.1) of $T$ in a flow box $U \simeq \mathbb{B} \times \mathbb{T}$, we can restrict $v$ in order to assume that $h_a \neq 0$ for $v$-almost every $a$. Assertion (1) follows easily from this decomposition.

We turn to assertion (2). Using again the decomposition (5.1) of $T$ in a flow box $U \simeq \mathbb{B} \times \mathbb{T}$, we see that $\tau = h_a^{-1} \partial h_a$ on the plaque passing through $a \in \mathbb{T}$ for $v$-almost every $a$. Then by the proof of [51, Proposition 3], we get
\[ i \tau \wedge \bar{\tau} \wedge T \leq T \wedge g_P. \tag{5.29} \]
Hence, assertion (2) follows (with $c = 1$ in Definition 5.27).

For the reader’s convenience, we give here a direct proof of (5.29). The following elementary result is needed.

**Lemma 5.29** Let $h$ be a positive harmonic function on $\mathbb{D}$. Then
\[ i \partial h(0) \wedge \bar{\partial} h(0) \leq h(0)^2 \cdot id\xi \wedge d\bar{\xi}. \]

**Proof of Lemma 5.29** We may assume without loss of generality that $h(0) = 1$. Since $h$ is positive harmonic, we see easily that $h$ belongs to the Hardy space $H^1(\mathbb{D})$. So there is a finite positive Borel measure $\lambda$ on $\partial \mathbb{D}$ such that $u$ is the Poisson integral of $\lambda$, that is,
\[ h(z) = \int_{\partial \mathbb{D}} \frac{1 - |z|^2}{|z - \xi|^2} \, d\lambda(\xi) \quad \text{for} \quad z \in \mathbb{D}. \]

Acting the derivative $\partial$ and then evaluating both sides at $z = 0$, we see easily that
\[ |\partial h(0)| \leq \int_{\partial \mathbb{D}} d\lambda(\xi) = h(0). \]
This implies the desired estimate. \qed
Now we come back the proof of (5.29). We may assume without loss of generality that $a \in \text{Hyp}({\mathcal{F}})$. Fix a point $x$ in the plaque $V_a := B \times \{a\}$, we only need to show that

$$\left(\frac{i \partial h_a \wedge \bar{\partial} h_a}{h_a^2}\right)(x) \leq g_P(x).$$

For every $R > 0$, covering $\phi_x(D_R)$ by a finite number of flow boxes, we can show that there is a positive harmonic function $h$ on $D_R$ such that $h \circ \phi_x = h$ on $D_R \cap \phi_x^{-1}(V_a)$, where $\phi_x$ is given in (2.1). Let $0 < r < 1$ be such that $r D = D_R$ by (5.30). Applying Lemma 5.29 to $h$ defined on $D_R$ and pushing forward, via $\phi_x$, the result on $\phi_x(D_R)$, we get that

$$\frac{i \partial h_a(x) \wedge \bar{\partial} h_a(x)}{h_a^2(x)} \leq \frac{1}{r^2} g_P(x).$$

Letting $R$ tend to infinity, we get $r \to 1$, and hence the last inequality implies the desired estimate.

Assertion (3) is a consequence of Proposition 4.8. \qed

We have the following lemma.

**Lemma 5.30** Assume that $T$ is of finite Poincaré mass. Then the bilinear forms $\tilde{q}$ and $\tilde{e}$ extend continuously to $H^1(T) \times H^1(T)$. If moreover $T$ is Poincaré-regular, then the same property holds for $q$ and $e$. Moreover, we have $q(u, u) = \tilde{q}(u, u)$ and $e(u, u) = \tilde{e}(u, u)$ for $u \in H^1(T)$.

**Proof** The first assertion is deduced from Lemma 5.26. Assume that $T$ is Poincaré-regular. Using Cauchy-Schwarz’s inequality, we have for $u, v \in \mathcal{D}(M)$

$$|q(u, v) - \tilde{q}(u, v)|^2 \leq \left|\int i \partial u \wedge \bar{v} \tau \wedge T\right|^2 \leq \left(\int i \partial u \wedge \bar{\partial} u \wedge T\right) \left(\int v^2 \tau \wedge \bar{\tau} \wedge T\right) \leq c \left(\int i \partial u \wedge \bar{\partial} u \wedge T\right) \left(\int v^2 g_P \wedge T\right) \leq c \|u\|^2_{H^1(T)} \|v\|^2_{L^2(T)},$$

where, in the third line we use the Poincaré regularity of $T$, and in the last line we use the finiteness of the Poincaré mass $\int g_P \wedge T < \infty$. This implies the second assertion. We also have for $u \in \mathcal{D}(M)$

$$q(u, u) - \tilde{q}(u, u) = \text{Re} \int (\nabla_p^2 u) u T \wedge g_P = \text{Re} \int i \partial u \wedge u \bar{\tau} \wedge T = \frac{1}{2} \text{Re} \int i u^2 \partial \bar{T} = 0.$$  

The identity $e(u, u) = \tilde{e}(u, u)$ follows readily from the last equality. \qed

**Remark 5.31** Define the domain $\text{Dom}(\pm \tilde{\Delta}_p)$ of $\pm \tilde{\Delta}_p$ (resp. $\text{Dom}(\pm \Delta_P)$ of $\pm \Delta_P$ when $T$ is Poincaré-regular) as the space of $u \in H^1(T)$ such that $\tilde{q}(u, \cdot)$ (resp. $q(u, \cdot)$) extends to a linear continuous form on $L^2(T)$. When $T$ is Poincaré-regular, we have seen in the proof of Lemma 5.30 that $q(u, v) - \tilde{q}(u, v)$ is continuous on $H^1(T) \times L^2(T)$. Therefore, we deduce from this discussion and Definition 5.6 and Remark 5.7 that

$$\text{Dom}(\pm \Delta_P) = \text{Dom}(\pm \tilde{\Delta}_p) = H_P(T).$$
where $H_p(T)$ is the completion of $\mathcal{D}(M)$ for the norm
\[ \|u\|_{H_p(T)} := \sqrt{\|u\|_{L^2(T)}^2 + \|\Delta_p u\|_{L^2(T)}^2}. \]
If moreover $T$ is directed by $\mathcal{F}$, then we deduce from the inclusion $\mathcal{D}(M \setminus E) \subset \mathcal{D}(M)$ and (5.6) that $H_p(\mu) \subset H_p(T)$.

The following result can be proved in the same way as Proposition 5.8.

**Proposition 5.32** Let $T$ be a positive $dd^c$-closed current of bidimension $(1, 1)$ in a complex manifold $M$ such that $T \wedge g_M$ is absolutely continuous with respect to $T \wedge g_P$ and the Poincaré mass of $T$ is finite. Then the associated operator $-\tilde{\Delta}_p$ (resp. $-\Delta_p$ when $T$ is Poincaré-regular) is maximal monotone on $L^2(T)$. In particular, it is the infinitesimal generator of semi-groups of contractions on $L^2(T)$ and its graph is closed.

The following result is an ergodic theorem associated to the abstract heat diffusions.

**Theorem 5.33** (Dinh-Nguyen-Sibony [32]) We keep the hypothesis of Proposition 5.32. Let $S(t)$, $t \in \mathbb{R}^+$, denote the semi-group of contractions associated with the operator $-\tilde{\Delta}_p$ (or $-\Delta_p$ if $T$ is Poincaré-regular) which is given by Theorem 5.1. Then
(1) The measure $\mu$ is $S(t)$-invariant (that is, $\langle S(t)u, \mu \rangle = \langle u, \mu \rangle$ for every $u \in L^p(T)$), and $S(t)$ is a positive contraction in $L^p(T)$ for all $1 \leq p \leq \infty$ (that is, $\|S(t)u\|_{L^p(T)} \leq \|u\|_{L^p(T)}$ for every $u \in L^p(T)$);
(2) For all $u_0 \in L^p(T)$, $1 \leq p < \infty$, the average
\[ \frac{1}{R} \int_0^R S(t)u_0 dt \]
converges pointwise $\mu$-almost everywhere and also in $L^p(T)$ to an $S(t)$-invariant function $u_0^*$ when $R$ goes to infinity. Moreover, $u_0^*$ is constant on the leaf $L_a$ for $\mu$-almost every $a$. If $T$ is an extremal directed current, then $u$ is constant $\mu$-almost everywhere.

The following result gives us the mixing for the operator $-\tilde{\Delta}_p$.

**Theorem 5.34** Under the hypothesis of Theorem 5.33, if $S(t)$ is associated with $-\tilde{\Delta}_p$ and if $T$ is extremal directed, $S(t)u_0 \to \langle \mu, u_0 \rangle$ in $L^p(T)$ when $t \to \infty$ for every $u_0 \in L^p(T)$ with $1 \leq p < \infty$.

The following result is similar to Proposition 5.14.

**Proposition 5.35** Let $T$ be an extremal directed positive $dd^c$-closed current of finite Poincaré mass. Then, the closures of $\Delta_p(\mathcal{D}(M))$ and of $\tilde{\Delta}_p(\mathcal{D}(M))$ in $L^p(T)$, $1 \leq p \leq 2$, are the hyperplane of $L^p(T)$ defined by the equation $\int vT \wedge g_P = 0$.

### 5.4 Geometric Ergodic Theorems

In this subsection, we will give an analogue of Birkhoff’s ergodic theorem in the context of a Riemann surface lamination $\mathcal{F} = (X, \mathcal{L}, E)$ with singularities. Our ergodic theorem is of geometric nature and it is close to Birkhoff’s averaging on orbits of a dynamical system. Here the averaging is on hyperbolic leaves and the time is the hyperbolic time.
Recall from the Main notation that for $0 < r < 1$, $r \mathbb{D}$ denotes the disc of center 0 and of radius $r$. In the Poincaré disc $(\mathbb{D}, g_P)$, $r \mathbb{D}$ is also the disc of center 0 and of radius $R := \log \frac{1 + r}{1 - r}$, which is also denoted by $\mathbb{D}_R$. So $r \mathbb{D} = \mathbb{D}_R$.

Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a Riemann surface lamination with singularities. Let $T$ be a positive harmonic current directed by $\mathcal{F}$. Assume that $T$ has no mass on $\text{Par}(\mathcal{F})$. Consider the positive measure $\mu = \Phi(T)$ on $X$ defined by (2.11). Assume in addition that $\mu$ is a probability measure.

For any point $x \in \text{Hyp}(\mathcal{F})$, let $\phi_x : \mathbb{D} \to L_x$ be given by (2.1). Following Fornæss-Sibony [49, 50], consider, for all $0 < R < \infty$, the measure $m_{x,R}$ and the current $\tau_{x,R}$ given by

$$m_{x,R} := \frac{1}{M_R} (\phi_x)_* \left( \log^+ \frac{r}{|\zeta|} g_P \right), \quad \tau_{x,R} := \frac{1}{M_R} (\phi_x)_* \left( \log^+ \frac{r}{|\zeta|} \right).$$

Here, $\log^+ := \max\{\log, 0\}$, $g_P$ denotes as usual the Poincaré metric on $\mathbb{D}$ and

$$M_R := \int \log^+ \frac{r}{|\zeta|} g_P = \int \log^+ \frac{r}{|\zeta|} \frac{2}{(1 - |\zeta|^2)^2} i d\zeta \wedge d\overline{\zeta}.$$

So, $m_{x,R}$ (resp. $\tau_{x,R}$) is a probability measure (resp. a directed positive current of bidimension $(1, 1)$) which depends on $x, R$, but does not depend on the choice of $\phi_x$. The current $\tau_{x,R}$ is called the Nevanlinna current of index $r$. See Fig. 5 for an illustration of the restriction of the universal covering map $\phi_x : \mathbb{D}_R \to \phi_x(\mathbb{D}_R) \subset L_x$; this map is, in general, finite-to-one.

**Theorem 5.36** (Dinh-Nguyen-Sibony [32]) We keep the above hypothesis and notation. Assume in addition that the current $T$ is extremal. Then for $\mu$-almost every point $x \in X$, the measure $m_{x,R}$ defined above converges to $\mu$ when $R \to \infty$.

To prove the theorem, our main ingredient is a delicate estimate on the heat kernel of the Poincaré disc (inequality (5.32) below, see also [32, p. 370, line 8-]). This estimate allows

![Fig. 5](image.png)

Fig. 5 Consider the restriction of the universal covering map $\phi_x : \mathbb{D}_R \to \phi_x(\mathbb{D}_R) \subset L_x$. The set of preimages of a point $a$ (resp. $b$) in $\phi_x(\mathbb{D}_R)$ is $\{a_1, a_2, a_3\}$ (resp. $\{b_1, b_2, b_3\}$)
us to deduce the desired result from the ergodic theorem associated with the abstract heat diffusions (Theorem 5.9). The remainder of the subsection is devoted to an outline of our proof.

For every $0 < R < \infty$, we introduce the operator $B_R$ on $L^1(\mu)$ by

$$B_Ru(a) := \frac{1}{M_R} \int_{|\xi| < 1} \log^+ \frac{r}{|\xi|}(\phi_a)^*(u g_P) = \langle m_{a,R}, u \rangle.$$

Note that for $u \in L^1(\mu)$, the function $B_Ru$ is defined $\mu$-almost everywhere. So, the convergence in Theorem 5.36 is equivalent to the convergence $B_Ru(a) \to \langle \mu, u \rangle$ for $u$ continuous and for $\mu$-almost every $a$.

**Proposition 5.37** Under the hypothesis of Theorem 5.36, for every $u \in L^1(\mu)$, we have

$$\int (B_Ru) d\mu = \int ud\mu.$$

In particular, $B_R$ is positive and of norm 1 in $L^p(\mu)$ for all $1 \leq p < \infty$.

**Proof** Fix an $R > 0$. The positivity of $B_R$ is clear. Since $B_R$ preserves constant functions, its norm in $L^p(\mu)$ is at least equal to 1. It is also clear that $B_R$ is an operator of norm 1 on $L^\infty(\mu)$. If $B_R$ is of norm 1 on $L^1(\mu)$, by interpolation [94], its norm on $L^p(\mu)$ is also equal to 1. So, the second assertion is a consequence of the first one. Observe that as in the proof of Theorem 2.9 (3), the operator $B_R$ can be obtained as an average of the operators $A_t$ on $t \leq R$, where $A_t$ is given by (2.12). So the first assertion follows from Lemma 2.10.

**Remark 5.38** In the above proposition which relies on Lemma 2.10, we make an essential use of the assumption that the current $T$ is directed.

We have the following ergodic theorem.

**Theorem 5.39** Under the hypothesis of Theorem 5.36, if $u$ is a function in $L^p(\mu)$, with $1 \leq p < \infty$, then $B_Ru$ converge in $L^p(\mu)$ towards a constant function $u^*$ when $R \to \infty$.

**Proof** We show that it is enough to consider the case where $p = 1$. By Proposition 5.37, it is enough to consider $u$ in a dense subset of $L^p(\mu)$, e.g., $L^\infty(\mu)$. For $u$ bounded, we have $\|B_Ru\|_\infty \leq \|u\|_\infty$. Therefore, if $B_Ru \to u^*$ in $L^1(\mu)$ we have $B_Ru \to u^*$ in $L^p(\mu)$, $1 \leq p < \infty$.

Now, assume that $p = 1$. Since $B_R$ preserves constant functions, by Proposition 5.14 applied to $p = 1$, it is enough to consider $u = \Delta_P v$ with $v \in \mathcal{D}(\mathcal{F})$. We have to show that $B_Ru$ converges to 0. Note that since $v$ is in $\mathcal{D}(\mathcal{F})$, the function $\Delta_P v$ is defined at every point on $\text{Hyp}(\mathcal{F})$ by the formula $(\Delta_P v)g_P := i \partial \bar{\partial} v$ on the leaves (see formulas (2.5) and (2.7)).

We deduce from (5.31) that $m_{a,R} = \tau_{a,R} \wedge g_P$ and

$$B_Ru(a) = B_R(\Delta_P v)(a) = \langle \tau_{a,R}, (\Delta_P v)g_P \rangle = \langle \tau_{a,R}, i \partial \bar{\partial} v \rangle = \langle i \partial \bar{\partial} \tau_{a,R}, v \rangle.$$

Now we show that the mass of $i \partial \bar{\partial} \tau_{a,R}$ tends to 0 uniformly on $a$. Indeed, by Jensen’s formula, we have that

$$M_R \cdot i \partial \bar{\partial} \tau_{a,R} = i \partial \bar{\partial}(\phi_a)^*(\log^+ \frac{r}{|\xi|}) = (\phi_a)^*(v_r) - \delta_a,$$
where \( \nu_r \) denotes the Lebesgue measure on the circle \( \partial(rD) \) which is the circle with center \( 0 \) and radius \( r \), and \( \delta_a \) is the Dirac mass at \( a \). Since a direct computation shows that \( M_R - 2\pi R = O(1) \), we get \( M_R \to \infty \). Hence, it is easy to see that the mass of \( i\partial\bar{\partial}r_{a,R} \) tends to 0 uniformly on \( a \). The result follows.

**Lemma 5.40** The leafwise heat diffusions \( D_t \) (see (2.3)) extends continuously to an operator of norm 1 on \( L^p(\mu) \) for \( 1 \leq p \leq \infty \). Moreover, there is a constant \( c > 0 \) such that for all \( \epsilon > 0 \) and \( u \in L^1(\mu) \), we have

\[
\mu \{ \tilde{D}u > \epsilon \} \leq c\epsilon^{-1}\| u \|_{L^1(\mu)},
\]

where the operator \( \tilde{D} \) is defined by

\[
\tilde{D}u(a) := \limsup_{R \to \infty} \frac{1}{R} \int_0^R D_t u(a) dt.
\]

**Proof** Recall from (2.3) and (2.4) that \( D_t \) is positive and preserves constant functions. Its norm on \( L^\infty(\mu) \) is equal to 1. On the other hand, by Theorem 2.9 (3), the norm of \( D_t \) on \( L^1(\mu) \) is equal to 1. By interpolation [94], its norm on \( L^p(\mu) \) is also equal to 1. The first assertion follows.

The second one is a direct consequence of Lemma VIII.7.11 in Dunford-Schwartz [47]. This lemma says that if \( D_t \) is a semi-group acting on \( L^1(\mu) \) for some probability measure \( \mu \) such that \( \| D_t \|_{L^1(\mu)} \leq 1 \), \( \| D_t \|_{L^\infty(\mu)} \leq 1 \) and \( t \mapsto D_t u \) is measurable with respect to the Lebesgue measure on \( t \), then

\[
\mu \{ \tilde{D}u > \epsilon \} \leq c\epsilon^{-1}\| u \|_{L^1(\mu)},
\]

where \( \tilde{D} \) is defined as above.

Consider also the operator \( \tilde{B} \) given by

\[
\tilde{B}u(a) := \limsup_{R \to \infty} |BRu(a)|.
\]

We have the following lemma.

**Lemma 5.41** There is a constant \( c > 0 \) such that for all \( \epsilon > 0 \) and \( u \in L^1(\mu) \) we have

\[
\mu \{ \tilde{B}u > \epsilon \} \leq c\epsilon^{-1}\| u \|_{L^1(\mu)}.
\]

**Proof** Since we can write \( u = u^+ - u^- \) with \( \| u \|_{L^1(\mu)} = \| u^+ \|_{L^1(\mu)} + \| u^- \|_{L^1(\mu)} \), it is enough to consider \( u \) positive with \( \| u \|_{L^1(\mu)} \leq 1 \). Write \( u = \sum_{i \geq 0} u_i \) with \( u_i \) positive and bounded such that \( \| u_i \|_{L^1(\mu)} \leq 4^{-i} \). We will show that \( \tilde{D}u_i = \tilde{B}u_i \). This, together with Lemma 5.40 applied to \( u_i \) and to \( 2^{-i-1}\epsilon \) give the result.

So, in what follows, assume that \( 0 \leq u \leq 1 \). We show that \( \tilde{D}u = \tilde{B}u \). This assertion will be an immediate consequence of the following estimate

\[
|BRu(a) - \frac{2\pi}{MR} \int_0^{\frac{2\pi}{M_R}} D_t u(a) dt| \leq cR^{-1/2} \sqrt{\log R} \tag{5.32}
\]

for \( \mu \)-almost every \( a \), where \( c \) is a constant independent of \( u \) and \( a, R \). Observe that the integral in the left hand side of (5.32) can be computed on \( \mathbb{D} \) in terms of \( \hat{u} := u \circ \phi_a \) and
the Poincaré metric $g_P$ on $\mathbb{D}$. So, in order to simplify the notation, we will work on $\mathbb{D}$. We have to show that

$$\left| B_R \tilde{u}(0) - \frac{2\pi}{M_R} \int_0^{\frac{M_R}{2\pi}} D_t \tilde{u}(0) dt \right| \leq c R^{-1/2} \sqrt{\log R},$$

where

$$B_R \tilde{u}(0) := \frac{1}{M_R} \int_{\mathbb{D}} \log \frac{r}{|\xi|} g_P \quad \text{and} \quad D_t \tilde{u}(0) := \int_{\mathbb{D}} p_{\mathbb{D}}(0, \cdot, t) \tilde{u} g_P.$$

In fact, the delicate inequality (5.32) follows from hard estimates based on the following identities for the heat kernel $p_{\mathbb{D}}(x, y, t)$ on the Poincaré disc $(\mathbb{D}, g_P)$ (see (2.6)): this is a positive function on $\mathbb{D}^2 \times \mathbb{R}^+$, smooth when $(x, y)$ is outside the diagonal of $\mathbb{D}^2$, and it satisfies

$$\int_{\mathbb{D}} p_{\mathbb{D}}(x, y, t) g_P(y) = 1 \quad \text{and} \quad \frac{1}{2\pi} \log \frac{1}{|y|} = \int_0^{\infty} p_{\mathbb{D}}(0, y, t) dt.$$

Moreover, the function $p_{\mathbb{D}}(0, \cdot, t)$ is radial (see, e.g., Chavel [22, p. 246]).

**Proof of Theorem 5.36** Let $u$ be a function in $L^1(\mu)$. It is enough to show that $B_R u(a) \to \langle \mu, u \rangle$ for $\mu$-almost every $a$. Since this is true when $u$ is constant, we can assume without loss of generality that $\langle \mu, u \rangle = 0$. Fix a constant $\epsilon > 0$ and define $E_\epsilon(u) := \{ \tilde{B}u \geq \epsilon \}$. To prove the theorem it suffices to show that $\mu(E_\epsilon(u)) = 0$.

By Proposition 5.14, $\Delta_P(\mathcal{D}(\mathcal{F}))$ is dense in the hyperplane of functions with mean 0 in $L^1(\mu)$. Consequently, for every $\delta > 0$ we can choose a smooth function $v$ such that $\|\Delta_P v - u\|_{L^1(\mu)} < \delta$. We have

$$E_\epsilon(u) \subset E_{\epsilon/2}(u - \Delta_P v) \cup E_{\epsilon/2}(\Delta_P v).$$

Therefore,

$$\mu(E_\epsilon(u)) \leq \mu(E_{\epsilon/2}(u - \Delta_P v)) + \mu(E_{\epsilon/2}(\Delta_P v)).$$

We have

$$B_R(\Delta_P v)(a) = \langle \tau_{a, R}, i \partial \bar{\partial} v \rangle = \langle i \partial \bar{\partial} \tau_{a, R}, v \rangle.$$

The last integral tends to 0 uniformly on $a$ since the mass of $i \partial \bar{\partial} \tau_{a, R}$ satisfies this property. Hence, $\mu(E_{\epsilon/2}(\Delta_P v)) = 0$ for $R$ large enough.

On the other hand, by Lemma 5.41, we have

$$\mu(E_{\epsilon/2}(u - \Delta_P v)) = \mu(\tilde{B}(u - \Delta_P v) > \epsilon/2) \leq 2c \epsilon^{-1} \|u - \Delta_P v\|_{L^1(\mu)} \leq 2c \epsilon^{-1} \delta.$$

Since $\delta$ is arbitrary, we deduce from the last estimate that $\mu(E_\epsilon(u)) = 0$. This completes the proof of the theorem.



### 6 Unique Ergodicity Theorems

#### 6.1 Case of Compact Nonsingular Laminations

A general principle for the unique ergodicity of a lamination $\mathcal{F}$ is that there exists only one directed positive harmonic current (up to a multiplicative constant).

Garnett establishes the unique ergodicity of the weak stable foliation of the geodesic flow of a compact surface of constant curvature $-1$ (see [54, Proposition 5]).
Consider a suspension $\mathcal{F}_h$ constructed in Example 2.31. C. Bonatti and X. Gomez-Mont [4] prove that either the group $h(\pi_1(S))$ has an invariant probability measure or the foliation $\mathcal{F}_h$ is uniquely ergodic and this is the generic case. They construct a probability measure on $M_h$ by an appropriate averaging process on the leaves and unique ergodicity means that the averaging process applied to an arbitrary leaf gives always the same limit.

In [30] Deroin and Kleptsyn investigate the case of minimal sets of a singular holomorphic foliation $\mathcal{F} = (X, \mathcal{L}, E)$. Recall that a set $M \subset X \setminus E$ is said to be minimal if it is leafwise saturated closed subset of $X$ which contains no proper subset with this property. Recall also the following Minimal Set Problem.

**Problem 6.1** Does there exist a $\mathcal{F} \in \mathcal{F}_d(P^2)$ with $d > 1$ which has a nontrivial minimal set, i.e., a minimal set which is not an algebraic curve?

This question seems to be asked by C. Camacho in the mid-1980’s, see [15]. It still remains open (see [50, Subsection 5.6] for a recent discussion).

Deroin and Kleptsyn in [30] prove the following result.

**Theorem 6.2** Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a singular holomorphic foliation in a compact complex surface $X$, and $M$ be a minimal set whose leaves are all hyperbolic. Assume that there is no positive closed current directed by $\mathcal{F}$ which is supported on $M$. Then there exists a unique directed positive harmonic current on $M$ (up to a multiplicative constant).

The first main step of their proof is the existence of at least one directed positive harmonic current whose associated Lyapunov exponent is strictly negative. The second step exploits this and the similarities between Brownian motions on different leaves in order to infer the unique ergodicity.

It is worth noting here that all the laminations considered in this subsection are compact nonsingular hyperbolic. We wish to address the unique ergodicity for holomorphic foliations. However, a general holomorphic foliation is often singular. In the presence of singularities, the machinery developed by the previous authors does not work. Let us look at the simple case where the ambient compact complex manifold is simply $P^2$.

### 6.2 Case of $P^2$

In [49] Fornæss and Sibony develop an energy theory for positive $dd^c$-closed currents of bidegree $(1, 1)$ on every compact Kähler manifold $(X, \omega)$ of arbitrary dimension $k \geq 2$. This allows them to define $\int_X T \wedge T \wedge \omega^{k-2}$ for every positive $dd^c$-closed current $T$ of bidegree $(1, 1)$ on $X$. This theory applies to directed positive $dd^c$-closed currents on singular holomorphic foliations on compact Kähler surfaces. A short digression will be presented in Section 7.

In [49, 51] Fornæss and Sibony also develop a geometric intersection theory for directed positive harmonic currents on singular holomorphic foliations on $P^2$.

Combining these two theories, they obtain the following remarkable unique ergodicity result for singular holomorphic foliations without invariant algebraic curves.

**Theorem 6.3** (Fornæss-Sibony [51]) Let $\mathcal{F}$ be a singular holomorphic foliation in $P^2$ whose singularities are all hyperbolic. Assume that $\mathcal{F}$ has no invariant algebraic curve. Then $\mathcal{F}$ has a unique directed positive $dd^c$-closed current of mass 1. Moreover, this unique
current $T$ is not closed. In particular, for every point $x$ outside the singularity set of $\mathcal{F}$, the current $\tau_{x,R}$ defined in (5.31) converges to $T$ when $R \to \infty$.

By [11] (see also Theorem 2.34 (3)), if all the singularities of $\mathcal{F} \in \mathcal{F}_d(\mathbb{P}^2)$ are hyperbolic and $\mathcal{F}$ does not possess any invariant algebraic curve, then $\mathcal{F}$ admits no directed positive closed current. So the conclusion of Theorem 6.3 is a typical property of the family $\mathcal{F}_d(\mathbb{P}^2)$. The proof in [51] is based on two ingredients. The first one is the energy theory for positive $dd^c$-closed currents which we mentioned previously. The second one is a geometric intersection calculus for these currents. For the second ingredient, the transitivity of the automorphism group of $\mathbb{P}^2$ is heavily used. Indeed, they define the geometric intersection of two directed positive $dd^c$-closed currents $T, S$ in $\mathbb{P}^2$ as the positive measure

$$T \wedge S := \lim_{\epsilon \to 0} (T \wedge \Phi^*_\epsilon S),$$

where $\Phi_\epsilon$ is a continuous family of automorphisms Aut($\mathbb{P}^2$) with $\Phi_0$ the identity. Moreover, the proof is quite technical. The computations needed to estimate the geometric intersections are quite involved. Using these techniques, Pérez-Garrandés [82] has studied the case where $X$ is a homogeneous compact Kähler surface.

The case where $\mathcal{F}$ possesses invariant algebraic curves has recently been solved as follows.

**Theorem 6.4** (Dinh-Sibony [45]) Let $\mathcal{F}$ be a singular holomorphic foliation in $\mathbb{P}^2$ whose singularities are all hyperbolic. Assume that $\mathcal{F}$ admits a finite number of invariant algebraic curves. Then any directed positive $dd^c$-closed current is a linear combination of the currents of integration on these curves. In particular, all directed positive $dd^c$-closed currents are closed.

Theorem 6.4 is surprising even in the special case where $\mathcal{F}$ admits the line at infinity $L_\infty$ as an invariant curve. Let $\mathcal{F}$ be a generic foliation of a given degree $d > 1$ with this property. By Khudai-Veronov [61], all leaves (except $L_\infty$) of $\mathcal{F}$ are dense. So by intuition from Theorem 6.3 one could expect that there should be a directed $dd^c$-closed current with the full support $\mathbb{P}^2$. However, Theorem 6.4 says that this intuition is false.

To prove Theorem 6.4 we need to show that if $T$ is a positive $dd^c$-closed current directed by $\mathcal{F}$ having no mass on any leaf, then $T$ is zero. For this purpose, Dinh and Sibony [45] develop a theory of densities of positive $dd^c$-closed $(1, 1)$-currents in a compact Kähler surface. The pioneering theory that laid down the foundation was previously introduced by these authors in [44] in the context of positive closed currents defined on compact Kähler manifolds. Applications of these theories in complex dynamics of higher dimension could be found in [36, 42, 43], etc.

Theorems 6.3 and 6.4 gives the complete dichotomy of the unique ergodicity for singular holomorphic foliations in $\mathbb{P}^2$ with hyperbolic singularities.

**Problem 6.5** Are there any versions of Theorem 6.3 and Theorem 6.4 in $\mathbb{P}^k$ with $k > 2$ when we assume that the singularities are all hyperbolic linearizable?

**Problem 6.6** Find analogous versions of Theorem 6.3 and Theorem 6.4 when the singularities are only linearizable in the case of $\mathbb{P}^2$, and then the general case $\mathbb{P}^k$ with $k > 2$. 
6.3 Case of Compact Kähler Surfaces

Our recent work in collaboration with Dinh and Sibony [35] gives a complete answer to the unique ergodicity for singular holomorphic foliations on general compact Kähler surfaces. Our results also hold for bi-Lipschitz laminations (by Riemann surfaces) (without singularities) in $X$. Recall that such lamination is a compact subset of $X$ which is locally a union of disjoint graphs of holomorphic functions depending in a bi-Lipschitz way on parameters (see Section 7.2 for a precise local description).

Let $H^{1,1}(X)$ denote the Dolbeault cohomology group of real smooth $(1, 1)$-forms on $X$. For a real smooth closed $(1, 1)$-form $\alpha$ on $X$, let $[\alpha]$ be its class in $H^{1,1}(X)$. The cup-product $\cup$ on $H^{1,1}(X) \times H^{1,1}(X)$ is defined by

$$( [\alpha], [\beta] ) \mapsto [\alpha] \cup [\beta] := \int_X \alpha \wedge \beta,$$

where $\alpha$ and $\beta$ are real smooth closed forms. The last integral depends only on the classes of $\alpha$ and $\beta$. The bilinear form $\cup$ is non-degenerate and induces a canonical isomorphism between $H^{1,1}(X)$ and its dual $H^{1,1}(X)^*$ (Poincaré duality). In the definition of $\cup$ one can take $\beta$ smooth and $\alpha$ a current in the sense of de Rham. So, $H^{1,1}(X)$ can be defined as the quotient of the space of real closed $(1, 1)$-currents by the subspace of $d$-exact currents. Recall that an $(1, 1)$-current $\alpha$ is real (resp. $dd^c$-closed) if $\alpha = \bar{\alpha}$ (resp. $dd^c \alpha = 0$).

Assume that $\alpha$ is a real $dd^c$-closed $(1, 1)$-current (this is the case when, for example, $\alpha$ is a directed positive harmonic current (see Theorem 2.9 (1)). Then by the $dd^c$-lemma, the integral $\int_X \alpha \wedge \beta$ is also independent of the choice of $\beta$ smooth and closed in a fixed cohomology class. So, using the above isomorphism, one can associate to such $\alpha$ a class $[\alpha]$ in $H^{1,1}(X)$.

We need to recall some terminology in Kähler geometry. A Kähler form on $X$ is a strictly positive closed smooth $(1, 1)$-form. The Kähler cone of $X$ is the set of the cohomology classes of Kähler forms on $X$. This is a cone in $H^{1,1}(X)$. We say that a cohomology class in $H^{1,1}(X)$ is nef if it belongs to the closure of the Kähler cone of $X$. We say that a cohomology class in $H^{1,1}(X)$ is big if it can be represented by a strictly positive closed (not necessarily smooth) $(1, 1)$-current.

Now we are in the position to state the first result of this subsection.

**Theorem 6.7 ([35, Theorem 1.1])** Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a singular holomorphic foliation with only hyperbolic singularities or a bi-Lipschitz lamination in a compact Kähler surface $(X, \omega)$. Assume that $\mathcal{F}$ admits no directed positive closed current. Then there exists a unique positive $dd^c$-closed current $T$ of mass 1 directed by $\mathcal{F}$. In particular, for an arbitrary point $x \in X \setminus E$, $\tau_{x,R} \to T$ in the sense of currents, as $R \to \infty$, where $\tau_{x,R}$ is the Nevanlinna current defined by (5.31). Moreover, the cohomology class $[T]$ of $T$ is nef and big.

The new idea in the proof of Theorem 6.7 is to introduce a more flexible tool which is a density theory for tensor products of positive $dd^c$-closed currents. It is worth noting that such tensor products are in general not $dd^c$-closed. So these currents go beyond the scope of previous theories of densities [44, 45]. The method allows us to bypass the assumption of homogeneity of $X$, which was frequently used in [51]. The proof is more conceptual and also far less technical. The strategy is as follows. Given a positive $dd^c$-closed current $T$ on a surface $X$, we consider the positive current $T \otimes T$ near the diagonal $\Delta$ of $X \times X$ which, in general, is not $dd^c$-closed. We study the tangent currents to $T \otimes T$ along the diagonal $\Delta$. As one can expect this is related to the self-intersection properties of the current $T$. It turns out
that the geometry of the tangent currents is quite simple. They are positive closed currents and are the pull-back of positive measures $\vartheta$ on $\Delta$ to the normal bundle of $\Delta$ in $X \times X$. We relate the mass of $\vartheta$ to a cohomology class of the current $T$ and its energy.

The foliation or lamination enters in the picture to prove that $\vartheta$ is zero when $T$ is directed by a foliation or lamination as above. This is done using the local properties of the foliation or lamination, the local description of $T$ and in particular, that the singularities are hyperbolic. The vanishing of $\vartheta$ gives easily the uniqueness using a kind of Hodge-Riemann relations.

Note that in Theorem 6.7, the current $T$ is necessarily extremal in the cone of all positive $ddc$-closed currents on $X$. Indeed, if $T'$ is such a current and $T' \leq T$, then $T'$ is necessarily directed by the foliation and according to the theorem, $T'$ is proportional to $T$. Note also that the nef property of $\{T\}$ is a consequence of a general result of independent interest (see Corollary 7.6 below). That corollary is a byproduct of our theory of densities of currents.

The following trichotomy gives us a more complete picture of the strong ergodicity obtained in the study of arbitrary compact Kähler surfaces. For the notion of invariant (closed) analytic curves see Definition 2.24.

**Theorem 6.8** ([35, Theorem 1.2]) Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a singular holomorphic foliation with only hyperbolic singularities or a bi-Lipschitz lamination in a compact Kähler surface $(X, \omega)$. Then one and only one of the following three possibilities occur:

(a) $\mathcal{F}$ admits invariant closed analytic curves and all positive directed $ddc$-closed $(1,1)$-currents are linear combinations, with non-negative coefficients, of the currents of integration on those curves. In particular, these currents are all closed.

(b) $\mathcal{F}$ admits a directed positive closed $(1,1)$-current $T$ of mass 1 having no mass on invariant closed analytic curves (this property holds when there is no such a curve). Every directed positive $ddc$-closed $(1,1)$-current is closed, and if it has no mass on invariant closed analytic curves, then it has no mass on each single leaf and its cohomology class is proportional to $\{T\}$. Moreover, $\{T\}$ is nef and $\{T\}^2 = 0$.

(c) $\mathcal{F}$ admits a unique directed positive $ddc$-closed and non-closed $(1,1)$-current $T$ of mass 1 having no mass on each single leaf. Every directed positive $ddc$-closed $(1,1)$-current is a combination, with non-negative coefficients, of $T$ and the currents of integration on invariant closed analytic curves (if there are any). Moreover, $\{T\}$ is nef and big.

Note that a general theorem by Jouanolou [62] says that either there are only finitely many invariant closed curves, or $\mathcal{F}$ admits a first meromorphic integral. When $\mathcal{F}$ admits a first meromorphic integral, all leaves are invariant closed analytic curves, and hence all directed positive $ddc$-closed currents are closed.

A polynomial vector field in $\mathbb{C}^2$ induces a holomorphic foliation in $\mathbb{P}^2$. When we fix the maximum of the degrees of its coefficients, if the vector field is generic, the line at infinity $L_\infty := \mathbb{P}^2 \setminus \mathbb{C}^2$ is an invariant curve (see Ilyashenko-Yakovenko [61]). The current $\{L_\infty\}$ is the only directed positive $ddc$-closed $(1,1)$-current of mass 1 by Theorem 6.4. So Property (a) holds in that case (see [45] for details and also Rebelo [86] for a related result).

If $\mathcal{F}$ is a smooth fibration on $X$, then the directed positive $ddc$-closed currents are all closed and are generated by the fibers of $\mathcal{F}$. They belong to the same cohomology class which is nef with zero self-intersection. So Property (b) holds in that case. Using a suspension one can also construct examples satisfying Property (b) which are not fibrations, see [52, Ex. 1] and replace the circle there by $\mathbb{P}^1$. In such examples, there are two invariant
closed curves and infinitely many directed positive closed \((1,1)\)-currents of mass 1 having no mass on those curves.

Property (c) implies in particular that for \(x \in X \setminus E\), a cluster point of the limit \(\tau_{x,R}\) as \(R\) tends to infinity is a current of the form \(\lambda T + \sum \lambda_j[V_j]\), where \(\lambda, \lambda_j \in \mathbb{R}^+\) and \(V_j\) are invariant closed analytic curves. Property (c) holds for foliations which are, in some sense, generic. There are many examples of such foliations in \(\mathbb{P}^2\) without invariant closed analytic curves. The cohomology class of the unique directed \(ddc\)-closed \((1,1)\)-current here is Kähler because \(H^2(\mathbb{P}^2, \mathbb{R})\) is of dimension 1. If we blow up the singularities of the foliation, we get examples satisfying the same property and having invariant closed analytic curves. Then, the cohomology class of the unique directed \(ddc\)-closed \((1,1)\)-current is no more Kähler but it is big. In fact, we have the following general result which is a direct consequence of Theorem 6.8.

**Corollary 6.9** Let \(\mathcal{F}\) be a singular holomorphic foliation with only hyperbolic singularities or a bi-Lipschitz lamination in a compact Kähler surface \(X\). Let \(T\) be a positive \(ddc\)-closed current directed by \(\mathcal{F}\) having no mass on invariant closed analytic curves. Then the following properties are equivalent:

1. \(T\) is not closed;
2. \(\{T\}\) is big;
3. \(\{T\}^2 > 0\); and
4. \(\{T\}^2 \neq 0\).

Note that the hyperbolicity of the singularities is necessary for this result. The foliation on \(\mathbb{P}^2\), given on an affine chart by the holomorphic 1-form \(x_2dx_1 - ax_1dx_2\) with \(a \in \mathbb{R}\), admits a non-hyperbolic singularity at 0 as well as diffuse invariant positive closed \((1,1)\)-currents whose cohomology classes are Kähler. See also Corollary 7.6 and Theorem 7.10 below which apply for foliations with arbitrary singularities.

**Remark 6.10** The results in this subsection give a complete answer to Problem 5.8 in our previous survey [79] (see also [28]).

**Problem 6.11** (Generalization of Problem 6.5) Are there any versions of Theorems 6.7 and 6.8 for singular holomorphic foliations with hyperbolic linearizable singularities on compact Kähler manifold \(X\) of dimension \(k > 2\)?

**Problem 6.12** (Generalization of Problem 6.6) Are there any versions of Theorems 6.7 and 6.8 for singular holomorphic foliations with linearizable singularities on compact Kähler manifold \(X\) of dimension \(k \geq 2\)?

### 7 Theory of Energy, Theory of Densities and Strategy for the Proof of the Unique Ergodicity

In this section, we outline the proof of Theorem 6.8. Before doing so, we present the main tools needed for the proof: Fornaess-Sibony theory of energy and our theory of densities for a class of non \(ddc\)-closed currents. We refer the reader to [44, 45] for the case of \(ddc\)-closed currents.

#### 7.1 Energy of Positive \(ddc\)-closed Currents

The following result was obtained by Fornaess-Sibony in [49].
Theorem 7.1 Let $T$ be a positive $dd^c$-closed current on $X$. Then it can be represented as

$$T = \Omega + \partial S + \overline{\partial S} + i \partial \bar{\partial} u$$

(7.1)

with $\Omega$ a smooth real closed $(1,1)$-form, $S$ a current of bidegree $(0,1)$ and $u$ a real function in $L^p$ for $p < 2$. Moreover, for every such a representation, the currents $\overline{\partial S}$ and $\partial \bar{\partial} S$ do not depend on the choice of $\Omega$, $S$, $u$ and they are forms of class $L^2$, uniquely determined by $T$.

Note that the representation (7.1) is not unique but the uniqueness of $\overline{\partial S}$ and $\partial \bar{\partial} S$ and their membership in the space $L^2$ allow Fornæss-Sibony in [49] to define the energy $\mathcal{E}(T)$ of $T$ as

$$\mathcal{E}(T) := \int_X \overline{\partial S} \wedge \partial \bar{\partial} S.$$  

(7.2)

This is a non-negative number which is independent of the choice of $\Omega$, $S$ and $u$. It is not difficult to see that $\mathcal{E}(T) = 0$ if and only if $\overline{\partial S} = 0$, and if and only if $T$ is closed (see [49] for details). The following result improves the regularity of the potential $u$ and its gradients.

Theorem 7.2 (see [35, Proposition B.4]) There is a representation as in (7.1) such that all currents $S$, $\overline{\partial S}$, $\partial \bar{\partial} S$, $\overline{\partial S}$, $\overline{\partial S}$ are forms of class $L^2$ and $u$ is a function of class $L^2$ and $\partial u$, $\bar{\partial} u$ are forms of class $L^p$ for every $1 \leq p < 2$.

The energy theory of Fornæss-Sibony has recently been developed in some new directions (see [31]).

### 7.2 Tangent Currents of Tensor Products of Positive $dd^c$-closed Currents

Let $(X,\omega)$ be a compact Kähler surface. Let $\Delta := \{(x,x) : x \in X\}$ be the diagonal of $X \times X$. We identify a chart of $X$ with the unit ball $\mathbb{B}$ in $\mathbb{C}^2$. On the chart $\mathbb{B} \times \mathbb{B}$ of $X \times X$, we use two local coordinate systems: the first system is the standard one $(x,y)$ and the second system is $(z,w) := (x-y,y)$ on which $\Delta$ is given by the equation $z=0$. The tangent bundles of $X \times X$ and $\Delta$ are denoted by $\text{Tan}(X \times X)$ and $\text{Tan}(\Delta)$. The normal vector bundle of $\Delta$ in $X \times X$ is denoted by $\mathbb{E} := \text{Tan}(X \times X)|_{\Delta}/\text{Tan}(\Delta)$, where $\Delta$ is also identified to the zero section of $\mathbb{E}$. Denote by $\pi : \mathbb{E} \rightarrow \Delta$ the canonical projection. The fiberwise multiplication by $\lambda \in \mathbb{C}^*$ on $\mathbb{E}$ is denoted by $A_{\lambda}$. Over $\Delta \cap (\mathbb{B} \times \mathbb{B})$, with the coordinates $(z,w)$, $\mathbb{E}$ is identified to $\mathbb{C}^2 \times \mathbb{B}$, $\pi$ is the projection $(z,w) \mapsto w$ and $A_{\lambda}$ is equal to the map $a_{\lambda}(z,w) := (\lambda z, w)$.

Consider two positive $dd^c$-closed $(1,1)$-currents $T_1$ and $T_2$ on $X$. We will study the density of $T_1 \otimes T_2$ near the diagonal $\Delta$ of $X \times X$ via a notion of “tangent cone” to $T_1 \otimes T_2$ along $\Delta$ that we introduce now.

Definition 7.3 A smooth admissible map is a smooth bijective map $\tau$ from a neighborhood of $\Delta$ in $X \times X$ to a neighborhood of $\Delta$ in $\mathbb{E}$ such that

1. The restriction of $\tau$ to $\Delta$ is the identity map on $\Delta$; in particular, the restriction of the differential $d\tau$ to $\Delta$ induces a map from $\text{Tan}(X \times X)|_{\Delta}$ to $\text{Tan}(\mathbb{E})|_{\Delta}$; since $\Delta$ is pointwise fixed by $\tau$, the differential $d\tau$ also induces two endomorphisms of $\text{Tan}(\Delta)$ and $\mathbb{E}$ respectively;
2. The differential $d\tau(x,x)$, at each point $(x,x) \in \Delta$, is a $\mathbb{C}$-linear map from the tangent space to $X \times X$ at $(x,x)$ to the tangent space to $\mathbb{E}$ at $(x,x)$;
3. The endomorphism of $\mathbb{E}$, induced by $d\tau$ (restricted to $\Delta$), is the identity map.
Note that the dependence of \( d\tau(x,x) \) in \( (x,x) \in \Delta \) is in general not holomorphic. Consider the exponential map from \( \mathbb{E} \) to \( X \times X \) with respect to any Hermitian metric on \( X \times X \). It defines a smooth bijective map from a neighborhood of \( \Delta \) in \( \mathbb{E} \) to a neighborhood of \( \Delta \) in \( X \times X \). The inverse map is smooth and admissible (see also [44, Lemma 4.2]). See Fig. 6.

Let \( \tau \) be any smooth admissible map as above. Define
\[
(T_1 \otimes T_2)_\lambda := (A_\lambda)_* \tau_*(T_1 \otimes T_2).
\]
This is a current of degree 4. Its domain of definition is some open subset of \( \mathbb{E} \) containing \( \Delta \) which increases to \( \mathbb{E} \) when \( |\lambda| \) increases to infinity.

Observe that \( (T_1 \otimes T_2)_\lambda \) is not a \((2,2)\)-current and we cannot speak of its positivity. Moreover, it is not \( d\dbar \)-closed in general and we cannot speak of its cohomology class. The present situation is more involved than the case where \( T_1 \) and \( T_2 \) are closed because in this case the current \( (T_1 \otimes T_2)_\lambda \) is also closed.

By (7.1) we can write for \( j \in \{1,2\} \),
\[
T_j = \Omega_j + \partial S_j + \overline{\partial S_j} + i\partial \bar{\partial} u_j,
\]
where \( \Omega_j \) is a closed real smooth \((1,1)\)-form, \( S_j \) is a current of bidegree \((0,1)\) and \( u_j \) is a real current of bidegree \((0,0)\). Note that \( \partial \overline{\partial} S_j \) and \( \overline{\partial} S_j \) are forms of class \( L^2 \) which are independent of the choice of \( \Omega_j, S_j, u_j \). It turns out that a crucial argument in the proof of Theorem 7.4 below is a result on the regularity of the potentials \( u_j \) and their gradients (see Theorem 7.2).

One of the main ingredients of the proof is the following theorem.

**Theorem 7.4** ([35, Theorem 2.2]) Let \( T_1 \) and \( T_2 \) be two positive \( d\dbar \)-closed \((1,1)\)-currents on a compact Kähler surface \((X, \omega)\). Assume that \( T_1 \) has no mass on the set \( \{ \nu(T_2, \cdot) > 0 \} \) and \( T_2 \) has no mass on the set \( \{ \nu(T_1, \cdot) > 0 \} \). Then, with the above notations, we have the following properties.

1. The mass of \((T_1 \otimes T_2)_\lambda\) on any given compact subset of \( \mathbb{E} \) is bounded uniformly on \( \lambda \) for \( |\lambda| \) large enough. If \( T \) is a cluster value of \((T_1 \otimes T_2)_\lambda\) when \( \lambda \to \infty \), then it is a positive closed \((2,2)\)-current on \( \mathbb{E} \) given by \( T = \pi^*(\vartheta) \) for some positive measure \( \vartheta \) on \( \Delta \).

---

**Fig. 6** Admissible map \( \tau \) sends a neighborhood \( U \) of \( \Delta \) in \( X \times X \) onto a neighborhood of the zero section \( \Delta \) in \( \mathbb{E} \). The restriction of \( \tau \) to \( \Delta \) is the identity map on \( \Delta \); the differential \( d\tau(x,x) \) at each point \((x,x) \in \Delta \) is a \( \mathbb{C} \)-linear map from \( \text{Tan}(X \times X) \) at \((x,x) \) to \( \text{Tan}(\mathbb{E}) \) at \((x,x) \); and the endomorphism of \( \mathbb{E} \), induced by \( d\tau \) (restricted to \( \Delta \)), is the identity map. For \( \lambda \in \mathbb{C}^* \), the dilatation \( A_\lambda \) is the fiberwise multiplication by \( \lambda \) on \( \mathbb{E} \). Here is the illustration with \( \lambda = 2 \)
Moreover, if $(\lambda_n)$ is a sequence tending to infinity such that $(T_1 \otimes T_2)_{\lambda_n} \to \mathbb{T}$, then $\mathbb{T}$ may depend on $(\lambda_n)$ but it does not depend on the choice of the map $\tau$.

2) (mass formula) The mass of $\vartheta$ does not depend on the choice of $T$ and it is given by

$$\|\vartheta\| = \int_X \Omega_1 \wedge \Omega_2 - \int_X \overline{\partial S}_1 \wedge \overline{\partial S}_2 - \int_X \overline{\partial S}_2 \wedge \overline{\partial S}_1.$$

In particular, if $T_1 = T_2 = T$ with $T = \Omega + \partial S + \overline{\partial S} + i \partial \overline{\partial u}$ as in (7.1), then

$$\|\vartheta\| = \int_X \Omega^2 - 2 \int_X \overline{\partial S} \wedge \overline{\partial S} = \int_X \Omega^2 - 2 \mathcal{E}(T).$$

(7.5)

Note that in general $\mathbb{T}$ is not unique as this is already the case for positive closed currents (see [44] for details). However, the mass formula shows that if one of such currents is zero then all of them are zero. We can now introduce the following notion.

**Definition 7.5** Any current $T$ obtained as in Theorem 7.4 is called a tangent current to $T_1 \otimes T_2$ along the diagonal $\Delta$.

Combining Theorem 7.4 with the works of Berndtsson–Sibony [2], Demailly-Păun [26] and Siu [90], we obtain the following result and refer to McQuillan [72] and Burns–Sibony [13] for some related results in the foliation setting.

**Corollary 7.6** ([35, Corollary 2.4]) Let $T$ be a positive $dd^c$-closed $(1, 1)$-current of a compact Kähler surface $X$. Assume that the set $\{v(T, \cdot) > 0\}$ is of Hausdorff 2-dimensional measure 0. Then the cohomology class $[T]$ of $T$ is nef, and when $T$ is not closed, $[T]$ is also big. In particular, if $T$ is a positive closed $(1, 1)$-current having no mass on proper analytic subsets of $X$, then $[T]$ is nef.

Recall that if a closed subset $Y$ of a complex surface $X$ is laminated by Riemann surfaces, then it admits an open covering $U_j$ and on each $U_j$ there is a homeomorphism $\varphi_j = (h_j, \lambda_j) : U_j \cap Y \to \mathbb{D} \times \mathbb{T}_j$, where $\mathbb{T}_j$ is a locally compact metric space and the maps $\varphi_j^{-1}(z, t)$, with $(z, t) \in \mathbb{D} \times \mathbb{T}_j$, are holomorphic in $z$. Moreover, on their domains of definition, the transition maps have the form

$$\varphi_k \circ \varphi_j^{-1}(z, t) = (h_{jk}(z, t), \lambda_{jk}(t)),$$

where $h_{jk}(z, t)$ is holomorphic with respect to $z$ and $\lambda_{jk}(t)$ do not depend on $z$. We can choose $\mathbb{T}_j$ as the intersection of a holomorphic disc with $Y$ and $\varphi_j$ such that its restriction to $\mathbb{T}_j$ is the canonical map from $\mathbb{T}_j$ to $\{0\} \times \mathbb{T}_j$. With this choice, when all $\varphi_j(z, t)$ are bi-Lipschitz maps, we say that the lamination is bi-Lipschitz.

The following result gives us the vanishing of the tangent currents in the setting of foliations and laminations.

**Theorem 7.7** ([35, Theorem 2.5]) Let $\mathcal{F}$ be either a singular holomorphic foliation with only hyperbolic singularities, or a bi-Lipschitz lamination, in a compact Kähler surface $X$. Then for every positive $dd^c$-closed current $T$ directed by $\mathcal{F}$ which does not give mass to any invariant closed analytic curve, zero is the unique tangent current to $T \otimes T$ along the diagonal $\Delta$. 


The last theorem expresses that the current $T \otimes T$ is not too singular along the diagonal of $X \times X$ as its density along the diagonal is zero. Now we outline the main steps of the proof of Theorem 7.7.

Consider a positive $dd^c$-closed $(1, 1)$-current $T$ directed by $\mathcal{F}$. We can show that if $T$ has positive mass on a leaf, then this leaf is an invariant closed analytic curve of $\mathcal{F}$ (see Theorem 7.10 below). So for Theorem 7.7, we can assume that $T$ has no mass on each single leaf of $\mathcal{F}$. Using Proposition 2.5 and Definition 2.16, a straightforward computation shows that $\nu(T, x) = 0$ for all $x$ outside the singularities of $\mathcal{F}$. Since positive $dd^c$-closed $(1, 1)$-currents have no mass on finite sets, we can apply Theorem 7.4 to the tensor product $T \otimes T$.

Consider a tangent current $T$ to $T \otimes T$ along $\Delta$. With the notation as in the above sections, there is a sequence $\lambda_n$ converging to infinity and a positive measure $\vartheta$ on $\Delta \simeq X$ such that

$$T = \lim_{n \to \infty} (T \otimes T)_{\lambda_n} = \pi^*(\vartheta).$$

We can identify $\vartheta$ with a positive measure on $X$. Recall that by Theorem 7.4 the mass $m$ of $\vartheta$ does not depend on the choice of $T$. Using Definition 7.3 and (7.3) outside the singularities $E$, we can show the following result (see [64] for a related situation).

**Proposition 7.8** ([35, Proposition 4.1]) For every choice of the tangent current $T$, the measure $\vartheta$ is supported on the singularities of $\mathcal{F}$.

For any function or more generally a current $f(s)$, depending on the parameter $s > 0$, we denote the expectation of $f(s)$ on the interval $(0, s]$ by $E(f(s))$. This is the mean value of $f$ on the interval $(0, s]$ which is given by the formula

$$E(f(s)) := s^{-1} \int_0^s f(t) dt.$$

The difficult part in the proof of Theorem 7.7 is the following:

**Proposition 7.9** ([35, Proposition 4.2]) We have

$$\lim_{s \to \infty} E((T \otimes T)_{es}) = 0$$

in a neighborhood of each point $(a, a) \in \Delta$, where $a$ is any singular point of $\mathcal{F}$.

To prove Proposition 7.9, we carry out a very delicate analysis of the current $(T \otimes T)_\lambda$ with $\lambda = e^s$ near the singular point $a$. This leads us to a geometric study of the leaves of $\mathcal{F}$ near the singular point $a$. Here we make an essential use of the hyperbolic nature of the point $a$.

**End of the proof of Theorem 7.7** Let $\mathcal{M}$ be the set of 2-current on $E$ with measure coefficients. We introduce natural semi-distances $\text{dist}_m$ ($m \in \mathbb{N}$) on $\mathcal{M}$ as follows. Let $E_m$ be an increasing sequence of domains in $E$ such that $E_0$ contains the image of the zero section of the vector bundle $E$ and $E_m \subset E_{m+1}$ and $\bigcup_{m \in \mathbb{N}} E_m = E$. Recall from Section 2.4 that $\mathcal{D}^2(E)$ (resp. $\mathcal{D}^2(E_m)$) is the space of 2-forms with compact support of class $C^1$ on $E$ (resp. on $E_m$). We fix a finite atlas of the vector bundle $E$. For $\alpha \in \mathcal{D}^2(E_m)$, let $\|\alpha\|_{C^1}$ be the sum
of $C^1$-norms of the coefficients of $\alpha$ with respect to the restriction of the above atlas to $\mathbb{E}_m$. If $T$ and $S$ are two $2$-currents in $\mathcal{M}$, define for $m \in \mathbb{N}$,

$$\text{dist}_m(T, S) := \sup_{\|\alpha\|_{C^1} \leq 1} |\langle T - S, \alpha \rangle|,$$

where $\alpha$ is a form in $\mathcal{D}^2(\mathbb{E}_m)$.

A current $T \in \mathcal{M}$ may be regarded as a $2$-form with measure coefficients. For $m \in \mathbb{N}$, let $\|T\|_m$ denote the sum of the mass of the variation of the coefficient measures of $T \in \mathcal{M}$ with respect to the restriction of the above atlas to $\mathbb{E}_m$.

We make the following observation. Let $m \in \mathbb{N}$ be a fixed number and $(T_n)$ a sequence in $\mathcal{M}$ such that $\|T_n\|_m$ is uniformly bounded in $n$. If $T_n |_{\mathbb{E}_m}$ converge weakly to $T |_{\mathbb{E}_m}$ as $n$ tends to infinity (i.e., $(T_n, \alpha) \rightharpoonup (T, \alpha)$ for all $\alpha \in \mathcal{D}^2(\mathbb{E}_m)$), then by Arzelà-Ascoli theorem we get that $\text{dist}_m(T_n, T) \to 0$. Conversely, if $\text{dist}_m(T_n, T) \to 0$, then $T_n |_{\mathbb{E}_m}$ converge to $T |_{\mathbb{E}_m}$.

For $\mathcal{A}, \mathcal{B} \subset \mathcal{M}$, let

$$\text{dist}_m(\mathcal{A}, \mathcal{B}) := \inf_{T \in \mathcal{A}, S \in \mathcal{B}} \text{dist}_m(T, S).$$

We say that a family $\mathcal{B} \subset \mathcal{M}$ is bounded if for every $m$, the set $\{\|T\|_m : T \in \mathcal{B}\}$ is bounded. It is clear that if $\mathcal{B}$ is bounded then its closure $\overline{\mathcal{B}}$ in $\mathcal{M}$ with respect to the weak topology is also bounded.

Now consider the family $\mathcal{B} := \{(T_1 \otimes T_2)_\lambda : \lambda \in \mathbb{C}, |\lambda| \geq 1 \} \subset \mathcal{M}$. By Theorem 7.4 (1), $\mathcal{B}$ is a bounded family. So $\overline{\mathcal{B}}$ is also bounded. Let $\mathcal{K}$ be the set of all tangent currents $T$. Also by Theorem 7.4 (1), $\mathcal{K}$ is a closed subset of $\overline{\mathcal{B}}$ and all elements of $\mathcal{K}$ are positive currents on $\mathbb{E}$. So $\mathcal{K}$ is a compact set in $\mathcal{M}$. Moreover, applying Theorem 7.4 (1) yields that

$$\lim_{\lambda \to \infty} \text{dist}_m((T_1 \otimes T_2)_\lambda, \mathcal{K}) = 0 \quad \text{for each} \quad m \in \mathbb{N}.$$

Otherwise, there would be $m \in \mathbb{N}$, $\epsilon > 0$ and a sequence $(\lambda_n) \to \infty$ such that $\text{dist}_m((T_1 \otimes T_2)_{\lambda_n}, \mathcal{K}) \geq \epsilon$ and $(T_1 \otimes T_2)_{\lambda_n} \rightharpoonup T \in \mathcal{K}$ weakly. But by the above observation, the last limit would imply that $\lim_{n \to \infty} \text{dist}_m((T_1 \otimes T_2)_{\lambda_n}, T) = 0$, which shows, in turn, that $\lim_{n \to \infty} \text{dist}_m((T_1 \otimes T_2)_{\lambda_n}, \mathcal{K}) = 0$, this is a contradiction.

For $\lambda = e^{i\theta}$ with $s > 0$, write

$$\mathbf{E}((T \otimes T)_{e^{i\theta}}) = \frac{1}{s} \int_0^s (T \otimes T)_{e^{i\theta}} dt.$$

Since $\mathbf{E}((T \otimes T)_{e^{i\theta}})$ is a convex combination of $(T \otimes T)_{e^{i\theta}}$ for $t \in [0, s]$, it follows that

$$\lim_{s \to \infty} \text{dist}_m(\mathbf{E}((T \otimes T)_{e^{i\theta}}), \mathcal{K}) = 0,$$

where $\mathcal{K}$ is the convex hull of $\mathcal{K}$ in $\mathcal{M}$. As $\mathcal{K}$ is closed, so is $\mathcal{K}$.

Let $T'$ be a limit current of $\mathbf{E}((T \otimes T)_{\lambda})$ when $\lambda > 0$ tends to infinity. We infer from the last limit that $\text{dist}_m(T', \mathcal{K}) = 0$ for every $m$. Hence by the above observation, for every $m \in \mathbb{N}$, there exists $T_m \in \mathcal{K}$ such that $T_m |_{\mathbb{E}_m} = T |_{\mathbb{E}_m}$. It is then easy to see that the Cesàro means $\frac{1}{m} \sum_{j=0}^{m-1} T_j$ tend weakly to $T'$ as $m$ tends to infinity. Since these means belong to $\mathcal{K}$, the current $T'$ belongs also to $\mathcal{K}$. So by Theorem 7.4 (1), we have $T' = \pi^*(\vartheta')$ for some positive measure $\vartheta'$ of mass $m$ on $\Delta \simeq X$. By Propositions 7.8 and 7.9, we have $\vartheta' = 0$. Therefore, we get $m = 0$ and hence, by the mass formula in Theorem 7.4, we have $T = 0$ for any choice of $T$. This proves the vanishing theorem. □
7.3 Sketchy Proof of the Unique Ergodicity for Singular Holomorphic Foliations

The following result holds in a more general setting but we only state it in the case we use (see also [45, 49]). Here, we do not need to assume that the singularities of the foliation are hyperbolic.

**Theorem 7.10** ([35, Theorem 2.6]) Let $T$ be a positive $dd^c$-closed $(1,1)$-current, on a compact Kähler surface $X$, which is directed by a singular holomorphic foliation or by a bi-Lipschitz lamination.

(a) If $T$ has a positive mass on a leaf $L$, then $L$ is a closed analytic curve and $L \setminus L$ is contained in the set of singularities of the foliation. Moreover, we can write $T = T' + T_{\text{an}}$, where $T'$ is a directed positive $dd^c$-closed $(1,1)$-current which is diffuse, i.e., having no mass on each single leaf, and $T_{\text{an}}$ is a finite or countable combination, with non-negative coefficients, of currents of integration on invariant closed analytic curves.

(b) Assume that $T$ gives no mass to any invariant closed analytic curve. Then $T$ is diffuse and its cohomology class $[T]$ is nef. Moreover, $[T]$ is also big when $T$ is not closed.

The first step of our proof consists in proving the following lemma.

**Lemma 7.11** ([35, Lemma 2.7]) Let $\mathcal{F}$ be either a singular holomorphic foliation with only hyperbolic singularities, or a bi-Lipschitz lamination in a compact Kähler surface $(X, \omega)$. Let $T_1$ and $T_2$ be two positive $dd^c$-closed currents of mass 1 directed by $\mathcal{F}$ such that neither of them gives mass to any invariant closed analytic curve. Then $T_1 - T_2$ is a closed current.

If both $T_1$ and $T_2$ are closed, then we have $\{T_1\}^2 = \{T_2\}^2 = \{T_1\} \sim \{T_2\} = 0$.

**Proof** Since both $T_1$ and $T_2$ do not give mass to any invariant closed analytic curve, it follows from Theorem 7.10 that $\nu(T_1, x) = \nu(T_2, x) = 0$ for all $x$ outside the singularities of $\mathcal{F}$. Since $T_1$ and $T_2$ do not give mass to this finite set, we see that $T_1$ and $T_2$ satisfy the assumption of Theorem 7.4.

By (7.4) and Stokes’ theorem, we have (the second integral is the mass of $T_j$ which is assumed to be 1)

$$\int_X \Omega_j \wedge \omega = \int_X T_j \wedge \omega = 1 \quad \text{for} \quad j = 1, 2. \quad (7.6)$$

Applying Theorems 7.4 and 7.7 to each one of the three directed positive $dd^c$-closed currents $T_1$, $T_2$ and $T_1 + T_2$, we obtain that all $T_1 \otimes T_1$, $T_2 \otimes T_2$ and $(T_1 + T_2) \otimes (T_1 + T_2)$ admit zero as the unique tangent current along the diagonal $\Delta$. This, combined with (7.4) and (7.5), implies that

$$\int_X \Omega_1^2 = 2 \int_X \overline{\partial} S_1 \wedge \partial \overline{S}_1, \quad \int_X \Omega_2^2 = 2 \int_X \overline{\partial} S_2 \wedge \partial \overline{S}_2$$

and

$$\int_X (\Omega_1 + \Omega_2)^2 = 2 \int_X \overline{\partial}(S_1 + S_2) \wedge \partial(\overline{S}_1 + \overline{S}_2). \quad (7.7)$$

If both $T_1$ and $T_2$ are closed, we deduce from the discussion after (7.2) that $\overline{\partial} S_1 = \overline{\partial} S_2 = 0$ and hence all integrals in (7.7) vanish. This implies $\{T_1\}^2 = \{T_2\}^2 = \{T_1\} \sim \{T_2\} = 0$ as stated in the second assertion of the lemma.
Let \( T := T_1 - T_2, \Omega := \Omega_1 - \Omega_2, S := S_1 - S_2 \) and \( u := u_1 - u_2 \). We infer from (7.4) and (7.6) that
\[
T = \Omega + \partial S + \overline{\partial S} + i \partial \overline{\partial u} \quad \text{and} \quad \int_X \Omega \wedge \omega = 0. \tag{7.8}
\]
Moreover, it follows from (7.7) that
\[
\int_X \Omega^2 = \int_X (\Omega_1 - \Omega_2)^2 = 2 \int_X \Omega_1^2 + 2 \int_X \Omega_2^2 - \int_X (\Omega_1 + \Omega_2)^2 = 2 \int_X \overline{\partial S} \wedge \partial \overline{S}. \tag{7.9}
\]
On one hand, since \( \partial S \) is an \( L^2(0, 2) \)-form, the current \( \overline{\partial S} \wedge \partial \overline{S} = \partial S \wedge \overline{\partial S} \) is a positive measure. So the last integral in (7.9) is non-negative and it vanishes if only if \( \partial S = 0 \) almost everywhere. On the other hand, since we know by (7.8) that \( \int_X \Omega \wedge \omega = 0 \), the cohomology class of \( \Omega \) is a primitive class of \( H^{1,1}(X, \mathbb{R}) \). Therefore, it follows from the classical Hodge–Riemann theorem that the first integral in (7.9) is non-positive (see, e.g., [96]). We conclude that \( \partial S = 0 \) almost everywhere. This and (7.8) imply that \( dT = 0 \). The proof of the lemma is thereby completed.

End of the proof of Theorem 6.8 (see also [49]) We only consider the case of a foliation because the case of lamination can be obtained in the same way. It is clear that not more than one property in the theorem holds. By Theorem 2.23, there exists a positive \( dd^c \)-closed current \( T_1 \) directed by \( \mathcal{F} \). We can assume that Property (a) in the theorem does not hold. So we can find a current \( T_1 \) of mass 1 which has no mass on each single leaf of \( \mathcal{F} \) (see Theorem 7.10). We show that either Property (b) or (c) holds.

Case 1 Assume that there is such a current \( T_1 \) which is not closed. We show that the foliation satisfies Property (c) in the theorem. By Theorem 7.10, the class \( \{ T_1 \} \) is nef and big. It remains to prove the uniqueness of \( T_1 \). Assume by contradiction that there is another positive \( dd^c \)-closed current \( T_2 \) of mass 1 directed by \( \mathcal{F} \). If there is such a current which is closed, then we assume that \( T_2 \) is closed. So we have
\[
\int_X T_1 \wedge \omega = \int_X T_2 \wedge \omega = 1.
\]
We need to find a contradiction.

Consider a flow box \( U \) away from the set of singularities \( E \) that we identify with \( \mathbb{D} \times \Sigma \), \( \mathbb{D} \) being as usual the unit disc and \( \Sigma \) being a transversal of \( U \). By Proposition 2.5, we have
\[
T_j = \int_{\Sigma} h_j^\alpha [V_\alpha] d\mu_j(\alpha), \quad j = 1, 2,
\]
where for \( \alpha \in \Sigma \), \( [V_\alpha] \) is the current of integration on the plaque \( V_\alpha \simeq \mathbb{D} \times \{\alpha\} \). Let \( \mu = \mu_1 + \mu_2 \) and write \( \mu_j = r_j \mu \) with a non-negative bounded function \( r_j \in L^\infty(\mu) \). Then we have
\[
T_1 - T_2 = \int_{\Sigma} (h_1^\alpha r_1(\alpha) - h_2^\alpha r_2(\alpha)) [V_\alpha] d\mu(\alpha).
\]
Since we know by Lemma 7.11 that \( T_1 - T_2 \) is a closed current, \( h_1^\alpha r_1(\alpha) - h_2^\alpha r_2(\alpha) \) is constant, for \( \mu \)-almost every \( \alpha \), that we will denote by \( c(\alpha) \).

We decompose \( c(\alpha)\mu(\alpha) \) on the space of plaques \( \Sigma \) and obtain that \( c(\alpha)\mu(\alpha) = v_1 - v_2 \) for mutually singular positive measures \( v_1 \) and \( v_2 \). Then
\[
T_1 - T_2 = [V_\alpha] v_1(\alpha) - [V_\alpha] v_2(\alpha) = T^+ - T^-
\]
for positive closed currents \( T^\pm \). These currents fit together to a global positive closed currents on \( X \setminus E \). Observe that the mass of \( T^\pm \) is bounded by the mass of \( T_1 + T_2 \). So the
mass of $T^\pm$ is bounded near $E$. Since $E$ is a finite set, $T^\pm$ extend as positive closed currents through $E$ (see [87, 91]). Recall that positive $dd^c$-closed currents of bidimension $(1, 1)$ have no mass on finite sets. Therefore, since we assumed above that $T_1 \neq T_2$, we have either $T^+ \neq 0$ or $T^- \neq 0$. It follows from our choice of $T_2$ that $T_2$ is closed and hence $T_1$ is closed as well. This is a contradiction which shows that such a current $T_2$ as above does not exist.

**Case 2** Assume now that all directed positive $dd^c$-closed $(1, 1)$-currents are closed. Consider arbitrary directed positive closed $(1, 1)$-currents $T_1$ and $T_2$ of mass 1 which are diffuse. So by Theorem 7.10 applied to $T_1, T_2$, the classes $\{T_1\}$ and $\{T_2\}$ are nef. By Lemma 7.11, we have $\{T_1\}^2 = \{T_2\}^2 = \{T_1\} \sim \{T_2\} = 0$. We show that Property (b) in the theorem holds. It is enough to show that $\{T_1\} = \{T_2\}$.

Since $T_1$ and $T_2$ are of mass 1, we have $\{(T_1) - (T_2)\} \sim \{\omega\} = 0$. So $\{T_1\} - \{T_2\}$ is a primitive class in the Hodge cohomology group $H^{1,1}(X, \mathbb{R})$ of $X$. By the classical Hodge-Riemann theorem, we have $\{(T_1) - (T_2)\}^2 < 0$ unless $\{T_1\} - \{T_2\} = 0$ (see, e.g., [96]). Using that $\{T_1\}^2 = \{T_2\}^2 = \{T_1\} \sim \{T_2\} = 0$, we deduce that $\{T_1\} = \{T_2\}$. This ends the proof of the theorem.

*End of the proof of Theorem 6.7* We only consider the case of a foliation because the case of a lamination can be proved in the same way. By hypothesis, the foliation has no invariant closed analytic curve. Moreover, by Theorem 6.8, Property (c) in that theorem holds. It follows that the foliation admits a unique directed positive $dd^c$-closed current $T$ of mass 1. This current is not closed and $\{T\}$ is nef and big. Since every cluster point of $\tau_{x, R}$ as $R$ tends to infinity is a positive $dd^c$-closed current of mass 1, $\tau_{x, R}$ converges necessarily to $T$ as $R$ tends to infinity. 

**8 Lyapunov–Oseledec Theory for Riemann Surface Laminations**

The purpose of this section is to present some recent results obtained in our works [75, 76].

### 8.1 Cocycles

The notion of (multiplicative) cocycles have been introduced in [75] for $(N$-dimensional real or complex) laminations. For the sake of simplicity we only formulate this notion for Riemann surface laminations with singularities in this article. In the rest of the section we make the following convention: $\mathbb{K}$ denotes either the field $\mathbb{R}$ or $\mathbb{C}$. Moreover, given any integer $d \geq 1$, $\text{GL}(d, \mathbb{K})$ denotes the general linear group of degree $d$ over $\mathbb{K}$ and $\mathbb{P}^d(\mathbb{K})$ denotes the $\mathbb{K}$-projective space of dimension $d$.

**Definition 8.1** (Nguyen [75, Definition 3.2]) Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a Riemann surface lamination with singularities and $\Omega := \Omega(\mathcal{F})$ be its sample-path space. A $\mathbb{K}$-valued cocycle (of rank $d$) is a map $A : \Omega \times \mathbb{R}^\times \to \text{GL}(d, \mathbb{K})$ such that

1. *(identity law)* $A(\omega, 0) = \text{id}$ for all $\omega \in \Omega$;
2. *(homotopy law)* if $\omega_1, \omega_2 \in \Omega$, and $t_1, t_2 \in \mathbb{R}^\times$ such that $\omega_1(t_1) = \omega_2(t_2)$ and $\omega_1|_{[0, t_1]}$ is homotopic to $\omega_2|_{[0, t_2]}$ (that is, the path $\omega_1|_{[0, t_1]}$ can be deformed continuously on $L_x$ to the path $\omega_2|_{[0, t_2]}$, the two endpoints of $\omega_1|_{[0, t_1]}$ being kept fixed during the deformation), then
   $$A(\omega_1, t_1) = A(\omega_2, t_2);$$
(3) (multiplicative law) \( A(\omega, s + t) = A(\sigma_t(\omega), s)A(\omega, t) \) for all \( s, t \in \mathbb{R}^+ \) and \( \omega \in \Omega \) (see (2.16) for \( \sigma_t \));

(4) (measurable law) the local expression of \( A \) on each laminated chart is Borel measurable. Here, the local expression of \( A \) on the laminated chart \( \Phi : U \to \mathbb{D} \times \mathbb{T} \), is the map \( A : D \times D \times \mathbb{T} \to \text{GL}(d, \mathbb{K}) \) defined by

\[
A(y, z, t) := A(\omega, 1),
\]

where \( \omega \) is any leafwise path such that \( \omega(0) = \Phi^{-1}(y, t), \omega(1) = \Phi^{-1}(z, t) \) and \( \omega[0, 1] \) is contained in the simply connected plaque \( \Phi^{-1}(\cdot, t) \).

A cocycle \( A \) on a smooth Riemann surface lamination with singularities \( F \) is called smooth if, for each laminated chart \( \Phi \) as above, the local expression \( A \) of \( A \) is smooth with respect to \( (y, z) \) and its partial derivatives of any total order with respect to \( (y, z) \) are jointly continuous in \( (y, z, t) \).

The cocycles of rank 1 have been investigated by several authors (see, for example, Candel [17], Deroin [27], etc.). The holonomy cocycle (or equivalently the normal derivative cocycle) of the regular part \( (X \setminus E, L) \) of a singular holomorphic foliation \( F = (X, L, E) \) with \( \dim_{\mathbb{C}} X = n \) is a typical example of \( \mathbb{C} \)-valued cocycles of rank \( n - 1 \). These cocycles capture the topological aspect of the considered foliations. Moreover, we can produce new cocycles from the old ones by performing some basic operations such as the wedge product and the tensor product (see [75, Section 3.1]).

### 8.2 Oseledec Multiplicative Ergodic Theorem

Now we are in the position to state the Oseledec multiplicative ergodic theorem for Riemann surface laminations with singularities.

**Theorem 8.2** (Nguyen [75, Theorem 3.11]) Let \( F = (X, L, E) \) be a Riemann surface lamination with singularities. Let \( \mu \) be a harmonic measure which is also ergodic. Consider a cocycle \( A : \Omega \times \mathbb{R}^+ \to \text{GL}(d, \mathbb{K}) \). Assume that the following integrability condition is satisfied for some real number \( t_0 > 0 \):

\[
\int_{x \in X} \left( \int_{\Omega_x} \sup_{t \in [0, t_0]} \log^+ \|A(\omega, t)\|dW_x(\omega) \right) d\mu(x) < \infty, \quad (8.1)
\]

where \( \log^+ := \max(0, \log) \). Then there exist a leafwise saturated Borel set \( Y \subset X \) of total \( \mu \)-measure and a number \( m \in \mathbb{N} \) together with \( m \) integers \( d_1, \ldots, d_m \in \mathbb{N} \) such that the following properties hold:

(i) For each \( x \in Y \) there exists a decomposition of \( \mathbb{K}^d \) as a direct sum of \( \mathbb{K} \)-linear subspaces

\[
\mathbb{K}^d = \bigoplus_{i=1}^m H_i(x),
\]

such that \( \dim H_i(x) = d_i \) and \( A(\omega, t)H_i(x) = H_i(\omega(t)) \) for all \( \omega \in \Omega_x \) and \( t \in \mathbb{R}^+ \). Moreover, \( x \mapsto H_i(x) \) is a measurable map from \( Y \) into the Grassmannian of \( \mathbb{K}^d \). For each \( 1 \leq i \leq m \) and each \( x \in Y \), let \( V_i(x) := \bigoplus_{j=1}^m H_j(x) \). Set \( V_{m+1}(x) \equiv \{0\} \).

(ii) There are real numbers

\[
\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1,
\]
and for each $x \in Y$, there is a set $F_x \subset \Omega_x$ of total $W_x$-measure such that for every $1 \leq i \leq m$ and every $v \in V_i(x) \setminus V_{i+1}(x)$ and every $\omega \in F_x$,

$$\lim_{t \to \infty, t \in \mathbb{R}^+} \frac{1}{t} \log \| A(\omega, t)v \|_{\|v\|} = \chi_i.$$  \hfill (8.2)

Moreover,

$$\lim_{t \to \infty, t \in \mathbb{R}^+} \frac{1}{t} \log \| A(\omega, t) \| = \chi_1$$  \hfill (8.3)

for each $x \in Y$ and for every $\omega \in F_x$.

Here $\| \cdot \|$ denotes the standard Euclidean norm of $\mathbb{K}^d$.

The above result is the counterpart, in the context of Riemann surface laminations with singularities, of the classical Oseledec multiplicative ergodic theorem for maps (see [66, 84]). In fact, Theorem 3.11 in [75] is much more general than Theorem 8.2. Indeed, the former is formulated for $l$-dimensional laminations endowed with leafwise Riemannian metrics, which satisfy some reasonable geometric conditions.

Assertion (i) above tells us that the Oseledec decomposition exists for all points $x$ in a leafwise saturated Borel set of total $\mu$-measure and that this decomposition is holonomy invariant. Observe that the Oseledec decomposition in (i) depends only on $x \in Y$, in particular, it does not depend on paths $\omega \in \Omega_x$.

The decreasing sequence of subspaces of $\mathbb{K}^d$ given by assertion (i):

$$\{0\} \equiv V_{m+1}(x) \subset V_m(x) \subset \cdots \subset V_1(x) = \mathbb{K}^d$$

is called the Lyapunov filtration associated to $A$ at a given point $x \in Y$.

The numbers $\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1$ given by assertion (ii) above are called the Lyapunov exponents of the cocycle $A$ with respect to the harmonic measure $\mu$. It follows from formulas (8.2) and (8.3) above that these characteristic numbers measure heuristically the expansion rate of $A$ along different vector-directions and along leafwise Brownian trajectories. In other words, the stochastic formulas (8.2)–(8.3) highlight the dynamical character of the Lyapunov exponents.

8.3 Estimates of the Lyapunov Exponents of Compact Smooth Hyperbolic Laminations

Let $A : \Omega \times \mathbb{R}^+ \to \text{GL}(d, \mathbb{K})$ be a smooth cocycle defined on a compact smooth hyperbolic Riemann surface lamination (without singularities) $\mathcal{F} = (X, \mathcal{L})$. Observe that the map $A^{-1} : \Omega \times \mathbb{R}^+ \to \text{GL}(d, \mathbb{K})$, defined by $A^{-1}(\omega, t) := (A(\omega, t))^{-1}$, is also a cocycle, where $A^t$ (resp. $A^{-1}$) denotes as usual the transpose (resp. the inverse) of a square matrix $A$.

We define two functions $\bar{\delta}(A)$, $\tilde{\delta}(A) : X \to \mathbb{R}$ as well as four quantities $\chi_{\max}(A)$, $\bar{\chi}_{\max}(A)$, $\chi_{\min}(A)$, $\bar{\chi}_{\min}(A)$ as follows. Fix a point $x \in X$, an element $u \in \mathbb{K}^d \setminus \{0\}$ and a simply connected plaque $K$ of $\mathcal{F}$ passing through $x$. Consider the function $f_{u,x} : K \to \mathbb{R}$ defined by

$$f_{u,x}(y) := \log \frac{\| A(\omega, 1)u \|}{\|u\|}, \quad y \in K, \ u \in \mathbb{K}^d \setminus \{0\},$$

where $\omega \in \Omega$ is any path such that $\omega(0) = x$, $\omega(1) = y$ and that $[0, 1]$ is contained in $K$.

Then define

$$\bar{\delta}(A)(x) := \sup_{u \in \mathbb{K}^d : \|u\|=1} (\Delta p f_{u,x})(x) \quad \text{and} \quad \tilde{\delta}(A)(x) := \inf_{u \in \mathbb{K}^d : \|u\|=1} (\Delta p f_{u,x})(x),$$

where $p$ is the projection onto $K$.
where $\Delta \rho$ is, as usual, the Laplacian on the leaf $L_x$ induced by the leafwise Poincaré metric $g_\rho$ (see formulas (2.5) and (2.7)). We also define

\[
\bar{\chi}_{\text{max}} = \bar{\chi}_{\text{max}}(\mathcal{A}) := \int_X \bar{\delta}(\mathcal{A})(x)d\mu(x),
\]

\[
\chi_{\text{max}} = \chi_{\text{max}}(\mathcal{A}) := \int_X \delta(\mathcal{A})(x)d\mu(x);
\]

\[
\chi_{\text{min}} = \chi_{\text{min}}(\mathcal{A}) := -\bar{\chi}_{\text{max}}(\mathcal{A}^\star - 1),
\]

\[
\bar{\chi}_{\text{min}} = \bar{\chi}_{\text{min}}(\mathcal{A}) := -\chi_{\text{max}}(\mathcal{A}^\star - 1).
\]

Note that our functions $\delta$, $\bar{\delta}$ are the multi-dimensional generalizations of the operator $\delta$ introduced by Candel [17].

We are in the position to state effective integral estimates on the Lyapunov exponents.

**Theorem 8.3** (Nguyen [75, Theorem 3.12]) Let $(X, \mathcal{L})$ be a compact smooth lamination by hyperbolic Riemann surfaces. Let $\mu$ be a harmonic measure which is ergodic. Let $\mathcal{A} : \Omega \times \mathbb{R}^+ \to \text{GL}(d, \mathbb{K})$ be a smooth cocycle. Let

\[
\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1
\]

be the Lyapunov exponents of the cocycle $\mathcal{A}$ with respect to $\mu$, given by Theorem 8.2. Then the following inequalities hold

\[
\bar{\chi}_{\text{max}} \leq \chi_1 \leq \chi_{\text{max}} \quad \text{and} \quad \chi_{\text{min}} \leq \chi_m \leq \bar{\chi}_{\text{min}}.
\]

This theorem generalizes some results of Candel [16] and Deroin [27] who treat the case $d = 1$, see also Ghys [55]. Under the assumption of Theorem 8.3, the integrability condition (8.1) follows from some well-known estimates of the heat kernels of the Poincaré disc and the fact that the lamination is compact and is without singularities. In fact, we improve the method of Candel in [17].

**9 Applications**

**9.1 Lyapunov Exponent of a Singular Holomorphic Foliation on a Compact Projective Surface**

We recall the holonomy cocycle of a singular holomorphic foliation $\mathcal{F} = (X, \mathcal{L}, E)$ on a Hermitian complex surface $(X, g)$. For each point $x \in X \setminus E$, let $\text{Tan}_x(X)$ (resp. $\text{Tan}_x(L_x) \subset \text{Tan}_x(X)$) be the tangent space of $X$ (resp. $L_x$) at $x$. For every transversal $S$ at a point $x$ (that is, $S$ is complex submanifold of a flow box and $S$ is transverse to every leaf of that flow box and $x \in S$), let $\text{Tan}_x(S)$ denote the tangent space of $S$ at $x$.

Now fix a point $x \in X \setminus E$ and a path $\omega \in \Omega_x$ and a time $t \in \mathbb{R}^+$, and let $y := \omega(t)$. Fix a transversal $S_x$ at $x$ (resp. $S_y$ at $y$) such that the complex line $\text{Tan}_x(S_x)$ is the orthogonal complement of the complex line $\text{Tan}_x(L_x)$ in the Hermitian space $(\text{Tan}_x(X), g(x))$ (resp. $\text{Tan}_y(S_y)$ is the orthogonal complement of $\text{Tan}_y(L_y)$ in $(\text{Tan}_y(X), g(y))$). Let $\text{hol}_{\omega, t}$ be the holonomy map along the path $\omega|_{[0, t]}$ from an open neighborhood of $x$ in $S_x$ onto an open neighborhood of $y$ in $S_y$, that is, let

\[
\text{hol}_{\omega, t} := \text{hol}_\gamma,
\]
where \( \gamma : [0, 1] \to L_x \) is the path given by \( \gamma(s) := \omega(ts) \) for \( s \in [0, 1] \) (see Definition 2.25).

The derivative \( D\text{hol}_{\omega,t} : \Tan_x(S_x) \to \Tan_y(S_y) \) induces the so-called holonomy cocycle \( H : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+ \) given by

\[
H(\omega, t) := \| D\text{hol}_{\omega,t}(x) \|.
\] (9.1)

The last map depends only on the path \( \omega|_{[0,t]} \), in fact, it depends only on the homotopy class of this path. In particular, it is independent of the choice of transversals \( S_x \) and \( S_y \). We see easily that

\[
H(\omega, t) = \lim_{z \to x, z \in S_x} \text{dist}(\text{hol}_{\omega,t}(z), y)/\text{dist}(z, x).
\]

On the other hand, we note the following multiplicative property which is an immediate consequence of the definition of \( H(\omega, t) \),

\[
H(\omega, t + s) = H(\omega, t)H(\sigma_t(\omega), s), \quad t, s \in \mathbb{R}^+, \omega \in \Omega,
\] (9.2)

where \( \sigma_t : \Omega \to \Omega \) is the shift-transformation given by (2.16).

The holonomy cocycle (or equivalently, the normal derivative cocycle) of a foliation is a very important object which encodes dynamical as well as geometric and analytic information of the foliation. Exploring this object allows us to understand more about the foliation itself.

The following fundamental question arises naturally:

**Question** Can one define the Lyapunov exponents of an ergodic harmonic measure \( \mu \) on a compact singular holomorphic hyperbolic foliation \( \mathcal{F} = (X, \mathcal{L}, E) \)?

By Theorem 2.9, this question can be rephrased for directed harmonic currents on the foliation. We have recently obtained the following affirmative answer to this question for generic foliations in dimension two.

**Theorem 9.1** ([78, Theorem 1.1]) Let \( \mathcal{F} = (X, \mathcal{L}, E) \) be a holomorphic foliation by Riemann surfaces defined on a Hermitian compact complex projective surface \( X \) satisfying the following two conditions:

- Its singularities \( E \) are all hyperbolic;
- \( \mathcal{F} \) is Brody hyperbolic (see Definition 4.2).

Let \( \mu \) be a harmonic measure which does not give mass to any invariant analytic curve. Assume, in addition, that \( \mu \) is ergodic.

Then

1. \( \mu \) admits the (unique) **Lyapunov exponent** \( \chi(\mu) \) given by the formula

\[
\chi(\mu) := \int_X \left( \int_{\Omega} \log \| H(\omega, 1) \| dW_x(\omega) \right) d\mu(x).
\] (9.3)

2. For \( \mu \)-almost every \( x \in X \setminus E \), we have

\[
\lim_{t \to \infty} \frac{1}{t} \log \| H(\omega, t) \| = \chi(\mu)
\]

for almost every path \( \omega \in \Omega \) with respect to \( W_x \).

In fact, assertion (1) is a consequence of the so-called integrability of the holonomy cocycle of singular holomorphic foliations. Assertion (2) says that the characteristic number \( \chi(\mu) \) measures heuristically the exponential rate of convergence of leaves towards each
other along leafwise Brownian trajectories (see Candel [17], Deroin [27] for the nonsingular case). Therefore, Theorem 9.1 gives a dynamical characterization of $\chi(\mu)$.

### 9.2 Sketchy Proof of the Existence of Lyapunov Exponent

Prior to the sketchy proof of Theorem 9.1, we discuss how to deal with the singularities.

To study $\mathcal{F}$ near a hyperbolic singularity $a \in E$, we use the following local model introduced in [33]. In this model, a neighborhood of $a$ is identified with the bidisc $\mathbb{D}^2$, and the restriction of $\mathcal{F}$ to $\mathbb{D}^2$, i.e., the leaves of $(\mathbb{D}^2, \mathcal{L}, \{0\})$ coincide with the restriction to $\mathbb{D}^2$ of the integral curves of a vector field

$$Z(z, w) = z \frac{\partial}{\partial z} + \lambda w \frac{\partial}{\partial w} \quad \text{with some} \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (9.4)$$

Notice that if we flip $z$ and $w$, we replace $\lambda$ by $\lambda^{-1}$. Since $(\text{Im} \lambda)(\text{Im} \lambda^{-1}) < 0$, we may assume below that the axes are chosen so that $\text{Im} \lambda > 0$.

For $x = (z, w) \in \mathbb{D}^2 \setminus \{0\}$, define the holomorphic map $\psi_x : \mathbb{C} \to \mathbb{C}^2 \setminus \{0\}$

$$\psi_x(\zeta) := \left(ze^{i\lambda \zeta}, we^{i\lambda \zeta}\right) \quad \text{for} \quad \zeta \in \mathbb{C}. \quad (9.5)$$

It is easy to see that $\psi_x(\mathbb{C})$ is the integral curve of $Z$ which contains $\psi_x(0) = x$. Write $\zeta = u + iv$ with $u, v \in \mathbb{R}$. The domain $\Pi_x := \psi_x^{-1}(\mathbb{D}^2)$ in $\mathbb{C}$ is defined by the inequalities

$$(\text{Im} \lambda)u + (\text{Re} \lambda)v > \log |w| \quad \text{and} \quad v > \log |z|.$$  

So, $\Pi_x$ defines a sector $S_x$ in $\mathbb{C}$. It contains 0 since $\psi_x(0) = x$. The leaf of $\mathcal{F}$ through $x$ contains the Riemann surface

$$\hat{L}_x := \psi_x(\Pi_x) \subseteq L_x.$$  

In particular, the leaves in a singular flow box are parametrized using holomorphic maps $\psi_x : \Pi_x \to L_x$.

Now we fix a finite cover $\mathcal{U}$ of flow boxes on $X$. We only consider flow boxes which are biholomorphic to $\mathbb{D}^2$. A regular flow box $U$ is a flow box with foliated chart $\Phi : U \to \mathbb{B} \times \Sigma$ outside the singularities, where $\mathbb{B}$ and $\Sigma$ are open sets in $\mathbb{C}$. For each $\alpha \in \Sigma$, the Riemann surface $V_\alpha := \Phi^{-1}(\mathbb{B} \times \{\alpha\})$ is called a plaque of $U$. Singular flow boxes are identified to their models $(\mathbb{D}^2, \mathcal{L}, \{0\})$ as described above. For $U := \mathbb{D}^2$ and $s > 0$, let $sU := (s\mathbb{D})^2$. For each singular point $a \in E$, we fix a singular flow box $U_a$ such that $2U_a \cap 2U_{a'} = \emptyset$ if $a, a' \in E$ with $a \neq a'$. We also cover $X \setminus \cup_{a \in E} U_a$ by a finite number of regular flow boxes $(U_q)$ such that each $U_q$ is contained in a larger regular flow box $U'_q$ with $U'_q \cap \cup_{a \in E}(1/2)U_a = \emptyset$. Thus we obtain a finite cover $\mathcal{U} := (U_p)_{p \in I}$ of $X$ consisting of regular flow boxes $(U_p)_{p \in I \setminus E}$ and singular ones $(U_a)_{a \in E}$.

Let $g_X$ be a Hermitian metric on $X$ and let $d$ denote the distance on $X$ induced by $g_X$. We often suppose without loss of generality that the ambient metric $g_X$ coincides with the standard Euclidean metric on each singular flow box $2U_a \simeq 2\mathbb{D}^2$, $a \in E$.

Now we discuss the proof of Theorem 9.1. It consists of two steps. In the first step, we show that Theorem 9.1 follows from the new integrability condition (9.6).

$$\int_X |\log \text{dist}(x, E)| \cdot d\mu(x) < \infty. \quad (9.6)$$

This new condition has the advantage over the old one (8.1), since the former does not involve the somewhat complicating Wiener measures, and hence it is easier to handle than the latter.
For this purpose we study the behavior of the holonomy cocycle near the singularities with respect to the leafwise Poincaré metric $g_{P}$. The following result gives an explicit expression for $H$ near a singular point $a$ using the above local model $(\mathbb{D}^2, \mathcal{L}, \{0\})$.

**Proposition 9.2** ([78, Proposition 3.1]) Let $\mathbb{D}^2$ be endowed with the Euclidean metric. For each $x = (z, w) \in \mathbb{D}^2$, consider the function $\Phi_x : \Pi_x \to \mathbb{R}^+$ as follows. For $\zeta \in \Pi_x$, consider a path $\omega \in \Omega$ (it always exists since $\Pi_x$ is convex and $0 \in \Pi_x$ as $x \in \mathbb{D}^2$) such that

$$\omega(t) = \psi_x(t\zeta) = (ze^{i\xi t}, we^{i\lambda \xi t}) \subset \mathbb{D}^2$$

for all $t \in [0, 1]$ (see (9.5) above). Define $\Phi_x(\zeta) := H(\omega, 1)$. Then

$$\Phi_x(\zeta) = |e^{i\xi}| |e^{i\lambda \xi}| \frac{\sqrt{|z|^2 + |\lambda w|^2}}{\sqrt{|ze^{i\xi}|^2 + |\lambda w e^{i\lambda \xi}|^2}}.$$  

Roughly speaking, Step 1 quantifies the expansion speed of the holonomy cocycle in terms of the ambient metric $g_X$ when one travels along unit-speed geodesic rays. The first main ingredient is Proposition 9.2. The second main ingredient of the first step is a detailed analysis of the behavior of the leafwise Poincaré metric near hyperbolic singularities which had previously been carried out in [32–34] and which culminates in Proposition 4.5.

The second main step is then devoted to the proof of inequality (9.6). The main difficulty is that known estimates (see, for example, [32]) on the behavior of $T$ near linearizable singularities, only give a weaker inequality

$$\int_X |\log \text{dist}(x, E)|^{1-\delta} d\mu(x) < \infty, \quad \forall \delta > 0.$$ (9.7)

So (9.6) is the limiting case of (9.7).

To prove (9.6), by Theorem 2.9 (1) write $\mu = \Phi(T)$ for a directed positive harmonic current $T$ giving no mass to $\text{Par}(\mathcal{F})$. Moreover, such a current $T$ may be identified with a directed positive $dd^c$-closed current giving no mass to $\text{Par}(\mathcal{F})$ by Proposition 2.17 and Definition 2.20.

The proof of (9.7) relies on the finiteness of the Lelong number of $T$ at every point which has been established in Proposition 2.15. Recall that Theorem 4.9 sharpens the last estimate by showing that the Lelong number of $T$ vanishes at every hyperbolic singular point $x \in E$. Nevertheless, even this better estimate does not suffice to prove (9.6). So the main difficulties in the proof are (1) the information of $T$ near the singularities is poor and (2) the Poincaré metric on leaves needs to be controlled near the singularities.

The situation near the singularities is not homogeneous. The new idea in [78] is that we use a cohomological argument which exploits fully the assumption that $X$ is projective. This assumption imposes a stronger mass-clustering condition on harmonic currents than (9.7). We need to divide the space $X$ into small cells where one uses different techniques to get the desired estimates. Some auxiliary quantities depending on $T$ are introduced and used to get a control, good enough, near the singularities. The final result is obtained using both a careful analysis of singular boxes and a global argument (roughly since the current sits in a projective surface, it cannot have a transcendental behavior). Delicate approximations are needed to deduce the global estimates from the local ones.

Now we explain in more details our proof of the integrability condition (9.6). Our approach is based on a cohomological invariance which says roughly that if two algebraic curves $\mathcal{C}$ and $\mathcal{D}$ on $X$ are cohomologous, that is, they are in the same cohomology class in $H^{1,1}(X)$ (for example, if they have the same algebraic degree when $X = \mathbb{P}^2$), then under
suitable assumptions, we can define the wedge product \( T \wedge [\mathcal{C}], T \wedge [\mathcal{D}] \) which are finite positive Borel measures and their masses are equal, i.e.,

\[
\int_X T \wedge [\mathcal{C}] = \int_X T \wedge [\mathcal{D}].
\] (9.8)

Before going further, let us explain why equality (9.8) could be true. Since \( \mathcal{C} \) and \( \mathcal{D} \) on \( X \) are cohomologous on \( X \), the \( \partial \bar{\partial} \)-lemma for compact Kähler manifolds provides us an integrable function \( u \) on \( X \) such that

\[
[\mathcal{C}] - [\mathcal{D}] = i \partial \bar{\partial} u
\]

in the sense of currents.

So we can write

\[
\int_X T \wedge [\mathcal{C}] - \int_X T \wedge [\mathcal{D}] = \int_X T \wedge i \partial \bar{\partial} u.
\]

The function \( u \) is, in general, not smooth near \( \mathcal{C} \) and \( \mathcal{D} \). However, if we could consider it like a smooth function, Stokes’ theorem would turn the right hand side of the last line into the following integral

\[
\int_X u(i \partial \bar{\partial} T) = 0,
\]

where the last equality holds since the \( dd^c \)-closedness of \( T \) implies that \( i \partial \bar{\partial} T = 0 \). Therefore, we may expect equality (9.8) to hold.

Resuming the sketchy proof of the integrability condition (9.6), let \( x_0 \in E \) and fix a coordinate system \((z, w)\) around \( x_0 \) such that the two separatrices of the hyperbolic singular point \( x_0 \) are \( \{z = 0\} \) and \( \{w = 0\} \). Then we can show that the vanishing of the Lelong number of \( T \) at 0 is equivalent to the following convergence

\[
\int_{B(0,2r)} T \wedge [z = r] \to 0 \quad \text{as} \quad r \to 0,
\] (9.9)

where \( B(0, s) \) is the ball in \( X \) with center \( x_0 = 0 \) and radius \( s \). More importantly, the integrability condition (9.6) is somehow equivalent to the statement that the convergence (9.9) has, in a certain very weak sense, a speed of order \( \frac{|\log r|^{-\delta}}{r^\rho} \) as \( r \to 0 \) for some \( \delta > 0 \). Note, however, that this speed does not at all mean that

\[
\int_{B(0,2r)} T \wedge [z = r] = O(|\log r|^{-\delta}).
\]

Now suppose for the sake of simplicity that \( X = \mathbb{P}^2 \) and \( N \in \mathbb{N} \) is large enough. We choose an algebraic curve \( \mathcal{C} \) of degree \( N \) which looks like the analytic curve \( \{z = w^N\} \) near 0. We also choose an algebraic curve \( \mathcal{D} \) of degree \( N \) which looks like the analytic curve \( \{r = z - w^N\} \) near 0. The following seven observations play a key role in our approach, where \( 0 < \delta < 1 \) is an exponent independent of \( r \) and \( N \), \( 0 < r < r_0 \) with \( r_0 > 0 \) a fixed small number.

(i) Outside a small ball \( B(0, r_0) \), the analytic curve \( \{z = w^N\} \) (and hence the algebraic curve \( \mathcal{C} \)) falls into a tubular neighborhood with size \( O(r^\rho) \) of the analytic curve \( \{r = z - w^N\} \) (and hence the algebraic curve \( \mathcal{D} \)), where \( \rho \) is a real number depending on \( N \) with \( 0 < \rho \leq 1 \). So we may expect

\[
\int_{X \setminus B(0, r_0)} T \wedge [\mathcal{C}] = \int_{X \setminus B(0, r_0)} T \wedge [\mathcal{D}] + O(r^\rho).
\]

(ii) Outside the ball \( B(0, r_1) \) and inside the small ball \( B(0, r_0) \), the analytic curve \( \{r = z - w^N\} \) (and hence the algebraic curve \( \mathcal{D} \)) behaves like the analytic curve

\[
\int_X T \wedge [\mathcal{D}] = \int_X T \wedge [\mathcal{D}] + O(r^\rho).
\]
\( z = w^N \) (and hence the algebraic curve \( \mathcal{C} \)) while intersecting the two curves with a general leaf. Indeed, when \( |w| \geq r^{1/N} |\log r|^{3/N} \), we have \( r \ll |w|^N \). So we may expect

\[
\int_{\mathbb{B}(0, r_0) \setminus \mathbb{B}(0, r^{1/N} |\log r|^{3/N})} T \wedge [\mathcal{C}] = \int_{\mathbb{B}(0, r_0) \setminus \mathbb{B}(0, r^{1/N} |\log r|^{3/N})} T \wedge [\mathcal{D}] + O(|\log r|^{-\delta}).
\]

(iii) The corona \( \mathcal{A}_{r,N} := \mathbb{B}(0, r^{1/N} |\log r|^{3/N}) \setminus \mathbb{B}(0, r^{1/N} |\log r|^{-3/N}) \) is, in some sense, small and it may be considered as negligible. So we may expect

\[
\int_{\mathcal{A}_{r,N}} T \wedge [\mathcal{C}] = O((|\log r|^{-\delta}) \quad \text{and} \quad \int_{\mathcal{A}_{r,N}} T \wedge [\mathcal{D}] = O(|\log r|^{-\delta}).
\]

(iv) Our next observation is the following partition of \( X \) for \( 0 < r \ll 1 \):

\[
X = (X \setminus \mathbb{B}(0, r_0)) \bigsqcup (\mathbb{B}(0, r_0) \setminus \mathbb{B}(0, r^{1/N} |\log r|^{3/N})) \bigsqcup \mathcal{A}_{r,N} \bigsqcup \mathbb{B}(0, r^{1/N} |\log r|^{-3/N}).
\]

This allows us to decompose both integrals of (9.8) into corresponding pieces.

Consequently, when the degree \( N \) is sufficiently high, by taking into account the observations (i)-(ii)-(iii)-(iv), and using (9.8), we see that

\[
\int_{\mathbb{B}(0, r^{1/N} |\log r|^{-3/N})} T \wedge [\mathcal{C}] - \int_{\mathbb{B}(0, r^{1/N} |\log r|^{-3/N})} T \wedge [\mathcal{D}] = O(|\log r|^{-\delta}).
\]

(v) Inside the ball \( \mathbb{B}(0, r^{1/N} |\log r|^{-3/N}) \), the analytic curve \( \{ z = w^N \} \) (and hence the algebraic curve \( \mathcal{C} \)) clusters around 0, in a certain sense, much more often than the analytic curve \( \{ z = r \} \) (and hence the algebraic curve \( \mathcal{D} \)). Indeed, we see in the equation \( z = w^N \) that both \( z \) and \( w \) can tend to 0, whereas in the equation \( z = r \), only \( w \) could tend to 0. So we may expect that in a certain sense,

\[
\int_{\mathbb{B}(0, r^{1/N} |\log r|^{-3/N})} T \wedge [\mathcal{D}] \ll \int_{\mathbb{B}(0, r^{1/N} |\log r|^{-3/N})} T \wedge [\mathcal{C}].
\]

This, combined with the estimate obtained just at the end of (iv), implies that both integrals

\[
\int_{\mathbb{B}(0, r^{1/N} |\log r|^{-3/N})} T \wedge [\mathcal{C}] \quad \text{and} \quad \int_{\mathbb{B}(0, r^{1/N} |\log r|^{-3/N})} T \wedge [\mathcal{D}]
\]

admit, in a certain sense, a speed of order \( |\log r|^{-\delta} \).

(vi) Inside the ball \( \mathbb{B}(0, r^{1/N} |\log r|^{-3/N}) \), the analytic curve \( \{ r = z - w^N \} \) (and hence the algebraic curve \( \mathcal{D} \)) behaves like the analytic curve \( \{ z = r \} \) while intersecting the two curves with a general leaf. Indeed, when \( |w| \leq r^{1/N} |\log r|^{-3/N} \), we have \( |w|^N \ll r \). So we may expect

\[
\int_{\mathbb{B}(0, r^{1/N} |\log r|^{-3/N})} T \wedge [\mathcal{D}] - \int_{\mathbb{B}(0, r^{1/N} |\log r|^{-3/N})} T \wedge [z = r] = O(|\log r|^{-\delta}).
\]

This, together with the estimate just obtained at the end of (v), yields that

\[
\int_{\mathbb{B}(0, r^{1/N} |\log r|^{-3/N})} T \wedge [z = r]
\]

has, in a certain sense, a speed of order \( |\log r|^{-\delta} \).
(vii) Our last observation is that one can show that there is a constant $c_N > 1$ independent of $r$ such that

$$c_N^{-1} \int_{B(0,r^{1/N})} T \wedge [z = r] \leq \int_{B(0,2r)} T \wedge [z = r] \leq c_N \int_{B(0,r^{1/N})} T \wedge [z = r].$$

This, together with the estimate just obtained at the end of (vi), implies that

$$\int_{B(0,2r)} T \wedge [z = r]$$

admits, in a certain sense, a speed of order $|\log r|^{-\delta}$. Hence, we get the convergence with speed (9.9). This is what we are looking for.

In fact, the factor $|\log r|^{3/N}$ appearing in the above observations comes from the degeneration of the Poincaré metric $g_P$ relative to the ambient metric $g_X$ (see formula (4.2) and Proposition 4.5). Moreover, the larger the degree $N$ is, the more evident the mass-clustering phenomenon in the previous observation becomes.

Our approach underlines several tasks. On the one hand, we need to define a geometric intersection of a directed positive harmonic current with a singular analytic curve defined on a neighborhood of a singular point of the foliation. On the other hand, we need to approximate some (local) analytic curves by global algebraic ones. The assumption of projectivity of $X$ is needed in order to ensure a good supply of algebraic curves.

Remark 9.3 The condition of Brody hyperbolicity seems to be indispensable for the integrability of the holonomy cocycle. Indeed, Hussenot [60, Theorem A] finds out the following remarkable property for a class of Ricatti foliations $\mathcal{F}$ on $\mathbb{P}^2$. For every $x \in \mathbb{P}^2$ outside invariant curves of every foliation in this class, it holds that

$$\limsup_{t \to \infty} \frac{1}{t} \log \| A(\omega, t) \| = \infty$$

for almost every path $\omega \in \Omega_x$ with respect to the Wiener measure at $x$ which lives on the leaf passing through $x$. By Theorem 2.34, these foliations are hyperbolic since all their singular points have non-degenerate linear part. Nevertheless, neither of them is Brody hyperbolic because they all contain integral curves which are some images of $\mathbb{P}^1$ (see Remark 4.3).

Remark 9.4 If one can prove the new integrability condition (9.6) without using that $X$ is projective, then Theorem 9.1 will hold without this assumption.

Remark 9.5 There is some growing interest in the study of Lyapunov exponents for surface group representations (see [5, 29] and the references therein).

Problem 9.6 Does Theorem 9.1 still hold if the ambient compact projective manifold $X$ is of dimension $> 2$?

Problem 9.7 Does Theorem 9.1 still hold if the ambient compact surface $X$ is only Kähler? If this is true, then it is a good question to investigate the general case of ambient compact Kähler manifolds of dimension $> 2$.

Problem 9.8 Does Theorem 9.1 still hold if the singularities of $\mathcal{F}$ are merely linearizable?
9.3 Negativity and Cohomological Formulas of Lyapunov Exponent

Suppose now that $\mathcal{F} = (X, \mathcal{L}, E)$ is a singular holomorphic foliation with only hyperbolic singularities in a compact projective surface $X$ such that $\mathcal{F}$ admits no directed positive closed current. So the assumptions of both Theorems 9.1 and 6.7 are fulfilled. By Theorem 6.7, let $T$ be the unique directed positive $dd^c$-closed current $T$ whose the Poincaré mass is equal to 1. So by Theorem 2.9, the measure $\mu = \Phi(T)$ given by Definition 2.8 is the unique probability harmonic measure of $\mathcal{F}$.

**Definition 9.9** The Lyapunov exponent of the foliation $\mathcal{F}$, denoted by $\chi(\mathcal{F})$, is by definition, the real number $\chi(\mu)$ given by Theorem 9.1.

When we explore the dynamical system associated with a foliation $\mathcal{F}$, the sign of its Lyapunov exponent is a crucial information. Indeed, the positivity/negativity of $\chi(\mathcal{F})$ corresponds to the repelling/attracting character of a typical leaf along a typical Brownian trajectory. Here is our second main result.

**Theorem 9.10** ([80, Theorem B]) Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a singular holomorphic foliation with only hyperbolic singularities in a compact projective surface $X$ such that $\mathcal{F}$ admits no directed positive closed current. Then $\chi(\mathcal{F})$ is a negative nonzero real number.

Roughly speaking, Theorem 9.10 says that in the sense of ergodic theory, generic leaves have the tendency to wrap together towards the support of the unique probability harmonic measure.

Recall from Section 2.9 that the foliation $\mathcal{F}$ is given by an open covering $\{U_j\}$ of $X$ and holomorphic vector fields $v_j \in H^0(U_j, \text{Tan}(X))$ with isolated singularities (i.e., isolated zeroes) such that

$$v_j = g_{jk}v_k \quad \text{on} \quad U_j \cap U_k$$

for some non-vanishing holomorphic functions $g_{jk} \in H^0(U_j \cap U_k, O^*_X)$.

The functions $g_{jk}$ form a multiplicative cocycle and hence give a cohomology class in $H^1(X, O^*_X)$, that is a holomorphic line bundle on $X$. This is the cotangent bundle $\text{Cotan}(\mathcal{F})$ of $\mathcal{F}$. Its dual $\text{Tan}(\mathcal{F})$, represented by the inverse cocycle $\{g^{-1}_{jk}\}$ is called the tangent bundle of $\mathcal{F}$. For a complex line bundle $E$ over $X$, let $c_1(E)$ denote the cohomology Chern class of $E$. This is an element in $H^{1,1}(X)$.

The next result gives cohomological formulas for $\chi(\mu)$ and $\|\mu\|$ in terms of the geometric quantity $T$ and some characteristic classes of $\mathcal{F}$.

**Theorem 9.11** ([80, Theorem A]) Under the assumption of Theorem 9.1, the following identities hold:

$$\chi(\mu) = -c_1(\text{Nor}(\mathcal{F})) \cdot \{T\}, \quad \|\mu\| = c_1(\text{Cotan}(\mathcal{F})) \cdot \{T\}.$$  

Here $\text{Nor}(\mathcal{F}) := \text{Tan}(X)/\text{Tan}(\mathcal{F})$ stands for the normal bundle of $\mathcal{F}$, where $\text{Tan}(X)$ (resp. $\text{Tan}(\mathcal{F})$ and $\text{Cotan}(\mathcal{F})$) is as usual the tangent bundle of $X$ (resp. the tangent bundle and the cotangent bundle of $\mathcal{F}$).

Now we apply the above results to the family $\mathcal{F}_d(\mathbb{P}^2)$ of singular holomorphic foliations on $\mathbb{P}^2$ with a given degree $d > 1$, which was previously introduced in Theorem 2.34. Consequently, Theorem 9.11 give us the following result.

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Corollary 9.12 Let \( \mathcal{F} = (\mathbb{P}^2, \mathcal{L}, E) \) be a singular foliation by curves on the complex projective plane \( \mathbb{P}^2 \). Assume that the degree of \( \mathcal{F} \) is \( d > 1 \) and that all the singularities are hyperbolic and that \( \mathcal{F} \) has no invariant algebraic curve. Then

\[
\chi(\mathcal{F}) = -\frac{d + 2}{d - 1}.
\]

So for a generic foliation \( \mathcal{F} \) of a given degree \( d > 1 \) in \( \mathbb{P}^2 \), we have \( \chi(\mathcal{F}) = -\frac{d + 2}{d - 1} \).

Remark 9.13 Theorem 9.10 gives a complete answer to Problem 7.7 in our previous survey [79] (see also Hussenot [60]).

Remark 9.14 It is of interest to know whether Theorems 9.10 and 9.11 still hold if \( X \) is merely a compact Kähler surface. This is definitely the case if we can relax the projectivity assumption in Theorem 9.1 (see Remark 9.4). The reader may find in [46, 80] some other open questions in the ergodic theory of singular holomorphic foliations.

Problem 9.15 Study the sign of Lyapunov exponents for a singular holomorphic foliation on a compact projective manifold of dimension \( k > 2 \) (if possible, more generally on a compact Kähler manifolds), when the singularities of the foliations are all hyperbolically linearizable. Can we relax the assumption that the singularities are hyperbolically linearizable?

Problem 9.16 Study the existence and the sign of Lyapunov exponents for a singular holomorphic foliation on a compact projective manifold of dimension \( k > 2 \) (if possible, more generally on a compact Kähler manifolds), when the singularities are of dimension \( \geq 1 \).

9.4 Normal Bundle, Curvature Density and Lyapunov Exponent

In this subsection, we provide preparatory results needed the proof of the first formula of Theorem 9.11. This preparation gives a bridge between the geometric aspect (normal bundle, curvature density) and the dynamical aspect (Lyapunov exponent) of a holomorphic foliation emphasizing the role of the singularities.

Let \( (L, h) \) be a singular Hermitian holomorphic line bundle on \( X \). If \( e_L \) is a holomorphic frame of \( L \) on some open set \( U \subset X \), then the function \( \varphi \) defined by \( |e_L|^2_h = \exp(-2\varphi) \) is called the local weight of the metric \( h \) with respect to \( e_L \). If the local weights \( \varphi \) are in \( L^1_{\text{loc}}(U) \), then (Chern) curvature current of \( (L, h) \) denoted by \( c_1(L, h) \) is given by \( c_1(L, h)|_U = dd^c\varphi \). This is a \((1, 1)\)-closed current. Its class in \( H^{1,1}(X) \) is called the Chern class of \( L \). If we fix a smooth Hermitian metric \( h_0 \) on \( L \), then every singular metric \( h \) on \( L \) can be written \( h = e^{-2\varphi}h_0 \) for some function \( \varphi \). We say that \( \varphi \) is the global weight of \( h \) with respect to \( h_0 \). Clearly, \( c_1(L, h) = c_1(L, h_0) + dd^c\varphi \).

Consider the normal bundle \( \text{Nor}(\mathcal{F}) = \text{Tan}(X)/\text{Tan}(\mathcal{F}) \) of \( \mathcal{F} \). For \( x \in X \setminus E \) and a vector \( u_x \in \text{Tan}_x(X) \), let \([u_x]\) denotes its class in \( \text{Nor}_x(\mathcal{F}) \). We also identify \([u_x]\) with the set \( u_x + \text{Tan}_x(\mathcal{F}) \subset \text{Tan}_x(X) \). Note that a (local) smooth section \( w \) of \( \text{Nor}(\mathcal{F}) \) can be locally written as \( w_x = [u_x] \) for some smooth vector field \( u \).

Consider the following metric \( g^X_\perp \) on the normal bundle \( \text{Nor}(\mathcal{F}) \):

\[
\|w_x\|_{g^X_\perp} := \min_{u_x \in [w_x]} \|u_x\|_{g^X} \quad \text{for} \quad w_x \in \text{Nor}(\mathcal{F})_x, \ x \in X \setminus E. \quad (9.10)
\]
Note that $u_x$ achieving the minimum in (9.10) is uniquely determined by $[w_x]$. $g^X_\omega$ is called transversal metric associated with $\mathcal{F}$ and the ambient metric $g_X$.

Fix a smooth Hermitian metric $g_0$ on $\text{Nor}(\mathcal{F})$. There is a global weight function $\varphi$ on $X$ such that $g^X_\omega = e^{-2\varphi}g_0$.

Next, we recall some notions and results from [75, Section 9.1]. Fix a point $x \in X$ and let $\phi_x : \mathbb{D} \to L = L_x$ be the universal covering map given in (2.1). Consider the function $\kappa_x : \mathbb{D} \to \mathbb{R}$ defined by

$$\kappa_x(\xi) := \log \|\mathcal{H}(\phi_x \circ \omega, 1)\|, \quad \xi \in \mathbb{D}, \quad (9.11)$$

where $\omega \in \Omega_0$ is any path such that $\omega(1) = \xi$. This function is well-defined because $\mathcal{H}(\phi_x \circ \omega, t)$ depends only on the homotopy class of the path $\omega|[0,t]$ and $\mathbb{D}$ is simply connected. Following [75], $\kappa_x$ is said to be the specialization of the holonomy cocycle $\mathcal{H}$ at $x$.

The following two conversion rules for changing specializations in the same leaf are useful (see [75]). For this purpose let $y \in L_x$ and pick $\xi \in \phi_x^{-1}(y)$. Since the holonomy cocycle is multiplicative (see (9.2)), the first conversion rule (see [75, identity (9.6)]) states that

$$\kappa_x(\frac{\xi - \xi}{1 - \xi^2}) = \kappa_x(\xi) - \kappa_y(\xi), \quad \xi \in \mathbb{D}. \quad (9.12)$$

Consequently, since $\Delta_p$ is invariant with respect to the automorphisms of $\mathbb{D}$, it follows that

$$\Delta_p \kappa_y(0) = \Delta_p \kappa_x(\xi). \quad (9.13)$$

By [75, identities (9.5) and (9.8)], we have that

$$\kappa_x(0) = 0 \quad \text{and} \quad \mathbb{E}_x[\log \|\mathcal{H}(\bullet, t)\|] = (D_t \kappa_x)(0), \quad t \in \mathbb{R}^+, \quad (9.14)$$

where $(D_t)_{t \in \mathbb{R}^+}$ is the family of diffusion operators associated with $(\mathbb{D}, g_p)$.

Consider the function $\kappa : X \setminus E \to \mathbb{R}$ defined by

$$\kappa(x) := (\Delta_p \kappa_x)(0) \quad \text{for} \quad x \in X \setminus E. \quad (9.15)$$

Now let $L = \text{Nor}(\mathcal{F})$. Suppose that $L$ is trivial over a flow box $U \simeq \mathbb{B} \times \Sigma$, i.e., $L|_U \simeq U \times \mathbb{C} \simeq \mathbb{B} \times \Sigma \times \mathbb{C}$. Consider the holomorphic section $e_L$ on $U$ defined by $e_L(x) := (x, 1)$. Let $\varphi$ be the local weight of $(L, g^X_\omega)$ with respect to $e_L$. Equality (9.1) is rewritten as follows

$$\frac{\|e_L(y)\|_{g^X_\omega}}{\|e_L(x)\|_{g^X_\omega}} = \frac{e^{-\varphi(y)}}{e^{-\varphi(x)}},$$

where $x$ and $y$ are on the same plaque in the flow box $U \simeq \mathbb{B} \times \Sigma$. Consequently,

$$dd^c_x \log \|\mathcal{H}(\omega, t)\|_{L_x} = \left(dd^c_x \log \left(\frac{\|e_L(y)\|_{g^X_\omega}}{\|e_L(x)\|_{g^X_\omega}}\right)\right)|_{L_x} = -dd^c_x \varphi(y)|_{L_x}. \quad (9.16)$$

Combining (9.11), (9.12), (9.13), (9.15) and (9.16), we obtain that

$$\kappa(x) = - (\Delta_p \varphi)(x) \quad \text{for} \quad x \in X \setminus E. \quad (9.17)$$

The function $\kappa$ is said to be the curvature density of $\text{Nor}(\mathcal{F})$.

For $x = (z, w) \in \mathbb{C}^2$, let $\|x\| := \sqrt{|z|^2 + |w|^2}$ be the standard Euclidean norm of $x$. Recall that $\log^*(\cdot) := 1 + |\log(\cdot)|$. Using Propositions 9.2 an 4.5, we obtain the following result which gives precise variations up to order 2 of $\mathcal{H}$ near a singular point $a$ using the local model $(\mathbb{D}^2, \mathcal{L}, \{0\})$ discussed in (9.4).
Proposition 9.17 ([81, Lemma 3.2]) Let $\mathbb{D}^2$ be endowed with the Euclidean metric. Then there is a constant $c > 1$ such that for every $x = (z, w) \in (\frac{1}{2} \mathbb{D})^2$, we have that
\[
 c^{-1} \log^* \|z, w\| \leq |d\kappa_x(0)|_P \leq c \log^* \|z, w\|,
\]
\[
 -c \frac{|z|^2 |w|^2}{(|z|^2 + |w|^2)^2} (\log^* \|z, w\|)^2 \leq \Delta_P \kappa_x(0) \leq -c^{-1} \frac{|z|^2 |w|^2}{(|z|^2 + |w|^2)^2} (\log^* \|z, w\|)^2,
\]
where the function $\kappa_x$ is defined in (9.11).

Following Proposition 9.1, consider a weight function $W : X \setminus E \to \mathbb{R}^+$ as follows. Let $x \in X \setminus E$. If $x$ belongs to a regular flow box then $W(x) := 1$. Otherwise, if $x = (z, w)$ belongs to a singular flow box $U_{a,a} \in E$, which is identified with the local model with coordinates $(z, w)$, then
\[
 W(x) := \log^* \|z, w\| + \frac{|z|^2 |w|^2}{(|z|^2 + |w|^2)^2} (\log^* \|z, w\|)^2.
\]
Note that
\[
 1 \leq \log^* \text{dist}(x, E) \leq W(x) \leq 2(\log^* \text{dist}(x, E))^2.
\]

Inspired by Definition 8.3 in Candel [17], we have the following:

Definition 9.18 A real-valued function $h$ defined on $\mathbb{D}$ is called weakly moderate if there is a constant $c > 0$ such that
\[
 \log |h(\xi) - h(0)| \leq c \text{dist}_P(\xi, 0) + c, \xi \in \mathbb{D}.
\]

Remark 9.19 The notion of weak moderateness is weaker than the notion of moderateness given in [17, 76].

The usefulness of weakly moderate functions is illustrated by the following Dynkin type formula.

Lemma 9.20 Let $f \in C^2(\mathbb{D})$ be such that $f$, $|df|_P$ (see the definition after (2.6)) and $\Delta_P f$ (see (2.5)) are weakly moderate functions. Then
\[
 (D_t f)(0) - f(0) = \int_0^t (D_s \Delta_P f)(0) ds, \quad t \in \mathbb{R}^+.
\]

Using Propositions 9.2, 9.17 and 4.5 as well as Lemma 9.20, we obtain an estimate on the expansion rate up to order 2 of $\mathcal{H}(\omega, \cdot)$ in terms of $\text{dist}_P(\cdot, 0)$ and the distance $\text{dist}(x, E)$.

Proposition 9.21 There is a constant $c > 0$ such that for every $x \in X \setminus E$ and every $\xi \in \mathbb{D}$,
\[
 |\kappa_x(\xi) - \kappa_x(0)| \leq c \log^* \text{dist}(x, E) \cdot \exp (c \text{dist}_P(\xi, 0)),
\]
\[
 |d\kappa_x(\xi)|_P - |d\kappa_x(0)|_P | \leq c \log^* \text{dist}(x, E) \cdot \exp (c \text{dist}_P(\xi, 0)),
\]
\[
 |\Delta_P \kappa_x(\xi) - \Delta_P \kappa_x(0)| \leq c(\log^* \text{dist}(x, E))^2 \cdot \exp (c \text{dist}_P(\xi, 0)).
\]

We keep the hypotheses and notation of Theorem 9.1. For $t \in \mathbb{R}^+$, consider the function $F_t : X \setminus E \to \mathbb{R}$ defined by
\[
 F_t(x) := \int_{\Omega} \log \|\mathcal{H}(\omega, t)\| dW_x(\omega) \quad \text{for} \quad x \in X \setminus E. \quad (9.18)
\]
By (9.3) the Lyapunov exponent $\chi(\mu)$ can be rewritten as

$$\chi(\mu) := \int_X F_1(x) d\mu(x) = \frac{1}{t} \int_X F_1(x) d\mu(x). \quad (9.19)$$

By [78, Proposition 3.3 and Lemma 4.1], we infer that there is a constant $c > 0$ such that

$$|F_1(x)| \leq c \log^\star \text{dist}(x, E) \quad \text{for} \quad x \in X \setminus E. \quad (9.20)$$

### 9.5 Sketchy Proof of the Cohomological Formula of Lyapunov Exponent

We only give here the proof of the first formula of Theorem 9.11 under the assumption that the ambient metric $g_X$ on $X$ is equal to the Euclidean metric in a local model near every singular point of $F$. For the proof of the general case (see [80]).

The following result relates the Lyapunov exponent $\chi(\mu)$ to the function $\kappa$ defined in (9.15). It plays the key role in the proof of formula (9.3).

**Proposition 9.22** ([80, Proposition 4.6]) Under the hypotheses and notations of Theorem 9.1, the integrals $\int_X |\kappa(x)| d\mu(x)$ and $\int_X W(x) d\mu(x)$ are bounded, and the following identity holds

$$\chi(\mu) = \int_X \kappa(x) d\mu(x).$$

**Remark 9.23** In fact, the $\mu$-integrability of the weight $W$ in Proposition 9.22 implies that of the curvature density $\kappa$. This is a crucial point of the proof of the first formula of Theorem 9.11. The novelty of this proposition is that the weight $W(x)$ behaves like $(\log^\star \text{dist}(x, E))^2$ when $x = (z, w)$ satisfies $|z| \approx |w|$, whereas the new integrability condition (9.6) only provides the $\mu$-integrability of the less singular weight $\log^\star \text{dist}(x, E)$.

However, for the sake of simplicity, we do not give a full argument of this important point in the sketchy proof below.

**Sketchy proof** By Proposition 9.21, $\kappa_x, |d\kappa_x|_P$ and $\Delta_P \kappa_x$ are weakly moderate functions on $D$. Consequently, applying Lemma 9.20 yields that

$$(D_1 \kappa_x)(0) - \kappa_x(0) = \int_0^1 (D_s (\Delta_P \kappa_x))(0) ds. \quad (9.21)$$

By (9.14) and (9.18), the left hand side of (9.21) is equal to

$$E_x [\log \mathcal{H}(\omega, 1)] = F_1(x),$$

which is finite because of (9.20). On the other hand, by (9.12) and (9.15), the right hand side of (9.21) can be rewritten as

$$\int_0^1 (D_s \kappa)(x) ds.$$

Consequently, integrating both sides of (9.21) with respect to $\mu$, we get that

$$\int_X F_1(x) d\mu(x) = \int_X \left( \int_0^1 (D_s \kappa)(x) ds \right) d\mu(x). \quad (9.22)$$

Since we know by (9.19), (9.20) and (8.1) that the left integral is bounded and is equal to $\chi(\mu)$, it follows that right-side double integral is also bounded.
Next, we make a full use of the above boundedness and the upper-bound of $\kappa(x)$ for $x$ close to the singularities (see Proposition 9.17). Consequently, we can show that $\int_X |\kappa(x)|d\mu(x) < \infty$ and $\int_X W(x)d\mu(x) < \infty$. We do not give here the full explanation for the last delicate argument, but refer the reader to [80, Proposition 4.6] for more details.

Hence, we infer from this and the fact that $\mu$ is weakly harmonic that

$$\int_X \left( \int_0^1 (D_s \kappa)(x)ds \right) d\mu(x) = \int_0^1 \left( \int_X (D_s \kappa)(x)d\mu(x) \right) ds = \int_X \kappa(x)d\mu(x).$$

This, combined with (9.22) and (9.19), implies

$$\chi(\mu) = \int_X F_1(x)d\mu(x) = \int_X \kappa(x)d\mu(x).$$

**Corollary 9.24** Under the hypotheses and notations of Theorem 9.1, the following identities hold

$$\kappa g_P = -c_1 \left( \text{Nor}(\mathcal{F}), g_X^\perp \right)$$

on $X \setminus E$, and

$$\int_X \kappa(x)d\mu(x) = -\int_X c_1 \left( \text{Nor}(\mathcal{F}), g_X^\perp \right) \land T.$$

**Proof** By (9.17) we obtain that

$$\kappa g_P = -(\Delta_P \varphi)g_P = -dd^c \varphi = -c_1 \left( \text{Nor}(\mathcal{F}), g_X^\perp \right)$$

on $U$.

The first identity follows. Since $\int_X |\kappa(x)|d\mu(x) < \infty$ by Proposition 9.22. Integrating both sides of the first identity over $X \setminus E$ gives the second identity.

**Proof of the first identity of Theorem 9.11 in a special case** We only consider the special case where the ambient metric $g_X$ is equal to the Euclidean metric in a local model near every singular point of $\mathcal{F}$. Fix a smooth Hermitian metric $g_0$ on the normal bundle $\text{Nor}(\mathcal{F})$ of $\mathcal{F}$. So there is a global weight function $f : X \to (-\infty, \infty)$ such that $g_X^\perp = g_0 \exp(-2f)$. We know that the weight function $f$ is smooth outside $E$. Now we investigate the behavior of $f$ near a singular point $a \in E$.

Consider the local holomorphic section $e_L$ given by $(z,w) \mapsto \partial_{\bar{z}}$ of $\text{Tan}(X)$ over $U_a \simeq \mathbb{D}^2$. This section induces a holomorphic section $\tilde{e}_L(x) = e_L(x)/\text{Tan}(\mathcal{F})_x$ of $\text{Nor}(\mathcal{F})$ over $U_a$.

Our special assumption on the ambient metric $g_X$ implies that on $U_a$, $g_X$ coincides with the Euclidean metric. Therefore, we have, for $x = (z,w) \in \mathbb{D} \times (\mathbb{D} \setminus \{0\})$,

$$\exp(-\varphi(x)) = |\tilde{e}_L(x)|_{g_X^\perp} = \frac{1}{\sqrt{|z|^2 + |\lambda w|^2}}.$$

Hence, for $x = (z,w) \in \mathbb{D}^2 \setminus \{(0,0)\}$,

$$c_1 \left( \text{Nor}(\mathcal{F}), g_X^\perp \right)(x) = dd^c \varphi(x) = dd^c \log \sqrt{|z|^2 + |\lambda w|^2}. \quad (9.23)$$

Moreover, in the local model with coordinates $(z,w)$ associated to the singular point $E \ni a \simeq (0,0) \in \mathbb{D}^2$, it follows from (9.23) that

$$c_1 \left( \text{Nor}(\mathcal{F}), g_0 \right) + dd^c f = c_1 \left( \text{Nor}(\mathcal{F}), g_X^\perp \right)(x) = dd^c \varphi(x) = \frac{1}{2} dd^c \log \left(|z|^2 + |\lambda w|^2 \right).$$
Consequently, for $x = (z, w) \in U_a \simeq \mathbb{D}^2$,
\[
f(x) = \frac{1}{2} \log (|z|^2 + |\lambda w|^2) + \text{a smooth function in } x. \tag{9.24}
\]
Using a finite partition of the unity on $X$ and using (9.24), we can construct a family of smooth functions $(f_\epsilon)_{0 < \epsilon \ll 1}$ on $X$ such that $f_\epsilon$ converges uniformly to $f$ in $C^2$-norm on each regular flow box as $\epsilon \to 0$ and that in a local model with coordinates $(z, w)$ associated to each singular point $a \in E$,
\[
f_\epsilon - \frac{1}{2} \log (|z|^2 + |\lambda w|^2 + \epsilon^2) = f - \frac{1}{2} \log (|z|^2 + |\lambda w|^2) \quad \text{on } \mathbb{D}^2. \tag{9.25}
\]
For every $0 < \epsilon \ll 1$ we endow $\text{Nor} (\mathcal{F})$ with the metric $g_\epsilon := g_0 \exp (-2f_\epsilon)$. Since $g_\epsilon$ is smooth and the current $T$ is $ddc$-closed, it follows that
\[
c_1 (\text{Nor} (\mathcal{F}), g_\epsilon) \wedge T \overset{\epsilon \to 0}{\longrightarrow} \int_X c_1 (\text{Nor} (\mathcal{F}), g_\epsilon) \wedge T. \tag{9.26}
\]
Let $\kappa_\epsilon : X \setminus E \to \mathbb{R}$ be the function defined by
\[
-c_1 (\text{Nor} (\mathcal{F}), g_\epsilon) (x) |_{L_x} = \kappa_\epsilon (x) g_P (x). \tag{9.27}
\]
This, combined with (2.11), implies that
\[
-c_1 (\text{Nor} (\mathcal{F}), g_\epsilon) \wedge T = \kappa_\epsilon d\mu.
\]
Since $f_\epsilon$ converges uniformly to $f$ in $C^2$-norm on compact subsets of $X \setminus E$ as $\epsilon \to 0$, it follows that $\kappa_\epsilon$ converge pointwise to $\kappa$ $\mu$-almost everywhere. Hence, we get that
\[
-c_1 (\text{Nor} (\mathcal{F}), g_\epsilon) \wedge T \overset{\epsilon \to 0}{\longrightarrow} \int_X \kappa (x) d\mu (x) \quad \text{as } \epsilon \to 0.
\]
We will show that on each singular flow box $U_a \simeq \mathbb{D}^2$,
\[
-c_1 (\text{Nor} (\mathcal{F}), g_\epsilon) \wedge T \overset{\epsilon \to 0}{\longrightarrow} \int_{U_a} \kappa (x) d\mu (x) \quad \text{as } \epsilon \to 0. \tag{9.28}
\]
Taking (9.28) for granted, we combine it with the previous limit and get that
\[
-c_1 (\text{Nor} (\mathcal{F}), g_\epsilon) \wedge T \overset{\epsilon \to 0}{\longrightarrow} \int_X \kappa (x) d\mu (x) = - \int_X c_1 (\text{Nor} (\mathcal{F}), g_\frac{\perp}{X}) \wedge T \quad \text{as } \epsilon \to 0,
\]
where the last equality follows from Corollary 9.24. We deduce from this and (9.26) that
\[
-c_1 (\text{Nor} (\mathcal{F})) \sim \{ T \} = \int_X \kappa (x) d\mu (x).
\]
By Proposition 9.22, the right hand side is $\chi (\mu)$. Hence, the last equality implies the desired identity of the theorem.

Now it remains to prove (9.28). We need the following result which gives a precise behavior of $\kappa_\epsilon$ near a singular point $a$ using the local model $(\mathbb{D}^2, \mathcal{L}, \{ 0 \})$ introduced in Section 9.2.
Lemma 9.25 There is a constant $c > 1$ such that for every $0 < \epsilon \ll 1$ and for every $x = (z, w) \in (\frac{1}{2} \mathbb{D})^2$, we have that

$$-c \left( \frac{|z|^2 |w|^2}{(|z|^2 + |w|^2 + \epsilon^2)^2} + (|z|^2 + |w|^2) \right) \left( \log^* \| (z, w) \| \right)^2 \leq \kappa_\epsilon (x)$$

$$\leq \left( -c^{-1} \frac{|z|^2 |w|^2}{(|z|^2 + |w|^2 + \epsilon^2)^2} + c(|z|^2 + |w|^2) \right) \left( \log^* \| (z, w) \| \right)^2.$$

Proof Since $g_\epsilon = g_0 \exp (-2 f_\epsilon)$ we get that

$$c_1(\text{Nor}(\mathcal{F}), g_\epsilon) = c_1(\text{Nor}(\mathcal{F}), g_0) + dd^c f_\epsilon = dd^c f_\epsilon + \text{a smooth } (1, 1)\text{-form independent of } \epsilon.$$

This, together with (9.24), (9.25) and (9.27), imply that

$$\kappa_\epsilon (x) g_P (x) = -\frac{1}{2} dd^c \log ((|\lambda|^2 + |\lambda w|^2 + \epsilon^2)(x)|_{\mathcal{L}}) + \text{a smooth } (1, 1)\text{-form independent of } \epsilon.$$

Using the parametrization (9.5) the pull-back of the first term of the right hand side by $\psi_x$ is

$$-\frac{1}{2} dd^c \log (|z e^{i\xi}|^2 + |\lambda w e^{i\lambda \xi}|^2 + \epsilon^2)(0),$$

whereas the pull-back of the second term of the right hand side by $\psi_x$ is $O(|z|^2 + |w|^2) d\xi \wedge d\bar{\xi}$. A straightforward computation shows that the former expression is equal to

$$\frac{|\lambda - 1|^2}{4\pi} \frac{|z|^2 |w|^2}{(|z|^2 + |\lambda w|^2 + \epsilon^2)^2} i d\xi \wedge d\bar{\xi}.$$

On the other hand, by (9.5) we get that

$$|d\psi_x (0)| \approx \| (z, w) \| = \text{dist}(x, E).$$

Using the last two estimates and applying Proposition 4.5, the result follows. 

We resume the proof of (9.28). By Proposition 9.17 and Lemma 9.25, there is a constant $c > 1$ such that for $(z, w) \in U_\alpha \simeq \mathbb{D}^2$ that

$$|\kappa_\epsilon (z, w)| \leq c \left( \frac{|z|^2 |w|^2}{(|z|^2 + |w|^2 + \epsilon^2)^2} + (|z|^2 + |w|^2) \right) \left( \log^* \| (z, w) \| \right)^2 \leq c^2 |\kappa(z, w)| + c^2.$$

Recall from Proposition 9.22 that $\int_{U_\alpha} |\kappa (x)| d\mu (x) < \infty$. On the other hand, $\kappa_\epsilon$ converge pointwise to $\kappa$ $\mu$-almost everywhere as $\epsilon \to 0$. Consequently, by Lebesgue dominated convergence,

$$\lim_{\epsilon \to 0} \int_{U_\alpha} \kappa_\epsilon d\mu = \int_{U_\alpha} \kappa d\mu.$$

This and (9.27) imply (9.28). The proof of formula (9.3) is thereby completed in the special case where the ambient metric $g_X$ is equal to the Euclidean metric in a local model near every singular point of $\mathcal{F}$. 

9.6 Geometric Characterization of Lyapunov Exponents

To find a geometric interpretation of these characteristic quantities, our idea consists in replacing the Brownian trajectories by the more appealing objects, namely, the unit-speed geodesic rays. These paths are parameterized by their length (with respect to the leafwise Poincaré metric $g_P$). Therefore, we characterize the Lyapunov exponents in terms of the expansion rates of $\mathcal{A}$ along the geodesic rays.
Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a Riemann surface lamination with singularities. Recall from (2.1) that $(\phi_x)_{x \in \text{Hyp}(\mathcal{F})}$ is a given family of universal covering maps $\phi_x : \mathbb{D} \to L_x$ with $\phi_x(0) = x$. For every $x \in \text{Hyp}(\mathcal{F})$, the set of all unit-speed geodesic rays $\omega : [0, \infty) \to L_x$ starting at $x$ (that is, $\omega(0) = x$), can be described by the family $(\gamma_{x, \theta})_{\theta \in [0, 1]}$, where

$$\gamma_{x, \theta}(R) := \phi_x(e^{2\pi i \theta} r_R), \quad R \in \mathbb{R}^+,$$

and $r_R$ is uniquely determined by the equation $r_R \mathbb{D} = \mathbb{D}_R$ (see (5.30)). The path $\gamma_{x, \theta}$ is called the unit-speed geodesic ray at $x$ with the leaf-direction $\theta$. Unless otherwise specified, the space of leaf-directions $[0, 1)$ is endowed with the Lebesgue measure. This space is visibly identified, via the map $[0, 1) \ni \theta \mapsto e^{2\pi i \theta}$, with the unit circle $\partial \mathbb{D}$ endowed with the normalized rotation measure.

Set $\Omega := \Omega(\mathcal{F})$ as usual. We introduce the following notions of expansion rates for cocycles.

**Definition 9.26** Let $A : \Omega \times \mathbb{R}^+ \to \text{GL}(d, \mathbb{K})$ be a $\mathbb{K}$-valued cocycle and $R > 0$ a time and $x$ a point in $\text{Hyp}(\mathcal{F})$.

The expansion rate of $A$ at $x$ in the leaf-direction $\theta$ at time $R$ along the vector $v \in \mathbb{K}^d \setminus \{0\}$ is the number

$$\mathcal{E}(x, \theta, v, R) := \frac{1}{R} \log \frac{\|A(\gamma_{x, \theta}, R)v\|}{\|v\|}.$$

The expansion rate of $A$ at $x$ in the leaf-direction $\theta$ at time $R$ is

$$\mathcal{E}(x, \theta, R) := \sup_{v \in \mathbb{K}^d \setminus \{0\}} \mathcal{E}(x, \theta, v, R) = \sup_{v \in \mathbb{K}^d \setminus \{0\}} \frac{1}{R} \log \frac{\|A(\gamma_{x, \theta}, R)v\|}{\|v\|} = \frac{1}{R} \log \|A(\gamma_{x, \theta}, R)\|.$$

Given a $\mathbb{K}$-vector subspace $\{0\} \neq H \subset \mathbb{K}^d$, the expansion rate of $A$ at $x$ at time $R$ along the vector space $H$ is the interval $\mathcal{E}(x, H, R) := [a, b]$, where

$$a := \inf_{v \in H \setminus \{0\}} \int_0^1 \left( \frac{1}{R} \log \frac{\|A(\gamma_{x, \theta}, R)v\|}{\|v\|} \right) d\theta \quad \text{and} \quad b := \sup_{v \in H \setminus \{0\}} \int_0^1 \left( \frac{1}{R} \log \frac{\|A(\gamma_{x, \theta}, R)v\|}{\|v\|} \right) d\theta.$$

Notice that $\mathcal{E}(x, \theta, v, R)$ (resp. $\mathcal{E}(x, \theta, R)$) expresses geometrically the expansion rate (resp. the maximal expansion rate) of the cocycle when one travels along the unit-speed geodesic ray $\gamma_{x, \theta}$ up to time $R$. Moreover, the integral $\int_0^1 d\theta$ means that we take the average over all possible directions $\theta$. On the other hand, $\mathcal{E}(x, H, R)$ represents the smallest closed interval which contains all numbers

$$\int_0^1 \left( \frac{1}{R} \log \frac{\|A(\gamma_{x, \theta}, R)v\|}{\|v\|} \right) d\theta,$$

where $v$ ranges over $H \setminus \{0\}$. Note that the above integral is the average of the expansion rate of the cocycle when one travels along the unit-speed geodesic rays along the vector-direction $v \in H$ from $x$ to the Poincaré circle with radius $R$ and center $x$ spanned on $L_x$.

We say that a sequence of intervals $[a(R), b(R)] \subset \mathbb{R}$ indexed by $R \in \mathbb{R}^+$ converges to a number $\chi \in \mathbb{R}$ and write $\lim_{R \to \infty} [a(R), b(R)] = \chi$, if $\lim_{R \to \infty} a(R) = \lim_{R \to \infty} b(R) = \chi$.

Now we are able to state the main result of this subsection.

**Theorem 9.27** (Nguyen [76]) Let $\mathcal{F} = (X, \mathcal{L})$ be a compact smooth hyperbolic Riemann surface lamination and $\mu$ a harmonic measure which is also ergodic. Consider a smooth cocycle $A : \Omega \times \mathbb{R}^+ \to \text{GL}(d, \mathbb{K})$. Then there is a leafwise saturated Borel set $Y$ of
total $\mu$-measure which satisfies the conclusion of Theorem 8.2 and the following additional geometric properties:

(i) For each $1 \leq i \leq m$ and for each $x \in Y$, there is a set $G_x \subset [0, 1)$ of total Lebesgue measure such that for each $v \in V_i(x) \setminus V_{i+1}(x)$,

$$\lim_{R \to \infty} \delta(x, \theta, v, R) = \chi_i, \quad \theta \in G_x.$$ 

Moreover, the maximal Lyapunov exponent $\chi_1$ satisfies

$$\lim_{R \to \infty} \delta(x, \theta, R) = \chi_1, \quad \theta \in G_x.$$ 

(ii) For each $1 \leq i \leq m$ and each $x \in Y$,

$$\lim_{R \to \infty} \delta(x, H_i(x), R) = \chi_i.$$ 

Here $\mathcal{K}^d = \oplus_{i=1}^m H_i(x), x \in Y$, is the Oseledec decomposition given by Theorem 8.2 and $\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1$ are the corresponding Lyapunov exponents.

Theorem 9.27 gives geometric meaning to the stochastic formulas (8.2)–(8.3).

Let $\mathcal{F} = (X, \mathcal{L}, E)$ be a transversally smooth (resp. holomorphic) singular foliation by Riemann surfaces in a Riemannian manifold (resp. Hermitian complex manifold) $X$. Consider a leafwise saturated set $M \subset X \setminus E$ which is compact in $X$ whose leaves are all hyperbolic. So the restriction of the foliation $(X \setminus E, \mathcal{L})$ to $M$ gives an inherited compact smooth/transversally holomorphic hyperbolic Riemann lamination $(M, \mathcal{L}|_M)$. Moreover, the holonomy cocycle of $(X \setminus E, \mathcal{L})$ induces, by restriction, an inherited smooth cocycle on $(M, \mathcal{L}|_M)$. Hence, Theorem 9.27 applies to the latter cocycle. In particular, when $(X, \mathcal{L}, E)$ is a singular holomorphic foliation on a compact Hermitian complex manifold $X$ of dimension $k$, the last theorem applies to the induced holonomy cocycle of rank $k-1$ associated with every minimal set $M$ whose leaves are all hyperbolic.

The proof of Theorem 9.27 (i) relies on the theory of Brownian trajectories on hyperbolic spaces. More concretely, some quantitative results on the boundary behavior of Brownian trajectories by Lyons [71] and Cranston [25] and on the shadow of Brownian trajectories by geodesic rays are our main ingredients. This allows us to replace a Brownian trajectory by a unit-speed geodesic ray with uniformly distributed leaf-direction. Hence, Part (i) of Theorem 9.27 will follow from Theorem 9.2.

To establish Part (ii) of Theorem 9.27 we need two steps. In the first step, we adapt to our context the so-called Ledrappier type characterization of Lyapunov spectrum which was introduced in [75]. This allows us to show that a similar version of Part (ii) of Theorem 9.27 holds when the expansion rates in terms of geodesic rays are replaced by some heat diffusions associated with the cocycle. The second step shows that the above heat diffusions can be approximated by the expansion rates. To this end, we establish a new geometric estimate on the heat diffusions which relies on the proof of the geometric Birkhoff ergodic theorem (Theorem 5.36).

**Problem 9.28** Is Theorem 9.27 still true if $\mathcal{F} = (X, \mathcal{L})$ is the whole regular part $(X' \setminus E', \mathcal{L}'|_{X' \setminus E'})$ of a singular holomorphic foliation $\mathcal{F}' = (X', \mathcal{L}', E')$ by hyperbolic Riemann surfaces on a compact complex manifold $X'$ and $\mathcal{A}$ is the holonomy cocycle? We can begin with the case where $X'$ is a surface.

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