Mass vs. Charge: 
Quantum Radiation from Zero Temperature Black Holes

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ABSTRACT

We study the mass-charge relation for the semiclassical extremal black hole of the $S$-wave sector Einstein-Maxwell theory coupled to $N$ conformal scalars. The classical ratio $M/|Q| = 1$ is shown to be modified to $M/|Q| \approx 1 - k/6$ for small $k \equiv N\hbar/(12\pi Q^2)$. Furthermore, numerical study for $k < 2$ shows that $M/|Q|$ is a monotonically decreasing function of $k$. We speculate on the consequence of such a modification in the 4-dimensional context.

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In connection with the physics of black hole evaporation, extremal black holes with vanishing temperature provide interesting theoretical laboratories. Immune to the Hawking’s thermal radiation, they are the first clues as to what the final stage of the evaporation process might be. But this does not mean we can consider the classical extremal black holes as the final product of the process. For one thing, the thermal behavior is already expected to break down for near extremal cases\cite{1}. Zero Hawking temperature simply means the leading quantum effect disappears. In order to address the questions of black hole quantum physics, we need a more systematic way of treating quantized matter in nontrivial geometries. In two-dimensional models, such a method has been adopted by Callan et al. (CGHS)\cite{3}, and used extensively to study 2-D black holes semiclassically.

In this letter, we want to concentrate on the case of the extremal Reissner-Nordström black hole and to study how semiclassical effects modify one of the classical properties, namely ADM mass $M$. The model we consider is dimensionally reduced Einstein-Maxwell theory. By restricting to the spherically symmetric sector we obtain the following 2-D action\footnote{3\textsuperscript{G} = c = 1 in this letter}

$$\mathcal{S}_g = \frac{1}{4} \int d^2 x \sqrt{-g^{(2)}} e^{-2\phi} \left( R^{(2)} + 2 (\nabla \phi)^2 + 2 e^{2\phi} - F^2 \right),$$

where the 4-D metric is split into 2-D metric $g^{(2)}$ and the dilaton part

$$g^{(4)} = g^{(2)} + e^{-2\phi} \, d\Omega^2.$$\hspace{1cm} (1)

The finite mass solutions with regular horizons are the well-known Reissner-Nordström solutions with mass $M$ and charge $Q$ satisfying the inequality $M \geq |Q|$.

$$g^{(4)} = -F(r) \, dt^2 + \frac{dr^2}{F(r)} + r^2 \, d\Omega^2, \quad F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \hspace{1cm} (3)$$

When the inequality is saturated, $F(r)$ has a double zero at the horizon $r = M = |Q|$ and the corresponding extremal black hole has zero Hawking temperature, hence no thermal radiation emanates from the horizon a long time after the black hole formation.
This implies that the usual late time estimate of the Bogoliubov transformation is not the leading quantum correction. It vanishes identically and we need to study next the nonvanishing contribution, which may or may not depend on the history of the collapse.

For this purpose, we can follow CGHS and couple $N$ conformal scalars to the above 2-D action. One can regard these 2-D scalars as the S-wave part of massless 4-D fermions, alternatively. Integrating them out completely, which is possible since we are in a two-dimensional toy world, produces the non-local Polyakov-Liouville action with a particular coefficient, which summarizes the effect of the quantized matter on gravity. Furthermore, this effective semi-classical gravity can be conveniently handled with the introduction of a scalar field $z$ with a background charge in the following manner,

$$S = S_g - \frac{N\hbar}{24\pi} \int d^2x \sqrt{-g^{(2)}} \left( (\nabla z)^2 - zR^{(2)} \right).$$

Since the field equation for $z$ reduces the second term to the original Polyakov-Liouville action of central charge $N$, solving this theory at tree level is equivalent to studying the semi-classical theory of 2-D gravity coupled to $N$ scalars. These are the leading terms in the large $N$ expansion of the full 2-D quantum theory, where $N$ is large but $N\hbar$ is of order one.

Notice that the scalar curvature $R^{(2)}$ acts as an external source coupled to the $z$ field. This effectively induces the usual Hawking radiation in a classical black hole geometry, i.e., a stationary point of $S_g$ only. Because the classical black holes radiate, the theory does not have any static solution with finite mass and regular horizon of finite temperature. The only static solutions of finite mass are those of zero temperature, which were first studied by S. Trivedi.

But first let us consider the effect of quantizing the original $N$ conformal scalars in a classical background. For example, given a classical geometry, the expectation value of the matter energy momentum tensor can be found simply by evaluating the classical energy momentum tensor of $z$-field on that classical background. A family of classical
geometries known as the Vaidya metric\(^7\) is particularly relevant to our discussion.

\[
g^{(4)} = -(1 - \frac{2m(v)}{r} + \frac{e^2(v)}{r^2})dv^2 + 2dv \, dr + r^2 d\Omega^2
\]  

(5)

It represents a collapsing massless shell whose cumulative energy and charge at retarded time \(v\) are \(m(v)\) and \(e(v)\). For smooth \(m\) and \(e^2\), the cosmic censorship is achieved by requiring the positive energy condition for the shell\(^8\).

For our purposes, however, it is appropriate to choose

\[
m(v) = M\theta(v - v_0), \quad e^2(v) = Q^2\theta(v - v_0),
\]

(6)

where \(\theta\) is the usual step function. The geometry is then that of an initial Minkowski spacetime glued to a Reissner-Nordström black hole across an ingoing null shock wave located at \(v = v_0\). Introducing a new coordinate \(u = v - 2 \int F^{-1}(r) \, dr\) with \(F(r)\) as in eq. (3), \((v, u)\) form a pair of light-cone coordinates above the shock,

\[
g^{(2)} = -F(r) \, dvdu, \quad v > v_0.
\]

(7)

In these coordinates, \(v \to \infty\) is the future null infinity, and \(u \to \infty\) is the future event horizon. Suppose we impose an initial condition on the \(N\) matter fields such that the expectation value of the energy-momentum tensor vanishes in the Minkowskian region. Using energy-momentum tensor conservation, this can be translated into

\[
<T_{uu}>|_{v=v_0} = \left(\frac{Nh}{12\pi}(\partial^2_u \rho - (\partial_u \rho)^2) + t_{uu}(u)\right)|_{v=v_0} = 0
\]

\[
\rho = \left(\frac{1}{2} \log F\right),
\]

(8)

where \(t_{uu}\) comes from the homogeneous part of the solution to field equation. This implies the following form of \(< T_{uu} >\) as we approach the future null infinity.

\[
<T_{uu}>|_{v \to \infty} = -t_{uu}(u) = \frac{Nh}{12\pi} \left(\frac{1}{16} F'(r)^2 - \frac{1}{8} F(r) F''(r)\right) |_{u=v_0-2 \int F^{-1} \, dr}.
\]

(9)

\(^4\)Of course, we have introduced an external charged matter source to create the shell itself.
As $u \to \infty$, this clearly shows a steady flux proportional to the temperature squared \((F'(r \to \text{horizon}) \sim T_{\text{Hawking}})\). Also as expected, this asymptotically steady flux is absent, if \(M\) is equal to \(|Q|\) so that \(F'(r \to \text{horizon}) = 0\). However, there is a finite integrated flux; the total energy radiated is

\[
\Delta M = \int_{-\infty}^{\infty} < T_{uu} >_{|v| \to \infty} du = \frac{N \hbar}{96\pi} \int_{|Q|}^{\infty} \frac{F'F'}{F} dr = \frac{k}{6}|Q|,
\]

where we used \(F'(r = |Q|) = 0\) for the \(M = |Q|\) case. Since we ignored the gravitational backreaction, this estimate is valid only for small \(k \equiv N\hbar/(12\pi Q^2)\), or equivalently for large black holes. Notice that \(\Delta M\) is positive for any \(F \geq 0\) with a double zero at the event horizon. \(\Delta M\) represents the energy radiated away by the quantized matter, and after properly taking into account the gravitational backreaction, the Bondi Mass of the system should approach as \(u \to \infty\)

\[
|Q|(1 - \frac{k}{6} + O(k^2)).
\]

One might assert that it is not clear whether the estimated loss depends on the particular history of the collapse chosen. After all, the metric chosen can never be realized, since one cannot assimilate the collapsing process by a smooth version of the shock wave. As shown in [8], for smooth \(m(v)\) and \(e^2(v)\) satisfying the positive energy condition, the extremality can never be achieved in finite time. It might be that a realistic collapse scenario produces different \(\Delta M\). We will show that the above estimate of energy loss is robust by finding numerically the ADM mass of the semi-classical analogue of the extremal black hole which must be the end stage of the process described so far.

Semi-classical static solutions with extremal horizons have been studied near the horizon[3]. The requirement of zero temperature horizon specifies a unique initial condition at the horizon for a given total charge, and the resulting static solution is known to be asymptotically flat. There is no known analytical form of the solution, but it is, in principle, possible to carry out numerical integration.

Before going into details of the simulations performed, it is helpful to discuss other static solutions of finite mass, all of which have naked singularities. Those with smaller
masses, to be called supercritical, are qualitatively similar to the classical ones with $M < Q$. The radius $e^{-\phi}$ monotonically decreases as we approach the naked singularity at near origin. On the other hand, solutions with larger masses, to be called subcritical, are quite different from classical analogues $M > Q$, which have curvature singularity at the center $e^{-\phi} = 0$ hidden by two layers of nonextremal horizons, since we assume no heat bath to support nonextremal horizons. (Heat bath makes ADM mass infinite.) More specifically, a semi-classical subcritical solution has a lower bound on the value of the radius $e^{-\phi}$ near would-be horizon. One can distinguish the two species by observing whether the simulation stops in the middle or continues all the way to the critical value of the radius $e^{-\phi_{cr}} \equiv \sqrt{kQ^2}$.

Coming back to the actual simulation, it turns out that static field equations can be decoupled to produce a single first order differential equation with the following gauge choice.

$$g^{(2)} = -A^2 \, dt^2 + B^2 \, Q^2 \, dr^2, \quad e^{-2\phi} = Q^2 \, r^2. \quad (12)$$

In this gauge we can extract two independent first order differential equations.

$$k\left(\frac{A'}{A}\right)^2 + 2r\left(\frac{A'}{A}\right) + (1 - B^2 + \frac{B^2}{r^2}) = 0$$

$$(r^2 - k)(\frac{A'}{A} - \frac{B'}{B}) + (r^2 + k)\frac{B^2}{r^3} - r(B^2 - 1) = 0. \quad (13)$$

Solving for $A'/A$ in terms of $B$ produces a first order differential equation for $1/B^2$. We performed two independent simulations. First, we started from the asymptotic region with the initial condition determined by $M/|Q|$, and searched for the range of $M/|Q|$ producing an extremal horizon (a double zero of $1/B^2$). Secondly, we integrate outward from the known behavior near the extremal horizon and extract the ADM mass by fitting the curve in the asymptotic region. Since the initial points are near, but not quite at the horizon or $r = \infty$, we needed to calculate accurate initial conditions. Symbolic expansions of $1/B^2$ in appropriate coordinates, solving the equation above approximately, are used
for this purpose. Fortunately, the nonanalytic behavior of the metric near the horizon emphasized in [6] does not occur for \(1/B^2\) as a function of \(r\). We used MATHEMATICA for all numerical and symbolic calculations as well as preparation of the plot. As we improved the accuracy of the numerical calculation by supplying more accurate initial data, and also by increasing the intrinsic accuracy of the program used, the results from each simulation converge to each other. The data for \(M/|Q|\) obtained by the two methods coincide to an accuracy of \(10^{-6}\).

The simulation is carried out only for \(k < 2\) because the extremal horizon disappears beyond \(k = 2\), when the horizon radius is equal to the critical value of the dilaton \(e^{-\phi_{cr}} = \sqrt{kQ^2}\). The plot of \(M/|Q|\) as a function of \(k \equiv N\hbar/(12\pi Q^2)\) (Fig. 1) clearly shows the initial slope of \(-1/6\) calculated above. Furthermore, up to \(k = 2\), the ratio continues to drop as we increase \(N\hbar\) or decrease the charge \(|Q|\).

What can we learn from this little demonstration? The first and foremost fact is that higher order corrections to Hawking’s calculation must be taken account into even for such a crude operation as mass measurement. One should expect that a similar mechanism works for four-dimensionally black holes and the classical bound \(M \geq |Q|\) is modified, unless some unbroken extended supersymmetry protects it. But the model we used gives few clues as to what the modification might be. While 2-D conformal scalars can be interpreted as the \(S\)-wave modes of 4-D massless fermions, we cannot regard our model as a quantitative approximation to the full 4-D physics. There is no generic mass gap present to separate \(S\)-wave fermions out from the rest.

Nevertheless this doesn’t prevent us from speculating on the effect of such a modified mass-charge relation in 4-D. In particular, suppose the same monotonic decreasing behavior is realized for the four-dimensionally extremal black holes. The possibility has been contemplated by J. Preskill for electrically charged extremal black holes with emphasis on charge renormalization [10]. The most immediate consequence would be to lift the well-known degeneracy for multi-extremal black hole configurations. Classically, a family
of solutions known as the Papapetrou-Majumdar space-time\[9\], describes many extremal black holes at rest relative to one another. The total ADM mass of such a solution is the sum of the individual masses,

$$M = \sum_{i} |Q_i|.$$  \hspace{1cm} (14)

This can easily be seen by imagining each hole separated from one another far away, so that whatever potential energy there might be becomes negligible. In fact, there is no potential between individual black holes and the total mass is given by eq. (14) for any finite separations. Therefore two different multi-black hole configurations in equilibrium have the same energy provided that the sum of absolute value of the charges are equal. But, with the modified $M/|Q|$ which decreases as $Q^2$ decreases, the same reasoning shows that it is energetically favorable to split one big black hole into many smaller ones. The classical degeneracy is lifted.

Classical physics forbids such a bifurcation process, since it violates the second law of black hole thermodynamics. However, there has been suggestions of possible finite action instantons mediating bifurcation of the extremal Reissner-Nordström black holes. In fact, D. Brill found an instanton of finite action interpolating between two Bertotti-Robinson metrics with different numbers of necks\[11\]. It is well known that a Bertotti-Robinson metric with a single neck approximates an extremal Reissner-Nordström black hole near the horizon. If the initial and the final states are of the same energy, the instanton will take infinite Euclidean time to make the transition, and the stationary state would be a linear combination of the two classical configurations. With the modified $M/|Q|$ relation however, a relevant Euclidean solution is a bounce solution and a big extremal Reissner-Nordström black would decay to many extremal black holes of smaller charges distantly separated.\[11\]

So far, we have completely ignored the possible presence of charged matter fields. Suppose there is an elementary charged particle of mass $m$ and charge $e$ and consider

\footnote{A similar observation has been made in the context of classical dilatonic black holes with a massive dilaton\[12\].}
an extremal black hole of mass $M$ and charge $Q$. For $m << |e|$, the Schwinger pair production near the horizon is always dominant over a possible bifurcation process and the black hole charge will eventually be wiped out. But for sufficiently large $m > m_{\text{min}}$, it will be kinematically impossible for an extremal black hole to lose its charge by emitting these charged particles\[10\]. For a large black hole $M \gg m$ in particular, we have $m_{\text{min}}/|e| \simeq M/|Q|$. Therefore the model we considered should be regarded as a possible scenario for magnetically charged extremal black holes in a world where the magnetic monopole comes with mass comparable to, or even larger than, its charge in Planck units.

In summary we have found that the classical inequality $M \geq |Q|$ can be modified through semi-classical effects. It would be most interesting to find out about similar effects in the context of 4-dimensional models but it is beyond the scope of this letter.

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Figure Caption

Fig.1: Plot of $M/|Q|$ versus $k \equiv N\bar{h}/(12\pi Q^2)$. The straight line shows the leading behaviour $M/|Q| = 1 - k/6$. The dots are the actual numerical results from the two independent simulations. Data points are at $k = n/10$ for $n = 1, ..., 19$ as well as $k = 0.001$. 
