Feynman Equation in Hamiltonian Quantum Field Theory

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Abstract

Functional Schrödinger equations for interacting fields are solved via rigorous non-perturbative Feynman type integrals.

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TO I.E.SEGAL, IN MEMORIAM

1 Introduction.

Semantically, quantum field theory means either a theory of quantum fields, or a quantum theory of fields. Mathematically, it describes either a classical evolution of operator fields, or an operator evolution of classical fields.

The Feynman equation in quantum theory of fields is

\[ \langle \phi'' , t'' | \phi' , t' \rangle = \sum_{\phi'} e^{i A_{\phi'}(\phi)} . \]

(We assume the Planck constant \( \hbar = 1 \).)

The left hand side of the equation is the probability amplitude of a quantum transition between classical fields \( \phi'(x) \) and \( \phi''(x) \) on the euclidean space \( \mathbb{R}^d \) at times \( t' \) and \( t'' \).

The right hand side of the equation is a sum over classical histories \( \phi = \phi(x,t) \) from \( t' \) to \( t'' \) on the Minkowski space-time \( \mathbb{R}^{d,1} \).
The action functional $A_t''$ is

$$A_t''(\phi) = \int_{t'}^{t''} dt \int_{\mathbb{R}^d} dx \, l(\phi, \nabla \phi, \dot{\phi}),$$

where the Lagrangean density $l$ is a Lorentz invariant real function of $\phi$ and its space and time derivatives $\nabla \phi, \dot{\phi}$.

A short-hand notation for the “sum” is the 1948 *Lagrangian Feynman integral*

$$\int_{\phi'}^{\phi''} D(\phi) \, e^{iA_t''(\phi)}.$$

This compact notation suggests the *standard algorithms of the elementary integral calculus*, including repeated integration, integration by parts, substitution rule, WKB approximation, Gaussian integrals and so on.

Another form of the Feynman integral is a *Hamiltonian Feynman integral* (R.Feynman [9] in 1951, W.Tobocman [19] in 1956)

$$\int_{\phi'}^{\phi''} D(\phi, \pi) \, e^{iA_t''(\phi, \pi)},$$

where $\pi = \partial l / \partial \dot{\phi}$ and $A_t''(\phi, \pi) = \int_{t'}^{t''} dt \int_{\mathbb{R}^d} dx \, [\pi \dot{\phi} - h(\phi, \pi)]$ with the *Hamiltonian function* $h(\phi, \pi) = \pi \dot{\phi} - l(\phi, \nabla \phi, \pi)$.

In 1960 J.Klauder [12] introduced the *Feynman integral over the coherent state histories* with $h$ being the Wick symbol of the quantum evolution.

In 1973 E.Lieb [15] modified the Klauder construction using the anti-Wick symbol of the quantum evolution.

Until now, in spite of their fundamental importance in quantum field theory, these Feynman integrals have been largely unjustified in rigorous mathematical terms.

According to canonical formalism, the quantum evolution in the left hand side of the Feynman equation is defined by a self-adjoint Hamiltonian operator $H$. However, until now even its domain has been largely problematic.

In this paper, for a wide variety of interaction Hamiltonians, we define rigorously both sides of the Feynman equation and show that they are equal under appropriate conditions.

In particular, we establish a rigorous equivalence of the corresponding canonical and path integral quantizations in such cases.
The construction of the Feynman type functional integral is fairly new. As in the Feynman original approach, it is of sequential type, but the Feynman-Tobocman semi-classical postulate for short time propagators is modified as in [8].

We use the Feynman-type integral to solve the corresponding functional temporal Schrödinger equation via a limit of multiple functional integrals over the infinite-dimensional phase space.

Because the solution is non-perturbative, all renormalization problems are circumvented.

The main results are in the section 4. They have been partially presented at the Conference on Feynman Integrals and Related Topics, July, 1999, Seoul, Korea, and at the Special Session on The Feynman Integral with Applications of the Annual Meeting of the American Mathematical Society, Washington, D.C., January, 2000.

This paper is an independent sequel of [8].

2 Review of Segal boson systems.

2.1 Segal boson system (cf.[2]).

The Segal boson system \((\mathcal{F}, \hat{\cdot}, \Omega, H_0)\) over a phase space \(\mathcal{H}\) is a universal model for concrete free boson fields of positive mass. According to I.Segal, this is a universal free boson field [2].

The phase space \(\mathcal{H}\) is a complex separable Hilbert space with a hermitean sesquilinear form \(\langle \cdot | \cdot \rangle\). By physicists convention, the form is antilinear on the left.

As a phase space, \(\mathcal{H}\) is a symplectic vector space with the symplectic form \(\Im\langle \psi_1 | \psi_2 \rangle\), the imaginary part of the Hermitean product on \(\mathcal{H}\).

The four constituents \((\mathcal{F}, \hat{\cdot}, \Omega, H_0)\) are defined axiomatically as follows.

- The abstract Fock space \(\mathcal{F}\) is a complex Hilbert space of quantum states \(\Psi\).
- The Heisenberg canonical commutation relation \(\hat{\cdot}\) (or CCR) is a continuous \(\mathcal{R}\)-linear mapping of phases \(\psi \in \mathcal{H}\) to self-adjoint operators \(\hat{\psi}\) on \(\mathcal{F}\) satisfying

\[
\hat{\psi}_1 \hat{\psi}_2 - \hat{\psi}_2 \hat{\psi}_1 = -i\Im\langle \psi_1 | \psi_2 \rangle \mathbf{1}.
\]
- The vacuum $\Omega \in F$ is a fixed fiducial quantum state, i.e., the linear span of $\exp(i\hat{\psi})\Omega$, $\psi \in \mathcal{H}$, is dense in $F$.
- The free Hamiltonian operator $H_0$ is a non-negative non-zero self-adjoint operator on $F$ such that $H_0\Omega = 0$ and
  
  \[ H_0\hat{\psi} - \hat{\psi}H_0 = i\hat{\psi}. \]

The continuity, linearity and the commutator relations for $\hat{\psi}$ are understood in terms of unitary operators $\exp(i\hat{\psi})$ (cf.[2]).

In spite of uncountably many unitary non-equivalent CCR, we have the following fundamental Segal’s theorem [2]:

- A Segal boson system $(F, \hat{\cdot}, \Omega, H_0)$ over a phase space $\mathcal{H}$ is unique up to unitary equivalence.
- For every self-adjoint operator $a$ on $\mathcal{H}$ there is a unique self-adjoint operator $\hat{a}$ on $F$ such that for all $\psi \in \mathcal{H}$
  
  \[ \hat{a}\hat{\psi} - \hat{\psi}\hat{a} = \hat{a}\psi. \]

In particular, $\hat{1} = H_0$.
- Moreover, $\hat{a} \geq 0$ if $a \geq 0$.

In view of Segal’s fundamental theorem, the Segal boson system is defined by the dimension of its phase space.

The Schrödinger formulation of the quantum mechanical harmonic oscillator is the Segal system over a finite-dimensional phase space. It is known as a first quantization of the classical harmonic oscillator. Its Fock space is a single particle space.

If one takes the infinite dimensional Fock space of the first quantization as a new phase space, the corresponding Segal system is the second quantization of the classical harmonic oscillator. Its free Hamiltonian $H_0$ is the number operator.

The Fock-Cook tensor representation of the CCR for a free relativistic boson system of positive mass (cf.[5]) is the second quantized Segal system.

Along with the Fock-Cook tensor representation of the Segal system, two Gaussian representations are most important (cf.[2]).
2.2 Real Gaussian representation.

A conjugation on the phase space $\mathcal{H}$ is an antilinear isometric involution $\psi \rightarrow \psi^*$. The invariant phases $\phi = \phi^*$ form the corresponding real part $\mathbb{R}^* \mathcal{H}$ of $\mathcal{H}$.

Let $D[\phi]$ be the functional measure on $\mathbb{R}^* \mathcal{H}$ defined as a weak inductive limit of the euclidean measures on real finite-dimensional subspaces with the euclidean scalar products $\Re \pi \langle \phi'' | \phi' \rangle$ (cf., e.g., [4], where the functional measure is defined on a pre-hilbert space and shown to satisfy the standard rules of elementary integral calculus).

The functional Gaussian measure $e^{-\langle \phi | \phi \rangle} D[\phi]$ is the corresponding weak inductive limit of Gaussian measures on real finite-dimensional subspaces of $\mathbb{R}^* \mathcal{H}$ (cf. [4]).

The Fock space $\mathcal{F}$ of the real Gaussian representation is the Hilbert space completion $L^2(\mathbb{R}^* \mathcal{H}, e^{-\langle \phi | \phi \rangle} D[\phi])$ of the complex span of the real-analytic polynomials $\Psi(\psi)$ on $\mathbb{R}^* \mathcal{H}$.

For real* phases $\phi = \phi^*$ the CCR representation is

$$(\hat{\phi} \Psi)(\phi') = \langle \phi | \phi \rangle \Psi(\phi'),$$

where $d_\psi$ is the functional derivative in the direction of $\psi$.

For imaginary* phases $\psi = i\phi$ the CCR representation is

$$(i \hat{\phi} \Psi)(\phi') = \frac{1}{i} (d_\phi \Psi)(\phi) + \frac{1}{i} \langle \phi | \phi' \rangle \Psi(\phi').$$

The vacuum vector $\Omega$ is the Gaussian vector $\exp(-\Re \langle \phi | \phi \rangle)$.

The free Hamiltonian $H_0$ is $d^\dagger d$, where $d$ is the functional differential and $d^\dagger$ is its Hermitian adjoint.

2.3 Complex Gaussian representation.

The Fock space of the complex Gaussian representation is the closure $\mathcal{B}^2(\mathcal{H})$ in the Hilbert space $L^2(\mathcal{H}, e^{-\langle \psi | \psi \rangle} D[\psi])$ of the complex span of the antiholomorphic polynomials $\Psi(\psi)$ on $\mathcal{H}$. The functional measure $D[\psi]$ is defined as the inductive limit of the euclidean measures on finite-dimensional complex subspaces with the euclidean scalar products $\pi \langle \psi'' | \psi' \rangle$.

The CCR representation is

$$(\hat{\psi} \Psi)(\psi') = \frac{1}{\sqrt{2}} \left[ \overline{d_\psi \Psi}(\psi') + \langle \psi' | \psi \rangle \Psi(\psi') \right],$$

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where $\partial_\psi$ is the anti-complex functional derivative.

The vacuum vector $\Omega$ is the constant 1.

The free Hamiltonian $H_0$ is $\bar{\mathcal{D}}^* \mathcal{D}$, where $\mathcal{D}$ is the (unbounded) functional anti-complex differential on $\mathcal{B}^2(\mathcal{H})$ and $\bar{\mathcal{D}}^*$ is its Hermitian adjoint.

### 2.4 Hilbert scales (cf.[16] and [2]).

A self-adjoint operator $s \geq 1$ on $\mathcal{H}$ is called a scaling operator. For $\rho \geq 0$ define the Hilbert spaces

$$\mathcal{H}_\rho = \{ \psi \in \mathcal{H} : ||\psi||_\rho = ||s^{\rho/2}\psi|| < \infty \},$$

and the Hilbert spaces $\mathcal{H}_{-\rho}$ which are the completion of $\mathcal{H}$ relative to the norm $||\psi||_\rho = ||(1+s)^{-\rho/2}\psi||$.

The family of $(\mathcal{H}_\rho, \rho \in \mathbb{R})$ form a Hilbert scale: $\mathcal{H}_{\rho+1}$ are densely and continuously imbedded in $\mathcal{H}_\rho$.

The topological intersection $\mathcal{H}_\infty = \cap \mathcal{H}_\rho$ is the core of the scale. The core is naturally a Frechet space.

A Hilbert scale is Hilbert-Schmidt if the inverse of the scaling operator is a Hilbert-Schmidt operator. The core of a Hilbert-Schmidt scale is nuclear.

The spaces $\mathcal{H}_\rho$ and $\mathcal{H}_{-\rho}$ are anti-dual relative to the basic hermitian form $\langle \cdot | \cdot \rangle$, and so are the core and the topological union $\mathcal{H}_{-\infty} = \cup \mathcal{H}_{-\rho}$.

The Hilbert scale construction is applicable to the state space $\mathcal{F}$ with the scaling operator $1 + \hat{s}$. This gives a Hilbert scale $(\mathcal{F}_\rho)$ with the core $\mathcal{F}_\infty$ and its anti-dual $\mathcal{F}_{-\infty}$.

**Henceforth, we deal with infinite-dimensional phase space $\mathcal{H}$ only.**

In its first quantized realization we choose the one-dimensional harmonic oscillator to be the scaling operator $s$. Now the core $\mathcal{H}_\infty$ is nuclear (though the core $\mathcal{F}_\infty$ is not), and $\Omega \in \mathcal{F}_\infty$.

Moreover, the vacuum vector $\Omega \in \mathcal{F}_{-\infty}$.

In the definitions of the functional measures $D[\phi]$ and $D[\psi]$ it is possible to choose the finite-dimensional subspaces from the core $\mathcal{R}^*\mathcal{H}_\infty$. Thus the integration over $\mathcal{R}^*\mathcal{H}$ and $\mathcal{H}$ coinside with the integration over $\mathcal{R}^*\mathcal{H}_\infty$ and $\mathcal{H}_{\infty}$.
Since the core $\mathcal{H}_\infty$ is nuclear, the Minlos theorem (cf.[10]) states that the integration relative to the functional Gaussian measures is equivalent to the integration relative to the Radon Gaussian measure on $\mathcal{H}_\infty$.

2.5 Coordinate representations.

Consider a topological subset $X \subset \mathcal{H}_{-\infty}$ with a Borel or a functional measure $\mu(\chi)$ on $X$.

By definition, $(X, \mu)$ is a Plancherel basis in $\mathcal{H}$ if $\int_X d\mu(\chi)|\chi\rangle\langle\chi|$ is an orthogonal resolution of the identity $1$ on $\mathcal{H}$. This means that if $\psi(\chi) = \langle\chi|\psi\rangle$ for $\psi \in \mathcal{H}_\infty$ then $\psi = \int_X d\mu(\chi)\psi(\chi)$ and $\langle\psi'\prime|\psi'\prime\prime\rangle = \int_X d\mu(\chi)\psi''(\chi)\psi'(\chi)$.

By extension, these two equations hold for all $\psi, \psi', \psi'' \in \mathcal{H}$. Then $\psi(\chi)$ are defined only for almost all $\chi$.

By a version of the spectral Gelfand-Kostuchenko theorem, a finite family of commuting self-adjoint operators on $\mathcal{H}$ has a common Plancherel eigenbase $X$ because $\mathcal{H}_\infty$ is nuclear.

Examples:
1) $\mathcal{H}$ is a functional space $L^2(\mathbb{R}^d, \gamma(x)dx)$ with $\gamma \in L^1(\mathbb{R}^d)$. The commuting self-adjoint operators are multiplications with coordinate functions. Then $(\mathbb{R}^d, \gamma(x)dx)$ is a Plancherel basis.

2) Let $X$ be the set of the eigenvectors of the scaling operator $s$ and $\nu$ be the counting measure on $X$. Then $(X, \nu)$ is a Plancherel basis. In the Plancherel expansion of $\mathcal{H}$ the scaling operator $s$ acts on $\psi$ as the multiplication $\lambda(\chi)\psi(\chi)$ with the eigenvalues $\lambda(\chi)$.

3) Suppose a conjugation on $\mathcal{H}$ commutes with $s$. Then it defines a conjugation on all $\mathcal{H}_\rho$, and we get real Hilbert scales $\mathbb{R}^*\mathcal{H}_\rho$.

Then in the associated Plancherel expansion $\mathcal{H} = L^2(X, \mu)$ the conjugation becomes the usual complex conjugation.

If $\mathcal{H} = L^2(\mathbb{R}^d, \gamma(x)dx)$, then the real phases represent classical fields.

2.6 Annihilation and Creation operators (cf.[1]).

For every $\psi$ in $\mathcal{H}$ define a closed annihilation operator on $\mathcal{F}$

$$A(\psi) = [\hat{\psi} + i(\hat{i}\psi)]/\sqrt{2}.$$ 

It is antilinear in the parameter $\psi$ and annihilates the vacuum $\Omega$. 

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The annihilation operators are continuous on $\mathcal{F}_\infty$ and commute. Moreover they are strongly continuous in the parameter $\psi$ relative to the topology of $\mathcal{H}_{-\infty}$. Then by continuous extension in $\psi$, they are defined for every $\psi \in \mathcal{H}_{-\infty}$ as commuting continuous operators on $\mathcal{F}_\infty$.

The sesquilinear map $(\psi, \Psi) \mapsto A(\psi)\Psi$ is jointly continuous from $\mathcal{H}_{-\infty} \times \mathcal{F}_\infty$ to $\mathcal{F}_\infty$.

In particular, the annihilation operators $A(\chi)$ are well defined on $\mathcal{F}_\infty$ and parametrically continuous on $X$.

The creation operators $C(\psi)$, $\psi \in \mathcal{H}_{-\infty}$, are the adjoints of $A(\psi)$ relative to the hermitian form $\langle \cdot | \cdot \rangle$. They are continuous operators on $\mathcal{F}_{-\infty}$ and commute. The creation operators $C(\psi)$ are linear in $\psi \in \mathcal{H}_{-\infty}$.

In the complex Gaussian representation $A(\psi) = \frac{\partial}{\partial \psi}$ and $C(\psi)$ is the multiplication with $\langle \psi | \cdot \rangle$.

In $(X, \mu)$-coordinates
\[ A(\psi) = \int_X d\mu(\chi) \overline{\psi(\chi)} A(\chi), \quad C(\psi) = \int_X d\mu(\chi) \psi(\chi) C(\chi). \]

in $\mathcal{F}_\infty$ and $\mathcal{F}_{-\infty}$, correspondingly.

The coherent states are $e^\psi = \sum_{n=0}^{\infty} \frac{1}{n!} C(\psi)^n \Omega$, $\psi \in \mathcal{F}$. Thus $\langle e^{\psi'} | e^{\psi''} \rangle = e^{\langle \psi'' | \psi' \rangle}$.

In the complex Gaussian representation, the coherent states are $e^\psi(\psi') = e^{\langle \psi | \psi' \rangle}$.

A coherent state $e^\psi$ belongs to $\mathcal{F}_\infty$ if and only if $\psi \in \mathcal{H}_\infty$.

Set $\Psi(\psi) = \langle \psi | \psi \rangle$ for $\Psi \in \mathcal{F}$. Then
\[ \langle \Psi'' | \Psi' \rangle = \int_{\mathcal{H}} D[\psi] e^{\langle \psi' | \psi'' \rangle - \langle \psi' | \psi \rangle} \overline{\Psi''(\psi)} \Psi'(\psi). \]

Thus $(\mathcal{H}, e^{-(\psi | \psi)} D[\psi])$ is a Plancherel basis in $\mathcal{B}^2(\mathcal{H})$.

3 Revision of the Lascar infinite-dimensional pseudodifferential operators (cf. [14]).

3.1 Wick and Berezin symbols.

Consider a continuous linear operator $Q$ from $\mathcal{F}_\infty$ to $\mathcal{F}_{-\infty}$. Among its various integral kernels we have the coherent state matrix element
\[ \langle e^{\psi''} | Q | e^{\psi'} \rangle : \]
\[ (Q \Psi)(\psi'') = \int D[\psi''] e^{(\psi'' - \psi'') \langle e^{\psi''} | Q | e^{\psi'} \rangle}. \]

As an entire function of \( \psi', \psi'' \) on \( \overline{\mathcal{H}_\infty} \times \mathcal{H}_\infty \), the coherent state matrix element is completely defined by its restriction \( \langle e^{\psi} | Q | e^{\psi} \rangle \) to the real diagonal.

The Wick (or normal) symbol \( Q^w \) of \( Q \) is
\[ Q^w(\psi) = e^{-\langle \psi | \psi \rangle} \langle e^{\psi} | Q | e^{\psi} \rangle. \]

Suppose \( \mathcal{F}_\infty \) is invariant under \( Q_1 \) and \( Q_2 \). Then the operator product \( Q_2 Q_1 \) is well defined on \( \mathcal{F}_\infty \) with the Wick symbol
\[ (Q_2 Q_1)^w(\psi) = \int D[\psi'] e^{-(\psi - \psi') \langle \psi' | Q_2(\psi, \psi') Q_1(\psi') \rangle}. \]

The Wick symbol of the adjoint operator \( Q^\dagger \) is \( \overline{Q^w(\psi)} \). Thus the operator \( Q \) is symmetric on \( \mathcal{F} \) if and only if its Wick symbol is real.

If an operator \( Q \) has a Toeplitz integral kernel:
\[ (Q \Psi)(\psi'') = \int D[\psi'] e^{(\psi'' - \psi') \langle e^{\psi'} | Q^b(\psi') \rangle} \Psi(\psi'), \]
with \( Q^b \in \mathcal{F}_\infty \) then \( Q^b(\psi) \) is the Berezin symbol of \( Q \). It is uniquely defined by \( Q \).

If \( Q^b \) exists then the Berezin symbol of the adjoint \( Q^\dagger \) exists and equals to \( \overline{Q^b(\psi)} \).

Thus the operator \( Q \) is symmetric on \( \mathcal{F} \) if and only if \( Q^b(\psi) \) is real.

The decisive advantage of the Berezin symbol is that the numerical range \( \{ \langle \psi | Q | \psi \rangle : \langle \psi | \psi \rangle = 1, \psi \in \mathcal{H}_\infty \} \) of \( Q \) is a subset of the closed convex hull of the range of \( Q^b \) in \( \mathbb{C} \).

It follows that the operator norm of \( Q \) is majorized by the supremum of \( |Q^B| \) on \( \mathcal{H} \).

Let \( \rho > 0 \), \( r \in \mathbb{R} \). A functional \( F = F(\psi) \) is of the class \( S^r_\rho \) if for any \( n \) there exists a constant \( C \) such that the \( n \)-th Frechet differential of \( F \)
\[ |||(d^n F)(\psi)|||_{-\rho} \leq C(1 + ||\psi||_{-\rho})^{r-n}, \]
where \( ||| \cdot |||_{-\rho} \) is the norm of a polynomial on \( \mathcal{H}_{-\rho} \).
Any such \( F \) is the Berezin symbol \( Q^b \) of a continuous linear operator \( Q : \mathcal{F}_\infty \to \mathcal{F}_{-\infty} \). The operator is a \( \rho \)-pseudodifferential operator of the class \( \text{Op}(\mathcal{S}_\rho^*) \) and order \( r \).

The product \( Q_2 Q_1 \) of two pseudodifferential operators \( Q_2 \) and \( Q_1 \) of orders \( r_2 \) and \( r_1 \) is a pseudodifferential operator of the order \( r_2 + r_1 \).

Its Berezin symbol has an asymptotic expansion (cf.\([14]\)):
\[
(Q_2 Q_1)^b(\psi) - \sum_{n=0}^{N} \frac{(-1)^n}{(2n)!} \left\langle \partial^n Q_2^b(\psi) | \partial^n Q_1^b(\psi) \right\rangle_n \in \mathcal{S}_{\rho + r_1}^{r_2 + r_1},
\]
where \( \langle \cdot | \cdot \rangle_n \) is the associated Hermitian form on the space of symmetric polynomials of order \( n \), and \( \partial^n \) are the complex differentials of order \( n \).

### 3.2 Polynomial operators.

For \( \psi_1, \ldots, \psi_{k+l} \in \mathcal{H}_{-\infty} \) define continuous \( kl \)-monomial operators
\[
M^{kl}(\psi_1, \ldots, \psi_{k+l}) = \prod_{j=1}^{k} C(\psi_j) \prod_{j=k+1}^{k+l} A(\psi_j) : \mathcal{F}_\infty \to \mathcal{F}_{-\infty}.
\]

With fixed \( \Psi', \Psi'' \in \mathcal{F}_\infty \), the \( kl \)-sesquilinear forms
\[
K^{kl}(\Psi''|\Psi') = \langle \Psi''| M^{kl}(\psi_1, \ldots, \psi_{k+l}) \Psi' \rangle
\]
are linear and symmetric in \( \psi_1, \ldots, \psi_k \) and anti-linear and symmetric in \( \psi_{k+1}, \ldots, \psi_{k+l} \). Moreover they belong to the core \( (\mathcal{H}^{\otimes k} \otimes \mathcal{H}^{\otimes l})_{-\infty} \) of the Hilbert scale associated with the scaling operator \( s^{\otimes(k+l)} \) on the Hilbert tensor product of symmetric Hilbert tensor powers of \( \mathcal{H}^{\otimes k} \) and \( \mathcal{H}^{\otimes l} \).

The contraction \( \langle c_{kl}| K^{kl}(\Psi''|\Psi') \rangle \) of \( K^{kl} \) with \( c_{kl} \in (\mathcal{H}^{\otimes k} \otimes \mathcal{H}^{\otimes l})_{-\infty} \) is a sesquilinear form of \( \Psi', \Psi'' \in \mathcal{F}_\infty \). It is an integral kernel of a continuous linear operator \( c_{kl} M^{kl}(\psi_1, \ldots, \psi_{k+l}) : \mathcal{F}_\infty \to \mathcal{F}_{-\infty} \).

If the coefficient \( c_{kl} \) is \( \rho \)-continuous with \( \rho > 0 \), i.e., \( c_{kl} \in (\mathcal{H}^{\otimes k} \otimes \mathcal{H}^{\otimes l})_{\rho} \) then \( c_{kl} M^{kl}(\psi_1, \ldots, \psi_{k+l}) \) transforms \( \mathcal{F}_{-\rho} \), to \( \mathcal{F}_{\rho} \).

A polynomial operator of order \( n \) with coefficients \( c_{kl} \) is a finite sum
\[
P = \sum_{k+l \leq n} c_{kl} M^{kl}.
\]
In \((X, \mu)\)-coordinates it can be written as

\[
P = \sum_{k+l \leq n} \int \prod_{j \leq k+l} d\mu(\chi_j) c_{kl}(\chi_1, \ldots, \chi_{k+l}) \prod_{j=1}^{k} C(\chi_j) \prod_{j=k+1}^{k+l} A(\chi_j).
\]

In particular, the free hamiltonian

\[
H_0 = \int d\mu(\chi') d\mu(\chi'') \tau(\chi', \chi'') C(\chi') A(\chi'') = \int_X d\mu(\chi) C(\chi) A(\chi),
\]

where \(\tau\) is the trace functional on the the symmetric product \(\mathcal{F} \otimes \mathcal{H}\).

A polynomial operator is called \(\rho\)-continuous if all its coefficients are \(\rho\)-continuous.

The coherent state matrix element a polynomial operator \(P\) is

\[
P(\psi''|\psi') = \sum_{k+l \leq n} c_{kl}(\psi'', \ldots, \psi''|\psi', \ldots, \psi').
\]

This is a holomorphic polynomial on \(\mathcal{F}_\infty \times \mathcal{H}_\infty\).

Its Wick symbol is the real-analytic polynomial

\[
P^w(\psi) = \sum_{k+l \leq n} c_{kl}(\psi, \ldots, \psi|\psi, \ldots, \psi).
\]

The Berezin symbol of a \(\rho\)-continuous polynomial operator \(P\) can obtained from its Wick symbol as the finite sum

\[
P^b(\psi) = \sum_j \frac{(-1)^j}{(2j)!} \int D[\psi'] e^{-(\psi'|\psi')} (d^{2j} P^w)(\psi'; \psi').
\]

This is well defined because the differentials \((d^{2j} P^w)(\psi; \psi')\) are continuous polynomials in \(\psi'\) of order \(2j\) on \(\mathcal{H}_\rho\) so that the integral can be understood as a Radon Gauss integral.

The principal parts of the Wick and Berezin symbols coincide:

\[
P^w_0(\psi) = P^b_0(\psi).
\]

The Wick symbol of the free hamiltonian operator is \(H^w_0(\psi) = \langle \psi|\psi \rangle\). Because it is not \(\rho\)-continuous for any \(\rho > 0\), the free hamiltonian \(H_0\) has no Berezin symbol.
4 Feynman equation for interacting Segal systems.

4.1 Elliptic polynomial operators.

A polynomial operator \( P \in \text{Op}(\mathcal{S}_\rho) \) is elliptic (cf.\cite{14}) if there are positive constants \( \rho \) and \( C \) such that the principal Wick symbol

\[
|P_0^w(\psi)| \geq C ||\psi||^{-\rho}_2.
\]

If, in addition, \( P^w(\psi) \) is real then \( P \) is essentially self-adjoint on \( \mathcal{F} \) and \( \mathcal{F}_\infty \) is its essential domain. (With the Lascar pseudodifferential calculus at hand, the proof is similar to the finite dimensional case (cf.\cite{18}).

Since \( P_0^w = \tilde{P}_0^b \), the spectrum of \( P \) is bounded from either side if and only if the principal Wick symbol \( P_0^w(\psi) \) is bounded from the same side.

A basic \( \rho \)-elliptic operator of order 2 is

\[
H_\rho = \int_X \nu(\chi) \lambda(\chi)^{-2\rho} C(\chi) A(\chi)
\]

associated with the spectral expansion of the scaling operator \( s \).

Its Wick and Berezin symbols both are equal to

\[
\int_X \nu(\chi) \lambda(\chi)^{-\rho} \langle \psi(\chi) | \psi(\chi) \rangle = ||\psi||^{2}_{-\rho}.
\]

In the next theorem we represent the Wick symbol of \( U(t) = e^{-iPt} \) via the Berezin symbol of the generator \( P \).

**Theorem 1** Suppose \( P \) is a \( \rho \)-continuous elliptic polynomial operator with the real Wick symbol \( P^w(\psi) \).

Then the coherent matrix element \( \langle e^{\psi''} | e^{-iPt} | e^{\psi'} \rangle \) of the quantum evolution operator \( e^{-iPt} \) is equal to the limit at \( N = \infty \) of the functional integrals over \( \mathcal{H}^{N-1} \)

\[
\int \prod_{j=1}^{N-1} D[\psi_j] \exp \sum_{j=1}^{N+1} \left[ \langle \Delta \psi_j | \psi_j \rangle - iP^b(\psi_j) t/N \right],
\]

where \( \Delta \psi_j = \psi_j - \psi_{j-1}, \psi_{N+1} = \psi'', \psi_0 = \psi' \).
The following proof is adapted from [8].

First, we have the Euler-Hille limit in the strong operator topology on $\mathcal{F}$

$$\exp(-iPt) = \lim_{N \to \infty} (1 + iPt/N)^{-N}.$$  

Next, let $Q_N$ be the $\rho$-pseudodifferential operator of order 0 with the Berezin symbol $Q^b_N(\psi) = [1 + iP^b(\psi)t/N]^{-1}$.

By the Lascar composition rule,

$$(1 + iPt/N)Q_N = 1 + (t/N)^2 R_N,$$

where $R_N$ is a pseudodifferential operator of zero order with the Berezin symbol bounded uniformly relative to $N$. By the fundamental property of the Berezin symbols, the norms of the operator $R_N$ on $\mathcal{F}$ are also uniformly bounded. Therefore, $Q_N$ approximate $(1 + iPt/N)^{-1}$ with the rate $(t/N)^2$ in the operator norm on $\mathcal{F}$.

Thus $(Q_N)^N$ strongly approximate $U(t)$ with the rate $t/N$, so that $(Q_N)^N$ strongly converge to $U(t)$.

The coherent matrix element

$$\langle e^{\psi'} | (Q_N)^N | e^{\psi''} \rangle \text{ is the } (N-1)\text{-fold kernel contraction of the Toeplitz kernels of } Q_N:\n$$

$$\int \prod_{j=1}^{N+1} D[\psi_j] \exp \left( \frac{\psi_j - \psi_{j-1}}{1 + iP^b(\psi_j)t/N} \right)$$

with $\psi_{N+1} = \psi''$, $\psi_0 = \psi'$.

Its limit at $N = \infty$ is the coherent matrix element of the evolution operator.

Replacement of $[1 + iP^b(\psi_j)t/N]^{-1}$ with $\exp \left[ -iP^b(\psi_j)t/N \right]$ in the $(N-1)$-fold kernel contraction makes an approximation with the rate $N(t/N)^2$.

This implies the theorem.

Setting $\psi_j = \psi(t_j)$, $t_j = jt/N$, $j = 0, 1, 2, \ldots, N$, and $\Delta t_j = t_{j+1} - t_j$, rewrite the multiple integral as

$$\int \prod_{j=1}^{N-1} D[\psi_j] \exp i \sum_{j=1}^{N} \left[ -i \langle \Delta \psi_j / \Delta t_j | \psi_j \rangle - P^b(\psi_j) \Delta t_j \right].$$

Its limit at $N = \infty$ is a rigorous mathematical definition of the heuristic Feynman-Lieb integral

$$\int_{\psi''}^{\psi'} \prod_{0 < s < t} D[\psi(s)] \exp \left[ i \int_0^t d\tau \left( -i \langle \dot{\psi}(\tau) | \psi(\tau) \rangle - P^b(\psi(\tau)) \right) \right].$$
Theorem 1 justifies the Feynman equation for ρ-continuous elliptic polynomial operators $P$.

4.2 Self-interacton of the Segal system.

A self-interaction of the Segal system is governed by a Hamiltonian operator $P$. Let us assume that $P$ is a ρ-continuous elliptic polynomial operator of arbitrary order with the principal Wick symbol bounded from below.

Let $c$ be a constant such that $P + c1 \geq 0$. Since $H_0$ is a non-negative self-adjoint operator on $F$, $H_0 + (P + c1)$ on $F_\infty$ has the Friedrichs self-adjoint extension $H_0 + (P + c1)$ on $F$ (cf.[17]). Denote $H = H_0 + P$ the self-adjoint operator $[H_0 + (P + c1)] - c1$. Certainly, it does not depend on the choice of $C$.

The evolution operator of the self-interacting Segal system is $e^{-iHt}$.

Since $H_0$, and therefore $H$, is not ρ-continuous, the theorem 1 above are not applicable to $H$ directly.

Consider the strong operator Trotter limit (cf.[17])

$$e^{-iHt} = \lim_{N \to \infty} \left( e^{-iH_0t/N} e^{-iPt/N} \right)^N.$$

Replacement of $e^{-iPt/N}$ with $(1 + iP/2)^{-1}$ does not change the limit (cf.[13]).

A further replacement with $Q_N$ from the previous section preserves the limit as well:

$$e^{-iHt} = \lim_{N \to \infty} \left( e^{-iH_0t/N} Q_N \right)^N.$$

The Wick kernel of $e^{-iH_0t/N}$ is $\exp \left( e^{-it/N} \langle \psi''|\psi' \rangle \right)$. Its kernel contraction with the Toeplitz kernel of $Q_N$ can be approximated with the rate $(t/N)^2$ by an integral kernel

$$K_N(\psi'', \psi') = \exp \left[ (i \langle \psi'' - \psi' | \psi' \rangle - iP^b(\psi'))t/N \right].$$

This implies

**Theorem 2** The coherent matrix element $\langle \psi''|e^{-iHt}|\psi' \rangle$ of the quantum evolution operator $e^{-iHt}$ is equal to the limit at $N = \infty$ of the
functional integrals over $\mathcal{H}^{N-1}$
\[
\int \prod_{j=1}^{N-1} D[\psi_j] \exp \sum_{j=1}^{N} \left[ (\Delta \psi_j | \psi_j) - i((\psi_{j+1} | \psi_j) + P^b(\psi_j))t/N \right],
\]
where $\Delta \psi_j = \psi_j - \psi_{j-1}$, $\psi_N = \psi''$, $\psi_0 = \psi'$.

4.3 $P(\phi)$-interaction.

The Wick symbol of a $P(\phi)$-interaction Hamiltonian $P$ of degree $2n$ satisfies
\[
P^w(\psi) = P^w(\phi), \quad \phi = (\psi + \psi^*)/2.
\]
With no dependence on complementary $\psi - \psi^*$, such operators may not be elliptic.

Nevertheless, suppose $P$ is a $\rho$-continuous polynomial operator of order $2n$ is elliptic on $\mathbb{R}^*\mathcal{H}$:
\[
|P^w_0(\phi)| \geq C(||\phi||_{-\rho})^{2n}, \phi \in \mathbb{R}^*\mathcal{H}.
\]
Consider the elliptic Hamiltonian $H_\rho \in Op(S^2_{2\rho})$ from Subsection 4.1
\[
H_\rho = \int_{X} d\nu(\chi) \lambda(\chi) - 2\rho C(\chi) A(\chi).
\]

The non-elliptic polynomial operator $Q = (1/2)H_\rho + P$ is hypoelliptic (cf.[18]). Indeed for any natural $m$ there is a constant $C$ such that the $m$-th Frechet differential
\[
|||d^mQ^b(\psi)|||_{-\rho} \leq C||Q^b(\psi)|| (1 + ||\psi||_{-\rho})^{-m},
\]
where $||| \cdot |||_{-\rho}$ is the norm of a polynomial on $\mathcal{H}_{-\rho}$.

Then $Q$ is essentially self-adjoint on $\mathcal{F}$ with $\mathcal{F}_\infty$ as an essential domain. (Again with the Lascar pseudodifferential calculus at hand, the proof is similar to the finite dimensional case (cf.[18]). Moreover, $Q$ is bounded from below because its Berezin symbol is.

On the other hand, since $\lambda(\chi) \leq 1$,
\[
|\langle \psi | (1/2)H_2\rho | \psi \rangle| \leq (1/2)|\langle \psi | H_0 | \psi \rangle|
\]
By the Kato-Rellich theorem (cf.[17]), $H_0 - (1/2)H_2\rho$ is a non-negative self-adjoint operator.

Finally, we have the Hamiltonian
\[
H_0 + P = (H_0 - (1/2)H_2\rho) + Q.
\]
the Friedrichs sum of a non-negative self-adjoint operator and a bounded from below self-adjoint operator (cf. previous section).

It follows that *Theorem 2 holds for* \( P(\phi) \)-*interactions that are elliptic on* \( \mathbb{R}^*\mathcal{H} \).

The theorem cannot be applied to \( \phi^{2n} \)-interactions because, though elliptic on \( \mathbb{R}^*\mathcal{H} \), they are not \( \rho \)-continuous. Still they may be made such if cut off by the contraction with a \( \rho \)-continuous coefficient.

### 4.4 Fermion Segal systems.

The theory of functional Schrödinger equations for boson systems has a natural corresponding theory of fermion systems.

The Segal axioms are the same with just one exception:

Instead of the Heisenberg commutation relations, the Segal fermion system satisfies the Clifford CCR:

\[
\hat{\psi}_1 \hat{\psi}_2 + \hat{\psi}_2 \hat{\psi}_1 = \Re \langle \psi_1 | \psi_2 \rangle \mathbf{1}.
\]

The fundamental uniqueness Segal theorem holds in the fermion case as well (cf.[2]).

A convenient complex Gaussian representations of the fermion Segal system is in [4] (but not in [2]). It is based on the Grassmannian Berezin functional integral.

The forced boundeness of the fermionic CCR implies that in the Nelson-Baez theory of annihilation and creation operators one may use the identity scaling operator.

The Lascar theory of pseudodifferential operators is valid for fermions, provided that the symmetry of the bosonic formulas should be replaced by the skew-symmetry of the fermionic ones.

The theorems of the previous section hold for fermions. Of course, one should use the Grassmannian Berezin Gaussian integrals.

The further generalization to Feynman equation for *supersymmetric* Segal systems is straightforward.

### 5 Appendix. Geometric Segal systems.

In concrete applications the Segal system has additional features.
E.g., to distinguish between particles and their anti-particles one may use geometric quantization (cf.\cite{11}).

Let $\mathcal{H}^\dagger$ be the anti-dual of $\mathcal{H}$. It may be represented as the space of all anti-linear mappings of the complex line $\mathbb{C}$ to $\mathcal{H}$. This is a complex Hilbert space (only the multiplication with a complex scalar is the multiplication with its complex-conjugate). In Dirac terms $\mathcal{H}$ is the space of ket-vectors and $\mathcal{H}^\dagger$ is the space of the bra-vectors.

Actually, $\mathcal{H}$ and $\mathcal{H}^\dagger$ may be identified as real vector spaces: a $\psi \in \mathcal{H}$ uniquely corresponds to that anti-linear mapping which takes $1 \in \mathbb{C}$ to $\psi$. Their symplectic forms differ by the sign only.

The Hilbert sum $\mathcal{H} \oplus \mathcal{H}^\dagger$ is symplectic as the sum of symplectic vector spaces.

There is a natural complex conjugation on $\mathcal{H} \oplus \mathcal{H}^\dagger$:

$$\overline{\psi^\prime \oplus \psi^\prime\prime} = \overline{\psi^\prime\prime} \oplus \overline{\psi^\prime}.$$

Lagrangian subspaces in a symplectic space are maximal among the subspaces on which the symplectic form vanishes.

A polarization $\mathcal{L}$ of $\mathcal{H}$ is a complex Hilbert Lagrangean subspace in $\mathcal{H} \oplus \mathcal{H}^\dagger$. Its anti-dual is identified with the complex-conjugate $\mathcal{L}^\dagger$.

A polarization $\mathcal{L}$ is real if $\mathcal{L} = \mathcal{L}^\dagger \cap \mathcal{L}$. Real polarizations are the complexifications of real Hilbert Lagrangean subspaces in $\mathcal{H}$.

Let $\sigma$ be the symplectic form on $\mathcal{H} \oplus \mathcal{H}^\dagger$. A polarization $\mathcal{L}$ is non-negative if the quadratic form $\beta(\psi^\prime, \psi^\prime\prime) = \sigma(\overline{\psi^\prime}, \overline{\psi^\prime\prime})$ is non-negative on $\mathcal{L}$. Real polarizations are non-negative.

If the quadratic form $\beta$ is positive definite on a polarization, then we have a Kähler polarization of $\mathcal{H}$.

The non-negative polarizations $\mathcal{L}$ of $\mathcal{H}$ define representations of the Segal system over $\mathcal{H}$. We call such representations the geometric Segal systems.

The real Gaussian representation is the geometric Segal system corresponding to the real diagonal polarization $\{\overline{\psi} \oplus \psi : \psi \in \mathcal{H}\}$.

The complex Gaussian representation is the geometric Segal system corresponding to the Kähler polarization $\mathcal{H} = 0 \oplus \mathcal{H}$.

In physics terms, the geometric Segal systems which correspond to anti-dual polarizations describe a pair of free anti-particles. One may think about the complex conjugation as a generalization of the CPT transformation.
Geometric Segal systems are geometrically equivalent if the corresponding polarizations are equivalent under transformations of \( \mathcal{H} \oplus \mathcal{H} \) induced by symplectic automorphisms of \( \mathcal{H} \).

Two non-negative polarizations are geometrically equivalent if and only if the nullity of the restriction of the quadratic form \( \beta \) is the same for both polarizations.

6 How constructive quantum field theory is possible.

There are three basic formulations of constructive quantum field theory (cf.[6]).

1. **Canonical formulation.**

   Quantum fields are operator-valued fields on the Minkowski space-time \( \mathbb{R}^{d,1} \) that satisfy the canonical commutation relations and solve the classical Hamiltonian equations. For interacting fields the equations are non-linear partial differential equations on \( \mathbb{R}^{d,1} \).

   Unfortunately for \( d > 3 \), the relativistic irreducible quantum fields, which satisfy the canonical commutation relations, are free by default (cf.[3]).

   Even the \( d = 3 \) case is troublesome. The simplest non-linearity in the perturbation theory is the \( \phi^4 \) interaction. Yet for \( d = 3 \) renormalization screens out the perturbation (cf.[7]).

2. **Feynman formulation.**

   The quantum propagators of classical fields are Feynman integrals over classical histories on the Minkowski space-time \( \mathbb{R}^{d,1} \).

   Since 1960’s the prevalent approach is the Lagrangean Feynman-Kac infinite-dimensional integral over the space of histories on the euclidean space \( \mathbb{R}^{d+1} \) with the aposteriory analytic continuation to the real time.

   This approach of K.Symanszik and E.Nelson has culminated in the work of J.Glimm and A.Jaffe [10]. However, its application to interacting fields in the space dimensions \( d > 2 \) is still open.

3. **Functional Schrödinger formulation.**

   The quantum states are functionals on the phase space propagated by the evolution operator of a linear functional differential Schrödinger equation.
For quite some time the functional Schrödinger formulation has been presumed mathematically unreasonable. To quote F.Berezin [4]:

...the mathematical problems occuring in the method of second quantization are somewhat removed from the problems of the traditional mathematical physics which are formulated in terms of partial differential equations. In particular, major roles in the method of second quantization are taken by purely algebraic questions, strange to classical mathematical physics...

Yet important analytic techniques for functional differential equations have been developed in the P. Krée seminar at the Institut Henri Poincaré in Paris during 70's. The B.Lascar theory [14] of infinite-dimensional pseudodifferential operators is its byproduct.

We have solved the linear functional Schrödinger equations for interacting Segal boson and fermion systems with $\rho$-continuous (hypo) elliptic polynomial interactions. The results are applicable to quantum fields on Minkowski space-times of any dimension.

The solutions are Feynman type sequential integrals defined as limits of multiple integrals over the phase space. Seemingly unpractical for concrete computations, they justify the basic integral calculus rules. In this respect the approximating multiple integrals play the role of the Riemann integral sums in elementary integral calculus providing a foundation for more sophisticated techniques.

Unfortunately, gauge fields do not admit elliptic interactions. However, presumably our approach is sound even in this case (the work is in progress).

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