THE HYDROSTATIC APPROXIMATION OF THE BOUSSINESQ EQUATIONS WITH ROTATION IN A THIN DOMAIN

XUEKE PU AND WENLI ZHOU

Abstract. In this paper, we improve the global existence result in [9] slightly. More precisely, the global existence of strong solutions to the primitive equations with only horizontal viscosity and diffusivity is obtained under the assumption of initial data \((v_0, T_0) \in H^1\) with \(\partial_z v_0 \in L^4\). Moreover, we prove that the scaled Boussinesq equations with rotation strongly converge to the primitive equations with only horizontal viscosity and diffusivity, in the cases of \(H^1\) initial data, \(H^1\) initial data with additional regularity \(\partial_z v_0 \in L^4\) and \(H^2\) initial data, respectively, as the aspect ratio parameter \(\lambda\) goes to zero, and the rate of convergence is of the order \(O(\lambda^{\eta/2})\) with \(\eta = \min\{2, \beta - 2, \gamma - 2\}(2 < \beta, \gamma < \infty)\). The convergence result implies a rigorous justification of the hydrostatic approximation.

1. Introduction

The primitive equations are considered as the fundamental model in geophysical flows (see, e.g., [47, 38, 43, 36, 46]). In large-scale ocean dynamics, an important feature is that the vertical scale of ocean is much smaller than the horizontal scale, which means that we have to use the hydrostatic approximation to simulate the motion of ocean in the vertical direction. Owing to this fact and the high accuracy of hydrostatic approximation, the three-dimensional viscous primitive equations of ocean dynamics can be formally derived from the three-dimensional Boussinesq equations with rotation (see [30, 11]).

The small aspect ratio limit from the Navier-Stokes equations to the primitive equations was studied first by Azérad-Guillén [1] in a weak sense, then by Li-Titi [32] in a strong sense with error estimates, and finally by Furukawa et al. [15] in a strong sense but under relaxing the regularity on the initial condition. Subsequently, the strong convergence of solutions of the scaled Navier-Stokes equations to the corresponding ones of the primitive equations with only horizontal viscosity was obtained by Li-Titi-Yuan [34]. Furthermore, the rigorous justification of the hydrostatic approximation from the scaled Boussinesq equations to the primitive equations with full viscosity and diffusivity was obtained by Pu-Zhou [39].

From a physical point of view, fluid flow is strongly influenced by effect of stratification (see, e.g., [35, 36, 40]). An important observation for effect of stratification is that the density of a fluid changes with depth. In some mathematical studies, considering the hydrodynamic equations with density stratification term can often obtain better results (see, e.g., [12, 6, 7, 8, 9, 10]). These two facts show that density stratification term is of great significance both physically and mathematically. The rigorous mathematical derivation for the governed equations describing the motion of stable stratified fluid, i.e., the viscous primitive equations with density stratification, is due to the work of Pu-Zhou [40]. Based on the ideas that follow from Li-Titi [32], Li-Titi-Yuan [34], and Pu-Zhou [40], We study the hydrostatic approximation of the Boussinesq equations with rotation in a thin domain.
1.1. The scaled Boussinesq equations with rotation in a thin domain. Let \( \Omega_\lambda = M \times (-\lambda, \lambda) \) be a \( \lambda \)-dependent domain, where \( M = (0, 1) \times (0, 1) \). Here, \( \lambda = H/L \) is called the aspect ratio, measuring the ratio of the vertical and horizontal scales of the motion, which is usually very small. Say, for large-scale ocean circulation, the ratio \( \lambda \sim 10^{-3} \ll 1 \).

Denote by \( \nabla_h = (\partial_x, \partial_y) \) the horizontal gradient operator. Then the horizontal Laplacian operator \( \Delta_h \) is given by

\[
\Delta_h = \nabla_h \cdot \nabla_h = \partial_{xx} + \partial_{yy}.
\]

Let us consider the anisotropic Boussinesq equations with rotation defined on \( \Omega_\lambda \)

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla \pi - \theta \vec{k} + f_0 \vec{k} \times v &= \mu_h \Delta_h u + \mu_z \partial_{zz} u, \\
\partial_t \theta + u \cdot \nabla \theta &= \kappa_h \Delta_h \theta + \kappa_z \partial_{zz} \theta, \\
\nabla \cdot u &= 0,
\end{align*}
\]

where the three dimensional velocity field \( u = (v, w) = (v_1, v_2, w) \), the pressure \( \pi \) and temperature \( \theta \) are the unknowns. \( f_0 \) is the Coriolis parameter and \( \vec{k} = (0, 0, 1) \) is unit vector pointing to the \( z \)-direction. \( \mu_h \) and \( \mu_z \) represent the horizontal and vertical viscosity coefficients, while \( \kappa_h \) and \( \kappa_z \) represent the horizontal and vertical heat diffusion coefficients, respectively.

In fact, the anisotropic Boussinesq equations with rotation \( 1.1 \) have an elementary exact solution

\[
(u, \theta, \pi) = (0, \bar{\theta}(z), \bar{\pi}(z)) = \left(0, N^2 z, \frac{N^2}{2} z^2\right),
\]

satisfying the hydrostatic approximation

\[
\frac{d\bar{\pi}(z)}{dz} - \bar{\theta}(z) = 0.
\]

Here, \( N > 0 \) is called the buoyancy or Brunt-Väisälä frequency and denotes the strength of stable stratification, which implies that the density of a fluid decreases with height and lighter fluid is above heavier fluid. Assume that

\[
p(x, y, z, t) = \pi(x, y, z, t) - \bar{\pi}(z),
\]

\[
T(x, y, z, t) = \theta(x, y, z, t) - \bar{\theta}(z).
\]

Then the anisotropic Boussinesq equations with rotation \( 1.1 \) become

\[
\begin{align*}
\partial_t v + (v \cdot \nabla_h)v + w \partial_z v + \nabla_h p + f_0 \vec{k} \times v &= \mu_h \Delta_h v + \mu_z \partial_{zz} v, \\
\partial_t w + v \cdot \nabla_h w + w \partial_z w + \partial_z p - T &= \mu_h \Delta_h w + \mu_z \partial_{zz} w, \\
\partial_t T + v \cdot \nabla_h T + w \partial_z T + N^2 w &= \kappa_h \Delta_h T + \kappa_z \partial_{zz} T,
\end{align*}
\]

\[
\nabla_h \cdot v + \partial_z w = 0.
\]

Firstly, we transform the anisotropic Boussinesq equations with rotation \( 1.2 \), defined on the \( \lambda \)-dependent domain \( \Omega_\lambda \), to the scaled Boussinesq equations with rotation defined on a fixed domain. To this end, we introduce the following new unknowns

\[
u_{\lambda} = (v_{\lambda}, w_{\lambda}), \quad w_{\lambda}(x, y, z, t) = \bar{w}(x, y, \lambda z, t),
\]

\[
u_{\lambda}(x, y, z, t) = \frac{1}{\lambda} \bar{w}(x, y, \lambda z, t), \quad p_{\lambda}(x, y, z, t) = p(x, y, \lambda z, t),
\]

\[
T_{\lambda}(x, y, z, t) = \lambda T(x, y, \lambda z, t), \quad \bar{\pi}_{\lambda}(z) = \bar{\pi}(\lambda z), \quad \bar{\theta}_{\lambda}(z) = \lambda^2 \bar{\theta}(\lambda z),
\]

for any \( (x, y, z) \in \Omega =: M \times (-1, 1) \) and for any \( t \in (0, \infty) \). Then the last two scalings allow us to write the pressure and temperature non-dimensionally as

\[
\bar{\pi}_{\lambda}(z) + p_{\lambda}(x, y, z, t) = \pi(\lambda z) + \bar{p}(x, y, \lambda z, t) = \pi(x, y, \lambda z, t)
\]
and
\[ \bar{\theta}_\lambda(z) + \lambda T_\lambda(x,y,z,t) = \lambda^2(\bar{\theta}(\lambda z) + T(x,y,\lambda z,t)) = \lambda^2\theta(x,y,\lambda z,t), \]
respectively.

Suppose that $\mu_h = \kappa_h = 1$, $\mu_z = \lambda^\beta$, and $\kappa_z = \lambda^\gamma$, with $2 < \beta, \gamma < \infty$. Under these scalings, the anisotropic Boussinesq equations with rotation (1.4) defined on $\Omega_\lambda$ can be written as the following scaled Boussinesq equations with rotation
\[ \begin{aligned}
&\partial_t v_\lambda + (v_\lambda \cdot \nabla_h)v_\lambda + w_\lambda \partial_z v_\lambda + \nabla_h p_\lambda + f_0 \vec{k} \times v_\lambda = \Delta_h v_\lambda + \lambda^{-2}\partial_{zz} v_\lambda, \\
&\frac{1}{\lambda^2}(\partial_t w_\lambda + v_\lambda \cdot \nabla_h w_\lambda + w_\lambda \partial_z w_\lambda) + \frac{1}{\lambda} (\partial_z p_\lambda - T_\lambda) = \lambda \Delta_h w_\lambda + \lambda^{-1}\partial_{zz} w_\lambda, \\
&\partial_t T_\lambda - \Delta_h T_\lambda - \lambda^{\gamma-2}\partial_{zz} T_\lambda + v_\lambda \cdot \nabla_h T_\lambda + w_\lambda \partial_z T_\lambda + w_\lambda = 0, \\
&\nabla_h \cdot v_\lambda + \partial_z w_\lambda = 0,
\end{aligned} \]
defined on the fixed domain $\Omega$.

Set $\lambda^2 N^2 = 1$, i.e., $N \sim 1/\lambda$, which means that the stratification effect is very strong as the aspect ratio $\lambda$ tends to zero. In such a case, the scaled Boussinesq equations with rotation (1.3) can be rewritten as
\[ \begin{aligned}
&\partial_t v_\lambda - \Delta_h v_\lambda - \lambda^{\beta-2}\partial_{zz} v_\lambda + (v_\lambda \cdot \nabla_h)v_\lambda + w_\lambda \partial_z v_\lambda + \nabla_h p_\lambda + f_0 \vec{k} \times v_\lambda = 0, \\
&\lambda^2(\partial_t w_\lambda - \Delta_h w_\lambda - \lambda^{\beta-2}\partial_{zz} w_\lambda + v_\lambda \cdot \nabla_h w_\lambda + w_\lambda \partial_z w_\lambda) + \partial_z p_\lambda - T_\lambda = 0, \\
&\partial_t T_\lambda - \Delta_h T_\lambda - \lambda^{\gamma-2}\partial_{zz} T_\lambda + v_\lambda \cdot \nabla_h T_\lambda + w_\lambda \partial_z T_\lambda + w_\lambda = 0, \\
&\nabla_h \cdot v_\lambda + \partial_z w_\lambda = 0.
\end{aligned} \]

Next, we supply the scaled Boussinesq equations with rotation (1.4) with the following boundary and initial conditions
\[ v_\lambda, w_\lambda, p_\lambda \quad \text{and} \quad T_\lambda \quad \text{are periodic in} \quad x, y, z, \]
\[ (v_\lambda, w_\lambda, T_\lambda)|_{t=0} = (v_0, w_0, T_0), \]
where $(v_0, w_0, T_0)$ is given. Moreover, we also equip the system (1.4) with the following symmetry condition
\[ v_\lambda, w_\lambda, p_\lambda \quad \text{and} \quad T_\lambda \quad \text{are even, odd, even and odd with respect to} \ z, \ \text{respectively}. \]

Noting that the above symmetry condition is preserved by the scaled Boussinesq equations with rotation (1.4), i.e., it holds provided that the initial data satisfies this symmetry condition. Due to this fact, throughout this paper, we always suppose that $v_0, w_0$ and $T_0$ are periodic in $x, y, z$, and are even, odd and odd in $z$, respectively.

In this paper, we will not distinguish in notation between spaces of scalar and vector-valued functions. Namely, we will use the same notation to denote both a space itself and its finite product spaces. For simplicity, we denote by notation $\|\cdot\|_p$ the $L^p(\Omega)$ norm.

The weak solutions of the scaled Boussinesq equations with rotation (1.4) are defined as follows.

**Definition 1.1.** Given $(u_0, T_0) = (v_0, w_0, T_0) \in L^2(\Omega)$, with $\nabla \cdot u_0 = 0$. We say that a space periodic function $(v_\lambda, w_\lambda, T_\lambda)$ is a weak solution of the system (1.4), subject to boundary and initial conditions (1.5)–(1.6) and symmetry condition (1.7), if
(i) $(v_\lambda, w_\lambda, T_\lambda) \in C([0,\infty);L^2(\Omega)) \cap L^2_{\text{loc}}([0,\infty);H^1(\Omega))$ and
(ii) $(v_\lambda, w_\lambda, T_\lambda)$ satisfies the following integral equality
\[ \int_0^\infty \int_{\Omega} \left\{ -v_\lambda \cdot \partial_t \varphi_h - \lambda^2 w_\lambda \partial_t \varphi_3 - T_\lambda \partial_t \psi - T_\lambda \varphi_3 + w_\lambda \psi + f_0(\vec{k} \times v_\lambda) \cdot \varphi_h \right\} = 0, \]
observe that it is not necessary to give the initial condition for vertical velocity system (1.10) with the same boundary and initial conditions (1.5)-(1.6) and symmetry term, providing additional dissipation for this system. Moreover, we supply the limiting that the weak solution (see Temam [45, Ch.III, Remark 4.1] and Robinson et al. v (1.10) satisfies the initial condition (1.6) just for convenience. Note that the initial value

\[
\phi \text{ for any spatially periodic function } (\phi, \psi) = (\phi_h, \varphi_3, \psi), \text{ with } \varphi_h = (\varphi_1, \varphi_2), \text{ such that } \nabla \cdot \phi = 0 \text{ and } (\phi, \psi) \in C^\infty_c(\Omega \times [0, \infty)).
\]

**Remark 1.2.** The proof of the existence of weak solutions to the scaled Boussinesq equations with rotation (1.14) follows from the similar argument in Lions-Temam-Wang [30, Part IV]. Specifically, for any initial data \((u_0, T_0) = (v_0, w_0, T_0) \in L^2(\Omega), \) with \(\nabla \cdot u_0 = 0,\) we can prove that there exists a global weak solution \( (v_\lambda, w_\lambda, T_\lambda) \) of the scaled Boussinesq equations with rotation (1.14), subject to boundary and initial conditions (1.5)-(1.6) and symmetry condition (1.7). Moreover, by the similar argument as Lions-Temam-Wang [30, Part IV], we can also show that it has a unique local strong solution \( (v_\lambda, w_\lambda, T_\lambda) \) for initial data \((u_0, T_0) = (v_0, w_0, T_0) \in H^1(\Omega), \) with \(\nabla \cdot u_0 = 0.\)

**Remark 1.3.** Similar to the theory of three-dimensional Navier-Stokes equations, e.g., see Temam [35 Ch.III, Remark 4.1] and Robinson et al. [32 Theorem 4.6], we can prove that the weak solution \( (v_\lambda, w_\lambda, T_\lambda) \) satisfies the following energy inequality

\[
\frac{1}{2} \left( \|v_\lambda(t)\|^2 + \lambda^2 \|w_\lambda(t)\|^2 + \|T_\lambda(t)\|^2 \right) \\
+ \int_0^t \left( \|\nabla_h v_\lambda\|^2 + \lambda^{2-2} \|\partial_z v_\lambda\|^2 + \lambda^2 \|\nabla_h w_\lambda\|^2 \right) ds \\
+ \int_0^t \left( \lambda^2 \|\partial_z w_\lambda\|^2 + \|\nabla_h T_\lambda\|^2 + \lambda^2 \|\partial_z T_\lambda\|^2 \right) ds \\
\leq \frac{1}{2} \left( \|v_0\|^2 + \lambda^2 \|w_0\|^2 + \|T_0\|^2 \right). \tag{1.9}
\]

for a.e. \( t \in [0, \infty), \) as long as the weak solution \( (v_\lambda, w_\lambda, T_\lambda) \) is obtained by Galerkin method.

### 1.2. The limiting system of the scaled Boussinesq equations with rotation

Now we discuss the limiting system of the scaled Boussinesq equations with rotation (1.14).

When \( 2 < \beta, \gamma < \infty, \) taking the limit \( \lambda \to 0 \) in system (1.14), then this system formally converges to the following primitive equations with only horizontal viscosity and diffusivity

\[
\begin{cases}
\partial_t v - \Delta_h v + (v \cdot \nabla_h) v + w \partial_z v + \nabla_h p + f_0 \vec{k} \times v = 0, \\
\partial_z p - T = 0, \\
\partial_t T - \Delta_h T + v \cdot \nabla_h T + w \partial_z T + w = 0, \\
\nabla_h \cdot v + \partial_z w = 0.
\end{cases} \tag{1.10}
\]

The term \( w \) in the third equation of system (1.10) is called the density stratification term, providing additional dissipation for this system. Moreover, we supply the limiting system (1.10) with the same boundary and initial conditions (1.5)-(1.6) and symmetry condition (1.7) as the system (1.4). In studying the well-posedness of system (1.10), we observe that it is not necessary to give the initial condition for vertical velocity \( w, \) since there is no dynamic equation for vertical velocity in the system. So we say that the system (1.10) satisfies the initial condition (1.6) just for convenience. Note that the initial value
$w_0$ for vertical velocity $w_\lambda$ is uniquely determined by the divergence-free condition and symmetry condition (1.7). Hence it can be represented as
\[
w_0(x, y, z) = -\int_0^z \nabla_h \cdot v_0(x, y, \xi) d\xi, \tag{1.11}
\]
for any $(x, y) \in M$ and $z \in (-1, 1)$. Therefore, only the initial conditions of horizontal velocity and temperature are given throughout the paper.

We remark that the limiting system of the scaled Boussinesq equations with rotation (1.4) is the primitive equations with full viscosity and diffusivity when $\beta = \gamma = 2$. This case was studied by the authors (see [40]). In consequence, the aim of this paper is to prove that the scaled Boussinesq equations with rotation (1.4) strongly converge to the primitive equations with only horizontal viscosity and diffusivity (1.10), in the cases of $H^1$ initial data, $H^1$ initial data with additional regularity $\partial_z v_0 \in L^4$ and $H^2$ initial data, respectively, as the aspect ratio parameter tends to zero. These convergence results are briefly described as follows:

- For $H^1$ initial data, the system (1.10) corresponding to (1.5)-(1.7) has a unique local strong solution $(v, T)$ (see [9]). Based on this local well-posedness result and Remark 1.2, we obtain the local strong convergence theorem (see Theorem 2.1).
- For $H^2$ initial data with additional regularity $\partial_z v_0 \in L^4$, the global existence of strong solutions to the system (1.10) with (1.5)-(1.7) is proved (see Theorem 2.2). Compared with [9], the condition $(v_0, T_0) \in L^\infty$ is removed by establishing $L^\infty_t L^4_x$ estimate on the vertical derivative of horizontal velocity that does not depend on $\|v_0, T_0\|_\infty$. Consequently, the improved result and Remark 1.2 yield the global strong convergence theorem (see Theorem 2.4).
- For $H^2$ initial data, there exists a unique global strong solution $(v, T)$ to the system (1.10) subject to (1.5)-(1.7) (see [8]). According to the energy estimate on the global strong solutions and Remark 1.2, we establish the corresponding global strong convergence result (see Theorem 2.5).

As can be seen from above, the well-posedness results of primitive equations with only horizontal viscosity and diffusivity (1.4) will play an important role in proving that the system (1.4) strongly converges to the system (1.10) as the aspect ratio parameter tends to zero. In order to make full use of these known well-posedness results, we must construct the primitive equations with density stratification. The way to construct it is to look for a suitable exact solution of the system (1.11), i.e.,
\[(u, \theta, \pi) = (0, \bar{\theta}(z), \bar{\pi}(z)) = \left(0, N^2 z, \frac{N^2}{2} z^2\right),\]
which satisfies the hydrostatic approximation
\[\frac{d\bar{\pi}(z)}{dz} - \bar{\theta}(z) = 0.\]

In addition to these well-posedness results, more results on the case of partial dissipation can be found in the work of Cao-Titi [12], Fang-Han [14], Li-Yuan [35], and Cao-Li-Titi [6, 7, 10].

Some other results for the primitive equations are as follows. The global existence of weak solutions of the primitive equations with full viscosity and diffusivity was first given by Lions-Temam-Wang [30, 29, 31], but the question of uniqueness to this mathematical model is still unclear. Only in some special cases are known results (see [4, 44, 25, 33, 22]). For arbitrarily large initial data belonging to $H^1$, the global existence of strong solutions of the full primitive equations with Neumann boundary conditions was
obtained by Cao-Titi[11]. In the case of mixed Dirichlet and Neumann boundary conditions, this result was also proved by Kukavica-Ziane[27][28]. Considering the same boundary conditions, the existence of global strong solutions of the primitive equations without temperature was established by Hieber-Kashiwabara[20] for $L^p$ initial data, and later by Giga et al.[16] for initial data in anisotropic $L^p$ space. In addition, the well-posedness result corresponding to the primitive equations without temperature in $L^p$ space can be extended to the full primitive equations, see Hieber et al.[19].

The inviscid primitive equations without temperature is called the hydrostatic Euler equations. Kukavica et al.[23] show that the solutions of the primitive equations converge to the solutions of the hydrostatic Euler equations, as viscosity coefficient goes to zero. The inviscid primitive equations with or without rotation is known to be ill-posed in Sobolev spaces, and its smooth solutions may develop singularity in finite time, see Renardy[11], Han-Kwan and Nguyen[18], Ibrahim-Lin-Titi[21], Wong[48], and Cao et al.[5]. However, the local well-posedness of the inviscid primitive equations was established by Kukavica et al.[23] for initial data in the space of analytic function, in which the maximal existence time of the analytical solutions depends on the rate of rotation $|f_0|$. Subsequently, this local well-posedness result was improved by Ghoul et al.[17], and then the long time existence of solutions was obtained. For more results on the inviscid primitive equations, we refer to the work of Brenier[3], Masmoudi-Wong[37], and Kukavica et al.[24].

The rest of this paper is organized as follows. Our main results are stated in Section 2. The strong convergence results of $H^1$ initial data, $H^1$ initial data with additional regularity, and $H^2$ initial data are presented in Section 3, section 4, and Section 5, respectively.

2. Main results

Now we are to state the main results of this paper. Assume that initial data $(v_0, T_0) \in H^1(\Omega)$. Then it deduces from (1.11) that $(v_0, w_0, T_0) \in L^2(\Omega)$, which implies that the system (1.4) subject to (1.5)-(1.7) has a global weak solution $(v_\lambda, w_\lambda, T_\lambda)$ by Remark 1.2, and the system (1.10) exists a unique local strong solution $(v, T)$ (see [9]). Denote by $t_0^*$ the maximal existence time of the local strong solution $(v, T)$ to the system (1.10). For this case, we have the following strong convergence theorem.

**Theorem 2.1.** Given a periodic function pair $(v_0, T_0) \in H^1(\Omega)$ with $\int_{\Omega} \nabla_h \cdot v_0 dz = 0$. Suppose that $(v_\lambda, w_\lambda, T_\lambda)$ is a global weak solution of the system (1.4), satisfying the energy inequality (1.9), and that $(v, T)$ is the unique local strong solution of the system (1.10), with the same boundary and initial conditions (1.5)-(1.6) and symmetry condition (1.7). Let

$$(V_\lambda, W_\lambda, \Phi_\lambda) = (v_\lambda - v, w_\lambda - w, T_\lambda - T).$$

Then the following estimate holds

$$\sup_{0 \leq t \leq t_0^*} \left( \left\| (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \right\|^2 \right) (t) + \int_0^{t_0^*} \left( \|\nabla_h V_\lambda\|^2_2 + \lambda^{\beta-2} \|\partial_2 V_\lambda\|^2_2 \right) dt$$

$$+ \int_0^{t_0^*} \left( \lambda^2 \|\nabla_h W_\lambda\|^2_2 + \|\nabla_h \Phi_\lambda\|^2_2 + \lambda^\beta \|\partial_2 W_\lambda\|^2_2 + \lambda^{\gamma-2} \|\partial_2 \Phi_\lambda\|^2_2 \right) dt$$

$$\leq C \lambda^\eta (t_0^* + 1) e^{C(t_0^* + 1)} \left[ 1 + \left( \|v_0\|^2_2 + \|w_0\|^2_2 + \|T_0\|^2_2 \right)^2 \right],$$

where $\eta = \min\{2, \beta - 2, \gamma - 2\}$ with $2 < \beta, \gamma < \infty$, and $C$ is a positive constant that does not depend on $\lambda$. As a result, we have the following strong convergences

$$(v_\lambda, \lambda w_\lambda, T_\lambda) \to (v, 0, T), \text{ in } L^\infty \left( (0, t_0^*); L^2(\Omega) \right),$$
Then the following estimate holds
\[
\left( \nabla_h v, \lambda^{(\beta-2)/2} \partial_z v, \lambda \nabla w, w \right) \to \left( \nabla_h v, 0, 0, w \right), \quad \text{in } L^2 \left( [0, t^*_0); L^2(\Omega) \right),
\]
\[
\left( \nabla_h T, \lambda^{\beta/2} \partial_z w, \lambda^{(\gamma-2)/2} \partial_z T \right) \to \left( \nabla_h T, 0, 0 \right), \quad \text{in } L^2 \left( [0, t^*_0); L^2(\Omega) \right),
\]
and the rate of convergence is of the order \( O(\lambda^{n/2}) \).

The authors in [9] obtain the global strong solutions of system (1.10), provided that initial data \((v_0, T_0) \in H^1(\Omega) \) has the additional regularity that \( \partial_z v_0 \in L^q(\Omega) \) with \( q \in (2, \infty) \) and \((v_0, T_0) \in L^\infty(\Omega) \). We improve this result slightly and then give the following theorem.

**Theorem 2.2.** Assume that \((v_0, T_0) \in H^1(\Omega) \) with \( \int_\Omega \nabla_h \cdot v_0 \, dz = 0 \), and that \( \partial_z v_0 \in L^4(\Omega) \). Then there exists a unique global strong solution \((v, T)\) to the system (1.10) subject to boundary and initial conditions (1.5)-(1.6) and symmetry condition (1.7) such that the following energy estimate holds
\[
\sup_{0 \leq s \leq t} \left( \| (v, T) \|_{H^1(\Omega)}^2 \right) (s) + \int_0^t \| \nabla_h v \|_{H^1(\Omega)}^2 \, ds + \int_0^t \| \nabla_h T \|_{H^1(\Omega)}^2 + \| (\partial_t v, \partial_t T) \|_{L^2(\Omega)}^2 \, ds \leq \mathcal{J}_2(t),
\]
for any \( t \in [0, \infty) \), where \( \mathcal{J}_2(t) \) is a nonnegative continuously increasing function defined on \([0, \infty)\).

**Remark 2.3.** The result of Theorem 2.2 still holds for \( \partial_z v_0 \in L^m(\Omega) \) with \( m \in (4, \infty) \), since \( \Omega = (0, 1)^2 \times (-1, 1) \) is a set of finite measure. For the case of \((v_0, T_0) \in H^1(\Omega) \) with \( \partial_z v_0 \in L^m(\Omega)(2 < m < 4) \), the global existence of strong solutions to the system (1.10) is unknown without assuming that \((v_0, T_0) \in L^\infty(\Omega) \).

Based on the global existence of strong solutions in Theorem 2.2, we have the corresponding global strong convergence theorem.

**Theorem 2.4.** Given a periodic function pair \((v_0, T_0) \in H^1(\Omega) \) satisfying \( \int_\Omega \nabla_h \cdot v_0 \, dz = 0 \) and \( \partial_z v_0 \in L^4(\Omega) \). Suppose that \((v_\lambda, w_\lambda, T_\lambda)\) is a global weak solution of the system (1.4), satisfying the energy inequality (1.9), and that \((v, T)\) is the unique global strong solution of the system (1.10), with the same boundary and initial conditions (1.5)-(1.6) and symmetry condition (1.7). Let
\[
(V_\lambda, W_\lambda, \Phi_\lambda) = (v_\lambda - v, w_\lambda - w, T_\lambda - T).
\]

Then the following estimate holds
\[
\sup_{0 \leq t \leq T} \left( \| (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_{L^2}^2 \right) (t) + \int_0^T \left( \| \nabla_h V_\lambda \|_{L^2}^2 + \lambda^{\beta-2} \| \partial_z V_\lambda \|_{L^2}^2 \right) dt + \int_0^T \left( \lambda^2 \| \nabla_h W_\lambda \|_{L^2}^2 + \| \nabla h \Phi_\lambda \|_{L^2}^2 + \lambda \| \partial_z W_\lambda \|_{L^2}^2 + \lambda^{\gamma-2} \| \partial_z \Phi_\lambda \|_{L^2}^2 \right) dt \leq \lambda^\eta \mathcal{J}_3(T),
\]
for any \( T > 0 \), where \( \eta = \min\{2, \beta - 2, \gamma - 2\} \) with \( 2 < \beta, \gamma < \infty \), and \( \mathcal{J}_3(t) \) is a nonnegative continuously increasing function defined on \([0, \infty)\). Therefore, the local strong convergences in Theorem 2.2 can be extended to the global strong convergences.

Finally, we suppose that initial data \((v_0, T_0)\) belongs to \( H^2(\Omega) \). Then from (1.11) it follows that \((v_0, w_0, T_0)\) belongs to \( H^1(\Omega) \). According to Remark 1.2 there exists a unique
local strong solution \((v_\lambda, w_\lambda, T_\lambda)\) to the system (1.4) corresponding to (1.3)-(1.7). Moreover, the system (1.10) has a unique global strong solution \((v, T)\) (see [8]). In this case, we also have the following strong convergence theorem.

**Theorem 2.5.** Given a periodic function pair \((v_0, T_0) \in H^2(\Omega)\) satisfying \(\int_{-1}^1 \nabla h \cdot v_0 dz = 0\). Suppose that \((v_\lambda, w_\lambda, T_\lambda)\) is the unique local strong solution of the system (1.4), and that \((v, T)\) is the unique global strong solution of the system (1.10), with the same boundary and initial conditions (1.5)-(1.7) and symmetry condition (1.7). Let \((V_\lambda, W_\lambda, \Phi_\lambda) = (v_\lambda - v, w_\lambda - w, T_\lambda - T)\).

Then, for any finite time \(T > 0\), there is a small positive constant \(\lambda(T) = \left(\frac{3\beta^2}{8\gamma_0(T)}\right)^{1/\eta}\) such that the system (1.4) exists a unique strong solution \((v_\lambda, w_\lambda, T_\lambda)\) on the time interval \([0, T]\), and that the system (5.2)-(5.6) (see Section 5, below) has the following estimate

\[
\sup_{0 \leq t \leq T} \left( \left\| (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \right\|^2_{H^1} (t) + \int_0^T \left( \left\| \nabla_h V_\lambda \right\|^2_{H^1} + \lambda^{\beta-2} \left\| \partial_z V_\lambda \right\|^2_{H^1} \right) dt \right.
\]

\[
+ \int_0^T \left( \lambda^2 \left\| \nabla_h W_\lambda \right\|^2_{H^1} + \left\| \nabla_h \Phi_\lambda \right\|^2_{H^1} + \lambda^2 \left\| \partial_z W_\lambda \right\|^2_{H^1} + \lambda^{\gamma-2} \left\| \partial_z \Phi_\lambda \right\|^2_{H^1} \right) dt \leq \lambda^\eta J_7(T),
\]

provided that \(\lambda \in (0, \lambda(T))\), where \(\eta = \min\{2, \beta - 2, \gamma - 2\}\) with \(2 < \beta, \gamma < \infty\), and \(J_7(t)\) is a nonnegative continuously increasing function that does not depend on \(\lambda\). As a result, we have the following strong convergences

\[
(v_\lambda, \lambda w_\lambda, T_\lambda) \rightarrow (v, 0, T), \text{ in } L^\infty([0, T]; H^1(\Omega)),
\]

\[
(\nabla_h v_\lambda, \lambda^{(\beta-2)/2} \partial_z v_\lambda, \lambda \nabla_h w_\lambda, w_\lambda) \rightarrow (\nabla_h v, 0, 0, w), \text{ in } L^2([0, T]; H^1(\Omega)),
\]

\[
(\nabla_h T_\lambda, \lambda^{\beta/2} \partial_z w_\lambda, \lambda^{(\gamma-2)/2} \partial_z T_\lambda) \rightarrow (\nabla_h T, 0, 0), \text{ in } L^2([0, T]; H^1(\Omega)),
\]

\[
w_\lambda \rightarrow w, \text{ in } L^\infty([0, T]; L^2(\Omega)),
\]

and the rate of convergence is of the order \(O(\lambda^{\beta/2})\).

3. Strong convergence for \(H^1\) initial data

In this section, assume that initial data \((v_0, T_0) \in H^1(\Omega)\), where initial velocity \(v_0\) satisfies

\[
\int_{-1}^1 \nabla h \cdot v_0(x, y, z) dz = 0, \text{ for all } (x, y) \in M,
\]

we prove that the scaled Boussinesq equations with rotation (1.4) strongly converge to the primitive equations with only horizontal viscosity and diffusivity (1.10) as the aspect ration parameter \(\lambda\) goes to zero.

As mentioned in the introduction, for initial data \((v_0, T_0) \in H^1(\Omega)\), the system (1.4) subject to (1.5)-(1.7) has a global weak solution \((v_\lambda, w_\lambda, T_\lambda)\), while the system (1.10) corresponding to (1.3)-(1.7) exists a unique local strong solution \((v, T)\). Denote by \(T_0\) the maximal existence time of this local strong solution. The well-posedness of strong solutions to the primitive equations with only horizontal viscosity and diffusivity (1.10) is as follows (see [9]).

**Proposition 3.1.** Let \(v_0, T_0 \in H^1(\Omega)\) be two periodic functions with \(\int_{-1}^1 \nabla h \cdot v_0(x, y, z) dz = 0, \text{ for all } (x, y) \in M\). Then the following assertions hold true:
Given a periodic function pair \((v, T)\) to the primitive equations with only horizontal viscosity and diffusivity \(\text{(1.10)}\) corresponding to \(\text{(1.3)}, \text{(1.7)}\), such that
\[
(v, T) \in L^\infty([0, t_0^*); H^1(\Omega)) \cap C([0, t_0^*); L^2(\Omega)),
\]
\[
(\nabla_h v, \nabla_h T) \in L^2([0, t_0^*); H^1(\Omega)),
\]
\[
(\partial_v v, \partial_t T) \in L^2([0, t_0^*); L^2(\Omega)),
\]
where \(t_0^*\) is the maximal existence time of this local strong solution;
(ii) The local strong solution \((v, T)\) to the system \(\text{(1.10)}\) satisfies the following energy estimate
\[
\sup_{0 \leq s \leq t} \left( \left\| (v, T) \right\|^2_{H^1(\Omega)} \right)(s) + \int_0^t \left\| \nabla_h v \right\|^2_{H^1(\Omega)} ds + \int_0^t \left( \left\| \nabla_h T \right\|^2_{H^1(\Omega)} + \left\| (\partial_v v, \partial_t T) \right\|^2_{L^2(\Omega)} \right) ds \leq C,
\]
for any \(t \in [0, t_0^*).\) Here \(C\) is a positive constant.

The following proposition is formally obtained by testing the scaled Boussinesq equations with rotation \(\text{(1.4)}\) with \((v, w, T)\). As for the rigorous justification for this proposition, we refer to the work of Li-Titi [32] and Bardos et al. [2].

**Proposition 3.2.** Given a periodic function pair \((v_0, T_0) \in H^1(\Omega)\) with
\[
\int_{-1}^1 \nabla_h \cdot v_0 dz = 0\text{ and } w_0(x, y, z) = -\int_0^z \nabla_h \cdot v_0(x, y, \xi) d\xi.
\]
Suppose that \((v_\lambda, w_\lambda, T_\lambda)\) is a global weak solution of the system \(\text{(1.4)}\), satisfying the energy inequality \(\text{(1.9)}\). Then the following integral equality holds
\[
\left( \int_{\Omega} (v_\lambda \cdot v + \lambda^2 w_\lambda w + T_\lambda T) \, dx dy dz \right) (r) + \int_0^r \int_{\Omega} f_0(k \times v_\lambda) \cdot v \, dx dy dz dt
\]
\[
+ \int_0^r \int_{\Omega} \left[ \nabla_h v_\lambda : \nabla_h v + \lambda^{3/2} (\partial_\lambda v_\lambda) \cdot \partial_\lambda v + \lambda^2 \nabla_h w_\lambda \cdot \nabla_h w \right] \, dx dy dz dt
\]
\[
+ \int_0^r \int_{\Omega} \left[ \lambda^3 (\partial_\lambda w_\lambda) \partial_\lambda w + \nabla_h T_\lambda \cdot \nabla_h T + \lambda^{-2} (\partial_\lambda T_\lambda) \partial_\lambda T \right] \, dx dy dz dt
\]
\[
= \|v_0\|^2_2 + \int_0^r \int_{\Omega} \left[ -(u_\lambda \cdot \nabla) v_\lambda \cdot v - \lambda^2 (u_\lambda \cdot \nabla w_\lambda) w - (u_\lambda \cdot \nabla T_\lambda) T \right] \, dx dy dz dt
\]
\[
+ \frac{\lambda^2}{2} \|w(r)\|^2_2 + \frac{\lambda^2}{2} \|w_0\|^2_2 + \lambda^2 \int_0^r \int_{\Omega} \left( \int_0^z (\partial_\lambda v(x, y, \xi, t) d\xi) \right) \cdot \nabla_h W_\lambda \, dx dy dz dt
\]
\[
+ \|T_0\|^2_2 + \int_0^r \int_{\Omega} (v_\lambda \cdot \partial_\lambda v + T_\lambda \partial_\lambda T + T_\lambda w - w_\lambda T) \, dx dy dz dt,
\]
for any \(r \in [0, t_0^*).\)

With the help of Proposition 3.2, we can estimate the difference function \((V_\lambda - v, W_\lambda, \Phi_\lambda) = (v_\lambda - v, w_\lambda - w, T_\lambda - T)\). Before this, we present a lemma (see \[13\]), which will be frequently used later.

**Lemma 3.3.** The following inequalities hold
\[
\int_M \left( \int_{-1}^1 \varphi(x, y, z) \, dz \right) \left( \int_{-1}^1 \psi(x, y, z) \phi(x, y, z) \, dz \right) \, dx dy
\]
\[
\leq C \left\| \varphi \right\|^2_{L^2} \left( \left\| \varphi \right\|^2_{L^2} + \left\| \nabla_h \varphi \right\|^2_{L^2} \right) \left\| \psi \right\|^2_{L^2} \left( \left\| \psi \right\|^2_{L^2} + \left\| \nabla_h \psi \right\|^2_{L^2} \right) \left\| \phi \right\|^2_{L^2},
\]
Replacing \( \phi \) respectively, and integrating over \( M \), we have

\[
\int_{M} \left( \int_{-1}^{1} \varphi(x, y, z) \, dz \right) \left( \int_{-1}^{1} \psi(x, y, z) \phi(x, y, z) \, dz \right) \, dx dy \\
\leq C \| \psi \|_{2}^{1/2} \left( \| \psi \|_{2}^{1/2} + \| \nabla_{h} \psi \|_{2}^{1/2} \right) \| \phi \|_{2}^{1/2} \left( \| \phi \|_{2}^{1/2} + \| \nabla_{h} \phi \|_{2}^{1/2} \right) \| \varphi \|_{2},
\]

for every \( \varphi, \psi, \phi \) such that the right-hand sides make sense and are finite, where \( C \) is a positive constant.

**Proposition 3.4.** Let \((V_{\lambda}, W_{\lambda}, \Phi_{\lambda}) = (v_{\lambda} - v, w_{\lambda} - w, T_{\lambda} - T)\). Under the same assumptions as in Proposition 3.2, the following estimate holds

\[
\sup_{0 \leq s \leq T} \left( \left\| (V_{s}, \lambda W_{s}, \Phi_{s}) \right\|_{2}^{2} \right)^{1/2} + \int_{0}^{T} \left( \| \nabla_{h} V_{s} \|_{2}^{2} + \lambda^{\beta-2} \| \partial_{z} V_{s} \|_{2}^{2} \right) ds \\
+ \int_{0}^{T} \left( \lambda^{2} \| \nabla_{h} W_{s} \|_{2}^{2} + \| \nabla_{h} \Phi_{s} \|_{2}^{2} + \lambda^{\beta} \| \partial_{z} W_{s} \|_{2}^{2} + \lambda^{\gamma} \| \partial_{z} \Phi_{s} \|_{2}^{2} \right) ds \\
\leq C(t + 1)e^{C(t+1)} \left[ \lambda^{2} + \lambda^{\beta-2} + \lambda^{\gamma-2} + \lambda^{2} \left( \| v_{0} \|_{2}^{2} + \lambda^{2} \| w_{0} \|_{2}^{2} + \| T_{0} \|_{2}^{2} \right) \right],
\]

for any \( t \in [0, t^{*}] \), where \( C \) is a positive constant that does not depend on \( \lambda \).

**Proof.** Multiplying the first three equation in system (1.10) by \( v_{\lambda} \), \( w_{\lambda} \) and \( T_{\lambda} \) respectively, and integrating over \( \Omega \times (0, r) \), then it follows from integration by parts that

\[
\int_{0}^{r} \int_{\Omega} \left( v_{\lambda} \cdot \partial_{t} v + T_{\lambda} \partial_{t} T + \nabla_{h} v_{\lambda} : \nabla_{h} v + \nabla_{h} T_{\lambda} \cdot \nabla_{h} T \right) \, dx dy dz dt \\
= \int_{0}^{r} \int_{\Omega} \left[ T w_{\lambda} - w T_{\lambda} - (u \cdot \nabla) v \cdot v_{\lambda} - (u \cdot \nabla T) T_{\lambda} \right] \, dx dy dz dt \\
+ \int_{0}^{r} \int_{\Omega} \left[ -f_{0}(k \times v) \cdot v_{\lambda} \right] \, dx dy dz dt. \tag{3.3}
\]

Replacing \((v_{\lambda}, w_{\lambda}, T_{\lambda})\) with \((v, w, T)\), a similar argument gives

\[
\frac{1}{2} \left( \| v(r) \|_{2}^{2} + \| T(r) \|_{2}^{2} \right) + \int_{0}^{r} \left( \| \nabla_{h} v \|_{2}^{2} + \| \nabla_{h} T \|_{2}^{2} \right) dt \\
= \frac{1}{2} \left( \| v_{0} \|_{2}^{2} + \| T_{0} \|_{2}^{2} \right), \tag{3.4}
\]

note that we have used the following fact

\[
\int_{0}^{r} \int_{\Omega} f_{0}(k \times v) \cdot v \, dx dy dz dt = 0.
\]

Thanks to Remark 1.3, the weak solution \((v_{\lambda}, w_{\lambda}, T_{\lambda})\) of the system (1.4) satisfies the following energy inequality

\[
\frac{1}{2} \left( \| v_{\lambda}(r) \|_{2}^{2} + \lambda^{2} \| w_{\lambda}(r) \|_{2}^{2} + \| T_{\lambda}(r) \|_{2}^{2} \right) \\
+ \int_{0}^{r} \left( \| \nabla_{h} v_{\lambda} \|_{2}^{2} + \lambda^{\beta-2} \| \partial_{z} v_{\lambda} \|_{2}^{2} + \lambda^{2} \| \nabla_{h} w_{\lambda} \|_{2}^{2} \right) dt \\
+ \int_{0}^{r} \left( \lambda^{\beta} \| \partial_{z} w_{\lambda} \|_{2}^{2} + \| \nabla_{h} T_{\lambda} \|_{2}^{2} + \lambda^{\gamma-2} \| \partial_{z} T_{\lambda} \|_{2}^{2} \right) dt \\
\leq \frac{1}{2} \left( \| v_{0} \|_{2}^{2} + \lambda^{2} \| w_{0} \|_{2}^{2} + \| T_{0} \|_{2}^{2} \right). \tag{3.5}
\]
Subtracting the sum of (3.2) and (3.3) from the sum of (3.4) and (3.5), we have
\[
\frac{1}{2} \left( \| V_\lambda (r) \|_2^2 + \lambda^2 \| W_\lambda (r) \|_2^2 + \| \Phi_\lambda (r) \|_2^2 \right) + \int_0^r \left( \| \nabla h V_\lambda \|_2^2 + \lambda^{\beta-2} \| \partial_2 V_\lambda \|_2^2 \right) dt \\
+ \int_0^r \left( \lambda^2 \| \nabla h W_\lambda \|_2^2 + \| \nabla h \Phi_\lambda \|_2^2 + \lambda^{\beta} \| \partial_2 W_\lambda \|_2^2 + \lambda^{\gamma-2} \| \partial_2 \Phi_\lambda \|_2^2 \right) dt \\
\leq \int_0^r \int_\Omega \left[ (u_\lambda \cdot \nabla T_\lambda) T + (u \cdot \nabla T) T_\lambda \right] dx dy dz dt \\
+ \int_0^r \int_\Omega \left[ (u_\lambda \cdot \nabla) v_\lambda \cdot v + (u \cdot \nabla) v \cdot v_\lambda \right] dx dy dz dt \\
+ \lambda^2 \int_0^r \int_\Omega \left[ - \left( \int_0^z \partial_t v(x, y, \zeta, t) d\xi \right) \cdot \nabla h W_\lambda \right] dx dy dz dt \\
+ \int_0^r \int_\Omega \left[ - \lambda^2 \nabla h W_\lambda \cdot \nabla h w - \lambda^{\gamma-2} (\partial_2 \Phi_\lambda) \partial_2 T \right] dx dy dz dt \\
+ \int_0^r \int_\Omega \left[ - \lambda^2 (\partial_2 W_\lambda) \partial_2 w - \lambda^{\beta-2} (\partial_2 \Phi_\lambda) \partial_2 v \right] dx dy dz dt \\
+ \lambda^2 \int_0^r \int_\Omega \left( u_\lambda \cdot \nabla w_\lambda \right) w dx dy dz dt \\
=: R_1 + R_2 + R_3 + R_4 + R_5 + R_6.
\] (3.6)

In order to estimate the first integral term \( R_1 \) on the right-hand side of (3.6), we use the divergence-free condition and integration by parts to obtain
\[
R_1 : = \int_0^r \int_\Omega \left[ (u_\lambda \cdot \nabla T_\lambda) T + (u \cdot \nabla T) T_\lambda \right] dx dy dz dt \\
= \int_0^r \int_\Omega \left[ [(u_\lambda - u) \cdot \nabla \Phi_\lambda) T \right] dx dy dz dt \\
= \int_0^r \int_\Omega \left[ (V_\lambda \cdot \nabla \Phi_\lambda) T + W_\lambda (\partial_2 \Phi_\lambda) T \right] dx dy dz dt \\
=: R_{11} + R_{12}. \] (3.7)

For the integral term \( R_{11} \) on the right-hand side of (3.7), noting that the fact that \( |T(z)| \leq \frac{1}{\tau} \int_{-1}^1 |T|dz + \int_{-1}^1 |\partial_2 T|dz \), and applying Lemma 3.3 and Young inequality, we have
\[
R_{11} : = \int_\Omega (V_\lambda \cdot \nabla \Phi_\lambda) T dx dy dz dt \leq \int_0^r \int_\Omega |V_\lambda| \| \nabla h \Phi_\lambda \| dx dy dz dt \\
\leq \int_0^r \int_M \left( \int_{-1}^1 (|T| + |\partial_2 T|)dz \right) \left( \int_{-1}^1 |V_\lambda| \| \nabla h \Phi_\lambda \|dz \right) dx dy dz dt \\
\leq C \int_0^r \left( \| \partial_2 T \|_2 + \| \partial_2 T \|_2^{1/2} \| \nabla h \partial_2 T \|_2^{1/2} \right) \left( \| V_\lambda \|_2^{1/2} \| \nabla h V_\lambda \|_2^{1/2} \| \nabla h \Phi_\lambda \|_2 dt \\
+ C \int_0^r \| V_\lambda \|_2 \| \nabla h \Phi_\lambda \|_2 \left( \| \partial_2 T \|_2 + \| \partial_2 T \|_2^{1/2} \| \nabla h \partial_2 T \|_2^{1/2} \right) dt \\
+ C \int_0^r \left( \| T \|_2 + \| T \|_2^{1/2} \| \nabla h T \|_2^{1/2} \right) \left( \| V_\lambda \|_2 + \| V_\lambda \|_2^{1/2} \| \nabla h V_\lambda \|_2^{1/2} \right) \| \nabla h \Phi_\lambda \|_2 dt
Next we need to estimate the integral term $R_{12}$ on the right-hand side of (3.7). Using the same method as the first integral term on the right-hand side of (3.7), this term can be bounded as

$$R_{12} = \int_0^r \int_\Omega \left( \nabla_h \cdot \Phi_{\lambda} \right) T \, dx dy dz dt$$

$$= \int_0^r \int_\Omega \left[ \int_0^z \left( |T| + |\partial_z T| \right) \right] \nabla_h \cdot \Phi_{\lambda} \, dx dy dz dt$$

$$\leq C \int_0^r \int_\Omega \left( \left| \int_0^z \left( |T| + |\partial_z T| \right) \right| \right) \nabla_h \cdot \Phi_{\lambda} \, dx dy dz dt$$

$$+ C \int_0^r \int_\Omega \left( \left| \int_0^z \left( \nabla_h \cdot \Phi_{\lambda} \right) T \right| \right) \, dx dy dz dt$$

$$\leq C \int_0^r \int_\Omega \left( \left| \int_0^z \left( |T| + |\partial_z T| \right) \right| \right) \nabla_h \cdot \Phi_{\lambda} \, dx dy dz dt$$

Adding (3.8) and (3.9) gives

The similar argument as the integral term $R_1$ yields

$$R_2 = \int_0^r \int_\Omega \left[ (u_{\lambda} \cdot \nabla) v_{\lambda} \cdot v + (u \cdot \nabla) v \cdot v_{\lambda} \right] \, dx dy dz dt$$

$$= \int_0^r \int_\Omega \left[ \int_0^z \left( |T| + |\partial_z T| \right) \right] \nabla_h \cdot \Phi_{\lambda} \, dx dy dz dt$$

$$\leq C \int_0^r \int_\Omega \left( \left| \int_0^z \left( |T| + |\partial_z T| \right) \right| \right) \nabla_h \cdot \Phi_{\lambda} \, dx dy dz dt$$

The similar argument as the integral term $R_1$ yields
By virtue of the Hölder inequality and Young inequality, the integral terms $R_3$, $R_4$ and $R_5$ on the right-hand side of (3.6) can be estimated as

$$
R_3 : = \lambda^2 \int_0^r \int_0^r \int_0^r \int_0^r \left[ - \left( \int_0^z \partial_t v(x, y, \xi, t) d\xi \right) \cdot \nabla_h W_\lambda \right] dxdydzdt \\
\leq \lambda^2 \int_0^r \int_0^r \int_0^r \left( \int_0^1 |\partial_x v| dz \right) \left( \int_0^1 |\nabla_h W_\lambda| dz \right) dxdydt \\
\leq C\lambda^2 \int_0^r \|\partial_t v\|_2^2 dt + \frac{1}{12} \int_0^r \lambda^2 \|\nabla_h W_\lambda\|_2^2 dt,
$$

$$
R_4 : = \int_0^r \int_0^r \left[ -\lambda^2 \nabla_h W_\lambda \cdot \nabla_h w - \lambda^{\gamma - 2}(\partial_x W_\lambda) \partial_x T \right] dxdydzdt \\
\leq C\lambda^2 \int_0^r \|\nabla_h w\|_2^2 dt + \frac{1}{12} \int_0^r \lambda^2 \|\nabla_h W_\lambda\|_2^2 dt \\
+ C\lambda^{\gamma - 2} \int_0^r \|\partial_x T\|_2^2 dt + \frac{1}{12} \int_0^r \lambda^{\gamma - 2} \|\partial_x W_\lambda\|_2^2 dt \\
\leq C\lambda^2 \int_0^r \|\nabla_h (\nabla_h \cdot v)\|_2^2 dxdydzdt + C\lambda^{\gamma - 2} \int_0^r \|\partial_x T\|_2^2 dt \\
+ \frac{1}{12} \int_0^r \left( \lambda^2 \|\nabla_h W_\lambda\|_2^2 + \lambda^{\gamma - 2} \|\partial_x W_\lambda\|_2^2 \right) dt
$$

and

$$
R_5 : = \int_0^r \int_0^r \left[ -\lambda^\beta (\partial_x W_\lambda) \partial_x w - \lambda^{\beta - 2} (\partial_x W_\lambda) \cdot \partial_x v \right] dxdydzdt \\
\leq C\lambda^\beta \int_0^r \|\partial_x w\|_2^2 dt + \frac{1}{12} \int_0^r \lambda^\beta \|\partial_x W_\lambda\|_2^2 dt \\
+ C\lambda^{\beta - 2} \int_0^r \|\partial_x v\|_2^2 dt + \frac{1}{12} \int_0^r \lambda^{\beta - 2} \|\partial_x W_\lambda\|_2^2 dt \\
\leq C\lambda^\beta \int_0^r \|\nabla_h v\|_2^2 dt + C\lambda^{\beta - 2} \int_0^r \|\nabla v\|_2^2 dt \\
+ \frac{1}{12} \int_0^r \left( \lambda^\beta \|\partial_x W_\lambda\|_2^2 + \lambda^{\beta - 2} \|\partial_x W_\lambda\|_2^2 \right) dt,
$$

respectively, noting that the divergence-free condition is used. Finally, it remains to deal with the last integral term $R_6$ on the right-hand side of (3.6). Thanks to the Lemma and Young inequality, we obtain

$$
R_6 : = \lambda^2 \int_0^r \int_\Omega (u_\lambda \cdot \nabla w_\lambda) wdx dydz \\
= \lambda^2 \int_0^r \int_\Omega \left[ w_\lambda (\nabla_h \cdot V_\lambda) \int_0^z \nabla_h \cdot v d\xi - v_\lambda \cdot \nabla_h W_\lambda \int_0^z \nabla_h \cdot v d\xi \right] dxdydzdt \\
\leq \lambda^2 \int_0^r \int_M \left( \int_{-1}^1 |\nabla_h v| dz \right) \left( \int_{-1}^1 |w_\lambda||\nabla_h V_\lambda| dz \right) dxdydt
$$
\[
+ \lambda^2 \int_0^r \int_M \left( \int_{-1}^1 |\nabla_h v| dz \right) \left( \int_{-1}^1 |v_\lambda| |\nabla_h W_\lambda| dz \right) \, dxdydt
\]
\[
\leq C\lambda^2 \|\nabla_h v\|_{2/2}^{2/2} \|\nabla_h v\|_{2/2}^{1/2} \|\nabla_h v\|_{2/2}^{1/2} \left( \|w_\lambda\|_{2/2}^{1/2} + \|\nabla_h w_\lambda\|_{2/2}^{1/2} \right) \|\nabla_h V_\lambda\|_2
\]
\[
+ C\lambda^2 \|\nabla_h v\|_{2/2}^{2/2} \|\nabla_h v\|_{2/2}^{1/2} \|\nabla_h v\|_{2/2}^{1/2} \left( \|v_\lambda\|_{2/2}^{1/2} + \|\nabla_h v\|_{2/2}^{1/2} \right) \|\nabla_h W_\lambda\|_2
\]
\[
\leq C\lambda^2 \int_0^r \left( \|\nabla_h v\|_2^2 + \|\nabla_h v\|_2^2 + \lambda^4 \|w_\lambda\|_2^4 + \lambda^4 \|w_\lambda\|_2^4 \|\nabla_h w_\lambda\|_2^2 \right) dt
\]
\[
+ C\lambda^2 \int_0^r \left( \|v_\lambda\|_2^4 + \|v_\lambda\|_2^4 \|\nabla_h v\|_2^2 \right) dt + \frac{1}{12} \int_0^r \left( \|\nabla_h V_\lambda\|_2^2 + \lambda^2 \|\nabla_h W_\lambda\|_2^2 \right) dt.
\]

Combining the estimates for $R_1$, $R_2$, $R_3$, $R_4$, $R_5$ and $R_6$, we reach

\[
\mathcal{F}(t) = \left( \|V_\lambda(t)\|_2^2 + \lambda^2 \|W_\lambda(t)\|_2^2 + \|\Phi_\lambda(t)\|_2^2 \right) + \int_0^t \left( \|\nabla_h V_\lambda\|_2^2 + \lambda^{\beta-2} \|\nabla_h V_\lambda\|_2^2 \right) ds
\]
\[
+ \int_0^t \left( \lambda^2 \|\nabla_h W_\lambda\|_2^2 + \|\nabla_h \Phi_\lambda\|_2^2 + \lambda^2 \|\nabla_h \Phi_\lambda\|_2^2 + \lambda^{\gamma-2} \|\nabla_h \Phi_\lambda\|_2^2 \right) ds
\]
\[
\leq C \int_0^t \left( \|\partial_z T\|_2^2 + \|\partial_z T\|_2 \|\nabla_h \partial_z T\|_2 + \|\partial_z T\|_2^4 + \|\partial_z T\|_2^4 \|\nabla_h \partial_z T\|_2^2 \right) \|V_\lambda\|_2^2 ds
\]
\[
+ C \int_0^t \left( \|\partial_z T\|_2^2 + \|\partial_z T\|_2 \|\nabla_h \partial_z T\|_2 + \|\partial_z T\|_2^4 + \|\partial_z T\|_2^4 \|\nabla_h \partial_z T\|_2^2 \right) \|\Phi_\lambda\|_2^2 ds
\]
\[
+ C \int_0^t \left( \|T\|_2^2 + \|T\|_2 \|\nabla_h T\|_2 + \|T\|_2^2 + \|T\|_2^2 \|\nabla_h T\|_2\right) \left( \|V_\lambda\|_2^2 + \|\Phi_\lambda\|_2^2 \right) ds
\]
\[
+ C \int_0^t \left( \|\partial_z v\|_2^2 + \|\partial_z v\|_2 \|\nabla_h \partial_z v\|_2 + \|\partial_z v\|_2^4 + \|\partial_z v\|_2^4 \|\nabla_h \partial_z v\|_2^2 \right) \|V_\lambda\|_2^2 ds
\]
\[
+ C \int_0^t \left( \|v_\lambda\|_2^2 + \|v_\lambda\|_2 \|\nabla_h v\|_2 + \|v_\lambda\|_2^4 + \|v_\lambda\|_2^4 \|\nabla_h v\|_2^2 \right) \|V_\lambda\|_2^2 ds
\]
\[
+ C \lambda^2 \int_0^t \left( \|\partial_z v\|_2^2 + \|\nabla_h v\|_2^2 \right) ds + C\lambda^{\gamma-2} \int_0^t \|\partial_z T\|_2^2 ds + C\lambda^\beta \int_0^t \|\nabla_h v\|_2^2 ds
\]
\[
+ C\lambda^2 \int_0^t \left( \|\nabla_h v\|_2^2 + \|\nabla_h v\|_2^2 \right) \|\nabla_h v\|_2^2 \right) ds + C\lambda^{\beta-2} \int_0^t \|\nabla_h v\|_2^2 ds =: G(t),
\]
for a.e. $t \in [0, t_0^*)$. Taking the derivative of $G(t)$ with respect to $t$ leads to

\[
G'(t) = C \left( \|\partial_z T\|_2^2 + \|\partial_z T\|_2 \|\nabla_h \partial_z T\|_2 + \|\partial_z T\|_2^4 + \|\partial_z T\|_2^4 \|\nabla_h \partial_z T\|_2^2 \right) \left( \|V_\lambda\|_2^2 + \|\Phi_\lambda\|_2^2 \right)
\]
\[
+ C \left( \|T\|_2^2 + \|T\|_2 \|\nabla_h T\|_2 + \|T\|_2^4 + \|T\|_2^4 \|\nabla_h T\|_2^2 \right) \left( \|V_\lambda\|_2^2 + \|\Phi_\lambda\|_2^2 \right)
\]
\[
+ C \left( \|\partial_z v\|_2^2 + \|\partial_z v\|_2 \|\nabla_h \partial_z v\|_2 + \|\partial_z v\|_2^4 + \|\partial_z v\|_2^4 \|\nabla_h \partial_z v\|_2^2 \right) \|V_\lambda\|_2^2
\]
\[
+ C \left( \|v_\lambda\|_2^2 + \|v_\lambda\|_2 \|\nabla_h v\|_2 + \|v_\lambda\|_2^4 + \|v_\lambda\|_2^4 \|\nabla_h v\|_2^2 \right) \|V_\lambda\|_2^2
\]
\[
+ C\lambda^2 \left( \|\partial_z v\|_2^2 + \|\nabla_h v\|_2^2 \right) + C\lambda^{\gamma-2} \|\partial_z T\|_2^2 + C\lambda^\beta \|\nabla_h v\|_2^2
\]
\[
+ C\lambda^2 \left( \|\nabla_h v\|_2^2 + \|\nabla_h v\|_2^2 + \lambda^4 \|w_\lambda\|_2^4 + \lambda^4 \|w_\lambda\|_2^4 \|\nabla_h w_\lambda\|_2^2 \right)
\]
This completes the proof of Proposition 3.4. Based on Proposition 3.4, we give the Proof of Theorem 2.1.

where we have used the inequality $F(t) \leq G(t)$. Noting that the fact that $G(0) = 0$, and applying the Gronwall inequality to the above inequality, we obtain

$$F(t) \leq \exp \left\{ C \int_0^t \left( \|\partial_z T\|_2^2 + \|\partial_z T\|_2 \|\nabla_h \partial_z T\|_2 + \|\partial_z T\|_2^2 + \|\partial_z T\|_2 \|\nabla_h \partial_z T\|_2^2 \right) ds ight\}$$

$$+ C \int_0^t \left( \|\partial_z v\|_2^2 + \|\partial_z v\| \|\nabla_h \partial_z v\|_2 + \|\partial_z v\|_2^2 + \|\partial_z v\|_2 \|\nabla_h \partial_z v\|_2^2 \right) ds$$

$$+ C \int_0^t \left( \|v\|_2^2 + \|v\|_2 \|\nabla_h v\|_2 + \|v\|_2^2 + \|v\|_2 \|\nabla_h v\|_2^2 \right) ds$$

$$\times \left\{ C\lambda^2 \int_0^t \left( \|\partial_z v\|_2^2 + \|\nabla_h v\|_2^2 \right) ds + C\lambda^{\gamma-2} \int_0^t \|\partial_z T\|_2^2 ds + C\lambda^\beta \int_0^t \|\nabla_h v\|_2^2 ds 
+ C\lambda^2 \int_0^t \left( \|\nabla_h v\|_2^2 + \lambda^2 \|w_\lambda\|_2^4 + \lambda^4 \|w_\lambda\|_2^2 \|\nabla_h w_\lambda\|_2^2 \right) ds 
+ C\lambda^2 \int_0^t \left( \|v_\lambda\|_2^4 + \|v_\lambda\|_2 \|\nabla_h v_\lambda\|_2^2 \right) ds + C\lambda^{\gamma-2} \int_0^t \|\nabla h v\|_2^2 ds \right\}.$$ (3.10)

From (8.1) and (8.15), it follows that

$$\left( \|V_\lambda(t)\|_2^2 + \lambda^2 \|W_\lambda(t)\|_2^2 + \|\Phi_\lambda(t)\|_2^2 \right) + \int_0^t \left( \|\nabla_h V_\lambda\|_2^2 + \lambda^\beta-2 \|\partial_z V_\lambda\|_2^2 \right) ds$$

$$+ \int_0^t \left( \lambda^2 \|\nabla_h W_\lambda\|_2^2 + \|\nabla_h \Phi_\lambda\|_2^2 + \lambda^\beta \|\partial_z W_\lambda\|_2^2 + \|\partial_z \Phi_\lambda\|_2^2 \right) ds$$

$$\leq C(t + 1)e^{C(t+1)} \left[ \lambda^2 + \lambda^{\beta-2} + \lambda^{\gamma-2} + \lambda^2 \left( \|v_0\|_2^2 + \|w_0\|_2^2 + \|f_0\|_2^2 \right) \right].$$

This completes the proof of Proposition 3.4. □

Based on Proposition 3.4 we give the Proof of Theorem 2.1.
Proof of Theorem 2.1. By the Proposition 3.4, we have the following estimate
\begin{equation}
\sup_{0 \leq t < t_0^*} \left( \left\| (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \right\|_2^2 \right) (t) + \int_0^{t_0^*} \left( \left\| \nabla h V_\lambda \right\|_2^2 + \lambda^{\beta - 2} \left\| \partial_z V_\lambda \right\|_2^2 \right) dt
+ \int_0^{t_0^*} \left( \lambda \left\| \nabla h W_\lambda \right\|_2^2 + \left\| \nabla h \Phi_\lambda \right\|_2^2 + \lambda^{\beta - 2} \left\| \partial_z W_\lambda \right\|_2^2 + \lambda^{\gamma - 2} \left\| \partial_z \Phi_\lambda \right\|_2^2 \right) dt
\leq C \lambda^n(t_0^* + 1) e^{C(t_0^*)^2} \left[ 1 + \left( \left\| v_0 \right\|_2^2 + \left\| w_0 \right\|_2^2 + \left\| T_0 \right\|_2^2 \right)^2 \right],
\end{equation}
where \( \eta = \min\{2, \beta - 2, \gamma - 2\} \) with \( 2 < \beta, \gamma < \infty \), and \( t_0^* \) is the maximal existence time of local strong solution \((v, T)\) to the system (1.10). Here \( C \) is a positive constant that does not depend on \( \lambda \). The above estimate implies that
\( (v_\lambda, \lambda w_\lambda, T_\lambda) \rightarrow (v, 0, T) \), in \( L^\infty ([0, t_0^*); L^2(\Omega)) \),
\( (\nabla_h v_\lambda, \lambda^{(\beta - 2)/2} \partial_z v_\lambda, \lambda \nabla_h w_\lambda) \rightarrow (\nabla_h v, 0, 0) \), in \( L^2 ([0, t_0^*); L^2(\Omega)) \),
\( (\nabla_h T_\lambda, \lambda^{\beta/2} \partial_z w_\lambda, \lambda^{(\gamma - 2)/2} \partial_z T_\lambda) \rightarrow (\nabla_h T, 0, 0) \), in \( L^2 ([0, t_0^*); L^2(\Omega)) \).
Owing to \( \nabla_h v_\lambda \rightarrow \nabla_h v \) in \( L^2 ([0, t_0^*); L^2(\Omega)) \), it deduces from the divergence-free condition that
\( w_\lambda \rightarrow w \) in \( L^2 ([0, t_0^*); L^2(\Omega)) \).
Finally, it can easily be seen that the rate of convergence is of the order \( O(\lambda^{n/2}) \). \( \square \)

4. STRONG CONVERGENCE FOR \( H^1 \) INITIAL DATA WITH ADDITIONAL REGULARITY

In this section, we study the strong convergence for the case of initial data \((v_0, T_0)\) in \( H^1(\Omega) \) with additional regularity \( \partial_z v_0 \in L^q(\Omega) \). The following global existence result is due to Cao-Li-Titi[9].

Proposition 4.1. Assume that \((v_0, T_0) \in H^1(\Omega) \cap L^\infty(\Omega) \) satisfying \( \int_{\partial \Omega} \nabla_h \cdot v_0 dz = 0 \), and that \( \partial_z v_0 \in L^q(\Omega) \) with \( q \in (2, \infty) \). Then the local strong solution \((v, T)\) of the system (1.10) subject to (1.5)-(1.7) can be extended uniquely to be a global one such that the following energy estimate holds
\begin{equation}
\sup_{0 \leq s \leq t} \left( \left\| (v, T) \right\|_{H^1(\Omega)}^2 \right) (s) + \int_0^t \left\| \nabla_h v \right\|_{H^1(\Omega)}^2 ds
+ \int_0^t \left( \left\| \nabla_h T \right\|_{H^1(\Omega)}^2 + \left\| (\partial_z v, \partial_z T) \right\|_{L^2(\Omega)}^2 \right) ds \leq \mathcal{J}_1(t),
\end{equation}
for any \( t \in [0, \infty) \), where \( \mathcal{J}_1(t) \) is a nonnegative continuously increasing function defined on \([0, \infty)\).

In order to prove the Theorem 2.2 we need the following proposition, which is a direct consequence of Lemma 2.2 with exponent (6, 6, 2) in [34].

Proposition 4.2. Let \((v, T)\) be the local strong solution to the system (1.10) corresponding to (1.5)-(1.7). Then the following inequalities hold
\begin{equation}
\sup_{(x, y, z) \in \overline{\Omega}} \left| v(x, y, z, t_\delta) \right| + \sup_{(x, y, z) \in \overline{\Omega}} \left| T(x, y, z, t_\delta) \right|
\leq C \left( \left\| \nabla_h v \right\|_6 + \left\| \partial_z v \right\|_2 + \left\| \nabla_h T \right\|_6 + \left\| \partial_z T \right\|_2 \right)(t_\delta)
\end{equation}
for some fixed time \( t_\delta \in (0, t_0^*) \).
With the help of Proposition 4.1 and 4.2, we give the proof of Theorem 2.2. The key to the proof is to establish $L^\infty([0, t_0^*); L^4(\Omega))$ estimate on the vertical derivative of horizontal velocity that does not depend on $\|\nu_0\|_\infty$.

**Proof of Theorem 2.2.** Thanks to the energy estimate (3.1), we obtain

$$\int_{t_0^*/4}^{t_0^*} \left( \|\nabla \nabla_h v\|_2^2 + \|\nabla \nabla_T\|_2^2 \right) ds \leq C,$$

which implies that there exists a fixed time $t_0 \in (t_0^*/4, t_0^*)$ such that

$$\|\nabla \nabla_h v\|_2^2 (t_0) + \|\nabla \nabla_T\|_2^2 (t_0) \leq C/t_0^*.$$  \hspace{1cm} (4.2)

Moreover, the following the energy estimate holds

$$\sup_{0 \leq s \leq t_0} \left( \|v(s, T)\|_{H^1(\Omega)}^2 \right) + \int_0^{t_0} \|\nabla \nabla_h v\|_{H^1(\Omega)}^2 ds + \int_0^{t_0} \left( \|\nabla \nabla_T\|_{L^2(\Omega)}^2 + \|\nabla \nabla_T\|_{L^2(\Omega)}^2 \right) ds \leq C.$$  \hspace{1cm} (4.3)

According to Proposition 4.2, it deduces from (4.2), (4.3) and Sobolev imbedding theorem that

$$\sup_{(x,y,z) \in \Omega} |v(x,y,z,t_0)| + \sup_{(x,y,z) \in \Omega} |T(x,y,z,t_0)|$$

$$\leq C \left( \|\nabla \nabla_h v\|_2 (t_0) + \|\nabla v\|_2 (t_0) + \|\nabla_\nu v\|_2 (t_0) \right)$$

$$+ C \left( \|\nabla \nabla_h T\|_2 (t_0) + \|\nabla \nabla_T\|_2 (t_0) + \|\nabla_\nu T\|_2 (t_0) \right)$$

$$\leq C \left( \|\nabla \nabla_h v\|_2^2 (t_0) + \|\nabla \nabla_T\|_2^2 (t_0) + \|\nabla \nabla_h T\|_2^2 (t_0) + \|\nabla \nabla_\nu T\|_2^2 (t_0) + 1 \right)$$

$$\leq C (1 + 1/t_0^*).$$

The above inequality leads to $(v(x,y,z,t_0), T(x,y,z,t_0)) \in L^\infty(\Omega)$.

Next, we show that $\partial_\nu v \in L^\infty([0, t_0^*); L^4(\Omega))$. Integrating the second equation of the system (1.10) with respect to $z$ gives

$$p(x,y,z,t) = p_\nu(x,y,t) + \int_0^z T(x,y,\xi,t) d\xi,$$

where $p_\nu(x,y,t)$ represents unknown surface pressure as $z = 0$. Based on the above relation, we can rewrite the first equation of the system (1.10) as

$$\partial_t v - \Delta_h v + (v \cdot \nabla_h) v - \left( \int_0^z \nabla_h \cdot v(x,y,\xi,t) d\xi \right) \partial_z v + \nabla_\nu p_\nu(x,y,t)$$

$$+ \int_0^z \nabla_h T(x,y,\xi,t) d\xi + f_0 \vec{k} \times v = 0,$$

note that the divergence-free condition is used. Differentiating the above equation with respect to $z$, we have

$$\partial_t (\partial_z v) - \Delta_h (\partial_z v) + (\partial_z v \cdot \nabla_h) v - (\nabla_h \cdot v) \partial_z v + \nabla_\nu T + f_0 \vec{k} \times (\partial_z v)$$

$$+ (v \cdot \nabla_h) \partial_z v - \left( \int_0^z \nabla_h \cdot v(x,y,\xi,t) d\xi \right) \partial_{zz} v = 0.$$  \hspace{1cm} (4.4)
Taking the $L^2(\Omega)$ inner product of the equation (4.4) with $|\partial_z v|^2 \partial_z v$, then it follows from integration by parts that

$$
\frac{1}{4} \frac{d}{dt} \|\partial_z v\|^4_4 + \int_{\Omega} (|\partial_z v|^2 |\nabla_h \partial_z v|^2 + 2|\partial_z v|^2 |\nabla_h |\partial_z v||^2) \, dx dy dz
$$

$$
= \int_{\Omega} [(\nabla_h \cdot v)\partial_z v - (\partial_z v \cdot \nabla_h) v] \cdot |\partial_z v|^2 \partial_z v \, dx dy dz
$$

$$
+ \int_{\Omega} [T|\partial_z v|^2(\nabla_h \cdot \partial_z v) + T(\partial_z v) \cdot (\nabla_h |\partial_z v|^2)] \, dx dy dz,
$$

where we have used the following facts that

$$
\int_{\Omega} \left[ (v \cdot \nabla_h) \partial_z v - \left( \int_0^t \nabla_h \cdot v(x, y, \xi, t) \, d\xi \right) \partial_{zz} v \right] \cdot |\partial_z v|^2 \partial_z v \, dx dy dz = 0,
$$

$$
\int_{\Omega} \left[ f_0 \tilde{k} \times (\partial_z v) \right] \cdot |\partial_z v|^2 \partial_z v \, dx dy dz = 0.
$$

For the first integral term on the right-hand side of (4.5), using the Lemma 3.3 and Young inequality yields

$$
\int_{\Omega} [(\nabla_h \cdot v)\partial_z v - (\partial_z v \cdot \nabla_h) v] \cdot |\partial_z v|^2 \partial_z v \, dx dy dz
$$

$$
\leq C \int_{\Omega} |\nabla_h v| |\partial_z v|^2 |\partial_z v|^2 \, dx dy dz
$$

$$
\leq C \int_{\Omega} \left( \int_{-1}^1 (|\nabla_h v| + |\nabla_h \partial_z v|) \, dz \right) \left( \int_{-1}^1 |\partial_z v|^2 |\partial_z v|^2 \, dz \right) \, dx dy
$$

$$
\leq C \left( \int_{\Omega} |\partial_z v|^4 \, dx dy dz \right) (\|\nabla_h v\|_2 + \|\nabla_h \partial_z v\|_2)
$$

$$
+ C(\|\nabla_h v\|_2 + \|\nabla_h \partial_z v\|_2) \|\partial_z v\|^2_4 \left( \int_{\Omega} |\partial_z v|^2 |\nabla_h \partial_z v|^2 \, dx dy dz \right)^{1/2}
$$

$$
\leq C \left( 1 + \|\nabla_h v\|^2_{H^1} \right) \|\partial_z v\|^4_4 + \frac{3}{8} \int_{\Omega} |\partial_z v|^2 |\nabla_h \partial_z v|^2 \, dx dy dz,
$$

note that the boundary condition (1.5) and symmetry condition (1.7) are used. Applying the Hölder inequality, Young inequality and Sobolev imbedding theorem, the last integral term on the right-hand side of (4.5) can be bounded as

$$
\int_{\Omega} [T|\partial_z v|^2(\nabla_h \cdot \partial_z v) + T(\partial_z v) \cdot (\nabla_h |\partial_z v|^2)] \, dx dy dz
$$

$$
\leq C \int_{\Omega} |T||\partial_z v|^2 |\nabla_h \partial_z v| \, dx dy dz
$$

$$
\leq C \int_{\Omega} |T|^2 |\partial_z v|^2 \, dx dy dz + \frac{3}{8} \int_{\Omega} |\partial_z v|^2 |\nabla_h \partial_z v|^2 \, dx dy dz
$$

$$
\leq C \|T\|^2_{H^1} \left( 1 + \|\partial_z v\|^4_4 \right) + \frac{3}{8} \int_{\Omega} |\partial_z v|^2 |\nabla_h \partial_z v|^2 \, dx dy dz.
$$

Adding (4.6) and (4.7), then it deduces from the Gronwall inequality and Proposition 3.1 that

$$
\sup_{0 \leq s \leq t_0} \|\partial_z v\|_4 (s) \leq \exp \left\{ C \int_0^{t_0} \left( 1 + \|\nabla_h v\|^2_{H^1} + \|T\|^2_{H^1} \right) \, ds \right\}
$$
\[ \times \left[ \|\partial_z v_0\|_4^4 + C \int_0^{t_0^+} \|T\|_{H^1}^2 \, ds \right]^{1/4} \leq C e^{C(1+t_0^+)} \left( t_0^+ + \|\partial_z v_0\|_4^4 \right)^{1/4}. \]

In particular, we have \((\partial_z v)(x, y, z, t_0) \in L^4(\Omega)\).

Recall that \((v(x, y, z, t_0), T(x, y, z, t_0)) \in L^\infty(\Omega),\) it follows from Proposition 4.4 that
\[
\sup_{t_0 \leq s \leq t} \left( \|\| v, T\|_{H^1(\Omega)}^2 \right) (s) + \int_{t_0}^t \|\nabla_h v\|_{H^1(\Omega)}^2 \, ds
\]
\[ + \int_{t_0}^t \left( \|\nabla_h T\|_{H^1(\Omega)}^2 + \|\partial_v, \partial_T\|_{L^2(\Omega)}^2 \right) \, ds \leq \mathcal{J}_1(t), \tag{4.8} \]
for any \(t \in [t_0, \infty)\). Combining (4.3) and (4.8) yields the energy estimate in Theorem 2.2. Hence the proof is completed. \(\Box\)

The proof Theorem 2.4 is shown below.

**Proof of Theorem 2.4.** Thanks to the proof of Proposition 3.4, substituting (3.5) and the energy estimate in Theorem 2.2 into (3.10) gives
\[
\sup_{0 \leq t \leq T} \left( \| (V_{\lambda}, \lambda W_{\lambda}, \Phi_{\lambda}) \|_{L^2}^2 \right) (t) + \int_0^T \left( \| \nabla_h v_0 \|_2^2 + \lambda \beta \| \partial_z v_{\lambda} \|_2^2 \right) \, dt
\]
\[ + \int_0^T \left( \lambda^2 \| \nabla_h W_{\lambda} \|_2^2 + \| \nabla_h \Phi_{\lambda} \|_2^2 + \lambda \beta \| \partial_z W_{\lambda} \|_2^2 + \lambda \gamma \| \partial_z \Phi_{\lambda} \|_2^2 \right) \, dt
\]
\[ \leq C \exp \left\{ C(T + 1) \left( \mathcal{J}_2(T) + \mathcal{J}_2(T) \right) \right\} \times \left\{ \left( \lambda^2 + \lambda \beta \right) \mathcal{J}_2(T) + \lambda \mathcal{J}_2(T) \right\}
\]
\[ + \left( \lambda \gamma + \lambda \beta \right) T \mathcal{J}_2(T) + \lambda^2 (T + 1) \left( \| v_0 \|_2^2 + \lambda \| w_0 \|_2^2 + \| T_0 \|_2^2 \right)^2 \}
\[ \leq C \lambda^\eta (T + 1) \left[ \mathcal{J}_2(T) + \mathcal{J}_2(T) \right] + \left( \| v_0 \|_2^2 + \| w_0 \|_2^2 + \| T_0 \|_2^2 \right)^2 \}
\]
\[ \times \exp \left\{ C(T + 1) \left( \mathcal{J}_2(T) + \mathcal{J}_2(T) \right) \right\} =: \lambda^\eta \mathcal{J}_2(T), \]
for any \(T > 0, \) where \(\eta = \min \{2, \beta - 2, \gamma - 2\} \) with \(2 < \beta, \gamma < \infty,\) and \(\mathcal{J}_2(t)\) is a nonnegative continuously increasing function defined on \([0, \infty)\). Here \(C\) is a positive constant that does not depend on \(\lambda.\) Obviously, the local strong convergences in Theorem 2.4 can be extended to the global strong convergences, and the rate of convergence is of the order \(O(\lambda^{\eta/2}).\) \(\Box\)

5. **Strong convergence for \(H^2\) initial data**

In this section, assume that the initial data \((v_0, T_0) \in H^2(\Omega)\) with
\[
\int_{-1}^1 \nabla_h \cdot v_0(x, y, z) \, dz = 0, \quad \text{for all } (x, y) \in M,
\]
we prove that the scaled Boussinesq equations with rotation (1.4) strongly converge to the primitive equations with only horizontal viscosity and diffusivity (1.10) as the aspect ratio parameter \(\lambda\) goes to zero.

With this assumption of the initial data, there is a unique local strong solution \((v_{\lambda}, w_{\lambda}, T_{\lambda})\) to the system (1.4), subject to (1.5)-(1.7). Denote by \(T_{\lambda}\) the maximal existence time of this local strong solution. Moreover, the system (1.10) corresponding to (1.5)-(1.7) has a unique global strong solution \((v, T)\). The global well-posedness of strong
solutions to the primitive equations with only horizontal viscosity and diffusivity (1.10) is as follows (see [8]).

**Proposition 5.1.** Let \( v_0, T_0 \in H^2(\Omega) \) be two periodic functions with \( \int_{-1}^{1} \nabla_h v_0(x, y, z) dz = 0 \), for all \((x, y) \in M\). Then the following assertions hold true:

(i) For any \( T > 0 \), there exists a unique global strong solution \((v, T)\) to the primitive equations with only horizontal viscosity and diffusivity (1.10) corresponding to (1.5)-(1.7), such that

\[
(v, T) \in L^\infty([0, T]; H^2(\Omega)) \cap C([0, T]; H^1(\Omega)),
\]

\[
(\nabla_h v, \nabla_h T) \in L^2([0, T]; H^2(\Omega)), (\partial_t v, \partial_t T) \in L^2([0, T]; H^1(\Omega));
\]

(ii) The global strong solution \((v, T)\) satisfies the following energy estimate

\[
\sup_{0 \leq s \leq t} \left( \| (v, T) \|_{H^2(\Omega)}^2 \right) (s) + \int_0^t \| \nabla_h v \|_{H^2(\Omega)}^2 \, ds
\]

\[
+ \int_0^t \left( \| \nabla_h T \|_{H^2(\Omega)}^2 + \| (\partial_t v, \partial_t T) \|_{H^1(\Omega)}^2 \right) \, ds \leq \mathcal{J}_4(t),
\]

for any \( t \in [0, \infty) \), where \( \mathcal{J}_4(t) \) is a nonnegative continuously increasing function defined on \([0, \infty)\).

Let

\[
(U_\lambda, \Phi_\lambda, P_\lambda) = (V_\lambda, W_\lambda, \Phi_\lambda, P_\lambda),
\]

\[
(V_\lambda, W_\lambda, \Phi_\lambda, P_\lambda) = (v_\lambda - v, w_\lambda - w, T_\lambda - T, p_\lambda - p).
\]

We subtract the system (1.10) from the system (1.4) and then lead to the following system

\[
\partial_t V_\lambda - \Delta_3 V_\lambda - \lambda^{\beta - 2} \partial_{zz} V_\lambda - \lambda^{\beta - 2} \partial_{zz} v + (u \cdot \nabla) V_\lambda + (U_\lambda \cdot \nabla) v
\]

\[
+ (U_\lambda \cdot \nabla) V_\lambda + \nabla_h P_\lambda + f_0 k \times V_\lambda = 0,
\]

\[
\lambda^2 (\partial_t W_\lambda - \Delta_3 W_\lambda - \lambda^{\beta - 2} \partial_{zz} W_\lambda + u \cdot \nabla w + u \cdot \nabla W_\lambda + U_\lambda \cdot \nabla w)
\]

\[
+ \lambda^2 U_\lambda \cdot \nabla W_\lambda + \partial_t P_\lambda - \Phi_\lambda + \lambda^2 (\partial_t w - \Delta_3 w + \lambda^{\beta - 2} \nabla_h \cdot \partial_z v) = 0,
\]

\[
\partial_t \Phi_\lambda - \Delta_3 \Phi_\lambda - \lambda^{\gamma - 2} \partial_{zz} \Phi_\lambda - \lambda^{\gamma - 2} \partial_{zz} T + u \cdot \nabla \Phi_\lambda
\]

\[
+ U_\lambda \cdot \nabla T + U_\lambda \cdot \nabla \Phi_\lambda + W_\lambda = 0,
\]

\[
\nabla_h \cdot V_\lambda + \partial_z W_\lambda = 0,
\]

defined on \( \Omega \times (0, T_\lambda^*) \).

**Proposition 5.2.** Suppose that \( (v_0, T_0) \in H^2(\Omega) \), with \( \int_{-1}^{1} \nabla_h \cdot v_0 dz = 0 \). Then the system (5.2)-(5.5) has the following basic energy estimate

\[
\sup_{0 \leq s \leq t} \left( \| (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_{L^2}^2 \right) (s) + \int_0^t \left( \| \nabla_h V_\lambda \|_{L^2}^2 + \lambda^{\beta - 2} \| \partial_z V_\lambda \|_{L^2}^2 \right) \, ds
\]

\[
+ \int_0^t \left( \lambda^2 \| \nabla_h W_\lambda \|_{L^2}^2 + \| \nabla_h \Phi_\lambda \|_{L^2}^2 + \lambda^2 \| \partial_z W_\lambda \|_{L^2}^2 + \lambda^{\gamma - 2} \| \partial_z \Phi_\lambda \|_{L^2}^2 \right) \, ds \leq \lambda^2 \mathcal{J}_5(t),
\]

for any \( t \in [0, T_\lambda^*) \), where

\[
\mathcal{J}_5(t) = C(t + 1) C(t + 1) \left( \mathcal{J}_4(t) + \mathcal{J}_4^2(t) \right) \left[ \mathcal{J}_4(t) + \mathcal{J}_4^2(t) + \left( \| v_0 \|_{L^2}^2 + \| w_0 \|_{L^2}^2 + ||T_0||_{L^2}^2 \right)^2 \right].
\]

Here \( \eta = \min\{2, \beta - 2, \gamma - 2\} \) with \( 2 < \beta, \gamma < \infty \) and \( C \) is a positive constant that does not depend on \( \lambda \).
The proof of Proposition 5.2 is similar to that of Theorem 2.4 and so is omitted. Note that the energy estimate (5.11) is used in this case. With the aid of Proposition 5.1 and 5.2, we can perform the first order energy estimate on the system (5.2)-(5.5) under some smallness condition.

**Proposition 5.3.** Suppose that \((v_0, T_0) \in H^2(\Omega), \text{ with } \int_{-1}^{1} \nabla_h \cdot v_0 dz = 0. \) Then there exists a small positive constant \( \ell_0 \) such that the system (5.2)-(5.5) has the following first order energy estimate

\[
\sup_{0 \leq s \leq t} \left( \| \nabla (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_2^2 \right) (s) + \int_0^t \left( \| \nabla \nabla_h V_\lambda \|_2^2 + \lambda^{3/2} \| \nabla \partial_3 V_\lambda \|_2^2 + \lambda^2 \| \nabla \nabla_h W_\lambda \|_2^2 \right) ds \\
+ \int_0^t \left( \| \nabla \nabla_h \Phi_\lambda \|_2^2 + \lambda^{3/2} \| \nabla \partial_3 W_\lambda \|_2^2 + \lambda^{5/2} \| \nabla \partial_3 \Phi_\lambda \|_2^2 \right) ds \leq \eta^2 J_6(t),
\]

for any \( t \in [0, T^*_\lambda) \), provided that

\[
\sup_{0 \leq s \leq t} \left( \| \nabla (V_\lambda, \Phi_\lambda) \|_2^2 + \lambda^2 \| \nabla W_\lambda \|_2^2 \right) (s) \leq \ell_0^2,
\]

where

\[
J_6(t) \leq C(t + 1) \left[ J_4(t) + J_4^2(t) + J_5(t) + J_5^2(t) + J_4(t) J_5(t) \right] \\
\times \exp \left\{ C(t + 1) \left[ J_4(t) + J_4^2(t) + J_5(t) + J_5^2(t) + 1 \right] \right\}.
\]

Here \( \eta = \min\{2, \beta - 2, \gamma - 2\} \) with \( 2 < \beta, \gamma < \infty \) and \( C \) is a positive constant that does not depend on \( \lambda \).

**Proof.** Taking the \( L^2(\Omega) \) inner product of the first three equation in system (5.2)-(5.5) with \(-\Delta V_\lambda, -\Delta W_\lambda \) and \(-\Delta \Phi_\lambda \), respectively, then it follows from integration by parts that

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_2^2 \right) + \left( \| \nabla \nabla_h V_\lambda \|_2^2 + \lambda^{3/2} \| \nabla \partial_3 V_\lambda \|_2^2 + \lambda^2 \| \nabla \nabla_h W_\lambda \|_2^2 \right) \\
+ \left( \| \nabla \nabla_h \Phi_\lambda \|_2^2 + \lambda^{3/2} \| \nabla \partial_3 W_\lambda \|_2^2 + \lambda^{5/2} \| \nabla \partial_3 \Phi_\lambda \|_2^2 \right) \\
= \int_\Omega (u \cdot \nabla \Phi_\lambda + U_\lambda \cdot \nabla T + U_\lambda \cdot \nabla \Phi_\lambda) \Delta \Phi_\lambda dx dy dz \\
+ \lambda^2 \int_\Omega (u \cdot \nabla w + u \cdot \nabla W_\lambda + U_\lambda \cdot \nabla w + U_\lambda \cdot \nabla W_\lambda) \Delta W_\lambda dx dy dz \\
+ \int_\Omega [(u \cdot \nabla) V_\lambda + (U_\lambda \cdot \nabla)v + (U_\lambda \cdot \nabla)V_\lambda] \cdot \Delta V_\lambda dx dy dz \\
+ \int_\Omega \left[ \lambda^2 (\partial_t w - \Delta h w) \Delta W_\lambda - \lambda^{5/2} (\partial_{zz} T) \Delta \Phi_\lambda \right] dx dy dz \\
+ \int_\Omega \left[ \lambda^{3/2} (\nabla_h \cdot \partial_3 v) \Delta W_\lambda - \lambda^{5/2} (\partial_{zz} v) \cdot \Delta V_\lambda \right] dx dy dz
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5,
\]

note that the resultants have been added up.

In order to estimate the first integral term \( I_1 \) on the right-hand side of (5.6), we split it into three parts, \( I_{11}, I_{12} \) and \( I_{13} \). By virtue of the divergence-free condition, Lemma 6.3 and Young inequality, these integral terms can be bounded as

\[
I_{11} := \int_\Omega (u \cdot \nabla \Phi_\lambda) \Delta \Phi_\lambda dx dy dz
\]
\[
I_{12} = \int_{\Omega} (v \cdot \nabla h) \Delta h \Phi_{2} + (\nabla h) \cdot \nabla \Phi_{2} + w(\nabla h) \Phi_{2} + \nabla (\nabla h) \Phi_{2} \, dx \, dy \, dz
\]
\[ \int_{\Omega} (U_\lambda \cdot \nabla \Phi_\lambda) \Delta \Phi_\lambda dx dy dz \]

\[ = \int_{\Omega} [(V_\lambda \cdot \nabla \Phi_\lambda) \Delta \Phi_\lambda + W_\lambda (\partial_z \Phi_\lambda) \Delta \Phi_\lambda] dx dy dz \]

\[ + \int_{\Omega} [(V_\lambda \cdot \nabla \Phi_\lambda) \partial_{zz} \Phi_\lambda + W_\lambda (\partial_z \Phi_\lambda) \partial_{zz} \Phi_\lambda] dx dy dz \]

\[ = \int_{\Omega} [(V_\lambda \cdot \nabla \Phi_\lambda) \Delta \Phi_\lambda + W_\lambda (\partial_z \Phi_\lambda) \Delta \Phi_\lambda] dx dy dz \]

\[ + \int_{\Omega} [(\nabla_\lambda \cdot V_\lambda)(\partial_z \Phi_\lambda \partial_z \Phi_\lambda - (\partial_z V_\lambda \cdot \nabla \Phi_\lambda) \partial_z \Phi_\lambda] dx dy dz \]

\[ \leq \int_{M} \left( \int_{-1}^{1} (|V_\lambda| + |\partial_z V_\lambda|) \right) \left( \int_{-1}^{1} |\partial_z | \partial_z \Phi_\lambda| \right) dx dy \]
\[ + \int_M \left( \int_{-1}^1 |\nabla_h V_\lambda(x,y)| \right) \left( \int_{-1}^1 |\partial_2 \Phi_\lambda(y)| \Delta_h \Phi_\lambda(x,y) \right) dxdy \\
+ \int_M \left( \int_{-1}^1 (|\nabla_h V_\lambda(x,y)| + |\nabla_h \partial_2 V_\lambda(x,y)|) \right) \left( \int_{-1}^1 |\partial_2 \Phi_\lambda(y)|^2 \right) dxdy \\
+ \int_M \left( \int_{-1}^1 (|\nabla_h \Phi_\lambda(x,y)| + |\nabla_h \partial_2 \Phi_\lambda(x,y)|) \right) \left( \int_{-1}^1 |\partial_2 V_\lambda(x,y)| \partial_2 \Phi_\lambda(y) \right) dxdy \]
\[
\leq C \left[ \|
abla_\lambda \|^2_2 \left( \|\nabla_\lambda \|^2_2 + \|\nabla_h \nabla \|^2_2 + \|\nabla \|^2_2 \right) + \|
abla_z \|^2_2 \left( \|\nabla \|^2_2 \right) \right] \\
\times \left( \|\nabla_\lambda \|^2_2 + \|\partial_2 \Phi_\lambda \|^2_2 \right) + \frac{2}{105} \left( \|\nabla \|^2_2 \right),
\]
respectively, where we have used the boundary condition (1.5) and symmetry condition (1.7). Combining the estimates for \(I_1, I_2\) and \(I_3\) gives
\[
I_1 : = \int_\Omega \left( (u \cdot \nabla \Phi_\lambda + U_\lambda \cdot \nabla T + U_\lambda \cdot \nabla \Phi_\lambda) \Delta \Phi_\lambda \right) dxdydz \\
\leq C \left\{ \left( \|v\|^2_2 + \|\nabla_h v\|^2_2 + \|\nabla \|^2_2 \right) + \left( \|\nabla \|^2_2 \right) \right\} \\
\times \left( \|\nabla_\lambda \|^2_2 + \|\partial_2 \Phi_\lambda \|^2_2 \right) + \frac{2}{35} \left( \|\nabla \|^2_2 \right). \tag{5.7}
\]
Using the similar method as the first integral term \(I_1\) on the right-hand side of (5.6), the integral terms \(I_2\) and \(I_3\) on the right-hand side of (5.6) can be estimated as
\[
I_2 : = \lambda^2 \int_\Omega \left( (u \cdot \nabla w + u \cdot \nabla W_\lambda + U_\lambda \cdot \nabla w + U_\lambda \cdot \nabla W_\lambda) \Delta W_\lambda \right) dxdydz \\
= \lambda^2 \int_\Omega \left[ (\nabla_h \cdot v) \int_0^z (\nabla_h \cdot v) d\xi - v \cdot \int_0^z \nabla_h (\nabla_h \cdot v) d\xi \right] \Delta_h W_\lambda dxdydz
\]
$$+ \lambda^2 \int_\Omega \left[ (\nabla_h \cdot v)(\nabla_h \cdot v) - \partial_z v \cdot \int_0^z \nabla_h(\nabla_h \cdot v) d\xi \right] (\nabla_h \cdot V_\lambda) dx dy dz$$

$$+ \lambda^2 \int_\Omega \left[ (\nabla_h \cdot \partial_z v) \int_0^z (\nabla_h \cdot v) d\xi - v \cdot \nabla_h(\nabla_h \cdot v) \right] (\nabla_h \cdot V_\lambda) dx dy dz$$

$$+ \lambda^2 \int_\Omega \left[ v \cdot \nabla_h W_\lambda + (\nabla_h \cdot V_\lambda) \int_0^z (\nabla_h \cdot v) d\xi \right] \Delta_h W_\lambda dx dy dz$$

$$+ \lambda^2 \int_\Omega \left[ \partial_z v \cdot \int_0^z \nabla_h(\nabla_h \cdot V_\lambda) d\xi - 2v \cdot \nabla_h \partial_z W_\lambda \right] \partial_z W_\lambda dx dy dz$$

$$+ \lambda^2 \int_\Omega \left[ (\nabla_h \cdot v) \int_0^z (\nabla_h \cdot V_\lambda) d\xi - V_\lambda \cdot \int_0^z \nabla_h(\nabla_h \cdot v) d\xi \right] \Delta_h W_\lambda dx dy dz$$

$$+ \lambda^2 \int_\Omega \left[ \partial_z V_\lambda \cdot \int_0^z \nabla_h(\nabla_h \cdot v) d\xi + (\nabla_h \cdot v) \partial_z W_\lambda \right] \partial_z W_\lambda dx dy dz$$

$$+ \lambda^2 \int_\Omega \left[ \partial_z V_\lambda \cdot \int_0^z \nabla_h(\nabla_h \cdot V_\lambda) d\xi - 2V_\lambda \cdot \nabla_h \partial_z W_\lambda \right] \partial_z W_\lambda dx dy dz$$

$$\leq C \left\{ [1 + \lambda^2 + ||v||^2_2 + ||\nabla_h v||^2_2 + (1 + \lambda^2) ||\nabla_h v||^2_2 + (1 + \lambda^2) ||\nabla^2 \nabla_h v||^2_2] 
+ \left[ (1 + \lambda^4) ||v||^2_2 + \lambda^2 ||\nabla^2 \nabla_h v||^2_2 + \lambda^4 ||\nabla^2 v||^2_2 \right] ||\nabla^2 \nabla_h v||^2_2] 
+ \left[ ||\nabla V_\lambda||^2_2 + ||\nabla_h V_\lambda||^2_2 + \lambda^2 ||\nabla_h W_\lambda||^2_2 + ||V_\lambda||^2_2 + ||\nabla V_\lambda||^2_2 \right] 
+ \left( ||\nabla \nabla_h V_\lambda||^2_2 + \lambda^2 ||\nabla \nabla_h W_\lambda||^2_2 \right) \right\} \times \left( ||\nabla V_\lambda||^2_2 + \lambda^2 ||\nabla W_\lambda||^2_2 \right)$$

$$+ C \lambda^2 \left( ||v||^2_2 + ||\nabla_h v||^2_2 + ||\nabla^2 \nabla_h v||^2_2 \right) \right\} \times \left( ||\nabla \nabla_h V_\lambda||^2_2 + \lambda^2 ||\nabla \nabla_h W_\lambda||^2_2 \right)$$

and

$$\mathcal{I}_b := \int_\Omega \left[ (u \cdot \nabla) V_\lambda + (U_\lambda \cdot \nabla) v + (U_\lambda \cdot \nabla) V_\lambda \right] \cdot \Delta V_\lambda dx dy dz$$

$$= \int_\Omega \left[ (v \cdot \nabla) V_\lambda - (\partial_z V_\lambda) \int_0^z (\nabla_h \cdot v) d\xi \right] \cdot \Delta_h V_\lambda dx dy dz$$

$$+ \int_\Omega \left[ (-\partial_z v \cdot \nabla_h) V_\lambda - 2(v \cdot \nabla_h) \partial_z V_\lambda \right] \cdot \partial_z V_\lambda dx dy dz$$

$$+ \int_\Omega \left[ (V_\lambda \cdot \nabla_h) v - (\partial_z v) \int_0^z (\nabla_h \cdot V_\lambda) d\xi \right] \cdot \Delta_h V_\lambda dx dy dz$$

$$+ \int_\Omega \left[ (\nabla_h \cdot V_\lambda) \partial_z v - (\partial_z V_\lambda \cdot \nabla_h) v \right] \cdot \partial_z V_\lambda dx dy dz$$

$$\int_0^z \left( \partial_z \Delta_h V_\lambda d\xi \right) \cdot \Delta_h V_\lambda dx dy dz$$
respectively. For the integral term $I_4$ on the right-hand side of (5.6), we apply the Hölder inequality and Young inequality to obtain

\[
I_4 := \int_\Omega \left[ \lambda^2 (\partial_t w - \Delta h w) \Delta W_\lambda - \lambda^{\gamma-2} (\partial_{zz} T) \Delta \Phi_\lambda \right] \, dx dy dz
= \int_\Omega \left[ \lambda^2 (\partial_t w - \Delta h w) \Delta h W_\lambda - \lambda^{\gamma-2} (\partial_{zz} T) \Delta h \Phi_\lambda \right] \, dx dy dz
+ \int_\Omega \left[ \lambda^{\gamma-2} (\partial_{zz} T) \partial_{zz} \Phi_\lambda + \lambda^2 (\partial_t w - \Delta h w) \partial_{zz} W_\lambda \right] \, dx dy dz
\leq \lambda^2 \int_\Omega \left( \int_{-1}^1 (|\nabla_h^3 v| + |\nabla_h \partial_t v|) \, dx dy dz \right) |\Delta h W_\lambda| \, dx dy dz
+ \lambda^{\gamma-2} |\partial_{zz} T|_2 \|\Delta h \Phi_\lambda|_2 + \lambda^{\gamma-2} \|\partial_{zz} T\|_2 \|\partial_{zz} \Phi_\lambda\|_2
+ \lambda^2 \int_\Omega \left( |\nabla_h^3 v| + |\nabla_h \partial_t v| \right) |\partial_{zz} W_\lambda| \, dx dy dz
\leq C\lambda^2 \int_\Omega \left( \int_{-1}^1 (|\nabla_h^3 v| + |\nabla_h \partial_t v|)^2 \, dx dy dz \right) + C\lambda^2 \|\partial_{zz} W_\lambda\|_2^2
+ C \left( \lambda^{\gamma-2} + \lambda^{2\gamma-4} \right) \|\partial_{zz} T\|_2^2 + C\lambda^2 \left( \|\nabla_h^3 v\|_2^2 + \|\nabla_h \partial_t v\|_2^2 \right)
+ \frac{2}{35} \left( \lambda^2 \|\Delta h W_\lambda\|_2^2 + \|\Delta h \Phi_\lambda\|_2^2 + \lambda^{\gamma-2} \|\partial_{zz} \Phi_\lambda\|_2^2 \right)
\leq C\lambda^2 \left( \|\nabla^2 \nabla_h v\|_2^2 + \|\nabla \partial_t v\|_2^2 + \|\nabla W_\lambda\|_2^2 \right) + C \left( \lambda^{\gamma-2} + \lambda^{2\gamma-4} \right) \|\nabla^2 T\|_2^2
+ \frac{2}{35} \left( \lambda^2 \|\nabla^2 \nabla_h W_\lambda\|_2^2 + \|\nabla^2 \Phi_\lambda\|_2^2 + \lambda^{\gamma-2} \|\nabla \Phi_\lambda\|_2^2 \right),
\]

(5.9)

where the divergence-free condition is used. A similar argument as the integral term $I_4$ on the right-hand side of (5.6) yields

\[
I_5 := \int_\Omega \left[ \lambda^\beta (\nabla_h \cdot \partial_{zz} v) \Delta W_\lambda - \lambda^{\beta-2} (\partial_{zz} v) \cdot \Delta \Phi_\lambda \right] \, dx dy dz
= \int_\Omega \left[ \lambda^\beta (\nabla_h \cdot \partial_{zz} v) \Delta h W_\lambda - \lambda^{\beta-2} (\partial_{zz} v) \cdot \Delta h \Phi_\lambda \right] \, dx dy dz
+ \int_\Omega \left[ \lambda^\beta (\nabla_h \cdot \partial_{zz} v) \partial_{zz} W_\lambda - \lambda^{\beta-2} (\partial_{zz} v) \cdot \partial_{zz} \Phi_\lambda \right] \, dx dy dz
\]
and choosing

Using the assumption given by the proposition

\[ \lambda^2 \left\| \nabla \partial_z v \right\|_2 + \lambda^{2\beta - 2} \left\| \partial_{zz} v \right\|_2 \leq C \left( \lambda^2 + \lambda^{2\beta - 2} \right) \left\| \nabla \nabla_h v \right\|_2^2 \]

Substituting (5.7) into (5.6) leads to

\[
\frac{1}{2} \frac{d}{dt} \left( \left\| \nabla(V_\lambda, \lambda W_\lambda, \Phi_\lambda) \right\|_2^2 \right) + \frac{5}{7} \left( \left\| \nabla \nabla_h V_\lambda \right\|_2^2 + \lambda^{2\beta - 2} \left\| \nabla \partial_z V_\lambda \right\|_2^2 + \lambda^2 \left\| \nabla \nabla_h W_\lambda \right\|_2^2 \right)
\]

\[
\leq C_0 \left\{ \left[ 1 + \lambda^2 + \left\| v \right\|_2^4 + \left\| \nabla_h v \right\|_2^2 + (1 + \lambda^2) \left\| \nabla v \right\|_2^2 + (1 + \lambda^2) \left\| \nabla \nabla_h v \right\|_2^2 \right] + \left[ \left\| \nabla v \right\|_2^2 + \left\| \nabla \nabla_h v \right\|_2^2 + \lambda^2 \left\| \nabla \nabla_h v \right\|_2^2 + \lambda^4 \left\| \nabla \nabla_h v \right\|_2^2 \right] \right\} \times \left( \left\| \nabla(V_\lambda, \lambda W_\lambda, \Phi_\lambda) \right\|_2^2 \right)
\]

Using the assumption given by the proposition

\[
\sup_{0 \leq s \leq t} \left( \left\| \nabla(V_\lambda, \Phi_\lambda) \right\|_2^2 + \lambda^2 \left\| \nabla W_\lambda \right\|_2^2 \right) \leq \epsilon_0^2,
\]

and choosing \( \epsilon_0 = \sqrt{\frac{3}{14 C_0}} \), it deduces from the above inequality that

\[
\frac{d}{dt} \left( \left\| \nabla(V_\lambda, \lambda W_\lambda, \Phi_\lambda) \right\|_2^2 \right) + \left( \left\| \nabla \nabla_h V_\lambda \right\|_2^2 + \lambda^{2\beta - 2} \left\| \nabla \partial_z V_\lambda \right\|_2^2 + \lambda^2 \left\| \nabla \nabla_h W_\lambda \right\|_2^2 \right)
\]

\[
\leq C_0 \left\{ \left[ \left\| v \right\|_2^4 + \left\| \nabla_h v \right\|_2^2 + (1 + \lambda^2) \left\| \nabla v \right\|_2^2 + (1 + \lambda^2) \left\| \nabla \nabla_h v \right\|_2^2 \right] + \left[ \lambda^2 \left\| \nabla \nabla_h v \right\|_2^2 + \lambda^4 \left\| \nabla \nabla_h v \right\|_2^2 \right] \right\}
\]
+ \left[ \left( \| \nabla_h T \|^2_2 + \| \nabla T \|^2_2 + \| \nabla T \|^4_2 + (1 + \lambda^2) \| \nabla v \|^2_2 \| \nabla_h v \|^2_2 \right) \right] \\
+ \left( \| \nabla^2 T \|^2_2 + \| \nabla \nabla_h T \|^2_2 + \| \nabla T \|^2_2 \| \nabla \nabla_h T \|^2_2 + \| V_\lambda \|^2_2 + \| \nabla_h V_\lambda \|^2_2 \right) \\
+ \left( \lambda^2 \| \nabla_h W_\lambda \|^2_2 + \| V_\lambda \|^2_2 + \| V_{\lambda} \|^2_2 \| \nabla_h V_{\lambda} \|^2_2 \right) \right) \times \left( \| \nabla (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|^2_2 \right) \\
+ C_0 \lambda^2 \left( \| v \|^2_2 + \| \nabla v \|^2_2 + \| \nabla \nabla v \|^2_2 + \| \nabla^2 \nabla_h v \|^2_2 \right) \\
+ C_0 \lambda^2 \left( \| v \|^2_2 + \| \nabla v \|^2_2 \| \nabla_h v \|^2_2 + \| \nabla \nabla v \|^2_2 + \| \nabla^2 \nabla_h v \|^2_2 \right) \\
+ C_0 \left( \| V_{\lambda} \|^4_2 + \| V_\lambda \|^2_2 \| \nabla_h V_\lambda \|^2_2 \right) + C_0 \left( \lambda^\gamma + \lambda^\gamma - 4 \right) \| \nabla^2 T \|^2_2 \\
+ C_0 \left( \lambda^{\beta - 2} + \lambda^{2\beta - 4} \right) \| \nabla^2 v \|^2_2 + C_0 \left( \lambda^{\beta} + \lambda^{2\beta - 2} \right) \| \nabla \nabla_h v \|^2_2 \\
+ C_0 \left( \| V_\lambda \|^2_2 + \| V_{\lambda} \|^2_2 \| \nabla \nabla_h v \|^2_2 + \| V_{\lambda} \|^2_2 \| \nabla \nabla_h T \|^2_2 \right) .

Noting that the fact that \((V_\lambda, W_\lambda, \Phi_\lambda)_{t=0} = 0\), and applying the Gronwall inequality to the above inequality, it follows from Proposition 5.1 and 5.2 that

\[
\left( \| \nabla (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|^2_2 \right) (t) + \int_0^t \left( \| \nabla \nabla_h V_\lambda \|^2_2 + \lambda^{\beta - 2} \| \nabla \partial_z V_\lambda \|^2_2 + \lambda^2 \| \nabla \nabla_h W_\lambda \|^2_2 \right) ds \\
+ \int_0^t \left( \| \nabla \nabla_h \Phi_\lambda \|^2_2 + \lambda^\beta \| \nabla \partial_z W_\lambda \|^2_2 + \lambda^{\gamma - 2} \| \nabla \partial_z \Phi_\lambda \|^2_2 \right) ds \\
\leq \exp \left\{ C' \int_0^t \left[ \| v \|^2_2 + \| \nabla v \|^2_2 + (1 + \lambda^2) \| \nabla v \|^2_2 + (1 + \lambda^2) \| \nabla \nabla v \|^2_2 \right] ds \\
+ C' \int_0^t \left[ \lambda^2 \| \nabla^2 \nabla v \|^2_2 + \lambda^4 \| \nabla^2 v \|^2_2 \| \nabla^2 \nabla v \|^2_2 + (1 + \lambda^4) \| \nabla \nabla v \|^2_2 \right] ds \\
+ C' \int_0^t \left[ (1 + \lambda^2) \| v \|^2_2 + \| \nabla^2 v \|^2_2 + \| \nabla v \|^2_2 \| \nabla_h v \|^2_2 + \| \nabla v \|^2_2 \| \nabla_h v \|^2_2 \right] ds \\
+ C' \int_0^t \left[ \| \nabla T \|^2_2 + \| \nabla \nabla_h T \|^2_2 + (1 + \lambda^4) \| \nabla v \|^2_2 \| \nabla \nabla_h v \|^2_2 \right] ds \\
+ C' \int_0^t \left( \| \nabla^2 T \|^2_2 + \| \nabla \nabla T \|^2_2 + \| \nabla T \|^2_2 \| \nabla \nabla_h T \|^2_2 + \| V_{\lambda} \|^2_2 \right) ds \\
+ C' \int_0^t \left( \| \nabla_v V_{\lambda} \|^2_2 + \lambda^2 \| \nabla_h W_\lambda \|^2_2 + \| V_\lambda \|^2_2 \| \nabla_h V_\lambda \|^2_2 \right] ds \right\} \\
\times \left\{ C' \lambda^2 \int_0^t \left[ \| v \|^2_2 + \| \nabla v \|^2_2 + \| \nabla \nabla v \|^2_2 + \| \nabla \nabla v \|^2_2 \| \nabla^2 \nabla v \|^2_2 \right] ds \\
+ C' \lambda^2 \int_0^t \left[ \| v \|^2_2 + \| \nabla v \|^2_2 \| \nabla_h v \|^2_2 + \| \nabla \nabla v \|^2_2 \| \nabla^2 \nabla_h v \|^2_2 + \| \nabla \partial v \|^2_2 \right] ds \\
+ C' \int_0^t \left( \| V_{\lambda} \|^4_2 + \| V_\lambda \|^2_2 \| \nabla_h V_\lambda \|^2_2 \right) ds + C' \left( \lambda^{\gamma - 2} + \lambda^{2\gamma - 4} \right) \int_0^t \| \nabla^2 T \|^2_2 ds \\
+ C' \left( \lambda^{\beta - 2} + \lambda^{2\beta - 4} \right) \int_0^t \| \nabla^2 v \|^2_2 ds + C' \left( \lambda^{\beta} + \lambda^{2\beta - 2} \right) \int_0^t \| \nabla \nabla_h v \|^2_2 ds \\
+ C' \int_0^t \left( \| V_{\lambda} \|^2_2 + \| V_\lambda \|^2_2 \| \nabla \nabla_h v \|^2_2 + \| V_\lambda \|^2_2 \| \nabla \nabla_h T \|^2_2 \right) ds \right\}.
\]
\[ \leq C' \lambda^\eta (t+1) \left[ J_1(t) + J_2^2(t) + J_3(t) + J_4(t) J_5(t) \right] \]
\[
\times \exp \left\{ C'(t+1) \left[ J_1(t) + J_2^2(t) + J_3(t) + J_4(t) J_5(t) + 1 \right] \right\},
\]
where \( \eta = \min\{2, \beta - 2, \gamma - 2\} \) with \( 2 < \beta, \gamma < \infty \). This completes the proof of Proposition 5.3. \( \square \)

By finding a small positive constant to eliminate the effect of the smallness condition in Proposition 5.3, we establish the \( H^1 \) estimate on difference function \( (V_\lambda, W_\lambda, \Phi_\lambda) \).

**Proposition 5.4.** Let \( T^*_\lambda \) be the maximal existence time of the strong solution \( (v_\lambda, w_\lambda, T_\lambda) \) to the system (1.4) corresponding to (1.5)-(1.7). Then, for any finite time \( T > 0 \), there exists a small positive constant \( \lambda(T) = \left( \frac{\lambda^2}{\eta \lambda_0^2} \right)^1 \) such that \( T^*_\lambda > T \) provided that \( \lambda \in (0, \lambda(T)) \). Furthermore, the system (5.3)-(5.7) has the following energy estimate

\[
\sup_{0 \leq t \leq T} \left( \| (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_2^2 \right) (t) + \int_0^T \left( \| \nabla V_\lambda \|_{H^1}^2 + \lambda^\beta - 2 \| \partial_z V_\lambda \|_{H^1}^2 \right) dt \\
+ \int_0^T \left( \lambda^2 \| \nabla W_\lambda \|_{H^1}^2 + \| \nabla \Phi_\lambda \|_{H^1}^2 + \lambda^\beta \| \partial_z W_\lambda \|_{H^1}^2 \right) dt \\
\leq \lambda^\eta \left( J_5(T) + J_6(T) \right),
\]

where \( \eta = \min\{2, \beta - 2, \gamma - 2\} \) with \( 2 < \beta, \gamma < \infty \). Here \( J_5(t) \) and \( J_6(t) \) are nonnegative continuously increasing functions that do not depend on \( \lambda \).

**Proof.** For any finite time \( T > 0 \), setting \( T^\delta_\lambda = \min \{ T^*_\lambda, T \} \), then from Proposition 5.2 it follows that

\[
\sup_{0 \leq t < T^\delta_\lambda} \left( \| (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_2^2 \right) (t) + \int_0^{T^\delta_\lambda} \left( \| \nabla V_\lambda \|_2^2 + \lambda^\beta - 2 \| \partial_z V_\lambda \|_2^2 \right) dt \\
+ \int_0^{T^\delta_\lambda} \left( \lambda^2 \| \nabla W_\lambda \|_2^2 + \| \nabla \Phi_\lambda \|_2^2 + \lambda^\beta \| \partial_z W_\lambda \|_2^2 \right) dt \leq \lambda^\eta J_5(T),
\]

where

\[
J_5(T) = C(T + 1) e^{C(T+1)(J_4(T)+J_4^2(T))} \left[ J_4(T) + J_4^2(T) + \left( \| v_0 \|_2^2 + \| w_0 \|_2^2 + \| T_0 \|_2^2 \right)^2 \right].
\]

Here \( \eta = \min\{2, \beta - 2, \gamma - 2\} \) with \( 2 < \beta, \gamma < \infty \), and \( C \) is a positive constant that does not depend on \( \lambda \).

Let \( \ell_0 \) be the constant from Proposition 5.3. Define

\[
t^\delta_\lambda := \sup \left\{ t \in (0, T^\delta_\lambda) \left| \sup_{0 \leq s \leq t} \left( \| \nabla (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_2^2 \right) (s) \leq \ell_0^2 \right. \right\}.
\]

By virtue of Proposition 5.3, the following estimate holds

\[
\sup_{0 \leq s \leq t} \left( \| \nabla (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_2^2 \right) (s) + \int_0^t \left( \| \nabla V_\lambda \|_2^2 + \lambda^\beta - 2 \| \nabla V_\lambda \|_2^2 + \lambda^2 \| \nabla W_\lambda \|_2^2 \right) ds \\
+ \int_0^t \left( \| \nabla \Phi_\lambda \|_2^2 + \lambda^\beta \| \nabla \partial_z W_\lambda \|_2^2 + \lambda^\gamma - 2 \| \nabla \Phi_\lambda \|_2^2 \right) ds \leq \lambda^\eta J_6(T),
\]

for any \( t \in (0, t^\delta_\lambda) \), where

\[
J_6(T) \leq C(T + 1) \left[ J_4(T) + J_4^2(T) + J_5(T) + J_5^2(T) + J_4(T) J_5(T) \right]
\]
\[ \times \exp \left\{ C(T + 1) \left[ J_4(T) + J_4^2(T) + J_5(T) + J_5^2(T) + 1 \right] \right\}. \]

Choosing \( \lambda(T) = \left( \frac{3\ell_0^2}{8s_0(T)} \right)^{1/\eta} \), it deduces from (5.13) that

\[ \sup_{0 \leq s \leq t} \left( \| \nabla (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_2^2 \right) (s) + \int_0^t \left( \| \nabla \nabla_h V_\lambda \|_2^2 + \lambda^{\beta-2} \| \nabla \partial_z V_\lambda \|_2^2 + \lambda^2 \| \nabla \nabla_h W_\lambda \|_2^2 \right) ds \]

\[ + \int_0^t \left( \| \nabla \nabla_h \Phi_\lambda \|_2^2 + \lambda^\beta \| \nabla \partial_z W_\lambda \|_2^2 + \lambda^{\gamma-2} \| \nabla \partial_z \Phi_\lambda \|_2^2 \right) ds \leq \frac{3\ell_0^2}{8}, \]

for any \( t \in [0, t_\lambda^*] \) and for any \( \lambda \in (0, \lambda(T)) \), which gives

\[ \sup_{0 \leq t < t_\lambda^*} \left( \| \nabla (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_2^2 \right) (t) \leq \frac{3\ell_0^2}{8} < \ell_0^2. \] (5.14)

According to the definition of \( t_\lambda^* \), (5.14) implies that \( t_\lambda^* = T_\lambda^* \). In consequence, the estimate (5.13) holds for \( t \in [0, T_\lambda^*] \) and for any \( \lambda \in (0, \lambda(T)) \).

We claim that \( T_\lambda^* > T \) for any \( \lambda \in (0, \lambda(T)) \). If \( T_\lambda^* \leq T \), then it is obvious that

\[ \lim_{t \to (T_\lambda^*)^-} \sup_{0 \leq t < T_\lambda^*} \left( \| \nabla (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_2^2 \right) = \infty. \]

Otherwise, the strong solution \((v_\lambda, w_\lambda, T_\lambda)\) to the system (1.4) can be extended beyond the maximal existence time \( T_\lambda^* \). However, the above result contradicts to (5.13). This contradiction leads to \( T_\lambda^* > T \), and hence \( T_\lambda^* = T \). Moreover, combining (5.12) with (5.13) yields the energy estimate in Proposition 5.4. This completes the proof. \( \square \)

Based on Proposition 5.4 we give the proof of Theorem 2.5.

**Proof of Theorem 2.5.** For any finite time \( T > 0 \), by virtue of Proposition 5.4, there exists a small positive constant \( \lambda(T) = \left( \frac{3\ell_0^2}{8s_0(T)} \right)^{1/\eta} \) such that \( T_\lambda^* > T \) provided that \( \lambda \in (0, \lambda(T)) \), which implies that the system (1.4) with (1.5)-(1.7) has a unique strong solution \((v_\lambda, w_\lambda, T_\lambda)\) on the time interval \([0, T]\) as long as \( \lambda \in (0, \lambda(T)) \). Let \( J_7(T) = J_5(T) + J_6(T) \), then the following estimate holds

\[ \sup_{0 \leq t \leq T} \left( \| (V_\lambda, \lambda W_\lambda, \Phi_\lambda) \|_{H^1} \right) (t) + \int_0^T \left( \| \nabla_h V_\lambda \|_{H^1}^2 + \lambda^{\beta-2} \| \partial_z V_\lambda \|_{H^1}^2 \right) dt \]

\[ + \int_0^T \left( \lambda^2 \| \nabla_h W_\lambda \|_{H^1}^2 + \| \nabla_h \Phi_\lambda \|_{H^1}^2 + \lambda^\beta \| \partial_z W_\lambda \|_{H^1}^2 + \lambda^{\gamma-2} \| \partial_z \Phi_\lambda \|_{H^1}^2 \right) dt \leq \lambda^\eta J_7(T), \]

where \( \eta = \min\{2, \beta - 2, \gamma - 2\} \) with \( 2 < \beta, \gamma < \infty \), and \( J_7(t) \) is a nonnegative continuously increasing function that does not depend on \( \lambda \). Finally, it is clear that the strong convergences stated in Theorem 2.5 are the direct consequences of the above estimate. The theorem is thus proved. \( \square \)

**Acknowledgment.** The work of X. Pu was supported in part by the National Natural Science Foundation of China (No. 11871172) and the Natural Science Foundation of Guangdong Province of China (No. 2019A1515012000). The work of W. Zhou was supported by the Innovation Research for the Postgraduates of Guangzhou University (No. 2021GDJC-D09).
RIGOROUS JUSTIFICATION OF THE HYDROSTATIC APPROXIMATION

REFERENCES

[1] P. Azérad, F. Guillén, Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics, SIAM J. Math. Anal., 33 (2001) 847-859.
[2] C. Bardos, M.C. Lopes Filho, D. Niu, H.J. Nussenzveig Lopes, E.S. Titi, Stability of two-dimensional viscous incompressible flows under three-dimensional perturbations and inviscid symmetry breaking, SIAM J. Math. Anal., 45 (2013) 1871-1885.
[3] Y. Brenier, Homogeneous hydrostatic flows with convex velocity profiles, Nonlinearity, 12 (1999) 495-512.
[4] D. Bresch, F. Guillén-González, N. Masmoudi, M.A. Rodríguez-Bellido, On the uniqueness of weak solutions of the two-dimensional primitive equations, Differ. Integral Equ., 16 (2003) 77-94.
[5] C. Cao, S. Ibrahim, K. Nakanishi, E.S. Titi, Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics, Commun. Math. Phys., 337 (2015) 473-482.
[6] C. Cao, J. Li, E.S. Titi, Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity, Arch. Ration. Mech. Anal., 214 (2014) 35-76.
[7] C. Cao, J. Li, E.S. Titi, Global well-posedness of strong solutions to the 3D primitive equations with horizontal eddy diffusivity, J. Differ. Equ., 257 (2014) 4108-4132.
[8] C. Cao, J. Li, E.S. Titi, Global well-posedness of the 3D primitive equations with only horizontal viscosity and diffusivity, Commun. Pure Appl. Math., 69 (2016) 1492-1531.
[9] C. Cao, J. Li, E.S. Titi, Strong solutions to the 3D primitive equations with horizontal dissipation: near $H^1$ initial data, J. Funct. Anal., 272 (2017) 4606-4641.
[10] C. Cao, J. Li, E.S. Titi, Global well-posedness of the 3D primitive equations with horizontal viscosity and vertical diffusivity, Phys. D, 412 (2020) 132606, 25 pp.
[11] C. Cao, E.S. Titi, Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics, Ann. Math., 166 (2007) 245-267.
[12] C. Cao, E.S. Titi, Global well-posedness of the 3D primitive equations with partial vertical turbulence mixing heat diffusion, Commun. Math. Phys., 310 (2012) 537-568.
[13] C. Cao, E.S. Titi, Global well-posedness and finite-dimensional global attractor for a 3-D planetary geostrophic viscous model, Commun. Pure Appl. Math., 56 (2003) 198-233.
[14] D. Fang, B. Han, Global well-posedness for the 3D primitive equations in anisotropic framework, J. Math. Anal. Appl., 484 (2020), 123714, 22 pp.
[15] K. Furukawa, Y. Giga, M. Hieber, A. Hussein, T. Kashiwabara, M. Wrona, Rigorous justification of the hydrostatic approximation for the primitive equations by scaled Navier-Stokes equations, Nonlinearity, 33 (2020) 6502-6516.
[16] Y. Giga, M. Gries, M. Hieber, A. Hussein, T. Kashiwabara, The hydrostatic Stokes semi-group and well-posedness of the primitive equations on spaces of bounded functions, J. Funct. Anal., 279 (2020), 108561, 46 pp.
[17] T.-E. Ghoul, S. Ibrahim, Q. Lin, E.S. Titi, On the effect of rotation on the life-span of analytic solutions to the 3D inviscid primitive equations, Arch. Ration. Mech. Anal., 243 (2022) 747-806.
[18] D. Han-Kwan, T. Nguyen, Ill-posedness of the hydrostatic Euler and singular Vlasov equations, Arch. Ration. Mech. Anal., 221 (2016) 1317-1344.
[19] M. Hieber, A. Hussein, T. Kashiwabara, Global strong $L^p$ well-posedness of the 3D primitive equations with heat and salinity diffusion, J. Differ. Equ., 261 (2016) 6950-6981.
[20] M. Hieber, T. Kashiwabara, Global strong well-posedness of the three dimensional primitive equations in $L^p$-spaces, Arch. Ration. Mech. Anal., 221 (2016) 1077-1115.
[21] S. Ibrahim, Q. Lin, E.S. Titi, Finite-time blowup and ill-posedness in Sobolev spaces of the inviscid primitive equations with rotation, J. Differ. Equ., 286 (2021) 557-577.
[22] N. Ju, On $H^2$ solutions and $z$-weak solutions of the 3D primitive equations, Indiana Univ. Math. J., 66 (2017) 973-996.
[23] I. Kukavica, M.C. Lombardo, M. Sammartino, Zero viscosity limit for analytic solutions of the primitive equations, Arch. Ration. Mech. Anal., 222 (2016) 15-45.
[24] I. Kukavica, N. Masmoudi, V.C. Vicol, T.K. Wong, On the local well-posedness of the Prandtl and the hydrostatic Euler equations with multiple monotonicity regions, SIAM J. Math. Anal., 46 (2014) 3865-3890.
[25] I. Kukavica, Y. Pei, W. Rusin, M. Ziane, Primitive equations with continuous initial data, Nonlinearity, 27 (2014) 1135-1155.
[26] I. Kukavica, R. Temam, V.C. Vicol, M. Ziane, Local existence and uniqueness for the hydrostatic Euler equations on a bounded domain, J. Differ. Equ., 250 (2011) 1719-1746.
[27] I. Kukavica, M. Ziane, The regularity of solutions of the primitive equations of the ocean in space dimension three, C. R. Math. Acad. Sci. Paris, 345 (2007) 257-260.
[28] I. Kukavica, M. Ziane, On the regularity of the primitive equations of the ocean, Nonlinearity, 20 (2007) 2739-2753.
[29] J.L. Lions, R. Temam, S. Wang, New formulations of the primitive equations of atmosphere and applications, Nonlinearity, 5 (1992) 237-288.
[30] J.L. Lions, R. Temam, S. Wang, On the equations of the large scale ocean, Nonlinearity, 5 (1992) 1007-1053.
[31] J.L. Lions, R. Temam, S. Wang, Mathematical theory for the coupled atmosphere-ocean models, J. Math. Pures Appl., 74 (1995) 105-163.
[32] J. Li, E.S. Titi, The primitive equations as the small aspect ratio limit of the Navier-Stokes equations: rigorous justification of the hydrostatic approximation, J. Math. Pures Appl., 124 (2019) 30-58.
[33] J. Li, E.S. Titi, Existence and uniqueness of weak solutions to viscous primitive equations for a certain class of discontinuous initial data, SIAM J. Math. Anal., 49 (2017) 1-28.
[34] J. Li, E.S. Titi, G. Yuan. The primitive equations approximation of the anisotropic horizontally viscous Navier-Stokes equations, J. Differ. Equ., 306 (2022) 492-524.
[35] J. Li, G. Yuan. Global well-posedness of $z$-weak solutions to the primitive equations without vertical diffusivity, J. Math. Phys., 63 (2022), 24 pp.
[36] A. Majda, Introduction to PDEs and Waves for the Atmosphere and Ocean, American Mathematical Society, Providence, RI, 2003.
[37] N. Masmoudi, T.K. Wong, On the $H^s$ theory of hydrostatic Euler equations, Arch. Ration. Mech. Anal., 204 (2012) 231-271.
[38] J. Pedlosky, Geophysical Fluid Dynamics, second edition, Springer, New York, 1987.
[39] X. Pu, W. Zhou, Rigorous derivation of the full primitive equations by scaled Boussinesq equations, arXiv: 2105.10621.
[40] X. Pu, W. Zhou, On the rigorous mathematical derivation for the viscous primitive equations with density stratification, arXiv: 2203.10529.
[41] M. Renardy, Ill-posedness of the hydrostatic Euler and Navier-Stokes equations, Arch. Ration. Mech. Anal., 194 (2009) 877-886.
[42] J.C. Robinson, J.L. Rodrigo, W. Sadowski, The Three-Dimensional Navier-Stokes Equations: Classical Theory, Cambridge University Press, Cambridge, 2016.
[43] D. Seidov, An intermediate model for large-scale ocean circulation studies, Dynam. Atmos. Oceans, 25 (1996) 25-55.
[44] T. Tachim Medjo, On the uniqueness of $z$-weak solutions of the three-dimensional primitive equations of the ocean, Nonlinear Anal. Real World Appl., 11 (2010) 1413-1421.
[45] R. Temam, Navier Stokes Equations: Theory and Numerical Analysis, North-Holland Publishing Co., Amsterdam New York Oxford, 1977.
[46] G.K. Vallis, Atmospheric and Oceanic Fluid Dynamics, Cambridge University Press, Cambridge, 2006.
[47] W.M. Washington, C.L. Parkinson, An Introduction to Three Dimensional Climate Modeling, Oxford University Press, Oxford, 1986.
[48] T.K. Wong, Blowup of solutions of the hydrostatic Euler equations, Proc. Amer. Math. Soc., 143 (2015) 1119-1125.

(X. Pu) School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
Email address: puxueke@gmail.com

(W. Zhou) School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
Email address: wywlzhou163.com