Zero-brane approach to quantization of biscalar field theory about topological kink-bell solution

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Abstract

We study the properties of the topologically nontrivial doublet solution arisen in the biscalar theory with a fourth-power potential introducing an example of the spontaneous breaking of symmetry. We rule out the zero-brane (non-minimal point particle) action for this doublet as a particle with curvature. When quantizing it as the theory with higher derivatives, we calculate the quantum corrections to the mass of the doublet which could not be obtained by means of the perturbation theory.

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1 Introduction

The kink solution is known to be the soliton (more correctly, solitary wave) solution appearing in the 2D relativistic $\varphi^4$ one-scalar model and admitting an interpretation in terms of a particle. Its classical and quantum properties are already studied in a number of works [1, 2, 3]. However, the majority of recent theories contains the multiscalar $\varphi^4$ theories (e.g., the scalar sector of the Weinberg-Salam model contains such a complex biscalar theory). Therefore, the increasing of internal space dimensions and studying of classical and quantum features of multiscalar theories seems to be of great necessity and interest. Thus, the aim of this paper is to study the topologically nontrivial solution of the biscalar $\varphi^4$ theory within the frameworks of the method developed (and applied to a one-scalar case) in Ref. [2]. This approach consists in the constructing of brane action where the non-minimal terms (first of all, depending on the world-volume curvature) are induced by the field fluctuations in the neighborhood of the static solution. Of course, these fluctuations are required to be reasonably small, and then the effective zero-brane action evidently arises after nonlinear reparametrization of an initial theory when excluding zero field oscillations.

The paper is arranged as follows. In Sec. 2 we study the kink-bell doublet solution and its properties on the classical level. In Sec. 3 we perform the nonlinear parametrization of the initial
action by means of the Bogolyubov transformation to the collective degrees of freedom. After this, minimizing the action with respect to field fluctuations, we remove zero modes and obtain the effective action having the minimal and non-minimal (curvature) terms. Sec. 4 is devoted to quantization of this action as a constrained theory with higher derivatives. We calculate the first excited level to obtain the mass of the doublet with quantum corrections.

2 Topological doublet

Let us consider the action describing $\varphi^4$ interaction of two spacetime scalar fields

$$S[\varphi] = \int L(\varphi) \, d^2x,$$

where

$$L(\varphi) = \frac{1}{2} \sum_{a=1}^{2} (\partial_m \varphi_a)(\partial^n \varphi_a) - U(\varphi),$$

and the potential part always can be rewritten in the following form:

$$U(\varphi) = \frac{1}{4}(\varphi_1^2 - 1)^2 + \frac{m^2}{2} \varphi_2^2 + \frac{\lambda}{4} \varphi_2^4 + \frac{1-2m^2}{2\eta^2} \varphi_2^2 (\varphi_1^2 - 1).$$

The corresponding equations of motion,

$$\partial^n \partial_m \varphi_a + U_a(\varphi) = 0,$$

where

$$U_a(\varphi) = \frac{\partial U(\varphi)}{\partial \varphi_a}, \quad U_{ab}(\varphi) = \frac{\partial^2 U(\varphi)}{\partial \varphi_a \partial \varphi_b},$$

admit in the class of solitary waves,

$$\varphi_a(\rho) = \varphi_a \left( \frac{x - vt}{\sqrt{1-v^2}} \right),$$

the following topologically nontrivial solution

$$\varphi_a^{(s)}(\rho) = \{ \tanh m\rho, \eta \text{sch} m\rho \},$$

thereby

$$\lambda \eta^4 = 1 - 2m^2(\eta^2 + 1),$$

and $\varphi_a^{(s)}$ are bound by the orbit equation

$$\varphi_2^{(s)2} + \eta^2 \left( \varphi_1^{(s)2} - 1 \right) = 0.$$ 

The first potential from the pair $\varphi_a^{(s)}$ is the well-known $\varphi^4$ kink whereas the second has a bell-like shape and seems to be topologically trivial as such. Nevertheless, the orbit equation provides strong stability of $\varphi_2^{(s)}$ by virtue of the topological non triviality of $\varphi_1^{(s)}$.

The field doublet has the localized energy density

$$\varepsilon(x, t) = m^2 \text{sch}^4 (m\rho) \left[ 1 + \eta^2 \text{sinh}^2 (m\rho) \right],$$

and can be interpreted as the relativistic point particle with the energy

$$E_{\text{class}} = \int_{-\infty}^{+\infty} \varepsilon(x, t) \, dx = \frac{\mu}{\sqrt{1 - v^2}}, \quad \mu = \frac{2}{3} m (\eta^2 + 2).$$
In subsequent sections we will deepen this interpretation both at classical (considering field fluctuations) and quantum levels.

Finally, some comments upon relations between one- and two-scalar $\varphi^4$ theories are in order. The single scalar field theory with the coupling constant of an arbitrary sign is described by the Lagrangian

$$L(\varphi) = \frac{1}{2} \left( \partial_m \varphi \right) \left( \partial^m \varphi \right) - \frac{\lambda}{4} \left( \varphi^2 - \eta^2 \right)^2 + \frac{\lambda \eta^4}{8} (1 - \xi),$$

$$\xi = \text{sign}(\lambda), \quad \lambda, \eta \in \mathbb{R},$$

where the last term is introduced in such a way that the potential energy would be equal zero in the appropriate local minimum point. At $\lambda < 0$ the state $\varphi = 0$ is the most energetically favorable whereas at $\lambda > 0$ the states $\varphi = \pm \eta$ turn to be favorable thus realizing the simplest spontaneous breaking of symmetry. Further, at $\lambda > 0$ this theory admits the kink as the only solution having localized energy. However, at $\lambda < 0$ the bell solution (like $\varphi_2^{(s)}$) appears to be the only solution with localized energy. With respect to energy the theory with $\lambda < 0$ has both the unphysical unbounded from below states and the physical ones (including the bell soliton) localized near a local minimum point which are stable against reasonably small field fluctuations.

Thus, in the one-scalar theory the kink and bell solutions cannot exist together, whereas one can see that in the biscalar theory they successfully coexist as a doublet within the same range of coupling parameters.

### 3 Effective action

In this Section we will construct the nonlinear effective action of the biscalar $\varphi^4$ theory about the kink-bell doublet. Let us change to the set of the collective coordinates $\{\sigma_0 = s, \sigma_1 = \rho\}$ such that

$$x^m = x^m(s) + \epsilon^m_{(1)}(s) \rho, \quad \varphi_a(x, t) = \bar{\varphi}_a(\sigma),$$

where $x^m(s)$ turn to be the coordinates of a (1+1)-dimensional point particle, $\epsilon^m_{(1)}(s)$ is the unit spacelike vector orthogonal to the world line. Hence, the action (1) can be rewritten in new coordinates as

$$S[\bar{\varphi}] = \int L(\bar{\varphi}) \Delta \, d^2 \sigma,$$

where

$$L(\bar{\varphi}) = \frac{1}{2} \sum_a \left[ \frac{(\partial_s \bar{\varphi}_a)^2}{\Delta^2} - (\partial_\rho \bar{\varphi}_a)^2 \right] - U(\bar{\varphi}),$$

$$\Delta = \det \left| \frac{\partial x^m}{\partial \sigma^k} \right| = \sqrt{\dot{x}^2(1 - \rho k)},$$

and $k$ is the curvature of a particle world line

$$k = \frac{\epsilon_{mn} \dot{x}^m \dot{x}^n}{(\sqrt{\dot{x}^2})^3},$$

where $\epsilon_{mn}$ is the unit antisymmetric tensor. This new action contains the two redundant degrees of freedom which eventually lead to appearance of the so-called “zero modes”. To eliminate them we must constrain the model by means of the condition of vanishing of the functional derivative with respect to the doublet field fluctuations about some chosen static solution (kink-bell in our case), and in result we will obtain the required effective action [2].
So, the fluctuations of the fields $\tilde{\varphi}_a(\sigma)$ in the neighborhood of the static solution $\varphi^{(s)}_a(\rho)$ are given by the expression

$$\tilde{\varphi}_a(\sigma) = \varphi^{(s)}_a(\rho) + \delta\varphi_a(\sigma).$$

Substituting them into eq. (11) and considering the static equation of motion (4) for $\varphi^{(s)}_a$ we have

$$S[\delta\tilde{\varphi}] = \int d^2\sigma \left\{ \Delta \left[ L(\varphi^{(s)}_a) + \frac{1}{2} \sum_a \left( \frac{\partial_a \delta\varphi_a}{\Delta} - \frac{\partial_\rho \delta\varphi_a}{\rho} \right)^2 - \sum_b U_{ab}(\varphi^{(s)}_a) \delta\varphi_a \delta\varphi_b \right] \right\} - k\sqrt{\Delta} \sum_a \varphi^{(s)}_a \delta\tilde{\varphi}_a + O(\delta\varphi^3) \right\} + \text{surf. terms},$$

where prime means the derivative with respect to $\rho$. Extremalizing this action with respect to $\delta\varphi_a$ one can obtain the system of equations in partial derivatives for field fluctuations:

$$\left( \partial_a \Delta^{-1} \partial_a - \partial_{\rho} \Delta \partial_{\rho} \right) \delta\varphi_a + \Delta \sum_b U_{ab}(\varphi^{(s)}_a) \delta\varphi_b + \varphi^{(s)}_a k \sqrt{\Delta} = O(\delta\varphi^2), \quad (14)$$

which has to be the constraint removing redundant degrees of freedom. Supposing $\delta\varphi_a(s, \rho) = k(s)f_a(\rho)$, in the linear approximations $\rho k \ll 1$ (which naturally guarantees also the smoothness of a world line at $\rho \to 0$) and $O(\delta\varphi^2) = 0$ we obtain the system of three ordinary derivative equations

$$\frac{1}{\sqrt{x^2}} \frac{d}{ds} \frac{1}{\sqrt{x^2}} \frac{dk}{ds} + ck = 0, \quad (15)$$

$$-f''_a + \sum_b \left( U_{ab}(\varphi^{(s)}_a) - c\delta_{ab} \right) f_b + \varphi^{(s)}_a \frac{k}{\sqrt{x^2}} = 0. \quad (16)$$

First of all, we are needed to find the solutions of (16) such that field fluctuations would be damping at both infinities (then the integral at a non-minimal term below will be finite). In other words, we suppose the boundary conditions

$$f_a(+\infty) = f_a(-\infty) = O(1), \quad (17)$$

which evidently correspond to the singular Stourm-Liouville problem describing bound states of some “quantum” system. Further, varying the orbit equation (3) we obtain the expression

$$\frac{f_2}{f_1} = -\eta^2 \frac{\varphi_1^{(s)}}{\varphi_2^{(s)}} \frac{\varphi_2^{(s)}}{\varphi_1^{(s)}},$$

one can separate $f_a$ in (16) by virtue of. Thus, we have the two independent Stourm-Liouville problems which can be resolved exactly and completely. The following theorem has to be very useful in this connection.

**Theorem.** Let us have the differential equation

$$-f''(u) + 2\xi \left( 3X_0^2 - \frac{\xi + 3}{4} \right) f(u) - cf(u) + X_0' = 0, \quad (18)$$

where $c$ is an arbitrary constant, $\xi^2 = 1$,

$$X_0 = \sqrt{\frac{1 + \xi}{2} - \xi\text{sch}^2 u}.$$
Then the corresponding Stourm-Liouville problem,

\[ f(+\infty) = f(-\infty) = O(1), \]  

has the only solution for each \( \xi \):

\[ f(\xi=+1) = \frac{K \sinh u + 1}{3\cosh^2 u}, \quad f(\xi=-1) = \frac{\sinh u + B}{3\cosh^2 u}, \]

thereby the corresponding eigenvalues are

\[ c = 3\xi, \]

where \( B \) and \( K \) are arbitrary integration constants.\(^\dagger\)

With the use of this theorem we obtain that the solutions of system (16) provided (17) are the two sets of eigenfunctions and eigenvalues which will be distinguished by virtue of a hat index

\[ f_{1n} = m \frac{\cosh^2 \rho}{c_n}, \quad f_{2n} = -\eta m \frac{\tanh m \rho}{c_n \cosh m \rho}, \]  

\[ c_n = 3(-1)^n m^2, \quad n = 0, 1. \]  

Substituting the found functions back in the action (13), we can rewrite it in the explicit p-brane form

\[ S_{\text{eff}} = S_{\text{eff}}^{(\text{class})} + S_{\text{eff}}^{(\text{fluct})} = -\int ds \sqrt{\dot{x}^2} \left( \mu + \alpha k^2 \right), \]  

where

\[ \mu = -\int_{-\infty}^{+\infty} d\rho (1 - \rho k) L(\bar{\varphi}^{(s)}) = \int_{-\infty}^{+\infty} \varepsilon(x, t) d\rho, \]

see eq. (8), and

\[ \alpha = \frac{1}{2} \sum_a \int_{-\infty}^{+\infty} f_{an} \varphi_a^{(s)} \rho d\rho = \frac{\mu}{2c_n} = (-1)^n \eta^2 + 2 \frac{9}{9m}. \]  

The action (21) yields the equations of motion for the kink-bell doublet field solution as a particle with curvature

\[ \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{ds} \left( \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{ds} k \right) + \frac{\mu - \alpha k^2}{2\alpha} k = 0, \]  

hence one can see that eq. (15) was nothing but this equation in the linear approximation \( k \ll 1 \), as was expected.

Thus, the considering of field fluctuations naturally leads to the splitting of the kink-bell doublet into the two subtypes in dependence on the constant before the non-minimal term \( k^2 \). However, when studying the quantum aspects of the model we will find that the absolute value of mass (even with quantum corrections) does not depend on the sign of \( c \), the energy of the non-minimal particle with \( n = 1 \) lies in the lower energy continuum and hence can be interpreted in terms of antiparticles. Therefore, below we will assume \( n = 0 \), i.e., \( \alpha = \mu/6m^2 \).

\(^\dagger\)The proof is not very complicated but too bulky hence it is not presented here (see ref. \( \dagger \) for the proof of similar theorem).
4 Quantization

In the previous section we obtained a classical effective action for the model in question. Thus, to quantize it we must consecutively construct the Hamiltonian structure of dynamics of the point particle with curvature. From eqs. (11) and (21) one can see that we have the theory with higher derivatives \[6, 7\]. Hence, below we will treat the coordinates and momenta as the canonically independent coordinates of phase space. Besides, the Hessian matrix, constructed from the derivatives with respect to accelerations, appears to be singular that says about the presence of the constraints on phase variables of the theory. Following the quantization procedure proposed in ref. 2 and developed in ref. 5 one can obtain the equation for discrete mass spectrum in the form

\[\varepsilon = \sqrt{B(B - 1)(n + 1/2) + O(h^2)}, \quad n = 0, 1, 2, \ldots,\]  

(24)

where

\[B = \frac{8\sqrt{2}}{3} \sqrt{\frac{\eta^2 + 2}{m} M}, \quad \varepsilon = \frac{8\mu^2}{3m^2} \left(1 - \frac{M}{\mu}\right),\]

and \(M\) is the total mass of the kink-bell doublet as a non-minimal point particle with curvature.

Further, again following [4], the first quantum correction to particle masses will be determined by the lower energy of oscillations:

\[\varepsilon = \frac{1}{2}\sqrt{B(B - 1) + O(h^2)},\]

(25)

that gives the algebraic equation for \(M\) as a function of \(m\) and \(\mu\).

We can easily resolve it in the approximation of weak coupling. Assuming \(\lambda \sim m^2/\eta^2 \to 0\) (or, equivalently, \(B \gg 1\)) in such a way that the value \(m^2\) remains to be nonzero we find eq. (25) in the form

\[\varepsilon = \frac{B}{2} + O(\lambda h^2),\]

(26)

which after all yields

\[(M - \mu)^2 = \frac{3m^2}{4}M + O(\lambda h^2).\]

(27)

Then one can seek for mass in the form \(M = \mu + \delta\) (\(\delta \ll \mu\)), and finally we obtain

\[M = \mu \pm \frac{\sqrt{3}}{2}m + O(\lambda h^2).\]

(28)

Thus, the mass of the doublet boson with quantum corrections reads in first approximation

\[M = \frac{4\eta^2 + 8 \pm 3\sqrt{3}}{6}m,\]

(29)

hence one can see that the main term in this expression turns to be singular at \(\lambda \to 0\), therefore, the obtained results are nonperturbative and can not be ruled out through the \(\lambda\)-series of the perturbation theory.

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Regrettably, ref. [7] was added when the paper had already been published in EPL.