Decomposition of scalar potentials of natural Hamiltonians into integrable and perturbative terms: a naive approach

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Abstract. The geometry of Stäckel systems is employed to find, for a given (classical) natural Hamiltonian, another natural Hamiltonian which is integrable by separation of variables and in some sense close to the first one. The study is intended as a first step for a perturbative analysis of a given Hamiltonian system. The method proposed here, still in development, is in large part coordinate independent and effective mainly on manifolds of constant curvature. Examples are given for the quadrupole field and Hénon-Heiles systems. Stäckel systems are associated with quadratic in the momenta first integrals which play a fundamental role in quantization of classical systems.

1. Introduction
In Hamiltonian perturbation theory, Hamiltonian functions are usually prepared to analysis by decomposing them as follows

\[ H = L_0(J) + \epsilon H_1(J, \theta) + \epsilon^2 H_2(J, \theta) + \ldots \]  

(1)

where \( L_0 \) is an integrable Hamiltonian and \((J, \theta)\) its action-angle coordinates, \( H_i \) the perturbative terms. We are searching for the best-fitting \( L_0 \) among natural Hamilton-Jacobi separable (Stäckel) systems.

Separable systems in that perspective are considered in literature (for example [5, 6, 2] about galactical dynamics) and analytic techniques are employed to find best-fitting separable systems for given problems. However, these approaches assume the separable coordinate systems are given and all computations for the search of the best approximation are performed by using the same coordinates. Here, most of the approximation and comparison work necessary to find the best fitting separable systems is done by considering tensorial objects and therefore in whatever coordinate system. While at a ”conjectural stage” the method seems effective, mainly in Riemannian manifolds of constant curvature like Euclidean spaces, \( n \)-dimensional spheres and pseudo-spheres and Minkowski spaces, where there is abundance of separable systems and their geometric characterization is very well understood. In two examples on the Euclidean plane, the quadrupole field and Hénon-Heiles systems, numerical evaluations of the method are provided. Stäckel systems are characterized by \( n \) independent quadratic in the momenta first integrals in involution and play a basic role in quantization of classical systems, as for example in the Born’s model of the hydrogen atom and more recently in [4] and references therein. Therefore, the
present approach naturally concerns quantum systems also. However, such an extension is not yet developed here.

1.1. The objective
Given a Riemannian manifold \( (Q^n, g) \), the cotangent bundle \( T^*Q \) with canonical coordinates \( (q^i, p_i) \) and a natural Hamiltonian function

\[
H = \frac{1}{2} g^{ij} p_i p_j + V(q^i)
\]

we want to find a function \( W_0(q^i) \) such that the natural Hamiltonian

\[
L_0 = \frac{1}{2} g^{ij} p_i p_j + W_0
\]

is Hamilton-Jacobi separable in some coordinate system and the integral curves associated with \( L_0 \) are close, in some sense, to those associated with \( H \) in some subset of \( T^*Q \).

The perturbative term is defined naturally by \( H_p = H - L_0 \). By writing \( H = L_0 + \epsilon H_p \) in action-angle coordinates for \( L_0 \) and expanding in Taylor series for \( \epsilon \) one obtain exactly the expression (1). The integration of \( L_0 \) by the Hamilton-Jacobi method naturally provides canonical coordinates (Jacobi coordinates) that can be transformed into action-angle by standard techniques [10]. Action-angle (or Jacobi) coordinates depend not only on \( L_0 \) but also on the other \( n-1 \) first integrals \( L_i \) that make \( L_0 \) integrable. Therefore, all the functions \( (L_0, L_i) \) must be taken in account in order to provide the best fit in (1). The idea here is to order first the separable systems according to the best fitting given by the first integrals \( L_i \) and then consider the associated separated Hamiltonians \( L_0 \). The Hamiltonian potentials \( W_0 \) can be found, given the separable system associated, by analytic methods [5, 6, 2], here, we will determine these functions just "by inspection" after writing the potential \( V \) in the separable coordinates we obtain by applying our method. While rough, this approach provides for the simple systems considered here good results.

2. Separable systems
We recall that a Hamiltonian \( L_0 \) is said to be (Stäckel-) separable if the (time independent) Hamilton-Jacobi equation

\[
L_0(q^i, p_i = \frac{\partial S}{\partial q^i}) - l = 0
\]

\( l \in R \), admits a solution \( S(q^i, a_i) \) depending on \( n \) real parameters \( (a_i) \) such that

\[
\det \left( \frac{\partial^2 S}{\partial q^i \partial a_j} \right) \neq 0
\]

and

\[
S = S_1(q^1, a_j) + \ldots + S_n(q^n, a_j)
\]

In this case \( S \) can be found by solving a system of \( n \) separated ODE’s. Once \( S \) has been found, the dynamics of \( L_0 \) can be determined by algebraic methods (see for example [1] and references therein).
2.1. Geometric characterization of separable Hamiltonians

The natural Hamiltonian (3) is separable in orthogonal coordinates if and only if [1] there exist \( n \) independent symmetric Killing tensors \( K_i, i = 0 \ldots n - 1, \) \( K_0 = g \), with common eigenvectors, satisfying

\[
[K_i, K_j] = 0, \quad d(K_i \, dW_0) = 0, \tag{4}
\]

where \([ , ]\) denotes the Schouten bracket. At each point the coordinate curves are the integral curves of the eigenvectors of the tensors \( K_i \). By assuming:

\[
L_i = \frac{1}{2} K_i^{ij} p_j p_h + W_i \tag{5}
\]

we see that (4) are equivalent to

\[
\{L_j, L_i\} = 0 \quad \{L_0, L_i\} = 0 \tag{6}
\]

where \( \{ , \} \) denotes the canonical Poisson bracket. Such \( L_i \) are then said to be in involution. After (5), the second of (6) are equivalent to

\[
\nabla W_i - K_i \, dW_0 = 0 \tag{7}
\]

2.2. Separable potentials

In orthogonal separable coordinates \((q^i)\) the potential \( W_0 \) is separable if and only if [1] there exist \( n \) functions \( \phi_i(q^i) \) depending each one on a single coordinate, such that we can write (Stäckel)

\[
W_0 = g^{11} \phi_1(q^1) + \ldots + g^{nn} \phi_n(q^n) \tag{8}
\]

\[
W_i = K_i^{jj} \phi_j(q^j) = \lambda_i^1 g^{11} \phi_1(q^1) + \ldots + \lambda_i^n g^{nn} \phi_n(q^n),
\]

where \( (\lambda_i^j) \) are the eigenvalues of \( K_i \).

If all the symmetric Killing two-tensors of a manifold are known, it follows that all the possible orthogonal Hamilton-Jacobi separable coordinate systems are (in principle) also known (the nonorthogonal ones are derivable from these ones [1]). It is necessary and sufficient to find the spaces of Killing tensors with the properties described above. In manifolds of constant curvature as Euclidean and Minkowski manifolds or \( n \)-Spheres these spaces, together with the associated coordinates, are well known (see for example [8]).

3. A comparison criterion between separable systems

For any Hamiltonian \( H \) and any function \( F \) on \( T^*Q \),

\[
\{H, F\} = \frac{dF}{dt}
\]

where \( t \) is the natural time parametrizing the integral curves of \( H \).

Given \( H \) as in (2) and \( n \) independent functions in involution \( L_i \) on \( T^*Q \) as in (5), (6), we define \( n \) functions \( m_i \) on \( T^*Q \) by

\[
m_i := \{H, L_i\}, \quad i = 0 \ldots n - 1.
\]

The functions \( m_i \) are linear homogeneous polynomials in \( p_i \).

Indeed, since the quantities \( K_i \) are Killing tensors, it follows that the cubic term in the momenta of the function \( \{H, L_i\} \) vanishes. Then

\[
m_i = m_i^j p_j.
\]
The $n^2$ components $m^j_i$ can be interpreted as the components of $n$ vector fields on $Q$

$$\mu_i = m^j_i \partial_j$$

in the coordinates $(q^i)$ canonically conjugate to the coordinates $(p_i)$. The vector fields $\mu_i$ express the ‘projection’ on each base point of the fibers of $T^*Q$ of the rate of change of each function $(L_0, L_i)$ along the integral curves of $H$.

We set

$$|\mu_i| = g^{h_i}m^h_im^i.$$ 

By expanding each $L_i$ in some neighborhood of a point $P \in T^*Q$ in Taylor series along the integral curves of $H$, we get

$$L_i(P_t) = L_i(P) + m_i(P)t + \frac{1}{2}\{\{L_i, H\}, H\}(P)t^2 + \Delta(P, t^3, t^4, \ldots). \quad (9)$$

Then, for $t$ small enough the change of $L_i$ on the integral curves of $H$ is given mainly by the functions $m_i$, homogeneous polynomial in $(p_i)$, whose coefficients are exactly the $m^j_i$ i.e. the components of the vectors $\mu_i$.

4. The comparison criterion

Let us consider for simplicity that the configuration manifold is properly Riemannian.

**Conjecture 1** Under the hypothesis that the series (9) are well defined in some neighborhood of $P \in T^*Q$ for the systems $(L_0, L_i)_1$ and $(L_0, L_i)_2$, for (positive) $t$ small enough, the trajectories of $H$ are better approximated by the system for which the quantity $|\mu_i| = |\mu_0| + \ldots + |\mu_{n-1}|$ in $P$ is smaller. In other words, the more $(L_0, L_i)$ are close to be constant along the trajectories of $H$, the more the trajectories of $L_0$ are close to those of $H$.

By considering the moduli $|\mu_i|$ we make the analysis independent from the coordinates we are using.

4.1. A necessary condition?

It is not easy to compute a priori and for all separable systems the functions $\mu_i$. We recall that, for $i \neq 0$,

$$d\mu_i = dK_idV$$

is independent from $(W_0, W_i)$. Moreover, it is the only obstruction to the separability of $H$.

Therefore, we try to minimize the obstruction by a suitable choice of the $(K_i)$ in the behalf that

**Conjecture 2** Under the same hypothesis of Conjecture 1 and for $i \neq 0$: $d\mu_i$ minimal is a necessary condition for $|\mu_i|$ minimal.

In dimension $2$ we have to evaluate the only non null component of $d\mu_i$. In the examples (Euclidean plane) this will be done according to its degree in Cartesian or Polar coordinates. In higher dimensions some suitable norm should be introduced in the space of the 2-forms (work in progress).

5. The practical strategy in 5 steps

Let $H$ be any natural Hamiltonian with potential $V$ in, for example, the 3-dim Euclidean space.

**Step 1:** Compute $dK_idV$ for the generic Killing Tensor $K$. It depends on 20 parameters.

**Step 2:** Find the subset $MK$ of the Killing tensors such that the obstructions $d\mu_i = d(K_idV)$ are minimal in some suitable neighborhood of $Q$. 

In our 2-dim examples, the degree of the component of $dK dV$ in $(x, y)$ or $r$ will be required maximum if in a neighborhood of the origin, minimum if $(x, y) \gg (1, 1)$ or $r \gg 1$.

**Step 3:** Find in MK all possible spaces of 3 (including the metric tensor) independent KT in involution with common eigenvectors. They are associated with separable coordinate systems. If no such subspace exists, return to step 2 after weakening the minimality requirement.

**Step 4:** If step 3 determines some well defined separable system, order them according to the functions $|\mu_0|$ and find for each system the functions $W_0$ through (8) such that $|\mu_0| = |\nabla(V - W_0)|$ is minimal, determine $\nabla W_1$ and $\nabla W_2$ through the $(K_i)$ and $W_0$ by (7) and compute the vectors $(\mu_0, \mu_1, \mu_2)$.

**Step 5:** Choose for the decomposition the system such that the associated function $|\mu|$ is minimal among the separable systems found in the neighborhood we are interested in. Remember that $(\nabla W_i - K_i dV)$ "minimal" does not assure that $(\nabla W_0 - K_0 dV) = \nabla(V_0 - V)$ also is; Conjecture 2 does not apply to $L_0$.

6. Examples
The examples we give here are both systems in the Euclidean plane. For this manifold we recall:

In dimension two, one only symmetric Killing two-tensor with distinct eigenvectors is sufficient to define a separable system.

In the Euclidean plane the space of the symmetric Killing two-tensors has dimension six.

In the Euclidean plane there are essentially four distinct separable systems: Cartesian, Parabolic, Polar and Elliptic.

7. The quadrupole field
Let $Q = E^2$ be the Euclidean plane, $(x, y)$ cartesian coordinates and $r = \sqrt{x^2 + y^2}$. Let us consider the Hamiltonian function

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V$$

with

$$V = \frac{G}{r} + \frac{D}{r^3}(3\frac{x^2}{r^2} - 1)$$

where $G, D$ are real constants. $H$ is the Hamiltonian function of one charged particle (of mass and electric charge = 1) in a gravitational (Coulomb) field centered into the origin of the coordinates plus an electrostatic quadrupole field corresponding to a three-dimensional distribution of static charges with the coordinate axis $x$ as symmetry axis [9]. We assume that the dipole momentum is zero. This system provides also a good approximation of the Keres-Israel potential, the newtonian approximation of the relativistic Kerr solution (see [3] where Stäckel approximations are considered). The motion will be restricted to a plane determined by the axes, the rotation around $X$ being a symmetry of the system. The potential function has also $Y$ as a symmetry axis.

**Step 1** We consider the space of the symmetric Killing two-tensors of $E^2$, defined by

$$K^{xx} = a - 2ey + cy^2, \quad K^{xy} = d + ex - fy - cxy, \quad K^{yy} = b + 2fx + cx^2,$$

where $a, b, c, d, e, f$ are real numbers.

**Steps 2 and 3** We calculate for such a $K$ the function $d(K dV)_{xy}$ and obtain a rational function whose numerator is a 6-degree polynomial in $x, y$ with coefficients depending on $a...f$ and whose denominator is $(x^2 + y^2)^{9/2}$. We are then led to choose a neighborhood $\Omega$ of $r = \infty$. 


By choosing the tensor $K$ in a suitable way it is possible to reduce the degree of the numerator polynomial to 4. Then the components of $K$ take the forms

$$K_{xx} = a + \frac{1}{2} \frac{(a - b)Gy^2}{D}, \quad K_{xy} = -\frac{1}{2} \frac{(a - b)Gxy}{D}, \quad (12)$$

$$K_{yy} = b + \frac{1}{2} \frac{(a - b)Gx^2}{D}, \quad a \neq b. \quad (13)$$

The corresponding separable coordinate system is of elliptic type, with foci on the $x$ axis if $D/G > 0$ with equation $x^2 = 2D/G$; with foci on the $y$ axis if $D/G < 0$ with equation $y^2 = -2D/G$.

Let us choose $D/G > 0$ and put $A^2 = 2D/G$. Elliptic coordinates $(u, v)$ are then defined by

$$u = r_1 - r_2, \quad v = r_1 + r_2,$$

where $r_1 = \sqrt{(x + A)^2 + y^2}$ and $r_2 = \sqrt{(x - A)^2 + y^2}$. We have

$$g_{uu} = -4 \frac{u^2 - 4A^2}{v^2 - u^2}, \quad g_{vv} = 4 \frac{v^2 - 4A^2}{v^2 - u^2},$$

and

$$r = \frac{1}{2} \frac{1}{\sqrt{u^2 + v^2 - 4A^2}}, \quad x = -\frac{uv}{4A}.$$

Step 4 Now we can try to approximate $V$ by a separable function:

$$W_0 = g_{uu} \phi_u(u) + g_{vv} \phi_v(v),$$

which is the most general separable potential in coordinates $(u, v)$. For $\phi_u = 0, \phi_v = \frac{1}{2(v^2 - 4A^2)} \left( Gv - 4 \frac{D}{v} \right)$ we have

$$W_0 = \frac{2}{v^2 - u^2} \left( Gv - 4 \frac{D}{v} \right)$$

which is in $\Omega$ a good approximation of $V$. Because $K$ is diagonalized in the elliptic coordinates $(u, v)$, we have $K_{uu} = \lambda_u g_{uu}$ and $K_{vv} = \lambda_v g_{vv}$, with $\lambda_u = \frac{1}{4}(v^2 - 4A^2) \lambda_v = \frac{1}{4}(u^2 - 4A^2)$, then, since $W = K^{ii} \phi_i$,

$$W_1 = \frac{1}{2} \frac{u^2 - 4A^2}{v^2 - u^2} \left( Gv - 4 \frac{D}{v} \right), \quad (14)$$

$$L_1 = \frac{2}{v^2 - u^2} \left( -(u^2 - 4A^2)p_1^2 + (v^2 - 4A^2)p_2^2 \right) + W_1. \quad (15)$$

Let us compare now the previous system with the one associated with the usual polar coordinates $(r, \theta)$, where

$$g_{rr} = 1, \quad g_{\theta \theta} = \frac{1}{r^2},$$

and

$$K_{rr} = 0, \quad K_{\theta \theta} = 1.$$

By following the same procedure as above, we find $\phi_r = \frac{G}{r} - \frac{D}{r^3}, \phi_\theta = 0$. Therefore

$$Wp_0 = \frac{G}{r} - \frac{D}{r^3}, \quad Wp_1 = 0.$$
It is evident that the elliptic system exhibits the same axial symmetry of the original system (axes $x$ and $y$), while the polar one exhibits a central symmetry.

**Step 5** The quantities $\mu = |\mu_0| + |\mu_1|$ are computed and plotted in cartesian coordinates for the elliptic and the polar systems of above, with $G = -1, D = -1/2$ (Figure 1). The non null component of $d\mu_1$ in cartesian coordinates is plotted for the same systems (Figure 2).

For the same systems, trajectories are numerically evaluated for $0 < t < 100$ at several values of the energy $h$. The original quadrupole in red, the elliptic system in green and the polar in blue (Figures 3 - 6).

For the same systems, the projection of the integral curves on the conjugate plane $(x, p_x)$ is plotted (Figures 7 - 10).

It appears that the value of the energy is not relevant on the level of the approximation given by the Stackel systems considered, at least for small values of $t$ as considered in our approach.

### 8. Hénon-Heiles systems

Let the Hénon-Heiles system be defined by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V$$
with

\[ V = \frac{1}{2}(Cx^2 + Dy^2) + Axy^2 - \frac{1}{3}Bx^3. \]

**Step 1** The generic Killing two-tensor of the Euclidean plane is again (11), the obstruction to separability is now

\[
dKdV_{12} = d(D - C) + [2A(a - b) + f(C - 4D)]y + [2d(A + B) +
+e(D - 4C)]x + e(5B + 2A)x^2 - 7eAy^2 +
+[4c(C - D) - 2f(B + 6A)]xy + 5cAy^3 - 5c(B + 2A)x^2y.
\]

**Steps 2 and 3** If we choose to approximate the system for \((x, y) >> (1, 1)\), we have to reduce the degree of \(dKdV_{12}\) to be linear in \((x, y)\). Then \(c = e = f = 0\), Killing tensors of this kind are associated with Cartesian coordinates. If we consider instead small neighborhoods of the origin, the Polar system centered in the origin, \(c = 1\) and the other parameters zero, provides the best \(d\mu_1\).

The separable system defined by \(c = e = f = a = b = 0, d = 1\) provides one of the best separable approximations to H-H (particularly good if \(A + B\) is small). The Cartesian coordinates associated with such a choice of Killing tensor are defined by

\[
q^1 = \frac{1}{\sqrt{2}}(x + y), \quad q^2 = \frac{1}{\sqrt{2}}(y - x).
\]

In these coordinates we have

\[
dKdV_{12} = D - C + \sqrt{2}(A + B)(q^1 - q^2)
\]

and

\[
V = \frac{\sqrt{2}}{4}(A + B)[(q^1)^2q^2 - q^1(q^2)^2] + \frac{1}{4}(C + D)(q^1)^2 + (q^2)^2] +
\frac{1}{2}(D - C)q^1q^2 + \sqrt{\frac{2}{3}}(B - A)[(q^2)^3 - (q^1)^3].
\]

**Step 4** We can extract from \(V\) the separable potential

\[
W_0 = \frac{1}{4}(C + D)[(q^1)^2 + (q^2)^2] + \sqrt{\frac{2}{3}}(B - A)[(q^2)^3 - (q^1)^3].
\]

which in \((x, y)\) is

\[
W_0 = \frac{1}{4}[(C + D)(x^2 + y^2) + (3A - B)xy^2 + (A - \frac{B}{3})x^3]
\]

In \((q^1, q^2)\) we have \(K^{11} = 1, K^{22} = -1, K^{12} = 0\) and \(g^{11} = g^{22} = 1, g^{12} = 0\), therefore, from \(W_0 = g^{\mu\nu}\phi_\mu(q^\nu)\) and \(W_1 = K^{\nu\mu}\phi_\nu(q^\mu)\) we obtain

\[
\phi_1 = \frac{1}{4}[(C + D)(q^1)^2 + \sqrt{2}(A - \frac{B}{3})(q^1)^3],
\]

\[
\phi_2 = \frac{1}{4}(C + D)(q^2)^2 - \sqrt{2}(A - \frac{B}{3})(q^2)^3,
\]

and

\[
W_1 = \phi_1 - \phi_2, \quad L_1 = \frac{1}{2}(p_1^2 - p_2^2) + W_1.
\]
Step 5 For the Polar system centered in the origin we have

\[ K^{11} = y^2, \quad K^{12} = -xy, \quad K^{22} = x^2, \]

and

\[ W_{p0} = \frac{1}{2}(x^2 + y^2), \quad W_{p1} = 0. \]

Therefore, the polar system coincides with the harmonic oscillator. We can now make the comparison between the original H-H, the cartesian and the polar systems. We will find that, after considering the whole \( \mu \), the Cartesian system can be a better choice for an approximation also in small neighborhoods of the origin. Indeed, (Figure 11) the quantities \( \mu = |\mu_0| + |\mu_1| \) are calculated and plotted for the systems cartesian and polar for \( C = D = 1, A = 1, B = 1 \) corresponding to the classical chaotic Hénon-Heiles system [7], in (Figure 12) the corresponding non null component of \( d\mu_1 \) are plotted.

From the (Figure 11) we should expect that, in the regions where the corresponding value for the polar system is smaller than that of the cartesian one, the polar system provide a better approximation, while in the remaining regions the contrary happens.
For the same systems, trajectories are numerically evaluated for several initial points with the same energy, for $0 < t < 0.8$. The original H-H system in red, the cartesian system in green and the polar in blue (Figures 13 - 16).
For the same systems, the projections of the integral curves on the conjugate plane \((y, p_y)\) are plotted (Figures 17 - 20).

It is remarkable that, in neighborhoods of the origin of radius \(<1\), even if \(d\mu_1\) is smaller for polar coordinates, the best \(\mu\) is given by the cartesian systems again, as appear in the example from the numerical evaluation of \(\mu\) plotted above. This is due to the fact that, in this case, the relevant part of \(\mu\) is provided by \(\mu_0\) and is not controlled by Conjecture 2, which is concerning \(\mu_i\) with \(i > 0\) only. Remark that trajectories developing closely to the positive \(x\) axis show a better approximation provided by the polar system, this is consistent with the evaluation of \(\mu\) plotted above. Similar comparisons with Parabolic systems have been made elsewhere verifying the two Conjectures.

9. Conclusions and future directions
Given a natural Hamiltonian, we have a strategy to follow for find natural separable systems whose dynamics is locally close to the given one. The method is in large part coordinate-independent and computations are easily done by computer-algebra systems. As future development of the approach, the two conjectures should be transformed, mutatis mutandis, into theorems. Moreover, \(n > 2\) dimensional manifolds and quantum mechanical systems can be considered, together with the characterization of those Hamiltonian systems admitting some "Stäckel approximation" of the kind treated here but over larger intervals of time.

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References
[1] Benenti S 1997 *J. Math. Phys.* 38 6578
[2] De Bruyne V, Leeuwin F and Dejonge H 2000 *MNRAS* 311 297
[3] de Felice F and Preti G 2000 *J. Phys. A: Math. Gen.* 33 2767
[4] Duval C and Valent G 2005 *J. Math. Phys.* 46
[5] de Zeeuw P T 1985 *MNRAS* 216 273
[6] de Zeeuw P T and Lynden-Bell D 1985 *MNRAS* 215 713
[7] Hénon M and Heiles C 1964 *Astr. J.* 69 73
[8] Kalnins E G 1986 *Separation of variables for Riemannian spaces of constant curvature* (Pitman Monographs and Surveys in P. A. Math.) 28
[9] Landau L D and Lifsits E M 1972 *Field theory* (Moscow: Mir) V 41
[10] Arnold V I 1976 *Les méthodes mathématiques de la mécanique classique* (Moscow: Mir)