Theory and applications of Marshall Olkin Marshall Olkin Weibull distribution

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Abstract. In probability theory generalizing distribution is an important area. Several distributions are inappropriate for data modeling, either symmetrical, semi-symmetrical, or heavily skewed. In this paper, a new compound distribution with four parameters called Marshall Olkin Marshall Olkin Weibull (MOMOWe) is introduced. Several important statistical properties of new distribution were studied and examined. The estimation of unknown four parameters was carried out according to the maximum likelihood estimation method. The flexibility of MOMOWe distribution is demonstrated by the adoption of two real datasets (semi-symmetric and right-skewed) with different information fitting criteria. Such flexibility allows using the new distribution in various application areas.

Keywords. Marshall Olkin, Weibull distribution, Compound distribution, Statistical properties, Maximum likelihood estimation.

1. Introduction
For many fields of use, including the physics, engineering, environmental, medical, biology, economics, finance, and insurance sectors, there is a clear need for new distributions that are more flexible for modeling data in these fields, as data can be semi-symmetric or skewed. In this area, an elastic family of distributions called Marshall Olkin-G (MO-G) proposed by Marshall and Olkin [1]. Recently, many distributions have been proposed based on MO-G family, such as the Marshall-Olkin Fréchet distribution introduced by Krishna et al. [2], the beta Marshall-Olkin Weibull distribution introduced by Alizadeh et al. [3], the Marshall-Olkin gamma-Weibull distribution introduced by Saboor and Pogány [4], the Marshall-Olkin extended Weibull-Exponential distribution introduced by Conleth et al. [5], the Marshall-Olkin exponential Gompertz introduced by Khaleel et al. [6], the Weibull Marshall-Olkin Lindley distribution introduced by Afify et al. [7], and the Marshall-Olkin Marshall-Olkin Gompertz introduced by Al-Noor and Khaleel [8].

In this paper, a newly proposed distribution called Marshall-Olkin Marshall-Olkin Weibull is introduced. The paper structure is as follows: In Section 2, the Marshall-Olkin Marshall-Olkin family
of distributions is discussed. In Section 3, Marshall Olkin Marshall Olkin Weibull distribution is proposed as a new compound continuous distribution. In Sections 4 and 5 statistical properties and maximum likelihood estimation method of their parameters are introduced. In Section 6, two real data set are adopted to illustrate the behavior of new distribution. Finally, conclusions are addressed in Section 7.

2. Marshall Olkin Marshall Olkin-G family

The MO-G family is built on a new reliability (or survival) function defined by giving the tilting parameter, say $\alpha$ as an additional non-negative parameter. Based on $H(x)$ and $h(x)$ which represents the cdf and pdf respectively for the specified baseline distribution, the reliability function of the MO-G family is defined by (see [6, 8, 9])

$$R(x)_{MO} = \frac{\alpha H(x)}{1 - \alpha H(x)} ; \bar{\alpha} = 1 - \alpha , \bar{H}(x) = 1 - H(x)$$

Consequently, the cdf and pdf of MO-G family are

$$F(x)_{MO} = \frac{H(x)}{\alpha + \bar{\alpha} H(x)}$$

$$f(x)_{MO} = \frac{\alpha h(x)}{\left(\alpha + \bar{\alpha} H(x)\right)^2}$$

In the same manner, defined $H(x)$ and $h(x)$ that point out above as a new baseline distribution with MO-G family based on non-negative tilt parameter $\theta$, i.e.

$$H(x)_{MO-G} = \frac{G(x)}{\theta + \bar{\theta} G(x)}$$

$$h(x)_{MO-G} = \frac{\theta g(x)}{\left(\theta + \bar{\theta} G(x)\right)^2}$$

where $G(x)$ and $g(x)$ defined as cdf and pdf respectively of any other baseline distribution.

By inserting (4) and (5) in (2) and (3), a new generated family known as Marshall Olkin Marshall Olkin-G (MOMO-G) is obtained by [8]. The cdf and pdf of MOMO-G family for $x > 0$ and $\alpha, \theta \geq 0$ are defined as

$$F(x)_{MOMO-G} = \frac{G(x)}{\alpha \theta + (\alpha \theta + \bar{\alpha})G(x)} ; \bar{\alpha} = 1 - \alpha , \bar{\theta} = 1 - \theta$$

$$f(x)_{MOMO-G} = \frac{\alpha \theta g(x)}{\left(\alpha \theta + (\alpha \bar{\theta} + \bar{\alpha})G(x)\right)^2}$$

When $\alpha = 1$ or $\theta = 1$ the original MO-G family will attend. So we can consider the MO-G family as a special case of the new proposed MOMO-G family.

The expanded formula of the pdf in (7) can be obtained by performing some simple steps with using $(1 - z)^{-b} = \sum_{i=0}^{\infty} \frac{\Gamma(b+i)}{\Gamma(b)} z^i ; |z| < 1, b > 0$ as

$$f^E(x)_{MOMO-G} = Mg(x)\left(G(x)\right)^i$$

where

$$M = \sum_{i=0}^{\infty} (-1)^i (i + 1)(\alpha \theta)^{1-i} (\alpha \bar{\theta} + \bar{\alpha})^i$$
3. Marshall Olkin Marshall Olkin Weibull distribution

Let \( G(x) \) and \( g(x) \) in (6), (7), and (8) be the cdf and pdf of Weibull distribution as [10]

\[
G(x) = 1 - e^{-\lambda x^\beta}; \quad x > 0, \lambda, \beta > 0 \tag{10}
\]

\[
g(x) = \lambda \beta x^{\beta-1} e^{-\lambda x^\beta} \tag{11}
\]

At that point, a new proposed distribution called Marshall Olkin Marshall Olkin Weibull (MOMOWe for short) distribution is attained as a special case of the MOMO-G family with the cdf, pdf, and expanded pdf given respectively by

\[
F(x)_{\text{MOMOWe}} = \frac{1 - e^{-\lambda x^\beta}}{a\theta + (a\tilde{\theta} + \bar{\alpha}) (1 - e^{-\lambda x^\beta})}; \quad x > 0, \alpha, \theta \geq 0, \lambda, \beta > 0 \tag{12}
\]

\[
f(x)_{\text{MOMOWe}} = \frac{a\theta \lambda \beta x^{\beta-1} e^{-\lambda x^\beta}}{(a\theta + (a\tilde{\theta} + \bar{\alpha}) (1 - e^{-\lambda x^\beta}))^2} \tag{13}
\]

\[
f^E(x)_{\text{MOMOWe}} = M \lambda \beta x^{\beta-1} e^{-\lambda x^\beta} \left(1 - e^{-\lambda x^\beta}\right)^i \tag{14}
\]

with \( M \) as in (9).

The cdf, pdf, and hazard function plots of the MOMOWe distribution for particular values of four parameters are showed in figures 1-3. It is noted that this new distribution is flexible to model positive data.

**Figure 1.** The cdf plot of MOMOWe with certain parameters values.

**Figure 2.** The pdf plot of MOMOWe with certain parameters values.
Figure 3. The increasing and decreasing hazard function plot of MOMOWe with certain parameters values.

4. Statistical properties of MOMOWe distribution

Here, the most needed statistical properties of MOMOWe distribution are present. For more recent papers that focused on examining the statistical properties, see [11-14].

**r-th moment:** The MOMOWe non-central r-th moment can be attained based on (14) as below

$$E(X^r)_{\text{MOMOWe}} = \int_0^\infty x^r f(x)_{\text{MOMOWe}} \, dx = \int_0^\infty x^r M \lambda \beta x^{\beta - 1} e^{-\lambda x^\beta} \left(1 - e^{-\lambda x^\beta}\right)^i \, dx$$

Now \(1 - e^{-\lambda x^\beta}\)^i can be rewritten according to \((1 - z)^b = \sum_{j=0}^\infty (-1)^j \binom{b}{j} z^j : |z| < 1, b > 0\) and \(\binom{b}{j}\) is a Binomial coefficient as

$$\left(1 - e^{-\lambda x^\beta}\right)^i = \sum_{j=0}^\infty (-1)^j \binom{b}{j} e^{-\lambda jx^\beta}$$

and then

$$E(X^r)_{\text{MOMOWe}} = \int_0^\infty x^r M \lambda \beta x^{\beta - 1} e^{-\lambda x^\beta} \sum_{j=0}^\infty (-1)^j \binom{b}{j} e^{-\lambda jx^\beta} \, dx$$

$$= M \sum_{j=0}^\infty (-1)^j \binom{b}{j} \int_0^\infty x^r \lambda \beta x^{\beta - 1} e^{-\lambda (j+1)x^\beta} \, dx$$

$$= M \sum_{j=0}^\infty (-1)^j \binom{b}{j} \frac{1}{j+1} \int_0^\infty x^r \lambda (j+1) \beta x^{\beta - 1} e^{-\lambda (j+1)x^\beta} \, dx$$

where \(\int_0^\infty x^r \lambda (j+1) \beta x^{\beta - 1} e^{-\lambda (j+1)x^\beta} \, dx = \left(\lambda (j+1)\right)^{-\frac{r}{\beta}} \Gamma\left(1 + \frac{r}{\beta}\right)\) represent the r-th moment of Weibull distribution. Therefore the r-th moment of the MOMOWe distribution is given by

$$E(X^r)_{\text{MOMOWe}} = M \sum_{j=0}^\infty (-1)^j \binom{b}{j} (j+1)^{-\frac{r}{\beta}} \lambda^{-\frac{r}{\beta}} \Gamma\left(1 + \frac{r}{\beta}\right)$$

with \(M\) as in (9).
The characteristic function: The MOMOWe characteristic function can be attained as

\[ Q_X(t)_{\text{MOMOWe}} = \sum_{j,r=0}^{\infty} \frac{(it)^r}{r!} E(X^r)_{\text{MOMOWe}} = \sum_{j,r=0}^{\infty} \frac{(it)^r}{r!} (-1)^j j!(j+1)^{-\frac{1}{\beta}} \lambda^{-\frac{r}{\beta}} \Gamma\left(1 + \frac{r}{\beta}\right) \]

with \( M \) as in (9).

The quantile function: Through inverting the MOMOWe cdf in (12), the quantile function can be attained as

\[ x_{q-\text{MOMOWe}} = \left(-\frac{1}{\lambda} \ln\left(1 - \frac{a\theta}{q^{-1} - a\theta - \bar{a}}\right)\right)^{\frac{1}{\beta}} \]

(17)

The median of MOMOWe random variable can be gained by setting \( q = 0.5 \). The MOMOWe data can be simulated by

\[ x_{\text{MOMOWe}} = \left(-\frac{1}{\lambda} \ln\left(1 - \frac{a\theta}{U^{-1} - a\theta - \bar{a}}\right)\right)^{\frac{1}{\beta}} \]

(18)

where \( U \) has the standard uniform distribution.

Shannon entropy (SH): In the information theory, SH plays an important role. It is well-defined as an uncertainty measure. The SH of the MOMOWe distribution can be obtained from

\[ -\int_0^\infty \ln(f(x)_{\text{MOMOWe}}) f(x)_{\text{MOMOWe}} \, dx \]

where the natural logarithm for the pdf in (13) is equal to

\[ \ln(f(x)_{\text{MOMOWe}}) = \ln(a\theta \lambda \beta) + (\beta - 1) \ln(x) - \lambda x^\beta - 2 \ln \left(1 + \frac{a\bar{\theta} + \bar{a}}{a\theta} (1 - e^{-\lambda x^\beta})\right) \]

Now, the SH of the MOMOWe distribution is

\[ SH_{\text{MOMOWe}} = \ln\left(\frac{1}{a\theta \lambda \beta}\right) - (\beta - 1)E(\ln(X)) + \lambda E(X^\beta) + 2E \left(\ln \left(1 + \frac{a\bar{\theta} + \bar{a}}{a\theta} (1 - e^{-\lambda x^\beta})\right)\right) \]

(19)

with

\[ E(\ln(X)) = M\Sigma_{j=0}^{\infty} (-1)^j C_j^1 \psi(1) \]

(20)

\[ E \left(\ln \left(1 + \frac{a\bar{\theta} + \bar{a}}{a\theta} (1 - e^{-\lambda x^\beta})\right)\right) = M\Sigma_{i,m,k=0}^{\infty} \frac{(-1)^i m^{i+k+1}}{i \, k!} C_m^i \left(\frac{a\bar{\theta} + \bar{a}}{a\theta}\right)^i (\lambda m)^k E(X^{k\beta}) \]

(21)

\[ E(X^\beta) \] and \( E(X^{k\beta}) \) as in (15) with \( r = \beta \) and \( r = k\beta \) respectively.

The relative entropy (RE): The RE of the MOMOWe distribution can be obtained from

\[ \int_0^\infty \ln\left(\frac{f(x)_{\text{MOMOWe}}}{f_1(x)_{\text{MOMOWe}}}\right) f(x)_{\text{MOMOWe}} \, dx \]

where the natural logarithm of \( f(x)_{\text{MOMOWe}} \) in (13) relative to \( f_1(x)_{\text{MOMOWe}} \) with \((\alpha_1, \theta_1, \lambda_1, \beta_1)\) is given by
\[
\ln \left( \frac{f(x)_{\text{MOMOWe}}}{f_1(x)_{\text{MOMOWe}}} \right) = \ln \left( \frac{\alpha \theta \lambda \beta}{\alpha_1 \theta_1 \lambda_1 \beta_1} \right) + (\beta - \beta_1) \ln(x) - \lambda x^\beta + \lambda_1 x^{\beta_1} \\
-2 \ln \left( 1 + \frac{\alpha \theta + \theta_1}{\alpha \theta} \left( 1 - e^{-\lambda x^\beta} \right) \right) + 2 \ln \left( 1 + \frac{\alpha_1 \theta_1 + \theta_1}{\alpha_1 \theta_1} \left( 1 - e^{-\lambda_1 x^{\beta_1}} \right) \right)
\]

Now the MOMOWe relative entropy is given by
\[
RE_{\text{MOMOWe}} = \ln \left( \frac{\alpha \theta \lambda \beta}{\alpha_1 \theta_1 \lambda_1 \beta_1} \right) + (\beta - \beta_1) E(\ln(x)) - \lambda E(X^\beta) + \lambda_1 E(X^{\beta_1}) \\
-2E \left( \ln \left( 1 + \frac{\alpha \theta + \theta_1}{\alpha \theta} \left( 1 - e^{-\lambda x^\beta} \right) \right) \right) + 2E \left( \ln \left( 1 + \frac{\alpha_1 \theta_1 + \theta_1}{\alpha_1 \theta_1} \left( 1 - e^{-\lambda_1 x^{\beta_1}} \right) \right) \right) \quad (22)
\]

with \( E(X^\beta) \) and \( E(X^{\beta_1}) \) as in (15) with \( r = \beta \) and \( \beta_1 \) respectively. The other expectations as in (20) and (21) with specified parameters.

**The stress strength:** The MOMOWe stress strength can be attained by
\[
SS_{\text{MOMOWe}} = P(Y < X) = \int_0^\infty f_X(x)_{\text{MOMOWe}} F_Y(x) \, dx
\]

where \( f_X(x)_{\text{MOMOWe}} \) is the pdf as in (13) and \( F_Y(x) \) is the cdf with parameters \((\alpha_1, \theta_1, \lambda_1, \beta_1)\) as
\[
F_Y(x) = \frac{1 - e^{-\lambda_1 x^{\beta_1}}}{\alpha_1 \theta_1 + (\alpha_1 \theta_1 + \theta_1) \left( 1 - e^{-\lambda_1 x^{\beta_1}} \right)}
= \frac{1}{\alpha_1 \theta_1 + \theta_1} \left( 1 + \frac{\alpha_1 \theta_1}{\alpha_1 \theta_1 + \theta_1} \left( 1 - e^{-\lambda_1 x^{\beta_1}} \right) \right)^{-1}
\]

Using \((1 - z)^{-b} = \sum_{i=0}^\infty \frac{\Gamma(b+i)}{i! \Gamma(b)} z^i \) for \(|z| < 1, b > 0\) and \( e^{-z} = \sum_{i=0}^\infty \frac{(-1)^i}{i!} z^i \) we get
\[
F_Y(x) = \frac{1}{\alpha_1 \theta_1 + \theta_1} \sum_{i,j,k=0}^\infty \frac{(-1)^{i+k}}{k!} \left( \frac{\alpha_1 \theta_1}{\alpha_1 \theta_1 + \theta_1} \right)^i \frac{\Gamma(i+j)}{i! \Gamma(i)} (\lambda_1 j)^k x^k \beta_1
\]

Therefor based on the above formula of \( F_Y(x) \), the stress strength of the MOMOWe distribution can be obtained by
\[
SS_{\text{MOMOWe}} = \frac{1}{\alpha_1 \theta_1 + \theta_1} \sum_{i,j,k=0}^\infty \frac{(-1)^{i+k}}{k!} \left( \frac{\alpha_1 \theta_1}{\alpha_1 \theta_1 + \theta_1} \right)^i \frac{\Gamma(i+j)}{i! \Gamma(i)} (\lambda_1 j)^k E(X^{k \beta_1}) \quad (23)
\]

where \( E(X^{k \beta_1}) \) as in (15) with \( r = k \beta_1 \).

5. **Maximum likelihood estimators (MLEs) of MOMOWe parameters**

Consider a MOMOWe complete random sample, say \( x_1, x_2, ..., x_n \) along with parameter vector \( \omega = (\alpha, \theta, \lambda, \beta)^T \).

Regarding to (13), the natural logarithm likelihood function \( \ell(\omega|x) \) is
\[
\ell(\omega|x) = n \ln(\alpha \theta \lambda \beta) + (\beta - 1) \sum_{i=1}^n \ln(x_i) - \lambda \sum_{i=1}^n x_i^\beta - 2 \sum_{i=1}^n \ln\left( \alpha \theta + (\alpha \theta + \lambda) \left( 1 - e^{-\lambda x_i^\beta} \right) \right) \quad (24)
\]
The MLEs of $\omega = (\alpha, \theta, \lambda, \beta)^T$ can be attained by solving the nonlinear natural logarithm likelihood system equations $\frac{\partial \ell(\omega|x)}{\partial \omega} = \left( \frac{\partial \ell(\omega|x)}{\partial \alpha}, \frac{\partial \ell(\omega|x)}{\partial \theta}, \frac{\partial \ell(\omega|x)}{\partial \lambda}, \frac{\partial \ell(\omega|x)}{\partial \beta} \right)^T = 0$ through computational iterative techniques.

6. Real Applications

The performance of the MOMOWe distribution proposed in this paper is evaluated by application on two real datasets (semi-symmetric and right-skewed).

**Real Dataset-1:** The data collection reflects the age of 155 breast tumor patients taken from June until November (2014) who entered the unit of breast tumor early detection, Benha Hospital University in Egypt [15,16].

"46, 32, 50, 46, 44, 42, 69, 31, 25, 29, 40, 42, 24, 17, 35, 48, 49, 50, 60, 26, 36, 56, 65, 48, 66, 44, 45, 30, 28, 40, 40, 50, 41, 39, 36, 63, 40, 42, 45, 31, 48, 36, 18, 24, 35, 30, 40, 48, 50, 60, 52, 47, 50, 49, 38, 30, 52, 12, 48, 50, 45, 50, 50, 53, 55, 38, 40, 42, 42, 34, 50, 38, 42, 43, 42, 36, 30, 28, 38, 54, 90, 80, 60, 45, 40, 50, 50, 50, 50, 50, 50, 34, 38, 38, 40, 42, 40, 38, 40, 50, 50, 50, 50, 36, 60, 90, 48, 58, 45, 35, 38, 32, 35, 38, 34, 43, 40, 35, 54, 60, 33, 35, 36, 40, 45, 45, 56".

**Real Dataset-2:** Here we consider a dataset of fatigue fracture life of Kevlar 373(epoxy) which is subject to constant pressure at a 90 percent stress level until all failures have occurred [8,16,17]. The 76 observations are as follows:

"0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5278, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960".

Through the R-software a comparison made between the new distribution and five known distributions namely Beta-Weibull (BWe), Kumaraswamy-Weibull (KuWe), Exponentiated Generalized-Weibull (EGWe), Weibull-Weibull (WeWe) and Marshal-Olkin-Weibull (MOWe) "for more details about these distributions see [18,19,20]". Then investigative measures including "Negative Log-Likelihood (NLL), Akaike Information Standard (AIC), Consistent Akaike Information Standard (CAIC), Bayesian Information Standard (BIC), Hannan-Quinn Information Standard (HQIC)" are used to verify the most appropriate distribution of the data under comparison. The MLEs for parameters and investigative measures for two sets of real data are presented in tables 1 and 2.

Based on the results, MOMOWe turns out to be the best to represent data sets (better fitting) compared to other distributions, it has the lowest value for analytical measures. Also, this better fitting can be seen through the plots of the histogram and empirical cdf of MOMOWe with the compared distributions for the two datasets given in figures 4-7.

**Table 1.** The values of MLEs and investigative measures to the real dataset-1

| Dist.  | $\hat{a}_{ML}$ | $\hat{b}_{ML}$ | $\hat{\lambda}_{ML}$ | $\hat{\beta}_{ML}$ | NLL   | AIC   | CAIC  | BIC   | HQIC  |
|-------|----------------|----------------|----------------------|-----------------|-------|-------|-------|-------|-------|
| MOMOWe| 63.87          | 63.87          | 0.34                 | 0.78            | 599.40| 1206.79| 1207.06| 1218.97| 1211.74|
| BWe   | 3.73           | 1.29           | 0.03                 | 2.06            | 603.25| 1214.49| 1214.76| 1226.67| 1219.44|
| KuWe  | 8.85           | 5.29           | 0.03                 | 1.06            | 602.90| 1213.81| 1214.07| 1225.98| 1218.75|
| EGWe  | 0.51           | 3.42           | 0.04                 | 2.19            | 603.26| 1214.51| 1214.78| 1226.69| 1219.46|
| WeWe  | 3.15           | 1.04           | 0.02                 | 1.17            | 610.30| 1228.59| 1228.86| 1240.77| 1233.54|
| MOWe  | 81.22          | 0.07           | 1.38                 | -               | 602.83| 1211.66| 1211.82| 1220.79| 1215.37|
Table 2. The values of MLEs and investigative measures to the real dataset-2

| Dist. | $\hat{\alpha}_{ML}$ | $\hat{\beta}_{ML}$ | $\hat{\lambda}_{ML}$ | $\hat{\beta}_{ML}$ | NLL   | AIC   | CAIC  | BIC   | HQIC  |
|-------|----------------------|----------------------|----------------------|----------------------|-------|-------|-------|-------|-------|
| MOMOWe| 13.39                | 13.39                | 29.77                | 0.42                 | 120.30| 248.61| 249.17| 257.93| 252.33|
| BWe   | 1.41                 | 0.76                 | 0.74                 | 1.12                 | 122.16| 252.31| 252.88| 261.63| 256.04|
| KuWe  | 1.89                 | 4.58                 | 0.24                 | 0.84                 | 122.07| 252.13| 252.70| 261.45| 255.86|
| EGWWe | 0.30                 | 1.44                 | 1.84                 | 1.10                 | 122.16| 252.33| 252.89| 261.65| 256.05|
| WeWe  | 0.53                 | 0.88                 | 0.46                 | 2.49                 | 122.41| 252.82| 253.38| 262.14| 256.55|
| MOWe  | 0.26                 | 0.27                 | 1.71                 | -                    | 122.23| 250.47| 250.80| 259.46| 253.26|

Figure 4. The dataset-1 histogram plot with the compared distributions.

Figure 5. The dataset-1 empirical cdf with the compared distributions.

Figure 6. The dataset-2 histogram plot with the compared distributions.

Figure 7. The dataset-2 empirical cdf with the compared distributions.
7. Conclusions

In this paper, a new four-parameter distribution called the Marshall Olkin Marshall Olkin Weibull (MOMOWe) is introduced. The explicit expressions for the r-th moment, characteristic function, quantile function, Shannon entropy, relative entropy, and stress-strength model are derived. The maximum likelihood estimation of the model parameters is discussed. The results of investigative measures with two real applications (semi-symmetric and right-skewed real data) illustrate that the MOMOWe distribution provides flexibility and consistently better fit compared with other distributions and this flexibility may allow using it in various application areas.

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