Relativistic flows on a spacetime lattice

N.L. Balazs1⁎, B.R. Schlei2,3†, and D. Strottman2‡

1 Department of Physics and Astronomy, SUNY at Stony Brook, NY 11794
2 Theoretical Division, DDT-DO, Los Alamos National Laboratory, NM 87545
3 Physics Division, P-25, Los Alamos National Laboratory, NM 87545

March 31, 2022

Abstract

The relativistic extension of non-relativistic hydrodynamics suffers from notorious difficulties. In non-relativistic hydrodynamics where difficulties also abound, it has proved a useful supplement to study lattice models which can imitate viscous fluid flow. In this paper we construct a relativistic spacetime lattice and construct a dynamics of points, thus a relativistic cellular automaton over it, to model relativistic fluid flow. A simple example is also explicitly studied, and some numerical results with figures are shown in the last section.

1 INTRODUCTION

A causal description of matter under extreme conditions is a very difficult task. A relativistic description of heavy ion physics suffers strongly from this predicament. A microscopic causal description is not available since we do not know sufficiently well the microscopic interactions among the constituents under these conditions. A proper relativistic interaction must be local and handled through fields. Usually, however, one attempts to formulate a theory entirely in terms of the matter degrees of freedom.

At the same time it is not ensured that a more macroscopic, causal and local description of nuclear matter alone, and in which the field degrees of freedom responsible for the interaction are entirely removed, exists. According to Hilbert[1], Bogoliubov[2] and others, such a separation and elimination of the degrees of freedom would require the coexistence of widely disparate time scales in the evolution of the system; one associated with the field degrees of freedom and the other with the matter degrees of freedom. The existence of

⁎E. Mail: balazs@sbhep.physics.sunysb.edu
†E. Mail: schlei@LANL.gov
‡E. Mail: dds@LANL.gov
different time scales in the field degrees of freedom and the matter degrees of freedom is not at all assured in the highly relativistic region! (Such a separation of the fields from the matter was given by van Kampen for a highly simplified, non-relativistic model and resulted in integral equations. For matter interacting through Boson fields, see de Groot et al., p.79.) If, however, we disregard these worries, we may attempt to use the simplest macroscopic causal description of the matter (with the exclusion of the fields), namely, hydrodynamics, supported with local thermodynamical relations, both adapted for extreme conditions, i.e., for high velocities and large internal energies. Such forms of relativistic hydrodynamics have been known for a very long time.

The tool of relativistic hydrodynamics has, alas, its own difficulties. In perfect fluid dynamics the mean free path in a fluid is zero. The introduction of dissipation (viscosity and heat conduction) provides for a non-zero mean free path. At relativistic energies where transparency effects become important, the introduction of such effects would appear essential. However, it is unclear how one can separate the dissipative effects from the others at relativistic energies. This separability also hinges on the existence of different time scales in the motion of relativistic matter alone. To uncover such a possibility one should translate Bogoliubov’s original ideas in an invariant manner to a relativistic description. This has not yet been done.

With dissipation, the Euler equations generalize, albeit with ambiguities, to something much more complicated. (A review of the basic problems with detailed literature references can be found in ref.) Two of the ambiguities are:

1) A result of introducing dissipation is that the energy flow will no longer be in the same direction as the matter flow or the entropy flow. One has then a dilemma in choosing a rest frame: a frame with zero energy flow may have a nonzero matter flow, and nonzero entropy flow. This ambiguity has led to different choices: Eckart chose a frame in which the matter flux was zero, Landau one in which the energy flux vanishes. The third natural choice, the vanishing of the entropy flux, could be exploited with equal
justification. Linear combinations of these fluxes could also be used. The resulting theories are not equivalent except in the limit of zero dissipation. This makes one wonder why one should introduce a rest frame at all! However, near equilibrium the instantaneous local rest frame is needed to specify the thermodynamical quantities which appear in the fluid equations.

2) Instabilities and acausal behavior may arise as demonstrated by Hiscock and Lindblom\cite{8} and by Geroch and Lindblom\cite{9,10}. In the Eckart frame all solutions have instabilities and are acausal. The instabilities grow exponentially with a time scale on the order of $10^{-25}$ seconds. In the Landau frame almost all solutions are unstable. A recent theory espoused by Carter\cite{11} has been found to have similar problems\cite{12}.

An alternative approach that attempts to include dissipation has been advanced by Israel and Stewart\cite{14,15,16,17} and is a generalization of a nonrelativistic theory by Müller\cite{18}. However, the five Euler equations are replaced in the Israel and Stewart model by fourteen equations and the three coefficients of viscosity and the coefficient of conduction in the Navier-Stokes equation is replaced by six new coefficients. Hiscock and Lindblom\cite{19,20} have also analyzed Israel’s theory. In the Eckart frame instabilities still persist, but in the Landau frame, solutions are stable and causal for modest deviations from equilibrium; for large deviations from equilibrium and for large values of internal energy, problems persist. A relativistic kinetic theory for dilute gases has also been developed, starting with the work of Synge \cite{21} and developed greatly by Israel and others. (See the detailed discussion in the book by de Groot et al.\cite{4} and in the articles by Israel\cite{22,23}). However, not all the difficulties and ambiguities have been successfully resolved.

A broad class of relativistic dissipative theories have been reviewed by Geroch\cite{24} who concludes that there are a vast number of competing theories whose physical usefulness is questionable in the regimes where they agree with each other, while their differences are important only in those regimes in which they already break down.

Past experience on non-relativistic hydrodynamics\cite{25,26} suggests that a clearer understanding of many features of the fluid flow may be gleaned
through the use of discrete models, lattice automata. In this note we describe how such lattice automata can be constructed in the relativistic domain. Since in these lattice automata the interactions are not handled through fields, but through contact interactions, the separation of the matter degrees of freedom from those of the field is not needed.

2 SPACETIME LATTICES

Non-relativistic lattice automata make use of the Galilean group. The Galilean group is the product of the Euclidean group (rotations and translations in space) and the (one dimensional) translation group along the time axis. Since the translations along time are independent of the spatial transformations, the structure of the lattice is entirely determined by the discrete subgroups of the Euclidean group, the standard crystallographic groups of the required spatial dimensions. The incorporation of the time translations are unnecessary. A space-like vector connects the lattice sites in Euclidean space, and its components transform corresponding to contravariant vectors under the action of the discrete rotations. Because the hopping speed from lattice point to lattice point is not restricted by any invariance requirement there will always be a Galilean transformation which can reduce any hopping velocity to rest.

The incorporation of the conservation laws in a non-relativistic lattice model simply means that the hopping rules must be such that a) if a point hops without encountering another one its velocity is unchanged; b) if two or more points meet, the rule determining the next step must satisfy the conservation laws. As the time translations play no role, the updating rules of a Newtonian automaton need to be concerned only with the crystallographic lattice.

The situation changes if the automaton is relativistic; now a full spacetime lattice must be used. As before, invariance requirements enter in the same two places, but with different consequences:

a) the spacetime lattice must now be invariant on a discrete subgroup of the
Poincaré group, the invariance group of relativistic dynamics;
b) the hopping rules must incorporate the relativistic energy-momentum
conservation laws.

The Poincaré group is formed by Lorentz transformations and spacetime
translations. It has an infinite number of discrete subgroups, producing an
infinite number of different spacetime lattices, in contradistinction to the
finite number of crystallographic lattices. The microscopic dynamics will
consist of hopping on such a spacetime lattice according to certain rules.
The discrete Lorentz transformations used in constructing such a lattice will
link the space and time variables. As a consequence, not all frame velocities
are permitted but only those which can be reached by the class of Lorentz
transformations present in the discrete subgroup envisaged. Accordingly,
one can preserve only as much of full Lorentz invariance (generated by the
continuous group) as allowed by the discretization. If we also desire - as
we shall do - that any microscopic one-step spacetime hop velocity could be
transformed away, then only those hop velocities are permitted which are
generated through boosts produced by the allowed class of discrete Lorentz
transformations. (This condition could conceivably be relaxed, arguing that
a local “rest” frame means the vanishing of some average flux, and not
a microscopic velocity. For example, the instantaneous speed of a Dirac
electron is always the speed of light, due to its Zitterbewegung, while its
average velocity can be anything less than c.)
The actual construction of the hopping rules require the construction of the
spacetime lattice in order to determine the allowed Lorentz transformations
and thereby the allowed world momenta. To do this we will study a spe-
cific example in greater detail. This simple example is too primitive to be
physically realistic, but it will exhibit explicitly the methods embodied.
An interesting previous attempt has been made without the use of the above
requirements by Hersbach[31]. There a one dimensional model was obtained
in which the particles move right or left with the velocity of light. This has
serious consequences: a) there is no rest frame for the microscopic particle
motion, b) in the collisions detailed balancing and time reversal invariance are violated.

3 THE CONSTRUCTION OF SPACETIME LATTICES

We have previously specified [27] that the construction advocated here fails in a (1+1) dimensional spacetime, but not in higher dimensions. Therefore we will describe the general approach in (3+1) and then in (2+1) spacetime dimensions. The actual transport model will use the latter.

An inertial frame specifies a coordinate system with straight axes along Minkowski (or M) orthogonal frame vectors \( e(a), a = 0, 1, 2, 3 \), satisfying the conditions

\[
\mathbf{e}(a) \cdot \mathbf{e}(b) = \eta(a, b), \tag{1}
\]

where \( \eta(a, b) \) is given by

\[
\eta(a, b) = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \tag{2}
\]

Since this spacetime is flat we can introduce a finite size radius vector \( r \) connecting a point in spacetime with the origin. This point has the coordinates \( x^a \) where

\[
r = \sum x^a e(a). \tag{3}
\]

One can similarly introduce an arbitrary world vector \( u \), expressed as a linear combination of frame vectors, the coefficients being the contravariant components \( u^a \),

\[
u = \sum u^a e(a). \tag{4}
\]

(It is useful to stress that changing frames means two things: changing the frame vectors \( e(a), a = 0, 1, 2, 3 \), and the associated components of the particular vectors. The components transform as contravariant components,
while the frame *vectors* change as if their names, *the labels (a)* were covariant indices.

The scalar product of two world vectors \( \mathbf{u} \) and \( \mathbf{v} \) is immediately given through their contravariant components as

\[
\mathbf{u} \cdot \mathbf{v} = -u^0 v^0 + u^1 v^1 + u^2 v^2 + u^3 v^3.
\]

(5)

Our notation will be slightly inconsistent, denoting sometimes a world vector in bold face as \( \mathbf{u} \), or by its components in a frame as \( u^i \), or just by \( u \). Where this would give rise to any ambiguity, we shall be precise.

The simplest lattices are cubic. Such a lattice can be constructed by selecting *lattice points* obtained through discrete translations of unit step-length along the coordinate axes [27]. Then a general lattice point, indexed by the integers \((k, l, m, n)\), will have a radius vector \( \mathbf{r} \)

\[
\mathbf{r} = k \mathbf{e}(0) + l \mathbf{e}(1) + m \mathbf{e}(2) + n \mathbf{e}(3).
\]

(6)

Letting \((k, l, m, n)\) run through all positive and negative integers, including zero, a spacetime lattice is generated.

When will the lattice sites be unchanged under Lorentz transformations? Consider a Lorentz transformation matrix with integer coefficients. The original quadruplet of integers \((k, l, m, n, )\) transforms into another quadruplet of integers \((k', l', m', n')\). Thus, from the active point of view the transformation shifts a lattice point into another lattice point; from the passive point of view it relabels a fixed lattice point. Thus the existence of an invariant spacetime lattice depends on the existence of Lorentz transformations specified by matrices with *integer* elements with respect to a particular set of basis vectors.

Let

\[
L = \begin{pmatrix}
a^0 & b^0 & c^0 & d^0 \\
a^1 & b^1 & c^1 & d^1 \\
a^2 & b^2 & c^2 & d^2 \\
a^3 & b^3 & c^3 & d^3 \\
\end{pmatrix}
\]

(7)

be a Lorentz matrix specified in an inertial frame with frame vectors.
March 31, 2022

\[ e(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e(1) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e(2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e(3) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \]

Applying \( L \) to these frame vectors we generate a new frame with the new frame vectors
\[ e'(0) = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}, \quad e'(1) = \begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{pmatrix}, \quad e'(2) = \begin{pmatrix} c^0 \\ c^1 \\ c^2 \\ c^3 \end{pmatrix}, \quad e'(3) = \begin{pmatrix} d^0 \\ d^1 \\ d^2 \\ d^3 \end{pmatrix}. \]

Hence the new frame vectors are the vectors \( a, b, c, d \) with components composed from the columns of the matrix. Since a Lorentz transformation must preserve \( \mathbf{M} \) orthonormality, the following conditions are imposed on the matrix elements, expressed as conditions on the frame vectors:

\[ \begin{align*}
-\mathbf{a} \cdot \mathbf{a} &= \mathbf{b} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{c} = \mathbf{d} \cdot \mathbf{d} = 1 \quad (8) \\
\mathbf{a} \cdot \mathbf{b} &= \mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{d} = 0 \\
\mathbf{b} \cdot \mathbf{c} &= \mathbf{b} \cdot \mathbf{d} = 0 \\
\mathbf{c} \cdot \mathbf{d} &= 0. \quad (9)
\end{align*} \]

These conditions lead to Diophantine equations.

In two dimensions there are no solutions unless we use null coordinates; in three dimensions – to which we now restrict the discussion – there are an infinity of solutions as indeed there are also in four dimensions. The one containing the smallest positive integers is given by the matrix
\[ L(1) = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \quad (10) \]

where the columns specify the frame vector components of the moving frame. (Other solutions can be found in [27, 28]). The first column corresponds to a world velocity with components \((1/\sqrt{1-v^2}, v_1/\sqrt{1-v^2}, v_2/\sqrt{1-v^2})\), hence, in the example above, to a velocity \(\sqrt{8}/3\) in units of the velocity of light.
How are the time and length units to be obtained in the moving frame? In the conventional case the marks denoting the units on the different coordinate axes are specified by the intersection of the coordinate axes with the unit calibration - hyperboloids. The time axis intercepts the hyperboloid of two sheets, while the two space-like axes intercept the hyperboloid of one sheet. In the present case there is a lattice point at each of these intersections, and these lattice points are the nearest to the origin located on the coordinate axes. Thus the conventional specification of space and time units is the same as taking the lattice point separation along each axis as the unit. (One may view the construction of the spacetime lattice as seeking all those points where the two length specifications coincide, the one through the calibration hyperboloids, and the other through spacing equally the lattice points on a coordinate axis, sliding them like beads.)

From $L(1)$ we can generate three other Lorentz transformations satisfying all conditions, Eqs. (8) and (9), by rotating the original spatial axes. This results in four $L$ transformations altogether; these are

\[
L(1) = \begin{pmatrix}
3 & 2 & 2 \\
2 & 1 & 2 \\
2 & 2 & 1
\end{pmatrix},
\]

\[
L(2) = \begin{pmatrix}
3 & -2 & 2 \\
-2 & 1 & -2 \\
2 & -2 & 1
\end{pmatrix},
\]

\[
L(3) = \begin{pmatrix}
3 & -2 & -2 \\
-2 & 1 & 2 \\
-2 & 2 & 1
\end{pmatrix},
\]

\[
L(4) = \begin{pmatrix}
3 & 2 & -2 \\
2 & 1 & -2 \\
-2 & -2 & 1
\end{pmatrix}.
\]

Our model will be built on these four transformations.
4 DYNAMICAL QUANTITIES

1) Invariance in this lattice is a reduced relativistic invariance. It shall mean invariance under transformations generated by the four discrete Lorentz transformations constructed above, their powers and their products.

2) In this frame the components of the four possible hopping world velocities, $u_A$, $(A = 1, 2, 3, 4; \text{modulo } 4)$ are given by the first column of each of these matrices. (Observe that capital letters refer to the name of the particular world velocity, and not to components.)

\[
\begin{align*}
  u_1 &= \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, &
  u_2 &= \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}, &
  u_3 &= \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}, &
  u_4 &= \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}.
\end{align*}
\]

Thus in this frame we find the following values for the dynamical quantities associated with a particle: energy $= 3mc^2$, speed $(\sqrt{8}/3)c$, magnitude of momentum $= \sqrt{8}mc$. This is a very relativistic speed indeed; all other solutions have even higher speeds. (We have chosen these world velocities as fundamental, since any one of them can be reduced to rest by using one of the basic Lorentz transformations.)

For the present case a sort of one particle (or reduced) phase space lattice can be constructed. (It is not a bona fide phase space since there is no symplectic form associated with this discrete manifold.) To each spacetime lattice point we attach as arrows to the vertices, $u_A, A = 1, 2, 3, 4$. Their tips will lie on the upper sheet of unit hyperboloids hovering over each lattice point. A microscopic dynamical state of a particle is specified by giving the location of the lattice point (to fix its spacetime position), and the tip of one of the attached $u_A$’s. The evolution is now conceived by hopping from tip to tip.

Three comments are in order. First, the previous considerations can be extended to a four dimensional spacetime. An infinity of solutions exist and are reported elsewhere [28]. Second, more complicated spacetime lattices can be generated that will increase isotropization in spacetime, and hence, also in the lattice points on the space-like cuts. Third, we can increase the isotropization in the world velocities (or world momenta) by applying
one or more allowed Lorentz transformations to any member of our original $u_A$ set and interpreting the generated time-like unit vectors as additional world velocities in the original frame (instead as the original vector in a new frame). (These new world velocities still can be reduced to rest applying products of our basic Lorentz transformations.) In the phase space picture this would produce a larger set of arrows attached to each lattice point with all tips touching each unit hyperboloid.

For these reasons the following Sections will be phrased in more general terms and not restricted to the special case described above. However, where appropriate we shall point out if the present model exhibits unexpected, special or odd features.

5 MICRODYNAMICS

Lattice points can be populated by mass points having one of the four world velocities; these mass points must jump to another lattice point at each iteration. The units used here make the mass and the speed of light unity; thus, the world velocity is numerically equal to the world momentum. (Notice that here the variable labeling the evolution is the invariant iteration number $q$, and not the time, which is one of the frame-dependent labels of $r$. In the future we shall specify the location and iteration number together, as $r, q$. The present simple model has the special feature that in this particular frame, each iteration advances the time by three units since the time component of each $u_A$ is 3.)

Let the scalar quantity $n_A(r, q)$ be unity if the lattice point $r$ is occupied at the q’th iteration by a point with world velocity $u_A$, and zero otherwise. The evolution equation of the four discrete functions is given by

$$n_A(r + u_A, q + 1) = n_A(r, q) + c_A(r, q).$$

The first term on the right describes pure streaming. The collision term, $c_A(r, q)$, determines the state of affairs at $r + u_A, q + 1$ arising from the collision at $r, q$. The collision must satisfy the conservation of world momenta.
In the present model this is enforced as follows. (For the non-relativistic situation see ref. [29].) Only two points can collide, and only if the collision is head on, and only if it results in a rebound with the spatial components of the incoming world velocities turned ninety degrees (in this special frame!). Thus, \( c_A(r,q) \) is equal to the configurations producing a Gain minus the configurations producing a Loss at \( r + u_A, q + 1 \). Loss occurs when at \( r \) the configurations with \( u_A, u_A + 2 \) are occupied to produce a head-on collision that will remove the point with \( u_A \). This can only happen if the configurations with \( u_A + 1, u_A + 3 \) are both unoccupied, to receive the points with the turned incoming world velocities. Gain occurs when at \( r, q \) the configurations with \( u_A + 1, u_A + 3 \) are occupied, and \( u_A, u_A + 2 \) are unoccupied. This gives

\[
\begin{align*}
\text{Loss} & = n_A n_{A+2} (1 - n_{A+1})(1 - n_{A+3}), \\
\text{Gain} & = n_{A+1} n_{A+3} (1 - n_A)(1 - n_{A+2}), \\
c_A(n) & = \text{Gain} - \text{Loss} \\
& = n_{A+1} n_{A+3} (1 - n_A)(1 - n_{A+2}) - n_A n_{A+2} (1 - n_{A+1})(1 - n_{A+3}).
\end{align*}
\] (12) (13) (14)

(All quantities are evaluated at \( r, q \).)

It is an easy exercise to show that the two conservation laws of particle number and world momentum

\[
\begin{align*}
\sum_A n_A c_A(n) & = 0, \\
\sum_A u_A^i n_A c_A(n) & = 0,
\end{align*}
\] (15) (16)

are satisfied. The solution, \( n_A(r,q) \), gives the most detailed microscopic description of the system at any iteration. Having \( n_A(r,q) \), all relevant quantities at \( r, q \) can be directly evaluated. These are the

\[
\text{number of particles} = \sum_A n_A,
\] (17)
total world momentum $= \sum_A u^i_A n_A$, \hspace{1cm} (18)

microscopic energy momentum tensor $= \sum_A u^i_A u^k_A n_A$. \hspace{1cm} (19)

6 KINETIC THEORY

A different description is provided by the precepts of kinetic theory using a probabilistic approach. We can average $n_A$ in many different ways, for example, averaging over different initial data, or averaging over spacetime regions of various sorts, or assigning a priori probabilities to the occurrence of the initial $n_A$’s, etc. One arrives at four distribution functions $f_A(r, q)$, one for each of the four world velocities, that specify the probability of finding a point with world velocity $u_A$ at the lattice point $r$ at a given iteration number $q$. Thus, the distribution functions are attached to lattice points in the discrete phase space, and consequently they are not densities in a phase space as in the usual continuum theories.

What are the evolution equations of these functions? If (following Boltzmann) we neglect correlations between the different $n_A$’s, and average the evolution equations of the $n_A$’s, these equations become identical in form with the evolution equations for the $n_A$’s, simply replacing the latter by the corresponding $f_A$’s, i.e.,

$$f_A(r + u_A, q + 1) = f_A(r, q) + c_A(r, q).$$ \hspace{1cm} (20)

where

$$c_A = f_{A+1}f_{A+3}(1 - f_A)(1 - f_{A+2})$$ \hspace{1cm} (21)

$$-f_Af_{A+2}(1 - f_{A+1})(1 - f_{A+3}).$$ \hspace{1cm} (22)

Just as for the $n_A$, here also the two conservation laws

$$\sum_A f_A c_A(f) = 0,$$ \hspace{1cm} (23)

$$\sum_A u^i_A f_A c_A(f) = 0,$$ \hspace{1cm} (24)
are satisfied. The four coupled equations in Eq. 21 correspond to the relativistic Boltzmann equation. (There are four equations only, since in our model there are only four world momenta, \( A = 1, 2, 3, 4 \) present instead of a continuous range. In general there will be as many equations as there are allowed world velocities.)

Given the starting data \( f_A(r,0) \) at the zeroth iteration, these equations give us \( f_A(r,q) \), the probability finding a particle with world velocity \( u_A \) at \( r, q \). Observe that the starting data refer to the zeroth iteration and not to time equal zero. The latter is a frame-dependent notion. It may happen, however, that the averaging process itself tacitly introduces an initial frame dependence; for example, averaging over initial data on a space-like, \( t = 0 \), plane. This often corresponds to the actual physical situation where experiments are performed with the same setup at different times in the laboratory frame, and assuming that the setup is the same at each starting time.

The relevant macroscopic, or mean quantities at \( r, q \) are as follows.

\[
\text{density} : \rho = \sum_A f_A \tag{25}
\]

world momentum flow, or particle flow : \( N^i = \sum_A u^i_A f_A \tag{26} \)

mean entropy flow : \( S^i = \sum_A u^i_A [f_A \log f_A + (1 - f_A) \log (1 - f_A)] \tag{27} \)

energy momentum tensor : \( T^{ik} = \sum_A u^i_A u^k_A f_A \tag{28} \)

where all \( f_A \)'s are evaluated at \( r, q \). The label density is a misnomer, since it does not have the physical dimensions of a density; we use it only to distinguish it from \( n \) – which we introduce below – the size of the particle flow world vector (which does not have the physical dimensions of a density either). Densities can naturally appear only in a continuum model or in approximations to it.

Each flow can also be written as the product of a scalar and a time-like unit world vector. The scalar specifies the strength of the flow and the world
vector its spacetime direction:

\[ N^i = nU^i, \quad (29) \]
\[ S^i = nsV^i. \quad (30) \]

This introduces the strength of the particle flow \( n \), and its direction \( U^i \); the strength of the entropy flow \( ns \); and its direction \( V^i \). It is important to stress that we now have two different and equally sensible invariant definitions of a scalar quantity which qualify as a “particle density”, the strength \( n \) of the particle flow, and \( \rho \), the latter being also equal to the (negative) trace of the energy-momentum tensor. It is unclear which one is better suited as a variable.

Other directions and strengths are specified through the eigenvectors and eigenvalues of the symmetric energy momentum tensor \( T^{ik} \). These are defined through the equations

\[ T_{ik}A^k = a\eta_{ik}A^k. \quad (31) \]

There are three eigenvectors \( A^k \), each with an associated eigenvalue \( a \). Of these, the most important is the direction \( W^i \) and the strength \( ne \), this being the time-like eigenvector and its eigenvalue, giving rise to the energy flow world vector \( neW^i \). The two other eigenvectors are space-like with eigenvalues giving the pressures \( P', P'' \). The two pressures are associated with forces on surface elements with normals along the spatial projection of the two space-like eigendirections. The quantities \( s \) and \( e \) are the invariant entropy per particle, and the invariant energy per particle, using \( n \) as the conversion factor between per lattice point and per particle, as is usual in the continuum theory. In lattice models one may use with equal justice \( \rho \) as the conversion factor, and specify another invariant entropy (per particle) and invariant energy (per particle) as \( s' = ns/\rho \), and \( e' = ne/\rho \).

7 EQUILIBRIUM CONSIDERATIONS

According to this description, three time-like unit world vectors, \( U^i, V^i, W^i \) vie in principle to specify preferred directions. We show now that in full
thermal equilibrium there can only be one preferred direction, provided there is only one set of dynamical conservation laws to be satisfied. In general, this is the conservation of world momenta. (In the present model, however, one has additional conservation laws.) Outside thermal equilibrium this need no longer be the case, and more preferred directions may exist.

In the absence of external fields of force we expect that the stationary states are homogeneous in spacetime, i.e., $f_A$ is independent of $r$. If this be so, and if $c_A$ vanishes, evolution ceases; a stationary state is present. If all possible stationary states are simultaneously present, presumably thermal equilibrium is reached. (The final test is whether the entropy has reached its maximum value given certain constraints. This, however, we will not investigate here but take it for granted.)

The vanishing of $c_A$ requires the validity of the scalar relation

$$ f_A f_{A+2} (1 - f_{A+1})(1 - f_{A+3}) = f_{A+1} f_{A+3} (1 - f_A)(1 - f_{A+2}), \quad (32) $$

or

$$ \log[f_A/(1 - f_A)] + \log[f_{A+2}/(1 - f_{A+2})] 
= \log[f_{A+1}/(1 - f_{A+1})] + \log[f_{A+3}/(1 - f_{A+3})]. \quad (33) $$

From this it follows that

$$ Q_A + Q_{A+2} = Q_{A+1} + Q_{A+3}, \quad (34) $$

where we have renamed the arguments of the log functions by $Q$. One particular solution is given by

$$ Q_A = \alpha, \quad (35) $$

where $\alpha$ is a constant, independent of $u_A$.

Consider now the conservation of world momentum in a pair collision,

$$ u^i_A + u^i_{A+2} = u^i_{A+1} + u^i_{A+3}. \quad (36) $$

Scalar multiplication of this relation with a world vector, say $\beta_i$, gives

$$ \beta \cdot u_A + \beta \cdot u_{A+2} = \beta \cdot u_{A+1} + \beta \cdot u_{A+3}. \quad (37) $$
Putting $Q_A = \beta \cdot u_A$ we recover the basic relation between the $Q_A$'s above. If there are no more conservation laws, these are all the particular solutions. Adding the two solutions we find that

$$\log\left[\frac{f_A}{1 - f_A}\right] = \alpha + (\beta \cdot u_A),$$

or

$$f_A = \frac{1}{1 + \exp(\alpha + (\beta \cdot u_A))},$$

where $\alpha$ is a scalar constant and $\beta^i$ is a constant world vector. The latter can, of course, be written as the product of its magnitude and a unit world vector.

Further statistical considerations are needed to determine the physical meaning and values of these parameters from the given conditions, as the invariant density $\rho$, (or $n$), and the invariant energy $ne$. One expects that $\alpha$ is proportional to the chemical potential, while the world vector $\beta^i$ is related to a time-like unit world vector describing the average flow direction in spacetime, its magnitude being the reciprocal temperature. The details, however, will depend on whether we define the invariant density via $\rho$, or via $n$. Moreover, at this stage it is not clear whether the flux determined in this manner can be reduced to rest through an allowed discrete Lorentz transformation, or any combination of them. (One may observe that the $f_A$ given above makes $c_A$ vanish even if $\alpha$ and $\beta^i$ are functions of $r$ and $q$. However, in this case the left hand side of Boltzmann’s equation need no longer vanish. These “local equilibrium functions” are useful in studying different possible linearizations.)

Perhaps it is useful to stress that this derivation shows why in strict thermal equilibrium, only one preferred direction, that of $\beta^i$, can exist; thus the spacetime directions of all fluxes must coincide. This is the consequence of the fact that in general only one vectorial conservation law can exist (the conservation of world momentum in a pair collision). Thus its conversion to a scalar expression requires the introduction of only one world vector. If, however, there are additional conservation laws, this need not be the
case. Outside equilibrium, with irreversible processes present, new preferred directions may arise.

An identical looking expression has been obtained in the non-relativistic case, see \[29\]. There, however, the scalar product appearing is between vectors in space, while here between world vectors. Consequently, the non-relativistic result is not Galilean invariant (though it should be), while the present one is Lorentz invariant.

In this simple model additional conserved quantities exist and they will modify the equilibrium distribution. This occurs as follows. In the special frame we find

\[
\begin{align*}
u_A + u_{A+2} &= \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}, & u_{A+1} + u_{A+3} &= \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}.
\end{align*}
\]

Thus, the line of collision of the incoming particles, and the lines of departure of the rebounding ones lie in one plane, the collision plane, in which the components of the sums of incoming and outgoing velocities vanish separately.

Multiply the first sum with a world vector \(K\) whose time-like component is zero but arbitrary otherwise; the second with the world vector \(K'\) whose time-like component is zero but arbitrary otherwise. (Thus the world vectors \(K, K'\) are space-like.) We get a new particular solution

\[
\begin{align*}
Q_A &= K \cdot u_A & (40) \\
Q_{A+2} &= K \cdot u_{A+2} & (41) \\
Q_{A+1} &= K' \cdot u_{A+1} & (42) \\
Q_{A+3} &= K' \cdot u_{A+3} & (43) \\
\end{align*}
\]

since the sum of the first pair of \(Q\)'s is equal to the sum of the second pair - as required - both being zero. Adding this new particular solution to the previous ones we now obtain

\[
\begin{align*}
f_A &= 1/[1 + \exp(\alpha + (\beta + K) \cdot u_A)], & (45) \\
f_{A+2} &= 1/[1 + \exp(\alpha + (\beta + K) \cdot u_A)], & (46)
\end{align*}
\]
\[ f_{A+1} = \frac{1}{1 + \exp(\alpha + ((\beta + K') \cdot u_A))}, \quad (47) \]
\[ f_{A+3} = \frac{1}{1 + \exp(\alpha + ((\beta + K') \cdot u_A))}. \quad (48) \]

As before, further considerations are needed to give physical meaning to the \( K, K' \) vectors. At this stage the only condition on them is that they should be space-like, lying in the collision plane. In the present model, therefore, we expect that the thermalization process generates two groups of particles, which thermalize within each group, but not the groups with each other. However, even the word ‘thermalization’ is inappropriate. Thermalization implies the existence of the notion of a temperature which manifests itself as a spread in the energy distribution among the particles. Here, in the special frame the zeroth component of all four world velocities are equal initially and stay equal during collisions, and thus there is no spread in this frame. Thermalization in the present model can only mean homogenization and isotropization.

Further shortcomings of our simple model appear as well. The magnitudes of the spatial velocities in this frame are all equal to each other, being \( 2\sqrt{2} \). Consequently, once a set of points separate from another set of points by uniform streaming, they cannot mix again (unless walls, or periodic conditions are applied), since no particle in one set is capable of catching up with particles in the other set. This latter simplification could be rectified.

\section{8 Numerical Studies}

The simplicity of the model, the associated microdynamics, and the kinetic theory can be exploited in numerical studies. Thus, one can study a variety of problems both from the microdynamical and the kinetic point of view and contrast the results. Moreover, in the latter we may use different types of averaging methods to study the similarities and differences. Here we shall give a few examples of each. (More detailed studies are in progress, and will be reported elsewhere). (In the figures the \( x \) position is always counted from right to left. This allows a simple transcription of a 32 \( x \) positions
occupancy into a binary long word that the computer uses while iterating particle propagation and collision.)

8.1 Frames

All figures, save the first, depict the evolution of systems given in a preferred frame in which the iteration number $q$ coincides with the time component of the radius vector of a point. By choosing another frame connected with the first by a Lorentz transformation, we would obtain a set of pictures where the points present in a $t' = \text{constant}$ plane would correspond to different iteration numbers. The fundamental world velocities in a general frame would appear quite different. Figure 1a shows the four fundamental world velocities from the point of view of an observer boosted so that $u_3$ is given by $(1,0,0)$, representing a particle at rest. In this figure we see clearly the relativistic effects. The other three boosted fundamental world velocities appear to move together and $u'_1$ lies nearly in the spacetime plane spanned by the vectors $u'_2$ and $u'_4$. (Of course it cannot lie in it, since momentum conservation requires that $u'_1 = u'_2 + u'_4 - u'_3$. Since $u'_1$ in this frame has components $(1,0,0)$, the last term in the sum appears to produce the smallest possible deviation from lying in this plane.) Figure 1b shows how the appearance of the evolution of the four fluids, based on the four fundamental world velocities, appear to shift with the boost.

The subsequent figures use the preferred frame. The points representing particles start initially from the $t = 0$ plane, and develop according to the equations given before. The result of each iteration therefore can be depicted in the corresponding $t = q$ plane. The appropriate iteration number, $q$, is given next to each picture. The arrows present in the pictures indicate the spatial directions of the four fundamental velocities.

8.2 Examples of microscopic evolution

On this level the relevant object to be investigated is $n_A(r,q)$. This describes the actual microdynamical evolution of particles on the spacetime lattice.
starting from $n_A(r,0)$. Figure 2a depicts the head-on collision of two sets of points initially separated. Each group occupies half the lattice points of a 96×96 square, chosen at random elaborate. Within each square the world velocities are uniform, advancing the squares towards each other, but with different edge orientations as shown at the zero’th iteration. After 40 iterations the collision of the squares is in full bloom, and after 80 iterations one finds four separate groups, each containing only one particular world velocity, and streaming apart accordingly. The pictures show the profound differences in the collisions generated by the interplay of the orientation – which regulates the arrangement of the overlap regions in the collision – and the preferred diagonal directions present in the collision mechanism where the incoming and outgoing directions are along the diagonals. The turning mechanism in the collision can, however, only become effective if the outgoing diagonal connects sites not yet occupied, thus depending on the overlap.

In edge-on collisions there is a large overlap region at the start of the collision which generates a spillout along the diagonal direction. This liberates sites to allow the turning process in the collision to operate well, generating increasing spillout, and so on. Finally we see the four well separated final regions. (We have briefly discussed at the end of Section 7 why this division into four, well separated, free streaming regions is the consequence of the model.)

In corner-on collisions the transverse flow is confined because in this configuration the initial target area is diminished and, in addition, the new incoming points impede the turning mechanism. Eventually the collisions do produce points with new world velocities, and the system again breaks up into four final parts, each homogeneous in the world velocities. (It is amusing to see how some of the sharp edges are preserved. This arises because during the collision of the squares these edges advance as a common front in each step, covering up precisely the new points spilling out due to the collision.)

Figure 2b shows the same arrangement in a four square collision. Here
the large degree of symmetry does not enable one to distinguish the final spillout regions from the others. Moreover, in the corner-on collision the conspiracy of the symmetries in the diagonals and edges finally generate only an exchange of world velocities in the diagonally opposite regions.

One may investigate how the occupancy in the initial configurations influences these results. Figure 3a demonstrates the effects of the initial occupation density on the post-collision configurations \((q = 80)\) in the two square collisions. The left column refers to edge-on collisions, the right column to the corner-on collisions. In the first row the occupation fraction in the initial squares was \(1/3\), in the second row \(2/3\), in the third row 1.

Figure 3b is arranged in an identical fashion, showing the effects of occupancy in a post collision configuration \((q = 80)\) for four square collisions. An interesting feature is the influence of the occupancy in the edge-on collisions. For very small occupancy the trailing regions are small (since for single occupancy this region must vanish). For full occupancy all edges are sharp, no trailing region exists. Consequently, there must exist one or more partial occupancy numbers for which the trailing region is a maximum.

8.3 Kinetic examples

There are many different questions and problems one can investigate using the kinetic approach. Some of these we shall mention later. Here we take up only one special case.

Figure 4a shows the microscopic evolution of points in a rectangle, i.e., the functions \(n_A(r, q)\). Initially \(1/8\) of the locations are occupied at random by each of the four different world velocities, also chosen at random. Thus half the points in the initial rectangle are occupied. After ten iterations the square increases in size, and nine regions can be distinguished in it, according to the number of different fundamental world velocities in it. The central region has points with all four velocities present; such a region we call phase \(III\). The regions at each four corners is phase \(I\), containing points with only one of the four basic velocities. These pure regions are connected
by regions of phase II with differing sizes, where the adjoining velocities are mixed pairwise. These regions separate during the evolution and the mixed regions progressively disappear. (Phases III and the small phase II disappear at \( q = 16 \), while the large phase II disappears at \( q = 32 \).)

We now conceive an ensemble where each member has different initial data chosen randomly in the same manner. Different averages over the ensemble will generate different distribution functions. From any one of them we can construct all the relevant average quantities discussed before. Hence, in principle, we can compare the different quantities obtained. There are the following quantities to be compared:

a) \( n_A(r, q) \), the solution of the microscopic equations, advancing the initial data \( n_A(r, 0) \);

b) \( f_A(r, q) \), the solution of the kinetic equation, advancing the initial data obtained by averaging \( n_A(r, 0) \);

c) \( F_A(r, q) \), the average of \( n_A(r, q) \).

The quantities \( n_A(r, q) \) give the complete microscopic description. (In the present case this is given in Figure 4a for one particular member of the ensemble). The functions \( F_A(r, q) \) give the most correct average description, through averaging the precise microscopic description for each member of the ensemble at the iteration \( q \). The functions \( f_A(r, q) \) give the kinetic description. Here, following Gibbs, we average over the initial data \( n_A(r, 0) \) within the ensemble, and then evolve this average via Boltzmann’s equation. (One could also use an initial average of one particular member of the ensemble over given “small, but not too small” spacetime regions, following Boltzmann’s ideas, and pursue the evolution of this distribution function. This we shall not do here.)

In Figures 4b, and 4c we show the behavior of some of these statistical averages computed in this manner. In principle, all ensemble averages still depend on the location \( r \), and iteration number \( q \). To simplify our pictures we will eventually also perform an additional averaging over sites occupied at each iteration. To gain further insight we shall do this averaging in two stages. First we average over the occupied sites in each phase separately.
These averages will be denoted by the square brackets, \([\cdot]\). Then we average over all the occupied sites by using the averages over the fluid in this phase and multiplying each with the fraction the phase was represented in the total number of occupied sites. These averages will be denoted by the pointed brackets, \(<\cdot>\).

In figure 4b there are several average quantities shown as a function of the iteration number \(q\). The solid curve indicates the quantity averaged over phase \(III\), the dashed curve an average over phase \(II\), the dotted curve an average over phase \(I\). (The solid and dashed lines terminate at those iterations where the overlap generating the phase ceases.) The last graph in the figure shows the fractional occupancy. The dashed line shows the fractional occupancy in Phase \(III\). There are two types of Phase \(II\) regions, large ones and small ones. (See, e.g., Figure 4a, \(q = 10\).) The small overlap regions disappear after 16 iterations, the large one after 32 iterations. Their fractional occupancy is shown by the dashed and by the dashed-dotted curves respectively. (Since both the large and small Phase \(II\) regions contribute equally to the overlap region in the \([\cdot]\) averages no distinction had to be made between these averages, hence no dashed-dotted lines appear.)

We notice that all average quantities jump immediately, with one iteration, to a stationary value. This arises since the initial values were so random that the phases immediately reached thermal equilibrium; only their mixing ratios could change during evolution.

Figure 4c shows as a function of \(q\), the average magnitudes per occupied site of the two differently defined invariant particle numbers, \(<\rho>\), \(<n>\), the entropy and energy flows \(<ns>, <ne>\), and the site-averaged pressures in the \(x\) and \(y\) directions, \(<P_x>, <P_y>\). (In this case the associated space-like eigendirections are along the \(x\) and \(y\) directions.) All quantities are averaged over the contributing sites, including all the different phases together. For these quantities, in this particular case, there are no differences visible whether we use the \(f\) functions or the \(F\) functions, showing the good quality of the kinetic approximation.

The situation is somewhat different for the normalized directions of the flows.
Instead of plotting them we shall only plot in Figure 4d the magnitudes of the squared differences. (These are symmetric in their arguments.) For example, the difference between the particle flow direction $U^i$ and entropy flow direction $V^i$ is defined as $-(U^i-V^i)(U_i-V_i) = 2(1+U^iV_i)$, with $(U^iU_i = V^iV_i = -1)$. This difference (averaged) is denoted by $<\delta(s,n)>$, or $<\Delta(s,n)>$, depending whether we evaluate it using the $f$, or $F$ distribution functions. The other differences are similarly defined. The labels $n, s, e$ in the figures refer to the associated flows.

8.4 Equation of state

Figures 4b and 4c have shown values for the pressures on planes normal to the two spatial directions. Figures 5a, b show typical graphs used in the construction of an equation of state associated with this model. Figure 5a uses the distribution function $f(r, q)$, while 5b uses the function $F(r, q)$. The two pressures $P_x, P_y$ and the energy density $ne$ are thus functions of $r$ and $q$. Here we see their relation at different lattice points at the $q = 5$ iteration. The values of both pressures are indicated on the vertical axis, the energy density on the horizontal one. In each picture four clouds of points appear associated with the regions of different phases (velocity mixtures). The small cloud at the origin depicts pure velocity phases, where both pressures vanish. (On the scale of the figure this cloud is invisible.) The large cloud has all phases present. The two regions where only two velocities mix are disjoint, since in these regions the energy densities are the same, but $P_x$ is zero while $P_y$ is not. These regions have a finite size due to the finite number of initial data in the ensemble. These sizes also depend whether we use $f(r, q)$, or $F(r, q)$ in the averaging, since each distribution function gives rise to different fluctuations. As the number of initial data increases, the different regions reduce in size, converging towards four points, given a density. Consequently, varying the density we finally generate four lines giving the equation of state. (Two of these line coincide, giving $P_x = 0$ for both.) We thus obtain
\[ P_x = 0, \quad \text{(49)} \]
\[ P_y = (4/5)ne, \quad \text{(50)} \]
\[ \rho = (1/5)ne, \quad \text{(51)} \]

for the two velocity region;

\[ P_x = P_y = \]
\[ P = (4/9)ne, \quad \text{(52)} \]
\[ \rho = (1/9)ne, \quad \text{(53)} \]

for the four velocity region;

\[ P_x = 0, \quad \text{(54)} \]
\[ P_y = 0, \quad \text{(55)} \]
\[ \rho = ne, \quad \text{(56)} \]

for the free streaming regions.

\section{9 OUTLOOK AND FURTHER QUESTIONS}

The above discussion and examples show that a relativistic lattice automaton describing relativistic fluid flows can be constructed. One is able to confront directly the microdynamics, \textit{i.e.}, the actual relativistic evolution with the one computed from the kinetic model. This may enable us to investigate both general and particular questions. Some of these are listed below.

a) To proceed, more realistic models and initial data are needed. We may improve the collision mechanism in order to destroy the additional constants of the motion, and to generate a spread in the particle energies. This will require the use of more than four fundamental world velocities, and, possibly, different spacetime lattices as well.
b) Given this, we can evaluate in more appealing models the various quantities containing sums over \( A \), using the three different distribution functions \((n_A, f_A, F_A)\) and compare them with each other to study the quality of the approximations involved in using averages, and using the kinetic equation (as we have done briefly in figure 3c). This may show the validity and range of the different assumptions entering the kinetic theory and its linearized approximations; it can also exhibit the fruitfulness (or otherwise) of the different definitions used. For example, what time-like unit vector should be chosen (if any) to describe the “hydrodynamical flow”; whether \( n \), or \( \rho \) should be used as the invariant occupation number on a site, etc.

c) Of special interest are questions connected with the entropy. The microscopic entropy flux is identically zero, because the logarithmic terms identically vanish (on account of \( n_A \) being either unity or zero). This vanishing is indeed appropriate, the entropy being a statistical notion. There remain, however, still two statistical entropy flux definitions, using either the \( F_A(r, q) \), or the \( f_A \) functions! The one defined through \( F \) is the actual entropy flux, the other is its kinetic approximation. How will these two different entropy fluxes differ, and how will they result in a different entropy production?

The entropy production itself is an important problem. We mention here two particular classes of questions.

For the entropy defined through the \( f \) functions, the Boltzmann equation insures the existence of an H theorem (for confined systems, or infinite systems) on the kinetic level. There exists, however, in this case no microscopic entropy flux using the \( n \) functions, and therefore no H theorem. What will then happen if we compare the long term solutions (large iteration numbers) of the microscopic equation, via an entropy defined through the \( F \) functions, \((i.e.,\) the correctly evolved, averaged \( n \) functions), with the equilibrium solutions? Disparities should exist due to the approximate nature of the kinetic evolution. Boltzmann himself conjectured long ago that fluctuations of the true entropy should arise even in equilibrium. Will the entropy defined through \( F \) verify this conjecture? We expect so.
All irreversible processes are linked to entropy production. To specify, however, macroscopic irreversible processes linked to this entropy production, macroscopic irreversible fluxes and its driving “forces” must be introduced. These are eventually connected by coefficients, displaying symmetry properties, the Onsager relations, exhibited by macroscopic time reversal arguments \[30\]. In conventional relativistic hydrodynamics this step and its consequences generate great difficulties, and contradictory results. The usual kinetic approach, based on the solutions of the kinetic equations near equilibrium has not resolved the difficulties. Another basic problem is to find the correct relativistic Onsager relations, which would then lead to hyperbolic equations describing the irreversible processes.

It seems to be essential to find an invariant small parameter. This is where much of the previous work, imitating Chapman and Enskog in a relativistic context, has difficulties. We conjecture that there exists a mean free iteration number corresponding to the mean free time in the non-relativistic case. The small parameter may then be the ratio of the iteration number over the mean free iteration number, or some other quantity related to them.

d) Conservation Orbits

The correct solutions of the kinetic equations can generate additional conserved quantities. For example, \( \rho(r, q) = \sum_A f_A(r, q) \) is such a quantity, implying that \( \sum_A f_A(r, q) = \sum_A f_A(r + u_A, q + 1) \). In other words there can be a point \( r' \), (or points) such that \( \rho(r', q + 1) = \rho(r, q) \). If there is only one such point at each iteration the successive points generate an orbit in spacetime. (These points need not be unique; for example if \( f_A \) is independent of \( r \), as in equilibrium, there will be an infinite number of them.) With each conserved quantity a conservation orbit could exist, but need not. If they do, different time-like “tangent” vectors can be defined by taking two adjacent points on an orbit, referring to iteration \( q \) and \( q - 1 \), taking their difference and normalizing them to \(-1\). What is their relation to the other unit time-like vectors specified through the fluxes?
10 SUMMARY

We have shown how one can construct spacetime lattices using the discrete subgroups of the Poincaré group, and define over them a dynamics that is as relativistic as possible replacing a continuous group of transformations with a discrete one. Then, using the simplest example we exhibit such a dynamics with contact interactions at the lattice sites, using both a detailed microdynamics and its kinetic approximation based on an associated Boltzmann equation. Apart from some analytical results we also showed numerical ones representing simple collisions. Finally, we have offered some questions for future work.

11 APPENDIX

One can establish a formal correspondence between the discrete formalism and the continuum one using the formally invariant relations

\[ \sum_A \rightarrow \int \frac{d^2p}{p^0} \]  \hspace{1cm} (57)
\[ r \rightarrow x^i \]  \hspace{1cm} (58)
\[ u_A^i \rightarrow p^i \]  \hspace{1cm} (59)
\[ f_A \rightarrow f(p, x) \]  \hspace{1cm} (60)
References

[1] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen* (Chelsea, 1953) 267, 281.

[2] N. N. Bogoliubov, in J. de Boer and G. E. Uhlenbeck, (Eds.) Studies in Statistical Mechanics (Interscience, N. Y., 1962).

[3] N. van Kampen, Physica **46**, 315 (1970).

[4] S. R. de Groot, W. A. van Leeuwen and Ch. G. van Weert, *Relativistic Kinetic Theory* (North-Holland Publishing Company, 1980).

[5] C. Eckart, Phys. Rev. **58**, 919 (1940).

[6] L. Landau and L. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, 1958) Section 127.

[7] N. L. Balazs, Los Alamos Report LA-UR-92-3676 (1992).

[8] W. A. Hiscock and L. Lindblom, Phys. Rev. **31D**, 725 (1985).

[9] R. Geroch and L. Lindblom, Phys. Rev. D **41**, 1855 (1990).

[10] R. Geroch and L. Lindblom, Ann. Phys. (NY) **207**, 416 (1991).

[11] B. Carter in A. Anile, Y. Choquet-Bruhat (Eds.), *Relativistic Fluid Mechanics*, Lecture Notes in Mathematics **1385**, (Springer-Verlag, Berlin, 1989), p. 1.

[12] T. S. Olson W. A. Hiscock, Phys. Rev. D **41**, 3687 (1990).

[13] W. Israel, Ann. Phys. (NY) **100**, 310 (1976).

[14] W. Israel and J. M. Stewart, Ann. Phys. (NY) **118**, 341 (1979).

[15] W. Israel, Ann. Phys. (NY) **151**, 181 (1988).

[16] J. M. Stewart, *Non-Equilibrium Relativistic Kinetic Theory*, Lecture Notes in Physics **10**, (Springer-Verlag, Berlin, 1971).
[17] J. M. Stewart, Proc. Roy. Soc. A357, 59 (1977).

[18] I. Müller, Z. Physik 198, 329 (1967).

[19] W. A. Hiscock and L. Lindblom, Ann. Phys. (NY) 151, 466 (1983).

[20] W. A. Hiscock and L. Lindblom, Contemp. Math. 71, 181 (1988).

[21] J. L. Synge, The Relativistic Gas, (North-Holland, Amsterdam, 1957).

[22] W. Israel in L. O’Raifeartaigh (Ed.), General Relativity, Papers in Honour of J. L. Synge, (Clarendon Press, Oxford, 1972), 201.

[23] W. Israel in A. Anile, Y. Choquet-Bruhat (Eds.), Relativistic Fluid Mechanics, Lecture Notes in Mathematics 1385, (Springer-Verlag, Berlin, 1989), 152.

[24] R. Geroch, J. Math. Phys. 36, 4226 (1995).

[25] G. D. Doolen, U. Frisch, B. Hasslacher, S. Orszag and S. Wolfram, (Eds.), Lattice Gas Methods for Partial Differential Equations, (Addison-Wesley Publishing Co. 1990).

[26] G. D. Doolen (Ed.), Lattice Gas Methods, (MIT Press, 1991).

[27] N. L. Balazs and D. Strottman, Heavy Ion Physics 1 139 (1995).

[28] D. Strottman et al., Los Alamos internal report (1997).

[29] U. Frisch, D. d’Humières, B. Hasslacher, P. L’Allemand, Y. Pomeau, J-P. Rivet in G. D. Doolen, U. Frisch, B. Hasslacher, S. Orszag and S. Wolfram, (Eds.), Lattice Gas Methods for Partial Differential Equations, (Addison-Wesley Publishing Co. 1990), 75.

[30] E. P. Wigner, J. Chem. Phys. 22, 1912 (1954).

[31] H. Hersbach, Complex Syst. 4, 251 (1990).
12 Figure Captions

Fig. 1a The appearance of the four fundamental world velocities $u_A' = L(1)u_A$ in the frame obtained from the special frame by the boost $L(1)$.

Fig. 1b The spacetime evolution of four fluids in the special frame (lower plot), and in the boosted frame.

Fig. 2a. History of edge-on and corner-on collisions of two squares populated by particles, as a function of iteration number $q$. (Detailed description in text.)

Fig. 2b. History of edge-on and corner-on collisions of four squares, populated by particles as a function of iteration number $q$. (Detailed description in text.)

Fig. 3a. Post-collision situations ($q = 80$) in edge-on and corner-on collisions of two squares, populated by particles with varying fractional occupancy, as indicated on the top of each picture. (Detailed description in Section VIII.)

Fig. 3b. Post-collision situations ($q = 80$) in edge-on and corner-on collisions of four squares, populated by particles with varying fractional occupancy, as indicated on the top of each picture. (Detailed description in VIII.)

Fig. 4a. History of collisions of particles situated originally in a rectangle as a function of iteration number $q$. (Detailed description in Section VIII.)

Fig. 4b. Averages of density $[\rho]$, magnitude $[n]$ of particle flow, magnitude $[ne]$ of energy flow, magnitude $[ns]$ of entropy flow, principal pressures $[P_x]$, $[P_y]$. These averages in regions III, II, I, denoted by [•] are restricted to sites in the different regions associated with different phases. (Detailed description in Section VIII.)

Fig. 4c. Average magnitudes for quantities in Fig. 4b, but averaged over all contributing site areas, irrespectively of the particular phase, denoted by $<\cdot>$. (Detailed description in Section VIII.)

Fig. 4d. Average deviation between pairs of flow directions indicated. Deviations $\delta$ are evaluated using the kinetic distribution function $f$; deviations $\Delta$ are evaluated using $F$, the microscopic distribution function averaged. The averages, denoted by $<\cdot>$ are over all contributing site areas with
deviations different from zero. (No such deviations can, of course, in the one fluid phase region. Detailed description in Section VIII.)

Fig. 5a. Graph showing relation between pressures and energy densities at different sites for iteration \( q = 5 \) using function \( f \). (Detailed description in Section VIII.)

Fig. 5b. Showing same, using function \( F \). (Detailed description in Section VIII.)
This figure "fig01.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9712018v1
This figure "fig02.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9712018v1
This figure "fig03.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9712018v1
This figure "fig04.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9712018v1
This figure "fig05.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9712018v1
This figure "fig06.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9712018v1
This figure "fig07.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9712018v1
This figure "fig08.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9712018v1
This figure "fig09.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9712018v1
This figure "fig10.gif" is available in "gif" format from:

http://arxiv.org/ps/physics/9712018v1