On the Complexity and Approximation of $\lambda_\infty$, Spread Constant and Maximum Variance Embedding

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Abstract

Twenty years ago in [BHT00], Bobkov, Houdré, and the last author introduced a
Poincaré-type functional graph parameter, $\lambda_\infty(G)$, of a graph $G$, and related it to the
vertex expansion of $G$ via a Cheeger-type inequality. This is analogous to the Cheeger-
type inequality relating the spectral gap, $\lambda_2(G)$, of the graph to its edge expansion.

While $\lambda_2$ can be computed efficiently, the computational complexity of $\lambda_\infty$ has
remained an open question. Following the work of the second author with Raghavendra
and Vempala [LRV13], wherein the complexity of $\lambda_\infty$ was related to the so-called
Small-Set Expansion (SSE) problem, it has been believed that computing $\lambda_\infty$ is a
hard problem. We settle this question by proving that computing $\lambda_\infty$ is indeed NP-
hard. Additionally, we use our techniques to prove NP-hardness of computing the
spread constant (of a graph), a geometric measure introduced by Alon, Boppana, and
Spencer [ABS98], in the context of deriving an asymptotic isoperimetric inequality on
cartesian products of graphs.

We complement our hardness results by providing approximation schemes for com-
puting $\lambda_\infty$ and the spread constant of star graphs, and investigate constant approx-
imability for weighted trees.

Finally, we provide improved approximation results for the maximum variance em-
bedding problem for general graphs, by replacing the optimal orthogonal projection
(PCA) with a randomized projection approach.

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1 Introduction

1.1 $\lambda_\infty$

One of the fundamental quantities in discrete optimization, spectral graph theory, and the theory of Markov chains is the spectral gap of a graph, denoted by $\lambda_2$. Applying Courant-Fischer-Weyl Theorem, $\lambda_2$ can be characterized as the second smallest eigenvalue of the normalized Laplacian matrix of the graph (see [CG97]), that can be efficiently computed.

One of the motivations to study $\lambda_2$ (in the context of isoperimetry) has been Cheeger-type inequalities [Che69] for graphs [AM85, Alo86, SJ89], that provide estimates for various isoperimetric constants of a graph. Such measures are of particular interest in theory and practice (e.g., see [HLW06, Kho16, ARV09, ZLM13] and references therein), yet are NP-hard to compute.

A decade after the seminal work of Alon [Alo86], that has lead to well known inequalities such as

$$\frac{\lambda_2}{2} \leq \phi^E(G) \leq \sqrt{2\lambda_2},$$

for edge expansion, $\phi^E$, of an undirected graph $G = (V, E)$, Bobkov, Houdré, and the third author [BHT00] introduced a novel Poincaré-type functional parameter, $\lambda_\infty$, and derived new Cheeger-type inequalities, such as

$$\frac{\lambda_\infty}{2} \leq \phi^V(G) \leq 4\lambda_\infty + 4\sqrt{\lambda_\infty},$$

where $\phi^V(G) = \phi^V(G, \pi)$ denotes vertex expansion of the graph for an arbitrary probability measure over the vertex set $\pi$, formally defined as

$$\phi^V(G) \overset{\text{def}}{=} \min_{S \subseteq V} \frac{\pi(N(S) \cup N(V \setminus S))}{\min\{\pi(S), \pi(V \setminus S)\}},$$

where for $S \subseteq V$, $N(S) \overset{\text{def}}{=} \{j \in V \setminus S : \exists i \in S \text{ such that } \{i, j\} \in E\}$.

For an undirected graph $G = (V, E)$ with non-negative edge weights given by $w : E \rightarrow \mathbb{R}_{\geq 0}$, $\lambda_2$ can be defined as

$$\lambda_2 \overset{\text{def}}{=} \inf_{f : V \rightarrow \mathbb{R}} \frac{\mathbb{E}_{v \sim \pi} \left[ \mathbb{E}_{u \sim \sigma_{N(v)}} \left[ |f(v) - f(u)|^2 \right] \right]}{\text{Var}_{\pi}[f]},$$

where $\sigma_{N(v)}$ denotes a transition probability over neighborhood of a vertex $v$ in which a vertex $u \in N(v)$ is sampled with probability proportional to $w(u, v)$, and $\pi$ is the stationary probability distribution over the vertices.

In contrast, Bobkov, Houdré, and the last author [BHT00] defined $\lambda_\infty$ as

$$\lambda_\infty \overset{\text{def}}{=} \inf_{f : V \rightarrow \mathbb{R}} \frac{\mathbb{E}_{v \sim \pi} \left[ \sup_{u \in N(v)} |f(v) - f(u)|^2 \right]}{\text{Var}_{\pi}[f]},$$

where $\pi$ is an arbitrary probability distribution over the vertices.

In general one can convert bounds for vertex and edge expansion at the cost of a degrade by a multiplicative factor of maximum degree of the graph, $\Delta = \Delta(G)$. However, directly
using (an estimate for) $\lambda_\infty$ that can range in $[\lambda_2/\Delta, \lambda_2]$ can yield better bounds on vertex expansion and relevant measures, as pointed out by [BHT00], in the context of refining some estimates from [AM85].

While computing the edge expansion of a graph is hard, good estimations in certain regimes of parameters are provided using Cheeger inequalities and efficient computation of $\lambda_2$. Computing vertex expansion is NP-hard as well, yet it was unknown whether $\lambda_\infty$ could be efficiently computed.

Although efficient computation of $\lambda_\infty$ is plausible as aforementioned, works due to [BHT00, LRV13] imply that $\lambda_\infty$ is SSE-hard (Small-set expansion hypothesis [RS10]) to approximate better than $O(\log \Delta)$ in certain parameter regimes, (refer to [LRV13] for formal statement).

Even though the Small-Set Expansion hypothesis remains unproven, this suggested that the computation of $\lambda_\infty$ is likely to be hard. Nevertheless, the fundamental question of proving NP-hardness of the computation of $\lambda_\infty$ remained unresolved since 2000. We settle this question by proving that computing $\lambda_\infty$ is NP-hard. We refer the reader to Theorem 2 for a formal statement.

1.2 Spread Constant, and Maximum Variance Embedding

We find our techniques likewise useful in providing first NP-hardness and new approximation bounds for a problem that has been discovered multiple times in the literature, due to applications in functional analysis [Fie73, Fie90, Fie93, GHW08], spectral graph theory [GBS08], theory of Markov chains [BDX04, SBXD06], machine learning [WS06b, WS06a] and data visualization [SGBS08].

The common ground for above citations is finding a Lipschitz embedding of vertices of a graph into a Euclidean space, such that the variance of the representation is maximized. The Lipschitzness constraint requires mapping of neighboring vertices, with respect to $G$, to nearby locations in the target space. Embedding into $\mathbb{R}^1$ was first introduced by Alon, Boppana, and Spencer [ABS98] as the spread constant. They showed that isoperimetric properties of Cartesian products of a graph [CT98], in asymptotic fashion, can be well-understood in a wide range by merely bounding the second moment over Lipschitz valuations of the vertices. The spread constant of a graph, $\mathcal{C} (= \mathcal{C}(G, \pi))$, is formally defined as

$$\mathcal{C} \equiv \sup_{f \in \mathcal{L}(G)} \text{Var}_\pi[f] ,$$

where $\mathcal{L}(G)$ denotes the set of Lipschitz functions $f : V \to \mathbb{R}$ with respect to the distance metric defined by $G$, i.e., satisfying $|f(u) - f(v)| \leq 1, \forall \{u, v\} \in E$.

This search for an embedding of maximum (normalized) second moment can be generalized to higher dimensional spaces, having

$$\mathcal{C}_k \equiv \sup_{f \in \mathcal{L}_k^p(G)} \mathbb{E}_{u \sim \pi} [||f(u) - \mathbb{E}_{v \sim \pi} [f(v)] ||^2] ,$$

where $\mathcal{L}_p^k(G)$, for $1 \leq p \leq \infty$, denotes the set of norm-$p$ Lipschitz functions $f : V \to \mathbb{R}^k$ with respect to $G$, i.e., satisfying $||f(u) - f(v)||_p \leq 1, \forall \{u, v\} \in E$.

This objective, namely, the Maximum Variance Embedding (MVE) has also appeared in machine learning literature as a heuristic for non-linear dimension reduction. Introduced
by Weinberger, Sha, and Saul [WSS04] as the Maximum Variance Unfolding (MVU) they showed well-sampled manifolds lying in high-dimensional spaces can be better represented in lower dimensional Euclidean spaces by an “unfolding” caused by the maximization of variance while (local) structure of the manifold is preserved thanks to the Lipschitzness constraint.

Anticipated to be further challenging in lower dimensions, which we confirm by providing first NP-hardness results in Theorem 3 spread constant and maximum variance embedding have been approached by a “lifting” of dimension. For embedding into $\mathbb{R}^n$, i.e., when $k = n$, the problem can be reformulated as a convex semidefinite program (SDP $\mathbb{M}$ from [MT05] and section 3 from [SBXD06] for more details.)

SDP 1.

$$\max_{u,v \sim \pi} \mathbb{E}_{u,v}[\|x_u - x_v\|_2^2] \quad s.t. \|x_u - x_v\|_2^2 \leq 1, \quad \forall \{u,v\} \in E, \quad x_v \in \mathbb{R}^{|V|}, \quad \forall v \in V.$$ 

It is worth mentioning that the SDP relaxation of maximum variance embedding is dual to the absolute algebraic connectivity of the graph, introduced by Fiedler [Fie90, GHW08]. This optimized spectral gap has found connections to the problem of speeding up a Markov chain with the same structure and only re-weighting of the edges, namely fastest mixing Markov process problem [SBXD06].

While the lifted relaxation can be reformulated as a convex program and solved up to arbitrary precision in strongly polynomial time, e.g., using interior point methods [NN94, Nes13], the challenge remains on how to retrieve a low dimensional solution from the lifted relaxation. This SDP is shown to have integrality gap $\Omega(\log n/(\log \log n))$ [Nao14], and has enabled approximation of spread constant up to a multiplicative factor of $O(\log n)$ [MT+06, SBXD06].

Previous studies show lifting is sufficient to attack this problem, e.g., despite non-convexity, all local (second order) optima for a Burer-Monteiro formulation [BM03] of MVU are globally optimal [KSP07]. We complement this by showing lifting is also necessary. For the first time, to the best of our knowledge, we show computing the spread constant, i.e., maximum variance embedding into $\mathbb{R}^1$, is NP-hard (Theorem 3). In particular, the special case of MVU for our gadget, for which we propose approximation schemes using dynamic programming, dismisses hopes for near-optimal performance of first order optimization methods in low dimensions.

On the other hand, we provide new worst-case guarantees for variance of the lowered (dimension) solution, proposing a randomized projection algorithm. Our algorithm can be a candidate alternative to the optimal orthogonal projection method, i.e., principal component analysis, which is the prevalent subroutine succeeding the semidefinite relaxation (e.g., see [WSS04]).

2 Summary of Results

We provide first NP-hardness results for $\lambda_\infty$ and the spread constant.

**Theorem 2.** Given a star graph $G$, and rational probability distribution $\pi$ over vertices, computing $\lambda_\infty(G, \pi)$ to polynomially many bits (to the size of input) is NP-hard.
Theorem 3. Given a star graph $G$, and rational probability distribution $\pi$ over vertices, computing $C(G, \pi)$ to polynomially many bits (to the size of input) is NP-hard.

We conclude the aforementioned hardness results by providing $(1 + \epsilon)$-approximation schemes for both, in Theorem 12 and Proposition 17.

Göring, Helmberg, and Wappler [GHW08] showed $C_n = C_k$, for $k > \text{tree-width of the input graph}$. Their algorithm reduces spanning dimension of the lifted ($n$-dimensional) solution down to width+1 of a valid (assumed given) tree decomposition of the graph, with no loss in terms of variance. In general, tree-width can be as large as $n - 1$ and moreover, computing it is NP-hard [ACP87]. In particular, their study complements our complexity result for stars (and trees), showing their tree-width+1 upper-bound on required lifting is tight.

On the other hand, the algorithm by [GHW08] relies on an SDP subroutine to optimally embed a tree in the Euclidean plane. This becomes a practical challenge not only because of exhaustive runtime but also due to a quadratic storage. We conclude the case for trees by devising a combinatorial algorithm for optimal embedding into plane, i.e., $C_2$, without further lifting, and using only linear storage (see Theorem 20).

Further in application side, PCA is the prevalent subroutine to reduce the dimension of the lifted (SDP) solution [WSS04]. PCA can lose variance linearly with respect to the dimension-shrinkage, e.g., guaranteeing only $\Omega(\frac{1}{\sqrt{n}})$ for one-dimensional embedding, we suggest a randomized rounding that preserves $\Omega(\frac{1}{\log n})$ along a single direction, and suffers negligible loss in $O(\log n)$ dimensions.

Theorem 4. There exists a polynomial time randomized algorithm which approximates $C_k$ to $O((\log n)/k)$ for $k \leq \log n$, and $1 + O\left(\frac{\sqrt{(\log n)}}{k}\right)$ for $k > \log n$. In particular, for $k = \Omega((\log n)/\epsilon^2)$, the approximation factor is $1 + \epsilon$.

3 Overview of Techniques

Discretization. Applying first and second order optimality conditions we searched for discrete characterizations of global optima for the problems under study. While in general this is not feasible for the non-convex, non-smooth, and non-integral problems under study, we achieved the desired discretization for the special case of a star graph (see Proposition 6 and Proposition 15). This paved the way for using the star as our gadget in providing a sound and complete reduction from Integer Partitioning problem at the heart of the NP-complete problem set.

Dynamic Programming for Approximation. Our (near) discrete characterizations of the optimal valuations for $\lambda_\infty$ and $C$, also enabled applying dynamic programming by defining appropriate sub-problems to be approximated.

Trees and Tree-Decomposition. While trees could be optimally embedded in two dimensions, in polynomial time, existing algorithms were initialized by the global optimum of a lifted (SDP) solution [GHW08]. Extending the separator-shadow characterization of the
optimal solution for trees, over the plane, allowed us to provide a combinatorial algorithm to directly solve this problem without any lifting and using linear storage.

Randomized Dimension Reduction. PCA is a prevalent subroutine to reduce the dimension of a lifted solution, as in [WSS04]. In the worst case, this can cause a loss in variance proportional to the reduction of dimension. While PCA is the optimal orthogonal projection with respect to maximizing the variance, we can replace this linear subroutine towards a non-linear dimension reduction objective. Johnson-Lindenstrauss lemma shows that random projections to $O(\log n/\epsilon^2)$ dimensions suffice for bounding $O(\epsilon)$ loss in variance, following which we suggest a randomized projection method with refined worst-case guarantees as in Theorem 4.

4 Notation

Let $n$ and $m$ denote the number of vertices and edges of the graph. We may abuse the notation for $\lambda_\infty(G, \pi)$ when $G$ and $\pi$ are clear from the context, denoting $\lambda_\infty$ as a function from valuation of vertices to the objective value of the optimization problem, i.e., $\lambda_\infty : \mathbb{R}^V \to \mathbb{R}$, $x \mapsto \mathbb{E}_{v \sim \pi} \left[ \max_{u \in N(v)} |x_u - x_v|^2 / \text{Var}_\pi[x] \right]$.

Similarly, when $G$ and $\pi$ are known, we denote by $\mathcal{C}$ and $\mathcal{C}_k$ as functions from $\mathbb{R}$ and $\mathbb{R}^k$ (corresponding to the embedding of vertices) to the (Euclidean) variance of the embedding, in case the embedding is valid (Lipschitz) and 0 otherwise.

$$\mathcal{C}_k(x) = \begin{cases} \text{Var}_\pi[x], & \|x_u - x_v\| \leq 1 \quad \forall(u, v) \in E \\ 0, & \text{otherwise} \end{cases}$$

Let us realize the vertices as points of mass, proportional to $\pi$, connected by ropes of unit length if there is an edge in between, corresponding to Lipschitzness constraints. The objective for MVU is maximizing the (weighted) second moment of the Euclidean distances from the barycenter of the embedding, i.e., $\mathbb{E}_{v \sim \pi}[y(v)]$.

The indicator function $1[\text{condition}]$ denotes whether the condition is satisfied, with $1[\text{True}] = 1$ and $1[\text{False}] = 0$. Sign function can be defined as

$$\text{sign} : \alpha \mapsto 1[\alpha > 0] - 1[\alpha < 0] \quad \forall \alpha \in \mathbb{R}.$$ 

Vectors of all ones and all zeros, where the dimension is clear from context, are denoted by $\mathbf{0}$ and $\mathbf{1}$.

Convex hull of a subset $S \subseteq \mathbb{R}^k$ is denoted by

$$\text{conv}(S) \overset{\text{def}}{=} \{ \alpha x + (1 - \alpha)y : x, y \in S, 0 \leq \alpha \leq 1 \}.$$ 

$S_n$ denotes the star graph with $n$ vertices, with $0 \in V$ as the central vertex, connected to all other (leaf) vertices $[n - 1]$. The edge set for $S_n$ would be $E_{S_n} = \{ \{0, i\} | i \in [n - 1] \}.$
5 \( \lambda_\infty \) and the Star Gadget

It is easy to show the infimum in the definition of \( \lambda_\infty \) is bounded and feasible for any connected graph; see Lemma 23 in the appendix. Therefore, \( \lambda_\infty \) is the optimal solution to the following optimization problem:

\[
\lambda_\infty = \min_{x: V \to \mathbb{R}} \mathbb{E}_{v \sim \pi} \left[ \max_{u \in N(v)} |x_u - x_v|^2 \right] \quad \text{s.t.} \quad \text{Var}_\pi [x] \geq 1.
\] (1)

5.1 Computing \( \lambda_\infty \) is NP-hard for Weighted Stars

From equation 1 we have

\[
\lambda_\infty(S_n, \pi) = \min_{x \in \mathbb{R}^V \setminus \{0\}} \frac{\mathbb{E}_{v \sim \pi} \left[ \max_{u \in N(v)} |x_u - x_v|^2 \right]}{\text{Var}_\pi [x]}
\]

uniform shift invariance

\[
= \min_{x \in \mathbb{R}^V \setminus \{0\}, x_0 = 0} \frac{\mathbb{E}_{v \sim \pi} \left[ \max_{u \in N(v)} |x_u - x_v|^2 \right]}{\text{Var}_\pi [x]}
\]

star structure

\[
= \min_{x \in \mathbb{R}^V \setminus \{0\}, x_0 = 0} \frac{\pi_0 \cdot \max_{i \in [n-1]} x_i^2 + \sum_{i \in [n]} \pi_i x_i^2}{\text{Var}_\pi [x]}
\]

\[
= \min_{x \in \mathbb{R}^V \setminus \{0\}, x_0 = 0} \frac{\pi_0 \cdot \max_{i \in [n-1]} x_i^2 + \mathbb{E}_{i \sim \pi} [x_i^2]}{\text{Var}_\pi [x] - (\mathbb{E}_{i \sim \pi} [x_i^2])^2}.
\] (2)

**Fact 5.** Given positive numbers \( a, b, c, d \in \mathbb{R} \), and \( \frac{a}{b} < \frac{c}{d} \), we have \( \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} \), i.e., mediant of two (positive) ratios falls in between. Consequently, \( \frac{c}{f} < \frac{c-g}{f-g} \) for \( e > f > g > 0 \).

What follows now is a useful characterization for optimal valuation of vertices of a star.

**Proposition 6.** Given star graph \( S_n \) and optimal valuation \( y: V \to \mathbb{R} \) for \( \lambda_\infty \) as

\[
y = \arg\min_{x \in \mathbb{R}^V \setminus \{0\}, x_0 = 0} \frac{\pi_0 \cdot \max_{i \in [n-1]} x_i^2 + \mathbb{E}_{i \sim \pi} [x_i^2]}{\text{Var}_\pi [x] - (\mathbb{E}_{i \sim \pi} [x_i^2])^2},
\] (3)

there exist at most one vertex \( i \neq 0 \) with \( |y_i| \neq \max_k |y_k| \).

**Proof.** Note that \( y_0 = 0 \). We prove the claim by contradiction, assuming \( \exists i \neq j \neq 0 \) for which

\[
y_i, y_j \in (-\max_k |y_k|, \max_k |y_k|).
\] (4)

We find a contradiction by showing \( \lambda_\infty(\cdot) \) can be further decreased, slightly moving from \( y \) along the direction \( v = \pi_i e_j - \pi_j e_i \), i.e., adding to one and decreasing another, while keeping the expectation intact. Namely for \( y' = y + \delta v \), we have

\[
\mathbb{E}_{k \sim \pi} [y'_k] = \sum_k \pi_k y'_k = \left( \sum_k \pi_k y_k \right) - \pi_i \pi_j \delta + \pi_j \pi_i \delta = \mathbb{E}_{k \sim \pi} [y_k].
\]
The assumption by equation \ref{eq:4} guarantees that for sufficiently small \( \delta \), we have

\[
\max_k y_k^2 = \max_k y_k'\).
\]

The only term in formulation of \( \lambda_\infty(\cdot) \) as of equation \ref{eq:2} that is affected by changing \( y \) to \( y' \) is in the second moment \( \gamma \triangleq \mathbb{E}_{k \sim \pi} [y_k^2] - \mathbb{E}_{k \sim \pi} [y_k']^2 \), for which we have

\[
\begin{align*}
\mathbb{E}_{k \sim \pi} [y_k^2] &= \mathbb{E}_{k \sim \pi} [y_k']^2 + \pi_i(-2y_i\pi_j\delta + (\pi_j\delta)^2) + \pi_j(2y_j\pi_i\delta + (\pi_i\delta)^2) \\
&= \mathbb{E}_{k \sim \pi} [y_k']^2 + 2\pi_i\pi_j(y_j - y_i) + \delta^2(\pi_i\pi_j^2 + \pi_j\pi_i^2) \\
&= \mathbb{E}_{k \sim \pi} [y_k']^2 + \gamma.
\end{align*}
\]

We can easily assure a \( \gamma > 0 \) by setting \( \delta = \epsilon(\text{sign}(y_j - y_i) + 1[y_j - y_i = 0]) \), for some small \( \epsilon > 0 \).

Computing \( \lambda_\infty \) at \( y' \) we have

\[
\begin{align*}
\lambda_\infty(y') &= \frac{\pi_0 \cdot \max_i \mathbb{E}_{k \sim \pi} [y_i^2] + \mathbb{E}_{k \sim \pi} [y_i']^2}{\mathbb{E}_{k \sim \pi} [y_i^2] - (\mathbb{E}_{k \sim \pi} [y_i'])^2} \\
&= \frac{\pi_0 \cdot \max_i \mathbb{E}_{k \sim \pi} [y_i^2] + \mathbb{E}_{k \sim \pi} [y_i']^2 + \gamma}{\mathbb{E}_{k \sim \pi} [y_i^2] - (\mathbb{E}_{k \sim \pi} [y_i'])^2 + \gamma} \\
&\in \left( \frac{\gamma}{\gamma}, \frac{\pi_0 \cdot \max_i \mathbb{E}_{k \sim \pi} [y_i^2] + \mathbb{E}_{k \sim \pi} [y_i']^2}{\mathbb{E}_{k \sim \pi} [y_i^2] - (\mathbb{E}_{k \sim \pi} [y_i'])^2} \right) = (1, \lambda_\infty(y)) \ . \quad \text{Fact \ref{fact:5}}
\end{align*}
\]

where we applied Fact \ref{fact:5} for \( a = b = \gamma \), \( c = \pi_0 \cdot \max_i \mathbb{E}_{k \sim \pi} [y_i^2] + \mathbb{E}_{k \sim \pi} [y_i']^2 \) and \( d = \mathbb{E}_{k \sim \pi} [y_i^2] - (\mathbb{E}_{k \sim \pi} [y_i'])^2 \). Equation \ref{eq:6} shows \( \lambda_\infty(y') \) is a median of two fractions \( c/d > 1 \), the last line provides the desired contradiction \( \lambda_\infty(y') < \lambda_\infty(y) \).

**Definition 7.** For a star graph, we call a valuation \( x : V \to \mathbb{R} \), binary, if all leaves are assigned numbers at the same absolute distance from the value assigned to the root, i.e.,

\[
|x_k - x_0| = \max_i |x_i - x_0|, \quad \forall k \in [n - 1],
\]

and call it balanced around vertex \( v \), if its first order moment with respect to \( x_v \) is zero, i.e.,

\[
\sum_{k \in V} \pi_k (x_k - x_v) = 0 \iff \mathbb{E}_{u \sim \pi} [x_u] = x_v.
\]

**Proposition 8.** For any star graph \( S_n \), probability distribution \( \pi : V \to [0, 1] \), and assignment \( x \in \mathbb{R}^V \setminus \{0\} \) we have \( \lambda_\infty(x) \geq \frac{1}{1 - \pi_0} \) and a valuation \( x \) achieves this lower bound if and only if it is (i) binary, and (ii) balanced around the root.

**Proof.** Considering a global optimal assignment \( y \), without loss of generality, we add the following sequence of assumptions

\[
y_0 = 0 \quad \text{uniform-shift invariance} \]
\[
\max_k |y_k| = 1 \quad \text{scale invariance} \]
\[
\exists i \in [n - 1] \forall k \neq i \ |y_k| = 1, \quad \text{for some Proposition \ref{prop:6}}
\]
Let \( \pi_- = \sum_{k \neq i} \pi_k \cdot 1[y_k = -1] \) and \( \pi_+ = \sum_{k \neq i} \pi_k \cdot 1[y_k = +1] \). We need to show

\[
\lambda_\infty = \frac{\pi_- \cdot (-1)^2 + \pi_+ \cdot (1)^2 + \pi_i y_i^2 + \pi_0 \cdot (1)^2}{\pi_- \cdot (-1)^2 + \pi_+ \cdot (1)^2 + \pi_i y_i^2 - (\pi_- \cdot (-1) + \pi_+ \cdot 1 + \pi_i y_i)^2} \geq \frac{1}{1 - \pi_0},
\]

and the equality holds if and only if the valuation is binary and balanced, in other words, \( y_i^2 = 1 \), and \( \pi_+ \cdot 1 + \pi_i \cdot y_i + \pi_- \cdot (-1) = 0 \).

We show equivalence between the target inequality \( \lambda_\infty \geq \frac{1}{1 - \pi_0} \) and inequality

\[
\pi_0 \pi_i (1 - y_i^2) + (\pi_+ - \pi_- + \pi_i y_i)^2 \geq 0,
\]

that is trivial due to the fact that \( y_i \in [0, 1] \). Re-writing the latter inequality as

\[
\pi_0 \pi_i - \pi_0 \pi_i y_i^2 + \pi_+^2 + \pi_-^2 + \pi_i^2 y_i^2 - 2 \pi_+ \pi_- - 2 \pi_- \pi_i y_i + 2 \pi_+ \pi_i y_i \geq 0
\]

and applying \( \pi_i = 1 - \pi_+ - \pi_- - \pi_0 \), we get

\[
\pi_0 (1 - \pi_+ - \pi_- - \pi_0) - \pi_0 \pi_i y_i^2 + \pi_+^2 + \pi_-^2 + \pi_i^2 y_i^2 - 2 \pi_+ \pi_- - 2 \pi_- \pi_i y_i + 2 \pi_+ \pi_i y_i \geq 0
\]

Rearranging and adding terms to both sides we get

\[
\pi_+ + \pi_- + \pi_i y_i^2 + \pi_0 - \pi_0 \pi_+ - \pi_0 \pi_- - \pi_0 \pi_i y_i^2 - \pi_0^2 \geq \\
\pi_+ + \pi_- + \pi_i y_i^2 - \pi_+^2 - \pi_-^2 - \pi_i^2 y_i^2 - 2 \pi_+ \pi_i y_i + 2 \pi_+ \pi_- + 2 \pi_+ \pi_i y_i
\]

Factorizing this expression, we get

\[
(\pi_+ + \pi_- + \pi_i y_i^2 + \pi_0) \cdot (1 - \pi_0) \geq (\pi_+ + \pi_- + \pi_i y_i^2) - (\pi_+ - \pi_- + \pi_i y_i)^2.
\]

A final rearrangement give us

\[
\lambda_\infty = \frac{\pi_+ + \pi_- + \pi_i y_i^2 + \pi_0}{(\pi_+ + \pi_- + \pi_i y_i^2) - (\pi_+ - \pi_- + \pi_i y_i)^2} \geq \frac{1}{1 - \pi_0},
\]

where derivations (holding in both directions) also hold in equality form, and so does for strict inequality. We also implicitly used the fact that \( \pi_i \in (0, 1), \forall i \) and that \( \text{Var}_\pi [x] > 0 \), particularly assuring the denominators are positive in the last inequality. \( \square \)

We reduce the following NP-complete problem to a decision form of computing \( \lambda_\infty \).

**Problem 9 (Integer Partition Problem).** Given positive integers \( p_1, \cdots, p_n \) we are to decide whether they can be partitioned into two subsets of equal sum.

**Lemma 10.** For an instance \( P = \{p_j \in \mathbb{N} : j \in [n-1]\} \) of Partition problem, and some \( \beta > 1 \), considering the following probability measure over vertices of a star graph \( S_n \),

\[
\pi_j = \begin{cases} 
\frac{\beta-1}{\beta^j}, & j = 0 \\
\frac{1}{\beta} \cdot \frac{p_j}{\sum_k p_k}, & j \geq 1
\end{cases}
\]

\[
\lambda_\infty = \beta \text{ if and only if } P \text{ can be Partitioned, otherwise } \lambda_\infty \geq \beta + \Omega \left( \frac{\beta-1}{\beta (\sum_k p_k)^2} \right).
\]
Proof. Applying Proposition \[8\] for the star, we have $\lambda_\infty = \beta$ for some normalized optimal valuation $y \in \mathbb{R}^V$, if and only if $y_k = \pm 1$ for every leaf $k$ and it is balanced w.r.t. $\pi$, i.e.,

$$
\sum_k \pi_k \cdot 1[y_k > 0] - \sum_k \pi_k \cdot 1[y_k < 0] = 0 \iff \sum_k p_k \cdot 1[y_k > 0] = \sum_k p_k \cdot 1[y_k < 0],
$$

which is proof of a YES answer to the original Partition problem with corresponding parts \{ $k | y_k < 0$ \} and \{ $k | y_k > 0$ \}.

The argument is clearly reversible and given a valid Partition, i.e., $S \subseteq [n - 1]$ where

$$
\sum_{j \in S} p_j = \sum_{j \notin S} p_j,
$$

the following assignment for $y$ (upper) bounds $\lambda_\infty$ by $\frac{1}{1 - \pi_0} = \beta$.

$$
y_k = \begin{cases} 
0, & k = 0 \\
1, & k \in S \\
-1, & \text{otherwise}
\end{cases}
$$

Considering the No Partition case, let $b^*$ be the optimal balance (minimum difference of sums) of two parts over all partitions of probabilities corresponding to leaves \{ $\pi_j : j \geq 1$ \}. Since the $p_j$’s are integers, we have that the optimal balance among them is at least 1, and therefore,

$$
b^* \geq \frac{1}{\beta} \cdot \frac{1}{\sum_j p_j}. \quad (7)
$$

Let $y$ be the vector corresponding to $\lambda_\infty$ of this star instance. As before, we will assume that $\max_k |y_k| = 1$, $y_0 = 0$, and let $i \in [n - 1]$ be the only (if any) index such that $|y_i| < 1$ (otherwise any index). By symmetry we can also assume $y_i \geq 0$. We are going to lower bound

$$
\lambda_\infty = \frac{\pi_+ + \pi_- + \pi_i y_i^2 + \pi_0}{(\pi_+ + \pi_- + \pi_i y_i^2) - (\pi_+ - \pi_- + \pi_i y_i)^2} = \frac{1 - \pi_i(1 - y_i^2)}{1 - \pi_0 - \pi_i(1 - y_i^2) - (-\pi_- + \pi_+ + \pi_i y_i)^2} \quad (\pi \cdot 1 = 1). \quad (8)
$$

Let $\kappa \in (0, \frac{1}{2})$ be a constant to be fixed later, and let $\epsilon = \kappa b^*$. 

9
Case 1  \[0 \leq y_i^2 < 1 - \epsilon.\] Continuing from equation [8]

\[
\lambda_\infty \geq \frac{1 - \pi_i(1 - y_i^2)}{1 - \pi_0 - \pi_i(1 - y_i^2)}
\]

neglecting second term in denominator

\[
> \frac{1 - \pi_i \epsilon}{1 - \pi_0 - \pi_i \epsilon}
\]

\[
= \frac{1}{1 - \pi_0} + \frac{\pi_0 \pi_i \epsilon}{(1 - \pi_0)(1 - \pi_0 - \pi_i \epsilon)}
\]

\[
\geq \frac{1}{1 - \pi_0} + \frac{\pi_0 \pi_i \epsilon}{(1 - \pi_0)(1 - \pi_0)}
\]

\[
= \beta + \beta(\beta - 1) \pi_i \epsilon
\]

\[
= \beta + \beta(\beta - 1) \pi_i \kappa b^*
\]

\[
\geq \beta + (\beta - 1) \frac{\pi_i \kappa}{\sum_i p_i}
\]

\[
\geq \beta + \frac{\beta - 1}{\beta} \cdot \frac{\kappa}{(\sum_i p_i)^2}
\]

\[
\pi_i \geq \frac{1}{\beta \sum_i p_i}
\]

Case 2  \[1 \geq y_i^2 \geq 1 - \epsilon \Rightarrow |y_i| = y_i > 1 - \epsilon\] we similarly have

\[
\lambda_\infty = \frac{1 - \pi_i(1 - y_i^2)}{1 - \pi_0 - \pi_i(1 - y_i^2) - (1 - \pi_0 + \pi_+ + \pi_i y_i)^2}
\]

\[
> \frac{1}{1 - \pi_0} \quad \text{equation [8]}
\]

\[
= \beta + \beta(-\pi_- + \pi_+ + \pi_i) - (1 - y_i)\pi_i)^2
\]

\[
\geq \beta + |\beta| - \pi_- + \pi_+ + \pi_i |(1 - \pi_0 + \pi_+ + \pi_i| - 2|1 - \pi_i y_i|)
\]

\[
\geq \beta + \beta b^*(b^* - 2\epsilon)
\]

\[
= \beta + \beta b^*(1 - 2\kappa)
\]

\[
\geq \beta + \frac{1}{\beta} \cdot \frac{(1 - 2\kappa)}{(\sum_i p_i)^2}.
\]

In inequality above, denoted by *, we applied \(|-\pi_- + \pi_+ + \pi_i| \geq b^*\) and further lower-bounded \(|-\pi_- + \pi_+ + \pi_i| - 2|1 - y_i)\pi_i| > 0\), considering \(|-\pi_- + \pi_+ + \pi_i| \geq b^*\) and \(2|1 - y_i)\pi_i| < 2\epsilon \pi_i \leq 2\epsilon < b^*\).

Substituting \(\kappa = 1/3\), we would have

\[
\lambda_\infty \geq \beta + \min \left\{ \frac{\beta - 1}{\beta} \cdot \frac{1}{3(\sum_i p_i)^2}, \frac{1}{\beta} \cdot \frac{1}{3(\sum_i p_i)^2} \right\}.
\]

\[\square\]

We are now ready to prove Theorem 2 using the following theorem.
Theorem 11. For any constant $\beta \in \mathbb{Q}_{>1}$, given a star graph $G$, and distribution $\pi$, verifying $\lambda_\infty(G) \leq \beta$ up to polynomially many bits is NP-hard.

Proof. Given a decision oracle for $\lambda_\infty \leq \beta$, Lemma 10 immediately provides a polynomial time reduction from Partition, which is NP-hard [Kar72].

To decide the Partition for $P = \{p_j : j \in [n-1]\}$ define a star $S_n$, with vertex set $V = \{0, \cdots, n\}$, and $E = \{(0, j)|j \in [n-1]\}$, and a probability measure $\pi : 2^V \rightarrow [0, 1]$ as follows, denoting $\pi(\{j\})$ by $\pi_j$.

$$\pi_j = \begin{cases} 
\frac{\beta - 1}{\beta}, & j = 0 \\
\frac{1}{\beta \cdot \sum_{k \neq j} p_k}, & j \geq 1
\end{cases}$$

To respond to the query, we forward the answer from the oracle on whether $\lambda_\infty(S_n, \pi) \leq \beta$. Note that this can be verified by checking a polynomially deep (with respect to length of the input) digit of the answer, considering the increase in $\lambda_\infty$ due to No-Partition case of Lemma 10. Specifically, assuming a base-2 Turing machine as the computational model, lemma 10 ensures $\lambda_\infty > \beta + \Omega((\beta - 1)/(\beta(\sum_i p_i)^2))$ that clearly increases a bit no deeper than $O(\log \sum_i p_i)$ that is polynomially bounded by the length of binary representation of the partition problem, i.e., $O(\sum_i \log p_i)$. \hfill $\square$

5.2 $1 + \epsilon$ approximability of $\lambda_\infty$ for a Star

We provide a fully polynomial time approximation scheme (FPTAS) for $\lambda_\infty$ of a star, complementing its NP-hardness.

Theorem 12. For $\epsilon \in (0, \min(0.1, \min_i \pi_i))$, there exists a poly($n, \epsilon^{-1}$) time algorithm computing a $(1 + \epsilon)$-approximation for $\lambda_\infty$ on star graphs.

Proof. Searching for a (near) optimal $y$, we can bound the search space using the following assumptions.

$$y_0 = 0 \quad \text{max} \ |y_k| = 1 \quad \text{uniform-shift invariance}$$

$$\exists i \in [n-1], \ |y_k| = 1, \forall k \neq i, y_i \in [-1, 1] \quad \text{scale invariance}$$

$$\text{by Proposition 6}$$

Without loss of generality we can assume the only (possibly) non-extreme leaf $i$ is known, suffering a multiplicative factor of $O(n)$ in the runtime. The decision to be made is on the value for $y_i \in [-1, 1]$ along partitioning of the remaining leaves (i.e., $y_k \in \{-1, +1\}$ for $0 < k \neq i$).

Namely for $\pi_- = \sum_{k \neq i} \pi_k \cdot 1[y_k = -1]$ and $\pi_+ = \sum_{k \neq i} \pi_k \cdot 1[y_k = +1]$, the objective is to minimize

$$\lambda_\infty = \min_{\pi_- \pi_+, y} \frac{\pi_- \cdot (-1)^2 + \pi_+ \cdot (1)^2 + \pi_i y_i^2 + \pi_0 \cdot (1)^2}{\pi_- \cdot (-1)^2 + \pi_+ \cdot (1)^2 + \pi_i y_i^2 - (\pi_- \cdot (-1) + \pi_+ \cdot 1 + \pi_i y_i)^2}$$

$$= \min_{\pi_- \pi_+ y_i} \frac{\pi_- + \pi_+ + \pi_i y_i^2 + \pi_0}{\pi_- + \pi_+ + \pi_i y_i^2 - (\pi_- + \pi_+ + \pi_i y_i)^2}, \quad \text{(9)}$$
Lemma 13. Given \( \lambda_\infty(\cdot) \) in a dense enough lattice over the search space for valid pairs of \(-\pi_- + \pi_+\) and \(y_i\), we have
\[
\frac{\pi_- + \pi_+ + \pi_i y'_i^2 + \pi_0}{\pi_- + \pi_+ + \pi_i y'_i^2 - (d + \pi_i y'_i)^2} \leq \frac{\pi_- + \pi_+ + \pi_i y_i^2 + \pi_0}{\pi_- + \pi_+ + \pi_i y_i^2 - (-\pi_- + \pi_+ + \pi_i y_i)^2 (1 + \epsilon)}.
\]

Proof. Since \( y'_i \leq y_i \) for each \( i \in [n-1] \), we have
\[
\pi_- + \pi_+ + \pi_i y'_i^2 + \pi_0 \leq \pi_- + \pi_+ + \pi_i y_i^2 + \pi_0.
\]

Therefore, the numerator of \( \lambda_\infty \) expression for \( y' \) (LHS above) can be upper bounded by the numerator of the \( \lambda_\infty \) expression for \( y \) (in the RHS.) Now we lower bound the denominator of the \( \lambda_\infty \) expression for \( y' \).

We first observe that
\[
y_i^2 \geq y'_i^2 - 2y_i \epsilon^2 / 100,
\]
which can be verified considering two cases; whether \( y_i \geq \epsilon^2 / 100 \) or not. In the first case
\[
y_i^2 \geq \max(0, y_i - \epsilon^2 / 100)^2 = (y_i - \epsilon^2 / 100)^2 \geq y_i^2 - 2y_i \epsilon^2 / 100.
\]

In the other case, \( 0 \leq y'_i \leq y_i \leq \epsilon^2 / 100 \), hence we have \( y_i^2 \geq 0 \geq y_i(y_i - 2\epsilon^2 / 100) = y_i^2 - 2y_i \epsilon^2 / 100 \).

Next, we can show
\[
(d + \pi_i y'_i)^2 \leq (| - \pi_- + \pi_+ + \pi_i y_i | + 2\epsilon^2 / 100)^2.
\]

This is due to the fact that given \(|a - a'| \leq \alpha \) and \(|b - b'| \leq \beta \) we have \((a + b)^2 \leq (|a' + b'| + \alpha + \beta)^2\), which we applied for \( a = d, a' = -\pi_- + \pi_+, \alpha = \beta = \epsilon^2 / 100, b = \pi_i y'_i, b' = \pi_i y_i \).
\[ D' \stackrel{\text{def}}{=} \pi_- + \pi_+ + \pi_i y_i^2 - (d + \pi_i y_i)^2 \\]
\[ \geq \pi_- + \pi_+ + \pi_i (y_i^2 - 2y_i \epsilon^2/100) - \left( | - \pi_- + \pi_+ + \pi_i y_i| + \frac{\epsilon^2}{50} \right)^2 \]
\[ \geq \left( \pi_- + \pi_+ + \pi_i y_i^2 - \frac{\epsilon^2}{50} \right) - (-\pi_- + \pi_+ + \pi_i y_i)^2 - \frac{\epsilon^2}{50} \left( 2| - \pi_- + \pi_+ + \pi_i y_i| + \frac{\epsilon^2}{50} \right) \]
\[ \geq (\pi_- + \pi_+ + \pi_i y_i^2 - (-\pi_- + \pi_+ + \pi_i y_i)^2) - \epsilon^2/10 \]
\[ = D - \epsilon^2/10. \]

Note that inequality 13 applied inequalities 11 and 12. Inequalities 14 and 15 apply trivial bounds \( \pi_i, y_i \in [0, 1] \) and \( \epsilon \in (0, 0.1) \).

Finally we have the desired
\[ (1 + \epsilon)D' \geq (1 + \epsilon)(D - \epsilon^2/10) \]
\[ \geq D + \epsilon D - \epsilon^2/10 - \epsilon^3/10 \]
\[ \geq D + \epsilon(D - \frac{1}{10} \epsilon(1 - \epsilon)) \]
\[ \geq D + \epsilon \cdot 0 \quad \text{inequality 10} \]

Algorithm 1 concludes this section with a dynamic programming solution. Enumerating over all choices for the special index \( i \), feasible balances of the remaining leaves are computed using dynamic programming over a dense quantization of the values, i.e., \( dp[\cdot] \) is True for entries corresponding to a valid \( \pi_- + \pi_+ \).

6 NP-Hardness of Spread Constant and MVU

Alon, Boppana, and Spencer [ABS98] showed there exist an integral optimal embedding \( y \in \mathbb{R}^V \) for the spread constant, with the following property. They showed there exists a set \( U \) of vertices and an assignment of sign \( s(C) \in \{ \pm 1 \} \) to every connected component in \( V \setminus U \), such that for every vertex \( v \) in \( C \),
\[ y_v = d_G(U, v) \cdot s(C) \]
and \( y_u \) being zero for \( u \in U \). Noticeably, this characterization facilitates an \( O(2^{O(n)} \cdot \text{poly}(n)) \) time algorithm to compute the spread constant. In the following lemma we show for star graphs, \( U \) can be chosen as a singleton set, while this need not hold in general (for other graphs).

Lemma 14. Given a star graph \( S_n = (V, E) \) and distribution \( \pi : V \to [0, 1] \), for any optimal assignment \( y \) achieving the spread constant (i.e., a Lipschitz \( y \) maximizing \( \text{Var}_\pi[y] \)) all leaf values are at unit distance from the root, i.e.,
\[ |y_j - y_0| = 1, \forall j \in [n - 1]. \]
Require: Star graph $G = (V, E), \pi \in \Delta^n, \epsilon \in (0, \min(0.1, \min_i \pi_i))$
Ensure: $\lambda \in \lambda_\infty(G) \cdot [1, 1 + \epsilon]$

$\lambda \leftarrow \infty$
$\pi' \leftarrow \left\lfloor \frac{100n \pi}{\epsilon^2} \right\rfloor \frac{\epsilon^2}{100n}$
for $i \in [n - 1]$
do
\hspace{1em} $dp \leftarrow$ all False for $-1: \frac{\epsilon^2}{100n} : 1$
\hspace{1em} $dp[0] \leftarrow$ True
\hspace{1em} for $j \in [n - 1] - \{i\}$
do
\hspace{2em} $dp[\cdot] = dp[\cdot] \cdot \delta[\cdot] - \pi'_j \vee dp[\cdot] \cdot \delta[\cdot] + \pi'_j$
\endfor
\hspace{1em} for $(d, y'_i)$ where $dp[d] =$ True and $y'_i \in 0: \frac{\epsilon^2}{100n} : 1$
do
\hspace{2em} $\lambda \leftarrow \min(\lambda, \frac{\pi_- + \pi_+ + \pi_i y'_i^2 + \pi_0}{\pi_- + \pi_+ + \pi_i y'_i^2 - (d + \pi_i y'_i)^2})$ \{where $\pi_- + \pi_+ = 1 - \pi_0 - \pi_i$\}
\endfor
\endfor
return $\lambda$

Algorithm 1: Approximating $\lambda_\infty$ of a weighted star

Proof. First, applying similar ideas as in proof of Proposition 6, we show the lemma holds for almost all of the leaves, i.e., except possibly one.

Translating $y$ uniformly, i.e., along $e = 1$, preserves its Lipschitzness and variance. Therefore, without loss of generality, we assume $y_0 = 0$. Proving by contradiction, consider leaves $i,j$ with $y_i, y_j \in (-1, 1)$. We can move $y(\cdot)$ by a sufficiently small non-zero amount along $v = \pi_i e_j - \pi_j e_i$, in either direction, preserving its Lipschitzness. Let $y' = y + \delta v$, for some small $\delta \in \mathbb{R}$. The change in variance can be derived as

$$\text{Var}_{\pi'}[y'] = \mathbb{E}_{\pi'}[y'^2] - \mathbb{E}_{\pi'}[y']^2$$
$$= \mathbb{E}_{\pi'}[y'^2] - \mathbb{E}_{\pi'}[y]^2$$
$$= \mathbb{E}_{\pi'}[y'^2] - \mathbb{E}_{\pi'}[y]^2 + \delta^2 \pi_i \pi_j^2 - 2y_i \delta \pi_i \pi_j + \delta^2 \pi_i^2 \pi_j + 2y_j \delta \pi_i \pi_j$$
$$= \text{Var}_{\pi'}[y] + \delta^2 \pi_i \pi_j^2 + 2y_i \delta \pi_i \pi_j + \delta^2 \pi_i^2 \pi_j + 2y_j \delta \pi_i \pi_j.$$ 

Now we can easily see if the gradient along $v$ is not zero, i.e., $-2y_i \pi_i \pi_j + 2y_j \pi_i \pi_j \neq 0$, the variance is changed by $\delta(-2y_i \pi_i \pi_j + 2y_j \pi_i \pi_j) + O(\delta^2)$ that is a positive number with the correct sign for $\delta$. On the other hand, for zero gradient, the positive curvature along $v$ makes the variance to increase either way, by $\delta^2 \pi_i \pi_j^2 + 2\delta^2 \pi_i^2 \pi_j > 0$, giving the desired contradiction. Therefore, we assume that there exists an index $i \in [n - 1]$ such that, $y_k \in \{-1, 1\} \forall k \neq i$ and $y_i \in [-1, 1]$. As before, define $\pi_- = \sum_{k \neq i} \pi_k \cdot 1[y_k = -1]$ and $\pi_+ = \sum_{k \neq i} \pi_k \cdot 1[y_k = +1]$. Without loss of generality, assume $\pi_- \geq \pi_+$. A simplifying observation at this point is that we should have $y_i > 0$, otherwise it can be easily checked negating $y_i$ increases the variance.

The final step towards characterization of our optimal solution is achieved due to first
order optimality condition, which must hold unless we’re at the boundary $y_i \in \{0, 1\}$,

$$0 = \frac{\partial \text{Var}_\pi [y]}{\partial y_i} = \frac{\partial}{\partial y_i} \sum_{j < k} \pi_j \pi_k (y_j - y_k)^2$$

$$= \frac{\partial}{\partial y_i} (\pi_+ \pi_i (1 + y_i)^2 + \pi_0 \pi_i y_i^2 + \pi_+ \pi_i (1 - y_i)^2)$$

$$= 2(\pi_+ \pi_i (1 + y_i) + \pi_0 \pi_i y_i - \pi_+ \pi_i (1 - y_i))$$

The last linear equation gives $y_i = \frac{\pi_+ \pi_i (1 + y_i)}{\pi_+ \pi_i + \pi_0} < 0$, which contradicts $y_i > 0$, hence $y_i \in \{0, 1\}$. Inspecting the remaining two cases and considering $\pi_- \geq \pi_+$, one can see that the variance achieved for $y_i = 1$ is not lower than that of $y_i = 0$.

Now we are ready for the counterpart of Proposition 8 for the spread constant.

**Proposition 15.** For any star graph $S_n$ and a probability distribution $\pi$ over its vertices, $C \leq 1 - \pi_0$. Moreover, an assignment $y$ achieves this upper bound if and only if it is (i) binary (with unit distance) and (ii) balanced around the root.

**Proof.** Lemma 14 resolves necessity of the first condition, allowing us to consider an optimal $y$ with

$$y_0 = 0$$

uniform-shift invariance

$$y_i \in \{-1, 1\}, \forall i \in [n - 1].$$

Redefining $\pi_- = \sum_i \pi_i \cdot 1[y_i = -1]$ and $\pi_+ = \sum_i \pi_i \cdot 1[y_i = +1]$, the spread constant is

$$C = \text{Var}_\pi [y] = \sum_{j < k} \pi_j \pi_k (y_j - y_k)^2$$

$$= 4\pi_- \pi_+ + (\pi_- + \pi_+)\pi_0$$

$$= 4\pi_- \pi_+ + (1 - \pi_0)\pi_0$$

$$\leq 4 \left(\frac{\pi_- + \pi_+}{2}\right)^2 + (1 - \pi_0)\pi_0$$

$$= (1 - \pi_0)^2 + (1 - \pi_0)\pi_0$$

$$= (1 - \pi_0)(1 - \pi_0 + \pi_0)$$

$$= 1 - \pi_0$$

We showed $C \leq 1 - \pi_0$ holding with equality if and only if Inequality (17) holds with equality, which happens only when $\pi_- = \pi_+$. □

**Proof sketch for Theorem 3.** Proposition 15 enables straightforward reduction (similar to and easier than that of $\lambda_\infty$) from Problem 9 i.e., Integer Partition. Defining $\pi_0 = 1 - \beta, \pi_i = \beta \frac{\pi_i}{\sum \pi_j}$, we can technically show the following.

**Proposition 16.** For any positive constant $\beta \in \mathbb{Q}_{<1}$, given a star graph $G$, and distribution $\pi$, verifying whether $C(G) \geq \beta$ up to polynomially many bits is NP-hard.
The input can be Partitioned if and only if \( C = \beta \), otherwise \( C < \beta - \Omega(\frac{1}{\sum_i p_i}) \).

It is worth mentioning, as with the case of \( \lambda_\infty \), that one can approximate the spread constant of a star graph, up to arbitrary precision.

**Proposition 17.** There is an FPTAS for computing the spread constant of a star.

**Proof idea.** Utilizing the FPTAS for a knapsack ([IK75]) of size \( \frac{1}{2} \sum p_i \), we can approximate the most balanced partitioning of leaves, corresponding to \( \pi_- \) and \( \pi_+ \) of the solution.

### 6.1 MVU for Trees

We study the problem of computing the spread constant for trees. An interesting study by Göring, Helmberg, and Wappler ([GHW08]) shed light over the lifted relaxation of spread constant.

**Theorem 18** ([GHW08]). Let \( y : V \to \mathbb{R}^n \) be a normalized (zero-mean) optimal solution to \( C_n \), and \( S \subseteq V \) be a separator, removing which creates disconnected components \( C_1, C_2 \subseteq V \). Then there exist \( i \in \{1, 2\} \) for which

\[
\text{conv}\{0, y_v\} \cap \text{conv}\{y_u : u \in S\} \neq \emptyset \quad \forall v \in C_i.
\]

In particular, ([GHW08]) shows \( C_2 = C_n \) for a tree yet relies on solving an SDP to report an optimal embedding of a tree into the plane. Further characterizing the optimum for trees, we derive a new combinatorial algorithm for this problem.

**Proposition 19.** Given a tree \( T = (V, E) \), for any \( C_2 \)-optimal embedding \( y : V \to \mathbb{R}^2 \), either (i) the barycenter overlaps a single vertex \( v \), and for every other vertex \( u \in V \) the graph distance between \( v \) and \( u \) matches that of the \( \ell^2 \) distance in the embedding, i.e.,

\[
d_T(u, v) = \|y(u) - y(v)\|_2
\]

or (ii) the barycenter belongs to a line segment corresponding to a single edge, i.e., \( \mathbb{E}_{y \sim \pi}[y_v] \in \text{conv}\{y_u, y(v)\} \) for some \( (u, v) \in E \), and the embedding spans only the line through \( y(u) \) and \( y(v) \) with all edges being stretched away from the barycenter.

**Proof.** First, note that all edges are stretched in the optimal embedding, i.e.,

\[
\|x(u) - x(v)\|_2 = 1 \quad \forall (u, v) \in E,
\]

that can be seen as either a complementary slackness condition or directly proved by contradiction, increasing the variance assuming otherwise, similar to Lemma [14]. Without loss of generality, assume the barycenter is at the origin, \( \mathbb{E}_{y \sim \pi}[x(v)] = 0 \). Applying Theorem [18] for every vertex \( v \in V \), for which \( x(v) \neq 0 \), for all neighbors \( u \in N(v) \), except at most one, we have

\[
x(u) = x(v) + \frac{1}{\|x(v)\|_2} x(v),
\]

as the shadow of separator vertex \( v \), with respect to the origin covers all and only the half line \( \{(1 + \alpha)x(v) : \alpha > 0\} \).
If more than a single edge (the segment between two embedded vertices) or a single vertex overlaps the origin Theorem 18 forces all other edges connected to them to remain in their starting direction moving away from the origin. This would dismiss any other edges in the graph that potentially connect such rays that contradicts connectivity of the graph.

Finally note that if the line through a segment corresponding to an edge \((u, v)\) does not include the origin, their neighbors and the rest of the tree falls in diverging rays from the origin into \(u\) and \(v\), and we will have a separating line (hyper-plane) between the origin and convex hull of \(x\), which contradicts \(0\) being the mean of \(x\).

So at least one of the cases in the statement happens.

**Theorem 20.** For a tree \(T = (V, E)\), \(C_2\) can be computed in \(O(n^2)\) time and \(O(n)\) storage.

**Proof Sketch.** We explore (and maximize the variance over) all \(O(n)\) cases rising from either of the scenarios of Proposition 19.

Scenario (i) with the barycenter overlapping vertex \(v\) : according to Proposition 19 there will be \(\text{deg}(v)\) branches stretched out of \(v\) corresponding to its neighbors and the variance squared equals sum of their second moments, i.e.,

\[
\mathbb{E}_{u \sim \pi} [||x(u) - \mathbb{E}_{w \sim \pi} [x(w)]||_2^2] = \mathbb{E}_{u \sim \pi} [||x(u) - x(v)||_2^2]
\]

(18)

\[
= \mathbb{E}_{u \sim \pi} [d_T(u, v)^2]
\]

(19)

Proposition 19

All that is left with this case is to check whether overlapping of the barycenter with \(x(v)\) is possible. Contribution of every branch \(b_i\) to the first moment vector, due to Proposition 19 has a fixed magnitude

\[
\sum_{u \in b_i} \pi_u d_T(v, u)
\]

towards an arbitrary direction, i.e., the problem becomes equivalent to balancing a weighted star. We need to decide whether there exists an arrangement of vectors (in the plane) that makes them sum up to zero. Using the triangle inequality, one can prove the necessary and sufficient condition for existence of the desired arrangement is the largest length for these vectors to be no larger than the sum of all the rest.

Scenario (ii) instances are more straightforward as the arrangement on the line is uniquely determined, given the edge from which every other edge is stretched away, and the mean and variance of each case can be computed in \(O(n)\).

7 Randomized Rounding Preserves More Variance

Principal Component Analysis is a prevalent subroutine to lower the dimension after solving a lifted relaxation. In this section we discuss an alternative solution, as Algorithm 2, that can guarantee negligible loss for embedding in dimensions logarithmic to the number of vertices. This follows the proof for the approximation bounds proposed by Theorem 4.

For each \(u \in V\), let \(y_u \overset{\text{def}}{=} G x_u\). Then, for any \(u, v \in V\),

\[
\mathbb{E}_G [||y_u - y_v||^2] = \sum_{i \in [k]} \mathbb{E}_G [(g_i, x_u - x_v)^2] = k ||x_u - x_v||^2.
\]
Fact 21 (LM00, Lemma 1). Let \( U \) be a \( \chi^2 \) random variable with \( D \) degrees of freedom. For any positive \( t \),
\[
\Pr[U - D \geq 2\sqrt{Dt} + 2t] \leq e^{-t}.
\]
Since \( \frac{\|y_u - y_v\|^2}{\|x_u - x_v\|^2} \) is a \( \chi^2 \)-random variable with \( k \) degrees of freedom, plugging in \( t = 3\log n \) in Fact 21 we get
\[
\Pr \left[ \frac{\|y_u - y_v\|^2}{\|x_u - x_v\|^2} - k \geq 2\sqrt{3k\log n} + 6\log n \right] \leq e^{-3\log n} = \frac{1}{n^3}.
\]
Recall that \( \tau_k = k + 2\sqrt{3k\log n} + 6\log n \). Using the union bound over all pairs of vertices in \( V \) we get
\[
\Pr \left[ \frac{\|y_u - y_v\|^2}{\|x_u - x_v\|^2} \leq \tau_k \quad \forall u, v \in V \right] \geq 1 - \frac{1}{n}.
\]
Hence, w.h.p., for \( \{u, v\} \in E, \frac{\|y_u - y_v\|^2}{\tau_k} \leq \|x_u - x_v\|^2 \leq 1 \). Therefore, w.h.p., \( \{y_u/\sqrt{\tau_k} : u \in V\} \) is a \( k \)-dimensional Lipschitz embedding of \( G \). From eq. (20), we get that
\[
\Pr \left[ \E u,v \sim V \left[ \frac{\|y_u - y_v\|^2}{\tau_k} \right] \geq \frac{k}{2\tau_k} \E u,v \sim V \left[ \|x_u - x_v\|^2 \right] \right] \geq \frac{1}{12}.
\]
Using the union bound over these two events, we get a \( O(\tau_k/k) \) approximation to \( \mathfrak{C}_k \), with constant probability. Note that
\[
\tau_k/k = \begin{cases} 
O((\log n)/k) & k \leq \log n \\
1 + O\left(\sqrt{(\log n)/k}\right) & k > \log n.
\end{cases}
\]
In particular, when \( k = \Omega((\log n)/\epsilon^2) \), \( \tau_k/k = 1 + \epsilon \).

Fact 22 (Folklore). Let \( g_1, \ldots, g_l \) be (not necessarily independent) Gaussian random variables each having mean 0, and \( \mathbb{E}[g_i^2] = \sigma_i^2 \). Then,
\[
\Pr \left[ \sum_{i \in [l]} g_i^2 \geq \frac{1}{2} \sum_{i \in [l]} \sigma_i^2 \right] \geq \frac{1}{12}.
\]
8 Conclusion

We provided the first NP-hardness results for $\lambda_\infty$, the spread constant, and maximum variance unfolding, showing NP-hardness of these problem for weighted stars. On the other hand, our approximation schemes suggest this special case to be easier than the general problem. We anticipate our methodology to be fruitful for complexity analysis of other graph functional constants.

$\lambda_\infty$ can range between $\frac{\lambda_2}{\Delta}$ and $\lambda_2$, while it is $O(\log \Delta)$ approximable [ST12]. The second author together with Raghavendra and Vempala [LRV13] showed that an asymptotic improvement on this bound would disprove the Small-set Expansion hypothesis (which in turn is closely related to the Unique Games Conjecture [RS10], see also [RST12]). Such consequences provide additional computational complexity motivation to study the approximability of $\lambda_\infty$.

$\lambda_2$ can help bounding mixing rate of a Markov chain while $\lambda_\infty$ is related to another dispersion processes [CLTZ18]. $\lambda_\infty$ does not only contrast $\lambda_2$ in terms of complexity, but also provides qualitatively different bounds for isoperimetric invariants of the graph. Such measures can provide information on clusters in a network [FCMR08], structure of a chemical molecule [Mer94], and spread of a rumor in a social network [GS12], to name but a few.

Last but not least, we provided new approximation results for maximum variance embedding, in particular, proposing a randomized rounding instead of PCA in reducing dimension of a lifted maximum variance unfolding. While worst-case guarantees for our algorithm are better than that of PCA, it would be interesting to investigate performance of this technique in real-world applications of MVU.

References

[ABS98] N. Alon, R. Boppana, and J. Spencer, An Asymptotic Isoperimetric Inequality, Geometric & Functional Analysis GAFA 8 (1998), no. 3, 411–436.

[ACP87] Stefan Arnborg, Derek G Corneil, and Andrzej Proskurowski, Complexity of finding embeddings in ak-tree, SIAM Journal on Algebraic Discrete Methods 8 (1987), no. 2, 277–284.

[Alo86] Noga Alon, Eigenvalues and expanders, Combinatorica 6 (1986), 83–96.

[AM85] Noga Alon and Vitali D Milman, $\lambda_1$, isoperimetric inequalities for graphs, and superconcentrators, Journal of Combinatorial Theory, Series B 38 (1985), no. 1, 73–88.

[ARV09] Sanjeev Arora, Satish Rao, and Umesh Vazirani, Expander flows, geometric embeddings and graph partitioning, Journal of the ACM (JACM) 56 (2009), no. 2, 1–37.

[BDX04] Stephen Boyd, Persi Diaconis, and Lin Xiao, Fastest mixing markov chain on a graph, SIAM review 46 (2004), no. 4, 667–689.
Sergey Bobkov, Christian Houdré, and Prasad Tetali, *lambda infinity, vertex isoperimetry and concentration*, Combinatorica 20 (2000), no. 2, 153–172.

Samuel Burer and Renato DC Monteiro, *A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization*, Mathematical Programming 95 (2003), no. 2, 329–357.

Fan RK Chung and Fan Chung Graham, *Spectral graph theory*, no. 92, American Mathematical Soc., 1997.

Jeff Cheeger, *A lower bound for the smallest eigenvalue of the laplacian*, Proceedings of the Princeton conference in honor of Professor S. Bochner, 1969, pp. 195–199.

T.-H. Hubert Chan, Anand Louis, Zhihao Gavin Tang, and Chenzi Zhang, *Spectral properties of hypergraph laplacian and approximation algorithms*, J. ACM 65 (2018), no. 3, 15:1–15:48.

Fan RK Chung and Prasad Tetali, *Isoperimetric inequalities for cartesian products of graphs*, Combinatorics, Probability and Computing 7 (1998), no. 2, 141–148.

Maurizio Filippone, Francesco Camastra, Francesco Masulli, and Stefano Rovetta, *A survey of kernel and spectral methods for clustering*, Pattern recognition 41 (2008), no. 1, 176–190.

Miroslav Fiedler, *Algebraic connectivity of graphs*, Czechoslovak mathematical journal 23 (1973), no. 2, 298–305.

Miroslav Fiedler, *Absolute algebraic connectivity of trees*, Linear and Multilinear Algebra 26 (1990), no. 1-2, 85–106.

Miroslav Fiedler, *Some minimax problems for graphs*, Discrete Mathematics 121 (1993), no. 1-3, 65–74.

Arpita Ghosh, Stephen Boyd, and Amin Saberi, *Minimizing effective resistance of a graph*, SIAM review 50 (2008), no. 1, 37–66.

Frank Göring, Christoph Helmberg, and Markus Wappler, *Embedded in the Shadow of the Separator*, SIAM Journal on Optimization 19 (2008), no. 1, 472–501 (en).

George Giakkoupis and Thomas Sauerwald, *Rumor spreading and vertex expansion*, Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms, SIAM, 2012, pp. 1623–1641.

Shlomo Hoory, Nathan Linial, and Avi Wigderson, *Expander graphs and their applications*, Bulletin of the American Mathematical Society 43 (2006), no. 4, 439–561.
[IK75] Oscar H. Ibarra and Chul E. Kim, *Fast Approximation Algorithms for the Knapsack and Sum of Subset Problems*, J. ACM 22 (1975), no. 4, 463–468.

[Kar72] Richard M. Karp, *Reducibility among Combinatorial Problems*, Complexity of Computer Computations, The IBM Research Symposia Series, Springer US, Boston, MA, 1972, pp. 85–103 (en).

[Kho16] Subhash Khot, *Hardness of approximation.*, ICALP, Citeseer, 2016, pp. 3–1.

[KSP07] Brian Kulis, Arun C Surendran, and John C Platt, *Fast low-rank semidefinite programming for embedding and clustering*, Artificial Intelligence and Statistics, 2007, pp. 235–242.

[LM00] Beatrice Laurent and Pascal Massart, *Adaptive estimation of a quadratic functional by model selection*, Annals of Statistics (2000), 1302–1338.

[LRV13] A. Louis, P. Raghavendra, and S. Vempala, *The Complexity of Approximating Vertex Expansion*, 2013 IEEE 54th Annual Symposium on Foundations of Computer Science, October 2013, pp. 360–369.

[Mer94] Russell Merris, *Laplacian matrices of graphs: a survey*, Linear algebra and its applications 197 (1994), 143–176.

[MT05] R Montenegro and P Tetali, *Mathematical Aspects of Mixing Times in Markov Chains*, Foundations and Trends in Theoretical Computer Science 1 (2005), no. 3, 237–354 (en).

[MT+06] Ravi Montenegro, Prasad Tetali, et al., *Mathematical aspects of mixing times in markov chains*, Foundations and Trends® in Theoretical Computer Science 1 (2006), no. 3, 237–354.

[Nao14] Assaf Naor, *Comparison of Metric Spectral Gaps*, Analysis and Geometry in Metric Spaces 2 (2014), no. 1.

[Nes13] Yuriii Nesterov, *Introductory lectures on convex optimization: A basic course*, vol. 87, Springer Science & Business Media, 2013.

[NN94] Yuriii. Nesterov and Arkadii. Nemirovskii, *Interior-Point Polynomial Algorithms in Convex Programming*, Studies in Applied and Numerical Mathematics, Society for Industrial and Applied Mathematics, January 1994.

[RS10] Prasad Raghavendra and David Steurer, *Graph Expansion and the Unique Games Conjecture*, Proceedings of the Forty-second ACM Symposium on Theory of Computing (New York, NY, USA), STOC ’10, ACM, 2010, event-place: Cambridge, Massachusetts, USA, pp. 755–764.

[RST12] Prasad Raghavendra, David Steurer, and Madhur Tulsiani, *Reductions between expansion problems*, 2012 IEEE 27th Conference on Computational Complexity, IEEE, 2012, pp. 64–73.
[SBXD06] Jun. Sun, Stephen. Boyd, Lin. Xiao, and Persi. Diaconis, The Fastest Mixing Markov Process on a Graph and a Connection to a Maximum Variance Unfolding Problem, SIAM Review 48 (2006), no. 4, 681–699.

[SGBS08] Le Song, Arthur Gretton, Karsten Borgwardt, and Alex J Smola, Colored maximum variance unfolding, Advances in neural information processing systems, 2008, pp. 1385–1392.

[SJ89] Alistair Sinclair and Mark Jerrum, Approximate counting, uniform generation and rapidly mixing markov chains, Information and Computation 82 (1989), no. 1, 93–133.

[ST12] David Steurer and Prasad Tetali, Personal communication, 2012.

[WS06a] Kilian Q Weinberger and Lawrence K Saul, An introduction to nonlinear dimensionality reduction by maximum variance unfolding, AAAI, vol. 6, 2006, pp. 1683–1686.

[WS06b] Kilian Q. Weinberger and Lawrence K. Saul, Unsupervised learning of image manifolds by semidefinite programming, International journal of computer vision 70 (2006), no. 1, 77–90.

[WSS04] Kilian Q Weinberger, Fei Sha, and Lawrence K Saul, Learning a kernel matrix for nonlinear dimensionality reduction, Proceedings of the twenty-first international conference on Machine learning, ACM, 2004, p. 106.

[ZLM13] Zeyuan Allen Zhu, Silvio Lattanzi, and Vahab Mirrokni, Local graph clustering beyond cheeger’s inequality, arXiv preprint arXiv:1304.8132 (2013).

A Omitted Proofs

Proof of Fact 22

\[
\mathbb{E} \left[ \left( \sum_{i \in [l]} g_i^2 \right)^2 \right] = \sum_{i \in [l]} \mathbb{E} [g_i^4] + 2 \sum_{\substack{i, j \in [l] \atop i \neq j}} \mathbb{E} [g_i^2 g_j^2] \\
\leq 3 \sum_{i \in [l]} \sigma_i^4 + 2 \sum_{\substack{i, j \in [l] \atop i \neq j}} \sqrt{\mathbb{E} [g_i^4]} \sqrt{\mathbb{E} [g_j^4]} \quad \text{(Cauchy-Schwarz inequality)} \\
= 3 \left( \sum_{i \in [l]} \sigma_i^4 \right) + 2 \left( \sum_{\substack{i, j \in [l] \atop i \neq j}} \sigma_i^2 \sigma_j^2 \right) = 3 \left( \sum_{i \in [l]} \sigma_i^2 \right)^2.
\]
Using the Paley-Zygmund inequality,

\[
\Pr \left[ \sum_{i \in [l]} q_i^2 \geq \frac{1}{2} \sum_{i \in [l]} \sigma_i^2 \right] \geq \left( 1 - \frac{1}{2} \right)^2 \cdot \frac{\left( \sum_{i \in [l]} \sigma_i^2 \right)^2}{3 \left( \sum_{i \in [l]} \sigma_i^2 \right)^2} = \frac{1}{12}.
\]

\[
\lambda_\infty = \inf_{f : V \to \mathbb{R}} \frac{\int_{\mathbb{R}} \sup_{y \in \mathbb{R}(x)} |f(x) - f(y)|^2 \, d\pi(x)}{\text{Var}_{\nu \sim \pi}[f(\nu)]}
= \inf_{x \in \mathbb{R} \setminus \{0\}} \frac{\mathbb{E}_{\nu \sim \pi} \left[ \max_{u \in \mathbb{N}(u)} |x_u - x_v|^2 \right]}{\text{Var}_{\nu \sim \pi}[x_v]}
= \inf_{x \in \mathbb{R} \setminus \{0\}, \mathbb{E}_{\nu \sim \pi}|x_u|=0} \frac{\mathbb{E}_{\nu \sim \pi} \left[ \max_{u \in \mathbb{N}(u)} |x_u - x_v|^2 \right]}{\mathbb{E}_{\nu \sim \pi}[x_v^2]}
= \inf_{x \in \mathbb{R} \setminus \{0\}, \mathbb{E}_{\nu \sim \pi}|x_u|=0, \mathbb{E}_{\nu \sim \pi}|x_u^2|=1} \mathbb{E}_{\nu \sim \pi} \left[ \max_{u \in \mathbb{N}(u)} |x_u - x_v|^2 \right]

\text{for } \rho^2 = \left( \mathbb{E}_{\nu \sim \pi}[x_v^2] \right)^{-1}.
\]

Therefore,

\[
\lambda_\infty = \inf_{x^T \pi = 0, x^T \text{diag}(\pi) = 1} \mathbb{E}_{\nu \sim \pi} \left[ \max_{u \in \mathbb{N}(u)} |x_u - x_v|^2 \right].
\]

Lemma 23 (Folklore). There is at least one optimal valuation \( f : V \to \mathbb{R} \), achieving \( \lambda_\infty \) in equation [1].

Proof. From the definition we have

\[
\lambda_\infty = \inf_{f : V \to \mathbb{R}} \frac{\int_{\mathbb{R}} \sup_{y \in \mathbb{R}(x)} |f(x) - f(y)|^2 \, d\pi(x)}{\text{Var}_{\nu \sim \pi}[f(\nu)]}
= \inf_{x \in \mathbb{R} \setminus \{0\}} \frac{\mathbb{E}_{\nu \sim \pi} \left[ \max_{u \in \mathbb{N}(u)} |x_u - x_v|^2 \right]}{\text{Var}_{\nu \sim \pi}[x_v]}
= \inf_{x \in \mathbb{R} \setminus \{0\}} \frac{\mathbb{E}_{\nu \sim \pi} \left[ \max_{u \in \mathbb{N}(u)} |(x_u + \nu) - (x_v + \nu)|^2 \right]}{\text{Var}_{\nu \sim \pi}[x_v]}
= \inf_{x \in \mathbb{R} \setminus \{0\}, \mathbb{E}_{\nu \sim \pi}|x_u|=0} \frac{\mathbb{E}_{\nu \sim \pi} \left[ \max_{u \in \mathbb{N}(u)} |x_u - x_v|^2 \right]}{\mathbb{E}_{\nu \sim \pi}[x_v^2]}
= \inf_{x \in \mathbb{R} \setminus \{0\}, \mathbb{E}_{\nu \sim \pi}|x_u|=0, \mathbb{E}_{\nu \sim \pi}|x_u^2|=1} \mathbb{E}_{\nu \sim \pi} \left[ \max_{u \in \mathbb{N}(u)} |x_u - x_v|^2 \right]

\text{for } \rho^2 = \left( \mathbb{E}_{\nu \sim \pi}[x_v^2] \right)^{-1}.
\]

Considering a pair of permutations \( p : V \to [n] \) and \( q : E \to [m] \), define the set

\[
S_{(p,q)} \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : x_u \leq x_v, \forall p(u) < p(v), \, |x_v - x_u| \leq |x_v' - x_u'|, \forall q(u, v) < q(u', v') \}.
\]

Note that \( \text{sign}(x_u - x_v) \) is uniform across \( S_{(p,q)} \) for every pair of vertices \( u, v \in V \). Therefore every \( S_{(p,q)} \) can be redefined without absolute values in the notation and hence is a polyhedron. Moreover, \( \cup_{p,q} S_{(p,q)} = \mathbb{R}^n \). Finally, due to the constraints corresponding to permutation of edges, in the definition of \( S_{(p,q)} \), the “farthest” neighbor \( v \) to each vertex \( u \) is constant for all \( x \in S_{(p,q)} \). In other words, for every vertex \( u \in V \) there exists a vertex \( v \in V \) such that

\[
|x_v - x_u| = \max_{w \in \mathbb{N}(u)} |x_w - x_u| \quad \forall x \in S_{(p,q)}.
\]

We denote this \( v \) by \( f_{p,q,u} \).
Since, $\mathbb{R}^n = \cup_{(p,q)\in[n]!\times[m]!} S_{(p,q)}$, we can rewrite $\lambda_\infty$ from equation 21 as

$$\lambda_\infty = \min_{(p,q)\in[n]!\times[m]!} \inf_{x\in S_{(p,q)}, x^T\pi=0, x^T\text{diag}(\pi)x=1} \mathbb{E}_{v\sim\pi} \left[ (x_u - x_{f_{p,q,u}})^2 \right].$$

(22)

For a fixed $S_{(p,q)}$, the function under infimum is a degree 2 polynomial with coefficients from $\mathbb{Z}[\pi]$, 

$$\mathbb{E}_{v\sim\pi} \left[ (x_u - x_{f_{p,q,u}})^2 \right] = \sum_u \pi_v (x_u - x_{f_{p,q,u}})^2,$$

hence is continuous.

Moreover, for each case $(p, q)$ the domain of infimum is the intersection of surface of an ellipsoid

$$x^T\text{diag}(\pi)x = 1,$$

a hyperplane

$$x^T\pi = 0,$$

and a polyhedron $S_{(p,q)}$. Since all three are compact sets, their intersection is also a compact set. Therefore, each inf in equation 22 computes the infimum of a continuous function over a compact set. Hence, applying the Extreme Value Theorem, we can replace these infima with min,

$$\lambda_\infty = \min_{(p,q)\in[n]!\times[m]!} \inf_{x\in P_{(p,q)}, x^T\pi=0, x^T\text{diag}(\pi)x=1} \mathbb{E}_{v\sim\pi} \left[ (x_u - x_{f_{p,q,u}})^2 \right],$$

allowing us to replace inf with min in either $\lambda_\infty$ formulation.

\[\square\]

**Corollary 24.** $\lambda_\infty$ is an algebraic number for rational inputs.

**Proposition 25.** Spread constant is a rational number, if not $\infty$, for rational inputs.

**Proof.** Following notation from Lemma 14, Without loss of generality, we may assume $y_i = 0$ for some arbitrary vertex $i$. Lipschitz conditions give constraints of form

$$y_u - y_v \leq 1, y_v - y_u \leq 1, \forall (u,v) \in E.$$

that defines a bounded polytope in $\mathbb{R}^V$ assuming the graph is connected. Vertices of the polytope, being intersection of $n$ constraints holding at equality (hyperplanes), are rational. Variance, being a linear combination of functions of form

$$(y_i - y_j)^2,$$

is strongly convex so is maximized at a vertex of our polytope and has rational value there, which equals the spread constant of the graph.

\[\square\]