SECCULAR INTERACTIONS BETWEEN INCLINED PLANETS AND A GASEOUS DISK

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ABSTRACT

In a planetary system, a secular particle resonance occurs at a location where the precession rate of a test particle (e.g., an asteroid) matches the frequency of one of the precessional modes of the planetary system. We investigate the secular interactions of a system of mutually inclined planets with a gaseous protostellar disk that may contain a secular nodal particle resonance. We determine the normal modes of some mutually inclined planet-disk systems. The planets and disk interact gravitationally, and the disk is internally subject to the effects of gas pressure, self-gravity, and turbulent viscosity. The behavior of the disk at a secular resonance is radically different from that of a particle, owing mainly to the effects of gas pressure. The resonance is typically broadened by gas pressure to the extent that global effects, including large-scale warps, dominate. The standard resonant torque formula is invalid in this regime. Secular interactions cause a decay of the inclination at a rate that depends on the disk properties, including its mass, turbulent viscosity, and sound speed. For a Jupiter-mass planet embedded within a minimum-mass solar nebula having typical parameters, dissipation within the disk is sufficient to stabilize the system against tilt growth caused by mean-motion resonances.

Subject headings: accretion, accretion disks — celestial mechanics — hydrodynamics — planets and satellites: general — solar system: general — waves

1. INTRODUCTION

The interaction between a young planetary system and its protoplanetary disk likely plays an important role in determining the orbital properties of the planets. Such interactions may be important for understanding the observed orbital properties of extra-solar planets (e.g., Marcy et al. 1999). Resonances within a gaseous disk likely play a key role in determining planetary eccentricities and inclinations. Much of the previous work, starting with Goldreich & Tremaine (1980), has emphasized the effects of mean-motion disk resonances, which involve frequencies that are comparable to the orbital frequencies of the planets.

Another potentially important class of resonances occurs where there is a matching of the precession frequency of a test particle (e.g., an asteroid) with the frequency of one of the precessional modes of the planetary system. The frequencies involved are much lower than in the mean-motion case. At such a resonance, the motion of a test particle can be strongly driven by the planets, resulting in a high orbital inclination (for a nodal resonance) or eccentricity (for an apsidal resonance). For example, in the solar system there is an important secular resonance that occurs near 2 AU caused by driving involving Jupiter and Saturn. This resonance is believed to be associated with the inner truncation of the asteroid belt (Tisserand 1882).

It was recognized by Ward, Colombo, & Franklin (1976) that these resonances must have swept across portions of the early solar system owing to the effects of the gaseous disk, even if the planets do not migrate. The reason is that even a minimum-mass solar nebula can have an important influence on the relevant precession rates. As the nebula disperses, the precession rates vary, along with the resonance locations (Ward 1981). Regions through which the resonances sweep may be driven into significantly eccentric or inclined orbits, such as are observed among the asteroids today (e.g., Nagasawa, Tanaka, & Ida 2000). The resonances can also have an important bearing on the conditions for planet formation by stirring the planetesimal disk.

However, collective effects can be important within the disk and may even result in the development of density waves or bending waves. Consequently, the response is less localized than would be indicated by test particles, as was recognized by Ward & Hahn (1998) and Tremaine (1998). These investigations concentrated on the collective effects of self-gravity in particle disks.

Secular interactions involving the gas disk of a young planetary system are potentially of importance, since much more mass is contained in the gas disk than in the planetesimal disk. The disk is a fluid body and is capable of wave motions through (at least) compressive, inertial, buoyancy, and self-gravitational forces. It is also capable of dissipating energy through a turbulent effective viscosity or through shocks. It is therefore important to understand the dynamical response of the disk to secular forcing and the implications for the evolution of the planetary orbits.

The purpose of this paper is to investigate the outcome of secular interactions in mutually inclined planet-disk systems. The relevant waves in the disk are long-wavelength bending waves with azimuthal wavenumber \( m = 1 \). The equations governing such waves in a protostellar disk are fairly well established, while those for long-wavelength eccentric density waves have received less attention (although see Ogilvie 2001). The importance of understanding the global \( m = 1 \) response of the disk to secular perturbations has been emphasized by Tremaine (1998).

The general outline of the paper is as follows. Sections 2–5 describe the physical model for nodal secular interactions and formulate the normal mode analysis. Sections 6 and 7 explore the limit of sufficiently slow modes, when the disk responds nearly rigidly. Section 8 describes how the effects of mean-motion resonances are included in the normal mode analysis. Sections 9 and 10 describe numerical results for Jupiter interacting with the solar nebula. Section

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11 provides a simple scaling analysis based on a nearly rigid tilt model. Section 12 discusses the relationship of the current approach with that of earlier work involving WKB theory. Section 13 describes the analysis of a close-orbiting planet in the central hole of a disk. Section 14 discusses Jupiter and Saturn interacting with the solar nebula at the \( v_{16} \) secular resonance.

2. MODELING SECULAR INTERACTIONS

Planets experience secular gravitational interactions on timescales much longer than their orbital periods. On such timescales, the planets may be considered as continuous rings representing their average mass density. These rings are in general elliptical and mutually inclined. Gravitational interactions between the rings lead to apsidal and nodal precession of the orbits.

In the case of small eccentricities and inclinations, the secular evolution of a system of \( n \) planets can be described in terms of normal modes of a linear dynamical system, \( n \) modes for eccentricity, and \( n \) for inclination (e.g., Murray & Dermott 1999). The eigenvector of a mode describes the relative distribution of eccentricity or inclination among the planetary orbits, while the eigenfrequency is the rate at which the pattern precesses.

In typical situations involving mean-motion resonances (Lindblad or vertical resonances), if a particle resonance is located at a certain radius in a fluid disk, the disk responds by launching a wave that carries energy and angular momentum away from the resonance. Such waves are usually assumed implicitly to be damped through some dissipative process before reaching an edge of the disk from which they might otherwise reflect. The gravitational torque between the disk and the perturber can then be calculated from a standard formula evaluated at the location of the resonance (Goldreich & Tremaine 1979).

However, in the case of a secular resonance, the frequency may be so small that the wavelength would be comparable to (or may even exceed) the size of the disk. In that case the global response of the disk must be computed, including explicit dissipation and the correct boundary conditions. The exact location of the resonance then ceases to have great importance, and the new possibility of a global secular resonance arises.

We focus on the case of inclination, so that the relevant waves in the disk are long-wavelength bending waves with azimuthal wavenumber \( m = 1 \). As noted above, the equations governing such waves in a protostellar disk are fairly well established.

3. BASIC EQUATIONS

Let \((r, \phi, z)\) be cylindrical polar coordinates with origin at the central object of mass \( M_\star \). We consider a protostellar disk of semithickness \( H(r) \) for which the turbulent viscosity parameter \( \alpha \) satisfies the condition \( \alpha \lesssim H/r \). The linearized equations for bending waves in such a disk have been derived in several papers (Papaloizou & Lin 1995; Masset & Tagger 1996; Demianski & Ivanov 1997). We present them in the form (Lubow & Ogilvie 2000)

\[
\Sigma r^2 \Omega \frac{\partial W}{\partial t} = \frac{\partial \Phi}{r \partial r} + T, \quad \frac{\partial \Phi}{\partial t} + \frac{\kappa^2 - \Omega^2}{2} \frac{\partial \Omega}{\partial t} = \frac{\mathcal{A}_r^2 \Omega^3}{4} \frac{\partial W}{\partial r}. \tag{1}
\]

Here \( W(r, t) = l_x + il_y \) is the complex tilt variable, which describes the warped shape of the disk. Essentially, the disk can be thought of as a continuum of concentric circular rings with radii \( r \) and unit normal vectors \( \hat{r} \). Also, \( \Phi(r, t) = G_x + iG_y \) is the complex internal torque variable. In a warped disk there is a horizontal internal torque \( 2\pi G(r, t) \) that is mediated by horizontal motions that are proportional to the distance \( z \) from the midplane. These stresses are responsible for the propagation of bending waves. Similarly, \( T(r, t) = T_e + iT_n \) is the complex horizontal external torque density acting on the disk. Finally, \( \Sigma(r) \) is the surface density, \( \Omega(r) \) the orbital angular velocity, \( \kappa(t) \) the epicycliche frequency, and \( \mathcal{A}(r, t) \) the second vertical moment of the density. These are defined by

\[
\Sigma(r) = \int_0^r \rho(z) \, dz, \quad \mathcal{A}(r, t) = \int_0^r \rho(z) z^2 \, dz, \tag{3}
\]

\[
r\Omega^2 = \frac{\partial \Phi}{\partial r} \bigg|_{r=0}, \quad \kappa^2 = 4\Omega^2 + 2\mathcal{A} \frac{d\Omega}{dr}, \tag{4}
\]

where \( \Phi(r, z) \) is the (axisymmetric component of the) gravitational potential experienced by the disk.

Consider a thin, uniform circular ring of mass \( m_1 \ll M_\star \) and radius \( r_i \), representing either an annulus of the disk or the time average of a planetary orbit. Its contribution to the potential in the plane \( z = 0 \) is

\[
\Phi_i = -\frac{G m_1}{2\pi} \int_0^{2\pi} \int_0^{r_i} (r^2 + r_i^2 - 2rr_i \cos \phi)^{-1/2} \, d\phi \, dr. \tag{5}
\]

Therefore, the angular velocity and epicyclic frequency experienced by the ring are obtained from summations over all such rings, in the forms

\[
\Omega_i^2 = \frac{Gm_1}{r_i^3} + \sum_{j \neq i} \frac{2Gm_j}{r_i r_j} \left[ K_0(r_i, r_j) - \frac{r_j}{r_i} K_1(r_i, r_j) \right], \tag{6}
\]

\[
\kappa_i^2 = \frac{Gm_1}{r_i^3} + \sum_{j \neq i} \frac{2Gm_j}{r_i r_j} \left[ K_0(r_i, r_j) - \frac{2r_j}{r_i} K_1(r_i, r_j) \right], \tag{7}
\]

where \( K_0 \) and \( K_1 \) are symmetric kernels with dimensions of inverse length, given by

\[
K_m(r_i, r_j) = \frac{r_i r_j}{4\pi} \int_0^{2\pi} (r_i^2 + r_j^2 - 2r_i r_j \cos \phi)^{-3/2} \cos^m \phi \, d\phi. \tag{8}
\]

Hereafter, we write \( K_1 \) simply as \( K \).

Now consider the tilt interaction between two such rings of sizes \( m_1 \) and \( m_2 \), radii \( r_i \) and \( r_j \), and tilt vectors \( \mathbf{l}_i \) and \( \mathbf{l}_j \). For small relative inclinations, the gravitational torque exerted by ring \( j \) on ring \( i \) is

\[
T_{ij} = Gm_i m_j K(r_i, r_j) \mathbf{l}_i \times \mathbf{l}_j. \tag{9}
\]

This expression can be obtained by averaging the torque exerted by a point mass on a circular ring (Lubow & Ogilvie 2000) over the orbital motion of the point mass. Similar expressions can be found in studies of galactic warps (e.g., Sparke & Casertano 1988). In the complex notation, for small inclinations from the \( x-y \) plane, the total external torque acting on ring \( i \) is

\[
T_i = \sum_{j \neq i} Gm_i m_j K(r_i, r_j) i[(W_j - W_i)]. \tag{10}
\]
The kernels can also be written as
\[ K_m(r_i, r_j) = \frac{r_i + r_j - h}{4r} b_{ji}^m, \]  
where \( r_\geq = \max(r_i, r_j) \), \( r_\leq = \min(r_i, r_j) \), and \( b_{ji}^m \) is the Laplace coefficient. In practice, we evaluate these in terms of elliptic integrals using Carlson’s algorithms (Press et al. 1992).

These kernels diverge as \( |r_i - r_j| \to 0 \). In reality, the gravitational interaction remains bounded because of the nonzero vertical thickness of the rings, which we have neglected. To take account of this in an approximate way, we soften the kernels by replacing
\[ K_m(r_i, r_j) \to K_m \left( \frac{r_i + r_j - h}{2}, \frac{r_i + r_j + h}{2} \right) \]  
whenever \( |r_i - r_j| < h \). Here the smoothing length \( h \) is to be understood as an approximate measure of the thickness of the rings.

4. COUPLED SYSTEMS OF PLANETS AND DISKS

We consider a general system consisting of multiple planets embedded in a gaseous disk. Provided the planets create gaps in the disk, the disk then becomes partitioned into a set of disks. The disks and planets interact through gravity.

We consider planets \( i = 1, 2, \ldots, n_p \) of masses \( m_{pi} \) in circular orbits of radii \( r_{pi} \) and angular velocities \( \Omega_{pi} \). We also consider disks \( k = 1, 2, \ldots, n_d \) of inner radii \( a_k \) and outer radii \( b_k \). The inclinations of the planetary orbits are described by the tilt variables \( W_{pi}(t) \). The tilt within a disk is described by \( W_{dk}(r, t) \). We assume throughout that the inclinations are small, so that linear theory applies.

If only secular nodal interactions are considered, the planets can be treated as inclined circular rings interacting with each other and with the disks through the torques described in § 3. We return later (§ 8) to the effect of mean-motion resonances on tilt evolution. However, we assume throughout that any evolution of the surface density of the disk or of the semimajor axes of the planetary orbits may be neglected.

The secular equations for the planets are then of the form
\[ m_{pi} r_{pi}^2 \Omega_{pi} \frac{dW_{pi}}{dt} = \sum_{j \neq i} G m_{pj} m_{pi} K(r_{pi}, r_{pj}) (W_{pj} - W_{pi}) + \sum_{k} \int_{a_k}^{b_k} G m_{pi} \Sigma_k K(r, r_{pj}) (W_{dk} - W_{pi}) 2\pi r dr, \]
while the equations for the disks are
\[ \Sigma_k r_{dk}^2 \frac{\partial W_{dk}}{\partial t} = \frac{1}{r} \frac{\partial \mathcal{G}_k}{\partial r} + \sum_{i} G m_{pi} \Sigma_k K(r, r_{pi}) (W_{pi} - W_{dk}) + \sum_{i} \int_{a_k}^{b_k} G \Sigma_k \Sigma_i K(r, r_{pi}) (W_{dp} - W_{dk}) 2\pi r dr', \]
\[ \frac{\partial \mathcal{G}_k}{\partial t} + \frac{\kappa^2 - \Omega^2}{\Omega^2} i \Omega \mathcal{G}_k + \frac{3}{4} \frac{r \Sigma_k \Omega^3}{\Omega^2} \frac{\partial W_{dk}}{\partial r}, \]
where \( \Sigma_j = \Sigma_j(r') \) and \( W_{dp} = W_{dp}(r', t) \). The final term in equation (14) represents the self-gravitation of each disk and the gravitational interactions between disks.

The (complex) total horizontal angular momentum of the system,
\[ \sum_i m_{pi} r_{pi}^2 \Omega_{pi} W_{pi} + \sum_k \int_{a_k}^{b_k} \Sigma_k r_{dk}^2 \Omega_{dk} 2\pi r dr, \]
is exactly conserved if, as we assume, the boundary conditions
\[ \mathcal{G}_k (a_k, t) = \mathcal{G}_k (b_k, t) = 0 \]
hold at all times. There is also a conservation law for the vertical angular momentum, which we derive in Appendix A.

5. NORMAL MODES

We have a set of coupled integrodifferential equations that are linear and homogeneous. Solutions may be sought in the form of normal modes,
\[ W_{pi}(t) = \mathcal{W}_{pi}(r) e^{i \omega t}, \quad W_{dk}(r, t) = \mathcal{W}_{dk}(r) e^{i \omega t}, \]
\[ \mathcal{G}_k (a_k, t) = \mathcal{G}_k (b_k, t) = 0, \]
where \( \omega \) is a complex frequency eigenvalue. The real part of \( \omega \) is the precession rate of the mode, while the imaginary part is the decay rate. Substituting this and omitting the tildes, we obtain the integral and integrodifferential equations
\[ i m_{pi} r_{pi}^2 \Omega_{pi} \mathcal{W}_{pi} = \sum_{j \neq i} G m_{pj} m_{pi} K(r_{pi}, r_{pj}) (\mathcal{W}_{pj} - \mathcal{W}_{pi}) + \sum_{k} \int_{a_k}^{b_k} G m_{pi} \Sigma_k K(r, r_{pj}) (\mathcal{W}_{dk} - \mathcal{W}_{pi}) 2\pi r dr, \]
\[ i \omega \Sigma_k r_{dk}^2 \mathcal{W}_{dk} = \frac{1}{r} \frac{\partial \mathcal{G}_k}{\partial r} + \sum_{i} G m_{pi} \Sigma_k K(r, r_{pi}) (\mathcal{W}_{pi} - \mathcal{W}_{dk}) + \sum_{i} \int_{a_k}^{b_k} G \Sigma_k \Sigma_i K(r, r_{pi}) (\mathcal{W}_{dp} - \mathcal{W}_{dk}) 2\pi r dr', \]
\[ i \omega \mathcal{G}_k + \frac{1}{2} \frac{\kappa^2 - \Omega^2}{\Omega^2} i \Omega \mathcal{G}_k + \frac{3}{4} \frac{r \Sigma_k \Omega^3}{\Omega^2} \frac{\partial \mathcal{W}_{dk}}{\partial r}, \]
subject to the boundary conditions
\[ \mathcal{G}_k (a_k) = \mathcal{G}_k (b_k) = 0. \]
Frequencies \( \Omega \) and \( \kappa \) are given by equations (6) and (7). All orbits are assumed to be prograde. There is always a trivial rigid-tilt mode with \( \omega = 0 \), \( \mathcal{G}_k = 0 \), and \( \mathcal{W}_{pi} = \mathcal{W}_{dk} \) is constant.

6. RIGID RESPONSE

Secular modes usually have very long periods. If a disk is easily able to maintain radial communication (through pressure, viscosity, or self-gravitation) on this long time-scale, it will participate in such a mode almost as a rigid body, and \( \mathcal{W}_{pi} \) will not vary much across the disk.

Suppose that each disk indeed behaves rigidly (\( \mathcal{W}_{dk} = \) constant). Then \( \mathcal{G}_k \) can be eliminated by multiplying equation (20) by \( 2\pi r \), integrating from \( a_k \) to \( b_k \), and using the
boundary conditions. We then find
\[ \omega J_{p_i} W_{p_i} = \sum_{j 
eq i} C_{1j} (W_{p_j} - W_{p_i}) + \sum_{k} C_{ik} (W_{d_k} - W_{p_i}), \]  
(23)
\[ \omega J_{d_k} W_{d_k} = \sum_{l} C_{ld} (W_{p_l} - W_{d_k}) + \sum_{l 
eq k} C_{kl} (W_{d_l} - W_{d_k}), \]  
(24)
where
\[ J_{pi} = m_{pi} r_{pi}^2 \omega_{pi} \]  
(25)
is the angular momentum of planet i and
\[ J_{dk} = \int_{a_k}^{b_k} \sum_{r} \Omega_{2\pi r} dr \]  
(26)
is the total angular momentum of disk k. The coupling coefficients are defined by
\[ C_{1j}^{pf} = G m_{p_i} m_{p_j} K(r_{pi}, r_{pj}), \]  
(27)
\[ C_{ik}^{ld} = \int_{a_k}^{b_k} G m_{p_i} \Sigma_k K(r, r)p_2 \pi r \ dr, \]  
(28)
\[ C_{kl}^{pd} = \int_{a_k}^{b_k} G \Sigma_k \Sigma_l K(r, r) \pi r d\pi r' dr'. \]  
(29)
Evidently, the disks can be treated formally as additional planets if they behave as rigid bodies. By introducing Greek indices that run from 1 to \( n_p + n_d \), we obtain the algebraic eigenvalue problem
\[ \omega J_{p_i} W_{p_i} = \sum_{\gamma \neq p_i} C_{\gamma p_i} (W_{\gamma} - W_{p_i}). \]  
(30)
This is formally identical to the multiple-planet problem described by, e.g., Murray & Dermott (1999). Since C is a symmetric matrix, we have
\[ \omega \sum_{\beta} J_{p_\beta} |W_{p_\beta}|^2 = -\frac{1}{2} \sum_{\beta, \gamma} C_{\beta \gamma} |W_{\beta} - W_{\gamma}|^2. \]  
(31)
Since, further, the coefficients \( C_{\beta \gamma} \) are real and positive, the eigenvalues are all real and nonpositive. Apart from the trivial rigid-tilt mode, all modes precess retrogradely without growth or decay.
Suppose the system contains only two components, e.g., one planet interacting with a connected disk. The rigid solution satisfies
\[ \begin{bmatrix} \omega J_1 + C & -C \\ -C & \omega J_2 + C \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]  
(32)
where \( C = C_{12} \) is the coupling coefficient between the planet and the disk. The eigenvalues and eigenvectors are
\[ \omega = 0, \quad \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \]  
(33)
a trivial solution corresponding to a rigid tilt, and
\[ \omega = -\frac{C(J_1 + J_2)}{J_1 J_2}, \quad \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \propto \begin{bmatrix} J_2 \\ -J_1 \end{bmatrix}, \]  
(34)
corresponding to a retrogradely precessing mode in which the tilt of each component is inversely proportional to its angular momentum (in order to conserve the total angular momentum).
In a three-component system, e.g., a planet interacting with interior and exterior disks, the rigid solutions satisfy
\[ \begin{bmatrix} \omega J_1 + C_{12} + C_{13} & -C_{12} & -C_{13} \\ -C_{12} & \omega J_2 + C_{12} + C_{23} & -C_{23} \\ -C_{13} & -C_{23} & \omega J_3 + C_{13} + C_{23} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \]  
(35)
where subscript 1 refers to the planet and subscripts 2 and 3 to the inner and outer disks.
In the general case, the eigenvalues and eigenfunctions are algebraically complicated. However, consider the case in which the planet is much less massive than either disk. The first nontrivial mode is the equivalent of equation (34), but involving the two dominant components:
\[ \omega \approx -\frac{C_{23}(J_2 + J_3)}{J_2 J_3}, \quad \begin{bmatrix} W_2 \\ W_3 \end{bmatrix} \propto \begin{bmatrix} J_3 \\ -J_2 \end{bmatrix}. \]  
(36)
The planet's tilt is then driven according to
\[ W_1 = \frac{C_{12} W_2 + C_{13} W_3}{\omega J_1 + C_{12} + C_{13}}. \]  
(37)
Secular resonance occurs here if the frequency is such that the denominator vanishes.
Secular resonance of this type could also occur, for example, in a system of two massive planets and one low-mass disk. In this case the disk would be responding globally as a rigid body. This is quite different from the situation in which a secular particle resonance is located somewhere within a disk.
The second nontrivial mode has
\[ \omega \approx -\frac{C_{12} + C_{13}}{J_1}, \quad |W_2|, |W_3| \ll |W_1|. \]  
(38)
Here only the planet's orbit is significantly tilted and precesses at a rate determined by the two disks.

7. NEARLY RIGID RESPONSE

The internal torque \( \mathcal{G}_k \) required for a disk to respond rigidly can be determined from equation (20). In general, equation (21) will then indicate that a slight warping is in fact required. The rigid solution can be understood as the leading term in an expansion of the solution in powers of a small parameter. If \( \epsilon = H/r \) is the angular semithickness of the disk and \( c_s \sim H \Omega \) the isothermal sound speed, the small parameter here is a characteristic value of \( \omega^2 c_s^2 \), or \( (\omega c_s^2)^2 \). This parameter is small when the disk can communicate effectively, i.e., when the time taken for a bending wave to propagate across the disk is less than the wave period.
Now suppose the disk behaves nearly rigidly, so that we may pose an expansion in powers of this parameter, of the
form
\[ \omega = \omega^{(0)} + \omega^{(1)} + \cdots, \]  
\[ W_{p i} = W_{p i}^{(0)} + W_{p i}^{(1)} + \cdots, \]  
\[ W_{d k} = W_{d k}^{(0)} + W_{d k}^{(1)}(r) + \cdots, \]  
\[ \mathcal{G}_k = \mathcal{G}_k^{(0)}(r) + \mathcal{G}_k^{(1)}(r) + \cdots. \]  

At leading order we obtain the rigid solution described in § 6, denoted by the superscript (0). At the next order we find

\[
\begin{align*}
\imath \omega^{(1)} J_{p i} W_{p i}^{(0)} + & \imath \omega^{(0)} J_{p i} W_{p i}^{(1)} \\
eq & \sum_{j \neq i} G m_{pj} m_{pi} K(r_{pj}, r_{pi}) \Pi(W_{p j}^{(1)} - W_{p i}^{(1)}) \\
& \quad + \sum_{k} \int_{a_k}^{b_k} G m_{pi} \Sigma_k K(r, r_{pi}) \Pi(W_{d k}^{(1)} - W_{d k}^{(1)}) \\
& \quad + \sum_{j} G \Sigma_k \sum_{r} G r(1) \Pi(W_{d j}^{(1)} - W_{d k}^{(1)}) 2 \pi r dr,
\end{align*}
\]

subject to the boundary conditions
\[ \mathcal{G}_k^{(1)}(a_k) = \mathcal{G}_k^{(1)}(b_k) = 0. \]

After some manipulations, we may eliminate \( \mathcal{G}_k^{(1)} \) by integration and obtain

\[
\begin{align*}
\omega^{(1)} \sum_{j \neq i} J_{p j} W_{p j}^{(0)} = & \sum_{k} \int_{a_k}^{b_k} \frac{4}{\mathcal{I}_k r^2 \Omega^3} \\
& \times \left( -\omega^{(0)} - \frac{\kappa^2 - \Omega^2}{\Omega^2} + i \alpha \Omega \right) \\
& \times | \mathcal{G}_k^{(0)} |^2 2 \pi r dr.
\end{align*}
\]

Since \( \mathcal{G}_k^{(0)} \) does not depend on \( \alpha \) but is proportional to \( \Sigma \), this shows that nearly rigid modes are damped at a rate \( \text{Im}(\omega^{(1)}) \) proportional to \( \alpha/\epsilon^2 \) (for a disk and planetary system of given dimensions). The imaginary part of equation (47) is a special limit of the conservation law derived in Appendix A.

It is important to note that the self-gravitational term in equation (20) has no effect on the results we have derived. The reason for this is that, if a disk behaves nearly rigidly, the self-precession of the disk is negligibly small even if the mass of the disk exceeds that of the planets.

8. MEAN-MOTION RESONANCES

So far we have treated only secular interactions between the planets and disks. However, a planet also interacts with a disk through mean-motion resonances (Lindblad resonances and vertical resonances). The launching of waves at these locations exerts a torque on the planet. If the planet’s orbit is inclined with respect to the resonant annulus of the disk, this inclination will evolve in time.
These quantities are listed in Tables 1 and 2 for all resonances satisfying $|r_r - r_p| > 0.1 r_p$. Note that the inner Lindblad resonances act in the opposite sense to the other resonances.

Although equation (49) formally requires the disk tilt $W$ to have a cusp at each resonance, these cusps will be very weak in a disk with good radial communication. For a disk that responds nearly rigidly, the effect of the mean-motion resonances is to add a small imaginary part to the planet-disk coupling coefficient $C^{pl}$:

$$C^{pl} \rightarrow C^{pl} + 2ni \sum_j \frac{G M_*^2}{r_j} s_{ij} r_{ij}.$$  

(54)

The sum over resonances is always positive for an exterior disk but usually negative for an interior disk. According to equation (31), positive sums of this kind tend to cause growth of the nontrivial normal modes. This must compete, however, with the viscous decay implied by equation (47).

### 9. MODEL FOR DISK AND NUMERICAL METHOD

We adopt a simplified model for the solar nebula. The disk model is similar to that described in Lubow & Ogilvie (2000). It is characterized by an angular semithickness $H/r = \epsilon$ constant and a surface density $\Sigma = \Sigma_0 f(r)$, where $\Sigma_0$ is a constant and $f(r)$ a function that is close to unity in most of the disk but tapers linearly to zero at the edges (including gaps around planets). The width of the tapers is equal to the local value of $H$. The vertical structure is that of a polytrope of index $n$, which gives $\Sigma = \Sigma_0 H^2/((2n+3))$ (as defined in eq. [3]) if $H$ is the true semithickness of the polytrope. The basic parameters of the model are then $\epsilon, \alpha, n$, together with the dimensions and mass of the disk. We normalize the surface density by quoting a nominal disk mass,

$$m_{\text{disk}} = \int_0^{r_{\text{out}}} \Sigma_0 r^{-3/2} 2\pi dr = 4\pi\Sigma_0 r_{\text{out}}^{-1/2},$$  

(55)

where $r_{\text{out}}$ is the outermost radius. The true disk mass is slightly less than $m_{\text{disk}}$ owing to the various edges.

We consider the nebula in the presence of a Jupiter-like planet. As standard parameters, we adopt $m_p = 0.001 M_*$, $m_{\text{disk}} = 0.01 M_*$, $\epsilon = 0.075$, $\alpha = 0.005$, and $n = 3/2$. The planet creates a gap in the nebula, leading to two separate disks whose parameters are taken to be $a_1 = 0.1 r_p$, $b_1 = (1 - g) r_p$, $a_2 = (1 + g) r_p$, and $b_2 = 20 r_p$. The standard relative half-width of the gap is taken to be $g = 0.2$. This situation may be compared with Jupiter in a solar nebula extending out to about 100 AU. The effect of varying the important parameters will be considered below.

Numerical solutions to the equations of § 5 are obtained by discretizing the disk, converting the integrals into sums, and converting derivatives into centered finite differences, with $W_{\text{disk}}$ defined on a set of $n_1$ rings for each disk and $\varphi_j$ on the boundaries between neighboring rings. Planet-disk and disk-disk interactions are computed with softened kernels $K_m$, as defined by equation (12). The smoothing length $h = \max(H_1, H_2)$ is taken to be the maximum of the semithicknesses of the two rings concerned. Equation (49) is implemented by identifying the ring in which each resonance falls and representing the delta function as $1/\delta r$, where $\delta r$ is the width of the ring. The problem then reduces to a generalized eigenvalue problem involving a non-Hermitian matrix of dimension $n_r + n_d(2n_1 - 1)$. This is solved numerically using the FORTRAN Sun Performance Library routine ZGEGV on a Sun Ultra 10, using $n_r = 200$. Numerical results are quoted in units such that $G = M_* = r_p = 1$. The modes are normalized such that $W_p = 1$.

### Table 1

| $r_r/r_p$ | $r_r$ | Type | $m$ |
|----------|-------|------|----|
| 0.48070  | 0.0133 | IVR  | 2  |
| 0.63000  | -1.1780 | ILR  | 2  |
| 0.63000  | 0.0696 | IVR  | 3  |
| 0.71144  | 0.1916 | IVR  | 4  |
| 0.72631  | -3.7521 | ILR  | 3  |
| 0.76314  | 0.4009 | IVR  | 5  |
| 0.79910  | 0.7200 | IVR  | 6  |
| 0.82550  | -7.6763 | ILR  | 4  |
| 0.82550  | 1.1716 | IVR  | 7  |
| 0.84570  | 1.7780 | IVR  | 8  |
| 0.86180  | -12.9485 | ILR  | 5  |
| 0.86180  | 2.5617 | IVR  | 9  |
| 0.87480  | 3.5451 | IVR  | 10 |
| 0.88550  | -19.5683 | ILR  | 6  |
| 0.88550  | 4.7508 | IVR  | 11 |
| 0.89460  | 6.2012 | IVR  | 12 |

### Table 2

| $r_r/r_p$ | $r_r$ | Type | $m$ |
|----------|-------|------|----|
| 2.08000  | 0.0191 | OVR  | 2  |
| 1.58740  | 1.4925 | OLR  | 1  |
| 1.58740  | 0.0880 | OLR  | 3  |
| 1.40570  | 0.2272 | OLR  | 4  |
| 1.31040  | 4.3077 | OLR  | 2  |
| 1.31040  | 0.4589 | OLR  | 5  |
| 1.25115  | 0.8055 | OLR  | 6  |
| 1.21140  | 8.4664 | OLR  | 3  |
| 1.21140  | 1.2895 | OLR  | 7  |
| 1.18240  | 1.9334 | OVR  | 8  |
| 1.16040  | 13.9710 | OLR  | 4  |
| 1.16040  | 2.7595 | OVR  | 9  |
| 1.14310  | 3.7904 | OVR  | 10 |
| 1.12920  | 20.8221 | OVR  | 5  |
| 1.12920  | 5.0485 | OVR  | 11 |
| 1.11778  | 6.5564 | OVR  | 12 |
| 1.10820  | 29.0200 | OVR  | 6  |
| 1.10820  | 8.3364 | OVR  | 13 |
| 1.10010  | 10.4110 | OVR  | 14 |
larger precession rates and damping rates (with or without mean-motion resonances). They are the proper bending modes of the disks, modified by gravitational coupling to the other components of the system.

When we proceed to the standard model, the numerical solution becomes significantly more complicated. The outer disk extending to $b_2 = 20$ is not able to maintain good radial communication for typical precession rates. The precession frequencies estimated from the nearly rigid theory are comparable to the frequencies of global bending modes in the outer disk. As a result, the nearly rigid theory is no longer accurate for the outer disk.

To simplify the presentation of results, we focus on three modes that are of greatest interest in that they involve the least warping and have the smallest precession frequencies and damping rates. “Mode 1” and “mode 2” are related to the nearly rigid modes mentioned above, while “mode 3” derives from a bending mode of the outer disk that interferes with the other two when the outer disk fails to behave rigidly. The structures of the three modes are plotted in Figure 1. The modes include the effects of mean-motion resonances.

In Figures 2 and 3 we plot the precession rates and decay rates of these three modes for the standard model, as we vary each of the parameters $\alpha$, $\epsilon$, $m_{\text{disk}}$, and $g$ about its standard value. Notice that the variation of decay rate with some parameters is nonmonotonic. In particular, the decay rate of mode 1 peaks at $m_{\text{disk}} \approx 0.02$.

In Appendix A, we derive an expression for the local decay rate of a warped disk due to dissipation. In Figure 4 we plot for mode 1 the decay rate, as given by equation (A3). Notice that the outer disk dissipation dominates over the inner disk dissipation. Although the dissipation peaks near the planet, it is broadly distributed throughout the outer disk. In the outer disk, the warp $|dW/dr|$ is greatest near the disk radial midpoint and is broadly distributed. Quantity $|dW/dr|$ must vanish at the inner and outer disk edges, because of boundary condition (22). The reason that the peak in Figure 4 in the outer disk occurs near the planet is that the magnitudes of various base-state quantities, such as $\Sigma$ and $\Omega$, are largest there.

In summary, an inclined planet gives rise to large-scale warps. The damping effects of secular interactions dominate over tilt excitation by mean-motion resonances in our model system, provided $x \gtrsim 0.001$. For the standard model, the decay time of the longest lived mode, mode 1, is about $2.7 \times 10^5$ yr. Although this timescale is short compared to typical disk lifetimes of several million years, it is comparable to or longer than the expected planetary migration timescale at this radius (Lin, Bodenheimer, & Richardson 1996; Ward 1997).

11. SIMPLE ESTIMATES

We compare the numerical results with the nearly rigid theory of § 7 for the standard disk model. To do this, we carry out some rough estimates and drop all factors of order unity. Consistent with our standard model, we assume below that the mass of the planet is less than or comparable to the masses of the two disks and also that the angular momentum of the outer disk is greater than or comparable to the angular momenta of the planet and inner disk.

The coupling coefficients are dominated by the parts of
the disk closest to the planet, where $|r - r_p|/r_p \sim g$. Here the kernel may be estimated as $K \sim 1/(g^2 r)$. Thus,

$$C^{pd} \sim \frac{1}{g} G m_p \Sigma r$$

(56)

and

$$C^{dd} \sim G \Sigma^2 r^3.$$  

(57)

Here $\Sigma$ is a typical surface density near the planet. The precession rate of a mode can be estimated from either equation (36) or equation (38) as

$$\omega \sim \frac{G \Sigma}{g r \Omega}.$$  

(58)

We then estimate $\mathcal{B}$ from equation (20) as

$$\mathcal{B} \sim i \omega \Sigma r^4 \Omega W.$$  

(59)

We apply these estimates to equation (47) to obtain an estimate of the viscous damping rate,

$$\text{Im}(\omega) \sim -\frac{\alpha}{\epsilon^2} \frac{(\Sigma r^2)^2}{M_\ast^2} \Omega.$$  

(60)

When mean-motion resonances are taken into account, the planet-disk coupling coefficient acquires a small imaginary part, as in equation (54). The sum over resonances can be estimated using arguments adapted from BGT. For $g \ll 1$, the number of resonances in the disk scales as $m \sim 1/g$. The strengths of the Lindblad resonances, which dominate here, scale as $s_r \sim m^2$. Thus,

$$\text{Im}(C^{pd}) \sim \frac{1}{g^3} \frac{G m_p \Sigma r}{M_\ast}.$$  

(61)

From equation (36) this would provide a resonant growth rate

$$\text{Im}(\omega) \sim \frac{1}{g^3} \frac{m_p \Sigma r^2}{M_\ast} \Omega.$$  

(62)

The ratio of viscous decay to resonant growth can then be estimated as

$$\frac{\text{decay}}{\text{growth}} \sim \frac{\alpha g \Sigma r^2}{\epsilon^2 m_p}.$$  

(63)

The first factor depends on the viscosity and temperature of the disk. The second is related to the ratio of the disk mass within (say) 2 gap widths of the planet to the mass of the planet itself. For our standard parameters, the first factor is close to unity, while the second is somewhat less than unity.
This favors tilt growth, provided that the disk indeed behaves nearly rigidly.

12. CONNECTION WITH WKB BENDING WAVE THEORY

Disks subject to forcing by a misaligned companion have been studied in the context of planetary rings, but with strikingly different results. Within Saturn's A ring, tightly wrapped waves are launched from a mean-motion vertical resonance by a misaligned satellite, as described by Shu, Cuzzi, & Lissauer (1983). Furthermore, the torque carried by the waves is independent of $\alpha$ and $\epsilon$, unlike the present case.

As emphasized earlier, the difference lies in the fact that the waves here are of low frequency such that the parameter $\omega r/c_s$ is typically less than unity for protostellar disks with secular resonances. For the mean-motion vertical resonances in planetary rings, this quantity is much greater than unity.

Nonetheless, a formal connection between the equations used in this paper and those used for planetary rings can be made by considering an artificially cold protostellar disk, such that $\omega r/c_s$ is much greater than unity. Furthermore, we consider the case that the disk mass is much greater than the planetary mass. Appendix B outlines the derivation of the dynamical equations in the WKB limit. In that limit we obtain

$$D(r)W - 2\pi \sigma_k G \Sigma \frac{dW}{dr}$$

$$+ \int \frac{\Omega^3}{2\Sigma \omega + (\kappa^2 - \Omega^2)/2\Omega - i\omega \Omega} \frac{d^2W}{dr^2} \approx \frac{2Gm_p}{r^2}$$

$$\times K(r_p, r)W_p,$$

(64)

where $\sigma_k = \pm 1$ with the sign depending on whether the
waves are trailing or leading, and

$$D(r) = 2\omega\Omega + \frac{2GM_p}{r^2} K(r_p, r) .$$

(65)

For $\omega \ll \Omega$, we can reexpress $D(r)$ in the standard form

$$D(r) = \mu^2 - (\Omega' - \omega)^2 ,$$

(66)

where

$$\mu^2 = \frac{\partial^2 \Phi'}{\partial z^2} \bigg|_{z=0} , \quad r\Omega'^2 = \frac{\partial \Phi'}{\partial r} \bigg|_{z=0} .$$

(67)

Potential $\Phi'$ is due to the star and axisymmetric contributions of the planet but does not include contributions due to the disk. This equation is identical in form to equation (20) of Shu et al. (1983) for $m = 1$, apart from the pressure term, the third term in equation (64).

The pressure term arises because the warping of the disk introduces horizontal pressure gradients. The resulting shearing horizontal motions give rise to the hydrodynamic torque $2\pi G$, which is the dominant factor in the propagation of bending waves in protostellar disks. The warping is described by the $n = 0$ mode in the notation Lubow & Pringle (1993). The dispersion relation for adiabatic oscillations of this mode in an exactly Keplerian disk, whose unperturbed state is vertically isothermal, follows from equation (54) of Lubow & Pringle (1993) with dimensionless frequencies $F = (\Omega - \omega)/\Omega \approx 1$ and $\kappa = 1$. In the low-frequency regime ($| \omega | \ll \Omega$), one obtains then that $\omega = \pm k\Omega/2$, independent of the exponent for adiabatic compression $\gamma$. Although the form of the dispersion relation is reminiscent of that for a compressive acoustic mode, this mode is largely incompressible, as suggested by the lack of dependence on $\gamma$ in the dispersion relation. The same dispersion relation (as was obtained by Papaloizou & Lin 1995) follows from equation (64), with $W$ varying as $\exp (ikr)$ and $J = (c_s/\Omega)^2 \Sigma$, and by taking $G = 0$ (ignoring gravity), $\kappa = \Omega$ and $\alpha = 0$.

For this limit where the planet mass is small compared to the disk mass, the resonance condition $D(r) = 0$ becomes

$$\frac{m_p}{m_d} K(r_p, r) \left( \frac{r_p}{r} \right)^{1/2} = 1 ,$$

(68)

where $\langle K \rangle$ denotes the mass-weighted average of $K$ over the disk (see Appendix B for more details). For the standard disk model, the resonance condition is satisfied at radial distances of about 0.7 and 1.3 within the inner and outer disks, respectively.

This system contains a secular particle resonance where the precession rate of a particle matches the precession rate of the disk-planet system. A free particle (e.g., an asteroid) will precess because of the gravitational effects of both the disk and the planet. That is, a free particle resonance will occur when $D(r) = 0$ in equation (66), but $\Phi'$ now has an additional contribution due to the disk. The particle resonance location is therefore not the same as the disk resonance location. In other words, in locating a particle secular resonance, the gravity of the gas disk must be included. Particles oscillating vertically in a nonoscillating gas disk have an additional vertical restoring force that alters their precession frequency and shifts their resonance site. This is a special property of the $m = 1$ bending modes (tilt modes), which is a consequence of the fact that there is no self-precession in a rigidly tilted disk. Near resonance, a tilt mode essentially behaves rigidly, because of its locally long wavelength.

In the case of a planetesimal (particle) disk, the nature of the pressure and viscous interaction may well differ from that of a gaseous disk. On the other hand, the secular resonance condition given by equation (68) likely applies to a low-mass planet interacting with a cold planetesimal disk.

13. PLANET IN CENTRAL HOLE

Current models for the early evolution of close-orbiting extrasolar planets, such as 51 Peg b (Mayor & Queloz 1995) with an orbital radius of about 0.05 AU, involve the planet migrating into a central hole in the disk (Lin et al. 1996). The planet is envisioned to migrate into the hole by disk tidal forces until its 2:1 outer Lindblad resonance lies just inside the inner edge of the disk. The planet would then remain at this radius, without further migration.

We consider the tilt stability involving both secular interactions and mean-motion resonances of such a planet-disk system, based on the standard model parameters in § 9. In this case, there is only an outer disk of mass 0.01 $M_*$, which extends from just outside the planet’s 2:1 outer Lindblad resonance 1.3$r_p$ (0.066 AU) to outer radius 2000$r_p$ (100 AU). Other disk parameters are the same as in the standard model ($e = 0.075, \alpha = 0.005$, and $n = 3/2$). The only mean-motion resonance present in the disk is the rather weak 1:3 resonance (2.08$r_p$), which acts to increase inclination.

Figure 5 shows the eigenfunction of the lowest mode of the system. The eigenfrequency is $-5.11 \times 10^{-3} + 4.6 \times 10^{-5}i$ in units of $\Omega_p$, corresponding to a decay timescale of only $3.9 \times 10^4$ yr. The inner part of the disk interacts nearly rigidly with the planet, owing to good radial communication $c_r/c_i \ll 1$ (Fig. 5, lower panel). However, the outer part of the disk is not in good communication ($c_r/c_i$ is of order unity), resulting in a warp and dissipation.

14. DISK AND TWO PLANETS

We consider here the case of Jupiter and Saturn interacting with the portion of the solar nebula that lies interior to the orbit of Jupiter, using the equations of § 5. This case is of interest because the classical $v_{16}$ nodal secular resonance lies within the disk. We ignore the effects of mean-motion resonances.

The rigid tilt model involving three objects (here the two planets plus disk) was considered at the end of § 6. As is mentioned there, a secular resonance of the two planets
We have carried out a normal-mode analysis of planet-disk systems that includes the effects of secular interactions and mean-motion resonances. The planets and disks interact gravitationally, and the disks communicate internally through gas pressure, self-gravity, and turbulent viscosity. Secular interactions of misaligned planet-disk systems give rise to global effects in the disks, owing mainly to the effects of gas pressure. The low frequencies ω associated with secular modes allow pressure to communicate effectively over radial distances in disks that are comparable to (and somewhat greater than) the planet’s orbital radius. Over such distances, disks behave rigidly with little dissipation. On the other hand, over larger distances (~c/ω), which may arise in a continuous disk exterior to the planet’s orbit, the radial communication breaks down and large-scale warps occur, along with enhanced dissipation (see Figs. 1 and 5). The protostellar disk response to secular interactions typically falls between that of a global nearly rigidly tilted disk (see §7) and that of a locally launched wave (see §§12 and 14).

Secular interactions act to decrease inclination by means of turbulent dissipation of horizontal shearing motions within the warp for a simple z disk prescription. The dissipation of the warp is peaked somewhat near the planet, but is broadly distributed throughout the disk (Fig. 4). Secular interactions are in competition with the effects of mean-motion resonances (BGT), which can act to increase inclination. For standard disk parameters, the secular interactions involving a single planet generally dominate, and the inclination decays in time. For a young Jupiter interacting with the solar nebula, tilt decay takes places for z ≥ 0.001, with a typical alignment timescale of order 10^5 yr (see Fig. 3). For a close-orbiting planet in the central hole of the disk, the alignment is more rapid. These results suggest that a planet formed within a disk will remain coplanar with the disk, at least as a consequence of disk interactions.

The classical ν16 particle secular resonance, due to Jupiter and Saturn, is expected to have resided within the solar nebula. Again, because of pressure effects, this resonance is greatly broadened in the gas disk. Consequently, the nebula responds nearly rigidly, with some decay in inclination. A very cold planetesimal disk might exhibit a genuinely wavelike response (see Fig. 6).

The analysis in this paper has focused mainly on the effects of a single planet interacting with its interior and exterior disks. For a system of a multiple planets that open multiple disk gaps, the situation is more complicated because of reflections that occur at the disk edges associated with the gaps. Such gaps can decrease the effective size of the disk region over which a planet can interact, thereby resulting in a weaker warp and weaker dissipation. The hydrodynamic torque cannot propagate across a gap, while the gravitational torque may at least partially do so. It is quite possible that much longer lived non-coplanar modes may result if the outer part of the nebula is almost disconnected in this way.

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APPENDIX A

DERIVATION OF THE LOCAL DECAY RATE OF BENDING DISTURBANCES

Starting from equations (13)–(15), we obtain the conservation law
\[
\frac{d}{dt} (-L_z) = - \sum_k \int_{\Omega_k} \frac{4\pi \rho_k |\mathcal{G}_k|^2}{\mathcal{J}_k r^4 \Omega^2} 2\pi r dr ,
\]
where
\[
-L_z = \sum_k \int_{\Omega_k} \left( \frac{1}{2} \sum_k r^2 \Omega^2 |\mathcal{W}_{\kappa}|^2 + \frac{2 |\mathcal{G}_k|^2}{\mathcal{J}_k r^4 \Omega^2} \right) 2\pi r dr .
\]

The quantity \( L_z \) is the vertical angular momentum associated with the bending disturbance. When an initially horizontal ring of matter having orbital angular momentum \( J \) is tilted through an angle \( \beta = |W| \), the change in the vertical component of angular momentum is \( J \cos \beta = -\frac{1}{2} J |W|^2 \) when \( |W| \ll 1 \). The term proportional to \( |\mathcal{G}|^2 \) represents an additional angular momentum stored in the bending wave. For a short-wavelength (WKB) bending wave in a non-self-gravitating disk, both terms in the integral for \(-L_z\) are equal. For a nearly rigidly tilted disk, the first term in the integral dominates. In the appropriate limit, this first term agrees with the expression for the angular momentum of a bending wave given by Bertin & Mark (1980, eq. [C12]).

In an inviscid system, \( L_z \) is conserved. When viscosity is present, the positive definite quantity \(-L_z\) decays monotonically to zero. The negative angular momentum of the bending disturbance is transferred to the disk through viscosity and causes accretion. (It is nevertheless consistent that we neglect the time dependence of \( \Sigma \) in a linear theory, because the induced accretion rate is quadratic in \( W \).)

We now define a local damping rate \( \gamma(r) \) by
\[
\gamma = \left( -\frac{1}{L_z} \right) \frac{2\pi \rho \mathcal{G}^2}{\mathcal{J} r^4 \Omega^2} 2\pi r \sum_k (b_k - a_k) .
\]

This has the property that
\[
\frac{\sum_k \int_{\Omega_k} \gamma dr}{\sum_k (b_k - a_k)} = - \frac{1}{2} \frac{d}{dt} \ln (-L_z) .
\]

In the case of a normal mode, the radial average of \( \gamma \) corresponds to the damping rate \( \text{Im} (\omega) \) of the mode.

APPENDIX B

DERIVATION OF THE BENDING WAVE EQUATIONS IN THE WKB LIMIT

In this appendix we derive equations (64) and (68) of the text. We consider equations (19)–(21) for a single planet interacting with a single disk:
\[
\text{i} \omega m_p r_p^2 \Omega_p W_p = \int G m_p \Sigma \mathcal{K}(r, r_p)(W - W_p) 2\pi r dr ,
\]
\[
\text{i} \omega \Sigma r^3 \Omega W = \frac{1}{r} \frac{d\mathcal{G}}{dr} + G m_p \Sigma \mathcal{K}(r, r_p)(W - W_p) + \int G \Sigma \Sigma' \mathcal{K}(r, r')(W' - W) 2\pi r' dr' ,
\]
\[
\text{i} \omega \mathcal{G} + \frac{\kappa^2 - \Omega^2}{2} \mathcal{G} + \alpha \Omega \mathcal{G} = \frac{\mathcal{J} r^3 \Omega^3}{4} \frac{dW}{dr} .
\]

We now apply the WKB approximation for a cold disk, \( \omega r \gg c_s \). In this regime we assume
\[
H^{-1} \gg \left| \frac{d\ln W}{dr} \right| > r^{-1} .
\]

The requirement on \( H^{-1} \) is used in the derivation of the warp equations (see Papaloizou & Lin 1995; Lubow & Ogilvie 2000). From equation (B3), we obtain
\[
\frac{1}{r} \frac{d\mathcal{G}}{dr} \approx - i \frac{\mathcal{J} r^3 \Omega^3}{4 \alpha + (\kappa^2 - \Omega^2)/2\Omega - i\alpha \Omega} \frac{d^2 W}{dr^2} .
\]
For the self-gravity term, we have that

\[
\int G\Sigma^j K(r, r') \langle W' - W \rangle 2\pi r'dr' \approx \pm \pi G^2 r^2 \frac{dW}{dr}.
\] (B6)

To obtain equation (B6), we recognize that self-gravity is dominated by local contributions in the WKB limit and apply the approximation that \(K(r, r') \approx r/\{2\pi (r - r')^2\}\) for \(r' \approx r\). The integral is computed by extending \(r'\) to the complex plane and integrating over a closed contour. The contour begins along the negative real axis, includes a small semicircle around the singularity at \(r' = r\), continues along the real axis, and closes on itself through a large semicircle. The large semicircle resides in either the upper or lower half-plane, depending on whether the waves are trailing or leading. Equation (B6) then follows from the residue theorem. This choice of trailing or leading waves determines the sign of the result. Substituting equations (B5) and (B6) into equation (B2), we obtain equation (64).

Integrating equation (B2) over the disk area and using equation (B1), we have that

\[
J_d \langle W \rangle = -J_p W_p,
\] (B7)

where \(\langle W \rangle\) denotes the angular momentum weighted average of \(W\). In the limit that \(J_p \ll J_d\) we expect that \(W\) on the right-hand side of equation (B1) can be ignored after integration. It then follows from equation (B1) that

\[
\omega = -\frac{\int G\Sigma K(r, r_p) 2\pi r dr}{r_p^2 \Omega_p} = -\frac{C_{pd}}{J_p}.
\] (B8)

Applying this equation for \(\omega\) to resonance condition \(D(r) = 0\), for \(D\) defined by equation (65), we obtain the resonance condition (68).

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