Almost Lie structures on an anchored Banach bundle

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Abstract

Under appropriate assumptions, we generalize the concept of linear almost Poisson structures, almost Lie algebroids, almost differentials in the framework of Banach anchored bundles and the relation between these objects. We then obtain an adapted formalism for mechanical systems which is illustrated by the evolutionary problem of the "Hilbert snake" as exposed in [PeSa].

1 Introduction

Recent developments about geometric formalism on anchor bundles on a finite dimensional manifold have helped to build a general framework for studying mechanical systems. Essentially, these geometric structures concern, linear almost Poisson structures, almost Lie algebroids and almost differentials (see for example [GLMM], [GLMM], [LMM], [Marl], [Marr], [PoPo] and all references inside these papers).

The purpose of this paper is to give a generalization of these geometric structures in the context of Banach anchored bundles. Of course, this framework leads to a lot of obstructions. At first, any local section of a Banach bundle cannot be extended to a global section without some properties of regularity of the typical fiber. So, in the setting of "Banach algebroid", we must impose that the Lie Bracket of sections (defined globally) has a property of "localization". Indeed, if the Banach manifold is regular, this property is always satisfied, as in finite dimension. However, in the general case, we must impose such a condition (see section 3.3). On the other hand, for a Poisson structure on a Banach manifold, we meet the same type of problem but also, if the typical Banach model is not reflexive, we must impose some other conditions (see section 4.1). Finally, the most important obstruction appears in the context of "Lie differential": not only we still meet the problem of localization of sections but, on the opposite of finite dimension, the graded algebra of forms on a Banach space is not generated by elements of degree zero and degree one, so, in general, such a differential is not characterized by its values on elements of these types.

However, by imposing appropriate assumptions, we defined the concept of almost Lie bracket, almost Lie algebroid which is a generalization of Lie algebroid on Banach manifold introduced in [Ana] and [Pel]. Now, recall that in finite dimension, on one hand there exists a bijection between Lie algebroid structures on an anchored bundle and Poisson structures on its dual, and a bijection between Lie algebroid structures and Lie differentials (see for instance [Marl] or [GLMM] among many references). In fact, in our context, if the typical fiber of the anchored bundle is not reflexive, we have a bijection with almost Lie algebroid on an anchored bundle but here with "sub almost Poisson structure" on its dual, that is, such a structure is only defined on the set of sections of a "canonical subbundle" of the dual bundle (see subsection 4.2). Of course, in the framework of paracompact Hilbert manifolds, all these obstructions do not exist and we recover the general setting of the finite dimensional context. On the opposite, in the infinite dimensional context, we do not have any bijection between almost Lie algebroid structures and almost Lie differentials even under appropriate assumptions (see subsection 4.3).

In the following section, we recall the concept of graded exterior algebra, some classical properties of Banach manifolds under which the mentioned previous assumptions can be avoid.
We also precise our notations in local context. The notion of almost Lie algebroid (AL algebroid in short) on a Banach anchored bundle is developed in section 3. Section 4 is devoted to the relation between "sub almost Poisson structure" and AL algebroid in one hand and almost Lie differential and AL algebroid on the other hand. At first we expose the context of sub almost Poisson morphism (sub AP morphism in short) (subsection 4.1). Then we look for the equivalence of AP morphism and AL algebroid structure (subsection 4.2). Finally, under strong appropriate assumption on an almost differential, we can associate an AL algebroid structure (subsection 4.3). As applications of our results, in section 5, we look for an adaptation of classical formalism for mechanical systems: Hamiltonian system, Hamilton-Jacobi equation, Lagrangian system and Euler-Lagrange equation. Finally, in section 6 we illustrate this formalism in the context of the evolution of the "head" of a "Hilbert snake" form the results of [PeSa].

2 Preliminaries and notations

2.1 Graded exterior algebra on a Banach space

Given a Banach space $E$, we denote by $\Lambda^k E^*$ the Banach space of exterior forms of order $k$ of $E$. More precisely, the set $\Lambda^k E^*$ can be identified with the closed subspace of $k$-multilinear skew-symmetric maps in the Banach space $\mathcal{L}_k(E)$ of $k$-multilinear forms on $E$. This space is the closure of the vector space generated by all exterior products of 1 forms $\{\xi_1 \wedge \cdots \wedge \xi_k, \ i_1 < \cdots < i_k\}$. (for a complete description, see [Ram]). Set

$$\Lambda E^* = \bigoplus_{k=0}^{\infty} \Lambda^k E^* = \{\omega = \sum_{k=0}^{\infty} \omega_k \in \Lambda^k E^*, \sup_k ||\omega_k|| < \infty\}.$$ 

Then, $\Lambda E^*$ is a Banach space which, provided with the exterior product, is an algebra.

If we consider the Banach space $E$, isometrically embedded in $E^{**}$, we can define, in the same way, the vector space $\Lambda^k E$ spanned by all exterior products $\{u_{i_1} \wedge \cdots \wedge u_{i_k}, \ i_1 < \cdots < i_k\}$. So, if we set:

$$\Lambda E = \Lambda^1 E = \bigoplus_{k=0}^{\infty} \Lambda^k E = \{u = \sum_{k=0}^{\infty} u_k \in \Lambda^k E, \sum_k ||u_k|| < \infty\}$$

then, $\Lambda E$ is a Banach space, which, provided with the exterior product is a Banach algebra and also a graded algebra. Moreover $\Lambda^k E^*$ is isomorphic to $(\Lambda^k E)^*$ for any $k \geq 0$ (see [Ram]).

Notice that we have $\Lambda^0 E^* = \Lambda^0 E = \mathbb{R}$, $\Lambda^1 E = E$, and $\Lambda^1 E^* = E^*$.

The interior product of $\omega \in \Lambda^k E^*$ by a vector $v \in E$, denoted by $i_v$, is characterized, as usual in the following way:

- if $k \leq 0$, $i_v = 0$ on $\Lambda^k E^*$
- if $k = 1$, then $i_v \omega = \langle \omega, v \rangle \in \mathbb{R}$
- if $k > 1$, for $v_1 \cdots v_{k-1} \in E$, then $i_v \omega(v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1})$
  and then $i_v \omega \in \Lambda^{k-1} E^*$.

In fact, given any fixed $v \in E$, the interior product $i_v$ can be clearly extended to a continuous endomorphism of $\Lambda E^*$ which is a derivation of degree $-1$ of $\Lambda E^*$ that is $i_v$ is send each factor $\Lambda^k E^*$ of the graduation into the factor $\Lambda^{k-1} E^* \Lambda E^*$.

The interior product $i_P$ by a multivector $P \in \Lambda^p E$ is defined in the following way:

- if $k < 0$, $i_P = 0$
- if $k = 0$, $P \in \mathbb{R}$ and for any form $\alpha \in \Lambda E^*$, then $i_P \omega = P \omega$
- if $k \geq 1$ and if $P$ is decomposable, i.e. $P = v_1 \wedge \cdots \wedge v_k$, we set $i_{v_1 \wedge \cdots \wedge v_k} = i_{v_1} \circ \cdots \circ i_{v_k}$.
We can extend, by linearity and continuity, the definition of $i_P$, for any $P \in \Lambda^kE$, to the graded algebra $\Lambda^*E$. For any fixed $P \in \Lambda^kE$ we then get an endomorphism $i_P$ of degree $-p$ of the graded algebra $\Lambda^*E$.

Let $\tau : E \rightarrow M$ be a Banach bundle of typical fiber $E$. In this situation, we denote by:
- $\mathcal{F}(E)$ the algebra of smooth functions on $E$;
- $\Gamma(\tau)$ the $\mathcal{F}$-module of smooth sections $C^\infty$ of this bundle;
- $\Lambda^k\Gamma^*(\tau)$ the $\mathcal{F}$-module of sections of the Banach bundle $\Lambda^kE^*$ of typical fiber $\Lambda^kE^*$;
- $\Lambda^k\Gamma(\tau)$ the $\mathcal{F}$-module of sections of the Banach bundle $\Lambda^kE$ of typical fiber $\Lambda^kE$;
- $\Lambda^\infty\Gamma(\tau)$ the $\mathcal{F}$-module of sections of the Banach bundle $\Lambda E^*$ of typical fiber $\Lambda E^*$;
- $\Lambda^\infty\Gamma(\tau)$ the $\mathcal{F}$-module of sections of the Banach bundle $\Lambda E$ of typical fiber $\Lambda E$.

For any $P$ in $\Lambda^k\Gamma^*(\tau)$, the integer $k$ is called the degree of $P$ and we set $k = \deg P$.

Notice that $\Lambda^0\Gamma(\tau) = \mathcal{F}$ and $\Lambda^1\Gamma(\tau) = \Gamma(\tau)$. For the exterior product of vectors (resp. of forms) we get a structure of graded exterior algebra on $\Lambda^\Gamma(\tau)$ (resp. $\Lambda^\infty\Gamma(\tau)$).

### 2.2 Some classical properties of Banach manifolds

First we recall some classical properties of Banach manifolds. The reader can find complete references about all these properties in [KrMi].

Let $M$ be a smooth Banach manifold modeled on the Banach space $\mathbb{M}$. The manifold $M$ is **paracompact** if and only if the topology of $M$ is metrizable. In particular the Banach space $\mathbb{M}$ must be also paracompact. This is always true for any (eventually non separable) Hilbert space. A Banach space which has $C^k$ partitions of unity is called $C^k$-**paracompact**, for $k \in \mathbb{N} \cup \infty$. Any paracompact manifold modeled on a $C^k$-paracompact Banach space has also $C^k$-partitions of unity and so is $C^k$-paracompact.

The Banach manifold $M$ is said $C^k$-**regular** (resp. **smooth regular**) if for any $x \in M$, there exists an open neighborhood $U$ of $x$ and a $C^k$ (resp. smooth) function $f : U \rightarrow \mathbb{R}$ such that $f(x) = 1$ and the closure of the set $\{z \mid f(z) \neq 0\}$ is contained in $U$; such a function is called a **bump function**. Notice that $M$ is smooth-regular if and only if $\mathbb{M}$ is $C^k$-regular for any $k \in \mathbb{N} \cup \infty$.

Of course not all Banach spaces are $C^k$-regular for $k > 0$: for instance, $l^1(\Gamma)$, for any set $\Gamma$, is not $C^1$ regular (see [KrMi]). But any $C^k$-regular Banach space is paracompact (for more details about regularity and paracompactness see also [Arn], [Kun], [Llo], [Vand] and [Ion]).

Notice that if $M$ is smooth paracompact, then $M$ is smooth regular.

### 2.3 Local coordinates in a Banach bundle

Consider a Banach bundle $(E, \tau, M)$ and the associated dual bundle $(E^*, \tau^*, M)$. Fix $x \in M$ and consider any open neighborhood $U$ of $x$ such that $E_U$ is isomorphic to the trivial bundle $U \times E$, which we always write $E_U \equiv U \times E$; we then say that $E_U$ is trivialized. Then we also have

- $E_U^* \equiv U \times E^*$;
- $T^*E_U^* \equiv U \times E^* \times M^* \times E^*$;
- $TE_U^* \equiv U \times E^* \times M \times E^*$.

Here we consider $E$ as a Banach subspace of $E^{**}$. Taking into account these equivalences, we get the following coordinates:

- $s = (x, u)$ on $E_U \equiv U \times E$;
- $\sigma = (x, \xi)$ on $E_U^* \equiv U \times E^*$;
- $(s, v) = (s, v_1, v_2)$ on $T^*E_U^* \equiv U \times E \times M \times E$;
- $(\sigma, w) = (\sigma, w_1, w_2)$ on $TE_U^* \equiv U \times E^* \times M \times E^*$.
\[(\sigma, \eta) = (\sigma_1, \eta_2) \text{ on } T^*E_{|U} \equiv U \times E^* \times M^* \times E^{**}\]

### 2.4 Local coordinates in basis

Suppose that the Banach space \(M\) has an unconditional, eventually uncountable, basis, \(\{\mu_i\}_{i \in I}\) and denote by \(\{\mu_i^*\}_{i \in I}\) the associated weak-* basis of the dual \(M^*\) (see \([\text{FiWo}]\)). So, each \(z \in M\) can be written in a unique way:

\[x = \sum_{i \in I} x^i \mu_i,\]

Note that we have \(x^i = \mu_i^*(x)\).

On the other hand, each \(\omega \in M^*\) can be "weak-*" written in a unique way:

\[\omega \equiv \sum_{i \in I} \omega_i \mu_i^*\]

which means that, for any \(u \in M\), we have:

\[< \sum_{i \in I} \omega_i \mu_i^*, u > = < \omega, u >\]

In fact we have \(\omega_i = < \omega, \mu_i >\).

Consider a chart \((U, \phi)\) on \(M\). Via the diffeomorphism \(\phi\), we can identify \(U\) with an open set of \(M\) and so any \(x \in U\) can be written in a unique way \(x = \sum_{i \in I} \mu_i^*(x) \mu_i\). We will say that the set of maps \(\{x^i := \mu_i^* : U \to \mathbb{R}\}_{i \in I}\) is the local system of coordinates on \(U\). As the tangent bundle \(TM_{|U}\) is isomorphic to \(U \times M\), we denote by \(\{\frac{\partial}{\partial x_i}\}_{i \in I}\) the basis of each fiber \(T_zM\), for \(z \in U\), canonically associated to \(\{\mu_i\}_{i \in I}\). So any vector field \(X\) on \(U\) can be written in a unique way as:

\[X = \sum_{i \in I} X_i \frac{\partial}{\partial x_i}\]

Moreover, as \(X\) can be identified with a map from \(U\) to \(M\) we have \(X_i = \mu_i^* \circ X\) and so each component \(X_i\) is a smooth function.

In the same way, the cotangent bundle \(T^*M_{|U}\) is isomorphic to \(U \times M^*\). We denote by \(\{dx_i\}_{i \in I}\) the weak-* basis on each fiber \(T^*_zM\), for \(z \in U\), canonically associated to \(\{\mu_i^*\}_{i \in I}\). Again each 1-form \(\omega\) on \(U\) can be weak-* written

\[\omega \equiv \sum_{i \in I} \omega_i dx_i\]

where of course we have

\[< \omega, X > = < \sum_{i \in I} \omega_i dx_i, X >\]

Again, each component \(\omega_i\) is a smooth function.

On the other hand, consider a Banach bundle \(\tau : E \to M\), and suppose that there exists an unconditional basis \(\{e_\alpha\}_{\alpha \in A}\) for \(E\). Consider an open set \(U \subset M\) which is a chart domain and such that \(E_{|U} \equiv U \times E\).

With the previous properties, we denote again by \(e_\alpha\) the constant section \(x \mapsto e_\alpha\) in \(E_{|U}\). Each section \(s \in \Gamma(\tau_{|U})\) can be written as:

\[s = \sum_{\alpha \in A} e_\alpha^*(s) e_\alpha\]

Again each "component" \(u_\alpha = e_\alpha^*(s)\) is a smooth function on \(U\).
So we have the following local coordinates on $E_U$:
- $(x,u) = (x^i,u^a)$ on $E_U$

if $M$ has also an unconditional basis, on $E_U$ the tangent space $T_*E_U$ is spanned by the basis

\[ \{ \frac{\partial}{\partial x^i} \}_i \in \mathbb{N} \text{ and } \{ \frac{\partial}{\partial u^a} \}_a \in \mathcal{A} \text{ where } \{ \frac{\partial}{\partial u^a} \}_a \text{ is identified with the basis } \{ \epsilon_a \}_a \in \mathcal{A} \text{ of } \{ s \} \times \mathbb{E} \]

So we have the following local coordinates on $TE_U$
- $(s,v_1,v_2) = (x^i,u^a,X^i,U^a)$

In the same way, for the dual bundle $\tau_* : E^* \to M$, on $E_U^*$ we have the constant sections $e^*_a : x \mapsto e^*_a$ and any section $\sigma$ of $E_U^*$, we also can write, in a "weak-*" way,

\[ \sigma = w \sum_{a \in \mathcal{A}} \xi_a e^*_a \]

and again each component $\xi_a$ is a smooth function on $U$.

So we have the following local coordinates on $E_U^*$
- $\sigma = (x,\xi) = (x^i,\xi_a)$ ("weak-* coordinates" for $\xi_a$)

if $M$ also has an unconditional basis, on $E_U^*$, the tangent space $T_*E_U^*$ is spanned by the basis

\[ \{ \frac{\partial}{\partial x^i} \}_i \in I \text{ and } \text{"weakly-}*\]

spanned by the basis \( \{ \frac{\partial}{\partial \xi_a} \}_a \in \mathcal{A} \text{ where again } \{ \frac{\partial}{\partial \xi_a} \}_a \text{ is identified with the weak-* basis } \{ e^*_a \}_a \in \mathcal{A} \).

So we have the following local coordinates on $TE_U^*$
- $(\sigma,w_1,w_2) = (x^i,\xi_a,X^i,\Xi_a)$

### 2.5 Derivations and vector fields

Let $M$ be a Banach manifold. Recall that a (global) derivation of $\mathcal{F}$ is a $\mathbb{R}$-linear map $\partial : \mathcal{F} \to \mathcal{F}$ such that:

\[ \partial(fg) = f\partial(g) + \partial(f)g \]

We denote by $\mathcal{D}$ the vector space of all derivations of $\mathcal{F}$.

An operational vector field $\partial$ at $x \in M$ is a derivation of $\mathcal{F}(U)$ for some neighborhood $U$ of $x$ which is compatible with restriction to open $V \subset U$ i.e. $\partial$ induces a unique derivation $\partial_V$ of $\mathcal{F}(V)$ such that

\[ \partial(f)|_V = \partial_V(f|_V) \]

Let $D_xM$ be the vector space of operational vector field at $x$ and $DM = \bigcup_{x \in M} D_xM$ the set of operational vector fields. In fact, the canonical projection $\hat{p}_M : DM \to M$ gives rise to a structure of Banach bundle (see [KrMl]). Unlike to the context of finite dimensional manifolds, if $M$ is not smooth regular, then the set of germs at $x \in M$ of elements of $\mathcal{D}$ can be smaller than $D_x$. On the other hand, any local vector field on $M$ gives rise to an operational vector field but there exist elements of $D_x$ which do not induce local vector fields (for more details see [KrMl]). In particular we have $TM \not\subset DM$.

### 3 Almost Banach Lie Algebroid

#### 3.1 Almost Lie bracket on an anchored Banach bundle

Let $\tau : E \to M$ be a Banach bundle of typical fiber $E$. We will denote by $E_x = \tau^{-1}(x)$ the fiber over $x \in M$.

A Banach morphism bundle $\rho : E \to TM$ is called an anchor. This morphism induces a map, again denoted by $\rho$ from $\Gamma(\tau)$ to $\Gamma(M)$ defined for any $x \in M$ and any section $s$ of $E$ by:

\[ \rho(s)(x) = \rho \circ s(x) \]

We say that $(E,\tau,M,\rho)$ is an anchored Banach bundle.
Local expressions:

In the context of local trivializations (see subsection 2.3), we have:

$$\rho(x, u) \equiv (x, u) \mapsto (x, R_x(u))$$ (1)

where $R : U \to L(\mathbb{E}, M)$.

Suppose that the Banach spaces $M$ and $E$ have basis. According to subsection 2.4, on any appropriate open $U$ in $M$, any anchor $\rho$ is locally characterized by a family $\{\rho^i_{\alpha}\}_{i \in I, \alpha \in A}$ of smooth functions such that:

$$\rho(e_\alpha) = \sum_{i \in I} \rho^i_{\alpha} \frac{\partial}{\partial x^i}$$ (2)

Definition 3.1

1. An almost Lie bracket (AL-bracket for short) on an anchored bundle $(E, \tau, M, \rho)$ is a bracket $[,]_\rho$ which satisfies the Leibniz property:

$$[s_1, fs_2]_\rho = f[s_1, s_2]_\rho + (\rho(s_1))(f) s_2$$

for any $f \in F$ and $s_1, s_2 \in \Gamma(\tau)$.

In this situation, $(E, \tau, M, \rho, [ , ]_\rho)$ is called an almost Lie Banach algebroid (AL-algebroid for short).

2. A Lie bracket (L-bracket for short) on an anchored bundle $(E, \tau, M, \rho)$ is an AL-bracket $[,]_\rho$ which satisfies the Jacobi identity: for all $s_1, s_2, s_3 \in \Gamma(\tau)$,

$$J(s_1, s_2, s_3) = [s_1, [s_2, s_3]] + [s_2, [s_3, s_1]] + [s_3, [s_1, s_2]] = 0$$

In this case $(E, \tau, M, \rho, [ , ]_\rho)$ is called a Lie Banach algebroid (L-algebroid for short)

When $(E, \tau, M, \rho, [ , ]_\rho)$ is a L-algebroid, then the bracket $[,]_\rho$ induces on $\Gamma(\tau)$ a Lie algebra structure. In this case $\rho : (\Gamma(\tau), [ , ]_\rho) \to (\Gamma(M), [ , ])$ is a Lie algebra morphism.

Examples 3.2

1. Let $(E, \tau, M)$ be a Banach subbundle of $(TM, p_M, M)$ which is complemented, i.e. there exists a Banach subbundle $(F, p, M)$ of $(TM, p_M, M)$ such that $T_xM = E_x \oplus F_x$. Let $\pi_1 : TM \to E$ be the Banach morphism associated to the projection of $T_xM$ onto $E_x$ whose kernel is $F_x$.

We define

$$[X, Y]_E = \pi_1[X, Y]$$

where $[,]$ is the usual Lie bracket on vector fields. Then $(E, \tau, M, \rho, [ , ]_E)$ is an AL-algebroid where the anchor is the natural inclusion $\rho$ of $E$ in $TM$. Notice that $(E, \tau, M, \rho, [ , ]_E)$ is a L-algebroid if and only if $(E, \tau, M)$ is involutive. In particular $(TM, p_M, M, Id, [ , ]_E)$ is a L-algebroid.

2. Consider a smooth right action $\psi : M \times G \to M$ of a connected Banach Lie group $G$ over a Banach manifold $M$. Denote by $\mathcal{G}$ the Lie algebra of $G$. We have a natural morphism $\rho$ from the trivial Banach bundle $M \times \mathcal{G}$ into $TM$ which is defined by

$$\rho(x, X) = T_{(x, e)}\psi(0, X)$$

For any $X$ and $Y$ in $\mathcal{G}$, we have:

$$\rho([X, Y]) = [\rho(X), \rho(Y)]$$
where \( \{ \cdot, \cdot \} \) denotes the Lie algebra bracket on \( \mathcal{G} \) (\cite{Bo}, \cite{KrMa}). On the trivial bundle \( M \times \mathcal{G} \), each section can be identified with a map \( \sigma : M \to \mathcal{G} \) we define a Lie bracket on the set of sections by

\[
\{\{\sigma, \sigma'\}\}(x) = \{\sigma(x), \sigma'(x)\} + d\sigma(\xi_{\sigma'(x)}) - d\sigma'(\xi_{\sigma(x)})
\]

We get an anchor \( \Psi : M \times \mathcal{G} \to TM \) by \( \Psi(x, X) = \xi_X(x) \). It follows that \( (M \times \mathcal{G}, \Psi, M, \{\cdot, \cdot\}) \) has a Banach Lie algebroid structure on \( M \).

3. Let \( \pi : N \to M \) be a submersion between Banach manifolds. The subspaces \( V_uN = T_u\pi^{-1}(x) \subset T_xuN \), denoted by \( VN \) defined a Banach sub-bundle of \( PN : TN \to N \) called the vertical subbundle. As the Lie bracket of two vertical vector fields is again a vertical vector field, we get a L-algebroid on \( (VE, \tau_{FE}VE, E) \).

4. Let \( \theta \) be a 1-form on a Banach manifold \( M \) such that \( d\theta \) is a weak symplectic form i.e. the canonical map \( \theta^* : TM \to T^*M \) defined by \( \theta^*(X) = i_Xd\theta \) for any \( X \in T_xM \) is injective (see \cite{OdRa2}). Assume that \( \theta^* \) is closed i.e. \( T^*M = \theta^*(TM) \) is closed in \( T^*_xM \). If \( \dot{\theta} : T^*M \to T^*M \) is the natural inclusion, then we set \( q_M^\sharp = q_M \circ \dot{\theta} : T^*M \to M \) the restriction of \( q_M : T^*M \to M \). Then \( (T^*M, q_M^\sharp, M) \) is a Banach subbundle of \( (T^*M, q_M, M) \). Denote by \( \Pi : T^*M \to TM \) the morphism \( (\theta^*)^{-1} \). We define a structure of L-algebroid on the anchored bundle \( (T^*M, q_M^\sharp, M, \Pi) \) by setting

\[
[\eta, \zeta]^\sharp = \theta^*(\Pi\eta, \Pi\zeta)
\]

where \([\cdot, \cdot]\) is the usual Lie bracket of the vector fields \( \Pi\eta \) and \( \Pi\zeta \). So \( (T^*M, q_M, M, [\cdot, \cdot]^\sharp) \) is a L-algebroid.

This situation precisely occurs on the cotangent bundle \( T^*M \) of any Banach manifold \( M \) where \( \theta \) is the Liouville 1-form on \( T^*M \) (see \cite{La}.

**Definition 3.3**

Let \( (E_i, \tau_i, M, \rho_i, [\cdot, \cdot]_{\rho_i}), \ i = 1, 2 \), be two AL-algebroids (resp. L-algebroids). A morphism \( \Psi \) from \( (E_1, \tau_1, M) \) to \( (E_2, \tau_2, M) \) (over \( Id_M \)) is called an AL-algebroid morphism (resp. L-algebroid morphism) if we have:

1. \( \Psi \circ \rho_2 = \rho_1 \)
2. \( [\Psi(s_1), \Psi(s_2)]_{\rho_2} = \Psi([s_1, s_2]_{\rho_2}) \) for any \( s_1, s_2 \in \Gamma(\tau_1) \)

Notice that if \( (E_i, \tau_i, M, \rho_i, [\cdot, \cdot]_{\rho_i}), \ i = 1, 2, \) are two L-algebroids, any AL-algebroid morphism \( \Psi \) from \( (E_1, \tau_1, M, \rho_1, [\cdot, \cdot]_{\rho_1}) \) to \( (E_2, \tau_2, M, \rho_2, [\cdot, \cdot]_{\rho_2}) \) induces a Lie algebra morphism from \( (\Gamma(\tau_1), [\cdot, \cdot]_{\rho_1}) \) to \( (\Gamma(\tau_2), [\cdot, \cdot]_{\rho_2}) \). In this case, we say that \( \Psi \) is a L-algebroid morphism.

### 3.2 Classical derivations on an AL-algebroid

In this subsection, \( (E, \tau, M, \rho, [\cdot, \cdot]_{\rho}) \) will be an AL-algebroid or L-algebroid.

#### 3.2.1 Lie derivative

Given any section \( s \in \Gamma(\tau) \), the **Lie derivative** with respect to \( s \) on \( \Lambda^0(\tau) \), denoted by \( L^s \), is the graded endomorphism, with degree 0, characterized by the following properties:

1. For any function \( f \in \Lambda^0(\tau) = \mathcal{F} \)

\[
L^s(f) = L_{\rho s}(f) = i_{\rho s}(df)
\]  

(\text{L0})

where \( L_X \) denote the usual Lie derivative with respect to the vector field \( X \)
2. For any $q$-form $\omega \in \Lambda^q \Gamma^*(\tau)$ (where $q > 0$)

\[(L^\rho_q \omega)(s_1, \ldots, s_q) = L^\rho_q (\omega(s_1, \ldots, s_q)) - \sum_{i=1}^{q} \omega(s_1, \ldots, s_{i-1}, [s, s_i], s_{i+1}, \ldots, s_q) \quad (Lq)\]

On the other hand, we can also define for any function $f \in \Lambda^0 \Gamma^*(\tau) = \mathcal{F}$ the element of $\Lambda^1 \Gamma^*(\tau)$, denoted $d_{\rho} f$, by

\[d_{\rho} f = \rho' \circ df \quad (d0)\]

where $\rho' : T^* M \to E^*$ is the transposed mapping of $\rho$.

The Lie derivative with respect to $s$ commute with $d_{\rho}$.

### 3.2.2 Almost exterior differential

The almost exterior differential on $\Lambda^* \Gamma^*(\tau)$, again denoted $d_{\rho}$, (A-differential for short), is the graded endomorphism of degree 1 characterized by the following properties:

1. For any function $f \in \Lambda^0 \Gamma^*(\tau) = \mathcal{F}$, $d_{\rho} f$ is the element of $\Lambda^1 \Gamma^*(\tau)$ defined by $d_{\rho} f = \rho' \circ df$

2. For any $\omega \in \Lambda^q \Gamma^*(\tau)$ ($q > 0$), $d_{\rho} \omega$ is the unique element of $\Lambda^{q+1} \Gamma^*(\tau)$ such that, for all $s_0, \ldots, s_q \in \Gamma(\tau)$,

\[(d_{\rho} \omega)(s_0, \ldots, s_q) = \sum_{i=0}^{q} (-1)^i L^\rho_{s_i} (\omega(s_0, \ldots, \hat{s}_i, \ldots, s_q)) + \sum_{0 \leq i < j \leq q} (-1)^{i+j} \left( \omega([s_i, s_j], s_0, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_q) \right) \]

We then have the following properties which are obvious or which can be proved as in finite dimension:

1. $d_{\rho}(\eta \wedge \zeta) = d_{\rho}(\eta) \wedge \zeta + (-1)^k \eta \wedge d_{\rho}(\zeta)$ for any $\eta \in \Lambda^k \Gamma^*(\tau)$ any $\zeta \in \Lambda^l \Gamma^*(\tau)$ and any $k, l$ in $\mathbb{Z}$

2. For a L-algebroid, we have $d_{\rho} \circ d_{\rho} = d_{\rho}^2 = 0$. In this case we say that $d_{\rho}$ is the exterior differential of the L-algebroid.

As in the context of finite dimension (cf. [Ana]), we can prove:

**Proposition 3.4**

Given two AL-algebroids (resp. L-algebroid) $(E_i, \tau_i, M, \rho_i, [\cdot, \cdot]_\rho_i)$, $i = 1, 2$, let $d_{\rho_i}$ be the associated A-differential (resp. exterior differential). For a bundle morphism $\Psi$ from $(E_1, \tau_1, M)$ to $(E_2, \tau_2, M)$ (over $\text{Id}_M$), we denote by $\Psi^* : \Lambda^p \Gamma^*(\tau_2) \to \Lambda^p \Gamma^*(\tau_1)$ the induced morphism on $p$-forms. Then, $\Psi$ is an AL (resp. L)-algebroid morphism if and only if

\[d_{\rho_2} \Psi^*(\omega) = \Psi^*(d_{\rho_1} \omega)\]

for any $\omega \in \Lambda^k \Gamma^*(\tau_1)$ and any integer $k > 0$.

Recall that, the bracket $[d_1, d_2]$ of derivations $d_1$ and $d_2$ of the graded algebra $\Lambda^* \Gamma^*(\tau)$ of degree $k_1$ and $k_2$ respectively is the derivation $d_1 \circ d_2 - (-1)^{k_1 k_2} d_2 \circ d_1$ of degree $k_1 + k_2$. On the graded algebra $\Lambda^* \Gamma^*(\tau)$ with the A-exterior derivation $d_{\rho}$ we have:

**Proposition 3.5**

For any $s_1$ and $s_2$ in $\Gamma(\tau)$, we have $i_{[s_1, s_2]}(\sigma) = [i_{s_1}, d_{\rho} i_{s_2}](\sigma)$ for any $\sigma \in \Gamma(\tau)$. 

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On one hand, a direct calculation gives the relation
\[ [[s_1, d_\rho], s_2]_\rho (\sigma) = L^\rho_{s_1} (\sigma (s_2)) - L^\rho_{s_2} (\sigma (s_1)) - d_\rho (\sigma (s_1, s_2)) \]
On the other hand according to the definition of \( d_\rho \), we get:
\[ [[s_1, s_2], d_\rho]_\rho (\sigma) = \sigma ([s_1, s_2]_\rho (\sigma)) - L^\rho_{s_1} (\sigma (s_2)) - L^\rho_{s_2} (\sigma (s_1)) - d_\rho (\sigma (s_1, s_2)). \]
\[ \Box \]

### 3.2.3 Almost Schouten-Nijenhuis bracket

An almost Schouten-Nijenhuis bracket (ASN-bracket for short) is an inner composition law in \( \Lambda^\Gamma (\tau) \) (again) denoted by \([., .]_\rho\), characterized by the following properties:

1. \([., .]_\rho\) is a bi-derivation of degree \(-1\), i.e. an \( R \)-bilinear map such that \( \deg [P, Q]_\rho = \deg P + \deg Q - 1 \) which fulfills the following property
   \[ [P, Q \wedge R]_\rho = [P, Q]_\rho \wedge R + (-1)^{(\deg P + 1) \deg Q} Q \wedge [P, R]_\rho \]
2. For all \( f, g \in \Lambda^0 \Gamma (\tau) = \mathcal{F}, \ [f, g]_\rho = 0 \)
3. For all \( s \in \Lambda^1 \Gamma (\tau) = \Gamma (\tau), \ p \in \mathbb{Z} \) and \( Q \in \Lambda^\tau \Gamma (\tau), \ [s, Q]_\rho = L^\tau_{s} Q \)
4. For all \( s_1, s_2 \in \Lambda^1 \Gamma (\tau) = \Gamma (\tau) \), \([s_1, s_2]_\rho\) corresponds to the bracket defined on the (A)L-algebroid
5. For all \( p, q \in \mathbb{Z}, \ p \in \Lambda^p \Gamma (\tau), \ Q \in \Lambda^q \Gamma (\tau), \ [P, Q]_\rho = (-1)^{pq} [Q, P]_\rho \)

The ASN-bracket \([., .]_\rho\) is called Schouten-Nijenhuis bracket (SN-bracket for short) if, for all \( p, q, r \in \mathbb{Z} \) and \( P \in \Lambda^p \Gamma (\tau), \ Q \in \Lambda^q \Gamma (\tau), \ R \in \Lambda^r \Gamma (\tau) \), the ASN-bracket \([., .]_\rho\) satisfies the graded Jacobi identity:
\[ (-1)^{pr} [P, Q]_\rho \wedge R + (-1)^{pq} [Q, R]_\rho \wedge P + (-1)^{qr} [R, P]_\rho \wedge Q = 0 \]

Notice that if we take the canonical L-algebroid \((TM, p_M, M, \text{Id}, [., .])\) the associated ASN-bracket \([., .]_{\text{Id}}\) is the usual Schouten-Nijenhuis bracket on the graded algebra \( \Lambda \Gamma (M) \).

### 3.3 Locality of an almost Lie bracket

In finite dimension it is classical that a an AL-bracket \([., .]_\rho\) on an anchored bundle \((E, \tau, M, \rho)\) respects the sheaf of sections \( \tau : E \to M \) or, for short, is localizable (see for instance \[Marl\]), if the following properties are satisfied:

(i) for any open set \( U \) of \( M \), there exists a unique bracket \([., .]_U\) on the space of sections \( \Gamma (\tau_U) \) such that, for any \( s_1 \) and \( s_2 \) in \( \Gamma (\tau_U) \), we have:
\[ [s_1|U, s_2|U]|U = ([s_1, s_2]_\rho)|U \]

(ii) (compatibility with restriction) if \( V \subset U \) are open sets, then \([., .]_U\) induces a unique AL-bracket \([., .]_V\) on \( \Gamma (\tau_V) \) which coincides with \([., .]_V\) (induced by \([., .]_\rho\)).

By the same arguments as in finite dimension, when \( M \) is smooth regular, we also have:

**Proposition 3.6**

If \( M \) is smooth regular then any AL-bracket \([., .]_\rho\) on an anchored bundle \((E, \tau, M, \rho)\) is localizable.
If $M$ is not smooth regular, we can no more used the arguments used in the proof of Proposition 3.6. Unfortunately, we have no example of Lie algebroid for which the Lie bracket is not localizable. Note that, according to [KoMi] sections 32.1, 32.4, 33.2 and 35.1, this problem is similar to the problem of localization (in an obvious sense) of global derivations of the module of smooth functions on $M$ or the module of differential forms on $M$. In [KoMi] and, to our known, more generally in the literature, there exists no example of such derivations which are not localizable. On the other hand, even if $M$ is not regular, the classical Lie bracket of vector fields on $M$ is localizable. So, there always exists an anchored bundle $\mathcal{A} = TM$ and a Lie bracket algebroid $(TM, Id, M, [\cdot, \cdot])$ for which its Lie bracket is localizable. Moreover, in Examples 3.2 we do not assume that $M$ is regular but, nevertheless, these Lie brackets are also localizable.

Convention 3.7 : in all this paper, from now, we will assume that either $M$ is smooth regular and if $M$ is not regular, then the Lie bracket $[\cdot, \cdot]_\rho$ is localizable.

Remark 3.8 :

1. If $M$ is smoothly paracompact, then $M$ is smooth regular and so any AL-bracket $[\cdot, \cdot]_\rho$ on an anchored bundle $(E, \tau, M, \rho)$ is localizable. On the contrary, when $M$ is paracompact, we can define some AL-bracket $[\cdot, \cdot]_\rho$ "locally": given a locally finite covering $\{U_i, i \in I\}$ of $M$ and a smooth partition of unity $\{\theta_i, i \in I\}$ subordinated to this covering, if $[\cdot, \cdot]_\rho$ is any AL-bracket on $E_{U_i}$ then:

$$[\cdot, \cdot]_\rho = \sum_{i \in I} \theta_i [\cdot, \cdot]_\rho$$

is an AL-bracket on $(E, \tau, M, \rho)$.

2. If $[\cdot, \cdot]_{\rho}$ satisfies the Jacobi identity, then for any open set $U$, $[\cdot, \cdot]_{U}$ satisfies also the Jacobi identity. In this case, $(E_{U}, \rho_{|U}, U, [\cdot, \cdot]_{U})$ is a L-algebroid.

Remark 3.9

Let $(E, \tau, M, \rho, [\cdot, \cdot]_{\rho})$ be a L-algebroid. As its bracket is localizable, then $D = \rho(E)$ is a weak distribution on $M$ (i.e. on each "fiber" $D_x$, $x \in M$, we have a Banach structure such that the identity map from $D_x$ as Banach space into $T_x M$ is continuous see [Pel]) . If the kernel of $\rho$ is complemented in each fiber, then $D$ is integrable (see [Pel]). This situation occurs when $\ker \rho$ is finite dimensional or finite codimensional, or when $M$ is a Hilbert manifold.

Proof of Proposition 3.6

Let $s_1 : M \to E$ be a smooth section of $\tau : E \to M$ which vanishes an open subset $U$ of $M$. We first show that, for any other smooth section $s_2$ of $\tau : E \to M$, the bracket $[s_1, s_2]_{\rho}$ vanishes on $U$. Indeed, choose any point $x \in U$ and choose a smooth bump function $f : M \to \mathbb{R}$ whose support is contained in $U$ and such that $f(x) = 1$. The section $fs_1$ is a (global section) which vanishes identically. Therefore, for any other smooth section $s_2$ we have:

$$0 = [fs_1, s_2]_{\rho} = -[s_2, fs_1]_{\rho} = -f [s_2, s_1]_{\rho} = df(\rho \circ s_2) s_1.$$ 

So at $x$ we have

$$f(x)[s_1, s_2]_{\rho}(x) = df(\rho \circ s_2)(x) = 0.$$ 

Since $f(x) = 1$, we obtain $[s_1, s_2]_{\rho}(x) = 0$. Given any open set $U$ in $M$, we must show that the bracket $[\cdot, \cdot]_{\rho}$ induces an unique bracket $[\cdot, \cdot]_{U}$ on $\Gamma(\tau_{|U})$ such that

$$[s_1|U, s_1|U]_{U} = ([s_1, s_2]_{\rho})_{U}$$

Choose any $x$ in $U$ and, as before, some bump function $f : M \to \mathbb{R}$ whose support is contained in $U$ and such that $f(x) = 1$. Then for any section $s_1$ and $s_2$ in $\Gamma(\tau_{|U})$, $fs_1$ and $fs_2$ are global
sections of $\tau : E \to M$. So $[fs_1, fs_2]_\rho(x)$ is well defined and from the previous argument, this value
do not depends of the choice of the bump function $f$. Therefore, we can set

$$[s_1, s_2]_{U}(x) = [fs_1, fs_2]_{\rho}(x)$$

for any choice of bump function as previously. It follows clearly from this construction that $[\cdot, \cdot]_{\rho}$
is localizable.

\[\triangle\]

Local expressions:
Let $(E, \tau, M, \rho, [[\cdot, \cdot]]_{\rho})$ be an AL-algebroid. In the context of local trivializations (subsection 2.3),
there exists a field

$$C : U \to L(\mathbb{E} \times \mathbb{E}, \mathbb{E})$$

such that for $s(z) = (z, u(z))$ and $s'(z) = (z, u'(z))$ we have:

$$[s, s']_{U}(z) = (z, C(z)(u(z), u'(z)))$$

(3)

Suppose that the typical fiber $E$ has an unconditional basis. According to subsection 2.4,
then for any $x \in M$, there exists an open neighborhood $U$ of $x$ and a set of smooth functions
$\{C_{\alpha\beta}, \alpha, \beta, \gamma \in A\}$ on $U$ such that $[[\cdot, \cdot]]_{U}$ is characterized by:

$$[e_{\alpha}, e_{\beta}]_{U} = \sum_{\gamma \in A} C_{\alpha\beta}^{\gamma} e_{\gamma}$$

(4)

More precisely, if $s = \sum_{\alpha \in A} s_{\alpha} e_{\alpha}$ and $r = \sum_{\alpha \in A} r_{\alpha} e_{\alpha}$, we have:

$$[s, r]_{U} = \sum_{\alpha, \beta, \gamma \in A} C_{\alpha\beta}^{\gamma} s_{\alpha} r_{\beta} e_{\gamma} + \sum_{\alpha \in A} (d\rho(\rho(s)) e_{\alpha} - ds_{\rho}(\rho(r)) e_{\alpha})$$

(5)

Note that, the almost exterior differential $d_{\rho}$ on $\Lambda^{*}(\tau_{U})$ is also localizable i.e. for any open set $U$ in
$M$, there exists a unique graded derivation $d_{U}$ of degree 1 on $\Lambda^{*}(\tau_{U})$, such that $(d_{\rho}\omega)_{U} = d_{U}(\omega|_{U})$
and which is compatible with restriction to open subsets $V \subset U$. So, as for an AL bracket, in
the context of local trivializations (subsection 2.3), given local sections $s(z) = (z, u(z))$ and
$s'(z) = (z, u'(z))$ of $\mathcal{E}_{U}$, and $\omega(z) = (z, \xi(z))$ any section of $\mathcal{E}_{U}$, according to (1) and (3) we have

$$d_{U}\omega(s, s') = \langle D\xi(\rho(s)), u' \rangle - \langle D\xi(\rho(s')), u \rangle - \langle \xi, [s, s']_{\rho} \rangle$$

$$< \langle D\xi(R(u)), u' \rangle - \langle D\xi(R(u')), u \rangle $$

(6)

where $D\xi$ denotes the differential of the map $\xi : U \to \mathbb{E}^{*}$.

In the same way, the Lie derivative $L^{\rho}_{s}$ is localizable.

For the sake of simplicity, for any open subset $U$ in $M$, we note $[[\cdot, \cdot]]_{\rho}$, $d_{\rho}$ and $L^{\rho}$ instead of
its restriction $[[\cdot, \cdot]]_{U}$, $(d_{\rho})_{U}$ and $L^{\rho}_{U}$, to $U$, respectively

4 Almost Lie algebroid defined by a sub AP-morphism or
an A-derivation

The notion of sub almost Poisson morphism is a generalization, in the context of Banach
manifolds, of Poisson morphism or equivalently Lie Poisson structure (see [OdRa2]). The canonical
situation corresponds to the weak symplectic structure on the cotangent bundle $T^{*}M$ of a Banach
manifold $M$ (see Example 5.3 4)
4.1 Sub almost Poisson morphism

Let $M$ be a Banach manifold. An almost Lie bracket (AL-bracket for short) on $\mathcal{F}$ is a $\mathbb{R}$-bilinear skew-symmetric pairing $\{.,.\}$ on $\mathcal{F}$ which satisfies the Leibniz rule, i.e. for any $f,g,h \in \mathcal{F}$
\[
\{f, gh\} = g\{f, h\} + h\{f, g\}.
\]

An almost Poisson morphism (AP-morphism for short) on $M$ is a bundle morphism $P : T^*M \to TM$ which is skew-symmetric according to the duality pairing (i.e. such that $\langle \eta, P\zeta \rangle = -\langle \zeta, P\eta \rangle$ for any $\eta, \zeta \in T^*M$).

We can associate to such a morphism a $\mathbb{R}$-bilinear skew symmetric pairing $\{.,.\}_P$ on $\Gamma^*(M)$ defined by:
\[
\{\eta, \zeta\}_P = \langle \zeta, P\eta \rangle.
\]
Moreover, for any $f \in \mathcal{F}$ we have:
\[
\{f \eta, \zeta\}_P = \langle \zeta, fP\eta \rangle = f\{\eta, \zeta\}_P.
\]
So, we get on $\mathcal{F}$ an almost Lie bracket $\{.,.\}_P$ defined by
\[
\{f, g\}_P = \{df, dg\}_P
\]
For any $f \in \mathcal{F}$ we can associate a unique vector field $\text{grad}_P(f) = -P(df)$ which is called the almost Hamiltonian vector field (A-Hamiltonian gradient for short).

Classically, to an AP-morphism $P$, we can associate a skew-symmetric tensor of type $(3,0)$ $[P,P] : \Gamma^*(\mathcal{T}) \times \Gamma^*(\mathcal{T}) \to \Gamma(\mathcal{T})$ defined by:
\[
[P,P](\eta, \zeta) = P( LP_{\eta} \zeta - LP_{\zeta} \eta + d \langle \eta, P\zeta \rangle) + [P\eta, P\zeta]
\]
for all $\eta, \zeta \in \Gamma^*(\mathcal{T})$.

As in finite dimension, $\{.,.\}_P$ satisfies the Jacobi identity if and only if the the tensor $[P,P]$ vanishes identically (see for instance [MaMo]). In this case, $F$ has a structure of Lie algebra and $(\mathcal{F}, \{.,.\}_P)$ is called a Banach Lie Poisson manifold (P-manifold for short) (see for instance [OdRa1] or [OdRa2]). If the Jacobi identity is not satisfied, we say that we have an almost Banach Lie Poisson manifold (an AP-manifold for short). In this case the vector field $\text{grad}_P(f)$ is called the hamiltonian gradient of $f$ and $P$ induces a morphism of Lie algebra between $(\mathcal{F}, \{.,.\}_P)$ and $(\Gamma(M), [.,.])$.

**Remark 4.1**

In finite dimension, a Poisson manifold is characterized by a bi-vector $\Lambda$ on a manifold $M$ such that the Schouten-Nijenhuis bracket $[\Lambda, \Lambda]$ vanishes identically. On a Banach manifold $M$, an AL-bracket on $\mathcal{F}$ gives rise to an element $\Lambda$ of $\Lambda T^{**}M$ such that
\[
\Lambda(df, dq) = \{f, g\}
\]
Such a bi-vector gives rise to a unique morphism $P : T^*M \to T^*M$ defined by the relation:
\[
\langle \zeta, P\eta \rangle = \Lambda(\eta, \zeta).
\]
for all $\eta, \zeta \in \Gamma^*(M)$ where $T^*M$ is the bidual tangent bundle of $M$.
To get a Poisson manifold, we need the additional condition: $P(T^*M) \subset TM$ (see [OdRa1], [OdRa2]).
Of course $[\Lambda, \Lambda] = 0$ if and only if $[P,P] = 0$.

For examples and more details about P-manifolds the reader can have a look at [OdRa1] or [OdRa2] and some references within these papers.

Let $q^*_M : T^*M \to M$ be a Banach subbundle of $q_M : T^*M \to M$. A bundle morphism $P : T^*M \to TM$ will be called a **sub almost Poisson morphism** (sub AP-morphism for short)

\[1\] this terminology is chosen in analogy to sub-riemannian structures on a manifold
if $P$ is skew-symmetric relatively to the duality pairing i.e. $<\alpha, P\beta> = -<\beta, P\alpha>$ for any $\alpha, \beta \in T^*M$. As before, we get a $\mathbb{R}$-bilinear skew-symmetric pairing $\{\ldots\}$ on $\Gamma(q^*_M)$. The set $F^\circ = \{f \in F : df \in \Gamma(q^*_M)\}$ is a sub-algebra of $F$. Using the same arguments as in [KoM], section 48, we can see that, as above, $P$ induces on $F^\circ$ an almost Lie bracket which will be again denoted by $\{\ldots\}_P$. We will say that $(M, F^\circ, \{\ldots\}_P)$ is a sub almost Banach Lie Poisson manifold (sub AP-manifold for short). Of course, when $\{\ldots\}_P$ satisfies the Jacobi identity, $(F^\circ, \{\ldots\}_P)$, has a Lie algebra structure and we say that $(F^\circ, \{\ldots\}_P)$ is a sub Banach Lie-Poisson manifold (sub P-manifold for short). Then for any $f \in F^\circ$ we can associate a sub almost Hamiltonian gradient $\text{grad}^P(f) = -P(df)$ (sub A-Hamiltonian gradient for short)

Of course, to a sub AP-morphism $P$ on $M$, we can also associate a skew symmetric tensor of type $(3, 0)$: $[P, P] : \Gamma(q^*_M) \times \Gamma(q^*_M) \to \Gamma(\tau)$ using the same definition as in [7], but for $\alpha, \beta \in T^*M$. Again, $\{\ldots\}_P$ satisfies the Jacobi identity if and only if $[P, P]$ vanishes identically. In this case, $(F^\circ, \{\ldots\}_P)$ has a Lie algebra structure and $P$ induces a Lie algebra morphism from $(F^\circ, \{\ldots\}_P)$ to $(\Gamma(M), \{\ldots\})$.

Consider two sub AP-morphisms $P_1$ on the manifolds $M_i$, $i = 1, 2$ and a map $\psi : M_1 \to M_2$. We say that $(M_1, P_1)$ and $(M_2, P_2)$ are $\psi$-related if we have:

$$P_2 = (T\psi)^* \circ P_1 \circ T\psi$$

In this case for the associated AL-bracket $\{\ldots\}_P$ on the algebra $F^\circ_1$ $i = 1, 2$ we have:

$$\{f \circ \psi, g \circ \psi\}_2 = \psi \circ \{f, g\}_1$$

for all $f, g \in F^\circ_1$. Moreover, in this case, $P_1$ is a sub P-morphism if and only if $P_2$ is a sub P-morphism and then $\psi$ gives rise to a Lie algebra morphism between $(F^\circ_1, \{\ldots\}_1)$ and $(F^\circ_2, \{\ldots\}_2)$.

A lot of results about AP-manifolds and P-manifolds can be extended to the context of sub AP-manifolds and sub-P-manifolds when considering the structure of Lie algebra of $(F^\circ, \{\ldots\}_P)$. We do not develop these aspects here.

**Example 4.2**

Let $\omega$ be a non degenerated 2-form on a manifold $M$. We denote by $\omega^\circ : TM \to T^*M$ the associated morphism defined by $\omega^\circ(X) = i_X\omega$ for $X \in T_xM$. Suppose that $T^*M = \omega^\circ(TM)$ is a Banach sub-bundle of $T^*M$. Then $\omega^\circ$ is an isomorphism from $TM$ onto $T^*M$. So $P = (\omega^\circ)^{-1} : T^*M \to TM$ is a sub AP-morphism. We get a sub P-morphism if and only if $\omega$ is closed. In particular, if $M$ is the cotangent bundle $T^*N$ of a Banach manifold $N$, if $\omega$ is the 2 fundamental form on $T^*N$, we get a sub Poisson structure on $T^*N$ which corresponds to the natural weak symplectic structure on $T^*N$ (see Example 4.4). Note that we get a Poisson structure on $T^*N$ if and only if $N$ is modeled on a reflexive Banach space.

**Remark 4.3**

Consider a sub-Poisson morphism $P : T^*M \to TM$, and denote by $D = P(T^*M)$ the associated (weak) distribution on $M$. If the kernel of $P$ is complemented in each fiber, then $D$ is integrable and each leaf is a weak symplectic manifold (see [FP2]). This situation is always satisfied when $\ker P$ is finite dimensional or finite co-dimensional (for instance if $P$ is Fredholm, or injective), or when $M$ is an Hilbert manifold.

Let $\tau : E \to M$ be a Banach bundle and $\tau_* : E^* \to M$ its associated dual Banach bundle. We denote by $F(E^*)$ the set of smooth functions on $E^*$. Any $f \in F(E^*)$ is called linear, if the restriction of $f$ to each fiber $E^*_x = \tau_*^{-1}(x)$ is a linear map.
Lemma 4.6
We then have the following properties:

- In the context of Example 4.2, if we take $w$ with our notations. We can consider which will be defined all "interesting" sub AP-morphisms (see Proposition 4.7 and subsection 4.2).
- Let $P$ on $E^*$ is called linear if for any linear functions $f$ and $g$ on $E^*$ which belong to $F(E^*)$ their bracket $[f,g]_P$ is linear.

Example 4.5
In the context of Example 4.2, if we take $E = TM$, the sub $P$-morphism $\Pi = (\omega^p)^{-1} : T^p(T^*M) \to T(T^*M)$ is a linear sub $P$-morphism on $T^*M$.

To a given Banach bundle $\tau : E \to M$, we will construct a canonical subbundle $T^p E^*$ on which will be defined all "interesting" sub AP-morphisms (see Proposition 4.7 and subsection 4.2).

First of all, for any section $s \in \Gamma(\tau)$, we associate the linear function $\Phi_s$ on $E^*$ defined by

$$\Phi_s(\xi) = \langle \xi, s \circ \tau_*(\xi) \rangle$$

We then have the following properties:

Lemma 4.6
1. The map $s \mapsto \Phi_s$ is linear and injective; we also have $\Phi_{fs} = (f \circ \tau_*)\Phi_s$ for any $s \in \Gamma(\tau)$ and $f \in F$.
2. In local trivializations (subsection 2.3) we have
   $$\langle d\Phi_s(\sigma), w_2 \rangle = \langle w_2, s \circ \tau_*(\sigma) \rangle$$
   for any $w_2 \in E^*$, considered as vertical fiber $T_\tau E^*$;
   if for some $f \in F$, we have $d(\Phi_s + f \circ \tau_*)(\sigma) = 0$ then $s \circ \tau_*(\sigma) = 0$.

Proof of Lemma 4.6

The first part is easy and left to the reader. With the previous notations, if $w_2$ belongs to the vertical part of $T_\tau E^*$ which is $\{\sigma\} \times E^*$ with our notations. We can consider $w_2$ as an element of $E^*_{\tau_*}(\sigma)$. We then have

$$\langle d\Phi_s(\sigma), w_2 \rangle = \lim_{t \to 0} 1/t \langle \langle \sigma + tw_2, s \circ \tau_*(\sigma) \rangle - \langle \sigma, s \circ \tau_*(\sigma) \rangle, w_2 \rangle.$$

Assume that $d(\Phi_s + f \circ \tau_*)(\sigma) = 0$. Given any $w_2 \in E^*_{\tau_*}(\sigma)$. As before, $w_2$ can be considered as a vector in the vertical part of $T_\tau E^*$. From the previous relation we get:

$$\langle d\Phi_s(\sigma), w_2 \rangle = -\langle df(\tau_*(\sigma)), T_\tau \tau_*(w_2) \rangle = 0$$

which ends the proof.

Proposition 4.7
The set

$$T^p E^* = \bigcup_{x \in M} \{ (\sigma, \eta) \in T^*_x E^*, \ \eta = d(\Phi_s + f \circ \tau_*) \circ s \in \Gamma(\tau_U), \ f \in F(U), \ U \ \text{neighbourhood of} \ x \}$$

is a well defined subset of $T^* E^*$ and if $q^p$ is the restriction of $q_E^*$ to $T^p E^*$ then

$$q^p : T^p E^* \to M$$

is a Banach bundle of typical fiber $M^* \times E$. In particular, $T^p E^* = T^* E^*$ if and only if $E$ is reflexive.
Banach subbundle. Of course, we get $T$-ivalence between Poisson structure on $E$.

Now we are able to give an adaptation to the Banach context of the classical result about equiv-

4.2 Relation between AP-algebroid and sub AP-morphism

Theorem 4.8

Let $P : T_0^* E^* \to T^* E^*$ be a linear sub AP-morphism on $E^*$. Then there exists a unique AL-algebroid structure $(E, \tau, M, \rho, [, .]_\rho)$ characterized by:

\[ \Phi_{[s_1, s_2]}\rho = \{ \Phi_{s_1}, \Phi_{s_2} \}_\rho, \text{ for any } s_1, s_2 \in \Gamma(\tau) \] \hspace{1cm} (8)

\[ \{ \Phi_s, f \circ \tau_s \}_\rho = L_s^\rho(f) \circ \tau_s, \text{ for any } f \in \mathcal{F}, s \in \Gamma(\tau) \] \hspace{1cm} (9)

Moreover, $(E, \tau, M, \rho, [, .]_\rho)$ is a L-algebroid if and only if $P$ is a sub-Poisson morphism. Conversely, each AL-algebroid structure $(E, \tau, M, \rho, [, .]_\rho)$ defines a unique linear bracket $[, .]_\rho$ on the sub-ring $\mathcal{F}^\rho(E^*)$ which is associated to a unique linear sub AP-morphism on $E^*$ which is characterized by relations (8) and (9). Moreover, $P$ is a sub-Poisson morphism on $E^*$ if and only if $(E, \tau, M, \rho, [, .]_\rho)$ has a L-algebroid structure.

Example 4.9

According to Example 4.2 and Example 4.5, as in finite dimension, the canonical sub P-morphism $\Pi$ induces on the bundle $(TM, p_M, M)$ a structure of L-algebroid which is, in fact, the canonical L-algebroid structure (see Example 3.2 1).

Local expressions:

In the context of local trivialization (subsection 2.3), recall that $TE^*_U \equiv U \times E^* \times M \times E^*$ and $T^0 E^*_U \equiv U \times E^* \times M^* \times E$. Given an AP-morphism $P$, recall that locally we have the following characterization (see (1) and (3))

$\rho(s) = (., R(u))$ for any $s(z) = (z, u(z))$

$[s, s']_\rho = (., C(u, u'))$ for any $s(z) = (z, u(z))$ and $s'(z) = (z, u'(z))$
so locally we have
\[
< (df', u'), P(\omega(x))(df, u) > = < \xi, [s, s']_\rho > (x) + < df', \rho(x, u) > - < df, \rho(x, u') > = < \xi, C(u, u') > + < df', R(u) > - < df, R(u') >
\]  
(10)
for any functions \( f \) and \( f' \) on \( U \), any sections \( s \) and \( s' \) any form \( \omega(z) = (z, \xi(z)) \).

If \( F \) is a function in \( \mathcal{F}^0(E_U^\ast) \), in local trivialization, we can write \( df = (D_1 F, D_2 F) \) on \( U \times \mathbb{E}^\ast \) where \( D_1 F \) (resp. \( D_2 F \)) is the partial derivative of \( F \) according to the first factor (resp. the second factor). Notice that, \( D_2 F(\sigma) \) belongs to \( \mathbb{E} \). So for any \( F, G \in \mathcal{F}^0(E_U^\ast) \), their Poisson bracket is given by:
\[
\{ F, G \} \rho(\sigma) = < D_1 F, R(D_2 G) > (\sigma) - < D_1 G, R(D_2 F) > (\sigma) - < \xi, [D_2 F, D_2 G]_\rho > (\sigma)
\]  
(11)
Suppose that each Banach space \( E \) and \( M \) has an unconditional basis. According to subsection 2.3 and also local trivializations (subsection 2.3), recall that we have the following local coordinates:
- \((x, u) = (x', u') \) on \( E_U \)
- \( \sigma = (x', \xi) = (x^i, \xi_\alpha) \) on \( E_U^\ast \) ("weak-* coordinates" for \( \xi_\alpha \))
- on \( E_U^\ast \), the tangent space \( T_x E_U^\ast \) is generated by the basis \( \{ \frac{\partial}{\partial x^i} \}_{i \in I} \) and "weakly-\(*" generated by the basis \( \{ \frac{\partial}{\partial \xi_\alpha} \}_{\alpha \in A} \);
- on \( E_U^\ast \), according to Lemma 4.10 part 1 and its proof, each fiber \( T_x^* E \) is generated by the basis \( \{ dx^i \}_{i \in I} \) and \( \{ e_\alpha \circ \tau \}_{\alpha \in A} \). Notice that, we have \( d\xi_\alpha = e_\alpha \circ \tau^* \).

With these notations, any sub AP-morphism \( P : T^0E^\ast \rightarrow TE^\ast \) associated to an AL-bracket \([.,.]_\rho \) (as in Theorem 4.8) are related in the following way:
the Lie bracket \([.,.]_P \) is locally characterized by \([e_\alpha, e_\beta]_U = \sum_{\gamma \in A} C_{\alpha\beta}^\gamma e_\gamma \) (see (4));
The anchor \( \rho \) can be characterized by \( \rho(e_\alpha) = \sum_{i \in I} \rho_{i\alpha} \frac{\partial}{\partial x^i} \) (see (2))
the AP-morphism \( P \) is characterized by:
\[
P(dx^i) = \sum_{\alpha \in A} \rho_{i\alpha}^\alpha \frac{\partial}{\partial \xi_\alpha}
\]
\[
P(d\xi_\alpha) = - \sum_{i \in I} \rho_{i\beta}^\alpha \frac{\partial}{\partial x^i} - \sum_{\gamma \in A} C_{\alpha\beta}^\gamma \xi_\gamma \frac{\partial}{\partial \xi_\beta}
\]
On \( E_U^\ast \), the associated algebra of functions \( \mathcal{F}^0(E_U^\ast) \) are functions on \( E_U^\ast \) which depend on variables \((x_i)_{i \in I} \) and \((\xi_\alpha)_{\alpha \in A} \). The AL-bracket \([.,.]_P \) on \( \mathcal{F}^0(E_U^\ast) \) is characterized by:
\[
\{ \xi_\alpha, \xi_\beta \}_P = - \sum_{\gamma \in A} C_{\alpha\beta}^\gamma \xi_\gamma, \quad \{ x^i, \xi_\alpha \}_P = \rho_{i\alpha}^\alpha, \quad \{ x^i, x^j \}_P = 0
\]
The Poisson Bracket of \( F, G \in \mathcal{F}^0(E_U^\ast) \) is given by:
\[
\{ F, G \}_P = \sum_{i \in I, \alpha \in A} \rho_{i\alpha} \left( \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial \xi_\alpha} - \frac{\partial G}{\partial x^i} \frac{\partial F}{\partial \xi_\alpha} \right) - \sum_{\alpha, \beta, \gamma \in A} C_{\alpha\beta}^\gamma \xi_\gamma \frac{\partial F}{\partial \xi_\alpha} \frac{\partial G}{\partial \xi_\beta}
\]
For the first part of the proof of Theorem 4.10, we need the following Lemma:

**Lemma 4.10**
Consider a linear sub AP-morphism \( P : T^0E^\ast \rightarrow TE^\ast \) on \( E^\ast \) and \([.,.]_P \) the associated bracket on \( \mathcal{F}^0(E^\ast) \).
1. for any section \( s \in \Gamma(\tau) \) and any \( f \in \mathcal{F} \) the bracket \( \{ \Phi_s, f \circ \tau^* \}_P \) belongs to \( \mathcal{F} \).
2. If \( f \) and \( g \) belong to \( \mathcal{F} \), then \( \{ f \circ \tau^*, g \circ \tau^* \}_P = 0 \).
Proof of Lemma 4.10

This proof is an adaptation of the proof of the analogous result in finite dimension, (cf [LMM] for instance)

Consider any \( s, s' \in \Gamma(\tau) \), and \( f \in \mathcal{F} \). Using the Leibniz rule for the bracket \( \{ \cdot, \cdot \}_P \) we get:

\[
\{ \Phi_s, (f \circ \tau_s)\Phi_{s'} \}_P = (f \circ \tau_s)\{ \Phi_s, \Phi_{s'} \}_P + \Phi_s\{ \Phi_s, f \circ \tau_s \}_P
\]

Moreover \( \{ \Phi_s, (f \circ \tau_s)\Phi_{s'} \}_P \) is a linear function on \( E^* \). Since \( (f \circ \tau_s)\{ \Phi_s, \Phi_{s'} \}_P \) is also a linear map, from (12) it follows that \( \Phi_s\{ \Phi_s, f \circ \tau_s \}_P \) is a linear function on \( E^* \) for any \( s' \in \Gamma(\tau) \). So \( \{ \Phi_s, f \circ \tau_s \}_P \) must be constant on each fiber and then we get 1.

On the other hand, by same argument, we have

\[
\{(f \circ \tau_s)\Phi_{s'}, g \circ \tau_s \}_P = (f \circ \tau_s)\{ \Phi_{s'}, g \circ \tau_s \}_P + \Phi_s\{ f \circ \tau_s, g \circ \tau_s \}_P
\]

So from part 1, we deduce that \( \Phi_{s'}\{ f \circ \tau_s, g \circ \tau_s \}_P \) belongs to \( \mathcal{F} \) for any \( s' \in \Gamma(\tau) \). It follows that we must have \( \{ f \circ \tau_s, g \circ \tau_s \}_P = 0 \) and we get 2.

\( \triangle \)

Proof of Theorem 4.8 (adaptation, in our context, of the proof of [LMM] of the same result in finite dimension)

Consider a linear sub AP-morphism \( P : T^*E^* \rightarrow TE^* \) on \( E^* \) and \( \{ \cdot, \cdot \}_P \) the associated bracket on \( \mathcal{F}(E^*) \). As \( \Phi \) is injective, the pairing \( \{ \cdot, \cdot \}_P \) defined by (8) is well defined and is \( \mathbb{R} \)-bilinear and skewsymmetric. From Lemma 4.10 part 1 and the Leibniz property of \( \{ \cdot, \cdot \}_P \) it follows that, for some fixed \( s \in \Gamma(\tau) \), the map \( \rho(s) : f \mapsto \{ \Phi_s, f \circ \tau_s \}_P \) defines a derivation on \( \mathcal{F} \). On the other hand, we have:

\[
\{ \Phi_s, f \circ \tau_s \}_P = d(f \circ \tau_s) \circ P(d\Phi_s) \geq df \circ \tau_s \circ P(d\Phi_s)
\]

It follows that \( \rho(s) = d\tau_s \circ P(d\Phi_s) \) is a vector field on \( M \). Notice that, from the properties of \( \Phi \) (Lemma 4.10 part 1), the Leibniz property of \( \{ \cdot, \cdot \}_P \), and Lemma 4.10 part 2, we have:

\[
\rho(f s) = f \rho(s)
\]

for any \( f \in \mathcal{F} \). As \( P \) is a bundle morphism, it follows that the bracket \( \{ \cdot, \cdot \}_P \) is localizable, so on one hand the same is true for the bracket \( \{ \cdot, \cdot \}_P \) and on the other hand, from (13) we get that \( \rho(s) \) only depends on the value of \( s \) at any point \( x \in M \) so we get a morphism bundle \( \rho : \mathcal{E} \rightarrow M \) defined by \( \rho(u) = \rho(s)(x) \) where \( s \) is any (local) section such that \( s(x) = u \).

From (8) and (9), and Lemma 4.10 part 1, for any \( s_1, s_2 \in \Gamma(\tau) \) and \( f \in \mathcal{F} \) we have:

\[
\Phi_{[s_1, f s_2]} = \{ \Phi_{s_1}, (f \circ \tau_s)\Phi_{s_2} \}_P = (f \circ \tau_s)\{ \Phi_{s_1}, \Phi_{s_2} \}_P + \Phi_{s_2}L_{\rho(s_1)}(f) \circ \tau_s
\]

So we get the following relation:

\[
[s_1, f s_2]_P = [s_1, s_2]_P + \rho(s_1)(f)s_2
\]

Thus we have proved that \( (E, \tau, M, P, \{ \cdot, \cdot \}_P) \) is an AL-algebroid. From the properties of \( \Phi \), it follows that if \( \{ \cdot, \cdot \}_P \) satisfies the Jacobi identity, it implies that \( \{ \cdot, \cdot \}_P \) condition is true if and only if \( P \) is a Poisson morphism.

Conversely, let \( (E, \tau, M, P, \{ \cdot, \cdot \}_P) \) be an AL-algebroid. Note that, from convention ?? the bracket \( \{ \cdot, \cdot \}_P \) is localizable. We want to associate a linear AP-morphism \( P : T^*E^* \rightarrow TE^* \) such that, according to the first part, the induced AL-algebroid structure on \( (E, \tau, M) \) is exactly \( (E, \tau, M, P, \{ \cdot, \cdot \}_P) \). For this we must have:

\[
\{ \Phi_s, f \circ \tau_s \}_P = L_{\rho(s)}(f) \circ \tau_s \quad \text{and} \quad \{ f \circ \tau_s, g \circ \tau_s \}_P = 0
\]

Locally, with the notations of Lemma 4.8, we define \( P_s : T_xE^* \rightarrow T_xE^* \) as follows:
for $\eta = d(\Phi_s + f \circ \tau_*)(\sigma)$, and $\eta' = d(\Phi'_s + f' \circ \tau_*)(\sigma)$ we set
\[ \Lambda(\sigma)(\eta, \eta') = \Phi_{[s,s']}(\sigma) + L_{\rho(s)}(f') \circ \tau_*(\sigma) - L_{\rho(s')}(f) \circ \tau_*(\sigma) \] (14)

As we have seen that $\eta$ (resp. $\eta'$) only depends on $s \circ \tau_*(\sigma)$ and $df \circ \tau_*(\sigma)$ (resp. $s' \circ \tau_*(\sigma)$ and $df' \circ \tau_*(\sigma)$), then it follows that $\Lambda$ is well defined at $\sigma$. Moreover, from its local definition, we get a smooth section of $\Lambda^2T^\circ E^*$. It follows that the map $\eta \mapsto \Lambda(\sigma)(\cdot, \eta)$ defines a linear map $P(\sigma) : T_\sigma^2E^* \to [T_\sigma^2E^*]^*$ directly given by:
\[ P_\sigma(\eta) = \Phi_{[s,\cdot]}(\sigma) + L_{\rho(s)}(\cdot) \circ \tau_*(\sigma) - L_{\rho(\cdot)}(f) \circ \tau_*(\sigma) \] (15)

Using our notations, we have $T_\sigma^2E^* \equiv \{ \sigma \} \times M^* \times E$ and so $[T_\sigma^2E^*]^* \equiv \{ \sigma \} \times M^{**} \times E^*$.

Recall that $P = d\Phi_s + df \circ \tau_*$. On one hand, any $\omega \in \{ \sigma \} \times M^*$ can be written as $\omega = dg \circ \tau_*(\sigma)$ for some $g : U \subset \mathbb{M} \to \mathbb{R}$; so we have
\[ <\omega, P_\sigma(\eta)> = \Lambda_\sigma(\eta, dg \circ \tau_*) = L_{\rho(s)}(g) \circ \tau_*(\sigma) \]

It follows that $P_\sigma(\eta)|_{\{ \sigma \} \times E^*}$ belongs to $\{ \sigma \} \times M$ considered as a subspace of $\{ \sigma \} \times M^{**}$. On the other hand we have
\[ <d\Phi_s', P_\sigma(\eta)> = \Lambda_\sigma(d\Phi_s, d\Phi_s) + \Lambda_\sigma(df \circ \tau_*, d\Phi_s') = \Phi_{[s,s']}(\sigma) - L_{\rho(s')}(f) \circ \tau_*(\sigma) \]

It follows that $P(\sigma)(\eta)|_{\{ \sigma \} \times E^*}$ belongs to $T_\sigma E^* \equiv \{ \sigma \} \times M^{**} \times E$. In conclusion, $P(\sigma)(\eta)$ belongs to $T_\sigma E^* \equiv \{ \sigma \} \times M^{**} \times E$.

Notice that the definition (15) of $P$ is in fact local so $P$ is a smooth bundle morphism.

Of course as usually, $P$ gives rise to a bracket on $\mathcal{F}^0(E^*)$ which is exactly given by (13). This bracket is denoted by $\{ \cdot, \cdot \}_P$. As $P$ is a bundle morphism, $\{ \cdot, \cdot \}_P$ is localizable. Moreover, it satisfies the relations
\[ \{ \Phi_{s_1}, \Phi_{s_2} \}_P = \Phi_{[s_1,s_2]}_P = <d\Phi_{s_1}, P(d\Phi_{s_2})> \quad \text{for any } s_1, s_2 \in \Gamma(\tau) \] (16)

\[ L_{\rho(s)}(f) \circ \tau_* = \{ \Phi_s, f \circ \tau_* \}_P = <df \circ \tau_*, P(d\Phi_s)>, \quad \text{for any } f \in F, s \in \Gamma(\tau) \] (17)

\[ \{ f_1 \circ \tau_*, f_2 \circ \tau_* \}_P = <df_2 \circ \tau_*, P(df_1) \circ \tau_* > = 0, \quad \text{for any } f_1, f_2 \in F. \] (18)

Let $F$ be a smooth linear function on $E^*$ which belongs to $\mathcal{F}^0$. Fix a point $\sigma = (x, \xi) \in E$ and $E^*_U \equiv U \times E^*$ a neighborhood of $\sigma$. We denote by $d_2F$ the partial differential of $F$ relative to $E^*$ in the product $U \times E^*$. As $F$ is linear, there exists a section $S : U \to E^*$ such that
\[ d_2F(\zeta) = <S, \zeta> \]

for any $\zeta \in E^*$. But $F$ belongs to $\mathcal{F}^0$ so $d_2F$ is a section of $T^\circ E^* \to E^*$; then we must have $S : U \to E$ and $d_2F = d_2\Phi_S$. Then, from (10) it follows that $P$ is a linear AP-morphism. On the other hand, the previous relations (10), (17) and (15) mean that the Lie bracket $\{ \cdot, \cdot \}_P$ induced by $P$ on $\Gamma(\tau)$ is exactly the original one $\{ \cdot, \cdot \}_P$. Finally, if $\{ \cdot, \cdot \}_P$ satisfies the Jacobi identity, the previous relation implies that $\{ \cdot, \cdot \}_P$ also satisfies the Jacobi identity for functions of type $\Phi_s + f \circ \tau_*$. So it follows that $[P, P]$ vanishes identically and then, the Jacobi identity is satisfied for any $f, g, h \in \mathcal{F}^0(E^*)$, which ends the proof.

\[ \triangle \]

4.3 Relation between AP-algebroid and A-derivations

Recall that an A-exterior differential is a graded derivation $\delta$ of degree 1 of $\Lambda^0(\tau)$ which is localizable i.e. for any open set $U$ in $M$, there exists a unique graded derivation $\delta_U$ of degree 1 of $\Lambda^0(\tau_U)$ such that
\[ (\delta\omega)|_U = \delta_U(\omega|_U) \]

which is compatible with restriction to open subsets $V \subset U$ and satisfies the following properties:

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1. \( \delta(\eta \wedge \zeta) = \delta(\eta) \wedge \zeta + (-1)^k \eta \wedge \delta(\zeta) \) for any \( \eta \in \Lambda^k \Gamma^\ast(\tau) \) any \( \zeta \in \Lambda^l \Gamma^\ast(\tau) \) and any \( k, l \in \mathbb{Z} \)

2. For a L-algebroid, we have \( \delta \circ \delta = \delta^2 = 0 \). In this case we say that \( \delta \) is a exterior differential.

We will adapt the classic result obtained in finite dimension: given a derivation \( \delta \), one associates a unique bracket \([\cdot, \cdot]_\delta\) on \( \Gamma(\tau) \) with anchor \( \rho \) such that the almost exterior derivative \( d_\rho \) associated is exactly \( \delta \). However, in finite dimension, any (local) derivation of \( \mathcal{F} \) is a vector field, but in infinite Banach context it is not true (see subsection 2.5). So, we must impose another condition on \( \delta \) to get an analogous result. Moreover, in finite dimension, the exterior algebra \( \Lambda^\ast \tau^\ast \) is locally generated by its elements of degrees 0 and 1 which is not true in the Banach framework, even when we have Schauder basis (see Remark 4.13).

In the context of the Proposition 3.5 first we have:

**Lemma 4.11**

Consider a graded A-derivation \( \delta \) of degree 1 of \( \Lambda^\ast \tau^\ast \) which is localizable. For any \( s_1 \) and \( s_2 \) in \( \Gamma(\tau) \), the bracket \([i_{s_1}, \delta], i_{s_2}\) is a derivation of degree \(-1\) and its restriction to \( \Gamma(\tau_s) \) can be identified with a section of the bi-dual \( \mathcal{E}^{\ast\ast} \rightarrow M \).

**Proof**

First by construction, the degree of \( D = [i_{s_1}, \delta], i_{s_2}\) is \(-1\) so \( D \) maps \( \Lambda^1 \Gamma(\tau_s) \) into \( \mathcal{F} \) and \( Df = 0 \) for any \( f \in \mathcal{F} \). It follows that \( Df(\sigma) = f D\sigma \) for any \( \sigma \in \Gamma(\tau_s) \) and \( f \in \mathcal{F} \) and so the map \( \sigma \mapsto D\sigma \) is \( \mathcal{F} \) is linear, which ends the proof.

\( \triangle \)

Now, in our context, we have

**Theorem 4.12**

Let \( \tau : E \rightarrow M \) be a Banach bundle and \( \tau_\ast : \mathcal{E}^\ast \rightarrow M \) its dual bundle. Consider a localizable graded derivation \( \delta \) of degree 1 of the graded algebra \( \Lambda^\ast \tau^\ast \). Assume that for any \( s_1 \) and \( s_2 \) in \( \Gamma(\tau) \), the bracket \([i_{s_1}, \delta], i_{s_2}\) in restriction to \( \Gamma(\tau_s) \) can be identified with a section of \( \mathcal{E} \subset \mathcal{E}^{\ast\ast} \rightarrow M \).

Then \( \delta \) defines a unique bracket \([\cdot, \cdot]_\delta\) on \( \Gamma(\tau) \) and there exists a unique morphism \( \rho : E \rightarrow M \) such that:

1. for any function \( f \in \mathcal{F} \) and any section \( s \in \Gamma(\tau) \) we have
   \[ \rho(s)(f) = \langle \delta f, s \rangle \] (19)
2. \([\cdot, \cdot]_\delta\) is characterized by
   \[ \sigma([s_1, s_2]_\delta) = \delta(\sigma(s_1))s_2 - \delta(\sigma(s_2))s_1 - \delta\sigma(s_1, s_2), \text{ for any } \sigma \in \Gamma(\tau_s) \] (20)

In particular \( (E, \tau, M, [\cdot, \cdot]_\delta) \) is an AL-algebroid and the almost exterior derivative associated to this structure coincides with \( \delta \) on \( \Lambda^0 \Gamma^\ast(\tau) = \mathcal{F} \) and \( \Lambda^1 \Gamma^\ast(\tau) = \Gamma(\tau_s) \). Moreover, \( (E, \tau, M, [\cdot, \cdot]_\delta) \) is a L-algebroid if and only if \( \delta^2 = 0 \).

**Proof.**

From (19) and our assumption, we get a linear map \( \rho : \Gamma(\tau) \rightarrow \Gamma(M) \) which gives rise to a linear map \( \rho_U : \Gamma(\tau_U) \rightarrow \Gamma(U) \) for any open \( U \) in \( M \). On the other hand, as \( \delta f \) is a 1-form on \( E \), for any smooth function \( h \) defined on an open \( U \) we have: \( \langle \delta f_U, hs > = h \langle \delta f_U, s \rangle \); it follows that \( \rho(s) \) only depends on the value of \( s \) at each point. So, we get a bundle morphism from \( E \) to \( TM \).

Notice that, according to the proof of Proposition 3.5 the RHS of (20) is exactly \( [i_{s_1}, \delta], i_{s_2}(\sigma) \). Taking into account the definition of the map \( s \mapsto \Phi_s \) in subsection 4.1, the LHS of (20) is exactly \( \Phi_{[s_1, s_2]_\delta}(\sigma) \). From our assumption, there is a section which we can denote by \([s_1, s_2]_\delta \) of \( \tau : E \rightarrow M \) such that \( \Phi_{[s_1, s_2]_\delta} = [i_{s_1}, \delta], i_{s_2}(\sigma) \). As \( s \mapsto \Phi_s \) is injective, \([s_1, s_2]_\delta \) is well defined. Moreover,
using again the injectivity of $\Phi$ the Leibniz property for $[,]_\delta$ is obtained from the following results:

$$\Phi_{[s_1,s_2],\delta} (\sigma) = \delta(\sigma(s_1))(f s_2) - \delta(\sigma(f s_2))(s_1) - \delta(s_1,f s_2)$$

$$= \delta(\sigma(s_1))(f s_2) - \delta(\sigma(s_2))(s_1) - \delta(s_1,s_2) + \sigma(s_2) \delta(f(s_1))$$

$$= \Phi_{f[s_1,s_2],\delta}(\sigma) + \Phi_{\sigma(s_1),\delta}(f(\sigma))$$

Now, if $\delta^2 = 0$ by “formal argument” as in finite dimension, used for instance in [Mar], we can prove that $[,]_\delta$ satisfies the Jacobi identity. From [19] we obtain that $\delta = d_\rho$ on $\mathcal{F}$. From Proposition 4.5 we obtain that $d_\rho = \delta$ on $\Gamma(\tau) = \Lambda^1 \Gamma^*(\tau)$. $\triangle$

Remark 4.13

1. Note that, in general, if $M$ is not regular, a derivation of the module of smooth functions $\mathcal{F}$ on a Banach manifold $M$ is not localizable (see for instance [KrMa] section 35.1). However, to our known there exists no example of such a derivation which is not localizable. So, in Theorem 4.12, if $M$ is not regular, we must impose that the $A$-exterior differential on $\Lambda^k \Gamma^*(\tau)$ is localizable.

2. In Theorem 4.12 we cannot assert that $\delta$ and $d_\rho$ coincides on $\Lambda^k \Gamma^*(\tau)$ for $k \geq 2$. Indeed, recall that the Banach space $\Lambda^k \mathcal{E}^*$ is generated by exterior products of 1 forms $\xi_1, \cdots, \xi_k$. However, even if $\mathcal{E}^*$ has a Schauder basis $\{\epsilon_\alpha\}_{\alpha \in \mathbb{N}}$ the family

$$\{\epsilon_{\alpha_1} \wedge \cdots \wedge \epsilon_{\alpha_k}, \ \alpha_1 < \cdots < \alpha_k\}$$

could not be a Schauder basis of $\Lambda^k \mathcal{E}^*$ for $k \geq 2$ (this is an unsolved problem see [Ham]). So, for infinite dimensional Banach spaces, locally, the module of local $k$-forms $\Lambda^k \Gamma^*(\tau)$ is not finitely generated but, moreover we have no “good topology” on $\Lambda^k \Gamma^*(\tau)$ such that each $k$ form $\xi$ can not be locally written as

$$\eta(x) = \sum_{i_1 < \cdots < i_k} \eta_{i_1 \cdots i_k}(x) \xi_{i_1} \wedge \cdots \wedge \xi_{i_k}$$

for some appropriate finite sequences of smooth functions $\xi_{i_1}, \cdots, \xi_{i_k}$ (for topologies on modules of sections see [KrMa] and [Lla]).

As a consequence, any two $A$-derivations which coincide on $\mathcal{F}(U) = \Lambda^0 \Gamma^*(\tau_U)$ and on $\Gamma^*(\tau_U) = \Lambda^1 \Gamma^*(\tau_U)$ can be different on $\Lambda^k \Gamma^*(\tau_U)$, for $k \geq 2$. Once more, unfortunately, we have no example of such a situation.

4.4 Set of AL structures on an anchored Banach bundle

All the essential previous results will be summarized in the next theorem.

Let $(E, \tau, M, \rho)$ be an anchored Banach bundle. We denote by $\text{ALB}(\tau, \rho)$ the set of (localizable) AL-brackets on $(E, \tau, M, \rho)$ with fixed anchor morphism $\rho$.

We have seen that to any sub AP-morphism on the dual bundle $E^*$ is associated an anchor morphism $\rho_P : E \rightarrow TM$ characterized by

$$to \ s \in \Gamma(\tau) one \ associates \ the \ derivation \ f \mapsto \{\Phi_s, f \circ \tau_s\} \ on \ \mathcal{F}.$$ 

We denote by $\text{AP}(\tau, \rho)$ the set of sub AP- Poisson morphisms on the dual bundle $E^*$ such that the associated anchor morphism is $\rho$.

Theorem 4.14

Let $(E, \tau, M, \rho)$ be an anchored Banach bundle. The set $\text{ALB}(\tau, \rho)$ has a natural structure of affine space in the following sense:

given any $[,]_E \in \text{ALB}(\tau, \rho)$ then we have: $\text{ALB}(\tau, \rho) = \{[,]_E + D, \ D \in \Lambda^2 \Gamma(\tau)\}$.

There exists a bijection from $\text{ALB}(\tau, \rho)$ to $\text{AP}(\tau, \rho)$ defined in the following way:

at any AL-bracket $[,]_E$ one associates the AL-bracket $[,]_E$ on $\mathcal{F}(E^*)$ characterized by

$$\Phi_{[s_1,s_2],E} = \{\Phi_{s_1}, \Phi_{s_2}\}$$

20
The only thing to prove is the structure of $\mathcal{ALB}(\tau, \rho)$. Consider two AL-brackets $\{., .\}_i$, $i=1,2$, on $(E, \tau, M, \rho)$. It easy to see that $D = \{., .\}_1 - \{., .\}_2$ is an element of $\Lambda^2 \Gamma(\tau)$. The others properties come from Theorem 4.8.

5 Mechanical systems on an almost Lie algebroid

5.1 Hamiltonian system and Hamilton-Jacobi equation

Let $(E, \tau, M, \rho, \{., .\}_p)$ be an AL-algebroid. Denote by $P : T^*E^* \to TE^*$ the sub AP-morphism associated to this AL-structure and $\{., .\}_p$ the associated AP-bracket on $\mathcal{F}(E^*)$ (see subsection 4.1). Any function $h \in \mathcal{F}(E^*)$ is called a Hamiltonian function. The triple $(E, \{., .\}_p, h)$ is called a Hamiltonian system. As we have already seen, to $h$ is associated a vector field $\text{grad}^P(h) = -P(\rho(h))$ called the sub A-hamiltonian gradient of $h$. As in finite dimension (see for instance [LMM]) we have the following result on Hamilton-Jacobi equation.

**Theorem 5.1**

Let $(E, \{., .\}_p, h)$ be a Hamiltonian system. Given any section $\omega \in \Gamma^*(\tau)$ we denote by $\vec{h}_\omega$ the vector field on $M$ defined by

$$\vec{h}_\omega(x) = T_{\omega(x)}\tau_* \vec{h}(\omega(x))$$

Assume that $d_\rho \omega = 0$; then the following properties are equivalent:
Lemma 5.2:

In our trivializations we have:

\[ \langle d\xi, v \rangle = 0 \quad \text{for all } \xi, v \in \mathbb{E}. \]

Proof of part 1

In the previous trivializations we have

\[ \langle (df', v'), (\rho(x, v), D\xi(\rho(x, v))) \rangle = \langle df', \rho(x, v) \rangle + \langle D\xi(\rho(x, v)), v' \rangle > 0. \]  \hspace{1cm} (24)

So for any \((df, v) \in [L_\omega(x)]^0\) according to 10 and 24, we have the following equality for any \((df', v') \in T^\omega(x)E^*\)

\[ \langle df', \rho(x, v) \rangle + \langle D\xi(\rho(x, v)), v' \rangle = \langle \xi, \langle s, s' \rangle_0 \rangle > (x) + \langle df', \rho(x, v) \rangle - \langle df, \rho(x, v') \rangle > 0. \]  \hspace{1cm} (25)

if and only if we have \(d_\rho \omega(s, s')(x) = 0\), using the expression 1, for any \(s'(z) = (z, v'(z)). \)

The proof will be completed if we have 25, for any given local section \(s : s(z) = (z, u(z)), \) there exists a function \(f_s\) such that \((df_s, v)(x)\) belongs to \([L_\omega(x)]^0\).

Indeed, fix such a section \(s\). We define \(f_s : \rho(E_x) \subset T_xM \equiv \mathbb{M} \to \mathbb{R} \) by

\[ f_s(\rho(x, u)) = \langle D\xi(\rho(x, u)), v(x) \rangle > 0. \]

for any \(u \in E_x\). So \(f_s\) is a linear form on \(\rho(E_x)\). From Hahn-Banach theorem, there exists on \(\mathbb{E}\) a continuous linear form \(f_s\) such that \(f_s = f_{\rho}\) on \(\rho(E_x)\). So, it follows that we have

\[ \langle df_s, \rho(x, u) \rangle > = \langle D\xi(\rho(x, u)), v(x) \rangle > 0 \quad \text{for all } u \in E_x \text{ i.e. } (df_s, v(x)) \text{ belongs to } [L_\omega(x)]^0. \]

Proof of part 2:

Consider \((df, v) \in \ker P(\omega(x)) \subset T^\omega(x)E^* \equiv \mathbb{M}^* \times \mathbb{E}\). So, for any \((df', v') \in T^\omega(x)E^* \equiv \mathbb{M}^* \times \mathbb{E}\), from 10 we have

\[ \langle \xi, \langle s, s' \rangle_0 \rangle > (x) + \langle df', \rho(x, v) \rangle - \langle df, \rho(x, v') \rangle = 0. \]  \hspace{1cm} (26)

Under the assumption \(d_\rho \omega = 0\) the relation 26 is equivalent to:

\[ \langle D\xi(\rho(x, v)), v' \rangle > - \langle D\xi(\rho(x, v')), v \rangle > + \langle df', \rho(x, v) \rangle - \langle df, \rho(x, v') \rangle = 0. \]  \hspace{1cm} (27)
So for any given \((x,v')\) choose \(f'\) such that \((df',v')\) belongs to \([\mathcal{L}_\omega(x)]^0\). For this choice, we get
\[
<\,D\xi\{\rho(x,v'),v\},v\,> + <df,\rho(x,v')> = 0.
\]
It follows that \((df,v)\) belongs to \([\mathcal{L}_\omega(x)]^0\)

\[\triangle\]

**We come back to the proof of Theorem 5.1.** The end of this proof follows exactly the same arguments as in Theorem 4.1 of [LMM] with some adaptations. First property (i) is clearly equivalent to

\[(i')\, \quad T\omega(h_\omega(x)) = h_\omega(\omega(x))\]

We begin by the implication \((i')\Rightarrow (ii)\).

With the previous notations, \(h\) is a function of the variables \((x,\xi) \in U \times E^*\). Denote by \(D_2h\) the partial derivative of \(h\) according to variable \(\xi\). As \(dh\) belongs to \(T^0E^*\), the differential \(D_2h\) gives rise to a section of \(E_U\). The component of \(h_\omega(x) = P_{\omega(x)}(dh) \in T_{\omega(x)}^2E^* \equiv M \times E^*\) on \(M\) is characterized by (see (22)):

\[
\overrightarrow{P}_{\omega(x)}(dh) = \rho(D_2h)(x)\,.
\]

From assumption \((i')\), we then have \(\overrightarrow{h}_\omega(\omega(x)) \in \mathcal{L}_\omega(x)\). From Lemma 5.2 part 1, there exists \(\eta \in \mathcal{L}_\omega(x)\) such that

\[
\overrightarrow{h}_\omega(\omega(x)) = P_{\omega(x)}(\eta)
\]

So \((\eta - dh)(x)\) belongs to \(\ker P_{\omega(x)}\) and, using Lemma 5.2 part 2 \((dh)(x)\) also belongs to \([\mathcal{L}_\omega(x)]^0\).

But

\[
<\,d(h \circ \omega), \rho(s)\,> = <dh,T_x\omega \circ \rho(s)>
\]

for any \(s \in E_x\). As \((dh)(x)\) belongs to \([\mathcal{L}_\omega(x)]^0\) and \(T_x\omega \circ \rho(s)\) belongs to \(\mathcal{L}_\omega(x)\) we get that \(\phi > 0\).

\[(ii)\Rightarrow (i')\]

Under the assumption \(d_{\phi}\omega = 0\), if we have \(<dh \circ \omega, \rho> = 0\), as previously, we can show that \((dh)(x)\) belongs to \([\mathcal{L}_\omega(x)]^0\), so \(\overrightarrow{h}_\omega(x) = P_{\omega(x)}(dh)\), belongs to \(P_{\omega(x)}([\mathcal{L}_\omega(x)]^0)\). But, using Lemma 5.2 part 1, there exists \(s \in E_x\) such that

\[
\overrightarrow{h}_\omega(\omega(x)) = T_x\omega(\rho(s))
\]

So we obtain

\[
\overrightarrow{h}_\omega(x) = T_{\omega(x)}(h_\omega(x)) = T_x(\tau_{\omega}(\rho(s)) = \rho(s)
\]

Using (28), we finally get

\[
T_\omega(h_\omega(x)) = \overrightarrow{h}_\omega(\omega(x))
\]

\[\triangle\]

### 5.2 Lagrangian and Euler-Lagrange equation on an AL-algebroid

Given an anchored Banach bundle \((E,\tau,M,\rho)\), a **semi spray** \(S\) is a vector field on \(E\) such that (see [Ana]):

\[p_E \circ S = Id_E\]

where \(p_E : TE \to E\) is the canonical projection;

\[T_\tau \circ S = \rho\]

where \(T_\tau : TE \to TM\) is the tangent map of \(\tau\).

On the other hand, a \(C^k\)-curve \((k \geq 1)\) \(c : [a,b] \to E\) is called admissible, if we have \(T_\mu \circ c(t) = \rho(\mu(t))\) for all \(t \in [a,b]\). According to [Ana], we have:

**Proposition 5.3**

A vector field \(S\) on \(E\) is a semi spray if and only if all integral curves of \(S\) are admissible curves.

Among the class of semi sprays the subclass of sprays takes an important place for applications: if we denote by \(h_\lambda : E \to E\), the homothety of factor \(\lambda > 0\) \((h_\lambda(u) = \lambda u\) for any \(u \in E_x\) and any

\[23\]
In local trivializations (subsection 2.3), a semispray can be written in the following way (see [Ana]):

\[
S(x,u) = (x,u,R_x(u),-2G(x,u))
\]

When \( M \) and \( E \) have unconditional basis, we have:

\[
S = \sum_{\alpha \in \mathcal{A}} \left( \sum_{i \in I} \rho^\alpha_i \frac{\partial}{\partial x^i} \right) - 2G^\alpha \frac{\partial}{\partial u^\alpha} \]

A Lagrangian on a Banach bundle \((E,\tau,M)\) is a smooth map \( L : E \to \mathbb{R} \). We say that \( L \) is homogenous of degree \( k \) if we have:

\[
L \circ h_\lambda = \lambda^k L.
\]

The following result is classical (see for instance [AbMa] section 3.5)

**Lemma 5.4**

Let \( L \) be a Lagrangian, we denote by \( L_x \) the restriction of \( L \) to the fiber \( E_x \). Then the map \( \Lambda_L : (x,u) \to (x,dL_x(u)) \) is a bundle morphism from \( E \) to \( E^* \).

We will say that \( L \) is regular (resp. strong regular) (resp. hyperregular) if \( \Lambda_L \) is an injective morphism (resp. isomorphism) (resp. diffeomorphism). Notice that when \( L \) is strong regular then \( \Lambda_L \) is a local diffeomorphism. So \( L \) is hyperregular if and only if \( L \) is strong regular and the restriction of \( \Lambda_L \) to each fiber is injective.

Denote by \( h_\lambda : E \to E \), the homothety of factor \( \lambda > 0 \) (\( h_\lambda(u) = \lambda u \) for any \( u \in E_x \) and any \( x \in M \)). As in finite dimension, let \( \Theta \) be the Liouville field on \( E \) which is the vector field whose flow is the homothety \( \{ h_\lambda \}_{\lambda \in \mathbb{R}} \). In local trivializations we have \( \Theta(x,u) = (x,u,0,u) \) and when \( M \) and \( E \) have unconditional basis, we have \( \Theta = \sum_{\alpha \in \mathcal{A}} u^\alpha \frac{\partial}{\partial u^\alpha} \).

We denote by \( H_L \) the Lagrangian energy associated to \( L \) i.e.

\[
H_L = dL(\Theta) - L
\]

Given a regular Lagrangian \( L \) on an AL-algebroid \((E,\tau,M,\rho,[..],\rho)\), \( \Lambda_L \) is a local diffeomorphism. So for any \((x,u) \in E \), there exists an open neighborhood \( U \times V \subset E_U \) of \((x,u)\) such that \( (\Lambda_L)_{|U \times V} \) is a diffeomorphism.

**Now suppose that \( E \) is reflexive.** Consider a regular Lagrangian \( L \) on \( E \) and \( U \times V \subset E \) an open set on which \( \Lambda_L \) is a diffeomorphism.

Let be the function \( h_L = H_L \circ \Lambda_L^{-1} \) on \( \Lambda_L(U \times V) \). Then \( dh_L(\sigma) \) belongs to \( T^*_\sigma E^* \) on \( \Lambda_L(U \times V) \) On \( U \times V \) we can define the vector field \( \vec{L} \) characterized by

\[
(\Lambda_L)_*(\vec{H}_L) = \vec{h}_L
\]

and which is called the local Euler-Lagrange vector field of \( L \) on \( U \times V \). In particular, when \( L \) is hyperregular, \( \vec{L} \) is globally defined and called Euler-Lagrange vector field of \( L \).

**Theorem 5.5**  Consider a regular Lagrangian \( L \) on \( E \).
1. Any curve \( c = (\gamma, \mu) : [a, b] \to U \times V \) is an integral curve of the local Euler-Lagrange vector field of \( L \) if and only if it is a solution of the Euler-Lagrange equations

\[
\dot{x} = R(u) \quad \frac{d}{dt}(D_2L) = R^l(D_1L) - D_2L(C(\cdot, u))
\] (31)

where \( R^l : U \to L(M, \mathbb{E}^n) \) is the field \( x \to (R_x)^l \) of transpose of \( R_x \in L(M, \mathbb{E}) \) and where \( C(\cdot, u) \) denote, for a fixed \( u \), the field of linear maps \( x \to [v \to C_x(v, u)] \) (recall that \( x \to C_x \) is a field of bilinear maps).

2. If \( L \) is hyperregular, the Euler-Lagrange vector field \( \overrightarrow{L} \) is a semi-spray. Moreover, if \( L \) is homogenous of degree 2 then \( \overrightarrow{L} \) is a spray.

Remark 5.6

If \( M \) and \( E \) have unconditional basis, then in the associated coordinate systems, the Euler-Equation can be written in the following way:

\[
\dot{x}^i = \sum_{\alpha \in A} \rho^a_\alpha u^a \quad \frac{d}{dt} \frac{\partial L}{\partial u^a} = \sum_{\nu \in I} \rho^a_\nu \frac{\partial L}{\partial x^\nu} - \sum_{\beta, \gamma \in A} C^\gamma_{\alpha \beta} u^\beta \frac{\partial L}{\partial u^\gamma}
\] (32)

for any \( i \in I \). When \( A \) and \( I \) are finite sets of indexes, \( \alpha_2 \) is the classical Euler-Lagrange equation on the Lie algebroid (see for instance [GMM]).

Proof

We again adopt the notations in local trivializations (subsection 2.3). So \( L \) is a function of variable \((x, u)\) and \( \Lambda_L \) is the map \((x, u) \to (x, D_2L(x, u))\). For simplicity, we will denote this map by \( \Lambda \) and for a fixed point \((x, u) \in E \) we denote by \((x, \xi) = \Lambda(x, u)\) The tangent map \( T\Lambda \) of \( \Lambda \) is:

\[
\begin{pmatrix}
Id & 0 \\
D_{12}L & D_{22}L
\end{pmatrix}
\] (33)

So \( [(T\Lambda)^*]^{-1} \)

\[
\begin{pmatrix}
Id & -D_{21}L \circ (D_{22}L)^{-1} \\
0 & (D_{22}L)^{-1}
\end{pmatrix}
\] (34)

On the other hand, we have \( H_L(x, u) = D_2L(x, u)(u) - L(x, u) \) so we get

\[
dH_L(x, u) = (D_{12}L(x, u)(u) - D_1L(x, u) \circ D_{22}L(x, u)(u)) \] (35)

So as \( dh_L = [(T\Lambda)^*]^{-1} \circ dH_L \), from (34) and (35) we get

\[
D_1h_L(x, \xi) = -D_1L(x, \xi) \quad \text{and} \quad D_2h_L(x, \xi) = u
\] (36)

From (21) the AL Hamiltonian \( \overrightarrow{r}_L \) is characterized by

\[
[\overrightarrow{r}^1_L](x, \xi) = Rx(u), \quad \text{and} \quad [\overrightarrow{r}^2_L] = -D_2L(x, \xi) \circ C_x(u) + D_1L(x, \xi) \circ \rho_x
\] (37)

Now as \( \overrightarrow{H}_L = T\Lambda(\overrightarrow{r}_L) \) from (34) and (35) we get

\[
[\overrightarrow{h}^1_L](x, u) = R_x(u)
\]

\[
D_{22}L(\overrightarrow{h}^2_L(x, u)) = -D_{12}L(x, u) \circ R_x(u) + D_1L(x, u) \circ R_x - D_2L(x, u) \circ C_x(u, \xi)
\]

We can easily see that these last equations are equivalent to the Euler-Lagrange equations.
5.3 Riemannian AL-algebroid and mechanical system

Let \((E, \tau, M)\) be a Banach bundle. Denote by \(S^2T^*M \to M\) the Banach bundle of symmetric bilinear form on \(TM\). Recall that a global section \(g\) of this bundle is called a **riemannian metric** on \(E\), if for for any \(x \in M\), the bilinear form \(g_x\) on \(T_xM\) is positive definite, i.e. \(g_x(u, u) > 0\) for any \(u \neq 0\).

To any riemannian metric \(g\) on \(E\), is associated a bundle morphism \(g^\flat : E \to E^\ast\) defined by \(g^\flat(X)(Y) = g(X, Y)\). Of course, \(g^\flat\) is always injective. We say that \(g\) is a **strong** riemannian metric if \(g^\flat\) is surjective. Note that, in these conditions, the Banach space \(E\) is isomorphic to a Hilbert space, and so \(E\) must be reflexive.

We will say that the AL-algebroid \((E, \tau, M, \rho, [\cdot, \cdot])\) is a **riemannian AL-algebroid** if there exists a riemannian metric \(g\) on \(E\). In this situation, as in finite dimension, we can consider the Lagrangian system map \(L : E \to \mathbb{R}\) of a **mechanical system** on \(E\) given by

\[
L(s) = \frac{1}{2}g(s, s) - V(\tau(s))
\]

where \(\frac{1}{2}g(s, s)\) is the ”kinetic energy” and \(V : M \to \mathbb{R}\) the ”potential energy” of the mechanical system. The associated Lagrangian energy is then:

\[
H_L(x, u) = \frac{1}{2}g(s, s) + V(\tau(s)).
\]

The Legendre transformation \(\Lambda_L\) is \(g^\flat\). Of course, the Lagrangian \(L\) is hyperregular, so the Lagrangian field \(\tilde{L}\) is well defined. Moreover if \(V \equiv 0\) then \(\tilde{L}\) is a spray.

**Local expressions**

In local trivialization (cf subsection 2.3), we also denote by \(x \to g_x\) the field of symmetric bilinear maps on \(E\) associated to \(g\) and \(x \to g^\flat_x : E \to E^\ast\) the field of associated isomorphisms. With these notations, we have

\[
L(x, u) = \frac{1}{2}g_x(u, u) - V(x).
\]

According to the local expression (29) we can write

\[
\tilde{L}(x, u) = (x, u, R(u), -2G(x, u))
\]

where \(G\) is characterized by:

\[
g^\flat_x(G(x, u)) = \frac{1}{2} \left[< R^\flat_x \circ D_1g^\flat_x(u, u) > - \frac{1}{2} < R^\flat_x \circ D_1g_x(u, u), > - < R^\flat_x \circ dV, > - < g^\flat_x(u, C_x, (., u)), > \right]
\]

The Euler-Lagrange equation is given by

\[
\begin{aligned}
\dot{x} &= R(u) \\
\dot{u} &= \frac{1}{2}(g^\flat_x)^{-1} \left[< R^\flat_x \circ D_1g^\flat_x(u, u), > - \frac{1}{2} < R^\flat_x \circ D_1g_x(u, u), > - < R^\flat_x \circ dV, > - < g^\flat_x(u, C_x, (., u)), > \right]
\end{aligned}
\]

In particular, when \(M\) and \(E\) have unconditional basis, the bilinear map can be written as a matrix \(g = (g_{\alpha \beta})_{\alpha, \beta \in A}\) and we have according to (30) can be written

\[
\sum_{(\alpha, \beta) \in A} g_{\alpha \beta} G^\beta = \frac{1}{2} \sum_{(\alpha, \beta, \gamma) \in A} \left[ \sum_{i \in I} \frac{\partial g_{\alpha \beta}}{\partial x^i} \rho^i_\gamma - \frac{1}{2} \sum_{i \in I} \frac{\partial g_{\beta \gamma}}{\partial x^i} \rho^i_\alpha - \sum_{(\delta, \gamma) \in A} C^\delta_{\alpha \beta \gamma} g_{\delta \alpha} \right]u^\delta u^\gamma - \sum_{i \in I} \frac{\partial V}{\partial x^i} \rho^i_\alpha,
\]

and, as in finite dimension, the Euler Lagrange equations have an analogue expression which is left to the reader.
Consider a riemannian L-algebroid \((E, \tau, M, \rho, [, .]_\rho)\) and \(g\) its riemannian metric. If \(\tau^*: F \rightarrow M\) is a Banach subbundle of \(\tau: E \rightarrow M\), we can define the complemented Banach subbundle \(F^\perp \rightarrow M\) whose fiber is \(F^\perp_x\) is the orthogonal in \(E_x\) of \(F_x\) relatively to the metric \(g\). Let be \(\Pi: E \rightarrow F\) is the natural morphism projection, we can define an AL-bracket on the set of section of \(F\) by:

\[ [s_1, s_2] = \Pi [s_1, s_2]_\rho\]

(see Example 3.2.3 [7]). So, if \(\rho' = \rho|_F\), for the induced metric \(g'\) on \(F\) induced by \(g\), \((F, \tau^*, M, \rho', [, .]'\) is a riemannian AL-algebroid.

In this context, denote by

- \(i_F: F \rightarrow E\) the canonical inclusion;
- \(i^*_F: E^* \rightarrow F^*\) the dual projection;
- \(\Pi^*: E^* \rightarrow F^*\) the dual morphism of \(\Pi\);
- \(P: T^* F^* \rightarrow TF\) the P-morphism on \(E\) associated to \([, .]_\rho\)
- \(P': T^* F^* \rightarrow TF\) the AP-morphism associated to \([, .]'\).

The Lagrangian \(L(x, u) = \frac{1}{2} g_{\rho}(u, u) - V(x)\) on \(E\) induces a Lagrangian \(L' = L \circ i_F\) on \(F\) which is a constrained Lagrangian on \(E\). As in finite dimensional case (see [Mar]), we associate a mechanical system on the AL-algebroid \((F, \tau^*, M, \rho', [, .]'\) called constrained mechanical system on \(E\) which is obtain from the unconstrained system associated to \(L'\) on \(E\) with the following relations:

- the Legendre transformation \(\Lambda_{L'}: TF \rightarrow T^* F\) satisfies \(\Lambda_{L'} = i^*_F \circ \Lambda_L \circ i_F\);
- The hamiltonian \(h_{L'} = h_L \circ \Pi^*\); the associated family \(\{S_t\}_{t \in \mathbb{R}}\) of snakes satisfies \(S_t(L) = c(t)\) for all \(t \in [0, 1]\) such that the family \(\{S_t\}\) has a minimal infinitesimal kinetic energy. This problem has the precise following formulation:

Each snake \(S\) of length \(L\) in \(\mathbb{R}^d\) can be given by a piecewise \(C^0\)-curve \(u: [0, L] \rightarrow S^{d-1}\) so that \(S(t) = \int_0^t u(\tau)d\tau\). We look for a 1-parameter family \(\{u_t\}_{t \in [0, 1]}\) so that the associated family \(S_t\) of snakes satisfies \(S_t(L) = c(t)\) for all \(t \in [0, 1]\) so that the infinitesimal kinetic energy

\[ \frac{1}{2} \int_0^L \| \frac{d}{dt} u_t(s) \| ds \]

is minimal.

A generalization of this problem in the context of an separable Hilbert space is developed in [PeSa]. More precisely, given a separable Hilbert space \(\mathbb{H}\) we consider the smooth hypersurface \(S^\infty\) of element of norm 1. As previously, an Hilbert snake of length \(L\) is a continuous piecewise \(C^1\)-curve \(S: [0, L] \rightarrow \mathbb{H}\) arc-length parameterized so that \(S(0) = 0\). Each such snake is again given by a piecewise \(C^0\)-curve \(u: [0, L] \rightarrow S^\infty\) so that \(S(t) = \int_0^t u(\tau)d\tau\). Given a fixed partition \(\mathcal{P}\) of \([0, L]\), the set \(C^0_{\mathcal{P}}\) of such curves will be called the configuration set and carries a natural structure of Banach manifold: when \(\mathcal{P} = \{0, L\}\), the set \(C^0_{\mathcal{P}}\) is a hypersurface of the Banach \(C([0, L], \mathbb{H})\) of continuous map from \([0, L]\) to \(\mathbb{H}\) with the classical norm \(\| . \|_\infty\); for the general case \(\mathcal{P} = \{0 = s_0, \cdots, s_N = L\}\) then \(C^0_{\mathcal{P}}\) is canonically homomorphic to the product \(\mathcal{C}([0, L], S^\infty)^N\) and so we put on \(C^0_{\mathcal{P}}\) the corresponding Banach structure product. Notice that, on each tangent space
$T_u C_P^L$ we also have an $L^2$ product:

$$< v, w >_{L^2} = \int_0^L < v(s), w(s) > ds$$

(42)

Where $< , >$ is the inner product in $\mathbb{H}$. Of course for the associated norm $|| ||_{L^2}$, the normed space $(T_u C_P^L, || ||_{L^2})$ is not complete.

To any "configuration" $u \in C_P^L$ is naturally associated the "end map" $E(u) = \int_0^L u(s) ds$.

This map is smooth and its kernel has a canonical complemented subspace which is the orthogonal of ker $T_u E$ in $T_u C_P^L$ according to the inner product (42). We then get a closed distribution $\mathcal{D}$ on $C_P^L$. As in finite dimension, for a one parameter family $\{ u_t \}_{t \in [0,1]}$ the associated family $S_t$ of snakes satisfies $S_t(L) = c(t)$ for all $t \in [0,1]$ so that the infinitesimal kinematic energy $\frac{1}{2} \int_0^L || \frac{d}{dt} u(s) || ds$

is minimal, if $c(t)$ has a "lift" $\tilde{c}$ in $C_P^L$ which is tangent to $\mathcal{D}$, called an "horizontal lift". So the problem for the head of the Hilbert snake to join an initial state $x_0$ to a final state $x_0$ can be transformed in the following "accessibility problem":

Given an initial (resp; final) configuration $u_0$ (resp. $u_1$) in $C_P^L$, so that $E(u_i) = x_i$, $i = 1, 2$, find a piecewise $C^1$ horizontal curve $\gamma : [0, T] \rightarrow C_P^L$ (i.e. $\gamma$ is tangent to $\mathcal{D}$) and which joins $u_0$ to $u_1$.

Given any configuration $u \in C_P^L$, we look for the accessibility set $A(u)$ of all configurations $v \in C_P^L$ which can be joined from $u$ by piecewise $C^1$ horizontal curves. In the context of finite dimension, in [Ro], using arguments about the action of the Mo"ebius group on $C_P^L$, it can be shown that $A(u)$ is the maximal integral manifold of a finite dimensional distribution on $C_P^L$. Unfortunately, in the context of Hilbert space, the same argument does not work. Moreover, as we are in the context of infinite dimension for $\mathbb{S}^\infty$, we cannot hope to get a finite dimensional distribution whose maximal integral manifolds is $A(u)$. However, we can construct a canonical distribution $\mathcal{D}$ modelled on Hilbert space, which is integrable and so that the accessibility set $A(u)$ is a dense subset of the maximal integral manifold through $u$ of $\mathcal{D}$ ([PeSa] Theorem 4.1). Moreover this distribution is minimal in some natural sense. In fact, when $\mathbb{H}$ is finite dimensional, $\mathcal{D}$ is exactly the finite distribution in [Ro] whose leaves are the accessibility sets.

6.2 AL - algebroid structure

To any Hilbert basis $\{ e_i, i \in \mathbb{N} \}$, we can associate a family of global vector fields $\{ E_i, i \in \mathbb{N} \}$ on $C_P^L$ which generates $\mathcal{D}$ and the announced distribution $\mathcal{D}$ is the Hilbert distribution generated by $\{ E_i, [E_j, E_l], i, j, l \in \mathbb{N} \}$. As $E$ is not a submersion everywhere, it follows that $\mathcal{D}$ is not a subbundle of $TC_P^L$.

On one hand if $\Lambda = \{ (i, j) \in \mathbb{N}^2, i < j \}$, we can consider the Hilbert space $\mathbb{G} = l^2(\mathbb{N}) \oplus l^2(\Lambda)$ Now, given any Hilbert basis $\{ e_i, i \in \mathbb{N} \}$ of $\mathbb{H}$ we define an anchored bundle $(C_P^L \times \mathbb{G}, \rho, C_P^L)$ by

$$\rho(\sigma, \xi) = \sum_{i \in \mathbb{N}} \sigma_i E_i + \sum_{(j, l) \in \Lambda} \xi_{jl}[E_j, E_l]$$

Of course $\rho$ is well defined surjective and do not depend of the choice of the Hilbert basis.

On the other hand, the Lie bracket of vector fields of the family $\{ E_i, [E_j, E_l], i, j, l \in \mathbb{N} \}$ satisfies the following relations: (Lemma 4.3 [PeSa])

$\{ E_i, E_j \}(u) = < e_j, u > E_i(u) - < e_i, u > E_j(u)$ for any $u \in C_P^L$ and any $i, j \in \mathbb{N}$;

$[E_i, [E_j, E_k]] = \delta_{jk} E_k - \delta_{ik} E_j$ for any $i, j, k \in \mathbb{N}$

$[[E_i, E_j], E_k] = \delta_{ik} [E_j, E_k] + \delta_{jk} [E_i, E_l] - \delta_{il} [E_j, E_k] - \delta_{jl} [E_i, E_k]$ for any $i, j, k, l \in \mathbb{N}$. 28
So, on $G$ we define a Lie algebra structure in the following way:

let be $(\epsilon_i)_{i \in \mathbb{N}}$ (resp. $(\epsilon_{ij})_{(i,j) \in \Lambda}$) the canonical basis of $(I^2(\mathbb{N}))$ (resp. $(I^2(\Lambda))$);

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according to the previous relations, we then define:

$$[\epsilon_i, \epsilon_j] = \omega_{ij}, \text{ for all } i, j \in \mathbb{N}$$

$$[\epsilon_i, \omega_{jk}] = \delta_{ij} \epsilon_k - \delta_{ik} \epsilon_j, \text{ for all } i \in \mathbb{N} \text{ and } (j, k) \in \Lambda$$

$$[\omega_{ij}, \omega_{kl}] = \delta_{il} \omega_{jk} + \delta_{jk} \omega_{il} - \delta_{ik} \omega_{jl} - \delta_{ij} \omega_{kl}, \text{ for all } (i, j)(kl) \in \Lambda.$$

For any $\sigma = \sum_i \sigma_i \epsilon_i, \sigma' = \sum_j \sigma'_j \epsilon_j$ in $I^2(\mathbb{N})$ and $\xi = \sum \xi_{ij} \omega_{ij}, \eta = \sum \eta_{kl} \omega_{kl}$ in $I^2(\Lambda)$, naturally we can define:

$$[\sigma, \sigma'] = \sum_{i,j \in \mathbb{N}} \sigma_i \sigma'_j [\epsilon_i, \epsilon_j]$$

$$[\sigma, \eta] = \sum_{i \in \mathbb{N}, (k, l) \in \Lambda} \sigma_i \eta_{kl} [\epsilon_i, \omega_{kl}]$$

$$[\xi, \eta] = \sum_{(i,j) \in \Lambda, (k,l) \in \Lambda} \xi_{ij} \eta_{kl} [\omega_{ij}, \omega_{kl}].$$

Coming back to the anchored bundle $(C^\mathbb{R}_P \times G, \rho, C^\mathbb{R}_P)$, each section $\varphi$ of the trivial bundle $C^\mathbb{R}_P \times G \to C^\mathbb{R}_P$ can be identified with a map $\varphi: C^\mathbb{R}_P \to G$. So, on the set $\Gamma(G)$ of section of this trivial bundle we can defined a Lie bracket by:

$$[[\varphi, \varphi'](u) = [\varphi(u), \varphi'(u)] + d\varphi(\rho(u, \varphi'(u)) - d\varphi(\rho(u, \varphi(u))$$

It follows that $(C^\mathbb{R}_P \times G, \rho, C^\mathbb{R}_P, [\ , \ ]\,)$ is a Banach Lie algebroid structure on $C^\mathbb{R}_P$.

In $G$ let be $\pi: G \to I^2(\mathbb{N})$ the canonical projection whose kernel is $I^2(\Lambda)$ and denote again by $\pi: C^\mathbb{R}_P \times G \to C^\mathbb{R}_P \times I^2(\mathbb{N})$ the associated projection bundle. Again any section of the trivial bundle $C^\mathbb{R}_P \times I^2(\mathbb{N}) \to C^\mathbb{R}_P$ can be identified with a map from $C^\mathbb{R}_P$ to $I^2(\mathbb{N})$. Of course the set $\Gamma(I^2(\mathbb{N}))$ of such sections is contained in $\Gamma(G)$. So, as in Example 6.1, on $\Gamma(I^2(\mathbb{N}))$, we can define an almost Banach Lie bracket by:

$$[[\varphi, \varphi'](u) = \pi([[\varphi, \varphi'](u)).$$

So, if we denote by $\theta$ the restriction of $\rho$ to $I^2(\mathbb{N}) \times C^\mathbb{R}_P$ we get an almost Banach Lie algebroid structure $(C^\mathbb{R}_P \times I^2(\mathbb{N}), \theta, C^\mathbb{R}_P, [\ , \ ]\,)$ on $C^\mathbb{R}_P$.

On the other hand, the distribution $D$ is a weak Hilbert integrable distribution, on $C^\mathbb{R}_P$, this means that, for any $u \in C^\mathbb{R}_P$, there exists an Hilbert manifold $N$ and a smooth injective map $f: N \to C^\mathbb{R}_P$ such that (see [Pa3]):

$u$ belongs to $f(N)$,

$T_x \pi: T_x N \to T_{f(x)} C^\mathbb{R}_P$ is injective,

$T_x f(T_x N) = D_{f(x)}$ for any $x \in N$.

We say that $N$ is an integral manifold through $u$.

Given such integral manifold which maximal (for the inclusion), the pull back $f_*\{C^\mathbb{R}_P \times G\}$ and $f_*\{C^\mathbb{R}_P \times I^2(\mathbb{N})\}$ can be identified with $N \times G$ and $N \times I^2(\mathbb{N})$ respectively; Then, $\rho$ (resp. $\theta$) induces an anchor $\rho_N: N \times G \to TN$ (resp. $\theta_N: N \times I^2(\mathbb{N}) \to TN$). The bracket $[[\ , \ ]]\,\text{ and (resp. the almost bracket }[[\ , \ ]]\,\text{ induces a bracket (resp. an almost bracket) again denoted }[[\ , \ ]]\,.\text{ So we get also a Banach Lie algebroid } (N \times G, \rho_N, N, [[\ , \ ]])\text{ and an almost Banach Lie algebroid structure } (N \times I^2(\mathbb{N}), \theta_N, N, [[\ , \ ]])\text{ on } N$.

Now recall the following result of [PeSa]:

**Proposition 6.1**

*Let be $N$ a maximal integral manifold of $\mathcal{D}$ and fix some $u \in N$. Then we have the following properties*
1. The set $\Sigma(E)$ at $E : C^{L}_{P} \to \mathbb{H}$ is not a submersion is a weak manifold of $C^{L}_{P}$ which is diffeomorphic to the projective space $\mathbb{P}^{\infty}$ of $\mathbb{H}$. Its complementary $R(E)$ is an open dense set of $C^{L}_{P}$. Moreover, $N$ is a maximal integral manifold of $D$.

(1) Assume that $u \in \Sigma(E)$ then $N = \Sigma(E)$. Let be $L_{u}$ the 1-codimensional Hilbert subspace $\ker[\theta_{N}]^{\perp} \subset \{v\} \times T^{2}(N)$ for any $v \in N$. Then $L = \cup_{v \in N} L_{v}$ is a 1-codimensional Hilbert subbundle of $N \times T^{2}(N)$ and the restriction $\psi_{N}$ of $\theta_{N}$ to $L$ is an isomorphism onto $TN$ and we have $D_{N} = TN$.

(2) Assume that $u \in R(E)$. Then $N$ is contained in $R(E)$. Let be $V_{u}$ the Hilbert subspace of $\mathbb{H}$ generated by the set 
\[ \{u(t) − u(0), t \in [0, L]\} \]
and choose an Hilbert basis $\{e_{a}^{\prime}, a \in A\}$ (resp $e_{b}^{\prime}, b \in B\}$ of $[V_{u}]^{\perp}$ (resp. $V_{u}$). If $\Lambda_{u}$ is the set of pair $(i, j) \in \Lambda$ such that that $i$ or $j$ do not belongs to $A$, then $N$ is a Hilbert manifold modeled on $L^{2}(N) \oplus L^{2}(\Lambda_{u})$ and is contained in $R(E)$.

Let be $L_{u}$ the orthogonal of $\ker[\rho]_{v} \subset \{v\} \times \mathbb{G}$. Then $L = \cup_{v \in N} L_{u}$ is a Hilbert subbundle of $N \times \mathbb{G}^{2}$ which contains $N \times L^{2}(N)$ and the restriction of $\psi_{N}$ of $\Phi_{N}$ to $L$ is an isomorphism on $TN$.

Moreover, $L$ contains $N \times L^{2}(N)$ and the restriction of $\theta_{N}$ to $N \times L^{2}(N)$ is an isomorphism on $D_{N}$.

So, on $N$, if we denote by $[\cdot, \cdot]$ the usual Lie bracket and $p_{N} : TN \to N$ the tangent bundle, we have the canonical $L$ algebroid $(TN, p_{N}, N, Id_{N}, [\cdot, \cdot])$. When $u \in R(E)$, on $N$, the distribution $F = D_{N}$ is an hilbert subbundle $p^{\prime} : F \to N$ of $p_{N} : TN \to N$. When $u \in \Sigma(E)$ we have $D_{N} = TN$ and so again we can consider $p^{\prime} : F = D_{N} \to N$ as an Hilbert subbundle of $TN$.

6.3 Constrained mechanical system

Let be $x$ and $y$ two states of the head of the Hilbert snake which can be joined by an optimal curve (in the sense of subsection 6.1) and consider the set $\Omega(x, y)$ the set of optimal curves $c$ which joins $x$ to $y$. Classically if $c \in \Omega(x, y)$ is defined on $[0, T]$ its kinematic energy is
\[ E(c) = \frac{1}{2} \int_{0}^{T} ||c(t)||^{2} dt \]

Let be $E(x, y) = \inf_{c \in \Omega(x, y)} E(c)$. Assume that there exists $\tilde{c} \in \Omega(x, y)$ such that $E(\tilde{c}) = E(x, y)$, then $\tilde{c}$ will be called an optimal minimizing curve which joins $x$ to $y$. Of course, such a curve is also an optimal minimizing curve between any pair of its points. So we can look for the existence of optimal minimizing curve which begin at a given original state $x$ of the head of the considered snake.

On one hand, each $c \in \Omega(x, y)$ has a horizontal lift $\gamma$ in $C^{L}_{P}$, and $\gamma$ lies in $N$ as $L \to N$ is an Hilbert subbundle of $N \times \mathbb{G} \to N$, we have a natural riemannian metric $g$ on $L$. From Proposition 6.1 the isomorphism $\psi_{N}$ gives rise to a riemannian metric - again denoted by $g$ - on $TN$ and induces a riemannian metric $g^{\prime}$ on $F = D_{N}$. Note that the inner product induces by $g^{\prime}$ on each fiber $D_{\tilde{u}}$, $\tilde{u} \in N$, is exactly the inner product induced by $\rho_{\tilde{u}} : D_{\tilde{u}} \to T_{\tilde{u}} \mathbb{H} \equiv \mathbb{H}$ of the canonical one on $\mathbb{H}$. So we can defined - as in subsection 6.2 - an AL bracket $[\cdot, \cdot, \cdot]$ on $F$ and we get an AL algebroid $(F, p^{\prime}, i_{F}, N, [\cdot, \cdot, \cdot])$

Now, coming back to our problem of optimal minimizing curve on $\mathbb{H}$. To each $c \in \Omega(x, y)$ has a horizontal lift $\gamma$ in $C^{L}_{P}$, and $\gamma$ lies in $N$ for such a lift $\gamma$ be an of some optimal curve $c : [0, T] \to \mathbb{H}$, we have then
\[ E(c) = \frac{1}{2} \int_{0}^{T} ||\gamma||^{2} dt. \]
Finally, it follows that if $\bar{c}$ is an optimal minimizing curve which joins $x$ and $y$, then the associated lift $\bar{\gamma}$ in $N$ is an extremal of the Lagrangian $L' : F \to \mathbb{R}$:

$$L'(v, \sigma) = \frac{1}{2} ||\sigma||^2$$

on the AL algebroid $(F, p', i_F, N, \left[~,~\right])$.

So we get a **constrained mechanical system** on the natural riemannian algebroid $(TN, p, N, Id_N, \left[~,~\right])$. It follows that such $\bar{\gamma}$ is a solution of the Euler-Lagrange equation of $L'$.

We will now give the differential system satisfied by such extremals in local coordinates.

Fix some $u \in \mathcal{R}(\mathcal{E})$. According to Proposition 6.1, $N$ is modelled on $L^2(\mathbb{N}) \oplus L^2(\Lambda_u)$. So we have a local coordinates $(\sigma, \xi) = \{(\sigma_i)_{i \in \mathbb{N}}, (\xi_{jl})_{(j,l) \in \Lambda_u}\}$. We denote by $\left\{\frac{\partial}{\partial \sigma_i}, \frac{\partial}{\partial \xi_{jl}}\right\}$ the local Hilbert basis of $TN$ associated to this coordinates system. With these notations, we have:

$$E_i = \frac{\partial}{\partial \sigma_i} + \sum_{(i,j) \in \Lambda_u} \sigma_j \frac{\partial}{\partial \xi_{jl}} - \sum_{(l,i) \in \Lambda_u} \sigma_l \frac{\partial}{\partial \xi_{li}}$$

So from (40) and (41), the components $(\sigma_i, \xi_{jl})$ of an extremal is given by

$$\begin{cases}
\ddot{\sigma}_i = 0, & i \in \mathbb{N} \\
\ddot{\xi}_{jl} = \dot{\sigma}_j \dot{\sigma}_l, & (j,l) \in \Lambda_u
\end{cases}$$

Now, for $u \in \Sigma(\mathcal{E})$, if we consider a basis $(e_i)_{i \in \mathbb{N}}$ of $\mathbb{H}$ such that $u = \pm e_1$, then $N$ is modelled on $e_1'$ (see Proposition 6.1). So, in local coordinates $\{\sigma_i, i > 1\}$ the component of an extremal satisfies

$$\ddot{\sigma}_i = 0, \quad i > 1$$

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