The homotopy type of spaces of real resultants with bounded multiplicity

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Abstract. For positive integers \(d, m, n \geq 1\) with \((m, n) \neq (1, 1)\) and \(K = \mathbb{R}\) or \(\mathbb{C}\), let \(Q_{d,m}^{n}(K)\) denote the space of \(m\)-tuples \((f_1(z), \ldots, f_m(z)) \in K[z]^m\) of \(K\)-coefficients monic polynomials of the same degree \(d\) such that polynomials \(\{f_k(z)\}_{k=1}^m\) have no common real root of multiplicity \(\geq n\) (but may have complex common root of any multiplicity). These spaces can be regarded as one of generalizations of the spaces defined and studied by Arnold and Vassiliev, and they may be also considered as the real analogues of the spaces studied by Farb-Wolfson. In this paper, we shall determine their homotopy types explicitly and generalize our previous results.

1. Introduction

Spaces of polynomials and the motivation. The principal motivation for this paper is derived from the two results obtained by Vassiliev [26] and Farb-Wolfson [5].

Vassiliev [26] described a general method for calculating cohomology of certain spaces of polynomials (more precisely, “complements of discriminants”) by using a spectral sequence (to which and its variants we shall refer as the Vassiliev spectral sequence). The most relevant example for us is the following. For \(K = \mathbb{R}\) or \(\mathbb{C}\), let \(P_d^n(K)\) denote the space of all \(K\)-coefficients monic polynomials \(f(z) \in K[z]\) of degree \(d\) which have no real root of multiplicity \(\geq n\) (but may have complex ones of arbitrary multiplicity). By identifying \(S^1 = \mathbb{R} \cup \{\infty\}\) and \(\mathbb{C} = \mathbb{R}^2\), we have the jet map

\[
(1.1) \quad j_{d,n,K}^d : P_d^n(K) \to \Omega_{[d]} \mathbb{R}P_d^d(K)^{n-1} \simeq \Omega S^d(K)^{n-1}
\]

defined by

\[
j_{d,n,K}^d(f(z))(\alpha) = \begin{cases} \{f(\alpha) : f'(\alpha) : \cdots : f^{(n-1)}(\alpha)\} & \text{if } \alpha \in \mathbb{R}, \\ \{1 : 1 : 1 : \cdots : 1\} & \text{if } \alpha = \infty \end{cases}
\]

for \((f(z), \alpha) \in P_d^n(K) \times S^1\), where \([d]_2 \in \{0, 1\}\) is the integer \(d\) mod 2, and \(d(K)\) denotes the positive integer defined by

\[
(1.2) \quad d(K) = \dim_{\mathbb{R}} K = \begin{cases} 1 & \text{if } K = \mathbb{R}, \\ 2 & \text{if } K = \mathbb{C}. \end{cases}
\]

For \(K = \mathbb{R}\), Vassiliev [26] obtained the following result:
The jet map $f_{n,R}^{d,1}: P_{n}^{d}(\mathbb{R}) \to \Omega_{\{d\}}^{n}\mathbb{R}P^{n-1}$ is a homotopy equivalence through dimension $(\frac{d}{2}) + 1)(n - 2) - 1$ for $n \geq 4$ and a homology equivalence through dimension $[\frac{d}{2}]$ for $n = 3$, where $[x]$ denotes the integer part of a real number $x$.

Remark 1.2. Let $X$ and $Y$ be based connected spaces. Then a based map $f : X \to Y$ is called a homotopy equivalence (resp. a homology equivalence) through dimension $N$ if the induced homomorphism $f_{*} : \pi_{k}(X) \to \pi_{k}(Y)$ (resp. $f_{*} : H_{k}(X;\mathbb{Z}) \to H_{k}(Y;\mathbb{Z})$) is an isomorphism for any integer $k \leq N$. Similarly, when $G$ is a group and $f : X \to Y$ is a $G$-equivariant map between $G$-spaces $X$ and $Y$, the map $f$ is called a $G$-equivariant homotopy equivalence through dimension $N$ (resp. a $G$-equivariant homology equivalence through dimension $N$) if the restriction map $f^{H} = f|X^{H} : X^{H} \to Y^{H}$ is a homotopy equivalence through dimension $N$ (resp. a homology equivalence through dimension $N$) for any subgroup $H \subset G$, where $W^{H}$ denotes the $H$-fixed subspace of a $G$-space $W$ given by $W^{H} = \{x \in W : h \cdot x = x \text{ for any } h \in H\}$.

Next, recall the recent result obtained by Farb and Wolfson [5]. For positive integers $m, n \geq 1$ with $(m,n) \neq (1,1)$ and a field $\mathbb{F}$ with its algebraic closure $\overline{\mathbb{F}}$, let $\text{Poly}_{n}^{d,m}(\mathbb{F})$ denote the space of $n$-tuples $(f_{1}(z), \cdots , f_{m}(z)) \in \mathbb{F}[z]^{m}$ of $\mathbb{F}$-coefficients monic polynomials of the same degree $d$ such that the polynomials $\{f_{k}(z)\}_{k=1}^{m}$ have no common root in $\overline{\mathbb{F}}$ with multiplicity $\geq 1$. They studied the spaces $\text{Poly}_{n}^{d,m}(\mathbb{F})$ from the point of view of algebraic geometry in the case when $\mathbb{F} = \mathbb{C}$ or $\mathbb{F}$ is a finite field $\mathbb{F}_{q}$. If $n = 1$ and $\mathbb{F} = \mathbb{C}$, the space $\text{Poly}_{1}^{1,m}(\mathbb{C})$ can be identified with the space $\text{Rat}^{m}_{1}(\mathbb{C}P^{1}, \mathbb{C}P^{m-1})$ of all based rational maps $f : \mathbb{C}P^{1} \to \mathbb{C}P^{m-1}$ of degree $d$ (it is well known that in this case, rational maps coincide with holomorphic ones). The space of holomorphic maps appears in various applications and has been quite extensively studied (e.g. [10], [24]).

In a different context, the same is true for the space $\text{Poly}_{n}^{d,1}(\mathbb{C})$ (for $m = 1$) of monic polynomials $f(z) \in \mathbb{C}[z]$ of degree $d$ without $n$-fold roots. This space can be viewed as the space of polynomial functions “without complicated singularities” and thus plays an important role in singularity theory ([2], [26]). The fact that these two spaces were closely related was noted and it was shown in [26] and [3] that they were stably homotopy equivalent. Recently this result was much improved and it was proved in [20] (cf. [10]) that there is a homotopy equivalence

\begin{equation}
\text{Poly}_{n}^{d,m}(\mathbb{F}) \simeq \text{Poly}_{1}^{1,n,m}(\mathbb{F}) \quad \text{for } \mathbb{F} = \mathbb{C} \text{ and } mn \geq 3.
\end{equation}

Farb and Wolfson showed that some of these topological results have algebraic analogues when $\mathbb{F}$ is a finite field $\mathbb{F}_{q}$. This leads them to ask if these spaces in (1.3) are isomorphic as algebraic varieties over arbitrary fields $\mathbb{F}$. The affirmative answer would imply that the underlying topological spaces in the cases $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$ are homeomorphic.\footnote{Recently H. Spink and D. Tseng [25] showed that they are not isomorphic as varieties for $\mathbb{F} = \mathbb{C}$.} Curtis McMullen indeed has shown that this is true in the simplest non-trivial case $(d,n) = (1,2)$ by constructing an explicit isomorphism.

Moreover, if $\mathbb{F} = \mathbb{R}$ and $n = 1$, the space $\text{Poly}_{1}^{d,m}(\mathbb{R})$ is precisely the space $\text{Rat}^{m}_{1}(\mathbb{F}P^{1}, \mathbb{F}P^{m-1})$ of real rational functions considered by Segal in [24]. This space is an
algebraic variety over $\mathbb{R}$ and one can ask if it is isomorphic (homeomorphic or homotopy equivalent?) to the variety $\text{Poly}_m^{d,1}(\mathbb{R})$ of real monic polynomials of degree $md$ without $m$ fold complex roots. Of course, the affirmative answer to the Farb-Wolfson question would also imply this. However, we have not been able to establish even homotopy equivalence in this case, and we will not consider this problem here.

On the other hand, there is another space of real rational maps which can be viewed as the real analogue of the space of complex ones and it was first studied by Mostovoy in [22]. Every such map can be represented by $n$-tuples of monic real polynomials of the same degree $d$ without a common real root (but possibly with common non-real roots). However, in this situation the space of rational maps and the space of tuples of polynomials are different and we need to distinguish them.\(^2\) This space contains Segal’s space of rational functions and is its closure in the space of all continuous maps. By analogy with the complex case, one can expect this space to be homotopy equivalent to the space of real monic polynomials of degree $md$ which do not have real roots of multiplicity $\geq n$. This is indeed true as was shown in [18] (cf. [28]). The spaces involved are not algebraic varieties; so the result is not implied by the positive answer to the Farb-Wolfson question.

The main purpose of this article is to generalize this result given in [18] for the space $Q_n^{d,m}(\mathbb{K})$, of $m$-tuples $(f_1(z), \cdots, f_m(z)) \in \mathbb{K}[z]^m$ of monic $\mathbb{K}$-coefficients polynomials of the same degree $d$, without $n$-fold common real roots (for $\mathbb{K} = \mathbb{C}$ of $\mathbb{R}$). We will also prove that an analogue of the homotopy equivalence (1.3) holds for the space $Q_n^{d,m}(\mathbb{K})$ (see Theorem 1.11 for the details).

### Basic definitions and notations.

For connected spaces $X$ and $Y$, let $\text{Map}(X,Y)$ (resp. $\text{Map}^+(X,Y)$) denote the space consisting of all continuous maps (resp. base-point preserving continuous maps) from $X$ to $Y$ with the compact-open topology, and let $\mathbb{R}P^N$ (resp. $\mathbb{C}P^N$) denote the $N$-dimensional real projective (resp. complex projective) space.

Note that the based loop space $\text{Map}^+(S^1, \mathbb{R}P^N) = \Omega \mathbb{R}P^N$ has two path-components $\Omega_1 \mathbb{R}P^N$ for $\varepsilon \in \{0, 1\}$ when $N \geq 2$. The space $\Omega_0 \mathbb{R}P^N$ is the path-component of null homotopic maps and $\Omega_1 \mathbb{R}P^N$ is the path-component which contains the natural inclusion of the bottom cell $S^1$ in $\mathbb{R}P^N$.

From now on, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $d, m, n \geq 1$ be positive integers such that $(m, n) \neq (1, 1)$, and we always assume that $z$ is a variable. Let $\mathbb{K}[z]^m$ denote the space of all $\mathbb{K}$-coefficients monic polynomials $f(z) = a_d z^d + \cdots + a_0 \in \mathbb{K}[z]$ of degree $d$.

**Definition 1.3.** (i) Let $Q_n^{d,m}(\mathbb{K})$ denote the space of $m$-tuples $(f_1(z), \cdots, f_m(z)) \in \mathbb{K}[z]^m$ of $\mathbb{K}$-coefficients monic polynomials of the same degree $d$ such that the polynomials $f_1(z), \cdots, f_m(z)$ have no common real root of multiplicity $\geq n$ (but they may have a common complex root of any multiplicity).

(ii) Let $(f_1(z), \cdots, f_m(z)) \in \mathbb{K}[z]^m$ be an $m$-tuple of monic polynomials of the same degree $d$. Then it is easy to see that $(f_1(z), \cdots, f_m(z)) \in Q_n^{d,m}(\mathbb{K})$ if and only if the derivative polynomials \( f_j^{(k)}(z) : 1 \leq j \leq m, \quad 0 \leq k < n \) have no common real root.

\(^2\)For example, let $(f_1(z), f_2(z), f_3(z)) \in \mathbb{R}[z]^3$ be a 3-tuple of monic polynomials of the same degree $d$ without common real root. Then the two 3-tuples $F = ((z^2 + 1)f_2(z), (z^2 + 1)f_1(z), (z^2 + 1)f_3(z))$ and $G = ((z^2 + z + 1)f_2(z), (z^2 + z + 1)f_1(z), (z^2 + z + 1)f_3(z))$ represent the same base-point preserving rational map from $S^1$ to $\mathbb{R}P^2$, although $F \neq G$. Although it seems very likely to be true, it has not been proved that the space of rational maps and the space of tuples of polynomials are homotopy equivalent.
Thus, by identifying $S^1 = \mathbb{R} \cup \infty$, one can define the jet map
\begin{equation}
(1.4) \quad j_{n,K}^{d,m} : Q_n^{d,m}(\mathbb{K}) \to \Omega_{[d]_n} \mathbb{R} \mathbb{P}^{d(\mathbb{K})mn-1} \simeq \Omega S^{d(\mathbb{K})mn-1}
\end{equation}

by
\begin{equation}
(1.5) \quad j_{n,K}^{d,m}(f_1(z), \ldots, f_m(z))(\alpha) = \begin{cases} \{f_1(\alpha) : \ldots : f_m(\alpha)\} & \text{if } \alpha \in \mathbb{R}, \\ [1 : 1 : \ldots : 1] & \text{if } \alpha = \infty \end{cases}
\end{equation}

for $(f_1(z), \ldots, f_m(z)) \in Q_n^{d,m}(\mathbb{K})$, where we identify $\mathbb{C} = \mathbb{R}^2$ in (1.5) if $\mathbb{K} = \mathbb{C}$, and $f_k(z)$ ($k = 1, \ldots, m$) is the $n$-tuple of monic polynomials of the same degree $d$ defined by
\begin{equation}
(1.6) \quad f_k(z) = (f_k(z), f_k(z) + f_k'(z), f_k(z) + f_k''(z), \ldots, f_k(z) + f_k^{(n-1)}(z)).
\end{equation}

**Remark 1.4.** Let $f = (f_1(z), f_2(z), \ldots, f_m(z)) \in Q_n^{d,m}(\mathbb{K})$ be any element. Then it is easy to see that $\tilde{f} = ((z^2 + 1)f_1(z), (z^2 + 1)f_2(z), \ldots, (z^2 + 1)f_m(z)) \in Q_n^{d+2,m}(\mathbb{K})$. Since the two maps $j_{n,K}^{d,m}(f)$ and $j_{n,K}^{d+2,m}(\tilde{f})$ are homotopic, the image of $j_{n,K}^{d,m}$ and that of $j_{n,K}^{d+2,m}$ are contained in the same path-component of $\Omega \mathbb{R} \mathbb{P}^{d(\mathbb{K})mn-1}$. We can check directly that $j_{n,K}^{d,m}(Q_n^{d,m}(\mathbb{K})) \subset \Omega_{[d]_n} \mathbb{R} \mathbb{P}^{d(\mathbb{K})mn-1}$ for $d \in \{1, 2\}$. Hence, the image $j_{n,K}^{d,m}(Q_n^{d,m}(\mathbb{K}))$ is contained in the component $\Omega_{[d]_n} \mathbb{R} \mathbb{P}^{d(\mathbb{K})mn-1}$ for any $d \geq 1$.

**Definition 1.5.** Note that $P_n^{d}(\mathbb{K}) = Q_n^{d,1}(\mathbb{K})$, and that the map $j_{n,K}^{d,m}$ coincides with the map $j_{n,K}^{d,1}$ given in (1.1) for $m = 1$.\footnote{Note that the space $Q_n^{d,m}(\mathbb{K})$ is also denoted by $Q_n^{d}(\mathbb{K})$ for $n = 1$ in [18].} Similarly, one can define a natural map
\begin{equation}
(1.7) \quad j_{n,K}^{d,m} : Q_n^{d,m}(\mathbb{K}) \to Q_1^{d,mn}(\mathbb{K})
\end{equation}

by
\begin{equation}
(1.8) \quad j_{n,K}^{d,m}(f_1(z), \ldots, f_m(z)) = (f_1(z), \ldots, f_m(z)).
\end{equation}

It is well-known that there is a homotopy equivalence ([16])
\begin{equation}
\Omega S^{N+1} \simeq S^N \cup e^{2N} \cup e^{3N} \cup \ldots \cup e^{kN} \cup e^{(k+1)N} \cup \ldots.
\end{equation}

We will denote the $kN$-skeleton of $\Omega S^{N+1}$ by $J_k(\Omega S^{N+1})$, i.e.
\begin{equation}
(1.10) \quad J_k(\Omega S^{N+1}) \simeq S^N \cup e^{2N} \cup e^{3N} \cup \ldots \cup e^{(k-1)N} \cup e^{kN}.
\end{equation}

This space is usually called the $k$-stage James filtration of $\Omega S^{N+1}$.

**Related known results.** Let $D(d; m, n, \mathbb{K})$ denote the positive integer defined by
\begin{equation}
(1.11) \quad D(d; m, n, \mathbb{K}) = \lfloor d(\mathbb{K})mn - 2 \rfloor \lfloor \frac{d}{n} \rfloor + 1 - 1
\end{equation}

Recall the following known results for the case $m = 1$ or $n = 1$.\footnote{Note that the space $Q_n^{d,m}(\mathbb{K})$ is also denoted by $Q_n^{d}(\mathbb{K})$ for $n = 1$ in [18].}
Theorem 1.6 ([18], [22], [26], [28]). (i) If $d(\mathbb{K})m \geq 4$ and $n = 1$, the jet map

$$J_{1,\mathbb{K}}^{d,m} : Q_1^{d,m}(\mathbb{K}) \to \Omega_d \mathbb{R}P^{d(\mathbb{K})m-1} \simeq \Omega S^{d(\mathbb{K})m-1}$$

is a homotopy equivalence through dimension $D(d; m, 1, \mathbb{K})$.

(ii) If $d(\mathbb{K})n \geq 4$ and $m = 1$, the jet map

$$J_{d,\mathbb{K}}^{d,1} : Q_n^{d,1}(\mathbb{K}) = P_n^d(\mathbb{K}) \to \Omega \mathbb{R}P^{d(\mathbb{K})n-1} \simeq \Omega S^{d(\mathbb{K})n-1}$$

is a homotopy equivalence through dimension $D(d; 1, n, \mathbb{K})$.

(iii) If $d(\mathbb{K})n \geq 4$, there are homotopy equivalences

$$Q_n^{d,1}(\mathbb{K}) = P_n^d(\mathbb{K}) \simeq J_{d,\mathbb{K}}(\Omega S^{d(\mathbb{K})n-1}) \quad \text{and} \quad Q_1^{d,n}(\mathbb{K}) \simeq J_d(\Omega S^{d(\mathbb{K})n-1}).$$

Thus, there is a homotopy equivalence $Q_n^{d,1}(\mathbb{K}) = P_n^d(\mathbb{K}) \simeq Q_1^{d,1}(\mathbb{K})$ if $d(\mathbb{K})m \geq 4$.

(iv) In particular, if $(\mathbb{K}, m) = (\mathbb{R}, 3)$ and $d \geq 1$ is an odd integer, there is a homotopy equivalence $Q_n^{d,1}(\mathbb{R}) \simeq J_d(\Omega S^2)$.

Note that the conjugation on $\mathbb{C}$ naturally induces a $\mathbb{Z}/2$-action on the space $Q_n^{d,m}(\mathbb{C})$. From now on, we regard $\mathbb{R}P^N$ as the $\mathbb{Z}/2$-space with trivial $\mathbb{Z}/2$-action, and recall the following result given in [18].

Theorem 1.7 ([18]). (i) If $m \geq 4$, then the jet map

$$J_{1,\mathbb{C}}^{d,m} : Q_n^{d,m}(\mathbb{C}) \to \Omega \mathbb{R}P^{2m-1} \simeq \Omega S^{2m-1}$$

is a $\mathbb{Z}/2$-equivariant homotopy equivalence through dimension $D(d; m, 1, \mathbb{C})$.

(ii) If $n \geq 4$, then the jet map

$$J_n^{d,1} : Q_n^{d,1}(\mathbb{C}) = P_n^d(\mathbb{C}) \to \Omega \mathbb{R}P^{2n-1} \simeq \Omega S^{2n-1}$$

is a $\mathbb{Z}/2$-equivariant homotopy equivalence through dimension $D(d; 1, n, \mathbb{C})$.

The main results. The main purpose of this paper is to determine the homotopy type of the space $Q_n^{d,m}(\mathbb{K})$ explicitly and generalize the above two theorems (Theorems 1.6 and 1.7) for the case $m \geq 2$ and the case $n \geq 2$. More precisely, the main results are as below.

Theorem 1.8. If $d(\mathbb{K})mn \geq 4$, the jet map

$$J_n^{d,m} : Q_n^{d,m}(\mathbb{K}) \to \Omega \mathbb{R}P^{d(\mathbb{K})mn-1} \simeq \Omega S^{d(\mathbb{K})mn-1}$$

is a homotopy equivalence through dimension $D(d; m, n, \mathbb{K})$.

Note that the conjugation on $\mathbb{C}$ naturally induces the $\mathbb{Z}/2$-action on the space $Q_n^{d,m}(\mathbb{C})$. Since the map $J_{n,\mathbb{C}}^{d,m}$ is a $\mathbb{Z}/2$-equivariant map and $Q_n^{d,m} \mathbb{Z}/2 = J_{n,\mathbb{K}}^{d,m}$, we also obtain the following result.
Corollary 1.9. If $mn \geq 4$, the jet map
\[ j_{d,n} : Q_{dn}^{d,m}(\mathbb{C}) \to \Omega_d \mathbb{R}^{2mn-1} \simeq \Omega S^{2mn-1} \]
is a $\mathbb{Z}/2$-equivariant homotopy equivalence through dimension $D(d; m, n, \mathbb{R})$.

Corollary 1.10. If $d(K)mn \geq 4$, the jet embedding
\[ i_{d,n} : Q_{dn}^{d,m}(K) \to Q_{d,mn}^d(K) \]
is a homotopy equivalence through dimension $D(d; m, n, K)$.

Finally we have the following result.

Theorem 1.11. If $d(K)mn \geq 4$, there is a homotopy equivalence
\[ Q_{dn}^{d,m}(K) \simeq J_{d-1}^{\frac{d}{2}}(\Omega S^{d(K)mn-1}). \]
Hence, in this situation, there are homotopy equivalences
\[ Q_{dn}^{d,m}(K) \simeq Q_{d-1}^{d,1}(K) \simeq Q_{1}^{d,1-dnm}(K). \]

This paper is organized as follows. In §2 we recall the simplicial resolutions and in §3 we construct the Vassiliev spectral sequences induced from the non-degenerate simplicial resolutions. In particular, by using this spectral sequence, we compute the homology $H_*(Q_{dn}^{d,m}(K), \mathbb{Z})$ explicitly. In §4, we recall the stabilization maps and prove the key unstable result (Theorem 4.5) by using comparison of the Vassiliev spectral sequence. In §5 we prove the stability result (Theorem 5.18) by using the horizontal scanning map. Finally in §6 we give the proof of the main results (Theorem 1.8, Corollary 1.10, Theorem 1.11) by using two key results (Theorems 4.5 and 5.18).

2. Simplicial resolutions

In this section, we give the fundamental definitions and summarize the basic facts about non-degenerate simplicial resolutions ([26, 27], cf. [23]).

Definition 2.1. (i) For a finite set $v = \{v_1, \ldots, v_l\} \subset \mathbb{R}^N$, let $\sigma(v)$ denote the convex hull spanned by $v$. Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, and let $i : X \to \mathbb{R}^N$ be an embedding. Let $X^\Delta$ and $h^\Delta : X^\Delta \to Y$ denote the space and the map defined by
\[ X^\Delta = \{(y, u) \in Y \times \mathbb{R}^N : u \in \sigma(i(h^{-1}(y)))\} \subset Y \times \mathbb{R}^N, \ h^\Delta(y, u) = y. \]
The pair $(X^\Delta, h^\Delta)$ is called the simplicial resolution of $(h, i)$. In particular, $(X^\Delta, h^\Delta)$ is called a non-degenerate simplicial resolution if for each $y \in Y$ any $k$ points of $i(h^{-1}(y))$ span a $(k-1)$-dimensional simplex of $\mathbb{R}^N$.

(ii) For each $k \geq 0$, let $X^\Delta_k \subset X^\Delta$ be the subspace given by
\[ X^\Delta_k = \{(y, u) \in X^\Delta : u \in \sigma(v), v = \{v_1, \ldots, v_l\} \subset i(h^{-1}(y)), l \leq k\}. \]
We make the identification \( X = X^\Delta \) by identifying \( x \in X \) with \((h(x), i(x)) \in X^i \Delta \), and we note that there is an increasing filtration

\[
\emptyset = X^0 \Delta \subset X = X^1 \Delta \subset X^2 \Delta \subset \cdots \subset X^k \Delta \subset \cdots \subset \bigcup_{k=0}^{\infty} X^k \Delta = X^\Delta.
\]

Since the map \( h^\Delta \) is a proper map, it extends to the map \( h^\Delta_+ : X^\Delta_+ \to Y_+ \) between one-point compactifications, where \( X_+ \) denotes the one-point compactification of a locally compact space \( X \).

**Theorem 2.2** ([26], [27], [23] (cf. [19])). Let \( h : X \to Y \) be a surjective map such that \( h^{-1}(y) \) is a finite set for any \( y \in Y \), \( i : X \to R^N \) an embedding, and let \((X^\Delta, h^\Delta)\) denote the simplicial resolution of \((h, i)\).

(i) If \( X \) and \( Y \) are semi-algebraic spaces and the two maps \( h, i \) are semi-algebraic maps, then \( h^\Delta_+ : X^\Delta_+ \to Y_+ \) is a homology equivalence.

(ii) If there is an embedding \( j : X \to R^M \) such that its associated simplicial resolution \((X^\Delta, h^\Delta)\) is non-degenerate, the space \( X^\Delta \) is uniquely determined up to homomorphism and there is a filtration preserving homotopy equivalence \( q^\Delta : X^\Delta \to X^\Delta \) such that \( q^\Delta_+|X = id_X \).

(iii) A non-degenerate simplicial resolution exists even if the map \( h \) is not finite to one.

**Proof.** The assertion (ii) follows from the universality of non-degenerate resolutions as in [23, pages 286-287], so it remains to show (i) and (iii). Note that (i) was already proved in [26, Lemma 1 (page 90)]. Indeed, by [7, Theorem (page 43)] semi-algebraic sets can be triangulated; therefore, there exists a cellular decomposition of the space \( Y_+ \) such that over each open cell the projection \( h^\Delta_+ \) is a trivializable bundle with simplex as a fiber. Filter the space \( Y_+ \) by the skeletons of this decomposition, and the space \( X^\Delta \) by the pre-images of these skeletons under the map \( h^\Delta_+ \). Let \( E^\Delta_+,q \) denote the spectral sequence induced from the filtration converging to the homology of the space \( X^\Delta_+ \). By construction of this filtration, we easily see that \( E^\Delta_+ = 0 \) for any \( q \geq 1 \), and that the complex \( \{E^\Delta_+;d_1\} \) is isomorphic to the cellular differential complex of the cellular decomposition of \( Y_+ \), the isomorphism being given by \((h^\Delta_+)_p \). Thus, \( h^\Delta_+ \) is a homology equivalence, which is the assertion (i).

Finally, we prove (iii). Since this result was already explained in [23, pages 286-287], we only give a sketch of the proof. Recall from [26, Lemma 1 (page 101)] that for each \( k \geq 1 \) there is an embedding \( J_k : R^N \to R^{N_k} \) such that no \( 2k \) points of the image \( J_k(R^N) \) lie in one \((2k - 2)\)-dimensional affine subspace of \( R^{N_k} \). Since there is an embedding \( i : X \to R^N \), the embedding \( i_k = J_k \circ i \) also has the same property. Thus we obtain the family \( I = \{i_k : X \to R^{N_k}\}_{k \geq 1} \) of embeddings satisfies the following condition:

\[
(2.3) \quad \text{For each pair } (k, y) \in N \times Y, \text{ any } t \text{ points of the set } i_k(h^{-1}(y)) \text{ span a } (t - 1)\text{-dimensional affine subspace of } R^{N_k} \text{ if } t \leq 2k.
\]

\[\text{Footnote: There are two slightly different editions of [26]. In the first edition this Lemma 1 appears on page 86, in the second on page 90. In this paper we always refer to the second (revised) edition of [26].}\]
We then let

\[(2.4) \quad X^\Delta_k = \{(y,u) \in Y \times \mathbb{R}^{N_k} : u \in \sigma(v), \, v = \{v_1, \ldots, v_l\} \subset i_k(h^{-1}(y)), \, l \leq k\}.\]

Since each space \(X^\Delta_k\) can be naturally identified with a subspace of \(X^\Delta_{k+1}\), we obtain a non-degenerate simplicial resolution \(X^\Delta\) of \(h\) by taking the union \(X^\Delta = \bigcup_{k \geq 1} X^\Delta_k\). \(\square\)

**Remark 2.3.** It is known that \(h^\Delta\) is actually a homotopy equivalence [27, page 156]. However, in this paper we do not need this stronger assertion.

**Definition 2.4.** Let \(i_k : X \to \mathbb{R}^{N_k}\) be an embedding. For a finite set \(\{u_1, \ldots, u_k\}\) denote the convex hull spanned by \(\{i_k(u_1), \ldots, i_k(u_k)\}\) and we denote by \(\text{Int}(\Delta(u_1, \ldots, u_k))\) the set of interior points of \(\Delta(u_1, \ldots, u_k)\).

**Lemma 2.5.** Let \(i_k : X \to \mathbb{R}^{N_k}\) be an embedding satisfying the condition \((2.3)\), and suppose that \(\{x_i\}_{i=1}^k \subset i_k((h^\Delta)^{-1}(y))\) and \(\{y_i\}_{i=1}^k \subset i_k((h^\Delta)^{-1}(y))\) for some \(y \in Y\). If \(\{x_i\}_{i=1}^k \neq \{y_i\}_{i=1}^k\), then \(\text{Int}(\Delta(x_1, \ldots, x_k)) \cap \text{Int}(\Delta(y_1, \ldots, y_k)) = \emptyset\).

**Proof.** Suppose that there is an element \(\alpha \in \text{Int}(\Delta(x_1, \ldots, x_k)) \cap \text{Int}(\Delta(y_1, \ldots, y_k))\). Then one can write \(\alpha = \sum_{i=1}^k \lambda_i i_k(x_i) = \sum_{i=1}^k \mu_i i_k(y_i)\) for \(\lambda_i, \mu_i > 0\) \((i = 1, \ldots, k)\) with \(\sum_{i=1}^k \lambda_i = \sum_{i=1}^k \mu_i = 1\). Since \(\{x_i\}_{i=1}^k \neq \{y_i\}_{i=1}^k\), by reindexing we may suppose that there is a positive integer \(1 \leq t \leq k\) such that \(\{x_i\}_{i=1}^t \cup \{y_i\}_{i=1}^t = \{x_1, \ldots, x_t, y_1, \ldots, y_t, z_{t+1}, \ldots, z_k\}\) satisfying the conditions \(\{x_i\}_{i=1}^t \cap \{y_i\}_{i=1}^t = \emptyset\) and \(x_i = y_i = z_i\) for any \(t+1 \leq i \leq k\). Since \(2t + (k - t) = k + t \leq 2k\), by using the condition \((2.3)\) we see that the elements \(i_k(x_i), i_k(y_i), i_k(z_i) : 1 \leq i \leq t < j \leq k\) are affinely independent. Since \(\sum_{i=1}^t \lambda_i i_k(x_i) = \sum_{i=1}^t \mu_i i_k(y_i) = \sum_{i=1}^t \lambda_i i_k(x_i) = \sum_{i=1}^t \mu_i i_k(y_i) + \sum_{i=t+1}^k (\lambda_i - \mu_i) i_k(z_i) = 0\) and \(\sum_{i=1}^t \lambda_i - \sum_{i=1}^t \mu_i + \sum_{i=t+1}^k (\lambda_i - \mu_i) = 0\), we see that \(\lambda_i = \mu_i = 0\) for any \(1 \leq i \leq t\). But this is a contradiction. This completes the proof. \(\square\)

3. The Vassiliev spectral sequence

In this section we construct a spectral sequence similar to the one frequently used by Vassiliev in [26] (which we will call simply “the Vassiliev spectral sequence”) converging to the homology of \(Q_n^{d,m}(\mathbb{K})\) and compute it explicitly.

**Definition 3.1.** (i) Let \(\mathbb{K} = \mathbb{R} or \mathbb{C}\), and let \(\Sigma_n^{d,m}(\mathbb{K})\) denote the discriminant of \(Q_n^{d,m}(\mathbb{K})\) in \(P^d(\mathbb{K})^m\) given by the complement

\[
\Sigma_n^{d,m} = P^d(\mathbb{K})^m \setminus Q_n^{d,m}(\mathbb{K}) = \{(f_1, \ldots, f_m) \in P^d(\mathbb{K})^m : f_1(x) = \cdots = f_m(x) = 0 \text{ for some } x \in \mathbb{R}\}.
\]

(ii) Let \(Z_n^{d,m} \subset \Sigma_n^{d,m} \times \mathbb{R}\) denote the tautological normalization of \(\Sigma_n^{d,m}\) given by

\[
Z_n^{d,m} = \{(f_1(z), \ldots, f_m(z), x) \in \Sigma_n^{d,m} \times \mathbb{R} : f_1(x) = \cdots = f_m(x) = 0\}.
\]

Let \(\pi_n^{d,m} : Z_n^{d,m} \to \Sigma_n^{d,m}\) denote the map given by the projection to the first factor, and let \((X^\Delta, \pi^\Delta : X^\Delta(\mathbb{K}) \to \Sigma_n^{d,m})\) be the non-degenerate simplicial resolutions of \(\pi_n^{d,m}\) as in
Theorem 2.2. Note that there is a natural increasing filtration
\[
\emptyset = \mathcal{X}_0^d \subset \mathcal{X}_1^d \subset \mathcal{X}_2^d \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathcal{X}_k^d = \mathcal{X}^d(\mathbb{K}).
\]
Since any \((f_1(z), \cdots, f_m(z)) \in \Sigma_{n,\mathbb{K}}^d, m \geq \lfloor \frac{d}{n} \rfloor\) has at most \(\lfloor \frac{d}{n} \rfloor\) distinct common real roots of multiplicity \(n\), the following equality holds:
\[
\mathcal{X}_k^d = \mathcal{X}^d(\mathbb{K}) \quad \text{if } k \geq \left\lfloor \frac{d}{n} \right\rfloor.
\]
By Theorem 2.2, the map \(\pi^d_{k+1} : \mathcal{X}^d(\mathbb{K})_+ \xrightarrow{\simeq} (\Sigma_{n,\mathbb{K}}^d)_+\) is a homology equivalence. Since \(\mathcal{X}_{k+1}^d / \mathcal{X}_{k-1}^d \cong (\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d)_+\), we have a spectral sequence
\[
\{E_{k,s}^{1,d} : E_{k,s}^{1,d} : E_{t,d}^{k+s,t+s+1-t} \Rightarrow H_c^{k+s}(\mathcal{X}_{k-1}^d, \mathbb{Z})\},
\]
such that \(E_{k,d}^{1,d} = H_c^{d+s}((\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d)_+ ; \mathbb{Z})\), where \(H_c^k(X; \mathbb{Z})\) denotes the cohomology group with compact supports given by \(H_c^k(X; \mathbb{Z}) = H^k(X_+ ; \mathbb{Z})\).
Since there is a homeomorphism \(P^d(\mathbb{K})^m \cong \mathbb{R}^{d(d+1)m}\), by Alexander duality there is a natural isomorphism
\[
\tilde{H}_k(Q_n^d(m)(\mathbb{K}) ; \mathbb{Z}) \cong H_c^{d(\mathbb{K})md-k-1}(\Sigma_{n,\mathbb{K}}^d, \mathbb{Z}) \quad \text{for any } k.
\]
By reindexing we obtain a spectral sequence
\[
\{E_{k,s}^{1,d} : E_{k,s}^{1,d} : E_{t,d}^{k+s,t+s+1-t} \Rightarrow \tilde{H}_{s-k}(Q_n^d(m)(\mathbb{K}) ; \mathbb{Z})\},
\]
where \(E_{k,s}^{1,d} = H_c^{d(\mathbb{K})md+k-s-1}((\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d)_+ ; \mathbb{Z})\).
For a space \(X\), let \(F(X, k) \subset X^k\) denote the ordered configuration space of distinct \(k\) points in \(X\) given by
\[
F(X, k) = \{(x_1, \cdots, x_k) \in X^k : x_i \neq x_j \text{ if } i \neq j\}.
\]
Let \(S_k\) be the symmetric group on \(k\) letters. Then the group \(S_k\) acts on \(F(X, k)\) by permuting coordinates and we let \(C_k(X)\) denote the orbit space
\[
C_k(X) := F(X, k)/S_k.
\]
**Lemma 3.2.** If \(1 \leq k \leq \lfloor \frac{d}{n} \rfloor\), \(\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d\) is homeomorphic to the total space of a real affine bundle \(\xi_{d,k}\) over \(C_k(\mathbb{R})\) of rank
\[
l_{d,k} := d(\mathbb{K})m(d-nk) + k - 1.
\]
**Proof.** The argument is exactly analogous to the one in the proof of [1, Lemma 4.4]. Namely, an element of \(\mathcal{X}_k^d \setminus \mathcal{X}_{k-1}^d\) is represented by an \((m+1)\)-tuple \((f_1(z), \cdots, f_m(z), u)\), where \((f_1(z), \cdots, f_m(z))\) is an \(m\)-tuple of monic polynomials of the same degree \(d\) in \(\Sigma_{n,\mathbb{K}}^d, m \geq \lfloor \frac{d}{n} \rfloor\) and \(u\) is an element of the interior of the span of the images of \(k\) distinct points \(\{x_1, \cdots, x_m\} \subset C_k(\mathbb{R})\) such that \(\{x_i\}_{i=1}^m\) are common roots of \(\{f_i(z)\}_{i=1}^m\) of multiplicity \(n\) under a suitable embedding \(i_k\) satisfying the condition (2.3). Note that the \(k\) distinct points
\{x_j\}_{j=1}^k$ are uniquely determined by $u$. Indeed, if there exists another set of common roots $\{y_i\}_{i=1}^k$ of $f_i(z)$ of multiplicity $n$ satisfying the same condition, then $u \in \text{Int}(\Delta(x_1, \ldots, x_k)) \cap \text{Int}(\Delta(y_1, \ldots, y_k))$. However, if $\{x_i\}_{i=1}^k \neq \{y_i\}_{i=1}^k$, then, by Lemma 2.5, $\text{Int}(\Delta(x_1, \ldots, x_k)) \cap \text{Int}(\Delta(y_1, \ldots, y_k)) = \emptyset$ and this is a contradiction. Thus, we have a projection map $\pi_{k,d} : A_k \setminus A_{k-1} \to C_k(\mathbb{R})$ defined by $((f_1, \ldots, f_m), u) \mapsto \{x_1, \ldots, x_k\}$.

Now suppose that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $1 \leq i \leq m$. Let $c = \{x_j\}_{j=1}^k \in C_k(\mathbb{R})$ be any fixed element and consider the fiber $\pi_{k,d}(c)$. It is easy to see that the condition for a polynomial $f_i(z) \in \mathbb{P}^d(\mathbb{K})$ to be divisible by the polynomial $\prod_{i=1}^k(z - x_j)^n$, is equivalent to the following:

\begin{equation}
(3.7) \quad f_i^{(t)}(x_j) = 0 \quad \text{for} \quad 0 \leq t < n, \quad 1 \leq j \leq k.
\end{equation}

In general, for each $0 \leq t < n$ and $1 \leq j < k$, the condition $f_i^{(t)}(x_j) = 0$ gives one linear condition on the coefficients of $f_i(z)$, and determines an affine hyperplane in $\mathbb{P}^d(\mathbb{K})$. For example, if we set $f_i(z) = z^d + \sum_{s=1}^d a_s z^{d-s}$, then $f_i(x_j) = 0$ for all $1 \leq j \leq k$ if and only if $A_1 \mathbf{x} = \mathbf{b}_1$, where we set

\[
A_1 = \begin{bmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{d-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{d-1} \\
& \vdots & \ddots & \ddots & \vdots \\
1 & x_k & x_k^2 & \cdots & x_k^{d-1}
\end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix}
a_d \\
a_{d-1} \\
\vdots \\
a_1
\end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix}
x_1^d \\
 x_2^d \\
\vdots \\
 x_k^d
\end{bmatrix}.
\]

Similarly, $f_i'(x_j) = 0$ for all $1 \leq j \leq k$ if and only if $A_2 \mathbf{x} = \mathbf{b}_2$, where we set

\[
A_2 = \begin{bmatrix}
0 & 1 & 2x_1 & 3x_1^2 & \cdots & (d-1)x_1^{d-2} \\
0 & 1 & 2x_2 & 3x_2^2 & \cdots & (d-1)x_2^{d-2} \\
& \vdots & \ddots & \ddots & \ddots \\
0 & 1 & 2x_k & 3x_k^2 & \cdots & (d-1)x_k^{d-2}
\end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix}
dx_1^{d-1} \\
 dx_2^{d-1} \\
\vdots \\
 dx_k^{d-1}
\end{bmatrix}.
\]

Analogously, $f_i''(x_j) = 0$ for all $1 \leq j \leq k$ if and only if $A_3 \mathbf{x} = \mathbf{b}_3$, where we set

\[
A_3 = \begin{bmatrix}
0 & 0 & 2 & 6x_1 & \cdots & (d-1)(d-2)x_1^{d-3} \\
0 & 0 & 2 & 6x_2 & \cdots & (d-1)(d-2)x_2^{d-3} \\
& \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 2 & 6x_k & \cdots & (d-1)(d-2)x_k^{d-3}
\end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix}
d(d-1)x_1^{d-2} \\
 d(d-1)x_2^{d-2} \\
\vdots \\
 d(d-1)x_k^{d-2}
\end{bmatrix}.
\]

and so on. Since $\{x_i\}_{i=1}^k \in C_k(\mathbb{R})$, by Gaussian elimination of rows of matrices, the matrix $A_1$ reduces to the matrix $B_1$, where $s_i(t) := \sum_{s_i+ \cdots + s_i = i} x_1^{s_1}x_2^{s_2} \cdots x_i^{s_i}$ and

\[
B_1 = \begin{bmatrix}
1 & x_1 & x_1^2 & x_1^3 & x_1^4 & \cdots & x_1^{d-2} & x_1^{d-1} \\
0 & 1 & s_1(2) & s_2(2) & s_3(2) & \cdots & s_{d-3}(2) & s_{d-2}(2) \\
0 & 0 & 1 & s_1(3) & s_2(3) & s_3(3) & \cdots & s_{d-4}(3) & s_{d-3}(3) \\
0 & 0 & 0 & 1 & s_1(4) & s_2(4) & \cdots & s_{d-5}(4) & s_{d-4}(4) \\
& \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & 1 & s_1(k) & s_2(k) & \cdots & s_{d-k}(k)
\end{bmatrix}.
\]
Similarly, by easy Gaussian elimination of rows of matrices, the matrix \( A_2 \) reduces to the matrix \( B_2 \), where

\[
B_2 = \begin{bmatrix}
0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 & \cdots & \cdots & \cdots & \cdots & (d - 1)x_1^{d-2} \\
0 & 0 & 2 & 3s_1(2) & 4s_2(2) & 5s_3(2) & \cdots & \cdots & \cdots & (d - 1)s_{d-3}(2) \\
0 & 0 & 0 & 3 & 4s_1(3) & 5s_2(3) & \cdots & \cdots & \cdots & (d - 1)s_{d-4}(3) \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & 0 & k - 1 & ks_1(k) & (k + 1)s_2(k) & \cdots & (d - 1)s_{d-k-1}(k)
\end{bmatrix}.
\]

Analogously, the matrix \( A_3 \) reduces to the matrix \( B_3 \), where

\[
B_3 = \begin{bmatrix}
0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 & \cdots & \cdots & \cdots & \cdots & d(d - 1)x_1^{d-3} \\
0 & 0 & 0 & 6 & 12s_1(2) & 20s_2(2) & \cdots & \cdots & \cdots & (d - 1)s_{d-k-2}(2) \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & 0 & (k - 2)(k - 3) & k(k - 1)s_1(k) & \cdots & \cdots & (d - 1)s_{d-k-3}(k)
\end{bmatrix}.
\]

If we repeat this process, we finally obtain the reduced \((k \times d)\) matrices \( \{B_t\}_{t=1}^n \) such that each \( A_t \) reduces to the matrix \( B_t \). Now define the \((k \times d)\) matrix \( C_t \) (for \( 1 \leq t \leq n \)) inductively by \( C_1 = B_1 \) and \( C_t = \left[ \begin{array}{c} C_{t-1} \\ B_t \end{array} \right] \) for \( 2 \leq t \leq n \). Then by the induction on \( t \) and easy Gaussian elimination of rows of matrices, we see that each matrix \( C_t \) has rank \( kt \) for each \( 1 \leq t \leq n \). Thus we see that the the condition (3.7) gives exactly \( nk \) affine independent conditions on the coefficients of \( f_i(z) \). Hence we see that the space of \( m \)-tuples \((f_1(z), \ldots, f_m(z))\) is a locally trivial bundle (and hence trivial since \( C_t(\mathbb{R}) \cong \mathbb{R}^k \) is contractible, unlike in the case considered in the proof of Proposition 6 of [23]). To see that it is locally trivial, consider a point \( c = \{x_1, \ldots, x_k\} \) in \( \mathcal{C}_k(\mathbb{R}) \), that is an unordered tuple of real numbers. Since the real numbers have a natural ordering, we can always represent such a tuple by an ordered one, and treat the numbers \( x_i \) as coordinates of \( x = (x_1, \ldots, x_k) \).

Suppose that we have a continuous surjective map \( f : X \to Y \), such that each fiber \( f^{-1}(y) \) is an affine space of dimension \( l \). Then to show that the map is a locally trivial affine bundle of dimension \( l \) it is enough to construct for each \( y \in Y \), a neighborhood \( V_y \subset Y \) and \( l \) continuous sections \( s_i : V_y \to X \ (1 \leq i \leq l) \), plus a section \( t : V_y \to X \) such that each point \( v \in f^{-1}(y) \) can be expressed in the form \( v = t(x) + \sum_{i=1}^l b_is_i(x) \) (where \( b_i \) are some real numbers).

Indeed, such an expression can be obtained by standard Gaussian elimination of rows of matrices. To see this, it suffices to consider the case \( m = 1 \). Let \( c = \{x_1, \ldots, x_k\} \in \mathcal{C}_k(\mathbb{R}) \) with \( x_1 < x_2 < \cdots < x_k \), and suppose that the \((k \times (d+1))\)-matrix \( [A_i, y_i] \) reduces to the matrix \( [B_i, c_i] \) for each \( 1 \leq i \leq n \) (by Gaussian elimination). Then the fiber \( \pi_{k,d}^{-1}(c) \) is homeomorphic to the product \( \mathcal{S} \times \mathbb{R}^{k-1} \), where \( \mathcal{S} = \{v \in \mathbb{K}^d : B_i\mathbf{v} = c_i \text{ for each } 1 \leq i \leq n \} \) and \( \mathbf{v} = (a_d, a_{d-1}, \ldots, a_1) \). Since each \( B_i \) has the form of a stepwise matrix,
by Gaussian elimination of rows of matrix, one can show that each element \( v \in S \) can be represented by the form \( v = t(c) + \sum_{i=1}^{d-nk} t_i s_i(c) \), where \( t(c) \) and \( s_i(c) \) are fixed \( \mathbb{R}^d \)-valued \( \mathbb{R} \)-coefficients polynomial maps with \( t_i \in \mathbb{K} \) (free parameter) for \( 1 \leq i \leq d - nk \). Thus \( \pi_{k,d} \) is locally trivial. For example, let us give an extremely simplified example for the case \((n,k) = (1,2)\). Consider one polynomial equation of degree \( d = 3 \) and the condition that it has at two real roots \( \{x_1, x_2\} \in C_2(\mathbb{R}) \). Consider the equation given by \( f(z) = 0 \), where \( f(z) = z^3 + a_2 z^2 + a_1 z + a_0 \). Solving the system \( f(x_1) = 0, f(x_2) = 0 \) for the coefficients \( a_0, a_1 \) and treating \( a_2 = t \) as a “free parameter” we obtain

\[
S = \{(x_1^2 x_2 + x_1 x_2^2, -x_1^2 - x_1 x_2 - x_2^2) + t(x_1 x_2, -x_1 - x_2) : t \in \mathbb{K}\}.
\]

This completes the proof. \( \square \)

**Lemma 3.3.** There is a natural isomorphism

\[
E_{k,s}^{1,d} \cong \begin{cases} 
\mathbb{Z} & \text{if } s = (d(\mathbb{K}) mn - 1)k \text{ and } 0 \leq k \leq \lfloor \frac{d}{n} \rfloor, \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** If \( k \leq 0 \), the assertion is trivial. If \( k > \lfloor \frac{d}{n} \rfloor \), the assertion easily follows from (3.1). So suppose that \( 1 \leq k \leq \lfloor \frac{d}{n} \rfloor \). Since there is a homeomorphism \( C_2(\mathbb{R}) \cong \mathbb{R}^k \) and it is contractible, the affine bundle \( \xi_{d,k} \) is trivial. Hence, there is a homeomorphism

\[
(\lambda_k^d)^+/(\lambda_{k-1}^d)^+ \cong (\lambda_k^d \setminus \lambda_{k-1}^d)^+ \cong (\mathbb{R}^{d,s} \times \mathbb{R}^k)^+ = S^{k+s,d,k}.
\]

Thus there is an isomorphism

\[
E_{k,s}^{1,d} \cong \tilde{H}(\mathbb{K}) mn + k - s - 1(S^{k+s,d,k}; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & \text{if } s = (d(\mathbb{K}) mn - 1)k, \\
0 & \text{otherwise.}
\end{cases}
\]

and this completes the proof. \( \square \)

**Corollary 3.4.** If \( d(\mathbb{K}) mn \geq 3 \), there is an isomorphism

\[
H_k(Q_d^m(\mathbb{K}); \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & \text{if } k = (d(\mathbb{K}) mn - 2)i \text{ and } 0 \leq i \leq \lfloor \frac{d}{n} \rfloor, \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** Consider the spectral sequence

\[
\{ E_{k,s}^{1,d}, E_{k,s}^{i,d} : E_{k,s}^{i,d} \to E_{k+s+t,i}^{i,d} \} \Rightarrow H_{k-s}(Q_d^m(\mathbb{K}); \mathbb{Z}).
\]

Then, for dimensional reasons and by Lemma 3.3, it is easy to see that \( E_{k,s}^{1,d} = E_{k+\infty,d}^{\infty,d} \) and the result follows. \( \square \)

**Lemma 3.5.** If \( d(\mathbb{K}) mn \geq 4 \), the space \( Q_d^m(\mathbb{K}) \) is simply connected.

**Proof.** If \( n = 1 \) or \( m = 1 \), the assertion follows from (i) and (ii) of Theorem 1.6. We therefore suppose that \( m \geq 2 \) and \( n \geq 2 \). An element of \( \pi_1(Q_d^m(\mathbb{K})) \) can be represented by strings with total multiplicity \( d \) of \( m \) different colors similarly to the classical representation of the elements of the braid group \( Br_d = \pi_1(C_d(\mathbb{C})) \) [13]. When
all strings of \( m \) different colors move continuously, the following case is not allowed to occur in this representation:

(i) All strings of multiplicity \( \geq n \) of \( m \) different colors pass through a single point on the real line.

By using this representation, one can show that any strings can intersect, pass through one another and thus can change the order (see also [9, §5. Appendix] for more details). Thus we see that \( a \cdot b = b \cdot a \) for any \( a, b \in \pi_1(\mathbb{Q}_n^{d,m}(\mathbb{K})) \) and we know that \( \pi_1(\mathbb{Q}_n^{d,m}(\mathbb{K})) \) is an abelian group. Moreover, since \( d(\mathbb{K})mn \geq 4 \), by Corollary 3.4 we see that \( H_1(\mathbb{Q}_n^{d,m}(\mathbb{K}); \mathbb{Z}) = 0 \). Hence, we have an isomorphism \( \pi_1(\mathbb{Q}_n^{d,m}(\mathbb{K})) \cong H_1(\mathbb{Q}_n^{d,m}(\mathbb{K}); \mathbb{Z}) = 0 \), and the assertion follows.

\[ \square \]

4. Stabilization maps

**Definition 4.1.** We identify \( \mathbb{C} = \mathbb{R}^2 \) by the identification \( x + y\sqrt{-1} \mapsto (x, y) \).

(i) For each integer \( d \geq 1 \) let \( \varphi_d : \mathbb{C} \to \mathbb{R}_d = \{ x \in \mathbb{C} : \Re (x) < d \} \) be any fixed homeomorphism such that \( \varphi(\alpha) = \varphi(\alpha) \) for any \( \alpha \in \mathbb{C} \). Note that \( \varphi(x) \in \mathbb{R} \) and \( \varphi(x) < d \) if \( x \in \mathbb{R} \). Let \( \bar{\varphi}_d : \mathbb{P}^d(\mathbb{C}) \to \mathbb{P}^d(\mathbb{C}) \) denote the map given by

\[
\bar{\varphi}_d \left( \prod_{j=1}^{d} (z - \alpha_j) \right) = \prod_{j=1}^{d} (z - \varphi_d(\alpha_j)).
\]

(ii) For each integer \( d \geq 1 \), let \( D_{d,\mathbb{K}} \subset \mathbb{K}^m \) denote the open set defined by

\[
D_{d,\mathbb{K}} = \begin{cases} \{ (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m : d < \alpha_k < d + 1, \alpha_i \neq \alpha_j \text{ if } i \neq j \} & \text{ if } \mathbb{K} = \mathbb{R}, \\ \{ (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m : d < \Re(\alpha_k) < d + 1, \alpha_i \neq \alpha_j \text{ if } i \neq j \} & \text{ if } \mathbb{K} = \mathbb{C} \end{cases}
\]

and fix some point \( x_d = (x_{d,1}, \ldots, x_{d,m}) \in D_{d,\mathbb{K}} \). Then as in [24, page 42] we can define the stabilization map \( s_{d,m}^{d,m} : \mathbb{Q}_n^{d,m}(\mathbb{K}) \to \mathbb{Q}_n^{d+1,m}(\mathbb{K}) \) by

\[
s_{d,m}^{d,m}(f) = ((z - x_{d,1})\bar{\varphi}_d(f_1(z)), \ldots, (z - x_{d,m})\bar{\varphi}_d(f_m(z)))
\]

for \( f = (f_1(z), \ldots, f_m(z)) \in \mathbb{Q}_n^{d,m}(\mathbb{K}) \). Note that the homotopy class of the map \( s_{d,m}^{d,m} \) does not depend on the choice of the homeomorphism \( \varphi_d \) and the point \( x_d \).

(iii) Let \( Q_{n,K}^{\infty,m}(\mathbb{K}) \) denote the colimit \( \lim_{d \to \infty} \mathbb{Q}_n^{d,m}(\mathbb{K}) \) constructed from the stabilization maps \( s_{d,m}^{d,m} \),

\[
\mathbb{Q}_n^{1,m}(\mathbb{K}) \xrightarrow{s_{n,K}^{1,m}} \mathbb{Q}_n^{2,m}(\mathbb{K}) \xrightarrow{s_{n,K}^{2,m}} \mathbb{Q}_n^{3,m}(\mathbb{K}) \xrightarrow{s_{n,K}^{3,m}} \mathbb{Q}_n^{4,m}(\mathbb{K}) \xrightarrow{s_{n,K}^{4,m}} \mathbb{Q}_n^{5,m}(\mathbb{K}) \xrightarrow{s_{n,K}^{5,m}} \ldots
\]

(iv) Let \( f_{n,K}^{d,m} : \mathbb{Q}_n^{d,m}(\mathbb{K}) \to \mathbb{Q}_n^{d+2,m}(\mathbb{K}) \) denote the map defined by

\[
f_{n,K}^{d,m}(f) = ((z^2 + 1)f_1(z), (z^2 + 1)f_2(z), \ldots, (z^2 + 1)f_m(z))
\]

for \( f = (f_1(z), \ldots, f_m(z)) \in \mathbb{Q}_n^{d,m}(\mathbb{K}) \).

**Lemma 4.2.** \( f_{n,K}^{d,m} \simeq s_{n,K}^{d+1,m} \circ s_{n,K}^{d,m} \) up to homotopy.
Proof. Let \( f = (f_1(z), \cdots, f_m(z)) \in Q_{d,m}^n(\mathbb{K}) \). Then the map \( s_{n,k}^{d+1,m} \circ s_{n,K}^{d,m}(f) \) is represented up to homotopy by \( ((z-x_{d,1})(z-x_{d+1,1}), \cdots, (z-x_{d,m})(z-x_{d+1,m})) \), where \( T_k = \tilde{\varphi}_d(f_k(z)) \) for \( 1 \leq k \leq m \).

Define the family of monic polynomials \( \{\phi_t(z) : 0 \leq t \leq 1\} \) of degree 2 in \( \mathbb{K}[z] \) by

\[
\phi_t(z) = \begin{cases} 
(z-x_{d,1})(z-2tx_{d,1}-(1-2t)x_{d+1,1}) & \text{if } 0 \leq t \leq \frac{1}{2}, \\
(z-(2-2t)x_{d,1}-(2t-1)\sqrt{1-t})(z-(2-2t)x_{d,1}+(2t-1)\sqrt{1-t}) & \text{if } \frac{1}{2} \leq t \leq 1
\end{cases}
\]

and consider the homotopy \( F \) given by

\[
F(f,t) = (\phi_t(z)T_1, (z-x_{d,2})(z-x_{d+1,2})T_2, \ldots, (z-x_{d,m})(z-x_{d+1,m})T_m).
\]

Since \( F(f,0) = s_{n,K}^{d+1,m} \circ s_{n,K}^{d,m}(f) \), the map \( s_{n,K}^{d+1,m} \circ s_{n,K}^{d,m}(f) \) is homotopic to the map

\[
F(f,1) = ((z^2+1)T_1, (z^2+1)T_2, (z-x_{d,3})(z-x_{d+1,3})T_3, \ldots, (z-x_{d,m})(z-x_{d+1,m})T_m).
\]

If we use this trick again, the map \( s_{n,K}^{d+1,m} \circ s_{n,K}^{d,m}(f) \) is homotopic to the map given by

\[
((z^2+1)T_1, (z^2+1)T_2, (z-x_{d,3})(z-x_{d+1,3})T_3, \ldots, (z-x_{d,m})(z-x_{d+1,m})T_m).
\]

By using this trick repeatedly, we see that the map \( s_{n,K}^{d+1,m} \circ s_{n,K}^{d,m}(f) \) is homotopic to the map \((z^2+1)T_1, (z^2+1)T_2, (z^2+1)T_3, \ldots, (z^2+1)T_m\), which is also homotopic to the map \( f_{n,K}^{d,m}(f) \). This completes the proof.

Corollary 4.3. Let \( \epsilon \in [0,1) \). Then the space \( Q_{d,m}^{\infty}(\mathbb{K}) \) is homotopy equivalent to the colimit constructed from the stabilization maps \( f_{n,K}^{d,m} \):

\[
Q_{d,m}^{n+\epsilon,m}(\mathbb{K}) \xrightarrow{f_{n,K}^{n+\epsilon,m}} Q_{d,m}^{n+\epsilon,m}(\mathbb{K}) \xrightarrow{f_{n,K}^{n+\epsilon,m}} Q_{d,m}^{n+\epsilon,m}(\mathbb{K}) \xrightarrow{f_{n,K}^{n+\epsilon,m}} Q_{d,m}^{n+\epsilon,m}(\mathbb{K}) \xrightarrow{f_{n,K}^{n+\epsilon,m}} \ldots.
\]

Proof. This follows from the definition of \( Q_{d,m}^{\infty}(\mathbb{K}) \) and Lemma 4.2. \( \Box \)

Let \( x_d = (x_{d,1}, \cdots, x_{d,m}) \in D_{d,K} \) and consider the stabilization map \( s_{n,K}^{d,m} \) given by (4.3). Note that this map clearly extends to a map \( s_{n,K}^{d,m} : \mathbb{K}^d \to \mathbb{K}^{d+1} \) by the same formula and its restriction gives a stabilization map \( s_{n,K}^{d,m} : \Sigma_{n,K}^d \to \Sigma_{n,K}^{d+1} \) between discriminants. If we choose a sufficiently small positive number \( \epsilon_0 > 0 \), then the following condition is satisfied:

\[ O(x_d) \cap O(x_d) = \emptyset \text{ if } i \neq j, \text{ where } O(x) = \{\alpha \in \mathbb{K} : |x - \alpha| < \epsilon_0\} \text{ for } x \in \mathbb{K}. \]

For each fixed element \( y = (y_1, \cdots, y_m) \in O(x_{d,1}) \times \cdots \times O(x_{d,m}) \), let us consider the map \( s_{n,K}^{d,m} : \Sigma_{n,K}^d \to \Sigma_{n,K}^{d+1} \) given by \( f_1(z), \cdots, f_m(z) \mapsto ((z-y_1)f_1(z)), \cdots, (z-y_m)f_m(z) \). Then it is easy to see that this map coincides with the map \( s_{n,K}^{d,m} \) if \( y = x_d \). Thus, by using a homeomorphism \( O(x_d) \cong \mathbb{K} \) for \( 1 \leq k \leq m \), we see that the map \( s_{n,K}^{d,m} : \Sigma_{n,K}^d \to \Sigma_{n,K}^{d+1} \) extends to an open embedding

\[
s_{n,K}^{d,m} : \Sigma_{n,K}^d \times \mathbb{K}^m \to \Sigma_{n,K}^{d+1}.
\]
Since one-point compactification is contravariant for open embeddings, it induces a map
\[
\hat{s}_{n,K}^d: (\Sigma_{n,K}^{d,m})_+ \to (\Sigma_{n,K}^{d,m} \times \mathbb{R}^m)_+ = (\Sigma_{n,K}^{d,m})_+ \wedge S^{d(K)m}
\]
(4.6)
between one-point compactifications and we obtain the following commutative diagram
\[
\begin{array}{ccc}
\hat{H}_k(Q^d_n(K) \mathbb{Z}) & \xrightarrow{\hat{s}_{n,K}^d} & \hat{H}_k(Q^d_{n+1}(K) \mathbb{Z}) \\
\downarrow A_l \cong & & \downarrow A_l \cong \\
\hat{H}_c^{d(K)dm-k-1}(\Sigma_{n,K}^{d,m} \mathbb{Z}) & \xrightarrow{\hat{s}_{n,K}^d} & \hat{H}_c^{d(K)(d+1)m-k-1}(\Sigma_{n,K}^{d,m} \mathbb{Z}),
\end{array}
\]
(4.7)
where $A_l$ denotes the Alexander duality isomorphism and $\hat{s}_{n+1}^d$, the composite of the suspension isomorphism with the homomorphism $(\hat{s}_{n,K}^d)_*$,
\[
H_c^{d(K)dm-k-1}(\Sigma_{n,K}^{d,m} \mathbb{Z}) \xrightarrow{\cong} H_c^{d(K)(d+1)m-k-1}(\Sigma_{n,K}^{d,m} \mathbb{Z}),
\]
(4.8)
Note that the map $\hat{s}_{n,K}^d$ induces the filtration preserving map
\[
\hat{s}_{n,K}^d: \mathcal{X}^d_n(K) \times \mathbb{R}^m \to \mathcal{X}^{d+1}_n(K)
\]
and it induces the homomorphism of spectral sequences
\[
(\theta_{k,s}^d : E^{d,s}_{k,s} \to E^{d+1,s}_{k,s}).
\]

**Lemma 4.4.** If $d(K)mn \geq 4$ and $1 \leq k \leq \lfloor \frac{d}{m} \rfloor$, $\theta_{k,s}^d : E^{d,s}_{k,s} \to E^{d+1,s}_{k,s}$ is an isomorphism for any $s$.

**Proof.** Since we use the result of Lemma 3.2 and $\mathcal{X}_d^m = \mathcal{X}_d^{m-1}$ if $k > \lfloor \frac{d}{m} \rfloor$, suppose that $1 \leq k \leq \lfloor \frac{d}{m} \rfloor$. If we set $\hat{s}_{n,K}^d = \hat{s}_{n,K}^d|_{\mathcal{X}_d^m \setminus \mathcal{X}_d^{m-1}}$, it follows from the construction of the map $\hat{s}_{n,K}^d$ that the following diagram is commutative:
\[
\begin{array}{ccc}
(\mathcal{X}_d^m \setminus \mathcal{X}_d^{m-1}) \times \mathbb{R}^m & \xrightarrow{\pi_{k,d}} & C_k(\mathbb{R}) \\
\downarrow \hat{s}_{n,K}^d & & \\
\mathcal{X}_d^{m+1} \setminus \mathcal{X}_d^{m+1-1} & \xrightarrow{\pi_{k,d+1}} & C_k(\mathbb{R}).
\end{array}
\]
Since one-point compactification is contravariant for open embeddings, the map $\hat{s}_{n,K}^d$ induces the map $(\hat{s}_{n,K}^d)_+ : (\mathcal{X}_d^{m+1} \setminus \mathcal{X}_d^{m+1-1})_+ \to ((\mathcal{X}_d^m \setminus \mathcal{X}_d^{m-1}) \times \mathbb{R}^m)_+ = (\mathcal{X}_d^m \setminus \mathcal{X}_d^{m-1})_+ \wedge S^{d(K)m}$ between one-point compactifications. Recall from the proof of Lemma 3.2 that $\xi_{d,K}$ (resp. $\xi_{d+1,K}$) is a trivial real affine bundle over $C_k(\mathbb{R})$ with rank $l_{d,K}$ (resp. $l_{d+1,K}$). Moreover, note that
\[
(d(K)md + k - s - 1) - l_{d,k} = (d(K)m(d+1) + k - s - 1) - l_{d+1,k} = d(K)mnk - s.
\]
Then from the above commutative diagram and the suspension isomorphism, we obtain
If (i) For an 

First, consider the case that 

is a homology equivalence for 

Let 

is induced from 

Note that 

Next assume that 

is a homology equivalence through 

In this case, by considering 

we easily see that 

Since 

is an isomorphism for any 

is induced from 

it follows from Lemma 3.3 and dimensional reasons that the spectral sequence collapses at 

Hence, 

is an isomorphism for any 

We prove the key result.

Theorem 4.5. If 

the stabilization map

is a homology equivalence for 

and it is a homology equivalence through dimension 

for 

Proof. First, consider the case 

In this case, by using Lemmas 3.3 and 4.4 we easily see that 

is an isomorphism for any 

is induced from 

it follows from (4.7) that the map 

is a homology equivalence.

Next assume that 

i.e. 

is a homology equivalence through dimension 

for 

in (4.1) or 

in (4.1) and 

in (4.1) and 

are positive integers, 

we easily see that 

is an isomorphism for any 

as long as the condition 

is satisfied. Hence, the map 

is a homology equivalence through dimension 

for 


\[
\begin{array}{ccc}
E^{d+1}_{k,s} & \xrightarrow{s^1_{k,s}} & E^{d+1}_{k,s} \\
\| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) & \xrightarrow{(s^d_{m,k})^*} & H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) \\
\| & & \| \quad \| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) & \xrightarrow{(s^d_{m,k})^*} & H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) \\
\| & & \| \quad \| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) & \xrightarrow{(s^d_{m,k})^*} & H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) \\
\| & & \| \quad \| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) & \xrightarrow{(s^d_{m,k})^*} & H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) \\
\| & & \| \quad \| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) & \xrightarrow{(s^d_{m,k})^*} & H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) \\
\| & & \| \quad \| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) & \xrightarrow{(s^d_{m,k})^*} & H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) \\
\| & & \| \quad \| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) & \xrightarrow{(s^d_{m,k})^*} & H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) \\
\| & & \| \quad \| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) & \xrightarrow{(s^d_{m,k})^*} & H^d_c((\mathcal{K}_k \setminus \mathcal{K}_{k-1}) \times \mathbb{K}^m; \mathbb{Z}) \\
\| & & \| \quad \| & & \|
\end{array}
\]
map $s^{D_{i}}_{n,n} : Q^{d_{i},\ldots,d_{m},m}_{n}(K) \rightarrow Q^{d_{i},\ldots,d_{i-1},d_{i+1},\ldots,d_{m},m}_{n}(K)$ by

\begin{equation}
\tag{4.11}
s^{D_{i}}_{n,n}(f) = (\tilde{T}_1(z), \ldots, \tilde{T}_{i-1}(z), (z-x_0)\tilde{T}_i(z), \tilde{T}_{i+1}(z), \ldots, \tilde{T}_m(z))
\end{equation}

for $f = (f_1(z), \ldots, f_m(z)) \in Q^{D, m}_{n}(K)$, where $\tilde{T}_k(z) = \tilde{\varphi}_d(f_k(z))$ for $1 \leq k \leq m$.

\textbf{Theorem 4.7.} If $d(K)mn \geq 4$, the stabilization map

$s^{D_{i}}_{n,n} : Q^{d_{i},\ldots,d_{m},m}_{n}(K) \rightarrow Q^{d_{i},\ldots,d_{i-1},d_{i+1},\ldots,d_{m},m}_{n}(K)$

is a homology equivalence for $\lfloor \frac{d_i}{n} \rfloor = \lfloor \frac{d_i+1}{n} \rfloor$, and it is a homology equivalence through dimension $D(d_i; m, n, K)$ for $\lfloor \frac{d_i}{n} \rfloor < \lfloor \frac{d_i+1}{n} \rfloor$.

\textbf{Proof.} The proof is completely analogous to that of Theorem 4.5. So we omit the details. \qed

\textbf{Lemma 4.8.} If $d(K)mn \geq 4$, $\pi_1(Q^{d_{i},\ldots,d_{m},m}_{n}(K))$ is an abelian group.

\textbf{Proof.} We can prove the assertion in the same way as in Lemma 3.5 by using the string representation. \qed

\section{Configuration spaces and horizontal scanning maps}

In this section, we prove the stable result (see Theorem 5.18 below) by using “horizontal scanning maps”. We continue to assume that $K = \mathbb{R}$ or $\mathbb{C}$.

\textbf{Definition 5.1.} (i) For a space $X$ let $SP^d(X)$ denote the $d$-th \textit{symmetric product} defined by the quotient space

\begin{equation}
SP^d(X) = X^d/S_d,
\end{equation}

where the symmetric group $S_d$ of $d$ letters acts on $X^d$ by the permutation of coordinates. Since $F(X, d)$ is an $S_d$-invariant subspace of $X^d$ and $C_d(X) = F(X, d)/S_d$, there is a natural inclusion $C_d(X) \subset SP^d(X)$.

(ii) It is easy to see that an element $\alpha \in SP^d(X)$ may be identified with the formal linear combination

\begin{equation}
\tag{5.2}
\alpha = \sum_{i=1}^{k} d_ix_i, \quad \text{where } \{x_i\}_{i=1}^{k} \in C_k(X) \quad \text{and } \sum_{i=1}^{k} d_i = d.
\end{equation}

We shall refer to $\alpha$ as a \textit{configuration} (or 0-cycle) having a \textit{multiplicity} $d_i$ at the point $x_i$.

(iii) Note that there is a natural homeomorphism

\begin{equation}
\tag{5.3}
\Phi : P^d(\mathbb{C}) \rightarrow SP^d(\mathbb{C}) \quad \text{given by } \Phi (\prod_{i=1}^{k} (z - \alpha_i)^{d_i}) = \sum_{i=1}^{k} d_i\alpha_i.
\end{equation}

Since there is a natural inclusion $P^d(\mathbb{R}) \subset P^d(\mathbb{C})$, we define the subspace $SP^d_{\mathbb{R}} \subset SP^d(\mathbb{C})$.
The element

\[(5.4) \quad SP^d = \Phi_d(P^d(\mathbb{R})).\]

Note that restriction gives a homeomorphism

\[(5.5) \quad \Phi_d(P^d(\mathbb{R})) : P^d(\mathbb{R}) \xrightarrow{\cong} SP^d.\]

(iv) For a subspace \(X \subset \mathbb{C}\), let \(SP^d_\mathbb{R}(X) \subset SP^d(X)\) denote the subspace of \(SP^d(X)\) given by

\[(5.6) \quad SP^d_\mathbb{R}(X) = SP^d \cap SP^d(X).\]

(v) Similarly, for an \(m\)-tuple \(D = (d_1, \cdots, d_m) \in \mathbb{N}^m\) of positive integers, define the subspaces \(Q^{D,m}_{n,\mathbb{C}}(X) \subset Q^{D,m}_{n,\mathbb{C}}(X)\) in \(SP^d(X) \times \cdots \times SP^d_n(X)\) by

\[Q^{D,m}_{n,\mathbb{C}}(X) = Q^{d_1,\cdots, d_m}_{n,\mathbb{C}}(X) = \{((\xi_1, \cdots, \xi_m)) \in SP^d(X) \times \cdots \times SP^d_n(X) : \text{\textasteriskcentered}\},\]

where the condition \(\text{\textasteriskcentered}\) is given by

\[\text{\textasteriskcentered}\]

The element \((\cap_{k=1}^m \xi_k) \cap \mathbb{R}\) contains no point of multiplicity \(\geq n\).

When \(d = d_1 = \cdots = d_m\) and \(D = (d_1, \cdots, d_m)\), we set \(Q^{d,m}_{n,\mathbb{R}}(X) = Q^{D,m}_{n,\mathbb{R}}(X)\).

**Remark 5.2.**

(i) For an open set \(V \subset \mathbb{C}\) and an \(m\)-tuple \(D = (d_1, \cdots, d_m) \in \mathbb{N}^m\), let \(Q^{D,m}_{n,\mathbb{R}}(V, \mathbb{K})\) denote the subspace of \(Q^{D,m}_{n,\mathbb{R}}(V)\) consisting of all \(m\)-tuples \((f_1(z), \cdots, f_m(z)) \in Q^{D,m}_{n,\mathbb{R}}(\mathbb{K})\) such that all roots of \(f_k(z)\) lie in \(V\) for each \(1 \leq k \leq m\). Then it is easy to see that there is a homeomorphism \(Q^{D,m}_{n,\mathbb{R}}(V, \mathbb{K}) = Q^{D,m}_{n,\mathbb{R}}(V)\) given by

\[(5.8) \quad \left(\prod_{k=1}^{s_1}(z - \alpha_k)^{d_k(i)}, \cdots, \prod_{k=1}^{s_m}(z - \alpha_k)^{d_k(m)}\right) \mapsto \left(\sum_{k=1}^{s_1} d_k(i) \alpha_k^{(1)}, \cdots, \sum_{k=1}^{s_m} d_k(m) \alpha_k^{(m)}\right),\]

where \(d_k(i) \in V\) and \(s_k = d_i\) for \(1 \leq k \leq s_i\) and \(1 \leq i \leq m\).

(ii) Let \(X \subset \mathbb{C}\) be a subspace. If we use the notation (5.2), an element \(\alpha \in SP^d_\mathbb{R}(X)\) can be represented as the formal linear combination

\[(5.9) \quad \alpha = \sum_{i=1}^{s} d_i x_i + \sum_{j=1}^{k} d_j(\alpha_j + \sigma_j),\]

where \(\{x_i\}_{i=1}^{s} \subset C_s(\mathbb{R}), \{\alpha_j\}_{j=1}^{k} \subset C_k(\mathbb{H} \cap (X \setminus \mathbb{R})), \sum_{i=1}^{s} d_i + 2 \sum_{j=1}^{k} d_j = d\) and \(\mathbb{H} = \{\alpha \in \mathbb{C} : \text{Im } (\alpha) > 0\}\).

If \(X \cap \mathbb{R} = \emptyset\) and \(D = (d_1, \cdots, d_m) \in \mathbb{N}^m\), \(Q^{D,m}_{n,\mathbb{R}}(X) = \prod_{k=1}^{m} SP^d_k(X)\), where we set \(SP^d_\mathbb{K} = SP^d(X)\) if \(\mathbb{K} = \mathbb{C}\).

**Definition 5.3.** Let \(X \subset \mathbb{C}\) and \(\emptyset \neq A \subset X\) be a closed subspace.
\[(i)\] Let \( SP^d(X, A) \) denote the quotient space \( SP^d(X) / \sim \), where the equivalence relation \( \sim \) on \( SP^d(X) \) is given by
\[
(5.10) \quad \xi \sim \eta \quad \text{if} \quad \xi \cap (X \setminus A) = \eta \cap (X \setminus A) \quad \text{for} \ \xi, \eta \in SP^d(X).
\]
Thus, the points in \( A \) are ignored. Note that there is a natural inclusion \( SP^d(X, A) \subset SP^{d+1}(X, A) \) by adding a point in \( A \). Let \( SP(X, A) \) denote the space given by the union
\[
(5.11) \quad SP(X, A) = \bigcup_{d \geq 0} SP^d(X, A), \quad \text{where we set} \ SP^0(X, A) = \{\emptyset\} \quad \text{for} \ d = 0.
\]
\[(ii)\] Similarly, let \( Q_{n, X}^{d,m}(X, A) \) be the quotient space of the space
\[
(5.12) \quad \{(\xi_1, \cdots, \xi_m) \in SP^{d_1}(X) \times \cdots \times SP^{d_m}(X) : (*)_n,A\},
\]
where the condition \( (*)_n,A \) is given by
\[
(*)_n,A \quad \text{The element} \ \cap_{k=1}^m \xi_k \cap R \cap (X \setminus A) \text{contains no point of multiplicity} \geq n.
\]
and the equivalence relation is given by
\[
(\xi_1, \cdots, \xi_m) \sim (\eta_1, \cdots, \eta_m) \quad \text{if} \quad \xi_i \cap (X \setminus A) = \eta_i \cap (X \setminus A) \quad \text{for each} \ 1 \leq i \leq m.
\]
Thus, points in \( A \) are ignored. Note that there is a natural inclusion
\[
(5.13) \quad Q_{n, X}^{d,m}(X, A) \subset Q_{n, X}^{d+1,m}(X, A)
\]
given by adding points in \( A \). Define the space \( Q_{n, X}^{m}(X, A) \) by the union
\[
(5.14) \quad Q_{n, X}^{m}(X, A) = \bigcup_{d \geq 0} Q_{n, X}^{d,m}(X, A), \quad \text{where we set} \ Q_{n, X}^{0,m}(X, A) = \{(\emptyset, \cdots, \emptyset)\}.
\]

**Remark 5.4.** As sets \( Q_{n, X}^{m}(X, A) \) and the disjoint union \( \coprod_{d \geq 0} Q_{n, X}^{d,m}(X \setminus A) \) are bijectively equivalent, but they are not homeomorphic topological spaces. For example, if \( X \) is connected, then \( Q_{n, X}^{m}(X, A) \) is connected while the disjoint union is, in general, disconnected.

We need two kinds of horizontal scanning maps. First, we define the scanning map for the configuration space of particles. From now on, we make the identification \( \mathbb{C} = \mathbb{R}^2 \).

**Definition 5.5.** For a rectangle \( X \) in \( \mathbb{C} = \mathbb{R}^2 \), let \( \sigma X \) denote the union of the sides of \( X \) which are parallel to the \( y \)-axis, and for a subspace \( Z \subset \mathbb{C} = \mathbb{R}^2 \), let \( Z \) be the closure of \( Z \). From now on, let \( I \) denote the interval \( I = [-1, 1] \) and let \( 0 < \epsilon < 1 \) be a fixed positive real number. For each \( x \in \mathbb{R} \), let \( V(x) \) be the set defined by
\[
(5.15) \quad V(x) = \{w \in \mathbb{C} : |\text{Re}(w) - x| < \epsilon, |\text{Im}(w)| < 1\} = (x - \epsilon, x + \epsilon) \times (-1, 1),
\]
and let us identify \( I \times I = I^2 \) with the closed unit rectangle \( \{t + s\sqrt{-1} \in \mathbb{C} : -1 \leq t, s \leq 1\} \) in \( \mathbb{C} \). Now define the horizontal scanning map
\[
(5.16) \quad sc_{n}^{d,m} : Q_{n}^{d,m}(\mathbb{R}) \to \Omega Q_{n, X}^{m}(I^2, \partial I \times I) = \Omega Q_{n, X}^{m}(I^2, \sigma I^2)
\]
as follows. For each \( m \)-tuple \( \alpha = (\xi_1, \cdots, \xi_m) \in Q_{n,\mathbb{R}}^{d,m}(C) \) of configurations, let \( sc_{n}^{d,m}(\alpha) : \mathbb{R} \to Q_{n,\mathbb{R}}^{d,m}(I^2, \partial I \times I) = Q_{n,\mathbb{R}}^{d,m}(I^2, \sigma I^2) \) denote the map given by

\[
\mathbb{R} \ni x \mapsto (\xi_1 \cap \nabla(x), \cdots, \xi_m \cap \nabla(x)) \in Q_{n,\mathbb{R}}^{m}(\nabla(x), \sigma \nabla(x)) \cong Q_{n,\mathbb{R}}^{m}(I^2, \sigma I^2),
\]

where we use the canonical identification \((\nabla(x), \sigma \nabla(x)) \cong (I^2, \sigma I^2) = (I^2, \partial I \times I)\).

Since \( \lim_{x \to \pm \infty} sc_{n}^{d,m}(\alpha)(x) = (\emptyset, \cdots, \emptyset) \), by setting \( sc_{n}^{d,m}(\alpha)(\infty) = (\emptyset, \cdots, \emptyset) \) we obtain a based map \( sc_{n}^{d,m}(\alpha) \in \Omega Q_{n,\mathbb{R}}^{m}(I^2, \sigma I^2) \), where we identify \( S^1 = \mathbb{R} \cup \infty \) and we choose the empty configuration \((\emptyset, \cdots, \emptyset)\) as the base point of \( Q_{n,\mathbb{R}}^{m}(I^2, \sigma I^2) \).

Hence, if we identify \( Q_{n,\mathbb{R}}^{d,m}(\emptyset) = Q_{n,\mathbb{R}}^{d,m}(C) \), we obtain a map \( sc_{n}^{d,m} : Q_{n,\mathbb{R}}^{d,m}(\emptyset) \to \Omega Q_{n,\mathbb{R}}^{m}(I^2, \sigma I^2) \).

Since \( sc_{n}^{d+1,m} \circ sc_{n}^{d,m} \simeq sc_{n}^{d,m} \), by setting \( S = \lim_{d \to \infty} sc_{n}^{d,m} \) we obtain the stable horizontal scanning map

\[(5.17) \quad S : Q_{n,\mathbb{R}}^{1,\infty}(\emptyset) = \lim_{d \to \infty} Q_{n,\mathbb{R}}^{d,m}(\emptyset) \to \Omega Q_{n,\mathbb{R}}^{m}(I^2, \partial I \times I) = \Omega Q_{n,\mathbb{R}}^{m}(I^2, \sigma I^2).\]

**Theorem 5.6** ([24], cf. [8]). If \( d(\emptyset)mn \geq 4 \), the stable horizontal scanning map

\[
S : Q_{n,\mathbb{R}}^{1,\infty}(\emptyset) \rightarrow \Omega Q_{n,\mathbb{R}}^{m}(I^2, \sigma I^2)
\]

is a homotopy equivalence.

**Proof.** Our proof is similar to those given in [24, Prop. 3.2, Lemma 3.4] and also in [8, Prop. 2]. However, as it appears to be probably the most difficult and least familiar part of the article [24], we have provided a detailed argument below (for a brief explanation of its relation to Segal’s original proof see Remark 5.8 below).

We identify \( C = \mathbb{R}^2 \) by means of the identification \( x + \sqrt{-1}y \mapsto (x, y) \) in the usual way. Let \( B \) and \( B^* \) denote the rectangles in \( \mathbb{R}^2 = C \) given by \( B^* = [-1, 2] \times [-1, 1] \) and \( B = (0, 1) \times (-1, 1) \), respectively. Let \( \{ V_t : 0 < t < 1 \} \) be a family of open rectangles in \( B \) given by \( V_t = (t - \epsilon(t), t + \epsilon(t)) \times (-1, 1) \), where \( \epsilon(t) \) denotes any continuous function defined on the open interval \( (0, 1) \) such that \( 0 < \epsilon(t) < \min\{t, 1 - t\} \) for any \( t \in (0, 1) \) with

\[
\lim_{t \to +0} \epsilon(t) = \lim_{t \to -0} \epsilon(t) = 0.
\]

Let \( \tilde{sc}_{d,H} : Q_{n,\mathbb{R}}^{d,m}(B) \times [0, 1] \to Q_{n,\mathbb{R}}^{m}(\overline{B}, \sigma \overline{B}) \) denote the map given by

\[
\tilde{sc}_{d,H}(\xi_1, \cdots, \xi_m, t) = (\xi_1 \cap V_t, \cdots, \xi_m \cap V_t) \in Q_{n,\mathbb{R}}^{m}(\nabla_t, \sigma \nabla_t) \cong Q_{n,\mathbb{R}}^{m}(\overline{B}, \sigma \overline{B}),
\]

where we use the canonical identification \( Q_{n,\mathbb{R}}^{m}(\overline{B}, \sigma \overline{B}) \cong Q_{n,\mathbb{R}}^{m}(\overline{B}, \sigma \overline{B}) \). Since \( \lim_{t \to +0} \tilde{sc}_{d,H}(\xi, t) = \lim_{t \to -0} \tilde{sc}_{d,H}(\xi, t) = (\emptyset, \cdots, \emptyset) \) for any \( \xi \in Q_{n,\mathbb{R}}^{d,m}(B) \), the adjoint of \( \tilde{sc}_{d,H} \) defines the map \( sc_{d,H} : Q_{n,\mathbb{R}}^{d,m}(B) \to \Omega Q_{n,\mathbb{R}}^{m}(\overline{B}, \sigma \overline{B}) \). If \( s^d : Q_{n,\mathbb{R}}^{d,m}(B) \to Q_{n,\mathbb{R}}^{d+1,m}(B) \) denotes the stabilization map defined by adding points from infinity as in Definition 4.1,
by using the identification (5.8) we obtain a homotopy commutative diagram

\[
\begin{array}{ccc}
Q_n^{d,m} (\mathbb{K}) & \xrightarrow{s_n^{d,m}} & Q_n^{d+1,m} (\mathbb{K}) \\
\cong & & \cong \\
Q_n^{d,m} (\mathbb{K}) & \xrightarrow{s^d} & Q_n^{d+1,m} (B).
\end{array}
\]

Hence, if \(Q_{n,K}^{\infty,m} (B)\) denotes the colimit \(Q_{n,K}^{\infty,m} (B) = \lim_{d \to \infty} Q_{n,K}^{d,m} (B)\) taken over the stabilization maps \(s^d\), there is a commutative diagram

\[
\begin{array}{ccc}
Q_n^{\infty,m} (\mathbb{K}) & \xrightarrow{s} & \Omega Q_n^{\infty,m} (I^2, \sigma I^2) \\
\cong & & \cong \\
Q_n^{\infty,m} (\mathbb{K}) & \xrightarrow{s'} & \Omega Q_n^{\infty,m} (\overline{\mathbb{B}}, \sigma \overline{\mathbb{B}}),
\end{array}
\]

where we set \(S' = \lim_{d \to \infty} s \circ d, H\). So it suffices to prove that \(S'\) is a homotopy equivalence.

To see this, let \(J = (0, 1)\) and let us choose the sequence \(\{c_d\}_{d=1}^\infty\) of \(m\)-tuples \(c_d = (c_{d,1}, \cdots, c_{d,m}) \in (J \times J)^m\) satisfying the following condition (*):

(*) \(c_{d,i} \neq c_{d,j}\) if \((d_1, i) \neq (d_2, j)\), and \(\lim_{d \to \infty} c_{d,i} = (1/2, 1) \in \partial \overline{\mathbb{B}}\) for each \(1 \leq i \leq m\).

Let \(\tilde{Q}_{n,K}^{m} (B^*, \sigma B^*)\) denote the space of \(m\)-tuples \((\xi_1, \cdots, \xi_m)\) of formal infinite divisors in \((B^*, \sigma B^*)\) satisfying the following two conditions:

(\(1\)) The element \((\bigcap_{k=1}^m \xi_k) \cap \mathbb{R} \cap (B^* \setminus \sigma B^*)\) contains no point of multiplicity \(\geq n\), where we identify \(\mathbb{R} = (-\infty, \infty) \times \{0\}\).

(\(2\)) Each divisor \(\xi_k\) is represented as the formal infinite sum of the form \(\xi_k = \sum_{d \geq 1} \xi_{k,d}\) such that \(\xi_{k,d} \in SP^1 (B^*, \sigma B^*)\) for each \(d \geq 1\), and it almost coincides (except finite sums) with \(\xi_k = \sum_{d \geq 1} (c_{d,k} + \overline{c_{d,k}})\). When \(K = \mathbb{R}\), the equality \(\xi_k = \xi_k\) also holds for each \(1 \leq k \leq m\), where we set \(\xi_k = \sum_{d \geq 1} \xi_{k,d}\).

Similarly, let \(\tilde{Q}_{n,K}^{m} (B)\) denote the space of \(m\)-tuples \((\xi_1, \cdots, \xi_m)\) of formal infinite divisors in \(B\) such that the element \((\bigcap_{k=1}^m \xi_k) \cap \mathbb{R}\) contains no point of multiplicity \(\geq n\) and satisfies the condition (\(1\))\(_2\).

Let us write \(X_0 = [-1,0] \times [-1,1]\) and \(X_1 = [1,2] \times [-1,1]\), and note that \(B^* = X_0 \cup B \cup X_1\). Then we define

\[
(5.18) \quad \tilde{q} : \tilde{Q}_{n,K}^{m} (B^*, \sigma B^*) \to Q_{n,K}^{m} (\overline{B}, \sigma \overline{B})^2
\]

to be the natural quotient map

\[
\tilde{Q}_{n,K}^{m} (B^*, \sigma B^*) \to Q_{n,K}^{m} (B^*, \sigma B^* \cup \overline{B}) \cong Q_{n,K}^{m} (X_0, \sigma X_0) \times Q_{n,K}^{m} (X_1, \sigma X_1) \cong Q_{n,K}^{m} (\overline{B}, \sigma \overline{B})^2.
\]

By using the Dold-Thom criterion [21, Lemma 3.3] (cf. [14, Lemma 4K.3]), one can show the following:
LEMMA 5.7. The map \( \hat{q} : \hat{Q}_{n,K}^m(B^*, \sigma B^*) \rightarrow \hat{Q}_{n,K}^m(\overline{B}, \sigma \overline{B})^2 \) is a quasifibration with fiber \( \hat{Q}_{n,K}^m(B) \).

We postpone the proof of Lemma 5.7 and complete the proof of Theorem 5.6.

For each \( d \geq 1 \), let \( Q_{n,K}^{d,m} \subset Q_{n,K}^m(B^*, \sigma B^*) \) denote the subspace of all \( m \)-tuples \( (\xi_1, \ldots, \xi_m) \in Q_{n,K}^m(B^*, \sigma B^*) \) such that \( (\xi_1 \cap B, \ldots, \xi_m \cap B) \in Q_{n,K}^{d,m}(B) \), and let \( q^d : Q_{n,K}^{d,m} \rightarrow Q_{n,K}^m(B^*, \sigma B^* \cup \overline{B}) \cong Q_{n,K}^m(B, \sigma B)^2 \) be the quotient map. While the map \( q^d \) is not a fiber bundle, one can easily show see that each fiber of \( q^d \) is homeomorphic to \( Q_{n,K}^{d,m}(B) \).

In fact, if we stabilize the map \( q^d \), it will become a quasifibration.

To see this, define the map \( f^d : Q_{n,K}^{d,m} \rightarrow Q_{n,K}^{d+2,m} \) by \( (\xi_1, \ldots, \xi_m) \mapsto (\xi_1 + c_d1 + c_d2, \ldots, \xi_m + c_d m + c_d m) \), and we denote by \( Q_{n,K}^m \) the colimit \( \lim_{k \rightarrow \infty} Q_{n,K}^{1+2k,m} \) taken from the maps \( \{ f^{1+2k} : k \geq 1 \} \). We also define another horizontal scanning map \( \hat{S} : Q_{n,K}^m \rightarrow \text{Map}(0,1, Q_{n,K}^m(\overline{B}, \sigma \overline{B})) \) by \( \hat{S}(\xi_1, \ldots, \xi_m)(t) = (\xi_1 \cap B_t, \ldots, \xi_m \cap B_t) \), where \( B_t \) denotes the open rectangle \( B_t = (2t - 1, 2t) \times (-1, 1) \) and we use the canonical identification \( Q_{n,K}^m(\overline{B}, \sigma \overline{B}) \cong Q_{n,K}^m(B, \sigma B) \).

To see that \( \hat{S} \) is a homotopy equivalence, consider the composite \( Q_{n,K}^m \rightarrow \text{Map}(0,1, Q_{n,K}^m(\overline{B}, \sigma \overline{B})) \rightarrow Q_{n,K}^m(\overline{B}, \sigma \overline{B}) \), where the first map is \( \hat{S} \) and the second is the evaluation at \( 1/2 \). The composite is obviously a homotopy equivalence hence so is \( \hat{S} \).

Furthermore, since \( q^{d+2} \circ f^d = q^d \), we obtain the stabilized map \( q^\infty = \lim_{k \rightarrow \infty} q^{1+2k} : Q_{n,K}^m \rightarrow Q_{n,K}^m(\overline{B}, \sigma \overline{B}) \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
Q_{n,K}^m & \xrightarrow{q^\infty} & Q_{n,K}^m(\overline{B}, \sigma \overline{B})^2 \\
\hat{S} & \cong & \cong \\
\text{Map}(0,1, Q_{n,K}^m(\overline{B}, \sigma \overline{B})) & \xrightarrow{\text{res}} & \text{Map}(0,1, Q_{n,K}^m(\overline{B}, \sigma \overline{B}))
\end{array}
\]

where \( \text{res} \) denotes the restriction map. Now consider the map \( g^d : Q_{n,K}^{d,m}(B) \rightarrow Q_{n,K}^{d+2,m}(B) \) given by \( (\xi_1, \ldots, \xi_m) \mapsto (\xi_1 + c_d1 + c_d2, \ldots, \xi_m + c_d m + c_d m) \). Then in the same way as in Lemma 4.2 we see that \( Q_{n,K}^m(B) \) can be identified with the colimit \( \lim_{k \rightarrow \infty} Q_{n,K}^{1+2k,m}(B) \) taken from the maps \( \{ g^{1+2k} : k \geq 1 \} \). Note that the space \( \hat{Q}_{n,K}^m(B^*, \sigma B^*) \) has \( Z^m \)-path components. An element \( q = (\xi_1, \ldots, \xi_m) \in \hat{Q}_{n,K}^m(B^*, \sigma B^*) \) belongs to the \( (n_1, \ldots, n_m) \)-th component if and only if \( \text{deg}(\xi_1 \cap (B^* \setminus \sigma B^*) - \xi_k) = n_k \) for each \( 1 \leq k \leq m \). Since we can identify \( \hat{Q}_{n,K}^m(B^*, \sigma B^*) \) with \( Z^m \times Q_{n,K}^m \), we can see that \( \hat{q}|Q_{n,K}^m = q^\infty \) and that \( Q_{n,K}^m = Z^m \times Q_{n,K}^m(B) \). Moreover, by using this identification, we see that \( \hat{S}|Q_{n,K}^m(B) = S' \). Thus, we finally obtain the quasifibration sequence:

\[
Q_{n,K}^m(B) \rightarrow Q_{n,K}^m(\overline{B}, \sigma \overline{B}) \rightarrow Q_{n,K}^m(\overline{B}, \sigma \overline{B})^2
\]
such that the following diagram is commutative:

\[
\begin{array}{ccc}
Q_{n,k}^{m,B}(B) & \longrightarrow & Q_{n,k}^{m,B} \\
S' \downarrow & & \downarrow q^\infty \\
\Omega Q_{n,k}^{m,B}(B, \sigma B) & \longrightarrow & \text{Map}([0,1],Q_{n,k}^{m,B}(B, \sigma B)) \quad \underset{\text{res}}{\longrightarrow} \text{Map}([0,1],Q_{n,k}^{m,B}(B, \sigma B))
\end{array}
\]

where the lower sequence is a fibration sequence. Thus \( S' \) is a homotopy equivalence. □

**Proof of Lemma 5.7.** Let us consider the family of closed subspaces \( \{Q_{d_1, \ldots, d_m} : d_j \geq 1 \text{ for } 1 \leq j \leq m \} \) of the base space \( Q_{n,k}^{m,B}(B^*, \sigma B^* \cup \bar{B}) \) defined by

\[ Q_{d_1, \ldots, d_m} = \{ (\xi_1, \ldots, \xi_m) \in Q_{n,k}^{m,B}(B^*, \sigma B^* \cup \bar{B}) : \deg(\xi_j \cap (B^* \setminus B')) = d_j \text{ for each } j \}, \]

where we write \( B' := \sigma B^* \cup \bar{B} \).

Suppose that \( \bar{q} \) is a quasifibration over the subspace \( Q' = Q_{d_1, \ldots, d_m} \). Using the Dold-Thom criterion [21, Lemma 3.3] (cf. [14, Lemma 4K.3]), it suffices show that \( \bar{q} \) is a quasifibration over \( Q = Q_{d_1, \ldots, d_{k-1}, d_k+2, d_{k+1}, \ldots, d_m} \). If we write \( Q'' = Q \setminus Q' \), it is easy to see that \( \bar{q} \) is a trivial bundle over \( Q'' \). Let \( U \) be an open neighborhood of \( \bar{q}^{-1}(Q') \) in \( \bar{q}^{-1}(Q) \) given by \( U = \bar{q}^{-1}(Q') \cup V \), where \( 0 < \delta < 10^{-10} \) denotes any fixed small number with \( I_\delta = ((-\delta,0) \cup (1+\delta)) \times (0,1) \), and \( V \) denotes the open subspace of \( \bar{q}^{-1}(Q'') \) consisting of all \( m \)-tuples \( (\xi_1, \ldots, \xi_k + x_0 + \bar{x}_0, \xi_{k+1}, \ldots, \xi_m) \) with \( x_0 \in I_\delta \) and \( (\xi_1, \ldots, \xi_m) \in \bar{q}^{-1}(Q') \), such that \( \min \{ |x_0 - x|, |\bar{x}_0 - x| : x \in \xi_i \} \geq 100 \delta \). By moving the points \( x_0 \) and \( \bar{x}_0 \) continuously to a point of \( \sigma B \) by suitable “gravitational” force, we obtain the homotopies \( H_t : \bar{q}^{-1}(Q) \to \bar{q}^{-1}(Q) \) and \( h_t : Q \to Q \) such that \( H_0 = \text{id}, h_0 = \text{id}, H_1(\bar{q}^{-1}(Q)) \subset \bar{q}^{-1}(Q') \) and \( q \circ H_t = h_t \circ q \) for each \( 0 \leq t \leq 1 \).

It remains to show that the Dold-Thom attaching map \( H_1 : \bar{q}^{-1}(x) \to \bar{q}^{-1}(h_1(x)) \) is a homotopy equivalence for any point \( x \in Q \). If we identify these fibers of \( \bar{q} \) with \( \bar{Q}_{n,k}^{m}(B) \), attaching this map can be regarded as the map \( \bar{Q}_{n,k}^{m}(B) \to \bar{Q}_{n,k}^{m}(B) \), and it is given by \( (\eta_1, \ldots, \eta_m) \mapsto (\eta_1, \ldots, \eta_{k-1}, \eta_k + x_0 + \bar{x}_0, \eta_{k+1}, \ldots, \eta_m) \). If we identify \( \bar{Q}_{n,k}^{m}(B) = \mathbb{Z}^m \times \bar{Q}_{n,k}^{m}(B) \), this map is the shift map on the component \( (\eta_1, \ldots, \eta_m) \mapsto (\eta_1, \ldots, \eta_{k-1}, \eta_k + 2, \eta_{k+1}, \ldots, \eta_m) \). Thus it induces a homology isomorphism for any local coefficients. Since \( \pi_1(\bar{Q}_{n,k}^{m}(B)) \) is an abelian group by Lemma 4.8, we see that this attaching map is a homotopy equivalence.

**Remark 5.8.** As we mentioned above, the technique used in the above proof was invented by Segal in [24]. In the notation analogous to the one we used, his argument can be sketched as follows. Segal defines the scanning map \( S_G : \hat{Q} \to \Omega^2(\mathbb{CP}^\infty \vee \mathbb{CP}^\infty) \) and proves that it is a homotopy equivalence. His proof consists of three steps. In the first step, he shows that there is a homotopy equivalence \( Q(S^2, \infty) \simeq \mathbb{CP}^\infty \vee \mathbb{CP}^\infty \) [24, Prop. 3.1], and in the second that the horizontal scanning map \( S_H : \hat{Q} \xrightarrow{\Xi} \Omega Q(I^2, \sigma I^2) \) is a homotopy equivalence [24, Lemma 3.4]. Lastly, he proves that the vertical scanning map \( S_V : Q(I^2, \sigma I^2) \xrightarrow{\Xi} \Omega Q(S^2, \infty) \) is also a homotopy equivalence in [24, Prop. 3.2]. Since the scanning map \( S_G \) is the composite \( \Omega S_V \circ S_H \) (up to homotopy), he concludes that it is a homotopy equivalence. The method or parts of it (e.g. using only the horizontal or the vertical scanning map) extend to other configuration spaces, including our case.
DEFINITION 5.9. (i) Let $\mathcal{P}^d(\mathbb{K})$ denote the space of all (not necessarily monic) polynomials $f(z) = \sum_{i=0}^{d} a_i z^i \in \mathbb{K}[z]$ of degree exactly $d$ and let $\text{Poly}^d_n(\mathbb{K})$ denote the space of all $m$-tuples $(f_1(z), \ldots, f_m(z)) \in \mathcal{P}^d(\mathbb{K})^m$ such that polynomials $\{f_1(z), \ldots, f_m(z)\}$ have no common real root of multiplicity $\geq n$.

(ii) For each nonempty subset $X \subset \mathbb{C}$, let $\text{Poly}^m_{n,\mathbb{K}}(X)$ denote the space of all $m$-tuples $(f_1(z), \ldots, f_m(z)) \in \mathbb{K}[z]^m$ of polynomials of the same degree such that polynomials $\{f_1(z), \ldots, f_m(z)\}$ have no common real root in $X$ of multiplicity $\geq n$, and let $\text{Poly}^m_{n,\mathbb{K}}(X) \subset \text{Poly}^m_{n,\mathbb{K}}$ denote the subspace of $(f_1(z), \ldots, f_m(z)) \in \text{Poly}^m_{n,\mathbb{K}}(X)$ such that no $f_i(z)$ is identically zero. When $X = \mathbb{C}$, we write

$$\text{(5.19)}$$

$$\text{Poly}^m_{n,\mathbb{K}} = \text{Poly}^m_{n,\mathbb{K}}(\mathbb{C}).$$

REMARK 5.10. It is easy to see that there are homeomorphisms

$$\mathcal{P}^d(\mathbb{K}) \cong \mathbb{K}^* \times \mathcal{P}^d(\mathbb{K}) \quad \text{and} \quad \text{Poly}^d_n(\mathbb{K}) \cong T^d_n \times Q^d_n(\mathbb{K}),$$

where we set $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$ and $T^d_n = (\mathbb{K}^*)^m$.

Next we consider horizontal scanning for algebraic maps.

DEFINITION 5.11. (i) We identify $\mathbb{C} = \mathbb{R}^2$ in a usual way. Let

$$U = \{ w \in \mathbb{C} : |\text{Re}(w)| < 1, |\text{Im}(w)| < 1 \} = (-1, 1) \times (-1, 1)$$

and define the horizontal scanning map for $\text{Poly}^d_{n,\mathbb{K}}$,

$$\text{(5.20)}$$

$$\text{sca}^d_{n} : \text{Poly}^d_{n,\mathbb{K}} \rightarrow \text{Map}(\mathbb{R}, \text{Poly}^d_{n,\mathbb{K}}(U))$$

by $\text{sca}^d_{n}(f_1(z), \ldots, f_m(z))((x)) = (f_1(x), \ldots, f_m(x))$ for $(f_1(z), \ldots, f_m(z)) \in \text{Poly}^d_{n,\mathbb{K}} \times \mathbb{R}$, where $V(x)$ denotes the space given by (5.15) and also use the canonical identification $U \cong V(x)$.

(ii) Let us identify $\overline{U} = I^2$ and let $q : \text{Poly}^m_{n,\mathbb{K}}(U) \rightarrow Q^m_{n,\mathbb{K}}(I^2, \partial I \times I)$ denote the map given by assigning to an $m$-tuples of polynomials their corresponding roots in $U$.

LEMMA 5.12. Any fiber of the map $q$ is homotopy equivalent to the space $T^m_{n,\mathbb{K}}$.

PROOF. Any fiber of the map $q$ is homeomorphic to the space $F(m)$ consisting of all $m$-tuples $(f_1(z), \ldots, f_m(z)) \in \mathbb{K}[z]^m$ of $\mathbb{K}$-coefficients polynomials such that each polynomial $f_i(z)$ has no root in $U$. It suffices to show that there is a homotopy equivalence $F(m) \cong T^m_{n,\mathbb{K}}$. First define the inclusion map $j_0 : T^m_{n,\mathbb{K}} \rightarrow F(m)$ by $j_0(x) = (x_1, \ldots, x_m)$ for $x = (x_1, \ldots, x_m) \in T^m_{n,\mathbb{K}}$. Next, let $f = (f_1(z), \ldots, f_m(z)) \in F(m)$ be any element. Since $0 \in U$, $(f_1(0), \ldots, f_m(0)) \in T^m_{n,\mathbb{K}}$. Hence, one can define the evaluation map $c_0 : F(m) \rightarrow T^m_{n,\mathbb{K}}$ by $c_0(f) = (f_1(0), \ldots, f_m(0))$ for $f = (f_1(z), \ldots, f_m(z)) \in F(m)$. It is easy to see that $c_0 \circ j_0 = \text{id}_{T^m_{n,\mathbb{K}}}$. 

Now consider the map $j_0 \circ c_0$. Note that if a polynomial $g(z) \in \mathbb{K}[z]$ has a root $\alpha \in \mathbb{C} \setminus U$ and $0 < t \leq 1$, the polynomial $g(tz)$ has a root $\alpha/t \in \mathbb{C} \setminus U$. Thus, one can define the homotopy $h : F(m) \times [0,1] \rightarrow F(m)$ by $h(f, t) = (f_1(tz), \ldots, f_m(tz))$ for $(f, t) = ((f_1(z), \ldots, f_m(z)), t) \in F(m) \times [0,1]$. It is easy to see that the map $F$...
The map \( V \) as before we identify \( V \) without common real \( n \), a sketch, we will give a more detailed argument. Since non-real roots of real polynomials must occur in conjugate pairs, the Dold-Thom homotopy equivalence. \( \alpha \) is shown to be a quasifibration as follows. We filter the base space \( SP \) in \( X \) without multiplicity larger than \( n \). The set \( SP_n(X) \) is topologized as the subspace of the space of all polynomials \( f(z) = \sum a_i z^i \) such that every root of \( f(z) \) in \( X \) has multiplicity less than \( n \). The set \( SP_n(X) \) is bijectively equivalent to the disjoint union \( \coprod_{d \geq 1} SP_d(C) \) of the spaces \( SP_d(C) \) of complex polynomials of degree exactly \( d \), is connected. For example the degree one polynomial \( az - b \) can be connected to the degree zero polynomial \( -b \) by the homotopy \( taz - b \), where \( t \in [0,1] \).

Let \( V = \{ x \in C : |x| < 1 \} \) and let \( SP_n(V, \partial V) \) denote the space of all configurations of points in \( V \) without multiplicity larger than \( n \), with two configurations identified if they differ only on the boundary (we can also think that points can enter the boundary and vanish).

Consider the natural projection map \( q : SP_n(V) \to SP_n(V, \partial V) \). In [11] this map is shown to be a quasi-fibration as follows. We filter the base space \( SP_n(V, \partial V) \) by the number of points in \( V \). It is easy to see that over the each successive difference in this filtration the map \( q \) is a locally trivial fiber bundle. We can now use the Dold-Thom method (see Lemma 3.3 and the proof of [24, Proposition 3.2]) to show that \( q \) is a quasifibration. The Dold-Thom attaching map on the fiber has the effect of multiplying \( \alpha \) by the \( taz - b \), where \( t \in [0,1] \).

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Next, turning to the case of \( m \)-tuples of configurations \( Q_{n,K}^m(I^2, \partial I \times I) \), we use essentially the same argument for proving Lemma 5.13. The difference is that we are now dealing with \( m \)-tuples of configurations. To avoid complexities of notation we shall only consider the case \( m = 2 \), and we write \( B = Q_{n,K}^m(I^2, \partial I \times I) \). The argument is essentially given in [24, Lemma 3.3], but as it is only a sketch, we will give a more detailed argument.

Recall that the base space \( B \) consists of pairs of divisors (or configurations) \( (\xi_1, \xi_2) \) without common real \( n \)-fold points in \( U \setminus \sigma U \). We filter the base space \( B \) by an increasing
family of subspaces \(\{B_{\leq d_i, \leq d_2}\}\), where \(B_{\leq d_i, \leq d_2}\) denotes the space of pairs \((\xi_1, \xi_2)\) \(\in B\) such that \(\deg(\xi_1 \cap \mathbb{R} \cap (U \setminus \sigma U)) \leq d_i\) for each \(i = 1, 2\).

We aim to prove that \(q\) restricted to \(B_{\leq d_1, \leq d_2}\) is a quasifibration for each pair \((d_1, d_2)\) of positive integers. For each \(0 \leq d \leq d_1\), let \(B_{d, \leq d_2}\) denote the subspace of \(B_{\leq d_1, \leq d_2}\) consisting of pairs \((\xi_1, \xi_2)\) for which \(\deg(\xi_1 \cap \mathbb{R} \cap (U \setminus \sigma U)) = d\). We want to prove that the map \(q\) restricted to this space is a quasifibration. We filter this space by subspaces \(\{B_{d, \leq d_2} : 0 \leq d' \leq d_2\}\). Over each difference \(B_{d,d'} = B_{d,\leq d} \setminus B_{d,d'-1}\) the map \(q\) is a locally trivial fiber bundle. We now use the Dold-Thom technique, exactly as in [24, Lemma 3.3] to stitch these quasifibrations, to one over \(B_{d,\leq d_2}\). We do this for each \(d \leq d_1\) and then turn to \(B_{\leq d_1, \leq d_2}\). We now filter it by subspaces \(\{B_{d_1,\leq d_2} : d \leq d_1\}\), where \(d_2\) is now fixed. The differences are now precisely the spaces \(B_{d,\leq d_2}\), over which we have already proved that the map \(q\) is a quasifibration. Repeating the Dold-Thom argument concludes the proof. The process in which we stitch a fiber bundle defined over an open subset and a quasifibration defined over a closed one, requires finding an open set containing the closed one and a suitable retraction that lifts to the total space and induces an isomorphism on fibers, exactly as in the proof of Lemma 5.7. 

\[\square\]

**Definition 5.14.** (i) Let \(ev : \text{Pol}_{n,K}^m(U) \to \mathbb{K}^{mn} \setminus \{0\}\) denote the evaluation map at \(z = 0\) given by

\[
ev(f_1(z), \ldots, f_m(z)) = (f_1(0), f_2(0), \ldots, f_m(0))
\]

for \((f_1(z), \ldots, f_m(z)) \in \text{Pol}_{n,K}^m(U)\), and let \(ev_0 : \text{Poly}_{n,K}^m(U) \to \mathbb{K}^{mn} \setminus \{0\}\) denote the restriction \(ev_0 = ev|\text{Pol}_{n,K}^m(U)\).

(ii) Let \(G\) be a group and \(X\) a \(G\)-space. Then we denote by \(X/\!/G\) the homotopy quotient of \(X\) by \(G\), \(X/\!/G = EG \times_G X\), where \(EG\) denotes the contractible free \(G\)-space.

**Remark 5.15.** Let \(T^n_K = (\mathbb{K}^n)^m\) and consider the diagonal \(T^n_K\)-action on the spaces \(\text{Poly}_{n,K}^m(U)\) and \(\mathbb{K}^{mn} \setminus \{0\}\) given by

\[
(g_1, \ldots, g_m) \cdot (f_1(z), \ldots, f_m(z)) = (g_1f_1(z), \ldots, g mf_m(z)) \quad \text{and} \quad (g_1, \ldots, g_m) \cdot (x_1, \ldots, x_m) = (g_1x_1, \ldots, g_mx_m)
\]

for \((g_1, \ldots, g_m) \in T^n_K\) and \((x_1, \ldots, x_m) \in \mathbb{K}^n\). Note that \(ev_0\) is a \(T^n_K\)-equivariant map.

**Lemma 5.16.** The map \(ev_0 : \text{Poly}_{n,K}^m(U) \to \mathbb{K}^{mn} \setminus \{0\}\) is a homotopy equivalence.

**Proof.** For each \(x = (x_0, \ldots, x_{n-1}) \in \mathbb{K}^n\), let \(\varphi_x(z) \in \mathbb{K}[z]\) denote the polynomial with coefficients in \(\mathbb{K}\) defined by \(\varphi_x(z) = x_0 + \sum_{k=1}^{n-1} \frac{z^{k-1} - x_k}{x_k} z^k\). Since the degree of the polynomial \(\varphi_x(z)\) is at most \(n-1\), it has no root of multiplicity \(\geq n\). Thus one can define the natural inclusion map \(i_0 : \mathbb{K}^{mn} \setminus \{0\} \to \text{Poly}_{n,K}^m(U)\) by

\[
i_0(x_1, \ldots, x_m) = (\varphi_{x_1}(z), \ldots, \varphi_{x_m}(z)) \quad \text{for} \quad (x_1, \ldots, x_m) \in \mathbb{K}^{mn} \setminus \{0\}.
\]

It is easy to see that \(ev \circ i_0 = id\). Next consider the homotopy \(F : \text{Pol}_{n,K}^m(U) \times [0,1] \to \text{Poly}_{n,K}^m(U)\) defined by \(F((f_1(z), \ldots, f_m(z), t) = (f_1(tz), \ldots, f_m(tz))\). Then \(F_1 = F(,1) = id\) and the map \(F_0 = F(,0)\) is given by \(F_0(f_1(z), \ldots, f_m(z)) = (f_1(z), \ldots, f_m(z))\).
(f_1(0), \ldots, f_m(0)). Note that for \( f = (f_1(z), \ldots, f_m(z)) \in \text{Pol}_{n, K}^m(U) \), we have
\[
(i_0 \circ ev)(f) = \left( f_1(0) + \frac{n-1}{k!} f_1^{(k)}(0) z^k, \ldots, f_m(0) + \frac{n-1}{k!} f_m^{(k)}(0) z^k \right).
\]

Define the homotopy \( G : \text{Pol}_{n, K}^m(U) \times [0, 1] \to \text{Pol}_{n, K}^m(U) \) by
\[
G((f_1(z), \ldots, f_m(z)), t) = \left( f_1(0) + t \sum_{k=1}^{n-1} \frac{f_1^{(k)}(0)}{k!} z^k, \ldots, f_m(0) + t \sum_{k=1}^{n-1} \frac{f_m^{(k)}(0)}{k!} z^k \right).
\]

Since this gives the homotopy between the maps \( F_0 \) and \( i_0 \circ ev \), the map \( i_0 \circ ev \) is homotopic to the identity map. Thus, the map \( ev \) is a homotopy equivalence.

Note that the complement \( \Sigma = \text{Pol}_{n, K}^m(U) \setminus \text{Poly}_{n, K}^m(U) \) is the space of all \( m \)-tuples \( (f_1(z), \ldots, f_m(z)) \in \text{Pol}_{n, K}^m(U) \) such that some \( f_i(z) \) is identically zero. Note that \( \text{Poly}_{n, K}^m(U) \) is an infinite dimensional manifold and that \( \Sigma \) is a finite union of linear subspaces, each of infinite codimension. Then by using the induction on the number of linear subspaces and by [4, Theorem 2], we can show that the inclusion \( \text{Poly}_{n, K}^m(U) \to \text{Poly}_{n, K}^m(U) \) is a homotopy equivalence. Thus the map \( ev_0 = ev|\text{Poly}_{n, K}^m(U) \) is also a homotopy equivalence. \( \square \)

**Definition 5.17.** Now recall the map \( d_{n, K}^{d, m} : Q_n^{d, m}(K) \to \Omega_{d, K}^n \mathbb{R}P^{d(m-1)} \simeq \Omega S^{d(m-1)} \) and the stabilized space \( Q_n^{\infty, m}(K) = \lim_{d \to \infty} Q_n^{d, m}(K) \) given in Definition 4.1. Since there is a homotopy commutative diagram
\[
\begin{array}{ccc}
Q_n^{d, m}(K) & \xrightarrow{d_{n, K}^{d, m}} & \Omega S^{d(m-1)} \\
\downarrow d_{n, K}^{d+1, m} & & \downarrow d_{n, K}^{d+1, m} \\
Q_n^{d+1, m}(K) & \xrightarrow{d_{n, K}^{d+1, m}} & \Omega S^{d(m-1)}
\end{array}
\]
for each \( d \geq 1 \), these maps induce the following map
\[
(5.24) \quad j_n^{\infty, m} : Q_n^{\infty, m}(K) \to \Omega S^{d(m-1)}.
\]

**Theorem 5.18.** If \( d(K)mn \geq 4 \), the map \( j_n^{\infty, m} : Q_n^{\infty, m}(K) \xrightarrow{\simeq} \Omega S^{d(m-1)} \) is a homotopy equivalence.

**Proof.** Note that the group \( T_{\mathbb{K}}^m \) acts on \( \text{Poly}_{n, K}^m(U) \) freely, but it does not act on \( \mathbb{K}^{mn} \setminus \{0\} \) freely. So we have to consider the homotopy quotient \( (\mathbb{K}^{mn} \setminus \{0\})/T_{\mathbb{K}}^m \) of the action. Since \( ev_0 \) is a \( T_{\mathbb{K}}^m \)-equivariant map, we obtain the following commutative diagram:
\[
\begin{array}{ccc}
T_{\mathbb{K}}^m & \longrightarrow & \text{Poly}_{n, K}^m(U) \\
\| & & \| \\
\text{T}_{\mathbb{K}}^m & \longrightarrow & \mathbb{K}^{mn} \setminus \{0\}
\end{array}
\]
\[
\begin{array}{ccc}
q_1 \quad & & q_2 \\
\| \quad & \simeq \quad & \| \\
\text{Poly}_{n, K}^m(U)/T_{\mathbb{K}}^m & \longrightarrow & (\mathbb{K}^{mn} \setminus \{0\})/T_{\mathbb{K}}^m,
\end{array}
\]
where two horizontal sequences are fibration sequences and and each \( q_i (i = 1, 2) \) is the
natural projection induced from the group action. Thus, we see that \( \tilde{ev}_0 \) is a homotopy equivalence. If \( q' : Poly_{d,m}^{n,R} / T^m_n \rightarrow Q_{n,R}^m(I^2, \partial I \times I) \) denotes the map induced from the map \( q \), then it is a homotopy equivalence by Lemma 5.13. Since \( \lim_{t \to \pm \infty} \text{sc}^{d,m}_n(f)(t) = (\emptyset, \cdots, \emptyset) \) for any \( f \in Poly_{d,m}^{n,R} \), the map \( \text{sc}^{d,m}_n \) can be extended to the based map \( \text{sc}^{d,m}_n : Poly_{d,m}^{n,R} \rightarrow \Omega Poly_{d,m}^{n,R}(U) \) by \( \infty \rightarrow \text{the constant loop at} \ (\emptyset, \cdots, \emptyset) \), where we identify \( S^1 = \mathbb{R} \cup \infty \) and we choose the points \( \infty \) and \( (\emptyset, \cdots, \emptyset) \) as base-points of \( S^1 \) and \( \Omega Poly_{d,m}^{n,R}(U) \), respectively. Now consider the following commutative diagram:

\[
\begin{array}{ccc}
Poly_{d,m}^{n,R} & \xrightarrow{\text{sc}^{d,m}_n} & \Omega Poly_{d,m}^{n,R}(U) \\
q_1 \downarrow & & \downarrow \Omega q_1 \\
Poly_{d,m}^{n,R} / T^m_n & \xrightarrow{\equiv} & \Omega(\text{Poly}_{d,m}^{n,R}(U) / T^m_n) \\
\equiv \downarrow & & \equiv \\
Q_{n,R}^{d,m}(\emptyset) & \xrightarrow{\text{sc}^{d,m}_n} & \Omega Q_{n,R}^{d,m}(I^2, \sigma I^2),
\end{array}
\]

where the map \( q_3 \) is induced from the corresponding group action. Now consider the map \( \gamma_R \) given by the second row of the above diagram. Since \( \text{sc}^{d,m}_n \) is a homotopy equivalence if \( d \rightarrow \infty \) (by Theorem 5.6), the map \( \gamma_R \) is a homotopy equivalence if \( d \rightarrow \infty \).

First, consider the case \( K = \mathbb{R} \). Since \( T^m_n = \{ \pm 1 \}^m \), the two maps \( \Omega q_i (i = 1, 2) \) are homotopy equivalences. However, since the map \( \gamma_R \) coincides the map \( j_{d,m}^{n,R} (if d \rightarrow \infty) \) up to homotopy equivalence, the map \( j_{d,m}^{\infty,m} \) is a homotopy equivalence.

Next consider the case \( K = \mathbb{C} \). Since \( Q_{n,R}^{d,m}(\mathbb{C}) \) is simply connected by Lemma 3.5, the space \( Q_{n,C}^{\infty,m}(\mathbb{C}) \) is also simply connected. Since \( \Omega S^{2mn-1} \) is simply connected, it suffices to prove that \( j_{n,m}^{\infty,m} \) induces an isomorphism on homotopy groups \( \pi_k( ) \) for any \( k \geq 2 \).

Since \( T^m_n \simeq (S^1)^m \), we see that the two maps \( \Omega q_i (i = 1, 2) \) induce isomorphisms on homotopy groups \( \pi_k( ) \) for any \( k \geq 2 \). Since \( \gamma_C \) is a homotopy equivalence if \( d \rightarrow \infty \), the map \( j_{n,C}^{\infty,m} \) also induces an isomorphism on homotopy groups \( \pi_k( ) \) for any \( k \geq 2 \).

6. Proof of the main results

In this section, we give the proofs of the main results (Theorem 1.8, Corollary 1.10 and Theorem 1.11).

Proof of Theorem 1.8. Suppose that \( d(K)mn \geq 4 \). Note that the two spaces \( Q_{d,m}^{d,m}(K) \) and \( \Omega S^{d(K)mn-1} \) are simply connected (by Lemma 3.5). Hence, the assertion follows from Theorems 4.5 and 5.18.

Proof of Corollary 1.10. It is easy to see that the following diagram is commutative:

\[
\begin{array}{ccc}
Q_{n,R}^{d,m}(K) & \xrightarrow{j_{d,m}^{n,R}} & \Omega S^{d(K)mn-1} \\
\downarrow j_{n,R}^{d,m} & & \| \\
Q_{1,R}^{d,m}(K) & \xrightarrow{j_{1,R}^{d,m}} & \Omega S^{d(K)mn-1}.
\end{array}
\]
By Theorem 1.8 the two maps \(j_{n,K}^{d,m}\) and \(j_{1,K}^{d,mn}\) are homotopy equivalences through dimension \(D(d; m, n, K)\) and \(D(d; mn, 1, K)\), respectively. Since \(D(d; m, n, K) < D(d; mn, 1, K)\), it follows from the above commutative diagram that the map \(i_{K}^{d,mn}\) is a homotopy equivalence through dimension \(D(d; m, n, K)\). \(\square\)

**Proof of Theorem 1.11.** Suppose that \(d(K)mn \geq 4\). It suffices to prove the first assertion. Since \(Q_{n}^{d,m}(K)\) is simply connected (by Lemma 3.5), it follows from Corollary 3.4 and the cellular approximation theorem that there is a map \(f : Q_{n}^{d,m}(K) \to J_{\left(\frac{d}{k}\right)}(\Omega S^{d(k)mn-1})\) such that the following diagram is homotopy commutative:

\[
\begin{array}{ccc}
Q_{n}^{d,m}(K) & \xrightarrow{j_{n,K}^{d,m}} & \Omega S^{d(k)mn-1} \\
\downarrow{j_{1,K}^{d,mn}} & & \downarrow{i_{C}} \\
J_{\left(\frac{d}{k}\right)}(\Omega S^{d(k)mn-1}) & \xrightarrow{i_{K}^{d,mn}} & \Omega S^{d(k)mn-1},
\end{array}
\]

where \(i\) is the natural inclusion map. Since the maps \(j_{n,K}^{d,m}\) and \(i\) are homotopy equivalences through dimension \(D(d; m, n, K)\), the map \(f\) is a homotopy equivalence through dimension \(D(d; m, n, K)\), too. Note that \(H_{k}(Q_{n,K}^{d,m}, \mathbb{Z}) = H_{k}(J_{\left(\frac{d}{k}\right)}(\Omega S^{d(k)mn-1}), \mathbb{Z}) = 0\) for any \(k > D(d; m, n, K)\). Hence, the map \(f\) is a homology equivalence. Since the spaces \(Q_{n}^{d,m}(K)\) and \(J_{\left(\frac{d}{k}\right)}(\Omega S^{d(k)mn-1})\) are simply connected, the map \(f\) is indeed a homotopy equivalence. \(\square\)

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