Metrics on Doubles as an Inverse Semigroup

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Abstract
For a metric space $X$ we study metrics on the two copies of $X$. We define composition of such metrics and show that the equivalence classes of metrics are a semigroup $M(X)$. Our main result is that $M(X)$ is an inverse semigroup. Therefore, one can define the $C^*$-algebra of this inverse semigroup, which is not necessarily commutative. If the Gromov–Hausdorff distance between two metric spaces, $X$ and $Y$, is finite then their inverse semigroups $M(X)$ and $M(Y)$ (and hence their $C^*$-algebras) are isomorphic. We characterize the metrics that are idempotents, and give examples of metric spaces for which the semigroup $M(X)$ (and the corresponding $C^*$-algebra) is commutative. We also describe the class of metrics determined by subsets of $X$ in terms of the closures of the subsets in the Higson corona of $X$ and the class of invertible metrics.

Keywords Metric · Inverse semigroup

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1 Introduction

Given metric spaces $X$ and $Y$, a metric $d$ on $X \sqcup Y$ that extends the metrics on $X$ and $Y$ depends only on the values of $d(x, y)$, $x \in X$, $y \in Y$, but it may be hard to check which functions $d : X \times Y \to (0, \infty)$ determine a metric on $X \sqcup Y$: one has to check the triangle inequality too many times. The problem of description of all such extended metrics is difficult due to the lack of a nice algebraic structure on the set of metrics. It was a surprise for us to discover that in the case $Y = X$, there is a nice
algebraic structure on the set $M(X)$ of quasi-isometry classes of extended metrics on the double $X \sqcup X$: it is an inverse semigroup.

Recall that a semigroup $S$ is an inverse semigroup if for any $u \in S$ there exists a unique $v \in S$ such that $u = uuv$ and $v = vuv$ [1]. Philosophically, inverse semigroups describe local symmetries in a similar way as groups describe global symmetries, and technically, the construction of the (reduced) group $C^*$-algebra of a group generalizes to that of the (reduced) inverse semigroup $C^*$-algebra [3].

Thus, one can associate a new (non-commutative) $C^*$-algebra to any metric space. In particular, all quasi-isometry classes of metrics on the double of $X$ are partial isometries. We characterize the metrics that are idempotents in $M(X)$ and show that any two idempotents commute (which proves that $M(X)$ is an inverse semigroup). We show that if the Gromov–Hausdorff distance between two metric spaces, $X$ and $Y$, is finite then their inverse semigroups $M(X)$ and $M(Y)$ (and hence the corresponding $C^*$-algebras) are isomorphic. We also describe the class of metrics determined by subsets of $X$ in terms of the closures of the subsets in the Higson corona of $X$ and the class of invertible metrics, and give examples of metric spaces for which the semigroup $M(X)$ is commutative.

Let $X = (X, d_X)$ be a metric space.

**Definition 1.1** A double of $X$ is a metric space $X \times \{0, 1\}$ with a metric $d$ such that

- the restriction of $d$ on each copy of $X$ in $X \times \{0, 1\}$ equals $d_X$;
- the distance between the two copies of $X$ is non-zero.

Let $M(X)$ denote the set of all such metrics.

We identify $X$ with $X \times \{0\}$, and write $X'$ for $X \times \{1\}$. Similarly, we write $x$ for $(x, 0)$ and $x'$ for $(x, 1)$, $x \in X$. Note that metrics on a double of $X$ may differ only when two points lie in different copies of $X$. To define a metric $d$ in $M(X)$ it suffices to define $d(x, y')$ for all $x, y \in X$.

Recall that two metrics, $d_1, d_2$, on the double of $X$ are quasi-isometric if there exist $\alpha > 0, \beta \geq 1$ such that

$$-\alpha + \frac{1}{\beta} d_1(x, y') \leq d_2(x, y') \leq \alpha + \beta d_1(x, y')$$

for any $x, y \in X$. We call two metrics, $d_1$ and $d_2$, on the double of $X$ equivalent if they are quasi-isometric. In this case we write $d_1 \sim d_2$, or $[d_1] = [d_2]$.

### 2 Composition of Metrics

The idea to consider metrics on the disjoint union of two spaces as morphisms from one space to another was suggested in [2].

**Lemma 2.1** Let $(X, d_X)$, $(Y, d_Y)$, and $(Z, d_Z)$ be metric spaces, let $d$ be a metric on $X \sqcup Y$, $\rho$ a metric on $Y \sqcup Z$ such that $d|_X = d_X$, $d|_Y = \rho|_Y = d_Y$, $\rho|_Z = d_Z$. Then
the formula

\[ b(x, z) = \inf_{y \in Y} [d(x, y) + \rho(y, z)], \quad x \in X, z \in Z, \]

defines a metric on \( X \sqcup Z \).

**Proof** Due to symmetry, it suffices to check the triangle inequality for the triangle \((x_1, x_2, z), x_1, x_2 \in X, z \in Z\). Fix \( \varepsilon > 0 \) and let \( y_1, y_2 \in Y \) satisfy

\[ d(x_1, y_1) + \rho(y_1, z) - b(x_1, z) < \varepsilon; \quad d(x_2, y_2) + \rho(y_2, z) - b(x_2, z) < \varepsilon. \]

Then

\[ d_X(x_1, x_2) \leq d(x_1, y_1) + \rho(y_1, z) + \rho(z, y_2) + d(y_2, x_2) \]
\[ \leq b(x_1, z) + b(x_2, z) + 2\varepsilon; \]
\[ b(x_2, z) \leq d(x_2, y_1) + \rho(y_1, z) \leq d_X(x_2, x_1) + d(x_1, y_1) + \rho(y_1, z) \]
\[ \leq d_X(x_2, x_1) + b(x_1, z) + \varepsilon. \]

Taking \( \varepsilon \) arbitrarily small, we obtain the triangle inequality. \( \square \)

We shall denote the metric \( b \) by \( \rho \circ d \), or \( \rho d \).

**Corollary 2.2** Let \( \rho, d \) be metrics on the double of \( X \). Then the formula

\[ \rho d(x, z') = \inf_{y' \in X} [d(x, y') + \rho(y, z')], \quad x, z \in X, \]

defines the composition of \( d \) and \( \rho \) on the double of \( X \).

**Lemma 2.3** The composition of metrics is associative.

**Proof** Obvious. \( \square \)

**Lemma 2.4** If \( d_1 \sim \tilde{d}_1 \) and \( d_2 \sim \tilde{d}_2 \) then \( \tilde{d}_1 \circ \tilde{d}_2 \sim d_1 \circ d_2 \).

**Proof** Suppose that there exist \( \alpha \geq 0, \beta \geq 1 \) such that \( \tilde{d}_1(x, y') \leq \alpha + \beta d_1(x, y') \) and \( \tilde{d}_2(x, y') \leq \alpha + \beta d_2(x, y') \) for any \( x, y \in X \).

Then

\[ \tilde{d}_1 \circ \tilde{d}_2(x, z') = \inf_{y' \in X} [\tilde{d}_2(x, y') + \tilde{d}_1(y, z')] \leq \inf_{y' \in X} [\alpha + \beta d_2(x, y') + \alpha + \beta d_1(y, z')] \]
\[ \leq \inf_{y' \in X} [2\alpha + \beta d_2(x, y') + d_1(y, z')] \]
\[ \leq 2\alpha + \beta d_1 \circ d_2(x, z'). \]

Lower bound is similar. \( \square \)

Thus, the multiplication is well defined on equivalence classes of metrics on the double of \( X \).

Denote the set of all equivalence classes of metrics on the double of \( X \) by \( M(X) = \mathcal{M}(X)/ \sim \). Then \( M(X) \) is a semigroup.

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**Example 2.5** If $X$ is discrete of finite diameter then all metrics on the double of $X$ are equivalent, so $M(X)$ consists of a single element.

**Example 2.6** Define a metric $I$ on the double of $X$ by $I(x, y') = d_X(x, y) + 1$. (The triangle inequality obviously holds.) Note that $I \circ d \sim d \circ I$ for any metric on the double of $X$, hence $[I]$ is the unit element in the semigroup $M(X)$.

For a metric $d$ on the double of $X$ define the adjoint metric in $M(X)$ by $d^*(x, y') = d(y, x'), x, y \in X$. Then $*$ is an involution: $(d^*)^* = d$ and $(d_1 \circ d_2)^* = d_2^* \circ d_1^*$, and it passes to the equivalence classes, making $M(X)$ a semigroup with involution.

A metric $d$ on the double of $X$ is selfadjoint if $d^* \in [d]$. Note that if $d$ is selfadjoint then there exists a metric $\tilde{d} \in [d]$ such that $\tilde{d}^* = \tilde{d}$. Indeed, we can set $\tilde{d}(x, y') = \frac{1}{2}(d(x, y') + d(y, x'))$, $x, y \in X$.

The following simple statement from [2] is the key observation allowing to see metrics as partial isometries.

**Proposition 2.7** The metrics $d$ and $d \circ d^* \circ d$ are equivalent for any metric $d$ on the double of $X$.

**Proof** Let $x, y \in X$. On the one hand, taking $t = y, s = x$, we get

$$(d \circ d^* \circ d)(x, y') = \inf_{t, s \in X} [d(x, t') + d^*(t, s') + d(s, y')] \leq 3d(x, y').$$

On the other hand, passing to infimum in the triangle inequality, we get

$$(d \circ d^* \circ d)(x, y') = \inf_{t, s \in X} [d(x, t') + d^*(t, s') + d(s, y')]$$

$$\geq \inf_{t, s \in X} [d(x, t') + d(t', s) + d(s, y')]$$

$$\geq d(x, y').$$

\qed

**Corollary 2.8** $[d^*d]$ is a selfadjoint idempotent for any metric $d$ on the double of $X$.

Recall that a semigroup $S$ is regular if for any $d \in S$ there is $b \in S$ such that $d = dbd$ and $b = bdb$.

**Corollary 2.9** $M(X)$ is a regular semigroup.

**Proof** Take $b = d^*$.

\qed

There are typically a lot of idempotent metrics, i.e., metrics representing idempotents in $M(X)$ (see Example 2.10) below. This means that, in general, $M(X)$ is not a cancellative semigroup. Indeed, if $d^2 \sim d$ then cancellation would imply $d \sim I$. 
Example 2.10  Let $X = \mathbb{Z}$ with the standard metric $d(n, m) = |n - m|$, $n, m \in \mathbb{Z}$. Set

$$d(n, m') = \begin{cases} n + m + 1, & \text{if } n, m \geq 0; \\ |n - m| + 1, & \text{otherwise}. \end{cases}$$

Then it is easy to see that $d^* = d$ and $[d \circ d] = [d]$, while $d$ is not quasi-isometric to $I$.

3 Idempotents

Denote by $d(x, X')$ the distance from $x \in X$ in the first copy of $X$ to the second copy $X'$ of $X$ in the double of $X$.

Theorem 3.1  Let $d^* = d$ be a metric on the double of $X$. Then $[d^2] = [d]$ if and only if there exist $\alpha \geq 0$, $\beta \geq 1$ such that $-\alpha + \frac{1}{\beta}d(x, x') \leq d(x, X')$ for any $x \in X$.

Proof  First, suppose that $[d^2] = [d]$. Then there exist $\alpha \geq 0$, $\beta \geq 1$ such that

$$d^2(x, x') \geq -\alpha + \frac{1}{\beta}d(x, x').$$

On the other hand,

$$d^2(x, x') = \inf_{y \in X} [d(x, y') + d(y, x')] = \inf_{y \in X} 2d(x, y') \leq 2d(x, X'),$$

hence $d(x, X') \geq -\frac{\alpha}{2} + \frac{1}{2\beta}d(x, x')$.

Second, suppose that there exist $\alpha \geq 0$, $\beta \geq 1$ such that $d(x, x') \leq \alpha + \beta d(x, X')$ for any $x \in X$. We need to estimate $d^2(x, z')$ both from below and from above. The estimate from above is given by

$$d^2(x, z') = \inf_{y \in X} [d(x, y') + d(y, z')] \leq d(x, x') + d(x, z') \leq \alpha + \beta d(x, X') + d(x, z') \leq \alpha + \beta d(x, z') + d(x, z') = \alpha + (\beta + 1)d(x, z').$$

Here we took $y = x$ and used that $d(x, X') \leq d(x, z')$ for any $z \in X$.

To obtain an estimate from below, note that

$$d^2(x, z') = \inf_{y \in X} [d(x, y') + d(y, z')] = \inf_{y \in X} [d(x, y') + d(y', z)] \geq d_X(x, z). \quad (3.1)$$

We also have

$$d^2(x, z') \geq d(x, X') + d(z, X') \geq -\alpha + \frac{1}{\beta}d(x, x') - \alpha + \frac{1}{\beta}d(z, z'), \quad (3.2)$$
It follows from (3.1) and (3.2) that
\[
d^2(x, z') \geq \frac{1}{2}d_x(x, z) - \alpha + \frac{1}{2\beta}(d(x, x') + d(z, z')) \\
\geq -\alpha + \frac{1}{2\beta}(d_x(x, z) + d(x, x') + d(z, z')). \tag{3.3}
\]

On the other hand, the triangle inequality shows that
\[
d(x, z') \leq d(x, x') + d_x(x', z') + d(z', z) = d(x, x') + d_x(x, z) + d(z, z'). \tag{3.4}
\]

Then (3.3) and (3.4) give
\[
d^2(x, z') \geq -\alpha + \frac{1}{2\beta}d(x, z').
\]

The next result shows that selfadjoint idempotents can be characterized only by the values \(d(x, x'), x \in X\).

We call two functions, \(\varphi, \psi : X \to [0, \infty)\), equivalent if there exist \(\alpha \geq 0, \beta \geq 1\) such that
\[-\alpha + \frac{1}{\beta}\psi(x) \leq \varphi(x) \leq \alpha + \beta\psi(x)\]
for any \(x \in X\).

**Proposition 3.2** Let \(d, \rho\) be two idempotent metrics on the double of \(X\), \(\rho^* = \rho\), \(d^* = d\). Then \(\rho \sim d\) if and only if the functions \(x \mapsto \rho(x, x')\) and \(x \mapsto d(x, x')\) are equivalent.

**Proof** One direction is trivial, so let us prove the non-trivial one.

Since \(d\) is a selfadjoint idempotent, there are \(\alpha \geq 0, \beta \geq 1\) such that
\[
d(x, x') \leq \alpha + \beta d(x, X') \leq \alpha + \beta d(x, z'), \tag{3.5}
\]
for any \(x, z \in X\).

If the functions \(x \mapsto \rho(x, x')\) and \(x \mapsto d(x, x')\) are equivalent then there exist \(\gamma > 0, \delta \geq 1\) such that
\[
\rho(x, x') \leq \gamma + \delta d(x, x') \leq \gamma + \delta(\alpha + \beta d(x, x')) = \alpha' + \beta' d(x, z'), \tag{3.6}
\]
where \(\alpha' = \gamma + \delta \alpha, \beta' = \delta \beta\).

Using (3.5) and the triangle inequality, we have
\[
d_x(x, z) = d(x', z') \leq d(x', x) + d(x, z') \leq \alpha + (1 + \beta) d(x, z'). \tag{3.7}
\]

Using the triangle inequality again, together with (3.6) and (3.7), we have
\[
\rho(x, z') \leq \rho(x, x') + \rho(x', z') = \rho(x, x') + d_x(x, z) \\
\leq \alpha' + \beta' d(x, z') + \alpha + (1 + \beta) d(x, z') \\
= (\alpha + \alpha') + (1 + \beta + \beta') d(x, z'),
\]

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i.e., \( d \) dominates \( \rho \). Symmetrically, \( \rho \) dominates \( d \), hence they are equivalent. \( \square \)

It would be interesting to find a characterization of functions on \( X \) which can be obtained from metrics on doubles. The next statement shows that selfadjoint idempotents in \( M(X) \) commute.

**Proposition 3.3** Let \( d, \rho \) be two idempotent metrics on the double of \( X \), \( \rho^* = \rho \), \( d^* = d \). Then \( d \rho \sim \rho d \).

**Proof** By definition, for any \( \varepsilon > 0 \) there exists \( y_0 \in X \) such that

\[
d \rho(x, z') = \inf_{y \in X} [\rho(x, y') + d(y, z')] \geq \rho(x, y'_0) + d(y_0, z') - \varepsilon. \tag{3.8}
\]

By Theorem 3.1, there exist \( \alpha \geq 0, \beta \geq 1 \) such that

\[
\rho(x, y'_0) \geq -\alpha + \frac{1}{\beta} \rho(y_0, y'_0); \quad d(y_0, z') \geq -\alpha + \frac{1}{\beta} d(y_0, y'_0).
\]

Then

\[
d \rho(x, z') + \varepsilon \geq \frac{1}{\beta} (\rho(y_0, y'_0) + d(y_0, y'_0)) - 2\alpha. \tag{3.9}
\]

The triangle inequality applied to the right-hand side of (3.8) gives

\[
d \rho(x, z') + \varepsilon \geq \rho(x, y_0) - \rho(y_0, y'_0) + d(z, y_0) - d(y_0, y'_0). \tag{3.10}
\]

On the other hand,

\[
\rho d(x, z') \leq d(x, y'_0) + \rho(z, y'_0) \leq d(x, y_0) + d(y_0, y'_0) + \rho(y_0, y'_0) + \rho(y_0, z). \tag{3.11}
\]

Denote \( d(x, y_0) + \rho(y_0, z) = d_X(x, y_0) + d_X(y_0, z) \) by \( r \) and \( d(y_0, y'_0) + \rho(y_0, y'_0) \) by \( s \). Then (3.9) and (3.10) can be written as

\[
d \rho(x, z') \geq \max \left( \frac{1}{\beta} s - 2\alpha, r - s \right) - \varepsilon,
\]

and (3.11) can be written as

\[
\rho d(x, z') \leq r + s.
\]

\( \square \)

To finish the argument, we need the following statement.

**Lemma 3.4** There exists \( \lambda > 1 \) such that \( r + s \leq \lambda (\max(\frac{1}{\beta} s - 2\alpha, r - s) + 2\alpha) \) for any \( r, s \geq 0 \).
**Proof** First, note that \(\max\left(\frac{1}{\beta}s, r - s\right) \leq \max\left(\frac{1}{\beta}s - 2\alpha, r - s\right) + 2\alpha\). It remains to show that
\[
r + s \leq \lambda \max\left(\frac{1}{\beta}s, r - s\right)
\]  
for some \(\lambda > 1\). Set \(s = tr, t \in [0, \infty)\). Then (3.12) becomes
\[
(1 + t)r \leq \lambda \max\left(\frac{t}{\beta}, (1 - t)r\right),
\]
or, simply,
\[
(1 + t) \leq \lambda \max\left(\frac{t}{\beta}, (1 - t)\right). \tag{3.13}
\]
Taking \(\lambda = \beta(2 + \beta)\), we can provide that (3.13) holds for any \(t \in [0, \infty)\). \(\square\)

Lemma 3.4 implies that \(\rho d(x, z') \leq \lambda d \rho (x, z') + 2\alpha + \varepsilon\). Symmetry implies that \(\rho d\) and \(d \rho\) are equivalent. This finishes the proof of Proposition 3.3.

## 4 Metrics from Subsets

**Example 4.1** Let \(A \subset X\) be a subset. Define a metric \(d_A\) on the double of \(X\) by
\[
d_A(x, y') = \inf_{z \in A} [d_X(x, z) + 1 + d_X(z, y)].
\]
Then \(d_A\) is selfadjoint, \(d_A(x, x') = \inf_{z \in A} [2d(x, z) + 1] = 2d(x, A) + 1\), and
\[
d_A(x, X') = \inf_{y \in X, z \in A} [d_X(x, z) + 1 + d_X(z, y)] = \inf_{z \in A} [d_X(x, z) + 1] = d(x, A) + 1,
\]
hence \([d_A]\) is an idempotent.

A special case of the above metrics \(d_A, A \subset X\), is the case \(A = \{x_0\}\) for a fixed point \(x_0 \in X\). It is clear that the equivalence class of the metric \(e_0 = d_{\{x_0\}}\) does not depend on the choice of the point \(x_0\).

Let \(A, B \subset X\) be closed subsets. In this section we establish when \([d_A] = [d_B]\) under the assumption that \(X\) is locally compact (and the metric \(d_X\) is proper).

Recall [4] that the Higson compactification \(hX\) of a locally compact metric space \(X\) is the Gelfand dual of the \(C^*\)-algebra \(C_h(X)\) of bounded continuous functions \(f\) on \(X\) such that \(\lim_{x \to \infty} \text{Var}_r(f)(x) = 0\) for any \(r > 0\), where
\[
\text{Var}_r(f)(x) = \sup_{y \in X, d_X(x, y) \leq r} |f(x) - f(y)|.
\]
The Gelfand dual of the quotient \(C_h(X)/C_0(X)\) is the Higson corona \(\nu X = hX \setminus X\).

Let \(J_A = \{f \in C_h(X) : f|_A = 0\}\). This is an ideal in \(C_h(X)\). Then the Gelfand dual of \(C_h(X)/J_A\) is the closure \(\bar{A}\) of \(A\) in \(hX\), and \(\bar{A} \setminus A = \bar{B} \setminus B\) if and only if \(J_A + C_0(X) = J_B + C_0(X)\).
Proposition 4.2 The following are equivalent:

1. \([d_A] = [d_B]\);
2. there exists \(C > 0\) such that \(A\) lies in the \(C\)-neighborhood of \(B\) and \(B\) lies in the \(C\)-neighborhood of \(A\);
3. \(A \setminus A = B \setminus B\) in the Higson corona.

Proof We begin with (1) \(\iff\) (2). Suppose first that \([d_A] = [d_B]\). Note that \(d_A(x, x') = 2d_X(x, A) + 1\). In particular, \(d_A(x, x') = 1\) when \(x \in A\). Then there exists \(C > 0\) such that \(d_B(x, x') = 2d_B(x, B) < C\) for any \(x \in A\), in other words, \(A\) lies in the \(C\)-neighborhood of \(B\). Similarly, \(B\) lies in the \(C\)-neighborhood of \(A\) (maybe with another \(C\)).

Assume now that (2) holds. Then \(d_X(x, A) - C \leq d_X(x, B) \leq d_X(x, A) + C\), hence the functions \(d_A(x, x') = 2d_X(x, A) + 1\) and \(d_B(x, x') = 2d_X(x, B) + 1\) are equivalent. By Proposition 3.2 we are done.

Now let us show that (2) \(\iff\) (3). Let (2) hold, and let \(f \in J_A\). Let \(x_0 \in X\). Let \(B \subset N_C(A)\). Let \(r : X \to [0, \infty)\) and \(\mu : X \to [0, \infty)\) be defined by \(r(x) = d_X(x, x_0)\) and by \(\mu(x) = d_X(x, A)\), respectively. Define the map \(\gamma : X \to [0, \infty) \times [0, \infty)\) by \(\gamma(x) = (r(x), \mu(x))\).

\[
F_0 = \gamma^{-1}([0, \infty) \times [0, C]); \quad F_1 = \gamma^{-1}([0, \infty) \times [2C, \infty));
\]
\[
D_k = \gamma^{-1}([k - 1, k] \times [C, 2C]).
\]

Then \(X = F_0 \cup F_1 \cup (\bigcup_{k=1}^{\infty} D_k)\).

For the function \(f \in J_A\), set \(f_n = \sup\{|f(x)| : x \in \bigcup_{k=1}^{n} D_k\}\). As \(\bigcup_{k=1}^{\infty} D_k \subset N_{2C}(A)\) and as \(f \in C_h(X)\) satisfies \(f|A = 0\), one has \(\lim_{n \to \infty} f_n = 0\).

Let us construct a function \(g \in C_h(X)\). Set \(g|F_0 = 0\) and \(g|F_1 = f|F_1\). Our aim is to extend \(g\) to the whole \(X\) with the following properties:

\[
\|g|D_n\| \leq 2f_n-1; \quad \|g|E_n\| \leq 2f_n,
\]

where \(E_n = \gamma^{-1}([n] \times [C, 2C])\). We construct such \(g\) inductively. Suppose that we have already extended \(g\) to \(F_0 \cup F_1 \cup (\bigcup_{k=1}^{n} D_k)\). By the Tietze Extension Theorem, extend \(g\) to \(D_{n+1}\), and denote this extension on \(D_{n+1}\) by \(\tilde{g}\). As \(\|g|E_n\| \leq 2f_n\) and \(|f(x)| \leq f_n\) for any \(x \in \gamma^{-1}([n, n + 1] \times [2C])\), we have \(\|\tilde{g}|D_{n+1}\| \leq 2f_n\).

As \(|\tilde{g}(x)| \leq f_{n+1}\) for any \(x \in \gamma^{-1}([n + 1] \times [C, 2C])\), there exists \(C_0 \in [C, 2C]\) such that \(|\tilde{g}(x)| \leq 2f_{n+1}\) for any \(x \in \gamma^{-1}([n + 1] \times [C_0, 2C])\). Let \(\varphi : [n, n + 1] \times [C, 2C] \to [0, 1]\) be a continuous function such that \(\varphi(r, 2C) = \varphi(n, \mu) = 1\) for any \(r \in [n, n + 1]\), \(\mu \in [C, 2C]\), and \(\varphi(n + 1, \mu) = 0\) for \(\mu \in [C, C_0]\). Then set \(g(x) = \tilde{g}(x)\varphi(\gamma(x))\) for \(x \in D_{n+1}\). Then \(g\) is continuous on \(F_0 \cup F_1 \cup D_1 \cup \cdots \cup D_{n+1}\), \(\|g|D_{n+1}\| \leq 2f_n\) and \(\|g|E_{n+1}\| \leq 2f_{n+1}\).

By construction, \(g|B = 0\), and it follows from (4.1) that \(f - g \in C_0(X)\), therefore \(g \in J_B + C_0(X)\), i.e., \(J_A \subset J_B + C_0(X)\). Symmetrically, \(J_B \subset J_A + C_0(X)\).

Now, suppose that (3) holds, i.e., \(J_A + C_0(X) = J_B + C_0(X)\). If \(A\) does not lie in a \(C\)-neighborhood of \(B\) for any \(C\) then there exists a sequence \(x_n \in A\), \(n \in \mathbb{N}\), such that \(\lim_{n \to \infty} d_X(x_n, B) = \infty\). Note that, necessarily, \(\lim_{n \to \infty} x_n = \infty\).
Passing to a subsequence of \((x_n)_{n \in \mathbb{N}}\), it may be arranged that \(d(x_n, B) \geq n\) and \(d(x_n, x_j) \geq n + j\) for all \(n, j \in \mathbb{N}\). Let \(h_n(x) = (1 - d(x, x_n)/n)_+\) be the positive part of \(1 - d(x, x_n)/n\). This function is Lipschitz with constant \(1/n\) and supported in the ball of radius \(n\) around \(x_n\). By assumption, these balls are all disjoint and disjoint from \(B\) as well. The sum \(f = \sum_{n \in \mathbb{N}} h_n\) belongs to \(C_h(X)\). As \(f|_B = 0\), we have \(f \in J_B\), hence \(f \in J_A + C_0(X)\). On the other hand, as \(f(x_n) = 1\) for any \(n \in \mathbb{N}\), hence \(f \notin J_A + C_0(X)\) (recall that all \(x_n, n \in \mathbb{N}\), lie in \(A\), and \(\lim_{n \to \infty} x_n = \infty\)). This contradiction finishes the proof. \(\square\)

Note that an arbitrary idempotent metric need not be equivalent to \(d_A\) for any \(A\).

## 5 Order Structure

Let \(\rho\) and \(d\) be metrics on the double of \(X\). We say that \([\rho] \leq [d]\) if there exists a metric \(d' \in [d]\) such that \(d'(x, z') \leq \rho(x, z')\) for any \(x, z \in X\). This gives a partial order on \(M(X)\).

**Lemma 5.1** Let \(d, \rho\) be selfadjoint idempotent metrics on the double of \(X\). Then \(d \leq \rho\) if and only if \([\rho][d] = [d]\).

**Proof** First, suppose that \([\rho d] = [d]\). Since both \(\rho\) and \(d\) are selfadjoint idempotents, there exist \(\alpha \geq 0, \beta \geq 1\) such that \(\rho(x, X') \geq \frac{1}{\beta} \rho(x, x') - \alpha\) and \(d(x, X') \geq \frac{1}{\beta} d(x, x') - \alpha\) for any \(x \in X\). Then

\[
\rho \circ d(x, x') \geq \inf_{y \in X} [d(x, y') + \rho(y, x')] \geq d(x, X') + \rho(x, X') \geq \rho(x, X') \\
\geq \frac{1}{\beta} \rho(x, x') - \alpha.
\]

It follows from \([\rho \circ d] = [d]\) that there exist \(\alpha' > 0, \beta' > 1\) such that \(\rho \circ d(x, x') \leq \beta'd(x, x') + \alpha'\). Combining the last two inequalities, we get \(\beta'd(x, x') + \alpha' \geq \frac{1}{\beta} \rho(x, x') - \alpha\), or \(d(x, x') \geq \frac{1}{\beta \beta'} \rho(x, x') - \alpha''\) for some \(\alpha'' > 0\). Using the proof of Proposition 3.2, we conclude that \(d \leq \rho\).

Second, suppose that \(d \leq \rho\). Without loss of generality, we may assume that \(d(x, z') \geq \rho(x, z')\) for any \(x, z \in X\). Then

\[
\rho \circ d(x, x') = \inf_{y \in X} [d(x, y') + \rho(y, x')] \leq d(x, x') + \rho(x, x') \leq 2d(x, x').
\]

This implies that \([d] \leq [\rho d]\). On the other hand,

\[
\rho \circ d(x, x') \geq d(x, X') + \rho(x, X') \geq d(x, X') \geq \frac{1}{\beta} (d(x, x') - \alpha),
\]

which implies \([\rho d] \leq [d]\). Thus, \([\rho d] = [d]\). \(\square\)
6 C*-Algebra of M(X)

Proposition 6.1 Let a ∈ M(X) be an idempotent. Then it is selfadjoint.

Proof Note that a* also must be an idempotent. Then use commutativity of selfadjoint idempotents to show that

\[ a^* = a^*aa^* = (a^*a)(aa^*) = (aa^*)(a^*) = aa^*a = a. \]

\[ \square \]

Corollary 6.2 Any two idempotents in M(X) commute.

Recall that a semigroup S is an inverse semigroup if for any a ∈ S there exists a unique b ∈ S such that a = aba and b = bab ([1], p. 6).

Theorem 6.3 M(X) is an inverse semigroup.

Proof In a regular semigroup, commutativity of idempotents is equivalent to being an inverse semigroup ([1], Theorem 3).

By Theorem 6.3, we can define the (reduced) semigroup C*-algebra \( C^*_r(M(X)) \) of the inverse semigroup M(X) ([3], Section 4.4).

Recall that if a ∈ M(X), \( V_a = \{ b ∈ M(X) : bb^* ≤ a^*a \} \), then the map \( b ↦ ab \) is injective on \( V_a \).

Let \( l_2(M(X)) \) denote the Hilbert space of square-summable functions on M(X) (as a discrete space) with the orthonormal basis (often uncountable) of delta-functions

\[ δ_b(c) = \begin{cases} 1, & \text{if } c = b; \\ 0, & \text{if } c ≠ b, \end{cases} \quad b, c ∈ M(X). \]

For a ∈ M(X), set

\[ \lambda_a(δ_b) = \begin{cases} δ_{ab}, & \text{if } b ∈ V_a; \\ 0, & \text{if } b ∉ V_a. \end{cases} \]

Then \( \lambda_a \) is a partial isometry for any \( a ∈ M(X) \), and the \( C^*-\)algebra \( C^*_r(M(X)) \) generated by all \( \lambda_a, a ∈ M(X) \), is the reduced \( C^*-\)algebra of M(X).

There is a special projection \( [e_0] \) in \( C^*_r(M(X)) \), given by the metric \( d_{\{x_0\}} \) for some \( x_0 ∈ X \). By definition, \( e_0(x, y') = d_X(x, x_0) + 1 + d_X(x_0, y) \), and it is easy to see that the equivalence class \( [e_0] \) does not depend on \( x_0 \).

Lemma 6.4 \( d ∘ e_0 \) and \( e_0 ∘ d \) are equivalent to \( e_0 \) for any metric \( d \) on the double of \( X \).

Proof Take \( y = x_0 \), and then use the triangle inequality to obtain

\[ e_0d(x, z') = \inf_{y ∈ X} [d(x, y') + d_X(y, x_0) + d_X(z, x_0) + 1] ≤ d(x, x_0') + d_X(z, x_0) + 1 \]

\[ ≤ d_X(x, x_0) + d(x_0, x_0') + d_X(z, x_0) + 1 ≤ e_0(x, z') + d(x_0, x_0'). \]
On the other hand, by the triangle inequality,

\[ d(x, y') + d_X(y, x_0) + d_X(z, x_0) + 1 = d(x, y') + d_X(y', x_0') + d_X(z, x_0) + 1 \]

\[ \geq d(x, x_0') + d_X(z, x_0) + 1. \]  \hfill (6.1)

Passing to the infimum in (6.1) with respect to \( y \in X \), we obtain \( e_0d(x, z') \geq d(x, x_0') + d_X(z, x_0) + 1 \). Another triangle inequality gives

\[ e_0d(x, z') \geq d(x, x_0') + d_X(z, x_0) + 1 \geq d_X(x, x_0) - d(x_0, x_0') + d_X(z, x_0) + 1 \]

\[ = e_0(x, z') - d(x_0, x_0'). \]

Thus, \( e_0(x, z') - \alpha \leq e_0d(x, z') \leq e_0(x, z') + \alpha \) for any \( x, z \in X \), where \( \alpha = d(x_0, x_0') \), hence \( e_0d \sim e_0 \). Similarly, \( de_0 \sim e_0 \). \hfill \( \square \)

Thus, \( [e_0] \) is the zero element in \( M(X) \).

**Proposition 6.5** The set \( V_{e_0} \) consists of a single element \( [e_0] \).

**Proof** Let \( s \in V_{[e_0]} \). Then \( ss^* \leq [e_0] \), hence \( ss^* = ss^*[e_0] = [e_0] \). Then \( s = ss^*s = [e_0]s = [e_0] \). \hfill \( \square \)

**Corollary 6.6** \( \lambda_{[e_0]} \) is a rank one projection in \( C_r^*(M(X)) \).

**Corollary 6.7** There is a direct sum decomposition of \( C^*-algebras \ C_r^*(M(X)) = C_0^*(M(X)) \oplus e_0 \mathbb{C} \), where \( C_0^*(M(X)) = \{ \lambda_a : a \in C_r^*(M(X)), a[e_0] = [e_0]a = 0 \} \).

Let \( A, B \subset X \), let \( d_X(A, B) = \inf_{x \in A, y \in B} d_X(x, y) \), and let \( B_R(x_0) \) denote the ball of radius \( R \) centered at \( x_0 \in X \).

**Proposition 6.8** Suppose that there exists \( \beta \geq 1 \) such that

\[ d_X(A \setminus B_R(x_0), B \setminus B_R(x_0)) > \frac{1}{\beta} R. \]  \hfill (6.2)

Then \( [d_Ad_B] = [e_0] \).

**Proof** By Proposition 3.2, it suffices to compare \( d_Ad_B(x, x') \) and \( e_0(x, x') \), \( x \in X \). Recall that

\[ d_Ad_B(x, x') = \inf_{u \in A, v \in B, y \in X} [d_X(x, u) + d_X(u, y) + d_X(y, v) + d_X(v, x) + 2]. \]

\[ e_0(x, x') = 2d_X(x, x_0) + 1. \]

By Proposition 6.5, \( [e_0] \leq [d_Ad_B] \), so it remains to show that \( [d_Ad_B] \leq [e_0] \).

Set \( L = \frac{1}{\beta} R \). Take \( x \in X \), and let \( R \) satisfy \( x \in B_{2R}(x_0) \) and \( x \notin B_{R+L}(x_0) \). Then

\[ e_0(x, x_0) = 2d(x, x_0) + 1 \leq 4R + 1. \]  \hfill (6.3)
Now let us estimate $d_Ad_B(x, x')$. By definition, there exist $u_0 \in A$, $v_0 \in B$, $y_0 \in X$ such that

$$d_Ad_B(x, x') \geq d_X(x, u_0) + d_X(u_0, y_0) + d_X(y_0, v_0) + d_X(v_0, x).$$

Consider the two cases:

(a) either $u_0 \in B_R(x_0)$ or $v_0 \in B_R(x_0)$;

(b) $u_0, v_0 \notin B_R(x_0)$.

In the case (a), if $u_0 \in B_R(x_0)$ and $x \notin B_{R+L}(x_0)$ then $d_X(x, u_0) \geq L$. Otherwise, if $v_0 \in B_R(x_0)$ then $d_X(v_0, x) \geq L$. Thus, $d_Ad_B(x, x') \geq L$.

In the case (b), by the triangle inequality and by (6.2),

$$d_Ad_B(x, x') \geq d_X(x, u_0) + d_X(u_0, y_0) + d_X(y_0, v_0) + d_X(v_0, x) \geq d_X(u_0, v_0) \geq L.$$

Thus, in both cases we have

$$d_Ad_B(x, x') \geq L = \frac{1}{\beta} R. \quad (6.4)$$

Combining (6.3) and (6.4), we get $e_0(x, x') \leq 4\beta d_Ad_B(x, x') + 1$, hence $[d_Ad_B] \leq [e_0]$. \hfill \Box

**Corollary 6.9** Under the assumption of Proposition 6.8, $(\lambda_{[d_A]} - \lambda_{[e_0]})(\lambda_{[d_B]} - \lambda_{[e_0]}) = 0$, i.e., the projections $\lambda_{[d_A]} - \lambda_{[e_0]}$ and $\lambda_{[d_B]} - \lambda_{[e_0]}$ are mutually orthogonal.

**Example 6.10** Let $X = \mathbb{R}^2$ with the standard metric. For $\varphi \in [0, 2\pi)$ let $A_\varphi$ be the ray from the origin with the angle $\varphi$ to the polar axis. If $\psi \in [0, 2\pi)$, $\psi \neq \varphi$, then the two rays $A_\varphi$ and $A_\psi$ satisfy the assumption of Proposition 6.8, hence the projections $\lambda_{[d_{A_\phi}]} - \lambda_{[e_0]}$ and $\lambda_{[d_{A_\psi}]} - \lambda_{[e_0]}$ are mutually orthogonal. Thus, the $C^*$-algebra $C^*(M(X))$ has uncountably many mutually orthogonal projections.

# 7 Examples

**Proposition 7.1** Let $X$ be a closed subset of $[0, \infty)$ with the induced metric. Then any $a \in M(X)$ is a selfadjoint idempotent. Hence $M(X)$ is commutative.

**Proof** First, let us show that any element of $M(X)$ is selfadjoint. Suppose the contrary. Then there exists a metric $d$ on the double of $X$ such that $d^*$ is not equivalent to $d$, and for any $n \in \mathbb{N}$ we can find points $y_n, z_n \in X$ such that

$$n \cdot d(y_n, z'_n) < d(y'_n, z_n). \quad (7.1)$$

Since $d(X, X') > 0$, the sequence $d(y'_n, z_n)$ is unbounded.
Passing to a subsequence, if necessary, we may assume without loss of generality that $y_n < z_n$ for any $n \in \mathbb{N}$. Then $d_X(x_0, z_n) = d_X(x_0, y_n) + d_X(y_n, z_n)$.

By the triangle inequality, we have

$$d(y'_n, z_n) \leq 2d_X(y_n, z_n) + d(y_n, z'_n), \tag{7.2}$$

so, (7.1) and (7.2) imply that

$$n \cdot d(y_n, z'_n) < 2d_X(y_n, z_n) + d(y_n, z'_n),$$

or, equivalently,

$$d(y_n, z'_n) < \frac{2}{n - 1} d_X(y_n, z_n). \tag{7.3}$$

Another application of the triangle inequality gives

$$d(y_n, z'_n) \geq d_X(x_0, z_n) - (d_X(y_n, x_0) + d(x_0, x'_0)).$$

Combining this with (7.3), we get

$$d_X(x_0, z_n) - (d_X(y_n, x_0) + d(x_0, x'_0)) < \frac{2}{n - 1} d_X(y_n, z_n). \tag{7.4}$$

By assumption, $d_X(x_0, z_n) = d_X(x_0, y_n) + d_X(y_n, z_n)$, so (7.4) implies that

$$d_X(y_n, z_n) - d(x_0, x'_0) < \frac{2}{n - 1} d_X(y_n, z_n),$$

holds, hence the values $d_X(y_n, z_n)$ are uniformly bounded. Let $C$ satisfy $d_X(y_n, z_n) < C$ for any $n \in \mathbb{N}$.

By the triangle inequality and (7.1), we have

$$n(d(y_n, y'_n) - C) < n(d(y_n, y'_n) - d_X(y'_n, z'_n)) \leq nd(y_n, z'_n)$$

$$< d(y'_n, z_n) \leq d(y'_n, y'_n) + d_X(y_n, z_n) < d(y_n, y'_n) + C,$$

hence $d(y_n, y'_n) < \frac{n + 1}{n - 1} C$, and the values $d(y_n, y'_n)$ are uniformly bounded.

Thus we get a contradiction: the left-hand side of the triangle inequality

$$d(y'_n, z_n) \leq d(y_n, y'_n) + d_X(y_n, z_n),$$

is unbounded, while both summands in the right-hand side are uniformly bounded.

Second, we have to show that any selfadjoint metric $d$ on the double of $X$ is an idempotent. Suppose the contrary: for any $n \in \mathbb{N}$ there exist points $y_n, z_n \in X$ such that

$$d(y_n, z'_n) < \frac{1}{n} d(y_n, y'_n). \tag{7.5}$$
Once again, we have two possibilities: either $d_X(x_0, z_n) = d_X(x_0, y_n) + d_X(y_n, z_n)$ or $d_X(x_0, y_n) = d_X(x_0, z_n) + d_X(z_n, y_n)$ for infinitely many numbers $n$’s, and let us assume that the first opportunity holds true.

Then, by the triangle inequality, we have

$$d_X(z_n, x_0) - (d_X(y_n, x_0) + d(x_0, x_0')) \leq d(y_n, z_n'),$$

or, equivalently,

$$d_X(y_n, z_n) - d(x_0, x_0') \leq d(y_n, z_n'),$$

which, together with (7.5), implies that

$$d_X(y_n, z_n) \leq d(y_n, z_n') + d(x_0, x_0') \leq \frac{1}{n}d(y_n, y_n') + d(x_0, x_0'). \quad (7.6)$$

Another triangle inequality combined with (7.5) and (7.6) gives

$$d(y_n, y_n') \leq d(y_n, z_n') + d(x_0, x_0'),$$

which holds for infinitely many $n$’s. The latter may be true only if $d(y_n, y_n')$ is bounded for these $n$’s, but this contradicts $d(X, X') > 0$. Indeed, if $d(y_n, y_n') < C$ for some $C > 0$ and for infinitely many $n$’s then the sequence $d(y_n, z_n')$ is not separated from 0. \qed

**Example 7.2** Let $X = \{(n, n, 0) : n \in \mathbb{N}\} \cup \{(n, -n, 0) : n \in \mathbb{N}\} \subset \mathbb{R}^3$ with the standard metric, and let $X' = \{(x, -y, 1) : (x, y, 0) \in X\}$. For $a_n = (n, n, 0) \in X$ we have $a_n' = (n, -n, 1) \subset X'$, and for $b_n = (n, -n, 0) \in X$ we have $b_n' = (n, n, 1) \subset X'$. Let the metric $d$ on the double of $X$ be inherited from the standard metric of $\mathbb{R}^3$. Then $d(a_n, a_n') = \sqrt{n^2 + 1}$, while $d(a_n, X') = d(a_n, b_n') = 1$, hence $[d]$ is selfadjoint, but not idempotent.

**Example 7.3** Let $X = \mathbb{R}$ with the standard metric, and let $A = [0, \infty), B = (-\infty, 0]$. For $x, y \in X$, set $d(x, y') = \begin{cases} |x + y| + 1, & \text{if } x \in A, y \in B; \\ |x| + |y| + 1, & \text{otherwise.} \end{cases}$

Then $d^*(x, y') = \inf_{z \in X} d(x, z') + d(y, z')$. If $x, y \in A$ then one may take $z = -x$, in this case $d(x, -x') + d(y, -x') = |x - y| + 2$. In other cases one may take $z = 0$, and $d^*(x, y') = |x| + |y| + 2$. Thus, $d^*(x, y') = d_A(x) + 1$, hence $[d^*] = [d_A]$. Similary, we can see that $[dd^*] = [d_B]$. Thus, $[d]$ is a partial isometry from $[d_A]$ to $[d_B]$.

**8 Examples from Extended Metrics**

When $X$ is non-compact, the inverse semigroup $M(X)$ is infinite. Here we consider the case when metrics are replaced by the so-called extended metrics, which are the
same as usual metrics, except that they are allowed to take infinite values. This gives a lot of examples with finite \( M(X) \).

Note that setting \( d(x, y') = \infty \) for any \( x, y \in X \) gives the zero element \( 0 \in M(X) \), as \( d0 = 0d = 0 \) for any metric \( d \) on the double of \( X \).

**Example 8.1** Let \( X \) be a one-point space, \( X = \{a\} \). Any two finite metrics on the double of \( X \) are equivalent, but an infinite metric with \( d(a, a') = \infty \) is not equivalent to a finite metric, so \( M(X) = \{I, 0\} \). We have \( V_I = \{I, 0\} \) and \( V_0 = \{0\} \). Then the \( C^* \)-algebra of \( X \) is a subalgebra in the algebra \( M_2(\mathbb{C}) \) of \( 2 \times 2 \) matrices, generated by the identity matrix and by a rank one projection, hence is isomorphic to \( \mathbb{C} \oplus \mathbb{C} \).

**Example 8.2** Let \( X \) be the space consisting of two points, \( a \) and \( b \), with \( d_X(a, b) = \infty \). Any metric in \( \mathcal{M}(X) \) is determined by the 4 values: \( d(a, a') \), \( d(a, b') \), \( d(b, a') \), and \( d(b, b') \). Metrics with any finite value are equivalent to those with this value equal to 1, so non-equivalent classes of metrics should take values 1 and \( \infty \). Taking into account the triangle inequality, there are 7 possible metrics in \( M(X) \):

1. \( 0(a, a') = 0(a, b') = 0(b, a') = 0(b, b') = \infty \);
2. \( I(a, a') = I(b, b') = 1 \), \( I(a, b') = I(b, a') = \infty \);
3. \( p(a, a') = 1 \), \( p(a, b') = p(b, b') = p(b, a') = \infty \);
4. \( q(b, b') = 1 \), \( q(b, a') = q(a, a') = q(a, b') = \infty \);
5. \( u(a, b') = 1 \), \( u(a, a') = u(b, a') = u(b, b') = \infty \);
6. \( u^*(b, a') = 1 \), \( u^*(b, b') = u^*(a, b') = u^*(a, a') = \infty \);
7. \( s(a, b') = s(b, a') = 1 \), \( s(a, a') = s(b, b') = \infty \).

Note that \( p \), \( q \) are idempotents, \( u \) and \( u^* \) are partial isometries, \( u^*u = p \), \( uu^* = q \), and \( s \) is a symmetry. Let \( L_0 = \{\delta_0\} \), \( L_1 = \{\delta_p, \delta_{u^*}\} \), \( L_2 = \{\delta_q, \delta_u\} \), \( L_3 = \{\delta_1, \delta_s\} \). Then \( V_0 = L_0 \), \( V_p = V_u = L_1 \oplus L_0 \), \( V_q = V_{u^*} = L_2 \oplus L_0 \), \( V_I = V_s = L_0 \oplus L_1 \oplus L_2 \oplus L_3 \).

We have \( \lambda_d|_{L_0} = \text{id} \) for any \( d \in M(X) \), \( \lambda_u|_{L_1} = L_2 \), \( \lambda_{u^*}|_{L_2} = L_1 \), and \( \lambda_s|_{L_1 \oplus L_2} = \lambda_u|_{L_1 \oplus L_2} + \lambda_{u^*}|_{L_1 \oplus L_2} \), so \( \lambda_u \), \( \lambda_{u^*} \), and \( \lambda_s \) restricted to the invariant subspace \( L_1 \oplus L_2 \) generate the algebra isomorphic to \( M_2(\mathbb{C}) \). Taking into account the invariant subspaces \( L_0 \) and \( L_3 \), where \( M(X) \) acts by scalars, we get \( C^*(M(X)) \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \subset M_7(\mathbb{C}) \).

### 9 Sufficient Condition for an Isomorphism \( M(X) \cong M(Y) \)

Given two metric spaces, \( X \) and \( Y \), consider all metrics \( d \) on the disjoint union \( X \sqcup Y \) such that

- \( d|_X = d_X, d|_Y = d_Y \);
- \( d(X, Y) \neq 0 \).

Let \( M(X, Y) \) denote the set of all such metrics.

Recall that, given a metric \( d \) on \( X \sqcup Y \), the Hausdorff distance between \( X \) and \( Y \) is \( d_H(X, Y) = \max(\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)) \), and the Gromov–Hausdorff distance between \( X \) and \( Y \) is \( \inf d_H(X, Y) \), where the infimum is over all metrics on \( X \sqcup Y \) that equal \( d_X \) and \( d_Y \) on \( X \) and \( Y \), respectively. Note that the Gromov–Hausdorff distance may be (and often is) infinite.
Lemma 9.1 The Gromov–Hausdorff distance between $X$ and $Y$ equals $\inf_{d \in \mathcal{M}(X,Y)} d_H(X, Y)$.

Proof If $d$ is a metric on $X \sqcup Y$ that equals $d_X$ and $d_Y$ on $X$ and $Y$, respectively, and $d_H(X, Y) = 0$, then for any $\epsilon > 0$, set $d^\epsilon|_X = d_X$, $d^\epsilon|_Y = d_Y$, and $d^\epsilon(x, y) = d(x, y) + \epsilon$ for $x \in X$, $y \in Y$. It is clear that $d^\epsilon$ is a metric in $\mathcal{M}(X,Y)$ and $d^\epsilon(X, Y) \geq \epsilon$, so it suffices to take the infimum over metrics for which the distance between $X$ and $Y$ is non-zero. \hfill \square

Proposition 9.2 Suppose that the Gromov–Hausdorff distance between $X$ and $Y$ is finite. Then $M(X)$ and $M(Y)$ are isomorphic.

Proof By assumption, there exists $\rho \in \mathcal{M}(X, Y)$ and $C > 0$ such that $\rho(x, Y) < C$ and $\rho(y, X)$ for any $x \in X$ and any $y \in Y$. Then $\rho^* \rho \in \mathcal{M}(X)$, and $\rho^* \rho(x, x') = \inf_{z \in X} 2 \rho(x, z') < 2C$ for any $x \in X$, hence $\rho^* \rho \sim I$, where $I \in \mathcal{M}(X)$ is defined in Example 2.6. Similarly, $\rho \rho^* \sim I$ in $\mathcal{M}(Y)$.

For $d \in \mathcal{M}(X)$, $b \in \mathcal{M}(Y)$, set $\varphi(d) = \rho d \rho^* \in \mathcal{M}(Y)$, $\psi(b) = \rho^* b \rho$. Clearly, $\varphi$ and $\psi$ pass to maps $\tilde{\varphi} : M(X) \to M(Y)$ and $\tilde{\psi} : M(Y) \to M(X)$, respectively. These maps are semigroup homomorphisms, as $[\rho^* \rho]$ and $[\rho \rho^*]$ are the unit elements in $M(X)$ and in $M(Y)$, respectively.

Finally, $\psi \circ \varphi(d) = \rho^* \rho d \rho^* \rho \sim d$, hence $\tilde{\psi} \circ \tilde{\varphi} = \text{id}_{M(X)}$. Similarly, $\tilde{\varphi} \circ \tilde{\psi} = \text{id}_{M(Y)}$. \hfill \square

10 Subgroup of Invertibles

An element $[d] \in M(X)$ is invertible if $[d^* d] = [dd^*] = [I]$. It is clear that the invertible elements form a group. Here we describe this group.

Definition 10.1 A map $f : X \to X$ is an almost isometry if

(i) there exists $C > 0$ such that

$$d_X(x, \tilde{x}) - C \leq d_Y(f(x), f(\tilde{x})) \leq d_X(x, \tilde{x}) + C,$$

for any $x, \tilde{x} \in X$;

(ii) there exist a map $g : X \to X$ and $D > 0$ such that $d_X(g \circ f(x), x) < D$ and $d_X(f \circ g(x), x) < D$ for any $x \in X$.

Note that if such a map $g$ exists then it automatically satisfies (i), possibly with different $C$.

Any isometry is patently an almost isometry. Another example of an almost isometry for $X = \Gamma$, where $\Gamma$ is a finitely generated group with the word-length metric, is provided by conjugation by a fixed element $g \in \Gamma$.

Given an almost isometry $f : X \to X$, set

$$d^f(x, y') = \inf_{z \in X} d_X(x, z) + C + d_X(f(z), y), \quad x, y \in X.$$
It was shown in [2] that \( d^f \) is a metric (one has to check four triangle inequalities).

If \( f, g : X \to X \) are almost isometries as in Definition 10.1 then (i2) implies that
\[
[d^f, d^g] = [d^g, d^f] = [I].
\]

**Proposition 10.2** Let \( d \in \mathcal{M}(X) \) and let \([d] \in M(X)\) be invertible. Then there exists an almost isometry \( f \) of \( X \) such that
\[
[d^f] = [d].
\]

**Proof** If \([d] \) is invertible then \([d^*d] = [dd^*] = [I] \), so there exists \( C > 0 \) such that
\[
\inf_{z \in X} [d(x, z') + d(z, x')] < C \text{ and } \inf_{z \in X} [d(x', z) + d(z, x')] < C \text{ for any } x \in X.
\]

Therefore there exist \( u, v \in X \) such that \( d(x, u') < C/2 \) and \( d(x', v) < C/2 \).

Set \( f(x) = u, g(x) = v \). Then \( d(x, f(x')) < C/2 \) and \( d(x', g(x)) < C/2 \) for any \( x \in X \), hence \( d(f(x'), g(f(x))) < C/2 \), and, by the triangle inequality,
\[
d_X(x, g \circ f(x)) \leq d(x, f(x')) + d(f(x'), g(f(x))) < C.
\]

Similarly one gets \( d_X(x, f \circ g(x)) < C \).

Let \( x, \tilde{x} \in X \). Then, by the triangle inequality,
\[
d_X(f(x), f(\tilde{x})) \leq d(f(x), x') + d_X(x', \tilde{x}') + d(\tilde{x}', f(\tilde{x})) \leq d_X(x, \tilde{x}) + C,
\]

and
\[
d_X(f(x), f(\tilde{x})) \geq -d(f(x), x') + d_X(x', \tilde{x}') - d(\tilde{x}', f(\tilde{x})) \geq d_X(x, \tilde{x}) - C,
\]

hence \( f \) is an almost isometry.

It remains to check that \([d^f] = [d] \). Taking \( z = x \) and using the triangle inequality, we get
\[
d^f(x, y') = \inf_{z \in X} [d_X(x, z) + d_X(f(z), y) + C]
\leq d_X(f(x), y) + C \leq d_X(f(x'), y') + C
\leq d(f(x'), x') + d(x, y') + C \leq d(x, y') + 3C/2.
\]

To prove the estimate from below, note that
\[
d_X(f(z), y) \geq d_X(g \circ f(z), g(y)) - C/2 \geq d_X(z, g(y)) - d_X(g \circ f(z), z) - C/2
\geq d_X(z, g(y)) - C - C/2 = d_X(z, g(y)) - 3C/2,
\]

hence, by the triangle inequality,
\[
d^f(x, y') = \inf_{z \in X} [d_X(x, z) + d_X(f(z), y) + C]
\geq \inf_{z \in X} [d_X(x, z) + d_X(z, g(y))] - C/2
\geq d_X(x, g(y)) - C/2 \geq d(x, y') - d(y', g(y)) - C - C/2
\geq d(x, y') - 3C/2.
\]

\[\square\]
11 Coarse Version

Two metrics, $d_1, d_2$, on $X$ are coarsely equivalent if there exists a monotone increasing function $f$ on $[0, \infty)$ with $\lim_{t \to \infty} f(t) = \infty$ such that

$$f^{-1}(d_2(x, y)) \leq d_1(x, y) \leq f(d_2(x, y)),$$

for any $x, y \in X$.

All our results hold also for the coarse equivalence classes of metrics on the double of $X$. This gives a smaller quotient inverse semigroup $M_c(X)$ of coarse equivalence classes. We may also use the fact that the image of an inverse semigroup, under a semigroup homomorphism, is an inverse semigroup.

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