Aspects of the category SKB of skew braces

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ABSTRACT
We examine the pointed protomodular category SKB of left skew braces. We study the notion of commutator of ideals in a left skew brace. Notice that in the literature, “product” of ideals of skew braces is often considered. We show that the so-called (Huq=Smith) condition holds for left skew braces. Finally, we give a set of generators for the commutator of two ideals, and prove that every ideal of a left skew brace has a centralizer.

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1. Introduction

Braces appear in connections to the study of set-theoretic solutions of the Yang-Baxter equation. A set-theoretic solution of the Yang-Baxter equation is a pair \((X, r)\), where \(X\) is a set, \(r: X \times X \to X \times X\) is a bijection, and \((r \times \text{id})(\text{id} \times r)(r \times \text{id}) = (\text{id} \times r)(r \times \text{id})(\text{id} \times r)\) [14]. Set-theoretic solutions of the Yang-Baxter equation appear, for instance, in the study of representations of braid groups, and form a category SYBE, whose objects are these pairs \((X, r)\), and morphisms \(f: (X, r) \to (X', r')\) are the mappings \(f: X \to X'\) that make the diagram

\[
\begin{array}{ccc}
X \times X & \xrightarrow{f \times f} & X' \times X' \\
\downarrow r & & \downarrow r \\
X \times X & \xrightarrow{f_{\times f}} & X' \times X'
\end{array}
\]

commute.

One way to produce set-theoretic solutions of the Yang-Baxter equation is using left skew braces.

Definition [16] A (left) skew brace is a triple \((A, *, o)\), where \((A, *)\) and \((A, o)\) are groups (not necessarily abelian) such that

\[
a \circ (b \ast c) = (a \circ b) \ast a^{-*} \ast (a \circ c) \quad \text{(B)}
\]

for every \(a, b, c \in A\). Here \(a^{-*}\) denotes the inverse of \(a\) in the group \((A, *)\). The inverse of \(a\) in the group \((A, o)\) will be denoted by \(a^{-o}\).
A brace is sometimes seen as an algebraic structure similar to that of a ring, with distributivity warped in some sense. But a better description of a brace is probably that of an algebraic structure with two group structures out of phase with each other.

For every left skew brace \((A, *, \circ)\), the mapping

\[ r: A \times A \to A \times A, \quad r(x, y) = (x^{-*} * (x \circ y), (x^{-*} * (x \circ y))^{-\circ} \circ x \circ y), \]

is a non-degenerate set-theoretic solution of the Yang-Baxter equation ([16, Theorem 3.1] and [20, p. 96]). Here “non-degenerate” means that the mappings \(\pi_1 r(x_0, -): A \to A\) and \(\pi_2 r(-, y_0): A \to A\) are bijections for every \(x_0 \in A\) and every \(y_0 \in A\), where \(\pi_1\) and \(\pi_2\) are the product projections.

The simplest examples of left skew braces are:

1. For any associative ring \((R, +, \cdot)\), the Jacobson radical \((J(R), +, \circ)\), where \(\circ\) is the operation on \(J(R)\) defined by \(x \circ y = x \cdot y + x + y\) for every \(x, y \in J(R)\).

2. For any group \((G, *)\), the left skew braces \((G, *, \circ)\) and \((G, * , *^{op})\).

Several non-trivial examples of skew braces can be found in [25]. A complete classification of braces of low cardinality has been obtained via computer [20].

A homomorphism of skew braces is a mapping which is a group homomorphism for both the first and second group structures. They both reflect isomorphisms. Since \(E\) is a category of skew braces, it naturally allows the lifting of the protomodular aspects of the category \(\text{op}Gp\) of groups to the category \(\text{DiGp}\). In turn, the left exact fully faithful embedding \(\text{SKB} \hookrightarrow \text{DiGp}\) makes \(\text{SKB}\) a pointed protomodular category as well. The protomodular axiom was introduced in [5] in order to extract the essence of the homological constructions and in particular to induce an intrinsic notion of exact sequence.

In this paper, after recalling the basic facts about protomodular categories, we study some “protomodular aspects” of left skew braces, in particular in relation to the category of digroups. We study the notion of commutator of ideals in a left skew brace (in the literature, “product” of ideals of skew braces is often considered). We show that the (Huq=Smith) condition holds for left skew braces. Notice that the (Huq=Smith) condition does not hold for digroups and near-rings [19]. We then show how, when the (Huq=Smith) condition holds, the Huq and Smith commutators determine each other in the homological context. And finally we give a set of generators for the commutator of two ideals, and prove that every ideal of a left skew brace has a centralizer.

### 2. Basic recalls and notations

In this work, any category \(\mathcal{E}\) will be supposed finitely complete, which implies that it has a terminal object 1. The terminal map from \(X\) is denoted \(\tau_X: X \to 1\). Given any map \(f: X \to Y\), the equivalence relation \(R[f]\) on \(X\) is produced by the pullback of \(f\) along itself:

\[
\begin{array}{ccc}
R[f] & \xrightarrow{d_0} & X \\
\downarrow & & \downarrow f \\
& \xrightarrow{d_1} & Y
\end{array}
\]

The map \(f\) is said to be a regular epimorphism in \(\mathcal{E}\) when \(f\) is the quotient of \(R[f]\). When it is the case, we denote it by a double head arrow \(\xrightarrow{\text{epi}}\). A monomorphism \(X \to Y\) is denoted with a tail. A morphism \(f: X \to Y\) is split if there exists a morphism \(s: Y \to X\) such that \(fs = 1_Y\); in this case, \(f\)
turns out to be a regular epimorphism. A given split epimorphism will be denoted by \((f, s) : X \rightrightarrows Y\). The category \(\mathbb{E}\) is said to be pointed when the terminal object 1 is initial as well, we then denote this zero object by 0.

### 2.1. Pointed protomodular categories

Let us recall that a pointed category \(\mathbb{A}\) is additive if and only if, given any split epimorphism \((f, s) : X \rightrightarrows Y\), the following downward pullback:

\[
\begin{array}{ccc}
\text{Ker} f & \xrightarrow{k_f} & X \\
\downarrow & & \downarrow f \\
0 & \xrightarrow{0_Y} & Y
\end{array}
\]

is an upward pushout, namely if and only if \(X\) is the direct sum (= coproduct) of \(Y\) and \(\text{Ker} f\).

Recall that a subobject of an object \(X\) in a category \(\mathbb{E}\) is the isomorphism class of a monomorphism \(u : U \rightarrow X\), where two monomorphisms \(u : U \rightarrow X\) and \(v : V \rightarrow X\) are isomorphic if there exists an isomorphism \(k : U \rightarrow V\) such that \(u = v k\). When the category \(\mathbb{E}\) is concrete, namely when its objects are sets (possibly with additional structure) and the morphisms are set functions (preserving the additional structure), we identify a subobject \(u : U \rightarrow X\) with its image by \(u\), namely as a subset. In such a context, when there is no ambiguity, we shall denote a subobject \(U \rightarrow X\) by its only domain \(U\). Let us recall the following:

**Definition 2.1.** [5] A pointed category \(\mathbb{E}\) is said to be protomodular when, given any split epimorphism as above, the pair \((k_f, s)\) of monomorphisms is jointly strongly epic.

This means that the only subobject \(u : U \rightarrow X\) containing the pair \((k_f, s)\) of subobjects is, up to isomorphism, \(1_X\). It implies that, given any pair \((f, g) : X \rightrightarrows Z\) of arrows which are equalized by \(k_f\) and \(s\), they are necessarily equal (take the equalizer of this pair), namely that \(k_f\) and \(s\) are jointly epicomorphic. Pulling back the split epimorphisms along the initial map \(0_Y : 0 \rightarrow Y\) being a left exact process, the previous definition is equivalent to saying that this process reflects isomorphisms, namely that the **split short five lemma holds**.

The category \(\text{Gp}\) of groups is clearly pointed protomodular. This is the case of the category \(\text{Rng}\) of rings as well, and more generally, given a commutative ring \(R\), of any category \(R\)-\text{Alg} of any given kind of \(R\)-algebras without unit, possibly non-associative. This is in particular the case of the category \(R\)-\text{Lie} of \(R\)-algebras. Even for \(R\) a non-commutative ring, in which case \(R\)-algebras have a more complex behavior (they are usually called \(R\)-rings, see [2, p. 36] or [15, p. 52]), one has that the category \(R\)-\text{Rng} of \(R\)-rings is pointed protomodular, as can be seen from the fact that the forgetful functor \(R\)-\text{Rng} \rightarrow \text{Ab}\) reflects isomorphisms and \(\text{Ab}\) is protomodular.

Any pointed protomodular category \(\mathbb{E}\) satisfies the following algebraic **Five Principles** which hold in \(\text{Gp}\), namely:

1. a morphism \(f\) is a monomorphism if and only if its kernel \(\text{Ker} f\) is trivial [5];
2. any regular epimorphism is the cokernel of its kernel, in other words any regular epimorphism produces an exact sequence, which determines an intrinsic notion of exact sequences in \(\mathbb{E}\) [5];
3. there a specific class of monomorphisms \(u : U \rightarrow X\), the normal monomorphisms [7], see next section;
4. there is an intrinsic notion of abelian object [7], see Section 4.1.1;
5. any reflexive relation in \(\mathbb{E}\) is an equivalence relation, i.e., the category \(\mathbb{E}\) is a Mal’tsev one [6].

Principle (5) is not directly exploited in \(\text{Gp}\); we shall show in Section 4.3.2 how importantly it works out inside a pointed protomodular category \(\mathbb{E}\). Pointed protomodular varieties of universal algebras are characterized in [13].
2.2. Normal monomorphisms

Given any map \( f : X \to Y \) and any equivalence relation \( S \) on \( Y \), the inverse image of \( S \) along \( f \) is given by the joint pullback of the pair \((d^0_S, d^1_S)\) along \( f \). It is an equivalence relation on \( X \) denoted by \( f^{-1}(S) \).

**Definition 2.2.** [7] In any category \( \mathbb{E} \), given a pair \((u, R)\) of a monomorphism \( u : U \hookrightarrow X \) and an equivalence relation \( R \) on \( X \), the monomorphism \( u \) is said to be \textit{normal to} \( R \) when the equivalence relation \( u^{-1}(R) \) is the indiscrete equivalence relation \( \nabla_X = R[\tau_X] \) on \( X \) and, moreover, the downward square indexed by 0 in the following induced diagram is a pullback:

\[
\begin{array}{ccc}
U \times U & \xrightarrow{\tilde{u}} & R \\
\downarrow d^U_0 & & \downarrow d^R \\
U & \xrightarrow{u} & X
\end{array}
\]

In the category \( \text{Set} \), provided that \( U \neq \emptyset \), these two properties characterize the equivalence classes of \( R \). In the category \( \text{Gp} \), a subgroup \( N \subset G \) is normal if and only if the equivalence relation \( RN \) on \( G \) defined by \( xRNy \iff x^{-1}y \in N \) is internal in \( \text{Gp} \) and \( N \) is the equivalence class of 1. In this case, \( RN \) is the only internal equivalence relation in \( \text{Gp} \) to which \( N \) is normal. By the Yoneda embedding, we have the following:

**Proposition 2.3.** Given any equivalence relation \( R \) on an object \( X \) in a category \( \mathbb{E} \) and any map \( x : 1 \to X \), the following upper monomorphism \( \tilde{x} = d^R_1 \bar{x} \) is normal to \( R \) where the left hand side square is a pullback:

\[
\begin{array}{ccc}
I_R & \xrightarrow{\tilde{x}} & R \\
\downarrow d^0_R & & \downarrow d^R_1 \\
1 & \xrightarrow{x} & X
\end{array}
\]

In a pointed category \( \mathbb{E} \), taking the initial map \( 0_X : 0 \to X \) gives rise to a monomorphism \( \iota_R : I_R \hookrightarrow X \) which is normal to \( R \), where \( I_R \) means \( I^0_R \) and can be interpreted as the equivalence class of \( 0_X \). So, this construction produces a preorder mapping \( j_X : \text{Equ}_X \mathbb{E} \to \text{Mon}_X \mathbb{E} \) from the preorder of the equivalence relations on \( X \) to the preorder of subobjects of \( X \); this mapping preserves intersections. Starting with any map \( f : X \to Y \), we get \( I_R[f] = \text{Ker} f \), which says that any kernel map \( k_f \) is normal to the kernel equivalence relation \( R[f] \) of the map \( f \). Of course, in a non-pointed context, the notion of normal monomorphism holds, while the notion of kernel does not. Principle (3) above is a consequence of the fact [7] that in a protomodular category a monomorphism is normal to at most one equivalence relation (up to isomorphism). So that being normal, for a monomorphism \( u \), becomes a property in this kind of categories, as occurs in \( \text{Gp} \). This uniqueness property is equivalent to saying that the preorder homomorphism \( j_X : \text{Equ}_X \mathbb{E} \to \text{Mon}_X \mathbb{E} \) reflects inclusions; in this case, the preorder \( \text{Norm}_X \) of normal subobjects of \( X \) is just the image \( j_X(\text{Equ}_X) \subset \text{Mon}_X \).

2.3. Regular and exact context

Let us recall from [1] the following:

**Definition 2.4.** A category \( \mathbb{E} \) is \textit{regular} when it satisfies the first two conditions, and \textit{exact} when it satisfies all the three conditions:

(1) regular epimorphisms are stable under pullbacks;
(2) any kernel equivalence relation \( R[f] \) has a quotient \( q_f \);
(3) any equivalence relation \( R \) is a kernel equivalence relation.
Then, in the regular context, given any map \( f : X \to Y \), the following canonical factorization \( m \), where \( qf \) is the quotient of the kernel equivalence relation \( R[f] \) is necessarily a monomorphism:

\[
\begin{array}{c}
X \\
^qf \downarrow \\
\downarrow m \\
\downarrow \\
Y
\end{array}
\]

This produces a canonical decomposition of the map \( f \) into a monomorphism and a regular epimorphism which is stable under pullbacks. Now, given any regular epimorphism \( f : X \twoheadrightarrow Y \) and any subobject \( u : U \hookrightarrow X \), set \( f(U) \) the direct image \( f(u) : f(U) \hookrightarrow Y \) of \( u \) along the regular epimorphism \( f \) is given by \( f(U) = \text{Im}_f u \hookrightarrow Y \) and it is described by the following diagram:

\[
\begin{array}{c}
U \\
^qf_u \downarrow \\
\downarrow f \downarrow \\
\downarrow \\
\downarrow f(u) \\
\downarrow \\
X \\
f(U) \\
\end{array}
\]

Any variety in the sense of Universal Algebra is exact and regular epimorphisms coincide with surjective homomorphisms.

### 2.4. Homological categories

The significance of pointed protomodular categories grows up in the regular context since, in this context, the split short five lemma can be extended to any exact sequence. Furthermore, the 3 × 3 lemma, Noether isomorphisms and snake lemma hold; they are all collected in [3]. This is the reason why a regular pointed protomodular category \( \mathbb{E} \) is called homological. Finally, a pointed protomodular category is called semi-abelian [18] when it is exact and has binary sums (also called coproducts), the choice of the terminology being based upon the fact that a pointed category \( \mathbb{E} \) is abelian if and only if both \( \mathbb{E} \) and \( \mathbb{E}^{op} \) are semi-abelian. So, any pointed protomodular variety in the sense of Universal Algebra is a semi-abelian category. Accordingly, so are \( \text{DiGp} \) and \( \text{SKB} \).

In an exact pointed protomodular category, and a fortiori in a semi-abelian one, normal subobjects coincide with kernels. This does not occur in homological categories: take a topological abelian group \( (G, \cdot) \) and the continuous map \( d : G \times G \to G \) defined by \( d(x, y) = x \cdot y^{-1} \); then consider any subgroup \( n : N \hookrightarrow G \) and endow it with a topology which is strictly finer than the one induced by the topology on \( G \); the continuous inclusion \( n : N \hookrightarrow G \) cannot be a kernel map in the category \( \text{GpTop} \) of topological groups; however, in this homological category, it is normal to the upper equivalence relation produced by the following pullback in \( \text{GpTop} \):

\[
\begin{array}{c}
R \\
\delta \downarrow \\
\downarrow d \\
\downarrow \\
N \\
^n \longrightarrow G
\end{array}
\]

### 3. Protomodular aspects of skew braces

We noticed in the Introduction that the fully faithful functors \( U_i : \text{DiGp} \to \text{Gp}, i \in \{0, 1\} \) and \( \text{SKB} : \text{DiGp} \), and hence \( \text{SKB} \), a protomodular category. Accordingly, in these categories a monomorphism is normal to at most one internal equivalence relation, and, for the monomorphisms, to be normal becomes a property. “Normal in \( \text{Gp} \)” below will mean normal subgroup. From [8], we get the characterization of normal monomorphisms in \( \text{DiGp} \):
Proposition 3.1. A subobject \( i : (G,*,o) \hookrightarrow (K,*,o) \) is normal in the category \( \text{DiGp} \) if and only if the three following conditions hold:

(1) \( i : (G,*) \hookrightarrow (K,*) \) is normal in \( \text{Gp} \),
(2) \( i : (G,o) \hookrightarrow (K,o) \) is normal in \( \text{Gp} \),
(3) for all \( (x,y) \in K \times K \), \( x^{-*} \ast y \in G \) if and only if \( x^{-o} \circ y \in G \).

3.1. First properties of skew braces

The following observation is very important:

Proposition 3.2. Let \( (G,*,o) \) be any skew brace. The mapping \( \lambda : (G,o) \rightarrow \text{Aut}(G,*) \) given by \( \lambda : a \mapsto \lambda_a \), where \( \lambda_a(u) = a^{-*} \ast (a \circ u) \), is a well-defined group homomorphism and this condition is equivalent to (B). Moreover, we have

\[
\lambda_{a^{-o}}(u) = a^{-o} \circ (a \ast u). \tag{1}
\]

Proof. For the first sentence, see [16]. For the moreover part, by (B) we have \( a^{-o} \circ (a \ast u) = (a^{-o} \circ a) \ast (a^{-o})^{-*} \ast (a^{-o} \circ u) \), and this implies that \( a^{-o} \circ (a \ast u) = (a^{-o})^{-*} \ast (a^{-o} \circ u) = \lambda_{a^{-o}}(u) \). \( \square \)

The following proposition is straightforward:

Proposition 3.3. \( \text{SKB} \) is a Birkhoff subcategory of \( \text{DiGp} \).

This means that any subobject of a skew brace in \( \text{DiGp} \) is a skew brace and that, given any surjective homomorphism \( f : X \rightarrow Y \) in \( \text{DiGp} \), the digroup \( Y \) is a skew brace as soon as so is \( X \). In this way, any equivalence relation \( R \) in \( \text{DiGp} \) on a skew brace \( X \) actually lies in \( \text{SKB} \) since it determines a subobject \( R \subset X \times X \) in \( \text{DiGp} \) and, moreover, its quotient in \( \text{SKB} \) is its quotient in \( \text{DiGp} \). The first part of this last sentence implies that any normal subobject \( u : U \hookrightarrow X \) in \( \text{DiGp} \) with \( X \in \text{SKB} \) is normal in \( \text{SKB} \).

We are now going to show that the normal subobjects in \( \text{SKB} \) coincide with the ideals of [16].

Proposition 3.4. A subobject \( i : (G,*,o) \hookrightarrow (K,*,o) \) is normal in the category \( \text{SKB} \) if and only if the three following conditions hold:

(1) \( i : (G,*) \hookrightarrow (K,*) \) is normal in \( \text{Gp} \),
(2) \( i : (G,o) \hookrightarrow (K,o) \) is normal in \( \text{Gp} \),
(3') \( \lambda_x(G) = G \) for all \( x \in K \).

Proof. Observe that (1) and (2) are the same as those of Proposition 3.1, so it suffices to show that (3) \( \iff \) (3'), with (3) given in Proposition 3.1.

Let \( x,y \in K \). Suppose that \( \lambda_x(G) \subset G \). If \( x^{-o} \circ y \in G \), then \( y = x \circ u \) for some \( u \in G \) and hence \( x^{-*} \ast y = x^{-*} \ast (x \circ u) = \lambda_x(u) \in G \). Viceversa, if \( x^{-*} \ast y \in G \), then \( y = x \ast v \) for some \( v \in G \) and hence \( x^{-o}y = x^{-o} \circ (x \ast v) = \lambda_{x^{-o}}(x) \in G \).

For the converse, suppose that for all \( (x,y) \in K \times K \), \( x^{-*} \ast y \in G \) if and only if \( x^{-o} \circ y \in G \). Given any \( u \in G \) and \( x \in K \), we can write \( u = x^{-*} \ast (x \ast u) \) and, by construction, \( u = x^{-o} \circ (x \circ u) \) belongs to \( G \) and hence \( x^{-*} \ast (x \circ u) \in G \).

Finally \( \lambda_x(G) \subset G \) for all \( x \in K \) is equivalent to \( \lambda_x(G) = G \) for all \( x \in K \). \( \square \)

Corollary 3.5. A subobject \( i : (G,*,o) \hookrightarrow (K,*,o) \) is normal in the category \( \text{SKB} \) if and only if it is an ideal in the sense of [16], namely is such that:

(1) \( i : (G,o) \hookrightarrow (K,o) \) is normal,
(2) \( G \ast a = a \ast G \) for all \( a \in K \),
(3) \( \lambda_a(G) \subset G \) for all \( a \in K \).
Proof. Straightforward.

We remark that for any ideal \( I \) of a skew brace \( G \), we have that \( x * I = x \circ I \) for every \( x \in G \), as Lemma 2.3 of [16] shows.

### 3.2. Internal skew braces

Given any category \( E \), the notion of internal group, digroup and skew brace is straightforward, determining the categories \( \text{Gp} E \), \( \text{DiGp} E \) and \( \text{SKB} E \). Since \( \text{Gp} E \) is protomodular, so are the two others. An important case is produced with \( E = \text{Top} \) the category of topological spaces. Although \( \text{Top} \) is not a regular category, so is the category \( \text{Gp} \text{Top} \), the regular epimorphisms being the open surjective homomorphisms. So \( \text{Gp} \text{Top} \) is homological but not semi-abelian.

Now let \( f : X \to Y \) be any map in \( \text{DiGp} \text{Top} \). Let us show that \( R[f] \) has a quotient in \( \text{DiGp} \text{Top} \). Take its quotient \( q_{R[f]} : X \to Q_{f} \) in \( \text{DiGp} \), then endow \( Q_{f} \) with the quotient topology with respect to \( R[f] \); then \( q_{R[f]} \) is an open surjective homomorphism since so is \( U_{0}(q_{R[f]}) \). Accordingly, a regular epimorphism in \( \text{DiGp} \text{Top} \) is again an open surjective homomorphism. Moreover this same functor \( U_{0} : \text{DiGp} \text{Top} \to \text{Gp} \text{Top} \) being left exact and reflecting the homeomorphic isomorphisms, it reflects the regular epimorphisms; so, these regular epimorphisms in \( \text{DiGp} \text{Top} \) are stable under pullbacks. Accordingly the category \( \text{DiGp} \text{Top} \) is regular. Similarly the category \( \text{SKB} \text{Top} \) is homological as well, without being semi-abelian. As any category of topological semi-abelian algebras, both \( \text{DiGp} \text{Top} \) and \( \text{SKB} \text{Top} \) are finitely cocomplete, see [4].

### 4. Skew braces and their commutators

#### 4.1. The Huq commutation and the Smith commutation

##### 4.1.1. Commutative pairs of subobjects

Given any pointed category \( E \), the protomodular axiom applies to the following specific downward pullback:

\[
\begin{array}{c}
X \\
\downarrow^{0_{X}} \\
U \end{array} \quad \begin{array}{c}
X \times Y \\
\downarrow^{p_{Y}} \\
0 \end{array} \quad \begin{array}{c}
Y \\
\downarrow^{0_{Y}} \\
V
\end{array}
\]

where the monomorphisms are the canonical inclusions. This restriction is the definition of a unital category [6]. In this kind of categories there is an intrinsic notion of commutative pair of subobjects:

**Definition 4.1.** Let \( E \) be a unital category. Given a pair \((u, v)\) of subobjects of \( X \), we say that the subobjects \( u \) and \( v \) cooperate (or commute) when there is a (necessarily unique) map \( \varphi \), called the cooperator of the pair \((u, v)\), making the following diagram commute:

\[
\begin{array}{ccc}
U \times V & \xrightarrow{\varphi} & X \\
\downarrow^{l_{U}} & & \downarrow^{u} \\
U & \xrightarrow{u} & X \\
\downarrow^{r_{U}} & & \downarrow^{v} \\
V & \xrightarrow{v} & X
\end{array}
\]

We denote this situation by \( [u, v] = 0 \) (or \( [U, V] = 0 \) when there is no ambiguity, in particular in a concrete setting) and we call it Huq commutation of \( u \) and \( v \) (see [17]). A subobject \( u : U \to Y \) is central when \( [u, 1_X] = 0 \). An object \( X \) is commutative when \( [1_X, 1_X] = 0 \).
Clearly \([1_X, 1_X] = 0\) gives \(X\) a structure of internal unitary magma, which, \(\mathcal{E}\) being unital, is necessarily underlying an internal commutative monoid structure. When \(\mathcal{E}\) is protomodular, this is actually an internal abelian group structure, so that we call \(X\) an abelian object [7]. This gives rise to a fully faithful subcategory \(\text{Ab}(\mathcal{E}) \hookrightarrow \mathcal{E}\), which is additive and stable under finite limits in \(\mathcal{E}\). From that we can derive:

**Proposition 4.2.** [7] A pointed protomodular category \(\mathcal{E}\) is additive if and only if any monomorphism is normal.

### 4.1.2. Connected pairs of equivalence relations

Since a protomodular category is necessarily a Mal’tsev one, we can transfer to it the following notions. Given any pair \((R, S)\) of equivalence relations on the object \(X\) in \(\mathcal{E}\), take the following rightward and downward pullback:

\[
\begin{array}{ccc}
R \times_X S & \xrightarrow{p_S} & S \\
\downarrow r_S & & \downarrow s_0^S \\
R & \xleftarrow{r_S} & X \\
\end{array}
\]

where \(l_R\) and \(r_S\) are the sections induced by the maps \(s_R^0\) and \(s_S^0\). Let us recall the following definition from [10]:

**Definition 4.3.** In a Mal’tsev category \(\mathcal{E}\), the pair \((R, S)\) of equivalence relations is said to be *connected* when there is a (necessarily unique) morphism

\[
p : R \rightarrow X, \ xRySz \mapsto p(xRySz)
\]

such that \(pr_S = d_1^S\) and \(pl_R = d_1^R\), namely such that the following identities hold: \(p(xRyS) = x\) and \(p(yRyS) = z\). This morphism \(p\) is then called the *connector* of the pair, and we denote the situation by \([R, S] = 0\) and we call it the Smith commutation of \(R\) and \(S\) (see [24]).

### 4.2. The (Huq-Smith) condition

From [11], let us recall that:

**Lemma 4.4.** Let \(\mathcal{E}\) be a Mal’tsev category, \(f : X \rightarrow Y\) any map, \((R, S)\) any pair of equivalence relations on \(X\), \((\tilde{R}, \tilde{S})\) any pair of equivalence relations on \(Y\) such that \(R \subset f^{-1}(\tilde{R})\) and \(S \subset f^{-1}(\tilde{S})\). Suppose moreover that \([R, S] = 0\) and \([\tilde{R}, \tilde{S}] = 0\). Then the following diagram necessarily commutes:

\[
\begin{array}{ccc}
R \times_X S & \xrightarrow{\tilde{f}} & \tilde{R} \times_Y \tilde{S} \\
\downarrow p(R, S) & & \downarrow p(\tilde{R}, \tilde{S}) \\
X & \xrightarrow{f} & Y
\end{array}
\]

where \(\tilde{f}\) is the natural factorization induced by \(f^{-1}(\tilde{R})\) and \(S \subset f^{-1}(\tilde{S})\).

A pointed Mal’tsev category is necessarily unital. From [10], in any pointed Mal’tsev category \(\mathcal{E}\), we have necessarily

\[
[R, S] = 0 \Rightarrow [I_R, I_S] = 0 \quad (2)
\]
In this way, the Smith commutation of two equivalence relations \( R \) and \( S \) implies the Huq commutation of the corresponding normal subobjects \( I_R \) and \( I_S \).

In general, the Huq commutation does not imply the Smith commutation, even if \( E \) is pointed protomodular, see Proposition 4.6. The category \( E \) is said to satisfy the \((Huq=Smith)\) condition when the Huq commutation implies the Smith commutation, see also [22]. Any pointed strongly protomodular category satisfies the \((Huq=Smith)\) condition, see [10]. The \((Huq=Smith)\) condition is true for \( Gp \) by the following straightforward:

**Proposition 4.5.** Let \((R, S)\) be a pair of equivalence relations in \( Gp \) on the group \((G, \ast)\). The following conditions are equivalent:

1. \([I_R, I_S] = 0\);
2. \(p(x, y, z) = x \ast y^{-1} \ast z \) defines a group homomorphism \( p : R \times_G S \rightarrow G; \)
3. \([R, S] = 0\).

The following result could be easily derived from a counterexample given by the first author with the contribution of G. Janelidze in [8] (they built a digroup homomorphism \( \pi_2 \) with \([\ker \pi_2, \ker \pi_2] = 0 \) and \([R[\pi_2], R[\pi_2]] \neq 0\), in order to prove that \( DiGp \) is not strongly protomodular), but it was not explicitly stated. We make now the result explicit and we refresh the proof:

**Proposition 4.6.** The category \( DiGp \) of digroups does not satisfy the \((Huq=Smith)\) condition.

**Proof.** Let \((A, +)\) be an abelian group and \( a \in A \) an object such that \(-a \neq a\). Then define \( \theta : A \times A \rightarrow A \times A \) as the involutive bijection which leaves fixed any object \((x, y)\) except \((a, a)\) which is exchanged with \((-a, a)\). Then define the group structure \((A \times A, \circ)\) on \( A \times A \) as the transformed along \( \theta \) of \((A \times A, +)\).

For any \((x, z) \circ (x', z') = \theta((x, z) + \theta(x', z'))\)

Clearly we have \((a, a)^{-\circ} = (a, -a)\). Since the second projection \( \pi_2 : A \times A \rightarrow A \) is such that \( \pi_2 \theta = \pi_2 \), we get a digroup homomorphism \( \pi_2 : (A \times A, +, \circ) \rightarrow (A, +, +) \) whose kernel map is, up to isomorphism, \( \iota_A : (A, +) \rightarrow (A \times A, +, \circ) \) defined by \( \iota_A(x) = (x, 0) \) for any \( x \in A \). The commutativity of the law \( \ast \) makes \( \iota_A, \iota_A \) = 0 inside \( DiGp \).

We are going to show that, however we do not have \([R[\pi_2], R[\pi_2]] = 0\). If it was the case, according to the previous proposition and considering the images by \( U_0 \) and \( U_1 \) of the desired ternary operation, we should have, for any triple \((x, y)R[\pi_2](x', y)R[\pi_2](x'', y)\):

\[
(x, y) - (x', y) + (x'', y) = (x, y) \circ (x', y)^{-\circ} \circ (x'', y)
\]

namely \((x, y) \circ (x', y)^{-\circ} \circ (x'', y) = (x - x' + x'', y)\).

Now take \( y = a = x' \) and \( a \neq x \neq -a \). Then we get:

\[
(x, a) \circ (a, a)^{-\circ} \circ (x'', a) = (x, a) \circ (a, -a) \circ (x'', a) = (x + a, 0) \circ (x'', a).
\]

If moreover \( a \neq x'' \neq -a \), then

\[
(x, a) \circ (a, a)^{-\circ} \circ (x'', a) = (x + a + x'', a).
\]

Now, clearly we get \( x + a + x'' \neq x - a + x'' \) since \( a \neq -a \).

**Proposition 4.7.** Given any pair \((U, V)\) of subobjects of \( X \) in \( SKB \), the following conditions are equivalent:

1. \([U, V] = 0\);
2. for all \((u, v) \in U \times V\), we get \( u \circ v = u \ast v, u \circ v = v \circ u \) and \( u \ast v = v \ast u \);
3. for all \((u, v) \in U \times V\), \( \lambda_u(v) = v, [U_0(U), U_0(V)] = 0 \) and \([U_1(U), U_1(V)] = 0 \) where \( U_i : SKB \rightarrow Gp \) are the two forgetful functors.
Proposition 4.8 (SKB does satisfy the (Huq-Smith) condition). Let $R$ and $S$ be two equivalence relations on an object $X \in \text{SKB}$. The following conditions are equivalent:

1. $[I_R, I_S] = 0$;
2. $[U_0(I_R), U_0(I_S)] = 0$, $[U_1(I_R), U_1(I_S)] = 0$ and $x \ast y^{-*} \ast z = x \circ y^{-\circ} \circ z$ for all $x R y S z$;
3. $[R, S] = 0$.

Proof. The identity $x \ast y^{-*} \ast z = x \circ y^{-\circ} \circ z$ is equivalent to

$$y^{-\circ} \circ z = x^{-\circ} \circ (x \ast y^{-*} \ast z) = (x^{-\circ} \circ x) \ast (x^{-\circ} \ast (x^{-\circ} \circ (y^{-*} \ast z))) = (x^{-\circ})^{-*} \ast (x^{-\circ} \circ (y^{-*} \ast z)),$$

which, in turn, is equivalent to

$$\lambda_{x^{-\circ}}(y^{-*} \ast z) = y^{-\circ} \circ z.$$

Suppose $x R y S z$. Setting $z = y \ast v, v \in I_S$, this is equivalent to $\lambda_{x^{-\circ}}(v) = y^{-\circ} \circ (y \ast v) = \lambda_{y^{-\circ}}(v)$ by Proposition 3.2. Moreover, Proposition 3.2 implies that $\lambda : (X, \circ) \to \text{Aut}(X, \ast)$ is a group homomorphism, hence $\lambda_{x^{-\circ}}(v) = \lambda_{y^{-\circ}}(v)$ is equivalent to $\lambda_y(\lambda_{x^{-\circ}}(v)) = \lambda_{y \circ x^{-\circ}}(v) = v, v \in I_S$. Setting $y = u \circ x, u \in I_R$, this is equivalent to $\lambda_{y^{-\circ}}(v) = v, (u, v) \in I_R \times I_S$.

Now, by Proposition 4.7, $[I_R, I_S] = 0$ is equivalent to: for all $(u, v) \in I_R \times I_S$, we get $\lambda_{y^{-\circ}}(v) = v,$ $[U_0(I_R), U_0(I_S)] = 0$ and $[U_1(I_R), U_1(I_S)] = 0$. So we get $[(1) \iff (2)]$.

Suppose (2). From $[U_0(I_R), U_0(I_S)] = 0$, we know by Proposition 4.5 that $p(x, y, z) = x \ast y^{-*} \ast z$ is a group homomorphism $(R \backslash X S, \ast) \to (X, \ast)$, and from $[U_1(I_R), U_1(I_S)] = 0$ that $q(x, y, z) = x \circ y^{-\circ} \circ z$ is a group homomorphism $(R \backslash X S, \circ) \to (X, \circ)$. If $p = q$, this produces the desired $R \backslash X S \to X$ in SKB showing that $[R, S] = 0$. Whence $[(2) \implies (3)]$. We have already noticed that the last implication $[(3) \implies (1)]$ holds in any Mal’tsev category, therefore the proof is concluded.

We are now going to show that any category $\text{SKB} E$ satisfies this condition as well. For that we need the following observations:

Proposition 4.9. When a pointed Mal’sev category $E$ satisfies the (Huq-Smith) condition, so does any functor category $F(C, E)$.

Proof. Let $(R, S)$ be a pair of equivalence relations on an object $F \in F(C, E)$. We have $[R, S] = 0$ if and only if for each object $C \in C$ we have $[R(C), S(C)] = 0$ since, by Lemma 4.4, the naturality follows. In the same way, if $(u, v)$ is a pair of subfunctors of $F$, we have $[u, v] = 0$ if and only if for each object $C \in C$ we have $[u(C), v(C)] = 0$. Suppose now that $E$ satisfies the (Huq-Smith) condition, and that $[I_R, I_S] = 0$. So, for each object $C \in C$ we have $[I_R(C), I_S(C)] = 0$, which implies $[R(C), S(C)] = 0$. Accordingly, we have $[R, S] = 0$.

Now let $T$ be any finitary algebraic theory, and denote by $T(E)$ the category of internal $T$-algebras in $E$. Let us recall that, given any variety of algebras $V(T)$, we have a Yoneda embedding for the internal $T$-algebras, namely a left exact fully faithful factorization of the Yoneda embedding for $E$.
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where $\mathcal{U} : \mathbb{V}(\mathbb{T}) \to \text{Set}$ is the canonical forgetful functor.

**Theorem 4.10.** Let $\mathbb{T}$ be any finitary algebraic theory such that the associated variety of algebras $\mathbb{V}(\mathbb{T})$ is pointed protomodular. If $\mathbb{V}(\mathbb{T})$ satisfies the (Huq=Smith) condition, so does any category $\mathbb{T}(\mathbb{E})$.

**Proof.** If $\mathbb{V}(\mathbb{T})$ satisfies the (Huq=Smith) condition, so does $\mathcal{F}(\mathbb{E}^{\text{op}}, \mathbb{V}(\mathbb{T}))$ by the previous proposition. Accordingly, $\bar{\psi}_{\mathbb{T}}$ being left exact and fully faithful, so does $\mathbb{T}(\mathbb{E})$. \qed

According to Theorem 4.10, we get the following:

**Corollary 4.11.** Given any category $\mathbb{E}$, the category $\text{SKB}(\mathbb{E})$ satisfies the (Huq=Smith) condition. This is the case in particular for the category $\text{SKB}(\mathbb{Top})$ of topological skew braces.

### 4.3. Abstract Huq and Smith commutators

It remains to make explicit the relationship between commutation and commutator, and to show how the (Smith=Huq) condition strengthens it.

#### 4.3.1. Abstract Huq commutator

Suppose now that $\mathbb{E}$ is any finitely cocomplete regular unital category. In this setting, we gave in [12], for any pair $u : U \to X$, $v : V \to X$ of subobjects, the construction of a regular epimorphism $\psi_{(u,v)}$ which universally makes their direct images cooperate. Indeed consider the following diagram, where $Q[u,v]$ is the limit of the plain arrows:

$$
\begin{array}{ccc}
U & \longrightarrow & \mathbb{X} \\
\downarrow \psi_{(u,v)} & & \downarrow \\
Q[u,v] & \longrightarrow & \mathbb{X} \\
\downarrow \psi_{(u,v)} & & \downarrow \\
V & \longrightarrow & \mathbb{X}
\end{array}
$$

The map $\psi_{(u,v)}$ is necessarily a regular epimorphism and the map $\bar{\psi}_{(u,v)}$ induces the cooperator of the direct images of the pair $(u, v)$ along the regular epimorphism $\psi_{(u,v)}$. So, this regular epimorphism $\psi_{(u,v)}$ measures the lack of cooperation of the pair $(u, v)$ in the sense that the map $\psi_{(u,v)}$ is an isomorphism if and only if $[u, v] = 0$. We then get a symmetric tensor product: $\iota_{R[\psi_{(-,-)}]} : \text{Mon}_X \times \text{Mon}_X \to \text{Mon}_X$ of preordered sets.

Since the map $\psi_{(u,v)}$ is a regular epimorphism, its distance from being an isomorphism is its distance from being a monomorphism, which is measured by the kernel equivalence relation $R[\psi_{(u,v)}]$. Accordingly, in the homological context (we need protomodularity to make the triviality of $R[\psi_{(u,v)}]$ asserted by the triviality of $\text{Ker}\psi_{(u,v)}$), it is meaningful to introduce the following definition, see also [21]:

**Definition 4.12.** Given any finitely cocomplete homological category $\mathbb{E}$ and any pair $(u, v)$ of subobjects of $X$, their abstract Huq commutator $[u, v]$ is defined as $\iota_{R[\psi_{(u,v)}]}$ or equivalently as the kernel $\text{Ker}\psi_{(u,v)}$.

By this universal definition, in the concrete category $\text{Gp}$, this Huq commutator $[u, v]$ coincides with the usual commutator $[U, V]$.
4.3.2. Abstract Smith commutator

Suppose \( E \) is a regular category. Then, given any regular epimorphism \( f : X \rightarrow Y \) and any equivalence relation \( R \) on \( X \), the direct image \( f(R) \rightarrow Y \times Y \) of \( R \rightarrow X \times X \) along the regular epimorphism \( f \times f : X \times X \rightarrow Y \times Y \) is reflexive and symmetric, but generally not transitive. Now, when \( E \) is a regular Mal’tsev category, this direct image \( f(R) \), being a reflexive relation, is an equivalence relation.

Suppose moreover that \( E \) is finitely cocomplete. Let \( (R, S) \) be a pair of equivalence relations on \( X \), and consider the following diagram, where \( Q[R, S] \) is the colimit of the plain arrows:

\[
\begin{array}{c}
\xymatrix{
R \ar[r]^{d_0} & Q[R, S] \ar[d] \ar[r] & X \\
\ar[r]_{d_1} & X(R,S) \ar@{.>}[u] & \\
S \ar[u]_{R(S)} & \\
\end{array}
\]

Notice that, here, in consideration of the pullback defining \( R \times_S X \), the role of the projections \( d_0 \) and \( d_1 \) have been interchanged. This map \( X(R,S) \) measures the lack of connection between \( R \) and \( S \), see [12]:

**Theorem 4.13.** Let \( E \) be a finitely cocomplete regular Mal’tsev category. Then the map \( X(R,S) \) is a regular epimorphism and is the universal one which makes the direct images \( X(R,S)(R) \) and \( X(R,S)(S) \) connected. The equivalence relations \( R \) and \( S \) are connected (i.e., \( [R, S] = 0 \)) if and only if \( X(R,S) \) is an isomorphism.

Since the map \( X(R,S) \) is a regular epimorphism, its distance from being an isomorphism is its distance from being a monomorphism, which is exactly measured by its kernel equivalence relation \( R[X(R,S)] \). Accordingly, we give the following definition:

**Definition 4.14.** Let \( E \) be any finitely cocomplete regular Mal’tsev category. Given any pair \( (R, S) \) of equivalence relations on \( X \), their abstract Smith commutator \( [R, S] \) is defined as the kernel equivalence relation \( R[X(R,S)] \) of the map \( X(R,S) \).

In this way, we define a symmetric tensor product \([-,-] = R[X(-,-)] : \text{Equ}_X \times \text{Equ}_X \rightarrow \text{Equ}_X \) of preordered sets. It is clear that, with this definition, we get \( [R, S] = 0 \) in the sense of connected pairs if and only if \( [R, S] = \Delta_X \) (the identity equivalence relation on \( X \)) in the sense of this new definition. This is coherent since \( \Delta_X \) is actually the 0 of the preorder \( \text{Equ}_X \).

We are now going to show how, in the homological context, the Huq and the Smith commutator are related when the (Huq=Smith) condition holds. For that, let us recall the following [9]:

**Proposition 4.15.** Let \( E \) be a pointed regular Mal’tsev category. Let \( f : X \rightarrow Y \) be a regular epimorphism and \( R \) an equivalence relation on \( X \). Then the direct image \( f(I_R) \) of the normal subobject \( I_R \) along \( f \) is \( I_{f(R)} \).

From that, we can assert the following:

**Proposition 4.16.** Let \( E \) be a finitely cocomplete homological category. Given any pair \( (R, S) \) of equivalence relations on \( X \), we have \( [I_R, I_S] \subset I_{[R,S]} \).

**Proof.** From (2), we get

\[
[X(R,S)(R), X(R,S)(S)] = 0 \Rightarrow [I_{X(R,S)(R)}, I_{X(R,S)(S)}] = 0
\]

By the previous proposition we have:

\[
0 = [I_{X(R,S)(R)}, I_{X(R,S)(S)}] = [X(R,S)(I_R), X(R,S)(I_S)]
\]
Accordingly, by the universal property of the regular epimorphism $\psi_{(I_R,J_S)}$ we get a factorization:

\[
\begin{array}{ccc}
X & \xrightarrow{\psi_{(I_R,J_S)}} & Q[I_R, I_S] \\
\downarrow X(R,S) & & \downarrow Q[R, S] \\
Q[R, S] & & \\
\end{array}
\]

which shows that $[I_R, I_S] \subset I_{[R,S]}$. \qedhere

**Theorem 4.17.** In a finitely cocomplete homological category $\mathbb{E}$ the following conditions are equivalent:

(1) $\mathbb{E}$ satisfies the (Huq=Smith) condition;
(2) $[I_R, I_S] = I_{[R,S]}$ for any pair $(R, S)$ of equivalence relations on $X$.

Under any of these conditions, the regular epimorphisms $\chi_{(R,S)}$ and $\psi_{(I_R,J_S)}$ do coincide.

**Proof.** Suppose 2). Then $[I_R, I_S] = 0$ means that $\psi_{(I_R,J_S)}$ is an isomorphism, so that $0 = [I_R, I_S] = I_{[R,S]}$. In a homological category $I_{[R,S]} = 0$ is equivalent to $[R, S] = 0$. Conversely suppose 1). We have to find a factorization:

\[
\begin{array}{ccc}
X & \xrightarrow{\psi_{(I_R,J_S)}} & Q[I_R, I_S] \\
\downarrow X(R,S) & & \downarrow Q[R, S] \\
Q[R, S] & & \\
\end{array}
\]

namely to show that $[\psi_{(I_R,J_S)}(R), \psi_{(I_R,J_S)}(S)] = 0$. By 1) this equivalent to $0 = [I_{\psi_{(I_R,J_S)}}(R), I_{\psi_{(I_R,J_S)}}(S)]$, namely to $0 = [\psi_{(I_R,J_S)}(I_R), \psi_{(I_R,J_S)}(I_S)]$ by **Proposition 4.15**. This is true by the universal property of the regular epimorphism $\psi_{(I_R,J_S)}$. \qedhere

### 4.4. Explicit description of the commutator in SKB

The categories SKB and SKBTOP are finitely cocomplete homological categories. Thanks to the previous theorem and to the (Huq=Smith) condition, the two notions of commutator determine each other. So, it remains now to make explicit the description of the Huq commutator (from which the description of Smith commutator will follow). The Huq commutator behind defined by means of the kernel of a map (Definition 4.12), the topology on the commutator $[I, J]$ in the category SKBTOP is the topology induced by the topological skew brace $(A, \ast, \circ)$ in question.

Recall that, by **Corollary 3.5**, ideals are normal subobjects in the category SKB.

We will determine a set of generators for the Huq commutator of two ideals in a skew brace.

**Proposition 4.18.** If $I$ and $J$ are two ideals of a left skew brace $(A, \ast, \circ)$, their Huq commutator $[I, J]$ is the ideal of $A$ generated by the union of the following three sets:

1. the set $\{ i \circ j \circ (j \circ i)^{-\circ} \mid i \in I, \ j \in J \}$, (which generates the commutator $[I, J]_{(A,\circ)}$ of the normal subgroups $I$ and $J$ of the group $(A,\circ)$);
2. the set $\{ i \ast j \ast (j \ast i)^{-\ast} \mid i \in I, \ j \in J \}$, (which generates the commutator $[I, J]_{(A,\ast)}$ of the normal subgroups $I$ and $J$ of the group $(A,\ast)$); and
3. the set $\{ (i \circ j) \ast (i \ast j)^{-\ast} \mid i \in I, \ j \in J \}$.

**Proof.** Assume that the mapping $\mu: I \times J \to A/K, \mu(i, j) = i \ast j \ast K$ is a skew brace morphism for some ideal $K$ of $A$. Then
of I

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This proves that the set (1) is contained in $K$.

Similarly,

$$(i \circ j) \circ K = (i \circ K) \circ (j \circ K) = (i \ast K) \circ (j \ast K) =
\mu((i, 1) \ast (1, j)) = \mu(i, j) =
\mu((i, 1) \ast (1, j)) = \mu(i, j) =
\mu((1, j) \ast (i, 1)) = \mu((1, j) \ast (1, j)) =
\mu((j \ast K) \circ (i \circ K)) = (j \ast K) \circ (i \circ K) = (j \circ i) \circ K.$$
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