Irregular Set Coloring of Certain Graphs

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Abstract. We determine irregular set chromatic number of comb, friendship, K-ary tree, n-sunlet, coconut tree and jelly fish graph.

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1. Introduction
Let G be a connected graph and let c:V(G)→\{1,2,3,...,k\} be a proper coloring of the vertices of G for some positive integer k. The color code of a vertex v of G (with respect to c) is the ordered (k+1)-tuple code\(c(v) = (a_0, a_1, a_2, ..., a_k)\), where \(a_0\) is the color assigned to v and for 1≤ j≤ k, \(a_j\) is the number of the vertices of G adjacent to that are colored i. The coloring c is irregular if distinct vertices have distinct color codes and the irregular chromatic number \(\chi_{irr}(G)\) of G is the minimum positive integer k for which G has an irregular k-coloring. Irregular coloring was introduced by Mary Radcliffe [2] and irregular set coloring was introduced by Roushini et.al [3]. The concept of k-ary, coconut tree, friendship and jelly fish are studied by Anitha et.al [1]. In this paper, We determine irregular set chromatic number of comb, friendship, K-ary tree, n-sunlet, coconut tree and jelly fish graph.

2. Preliminaries
In this section, we give some basic definitions relevant to this paper.

Definitions:1 A proper coloring c: V(G)→\{1,2,3,...,t\} is said to be irregular set coloring if the set color code of V(G) is \(\{c_0, c_1, c_2, ..., c_k\}\), where \(c_0\) is the color assigned to v and for 1≤ j≤ k, \(c_j\) is the number of vertices of G adjacent to v that are colored j. The length of each set color code is (t+1)-tuple which occurs in any order at most once in G and the irregular set chromatic number is denoted by \(\chi_{irs}(G)\)

Definitions:2 Comb graph \(P_n \square K_1\) is obtained by joining a single pendant edge to each vertex of a path.

Definitions:3 The friendship graph \(F_n\) is a set of n triangles having a common central vertex and V\((F_n) = \{v_1, v_2, ..., v_{2n+1}\}\) with \(v_1\) as the central vertex and E\((F_n) = \{v_1v_1/ 2\leq i \leq 2n+1\}\)U\(\{v_2v_{2i+1}/ 1\leq i \leq n\}\) respectively.

 Definitions:4 A graph G is called an K-ary tree if G is a rooted tree such that the root has degree k and all the other vertices have degree k+1.

Definitions:5 The n-sunlet graph \(S_n\) is obtained by attaching n pendant edges to a cycle \(C_n\).

Definitions:6 A coconut tree CT(m,n) is the graph obtained from the path \(P_m\) by appending a new pendant n edges at an end vertex of \(P_m\).
Definitions: The Jelly fish graph $J(m,n)$ is obtained from a 4-cycle $v_1,v_2,v_3,v_4$ by joining $v_1$ and $v_3$ with an edge and appending $m$ pendent edges to $v_2$ and $n$ pendent edges to $v_4$.

3. Main Results

Theorem 1
For any comb graph $P_n \boxtimes K_1$, $\chi_{irs}(P_n \boxtimes K_1) = n$.

Proof:
Let $G=P_n \boxtimes K_1$ be a comb graph and $v_1,v_2,\ldots,v_n$ be the vertices of $P_n$ and $u_1,u_2,\ldots,u_n$ be the vertices of $nK_1$ and $|V(K_1)|=n$. Therefore $\chi_{irs}(P_n \boxtimes K_1) \geq n$, since the pendant vertices are not in same color.

Let $c$ be an irregular set coloring of $P_n \boxtimes K_1$. The irregular set coloring of $P_n \boxtimes K_1$ is given as follows: $c(u_i)=i$ for $1 \leq i \leq n$, $c(v_i)=i+1$ for $1 \leq i \leq n-1$ and $c(v_n)=n-1$. In the way of minimum irregular set coloring, $\chi_{irs}(P_n \boxtimes K_1) \leq n$ and all the color codes of the vertices are distinct. Hence $\chi_{irs}(P_n \boxtimes K_1) = n$.

Example 1
Irregular set coloring of comb graph with $n=6$ is shown in figure (i).

![Figure (i)](image1)

Theorem 2
For any friendship graph $F_n$, $\chi_{irs}(F_n) = 2n + 1$.

Proof:
Let $G=F_n$ be a friendship graph. Let the vertices of $F_n$ be $v_1,v_2,\ldots,v_{2n+1}$ with $v_1$ as common central vertex and $d(v_1)=2n$, $d(v_i)=2$ for $2 \leq i \leq 2n+1$. Let $c$ be an irregular set coloring of $F_n$. Let us assign $c(v_1)=1$, $c(v_i) \neq 1$ since $N(v_1)=v_i$ for $2 \leq i \leq 2n$ and $c(v_2) \neq c(v_{2n+1})$ since $N(v_2)=v_{2n+1}$ for $1 \leq i \leq n$.

Let $x$ and $y$ be two non adjacent vertices of degree two, but both vertices are adjacent to $v_1$. Suppose $N(x)=\{v_1,s\}$ and $N(y)=\{v_1,t\}$, where $s$ and $t$ are also non adjacent vertices. If $c(x)=c(y)=2$, then the code$(x)=\{2,1,s\}$ and code$(y)=\{2,1,t\}$. If $s=t$ or $s \neq t$, then the code$(x)=\text{code}(y)$. Hence a color can be assigned only one time in $F_n$ and $|\text{v}(F_n)|=2n+1$. Hence $\chi_{irs}(F_n) = 2n + 1$.

Example 2
Irregular set coloring of friendship graph with $n=4$ is shown in figure (ii).

![Figure (ii)](image2)
Theorem-3
For any K-ary graph, \(\chi_{irs}(K_a) = n^2\).
Proof:
Let \(K_a\) be a K-ary tree graph and \(\{v, v_i, u_j/ 1 \leq i \leq n, 1 \leq j \leq n^2\}\) be the vertices of \(K_a\) and \(d(v)=n, d(v_i)=n+1\) for \(1 \leq i \leq n\), \(d(u_j)=1\) for \(1 \leq j \leq n^2\). Let \(P\) be the pendant vertices in K-ary tree graph and \(|P|=n^2\). Therefore \(\chi_{irs}(K_a) \geq n^2\) since the pendant vertices are not in same color. Let \(c\) be an irregular set coloring of \(K_a\). The irregular set coloring of \(K_a\) is given as follows: \(c(u_j)=j\) for \(1 \leq j \leq n^2\), \(c(v_i)=ni+1\) for \(1 \leq i \leq n-1\), \(c(v_n)=1\) and \(c(v)=n^2\). In the way of minimum irregular set coloring, \(\chi_{irs}(K-ary\, tree) \leq n\) and all the color codes of the vertices are distinct. Hence \(\chi_{irs}(K-ary\, tree) = n^2\).

Example-3: Irregular set coloring of K-ary tree graph with \(n=3\) is shown in figure (iii)

![Figure (iii)](image)

Theorem-4
For \(n \geq 3\), \(\chi_{irs}(S_n) = n\).
Proof:
Let \(S_n\) be a n-sunlet graph. Let the pendant vertices of \(S_n\) are \(u_1, u_2, \ldots, u_n\) and \(v_1, v_2, \ldots, v_n\) be the vertices of the cycle \(C_n\) in sunlet graph. We know that the pendant vertices are not in same color. Therefore \(\chi_{irs}(S_n) \geq n\). The colors of the pendant vertices are as follows: \(c(u_i)=i\) for \(1 \leq i \leq n\), \(c(v_i)=i+1\) for \(1 \leq i \leq n-1\) and \(c(v_n)=1\). In the way of minimum irregular set coloring, the vertices of \(S_n\) are colored by \(\{1, 2, \ldots, n\}\) and all the color codes of the vertices are distinct. Hence \(\chi_{irs}(S_n) = n\).

Example-4: Irregular set coloring of sunlet graph with \(n=5\) is shown in figure (iv)

![Figure (iv)](image)
Theorem 5
For \( m \geq 3 \), \( \chi_{irs}(CT(m, m)) = m + 1 \).

Proof:
Let \( CT(m, m) \) be a coconut tree graph with \( V(CT(m, m)) = \{u_i, v_i/ 1 \leq i \leq m\} \). In \( CT(m, m) \), \( u_i, u_2, \ldots, u_m \) be the vertices of path and \( v_1, v_2, \ldots, v_m \) be the pendant vertices. Since the pendant vertices are not in same color, therefore \( \chi_{irs}(CT(m, m)) \geq m + 1 \). Let \( c \) be an irregular set coloring of \( CT(m, m) \). The irregular set coloring of the vertices of \( P_m \) are as follows: \( c(u_1) = c(u_m) = m + 1 \), \( c(u_{i+1}) = i \) for \( 1 \leq i \leq m-2 \). In the way of minimum irregular set coloring, the vertices of \( CT(m, m) \) are colored by \( \{1, 2, \ldots, m+1\} \) and all the color codes of the vertices are distinct. Hence \( \chi_{irs}(CT(m, m)) = m + 1 \).

Example 5 Irregular set coloring of coconut tree graph with \( n = 5 \) is shown in figure (v).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure_v.png}
\caption{Figure (v)}
\end{figure}

Theorem 6
For \( m, n \geq 1 \), \( \chi_{irs}(J(m, n)) = \begin{cases} 4 & \text{if } m + n < 4 \\ m + n & \text{if } m + n \geq 4 \end{cases} \)

Proof:
Let \( J(m, n) \) be a jelly fish graph with \( V(J(m, n)) = \{x, y, z, w, v_i, u_j/ 1 \leq i \leq m, 1 \leq j \leq n\} \) and \( E(J(m, n)) = \{xy, xz, xw, zw, yz, yv_i, wu_j/ 1 \leq i \leq m, 1 \leq j \leq n\} \). Define a coloring \( c: V \rightarrow N \).

Case (i) if \( m+n < 4 \):
Let \( c(x) = 1 \), \( c(y) = 2 \), since \( y \) is adjacent to \( x \) and \( c(z) = 3 \), since \( z \) is adjacent to \( x \) and \( y \). Suppose \( m+n = 2 \), the colors of the pendant vertices are as follows: \( c(v_1) = 1 \), \( c(u_1) = 2 \), since the pendant vertices are not in same color. Suppose \( m+n = 3 \), the pendant vertices of \( J(m, n) \) are \( \{v_1, v_2, u_1\} \) or \( \{u_1, u_2, v_1\} \). The possible color pattern of the pendant vertices are \( c(v_1) = 1 \), \( c(v_2) = 3 \), \( c(u_1) = 2 \) or \( c(u_1) = 1 \), \( c(u_2) = 2 \), \( c(v_1) = 3 \). The neighbouring colors of the remaining vertex \( w \) is \( 1, 2, \) and \( 3 \). Therefore \( c(w) = 4 \). Hence \( \chi_{irs}(J(m, n)) = 4 \).

Case (ii) if \( m+n \geq 4 \):
The pendant vertices of \( G \) are not in same color. Therefore \( \chi_{irs}(J(m, n)) \geq 4 \). Without loss of generality, \( c(x) = 1 \), \( c(y) = 2 \), \( c(z) = 3 \), \( c(w) = 4 \), \( c(v_1) = 1 \), \( c(u_1) = 2 \), \( c(v_{i+1}) = i+2 \) for \( 1 \leq i \leq m-1 \), \( c(u_{i+1}) = m+1+j \) for \( 1 \leq j \leq n-1 \). Therefore \( \chi_{irs}(J(m, n)) \leq 4 \). Hence \( \chi_{irs}(J(m, n)) = m+n \) for \( m+n \geq 4 \).

Example 6 Irregular set coloring of jelly fish graph with \( m,n = 3 \) is shown in figure (iv).
4. Conclusion

In this paper, we have determined the irregular set chromatic number of comb, friendship, K-ary tree, n-sunlet, coconut tree and jelly fish.

5. Reference

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