A correspondence between a class of cone structures and contact forms

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Abstract

In the spirit of Sullivan’s paper *Cycles for the Dynamical Study of Foliated Manifolds and Complex Manifolds*, existence of a contact structure on a closed manifold $M$ is shown to be equivalent to existence of an ample $S^1$-invariant cone structure with no nontrivial exact structure cycles on the manifold $S^1 \times M$.

Introduction

In [Sul76], Sullivan establishes, among other results, a correspondence between symplectic structures on a closed manifold and certain cone structures of bivectors. Such a structure is a continuous distribution of compact convex cones $(C_x)_{x \in M}$, each $C_x$ contained in $\Lambda^2 T_x M$. (The cone $C_x$ is compact when its quotient by the action of $\mathbb{R}_{>0}$ is compact.) To such a cone structure is associated a cone $\mathcal{C}$ in the space of 2-currents. It is defined to be the closed convex hull of the space of Dirac currents $\delta_P : \omega \mapsto \omega(P)$, for $P$ in the cone structure. When the manifold is closed, this cone is compact and plays an important role through the notion of positive differential forms, that is to say forms $\beta$ for which $c(\beta) > 0$ for all nontrivial elements $c$ in $\mathcal{C}$.

A symplectic structure $\omega$ induces a cone structure through the choice of a compatible almost complex structure $J$. It is ample and $\mathcal{C}$ does not contain any nontrivial exact current. Conversely, any ample cone structure $\mathcal{C}$ on $M$ for which $\mathcal{C}$ does not contain any nontrivial exact current admits a contractible collection of positive forms that are symplectic. The proof uses the Hahn Banach separation theorem, as well as the duality between forms and currents.
The purpose of this paper, whose motivation is explained in the next paragraph, is to propose a correspondence between contact structures and certain cone structures. Our idea is to apply Sullivan’s correspondence to the symplectization of a contact manifold. There are a number of issues to be dealt with. One is that the usual symplectization yields an open manifold: $\mathbb{R} \times N$, while the assumption that $M$ is closed in Sullivan’s correspondence is necessary since it implies that $\mathcal{C}$ is compact, whence that Hahn Banach may be applied. Another issue is that the symplectic structures we recover on $\mathbb{R} \times N$ may not yield contact structures on $N$. Indeed, only $\mathbb{R}$-equivariant symplectic structures on $\mathbb{R} \times N$ naturally induce contact structures on $N$. These two problems can be bypassed by considering another version of the symplectization process that builds from a contact form on $N$ an $S^1$-invariant nondegenerate 2-form on $S^1 \times N$ that is closed for a twisted differential $D\beta = ds \wedge \beta + d\beta$. Such forms are called hereafter $S^1$-invariant $D$-symplectic forms. Now such a form yields an $S^1$-invariant cone structure through the choice of a compatible $S^1$-invariant almost complex structure. Conversely, an invariant version of the Hahn Banach separation theorem, applied to the closed subspace of $D^*$-exact currents and to an invariant basis for $\mathcal{C}$, yields an $S^1$-invariant $D$-symplectic form on $S^1 \times N$, whence a contact form on $N$.

Our motivation to transpose Sullivan’s correspondence to the contact world originated in our interest for symplectic and contact structures of weaker than smooth regularity. There is, for instance, a growing interest for the notion of $C^0$-symplectic structures. Those are given by an atlas whose transition maps are symplectic homeomorphisms. The latter can be defined in essentially two ways: either as homeomorphisms that preserve a fixed symplectic capacity for open subsets of $\mathbb{R}^{2n}$ or as $C^0$-limits of symplectic diffeomorphisms. Of course other categories of symplectic and contact manifolds could also be considered, as for instance the PL or bi-Lipschitz ones. Now one would like to compare the various notions: smooth versus $C^0$, PL or bi-Lipschitz structures. Amongst fundamental questions is the following one: Do the even-dimensional spheres admit $C^0$-symplectic structures? An idea to prove that certain non-smooth symplectic structures induce smooth ones is to construct from the non-smooth structure a cone structure with the right hypotheses so as to induce a smooth structure. The hope is that obtaining the cone structure is less difficult than constructing a smooth structure directly. Now the reason for wanting a contact version of Sullivan’s correspondence is simply that after the dimension 2, for which most phenomena are simple, comes the dimension 3, that belongs to the contact world.

The paper is organized as follows. The first section describes Sullivan’s original construction. The second one presents a version of the symplectization process that is useful to us. The third section discusses the invariant Hahn Banach separation theorem and the last one gathers results from the previous sections to establish the promised correspondence.
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1 Sullivan’s construction

Let us recall some basic notions about convex sets.

Definition 1.1. Let $V$ denote a topological vector space over the reals.

1. A cone in $V$ is a subset $C$ that is invariant under multiplication by positive reals. It is said to be convex when it is a convex subset of $V$.

2. A continuous linear form $\alpha \in V'$ is said to be positive on a cone $C \subset V$ if $\alpha(v) > 0$ for each $v \in C - \{0\}$.

3. If a cone $C \subset V$ admits a positive linear form $\alpha$, the subset $C = \alpha^{-1}(1) \subset C - \{0\}$ is called a base for $C$. The set $C$ is in bijective correspondance with the collection of rays in $C$.

4. A cone is compact when it admits a positive linear form $\alpha$ and $C$ is compact (notice that all bases are homeomorphic).

The following definition is due to Sullivan’s (cf. [Sul76]).

Definition 1.2. A cone structure of $k$-vectors on a manifold $M$ is a continuous field $C = (C_x)_{x \in M}$ of compact convex cones $C_x \subset \Lambda^k T_x M$.

To make sense of continuity, one chooses a Riemannian metric $g$ on $\Lambda^k \mathbb{R}^n$, the sphere $\left(\Lambda^k \mathbb{R}^n - \{0\}\right)/\mathbb{R}_0^+$ of $\Lambda^k \mathbb{R}^n$. The metric $g$ induces a distance $d$ and a Hausdorff distance $\rho$ on the collection of non-empty compact subsets of $\Lambda^k \mathbb{R}^n$:

$$\rho(S_1, S_2) = \max \left\{ \sup_{x \in S_1} d(x, S_2), \sup_{y \in S_2} d(S_1, y) \right\}.$$  

Given a trivialization of $\Lambda^k T M$ on a neighborhood $U$ of a point $x$, the cone structure on $U$ can be seen as a map from $U$ to the collection of non-empty compact subsets of $\Lambda^k \mathbb{R}^n$ with $n = \dim M$ and continuity of $C$ at $x$ can thus be defined by means of $\rho$.

Definition 1.3. A $k$-form $\beta$ is said to be transverse to a cone structure $C$ of $k$-vectors, or positive on $C$, if $\beta_x(P) > 0 \quad (\text{if } P = v_1 \wedge \ldots \wedge v_k, \text{ then } \beta(P) = \beta(v_1, \ldots, v_k).}$
forms $\beta_x$ may be glued into a global positive $k$-form by means of a partition of unity.

**Example 1.4.** Examples of cone structures that are essential here are the cone structures induced by symplectic forms. Let $\omega$ be such a form on a manifold $M$ and consider an almost complex structure $J$ compatible with the symplectic structure $\omega$. Define the cone structure $C^J = (C^J_x)_{x \in M}$ whose fiber at $x$ is defined to be the convex closure of the cone

$$\{ v \wedge Jv \mid v \in TM \}.$$  

The proof of Lemma 4.1 shows that $C^J$ is indeed a cone structure. The original symplectic form $\omega$ is a positive form on $C^J$. Alternatively, one could consider, instead of $C^J_x$, the convex closure of the collection of bivectors $v \wedge w \in \Lambda^2 T_x M$ such that $\omega_x(v, w) > 0$, but that cone is not compact and its topological closure is too large in the sense that it contains isotropic bivectors as well.

Let us now recall some basic facts about currents that will be needed in the sequel. Let $M$ be a manifold. For $k \in \mathbb{N}$, a $k$-current on $M$ is a continuous linear form on the space $\Omega^k_{c}(M)$ of compactly supported $k$-forms on $M$ endowed with its standard Fréchet topology. The collection of $k$-currents, that is, the topological dual of $\Omega^k_{c}(M)$, is denoted by $D_k(M)$ and endowed with the strong topology. The transpose of the exterior differential $d : \Omega^k_{c}(M) \to \Omega^{k+1}_{c}(M)$ is a continuous differential $d^* : D_{k+1}(M) \to D_k(M)$.

It is a standard fact (see [dR84] Théorème 13 p. 89) that $\Omega^k_{c}(M)$ is reflexive which means that it is isomorphic to the strong dual of its strong dual through the evaluation map. In particular, any continuous linear form on currents corresponds to a unique differential form. Moreover, the forms vanishing on the space of exact currents are the closed forms. This is a consequence of reflexivity and of the following relation satisfied by the transpose of a continuous linear map $F$ between topological vector spaces (see [Trè06] Formula (23.2) p 241):

$$\text{Ker } F^* = (\text{Im } F)^\perp.$$  

Another key point is that the set of exact currents is a closed subspace of the space of all currents or equivalently the differential $d^*$ is homomorphism. It is a consequence of the standard result recalled in Proposition 4.4 and of the closedness of the space of exact forms, itself a consequence of de Rham’s theorem. The trivial observation that a subspace that is closed for the weak topology is also closed for the strong one is also needed.

Coming back to cone structures, to any such is associated a compact convex cone in the space of currents as described below. First observe that an element $P$ in some $\Lambda^k T_x M$ induces the Dirac current $\delta_P$ mapping a $k$-form $\beta$ to $\beta(P)$. Given

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a cone structure $C$ on $M$, consider the collection $\mathcal{D}_C = \{ \delta P \mid P \in C \}$ of Dirac currents. It is a (non-convex) cone in $\mathcal{D}_k(M)$.

**Definition 1.5.** To a cone structure $C$ on $M$ is associated the cone of structure currents, that is, the topological closure of the convex closure of the cone $\mathcal{D}_C$. It is denoted by $\mathcal{C}$. A structure cycle is a closed current that belongs to $\mathcal{C}$.

The subset $\mathcal{C}$ is of course a closed convex cone and, when $M$ is a closed manifold, it is also compact (cf. [Sul76], Proposition I.5, p. 230).

Now one of Sullivan’s main points in [Sul76] is that certain properties of transverse $k$-forms may be encoded in the structure cone. For instance, whether exact or closed transverse forms exist is determined by the position of the structure cone relative to the subspaces of closed and exact currents, denoted respectively by $\mathcal{Z}_k(M)$ and $\mathcal{B}_k(M)$. That relative position can be of either one of the following types:

1. $\mathcal{C}$ intersects $\mathcal{B}_k(M)$ non-trivially,
2. $\mathcal{C}$ intersects $\mathcal{B}_k(M)$ trivially and $\mathcal{Z}_k(M)$ non-trivially,
3. $\mathcal{C}$ intersects $\mathcal{Z}_k(M)$ trivially.

Here is a precise statement:

**Theorem 1.6.** (Sullivan [Sul76] Theorem I.7, p. 231) Let $C$ denote a cone structure of $k$-vectors on a closed manifold $M$. Then

1. There are either non-trivial closed structure currents or closed transverse forms.
2. If there are no closed transverse forms, then there are non-trivial exact structure currents (this corresponds to case I).
3. If there are no non-trivial closed structure currents, then there are exact transverse forms (this corresponds to case III).

More is said in [Sul76] about case II, but we do not need this here. The main ingredient in the proof is the Hahn Banach separation theorem (see [Tre06], Proposition 18.2, p. 191 for instance), one version of which we recall below:

**Hahn Banach separation theorem.** Let $W$ be a closed subspace in a locally convex topological vector space $V$ that does not intersect a compact convex subset $C \subset V$. Then $W$ can be extended to a closed hyperplane $H$ that does not meet $C$ either.

The first point of Theorem 1.6 is a consequence of the other two. The idea for the second point is that if the structure cone does not intersect $\mathcal{B}_k(M)$ (except along 0), then, because $C$ is compact, the separating Hahn Banach theorem implies that the closed subspace $\mathcal{B}_k(M)$ is contained in a closed hyperplane $H$ that does not intersect a base for $\mathcal{C}$. That closed hyperplane is the kernel of a
continuous linear functional on $D_k(M)$, determined up to a non-zero factor. By
duality between forms and currents, that functional determines a $k$-form which
can be chosen to be positive on $C$, whence on $C$ since $D_C \subset C$, and which is closed
because it vanishes on $B_k(M)$. If $C$ is disjoint not only from $B_k(M)$ but also from
$Z_k(M)$, we obtain likewise exact transverse $k$-forms, proving the third point.

Coming back to the symplectic world, the question is how to encode the defining
properties of a symplectic form into properties of an associated structure cone
to which it is transverse. Recall the cone structure $C_J$ (introduced in Example $[1.4]$)
associated to an almost complex structure $J$, itself compatible with a given sym-
plectic structure $\omega$. The previous discussion shows that closedness of $\omega$ corresponds
to absence of exact structure currents. How about nondegeneracy? As explained
hereafter, it corresponds to the property of ampleness for cone structures that is
stated in Definition $[1.8]$ below.

As a preliminary, let us recall that the Schubert variety of a 2-plane $\tau$ in a
vector space $V$ is the collection, denoted by $S_\tau$, of all 2-planes intersecting $\tau$ non-
trivially. Observe also that to a 2-plane $\tau$ corresponds the line in $\Lambda^2 V$ consisting
of the bivectors $v \wedge w$ for $\{v, w\} \subset \tau$. This allows us to think of the Schubert
variety $S_\tau$ as being a cone in $\Lambda^2 V$.

**Definition 1.7.** A cone $C$ in $\Lambda^2 V$ is said to be ample if for any 2-plane $\tau \subset V$,
the Schubert variety of $\tau$ intersects $C$ non-trivially. A cone structure $C = (C_x)_{x \in M}$
is ample when each $C_x$ is ample in $\Lambda^2 T_x M$.

**Lemma 1.8.** The cone structure $C^J$ associated to an almost complex structure $J$
(cf. Example $[1.4]$) is ample. Any 2-form transverse to an ample cone structure of
2-vectors is non-degenerate.

**Proof.** For the proof that $C^J_x$ is ample, let $\tau$ be a 2-plane in $T_x M$ and let $v \in \tau$.
The bivector $v \wedge Jv$ belongs to $C^J_x$, which implies that $S_\tau$ intersects $C^J_x$
non-trivially.

For the second statement, let us suppose, on the contrary, that a 2-form $\omega$
transverse to a cone structure $C$ is degenerate, that is, admits a 2-plane $\tau$ in its
radical. Let $\tau' \in S_\tau$ and let $\{v, w\}$ be a basis for $\tau'$ with $v \in \tau \cap \tau'$. We have
$\omega(v, w) = 0$ which implies that the bivector $v \wedge w$ cannot belong to $C_x$, whence
that $C_x \cap S_\tau = \emptyset$. Thus $C_x$ is not ample.

Now Theorem $[1.6]$ and Lemma $[1.8]$ imply the following proposition.

**Proposition 1.9.** (Sullivan, [Sul76] Theorem III.2, p. 249) Let $M$ be a closed
manifold. If a symplectic structure $\omega$ is given on $M$, then any choice of compat-
ible almost complex structure $J$ induce an ample cone structure $C^J$ that has no
exact structure cycles and admits $\omega$ as a transverse form. Conversely, an ample
cone structure $C$ without exact structure cycles admits a non-empty contractile
collection of transverse forms that are symplectic.
The statement about contractibility is easily seen. Indeed, the collection of forms that are positive on the cone structure and vanish on $B^2(M)$ is convex.

2 An invariant version of the symplectization process

The symplectization of a contact manifold $(M, \xi)$ consists of the open manifold $\mathbb{R} \times M$ endowed with an $\mathbb{R}$-equivariant symplectic form. We present, in this section, a variant of the symplectization process which yields a bijective correspondence between contact forms on a closed manifold $M$ and invariant non-degenerate 2-forms on the manifold $S^1 \times M$ that are closed for a twisted differential.

Let us first recall the usual symplectization process. If $(M, \alpha)$ is a closed co-oriented contact manifold then $(\mathbb{R} \times M, d(e^s \pi^* \alpha))$, where $\pi$ is the canonical projection of $\mathbb{R} \times M$ onto $M$ and $s$ is the function $\mathbb{R} \times M \to \mathbb{R} : (s, x) \mapsto s$, is an open symplectic manifold whose symplectic form is equivariant with respect to the standard action $\rho(t, (s, x)) = \rho_t(s, x) = (t + s, x)$ of $\mathbb{R}$ on $\mathbb{R} \times M$. That is, the form $\omega = d(e^s \pi^* \alpha)$ satisfies the relation

$$\rho_t^* \omega = e^t \omega, \quad \forall t \in \mathbb{R}.$$

Conversely, an equivariant symplectic form on $\mathbb{R} \times M$ yields a contact form. Indeed, an equivariant 2-form on $\mathbb{R} \times M$ is of the type

$$\beta = e^s \pi^* \beta_0 + e^s ds \wedge \pi^* \alpha_0,$$

for a 2-form $\beta_0$ and 1-form $\alpha_0$ on $M$. It is closed if and only if $d\alpha_0 = \beta_0$ and non-degenerate exactly when $\alpha_0 \wedge \beta_0 \wedge ... \wedge \beta_0$ does not vanish.

This yields a one-to-one correspondence between the collection $\text{Cont}(M)$ of contact forms on $M$ and the collection $\text{ESymp}(\mathbb{R} \times M)$ of equivariant symplectic forms on $\mathbb{R} \times M$. Since the aim is a correspondence between contact forms and invariant forms on $S^1 \times M$, we are interested in invariant rather than equivariant forms on $\mathbb{R} \times M$, as they pass to the quotient.

**Remark 2.1.** An invariant 2-form on $\mathbb{R} \times M$, that is, a form $\beta \in \Omega^2(\mathbb{R} \times M)$ that satisfies $\rho_t^* \beta = \beta$ for all $t \in \mathbb{R}$, induces a 2-form $\beta_0$ and 1-form $\alpha_0$ on $M$ such that

$$\beta = \pi^* \beta_0 + ds \wedge \pi^* \alpha_0.$$

Notice that $\beta$ is closed if and only if both $\beta_0$ and $\alpha_0$ are closed.

Thus an invariant symplectic 2-form is far from inducing a contact form. On the other hand, an equivariant symplectic form induces a non-degenerate 2-form that is closed for a twisted differential. Indeed, observe that the map $\varphi : \beta \mapsto e^s \beta$ induces a one-to-one correspondence between invariant and equivariant forms on $\mathbb{R} \times M$. The property of non-degeneracy is of course preserved. Moreover,

$$d(e^s \beta) = e^s (ds \wedge \beta + d\beta).$$
This means that the map $\varphi$ intertwines the exterior differential $d$ with the differential operator

$$D : \Omega^i(\mathbb{R} \times M) \to \Omega^{i+1}(\mathbb{R} \times M) : \beta \mapsto D\beta = ds \wedge \beta + d\beta.$$ 

With symbols:

$$d \circ \varphi = \varphi \circ D.$$

**Lemma 2.2.** The operator $D$ obviously satisfies the following properties:

- $D^2 = 0$,
- $\rho_t^* \circ D = D \circ \rho_t^*$, for all $t \in \mathbb{R}$,
- $D$ is continuous.

The second relation implies that $D$ induces a differential operator on the collection $\Omega^*(S^1 \times M)$ of forms defined on $S^1 \times M$ that satisfies identical relations. It will be denoted by $D$ as well.

Introducing the collection $\text{ISymp}^D(\mathbb{R} \times M)$ of non-degenerate invariant 2-forms $\beta$ on $\mathbb{R} \times M$ satisfying $D\beta = 0$, called hereafter $\mathbb{R}$-invariant $D$-symplectic forms, the previous discussion implies that the following map is a bijection:

$$\psi : \text{Cont}(M) \to \text{ISymp}^D(\mathbb{R} \times M) : \alpha \mapsto e^{-s}d(e^s\pi^*\alpha) = ds \wedge \pi^*\alpha + d(\pi^*\alpha).$$

Now we would like to pass from $\mathbb{R} \times M$ to $S^1 \times M$. The map $p : \mathbb{R} \times M \to S^1 \times M : (s,x) \mapsto (e^{is},x)$ induces a push-forward $p_*$ from the space of $\mathbb{R}$-invariant forms on $\mathbb{R} \times M$ to the space of $S^1$-invariant forms on $S^1 \times M$ defined by

$$[p_*(\beta)](e^{is},x)(v_1, \ldots, v_s) = \beta(s,x)(\overline{v_1}, \ldots, \overline{v_s}),$$

where $\overline{v_i} \in T(\mathbb{R} \times M)$ is the lift of $v_i$ through the point $(s,x)$. It is of course independent on the choice of $(s,x)$ in $p^{-1}(e^{is},x)$.

We have thus obtained a bijective correspondence between contact forms on $M$ and $S^1$-invariant $D$-symplectic forms on $S^1 \times M$. More precisely, the map

$$S : \text{Cont}(M) \to \text{Symp}_{D}^S(S^1 \times M) : \alpha \mapsto S(\alpha) = p_*(e^{-s}d(e^s\pi^*\alpha))$$

is one-to-one, where $\text{Symp}_{D}^S(S^1 \times M)$ denotes the collection of all $S^1$-invariant $D$-symplectic forms on $S^1 \times M$.

### 3 An invariant version of the Hahn Banach separation theorem

In this section appears a proof of the invariant separation Hahn Banach theorem that relies on the invariant analytic Hahn-Banach theorem. Surprisingly, a proof of the precise statement we need is not so easily accessible in the literature whence
our decision to include it in the text. See nevertheless [Lau74] Theorem 1 in Section 3, as well as [DG14] Théorème 2.20 p 32.

Let us first recall the statement, due to R. Agnew and A. Morse [AM38], of the invariant analytic Hahn Banach theorem.

**Theorem 3.1.** (Agnew & Morse) Let $V$ denote a real topological vector space and $W$ a linear subspace. Let $p : V \to \mathbb{R}^+$ be a positively homogeneous subadditive functional\(^{IV}\) and let $f$ be a linear form on $W$ such that $f(w) \leq p(w)$ for all $w \in W$. If $G$ is a solvable group acting continuously on $V$, preserving $W$ and leaving $p$ and $f$ invariant, then there exists a linear extension of $f$ to $V$ that is invariant under $G$ and satisfies $f(v) \leq p(v)$ for all $v \in V$.

Here is the version of the invariant geometric Hahn-Banach theorem used hereafter:

**Theorem 3.2.** Let $G$ be a solvable group acting continuously on a real Hausdorff topological vector space $V$. Suppose $K$ is a compact convex subset of $V$ invariant under $G$ and $W$ is a closed linear subspace of $V$ also invariant under $G$ and that does not meet $K$. If $K$ contains a fixed point $c_0$ for the action of $G$ and if the complement of $W - K$ contains a convex invariant open neighborhood of 0, then the subspace $W$ can be extended to a closed invariant hyperplane that does not meet $K$.

**Proof.** Let $A$ be a convex invariant open neighborhood of 0 contained in $W - K$. Consider the subset $O = A + K - c_0$. It is convex, invariant and open as a union $O = \bigcup_{c \in K} (A + (c - c_0))$ of open sets. Consider its gauge

$$p_O : V \to \mathbb{R}^+ : v \mapsto \inf \{ \lambda > 0 \mid v \in \lambda O \}.$$  

The functional $p_O$ is positively homogeneous subadditive and invariant because $O$ is invariant. Moreover $O = p_O^{-1}([0,1))$ since $O$ is open. Thus $p_O$ is continuous. Now consider the subspace $W'$ generated by $W$ and $c_0$ and the linear functional $\phi : W' \to \mathbb{R} : w - tc_0 \mapsto t$. Observe that $W'$ is invariant and that $\phi$ is continuous (because $W'$ in closed), invariant and satisfies $\phi(w') \leq p_O(w')$ for all $w' \in W'$. Indeed, for $t > 0$, we have

$$p_O(w - tc_0) = tp_O(\frac{w}{t} - c_0) \geq t$$

since $\frac{w}{t} - c_0 \in W - K$ which is disjoint from $O$.

The Agnew-Morse theorem above implies that $\phi$ can be extended to an invariant linear functional $\overline{\phi} : V \to \mathbb{R}$ such that $\overline{\phi}(v) \leq p_O(v)$ for all $v \in V$. Observe that, as implied by the previous inequality and the continuity of $p_O$, the functional $\overline{\phi}$ is continuous. Let $H$ denote the kernel of $\overline{\phi}$. It is a closed invariant hyperplane

\(^{IV}\)This means that $p(\lambda v) = \lambda p(v)$ $\forall \lambda \in \mathbb{R}^+$, $v \in V$ and $p(v + w) \leq p(v) + p(w)$ $\forall v, w \in V$. 

Lemma 3.3 applied to the additive action of $V$ therefore a convex open neighborhood $A$ of $W$. Theorem 3.2, is guaranteed. Indeed, the subset $\{0\}$ is closed. It is of course convex as well.

The awkward assumption of existence of a convex invariant open neighborhood of $0$ contained in $W - K$ is satisfied. To prove this we need the following standard result about actions of topological groups.

**Lemma 3.3.** Consider a continuous action of a topological group $G$ on a topological space $X$. Then, for a compact subspace $\Theta$ of $G$ and a closed subset $C$ of $X$, the set $\Theta \cdot C = \{ \theta \cdot x \mid \theta \in \Theta, x \in C \}$ is closed.

**Proof.** The aim is to show that any point $x$ in $X - \Theta \cdot C$ admits a neighborhood contained in $X - \Theta \cdot C$. Because $x$ does not belong to $\Theta \cdot C$, the orbit of $x$ under $\Theta^{-1} = \{ \theta^{-1} \mid \theta \in \Theta \}$ does not meet $C$. For each $y = \theta^{-1} \cdot x \in \Theta^{-1} \cdot x$, let $O_y$ be a neighborhood of $y$ in $X - C$. Because the action of $g$ on $X$ is continuous we may assume, without loss of generality, that $O_y = U_y \cdot V_y$, where $U_y$ is a neighborhood of $\theta^{-1}$ in $G$ and $V_y$ is a neighborhood of $x$ in $X$. Since $\Theta$ is compact, the set $\Theta^{-1}$ is compact and the open cover $\{ U_y \mid y \in \Theta^{-1} \cdot x \}$ of $\Theta^{-1}$ admits a finite subcover, say $\{ U_{y_1}, \ldots, U_{y_k} \}$. The open neighborhood

$$V = \bigcap_{1 \leq i \leq k} V_{y_i}$$

of $x$ does not intersect $\Theta \cdot C$. Indeed, if, on the contrary, some $x'$ in $V$ is of the type $\theta \cdot c$ for some $\theta$ in $\Theta$ and some $c$ in $C$, then $\theta^{-1} \cdot x'$ belongs to $C$. But $\theta^{-1} \in U_{y_i}$ for some $1 \leq i \leq k$. Thus $\theta^{-1} \cdot x' \in U_{y_i} \cdot V_{y_i} \subset X - C$, yielding a contradiction.

**Corollary 3.4.** Let $G$ be a compact solvable topological group acting continuously on a real locally convex Hausdorff topological vector space $V$. Suppose $K$ is a compact convex subset of $V$ invariant under $G$ and $W$ is a closed linear subspace of $V$ also invariant under $G$ and that does not meet $K$. If $K$ contains a fixed point for the action of $G$ then the subspace $W$ can be extended to a closed invariant hyperplane that does not meet $K$.

**Proof.** Observe that, under the assumptions of Corollary 3.4, the existence of the convex invariant open neighborhood of $0$ in the complement of $W - K$, needed in Theorem 3.2, is guaranteed. Indeed, the subset $W - K$ is closed, as implied by Lemma 3.3, applied to the additive action of $V$ on itself. Its complement contains therefore a convex open neighborhood $A_0$ of $0$. Then Lemma 3.3 again, implies that the invariant subset $A = \bigcap_{g \in G} (g \cdot A_0)$ is open (since its complement $G \cdot (V - A_0)$ is closed). It is of course convex as well.
4 A correspondance between contact forms and specific cone distributions

We are now ready to establish a correspondance between contact forms and a certain type of cone structures. Recall from Section 2 that we may think of contact forms on $M$ as being $S^1$-invariant $D$-symplectic forms on $S^1 \times M$. The idea now is to extend Sullivan’s result to forms and cone structures that are invariant under an $S^1$-action.

Set $P = S^1 \times M$. For an element $\omega$ in $\text{Symp}_{S^1}^D(P)$, consider an invariant almost complex structure $J$ that is compatible with $\omega$. There is a natural class of such structures. Indeed, consider the contact form $\alpha$ associated with $\omega$ and its Reeb vector field $R_\alpha$ (defined by the relations $d\alpha(R_\alpha, \cdot) \equiv 0$ and $\alpha(R_\alpha) \equiv 1$). Choose a compatible almost complex structure $J_0$ on the symplectic vector bundle $(\xi = \text{Ker} \, d\alpha, d\alpha|_\xi \oplus \xi)$ consisting of the contact distribution endowed with the fiberwise linear symplectic structure determined by $\alpha$. Now define a vector bundle morphism $J : TP \to TP$ by setting

\[
\begin{align*}
J(v) &= J_0(v) \text{ for } v \in \xi \\
J(R_\alpha) &= \frac{\partial}{\partial t} \\
J(\frac{\partial}{\partial t}) &= -R_\alpha.
\end{align*}
\]

The morphism $J$ is an $S^1$-invariant almost complex structure compatible with $\omega$, as is easily verified.

Now we construct the field $C^J$ of cones of 2-vectors on $P$ associated to the almost complex structure $J$ as is done in Remark 1.4. The cone at $p$ is thus the convex hull of the set

\[\{ v \wedge Jv \mid v \in T_pP \} \]

Lemma 4.1. The field $C^J$ is continuous, compact, ample and invariant.

The proof is elementary but is nevertheless included.

Proof. The fact that $C^J$ is continuous follows directly from the fact that one can find local trivializations of the vector bundle $TP$ relative to which the almost complex structure $J$ and thus also the cone field $C^J$ are constant.

To prove compactness of $C$, let $p \in P$ and consider pointwise linearly independent local sections $e_1, ..., e_{2n}$ of $TP$ near $p$ such that $J(e_i) = e_{i+n}$, $i = 1, ..., n$. Define the local 2-form

\[\beta = \sum_{i=1}^{n} e_i^\ast \wedge e_{i+n}^\ast,\]
where \( \{ e_1^*, ..., e_{2n}^* \} \) is the basis of \( T^*P \) dual to \( \{ e_1, ..., e_{2n} \} \). Then

\[
\beta(v \wedge Jv) = \sum_{i=1}^{2n} v_i^2,
\]

where \( v_1, ..., v_{2n} \) are the coordinates of \( v \). It is now obvious that \( \beta_p^{-1}(1) \subset \Lambda^2 T_p P \) is compact.

Ampleness of \( C^J \) has been verified in the proof of Lemma 1.8 and invariance follows directly from that of \( J \).

Lemma 4.2. The structure cone \( C^J \) associated to \( C^J \) is compact and invariant. Moreover \( C^J \) admits an invariant basis.

Proof. For the compactness of \( C^J \), we refer to [Sul76]. Invariance follows directly from that of \( C^J \). To show that it admits an invariant basis, it suffices to construct an invariant transverse form, which is easily done as follows. Let \( \beta \) be a transverse form. Define \( \beta' \) to be the invariant extension of the restriction of \( \beta \) to \( T\{0\} \times M P \).

More explicitly :

\[
\beta'_{(t,x)} = \rho_{-t}^*(\beta_{(0,x)}).
\]

The form \( \beta' \) is invariant and remain transverse because \( C^J \) is invariant.

To pursue, we have to know a little more about the properties of the operator \( D \), introduced in Section 2. More specifically, to be able to apply the invariant Hahn Banach separation theorem it is necessary to know that the space of currents that are exact for the adjoint \( D^* \) of \( D \) is a closed subspace of \( D^*(M) \).

Let us denote by \( Z^k_D(P) \) (respectively \( B^k_D(P) \)) the subspace of \( \Omega^k(P) \) consisting of \( D \)-closed (respectively \( D \)-exact) \( k \)-forms and by \( Z^k_{D^*}(P) \) (respectively \( B^k_{D^*}(P) \)) the space of closed (respectively exact) currents for \( D^* \). Because \( \rho^*_t \circ D = D \circ \rho^*_t \) for all \( t \in S^1 \) (cf. Lemma 2.2), all those spaces are \( S^1 \)-invariant.

Lemma 4.3. The space \( B^k_{D^*}(P) \) is a closed subspace of \( D_k(P) \).

Proof. First show that \( B^k_{D^*}(P) = \text{Im} D \) is a closed subspace of \( \Omega^{k+1}(P) \) and then invoke the following classical result (e.g. [Tre06] Proposition 35.7 p. 366) :

Proposition 4.4. For a continuous linear map \( u \) between two real locally convex Hausdorff topological vector spaces \( E \) and \( F \), the following properties are equivalent

- \( u(E) \) is closed in \( F \);
- the transpose \( u^* \) of \( u \) is a homomorphism of \( F' \) onto \( u^*(F') \subset E' \) when \( F' \) and \( E' \) are endowed with their weak topology.

Supposing that \( B^k_{D^*}(P) \) is closed and applying this proposition to \( D : \Omega^k(P) \to \Omega^{k+1}(P) \) implies that \( D^* \) is a homomorphism. Because the space \( \Omega^k(P)/\text{Ker} D \) is complete, the space \( \text{Im} D^* \) is also complete, whence closed. This is true for the
weak topology and thus for the strong topology too.

It remains now to show that $B_{D}^{k}(P)$ is closed for all $k$. Recall from Section\[2\] the map

$$\varphi: \Omega^{k}(\mathbb{R} \times M) \to \Omega^{k}(\mathbb{R} \times M): \beta \mapsto e^{s} \beta.$$  

It is a homeomorphism that intertwines $d$ with $D$ and, therefore, that induces a bijection between their images $B_{D}^{k}(\mathbb{R} \times M)$ and $B^{k}(\mathbb{R} \times M)$. This shows that $B_{D}^{k}(\mathbb{R} \times M)$ is closed in $\Omega^{k}(\mathbb{R} \times M)$. Besides, the map

$$p^{*}: \Omega^{k}(S^{1} \times M) \to \Omega^{k}(\mathbb{R} \times M): \beta \mapsto p^{*}(\beta).$$  

induces a homeomorphism between $\Omega^{k}(S^{1} \times M)$ and $\Omega^{k}(\mathbb{R} \times M)$. The latter being continuous, the set $\Omega^{k}(\mathbb{R} \times M)$ is a closed subspace of $\Omega^{k}(\mathbb{R} \times M)$. Thus $B_{D}^{*}(P)$ appears to correspond, under $p^{*}$, to an intersection $\Omega^{k}(\mathbb{R} \times M) \cap B_{D}^{*}(\mathbb{R} \times M)$ of closed subspaces of $\Omega^{k}(\mathbb{R} \times M)$.

Now to apply the invariant version of the Hahn Banach separation theorem, we need to have a fixed point for the action of $S^{1}$ on a basis for the structure cone. Let us recall Tychonoff’s fixed point theorem.

**Theorem 4.5.** Let $E$ be a locally convex topological vector space, let $C$ be a compact convex subset of $E$ and let $f: C \to C$ be a continuous map. Then $f$ has a fixed point.

Consider an element $g \in S^{1}$ that generates a dense subgroup in $S^{1}$. Tychonoff’s fixed point theorem implies that $\rho_{g}: \mathcal{C} \to \mathcal{C}$ has a fixed point $c$. Now $c$ is fixed under the action of the subgroup generated by $g$ as well. Since that subgroup is dense in $S^{1}$ and the action of $S^{1}$ on $\mathcal{C}$ is continuous, all elements of $S^{1}$ fix $c$.

Alternatively, one may use the Markov-Kakutani fixed point theorem whose statement is recalled hereafter.

**Theorem 4.6.** (Markov [Mar36] and Kakutani [Kak38]) Let $C$ be a compact convex subset of a Hausdorff topological vector space $E$ and let $G$ be a collection of commuting continuous affine transformations of $E$ that preserve $C$. Then there exists a point in $C$ that is fixed under all elements of $G$.

Existence of a fixed point allows us to apply the geometric Hahn-Banach theorem.

**Theorem 4.7.** Let $M$ be a closed manifold. A contact form $\alpha$ on $M$ induces a non-empty collection of ample $S^{1}$-invariant cone structures on $P = S^{1} \times M$ with no non-trivial $D^{*}$-exact structure currents. Conversely, an ample $S^{1}$-invariant cone structure on $P$ with no non-vanishing $D^{*}$-exact structure current induces a non-empty contractible collection of contact structures on $M$.

**Proof.** Given a contact form $\alpha$ on $M$, consider an invariant almost complex structure $J$ on $P$ compatible with the symplectization $\omega = S(\alpha)$ of $\alpha$, described
in Section 2 and the associated cone structure $C^J$. Then $C^J$ is ample, invariant (cf. Lemma 4.1) and $C^J$ does not contain any $D^*$-exact structure cycle. Indeed, the form $\omega$ is positive on $C^J$ but a $D$-closed form may not be positive on $D^*$-exact currents.

Conversely, let $C$ denote an ample $S^1$-invariant cone structure on $P$ without non-vanishing $D^*$-exact structure current. Consider an invariant basis $C^J$ for the structure cone $C^J$. Tychonoff’s fixed point theorem implies existence of a fixed point $c$ in $C^J$ for the action of $S^1$ and thus Corollary 3.4 implies that the closed subspace $B_2^D(P)$ may be extended to a closed hyperplane that does not meet $C^J$. That closed hyperplane is the kernel of a continuous linear functional $\alpha$ on $D_2(P)$ positive on $C^J$. The presence of the fixed point $c$ implies that $\alpha$ is invariant. The space $\Omega^2(P)$ being reflexive, that functional is induced by a 2-form $\omega$ which is invariant and $D$-closed because it vanishes on $B_2^D(P)$ (argument identical to the one that shows that closed forms are the forms vanishing on exact currents and that uses reflexivity of $\Omega^k(P)$ together with formula (1)). Finally, such a form is the symplectization $S(\alpha)$ of a contact form $\alpha$ on $M$.

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