ANALYSIS OF AMNESIAC FLOODING

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ABSTRACT

The purpose of the broadcast operation in distributed systems is to spread information located at some nodes to all other nodes. The broadcast operation is often realized by flooding. With flooding the source nodes send a message containing the information to all their neighbors. Each node receiving the message for the first time forwards it to all other neighbors. A stateless variant of flooding for synchronous systems is called amnesiac flooding. In this case a node after receiving a message, forwards it only to those neighbors from which it did not receive the message in the current round. In this paper we analyze the termination time of amnesiac flooding. We define the k-flooding problem. The objective is to find a set S of size k, such that amnesiac flooding when started concurrently by all nodes of S terminates in a minimal number of rounds. We provide sharp upper and lower bounds for the termination time. We prove that for every non-bipartite graph there exists a bipartite graph such that the execution of amnesiac flooding on both graphs is strongly correlated and that the termination times coincide. This construction considerably simplifies existing proofs for amnesiac flooding and gives more insight into the flooding process.

Keywords Distributed Algorithms, Flooding

1 Introduction

The most basic algorithm to disseminate information in a distributed system is the deterministic flooding algorithm. The originator of the information sends a message containing the information to all neighbors and whenever a node receives this message for the first time, it sends it to all its neighbors in the communication graph G. This algorithm uses 2|E| messages and terminates in \( \epsilon_G(v_0) \) rounds, where \( v_0 \) is the originating node and \( \epsilon_G(v_0) \) is the maximal distance of \( v_0 \) to any other node. These bounds hold in the synchronous and in the asynchronous case [1].

The flooding algorithm requires each node to maintain for each message a marker that the message has been forwarded. This requires storage proportional to the number of disseminated messages per node, which is a problem for resource-constrained devices. Another issue is how long these markers are kept. A variant of this algorithm for synchronous systems that goes without such markers is called amnesiac flooding. In this algorithm a node after receiving a message, forwards it only to those neighbors from which it did not receive the message in the current round. Hence, this variant of flooding is stateless and avoids the above mentioned storage issues. Hussak and Trehan have analyzed the termination time of amnesiac flooding with a single originating node \( v_0 \) [2, 3, 4]. They show that synchronous amnesiac flooding always terminates on any finite graph. For bipartite graphs amnesiac flooding terminates after \( \epsilon_G(v_0) \) rounds, i.e., the same number of rounds as the marker based algorithm. In the non-bipartite case amnesiac flooding requires at least \( \epsilon_G(v_0) + 1 \) and at most \( \epsilon_G(v_0) + Diam(G) + 1 \) rounds, where \( Diam(G) \) denotes the diameter of \( G \). The proof of this result in [4] is rather technical and does not give much insight into the problem.

In this paper we analyze amnesiac flooding for any number of originating nodes. The contribution of this paper is twofold. First of all we prove that for every non-bipartite graph there exists a bipartite graph such that the execution of amnesiac flooding on both graphs is strongly correlated and that the termination times coincide. This construction
A problem related to broadcast is rumor spreading that describes the dissemination of information in large and complex networks through pairwise interactions. A simple model for rumor spreading is to assume that in each round, each node that receives the message for the first time forwards it to all other nodes it is connected to, as long as it has not already received the message from that neighbor. This leads to a simple model for broadcast where each node that receives the message for the first time forwards it to all other nodes it is connected to, as long as it has not already received the message from that neighbor.

The standard flooding algorithm, where each node that receives the message for the first time forwards it to all other nodes it is connected to, is known as the basic flooding algorithm. However, this algorithm can be problematic in large networks, as it can lead to a large number of messages being sent and received, and can lead to network congestion. A stateless version of flooding was proposed by Hussak and Trehan [2]. Their algorithm – called amnesiac flooding – introduces an auxiliary graph, which is used to keep track of nodes that have already received the message for the first time. This allows nodes to forward messages only to nodes that have not already received the message, which reduces the number of messages being sent and received. Furthermore, this approach opens more possibilities for more general problems related to amnesiac flooding.

Broadcast in computer networks has been the subject of extensive research. The survey paper [7] covers early work. The standard flooding algorithm, where each node that receives the message for the first time forwards it to all other nodes it is connected to, is known as the basic flooding algorithm. However, this algorithm can be problematic in large networks, as it can lead to a large number of messages being sent and received, and can lead to network congestion. A stateless version of flooding was proposed by Hussak and Trehan [2]. Their algorithm – called amnesiac flooding – introduces an auxiliary graph, which is used to keep track of nodes that have already received the message for the first time. This allows nodes to forward messages only to nodes that have not already received the message, which reduces the number of messages being sent and received. Furthermore, this approach opens more possibilities for more general problems related to amnesiac flooding.

In the following section, we reduce the case $|S| > 1$ to the standard case. In this section we introduce the auxiliary graph $G$. The eccentricity of any node in $G$ is defined as $\epsilon_G(v) = \delta(v)$. The radius $Rad(G)$ of $G$ is the minimum eccentricity of any vertex of $G$. The diameter $Diam(G)$ of $G$ is the maximum eccentricity of any vertex in $G$. A central node in $G$ is a node $v$ with $\epsilon_G(v) = Rad(G)$. An edge $(u, w) \in E$ is called a forward edge with respect to a node $v_0$ if $d_G(v_0, u) = d_G(v_0, w)$. Any edge of $G$ that is not a cross edge with respect to $v_0$ is called a forward edge for $v_0$.

Let $n \geq k \geq 1$ be an integer. We call $r_k(G) = \min \{ \delta(U) \mid |U| = k \}$ the $k$-radius of $G$. Thus, $r_1(G) = Rad(G)$. Each subset $U \subseteq V$ with $|U| = k$ and $r_k(G) = \delta(U)$ is called a k-center of $G$. Similarly, we call $r^*_k(G) = \min \{ \delta(U) \mid |U| = k \}$ the non-isolated $k$-radius of $G$. Clearly $r_k(G) \leq r^*_k(G)$. Each subset $U \subseteq V$ with $|U| = k$ such that $G[U]$ has no isolated node and $r^*_k(G) = \delta(U)$ is called a non-isolated $k$-center of $G$.

Throughout the paper we consider a synchronous distributed system. This means that algorithms are executed in rounds of fixed lengths and all messages sent by all nodes in a particular round are also received in this round. Furthermore, no messages are lost or corrupted. For a discussion of asynchronous amnesiac flooding we refer to [3].

### 3 State of the Art

Broadcast in computer networks has been the subject of extensive research. The survey paper [7] covers early work. The standard flooding algorithm, where each node that receives the message for the first time forwards it to all other neighbors, requires in the worst case $Diam(G)$ rounds until all nodes have received the message and uses $O(m)$ messages [1]. This result holds both in the synchronous and in the asynchronous model. The number of messages can be reduced if flooding is performed via the edges of a spanning tree only.

The flooding algorithm is a stateful algorithm. Each node needs to maintain for each message a marker that the message has been forwarded. This requires storage per node proportional to the number of disseminated messages. This communication pattern is therefore not suitable for resource-constrained devices as those used in the Internet of Things. A stateless version of flooding was proposed by Hussak and Trehan [2]. Their algorithm – called amnesiac flooding – forwards a newly received message only to those neighbors from which it did not receive the message in the current round. Amnesiac flooding has a much lower memory requirement since markers are only kept for one round. Note, that a node may forward a message more than once. They prove that in synchronous networks amnesiac flooding when started by a node $v_0$ terminates after at most $\epsilon_G(v_0) + Diam(G) + 1$ rounds. Their proof is based on an analysis of the forwarding process on a round by round basis, whereas the analysis in this work is based on an auxiliary graph. We believe that this approach opens more possibilities for more general problems related to amnesiac flooding. To the best of our knowledge, the problem of simultaneously starting the flooding process from many nodes has not been covered in the literature.

The $k$-center problem received a lot of attention since it was first proposed [5]. The task is to find a $k$-center of a graph. The problem and many variants of it including some approximations are known to be NP-hard [8, 6].
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Figure 1: Three executions (one per row) of algorithm $A_{AF}$ for different choices of $S$.

vertex that knows the rumor, forwards it to a randomly chosen neighbor. For many network topologies, this strategy is a very efficient way to spread a rumor. With high probability the rumor is received by all vertices in time $\Theta(\log n)$, if the graph is a complete graph or a hypercube [9, 10]. New results about rumor spreading can be found in [11].

4 Amnesiac Flooding: Algorithm $A_{AF}$

The goal of amnesiac flooding is to distribute a message – initially stored at all nodes of a set $S$ – to all nodes of the network. In the first round each node of $S$ sends the message to all its neighbors. In each of the following rounds each node that receives at least one message forwards the message to all of its neighbors from which it did not receive the message in this round. The algorithm terminates, when no more messages are sent. Algorithm 1 shows a formal definition of algorithm $A_{AF}$ as considered in this paper.

Algorithm 1: Algorithm $A_{AF}$ distributes a message in the graph $G$

input : A graph $G = (V, E)$, a subset $S$ of $V$, and a message $m$.

In round 1 each node $v \in S$ sends message $m$ to each neighbor in $G$; Each node $v$ executes in every round $i > 1$

- $M := N(v)$;
- foreach receive($w, m$) do
  - $M := M \setminus \{w\}$
  - if $M \neq N(v)$ then
    - forall $u \in M$ do send($u, m$);

To illustrate the flow of messages of algorithm $A_{AF}$ we consider a graph with four nodes as depicted in Fig. 1 (nodes in $S$ are depicted in black). The top two rows show the flow of messages for two different choices for $S$ with $|S| = 1$. In the first case $A_{AF}$ terminates after three rounds and in the second case after five. The last line of this figure shows an example with $|S| = 2$. In this case the algorithm also terminates after three rounds.

These examples demonstrate that the termination time of $A_{AF}$ highly depends on $S$. This is captured by the following definition.

Definition 1. For $S \subseteq V$ denote by $\text{Flood}_{G}(S)$ the number of rounds algorithm $A_{AF}$ requires to terminate when started by all nodes in $S$. For $1 \leq k \leq n$ define

$$\text{Flood}_{k}(G) = \min\{\text{Flood}_{G}(S) \mid S \subseteq V \text{ with } |S| = k\}.$$ 

Obviously, $\text{Flood}_{n}(G) = 1$ for any graph $G$. For a complete graph $K_n$ with $n > 2$ we have $\text{Flood}_{1}(K_n) = 2$ for $1 < i < n$ and $\text{Flood}_{1}(K_n) = 3$. For a cycle graph $C_n$ we have $\text{Flood}_{k}(C_n) = \lceil n/k \rceil$ if $n \equiv 1(2)$ and otherwise

$$\text{Flood}_{k}(C_n) = \begin{cases} \lceil n/(2k) \rceil & \text{if } k \leq n/2 \\ 1 & \text{if } k = n \\ 2 & \text{otherwise.} \end{cases}$$

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Whereas the $k$-center of a graph is a good choice for $S$ to minimize $\text{Flood}_k(S)$. This is not the case. For the graph depicted in Fig. 2 with $n \equiv 0(2)$ the 1-center consists of the node with distance $n/2 - 1$ to the rightmost node. Algorithm $A_{AF}$ started in this central node terminates after $3n/2 - 2$ rounds. Whereas the minimal value of $n - 1$ rounds is independently of $n$ achieved for each of the two leftmost nodes, i.e. $\text{Flood}_1(G) = n - 1$. Note that $\text{Rad}(G) = n/2 - 1$.

![Figure 2: A graph $G$ with $\text{Flood}_1(G) = n - 1$ and $\text{Flood}(v_0) = 3n/2 - 2$ where $v_0$ is a central node.](image)

Whereas the $k$-radius monotonically decreases with increasing $k$ for a fixed graph, this is not generally true for the value of $\text{Flood}_k(G)$. For example for $n \equiv 0(2)$ we have $\text{Flood}_{n/2}(C_n) = 1$ but $\text{Flood}_{n/2+1}(C_n) = 2$. We will show that monotony holds if $G$ is non-bipartite.

## 5 The Main Results

The following two theorems summarize the main results of this paper. The proofs are contained in the following sections. The first theorem shows that the value of $\text{Flood}_k(G)$ significantly depends on whether $G$ is bipartite or not. It also provides upper and lower bounds for $\text{Flood}_k(G)$. The second theorem characterizes graphs with $\text{Flood}_k(G) \leq 2$.

**Theorem 2.** Let $G = (V, E)$ be a finite connected graph and $1 < k < n$.

1. For every $S \subseteq V$ there is a bipartite graph $G(S)$ with a node $v^*$ such that $\text{Flood}_k(G(S)) = \text{Flood}_{\bar{G}(S)}(v^*) - 1$.

2. If $G$ is bipartite with $V = V_1 \cup V_2$ then $\text{Flood}_k(G) \in \{r_k(G), r_k(G) + 1\}$ and $\text{Flood}_k(G) = r_k(G)$ if and only if $G$ has a $k$-center that is either contained in $V_1$ or in $V_2$.

3. If $G$ is non-bipartite then $\text{Flood}_{k+1}(G) \leq \text{Flood}_k(G)$ and $\text{Rad}(G)/k + 1/2 \leq \text{Flood}_k(G) \leq r_k^{ni}(G) + 1 \leq r_{(k/2)}(G) + 1$.

**Theorem 3.** Let $G = (V, E)$ be a finite connected graph and $k \geq 1$.

1. $\text{Flood}_k(G) = 1$ if and only if $n = k$ or $G$ is bipartite with $|V_1| = k$ or $|V_2| = k$.

2. Let $G$ be non-bipartite graph and $k > 1$. Then $\text{Flood}_k(G) = 2$ if and only if $r_k^{ni}(G) = 1$.

## 6 The Case $|S| = 1$

Let $S = \{v_0\}$ for a node $v_0 \in V$. The way messages are forwarded by algorithm $A_{AF}$ implies that a message can arrive multiple times at a node. Clearly, the first time that a message arrives at a node is along the shortest path from $v_0$ to this node. The following lemma is easy to prove.

**Lemma 4.** In round $i > 0$ of algorithm $A_{AF}$ each node $v$ with $d_G(v_0, v) = i - 1$ sends a message to all neighbors $u$ with $d_G(v_0, u) = i$.

Edges that do not belong to a shortest path affect the flow of messages along shortest paths. Such edges are cross edges with respect to a depth-first search starting in $v_0$. Since bipartite graphs have no cross edges Lemma 4 can be strengthened for bipartite graphs. The proof of the next lemma is by induction on $i$.

**Lemma 5.** Let $G$ be a bipartite graph and $v$ a node with $d_G(v_0, v) = i$. In round $i + 1$ of algorithm $A_{AF}$ node $v$ sends a message to all neighbors $u$ with $d_G(v_0, u) = i + 1$ and to no other neighbor. In all other rounds $v$ does not send a message.

**Corollary 6.** If $G$ is bipartite then $\text{Flood}_G(v_0) = \epsilon_G(v_0)$ and $\text{Flood}_1(G) = \text{Rad}(G)$.

To analyze the behavior of $A_{AF}$ for non-bipartite graphs we introduce an auxiliary graph.
6.1 The Auxiliary Graph $\mathcal{G}(v_0)$

Next we define for a given graph $G$ and a starting node $v_0$ an auxiliary graph $\mathcal{G}(v_0)$. The executions of $A_{AF}$ on these two graphs are tightly coupled. Since $\mathcal{G}(v_0)$ is bipartite we can apply Corollary 6 to compute $\text{Flood}_G(v_0)$.

**Definition 7.** Denote by $\mathcal{F}(v_0)$ the subgraph of $G$ with node set $V$ and all edges of $G$ that are not cross edges with respect to $v_0$.

Obviously $\mathcal{F}(v_0)$ is always bipartite. Fig. 3 demonstrates this definition.

**Definition 8.** Denote by $\mathcal{G}(v_0)$ the graph that consists of two copies of $\mathcal{F}(v_0)$ with node sets $V$ and $V'$ and additionally for any cross edge $(u, w)$ of $G$ the edges $(u, u')$ and $(w, w')$.

In the following we denote for every $v \in V$ the copy of $v$ in $V'$ by $v'$. $\mathcal{G}(v_0)$ consists of $2|V|$ nodes and $2|E|$ edges. Note that every additional edge connects a node from $V$ with a node from $V'$. Fig. 4 demonstrates this construction. For each $v \in V$ we have $\text{deg}_G(v) = \text{deg}_{\mathcal{G}(v_0)}(v) = \text{deg}_{\mathcal{G}(v_0)}(v')$. Furthermore, $d_{\mathcal{G}(v_0)}(v_0, v) = d_{\mathcal{F}(v_0)}(v_0, v) = d_{\mathcal{G}}(v_0, v)$ for $v \in V$.

![Figure 3: On the left the graph $G$ with $v_0$ marked and $\mathcal{F}(v_0)$ on the right.](image)

![Figure 4: The graph $G$ on the left has two cross edges (dotted lines), $\mathcal{G}(v_0)$ is shown on the right.](image)

**Lemma 9.** A shortest path in $\mathcal{G}(v_0)$ from $v_0$ to a node $w' \in V'$ uses exactly one edge from $V$ to $V'$.

**Proof.** Assume that a shortest path $P$ from $v_0$ to $w'$ in $\mathcal{G}(v_0)$ uses more than one edge from $V$ to $V'$. Let $(a, b')$ and $(c', d)$ be the first two such edges. The length of $P$ from $v_0$ to $d$ in $\mathcal{G}(v_0)$ is $d_{\mathcal{F}(v_0)}(v_0, a) + 1 + d_{\mathcal{F}(v_0)}(b, c) + 1$. On the other hand $d_{\mathcal{F}(v_0)}(v_0, d) = d_{\mathcal{F}(v_0)}(v_0, a) + d_{\mathcal{F}(v_0)}(b, c)$. Hence $P$ is not a shortest path in $\mathcal{G}(v_0)$. Contradiction.

**Lemma 10.** $\mathcal{G}(v_0)$ is bipartite.

**Proof.** It suffices to prove that $d_{\mathcal{G}(v_0)}(v_0, u) \neq d_{\mathcal{G}(v_0)}(v_0, w)$ for every edge $(u, w)$ of $\mathcal{G}(v_0)$. There are three cases to consider. If $u, w$ are both nodes of $V$ then any shortest path from $v_0$ to $u$ (resp. to $w$) in $\mathcal{G}(v_0)$ is also a shortest path in $\mathcal{F}(v_0)$. Since $\mathcal{F}(v_0)$ is bipartite we have $d_{\mathcal{G}(v_0)}(v_0, u) \neq d_{\mathcal{G}(v_0)}(v_0, u)$.

If $u$ is a node of $V$ and $w$ a node of $V'$, i.e., $w = v'$ for some node $v \in V$, then $(u, v)$ is a cross edge of $G$ with respect to $v_0$ and hence $d_{\mathcal{F}(v_0)}(v_0, u) = d_{\mathcal{F}(v_0)}(v_0, v)$. Thus, Lemma 9 yields

$$d_{\mathcal{G}(v_0)}(v_0, u) = d_{\mathcal{F}(v_0)}(v_0, u) + 1 = d_{\mathcal{G}(v_0)}(v_0, u) + 1 = d_{\mathcal{G}(v_0)}(v_0, u).$$

Finally consider the case that $u, w$ are both nodes of $V'$. Thus, $d_{\mathcal{F}(v_0)}(v_0, u) \neq d_{\mathcal{F}(v_0)}(v_0, w)$ since $\mathcal{F}(v_0)$ is bipartite and $(u, w)$ is an edge of $\mathcal{F}(v_0)$. Let $(w_1, w_2')$ (resp. $(w_1', w_2)$) be the cross edge on a shortest path from $v_0$ to $u$ (resp. $w$) in $\mathcal{G}(v_0)$ (Lemma 9). Then $d_{\mathcal{G}(v_0)}(v_0, u) = d_{\mathcal{F}(v_0)}(v_0, u_1) + d_{\mathcal{F}(v_0)}(u_2, u) = d_{\mathcal{F}}(v_0, u) + d_{\mathcal{F}(v_0)}(w_2, w) + 1 = d_{\mathcal{F}(v_0)}(v_0, w) + 1$. Hence, again $d_{\mathcal{G}(v_0)}(v_0, u) \neq d_{\mathcal{G}(v_0)}(v_0, u)$. \qed
Let \((u, w')\) be an edge of \(\mathcal{G}(v_0)\) with \(u, w \in V\). Lemma 5 and 10 imply that in \(\mathcal{G}(v_0)\) node \(w'\) never sends a message to \(u\) via edge \((u, w')\) but \(u\) sends a message via this edge to \(w'\). Fig. 5 depicts an execution of \(\mathcal{A}_{\text{AF}}\) on \(\mathcal{G}(v_0)\) for the graph \(G\) shown in Fig. 4. The next lemma shows the relationships between the execution of \(\mathcal{A}_{\text{AF}}\) on \(G\) and \(\mathcal{G}(v_0)\).

**Figure 5:** The labels of the nodes state the round a message is received by the node in \(\mathcal{G}(v_0)\).

**Lemma 11.** Let \(v, w \in V\). Node \(v\) receives a message from \(w\) in round \(i\) in \(G\) if and only if in round \(i\) node \(v\) receives a message from \(w\) in \(\mathcal{G}(v_0)\), or \(v'\) receives a message from \(w\) or from \(w'\) in \(\mathcal{G}(v_0)\).

**Proof.** The proof is by induction on \(i\). The statement is true for \(i = 1\). Note that from the three conditions for \(\mathcal{G}(v_0)\) only the first can occur in round 1. Assume \(i > 1\).

First suppose that \(w\) sends in \(G\) a message to \(v\) in round \(i\). Then \(w\) received in \(G\) a message from a neighbor \(z\) with \(v \neq z\). By induction in round \(i - 1\) in \(\mathcal{G}(v_0)\) node \(v\) did not send a message neither to \(w\) nor to \(w'\) and \(v'\) did not sent a message to \(w'\). Also, in round \(i - 1\) in \(\mathcal{G}(v_0)\) node \(z\) did sent a message to \(w\) or \(w'\) or \(z'\) did sent a message to \(w'\). Thus, in round \(i\) in \(\mathcal{G}(v)\) either node \(w\) sends a message to \(v\) or \(w'\) sends a message to \(v\) or \(v'\).

Conversely suppose that one of the three events happens in \(\mathcal{G}(v_0)\) in round \(i\). First assume that \(v\) received a messages from \(w\) in \(\mathcal{G}(v_0)\). Then in round \(i - 1\) node \(v\) did not send a message to \(w\) and \(w\) received a message from a node \(z \neq v\) in \(\mathcal{G}(v_0)\). Then by induction in \(G\) node \(z\) sent a message to \(w\) and \(w\) did not receive a message from \(v\). This yields that in round \(i\) in \(G\) node \(w\) sends a message to \(v\). Next suppose \(v'\) receives a message from \(w\) in \(\mathcal{G}(v_0)\) in round \(i\). Then \((v, w)\) is a cross edge of \(G\). Thus, \(d_{\mathcal{G}(v_0)}(v_0, v) = d_{\mathcal{G}(v_0)}(v_0, v) = i - 1\) and hence, \(d_{\mathcal{G}(v_0, w)} = d_{\mathcal{G}(v_0, v)} = i - 1\). This implies that \(v\) and \(w\) do not send messages before round \(i\) in \(G\) and in round \(i\) they send messages to each other in \(G\).

Finally suppose that \(v'\) received a message from \(w'\) in \(\mathcal{G}(v_0)\) in round \(i\). Lemma 5 gives \(i = d_{\mathcal{G}(v_0)}(v_0, v')\) and \(d_{\mathcal{G}(v_0)}(v_0, w') = i - 1\). Let \(P\) be a shortest path in \(\mathcal{G}(v_0)\) from \(v_0\) to \(w'\) that uses the cross edge (i.e., from \(B\) to \(B'\)) with the smallest distance \(d\) to \(v_0\). Let \((x, y')\) be the cross edge. Thus, in \(G\) both nodes \(x\) and \(y\) receive a message in round \(d_{\mathcal{G}(v_0, x)}\), but not from each other. Hence, in round \(d_{\mathcal{G}(v_0, x)}\) they receive a message. After another \(d_{\mathcal{G}(y, v)}\) rounds, \(w\) sends a message to \(v\). Thus, in \(G\) node \(v\) receives in round \(d_{\mathcal{G}(v_0, x)} + 1 + d_{\mathcal{G}(y, v)} = d_{\mathcal{G}(v_0, v')}\) a message from \(w\).

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\Box
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This lemma proves that if no node in \(G\) receives a message in a specific round then no node in \(\mathcal{G}(v_0)\) receives a message in this round and vice versa. Thus, this yields the following theorem.

**Theorem 12.** \(\text{Flood}_{G}(v_0) = \text{Flood}_{\mathcal{G}(v_0)}(v_0)\) for every \(v_0 \in V\).

Note that if \(G\) is bipartite then \(\mathcal{G}(v_0)\) is disconnected and the connected component of \(\mathcal{G}(v_0)\) containing \(v_0\) is just \(\mathcal{F}(v_0)\), i.e., \(\epsilon(\mathcal{G}(v_0)) = \epsilon(\mathcal{G}(v_0))\) in this case. Lemma 10 and Theorem 12 together with Lemma 11 imply the following result.

**Theorem 13.** Let \(G(V, E)\) be a graph. Then \(\text{Flood}_{G}(v_0) = \epsilon(\mathcal{G}(v_0))\). Algorithm \(\mathcal{A}_{\text{AF}}\) sends \(|E|\) messages if \(G\) is bipartite and \(2|E|\) otherwise.

Note that in case \(G\) is non-bipartite for some edges messages are sent in both directions, while for other edges two messages are sent in one direction. Next we give an upper bound for \(\epsilon(\mathcal{G}(v_0))\). With the introduced technique the following theorem that is already contained in [2] can be easily proved.

**Theorem 14.** Let \(G\) be a non-bipartite graph and \(v_0 \in V\). Then \(\text{Rad}(G) < \text{Flood}(v_0) \leq \epsilon(G(v_0)) + \text{Diam}(G) + 1\). Furthermore, \(\text{Rad}(G) < \text{Flood}(G) \leq \text{Rad}(G) + \text{Diam}(G) + 1\). \(\text{Rad}(G) = \text{Flood}(G)\) if and only if \(G\) is bipartite.
Figure 6: On the left a graph with $|S| = 3$. The auxiliary graph $G^*(S)$ is depicted on the right.

**Theorem 2.1** Let $G$ be a connected graph and $S \subseteq V$. Then $Flood_G(S) = Flood_{G^*(S)}(v^*) - 1$.

**Proof.** Let $u \in V$. Then $d_{G(v_0)}(v_0, u) \leq Rad(G)$. Thus, it suffices to give a bound for $d_{G(v_0)}(v_0, u')$. Since $G$ is non-bipartite there exist cross edges with respect to $v_0$. Among all cross edges of $G$ choose $(u, w)$ such that $\min\{d_G(u, v), d_G(u, w)\}$ is minimal. WLOG assume that $d_G(u, v) \leq d_G(u, w)$. Then the shortest path from $v$ to $u$ does not contain a cross edge (by choice of $(v, w)$). Thus, the distance from $v'$ to $u'$ in $G(v_0)$ is at most $Diam(G)$. Hence,

$$d_{G(v_0)}(v_0, u') \leq d_G(v_0, u) + 1 + Diam(G) \leq \epsilon_G(v_0) + Diam(G) + 1.$$  

Hence, Theorem 13 implies the upper bound. Let $v$ be a node with $dist_G(v_0, v) \geq Rad(G)$. Then $d_{G(v_0)}(v_0, u') \geq Rad(G) + 1$. This yields the lower bound. Since this is true for all $v_0 \in V$ the second statement also holds. Now the last statement follows from Corollary 6.

The above upper bound is sharp as can be seen for $G = C_n$ with $n = 1(2)$. In this case $Rad(C_n) = Diam(C_n) = (n - 1)/2$ and $Flood_1(C_n) = n$. Fig. 7 shows on the left a non-bipartite graph with $Rad(G) + 1 = Flood_1(G)$.

**7 The Case $|S| > 1$**

The case $|S| > 1$ requires a slightly different definition of the auxiliary graph $G$. First, a new virtual source $v^*$ that is connected by edges to all source nodes in $S$ is introduced. Call this graph $G^*(S)$. Obviously, $Flood_G(S) = Flood_{G^*(S)}(v^*) - 1$. Note that even in case $G$ is bipartite $G^*(S)$ is not necessarily bipartite. Fig. 6 shows an example for the graph $G^*(S)$. The auxiliary graph $G(S)$ is the graph $G(v^*)$ constructed from $G^*(S)$ as in section 6.1 with the only difference that the copy of node $v^*$ in the second copy of $F(S)$ is removed. We simply call this graph in the following $G(S)$. Clearly, Theorem 12 implies the following lemma.

**Lemma 15.** Let $G$ be a connected graph and $S \subseteq V$. Then $Flood_G(S) = Flood_{G(S)}(v^*) - 1$.

**Lemma 10 and Lemma 15** prove Theorem 2.1. In the following we separately analyze bipartite and non-bipartite graphs.

**7.1 Bipartite Graphs**

For $k = 1$ we have $Flood(G) = r_1(G)$ provided $G$ is bipartite and vice versa (Theorem 14, Theorem 3.5 [4]). For $k > 1$ we have a slightly different situation.

**Lemma 16.** Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. Then $Flood_k(G) = r_k(G)$ if and only if $G$ has a $k$-center that is completely contained in $V_1$ or $V_2$.

**Proof.** If $G$ has a $k$-center $S$ that is contained in $V_1$ or $V_2$ then the graph $G^*(S)$ has no cross edge with respect to $v^*$ (nodes with the same distance to $v^*$ are either in $V_1$ or $V_2$). Thus, Lemma 15 implies $Flood_k(G) = r_k(G)$.

Next assume that $Flood_k(G) = r_k(G)$. Let $S \subseteq V$ with $|S| = k$ and $Flood(S) = Flood_k(G)$. Then $r_k(G) \leq d_G(S, V) \leq Flood(S) = r_k(G)$, i.e., $d_G(S, V) = r_k(G)$. Since $Flood(G) = r_k(G)$, $G^*(S)$ does not contain a cross edge with respect to $v^*$. Let $S_1 = S \cap V_1$. Denote by $V^i$ the set of descendants of $S_i$ (including $S_i$). Since there are no cross edges, there exist no edge connecting a node from $V^1$ with a node from $V^2$. Since $G$ is connected, either $S_1 = \emptyset$ or $S_2 = \emptyset$. This implies the result.

Note that Lemma 16 implies Corollary 6.

**Lemma 17.** If $G = (V_1 \cup V_2, E)$ is bipartite then $Flood_k(G) \in \{r_k(G), r_k(G) + 1\}$.
While for bipartite graphs Rad does not hold for a bipartite graph this argument shows that there is a shortest path in Flood that is not longer then Flood + 1. Clearly, Flood ≤ Flood + 1. Obviously, Flood does not contain a cross edge, since the graph Flood is bipartite. Hence, the result follows from Lemma 15 and Corollary 6. □

Lemma 16 and Lemma 17 prove Theorem 2.2. Fig. 7 shows on the right a bipartite graph G with Flood = 2 and Flood = 3.

7.2 Non-Bipartite Graphs

As stated in Section 4 the value of Flood does not necessarily decrease with increasing values for k. Next we prove that this property holds for non-bipartite graphs.

Lemma 18. Let G be a connected non-bipartite graph, 0 ≠ S ⊆ V, and v ∈ V \ S. Then Flood(S ∪ {v}) ≤ Flood(S).

Proof. Let S = S ∪ {v}. It suffices to prove that d(S, v) ≤ d(S, v) for all u ∈ V \ V. Let P a shortest path from v to u in G(S). First consider the case u ∈ V. Then P is a shortest path in F(S). If an edge (v, v) of P is a cross edge in G then there exits a path in F(S) from v to v that is not longer then d(S, v, v). Repeating this argument shows d(S, v, u) ≤ d(S, v, u). Next consider the case u ∈ V, i.e., there exists w ∈ V with u = w.

By Lemma 9 P contains a single cross edge (x, y). If (x, y) is not a cross edge of G, then there exists a path from w via edge (w, v) to y in F(S) with the same length as the corresponding section of P. The same arguments show that there is a shortest path in F(S) from y to u that is at most as the length of the shortest path from y to u in F(S). The proof for the case that (x, y) is a cross edge of G(S) is similar. □

The reason that Lemma 18 does not hold for a bipartite graph G is that depending on S and v the graph G(S) may be bipartite while G(S) is non-bipartite.

Lemma 19. For k = 1, . . . , n − 1 we have Floodk+1(G) ≤ Floodk(G) for all connected non-bipartite graphs G.

Proof. Let S ⊆ V with |S| = k such that Flood(S) = Floodk(G). Let v ∈ V \ S. Then Lemma 18 yields Floodk+1(G) ≤ Flood(S ∪ {v}) ≤ Flood(S) = Floodk(G). □

7.3 Upper Bounds

While for bipartite graphs Floodk(G) is either r(G) or r(G) + 1 the situation is more complex for non-bipartite graphs. In this section we prove upper bounds for Floodk(G) for non-bipartite graphs. For k = 1 Theorem 14 provides with Rad + 1 and Rad + Diam + 1 sharp upper and lower bounds.

Lemma 20. Let G be a non-bipartite Graph and k > 1. Then Floodk(G) ≤ r(G) + 1 ≤ Rad(G) + 1.

Proof. Let U be a non-isolated k-center of G and v ∈ V. Then there exists u ∈ U such that d(u, v) ≤ r(U) and the path from u to v in G(U) does not use a cross edge with respect to v. Also there exists w ∈ N(u) ∩ U. Hence, the path v, w, u exists in G(U). Therefore, the distance from v to v in G(U) is at most 2 + distC(u, v). This yields the result. □
The bound is sharp. For $C_n$ with $n = 3(4)$ we have $(n - 3)/4 + 1 = \text{Flood}_4(C_n) = r_n^4(C_n) + 1$. On the other hand $r_n^4(C_n) = \text{Flood}_3(C_n)$ with $n \equiv 1(2)$ is unbounded for growing $n$.

**Corollary 21.** Let $G$ be a non-bipartite Graph and $k > 1$. Then $\text{Flood}_k(G) \leq r_{\lfloor k/2 \rfloor}(G) + 1$.

**Proof.** Let $S \subset V$ with $|S| = \lfloor k/2 \rfloor$ and $d_G(S, V) = r_{\lfloor k/2 \rfloor}(G)$. Obviously, there exists a subset $S'$ of $V$ such that $|S \cup S'| = k$ and $G[S \cup S']$ has no isolated node. Let $\hat{S} = S \cup S'$. Then, $r_n^k(G) \leq \text{dist}(\hat{S}, V) \leq r_{\lfloor k/2 \rfloor}(G)$. The last inequality follows from Lemma 20.

A naive approach to determine $\text{Flood}_k(G)$ requires $O(n^k m)$ time. Corollary 21 suggests that the well-known greedy algorithm for the metric $k$-center might be a good heuristic to determine a set $\hat{S}$ with small value of $\text{Flood}_k(G)$. Unfortunately, the bound can be arbitrarily bad as the graphs $C_n$ with $n \equiv 1(2)$ show. For $k = 3$ the sequence $r_1(C_n) = \text{Flood}_3(C_n)$ is unbounded.

Dankelmann et al. provide several upper bounds for $r_k(G)$ in terms of $n$ and $\delta$ [12]. These can be used to state bounds for $\text{Flood}_k(G)$ in terms of $n$ and $\delta$. The following result is a consequence of Theorem 14 of [12] and the last Corollary.

**Corollary 22.** Let $G$ be a connected triangle-free non-bipartite graph and $1 < k < n$. Then

$$\text{Flood}_k(G) \leq \frac{2(n-1)}{\delta(\lfloor k/2 \rfloor + 1)} + 5.$$ 

### 7.4 Lower Bounds

**Lemma 23.** For $k > 0$ and all trees $T$ we have $kr_k(T) \geq \text{Rad}(T) - k/2$.

**Proof.** Let $v, w \in V$ such that $d_G(v, w) = \text{Diam}(T)$. In the best case two consecutive nodes on the path from $v$ to $w$ that belong to a $k$-center have distance $2r_k(T)$. Thus, $2kr_k + k - 1 \geq \text{Diam}(T)$. This yields $2kr_k + k - 1 \geq 2\text{Rad}(T) - 1$ which proves the result.

Note that the bound of this lemma is sharp. Consider a path $P$ of length $k(2c + 1)$ for $c > 0$. Then $r_k(P) = c$ and $\text{Rad}(P) = k(2c + 1)/2$.

**Lemma 24.** Let $G$ be a connected graph. Then there exists a spanning tree $T$ of $G$ such that $r_k(G) = r_k(T)$ for all $k \geq 1$.

**Proof.** Let $U = \{u_1, \ldots, u_k\}$ be a $k$-center of $G$. Let $V_i = \{v \in V \mid d_G(v, u_j) = d_G(v, u_j) \text{ for } j = 2, \ldots, k\}$. For $i = 2, \ldots, k$ let $V_i = \{v \in V \mid d_G(v, u_j) \leq d_G(v, u_j) \text{ for } j = i + 1, \ldots, k\}$. Note that $u_i \in V_i$. Clearly, the $V_i$ form a partitioning of $V$. Let $v \in V_i$ and $u$ a node on the shortest path from $u_i$ to $v$. Thus, $d_G(u_i, v) \leq d_G(u_j, v)$. Assume that $u \in V_j$. Then $d_G(u_j, u) \leq d_G(u_i, u)$. This implies that $d_G(u_j, u) = d_G(u_i, u)$ and $d_G(u_j, v) = d_G(u_i, v)$. Since $v \in V_i$ we have that $i < j$ and hence $u \in V_i$. Thus, $u \in V_i$ and hence $G_i = G[V_i]$ is a connected graph. Let $T_i$ be a breadth-first tree of $G_i$ rooted in $u_i$. By adding $k - 1$ edges the $T_i$’s can be combined into a spanning tree $T$ of $G$. Let $u_i$ be a central node of $T_i$ and $U_T = \{u_1, \ldots, u_k\}$. For each $j = 1, \ldots, k$ we have

$$\text{Rad}(T_j) = \max_{v \in V_j} d_{T_j}(v, w_j) \leq \max_{v \in V_j} d_{T_j}(v, u_j) = \max_{v \in V_j} d_G(v, u_j) \leq r_k(G).$$

This yields

$$r_k(T) \leq d_T(U_T, V) \leq \max_{j=1,\ldots,k} \text{Rad}(T_j) \leq r_k(G) \leq r_k(T).$$

This completes the proof.

**Lemma 25.** If $G$ is non-bipartite then $\text{Flood}_k(G) \geq \text{Rad}(G)/k + 1/2$.

**Proof.** By Lemma 24 there exists a spanning tree $T$ of $G$ such that $r_k(G) = r_k(T)$. Note that $r_k(G) + 1 \leq \text{Flood}_k(G)$. By Lemma 23 $kr_k(T) \geq \text{Rad}(T) - k/2$. Hence

$$k\text{Flood}_k(G) \geq kr_k(G) + k = kr_k(T) + k \geq \text{Rad}(T) + k/2$$

since $\text{Rad}(T) \geq \text{Rad}(G)$.

Lemma 19, Lemma 20, Corollary 21, and Lemma 25 prove Theorem 2.3.

We suspect that the bound stated in Lemma 25 is not sharp. Instead we have the following conjecture.
Assume there exists $v$. There are a few open problems related to amnesiac flooding. Firstly, Conjecture 26 with an improved lower bound. Theorem 2.2 implies that The results of section 7.3 can be used to characterize graphs with betweenness, eigenvector centrality etc.). The question is whether it coincides with any of the known centrality indices. i.e., it defines a centrality index [14]. There are many centrality indices proposed in the literature (degree, closeness, betweenness, eigenvector centrality etc.). The question is whether it coincides with any of the known centrality indices.

Conjecture 26. If $G$ is non-bipartite then $k\text{Flood}_k(G) \geq \text{Rad}(G) + k - 1$.

If $\text{Flood}_k(G) \geq r_k(G) + 2$ then the proof of Lemma 25 shows the new bound. Thus, in proving the conjecture one can assume $\text{Flood}_k(G) = r_k(G) + 1$. This new bound would be sharp. Let $H_{12}$ be the graph with 12 nodes as depicted in Fig. 8. Connect eight copies of $H_{12}$ by adding 7 edges connecting the copies one after the other at the end nodes. The resulting graph $G$ has 96 nodes, $\text{Rad}(G) = 40$, and $\text{Flood}_3(G) = 14$.

8 Special Cases

In this section graphs with $\text{Flood}_k(G) = 1$ or $\text{Flood}_k(G) = 2$ are characterized as stated in Theorem 3.

Proof. (Theorem 3.1) Let $\text{Flood}_k(G) = 1$. Then each node $v \in V$ must have all its neighbors in $S$ or none. Let $S_1$ be the set of nodes that have all their neighbors in $S$. Let $v \in V \setminus S_1$. Then $N(v) \cap S = \emptyset$. Hence, $v$ does not receive a message in the first round. Since $\text{Flood}_k(G) = 1$ and $G$ is connected, we have $v \in S$. Thus $V \setminus S_1 \subseteq S$.

Assume there exists $v \in S_1$ with $N(v) \subseteq S_1$. Then $v \in S$. This yields $N(N(v)) \subseteq S_1$ and consequently $V = S_1$ and thus $V = S$ since $G$ is connected, i.e., $n = k$. Next assume $N(v) \not\subseteq S_1$ for all $v \in S_1$. Thus, for $v_1 \in S_1$ there exists a neighbor $v_2$ that is not in $S_1$, i.e., $N(v_2) \cap S = \emptyset$. Then $v_2 \in S$ because $N(v_1) \subseteq S$. If $v_1$ would be in $S$, then all neighbors of $v_2$ would be in $S$ and thus, $v_2 \in S_1$. Thus, $v_1 \not\in S_1$. Hence, $S_1 \cap S = \emptyset$. Therefore, $S$ and $S_1$ are independent sets. Also $S_1 \cup S = V$. Thus, $G$ is bipartite. Since the opposite direction is trivially true, the proof is complete.

The results of section 7.3 can be used to characterize graphs with $\text{Flood}_k(G) = 2$.

Proof. (Theorem 3.2) If $r_k^{ni}(G) = 1$ then $\text{Flood}_k(G) \leq 2$ by Lemma 20 and thus $\text{Flood}_k(G) = 2$ by Theorem 3.1. Next suppose that $\text{Flood}_k(G) = 2$ and let $S \subseteq V$ be a $k$-center. Assume that $G[S]$ contains an isolated node $v$. The shortest path in $G[S]$ from $v^*$ to $v'$ has length at least 4. Thus, $\text{Flood}_k(G) \geq 3$ by Theorem 12 and Lemma 15. This contradiction proves that $r_k(G) = r_k^{ni}(G)$. Since $2 = \text{Flood}_k(G) \geq r_k(G) + 1$ we have $r_k^{ni}(G) = 1$.

The last result does not hold for bipartite graphs as a path $P$ of length 12 demonstrates, $\text{Flood}_3(P) = 2$ and $r_3^{ni}(P) = 5$. Theorem 2.2 implies that $r_k(G) \in \{1, 2\}$ if $G$ is bipartite and $\text{Flood}_k(G) = 2$. Thus, $r_k(G) = 1$ implies $\text{Flood}_k(G) \leq 2$. Bipartite graphs with $r_k(G) = 2$ can have $\text{Flood}_k(G) > 2$ as the example in Figure 7 shows.

9 Conclusion and Future Work

In this paper we analyzed amnesiac flooding for a set $S$ of $k$ initiators and introduced the $k$-flooding problem. The main result is the construction of a bipartite graph $G(S)$ such that the executions of amnesiac flooding on $G$ and $G(S)$ are equivalent. This allows us to prove upper and lower bounds for the termination time of amnesiac flooding on non-bipartite graphs. Furthermore, we showed the relationship between the $k$-center and $k$-flooding for bipartite graphs.

There are a few open problems related to amnesiac flooding. Firstly, Conjecture 26 with an improved lower bound is still open. Secondly, by Theorem 2.2 $\text{Flood}_k(G)$ assumes one of two values in case $G$ is bipartite. Is it possible to infer from structural parameters of $G$ the value of $\text{Flood}_k(G)$ in case $G$ is bipartite? There exists a simple greedy algorithm with approximation ratio 2 for the metric $k$-center problem. Is there a similar approximation algorithm for the $k$-flooding problem?

Denote by $d(v, w)$ the number of the round in which node $w$ receives the last message when amnesiac flooding is started in node $v$. It is straightforward to prove that for non-bipartite graphs this does not define a metric but a meta-metric in the sense of [13]. Hence it can be used to quantify the importance of a node in a given network, i.e., it defines a centrality index [14]. There are many centrality indices proposed in the literature (degree, closeness, betweenness, eigenvector centrality etc.). The question is whether it coincides with any of the known centrality indices.
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