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Local Aronson-Bénilan type gradient estimates for the porous medium type equation under Ricci flow

Wen Wang Hui Zhou Dapeng Xie

Abstract. In this paper, we investigate some new local Aronson-Bénilan type gradient estimates for positive solutions of the porous medium equation

$$u_t = \Delta u^m,$$

under Ricci flow. As application, the related Harnack inequalities are derived. Our results generalize known results. These results in the paper can be regard as generalizing the gradient estimates of Lu-Ni-Vázquez-Villani and Huang-Huang-Li to the Ricci flow.

1. Introduction

In the paper, we mainly derive the parabolic version gradient estimates and Harnack inequality for positive solutions to the porous medium equation (PME for short)

$$u_t = \Delta u^m, \quad m > 1 \quad (1.1)$$

under Ricci flow.

Let $(M^n, g)$ be a complete Riemannian manifold. Li and Yau [9] established a famous gradient estimate for positive solutions to the heat equation. In 1991, Li in [10] deduced gradient estimates and Harnack inequalities for positive solutions to the nonlinear parabolic equation on $\mathbb{M} \times [0, \infty)$. In 1993, Hamilton in [5] generalized the constant $\alpha$ of Li and Yau’s result to the function $\alpha(t) = e^{2Kt}$. In 2006, Sun [17] also proved gradient estimates of different coefficient. In 2011, Li and Xu in [11] further promoted Li and Yau’s result, and found two new functions $\alpha(t)$. Recently, first author and Zhang in [18] further generalized Li and Xu’s results to the nonlinear parabolic equation. Related results can be found in [4, 14, 19].

In 2009, Lu, Ni, Vázquez and Villani in [13] studied the PME on manifolds, and obtained the results below.

Theorem A (Lu, Ni, Vázquez and Villani). Let $(M^n, g)$ be an $n$-dimensional complete Riemannian manifold with $\text{Ric}(B_p(2R)) \geq -K$, $K > 0$. Assume that $u$ is a positive solution of (1.1). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \max_{B_p(2R) \times [0,T]} v$. Then for any $\alpha > 1$, we have

$$\frac{\nabla v^2}{v} - \alpha \frac{v_t}{v} \leq \frac{CMa^2}{R^2} \left( \frac{a^2}{\alpha - 1}am^2 + (m - 1)(1 + \sqrt{KR}) \right) + \frac{a^2}{\alpha - 1}a(m - 1)MK + \frac{aa^2}{t}$$

on the ball $B_p(2R)$, where $a = \frac{n(m-1)}{n(m-1)+2}$ and the constant $C$ depends only on $n$.

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Moreover, when $R \to \infty$, the following gradient estimate on complete noncompact Riemannian manifold $(M^n, g)$ can be deduced:

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha^2}{\alpha - 1} a(m-1)MK + \frac{aa^2}{t}$$

Huang, Huang and Li in [7] generalized the results of Lu, Ni, Vázquez and Villani, obtained Li-Yau type, Hamilton type and Li-Xu type gradient estimates.

Recently, above these results had been generalized to the Ricci flow. The Ricci flow

$$\partial_t g(x, t) = -2\text{Ric}(x, t)$$

was first introduced by Hamilton [6] to investigate the Poincaré conjecture on compact three dimensional manifolds of positive Ricci curvature. In 2008, Kuang and Zhang [8] proved a gradient estimate for positive solutions to the conjugate heat equation under Ricci flow on a closed manifold. Soon afterwards, gradient estimate for positive solutions to the heat equation under Ricci flow were further studied, one can see [1, 12, 14, 16]. Recently, Cao and Zhu [3] derived some Aronson and Bénilan estimates for PME (1.1) under Ricci flow.

We first introduce three $C^1$ functions $\alpha(t)$, $\phi(t)$ and $\gamma(t)$ : $(0, +\infty) \to (0, +\infty)$. Suppose that three $C^1$ functions $\alpha(t)$, $\phi(t)$ and $\gamma(t)$ satisfy the following conditions:

(C1) $\alpha(t) > 1$,

(C2) $\phi(t)$ and $\gamma(t)$ are non-decreasing.

Our results state as follows.

**Theorem 2.1.** Let $(M^n, g(x, t))_{t \in [0, T]}$ be a complete solution to the Ricci flow (1.2). Let $M^n$ be complete under the initial metric $g(x, 0)$. Assume that $|\text{Ric}(x, t)| \leq K$ for some $K > 0$ and all $t \in [0, T]$. Suppose that there exist three functions $\alpha(t)$, $\phi(t)$ and $\gamma(t)$ satisfy conditions (C1), (C2), (C3) and (C4).

Given $x_0 \in M$ and $R > 0$, let $u$ be a positive solution of the nonlinear parabolic equation (1.1) in the cube $B_{2R, T} := \{(x, t)|d(x, x_0, t) \leq 2R, 0 \leq t \leq T\}$. Let $v = \frac{u}{R^{m-1}}$ and $v \leq M$.

If $\frac{2\alpha}{\alpha - 1} \leq C_2$ for some constant $C_2$, Then for any $(x, t) \in B_{2R, T},$

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq C_2a^2 \left[ \frac{1}{R^2} \left(1 + \sqrt{KR}\right) + K \right] + \frac{CaMm^2}{R^2\gamma}$$
\[ + \alpha^2 K \sqrt{a(m-1)} + \frac{K \alpha^2}{m-1} \sqrt{\alpha n}. \]  

(2.1)

If \( \frac{\gamma}{\alpha - 1} \leq C_2 \) for some constant \( C_2 \), then for any \((x, t) \in B_{2R,T}\),

\[
\frac{\nabla v}{v} - \frac{v_t}{v} \leq Ca\alpha^2 \left[ \frac{1}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] + \frac{CaMm^2\alpha^4}{R^2\gamma} \\
+ \alpha^2 K \sqrt{a(m-1)} + \frac{K\alpha^2}{m-1} \sqrt{\alpha n}. 
\]  

(2.2)

holds on \( B_{2R,T} \), where \( a = \frac{n(m-1)}{n(m-1)+1} \), and the constant \( C \) depends only on \( n \).

Let us show some special functions to illustrate the theorem 2.1 holds for different circumstances and see appendix in section 5 for detailed calculation process.

**Remarks:**

1. Li-Yau type:
   \[
   \alpha(t) = \text{constant}, \quad \varphi(t) = \frac{\alpha n(m-1)}{t} + \frac{n(m-1)^2MK}{\alpha - 1},
   \quad \gamma(t) = t^\theta \quad \text{with} \quad 0 < \theta \leq 2.
   \]

   Then
   \[
   \frac{\nabla v}{v} - \frac{v_t}{v} \leq Ca\alpha^2 \left[ \frac{1}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] + \frac{CaMm^2\alpha^4}{R^2\gamma} \\
   + \alpha^2 K \sqrt{a(m-1)} + \frac{K\alpha^2}{m-1} \sqrt{\alpha n}.
   \]

2. Hamilton type:
   \[
   \alpha(t) = e^{2(m-1)MKt}, \quad \varphi(t) = \frac{n(m-1)}{t}e^{4(m-1)MKt}, \quad \gamma(t) = te^{2(m-1)MKt}.
   \]

   Then
   \[
   \frac{\nabla v}{v} - \frac{v_t}{v} \leq Ca\alpha^2 \left[ \frac{1}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] + \frac{CaMm^2\alpha^4}{R^2\gamma} \\
   + \alpha^2 K \sqrt{a(m-1)} + \frac{K\alpha^2}{m-1} \sqrt{\alpha n}.
   \]

3. Li-Xu type:
   \[
   \alpha(t) = 1 + \frac{\sinh((m-1)MKt) \cosh((m-1)MKt) - (m-1)MKt}{\sinh^2((m-1)MKt)}, \\
   \varphi(t) = 2n(m-1)^2MK[1 + \coth((m-1)MKt)], \quad \gamma(t) = \tanh((m-1)MKt).
   \]

   Then
   \[
   \frac{\nabla v}{v} - \frac{v_t}{v} \leq Ca \left[ \frac{1}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] + \frac{Ca^2M}{R^2\gamma} \\
   + \alpha^2 K \sqrt{a(m-1)} + \frac{K\alpha^2}{m-1} \sqrt{\alpha n},
   \]

where \( \alpha(t) \) is bounded uniformly.

4. Linear Li-Xu type:
   \[
   \alpha(t) = 1 + (m-1)MKt, \quad \varphi(t) = \frac{n(m-1)}{t} + n(m-1)^2MK, \\
   \gamma(t) = (m-1)MKt.
   \]
Then
\[
\frac{\|v\|^2}{v} - \frac{v_t}{v} \leq C\alpha^2 \left[ \frac{1}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] + \frac{C\alpha^2 a^4 M}{R^2 \gamma} + \alpha^2 K \sqrt{a(m-1)} + \frac{K\alpha^2}{m-1} \sqrt{an}.
\]

The local estimates above imply global estimates.

Corollary 2.1. Let \((M^n, g(0))\) be a complete noncompact Riemannian manifold without boundary, and assume \(g(t)\) evolves by Ricci flow in such a way that \(|\text{Ric}| \leq K\) for \(t \in [0, T]\). Let \(u(x,t)\) be a positive solution to the equation (1.1), and let \(v = \frac{m}{m-1} u^{m-1}\) and \(v \leq M\).

If \(l \leq 1\) and for \((x,t) \in M^n \times (0, T]\), then
\[
\frac{\|v\|^2}{v} - \frac{v_t}{v} \leq C\alpha^2 K + \alpha^2 K \sqrt{a(m-1)} + \frac{K\alpha^2}{m-1} \sqrt{an}.
\]

3. Preliminary

Let \(v = \frac{m}{m-1} u^{m-1}\) and put into equation (1.1), we get
\[
v_t = (m-1)v \Delta v + |\nabla v|^2,
\]
which is equivalent to the following form:
\[
\frac{v_t}{v} = (m-1)\Delta v + \frac{|\nabla v|^2}{v}.
\]

Lemma 3.1. Assume that \((M^n, g(x,t))\) satisfies the hypotheses of Theorem 2.1. We introduce the differential operator
\[
\mathcal{L} = \partial_t - (m-1) \Delta v.
\]

Let \(F = \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} - \alpha \varphi\), where \(\alpha = \alpha(t) > 1\). Then we have
\[
\mathcal{L}(F) \leq -(m-1) \sum_{i,j} e_{ij}^2 + (m-1) \alpha^2 K^2 + 2(m-1)K|\nabla v|^2 + 2m \nabla v \nabla F
\]
\[
-[(m-1)\Delta v]^2 + 2(\alpha - 1)K \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} - \alpha \varphi - \alpha \varphi'.
\]

Proof. Simple calculation shows
\[
\partial_t \left( \frac{v_t}{v} \right) = \partial_t \left[ (m-1)\Delta v + \frac{|\nabla v|^2}{v} \right]
\]
\[
= (m-1)(\Delta v)_t + \frac{2\nabla v \nabla v_t}{v} + \frac{2}{v} \text{Ric}(\nabla v, \nabla v) - \frac{|\nabla v|^2 v_t}{v^2}
\]
\[
= (m-1)(\Delta v)_t + \frac{2\nabla v \nabla [ (m-1)v \Delta v + |\nabla v|^2 ]}{v} - \frac{|\nabla v|^2}{v} [ (m-1)v \Delta v + |\nabla v|^2 ] + \frac{2}{v} \text{Ric}(\nabla v, \nabla v)
\]
\[
= (m-1)(\Delta v)_t + (m-1) \frac{|\nabla v|^2 \Delta v}{v} + 2(m-1) \nabla v \nabla (\Delta v)
\]
\[
+ \frac{2 \nabla v \nabla |\nabla v|^2}{v} + (m-1) \lambda |\nabla v|^2 - \frac{|\nabla v|^4}{v^2} + \frac{2}{v} \text{Ric}(\nabla v, \nabla v),
\]
and
\[ \Delta \left( \frac{v_t}{v} \right) = \frac{\Delta (v_t)}{v} + 2 \sum_{i,j} R_{ij} f_{ij} - \frac{2 \nabla v \nabla v_t}{v^2} - \frac{v_t \Delta v}{v^2} + \frac{2 |\nabla v|^2 v_t}{v^3}, \tag{3.5} \]
where we use the fact that
\[ (\Delta v)_t = \Delta (v_t) + 2 \sum_{i,j} R_{ij} v_{ij}, \tag{3.6} \]
\[ (|\nabla v|^2)_t = 2 \nabla v \nabla (v_t) + 2 Ric(\nabla v, v). \tag{3.7} \]
Combining (3.4) and (3.5), we have
\[ L \left( \frac{v_t}{v} \right) = (m - 1) \frac{|\nabla v|^2 \Delta v}{v} + 2(m - 1) \frac{\nabla v \nabla (\Delta v)}{v} + 2 \frac{|\nabla v|^2 |\nabla v|}{v^2} \]
\[ - \frac{|\nabla v|^4}{v^2} + 2(m - 1) \frac{\nabla v \nabla v_t}{v} + (m - 1) \frac{v_t \Delta v}{v} - 2(m - 1) \frac{|\nabla v|^2 v_t}{v^2} \]
\[ + 2(m - 1) \sum_{i,j} R_{ij} v_{ij} + \frac{2}{v} Ric(\nabla v, \nabla v). \tag{3.8} \]
Since
\[ \nabla v_t = (m - 1) \nabla v \Delta v + (m - 1) \nabla (\Delta v) + \nabla |\nabla v|^2, \]
then
\[ 2(m - 1) \frac{\nabla v \nabla (\Delta v)}{v} + 2(m - 1) \frac{|\nabla v|^2 v_t}{v^2} = 2(m - 1) \nabla \left( \frac{v_t}{v} \right) \nabla \log v \]
into (3.8), we have
\[ L \left( \frac{v_t}{v} \right) = \frac{2}{v} \nabla v \nabla v_t - \frac{|\nabla v|^2 v_t}{v^2} + 2(m - 1) \frac{\nabla v \nabla \left( \frac{v_t}{v} \right)}{v} \nabla \log v \]
\[ + (m - 1) \frac{v_t \Delta v}{v} + (m - 1) \sum_{i,j} R_{ij} v_{ij} + \frac{2}{v} Ric(\nabla v, \nabla v). \tag{3.9} \]
On the other hand, similar calculations show
\[ \partial_t \left( \frac{|\nabla v|^2}{v} \right) = \frac{2 \nabla v}{v} \nabla \left[ (m - 1)v \Delta v + |\nabla v|^2 \right] + \frac{2 Ric(\nabla v, \nabla v)}{v} \]
\[ - \frac{|\nabla v|^2}{v} \left( (m - 1) \Delta v + \frac{|\nabla v|^2}{v} \right) \]
\[ = 2(m - 1) \Delta v \frac{|\nabla v|^2}{v} + 2(m - 1) \nabla v \nabla (\Delta v) + \frac{2}{v} \nabla v \nabla |\nabla v|^2 \]
\[ -(m - 1) \Delta v \frac{|\nabla v|^2}{v} - \frac{|\nabla v|^4}{v^2} + \frac{2 Ric(\nabla v, \nabla v)}{v}, \tag{3.10} \]
where we utilize the formula (3.7) in (3.10).

By utilize Bochner’s formula, we have

\[
\Delta \left( \frac{\left| \nabla v \right|^2}{v} \right) = \frac{2v_i^2}{v} + \frac{2\nabla v \Delta \left( \nabla v \right)}{v} - \frac{2\nabla v \nabla \left| \nabla v \right|^2}{v^2} - \Delta v \frac{\left| \nabla v \right|^2}{v^2} + \frac{2\left| \nabla v \right|^4}{v^3}
\]

\[
= \frac{2v_i^2}{v} + 2R_{ij} \frac{\left| \nabla v \right|^2}{v} + \frac{2\nabla v \nabla \left( \Delta v \right)}{v} - \Delta v \frac{\left| \nabla v \right|^2}{v^2}
- 2\nabla \left( \frac{\left| \nabla v \right|^2}{v} \right) \nabla \left( \log v \right).
\]

(3.11)

From (3.10) and (3.11), we obtain

\[
\mathcal{L} \left( \frac{\left| \nabla v \right|^2}{v} \right) = 2(m - 1)\Delta v \frac{\left| \nabla v \right|^2}{v} + \frac{2\nabla v \nabla \left| \nabla v \right|^2}{v^2} - 2(m - 1)R_{ij} \nabla v_i \nabla v_j
- 2(v - 1)v_i^2 \nabla \left( \frac{\left| \nabla v \right|^2}{v^2} \right) \nabla \left( \log v \right)
+ 2Ric \left( \nabla v, \nabla v \right) \frac{v}{v}. \tag{3.12}
\]

By utilize (3.9) and (3.12), we have

\[
\mathcal{L}(F) = \mathcal{L} \left( \frac{\left| \nabla v \right|^2}{v} \right) - \alpha \mathcal{L} \left( \frac{v_t}{v} \right) - \alpha' \frac{v_t}{v} - \alpha \nabla \frac{v_t}{v} - \alpha' \nabla \frac{v_t}{v}
= 2(m - 1)\Delta v \frac{\left| \nabla v \right|^2}{v} + \frac{2\nabla v \nabla \left| \nabla v \right|^2}{v^2} - 2(m - 1)R_{ij} \nabla v_i \nabla v_j
- 2(v - 1)v_i^2 \nabla \left( \frac{\left| \nabla v \right|^2}{v} \right) \nabla \left( \log v \right)
\]

\[
- \alpha \frac{\nabla v \nabla v_t}{v} + 2m \left( \frac{\left| \nabla v \right|^2}{v^2} \right) \nabla \left( \log v \right)
- \alpha (m - 1) \Delta v \frac{v_t}{v} - 2\alpha (m - 1) \nabla \frac{v_t}{v} \nabla \left( \frac{\left| \nabla v \right|^2}{v} \right) \nabla \left( \log v \right)
- \alpha (m - 1) \Delta v \frac{v_t}{v} - \frac{2\left( \alpha - 1 \right)}{v} \nabla \left( \frac{\left| \nabla v \right|^2}{v} \right) \nabla \left( \log v \right).
\]

(3.13)

It is not difficult to calculate that

\[
2(m - 1)\nabla \left( \frac{\left| \nabla v \right|^2}{v} \right) \nabla \left( \log v \right) - 2\alpha (m - 1) \nabla \left( \frac{v_t}{v} \right) \nabla \left( \log v \right)
= 2(m - 1)\nabla v \nabla \left[ \frac{\left| \nabla v \right|^2}{v} - \alpha \frac{v_t}{v} \right]
= 2(m - 1) \nabla v \nabla F, \tag{3.14}
\]

and

\[
\frac{2}{v} \nabla v \nabla \left| \nabla v \right|^2 - \alpha \frac{2}{v} \nabla v \nabla v_t = \frac{2}{v} \nabla v \nabla \left( \left| \nabla v \right|^2 - \alpha v_t \right)
= \frac{2}{v} \nabla v \nabla (Fv)
= 2\nabla v \nabla F + 2F \frac{\left| \nabla v \right|^2}{v}. \tag{3.15}
\]

We deduce from (3.14) and (3.15) that

\[
2(m - 1)\nabla \left( \frac{\left| \nabla v \right|^2}{v} \right) \nabla \left( \log v \right) - 2\alpha (m - 1) \nabla \left( \frac{v_t}{v} \right) \nabla \left( \log v \right)
\]
suppose that $2m \leq 2F t_1 (3.19)$, we conclude we complete the proof of Lemma 3.1

Further, applying Young’s inequality from (3.16) and (3.17), we have

$$2(m - 1) \Delta v \frac{|\nabla v|^2}{v} - \frac{|\nabla v|^4}{v^2} - \alpha (m - 1) \Delta v \frac{v_t}{v} + \alpha \frac{v_t}{v} |\nabla v|^2$$

$$= 2 \frac{v_t}{v} \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right) - \frac{|\nabla v|^4}{v^2} - \alpha \frac{v_t}{v} \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right) + \alpha \frac{v_t}{v} |\nabla v|^2$$

$$= (2 \alpha + 2) \frac{v_t}{v} |\nabla v|^2 - 3 |\nabla v|^4 - \alpha \left( \frac{v_t}{v} \right)^2.$$  

From (3.16) and (3.17), we have

$$2(m - 1) \Delta v \frac{|\nabla v|^2}{v} = 2 \alpha (m - 1) \Delta v \frac{(v_t)}{v} \nabla (\log v) + \frac{2}{v} \nabla v \nabla |\nabla v|^2$$

$$- \alpha |\nabla v|^2 \nabla v + 2(m - 1) \Delta v \frac{|\nabla v|^2}{v} - \frac{|\nabla v|^4}{v^2} - \alpha (m - 1) \Delta v \frac{v_t}{v} + \alpha \frac{v_t}{v} |\nabla v|^2$$

$$= 2m \nabla v \nabla F - \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right)^2 + (1 - \alpha) \left( \frac{v_t}{v} \right)^2$$

$$\leq 2m \nabla v \nabla F - [(m - 1) \Delta v]^2 \text{ for } \alpha > 1. \quad (3.18)$$

Substituting (3.18) into (3.13), we arrive at

$$\mathcal{L}(F) \leq -2(m - 1)(v^2_i + \alpha R_{ij} v_{ij}) - 2(m - 1)R_{ij} |\nabla v|^2$$

$$+ 2m \nabla v \nabla F - [(m - 1) \Delta v]^2 - (2(\alpha - 1)R_{ij} \frac{|\nabla v|^2}{v} - \alpha' \varphi - \alpha \varphi'). \quad (3.19)$$

Further, applying Young’s inequality

$$|R_{ij}| v_{ij} | \leq \frac{\alpha}{2} R_{ij}^2 + \frac{1}{2 \alpha} v_{ij}^2$$

to (3.19), we conclude We complete the proof of Lemma 3.1. \qed

**Lemma 3.2.** Suppose that $(M^n, g(t))_{t \in [0, T]}$ satisfies the hypotheses of Theorem 2.1. We also assume that $\alpha(t) > 1$ and $\varphi(t) > 0$ satisfy the following system

$$\begin{cases}
\frac{2 \varphi}{n(m - 1)} - 2(m - 1)MK \geq \left( \frac{2 \varphi}{n(m - 1)} - \alpha' \right) \frac{1}{\alpha}, \\
\frac{2 \varphi}{n(m - 1)} - \alpha' > 0, \\
\frac{\varphi^2}{n(m - 1)} + \alpha \varphi' \geq 0,
\end{cases} \quad (3.20)$$

and $\alpha(t)$ is non-decreasing. Then

$$\mathcal{L}F \leq - (m - 1) \sum_{i,j}^n \left[ v_{ij} + \frac{\varphi}{n(m - 1)} \delta_{ij} \right]^2 - \left[ \frac{2 \varphi}{n(m - 1)} - \alpha' \right] \frac{1}{\alpha} F^2.$$
\[ + (m - 1)\alpha^2 K^2 + 2(\alpha - 1)K \frac{\nabla v^2}{v} + 2m\nabla v \nabla F - [(m - 1)\Delta v]^2. \quad (3.21) \]

**Proof.** By utilizing the unit matrix \((\delta_{ij})_{n \times n}\) and (3.3), we obtain

\[
\mathcal{L}(F) \leq -(m - 1) \sum_{i,j} \left[ v_{ij}^2 + \frac{\varphi}{n(m-1)} \delta_{ij}^2 \right] + \frac{2\varphi}{n(m-1)} + \frac{2\varphi}{n} \Delta v \\
+ (m - 1)\alpha^2 K^2 + 2(\alpha - 1)K \frac{\nabla v^2}{v} + 2m\nabla v \nabla F \\
- [(m - 1)\Delta v]^2 + 2(\alpha - 1)K \frac{\nabla v^2}{v} - \alpha' v_i - \alpha' \varphi - \alpha \varphi'.
\]

Applying (3.2) to above inequality, we have

\[
\mathcal{L}(F) \leq -(m - 1) \sum_{i,j} \left[ v_{ij}^2 + \frac{\varphi}{n(m-1)} \delta_{ij}^2 \right] - \left[ \frac{2\varphi}{n(m-1)} - 2(\alpha - 1)MK \right] \frac{\nabla v^2}{v} \\
+ \left[ \frac{2\varphi}{n(m-1)} \right] v_i + \left[ \frac{2\varphi}{n(m-1)} \right] \frac{\alpha \varphi}{\alpha} + \left[ \frac{2\varphi}{n(m-1)} \right] \frac{\varphi}{\alpha} \\
- \left[ \frac{2\varphi}{n(m-1)} \right] \frac{\alpha \varphi}{\alpha} + (m - 1)\alpha^2 K^2 + 2(\alpha - 1)K \frac{\nabla v^2}{v} \\
+ 2m\nabla v \nabla F - [(m - 1)\Delta v]^2 + 2(\alpha - 1)K \frac{\nabla v^2}{v} - \alpha' \varphi - \alpha \varphi'.
\]

Again using (3.20), we follows (3.21). \qed

**Lemma 3.3.** Let \(G = \gamma(t) F\). Then

\[
\mathcal{L}G \leq -\frac{1}{n\alpha^2 \gamma} G^2 + \left[ \frac{\gamma'}{\gamma} - \left( \frac{2\varphi}{n(m-1)} - \alpha' \right) \frac{1}{\alpha} \right] G \\
- \frac{2(\alpha - 1)\nabla v^2}{v} G - \frac{\gamma(m - 1)(\alpha - 1)^2}{n\alpha^2} \frac{\nabla v^4}{v^2} \\
+ (m - 1)\alpha^2 \gamma K^2 + 2\gamma(\alpha - 1)K \frac{\nabla v^2}{v} + 2m\nabla v \nabla G, \quad (3.22)
\]

where \(a = \frac{n(m-1)}{n(m-1)+1}\).

**Proof.** Sample calculation gives

\[
\mathcal{L}G = \gamma \mathcal{L}F + \gamma' F \\
\leq -(m - 1)\gamma \left[ v_{ij}^2 + \frac{\varphi}{n(m-1)} \delta_{ij}^2 \right] + \left[ - \left( \frac{2\varphi}{n(m-1)} - \alpha' \right) \frac{1}{\alpha} + \frac{\gamma'}{\gamma} \right] G \\
+ (m - 1)\alpha^2 \gamma K^2 + 2\gamma(\alpha - 1)K \frac{\nabla v^2}{v} + 2m\nabla v \nabla G \\
- \gamma[(m - 1)\Delta v]^2. \quad (3.23)
\]

Since

\[
\left[ v_{ij}^2 + \frac{\varphi}{n(m-1)} \delta_{ij}^2 \right] \geq \frac{1}{n} (\Delta v + \varphi) \geq \frac{1}{n a^2 (m-1)^2} \left[ F + (m - 1)(\alpha - 1)\frac{\nabla v^2}{v} \right]^2, \quad (3.24)
\]
and

\[(m - 1)\Delta v = \frac{F}{\alpha} - \frac{\alpha - 1}{\alpha} \frac{|\nabla v|^2}{v} - \varphi \leq -\frac{F}{\alpha}.\]  

Therefore, we follow that from (3.23), (3.24) and (3.25)

\[LG \leq -\frac{\gamma}{n\alpha^2(m - 1)} \left[ F + (m - 1)(\alpha - 1)\frac{|\nabla v|^2}{v} \right]^2 \]

\[+ \left[ -\left( \frac{2\varphi}{n(m - 1)} - \frac{1}{\alpha} + \frac{\gamma'}{\gamma} \right) G + (m - 1)\alpha^2\gamma^2 \right. \]

\[+ 2\gamma(\alpha - 1)K \frac{|\nabla v|^2}{v} + 2m\nabla v \nabla G - \frac{C^2}{\alpha^2\gamma}.\]  

(3.26)

From (3.26), we infer (3.22). The proof is complete. \(\square\)

4. Proof of Main Results

In this section, we will prove our main results.

Proof of Theorem 2.1. Now let \(\varphi(r)\) be a \(C^2\) function on \([0, \infty)\) such that

\[\varphi(r) = \begin{cases} 
1 & \text{if } r \in [0, 1], \\
0 & \text{if } r \in [2, \infty),
\end{cases}\]

and

\[0 \leq \varphi(r) \leq 1, \quad \varphi'(r) \leq 0, \quad \varphi''(r) \leq 0, \quad \frac{|\varphi'(r)|}{\varphi(r)} \leq C,\]

where \(C\) is an absolute constant. Let define by

\[\phi(x, t) = \varphi(d(x, x_0, t)) = \varphi \left( \frac{d(x, x_0, t)}{R} \right) = \varphi \left( \frac{\rho(x, t)}{R} \right),\]

where \(\rho(x, t) = d(x, x_0, t)\). By using the maximum principle, the argument of Calabi [2] allows us to suppose that the function \(\phi(x, t)\) with support in \(B_{2R,T}\), is \(C^2\) at the maximum point. By utilize the Laplacian theorem, we deduce that

\[\frac{|\nabla \phi|^2}{\phi} \leq \frac{C}{R^2},\]

(4.1)

\[-\Delta \phi \leq \frac{C}{R^2}(1 + \sqrt{KR}),\]

(4.2)

For any \(0 \leq T_1 \leq T\), let \(H = \phi G\) and \((x_1, t_1)\) be the point in \(B_{2R,T_1}\) at which \(G\) attain its maximum value. We can suppose that the value is positive, because otherwise the proof is trivial. Then at the point \((x_1, t_1)\), we infer

\[\mathcal{L}(H) \geq 0, \quad \nabla G = -\frac{G}{\phi} \nabla \phi.\]

(4.3)

By the evolution formula of the geodesic length under the Ricci flow [4], we calculate

\[\phi_t G = -G\phi' \left( \frac{\rho}{R} \right) \frac{1}{R} \frac{d\rho}{dt} = G\phi' \left( \frac{\rho}{R} \right) \int_{\gamma_{t_1}} \text{Ric}(S, S)ds\]

\[\leq G\phi' \left( \frac{\rho}{R} \right) \frac{1}{R} K_2 \rho \leq G\phi' \left( \frac{\rho}{R} \right) K_2 \leq G\sqrt{C}K_2,\]
where $\gamma_{t,1}$ is the geodesic connecting $x$ and $x_0$ under the metric $g(t_1)$, $S$ is the unite tangent vector to $\gamma_{t,1}$, and $ds$ is the element of the arc length. Hence, by applying (4.2), we have

$$0 \leq \mathcal{L}(H) \leq \phi \mathcal{L}G - (m - 1)G \left( \Delta \phi - 2 \frac{\nabla \phi^2}{\phi} \right) + \phi G$$

$$\leq - \frac{1}{a^2 \gamma} \phi G^2 + \left[ \frac{\gamma'}{\gamma} - \left( \frac{2 \varphi}{n(m - 1) - \alpha'} \right) \frac{1}{\alpha} \right] \phi G$$

$$- \frac{2(\alpha - 1)}{n \alpha^2} \frac{\nabla v^2}{v} \phi G - \frac{\gamma(m - 1)(\alpha - 1)^2}{n \alpha^2} \frac{\nabla v^4}{v^2} \phi$$

$$+(m - 1)\alpha^2 \gamma \phi K^2 + 2\gamma \phi(\alpha - 1)K \frac{\nabla v^2}{v} + 2m \phi \nabla v \nabla G$$

$$- \frac{(m - 1)G}{\phi} \left( \Delta \phi - 2 \frac{\nabla \phi^2}{\phi} \right) + \sqrt{C} K \phi G \quad (4.4)$$

Multiply $\phi$, we have

$$0 \leq - \frac{1}{a^2 \gamma} \phi^2 G^2 + \left[ \frac{\gamma'}{\gamma} - \left( \frac{2 \varphi}{n(m - 1) - \alpha'} \right) \frac{\phi}{\alpha} \right] \phi G$$

$$- \frac{2(\alpha - 1)}{n \alpha^2} \frac{\nabla v^2}{v} \phi G - \frac{\gamma(m - 1)(\alpha - 1)^2}{n \alpha^2} \frac{\nabla v^4}{v^2} \phi$$

$$+ (m - 1)\alpha^2 \gamma \phi^2 K^2 + 2\gamma \phi(\alpha - 1)K \frac{\nabla v^2}{v} + 2m \phi \nabla v \nabla G - (m - 1)G$$

Further using the inequality $Ax^2 + Bx \geq -\frac{B^2}{4A}$ with $A > 0$, we have

$$- \frac{2(\alpha - 1)}{n \alpha^2} \frac{\nabla v^2}{v} \phi G - 2m \phi \nabla v \nabla G \leq \frac{nm \alpha^2}{2(\alpha - 1)} \frac{\nabla \phi^2}{\phi} G,$$

$$- \frac{\gamma(m - 1)(\alpha - 1)^2}{n \alpha^2} \frac{\nabla v^4}{v^2} \phi + 2\gamma \phi(\alpha - 1)K \frac{\nabla v^2}{v} \leq \frac{m \alpha^2 K^2}{m - 1} \phi^2 \gamma.$$ 

Hence, we deduce that

$$0 \leq - \frac{1}{a^2 \gamma} \phi^2 G^2 + \left[ \frac{\gamma'}{\gamma} - \left( \frac{2 \varphi}{n(m - 1) - \alpha'} \right) \frac{\phi}{\alpha} \right] \phi G$$

$$+ (m - 1) \left( \Delta \phi - 2 \frac{\nabla \phi^2}{\phi} \right) + \sqrt{C} K \phi G$$

$$+ (m - 1)\alpha^2 K^2 \phi^2 \gamma + \frac{m \alpha^2 K^2}{m - 1} \phi^2 \gamma. \quad (4.5)$$

Combine (4.1), (4.2) and (4.5), we have

$$0 \leq - \frac{1}{a^2 \gamma} \phi^2 G^2 + \left[ \frac{\gamma'}{\gamma} - \left( \frac{2 \varphi}{n(m - 1) - \alpha'} \right) \frac{\phi}{\alpha} \right] \phi G$$

$$+ \frac{C(m - 1)}{R^2} (1 + \sqrt{KR} + \sqrt{C} K) \phi G$$

$$+ (m - 1)\alpha^2 K^2 \phi^2 \gamma + \frac{m \alpha^2 K^2}{m - 1} \phi^2 \gamma.$$
This inequality becomes
\[
\frac{1}{aa^2\gamma}\phi^2G^2 - \left[\frac{\gamma'}{\gamma} - \left(\frac{2\varphi}{n(m-1)} - \alpha'\right)\frac{\phi}{\alpha} + \frac{Cm^2\alpha^2}{R^2(\alpha - 1)}\right] + \frac{C(m - 1)}{R^2}(1 + \sqrt{KR}) + \sqrt{C}G = \frac{(m - 1)\alpha^2K^2\phi^2\gamma}{} + \frac{n\alpha^2K^2}{m - 1}\phi^2\gamma.
\]

For the inequality \(Ax^2 - 2Bx \leq C\), one has \(x \leq \frac{2B}{A} + \left(\frac{C}{A}\right)^{\frac{1}{2}}\), where \(A, B, C > 0\).

If \(\gamma\) is nondecreasing which satisfies the system
\[
\begin{cases}
\gamma' - \frac{2\varphi}{n} - \alpha'\frac{1}{\alpha} \leq 0, \\
\gamma\alpha^{\alpha - 1} \leq C.
\end{cases}
\]

Recall that \(\alpha(t)\) and \(\gamma(t)\) are non-decreasing and \(t_1 < T_1\). Hence, we have
\[
\phi G(x, T_1) \leq (\phi G)(x_1, t_1) \leq Caa^2(T_1)\gamma(T_1) \left[1\frac{R^2}{R^2} \left(1 + \sqrt{KR}\right) + K\right] + \frac{Cam^2}{R^2} + \alpha^2(T_1)K\gamma(T_1)\phi\sqrt{a(m - 1)} + \frac{K\alpha^2(T_1)\gamma(T_1)}{m - 1}\phi\sqrt{an}.
\]

Hence, we have for \(\phi \equiv 1\) on \(B_{R,T}\),
\[
F(x, T_1) \leq Caa^2(T_1)\gamma(T_1) \left[1\frac{R^2}{R^2} \left(1 + \sqrt{KR}\right) + K\right] + \frac{Cam^2}{R^2\gamma(T_1)} + \alpha^2(T_1)K\sqrt{a(m - 1)} + \frac{K\alpha^2(T_1)}{m - 1}\sqrt{an}.
\]

If \(\gamma\) is nondecreasing which satisfies the system
\[
\begin{cases}
\gamma' - \frac{2\varphi}{n} - \alpha'\frac{1}{\alpha} \leq 0, \\
\gamma\alpha^{\alpha - 1} \leq C.
\end{cases}
\]

Recall that \(\alpha(t)\) and \(\gamma(t)\) are non-decreasing and \(t_1 < T_1\). Hence, we have
\[
\phi G(x, T_1) \leq (\phi G)(x_1, t_1) \leq Caa^2(T_1)\gamma(T_1) \left[1\frac{R^2}{R^2} \left(1 + \sqrt{KR}\right) + K\right] + \frac{Cam^2}{R^2}.
\]
\[ + \alpha^2(T_1)K \gamma(T_1)\sqrt{\alpha(m-1)} + \frac{K\alpha^2(T_1)\gamma(T_1)}{m-1} \sqrt{\alpha n}. \]

Hence, we have for \( \phi \equiv 1 \) on \( B_{R,T} \),
\[
F(x,T_1) \leq C\alpha^2(T_1) \left[ \frac{1}{R^2} \left( 1 + \sqrt{KR} \right) + K \right] + \frac{Cm^2\alpha^4(T_1)}{R^2\gamma(T_1)} \\
+ \alpha^2(T_1)K \sqrt{\alpha(m-1)} + \frac{K\alpha^2(T_1)}{m-1} \sqrt{\alpha n}. \]

Because \( T_1 \) is arbitrary, so the conclusion is valid. \( \square \)

5. Appendix

We will check some functions \( \alpha(t) > 1, \varphi(t) > 0 \) and \( \gamma(t) > 0 \) in Remark satisfy the following two systems

\[
\begin{cases}
2\varphi(n(m-1)) - 2(m-1)MK \geq \left( \frac{2\varphi}{n(m-1)} - \alpha' \right) \frac{1}{\alpha}, \\
\frac{2\varphi}{n(m-1)} - \alpha' > 0, \\
\frac{\varphi^2}{n(m-1)} + \alpha\varphi' \geq 0.
\end{cases}
\]

and

\[
\begin{cases}
\gamma' - \frac{2\varphi}{n(m-1)} - \alpha' \frac{1}{\alpha} \leq 0, \\
\frac{\gamma\alpha^4}{\alpha - 1} \leq C, \text{ or } \frac{\gamma}{\alpha - 1} \leq C.
\end{cases}
\]

Besides, \( \alpha(t) \) and \( \gamma(t) \) are non-decreasing.

(1) Let \( \alpha(t) = 1 + (m-1)MKt, \varphi(t) = \frac{n(m-1)}{t} + n(m-1)^2MK \) and \( \gamma(t) = (m-1)MKt. \)

Direct calculation shows

(i) \[ \frac{2\varphi}{n(m-1)} - \alpha' \]
\[
= \frac{2}{t} + 2(m-1)MK - (m-1)MK > 0,
\]
(ii) \[ \frac{\varphi^2}{n(m-1)} + \alpha\varphi' = \frac{n(m-1)}{t^2} + n(m-1)^3MK^2 + \frac{2}{t}n(m-1)^2MK \]
\[
+ \left[ 1 + (m-1)MKt \right] \left[ - \frac{n(m-1)}{t^2} \right] > 0
\]
(iii) \[ \alpha \left( \frac{2\varphi}{n(m-1)} - 2(m-1)MK \right) - \left( \frac{2\varphi}{n(m-1)} - \alpha' \right) \]
\[
= (\alpha - 1) \frac{2\varphi}{n(m-1)} - 2(m-1)MK\alpha + \alpha' \]
\[
= 2(m-1)MK + 2(m-1)^2MK^2t - 2(m-1)MK \\
- 2(m-1)^2MK^2t + \alpha' > 0
\]

Hence, \( \alpha(t) = 1 + \frac{1}{3}(m-1)MKt, \varphi(t) = \frac{n(m-1)}{t} + \frac{1}{3}n(m-1)^2MK \) satisfy the system (5.1).
On the other hand, one has
\[
\frac{\gamma'}{\gamma} - \left(\frac{2\varphi}{n(m-1)} - \alpha'\right) \frac{1}{\alpha} = \frac{1}{t} - \left[\frac{2}{t} + \frac{2}{3}(m-1)MK - \frac{1}{3}(m-1)MK\right] \frac{1}{1 + \frac{4}{3}(m-1)MKt} \\
= \frac{1}{t} - \frac{1}{t(1 + \frac{4}{3}(m-1)MKt)} \\
\leq 0, \quad \text{for} \quad t \geq 0.
\]
and \(\frac{n}{\alpha} = 1\). So, \((5.2)\) is also satisfied.

\[\text{(2) } \alpha(t) = e^{2(m-1)MKt}, \varphi(t) = \frac{n(m-1)}{t} e^{4(m-1)MKt} \text{ and } \gamma(t) = te^{2(m-1)MKt}.\]

Direct calculation shows
\[\text{(i) } \frac{2\varphi}{n(m-1)} - \alpha' = \frac{2}{t} e^{2(m-1)MKt}(e^{2(m-1)MKt} - (m-1)MKt) > 0;\]
\[\text{(ii) } \frac{\varphi^2}{n(m-1)} + \alpha \varphi' = \frac{n(m-1)}{t^2} e^{6(m-1)MKt}(e^{2(m-1)MKt} - 1 + 4(m-1)MKt) > 0,\]
\[\text{(iii) } \frac{2\varphi}{n(m-1)} - 2(m-1)MK - \left(\frac{2\varphi}{n(m-1)} - \alpha'\right) \frac{1}{\alpha} = \frac{2}{t} e^{4(m-1)MKt} - 2(m-1)MK - \frac{2}{t} e^{2(m-1)MKt} + 2(m-1)MK \\
= \frac{2}{t} e^{2(m-1)MKt}(e^{2(m-1)MKt} - 1) \geq 0.\]

Hence, \(\alpha(t) = e^{2(m-1)MKt}\) and \(\varphi(t) = \frac{n(m-1)}{t} e^{4(m-1)MKt}\) satisfy the system \((5.1)\).

Besides, we have
\[
\frac{\gamma'}{\gamma} - \left(\frac{2\varphi}{n(m-1)} - \alpha'\right) \frac{1}{\alpha} = \frac{1}{t} - \left(\frac{2}{t} e^{2(m-1)MKt} - 2(m-1)MK\right) \\
= \frac{1}{t}(1 + 4(m-1)MKt - 2e^{2(m-1)MKt}) \\
\leq 0, \quad \text{for} \quad t \geq 0.
\]
and as \(t \to 0^+\), \(\frac{n}{\alpha(t)} = \frac{2\varphi}{e^{2MKt}} \to \frac{1}{MK}\). This implies \(\frac{n}{\alpha(t)} \leq C\). So, \((5.2)\) is also satisfied.

\[\text{(3) } \alpha(t) = 1 + \frac{\sinh((m-1)MKt)}{\sinh((m-1)MKt)} \text{ and } \gamma(t) = \tanh((m-1)MKt).\]

Direct calculation shows
\[\alpha'(t) = 2(m-1)MK - 2(\alpha - 1)(m-1)MK \coth((m-1)MKt).\]

Then
\[\text{(i) } \frac{2\varphi}{n(m-1)} - \alpha' = 2(m-1)MK[1 + \coth((m-1)MKt)] - 2(m-1)MK \\
+ 2(\alpha - 1)(m-1)MK \coth((m-1)MKt) > 0,\]
\[\text{(ii) } \alpha = \frac{2\varphi}{n(m-1)} - (\frac{2\varphi}{n(m-1)} - \alpha') \\
= 2(m-1)MK[\alpha[1 + \coth((m-1)MKt)] - 2(m-1)MK \alpha.\]
where

\[-2(m - 1)MK[1 + \coth(m - 1)MKt] + \alpha'+
\]

\[= 2(m - 1)MK(\alpha - 1)[1 + \coth((m - 1)MKt)] - 2(m - 1)MK + \alpha' + (m - 1)MK(\alpha - 1)\coth((m - 1)MKt) - 2(m - 1)MK + \alpha' = 0\]

Let \(x = (m - 1)MKt\), then

\[\frac{\varphi^2}{n(m - 1)} + \alpha \varphi' = \frac{n(m - 1)^3 M^2 K^2}{\sinh^2 (m - 1)MKt} \left( [1 + \coth((m - 1)MKt)]\sinh^2 (m - 1)MKt - \alpha \right).\]

Let \(x = 1 + \coth((m - 1)MKt)\) and \(\varphi(t) = 2n(m - 1)^2 MK[1 + \coth((m - 1)MKt)]\) satisfy the system (5.1).

On the other hand, as \(t \to 0\), we have \(\frac{\alpha^4}{\alpha^2 - 1} \to 2\); \(\frac{\alpha^4}{\alpha - 1} \to 1\) for \(t \to \infty\). These imply \(\frac{\alpha^4}{\alpha^2 - 1} \leq C\), here \(C\) is a universal constant.

Besides, we have

\[
\frac{\gamma'}{\gamma} - \frac{2\varphi}{n(m - 1)} - \alpha' \frac{1}{\alpha} = \frac{1}{\alpha} \left[ \frac{x\alpha}{\sinh(xt) \cosh(xt)} - 2x - 2x(1 + \alpha) \coth(xt) \right] = \frac{1}{\alpha} \left[ \frac{x}{\sinh(xt) \cosh(xt)} [\alpha - 2(1 + \alpha) \cosh^2(xt)] - 2x \right] = \frac{1}{\alpha} \left[ \frac{x}{\sinh(xt)} [\alpha(1 - 2 \cosh(xt)) - 2 \cosh(xt) - 2K] \right] \leq 0, \quad \text{for } t \geq 0,
\]

where \(x = (m - 1)MK\). So, (5.2) is also satisfied.

(4) \(\alpha(t) = \text{constant}, \varphi(t) = \frac{\alpha n(m - 1)}{t} + \frac{n(m - 1)^2 MK}{\alpha - 1}\) and \(\gamma(t) = t^\theta\) with \(0 < \theta \leq 2\).

Direct calculation gives

\[
(i) \quad \frac{2\varphi}{n(m - 1)} - \alpha' = \frac{2\alpha}{t} + \frac{(m - 1)MK}{\alpha - 1} > 0,
\]

\[
(ii) \quad \frac{\varphi^2}{n(m - 1)} + \alpha \varphi' = \frac{n(m - 1)\alpha^2}{t^2} - \frac{n(m - 1)\alpha^2}{t^2} + \frac{n^2(m - 1)^4 M^2 K^2}{(\alpha - 1)^2} + \frac{2\alpha n^2(m - 1)^3 MK}{(\alpha - 1)t} > 0,
\]
\[
(iii) \quad \alpha \left(\frac{2\phi}{n(m-1)} - 2(m-1)MK\right) - \left(\frac{2\phi}{n(m-1)} - \alpha'\right)
\]
\[
= (\alpha - 1) \frac{2\phi}{n(m-1)} - 2(m-1)MK > 0.
\]

Hence, \(\alpha(t) = \text{constant}\), and \(\varphi(t) = \frac{\alpha n(m-1)}{t} + \frac{n(m-1)^{2}MK}{\alpha-1}\) satisfy the system (5.1).

On the other hand, we have
\[
\gamma' - \left(\frac{2\phi}{n(m-1)} - \alpha'\right) \frac{1}{\alpha} = \frac{\theta}{t} - \frac{2}{t} \left(\frac{m-1}{\alpha-1}\right) MK \leq 0, \text{ for } t \geq 0 \text{ and } 0 < \theta \leq 2.
\]

So, (5.2) is also satisfied.

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(W. Wang\textsuperscript{1,2}, H. Zhou\textsuperscript{1,2}, D. Xie\textsuperscript{1}) 1. \textsc{School of Mathematics and Statistics, Hefei Normal University, Hefei 230601, P.R.China};

2. \textsc{School of mathematical Science, University of Science and Technology of China, Hefei 230026, China}

\textit{E-mail address: wwen2014@mail.ustc.edu.cn}