Existence of gravity-capillary Crapper waves with concentrated vorticity

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Abstract. The aim of this paper is to prove the existence of gravity-capillary Crapper waves with the presence of vorticity. In particular, we consider a concentrated vorticity: point vortex and vortex patch. We show that for small gravity and small vorticity it is possible to demonstrate that the waves are overhanging.

1. Introduction

This paper is devoted to the study of perturbations of the Crapper waves through gravity and concentrated vorticity. The problem that we analyze is the free-boundary stationary Euler equation with vorticity

\[
\begin{align*}
\mathbf{v} \cdot \nabla \mathbf{v} + \nabla p + g e_2 &= 0 \quad \text{in } \Omega \\
\nabla \cdot \mathbf{v} &= 0 \quad \text{in } \Omega \\
\nabla_\perp \cdot \mathbf{v} &= \mathbf{\omega} \quad \text{in } \Omega \\
v \cdot \mathbf{n} &= 0 \quad \text{on } S \\
p &= TK \quad \text{on } S
\end{align*}
\]

Here \( v \) and \( p \) are the velocity and the pressure, respectively; \( g \) is the gravity, \( e_2 \) is the second vector of the Cartesian basis, \( K \) is the curvature of the free boundary, \( T \) the surface tension and \( \mathbf{\omega} \) the vorticity, that we will specify later. Moreover, since \( \Omega \) is defined as a fluid region and \( \mathbb{R}^2 \setminus \Omega \) as a vacuum region, there exists an interface \( S \) that separates the two regions and \( n \) is the normal vector at the interface. We parametrize the interface with \( z(\alpha) = (z_1(\alpha), z_2(\alpha)) \), for \( \alpha \in [-\pi, \pi] \). Thus \( \Omega \) is defined for \( -\pi < x < \pi \) and \( y \) below the interface \( z(\alpha) \) with \( z_2(\pm\pi) = 1 \).

The Crapper waves are exact solutions of the water waves problem with surface tension at infinite depth. In [11], Crapper proves the existence of pure capillary waves with an overhanging profile. Its result has been extended in [16] by Kinnersley for the finite depth case.

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and in [1] by Akers-Ambrose-Wright by adding a small gravity. In [2] Ambrose-Strauss-Wright analyze the global bifurcation problem for traveling waves, considering the presence of two fluids and in [9] and [10], Córdoba-Enciso-Grubic add beyond the small gravity a small density in the vacuum region in order to prove the existence of self-intersecting Crapper solutions with two fluids.

In the present paper we will deal with rotational waves. The literature about these waves is very recent and the first important result is the one by Constantin and Strauss [5]. They study the rotational gravity water waves problem without surface tension at finite depth and they are able to prove the existence of large amplitude waves. Later, in [7], Constantin and Varvaruca extend the Babenko equation for irrotational flow [3] to the gravity water waves with constant vorticity at finite depth. They remark that the new formulation opens the possibility of using global bifurcation theory to show the existence of large amplitude and possibly overhanging profiles. Furthermore, in a recent paper [6], the same authors construct waves of large amplitude via global bifurcation. Such waves could have overhanging profiles but their explicit existence is still an open problem.

Furthermore, there are some new results by Hur and Vanden-Broeck [14] and by Hur and Wheeler [15], where the authors prove the numerical and further analytical existence of a new exact solution for the periodic traveling waves in a constant vorticity flows of infinite depth, in the absence of gravity and surface tension. They show that the free surface is the same as that of Crapper’s capillary waves in an irrotational flow.

Concerning the presence of surface tension in a rotational fluid we recall the works by Wahlén, in [20], where the author proves the existence of symmetric regular capillary waves for arbitrary vorticity distributions, provided that the wavelength is small enough and in [19], he adds a gravity force acting at the interface and proves the existence of steady periodic capillary-gravity waves. As far as we know, there is not a proof of the existence of overhanging waves in both capillary and gravity-capillary rotational settings, with a fixed period. In [12], De Boeck shows that Crapper waves are limiting configuration for both gravity-capillary water waves in infinite depth (see also [1]) and gravity-capillary water waves with constant vorticity at finite depth. His formulation comes from the one introduced in [7] and the idea is based on taking a small period, which implies that Crapper’s waves govern both gravity-capillary and gravity-capillary with constant vorticity at finite depth. Differently from his work, we will consider a fixed period and small and concentrated vorticity as the point vortex and the vortex patch.

In [18], Shatah, Walsh and Zheng study the capillary-gravity water waves with concentrated vorticity and they extend their work in [13] by considering an exponential localized vorticity; in both cases they perturb from the flat and they do not consider overhanging profiles.

However, the technique we will use is completely different from the cited papers since we would like to show the existence of a perturbation of Crapper’s waves with both small concentrated vorticity and small gravity.

1.1. Outline of the paper. In section 2 we describe the setting in which we work and we introduce a new formulation for the problem (1), through the stream function and a proper change of coordinates to fix the domain. In section 3 we describe the point vortex formulation and the principal operators that identify our problem. In the end of the section we will prove the main theorem 3.1 which shows the existence of a perturbation of Crapper’s waves with a small point vortex. In the last section we introduce the problem (1)
with a vortex patch, which we identify through three operators and the implicit function theorem allow us to prove the existence of a perturbation of Crapper’s waves also with a small vortex patch, theorem 4.1.

2. Setting of the problem

The interface $S = \partial \Omega$, between the fluid region with density $\rho = 1$ and the vacuum region, has a parametrization $z(\alpha)$ which satisfies the periodicity conditions

$$z_1(\alpha + 2\pi) = z_1(\alpha) + 2\pi, \quad z_2(\alpha + 2\pi) = z_2(\alpha),$$

and it is symmetric with respect to the $y-$axis

$$z_1(\alpha) = -z_1(-\alpha), \quad z_2(\alpha) = z_2(-\alpha).$$

The aim of this paper is to prove the existence of perturbations of the Crapper waves with vorticity through the techniques developed in [17], in [1] and [9]. First of all we will rewrite the system (1) in terms of the stream function and then we will do some changes of variables in order to modify the fluid region and to analyse the problem in a more manageable domain. The key point is the use of the implicit function theorem to show that in a neighborhood of the Crapper solutions there exists a perturbation due to the presence of the gravity and the vorticity.

2.1. The stream formulation with vorticity. The fluid flow is governed by the incompressible stationary Euler equations (1). The incompressibility condition (1b) implies the existence of a stream function $\psi : \Omega \to \mathbb{R}$, with $v = \nabla \perp \psi$ and the kinematic boundary condition (1d) implies $\psi = 0$ on $S$. In addition we can rewrite the equation (1a) at the interface by using the condition (1e) and the fact that the vorticity we consider is concentrated in the domain $\Omega$, we end up in the Bernoulli equation.

$$\frac{1}{2} |v|^2 + TK + gy = \text{constant.}$$

We can write the system (1) in terms of the stream function as follows

$$\begin{cases} 
\Delta \psi = \omega & \text{in } \Omega \\
\psi = 0 & \text{on } S \\
\frac{1}{2} |\nabla \psi|^2 + gy + TK = \text{constant} & \text{on } S \\
\frac{\partial \psi}{\partial x} = 0 & \text{on } x = \pm \pi \\
\lim_{y \to 0} \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right) = (c, 0) & \text{on } y = 0 \end{cases}$$

where, the condition (4d) comes from the periodic and symmetric assumptions and the condition (4e) means that the flow becomes uniform at the infinite bottom and $c \in \mathbb{R}$ is
the wave speed. The main problem we have to face is the absence of a potential and is due to the rotationality of the problem. We will treat the point vortex and the vortex patch in two different ways, since the singularity of the problem is distinct, but before dealing with our problem we will focus on the general framework.

2.2. The general vorticity case. The main difficulties of the problem (4) are the presence of a moving interface and the absence of a potential, since the fluid is not irrotational. We recall the Zeidler theory [21] about pseudo-potential, so we introduce the function $\phi$, which satisfies the following equations

\[
\begin{align*}
\frac{\partial \phi}{\partial x} &= W(x, y) \frac{\partial \psi}{\partial y}, \\
\frac{\partial \phi}{\partial y} &= -W(x, y) \frac{\partial \psi}{\partial x},
\end{align*}
\]

(5)

where $W(x, y)$ is exactly equal to 1 when the fluid is irrotational and satisfies

\[
\frac{\partial W}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial W}{\partial y} \frac{\partial \psi}{\partial y} + W \Delta \psi = 0.
\]

(6)

We transform the problem from the $(x, y)$-plane into the $(\phi, \psi)$-plane, by taking the advantage of the fact that the stream function is zero at the interface, see fig. 1. Furthermore, we consider the case of symmetric waves, then it follows that

\[
\begin{align*}
\phi(x, y) &= -\phi(-x, y) \\
\psi(x, y) &= \psi(-x, y),
\end{align*}
\]

(7)

and they satisfy the following relations, coming from (5).

\[
\begin{pmatrix}
\frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \psi} \\
\frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \psi}
\end{pmatrix} = \frac{1}{W(v_1^2 + v_2^2)} \begin{pmatrix} v_1 & -Wv_2 \\
v_2 & Wv_1 \end{pmatrix},
\]

(8)

where $v_1, v_2$ are the components of the velocity field. Moreover, we want to write the system in a non-dimensional setting, thus the new variables are

\[
(\phi, \psi) = \frac{1}{c}(\phi^*, \psi^*), \quad (v_1, v_2) = \frac{1}{c}(v_1^*, v_2^*), \quad \omega = \frac{1}{c}\omega^*,
\]

where the variables with the star are the dimensional one and $c$ is the wave speed. The properties of our problem allow us to pass from $\Omega$ into $\tilde{\Omega}$, defined as follows

\[
\tilde{\Omega} := \{(\phi, \psi) : -\pi < \phi < \pi, -\infty < \psi < 0\}.
\]

(9)

We have to transform the system (4) and the equation (6) in the new coordinates. So we take the derivative with respect to $\phi$ of the condition (4c) and we get
\[ \frac{\partial}{\partial \phi} \left( \frac{v_1^2 + v_2^2}{2} \right) + p \frac{v_2}{v_1^2 + v_2^2} - q \frac{\partial}{\partial \phi} \left[ \frac{W}{\sqrt{v_1^2 + v_2^2}} \left( \frac{v_1}{v_1^2 + v_2^2} \frac{\partial v_2}{\partial \phi} - \frac{v_2}{v_1^2 + v_2^2} \frac{\partial v_1}{\partial \phi} \right) \right] = 0, \] (10)

where \( p = \frac{g}{c^2} \) and \( q = \frac{T}{c^2} \) and (6) becomes
\[ (v_1^2 + v_2^2) \frac{\partial W}{\partial \psi} = W \omega. \] (11)

The problem we study is periodic, so it is more natural to do the analysis in a circular domain. We introduce the independent variable \( \zeta = e^{-i\phi+i\psi} \), where \( \phi + i\psi \) runs in \( \tilde{\Omega} \) and \( \zeta \) in the unit disk, so \( \zeta = \rho e^{i\alpha} \). The relation between \( (\phi,\psi) \) and the variable in the disk \( (\alpha,\rho) \) is the following \( (\phi,\psi) = (\alpha,\log(\rho)) \), where \(-\pi < \alpha < \pi\) and \(0 < \rho < 1\). Thus, we pass from \( \tilde{\Omega} \) into the unit disk, see fig. 1.

![Figure 1. The domains \( \Omega, \tilde{\Omega} \) and the disk.](image)

Furthermore we define the dependent variables \( \tau(\alpha,\rho) \) and \( \theta(\alpha,\rho) \) as follows
\[ \tau = \frac{1}{2} \log(v_1^2 + v_2^2), \quad \theta = \arctan \left( \frac{v_2}{v_1} \right) \] (12)

Thanks to (12), the equation (11) for \( W \) becomes
\[ e^{2\tau} \rho \frac{\partial W}{\partial \rho} = W \omega, \]
then we have
\[ W(\alpha,\rho) = \exp \left( \int_0^\rho \omega \frac{e^{-2\tau(\alpha,\rho')}}{\rho'} d\rho' \right). \] (13)

The derivative of the Bernoulli equation (10), computed at the interface \( z(\alpha) \) which corresponds to \( \rho = 1 \), becomes
\[
\frac{\partial}{\partial \alpha} \left( \frac{1}{2} e^{2\tau(\alpha,1)} \right) - p \frac{e^{-\tau(\alpha,1)} \sin(\theta(\alpha,1))}{W(\alpha,1)} + q \frac{\partial}{\partial \alpha} \left( W(\alpha,1) e^{\tau(\alpha,1)} \frac{\partial \theta}{\partial \alpha} \right) = 0. \tag{14}
\]

3. The point vortex case

3.1. The point vortex framework. We consider a point of constant vorticity, that does not touch the interface \( z(\alpha) \), defined as \( \omega = \omega_0 \delta((x, y) - (0, 0)) \), where \( \delta((x, y) - (0, 0)) \) is a delta distribution taking value at the point \((0, 0)\) and \( \omega_0 \) is a small constant. In addition, since we have a fluid with density 1 inside the domain \( \Omega \) and the vacuum in \( \mathbb{R}^2 \setminus \Omega \), then there is a discontinuity of the velocity field at the interface and a concentration of vorticity \( \tilde{\omega}(\alpha) \delta((x, y) - (z_1(\alpha), z_2(\alpha))) \), where \( \tilde{\omega}(\alpha) \) is the amplitude of the vorticity along the interface. This implies the stream function \( \psi \in \Omega \) to be the sum of an harmonic part

\[
\psi_H(x, y) = \frac{1}{2\pi} \int_0^\pi \log |(x, y) - (z_1(\alpha'), z_2(\alpha'))| \tilde{\omega}(\alpha') \, d\alpha', \tag{15}
\]

which is continuous over the interface and another part related to the point vortex. The velocity can be obtained by taking the orthogonal gradient of the stream function and we have

\[
v(x, y) = (\partial_y \psi_H(x, y), -\partial_x \psi_H(x, y)) + \frac{\omega_0 (y, -x)}{2\pi x^2 + y^2}. \tag{16}
\]

However, in order to describe the point vortex problem we have to adapt the kinematic boundary condition \( [13] \) and the Bernoulli equation \( [14] \), equivalent to \( [15] \). At first, let us compute the velocity at the interface by taking the limit in the normal direction and we get

\[
v(z(\alpha)) = (\partial_z \psi_H, -\partial_{z_1} \psi_H) + \frac{1}{2} \frac{\tilde{\omega}(\alpha)}{|\partial_\alpha z|^2} \partial_\alpha z + \frac{\omega_0}{2\pi} \frac{(z_2(\alpha), -z_1(\alpha))}{|z(\alpha)|^2}
\]

\[
= BR(z(\alpha), \tilde{\omega}(\alpha)) + \frac{1}{2} \frac{\tilde{\omega}(\alpha)}{|\partial_\alpha z|^2} \partial_\alpha z + \frac{\omega_0}{2\pi} \frac{(z_2(\alpha), -z_1(\alpha))}{|z(\alpha)|^2}, \tag{17}
\]

where \( BR(z(\alpha), \tilde{\omega}(\alpha)) \) is the Birkhoff-Rott integral

\[
BR(z(\alpha), \tilde{\omega}(\alpha)) = \frac{1}{2\pi} \text{P.V.} \int_{-\pi}^\pi \frac{(z(\alpha) - z(\alpha'))^\perp}{|z(\alpha) - z(\alpha')|^2} \cdot \tilde{\omega}(\alpha') \, d\alpha'.
\]

Thus the condition \( [16] \) becomes

\[
v(z(\alpha)) \cdot (\partial_\alpha z(\alpha))^\perp = \left( BR(z(\alpha), \tilde{\omega}(\alpha)) + \frac{\omega_0}{2\pi} \frac{(z_2(\alpha), -z_1(\alpha))}{|z(\alpha)|^2} \right) \cdot \partial_\alpha z(\alpha)^\perp = 0. \tag{18}
\]
To deal with the Bernoulli equation and to reach a manageable formulation, we have to use the change of variables described in section 2.2. We pass from the domain $\Omega$ in $(x, y)$ variables, fig. 1 into $\tilde{\Omega}$ in $(\phi, \psi)$ and finally in the unit disk. In order to pass from $\Omega$ into $\tilde{\Omega}$ we use the pseudo-potential defined in (5) and (6). Moreover, as one can see in fig. 2 (left), the interface $z(\alpha)$ is sending in the line $\psi = 0$, thanks to condition (4b), and the point vortex is still a point $(0, \psi_0)$ on the vertical axis due to the oddness of $\phi$. In order to pass from $\tilde{\Omega}$ into the unit disk, see fig. 2 (right), we use the function $e^{\psi - i\phi} = \rho e^{i\alpha}$ and the point vortex $(0, \psi_0)$ becomes a point $(0, \rho_0)$, it does not depend on the angle $\alpha$.

After this change of variables, we rewrite the equation (13) for $W(\alpha, \rho)$, by substituting $\omega = \omega_0 \delta((\alpha, \rho) - (0, \rho_0))$. And we have

$$W(\alpha, \rho) = \begin{cases} 1 & \alpha \neq 0 \\ \exp\left(\frac{\omega_0 e^{-2\tau(\alpha, \rho)}}{\rho_0}\right) & \alpha = 0, \rho_0 \in (0, \rho). \end{cases}$$

We immediately point out that in this case the function $W(\alpha, \rho) = W_{\omega_0, \rho_0} = W_0 \in \mathbb{R}$ and the constant is exactly one when there is no vorticity. The derivative of the Bernoulli equation (14) becomes

$$\frac{\partial}{\partial \alpha} \left(\frac{1}{2} e^{2\tau(\alpha, 1)} \right) - p e^{-\tau(\alpha, 1)} \sin(\theta(\alpha, 1)) W_0 + q W_0 \partial_{\alpha} \left( e^{\tau(\alpha, 1)} \frac{\partial \theta}{\partial \alpha}(\alpha, 1) \right) = 0$$

By integrating with respect to $\alpha$, we get

$$\frac{1}{2} e^{2\tau(\alpha, 1)} - p \frac{\int_{-\pi}^{\alpha} e^{-\tau(\alpha', 1)} \sin(\theta(\alpha', 1)) d\alpha'}{W_0} + q W_0 e^{\tau(\alpha, 1)} \frac{\partial \theta}{\partial \alpha}(\alpha, 1) = \tilde{\gamma}.$$
\[
\sinh(\tau(\alpha, 1)) - \frac{p}{W_0} e^{-\tau(\alpha, 1)} \left( \int_{-\pi}^{\alpha} e^{-\tau(\alpha', 1)} \sin(\theta(\alpha', 1)) d\alpha' - 1 \right) \\
+ q W_0 \frac{\partial \theta(\alpha, 1)}{\partial \alpha} - B e^{-\tau(\alpha, 1)} = 0.
\]

(21)

We can solve our problem by finding \(2\pi\) periodic functions \(\tau(\alpha)\) even and \(\theta(\alpha)\) odd, a function \(\tilde{\omega}(\alpha)\) even, that satisfy the following equations (18) and (21). However, in subsection 2.2 we explain the necessary change of variables to fix the domain. We observe that at the interface \(z(\alpha)\), we have \(\psi(z(\alpha)) = 0\) and \(\rho = 1\), respectively. Thus,

\[
\phi(z(\alpha)) = -\alpha \quad \Rightarrow \quad \nabla \phi(z(\alpha)) \cdot \partial_\alpha z(\alpha) = -1,
\]

(22)

from (5) we get that (22), can be written as follows

\[
W_0 v(z(\alpha)) \cdot \partial_\alpha z(\alpha) = -1.
\]

(23)

Since the equation (23) has been obtained by using the kinematic boundary condition (18), then to solve our problem we will use the Bernoulli equation (21) and the equation (23).

3.2. Crapper formulation. Our goal is to prove the existence of overhanging waves with the presence of concentrated vorticity, such as a point vortex or a vortex patch (Section 4). It is well-known that without vorticity \((\omega_0 = 0\) equivalent to \(W_0 = 1\)), in [1] the authors prove the existence of gravity-capillary overhanging waves. If we remove also the gravity then there is the pillar result of Crapper [11], where the problem was to find a \(2\pi\) periodic, analytic function \(f_c = \theta_c + i\tau_c\) in the lower half plane which solves the Bernoulli equation

\[
\sinh(\tau_c) + q \frac{\partial \theta_c}{\partial \alpha} = 0,
\]

(24)

where \(q = \frac{T}{2}\). Furthermore, the analyticity of the function \(f\) implies that \(\tau_c\) can be written as the Hilbert transform of \(\theta_c\) at the boundary \(\rho = 1\), so the equation above reduces to an equation in the variable \(\theta_c\),

\[
\sinh(H\theta_c) + q \frac{\partial \theta_c}{\partial \alpha} = 0.
\]

(25)

This problem admits a family of exact solutions,

\[
f_c(w) = 2i \log \left( \frac{1 + Ae^{-iw}}{1 - Ae^{-iw}} \right),
\]

(26)
where \( w = \phi + i \psi \) and in this case \((\phi, \psi)\) are harmonic conjugates. The parameter \( A \) is defined in \((-1, 1)\) and for \(|A| < A_0 = 0.45467 \ldots\), the interface do not have self-intersection. Moreover, by substituting (26) into (25) for \( \rho = 1 \), we get \( q = \frac{1+A^2}{1-A^2} \). This implies

\[
T = \frac{1 + A^2}{1 - A^2}.
\]  

(27)

By using (8) in the Crapper case so with \( W = 1 \), coupled with \( \phi = -\alpha \) and \( \rho = 1 \), we get

\[
\partial_\alpha z^c(\alpha) = -e^{-\tau c(\alpha) + \theta c(\alpha)}.
\]  

(28)

We focus on this kind of waves because for some values of the parameter \( A \), these waves are overhanging.

### 3.3. Perturbation of Crapper waves with a point of vorticity

In our formulation, the main difference with respect to the Crapper [11] waves is in the function \( f = \tau + i\theta \) which is not analytic because of the presence of vorticity. The idea is to prove that our solutions are perturbation of the Crapper waves. If we recall the Crapper solution with small gravity but without vorticity, \((\theta_A, \tau_A)\), we know that \( f_A = \theta_A + i\tau_A \) is now analytic and \((\theta_A, \tau_A)\) satisfy the following relations in both \((\phi, \psi)\) and \((\alpha, \rho)\) variables.

\[
\begin{align*}
\frac{\partial \theta_A}{\partial \phi} &= \frac{\partial \tau_A}{\partial \psi} \\
\frac{\partial \theta_A}{\partial \psi} &= -\frac{\partial \tau_A}{\partial \phi}
\end{align*}
\Rightarrow
\begin{align*}
\frac{\partial \theta_A}{\partial \alpha} &= -\rho \frac{\partial \tau_A}{\partial \rho} \\
\frac{\partial \theta_A}{\partial \rho} &= \frac{\partial \tau_A}{\partial \alpha}.
\end{align*}
\]  

(29)

Moreover, \( \tau_A = \mathcal{H}\theta_A \) at the interface. The idea is to write our dependent variables \( \tau \) and \( \theta \) as the sum of a Crapper part and a small perturbation, due to the small vorticity. So we have

\[
\tau = \tau_A + \omega_0 \tilde{\tau}, \quad \theta = \theta_A + \omega_0 \tilde{\theta}.
\]  

(30)

So the Bernoulli equation [21], reduces

\[
\sinh(\mathcal{H}\theta_A + \omega_0 \tilde{\tau}) - pe^{-\mathcal{H}\theta_A - \omega_0 \tilde{\tau}} \left( \frac{1}{W_0} \int_{-\pi}^\alpha e^{-\mathcal{H}\theta_A - \omega_0 \tilde{\tau}} \sin(\theta_A + \omega_0 \tilde{\theta}) d\alpha' - 1 \right) + q \frac{\partial (\theta_A + \omega_0 \tilde{\theta})}{\partial \alpha} W_0 - Be^{-\mathcal{H}\theta_A - \omega_0 \tilde{\tau}} = 0 \quad \text{at} \quad \rho = 1.
\]  

(31)

However, we will figure out that \( \tilde{\tau} \) and \( \tilde{\theta} \) are functions of \( \theta_A \) and \( \tilde{\theta} \) and so (31) will be an equation in the variable \( \theta_A \). In order to end up with this statement we need to use some properties of our problem. We use the incompressibility and rotational conditions and we get the following relations for \((\tau, \theta)\)
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\[
\begin{aligned}
\frac{\partial \theta}{\partial \psi} &= -W_0 \frac{\partial \tau}{\partial \phi} \\
\frac{\partial \theta}{\partial \phi} &= \omega_0 e^{-2\tau(0,\psi_0)} \delta((\phi, \psi) - (0, \psi_0)) + \frac{1}{W_0} \frac{\partial \tau}{\partial \psi}
\end{aligned}
\]  

(32)

By substituting (30) in (32), we get

\[
\begin{aligned}
\omega_0 \frac{\partial \tilde{\theta}}{\partial \psi} &= -W_0 \omega_0 \frac{\partial \tau_A}{\partial \phi} - W_0 \frac{\partial \tau_A}{\partial \phi} - \frac{\partial \theta_A}{\partial \phi} \\
\omega_0 \frac{\partial \tilde{\theta}}{\partial \phi} &= \omega_0 e^{-2H(0,\psi_0)} \delta((\phi, \psi) - (0, \psi_0)) + \frac{1}{W_0} \left( \omega_0 \frac{\partial \tau_A}{\partial \psi} + \frac{\partial \theta_A}{\partial \phi} \right)
\end{aligned}
\]  

(33)

If we cross systems (29) with system (33) then we obtain

\[
\begin{aligned}
\omega_0 \frac{\partial \tilde{\theta}}{\partial \psi} &= -W_0 \omega_0 \frac{\partial \tau_A}{\partial \phi} + (W_0 - 1) \frac{\partial \theta_A}{\partial \phi} \\
\omega_0 \frac{\partial \tilde{\theta}}{\partial \phi} &= \omega_0 e^{-2H(0,\psi_0)} - 2\omega_0 \tau(0,\psi_0) \delta((\phi, \psi) - (0, \psi_0)) + \frac{\partial \tau_A}{\partial \psi} + \left( \frac{1}{W_0} \right) \frac{\partial \theta_A}{\partial \phi}
\end{aligned}
\]  

(34)

By taking the derivative with respect to $\phi$ in the first equation and the derivative with respect to $\psi$ in the second equation and then the difference we get an elliptic equation

\[
W_0 \frac{\partial^2 \tau}{\partial \phi^2} + 1 \frac{\partial^2 \tau}{\partial \psi^2} + \frac{\omega_0 e^{-2\tau(0,\psi_0)}}{W_0} \frac{\partial}{\partial \psi} \delta(\phi, \psi - \psi_0) + \left( \frac{1 - W_0^2}{W_0} \right) \frac{\partial^2 \theta_A}{\partial \phi \partial \psi} = 0
\]  

(35)

We can do the same as (32), (33) and (34) also in the variables $(\alpha, \rho)$ and the elliptic equation is the following

\[
\rho \frac{\omega_0}{W_0} \frac{\partial^2 \tau}{\partial \rho^2} + \frac{1}{W_0} \omega_0 \frac{\partial \tau}{\partial \rho} + \frac{W_0}{\rho} \omega_0 \frac{\partial^2 \tau}{\partial \alpha^2} + \frac{\partial}{\partial \rho} \left( \frac{\omega_0 e^{-2\theta_A - 2\omega_0 \tau}}{W_0} \right) + \left( \frac{1 - W_0^2}{W_0} \right) \frac{\partial^2 \theta_A}{\partial \alpha \partial \rho} = 0.
\]

Once we solve the elliptic equation we have a solution $\tilde{\tau}$ as a function of $\theta_A$ and thanks to the relations (34) also $\tilde{\theta}$ is a function of $\theta_A$.

3.4. The elliptic problem. In this section we want to show how to solve the elliptic problem. For simplicity, we will study the problem in the $(\phi, \psi)$ coordinates thus, from (35), the system is

\[
W_0 \omega_0 \frac{\partial^2 \tau}{\partial \phi^2} + 1 \frac{\partial^2 \tau}{\partial \psi^2} + \frac{\omega_0 e^{-2\tau(0,\psi_0)}}{W_0} \frac{\partial}{\partial \psi} \delta(\phi, \psi - \psi_0) + \left( \frac{1 - W_0^2}{W_0} \right) \frac{\partial^2 \theta_A}{\partial \phi \partial \psi} = 0.
\]
The equation above is a linear elliptic equation with constant coefficients $W_0, \frac{1}{W_0}$. If we do a change of variables we obtain a Poisson equation. In the specific if we define $\phi = W_0 \phi'$, then we have

$$\frac{\partial f}{\partial \phi'}(\phi', \psi) = \frac{\partial f}{\partial \phi} \frac{\partial \phi}{\partial \phi'} = W_0 \frac{\partial f}{\partial \phi} \Rightarrow \frac{\partial f}{\partial \phi} = \frac{1}{W_0} \frac{\partial f}{\partial \phi'}$$

$$\frac{\partial^2 f}{\partial \phi^2} = \frac{1}{W_0^2} \frac{\partial^2 f}{\partial \phi'^2}.$$  

And the domain $\tilde{\Omega}' = \{(\phi', \psi) : -\frac{\pi}{W_0} < \phi < \frac{\pi}{W_0}, -\infty < \psi < 0\}$. By substituting in (35), we have

$$\omega_0 \frac{\partial^2 \tilde{\tau}}{\partial \phi'^2} + \omega_0 \frac{\partial^2 \tilde{\tau}}{\partial \psi'^2} + \omega_0 e^{-2\tau(\phi', \psi)} \frac{\partial}{\partial \psi} \delta(\phi, \psi - \psi_0) + \left(1 - \frac{W_0^2}{W_0}\right) \frac{\partial^2 \theta_A}{\partial \phi' \partial \psi} = 0 \quad (36)$$

Since we are looking for $\tau \in H^2$ and we know that $\tilde{\tau} \in H^2$ so that its Laplacian is in $L^2(\tilde{\Omega}')$; then by the elliptic theory there exists a weak solution and so we can invert the Laplace operator, [4, Theorem 9.25]. We have

$$\omega_0 \tilde{\tau} = \left(-\omega_0 e^{-2\tau(0, \psi_0)} \frac{\partial}{\partial \psi} \delta(\phi, \psi - \psi_0) - \left(1 - \frac{W_0^2}{W_0}\right) \frac{\partial^2 \theta_A}{\partial \phi' \partial \psi}\right) \ast G_2(\phi', \psi), \quad (37)$$

where $G_2$ is the Green function of the Poisson equation in $\tilde{\Omega}'$.

3.5. Existence of gravity rotational perturbed Crapper waves. The main theorem we want to prove is the following

**Theorem 3.1.** Let us consider the water waves problem [1], with a small point vortex and a small gravity $g$. Then, for some values of $A < A_0$, defined in (26), there exist periodic solutions to (1) with overhanging profile.

In order to prove the existence of perturbed rotational Crapper waves we will apply the implicit function theorem around the Crapper solutions.

**Theorem 3.2** (Implicit function theorem). Let $X, Y, Z$ be Banach spaces and $\zeta : X \times Y \to Z$ is a $C^k$, with $k \geq 1$. If $\zeta(x_*, y_*) = 0$ and $D_x \zeta(x_*, y_*)$ is a bijection from $X$ to $Z$, then there exists $\varepsilon > 0$ and a unique $C^k$ map $\chi : Y \to X$ such that $\chi(y_*) = x_*$ and $\zeta(\chi(y_*) , y_*) = 0$ when $\|y - y_*\|_Y \leq \varepsilon$.

The operators that identify the water waves problem with a point vortex are the following
\[ F_1(\theta_A, \tilde{\omega}; B, p, \omega_0) := \sinh(\mathcal{H}\theta_A + \omega_0 \tilde{\omega}(\theta_A)) - pe^{-\mathcal{H}\theta_A - \omega_0 \tilde{\omega}(\theta_A)} \left( \frac{1}{W_0} \int_{-\pi}^{\alpha} e^{-\mathcal{H}\theta_A - \omega_0 \tilde{\omega}(\theta_A)} \sin(\theta_A + \omega_0 \tilde{\omega}(\theta_A)) d\alpha' - 1 \right) \] (38a)

\[ + q\frac{\partial(\theta_A + \omega_0 \tilde{\omega}(\theta_A))}{\partial \alpha} W_0 - Be^{-\mathcal{H}\theta_A - \omega_0 \tilde{\omega}(\theta_A)} \] (38b)

\[ F_2(\theta_A, \tilde{\omega}; B, p, \omega_0) := W_0 (2BR(z(\alpha), \tilde{\omega}(\alpha))) \cdot \partial_\alpha z(\alpha) + \tilde{\omega}(\alpha) \]

\[ + \frac{\omega_0 \left( z_2(\alpha), -z_1(\alpha) \right)}{\pi \left| z(\alpha) \right|^2} \cdot \partial_\alpha z(\alpha) + 2 \]

We have that

\[ (F_1, F_2)(\theta_A, \tilde{\omega}; B, p, \omega_0) : H^2_{odd} \times H^1_{even} \times \mathbb{R}^3 \rightarrow H^1_{even} \times H^1_{even}. \]

3.5.1. **Proof of Theorem 3.1.** We have to analyze the two operators. First we have to show that the operators are zero when computed at the point \((\theta_c, \tilde{\omega}_c; 0, 0, 0)\).

\[ F_1(\theta_c, \tilde{\omega}_c; 0, 0, 0) = \sinh(\mathcal{H}\theta_c) + q\frac{\partial \theta_c}{\partial \alpha} = 0, \] (39)

since this is exactly \[[25]\].

The second operator related to the kinematic boundary conditions satisfies

\[ F_2(\theta_c, \tilde{\omega}_c; 0, 0, 0) = 2BR(z^c(\alpha), \tilde{\omega}_c(\alpha)) + \tilde{\omega}_c(\alpha) + 2 = 0, \] (40)

where \(z^c(\alpha)\) is the parametrization of the Crapper interface and it is zero by construction \[[22]\].

Now, we compute all the Fréchet derivatives. We will take the derivatives with respect to \(\theta_A\) and \(\tilde{\omega}\), then we will compute them at the point \((\theta_c, \tilde{\omega}_c; 0, 0, 0)\) and we will show their invertibility. For the operator \(F_1\) we observe that

\[ D_{\tilde{\omega}} F_1(\theta_c, \tilde{\omega}_c; 0, 0, 0) = 0. \]

It remains to compute the derivative with respect to \(\theta_A\).
\[
D_{\theta_A} F_1 = \left[ \frac{d}{d\mu} \mathcal{F}_1(\theta_A + \mu \theta_1, \psi_H, c; B, p, \varepsilon, \omega_0) \right]_{\mu=0} = \left[ \frac{d}{d\mu} \left[ \sinh(\mathcal{H}\theta_A + \mu \mathcal{H}\theta_1 + \omega_0 \tilde{\tau}(\theta_A + \mu \theta_1)) \right] \right]_{\mu=0} - p e^{-\mathcal{H}\theta_A - \mu \mathcal{H}\theta_1 + \omega_0 \tilde{\tau}(\theta_A + \mu \theta_1)} \left( \frac{1}{W_0} \int_{-\pi}^{\alpha} e^{-\mathcal{H}\theta_A + \mu \mathcal{H}\theta_1 + \omega_0 \tilde{\tau}(\theta_A + \mu \theta_1)} \sin(\theta_A + \mu \theta_1 + \omega_0 \tilde{\theta}) \, d\alpha' - 1 \right) + q \frac{\partial \theta_1}{\partial \alpha} + B \left[ \frac{d}{d\mu} \left[ e^{-\mathcal{H}\theta_A - \mu \mathcal{H}\theta_1 + \omega_0 \tilde{\tau}(\theta_A + \mu \theta_1)} \right] \right]_{\mu=0} =
\]

\[
= \cosh(\mathcal{H}\theta_A + \omega_0 \tilde{\tau}(\theta_A)) \left( \mathcal{H}\theta_1 + \omega_0 \left[ \frac{d}{d\mu} \tilde{\tau}(\theta_A + \mu \theta_1) \right]_{\mu=0} \right) - p \left[ \frac{d}{d\mu} \left[ e^{-\mathcal{H}\theta_A - \mu \mathcal{H}\theta_1 + \omega_0 \tilde{\tau}(\theta_A + \mu \theta_1)} \right] \right]_{\mu=0} \cdot \left( \frac{1}{W_0} \int_{-\pi}^{\alpha} e^{-\mathcal{H}\theta_A + \mu \mathcal{H}\theta_1 + \omega_0 \tilde{\tau}(\theta_A + \mu \theta_1)} \sin(\theta_A + \mu \theta_1 + \omega_0 \tilde{\theta}) \, d\alpha' - 1 \right) \bigg|_{\mu=0} + q W_0 \frac{\partial \theta_1}{\partial \alpha} - B \left[ \frac{d}{d\mu} \left[ e^{-\mathcal{H}\theta_A - \mu \mathcal{H}\theta_1 + \omega_0 \tilde{\tau}(\theta_A + \mu \theta_1)} \right] \right]_{\mu=0}
\]

In order to compute \( \frac{d}{d\theta} \tilde{\tau} \) we refer to (37) and since it is multiply by \( \omega_0 \) that we will take equal to zero, then it will desapper as well as the terms multiplied by \( p, B \). Thus the Fréchet derivative computed at \((\theta_c, \tilde{\omega}_c; 0, 0, 0)\) is

\[
D_{\theta_A} F_1(\theta_c, \tilde{\omega}_c; 0, 0, 0) = \cosh(\mathcal{H}\theta_c) \mathcal{H}\theta_1 + q \frac{\partial \theta_1}{\partial \alpha}. \tag{41}
\]

The Fréchet derivative with respect to \( \theta_A \), can be obtained by substituting the definition of the interface \( z(\alpha) \) to the operator. Indeed, from the equations (43), we get

\[
\begin{align*}
\frac{\partial z_1}{\partial \alpha} &= -\frac{e^{-\tau(\alpha, 1) \cos(\theta(\alpha, 1))}}{W(\alpha, 1)} \\
\frac{\partial z_2}{\partial \alpha} &= -\frac{e^{-\tau(\alpha, 1) \sin(\theta(\alpha, 1))}}{W(\alpha, 1)},
\end{align*}
\tag{42}
\]

By substituting the value of \( W(\alpha, 1) \) for the point vortex and by rewriting \( \tau, \theta \) as the sum of Crapper and a perturbation, then we have

\[
\begin{align*}
z_1(\alpha) &= -\frac{1}{W_0} \int_{-\pi}^{\alpha} e^{-\mathcal{H}\theta_A - \omega_0 \tilde{\tau}(\theta_A)} \cos(\theta_A + \omega_0 \tilde{\theta}(\theta_A)) \, d\alpha' \\
z_2(\alpha) &= -\frac{1}{W_0} \int_{-\pi}^{\alpha} e^{-\mathcal{H}\theta_A - \omega_0 \tilde{\tau}(\theta_A)} \sin(\theta_A + \omega_0 \tilde{\theta}(\theta_A)) \, d\alpha' - 1 \tag{43}
\end{align*}
\]

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In a compact way, the interface $z(\alpha)$ is
\[
z(\alpha) = -\frac{1}{W_0} \int_{-\pi}^{\alpha} e^{-H\theta_A - \omega_0 \tau(\theta_A) + i(\theta_A + \omega_0 \theta_A)} d\alpha' - e_2.
\]  
(44)

The main Fréchet derivative for the operator $\mathcal{F}_2$ is with respect to $\tilde{\omega}$.
\[
D_{\tilde{\omega}} \mathcal{F}_2 = \left[ \frac{d}{d\mu} \mathcal{F}_2(\theta_A, \tilde{\omega} + \mu \omega_1; B, \rho, \omega_0) \right]_{\mu=0} = \left[ \frac{d}{d\mu} [2W_0 BR(z(\alpha), \tilde{\omega}(\alpha) + \mu \omega_1) \cdot \partial_\alpha z(\alpha)
\right.
\]
\[
+ W_0 (\tilde{\omega}(\alpha) + \mu \omega_1(\alpha)) + W_0 \frac{\omega_0}{\pi} \frac{(z_2(\alpha), -z_1(\alpha))}{|z(\alpha)|^2} \cdot \partial_\alpha z(\alpha) + 2] \right]_{\mu=0}
\]
\[
= 2W_0 P.V. \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha'))^\perp}{|z(\alpha) - z(\alpha')|^2} \cdot \omega_1(\alpha') d\alpha' \cdot \partial_\alpha z(\alpha) + W_0 \omega_1(\alpha).
\]

When we compute this derivative at the point $(\theta_c, \tilde{\omega}; 0, 0, 0)$ we get
\[
D_{\tilde{\omega}} \mathcal{F}_2(\theta_c, \tilde{\omega}; 0, 0, 0) = 2BR(z^c(\alpha), \omega_1(\alpha)) \cdot \partial_\alpha z^c(\alpha) + \omega_1(\alpha),
\]  
(45)

where $z^c(\alpha)$ is the parametrization of the Crapper interface coming from (28).

The final step of this proof is to show the invertibility of the derivative’s matrix, defined as follows
\[
D \mathcal{F}(\theta_c, \tilde{\omega}; 0, 0, 0) = \begin{pmatrix}
D_{\theta_A} \mathcal{F}_1 & 0 \\
D_{\theta_A} \mathcal{F}_2 & D_{\tilde{\omega}} \mathcal{F}_2
\end{pmatrix} = \begin{pmatrix}
\Gamma & 0 \\
D_{\theta_A} \mathcal{F}_2 & \mathcal{A}(z^c(\alpha)) + \mathcal{I}
\end{pmatrix} \cdot \begin{pmatrix}
\theta_1 \\
\omega_1
\end{pmatrix}
\]  
(46)

where
\[
\Gamma \theta_1 = \cosh(\mathcal{H} \theta_c) \mathcal{H} \theta_1 + q \frac{d}{d\alpha} \theta_1
\]
\[
(\mathcal{A}(z^c(\alpha)) + \mathcal{I}) \omega_1 = 2BR(z^c(\alpha), \omega_1) \cdot \partial_\alpha z^c(\alpha) + \omega_1.
\]

The invertibility of (46) is related with the invertibility of the diagonal, since the matrix is triangular. Hence we have to analyze the invertibility of the operators $\Gamma$ and $\mathcal{A} + \mathcal{I}$, where $\mathcal{I}$ stays for the identity operator. Below, we resume the properties of the $\Gamma$ operator, for details, see in [1] and [9].

**Lemma 3.3.** The operator
\[
D_{\theta_A} \mathcal{F}_1(\theta_c, \tilde{\omega}; 0, 0, 0) = \cosh(\mathcal{H} \theta_c) \theta_1 + q \frac{d}{d\alpha} \theta_1 = \Gamma \theta_1,
\]
defined $\Gamma : H^1_{odd} \to L^2_{even}$ is injective.
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**Proof.** The injectivity follows from the fact that $\Gamma \theta_1 = 0$ if and only if $\theta_1 = \frac{d\theta_c}{d\alpha}$, see [17, Lemma 2.1]. Moreover, we know that $\theta_c$ is an odd function then $\frac{d\theta_c}{d\alpha}$ is even. This statement implies that the constants are the only trivial solutions of $\Gamma \theta_1 = 0$.

The problem concerning the invertibility of this operator is related with its surjectivity.

**Lemma 3.4.** Let $f \in L^2_{\text{even}}$. Then there exists $\theta_1 \in H^1_{\text{odd}}$ with $\Gamma \theta_1 = f$ if and only if

$$(f, \cos \theta_c) = \int_{-\pi}^{\pi} f(\alpha) \cos \theta_c(\alpha) \, d\alpha = 0$$

**Proof.** The complete proof can be found in [1, Proposition 3.3]. Here we will prove that the cokernel has dimension one and it is spanned by $\cos \theta_c$.

If we consider the operator $\mathcal{F}_1$ with $(p, \omega_0, B) = (0, 0, 0)$, we have

$$\int_{-\pi}^{\pi} \mathcal{F}_1 \cos \theta \, d\alpha = \int_{-\pi}^{\pi} \left( \sin \mathcal{H} \theta + q \frac{d\theta}{d\alpha} \right) \cos \theta \, d\alpha = 0,$$

because the second term is the $q$ multiplied by $\sin \theta$ in the interval $(-\pi, \pi)$ and the first term is 0 because of the Cauchy integral theorem. In particular, if we take the derivative with respect to $\theta$ and we compute it in $\theta_c$ we get

$$\int_{-\pi}^{\pi} \Gamma \theta_1 \cos \theta_c \, d\alpha + \int_{-\pi}^{\pi} \mathcal{F}_1 \sin \theta_c \, d\alpha = \int_{-\pi}^{\pi} \Gamma \theta_1 \cos \theta_c \, d\alpha + \int_{-\pi}^{\pi} \left( q \frac{d\theta_c}{d\alpha} + \sinh \mathcal{H} \theta_c \right) \sin \theta_c \, d\alpha = 0,$$

since the quantity in the brackets is 0 for (24) thus it follows

$$\int_{-\pi}^{\pi} \Gamma \theta_1 \cos \theta_c \, d\alpha = 0.$$  \hfill (47)  

For the operator $\mathcal{A} + \mathcal{I}$ we have the following result, proved in [8].

**Lemma 3.5.** Let $z \in H^3$ be a curve without self-intersections. Then

$$\mathcal{A}(z) \omega = 2B R(z, \omega) \cdot \partial_{\alpha} z$$

defines a compact linear operator

$$\mathcal{A}(z) : H^1 \to H^1$$

whose eigenvalues are strictly smaller than 1 in absolute value. In particular, the operator $\mathcal{A} + \mathcal{I}$ is invertible.

In conclusion, the equations

$$\begin{pmatrix} \Gamma & 0 \\ D_{\theta, \mathcal{F}_2} & D_{\omega, \mathcal{F}_2} \end{pmatrix} \cdot \begin{pmatrix} \theta_1 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$  \hfill (48)
computed at the point \((\theta_c, \tilde{\omega}_c; 0, 0, 0)\) has a solution if and only if \(|A| < A_0\) and \((f, \cos \theta_c) = 0\).

To prove Theorem 3.1, we cannot use directly the implicit function theorem 3.2 since the Fréchet derivative \(D\mathcal{F}\) is not surjective. Following [1] and also [9], we use an adaptation of the Lyapunov-Schmidt reduction argument. Define

\[
\Pi \theta_1 := (\cos \theta_c, \theta_1) \frac{\cos \theta_c}{\|\cos \theta_c\|_2^2},
\]

where \(\Pi\) is the \(L^2\) projector onto the linear span of \(\cos \theta_c\) and from (47) we have \(\Pi \Gamma = 0\).

Thus, we define the projector on \(\Gamma(\mathcal{H}^2_{\text{odd}})\), as

\[
\bar{\mathcal{F}} = ((\mathcal{I} - \Pi)\mathcal{F}_1, \mathcal{F}_2): H^2_{\text{odd}} \times H^1_{\text{even}} \times \mathbb{R}^3 \to \Gamma(\mathcal{H}^2_{\text{odd}}) \times L^2,
\]

where \(\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)\) is defined in (38). The Fréchet derivatives of (49) in \((\theta_A, \tilde{\omega}_A)\) at the Crapper point \((\theta_c, \tilde{\omega}_c; 0, 0, 0)\) is now invertible. So we can apply the implicit function theorem to \(\bar{\mathcal{F}}\) then there exists a smooth function \(\Theta_c: U_{B,p,\omega_0} \to H^2_{\text{odd}} \times H^1_{\text{even}}\), where \(U_{B,p,\omega_0}\) is a small neighborhood of \((0, 0, 0)\) such that \(\Theta_c(0, 0, 0) = (\theta_c, \tilde{\omega}_c)\) and for all \((B, p, \omega_0) \in U_{B,p,\omega_0}\),

\[
\bar{\mathcal{F}}(\Theta_c(B, p, \omega_0); B, p, \omega_0) = 0.
\]

But now, if we consider \(\mathcal{F}(\Theta_c(B, p, \omega_0); B, p, \omega_0)\), defined in (38), then it could not be 0. So we introduce a differentiable function on \(U_{B,p,\omega_0}\):

\[
f(B; p, \omega_0) = (\cos \theta_c, \mathcal{F}_1(\Theta_c(B, p, \omega_0); B, p, \omega_0)).
\]

We have that \(\Pi \mathcal{F}_1 = f(B; p, \omega_0) \frac{\cos \theta_c}{\|\cos \theta_c\|_2^2}\) and if we find a point \((B^*, p^*, \omega_0^*)\) such that \(f(B^*, p^*, \omega_0^*) = 0\), then \(\mathcal{F}_1(\Theta_c(B^*, p^*, \omega_0^*); B^*, p^*, \omega_0^*)) = 0\) and so our problem is solved.

We note that choosing \((B^*, p^*, \omega_0^*) = (0, 0, 0)\), then \(f(0; 0, 0) = 0\). Its derivative with respect to \(B\) is

\[
D_B f(0, 0, 0) = (\cos \theta_c, \Gamma \partial_k \Theta_c + e^{-H\theta_c}) = (\cos \theta_c, e^{-H\theta_c}) = -2\pi,
\]

where we have used (47) and the Cauchy integral theorem. Hence, we can apply the implicit function theorem 3.2 to the function \(f\) and there exists a smooth function \(B^*(p, \omega_0)\) that satisfies \(f(B^*(p, \omega_0); p, \omega_0) = 0\), for \((p, \omega_0)\) in \(U_{p,\omega_0}\), a small neighborhood of \((0, 0)\).

We can resume these results in the following theorem.
Theorem 3.6. Let $|A| < A_0$. There exist $(B, p, \omega_0)$ and a unique smooth function $B^* : U_{p,\omega_0} \rightarrow U_B$, such that $B^*(0, 0) = 0$ and a unique smooth function

$$\Theta_c : U_{p,\omega_0} \rightarrow H^2_{odd} \times H^1_{even},$$

such that $\Theta_c(0, 0, 0) = (\theta_c, \tilde{\omega}_c)$ and satisfy

$$\mathcal{F}(\Theta_c(B^*(p, \omega_0), p, \omega_0); B^*(p, \omega_0), p, \omega_0, ) = 0.$$

The main Theorem 3.1 is a direct consequence of the theorem above.

4. The vortex patch case

4.1. Framework. We consider a patch of vorticity $\omega(x, y) = \omega_0 \chi_D(x, y)$, where $\omega_0 \in \mathbb{R}$ and $\chi_D$ is the indicator function of the vortex domain $D$ near the origin, symmetric with respect to the $y$-axis, satisfying

$$\max\{\text{dist}((x, y), (0, 0))\} << 1, \quad \forall (x, y) \in \partial D,$$

(50)
equivalent to consider a small vortex patch. In this case, as for the previous one, the fluid is incompressible then we introduce a stream function, which is the sum of an harmonic part $\psi_H$, defined in (15) and a part related to the vortex patch

$$\psi_{VP}(x, y) = \frac{\omega_0}{2\pi} \int_D \log |(x, y) - (x', y')| \, dx' \, dy'.$$

(51)

We introduce the parametrization of the boundary $\partial D = \{\gamma(\alpha), \alpha \in [-\pi, \pi]\}$, that for now is a generic parametrization that satisfies the condition (50). The representation of $D$ in the fig. 3 is just an example, since it depends on the choice of the parametrization. Additionally, we obtain the velocity by taking the orthogonal gradient of the stream function, then the velocity associated to (51) is

$$v_{VP}(x, y) = \frac{\omega_0}{2\pi} \int_{\partial D} \log |(x, y) - (\gamma_1(\alpha'), \gamma_2(\alpha'))| \partial_\alpha \gamma(\alpha') \, d\alpha'.$$

(52)

The velocity is the sum of (52) and the orthogonal gradient of the harmonic stream function (15)

$$v(x, y) = (-\partial_y \psi_H(x, y), \partial_x \psi_H(x, y)) + v_{VP}(x, y).$$

(53)

As for the case of the point vortex we have to adapt the problem (1). One of the conditions is the kinematic boundary condition, so we need the velocity at the interface,

$$v(z(\alpha)) = BR(z(\alpha), \tilde{\omega}(\alpha)) + \frac{1}{2} \frac{\tilde{\omega}(\alpha)}{|\partial_\alpha z|^2} \partial_\alpha z + \frac{\omega_0}{2\pi} \int_{-\pi}^\pi \log |z(\alpha) - \gamma(\alpha')| \partial_\alpha \gamma(\alpha') \, d\alpha',$$

(54)

where $z(\alpha)$ is the parametrization of the interface $\partial \Omega$ and we can write the kinematic boundary condition as follows
\[ v(\alpha) \cdot (\partial_\alpha z(\alpha))^\perp = BR(z(\alpha), \omega(\alpha)) \cdot \partial_\alpha z(\alpha)^\perp \]
\[ + \frac{\omega_0}{2\pi} \int_{-\pi}^{\pi} \log |z(\alpha) - \gamma(\alpha')| \partial_\alpha \gamma(\alpha') \, d\alpha' \cdot (\partial_\alpha z(\alpha))^\perp = 0. \]  

(55)

In the analysis of this case we observe that the patch satisfies an elliptic equation and it has a moving boundary. We impose the patch to be fixed by the following condition

\[ v(\gamma(\alpha)) \cdot (\partial_\alpha \gamma(\alpha))^\perp = 0. \]

This is equivalent to require

\[ v(\gamma(\alpha)) \cdot (\partial_\alpha \gamma(\alpha))^\perp = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\gamma(\alpha) - z(\alpha')^\perp \cdot \omega(\alpha') \cdot \partial_\alpha \gamma(\alpha')}{|\gamma(\alpha) - z(\alpha')|^2} \cdot \partial_\alpha \gamma(\alpha) \cdot \partial_\alpha \gamma(\alpha') \, d\alpha' + \frac{\omega_0}{2\pi} \text{P.V.} \int_{\partial D} \log |\gamma(\alpha) - \gamma(\alpha')| \partial_\alpha \gamma(\alpha') \, d\alpha' \cdot \partial_\alpha \gamma(\alpha)^\perp = 0 \]

(56)

Furthermore, we need another condition for identify completely our problem. This condition is related to the Bernoulli equation (1e), equivalent to (4c). The most important issue is to fix the interface \( \partial \Omega \), but for the vortex patch case we will slightly change the idea presented in subsection 2.2 and used for the point vortex case. To pass from the domain \( \Omega(x, y) \) into \( \Omega(\phi, \tilde{\psi}) \) we will consider an approximated stream function \( \tilde{\psi} \) such that

\[
\begin{align*}
\frac{\partial \tilde{\psi}}{\partial x} &= W(x, y) \frac{\partial \psi}{\partial x} \\
\frac{\partial \tilde{\psi}}{\partial y} &= W(x, y) \frac{\partial \psi}{\partial y},
\end{align*}
\]

(57)

hence, in this way \( (\phi, \tilde{\psi}) \) are in relation through the Cauchy-Riemann equations

\[
\begin{align*}
\frac{\partial \phi}{\partial x} &= \frac{\partial \tilde{\psi}}{\partial y} = Wv_1 \\
\frac{\partial \phi}{\partial y} &= - \frac{\partial \tilde{\psi}}{\partial x} = Wv_2,
\end{align*}
\]

(58)

where \( W(x, y) \) has to satisfy (6) and it is equal to 1 in the case of irrotational fluid and \( \Delta \tilde{\psi} = 0 \). Moreover, we point out that the new domain \( \Omega(\phi, \tilde{\psi}) \) is also the lower half plane, see fig. 3, because the approximated stream function \( \tilde{\psi}(z(\alpha)) = 0 \), due to the kinematic boundary condition (1d) and the positivity of \( W \).

In view of the fact that we will use the new coordinate system \((\phi, \tilde{\psi})\), we have to rewrite (8) and (10). Let us start by writing the relation between \((x, y)\) and \((\phi, \tilde{\psi})\), so the system (8) becomes
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\[
\begin{pmatrix}
\frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \tilde{\psi}} \\
\frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \tilde{\psi}}
\end{pmatrix}
= \frac{1}{W(v_1^2 + v_2^2)} \begin{pmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{pmatrix},
\]
(59)

Now, we have to rewrite the Bernoulli equation (4c) in the new coordinates. We will derive (4c) with respect to \(\phi\), such that the constant on the RHS will disappear. Thus we have

\[
\frac{1}{2} \frac{\partial}{\partial \phi} (v_1^2 + v_2^2) + p \frac{v_2}{W(v_1^2 + v_2^2)} - q \frac{\partial}{\partial \phi} \left( \frac{W}{\sqrt{v_1^2 + v_2^2}} \left( v_1 \frac{\partial v_2}{\partial \phi} - v_2 \frac{\partial v_1}{\partial \phi} \right) \right) = 0.
\]
(60)

Finally, it is natural to bring the equations into a disk, because of the periodicity of the problem. As we did for the point and we define

\[
\begin{cases}
\phi = -\alpha \\
\tilde{\psi} = \log \rho,
\end{cases}
\]
(61)

And we pass from \(\tilde{\Omega}(\phi, \tilde{\psi})\) into the unit disk, see fig. 3. We observe that the patch has been chosen symmetric with respect to the vertical axis, so in the coordinates \((\phi, \tilde{\psi})\) it remains symmetric with respect to the \(\tilde{\psi}\)-axis (due to the symmetry of the functions (7)). And in the coordinates \((\alpha, \rho)\), by using (61), it will be symmetric with respect to the horizontal axis and contained in a circular sector, where \(\pm \alpha_1\) are defined through \(\pm \phi_1\), in this way

\[
\text{dist}((\pm \phi_1, \tilde{\psi}_1), (0, 0)) > \text{dist}((\phi, \tilde{\psi}), (0, 0)), \quad \forall (\phi, \tilde{\psi}) \in \partial \tilde{D}.
\]

**Figure 3.** The transformation of the patch \(D(x, y), \tilde{D}(\phi, \tilde{\psi})\) and \(A\).

In addition, by using the independent variables \((\tau, \theta)\), defined in (12), we write the equation \(\Delta \tilde{\psi} = 0\) in the new coordinates \((\phi, \tilde{\psi})\), by using the relations (57) and (58) and we get an equation for \(W(\phi, \tilde{\psi})\).
\[
\frac{\partial W}{\partial \tilde{\psi}} (v_1^2 + v_2^2) - \omega_0 \chi_D = 0 \quad \Rightarrow \quad W(\phi, \tilde{\psi}) - W(\phi, -\infty) = \omega_0 \int_{-\infty}^{\tilde{\psi}} e^{-2\tau(\phi, \tilde{\psi})} \chi_D(\phi, \tilde{\psi}) d\tilde{\psi}'.
\]

Since the value of \( W \) at infinite is 1, then in the variables \((\phi, \tilde{\psi})\), we have

\[
W(\phi, \tilde{\psi}) = 1 + \omega_0 \int_{-\infty}^{\tilde{\psi}} e^{-2\tau(\phi, \tilde{\psi})} \chi_D(\phi, \tilde{\psi}) d\tilde{\psi}'.
\]  

(62)

By using the change of variables (61), we have

\[
W(\alpha, \rho) = 1 + \omega_0 \int_{0}^{\rho} \frac{e^{-2\tau(\alpha, \rho')}}{\rho'} \chi_A(\alpha, \rho') d\rho'
\]

(63)

Concerning the derivative of the Bernoulli equation (60), we get

\[
\frac{\partial}{\partial \alpha} \left( \frac{1}{2} e^{2\tau} \right) - p \frac{e^{-\tau} \sin \theta}{W} + q \frac{\partial}{\partial \alpha} \left( W e^\tau \frac{\partial \theta}{\partial \alpha} \right) = 0.
\]

(64)

By integrating with respect to \( \alpha \) and taking the constant that appears on the RHS as \( \frac{1}{2} + B \), as explained for the point vortex case, we get

\[
\frac{1}{2} e^{2\tau} - p \left( \int_{-\pi}^{\alpha} \frac{e^{-\tau} \sin \theta}{W(\alpha, 1)} d\alpha' - 1 \right) + q W(\alpha, 1) e^\tau \frac{\partial \theta}{\partial \alpha} = \frac{1}{2} + B,
\]

We multiply by \( e^{-\tau} \) and we obtain the equation

\[
\sinh(\tau(\alpha, 1)) - p e^{-\tau(\alpha, 1)} \left( \int_{-\pi}^{\alpha} \frac{e^{-\tau(\alpha', 1)} \sin \theta(\alpha', 1)}{W} d\alpha' - 1 \right) + q W \frac{\partial \theta(\alpha, 1)}{\partial \alpha} - B e^{-\tau(\alpha, 1)} = 0.
\]

(65)

We can solve our problem by finding a \( 2\pi \) periodic functions \( \tau(\alpha) \) even and \( \theta(\alpha) \) odd, an even function \( \tilde{\omega}(\alpha) \) and a curve \( \gamma(\alpha) \), which is the parametrization of the vortex patch that satisfy (55), (56) and (65). As we did for the point vortex, the kinematic boundary condition (55), can be replaced by

\[
W(\alpha, 1) v(z(\alpha)) \cdot \partial_\alpha z(\alpha) = -1,
\]

(66)

then our problem reduced to analyze the equation (56), (65) and (66).
4.2. Perturbation of the Crapper formulation with a vortex patch. In this section, we want to write our variables as a perturbation of Crapper variables. First of all, we get a relation between \((\tau, \theta)\) in both \((\phi, \tilde{\psi})\) and \((\alpha, \rho)\) variables, by using the rotational and the divergence free conditions,

\[
\begin{align*}
\frac{\partial \theta}{\partial \phi} &= \frac{\omega e^{-2\tau}}{W} + \frac{\partial \tau}{\partial \tilde{\psi}} \\
\frac{\partial \theta}{\partial \tilde{\psi}} &= -\frac{\partial \tau}{\partial \phi}
\end{align*}
\Rightarrow \begin{cases}
\frac{\partial \theta}{\partial \alpha} &= \frac{\omega e^{-2\tau}}{W} + \frac{\partial \tau}{\partial \rho} \\
\frac{\partial \theta}{\partial \rho} &= -\frac{\partial \tau}{\partial \alpha}.
\end{cases}
\tag{67}
\]

Once we find the values of \((\tau, \theta)\), we can use the relations (59) and (61) to obtain the parametrization of the interface

\[
\begin{align*}
\frac{\partial z_1}{\partial \alpha} &= -e^{-\tau(a,1)} \cos(\theta(a,1)) \frac{\omega_0 \chi D e^{-2\tau}}{W(a,1)} \\
\frac{\partial z_2}{\partial \alpha} &= -e^{-\tau(a,1)} \sin(\theta(a,1)) \frac{\omega_0 \chi D e^{-2\tau}}{W(a,1)},
\end{align*}
\tag{68}
\]

where \(W(a,1)\) is defined in (63).

In the case of rotational waves \((\tau, \theta)\) do not satisfy the Cauchy-Riemann equations. For this reason we define \(\tau = \tau_A + \omega_0 \tilde{\tau}\) and \(\theta = \theta_A + \omega_0 \tilde{\theta}\), such that \((\tau_A, \theta_A)\) is the Crapper solution with small gravity but without vorticity thus it is incompressible and irrotational and satisfies the Cauchy-Riemann equations in the variables \((\phi, \tilde{\psi})\), as explained in (29)

\[
\begin{align*}
\frac{\partial \theta_A}{\partial \phi} &= \frac{\partial \tau_A}{\partial \tilde{\psi}} \\
\frac{\partial \theta_A}{\partial \tilde{\psi}} &= -\frac{\partial \tau_A}{\partial \phi}.
\end{align*}
\]

This implies that on the interface \(S\), i.e. \(\tilde{\psi} = 0\), one variable can be written as the Hilbert transform of the other \(\tau_A = \mathcal{H} \theta_A\). Hence, in the \((\phi, \tilde{\psi})\) variables, we have

\[
\begin{align*}
\frac{\partial \theta_A}{\partial \phi} &= W \frac{\partial \mathcal{H} \theta_A}{\partial \tilde{\psi}} \\
\frac{\partial \theta_A}{\partial \tilde{\psi}} &= -\frac{1}{W} \frac{\partial \mathcal{H} \theta_A}{\partial \phi}
\end{align*}
\Rightarrow \begin{cases}
\frac{\partial \theta_A}{\partial \alpha} &= W \rho \frac{\partial \mathcal{H} \theta_A}{\partial \rho} \\
\frac{\partial \theta_A}{\partial \rho} &= -\frac{1}{W} \frac{\partial \mathcal{H} \theta_A}{\partial \alpha}. \tag{69}
\end{cases}
\]

By substituting (69) in (67), we have

\[
\begin{align*}
\frac{\omega_0 \partial \tilde{\theta}}{\partial \phi} &= \left(\frac{1}{W} - 1\right) \frac{\partial \theta_A}{\partial \phi} + \frac{\omega_0 \chi D e^{-2\tau}}{W} + \omega_0 \frac{\partial \tilde{\tau}}{\partial \tilde{\psi}} \\
\frac{\omega_0 \partial \tilde{\theta}}{\partial \tilde{\psi}} &= -\omega_0 \frac{\partial \tilde{\tau}}{\partial \phi} + (W - 1) \frac{\partial \theta_A}{\partial \tilde{\psi}} \tag{70}
\end{align*}
\]
By deriving with respect to the opposite variable and taking the difference, we have the following elliptic equation in \((\alpha, \rho)\)

\[
\omega_0 \frac{\partial^2 \tilde{\tau}}{\partial \rho^2} + \frac{1}{\rho^2} \omega_0 \frac{\partial^2 \tilde{\tau}}{\partial \alpha^2} + \frac{1}{\rho} \omega_0 \frac{\partial \tilde{\tau}}{\partial \rho} = - \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\omega_0 \chi_A(\alpha, \rho)e^{-2\tau}}{W} \right) + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{1}{W} \right) \frac{\partial \theta_A}{\partial \alpha} + \frac{1}{\rho} \frac{\partial^2 \theta_A}{\partial \alpha \partial \rho} \left( 1 - \frac{W^2}{W} \right). \tag{72}
\]

But we are interested in the elliptic equation in \((\phi, \tilde{\psi})\)-coordinates, since it will be easy to study and we have

\[
\omega_0 \Delta \tilde{\tau} = - \frac{\partial}{\partial \tilde{\psi}} \left( \frac{\omega_0 \chi_D(\phi, \tilde{\psi})e^{-2\tau}}{W} \right) + \frac{1}{W^2} \frac{\partial W}{\partial \psi} \frac{\partial \theta_A}{\partial \phi} + \frac{\partial W}{\partial \psi} \frac{\partial \theta_A}{\partial \phi} + \left( \frac{W^2 - 1}{W} \right) \frac{\partial^2 \theta_A}{\partial \phi \partial \tilde{\psi}}. \tag{73}
\]

We want to find a solution \(\tilde{\tau}\) of the elliptic problem \((73)\). First of all let us rewrite the equation with all the explicit terms.

\[
\Delta \tilde{\tau}(\phi, \tilde{\psi}) = - \frac{\partial}{\partial \tilde{\psi}} \left( \frac{\chi_D(\phi, \tilde{\psi})e^{(-2\theta_A - 2\omega_0 \tilde{\tau})(\phi, \tilde{\psi})}}{1 + \omega_0 \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}')e^{(-2\theta_A - 2\omega_0 \tilde{\tau})(\phi, \tilde{\psi}')} \, d\tilde{\psi}'} \right)
\]

\[
+ \frac{\chi_D(\phi, \tilde{\psi})e^{(-2\theta_A - 2\omega_0 \tilde{\tau})(\phi, \tilde{\psi})}}{1 + \omega_0 \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}')e^{(-2\theta_A - 2\omega_0 \tilde{\tau})(\phi, \tilde{\psi}')} \, d\tilde{\psi}'} \left( \frac{\partial \theta_A}{\partial \phi} \right)^2
\]

\[
+ \frac{\partial}{\partial \phi} \left( \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}')e^{(-2\theta_A - 2\omega_0 \tilde{\tau})(\phi, \tilde{\psi}')} \, d\tilde{\psi}' \right) \frac{\partial \theta_A}{\partial \tilde{\psi}}
\]

\[
+ \frac{2 \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}')e^{(-2\theta_A - 2\omega_0 \tilde{\tau})(\phi, \tilde{\psi}')} \, d\tilde{\psi}'}{1 + \omega_0 \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}')e^{(-2\theta_A - 2\omega_0 \tilde{\tau})(\phi, \tilde{\psi}')} \, d\tilde{\psi}'} \frac{\partial^2 \theta_A}{\partial \phi \partial \tilde{\psi}}
\]

\[
+ \frac{\omega_0 \left( \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}')e^{(-2\theta_A - 2\omega_0 \tilde{\tau})(\phi, \tilde{\psi}')} \, d\tilde{\psi}' \right)^2}{1 + \omega_0 \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}')e^{(-2\theta_A - 2\omega_0 \tilde{\tau})(\phi, \tilde{\psi}')} \, d\tilde{\psi}'} \frac{\partial^2 \theta_A}{\partial \phi \partial \tilde{\psi}} = f((\phi, \tilde{\psi}), \tilde{\tau}).
\]
where we use that \( \frac{\partial W}{\partial \psi} = \omega_0 \chi_D(\phi, \tilde{\psi}) e^{(-2\theta_A - 2\omega_0 \tau)(\phi, \tilde{\psi})} \). Now, we define a solution in the following way

\[
\tilde{\tau}(\phi, \tilde{\psi}) = f((\phi, \tilde{\psi}), \tilde{\tau}) * G_2(\phi, \tilde{\psi}),
\]

where \( G_2(\phi, \tilde{\psi}) \) is the Green function in the domain \( \tilde{\Omega} \). We will show that (74) solves the elliptic equation, thanks to the smallness of the parameters involved.

If we use the properties of commutativity and differentiation of the convolution; the integration by parts with the fact that \( W(\pm \pi, \tilde{\psi}') = 1 \), then we are able to eliminate the derivative of \( \tilde{\tau} \) and we have

\[
\tilde{\tau}(\phi, \tilde{\psi}) = - \frac{\chi(\phi, \tilde{\psi}) e^{(-2\theta_A - 2\omega_0 \tau)(\phi, \tilde{\psi})}}{1 + \omega_0 \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}') e^{(-2\theta_A - 2\omega_0 \tau)(\phi, \tilde{\psi}')} d\tilde{\psi}'} \frac{\partial}{\partial \psi} G_2(\phi, \tilde{\psi})
\]

\[
+ \frac{\chi(\phi, \tilde{\psi}) e^{(-2\theta_A - 2\omega_0 \tau)(\phi, \tilde{\psi})}}{(1 + \omega_0 \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}') e^{(-2\theta_A - 2\omega_0 \tau)(\phi, \tilde{\psi}')} d\tilde{\psi}')^2} \frac{\partial \theta_A}{\partial \phi} * G_2 \]

\[
- \frac{\partial G_2}{\partial \phi} \frac{\partial \theta_A}{\partial \psi} \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}') e^{(-2\theta_A - 2\omega_0 \tau)(\phi, \tilde{\psi}')} d\tilde{\psi}'} \]

\[
+ \frac{2 \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}') e^{(-2\theta_A - 2\omega_0 \tau)(\phi, \tilde{\psi}')} d\tilde{\psi}'}{1 + \omega_0 \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}') e^{(-2\theta_A - 2\omega_0 \tau)(\phi, \tilde{\psi}')} d\tilde{\psi}'} \frac{\partial^2 \theta_A}{\partial \phi \partial \psi} * G_2 \]

\[
+ \frac{\omega_0 \left( \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}') e^{(-2\theta_A - 2\omega_0 \tau)(\phi, \tilde{\psi}')} d\tilde{\psi}' \right)^2}{1 + \omega_0 \int_{-\infty}^{\tilde{\psi}} \chi_D(\phi, \tilde{\psi}') e^{(-2\theta_A - 2\omega_0 \tau)(\phi, \tilde{\psi}')} d\tilde{\psi}'} \frac{\partial^2 \theta_A}{\partial \phi \partial \psi} * G_2 \]

Since we are looking for a solution with small \( \omega_0 \), we rewrite (75) around \( \omega_0 = 0 \), we write just the first order

\[
\tilde{\tau} = -\chi(\phi, \tilde{\psi}) e^{2\theta_A(\phi, \tilde{\psi})} \frac{\partial G_2}{\partial \psi} + \chi(\phi, \tilde{\psi}) e^{2\theta_A(\phi, \tilde{\psi})} \frac{\partial \theta_A}{\partial \phi} \]

\[
- \frac{\partial G_2}{\partial \phi} \frac{\partial \theta_A}{\partial \psi} \int_{-\infty}^{\tilde{\psi}} \chi(\phi, \tilde{\psi}') e^{2\theta_A(\phi, \tilde{\psi}')} d\tilde{\psi}'
\]

(76)
We define the operator

\[ ω \frac{∂^2 θ_A}{∂φ∂ψ} \int_{-∞}^{∞} χ(φ, ψ') e^{-2Hθ_A(φ, ψ')} dψ' + 2 \int_{-∞}^{ψ} χ(φ, ψ') e^{-2Hθ_A(φ, ψ')} dψ' \cdot \frac{∂^2 θ_A}{∂φ∂ψ} + G_2 \]

\[ + ω_0 2\chi(φ, ψ') ϵ^{-2Hθ_A(φ, ψ')} \tilde{τ}(φ, ψ') \cdot \frac{∂G_2}{∂ψ} \]

\[ + ω_0 \chi(φ, ψ') ϵ^{-2Hθ_A(φ, ψ')} \int_{-∞}^{ψ} \chi(φ, ψ') e^{-2Hθ_A(φ, ψ')} dψ' \cdot \frac{∂G_2}{∂ψ} \]

\[ - 2ω_0 \chi(φ, ψ') ϵ^{-2Hθ_A(φ, ψ')} \tilde{τ}(φ, ψ') \cdot \frac{∂θ_A}{∂φ} + G_2 \]

\[ - 2ω_0 \chi(φ, ψ') ϵ^{-2Hθ_A(φ, ψ')} \int_{-∞}^{ψ} \chi(φ, ψ') e^{-2Hθ_A(φ, ψ')} dψ' \cdot \frac{∂θ_A}{∂φ} + G_2 \]

\[ + ω_0 \frac{∂G_2}{∂φ} \cdot \frac{∂θ_A}{∂ψ} \cdot 2 \int_{-∞}^{ψ} \chi(φ, ψ') e^{-2Hθ_A(φ, ψ')} \tilde{τ}(φ, ψ') dψ' \]

\[ + ω_0 G_2 \cdot \frac{∂^2 θ_A}{∂φ∂ψ} \cdot 2 \int_{-∞}^{ψ} \chi(φ, ψ') e^{-2Hθ_A(φ, ψ')} \tilde{τ}(φ, ψ') dψ' \]

\[ - 4ω_0 \int_{-∞}^{ψ} \chi(φ, ψ') e^{-2Hθ_A(φ, ψ')} \tilde{τ}(φ, ψ') dψ' \cdot \frac{∂^2 θ_A}{∂φ∂ψ} + G_2 \]

\[ - ω_0 \left( \int_{-∞}^{ψ} \chi(φ, ψ') e^{-2Hθ_A(φ, ψ')} dψ' \right)^2 \cdot \frac{∂^2 θ_A}{∂φ∂ψ} + G_2 + o(ω_0^2) \]

\[ \equiv ω_0 A_1(\tilde{τ}, θ_A) + ω_0 A_2(θ_A) + b(θ_A) + o(ω_0^2), \]

We define the operator

\[ G(\tilde{τ}; ω_0, θ_A) = \tilde{τ} - ω_0 A_1(\tilde{τ}, θ_A) - ω_0 A_2(θ_A) - b(θ_A) + o(ω_0^2), \]

(77)

where \( G(\tilde{τ}; ω_0, θ_A) : H^2_{even} × \mathbb{R} × H^2_{odd} \rightarrow H^2 \) and to invert this operator in a neighborhood of \( ω_0 = 0 \) we will use the Implicit function theorem [3.2]. We observe that

\[ \left\{ \begin{array}{l}
G(\tilde{τ}; 0, θ_A) = 0 \\
D_1 G(\tilde{τ}; 0, θ_A) = τ_1.
\end{array} \right. \]

(78)
The equations (78) guarantees that in a neighborhood of \((\omega_0 = 0, \theta_A)\), there exists a smooth function \(\tilde{\tau}^*(\omega_0, \theta_A)\), such that \(\tilde{\tau}^*(0, \theta_A) = \tilde{\tau}\).

4.3. **Existence of Crapper waves in the presence of a small vortex patch.** In this section we prove the existence of a perturbation of the Crapper waves, with small gravity and small vorticity. We will prove the existence theorem (Theorem 4.1), by means of the implicit function theorem. However, to prove it we need an explicit parametrization for \(\gamma(\alpha)\) in such a way that the operator, related to (56), fulfils the hypothesis of the implicit function theorem. We define \(\gamma(\alpha)\) as follows

\[
\gamma(\alpha) = \begin{cases} 
 r \left( \frac{\alpha + \pi}{\sin \alpha} \cos \alpha, -\alpha - \pi \right) & -\pi \leq \alpha < -\frac{\pi}{2} \\
 r \left( \frac{\alpha}{\sin \alpha} \cos \alpha, \alpha \right) & -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \\
 r \left( \frac{\alpha - \pi}{\sin \alpha} \cos \alpha, -\alpha + \pi \right) & \pi \frac{2}{2} < \alpha \leq \pi,
\end{cases}
\]

where \(r \in \mathbb{R}\) the small radius. So that its derivative is

\[
\partial_\alpha \gamma(\alpha) = \begin{cases} 
 r \left( \frac{(\alpha + \pi) - \cos \alpha \sin \alpha}{\sin^2 \alpha}, -1 \right) & -\pi \leq \alpha < -\frac{\pi}{2} \\
 r \left( \frac{\cos \alpha \sin \alpha - \alpha}{\sin^2 \alpha}, 1 \right) & -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \\
 r \left( \frac{(\alpha - \pi) - \cos \alpha \sin \alpha}{\sin^2 \alpha}, -1 \right) & \frac{\pi}{2} < \alpha \leq \pi
\end{cases}
\]

Figure 4. The choice of \(\gamma(\alpha)\).
However, we will work in a neighborhood of $\omega_0 = 0$, then we substitute $\tau = H\theta_A + \omega_0 \tau^*$. 

$$F_1(\theta_A, \omega, r; B, p, \omega_0) = \sinh(H\theta_A + \omega_0 \tau^*)$$ 

$$-pe^{-H\theta_A - \omega_0 \tau^*} \left( \int_{-\pi}^{\alpha} e^{-H\theta_A - \omega_0 \tau^*} \sin \theta(\alpha', 1) \, d\alpha' - 1 \right)$$ 

$$+ qW(\alpha, 1) \frac{\partial(\theta_A + \omega_0 \tilde{\theta}^*)}{\partial \alpha} - Be^{-H\theta_A - \omega_0 \tau^*}.$$ 

$$F_2(\theta_A, \omega, r; B, p, \omega_0) := W(\alpha, 1) (2BR(z(\alpha), \tilde{\omega}(\alpha)) \cdot \partial_\alpha z(\alpha) + \tilde{\omega}(\alpha))$$ 

$$+ \frac{\omega_0}{2\pi} \int_{-\pi}^{\pi} \log |z(\alpha) - \gamma(\alpha')| \partial_\alpha \gamma(\alpha') \, d\alpha' + 2$$ 

$$F_3(\theta_A, \omega, r; B, p, \omega_0) := \frac{1}{2\pi} \int_{-\pi}^{\pi} (\gamma(\alpha) - z(\alpha')) \partial_\alpha \gamma(\alpha') \, d\alpha' \cdot \partial_\alpha \gamma(\alpha')$$ 

$$+ \frac{\omega_0}{2\pi} PV \int_{-\pi}^{\pi} \log |\gamma(\alpha) - \gamma(\alpha')| \partial_\alpha \gamma(\alpha') \, d\alpha' \cdot \partial_\alpha \gamma(\alpha').$$ 

We have that 

$$(F_1, F_2, F_3)(\theta_c, \tilde{\omega}, r; B, p, \omega_0) : H^2_{odd} \times H^1_{even} \times \mathbb{R}^4 \to H^1_{even} \times H^1_{even} \times H^1.$$ 

The main theorem we want to prove is the following 

**Theorem 4.1.** Let us consider the water waves problem (1), with a small vortex patch and a small gravity $g$. Then, for some values of $A < A_0$, defined in (26), there exist periodic solutions to (1) with overhanging profile.

4.3.1. **Proof of Theorem 4.1.** We will analyse the three operators (81) that identify our problem. And we will show they satisfy the hypothesis of the implicit function theorem. First of all we have to show that 

$$(F_1, F_2, F_3)(\theta_c, \tilde{\omega}, r; B, p, \omega_0) = (0, 0, 0).$$ 

For $F_1$, we use (24) 

$$F_1(\theta_c, \tilde{\omega}, 0; 0, 0, 0) = \sinh(H\theta_c) + q \frac{\partial \theta_c}{\partial \alpha} = 0$$ 

For $F_2$ it holds by construction (66). For $F_3$, we write explicitly $\gamma(\alpha)$ as in (79) and by taking the radius $r$ to be 0. Thus $F_3(\theta_c, \tilde{\omega}, 0; 0, 0, 0)$ satisfies (82).
The most considerable part is to prove the invertibility of the derivatives. We observe that
\( D_{\omega} F_1 = D_{\omega} F_1 = 0 \), so it remains to compute \( D_{\theta_A} F_1 \).

\[
D_{\theta_A} F_1 = \frac{d}{d\mu} \left[ \sinh(\mathcal{H}_A + \mu \mathcal{H}_1 + \omega_0 \tilde{\pi}^*(\theta_A + \mu \theta_1)) + q W_{\mu} \frac{\partial(\theta_A + \mu \theta_1 + \omega_0 \tilde{\pi}^*(\theta_A + \mu \theta_1))}{\partial \alpha} \right]
\]

\[
- p e^{-\mathcal{H}_A - \mu \mathcal{H}_1 - \omega_0 \tilde{\pi}^*(\theta_A + \mu \theta_1)} \int_{-\pi}^{\alpha} e^{-\mathcal{H}_A - \mu \mathcal{H}_1 - \omega_0 \tilde{\pi}^*(\theta_A + \mu \theta_1)} \sin(\theta_A + \mu \theta_1 + \omega_0 \tilde{\pi}^*(\theta_A + \mu \theta_1)) \frac{d\alpha}{W}
\]

\[
- B e^{-\mathcal{H}_A - \mu \mathcal{H}_1 - \omega_0 \tilde{\pi}^*(\theta_A + \mu \theta_1)} \left|_{\mu=0} \right.
\]

\[
= \cosh(\mathcal{H}_A + \omega_0 \tilde{\pi}^*(\theta_A)) \cdot \left( \mathcal{H}_1 + \omega_0 \left[ \frac{d}{d\mu} \tilde{\pi}^*(\theta_A + \mu \theta_1) \right] \right) + q W_{\mu} \frac{\partial(\theta_A + \mu \theta_1 + \omega_0 \tilde{\pi}^*(\theta_A + \mu \theta_1))}{\partial \alpha} \left( \theta_1 + \omega_0 \left[ \frac{d}{d\mu} \tilde{\pi}^*(\theta_A + \mu \theta_1) \right] \right)
\]

\[
+ p e^{-\mathcal{H}_A - \omega_0 \tilde{\pi}^*(\theta_A)} \left( \mathcal{H}_1 + \omega_0 \left[ \frac{d}{d\mu} \tilde{\pi}^*(\theta_A + \mu \theta_1) \right] \right) \int_{-\pi}^{\alpha} e^{-\mathcal{H}_A - \omega_0 \tilde{\pi}^*(\theta_A)} \sin(\theta_A + \omega_0 \tilde{\pi}^*(\theta_A)) \frac{d\alpha}{W}
\]

\[
- p e^{-\mathcal{H}_A - \omega_0 \tilde{\pi}^*(\theta_A)} \int_{-\pi}^{\alpha} e^{-\mathcal{H}_A - \omega_0 \tilde{\pi}^*(\theta_A)} \cos(\theta_A + \omega_0 \tilde{\pi}^*(\theta_A)) \left( \theta_1 + \omega_0 \left[ \frac{d}{d\mu} \tilde{\pi}^*(\theta_A + \mu \theta_1) \right] \right) \frac{d\alpha}{W}
\]

\[
+ p e^{-\mathcal{H}_A - \omega_0 \tilde{\pi}^*(\theta_A)} \int_{-\pi}^{\alpha} e^{-\mathcal{H}_A - \omega_0 \tilde{\pi}^*(\theta_A)} \sin(\theta_A + \omega_0 \tilde{\pi}^*(\theta_A)) \left[ \frac{d}{d\mu} W_{\mu} \right] \frac{d\alpha}{W^2}
\]

\[
- B e^{-\mathcal{H}_A - \omega_0 \tilde{\pi}^*(\theta_A)} \left( \mathcal{H}_1 - \omega_0 \left[ \frac{d}{d\mu} \tilde{\pi}^*(\theta_A + \mu \theta_1) \right] \right) \left|_{\mu=0} \right.
\]

Remark 4.2. The equation for \( W_{\mu}(\alpha, 1) \) is the following

\[
W_{\mu}(\alpha, 1) = 1 + \int_{0}^{1} \omega_0 \chi A(\alpha, \rho') e^{-2\mathcal{H}_A - 2\mu \mathcal{H}_1 - 2\omega_0 \tilde{\pi}^*(\theta_A + \mu \theta_1)} \frac{d\rho'}{\rho'}
\]  \hspace{1cm} (83)

Now we have to compute \( \frac{dW_{\mu}}{d\mu} \) that is
It remains to observe that for our purpose it is sufficient to have the existence of 
\[ dW_\mu \left|_{\mu=0} \right. \] coming from the elliptic equation (72). Indeed we must compute the Fréchet derivative at the point \((\theta_c, \tilde{\omega}_c, 0; 0, 0, 0)\) and, as we can see in \(D\theta A F_1\), the term 
\[ \frac{d}{d\mu} \tilde{\tau}^*(\theta_A + \mu\theta_1) \] is always multiplied by \(\omega_0\) that it is taken equal to zero. And we can state that also 
\[ \left[ \frac{dW_\mu}{d\mu} \right]_{\mu=0} \] is zero.

The remark 4.2 implies that 
\[ D\theta A F_1(\theta_c, \tilde{\omega}_c, 0; 0, 0, 0) = \cosh(\mathcal{H}\theta_c) \cdot \mathcal{H}\theta_1 + q\frac{\partial\theta_1}{\partial\alpha}. \]

For the second operator, we observe that for computing \(D\theta A F_2\), we need to use the equation for \(z(\alpha)\) can be obtained by integrating (68) and we define 
\[ z_\mu(\alpha) = - \int_{-\pi}^\alpha e^{-\mathcal{H}\theta_A - \mu\mathcal{H}\theta_1 - \omega_0\tilde{\tau}^*(\theta_A + \mu\theta_1) + i(\theta_A + \mu\theta_1 + \omega_0\tilde{\tau}^*(\theta_A + \mu\theta_1))} \frac{W_\mu(\alpha', 1)}{W_\mu(\alpha', 1)} d\alpha' - e_2 \]

where \(W_\mu(\alpha', 1)\) is defined in (83).

In the same way, we did for computing \(D\omega F_1\), we can compute \(D\theta A F_2\) and then at the point \((\theta_c, \tilde{\omega}_c, 0; 0, 0, 0)\) we will get \(D\theta A F_2(\theta_c, \tilde{\omega}_c, 0; 0, 0, 0)\).

Instead, it is important to compute \(D\omega F_2\).
D_\alpha F_2 = \left[ \frac{d}{d\mu} F_2(\theta_A, \omega + \mu \omega_1, r; p, \epsilon, \omega_0, B) \right]_{\mu=0} = \left[ \frac{d}{d\mu} (2W(\alpha, 1)BR(z(\alpha), \omega(\alpha) + \mu \omega_1) \cdot \partial_\alpha z(\alpha) + W(\alpha, 1)\omega(\alpha)) \right]_{\mu=0}

+ W(\alpha, 1)(\omega(\alpha) + \mu \omega_1(\alpha)) + W(\alpha, 1)\frac{\omega_0}{2\pi} \int_{-\pi}^{\pi} \log |z(\alpha) - \gamma(\alpha')| \partial_\alpha \gamma(\alpha') \, d\alpha' \cdot \partial_\alpha z(\alpha) + 2 \right]_{\mu=0}

= 2W(\alpha, 1)P.V. \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha'))}{|z(\alpha) - z(\alpha')|^2} \cdot \omega_1(\alpha') \, d\alpha' \cdot \partial_\alpha z(\alpha) + W(\alpha, 1)\omega(\alpha).

At the Crapper point we have

D_\omega F_2(\theta_c, \omega_c, 0; 0, 0, 0, 0) = 2P.V. \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(z(\alpha) - z(\alpha'))}{|z(\alpha) - z(\alpha')|^2} \cdot \omega_1(\alpha') \, d\alpha' \cdot \partial_\alpha z(\alpha) + \omega(\alpha).

It remains to compute the last derivative

D_\alpha F_2 = \left[ \frac{d}{d\mu} F_2(\theta_A, \omega + \mu \omega_1; r, B, p, \omega_0) \right]_{\mu=0}

= \left[ \frac{d}{d\mu} (2W(\alpha, 1)BR(z(\alpha), \omega(\alpha)) \cdot \partial_\alpha z(\alpha) + W(\alpha, 1)\omega(\alpha)) \right]_{\mu=0}

+ W(\alpha, 1)\frac{\omega_0}{2\pi} \int_{-\pi}^{\pi} \log \left( \frac{z_1(\alpha) + (r + \mu r_1)\frac{\alpha' + \pi}{\sin \alpha'} \cos \alpha'}{r + \mu r_1} \right)^2 + \left( \frac{z_2(\alpha) + (r + \mu r_1)(\alpha' + \pi)}{r + \mu r_1} \right)^2

\cdot (r + \mu r_1) \left( \frac{\alpha' + \pi - \cos \alpha' \sin \alpha'}{\sin^2 \alpha'}, -1 \right) \, d\alpha' \cdot \partial_\alpha z(\alpha)

+ W(\alpha, 1)\frac{\omega_0}{2\pi} \int_{-\pi}^{\pi} \log \left( \frac{z_1(\alpha) - (r + \mu r_1)\frac{\alpha'}{\sin \alpha'} \cos \alpha'}{r + \mu r_1} \right)^2 + \left( \frac{z_2(\alpha) - (r + \mu r_1)(\alpha')}{r + \mu r_1} \right)^2

\cdot (r + \mu r_1) \left( \frac{\cos \alpha' \sin \alpha' - \alpha'}{\sin^2 \alpha'}, 1 \right) \, d\alpha' \cdot \partial_\alpha z(\alpha)

+ W(\alpha, 1)\frac{\omega_0}{2\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \log \left( \frac{z_1(\alpha) + (r + \mu r_1)\frac{\alpha' - \pi}{\sin \alpha'} \cos \alpha'}{r + \mu r_1} \right)^2 + \left( \frac{z_2(\alpha) + (r + \mu r_1)(\alpha' - \pi)}{r + \mu r_1} \right)^2

\cdot (r + \mu r_1) \left( \frac{(\alpha' - \pi) - \cos \alpha' \sin \alpha'}{\sin^2 \alpha'}, -1 \right) \, d\alpha' \cdot \partial_\alpha z(\alpha) \right]_{\mu=0}
\[= W(\alpha, 1) \frac{\omega_0}{2\pi} r \cdot r_1 \int_{-\pi}^{\pi} \frac{z_1(\alpha) + r \frac{\alpha' + \pi}{\sin \alpha} \cos \alpha' + z_2(\alpha) + r(\alpha' + \pi)}{(z_1(\alpha) + r \frac{\alpha' + \pi}{\sin \alpha} \cos \alpha')^2 + (z_2(\alpha) + r(\alpha' + \pi))^2} \cdot \left( \frac{(\alpha' + \pi) - \cos \alpha' \sin \alpha'}{\sin^2 \alpha'}, -1 \right) \, d\alpha' \cdot \partial_\alpha z(\alpha) \]

\[+ W(\alpha, 1) \frac{\omega_0}{2\pi} r \cdot r_1 \int_{-\pi}^{\pi} \log \sqrt{ \left( z_1(\alpha) + r \frac{\alpha'}{\sin \alpha} \cos \alpha' \right)^2 + \left( z_2(\alpha) - r \alpha' \right)^2} \cdot \left( \frac{(\cos \alpha' \sin \alpha' - \alpha')}{\sin^2 \alpha'}, 1 \right) \, d\alpha' \cdot \partial_\alpha z(\alpha) \]

\[+ W(\alpha, 1) \frac{\omega_0}{2\pi} r \cdot r_1 \int_{\pi}^{\pi} \log \sqrt{ \left( z_1(\alpha) + r \frac{\alpha'}{\sin \alpha} \cos \alpha' \right)^2 + \left( z_2(\alpha) + r \alpha' \right)^2} \cdot \left( \frac{(\alpha' - \pi) - \cos \alpha' \sin \alpha'}{\sin^2 \alpha'}, -1 \right) \, d\alpha' \cdot \partial_\alpha z(\alpha) \]

\[+ W(\alpha, 1) \frac{\omega_0}{2\pi} r \cdot r_1 \int_{\pi}^{\pi} \log \sqrt{ \left( z_1(\alpha) + r \frac{\alpha'}{\sin \alpha} \cos \alpha' \right)^2 + \left( z_2(\alpha) + r(\alpha' - \pi) \right)^2} \cdot \left( \frac{(\alpha' - \pi) - \cos \alpha' \sin \alpha'}{\sin^2 \alpha'}, -1 \right) \, d\alpha' \cdot \partial_\alpha z(\alpha) \]

When we evaluate this derivative at \((\theta_c, \tilde{\omega}_c, 0; 0, 0, 0)\), we get

\[D_r \mathcal{F}_2(\theta_c, \tilde{\omega}_c, 0; 0, 0, 0) = 0.\]

For the last operator \(\mathcal{F}_3\) we have to compute the derivates, but for \(D_{\theta_3} \mathcal{F}_3\) and \(D_{\tilde{\omega}} \mathcal{F}_3\), we have just to substitute \(\theta_A \rightarrow \theta_A + \mu \theta_1\) and \(\tilde{\omega} \rightarrow \tilde{\omega} + \mu \omega_1\), respectively and compute the derivatives as we did for the previous operators. Then we will compute them at the Crapper point, so that we get \(D_{\theta_3} \mathcal{F}_3(\theta_c, \tilde{\omega}_c, 0; 0, 0, 0)\) and \(D_{\tilde{\omega}} \mathcal{F}_3(\theta_c, \tilde{\omega}_c, 0; 0, 0, 0)\). In order
to apply the implicit function theorem the relevant derivative for the third operator is the one with respect to \( r \). The presence of \( r \) is in the definition of \( \gamma(\alpha) \) in \([79]\), so we rewrite \( F_3 \) in a convenient way.

\[
F_3(\theta_A, \hat{\omega}, r; B, p, \omega_0) = -\frac{\partial_\alpha \gamma_2(\alpha)}{2\pi} \int_{-\pi}^{\pi} \frac{-\gamma_2(\alpha) + z_2(\alpha')}{(\gamma_1(\alpha) - z_1(\alpha'))^2 + (\gamma_2(\alpha) - z_2(\alpha'))^2} \hat{\omega}(\alpha') d\alpha' \\
+ \frac{\partial_\alpha \gamma_1(\alpha)}{2\pi} \int_{-\pi}^{\pi} \frac{\gamma_1(\alpha) - z_1(\alpha')}{(\gamma_1(\alpha) - z_1(\alpha'))^2 + (\gamma_2(\alpha) - z_2(\alpha'))^2} \hat{\omega}(\alpha') d\alpha' \\
- \frac{\omega_0}{2\pi} \partial_\alpha \gamma_2(\alpha) \text{P.V.} \int_{-\pi}^{\pi} \log \sqrt{(\gamma_1(\alpha) - \gamma_1(\alpha'))^2 + (\gamma_2(\alpha) - \gamma_2(\alpha'))^2} \partial_\alpha \gamma_1(\alpha') d\alpha' \\
+ \frac{\omega_0}{2\pi} \partial_\alpha \gamma_1(\alpha) \text{P.V.} \int_{-\pi}^{\pi} \log \sqrt{(\gamma_1(\alpha) - \gamma_1(\alpha'))^2 + (\gamma_2(\alpha) - \gamma_2(\alpha'))^2} \partial_\alpha \gamma_2(\alpha') d\alpha'.
\]

In order to simplify the computation we will define \( \gamma(\alpha) = r(\tilde{\gamma}_1(\alpha), \tilde{\gamma}_2(\alpha)) \) and \( \partial_\alpha \gamma(\alpha) = r(\partial_\alpha \tilde{\gamma}_1(\alpha), \partial_\alpha \tilde{\gamma}_2(\alpha)) \).

\[
D_r F_3 = \frac{d}{d\mu} \left| F_3(\theta_A, \hat{\omega}, r + \mu r_1; B, p, \omega_0) \right|_{\mu=0} = \\
= -\frac{r_1 \partial_\alpha \tilde{\gamma}_2(\alpha)}{2\pi} \int_{-\pi}^{\pi} \frac{-r_1 \tilde{\gamma}_2(\alpha) + z_2(\alpha') + 1}{(r_1 \tilde{\gamma}_1(\alpha) - z_1(\alpha'))^2 + (r_1 \tilde{\gamma}_2(\alpha) - z_2(\alpha') - 1)^2} \hat{\omega}(\alpha') d\alpha' \\
- \frac{r \partial_\alpha \tilde{\gamma}_2(\alpha)}{2\pi} \left\{ \int_{-\pi}^{\pi} \frac{-r_1 \tilde{\gamma}_2(\alpha)}{(r_1 \tilde{\gamma}_1(\alpha) - z_1(\alpha'))^2 + (r_1 \tilde{\gamma}_2(\alpha) - z_2(\alpha') - 1)^2} \hat{\omega}(\alpha') d\alpha' \\
- \int_{-\pi}^{\pi} \frac{-r_1 \tilde{\gamma}_2(\alpha) + z_2(\alpha') + 1}{(r_1 \tilde{\gamma}_1(\alpha) - z_1(\alpha'))^2 + (r_1 \tilde{\gamma}_2(\alpha) - z_2(\alpha') - 1)^2} \hat{\omega}(\alpha') \cdot [2(r_1 \tilde{\gamma}_1(\alpha) - z_1(\alpha'))r_1 \tilde{\gamma}_1(\alpha) + 2(r_1 \tilde{\gamma}_2(\alpha) - z_2(\alpha') - 1)r_1 \tilde{\gamma}_2(\alpha)] d\alpha' \right\} \\
+ \frac{r_1 \partial_\alpha \tilde{\gamma}_1(\alpha)}{2\pi} \int_{-\pi}^{\pi} \frac{r_1 \tilde{\gamma}_1(\alpha) - z_1(\alpha')}{(r_1 \tilde{\gamma}_1(\alpha) - z_1(\alpha'))^2 + (r_1 \tilde{\gamma}_2(\alpha) - z_2(\alpha') - 1)^2} \hat{\omega}(\alpha') d\alpha' \\
+ \frac{r \partial_\alpha \tilde{\gamma}_1(\alpha)}{2\pi} \left\{ \int_{-\pi}^{\pi} \frac{r_1 \tilde{\gamma}_1(\alpha)}{(r_1 \tilde{\gamma}_1(\alpha) - z_1(\alpha'))^2 + (r_1 \tilde{\gamma}_2(\alpha) - z_2(\alpha') - 1)^2} \hat{\omega}(\alpha') d\alpha' \\
- \int_{-\pi}^{\pi} \frac{r_1 \tilde{\gamma}_1(\alpha) - z_1(\alpha')}{(r_1 \tilde{\gamma}_1(\alpha) - z_1(\alpha'))^2 + (r_1 \tilde{\gamma}_2(\alpha) - z_2(\alpha') - 1)^2} \hat{\omega}(\alpha') \cdot [2(r_1 \tilde{\gamma}_1(\alpha) - z_1(\alpha'))r_1 \tilde{\gamma}_1(\alpha) + 2(r_1 \tilde{\gamma}_2(\alpha) - z_2(\alpha') - 1)r_1 \tilde{\gamma}_2(\alpha)] d\alpha' \right\}.
\]
Then we end up in

$$-\frac{\omega_0}{2\pi} 2rr_1 \partial_\alpha \tilde{\gamma}_2(\alpha) \text{P.V.} \int_{-\pi}^\pi \log \left( r \sqrt{(\tilde{\gamma}_1(\alpha) - \tilde{\gamma}_1(\alpha'))^2 + (\tilde{\gamma}_2(\alpha) - \tilde{\gamma}_2(\alpha'))^2} \right) \partial_\alpha \tilde{\gamma}_1(\alpha') \, d\alpha'$$

$$-\frac{\omega_0}{2\pi} rr_1 \partial_\alpha \tilde{\gamma}_2(\alpha) \int_{-\pi}^\pi \partial_\alpha \tilde{\gamma}_1(\alpha') \, d\alpha'$$

$$+ \frac{\omega_0}{2\pi} 2rr_1 \partial_\alpha \tilde{\gamma}_1(\alpha) \text{P.V.} \int_{-\pi}^\pi \log \left( r \sqrt{(\tilde{\gamma}_1(\alpha) - \tilde{\gamma}_1(\alpha'))^2 + (\tilde{\gamma}_2(\alpha) - \tilde{\gamma}_2(\alpha'))^2} \right) \partial_\alpha \tilde{\gamma}_2(\alpha') \, d\alpha'$$

$$+ \frac{\omega_0}{2\pi} rr_1 \partial_\alpha \tilde{\gamma}_1(\alpha) \int_{-\pi}^\pi \partial_\alpha \tilde{\gamma}_2(\alpha') \, d\alpha'$$

**Remark 4.3.** We notice that all the terms above for $r = 0$ disappear except for the first one and the third one. Moreover, by computing them at the Crapper point $(\theta_c, \tilde{\omega}_c)$ it follows that also the third will be zero because of the parity of the Crapper curve $z^c(\alpha)$ (see (2)) and of $\tilde{\omega}(\alpha)$ which is even. Hence, in order to have the Fréchet derivative different from zero for every $\alpha \in [-\pi, \pi]$, we will choose $\gamma(\alpha)$ as (79) so that the first term will always be different from zero.

Then we end up in

$$D_r \mathcal{F}_3(\theta_c, \tilde{\omega}_c, 0; 0, 0, 0) = -\frac{r_1 \partial_\alpha \tilde{\gamma}_2(\alpha)}{2\pi} \int_{-\pi}^\pi \frac{z_2^c(\alpha') + 1}{z_1^c(\alpha')^2 + (z_2^c(\alpha') + 1)^2} \omega(\alpha') \, d\alpha'.$$  

It remains to prove the invertibility of the derivatives. In particular, the derivatives’ matrix is the following

$$DF(\theta_c, \tilde{\omega}_c, 0; 0, 0, 0) = \begin{pmatrix} D_{\theta_1} \mathcal{F}_1 & 0 & 0 \\ D_{\theta_1} \mathcal{F}_2 & D_\omega \mathcal{F}_2 & 0 \\ D_{\theta_1} \mathcal{F}_3 & D_\omega \mathcal{F}_3 & D_r \mathcal{F}_3 \end{pmatrix} = \begin{pmatrix} \Gamma & 0 & 0 \\ D_{\theta_1} \mathcal{F}_2 & A(z^c(\alpha)) + I & 0 \\ D_{\theta_1} \mathcal{F}_3 & D_\omega \mathcal{F}_3 & D_r \mathcal{F}_3 \end{pmatrix} \cdot \begin{pmatrix} \theta_1 \\ \omega_1 \\ r_1 \end{pmatrix}$$

(85)

where

$$\Gamma \theta_1 = \cosh(\mathcal{H} \theta_c) \mathcal{H} \theta_1 + \frac{d}{d\alpha} \theta_1$$

$$(A(z^c(\alpha)) + I) \omega_1 = 2BR(z^c(\alpha), \omega_1) \cdot \partial_\alpha z^c(\alpha) + \omega_1.$$
We put in evidence only these three operators since the matrix is diagonal and it will be invertible if the diagonal is invertible.

We observe immediately that the choice of the curve $\gamma(\alpha)$ is crucial since the second component of $\partial_\alpha \gamma(\alpha) \neq 0$, for every $\alpha \in [-\pi, \pi]$. So we can invert $D_r \mathcal{F}_3(\theta_c, \tilde{\omega}_c; 0, 0, 0)$, as required. For the other two operators we have to use Lemma 3.4 and Lemma 3.5 to overcome the problem of the non invertibility of $\Gamma$, see section 3.5.1. Hence, we state the following result.

**Theorem 4.4.** Let $|A| < A_0$. Then

1. there exists $(\omega_0, \theta_A)$ and a unique smooth function $\tilde{\tau}^* : U_{\omega_0, \theta_A} \to H^2_{even}$, such that $\tilde{\tau}^*(0, \theta_A) = \tilde{\tau}$ (see (77)),

2. there exists $(B, p, \omega_0)$ and a unique smooth function $B^* : U_{p, \omega_0} \to U_B$, such that $B^*(0, 0) = 0$,

3. there exists a unique smooth function $\Theta_c : U_{B, p, \omega_0} \to H^2_{odd} \times H^1_{even} \times \mathbb{R}$, such that $\Theta_c(0, 0, 0) = (\theta_c, \tilde{\omega}_c, 0)$

and satisfy

$$\mathcal{F}(\Theta_c(B^*(p, \omega_0), p, \omega_0), B^*(p, \omega_0), p, \omega_0) = 0.$$ 

The proof of Theorem 4.1 holds directly from this Theorem.

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