FAST TRACK COMMUNICATION

Monte-Carlo simulations of superradiant lasing

Yuan Zhang, Yu-Xiang Zhang and Klaus Mølmer
Department of Physics and Astronomy, Aarhus University, Ny Munkegade 120, DK-8000 Aarhus C, Denmark
E-mail: yzhang@phys.au.dk

Keywords: superradiance, lasing, quantum jumps

Abstract
We simulate the superradiant dynamics of ensembles of atoms in the presence of collective and individual atomic decay processes. We apply the Monte-Carlo wave-function method and identify quantum jumps in a reduced Dicke state basis, which reflects the permutation symmetry of the system. While the number of density matrix elements in the Dicke representation increases polynomially with atom number, the quantum jump dynamics populates only a single Dicke state at the time and thus efficient simulations can be carried out for tens of thousands of atoms. The superradiant pulses from initially excited atoms agree quantitatively with recent experimental results of strontium atoms but rapid atom loss in these experiments does not permit steady-state superradiance. By introducing an incident flux of new atoms to maintain a large average atom number, our theoretical calculations predict lasing with a millihertz linewidth despite rapid atom number fluctuations.

1. Introduction

Superradiance is caused by collective interaction of atoms with a radiation field and has been the subject of interest since the early proposal by Dicke [1], (see also [2, 3]). Recent experiments with tens of thousands of atoms in optical cavities [4–6] explore the possibility of achieving superradiant lasing in a bad cavity. Opposite to the Schawlow–Townes limit of a normal laser, it has a linewidth set by the energy transfer rate from single atoms to the cavity mode and can in principle reach millihertz level [7]. So far, such performance has been hindered by fast atom loss depleting the atomic ensemble before the coherence is established [8]. To solve this problem, we might feed new atoms to the system to stabilize the atom number.

In this article, we study the system depicted in figure 1(a), where $N$ atoms trapped in an one-dimensional optical lattice interact collectively with a lossy fundamental cavity mode while being subject to individual decay, dephasing and excitation (pumping), loss as well as feeding due to the interaction with their local environment. This system can be studied with either a laser master equation or an atomic superradiance master equation [8]. However, since the number of density matrix elements scales exponentially with the atom number, calculations with the atomic product states are restricted to only tens of atoms [9]. Since the atoms feel almost the same environment, it is reasonable to assume that they are identical. Then, using permutation symmetry among the atoms, calculations can be carried out in a basis of Dicke states $|J, M\rangle$, where $J \leq N/2$ are collective spin quantum numbers and $M = -J, ... , J$ represent the total number of excited atoms $J + M$ [10–12], or of collective numbers [13–15] (exploiting SU(4) group theory [16]), where the complexity scales only cubically and permits simulations for hundreds of atoms. Here, we demonstrate that by unraveling the master equation in the Dicke state basis with the Monte-Carlo wave-function method (MCWF), we can simulate tens of thousands of atoms as encountered in experiments and gain novel insights into the evolution dynamics of the system quantum states.

The MCWF method [17, 18] uses ensembles of wave-functions instead of a density matrix to represent a quantum system, and applies random quantum jumps to describe the effect of dissipation. The jumps can also describe measurement back-action, and thus form an essential component of so-called quantum trajectories [19] with applications in quantum measurement and control [20]. When a system is allowed to explore all states
of a high dimensional Hilbert space, the conditional wave-functions are lower dimensional objects than density matrices. This renders the MCWF method an efficient numerical tool for complex system simulations. The symmetries mentioned above restrict the dynamics to sub-spaces and further speed up the wave-function calculations. In the following, we will review briefly the master equation in the Dicke state basis and then demonstrate its unraveling and effective simulation with the MCWF method.

2. Superradiance master equation and MCWF simulations

The master equation for the density matrix of \( N \) two-level atoms with identical transition frequency \( \omega_a \) can be written as

\[
\frac{\partial}{\partial t} \rho = -\frac{i}{\hbar} [H, \rho] + D^\dagger [\rho] + D [\rho],
\]

(1)

where \( H = \hbar (\omega_a/2) \sigma^z \) with the collective Pauli operator \( \sigma^z = \sum_{i=1}^N \sigma_i^z \). The cavity mode with frequency \( \omega_c \) and loss rate \( \gamma_c \) provides a channel for the collective decay, and in the bad cavity limit the adiabatic elimination of the cavity field operators yields \[ D^\dagger [\rho] = -i[\omega_c \sigma^z, \rho] - \Gamma_1 D(\sigma^+) \rho \text{ with collective raising (lowering) operator } \sigma^z = \sum_i \sigma_i^z \text{ and superoperator } D[a] = \{a^\dagger a, \rho\} / 2 - a\rho a^\dagger. \]

Here, \( \omega_c = g^2 \chi (\chi^2 + \gamma_c^2/4)^{-1} \) and \( \Gamma_1 = (\gamma_c^2/2)(\chi^2 + \gamma_c^2/4)^{-1} \) are the collective Lamb shift and decay rate, respectively, and they are determined by \( \chi = \omega_i - \omega_c - \gamma_c \) and the atom–cavity mode coupling \( g \). The last term in equation (1) represents dissipation of individual atoms by three Lindblad terms \[ D(\sigma^+) \rho = -\gamma_{\sigma^+}^\dagger D[\sigma^+] \rho - \gamma_{\sigma^-}^\dagger D[\sigma^-] \rho - \gamma_{I_d}^\dagger D[I_d] \rho \text{ with a decay rate } \gamma_{\sigma^+}, \text{ pumping rate } \gamma_{\sigma^-}, \text{ and a dephasing rate } \gamma_{I_d}. \]

To realize the incoherent pumping with \( \gamma_{\sigma^-} \), the atoms may be excited from the lower level via a higher excited level from which they decay rapidly to the upper level of our atoms [7, 9].

For general systems like atomic gas in an optical cavity the atoms might have different frequencies and dissipative processes due to their different environment. However, for the experimental setup [4] as shown in figure 1(a) the master equation (1) is sufficient to capture the main physics. In this case, we can describe the system with Dicke states. These states \( |JM\rangle \) are eigenstates of collective operators \( j^2 = \sum_{j=\pm,\pm} \hat{j}^2 \) and \( j^\dagger j \) with eigenvalues \( J(J+1) \) and \( M \), where \( j^x = (\sigma^x + \sigma^-)/2 \), \( j^y = i(\sigma^- - \sigma^x)/2 \), and \( j^z = \sigma^z/2 \). For \( J < N/2 \), the states with same \( J \) and \( M \) are degenerate, and may be labeled by an additional quantum number \( \alpha = 1, \ldots, d_{JM} \), which specifies \( d_{JM}^\alpha = N!/((N-J)!(N/2-J)!(N/2+J+1)!) \) different symmetric linear combinations of atomic product states [21]. For the atoms with identical dissipation, while the dissipative processes of a single atom break the symmetry of the many body state, the sum of these processes over all the atoms in equation (1) preserves the symmetry [10, 11]. These processes do not build coherence between states of different \( \alpha \) or \( J \), and the equality of density matrix elements \( \rho_{JM\alpha JM\alpha'} \) for different \( \alpha \) motivates us to introduce a single term \( \rho_{JM\alpha}^I = (1/d_{JM}^\alpha) \sum_{\alpha'} \rho_{JM\alpha JM\alpha'} \) to represent all of them. The resulting effective density matrix is normalized as \( \text{Tr} [\rho] = \sum_{J,M} d_{JM} d_{JM}^\alpha \rho_{JM\alpha}^I = 1. \)

The equations for \( \rho_{JM\alpha}^I \) are derived in [10, 11]. Now we proceed to unravel them with the MCWF method.

To this end we introduce an ensemble of wave-functions \( |\psi_i(t)\rangle = \sum_J C_{JM}^i(t) |JM\rangle \). Here, the effective states \( |JM\rangle \) are not real states, but symbolic constructions, which disregard the degeneracy of the subspace with same \( J \) and \( M \), but allow sampling of \( \rho_{JM\alpha}^I \) as \( \sum_{J,M} d_{JM}^\alpha C_{JM}^i \sum_{\alpha'} C_{JM}^{i\alpha'} \). The normalization of the wave-functions is \( \sum_J d_{JM} C_{JM}^i C_{JM}^{i\dagger} = 1 \). The absence of coherence between different \( J \) allows us to consider only amplitudes of the wave-functions for each separate value of \( J \).
In the appendix A, we detail the evaluation of the MCWF. Here, we emphasize the main procedures and highlight the related quantum jumps. The wave-functions are represented as state vectors in the Dicke states space and are propagated with a non-Hermitian Hamiltonian including the atomic Hamiltonian and the anti-commutator of the dissipative superoperators. The $G_{JM}$ amplitudes follow equations including a reduction of norm associated with different physical processes that we simulate by discrete quantum jumps. The collective decay applies the jump operator $\sigma^-$ on the state-vectors, which leads to the jumps to the Dicke states of the same $J$ but $M - 1$, see figure 1(b), with a probability proportional to $|A_{JM}|^2$ [with $A_{JM} = \sqrt{(J + M)(J - M + 1)}$].

For the individual decay, we use the Clebsch–Gordan expansion (10, 11) to write the Dicke states of $N$ atoms as the product of those of $N - 1$ atoms and a single atom, and then apply the operator $\sigma_i^-$ on the single atom state. Finally, we use the inverse CG expansion to convert the state back to the Dicke states of all $N$ atoms. This introduces jumps to Dicke states with quantum numbers $M - 1$ and $J + s$ ($s = 0, +1, -1$). The branching ratio of the jumps are given in the table A1 in appendix A and favors jumps to states with quantum number $J - 1$, see figure 1(c). We apply the same treatment to the individual pumping (dephasing) by the operator $\sigma_i^-$, yielding jumps to states with quantum numbers $J + s$ and $M + 1$ ($M$), see figures 1(d), (e) and the table A1.

For the description of atom loss and feeding, we assume the loss and feeding rates, $\gamma_{\text{loss}}$ and $\kappa_{\text{feed}}$, are independent of the atomic internal state. The average atom number follows the rate equation $\frac{dN}{dt} = -\gamma_{\text{loss}}N + \kappa_{\text{feed}}$, which yields a steady-state mean atom number $N_i = \frac{\kappa_{\text{feed}}}{\gamma_{\text{loss}}}$. To simulate the loss of an atom we use the CG expansion to decouple the Dicke states of $N$ atoms as a product of states of $N - 1$ atoms and of a single atom, and then we discard the state of the single atom by a partial trace, simulated by quantum jumps from Dicke states of $N$ atoms to those of $N - 1$ atoms with probabilities proportional to squared CG coefficients, see figure 1(f) and the table A1. To incorporate a new atom, we use the inverse CG expansion and simulate a quantum jump from the Dicke state of $N$ atoms to one of $N + 1$ atoms, see figure 1(g) and the table A1.

After the implementation of any of the above quantum jumps, the state vectors are re-normalized, and the wave-function propagation and further random jumps proceed. With multiple simulated wave-function trajectories we can evaluate ensemble averages of physical observables, such as the Dicke states population $P_{JM} = \langle \beta_{JM}^2 \rangle$, the radiation through the cavity $\Gamma_t \langle \sigma^+ \sigma^- \rangle = \Gamma_t \sum_{JM} d_{JM}^2 |A_{JM}|^2 P_{JM}$, and the radiation into free space $\gamma_t \sum_{JM} \langle \sigma^+ \sigma^- \rangle = \gamma_t \sum_{JM} d_{JM}^2 (N/2 + M) P_{JM}$.

If the system has no initial coherence between states of different $J$, such coherences will never appear since the jumps happen either within the same ladder or from one to a neighboring ladder, see figures 1(b)–(g). Thus, our wave-functions can at any time be expanded on only a single ladder of states, restricting the dimension to $2J + 1 \leq N + 1$ elements. Notice the atoms can be prepared in a superposition of the fully symmetric Dicke states by driving the cavity mode [22]. If the system is initially prepared in a single Dicke state, e.g. the ground or fully-excited state, all coherences between states with different $M$ vanish for all times, and the system evolution can be effectively simulated as an incoherent jump process between states $|JM\rangle$. In this case, the dynamics of a stochastic wave-function can be visualized as a trajectory in a $(N, J, M)$ coordinate system, and we only need to evaluate the different jump probabilities to simulate the random evolution of $N(t), J(t)$, and $M(t)$. Notice that the atoms can be prepared in a fully excited state by choosing proper duration of driving field [4].

3. Superradiant dynamics of fifty atoms

We first illustrate our simulations with a small system of fifty atoms (in the absence of atom loss and feeding), which are initially all excited, i.e. $|J| = 25, M = 25$, see figures 2(a), (b), or initially all in the ground state, i.e. $|J| = 25, M = -25$, but pumped incoherently, see figures 2(c), (d). The main panels show the time evolution of the radiation intensity, while the insets show quantum jump trajectories among Dicke states.

To study the influence of individual decay with rate $\gamma$, we set $\gamma_i = \Gamma_i$ in figure 2(a) and $\gamma_i = 10\Gamma_i$ in figure 2(b). For weak individual decay the system remains close to the fully symmetric Dicke states with $J = N/2$, while for larger individual decay it explores states with lower values of $J$. In all cases, all atoms eventually end up in the ground state in the lower right corner in both insets. This dynamics is also reflected in the radiation of the system. Figure 2(a) shows a minor radiation into the free-space (blue curve), and a dominant radiation through the cavity, i.e. superradiance. The density matrix curves reflect rather large shot-to-shot variations in our calculated superradiance from the system with small atom number. The average superradiance (black solid curve) agrees well with the master equation results (red dashed curve), verifying our MCWF method. Figure 2(b) shows that for strong individual decay, the superradiance is suppressed and weaker than the exponentially decreasing radiation into the free-space (blue curve).

To study the effect of individual pumping with rate $\kappa$, we set $\kappa_i = \Gamma_i$ in figure 2(c) and $\kappa_i = 10\Gamma_i$ in figure 2(d). For weak pumping the trajectories start from the lower right corner and propagate along the lower boundary of the Dicke ladders, and finally wander around the left corner with small values of $J$ and $M$, while the
radiation through the cavity fluctuates around $25\Gamma_c$. For strong pumping the trajectories explore more states and end up in a region around $J = 15$ and $M = 8$. As a consequence of the higher symmetry and excitation of the states, the collectively enhanced emission probabilities are higher and both the fluctuations and the averaged radiation increase. In all cases, the averaged radiation trajectories are in perfect agreement with the exact master equation calculations. The reduced radiation at strong pumping can be ascribed to the pump-induced noise and may also be understood as a suppression of the superradiant transition matrix elements $\mu_I^M$ for $M \approx J$.

4. Superradiant lasing

After having verified our method with a toy model calculation for tens of atoms, we now turn to the simulation of much larger ensembles, see figures 3 and 4, as studied by Norcia et al [4]. In their experiment, more than $10^5$ strontium atoms are trapped in an optical lattice inside an optical cavity and coupled to a cavity mode through their ultra-narrow $S_{1/2}^1 - P_{3/2}^0$ optical clock transition.

For this problem, the many independent trajectories explore ranges of $J$ and $M$ according to the same diffusion-like process as observed in figure 2. However, for $N > 10^5$ the relative fluctuations among trajectories are small, and we can simulate the system and obtain the information of interest with only a few hundred quantum trajectories. In figure 3 we show the superradiance pulses for different initial numbers of atoms $N(0)$ in the presence of atom loss. The inset shows that the atom number decreases exponentially (blue dotted curve), which reduces the population of the upper (lower) level before (after) the superradiance pulse. The calculated pulses agree well with the experimental results (noisy curves) with a fitted atom loss rate of $\gamma_{\text{loss}} = 5.58 \text{ Hz}$ (compatible with the magnitude estimated in [4]) and negligible dephasing rate $d_0$. If we increase $d_0$ (up to $\gamma_{\text{loss}}$), the calculated pulses are reduced and shifted to earlier time and thus do not match the experimental results. A ten-fold increase of $\gamma_I$ only changes the pulse slightly. These results confirm the assessment in [4] that the dynamics is dominated by the collective decay and the atom loss.

In our simulations all the atoms are lost in about 0.2 s and there is no time to establish the coherence leading to lasing with millihertz linewidth. However, by feeding new atoms, we can achieve steady-state superradiance. To calculate the corresponding spectrum, we apply the quantum regression theorem and calculate

**Figure 2.** Dynamics and radiation of 50 atoms initially excited (a), (b) and initially unexcited but continuously pumped (c), (d). The main panels show the radiation intensity (relative to $\Gamma_c$) versus dimensionless time $t\Gamma_c$, while their insets show quantum jump trajectories among the $|JM\rangle$ Dicke states. The gray areas are explored by 512 simulated trajectories, while single sample trajectories and average trajectories are indicated by the dashed and solid black curves, respectively. The red dashed curves are computed with the master equation and match perfectly with the black solid curves. The blue solid curves show the averaged radiation into free-space. $\gamma_I/T_c = 1, 10$ in panels (a), (b) while $\kappa/\Gamma_c = 1, 10$ in panels (c), (d).
with the two-time correlation function $S(t, t')$. We follow [17] to calculate this function with the following procedure: we first propagate the stochastic wave-functions to long time (to simulate steady-state), and then apply four different combinations of the collective lowering operator and the unit operator to initialize four ancillary wave-functions that we propagate with the MCWF method to obtain correlation function trajectories. The average of many such trajectories yields the exact correlation function, see the appendix B. During the evolution the ancillary wave-functions are coherent superposition of two adjacent states (with identical $J$). They can thus be propagated by solution of only two coupled equations, and their quantum jumps are also readily implemented and retain their simple two-component form at all times.

The left panel of figure 4 shows the intensity and linewidth of the steady-state superradiant spectrum from $10^5$ strontium atoms, which are pumped individually with increasing rate $\kappa_l$ in the absence of atom loss and feeding. We see that the intensity increases linearly with $\kappa_l$, while the linewidth approaches a minimum and then increases due to the pump-induced noise. The intensity agrees qualitatively with the prediction in [7], while the linewidth broadening occurs for weaker pumping because of the negligible dephasing in our simulations. The minimal linewidth achieved is about 16 mHz and may be much smaller for systems with larger atom number.

$S(\omega) \propto \Gamma_i \Re \int_0^\infty d\tau e^{-i\omega\tau} \langle \sigma^+ (\tau) \sigma^- (0) \rangle$ with the two-time correlation function $\langle \sigma^+ (\tau) \sigma^- (0) \rangle$. We follow [17] to calculate this function with the following procedure: we first propagate the stochastic wave-functions to long time (to simulate steady-state), and then apply four different combinations of the collective lowering operator and the unit operator to initialize four ancillary wave-functions that we propagate with the MCWF method to obtain correlation function trajectories. The average of many such trajectories yields the exact correlation function, see the appendix B. During the evolution the ancillary wave-functions are coherent superposition of two adjacent states (with identical $J$). They can thus be propagated by solution of only two coupled equations, and their quantum jumps are also readily implemented and retain their simple two-component form at all times.

The left panel of figure 4 shows the intensity and linewidth of the steady-state superradiant spectrum from $10^5$ strontium atoms, which are pumped individually with increasing rate $\kappa_l$ in the absence of atom loss and feeding. We see that the intensity increases linearly with $\kappa_l$, while the linewidth approaches a minimum and then increases due to the pump-induced noise. The intensity agrees qualitatively with the prediction in [7], while the linewidth broadening occurs for weaker pumping because of the negligible dephasing in our simulations. The minimal linewidth achieved is about 16 mHz and may be much smaller for systems with larger atom number.
The right panel of figure 4 shows the results in the presence of atom loss $\gamma_{\text{loss}} = 5.58$ Hz and feeding

$\kappa_{\text{feed}} = \gamma_{\text{loss}} N_t$ being able to maintain large steady-state atom number, $N_t = 10^5$. As each atom typically stays for only a fraction of a second in the cavity, the incoherent pumping has to be fast to ensure excited state population available for the lasing process. Our calculations, indeed, show a threshold effect, requiring a pumping rate $\kappa_s > 2$ Hz. Since the adjacent Dicke states in the ancillary wave-functions are exposed to the same jumps during the correlation function evolution, their coherence survives longer than the atoms, leading to sub-hertz linewidth for pumping smaller than 9 Hz and a minimal 60 mHz linewidth for the 2 Hz pumping rate.

5. Conclusions

To compensate the atoms loss hindering the current exploration of superradiant lasing, we propose to feed new atoms. To address the question if the super-narrowing lasing can be sustained in the presence of these processes, we developed a MCWF method for superradiance of two-level atoms in a cavity. Our treatment incorporates atomic collective decay, individual decay, pumping and dephasing as quantum jumps within or among Dicke ladders of fixed atom number, and treats the atom loss and feeding as jumps between Dicke ladders with reduced and increased atom number. Our method is verified by comparison with the exact master equation calculations for fifty atoms and our calculated superradiance pulses by more than $10^5$ strontium atoms agree with the experimental results with a number. Our method is verified by comparison with the exact master equation calculations for fifty atoms and our calculated superradiance pulses by more than $10^5$ strontium atoms agree with the experimental results with a fitted atom loss rate. By feeding atoms in the lower-level into the cavity, we can compensate the atom loss and obtain steady-state spectra with a minimal linewidth about 60 mHz. The corresponding coherence time is about 16 s and is thus hundred times longer than the time spent by any typical atom in the cavity. The MCWF unraveling of the density matrix may also be applied to extract important insight into dynamical evolution of the quantum states in superradiant beats [22], synchronization [23], and spin-squeezing [24] of atomic ensembles.

Acknowledgments

This work was supported by the Villum Foundation (Y Zhang and K Mølmer) and the European Union’s Horizon 2020 research and innovation program (No. 712721, NanoQtech, Y X Zhang and K Mølmer).

Appendices

In the main text, we summarize briefly how MCWF $|\psi(t)\rangle$, defined as superposition states in an effective symmetric Dicke state $|J, M\rangle$ basis, are propagated under the influence of collective and individual dissipation. As indicated there, in the special case, where there is no initial superposition between states of different $J$, such superposition will never appear because of the characteristics of the quantum jumps and thus the wave-functions can always be expanded in only a single ladder of Dicke states. In the more peculiar case as studied in the main text, where the system occupies one Dicke state initially, the wave-functions will always populate one state and thus their dynamics can be effectively visualized as incoherent jumps of the numbers $J(t), M(t)$ (and the atom number $N(t)$). In the following appendices, we offer details of the wave-function propagation for the general case rather than the special cases to ensure that the formalism is ready to apply to other problems as indicated in the conclusion. For simplicity we will suppress the index $J$ in the following.

Appendix A. Monte-Carlo wave-function evaluation

The wave-functions are propagated according to the Schrödinger equation with a non-Hermitian Hamiltonian $\hat{H}$ incorporating the atomic Hamiltonian $H$ and the anti-commutator part of dissipative superoperators $D^\dagger[\rho]$ and $D[\rho]$ in the master equation (1). In first order of the time step $\Delta t$ we have the explicit equations for the state amplitudes in the Dicke state basis

$$C_M(t + \Delta t) = (1 - i\omega_{JM}\Delta t)C_M(t).$$  \hspace{1cm} (A.1)

Here, the complex frequency term $\omega_{JM}$ has the real part $\omega_{JM} = \omega_J + \omega_{JM}^\dagger$, where $\omega_J$ is the atomic transition frequency and $\omega_{JM}$ is the Lamb shift. The imaginary part

$$-\frac{1}{2}[\Gamma_{\alpha}(A_{JM}^\dagger A_{JM})^2 + \gamma(N/2 + M) + \kappa_s(N/2 - M) + d_sN]/2,$$

where, $\Gamma_{\alpha}, \gamma, \kappa_s, d_s$ denote the collective decay rate, and the individual decay, pumping and dephasing rate, respectively. The factors

$$A_{JM} = \sqrt{(J + M)(J + M + 1)}$$

depend on the Dicke state quantum numbers. These numbers govern the so-called no-jump dynamics of the wave-function. The imaginary part reduces the norm of the state vector by a sum of infinitesimal probabilities $\sum_{\alpha} \beta_{\alpha}$, which are, in the master equation, compensated by (sandwich) feeding terms, e.g. $\sigma^\dagger(\sigma^\dagger)^{\alpha}$ in $D^\dagger[\rho]$. In the MCWF method, the feeding terms are represented by the return of the population by quantum jumps into different final states. $\beta$ enumerates eighteen different quantum
Table A1. Comprehensive list of quantum jumps associated with the different dissipative processes. The table lists the change in number of atoms and Dicke collective spin quantum numbers, the respective jump probabilities in a short time interval $\delta t$, and explicit expressions of the abbreviations used. $P_+$ and $P_-$ are the upper and lower state occupation of atoms in the feeding process.

| Dissipations | Jumps | Probabilities |
|--------------|-------|---------------|
| Collective decay | $(N, J, M) \rightarrow (N, J, M - 1)$ | $A_{JM}^{LM} = \frac{(J + M + 1)(J - M + 1)}{2N + 1}$ |
| Individual decay | $(N, J, M) \rightarrow (N, J, M - 1)$ | $A_{JM}^{LM} = \frac{(N - 1)(J - M)(J - M + 1)}{2N + 1}$ |
| Atom loss | $(N, J, M) \rightarrow (N - 1, J + 1, M + 1)$ | $A_{JM}^{LM} = \frac{(J + M + 1)(J - M + 1)}{2N + 1}$ |
| Atom feeding | $(N, J, M) \rightarrow (N + 1, J + 2, M)$ | $A_{JM}^{LM} = \frac{(J + M + 1)(J - M + 1)}{2N + 1}$ |

The above expressions for collective and individual decay are derived from the master equation formulation. For individual pumping and atom loss, the expressions are obtained from the symmetric Dicke states expansion. For atom feeding, the expressions are derived from the master equation formulation.

Jump channels, depicted in figures 1(b)–(g), with the probabilities summarized in table A1. The values of the different $\delta P_{JM}$ and the corresponding quantum jump actions, will be detailed in the following.

For the collective decay, we apply the collective jump operator $\sigma^-$ on the wave function to yield $|\psi(t + \delta t)\rangle = (\delta t \Gamma) \sigma^- |\psi(t)\rangle$, which leads to the relation for the state amplitudes $C_{JM \rightarrow (t + \delta t) = (\delta t \Gamma)^{LM} C_{JM}(t)}$.

This quantum jump, which lowers $M$ and retains $J$ as illustrated in figure 1(b), occurs with the probability $\delta P_{JM} = \delta t \Gamma |\lambda\rangle \sum A_{JM}^{LM} C_{JM}^2$. Since the system has vanishing coherences between states with different $J$, we can restrict the expansion of the quantum state to a single value of $J$, which is unchanged by the collective decay process (and by the no-jump dynamics).

The dissipation terms $D[\sigma_+^j \rho_j]$, $D[\sigma_-^j \rho_j]$ and $D[\sigma_0^j \rho_j]$ acting on the individual atoms can be treated by expanding the symmetric Dicke states $|J, M\rangle$ in the tensor product basis of the single atom affected and all the other atoms, i.e. products states of a single spin $|j_1 = \pm \frac{1}{2}, m_1 = \pm \frac{1}{2}\rangle$ and symmetric Dicke states $|j_2 = J \pm \frac{1}{2}, m_2 = M - m_2\rangle$. The action of the individual dissipation operators affects only the $|J, M\rangle$ component, and the irreducible representation of the resulting product states yields a new expansion on Dicke states $|J, M\rangle$, all of atoms. This expansion will include states with $J_1 = J + s (s = 0, \pm 1)$ and $M_0 = M + q$, where $q = -1$ for decay, $q = 1$ for incoherent excitation, and $q = 0$ for atomic dephasing. The careful consideration of the sum of identical terms over all atoms restricts the coupling of density matrix elements $P^j_{JM}$ to other similar matrix elements $P^j_{J'M'}$ with branching ratios determined by the dimensionality factors $d_j$ and CG coefficients for the angular momentum coupling of the spin states. We refer the reader to [10, 11] for the technical details of this derivation. In our work the dissipation terms in the master equation are simulated by corresponding quantum jumps of our effective Dicke state vector. Note that the jumps do not correspond to a definite measurement process and outcome but are utilized here merely as a computational tool to simulate the individual dissipation of the atoms while exploiting the permutation symmetry of the density matrix.

In our symmetrical average evolution, the decay of individual atoms thus leads to the transformation of state amplitudes $C_{JM \rightarrow (t + \delta t) = (\delta t \Gamma)^{LM} C_{JM}(t)}$, which lowers $M$ by unity and changes $J$ by 0 or ±1, as sketched in figure 1(c), with the jump probabilities $\delta P_{JM} = \delta t \Gamma |\lambda\rangle \sum A_{JM}^{LM} C_{JM}(t)^2$. Here, the values of $P^j_{JM}$ are presented in table A1. The incoherent excitation

\[ C_{JM \rightarrow (t + \delta t) = (\delta t \Gamma)^{LM} C_{JM}(t)}, \]
(pumping) of individual atoms leads to a similar transformation of state amplitudes
\[ C_{LM}(t + \Delta t) = \sqrt{\delta \text{tr}_{\text{fi}} \rho_{LM}(t)} C_{LM}(t), \]
which however increases M by unity and changes J by 0 or ±1, as sketched in figure 1(d), with the jump
probabilities \( \delta \rho_{LM} = \delta t \text{ri} d_k^L \sum_{\text{at}} P_{LM} C_{LM}(t) \). The values of \( P_{LM} \) are also presented in table A1. The dephasing of
individual atoms leads to the transformation of state amplitudes
\[ C_{LM}(t + \Delta t) = \sqrt{\delta \text{tr}_{\text{fi}} \rho_{LM}(t)} C_{LM}(t), \]
which retains the value of M and changes J by 0, ±1, as sketched in figure 1(e), with the jump
probabilities \( \delta \rho_{LM} = \delta t \text{ri} d_k^L \sum_{\text{at}} P_{LM} C_{LM}(t) \). The values of \( P_{LM} \) are again presented in table A1.

To describe the atom loss, we use the CG expansion \( |JM\rangle = \sum_{j_{\text{jm}}} \sum_{\text{at},m_{\text{jm}}} |j_{\text{jm}}\rangle_{\text{at}} |jm\rangle_{\text{jm}} \) with the CG
coefficients \( C_{j_{\text{jm}}}^{jm} \) and interpret each term in the expansion as one quantum jump, which removes \( 2j \) atoms in the states \( j_{\text{jm}} \) from the system and leave the \( 2j_2 = N - 2j_1 \) atoms in the states \( j_{\text{jm}} \). The different jumps will
take care of all possible ways of removing the atoms in all possible states. To describe the feeding of new and
uncorrelated atoms into the system, we use the inverse CG expansion \( |j_{\text{jm}}\rangle_{\text{at}} |JM\rangle = \sum_{j_{\text{jm}}} \sum_{\text{at},m_{\text{jm}}} |j_{\text{jm}}\rangle_{\text{at}} C_{j_{\text{jm}}}^{jm} |j_{\text{jm}}\rangle_{\text{at}} \) and interpret each term in the expansion as one quantum jump, which adds \( 2j_1 \) atoms in the state \( j_{\text{jm}} \) to the states \( |JM\rangle \) of N atoms to form the new states \( |j_{\text{jm}}\rangle_{\text{at}} \). The different jumps will account
for all possible final states by adding atoms in one specified state. Since the probabilities of the jumps are
proportional to the squared CG coefficients, the jumps of removing and feeding a single atom are more favored
than other jumps. In the case of a single atom loss, i.e. \( j_1 = 1/2 \), we will have four different quantum jumps,
which remove the single atom in the upper (lower) level and leave the \( N - 1 \) atoms in the Dicke states of \( J = 1/2 \)
and \( M = 1/2 (M + 1/2) \), see figure A(f). These jumps can be implemented with
\[ C_{LM}(t + \Delta t) = \sqrt{\delta \text{tr}_{\text{loe}} \rho_{LM}} C_{LM}(t). \]
Here, we have introduced \( J_{\text{loe}} = J + 1/2 \) and \( M_{\text{loe}} = M + 1/2 \), and the values of \( L_{\text{loe}} \) are provided in table A1. The jump probabilities are \( \delta \rho_{LM} = \delta t \text{loe} d_k^L \sum_{\text{at}} \rho_{LM}(t) \). In the case of single atom feeding, i.e.
\( j_1 = 1/2 \), we will also have four different quantum jumps, which add the single atom in the upper (lower) level
and leave the \( N + 1 \) atoms in the Dicke states of \( J = 1/2 \) and \( M = 1/2 (M + 1/2) \), see figure A(g). These jumps can be implemented with
\[ C_{LM}(t + \Delta t) = \sqrt{\delta \text{tr}_{\text{fe}} \rho_{LM}} C_{LM}(t), \]
where the coefficients \( \rho_{LM} \) are given in table A1. The different processes occur with the jump probabilities
\( \delta \rho_{LM} = \delta t \text{fe} d_k^L \sum_{\text{at}} \rho_{LM}(t) \). For the atom loss we can not control the state of the lost atom and thus should
consider all the possibilities. However, for the atom feeding, we can control the jumps by preparing the
fed atom in different level. For this reason, we introduce the probability \( P_j \) to describe the fed atom in the upper
\( (s = +) \) or lower \( (s = -) \) level, respectively.

Appendix B. Steady-state spectrum calculation

In the main text we indicate how to calculate the steady-state spectrum with the MCWF method. As outlined there,
the spectrum can be obtained from Fourier transformation of two-time correlation function \( \langle \sigma^+(t) \sigma^-(0) \rangle \).
According to [17], this function can be calculated as \( \langle 1/4 \rangle \sum_{k} k \{ \mu_k \xi_k(t) \} - iv_k d_k(t) \rangle \), where \( \xi \) denote the
average over many trajectories of the functions \( c_k(t) = \langle \chi_k(\tau) | \sigma^+ | \chi_k(\tau) \rangle \) and \( d_k(t) = \langle \chi_k(\tau) | \sigma^- | \chi_k(\tau) \rangle \). These functions are simulated with four ancillary wave-functions \( |\chi_k(\tau)\rangle \) and \( |\chi_k(\tau)\rangle \) that are specified by the initial conditions \( \langle 0 | \chi_k(0) \rangle = \sqrt{1/\mu_k} \{ 1 - \sigma_+ \} |\psi(t)\rangle \), \( |\chi_k(0)\rangle = \sqrt{1/\mu_k} \{ 1 - \sigma_- \} |\psi(t)\rangle \). Here, \( \mu_\pm, v_\pm \) are normalization coefficients. In the Dicke state basis, these functions are evaluated as \( \sum_k d_k^L \sum_{\text{at}} |A_k^L D_{LM} |D_{LM}^H |(\tau)D_{LM}^H (\tau) \rangle \), where the coefficients \( D_{LM} \) are the state amplitudes of the ancillary wave-functions.

In the peculiar case as studied in the main text, the ancillary wave-functions have only two non-vanishing amplitudes at any point during the stochastic wave-function evolution. In the evaluation, these
amplitudes, which reflect superposition of two adjacent Dicke states, persist despite of the quantum jumps,
which ensures that the coherence time can be much longer than the time spent in the system by any typical atom.
This guarantees the lasing with millihertz linewidth as shown in the main text.

References

[1] Dicke R H 1954 Phys. Rev. 93 99
[2] Andreev A V, Erem’yanov V and I’inskii Y A 1980 Sov. Phys.—Usp. 23 493
[3] Garraway B M 2011 Phil. Trans. R. Soc. A 369 1137–55
[4] Norcia M A, Winchester M N, Cline J R K and Thompson J K 2016 Sci. Adv. 2 e1601231
[5] Norcia M A and Thompson J K 2016 Phys. Rev. X 6 011025
[6] Bohnet J G, Chen Z, Weiner J M, Meiser D, Holland M J and Thompson J K 2012 Nature 484 78
[7] Meiser D, Ye J, Carlson D R and Holland M J 2009 Phys. Rev. Lett. 102 163601
[8] Bonifacio R, Schwendimann P and Haake F 1971 Phys. Rev. A 4 302
Bonifacio R, Schwendimann P and Haake F 1971 Phys. Rev. A 4 854
[9] Meiser D and Holland M J 2010 Phys. Rev. A 81 033847
[10] Chase B A and Geremia J M 2008 Phys. Rev. A 78 052101
[11] Baragiola B Q, Chase B A and Geremia J M 2010 Phys. Rev. A 81 032104
[12] Shammah N, Ahmed S, Lambert N, Liberato S D and Nori F 2018 arXiv:1805.05129v3
[13] Richter M, Gegg M, Theuerholz T S and Knorr A 2015 Phys. Rev. B 91 035306
[14] Gegg M and Richter M 2016 New J. Phys. 18 043037
[15] Zhang Y and May V 2015 J. Chem. Phys. 142 224702
[16] Xu M, Tieri D A and Holland M J 2013 Phys. Rev. A 87 062101
[17] Mølmer K, Castin Y and Dalibard J 1993 J. Opt. Soc. Am. B 10 524
[18] Plenio M B and Knight P L 1998 Rev. Mod. Phys. 70 101
[19] Wiseman H M and Milburn G J 2010 Quantum Measurement and Control (Cambridge: Cambridge University Press)
[20] Mandel L and Wolf E 1995 Optical Coherence and Quantum Optics (Cambridge: Cambridge University Press) p 838
[21] Norcia M A, Cline J R K, Muniz J A, Robinson J M, Huston R B, Goban A, Marti G E, Ye J and Thompson J K 2018 Phys. Rev. X 8 021036
[22] Weiner J M, Cox K C, Bohnet J G and Thompson J K 2017 Phys. Rev. A 95 033808
[23] Hosten O, Engelsen N J, Krishnakumar R and Kasevich M A 2016 Nature 529 505