Dynamics of a trapped 2D Bose-Einstein condensate with periodically and randomly varying atomic scattering length

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Abstract

In this work we consider the oscillations and associated resonance of a 2D Bose-Einstein condensate under periodic and random modulations of the atomic scattering length. For random oscillations of the trap potential and of the atomic scattering length we are able to calculate the mean growth rate for the width of the condensate. The results obtained from the reduced ODE’s for oscillations of the width of condensate are compared with the numerical simulations of the full 2D Gross-Pitaevskii equation with modulated in time coefficients.

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I. INTRODUCTION

The problem of the oscillations of two and three dimensional trapped Bose-Einstein condensates (BEC) have attracted a great deal of recent attention [1], particularly the problem where the trapping potential is temporally modulated. The main physical motivation for considering such a problem is the experiments of Jin, et. al. [2], which observed resonant response of the BEC oscillations to periodically perturbed traps. Some theoretical investigations of the problem have included the work of Castin and Dum [3], in which the resonances in the width oscillations have been analyzed using scaling theory; the work of Garcia [4], in which it was argued that the oscillations of 2D condensate can be described by the parametric resonances of linear oscillator; and the papers by Pitaevski and Rosch [5],[6] and Kagan et. al. [7], where it was shown that a 2D condensate exhibits a harmonic mode with frequency $2\omega_0$ when the harmonic trap is driven with frequency $\omega_0$. Note that the related problem of a soliton interacting with an impurity in 2D molecular crystals had been studied earlier by Gadidei et al. [7].

The excitation of resonant oscillations is also interesting as a possible mechanism for the stabilization of a BEC with attractive interactions. Such a possibility has been demonstrated for the BEC with negative atomic scattering length under temporally modulated trap potential if the surface mode is exited [8]. In this work a variational approach is used, and the trap was anisotropic.

It is also interesting to consider the response of a condensate to temporally periodic variations in the atomic scattering length. Such a variation of the atomic scattering length can be achieved experimentally by varying the magnetic field [9] or using optically induced Feshbach resonances [10]. The case where the scattering length varies monotonically in time, in particular the case where the sign changes from repulsive to attractive, has been studied recently by Dalfovo et. al [11], and Fedichev, et. al. [9]. Similarly the effect of a periodic variation of atomic scattering length on the tunneling between two condensates in the double-well trap, a resonant tunneling, has been studied by Abdullaev and Krankel [10].
These observations suggest the possibility of new phenomenon in Bose-Einstein condensate width oscillations when the atomic scattering length $a_s$ is allowed to fluctuate about some mean value. In this paper we study the influence of both periodic and random fluctuations of the atomic scattering length on the condensate dynamics. As noted in [6], at temperatures $T \gg n g, g = 4\pi\hbar^2 a_s/m$, (where $n$ is the gas density, $a_s$ is the atomic scattering length, $m$ is the atom mass), only the condensate evolution is pronounced. The perturbation of the thermal cloud is small at these temperatures, and in this case the eigenfrequencies of small oscillations of the thermal cloud are close to those of the condensate.

The specific problem that we consider is the 2D Gross-Pitaevskii equation with periodic or random variations in the coefficient of the nonlinear term. This problem is interesting not only for BEC but also for nonlinear optics too - for example the dynamics of optical beams in nonlinear layered waveguides [13].

The structure of the remainder of the article is follows: In Sect.2 we describe the model and derive the reduced ODE system to describe the dynamics of the condensate width. In Sect.3 we analyze the effect of time-period perturbations on the width dynamics using the action-angle variables for the reduced ODEs. We explore the resonances in condensate oscillations in Sect.4, and the dynamics of width oscillations under fluctuating trap potential and atomic scattering length in last section.

II. THE REDUCED ODE MODEL

It is well-known that wavefunction for a 2D Bose-Einstein condensate in a trap potential $V(r)$ is described by the Gross-Pitaevskii equation:

$$i\hbar \psi_t = -\frac{\hbar^2}{2m} \Delta \psi + V_{tr}(r)\psi + g(t)|\psi|^2\psi.$$  

(1)

Here $V_{tr}(r) = m\omega^2 r^2/2$ is the trap potential and $g(t) = 4\pi\hbar^2 a_s/m$, with $a_s$ being the atomic scattering length. We will assume that the time-dependent scattering length is constant to leading order, with

$$a_s = a_0(1 + \epsilon_0(t)).$$
In this paper we analyze the case where $\epsilon_0(t)$, representing the small fluctuations, is a periodic function of time as well as the case where $\epsilon_0(t)$ is a mean zero white noise random process: $<\epsilon_0(t)\epsilon_0(t')> = 2\sigma^2\delta(t-t')$.

Under the scaling $\tau = t\omega, x' = x\sqrt{\hbar/(2m\omega)}, u = \psi\sqrt{4\pi a_0/\omega}$ the Gross-Pitaevskii equation can be written in the dimensionless form

$$iu_{\tau} + \Delta u - \frac{r^2}{2} u + s|u|^2u = 0,$$

where $s > 0, s < 0$ correspond to attractive or repulsive interactions between atoms in the condensate respectively.

A number of methods can be employed to investigate the dynamics of the Bose-Einstein condensate under temporal variations of the atomic scattering length. One of the simplest to apply is the averaged Lagrangian approach. According to this method we take the Gaussian anzatz for the field \[17,13,4\],

$$u(r, \tau) = A(\tau) \exp\left(-\frac{r^2}{2a(\tau)^2} + \frac{ib(\tau)r^2}{2} + i\phi(\tau)\right),$$

To derive the equations for the wavepacket parameters $A(\tau), a(\tau), b(\tau), \phi(\tau)$ one should calculate the averaged Lagrangian

$$\bar{L}(\tau) = \int rdrL(r, \tau).$$

For the Gaussian ansatz in Eq.(3) the averaged Lagrangian is given by

$$\bar{L} = -\frac{\pi}{2} A^2 a^2 (a^2 b_r + 2\phi_r + \frac{2}{a^2} + 2a^2 b^2 + a^2 - \frac{s(\tau)A^2}{2}).$$

The Euler-Lagrange equations for the functional $\bar{L}(\tau)$ lead to the following equations for the the width $a(\tau)$ and chirp $b(\tau)$.

$$a_{\tau} = 2ab$$

$$b_{\tau} = \frac{2 - s(\tau)N}{a^4} - 2b^2 - 1,$$

with $N = \frac{1}{2\pi} \int |u|^2 d^2r = \frac{a^2 A^2}{2}$. It is easy to see that the above is a Hamiltonian system, with the width $a$ playing the role of a position variable, and the chirp $b$ the conjugate momentum.
One can eliminate $b$ from the above system which leads to the following evolution equation for $a$:

$$a_{tt} + a = \frac{2 - s(t)N}{a^3},$$

where $t = \sqrt{2\tau}$. For the attractive problem with constant atomic scattering length $s(t) = 1$ the variational approach predicts the critical threshold for collapse $N_c = 2$, compared with the the exact value $N_c = 1.862$.

We comment that, if the initial condition is close to that of the ground state (Townes) soliton, $|N - N_c| \sim \epsilon \ll 1$, then the modulation theory of Fibich and Papanicolaou [14] leads naturally to a modulation equation of the same form as the above, where again the constant 2 is replaced by the critical value 1.862. The qualitative types of behavior exhibited by the above equation do not, of course, depend on the exact values of the constants, since all such constants can be scaled out.

It is useful to rewrite Eq.(6) in the form

$$a_{tt} + a = \frac{Q(t)}{a^3},$$

where $Q(t) = Q(1 + \epsilon(t)), Q = 2 - N, \epsilon = N\epsilon_0/(2 - N)$. In the next section we construct the action-angle variables for the above problem.

III. PERTURBATION THEORY IN THE ACTION-ANGLE VARIABLES

We begin our analysis of the Eq.(6) by constructing the action-angle variables. Of course any two dimensional Hamiltonian system can always be reduced to quadrature, but the above system is particularly nice because the action-angle variables can be expressed in terms of elementary functions. First we note that Eq.(6) is Hamiltonian:

$$H(a_t, a) = \frac{a_t^2}{2} + U(a), U(a) = \frac{a^2}{2} + \frac{Q}{2a^2}.$$

For $N < 2, Q > 0$ and the motion of the effective particle is bounded, and bounded away from 0 due to the “angular momentum” barrier $\frac{Q}{2a^2}$. If the energy is $E$ then the width
oscillates between \( a_{\text{min}} = \sqrt{E - \sqrt{E^2 - Q}} \) and \( a_{\text{max}} = \sqrt{E + \sqrt{E^2 - Q}} \). The minimum of the potential \( U \) occurs at \( a_c = Q^{1/4} \), with a minimum energy of \( E = U_c = \sqrt{Q} \). When \( N > 2, Q < 0 \) there is no local minimum, and there exist solutions for which \( a \to 0 \), corresponding to collapse of a condensate (see Fig.1).

Since the Hamiltonian is conserved, \( H = E \), we have

\[
\int \frac{da}{\sqrt{2E - a^2 - Qa^2}} = \int dt
\]

which can be integrated up to the following solution for \( Q > 0 \):

\[
a(t) = \sqrt{E + \sqrt{E^2 - Q}} \sin(2t + \psi_0). \tag{9}
\]

The action variable for a Hamiltonian system is defined to be

\[
J = \frac{1}{2\pi} \oint pdq, \tag{10}
\]

where \( q \) is the position variable and \( p \) the conjugate momentum. For the above oscillator the position variable is \( a \), with momentum \( p = \sqrt{2Ea^2 - a^4 - Q/a} \). The integration is easily done via contour integration in the complex \( a \) plane leading to the following expression for the action variable:

\[
J = \frac{1}{2}(E - \sqrt{Q}). \tag{11}
\]

The bottom of the potential well, \( E = \sqrt{Q} \), corresponds to \( J = 0 \). Since the energy is linear in the action, \( E = 2J + \sqrt{Q} \), the frequency of the unperturbed oscillations is constant

\[
\omega = \frac{dE}{dJ} = 2. \tag{12}
\]

This can, of course, also be seen directly from the solution given in Eq. (9).

The Hamiltonian for the perturbed problem is given by

\[
H = \frac{p^2}{2} + U(a) + \epsilon(t)V = \frac{p^2}{2} + U(a) + \frac{Q\epsilon(t)}{2a^2}
\]
, with $V(a)$ the perturbation Hamiltonian. In the action-angle coordinates $J, \theta$ the perturbation becomes

$$V_\theta = -\frac{Q\sqrt{E^2 - Q \cos(\theta)}}{2(E + \sqrt{E^2 - Q \sin(\theta)})^2}.$$ 

and the perturbed equations of motion are given by

$$\frac{dJ}{dt} = -\epsilon(t) \frac{\partial V}{\partial \theta},$$

$$\frac{d\theta}{dt} = 2 + \epsilon(t) \frac{\partial V}{\partial J}. \tag{14}$$

**IV. RESONANCES OF BEC OSCILLATIONS UNDER PERIODICALLY VARYING ATOMIC SCATTERING LENGTH**

In this section we consider perturbations $\epsilon(t)$ which are periodic in time. For simplicity we discuss the case $\epsilon(t) = \sin(\Omega t)$, though this can easily be generalized. To analyze the resonances of a BEC under such perturbations we use the multiscale expansion method. We introduce the slow time $T = \epsilon t$ and assume a multiple-scales ansatz of the form

$$\theta = \theta^{(0)}(t, T) + \epsilon\theta^{(1)}(t, T) + ..., \tag{15}$$

$$J = J^{(0)}(t, T) + \epsilon J^{(1)}(t, T) + ... \tag{16}$$

The solution to the evolution on the fast scale is obviously $\theta^{(0)} = 2t + \Phi, J = \text{constant}$. At the next order we find the following evolutions for the action $J$ and the slow angle $\Phi$ on the slow scale:

$$\frac{dJ}{dT} = -\frac{\partial \bar{H}}{\partial \Phi}, \tag{17}$$

$$\frac{d\Phi}{dT} = \frac{\partial \bar{H}}{\partial J}, \tag{18}$$

where $\bar{H}$, the slow Hamiltonian, is given by

$$\bar{H}(J, \Phi) = \frac{1}{2\pi} \int_0^\pi \frac{\sin(\Omega t)dt}{(2J + Q^{1/2} + 2J^{1/2}(J + Q^{1/2})^{1/2}\sin(2t + \Phi))}. \tag{19}$$
It is clear that we have a resonant response whenever \( \omega = 2n \), and the period of the perturbation is commensurate with the natural period of the condensate. Note that, even though this is a nonlinear oscillator the period is *independent* of the amplitude. The effect of this is that there is no ”detuning” from the resonant frequency as the amplitude increases. Of course when \( J \) becomes large and the width oscillations become significant we no longer expect the averaged Lagrangian ODE’s to provide an accurate description of the dynamics of the condensate width.

The first (nontrivial) resonance occurs for \( \Omega = 2 \). In this case the slow Hamiltonian is given by

\[
\bar{H}(J, \Phi) = -\frac{\cos(\Phi)J^{1/2}}{2(J + Q^{1/2})^{1/2}Q^{1/2}}. \tag{19}
\]

and the equations of motion are

\[
\frac{dJ}{dt} = -\frac{\partial \bar{H}}{\partial \Phi} = -\frac{\sin(\Phi)J^{1/2}}{2(J + Q^{1/2})^{1/2}Q^{1/2}}, \tag{20}
\]

\[
\frac{d\Phi}{dt} = \frac{\partial \bar{H}}{\partial J} = -\frac{\cos(\Phi)}{4J^{1/2}(J + Q^{1/2})^{3/2}}. \tag{21}
\]

The phase plane for the oscillator is depicted in Fig. (2).

Note that the line \( \Phi = -\pi/2 \) is invariant under the dynamics, and along this line \( J \) evolves according to

\[
\frac{dJ}{dt} = \frac{J^{1/2}}{2(J + Q^{1/2})^{1/2}Q^{1/2}}. \tag{22}
\]

This solution corresponds to a resonant driving, where the variations in the scattering length reinforce the width oscillations. It is clear that for large \( t \) the action grows linearly, \( J \propto t/2Q^{1/2} \) and thus amplitude of oscillations grows like \( a \sim \sqrt{t} \). It is also clear from the phase portrait that all orbits for which \( \Phi \neq \pi/2 \) are asymptotic to the invariant manifold \( \Phi = -\pi/2 \), so for generic initial conditions one expects that the width will grow like \( a \sim \sqrt{t} \).

There is, of course, also the solution \( \Phi = \pi/2 \) in which the variations in the scattering length are anti-resonant with the variations in the width of the condensate, and act to *damp* the width oscillations. It is easy to see that when \( \Phi = \pi/2 \) the action \( J \) goes to zero in finite
time. Since generic initial conditions are asymptotic to \( \Phi = -\pi/2 \) the \( \Phi = \pi/2 \) solutions are unlikely to be observed experimentally, though it is possible that they could be realized with some kind of control.

Next resonance occurs at \( \Omega = 4 \). In this case the Hamiltonian is given by

\[
\tilde{H} = \frac{J \sin(2\Phi)}{(J + Q^{1/2})Q^{1/2}}. \tag{23}
\]

Again it is easy to see that the line \( \Phi = \pi/2 \) is invariant under the dynamics, corresponding to width oscillations which are in phase with the variation of the scattering length. Since the action evolves according to

\[
\frac{dJ}{dT} = -\frac{2J \cos(2\Phi)}{(J + Q^{1/2})Q^{1/2}} \tag{24}
\]

when \( \Phi = \pi/2 \) we have that the growth of the action \( J \) is asymptotically linear - \( J \approx 2t/Q^{1/2} \).

A. Numerical simulations

We have conducted some numerical simulations to test the validity of the calculations of the last section. We discretize the problem in the standard way, with time step \( \Delta t \) and spatial step \( h \), so the \( u^k_j \) approximates \( u(jh, k\Delta t) \). More specifically we approximate Eq.(2) with the following second order accurate semi-implicit Crank-Nicholson scheme,

\[
\frac{i(u^k_{j+1} - u^k_j)}{\Delta t} = -\frac{1}{2h^2}[(u^k_{j-1} - 2u^k_j + u^k_{j+1}) + (u^k_{j-1} - 2u^k_{j+1} + u^k_{j+2})] - \frac{1}{4r_jh}[(u^k_{j+1} + u^k_{j-1}) + (u^k_{j+1} - u^k_{j-1})] + \frac{1}{2}[r_j^2 - (1 + \epsilon(t))s|u^k_j|^2](u^k_j + u^k_{j+1}), \tag{25}
\]

where \( \epsilon(t) \) is the perturbation term. In the numerical simulations, as in the analysis, the perturbation was chosen to be \( \epsilon(t) = \epsilon \sin(\Omega t) \). Eq. (25) represents a tridiagonal set of equations for unknowns \( u^{k+1}_{j-1}, u^{k+1}_j \) and \( u^{k+1}_{j+1} \) \([j = 1, 2... (N-1)]\) in a lattice of \( N \) points, with the values of \( u^k_0 \) and \( u^k_{N+1} \) being determined from the boundary conditions \( \frac{\partial u}{\partial r} |_{r=0} = 0 \) and \( u(r) |_{r=\infty} \rightarrow 0 \).
The set of algebraic equations (25) is solved by the vectorial sweep method. In actual calculations the typical space step $h$ ranged from 0.01 to 0.005 and time step $\Delta t$ from 0.005 to 0.002 depending on the closeness of $\Omega$ to the points of resonance.

The first experiment, shown in Fig. 3, depicts the solution of the Gross-Pitaevski equation when the driving frequency is slightly off resonance: the natural frequency of the width oscillations is $\Omega_0 = 2$, while the perturbation has frequency $\Omega = 1.9$ and amplitude $\epsilon = 0.1$. In this plot, as well as all subsequent plots, the solid line represents the solution to the full PDE while the dotted line represents the solution to the reduced ODEs. We observe oscillations with beats, representing the superposition of low and high frequency oscillations. From simulations we seen the good agreement between the full PDE and the reduced ODE model.

Fig.4 represents numerical simulations of the ODE and PDE models at the resonance point $\Omega = \Omega_0 = 2$. The agreement is quite good for the period of the width oscillations, although there is clearly some discrepancy in the actual value of oscillation amplitude. Note that the same phenomenon has been observed in numerical simulations of the resonances in condensate oscillations under periodically varying trap potential [4]. The graph in Fig.4 depicts the energy versus time for the ODE (dotted line) and PDE (solid line) simulations for the same values of parameters as in Fig.4. The agreement between simulations of the full PDE and ODE is very good for time less than 30 or so, though for times between 30 and 40 the oscillations in the energy of the PDE are smaller than the analogous oscillations of the ODE, probably due to radiative damping. As was argued earlier the variational approach is unlikely to be valid when the oscillations have large amplitude and other effects, such as radiative damping, become important.

Fig.5 depicts the oscillations of the square width of the condensate at the 2 : 1 resonance $\Omega = 2\Omega_0$, where the frequency of the perturbation is twice the natural period of the width oscillation. As in previous case the frequencies agree very well, but the amplitude of oscillations is larger for PDE in comparison with ODE. This discrepancy grows with time. In this case it seems clear that the Gaussian anzatz does not correctly capture the behavior of
the underlying PDE. The rate of growth of the energy, which is not depicted, is still linear, in agreement with the theoretical estimates.

V. EVOLUTION OF BEC UNDER RANDOM FLUCTUATIONS

The action-angle formulation also provides a suitable framework for the analysis of condensate oscillations under random (white-noise) perturbations. One interesting quantity which can be calculated is the mean time to achieve a given distortion under a stochastic perturbation. From the ODE point of view this problem is equivalent to the problem of the mean time to achieve the given level of the oscillations amplitude in the effective potential.

The starting point for this calculation is Eqn(13,14), where \( \epsilon(t) \) is taken to be a white noise process \( \epsilon(t)dt = dB \), with \( dB \) the increment of a Brownian motion \( B \). This pair of stochastic ordinary differential equations has an associated Fokker-Planck equation for \( P(J,\theta,t) \), the probability of having values \( J, \theta \) at time \( t \). In the weak noise limit one can do a straightforward multiple scale expansion on this Fokker-Planck equation. Upon doing so and averaging over the fast angle variable one is lead to the following equation for \( P(J,t) \):

\[
\frac{\partial P}{\partial t} = \sigma^2 \frac{\partial}{\partial J} \left( A(J) \frac{\partial P}{\partial J} \right) \quad P(J,0) = \delta(J - J_0),
\]

where \( A(J) \) the average diffusivity for diffusion across energy levels is given by

\[
A(J) = \frac{1}{2\pi} \int_0^{2\pi} V_\theta^2 d\theta.
\]

For details of this calculation see the paper of Abdullaev, et. al [18] This one-dimensional diffusion can be analyzed in some detail. For instance the mean time to reach action \( J \) staring from \( J_0 \) is given by

\[
<t>_{J_0,J} = \int_{J_0}^J \frac{JdJ}{A(J)}.
\]

Note that if \( A(J) \) grows sufficiently rapidly, so that the above integral converges, then the mean time to random walk to \( J = \infty \) is actually finite.
A. Fluctuating trap potential

One physically interesting random perturbation is when the strength of the trap is allowed to fluctuate randomly. Since the trapping potential is imposed optically, by a laser, it is important to take into account the fluctuations of the effective trap potential due to the fluctuations of the laser field intensity. The fluctuations of the laser field intensity lead to random variations of the frequency of the effective harmonic trap \( \omega^2 = \omega_0^2(1 + \epsilon(t)) \), where \( \epsilon(t) \) is the white noise process [19]. The function \( \epsilon(t) \) should be thought of as the fluctuations of the laser intensity around its mean value \( E_0 \):

\[
\epsilon(t) = \frac{E(t) - E_0}{E_0}
\]

This problem was studied in [20] using a moment expansion, though without numerical simulations of the full stochastic GP equation. Here we consider this problem both as an illustration of effectiveness of our technique, and to provide numerics for this problem. In this case the perturbation Hamiltonian is

\[
V(a) = \frac{1}{2}(E + \sqrt{E^2 - Q \sin \theta}).
\]

Then from Eqn. (11) the diffusivity is given by

\[
A(J) = \frac{J(J + \sqrt{Q})}{2}.
\]

Calculating the mean exit time \( <t> \) we find the energy as a function of the mean exit time

\[
E = 2\sqrt{Q}(e^{\sigma^2 <t>/2} - \frac{1}{2}). \quad (28)
\]

While the mean exit time is not the same as the physical time in the limit of weak noise we expect that, to leading order in the noise parameter, the mean energy should have the same dependence on \( t \). In Fig. 7 we present the results of comparison of the theory and the numerical simulations of the full stochastic NLS equation (2) for \( \sigma = 0.04, N = 1.0 \). For the PDE we have averaged over 50 realizations, and for ODE we have averaged over
1000 realizations. The figure shows good agreement between predictions of the theory and numerical simulations. For large times the solution to the ODE overestimates the actual energy. This discrepancy is probably due to effects such as dispersive radiation, excitation of higher modes, etc. which are difficult to incorporate into the variational ansatz.

From the above we can easily estimate the mean exit time from the potential. In optical traps the depth of the effective potential is $U_0 = 2E_R = 2\hbar^2k^2/2m$, with $k = 2\pi/\lambda$, where $\lambda$ is the wavelength, The condensate lifetime can be estimated as the time when become $E \sim U_0$. Then

$$t_{BEC} \approx \frac{2}{\sigma^2} \ln\left(\frac{U_0}{2\sqrt{Q}} + \frac{1}{2}\right).$$

Substituting the typical values for the laser intensity fluctuations we find that the mean exit time is of the order of seconds, $t_{BEC} \sim \text{sec}$.

### B. Fluctuating atomic scattering length

In the case where the atomic scattering length is allowed to fluctuate we find that the perturbation Hamiltonian is given by

$$V_\theta = -\frac{Q\sqrt{E^2 - Q}\cos(\theta)}{2(E + \sqrt{E^2 - Q}\sin(\theta))^2}. $$

From this is follows that the the effective diffusivity is

$$A(J) = \frac{Q^2(E^2 - Q)}{8\pi} \int_0^{2\pi} \frac{\cos^2(\theta)}{(E + \sqrt{E^2 - Q}\sin(\theta))^4} d\theta = \frac{E(E^2 - Q)}{8\sqrt{Q}}, \quad (29)$$

and expressing the energy $E$ in terms of the action $J$ we find

$$A(J) = \frac{J(2J + \sqrt{Q})(J + \sqrt{Q})}{2\sqrt{Q}}. $$

Note that the expected time to random walk to infinity is actually finite, since $A(J) \propto J^3$. Substituting this expression for into Eq. $(27)$ we obtain for the time to pass from the bottom of the potential well where the action $J_0 = 0$ to the state with the action $J$

$$J = \frac{\sqrt{Q}(e^y - 1)}{2 - e^y}, \quad y = \frac{\sigma^2 t}{2}, \quad (30)$$
or in the terms of the total energy

\[ E = \frac{\sqrt{Q e^y}}{2 - e^y} \]  \hspace{1cm} (31)

From the above it is easy to see that the expected time for the width of the condensate to grow to infinity - the mean time for the condensate to break up - is given by

\[ t^* = \frac{2 \ln 2}{\sigma^2}. \]  \hspace{1cm} (32)

In Fig. 8 we compare the theoretical expression Eq.(31) with numerical simulations of the full stochastic GP equation with fluctuating scattering length in the case where \( \sigma = 0.04 \) and \( N = 1.0 \). The energy has been numerically calculated for the localized part of the condensate wavefunction. We observe good agreement between the theory and numerical simulations for \( t \leq 80 \).

VI. CONCLUSIONS

We have considered the oscillations of a 2D BEC with radial symmetry under periodic and random modulations of the atomic scattering length, as well as random fluctuations of trap potential. We have calculated the position of resonances and the energy growth using a reduced ODE for the condensate width. We have also confirmed the analytical predictions with numerical simulations of the 2D Gross-Pitaevskii equation. In the resonant case the frequency of oscillations agrees very well with the predictions of ODE, though the amplitude shows some discrepancy at large times that is probably due to the approximate nature of the variational approach. For the random modulations of the trap potential and the atomic scattering length we have calculated the mean exit time corresponding to the time for the amplitude of oscillations to exceed a given value and estimated the magnitude of the escape time for real experiments with 2D Bose-Einstein condensates.
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FIGURES

FIG. 1. Effective potential to describe evolution of the BEC width.

FIG. 2. Phase plane for the perturbation Hamiltonian at the resonance $\Omega = 2$.

FIG. 3. Oscillations of the condensate width for $\Delta \Omega/\Omega = 0.1$. The solid line is the variational approximation, the dotted line the numerical simulations of 2D Gross-Pitaevskii equation (I).

FIG. 4. The resonant oscillations when $\Omega = 2$.

FIG. 5. The growth of the energy in the resonant point.

FIG. 6. The oscillations of width at the second resonance $\Omega = 4$

FIG. 7. Fluctuating trap: the growth of the energy when $\sigma = 0.04$ and $N = 1.0$. Solid, dot and dash lines are for the solution of PDE, ODE and Fokker-Planck equation respectively.

FIG. 8. Fluctuating scattering length: the growth of the energy when $\sigma = 0.04$ and $N = 1.0$. Solid, dot and dash lines represent the solutions of PDE, ODE and Fokker-Planck equation respectively.
$Q > 0$  
Oscillatory motion

$Q < 0$  
Incidence upon the center
Energy, $E$ vs. $t$
