Monopoles At Finite Volume and Temperature in SU(2) Lattice Gauge Theory

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Abstract

We resolve a discrepancy between the $SU(2)$ spacial string tension at finite temperature, and the value obtained by monopoles in the maximum Abelian gauge. Previous work had incorrectly omitted a term due to Dirac sheets. When this term is included, the monopole and full $SU(2)$ determinations of the spacial string tension agree to within the statistical errors of the monopole calculation.
The string tension in pure $SU(2)$ lattice gauge theory has recently been calculated from monopoles, giving results that agree with the full $SU(2)$ string tension, to within the statistical errors of the monopole calculations \cite{1}. Given this success, it is disappointing that at finite temperature, monopoles apparently fail to explain the fundamental representation spacial string tension, $\sigma_s$. In the region of temperatures above the deconfining transition, the values for $\sigma_s$ obtained from monopoles are too small, by an amount well outside of statistical errors, and the discrepancy increases with increasing temperature \cite{2,3}.

In this paper we resolve this discrepancy. In the part of the calculation in which Wilson loops are calculated from monopoles, a term involving Dirac sheet variables had been dropped. When this term is included as it should be, the monopole and full $SU(2)$ determinations of the fundamental spacial string tensions agree to within statistical errors. The present calculations were carried out on lattices of size $16^3 \times N_t$ at $\beta = 2.5115$, for $N_t = 4, 6, 8, 12$. At this value of $\beta$, the deconfining transition corresponds to $N_t = 8$ \cite{4}, so $N_t = 4, 6$ are in the high-temperature phase. We compare to the full $SU(2)$ results for $\sigma_s$ obtained by Bali et al \cite{5} at this same $\beta$ value on $32^3 \times N_t$ lattices. Before presenting our results, we briefly review the way the monopole calculations are done, and discuss the correction due to Dirac sheets.

A summary of the steps \cite{1} involved in the monopole calculations is as follows: (1) $SU(2)$ configurations are projected into the maximum Abelian gauge. (2) The resulting $SU(2)$ links are factored into Abelian and charged parts. (3) The Abelian link angles are used to locate the magnetic current of the monopoles. (4) Monopole Wilson loops are calculated. (5) The string tension is extracted from fits to the monopole Wilson loops. Steps (3)-(5) proceed exactly as they would in $U(1)$ lattice gauge theory.

For $SU(2)$ lattice gauge theory, these steps involve strong assumptions. Briefly, these are that in the maximum Abelian gauge (MAG), long range physics is Abelian and controlled by monopoles. Each link of a configuration in the MAG is approximated by its
Abelian part, \( \exp(i\phi_\mu^3 \tau_3) \), where \( \tau_3 = \sigma_3/2 \) is the isospin generator, and \(-2\pi \leq \phi_\mu^3 \leq 2\pi\).

The key assumption for Wilson loops is

\[
< W_{SU(2)} > \sim < W_\phi > ,
\]

where the \( \sim \) sign means equivalent at long range. The Abelian Wilson loop \( W_\phi \) is given by

\[
W_\phi = \exp(i \sum_{x, \mu} \phi_\mu(x) J_\mu(x)). \tag{1}
\]

In Eq.(1), \( \phi_\mu \) is a rescaled link angle, \( \phi_\mu = \phi_\mu^3/2 \), while \( J_\mu \) is an integer-valued current describing the path traversed by the heavy quark. Motivated by results in \( U(1) \) lattice gauge theory [6], the Abelian Wilson loop \( W_\phi \) is further assumed to factor into a short range part involving the exchange of neutral gluons, times a term \( W_{\text{mon}} \) arising from monopoles. Thus for the confining part of the heavy quark potential

\[
< W_{SU(2)} > \sim < W_\phi > \sim < W_{\text{mon}} > \tag{2}
\]

is supposed to hold. In particular, \( W_{SU(2)} \), \( W_\phi \), and \( W_{\text{mon}} \) should all produce the full \( SU(2) \) string tension. Here we are concerned with the correct calculation of \( W_{\text{mon}} \), and therefore with the second equivalence of Eq.(2), \( < W_\phi > \sim < W_{\text{mon}} > \). The assumption of Abelian dominance and the use of the MAG are mainly concerned with the first equivalence in Eq.(2).

To calculate \( W_{\text{mon}} \), monopole variables must be located in a \( \phi_\mu \) configuration [7]. Abelian plaquettes \( \phi_{\mu\nu} \) are formed from the link angles \( \phi_\mu \), and expressed as \( \phi_{\mu\nu} = \phi'_{\mu\nu} + 2\pi m_{\mu\nu} \), where \( \phi'_{\mu\nu} \in [-\pi, \pi] \), and \( m_{\mu\nu} \) is an integer. The surface formed by the dual variables \( m^*_{\mu\nu} \) describes the Dirac sheets which are present. [7] Two currents can be

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1 Our conventions are that direct lattice link variables \( (j_\mu, A^e_\mu) \), originate from the labelling site, while dual lattice link variables \( (m_\mu, A^m_\mu) \) terminate at the the labelling site. Direct lattice plaquette variables \( (m_{\mu\nu}, F^e_{\mu\nu}, F^m_{\mu\nu}) \) have the site at the lower left corner of the plaquette, while dual lattice plaquette variables \( (m^*_\mu, F^e^*_\mu, F^m^*_\mu) \) have the site at the upper right corner. In going from the direct to the dual lattice, the discrete difference \( \partial_\mu^\pm \) is replaced by \( \partial_\mu^\mp \), and vice versa.
derived from $m_{\mu\nu}$, one the magnetic current of the monopoles, $m_\mu = \partial^+ m^\ast_{\mu\nu}$, the other the electric Dirac sheet current, $j_\mu = \partial^- m_{\mu\nu}$. Geometrically, $m_\mu$ flows on the edge of the open surface defined by $m^\ast_{\mu\nu}$. To obtain $W_{\text{mon}}$, we start from an expression which can be derived analytically in (Villain) $U(1)$ lattice gauge theory \([6]\),

$$W_{\text{mon}} = \exp(2\pi i \sum_{x,\mu} J_\mu(x) A^e_\mu(x)). \quad (3)$$

Here $A^e_\mu$ is an electric vector potential whose source is $j_\mu$. The subtlety with which the present paper is concerned arises when an attempt is made to express $W_{\text{mon}}$ in terms of the magnetic current $m_\mu$. This cannot be done using vector potentials, since the magnetic current produces a magnetic vector potential $A^m_\mu$, whereas the electric current of the quark can only couple to an electric vector potential, in this case $A^e_\mu$. Progress can be made by going to field strengths. We introduce a surface $D_{\mu\nu}$ of plaquettes whose boundary is the heavy quark current $J_\mu$, so that $J_\mu = \partial^- D_{\mu\nu}$. Then using the lattice form of Stoke’s theorem, $W_{\text{mon}}$ can be written as the exponential of a flux integral

$$W_{\text{mon}} = \exp\left(\frac{2\pi i}{2} \sum_{x,\mu,\nu} D_{\mu\nu}(x) F^e_{\mu\nu}(x)\right), \quad (4)$$

where the field strength is given by $F^e_{\mu\nu} \equiv \partial^+ A^e_\nu - \partial^- A^e_\mu$, and satisfies the electric Maxwell equation, $\partial^- F^e_{\mu\nu} = j_\mu$. Analogous to $A^e_\mu$, there is a magnetic vector potential $A^m_\mu$, whose source is $m_\mu$. The field strength given by $F^m_{\mu\nu} = \partial^- A^m_\nu - \partial^+ A^m_\mu$, satisfies the magnetic Maxwell equation, $\partial^+ F^m_{\mu\nu} = m_\mu$.

In a finite volume, it is not valid to replace $F^e_{\mu\nu}$ in the exponent of Eq.(4) by the dual of $F^m_{\mu\nu}$, defined as usual by $F^{\ast m}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^m_{\alpha\beta}$. For the case of periodic boundary condition in all directions, the correct equation relating $F^e_{\mu\nu}$ and $F^{\ast m}_{\mu\nu}$ is

$$F^e_{\mu\nu}(x) + F^{\ast m}_{\mu\nu}(x) = m_{\mu\nu}(x) - \bar{m}_{\mu\nu}, \quad (5)$$

where $\bar{m}_{\mu\nu}$ is the space-time average of $m_{\mu\nu}$,

$$\bar{m}_{\mu\nu} = \frac{1}{V} \sum_x m_{\mu\nu}(x). \quad (6)$$
It is straightforward to derive Eq.(5). Here we simply list various consistency checks. Applying $\partial_\nu$ to Eq.(5) gives an identity since $\partial_\nu F^{\ast}_{\mu\nu} = 0$, and $\partial_\nu F^e_{\mu\nu} = \partial_\nu m_{\mu\nu} = j_\mu$. Likewise, the equation obtained by applying $\partial^+_\nu$ to the dual of Eq.(5) is identically satisfied. The constant $-\bar{m}_{\mu\nu}$ on the right side of Eq.(5) is needed to make the space-time average of the right hand side vanish. The space-time average of the left hand side vanishes for periodic boundary conditions, since $F^e_{\mu\nu}$ and $F^{\ast}_{\mu\nu}$ are linear combinations of gradients in their respective vector potentials, so the sum over all $x$ of $F^e_{\mu\nu} + F^{\ast}_{\mu\nu}$ vanishes.

Using Eq.(5) to replace $F^e_{\mu\nu}$ in Eq.(4), we have

$$W_{\text{mon}} = \exp(-\frac{2\pi i}{2} \sum_{x,\mu,\nu} D_{\mu\nu}(x)(F^{\ast}_{\mu\nu}(x) + \bar{m}_{\mu\nu})).$$

(7)

The integer $m_{\mu\nu}$ term on the right hand side of Eq.(5) has no effect on $W_{\text{mon}}$. However, the presence of the non-integer term $\bar{m}_{\mu\nu}$ in Eq.(7) means that in addition to the magnetic current, the six numbers $\bar{m}_{\mu\nu}$ must be specified to obtain $W_{\text{mon}}$. Since Eq.(7) is equivalent to Eq.(4), and therefore to Eq.(5), the necessary property that $W_{\text{mon}}$ is unity for a plane-filling loop is now guaranteed. This is false if the $\bar{m}_{\mu\nu}$ term is omitted in the exponent of Eq.(7). While these three forms for $W_{\text{mon}}$ are completely equivalent, there are reasons for preferring the representation of Eq.(7). If monopoles are the cause of non-perturbative phenomena, it is desirable to express physical quantities in terms of $m_{\mu}$ to the greatest extent possible. Further, the magnetic current is sparse, occupying only a small percentage of the links of the lattice, and is essentially limited to the values $\pm 1$. The electric Dirac sheet current, $j_\mu$, on the other hand, is dense, occupying a large fraction of the links of the lattice, and is not dominated by the values $\pm 1$. Finally, $j_\mu$ varies under local deformations of Dirac sheets. The deformation $m_{\mu\nu} \to m_{\mu\nu} + \partial^+_{\mu} n_\nu - \partial^+_{\nu} n_\mu$, $n_\mu$ an integer, leaves $m_{\mu}$ and $\bar{m}_{\mu\nu}$ unchanged, while $j_\mu$ does change, $j_\mu \to j_\mu + \partial^+_{\mu}(\partial^- \cdot n) - (\partial^+ \cdot \partial^-) n_\mu$. Although $W_{\text{mon}}$ is invariant under local Dirac sheet deformations, there remains a dependence on the overall Dirac sheet topology. For example, a loop of magnetic current could have a Dirac sheet consisting of plaquettes with non-vanishing $m^*_{\mu\nu}$, which tile the “inner” area.
Table 1: Spacial string tensions from magnetic current, magnetic current + Dirac sheets, and for full \(SU(2)\)

| \(N_t\) | \(\sigma_s(m_\mu)\) | \(\sigma_s(m_\mu, \bar{m}_{\mu\nu})\) | \(\sigma_s(SU(2))\) |
|--------|---------------------|-------------------------------|-------------------|
| 4      | 0.049(3)            | 0.065(3)                      | 0.0643(6)         |
| 6      | 0.026(1)            | 0.040(4)                      | 0.0381(4)         |
| 8      | 0.022(2)            | 0.031(3)                      | 0.0325(7)         |
| 12     | 0.029(1)            | 0.034(1)                      | ————             |

Table 1: Spacial string tensions from magnetic current, magnetic current + Dirac sheets, and for full \(SU(2)\)

of the loop, or by virtue of the periodic boundary conditions, the “outer” area. The difference between the value of \(W_{mon}\) for these two cases comes entirely from \(\bar{m}_{\mu\nu}\).

We now turn to our results. The methods we used for generating \(SU(2)\) configurations and our gauge-fixing criterion are described in detail in [4]. After equilibration, we saved configurations every 20 lattice updates, resulting in a total of 500 configurations of magnetic current \(m_\mu\), and the six Dirac sheet space-time averages, \(\bar{m}_{\mu\nu}\) [8]. As we previously found at zero temperature, the magnetic current is sparse. The average fraction of links carrying magnetic current can be conveniently written as \(f \times 10^{-2}\). For our lattices of size \(16^3 \times N_t\) the values of \(f\) were 1.30(1), 1.07(1), 1.18(1), 1.25(1), for \(N_t = 4, 6, 8, 12\) respectively. The averages of \(\bar{m}_{\mu\nu}\) are statistically zero, so as a measure of the effect of \(\bar{m}_{\mu\nu}\) in Eq.(7), we give \(2\pi\) times the standard deviation of \(\bar{m}_{\mu\nu}\), over the three purely spacial planes of the lattice. Expressing this as \(h \times 10^{-3}\), we have \(h = 11(2), 8(2), 9(2), 5(1)\), for \(N_t = 4, 6, 8, 12\).

Monopole Wilson loops were calculated using Eq.(7). These loops are located in the purely spacial planes of the \(16^3 \times N_t\) lattice. One dimension of the loop can be regarded as a separation \(R\), while the other \(S\) is a pseudo-time, and by the same arguments as used
in the usual case of a symmetric lattice, \( W_{\text{mon}}(R, S) \) should approach \( \exp(-SV_{ps}(R)) \), as \( S \) becomes large, where \( V_{ps}(R) \) is the pseudo-potential. The values of \( V_{ps}(R) \) were determined by fitting \(-\ln(W_{\text{mon}})\) to a straight line in \( S \), over the interval \( S_{\text{min}} = R + 2 \) to \( S_{\text{max}} = 13 \). Then linear-plus-Coulomb fits were performed on \( V_{ps}(R) \) over the interval \( R = 2 \) to \( R = 7 \). The coefficient of the linear term in these fits gives our estimate of the spacial string tension due to monopoles. We denote the spacial string tension deduced from \( W_{\text{mon}} \) values obtained from Eq.(7) as \( \sigma_s(m_\mu, \bar{m}_\mu) \). For our old results where the \( \bar{m}_\mu \) term was omitted in Eq.(7), we use the symbol \( \sigma_s(m_\mu) \). In table I, we compare these two determinations of \( \sigma_s \) to the full \( SU(2) \) results of [3] on \( 32^3 \times N_t \) lattices. As can be seen by a glance at the table, the \( \sigma_s(m_\mu, \bar{m}_\mu) \) values agree well with the full \( SU(2) \) results, whereas the values obtained by omitting the contribution due to \( \bar{m}_\mu \) are clearly too small.

The values of the \( \bar{m}_\mu \) are of course highly correlated with the magnetic current in a given configuration. However, the \( \bar{m}_\mu \) cannot be constructed from knowledge of the magnetic current alone. A useful equation relating \( \bar{m}_\mu \) to \( F_{\mu \nu}^{m*} \) can be derived by summing Eq.(5) over the \( \mu - \nu \) plane. It is possible to show that \( \bar{m}_\mu \) can be recovered from \( F_{\mu \nu}^{m*} \) using this equation only if \( |\sum_s m_{\mu \nu}| < \frac{1}{2} A_{\alpha \beta} \), where \( A_{\alpha \beta} \) is the area in lattice units of the plane dual to the \( \mu - \nu \) plane. If this restriction is ignored and the resulting estimate of \( \bar{m}_\mu \) is used in Eq.(7), values of \( \sigma_s \) which are intermediate between those of columns 1 and 2 of Table I are obtained. We have also checked that using the \( \bar{m}_\mu \) term by itself to calculate \( W_{\text{mon}} \) does not produce an area law for Wilson loops, so the effect of \( \bar{m}_\mu \) is not a simple additive term in the string tension.

In summary, the apparent serious discrepancy between monopole and full \( SU(2) \) answers for the spacial string tension has been removed. The price paid is that a rather large contribution from Dirac sheets must be included. Dirac sheets are no longer “invisible” in a finite volume. It will clearly be of great interest to further explore this effect as a function of spacial volume and temperature, including zero temperature.
Part of this work was carried out when two of us (J. Stack and R. Wensley) were visitors at the University of Wales, Swansea. This work was supported in part by the National Science Foundation under Grant No. NSF PHY 94-12556 and Grant No. NSF PHY 94-03869, and the Higher Education Funding Council for Wales (HEFCW). The calculations were carried out on the Cray C90 system at the San Diego Supercomputer Center (SDSC), supported in part by the National Science Foundation.

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[8] We take one MAG gauge-fixed configuration for each original $SU(2)$ configuration. Based on recent work by A. Hart and M. Teper, (to be published) our conclusion that inclusion of Dirac sheets effects is essential is likely to survive in a more elaborate calculation involving several gauge copies for each configuration.