CONCENTRATION PHENOMENA AT SADDLE POINTS OF POTENTIAL FOR SCHRÖDINGER-POISSON SYSTEMS

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ABSTRACT. We consider in $\mathbb{R}^3$ the singularly perturbed Schrödinger-Poisson system

$$
\begin{align*}
-\varepsilon^2 \Delta v + V(x)v + \phi(x)v &= f(v) \\
-\Delta \phi &= v^2.
\end{align*}
$$

Using variational techniques, we construct solutions which concentrate around the saddle points of the external potential $V$, as $\varepsilon \to 0$.

1. Introduction. We are concerned with the existence of solutions for the following strong coupling of nonlinear Schrödinger equation with Poisson equation

$$
\begin{align*}
-\varepsilon^2 \Delta v + V(x)v + \phi(x)v &= f(v) \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= v^2 \quad \text{in } \mathbb{R}^3.
\end{align*}
$$

System $(SP)$ arises in the study of particles/field-functions $v$ in equilibrium with the electrostatic field $\phi$ which their own motion generates, as prescribed by Maxwell’s equation (and for this reason sometimes called Schrödinger-Maxwell systems) and once the coupling electrostatic field–particle agrees with the so-called minimal coupling rule. We refer mainly to the monograph by V. Benci and D. Fortunato [5], and references therein, for the Physics involved and related mathematical models. After their pioneering work in the ’90s, an impressive literature has been devoted to the subject in the last few years (apparently in the last two years more than one hundred papers have been published on this topic). From one side, this is an evidence of the importance of this research field within the scientific community, on the other side this prevents one the possibility of giving exhaustive references.

The singular perturbation introduced by the presence of the parameter $\varepsilon$, the adimensionalized Planck constant, has an important physical interest: the asymptotic behavior of solutions as $\varepsilon \to 0$, the so-called semiclassical states, can be seen

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as a threshold between quantum and classical mechanics. In this case the field function $v$ concentrates, exhibiting a spike shape and thus behaving as a classical particle. See also [1] for further interesting applications. Let us mention that in the existing literature sometimes the singular perturbation $\varepsilon^2$ has been considered also in the Poisson equation, see [6, 18, 19, 22, 25]. In the case of a sphere constraint for the field, then a natural singular perturbation of order $\varepsilon$ appears in the Poisson equation, see [16, 11].

The nonlinearity $f$, besides the mathematical interest, simulates the interaction with other particles. The external Schrödinger potential $V$ in the semiclassical limit procedure, plays the role of localizing the concentration points of the field function $v$ which usually turn out to be local minima, in presence of a standard potential well. However, critical points for the potential other than minima have this confining property but not all of them, as we know since the seminal paper [9].

Concentration phenomena for Schrödinger-Poisson systems have been investigated by many authors [2, 7, 12, 15, 21, 23, 24] essentially in the case in which the potential has a minimum point which attracts the concentrating field.

Here, inspired by [8] where the authors study the singular perturbation of the nonlinear Schrödinger equation, our assumptions on the potential $V : \mathbb{R}^3 \to \mathbb{R}$ allow for degenerate cases in which Lyapunov-Schmidt reduction methods cannot be used. Namely, we assume as in [8] without loss of generality that $0$ is a critical point of $V$ and $V(0) = 1$, that the potential is bounded, i.e.

$(V_0)$ there exist $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 \leq V(x) \leq \alpha_2$ for all $x \in \mathbb{R}^3$,

and that, in addition, one of the following conditions holds true:

$(V_1)$ $V \in C^1$ in a neighborhood of $0$ which is an isolated local maximum of $V$;

$(V_2)$ $V \in C^2$ in a neighborhood of $0$ which is a non-degenerate saddle point of $V$;

$(V_3)$ $V \in C^2$ in a neighborhood of $0$ which is an isolated critical point of $V$ and there exists a vector space $E$ such that $V|_E$ has a local maximum at $0$ and $V|_{E^\perp}$ has a local minimum at $0$.

Here the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

$(f_0)$ $f \in C^1([0, +\infty), [0, +\infty))$ and $f(t) = 0$ for $t \leq 0$;

$(f_1)$ $f(t) = o(t)$, as $t \to 0$;

$(f_2)$ $f(t) \leq C(1 + t^p)$ for some constant $C > 0$ and $p \in (1, 5)$, for all $t \geq 0$;

$(f_3)$ $0 < \mu F(t) \leq tf(t)$, for some $\mu > 4$, for all $t > 0$.

The main result of the paper is the following

**Theorem 1.1.** Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $(f_0) - (f_2)$ and $V : \mathbb{R}^3 \to \mathbb{R}$ satisfies $(V_0)$ and either $(V_1)$ or $(V_2)$ or $(V_3)$. Then, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the system $(SP)$ admits a positive solution $(u_\varepsilon, \phi_\varepsilon) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$. Moreover, there exists $\{y_\varepsilon\} \subset \mathbb{R}^3$ such that $\varepsilon y_\varepsilon \to 0$ and

$$u_\varepsilon(\varepsilon \cdot + y_\varepsilon) \to U \quad \text{in } H^1(\mathbb{R}^3), \quad \text{as } \varepsilon \to 0,$$

where $U$ is a ground state solution to

$$-\Delta u + u = f(u) \quad \text{in } \mathbb{R}^3.$$

**Overview.** In Section 2 we recall some known facts on the Schrödinger-Maxwell coupling and introduce the setting which consists of two main ingredients: reduction of the system to one single equation containing a nonlocal operator and a penalization argument. In Section 3 we prove the existence of solutions to an auxiliary penalized problem by means of a topological argument introduced in [8] and which exploits a crucial barycenter constraint in order to get energy estimates in the case of
non-minima critical points of the potential. With respect to the single Schrödinger equation studied in [8], here everything undergoes the solution map in the Poisson equation which turns the system into a nonlocal equation, in particular some new regularity issues show up in this different context. Some sharp asymptotic of the Lagrange multiplier in the barycenter constraint is also needed. Finally, in Section 4 we prove that actually solutions to the penalized problem converge to a solution of the original problem, as the parameter $\varepsilon \to 0$. Moreover, the solution exhibits a concentrating behavior having the precise asymptotic profile of the ground state solution to the semi-linear Schrödinger equation with constant potential, as in the case of the single equation. In this sense it turns out that the effect of the electrostatic gauge field disappears in the semiclassical limit procedure, which is somehow interesting from the mathematical as well as physical point of view.

Notation. In what follows for simplicity we will write $u$ in place of $u_\varepsilon$, dropping the dependence on $\varepsilon$ when this does not yield confusion. Moreover, we will make use of the following notations: on $H^1(\mathbb{R}^3)$ we will consider the standard norm
\[
\|u\|^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx
\]
and $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the Dirichlet norm
\[
\|u\|^2_{D^{1,2}} := \int_{\mathbb{R}^3} |\nabla u|^2 \, dx .
\]

The $L^p$-norm of a function $u \in L^p(\mathbb{R}^3)$ will be denoted by $\|u\|_p$, $p \in [1, +\infty]$. $B(x, R)$ denotes the open ball in $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$ and with radius $R > 0$. Given any set $\Omega \subset \mathbb{R}^3$ and $\varepsilon > 0$, let us set
\[
\Omega^\varepsilon := \{ x \in \mathbb{R}^3 : \varepsilon x \in \Omega \}.
\]
The symbols $c, C$ will denote positive constants whose exact value may change from line to line still being independent of other quantities involved.

2. Preliminaries. Observe first that system $(SP)$ turns out to be equivalent to a non-local single equation. Indeed, given $v \in H^1(\mathbb{R}^3)$, Lax-Milgram’s theorem implies that there exists a unique $\phi_v \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta \phi_v = v^2$ and the following representation formula holds:
\[
\phi_v(x) = \frac{1}{4\pi} \cdot \frac{1}{|x|} * v^2(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{v^2(y)}{|x - y|} \, dy .
\] (2.1)
Substituting (2.1) into $(SP)$ one has
\[
-\varepsilon^2 \Delta v + V(x)v + \phi_v(x)v = f(v), \quad v \in H^1(\mathbb{R}^3),
\] (2.2)
and setting $u_\varepsilon(x) := v(\varepsilon x)$ and $V_\varepsilon(x) := V(\varepsilon x)$, equation (2.2) turns into the following
\[
-\Delta u_\varepsilon + V_\varepsilon(x)u_\varepsilon + \varepsilon^2 \phi_{u_\varepsilon}(x)u_\varepsilon = f(u_\varepsilon), \quad u_\varepsilon \in H^1(\mathbb{R}^3) .
\] (2.3)
It is well known that weak solutions to (2.3) correspond to critical points of the functional $I_\varepsilon : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by
\[
I_\varepsilon(u) := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V_\varepsilon(x)u^2) \, dx + \frac{\varepsilon^2}{4} \int_{\mathbb{R}^3} \phi_{u_\varepsilon}u^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx ,
\] (2.4)
for all $u \in H^1(\mathbb{R}^3)$. 
In particular, some properties of the operator \( \Phi : H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3) \) defined by \( \Phi(u) := \phi_u \) for any \( u \in H^1(\mathbb{R}^3) \), are summarized below.

**Proposition 1.** The following properties hold true:

(i) \( \Phi \) is continuous and maps bounded sets into bounded sets;

(ii) \( \Phi(u) \geq 0, \| \nabla \Phi(u) \|_2 \leq c\| u \|^2 \) and \( T(u) \leq C\| u \|^4 \) for some \( c, C > 0 \), where

\[
T(u) := \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 \, dx, \quad u \in H^1(\mathbb{R}^3);
\]

(iii) if \( u_n \rightharpoonup u \) in \( H^1(\mathbb{R}^3) \), then \( \Phi(u_n) \rightharpoonup \Phi(u) \) in \( D^{1,2}(\mathbb{R}^3) \);

(iv) if \( u_n \rightharpoonup u \) in \( H^1(\mathbb{R}^3) \), then \( T(u_n) = T(u) + T(u_n - u) + o_n(1) \) and \( T'(u_n)v = T'(u)v + o_n(1) \) for all \( v \in H^1(\mathbb{R}^3) \).

Let us remark that, thanks to the maximum principle, by (\( f_0 \)) any nontrivial solution to (2.3) is positive.

Now, following [8] we set

\[
\tilde{f}(t) := \begin{cases} 
\min\{ f(t), at \} & \text{if } t \geq 0, \\
0 & \text{if } t < 0,
\end{cases}
\]

with

\[
0 < a < \left( 1 - \frac{2}{\mu} \right) \alpha_1,
\]

and \( B_i := B(0, R_i) \), where \( 0 < R_i < R_{i+1}, \ i = 0, 1, 2, 3, 4 \), will be chosen in what follows. Moreover, we choose \( R_1 \) such that

\[
\frac{\partial V(x)}{\partial \tau} \neq 0, \quad \text{for all } x \in \partial B_1 : V(x) = 1, \tag{2.7}
\]

where \( \tau \) is the unit tangent vector to \( \partial B_1 \) at \( x \). Next, define \( \chi : \mathbb{R}^3 \to \mathbb{R} \) by

\[
\chi(x) := \begin{cases} 
1, & x \in B_1, \\
\frac{R_2 - |x|}{R_2 - R_1}, & x \in B_2 \setminus B_1, \\
0, & x \in \mathbb{R}^3 \setminus B_2,
\end{cases}
\]

and

\[
g(x, t) := \chi(x) f(t) + (1 - \chi(x)) \tilde{f}(t),
\]

\[
G(x, t) := \int_0^t g(x, s) \, ds = \chi(x) F(t) + (1 - \chi(x)) \tilde{F}(t), \tag{2.9}
\]

for all \( (x, t) \in \mathbb{R}^3 \times \mathbb{R} \), where \( \tilde{F}(t) := \int_0^t \tilde{f}(s) \, ds \). In the following, we will denote by \( \chi_\varepsilon \) and \( g_\varepsilon \) the \( \varepsilon \)-scaling (with respect to the variable \( x \)) of \( \chi \) and \( g \), respectively, namely

\[
\chi_\varepsilon(x) := \chi(\varepsilon x), \quad g_\varepsilon(x, t) := g(\varepsilon x, t),
\]

and set \( G_\varepsilon(x, t) := \int_0^t g_\varepsilon(x, s) \, ds \).

We consider first the penalized problem

\[
- \Delta u + V_\varepsilon(x) u + \varepsilon^2 \phi_u(x) u = g_\varepsilon(x, u) \quad \text{in } \mathbb{R}^3, \tag{2.10}
\]

with corresponding energy given by

\[
\tilde{I}_\varepsilon(u) := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V_\varepsilon(x) u^2) \, dx + \frac{\varepsilon^2}{4} \int_{\mathbb{R}^3} \phi_u u^2 \, dx - \int_{\mathbb{R}^3} G_\varepsilon(x, u) \, dx \tag{2.11}
\]

for all \( u \in H^1(\mathbb{R}^3) \).
Proposition 2. For any \( \varepsilon > 0 \), the functional \( \tilde{I}_\varepsilon \) satisfies the \((PS)\)-condition.

Proof. Let \( \{u_n\} \subset H^1(\mathbb{R}^3) \) such that

\[
\tilde{I}_\varepsilon(u_n) \to c, \quad \tilde{I}_\varepsilon'(u_n) \to 0, \quad \text{as } n \to \infty.
\]

Thanks to \((f_3)\) and \((2.6)\), one has

\[
\mu \tilde{I}_\varepsilon(u_n) - \tilde{I}_\varepsilon'(u_n)(u_n)
= \left( \frac{\mu}{2} - 1 \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 + V_\varepsilon(x) u_n^2 \, dx + \varepsilon^2 \left( \frac{\mu}{4} - 1 \right) \int_{\mathbb{R}^3} \phi_n u_n^2 \, dx
\]

\[
- \int_{\mathbb{R}^3} \chi_\varepsilon(x) \left( \mu F(u_n) - f(u_n) u_n \right) \, dx - \int_{\mathbb{R}^3} (1 - \chi_\varepsilon(x)) \left( \mu \tilde{F}(u_n) - \tilde{f}(u_n) u_n \right) \, dx
\]

\[
\geq \left( \frac{\mu}{2} - 1 \right) \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx + \left( \frac{\mu - 2}{2} \alpha_1 - \frac{\mu}{2} \right) \int_{\mathbb{R}^3} u_n^2 \, dx \geq c \|u_n\|^2,
\]

and so \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \) and there exists \( u \in H^1(\mathbb{R}^3) \) such that, up to a subsequence, \( u_n \rightharpoonup u \). Let us prove that the convergence is actually strong by showing that, for any \( \delta > 0 \), there exists \( R > 0 \) such that

\[
\limsup_{n \to \infty} \|u_n\|_{H^1(\mathbb{R}^3 \setminus B(0,R))} < \delta.
\]

Let \( R > 0 \) such that \( B_2^+ \subset B(0,R/2) \) and let \( \psi_R \) be a cut-off function such that \( \psi_R = 0 \) in \( B(0,R/2) \), \( \psi_R = 1 \) in \( \mathbb{R}^3 \setminus B(0,R) \), \( 0 \leq \psi_R \leq 1 \) and \( |\nabla \psi_R| \leq C/R \). We have

\[
\tilde{I}_\varepsilon'(u_n)(\psi_R u_n) = \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla (\psi_R u_n) \, dx + \int_{\mathbb{R}^3} V_\varepsilon(x) u_n^2 \psi_R \, dx + \varepsilon^2 \int_{\mathbb{R}^3} \phi_n u_n^2 \psi_R \, dx
\]

\[
- \int_{\mathbb{R}^3} g_\varepsilon(x,u_n) u_n \psi_R \, dx = o_n(1).
\]

Hence,

\[
\int_{\mathbb{R}^3 \setminus B(0,R)} (|\nabla u_n|^2 + V_\varepsilon(x) u_n^2) \, dx
\]

\[
\leq \int_{\mathbb{R}^3} \psi_R (|\nabla u_n|^2 + V_\varepsilon(x) u_n^2) \, dx
\]

\[
\leq - \int_{\mathbb{R}^3} u_n \nabla u_n \cdot \nabla \psi_R \, dx - \varepsilon^2 \int_{\mathbb{R}^3} \phi_n u_n^2 \psi_R \, dx + \int_{\mathbb{R}^3} f(u_n) \psi_R u_n \, dx + o_n(1)
\]

\[
\leq \frac{C}{R} \|u_n\|_2 \|\nabla u_n\|_2 + a \int_{\mathbb{R}^3} u_n^2 \, dx + o_n(1),
\]

and

\[
\int_{\mathbb{R}^3 \setminus B(0,R)} (|\nabla u_n|^2 + (\alpha_1 - a) u_n^2) \, dx \leq \frac{C}{R} + o_n(1),
\]

which yields the conclusion. \( \square \)

The following limit problem for \((2.3)\)

\[
- \Delta u + ku = f(u), \quad k > 0,
\]

will play a crucial role in the sequel. The associated energy functional is given by

\[
L_k(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{k}{2} \int_{\mathbb{R}^3} u^2 \, dx - \int_{\mathbb{R}^3} F(u) \, dx, \quad u \in H^1(\mathbb{R}^3).
\]
Proposition 3. There exist $\Gamma_k := \{ \gamma \in C^0([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, L_k(\gamma(1)) < 0 \}$.

Moreover, for any $k > 0$ and $U \in S_k$, there exists $\gamma \in \Gamma_k$ such that $m = \max_{t \in [0,1]} L_k(\gamma(t))$, with $U \in \gamma([0,1])$. The map $(0, +\infty) \ni k \mapsto m_k \in (0, +\infty)$ is strictly increasing and continuous, see [8]. Let us set $L := L_1$, $m := m_1$ and $S := S_1$.

3. Existence results for the penalized problem (2.10). In the spirit of [8], define the topological cone

$$C_\varepsilon := \{ \gamma(t, -\varepsilon) : t \in [0,1], \varepsilon \in \mathcal{B}_0 \cap E \}$$

where, following the notation of the previous Section, the curve $\gamma_t = \gamma(t)$ is such that $m = \max_{t \in [0,1]} L(\gamma(t))$ and $\partial C_\varepsilon$ is the topological boundary of $C_\varepsilon$, and a family of deformations of $C_\varepsilon$ as follows

$$\Gamma_\varepsilon = \{ \eta : C_\varepsilon \rightarrow H^1(\mathbb{R}^3) : \eta \text{ is a homeomorphism and } \eta(u) = u \text{ for all } u \in \partial C_\varepsilon \} .$$

In this framework, consider the following min-max level

$$m_\varepsilon := \inf_{\eta \in \Gamma_\varepsilon} \max_{u \in C_\varepsilon} \tilde{I}_\varepsilon(\eta(u)).$$

Proposition 3. There exist $\varepsilon_0, \delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\tilde{I}_{\gamma(\cdot, -\varepsilon)} \leq m - \delta.$$

Proof. Set $\bar{U}(\cdot) := \gamma_1(\cdot, -\varepsilon)$, for some $\varepsilon \in B_0 \cap E$. One has

$$\tilde{I}_\varepsilon(\bar{U}) \leq \int_{B_1} \left( \frac{1}{2} |\nabla \bar{U}|^2 + \frac{1}{2} V_\varepsilon(x)\bar{U}^2 - F(\bar{U}) \right) dx + \int_{\mathbb{R}^3 \setminus B_1} \left( \frac{1}{2} |\nabla \bar{U}|^2 + \frac{1}{2} V_\varepsilon(x)\bar{U}^2 \right) dx$$

$$+ \frac{\varepsilon^2}{4} \int_{\mathbb{R}^3} \phi_0(x)\bar{U}^2 dx$$

$$\leq \int_{B_1} \left( \frac{1}{2} |\nabla \bar{U}|^2 + \frac{1}{2} V_\varepsilon(x)\bar{U}^2 - F(\bar{U}) \right) dx + \int_{\mathbb{R}^3 \setminus B_1} \left( \frac{1}{2} |\nabla \bar{U}|^2 + \frac{1}{2} V_\varepsilon(x)\bar{U}^2 \right) dx$$

$$+ \varepsilon^2 C \left( \int_{\mathbb{R}^3} (|\nabla \bar{U}|^2 + \bar{U}^2) \right)^2 .$$

Since

$$|D^\alpha \bar{U}(x)| \leq ce^{-\beta|x|},$$

for some $c, \beta > 0$ and for all $|\alpha| \leq 2$, $x \in \mathbb{R}^3$, we get

$$\tilde{I}_\varepsilon(\bar{U}) \leq L_v(\bar{U}) + o_v(1),$$

where $v := \max_{x \in \mathcal{B}_1} V(x)$, and then, possibly shrinking $B_1$,

$$\tilde{I}_\varepsilon(\gamma_1(\cdot, -\varepsilon)) < 0, \quad \forall \varepsilon \in B_0 \cap E .$$

Next, observe that $V(x) < 1 - \sigma$ for every $x \in \partial B_0 \cap E$ and for some $\sigma > 0$. Then

$$\tilde{I}_\varepsilon(\gamma_t(\cdot, -\varepsilon)) \leq \int_{B(0,1/\sqrt{\varepsilon})} \left( \frac{1}{2} |\nabla \gamma_t(x)|^2 + \frac{1}{2} V(\varepsilon(x + \varepsilon))\gamma_t^2(x) - F(\gamma_t(x)) \right) dx$$
+ \int_{R^3 \setminus B(0,1/\sqrt{\tau})} \left( \frac{1}{2} | \nabla \gamma_t(x) |^2 + \frac{1}{2} \alpha_2 \gamma_t^2(x) \right) dx \\
+ \varepsilon^2 C \left( \int_{R^3} (| \nabla \gamma_t(x) |^2 + \gamma_t^2(x)) dx \right)^2.

By the exponential decay of \( \gamma_t \) and the dominated convergence theorem we get

\[ \tilde{I}_\varepsilon(\gamma_t(\cdot - \xi)) \leq L_{1-\sigma}(\gamma_t) + o_\varepsilon(1). \]

The monotonicity of the map \( k \mapsto L_k \), \( k > 0 \), implies

\[ \max_{t \in [0,1]} L_{1-\sigma}(\gamma_t) < \max_{t \in [0,1]} L(\gamma_t) = m \]

and hence

\[ \tilde{I}_\varepsilon(\gamma_t(\cdot - \xi)) < m - \delta, \ \forall \xi \in \partial B^c_0 \cap E, \ t \in [0,1]. \quad (3.4) \]

Inequalities (3.3) and (3.4) yield the conclusion.

**Proposition 4.** One has

\[ \limsup_{\varepsilon \to 0} m_\varepsilon \leq m. \]

**Proof.** Clearly

\[ m_\varepsilon \leq \max_{u \in L_\varepsilon} \tilde{I}_\varepsilon(u). \]

Take a sequence \( \{ \varepsilon_n \} \) of positive numbers approaching zero and denote it simply by \( \{ \varepsilon \} \). Then, there exist \( \{ t_\varepsilon \} \) and \( \{ \xi_\varepsilon \} \), with \( \xi_\varepsilon \in \partial B^c_0 \cap E \), such that

\[ \max_{u \in L_\varepsilon} \tilde{I}_\varepsilon(u) = \tilde{I}_\varepsilon(\gamma_t(\cdot - \xi_\varepsilon)) \]

\[ \leq \int_{B(0,1/\sqrt{\tau})} \left( \frac{1}{2} | \nabla \gamma_{t_\varepsilon}(x) |^2 + \frac{1}{2} V(\varepsilon(x + \xi_\varepsilon)) \gamma_{t_\varepsilon}^2(x) - F(\gamma_{t_\varepsilon}(x)) \right) dx \]

\[ + \int_{R^3 \setminus B(0,1/\sqrt{\tau})} \left( \frac{1}{2} | \nabla \gamma_{t_\varepsilon}(x) |^2 + \frac{\alpha_2}{2} \gamma_{t_\varepsilon}^2(x) \right) dx \]

\[ + \varepsilon^2 C \left( \int_{R^3} (| \nabla \gamma_{t_\varepsilon}(x) |^2 + \gamma_{t_\varepsilon}^2(x)) dx \right)^2. \]

Passing to a further subsequence, there exist \( t_0 \in [0,1] \) and \( x_0 \in \overline{B}_0 \cap E \) such that \( t_\varepsilon \to t_0 \) and \( \varepsilon \xi_\varepsilon \to x_0 \). Hence

\[ \tilde{I}_\varepsilon(\gamma_{t_0}(\cdot - \xi_\varepsilon)) \to L_{V(x_0)}(\gamma_{t_0}). \]

The fact that \( V(x_0) \leq 1 \) and that the map \( k \mapsto L_k \) is increasing for \( k > 0 \), provide the desired conclusion.

Next we show that \( m_\varepsilon \) is a critical value for \( \tilde{I}_\varepsilon \). We will give first an estimate from below on \( m_\varepsilon \). We recall that, under our assumptions on \( V \), there exists a vector space \( E \) such that \( V|_E \) has a strict local max at 0, \( V|_{E^\perp} \) has a strict local min at 0. In particular: \( E = \mathbb{R}^3 \) in case \( (V_1) \), \( E \) is the space spanned by eigenvectors associated with negative eigenvalues of \( D^2 V(0) \) in case \( (V_2) \).

Let \( \pi_E \) be the orthogonal projection on \( E \), let \( h_\varepsilon : \mathbb{R}^3 \to E \) be defined by

\[ h_\varepsilon(x) := \pi_E(x) \chi_{B^c_0}(x), \]
\( \chi_{B_3^c} \) being the characteristic function of \( B_3^c \), and let \( \beta_\varepsilon : H^1(\mathbb{R}^3) \setminus \{0\} \to E \) be the barycenter-type map

\[
\beta_\varepsilon(u) := \frac{\int_{\mathbb{R}^3} h_\varepsilon(x) u^2 \, dx}{\int_{\mathbb{R}^3} u^2 \, dx}.
\]

For a fixed \( \delta > 0 \), sufficiently small, define a new min-max level by

\[
b_\varepsilon := \inf_{\Xi \in \Xi_\varepsilon} \max_{u \in \Sigma} \bar{I}_\varepsilon(u),
\]

where

\[
\Xi_\varepsilon := \left\{ \Sigma \subset H^1(\mathbb{R}^3) \setminus \{0\} : \exists u_0, u_1 \in \Sigma : \|u_0\| \leq \delta, \bar{I}_\varepsilon(u_1) < 0; \beta_\varepsilon(u) = 0, \forall u \in \Sigma \right\}.
\]

The following facts follow from [8], so we omit the proof.

**Proposition 5.** There exists \( \varepsilon_0 > 0 \) such that, for any \( \varepsilon \in (0, \varepsilon_0) \), \( m_\varepsilon \geq b_\varepsilon \).

**Proposition 6.** There exists \( \varepsilon_0 > 0 \) such that, for any \( \varepsilon \in (0, \varepsilon_0) \), there exists \( u_\varepsilon \in H^1(\mathbb{R}^3) \setminus \{0\} \) with \( \beta_\varepsilon(u_\varepsilon) = 0 \), and \( \lambda_\varepsilon \in E \) such that \( u_\varepsilon \) is a solution to

\[
- \Delta u + V_\varepsilon(x)u + \varepsilon^2 \phi_\varepsilon u = g_\varepsilon(x,u) + \lambda_\varepsilon \cdot h_\varepsilon(x)u
\]

and

\( \bar{I}_\varepsilon(u_\varepsilon) = b_\varepsilon \).

Moreover, the sequence \( \{u_\varepsilon\} \) is bounded in \( H^1(\mathbb{R}^3) \).

Now we provide a regularity result for the solution \( u_\varepsilon \) of Proposition 6.

**Proposition 7.** Let \( u_\varepsilon \) be a solution to (3.6). Then \( u_\varepsilon \in H^2_{\text{loc}}(\mathbb{R}^3) \).

**Proof.** Step 1. By Hardy’s inequality, for any \( x \in \mathbb{R}^3 \),

\[
4\pi \phi_\varepsilon(x) = \int_{\mathbb{R}^3} \frac{u_\varepsilon^2(y)}{|x-y|} \, dy \leq \int_{|x-y| > 1} u_\varepsilon^2(y) \, dy + \int_{|x-y| \leq 1} \frac{u_\varepsilon^2(y)}{|x-y|^2} \, dy
\]

\[
\leq \|u_\varepsilon\|_2^2 + \int_{\mathbb{R}^3} \frac{u_\varepsilon^2(x+y)}{|y|^2} \, dy \leq \|u_\varepsilon\|_2^2 + 4 \int_{\mathbb{R}^3} |\nabla u_\varepsilon(y)|^2 \, dy,
\]

so \( \phi_\varepsilon \in L^\infty(\mathbb{R}^3) \). Now, setting \( v = |u_\varepsilon| \), we claim that

\[
- \Delta v + V_\varepsilon(x)v + \phi_\varepsilon v \leq g_\varepsilon(x,v) + |\lambda_\varepsilon : h_\varepsilon(x)|v
\]

in the weak sense. Borrowing some ideas from [14], the function \( v_\delta := (u_\varepsilon^2 + \delta^2)^{1/2} - \delta \), \( \delta > 0 \), converges strongly to \( v \) in \( H^1(\mathbb{R}^3) \) as \( \delta \to 0 \). So, if \( \varphi \in C_0^\infty(\mathbb{R}^3) \), \( \varphi \geq 0 \), we have

\[
\int_{\mathbb{R}^3} \nabla v_\delta \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^3} \nabla \left( \frac{u_\varepsilon}{(u_\varepsilon^2 + \delta^2)^{1/2}} \right) \cdot \nabla \varphi \, dx
\]

\[
= \int_{\mathbb{R}^3} \nabla u_\varepsilon \cdot \nabla \left( \frac{u_\varepsilon}{(u_\varepsilon^2 + \delta^2)^{1/2}} \varphi \right) \, dx - \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \frac{\delta^2}{(u_\varepsilon^2 + \delta^2)^{3/2}} \varphi \, dx
\]

\[
\leq \int_{\mathbb{R}^3} \nabla u_\varepsilon \cdot \nabla \left( \frac{u_\varepsilon}{(u_\varepsilon^2 + \delta^2)^{1/2}} \varphi \right) \, dx
\]

\[
= - \int_{\mathbb{R}^3} V_\varepsilon(x) \frac{u_\varepsilon^2}{(u_\varepsilon^2 + \delta^2)^{1/2}} \varphi \, dx - \int_{\mathbb{R}^3} \phi_\varepsilon \frac{u_\varepsilon}{(u_\varepsilon^2 + \delta^2)^{1/2}} \varphi \, dx
\]

\[
+ \int_{\mathbb{R}^3} g_\varepsilon(x,u_\varepsilon) \frac{u_\varepsilon}{(u_\varepsilon^2 + \delta^2)^{1/2}} \varphi \, dx + \int_{\mathbb{R}^3} |\lambda_\varepsilon : h_\varepsilon(x)| \frac{u_\varepsilon^2}{(u_\varepsilon^2 + \delta^2)^{3/2}} \varphi \, dx.
\]
\[
\leq - \int_{\mathbb{R}^3} V_{\varepsilon}(x) \frac{u_2^2}{(u_2^2 + \delta^2)\varepsilon} \varphi dx - \int_{\mathbb{R}^3} \phi_{u_2} \frac{u_2^2}{(u_2^2 + \delta^2)\varepsilon} \varphi dx \\
+ \int_{\mathbb{R}^3} g_{\varepsilon}(x,v) \frac{v}{(u_2^2 + \delta^2)\varepsilon} \varphi dx + \int_{\mathbb{R}^3} |\lambda_{\varepsilon} \cdot h_{\varepsilon}(x)| \frac{u_2^2}{(u_2^2 + \delta^2)\varepsilon} \varphi dx.
\]

Taking the limit as \(\delta \to 0\), by dominated convergence we obtain (3.7) for any non-negative \(\varphi \in C_0^\infty(\mathbb{R}^3)\) and, by density, for any non-negative \(\varphi \in H^1(\mathbb{R}^3)\).

**Step 2.** We next use the Moser iteration technique to show that \(v \in L^\infty(\mathbb{R}^3)\), see [10, 17, 4]. Obviously, for any fixed \(\varepsilon > 0\), \(|\lambda_{\varepsilon} \cdot h_{\varepsilon}(\cdot)| \in L^\infty(\mathbb{R}^3)\). Since \(v, \phi_v \geq 0\) in \(\mathbb{R}^3\), from (\(f_1\)) and (\(f_2\)) there exists \(C > 0\) such that

\[-\Delta v \leq C v + v^5 \quad \text{in} \; \mathbb{R}^3,
\]

in the weak sense, that is, for any non-negative function \(\varphi \in H^1(\mathbb{R}^3)\), there holds

\[
\int_{\mathbb{R}^3} \nabla v \cdot \nabla \varphi dx \leq \int_{\mathbb{R}^3} (C v + v^5) \varphi dx.
\]

If \(q > 0\), for any \(k \in \mathbb{N}\) consider the sets \(A_k := \{x \in \mathbb{R}^3 : v \leq k\}\) and \(B_k := \mathbb{R}^3 \setminus A_k\) and define \(v_k : \mathbb{R}^3 \to \mathbb{R}\) by

\[
v_k := \begin{cases} 
  v^{2q+1} & \text{in} \; A_k, \\
  k^q v & \text{in} \; B_k.
\end{cases}
\]

Thus, \(v_k \in H^1(\mathbb{R}^3), 0 \leq v_k \leq v^{2q+1}\), and

\[
\nabla v_k = \begin{cases} 
  (2q+1)v^{2q}\nabla v & \text{in} \; A_k, \\
  k^q \nabla v & \text{in} \; B_k.
\end{cases}
\]

Now, using \(v_k\) as a test function in (3.8), we have

\[
\int_{\mathbb{R}^3} \nabla v \cdot \nabla v_k dx \leq \int_{\mathbb{R}^3} (C v + v^5)v_k dx,
\]

that is

\[
(2q + 1) \int_{A_k} v^{2q} |\nabla v|^2 dx + k^q \int_{B_k} |\nabla v|^2 dx \leq \int_{\mathbb{R}^3} (C v + v^5)v_k dx. \tag{3.9}
\]

Setting

\[
w_k := \begin{cases} 
  v^{q+1} & \text{in} \; A_k, \\
  k^q v & \text{in} \; B_k,
\end{cases}
\]

one has \(w^2_k = v v_k \leq v^{2q+2}\) and

\[
\nabla w_k = \begin{cases} 
  (q+1)v^q \nabla v & \text{in} \; A_k, \\
  k^q \nabla v & \text{in} \; B_k,
\end{cases}
\]

and therefore

\[
\int_{\mathbb{R}^3} |\nabla w_k|^2 dx = (q + 1)^2 \int_{A_k} v^{2q} |\nabla v|^2 dx + k^q \int_{B_k} |\nabla v|^2 dx. \tag{3.10}
\]

Combining (3.9) and (3.10), we obtain

\[
\frac{2q + 1}{(q + 1)^2} \int_{\mathbb{R}^3} |\nabla w_k|^2 dx \leq \int_{\mathbb{R}^3} (C + v^4)w^2_k dx. \tag{3.11}
\]
From [3, Lemma 2.1] it follows that, for every $\varepsilon > 0$, there exists $\sigma(\varepsilon, v) > 0$, such that
\[
\int_{\mathbb{R}^3} v^4 w_k^2 dx \leq \varepsilon \int_{\mathbb{R}^3} |\nabla w_k|^2 dx + \sigma(\varepsilon, v) \int_{\mathbb{R}^3} w_k^2 dx,
\]
and hence, choosing $\varepsilon = \frac{2^{q+1}}{2(q+1)^2}$, we get from (3.11)
\[
\int_{\mathbb{R}^3} |\nabla w_k|^2 dx \leq \tilde{C}_q \int_{\mathbb{R}^3} w_k^2 dx, \tag{3.12}
\]
where $\tilde{C}_q = \frac{2(q+1)^2}{2} (C_1 + \sigma(\varepsilon, v))$. If $v \in L^{2(q+1)}(\mathbb{R}^3)$ for some $q \geq 2$, by Sobolev's embedding theorem,
\[
\left( \int_{A_k} v^6 dx \right)^{\frac{1}{6}} \leq S \tilde{C}_q \int_{\mathbb{R}^3} w_k^2 dx,
\]
where $S$ is the best constant of the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$. Therefore,
\[
\left( \int_{A_k} v^{6(q+1)} dx \right)^{\frac{1}{6}} \leq S \tilde{C}_q \int_{\mathbb{R}^3} v^{2(q+1)} dx,
\]
and letting $k \to \infty$, we have
\[
\left( \int_{\mathbb{R}^3} v^{6(q+1)} dx \right)^{\frac{1}{6}} \leq S \tilde{C}_q \int_{\mathbb{R}^3} v^{2(q+1)} dx. \tag{3.13}
\]
That is, for any $q \geq 2$ such that $v \in L^{2(q+1)}(\mathbb{R}^3)$, there exist $C_q > 0$ such that $v \in L^{6(q+1)}(\mathbb{R}^3)$ and
\[
\|v\|_{6(q+1)} \leq C_q \|v\|_{2(q+1)}. \tag{3.14}
\]
Now we carry on an iteration argument. Taking $q_1 = 2$ and noting that $v \in L^6(\mathbb{R}^3)$, by (3.14), there exist $C_1 > 0$, such that $v \in L^{6(q_1+1)}(\mathbb{R}^3)$ and
\[
\|v\|_{6(q_1+1)} \leq C_1 \|v\|_{2(q_1+1)}.
\]
Choosing $q_2$ satisfying $2(q_2+1) = 6(q_1+1)$, we see that $q_2 > q_1$ and $v \in L^{2(q_2+1)}(\mathbb{R}^3)$. Thus, by (3.14), there exists $C_2 > 0$ such that $v \in L^{6(q_2+1)}(\mathbb{R}^3)$ and
\[
\|v\|_{6(q_2+1)} \leq C_2 \|v\|_{2(q_2+1)}.
\]
Continuing with this iteration, we obtain a sequence $\{C_k\} \subset \mathbb{R}^+$ and an increasing sequence $\{q_k\} \subset [2, +\infty)$, with $2(q_k+1) = 6(q_k+1)$, such that $v \in L^{6(q_k+1)}(\mathbb{R}^3)$ and
\[
\|v\|_{6(q_k+1)} \leq C_k \|v\|_{2(q_k+1)}.
\]
Obviously, $q_k = 3^k - 1$. It follows that, for any $q \geq 2$, there exists $C_q > 0$ such that
\[
\|v\|_q \leq C_q \|v\|_6. \tag{3.15}
\]
By virtue of [10, Theorem 8.17], for any $y \in \mathbb{R}^3$,
\[
\sup_{B(y, 1)} v \leq C \left( \|v\|_{L^2(B(y, 2))} + \|v\|_{L^3(B(y, 2))} + \|v\|_{L^5(B(y, 2))} \right),
\]
where $C > 0$ is independent of $y$. Thus, $\|v\|_{\infty} < \infty$.

Step 3. Letting $\tilde{V}(x) := V_\varepsilon(x) + \varphi_\varepsilon(x)$ and $\tilde{h}(x) := g_\varepsilon(x, u_\varepsilon) + \lambda_\varepsilon \cdot h_\varepsilon(x) u_\varepsilon$, one has
\[
-\Delta u_\varepsilon + \tilde{V}(x) u_\varepsilon = \tilde{h}(x), \quad x \in \mathbb{R}^3.
\]
Since $u_\varepsilon \in L^\infty(\mathbb{R}^3)$, then $\tilde{h} \in L^2_{\text{loc}}(\mathbb{R}^3)$. Moreover, by $\tilde{V} \in L^\infty(\mathbb{R}^3)$, it follows from the interior $H^2$-regularity that $u_\varepsilon \in H^2_{\text{loc}}(\mathbb{R}^3)$.
We next investigate the asymptotic behavior, as \( \varepsilon \to 0 \), of some norms of \( u_\varepsilon \) and of the Lagrange multiplier \( \lambda_\varepsilon \) of Proposition 6.

**Proposition 8.** As \( \varepsilon \to 0 \) one has:

(i) \( \|u_\varepsilon \chi_{B_2^\varepsilon}\|_2 \not\to 0 \);

(ii) \( \|u_\varepsilon\|_{H^1(\mathbb{R}^3 \setminus B_2^\varepsilon)} \to 0 \);

(iii) \( \lambda_\varepsilon = O(\varepsilon) \).

**Proof.** (i) Due to \((f_1)\) and \((f_2)\), fixed \( \delta > 0 \), there exists \( c_\delta > 0 \) such that

\[
|f(t)| \leq \delta |t| + c_\delta |t|^p, \quad \forall t \in \mathbb{R}.
\]

Recalling that \( u_\varepsilon \) weakly solves \((3.6)\) and that \( \beta(u_\varepsilon) = 0 \), multiplying \((3.6)\) by \( u_\varepsilon \) and using the above inequality for \( \delta > 0 \) small enough, we get

\[
\int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 + \alpha_1 u_\varepsilon^2) dx \leq \int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 + V_\varepsilon(x)u_\varepsilon^2) dx + \varepsilon^2 \int_{\mathbb{R}^3} \phi_\varepsilon u_\varepsilon^2 dx
\]

\[
\leq \int_{\mathbb{R}^3} (a + \delta) u_\varepsilon^2 dx + c_\delta \int_{B_2^\varepsilon} u_\varepsilon^{p+1} dx.
\]

From this we deduce that

\[
\int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 + (\alpha_1 - a - \delta) u_\varepsilon^2) dx \leq c_\delta \int_{B_2^\varepsilon} u_\varepsilon^{p+1} dx
\]

and hence

\[
\|u_\varepsilon\|_{L^{p+1}(B_2^\varepsilon)}^2 \leq c \|u_\varepsilon\|_2^2 \leq c \|u_\varepsilon\|_{L^{p+1}(B_2^\varepsilon)}^{p+1}.
\]

The previous inequalities yield

\[
 u_\varepsilon \chi_{B_2^\varepsilon} \not\to 0 \quad \text{in } L^{p+1}(\mathbb{R}^3) \quad \text{as } \varepsilon \to 0.
\]

Since \( \{u_\varepsilon\} \) is bounded in \( H^1(\mathbb{R}^3) \), it will be bounded in \( L^s(\mathbb{R}^3) \) as well, for some \( s > p + 1 \). Then, by interpolation, for some \( \vartheta \in (0,1) \),

\[
0 < c_1 \leq \|u_\varepsilon\|_{L^{p+1}(B_2^\varepsilon)} \leq \|u_\varepsilon\|_{L^s(\mathbb{R}^3)}^{\vartheta} \|u_\varepsilon\|_{L^2(B_2^\varepsilon)}^{1-\vartheta} \leq c_2 \|u_\varepsilon\|_{L^2(B_2^\varepsilon)}^{\vartheta},
\]

and thus the conclusion follows.

(ii) Let \( \psi_\varepsilon : \mathbb{R}^3 \to \mathbb{R} \) a smooth function satisfying

\[
\psi_\varepsilon(x) = \begin{cases} 
0 & \text{in } B_2^\varepsilon, \\
1 & \text{in } \mathbb{R}^3 \setminus B_4^\varepsilon,
\end{cases}
\]

as well as \( \psi_\varepsilon(x) \in [0,1] \) and \( |\nabla \psi_\varepsilon(x)| \leq c \varepsilon \) for any \( x \in \mathbb{R}^3 \). Being \( u_\varepsilon \) a weak solution to \((3.6)\) and \( \psi_\varepsilon h_\varepsilon \equiv 0 \), we deduce that

\[
\int_{\mathbb{R}^3} \nabla u_\varepsilon \nabla (\psi_\varepsilon u_\varepsilon) dx + \int_{\mathbb{R}^3} V_\varepsilon(x) u_\varepsilon^2 \psi_\varepsilon dx + \varepsilon^2 \int_{\mathbb{R}^3} \phi_\varepsilon u_\varepsilon^2 dx = \int_{\mathbb{R}^3} g_\varepsilon(x,u_\varepsilon) u_\varepsilon \psi_\varepsilon dx,
\]

and hence

\[
\int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 + V_\varepsilon(x)u_\varepsilon^2) \psi_\varepsilon dx + \int_{\mathbb{R}^3} u_\varepsilon \nabla u_\varepsilon \nabla \psi_\varepsilon dx \leq a \int_{\mathbb{R}^3} u_\varepsilon^2 \psi_\varepsilon dx.
\]

One has

\[
\int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 + (V_\varepsilon(x) - a)u_\varepsilon^2) \psi_\varepsilon dx \geq \int_{\mathbb{R}^3 \setminus B_4^\varepsilon} (|\nabla u_\varepsilon|^2 + (\alpha_1 - a)u_\varepsilon^2) dx
\]

and

\[
\int_{\mathbb{R}^3} u_\varepsilon \nabla u_\varepsilon \nabla \psi_\varepsilon dx \leq c \varepsilon \|u_\varepsilon\|_2 \|\nabla u_\varepsilon\|_2 \leq C \varepsilon,
\]
from which the fulfillment of the second statement is true.

(iii) If $\lambda_\varepsilon = 0$ the conclusion is of course true, so assume $\lambda_\varepsilon \neq 0$ and set $\hat{\lambda}_\varepsilon = \lambda_\varepsilon / |\lambda_\varepsilon|$. Let $\psi_\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ a smooth function satisfying

$$\psi_\varepsilon(x) = \begin{cases} 1 & \text{in } B^\varepsilon_2, \\ 0 & \text{in } \mathbb{R}^3 \setminus B^\varepsilon_3, \end{cases}$$

as well as $\psi_\varepsilon(x) \in [0,1]$ and $|\nabla \psi_\varepsilon(x)| \leq c\varepsilon$ for any $x \in \mathbb{R}^3$. Since $u_\varepsilon \in H^2_{loc}(\mathbb{R}^3)$, we are allowed to multiply (3.6) by $\psi_\varepsilon \partial_{\hat{\lambda}_\varepsilon} u_\varepsilon$ and integrate by parts:

$$\int_{B^\varepsilon_3} \nabla u_\varepsilon \nabla (\psi_\varepsilon \partial_{\hat{\lambda}_\varepsilon} u_\varepsilon) dx + \int_{B^\varepsilon_3} V_\varepsilon(x) u_\varepsilon \psi_\varepsilon \partial_{\hat{\lambda}_\varepsilon} u_\varepsilon dx + \varepsilon^2 \int_{B^\varepsilon_3} \phi_\varepsilon u_\varepsilon \psi_\varepsilon \partial_{\hat{\lambda}_\varepsilon} u_\varepsilon dx$$

$$- \int_{B^\varepsilon_3} g_\varepsilon(x, u_\varepsilon) u_\varepsilon \psi_\varepsilon \partial_{\hat{\lambda}_\varepsilon} u_\varepsilon dx \leq \int_{B^\varepsilon_3} (\lambda_\varepsilon \cdot h_\varepsilon) u_\varepsilon \psi_\varepsilon \partial_{\hat{\lambda}_\varepsilon} u_\varepsilon dx = 0. \quad (3.16)$$

By the same reasoning as [8], we know that

$$\int_{B^\varepsilon_3} \nabla u_\varepsilon \nabla (\psi_\varepsilon \partial_{\hat{\lambda}_\varepsilon} u_\varepsilon) dx = \int_{B^\varepsilon_3} V_\varepsilon(x) u_\varepsilon \psi_\varepsilon \partial_{\hat{\lambda}_\varepsilon} u_\varepsilon dx = \int_{B^\varepsilon_3} g_\varepsilon(x, u_\varepsilon) u_\varepsilon \psi_\varepsilon \partial_{\hat{\lambda}_\varepsilon} u_\varepsilon dx = O(\varepsilon)$$

as $\varepsilon \to 0$, while

$$\int_{B^\varepsilon_3} (\lambda_\varepsilon \cdot h_\varepsilon) u_\varepsilon \psi_\varepsilon \partial_{\hat{\lambda}_\varepsilon} u_\varepsilon dx = -\frac{1}{2}|\lambda_\varepsilon| \int_{B^\varepsilon_3} u^2_\varepsilon \psi_\varepsilon dx + O(\varepsilon).$$

Let us estimate the third term of (3.16). First of all, let us observe that

$$\int_{\mathbb{R}^3} \partial_{\hat{\lambda}_\varepsilon} (\phi_\varepsilon u^2_\varepsilon \psi_\varepsilon) dx = 0. \quad (3.17)$$

Indeed, fixed $\varepsilon > 0$ and $R > 0$ large enough, choose $\varphi_R \in C_0^\infty(\mathbb{R}^3, [0,1])$ satisfying

$$\varphi_R(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{R}{\varepsilon}, \\ 0 & \text{if } |x| \geq R, \end{cases}$$

and $|\nabla \varphi_R(x)| \leq \frac{2}{R}$ for any $x \in \mathbb{R}^3$. Being $u_\varepsilon \in H^2_{loc}(\mathbb{R}^3)$ and $\psi_\varepsilon \in C_0^\infty(\mathbb{R}^3)$, one has, in the weak sense,

$$\partial_{\hat{\lambda}_\varepsilon} (\phi_\varepsilon u^2_\varepsilon \psi_\varepsilon) = \nabla (\phi_\varepsilon u^2_\varepsilon \psi_\varepsilon) \cdot \hat{\lambda}_\varepsilon$$

So, by the definition of $\psi_\varepsilon$, we deduce

$$\int_{\mathbb{R}^3} \frac{\partial}{\partial x_i} (\phi_\varepsilon u^2_\varepsilon \psi_\varepsilon) dx$$

$$= \int_{B^\varepsilon_3} \frac{\partial}{\partial x_i} (\phi_\varepsilon u^2_\varepsilon \psi_\varepsilon) \varphi_R dx = \int_{\mathbb{R}^3} \frac{\partial}{\partial x_i} (\phi_\varepsilon u^2_\varepsilon \psi_\varepsilon) \varphi_R dx = -\int_{\mathbb{R}^3} \phi_\varepsilon u^2_\varepsilon \psi_\varepsilon \frac{\partial \varphi_R}{\partial x_i},$$

and hence

$$\left| \int_{\mathbb{R}^3} \frac{\partial}{\partial x_i} (\phi_\varepsilon u^2_\varepsilon \psi_\varepsilon) dx \right| \leq \frac{2}{R} \int_{\mathbb{R}^3} \phi_\varepsilon u^2_\varepsilon \psi_\varepsilon dx.$$

Taking the limit as $R \to +\infty$, we easily derive (3.17). Thus we have

$$0 = \int_{\mathbb{R}^3} \partial_{\hat{\lambda}_\varepsilon} (\phi_\varepsilon u^2_\varepsilon \psi_\varepsilon) dx$$

$$= \int_{\mathbb{R}^3} (\partial_{\hat{\lambda}_\varepsilon} \phi_\varepsilon) u^2_\varepsilon \psi_\varepsilon dx + 2 \int_{\mathbb{R}^3} (\partial_{\hat{\lambda}_\varepsilon} u_\varepsilon \phi_\varepsilon) u_\varepsilon \psi_\varepsilon dx + \int_{\mathbb{R}^3} (\partial_{\hat{\lambda}_\varepsilon} \psi_\varepsilon) \phi_\varepsilon u^2_\varepsilon dx,$$
Moreover
\(2.10\):
\[
\lim \inf \text{Proposition 9.}
\]
and following the argument of \([8]\), up to extracting a subsequence, there exist
\[
\text{Proof. In the light of Proposition 8 - (iii), consider a sequence } \varepsilon_n \to 0 \text{ such that}
\]
\[
\frac{\lambda_1}{\varepsilon_n} \to \tilde{\lambda} \in E.
\]
For brevity, such a sequence will be denoted simply by \(\varepsilon\). Now, setting
\[
H_\varepsilon = \left\{ x \in \mathbb{R}^3 : \tilde{\lambda} \cdot x \leq \frac{\alpha_1}{2\varepsilon} \right\},
\]
and following the argument of \([8]\), up to extracting a subsequence, there exist \(n \in \mathbb{N}\), \(c > 0\) and, for any \(i = 1, \ldots, n\), \(y_i^\varepsilon \in H_\varepsilon \cap H_{\varepsilon_i}\), \(y_i \in B_2\) and \(u_i \in H^1(\mathbb{R}^3) \setminus \{0\}\) satisfying
\[
-\Delta u_i + V(y_i)u_i = g(y_i, u_i) + \tilde{\lambda} \cdot y_i u_i
\]
and such that, as \(\varepsilon \to 0\):
\[
\begin{align*}
&\varepsilon y_i^\varepsilon \to \tilde{y}_i; \\
&|y_i^\varepsilon - y_j^\varepsilon| \to +\infty \text{ if } i \neq j; \\
&u_i(\cdot + y_i^\varepsilon) \rightharpoonup u_i \text{ in } H^1(\mathbb{R}^3); \\
&\|u_i\| \geq c; \\
&\|u_i - \sum_{i=1}^n u_i(\cdot - y_i^\varepsilon)\|_{H^1(H_\varepsilon)} \to 0.
\end{align*}
\]
As a consequence we get
\[
b_\varepsilon = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u_\varepsilon|^2 + V_\varepsilon(x)u_\varepsilon^2)dx + \frac{\varepsilon^2}{4} \int_{\mathbb{R}^3} \phi_{u_\varepsilon}u_\varepsilon^2dx - \int_{\mathbb{R}^3} G_\varepsilon(x, u_\varepsilon)dx
\geq \sum_{i=1}^n \left( \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u_i|^2 + V(y_i)u_i^2)dx - \int_{\mathbb{R}^3} G(y_i, u_i)dx \right) + o_\varepsilon(1)
\geq \sum_{i=1}^n L_{V(y_i)}(u_i) + o_\varepsilon(1) \geq m + o_\varepsilon(1).
\]
Collecting the results of this Section we obtain the existence result for problem (2.10):
Finally we prove that the solution admits a positive solution $u_\varepsilon$ to solve the original problem (2.3), as $\varepsilon \to 0$. By Proposition 3, this implies the existence of some $\varepsilon_0 > 0$ such that $m_\varepsilon > \max_{\partial Q_\varepsilon} I_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0)$. Since by Proposition 2 $I_\varepsilon$ enjoys the $(PS)$-condition, thanks to a standard minimax argument, see [20], $I_\varepsilon$ admits a critical point $u_\varepsilon$ with critical value $m_\varepsilon$.

4. Existence and concentration results for problem (2.3): proof of Theorem 1.1. Finally we prove that the solution $u_\varepsilon$ obtained in Theorem 3.1 turns out to solve the original problem (2.3), as $\varepsilon \to 0$.

Observe that due to $(V_0)$ the following norms for $u \in H^1(\mathbb{R}^3)$ are equivalent:

$$
\|u\|_2^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + V_\varepsilon(x)u^2)dx, \quad \varepsilon > 0
$$

and

$$
\|u\|_2^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + V(y)u^2)dx, \quad \forall y \in \mathbb{R}^3.
$$

A key ingredient in what follows is the concentration-compactness type result below, where we will denote a sequence (and subsequences) $\{\varepsilon_n\}$ just by $\{\varepsilon\}$.

**Proposition 10.** Given a sequence $\varepsilon \to 0$, up to a subsequence, there exist $l \in \mathbb{N}$, sequences $\{y_\varepsilon^l\} \subset \mathbb{R}^3$, $y_\varepsilon^l \in B_2$, $U_i \in H^1(\mathbb{R}^3) \setminus \{0\}$, $i = 1, 2, \ldots, l$, such that

(i) $|y_\varepsilon^l - y_\varepsilon^j| \to +\infty$, if $i \neq j$;

(ii) $\varepsilon y_\varepsilon^l \to y$;

(iii) $\left\|u_\varepsilon - \sum_{i=1}^l U_i(\cdot - y_\varepsilon^i)\right\|_\varepsilon \to 0$;

(iv) $L_{V(y_\varepsilon^l)}(U_i) = 0$;

(v) $I_\varepsilon(u_\varepsilon) \to \sum_{i=1}^l L_{V(y_\varepsilon^l)}(U_i)$.

**Proof.** Obviously, $\{u_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^3)$. Without loss of generality, we may assume $u_\varepsilon \to \bar{u}$ in $H^1(\mathbb{R}^3)$ and a.e. in $\mathbb{R}^3$, as $\varepsilon \to 0$. If $\bar{u} \neq 0$, then $\bar{u}$ is a nontrivial solution to

$$
-\Delta u + V(0)u = g(0, u) \quad \text{in } \mathbb{R}^3,
$$

namely

$$
-\Delta u + u = f(u) \quad \text{in } \mathbb{R}^3,
$$

and then, by Fatou’s Lemma,

$$
m \leq L(\bar{u}) = L(\bar{u}) - \frac{1}{\mu} L'(\bar{u})(\bar{u})
$$

$$
= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^3} (|\nabla \bar{u}|^2 + \bar{u}^2)dx + \int_{\mathbb{R}^3} \left(\frac{1}{\mu} \bar{u}f(\bar{u}) - F(\bar{u})\right) dx
$$

$$
\leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^3} \left(\frac{1}{2} - \frac{1}{\mu}\right) (|\nabla u_\varepsilon|^2 + V_\varepsilon(x)u_\varepsilon^2) + \frac{1}{\mu} u_\varepsilon g_\varepsilon(x, u_\varepsilon) - G_\varepsilon(x, u_\varepsilon) dx
$$

$$
= \liminf_{\varepsilon \to 0} \left(I_\varepsilon(u_\varepsilon) - \frac{1}{\mu} I'_\varepsilon(u_\varepsilon)u_\varepsilon + \left(\frac{1}{\mu} - \frac{1}{4}\right) \varepsilon^2 \int_{\mathbb{R}^3} \phi_{u_\varepsilon} u_\varepsilon^2 dx\right)
$$

$$
= \lim_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) = m,
$$

Finally, we have

$$
I_\varepsilon(u_\varepsilon) \leq \liminf_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) = m.
$$

Therefore, $m \leq I_\varepsilon(u_\varepsilon) \leq m$, which completes the proof.
where we have also used the fact
\[(x,t) \mapsto \left(\frac{1}{2} - \frac{1}{\mu}\right) \alpha_1 t^2 + \left(\frac{1}{\mu} x g(x,t) - G(x,t)\right) \geq 0, \quad x \in \mathbb{R}^3, t \in \mathbb{R}.
\]
This implies that \(L(\bar{u}) = m\) and \(u_\varepsilon \to \bar{u}\) strongly in \(H^1(\mathbb{R}^3)\). So, in this case, choosing \(l = 1, y_1 = 0\) and \(U_1 = \bar{u}\), we are done.

Otherwise, if \(u_\varepsilon \to 0\) in \(H^1(\mathbb{R}^3)\), as \(\varepsilon \to 0\), let us proceed in three steps.

**Step 1.** If for some \(R > 0\) one has
\[
\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^3} \int_{B(x,R)} |u_\varepsilon|^2 dx = 0,
\]
by Lions’ Lemma, \(u_\varepsilon \to 0\) in \(L^q(\mathbb{R}^3)\) for \(q \in (2,6)\). Since \(\tilde{L}'(u_\varepsilon)(u_\varepsilon) = 0\), it is easy to check that \(u_\varepsilon \to 0\) in \(H^1(\mathbb{R}^3)\), which contradicts the fact that \(m > 0\). As a consequence, there exists \(y_1^\varepsilon \in \mathbb{R}^3\) such that
\[
\lim_{\varepsilon \to 0} \int_{B(y_1^\varepsilon, R)} |u_\varepsilon|^2 dx > 0
\]
and \(|y_1^\varepsilon| \to +\infty\), as \(\varepsilon \to 0\). Let
\[
v_1^\varepsilon(\cdot) := u_\varepsilon(\cdot + y_1^\varepsilon).
\]
Then, up to a subsequence, \(v_1^\varepsilon \to U_1 \neq 0\) in \(H^1(\mathbb{R}^3)\) and a.e. in \(\mathbb{R}^3\) as \(\varepsilon \to 0\), and \(v_\varepsilon\) satisfies
\[
-\nabla v_1^\varepsilon + V(\varepsilon x + \varepsilon y_1^\varepsilon) v_1^\varepsilon + \varepsilon^2 \phi_{v_1} v_1^\varepsilon = g(\varepsilon x + \varepsilon y_1^\varepsilon, v_1^\varepsilon), \quad x \in \mathbb{R}^3. \tag{4.1}
\]
If, up to a subsequence, \(\varepsilon y_1^\varepsilon \subset \mathbb{R}^3 \setminus B_2\) and \(V(\varepsilon y_1^\varepsilon) \to \tilde{V}\) as \(\varepsilon \to 0\), then \(U_1\) would be a weak solution to
\[
-\nabla u + \tilde{V} u = \tilde{f}(u), \quad \text{in } \mathbb{R}^3.
\]
Since \(\tilde{V} \geq \alpha_1\) and \(\tilde{f}(t) < \alpha_1 t\) for all \(t > 0\), we would get \(U_1 \equiv 0\), again a contradiction. Thus, \(\varepsilon y_1^\varepsilon \subset B_2\) for small \(\varepsilon > 0\). If \(\varepsilon y_1^\varepsilon \to y_1^\varepsilon \subset B_2\) as \(\varepsilon \to 0\), it follows from (4.1) that \(L_{V(y_1^\varepsilon)}(U_1) = 0\). Now, setting
\[
w_1^\varepsilon(\cdot) := u_\varepsilon(\cdot) - U_1(\cdot - y_1^\varepsilon),
\]
one has \(w_1^\varepsilon \to 0\) in \(H^1(\mathbb{R}^3)\), as \(\varepsilon \to 0\). Moreover, by Brezis-Lieb’s Lemma we have:
- \(|w_1^\varepsilon|^2 = ||U_1||_{y_1^\varepsilon}^2 + ||w_1^\varepsilon||_{y_1^\varepsilon}^2 + \alpha_\varepsilon(1);
- \(\tilde{I}_\varepsilon(w_1^\varepsilon) = L_{V(y_1^\varepsilon)}(U_1) + \tilde{I}_\varepsilon(w_1^\varepsilon) + \alpha_\varepsilon(1);
- \(\tilde{I}_\varepsilon(w_1^\varepsilon) = \alpha_\varepsilon(1).
\]
The proof will be complete once we show that \(w_1^\varepsilon \to 0\) in \(H^1(\mathbb{R}^3)\), as \(\varepsilon \to 0\).

**Step 2.** By contradiction assume that \(w_1^\varepsilon \not\to 0\) in \(H^1(\mathbb{R}^3)\), as \(\varepsilon \to 0\). Similarly as above, by Lions’ Lemma, for some \(R > 0\) there exists \(y_2^\varepsilon \in \mathbb{R}^3\) such that
\[
\lim_{\varepsilon \to 0} \int_{B(y_2^\varepsilon, R)} |w_1^\varepsilon|^2 dx > 0
\]
and \(|y_2^\varepsilon| \to \infty\), as \(\varepsilon \to 0\). Letting
\[
v_2^\varepsilon(\cdot) := w_1^\varepsilon(\cdot + y_2^\varepsilon) = u_\varepsilon(\cdot + y_2^\varepsilon) - U_1(\cdot + y_1^\varepsilon - y_2^\varepsilon),
\]
up to a subsequence \(v_2^\varepsilon \to U_2\) in \(H^1(\mathbb{R}^3)\) and a.e. in \(\mathbb{R}^3\), as \(\varepsilon \to 0\). Recalling that \(\tilde{I}_\varepsilon(w_1^\varepsilon) = \alpha_\varepsilon(1)\), arguing as before one gets \(\varepsilon y_2^\varepsilon \to y_2^\varepsilon \in B_2\) as \(\varepsilon \to 0\) and \(L_{V(y_2^\varepsilon)}(U_2) = 0\).

We claim that
\[
|y_2^\varepsilon - y_2^\varepsilon| \to \infty, \quad \text{as } \varepsilon \to \infty.
\]
Indeed, if not there would exist $R_1 > 0$ such that $B(y^2_ε, R) \subset B(y^1_ε, R_1)$ for small $ε$ and then
\[
\lim_{ε → 0} \int_{B(y^2_ε, R)} |w^1_ε|^2 dx \leq \lim_{ε → 0} \int_{B(y^1_ε, R_1)} |w^1_ε|^2 dx = \lim_{ε → 0} \int_{B(0, R_1)} |w^1_ε - U_1|^2 dx = 0,
\]
which is a contradiction. Set
\[
w^2_ε := w^1_ε - U_2(\cdot - y^2_ε) = u_ε - U_1(\cdot - y^1_ε) - U_2(\cdot - y^2_ε).
\]
Arguing as usual, up to a sequence, $w^2_ε \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ and a.e. in $\mathbb{R}^3$, as $ε → 0$ and we have:

- $||u_ε||^2 = ||U_1||^2_{y^1} + ||U_2||^2_{y^2} + ||w^2_ε||^2 + o_ε(1)$;
- $I_ε(u_ε) = L_{V(y^1)}(U_1) + L_{V(y^2)}(U_2) + I_ε(w^2_ε) + o_ε(1)$;
- $I_ε^r(w^2_ε) = o_ε(1)$.

Once again, if $w^2_ε \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, as $ε → 0$, the proof is done.

**Step 3.** Finally, if $w^2_ε \not \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, as $ε → 0$, there exist $l ∈ \mathbb{N}$, $y^k_ε ∈ \mathbb{R}^3$, $y^k_ε \in B_2$ and $U_i ∈ H^1(\mathbb{R}^3) \setminus \{0\}$, $i = 3, 4, \ldots, l$ such that:

- $|y^k_ε - y^k_ε| → ∞$ as $ε → 0$ for $i \neq j$;
- $ε y^k_ε → y^k$ as $ε → 0$;
- $L_{V(y^k)}(U_i) = 0$;
- $||u_ε||^2 = \sum_{i=1}^l ||U_i||_{y^k}^2 + ||w^k_ε||^2 + o_ε(1)$, $j = 1, 2, \ldots, l$, where
\[
w^k_ε := u_ε - \sum_{i=1}^j U_i(\cdot - y^k_ε);
\]

- $I_ε(u_ε) = \sum_{i=1}^l L_{V(y^k)}(U_i) + I_ε(w^k_ε) + o_ε(1)$, $j = 1, 2, \ldots, l$;
- $I_ε^r(w^k_ε) = o_ε(1)$, $j = 1, 2, \ldots, l$.

Since for some $C > 0$ (independent of $i$),
\[
g(y^k, t)t ≤ \frac{α_1}{2} t^2 + Ct^6, \quad t ≥ 0
\]
and $U_i$ satisfies
\[
-ΔU_i + V(y^k)U_i = g(y^k, U_i), \quad \text{in} \; \mathbb{R}^3,
\]
\[
||U_i||_{y^k}^2 = \int_{\mathbb{R}^3} g(y^k, U_i) U_idx ≤ \frac{α_2}{2} ||U_i||_{y^k}^2 + C ||U_i||_{y^k}^6,
\]
there exists $c > 0$ (independent of $i$) such that
\[
\min_i ||U_i||_{y^k} ≥ c.
\]

Thanks to the fact that $\{u_ε\}$ is bounded in $H^1(\mathbb{R}^3)$, one has $w^l_ε \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, as $ε → 0$ for some $l$ and the proof is now complete. \(\square\)

**Proposition 11.** If $ε_j → 0$, up to a subsequence there exists $\{y_{ε_j}\} ⊂ \mathbb{R}^3$ such that
\[
ε_j y_{ε_j} → 0, \quad ||u_{ε_j} - U(\cdot - y_{ε_j})|| → 0, \quad j → ∞
\]
where $U ∈ S$.

**Proof.** We exploit the decomposition obtained in the previous Proposition and observe that $L_{V(j)} ≥ m_{V(j)}$. Moreover, monotonicity and continuity of the map $i → m_i$ imply that $m_{V(j)} ≥ m - δ$ for any $y^i ∈ B_2$ and sufficiently small $δ > 0$. This implies that $l = 1$. Thus it is enough to show that $y^1 = 0$. 

Following the above calculations, since \( \{ u_\varepsilon \} \subset H^2_{\text{loc}}(\mathbb{R}^3) \) and it is bounded, choose \( r > 0 \) and a smooth function \( \psi_\varepsilon : \mathbb{R}^3 \to \mathbb{R} \) satisfying
\[
\psi_\varepsilon(x) = \begin{cases} 
1 & \text{in } B(y_1^1, r \varepsilon^{-1}), \\
0 & \text{in } \mathbb{R}^3 \setminus B(y_1^1, 2r \varepsilon^{-1}),
\end{cases}
\]
with \( |\nabla \psi_\varepsilon(x)| \leq c \varepsilon \) for any \( x \in \mathbb{R}^3 \). Multiplying (2.10) by \( (\partial_\nu u_\varepsilon) \psi_\varepsilon \), integrating and taking into account the identity
\[
\varepsilon^2 \int_{B(y_1^1, r \varepsilon^{-1})} (\partial_\nu u_\varepsilon) u_\varepsilon \partial_\nu \psi_\varepsilon \, dx
= - \frac{\varepsilon^2}{2} \int_{B(y_1^1, r \varepsilon^{-1})} (\partial_\nu \phi_\varepsilon) u_\varepsilon^2 \, dx - \frac{\varepsilon^2}{2} \int_{B(y_1^1, r \varepsilon^{-1})} (\partial_\nu \psi_\varepsilon) \phi_\varepsilon u_\varepsilon^2 \, dx,
\]
(see Proposition 8 - (iii) for details), one has
\[
\int_{B(y_1^1, r \varepsilon^{-1})} \partial_\nu V(\varepsilon x) u_\varepsilon^2 \, dx - \varepsilon \int_{B(y_1^1, r \varepsilon^{-1})} \partial_\nu \chi(\varepsilon x)(F(u_\varepsilon) - \tilde{F}(u_\varepsilon)) \, dx = o(\varepsilon).
\]
From this point, the proof is the same as in [8] which we refer to.

We are now in the position to prove the main result of this paper:

**Proof of Theorem 1.1.** Of course we intend to show that \( u_\varepsilon \) from Theorem 3.1 solves for the original problem. If \( u_\varepsilon(x) \to 0 \) uniformly in \( \mathbb{R}^3 \setminus B_1^1 \), as \( \varepsilon \to 0 \), then \( g_\varepsilon(x, u_\varepsilon) = f(u_\varepsilon) \) and the conclusion follows. By the previous Proposition we have
\[
\|u_\varepsilon\|_{H^1(\mathbb{R}^3 \setminus B_1^1)} \to 0, \quad \text{as } \varepsilon \to 0.
\]
Moreover, for any \( x \in \mathbb{R}^3 \setminus B_1^1 \), one has that \( B(x, 2) \subset \mathbb{R}^3 \setminus B_0^1 \) and therefore, by elliptic estimates, there exists \( c > 0 \) (independent of \( x \)) such that
\[
u_\varepsilon(x) \leq c \|u_\varepsilon\|_{H^1(B(x, 2))} \to 0, \quad \text{as } \varepsilon \to 0.
\]

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