Reconfiguration of Vertex Colouring and Forbidden Induced Subgraphs

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March 7, 2023

Abstract

The reconfiguration graph of the $k$-colourings, denoted $R_k(G)$, is the graph whose vertices are the $k$-colourings of $G$ and two colourings are adjacent in $R_k(G)$ if they differ in colour on exactly one vertex. In this paper, we investigate the connectivity and diameter of $R_{k+1}(G)$ for a $k$-colourable graph $G$ restricted by forbidden induced subgraphs. We show that $R_{k+1}(G)$ is connected for every $k$-colourable $H$-free graph $G$ if and only if $H$ is an induced subgraph of $P_4$ or $P_3 + P_1$. We also start an investigation into this problem for classes of graphs defined by two forbidden induced subgraphs. We show that if $G$ is a $k$-colourable $(2K_2, C_4)$-free graph, then $R_{k+1}(G)$ is connected with diameter at most $4n$. Furthermore, we show that $R_{k+1}(G)$ is connected for every $k$-colourable $(P_5, C_4)$-free graph $G$.

Keywords: reconfiguration graph, forbidden induced subgraph, $k$-colouring, $k$-mixing, frozen colouring.

1 Introduction

Let $G$ be a finite simple graph with vertex-set $V(G)$ and edge-set $E(G)$. We use $n = |V(G)|$ to denote the number of vertices of $G$ when the context is clear. For a positive integer $k$, a $k$-colouring of $G$ is a mapping $\alpha: V(G) \to \{1, 2, \ldots, k\}$ such that $\alpha(u) \neq \alpha(v)$ whenever $uv \in E(G)$. We say that $G$ is $k$-colourable if it admits a $k$-colouring and the chromatic number of $G$, denoted $\chi(G)$, is the smallest integer $k$ such that $G$ is $k$-colourable. The set of vertices that are assigned the same colour is called a colour class. We say that the colour classes of a colouring $\alpha$ of $G$ matches the colour classes of a colouring $\beta$ of $G$ if both colourings induce the same colour classes.

The reconfiguration graph of the $k$-colourings, denoted $R_k(G)$, is the graph whose vertices are the $k$-colourings of $G$ and two colourings are joined by an edge if they differ in colour on exactly one vertex. We say that $G$ is $k$-mixing if $R_k(G)$ is connected and the $k$-recolouring diameter of $G$ is the diameter of $R_k(G)$. Given two $k$-colourings $\alpha$ and $\beta$ of $G$, deciding whether there exists a path between the two colourings in $R_k(G)$ was proved to be PSPACE-complete for all $k > 3$ [5]. The problem remains PSPACE-complete for graphs with bounded bandwidth and hence bounded treewidth [13].

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A k-colouring of a graph $G$ is called frozen if it is an isolated vertex in $R_k(G)$. In other words, for every vertex $v \in V(G)$, each of the $k$ colours appears in the closed neighbourhood of $v$. One way to show that a graph $G$ is not $k$-mixing is to exhibit a frozen $k$-colouring of $G$. Since every $k$-colouring of $K_2$ is frozen, it is common to study $R_{k+1}(G)$ for a $k$-colourable graph $G$. For the rest of this paper, we assume that $\ell \geq \chi(G) + 1$.

A graph $G$ is $H$-free if no induced subgraph of $G$ is isomorphic to $H$. For a collection of graphs $\mathcal{H}$, $G$ is $\mathcal{H}$-free if $G$ is $H$-free for every $H \in \mathcal{H}$. Let $P_n$, $C_n$, and $K_n$ denote the path, cycle, and complete graph on $n$ vertices, respectively. For two vertex-disjoint graphs $G$ and $H$, the disjoint union of $G$ and $H$, denoted by $G+H$, is the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. For a positive integer $r$, we use $rG$ to denote the graph obtained from the disjoint union of $r$ copies of $G$. The goal of this paper is to answer the following question.

**Question 1.** For which $H$ is every $H$-free graph $\ell$-mixing?

Question 1 has been answered completely when $H$ is a 3-vertex graph, namely $3K_1$, $P_2 + P_1$, $P_3$, and $K_3$. The last author [11] showed that every $3K_1$-free graph is $\ell$-mixing and the $\ell$-recolouring diameter is at most $4n$. The class of $(P_2 + P_1)$-free graphs is a subclass of $P_3$-free graphs, and so we refer to the results on $P_3$-free graphs in this case (see Theorem 1). A graph is $P_3$-free if it is the disjoint union of cliques and it is well known that every $P_3$-free graph is $\ell$-mixing (see e.g. [4]). Bonamy and Bousquet [3] showed that the $\ell$-recolouring diameter of a $P_3$-free graph is at most $2n$. Cereceda, van den Heuvel, and Johnson [6] showed that for all $\ell \geq 3$, there is a bipartite graph that is not $\ell$-mixing (see Figure 3). Since bipartite graphs are a subclass of $K_3$-free graphs, it follows that not every $K_3$-free graph is $\ell$-mixing. See Table 1 for a summary of these results.

| $H$         | Always $\ell$-mixing | Upper bound on the $\ell$-recolouring diameter |
|------------|-----------------------|-----------------------------------------------|
| $3K_1$     | YES [11]              | $4n$ [11]                                     |
| $P_2 + P_1$| YES [3]               | $4n$ [1] (Theorem 1)                          |
| $P_3$      | YES [4]               | $2n$ [3]                                      |
| $K_4$      | NO [6]                | $\infty$ [6] (see Figure 3)                  |

**Table 1:** Summary of recolouring an $H$-free graph for a 3-vertex graph $H$.

Next we survey results on Question 1 when $H$ is a 4-vertex graph. There are 11 graphs on 4 vertices (see Figure 1). Bonamy and Bousquet [3] showed that every $P_4$-free graph $G$ is $\ell$-mixing and the $\ell$-recolouring diameter is at most $2 \cdot \chi(G) \cdot n$. Biedl, Lubiw, and Merkel [1] investigated a class of graphs that generalizes $P_4$-free graphs. The following theorem was not directly stated in [1] but follows from the proof of Theorem 1 in [1] and improves the bound on the $\ell$-recolouring diameter of a $P_4$-free graph.

**Theorem 1 ([1]).** The $\ell$-recolouring diameter of a $P_4$-free graph is at most $4n$.

Feghali and Merkel [8] showed that for all $p \geq 1$, there exists a $k$-colourable $2K_2$-free graph with a frozen $(k + p)$-colouring. This result together with the result of Bonamy and Bousquet [3] answers Question 1 when $H$ is a path.

**Theorem 2 ([8]).** Every $P_t$-free graph is $\ell$-mixing if and only if $t \leq 4$.

The connectivity and diameter of $R_{\ell}(G)$ has also been investigated for classes of graphs defined by more than one forbidden induced subgraph. Bonamy, Johnson, Lignos, Patel, and Paulusma [4] showed that every chordal graph and every chordal bipartite graph is $\ell$-mixing and the $\ell$-recolouring diameter is at most $2n^2$. Feghali and Fiala [7] showed that every co-chordal graph is $\ell$-mixing and the $\ell$-recolouring diameter is at most $2n^2$. They also showed that every
3-colourable \((P_3, \text{co-}P_3, C_5)\)-free graph is \(\ell\)-mixing and the \(\ell\)-recolouring diameter is at most \(2n^2\) [7]. Biedl, Lubiw, and Merkel [1] showed that every \(P_4\)-sparse graph is \(\ell\)-mixing and the \(\ell\)-recolouring diameter is at most \(4n^2\). The last author [11] showed that for all \(p \geq 1\), there exists a \(k\)-colourable weakly chordal graph that is not \((k + p)\)-mixing.

**Our contributions**

We completely answer Question 1 with the following theorem.

**Theorem 3.** Every \(H\)-free graph is \(\ell\)-mixing if and only if \(H\) is an induced subgraph of \(P_4\) or \(P_3 + P_1\).

We use the following theorems in the proof of Theorem 3.

**Theorem 4.** For all \(p \geq 1\), there exists a \(k\)-colourable \((4K_1, C_4, \text{claw})\)-free graph that is not \((k + p)\)-mixing.

**Theorem 5.** For all \(p \geq 1\), there exists a \(k\)-colourable \((K_4, \text{diamond}, \text{paw}, \text{co-claw}, \text{co-diamond})\)-free graph that is not \((k + p)\)-mixing.

**Theorem 6.** Every \((P_3 + P_1)\)-free graph is \(\ell\)-mixing and the \(\ell\)-recolouring diameter is at most \(6n\).

The proof of Theorem 6 leads to a polynomial-time algorithm to find a path of length at most \(6n\) between any two \(\ell\)-colourings of \(G\). See Table 2 for a summary of these results.

| \(H\)          | Always \(\ell\)-mixing | Upper bound on the \(\ell\)-recolouring diameter |
|-----------------|-------------------------|-------------------------------------------------|
| \(4K_1\)        | NO                      | \(\infty\) (Theorem 4)                          |
| co-diamond      | NO                      | \(\infty\) (Theorem 5)                          |
| \(2K_2\)        | NO \([8]\)              | \(\infty\) \([8]\)                              |
| \(P_3 + P_1\)   | YES                     | \(6n\) (Theorem 6)                              |
| claw            | NO                      | \(\infty\) (Theorem 4)                          |
| \(P_4\)         | YES \([3]\)             | \(4n\) \([1]\) (Theorem 1)                     |
| co-claw         | NO                      | \(\infty\) (Theorem 5)                          |
| paw             | NO                      | \(\infty\) (Theorem 5)                          |
| \(C_4\)         | NO                      | \(\infty\) (Theorem 4)                          |
| diamond         | NO                      | \(\infty\) (Theorem 5)                          |
| \(K_4\)         | NO                      | \(\infty\) (Theorem 5)                          |

**Table 2:** Summary of recolouring an \(H\)-free graph for a 4-vertex graph \(H\).
We also start an investigation into classes of graphs defined by two forbidden induced subgraphs.

**Theorem 7.** Every $(2K_2, C_4)$-free graph is $\ell$-mixing and the $\ell$-recolouring diameter is at most $4n$.

Note that the class of split graphs, equivalently the class of chordal and co-chordal graphs, is a subclass of $(2K_2, C_4)$-free graphs. Theorem 7 improves the upper bound on the $\ell$-recolouring diameter for a split graph from $2n^2$ to $4n$. We also investigate the superclass of $(P_5, C_4)$-free graphs and prove the following.

**Theorem 8.** Every $(P_5, C_4)$-free graph is $\ell$-mixing.

We note that there are $P_5$-free graphs and $C_4$-free graphs that are not $\ell$-mixing.

The rest of the paper is organized as follows. In Section 2 we give definitions and terminology used throughout the paper. We prove Theorems 4 and 5 in Section 3 and we prove Theorem 6 in Section 4. We prove Theorem 3 in Section 5 and we prove Theorems 7 and 8 in Section 6. We end with some open problems in Section 7.

## 2 Preliminaries

For a graph $G$, the *complement* of $G$ is the graph with vertex-set $V(G)$ such that the edges of the complement are exactly the non-edges of $G$. A component of $G$ is a maximal connected subgraph and an anticomponent of $G$ is a component of the complement of $G$. For a vertex $v \in V(G)$, the open neighbourhood of $v$ is the set of vertices adjacent to $v$ in $G$. The closed neighbourhood of $v$ is the set of vertices adjacent to $v$ in $G$ together with $v$. For $X, Y \subseteq V(G)$, we say that $X$ is complete to $Y$ if every vertex in $X$ is adjacent to every vertex in $Y$. If no vertex of $X$ is adjacent to a vertex of $Y$, we say that $X$ is anticomplete to $Y$. Let $G$ and $H$ be vertex-disjoint graphs and let $v \in V(G)$. By substituting $H$ for the vertex $v$ of $G$, we mean taking the graph $G - v$ and adding an edge between every vertex of $H$ and every vertex of $G - v$ that is adjacent to $v$ in $G$.

For a colouring $\alpha$ of $G$ and $X \subseteq V(G)$, we say that the colour $c$ appears in $X$ if $\alpha(x) = c$ for some $x \in X$, and we use $\alpha(X)$ for the set of colours appearing in $X$ and $|\alpha(X)|$ for the number of colours appearing on $X$. A $k$-colouring of a graph $G$ is called frozen if it is an isolated vertex in the recolouring graph $\mathcal{R}_k(G)$. In other words, for every vertex $v \in V(G)$, each of the $k$ colours appears in the closed neighbourhood of $v$.

## 3 Frozen colourings

We use frozen colourings to prove Theorems 4 and 5, answering Question 1 in the negative for 8 out of the 11 cases when $H$ is a 4-vertex graph. To prove Theorem 4, we construct a family of graphs $\{G_p \mid p \geq 1\}$ such that $G_p$ is a 2$p$-colourable $(4K_1, C_4, claw)$-free graph that has a frozen 3$p$-colouring.

**Lemma 1.** For $p \geq 1$, let $G_p$ be the graph obtained from $C_6$ by substituting the complete graph $K_p$ into each vertex. Then $G_p$ is $(4K_1, C_4, claw)$-free, is 2$p$-colourable, and has a frozen 3$p$-colouring.

**Proof.** See Figure 2 for a 2$p$-colouring and frozen 3$p$-colouring of $G_p$. Notice that the vertices of $G_p$ can be partitioned into 3 cliques, and so $G_p$ must be $4K_1$-free. Also notice that every vertex of $G_p$ is bisimplicial. That is, the closed neighbourhood of every vertex can be partitioned into
two cliques, and so $G_p$ must be claw-free. Finally, suppose $G_p$ contains an induced $C_4$. Then $G_p$ contains an induced $P_3$, call it $P$, with vertices $w, x, y$ in order. Notice that each vertex of $P$ must belong to a distinct copy of $K_p$. But then any other vertex of $G_p$ that is adjacent to both $w$ and $x$ must also be adjacent to $y$, a contradiction.

To prove Theorem 5, we use the family of bipartite graphs $\{B_p \mid p \geq 3\}$ introduced by Cereceda, van den Heuvel, and Johnson [6]. For $p \geq 3$, the graph $B_p$ is 2-colourable, and has a frozen $p$-colouring.

**Lemma 2.** For $p \geq 3$, let $B_p$ be the graph obtained from the complete bipartite graph $K_{p,p}$ by deleting the edges of a perfect matching. Then $B_p$ is ($K_4$, diamond, paw, co-claw, co-diamond)-free, 2-colourable, and has a frozen $p$-colouring.

**Proof.** See Figure 3 for a frozen $p$-colouring of $B_p$. Since $B_p$ is bipartite, it is also $K_3$-free and so it must be ($K_4$, diamond, paw, co-claw)-free. To show that $B_p$ is co-diamond-free, we show that the complement of $B_p$ is diamond-free. The complement of $B_p$ consists of two disjoint cliques with $p$ vertices, call them $Q_1$ and $Q_2$, and the edges between $Q_1$ and $Q_2$ are a perfect matching. Suppose by contradiction that the complement of $B_p$ contains an induced diamond $D$. Let $w, x$ be the adjacent vertices of degree 3 in $D$ and let $y, z$ be the non-adjacent vertices of degree 2 in $D$. Since each vertex in $Q_i$, for $i = 1, 2$, can be adjacent to at most one vertex in the other clique, either $x, w, y \in Q_1$ or $x, w, z \in Q_1$. Without loss of generality, let $x, w, y \in Q_1$. Since $z$ is not adjacent to $y$, $z$ must be in $Q_2$. But then $z \in Q_2$ must be adjacent to two vertices in $Q_1$, a contradiction.
4 Recolouring \((P_3 + P_1)\)-free graphs

In this section, we investigate the connectivity and diameter of \(R_\ell(G)\) for a \((P_3 + P_1)\)-free graph \(G\). Our strategy takes advantage of the anticomponents of \(G\). Since there are all possible edges between the anticomponents of a graph, in any colouring of the graph, the colours that appear on each anticomponent must be pairwise distinct. Thus if we can recolour a vertex in some anticomponent of \(G\), that recolouring step would extend to a recolouring step for all of \(G\).

Given two arbitrary \(\ell\)-colourings \(\alpha\) and \(\beta\) of \(G\), we show how to recolour \(\alpha\) into \(\beta\) using the following strategy. First fix an arbitrary \(\chi(G)\)-colouring \(\gamma\) of \(G\) by finding an optimal colouring of each anticomponent of \(G\). Next recolour \(\alpha\) into a \(\chi(G)\)-colouring \(\alpha'\) of \(G\) by recolouring each anticomponent \(A\) of \(G\) so that the colour classes of \(\alpha_A\) match the colour classes of \(\gamma_A\). Similarly recolour \(\beta\) into a \(\chi(G)\)-colouring \(\beta'\) of \(G\) by recolouring each anticomponent \(A\) of \(G\) so that the colour classes of \(\beta_A\) match the colour classes of \(\gamma_A\). Once we have the colourings \(\alpha'\) and \(\beta'\), we use the following Renaming Lemma.

**Lemma 3 (Renaming Lemma [3]).** Let \(\alpha'\) and \(\beta'\) be two \(k\)-colourings of \(G\) that induce the same partition of vertices into colour classes and let \(\ell \geq k + 1\). Then \(\alpha'\) can be recoloured into \(\beta'\) in \(R_\ell(G)\) by recolouring each vertex at most 2 times.

Olarin [12] proved the following useful result characterizing the anticomponents of a \((P_3 + P_1)\)-free graph.

**Theorem 9 ([12]).** Each connected component of a paw-free graph is either \(K_3\)-free or \((P_2 + P_1)\)-free.

Therefore each anticomponent of a \((P_3 + P_1)\)-free graph is either \(3K_1\)-free or \(P_3\)-free. The last author proved the following theorem on recolouring \(3K_1\)-free graphs.

**Theorem 10 ([11]).** Every \(3K_1\)-free graph is \(\ell\)-mixing and the \(\ell\)-recolouring diameter is at most \(4n\).

The proof of Theorem 10 uses the following lemma which we will use in the proof of Theorem 6.

**Lemma 4 ([11]).** Let \(\gamma\) be a \(\chi(G)\)-colouring of a \(3K_1\)-free graph \(G\). Any \(\ell\)-colouring of \(G\) can be recoloured into a \(\chi(G)\)-colouring \(\gamma'\) such that the colour classes of \(\gamma'\) match the colour classes of \(\gamma\) by recolouring each vertex at most once.

We are now ready to prove Theorem 6.

**Proof of Theorem 6.** Let \(G\) be a \(k\)-colourable \((P_3 + P_1)\)-free graph and let \(\alpha\) and \(\beta\) be two \(\ell\)-colourings of \(G\) with \(\ell \geq k + 1\). Fix a \(\chi(G)\)-colouring \(\gamma\) of \(G\). We note that this colouring can be found in polynomial time by optimally colouring each anticomponent of \(G\) [10].

Let \(A_1, A_2, \ldots, A_p\) be the anticomponents of \(G\). Since the anticomponents of any graph are pairwise complete, the colours used on each anticomponent are pairwise distinct, and thus \(\chi(G) = \sum_{1 \leq i \leq p} \chi(A_i)\). We proceed by recolouring each anticomponent \(A_i\) of \(G\) so that the colouring of \(A_i\) matches the colour classes of \(A_i\) in \(\gamma\). The anticomponent that is selected to be recoloured is determined based on the following claim.

**Claim 1.** In any \(\ell\)-colouring \(\alpha\) of \(G\), either some colour does not appear in \(\alpha\) or there is some anticomponent \(A\) of \(G\) such that \(\alpha\) uses at least \(\chi(A) + 1\) colours on \(A\).

**Proof of claim:** Suppose \(\alpha\) colours each anticomponent \(A_i\) of \(G\) with \(\chi(A_i)\) colours. Since \(\chi(G) = \sum_{1 \leq i \leq p} \chi(A_i)\), only \(\chi(G)\) colours appear in \(\alpha\) and since \(\ell \geq \chi(G) + 1\), some colour does not appear in \(\alpha\). This completes the proof of the Claim 1.
If some colour \( c \) is not being used, we take some anticomponent \( A_i \) of \( G \) that has not been recoloured. Recolour \( A_i \) as described below depending on whether \( A_i \) is \( P_3 \)-free or \( 3K_1 \)-free using \( c \) together with the \( \chi(A_i) \) colours appearing on \( A_i \).

Now suppose that there is some anticomponent \( A_i \) where the current colouring uses at least \( \chi(A_i) + 1 \) colours on \( A_i \). If \( A_i \) is \( P_3 \)-free then since each component of \( A_i \) is a clique, we use the Renaming Lemma to recolour each component of \( A_i \) to match the colour classes of \( \gamma \) by recolouring each vertex at most twice. If \( A_i \) is \( 3K_1 \)-free, then we use Lemma 4 to recolour the current colouring of \( A_i \) to match the colour classes of \( \gamma \).

We have recoloured \( \alpha \) into a \( \chi(G) \)-colouring \( \alpha' \) whose colour classes correspond to the colour classes of \( \gamma \) by recolouring each vertex at most twice. Similarly, we recolour \( \beta \) into a \( \chi(G) \)-colouring \( \beta' \) whose colour classes correspond to those of \( \gamma \) by recolouring each vertex at most twice. By the Renaming Lemma, \( \alpha' \) can be recoloured into \( \beta' \) by recolouring each vertex at most twice. Thus, we have found a path from \( \alpha \) to \( \beta \) in \( \mathcal{R}_\ell(G) \) by recolouring each vertex at most 6 times.

\[ \square \]

5 Proof of Theorem 3

Theorem 3 follows from the following lemma.

**Lemma 5.** There is no graph \( H \) on five vertices such that every \( H \)-free graph \( G \) is \( (\chi(G)+1) \)-mixing.

*Proof.* There are 34 graphs on 5 vertices. We partition them into four groups: Group 1 consists of the 21 graphs which are not bipartite. Group 2 consists of the 7 graphs which are bipartite and contain the co-diamond. Group 3 consists of \( P_5 \). Group 4 consists of the remaining 5 graphs, which are \( 5K_1, K_{1,4}, \text{claw}+K_1, K_{2,3} \) and the banner, where the banner consists of the cycle \( C_4 \) on four vertices together with a vertex joined to exactly one vertex of the \( C_4 \).

The graph \( B_p \) is bipartite and, as mentioned in Lemma 2, is co-diamond-free, and thus is \( H \)-free for any non-bipartite graph \( H \) and for any graph \( H \) containing the co-diamond. By Lemma 2, \( B_p \) has a frozen \( p \)-colouring, for every \( p \geq 3 \).

Theorem 2 [8] says that not every \( P_5 \)-free graph is \( (\chi+1) \)-mixing.

Lemma 1 says that there are frozen colourings of the graph \( G_p \) which is \( (4K_1, C_4, \text{claw}) \)-free, and thus also \( (5K_1, K_{1,4}, \text{claw}+K_1, K_{2,3}, \text{banner}) \)-free. Thus, for every graph \( H \) in Group 4, there is an \( H \)-free graph which is not \( (\chi+1) \)-mixing.

\[ \square \]

6 Recolouring \((P_5, C_4)\)-free graphs

A blow-up of a graph \( G \) with vertices \( v_1, v_2, \ldots, v_n \) is any graph \( H \) that can be obtained by substituting a clique \( A_i \) for \( v_i \) in \( G \), one at a time for all \( i \in [n] \). A clique cutset \( Q \) of a graph \( G \) is a clique in \( G \) such that \( G-Q \) has more components than \( G \).

A split graph is one whose vertex-set can be partitioned into a clique and an independent set. Equivalently, split graphs are the class of \((2K_2, C_4, C_5)\)-free graphs or the class of graphs which are both chordal and cochordal. Blázsik et al [2] gave the following characterization of \((2K_2, C_4)\)-free graphs.

**Theorem 11** ([2]). A graph \( G \) is \((2K_2, C_4)\)-free if and only if its vertex-set can be partitioned into three possibly empty sets, \( Q, C \) and \( I \) such that \( Q \) is a complete graph, \( I \) is an independent set and \( C \) (if nonempty) induces a \( C_5 \), and \( C \) is complete to \( Q \) and anticOMPlete to \( I \).

First we consider the subclass \((2K_2, C_4)\)-free graphs and prove Theorem 7.
Proof of Theorem 7. Let $\gamma_1$ and $\gamma_2$ be any two colourings of a $(2K_2, C_4)$-free graph $G$. We prove the theorem by providing a path between the two colourings of length at most $4n$. First we obtain two colourings $\alpha$ and $\beta$ from $\gamma_1$ and $\gamma_2$, respectively, such that $\alpha$ and $\beta$ induce the same partition of the vertices into colour classes. We consider two cases.

Case 1: When $G$ has an induced $C_5$. By Theorem 11, $V(G)$ can be partitioned into three subsets $C$, $Q$, and $I$, where $C$ induces a $C_5$, $Q$ is a clique and $I$ is an independent set. Since $C$ is complete to $Q$ and anticomplete to $I$, we have $\chi(G) = 3 + |Q|$ and $\gamma_i(C) \cap \gamma_i(Q) = \emptyset$, for $i = 1, 2$. Let $C = \{v_1, v_2, \ldots, v_5\}$ such that $v_j v_{j+1} \in E(G)$ for all $j \pmod 5$. Since $\ell \geq \chi(G) + 1 = 4 + |Q|$, if $\gamma_i$ uses three colours on $C$ then there is a colour that does not appear on $C \cup Q$ and we recolour a vertex of $C$ with this colour to use at least four colours on $C$. There exist three vertices, say $v_1, v_2, v_3$, such that $\gamma_i(v_1) \neq \gamma_i(v_2) \neq \gamma_i(v_3)$ and $\gamma_i(v_1) \neq \gamma_i(v_3)$, for $i = 1, 2$. We define colourings $\alpha$ and $\beta$, from $\gamma_1$ and $\gamma_2$, respectively, as follows:

$$
\alpha(v) = \begin{cases} 
\gamma_1(v_1) & \text{if } v = v_4 \\
\gamma_1(v_2) & \text{if } v = v_5 \\
\gamma_1(v_1) & \text{if } v \in I \\
\gamma_1(v) & \text{if } v \in \{v_1, v_2, v_3\} \cup Q;
\end{cases} \\
\beta(v) = \begin{cases} 
\gamma_2(v_1) & \text{if } v = v_4 \\
\gamma_2(v_2) & \text{if } v = v_5 \\
\gamma_2(v_1) & \text{if } v \in I \\
\gamma_2(v) & \text{if } v \in \{v_1, v_2, v_3\} \cup Q;
\end{cases}
$$

Case 2: When $G$ is $C_5$-free. Since $G$ is $(2K_2, C_4, C_5)$-free, $G$ is a split graph. Hence, $V(G)$ can be partitioned into subsets $Q$ and $I$, where $Q$ is a maximum clique and $I$ is an independent set. Let $Q = \{u_1, u_2, \ldots, u_p\}$. We partition $I$ into subsets $I_1, I_2, \ldots, I_p$, where $I_j = \{v \in I \mid v$ is non-adjacent to $u_j$ and adjacent to $u_1, \ldots, u_{j-1}\}$, for all $j \in [p]$. We define colourings $\alpha$ and $\beta$, from $\gamma_1$ and $\gamma_2$, respectively, as follows:

$$
\alpha(v) = \begin{cases} 
\gamma_1(v_j) & \text{if } v \in I_j \\
\gamma_1(v) & \text{if } v \in Q;
\end{cases} \\
\beta(v) = \begin{cases} 
\gamma_2(v_j) & \text{if } v \in I_j \\
\gamma_2(v) & \text{if } v \in Q;
\end{cases}
$$

Therefore, in both cases, we obtain $\alpha$ and $\beta$ from $\gamma_1$ and $\gamma_2$, respectively, by recolouring a vertex at each step in at most $n$ steps. Since $\alpha$ and $\beta$ induce the same partition of the vertices into colour classes and use $\chi(G)$ colours, by the Renaming Lemma there exists a path between the colourings of length at most $2n$ in $R_\ell(G)$. Hence there exists a path between $\gamma_1$ and $\gamma_2$ in $R_\ell(G)$ of length at most $4n$.

A clique cutset $Q$ in a graph $G$ is called a tight clique cutset if there exists a component $H$ of $G$-Q which is complete to $Q$, and then $H$ is called a tight component.

Lemma 6. Suppose $G$ is a graph with a tight clique cutset $Q$ and let $H$ be a tight component of $G$-Q. Let $V_2 = V(G) \setminus V(H)$. Let $\alpha$ and $\beta$ be two $\ell$-colourings of $G$ such that $\alpha v_2 = \beta v_2$. Suppose $H$ is $\ell_1$-mixing for all $\ell_1 \geq \chi(H) + 1$. Then there exists a path between $\alpha$ and $\beta$ in $R_\ell(H)$.

Proof. Let $V_1 = V(H)$ and $V_2 = V(G) \setminus V(H)$. Since $\ell > \chi(G) \geq \chi(H) + |Q|$ and $\alpha(Q) = \beta(Q)$, there exist at least $\ell_1$ colours that do not appear in $\alpha(Q)$. Let $A$ be the set of all colours that does not appear in $\alpha(Q)$.

Since $H$ is $\ell_1$-mixing, there exists a path between $\alpha_1$, the restriction of $\alpha$ on $V_1$, and $\beta_1$, the restriction of $\beta$ on $V_1$, in $R_{\ell_1}(H)$ such that every colouring in the path uses only colours in $A$. We know that none of the colours in $A$ appear in $\alpha(Q)$, so we can extend every colouring in the path by colouring every vertex $v \in V_2$ with the colour $\alpha(v) = \beta(v)$.

Lemma 7. Suppose $G$ has a tight clique cutset $Q$. If every induced subgraph $F$ of $G$ is $\chi(F) + c$-mixing, for all $c \geq 1$, then $G$ is $\ell$-mixing.
Claim 2. There exists a path between any two $\ell$-colourings $\epsilon$ and $\psi$ of $G$, whose restrictions on $V_2$ are $\epsilon_2$ and $\psi_2$, respectively.

By Lemma 6, it is sufficient to prove that there exists a path between $\epsilon$ and some colouring $\psi$ whose restriction on $V_2$ is $\psi_2$. We may also assume that $\epsilon_1$, the restriction of $\epsilon$ on $V_1$, uses $\chi(H)$ colours.

Since $\epsilon_2$ and $\psi_2$ are adjacent in the path $P$, there is a unique vertex $v \in V_2$ such that $\epsilon_2(v) \neq \psi_2(v)$, if $v \notin Q$ or $\psi(v) \notin \epsilon(V_1)$, we can recolour $v$ in $\epsilon$ with the colour $\psi(v)$ to obtain $\psi$.

Let $v \in Q$ and $\psi(v) \in \epsilon(V_1)$. Since $\ell > \chi(H) + |Q|$, there exists a colour, say $r$, that does not appear in $\epsilon(V_1 \cup Q)$. Starting with $\epsilon$ recolour every vertex in $V_1$ coloured $\psi(v)$ with the colour $r$, then recolour $v$ with the colour $\psi(v)$ to obtain the colouring $\psi$. □

Proof of Theorem 8. Let $G$ be a connected $(P_5, C_4)$-free graph. The proof is by induction on the number of vertices. If there is no induced $C_5$ in $G$, then $G$ is a chordal graph and hence $R_r(G)$ is connected [4]. If there is an induced $C_5$ in $G$, then by the structure of $(P_5, C_4)$-free graphs given in [9] there exists a blowup of $C_5$, say $C$, and a clique $Q$ such that $C$ is complete to $Q$ and $C$ is a component of $G-Q$. Hence the proof follows from Lemma 7. □

Note that there are other classes of graphs that admit a tight clique cutset, for example, chordal graphs.

7 Conclusion

We have proved a dichotomy theorem: The reconfiguration graph of the $(k+1)$-colourings of a graph $G$, $R_{k+1}(G)$, is connected for every $k$-colourable $H$-free graph $G$ if and only if $H$ is an induced subgraph of $P_4$ or $P_3 + P_1$. It is interesting to note that these are exactly the family of $H$-free graphs for which the colouring problem is polynomial-time solvable [10]. Next it would be natural to do try to do the same for $(H_1, H_2)$-free graphs.

One might first consider graphs on four vertices. There are 11 such graphs, however, by the dichotomy result, we do not need to consider $P_4$ or $P_3 + P_1$. Of the remaining 36 pairs of the other 9 graphs on four vertices, the recolouring diameter is infinite for 3 pairs by Theorem 4 and for 10 other pairs by Theorem 5, and finite for $(2K_2, C_4)$-free graphs. Further, it is easy to see that $C_6$ has a frozen 3-colouring. The only graphs on four vertices which are induced subgraphs of $C_6$ are $P_4$, $P_3 + P_1$ and $2K_2$. Thus it is of particular interest to study the connectivity of the reconfiguration graph of the $k$-colourings of $(2K_2, H)$-free graphs, where $H$ is a graph with at least four vertices and is not $P_4$, $P_3 + P_1$ or $C_4$.

Theorem 8 states that the class of $(P_5, C_4)$-free graphs is $\ell$-mixing, however the $\ell$-recolouring diameter is still unknown. It is interesting to find other classes of $(H_1, H_2)$-free graphs which are $\ell$-mixing and the $\ell$-recolouring diameter is linear or polynomial. In [8], Peghali and Merkel asked if there is a $k$-colourable graph $G$ such that $R_{k+1}(G)$ is disconnected but every component of $R_{k+1}(G)$ has at least two vertices. When $k = 2$, L. Cereceda et.al [6] proved that there exist bipartite graphs that are not 3-mixing and do not admit a frozen 3-colouring, for example $C_8$. Furthermore, let $H$ be any graph with $V(H) = \{v_1, v_2, \ldots, v_n\}$ which admits a frozen
ell-colouring, \( \ell \geq 3 \). We can construct a connected graph \( G \) from \( H \) by adding vertices \( u_1, \ldots, u_i \), \( 1 < i \leq n \), and edges \( v_ju_j \) for all \( j \in [i] \). Note that \( \chi(G) = \chi(H) \) and \( G \) contains \( H \) as an induced subgraph, but \( G \) is not \( \ell \)-mixing and does not admit a frozen \( \ell \)-colouring. We ask the following:

**Question 2.** Is there a graph \( G \) which is not \( \ell \)-mixing, \( \ell \geq 4 \), and does not contain an induced subgraph which admits a frozen \( \ell \)-colouring?

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