COMMENT

Comment on: Harmonic oscillator in an environment with a pointlike defect. (2019 Phys. Scr. 94 125301)

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Abstract
We analyze recent results for a harmonic oscillator in an environment with a pointlike defect. We show that the allowed oscillator frequencies predicted by the authors stem from a misinterpretation of the exact solutions of a conditionally solvable eigenvalue equation. Also the exact eigenvalues derived by those authors are meaningless because they belong to different quantum-mechanical models.

In a recent paper Vitória and Belich [1] investigated the topology effects of the medium on a harmonic oscillator and the effects of a Coulomb and linear central potentials on the harmonic oscillator in an environment with a pointlike defect. The Schrödinger equation is separable in spherical coordinates and they solved the eigenvalue equation for the radial part by means of the Frobenius method. Since the coefficients of the expansion satisfy a three-term recurrence relation they could obtain exact polynomial solutions and eigenvalues by truncation of the series. They concluded that the angular frequency of the harmonic oscillator has restricted values determined by the quantum numbers of the system. The purpose of this comment is to analyze the effect of the truncation approach on the results obtained by Vitória and Belich and on the physical conclusions drawn from them.

The eigenvalue equation for the third model discussed by the authors (given by the potential, \( V(r) = k/r + m \omega^2 r^2/2 \)) is

\[
u''(s) + \frac{1}{s} u'(s) - \frac{\gamma^2}{s^2} u(s) - \frac{2}{s} u(s) - \frac{s^2}{s} u(s) + \delta u(s) = 0,
\]

\[
\gamma = \frac{2k}{(\alpha \hbar)^3/2 \sqrt{m/\omega}}, \quad \beta = \frac{2E}{\alpha/\omega}, \quad \epsilon^2 = \frac{4l(l + 1) + \alpha^2}{\alpha^2}, \tag{1}
\]

where \( l = 0, 1, \ldots \) is the angular momentum quantum number, \( m \) the mass of the particle, \( \alpha \) the parameter associated to the pointlike global monopole and \( \mathcal{E} \) the energy. By means of the truncation method that we discuss below the authors obtained an expression for the energy \( \varepsilon_{l,\bar{n}} = \alpha/\omega_{l,\bar{n}}(1 + \bar{n} + |l|) \), where \( \bar{n} = 1, 2, \ldots \) denotes the radial modes and \( \omega_{l,\bar{n}} \) the allowed angular frequency that, according to the authors, depends on the quantum numbers. By straightforward inspection one immediately suspects that something is amiss here because the eigenvalue equation (1) exhibits bound states for all \( -\infty < \gamma < \infty \); therefore, there is no room for such discrete values of \( \omega \). One expects, of course, allowed values of \( \delta \) (or \( \mathcal{E} \)) that is the eigenvalue in this equation.

The fourth example comes from the potential \( V(r) = \eta r + m \omega^2 r^2/2 \) and the eigenvalue equation for the radial part is

\[
u''(s) + \frac{1}{s} u'(s) - \frac{\theta^2}{s^2} u(s) - \theta s u(s) - s^2 u(s) + \delta u(s) = 0,
\]

\[
\theta = \frac{2\eta}{\sqrt{\alpha/m \omega^3}}. \tag{2}
\]

The authors obtained the exact energies \( \varepsilon_{l,\bar{n}} = \alpha/\omega_{l,\bar{n}}(1 + \bar{n} + |l|) - \eta^2/(2m \omega_{l,\bar{n}}) \) in terms of the allowed frequencies \( \omega_{l,\bar{n}} \). Since the eigenvalue equation (2) supports bound states for all \( -\infty < \theta < \infty \) we realize that something is amiss here too.

The potential-energy function for the fifth example is \( V(r) = \eta r + k/r + m \omega^2 r^2/2 \) and the resulting eigenvalue equation reads

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\[
  u''(s) + \frac{1}{s}u'(s) - \frac{s^2}{s^2 - s^2}u(s) - \frac{\gamma}{s} u(s) - \theta s u(s) - s^2 u(s) + \delta u(s) = 0.
\]

Also in this case the authors found allowed values of the oscillator frequency that, as argued above, is unexpected.

Vitória and Belich [1] argued that the solutions to equations (1)–(3) can be expressed in terms of the solutions to a biconfluent Heun equation; however, they never used the mathematical properties of the latter equation explicitly and tried the Frobenius method outlined below.

All the examples discussed by Vitória and Bakke [1] are particular cases of the eigenvalue equation
\[
  u''(x) + \frac{1}{x}u(x) - \frac{\gamma^2}{x^2}u(x) - \frac{a}{x} u(x) - bx u(x) - x^2 u(x) + W u(x) = 0,
\]
where \(\gamma, a \) and \(b \) are real model parameters (notice that \(\gamma \) has nothing to do with the parameter in equations (1) and (3)). This eigenvalue equation has square integrable solutions
\[
  \int_0^\infty |u(x)|^2 x \, dx < \infty,
\]
for all \(-\infty < a, b < \infty\) for an infinite number of allowed values of \(W(a, b)\). Such eigenvalues satisfy the Hellmann-Feynman theorem [2, 3]
\[
  \frac{\partial W}{\partial a} = \left\{ \frac{1}{x} \right\}_{x>0}, \quad \frac{\partial W}{\partial b} = \langle x \rangle > 0.
\]

It is worth noticing that Vítoria and Belich [1] carried out the transformation \(u(r) = \sqrt{r} R(r)\) so that the volume element in spherical coordinates \(r^2 \, dr\) for the radial part of the wavefunction becomes \(r \, dr\) as shown in equation (3).

In what follows we apply the Frobenius method to the eigenvalue equation (4) by means of the ansatz
\[
  u(x) = x^{\gamma} \exp\left(-\frac{b}{2}x - \frac{x^2}{2}\right) P(x), \quad P(x) = \sum_{j=0}^{\infty} c_j x^j, \quad s = |\gamma|.
\]
The expansion coefficients \(c_j\) satisfy the three-term recurrence relation
\[
  c_{j+2} = A_j c_{j+1} + B_j c_j, \quad j = -1, 0, 1, 2, \ldots, c_{-1} = 0, \quad c_0 = 1,
\]
\[
  A_j = \frac{2a + b(2j + 2s + 3)}{2(j + 2)[j + 2(s + 1)]}, \quad B_j = \frac{4(2j + 2s - W + 2) - b^2}{4(j + 2)[j + 2(s + 1)]}.
\]

If the truncation condition \(c_{n+1} = c_{n+2} = c_n = 0, n = 0, 1, \ldots\), has physically acceptable solutions for \(a, b\) and \(W\) then we obtain exact eigenfunctions because \(c_j = 0\) for all \(j > n\). This truncation condition is equivalent to \(B_n = 0, c_{n+1} = 0\) or
\[
  W_s^{(n)} = 2(n + s + 1) - \frac{b^2}{4}, \quad c_{n+1}(a, b) = 0,
\]
where the second condition determines a relationship between the parameters \(a\) and \(b\). On setting \(W = W_s^{(n)}\) the coefficient \(B_j\) takes a simpler form:
\[
  B_j = \frac{2(j - n)}{(j + 2)[j + 2(s + 1)]}.
\]

Notice that the truncation condition does not provide all the solutions but only those for which the parameters \(a\) and \(b\) exhibit certain relations. The reason is that this problem is not exactly solvable, as Vítória and Belich appear to believe, but quasi-exactly solvable or conditionally solvable (see [4–7] and, in particular, the remarkable review [8] and references therein for more details).

As a first example we consider the eigenvalue equation (4) with \(b = 0\) that is defined by the potential \(V(a, x) = a/x + x^2\). In this case \(c_{n+1}(a, 0) = 0\) is a polynomial function of \(a\) of degree \(n + 1\) and it can be proved that all the roots \(a_{n+1}^{(n, i)}, i = 1, 2, \ldots, n + 1\), are real [3, 5]. Besides, \(c_{n+1}(a, 0) = a^2 Q_n(a^2)\), where \(Q_n(a)\) is a polynomial function of \(a^2\) of degree \((n + 1)/2\) for \(n\) even for \(n\) odd, \(Q_n(a)\) is a polynomial function of \(a^2\) of degree \((n + 1)/2\) for \(n\) odd or \((n-1)/2\) for \(n\) even. For convenience we arrange the roots so that \(a_{n+1}^{(n, i)} > a_{n+1}^{(n, i+1)}\) and stress the point that all of them correspond to the same eigenvalue \(W_s^{(n, i)} = W_s^{(n, n)}\). It is important to realize that the eigenvalue \(W_s^{(n)}\) is common to a set of different quantum-mechanical problems given by \(V_s^{(n, i)}(x) = V(a^{(n, i)}, x)\). Part of the authors’ mistakes stem from overlooking this obvious fact. For example, the eigenvalues \(E_{\ell, n}\) obtained by them correspond to different quantum-mechanical problems \(V_s^{(n, i)}(x)\). Another part of the authors’ mistakes comes from the belief that these polynomial solutions are the only square integrable eigenfunctions supported by equation (4).
The actual eigenvalues $W_{n,0}(a)$, $\nu = 0, 1, \ldots$, $W_{n,0} < W_{n+1,0}$ of equation (4) with $b = 0$ are curves in the $a - W$ plane. It follows from the Hellmann-Feynman theorem (6) that $(a^{0,0}_{\nu}, W^{0}_{\nu})$ is a point on the curve $W_{n,0}(a)$. In order to verify this fact we need the actual eigenvalues $W_{n,0}$ that we have to obtain by means of a suitable approximate method because the eigenvalue equation (4) is not exactly solvable [5, 8]. Here, we resort to the well known Rayleigh-Ritz variational method that is known to yield upper bounds to all the eigenvalues [9] and, for simplicity, choose the non-orthogonal basis set of Gaussian functions \[ \phi_{j}(x) = x^{j} \exp \left( -\frac{x^{2}}{2} \right), \quad j = 0, 1, \ldots \].

In order to make the variational calculations simpler we choose $s = 0$ in what follows. Figure 1 shows several eigenvalues $W_{0}^{(n)}$ given by the truncation condition (red points) and the lowest actual eigenvalues $W_{n,0}(a)$ obtained from the variational method (blue lines). We see that there are solutions to the eigenvalue equation (4) for all values of $a$, that each $W_{n,0}(a)$ is a continuous function of $a$ that satisfies the Hellmann-Feynman theorem (6) and that each pair $(a^{0,0}_{\nu}, W^{0}_{\nu})$ is a point on those curves as argued above. This figure also shows the horizontal line (green, dashed) for the mode $n = 10$. Any vertical line starting from a given value of $a$ will pass through no more that one red point. It means that the truncation condition yields only one eigenvalue and just for a particular model potential $V^{(n,0)}(x)$. An exception should be made for the trivial case $a = 0$ (harmonic oscillator) for which the truncation method yields the whole spectrum. This particular case is the only one in which the expansion coefficients satisfy a two-term recurrence relation and the truncation approach is known to produce the actual spectrum of the exactly-solvable quantum-mechanical model [9]. We realize that the radial modes defined by Vitória and Belich have no physical meaning unless one connects the points $(a^{0,0}_{\nu}, W^{0}_{\nu})$ properly. Since these authors were unaware of such connection they drew nonsensical physical conclusions like the existence of allowed oscillator frequencies.

As a second illustrative example we consider the eigenvalue equation (4) with $a = 0$ and denote $V(b, x) = bx + x^{2}$ the model potential. In this case $c_{n+1}(0, b)$ is a polynomial function of $b$ of degree $n + 1$ with roughly the same features discussed above for the preceding example. There are also $n + 1$ real roots $b^{(n,0)}_{i}$, $i = 1, 2, \ldots, n + 1$ that we arrange in the same way: $b^{(n,0)}_{1} > b^{(n,0)}_{2} > \cdots > b^{(n,0)}_{n+1}$. Each of them gives rise to a model with the potential $V^{(n,0)}(x) = V(b^{(n,0)}_{i}, x)$. The main difference with respect to the preceding case is that the eigenvalues $W^{(n,0)}_{j} = 2(n + s + 1) - \frac{(b^{(n,0)}_{i})^{2}}{4}$ lie on an inverted parabola instead of on an horizontal straight line. As in the preceding example $(b^{(n,0)}_{i}, W^{(n,0)}_{j})$ is a point on the curve $W_{n-1,0}(b)$.

Figure 2 shows some of the eigenvalues $W^{(n,0)}_{j}$ given by the truncation condition (red points) and the lowest actual eigenvalues $W_{n,0}(b)$ obtained by means of the variational method with the same Gaussian basis set indicated above (blue lines). As in the preceding case, we appreciate that the true eigenvalues of equation (4) with $a = 0$ are continuous functions of $b$ that satisfy the Hellmann-Feynman theorem (6). The allowed oscillator frequencies conjectured by Vitória and Belich [1] are a consequence of misunderstanding the meaning of the eigenvalues $W^{(n,0)}_{j}$ given by the truncation condition. Since they failed to connect them properly they could not understand that such eigenvalues are just points on the curves $W_{n,0}(b)$ as clearly shown in figure 2 for $s = 0$. Notice that the truncation condition yields at most one eigenvalue and just for a particular model potential $V^{(n,0)}(x)$, exception being made for the trivial case $b = 0$ as argued above. Figure 2 also shows the inverted parabola (green, dashed line) for the mode $n = 15$ that connects the points $(b^{(15,0)}_{0}, W^{(15,0)}_{0})$.

When $a > 0$ and $b > 0$ the true eigenvalues $W_{n,0}(a, b)$ of equation (4) are surfaces in the three dimensional space $(a, b, W)$. For simplicity we choose $b = 1$ so that $W_{n,0}(a, 1)$ is a curve as before. Figure 3 shows eigenvalues $W_{0}^{(n)}$ for $b = 1$ and $s = 0$ (red points) and variational results $W_{n,0}(a, 1)$ (blue lines). If we compare the distribution of the eigenvalues $W_{0}^{(n)}$ for $b = 0$ and $b = 1$, given by the truncation method, we appreciate that in the latter case the symmetry is lost and that the exact results for the harmonic oscillator do not appear. In this
case the truncation method only yields one eigenvalue and just for some model potentials given by particular values of $a$ (the roots of $c_{n+1}(a, 1) = 0$).

**Summarizing:** It follows from present analysis that the eigenvalues $W_{n,0}(a, b)$ of equation (4) are continuous functions of $-\infty < a < \infty$ and $-\infty < b < \infty$; therefore, the allowed or permitted oscillator frequencies $\omega_{n,0}$ were fabricated by Vitória and Belich [1] by means of a wrong interpretation of the meaning of the roots provided by the truncation method. The analytical eigenvalues presented by these authors are meaningless because they correspond to different model potentials. More precisely, the roots of the truncation condition are meaningless unless one arranges and connects them in a suitable way as shown in present figures 1 and 2.

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