Supersymmetric features of the Maxwell fish-eye lens

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ABSTRACT

We provide a supersymmetric analysis of the Maxwell fisheye (MF) wave problem at zero energy. Working in the so-called $R_0 = 0$ sector, we obtain the corresponding superpartner (fermionic) MF effective potential within Witten’s one-dimensional (radial) supersymmetric procedure.

Keywords: fisheye lens, supersymmetry

1. INTRODUCTION

Since it is one of the most symmetric systems in the world, the MF lens has been tackled by many authors. The well-known analogs of the optical MF are the Kepler problem\(^1\) and the hydrogen atom\(^2\),\(^3\). The classical mechanical counterpart of the MF spherical waveguide/lens, \(n(r) = \frac{2R^2}{R^2 + r^2}\), \((r \leq R)\), is the motion of a particle with zero energy (i.e., zero velocity at infinity) in the spherically symmetric potential \(U(r) = -\frac{w}{2R^2(1+(r/R)^2)}\). In this paper we shall consider the MF wave/quantum problem at fixed null energy, (or zero-binding energy), \[-\frac{\hbar^2}{2m} \nabla_r^2 + U_{MF}(r)\] \(\psi(r) = 0\), with \(U_{MF}(r) = -\frac{wE_0}{[1+(r/R)^2]^2}\), where \(w > 0\) is a coupling constant, and \(R > 0\) is the radius of the lens. An energy scale \(E_0 = \hbar^2/2mR^2\) has been introduced for the potential part\(^4\).

Demkov and Ostrovsky included the MF wave problem in a more general type of focusing potentials\(^1\) bearing their name\(^5\) and characterized by a parameter \(\kappa\), which is unity for the fisheye. They have shown that for the cases \(\kappa = k_1/k_2\), with \(k_1\) and \(k_2\) integers, i) the classical trajectories of a zero-energy (i.e., zero velocity at infinity) particle close after \(k_2\) revolutions around the force centre, and ii) all the trajectories passing through a given point come to a focus after \(k_2/2\) revolutions.

The above wave equation can be written in terms of the scaled variable \(\rho = r/R\), (hereafter the energy scale is to be understood), as follows
\[-\frac{\partial^2}{\partial \rho^2} - \frac{2}{\rho} \frac{\partial}{\partial \rho} + \frac{l(l+1)}{\rho^2} - \frac{w}{(1+\rho^2)^2}]\psi(\rho) = 0. \tag{1}\]

It is quite straightforward to solve the Sturm-Liouville problem, Eq. (1). Moreover, it can be turned into an eigenvalue problem for the coupling constant, \(w\), and can also be written as a Laplace equation on the four-dimensional sphere\(^1\). The known results are the following. When \(w\) assumes the quantized values, \(w_n = 4n^2 - 1\), where \(n\) is the MF principal quantum number (see below), the regular, normalizable (“bound”) solutions, decreasing at infinity, read

\[
\psi_{n\ell m}(\rho) = R_{n\ell}(\rho)Y_{\ell m}(\theta, \phi) = \frac{N_{n\ell}}{\rho^{-l}(1+\rho^2)^{(2\ell+1)/2}}C_{n-l-\ell}^{\ell+1}(\xi)Y_{\ell m}(\theta, \phi), \tag{2}\]

where \(\xi = \frac{1-\rho^2}{1+\rho^2}\), \(n = n_r + \ell + 1\), \(n_r = 0, 1, 2, \ldots\), are the MF principal and radial quantum numbers, respectively, \(\ell\) and \(m\) are the spherical harmonic numbers, \(C_p^q(\xi)\) are the Gegenbauer polynomials, i.e., the solutions of the corresponding ultraspherical equation (see Eq. (5) below), and \(N_{n\ell}\) are the normalization constants. What one gets when the \(w\) parameter becomes larger and larger is an increase of the degeneracy of the “bound”, \(E=0\) state, but only for the quantized values \(w_n\). The degree of degeneracy of such a group of states is \(n^2\), \((n = 1, 2, 3, \ldots)\), similar to the electron energy levels in a Coulomb field.

### 2. MF SUPERSYMMETRY IN THE \(R_0\) SECTOR

Since we want to apply the supersymmetric Natanzon-like scheme as discussed for example by Léval\(^6\), we pass to the functions \(u_{n\ell} = \rho R_{n\ell}\) fulfilling the one-dimensional radial equation

\[
[-\frac{\partial^2}{\partial \rho^2} + U_{\text{eff}}(\rho)]u \equiv \left[-\frac{\partial^2}{\partial \rho^2} + \frac{l(l+1)}{\rho^2} - \frac{4n^2 - 1}{(1+\rho^2)^2}\right]u \equiv H^- u = 0, \tag{3}\]

in which we have already included supersymmetric superscripts. The functions \(u_{n\ell}\) are of the type \(f_\ell(\rho)C_{n-\ell-1}^{l+1}(\xi(\rho))\), where \(f_\ell(\rho)\) reads

\[
f_\ell(\rho) = \frac{\rho^{l+1}}{(1+\rho^2)^{(2l+1)/2}}, \tag{4}\]

and the Gegenbauer polynomials, \(C_p^q\), of degree \(p = n_r\) and parameter \(q = \ell + 1\), are the solutions of a second-order differential (ultraspherical) equation of the type

\[
P(\xi) \frac{d^2C}{d\xi^2} + Q_l(\xi) \frac{dC}{d\xi} + R_p(\xi)C(\xi) = 0, \tag{5}\]

with \(P(\xi) = 1\), \(Q_l(\xi) = \frac{(2l+3)\xi}{\xi^2 - 1}\) and \(R_p(\xi) = -\frac{p(2q+p)}{\xi^2 - 1}\), in which we emphasized the indexing of the \(R\)-functions according to the various sectors \(p = n_r (0, 1, 2, \ldots)\), which is a quite general characteristic of orthogonal polynomials\(^6\).

In the Natanzon-like scheme, the following equations can be obtained

\[
\frac{\xi''}{(\xi')^2} + \frac{2f_\ell}{\xi f_\ell} = Q_l(\xi(\rho), \tag{6}\]

and

\[
\frac{f_\ell''}{(\xi')^2} - \frac{U_{\text{eff}}}{(\xi')^2} = R_p(\xi(\rho), \tag{7}\]

where

\[
\xi = \frac{1-\rho^2}{1+\rho^2}. \tag{2}\]
where $U_{eff}^-$ is given in Eq. (3). From Eq. (6) the function $f_l(\rho)$ can be written as follows

$$f_l(\rho) \sim (\xi'(\rho))^{-1/2} \exp\left[\frac{1}{2} \int_{\xi(\rho)}^{\xi(\rho)} Q_l(\xi) d\xi \right].$$

(8)

One can then define the ‘ground state’ by means of the $R_0(0) = 0$ sector, within which the Gegenbauer polynomials are $C_q^0 = 1$ for any $q$. In this simple case, from Eq. (7) one gets

$$U_{eff}^- = -W^2_l(\rho) - \frac{dW_l}{d\rho},$$

(9)

with $W_l(\rho) = -\frac{d}{d\rho} \ln f_l(\rho)$. Eq. (9) is the standard Riccati equation in ordinary (Witten) 1D supersymmetric quantum mechanics. Since we actually know from Eq. (4) the function $f_l(\rho)$ in the MF case (one can check that Eq. (8) leads to the same function), a short calculation gives the MF superpotential

$$W_{MF}(\rho) = \frac{l - 2l + 1}{\rho(1 + \rho^2)}.$$  

(10)

The effective MF superpartner in the $R_0 = 0$ sector is obtained by changing the sign of the derivative in the Riccati equation

$$U_{eff}^+ = + \frac{dW_{MF}(\rho)}{d\rho} + W^2_{MF}(\rho).$$

(11)

Thus,

$$U_{eff}^- = \frac{l(l + 1)}{\rho^2} - \frac{(2l + 1)(2l + 3)}{(1 + \rho^2)^2},$$

(12)

and

$$U_{eff}^+ = \frac{l(l - 1)}{\rho^2} - \frac{(2l + 1)(2l - 3)}{(1 + \rho^2)^2} + \frac{2(2l + 1)}{\rho^2(1 + \rho^2)^2}.$$  

(13)

We have plotted $U_{eff}^-$ and $U_{eff}^+$ for some values of the parameters in Fig. 1. The MF factorization operators $A_{MF} = \frac{d}{d\rho} + W_{MF}$, and $A_{MF}^+ = -\frac{d}{d\rho} + W_{MF}$ can be used to write the MF fermionic equation. From the plot of the MF fermionic potentials one can notice their repulsive nature. In the supersymmetric sense this means the disappearance of the zero-energy ‘ground state’ for the superpartner Hamiltonian. Consequently, the fermionic equation should be written in the continuum

$$H^+ u_1 \equiv A^* A^+ u_1 \equiv (-\frac{d^2}{d\rho^2} + U_{eff}^+) u_1 = k^2 u_1, \quad k \in (0, \infty).$$  

(14)

This equation will be studied elsewhere. Here we remark that in order to get the supersymmetric increment in the effective potential we used only the particular solution of the Riccati equation, Eq. (9). On the other hand, the connection with the Gel’fand-Levitan inverse scattering method requires the general Riccati solution, which we have worked out recently.

In conclusion, we have presented the supersymmetric structure of the $R_0 = 0$ sector of the MF problem. In this way, we were able to introduce the MF fermionic effective potentials. Moreover, we have found numerically their trapping region (see Fig. 1b).

3. ACKNOWLEDGEMENTS

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Fig. 1.

MF superpartners of the $R_0 = 0$ sector in $\mathcal{E}_0$ units: (a), $U_{eff}^-$ for $l=1, 5, 10$, and (b), $U_{eff}^+$ for $l=6, 7, 8$. We have plotted $U_{eff}^+$ in the region of the critical (inflexion) angular number, $l_{cr}$, that we have found numerically to be $l_{cr}=6.876$ for $\rho_{cr}=1.599$. The critical $l$ is the entry point toward a pocket (trapping) region of $U_{eff}^+$ for $l > l_{cr}$. 
This figure "fig1-1.png" is available in "png" format from:

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