We replace trees by multi-indices as an index set of the abstract model space to tackle quasi-linear singular stochastic partial differential equations. We show that this approach is consistent with the postulates of regularity structures when it comes to the structure group, which arises from a Hopf algebra and a comodule.

Our approach, where the dual of the abstract model space naturally embeds into a formal power series algebra, allows to interpret the structure group as a Lie group arising from a Lie algebra consisting of derivations on this power series algebra. These derivations in turn are the infinitesimal generators of two actions on the space of pairs (nonlinearities, functions of space-time mod constants).

We also argue that there exist pre-Lie algebra and Hopf algebra morphisms between our structure and the tree-based one in the cases of branched rough paths (Grossman-Larson, Connes-Kreimer) and of the stochastic heat equation.
1. Introduction

In this article, we connect the regularity structure \((\mathcal{A}, \mathcal{T}, \mathcal{G})\) introduced by the second author, Sauer, Smith and Weber in [27] for a simple class of quasi-linear equations to the general Hopf-algebraic framework formulated by Hairer [16] and later expanded in [6, 10, 4]. The main difference between [27] on the one hand, and the output of the general strategy in [16] applied to this class of equations on the other hand, lies in the effectively smaller abstract model space \(\mathcal{T}\): The basis elements in [27] amount to specific linear combinations of the basis in [16], which is indexed by trees. Trees do not
play any role in the contribution of this paper; thus, the Hopf algebras underlying rough paths [24, 14] as worked out in [15, 18], and regularity structures [16, 6] are not at our disposal. The goal of this paper is to unveil this Hopf structure in the tree-free set-up of [27]; loosely speaking, this amounts to replacing combinatorics by Lie geometry. For an introduction to our framework, we refer to the series of lectures [26] and the notes [22]; for an introduction to classical regularity structures, we refer to the review article [17].

In our approach to the regularity structure \((A, T, G)\), we start from the space of tuples \((a, p)\) of (polynomial) nonlinearities \(a\) and space-time polynomials \(p\), which we think of parameterizing the entire manifold of solutions\(^1\) \(u\). We consider the actions of shift by a space-time vector \(h \in \mathbb{R}^{d+1}\) and of tilt by a space-time polynomial \(q\) on \((a, p)\)-space, where, crucially, the tilt by a constant is encoded as a shift of the (one-dimensional) \(u\)-space because we think of \(p\) as \(p \mod\) constants. We consider the infinitesimal generators of these actions, and pull them back as derivations on the algebra of formal power series \(\mathbb{R}[[z_k, z_n]]\) in the natural coordinates \(\{z_k\}_{k \geq 0}\) and \(\{z_n\}_{n \in \mathbb{N}^{d+1}\setminus\{0\}}\) of \((a, p)\)-space\(^2\), which give rise to an index set of multi-indices. These derivations define a Lie algebra \(L\); the corresponding Lie group coincides with the pointwise dual \(G^*\) of the structure group \(G\). However, we take a completely algebraic route to construct \(G\), which passes via the universal envelope \(U(L)\) of \(L\), and a module structure which identifies \(U(L)\) with a space of endomorphisms of \(T^*\), the algebraic dual of the model space \(T\).

The algebraic construction of the present paper is similar in spirit to the recent work by Bruned and Manchon [8], who construct Hopf algebras starting from a (multi) pre-Lie algebra that encodes grafting of decorated trees, following the general theory developed by Guin and Oudom [28]. Also our Lie structure comes from a natural pre-Lie product on \(L\), which however is not closed (see Subsection 3.8 for details); more recently, and motivated by the present paper, Bruned and Katsetsiadis [7] have interpreted this structure as a post-Lie algebra. Like in [28] we use it to canonically identify the enveloping algebra \(U(L)\) with the symmetric algebra \(S(L)\), which we implement through the choice of a specific basis\(^3\) for \(U(L)\). This basis is crucial for recovering the intertwining property that relates the coproduct and coaction from regularity structures [17, (4.14)].

While in [27] the regularity structure \((A, T, G)\) was introduced for quasi-linear equations of the form

\[
\left(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2}\right)u = a(u)\frac{\partial^2 u}{\partial x_1^2} + \xi,
\]

\(^1\)satisfying the equation up to space-time polynomials
\(^2\)For the sake of clarity, we will fix \(d = 1\), though no fundamental changes appear when increasing the spatial dimension.
\(^3\)This basis is different from the standard basis used in the Poincaré-Birkhoff-Witt theorem, which relies on a non-canonical ordering of the index set of \(L\).
the structures defined in this paper cover other (semi-linear) equations relying on a single scalar nonlinearity $a(u)$. In fact, the algebraic structure we build is oblivious to the form of the equation and just relies on the solution to the linearized problem being of positive regularity, as will become apparent in Section 6 and, more importantly, Section 7.\footnote{Our structure would also work, for example, for a generalized KPZ equation with only one nonlinearity, i.e.} In Section 6 we consider a driven ODE of the form
\begin{equation}
\frac{du}{dx_2} = a(u)\xi,
\end{equation}
provide a dictionary $\phi$ between our index set of multi-indices and linear combinations of trees in the Connes-Kreimer Hopf algebra (which is at the basis of branched rough paths), and prove that $\phi$ generates a Hopf algebra morphism. Section 7 is devoted to the stochastic heat equation (SHE)
\begin{equation}
\left(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2}\right)u = a(u)\xi,
\end{equation}
where now the morphism property is established with respect to the Hopf algebra in regularity structures [17]. While the morphism $\phi$ between our model space and the one based on decorated trees changes from one equation to another, the consistency between our geometric definition and the combinatorial definitions persists, and we expect it to hold as well for the class (1.1).

Working with our more parsimonious regularity structure $(A, T, G)$ and model $(\Pi_x, \Gamma_{xy})$ has the potential advantage of reducing the number of counter-terms in renormalization. In joint work with P. Tsatsoulis [23], we show that algebraic renormalization of (1.1) combines well with our greedier setting: we show that under a natural symmetry condition on the noise $\xi$, a BPHZ-type choice of renormalization can be performed, leading to a renormalized equation of the form $\left(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2}\right)u = a(u)\frac{\partial u}{\partial x_1} + h(u) + \xi$ with a deterministic and local counter-term $h$, as postulated in [27, Theorem 1], and – most importantly – we show that the greedy model $(\Pi_x, \Gamma_{xy})$ can be naturally estimated without resorting to trees. We believe this to be a general principle, namely that multi-indices provide a more efficient bookkeeping of the renormalization constants; in Subsection 6.6 we show this for (1.2), connecting to translation of rough paths [5].

Our Ansatz has two invariances built-in, with beneficial effects for renormalization. The first invariance is the independence on the choice of an origin in $u$-space, which is ensured by the prominent role of the infinitesimal generator of $u$-shifts. The second invariance relates to the more specific class of quasi-linear equations (1.1). Namely, our theory is not affected by interpreting (1.1) as a perturbation around $\left(\frac{\partial}{\partial x_2} - 2\frac{\partial^2}{\partial x_1^2}\right)u = \xi$, i.e. by
\begin{equation}
\left(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2}\right)u = a(u)\left(\frac{\partial}{\partial x_1}u\right)^2 + \xi.
\end{equation}
rewriting (1.1) as \((\frac{\partial^2}{\partial x_2^2} - 2\frac{\partial^2}{\partial x_1^2})u = (a(u) - 1)\frac{\partial^2}{\partial x_1^2}u + \xi\). In other words, our approach is invariant under changing the reference value in \(a\)-space. Insisting on such (collective) in- or rather covariances in renormalization has been implemented in a much broader way in [21].

2. Motivation and interpretation of the main result

2.1. Modding out constants.

We take the perspective Butcher [9] introduced on the level of ODEs, and which was extended in [15] to driven ODEs of the form\(^\text{6}\) (1.2), of viewing the solution of the homogeneous initial value problem, i.e. with \(u(x_2 = 0) = 0\), as a function(al) of the nonlinearity \(a\), i.e. \(u = u[a](x_2)\). Obviously, the solution \(\tilde{u}\) for an (inhomogeneous) initial datum \(u_0\) can then be recovered by a \(u\)-shift:

\[
\tilde{u} = u[a(\cdot + u_0)] + u_0. \tag{2.1}
\]

In particular, re-centering in the sense of imposing homogeneous initial conditions at some other time instance, say \(u_1(x_2 = 1) = 0\), can be recovered by a suitable variable \(u\)-shift \(\pi = \pi[a]\) in the form of the Ansatz \(u_1[a] = u[a(\cdot + \pi[a])] + \pi[a]\).

The extension to a driven PDE, e.g. (1.3), is more subtle, since even for fixed \(a\), the solution manifold is infinite-dimensional. Relaxing the equation to hold only modulo space-time polynomials, one expects that the solution manifold can be (locally) parameterized by all space-time polynomials \(p\). It is therefore natural to think in terms of \(u = u[a, p](x)\), like is implicitly done in\(^\text{7}\) [4, p.879]. However, this is an over-parameterization in the sense that it does not take advantage of \(u\)-shifts, cf. (2.1). A key feature of our approach, which will be spelled out in the upcoming Subsections 2.2 and 2.3 is to consider \(p\) only modulo constants (and to keep track of \(u\) only modulo constants). In Subsection 3.4 we argue that this greedy approach to the regularity structure is actually truthful.

2.2. The \((a, p)\)-space.

At the basis of our construction is the space \(\mathbb{R}[u] \times (\mathbb{R}[x_1, x_2]/\mathbb{R})\), which is the set\(^\text{8}\) of pairs \((a, p)\), where \(a\) is a polynomial in a single variable \(u\), and \(p\) is a polynomial in two variables \((x_1, x_2) = x\). As indicated by the quotient, we consider \(p\)'s only up to additive constants. Note that \(\mathbb{R}[u] \times (\mathbb{R}[x_1, x_2]/\mathbb{R})\) is the direct sum indexed by the disjoint union of \(\mathbb{N}_0\) and \(\mathbb{N}_0^2 \setminus \{0\}\); we often write \(\{k \geq 0\} \cup \{n \neq 0\}\).

We recall that the polynomial \(a\) plays the role of the nonlinearity in case of the quasi-linear class (1.1), its argument \(u\) is a placeholder for the solution. The polynomial \(p\) plays the role of a (local) parameterization of the

---

\(^5\)There, in the absence of a linear structure, the shift of \(u\) and \(a\) not just by a constant, but by a (regular) field is considered.

\(^6\)We consistently denote by \(x_2\) the time-like variable.

\(^7\)There, \(\mathcal{O}\) corresponds to the space of all jets \(p\)'s.

\(^8\)Our approach ignores the linear structure of \(a\)-space, and only appeals to the affine structure of \(p\)-space.
manifold of solutions; the values of \( u \) and \( p \) are thus thought to be in the same space, i.e. the real line, whereas the argument \( x \) of \( p \) is in space-time.

2.3. Actions of shift and tilt.
There are two natural actions on the \((a, p)\)-space \( \mathbb{R}[u] \times (\mathbb{R}[x_1, x_2]/\mathbb{R}) \), which we shall call “shift” and “tilt”. We start by introducing the shift, by which we think of shifts of space \( x_1 \) and time \( x_2 \). We seek an action\(^9\) of the additive group \( \mathbb{R}^2 \ni h \) on \((a, p)\)-space. We (momentarily) identify \( \mathbb{R}[x_1, x_2]/\mathbb{R} \cong \{ p \in \mathbb{R}[x_1, x_2] \mid p(0) = 0 \} \), which in particular allows to define the composition \( a \circ p \in \mathbb{R}[x_1, x_2] \) on \((a, p)\)-space \( \mathbb{R}[u] \times (\mathbb{R}[x_1, x_2]/\mathbb{R}) \). Then for \( h \in \mathbb{R}^2 \), the transformation
\[
(a, p) \mapsto \left( a(\cdot + p(h)), p(\cdot + h) - p(h) \right)
\] (2.3)
is well-defined. The action on the \( p \)-component is such that it corresponds to shift projected onto (2.2). The action on the \( a \)-component is made such that the composition \( a \circ p \) is mapped onto its (unprojected) shift \( x \mapsto (a \circ p)(x + h) \). Thus under the lens of \( a \), this action corresponds to the plain shift by \( h \). It is easy to check that (2.3) is indeed an action. The presence of the composition \( a \circ p \) connects to the Faà di Bruno formula, cf. [13], which expresses composition in terms of coefficients and thus encodes the chain rule. It explicitly enters in the exponential formula (5.16) via (5.17) and (A.1).

We now turn to tilt. By this we momentarily\(^10\) think of an action on \((a, p)\)-space of the polynomial space \( \mathbb{R}[x_1, x_2] \) (now including the constants). It is defined by writing \( \mathbb{R}[x_1, x_2] \ni q = \sum_n \pi^{(n)} x^n \), where\(^11\) \( x^n = x_1^n x_2^n \), and considering
\[
(a, p) \mapsto \left( a(\cdot + Q), p + \sum_{n \neq 0} \pi^{(n)} x^n \right).
\] (2.4)
This treatment of the \( p \)-component ensures that the transformation (2.4) is well-defined in view of (2.2). The treatment of the \( a \)-component is such that the composition \( a \circ p \) is mapped onto \( a \circ (p + q) \) under (2.4). So once more, under the lens of \( a \), this action corresponds to the tilt of \( p \) by \( q \). Note that the addition of a polynomial is involved in centering, by analogy with the addition of constants in the ODE case at the beginning of Subsection 2.1.

2.4. Seeking a representation as algebra endomorphisms.
We are interested in the group \( G \) of transformations on \((a, p)\)-space generated by the two actions (2.3) and (2.4). We seek a representation of \( G \) as a matrix group, i.e. as a subgroup of \( \text{Aut}(T) \) for a suitable linear space \( T \). The natural approach is to lift (2.3) and (2.4) to an action on a space

\(^9\)By action one means that addition in the group \( \mathbb{R}^2 \) is compatible with composition of the transformations of \((a, p)\)-space.

\(^10\)We need an extension later on.

\(^11\)With the implicit understanding that \( n \in \mathbb{N}_0^2 \) if not stated otherwise.
of nonlinear functionals \( \pi \) on \((a, p)\)-space by “pull-back”. Indeed, it is tautological that (2.3) defines an algebra endomorphism \( \Gamma^* \) of the algebra of functions \( \pi \) on \((a, p)\)-space via

\[
\Gamma^* \pi[a, p] = \pi \left[ a \left( \cdot +p(h) \right), p(\cdot + h) - p(h) \right].
\]

Similarly for (2.4)

\[
\Gamma^* \pi[a, p] = \pi \left[ a \left( \cdot +\pi^{(0)}(\cdot) \right), p + \sum_{n \neq 0} \pi^{(n)}(\cdot) x^n \right].
\]

For the moment, the notation \( \Gamma^* \) is just suggestive; it will become meaningful when we identify this object with the dual of an element of \( G \). The same remark applies to the forthcoming \( T^* \) and \( G^* \).

This pull-back also suggests to naturally extend (2.6) from constant tilt \( \pi^{(n)} \in \mathbb{R} \) to variable tilt, meaning that \( \pi^{(n)} \) itself is a function on \((a, p)\)-space:

\[
\Gamma^* \pi[a, p] = \pi \left[ a \left( \cdot +\pi^{(0)}[a, p] \right), p + \sum_{n \neq 0} \pi^{(n)}[a, p] x^n \right].
\]

Note that also (2.5) has this form.

We use the notation \( \pi \) for a generic function on \((a, p)\)-space, since it acts as a placeholder for the model \( \Pi = \Pi[a, p]\), which indeed can be considered as a parameterization of the solution manifold by \( p \) and depending on \( a \) (next to depending on space-time \( x \)).

2.5. **Seeking a group structure.**

Obviously, (2.7) no longer is an action of the additive group of functions \( \{\pi^{(n)}[a, p]\} \); however, it can be interpreted as an action of the monoid given by the (non-Abelian) group operation

\[
\bar{\pi}^{(n)} = \pi^{(n)} + \Gamma^* \pi^{(n)},
\]

in the sense that the corresponding\(^{12}\) \( \bar{\Gamma}^* \) satisfies \( \bar{\Gamma}^* = \Gamma^* \bar{\Gamma}^* \). The argument\(^{13}\) for (2.8) is a straightforward computation from (2.7). The relation \( \{\pi^{(n)}\}_n \mapsto \Gamma^* \) given by (2.7) and the composition rule (2.8) reflect [5, Definition 14] on the level of branched rough paths.

While according to (2.8), the set of \( \Gamma^* \)'s defined through (2.7), where \( \pi^{(n)} \) runs through all functions on \((a, p)\)-space, is closed under composition, there is in general no inverse. For this, we will have to pass to a more restricted space for the \( \pi^{(n)} \)'s. Incidentally, while (2.5) is contained in (2.7) when \( \pi^{(n)} \) runs through all functions on \((a, p)\)-space, this will not be the case for the restricted space.

\(^{12}\)Here, \( \Gamma^* \), \( \Gamma^{**} \) and \( \bar{\Gamma}^* \) are defined through (2.7) by \( \{\pi^{(n)}\}_n \), \( \{\pi^{(n)}\}_n \) and \( \{\bar{\pi}^{(n)}\}_n \), respectively.

\(^{13}\)We will provide a rigorous proof in the context of Proposition 5.1.
2.6. **Seeking a matrix representation.**

A reason for not only restricting the space of \( \pi^{(n)} \)'s but also the one of \( \pi \)'s in (2.7) is that the algebra of all functions on \((a, p)\) is too large for a representation in terms of countably many coordinates. Let us therefore start from the following coordinates on \((a, p)\)-space:

\[
\begin{align*}
    z_k[a, p] &= \frac{1}{k!} \frac{d^k a}{du_k^k}(0), \quad k \geq 0 \text{ and } \\
    z_n[a, p] &= \frac{1}{n!} \frac{d^n p}{dx_n^n}(0), \quad n \neq 0.
\end{align*}
\]

In (2.9) we use the standard abbreviation \( n! = n_1! n_2! \) and \( \frac{d^n}{dx_k^k} = \frac{d^n}{dx_1^1} \frac{d^n}{dx_2^2}. \)

Note that \( \{z_k, z_n\}_{k, n} \) can be considered as the dual basis to the standard monomial basis \( \{u^k, x^n\}_{k, n} \) of \((a, p)\)-space. In particular, these coordinates arbitrarily fix an origin of the affine \( u \)-space and the affine \( x \)-space. The effect of changing these origins is considered in Subsection 3.2. Clearly, every polynomial expression in (2.9) can be identified with a function on \((a, p)\)-space. This allows us to identify the polynomial algebra \( \mathbb{R}[z_k, z_n] \) with a sub-algebra of the algebra of functions on \((a, p)\)-space. Note that \( \mathbb{R}[z_k, z_n] \) is the direct sum over the index set of multi-indices\(^{14}\) \( \gamma \). In particular, the monomials

\[
z^\gamma := \prod_{k \geq 0, n \neq 0} z_k^{(k)} z_n^{(n)}
\]

form a countable basis of \( \mathbb{R}[z_k, z_n] \). We will denote by \( e_k \) and \( e_n \) the multi-indices such that \( z^{e_k} = z_k \) and \( z^{e_n} = z_n \), respectively.

However, \( \mathbb{R}[z_k, z_n] \) is not preserved by the \( \Gamma^* \) defined through (2.6): Taking \( \pi^{(0)} = v \in \mathbb{R} \setminus \{0\} \) and \( \pi^{(1)} = 0 \) for \( n \neq 0 \), and considering the function \( \pi = z_0 \), we have \( \Gamma^* \pi[a, p] = a(v) \). Now \( a(v) \) cannot be expressed as a finite linear combination of \( z \)'s. Actually, it follows from Taylor’s formula that \( a(v) \) can be written as

\[
a(v) = \sum_{k \geq 0} z_k[a] v^k,
\]

so that the function \( \Gamma^* z_0 \) can be identified with a formal power series in the variables (2.9), that is, an element of \( \mathbb{R}[[z_k, z_n]] \).

Hence in coordinates, we a priori only know that (2.7) defines an algebra morphism from \( \mathbb{R}[z_k, z_n] \) into the larger \( \mathbb{R}[[z_k, z_n]] \). Loosing the endomorphism property of course obscures the group structure. We thus seek an extension of the above \( \Gamma^* \)'s to endomorphisms of \( \mathbb{R}[[z_k, z_n]] \). This will require restricting \( \Gamma^* \) to a (linear) subspace \( T^* \) of \( \mathbb{R}[[z_k, z_n]] \), which amounts to the restriction of the space of \( \pi \)'s mentioned at the beginning of this subsection.

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\(^{14}\)This means that \( \gamma \) is a map from the above index set into \( \mathbb{N}_0 \) that is zero for all but finitely many indices.
2.7. Main result.
Our main results are: i) The goals outlined in Subsections 2.4, 2.5, and 2.6 can be achieved, provided we restrict to a suitable subspace $T^* \subset \mathbb{R}[[z_k, z_n]]$ and restrict the admissible $\pi^{(n)}$’s. ii) The objects are dual to a regularity structure. iii) They arise from a natural Hopf algebra structure based on a pre-Lie algebra structure.

**Theorem 2.1.** For arbitrary yet fixed $\alpha > 0$ introduce the homogeneity\(^{15}\) of a multi-index $\gamma$ by

$$|\gamma| := \alpha(|\gamma| + 1) + \sum_{n \neq 0} |n| \gamma(n),$$

where $[\gamma] := \sum_{k \geq 0} k \gamma(k) - \sum_{n \neq 0} \gamma(n),$$

and let\(^6\) $N_0^2 \ni n \mapsto |n| \in \mathbb{R}_+$ be additive and satisfy $|(1, 0)|, |(0, 1)| > 0$. Moreover, introduce the linear subspace of $\mathbb{R}[[z_k, z_n]]$

$$T^* := \{ \pi | \pi_n = 0 \text{ unless } [\gamma] \geq 0 \text{ or } \gamma = e_n \text{ for some } n \neq 0 \}.$$  

i) Suppose the tilt $\{\pi^{(n)}\}_n$ satisfies

$$\pi^{(n)} \in \{ \pi | \pi_n = 0 \text{ unless } [\gamma] \geq 0 \text{ and } |\gamma| > |n| \}.$$  

Then $\Gamma^*$ defined through (2.7) extends to an automorphism of $T^*$, which respects the algebra structure of the ambient $\mathbb{R}[[z_k, z_n]]$, cf. (5.19). The same holds true for the $\Gamma^*$ defined through (2.5). These two types of $\Gamma^*$’s generate a group $G^* \subset \text{End}(T^*)$ consistent with (2.8). As a set, $G^*$ is parameterized by a shift $h \in \mathbb{R}^2$ and a tilt $\{\pi^{(n)}\}_n$ through an exponential formula\(^7\), cf. (5.16).

ii) There exists a linear space $T$ of which $T^*$ is the algebraic dual; for every $\Gamma^* \in G^*$ there exists a $\Gamma \in \text{End}(T)$ of which $\Gamma^*$ is the dual, thereby defining a group $G \subset \text{End}(T)$. Letting $A := (\alpha N_0 + N_0) \setminus \{0\}$, the triple $(A, T, G)$ forms a regularity structure.

iii) Consider the Lie algebra $L \subset \text{End}(T^*)$ spanned by the infinitesimal generators of shift and tilt. Consider its universal enveloping algebra $U(L)$ with its canonical algebra morphism $U(L) \to \text{End}(T^*)$. There exists a non-degenerate pairing between $U(L)$ and a linear space $T^+$ such that the Hopf algebra structure on $U(L)$ defines a Hopf algebra structure on $T^+$. Likewise, the pairing allows to lift the action given through the algebra morphism $U(L) \to \text{End}(T^*)$ to a coaction $\Delta: T \to T^+ \otimes T$. In line with regularity structures, the group $G \subset \text{End}(T)$ then arises from the Hopf algebra structure of $T^+$ together with $\Delta$. The exponential formula arises from choosing a specific basis in $U(L)$, which is based on a pre-Lie algebra structure on $L$.

\(^{15}\)See Subsection 3.9 for a motivation of this expression which is targeted to the application for (1.1).

\(^{16}\)The definition of $|n|$ is related to the scaling of the differential operator. See Subsection 3.5 below.

\(^{17}\)which is distinct from the matrix exponential in $\text{End}(T^*)$
This basis determines the pairing and ensures the intertwining of $\Delta^+$ and $\Delta$ modulo the re-centering maps $\mathcal{J}_n$, cf. (4.49).

2.8. Outline of the paper.

Section 3 introduces and motivates the main objects. More precisely, in Subsection 3.2, we will introduce the infinitesimal generators of shift $\{\partial_i\}_{i=1,2}$ and (constant) tilt $\{D^{(n)}\}_{n \in \mathbb{N}}$ as derivations on the algebra $\mathbb{R}\llbracket z_k, z_n \rrbracket$. In Subsection 3.3, the polynomial sector $\mathcal{T}$ will be defined; in Subsection 3.6, we define the space $\mathcal{T}^* \subset \mathbb{R}\llbracket z_k, z_n \rrbracket$, which turns out to be dual to the abstract model space $\mathcal{T}$. The corresponding mapping properties of $\{\partial_i\}_{i}, \{D^{(n)}\}_{n}$, and their respective transposed versions are characterized.

In Subsection 3.4, we point out that the commutators of $\{\partial_i\}_{i}$ and $\{D^{(n)}\}_{n}$ behave in the same way shift and tilt operators would act on polynomials including the constants. In Subsection 3.7, we extend from constant to variable tilt parameters $\pi^{(n)}$, by introducing the infinitesimal generator $z^\gamma D^{(n)}$ of variable tilt. In Subsection 3.8, we explore the natural pre-Lie algebra structure of the set of generators and a bigrading. In Subsection 3.9, the homogeneity $|\gamma|$ of a multi-index $\gamma$ and thus the set of homogeneities $A$ and the ensuing grading of $\mathcal{T}$ will be introduced. In Subsection 3.10, we define the Lie algebra $\mathcal{L}$ as the subspace of $\text{End}(\mathcal{T}^*)$ spanned by $\{\partial_i\}_{i=1,2}$ and $\{z^\gamma D^{(n)}\}_{|\gamma|>|n|}$.

While Section 3 is mostly about definitions and elementary properties, Section 4 states the main, partially technical, results that require a proof. In Subsection 4.1, we appeal to the general theory of Hopf algebras: We consider the universal enveloping algebra $U(\mathcal{L})$ of the Lie algebra $\mathcal{L}$, which is obtained from the tensor algebra factorized by the ideal generated by the Lie bracket, and which naturally is a Hopf algebra. Moreover, since $\mathcal{L} \subset \text{End}(\mathcal{T}^*)$, there is a canonical algebra morphism $\rho : U(\mathcal{L}) \to \text{End}(\mathcal{T}^*)$ and the concatenation product on $U(\mathcal{L})$ coincides with the composition in $\text{End}(\mathcal{T}^*)$. This action naturally defines a (left) module structure $U(\mathcal{L}) \otimes \mathcal{T}^* \to \mathcal{T}^*$. In Subsection 4.2, the pre-Lie product of Subsection 3.8 is extended to an operation of $\{z^\gamma D^{(n)}\}_{|\gamma|>|n|}$ on $U(\mathcal{L})$, cf. (4.6), which is shown to be consistent with the Hopf algebra structure. This operation will allow us, in Subsection 4.3, to select a basis that is natural, but different from the typical bases considered in the Poincaré-Birkhoff-Witt theorem, cf. (4.15). Such a basis also provides a non-degenerate pairing between $U(\mathcal{L})$ and a space $\mathcal{T}^+$, see (4.42), which is introduced in Subsection 4.5. Under this pairing, the coproduct on $U(\mathcal{L})$ turns into a product on $\mathcal{T}^+$ that allows to identify $\mathcal{T}^+$ with the polynomial algebra in variables indexed by the index set of $\mathcal{L}$, cf. (4.19).

Next, we embark on the more subtle part of the dualization. This heavily relies on finiteness properties stated in (4.35) and (4.41), which in turn are an outcome of extending the bigrading of $\mathcal{L}$ to $U(\mathcal{L})$; this is carried out in Subsection 4.4. In order to obtain these finiteness properties, it is

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18. in the jargon of regularity structures

19. This subsection is logically not needed, but provides a key intuition.
crucial to pass from $\mathbb{R}[z_k, z_n]$ to $T^*$. As a consequence, the action and the product of $U(L)$, in terms of their coordinate representation with respect to our basis, turn into coaction and coproduct, respectively, for the couple $T$ and $T^+$, see Proposition 4.11. More precisely, the action $U(L) \otimes T^* \rightarrow T^*$ gives rise to a coaction $\Delta: T \rightarrow T^+ \otimes T$, and the concatenation product $U(L) \otimes U(L) \rightarrow U(L)$ gives rise to a coproduct $\Delta^+: T^+ \rightarrow T^+ \otimes T^+$. In particular, $T^+$ carries the structure of a (graded connected) Hopf algebra.

In Subsection 4.6 we argue that $\Delta$ and $\Delta^+$ intertwine as postulated by regularity structures.

Section 5 deals with the group structure and connects to the goals of Theorem 2.1. In Subsection 5.1, we apply general Hopf algebra theory to $T^+$. This allows to endow the space of multiplicative linear forms $\text{Alg}(T^+, \mathbb{R}) \subset (T^+)^*$ with a group structure, with help of the (convolution) product coming from the coproduct $\Delta^+$. Together with the coaction $\Delta$, this gives rise to our $G \subset \text{End}(T)$, establishing part $iii)$ of Theorem 2.1. We also state that $G$ is consistent with the requirements of regularity structures with respect to the gradedness of $T$, cf. (5.10), and the polynomial sector $\tilde{T}$, cf. (5.11). This concludes part $ii)$ of Theorem 2.1. In Subsection 5.2, we connect back to Section 2 by establishing part $i)$ of Theorem 2.1. Namely, we show that the $\Gamma^*$’s extend the definition (2.7) (Proposition 5.1 $ii)$), that they respect the algebra structure of $\mathbb{R}[z_k, z_n]$ (Proposition 5.1 $v)$), and that they respect the group structure (2.8) (Proposition 5.1 $vi)$).

In Subsection 5.3 we enlarge our regularity structure to meet exactly Hairer’s axioms, showing that our smaller structure contains all the relevant information.

At this stage, the reader may wonder why the passage from the Lie algebra $L \subset \text{End}(T^*)$ to the corresponding group $G^* \subset \text{Aut}(T^*)$, which both live on the dual side, has to pass via the primal side in form of $T$ and $T^+$. The reason resides in our purely Hopf-algebraic approach, which prevents us from appealing to the matrix exponential in $\text{End}(T^*)$ that analytically links the Lie algebra $L$ to its (Lie) group $G^*$, even if, as in our case, the exponential sum is effectively finite because of gradedness. Indeed, the universal enveloping algebra $U(L)$, as finite linear combinations of products of elements of $L$, is obviously too small to contain matrix exponentials of elements of $L$, even if $T^+$ were finite dimensional. First passing to $T^+$, which as a linear space is isomorphic to $U(L)$, and then to its algebraic dual $(T^+)^*$, which as a linear space is much larger than $U(L)$, is an algebraic way of extending $U(L)$. It turns out to contain the matrix exponentials of $\{D^\dagger \mid D \in L \subset \text{End}(T^*)\} \subset \text{End}(T)$, namely in form of $\text{Alg}(T^+, \mathbb{R}) \subset (T^+)^*$ as seen through the coaction $\Delta$. Note that the primal $G$ is more valuable than its dual $G^*$, since one may always pass from $\Gamma \in \text{End}(T)$ to $\Gamma^* \in \text{End}(T^*)$, while the opposite is only possible in finite dimensions.

---

$^{20}$ not just up to the constant part, and including an abstract integration map
$^{21}$ meaning that it is finite for a given matrix element
$^{22}$ The elements $D^\dagger$ are well defined by (3.3), since we have the stronger (3.42).
Sections 6 and 7 are logically independent of the rest of the paper, but connect the combinatorial structures to our Lie-geometric construction. More precisely, we make this connection in the well-studied cases of branched rough paths (1.2) and the stochastic heat equation (1.3).

In Section 6, we consider the driven ODE example (1.2). Here at a fixed $a$ the solution manifold is parameterized by $\mathbb{R}$, the space of initial conditions. This allows us to restrict the $(a, p)$-space to the space of nonlinearities $a$, hence the index set to multi-indices $\gamma$ over $\{k \geq 0\}$, and thus the Lie algebra to $\{z^\gamma D^{(0)}\}$. We show that the construction of Sections 3 and 4 is compatible with the Connes-Kreimer Hopf algebra underlying branched rough paths [15, 18]. More specifically, we associate our multi-indices (2.10) with linear combinations of (undecorated) trees in the Connes-Kreimer framework via a map $\phi$, and show that it is a Hopf algebra morphism. To do so, Lemma 6.2 establishes a pre-Lie algebra morphism property of the transpose of $\phi$ with respect to the grafting pre-Lie product (after a suitable normalization), which is at the core of the construction of Connes and Kreimer [12, 20]. This morphism property is shown to be related to the one involving the map $\Upsilon$ in [2] in Subsection 6.5. In addition, in Subsection 6.6 we discuss how renormalization fits our setting and show that the transpose of $\phi$ intertwines with the translation maps of rough paths defined in [5], cf. Lemma 6.5.

In Section 7, we connect to tree-based regularity structures [16, 17, 4] for SHE. This example is simple in the sense that it only involves one integration kernel so that no edge decorations appear in the tree-based approach. We adopt the two perspectives in [4] when it comes to the treatment of the polynomial decorations. In Subsections 7.1 to 7.3, we consider a detailed description of the polynomials, in line with the space $\mathcal{B}$ in [4, Subsection 4.1], and show that our Lie algebra $L$ reflects the grafting operations in [4, Definition 4.7]; as in Section 6, a pre-Lie morphism property of our dictionary is shown, connecting to the morphism properties of $\Upsilon$ in [4]. In Subsection 7.4, we relate to the coarser description which contracts all polynomials by multiplication giving rise to the model space $T_H$; the morphism $\phi$ between $T$ and $T_H$ will no longer be one-to-one. In Proposition 7.5 we establish that $\phi$ induces a morphism $\Phi$ between $T^\dagger$ and the Hopf algebra $T_H^\dagger$ in [17] (without appealing to a pre-Lie structure).

3. The Lie algebra structure

3.1. Duality and transposition $\dagger$.
Recall that the monomials $z^\gamma$, defined in (2.10), can also be considered as elements of $\mathbb{R}[z_k, z_n]$ and that $\mathbb{R}[z_k, z_n]$ is the direct product over the index set of multi-indices $\gamma$. Denoting the direct sum over the same index set by $\mathbb{R}[[z_k, z_n]]$ and its basis elements – to which the monomials of

\footnote{which is isomorphic to $\mathbb{R}[z_k, z_n]$, but we choose a different notation to distinguish $\mathbb{R}[z_k, z_n]$ as a subspace of the space of formal power series from $\mathbb{R}[[z_k, z_n]]^\dagger$ as the pre-dual of the space of formal power series.}
\( \mathbb{R}[z_k, z_n] \) are dual - by \( z_\beta \), we have \( (\mathbb{R}[z_k, z_n])^* = \mathbb{R}[z_k, z_n] \) with the canonical pairing
\[
\langle z_\gamma, z_\beta \rangle = \delta_{\gamma, \beta}.
\]
For \( D \in \text{End}(\mathbb{R}[z_k, z_n]) \) we may consider the components of the sequence \( Dz^\gamma \) as a matrix representation \( \{D^\gamma_\beta\}_{\beta, \gamma} \). Since the \( D_\beta \)'s we construct below will have the finiteness property
\[
\{ \beta \mid D^\gamma_\beta \neq 0 \}
\]
is finite for all \( \gamma \), we may write \( Dz^\gamma = \sum_\beta D^\gamma_\beta z_\beta \). Moreover, they also will have the dual finiteness property
\[
\{ \gamma \mid D^\gamma_\beta \neq 0 \}
\]
is finite for all \( \beta \).

This second finiteness property is just the one needed to have a unique \( D^\dagger \in \text{End}(\mathbb{R}[z_k, z_n]^*) \) such that \((D^\dagger)^* = D\); on the level of the matrix representation this just means
\[
D^\dagger z_\beta = \sum_\gamma D^\gamma_\beta z_\gamma.
\]

Passing from the polynomial space \( \mathbb{R}[z_k, z_n] \) to the formal power series space \( \mathbb{R}[\![z_k, z_n]\!] \) serves us well in the actual application, cf. [23, 22], where we think of (2.9) as coordinates on the manifold of all solutions \( u \). As a consequence, the general solution \( u \) can be seen as a function of the variables \( z_k, z_n \) (with values in the space of functions of \( x \)), or rather as a formal power series in these variables, described by its coefficients \( \Pi_\beta \) indexed by our multi-indices \( \beta \). Formally, \( \Pi_\beta \) is, up to the combinatorial factor \( \beta! \), a partial derivative of the general solution \( u \) w. r. t. the above variables. By Leibniz’ rule, (1.1) gives rise to the following family of linear equations indexed by \( \beta \):
\[
\left( \frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2} \right) \Pi_\beta = \sum_{k \geq 0} \sum_{\epsilon_k + \beta_1 + \cdots + \beta_{k+1} = \beta} \Pi_{\beta_1} \cdots \Pi_{\beta_k} \frac{\partial^2}{\partial x_1^2} \Pi_{\beta_{k+1}} + \delta_{\beta}^0 \xi.
\]

It turns out that this \( \{\Pi_\beta\}_\beta \) can be inductively rigorously constructed\(^\text{24}\), and thus naturally gives rise to the formal power series \( \Pi(x) \in \mathbb{R}[\![z_k, z_n]\!] \). In particular, the algebra structure of \( \mathbb{R}[\![z_k, z_n]\!] \), which will contain the dual \( T^* \) of the abstract model space \( T \), is inherent to our approach. This dual perspective is consistent with the definition of the model as a distribution with values in \( T^* \) [17, Definition 3.3].

In the solution theory of regularity structures, one considers truncations of this formal power series as approximations of the actual solution; such truncations, in turn, shall be seen as (coherent) modelled distributions, in the language of [16, 4], after the application of the model\(^\text{25}\) \( \Pi \).

\(^{24}\)Here, we think of a smoothed-out noise so that the model is not a distribution but actually a smooth function in \( x \).

\(^{25}\)We invite the reader to check [27], where these objects are used to obtain a priori bounds for (1.1).
3.2. **The infinitesimal generators of shift \{\partial_{i}\}_{i} and constant tilt \{D^{(n)}\}_{n}**

We now come to the definition of those derivations\[D \in \text{End}(\mathbb{R}[z_{k}, z_{n}])\] that form the building blocks for the Lie algebra \(L\). The definitions capture the infinitesimal generators of the actions of shift and tilt on \((a, p)\)-space, see (2.3) and (2.4) in Section 2. Let us now introduce the main building blocks \(D^{(0)}, \{D^{(n)}\}_{n \neq 0}\) and \(\{\partial_{i}\}_{i=1, 2}\). We start with \(D^{(0)}\), which is to capture the action of \(\mathbb{R}\) onto \((a, p)\)-space by tilt by constants, which in view of (2.4) amounts to a shift of \(u\)-space

\[(a, p) \mapsto (a(s), p - v)\]

Here, the action on the \(p\)-component, which is made such that \(a \circ p\) stays invariant, is immaterial because we “mod out” constants. As for the shift (2.3) of \(x\), this action lifts by pull-back to functions \(\pi\) of \((a, p)\). We formally define \(D^{(0)}\) to be the infinitesimal generator of this action\[D^{(0)} \pi [a, p] = \frac{d}{dv}|_{v=0} \pi [a(s) + v, p - v].\]

This can be given sense for \(\pi \in \{z_{k}, z_{n}\}\), cf. (2.9), and yields

\[D^{(0)} z_{k} = (k + 1) z_{k+1}, \quad D^{(0)} z_{n} = 0.\]

In addition, (3.7) suggests that \(D^{(0)}\) is a derivation, which we postulate. This and (3.8) yield that, on the space \(\mathbb{R}[z_{k}, z_{n}]\), \(D^{(0)}\) assumes the form

\[D^{(0)} = \sum_{k \geq 0} (k + 1) z_{k+1} \partial_{z_{k}};\]

the sum is obviously effectively finite on \(\mathbb{R}[z_{k}, z_{n}]\). From (3.9) we infer the matrix representation with respect to the monomial basis:

\[(D^{(0)})^{\gamma}_{\beta} = \sum_{k \geq 0} \left\{ \begin{array}{ll}
(k + 1) \gamma(k) & \text{if } \gamma + e_{k+1} = \beta + e_{k} \\
0 & \text{otherwise}
\end{array} \right\}.\]

Note that the finiteness property (3.3) is satisfied so that \(D^{(0)}\) is well defined as a derivation of \(\mathbb{R}[z_{k}, z_{n}]\). We also see that the finiteness property (3.2) holds, so that (3.4) defines an endomorphism \((D^{(0)})^\dagger\) on \(\mathbb{R}[z_{k}, z_{n}]^\dagger\).

After this representation (3.10) of infinitesimal shifts of \(u\), we turn to the shifts of space \(x_{1}\) and time \(x_{2}\), that is, the action (2.3) of \(\mathbb{R}^{2}\) on \((a, p)\)-space. Again, this action extends by pull-back to functions \(\pi\) on \((a, p)\), see (2.5). We formally consider its infinitesimal generators\[\partial_{1} \pi [a, p] = \frac{d}{dy_{1}}|_{y_{1}=0} \pi \left[ a \left( s + p(y_{1}, 0) \right), p \left( s + (y_{1}, 0) \right) - p(y_{1}, 0) \right], \quad \partial_{2} \pi [a, p] = \frac{d}{dy_{2}}|_{y_{2}=0} \pi \left[ a \left( s + p(0, y_{2}) \right), p \left( s + (0, y_{2}) \right) - p(0, y_{2}) \right].\]
By the chain rule and (3.8) for $z_k$, and using the same argument (with $p$ playing the role of $a$) that led to (3.8) for $z_n$, we formally derive

$$\partial_1z_k = z_{(1,0)}D^{(0)}z_k, \quad \partial_1z_n = (n_1 + 1)z_{n+(1,0)},$$

which we now postulate. Together with the postulate that $\partial_1$ be a derivation, this implies that on the sub-algebra $\mathbb{R}[z_k, z_n]$ we have

$$(3.12) \quad \partial_1 = \sum_n (n_1 + 1)z_{n+(1,0)}D^{(n)} \text{ with } D^{(n)} := \partial_{z_n} \text{ for } n \neq 0.$$ 

The notation $D^{(n)} = \partial_{z_n}$ is redundant, but very convenient; we obviously have the matrix representation

$$(3.13) \quad (D^{(n)})_{\gamma \beta} = \begin{cases} \gamma(n) & \text{if } \gamma = \beta + e_n \\ 0 & \text{otherwise} \end{cases} \quad \text{for } n \neq 0.$$ 

Incidentally, still for $n \neq 0$, we have

$$(3.14) \quad D^{(n)}\pi[a, p] = \frac{d}{dt}\big|_{t=0} \pi[a, p + tx^n],$$

which can be given a sense as an endomorphism on both $\mathbb{R}[[z_k, z_n]]$ and $\mathbb{R}[z_k, z_n]$. Inserting (3.10) and (3.13) into (3.12) we obtain the matrix representation

$$(3.15) \quad (\partial_1)_{\gamma \beta} = \sum_{k \geq 0} \begin{cases} (k + 1)\gamma(k) & \text{if } \gamma + e_{k+1} + e_{(1,0)} = \beta + e_k \\ 0 & \text{otherwise} \end{cases} \quad + \sum_{n \neq 0} \begin{cases} (n_1 + 1)\gamma(n) & \text{if } \gamma + e_{n+(1,0)} = \beta + e_n \\ 0 & \text{otherwise} \end{cases};$$

we again learn from (3.15) that $\partial_1$ satisfies the finiteness properties (3.2) and (3.3). Hence $\partial_1$ is also well defined as an element of $\text{End}(\mathbb{R}[[z_k, z_n]])$ and $\partial_1^\dagger$ as an element of $\text{End}(\mathbb{R}[[z_k, z_n]])^\dagger$. The same applies to $\partial_2$, with $(1,0)$ and $n_1$ replaced by $(0,1)$ and $n_2$ in (3.15).

We now have defined the building blocks, which are derivations on $\mathbb{R}[[z_k, z_n]]$

$$(3.16) \quad \{D^{(n)}\}_n \cup \{\partial_1\} \subset \text{Der}(\mathbb{R}[[z_k, z_n]])$$

satisfying the finiteness properties (3.2) and (3.3).

### 3.3. The polynomial sector $\mathcal{T}$.

In view of the second item of (2.9), which identifies the coordinate $z_n = z^n$ of $\mathbb{R}[[z_k, z_n]]$ with the derivative $\frac{d}{dt}$, it is natural to identify the element $z_n \in \mathbb{R}[[z_k, z_n]]^\dagger$ with the polynomial $x^n = x_1^{n_1}x_2^{n_2}$. Hence we identify

$$(3.17) \quad \mathcal{T} := \text{span}\{z_n\}_n \neq 0 \subset \mathbb{R}[[z_k, z_n]]^\dagger$$

with $\mathbb{R}[x_1, x_2] / \mathbb{R}$, the space of polynomials in the variables $x_1, x_2$ quotiented by the constants. Following [17, Assumption 3.20], we call $\mathcal{T}$ the polynomial
sector. We note that the transposed endomorphisms of (3.16) preserve this polynomial sector\textsuperscript{29} $\overline{T}$

\begin{equation}
D^\dagger \overline{T} \subset \overline{T} \quad \text{for} \quad D \in \{D^{(n)}\}_n \cup \{\partial_i\}_i
\end{equation}

which on the level of the matrix representation amounts to

\begin{equation}
D^\gamma = 0 \quad \text{for} \quad \beta \in \{e_n\}_{n \neq 0} \quad \text{and} \quad \gamma \notin \{e_n\}_{n \neq 0}.
\end{equation}

and can be inferred from (3.10), (3.13), and (3.15). We note that $\partial^\dagger_1, \partial^\dagger_2$ almost act as partial derivatives on the polynomial sector\textsuperscript{30} $\overline{T}$, which (in case of $\partial^\dagger_1$) means

\begin{equation}
\partial^\dagger_1 x^n = \begin{cases} n_1 x^{n-(1,0)} & \text{if } n > (1,0), \\ 0 & \text{otherwise} \end{cases}
\end{equation}

for $n \neq 0$.

In terms of the matrix representation, this means

\begin{equation}
(\partial_1)^\gamma e_n = \begin{cases} n_1 & \text{if } \gamma = e_n-(1,0), \ n > (1,0), \\ 0 & \text{otherwise} \end{cases}
\end{equation}

which in turn can be read off from (3.15). The reason why the case $n = (1,0)$ (and analogously for $\partial_2$ the case $n = (0,1)$) is excluded is that we modded out constants in the polynomial $p$, cf. (2.2); see however the upcoming Subsection 3.4.

3.4. **Commutators of $\{D^{(n)}\}_n$ and $\{\partial_i\}_i$ behave naturally.**

We now make a connection between $\{D^{(n)}\}_n \cup \{\partial_i\}_i$ and the classical Lie algebra of tilt and shift on polynomials. We start noting that

\begin{equation}
[D^{(n)}, D^{(n')}]= 0,
\end{equation}

which is obvious in case of $n \neq 0$ and $n' \neq 0$, and can be easily inferred from (3.9) for $n \neq 0$ and $n' = 0$. We next argue that (3.11) implies

\begin{equation}
[\partial_1, \partial_2] = 0.
\end{equation}

Indeed, by the finiteness property (3.2), the monomial $z^\gamma$ is mapped by $\partial_1, \partial_2$ onto finite linear combinations of monomials. Hence we may indeed appeal to (3.11) when computing $(\partial_1 \partial_2 - \partial_2 \partial_1)z^\gamma$, which shows that this expression vanishes by the symmetry of second derivatives. Turning to the commutator between $D^{(n)}$’s and $\partial_i$’s, we first observe that by the characterization (3.12) and the commutation relation (3.21) we have

\begin{equation}
[D^{(0)}, \partial_1] = 0
\end{equation}

by the second item in (3.8). Likewise, for $n \neq 0$, we have

\begin{equation}
[D^{(n)}, \partial_1] = n_1 D^{(n-1,0)}
\end{equation}

(with the understanding that this expression vanishes if $n_1 = 0$) by the second item in (3.12). We retain that

\begin{equation}
[D^{(n)}, \partial_1] = n_1 D^{(n-(1,0))} \quad \text{and} \quad [D^{(n)}, \partial_2] = n_2 D^{(n-(0,1))}
\end{equation}

for all $n$.\textsuperscript{31}

\textsuperscript{29}which is the Lie algebra version of the corresponding postulate on $\Gamma \in G$ in [17, Assumption 3.20]

\textsuperscript{30}which is the Lie algebra version of the corresponding Lie group postulate in [17, Assumption 3.20]

\textsuperscript{31}where $m < n$ means $m \leq n$ component-wise and $m \neq n$
The identities (3.21), (3.22), and (3.23) mean that the derivations $\partial_1$, $\partial_2$, $\{D^{(n)}\}_n$ on $\mathbb{R}[z_k, z_n]$, when it comes to their commutators, precisely behave like certain endomorphisms on $\mathbb{R}^2$. The important fact here is that this is the full space $\mathbb{R}^2$, not just the space $\mathbb{R}^2/\mathbb{R}$ with the constants factored out, which was our starting point in Section 2. The corresponding endomorphisms on $\mathbb{R}^2$ are just the infinitesimal generators of shift and tilt. In particular, the subtle $D^{(0)}$ has a simple analogue in the infinitesimal generator of the “tilt” by a constant polynomial. This shows that the incorporation of constants into the $a$-part in (2.3) and (2.4) did not lead to a loss of information.

3.5. Triangular structure.

We now point out that the building blocks $\{D^{(n)}\}_n \cup \{\partial_i\}_i$, are strictly triangular with respect to the following two additive functionals on multi-indices $\gamma$

\[
(3.24) \quad [\gamma] := \sum_{k \geq 0} k\gamma(k) - \sum_{n \neq 0} \gamma(n) \quad \text{and} \quad \sum_{n \neq 0} |n| \gamma(n).
\]

Here $\mathbb{N}_0^2 \ni n \mapsto |n| \in \mathbb{N}_0$ denotes a scaled length of $n$. In applications to parabolic equations of the form (1.1), one considers $|n| = n_1 + 2n_2$; in general, $|n|$ is an additive and coercive map which is determined by the scaling of the differential operator. We note that the combination of $\sum_{k \geq 0} k\gamma(k)$ and $\sum_{n \neq 0} \gamma(n)$ in (3.24) is natural: Like we identified $z_m$ with $p(x) = x^m$ at the beginning of Subsection 3.3, we may identify $z_m$ with $a(u) = u^k$; hence while $\sum_{k \geq 0} k\gamma(k)$ measures the homogeneity in the $u$-variable, $\sum_{n \neq 0} \gamma(n)$ measures the homogeneity in the polynomial $p$; $u$ and $p$-values have the same “physical” dimension. Considering the difference $[\gamma]$ is forced upon us by the following.

**Lemma 3.1.** It holds for $n \neq 0$

\[
(D^{(0)})_\beta \neq 0
\]

\[
(3.25) \quad \implies \left\{ \begin{array}{l}
[\gamma] + 1 = [\beta] \quad \text{and} \\
\sum_{m \neq 0} |m| \gamma(m) = \sum_{m \neq 0} |m| \beta(m)
\end{array} \right\},
\]

\[
(D^{(n)})_\beta \neq 0
\]

\[
(3.26) \quad \implies \left\{ \begin{array}{l}
[\gamma] + 1 = [\beta] \quad \text{and} \\
\sum_{m \neq 0} |m| \gamma(m) = \sum_{m \neq 0} |m| \beta(m) + |n|
\end{array} \right\}
\]

\[
(\partial_i)_\beta \neq 0
\]

\[
(3.27) \quad \implies \left\{ \begin{array}{l}
\sum_{m \neq 0} |m| \gamma(m) + |(1,0)| = [\beta] \quad \text{and} \\
\sum_{m \neq 0} |m| \beta(m)
\end{array} \right\}
\]

Note that (3.25) and (3.27) are of similar character: If the matrix element does not vanish, both functionals (3.24) are ordered and one of them is strictly ordered. However, (3.26) is of a different character, and we will get back to this in (3.43).
or rather its pointwise dual $G^\ast$. Start with a motivation: The purpose of the structure group $G$ see [17, Definition 3.3]. This re-centering involves subtracting a Taylor $T$ of the

After introducing the building blocks (3.16), we now specify the full collection of derivations on $\mathfrak{L}$ and follows from (3.25), (3.26), and (3.27).

We define $T^\ast$ to be the direct product over the multi-indices $\gamma$ with

$$\gamma \geq 0 \quad \text{or} \quad \gamma \in \{e_n\}_{n \neq 0}.$$  

This restriction of $\mathbb{R}\llbracket z_k, z_n \rrbracket$ to $T^\ast$ is motivated by the fact that the model component $\Pi_\gamma$ is only non-vanishing when $[\gamma] \geq 0$ or $\gamma \in \{e_n\}_{n \neq 0}$. This can be read off from (3.5); the same holds for (1.2) and (1.3), see (6.3) and (7.2), respectively. We denote by $\tilde{T}^\ast$ the subspace of elements of the dual space $T^\ast$ that vanish on the space $\tilde{T}$ introduced in (3.17); in particular $\tilde{T}^\ast$ is the direct product over the multi-indices satisfying $[\gamma] \geq 0$. Then $T^\ast = \tilde{T}^\ast \oplus \tilde{T}^*$, where $\tilde{T}^\ast$ is the direct product over the multi-indices $\gamma \in \{e_n\}_{n \neq 0}$. Thus we can identify the model space $T$ with the direct sum of the polynomial sector $\tilde{T}$ introduced in (3.17) and the space $\tilde{T}$ spanned by all monomials $z_\gamma$ with $[\gamma] \geq 0$. Since $[\gamma] \geq 0$ is closed under addition of multi-indices, and thus under multiplication, $\tilde{T}^\ast$ is a sub-algebra. We note that the derivations (3.16) map $\tilde{T}^\ast$ into $\tilde{T}^\ast$:

$$(3.29) \quad D\tilde{T}^\ast \subset \tilde{T}^\ast \quad \text{for} \quad D \in \{D^{(n)}\}_n \cup \{\partial_i\}_i,$$

which on the level of the coordinate representation means

$$(3.30) \quad D^\gamma_\beta = 0 \quad \text{for} \quad [\beta] < 0 \quad \text{and} \quad [\gamma] \geq 0,$$

and follows from (3.25), (3.26), and (3.27).

### 3.6. The abstract model space $T^\ast$

Now is a good moment to introduce the model space $^32 T^\ast$ and its dual $T^\ast$. We define $T^\ast \subset \mathbb{R}\llbracket z_k, z_n \rrbracket$ to be the direct product over the multi-indices $\gamma$ with

The infinitesimal generators of variable tilt $\{z^\gamma D^{(n)}\}_{\gamma, n}$.

After introducing the building blocks (3.16), we now specify the full collection of derivations on $\mathbb{R}\llbracket z_k, z_n \rrbracket$ that will act as the basis of $\mathfrak{L}$. Again, we start with a motivation: The purpose of the structure group $G \subset \text{End}(T^\ast)$, or rather its pointwise dual $G^\ast \subset \text{End}(T^\ast)$, is to provide the transformations of the $T^\ast$-valued model $\Pi_\gamma$ when passing from one base-point $x$ to another, see [17, Definition 3.3]. This re-centering involves subtracting a Taylor $\tilde{T}^\ast$.
polynomial. Denoting the coefficients of such a polynomial by \( \{ \pi^{(n)} \}_n \), and treating the constant part (i.e. the part with \( n = 0 \)) differently in line with (3.6) and (2.3), this corresponds to the action (2.4).

In the inductive construction of the \( \hat{T}^* \)-valued centered model \( \Pi_x \), the coefficients \( \pi^{(n)} \) depend on the \( \hat{T}^* \)-valued \( \Pi_x \) itself, which for us means \( \pi^{(n)} \in \hat{T}^* \subset \mathbb{R}[\{ z_k, z_n \}] \). We pass from \( \pi^{(n)} \in \hat{T}^* \) to a finite linear combination\(^{34}\) of monomials \( z^\gamma \) with \( [\gamma] \geq 0 \). Hence on an infinitesimal level, in view of the characterization (3.7) of \( D^{(0)} \) and the definition (3.12) of \( D^{(n)} \) for \( n \neq 0 \), transformations of the type (2.4) give rise to the derivations

\[
(3.31) \quad z^\gamma D^{(n)} \quad \text{for} \quad [\gamma] \geq 0 \quad \text{and} \quad n.
\]

Since \( \hat{T}^* \) is closed under multiplication, it follows that (3.29) is preserved:

\[
(3.32) \quad D\hat{T}^* \subset \hat{T}^* \quad \text{for} \quad D \in \{ z^\gamma D^{(n)} \}_{[\gamma] \geq 0, n} \cup \{ \partial_i \}_i.
\]

However, even for \( [\gamma] \geq 0 \), multiplication with \( z^\gamma \) does not map \( \hat{T}^* \) into \( T^* \). Luckily, the composition \( z^\gamma D^{(n)} \) does; we have

\[
(3.33) \quad DT^* \subset T^* \quad \text{for} \quad D \in \{ z^\gamma D^{(n)} \}_{[\gamma] \geq 0, n} \cup \{ \partial_i \}_i.
\]

Indeed, this is an immediate consequence of (3.9) and (3.12) together with (3.32).

On the level of the matrix representation,

\[
(3.34) \quad (z^\gamma D^{(n)})^\gamma' = (D^{(n)})^\gamma'_{\gamma'-\gamma};
\]

this implies that for all these operators, and not just for \( \partial_1 \) and \( \partial_2 \), the finiteness property (3.3) holds. Moreover, when passing from \( D = D^{(n)} \) to \( D = z^\gamma D^{(n)} \), (3.19) is preserved so that (3.18) can be upgraded to

\[
(3.35) \quad D\hat{T}^* \subset \hat{T}^* \quad \text{for} \quad D \in \{ z^\gamma D^{(n)} \}_{[\gamma] \geq 0, n} \cup \{ \partial_i \}_i.
\]

### 3.8. A pre-Lie structure \( \triangleright \) and bigrading bi.

The Lie algebra \( \text{Der}(\mathbb{R}[\{ z_k, z_n \}]) \) of derivations on the algebra \( \mathbb{R}[\{ z_k, z_n \}] \) can be seen as the space of vector fields on the linear span of \( \{ z_k, z_n \} \). Since \( \mathbb{R}[\{ z_k, z_n \}] \) as an affine space is flat, the Lie bracket \( [\cdot , \cdot] \) arises from the pre-Lie product \( \triangleright \) that is given by the covariant derivative of one vector field along another vector field, see e.g. [25]; the relation between the bracket and the product is given by \( [D, D'] = D \triangleright D' - D' \triangleright D \). In case of our derivations we find for arbitrary \( D \in \text{Der}(\mathbb{R}[\{ z_k, z_n \}]) \)

\[
(3.36) \quad D \triangleright z^\gamma D^{(n)} = (Dz^\gamma) D^{(n)} \quad \text{and} \quad z^\gamma D^{(n)} \triangleright \partial_1 = n_1 z^\gamma D^{(n-\{1,0\})}
\]

and an analogous formula with \( \partial_1 \) replaced by \( \partial_2 \). However, \( \partial_1 \triangleright \partial_1 \) cannot be expressed in terms of a linear combination of \( \{ \partial_i \}_i \cup \{ z^\gamma D^{(n)} \}_{\gamma, n} \), so that the span of the latter is not closed under \( \triangleright \). Note that it is not possible to fix this by postulating \( \partial_1 \triangleright \partial_1 = 0 \), since then the (left) pre-Lie identity is not satisfied.\(^{35}\) Nevertheless, it follows from (3.36) and (3.22) that the span

\(^{34}\)We will free ourselves from this restriction later.

\(^{35}\)A simple counterexample is \((z^\gamma D^{(n)} \triangleright \partial_1) \triangleright \partial_1 - z^\gamma D^{(n)} \triangleright (\partial_1 \triangleright \partial_1) = n_1(n_1 - 1)z^\gamma D^{(n-\{2,0\})} \), whereas \((\partial_1 \triangleright z^\gamma D^{(n)}) \triangleright \partial_1 - \partial_1 \triangleright (z^\gamma D^{(n)} \triangleright \partial_1) = (\partial_1 z^\gamma) D^{(n)} \triangleright \partial_1 - \partial_1 \triangleright n_1 z^\gamma D^{(n-\{1,0\})} = 0 \).
of \(\{\partial_i\} \cup \{z^\gamma D^{(n)}\}_{\gamma,n}\) is closed under \([\cdot, \cdot]\), which will be used in Subsection 3.10.

The presence of a pre-Lie structure connects to the pre-Lie algebras in rough paths [5] and regularity structures [4]. Indeed, as we shall see in Section 6 in the specific case of driven ODEs, \(\triangleright\) is related to the grafting pre-Lie product (up to combinatorial factors, see Subsection 6.3 for a detailed discussion).

We now come to an important observation: There is a bigrading\(^{36}\) on the index set \(\{1, 2\} \cup \{(\gamma, n) | [\gamma] \geq 0, n \neq 0\}\) of our (linearly independent) family of derivations that is compatible with the pre-Lie product \(\triangleright\). Indeed, we associate a pair of integers to every index by the following map \(bi:\)

\[
bi(\gamma, n) := (1 + [\gamma], \sum_{m \neq 0} |m|\gamma(m) - |n|),
bi 1 := (0, |(1, 0)|), \quad bi 2 := (0, |(0, 1)|).
\]

By compatibility we mean that for any two elements \(D, D'\) of our family, provided not both are of the form \(\partial_i\), the product \(D \triangleright D'\) is a linear combination of elements of our family that only correspond to indices such that their bigrading is the sum of the bigradings of the index for \(D\) and for \(D'\). This is obvious for the second item in (3.36). Expanding the first item in (3.36) as

\[
z^\gamma D^{(n')} \triangleright z^\gamma D^{(n)} = \sum_\beta (z^\gamma D^{(n')})_\beta z^\beta D^{(n)},
\]

\[
\partial_1 \triangleright z^\gamma D^{(n)} = \sum_\beta (\partial_1)_\beta z^\beta D^{(n)}
\]

and appealing to definition (3.37) we see that our claim amounts to

\[
(z^\gamma D^{(n')})_\beta \neq 0 \quad \implies \quad (D^{(n')})_{\gamma - \gamma'} \neq 0
\]

\[
(3.38)
\]

\[
\implies \sum_{m \neq 0} |m|\beta(m) = \sum_{m \neq 0} |m|\gamma(m) + \sum_{m \neq 0} |m|\gamma'(m) - |n'|
\]

and to (3.27). The second part of this implication also follows from Lemma 3.1.

Bigraded spaces appear in the context of regularity structures in [6]. In the tree-based setting, one chooses a bigrading [6, (2.4)] which encodes the size of the tree, on the one hand, and the decorations, on the other. The same guiding principle is present in (3.37): the quantity \(1 + [\gamma]\) is the number of edges of the trees represented by the multi-index \(\gamma\), whereas the second component is, roughly speaking, counting the polynomial decorations. We refer to Sections 6 and 7 for more details.

---

\(^{36}\)This is just a compact way of saying that there exist two gradings, which we put together in a two-component vector, cf. (3.37).
3.9. **Homogeneities** \(|\cdot| \in A, \text{ and gradedness of } T.\)
We now return to the strict triangular structure with respect to (3.24) and in particular the deficiency of (3.26). The choices we make now are guided by the application to the quasi-linear equation (1.1) with a driver \(\xi\) of regularity \(\alpha - 2\). Inspired by (3.37) we choose an \(\alpha > 0\) and define the homogeneity of a multi-index \(\gamma\) as

\[
|\gamma| = \alpha(|\gamma| + 1) + \sum_{n \neq 0} |n|\gamma(n),
\]

where the normalization \(|0| = \alpha\), which destroys additivity, is made such that, in line with [17, Assumption 3.20],

\[
|\epsilon_n| = |n| \quad \text{for } n \neq 0;
\]
in particular, on the index set \(\{\epsilon_n\}_{n \neq 0}\) of \(\tilde{T}\) we have \(|\epsilon_n| \in \mathbb{N}\). On the index set \(\{\gamma \mid |\gamma| \geq 0\}\) of \(\tilde{T}\), we have that \(|\gamma| \in \alpha\mathbb{N} + \mathbb{N}_0\). Hence \(A := \{\gamma \mid z, \in T\}\) satisfies the assumptions of [17, Definition 3.1] of being bounded from below (namely by \(\min\{\alpha, 1\}\)) and locally finite. Note that in our setting, the entire index set of \(T\) has positive homogeneity, and only corresponds to the “integrated” part of the model; for a detailed connection to Hairer’s set of homogeneities see Subsection 5.3.

Provided the monomials that appear as multiplication operators in (3.31) are constrained by

\[
z^\gamma D^{(n)} \quad \text{for } |\gamma| \geq 0 \text{ and } |\gamma| > |n|,
\]
as a consequence of combining (3.10), (3.16) and (3.34), the finiteness property (3.3) is uniform over the entire collection \(\{z^\gamma D^{(n)}\}_{|\gamma| \geq 0} \cup \{\partial_i\}_i\):

\[
\{(\gamma', (\gamma, n)) \mid (z^\gamma D^{(n)})^{\gamma'} \neq 0\} \text{ is finite for all } \gamma'.
\]
This strengthening of (3.3) will be crucial when constructing \(\Delta\). Moreover, we have the following strict triangular structure:

**Lemma 3.2.** *For \(D \in \{z^\gamma D^{(n)}\}_{|\gamma| \geq 0, |\gamma| > |n|} \cup \{\partial_i\}_i\) it holds

\[
D^{\gamma'}_{\beta'} \neq 0 \implies |\gamma'| < |\beta'|.
\]

**Proof.** For \(D = \partial_1\) (and \(\partial_2\)), this follows immediately from (3.27) by definition (3.39). We turn to \(D = z^\gamma D^{(0)}\). From (3.25) and (3.34) we read off that \((z^\gamma D^{(0)})^{\gamma'}_{\beta'} \neq 0\) implies \(|\gamma' + \gamma| < |\beta'|\) and \(\sum_{n \neq 0} |n|(|\gamma' + \gamma|)(n)\leq \sum_{n \neq 0} |n|\beta'(n)\). The latter, due to \(\alpha > 0\), implies \(\alpha(|\gamma' + \gamma| + 1) + \sum_{n \neq 0} |n|(|\gamma' + \gamma|)(n) < \alpha(|\beta'| + 1) + \sum_{n \neq 0} |n|\beta'(n)\), which because of \(|\gamma| \geq 0\) in turn yields the desired \(\alpha(|\gamma'| + 1) + \sum_{n \neq 0} |n|\gamma'(n) < \alpha(|\beta'| + 1) + \sum_{n \neq 0} |n|\beta'(n)\). By definition (3.39), this establishes (3.43) for \(D = z^\gamma D^{(0)}\).

We now turn to \(D = z^\gamma D^{(n)}\) with \(n \neq 0\) and note that by (3.13) and (3.34) we have \(D^{\gamma'}_{\beta'} \neq 0\) only for \(\gamma' + \gamma = \beta' + \epsilon_n\), which by (3.39) implies \(|\gamma'| + |\gamma| = |\beta'| + |\epsilon_n|\). By (3.40) and the condition in (3.41), this yields as desired \(|\gamma'| < |\beta'|\). \(\square\)
Property (3.43) results in the following gradedness: For $\kappa \in A$ let $T_\kappa \subset T$ denote the subspace corresponding to the indices $\gamma$ with $|\gamma| = \kappa$; we obviously have

$$T = \bigoplus_{\kappa \in A} T_\kappa,$$

in line with [17, Definition 3.1]. Then (3.43) can be reformulated as

$$D^\dagger T_\kappa \subset \bigoplus_{\kappa' < \kappa} T_{\kappa'},$$

(3.44)

with the implicit understanding that $\kappa, \kappa' \in A$. We note that because of the presence of the $z_0$-variable, and thus the $\gamma(k = 0)$-component on which $|\gamma|$ is not coercive, $T_\kappa$ is not finite dimensional. However, in the practice of (1.1), this is of no concern since the model $\Pi(x) \in \mathbb{R}[[z_0, z_n]]$, which a priori is a formal power series, actually is analytic in $z_0$, which plays the role of a constant coefficient in $\partial_2 - (1 + z_0)\partial_1^2$.

3.10. The Lie algebra $L$.

**Lemma 3.3.** The span of

$$(3.45) \{ z^\gamma D^{(n)} \}_{|\gamma| \geq 0, |\gamma| > |n|} \cup \{ \partial_i \},$$

as derivations on $\mathbb{R}[[z_k, z_n]]$, defines a bigraded Lie algebra $L$.

**Proof.** We need to show that this sub-space of $\text{Der}(\mathbb{R}[[z_k, z_n]])$ is closed under taking the commutator $[D, D'] = D \circ D' - D' \circ D$, cf. Subsection 3.8. To this purpose, for any $D, D' \in \{ z^\gamma D^{(n)} \}_{|\gamma| \geq 0, |\gamma| > |n|} \cup \{ \partial_i \}$, we have to identify $[D, D']$ as a linear combination of elements of this set.

We first note that by (3.22) we have $[\partial_1, \partial_2] = 0$. By (3.36), written in its component-wise form, we obtain

$$(3.46) [z^\gamma D^{(n)}, z^\gamma' D^{(n')} - \sum_{\beta'} (z^\gamma D^{(n)})_{0}^{\beta'} z^{\beta'} D^{(n')} - \sum_{\beta} (z^\gamma' D^{(n')})_{\beta}^{0} z^{\beta} D^{(n)},$$

where both sums are finite due to (3.2). By (3.30), we learn that because of $[\gamma], [\gamma'] \geq 0$, the sums restrict to $[\beta], [\beta'] \geq 0$. Due to (3.43) they restrict to $|\beta| > |\gamma|$ and $|\beta'| > |\gamma'|$. Moreover, by assumption (3.41) we have $|\gamma| > |n|$ and $|\gamma'| > |n'|$; hence as desired, the sums in (3.46) involve only multi-indices with $|\beta'| > |n'|$ and $|\beta| > |n|$.

Finally, again by (3.36), we have

$$(3.47) [z^\gamma D^{(n)}, \partial_1] = n_1 z^\gamma D^{(n - (1, 0))} - \sum_{\beta} (\partial_1)^{\gamma}_{\beta} z^{\beta} D^{(n)},$$

where (3.2) again ensures the effective finiteness of the sum. We note that (3.47) has the desired form: The first r. h. s. term, which only is present for $n_1 \geq 1$, is admissible since obviously $|\gamma| > |n| > |n - (1, 0)|$. For the second r. h. s. term we note that by (3.30) the sum is limited to $|\beta| \geq 0$, and by (3.43) it is limited to $|\beta| > |\gamma| > |n|$.
It remains to show that $L$ is bigraded; this is clear from (3.27) and (3.38), which show that the pre-Lie product (and thus the Lie bracket) is compatible with (3.37), together with the commutation relation $[\partial_1, \partial_2] = 0$. □

4. The Hopf algebra structure

4.1. The universal enveloping algebra $U(L)$, $T^*$ as a module over $U(L)$.

We now adopt a more abstract point of view and consider the elements of the Lie algebra $L$ as mere symbols rather than endomorphisms, and we interpret (3.22), (3.46) and (3.47) as a coordinate representation of the Lie bracket in terms of the basis (3.45). We denote by $U(L)$ the corresponding universal enveloping algebra [1, p. 28], an algebra which is based on the tensor algebra formed by $L$ and quotiented through the ideal generated by the relations defining the Lie bracket. We may think of the tensor algebra as the direct sum indexed by words.

Due to the mapping properties (3.33), the canonical Lie algebra morphism $\rho : L \to \text{End}(T^*)$, which replaces every abstract symbol $D \in L$ with its corresponding endomorphism, is well defined, and as a consequence of Subsection 3.10, $\rho$ is a Lie algebra morphism. By the universality property [1, (U), p. 29], such $\rho$ extends in a unique way to an algebra morphism $\rho : U(L) \to \text{End}(T^*)$; in particular, concatenation of words turns into composition of endomorphisms. However, this representation is not faithful\(^{37}\). In a canonical way, we may rewrite $\rho$ as a map $U(L) \otimes T^* \to T^*$, so that $T^*$ as a linear space becomes a left module over $U(L)$.

The universal enveloping algebra $U(L)$ is naturally a Hopf algebra, cf. [1, Examples 2.5, 2.8]; the product is given by the concatenation of words, whereas the coproduct is characterized by its action on the elements $D \in L$ (which we call primitive elements), namely

\[(4.1) \quad \text{cop}D = 1 \otimes D + D \otimes 1,\]

and in general by the compatibility with the product, meaning that for all $U, U' \in U(L)$

\[(4.2) \quad \text{cop}UU' = (\text{cop}U)(\text{cop}U').\]

4.2. The derived algebra $\tilde{L}$ and the pre-Lie structure $\triangleright$ revisited.

As mentioned in Subsection 3.8, the Lie algebra $L$ is not closed under the pre-Lie product $\triangleright$. However, the only failure, namely $\partial_i \triangleright \partial_j \notin \tilde{L}$, turns out to be peripheral. This follows from the fact that $[D, D']$ does not have a $\partial_i$-component, see (3.46) and (3.47). In other words we have for the derived algebra $[L, L] \subset \tilde{L}$, where the Lie sub-algebra $\tilde{L} \subset L$ is defined as

\[(4.3) \quad \tilde{L} := \text{span}\{z^\gamma D^{(n)}\}_{|\gamma| \geq 0, |\gamma| > |n|}.\]

Since $\tilde{L}$ is also an ideal, the quotient Lie algebra $L/\tilde{L}$ is Abelian, see [19, Lemma 1.2.5], and thus is isomorphic to $\{\partial_1, \partial_2\}$. Moreover, the

\[^{37}\text{i.e. one-to-one: consider} z^{e_1}D^{(1,0)}z^{e_2}D^{(1,0)} \text{ and } z^{e_1}D^{(1,0)}z^{e_1+e_2}D^{(1,0)}, \text{ which are different words in } U(L), \text{ but the same as endomorphisms.}\]
Lie algebra morphism \( L \to L/\tilde{L} \cong \{\partial_1, \partial_2\} \) induces an algebra morphism \( U(L) \to U(L/\tilde{L}) \cong \{\partial^m\}_{m \in \mathbb{N}_0^2} \), see [1, p. 29]. This algebra morphism in turn induces the decomposition
\[
U(L) = \bigoplus_{m \in \mathbb{N}_0^2} U_m.
\]

(4.4)

By definition, \( U_0 \) is canonically isomorphic to \( U(\tilde{L}) \). Since \( \tilde{L} \) is closed under \( \triangleright \), the pre-Lie structure provides a canonical isomorphism, as co-commutative coalgebras, between \( U(\tilde{L}) \) and the symmetric tensor algebra \( S(\tilde{L}) \), see [28, Theorem 2.12]. Via the definition (4.5), the pre-Lie structure \( \triangleright : L \times \tilde{L} \to L \) provides a natural isomorphism between the linear spaces \( U_m \) and \( U(\tilde{L}) \), as will become apparent in Subsection 4.3. These natural isomorphisms, of which we will make no explicit use, will guide our construction of a basis in Subsection 4.3.

We now will be more precise on how we salvage the pre-Lie structure \( \triangleright \). We use \( \triangleright \) in terms of the product
\[
L \times \tilde{L} \ni (D, \tilde{D}) \mapsto D \tilde{D} - D \triangleright \tilde{D} \in U(L).
\]
(4.5)

Fixing the second factor \( \tilde{D} = z^\gamma D^{(n)} \), we extend this product from \( D \in L \) to \( U \in U(L) \)
\[
U(L) \ni U \mapsto z^\gamma UD^{(n)} \in U(L),
\]
(4.6)

The map (4.6) is inductively defined in the length of \( U \) by anchoring through \( z^\gamma 1D^{(n)} = z^\gamma D^{(n)} \) and postulating for any \( D \in L \subset U(L) \)
\[
z^\gamma UD^{(n)} = Dz^\gamma UD^{(n)} - \sum_{\beta} D^\beta z^\beta UD^{(n)}.
\]
(4.7)

Let us comment on (4.7): First of all, the identity (4.7) is consistent with the map \( \rho : U(L) \to \text{End}(T^*) \) in the sense of
\[
\rho z^\gamma UD^{(n)} = z^\gamma (\rho U) D^{(n)},
\]
(4.8)

since \( D \) as an element of \( \text{End}(T^*) \) is a derivation. As an identity in \( U(L) \), it is to be read as follows: On the l. h. s., we first multiply \( U \) by \( D \) via concatenation, and then apply (4.6). For the first r. h. s. term, we reverse this order. The second r. h. s. term is a linear combination of several versions of (4.6) (with \( \gamma \) replaced by \( \beta \)); the coefficients are given by identifying \( D \in L \) with \( D \in \text{End}(T^*) \), and (3.43) shows that \( (\beta, n) \) is in the index set of \( L \). Hence (4.7) indeed provides an inductive definition of (4.6).

A first crucial observation is that the maps (4.6) commute\(^{38}\):

**Lemma 4.1.** It holds
\[
z^\gamma z^\gamma UD^{(n)} D^{(n')} = z^\gamma z^\gamma UD^{(n')} D^{(n)}.
\]
(4.9)

\(^{38}\)which is at the basis of the canonical identification of \( S(L) \) with \( U(L) \) in [28, Theorem 3.14]
Proof. We argue by induction. The base case of $U = 1$ follows from using (4.7) twice (once for $D = z^\gamma D^{(n')}$ and $U = 1$) and connecting the outcomes via (3.46). We now assume that (4.9) is satisfied for some $U$ and give ourselves an element $D \in L$. Applying (4.7) twice, we obtain

\[
z^\gamma z^\gamma D U D^{(n')} D^{(n')} = Dz^\gamma z^\gamma U D^{(n')} D^{(n')} - \sum_{\beta} D^\gamma_\beta z^\beta z^\gamma U D^{(n')} D^{(n')} - \sum_{\beta} D^\gamma_\beta z^\gamma z^\beta U D^{(n')} D^{(n')},
\]

and the analogous expression in case of $z^\gamma z^\gamma D U D^{(n')} D^{(n')}$. By the induction hypothesis, both are equal. \( \square \)

A second crucial observation is that the maps (4.6) commute with the coproduct on $U(L)$ in the following sense:

**Lemma 4.2.** If

\[ \text{cop} U = \sum_{\{U\}} U_{(1)} \otimes U_{(2)} \]

then

\[ \text{cop} z^\gamma U D^{(n)} = \sum_{\{U\}} (z^\gamma U_{(1)} D^{(n)} \otimes U_{(2)} + U_{(1)} \otimes z^\gamma U_{(2)} D^{(n)}). \]

Proof. Once more we argue by induction. The base case of $U = 1$ is included in (4.1) and our definition $z^\gamma 1 D^{(n)} = z^\gamma D^{(n)}$. We now assume that (4.11) is satisfied for some $U$ and give ourselves an element $D \in L$. Using the inductive definition (4.7)

\[
\text{cop} z^\gamma D U D^{(n)} = \text{cop} D z^\gamma U D^{(n)} - \sum_{\beta} D^\gamma_\beta \text{cop} z^\beta U D^{(n)} = \]

\[
(1 \otimes D + D \otimes 1) \text{cop} z^\gamma U D^{(n)} - \sum_{\beta} D^\gamma_\beta \text{cop} z^\beta U D^{(n)},
\]

we may feed in the induction hypothesis, leading to

\[
\text{cop} z^\gamma D U D^{(n)}
\]

\[
= \sum_{\{U\}} \left( z^\gamma U_{(1)} D^{(n)} \otimes D U_{(2)} + U_{(1)} \otimes D z^\gamma U_{(2)} D^{(n)} 
+ Dz^\gamma U_{(1)} D^{(n)} \otimes U_{(2)} + DU_{(1)} \otimes z^\gamma U_{(2)} D^{(n)} 
- \sum_{\beta} D^\gamma_\beta (z^\beta U_{(1)} D^{(n)} \otimes U_{(2)} + U_{(1)} \otimes z^\beta U_{(2)} D^{(n)}) \right),
\]

which by the inductive definition (4.7) compactifies to

39Here and in the sequel we use Sweedler’s notation, see e.g. [1, p. 56].
\[ \text{cop } z^\gamma D U D^{(n)} = \sum_{(U)} \left( z^\gamma U_{(1)} D^{(n)(1)} \otimes D^{(n)(2)} + U_{(1)} \otimes z^\gamma U_{(2)} D^{(n)} \right) + \left( z^\gamma D U_{(1)} D^{(n)} \otimes U_{(2)} + DU_{(1)} \otimes z^\gamma U_{(2)} D^{(n)} \right). \]

Since by (4.1) and (4.2)

\[ (4.12) \quad \text{cop } DU = \sum_{(U)} (DU_{(1)} \otimes U_{(2)} + U_{(1)} \otimes DU_{(2)}), \]

the proof is complete. \[ \square \]

A third crucial observation is that the maps (4.6) connect product and coproduct in the following sense:

**Lemma 4.3.** Under the assumption (4.10),

\[ (4.13) \quad U z^\gamma D^{(n)} = \sum_{(U), \beta} (U_{(1)})_\beta^\gamma z^\beta U_{(2)} D^{(n)}. \]

**Proof.** Again we argue by induction. For \( U = 1 \) the identity follows by noting that \( \text{cop } 1 = 1 \otimes 1 \). Assume now that (4.13) is satisfied for some \( U \), then for \( D \in L \) by the induction hypothesis

\[ DUz^\gamma D^{(n)} = D \sum_{(U), \beta} (U_{(1)})_\beta^\gamma z^\beta U_{(2)} D^{(n)}, \]

which by (4.7) leads to

\[ DUz^\gamma D^{(n)} = \sum_{(U), \beta} (U_{(1)})_\beta^\gamma \left( z^\beta D U_{(2)} D^{(n)} + \sum_{\beta'} D_{\beta'}^\beta z^{\beta'} U_{(2)} D^{(n)} \right). \]

Using \( \sum_{\beta} D_{\beta'}^\beta (U_{(1)})_\beta^\gamma = (DU_{(1)})_{\beta'}^\gamma \) together with (4.12) finishes the proof. \[ \square \]

A final observation is an intertwining of the maps (4.6) with \( \partial_1 \):

**Lemma 4.4.** It holds

\[ (4.14) \quad z^\gamma U D^{(n)} \partial_1 = z^\gamma U \partial_1 D^{(n)} + n_1 z^\gamma U D^{(n-1,0)}, \]

and an analogous statement holds for \( \partial_1 \) replaced by \( \partial_2 \).

**Proof.** We argue by induction. If \( U = 1 \),

\[ (3.47) \quad z^\gamma D^{(n)} \partial_1 = \partial_1 z^\gamma D^{(n)} + n_1 z^\gamma D^{(n-1,0)} - \sum_{\beta} (\partial_1)_\beta^\gamma z^\beta D^{(n)} \]

\[ (4.7) \quad = z^\gamma \partial_1 D^{(n)} + n_1 z^\gamma D^{(n-1,0)}. \]
We now assume that (4.14) is true for a given \( U \) and aim to prove it for \( DU \), where \( D \in L \). Then by (4.7)

\[
z^\gamma DU D^{(n)} \partial_1 = Dz^\gamma UD^{(n)} \partial_1 - \sum_\beta D_\beta z^\beta UD^{(n)} \partial_1,
\]

and appealing to the induction hypothesis yields

\[
z^\gamma DU D^{(n)} \partial_1 = Dz^\gamma U \partial_1 D^{(n)} + n_1 Dz^\gamma UD^{(n-(1,0))}
\]

\[
- \sum_\beta D_\beta z^\beta UD^{(n)} - n_1 \sum_\beta D_\beta z^\beta UD^{(n-(1,0))},
\]

which by a second application of (4.7) takes the desired form. \( \square \)

4.3. **The choice of basis** \( \{D_{(J,m)}\}_{(J,m)} \).

We now define a basis in \( U(L) \) based on the structure derived from the pre-Lie structure in the previous Subsection 4.2. Applying iteratively the map (4.6) to an element \( \partial^m \), and making use of Lemma 4.1 we may define

(4.15) \( D_{(J,m)} := \frac{1}{J!m!} \prod_{(\gamma,n)} (z^\gamma)^{J(\gamma,n)} \partial_1^{m_1} \partial_2^{m_2} \prod_{(\gamma,n)} (D^{(n)})^{J(\gamma,n)} \in U(L). \)

Here \( J \) denotes a multi-index on tuples \( (\gamma,n) \) with \( |\gamma| \geq 0 \) and \( |\gamma| > |n| \); we set \( J! := \prod_{(\gamma,n)} J(\gamma,n)! \), so that the normalization constant \( J!m! \) may be seen as the multi-index factorial \( (J,m)! \). This normalization is chosen such that the basis representation of the coproduct is standard, see (4.19) below. The collateral damage of this normalization is that the basis representation of (4.6) acquires a combinatorial factor:

(4.16) \( z^\gamma D_{(J,m)}D^{(n)} = (J(\gamma,n) + 1)D_{(J+e(\gamma,n),m)}. \)

For \( J = 0 \), (4.15) reduces to the standard basis \( \{\frac{1}{m!} \partial^m\}_{m \in \mathbb{N}_0} \) for the coalgebra of differential operators, characterized as dual to the standard basis \( \{x^m\}_{m \in \mathbb{N}_0} \) of the algebra \( \mathbb{R}[x_1, x_2] \) under the pairing of [18, Example 2.2]. The concatenation of these elements leads as well to a combinatorial factor:

(4.17) \( D_{(0,m')}D_{(0,m'')} = (m'+m'')D_{(0,m'+m'')} \).

For later purpose, let us define the length of \( (J,m) \) by

\[
|(J,m)| := \sum_{(\gamma,n)} J(\gamma,n) + m_1 + m_2.
\]

**Lemma 4.5.** The set \( \{D_{(J,m)}\}_{(J,m)} \) is a basis of \( U(L) \).

**Proof.** As a consequence of the Poincaré-Birkhoff-Witt Theorem, cf. [19, Theorem 1.9.6], after a choice of an order \( \prec \) on the set of pairs \( (\gamma,n) \), the set of elements of the form

(4.18) \( B_{(J,m)} := \frac{1}{J!m!} \partial_1^{m_1} \partial_2^{m_2} z^\gamma D^{(n_1)} \ldots z^\gamma_k D^{(n_k)}, \)

where \( J = e(\gamma_1,n_1) + \ldots + e(\gamma_k,n_k) \)

and \( (\gamma_1,n_1) \preceq \ldots \preceq (\gamma_k,n_k) \)
is a basis of \( U(\mathbf{L}) \). Applying (4.7) iteratively, one can show the representation
\[
D_{(J, m)} = B_{(J, m)} + \text{span}\{B_{(J', m')} \mid |(J', m')| < |(J, m)|\}.
\]
Since \( \{B_{(J, m)}\}_{(J, m)} \) is a basis, it is easy to deduce from this identity that also \( \{D_{(J, m)}\}_{(J, m)} \) is a basis.

The advantage of the basis (4.15) over a Poincaré-Birkhoff-Witt basis of the form (4.18) is that the former does not rely on the choice of an order in \( \mathbf{L} \), cf. (4.9), whereas the latter crucially does. The only choice to be made is the order of the three symbols: having first the \( z \)'s, then the \( \partial \)'s and last the \( D^{(n)} \)'s generates the only basis for which the analogue of [17, (4.14)], namely (4.49), is true. In addition, with the basis (4.15) we obtain the most direct identification of our group elements as exponentials of shift and tilt parameters, cf. Proposition 5.1.

The coproduct has the following simple structure in the basis (4.15), which is reminiscent of the Hopf algebra of constant-coefficient differential operators over the algebra of smooth functions, cf. [3].

**Lemma 4.6.** It holds
\[
(4.19) \quad \text{cop} \ D_{(J, m)} = \sum_{(J', m')+(J'', m'')=(J, m)} D_{(J', m')} \otimes D_{(J'', m'')}.
\]

**Proof.** We proceed by induction in the length \(|(J, m)|\). The base case \(|(J, m)| = 0\) reduces to the trivial \( \text{cop} \ 1 = 1 \otimes 1 \). For the induction step, we give ourselves an element \( D_{(J, m)} \) and distinguish two cases. If \( J = 0 \), we assume without loss of generality \( m_1 \neq 0 \) and with help of (4.17) we write
\[
m_1 D_{(0, m)} = D_{(0, m-1, 0)} D_{(0, 1, 0)}
\]
in order to access the induction hypothesis. Then by (4.1), (4.2) and the induction hypothesis,
\[
\text{cop} \ D_{(0, m-1, 0)} D_{(0, 1, 0)} = \sum_{m'+m''=m-1, 0} D_{(0, m')} D_{(0, m'')} D_{(0, 1, 0)} = \sum_{m'+m''=m-1, 0} m_1' D_{(0, m'+1, 0)} D_{(0, m'')} + m_1'' D_{(0, m')} D_{(0, m''+1, 0)}.
\]
By (4.15) and Lemma A.1, this equals \( m_1 \sum_{m'+m''=m} D_{(0, m')} D_{(0, m'')} \), which proves (4.19) in the case \( J = 0 \).

We now address the case \( J \neq 0 \). We take a pair \((\gamma, \mathbf{n})\) such that \( J(\gamma, \mathbf{n}) \neq 0 \) and use (4.16) to write
\[
(\gamma, \mathbf{n}) D_{(J, m)} = z^\gamma D_{(J-\epsilon(\gamma, \mathbf{n}), m)} D^{(\mathbf{n})}.
\]
Then by (4.11) and the induction hypothesis,
\[
\text{cop} z^\beta D_{(J-(\epsilon_n, m), m)} D^{(n)} = \sum_{(J', m')} z^{J'(\gamma, n)} D^{(n)} \otimes D_{(J'+\epsilon_n, m')} + D_{(J', m')} \otimes z^{J' D_{(J'+\epsilon_n, m')}} D^{(n)}
\]
\[
= \sum_{(J', m')} (J'(\gamma, n) + 1) D_{(J'+\epsilon_n, m')} \otimes D_{(J'+\epsilon_n, m')} + (J'(\gamma, n) + 1) D_{(J', m')} \otimes D_{(J'+\epsilon_n, m')}
\]
By (4.15) and Lemma A.1, this as desired reduces to
\[
(J(\gamma, n) + 1) \sum_{(J', m')} D_{(J', m')} \otimes D_{(J', m')}
\]
Recall that by construction, the concatenation product on \( U(L) \) is an abstract lifting of the composition product on \( \text{End}(T^*) \), as can be seen by applying \( \rho \). The upcoming lemma is a projection of this fact onto \( \tilde{L} \), see (4.3). More precisely, we resolve \( \tilde{L} \) in terms of \( n \) by introducing the family of maps \( \iota_n : U(L) \to T^* \), determined by how it acts on elements of the basis (4.15):
\[
(4.20) \quad \iota_n D_{(J, m)} = \begin{cases} 
z^\beta & \text{if } (J, m) = (e(\beta, n), 0) \\ 0 & \text{otherwise} \end{cases}
\]
As it turns out, the next lemma involves generalized counits\(^{40}\) \( \varepsilon_n : U(L) \to \mathbb{R} \) defined on the basis through
\[
(4.21) \quad \varepsilon_n D_{(J, m)} = \begin{cases} 
1 & \text{if } (J, m) = (0, n) \\ 0 & \text{otherwise} \end{cases}
\]
\[\text{Lemma 4.7.} \quad \text{It holds as an identity in } T^*
\]
\[
(4.22) \quad \iota_n U_1 U_2 = (\rho U_1) \iota_n U_2 + \sum_m \binom{n+m}{m} (\varepsilon_m U_2) \iota_{n+m} U_1.
\]
This identity should be seen as the dual of the forthcoming intertwining relation of \( \Delta^+ \) and \( \Delta \) via \( J_n \), cf. (4.49).

\[\text{Proof.} \quad \text{By linearity of } \iota_n, \text{ it is enough to show (4.22) for } U_1 \text{ and } U_2 \text{ in the set of basis elements (4.15), namely } U_1 = D_{(J_1, m_1)} \text{ and } U_2 = D_{(J_2, m_2)}. \text{ Moreover, we assume } (J_1, m_1) \neq (0, 0); \text{ otherwise the statement is trivial, since } \rho 1 = \text{id} \text{ and } \iota_n 1 = 0, \text{ cf. (4.20)).}
\]
We argue by induction in the length \(|(J_2, m_2)|\); the base case amounts to \( U_2 = 1 \), which is immediate because of \( \iota_n 1 = 0 \) and \( \varepsilon_m 1 = \delta_0^m \). For the induction step, we distinguish the cases \( J_2 = 0 \) and \( J_2 \neq 0 \). The former implies \( U_2 = D_{(0, m_2)} \), so that, by \( \iota_n D_{(0, m_2)} = 0 \) and \( \varepsilon_m D_{(0, m_2)} = \delta_m^{m_2} \), (4.22) assumes the form
\[
(4.23) \quad \iota_n D_{(J_1, m_1)} D_{(0, m_2)} = \binom{n+m_2}{m_2} \iota_{n+m_2} D_{(J_1, m_1)}.
\]
\(^{40}\)\( \varepsilon_0 \) is the plain counit of \( U(L) \)
We assume without loss of generality $m_2 \geq (1, 0)$. Recalling that $D_{(0,m)} = \frac{1}{m!} \partial^m$, cf. (4.15), we rewrite the l. h. s. of (4.23) as
\begin{equation}
(4.24) \quad t_n D_{(J_1,m_1)} D_{(0,m_2)} = \frac{1}{(m_2)!} t_n D_{(J_1,m_1)} D_{(0,m_2-(0,1))} \partial_1.
\end{equation}

We will now use the following identity, which holds for all $U \in U(L)$:
\begin{equation}
(4.25) \quad t_n U \partial_1 = (n_1 + 1) t_n D_{(0,1)} U.
\end{equation}

To prove (4.25), by linearity it is enough to take $U$ of the form (4.15). If $J = 0$, then (4.25) is clear since $t_n D_{(0,m)} \partial_1 = (m_1 + 1) t_n D_{(0,m+(1,0))}$, cf. (4.17), and that vanishes by construction (4.20). If $J \neq 0$, we choose a pair $(\gamma', n')$ such that $U = \frac{1}{\beta(J, \gamma', n')} D_{(J,m)} D^{(n')}$ and write, with help of (4.14),
\begin{equation}
(4.26) \quad t_n z^{\gamma'} D_{(J,m)} D^{(n')} \partial_1 = t_n z^{\gamma'} D_{(J,m)} \partial_1 D^{(n')} + n'_1 t_n z^{\gamma'} D_{(J,m)} D^{(n'-(1,0))}.
\end{equation}

We now note that (4.20) implies the following:
\begin{equation}
(4.27) \quad t_n z^{\gamma'} D_{(J,m)} D^{(n)} \neq 0 \implies (J, m) = 0.
\end{equation}

Then by (4.27) the first r. h. s. contribution of (4.26) is always vanishing, since terms coming from $D_{(J,m)} \partial_1$ have strictly positive length, cf. (4.14). The second r. h. s. contribution of (4.26) yields the output of (4.25).

We now combine (4.24), (4.25) and the induction hypothesis, yielding (4.23) in form of
\begin{equation}
(4.28) \quad t_n D_{(J_1,m_1)} D_{(0,m_2)} = \frac{n_1 + 1}{(m_2)!} t_n D_{(J_1,m_1)} D_{(0,m_2-(0,1))}.
\end{equation}

In the case of $J_2 \neq 0$, and thus $\epsilon_m D_{(J_2,m_2)} = 0$, once more we choose a pair $(\gamma', n')$ such that $J_2(\gamma', n') \neq 0$ and use (4.15) to write $U_2 = \frac{1}{\beta(J_2, \gamma', n')} D_{(J_2,m_2)} D^{(n')}$. By (4.13) applied to $U = D_{(J_2,m_2)}$, combined with (4.19), we have
\begin{equation}
(4.29) \quad t_n D_{(J_1,m_1)} z^{\gamma'} D_{(J_2,m_2)} D^{(n')} = t_n D_{(J_1,m_1)} \left( D_{(J_2,m_2)} z^{\gamma'} D^{(n')} - \sum_{\substack{(J_2', m_2') \in (J_2, m_2), (J_2', m_2') \neq (J_2, m_2) \neq (0,0)}} (D_{(J_2', m_2')})^{(n')} z^{\beta'} D_{(J_2', m_2')} D^{(n')} \right).
\end{equation}

On the first r. h. s. term we apply once more (4.13) to $U = D_{(J_1,m_1)} D_{(J_2', m_2')}$. Since by assumption $(J_1, m_1) \neq (0,0)$, the product $D_{(J_1,m_1)} D_{(J_2', m_2')}$ is a linear combination of basis elements (4.15) with strictly positive length, as may be seen by an iterative application of (4.13). We thus appeal to (4.27) to the effect of
\begin{equation}
(4.30) \quad t_n D_{(J_1,m_1)} D_{(J_2', m_2')} z^{\gamma'} D^{(n')} = (\rho D_{(J_1,m_1)} D_{(J_2', m_2')}) t_n z^{\gamma'} D^{(n')}.
\end{equation}

For the second r. h. s. term, we note that if $(J_2', m_2') = (0,0)$ the sum is empty, so that (4.22) follows from (4.29). If $(J_2', m_2') = (0,0)$, the length
Recall that (4.4).

The bigrading revisited and finiteness properties.

Let us fix a pair \((J, m)\). We adopt the same notation \(b_i\) as in (3.37), without any risk of confusion. Note that this decomposition is different from (4.4). It turns out that our basis elements (4.15) are homogeneous:

\[
\rho D(J, m) = \sum b_i \cdot D(J, m) \cdot b_i.
\]

We will show that (4.27) \(\implies\) (4.28). Since \((J', m_2') \neq (0, 0)\), so (4.29) further reduces to

\[
\sum b_i \cdot D(J, m) \cdot b_i = 0,
\]

which cancels with (4.28). Since \((J', m_2') \neq (0, 0)\) implies by (4.27) that \(\sum b_i \cdot D(J, m) \cdot b_i = 0\), this shows that (4.22) holds.

4.4. The bigrading revisited and finiteness properties.

Recall that \(L\) is a bigraded Lie algebra with respect to (3.37), and thus \(U(L)\) becomes a bigraded Hopf algebra. This means that there exists a decomposition \(U(L) = \bigoplus_{b \in \mathbb{N} \times \mathbb{Z}} U_b\) such that the concatenation product maps \(U_{b'} \otimes U_{b''} \to U_{b' + b''}\) and the coproduct cop maps \(U_b \to \bigoplus_{b' + b'' = b} U_{b'} \otimes U_{b''}\). Note that this decomposition is different from (4.4). It turns out that our basis elements (4.15) are homogeneous:

**Lemma 4.8.** \(D(J, m) \in U_{\text{bi}(J, m)}\), where

\[
\text{bi}(J, m) := \sum_{(\gamma, n)} J(\gamma, n) \text{bi}(\gamma, n) + (0, |m|).
\]

We adopt the same notation \(\text{bi}\) as in (3.37), without any risk of confusion.

**Proof.** Let us fix a pair \((\gamma, n)\). We will show that

\[
U \in U_b \implies z^\gamma UD^{(n)} \in U_{b + \text{bi}(\gamma, n)};
\]

this clearly proves the lemma, since \(D(J, m)\) is built starting from \(\frac{1}{m!} \partial^{m} \in U_{(0, |m|)}\) and applying (4.6) iteratively. We argue in favor of (4.31) by induction. The case \(U = 1\) holds since by construction \(1 \in U_{(0, 0)}\). We now assume (4.31) to be true for a given \(U\) and give ourselves a \(D \in L\) such that \(D \in U_{b'}\). We express \(z^\gamma UD^{(n)}\) using (4.7). Since \(DU \in U_{b + b'}\), our goal is to prove \(z^\gamma UD^{(n)} \in U_{b + b' + \text{bi}(\gamma, n)}\). Indeed, for the first r. h. s. term of (4.7), by the induction hypothesis,

\[
Dz^\gamma UD^{(n)} \in U_{b' + b + \text{bi}(\gamma, n)}.
\]

For the second r. h. s. term, we note that by the compatibility of \(\text{bi}\) and \(\triangleright\), see Subsection 3.8, \(D_\gamma \neq 0\) implies

\[
\text{bi}(\beta, n) = b' + \text{bi}(\gamma, n),
\]
which combined with the induction hypothesis yields
\[ D_\beta^\gamma z^\beta UD^{(n)} \in U_{b^r + b + bi(\gamma, n)}. \]

Any linear combination of the bigrading defines a new grading compatible with the Hopf algebra structure of \( U(L) \). In view of the definition (3.39) of the homogeneity, a natural choice is to consider the first component weighted by \( \alpha \) and the second weighted by 1; by (3.39), this defines
\[ |(J, m)|_{gr} := \sum_{(\gamma, n)} J(\gamma, n)(|\gamma| - |n|) + |m|. \]

Thanks to our restriction \(|\gamma| > |n|\), cf. (3.41), it holds that
\[ |(J, m)|_{gr} \geq 0, \quad \text{and} \quad |(J, m)|_{gr} = 0 \iff (J, m) = (0, 0). \]

With help of the bigrading (4.30) and the grading (4.32) we will now establish finiteness properties of the action and the product. For this, we first write the basis representations of both maps. It is tautological that the basis representation of the action \( U(L) \otimes T^* \rightarrow T^* \) with respect to (4.15) and the monomial “basis”\(^{41}\), i.e.
\[ D_{(J,m)} \otimes z^\gamma 
\rightarrow \sum_{\beta} \Delta_{\beta}^\gamma (J, m) z^\beta, \]
is given by
\[ \Delta_{\beta}^\gamma (J, m) = (D_{(J,m)})_\beta^\gamma, \]
where \( (D_{(J,m)})_\beta^\gamma \) is the matrix representation of \( \rho D_{(J,m)} \in \text{End}(T^*) \). We choose the notation \( \Delta \) since it will give rise to a coaction, cf. (4.44), that plays the role of the one in [17, Subsection 4.2].

**Lemma 4.9.**
\[ \{(J, m, \gamma) \text{ with } |\gamma| \geq 0 \text{ or } \gamma = e_n | \Delta_{\beta}^\gamma (J, m) \neq 0 \} \]
is finite for all \( \beta \).

Moreover, for \( (J, m) \neq (0, 0) \) we have the triangular structure
\[ \Delta_{\beta}^\gamma (J, m) \neq 0 \iff |\gamma| < |\beta|. \]

We stress that the restriction of \( \gamma \) is crucial in our approach, as will become apparent in the proof. Incidentally, by (3.33), it implies the same restriction for \( \beta \).

**Proof.** We first show by induction in the length \(|(J, m)|\) that
\[ (D_{(J,m)})_\beta^\gamma \neq 0 \iff (1 + [\beta], \sum_{n' \neq 0} |n'| \beta(n')) \]
\[ = (1 + [\gamma], \sum_{n' \neq 0} |n'| \gamma(n')) + bi(J, m). \]

\(^{41}\) with a slight abuse of language, since the elements \( z^\gamma \) do not constitute a basis of \( T^* \).
The base case \(|(J, m)| = 0\) is trivial, since this implies \(\beta = \gamma\). In the induction step, we fix \((J, m)\) and distinguish two cases: if \(J = 0\), then the claim follows from \((\partial^m)^\gamma_\beta = \sum_\gamma (\partial^{m_{\text{even}}})^\gamma_\beta (\partial_1)^\gamma_{\gamma'}\) via (3.27) and the induction hypothesis in form of
\[
(\partial^{m_{(1,0)}})^\gamma_{\beta} \neq 0 \implies (1 + [\beta], \sum_{n' \neq 0} |n'| \beta(n')) = (1 + [\gamma'], \sum_{n' \neq 0} |n'| \gamma'(n')) + (0, |m| - (1, 0)).
\]

If \(J \neq 0\), the claim likewise follows via (4.13), which we may use thanks to (4.19), into which we insert (3.38).

We now claim that (4.35) holds true when restricting \((J, m)\) to be of fixed length. Indeed, this is again established by induction in the length \(|(J, m)|\): the argument is the very same as for (4.37), just starting from (3.15) and (3.42) instead of (3.27) and (3.38). The next step is to show that the length \(|(J, m)|\) is bounded. To this end, we first note that for \((\gamma', n')\) with \(J(\gamma', n') \neq 0\) we have \([\gamma'] \geq 0\), so that the first component of \(bi(J, m)\) dominates \(|J|\), cf. (4.30). Therefore, assuming that \((D_{(J, m)})_\beta^\gamma \neq 0\), the first component in (4.37) yields \(|J| \leq 1 + [\beta]\), where we used \([\gamma] \geq -1\). Moreover, taking in (4.37) a linear combination of the first and the second component weighted by \(\alpha\) and \(1\), respectively, yields by definitions (3.39) and (4.32)

\[
(\text{4.38}) \quad |\beta| = |\gamma| + |(J, m)|_{\text{gr}}.
\]

Since \(|\gamma'| > |n'|\) for pairs with \(J(\gamma', n') \neq 0\), and since \(|\gamma| > 0\), we now obtain \(|m| < |\beta|\). Summing up, we find \(|(J, m)| \leq 1 + [\beta] + |\beta|\), which finishes the proof of (4.35). Finally, (4.36) is a straightforward consequence of (4.38) and the positivity of \(\cdot|_{\text{gr}}\), cf. (4.33).

We shall now give a characterization of the basis representation of the concatenation product, i. e.

\[
(\text{4.39}) \quad D_{(J, m')} D_{(J'', m'')} = \sum_{(J, m)} (\Delta^+)^{(J, m)}_{(J', m')} (J'', m'') D_{(J, m)}.
\]

Writing (4.2) in coordinates and using (4.19), we see that the numbers \((\Delta^+)^{(J, m)}_{(J', m')}\) are determined by the special case where the multi-index \((J, m)\) is of length one. This means either \(m \in \{(1, 0),(0, 1)\}\) and \(J = 0\) or \(m = 0\) and the multi-index \(J\) having just one non-trivial entry – equal to one – at \((\gamma, n)\); for this, we write \(J = e_{(\gamma, n)}\). The former case is easy; indeed, by (4.7) and (4.14), we see that \(J = 0\) implies \(J' = J'' = 0\), which reduces all possible situations to formula (4.17). In particular, this yields

\[
(\Delta^+)^{(0, (1, 0))}_{(J', m')} (J'', m'') = \delta^{(0, (1, 0))}_{(J', m') + (J'', m')},
\]

and a similar statement for \(m = (0, 1)\). The case \(J = e_{(\gamma, n)}\) is characterized by the application of (4.22); indeed, \((\Delta^+)^{(e_{(\gamma, n)}, 0)}_{(J', m')} (J'', m'')\) is the coefficient of \(z^n\) in \(e_n D_{(J', m')} D_{(J'', m'')}\).
Lemma 4.10.

\[(J', m'), (J'', m'') \mid (\Delta^+)^{(J,m)}_{(J',m')(J'',m'')} \neq 0\]

is finite for all \((J, m)\).

Proof. Once more by (4.2) and (4.19) it is enough to show (4.41) for \(|(J, m)| = 1\), see the discussion after (4.39). The case \(J = 0\) and \(|m| = 1\) is trivial from (4.40). For \(|J| = 1\) and \(m = 0\), we write \(J = e(\beta, n)\) and claim that (4.41) follows from (4.22). For this, we apply (4.22) with \(U_1 = D(J', m')\) and \(U_2 = D(J'', m'')\), and consider the coefficient of the \(z^\beta\)-term, which is non-vanishing by assumption. By (4.27), the first r. h. s. term in (4.22) is non-vanishing only if \(U_2 = z^\gamma D(n)\) for some \(\gamma\); applying (4.35) to the first factor \(\rho U_1\), we see that there are only finitely many \(\gamma\)'s and \((J', m')\)'s which give a non-vanishing contribution to the \(z^\beta\)-coefficient. Turning to the second r. h. s. term of (4.22), its \(z^\beta\)-coefficient is non-zero unless \(U_1 = z^\beta D(n+m)\) and \(U_2 = D(0, m)\) for some \(m\). The constraint \(|n + m| < |\beta|\) only allows for finitely many \(m\)'s. \(\square\)

4.5. Dualization leading to \(T^+, \Delta^+\) and \(\Delta\).

We consider the linear space \(T^+\), with basis \(\{Z^{(J,m)}\}_{(J,m)}\) indexed by \((J, m)\) and the canonical non-degenerate pairing between \(U(L)\) and \(T^+\) given by

\[(D(J', m'), Z^{(J,m)}) = \delta^{(J,m)}_{(J',m')}\]

As a consequence of (4.42), \(U(L)\) canonically is a subspace of \((T^+)\); note that the latter is much larger, since it is the direct product over the index set of all \((J, m)\)'s, whereas \(U(L)\) is just the direct sum.

Our next goal is to provide a structure for \(T^+\) by dualization of the Hopf algebra and the module structures of \(U(L)\). First, we note that the basis representation of a coproduct has the algebraic properties of a product, and thus (4.19) defines a product in \(T^+\) given by

\[Z^{(J',m')} Z^{(J'',m'')} = Z^{(J',m') + (J'',m')}\]

This way \((T^+, \cdot)\) becomes the (commutative) polynomial algebra over variables indexed by the index set of \(L\).

In a similar way, we want to transpose the action and the coproduct mentioned in the previous subsection. The transposition in these two cases is possible thanks to the finiteness properties which were stated in Lemmas 4.9 and 4.10. Starting with the action, analogously to (3.4), from the basis representation (4.34) we define a map \(\Delta : T \rightarrow T^+ \otimes T\) by

\[\Delta z_\beta = \sum_{\gamma, (J,m)} \Delta^\gamma_\beta (J,m) Z^{(J,m)} \otimes z_\gamma.\]

\(^{42}\)unique up to linear isomorphisms
\(^{43}\)We will omit the dot in the notation.
The sum is finite due to (4.35), and hence $\Delta$ is well-defined. We stress that the restriction to $T$ is crucial for our argument; it does not seem possible to extend $\Delta$ to $\mathbb{R}[[z_0, z_n]]^! \to T^+ \otimes \mathbb{R}[[z_0, z_n]]^!$.

We now turn to the product; by the basis representation (4.39), we define a map $\Delta^+: T^+ \to T^+ \otimes T^+$ via
\begin{equation}
\Delta^+ Z^{(J,m)} = \sum_{(J',m'), (J'',m'')} (\Delta^+)^{(J,m)}_{(J',m') (J'',m'')} Z^{(J',m')} \otimes Z^{(J'',m'')}.
\end{equation}
Such a map has the algebraic properties of a coproduct in $T^+$. The fact that this map is well-defined is a consequence of the finiteness property (4.41).

The only missing ingredient to make $T^+$ a Hopf algebra is an antipode $S$: since (4.32) and (4.33) make $T^+$ a connected graded bialgebra, this is guaranteed by general theory, see [19, Proposition 3.8.8].

These observations are collected in the following result.

**Proposition 4.11.** Let $\Delta^+: T^+ \to T^+ \otimes T^+$ be given by (4.45). Then there exists a map $S$ such that $(T^+, \cdot, \Delta^+, S)$ is a Hopf algebra with antipode $S$. Moreover, let $\Delta: T \to T^+ \otimes T$ be given by (4.44). Then $(T, \Delta)$ is a (left-) comodule over $T^+$, i.e.
\begin{equation}
(id \otimes \Delta) \Delta = (\Delta^+ \otimes id) \Delta.
\end{equation}
Note that (4.46) is the dualization of the morphism property of $\rho$.

We now have introduced all the objects required to construct a structure group $G \subset \text{End}(T)$ according to [17, Section 4.2]. A minor difference is that, in our case, $(T, \Delta)$ is a left comodule while in [17, (4.15)] it is a right comodule, a fact that transfers to (4.49) and more upcoming identities. This does not affect the construction. In fact, with a similar (though more cumbersome) definition of the Lie algebra $L$, working at the level of the transposed endomorphisms from the beginning, we would have been able to recover the same structure, but paying the price of blurring the connection to the actions on $(a, p)$-space that served as a motivation in Section 2.

### 4.6. Intertwining of $\Delta$ and $\Delta^+$ through $J_n$.

Let us define for every $n$ a map $J_n : T \to T^+$ in coordinates by
\begin{equation}
J_n Z_{\gamma} = \begin{cases} 
  n! Z^{(e,0,0)}_{(\gamma, n, 0)} & \text{if } [\gamma] \geq 0, \ |\gamma| > |n| \\
  0 & \text{otherwise}
\end{cases}
\end{equation}
Note that in view of (4.20) $J_n$ is the transposition of $\iota_n$ up to a combinatorial factor:
\begin{equation}
J_n = n! \iota_n^t.
\end{equation}

---

44 The existence of unit and counit maps easily follows from transposing the counit and unit, respectively, of $U(L)$. No finiteness properties are required.

45 i.e. the zero-degree subspace is $\mathbb{R}$, cf. [19, Definition 2.10.6]
The normalization with $n!$ is made such that the dualization of (4.22) takes

the form of the following intertwining relation between the coaction $\Delta$ and

the coproduct $\Delta^+$, which is an identity in $T^+ \otimes T^+$:

$$\Delta^+ J_n z_\gamma = (\text{id} \otimes J_n) \Delta z_\gamma + \sum_m J_{m+n} z_\gamma \otimes \frac{Z^{(0,m)}}{m!}. \quad (4.49)$$

Indeed, (4.49) amounts to (4.22) once tested with $U_1 \otimes U_2$, where we use

the pairings (3.1) and (4.42), the definitions (4.39) and (4.34), and the fact

that $\langle U, Z^{(0,m)} \rangle = \varepsilon_m U$ in view of (4.21). Combined with

$$\Delta^+ Z^{(0,(1,0))} = Z^{(0,(1,0))} \otimes 1 + 1 \otimes Z^{(0,(1,0))}, \quad (4.50)$$

which follows from (4.40), we see that $\Delta^+$ is determined by $\Delta$ through $J_n$ in

agreement with regularity structures, cf. [17, (4.14)]. Let us also mention

that the coaction applied to the polynomial sector $\tilde{T}^*$ is in agreement

with [17, p. 23],

$$\Delta z_\gamma e_n = \sum_{n',n''} \left( \begin{array}{c} n' \\ n'' \end{array} \right) Z^{(0,n')} \otimes z_{e_{n''}}. \quad (4.51)$$

This may be seen by (4.44) and (4.34). First note that $\rho D(J,m')$ preserves

$\tilde{T}^*$, as a consequence of the same property of $L$, see (3.32), and that

$\rho D(J,m')$ maps to $\bar{T}^*$ only if $J = 0$ as can be read off (4.15). Therefore, (4.51) follows

from $D_{(0,n')}(z_{n''}) = \left( \begin{array}{c} n'' \\ n' \end{array} \right) z_{n'-n''}$, which is a consequence of (4.15) and (3.12).

5. The group structure

5.1. The structure group $G$.

With all the algebraic objects defined in Section 4, we follow [17, Subsection

4.2] in the construction of the structure group. Let us consider the space

of multiplicative linear functionals on $T^+$, which we denote by $\text{Alg}(T^+, \mathbb{R})$.

Writing

$$f^{(J,m)} := f Z^{(J,m)} \quad (5.1)$$

for $f \in (T^*)^*$, by (4.43) the space $\text{Alg}(T^+, \mathbb{R})$ is characterized by

$$f^{(J',m')+(J'',m'')} = f^{(J',m')} f^{(J'',m'')} \quad \text{and} \quad f^{(0,0)} = 1. \quad (5.2)$$

Due to this property, the elements $f \in \text{Alg}(T^+, \mathbb{R})$ are parameterized by

$$f^{(J,m)} = h^m \prod_{\gamma,n} (\pi^{(n)}_{\gamma})^{J(\gamma,n)}, \quad (5.3)$$

where $h \in \mathbb{R}^2$ and $\{\pi^{(n)}\}_{n} \subset \bar{T}^*$ is constrained by

$$\pi^{(n)}_{\gamma} \neq 0 \quad \iff \quad |\gamma| > |n|. \quad (5.4)$$

From the Hopf algebra structure of $T^+$, the space $\text{Alg}(T^+, \mathbb{R})$

inherits a natural group structure, namely the convolution product of functionals:

$$fg := (f \otimes g) \Delta^+. \quad (5.5)$$

46up to the constant in $\tilde{T}$ which we modded out, cf. (2.2) and (3.17)

47By (4.47) and (5.3), the coefficients $\pi^{(n)}_{\gamma}$ may be identified with [17, (4.8)].
The neutral element $e$ of this group, which is the counit of $\mathcal{T}^+$, maps $Z^{(0,0)}$ to 1 and every other basis element of $\mathcal{T}^+$ to 0, and the inverse elements are given by $f^{-1} = fS$, cf. [1, Theorem 2.1.5].

Following [17, Subsection 4.2], we now define a map $\Gamma : (\mathcal{T}^+)^* \rightarrow \text{End}(\mathcal{T})$ by

$$\Gamma_f := (f \otimes \text{id}) \Delta.$$  

Then the set

$$G := \{\Gamma_f \mid f \in \text{Alg}(\mathcal{T}^+, \mathbb{R})\} \subset \text{End}(\mathcal{T})$$

inherits the group structure of $\text{Alg}(\mathcal{T}^+, \mathbb{R})$, where

$$\Gamma_e = \text{id}, \quad \Gamma_{fg} = \Gamma_f \Gamma_g \quad \text{and} \quad \Gamma_{f^{-1}} = (\Gamma_f)^{-1};$$

we call $G$ structure group. Applying definition (5.6) to $z_\beta$, plugging in (4.44), (5.1) and (4.34), we obtain from (3.4) the representation

$$\Gamma_f = \sum_{(J,m)} f^{(J,m)} D_{(J,m)}^{\dag},$$

and note that this sum is effectively finite because of (4.35). Moreover, as a consequence of (4.36),

$$(\Gamma_f - \text{id})_{\beta} \neq 0 \implies |\gamma| < |\beta|;$$

this may be rewritten more in line with the corresponding requirement in [17, Definition 3.1]:

$$(\Gamma_f - \text{id})_{T_\kappa} \subset \bigoplus_{\kappa' < \kappa} T_{\kappa'}.$$  

The elements of the structure group behave nicely with the polynomial sector $T$, see Subsection 3.3, in the sense that for $f \in \text{Alg}(\mathcal{T}^+, \mathbb{R})$ with $h \in \mathbb{R}^2$ being its parameter according to (5.3), and for all $n \neq 0$,

$$\Gamma_f x^n = \sum_{m<n} \binom{n}{m} h^m x^{n-m},$$

which follows from (5.6), (4.51) and (5.3). In Subsection 5.3, we argue that this is in line with [17, Assumption 3.20].

5.2. Consistency of $G^*$ with our goals. 

Our final goal is to identify the construction of Subsection 5.1 with the group of transformations on $(a, p)$-space heuristically described in Section 2. We define $G^* := \{\Gamma^* \mid \Gamma \in G\}$, which due to (5.8) is formed by the maps

$$\Gamma_f^* = \sum_{(J,m)} f^{(J,m)} D_{(J,m)} \in \text{End}(\mathcal{T}^*) \quad \text{where} \quad f \in \text{Alg}(\mathcal{T}^+, \mathbb{R}).$$

From (5.7), $G^*$ inherits a group structure given by

$$\Gamma_e^* = \text{id}, \quad \Gamma_{fg}^* = \Gamma_g^* \Gamma_f^* \quad \text{and} \quad \Gamma_{f^{-1}}^* = (\Gamma_f^*)^{-1};$$

\[\text{Here and in the sequel we identify } D_{(J,m)} \text{ with the corresponding endomorphism } \rho_{D_{(J,m)}}.\]
note that the order in the composition rule is reversed as a consequence of transposition.

We gather all our results in the following proposition.

**Proposition 5.1.** Let $h \in \mathbb{R}^2$ and \{\(\pi^{(n)}_\gamma\)\}_{(\gamma, n)} \subset \mathbb{R}$ generate $f$ through the characterization (5.3). Let $\pi^{(n)} \in T^*$ for every $n \in \mathbb{N}_0$ be given by

\[
\pi^{(n)} = \sum_{\lceil n \rceil \geq 0} \pi^{(n)}_\gamma z^\gamma + \sum_{m \neq 0} \pi^{(m)}_\gamma z_m,
\]

where

\[
\pi^{(m)}_\gamma := \begin{cases} 
(m \choose n) h^{m-n} & \text{if } n < m \\
0 & \text{otherwise}
\end{cases}
\]

i) The following formula holds

\[
\Gamma_f^* = \sum_{k \geq 0} \frac{1}{k!} \sum_{n_1, \ldots, n_k} \pi^{(n_1)} \cdots \pi^{(n_k)} D^{(n_k)} \cdots D^{(n_1)}.
\]

In particular,

\[
\Gamma_f^* z_k = \sum_{l \geq 0} {k+l \choose k} (\pi^{(0)})^l z_{k+l} \quad \text{for all } k \geq 0,
\]

\[
\Gamma_f^* z_n = z_n + \pi^{(n)} \quad \text{for all } n \neq 0.
\]

ii) For all $\pi_1, \ldots, \pi_k \in T^*$ such that $\pi_1 \cdots \pi_k \in T^*$,

\[
\Gamma_f^* \pi_1 \cdots \pi_k = (\Gamma_f^* \pi_1) \cdots (\Gamma_f^* \pi_k).
\]

iii) The composition rule (2.8) holds.

iv) If \(\pi^{(n)}\) \(\in T^* \cap \mathbb{R}[z_k, z_n]\), then for all \((a, p)\) and $\pi \in T^* \cap \mathbb{R}[z_k, z_n]$,

\[
\Gamma_f^* \pi[a, p] = \pi \left[ a \left( \cdot + \pi^{(0)}[a, p] \right), p + \sum_{n \neq 0} \pi^{(n)}[a, p] x^n \right].
\]

v) The subset $\tilde{G}^* \subset G^*$ generated via (5.12) by $f \in \text{Alg}(T^+, \mathbb{R})$ of the form (5.3) with $\pi^{(n)}_\gamma = 0$ for all $(\gamma, n)$ is a subgroup isomorphic to $(\mathbb{R}^2, +)$. Moreover, if $\Gamma_f^* \in \tilde{G}^*$ with $h \in \mathbb{R}^2$ as in (5.3), then for all $(a, p)$ and $\pi \in T^* \cap \mathbb{R}[z_k, z_n]$,

\[
\Gamma_f^* \pi[a, p] = \pi \left[ a \left( \cdot + p(h) \right), p(\cdot + h) - p(h) \right].
\]

vi) The subset $\tilde{G}^* \subset G^*$ generated via (5.8) by $f \in \text{Alg}(T^+, \mathbb{R})$ of the form (5.3) with $h = (0, 0)$ is a subgroup.

The reader should see (5.20) as a variant of (2.7) for the functions on $(a, p)$-space given by polynomials $\pi \in T^* \cap \mathbb{R}[z_k, z_n]$, where we interpret $\pi^{(n)}$’s as in (5.14). The subgroups $G^*$ and $\tilde{G}^*$ correspond to shifts and $((a, p)$-dependent) tilts, respectively: the shift (2.5) is recovered by (5.21), whereas (5.20) translates into (2.7) (since also (2.7) includes (2.5)). It is however not possible to recover the tilt by an $(a, p)$-independent polynomial, namely (2.6), because the $\pi^{(n)}$’s are restricted by (3.41) which does not allow $(a, p)$-independent expressions for large $|n|$. Finally, although (5.20) and (5.21)
hold for all possible $\pi \in T^* \cap \mathbb{R}[z_k, z_n]$, there is no hope to extend them to $T^*$ because generic elements of $\mathbb{R}[[z_k, z_n]]$ cannot be identified with functions of $(a, p)$; the same applies for $\{\pi^{(n)}\}_n$ in (5.20).

**Proof.** We first show (5.19); indeed, it is a direct consequence of (5.2) and the following generalized Leibniz rule: For all $\pi_1, \ldots, \pi_l \in \mathbb{R}[z_k, z_n]$

\[(5.22) D_{(j,m)} \pi_1 \cdots \pi_l = \sum (D_{(j_1,m_1)} \pi_1) \cdots (D_{(j_l,m_l)} \pi_l),\]

where the sum runs over all $(j_1, m_1), \ldots, (j_l, m_l)$ with $(j_1, m_1) + \ldots + (j_l, m_l) = (j, m)$. It is easy to see that by induction, (5.22) for general $l \in \mathbb{N}$ follows from the case $l = 2$. In view of (4.19), this case reduces to

\[(5.23) U \pi_1 \pi_2 = \sum_{(U)} (U(1) \pi_1)(U(2) \pi_2)\]

for $U \in U(L)$, where we do not distinguish between $\rho U \in \text{End}(T^*)$ and $U$. Formula (5.23) is trivial for $U = 1$; it is obvious for $U \in L \subset \text{Der}(T^*)$. It remains to pass from $U$ to $UD$ for some $D \in L$. For the l. h. s. of (5.23) we note that by induction hypothesis (and base case) we have

\[UD \pi_1 \pi_2 = \sum_{(U)} ((U(1)D \pi_1)(U(2) \pi_2) + (U(1) \pi_1)(U(2)D \pi_2)).\]

For the r. h. s. of (5.23) we have by the compatibility of the coproduct with concatenation (composition) and (4.1)

\[\text{cop}UD = \sum_{(U)} (U(1)D \otimes U(2) + U(1) \otimes U(2)D).\]

We now turn to the proof of (5.16). We first argue that the r. h. s. of (5.16), when interpreted as an endomorphism of $T^*$, is effectively finite (note that we already know that the l. h. s. is effectively finite from (4.35)). For this, we note that the r. h. s. of (5.16) is an infinite sum of terms of the form

\[z^{\gamma_1} \cdots z^{\gamma_k} D^{(n_k)} \cdots D^{(n_1)},\]

where either $[\gamma_i] \geq 0$ or $\gamma_i \in \{e_n\}_{n \neq 0}$ for $i = 1, \ldots, k$. We extend the family of derivations $\hat{L}$ by incorporating purely polynomial multi-indices, so that we consider the set $\{z^{\gamma} D^{(n)}\}_{[\gamma] \geq 0, [\gamma] > |n|} \cup \{z_m D^{(n)}\}_{m > |n|}$. It can be easily checked that this family is closed under the standard pre-Lie product $\lhd$ given by the first item in (3.36): For the mixed terms this follows from $(z_m D^{(0)})z^{\gamma} = \sum_{k \geq 0} (k + 1) \gamma(k) z^{\gamma - e_k + e_{k+1} + e_m}$ with $|\gamma - e_k + e_{k+1} + e_m| = |\gamma| + \alpha + |m| > |\gamma|$, cf. (3.10); from $(z_m D^{(n)})z^{\gamma'} = \gamma'(n) z^{\gamma + \gamma'} + e_n$ with $|e_m + \gamma - e_n| = |\gamma'| + |m| - |n| > |\gamma| > |n|$, cf. (3.12); and from $(z^{\gamma'} D^{(n')})z_m = \delta_{m,z}^{n,z}$. With this extension at hand, we follow the strategy of Lemma 4.9. For this we need the two following properties:
showing (5.16) amounts to showing (5.17) and (5.18). We start with (5.17).

Using (5.12) and applying (A.1) (with the roles of \( \gamma \) it is enough to consider

Here we interpret \( \iota \)

that it is enough to show that

By (5.3), (4.20) and (4.21), this equals

which by the binomial formula and (5.14) is seen to coincide with

To show (5.18), we appeal to (5.12) to see

Indeed, both follow from (3.10) and (3.12). Under these hypotheses, the
inductive argument in the proof of Lemma 4.9 may be used here to show
the analogue of (4.35), and thus effective finiteness of the r. h. s. of (5.16).

Hence, to show (5.16) it is enough to apply both sides to a monomial
\( z^\gamma \) with \( [\gamma] \geq 0 \) or \( \gamma \in \{e_n\}_{n \neq 0} \). Note that both sides are multiplicative\(^{49}\), so it is enough to consider \( \gamma \)'s of length one. Thus, in view of (3.9) and (3.12),
showing (5.16) amounts to showing (5.17) and (5.18). We start with (5.17).

Using (5.12) and applying (A.1) (with the roles of \( l \) and \( k \) flipped), we see
that it is enough to show that

Here we interpret \( \iota_0 + \sum_{n \neq 0} e_n \otimes z_n \) as a linear map from \( U(L) \), a space endowed with coproduct, into the algebra \( \mathbb{R}[[z_k, z_n]] \), so that powers make sense. By the binomial formula\(^{50}\) applied to \( \iota_0 + \sum_{n \neq 0} e_n \otimes z_n \)\(^k\) and (4.19), the l. h. s. equals

By (5.3), (4.20) and (4.21), this equals

which by the binomial formula and (5.14) is seen to coincide with \( (\pi(0))^k \).
To show (5.18), we appeal to (5.12) to see

\( \Gamma \) is finite for all \( \beta' \).

Indeed, this holds for any effectively finite expression of the form of the r. h. s.
of (5.16) with \( D^{(n)} \)'s being commuting derivations and \( \pi^{(n)} \)'s being multiplication operators.

\( \)\(^{49}\)Note that the coproduct in \( U(L) \) is co-commutative, see (4.19), therefore the product
of linear maps from \( U(L) \) to any commutative algebra is Abelian.

\( \)\(^{50}\)
since \( D_{(J,m)} \) with \( J \neq 0 \) and \( m \neq 0 \) would annihilate \( z_n \), cf. (4.15). The first sum has contributions only from \( J = e_{(\gamma,n)} \), therefore by (3.12) and (5.3) we have

\[
\Gamma_j^* z_n = z_n + \sum_{\gamma} \pi_\gamma^{(n)} z_\gamma + \sum_{m \neq 0} (n+m) h^m z_{n+m},
\]

which together with (5.14) yields (5.18).

We now turn to the proof of the composition rule (2.8). Given \( \pi^{(n)}_\gamma, \pi^{(n)}_\gamma \) with corresponding \( \Gamma^*, \Gamma'^* \), we define \( \pi^{(n)} := \pi^{(n)} + \Gamma^* \pi^{(n)} \) and have to show that the corresponding \( \Gamma^* \) satisfies \( \Gamma^* = \Gamma^* \Gamma'^* \). By multiplicativity (5.19), it is enough to show that \( \Gamma^* \) coincides with \( \Gamma^* \Gamma'^* \) on the coordinates \( \{ z_k, z_n \}_{k \geq 0, n \neq 0} \). For \( n \neq 0 \), we obtain by applying (5.18) three times

\[
\Gamma^* \Gamma'^* z_n = \Gamma^* (z_n + \pi^{(n)}) = z_n + \pi^{(n)} + \Gamma^* \pi^{(n)} = \Gamma^* z_n.
\]

For \( k \geq 0 \) we apply (5.17) twice and use multiplicativity (5.19) to obtain

\[
\Gamma^* \Gamma'^* z_k = \Gamma^* \sum_{l \geq 0} \binom{k+l}{k} (\pi^{(0)})^l z_{k+l} = \sum_{l \geq 0} \binom{k+l}{k} (\Gamma^* \pi^{(0)})^l \sum_{l \geq 0} \binom{k+l+l}{k+l} (\pi^{(0)})^l z_{k+l+l}.
\]

A re-summation together with the binomial formula and applying once more (5.17) yield

\[
\Gamma^* \Gamma'^* z_k = \sum_{l \geq 0} \binom{k+l}{k} (\pi^{(0)} + \Gamma^* \pi^{(0)})^l z_{k+l} = \Gamma^* z_k,
\]

which finishes the proof of (2.8).

To prove (5.20), by (5.19) it is enough to show it for \( \pi \in \{ z_k, z_n \}_{k \geq 0, n \neq 0} \). These special cases are a consequence of (5.17), (5.18) and Taylor’s formula.

We now argue in favor of \( v \). By (5.13), we have to show that for \( f \) of the form

\[
(5.24) \quad f^{(J,m)} = \begin{cases} h^m & \text{if } J = 0 \\ 0 & \text{otherwise} \end{cases}
\]

the product (5.5) amounts to addition of \( h \in \mathbb{R}^2 \). Indeed,

\[
(5.5) \quad (f' f'')^{(J,m)} \equiv (5.5), (5.24) \sum_{m',m''} (\Delta^+)^{(J,m)} (0,m')(0,m'') (h')^{m'} (h'')^{m''}
\]

\[
(4.17), (4.30) \delta_0^J \sum_{m'+m''=m} \binom{m'+m''}{m'} (h')^{m'} (h'')^{m''},
\]

and we conclude by the binomial formula. For \( f \) of the form (5.24), and using (2.9), (5.14) assumes the form \( \sum_n \pi^{(n)}[a, p] x^n = p(x + h) \), so that (5.20) turns into (5.21).

We finally turn to the proof of \( vi \). By (5.13), it suffices to show that the set of \( f \in \text{Alg}(T^+, \mathbb{R}) \) such that

\[
f^{(J,m)} = 0 \quad \text{for } m \neq 0
\]
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is a subgroup; this is a direct consequence of $U(\tilde{L})$, cf. (4.3), being a sub-Hopf algebra of $U(L)$.

5.3. Relation to Hairer’s regularity structure.
In this subsection, while keeping $G$ as an abstract group, we enlarge the abstract model space $T$ on which it acts. We do so in order to draw a closer connection to [17]. We will proceed in two steps, first enlarging the abstract model space by a placeholder for the omitted constants, and then by a placeholder for the right-hand side of the equation.

While thinking of $p$ only modulo constants was an important guiding principle in uncovering the algebraic structure, see Section 2, we will now reintroduce constants into the polynomial sector $\bar{T}$, cf. Subsection 3.3, by augmenting its basis $\{x^n\}_{n \neq 0}$ by the element $x^0$. As a consequence, we pass from $T$ to $\mathbb{R} \oplus T$. We now argue that the action of $G$ naturally extends to $\mathbb{R} \oplus T$, where we first adopt the point of view of Subsection 5.1: Indeed, given $h \in \mathbb{R}^d$ and $\{\pi^{(n)}\}_n \subset \tilde{T}$, which gives rise to $\Gamma \in \text{End}(T)$, the extension to an endomorphism of $\mathbb{R} \oplus T$ is visualized by the block structure

\[(5.25) \quad \begin{pmatrix} \text{id} & \pi^{(0)} + \sum_{m \neq 0} h^m z_m \\ 0 & \Gamma \end{pmatrix}.\]

This form of extension completes the action (5.11) on the (extended) polynomial sector $\mathbb{R} \oplus \tilde{T}$ in the sense of [17, Assumption 3.20]: Since an element of $\tilde{T}^*$, like $\pi^{(0)}$, is characterized by vanishing on $\tilde{T}$, (5.25) maps the basis element $x^n$ onto $\sum_m \binom{n}{m} h^m x^{n-m}$, which formally can be written as $(x+h)^n$.

We will motivate the presence of $\pi^{(0)}$ in (5.25) below.

For (5.25) to define an action, we need to check that the composition of two endomorphisms of the form of (5.25) preserves this form. This is more easily seen for the induced dual action on $(\mathbb{R} \oplus T)^* \cong \mathbb{R} \oplus T^*$, which is of the block form

\[(5.26) \quad \begin{pmatrix} \text{id} & \pi^{(0)} + \sum_{m \neq 0} h^m z_m \\ \pi^{(0)} + \sum_{m \neq 0} h^m z_m & \Gamma^* \end{pmatrix}.\]

It is now convenient to adopt the point of view of Subsection 5.2, which amounts to viewing the lower left entry of (5.26) as a single object, as done in (5.14) and (5.15), still labelled by $\pi^{(0)}$. The desired statement then follows from (2.8), which was rigorously established in part iii) of Proposition 5.1.

It is also on the level of this extended definition (5.14) of $\pi^{(0)}$ that we may motivate (5.26): The purpose of $G^*$ is to contain elements $\Gamma_{xy}$ that “algebrize” the re-centering of the model, which is a $T^*$-valued function \footnote{The sum in the upper right entry is effectively finite and thus defines an element of $T^*$, in line with the meaning of this block} of space-time, from its version $\Pi_y$ centered at one base point $y$ to its version $\Pi_x$ centered at another base point $x$, see [17, Definition 3.3]. In the application

\footnote{Or distribution, depending on the application}
of our setting, this holds only up to a space-time constant 
\( \pi_{xy}^{(0)} \in T^* \), that is,
\[
\Pi_x = \Gamma_{xy}^* \Pi_y + \pi_{xy}^{(0)},
\]
which however is tied to \( \Gamma_{xy}^* \) via (5.17). Now (5.26), with \( \Gamma^* \) and \( \pi^{(0)} \)
specified to \( \Gamma_{xy}^* \) and \( \pi_{xy}^{(0)} \), is made such that (5.27) exactly assumes the
form of [17, Definition 3.1] provided we augment the model by the constant
space-time function of value 1, which lives in the \( \mathbb{R} \)-component of \( \mathbb{R} \oplus T^* \).

We now come to the second extension. Hairer’s construction of a regu-
lar-ity structure is bottom-up and combinatorial, in the sense that the index
set of the abstract model space encodes all combinations of integ-
ration and multiplication which are relevant for the equation. In particular, this
implies that for every model component \( \Pi_{x\beta} \), also \( \Pi_{-x\beta} := (\partial_2 - \partial_1^2)\Pi_{x\beta} \)
is a component of the model. Since in regularity structures one only cares
for the equation up to polynomials, it is natural to think of \( \Pi_{-x} \) as a
\( \tilde{T}^* \)-valued Schwartz distribution; recall that \( \tilde{T}^* \) is canonically charac-
terized as the space of all linear functionals \( \pi \in T^* \) that vanish on the \( x^n \)'s. In order
to capture this on the level of our abstract model space, or rather its dual,
we pass from \( (\mathbb{R} \oplus T)^* \) to \( (\mathbb{R} \oplus T)^* \oplus \tilde{T}^* \).

We extend (5.26) on this larger space \( (\mathbb{R} \oplus T)^* \oplus \tilde{T}^* \) as follows
\[
(5.28)
\left( \sum_m \pi^{(m)} \otimes (\partial_2 - \partial_1^2)^{1}x^m \quad 0 \quad \Gamma^* \right).
\]
In formulating (5.28), we return to the perspective of Subsection 5.1 in the
sense that \( \pi^{(m)} \in \tilde{T}^* \), so that together with \( (\partial_2 - \partial_1^2)^{1}x^m \in \mathbb{R} \oplus T \), which
canonically embeds into the bi-dual \( (\mathbb{R} \oplus T)^* \), the bottom left entry indeed
defines a linear map from \( (\mathbb{R} \oplus T)^* \) into \( \tilde{T}^* \). Here, we extended the definition of \( \partial_1^1 \), see (3.20), to \( x^0 \) in the obvious way, so that \( m = (0,0),(1,0) \) do
not contribute, and the contribution of \( m = (0,1),(2,0) \) renders \( x^0 \) and
\(-2x^0 \), respectively. The sum is effectively finite according to the population
condition (5.4). Since as a consequence of (3.32), \( \Gamma^* \) preserves \( \tilde{T}^* \), it is
legitimate as a bottom right entry.

The motivation for the extension (5.28) is again given by the application
[23, (2.40)] of our abstract structure. Indeed, the new model components
transform according to \( \Gamma_{xy}^* \) up to a \( \tilde{T}^* \)-valued (formal) power series in space-
time
\[
(5.29)
\Pi_x = \Gamma_{xy}^* \Pi_y + \sum_m \left( \frac{\partial}{\partial z_2} - \frac{\partial^2}{\partial z_1^2} \right)(\cdot - y)^m \pi_{xy}^{(m)},
\]
where again the coefficients are tied to \( \Gamma_{xy}^* \) via (5.18); note that the terms
\( m = (0,0),(1,0) \) do not contribute. We note that by [23, (2.23)], which
is in line with the axioms [17, first item in (3.11)], and (3.20), we may

\[\text{which in case of (1.1) involves the two kernels } \left( \frac{\partial}{\partial z_2} - \frac{\partial^2}{\partial z_1^2} \right)^{-1} \text{ and } \left( \frac{\partial}{\partial z_2} - \frac{\partial^2}{\partial z_1^2} \right)^{-1} \frac{\partial^2}{\partial z_1^2}; \]

here we denote by \( z \) the active variable of \( \Pi_x \).
write \( \left( \frac{\partial^2}{\partial y_2} - \frac{\partial^2}{\partial y_1} \right)(\cdot - y)^m = (\Pi_y (\partial_2 - \partial_1^2)^t x^m) \). Hence we see that (5.29) assumes the axiomatic form [17, Definition 3.3], which is free of polynomial corrections, under the extension (5.28).

In order to establish that (5.28) provides indeed a representation of \( \mathcal{G} \), we now argue that it is compatible with composition. By the compatibility of (5.26), and of the bottom right block of (5.28), we are left with the bottom left block of the product, which consists of the summands

\[
(5.30) \quad \pi^{(m)} \otimes ((5.25)' \text{ applied to } (\partial_2 - \partial_1^2)^t x^m) + \Gamma^* \pi^{(m)} \otimes (\partial_2 - \partial_1^2)^t x^m,
\]

where (5.25)' stands for (5.25) with \( \Gamma \) and \( \pi^{(0)} \) replaced by \( \Gamma' \) and \( \pi^{(0)}' \), respectively. We note that \( p_m := (\partial_2 - \partial_1^2)^t x^m \) is an element of the (extended) polynomial sector \( \mathbb{R} \otimes \mathbb{T} \); hence applying (5.25)' to it, we obtain

\[
(5.31) \quad \pi^{(m)} \otimes p_m(\cdot + h') + \Gamma^* \pi^{(m)} \otimes p_m \quad \text{with} \quad p_m = (\partial_2 - \partial_1^2)^t x^m.
\]

We need to re-express (5.31) in terms of the extended definition (5.14) in order to (eventually) apply (2.8). To this purpose we introduce

\[
\text{id} - P = \sum_{n \neq 0} z_n \otimes x^n,
\]

which since \( \{z_n\}_{n \neq 0} \) is dual to \( \{x^n\}_{n \neq 0} \) defines a projection \( P \) from \( \mathcal{T}^* \) onto \( \mathbb{T}^* \) that allows to pass from the extended definition (5.14) of \( \pi^{(m)} \) to the original one used in (5.31). Clearly, the second summand in (5.31) calls for the commutator of \( \Gamma^* \) and \( P \), of which we need a representation: Using once more that \( \Gamma^* \) preserves \( \mathbb{T}^* \), which can be written as \( \Gamma^* P = P(\Gamma^* - \Gamma^*(\text{id} - P)) \), we obtain by (5.18)

\[
\Gamma^* P = P(\Gamma^* - \sum_{n \neq 0} \pi^{(n)} \otimes x^n).
\]

In view of (5.31), we apply this to \( \pi^{(m)}' \), which by (5.15) yields

\[
\Gamma^* P \pi^{(m)}' = P(\Gamma^* \pi^{(m)} - \sum_{n > m} \frac{n}{m} h^{m-n} \pi^{(n)}).
\]

Hence in terms of the extended definition (5.14), (5.31) assumes the form

\[
(5.32) \quad P \left( \pi^{(m)} \otimes p_m(\cdot + h') + (\Gamma^* \pi^{(m)} - \sum_{n > m} \frac{n}{m} h^{m-n} \pi^{(n)}) \otimes p_m \right)
\]

\[
(5.33) \quad + \pi^{(m)} \otimes (p_m(\cdot + h') - p_m) - \sum_{n > m} \frac{n}{m} h^{m-n} \pi^{(n)} \otimes p_m.
\]

\[^{54} \text{to be interpreted in a formal sense.} \]
According to (2.8), the term in line (5.32) is the desired output. Hence it remains to argue that the term in (5.33) vanishes when summed over \(m\), which by resummation and relabelling amounts to

\[
p_m(\cdot + h') - p_m = \sum_{n < m} (m_n) h^{m-n} p_n.
\]

Recalling that \(p_n = (\partial_2 - \partial_1^2)^t x^n\), and by (3.20), this follows from the same formula with \(p_n\) replaced by \(x^n\), which amounts to Leibniz' rule.

We note that \(\tilde{T} \subset T\) is canonically defined as consisting of those elements that are annihilated by \(\{z_n\}_{n \neq 0} \subset T^*\); it is a natural complement of \(\bar{T}\) in \(T\). Moreover, \((\mathbb{R} \oplus T)^* \oplus \tilde{T}^*\) is the dual of \((\mathbb{R} \oplus T) \oplus \tilde{T}\). Passing to this primal side \((\mathbb{R} \oplus T) \oplus \tilde{T}\), (5.28) turns into

\[
\Gamma = \left( \begin{array}{c} (5.25) \sum_m \pi^{(m)} \otimes (\partial_2 - \partial_1^2)^t x^m \\ P^\dagger \Gamma \end{array} \right),
\]

since the bottom r. h. s. of (5.28) can be rewritten as \(\Gamma^* P\), and since \(P^\dagger\) defined through

\[
\text{id} - P^\dagger = \sum_{n \neq 0} x^n \otimes z_n
\]

is the dual of \(P\), and a projection from \(T\) onto \(\tilde{T}\). Hairer's integration map [17, Assumption 3.21] has the block form

\[
I = \left( \begin{array}{cc} 0 & \iota \\ 0 & 0 \end{array} \right),
\]

where \(\iota\) is the injection \(\tilde{T} \subset T\). By definition (5.25), the commutator \(\Gamma I - I \Gamma\) has the block form

\[
\Gamma I - I \Gamma = \left( \begin{array}{cc} 0 & (\text{id} - P^\dagger) \Gamma \\ 0 & 0 \end{array} \right),
\]

so that by (5.35), the image of \(\Gamma I - I \Gamma\) is contained in \(\tilde{T}\), in line with the axiom [17, (3.9)].

We close this subsection by arguing, in line with the axiom [17, Definition 3.1], that the matrix representation of \(\Gamma - \text{id}\) is strictly triangular, provided we extend the basis \(\{z_\beta\}_{(\beta) \geq 0} \cup \{\beta = e_m\}\) of \(T\) to a basis of \(\mathbb{R} \oplus T \oplus \tilde{T}\), and extend the homogeneity of its index set as follows: On the \(\mathbb{R}\)-component, we take the dual basis vector to \(x_0\), and endow this single index, which we (momentarily) denote by 0, with homogeneity 0; on the \(\tilde{T}\)-component, we take the basis \(\{z_\beta\}_{(\beta) \geq 0}\), endowing the index \(\beta\) with the homogeneity \(|\beta| - 2\). In particular, the integration map (5.36) increases the homogeneity by 2, which is the order of \(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2}\). We first turn to (5.25), where it remains to consider the upper right entry, which is given by

\[
\Gamma_\beta^0 = \left\{ \begin{array}{ll} \pi_\beta^{(0)} & \text{for } [\beta] \geq 0 \\ h^m & \text{for } \beta = e_m \text{ for some } m \neq 0 \end{array} \right\},
\]
and satisfies triangularity because of $|\beta| > 0 = |0|$. We now turn to (5.34), where once more it suffices to consider the upper right entry, which is given by

$$\Gamma^\gamma_\beta = \sum_m \pi^{(m)}_{\gamma}(z^\gamma, (\partial_2 - \partial_2^2)^{\dagger} x^m).$$

This expression vanishes unless $\gamma \in \{e_{m-(0,1)}, e_{m-(2,0)}\}$ and thus $|\gamma| + 2 = |m|$, and unless $|m| < |\beta|$, by the population condition (5.4). Hence (5.38) vanishes unless $|\gamma| < |\beta| - 2$.

6. Homomorphism to the Connes-Kreimer Hopf algebra

In this logically independent and rather combinatorial section, we specify to the particularly simple case of scalar branched rough paths. We will argue that our coproduct $\Delta^+$ on $T^+$, when suitable restricted, arises from the Connes-Kreimer coproduct. More precisely, there is a linear subspace $T_{RP} \subset T$ and a pre-Lie subalgebra $L_{RP} \subset L$ (which as a linear space is isomorphic to $T_{RP}$) of derivations $D$ (which are such that $D^\dagger$ preserves $T_{RP}$) such that the corresponding restriction of $\Delta^+$ intertwines with the Connes-Kreimer coproduct on forests, see Subsection 6.4. The intertwining is provided by the linear one-to-one map $\phi$ that relates our model, which is indexed by multi-indices, to branched rough paths indexed by trees, see Subsection 6.2.

6.1. Relating the model $\Pi$ to branched rough paths.

Since it does not affect the algebraic insight of this section, we consider a qualitatively smooth driver $\xi$ to avoid renormalization. Following our initial discussion in Subsection 2.1, we consider the solution of the initial value problem

$$\frac{du}{dx^2} = a(u)\xi, \quad u(x_2 = 0) = 0$$

as a functional $u = u[a](x_2)$ of the (polynomial) nonlinearity $a$. It lifts to a function of the coordinates $\{z_k\}_{k \geq 0}$ introduced in (2.9). Hence we may take derivatives with respect to these coordinates evaluated at $z_k = 0$; these partial derivatives are indexed by multi-indices $\beta$. It is easy to (formally) verify that the resulting partial derivatives $\Pi_\beta$ satisfy

$$\frac{d\Pi_\beta}{dx^2} = \sum_{k \geq 0} \sum_{e_k + \beta_1 + \cdots + \beta_k = \beta} \Pi_{\beta_1} \cdots \Pi_{\beta_k} \xi, \quad \Pi_\beta(x_2 = 0) = 0,$$

with the understanding that $\Pi_{e_0}(x_2) = \int_0^{x_2} \xi$. These components combine to the centered model $\Pi = \{\Pi_\beta\}_\beta$. Incidentally, interpreting $x_2 \mapsto \Pi(x_2) \in \mathbb{R}[\{z_k\}]$, (6.2) can be compactly written as $\frac{d\Pi}{dx^2} = \sum_{k \geq 0} z_k \Pi^k \xi$. While this derivation of (6.2) is formal, $\Pi = \{\Pi_\beta\}_\beta$ can be, inductively in the length of $\beta$, constructed rigorously for sufficiently regular $\xi$. \footnote{which are maps $\mathbb{N}_0 \ni k \mapsto \beta(k) \in \mathbb{N}_0$ with finitely many non-zero values}

\footnote{Centered at time $x_2 = 0$, which however we suppress in our notation.}

\footnote{i.e. in a one-dimensional state space}
Based on (6.2) we may read off that not all the multi-indices are populated. More precisely, we claim that $\Pi_\beta \neq 0$ implies

$$\sum_{k \geq 0} (k-1)\beta(k) = -1. \tag{6.3}$$

We establish (6.3) in its negated form by induction in $\sum_{k \geq 0} k\beta(k)$. In the base case $\sum_{k \geq 0} k\beta(k) = 0$, which is equivalent to $\beta \in \mathbb{N}_0 e_0$, in which case the r. h. s. of (6.2) reduces to $k = 0$ and thus $\beta = e_0$, which satisfies (6.3).

Turning to the induction step, we note that the r. h. s. of (6.1) restricts to $k \geq 1$ so that the induction hypothesis can be applied to $\beta_1, \ldots, \beta_k$.

Hence the induction step follows from the fact that (6.3) is preserved when passing from $\beta_1, \ldots, \beta_k$ to $\beta = e_k + \beta_1 + \cdots + \beta_k$.

We now compare (6.2) to the standard definition of branched rough paths, which is based on (if not otherwise stated: rooted and thus non-empty and undecorated) trees $\tau$ instead of multi-indices $\beta$. We recall that for a collection $\tau_1, \ldots, \tau_k, \tau$ of such trees, the notation

$$\tau = B_+(\tau_1 \cdots \tau_k) \tag{6.4}$$

means that $\tau$ is the tree that is obtained from attaching an edge to each of the trees $\tau_1, \ldots, \tau_k$ and merging them in a common root, with the understanding that $B_+(\emptyset)$ gives the tree with a single node\(^{58}\), denoted by $\bullet$.

We recall from [15, Section 4] that the branched rough path\(^{59}\) $\{X_\tau\}_\tau$ is, inductively in the number of edges, defined through

$$\frac{dX_\tau}{dx_2} = X_{\tau_1} \cdots X_{\tau_k}\xi, \quad X_\tau(x_2 = 0) = 0 \quad \text{provided (6.4) holds}, \tag{6.5}$$

which includes $X_\bullet(x_2) = \int_0^{x_2} \xi$.

It is clear from (6.2) and (6.5) that every $\Pi_\beta$ is a linear combination of the $X_\tau$'s.

**Lemma 6.1.** For every multi-index $\beta$,

$$\Pi_\beta = \sum_{\tau \in \mathcal{T}_\beta} \frac{\sigma(\beta)}{\sigma(\tau)} X_\tau. \tag{6.6}$$

Here $\mathcal{T}_\beta$ is the set of trees that have $\beta(k)$ nodes with $k$ children\(^{60}\), and where $\sigma(\beta)$ and $\sigma(\tau)$ are symmetry factors defined as follows:

$$\sigma(\beta) := \prod_{k \geq 0} (k!)^{\beta(k)} \tag{6.7}$$

is the size of the group of all transformations of a tree $\tau \in \mathcal{T}_\beta$ that are obtained by permuting the children (with their descendants attached) at every node; $\sigma(\tau)$ is the size of the subgroup that leaves a particular tree

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\(^{58}\) which is the root

\(^{59}\) the canonical lift of $\xi$

\(^{60}\) Note that thanks to the restriction (6.3), this set is not empty.
\( \tau \in T_\beta \) invariant\(^{61}\), hence \( \frac{\sigma(\beta)}{\sigma(\tau)} \) is the size of the orbit of \( \tau \) under all above transformations\(^{62}\).

**Proof.** We proceed by induction in the number of edges \( \sum_{k \geq 0} k \beta(k) \). In the base case when \( \sum_{k \geq 0} k \beta(k) = 0 \), (6.3) implies that \( \beta = e_0 \), so that \( T_\beta = \{ \bullet \} \) and \( \sigma(\beta) = \sigma(\bullet) = 1 \), and hence the claim follows. For the induction we give ourselves a \( \beta \) and consider \( \tilde{\Pi}_\beta := \sum_{\tau \in T_\beta} \sigma(\beta) \tau \cdot \cdots \cdot \tau \xi \), so that by (6.5)

\[
\frac{d\tilde{\Pi}_\beta}{dx_2} = \sum_{\tau \in T_\beta} \frac{\sigma(\beta)}{\sigma(\tau)} \tau_1 \cdots \tau_k \xi,
\]

where \( k \) and the \( k \)-tuple \( (\tau_1, \ldots, \tau_k) \) of trees is, uniquely up to permutation, determined by (6.4). More precisely, it is the multi-index \( J = e_{\tau_1} + \cdots + e_{\tau_k} \) of trees. Note that \( \tau = B_+(\tau_1 \cdots \tau_k) \) implies \( \sigma(\tau) = J! \sigma(\tau_1) \cdots \sigma(\tau_k) \), cf. [3, p. 430]; it also yields

\[
\beta = e_k + e_1 + \cdots + e_k,
\]

where \( \tau_j \in T_{\beta_j} \), which in turn implies by definition (6.7)

\[
\sigma(\beta) = k! \sigma(\beta_1) \cdots \sigma(\beta_k).
\]

We now appeal to the re-summation in Lemma A.2, which yields

\[
\sum_{\tau \in T_\beta} \frac{\sigma(\beta)}{\sigma(\tau)} \tau_1 \cdots \tau_k \xi
= \sum_{k \geq 0} e_k + e_1 + \cdots + e_k = \beta
\sum_{\tau_1 \in T_{\beta_1}} \left( \sum_{\tau \in T_{\beta_1}} \frac{\sigma(\beta_1)}{\sigma(\tau_1)} \tau_1 \right) \cdots \left( \sum_{\tau_k \in T_{\beta_k}} \frac{\sigma(\beta_k)}{\sigma(\tau_k)} \tau_k \right) \xi.
\]

By induction hypothesis, the r. h. s. assumes the desired form of (6.2). \( \square \)

### 6.2. Relating the abstract model space \( T_{RP} \) to \( B \).

According to the previous subsection, in our setting, the abstract model space \( T_{RP} \) relevant for branched rough paths is the linear sub-space of \( T \), see Subsection 3.6, corresponding to the multi-indices \( \beta \) that satisfy (6.3) and only depend on \( k \) (and thus trivially satisfies \( [\beta] \geq 0 \)). In the standard setting, the abstract model space \( B \) relevant for branched rough paths is the direct sum indexed by all \( \tau \)'s. Following [17, Definition 3.3], we think of the model\(^{63}\) as a linear map from the abstract model space into the space of distributions \( S'(\mathbb{R}) \). Then the relation (6.6) between the components of the two models defines a linear map \( \phi: T_{RP} \to B \) such that

\[
\Pi = X \phi.
\]

---

\(^{61}\)For a more explicit definition, see [3, p. 430].

\(^{62}\)Thus it is an integer.

\(^{63}\)centered in one point, here \( x_2 = 0 \)
Applying this identity to the (dual) basis vector \( z_\beta \), we read off from (6.6) that \( \phi \) acts as

\[
\phi z_\beta = \sum_{\tau \in \mathcal{T}_\beta} \frac{\sigma(\beta)}{\sigma(\tau)} z_\tau,
\]

from which we learn that \( \phi \) is one-to-one (but not onto). Hence the matrix representation of \( \phi \) is given by

\[
\phi^\beta_\tau = \begin{cases} 
\frac{\sigma(\beta)}{\sigma(\tau)} & \text{if } \tau \in \mathcal{T}_\beta \\
0 & \text{otherwise}
\end{cases}
\]

(6.12)

6.3. Relating the pre-Lie algebra \( L_{RP} \) to \( L^1 \).

In our setting, the subspace \( L_{RP} \) of \( L \) relevant for branched rough paths is spanned by \( z^\gamma D(0) \subset \text{Der}(\mathbb{R}[[z]]) \), with multi-indices \( \gamma \) only depending on \( k \) and satisfying (6.3). As opposed to \( L \), \( L_{RP} \) is closed under the pre-Lie product \( \triangleright \) on \( \text{Der}(\mathbb{R}[[z]]) \). Indeed, this follows from fact that by (3.10), \( z^\gamma D(0)z^\beta \) is a linear combination of \( z^\gamma \)'s with \( \beta + e_k = \gamma + e_{k+1} \) for some \( k \geq 0 \), so that (6.3) is preserved. Moreover, for \( D \in L_{RP} \) we have that \( D^\dagger \) preserves \( T_{RP} \). Indeed, this follows from (3.10) and (3.34) by the same reasoning. Finally, there is a canonical (non-degenerate) pairing between \( T_{RP} \) and \( L_{RP} \), which are isomorphic as linear spaces, defined through

\[
\langle z_\gamma, z^\gamma D(0) \rangle = \delta^\gamma_\gamma.
\]

(6.13)

On the classical side, we consider the (pre-)Lie algebra \( L^1 \) introduced by Connes and Kreimer [12], which as a linear space is the direct sum indexed by trees \( \tau \) and thus isomorphic to \( B \); we denote by \( Z_\tau \) the standard basis. It comes with the pre-Lie product

\[
Z_{\tau_1} \triangleright Z_{\tau_2} = \sum_{\tau} n(\tau_1, \tau_2; \tau) Z_\tau,
\]

(6.14)

where \( n(\tau_1, \tau_2; \tau) \) is the number of single cuts of \( \tau \) such that the branch is \( \tau_1 \) and the trunk is \( \tau_2 \). In other words, \( n(\tau_1, \tau_2; \tau) \) is the number of edges of \( \tau \) that when removed yields the trees \( \tau_1 \) and \( \tau_2 \), where the second one is defined to be the one containing the root of \( \tau \). The fact that (6.14) satisfies the axiom of a pre-Lie product is established in [12, (103)]. We note that the pre-Lie product can also be recovered from

\[
(\sigma(\tau_1) Z_{\tau_1}) \triangleright (\sigma(\tau_2) Z_{\tau_2}) = \sum_{\tau} m(\tau_1, \tau_2; \tau) \sigma(\tau) Z_\tau,
\]

(6.15)

where \( m(\tau_1, \tau_2; \tau) \) is the number of nodes of \( \tau_2 \) to which \( \tau_1 \) may be attached to yield \( \tau \). Passing from (6.14) to (6.15) relies on the combinatorial identity\(^{65} \)

\[
n(\tau_1, \tau_2; \tau) \sigma(\tau_1) \sigma(\tau_2) = m(\tau_1, \tau_2; \tau) \sigma(\tau),
\]

which is established in [20, Proposition 4.3]. We note that it is the product

\[
\tau_1 \cap \tau_2 := \sum_{\tau} m(\tau_1, \tau_2; \tau) \tau
\]

64Where we denote the basis elements of \( B \) by \( z_\tau \).

65The notation in [20, Proposition 4.3] is opposite to ours, which is the one of [12].
that comes with the intuition of grafting the tree $\tau_1$ onto the tree $\tau_2$; it can be extended to a pre-Lie product on $\mathcal{B}$ by linearity. While (6.15) shows that the two pre-Lie structures on $\mathcal{L}^1$ and on $\mathcal{B}$ are isomorphic it is helpful to distinguish them here. This is related to the fact that we consider the standard pairing between $\mathcal{B}$ to distinguish them here. This is related to the fact that we consider the standard pairing between $\mathcal{B}$ and $\mathcal{L}^1$, i.e. we think of $z_\tau \in \mathcal{B}$ and $Z_\tau \in \mathcal{L}^1$ as dual bases\footnote{Alternatively, one could work with $\wedge$ but impose the pairing $Z_\tau \cdot z_{\tau'} = \sigma(\tau)\delta_{\tau\tau'}^\tau$, see more in \cite[Subsection 3.3]{B}, on the choice of pairings viz. inner products.}. These pre-Lie products have been evoked in branched rough paths \cite[Subsection 3.2.2]{B} and in regularity structures \cite[Remark 4.1]{B}.

In view of the obvious finiteness properties of $\phi$, see (6.12), the two above-mentioned non-degenerate pairings define a linear map $\phi^\dagger : \mathcal{L}^1 \to \mathcal{L}_{RP}$ by duality. From (6.12) we learn that it acts as

\begin{equation}
\phi^\dagger \sigma(\tau) Z_\tau = \sigma(\beta) z^\beta D^{(0)} \quad \text{provided } \tau \in \mathcal{T}_\beta.
\end{equation}

**Lemma 6.2.** $\phi^\dagger$ is a pre-Lie algebra morphism:

\begin{equation}
\phi^\dagger (Z_{\tau_1} \sim Z_{\tau_2}) = (\phi^\dagger Z_{\tau_1} ) \triangleright (\phi^\dagger Z_{\tau_2}).
\end{equation}

**Proof.** Multiplying (6.18) by $\sigma(\tau_1)\sigma(\tau_2)$, appealing to (6.15) for the l. h. s. and then to (6.17), we see that (6.18) follows from

\begin{equation}
\sum_\tau m(\tau_1, \tau_2; \tau) \sigma(\beta) z^\beta = \sigma(\beta_1) \sigma(\beta_2) z^{\beta_1} D^{(0)} z^{\beta_2},
\end{equation}

with the understanding that $\beta$, $\beta_1$ and $\beta_2$ are determined by $\tau \in \mathcal{T}_\beta$, $\tau_1 \in \mathcal{T}_{\beta_1}$ and $\tau_2 \in \mathcal{T}_{\beta_2}$. We note that $m(\tau_1, \tau_2; \tau) \neq 0$ only if the fertilities are related by $\beta_1 + \beta_2 + \varepsilon_{k+1} = \beta + \varepsilon_k$ for some $k \geq 0$, which amounts to taking a node of $\tau_2$ with $k$ children and attaching $\tau_1$ to it (via a new edge). Hence the l. h. s. naturally decomposes into a sum over $k \geq 0$ – and so does the r. h. s. in view of (3.9). Hence (6.19) follows from

\[ \sum_{\tau: \beta_1 + \beta_2 + \varepsilon_k = \beta + \varepsilon_k} m(\tau_1, \tau_2; \tau) \sigma(\beta) z^\beta = (k + 1) \sigma(\beta_1) \sigma(\beta_2) z^{\beta_1} Z_{k+1} \partial_k z^{\beta_2}, \]

which, upon multiplication by $z_k$, reduces to the purely combinatorial

\[ \sum_{\tau: \beta_1 + \beta_2 + \varepsilon_k = \beta + \varepsilon_k+1} m(\tau_1, \tau_2; \tau) \sigma(\beta) = (k + 1) \sigma(\beta_1) \sigma(\beta_2) \beta_2(k). \]

According to the definition (6.7) of $\sigma(\beta)$ this reduces to

\[ \sum_{\tau: \beta_1 + \beta_2 + \varepsilon_k = \beta + \varepsilon_k+1} m(\tau_1, \tau_2; \tau) = \beta_2(k). \]

This last identity holds because also the l. h. s. is the number of nodes of $\tau_2$ with $k$ children on which $\tau_1$ can be attached (via a new edge). □

The fact that $\phi^\dagger$ is not one-to-one reflects that our (pre-)Lie algebra is not free, as opposed to $\mathcal{L}^1$ (cf. \cite[Theorem 1.9]{B}). Moreover, $\mathcal{L}_{RP}$ is isomorphic to $\mathcal{L}_1$ quotiented by an ideal.
Corollary 6.3. As pre-Lie algebras, \((L_{RP}, \triangleright)\) and \((L^1 / R, \rightsquigarrow)\) are isomorphic, where

\[
R := \text{span}\{\sigma(\tau_1)Z_{\tau_1} - \sigma(\tau_2)Z_{\tau_2} \mid \tau_1, \tau_2 \in T_\beta\}_\beta.
\]

Proof. By Lemma 6.2, \(\phi^\dagger : L^1 \to L_{RP}\) is a morphism. Thanks to the restriction (6.3), it is onto. It only remains to show that \(\ker \phi^\dagger = R\). By (6.17), \(\phi^\dagger\) vanishes on the generating set (6.20), hence \(R \subset \ker \phi^\dagger\). To show the opposite inclusion, we fix \(\sum_{\tau} c_{\tau} Z_{\tau} \in \ker \phi^\dagger\), and by linearity it is enough to show that \(\sum_{\tau \in T_\beta} c_{\tau} Z_{\tau} \in R\) for every \(\beta\). By the representation (6.17), \(\sum_{\tau} c_{\tau} Z_{\tau}\) is in the kernel of \(\phi^\dagger\) if and only if \(\sum_{\tau \in T_\beta} \frac{c_{\tau}}{\sigma(\tau)} = 0\) for all \(\beta\). \(\square\)

Remark 6.4. Note that, although freeness is lost, the generation property is preserved; in this setting, \(z_0 D^{(0)}\) is the generator of \(L_{RP}\).

6.4. Relating the coproduct \(\Delta^+_{RP}\) to Butcher’s.

The pre-Lie algebra morphism property (6.18) of \(\phi^\dagger\) obviously implies that it is also a Lie-algebra morphism between \(L^1\) and \(L_{RP}\). By the characterizing property of universal envelopes, \(\phi^\dagger\) lifts to a morphism between the Hopf algebras \(U(L^1)\) and \(U(L_{RP})\). According to [12, Theorem 3 b)], the standard pairing [12, (105)] between the Hopf algebra \(U(L^1)\) and the Connes-Kreimer Hopf algebra \(\mathcal{H}\) respects the Hopf algebra structures. We recall that as an algebra, \(\mathcal{H}\) is the polynomial algebra \(\mathbb{R}[\tau]\) over trees \(\tau\), and the coproduct \(\Delta_B\) is defined according to Butcher via cutting-off sub-trees (“pruning”), see e. g. [3, Section 3]. Defining \(T^+_{RP}\) and \(\Delta^+_{RP}\), based on the Lie algebra \(L_{RP}\), in analogy to \(T^+\) and \(\Delta^+\), see Subsection 4.5, we thus obtain that \(\phi : T^+_{RP} \to \mathcal{H}\) is a Hopf algebra morphism, in particular

\[
(\phi \otimes \phi)\Delta^+_{RP} = \Delta_B \phi.
\]

Here, we used that on \(T^+_{RP} \otimes T^+_{RP}\), which is naturally a subspace of \((U(L_{RP}) \otimes U(L_{RP}))^*\), we have that \((\phi^\dagger \otimes \phi^\dagger)^* = \phi \otimes \phi^\dagger\).

6.5. Relating \(\phi^\dagger\) to \(\Upsilon\).

The morphism property (6.18) is closely related to the ones that appear in regularity structures [4, Corollary 4.15] and branched rough paths [2, Lemma 3.7], as we shall explain in this subsection for the latter: In view of its canonical pairing (6.13) with \(T_{RP}\), \(L_{RP}\) can be canonically identified with a subspace\(^{67}\) of \(T_{RP}\), so that we may think of \(\phi^\dagger\) as mapping into \(T_{RP}\) and then interpret (6.18) as the following identity in \(T_{RP} \subset \mathbb{R}[[z_k]]\):

\[
\phi^\dagger(Z_{\tau_1} \rightsquigarrow Z_{\tau_2}) = (\phi^\dagger Z_{\tau_1})(D^{(0)} \phi^\dagger Z_{\tau_2}).
\]

We note that the image of \(\phi^\dagger\) is actually contained in the polynomial subspace \(\mathbb{R}[z_k]\), and thus in view of (2.9) in the space of functions on \(a\)-space. Hence we may apply \(\phi^\dagger Z_{\tau}\) to a polynomial\(^{68}\) \(a\), and thus also to \(a(\cdot + u)\) for

\(^{67}\)Both have the same index set, but while \(L_{RP}\) is a direct sum, \(T^+_{RP}\) is a direct product.

\(^{68}\)Even to a formal power series.
some shift $u \in \mathbb{R}$. We also note that $D^{(0)}$ preserves $\mathbb{R}[z_k]$, see (3.9). Hence we may “test” (6.21) with $a(\cdot + u)$ and obtain by definition (3.7) of $D^{(0)}$

$$\phi^\dagger(Z_{\tau_1} \hookrightarrow Z_{\tau_2})[a(\cdot + u)] = (\phi^\dagger Z_{\tau_1})[a(\cdot + u)] \left( \frac{d}{du} \phi^\dagger Z_{\tau_2}[a(\cdot + u)] \right).$$

With the abbreviation

$$\Upsilon^a[\tau](u) := \phi^\dagger(\sigma(\tau)Z_{\tau}[a(\cdot + u)])$$

and the help of (6.15) and (6.16), (6.22) turns into the following simple version of [2, Lemma 3.4]

$$\Upsilon^a[\tau_1 \cdot \tau_2] = \Upsilon^a[\tau_1] \left( \frac{d}{du} \Upsilon^a[\tau_2] \right),$$

which states that for fixed $a$, $\Upsilon^a$ is a pre-Lie algebra morphism from $(B, \cdot \cdot)$ into the pre-Lie algebra of functions of $u \in \mathbb{R}$.

We remark that the object (6.23) coincides with the one recursively defined in [2, Definition 2.13]. This follows from the fact that under the assumption (6.4), we learn that by (6.17) and (6.9), (6.10) translate into the following identity in $T_{RP}^* \subset \mathbb{R}[\mathbb{R}[z_k]]$

$$\phi^\dagger(\sigma(\tau)Z_{\tau}) = k!z_k(\phi^\dagger(\sigma(\tau_1)Z_{\tau_1}) \cdots (\phi^\dagger(\sigma(\tau_k)Z_{\tau_k})).$$

This identity, when tested with $a(\cdot + u)$, by definitions (2.9) and (6.23), turns into

$$\Upsilon^a[\tau] = \left( \frac{d^k a}{du^k} \right) \Upsilon^a[\tau_1] \cdots \Upsilon^a[\tau_k],$$

which coincides with the induction [2, (2.11)]. The base case is obvious: For $\tau = \cdot$, we learn from (6.17) that $\phi^\dagger(\sigma(\tau)Z_{\tau}) = z_0$, and from (2.9) that $z_0[a(\cdot + u)] = a(u)$.

### 6.6. Renormalization of rough paths via multi-indices.

We now give some details on future directions of our research, namely that renormalization can be carried out within the multi-index description without passing via trees. From the analytic and stochastic viewpoint, this is carried out in the case of quasi-linear SPDEs in the work [23]. In this section, we reveal the algebraic structure that guides renormalization in the simple case of branched rough paths, in line with [5].

Let us consider the following generalized version of (6.1):

$$\frac{du}{dx_2} = a_0(u) + a_1(u)\xi, \quad u(x_2 = 0) = 0.$$ 

Following Subsection 2.1, we see the solution $u$ as a function of both non-linearities, $u = u[a_0, a_1]$. For simplicity, we will only study transformations

---

69Simple because our scalar setting does not require node decorations, and since we did not extend to forests.
in $a_0$ by addition of a function of $a_1$, i.e.

$$ (a_0, a_1) \mapsto (a_0 + c[a_1], a_1); $$

here,

$$ c[a_1](u) := c[a_1(\cdot + u)], $$

so that shift-covariance is built in. As in Section 2, we lift (6.25) to the space of functions of $(a_0, a_1)$ by means of an algebra morphism $M_c$:

$$ M_c \pi[a_0, a_1] = \pi[a_0 + c[a_1], a_1]. $$

In analogy with (2.9), we introduce coordinates on $(a_0, a_1)$-space

$$ z_{0,k}[a_0, a_1] = \frac{1}{k!} \frac{d^k a_0}{dv^k}(0), \quad z_{1,k}[a_0, a_1] = \frac{1}{k!} \frac{d^k a_1}{dv^k}(0), $$

and multi-indices which now involve both families of variables, i.e.

$$ z^\beta := \prod_{k \geq 0} (z_{0,k}^{\beta(0,k)} z_{1,k}^{\beta(1,k)}). $$

We still denote by $T^{\ast}_{RP}$ the dual of the model space, now characterized by the population condition

$$ \sum_{k \geq 0} (k + 1)(\beta(0, k) + \beta(1, k)) = -1; $$

the argument is the same as that of (6.3).

We moreover define the infinitesimal generator of tilt by a constant in line with (3.7):

$$ D^{(0)} \pi[a_0, a_1] = \frac{d}{dv|_{v=0}} \pi[a_0(\cdot + v), a_1(\cdot + v)]. $$

Then (6.26) acts on the coordinate functionals (6.27) as

$$ M_c z_{k}^0 = z_{k}^0 + \frac{1}{k!} (D^{(0)})^k c, \quad M_c z_{k}^1 = z_{k}^1, $$

which thanks to the algebra morphism property defines the action of $M_c$ on the polynomial algebra $\mathbb{R}[z_{k}^0, z_{k}^1]$. Since $M_c$ amounts to plugging a power series into a power series, cf. (6.26), this action extends to the full power series space $\mathbb{R}[[z_k, z_n]]$ as long as we impose

$$ c_{\beta=0} = 0. $$

The following result gathers the properties of the map $M_c$.

---

70This class includes the Itô-Stratonovich conversion in SDEs: Assume that $\xi$ is the time derivative of a standard Brownian motion and take $c = \frac{1}{2} z_{1} z_{1}'$, so that $c[a_1](u) = \frac{1}{4} a_1(u) a_1'(u)$. Then (6.24) transforms into

$$ \frac{du}{dx_2} = a_0(u) + a_1(u) \xi + \frac{1}{2} a_1(u) a_1'(u). $$
Lemma 6.5. Let \( c \in T^*_{RP} \cap \mathbb{R}[[z_k^1]] \). Then \( M_c \in \text{End}(T^*_{RP}) \). In addition, for all \( \pi_1, \pi_2 \in T^*_{RP} \),

\[
M_c((\pi_1 D^{(0)}) \pi_2) = (M_c \pi_1) D^{(0)}(M_c \pi_2),
\]

which implies that \( M_c \) is a pre-Lie morphism in \( L_{RP} \subset T^*_{RP} \). Moreover, the following composition rule holds:

\[
M_c_1 M_c_2 = M_{c_1 + c_2},
\]

which yields an Abelian group structure.

Proof. Since the condition (6.31) is contained in (6.28), \( M_c \) is well-defined. We first argue that \( M_c \) commutes with \( D^{(0)} \); indeed, this follows from (3.9) and (6.30) via

\[
D^{(0)} M_c z_k^0 = (k + 1) z_{k+1}^0 + \frac{1}{k!} (D^{(0)})^{k+1} c = (k + 1) M_c z_{k+1}^0 = M_c D^{(0)} z_k^0,
\]

and is tautological for \( z_k^1 \). It then extends to the general case by the algebra morphism property of \( M_c \) and Leibniz rule for \( D^{(0)} \). As a consequence, (6.32) is satisfied:

\[
M_c((\pi_1 D^{(0)}) \pi_2) = (M_c \pi_1) M_c(D^{(0)} \pi_2) = (M_c \pi_1) D^{(0)}(M_c \pi_2).
\]

We now argue that \( M_c T^*_{RP} \subset T^*_{RP} \). It is enough to show it for the space of finite sums, which is isomorphic to \( L_{RP} \).

Since (6.32) implies that \( M_c \) is a pre-Lie morphism, and since the elements \( z_0^0, z_1^0 \) generate \( L_{RP} \), cf. Remark 6.4, it suffices to show \( M_c z_0^0, M_c z_1^0 \in T^*_{RP} \); this follows from (6.30) under the assumption \( c \in T^*_{RP} \). Finally, the composition rule (6.33) may be read off from (6.26).

Combining (6.30) and (6.32), the map \( M_c \) may be seen as a shift of the form \( z_0^0 \mapsto z_0^1 + c \) which, in addition, is a pre-Lie morphism. This connects the approach to the translation of (branched) rough paths as described in [5, Definition 14]. In the specific setting of this section, given an element \( v \in L^1 \), its associated translation map, which we denote by \( M_v^{BCFP} \), is defined as the unique pre-Lie morphism that extends

\[
M_v^{BCFP} Z_{\bullet_0} = Z_{\bullet_0} + v, \quad M_v^{BCFP} Z_{\bullet_1} = Z_{\bullet_1}.
\]

We follow the construction of previous subsections and build a dictionary \( \phi \) from our model space \( T_{RP} \) to the linear space of trees decorated by 0 and 1, so that (6.11) still holds with \( T_\beta \) given by the set of trees which contain \( \beta(i, k) \) nodes decorated by \( i \) and with \( k \) children. This dictionary intertwines with the translation maps:

---

71 See the discussion at the beginning of Subsection 6.5.

72 The general setting of [5] allows \( v \) to be a Lie series in the Grossman-Larson Hopf algebra generated by two nodes distinguished by decorations corresponding to each non-linearity (in the specific setting of transformations of the form (6.25) only one decoration matters, so it is legitimate to think of \( v \) as non-decorated). Although we can also work with (infinite) Lie series, for notational convenience we restrict to finite sums and write \( v \in L^1 \).

73 This time we consider \( \phi^\dagger : L^1 \to T_{RP} \) instead of \( \phi^\dagger : L^1 \to L_{RP} \), in line with Subsection 6.5.
Lemma 6.6.
\begin{equation}
\phi^\dagger M^\text{BCFP}_v = M_{\phi^\dagger v} \phi^\dagger.
\end{equation}

Proof. Since both $\phi^\dagger M^\text{BCFP}_v$ and $M_{\phi^\dagger v} \phi^\dagger$ are pre-Lie morphisms, cf. (6.21) and (6.32), it is enough to show (6.35) for the generators $Z_{\bullet 0}$ and $Z_{\bullet 1}$.

The case $Z_{\bullet 0}$ follows from
\begin{equation}
\phi^\dagger M^\text{BCFP}_v Z_{\bullet 0} = \phi^\dagger (Z_{\bullet 0} + v) = z_0^0 + \phi^\dagger v = M_{\phi^\dagger v} z_0^0 = M_{\phi^\dagger v} \phi^\dagger Z_{\bullet 0}.
\end{equation}

The case $Z_{\bullet 1}$ is trivial from (6.34).

The map $M_c$ induces a (purely algebraic) transformation of the model. More precisely, if $\Pi$ is the model constructed inductively from the ODE in $\mathbb{R}[z_k^0, z_k^1]$,
\begin{equation}
\frac{d}{dx_2} \Pi = \sum_{k \geq 0} z_k^0 \Pi^k + \sum_{k \geq 0} z_k^1 \Pi^k \xi,
\end{equation}
which arises from (6.24), then thanks to the morphism property and (6.30), $\tilde{\Pi} := M_c \Pi$ solves
\begin{equation}
\frac{d}{dx_2} \tilde{\Pi} = \sum_{k \geq 0} z_k^0 \tilde{\Pi}^k + \sum_{k \geq 0} z_k^1 \tilde{\Pi}^k \xi + \sum_{k \geq 0} \frac{1}{k!} \tilde{\Pi}^k (D^{(0)})^k c.
\end{equation}

The form of the last r. h. s. term in (6.36) connects to the form of the counter-term in the model equations for quasi-linear SPDEs which was postulated in [27, Subsection 1.1] and systematically constructed in [23].

Finally, we address the transformation of the equation (6.24); under the action (6.25), it tautologically turns into
\begin{equation}
\frac{du}{dx_2} = a_0(u) + a_1(u) \xi + c[a_1(\cdot + u)].
\end{equation}

We now argue that any translation in the sense of [5], which is expressed in terms of trees, may be expressed in terms of multi-indices in the form of (6.37). For this purpose, we note that, in the notation of [5, p. 37],
\begin{equation}
a_v(u) = \phi^\dagger v[a_0(\cdot + u), a_1(\cdot + u)].
\end{equation}

Indeed, this follows from (6.27) for $v = Z_{\bullet 0}, Z_{\bullet 1}$, and is extended by the morphism property (6.22). Therefore by [5, Theorem 38 (ii)] equation (6.24) assumes the form
\begin{equation}
\frac{du}{dx_2} = a_0(u) + a_1(u) \xi + \phi^\dagger v[a_1(\cdot + u)].
\end{equation}

Comparing (6.37) and (6.38), we see that the greedier setting of multi-indices loses no information with respect to the tree-based approach; on the contrary, it reduces the complexity by grouping trees which give rise to the same renormalization procedure into a single multi-index. We expect this to extend to SPDEs, reducing the size of the renormalization group.
7. Homomorphism to the SHE structure

In the spirit of the previous section we show that our algebraic structure is compatible with the ones in regularity structures when it comes to semi-linear SPDEs. More precisely, in Subsections 7.1 to 7.3 we connect our model space and pre-Lie structure to [4, Subsection 4.1]; in Subsection 7.4, we show compatibility of our Hopf algebra \( T^+ \) with the one in [17, Subsection 4.2]. We will establish this in the specific case of the stochastic heat equation (1.3). In this specific case, the model is defined through the hierarchy of linear PDEs

\[
\left( \frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2} \right) \Pi_{\beta} = \sum_{k \geq 0} \sum_{e_k + \beta_1 + \cdots + \beta_k = \beta} \Pi_{\beta_1} \cdots \Pi_{\beta_k} \xi,
\]

for \( [\beta] \geq 0 \), together with \( \Pi_{e_n}(x) = x^n \). This recursive definition leads to the following population condition: \( \Pi_{\beta} \neq 0 \) implies

\[
\sum_{k \geq 0} (k - 1) \beta(k) - \sum_{n \neq 0} \beta(n) = -1.
\]

7.1. Relating the abstract model space \( T \) to \( B \).

We first fix the dictionary between our model space \( T \) and the space \( B \) of linear combinations of trees \( \tau \) with expanded polynomial decorations defined in [4, Subsection 4.1]. In the specific case of SHE, the set of trees [4, (4.3)] is inductively defined through

\[
\tau = \cdot (\prod_{i \in I} \mathcal{F} X^{n_i})(\prod_{j \in J} \mathcal{I} \tau_j)
\]

and its integrated version \( \mathcal{I} \tau \), where \( \mathcal{I} \) is the placeholder for integration, i.e. application of the kernel of the solution operator, \( \cdot \) is the placeholder for the noise and with the understanding that \( n_i \neq 0 \) for all \( i \in I \). Although the factor \( \cdot (\prod_{i \in I} \mathcal{F} X^{n_i}) \) is considered in [4] as a node decoration,

\[
\text{Since in our setting we work not with kernels but directly with the PDE, in order to guarantee uniqueness of the model we need to impose some extra conditions. Such conditions, which are irrelevant in the algebraic context of this article, amount to growth bounds that in turn allow for the application of Liouville principles; cf. e.g. [23, Proposition 5.3] or [22, Lemma 4.9].}
\]

\[
\text{The reason why we do not restrict the model space \( T \) to the populated subspace, as we did in the rough path case, is that even then, our dictionary \( \phi \) will no longer be one-to-one, see Subsection 7.4.}
\]

\[
\text{As opposed to the space \( \mathcal{V} \) of combinatorial decorated trees [4, (3.5)], which have simpler polynomial decorations, and contain the actual model space in the tree-based framework; see Subsection 7.4.}
\]

\[
\text{In order to connect to [17], we switch the notation in [4] and use \( \mathcal{I} \) for the abstract integration and \( \mathcal{F} \) for the polynomial labelling. Here, \( \mathcal{F} \) serves as a reminder that we cannot multiply polynomials. It is understood that (7.3) is independent of the order.}
\]

\[
\text{denoted in [4] as \( \Xi \)}
we will instead think of $\mathcal{B}X^n$ as a subtree; for example, the graphical representation of $\star(\mathcal{B}X^n)(\mathcal{B}X^n)\mathcal{I}\star$ is given by
\[
\begin{array}{c}
X^{n_1} \\
\downarrow \\
X^{n_2}
\end{array}
\]
(7.4)

The symmetry factor is defined accordingly, so that, for example,
$$\sigma\left(X^n\begin{array}{c} \\
\downarrow \\
X^n\end{array}\right) = 2.$$

For later purpose, we recursively define $N(\bullet) = 1$ and
\[
N(\tau) = \prod_{i \in I} n_i! \prod_{j \in J} N(\tau_j),
\]
(7.5)
so that $N(\tau)$ is the product of the factorials of all polynomial decorations of $\tau$.

Let us now define the two linear maps $\hat{\phi}_-$ and $\hat{\phi}$ by
\[
\hat{\phi}_- z_\beta = 0, \quad \hat{\phi} : T \to \mathcal{B}
\]
and
\[
\hat{\phi} z_\beta = \sum_{k \geq 0} \sum_{e_k + \beta_1 + \cdots + \beta_k = \beta} \hat{\phi}_-^k \cdots \hat{\phi}_- \hat{\phi}_+ \cdot
\]
(7.6)
\[
(\hat{\phi}_-) z_\beta = \begin{cases} 
\mathcal{B}X^n & \text{if } \beta = e_n \\
\mathcal{I} \hat{\phi}_- z_\beta & \text{otherwise}
\end{cases}
\]
(7.7)

As pointed out in Subsection 5.3, the reason why we need both maps $\hat{\phi}_-$ and $\hat{\phi}$ is that the traditional model space in regularity structures relates to ours via $\mathcal{B} \oplus \mathcal{R} \oplus T$; in trees, this means that every non-purely polynomial multi-index encodes both a linear combination of rooted and of integrated (planted) trees. In line with branched rough paths in Section 6, when restricted to populated multi-indices, cf. (7.2), both maps $\hat{\phi}_-$ and $\hat{\phi}$ are one-to-one but not onto. Moreover we have the following characterization:

**Lemma 7.1.** For every multi-index $\beta$,
\[
\hat{\phi}_- z_\beta = \frac{\sigma(\beta)}{\sigma(\tau)} T, 
\]
(7.8)
Here $T_\beta$ is the set of trees with $\beta(k)$ nodes with $k$ children and $\beta(n)$ decorations $\mathcal{B}X^n$. Recall that we think of the latter as subtrees, so they count as children. For example, (7.4) belongs to $T_\beta$ for $\beta = e_0 + e_3 + e_{n_1} + e_{n_2}$. The number $\sigma(\beta)$ is the same as (6.7); in particular, it does not depend on $\beta(n)$ for any $n$. The proof of Lemma 7.1 is similar to that of Lemma 6.1, and thus we omit it.

Identity (7.8) establishes the matrix representation of $\hat{\phi}_-$, namely
\[
(\hat{\phi}_-)^\beta = \begin{cases} 
\frac{\sigma(\beta)}{\sigma(\tau)} & \text{if } \tau \in T_\beta \\
0 & \text{otherwise}
\end{cases}
\]
(7.9)

Following [4], we adopt the notation $\circ$ to refer to the setting of expanded polynomial decorations; we will drop it in the contracted setting of Subsection 7.4.
this, in turn, defines the transposed map $\phi^\dagger_\tau$.

7.2. Relating $\mathcal{L}$ to grafting operators on $\mathcal{B}$.

There are two operations on $\mathcal{B}$, namely grafting and increasing polynomial decorations, cf. [4, Definition 4.7]. Let us start with grafting, and define\textsuperscript{80} in line with (6.16)

\begin{equation}
(7.10) \quad \tau_1 \bowtie_n \tau_2 = \sum_{\tau} m_n(\tau_1, \tau_2; \tau),
\end{equation}

where $m_0(\tau_1, \tau_2; \tau) = m(\tau_1, \tau_2; \tau)$, cf. (6.15), and $m_{n \neq 0}(\tau_1, \tau_2; \tau)$ is the number of decorations $\mathcal{J}X^n$ in $\tau_2$ that when replaced by $\mathbb{L} \tau_1$ yield $\tau$. As in Subsection 6.3, let us denote by $\mathcal{Z}_\tau$ the standard basis of a linear space indexed by trees $\tau$. We define for every $n$

\begin{equation}
(7.11) \quad \mathcal{Z}_{\tau_1} \bowtie_n \mathcal{Z}_{\tau_2} = \sum_{\tau} n_n(\tau_1, \tau_2; \tau) \mathcal{Z}_\tau,
\end{equation}

where $n_0(\tau_1, \tau_2; \tau) = n(\tau_1, \tau_2; \tau)$, cf. (6.14), and $n_{n \neq 0}(\tau_1, \tau_2; \tau)$ is the number of single cuts of $\tau$ such that the branch is $\tau_1$ and, after adjoining the decoration $\mathcal{J}X^n$ to the trunk at the place the cut was made, yield $\tau_2$. The combinatorial identity $n_n(\tau_1, \tau_2; \tau)\sigma(\tau_1)\sigma(\tau_2) = m_n(\tau_1, \tau_2; \tau)\sigma(\tau)$ still holds\textsuperscript{81}, so that (7.11) may be rewritten as

\begin{equation}
(7.12) \quad \sigma(\tau_1)\mathcal{Z}_{\tau_1} \bowtie_n \sigma(\tau_2)\mathcal{Z}_{\tau_2} = \sum_{\tau} m_n(\tau_1, \tau_2; \tau)\sigma(\tau)\mathcal{Z}_\tau.
\end{equation}

In particular, (7.11) defines a pre-Lie product which is isomorphic to (7.10).

**Lemma 7.2.** For every $n$ and every $\tau_1, \tau_2$

\begin{equation}
(7.13) \quad \phi^\dagger_\tau(\mathcal{Z}_{\tau_1} \bowtie_n \mathcal{Z}_{\tau_2}) = (\phi^\dagger_\tau \mathcal{Z}_{\tau_1}) (D^{(n)} \phi^\dagger_\tau \mathcal{Z}_{\tau_2}).
\end{equation}

Identity (7.13) may be regarded as a pre-Lie morphism into $\hat{\mathcal{L}}$, cf. (4.3), namely

$$
\phi^\dagger_\tau(\mathcal{Z}_{\tau_1} \bowtie_n \mathcal{Z}_{\tau_2})D^{(n^2)} = (\phi^\dagger_\tau \mathcal{Z}_{\tau_1}) D^{(n)} \triangleright (\phi^\dagger_\tau \mathcal{Z}_{\tau_2}) D^{(n^2)}.
$$

Note that, unlike [4, Proposition 4.21] and in line with Subsection 6.3, our construction does not define a free (multi) pre-Lie algebra.

**Proof.** The case $n = 0$ follows from the arguments of Lemma 6.2. Let us now fix $n \neq 0$; multiplying (7.13) by $\sigma(\tau_1)\sigma(\tau_2)$ and combining (7.8) and (7.12), we see that (7.13) follows from

\begin{equation}
(7.14) \quad \sum_{\tau} m_n(\tau_1, \tau_2; \tau)\sigma(\beta)z^\beta = \sigma(\beta_1)\sigma(\beta_2)(z^{\beta_1}D^{(n)}z^{\beta_2}),
\end{equation}

with the understanding that $\beta_1$ and $\beta_2$ are determined by $\tau_1 \in \mathcal{T}_{\beta_1}$ and $\tau_2 \in \mathcal{T}_{\beta_2}$. Note that $m_n(\tau_1, \tau_2; \tau) \neq 0$ only if $\beta_1 + \beta_2 = \beta + e_n$; moreover, in

\textsuperscript{80}It is straightforward to check that this definition is the non-recursive version of [4, Definition 4.7].

\textsuperscript{81}The proof follows the argument in [20, Proposition 4.3].
such a case $\sigma(\beta_1)\sigma(\beta_2) = \sigma(\beta)$. Since $(z^{\beta_1} D^{(n)} z^{\beta_2}) z^{\beta_1 + \beta_2 - e_\alpha} = 0$, (7.16) reduces to
\[ \sum_{\tau;\beta_1+\beta_2=\beta+n} m_n(\tau_1, \tau_2; \tau) = \beta_2(n). \]
This clearly holds because $\beta_2(n)$ is the number of decorations $\mathcal{S} X^n$ that $\tau_2$ contains.

We now turn to the second operation, namely $\uparrow_1$, which increases polynomial decorations. As for $\cap_n$, there is a non-recursive expression of $[4$, Definition 4.7] given in this case by\footnote{The coefficient $m_n(X^{n+1}, \tau; \tau')$ is meaningful because we think of the decoration $\mathcal{S}[X^{n}(1,0)]$ as a subtree, and thus grafting $X^{n+1}$ makes sense.}
\[ (7.15) \quad \uparrow_1 \tau = \sum_n \sum_{\tau'} m_n(X^{n+1,0}, \tau; \tau') \tau', \]
and a similar expression for $\uparrow_2$ (in the sequel, we will focus on $\uparrow_1$). On the dual side, we define the operator $\xi_1$ as
\[ (7.16) \quad \xi_1 Z_{\tau} = \sum_n (n_1 + 1) \sum_{\tau'} n_n(X^{n+1,0}, \tau; \tau') Z_{\tau'}, \]
and analogously for $\xi_2$. Since by (7.5)
\[ n_n(X^{n+1,0}, \tau; \tau') \neq 0 \implies (n_1 + 1) N(\tau) = N(\tau'), \]
and thanks to $n_n(X^{n+1,0}, \tau; \tau') \sigma(\tau) = m_n(X^{n+1,0}, \tau; \tau') \sigma(\tau')$, we may pass from (7.16) to (7.15) by
\[ (7.17) \quad \xi_1 \sigma(\tau) N(\tau) Z_{\tau} = \sum_n \sum_{\tau'} m_n(X^{n+1,0}, \tau; \tau') \sigma(\tau') N(\tau') Z_{\tau'}. \]

Lemma 7.3. For every $\tau$
\[ (7.18) \quad \partial_1 \xi_1 Z_{\tau} = \partial_1 \phi_1 \xi_1 Z_{\tau}. \]

Proof. By definitions (3.12) of $\partial_1$ and (7.16) of $\xi_1$, it suffices to establish for fixed $n$ (and $\tau$)
\[ \partial_1 \phi_1 \sum_{\tau'} n_n(X^{n+1,0}, \tau; \tau') Z_{\tau'} = z_{n+(1,0)}(D^{(n)} \phi_1 Z_{\tau}). \]
Denoting by $\beta$ and $\beta'$ the multi-indices with $\tau \in T_\beta$ and $\tau' \in T_{\beta'}$, respectively, and appealing to the definition (7.8) of $\phi_-$ (and thus its transpose) this reduces to the following identity in $T^*$
\[ \sum_{\tau} n_n(X^{n+1,0}, \tau; \tau') \frac{\sigma(\beta')}{\sigma(\tau')} \beta' = \frac{\sigma(\beta)}{\sigma(\tau')} z_{n+(1,0)}(D^{(n)} z_\beta), \]
which by the combinatorial identity can be reformulated as
\[ (7.19) \quad \sum_{\tau'} m_n(X^{n+1,0}, \tau; \tau') \beta' = \sigma(\beta) z_{n+(1,0)}(D^{(n)} z_\beta). \]
We distinguish the case $n \neq 0$ and the remaining case of
\begin{equation}
\sum_{\tau'} m(X^{(1,0)}, \tau; \tau') \sigma(\beta') z^{\beta'} = \sigma(\tau) z_{(1,0)}(D^{(0)} z^\beta),
\end{equation}
which we treat first.

We note that by definition, $m(X^{(1,0)}, \tau; \tau') \neq 0$ implies that there exists a $k \geq 0$ such that
\begin{equation}
\beta' = \beta - e_k + e_{k+1} + e_{(1,0)}.
\end{equation}
Hence by definition (3.9) of $D^{(0)}$, for (7.20) it suffices to show for fixed $k$
\[\sum_{\tau':k} m(X^{(1,0)}, \tau; \tau') \sigma(\beta') z^{\beta'} = \sigma(\beta) z_{(1,0)}(k + 1) z_{k+1} (\partial z^\beta),\]
where the sum is over all trees $\tau'$ that arise from attaching the decoration $X^{(1,0)}$ to a node of the tree $\tau$ with $k$ children. By (7.21) and definition (6.7)
of $\sigma(\beta)$, this reduces to the combinatorial identity
\[\sum_{\tau':k} m(X^{(1,0)}, \tau; \tau') = \beta(k),\]
which is tautological by definition of $m$.

We now turn for (7.19) for $n \neq 0$. By definition, $m_n(X^{n+(1,0)}, \tau; \tau') \neq 0$ implies $\beta' = \beta - e_n + e_{n+1}$. Thus by definition (3.12) of $D^{(n)}$, and by definition (6.7) of $\sigma(\beta)$, (7.19) reduces to
\[\sum_{\tau'} m_n(X^{n+(1,0)}, \tau; \tau') = \beta(n),\]
which again is tautological by definition of $m_n$. \hfill \Box

### 7.3. Relating $\tilde{\phi}^\dagger$ to $\tilde{Y}$.

In the spirit of Subsection 6.5, we want to relate (7.13) and (7.18) with the morphism properties established in [4, Lemma 4.8]. Fixing a nonlinearity $a$, for arbitrary polynomial\textsuperscript{83} $u$ and shift\textsuperscript{84} $y$, we take inspiration from (2.3) to generalize definition (6.23) to
\begin{equation}
\tilde{Y}^a[\tau](u, y) := \tilde{\phi}^\dagger N(\tau) \sigma(\tau) Z \left[ a(\cdot + u(y)) , u(\cdot + y) - u(y) \right].
\end{equation}

By linearity, (7.22) extends from trees $\tau$ to linear combinations thereof. It is an easy consequence of Lemmas 7.2 and 7.3 that for fixed $a$, this linear map $\tilde{Y}^a$ from $\mathcal{B}$ into the space of functions of polynomial/base-point $(u, y)$ is a morphism w. r. t. to $\bowtie$ and $\uparrow$, in line with [4, Lemma 4.8]:

**Corollary 7.4.**

\begin{align*}
(7.23) & \quad \tilde{Y}^a[\tau_1 \bowtie_n \tau_2] = \tilde{Y}^a[\tau_1] \left( \frac{d}{du_n} \tilde{Y}^a[\tau_2] \right), \\
(7.24) & \quad \tilde{Y}^a[\uparrow_1 \tau] = \frac{d}{dy_1} \tilde{Y}^a[\tau],
\end{align*}

\textsuperscript{83}This notation is chosen to agree with [4]; we would rather replace $u$ by $p$.

\textsuperscript{84}Again, the notation is aligned upon [4]; for us, $y = h$ would be more natural.
where the coefficients $u^{(n)}$ are defined through $u(x) = \sum_n \frac{1}{n!} u^{(n)} x^n$.

Proof. We start with (7.23) by inserting definition (7.10) of $\bowtie_n$ and then use the linearity and definition (7.22) of $\Upsilon_a[\tau]$ to obtain

$$\text{l. h. s. of (7.23)} = \phi_{\perp} N(\tau) \sigma(\tau) m_n(\tau_1, \tau_2; \tau) Z_\tau,$$

where here and in the sequel we suppress the argument $[a(\cdot + u(y)), u(\cdot + y) - u(y)]$. Since $m_n(\tau_1, \tau_2; \tau) \neq 0$ implies $N(\tau_1) N(\tau_2) = n! N(\tau)$, this yields by (7.12)

$$\text{l. h. s. of (7.23)} = \phi_{\perp} \sum N(\tau_1) \sigma(\tau_1) Z_{\tau_1} \sim_n N(\tau_2) \sigma(\tau_2) Z_{\tau_2}.$$

This allows us to appeal to (7.13) yielding

(7.25) $\text{l. h. s. of (7.23)} = (\phi_{\perp} N(\tau_1) \sigma(\tau_1) Z_{\tau_1}) (D^{(n)} \phi_{\perp} N(\tau_2) \sigma(\tau_2) Z_{\tau_2}).$

Using that the argument of $\Upsilon_a$, see (7.22), can also be written as $[a(\cdot + u(y)), \sum_{n \neq 0} \frac{1}{n!} u^{(n)} x^n]$, we learn from the definitions (3.7) and (3.14) of $D^{(n)}$ that (7.25) is identical to the r. h. s. of (7.23).

We now turn to the l. h. s. of (7.24). By definition (7.15) of $\uparrow_1$, definition (7.22) of $\Upsilon_a$, and (7.17) we have (still with suppressed argument)

$$\text{l. h. s. of (7.24)} = \phi_{\perp} N(\tau) \sigma(\tau) Z_\tau,$$

so that we may appeal to (7.18) to the effect of

$$\text{l. h. s. of (7.24)} = (\partial_1 \phi_{\perp} N(\tau) \sigma(\tau) Z_\tau),$$

so that we may appeal to (7.18) to the effect of

$$\text{l. h. s. of (7.24)} = (\partial_1 \phi_{\perp} N(\tau) \sigma(\tau) Z_\tau) [a(\cdot + u(y)), u(\cdot + y) - u(y)].$$

By the characterization (3.11) of $\partial_1$ and definition (7.22) of $\Upsilon_a$, this turns into the l. h. s. of (7.24). $\square$

7.4. Relating $(\Delta, \Delta^\perp)$ to $(\Delta_H, \Delta_H^\perp)$.

Our final goal is to connect our Hopf algebra structure to that of [17]. A first strong hint that the structures are compatible is (4.49); however, the analogue (7.31) for the coaction $\Delta$ is missing.

We will pass to a coarser tree-based description, which no longer distinguishes different polynomial decorations of a given node but contracts them by multiplication. We identify the space of linear combinations of trees with contracted decorations with the model space in [17, Subsection 4.2], and denote it by $T_H$. More precisely, the model space $T_H$ consists of linear combinations of trees of the inductive form

$$\tau = \cdot X^n (\prod_{j \in J} I_{\tau_j}),$$

85In the more general context of [4], this contracted space corresponds to $V$, but we will directly work with the restriction to relevant trees.
as well as their integrated versions $\mathcal{I}\tau$. The passage from the detailed to the contracted description is encoded, as in [4, Subsection 4.1], by a linear map $Q: \mathcal{B} \to \mathcal{T}_H$ recursively given, for $\tau$ as in (7.3), by

$$Q\tau = \cdot X^n \prod_{j \in J} (\mathcal{I}Q\tau_j).$$

For example, (7.4) turns into

$$Q = X^{n_1} \downarrow X^{n_2} = I_{X^{n_1+n_2}}.$$  

The map $Q$ defines a new dictionary $\phi_-, \phi: \mathcal{B} \to \mathcal{T}_H$ by means of\footnote{Here, we are implicitly extending $\phi_-$ to the whole model space $\mathcal{B}$ by projecting onto the complement of the polynomial sector, i.e. $\phi_- z_e = 0$.}

$$\phi_- = Q\phi_-, \quad \phi = Q\phi_+.$$  

Equivalently, due to (7.6) and (7.7), $\phi_-$ and $\phi$ are determined by $\phi_- z_{\beta=0} = 0$ and then recursively in the length of $\beta$ by

$$\phi_- z_\beta = \sum_{k \geq 0} \sum_{e_k + \beta_1 + \cdots + \beta_k = \beta} \phi z_{\beta_1} \cdots \phi z_{\beta_k},$$

and

$$\phi z_\beta = \begin{cases} X^n & \text{if } \beta = e_n \\ \mathcal{I} \phi_- z_\beta & \text{otherwise} \end{cases}.$$  

For example, we have

$$\phi_- z_{\epsilon_0} = \cdot, \quad \phi z_{\epsilon_0} = 1,$$

$$\phi z_{\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_{(2,0)}} = 2 X^2 + 2 I X^2 + 2 I X^2.$$  

Moreover, $\phi_- z_\beta$ and $\phi z_\beta$ vanish unless (7.2) is satisfied. Due to $Q$, unlike $\phi_-$ and $\phi$, $\phi_-$ and $\phi$ are not one-to-one even if we restrict to populated indices; e.g. $z_{e_1+\epsilon_{(2,0)}}$ and $z_{e_2+\epsilon_{(1,0)}}$ are both mapped to $\cdot X^2$ by $\phi_-$.  

We denote by $| \cdot |_H$ the homogeneity in [17, p.23], which is defined as follows: $|X^n|_H := |n|, |\cdot|_H := \alpha - 2$ and then inductively via $|\tau_1 \tau_2|_H := |\tau_1|_H + |\tau_2|_H$ and $|I\tau|_H := |\tau|_H + 2$. We now argue that $\phi_- z_\beta$ is a linear combination of trees $\tau$ satisfying $|\tau|_H = |\beta| - 2$. The analogue for $\phi$ holds true with $|\tau|_H = |\beta|$. Indeed, this can be seen by induction in the length of $\beta$, where we may restrict to $\beta$’s satisfying (7.2). For $\beta$’s of length one we only have to consider $\phi_- z_{\epsilon_0}, \phi z_{\epsilon_0}$ and $\phi z_{\epsilon_1}$, for which the statement is clear by (7.28), (7.29) and recalling that $|\epsilon_0| = \alpha$ and $|\epsilon_1| = |n|$, cf. (3.39). In the induction step, the statement for $\phi_- z_\beta$ follows from (7.27), by using the induction hypothesis, the definition of $| \cdot |_H$ and $|\beta| = |e_k + \beta_1 + \cdots + \beta_k| = \alpha + |\beta_1| + \cdots + |\beta_k|$ in every summand of (7.27). As a consequence we obtain the corresponding statement for $\phi z_\beta$ from (7.28).
vanishes for $\tau = X^\nu$ for every $\nu'$, and for $|\nu| \geq |\tau|_H + 2$. Moreover, we define $\Phi : T^+ \rightarrow T_H^+$ by postulating it to be multiplicative and

$$(7.30) \quad \Phi Z^{(0,(1,0))} = X_1, \quad \Phi Z^{(0,(0,1))} = X_2, \quad \Phi J_n z_\beta = J_n^H \phi_- z_\beta,$$

for $|\nu| < |\beta|$. By the compatibility of the homogeneities $|\cdot|_H$ and $|\cdot|$, see above, the last equality in (7.30) can also be seen to hold for arbitrary $\beta$.

Finally, we recall the definition of the flipped\textsuperscript{87} coaction $\Delta_H$ and coproduct $\Delta_H^+$ of [17, pp. 25-26]. The coaction $\Delta_H : T_H \rightarrow T_H^+ \otimes T_H$ is by multiplicativity $\Delta_H \tau \tau' = (\Delta_H \tau) (\Delta_H \tau')$ recursively defined via

$$\Delta_H 1 = 1 \otimes 1, \quad \Delta_H \cdot = 1 \otimes \cdot, \quad \Delta_H X_i = 1 \otimes X_i + X_i \otimes 1,$$

(7.31)

$$\Delta_H I_\tau = (\text{id} \otimes I) \Delta_H \tau + \sum_n J_n^H \tau \otimes X_n^n/n!.$$

The coproduct $\Delta_H^+ : T_H^+ \rightarrow T_H^+ \otimes T_H^+$ is by multiplicativity $\Delta_H^+ \tau \tau' = (\Delta_H^+ \tau)(\Delta_H^+ \tau')$ via the coaction recursively defined by

$$\Delta_H^+ 1 = 1 \otimes 1, \quad \Delta_H^+ X_i = 1 \otimes X_i + X_i \otimes 1,$$

(7.32)

$$\Delta_H^+ J_n^H \tau = (\text{id} \otimes J_n^H) \Delta_H \tau + \sum_m J_m^H \cdot \otimes X_m^m/m!.$$

With this definitions at hand we may formulate the following intertwining property, which is the main result of this section.

**Proposition 7.5.**

(7.33) \quad $\Delta_H \phi_- = (\Phi \otimes \phi_-) \Delta,$

(7.34) \quad $\Delta_H^+ \Phi = (\Phi \otimes \Phi) \Delta^+.$

Since by definition, $\Phi$ is an algebra morphism, and bialgebra morphisms between Hopf algebras are automatically Hopf algebra morphisms, $\Phi$ is in particular a Hopf algebra morphism.

**Proof.** Step 1. From coaction to coproduct by intertwining. We claim that (7.33) implies (7.34). By multiplicativity of $\Phi$, $\Delta_H^+$, and $\Delta^+$, it is enough to establish (7.34) when applied to $Z^{(\varepsilon,(\nu,\alpha))^0}$, $Z^{(0,(1,0))}$, and $Z^{(0,(0,1))}$. The two latter cases follow from (4.50), (7.30), and (7.31). For the former case we start from (4.49) to which we apply $\Phi \otimes \Phi$, so that by (7.30) (and the multiplicativity of $\Phi$)

$$(\Phi \otimes \Phi) \Delta^+ J_n z_\beta = (\text{id} \otimes J_n^H) (\Phi \otimes \phi_-) \Delta z_\beta + \sum_m J_m^H \phi_- z_\beta \otimes X_m^m/m!.$$ 

On the other hand, we use (7.32) for $\tau = \phi_- z_\beta$ so that by (7.30)

$$\Delta_H^+ \Phi J_n z_\beta = (\text{id} \otimes J_n^H) \Delta_H \phi_- z_\beta + \sum_m J_m^H \phi_- z_\beta \otimes X_m^m/m!.$$ 

We now see that (7.33) implies $\Delta_H^+ \Phi J_n = (\Phi \otimes \Phi) \Delta^+ J_n$.

\textsuperscript{87}i. e. $\text{tw} \circ \Delta_H$ is the coaction in [17], where $\text{tw}(x \otimes y) = y \otimes x$, and the same for the coproduct.
STEP 2. Taking care of the polynomial sector. We claim that (7.33) implies
\begin{equation}
\Delta_H \phi = (\Phi \otimes \phi) \Delta + \Phi(J_0 + \sum_{n \neq 0} Z^{(0, n)} \otimes z_n) \otimes 1
\end{equation}
componentwise, by which we mean that for fixed $\beta$ with $[\beta] \geq 0$, (7.33) applied on $z_\beta$ implies (7.35) applied on $z_\beta$, and that (7.35) automatically holds when applied to $z_\beta$. We split (7.35) into
\begin{align}
\Delta_H \phi &= (\Phi \otimes \phi) \Delta + \Phi J_0 \otimes 1 \quad \text{on } \tilde{T}, \\
\Delta_H \phi &= (\Phi \otimes \phi) \Delta + \Phi \left( \sum_{n \neq 0} Z^{(0, n)} \otimes z_n \right) \otimes 1 \quad \text{on } \bar{T}.
\end{align}

By (7.28) we have $\phi = I \phi_-$ on $\tilde{T}$, by (7.30) we have $\Phi J_n = J_n H \phi_-$, so that by (7.31) we obtain
\begin{equation}
\Delta_H \phi = (\text{id} \otimes I) \Delta_H \phi_- + \sum_n \Phi J_n \otimes \frac{X^n}{n!} \quad \text{on } \tilde{T}.
\end{equation}
We now insert (7.33):
\begin{equation}
\Delta_H \phi = (\Phi \otimes I) \Delta_H \phi_- + \sum_n \Phi J_n \otimes \frac{X^n}{n!} \quad \text{on } \tilde{T}.
\end{equation}
Hence (7.36) follows from
\begin{equation}
(id \otimes (\phi - I \phi_-)) \Delta = \sum_{n \neq 0} J_n \otimes \frac{z_n}{n!} \quad \text{on } \bar{T}.
\end{equation}
Here comes the argument for (7.38): By (7.28), $\phi - I \phi_-$ is the projection onto the polynomial sector $\bar{T}_H$. Hence (7.38) assumes the form
\begin{equation}
(id \otimes (1 - P^\dagger)) \Delta = \sum_{n \neq 0} J_n \otimes \frac{z_n}{n!} \quad \text{on } \bar{T},
\end{equation}
where $P^\dagger$ denotes the projection from $T$ onto $\bar{T}$ (defined through the direct sum $T = \bar{T} \oplus \tilde{T}$). In view of (4.34) and (4.48), the last statement amounts to
\begin{equation}
((\rho U) z_n) = \iota_n U \mod \bar{T}^* \quad \text{for all } U \in U(L) \text{ and } n \neq 0.
\end{equation}
We check (7.39) on the basis elements $U = D_{(j, m)}$, cf. (4.15): By definition (4.20) of $\iota_n$, the r. h. s. of (7.39) does not vanish iff $(J, m) = (e_{(\beta, n)}, 0)$, in which case both sides coincide with $z^\beta$. The l. h. s. of (7.39) is also non-zero for $J = 0$, it is then given by $(\frac{1}{m!} \partial^m z_n) \in \bar{T}^*$, see (3.12), so that the statement is not affected.

We now turn to (7.37), which can be rephrased as
\begin{equation}
\Delta_H \phi z_n = (\Phi \otimes \phi) \Delta z_n + \Phi Z^{(0, n)} \otimes 1 \quad \text{for all } n \neq 0.
\end{equation}
In view of (7.31) and the multiplicativity of $\Delta_H$ we have
\[ \Delta_H X^n = \sum_{n' + n'' = n} \binom{n}{n'} X^{n'} \otimes X^{n''}. \]

In view of the multiplicativity of $\Phi$ in conjunction with (7.30) we have $\Phi Z^{(0,n)} = X^n$ (tautologically including $n = 0$). Together with (7.28), we thus see that (7.40) reduces to (4.51).

**Step 3.** Inductive proof of (7.33). We carry out an induction in the length of $\beta$ with $|\beta| \geq 0$, the component of (7.33). We start with the base case of $\beta = 0$. The l. h. s. vanishes since (7.27) includes $\phi - z_{\beta=0} = 0$. For the r. h. s. to also vanish, we just need to convince ourselves that $\Delta z_{\beta=0} = 0 = \sum_{n \neq 0} Z(e_{(0,n)}, 0) \otimes z_e_{n}$. By (4.44) and (4.34), this amounts to the fact that the image of $\rho_{D(J,m)} \in \text{End}(T^*)$, see (4.15), contains 1 iff $(J, m) = (e_{(0,n)}, 0)$ for some $n \neq 0$, in which case the pre-image must be $z_n$.

Obviously, the pre-dual $\mathbb{R}[[z_k, z_n]]^\dagger$ of $\mathbb{R}[[z_k, z_n]]$, which is isomorphic to $\mathbb{R}[z_k, z_n]$, is endowed with a coproduct; since $T_H$ is endowed with a product, we may multiply linear maps from $\mathbb{R}[[z_k, z_n]]^\dagger$ into $T_H$. Extending $\phi$ (next to $\phi_-$) trivially from $T$ to $\mathbb{R}[[z_k, z_n]]^\dagger$, we may apply this to $\phi$ in order to give a sense to $\phi_k$. Furthermore, we may identify elements of $\mathbb{R}[[z_k, z_n]]$, like $z_k$, with a linear map (of rank 1) from $\mathbb{R}[[z_k, z_n]]^\dagger$ into $T_H$. Note finally that (7.27) extends from populated, cf. (3.28), to all $\beta$; indeed, if $\beta$ is not populated, (7.2) is violated, hence if $e_k + \beta_1 + \cdots + \beta_k = \beta$, one of the $\beta_i$’s violates (7.2) and thus the r. h. s. of (7.27) vanishes. This allows us to re-interpret (7.27) in the more compact way
\[ \phi_- = \cdot \sum_{k \geq 0} z_k \phi_k \]

as an identity in $\mathcal{L}(\mathbb{R}[[z_k, z_n]]^\dagger, T_H)$. By the multiplicativity of $\Delta_H$ and (7.31), this implies
\[ \Delta_H \phi_- = (1 \otimes \cdot) \sum_{k \geq 0} z_k (\Delta_H \phi)^k, \]

an identity in the space of linear maps from $\mathbb{R}[[z_k, z_n]]^\dagger$ into the algebra $T_H^+ \otimes T_H$.

It will be convenient to test this identity with elements $Z$ of a space endowed with a non-degenerate pairing with $T_H^+$, which we choose to be the unique up to linear isomorphisms direct sum indexed by the basis elements of $T_H^+$ with the corresponding canonical pairing. Since $T_H^+$ is endowed with the polynomial product, this space carries a coproduct, so that in Sweedler’s notation
\[ \langle Z, \Delta_H \phi_- \rangle = \cdot \sum_k z_k \sum_{(Z)} \langle Z(1), \Delta_H \phi \rangle \cdots \langle Z(k), \Delta_H \phi \rangle, \]

\[ 89\text{unique up to linear isomorphisms} \]
with the understanding that the pairing acts on the $T^+_H$ component, and for $k = 0$ the $\sum (\epsilon)$-term is the counit applied to $Z$. Due to the presence of $z_k$, the r. h. s. of the $\beta$-component of this identity only involves components $\gamma$'s of smaller length so that we may appeal to the induction hypothesis (7.33). We use (7.33) in its upgraded form (7.35), tested with $Z$. Note that the matrix representation of $\Phi$ in the standard bases of the polynomial algebras $T^*$ and $T^+_H$ has the finiteness property that allows us to define $\Phi^*Z \in U(L)$. Indeed, this finiteness property is inherited from the corresponding one of $\phi_-$, see (7.9), via (7.30) and (7.26). Using the argument from the end of Subsection 6.4, we have $\langle Z, (\Phi \otimes \phi) \Delta \rangle = \phi(\rho U)^\dagger$, where $U := \Phi^*Z \in U(L)$. Hence (7.35) tested with $Z$ takes the form of an identity in $L(\mathbb{R}[[z_k, z_n]]^\dagger, T_H^*) \supset T^* \otimes T_H$

\begin{equation}
(7.43) \quad \langle Z, \Delta_H \phi \rangle = \phi(\rho U)^\dagger + (\iota_0 U + \sum_{n \neq 0} (\epsilon_n U)z_n) \otimes 1 \quad \text{on } T.
\end{equation}

Extending $(\rho U)^\dagger$ trivially, we may think of (7.43) as holding not just on $T$ but all of $\mathbb{R}[[z_k, z_n]]^\dagger$, which allows us to insert (7.43) into (7.42). Since $\Phi$ is multiplicative, $\Phi^*$ preserves the coproduct, so that we obtain

\begin{equation}
(7.44) \quad \langle Z, \Delta_H \phi_- \rangle = \cdot \sum_k z_k \prod_{(U)} \phi(\rho U_{(U)})^\dagger + (\iota_0 U_{(U)} + \sum_{n \neq 0} (\epsilon_n U_{(U)})z_n) \otimes 1),
\end{equation}

again with the understanding that for $k = 0$ the term $\sum_{(U)}$ reduces to $\epsilon_0 U$. Here we interpret $\iota_0 + \sum_{n \neq 0} \epsilon_n \otimes z_n$ as a linear map from $U(L)$, a space endowed with a coproduct, into the algebra $\mathbb{R}[[z_k, z_n]]$, so that powers make sense. To further rewrite (7.44), we first claim that linear maps $\phi_1, \phi_2 \in L(\mathbb{R}[[z_k, z_n]]^\dagger, T_H^*)$ satisfy a generalized Leibniz rule, i. e. for $U \in U(L)$ we have

\begin{equation}
(7.45) \quad (\phi_1 \phi_2)(\rho U)^\dagger = \sum_{(U)} (\phi_1(\rho U_{(1)})^\dagger)(\phi_2(\rho U_{(2)})^\dagger),
\end{equation}

which we show by duality. Therefore, we consider elements $Z$ of a space endowed with a non-degenerate pairing with $T_H^*$, which we choose to be the direct sum indexed by the basis elements of $T_H$ with the corresponding canonical pairing. This space carries a coproduct, since the product on $T_H$ satisfies the right finiteness property.\textsuperscript{90} Applying the l. h. s. of (7.45) to $z_\beta \in \mathbb{R}[[z_k, z_n]]^\dagger$ and testing with $Z$, we obtain using the argument from the end of Subsection 6.4 by duality

\begin{equation}
\langle Z, (\phi_1 \phi_2)(\rho U)^\dagger z_\beta \rangle = \langle (\rho U)(\phi_1^* \phi_2^*)Z, z_\beta \rangle.
\end{equation}

\textsuperscript{90}Compare to $U(L)$ and $T^+$ in Section 4.
Using (5.23) and again duality yields
\[
\langle (\rho U_1^*) (\rho U_2^*) Z, z_\beta \rangle = \langle \sum_{(U)} ((\rho U_1^*) (\rho U_2^*) ) Z, z_\beta \rangle = \langle Z, \sum_{(U)} (\phi_1 (\rho U_1^*) ) (\phi_2 (\rho U_2^*) ) z_\beta \rangle,
\]
finishing the argument for (7.45). By (7.45) in its multi-linear form, (7.44) thus assumes the form
\[
\langle Z, \Delta H \phi \rangle = \sum_k z_k \sum_{l=0}^k (k^l_1 A) (\phi_1^k (\rho U_1^*) ) (\phi_2^l (\rho U_2^*) ) z_\beta.
\]
(7.46)
Changing the order of summation in \( k \geq l \) and applying (A.1) with \( U \) replaced by \( U_2 \), this collapses to
\[
\langle Z, \Delta H \phi \rangle = \sum_l \sum_{(U)} \phi_1^l (\rho U_1^*) z_l.
\]
Using once more (7.45), this yields
\[
\langle Z, \Delta H \phi \rangle = \sum_l \phi_1^l (\rho U_1^*) z_l,
\]
which is the tested/dual form of (7.33).

**Appendix A.**

We first provide summation formulas for multi-indices which are repeatedly used throughout the text. We assume that \( J, J', J'' \) are multi-indices over a set \( I \) and \( A, B \) are finite sequences indexed by multi-indices over \( I \) in a real vector space.

**Lemma A.1.** It holds
\[
(J(i) + 1) \sum_{J' + J'' = J + e_i} \frac{1}{J'!} A_{J'} \otimes \frac{1}{J''!} B_{J''} = \sum_{J' + J'' = J} \left( \frac{1}{J'!} A_{J' + e_i} \otimes \frac{1}{J''!} B_{J''} + \frac{1}{J'!} A_{J'} \otimes \frac{1}{J''!} B_{J'' + e_i} \right).
\]

**Proof.** We rewrite each of the terms of the r. h. s as
\[
(J' + 1) \frac{1}{J'! + e_i!} A_{J' + e_i} \otimes \frac{1}{J''!} B_{J''} + (J'' + 1) \frac{1}{J''! + e_i!} A_{J'' + e_i} \otimes \frac{1}{J'!} B_{J'}.
\]
The sum then turns into
\[
\sum_{J' + J'' = J + e_i} J' \frac{1}{J'!} A_{J'} \otimes \frac{1}{J''!} B_{J''} + \sum_{J' + J'' = J + e_i} J'' \frac{1}{J''!} A_{J''} \otimes \frac{1}{J'!} B_{J'}.
\]
Note that the restrictions $J' \geq e_i$ and $J'' \geq e_i$ are immaterial because of the factors $J'(i)$ and $J''(i)$. We may then combine the sums, and since $J'(i) + J''(i) = J(i) + 1$, this concludes the proof. \hfill \Box

**Lemma A.2.** It holds

$$\sum \frac{1}{J!} A_J = \sum_{k \geq 0} \sum_{i_1, \ldots, i_k \in I} A_{e_{i_1} + \ldots + e_{i_k}}.$$

**Proof.** We split the sum of the l. h. s. according to $k = |J|$, parametrized according to $J = \sum_{j=1}^k e_{i_j}$ and count repetitions to obtain

$$\sum \frac{1}{J!} A_J = \sum_{k \geq 0} \sum_{|J| = k} \frac{1}{k!} \sum_{i_1, \ldots, i_k \in I} A_{e_{i_1} + \ldots + e_{i_k}}$$

$$= \sum_{k \geq 0} \frac{1}{k!} \sum_{i_1, \ldots, i_k \in I} A_{e_{i_1} + \ldots + e_{i_k}}. \hfill \Box$$

Finally, we show an extended chain rule, which connects to Faà di Bruno’s formula.

**Lemma A.3.** For $l \geq 0$ and $U \in U(L)$ it holds

$$(\rho U) z_l = \sum_{k \geq l} \binom{k}{l} z_k (t_0 + \sum_{n \neq 0} \varepsilon_n \otimes z_n)^{k-l} U.$$ (A.1)

Recall that (A.1) is short for

$$(\rho U) z_l = \sum_{k \geq l} \binom{k}{l} z_k$$

$$(A.2) \times \sum_{(U)} \left( t_0 U_{(1)} + \sum_{n \neq 0} (\varepsilon_n U_{(1)}) z_n \right) \cdots \left( t_0 U_{(k-l)} + \sum_{n \neq 0} (\varepsilon_n U_{(k-l)}) z_n \right).$$

**Proof.** Assume we have established (A.1) for $U = \frac{1}{m!} \partial^m$ for any $m$. It then remains to argue for admissible $(\beta, n)$ that if $U$ satisfies (A.2), also $z^{\beta} U D^{(m)}$ satisfies (A.2), which we shall do now. We first address the case of $m \neq 0$, in which case the l. h. s. of (A.2) (with $U$ replaced by $z^{\beta} U D^{(m)}$) vanishes. According to Lemma 4.2, the rank-one constituents of the coproduct of $z^{\beta} U D^{(m)}$ always involve one factor of the form $z^{\beta} U D^{(m)}$, which is annihilated by both $t_0$ and $\varepsilon_n$. Hence it remains to consider $m = 0$. For the l. h. s. of (A.2) we have by (4.8) and (3.9)

$$(\rho z^{\beta} U D^{(0)}) z_l = (l+1) z^{\beta} ((\rho U) z_{l+1}).$$ (A.3)

Once more according to Lemma 4.2, the rank-one constituents of the coproduct of $z^{\beta} U D^{(0)}$ are those of $U$ besides exactly one factor of the form $z^{\beta} U D^{(0)}$, which is annihilated by $\varepsilon_n$, and is annihilated by $t_0$ unless $U' = 1,$
in which case it gives rise to $z^\beta$. Hence the r. h. s. of (A.2) (with $U$ replaced by $z^\beta UD^{(0)}$) assumes the form

\begin{equation}
\sum_{k \geq l} \binom{k}{l} z_k (k-l) \times \sum_U \left( (\varepsilon U_1(z) + \sum_{n \neq 0} \varepsilon_n U_{(1)}(z^n)) \cdots (\varepsilon U_{(k-l-1)}(z^n) + \sum_{n \neq 0} \varepsilon_n U_{(k-l-1)}(z^n)) \right) .
\end{equation}

It follows from (A.2) (with $l$ replaced by $l+1$) and the combinatorial identity $\binom{k-l}{l} = \binom{l+1}{k+l}$ that (A.3) and (A.4) coincide.

We now turn to the proof of (A.2) for $U = \frac{1}{m!} \partial^m$. Since the factors of the coproduct of $U$ are of the form $U' = \frac{1}{m'!} \partial^{m'}$, see Lemma 4.6, on which $\iota_0$ vanishes and on which $\varepsilon_n$ renders 1 provided $n = m' \neq 0$, (A.2) assumes the form

\begin{equation}
\left( \frac{1}{m!} \partial^m z_l \right) = \sum_{k \geq l} \binom{k}{l} z_k \sum_{m_1 + \cdots + m_{k-l} = m} z_{m_1} \cdots z_{m_{k-l}},
\end{equation}

which, relabelling $k-l$ as $k$, we rewrite as

\begin{equation}
\left( \frac{1}{m!} \partial^m \partial^l z_l \right) = \sum_{k \geq 0} \frac{1}{k!} \partial^{k+l} z_k \sum_{m_1 + \cdots + m_k = m} z_{m_1} \cdots z_{m_k}.
\end{equation}

From definitions (2.9) and (3.11) we see that this amounts to the Faà di Bruno formula

\begin{equation}
\frac{1}{m!} \frac{d^m}{d y^m} \frac{d^l}{d u^l} (p(y)) = \sum_{k \geq 0} \frac{1}{k!} \frac{d^{k+l}}{d u^{k+l}} (0) \sum_{m_1 + \cdots + m_k = m} \frac{1}{m_1!} \frac{d^{m_1} p(0)}{dy^{m_1} (0)} \cdots \frac{1}{m_k!} \frac{d^{m_k} p(0)}{dy^{m_k} (0)},
\end{equation}

cf. [13, (2.1)] up to re-summation.

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The authors thank Yvain Bruned and Stefan Hollands for fruitful discussions. PL thanks Nikolas Tapia for discussions on free pre-Lie algebras. MT thanks Lorenzo Zambotti for suggestions.

\section*{Symbolic index}

| Symbol | Description | Page |
|--------|-------------|------|
| $\bowtie$ | Grafting of trees | 49 |
| $\bowtie_n$ | Generalized grafting pre-Lie product in [4, Definition 4.7] | 58 |
| $\uparrow_i$ | Operator of increasing decorations in [4, Definition 4.7] | 59 |
| $\triangleright$ | Pre-Lie product on $\text{Der}(\mathbb{R}[[z_k, z_n]])$ | 19 |
| $\rightsquigarrow$ | Pre-Lie product on $\mathcal{L}^1$ | 49 |
\( \sim_n \) Generalized grafting pre-Lie product on \( L^1 \) 58
\( \sharp_i \) Operator of increasing decorations on \( L^1 \) 59
\( [\gamma] \) Noise-homogeneity of the multi-index \( \gamma \) 17
\( |\gamma| \) Homogeneity of the multi-index \( \gamma \) 21
\( |\tau|_H \) Homogeneity of the tree \( \tau \) of [17] 62
\( |n| \) Scaled length of \( n \) 9
\( |(J, m)| \) Length of \( (J, m) \) 27
\( |\cdot|_{\text{gr}} \) Grading on \( (J, m) \) 32
A Set of homogeneities 21
\( B \) Model space for branched rough paths 48
\( B \) Space of trees with expanded polynomial decorations in [4] 56
\( \text{bi} \) Bigrading on pairs \((\gamma, n)\) and \((J, m)\) 20, 31
\( \text{cop} \) Coproduct in \( U(L) \) 23
\( D_{(J,m)} \) Basis element of \( U(L) \) 27
\( D^{(n)} \) Infinitesimal generator of constant tilt by \( x^n \) 14
\( \dagger \) Transposition of linear maps 13
\( \partial_i \) Infinitesimal generator of shift 14
\( \Delta \) Coaction on \( T \) over \( T^+ \) 34
\( \Delta_B \) Coproduct on trees of [3] 51
\( \Delta_H \) Coaction on \( T_H \) over \( T_H^+ \) of [17] 63
\( \Delta^+ \) Coproduct in \( T^+ \) 35
\( \Delta_H^+ \) Coproduct on \( T_H^+ \) of [17] 63
\( \Delta_{RP}^+ \) Restriction of \( \Delta^+ \) relevant for rough paths 51
\( \varepsilon_n \) Generalized counit in \( U(L) \) 29
\( f \) Generic element of \( \text{Alg}(T^+, \mathbb{R}) \) 36
\( G \) Structure group 37
\( \Gamma_f \) Generic element of the structure group \( G \) 37
\( \mathcal{I} \) Abstract integration map in [17, 4] 56
\( \iota_n \) Projection of \( \{z^n D^{(n)}\}_{\gamma} \) to \( T^* \) 29
\( J_n \) Embedding of \( \{\gamma \in \tilde{T} \mid |\gamma| > |n|\} \) into \( T^+ \) 35
\( J_n^H \) Abstract placeholder for Taylor coefficients in [17] 63
\( L \) Lie algebra of \( \{z^n D^{(n)}\}_{(\gamma, n)} \cup \{\partial_i\}_i \) 22
\( L^1 \) Lie algebra of \( \{z^n D^{(n)}\}_{(\gamma, n)} \) 23
\( L_{\text{RP}}^1 \) (pre-)Lie algebra of Connes-Kreimer in [12] 49
\( L_{\text{RP}} \) Restriction of \( L \) relevant for rough paths 49
\( M_{\epsilon} \) Renormalization map for rough paths 53
\( M_{\epsilon}^{BCFP} \) Translation map for rough paths in [5, Definition 14] 54
\( P \) Projection of \( T^* \) on \( \tilde{T}' \) 44
\( P^\dagger \) Projection of \( T \) on \( \tilde{T} \) 45
\( \phi \) Dictionary between multi-indices and trees 48
\( \phi \) Dictionary between \( T \) and \( B \) 57
\( \phi_\dagger \) Dictionary between \( \tilde{T} \) and \( B \) 57
THE STRUCTURE GROUP FOR QUASI-LINEAR EQUATIONS

Φ Dictionary between $T^+$ and $T_H^+$ 63
Π Model of a regularity structure 46
Q Decorations contraction operator in [4, p. 911]
$\rho$ Representation of $U(L)$ 23
$T$ Model space 18
$T$ Polynomial sector of $T$ 15
$\tilde{T}$ Non-polynomial subspace of $T$ 18
$T_H$ Model space of [17] 61
$T_{RP}$ Restriction of $T$ relevant for rough paths 48
$T^+$ Dual structure of $U(L)$ 34
$T_H^+$ Formal expressions representing Taylor coefficients of [17] 62
$T_{RP}^+$ Restriction of $T^+$ relevant for rough paths 51
$\tau$ Generic tree 47
$U(L)$ Universal enveloping algebra of $L$ 23
$\Upsilon$ Linear map from $B$ into functions, cf. [2, Definition 2.13] 52
$\hat{\Upsilon}$ Linear map from $B$ into functions, cf. [4, (4.4)] 60
$\mathcal{X}$ Model for branched rough paths 48
$z_{k}, z_{n}$ Coordinate functionals on $(a, p)$ 8
$z_{\gamma}$ Monomial in $T$ 18
$z^\gamma$ Monomial in $T^*$ 12
$z^\gamma D^{(n)}$ Infinitesimal generator of variable tilt 19
$Z^{(J,m)}$ Basis element of $T^+$ 34
$Z_{\tau}$ Basis element of $L^1$ 49

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