TEACHER EDUCATION & DEVELOPMENT | RESEARCH ARTICLE

What teachers understand of model lessons

Scott A. Courtney1*

Abstract: Over the past two decades, researchers in mathematics teacher education have identified characteristics of high quality professional development (PD). This report describes an investigation of a common approach to PD with secondary mathematics teachers, providing teachers with opportunities to experience reform-oriented model lessons as students of the lesson's mathematics. Teachers experienced a lesson, as students, on the design of displays of statistical data as a method to investigate statistical questions. The investigation then examined the ways in which teachers assimilated the lesson by having them attempt to reconstruct the lesson for their own use. The author points to ways in which teachers’ current meanings and ways of thinking about statistics in particular, and mathematics in general, interfered with their ability to create a coherent reconstruction of the lesson and identifies two potential sources that account for the difficulties teachers exhibited in their attempts to reconstruct the lesson.

Subjects: Mathematics Education; Mathematics; Theories of Learning; Teachers & Teacher Education; Teaching & Learning

Keywords: mathematics education; professional development; teacher cognition; mathematical meanings

ABOUT THE AUTHOR

Scott A. Courtney is an assistant professor in Mathematics Education at Kent State University. Dr Courtney’s research interests include teachers’ conceptions of mathematics within grades 6-12, instructional engagements propitious for student development of intended ideas and ways of thinking and teachers’ conceptions, and ways of thinking that support or constrain their capacity to transform their cognitions with cognitive structures that are more conceptually oriented.

PUBLIC INTEREST STATEMENT

Providing mathematics teachers with opportunities to act as students in classroom activities, where teacher educators or professional developers model effective instructional practices or changes to mathematics content or process standards, is an activity meeting several characteristics of high quality professional development. Although providing teachers with opportunities to engage in and reflect on such experiences is viewed as a best practice, there has been little research into the meanings teachers make of such experiences. This report describes the meanings 16 United States high school mathematics teachers (with students ages 14–18 years) made of a lesson they experienced as students on the design of displays of statistical data as a method to investigate statistical questions. In addition, the report operationalizes Thompson’s interpretive framework for the development of powerful Mathematical Meanings for Teaching (MMT). Explication of teachers’ personal and pedagogical mathematical meanings, as presented here, supports future investigations employing Thompson’s MMT framework.

© 2017 The Author(s). This open access article is distributed under a Creative Commons Attribution (CC-BY) 4.0 license.
1. Introduction
Over the past two decades, researchers in teacher education and teacher change have identified characteristics of high quality professional development (PD) (e.g. Guskey & Yoon, 2009; Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2010; Penuel, Fishman, Yamaguchi, & Gallagher, 2007). According to Desimone (2009), “Research reflects a consensus about at least some of the characteristics of PD that are critical to increasing teacher knowledge and skills and improving their practice, and which hold promise for increasing student achievement” (p. 183). Such characteristics include: activities that focus on subject matter content and how students learn that content; active learning experiences; and, interaction and discourse with teachers from the same school, grade, or subject area (Desimone, 2009, p. 184). Providing teachers with opportunities to act as students in classroom activities, where professional developers “model desired classroom approaches during in-service sessions to project a clearer vision of the proposed changes” (Clarke & Clarke, 2005) is a PD activity meeting all three characteristics. Borasi and Fonzi (2002) assert such experiences should “model effective instructional and/or learning practices promoted by school mathematics reform” (p. 35), and allow for teacher reflection following the mathematical learning experience to help initiate “teachers' rethinking of their views of mathematics, teaching and learning” (p. 44).

Although providing teachers with opportunities to engage in or observe model (or demonstration) lessons and opportunities to reflect on such experiences are viewed as best practices for PD and teacher education, there has been little research into the meanings teachers make of such experiences. With few exceptions (e.g. Tzur, Simon, Heinz, & Kinzel, 2001), research suggesting the benefits of teachers experiencing mathematics as learners (e.g. Borasi, Fonzi, Smith, & Rose, 1999; Saxe, Gearhart, & Nasir, 2001) do not make explicit how teachers’ meanings and ways of thinking influence their: (1) understanding of the lesson and its purposes and (2) abilities to reflect on the lesson in relation to their beliefs, knowledge, or practices.

Furthermore, the growing body of research suggesting the benefits of providing teachers with opportunities to observe (in-person or via videotape) model lessons focus analyses on teachers’ self-reported lesson perceptions (e.g. Hudson, 2007) or teachers’ capacities to notice particular aspects or features of the lesson or lesson segments (e.g. Kersting, 2008; Sherin, Jacobs, & Philipp, 2011). Descriptions of how teachers might have assimilated such experiences are missing from such analyses.

The current report addresses this gap by examining teachers' understandings and ways of thinking as they engage in a model lesson as students of mathematics. The model lesson described in this report provided teachers the opportunity to experience a mathematics lesson that is grounded in meanings and the development of significant mathematical ideas. Such a lesson is considered a reform lesson in the United States, and "is likely to be a new, and perhaps, alien experience for many [American] teachers" (Conference Board of the Mathematical Sciences, 2012, p. 11). Furthermore, the lesson was designed to support students' (and teachers') development of understandings and ways of thinking that lend coherence and practicality to their mathematical thinking.

This report takes as foundational the assertion, “Teachers’ current conceptions frame their interpretations and assimilation of the PD experiences in which they engage ... not the conceptions of those who designed or implemented the teacher education intervention” (Heinz, Kinzel, Simon, & Tzur, 2000, pp. 102–103). As such, the study attempted to address the following questions:

(1) What was the model lesson that teachers experienced?
(2) What meanings did teachers utilize as they experienced alternative approaches to teaching and learning mathematics?

In the following sections, the author describes a model lesson for in-service secondary mathematics teachers and attempts to identify what teachers understood the experience to have been.
Specifically, the author describes teachers’ understandings and ways of thinking as they participated in the lesson and as they attempted to reconstruct the experience they had.

2. Methodology

The framework underlying the study is Pat Thompson’s interpretive framework for the development of powerful Mathematical Meanings for Teaching (MMT). Thompson (2013) developed his framework as an elaboration of the earlier work of Thompson, Philipp, Thompson, and Boyd (1994), Thompson and Thompson (1996), and Silverman and Thompson (2008).1 Although Thompson’s MMT framework has been utilized elsewhere, it was not articulated in a manner sufficient for the current study. Therefore, in order to attempt to identify what teachers understood the model lesson experience to have been, it was critical for the author to operationalize the work of Thompson and his colleagues (e.g. Silverman, 2005; Silverman & Thompson, 2008; Thompson, 1994; Thompson et al., 1994).

2.1. Thompson’s MMT

Thompson’s framework is both developmental and cognitive, and grounded in the idea that teachers “teach what they know,” where “to know” means to have a scheme of meanings (i.e. understandings and ways of thinking) that express themselves in action (Thompson, 1994). Therefore, for Thompson (2013), “to know” entails both a teacher’s personal and pedagogical understandings of the mathematics they teach and their “images of mathematical activity and beliefs about the enterprise of learning and teaching mathematics” (Thompson, 1994, p. 3). As described in Silverman and Thompson (2008), powerful MMT involves pedagogical understandings that are grounded in significant, coherent personal understandings, which, in turn produce robust ways of thinking about mathematics and learning mathematics.

Powerful personal understandings are defined as key developmental understandings, or KDUs (Simon, 2006), and describe understandings “that carry through an instructional sequence, are foundational for learning other ideas, and play into a network of ideas that does significant work in students’ reasoning” (Thompson, 2008, p. 46). Therefore, KDUs are useful goals of mathematics instruction (Simon, 2006) and can act as “powerful springboards for learning” (Silverman & Thompson, 2008, p. 502). An individual (e.g. teacher, student) has developed a KDU when they have constructed a scheme of meanings that proves foundational for understanding a broad array of mathematical ideas and methods. An example of a KDU is the scheme of meanings comprised by quantity, variation, covariation, constant rate of change, average rate of change, and quantitative relationships (e.g. Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson & Thompson, 1996). KDUs need not be as large as this example and are not mutually exclusive. An understanding that equal partitioning creates specific units of quantity (Simon, 2006) and a covariational conception of a function (Silverman, 2005) are two “smaller scale” KDU examples. As schemes, KDUs can participate in other KDUs.

Key developmental understandings are about personal knowledge. An individual (e.g. teacher, student) can have a KDU and be unaware of it; they are aware that things make sense and they can make connections. A teacher with a KDU could have powerful understandings for the mathematics they teach without expressing that KDU in his or her instructional actions. Key developmental understandings may be transformed from an understanding having pedagogical potential to an understanding that is pedagogically powerful.

Pedagogically powerful understandings, defined as key pedagogical understandings (KPUs) (Silverman, 2005), involve a teacher’s transformation of a KDU from a way of personally understanding a particular mathematical concept to a way of understanding how this KDU could empower their students’ learning of related ideas they have to have it (Silverman, 2005). Silverman (2005) asserts that, “It is against the background of the images that a teacher holds with regard to their own understandings and of the understandings they hope students will have that they select tasks, pose questions, and make other pedagogical decisions” (p. 3).
According to Silverman and Thompson (2008), both the personally and pedagogically powerful understandings (i.e. KDUs and KPUs) develop via a process Piaget (2001) called reflective abstraction—through the construction and reorganization of schemes of meanings. For Thompson, powerful MMT is constituted by a structure of image-informed schemes (i.e. KPUs) as shown in Figure 1, which orient a teacher’s actions to be driven by:

- An image of a coherent system of ideas and ways of thinking that she intends for the students to develop (Thompson & Thompson, 1996);
- An image of how these ideas and ways of thinking can develop in a learner (Thompson & Thompson, 1996);
- Images about what her students might understand;
- An image of ways of knowing that might be problematic to students’ understanding (Thompson & Thompson, 1996);
- Ideas related to materials and activities that promote and support reflective discourse around the desired mathematical ideas which itself might support students developing those ideas (Thompson, 2002); and,
- An image of the students’ passage through an entire curriculum (Thompson, 1985).

### 2.2. Participants and data

Data were generated from a group of 16 high school mathematics teachers in the southwestern United States participating in a sequence of PD sessions on conceptual approaches to teaching major ideas in secondary mathematics. PD sessions were a major component of a larger project, conducted by Pat Thompson and his research team at Arizona State University, designed to help teachers move from a very teacher-centered orientation to a very student-centered orientation. All participants provided appropriate research ethics clearance.

PD sessions were designed and conducted to investigate teachers’ understandings and ways of thinking that supported or constrained their capacity to reflect on their practice and were conducted as much as teaching experiments as PD sessions. Such an assertion does not indicate PD sessions did not attempt to induce learning. Rather, session activities were designed explicitly to create
contexts for which teachers would reflect on their own activity and to investigate the understandings teachers made of such opportunities (i.e. investigate teachers' development of KPUs).

The complete PD data corpus consisted of audio- and video-recordings of all 14 PD sessions, teachers' written work, and field (e.g. observation) notes made by project research assistants. All documents were collected at the end of each PD session. Copies of teachers' written work were photocopied and returned to teachers during subsequent PD sessions. Data for this article comes from one particular PD session (Session #13), one of two sessions that explicitly engaged teachers in a model lesson activity. Data consisted of audio- and video-recording of the session, teachers' written work, the author’s observation notes, and a Lesson Logic (Thompson, 2006)—an artifact created by teachers as a group, described in detail below.

2.3. Teaching experiment PD design
The PD session was designed to employ teaching as a scientific tool (i.e. a teaching experiment), as a method for probing teachers' meanings and ways of thinking (Steffe & Thompson, 2000b). As described by Steffe and Thompson (2000b), “A primary goal of the teacher in a teaching experiment is to establish living models of students' mathematics” (p. 284). As such, the PD session (i.e. model lesson activity) was designed to bring forth, establish, and explore the conceptual boundaries on teachers' ways and means of operating.

The model lesson activity comprised two components: teachers first engaged in a model lesson as students of mathematics (identified as Phase 1); the second, reflective, component asked teachers to reconstruct the lesson as if they were designing it (Phase 2). Although the activity was presented as a model lesson designed to develop coherence among ideas—a key developmental idea or KDU—related to data analysis (i.e. how organizational display influences what can be analyzed, the statistical case, constructing hypotheses, distributional analysis, and conditional probability), the focus of the PD session was on probing teachers’ meanings and ways of thinking as they engaged with instruction.

The second phase of the activity provoked reflection on the part of teachers as they attempted to reconstruct the logic of the very lesson in which they had participated as students. Specifically, this phase was designed to provoke reflection on the instructional actions teachers might take to support student development of the intended meanings and ideas, and the reasons why those actions might work. Such images included the tasks and classroom discourse teachers would employ. In short, the activity was designed to probe teachers’ development of a KPU related to the teaching and learning of those data analysis ideas indicated above.

2.3.1. Lesson Logic
The lesson reconstruction (Phase 2) involved teachers creating a Lesson Logic of the lesson they had just experienced. The Lesson Logic (Thompson, 2006) was developed as a way for a teacher to outline how she will develop and interconnect a lesson’s main ideas, and focuses teacher’s attention on the ideas she intends for students to develop, the way she intends for students to develop these ideas, and why she takes the approach she does. A Lesson Logic is not a plan for a day, as in a typical lesson plan or Thinking Through a Lesson Protocol (Smith, Bill, & Hughes, 2008), but a plan for supporting students' development of robust meanings and significant mathematical ideas, and can therefore cover several encounters with students.

A Lesson Logic requires the listing of major ideas of the lesson, listed in a way that summarizes the logic. In addition, the Lesson Logic should list meanings that students must understand at the outset of the lesson if they are to be productive participants in the lesson. Finally, the Lesson Logic details each action to take and the reason for taking it, where actions include statements and questions, and the reason explains why the teacher believes each action will likely advance the students’ understanding one more step toward grasping the lesson’s main ideas.
2.4. Qualitative methods
Since the study attempted to reveal teachers’ understandings and ways of thinking and explore how these conceptions supported or constrained their capacity to productively reflect on the lesson, it was necessary to make models of teachers’ conceptions. As such, Glasersfeld’s (1995) method of conceptual analysis was adopted to create models of teachers’ thinking throughout both phases of the activity. These models served as the author’s interpretations of teachers’ mathematical realities, and were considered viable so long as they were not countermanded by teachers’ actions or inactions.

Although conceptual analysis provided a means with which to construct models of teachers’ understandings, it was necessary to ascertain the viability of these models. For this, a method consistent with Cobb and Whitenack’s (1996) method for retrospective analysis of qualitative video data and grounded theory’s (Strauss & Corbin, 1998) iterative process of continual review, constant comparison, and regeneration was employed. This analytical approach consisted of two levels: (1) provide descriptions of and rationale for the activity—this provided context for analysis and models of the PD instructor’s (Pat) intended instructional outcomes; and (2) construct models of teachers’ ways of operating as they engaged in, and attempted to reconstruct, the lesson.

3. Analysis
Since the study attempted to characterize the model lesson as teachers assimilated it, by developing models of teachers’ thinking throughout the activity, it was necessary to interpret teachers’ actions throughout each phase of the activity in an attempt to create viable and stable hypotheses of teachers’ conceptions. Such analysis requires an ongoing narrative of teacher engagement and discourse.

3.1. Model lesson activity
Prior to initiating the lesson, Pat (PD instructor) informed teachers they were first going to engage in a lesson as students of mathematics and they were then going to talk about the logic of the lesson. A chronological overview of the activity’s phases and sub-phases is provided in Table 1.

| Table 1. Chronological overview of model lesson activity from PD Session #13 |
|---------------------------------|-----------------|
| Phase                           | Duration (minutes) |
| Phase 1—Teachers as students of the lesson |
| Phase 1.1—Unorganized data is not very useful | 3.3 |
| Phase 1.2—Giving meaning to the role organization plays in data analysis and generating hypotheses | 19 |
| Phase 1.3—Reinforcing the role organization plays in analyzing data | 11.8 |
| Phase 1.4—Introducing Tinkerplots: reinforcing the role organization plays in analyzing data (group work) | 14.3 |
| Phase 1.5—Constructing images of data organization that are propitious for solving problems | 16.9 |
| Phase 1.6—Addressing issues related to data analysis and instructional design | 28.8 |
| Phase 2—Teachers as instructional designers of the lesson |
| Phase 2.1—Lesson Logic review | 4.5 |
| Phase 2.2—Teachers’ images of Pat’s initial instructional action and the reason for that action | 6.7 |
| Phase 2.3—Trying to identify ideas and meanings | 6.6 |
| Phase 2.4—The type of organization matters | 5.4 |
| Phase 2.5—Diverging from the lesson that was presented | 3.3 |
| Phase 2.6—Attempting to return to the lesson that was presented | 3.7 |
| Phase 2.7—Uncertainty in recalling the lesson that was presented | 4.7 |
| Phase 2.8—The lesson that was experienced was not the logic of the presented lesson | 3.8 |
3.1.1. Phase 1—teachers as students of the lesson

Phase 1 was designed to develop and give meaning to the role organization plays in data analysis. Throughout Phase 1, Pat attempted to focus teachers’ attention on the role organization plays in generating and testing hypotheses, and consistently moved teachers to construct images of productive data display.

3.1.2. Phase 1.1—unorganized data is not very useful

Pat initiated Phase 1 by stating that a recent study had been conducted at a high school where students were given an anxiety assessment just before taking a mathematics test. Pat then distributed (to teachers) a two-sided sheet containing unorganized data, explaining that it came from this study. The sheet contained four columns listing test scores on one side and four columns listing anxiety level scores on the reverse side (Figure 2). He asked teachers what information they could make from the data. This action enabled Pat to lead the discussion to the conclusion that little could be made of unorganized data.

Teachers said they could not make sense of the data. Though Teacher 1 suggested there might be some relationship between the variables, Pat steered the conversation toward the need for a better organization of the data by asking if they could be sure the nth score on one side of the sheet corresponded with the nth score on the other. This instructional action anticipated future discourse regarding the role organization could play in identifying relationships, and subsequently in generating hypotheses, based on the data.

3.1.3. Phase 1.2—giving meaning to the role organization plays in data analysis and generating hypotheses

Pat next distributed a second data sheet, which showed anxiety level scores together with test scores (Figure 3). Pat asked teachers whether they felt this organizational display was better than the first. This action focused teachers’ attention on identifying the benefits organized data provided and constructing images of more propitious displays. In addition, the conversation anticipated the notion of statistical case, by emphasizing the necessity for teachers to assume each row of data went with one and only one student.

As illustrated in the following excerpt, several teachers impeded their reasoning by focusing, albeit reasonably, on the development of plausible explanations to relationships within the data, based on the quantities under consideration, rather than maintaining their focus on developing hypotheses based on the numbers themselves.
Pat: On this one, what are some of the things that you can tell?
Teacher 2: Still not a lot because the data’s not organized.
Teacher 3: You still don’t know if high anxiety … this number’s a good thing or a bad thing.
Teacher 1: Does a high anxiety score mean that you're anxious or not anxious?

Teachers 1 and 3 indicated that in order to develop a hypothesis, they first needed to be able to interpret the anxiety level score. This suggests each might have intended to develop their hypothesis based on what made sense (e.g. high test scores go with low anxiety scores)—a natural intent given the constraints of the information provided. Teachers 1 and 3 did raise the very important issue of, “What do the numbers mean? How are the variables being measured defined?” Pat explained that the variables were defined so that high anxiety score meant high anxiety.

Pat next encouraged teachers to generate hypotheses pertaining to the data, but insisted they base their hypotheses on the numbers rather than on what they imagined would be a natural relationship between test score and anxiety level.

Teacher 4: It seems … a lot of them … the ones with a higher [test] scores … seem to have the score for the anxiety level is a little bit lower.
Teacher 2: I don’t think I agree with that.
Teacher 4: Not all of them, but some of them … the ones where … your scores are in the 100’s, and 90’s, the anxiety level is quite a bit lower.
Pat: Okay, we hear some disagreement … so, what’s your general hypothesis?
Teacher 4: The higher the test score, the lower the anxiety.
Teacher 5: Well, if you think you’re prepared for the test, then you don’t have anxiety over the test.
Pat: Okay, that’s an explanation for the hypothesis, but not a hypothesis.
Teacher 2: But then you have a couple of kids that don’t care.
Teacher 6: It could be that they’re clueless, and don’t know what they are doing.
Pat: Stay to the numbers ... what is keeping you from moving forward [in generating claims]?

As illustrated in the preceding excerpt, several teachers attempted to justify Teacher 4’s claim that many of the higher test scores were associated with lower anxiety level scores by focusing on plausible non-cognitive aspects of the students, rather than the numbers themselves. These teachers’ reasonable attempts to explain why Teacher 4’s hypothesis might be true, constrained their capacity to build meaning for the role data’s organization plays in making interpretations of it.

Pat continued to question teachers as to why they believed they were experiencing difficulty in generating claims. This action led to the suggestion (by Teacher 11) they order the test scores from highest to lowest, and look for a relationship between the variables. Pat used Teacher 11’s suggestion to move the conversation toward discussing what teachers envisioned such a display would provide. This conversation led to the notion that particular types of organization offer particular types of information and again highlighted the importance of assuming each row represented data from a unique individual. In addition, this conversation explicitly highlighted the idea of a statistical case as the fundamental unit of analysis in statistics.

Pat next moved the conversation back to generating hypotheses. Teachers constructed the following two hypotheses: (1) Higher anxiety scores tend to go with lower test scores, and (2) Higher test scores tend to go with lower anxiety score. Pat pushed teachers to imagine organizations of the data that would enable them to judge the hypotheses’ veracity. This action emphasized the role data organization plays in generating and testing hypotheses, thus giving meaning to the idea that how a display organizes the data matters greatly in data analysis.

3.1.4. Phase 1.3—reinforcing the role organization plays in analyzing data
Pat next introduced a third variable (the name of each student’s teacher) to the data-set (Figure 4) and asked what hypotheses participating teachers could make with the addition of the third variable. This action not only required teachers to coordinate their notion of statistical case with their images of organizational displays to construct hypotheses based on the data, but also served to reinforce the idea that particular organizations provide particular kinds of information.
As with the earlier two data sheets, teachers focused their attention on creating and explaining hypotheses based on the meanings of the quantities, rather than on the data.

Teacher 3: Now you can see how test anxiety relates to a certain teacher ... or if test [scores] relate to a different teacher.

Teacher 1: Now, does the teacher control the anxiety? Do certain teachers make more anxiety?

Pat: How can we say that ... without any value statement?

Teacher 3: Students ... pick a teacher that they know could help them. So, I know maybe I always have high anxiety, so I pick this particular teacher ... so they [teacher] didn’t cause it ...

Pat: Okay, you're trying to explain a hypothesis before we've formulated it.

Pat next sorted the data, as requested by teachers, using an Excel spreadsheet with the data embedded. The sorted data was displayed on a computer projector so teachers could see the results of their requested organization and attempt to generate hypotheses. This visual display again highlighted the importance of the idea of a statistical case and a need for a better way to organize the data, thus reinforcing the role organization plays in analyzing data.

Pat next introduced TinkerPlots (Phase 1.4), an exploratory data analysis computer program, by constructing a variety of organizational displays of the data provided in the third data-set. Although Phases 1.4–1.6 were important to the overall coherence of the lesson presented to teachers (in their role as students of mathematics), teachers were unable to reach the point in their lesson reconstruction (Phase 2) where TinkerPlots was introduced. Therefore, the author will not describe Phases 1.4–1.6 in any detail.

After concluding the TinkerPlots overview, Pat partitioned teachers into four groups and distributed a sheet with six Test-Anxiety questions pertaining to the data. The questions were designed to reinforce teachers' images of the role organization plays in data analysis and provide a natural way in which the idea of conditional probability could arise in the context of instruction (e.g. Suppose you talk to a highly anxious student. How likely are they to have a low test score?). Each group selected a distinct question and worked to develop a sketch of the organizational display they believed would allow them to address their question. During the last part of Phase 1 (i.e. Phase 1.6), Pat generated each group's proposed organizational display on the projector screen using TinkerPlots while the group described how and why they believed their display: (1) allowed them, in principle, to answer their question, and (2) supported their specific answer to their question.

3.1.5. Phase 1—summary
Throughout Phase 1, Pat placed an emphasis on the meanings and ideas teachers were intended to develop. In addition, Pat modeled instruction designed to generate discussions around the intended meanings and ideas, although he did not draw attention to this fact. Teachers gave no indication that they attended to the fact that the lesson might have a logic. Rather, they focused on their participation in the lesson. Put another way, teachers were focused on their activities as learners, not as managers of others' learning—a justifiable focus, given their assigned role as students of the lesson. Furthermore, rather than focusing their attention on their meanings and reasoning (e.g. the role that organization plays or can play in data analysis, the idea of a statistical case, the role and nature of hypotheses in statistical reasoning), several teachers focused on the products of their reasoning (e.g. explaining hypotheses).

3.1.6. Phase 2—teachers as instructional designers of the lesson
The second phase of the model lesson activity began after a short break at the conclusion of Phase 1.
3.1.7. Phase 2.1—Lesson Logic review
Pat initiated the second phase by giving a brief review of a Lesson Logic. Lesson Logics were not new to teachers, since they had developed Lesson Logics on several other occasions, including the previous PD session (Session #12).

3.1.8. Phase 2.2—teachers’ images of Pat’s initial instructional action and the reason for that action
One of the teachers, Teacher 4, was chosen to act as recorder and instructed to record the groups’ responses as they attempted to reconstruct the lesson’s logic. Teacher 4 recorded the groups’ responses on a blank Lesson Logic form projected onto the classroom projector screen. After Teacher 4 asked where the group wanted to begin, there were seven seconds of silence before Pat interjected in an attempt to get teachers moving. Pat emphasized the fact that teachers were not being asked to create a Lesson Logic from scratch, but rather to describe the lesson they had just experienced.

Teacher 7 commented that Pat handed out raw data and did so in order to establish a need to analyze the data. Teacher 4 typed Teacher 7’s suggestions onto the Lesson Logic form (Figure 5).

When it became apparent teachers intended to move to the next action (step 2) without discussing what Teacher 7 had offered, Pat interjected and asked whether everyone agreed with what Teacher 7 had suggested.

Teacher 8: Should it be more like for categorizing the data or organizing?
Teacher 7: That’s probably a better word actually … organizing the data. The first thing was he [Pat] had to establish why are we doing any of this? Here’s raw data, why do any of it? We had to have a need established for organizing the information.

The preceding excerpt centered on questioning whether the reason for the instructional action should be to establish a need to analyze or to organize. Although this brings the notion of organization into the discussion, there was no mention of the thinking or ideas that the action intended to promote. Rather, that some need to organize would be established. In addition, teachers did not indicate that instruction first needed to give a sense as to the unhelpfulness of unorganized data. Teacher 7’s comment, “We had to have a need established for organizing the information,” gives the sense that organizing the data was a task that needed to be done, rather than an idea (i.e. organization) that needed to be developed.

Next, Teacher 1 asserted that he believed Teacher 7’s action did not support the given reason, because Teacher 1 felt their reason needed to answer the question as to why Pat handed out a data-set.

Teacher 7: Well, they [students] need to see that they can’t get anything from it. Is that an action or not?
Teacher 4: When he handed it out were we looking at analyzing it or … was it to organize it?
Teacher 9: He [Pat] said … ‘Can you determine anything from the data?’
Teacher 7: Maybe a better action then would be, ‘Can we find out anything from unorganized data?’
Teacher 3: How is your reason a question?
Teacher 4: What do you want to say?
Teacher 8: Can we organize the data so that we could analyze it?
Teacher 4: Are we talking about action or reason?

In the preceding exchange, teachers appeared disinclined to focus on the thinking that their actions might engender in students. This is not to say that teachers purposefully neglected such a focus; rather, that such reflection was not part of their way of operating at that moment. Teachers 4 and 7’s comments continued to focus on the instructional actions as being undertaken with the intent that students do something, rather than to think in some manner. For teachers, organization appeared to be a task to perform rather than an idea that had been reasoned with as a means to assist in solving a problem.

In addition, teachers focused their attention on attempting to recall what they had perceived, rather than thinking about the lesson’s logic. Rather than asking, “What would have been a natural thing to do given what we’ve done and where we’re trying to head,” teachers focused exclusively on actions they had performed and the sequence they could recall performing these actions. Furthermore, there appeared to be general confusion among some teachers as to what they were actually discussing, an action or a reason.

Teachers agreed to change the first action to, “Hand out raw data and ask ‘Can we analyze this data?’” and their initial reason to, “Establish need to organize data.” At this point, teachers had spent almost seven minutes attempting to reconstruct the initial action and reason for the lesson they had just experienced.

3.1.9. Phase 2.3—trying to identify ideas and meanings
Teacher 4 asked what the group wanted to say next and Teacher 9 stated that Pat had given them another sheet. The ensuing discussion centered on teachers’ attempts to recall what they perceived as the lesson’s sequence of actions, rather than on focusing on the lesson’s logic. Furthermore, the focus continued to be on teachers’ activities as learners, not as managers of others’ learning of ideas.

Teacher 1: Wasn’t there a couple of more things that we did with this one [data sheet]?
Teacher 2: It’s the same thing.
Teacher 4: Couldn’t you put those together, because they were sort of asking the same question too?

In the preceding exchange, Teacher 1 stated there were a couple of more things that were “done” with the first data sheet, but gives no indication that any ideas were being promoted or that there might have been a logic to how Pat had distributed the data sheets. In addition, Teacher 4 stated the two data sheets were “sort of asking the same question,” or asking them to perform a similar action, which makes unclear what she felt the difference was. If teachers conceived of organization as an action to be performed, then the two data sheets would have tended to reinforce the importance of performing the act of organizing, without giving any meaning to how or why organization was important.

As the lesson reconstruction continued, teachers struggled to make explicit both the ideas instruction intended to promote and how the lesson developed them. To overcome this barrier teachers appeared to have attempted to recall the order that actions were performed. This suggests that, throughout their participation in Phase 1, teachers were focused on the products of their reasoning as students and were unaware of their reasoning. Their unawareness of the reasoning Pat’s actions had engendered in them constrained their capacity to reflect on their reasoning—their reasoning was hidden in its products. Furthermore, teachers focused on their in-the-moment participation in
the lesson to the exclusion of thinking that the lesson might have a logic. Therefore, teachers’ egocentricty and inattention to meaning during their engagement as students—teachers’ natural focus on performing actions as students—constrained their capacity, later, to imagine the lesson in a way they might lead it.

Teachers next refocused their attention on the reason for their initial action.

Teacher 10: Can we add then to the reason, ‘cause I think then if we’re going to include all the data that he gave us, it wasn’t just to organize data but also to establish a ...

Teacher 9: Relationship.
Teacher 10: Case.
Teacher 2: Establish a hypothesis.
Teacher 4: Case, hypothesis?
Teacher 8: A hypothesis, right? Because a case is just one ...
Teacher 10: Yeah, but see that was the whole thing between the first [hand-out] and the second and the third ...
Teacher 4: So case?
Teacher 10: Maybe both?
Teacher 8: I think a hypothesis.
Teacher 7: A case then a hypothesis

In the preceding exchange, Teacher 10 indicated some sense of a logic between hypothesis and case with her statement, “The whole thing between the first and the second and the third.” At this point, perhaps some teachers, in reflecting on the lesson, were beginning to construct an image of the lesson that involved the development of ideas. In fact, Teacher 7 stated that a case was established first, then a hypothesis, but did not indicate how one promoted the other, suggesting he (and possibly others) were having difficulty coordinating their meanings for statistical case, relationship, and hypothesis. It is unclear whether such difficulties were due to Teacher 7 (and perhaps other teachers) not having developed a key developmental understanding (KDU) among ideas related to organizational display, the statistical case, and constructing hypotheses. Alternatively, as indicated earlier, such difficulties could have arisen due to teachers' egocentricity and inattention to meaning during their engagement as students.

The way the terms “case” and “hypothesis” were mentioned and ultimately included in the reason (Figure 6), which seems to imply that “hypothesis” and “case” are nearly equivalent, suggests these terms were merely facts being established, not coherent meanings and ideas that were being discussed among participants in the lesson.

As teachers were discussing whether to include case or hypothesis (or both) as a reason, Pat asked whether they believed any ideas or meanings had come up in their discussion that needed to be addressed. Teacher 1 indicated, “One of the ideas was the fact that … data has attributes,” that he believed it was a major idea, but was uncertain as to whether it needed to be included in the lesson’s logic. Although Teacher 1 asserted “data has attributes” was an idea, he did not to give any meaning to what this idea entailed, and seemed unclear as to what meaning he had regarding attributes.

![Figure 6. Adding hypothesis and case to the reason in step 1. This figure illustrates the additions and modifications teachers provided to their initial action and corresponding reason for that action.](image-url)
Teacher 4: When we talked about when we move the numbers, if we move one, all those values have to move, you can’t just change that one column.
Teacher 1: Yeah … the attributes all linked to each individual student.
Teacher 4: So, is that a meaning? That they [students] need to understand before they have this lesson?
Teacher 7: That will unfold in the lesson.
Several: Yeah.
Teacher 4: So, do you want to just go back to the … steps?
Teacher 7: Have we decided … if what Teacher 1 was talking about is a major idea?
Teacher 10: The attributes.
Teacher 7: The attributes, is that something important?
Teacher 8: I think so.

Teachers 1 and 4’s initial focus appeared to be on how the connection of the attributes (to each specific case) limits what can be done, rather than a focus on the cases (the students) as the unit of analysis, and that these cases have attributes.

Teacher 1: When you start doing the sorting, everything has got to stay … connected.
Teacher 7: Then, if I think that the answer to that is ‘yes,’ then we put it up there [under major ideas].
Teacher 4: Where? Which one?
Teacher 7: Under the … major ideas.
Teacher 1: Major ideas, yeah, see I’m thinking it’s a major idea.
Teacher 5: Sorting the data.
Teacher 1: Data has attributes … relational attributes.
Teacher 9: The attributes stay linked to the student.
Teacher 3: Data has specific cases.
Teacher 8: i.e. teacher, test score, …
Teacher 4: So, what do you want to say?
Teacher 7: Data has cases and each case can have several attributes.

Although teachers listed Teacher 7’s statement as an idea, they concluded the discussion regarding case and attribute without delving into the development of these ideas, or discussing how this idea is connected to or helped to develop other ideas. Furthermore, although teachers discussed what they wanted students to understand regarding the idea that when you sort everything must stay connected, they did not include this as part of their description of the idea (Figure 7).

Teacher 4 next focused the discussion on meanings students might need prior to the lesson. Teacher 5 indicated students should know what is meant by a case and by a hypothesis to which Teacher 7 responded, “I think just hypothesis, because case is something that will develop in the lesson.” Although teachers listed hypothesis as an understanding that was required for students to participate productively in the lesson (Figure 8), they did not discuss the actual meaning for hypothesis that students must have.

Figure 7. Listing the idea that data has cases with attributes. This figure illustrates the initial lesson idea that teachers provided to the Lesson Logic.

The following lesson logic provides a structure in which the surrounding conversation unfolds these ideas:

1. Data has cases and each case has several attributes.
2. (Major ideas of the lesson, listed in a way that summarizes the logic)
For example, hypothesis can be defined as:

1. A statement that is testable and falsifiable (CK-12 Foundation, 2016).
2. A proposition or supposition developed to provide a basis for further investigation or research (Smarter Balanced Assessment Consortia, 2012).
3. The ‘if’ part of a conditional statement (Demana, Waits, Foley, & Kennedy, 2004).
4. An assumed statement used as a premise in a proof (New York State Department of Education, 2006).

The first two definitions (above) provide a much more productive meaning for hypothesis in the lesson described in this study than do the third and fourth definitions.

Although teachers had earlier agreed to place “case” in the ideas sections of the Lesson Logic, no one questioned Teacher 5’s suggestion that case was a meaning students must have prior to the lesson, or addressed in what way a meaning for hypotheses was necessary for students to participate in the lesson or its role in developing any ideas. Not only did teachers fail to make explicit the meanings they believed were needed to participate productively in the lesson, they did not give meaning to the ideas they viewed instruction as having addressed. Teachers next agreed to move back to the steps of the Lesson Logic.

3.1.10. Phase 2.4—the type of organization matters

Teacher 4 moved the cursor back to the action for step 2 in the Lesson Logic and 24 s quietly passed before Teacher 7 asked when they wanted to mention that the kind of organization mattered. Teacher 3 stated that this was where Pat had gone to next (Phase 1.3) and pointed to the white board where Pat had written:

“Particular organizations can provide particular kinds of information,” such as:

• Ordering test scores from highest to lowest

Teacher 4 asked what the action for this reason should be and Teacher 8 suggested they write, “Discuss organization of data.” Teacher 3 suggested their action be stated, “Discuss why organizing data in certain ways matters.” Teachers agreed with this suggestion to use what Pat had written in Phase 1 (including the example) as their reason (Figure 9), without any discussion of how they...
understood this as fitting into the lesson’s logic. Teachers’ focus continued to remain on the products of their reasoning—on establishing the fact that particular organizations can provide particular kinds of information.

3.1.11. Phase 2.5—diverging from the lesson that was presented
Teacher 8 stated they had next developed a hypothesis. Teacher 7 indicated that at some point they had talked about the case, and asked whether that was when they had discussed hypothesis. Teacher 4 stated they had discussed case just before hypothesis, when Pat had asked if they could just change one variable as teachers had attempted to organize the data. In the conversation that ensued, teachers agreed that case was a big idea and that they did need to address it in the steps.

Teacher 5 stated that she liked Teacher 4’s earlier assertion (Phase 2.3), “You can’t just change that one column.” Teacher 4 asked whether they should include this in Step 2 of their Lesson Logic and the group agreed to move back to Step 2. Teacher 1 stated that if they were going to organize data, then they needed to make sure everything stayed connected, because each case had several attributes and they needed to make certain all of the information for one case stayed with that case.

Teacher 7: So, is that a separate step or is it like 2a, or does it matter?
Teacher 4: You could almost … [use] a spreadsheet that has the data and just move one piece of information and have them [students] see what’s going on … some of them [students] may not be able to pick that up if they don’t see that.

Teacher 7: Actually see that happen, yeah.
Teacher 4: So, how should I say that?
Teacher 9: Well, you’re … making certain the attributes stay with the case.
Teacher 7: Right, but we want them to see that.

The sequence of actions described in the preceding excerpt is designed to help students see they cannot sort a single variable (column of data in the spreadsheet) without moving all of the other values that are connected to each specific case. This suggests teachers were thinking about a way to make their meaning of case visible to their students. Specifically, that everything associated with each case must stay connected. This was the first, and ultimately only, occurrence where teachers appeared to be concerned with promoting an explicitly discussed and detailed understanding and how that understanding might be promoted to their students.

This occurrence is also an indication of teachers’ decentering (Steffe & Thompson, 2000b)—attempting to see mathematical understandings from the perspective of another (their students). It is important to note that teachers spoke of this activity as if they did not recall Pat having used Excel to perform something similar during Phase 1. In addition, although Pat used Excel to sort the data, he did not place an emphasis on the sorting, but on what teachers imagined sorting would do and as a means to emphasize the importance of organization.

3.1.12. Phase 2.6—attempting to return to the lesson that was presented
Teacher 3 pointed out that if they changed Step 2’s reason they would need to change the reason in Step 1, because they then would not have discussed hypothesis and case until Step 3, suggesting Teacher 3 was thinking about the lesson’s logic. Teacher 4 moved “Establish a hypothesis/case” from the reason in Step 1 to the reason for Step 3, and asked what the corresponding action should be for Step 3.

Teacher 8 suggested the action be to get kids to use software. Teacher 10 stated that she felt they were getting ahead of the game and that they had not touched software yet (during Phase 1). Teacher 8 stated that she believed what they were discussing was new, not what they had done, but something they could do at that point with their own students.
3.1.13. Phase 2.7—uncertainty in recalling the lesson that was presented

Teacher 3 suggested the action in Step 3 be that Pat asked whether they could create a hypothesis based on the data they had and in the form the data was in. Teacher 4 added this to Step 3 (Figure 10).

The following exchange between Teachers 1 and 3 exhibits Teacher 1’s difficulty with coordinating his meanings regarding organization, statistical case, and hypotheses as he attempted to recall the order of the events. This suggests that Teacher 1, and perhaps others, had developed isolated meanings, rather than a scheme of meanings (i.e. key developmental understanding, KDU).

Teacher 1: I’m just wondering ... we were given the data set like this and we were asked some questions about, you know, what kinds of things we could see and all that. And I think maybe some of us were already making connections, but we already know you got to keep them connected.

Teacher 3: Yeah, but we knew that.

Teacher 1: But when we got it organized in this way, weren’t we also getting to that point right away with ... each case being connected to the attributes...

Teacher 3: But it wasn’t brought out yet. It wasn’t anything you had addressed, it might have been something we thought about but we didn’t address it or bring it out as part of our logic yet.

Teacher 1: I’m just trying to remember when ... we did this what were we putting this form first for?

3.1.14. Phase 2.8—the lesson that was experienced was not the logic of the presented lesson

Prior to the group moving on to Step 4 of the Lesson Logic, Pat interrupted and stated that at the rate they were moving, teachers were going to be there a long time. At this point in the activity teachers had spent almost 35 min attempting to reconstruct the lesson, and had only accounted for the first three sub-phases (at most) of the lesson they had just experienced. Pat stated that what teachers felt they had gotten from the lesson was not the logic of the lesson. Pat asserted that constructing a logic of a lesson required the structuring of ideas, not the structuring of activities, and required teachers to structure how they anticipated these ideas might unfold.

3.1.15. Phase 2—summary

Although teachers understood every idea discussed in the data analysis lesson without difficulty, they could not reconstruct the lesson’s logic. As teachers began to disagree about steps in the lesson and the logic of successive steps. Rather than reflecting on their reasoning, teachers attempted to recall the products of their reasoning. As the lesson reconstruction progressed, teachers began to disagree about the sequencing of the outcomes (e.g. “Did we first establish a case or a hypothesis?”). Further on in their attempt to reconstruct the lesson, teachers began to introduce actions that had not occurred in the actual lesson, actions pertaining to how they would teach the lesson (e.g. focusing on using software to make the idea of statistical case visible to students). As time ran out for PD Session #13, teachers had segued into constructing a lesson they themselves would have taught.
Even though Pat had emphasized that the construction of a lesson’s logic (see Appendix 1 for Lesson Logic constructed by teachers) required teachers think about and construct images of the understandings they want their students to develop, teachers did not discuss what it was that students needed to understand in order to participate productively in the lesson or the understandings they intended students develop as the lesson progressed. Throughout Phase 2, teachers appeared disinclined to reflect on the meanings they held, the meanings they were building, or the ideas that were being developed—the logic of the lesson. Rather, teachers focused on the order of the activities they could recall, those acts that Pat’s actions required of them as students of the lesson.

In order to reconstruct the lesson’s logic, teachers would have needed to do more than simply recall the lesson, or more specifically, recall the order of the non-cognitive actions they engaged in as students of the lesson. They would instead need to summarize the meanings and ideas they understood as being promoted, and the activities and conversations that were designed to promote the development of those meanings and ideas. In order to construct the Lesson Logic, by reflecting on the lesson they had experienced as students, it would have been necessary for teachers to take their thinking and meanings as objects of thought.

4. Conclusions
This study investigated the understandings teachers made of an opportunity to experience a mathematics lesson grounded in meanings and the development of significant mathematical ideas. The two research questions will be addressed based on the findings indicated in the analysis. What was the model lesson that teachers experienced? As characterized in the analysis, rather than a lesson focused on developing ideas related to data analysis (i.e. how organizational display influences what can be analyzed, the statistical case, constructing hypotheses, distributional analysis, and conditional probability), the model lesson teachers experienced focused on performing a sequence of actions and obtaining a sequence of outcomes: hand out raw data (establish need to organize), discuss why organization matters (particular organizations provide particular kinds of information), and create a hypothesis (establish case and see relationship). What meanings did teachers utilize as they experienced alternative approaches to teaching and learning mathematics? As described in the analysis, rather than focusing their attention on their meanings and reasoning, several teachers focused on the products of their reasoning. As such, attempts to characterize teachers’ meanings for organization, statistical case, and hypothesis were constrained. Teachers’ pedagogical meanings can be characterized as being comprised of performing a sequence of actions with the intent to obtain a specific sequence of outcomes, rather than a coherent system of ideas and ways of thinking he or she intended for students to develop and an image for how these ideas and ways of thinking might be developed.

As indicated in the analysis, when trying to reconstruct the lesson in which they participated as students, teachers focused on their memory of their participation largely to the exclusion of thinking that the lesson might have a logic. As a result, teachers’ capacities to focus on their activities from the point of view of managers of others’ learning these ideas was highly constrained by their lack of access to the meanings, thinking, and reasoning they employed as students of the lesson. Although the author was unable to observe teachers attempting to replicate the lesson, as the PD instructor (Pat Thompson) had designed it, in their own classrooms, data provides little reason to suspect they could. Rather, data suggests the lessons they would teach would be focused on performing a sequence of actions that emphasized the products of students’ reasoning.

Although this study focused on the understandings and ways of thinking practicing mathematics teachers utilized as they experienced alternative approaches to teaching and learning mathematics, the implications derived from analysis do not limit themselves to in-service teacher education. The author sees little reason to believe pre-service mathematics teachers would be likely to experience greater success in extracting the underlying logic of a model lesson in which they participate than did the in-service teachers in this study.
The lesson reconstruction was designed to move teachers to take their meanings and reasoning as objects of thought. More specifically, the activity was designed to move teachers to think about how particular understandings and ways of thinking could empower students’ learning of related ideas, and how their actions (as teachers) might support students’ development of these ideas and ways of thinking (Silverman & Thompson, 2008); that is, to develop a KPU. To engage in such reflective abstraction not only requires meanings sufficiently robust to sustain productive reflection, but also what Piaget called decentering—the attempt to imagine one’s experience from another perspective (Steffe & Thompson, 2000a).

When looked at from this perspective, it is understandable why teachers experienced the difficulties they did. Their difficulties could have come from two sources. At the time of their participating in the lesson as students (Phase 1), teachers did not attend to the meanings that were being developed. That is, teachers’ participation itself might have been unreflective. If this were the case, then the only records of experience upon which teachers could reflect were their memories of events according to the events’ superficial features. A second reason for teachers’ difficulties might have been that the meanings they created during instruction (Phase 1) were insufficient to support reflection. Put another way, if the meanings teachers built during instruction were procedural, then those would be the meanings that supported their later reflection. Therefore, if teachers had not developed a key developmental understanding (KDU) during instruction (Phase 1), this would have constrained their capacity to transform this KDU into a KPU during Phase 2. Future research must explore each of these hypotheses if model lessons are to serve a productive role in PD and teacher education. Teachers’ constrained attempts to reconstruct the lesson suggest that an instructor-led “debriefing” of the lesson’s logic might have led teachers to internalizing a more coherent structure of the lesson in terms of the ideas that were developed and how they fit. This, again, is an empirical question that will need to be explored through future research.

Results presented here inform PD and mathematics teacher education’s efforts to transform teachers’ personal and pedagogical mathematical meanings. As described in this report, efforts to provide teachers with an opportunity to develop powerful pedagogical understandings (i.e. a KPU) were fundamentally constrained by teachers’ personal mathematical meanings (i.e. KDUs). According to Silverman and Thompson (2005), “PD efforts must be grounded in helping [teachers] develop particularly powerful pedagogical conceptualizations of the mathematics that they … teach” (p. 1). Research has shown that teachers who have been successful at developing robust and coherent personal mathematical understandings for the mathematics they teach, or will teach (i.e. KDUs), from their participation in sustained PD opportunities, are those “who coordinated [new meanings] at a micro level [and] built new meanings into a coherent whole” (Thompson, Carlson, & Silverman, 2007, p. 23). Furthermore, these teachers also “saw implications of their own reasoning for student learning [and] were … successful at expressing that reasoning in natural language” (Thompson et al., 2007, p. 23). Such results demonstrate a need for further investigation into the development of key developmental understandings, KPUs, and image-informed schemes of meanings. Operationalization of Thompson’s MMT framework, as presented here, should support such investigation.

Funding
This work was supported by National Science Foundation [grant number EHR-0353470].

Author details
Scott A. Courtney
E-mail: scourtn5@kent.edu
1 School of Teaching, Learning, and Curriculum Studies, Kent State University, 401 White Hall, P.O. Box 5190, Kent, OH 44242-0001, USA.

Citation information
Cite this article as: What teachers understand of model lessons, Scott A. Courtney, Cogent Education (2017), 4: 1296528.

Notes
1. In Thompson and Thompson (1996) and Silverman and Thompson (2008), the term Mathematical Knowledge for Teaching (MKT) was used to describe mathematical knowledge that is specifically useful in teaching mathematics. Thompson (2013) now uses the term MMT to emphasize his framework’s focus on teachers’ schemes of meanings and to help distinguish his framework from that of Deborah Loewenberg Ball and her colleagues (e.g. Loewenberg Ball, Thames, & Phelps, 2008).
2. The use of the term key developmental understanding employed here is Silverman and Thompson’s (2008) interpretation of Simon’s construct. Simon (personal communication, April, 2010) asserts his original conceptualization of the term involves both a teacher’s personal
and pedagogical understandings of an idea (from an observer's perspective), which is more aligned with what Silverman and Thompson identify as a KPU.

3. In the excerpt sections, "Pat" stands for utterances made by the PD observer; "Teacher 1"..."Teacher 2"..."Teacher 11"... stand for utterances made by participating teachers. The symbol "..." signifies either the speaker paused during the utterance, or the speaker did not complete the utterance due to an interruption or because they simply stopped speaking.

References

Borasi, R., & Fonzi, J. (2002). Foundations: Professional development that supports school mathematics reform, 3 (NSF 02-084). Arlington, VA: National Science Foundation.

Borasi, R., Fonzi, J., Smith, C. F., & Rose, B. J. (1999). Beginning because they simply stopped speaking. A method for conducting Mathematics Teacher Education, 2, 49–78. http://dx.doi.org/10.1023/A:1009988606120

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. Journal for Research in Mathematics Education, 33, 352–378. http://dx.doi.org/10.2307/749958

Demana, F., Fo克斯, B. G., & Kennedy, D. (2004). Precalculus: Graphical, numerical, algebraic (6th ed.). New York, NY: Pearson Addison Wesley.

Desimone, L. M. (2009, April). Improving impact studies of teachers’ professional development: Toward better conceptualizations and Measures. Educational Researcher, 38, 181–199. http://dx.doi.org/10.3102/0013189X03811140

Glaserfeld, E. v. (1995). Radical Constructivism. London: Falmer Press. http://dx.doi.org/10.4324/9780203454422

Guskey, T. R., & Voon, K. S. (2003). What works in professional development? Phi Delta Kappan, 90, 495–500. http://dx.doi.org/10.1177/0031721709000707

Heinz, K., Kinzel, M., Simon, M., & Tzur, R. (2000). Moving students through steps of mathematical knowing. The Journal of Mathematical Behavior, 19, 83–107. http://dx.doi.org/10.1016/S0732-3123(00)00037-7

Hudson, P. B. (2007). Examining mentors’ practices for enhancing preservice teachers’ pedagogical development in mathematics and science. Mentoring & Tutoring: Partnership in Learning, 15, 201–217. http://dx.doi.org/10.1080/136112611860108639

Kersting, N. (2008). Using video clips as item prompts to measure teachers’ knowledge of teaching mathematics. Educational and Psychological Measurement, 68, 845–861.

Loewenberg Ball, D., Thomas, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? Journal of Teacher Education, 59, 389–407. http://dx.doi.org/10.1177/0022487108324654

Louches-Horsley, S., Stiles, K. E., Mundry, S., Love, N., & Hewson, R. W. (2010). Designing professional development for teachers of science and mathematics (3rd ed.). Thousand Oaks, CA: Corwin.

New York State Department of Education. (2006). Commencement-level glossary of mathematical terms. Albany, NY: NYSED. Retrieved from http://www.p12.nysed.gov/ciai/msth/glossary/glossaryHS.doc

Peniel, W. R., Fishman, B., Yamaguchi, R., & Gallagher, L. P. (2007). What makes professional development effective? Strategies that foster curriculum implementation. American Educational Research Journal, 44, 921–958. http://dx.doi.org/10.3102/0021931407302821

Piaget, J. (2001). Studies in reflecting abstraction. (R. Campbell, Trans.). London: Taylor and Francis.

Saxe, G., Geogan, M., & Nasi, N. S. (2001). Enhancing students’ understanding of mathematics: A study of three contrasting approaches to professional support. Journal of Mathematics Teacher Education, 4, 55–79.

Shein, M. G., Jacobs, V. R., & Philp, R. A. (Eds.). (2011). Mathematics teacher noticing: Seeing through teachers’ eyes. New York, NY: Routledge.

Silverman, J. (2005). Examining the relationships between teachers’ understandings of mathematics content and their developing pedagogy (Unpublished doctoral dissertation). Vanderbilt University, Nashville, TN.

Silverman, J., & Thompson, P. W. (2005). Investigating the relationship between mathematical understanding and teaching mathematics. In S. Wilson (Ed.), Proceedings of the 27th Annual Meeting of the International Group for the Psychology of Mathematics Education, Roanoke, VA: Vicksburg, VA: Virginia Tech. Retrieved from http://bit.ly/1SI1EhYe

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. Journal of Mathematics Teacher Education, 11, 499–511. http://dx.doi.org/10.1007/s10857-008-9089-5

Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. Mathematical Thinking and Learning, 8, 356–371.

Smarter Balanced Assessment Consortia. (2012, April). Smarter Balanced Assessment Consortium: Mathematics item specifications high school. Olympia, WA: Author.

Smith, M. S., Bill, V., & Hughes, E. K. (2008). Thinking through a lesson: Successfully implementing high-level tasks. Mathematics Teaching in the Middle School, 14, 132–138.

Steffe, L. P., & Thompson, P. W. (2000a). Interaction or intersubjectivity? A reply to Lerman. Journal for Research in Mathematics Education, 31, 191–209. http://dx.doi.org/10.2307/37751

Steffe, L. P., & Thompson, P. W. (2000b). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), Research design in mathematics and science education (pp. 267–307). Mahwah, NJ: Lawrence Erlbaum Associates.

Strauss, A. L., & Corbin, J. M. (1998). Basics of qualitative research: Techniques procedures for developing grounded theory. Thousand Oaks, CA: Sage Publications.

Thompson, A. G., Philp, R. A., Thompson, P. W., & Boyd, B. A. (1994). Calculational and conceptual orientations in teaching mathematics. In A. Cofærd (Ed.), 1994 Yearbook of the NCTM (pp. 1979–1992). Reston, VA: NCTM.

Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. Journal for Research in Mathematics Education, 27, 2–24. http://dx.doi.org/10.2307/749194
Thompson, P. W. (1985). Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula. In E. Silver (Ed.), Teaching and learning mathematical problem solving: Multiple research perspectives (pp. 189–243). Hillsdale, NJ: Erlbaum.

Thompson, P. W. (1994). Bridges between mathematics and science education. Paper presented at the Research blueprint for science education conference, New Orleans, LA.

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeier, R. Lehrer, B. v. Oers, & L. Verschaffel (Eds.), Symbolizing and modeling in mathematics education (pp. 191–212). Dordrecht: Kluwer.

Thompson, P. W. (2006). Lesson Logic form. Retrieved from http://tpc2.net/Resources/index.html

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. Paper presented at the 32nd Annual Meeting of the International Group for the Psychology of Mathematics Education. Morelia: PME.

Thompson, P. W. (2013). In the absence of meaning .... In K. Leatham (Ed.), Vital directions for research in mathematics education (pp. 57–93). New York, NY: Springer.

Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers’ development of coherent mathematical meanings. Journal of Mathematics Teacher Education, 10, 415–432. Retrieved from http://bit.ly/11hHe2

Tzur, R., Simon, M. A., Heinz, K., & Kinzel, M. (2001). An account of a teacher’s perspective on learning and teaching mathematics: Implications for teacher development. Journal of Mathematics Teacher Education, 4, 227–254. http://dx.doi.org/10.1023/A:1011493204582
Appendix 1

(Lesson Name):
Logic of the Lesson

(Author)

The following is a lesson logic for teaching (the major idea or ideas) in (course, topic, or grade level).

A lesson logic is the outline of how you will develop the lesson's main ideas. It does not pay attention to time, meaning that the "lesson" may transcend several class periods. It does not give the level of detail that a lesson plan gives, meaning it might not say how you will organize the classroom, how you will transition from one activity to another, etc. Instead, it focuses on the ideas you will develop, the way you develop them, and why you take the approach you take.

The following lesson logic provides a structure in which the surrounding conversation unfolds these ideas:
1. Data has cases and each case has several attributes.
2. (Major ideas of the lesson, listed in a way that summarizes the logic)

Meanings students must have before the lesson:
1) What is meant by a hypothesis.
2) (Things students must understand at the outset if they are to participate productively in the lesson. This is not the same as things they must be able to do.)

Steps in the Lesson Logic

| Step | Action | Reason |
|------|--------|--------|
| 1.   | Hand out raw data and ask “Can we analyze this data?” | Establish need to organize data |
| 2.   | Discuss why organization of data matters. | Particular organization can provide particular kinds of information. (i.e. ordering test scores from high to low) |
| 3.   | Create a hypothesis based on the data we have. | Establish a case and see a relationship between the data |
| 4.   | | |

1 Your description of the major ideas that this lesson addresses should evolve from your attempts to create the lesson logic for teaching them. In other words, someone reading your lesson logic will read your description of the big ideas before reading your steps for teaching them. But, you will have created your description of big ideas after having created the steps for teaching them.

2 Your list of "meanings students must have" should evolve while you write the steps in your lesson logic. Write them as they occur to you, and be alert to when you are plan a step that presumes students have a meaning that is essential for them to participate.
