M-Branes and Their Interactions in Static Matrix Model

Amir H. Fatollahi\textsubscript{a,b}*, Kamran Kaviani\textsubscript{a,c}, Shahrokh Parvizi\textsubscript{a}†

\textit{a)Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O.Box 19395-5531, Tehran, Iran.}

\textit{b)Department of Physics, Sharif University of Technology, P.O.Box 11365-9161, Tehran, Iran.}

\textit{c)Department of Physics, Az-zahra University, P.O.Box 19834, Tehran, Iran.}

**ABSTRACT**

Different BPS M-brane configurations including single and two parallel $M_p$-branes ($p=\text{even}$) and $M_5$-branes are introduced as the classical solutions of the recently proposed Static Matrix Model. Also the long range interactions of two relatively rotated $M_p$-branes (one and two angles) and $M_p$-brane–anti-$M_p$-brane are calculated. The results are in agreement with 11 dimensional supergravity results.

*E-mail: fath@theory.ipm.ac.ir
†E-mail: kaviani@theory.ipm.ac.ir
‡E-mail: parvizi@netware2.ipm.ac.ir
1 Introduction

M-theory [1] played a central role in string theory unifications in the last two years. This approach to string theory unification became more concrete [2, 3, 4, 5, 6] since the conjecture that M-theory is a M(atrix) model [7], a dimensional reduction of the 9+1 dimensional $\mathcal{N} = 1$, $U(N)$ SYM to 0+1 dimension [8].

In the recent paper [9] a Static Matrix Model was proposed for static configurations of M-theory. The main idea was based on the conjecture of [7] about equivalency of regularized supermembrane theory in the light-cone gauge and M-theory in infinite momentum frame, which suggests generalization of this equivalency to other gauges [9, 10]. The proposed model is defined in ten spatial dimensions and does not have any stringy parameter ($g_s$, $l_s$, ...), but only the supermembrane tension $T_M$. Furthermore the model can not be considered as a dimensionally reduced SYM theory.

In [9] the long range interaction of M2-brane and anti-M2-brane was calculated and found to be in agreement with the 11 dimensional supergravity results (i.e. $V(r) \sim \frac{1}{r^6}$). On the other hand, in [11] the result for the same problem appeared in compactified limit of the 11 dimensional supergravity (i.e. $V(r) \sim \frac{1}{r^5}$), in the context of the light cone M(atrix) model. This may be considered as the advantage of Static Matrix Model which is defined in 10 spatial dimensions.

Here we would like to put the Static Matrix Model to further tests, including interaction of rotated $M_p$-branes, $M_p$-branes and anti-$M_p$-branes.

The article is organized as follows. Section 2 is devoted to a short review of Static Matrix Model. In section 3 we introduce some solutions of the equations of motion which preserve half of SUSY. In section 4 we calculate long range interaction of $M$-branes including relatively rotated $M$-branes, with one or two angles, and $M$-anti-$M$-brane configurations.

2 Static Matrix Model

Here we give a short review of the Static Matrix Model [9]. The starting point is the supermembrane action in 11 dimensions [13, 12]

$$S = -\frac{1}{2} \int d^3\eta \left( 2\sqrt{-g} + \epsilon^{abc} \bar{\theta} \Gamma_{\mu\nu} \partial_a \theta \times (\Pi^\mu_b \Pi_c X^\nu + \frac{1}{3} \bar{\theta} \Gamma^\mu \partial_b \theta \bar{\theta} \Gamma^\nu \partial_c \theta) \right), \quad (2.1)$$

where $\Pi$ and $g$ are

$$\Pi^\mu_a = \partial_a X^\mu + \bar{\theta} \Gamma^\mu \partial_a \theta,$$

$$g_{ab} = \Pi_a \cdot \Pi_b; \quad (2.2)$$
and $\theta$ is an eleven dimensional Majorana spinor (in this section $a, b = 0, 1, 2$). We use the following notations everywhere:

$$\mu, \nu = 0, 1, ..., 9, 10; \ I, J, K = 1, 2, ..., 9, 10; \ i, j, k = 1, 2, ..., 9.$$  

By decomposition of the coordinates $\eta_a = (\tau, \sigma_r), \ r = 1, 2$, and going to the static regime defined by

$$X^0 \equiv \tau, \quad \dot{X}^I \equiv \dot{\theta} \equiv 0; \quad (2.3)$$  

the components of $g$ are found to be

$$g_{00} = -1, \quad f_r \equiv g_{0r} = \bar{\theta} \Gamma^0 \partial_r \theta, \quad g_{rs} = \bar{g}_{rs} - f_r f_s, \quad \bar{g}_{rs} \equiv \Pi_r I \Pi_s I; \quad (2.4)$$  

and it can easily be shown that

$$g = -\bar{g}, \quad \bar{g} = det \bar{g}_{rs} = \frac{1}{2} \epsilon^{rs} \epsilon^{r's'} \bar{g}_{rr} \bar{g}_{ss'}, \quad (2.5)$$

Putting all the above relations in (2.1), we obtain

$$S = \frac{1}{2} \int d\tau d^2\sigma \left(-e^{-1} - e \bar{g} - 2\epsilon^{rs} \bar{\theta} \Gamma_{0j} \partial_r \partial_s X^I - \epsilon^{rs} \bar{\theta} \Gamma_{0j} \partial_r \partial_s \bar{\theta} \bar{g} \Gamma^I \partial_s \theta\right); \quad (2.6)$$  

where $e$ appears as an auxiliary field for linearising the action; its equation of motion gives

$$e^2 \bar{g} = 1, \quad (2.7)$$

which can be used for eliminating $e$. Due to (2.7), configurations with $\bar{g} = 0$ are unacceptable.

The action (2.1) has a local fermionic $\kappa$-symmetry, which allows one to gauge away half of the fermionic degrees of freedom. $\theta$ is a 32-component 11-dimensional Majorana spinor and in a real representation of $\Gamma$ matrices is real. We fix the $\kappa$-symmetry as for the light cone gauge, i.e. $(\Gamma^0 + \Gamma^{10})\theta = \Gamma^+ \theta = 0$. After integration over $\tau$, the action (2.6) takes the following form

$$S = -\frac{1}{2} T \int d^2\sigma e^{-1} \left(\frac{1}{2} \{X^i, X^j\}^2 + \{\{X^i, X^{10}\} - \frac{1}{2} \lambda^T \{X^i, \lambda\}\}^2 + \lambda^T \gamma_i \{X^i, \lambda\} + 1\right), \quad (2.8)$$
where $T$ is the time interval and $\lambda$ is the remaining 16 component part of $\theta$ and
\[
\{a, b\} = e (\partial_{\sigma_x} a \partial_{\sigma_y} b - \partial_{\sigma_y} a \partial_{\sigma_x} b) = e \epsilon^r s \partial_r a \partial_s b.
\] (2.9)

By the known substitutions $[13, 7]$ \[
\{a, b\} \Rightarrow -i [a, b],
\]
\[
\int e^{-1} d^2\sigma \Rightarrow Tr,
\] (2.10)
one finds
\[
S = - \frac{1}{2} T T^{5/3}_M T_r \left( \frac{1}{2} [X^i, X^j]^2 + ([X^i, X^{10}] - \frac{1}{2} \lambda^T [X^i, \lambda])^2 + i \lambda^T \gamma_i [X^i, \lambda] \right)
+ 6 \pi T T^{1/3}_M T_r (1),
\] (2.11)
in which $T_M$ is the supermembrane tension (in [2] the numerical factors were fixed by interaction considerations).

The action (2.11) has a gauge symmetry defined by:
\[
\delta_{gauge} X^i = i [X^i, \alpha],
\]
\[
\delta_{gauge} \lambda = i [\lambda, \alpha],
\]
\[
\delta_{gauge} X^{10} = i [X^{10}, \alpha].
\] (2.12)

The action (2.11) is invariant under the two SUSY transformations defined by two real anti-commuting $SO(9)$ spinors $\epsilon_1$ and $\epsilon_2$
\[
\delta^{(1)} X^i = 0,
\]
\[
\delta^{(1)} \lambda = \epsilon_1,
\]
\[
\delta^{(1)} (A^i A_i) = 0 \ (\Rightarrow \delta^{(1)} X^{10} = \frac{1}{2} \epsilon_1^T \lambda),
\] (2.13)
and
\[
\delta^{(2)} X^i = i^T_2 \gamma^i \lambda,
\]
\[
\delta^{(2)} \lambda = \frac{i}{2} [X^i, X^j] \gamma_{ij} \epsilon_2,
\]
\[
\delta^{(2)} (A^i A_i) = -i \lambda^T [\lambda, \lambda^T \epsilon_2] \ (\Rightarrow \delta^{(2)} X^{10} = \cdots),
\] (2.14)
where
\[
A^i = [X^i, X^{10}] - \frac{1}{2} \lambda^T [X^i, \lambda].
\]
The last line of these transformations can be used for defining the SUSY transformations of $X^{10}$. The variations of $A^i A_i$ are the same as those of the auxiliary fields in SUSY theories.
Note that the balance between bosonic and fermionic degrees of freedom. There are ten $X^I$; one of them, e.g. $X^{10}$ can be gauged away by the gauge symmetry (2.12) and there remain 9 bosonic degrees of freedom. Because of gauge fixing one complex ghost must be introduced; so there are $16+2=18$ (real) fermionic degrees of freedom which gives the correct balance [13].

3 BPS M-Brane Configurations

In this section we search for different BPS M-brane configurations of the model which are described by solutions of the classical equations of motion. The classical equations of motion with the condition $\lambda = 0$ are

$$\sum [X^I, [X^I, X^J]] = 0.$$  \hspace{1cm} (3.1)

Every configuration with $[X^I, X^J] \sim 1$ and with the other $X$’s vanishing are solutions of (3.1).\footnote{The point like (fully commuting) configurations which may be represented by $X^i = \text{diag}(x^1, x^2, \ldots, x^n), \quad X^{10} = \lambda = 0$, are not acceptable because of vanishing $\bar{g}$ in (2.7). It is in agreement with the fact that the individual 11 dimensional supergravitons which are candidates for ”quark” substructure of our model (due to their role in infinite momentum frame M(atrix) model as ”partons”) can not be studied as static configurations in 11 dimensions, because they are massless. This argument also will be supported by the equation of motion of $n$, the size of matrices. By inserting solutions introduced above in the action one finds, $S = 0 + n$. The equation of motion for $n$ has no solutions (gives 1 = 0).}

Solutions which can be interpreted as $M_p$-brane have the form

$$X^c_I = (B_1, B_2, \ldots, B_p, 0, \ldots, 0), \quad \lambda = 0,$$ \hspace{1cm} (3.2)

where $B_1, \ldots, B_p$ are $n \times n$ matrices with $n$ large, with the commutation relations

$$[B_a, B_b] = ic_{ab} 1,$$ \hspace{1cm} (3.3)

and $a, b = 1, \ldots, p$, and for $2l \equiv p = 2, 4, 6, 8$ \footnote{Interpretation of $p = 4, 6$ solutions as those found in [13] is obscure, because remaining SUSY in each case, these preserve half of SUSY but those $\frac{1}{4}$ and $\frac{1}{4}$ for $p = 4, 6$ respectively. Although the ”stack” behaviour of these solutions suggests that the conditions introduced in [13] on Killing spinors may be reduced to one condition which preserves half of SUSY. By all of the above $p = 8$ remains a problem [17].}.

\[ \]
By a proper rotation the anti-symmetric matrix $c_{ab}$ can be brought to the Jordan form
\[
    c_{ab} = \begin{pmatrix}
    0 & \omega_1 \\ -\omega_1 & 0 \\
    \vdots & \vec{\ddots} & \ddots & \ddots & \omega_l \\
    -\omega_l & 0
    \end{pmatrix}.
\] (3.4)

These solutions can be represented by $(2l = p)$
\[
    \begin{cases}
    X_{2i-1} = 1_{n_1} \otimes 1_{n_2} \otimes \ldots \otimes \frac{L_{2i-1}}{2\pi n_i} q_i \otimes 1_{n_{i+1}} \otimes \ldots \otimes 1_{n_l}, \\
    X_{2i} = 1_{n_1} \otimes 1_{n_2} \otimes \ldots \otimes \frac{L_{2i}}{2\pi n_i} p_i \otimes 1_{n_{i+1}} \otimes \ldots \otimes 1_{n_l}, \\
    X^i = 0, \quad i > 2l = p,
    \end{cases}
\] (3.5)

where $n_1n_2\ldots n_l = n$ and $L_a$’s are compactification radii, with commutation relations
\[
    [q_i, p_j] = i\delta_{ij} 1_{n_i}.
\]

The eigenvalues of $q$, $p$ are uniformly distributed as
\[
    -\sqrt{\frac{\pi n_i}{2}} \leq q_i, p_i \leq \sqrt{\frac{\pi n_i}{2}}.
\]

So the extension of solutions along $X_i$ axis is $L_i \to \infty$. Thus one can obtain
\[
    [X_{2i-1}, X_{2i}] = \frac{i}{2\pi n_i} L_{2i-1} L_{2i} 1_{n_i}, \quad 0 \leq i \leq l,
\] (3.6)

and correspondingly
\[
    \frac{n_i}{L_{2i-1} L_{2i}} = \frac{1}{2\pi \omega_i},
\] (3.7)

by $n_i \sim n_i^{\frac{1}{4}}$ [17, 19].

By putting these solutions in (2.11) and assuming\footnote{The only relevant scale in the theory is 11 dimensional Planck length, $l_p$, ($T_M \sim l_p^{-3}$).}
\[
    \frac{L_{2i-1} L_{2i}}{2\pi n_i} \sim T_M^{\frac{5}{2}},
\]

one finds
\[
    S \sim T^\frac{1}{3} \nu \quad \Rightarrow \quad S \sim T_M^\frac{1}{3} L_1 L_2 \ldots L_p T_M^{\frac{2l}{3}},
\]
\[
    \sim T_p (T L_1 L_2 \ldots L_p),
\] (3.8)

where the second line is the action of $M_p$-brane after passing $T$ of time. So one finds
\[
    T_p \sim T_M^{\frac{2l+1}{5}}.
\]
for the tension of a M\(_p\)-brane.

Also one can introduce parallel M\(_p\)-brane configurations. The configurations with two parallel M\(_p\)-branes can be obtained from the block-diagonal matrix with two identical blocks describing a pair of M\(_p\)-branes. Translating along the \((p + 1)\)-th axis by the distance \(r\) from each other we obtain the configuration of two parallel M\(_p\)-branes \(2l \equiv p\)

\[
X_a^{cl} = \begin{pmatrix}
B_a & 0 \\
0 & B_a \\
\end{pmatrix}, \quad a = 1, \ldots, p,
\]

\[
X_{p+1}^{cl} = \begin{pmatrix}
r/2 & 0 \\
0 & -r/2 \\
\end{pmatrix},
\]

\[
X_{i+10}^{cl} = X_i^{cl} = 0, \quad i = p + 2, \ldots, 9.
\]

(3.9)

It can be shown that all of the above solutions preserve half of the SUSY: they are BPS. By combining the two SUSY variations of \(\lambda\) with the solutions as background (i.e. \([X_a, X_b] \sim c_{ab}\)) one can see that the solutions are invariant under half of the SUSY’s by requiring the condition

\[
\epsilon_1 \sim \frac{i}{2} c_{ab} \gamma_{ab} \epsilon_2
\]

(3.10)
on the SUSY parameters \(\epsilon_{1,2}\). The rank of the \(16 \times 16\) matrix which appeared in RHS of (3.10) is 8 and 4 in cases \(p = 4, 6\) respectively. This supports the argument of the previous footnote about identification of these solutions and those of [15].

The above argument is in agreement with the one-loop effective action considerations. The one-loop effective action is calculated with the above backgrounds in [9]

\[
W = \frac{1}{2} Tr\log \left( P^2 K^{\delta_{I,J}} - 2i F_{I,J} \right) - \frac{1}{4} Tr\log \left( (P^2_I + \frac{i}{2} F_{J,I} \Gamma_{J,I}) \left( \frac{1 + \Gamma_{11}}{2} \right) \right) - Tr\log (P^2_I),
\]

(3.11)

with \(P_I * = [X_I^{cl}, *], F_{I,J} * = [f_{I,J}, *], f_{I,J} = i[X_I^{cl}, X_J^{cl}]\).

For the above solutions one has \(F_{I,J} = 0\), so

\[
W \sim \left( \frac{1}{2} \cdot 10 - \frac{1}{4} \cdot 16 - 1 \right) Tr\log (P^2_I) = 0.
\]

(3.12)

This indicates that solutions won’t take quantum corrections, as one expects for a BPS state.

Before ending this section it is interesting to consider possible odd dimensional solu-
tions. For example in 5 dimensions, a solution can be represented by:

\[
\begin{aligned}
X_1 &= \frac{L_1}{\sqrt{2\pi n_1}} q_1 \otimes 1_{n_2} \otimes 1_{n_3}, \\
X_2 &= \frac{L_2}{\sqrt{2\pi n_1}} p_1 \otimes 1_{n_2} \otimes 1_{n_3}, \\
X_3 &= 1_{n_1} \otimes \frac{L_3}{\sqrt{2\pi n_2}} q_2 \otimes 1_{n_3}, \\
X_4 &= 1_{n_1} \otimes \frac{L_4}{\sqrt{2\pi n_2}} p_2 \otimes 1_{n_3}, \\
X_5 &= 1_{n_1} \otimes 1_{n_2} \otimes \frac{L_5}{(2\pi n_3)^x} q_3, \\
X_i &= X_{10} = 0, \quad i > 5,
\end{aligned}
\]

where \( n_1 n_2 n_3 = n \), and we have let the power \( x \) in \( X_5 \) be unknown to be determined later. The desired commutation relations are

\[
[q_i, p_i] = i 1_{n_i}; \quad [q_3, q_1] = [q_3, p_1] = 0, \quad i = 1, 2,
\]

and the following eigenvalue distributions

\[
-\sqrt{\frac{\pi n_i}{2}} \leq q_i, p_i \leq \sqrt{\frac{\pi n_i}{2}}, \\
-(\frac{\pi n_3}{2})^x \leq q_3 \leq (\frac{\pi n_3}{2})^x.
\]

So the extensions of the solution in 12345 directions are \( L_{12345} \). To understand this solution further it is convenient to diagonalize \( q_3 \). The solution can be interpreted by two infinite stacks of M2-branes as a 4 dimensional object, stacked along 5th direction. By ignoring the 5th direction one notes the similar stack behaviour of this solution and the longitudinal 5-branes of M(atrix) theory \[17\]. By putting this solution in (2.11) one finds:

\[
S \sim T^{1/3} T M \quad n \Rightarrow S \sim T^{1/3} T L_1 L_2 L_3 L_4 L_5 \frac{1}{T M^{x+1}},
\]

\[
\sim T_5 (T L_1 L_2 L_3 L_4 L_5), \quad \text{(3.14)}
\]

where the second line is the action of M5-brane extended in the 12345 directions and after passing \( T \) of time. Therfore, tension becomes

\[
T_5 \sim T^{5+1} M^{1-x} L_5^{1-x}.
\]

** Solutions like (3.13) can not be interpreted as 5 dimensional objects in the context of the light cone M(atrix) theory. This can be understood by studying the related supercharge which was found in \[17\],

\[
Z_{1234} \sim R_{111} L_1 L_2 L_3 L_4,
\]

which indicates extension along the longitudinal direction, \( R_{11} \). So by considering the 5th direction one finds a 6 dimensional object.
To have finite and non-zero tension as \( L_5 \to \infty \) one must put \( x = 1 \). Then the tension is found to be

\[
T_5 \sim T_M^2,
\]
as one expected for a M5-brane \([16]\).

The other simple solution, which is consistent with the equation of motion of \( n \), is a 3 dimensional solution represented by

\[
\begin{align*}
X_1 &= \frac{L_1}{\sqrt{2\pi n_1}} q_1 \otimes 1_{n_2}, \\
X_2 &= \frac{L_2}{\sqrt{2\pi n_1}} p_1 \otimes 1_{n_2}, \\
X_3 &= 1_{n_1} \otimes \frac{L_3}{2\pi n_2} q_2, \\
X^i &= X_{10} = 0, \quad i > 3,
\end{align*}
\]

where \( n_1 n_2 = n \) and

\[
[q_1, p_1] = i1_{n_1}; \quad [q_2, q_1] = [q_2, p_1] = 0.
\]
The operators \( q_1 \) and \( p_1 \) represent extensions in two directions and continuous spectrum of \( q_2 \) represents an infinite continuous "stack" of two dimensional objects along the 3rd direction.

### 4 M-Brane Long Range Interactions

In this section we calculate the long range interaction between different M-brane configurations \([4]\).

The one-loop effective action \( W \) was introduced in the previous section (and calculated in \([3]\)) with the backgrounds \( \lambda = X_{10}^{cl} = 0 \), (writing in ten dimensional language)

\[
W = \frac{1}{2} Tr \log \left( P^2 + 2iF_{IJ} \right) - \frac{1}{4} Tr \log \left( \left( P^2 + \frac{i}{2} F_{IJ} \Gamma^{IJ} \left( \frac{1 + \Gamma_{11}}{2} \right) \right) - Tr \log (P_I^2), \right)
\]
with \( P_I^2 = [X_I^{cl}, \ast], F_{IJ} \ast = [f_{IJ}, \ast], f_{IJ} = i[X_I^{cl}, X_J^{cl}] \) and

\[
\Gamma_{32}^i = \begin{pmatrix} 0 & \gamma^i_{16} \\ \gamma^i_{16} & 0 \end{pmatrix}, \quad \Gamma_{32}^{10} = \begin{pmatrix} 1_{16} & 0 \\ 0 & -1_{16} \end{pmatrix}, \quad \Gamma_{11} = i\Gamma_{123456789,10}.
\]

In the cases of our interest we have \( [X_I^{cl}, f_{IJ}] = c - number \), and so \( P_I^2 \) and \( F_{IJ} \) are simultaneously diagonalizable. By setting \( F_{IJ} \) to be in the Jordan form (because of \( X_{10}^{cl} = 0 \) we put \( a_5 = 0 \))

\[\text{†† Because of some similarities, at one loop order, between Static Matrix Model and the matrix theory approach to the type IIB superstring theory (called IKKT) \([18]\), there is a short cut for us to use the techniques developed in that framework \([18, 19, 20]\). Although for completion we try to give the required details.}\]
\[
F_{IJ} = \begin{pmatrix}
0 & -a_1 \\
a_1 & 0 \\
\ddots & \ddots \\
0 & -a_4 \\
a_4 & 0 \\
0 & -a_5 (= 0) \\
a_5 (= 0) & 0
\end{pmatrix},
\]  
and noting that eigenvalues of $\Gamma_{IJ}$'s are $\pm i$ one finds the following expression for $W$:

\[
W = \frac{1}{2} \sum_{i=1}^{5} Tr \log \left( 1 - \frac{4a_i^2}{(P_i^2)^2} \right) - \frac{1}{4} \sum_{s_1, \ldots, s_5 = \pm 1} Tr \log \left( 1 - \frac{a_1s_1 + \ldots + a_5s_5}{P_i^2} \right),
\]  

(4.3)

where the sum over $s_a$'s is restricted because of projecting on 16 dimensional subspace of $\Gamma$ matrices by tracing only on that part of basis which give 1 by acting the operator $i\Gamma^{12}\Gamma^{34}\Gamma^{56}\Gamma^{78}\Gamma^{9,10}$.

### 4.1 Rotated M-Branes Interaction (two angles)

The configuration with two rotated $Mp$-branes can be obtained from the block-diagonal matrix with two identical blocks describing a pair of $Mp$-branes. Translating along the $(p + 3)$-th axis by the distance $r$ from each other and rotating in opposite directions in $(p, p + 1)$ plane and $(p - 1, p + 2)$ through the angles $\phi/2$ and $\psi/2$, we obtain the configuration of two rotated $Mp$-branes ($2l \equiv p=2,4,6$)

\[
X_{cl}^{a} = \begin{pmatrix}
B_a & 0 \\
0 & B_a
\end{pmatrix}, \quad a = 1, \ldots, p - 2,
\]

\[
X_{p-1}^{cl} = \begin{pmatrix}
B_{p-1} \cos \frac{\psi}{2} & 0 \\
0 & B_{p-1} \cos \frac{\phi}{2}
\end{pmatrix},
\]

\[
X_{p}^{cl} = \begin{pmatrix}
B_p \cos \frac{\phi}{2} & 0 \\
0 & B_p \cos \frac{\phi}{2}
\end{pmatrix},
\]

\[
X_{p+1}^{cl} = \begin{pmatrix}
B_p \sin \frac{\phi}{2} & 0 \\
0 & -B_p \sin \frac{\phi}{2}
\end{pmatrix},
\]

\[
X_{p+2}^{cl} = \begin{pmatrix}
B_{p-1} \sin \frac{\psi}{2} & 0 \\
0 & -B_{p-1} \sin \frac{\psi}{2}
\end{pmatrix},
\]
\[ X_{p+3}^{cl} = \begin{pmatrix} \frac{r}{2} & 0 \\ 0 & -\frac{r}{2} \end{pmatrix}, \]
\[ X_{10}^{cl} = X_i^{cl} = 0, \quad i = p + 4, \ldots, 9. \quad (4.4) \]

So one finds
\[
\begin{align*}
 f_{p-1,p} &= -\omega_l \cos \frac{\psi}{2} \cos \frac{\phi}{2} \otimes 1, \\
 f_{p-1,p+1} &= -\omega_l \cos \frac{\psi}{2} \sin \frac{\phi}{2} \otimes \sigma_3, \\
 f_{p,p+2} &= \omega_l \cos \frac{\phi}{2} \sin \frac{\psi}{2} \otimes \sigma_3, \\
 f_{p+1,p+2} &= \omega_l \sin \frac{\phi}{2} \sin \frac{\psi}{2} \otimes 1,
\end{align*}
\]
otherwise \( f_{ab} = 0, \quad (4.5) \)

where \( \sigma_3 \) is the Pauli matrix. Then
\[
\begin{align*}
 [P_{p-1}, P_{p+1}] &= i\omega_l \cos \frac{\psi}{2} \sin \frac{\phi}{2} \otimes \Sigma_3, \\
 [P_{p}, P_{p+2}] &= -i\omega_l \cos \frac{\phi}{2} \sin \frac{\psi}{2} \otimes \Sigma_3, \\
 P_{p+3} &= \frac{r}{2} \otimes \Sigma_3, \\
 otherwise \quad F_{ab} &= 0, \quad (4.6) \)
\]

where
\[
\Sigma_3^* = [1 \otimes \sigma_3, *].
\]

\( \Sigma_3 \) has 2, -2, 0, 0 as eigenvalues. Zero eigenvalues won’t have any contribution to the effective action because of (3.12). The other two force \( P_{p-1}, P_{p+1} \) and \( P_p, P_{p+2} \) to behave as harmonic oscillators with their related frequencies to be read from (4.6). So the eigenvalues of \( P_I^2 \) are
\[
E_{q,p,k,k'} = r^2 + 2 \sum_{i=1}^{l-1} (q_i^2 + p_i^2) + 4\omega_l \cos \frac{\psi}{2} \sin \frac{\phi}{2} (k + \frac{1}{2}) + 4\omega_l \sin \frac{\psi}{2} \cos \frac{\phi}{2} (k' + \frac{1}{2}),
\]
\[
(4.7)
\]

where \( q_i, p_i \) are eigenvalues of \( P_i; \ a = 1, \ldots, p-2 \) and \( k, k' \) are harmonic oscillator numbers.

So one finds for (4.3)
\[
W = \prod_{a=1}^{p-2} \left( \frac{n_a^+}{L_a} \right) \int d^{l-1}q \ d^{l-1}p \times \\
\sum_{k',k=0}^{\infty} \ln \left( 1 - \frac{16\omega_l^2 \cos^2 \frac{\psi}{2} \sin^2 \frac{\phi}{2}}{E_{q,p,k,k'}^2} \right) + \ln \left( 1 - \frac{16\omega_l^2 \cos^2 \frac{\phi}{2} \sin^2 \frac{\psi}{2}}{E_{q,p,k,k'}^2} \right)
\]
\[\frac{1}{2} \sum_{s_1, \ldots, s_5 = \pm 1 \atop s_1 \ldots s_5 = 1} \ln \left( 1 - \frac{2s_1 \omega_l \cos \frac{\psi}{2} \sin \frac{\phi}{2} + 2s_2 \omega_l \cos \frac{\phi}{2} \sin \frac{\psi}{2}}{E_{q,p,k,k'}} \right). \tag{4.8}\]

The last term in the above can be rewritten as

\[\frac{1}{2} \sum_{s_1, \ldots, s_5 = \pm 1 \atop s_1 \ldots s_5 = 1} \ln \left( 1 - \frac{2s_1 \omega_l \cos \frac{\psi}{2} \sin \frac{\phi}{2} + 2s_2 \omega_l \cos \frac{\phi}{2} \sin \frac{\psi}{2}}{E_{q,p,k,k'}} \right) =
\]

\[2 \ln \left( 1 - \frac{4\omega_l^2 \sin^2 \frac{\phi + \psi}{2}}{E_{q,p,k,k'}^2} \right) + 2 \ln \left( 1 - \frac{4\omega_l^2 \sin^2 \frac{\phi - \psi}{2}}{E_{q,p,k,k'}^2} \right). \tag{4.9}\]

It is convenient to represent the logarithms in (4.8) in the form:

\[\ln \frac{u}{v} = \int_0^\infty \frac{ds}{s} \left( e^{-us} - e^{-us} \right). \tag{4.10}\]

The sums over \(k, k'\) and the integrals over \(q\) and \(p\) can be calculated as follows

\[\int d^{d-1} q \, d^{d-1} p \sum_{k,k'=0}^{\infty} e^{-sE_{q,p,k,k'}} = \]

\[\left( \frac{\pi}{2s} \right)^{L-1} \frac{\omega_l}{4 \sinh \left( 2s \omega_l \cos \frac{\psi}{2} \sin \frac{\phi}{2} \right) \sinh \left( 2s \omega_l \sin \frac{\psi}{2} \cos \frac{\phi}{2} \right)} \tag{4.11}\]

We finally obtain the following form

\[W = \prod_{a=1}^{p-2} \left( \frac{n^L}{L_a} \right) J_0^\infty \frac{ds}{s} \left( \frac{\pi}{2s} \right)^{L-1} e^{-s} \times \]

\[\left[ \frac{2 \sinh^2 (\omega_l s \sin \frac{\phi + \psi}{2}) + 2 \sinh^2 (\omega_l s \sin \frac{\phi - \psi}{2})}{\sinh (2\omega_l s \sin \frac{\psi}{2} \cos \frac{\phi}{2}) \sinh (2\omega_l s \sin \frac{\psi}{2} \cos \frac{\phi}{2})} - \]

\[\frac{\sinh^2 (2\omega_l s \sin \frac{\phi}{2} \cos \frac{\psi}{2}) + \sinh^2 (2\omega_l s \sin \frac{\psi}{2} \cos \frac{\phi}{2})}{\sinh (2\omega_l s \sin \frac{\psi}{2} \cos \frac{\phi}{2}) \sinh (2\omega_l s \sin \frac{\psi}{2} \cos \frac{\phi}{2})} \right]. \tag{4.12}\]

For large separation between branes we find:

\[W = \frac{2}{3} \left( \frac{\pi}{2} \right)^{L-1} \omega_l^2 (2 - l) \prod_{a=1}^{p-2} \left( \frac{n^L}{L_a} \right) \times \]

\[\frac{\sin^4 \frac{\phi + \psi}{2} + \sin^4 \frac{\phi - \psi}{2} - 8 \cos^4 \frac{\psi}{2} \sin^4 \frac{\phi}{2} - 8 \cos^4 \frac{\phi}{2} \sin^4 \frac{\psi}{2}}{\sin \psi \sin \phi} \left( \frac{1}{\rho^{6-p}} \right) + \ldots \]

\[= -\left( \frac{\pi}{2} \right)^{L-1} \omega_l^2 (2 - l) \prod_{a=1}^{p-2} \left( \frac{n^L}{L_a} \right) \times \left( \frac{\cos \phi - \cos \psi}{\sin \psi \sin \phi} \right) \left( \frac{1}{\rho^{6-p}} \right) + \ldots, \tag{4.13}\]
which is in agreement with supergravity results both in angular and $r$ dependences \[21\].

The above interaction vanishes in $\psi = \phi$ cases, signalling enhancement of SUSY. An equivalent result is obtained in \[21\] by considering interactions of rotated D$p$-branes, and in \[22\] by studying the SUSY algebra for rotated M-objects.

### 4.2 Rotated M-Branes Interaction (one angle)

By putting $\psi = 0$ in two rotated angle configurations of the previous subsection one finds the following for one angle case:

$$X^{cl}_a = \begin{pmatrix} B_a & 0 \\ 0 & B_a \end{pmatrix}, \quad a = 1, \ldots, p-1,$$

$$X^{cl}_p = \begin{pmatrix} B_p \cos \phi & 0 \\ 0 & B_p \cos \phi \end{pmatrix},$$

$$X^{cl}_{p+1} = \begin{pmatrix} B_p \sin \phi & 0 \\ 0 & -B_p \sin \phi \end{pmatrix},$$

$$X^{cl}_{p+2} = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix},$$

$$X^{cl}_{10} = X^{cl}_i = 0, \quad i = p+3, \ldots, 9.$$  \[4.14\]

So one finds

$$f_{p-1,p} = -\omega_1 \cos \frac{\phi}{2} \otimes 1,$$

$$f_{p-1,p+1} = -\omega_1 \sin \frac{\phi}{2} \otimes \sigma_3,$$

otherwise $f_{ab} = 0$, \[4.15\]

and then

$$[P_{p-1}, P_{p+1}] = i\omega_1 \sin \frac{\phi}{2} \otimes \Sigma_3,$$

$$P_{p+3} = r \frac{1}{2} \otimes \Sigma_3,$$

otherwise $F_{ab} = 0$. \[4.16\]

Again we have an harmonic oscillator and the eigenvalues of $P_I^2$ are

$$E_{q,p,k} = r^2 + 2 \sum_{i=1}^{l-1} (q_i^2 + p_i^2) + 2 \cos^2 \frac{\phi}{2} q_l^2 + 4\omega_1 \sin \frac{\phi}{2} (k + \frac{1}{2}).$$ \[4.17\]
So one finds for (4.3)

\[
W = \prod_{a=1}^{p-1} \left( \frac{n^t_a}{L_a} \right) \int d^q d^{l-1} p \sum_{k=0}^{\infty} \ln \left( 1 - \frac{16\omega^2 \sin^2 \frac{q}{2}}{E_{q,p,k}^2} \right) \]

\[
- \frac{1}{2} \sum_{s_1, \ldots, s_5 = \pm 1} \ln \left( 1 - \frac{2s_1\omega \sin \frac{q}{2}}{E_{q,p,k}} \right). \quad (4.18)
\]

The last term which originates from the integration over fermions can be rewritten as

\[
\frac{1}{2} \sum_{s_1, \ldots, s_5 = \pm 1} \ln \left( 1 - \frac{2s_1\omega \sin \frac{q}{2}}{E_{q,p,k}} \right) = 4 \ln \left( 1 - \frac{4\omega^2 \sin^2 \frac{q}{2}}{E_{q,p,k}^2} \right). \quad (4.19)
\]

The sum over \( k \) and the integrals over \( q \) and \( p \) then can be evaluated using the formula

\[
\int d^q d^{l-1} p \sum_{k=0}^{\infty} e^{-sE_{q,p,k}} = \frac{\left( \frac{\pi}{2s} \right)^{p-1}}{2 \cos \frac{q}{2} \sinh \left( 2\omega \sin \frac{q}{2} \right)}. \quad (4.20)
\]

We finally obtain the following form

\[
W = \prod_{a=1}^{p-1} \left( \frac{n^t_a}{L_a} \right) \int_0^{\infty} \frac{d s}{s} \left( \frac{\pi}{2s} \right)^{p-1} e^{-s^2 s} \frac{4 \sinh^2 (\omega s \sin \frac{q}{2}) - \sin^2 (2\omega s \sin \frac{q}{2})}{\cos \frac{q}{2} \sinh(2\omega s \sin \frac{q}{2})}. \quad (4.21)
\]

For large separation between branes we find:

\[
W = -4 \left( \frac{\pi}{2} \right)^{p-1} (5/2 - l)! \prod_{a=1}^{p-1} \left( \frac{n^t_a}{L_a} \right) \omega^3 \tan \frac{q}{2} \sin^2 \frac{q}{2} \frac{1}{r^{p-2}} + \cdots, \quad (4.22)
\]

again in agreement with supergravity results both in angular and in \( r \) dependences [21].

### 4.3 M-Brane and Anti-M-Brane Interaction

The M2-brane and anti-M2-brane long range interaction have been studied in the framework of M(atrix) theory in [11, 23] which their results appeared in compactified limit of 11 dimensional supergravity (i.e. \( V(r) \sim \frac{1}{r^3} \)). Here we want to consider the same problem but for Mp-brane and anti-Mp-brane in the context of our model. The long range interaction for two anti-parallel Mp-branes was calculated in [1] for M2-branes.
The classical solution describing anti-parallel $M_p$-branes at the distance $r$ from each other are represented by block–diagonal matrices (this solution can be obtained by setting $\phi = \pi$ in the previous subsection)

\[
X_{a}^{cl} = \begin{pmatrix} B_a & 0 \\ 0 & B_a \end{pmatrix}, \quad a = 1, \ldots, p - 1
\]

\[
X_{p}^{cl} = \begin{pmatrix} B_a & 0 \\ 0 & -B_a \end{pmatrix},
\]

\[
X_{p+1}^{cl} = \begin{pmatrix} r/2 & 0 \\ 0 & -r/2 \end{pmatrix},
\]

\[
X_{i}^{cl} = 0, \quad i = p + 2, \ldots, 9, (4.23)
\]

So one finds

\[
f_{p-1,p} = -\omega l \otimes \sigma_3,
\]

otherwise $f_{ab} = 0$, (4.24)

and then

\[
[P_{p-1}, P_p] = i\omega l \otimes \Sigma_3,
\]

\[
P_{p+3} = \frac{r}{2} \otimes \Sigma_3,
\]

otherwise $F_{ab} = 0$. (4.25)

Again we have harmonic oscillators and the eigenvalues of $P_i^2$ are

\[
E_{q,p,k} = r^2 + 2 \sum_{i=1}^{l-1} (q_i^2 + p_i^2) + 4\omega_l (k + \frac{1}{2}). (4.26)
\]

So one finds for (4.3)

\[
W = \prod_{a=1}^{p-1} \left( \frac{n^4}{L_a} \right) \int d^{d-1}q \int d^{d-1}p \sum_{k=0}^{\infty} \ln \left( 1 - \frac{16\omega_l^2}{E_{q,p,k}^2} \right) 
\]

\[
- \frac{1}{2} \sum_{s_1, \ldots, s_5 = \pm 1, s_1 \ldots s_5 = 1} \ln \left( 1 - \frac{2s_l\omega_1}{E_{q,p,k}} \right). (4.27)
\]

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The last term may be rewritten as
\[
\frac{1}{2} \sum_{s_1,\ldots,s_5 = \pm 1} \ln \left( 1 - \frac{2 s_1 \omega_l}{E_{q,p,k}} \right) = 4 \ln \left( 1 - \frac{4 \omega_l^2}{E_{q,p,k}^2} \right) .
\] (4.28)

The sum over \(k\) and the integrals over \(q\) and \(p\) then can be calculated
\[
\int d^{d-1}q \ d^{d-1}p \sum_{k=0}^{\infty} e^{-s E_{q,p,k}} = \left( \frac{\pi}{2s} \right)^{\frac{d}{2}-1} \frac{e^{-r^2 s}}{\sinh(2\omega_l s)}. \] (4.29)

We finally obtain the following form
\[
W = \prod_{a=1}^{p-1} \left( \frac{\eta^4}{L_a} \right) \int_0^\infty \frac{ds}{s} \left( \frac{\pi}{2s} \right)^{\frac{d}{2}-1} e^{-r^2 s} \frac{4 \sinh^2(\omega_l s) - \sinh^2(2\omega_l s)}{\sinh(2\omega_l s)}. \] (4.30)

For large separation between branes we find:
\[
W = -4 \left( \frac{\pi}{2} \right)^{\frac{d}{2}} (3 - l) ! \prod_{a=1}^{p-1} \left( \frac{\eta^4}{L_a} \right) \omega_l^3 \frac{1}{r^{8-p}} + \cdots ,
\] (4.31)
in agreement with supergravity results, especially for \(p = 2\) case \[14, 11\].

5 Conclusion

In this article we studied some aspects of M-branes of Static Matrix Model. First we introduced certain BPS configurations of the model, including single Mp-brane (\(p=\text{even}\)) and two parallel Mp-branes. It was shown that these configurations do not take quantum corrections at one-loop.

Then the long range interactions of Mp-branes were studied. In particular, we calculated the long range interaction between two relatively rotated (in one and two angles) Mp-branes and Mp-anti-Mp-branes. The results were in agreement with the 11 dimensional supergravity. Moreover we found new BPS configurations in the two angle case which correspond to two Mp-branes rotated with equal angles with respect to each other, as the same of \[21, 22\].

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