The Group of Symmetries of the shorter Moonshine Module

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Abstract
It is shown that the automorphism group of the shorter Moonshine module $VB^\natural$ constructed in [Höh95] is the direct product of the finite simple group known as the Baby Monster and the cyclic group of order 2.

1 Introduction

The shorter Moonshine module or Baby Monster vertex operator superalgebra denoted by $VB^\natural$ was constructed in [Höh95]. It is a vertex operator superalgebra of central charge $23\frac{1}{2}$ on which a direct product of the cyclic group of order 2 and Fischer’s Baby Monster $B$ acts by automorphisms ([Höh95], Th. 2.4.7). Furthermore, it was shown loc. cit. that the complete automorphism group $\text{Aut}(VB^\natural)$ of $VB^\natural$ is finite. It was conjectured that this is already the full automorphism group, i.e. $\text{Aut}(VB^\natural) = 2 \times B$. The purpose of this note is to present a proof for this conjecture.

Theorem 1 The automorphism group of the shorter Moonshine module $VB^\natural$ is the direct product of the Baby Monster $B$ and a cyclic group of order 2.

The Baby Monster was discovered by Bernd Fischer in 1973 (unpublished) and a computer proof of its existence and uniqueness was given by J. S. Leon and C. C. Sims in 1977 (for an announcement see [LS77]). An independent computer-free construction was given by R. Griess during his construction of the Monster (see [Gri82]). The Baby Monster is the second largest sporadic simple group and has order $2^{41}3^{13}5^67^{11} \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$.

The paper is organized as follows. In the rest of the introduction we will recall the definition of $VB^\natural$ from [Höh95] and discuss the original proof of the corresponding theorem for the Moonshine module:

Theorem 2 The automorphism group of the Moonshine module $V^\natural$ is the Monster $M$.

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The second section contains the proof of Theorem 1 which is divided into two parts. In the first step, we show that $B$ equals the automorphism group of the algebra $B'$, the subspace of the shorter Moonshine module $V B'$ spanned by the vectors of conformal weight 2. This is done by identifying certain idempotents in $\mathfrak{B}$ and $\mathfrak{B}'$ with certain conjugacy classes of involutions in $M$ resp. $B$. The second step consists in showing that the vertex operator subalgebra $V B_{(0)}$ of even vectors in $V B'$ is generated as a vertex operator algebra by the conformal weight 2 component $\mathfrak{B}'$. For this, we use the theory of framed vertex operator algebras as developed in [DGH98].

The Moonshine module $V^2$ is a vertex operator algebra of central charge 24 having as conformal character $\chi_{V^2} = q^{-1} \sum_{n=0}^{\infty} \text{dim} V^2_n q^n$ the elliptic modular function $j = q^{-1} + 196884 q + 21493760 q^2 + \cdots$. It was constructed in [FLM88] (see also [Bor86]) as a $\mathbb{Z}_2$-orbifold of the lattice vertex operator algebra $V_\Lambda$ associated to the Leech lattice $\Lambda$. The algebra induced on the conformal weight 2 part $V^2_2$ can be identified with the Griess algebra $\mathfrak{B}$ (see [FLM88], Prop. 10.3.6).

We outline the proof of Theorem 2: From the construction of $V^2$ one can find a subgroup of type $2_1^{+24}.\text{Co}_1$ as well as an extra triality involution $\sigma$ inside $\text{Aut}(V^2)$. It was shown in [Gri82] that $\text{Aut}(\mathfrak{B})$ is finite and certain theorems from finite group theory ([Smi79, GMS89]) were invoked to conclude that the subgroup $\langle 2_1^{+24}.\text{Co}_1, \sigma \rangle$ of $\text{Aut}(\mathfrak{B})$ must be the finite simple group called the Monster. The finiteness proof for $\text{Aut}(\mathfrak{B})$ was simplified in [Con84, Tit84]. Furthermore, Tits showed in [Tit84] that the Monster is the full automorphism group of $\mathfrak{B}$. The proof of Theorem 2 is completed by showing that $V^2$ is generated as a vertex operator algebra by $\mathfrak{B}$ (see [FLM88], Th. 12.3.1 (g)). An independent proof of Theorem 2 was given by Miyamoto [Miy04].

The Monster group has two conjugacy classes of involutions, denoted by $2A$ and $2B$ in the notation of the Atlas of finite groups (cf. [CCN+85]). For any involution in class $2A$ there exists an associated idempotent $e$ in the Griess algebra $\mathfrak{B}$ called the transposition axis of that involution in $\text{Con84}$. (In $\text{Con84}$, 64 $e$ was used, but we prefer to work with the idempotents.) The vertex operator subalgebra of $V^2$ generated by $e$ is isomorphic to the simple Virasoro vertex operator algebra $L(1/2,0)$ of central charge 1/2 and has the Virasoro element $2e$ (cf. [DMZ94], see also [Höh95], Th. 4.1.2 or [Miy96]).

The shorter Moonshine module $V B^2$ is a vertex operator superalgebra of central charge $33\frac{1}{2}$ with conformal character

$$\chi_{V B^2} = q^{-17/48}(1 + 4371 q^{3/2} + 96256 q^2 + 1143745 q^{5/2} + \cdots).$$

We recall its definition from [Höh95], Sec. 3.1 and 4.2: Let $L(1/2,0)$ be the vertex operator subalgebra of $V^2$ generated by a transposition axis $e$. The Virasoro vertex operator algebra $L(1/2,0)$ has three irreducible modules $L(1/2,0)$,
Let $L(\frac{1}{2}, \frac{1}{2})$ and $L(\frac{1}{2}, \frac{1}{16})$ of conformal weights 0, $\frac{1}{2}$ and $\frac{1}{16}$ (cf. [DMZ94]). Let $VB^2_{(0)}$ be the commutant of $L(\frac{1}{2}, 0)$ in $V^\natural$. One has a decomposition

$$V^\natural = VB^2_{(0)} \otimes L(\frac{1}{2}, 0) \oplus VB^2_{(1)} \otimes L(\frac{1}{2}, \frac{1}{2}) \oplus VB^2_{(2)} \otimes L(\frac{1}{2}, \frac{1}{16})$$

with $VB^2_{(0)}$-modules $VB^2_{(0)}$, $VB^2_{(1)}$ and $VB^2_{(2)}$. The shorter Moonshine module is defined as

$$VB^2 = VB^2_{(0)} \oplus VB^2_{(1)}.$$ 

A vertex operator superalgebra structure on $VB^2$ respecting the vertex operator superalgebra structure such that the central involution acts trivially (see [Höh95], Th. 3.1.3). The centralizer of the class 2A involution associated to the idempotent $e$ is $2.B$, a 2-fold cover of the Baby Monster. There is an induced action of $2.B$ on $VB^2$ acting trivially on $VB^2_{(0)}$ and by multiplication with $-1$ on $VB^2_{(1)}$.

A construction of $VB^2$ using the 2A-twisted module of $V^\natural$ was explained in [Yam02].

Remarks: The shorter Moonshine module can be understood as the vertex operator algebra analog of the shorter Leech lattice (which has 2.Co2 as automorphism group, where Co2 is the second Conway group) and the shorter Golay code (which has $M_{22}:2$ as automorphism group, where $M_{22}$ is a Mathieu group) (cf. [CCN+85]).

As a framed vertex operator superalgebra (for the definition see the next section), $VB^2$ can in principle be constructed without reference to $V^\natural$. This suggests that a natural way to define the Baby Monster would be to use the symmetry group of $VB^2$.

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2 Proof of Theorem

Let $V$ be a nonnegatively graded rational vertex operator algebra having a Virasoro vertex operator algebra $L(\frac{1}{2}, 0)$ as subalgebra. It was shown in [Miy99], Sect. 4, that in this situation a nontrivial automorphism of $V$ can often be constructed: There is the decomposition $V = W(0) \oplus W(\frac{1}{2}) \oplus W(\frac{1}{16})$, where $W(h)$ is the subspace of $V$ generated by all $L(\frac{1}{2}, 0)$-modules isomorphic to $L(\frac{1}{2}, h)$. If $W(\frac{1}{16}) \neq 0$, one defines an involution $\tau$ as the identity on $W(0) \oplus W(\frac{1}{2})$ and the multiplication with $-1$ on $W(\frac{1}{16})$. If $W(\frac{1}{16}) = 0$ but $W(\frac{1}{2}) \neq 0$, an involution $\sigma$ is defined as the identity on $W(0)$ and the multiplication with $-1$ on $W(\frac{1}{2})$. The nontrivial fusion rules of the fusion algebra for $L(\frac{1}{2}, 0)$
are $L(1/2,1/2) \times L(1/2,1/2) = L(1/2,0)$, $L(1/2,1/2) \times L(1/2,1/16) = L(1/2,1/16)$ and $L(1/2,1/16) \times L(1/2,1/16) = L(1/2,0) + L(1/2,1/2)$ (cf. [DNZ94]). This is the reason that $\tau$ resp. $\sigma$ are indeed automorphisms of $V$. The following result was mentioned in [Miy96], p. 547, but no explicit proof was given.

**Lemma 3** Every idempotent of $\mathfrak{B}$ generating a simple Virasoro vertex operator algebra of central charge $\frac{1}{2}$ is a transposition axis. In particular, the map which associates to such an idempotent its Miyamoto involution defines a bijection between the set of such idempotents and the class $2A$ involutions of the Monster.

**Proof.** Let $e$ be an idempotent of $\mathfrak{B}$ as in the lemma, let $L(1/2,0) \subset V^2$ be the simple Virasoro vertex operator algebra of central charge $\frac{1}{2}$ it generates, and let $V^2 = W(0) \oplus W(1/2) \oplus W(1/16)$ be the decomposition of $V^2$ into $L(1/2,0)$-modules as described above.

First we observe that the case $W(1/2) = W(1/16) = 0$ cannot occur. In this case, the algebra $\mathfrak{B}$ would be the direct orthogonal sum of the one dimensional algebra $C \cdot e$ and a complementary subalgebra. Let $e' = x \cdot e + u$ be another element in the Monster orbit of $e$ where $u$ is in the complementary subalgebra. We get $e'^2 = x^2 \cdot e + u^2 = x \cdot e + u = e'$ and thus either $x = 1$ or $x = 0$. In the first case, we would have $\langle e', e' \rangle = \langle e, e \rangle + \langle u, u \rangle$ and thus $u = 0$ and $e' = e$. It follows that $x = 0$ and $\langle e', e \rangle = 0$. Thus the Monster orbit of $e$ would consist of pairwise orthogonal idempotents generating simple Virasoro vertex operator algebras of central charge $1/2$. Since the central charge of $V^2$ is 24, there can be at most 48 such idempotents. However, no non-trivial factor group of the Monster is a subgroup of any symmetric group of degree $n \leq 48$. Hence the Monster fixes $e$ and thus also $1 - e$ where 1 is the identity element of $\mathfrak{B}$. Since the trivial Monster representation occurs only with multiplicity one, this is impossible. Therefore this case cannot occur and the idempotent $e$ defines an involution $t \in \text{Aut}(V^2) = M$.

Let $\mathfrak{B} = X^+ \oplus X^-$ be the decomposition of $\mathfrak{B}$ into the $+1$ and $-1$ eigenspaces of $t$ and let $C$ be the centralizer of $t$ in the Monster. The fusion rules for $L(1/2,0)$ imply that the algebra multiplication restricts to a $C$-equivariant map $\mu : X^+ \to \text{End}(X^-)$ defined by $\mu(a)x = a(1)x$ for $a \in X^+$ and $x \in X^-$. We claim that $\text{Ker} \, \mu = 0$. For, consider a $C$-irreducible component of $\text{Ker} \, \mu$. There are two cases:

1. The involution $t$ is in class $2B$ and $C \cong 2^{1+24}.Co_1$. The irreducible $C$-components of $X^+$ have dimensions 1, 299 and 98280. By the construction of the Griess algebra [Gri82], the 1-, 299- and 98280-dimensional component of $X^+$ act nontrivially on the 98304-dimensional component $X^-$. This proves the claim in the $2B$ case.

2. The involution $t$ is in class $2A$ and $C \cong 2.\mathfrak{B}$. The decomposition of the Griess algebra $\mathfrak{B}$ into irreducible components for the centralizer $2.\mathfrak{B}$ is shown in table [1] (cf. [Gri82, MN93]). The components of $X^+$ have dimensions 1, 1, 4371 and 96255. Again, it can easily been seen that the 4371- and the 96255-dimensional component act nontrivially on $X^-$. For example, one can use the
Table 1: Decomposition of the Griess algebra \( \mathfrak{B} \) under \( 2 \mathfrak{B} \)

| \( 2 \mathfrak{B} \) representations: | \( 1 \) | \( 1 \) | \( 1 \) | \( 96255 \) | \( 4371 \) | \( 96256 \) |
|---------------------|-----|-----|-----|-----|-----|-----|
| Eigenvalues of \( \mu(2e) \): | 2   | 0   | 0   | \( \frac{1}{2} \) | \( \frac{1}{2} \) | \( \frac{1}{16} \) |
| Eigenvalues of \( \tau \):      | +1  | +1  | +1  | +1  | -1  |

Parts of \( VB^2 \):

\[
VB_2^2 = \mathfrak{B'} \\
VB_{3/2}^2
\]

Virasoro frame decomposition of \( V^2 \) given in \[DGH98\] (for the notation, see the discussion before Proposition 3). Let \( F = \{\omega_1, \ldots, \omega_{48}\} \) be the Virasoro frame for \( V^2 \) as in \[DGH98\]. We can choose \( \omega_1 = 2e \). As seen from table I \( \omega_1 \) and \( \omega_1 + \cdots + \omega_{48} \) generate the trivial isotypical \( 2 \mathfrak{B} \)-module component of \( X^- \) and act by multiplication with \( 1/16 \) resp. 2 on \( X^- \). The element \( \omega_2 \) which is contained in the direct sum of the trivial with the 96255-dimensional component acts on \( X^- \). On the Viraroso submodule \( L(1/2, 1/16) \otimes L(1/2, 1/2) \otimes (L(1/2, 1/16) \otimes L(1/2, 0))^{\otimes 23} \) generated by a vector \( v \) in \( X^- \) the action of \( \omega_2 \) is multiplication with 1/2, on the Viraroso submodule \( L(1/2, 1/16) \otimes L(1/2, 0) \otimes L(1/2, 1/16) \otimes L(1/2, 1/2) \otimes (L(1/2, 1/16) \otimes L(1/2, 0))^{\otimes 22} \) generated by a vector \( v' \) in \( X^- \) the action of \( \omega_2 \) is multiplication with 0. Thus \( \mu(\omega_2) \not\in \mathfrak{C} \cdot \text{id}_{X^-} \) and we see that the action of the 96255-dimensional component must be nontrivial. For the 4371-dimensional component of \( X^+ \), choose again a Viraroso submodule \( L(1/2, 1/2) \otimes L(1/2, 1/2)^{\otimes 3} \otimes L(1/2, 0)^{\otimes 44} \) which is generated by a highest weight vector \( w \) in this component. For \( X^- \), choose a Viraroso submodule \( L(1/2, 1/16) \otimes L(1/2, 1/2) \otimes (L(1/2, 1/16) \otimes L(1/2, 0))^{\otimes 23} \) which is generated by a vector \( w' \) in \( X^- \). One sees from the fusion rules that the algebra product \( w \cdot w' \) is nontrivial. This finishes the proof of the claim in the \( 2A \) case.

Let \( U = \mathfrak{C} \cdot e \) be the \( \mathfrak{C} \)-submodule of \( X^+ \) generated by \( e \). Since \( e \) acts by multiplication with a non-zero scalar \( \lambda \) on \( X^+ \) (one has \( \lambda = \frac{1}{3} \) if \( t \) is of type \( \tau \) and \( \lambda = \frac{1}{3} \) if \( t \) is of type \( \sigma \) ), all elements of \( U \) are mapped by \( \mu \) to \( \mathfrak{C} \cdot \text{id}_{X^-} \subset \text{End}(X^-) \), and \( \mu(e) = \lambda \cdot \text{id}_{X^-} \) with \( \lambda \not= 0 \). It follows that \( U \) must be a trivial representation of \( \mathfrak{C} \). In case (1), this implies that \( e \) must be the identity of \( \mathfrak{B} \). Since the identity corresponds to the Viraroso element of central charge 24, this case is impossible. In case (2), we see that \( e \) is contained in the 2-dimensional trivial \( \mathfrak{C} \)-component \( \mathfrak{C} \oplus \mathfrak{C} \), and this implies that \( e \) must be an axis since this component is spanned by \( e \) and the identity of \( \mathfrak{B} \).

Denote with \( \mathfrak{B'} \) the subspace of \( VB^2 \) spanned by the vectors of conformal weight 2. The vertex operator algebra structure on \( VB^2_{(0)} \) induces an algebra structure on \( \mathfrak{B'} \).

**Lemma 4** The map which associates to an idempotent of \( \mathfrak{B'} \) generating a simple Virasoro vertex operator algebra of central charge \( \frac{1}{2} \) its Miyamoto involution
defines a bijection between the set of transposition axis contained in \( \mathcal{B}' \) and the class \( 2\mathcal{B} \) involutions of the Baby Monster.

**Proof.** Since, by construction, \( \mathcal{B}' \) is a subalgebra of \( \mathcal{B} \), every idempotent of \( \mathcal{B}' \) is also an idempotent of \( \mathcal{B} \). By Lemma 3, the idempotents which generate a simple Virasoro vertex operator algebra of central charge \( \frac{1}{2} \) must be transposition axes.

The set of \( 2\mathcal{A} \) involutions of \( \mathcal{M} \) decomposes under the conjugation action of the centralizer \( 2.\mathcal{B} \) of a fixed \( 2\mathcal{A} \)-involution \( \tau \) into nine orbits \([\text{Nor82, GMS89}]\). These orbits can be labeled by the Monster conjugacy class of the product of an involution in an orbit with the fixed \( 2\mathcal{A} \)-involution \( \tau \) corresponding to an axis \( e \). The orbit for the class \( 2\mathcal{B} \) corresponds to the axis of \( \mathcal{B} \) contained in the \( \mathcal{B}' \) component of \( \mathcal{B} \) (see [MN93], Th. 5). The stabilizer in \( \mathcal{B} \) of an involution in this orbit (the central element \( 2 \) in \( 2\mathcal{B} \) acts trivial) is \( 2^{1+22}.\text{Co}_2 \) (cf. [GMS89]) and the action of \( \mathcal{B} \) on the orbit can be identified with the conjugation action of \( \mathcal{B} \) on the class of \( 2\mathcal{B} \) involutions of \( \mathcal{B} \).

Using the map \( 2.\mathcal{B} = \text{Cent}(\tau) = \text{Stab}_\mathcal{M}(e) \rightarrow \text{Aut}(VB^3_{(0)}), \ g \mapsto g|_{\text{Aut}(VB^3_{(0)})} \), with kernel \( \langle \tau \rangle \), we see that the involution \( \tau' \) on \( VB^3_{(0)} \) associated to an idempotent \( e \) in \( \mathcal{B}' \subset \mathcal{B} \) is the restriction to \( VB^3_{(0)} \) of the associated involution \( \tau \) of \( V^3 \).

**Proposition 5** The automorphism group of the algebra \( \mathcal{B}' \) is the Baby Monster.

**Proof.** Let \( G = \text{Aut}(\mathcal{B}') \geq \mathcal{B} \). Since all the idempotents \( e \) of \( \mathcal{B}' \) generating a central charge \( \frac{1}{2} \) Virasoro algebra correspond to the class \( 2\mathcal{B} \) involutions of \( \mathcal{B} \), the conjugation action of \( G \) leaves the set of \( 2\mathcal{B} \) involutions invariant. Since \( \mathcal{B} \) is generated by its \( 2\mathcal{B} \) involutions it is a normal subgroup of \( G \). Since \( \text{Aut}(\mathcal{B}) = \mathcal{B} \) (cf. [CCN+85]), we see \( G = A.\mathcal{B} \) and \( A \) centralizes \( \mathcal{B} \). Recall from Table \( \mathcal{B} \) that \( \mathcal{B}' = 1 \oplus 96255 \) as \( \mathcal{B} \)-module. Since \( G \) fixes the Virasoro element \( \omega \) of \( VB^3_{(0)} \), this is also a decomposition of \( G \)-modules. Let \( a \in A \). Then \( a \) must act by a scalar \( \lambda \) on the 96255-dimensional component. We write an idempotent \( e \) as above in the form \( e = x \cdot \omega + u \) where \( \omega \) is the Virasoro element and \( u \) is in the 96255-dimensional component. Clearly, \( u \neq 0 \). We obtain \( a(e) = x \cdot \omega + \lambda u \) since \( a \) fixes \( \omega \). As \( e = a(e) \) it follows that \( \lambda = 1 \). Thus \( a = 1 \) and \( A \) is trivial.

**Remark:** In a similar way, one could prove that \( \text{Aut}(\mathcal{B}) = \mathcal{M} \). However, the proof of Lemma already assumes that the Monster is the whole automorphism group of \( \text{Aut}(\mathcal{B}) \).

For proving that \( VB^3_{(0)} \) is generated by the subspace \( \mathcal{B}' \), we recall some results about framed vertex operator algebras from [DCH98]. A subset \( F = \{\omega_1, \ldots, \omega_r\} \) of a simple vertex operator algebra \( V \) is called a Virasoro frame...
if the $\omega_i$ for $i = 1, \ldots, r$ generate mutually commuting simple Virasoro vertex operator algebras $L(1/2, 0)$ of central charge $1/2$ and $\omega_1 + \cdots + \omega_r$ is the Virasoro element of $V$. Such a vertex operator algebra $V$ is called a framed vertex operator algebra. Under $L(1/2, 0)^{\otimes r}$, the vertex operator subalgebra spanned by the Virasoro frame, $V$ decomposes as

$$V = \bigoplus_{h_1, \ldots, h_r \in \{0, 1/2, 1/16\}} n_{h_1, \ldots, h_r} L(h_1, \ldots, h_r),$$

into isotypical components of modules $L(h_1, \ldots, h_r) = L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_r)$. Using this decomposition, one defines the two binary codes

$$C = \{ c \in F_2^r \mid n_{c_1/2, \ldots, c_r/2} \neq 0 \}$$

and

$$D = \{ d \in F_2^r \mid \sum_{h_1, \ldots, h_r} n_{h_1, \ldots, h_r} \neq 0 \},$$

where the sum in the definition of $D$ runs over those $h_i$ with $h_i \in \{0, 1/2\}$ if $d_i = 0$ and $h_i = 1/16$ if $d_i = 1$. It was shown in [DGH98] that both codes are linear, $D \subset C^\perp$ ($C^\perp$ denotes the code orthogonal to $C$), $C$ is even, and all weights of vectors in $D$ are divisible by $8$. For $c \in C$, we denote the sum of all irreducible modules isomorphic to $L(c_1/2, \ldots, c_r/2)$ by $V(c)$; for $I \in D$, we denote the sum of all irreducible modules isomorphic to $L(h_1, \ldots, h_r)$ such that $h_i \in \{0, 1/2\}$ if $I_i = 0$ and $h_i = 1/16$ if $I_i = 1$ for $i = 1, \ldots, r$ by $V^I$.

**Proposition 6** The even part $\text{VB}_2^{(0)}$ of the shorter Moonshine module $\text{VB}_2^3$ is generated by $\mathcal{B}'$, the subspace of $\text{VB}_2^3$ consisting vectors of conformal weight 2.

**Proof.** The Moonshine module $V^3$ is a framed vertex operator algebra of central charge 24 (cf. [DMZ94]). To any Virasoro frame in $V^3$ there are the two associated binary codes $C$ and $D$ of length 48. For the construction of the shorter Moonshine module, one has to fix a simple central charge $1/2$ Virasoro vertex operator subalgebra generated by an axis. It follows from Lemma 8 that all such vertex operator subalgebras are equivalent, one can choose an arbitrary $L(1/2, 0)$ of a given Virasoro frame in $V^3$ and the remaining 47 factors $L(1/2, 0)$ define a Virasoro frame of $\text{VB}_2^{(0)}$. The associated binary codes $C'$ and $D'$ for the framed vertex operator algebra $\text{VB}_2^{(0)}$ are obtained from the codes $C$ resp. $D$ by taking those vectors of $C$ resp. $D$ which have the value 0 at the coordinate corresponding to the first chosen $L(1/2, 0)$ and then omitting this coordinate.

We fix now the Virasoro frame of $V^3$ studied in [DGH98], Sect. 5, Example II. The code $D$ has in this case the generator matrix

$$
\begin{pmatrix}
1111111111111111 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 1111111111111111 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111111111111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 1111111111111111 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 1111111111111111 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0000000000000000 & 0000000000000000 & 0000000000000000 & 0000000000000000 \\
0101010101010101 & 0101010101010101 & 0101010101010101 & 0101010101010101 \\
0101010101010101 & 0101010101010101 & 0101010101010101 & 0101010101010101 \\
0101010101010101 & 0101010101010101 & 0101010101010101 & 0101010101010101 \\
0101010101010101 & 0101010101010101 & 0101010101010101 & 0101010101010101
\end{pmatrix}
$$

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and $C$ is the dual code of $D$, i.e., $C = \{ c \in \mathbf{F}_2^{48} \mid \sum_{i=1}^{48} c_i \cdot d_i = 0 \text{ for all } d \in D \}$ (Theorem 5.8 of [DGH98]). In particular, $C$ is even. It was shown in [DGH98] that $C$ has minimal weight 4.

We claim that the codewords of weight 4 in $C$ span the code: The 48 co-ordinates of $C$ are naturally partitioned into three blocks of 16 coordinates by the first three rows of the above matrix. Let $c$ be a codeword of $C$ of weight 6 or larger. By the pigeon hole principle, we can find two blocks such that the support of $c$ has at least four coordinates at the 32 coordinates of these two blocks. The codewords of $C$ having the value 0 at the remaining third block can be identified with the extended Hamming code $H_{32}$ of length 32 since its dual code is the extended simplex code of that length whose generator matrix equals the restriction of $D$ to the two blocks. The code $H_{32}$ has the property that for any set of three of its 32 coordinates there is a codeword $v$ of weight 4 in $H_{32} \subset C$ containing this set of coordinates in its support, i.e., the codewords of weight 4 define a Steiner system $S(3, 4, 32)$ (cf. [MC77], Ch. 2, Thm. 15). By choosing the three coordinates from the support of $c$ contained in the two selected blocks, we see that $c + v$ is a codeword of $C$ of smaller weight then the weight of $c$. This implies the claim.

Now we claim that the codewords of weight 4 in $C'$ span $C'$: By using the claim for $C$, we see that a codeword $c$ of $C$ with value 0 at a fixed coordinate is the sum of codewords in $C$ of weight 4 such that in the sum an even number of the codewords have the value 1 at the fixed coordinate. The sum $d$ of two different weight 4 codewords having the value 1 at the fixed coordinate is a codeword of weight 4 or 6. We must show that $d$ is the sum of weight 4 codewords of $C'$. So if $d$ has weight 4 we are done. If $d$ has weight 6, let $i \in \{0, 1, \ldots, 6\}$ be the number of coordinates of the support of $d$ which are in the same block as the distinguished coordinate used in the definition of $C'$.

For $i \leq 3$, the support of $d$ contains $6 - i \geq 3$ coordinates in the two other blocks. In this case we can choose a weight 4 vector $v$ as above with support in these two blocks and obtain with $v$ and $d + v$ two weight 4 vectors in $C'$ with sum $d$.

For $i \geq 4$, one observes from the definition of $C$ that the codewords of $C$ having the value 0 at the two other block can be identified with the extended Hamming code $H_{16}$ of length 16: its dual code is the extended simplex code of that length whose generator matrix equals the restriction of $D$ to the remaining block. We choose now in two different ways 3 coordinates out of a set of 4 coordinates contained in the $i \geq 4$ coordinates which form the intersection of the support of $d$ with the block containing the distinguished coordinate. Since the weight 4 codewords of $H_{16}$ form a Steiner system $S(3, 4, 16)$, there are two codewords $v$ and $v'$ of weight 4 in $H_{16}$ containing these two sets of 3 coordinates in their support. Not both of them can have the value 1 at the distinguished coordinate since otherwise their sum would have weight 2. So we have found with $v$ and $d + v$ or with $v'$ and $d + v'$ two vectors of weight 4 in $C'$ with sum $d$ and we are done again.
It is clear that $D$ and $D'$ are both generated by their codewords of weight 16 and 24.

In [DGH98], Prop. 2.5. (5) and (6), it was shown that for a framed vertex operator algebra $V$ the module $V(c)$ generated by the isotypical component corresponding to a codeword $c$ in the associated binary code $C$ is irreducible and the span of the images of the maps from $V(c) \times V(c')$ to $V(c + c')$, $c, c' \in C$, defined by the components of the vertex operator generate $V(c + c')$. Since the Virasoro frame is contained in $V_2$ and the conformal weight of a Virasoro highest weight vector in $V(c)$ for $c \in C$ is half the weight of $c$, it follows from the above discussion that the vertex operator subalgebra $V_0 = \bigoplus_{c \in C} V(c)$ for $V$ equal to $V^\natural$ resp. $V_{B}^\natural(0)$ is generated by $B$ resp. $B'$. Similarly, using [DGH98], Prop. 2.5. (1) and (2), one has: $V = \bigoplus_{I \in D} V^I$, the $V^I$ are irreducible $V^0$-modules, and the span of the images of the maps from $V^I \times V^{I'}$ to $V^{I + I'}$, $I, I' \in D$, defined by the components of the vertex operator generate $V^{I + I'}$. By direct inspection of the Virasoro decomposition of the moonshine module given explicitly in [DGH98], one finds for any codeword $I$ of weight 16 or 24 in $D$ a Virasoro highest weight vector in $(V^\natural)^2 I$. For example, there is the Virasoro highest weight module $L(1/2, 15, 1/2, 15, (1/16))$. (In more detail, one can use the decomposition polynomial given in [DGH98], Cor. 5.9 and observe that the situation is symmetric for all codewords in $D$ of fixed weight.)

For $V_{B}^\natural(0)$, one observes that for all $I \in D'$ the Virasoro highest weight module can be chosen to have conformal weight 0 at the fixed coordinate. (Again, this can be seen from the decomposition polynomial.)

Putting the above arguments together, we have proven that $V_{B}^\natural(0)$ is generated by $B'$.

**Remark:** The proof of Prop. 6 also shows that the Moonshine module $V^\natural$ is generated by the Griess algebra $B$, a result first proven in [FLM88], Prop. 12.3.1 (g) by using similar ideas.

Theorem 1 follows now from Prop. 5 and Prop. 6 as explained in [Höh95], Th. 4.2.7: Any automorphism $g$ of $V_{B}^\natural$ must fix the even and odd components $V_{B}^\natural(0)$ and $V_{B}^\natural(1)$, i.e., can be written in the form $g = g_0 \oplus g_1$ and one has $g_0 \in \text{Aut}(V_{B}^\natural(0)) = B$. Let $g' = g_0 \oplus g_1'$ be another automorphism with $g_0 = g_0'$. Since $V_{B}^\natural(1)$ is an irreducible $V_{B}^\natural(0)$-module ([Höh95], Th. 4.2.2) and $g^{-1}g' = \text{id} \oplus g_1^{-1}g_1'$ is a $V_{B}^\natural(0)$-module homomorphism, there is a scalar $s$ such that $g' = g_0 \oplus s \cdot g_1$. Since $V_{B}^\natural$ is a vertex operator superalgebra, one gets $s = \pm 1$.

**References**

[Bor86] R. E. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Natl. Acad. Sci. USA 83 (1986), 3068–3071.
[CCN⁺85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *ATLAS of finite groups*, Clarendon Press, Oxford, 1985.

[Con84] J. H. Conway, *A simple construction of the Fischer-Griess Monster group*, Invent. Math. 79 (1984), 513–540.

[DGH98] Chongying Dong, Robert Griess, and Gerald Höhn, *Framed Vertex Operator Algebras, Codes and the Moonshine Module*, Comm. Math. Phys. 193 (1998), 407–448, [arXiv:q-alg/9707008].

[DMZ94] Chongying Dong, Geoffrey Mason, and Yongchang Zhu, *Discrete series of the Virasoro algebra and the moonshine module*, Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991), pp. 295–316, Proc. Sympos. Pure Math., 56, Part 2, Amer. Math. Soc., Providence, RI, 1994.

[FLM88] Igor Frenkel, James Lepowsky, and Arne Meuerman, *Vertex Operator Algebras and the Monster*, Academic Press, San Diego, 1988.

[GMS89] Robert L. Griess, Ulrich Meierfrankenfeld, and Yoav Segev, *A uniqueness proof for the Monster*, Annals of Mathematics 130 (1989), 567–602.

[Gri82] R. L. Griess, *The friendly giant*, Invent. Math. 69 (1982), 1–102.

[Höh95] Gerald Höhn, *Selbstduale Vertexoperatorsuperalgebren und das Babymonster*, Ph.D. thesis, Universität Bonn, 1995, see: Bonner Mathematische Schriften 286; also available as [arXiv:0706.0236](http://arxiv.org/abs/0706.0236).

[LS77] Jeffrey S. Leon and Charles C. Sims, *The Existence and Uniqueness of a simple group generated by *{3,4}*-Transpositions*, Bull. Amer. Math. Soc. 83 (1977), 1039–1040.

[McS77] F. J. MacWilliams and N. Sloane *The Theory of Error Correcting Codes*, Elsevier, Amsterdam, 1977.

[Miy96] Masahiko Miyamoto, *Griess Algebras and Conformal Vectors in Vertex Operator Algebras*, Journal of Algebra 179 (1996), 523–548.

[Miy04] Masahiko Miyamoto, *A new construction of the Moonshine vertex operator algebra over the real number field*, Ann. of Math. (2) 159 (2004), 535–596.

[MN93] Werner Meyer and Wolfram Neutsch, *Associative Subalgebras of the Griess Algebra*, Journal of Algebra 158 (1993), 1–17.

[Nor82] S. P. Norton, *The uniqueness of the Monster*, Contemp. Math. Vol. 45, Amer. Math. Soc., Providence, 1982, pp. 271–285.

[Smi79] Stephen D. Smith, *Large Extraspecial Subgroups of Widths 4 and 6*, Journ. of Algebra 58 (1979), 251–281.

[Tit84] J. Tits, *On R. Griess’ “Friendly giant”*, Invent. Math. 78 (1984), 491–499.

[Yam02] Hiroshi Yamauchi, *On Z²-twisted representation of vertex operator superalgebras and the Ising model SVOA*, preprint, [math.QA/0203086](http://arxiv.org/abs/math.QA/0203086) (2002).