HEREDITARILY NON-PYTHAGOREAN FIELDS

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Abstract. We prove for a large class of fields $F$ that every proper finite extension of $F_{pyth}$, the pythagorean closure of $F$, is not a pythagorean field. This class of fields contains number fields and fields $F$ that are finitely generated of transcendence degree at least one over some subfield of $F$.

1. Introduction

A field of characteristic different from 2 is called pythagorean if every sum of squares is a square, and is called quadratically closed if every element is a square. A field is quadratically closed if and only if the field is pythagorean and nonreal.

All fields that are not residue fields of valuations are assumed to have characteristic different from 2. Let $E_{pyth}$ be the direct limit of finite extensions of $E$ that consist of chains of quadratic extensions, obtained by iteratively adjoining square roots (inside a fixed algebraic closure of $E$) of sums of two squares in the predecessor. The field $E_{pyth}$ is called the pythagorean closure of $E$, and is a pythagorean field. If $E$ is nonreal, then the field $E_{pyth}$ is the quadratic closure of $E$. When $E$ is a real field and we discuss results that apply to both the pythagorean closure and quadratic closure of $E$, then we often write $E_0$ to denote either of these fields.

We say that a field $k$ is hereditarily non-pythagorean if every proper finite extension $F$, with $F/k$ not purely inseparable, is non-pythagorean. By [La-05, Chapter VIII, Theorem 5.7], every non-pythagorean field is hereditarily non-pythagorean. The notion is thus only interesting for pythagorean fields. Similarly, a field $k$ is hereditarily non-quadratically closed if every proper finite extension $F$, with $F/k$ not purely inseparable, is not quadratically closed. By [La-05, Chapter VIII, Corollary 5.11] and its proof, every nonreal, non-quadratically closed field is hereditarily non-quadratically closed. Again, the notion is interesting only for quadratically closed fields. We exclude the case when $F/k$ is purely inseparable because by Lemma 4.1 if $k$ is quadratically closed and $F/k$ is a finite purely inseparable extension, then $F$ is also quadratically closed.

The main results of this paper (Theorem 3.4 and Theorem 5.3) are that if $E$ is either an algebraic number field (a finite extension of $\mathbb{Q}$) or finitely generated of transcendence degree at least one over some subfield, then $E_{pyth}$ is hereditarily non-pythagorean.

Previously, a version of this problem was studied for the quadratic closure of a number field. It was shown in [La-05, Chapter VII, Corollary 7.11] that if $F$ is the quadratic closure of a number field $k$, then every proper finite extension of $F$ has

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ininitely many square classes. In particular, $F$ is hereditarily non-quadratically closed.

We further discuss the cases when $E$ is an infinite dimensional algebraic extension of a number field or an infinite dimensional algebraic extension of a function field. We prove in Theorem 6.1 that the same result as above holds under the additional assumption that $E$ is a Galois extension over some number field or over some function field. In Example 6.2, we construct a simple example of a nonreal infinite number field $E$ that is not quadratically closed and not Galois over some number field, and such that every finite extension of $E_{pyth}$ is quadratically closed. We prove some results in Section 7 that give a construction of a real infinite number field whose pythagorean closure is not hereditarily non-pythagorean.

Hereditary properties of quadratic closures and pythagorean closures have been studied for other fields. In [La-05, Chapter VII, Theorems 7.17, 7.18], it was shown that if $F$ is the quadratic closure of a local field $k$ then $F$ is hereditarily quadratically closed if $k$ is nondyadic, and is hereditarily non-quadratically closed if $k$ is dyadic. Becker wrote an extensive treatment of hereditarily pythagorean fields in [B-78].

We let $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the field of rational numbers, respectively. A number field is a finite algebraic extension of $\mathbb{Q}$. An infinite number field will mean an infinite algebraic extension of $\mathbb{Q}$. A local field is a completion of a number field with respect to some nonarchimedean absolute value. A local field is dyadic if its residue field has characteristic 2, and is nondyadic if its residue field has characteristic different from 2. For a prime number $p$, we let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ with respect to the $p$-adic absolute value.

For a field $K$, we let $K^\times = K \setminus \{0\}$ and we let $v$ denote the characteristic of $K$. We let $K^2$ denote the set of squares of elements of $K$, and we let $\sum K^2$ denote the set of sums of squares of elements in $K$. Let $(\sum K^2)^\times = (\sum K^2) \setminus \{0\}$.

The pythagoras number of a field $K$, written $p(K)$, is defined as the smallest integer $n$ such that each element in $\sum K^2$ can be written as a sum of $n$ squares of elements in $K$. If no such integer exists, we set $p(K) = \infty$.

A field $K$ is nonreal if $-1 \in \sum K^2$. Otherwise, a field is real. A field $K$ is real if and only if $K$ has an ordering by [La-05, Chapter VIII, Theorem 1.10]. If $K$ is nonreal and char $K \neq 2$, then $K = \sum K^2$. A field $K$ is pythagorean if $\sum K^2 = K^2$, and is quadratically closed if $K = K^2$. Thus if $K$ is a pythagorean field, then $p(K) = 1$.

We use [La-05] as a standard reference for other undefined terms and standard results.

2. Some general results

In this section we establish some general results and strategies that allow us to prove that certain finite extensions of pythagorean closures are not pythagorean. One of the tools we use is valuation theory. We will say that $v$ is a nonreal valuation on a field $K$ if the residue field of $(K, v)$ is a nonreal field.

**Lemma 2.1.** Let $K$ be a field with char $K \neq 2$. Let $v$ be a nonreal discrete valuation on $K$ with value group $\mathbb{Z}$. Then there exists an element $a \in \sum K^2$ with $v(a) = 1$.

**Proof.** Let $R$ be the valuation ring of $(K, v)$, $m$ the maximal ideal, $\pi$ the uniformizer, and $k_v$ the residue field of $v$. Since $k_v$ is nonreal, there is a nontrivial representation $-1 = x_1^2 + \cdots + x_s^2$ where each $x_i \in R$ and $x_i \in k_v$. Then $1 + x_1^2 + \cdots + x_s^2 \in \sum K^2$. 


m. First suppose that char $\kappa_v \neq 2$. Then either $v(1 + x_1^2 + \cdots + x_4^2) = 1$ or $v((1 + \pi)^2 + x_1^2 + \cdots + x_4^2) = 1$.

Now assume that char $\kappa_v = 2$. Let $K_v$ denote the completion of $K$ with respect to the metric induced by $v$. Then $K_v$ is a nonreal field because $\kappa_v$ is nonreal. To see this, consider the equation $2^2 + 1^2 + 1^2 + 1^2 + 1^2 = 2^3$. Suppose that $(2) = (\pi^e)$. We can apply Hensel’s lemma to the quadratic form $f = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$ because $f(2, 1, 1, 1, 1) \equiv 0 \bmod \pi^{3e}$, $f'(2, 1, 1, 1, 1) = 2 \neq 0 \bmod \pi^{e+1}$, and $3e \geq 2e + 1$. (For example, see [La-05, Chapter VI, Theorem 2.18].) Thus $f$ is isotropic over $K_v$. Then $K_v$ is a nonreal field and every element of $K_v$ is a sum of squares of elements from $K$. In particular, $\tau$ is a sum of squares of elements from $K_v$. Since $K$ is dense in $K_v$, it follows that $K$ contains an element $a \in \sum K^2$ such that $v(a) = 1$. □

**Lemma 2.2.** Let $K$ be a field and let $v_1, \ldots, v_n$ be a finite number of distinct nonreal discrete valuations on $K$ each with value group $\mathbb{Z}$. Then the induced homomorphism of groups

$$\tau : \left( \sum K^2 \right)^n \to \mathbb{Z}^n$$

given by $\sigma \mapsto (v_1(\sigma), \ldots, v_n(\sigma))$ is surjective.

**Proof.** It is sufficient to show that the canonical $\mathbb{Z}$-basis of $\mathbb{Z}^n, \{e_1, \ldots, e_n\}$, is contained in the image. We first show that $(2\mathbb{Z})^n$ is contained in the image. Let $(2b_1, \ldots, 2b_n) \in (2\mathbb{Z})^n$ be given. By the weak approximation theorem for discrete valuations, there exists $a \in K^\times$ such that $(v_1(a), \ldots, v_n(a)) = (b_1, \ldots, b_n)$. Then $\tau(a^2) = (2b_1, \ldots, 2b_n)$.

By Lemma 2.1, there exists $y \in \sum K^2$ such that $v_1(y) = 1$. Now choose $m_1, \ldots, m_n \in \mathbb{Z}$ such that $m_1 > \frac{1}{2}v_1(y)$ and $m_i < \frac{1}{2}v_i(y)$ for $2 \leq i \leq n$.

By weak approximation, there exists $z \in K$ such that $v_i(z) = m_i$ for $1 \leq i \leq n$. Then $v_1(y + z^2) = v_1(y) = 1$ and $v_i(y + z^2) = v_i(z^2) \in 2\mathbb{Z}$ for all $2 \leq i \leq n$. Then $\tau(y + z^2) \in e_1 + (2\mathbb{Z})^n$. Since $(2\mathbb{Z})^n \subset \text{im}(\tau)$, it follows that $e_1 \in \text{im}(\tau)$. The same holds for each other basis element $e_i$, $i \geq 2$. □

**Lemma 2.3.** Let $K/E$ be a proper finite field extension of degree $n$. Suppose there exists a nontrivial, nonreal discrete valuation $v$ on $E$ that extends to $n$ distinct valuations on $K$. Then there exists $\sigma \in \left( \sum K^2 \right) \setminus EK^2$.

**Proof.** Let $w_1, \ldots, w_n$ denote the $n$ distinct valuations on $K$ that extend $v$. We may assume that the value group of each $w_i$ is equal to $\mathbb{Z}$. By Lemma 2.2, there exists $\sigma \in \sum K^2$ such that $w_i(\sigma)$ is odd and $w_i(\sigma)$ is even for $i \geq 2$. Suppose that $\left( \sum K^2 \right) \subseteq EK^2$. Then we can find $\sigma$ with the same properties and such that $\sigma \in E$. But this yields a contradiction, since on the one hand $w_i(\sigma) = v(\sigma) = w_i(\sigma)$ for $i \geq 2$, and on the other hand $w_i(\sigma) \neq w_i(\sigma)$ for $i \geq 2$. □

Remark: The proof above requires only that $v$ has at least two distinct unramified extensions on $K$. In fact, P. Gupta observed that the two extensions do not even need to be unramified. However, in view of our applications later, it is convenient to assume the stronger hypothesis that $v$ splits completely in $K$.

**Proposition 2.4.** Let $E/F$ be a finite extension of fields with $E$ is pythagorean. Then $F$ is pythagorean.

**Proof.** See [La-05] Chapter VIII, Theorem 5.7 □
Let $L/F$ be any finite algebraic extension. Then there is a unique field $K$ such that $F \subseteq K \subseteq L$ where $K/F$ is separable and $L/K$ is purely inseparable. We use the convention that a field is purely inseparable over itself.

The following result gives a simpler criterion for a field to be hereditarily non-pythagorean

**Lemma 2.5.** A field $k$ is hereditarily non-pythagorean if and only if every proper finite separable extension $F$ of $k$ is non-pythagorean.

**Proof.** Assume that every proper finite separable extension of $k$ is non-pythagorean. Let $L$ be a proper finite extension of $k$ that is not purely inseparable over $k$. There is a subfield $F$ of $L$ satisfying $k \subseteq F \subseteq L$ such that $F/k$ is a proper separable extension and $L/F$ is a purely inseparable extension. By hypothesis, $F$ is non-pythagorean. By Proposition 2.4, $L$ is non-pythagorean. Thus $k$ is hereditarily non-pythagorean. □

**Proposition 2.6.** Let $F$ be a field. Assume that for every proper finite separable extension $K/F$ and for each chain of fields $F \subseteq \tilde{F} \subseteq K$, that $\sum K^2 \not\subseteq \tilde{F} K^2$.

Let $E_0/F$ be a (possibly infinite) Galois extension and assume that $E_0$ is either a pythagorean field or a quadratically closed field. Then $E_0$ is hereditarily non-pythagorean.

**Proof.** Let $L$ be a proper finite separable extension of $E_0$. Let $\beta \in L$ be a primitive element for $L/E_0$. The irreducible polynomial of $\beta$ over $E_0$ is defined over a finite extension $\tilde{F}/F$ contained in $E_0$. Let $K = \tilde{F}(\beta)$. Then $\tilde{F} \subseteq K$. Note that $L = E_0 K$ and that the fields $E_0$ and $K$ are linearly disjoint over $\tilde{F}$. Since $E_0/\tilde{F}$ is a Galois extension, it follows that $L/K$ is a Galois extension. We have an isomorphism $Gal(L/K) \rightarrow Gal(E_0/\tilde{F})$ given by $\varphi \mapsto \varphi|_{E_0}$.

By hypothesis, there exists $\sigma \in \sum K^2$ with $\sigma \not\in \tilde{F} K^2$. We will show that $\sigma \not\in L^2$. Suppose on the contrary that $K(\sqrt{\delta}) \subseteq L$. Then by Galois theory, there exists a quadratic extension $\tilde{F}(\sqrt{\delta})/\tilde{F}$ with $\delta \in \tilde{F}$ such that $K(\sqrt{\delta}) = K(\sqrt{\sigma})$. This gives $\delta \sigma \in K^2$. Then $\sigma \in \delta K^2 \subseteq \tilde{F} K^2$, which is a contradiction. Hence, $\sigma \in \sum K^2 \subseteq \sum L^2$, but $\sigma \not\in L^2$. Thus $L$ is not pythagorean and so $E_0$ is hereditarily non-pythagorean by Lemma 2.5. □

**Proposition 2.7.** Let $F$ be a field and assume that for each proper finite separable extension $K/F$ there is a nontrivial, nonreal discrete valuation $v$ on $F$ that splits completely in $K$.

Let $E_0/F$ be a (possibly infinite) Galois extension and assume that $E_0$ is either a pythagorean field or a quadratically closed field. Then $E_0$ is hereditarily non-pythagorean.

**Proof.** Let $K/F$ be a proper finite separable extension and let $F \subseteq \tilde{F} \subseteq K$ be a chain of fields. The hypothesis implies that there is a nontrivial, nonreal discrete valuation $v$ on $\tilde{F}$ that splits completely in $K$. Lemma 2.3 implies that $\sum K^2 \not\subseteq \tilde{F} K^2$. The result now follows from Proposition 2.6. □

3. Pythagorean and quadratic closures of number fields

In this section we deal with the case of number fields. The key ingredient for the main result (Theorem 3.4) is based on the following result.
Proposition 3.1. Let $K/E$ be a proper finite extension of number fields. Then $\sum K^2 \not\subset EK^2$.

This result follows easily from \cite[Chapter VII, Theorem 7.12]{La-05}, which states (with an easily corrected typo) that $K^x/E^x(K^x)^2$ is an infinite group. Since $K$ has finitely many orderings $P_1, \ldots, P_n$ and $P_1 \cap \cdots \cap P_n = \sum K^2$, it follows that $K^x/(\sum K^2)^x$ injects into $(K^x/P_1^x) \times \cdots \times (K^x/P_n^x)$, and therefore $|K^x/(\sum K^2)^x| \leq 2^n$. If $\sum K^2 \subset EK^2$, then $|K^x/E^x(K^x)^2| \leq |K^x/(\sum K^2)^x| \leq 2^n$, a contradiction. Thus $\sum K^2 \not\subset EK^2$.

The proof of \cite[Chapter VII, Theorem 7.12]{La-05} is based on the number theoretic result that in a finite extension of number fields there exist infinitely many prime ideals that do not remain prime in the extension field. We use this opportunity to give an alternative proof of Proposition 3.1 which reduces to the local dyadic ideals that do not remain prime in the extension field. We use this opportunity to give an alternative proof of Proposition 3.1 which reduces to the local dyadic case, and where no result on the existence of splitting primes is needed. Instead we require only the following result.

Lemma 3.2. Let $E/Q_2$ be a finite field extension and $K/E$ a proper finite extension. Then $E^x(K^x)^2 \not\subset K^x$.

Proof. Let $[E : Q_2] = m$ and $[K : E] = n > 1$, and thus $[K : Q_2] = mn$. By \cite[Chapter VI, Corollary 2.23]{La-05}, we have

$$|E^x/(E^x)^2| = 2^{m+2} < 2^{mn+2} = |K^x/(K^x)^2|.$$  

This gives

$$|E^x(K^x)^2/(K^x)^2| = |E^x/(E^x \cap (K^x)^2)| \leq |E^x/(E^x)^2| < |K^x/(K^x)^2|.$$  

\[\square\]

Alternative proof of Proposition 3.1. Let $K/E$ be a proper finite extension of number fields. Let $p$ be a dyadic prime ideal in the ring of integers of $E$. Let $E_p$ be the completion of $E$ with respect to $p$. If the hypotheses of Lemma 2.3 hold, then $\sum K^2 \not\subset EK^2$. If the hypotheses of Lemma 2.3 do not hold, then there exists a dyadic prime ideal $P$ in the ring of integers $K$ such that $p = E \cap P$ and $[K_P : E_p] > 1$, where $K_P$ the completion of $K$ with respect to $P$. Then Lemma 3.2 implies that there exists $\sigma_p \in K_P^x \setminus E_p^x(K_P^x)^2$.

Since $K_P$ is nonreal, we have $K_P^x = (\sum K_P^2)^x$. Thus $\sigma_p \in (\sum K_P^2)^{x} \setminus E_p^x(K_P^x)^2$. Let $\sigma_{p,i} = x_{P,1}^{2} + \cdots + x_{P,m}^{2}$ for some $m \in \mathbb{N}$ and $x_{P,i} \in K_P$. For every $i \in \{1, \ldots, m\}$ let $(x_{P,i}^{(t)})_{t \in \mathbb{N}}$ be a sequence in $K$ that converges to $x_{P,i}$ in the $P$-adic metric. Let $\sigma_{p,i}^{(t)} = x_{P,1}^{2} + \cdots + x_{P,m}^{2}(t)$. Then $(\sigma_{p,i}^{(t)})_{t \in \mathbb{N}}$ is a sequence in $\sum K^2$ that converges to $\sigma_p$ in the $P$-adic metric. Since $\sigma_p \neq 0$, for $\ell$ sufficiently large, $(\sigma_{p,i}^{(t)})_{t \in \mathbb{N}}$ is a sequence in $(\sum K^2)^x$. We finish the proof by showing that for $\ell$ sufficiently large, $\sigma_{p,i}^{(t)}$ is not contained in $E_{p,\ell}^x K_P^{x^2}$, which contains $E^x K_P^{x^2}$ as a subset. This will follow easily after we show that $E_{p,\ell}^x K_P^{x^2}$ is a closed subset of $K_P^x$ in the $P$-adic metric.

So, let $(z_n)_{n \in \mathbb{N}}$ be a sequence in $E_{p,\ell}^x K_P^{x^2}$ that converges to some $z \in K_P^x$. For every $n \in \mathbb{N}$ there exist $c_n \in E_{p,\ell}^x$ and $x_n \in K_P^x$ such that $z_n = c_n x_n^{2}$. After possibly multiplying $c_n$ with a square in $E_{p,\ell}^{x^2}$ and $x_n^2$ with its inverse, if necessary, we can arrange that the sequence $(c_n)_{n \in \mathbb{N}}$ is bounded in the $P$-adic metric, and since $(z_n)_{n \in \mathbb{N}}$ converges to a non-zero element and hence is also bounded, we conclude that $x_n$ is also a bounded sequence. Since closed balls in the $P$-adic metric are
compact, every sequence in a closed ball has a convergent subsequence. Then there exist subsequences \((e_{n_k})_{k \in \mathbb{N}}\) and \((x_{n_k})_{k \in \mathbb{N}}\) that converge to some element \(e \in E_p\) and \(x \in K_p\), and thus \(z_{n_k} = e_{n_k}^2 x_{n_k}^2\) converges to \(ex^2\) for \(k \to \infty\). It follows that 
\[
z = ex^2,
\]
and thus \(z \in E_p K_p^×\), showing that \(E_p K_p^×\) is a closed subset of \(K_p^×\). \(\square\)

**Lemma 3.3.** Let \(K\) be a number field and let \(E_0\) be an infinite dimensional number field that is either pythagorean or quadratically closed. Let \(p\) be a prime ideal of \(K\) and \(K_p\) the \(p\)-adic completion of \(K\). Let \(E_0K_p\) be a compositum in some algebraic closure of \(K_p\). Then \(E_0K_p\) is an infinite algebraic extension of \(K_p\).

**Proof.** Let \(p = \mathbb{Q} \cap p\). Consider the following two chains of fields.

\[
\mathbb{Q}_p \subseteq E_0 \mathbb{Q}_p \subseteq E_0 K_p
\]
\[
\mathbb{Q}_p \subseteq K_p \subseteq E_0 K_p
\]

We can use Lemma 2.4 to show that the value group of any extension of the \(p\)-adic valuation to \(E_0\) is 2-divisible, but the extension of the \(p\)-adic valuation to \(K_p\) is discrete. Therefore \([E_0 \mathbb{Q}_p : \mathbb{Q}_p] = \infty\) and \([K_p : \mathbb{Q}_p]\) is finite. It follows that 
\[
[E_0 K_p : K_p] = \infty.
\]

\(\square\)

We now prove the main result of this section.

**Theorem 3.4.** Let \(F\) be a number field and let \(E_0/F\) be an infinite Galois extension. Assume that \(E_0\) is either a pythagorean field or a quadratically closed field. Then \(p(L) = 2\) for every proper finite extension \(L/E_0\). In particular, \(E_0\) is hereditarily non-pythagorean.

**Proof.** First we show that \(L^2 \subseteq \sum L^2\). Let \(F \subseteq \tilde{F} \subseteq K\) be a chain of fields with \([K : F] < \infty\). By Proposition 3.1, \(\sum K^2 \not\subseteq \tilde{F}K^2\). The result now follows from Proposition 2.6.

Next we prove that \(\sum L^2 = L^2 + L^2\). Let \(\sigma \in (\sum L^2)^×\). Write \(\sigma = \sum_{i=1}^m \alpha_i^2\), where each \(\alpha_i \in L\). Then \(\{\alpha_1, \ldots, \alpha_m\}\) is algebraic over some finite extension \(F_1\) of \(F\), where \(F \subseteq F_1 \subseteq E_0\). Let \(K = F_1(\alpha_1, \ldots, \alpha_m)\). The field composite \(K E_0\) is an infinite field extension of \(K\) contained in \(L\), and \(\sigma \in \sum K^2\). We will show that there exists a finite extension \(\tilde{F}\) of \(F_1\) contained in \(E_0\) such that the quadratic form \(q = (1, 1, -\sigma)\) is isotropic over \(K \tilde{F}\). This quadratic form is totally indefinite over \(K \tilde{F}\) for every such extension \(\tilde{F}/F_1\) because \(q\) is totally indefinite over \(K\). There are only finitely many prime ideals \(p\) in \(K\) such that \(v_p(\sigma) \neq 0\). Let \(S\) denote the set of prime ideals \(p\) in the ring of integers of \(K\) such that \(v_p(\sigma) \neq 0\) together with the finitely many prime ideals containing 2 (the dyadic primes). By Springer’s theorem, we have for any prime ideal \(p \notin S\) that \(q\) is isotropic over the \(p\)-adic completion \(K_p\), and the same is then true for \((K \tilde{F})_p\) for any finite extension \(\tilde{F}/F\) and for any prime ideal \(P\) of \(K \tilde{F}\) extending \(p\). Now let \(p \in S\). Let \(F_0\) be either the pythagorean closure or the quadratic closure of \(F\) so that \(F_0 \subseteq E_0\). By Lemma 3.3, a compositum \(K_p F_0\) is an infinite algebraic extension of \(K_p\). There exists a finite 2-extension \(\tilde{F}/F_1\) such that \(K_p \tilde{F}\) is a proper finite extension of \(K_p\). Since \(S\) contains only finitely many prime ideals, one can find \(\tilde{F}\) such that this is the case for all primes \(p \in S\). For any prime \(P\) of \(K \tilde{F}\) lying over some prime \(p\), we have that \((K \tilde{F})_P\) is \(K_p\)-isomorphic to \(K_p \tilde{F}\). If \(p \notin S\), we already have that \(q\) is isotropic over \(K_p \tilde{F}\). If \(p \in S\) then \((K \tilde{F})_P\) is a proper finite 2-extension over \(K_p\). Then \(q\) is isotropic over \((K \tilde{F})_P\) by
In this section, we will let \(E\) be a field with characteristic different from two. Let \(E\) be a separable algebraic extension. Let \(M\) be separable over \(K\) and \(M/K\) is purely inseparable. It follows that \(N/M\) is purely inseparable, and so \(N\) is purely inseparable. Then \(\tilde{N}\) is purely inseparable because \(\tilde{N}\) is a finite purely inseparable extension. By repeating this for a tower of quadratic extensions that generates \(N/L\), it follows from (1) that \(\tilde{N} = LM\) where \(M\) is some finite 2-extension of \(K\). Since \(M \subseteq K_0\), we have that \(\alpha \in N = LM \subseteq LK_0\). Therefore, \(L_0 = K_0L\). It follows that \(L_0/K_0\) is purely inseparable because \(L/K\) is purely inseparable.

**Corollary 3.5.** Let \(F\) be a number field and let \(E_0\) be either the pythagorean closure of \(F\) or the quadratic closure of \(F\). Then \(E_0\) is either hereditarily non-pythagorean or hereditarily non-quadratically closed, respectively.

**Proof.** This result follows immediately from Theorem 3.4 because \(E_0/F\) is a Galois extension by [La-05, Chapter VII, pp. 219-220] or by [La-05, Chapter VIII, p. 258]. □

4. **Inseparable extensions and quadratic closures of fields**

We collect some results on separable and inseparable extensions with connections to quadratic closures of fields that will be needed in the following section.

All fields in this section have positive characteristic different from two. Let \(E\) be a field with \(\text{char} E = p > 0, p \neq 2\). Then \(E\) is nonreal and the pythagorean closure \(E_{\text{pyth}}\) of \(E\) is the same as the quadratic closure of \(E\). Since fields are nonreal in this section, we will let \(E_0\) denote the quadratic closure of a field \(E\).

**Lemma 4.1.** Let \(L/K\) be a finite purely inseparable algebraic extension.

(1) The canonical map of square classes \(K^\times/(K^\times)^2 \to L^\times/(L^\times)^2\) is an isomorphism. In particular, \(K\) is quadratically closed if and only if \(L\) is quadratically closed.

(2) Let \(K_0\) be the quadratic closure of \(K\) and \(L_0\) be the quadratic closure of \(L\). Then \(L_0 = K_0L\) and \(L_0/K_0\) is a finite purely inseparable extension.

**Proof.** (1) The map is injective because \([L : K]\) is odd. Let \(\alpha \in L^\times\). Then \(\alpha^{p^i} \in K^\times\) for some \(i \geq 0\) because \(L/K\) is purely inseparable. Then \(a := \alpha^{p^i} = \alpha\alpha^{p^{i-1}} \in \alpha(L^\times)^2\). Thus the map is also surjective.

(2) It is clear that \(K_0L \subseteq L_0\). Let \(\alpha \in L_0^\times\). Then \(\alpha\) lies in some finite 2-extension \(N\) of \(L\). Any 2-extension of \(K\) is a separable extension because \(\text{char} K \neq 2\). Let \(N_1/L\) be a quadratic extension. It follows from (1) that \(N_1 = L(\sqrt{\alpha})\) for some \(a \in K\). Then \(N_1 = K(\sqrt{\alpha})L\) and \(N_1/K(\sqrt{\alpha})\) is purely inseparable. By repeating this for a tower of quadratic extensions that generates \(N/L\), it follows from (1) that \(\tilde{N} = LM\) where \(M\) is some finite 2-extension of \(K\). Since \(M \subseteq K_0\), we have that \(\alpha \in N = LM \subseteq LK_0\). Therefore, \(L_0 = K_0L\). It follows that \(L_0/K_0\) is purely inseparable because \(L/K\) is purely inseparable. □

**Lemma 4.2.** Let \(L/K\) be a finite purely inseparable algebraic extension. Let \(N/L\) be a separable algebraic extension. Let \(M\) be the maximal subfield of \(N\) that is separable over \(K\). Then the following statements hold.

(1) \(N = LM\) and \(N/M\) is purely inseparable.

(2) \(N/L\) is a Galois extension if and only if \(M/K\) is a Galois extension.

(3) \(N\) is quadratically closed if and only if \(M\) is quadratically closed.

**Proof.** (1) Then \(N/M\) is purely inseparable, and so \(N/LM\) is both separable and purely inseparable. It follows that \(N = LM\). Since \(L/K\) is purely inseparable, it follows that \(N/M = LM/M\) is purely inseparable.

[La-05] Chapter VI, Lemma 2.14. The Hasse Minkowski theorem implies that \(q\) is isotropic over \(KF\), and hence over \(L\). Thus \(\sigma \in L^2 + L^2\). □

Remark: In the previous theorem when \(E_0\) is a quadratically closed field, it also follows from [La-05] Chapter VII, Corollary 7.11 that \(E_0\) is hereditarily non-quadratically closed.
Suppose that $M/K$ is a Galois extension. Let $\varphi : N \to L^{alg}$ be an $L$-isomorphism. Then $\varphi$ is also a $K$-isomorphism. We have $\varphi(L) = L$ because $L/K$ is purely inseparable. Since $M/K$ is a Galois extension, we have $\varphi(M) = M$. Thus $\varphi(N) = \varphi(LM) = \varphi(L)\varphi(M) = LM = N$ and so $N/L$ is a Galois extension.

Now suppose that $N/L$ is a Galois extension and let $\varphi : M \to K^{alg}$ be a $K$-isomorphism. There exists an extension to a $K$-isomorphism $\varphi : N \to K^{alg}$. Since $L/K$ is purely inseparable, it follows that $\varphi : N \to K^{alg}$ is an $L$-isomorphism. Since $N/L$ is a Galois extension, we have $N = \varphi(N) = \varphi(L)\varphi(M) = L\varphi(M)$. Since $M/K$ is a separable extension, it follows that $\varphi(M)/K$ is a separable extension. Since $\varphi(M) \subset N$ and $M$ is the maximal subfield of $N$ that is separable over $K$, we have $\varphi(M) \subseteq M$. In a similar way, we have $M \subseteq \varphi(M)$, and thus $\varphi(M) = M$. Therefore, $M/K$ is a Galois extension.

(3) Since $N/M$ is purely inseparable, the result follows from Lemma 4.1 (1).

**Proposition 4.3.** Let $L/K$ be a finite purely inseparable algebraic extension. Let $L_0/L$ be a separable extension and assume that $L_0$ is quadratically closed. Let $K_0$ be the maximal subfield of $L_0$ that is separable over $K$.

Then every purely inseparable extension of $K_0$ and $L_0$ is quadratically closed. Moreover, $K_0$ is hereditarily non-pythagorean if and only if $L_0$ is hereditarily non-pythagorean.

**Proof.** By Lemma 4.2 (3), $K_0$ is quadratically closed. Lemma 4.1 (1) implies that every purely inseparable extension of $K_0$ and $L_0$ is quadratically closed.

Assume that $K_0$ is hereditarily non-pythagorean. Then every proper finite separable extension of $K_0$ is not quadratically closed. Let $N$ be a proper finite separable extension of $L_0$. Since $L_0/K_0$ is purely inseparable by Lemma 4.2 (1), Lemma 4.2 implies that there exists a proper finite separable extension $M/K_0$ such that $N = L_0M$. Since $M$ is not quadratically closed by assumption and $N/M$ is purely inseparable, Lemma 4.1 (1) implies that $N$ is not quadratically closed. Thus $L_0$ is hereditarily non-pythagorean by Lemma 2.5.

Assume that $L_0$ is hereditarily non-pythagorean. Then every proper finite separable extension of $L_0$ is not quadratically closed. Let $M$ be a proper finite separable extension of $K_0$. Then $L_0M$ is a proper finite separable extension of $L_0$. Thus $L_0M$ is not quadratically closed. Since $L_0M/M$ is purely inseparable, Lemma 4.1 (1) implies that $M$ is not quadratically closed. Thus $K_0$ is hereditarily non-pythagorean by Lemma 2.5.

5. **Pythagorean and Quadratic Closures of Function Fields**

The main result of this section, Theorem 5.3, shows that the pythagorean closure of a field that is finitely generated of transcendence degree at least one over a subfield is hereditarily non-pythagorean. In order to be able to apply Proposition 2.4 in the case of a general base field, we need to show a result on extensions of nonreal valuations, inspired by the statement from number theory that “there are infinitely many primes that split completely in a finite extension”.

**Theorem 5.1.** Let $k$ be field. Let $E/k$ be an algebraic function field in one variable and let $F/E$ be a finite separable extension. Assume that $E$ is separable over some rational function field of $k$. Then there are infinitely many $k$-valuations $v$ on $E$ that extend to $[F : E]$ valuations on $F$. 

Proof. By enlarging $F$ and taking a subfield of $E$ if necessary, it is sufficient to show the claim in the case where $F/E$ is a Galois extension and where $E = k(X)$ is a rational function field. In this case, all $k$-valuations on $F$ extending a given $k$-valuation on $E$ are conjugate by the Galois group of $F/E$, and in order to show that there are $d := [F : E]$ such extensions, it is equivalent to show that any one of them is unramified over $v$ and its residue field equals the residue field of $v$.

Let $f(T) ∈ E[T]$ be the minimal polynomial of a primitive element for $F/E$. By choosing the primitive element appropriately, we may assume that the coefficients of $f$ lie in $k[X]$. Write $f = f(X, T) = T^d + \sum_{i=0}^{d-1} a_i(X)T^i$, where each $a_i(X) ∈ k[X]$. Let $Δ ∈ k[X]$ be the discriminant of $f$ considered as a polynomial in one variable over $k[X]$. Any monic irreducible polynomial $p ∈ k[X]$ that does not divide $Δ$ in $k[X]$ and that divides $f(X, g(X))$ for some $g(X) ∈ k[X]$ yields a $k$-valuation $v_p$ on $E = k(X)$ that extends to $[F : E]$ distinct $k$-valuations on $F$. To see this, let $L = k[x]/(p(x))$. Then $L$ is a finite algebraic extension of $k$. Let $α ∈ L$ be a root of $p$. Let $v_p$ denote the valuation on $k(x)$ associated to the monic irreducible polynomial $p$. The element $g(α) ∈ L$ is a root of the image of the polynomial $f(α, T)$ in $k[x]/(p)$ because $p ∣ f(X, g(X))$. Thus the valuation $v_p$ extends to a valuation on $F$ that is unramified (because $p ∤ Δ$) with residue field equal to $L$ (because $g(α) ∈ L$). Since $F/E$ is Galois, it follows that $v_p$ extends to $d$ distinct valuations on $F$.

We denote by $P_0$ the set of such $p$. We further denote by $P_Δ$ the finite set of monic irreducible polynomials $p ∈ k[X]$ that divide $Δ$. Let $P$ be any nonempty set of monic irreducible polynomials that contains $P_0 ∪ P_Δ$ and at most finitely many other monic irreducible polynomials. We will show that $P_0$ is infinite. Suppose on the contrary that $P_0$ is finite, and thus $P$ is finite. Write $P = \{p_1, \ldots, p_n\}$. For any $r ∈ \mathbb{N}$, let

$$g_r(X) = \prod_{j=1}^n \left( p_j(X)^r \right) \in k[X].$$

We will show that for $r$ sufficiently large, there exists a monic irreducible $q ∈ k[X]$, $q ∉ P$, such that $q$ divides $f(X, g_r(X))$ in $k[X]$, thereby yielding the contradiction that $q ∈ P_0 ⊆ P$.

For $r$ sufficiently large, we have that

$$v_{p_j}(f(X, g_r(X))) = v_{p_j}(f(X, 0))$$

for every $j ∈ \{1, \ldots, n\}$. Thus, for $r$ sufficiently large, the monic irreducible polynomials $\{p_1, \ldots, p_n\}$ appear always with the same multiplicity in the factorization of $f(X, g_r(X))$ in $k[X]$. On the other hand, since $P ≠ \emptyset$, we have that for sufficiently large $r$, $\deg_X(f(X, g_r(X))) = rd \sum_{j=1}^n \deg_X(p_j)$, hence growing linearly in $r$. In particular, for all sufficiently large $r$, the polynomial $f(X, g_r(X)) ∈ k[X]$ has a monic irreducible factor $q$ distinct from $p_1, \ldots, p_n$.

Corollary 5.2. Let $k$ be field. Let $E/k$ be an algebraic function field in one variable and let $F/E$ be a finite separable extension. Assume that $E$ is separable over some rational function field of $k$. Then there are infinitely many nonreal $k$-valuations $v$ on $E$ that extend to $[F : E]$ valuations on $F$.

Proof. We apply Theorem 5.1 to the extension $F(\sqrt{-1})/E$. Suppose that $v$ is a $k$-valuation on $E$ that splits completely in $F(\sqrt{-1})$. Then any extended $k$-valuation is unramified over $E$ and has the same residue field as $v$. Since $\sqrt{-1}$ is a unit in
any valuation ring, the residue field contains $\sqrt{-1}$. In particular, the residue field of $v$ is nonreal.

By [S, Chapter III, Proposition 9.2], if $k$ is perfect, then every function field in one variable over $k$ is separable over some rational function field of $k$.

**Theorem 5.3.** Let $E$ be a field that is finitely generated of transcendence degree at least one over a subfield. Let $E_0$ be either a pythagorean field or a quadratically closed field and assume that $E_0/E$ is a Galois extension. Then $E_0$ is hereditarily non-pythagorean.

**Proof.** One can find a subfield $k \subset E$ such that $E$ is an algebraic function field in one variable over $k$. There is a subfield $E' \subseteq E$ such that $E'$ is a finite separable extension of a rational function field of $k$ and $E$ is a finite purely inseparable extension of $E'$. Let $E'_0$ be the maximal subfield of $E_0$ that is separable over $E'$. By Lemma 4.2, $E_0 = E E'_0$, $E'_0/E'$ is Galois, and $E'_0, E_0$ are either both pythagorean or both quadratically closed. By Corollary 5.2 and Proposition 2.7, $E'_0$ is hereditarily non-pythagorean.

Since $E/E'$ is purely inseparable, it follows that $E_0/E'_0$ is purely inseparable by Lemma 4.2 (1). Thus $E_0$ is hereditarily non-pythagorean by Proposition 4.3. □

**Corollary 5.4.** Let $E$ be a field that is finitely generated of transcendence degree at least one over a subfield. Let $E_0$ be either the pythagorean closure of $E$ or the quadratic closure of $E$. Then $E_0$ is either hereditarily non-pythagorean or hereditarily non-quadratically closed, respectively.

**Proof.** As in Corollary 3.5, the result follows immediately from Theorem 5.3 because $E_0/E$ is a Galois extension. □

6. SOME RESULTS FOR INFINITE ALGEBRAIC EXTENSIONS OF NUMBER FIELDS AND FUNCTION FIELDS

The next result extends Theorem 3.4 and Theorem 5.3.

**Theorem 6.1.** Let $F$ be either a number field or a field that is finitely generated of transcendence degree at least one over a subfield. Let $E/F$ be an infinite Galois extension. Let $E_0$ be the pythagorean closure or quadratic closure of $E$. Then $E_0$ is hereditarily non-pythagorean.

**Proof.** We first show that $E_0$ is a Galois extension of $F$. Let $\varphi \in Gal(F^{sep}/F)$. We claim that $\varphi(E_0) = E_0$. It is straightforward to verify that $\varphi(E_0)$ is the pythagorean or quadratic closure of $\varphi(E)$. Since $E/F$ is a Galois extension, we have that $\varphi(E) = E$. Thus $\varphi(E_0)$ is the pythagorean or quadratic closure of $E$, and so $\varphi(E_0) = E_0$.

The result now follows from Theorem 3.4 and Theorem 5.3. □

**Example 6.2.** Let $F$ be a field that is not separably algebraically closed or real closed and let $F^{sep}$ be its separable algebraic closure. Then there exists an automorphism $\varphi \in Gal(F^{sep}/F)$ having infinite order. Let $E$ be the fixed field of $\varphi$. Then $Gal(F^{sep}/E)$ is the pro-finite closure of the cyclic group generated by $\varphi$, which is isomorphic to $\hat{\mathbb{Z}}$, the inverse limit of all $\mathbb{Z}/n\mathbb{Z}$. Suppose that $E$ is a real field. Then $E$ would be contained in a real closure. However, $\hat{\mathbb{Z}}$ does not contain an element of order two. Thus $E$ is a nonreal field. Moreover $E$ is not quadratically closed.
closed, since \( \hat{E} \) contains a subgroup of index 2. Let \( E_0 \) denote the quadratic closure of \( E \), which is a Galois extension of \( E \). Then
\[
\text{Gal}(F^{\text{sep}}/E_0) \cong \bigoplus_{p \neq 2} \mathbb{Z}_p
\]
It follows that every finite extension of \( E_0 \) has odd degree. In particular, \( E_0 \) is hereditarily quadratically closed.

If \( F \) is either a number field or a field that is finitely generated of transcendence degree at least one over a subfield, then Theorem 6.1 implies that \( E \) is not a Galois extension of \( F \).

**Proposition 6.3.** Let \( E/Q \) be a (possibly infinite) algebraic extension and assume that \( E \) is real but not pythagorean. Let \( E_{\text{pyth}} \) be the pythagorean closure of \( E \). Let \( L/E_{\text{pyth}} \) be a quadratic extension of \( E_{\text{pyth}} \). Then \( L \) is not pythagorean. In particular, no quadratic extension of \( E_{\text{pyth}} \) is pythagorean.

**Proof.** Let \( L = E_{\text{pyth}}(\sqrt{\alpha}) \). Note that \( \alpha \) is negative with respect to at least one ordering of \( E_{\text{pyth}} \). Since \( E_{\text{pyth}}/E \) is an infinite totally real extension (every ordering of \( E \) extends to \( n \) orderings in every finite extension of degree \( n \) in \( E_{\text{pyth}} \)), and since the element \( \alpha \) is defined over a finite extension of \( E \) in \( E_{\text{pyth}} \), we conclude that \( \alpha \) is negative at an infinite number of orderings.

Let \( E'/Q \) be a finite extension contained in \( E_{\text{pyth}} \) such that \( \alpha \in E' \) and such that two of the orderings of \( E_{\text{pyth}} \) with respect to which \( \alpha \) is negative, restrict to two distinct orderings \( <_1 \) and \( <_2 \) on \( E' \). Note that \( L' = E'((\sqrt{\alpha}) \) is a quadratic extension contained in \( L \).

By weak approximation in the number field \( E' \), there exist \( x, y \in E' \) such that for \( \beta := x^2 + y^2\alpha \), we have \( \beta > 0 \) and \( \beta < 0 \). As a consequence, we have that both \( \beta \) and \( \beta\alpha \) are not totally positive in \( E_{\text{pyth}} \). Hence
\[
\beta \notin (E_{\text{pyth}}^\times)^2 \cup \alpha(E_{\text{pyth}}^\times)^2 = E_{\text{pyth}} \cap L^2.
\]
This shows that \( \beta \notin L^2 \). Since \( \beta = x^2 + y^2\alpha \) and \( \alpha \in L^2 \), it follows that \( \beta \in \sum L^2 \).
Hence \( L \) is not pythagorean. \( \square \)

**Theorem 6.4.** Let \( E/Q \) be a (possibly infinite) algebraic extension and assume that \( E \) is real but not pythagorean. Let \( E_{\text{pyth}} \) be the pythagorean closure of \( E \). Let \( L/E_{\text{pyth}} \) be a finite proper extension that can be written as a chain of field extensions in which one intermediate extension is a quadratic extension. Then \( L \) is not pythagorean.

**Proof.** By Proposition 2.4, we may assume that the quadratic extension is the last extension in the chain. Thus we can assume that \( E_{\text{pyth}} \subseteq F_0 \subseteq L \) where \( L = F_0(\sqrt{\alpha}) \) for some \( \alpha \in F_0 \).

First suppose that \( L \) is nonreal pythagorean (i.e. quadratically closed). Since \( L/E_{\text{pyth}} \) is a finite extension and \( E_{\text{pyth}} \) is a real field, it follows from [La-05, Chapter VIII, Corollary 5.11] that \( E_{\text{pyth}} \) is euclidean. In particular, \( E_{\text{pyth}} \) is uniquely ordered. However, since \( E \) is not pythagorean, \( E_{\text{pyth}} \) is an infinite extension by Proposition 2.4, which is given by iteratively adjoining square roots of sums of squares, whereby \( E_{\text{pyth}} \) has infinitely many orderings. This is a contradiction.

Suppose now that \( L \) is real pythagorean. Let \( \hat{E}/E \) be a finite extension in \( E_{\text{pyth}} \) such that the minimal polynomial of some primitive element \( \beta \) for \( F_0/E_{\text{pyth}} \) is
defined over \( \tilde{E} \) and such that \( \alpha \in \tilde{F} := \tilde{E}(\beta) \). The second property is possible to arrange, since \( F_0 = E_{\text{pyth}}(\beta) \) is the direct limit of \( E'(\beta) \) where \( E' \) runs through the finite extensions \( E' \) inside \( E_{\text{pyth}} \) of any \( \tilde{E} \) with the first property.

Since \( F_0 \) is also real pythagorean by Proposition 2.3, we have that \( F_0 \) is  the pythagorean closure of \( F \) (which is not pythagorean). We are now in the situation of Proposition 6.3 where we only consider a quadratic extension of the pythagorean closure of a real number field that is not pythagorean. We showed in Proposition 6.3 that in this case \( L \) is not pythagorean. Hence we have a contradiction also in this case. \( \square \)

In the next section, we prove some general results that imply the existence of real infinite number fields that are not pythagorean and whose pythagorean closures are not hereditarily non-pythagorean.

### 7. Pythagorean closures admitting pythagorean finite extensions

**Lemma 7.1.** Let \( F/E \) be a finite extension of fields with \( [F : E] = n, n \) odd. The following statements are equivalent.

1. \( \sum F^2 = (\sum E^2)^2 \).
2. \( \sum F^2 \subseteq EF^2 \).
3. \( \alpha N_{F/E} : (\sum F^2)^\times /((\sum F^2))^2 \to (\sum E^2)^\times /((\sum E^2))^2 \) is injective.
4. \( \alpha N_{F/E}(\alpha) \in F^2 \) for all \( \alpha \in \sum F^2 \).

**Proof.** (1) \( \Rightarrow \) (2). This is trivial.

(2) \( \Rightarrow \) (3). Assume that (2) holds and let \( \alpha \in (\sum F^2)^\times \). Then \( \alpha = a\beta^2 \) where \( a \in E \) and \( \beta \in F \). Assume that \( N_{F/E}(\alpha) \in E^2 \). Then \( N_{F/E}(\alpha) = a^n(N_{F/E}(\beta))^2 \in E^2 \), which implies that \( a^n \in E^2 \). Since \( n \) is odd, we have \( a \in E^2 \), and thus \( \alpha \in F^2 \).

(3) \( \Rightarrow \) (4). Assume that (3) holds. Let \( \alpha \in (\sum F^2)^\times \). Then \( N_{F/E}(\alpha N_{F/E}(\alpha)) = (N_{F/E}(\alpha))^{n+1} \in (\sum E^2)^2 \) because \( n + 1 \) is even. Therefore \( \alpha N_{F/E}(\alpha) \in F^2 \).

(4) \( \Rightarrow \) (1). Assume that (4) holds and let \( \alpha \in \sum F^2 \). Then \( N_{F/E}(\alpha) \in \sum E^2 \) and \( \alpha \in N_{F/E}(\alpha)F^2 \subseteq (\sum E^2)F^2 \). Thus \( \sum F^2 \subseteq (\sum E^2)F^2 \). The other inclusion is obvious. \( \square \)

**Lemma 7.2.** Let \( F/E \) be a finite extension of fields with \( [F : E] = n, n \) odd. Let \( K = F(\sqrt{d}) \) where \( d \in \sum E^2 \), and assume that \( [K : E] = 2 \). Let \( L = KF = F(\sqrt{d}) \). Then \( \sum F^2 \subseteq EF^2 \) if and only if \( \sum L^2 \subseteq KL^2 \).

**Proof.** First assume that \( \sum F^2 \subseteq EF^2 \). Let \( \beta \in \sum L^2 \). Let \( \alpha = N_{L/F}(\beta) \in \sum F^2 \). Then

\[
N_{L/F}(\beta N_{L/K}(\beta)) = N_{L/F}(\beta)N_{K/E}(N_{L/K}(\beta))
\]

\[
= N_{L/F}(\beta)N_{F/E}(N_{L/F}(\beta)) = \alpha N_{F/E}(\alpha) \in F^2;
\]

by Lemma 7.1. It follows that \( \beta N_{L/K}(\beta) \in FL^2 \) by [La-05] Chapter VII, Theorem 3.8.

Since \( \beta \in \sum L^2 \), we have \( N_{L/K}(\beta) \in \sum K^2 \subseteq \sum L^2 \). Since \( d \in \sum E^2 \subseteq \sum F^2 \) and \( L = F(\sqrt{d}) \), we have \( F \cap \sum L^2 \subseteq \sum F^2 \). These two observations give

\[
\beta N_{L/K}(\beta) \in \left( F \cap \sum L^2 \right)L^2 = \left( \sum F^2 \right)L^2 \subseteq EF^2L^2 \subseteq KL^2.
\]

Thus \( \beta \in KL^2 \) because \( N_{L/K}(\beta) \in K \). Therefore \( \sum L^2 \subseteq KL^2 \).
Now assume that $\sum L^2 \subseteq KL^2$. Let $a \in \sum F^2$. Then $a \in \sum L^2$ and Lemma 7.1 implies that $aN_{L/K}(a) \in L^2$. Since $N_{L/K}(a) = N_{F/E}(a)$, we have $aN_{F/E}(a) \in L^2 \cap F = F^2 \cup dF^2 \subseteq EF^2$. Then $a \in EF^2$ because $N_{F/E}(a) \in E$. Therefore, $\sum F^2 \subseteq EF^2$. □

Proposition 7.3. Let $F/E$ be a finite extension of fields with $[F : E] = n$, $n$ odd. Assume that $\sum F^2 \subseteq EF^2$. Then $F_{\text{pyth}} = E_{\text{pyth}}F$ and $[F_{\text{pyth}} : E_{\text{pyth}}] = n$.

Proof. Let $K/E$ be any finite extension where $K \subseteq E_{\text{pyth}}$. Then there exists a finite chain $E = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m = K$ where $E_j/E_{j-1}$ is a quadratic extension with $E_j = E_{j-1}(\sqrt{d_j})$ and $d_j \in \sum E_{j-1}^2$, $1 \leq j \leq m$. Let $L = KF$. Then $[L : K] = n$ and by induction, Lemma 7.2 implies that $\sum L^2 \subseteq KL^2$. Lemma 7.1 implies that $\sum L^2 = (\sum K^2) L^2$.

Since this holds for every finite extension $K/E$ where $K \subseteq E_{\text{pyth}}$, it follows that by setting $M = E_{\text{pyth}}F$ we have $\sum M^2 = \left(\sum E_{\text{pyth}}^2\right) M^2 = E_{\text{pyth}}^2 \subseteq M^2$. Thus $M$ is pythagorean, and so $M = F_{\text{pyth}}$. □

Proposition 7.4. Let $p$ be an odd prime, and let $F/E$ be a finite separable extension with $[F : E] = p$.

1. There is an algebraic extension $E_1/E$ such that with $F_1 = E_1 F$ the following statements hold.
   (a) $E_1$ and $F$ are linearly disjoint over $E$ so that $[F_1 : E_1] = [F : E] = p$.
   (b) $E_1$ and $F_1$ are pythagorean.

2. Assume that $E$ is not pythagorean and let $a \in \sum E^2$, $a \notin E^2$. Assume that $F/E$ is a Galois extension. There is an algebraic extension $E_2/E$ such that with $F_2 = E_2 F$ the following statements hold.
   (a) $E_2$ and $F$ are linearly disjoint over $E$ so that $[F_2 : E_2] = [F : E] = p$.
   (b) $a \notin E_2^2$, and thus $a \notin F_2^2$, so that $E_2$ and $F_2$ are each not pythagorean.
   (c) $\sum F_2^2 = F_2^2 \cup aF_2^2 \subseteq E_2 F_2^2$.

3. If $F/E$ is a Galois extension and $E$ is formally real, then it can be arranged in (1) and (2) that $F_1$ and $F_2$ are each formally real.

Proof. (1) Let $E_1$ be a maximal algebraic extension of $E$ such that $E_1$ and $F$ are linearly disjoint over $E$. Let $F_1 = E_1 F$. Then $[F_1 : E_1] = [F : E] = p$. We will show that $F_1$ is pythagorean, and thus $E_1$ is also pythagorean.

Suppose that $a \in \sum F_1^2$, $a \notin F_1^2$. Let $M$ be the Galois closure of $F_1(\sqrt{a})/E_1$. We now show that $p \mid [M : E_1]$ but $p^2 \nmid [M : E_1]$. Let $L$ be the Galois closure of $F_1/E_1$. Then $[L : E_1] \mid p!$, and thus $p^2 \nmid [L : E_1]$. Since $L/E_1$ is Galois and $[F_1(\sqrt{a})L : L] = 1$ or 2, it follows that $[M : L]$ is a 2-power, and so $p^2 \nmid [M : E_1]$.

Let $K$ be the fixed field of a Sylow $p$-subgroup of $\text{Gal}(M/E_1)$. Then $[M : K] = p$ and $p \nmid [K : E_1]$. It follows that $K$ and $F_1$ are linearly disjoint over $E_1$ and $M = KF_1$. Then $K$ and $F$ are linearly disjoint over $E$. The maximality of $E_1$ implies that $K = E_1$. Then $M = KF_1 = F_1$. Since $F_1 \subseteq F_1(\sqrt{a}) \subseteq M$, we obtain a contradiction, and thus $F_1$ is pythagorean.

(2) The proof of (2) is similar to the proof of (1). Let $E_2$ be a maximal algebraic extension of $E$ such that $E_2$ and $F$ are linearly disjoint over $E$ and such that $a \notin E_2$. Let $F_2 = E_2 F$. Then $F_2/E_2$ is a Galois extension and $[F_2 : E_2] = [F : E] = p$. Also, $E_2$ is not pythagorean, and thus $F_2$ is not pythagorean. We now show that $\sum F_2^2 = F_2^2 \cup aF_2^2$. Let $\beta \in \sum F_2^2$, $\beta \notin F_2^2$. 


Assume first that \( N_{F_2/E_2}(\beta) \in E_2^2 \). Let \( M \) be the Galois closure of \( F_2(\sqrt{\beta})/E_2 \). We show as in (1) that \( p \mid [M : E_2] \) but \( p^2 \nmid [M : E_2] \). Let \( K \) be the fixed field of a Sylow-\( p \)-subgroup of \( \text{Gal}(M/E_2) \). Then \([M : K] = p\) and \( p \nmid [K : E_2] \). Then \( K \) and \( F_2 \) are linearly disjoint over \( E_2 \), and thus \( K \) and \( F \) are linearly disjoint over \( E \). If \( a \notin K^2 \), then the maximality of \( E_2 \) implies that \( K = E_2 \), which leads to a contradiction, as in (1). Thus \( a \in K^2 \subset M^2 \). Since \( a \notin E_2^2 \) and \([F_2 : E_2] \) is odd, we have \( a \notin F_2^2 \). Let \( \text{Gal}(F_2/E_2) = \{ \sigma_1, \ldots, \sigma_p \} \). Since \( F_2/E_2 \) is Galois, \( M = F_2(\sqrt{\sigma_1(\beta)}, \ldots, \sqrt{\sigma_p(\beta)}) \) and \( a \in F_2 \cap M^2 = \cup \sigma_1(\beta)^{e_1} \cdots \sigma_p(\beta)^{e_p} F_2^2 \), where \( e_1, \ldots, e_p \in \{0,1\} \). This is a contradiction because \( N_{F_2/E_2}(a) \in aE_2^2 \neq E_2^2 \) but \( N_{F_2/E_2}(\sigma_1(\beta)^{e_1} \cdots \sigma_p(\beta)^{e_p}) \in E_2^2 \).

It follows that \( \sum F_2^2 \subset E_2F_2^2 \) by Lemma 7.4 (3) \( \Rightarrow \) (2). Thus we may assume that \( \beta \in E_2 \). Then \( M = F_2(\sqrt{\beta}) \) and so \( K = E_2(\sqrt{\beta}) \). Since \( a \in E_2 \cap K^2 = E_2^2 \cup \beta E_2^2 \), it follows that \( \beta \in aE_2^2 \). Thus \( \beta \in aE_2^2 \subset aF_2^2 \). Therefore \( \sum F_2^2 = F_2^2 \cup aF_2^2 \).

(3) Suppose that \( F/E \) is a Galois extension and that \( E \) is formally real, and thus \( F \) is formally real. Let \( R \) be a real closure of \( E \). Modify the proofs of (1) and (2) by choosing \( E_1 \) and \( E_2 \) maximal algebraic extensions of \( E \) contained in \( R \) that are linearly disjoint from \( F \) over \( E \), and for \( E_2 \) additionally require that \( a \notin E_2^2 \). Since \( F/E \) is Galois, we have that \( F \subset R \), and thus it follows that \( M \), the Galois closure of \( F_1(\sqrt{\beta})/E_1 \) or \( F_2(\sqrt{\beta})/E_2 \), also lies in \( R \). The rest of the proof of (3) now follows the proof of (1) and (2).

Corollary 7.5. Let \( p \) be an odd prime and let \( E \) be any non-pythagorean field that admits a Galois extension \( F/E \) with \([F : E] = p\).

(1) Then there exists a non-pythagorean algebraic extension \( E_0/E \) such that \( E_0 \) and \( F \) are linearly disjoint over \( E \) and setting \( F_0 = E_0 F \), we have \( [F_0 : E_0]_{\text{pyth}} = (E_0)_{\text{pyth}} F \) and \([F_0 : E_0]_{\text{pyth}} = [F : E] = p\).

(2) If \( E \) is real, then it can be arranged that \( E_0 \) is real in addition to the conditions in (1). In particular, \( (E_0)_{\text{pyth}} \) is not hereditarily non-pythagorean.

Proof. By Proposition 7.4 (2), there exists an algebraic extension \( E_0/E \) such that with \( F_0 = E_0 F \), we have \( E_0 \) and \( F \) are linearly disjoint over \( E \), \([F_0 : E_0] = p\), \( E_0 \) is non-pythagorean, and \( \sum (F_0)^2 \subset E_0 F_0^2 \). If \( E \) is real, then by Proposition 7.4 we can arrange that \( F_0 \) is real. Proposition 7.4 implies that \( (F_0)_{\text{pyth}} = (E_0)_{\text{pyth}} F \) and \([F_0 : E_0]_{\text{pyth}} = [F_0 : E_0] = p\). 

Remark: Proposition 7.4 (2), (3), as well as Corollary 7.4 can be extended to the situation where the Galois extension \( F/E \) has any odd degree (not necessarily prime). The Feit-Thomas theorem (1) implies that the Galois group of \( F/E \) is solvable. Since \( M/F_2 \) is a (solvable) 2-extension and \( F_2/E_2 \) is a solvable extension, it follows that \( M/E_2 \) is a solvable Galois extension. Since \([M : F_2] \) is a 2-power, \( n \) is odd, and \( \text{Gal}(M/E_2) \) is a solvable group, a theorem of P. Hall (2, p. 141) implies that \( \text{Gal}(M/E_2) \) has a subgroup of order \( n \). We let \( K \) be the subfield of \( M \) corresponding to this subgroup of order \( n \). The rest of the proof of Proposition 7.4 (2), (3) and Corollary 7.4 proceeds as above.

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