Optimal covariant fitting to a Robertson-Walker metric and smallness of backreaction

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Abstract: We define a class of “optimal” coordinate systems by requiring that the deviation from an exact Robertson-Walker metric is “as small as possible” within a given four dimensional volume. The optimization is performed by minimizing several volume integrals which would vanish for an exact Robertson-Walker metric. Covariance is automatic. Foliation of space-time is part of the optimization procedure. Only the metric is involved in the procedure, no assumptions about the origin of the energy-momentum tensor are needed. A scale factor does not show up during the optimization process, the optimal scale factor is determined at the end. The general formulation is non perturbative. An explicit perturbative treatment is possible. The shifts which lead to the optimal coordinates obey Euler-Lagrange equations which are formulated and solved in first order of the perturbation. The extension to second order is sketched, but turns out to be unnecessary. The only freedom in the choice of coordinates which finally remains are the rigid transformations which keep the form of the Robertson-Walker metric intact, i.e. translations in space and time, spatial rotations, and spatial scaling. Spatial averaging becomes trivial. In first order of the perturbation there is no backreaction. A simplified second order treatment results in a very small effect, excluding the possibility to mimic dark energy from backreaction. This confirms (as well as contradicts) statements in the literature.

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1 Introduction

The averaging problem, i.e. the problem of averaging a realistic inhomogeneous metric into a smooth one, as well as the fitting problem, the fitting of an “optimal” Robertson-Walker (R-W) metric to a realistic inhomogeneous metric, are both non-trivial due to the freedom of choosing arbitrary coordinates in general relativity. Most papers focus on “gauge transformations”, where the R-W background metric $\bar{g}_{\mu\nu}$ is given and fixed, and only the perturbation $h_{\mu\nu}$ is transformed. To fix the background one usually resorts to a flow of matter. This is unsatisfactory for two reasons. Firstly one has to make rather stringent assumptions concerning the flow, like only one single component and absence of rotation. The second point, although quite obvious but nevertheless hard to find mentioned in the literature, appears even more drastic: The flow which is used is, of course, not the real flow of matter but already some average over an irregular flow. The result of this averaging clearly depends on the choice of coordinates. One is faced with the bizarre situation that one starts some sophisticated “gauge invariant” averaging procedure on the basis of a background obtained from an ambiguous and unspecified averaging. This makes the whole procedure quite dubious.

There is an extensive literature on both topics which can only be briefly addressed here. We refer e.g. to the monograph of Krasiński [1] and the comprehensive review of Buchert [2]. A careful analysis of the fitting problem was given by Ellis and Stoeger [3]. We only briefly mention some aspects here. The averaging problem was first raised by Shirokov and Fisher [4] in 1963. The authors suggested to integrate the metric tensor over a four dimensional volume with the familiar factor $\sqrt{-g}$ in the measure. Such an expression is, however, not covariant for a tensor due to the freedom of performing local transformations. A covariant averaging prescription can be constructed by introducing a bivector $g^\alpha_\beta(x, x')$ of geodesic parallel displacement, as discussed in the appendix of [5]. This transforms as a vector with respect to coordinate transformations at either $x$ or $x'$ and maps a vector $A_\beta(x')$ to $\bar{A}_\alpha(x) = g^\alpha_\beta(x, x') A_\beta(x')$, analogously for higher order tensors. An averaging with the help of bivectors is also used in the work of Zalaletdinov [6] where the emphasis was on the commutativity of averaging and covariant differentiation. As remarked by Stoeger, Helmi, and Torres [7] the method of using a covariantly conserved bivector is not applicable to the metric, because the covariant derivative of the metric vanishes. The metric is therefore invariant under this averaging procedure. Another popular method due to Bardeen [8] is to work with gauge invariant (in first order of the perturbation and for static transformations only) quantities. The most general covariant and translation invariant first order averaging scheme has been given in [9]. But any such an averaging has the principal problem that a plane wave, instead of being averaged to zero, will always stay a plane wave, albeit with reduced amplitude: $\exp(ikx) \rightarrow \int f(x - y) \exp(iky) d^3y = \{ \int f(z) \exp(-ikz) d^3z \} \cdot \exp(ikx)$.

Instead of attempting an averaging, it appears therefore more promising to determine directly an “optimal” approximating smooth metric. Our approach is conceptually simple. We fix the coordinate system as far as principally possible, so that no unphysical gauge freedom remains. The coordinate system is chosen in such a way that, in a given four dimensional volume, the metric is as close to an exact R-W metric as possible.
Before going into details one should recall that an exact R-W metric (with \( k = 0 \)) keeps its form under an eight parameter group of global symmetry transformations: Rigid translations in space and time, rigid spatial rotations, and rigid scaling of the spatial coordinates. (For \( k = \pm 1 \) there is only a seven parameter group, scaling is not allowed.) In the case of translations in time, and of scaling in space, the scale function \( a(t) \) changes.

This freedom in the choice of coordinates cannot and need not be fixed. It is inevitably connected with the symmetry of the R-W metric. Any covariant fitting procedure will necessarily share this freedom of transformations. Therefore a maximal fixing of the coordinate system means that the coordinates are fixed up to the above rigid transformations, while no further transformations are allowed anymore. The coordinate system obtained at the end should fulfill two criteria:

**Covariance:** Let two observers \( A, B \) describe the same realistic inhomogeneous space in different and completely arbitrary coordinate systems \( S_A \) and \( S_B \). Both of them apply the same definite method to transform to “optimized” systems \( S_A' \) and \( S_B' \) respectively. Then the systems \( S_A' \) and \( S_B' \) thus obtained can only be related by a transformation from the eight parameter group above.

**Optimization:** The metric in the optimized system should be “as close to an exact R-W metric as possible” within a given four dimensional volume. Since any perturbative treatment is performed around a R-W background, this requirement guarantees that the perturbation becomes as “small” as possible. One has to define the conditions of this optimization and to construct the “optimal” coordinate system.

Our covariant optimization proceeds via a series of minimizations of four dimensional volume integrals. Expressions which vanish for the exact R-W metric are minimized by choosing an optimal gauge. This gauge fixing is performed as far as principally possible. Starting from an arbitrary system \( S \) with metric \( g_{\mu\nu}(x) \), one constructs an ”optimal” system \( S' \) with the transformed metric \( g'_{\mu\nu}(x') \). At the end one can define the ”optimal” approximating R-W metric \( g_{\mu\nu}(x') \).

The method has the following properties and advantages.

- It only uses the metric \( g_{\mu\nu} \), no assumptions about the origin of the energy momentum tensor are necessary. A scale factor does not show up during the procedure, the optimal scale factor is determined at the end.
- Foliation of space-time is part of the procedure and is obtained in a unique way, again without resorting to any assumptions concerning a flow of matter.
- The general formulation is non perturbative.
- The procedure can be explicitly applied in perturbation theory if the metric is a small deviation from an exact R-W metric. We will present the explicit formulae in first order of the perturbation and sketch the procedure for the second order.
- The four dimensional volume over which the minimization is performed is arbitrary. For the perturbative treatment we will specialize to simple volumes.
• Covariance is an immediate consequence of the method. Since the variation is taken over all coordinate systems, it does not matter in which coordinate system one starts.

• Spatial averages of arbitrary tensor fields can be naively performed when using the optimal coordinate system. There is no need to decompose into tensor structures or to restrict to averaging of scalar quantities.

• There is no backreaction in first order.

The paper is organized as follows. In sect. 2 we present the general nonperturbative method. In sect. 3 this is explicitly applied in first order of the perturbation. The Euler-Lagrange equations for the coordinate shift which leads to the optimal system are formulated and solved. Boundary effects turn out to be irrelevant if the wavelength of the perturbation is small compared to the spatial extension of the volume. In sect. 4 we present the rather simple extension to second order. The short sect. 5 describes averaging which has become trivial. Sect. 6 deals with backreaction. There is no backreaction in first order. In a simplified static treatment of the second order it turns out that the second order of the transformation is not needed, only the first order perturbation of the metric introduced into the second order Einstein tensor enters. We consider the contributions of galaxy clusters, galaxies, and stars. We give arguments why neither a concentration of clusters and galaxies in bubble walls which surround large voids nor retardation effects are relevant. We find that the ratio $\rho_b/\rho$ between the density $\rho_b$ mimicked by backreaction, and the averaged matter density $\rho$ is small, of the order of $10^{-4}$ to at most $10^{-2}$. To mimic dark energy from backreaction appears practically impossible. Sect. 7 gives a summary.

2 General conditions for the optimal coordinates

For an exact spatially flat ($k = 0$) Robertson-Walker (R-W) metric $g_{\mu\nu}$ one has

$$g_{mn} = a^2(t)\delta_{mn}, \ g_{m0} = 0, \ g_{00} = -1.$$ (2.1)

Consider now a realistic metric with metric tensor $g_{\mu\nu}(x)$, and a given four dimensional volume. We want to define “optimal” coordinates $x'^\mu$ in which the metric within this volume becomes, in a sense to be defined, as close as possible to an exact R-W metric $\bar{g}_{\mu\nu}$ with $k = 0$. It would be impractical to combine all conditions into a single variation problem by minimizing an integral over the sum of appropriate squares. This would lead to rather complicated Euler-Lagrange equations even in a perturbative treatment. It is technically much simpler to proceed in steps. Each of the first four steps approximates a certain property of the exact R-W metric. A scale factor does not show up in the conditions. The optimal scale factor $a(t)$ associated with the given realistic metric is determined at the end of the procedure in step 5.

All integrals in the four steps below are taken over the four dimensional volume under consideration. In step 1 the time coordinate is fixed but arbitrary, and the variation is taken over all primed systems which are time independent coordinate transformations.
of the original one. In steps 2 - 4 we also allow time dependent transformations. These transformations have to respect the restrictions obtained in the previous steps. Because all coordinate systems are admitted in the variation procedure the covariance of the method is automatically guaranteed. Each step restricts the freedom of choice of coordinates more and more, at the end the coordinates are fixed as far as principally possible. Step 1 leads to a transversal perturbation. Usually one will start already with some "reasonable" coordinate system. In this case steps 2, 3, 4 become trivial. In step 5, finally, we define the optimal scale factor \( a(t') \) which gives the "best" approximation of the given metric to an exact R-W metric. Here and in the following

\[
\langle f \rangle = \langle f \rangle(t) = \frac{\int f(y, t)\sqrt{g(y, t)}d^3y}{\int \sqrt{3}g(y, t)d^3y}
\]

(2.2)

denotes the spatial average of \( f(x, t) \).

**Step 1:**

\[
\int \left( \frac{g'_{mn}(x') - \langle g'_{ii}/3 \rangle \delta_{mn}}{\langle g'_{jj}/3 \rangle} \right)^2 \sqrt{-g'(x')}d^4x' = \text{Minimum},
\]

where time is fixed but arbitrary, and the variation is over all time independent coordinate transformations.

(2.3)

**Step 2:**

\[
\int \left( \frac{g'_{n0}(x')}{\sqrt{-g'_{00}\langle g'_{jj}/3 \rangle}} \right)^2 \sqrt{-g'(x')}d^4x' = \text{Minimum},
\]

with the variation taken over all coordinate transformations which respect the restrictions obtained in step 1.

(2.4)

**Step 3:**

\[
\int \left( g'_{00}(x') + 1 \right)^2 \sqrt{-g'(x')}d^4x' = \text{Minimum},
\]

with the variation taken over all coordinate transformations which respect the restrictions obtained in steps 1,2.

(2.5)

**Step 4:**

\[
\int \left( \frac{\partial}{\partial t'}g'_{00}(x') \right)^2 \sqrt{-g'(x')}d^4x' = \text{Minimum},
\]

with the variation taken over all coordinate transformations which respect the restrictions obtained in steps 1,2,3.

(2.6)

**Step 5:**

Define the optimal scale factor \( a(t') \) by

\[
a^2(t') \equiv \langle g'_{ii}/3 \rangle(t').
\]

(2.7)

Summation convention is always understood, also for identical lower indices.

The meaning of the conditions should be obvious. For an exact R-W metric all the integrands would vanish. In steps 1 and 2 we introduced normalization factors in the denominator. This is necessary, because otherwise one would run into an unphysical minimum by a simple scaling of the metric.

The conditions above do not fix the metric completely. They yield a whole class of optimal coordinate systems. This class is, by construction, independent of the system with which one has started. The freedom which remains are the transformations from the eight parameter group of rigid translations in space and time, rigid rotations in space,
and rigid scaling of the space coordinates. This is just the invariance group of coordinate transformations mentioned in the introduction which keep the form of the exact R-W metric intact.

3 First order

We consider a perturbed Robertson-Walker metric of the form

\[ g_{mn}(x) = \tilde{a}^2(t)\delta_{mn} + h_{mn}(x), \quad g_{m0}(x) = h_{m0}(x), \quad g_{00}(x) = -1 + h_{00}(x). \]  (3.1)

The gauge and the way of splitting into background and perturbation is completely arbitrary, except that the perturbation \( h_{\mu\nu} \) should be small. The scale factor \( \tilde{a}(t) \) is in general not identical with the optimal scale factor \( a(t) \) obtained at the end.

Introduce the perturbed metric in the primed system into the integrands in (2.3) - (2.6). The primed system is connected to the old one by an infinitesimal transformation \( x^\mu = x'^\mu + \xi^\mu \). Because all the brackets vanish for the unperturbed metric it is sufficient to expand these up to first order in \( h'_{\mu\nu} \) and \( \xi^\mu \). Furthermore we can put \( \sqrt{-g} = \tilde{a}^3(t) \) and restrict to the leading order in the denominators. Express the \( h'_{\mu\nu} \) in the primed system by the \( h_{\mu\nu} \) in the old one and the shifts \( \xi^\mu \). In lowest order one has \( \xi_m = \tilde{a}^2(t)\xi^m \), \( \xi_0 = -\xi^0 \). The well known transformation laws for the metric in lowest order of \( \xi^\mu \) are:

\[ g'_{mn}(x') = g_{mn}(x) + \xi_m, n + \xi_n, m \]
\[ = \tilde{a}^2(t)\delta_{mn} + h_{mn} + \xi_m, n + \xi_n, m \]  (3.2)
\[ = \tilde{a}^2(t')\delta_{mn} + h_{mn} + \xi_m, n + \xi_n, m - 2\tilde{a}(t')\dot{\tilde{a}}(t')\xi_0 \delta_{mn}, \]  (3.3)
\[ g'_{m0}(x') = g_{m0}(x) + \xi_{m,0} + \xi_{0,m} - 2(\dot{\tilde{a}}(t)/\tilde{a}(t))\xi_m \]
\[ = h_{m0} + \xi_{0,m} + \tilde{a}^2(t')\xi_{m,0}, \]  (3.4)
\[ g'_{00}(x') = g_{00}(x) + 2\xi_{0,0} \]
\[ = -1 + h_{00} + 2\xi_{0,0}. \]  (3.5)

The transformation of the averages \( \langle g'_{\mu\nu}(t') \rangle \), which now refer to a different time \( t' \), is most easily obtained by writing \( d^3y' = \delta(y^0 - t')d^3y' \). This results in

\[ \langle g'_{ii}/3 \rangle = \tilde{a}^2(t) + \langle h_{ii}/3 \rangle + \frac{2}{3}\xi_{ii} + 2\tilde{a}\dot{\tilde{a}}(\xi_0 - \langle \xi_0 \rangle) \]  (3.6)
\[ = \tilde{a}^2(t') + \langle h_{ii}/3 \rangle + \frac{2}{3}\xi_{ii} - 2\tilde{a}\dot{\tilde{a}}(\xi_0). \]  (3.7)

In this way we obtain the integrals which have to be minimized.

Step 1:

We have to minimize

\[ \frac{1}{2} \int d^3y' \delta(y^0 - t')d^3y' \]
\[
\int \left( \frac{g_{,mn}(x') - (g_{,ii}/3)\delta_{mn}}{g'_{,ij}/3} \right)^2 \sqrt{-g'(x')} d^4x' = \quad (3.8)
\]

\[
\int \frac{1}{\hat{a}(t)} \left( h_{mn} - \frac{1}{3}(h_{ii})_{,mn} + \xi_{m,n} + \xi_{n,m} - \frac{2}{3}(\xi_{i,i})_{,mn} - 2\hat{a}(t)\hat{a}(t)(\xi_0 - (\xi_0))_{,mn} \right)^2 d^4x
\]

with respect to \( \xi_m \) while keeping \( \xi_0 \) arbitrary but fixed. Here and in the following it is irrelevant whether we consider the expressions in the old or in the new system. In all brackets the leading terms cancel, changes in the boundary of the volume only contribute to higher order.

From a variation \( \delta \xi_m \) in the interior we obtain the Euler-Lagrange (E-L) equations

\[
\xi_{m,nn} + \xi_{n,mn} + h_{mn,mn} - 2\hat{a}(t)\hat{a}(t)\xi_{0,m} = 0. \quad (3.9)
\]

We use the usual decompositions for \( h_{mn} \) and \( h_{m0} \):

\[
\begin{align*}
    h_{mn} &= \bar{a}^2(t)[A_\delta_{mn} + B_{mn} + C_{m,n} + C_{n,m} + D_{mn}], \\
    h_{m0} &= \bar{a}(t)[F_{m} + G_{m}],
\end{align*}
\]

with

\[
C_{m,m} = 0, \quad D_{mn} = D_{nm}, \quad D_{mn,n} = 0, \quad D_{mn,m} = 0, \quad G_{m,m} = 0. \quad (3.12)
\]

A special solution of the E-L equations (3.9) is then (here \( \Delta \equiv \partial_m \partial_n \))

\[
\xi_{m}^{(s)}(x, t) = -\bar{a}^2(t)\left\{ \frac{1}{2}\Delta^{-1}(A - \frac{2}{\hat{a}(t)}\xi_0)_{,m} + \frac{1}{2}B_{m}(x, t) + C_{m}(x, t) \right\}. \quad (3.13)
\]

Introducing this into the transformation gives \( h'_{mn} \) as in (3.10), where now

\[
\begin{align*}
    A' &= A - \frac{2}{\hat{a}(t)}\xi_0, \quad B' = \Delta^{-1}(A - \frac{2}{\hat{a}(t)}\xi_0), \quad C'_m = 0, \quad D'_{mn} = D_{mn}, \text{ i.e.} \\
    h'_{mn} &= \bar{a}^2(t')[(\partial_m \partial_n - \delta_{mn}\Delta)B' + D_{mn}]. \quad (3.14)
\end{align*}
\]

We have \( A' + \Delta B' = 0 \), which implies that \( h'_{mn} \) is transversal, \( h'_{mn,m} = 0 \).

The solution \( \xi_{m}^{(s)} \) in (3.13) is not unique, because neither the operator \( \Delta^{-1} \) nor the decomposition (3.10) is unique. To see this more explicitly, let \( c \) be a constant and \( \varphi, \psi \) functions with \( \Delta \varphi = \Delta \psi = 0 \). Then one can replace \( A \rightarrow A + c, B \rightarrow B - cx^2/2 - \psi, C_m \rightarrow C_m - \varphi_m, D_{mn} \rightarrow D_{mn} + 2\varphi_m,n + \psi_{mn}, \) without changing \( h_{mn} \). In particular one can always remove a constant \( D_{mn} = D_{mn} \) by putting it into \( B_{mn} \) with \( B = D_{ij}x^i x^j / 2 \). We assume that this has been performed if necessary. For a detailed discussion we write the most general solution of (3.9) as

\[
\xi_{m}(x, t) = \xi_{m}^{(s)}(x, t) + \eta_{m}(x, t). \quad (3.15)
\]
Introducing this into (3.8) leads to a variation problem for $\eta_m$:

$$\int \left( \eta_{m,n} + \eta_{n,m} - \frac{2}{3} \langle \eta_{i,i} \rangle \delta_{mn} + h'_{mn} - \frac{1}{3} \langle h'_{ii} \rangle \delta_{mn} \right)^2 d^3x = \text{Minimum.}$$  \hspace{1cm} (3.16)

A variation in the interior results in the homogeneous equations associated with (3.9), i.e.

$$\eta_{m,n} + \eta_{n,m} = 0.$$  \hspace{1cm} (3.17)

Two tasks have to be done. Firstly one has to consider variations which also involve changes at the boundary. This will yield a special solution for $\eta_m(x,t)$ which fulfills the boundary conditions. Secondly one has to classify the whole set of solutions, in order to determine the remaining freedom.

Concerning the first point it will turn out that the solution is concentrated in a strip along the boundary, with an extension of the wave length of the perturbation $h'_{mn}$. If the wave length is small compared to the extension of the volume it is therefore irrelevant.

As for the second point we will find that the freedom consists in rigid translations, rotations, and scaling, at this stage still with an arbitrary time dependence.

If these results appear obvious, one may skip the following lengthy derivations and proceed directly to (3.24), (3.25) at the end of step 1.

We now treat general variations $\eta_m \rightarrow \eta_n + \delta \eta_m$ which also involve changes at the boundary. This poses a delicate problem, because $\eta_m$ has to fulfill the homogeneous E-L equations in the interior. Therefore it would not help to consider the boundary terms from the partial integration because $\eta_m$ cannot be chosen completely free at the boundary. We therefore proceed in the following way.

We know that $\eta_m$ has to fulfill the homogeneous E-L equations. Consider therefore a complete set of solutions $\eta^{[\alpha]}_m$ of (3.17), expand $\eta_m = \sum_\alpha c_\alpha \eta^{[\alpha]}_m$, and introduce into the variation problem (3.16). Differentiation with respect to the coefficients $c_\alpha$ leads to the linear system of equations

$$I^{[\alpha\beta]}_{C\beta} + \int \left( \eta^{[\alpha]}_{m,n} + \eta^{[\alpha]}_{n,m} - \frac{2}{3} \langle \eta^{[\alpha]}_{i,i} \rangle \delta_{mn} \right) (h'_{mn} - \frac{1}{3} \langle h'_{ii} \rangle \delta_{mn}) d^3x = 0,$$

with

$$I^{[\alpha\beta]} = \int \left( \eta^{[\alpha]}_{m,n} + \eta^{[\alpha]}_{n,m} - \frac{2}{3} \langle \eta^{[\alpha]}_{i,i} \rangle \delta_{mn} \right) (\eta^{[\beta]}_{m,n} + \eta^{[\beta]}_{n,m} - \frac{2}{3} \langle \eta^{[\beta]}_{j,j} \rangle \delta_{mn}) d^3x.$$  \hspace{1cm} (3.18)

We may assume that $I^{[\alpha\beta]}$ is diagonalized, and normalized such that it has eigenvalues 0 and $L^3$ only, where $L$ is some length introduced for dimensional reasons. Eigenvalues 0 belong to solutions of (3.17) which in addition fulfill

$$\eta^{[\alpha]}_{m,n} + \eta^{[\alpha]}_{n,m} - \frac{2}{3} \langle \eta^{[\alpha]}_{i,i} \rangle \delta_{mn} = 0.$$  \hspace{1cm} (3.19)

Contributions of this type solve (3.18) trivially for arbitrary $c_\alpha$ and may always be added. The only solutions of these equations are rigid translations, rotations, and scaling, at this stage still with an arbitrary time dependence. To show this formally, we first observe...
that (3.19) implies \( n_{m,n,k}^{[\alpha]} + n_{n,m,k}^{[\alpha]} = 0 \) (at this point it becomes apparent why we used the average \( \langle g_{\ell i}/3 \rangle \) instead of \( g_{\ell i}/3 \) in (2.3), (3.8)). Applying (3.19) again in order to exchange \( m, k \) and \( n, k \), respectively, implies \(-2n_{k,mn}^{[\alpha]} = 0\), i.e. all second derivatives vanish. Thus \( n_{m}^{[\alpha]} \) can only contain constant and linear contributions. Introducing a last time into (3.19), one finds that the linear part is restricted to a scaling and a rotation.

The coefficients which refer to the non trivial solutions associated with the eigenvalues \( L^3 \) are uniquely fixed, namely

\[
c_{\alpha} = -\frac{1}{L^3} \int (n_{m,n}^{[\alpha]} + n_{m,m}^{[\alpha]} - \frac{2}{3} \langle n_{i,i}^{[\alpha]} \rangle \delta_{mn}^3 (h_{mn}^3 - \frac{1}{3} \delta_{mn}) d^3 x
\]

\[
= -\frac{2}{L^3} \int n_{m}^{[\alpha]} (h_{mn}^3 - \frac{1}{3} \delta_{mn}) n_{n} dA, \quad (3.20)
\]

with \( n_{n} \) the normal vector at the boundary. The term \( \sim \langle n_{i,i}^{[\alpha]} \rangle \delta_{mn} \) in the first line does not contribute, in the second line we performed a partial integration, making use of \( h_{mn}^3 = 0 \).

We give examples with plane waves in \( z \)-direction. Consider first a gravitational wave \( h_{mn}^3 = h_{mn} \cos k z \), with \( h_{11} = -h_{22} = h = \text{const.} \) as the only non vanishing components. This implies \( h_{jj}^3 = 0 \). It is convenient to choose the volume as a cylinder with radius \( r_0 \) and \( 0 \leq z \leq L \). Introduce cylindrical coordinates \( \rho, \phi, z \), together with the corresponding unit vectors \( e_m^{(\rho)} = (\cos \phi, \sin \phi, 0) \), \( e_m^{(\phi)} = (-\sin \phi, \cos \phi, 0) \), \( e_m^{(z)} = (0, 0, 1) \). Because of \( h_{mn}^3 = 0 \) the only surface which contributes in (3.20) is \( \rho = r_0 \), where \( n_{n} = e_n^{(\rho)} \). One has \( e_m^{(\rho)} h_{mn} e_n^{(\rho)} = h \cos 2\phi \), \( e_m^{(\phi)} h_{mn} e_n^{(\rho)} = -h \sin 2\phi \), \( e_m^{(z)} h_{mn} e_n^{(\rho)} = 0 \). This implies that one only needs to consider solutions \( n_{m}^{[\alpha]} \) with a corresponding structure, such that the surface integral in (3.20) is non vanishing. Define the vectors in cylindrical coordinates \( n_{m}^{[\gamma]} = n_{m} e_m^{(\gamma)} \), \( \gamma = \rho, \phi, z \). Then a basis of relevant solutions of the free equation (3.17) is obtained by an ansatz of the form

\[
n_{m}^{[\gamma]} = \begin{pmatrix} f(\rho) \cos 2\phi \cos k z \\ g(\rho) \sin 2\phi \cos k z \\ h(\rho) \cos 2\phi \sin k z \end{pmatrix}, \quad (3.21)
\]

where the index \( (\gamma) \) denotes the components in \( \rho, \phi, z \)-direction. Obviously there is no need to consider any other contributions of \( \cos n\phi \), \( \sin n\phi \). Either they cannot fulfill (3.17), or they give a vanishing \( c_{\alpha} \). If \( L \) is a multiple of the wave length \( \lambda = 2\pi/k \), other modes in \( k z \) will also not contribute to \( c_{\alpha} \).

Introducing into (3.17) leads to three coupled differential equations for the components \( \rho, \phi, z \):

\[
2f'' + 2\frac{f'}{\rho} - 6\frac{f}{\rho^2} - k^2 f + 2g' + 6 \frac{g}{\rho^2} - 2 k h' = 0,
\]

\[
g'' + \frac{g'}{\rho} - \frac{9g}{\rho^2} - k^2 g - 2 \frac{f'}{\rho} - 6 \frac{f}{\rho^2} - 2 k h = 0,
\]

\[
h'' + \frac{h'}{\rho} - \frac{4h}{\rho^2} - 2 k^2 h - k [f' + \frac{f}{\rho} + 2 \frac{g}{\rho}] = 0.
\]
A (non orthonormalized) basis for the regular solutions is

\[
\begin{pmatrix}
  f^{[1]} \\
  g^{[1]} \\
  h^{[1]}
\end{pmatrix} = \begin{pmatrix}
  I_1 & I_1 & 0 \\
  -I_1 & I_1 & 0 \\
  -I_2 & -I_2 & k\rho I_1 - 3I_2
\end{pmatrix},
\]

with \( I_m \equiv I_m(\kappa \rho) \) the modified Bessel functions with argument \( \kappa \rho \).

There is no need to go into more details, the qualitative behavior can be read off immediately. The solutions \( \eta^{[a]}_m \) will be combinations of modified Bessel functions \( I_m(\kappa \rho) \). These are monotonically increasing and behave like \((\kappa \rho/2)^m/m!\) for small \( \kappa \rho \), and like \( \exp(\kappa \rho)/\sqrt{2\pi \kappa \rho} \) for large \( \kappa \rho \). If the radius of the averaging volume is large compared to the wave length, i.e. if \( \kappa \rho_0 \gg 1 \), the modified Bessel functions as well as the shifts \( \eta^{(\gamma)} \) are only relevant in a small strip along the boundary \( \rho = \rho_0 \), with an extension of the order of \( \lambda = 2\pi/k \).

One can normalize \( I^{[\alpha][\beta]} \) by multiplying \( \eta^{[\alpha]}_m \) in (3.23) by factors \( \sim L \exp(-\kappa \rho_0) \).

For the orthonormalized solutions one gets \( c\eta^{[\alpha]}_m \sim (\tilde{h}/k)\sqrt{\rho_0/\rho} \exp(-\kappa \rho_0) \exp(\kappa \rho), \) i.e. \( c\eta^{[\alpha]}_m \sim \tilde{h}/k \) at the boundary, independent of \( \rho_0 \) and \( L \). Therefore, if desired, one may perform the limit \( \rho_0 \to \infty \), and/or \( L \to \infty \) and one is sure that the solution stays finite.

If the boundaries for \( z \) are less convenient, i.e. if \( L \) is not a multiple of \( \lambda \), the situation is slightly more complicated. Instead of (3.21) one needs a superposition of terms with \( \cos 2\pi nz/L \) and \( \sin 2\pi nz/L, \ n = 0, 1, 2, \cdots \). The coefficients in front are only sufficiently large for \( n \) with \( n \sim kL/2\pi \), the coefficients for smaller \( n \) decrease in the same way as the extension of the modified Bessel functions increases, again the solution is concentrated at the boundary \( \rho = \rho_0 \).

In our next example we consider a wave with \( B' = B \cos kz \), which implies \( h'_{mn} = \tilde{h}_{mn} \cos kz \), with \( \tilde{h}_{11} = \tilde{h}_{22} = \tilde{h} = \tilde{B}k^2 \) as the only non vanishing components. Again, for a wave, \( \langle h'_{ij} \rangle \approx 0 \). We now have \( e^{(\rho)}_m \tilde{h}_{mn} e^{(\rho)}_n = \tilde{h}, \ e^{(\varphi)}_m \tilde{h}_{mn} e^{(\varphi)}_n = e^{(\varphi)}_{mn} \). Therefore one can use an ansatz like (3.21), with \( \cos 2\varphi \) and \( \sin 2\varphi \) replaced by 1. The further calculation proceeds as before with an analogous result.

The previous findings are very convenient. For a large volume one can neglect the boundary effects, or alternatively, apply the result of the transformation only within a slightly smaller volume.

After step 1 we have transformed to a system where

\[
g^{(1)}_{mn} = \bar{a}^2(t')\delta_{mn} + h'_{mn} = \bar{a}^2(t')[\delta_{mn} + (\partial_m \partial_n - \delta_{mn}\Delta)B' + D'_{mn}],
\]

i.e. \( A' + \Delta B' = 0 \), \( C_m = 0 \) \( \Rightarrow h'_{mn;n} = 0 \).

The freedom of remaining transformations in the new system, due to the freedom in the homogeneous solution \( \eta^m \), is now restricted to

\[
\begin{align*}
\xi^m &= \xi_m/\bar{a}^2(t) = b^m(t) + S(t)x^m + [\omega(t) \times x]^m, \\
\xi_0 &= \text{completely arbitrary}.
\end{align*}
\]
The functions \( b^m(t), S(t), \omega^k(t) \) still have an arbitrary time dependence at this stage.

Before proceeding we will assume that the above transformations have been performed, such that (3.24), (3.25) hold. These conditions have to be maintained in the following steps. We now drop the primes for the optimized system obtained after step 1, and use the prime for the optimized system of step 2.

**Step 2:**

According to (2.4) we have to minimize

\[
\int \left( \frac{g'_m(x')}{{\sqrt{g''(x')}}} \right)^2 \sqrt{-g'(x')} d^4x' = \int \tilde{a}(t) \left( h_{m0} + \xi_0^m + \bar{a}^2 \xi^m \right)^2 d^4x, \tag{3.26}
\]

where \( \xi^m \) is now restricted to the special form (3.25). The functions in the minimization procedure are \( b^m(t), S(t), \omega^k(t), \) and \( \xi_0(x,t) \).

Introducing the decomposition (3.11) and writing the terms with \( b^m(t) \) and \( S(t)x^m \) in \( \xi^m \) as gradients one obtains

\[
h'_m = h_{m0} + \xi_0^m + \bar{a}^2(t) \xi^m
= \left\{ \xi_0 + \bar{a}(t) F + \bar{a}^2(t) \left[ \hat{b}^k(t)x^k + \dot{S}(t)x^2/2 \right] \right\}^m + \bar{a}(t) G_m + \bar{a}^2(t) [\hat{\omega}(t) \times x]^m. \tag{3.27}
\]

Obviously one cannot determine \( \xi_0, \hat{b}^k(t), \dot{S}(t) \) separately because they only enter in the combination

\[
\tilde{\xi}_0(x,t) = \xi_0(x,t) + \bar{a}(t) F + \bar{a}^2(t) \left[ \hat{b}^k(t)x^k + \dot{S}(t)x^2/2 \right]. \tag{3.28}
\]

Therefore we may, at this stage, choose \( \hat{b}^k(t) \) and \( \dot{S}(t) \) arbitrary and vary only \( \tilde{\xi}_0(x,t) \) and \( \hat{\omega}^k(t) \).

Only for simplicity and the sake of obtaining more transparent formulae, from now on we specialize to the case that the volume is a sphere of radius \( r_0 = r_0(t) \) around the origin. Obviously this condition is coordinate dependent, nevertheless it does not destroy the covariance of the procedure. In any other system, obtained by an infinitesimal transformation, the volume is a distorted sphere with an infinitesimal modification of the boundary. This only introduces irrelevant boundary effects of higher order.

The E-L equation and boundary condition (\( n_m \) denotes the normal vector at the boundary of the three dimensional sphere) for \( \tilde{\xi}_0 \) become

\[
\Delta \tilde{\xi}_0 = 0, \tag{3.29}
\]

\[
(\tilde{\xi}_0^m + \bar{a}(t) G_m) n_m = 0 \text{ at the boundary}.
\]

We dropped the boundary term \( \bar{a}^2(t)[\hat{\omega}(t) \times x]^m n_m \) which vanishes for a sphere.

The situation is similar to that in the first step, but simpler because we now can use the Neumann type boundary conditions for \( \tilde{\xi}_0 \). As an example consider a plane transversal wave \( G_m = \bar{G} \delta_{mx} \cos kz \). This gives \( G_m n_m = \bar{G} \sin \theta \cos \varphi \cos(kr_0 \cos \theta) \).
solution of (3.29) can then be expanded as \( \tilde{\xi}_0 = \overline{G} \sum_{l=1}^{\infty} c_l(r/r_0)^l P_l^l(\cos \theta) \cos \varphi \), with
\[
c_i = -[\bar{a}(t) r_0 (2l+1)/(2l+1)] \int_0^r a^2(t) \Theta^l \cos \Theta \cos(kr_0 \cos \Theta) d\Theta.
\]
If \( kr_0 = 2\pi r_0/\lambda \gg 1 \), the factor \( \cos(kr_0 \cos \Theta) \) oscillates rapidly, therefore the integral is only important if this oscillation matches with the oscillation of the Legendre polynomial, i.e. if \( l \approx 2kr_0/\pi \). This is large, therefore the factor \( (r/r_0)^l \) is only relevant near the boundary. In the interior the solution is essentially zero. We may therefore replace (3.29) by the corresponding Neumann problem with \( G_m = 0 \) which has the unique solution \( \tilde{\xi}_0 = -\tau(t) \), where \( \tau \) is constant in space but may depend on time. Therefore (3.28) gives
\[
\xi_0(x,t) = -\tau(t) - \bar{a}(t) F - \bar{a}^2(t) [\dot{\bar{b}}^k(t)x^k + \dot{\bar{S}}(t)x^2/2].
\]
Next we vary \( \dot{\omega}^k(t) \). Because this is a function of \( t \) only, the \( x \)-integration remains and we obtain
\[
\dot{\omega}^k(t) = - \frac{3}{2\bar{a}(t) \langle x^2 \rangle} \epsilon_{kln} \langle x^l G_n \rangle.
\]
We used some simplifications for the case of a sphere, and the result just obtained for \( \tilde{\xi}_0 \).

After applying the transformation in (3.27) with the results (3.30) and (3.31) we have transformed to a system where
\[
h_{m0}' = \bar{a}(t') G_m', \quad \text{with} \quad G_m' = G_m + 3\bar{a}(t') \langle x^m G_n - x^n G_m \rangle x^n/2,
\]
i.e.
\[
F' = 0, \quad \Rightarrow \quad h_{m0;m} = 0.
\]

The remaining freedom for coordinate transformations in the new system is now
\[
\xi^m = \xi_m/\bar{a}^2(t) = b^m(t) + S(t)x^m + [\omega \times x]^m,
\]
\[
\xi^0 = -\dot{\xi}_0 = \tau(t) + \bar{a}^2(t) [\dot{\bar{b}}^k(t)x^k + \dot{\bar{S}}(t)x^2/2].
\]

We are no longer free to perform different rotations at different times, \( \omega^k = \text{const.} \), while translations \( b^m(t) \), scalings \( S(t) \), as well as \( \tau(t) \), can still be time dependent.

Again we assume that the transformations have been performed and that the following steps respect (3.32), (3.33). The primes will be dropped.

**Step 3:**
We have to minimize
\[
\int \left( g_{00}'(x') + 1 \right)^2 \sqrt{-g'(x')} d^4x' = \int \bar{a}^3(t) \left( h_{00} + 2\xi_{0;0} \right)^2 d^4x.
\]

Using the result (3.33) for \( \xi_0 \), the bracket becomes
\[
h_{00} + 2\xi_{0;0} = h_{00} - 2\frac{\partial}{\partial t} \left\{ \tau(t) + \bar{a}^2(t) [\dot{\bar{b}}^k(t)x^k + \dot{\bar{S}}(t)x^2/2] \right\}.
\]

Variation of \( \tau(t), \dot{\bar{b}}^k(t), \dot{\bar{S}}(t) \) gives the equations (3.36) - (3.38) below, in which we already performed the spatial integrations where possible. In the derivation we used some
The solutions are

\[
\begin{align*}
\hat{\tau}(t) &= \frac{1}{2} \frac{(x^2)^2 \langle h_{00} \rangle - \langle x^2 \rangle \langle x^2 h_{00} \rangle}{2 (\langle x^2 \rangle - (x^2)^2)}, \\
\frac{\partial}{\partial t} (\bar{a}^2(t) \hat{b}(t)) &= \frac{3}{2} \frac{\langle x^2 h_{00} \rangle}{\langle x^2 \rangle}, \\
\frac{\partial}{\partial t} (\bar{a}^2(t) \hat{S}(t)) &= \frac{\langle x^2 h_{00} \rangle - \langle x^2 \rangle \langle h_{00} \rangle}{\langle (x^2)^2 \rangle - (x^2)^2}.
\end{align*}
\]

The function \(\tau(t)\) is now fixed up to an additive constant \(\tau\), while \(\hat{b}(t)\) and \(\hat{S}(t)\) are fixed up to two integration constants \(b^m, \beta^m\), and \(S, \Sigma\), respectively.

\[
b^m(t) = \hat{b}^m(t) + b^m + \beta^m \int t \frac{dt'}{\bar{a}^2(t')}, \quad S(t) = \hat{S}(t) + S + \Sigma \int t \frac{dt'}{\bar{a}^2(t')},
\]

with \(\hat{b}^m(t)\) and \(\hat{S}(t)\) obtained from integrating (3.40) and (3.41).

After having performed the transformations of step 3 one has the equations \(\langle h'_{00} \rangle = \langle x^m h'_{00} \rangle = \langle x^2 h'_{00} \rangle = 0\). The remaining freedom for transformations is now

\[
\xi^m = \xi_m \bar{a}^2(t) = b^m + S x^m + [\omega \times x]^m + (\beta^m + \Sigma x^m) \int t \frac{dt'}{\bar{a}^2(t')},
\]

\[
\xi^0 = -\xi_0 = \tau + \beta^k x^k + \Sigma x^2 / 2,
\]

where \(b^m, S, \omega^k, \tau, \beta^m, \Sigma\) are all constant. As before we assume that the transformations have been performed and that the form of further transformations is restricted to (3.43).

**Step 4:**

Before proceeding with this step we look at the meaning of the constants \(\beta^m\) and \(\Sigma\). Obviously \(\beta^m\) describes an infinitesimal boost (slightly modified because \(\bar{a}^2(t) \neq \text{const.}\)), while \(\Sigma\) describes a time dependent scaling. A transformation with (3.43) gives a space dependent contribution \(2\bar{a} \hat{a} \beta^k x^k + \Sigma x^2 / 2\) \(\delta_{mn}\) in \(g'_{mn}\), which illustrates that boosts do not leave the standard form of the R-W metric invariant. This contribution is, however,
suppressed by the factor $\tilde{a}$, therefore a minimization along the previous lines appears ineffective. We therefore choose condition (2.6) in step 4, and minimize

$$\int \left( \frac{\partial}{\partial t'} g_{00}(x') \right)^2 \sqrt{-g'(x')} d^4x' = \int \tilde{a}^3(t) \left( h_{00,0} + \frac{1}{\tilde{a}^2(t)} (\beta^k + \Sigma x^k) h_{00,k} \right)^2 d^4x.$$  

We are left with an ordinary minimization problem for the four constants $\beta^k$ and $\Sigma$. Define the integrals

$$\begin{align*}
\{ I \} &= \int \frac{1}{\tilde{a}(t)} \left\{ \begin{array}{c} x^m x^n \\ x^n \\ 1 \end{array} \right\} h_{00,m} h_{00,n} d^4x, \\
\{ J \} &= \int \tilde{a}(t) \left\{ \begin{array}{c} x^m \\ 1 \end{array} \right\} h_{00,m} h_{00,0} d^4x. 
\end{align*}$$

The linear system for $\beta^k$ and $\Sigma$ then becomes

$$\begin{align*}
I \Sigma + I_n \beta^n &= -J \\
I_m \Sigma + I_{mn} \beta^m &= -J_m. 
\end{align*}$$

This can be easily solved after $h_{00}$ is specified. Practically it is even simpler. For a large averaging sphere the volume integrals are rotation invariant, which implies $I_m = J_m = 0$ and $I_{mn} = \delta_{mn} I_{ii}/3$, leading to $\beta^m = 0, \Sigma = -J/I$.

In the transformed system we will then have $J' = 0, J'_m = 0$.

The nature of step 4 is somewhat different from the steps before. While steps 1, 2, 3 only require that $h_{\mu \nu}$ is small, step 4 requires in addition that time variations are small compared to spatial variations.

Our optimization procedure has now come to an end. The integration constants $\beta^m, \Sigma$ are fixed. After step 4 the only allowed transformations which remain are those which keep the form of the R-W metric invariant.

$$\begin{align*}
\xi^m &= \xi_m/\tilde{a}^2(t) = b^m + S x^m + [\omega \times x]^m, \\
\xi^0 &= -\xi_0 = \tau,
\end{align*}$$

with $b^m, S, \omega^k, \tau$ all constant.

**Step 5:**

In the last step we determine the optimal scale factor $a(t)$. In (3.24) we obtained the form $g_{mn}^{(1)}(t) = \tilde{a}^2 (t) [(1 - \Delta B) \delta_{mn} + B_{mn} + D_{mn}]$, where we denote the first order result for the optimized metric by an index (1). According to (2.7) we define

$$a^2(t) \equiv \langle g_{ii}^{(1)} / 3 \rangle(t) = \tilde{a}^2(t) \left[ 1 - \frac{2}{3} \langle \Delta B \rangle \right].$$

(3.48)
Eliminating $a^2(t)$ gives \( g^{(1)}_{mn} = a^2(t)[(1 - \Delta B + \frac{2}{3}\langle\Delta B\rangle)\delta_{mn} + B_{mn} + D_{mn}] \), which can finally be written in the form

\[
g^{(1)}_{mn} = a^2(t)\left[\delta_{mn} + (\partial_m\partial_n - \delta_{mn}\Delta)B^{(1)} + D^{(1)}_{mn}\right],
\]

with \( B^{(1)} = B - \langle\Delta B\rangle x^2/6 \), \( D^{(1)}_{mn} = D_{mn} \). One now has \( \langle h^{(1)}_{mn}\rangle = -2a^2\langle\Delta B^{(1)}\rangle = 0 \).

The whole procedure is simpler than it appears. Usually one starts already with an ansatz for the fluctuations in a "reasonable" coordinate system. Then steps 2 - 5 may become trivial from simple symmetry arguments, i.e. one is already using the optimal coordinates and no further transformations are necessary.

**Summary of the first order transformation**

We found that the transformed metric \( g^{(1)}_{\mu\nu} \) after the first order optimization has the form

\[
g^{(1)}_{mn} = a^2(t)\delta_{mn} + h_{mn} = a^2(t)[\delta_{mn} + (\partial_m\partial_n - \delta_{mn}\Delta)B + D_{mn}],
\]

\[
g^{(1)}_{m0} = h_{m0} = a(t)G_m,
\]

\[
g^{(1)}_{00} = -1 + h_{00}.
\]

The approximating R-W metric \( \overline{g}_{\mu\nu} \) is obtained by dropping the perturbations \( B, D_{mn}, G_m, h_{00} \). This provides a natural basis for splitting into background and perturbation, and for performing perturbative calculations.

The quantities which could not be removed by gauge transformations nevertheless share some properties of the unperturbed metric. Several relations can be derived by appropriate partial integrations and some simplifications which hold for a sphere:

\[
h_{mn,n} = 0, \quad \langle h_{ii}\rangle = 0, \quad \langle x^n h_{mn}\rangle = 0, \quad \langle (x^m x^n + x^2 \delta^{mn}/2) h_{mn}\rangle = 0,
\]

\[
h_{m0,m} = 0, \quad \langle x^m h_{m0}\rangle = 0, \quad \langle x_n h_{m0} - x_m h_{n0}\rangle = 0,
\]

\[
\langle h_{00}\rangle = 0, \quad \langle x^m h_{00}\rangle = 0, \quad \langle x^2 h_{00}\rangle = 0,
\]

\[
\int a^3(t) \left\{ x^m \right\}_{1} h_{00,m} h_{00,0} d^4 x = 0.
\]

The relations \( \langle h_{mn}\rangle = 0, \quad \langle h_{00}\rangle = 0 \), will lead to the absence of backreaction in first order.

### 4 Second order

We will see in sect. 5 that the second order of the transformation to the optimal gauge is not needed if one neglects time derivatives in the background metric and in the perturbed density. Nevertheless, from a principle point of view, it is instructive to show how the procedure can be extended to second order in a quite simple way.
Rather than trying an approach by brute force, one should proceed in an iterative way. Determine the shift $\xi_\mu$ which leads to the optimal system in first order. If one uses, instead of (3.2) - (3.5), the exact transformation formulae, or at least, considers $\xi_\mu$ up to second order, one obtains a perturbation of the form $h_{\mu\nu} = h^{(1)}_{\mu\nu} + \tilde{h}_{\mu\nu}$ in the optimal system. All observers, irrespective of their original gauge, have ended up with the same $h^{(1)}_{\mu\nu}$, up to the remaining 8-parameter group of rigid transformations. For the higher order contributions $\tilde{h}_{\mu\nu}$ which have not been optimized this is not the case. Unlike $h^{(1)}_{\mu\nu}$ they do depend on the original gauge. An explicit calculation would be tedious. We will, however, see that $\tilde{h}_{\mu\nu}$ does not contribute to backreaction, therefore there is no need to calculate it.

We now start the second order calculation with the metric $h_{\mu\nu} = h^{(1)}_{\mu\nu} + \tilde{h}_{\mu\nu}$. Within the brackets in (2.3) - (2.6) we have to expand the metric $g'_{\mu\nu}$ up to second order in the perturbation and to perform the minimizations in the four steps. There is a considerable simplification due to the iterative procedure. Because $h^{(1)}_{\mu\nu}$ is the solution of the first order problem there are no first order terms in $\xi_\mu$. Consequently $\xi_\mu$ needs only to be considered in lowest order, i.e. one can again apply the simple transformation formulae (3.2) - (3.5).

It is convenient to include the factor $\sqrt{-g'}$ into the squares by writing

$$\sqrt{-g'} = a^3 \left[ 1 + \frac{1}{4} \left( \frac{h^{(1)}_{ii}}{a^2} - h^{(1)}_{00} \right) \right]^2. \quad (4.1)$$

The scale factor $\tilde{a}$ has been replaced by the optimized scale factor $a$ of the first order. The brackets have now to be considered in second order of the perturbation. Due to the properties $\langle h^{(1)}_{ii} \rangle = 0$ and $\langle h^{(1)}_{00} \rangle = 0$, there are no corrections from the averages in the denominators of steps 1,2.

The shift $\xi_\mu$ is a superposition of two terms, $\xi_\mu = \tilde{\xi}_\mu + \xi^{(Q)}_\mu$. The first term, $\tilde{\xi}_\mu$, corresponds to the solution of the E-L equations with $h_{\mu\nu}$ replaced by $\tilde{h}_{\mu\nu}$. This part transforms $\tilde{h}_{\mu\nu}$ into $\tilde{h}^{(2)}_{\mu\nu}$ in the same way as previously it transformed $h_{\mu\nu}$ into $h^{(1)}_{\mu\nu}$. Therefore $\tilde{h}^{(2)}_{\mu\nu}$ shares the properties (3.53), (3.55), i.e. $\langle \tilde{h}^{(2)}_{ii} \rangle = 0$, $\langle \tilde{h}^{(2)}_{00} \rangle = 0$, consequently it will not lead to any backreaction, and one does not need $\tilde{\xi}^{(2)}_\mu$ and $\tilde{h}^{(2)}_{\mu\nu}$.

The second contribution, $\xi^{(Q)}_\mu$, is due to the quadratic terms in $h^{(1)}_{\mu\nu}$ in the brackets in (2.3) - (2.6), which now have to be inserted into the E-L equations. This part could, in principal, lead to a backreaction. As mentioned before, we will however see that this does not happen if we drop the time dependence.

### 5 Averaging

In first order of the perturbation we finally obtained an optimal coordinate system with a metric of the special form (3.50) - (3.52), and the approximating R-W metric $\overline{g}_{\mu\nu}$ obtained by dropping the perturbations. One can now simply define spatial averages of arbitrary tensors in the naive way as in (2.2), i.e.
\[ \langle A_{\mu\nu...} \rangle(t) = \frac{\int A_{\mu\nu...}(y, t)\sqrt{\det g(y, t)}d^3y}{\int \sqrt{\det g(y, t)}d^3y}. \] (5.1)

The freedom in the choice of coordinates is restricted to the eight parameter group of global transformations described by the parameters \( b_m, S, \omega_k, \tau \), and these transformations commute with the operation (5.1) of averaging.

It is not necessary to restrict to averaging of scalar quantities or to decompose into invariants (usually only with respect to purely spatial transformations) and perform the averages for the latter. In fact such an approach can be problematic. The expansion tensor \( \Theta_{\mu\nu} = \frac{g_{\mu\nu}}{2} \) is often decomposed into invariants, and quantities which enter linearly and quadratically are averaged separately (see e.g. [2]). This procedure can lead to quite strange consequences like negative averages of positive definite expressions. Nothing of this kind can happen in our case.

For the special case of the metric we note that \( \langle g_{\mu\nu} \rangle \) is not necessarily identical with \( g_{\mu\nu} \). Due to (3.53), (3.55) the relation is, however, true for \( g_{00} \) and for \( g_{mm} \) in first order, i.e. \( \langle g_{00}^{(1)} \rangle = \bar{g}_{00} \) and \( \langle g_{mm}^{(1)} \rangle = \bar{g}_{mm} \). These are the relevant quantities for backreaction which we discuss now.

### 6 Backreaction

In the same way as the perturbed metric, \( g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \), we split the Einstein tensor, \( G_{\mu\nu} = \bar{G}_{\mu\nu} + \delta G_{\mu\nu} \), with \( \bar{G}_{\mu\nu} \) the Einstein tensor associated with \( \bar{g}_{\mu\nu} \). Using the Einstein equations (we include a cosmological constant \( \Lambda \)) for \( G_{\mu\nu} \) one has

\[
\bar{G}_{\mu\nu} = \langle G_{\mu\nu} \rangle = \langle G_{\mu\nu} \rangle - \langle \delta G_{\mu\nu} \rangle = \kappa \langle T_{\mu\nu} \rangle - \Lambda \langle g_{\mu\nu} \rangle - \langle \delta G_{\mu\nu} \rangle. \] (6.1)

The first two terms, \( \kappa \langle T_{\mu\nu} \rangle - \Lambda \langle g_{\mu\nu} \rangle \), describe the equations which one would expect from the averaged energy-momentum tensor and metric, the third one, \( -\langle \delta G_{\mu\nu} \rangle \), is the deviation from this, i.e. the backreaction. The essential quantities associated with density and pressure mimicked by backreaction are \( \kappa \rho_b = -\langle \delta G_{00} \rangle \) and \( \kappa p_b = -\langle \delta G_{mm} \rangle / 3 \). In first order of the perturbation one has (indices are raised and lowered with the background metric \( \bar{g}_{\mu\nu} \))

\[
\delta G_{00}^{(1)} = \frac{1}{2} (h_{ij}^{ij} - h_{ij}^{ij}) - 2 \frac{\dot{a}}{a^2} h_i^i + \frac{\dot{a}}{a} h_{i0}^0 - 2 \frac{\dot{a}}{a} h_{0i}^i, \] (6.2)
\[
\delta G_{mm}^{(1)} = \frac{1}{2} (h_{ij}^{ij} - h_{ij}^{ij}) - 2 \frac{\dot{a}}{a^2} h_i^i + \frac{\dot{a}}{a} h_{i0}^0 - h_{i0}^i + 2 \frac{\dot{a}}{a} h_{0i}^0 + 2h_{00}^i \\
-3(\frac{\ddot{a}}{a^2} + 2\frac{\dot{a}}{a}) h_{00}^0 - h_{00}^i - 3 \frac{\dot{a}}{a} h_{00,0}. \] (6.3)

If one inserts \( h_{\mu\nu}^{(1)} \) and uses the properties (3.53) - (3.55) one observes that there are three types of terms in \( \delta G_{00}^{(1)} \) and \( \delta G_{mm}^{(1)} \):
a) terms which vanish,
b) terms where the spatial average vanishes,
c) terms which are spatial derivatives and can be written as surface contributions in the integral which are irrelevant for large volumes.

Therefore \( \langle \delta G^{(1)}_{00} \rangle = \langle \delta G^{(1)}_{mm} \rangle = 0 \), there is no backreaction in first order. This is in fact a property which one expects from any reasonable lowest order averaging prescription, where positive and negative contributions of the fluctuations should cancel.

We now come to the second order. Here one has two types of contributions. The first one arises from introducing the second order correction \( h^{(2)}_{\mu\nu} \) of the metric into the first order correction \( \langle \delta G^{(1)}_{\mu\nu} \rangle \) of the Einstein tensor. In general this does not vanish because \( h^{(2)}_{\mu\nu} \) does not fulfill the properties (3.53) - (3.55). For an estimate one can, however, use a simple static approximation (we will comment on retardation below) which is appropriate for the present day universe. The peculiar velocities of galaxies are small, with \( v/c \) of the order of \( 10^{-3} \). We also do not have sizeable perturbations with extremely short wave length which could contribute large derivatives. Let the extension in time of the averaging volume be small compared to the Hubble time. Then the variation of \( a(t) \), can be neglected in comparison with the variation in space. It is convenient to fix \( a(t_0) = 1 \) for the present time \( t_0 \). The surviving parts of \( \langle \delta G^{(1)}_{00} \rangle \) and \( \langle \delta G^{(1)}_{mm} \rangle \) in (6.2), (6.3) only contain spatial derivatives, i.e. the averages vanish up to irrelevant boundary terms. This holds irrespective of the special form of \( h^{(2)}_{\mu\nu} \) and is, of course, very convenient. There is no need to calculate \( h^{(2)}_{\mu\nu} \).

The second contribution arises from introducing the first order correction \( h^{(1)}_{\mu\nu} \) of the metric into the second order correction \( \langle \delta G^{(2)}_{\mu\nu} \rangle \) of the Einstein tensor. The rather lengthy expression for \( \delta G^{(2)}_{\mu\nu} \) can be found in Wetterich [10] in the approximation that derivatives acting on the background metric are neglected. It will not be written down here. We only give the two quantities needed for the effective density and pressure, anticipating \( h_{m0} = 0 \), neglecting the time dependence in the perturbation, and performing some spatial partial integrations.

\[
\langle \delta G^{(2)}_{00} \rangle = \left\{ \begin{aligned}
\frac{1}{2} h_{00} h_{i, j} + & \frac{1}{2} h_{00} h_{ij, ij} + \frac{1}{8} h_i^j h^j k_k + \frac{1}{8} h_i^j h_{ij, k} + \frac{1}{4} h_i^j h_{ik, j} \end{aligned} \right\}, \\
\langle \delta G^{(2)}_{mm} \rangle = \left\{ \begin{aligned}
-\frac{1}{2} h_{00} h_{00, i} + & \frac{1}{2} h_{00} h_{ij, ij} + \frac{3}{8} h_i^j h^j k_k - \frac{5}{8} h_{ij, ij, k} - h_i^j h_{jk, jk} + \frac{5}{4} h_{ij, jk, k_i} \end{aligned} \right\}.
\tag{6.4}
\tag{6.5}
\]

In order to estimate \( h_{\mu\nu} \equiv h^{(1)}_{\mu\nu} \) from the sources we use again the simple approximation above, i.e. neglect any time dependence. This is essentially the model of Wetterich [10]. It has the advantage that it is transparent and leads to explicit formulae.

Consider a dust universe with \( \rho \) only weakly time dependent and \( p = 0 \). The solution for the perturbed metric is only needed in lowest order and most conveniently first derived in the harmonic gauge (marked by a hat)

\[
\hat{h}_{\mu, \nu} = \frac{1}{2} \hat{h}^{\nu, \mu},
\tag{6.6}
\]
where the perturbed Friedmann equations have the simple form
\[ \hat{h}_{\mu\nu}^\rho = -2\kappa(\delta T_{\mu\nu} - \frac{1}{2}\delta T^\rho_{\rho} g_{\mu\nu}). \] (6.7)

In our simple model \( \delta \hat{T}_{00} = \delta \rho \) is time independent while all the other components of \( \delta \hat{T}_{\mu\nu} \) vanish. This implies
\[ \Delta \hat{h}_{00} = -\kappa \delta \rho, \] furthermore \( \hat{h}_m^m = \delta_m^0 \hat{h}_{00}, \hat{h}_{m0} = 0. \] (6.8)

One can transform back from the harmonic gauge to our optimal (transversal) gauge by a purely spatial shift \( \xi^m = -\Delta^{-1} \hat{h}_{00,m}/2 \) which leads to
\[ h_{00} = \hat{h}_{00}, \quad h_m^n = (\delta_m^n - \Delta^{-1} \partial_m \partial^n)h_{00}, \quad h_{m0} = 0, \quad \delta T_{\mu\nu} = \delta \hat{T}_{\mu\nu}. \] (6.9)

Introducing (6.9) into (6.4), (6.5), all terms can be expressed by
\[ \langle h_{00} \Delta h_{00} \rangle = -\kappa \langle h_{00} \delta \rho \rangle, \] and one obtains
\[ \rho_b = -\frac{1}{\kappa} \langle \delta G^{(2)}_{00} \rangle = \frac{7}{4} \langle h_{00} \delta \rho \rangle \] (6.10)
\[ p_b = -\frac{1}{3\kappa} \langle \delta G^{(2)}_{mm} \rangle = -\frac{1}{12} \langle h_{00} \delta \rho \rangle = -\frac{1}{21} \rho_b. \] (6.11)

The pressure term coincides with the result of [10], the density term has a factor 7/4 instead of 9/4 in [10]. This shows that the correction is smaller in our “optimal” gauge as one would expect. It also illustrates that differences between reasonable gauges are small.

We now consider a more specific simple model. We start with a hierarchy of clusters, composed of galaxies, composed of dark matter and stars. Subsequently we will also discuss the modifications due to the presence of voids.

Consider first a space filled with homogeneous spherical objects of radius \( L \), density \( \hat{\rho} \), and mass \( m = 4\pi \hat{\rho}L^3/3 \), which are roughly uniformly distributed in space at positions \( \mathbf{r}_i \). The average density is denoted by \( \bar{\rho} \). The corresponding density fluctuation is
\[ \delta \rho(\mathbf{r}) \equiv \rho(\mathbf{r}) - \bar{\rho} = \hat{\rho} \sum_i \Theta(L - |\mathbf{r} - \mathbf{r}_i|) - \bar{\rho}, \] (6.12)
such that the average \( \overline{\delta \rho} \) vanishes. To find a useful approximation to the corresponding \( h_{00}(\mathbf{r}) \) we first define a distance \( D \) by the requirement that the average of \( \delta \rho \) vanishes within a sphere of radius \( D \) around a source, i.e. \( D^3/L^3 = \hat{\rho}/\bar{\rho} \). This \( D \) is roughly half of the average distance between the spheres. For well separated sources one has \( D \gg L \). A solution of (6.8) within this sphere (chosen around the origin for simplicity) is then
\[ h_{00}(\mathbf{r}) = \kappa \left\{ \hat{\rho} \left( \frac{L^2}{2} - \frac{r^2}{6} \right) \Theta(L - r) + \hat{\rho} \frac{L^3}{3r} \Theta(r - L) + \bar{\rho} \frac{r^2}{6} + \hat{\rho} c \right\}. \] (6.13)

This is just the well known potential of a uniformly charged sphere in a constant background. The constant \( c \) has been introduced in order to achieve \( \int_{r \leq D} h_{00} d^3x = 0 \). It is of the order \( L^3/D \) and will turn out to be irrelevant. Of course (6.13) is not an exact
solution of (6.8), (6.12). The spheres of radius $D$ around the sources overlap in some areas and leave empty regions elsewhere. Furthermore $h_{00} \sim \kappa \hat{\rho} L^2 / 3 \neq 0$ at the boundary of the circle, thus introducing boundary contributions there. The average of the fictitious mass distribution associated with these corrections vanishes, and the location is a distance $\approx D \gg L$ away from the sources where $h_{00}$ enters in (6.10), (6.11). Therefore the resulting corrections to $h_{00}$ are suppressed compared to (6.13) and can be ignored. The average $\langle h_{00} \rangle$ will already vanish when taken over regions involving only a modest number of sources. This implies that also $\langle x^m h_{00} \rangle = 0$ and $\langle x^2 h_{00} \rangle = 0$, as required in (3.55).

We are interested in the ratio $\rho_b / \bar{\rho} = (7/4) \langle h_{00} \delta \rho \rangle / \bar{\rho}$. When calculating $\langle h_{00} \delta \rho \rangle$ one can drop the constant $-\bar{\rho}$ in (6.12) because $\langle h_{00} \rangle = 0$, i.e. one can replace $\delta \rho(r)$ by $\rho(r)$. Only the regions $|r - r_i| \leq L$ where $\rho(r) \neq 0$ contribute in the product. This results in a considerable simplification. It is now straightforward to calculate $\langle h_{00} \delta \rho \rangle$ from (6.12) and (6.13). Dropping corrections which are suppressed by higher powers of $L / D$ one finds the following result which can be written in various useful ways.

$$
\rho_b / \bar{\rho} = \frac{7}{4} \langle h_{00} \delta \rho \rangle / \bar{\rho} = \frac{7}{10} \kappa \hat{\rho} L^2 = \frac{7}{10} \kappa \bar{\rho} \frac{D^3}{L} = \frac{21}{40} \frac{m}{\kappa L} = \frac{21}{10} h_{00}(L).
$$

(6.14)

Before proceeding we give a more careful justification for our neglect of time dependence. It has been argued by Kolb, Matarrese, Notari, Riotto [11], as well as by Bochner [12], that retardation effects, though irrelevant for nearby sources, may become important when taking into account contributions from distant regions. We thus look at retardation effects. Let $D$ be the average distance between the sources, $v \ll c$ their average velocity, and define a distance $\overline{D}$ such that $(v/c)D \ll \overline{D} \ll D$. Retardation is negligible as long as the sources can only move a small fraction of $D$ within the retardation time, i.e. for $(v/c)|r - r'| \leq \overline{D} \ll D$. In the additional contribution $\tilde{h}_{00}$, where retardation might become relevant, one has the retarded solution $\tilde{h}_{00}(r, t) = (\kappa / 4\pi) \int \delta \rho(r + r''), t - r'' / c) d^3x'' / r''$, where the integration is only over the region $r'' \equiv |r - r'| \geq (c/v)\overline{D} \gg D$. Because $r''$ is large compared to the distance of the sources, and $\delta \rho$ homogeneous on average, this expression is independent of $r$, i.e. it would contribute a non trivial, time dependent but space independent $\tilde{h}_{00}$. But this is impossible in our gauge because it would contradict the condition $\langle h_{00} \rangle = 0$.

Here it becomes clear that the question of backreaction is not only a problem of using a reliable approximation for the distribution of matter and a consistent mathematical treatment. It is also crucial to connect observations with statements about the metric, in particular about the scale factor which describes the expansion. If observations refer to something like our “optimal gauge” there is no back reaction from retardation. If, on the other hand, they refer to some “bad” gauge in the past, there might be backreaction effects. Observations use light which essentially moves through a space which expands according to the average density $\overline{\rho}$, the corresponding average fluctuation $\overline{\delta \rho}$ vanishes. Therefore we don’t expect a sizable additional contribution $\tilde{h}_{00}$.

If the universe would be only built up from clusters, and if clusters would be homogeneous objects, (6.14) would be the final result and one should use it with the assignment
\( \dot{\rho} \to \rho_C, \ L \to L_C, \ D \to D_C, \ m \to m_C. \) We keep this contribution and next consider the additional effect that clusters are made up from galaxies. Because the average effect of clusters has already been considered, the additional density fluctuation within a cluster is described by an expression like (6.12), where now \( r_i \) denotes the position of galaxies, \( \dot{\rho} \) has to be taken as the density within a galaxy, and \( \overline{\rho} \) is replaced by the average density \( \rho_C \) of a cluster. The product of \( h_{00} \) and \( \delta \rho \) in \( \langle h_{00} \delta \rho \rangle \) does not contain mixed terms between the expressions for clusters and for galaxies. The reason is that the terms referring to galaxies contain contributions which are located at distances \( \approx D_G \) which are small compared to the extension \( L_C \) of the clusters. Therefore one may replace them by their averages over the cluster which vanishes. Within one cluster we thus obtain again the contribution (6.14) where now the parameters are those for galaxies, and the average as well as the average density refer to a single cluster. Because of \( \overline{\rho} = \rho_C V_C / V_{\text{tot}} \) one has \( \langle h_{00} \delta \rho \rangle_C / \rho_C = \langle h_{00} \delta \rho \rangle / \overline{\rho} \). If only a fraction \( \eta \) of galaxies is located in clusters and the fraction \( 1 - \eta \) roughly uniformly distributed, the expression for the clusters is multiplied by a factor \( \eta \), while \( \eta \) cancels in the two contributions for galaxies.

Finally consider that galaxies are made up of stars and dark matter. Because only a fraction of matter is made up of stars, while dark matter is assumed uniformly distributed, there is a factor \( \Omega_b / \Omega_m \). Thus the contributions of clusters, galaxies, and stars add up to

\[
\rho_b / \overline{\rho} \approx \frac{21}{40\pi} \kappa \left\{ \eta \frac{m_C}{L_C} + \frac{m_G}{L_G} + \frac{\Omega_b}{\Omega_m} \frac{m_S}{L_S} \right\}, \quad (6.15)
\]

\[
w_b \equiv \rho_b / \rho_b = -\frac{1}{21}. \quad (6.16)
\]

This expression has the same structure as the result in [10], but with the definite factor \( 21 / 40\pi = 0.17 \) in front, which, of course, should only be considered as a rough estimate. Our ratio \( w_f = p_f / \rho_f = -1/21 \) is slightly different from the ratio \( -1/27 \) obtained in [10], which is due to the different gauges.

Galaxy clusters and galaxies are not distributed uniformly, but there exist large voids, surrounded by bubble walls. Because this fact plays some role in the discussion on backreaction, let us discuss the implications. Again we use a simple model which shows the essential features and can be treated explicitly. We describe the bubble wall by a uniform distribution of matter in a shell of radius \( R \), thickness \( 2L \), density \( \hat{\rho} \), and total mass \( m \). For \( L \ll R \), which we assume, one has \( \hat{\rho} = m / 8\pi R^2 L \). Let the bubbles lie close together with only little space left in between. The density fluctuation for a bubble centered around the origin is then

\[
\delta \rho(r) = \hat{\rho} \Theta(L - |r - R|) - \overline{\rho}, \quad (6.17)
\]

with \( \overline{\rho} = \hat{\rho} \frac{6L}{R} \). The solution of (6.8) with \( \langle h_{00} \rangle = 0 \) becomes

\[
h_{00}(r) = \kappa \hat{\rho} \left\{ - \frac{3}{5} R L \Theta(R - L - r)
+ \left( \frac{R^2 + L^2}{2} - \frac{8}{5} R L - \frac{r^2}{6} - \frac{(R - L)^3}{3r} \right) \Theta(L - |r - R|) \right\}
\]

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\[+ \left( \frac{2R^2 L + 2L^3/3}{r} - \frac{13}{5} RL \right) \Theta(r - R - L) + \frac{L}{R} r^2 + c \right), \tag{6.18} \]

with \( c = O(L^2) \). In contrast to the case of matter located inside a sphere, one can here perform the limit \( L \to 0 \), while keeping the total mass fixed. This results in

\[
h_{00}(r) = \frac{\kappa m}{8\pi R} \left\{ - \frac{3}{5} \Theta(R - r) + \left( \frac{2R}{r} - \frac{13}{5} \right) \Theta(r - R) + \frac{r^2}{R^2} \right\}. \tag{6.19} \]

Following the same steps as before one arrives at the back reaction due to bubble walls,

\[
\left( \frac{\rho_b}{\bar{\rho}} \right)_{BW} = \frac{7}{4} \langle h_{00} \delta \rho \rangle / \bar{\rho} = \frac{7}{10} \kappa \hat{\rho} R L \frac{7}{60} \kappa \bar{\rho} R^2 = \frac{7}{80\pi} \frac{m}{R} = \frac{7}{4} h_{00}(R). \tag{6.20} \]

We next estimate the various contributions by inserting some standard values.

For clusters we use \( m_C = 10^{15} m_\odot = 2 \cdot 10^{48} g \), and \( L_C = 5 \text{Mpc} = 1.5 \cdot 10^{25} \text{cm} \).

For galaxies we insert the values for the milky way, \( m_G = 10^{12} m_\odot = 2 \cdot 10^{45} g \), and \( L_G = 100 \text{kpc} = 3 \cdot 10^{23} \text{cm} \), where both numbers include, of course, both baryonic and dark matter. For the contribution of the stars, finally, we insert the values of the sun, \( m_S = m_\odot = 2 \cdot 10^{33} g \), \( L_S = L_\odot = 7 \cdot 10^{10} \text{cm} \).

For the ratios mass over radius one obtains

\[
m_C / L_C = 1.3 \cdot 10^{23} g \text{ cm}^{-1} \tag{6.21} \]
\[
m_G / L_G = 0.7 \cdot 10^{22} g \text{ cm}^{-1} \tag{6.22} \]
\[
m_S / L_S = 3 \cdot 10^{22} g \text{ cm}^{-1}. \tag{6.23} \]

We further use \( \kappa = 2 \cdot 10^{-27} g^{-1} \text{cm} \) and \( \Omega_b/\Omega_m = 0.17 \). For all three cases one has \( |h_{00}(r)| \leq h_{00}(0) < 0.3 \cdot 10^{-4} \), furthermore \( h_{00} \) is smoother than \( \delta \rho \). Serious doubts concerning the validity of perturbation theory appear inappropriate. Nevertheless one can often read the argument that a perturbative expansion would be inadmissible because \( \delta \rho / \bar{\rho} \) is large. One should take a closer look at this “argument”. In our model \( \delta \rho_{\text{max}} = 3m / 4\pi L^3 \), \( \bar{\rho} \approx 3m / 4\pi D^3 \), therefore indeed \( \delta \rho_{\text{max}} / \bar{\rho} \approx D^3 / L^3 \) is large. But there is no reason to panic, because this quantity has nothing to do with the perturbation expansion. The relevant dimensionless quantity which enters is not \( \delta \rho / \bar{\rho} \), but \( \kappa m / L \) which is small, less than \( 3 \cdot 10^{-4} \) in all cases!

The three contributions in (6.15) become

\[
\left( \frac{\rho_b}{\bar{\rho}} \right)_C = \eta \cdot 0.4 \cdot 10^{-4} \tag{6.24} \]
\[
\left( \frac{\rho_b}{\bar{\rho}} \right)_G = 2 \cdot 10^{-6} \tag{6.25} \]
\[
\left( \frac{\rho_b}{\bar{\rho}} \right)_S = 1.5 \cdot 10^{-6}. \tag{6.26} \]

All these corrections due to backreaction are very small.

Let us look at some other stellar objects with a larger ratio \( m / L \) which might be relevant.
For white dwarfs the ratio $m/L$ is about a factor of $30 \div 100$ larger than that of the sun, the number of white dwarfs is estimated as about 10% of all stars. Therefore their contribution is also unimportant.

For neutron stars and black holes one has $m/L = 6 \cdot 10^2 g \text{ cm}^{-1}$, which is a factor of $2 \cdot 10^5$ larger than the ratio for the sun. If $\eta_{NS}$ and $\eta_{BH}$ denote the fraction of neutron stars and black holes one obtains a contribution $(\rho_b/\bar{\rho})_{NS+BH} = (\eta_{NS} + \eta_{BH}) \cdot 0.3$. This is still a moderate contribution, but the number of these objects becomes relevant. Neutron stars are supposed to provide a portion of less than 1% of all stars, one expects a contribution not larger than $10^{-2}$. Unless there is an extremely large number of black holes the latter will also only give a small contribution.

Finally let us look at the implications of voids, surrounded by bubble walls. The average density of matter is $\bar{\rho} = 2 \cdot 10^{-30} g \text{ cm}^{-3}$, for the radius of the voids we take $R = 20 \text{ Mpc} = 6 \cdot 10^{25} \text{ cm}$. From (6.20) we then obtain $(\rho_b/\bar{\rho})_{BW} = 2 \cdot 10^{-6}$. The smallness of this contribution is surprising at first sight, therefore it deserves a comment. The result can be easily understood. The distribution of matter in the bubble wall is homogeneous in the two tangential directions. Only in radial direction it is concentrated in a small shell of thickness $2L$. The situation is therefore essentially one-dimensional. But while in three or two dimensions the Green function of the Laplacian is $\sim 1/r$ and $\sim \ln r$ respectively, i.e. singular at the origin, in one dimension it is $\sim r$, which is finite. Therefore one can perform the limit $L \to 0$ in the latter case. No problems arise if one squeezes the matter in the surface $r = R$ by taking $\rho(r) = m\delta(r - R)/4\pi R^2$. The solution $h_{00}(r)$ in (6.19) is finite at $r = R$, it just has a kink there. The absence of any singular behavior for small $L$ for matter concentrated in walls explains why the effect is so small.

We are aware that our result is in striking contrast to the statements of Wiltshire [13] who claims that the presence of large voids should lead to a considerable backreaction. We don’t feel in a position to comment on this work. But we doubt whether a separate treatment of the metric within and outside the voids is appropriate or even legitimate.

The relative unimportance of backreaction is, of course, by no means a new result. It was already found (in a non covariant calculation) by Nelson [14] in 1972, also by Wetterich [10] who’s model we essentially used, as well as by many others. But certainly the discussion on the (un)importance of back reaction will not end in the near future.

7 Summary and conclusions

If one wants to go beyond the simple static approximation which we used for the calculation of backreaction one is faced with the problem that the metric is not known from the beginning. It has to be determined e.g. from a perturbative solution of the field equations which couple metric and matter. This calculation can be done in any reasonable gauge, e.g. in the harmonic gauge. Subsequently one can transform to our optimal gauge, thus removing all unphysical gauge modes.

The covariant fitting procedure presented here is, of course, not unique because one could modify the minimization steps which define the optimization. But it is not at all trivial to formulate conditions which do not involve the initially unknown scale factor,
can be simply treated perturbatively, and lead to a maximal fixing of the coordinates. One could e.g. exchange the order of the steps. We did not perform a systematic investigation of all possibilities but chose an order which was motivated by technical simplicity. If one would perform steps 2 and 3 at the beginning one would obtain the synchronous gauge. The E-L equations for the former step 1 would subsequently become rather ugly. There is now no reason for $\langle \delta G(1)_{00} \rangle$ and $\langle \delta G(1)m \rangle$ to vanish, one could obtain a (small) backreaction already in first order.

It is worthwhile to emphasize that familiar gauge fixing prescriptions in the literature do not fix the gauge in the maximally possible way as it was achieved here. This is well known for the synchronous gauge, but it is also true for the Newtonian gauge, even if one restricts to time independent transformations. As an example consider the nonlinear transformation $\xi_0 = 0$, $\xi^m = \alpha^m r^2 - 2x^m \sum_k \alpha^k x^k$, with $\alpha^k$ a constant vector. This leaves $g_{00}$ and $g_{m0}$ unchanged, and only changes $A$ in the decomposition (3.10) to $A - 4 \sum \alpha^k x^k$. Therefore demanding e.g. $B = C_m = F = 0$ is not sufficient for fixing the gauge.

We came to the conclusion that dark energy cannot be mimicked from backreaction. The only quantities which agree are the signs $\rho_b > 0$ and $p_b < 0$. But the relative importance of backreaction in the present day universe turned out to be of order $10^{-4}$ to at most $10^{-2}$. Somewhat different parameters than the ones used above, or more realistic density distributions inside the sources, would not change the numbers considerably. We also gave arguments why a perturbative expansion makes sense, and why neither the presence of large voids nor retardation effects should seriously modify the results. Only if one widely gives up the cosmological principle one may evade these conclusions, but then it becomes hard to derive even semiquantitative statements.

In order to mimic dark energy from backreaction the small ratio $\rho_b/\rho$ should somehow increase to the observed ratio $\rho_{DE}/\rho \approx 7/3$ between dark energy and matter. It appears miraculous how this could happen. An even greater miracle would be needed to change the ratio $w_b = -1/21$ to the observed $w_{DE} \approx -1$ for dark energy.

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