The Problem of Projecting the Origin of Euclidean Space onto the Convex Polyhedron

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Abstract. This paper is aimed at presenting a systematic survey of the existing now different formulations for the problem of projection of the origin of the Euclidean space onto the convex polyhedron (PPOCP). In the present paper, there are investigated the reduction of the projection program to the problems of quadratic programming, maximin, linear complementarity, and nonnegative least squares. Such reduction justifies the opportunity of utilizing a much more broad spectrum of powerful tools of mathematical programming for solving PPOCP. The paper’s goal is to draw the attention of a wide range of research at the different formulations of the projection problem, which remain largely unknown due to the fact that the papers (addressing the subject of concern) are published even though on the adjacent, but other topics, or only in the conference proceedings.

Keywords: projection, convex polyhedron, quadratic programming, maximin problem, complementarity problem, nonnegative least squares problem

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1 Introduction

The problem of projecting the origin of the Euclidean space onto a convex polyhedron plays an invaluable role in the differentiable and nondifferentiable convex optimization, in the theory of linear separation the convex polyhedra, data classification and identification. For many years, the numerous applications of the problem of projecting the origin onto a convex polyhedron motivates intensive studying the possibility of reducing this problem (in the original formulation with an abstract constraint) to the other related ones of mathematical programming [1]–[4]. Such a reduction, as a rule, tends to encourage the appearance of new methods for solving the original problem and finding new applications for the already existing methods.

Indeed, the class of projection techniques is successfully used for solving a variety of the theoretical and applied mathematical problems. The methods applying the concept of projection are commonly referred to as the projection methods [5]. These algorithms use different kinds of projections onto the convex sets (including orthogonal projections which are the subject of this article), and serve for solving the optimization problems as well as the so-called feasibility problems (the problems of finding the point belonging to some set). The adequate for the most applications of the projection methods up-to-date overview of the literature is presented in [3]–[6]. In these sources, there is annotated the bibliography on the iterative projective techniques that use a projection onto the convex and closed sets specified by a finite family of sets (for example, the intersection of the individual sets from this family).
The computational success of these methods of projection is determined by the fact that, as a rule, it is always easier to implement the projection onto the individual sets with a simpler structure than, for example, at their intersection.

Finding the projection of points of the space onto a convex set, including a convex polyhedron, is used in particular in the method of cyclic projections. The cyclic orthogonal projection method can serve, for example, to find a point belonging to the set \( \Phi = \Phi_1 \cap \Phi_2 \cap \ldots \cap \Phi_N \), where \( \Phi_i, \ i = 1, N \) are the polyhedra specified by the convex hulls of the finitely many vectors \([6]–[7]\). We further note that the problem of projection of some point \( p \in \mathbb{R}^n \) onto the convex polyhedron \( L \) may be reduced to the problem of projecting the origin onto the Minkowski difference \( L - p \), as it was shown in \([8]\) (see p.565). In \([3]–[4], [8]–[9]\), there are investigated the applications of the problem under study for solving the related problems such as

- The problem of the strong linear separation of the convex polyhedra,
- The problem of determining the distance between the convex polyhedra,
- The problem of concretizing the nearest points of the convex polyhedra,
- The problem of estimating the thickness of the separator between the two convex polyhedra,
- The problem of solving the system of the linear (or quadratic) inequalities,
- The variational inequalities associated with the linear separation of the convex polyhedra.

PPOCP is also widely used in nonsmooth convex optimization \([10]\) on the whole and the convex minimax problems in particular \([11]\). In minimization of the nondifferentiable convex functions, the learnt problem is used for the choice of the descent direction as well as for verification the stop criterion of optimization algorithms. With the same purpose, there can be utilized PPOCP in constrained minimization of the differentiable pseudo-convex function (see, for instance \([12]\)). Due to a well known connectedness of PPOCP with the program of computing the distance between the convex polyhedra (through the Minkowski difference), all enumerated \([13]\) applications such as computer graphics, robotics, collision detection, and path finding in presence of obstacles are closely relevant to the problem under consideration. The role of PPOCP is also valuable for the mathematical methods in biomedical imaging and intensity-modulated radiation therapy \([14]\).

In present paper, our goal is to provide a systematic exposition of the existing now various settings for PPOCP. Here, we study the questions related to the issue of reducing the projection program to the problems of quadratic programming, maximin, linear complementarity, and nonnegative least squares. Our prime aim is to draw the attention of a broad range of research at the different formulations of the projection problem, which remain largely unknown due to the fact that the papers (addressing the subject of concern) are published even though on the relevant, but other topics, or only in the conference proceedings.

2 Projection problem in Original Setting

In this section of the paper, we study the considered projection problem in classical setting.
Let the polyhedron is represented as convex hull of finitely many vectors from the $n$-dimensional Euclidean space: 

$$L := \text{conv}\{z_i\}_{i \in I},$$

where $I = \{1, 2, \cdots, m\}$, i.e.

$$L = \{z \in \mathbb{R}^n : z = \sum_{i \in I} \alpha_i z_i \mid \alpha \in \mathbb{R}_+, \sum_{i \in I} \alpha_i = 1\}.$$

By construction, the polyhedron $L \subset \mathbb{R}^n$ is convex and compact.

Let us recall a widely known and traditional formulation of PPOCP:

$$\min_{z \in L} \left( f(z) = \|z\|^2 \right),$$

(see, for instance, \[15\], \[11\]).

Owing to convexity and closedness of $L$, and strict convexity of the objective function, a solution of the problem (1) exists, and moreover it is unique (see, for instance, Theorem 1 from \[16\], p. 193).

If $\min_{z \in L} \|z\|^2 = \|\rho\|^2$, then the point $\rho \in L$ is used to call a projection of the origin of Euclidean space $\mathbb{R}^n$ onto $L$, and namely the problem (1) is commonly referred to as a problem of projecting the origin onto the convex polyhedron $L$.

An optimality criterion for the solution of the problem (1) is performed in the following statement.

**Lemma 2.1** For $\rho \in L$ to be a nearest to the origin of $\mathbb{R}^n$ point of the polyhedron $L \subset \mathbb{R}^n$, it is necessary and sufficient to hold the following variational inequality:

$$\langle z - \rho, \rho \rangle \geq 0 \quad \forall z \in L.$$  (2)

**Proof.** Due to the known theorem specifying a necessary and sufficient condition for the smooth convex function $f(z)$ to attain a minimum on the convex set $L \subset \mathbb{R}^n$ at the point $\rho$, it is fulfilled the following correlation: $\langle f'(\rho), z - \rho \rangle \geq 0 \quad \forall z \in L$. $\square$

The previous lemma demonstrates that the problem (1) can be reduced to the problem of finding the point $\rho \in L$ satisfying to the variational inequality (2).

**Lemma 2.2** For the point $y \in \mathbb{R}^n$ to satisfy the variational inequality

$$\langle z - y, y \rangle \geq 0 \quad \forall z \in L,$$

it is necessary and sufficient for the vector $y$ to be a solution of the system of the quadratic inequalities:

$$\langle z_i - y, y \rangle \geq 0 \quad \forall i \in I.$$

The assertion of the Lemma 2.2 can be easily proved by taking into account that, on the one hand, any point $z = z_i, i = 1, m$ belongs to $L$. On the other hand, any point $z \in L$ can be represented as a convex combination of the vectors $z_i, i = 1, m$.

By help of Lemmas 2.1, 2.2 we can reformulate the optimality criterion for the solution of problem (1) in the following constructive (computationally realized) form:

The vector $\rho \in L$ is a nearest to the origin of $\mathbb{R}^n$ point of $L$ if and only if the following system of the quadratic inequalities is consistent:

$$\langle z_i - \rho, \rho \rangle \geq 0 \quad \forall i \in I.$$  (3)
We underline that the formulation of the problem (1) (in contrast to the other formulations of PPOCP) has an abstract constraint $z \in L$ which does not cause inconvenience to the development of some optimization framework. In this regard, there should be mentioned the classical nearest point algorithms such as Gilbert’s method [13], MDM-method [15], and more recent suitable affine subspace method for the case of a simplex [17].

3 Reduction to the Quadratic Programming Problems

This section is devoted to a treatment of how PPOCP can be posed as the different quadratic programming problems (QPP). For solving of QPP, there is a wide range of the effective algorithms [18]–[27]. Let us note that QPP may be also solved by using a package Optimization Toolbox of MATLAB. For instance, the function quadprog allows to minimize the quadratic objective function subject to the linear restrictions.

The rest of this section is organized as follows. In subsections 3.1–3.2, we collect the QPP having the obvious close association with the cone of generalized strong support vectors [4] of the given polyhedron $L$. Subsection 3.3 deals with the optimization problem which consists in finding the convex combination of the vectors $z_i, i = 1, m$ with the least norm.

3.1

In this subsection of the paper, we will justify a reduction of the problem (1) to the following quadratic programming problems formulated in [3]:

$$\max_{y \in \Omega} \|y\|^2, \quad \text{(4)}$$

$$\min_{y \in D} \|y\|^2, \quad \text{(5)}$$

where $D = \{y \in \mathbb{R}^n : \langle z_i, y \rangle \geq 1, i = 1, m\}$, $\Omega = \{y \in \mathbb{R}^n : \langle z_i, y \rangle \geq \|y\|^2, i = 1, m\}$. We further illustrate of how one can solve PPOCP by means of normalizing of the best strong support vectors from the sets $D$ and $\Omega$. Let us remark that unlike the polyhedron $L$ having the inner description, the set $D$ is given by the outer representation. First we also note that, by construction, the set $D$ is convex and closed. It is well known in optimization theory that a solution for the problem of minimizing the strictly convex function on the nonempty convex and closed set exists, and its uniqueness is justified (see, for instance, [28], p. 41). In [3], there was proved that the set $\Omega$ is an intersection of the $m$ $n$-dimensional balls of radii

$$R_i = \frac{\|z_i\|}{2} \quad \text{with centers at the points } O_i = \frac{z_i}{2}.$$ 

Consequently, the set $\Omega$ is convex and compact. Note that, by construction of the set $\Omega$, it always holds: $0 \in \Omega$. According to the formula (3), we obviously have $\rho \in \Omega$.

The following two statements establish a natural and useful connection between the feasible and optimal solutions of (4) and (5).
Lemma 3.1 If \( y^* \in \Omega, \ y^* \neq 0; \ \tilde{y} \in D, \)

then: 1) \( \hat{y} \in \Omega \) with respect to \( \hat{y} = \tilde{y} / \| \tilde{y} \|^2 \),

2) \( \tilde{y} \in D \) with respect to \( \tilde{y} = y^* / \| y^* \|^2 \).

The lemma can be proved by help of substitution.

Lemma 3.2 If \( y^* \neq 0 \) is a solution of the problem (4), \( \tilde{y} \) is a solution of the problem (5), then

1) the vectors \( \hat{y} = \tilde{y} / \| \tilde{y} \|^2 \) and \( \tilde{y} = y^* / \| y^* \|^2 \) are the solutions of the problems (4), (5), respectively;

2) a solution of (4) is unique.

Proof. Using Lemma 3.1 one can prove the first two statements by reductio ad absurdum. Taking into account the uniqueness of the solution to the problem (5), by help of the first assertion of this lemma, it is easy to verify that the solution of (4) is unique, too. \( \square \)

Due to Lemmas 3.1–3.2, by reductio ad absurdum, it is not hard to prove the following statement.

Lemma 3.3 If the vector \( y^* = 0 \) is the solution to the problem (4), then the problem (5) has no solutions. If the domain of the feasible solutions for (5) is empty, then \( y^* = 0 \) is the solution to the problem (4).

The proof of this assertion can be found in [3], p.24.

Lemma 3.4 If \( y^* \) is the solution for the problem (4), then \( \rho = y^* \).

Lemma 3.5 \( y^* = 0 \) is the solution to the problem (4) if and only if \( 0 \in L \).

Proof. Since it holds \( \rho = 0 \) if and only if \( \tau 0 \in L \); then the statement of the lemma follows directly from the Lemma 3.4 \( \square \)

The assertions of Lemmas 3.3, 3.5 implies the following statement.

Lemma 3.6 The domain of the feasible solutions for the problem (5) is empty if and only if \( 0 \in L \).

Therefore, for solving the problem of projecting the origin of the space onto the convex polyhedron, it suffices to find the solution \( y^* \) for the problem (5) or \( \tilde{y} \) for (4). The assertion of Lemma 3.4 thereby provides that \( \rho = y^* \). Due to Lemmas 3.4, 3.2 the formula \( \rho = \tilde{y} / \| \tilde{y} \|^2 \) is made obvious.

3.2

In this subsection, there is studied the reduction of the projection problem to the dual one for the problem (5). We rewrite first the constraints of the problem (5) in the vector-matrix form as follows:

\[
\begin{align*}
\min & \| y \|^2 \\
Cy + \varepsilon & \leq 0,
\end{align*}
\]
where $C$ is an $m \times n$ matrix, the rows of which are the vectors $- [z_i]^T$, $i \in I$; $e$ is an $m$-dimensional vector, $e^T = (1, \ldots, 1)$. We further construct the dual program to (6)–(7).

To this end, we construct the Lagrange function for the problem (6)–(7):

$$
\Lambda(y, u) = \|y\|^2 + \langle u, Cy + e \rangle, \ y \in \mathbb{R}^n, \ u \in \mathbb{R}_+^m.
$$

In what follows, we use the following notation: $h(u) = \min_{y \in \mathbb{R}^n} \Lambda(y, u)$. Then, the program $\max_{u \in \mathbb{R}_+^m} h(u)$ is called a dual program to (6)–(7), and we will call its variables $u_1, \ldots, u_l$ the dual ones. The fact that $\Lambda(y, u)$ is a strictly convex and quadratic function with respect to $y$, provides the attainment of its minimum at the point $y(u)$ such that:

$$
\Lambda_y(y, u) = 2y + C^T u = 0 \Rightarrow y(u) = -\frac{1}{2} C^T u.
$$

Therefore, the function $\Lambda(y(u), u)$ will have the form:

$$
\Lambda(y(u), u) = -\frac{1}{4} \langle u, CC^T u \rangle + \langle e, u \rangle.
$$

For this reason, the dual program may be expressed as follows (see, for instance, [3]):

$$
\min_{u \geq 0} (f(u) = \frac{1}{4} \langle u, CC^T u \rangle - \langle e, u \rangle).
$$

Let us note that this expression of the dual problem has an interesting role in establishing the connection of the original problem with the nonnegative least squares problem and linear complementarity one, discussed below.

**Theorem 3.1** For the vector $y^* \in D$ to be a solution of the program (6)–(7), it is necessary and sufficient to exist the vector $u^* \in \mathbb{R}_+^m$ satisfying

$$
2y^* + C^T u^* = 0,
$$

$$
u^*(Cy^* + e) = 0.
$$

The previous theorem represents the application of Theorem 10.4 from [29] (p.180) for the concrete setting of (6)–(7).

**Theorem 3.2** For the system (7) to be inconsistent, it is necessary and sufficient that the function $h(u)$ be unbounded from above.

**Proof.** Since the domain of the feasible solutions for the problem (8) is nonempty, the statement of this theorem immediately follows from the second statement of Theorem 10.3 from [30], p.115. 

Theorem 3.2 together with Lemma 3.6 yield that the function $f(u)$ is unbounded from below if and only if $0 \in L$.

For finding $y^*$, we need to solve the problem (8), which consist of minimizing the quadratic convex function on the set with a simple structure, or equivalently to solve the following problem:

$$
\max_{u \geq 0} \Lambda(y(u), u).
$$

By help of the next theorem, it is easily be shown the convexity of the objective function for the problem (8).
Theorem 3.3 ([16], p. 173) Let $U$ be a convex set in $\mathbb{R}^n$, $\text{int}(U) \neq \emptyset$, $f(u) \in C^2(U)$. Then, for function $f(u)$ to be convex on $U$, it is necessary and sufficient to hold

$$
(f''(u)\xi, \xi) \geq 0
$$

for all $\xi = (\xi^1, \cdots, \xi^n) \in \mathbb{R}^n$ and $u \in U$.

Lemma 3.7 The function $f(u)$ is convex on the set $U = \mathbb{R}^m_+$. 

Proof. The direct calculations with the following formulas:

$$f'(u) = \frac{1}{2}CC^T u - e, \quad f''(u) = \frac{1}{2}CC^T$$

gives us

$$
(f''(u)\xi, \xi) = \frac{1}{2} (CC^T \xi, \xi) = \frac{1}{2} ||C^T \xi||^2 \geq 0
$$

for all $\xi \in \mathbb{R}^m, u \in \mathbb{R}^m_+$. From Theorem 3.3 it obviously follows that the objective function $f(u)$ of the problem (3) is convex in $\mathbb{R}^m_+$. □

Let the problem (3) be solved, and the vector $u^*$ be its solution. Then, according to Theorem 3.1, one may determine the solution of the prime problem (6)--(7) by the explicit formula

$$\bar{y} = -\frac{1}{2}C^T u^*. \quad (9)$$

It remains only to calculate the solution of the problem (11) by the formula $\rho = \bar{y}/||\bar{y}||^2$.

3.3

In the subsection, a subject of our investigation is the optimization problem often used for the construction of the data classifier in machine learning [31]. To the best of our knowledge, the use of this optimization problem was pioneered by Wolfe [1] for finding the closest to the origin of $\mathbb{R}^n$ point on the polyhedron $L = \text{conv}\{z_i\}_{i \in I}$. The projection problem in this setting becomes especially of interest later in the development of SVM (Support Vector Machines). It is widely used for determining the nearest points of the convex hulls for the two given sets of the points (see, for instance, [32]).

We now consider the projection problem in above-mentioned setting:

$$\min \left( \varphi(\alpha) = \left\| \sum_{i \in I} \alpha_iz_i \right\|^2 \right), \quad (10)$$

$$\sum_{i \in I} \alpha_i = 1, \quad (11)$$

$$\alpha_i \geq 0, \quad i \in I. \quad (12)$$

Convert the problem (10)--(12) to a standard vector-matrix form:

$$\min \left( \varphi(\alpha) = \langle \alpha, B\alpha \rangle \right), \quad (13)$$

$$\langle e, \alpha \rangle = 1, \quad (14)$$

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α ≥ 0, \hspace{1cm} (15)

where \( B \) is \( m \times m \)-dimensional matrix, whose element indexed by \( j \) in the row indexed by \( i \) is equal to \( \langle z_i, z_j \rangle \), \( i \in I, \ j \in I, \ e^T = (1, \cdots, 1) \). Obviously, \( B = C \cdot C^T \). This property of the matrix \( B \) allows to reveal that the function \( \varphi(\alpha) \) is convex.

**Lemma 3.8** The problem (13)–(15) is the convex quadratic programming problem.

**Proof.** Since \( \varphi''(\alpha) = 2B \), it holds

\[
\langle \varphi''(\alpha)\xi, \xi \rangle = 2\langle B\xi, \xi \rangle = 2\|C^T\xi\|^2 \geq 0 \; \forall \xi \in \mathbb{R}^m, \forall \alpha \in \mathbb{R}^m.
\]

Then, due to Theorem 3.3 the function \( \varphi(\alpha) \) is convex in \( \mathbb{R}^m_+ \). Consequently, \( \varphi(\alpha) \) is also convex on the convex set \( \{ \alpha \in \mathbb{R}^m : \langle e, \alpha \rangle = 1 \} \). \( \square \)

By virtue of the non-emptiness of the set \( \{ \alpha \in \mathbb{R}^m_+ : \langle e, \alpha \rangle = 1 \} \) and boundedness from below of the function \( \varphi(\alpha) \) on this set, the problem (13)–(15) is solvable in the case of any convex polyhedron \( L = \text{conv}\{z_i\}_{i \in I} \).

On the simple examples, it will be further illustrated that the matrix \( B \) may be singular in some cases. In this events, the function \( \varphi(\alpha) \) is not strictly convex a fortiori strongly convex.

**Example 3.1** Let \( z_1 = (1, 0), \ z_2 = (2, 0) \), i.e. the vectors \( z_i, i = 1, 2 \) be linearly dependent. Then the matrix \( B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \) is singular.

**Example 3.2** Let \( z_1 = (-1, 0), \ z_2 = (1, 0) \), then \( 0 \in \text{conv}\{z_i\}_{i \in \{1,2\}} \). Having constructed the matrix \( B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \), one can easily see that it is also singular.

**Example 3.3** Now, we consider the example for which it holds \( m \gg n \). Let \( n = 1, \ m = 3, \ z_1 = -1, z_2 = 1, z_3 = 5 \). In this case, \( C = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix} \). Consequently, the matrix \( C \cdot C^T = \begin{pmatrix} 1 & -1 & -5 \\ -1 & 1 & 5 \\ -5 & 5 & 25 \end{pmatrix} \) is singular, because \( \det C \cdot C^T = 0 \).

By construction, any point from \( L := \text{conv}\{z_i\}_{i \in I} \) is representable as a convex combination of the vectors \( z_i \). This implies that the problem (10)–(12) is equivalent to the program (1).

Let \( \alpha^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_m^*) \) be an optimal solution of the problem (10)–(12). Then, the following statement is evident.

**Lemma 3.9** \( \rho = \sum_{i \in I} \alpha_i^* z_i \).

We remind that \( \rho = 0 \) if and only if \( 0 \in L \). As a consequence of this fact, Lemma 3.9 gives the obvious justification of the next result.

**Lemma 3.10** \( \varphi(\alpha) = 0 \) if and only if \( 0 \in L \).
On account of the specificity of the problem setting for (10)–(12), we formulate an optimality criterion of its solutions. Having gotten the solution \( \alpha \), we need to check the fulfillment of the inequalities

\[
\langle z_i, \sum_{j=1}^{m} \alpha_j z_j \rangle \geq \| \sum_{j=1}^{m} \alpha_j z_j \|_2 = \langle \alpha, B\alpha \rangle \quad \forall i = 1, m, \text{ i.e. } \sum_{j=1}^{m} \alpha_j b_{ij} \geq \langle \alpha, B\alpha \rangle \quad \forall i = 1, m,
\]

where \( b_{ij} = \langle z_i, z_j \rangle \).

Thus, for solving the problem of projecting the origin onto the convex polyhedron \( L \), it suffices to find the solution \( \alpha^* \) to the problem (13)–(15). Then, according to Lemma 3.9, we calculate \( \rho = \sum_{i \in I} \alpha_i^* z_i \).

Let us carry out some comparison of the problems (8) and (10)–(12). Both problems have the constraint on the nonnegativity of their variables. The problem (8), in contrast to (10)–(12), has no constraints except one which ensures that its decision variables are nonnegative.

The functions \( f(u) \) and \( \varphi(\alpha) \) have the same squared part \( \langle x, CC^T x \rangle \). In the problem (10)–(12), the linear function \( \langle e, x \rangle \) involved in the system of constraints, while in the problem (8) it is in the objective function. The function \( \varphi(\alpha) \) is bounded from below: \( \varphi(\alpha) \geq 0 \quad \forall \alpha \geq 0 \). The equality \( \varphi(\alpha) = 0 \) is fulfilled if and only if \( 0 \in L \). The same condition is in fact necessary and sufficient for the function \( f(u) \) to be unbounded from below on the domain of the feasible solutions.

Finally, we note that in [2], there was modified the problem (10)–(12) with the purpose of getting an equivalent nonsmooth problem and further effective solving it by the nonsmooth penalty and subgradient algorithms.

### 4 Reduction to the Maximin Problem

This section is devoted to the investigation of how the reduction of the learnt projection problem can be made to the maximin problem. Solving of the maximin problem may be carried out by simple transition to the minimax problem and the subsequent application of the software package. For instance, a package Optimization Toolbox in MATLAB includes the function fminimax which is used to solve the minimax constraint problem.

Let \( c^* \) be a solution of the program

\[
\max_{\| c \| = 1} t_L(c), \tag{16}
\]

where \( t_L(c) = \inf_{z \in L} \langle c, z \rangle \). By virtue of continuity on the whole space, the linear function \( \langle c, x \rangle, c \in \mathbb{R}^n \) attains its infimum on the compact set \( L \). Consequently, it holds \( \inf_{z \in L} \langle c, z \rangle = \min_{z \in L} \langle c, z \rangle \). Let \( z^* \) be the optimizer of the following problem

\[
\min_{z \in L} \langle c, z \rangle \tag{17}
\]

for some \( c \in \mathbb{R}^n \). Note that, from the above, it obviously follows \( z^* \in L \).

**Lemma 4.1** If \( z^* \in L \) is the solution of (17), then \( \langle c, z^* \rangle = \min_{i \in I} \langle c, z_i \rangle \).

**Proof.** Suppose that the assertion of the lemma is not true, i.e. for all \( i \in I \) it takes place the correlation \( \langle c, z_i \rangle > \langle c, z^* \rangle \). By definition of the set \( L \), any its point can be performed as the convex
combination of $z_i, i \in I$. Then, it is fulfilled

$$\langle c, z^* \rangle = \langle c, \sum_{i \in I} \alpha_i z_i \rangle = \sum_{i \in I} \alpha_i \langle c, z_i \rangle > \langle c, z^* \rangle, \text{ for } \alpha_i \geq 0, \sum_{i \in I} \alpha_i = 1.$$  

The obtained contradiction proves that our assumption is not true, i.e. there can be found at least one index $i_0 \in I$ such that $\langle c, z_{i_0} \rangle = \langle c, z^* \rangle$. □

The statement of the preceding lemma means that a minimum of the linear function $\langle c, z \rangle, c \in \mathbb{R}^n$ on the polyhedron, specified as the convex hull of a finite number of points, is reached at least at one of the points. Then, for the problem (16), on account of the next equality

$$\min_{z \in L} \langle c, z \rangle = \min_{i \in I} \langle c, z_i \rangle, \forall c \in \mathbb{R}^n,$$

we have $t_L(c) = \min_{i \in I} \langle c, z_i \rangle$.

**Lemma 4.2** 1. If $t_L(c^*) > 0$, $v = c^* \cdot t_L(c^*)$, then it holds $P_L(0) = v$.  
2. If $t_L(c^*) \leq 0$, then $P_L(0) = 0$.

The proof of the previous lemma presented in [4] (see the proof of Lemma 3.8, p. 167). In [4], this statement was proved for more general setting, namely for the case when the set, onto which there is carried out the projection, is convex and closed on $\mathbb{R}^n$. In [4], there is justified the solvability of the problem (16) (see p.149) and uniqueness of its solution (see p.167). The problem (16) may be reduced to the following problem:

$$\max_{\|c\| \leq 1} t_L(c). \quad (18)$$

The interconnection of the solutions for problems (16) and (18) is described in the next lemma.

**Lemma 4.3** If $\hat{c} \neq 0$, and $c^*$ are the solutions of the problems (18) and (16), respectively, then it holds

$$t_L(c^*) = t_L\left(\frac{\hat{c}}{\|\hat{c}\|}\right).$$

For details of the lemma’s proof in more general setting (for the case when $L$ is any convex and closed subset of $\mathbb{R}^n$), we refer the interested reader to [4], p.167. Thus, the previous lemma proves that it is possible to reduce the problem (16) to (18).

For justifying of some additional results, we shall use the lemma on the necessary and sufficient conditions for emptiness of the cone of generalized strong support vectors of the set $L$:

$$V_L := \{c \in \mathbb{R}^n : \min_{z \in L} \langle c, z \rangle > 0\}.$$ 

Due to Lemma 4.1, it is obviously fulfilled $V_L = \{c \in \mathbb{R}^n : \min_{i \in I} \langle c, z_i \rangle > 0\}$.

**Lemma 4.4** $V_L = \emptyset$ if and only if $0 \in L$.

The statement of the lemma is the particular case for the statement of Lemma 3.7 from [9].

**Lemma 4.5** If $0 \notin L, \hat{c}$ is the solution of (18), then there are fulfilled the following statements:

1. $t_L(\hat{c}) > 0,$
2. the solution of the problem (18) is unique.
Lemma 4.6 For $0 \in L$ to be fulfilled, it is necessary and sufficient to have $t_L(\hat{c}) = 0$.

Proof. Sufficiency can be proved by reductio ad absurdum. Suppose that $0 \notin L$. Then, by force of Lemma 4.5, we have $t_L(\hat{c}) > 0$. This contradicts to the condition of the lemma. Consequently, our assumption is not true, i.e. it holds $0 \in L$.

Necessity. Owing to Lemma 4.3, the statement that $0 \in L$ is the same as saying that the cone of generalized strong support vectors for $L$ is empty. This, in its own turn, means that the origin of $\mathbb{R}^n$ is not strongly separable from the polyhedron $L$. We apply further the afore-mentioned Theorem 2.1 from [4] (see the proof of the first statement of the previous lemma). Owing this theorem, we have $t_L(c^*) \leq 0$. By the formulation the problem (18), it always holds $t_L(\hat{c}) > 0$. This contradiction allows to complete the proof. □

Lemma 4.7 If $t_L(\hat{c}) > 0$, then $0 \notin L$.

Proof. Suppose that the assertion of the lemma is not true, i.e. $0 \in L$. Then, due to Lemma 4.6, it holds $t_L(\hat{c}) = 0$. This contradicts the condition of the lemma. □

Lemma 4.8 Let the vector $\hat{c} \neq 0$ be the solution of the problem (18), then $\hat{c}$ belongs to surface of the unit ball, i.e. $\|\hat{c}\| = 1$.

Proof. Suppose the converse, i.e. $\|\hat{c}\| < 1$. According to Lemma 1.3, we then obtain

$$ t_L(c^*) = t_L\left(\frac{\hat{c}}{\|\hat{c}\|}\right) = \frac{1}{\|\hat{c}\|} \cdot t_L(\hat{c}) > t_L(\hat{c}). \quad (19) $$

We note that the vector $c^*$ belongs to the domain of the feasible solutions for (18). Therefore, the inequality (19) contradicts to the fact that $\hat{c}$ is the solution of (18). The obtained contradiction allows to see that our assumption is not true, i.e. it holds $\|\hat{c}\| = 1$. □

From Lemmas 4.3, 4.5, 4.8, we get the following corollary.

Corollary 4.1 Let the vector $\hat{c} \neq 0$ be the solution of (18), then $c^* = \hat{c}$. 

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Note that the previous Lemmas 4.5, 4.7 describe a necessary and sufficient condition for holding $0 \not\in L$. Lemmas 4.3–4.7 provide the possibility of using the problem (18) (instead the problem (16)) for projecting the origin onto the convex polyhedron. The problem (18) compares favorably with the problem (16) in the fact that, unlike the problem (16), the objective function in (18) is optimized on the whole $n$–dimensional unit ball, not only on its surface. As is widely known, the numerical methods usually better work in the case of the problems with the inequality constraints, not with the equality ones.

Owing to Lemma 4.2 and Corollary 4.1, the assertion of the following lemma is true.

**Lemma 4.9**

1. If $t_L(\hat{c}) > 0$, $v = \hat{c} \cdot t_L(\hat{c})$, then it holds $P_L(0) = v$.
2. If $t_L(\hat{c}) = 0$, then $P_L(0) = 0$.

### 5 Reduction to the Linear Complementarity Problem

In this section, we realize the reduction of the problem of projecting the origin onto the convex polyhedron to the linear complementarity problem (LCP).

Using the $n \times n$–dimensional unit matrix $E$, we rewrite first the quadratic programming problem (6)–(7) in the vector-matrix form

\[
\begin{align*}
\min & \; \langle y, Ey \rangle, \\
\text{subject to} & \; -Cy \geq e.
\end{align*}
\]

We need to convert the problem (20)–(21) to the canonical form. The following form of the quadratic programming problem is considered to be canonical:

\[
\begin{align*}
\min f(\hat{x}) &= \langle \hat{x}, \hat{D}\hat{x} \rangle + \langle \hat{p}, \hat{x} \rangle, \\
\hat{A}\hat{x} &\geq \hat{b}, \\
\hat{x} &\geq 0,
\end{align*}
\]

where $\hat{D}$ is an $n \times n$–dimensional matrix, $\hat{A}$ is an $l \times n$–matrix of the constraint coefficients, $\hat{b}$ is the $l$–dimensional vector, $\hat{p}, \hat{x} \in \mathbb{R}^n$.

Carry out the change of the variables in the problem (20)–(21): $y_1 = s_1 - s_i^1$, $y_2 = s_2 - s_i^2$, . . . , $y_i = s_i - s_i^i$, . . . , $y_n = s_n - s_n^i$ using the nonnegative variables $s_i, s_i^i, i = 1, 2, \ldots, n$. Then, it is easy to see that the objective function takes a form:

\[
\langle y, Ey \rangle = s_1^2 - 2s_1s_i^i + (s_i^i)^2 + \ldots + s_n^2 - 2s_ns_n^i + (s_n^i)^2 = \langle x, Dx \rangle + \langle p, x \rangle,
\]

where $x \in \mathbb{R}^{2n}$, $x = (s_1, s_2, \ldots, s_n, s_i^1, s_i^2, \ldots, s_n^i)$, $p = (0, \ldots, 0)$, $D$ has the dimension $2n \times 2n$, $D = \begin{pmatrix} E & -E \\ -E & E \end{pmatrix}$. Clearly, the matrix $D$ is symmetrical, i.e. $D = D^T$. We further substitute the variables in the system of constraints in the problem (20)–(21):

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\[-C \cdot y = -C \cdot \begin{pmatrix} s_1 - s'_1 \\ s_2 - s'_2 \\ \vdots \\ s_n - s'_n \end{pmatrix} = A \cdot x \geq b,\]
where the matrices \(C\) and \(A\) have the dimension \(m \times n\) and \(m \times 2n\), respectively, \(A = \begin{pmatrix} -C \mid C \end{pmatrix}\), \(b = (1, \ldots, 1)\), \(x \geq 0\). Under the change of variables, the problem (20)–(21) is transformed to the following canonical form:

\[
\begin{align*}
\min f(x) &= \langle x, Dx \rangle + \langle p, x \rangle, \\
Ax &\geq b, \\
 x &\geq 0.
\end{align*}
\] (25) (26) (27)

**Example 5.1** For \(n = 2\), the matrix of the quadratic form \(\langle x, Dx \rangle\) can be written as:

\[
D = \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix}.
\]

As is known, the type of the problem (25)–(27) depends on the sign-definiteness of the matrix corresponding to the quadratic form \(\langle x, Dx \rangle\). If the matrix \(D\) is non-negatively (non-positively) definite, then (25)–(27) is the convex (nonconvex) programming problem. In the case of the positive-definiteness of the matrix \(D\), the objective function of the problem is strongly convex. In this event, the problem (25)–(27) has the unique solution.

Let us remind (see [33]) that the real-valued quadratic form \(\langle x, Dx \rangle\) is called positively definite, if for real \(x \neq 0\) it holds \(\langle x, Dx \rangle > 0\). The real-valued quadratic form is called non-negatively definite if for \(x \in \mathbb{R}^n\) it is fulfilled \(\langle x, Dx \rangle \geq 0\). The positive (nonnegative) definiteness of the quadratic form corresponds to the same sign-definiteness of the associated matrix.

We further clarify what is the sign-definiteness of the block-structural matrix

\[
D = \begin{pmatrix} E & -E \\ -E & E \end{pmatrix}.
\]

By construction, \(\langle x, Dx \rangle = (s_1 - s'_1)^2 + \cdots + (s_n - s'_n)^2 \geq 0\). For all cases, even when it simultaneously holds \(s_i \neq 0, s'_i \neq 0, i = 1, \ldots, n\), but \(s_i = s'_i, \forall i = 1, \ldots, n\) we will obtain \(\langle x, Dx \rangle = 0\). Therefore, \(\langle x, Dx \rangle\) satisfies the definition of the non-negatively definite quadratic form. As a consequence, the matrix \(D\) is also nonnegative definite.

Write the optimality conditions of Kung–Tucker for the problem of minimizing the convex function subject to the linear constraints ((25)–(27)): it is necessary to find the vectors \(x, z, u, r\) such that

\[
u = 2Dx - A^T z + p^T = 2Dx - A^T z,
\]
\[
r = Ax - b,
\]
\[ x, z, u, r > 0, \]
\[ w^T x + r^T z = 0. \]

These are the necessary and sufficient conditions for the solution of the problem [26] to be optimal. Following to [34], we represent this system as LCP which consists in finding the vectors \( w \) and \( v \) such that

\[
\begin{align*}
w & = Mv + q, \quad (28) \\
w & \geq 0, v \geq 0, \quad (29) \\
\langle w, v \rangle & = 0, \quad (30)
\end{align*}
\]

where \( M \) is an \( k \times k \)-dimensional matrix; \( w, v, q \) are the \( k \)-dimensional vectors,

\[
w = \begin{pmatrix} u \\ r \end{pmatrix}, \quad v = \begin{pmatrix} x \\ z \end{pmatrix},
\]

\[
M = \left( \begin{array}{c|c}
2D & -A^T \\
A & \Theta
\end{array} \right), \quad k = 2n + m, \quad q = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -b \\ -a \end{pmatrix}, \quad \Theta \text{ is the null matrix with dimension } m \times m.
\]

It is known that to any optimal solution of LCP (by virtue of the connection through the conditions of Kung–Tucker) corresponds the optimal solution of the quadratic programming problem. However, the converse is not true (except the case when the value of the objective function of the quadratic programming problem is equal to zero). In [35], there are studied the conditions to which must satisfy the matrix \( M \) in order to guaranteed that the solutions of LCP are identical to the points of Kung–Tucker for the quadratic programming problem (associated with this LCP). It is known that if the matrix \( D \) is non-negatively definite, then the matrix \( M \) is also non-negatively definite [34]. For this case, it is proved that if there exists the solution of the complementarity problem, then Lemke’s method [36] allows to find it for the finite number of iterations. If the linear complementarity problem is without solutions, then the quadratic programming problem has no solutions, too. In our case, this means that the domain of the feasible solutions for the problem (20)–(21) is empty. According to Lemma 3.6, we then have \( 0 \in L \).

The results of the experimental research published in [37] have shown that, for the case when the matrix \( \hat{D} \) of the problem (22)–(24) is non-negatively definite, Lemke’s method has several advantages in comparison to the majority of the quadratic programming methods. Let us note that the function LCPSolve in MATLAB solves the linear complementarity problem using a pivoting algorithm.

Further, we briefly consider the representation of the optimization problem (13)–(15) as LCP (28)–(30). In this case, \( M = \left( \begin{array}{c|c}
2B & -A^T \\
A & \Theta
\end{array} \right), \quad q = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{pmatrix}, \) where \( M \) is a quadratic matrix with
dimension \((m + 2) \times (m + 2)\); \(w, v, q\) are the \((m + 2)\)-dimensional vectors, \(\Theta\) is the null matrix of dimension \(2 \times 2\), \(B = C \cdot C^T\), \(A = \begin{pmatrix} e^T \\ -e^T \end{pmatrix}\).

The problem (8) may also be represented as LCP (28)–(30) with

\[
M = \frac{1}{2} B, \quad q = -e,
\]

\(M\) is an \(m \times m\)-dimensional matrix; \(w, v, q\) are the \(m\)-dimensional vectors.

Thus, for the considered three quadratic programming problems, LCP associated with the problem (8) compares favorably with the others. The advantage of this complementarity problem is that it has the least dimension.

6 Reduction to the Nonnegative Least Squares Problem.

In this section, we investigate how the problem of projecting the origin onto the convex polyhedron can be reduced to the nonnegative least squares problem (NLSP).

In that scientific and practical applications, where one needs to estimate some vector of observations \(b \in \mathbb{R}^m\) by the \(n\) basis factors or measures

\[
A_i, \quad i = 1, n, \quad A = [A_1, A_2, \ldots, A_n] \in \mathbb{R}^{m \times n},
\]

there arises the least squares problem. The nonnegative least squares problem is usually formulated as the following optimization problem:

\[
\min_{x \geq 0} \frac{1}{4} \|Ax - b\|^2, \quad \text{(31)}
\]

\(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m\),

which consists in finding the nonnegative vector \(x\) maximizing the closeness of the vector of deviations to the origin of the space (in sense of the Euclidean distance). Clearly, the problem (31) may be converted to the quadratic programming problem with the nonnegative variables:

\[
\min_{x \geq 0} f(x) = \frac{1}{4} x^T Q x + p^T x, \quad \text{(32)}
\]

where \(Q = A^T A, p = -\frac{1}{2} A^T b\).

Let us note that (31) is the convex programming problem, since the matrix \(Q = A^T A\) is non-negatively definite and the non-negativity constraints specify the convex set of the feasible solutions \(x\).

To establish a connection of the problem of projecting the origin onto the convex polyhedron with NLSP, we further consider the problem (8). The problem (8) may be reduced to (32) under conditions:

1) \(A = C^T\), \quad 2) \(A^T b = 2e\). \quad \text{(33)}

On the simple examples, we further demonstrate that it is not hard to achieve the fulfillment of the second condition in (33).
Example 6.1  \[ A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, A^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 4 \end{pmatrix}. \]

Example 6.2  \[ A = A^T = \begin{pmatrix} -10 & 0 \\ 0 & 5 \end{pmatrix}, b = \begin{pmatrix} -0.2 \\ 0.4 \end{pmatrix}. \]

Below, we consider the rule of construction of the vector \( b \) satisfying the second condition of (33) for the case of the quadratic diagonal matrix \( A \in \mathbb{R}^{n \times n} \) (with zero non-diagonal elements and nonzero elements of the principal diagonal):

\[ b_i = \frac{2}{a_{ii}}, i = 1, n, a_{ii} \neq 0. \]

For this case, we consider the vector \( x^* \) having coordinates:

\[ x^*_i = \frac{b_i}{a_{ii}} = \frac{2}{a_{ii}^2}, i = 1, n, a_{ii} \neq 0. \]

By construction, we have \( x^*_i > 0, i = 1, n \). Consequently, the vector \( x^* \) is the solution of the equivalent problems (31) and (32). If the vector \( b \) and matrix \( A \) of the problem (32) satisfy the conditions (33), then the obtained vector \( x^* \) is also the solution of the problem (8). Then, the solution of (6)–(7) may be calculated by the following formula:

\[ y^* = -\frac{1}{2} C T x^*. \quad (34) \]

We further determine the solution of the original problem (1) by direct calculation as follows: \( \rho = y^*/\|y^*\|^2 \).

Let \( A = C T \) be non-singular quadratic matrix of dimension \( n \times n \). In this case, the vector \( b \) satisfying the second condition of (33) can be determined as follows:

\[ b = 2(A^T)^{-1} e = 2C^{-1} e. \]

Based on linear algebra, we may state that a necessary and sufficient condition for the matrix \( A \) to be nonsingular is that the given polyhedron \( L \) be specified by the system of the linearly independent vectors \( z_i, i = 1, n \).

The first known method for solving NLSP has been proposed in [38]. Nowadays, this method is known as the active-set method. For instance, the afore-mentioned method is realized in MATLAB by help of function lsqnonneg. In the literature, there are known the fast modifications and parallel variants of the active-set method [39], [41]. Comparing analysis of the methods for solving NLSP can be found, for instance, in [40].

7 Conclusions

The main contributions of the present paper may be briefly summarized as follows:
We have treated the opportunity to reduce PPOCP to the different relevant problems of mathematical programming (MP) such as QPP, maximin problem, LCP, and NLSP. Such reduction makes it possible to utilize a much more broad spectrum of powerful tools of MP for solving PPOCP.
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