Monotonic decrease of the quantum nonadditive divergence by projective measurements

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Nonadditive (nonextensive) generalization of the quantum Kullback-Leibler divergence, termed the quantum $q$-divergence, is shown not to increase by projective measurements in an elementary manner.

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In recent papers [1-3], we have developed a nonadditive generalization of information theory and have discussed its distinguished roles in the study of quantum entanglement extensively (see also, [4-8]). These works have primarily been concerned with the Tsallis nonadditive (nonextensive) entropy [9] and the associated generalized conditional entropy [1]. On the other hand, quite recently, the role of the generalized Kullback-Leibler divergence, termed the quantum $q$-divergence, has been examined as a measure of the degree of state purification [10]. There, an advantageous point of the quantum $q$-divergence over the ordinary quantum Kullback-Leibler divergence has been clarified [see the discussion after Eq. (5) below].

In this article, we study the behavior of the quantum $q$-divergence under measurements, i.e., quantum operations. In particular, we present an elementary proof that the quantum $q$-divergence does not increase by projective measurements.

The quantum $q$-divergence is the relative entropy associated with the Tsallis entropy. The Tsallis entropy reads

$$S_q[\rho] = -\text{Tr}(\rho^q \ln_q \rho).$$

Here, $\rho$ is the normalized density matrix of the quantum system under consideration and $q$ is the entropic index which can be an arbitrary positive number at this level. $\ln_q x$ stands for the $q$-logarithmic function [11] defined by $\ln_q x = (x^{1-q}-1)/(1-q)$,
which tends to the ordinary logarithmic function, \( \ln x \), in the limit \( q \to 1 \). Then, the quantum \( q \)-divergence of \( \rho \) with respect to the reference density matrix \( \sigma \) is given by

\[
K_q[\rho \| \sigma] = \text{Tr}\left[\rho^q \left( \ln_q \rho - \ln_q \sigma \right) \right].
\]

(The classical counterpart of this quantity has been introduced independently and almost simultaneously in [12-14].) Using the definition of the \( q \)-logarithmic function, Eq. (2) can also be written in the following compact form:

\[
K_q[\rho \| \sigma] = \frac{1}{1-q} \left[ 1 - \text{Tr} \left( \rho^q \sigma^{1-q} \right) \right].
\]

Since this quantity should not be too sensitive to small eigenvalues of the density matrices, the range of \( q \) is taken to be

\[
0 < q < 1.
\]

Let \( s_\rho \) and \( s_\sigma \) be the supports of \( \rho \) and \( \sigma \), respectively. In the case when \( s_\rho \leq s_\sigma \), \( K_q[\rho \| \sigma] \) has the well-defined limit \( q \to 1-0 \), which yields the ordinary quantum Kullback-Leibler divergence introduced by Umegaki [15]
\[ K[\rho \parallel \sigma] = \text{Tr}[\rho (\ln \rho - \ln \sigma)]. \quad (5) \]

Here, the condition, \( s_\rho \leq s_\sigma \), is crucial. In fact, \( K[\rho \parallel \sigma] \) becomes singular when \( s_\rho > s_\sigma \). Therefore, \( K[\rho \parallel \sigma] \) cannot be defined if, for example, \( \sigma \) is a pure state (i.e., an idempotent operator), since \( \ln \sigma = (\sigma - I) \zeta'(1) \), which is divergent, where \( I \) and \( \zeta(s) \) are the identity operator and the Riemann zeta function, respectively. In marked contrast to this, \( K_q[\rho \parallel \sigma] \) with \( q \in (0, 1) \) remains well-defined even in such a case [10].

In Ref. [10], it has been shown that (i) \( K_q[\rho \parallel \sigma] \geq 0 \) and \( K_q[\rho \parallel \sigma] = 0 \) if and only if \( \rho = \sigma \), (ii) for product states, \( \rho(A, B) = \rho_1(A) \otimes \rho_2(B) \) and \( \sigma(A, B) = \sigma_1(A) \otimes \sigma_2(B) \), of a bipartite system \((A, B)\), \( K_q[\rho \parallel \sigma] \) satisfies pseudoadditivity:

\[
K_q[\rho_1 \otimes \rho_2 \parallel \sigma_1 \otimes \sigma_2] = K_q[\rho_1 \parallel \sigma_1] + K_q[\rho_2 \parallel \sigma_2] + (q - 1) K_q[\rho_1 \parallel \sigma_1] K_q[\rho_2 \parallel \sigma_2]
\]

and (iii) \( K_q \) can be observed as the \( q \)-analog (i.e., \( q \)-deformation) of \( K \) in the sense in [16].

In addition to the properties (i)-(iii), we wish to notice another important one anew here. That is, \( K_q[\rho \parallel \sigma] \) is jointly convex

\[
K_q \left[ \sum_i \lambda_i \rho^{(i)} \parallel \sum_i \lambda_i \sigma^{(i)} \right] \leq \sum_i \lambda_i K_q[\rho^{(i)} \parallel \sigma^{(i)}]. \quad (6)
\]

where \( \lambda_i > 0 \) and \( \sum_i \lambda_i = 1 \). This directly follows from the expression in Eq. (3) as well as Lieb’s theorem [17] stating that \( \text{Tr}\left(L^{1-x} M^x\right) \) with \( x \in (0, 1) \) is jointly concave.
in any positive operators, \( L \) and \( M \). Eq. (6) generalizes joint convexity of the ordinary quantum divergence (see [18], for example).

Now, let us discuss the behavior of \( K_q[\rho \| \sigma] \) under projective measurement of \( \rho \) and \( \sigma \). This measurement can be regarded as a particular kind of positive trace-preserving quantum operation, but is quite common from the experimental viewpoint [19]. Let \( Q \) be an observable with eigenspaces defined by orthogonal projections \( P_k \) and \( \{q_k\} \) be its measured values. Then, \( Q = \sum_k q_k P_k \), \( P_k P_k' = \delta_{kk'} P_k \) and \( \sum_k P_k = I \). The finite probability \( p_k \) of obtaining the value \( q_k \) of \( Q \) in a state \( \rho \) of the system through the projective measurement is \( p_k = \text{Tr}(\rho P_k) \). From this, \( \rho \) is transformed to \( \rho_k = p_k^{-1} P_k \rho P_k \). Averaging over all possible outcomes, we have

\[
\Pi(\rho) = \sum_k p_k \rho_k = \sum_k P_k \rho P_k.
\]

(7)

Clearly, \( \Pi \) is a positive trace-preserving operation.

Let us employ the diagonal representations of \( \rho \) and \( \sigma \):

\[
\rho = \sum_a r(a) |a\rangle\langle a|, \quad \sigma = \sum_b s(b) |b\rangle\langle b|,
\]

(8)

where \( r(a) \geq 0 \), \( \sum_a r(a) = 1 \), \( \langle a|a' \rangle = \delta_{aa'} \), \( \sum_a |a\rangle\langle a| = I \) and so on. Under the operation of a projective measurement, they are replaced by
\[ \Pi(\rho) = \sum_a r(a) \Pi(|a\rangle\langle a|), \quad \Pi(\sigma) = \sum_b s(b) \Pi(|b\rangle\langle b|), \quad (9) \]

respectively. Let us further use the diagonal representations

\[ \Pi(|a\rangle\langle a|) = \sum_{\alpha} \mu(\alpha, a) |\alpha\rangle\langle \alpha|, \quad \Pi(|b\rangle\langle b|) = \sum_{b} \nu(\beta, b) |\beta\rangle\langle \beta|, \quad (10) \]

where \( \mu(\alpha, a) = \sum_k |\langle \alpha | P_k | a \rangle|^2 \geq 0, \quad \sum_{\alpha} \mu(\alpha, a) = \sum_{\alpha} \mu(\alpha, a) = 1, \quad \langle \alpha | \alpha' \rangle = \delta_{\alpha \alpha'}, \)
\[
\sum_{\alpha} |\alpha\rangle\langle \alpha| = I \quad \text{and so on. These decompositions in Eq. (10) are valid if the projection operator is identified with} \quad P_{k=\alpha} = |\alpha\rangle\langle \alpha|. \quad \text{Henceforth, the projection operator is simply written as} \quad P_k = |k\rangle\langle k|, \quad \text{and accordingly Eq. (10) may be reexpressed as follows:}
\]

\[ \Pi(|a\rangle\langle a|) = \sum_{k} \mu(k, a) |k\rangle\langle k|, \quad \Pi(|b\rangle\langle b|) = \sum_{k} \nu(k, b) |k\rangle\langle k|, \quad (11) \]

where \( \mu(k, a) = |\langle a | k \rangle|^2, \quad \nu(k, b) = |\langle k | b \rangle|^2, \quad \langle k | k' \rangle = \delta_{kk}, \quad \text{and} \quad \sum_k |k\rangle\langle k| = \sum_k P_k = I. \]

Therefore, we have

\[ [\Pi(\rho)]^q = \sum_k \left[ \sum_a r(a) \mu(k, a) \right]^q P_k, \]
\[
[\Pi(\sigma)]^{1-q} = \sum_k \left[ \sum_b s(b) v(k, b) \right]^{1-q} P_k, \tag{12}
\]

which lead to

\[
\text{Tr} \left\{ \left[ \Pi(\rho) \right]^q [\Pi(\sigma)]^{1-q} \right\} = \sum_k \left[ \sum_a r(a) \mu(k, a) \right]^q \left[ \sum_b s(b) v(k, b) \right]^{1-q}. \tag{13}
\]

Since \( f(x) = x^p \) \( (x > 0, \ 0 < p < 1) \) is a concave function, holds \( f\left( \sum_i \lambda_i a_i \right) \geq \sum_i \lambda_i f(a_i) \) for \( \lambda_i > 0 \) and \( \sum_i \lambda_i = 1 \). Therefore, it follows that

\[
\text{Tr} \left\{ \left[ \Pi(\rho) \right]^q [\Pi(\sigma)]^{1-q} \right\} \\
\geq \sum_k \left( \sum_a \mu(k, a) [r(a)]^q \right) \left( \sum_b v(k, b) [s(b)]^{1-q} \right) \\
= \sum_{a,b} [r(a)]^q [s(b)]^{1-q} \text{Tr} [\Pi(|a\rangle\langle a|)] b\langle b|]. \tag{14}
\]

So far, no peculiar assumptions have been made on the algebraic structure of \( \Pi \) in connection with \( \sigma \). To the best of our knowledge, to proceed further, it seems necessary to assume that \( \Pi \) is an “expectation” [18]: \( \Pi(|a\rangle\langle a|) b\langle b| = \Pi(|a\rangle\langle a| b\langle b|) \). This essentially implies that \( P_k \) \( (\forall \ k) \) can commute with \( \sigma \). Then, Eq. (14) yields
\[
\text{Tr}\left\{[\Pi(\rho)]^q [\Pi(\sigma)]^{1-q}\right\} \geq \text{Tr}(\rho^q \sigma^{1-q}).
\] (15)

leading to

\[
K_q[\Pi(\rho)\|\Pi(\sigma)] \leq K_q[\rho\|\sigma].
\] (16)

Consequently, we obtain the main result that the quantum \(q\)-divergence does not increase by projective measurements.

In conclusion, we have shown that the quantum \(q\)-divergence is jointly convex and does not increase by projective measurements. Physically, this implies (Tsallis) entropy production by the measurements. Quite recently, it has been shown [20] that Clausius’ inequality can be established in nonextensive quantum thermodynamics by making use of the quantum \(q\)-divergence and its monotonicity with respect to trace-preserving completely positive unital quantum operations. Further investigation in this direction may be important for developing thermodynamics of small systems [20].
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