Mean convex properly embedded \([\varphi, \vec{e}_3]\)-minimal surfaces in \(\mathbb{R}^3\)

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Abstract

We establish curvature estimates and a convexity result for mean convex properly embedded \([\varphi, \vec{e}_3]\)-minimal surfaces in \(\mathbb{R}^3\), i.e., \(\varphi\)-minimal surfaces when \(\varphi\) depends only on the third coordinate of \(\mathbb{R}^3\). Led by the works on curvature estimates for surfaces in 3-manifolds, due to White for minimal surfaces, to Rosenberg, Souam and Toubiana, for stable CMC surfaces, and to Spruck and Xiao for stable translating solitons in \(\mathbb{R}^3\), we use a compactness argument to provide curvature estimates for a family of mean convex \([\varphi, \vec{e}_3]\)-minimal surfaces in \(\mathbb{R}^3\). We apply this result to generalize the convexity property of Spruck and Xiao for translating solitons. More precisely, we characterize the convexity of a properly embedded \([\varphi, \vec{e}_3]\)-minimal surface in \(\mathbb{R}^3\) with non positive mean curvature when the growth at infinity of \(\varphi\) is at most quadratic.

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1 Introduction.

From a physical point of view, a \(\varphi\)-minimal surface \(\Sigma\) in a domain \(\Omega\) of \(\mathbb{R}^3\) arises as a surface in equilibrium under a conservative force field \(\mathcal{F}\), with potential \(e^\varphi\) for some smooth function \(\varphi\) on \(\Omega \subseteq \mathbb{R}^3\) (see [15, pp. 173-187]). It can be also viewed (see [13]) either as a critical point of the weighted area functional

\[
A^\varphi(\Sigma) = \int_{\Sigma} e^\varphi \, d\Sigma,
\]

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where $d\Sigma$ is the area element of $\Sigma$, or as a minimal surface in $\mathbb{R}^3$ with the following conformally changed metric

$$\langle \cdot, \cdot \rangle^\varphi := e^{\varphi} \langle \cdot, \cdot \rangle.$$  

When $\varphi$ only depends on the third coordinate in $\mathbb{R}^3$, $\Sigma$ will be called $[\varphi, \vec{e}_3]$-minimal surface and then, the equilibrium condition is given in terms of the mean curvature vector $H$ of $\Sigma$ as follows

$$H = \dot{\varphi} \vec{e}_3,$$

where $\dot{\cdot}$ denotes the derivative with respect to the third coordinate and $\perp$ is the projection to the normal bundle of $\Sigma$. The condition (1.3) is the equation of a heavy surface in a gravitational field $\mathcal{F} = (0, 0, gE(z))$, where $g$ is the gravitational constant and $E(z)$ a density function on the surface.

Minimal surfaces are obtained for $\dot{\varphi} \equiv 0$. The case $\dot{\varphi} = \text{constant} \neq 0$ gives translating solitons, that is, surfaces $\Sigma$ such that $t \mapsto \Sigma + t\vec{e}_3$ is a mean curvature flow. The case $\dot{\varphi}(z) = \frac{\alpha}{z}$, $z > 0$, $\alpha \in \mathbb{R}$, includes the family of “perfect domes” (when $\alpha = 1$, see [2]) and the family of minimal surfaces in the hyperbolic space $\mathbb{H}^3$ (when $\alpha = -2$).

A $[\varphi, \vec{e}_3]$-minimal surface $\Sigma$ is said to be stable if it is stable as minimal surface in the Ilmanen’s space $\Omega^\varphi := (\Omega, \langle \cdot, \cdot \rangle^\varphi)$, that is (see [4, Appendix]), if for any compactly supported smooth function $v \in C^\infty_0(\Sigma)$, it holds that

$$\int_\Sigma e^\varphi \left( |\nabla v|^2 - (|S|^2 - \varphi \eta^2) v^2 \right) d\Sigma \geq 0,$$

where $\nabla$ denotes the gradient operator, $|S|^2 = H^2 - 2K$ denotes the length of the second fundamental form and $H$ and $K$ stand for the mean and Gaussian curvatures of $\Sigma$ in $\mathbb{R}^3$. Furthermore, we denote by $\eta := \langle N, \vec{e}_3 \rangle$ the angle function respect to the direction $\vec{e}_3$.

In 1983, Schoen [19] obtained an estimate for the length of the second fundamental form of stable minimal surfaces in a 3-manifold. In particular, in $\mathbb{R}^3$, he proved the existence of a constant $C$ such that

$$|S(p)| \leq \frac{C}{d_{\Sigma}(p, \partial \Sigma)}, \quad p \in \Sigma,$$

for any stable minimal surface $\Sigma$ in $\mathbb{R}^3$ where $d_{\Sigma}$ stands for the intrinsic distance of $\Sigma$. Later, in 2010, Rosenberg, Souam and Toubiana [17] obtained an estimate for the length of the second fundamental form, depending on the distance to the boundary, for any stable $H$-surface $\Sigma$ in a complete Riemannian 3-manifold of bounded sectional curvature $|K| \leq \beta < +\infty$. More precisely, they proved the existence of a constant $C > 0$ such that

$$|S(p)| \leq \frac{C}{\min\{d_{\Sigma}(p, \partial \Sigma), \pi/2\sqrt{\beta}\}}, \quad p \in \Sigma.$$
More recently, in 2016, White \[22\] obtained an estimate for the length of the second fundamental form for minimal surfaces with finite total absolute curvature less than $4\pi$ in 3-manifolds, depending of the distance to the boundary, of the sectional curvature and of the gradient of the sectional curvature of the ambient space.

Following Colding and Minicozzi method, \[5, 6\], Spruck and Xiao \[20\] have also obtained area and curvature bounds for complete mean convex translating solitons in $\mathbb{R}^3$. As application and using the Omori-Yau Theorem (see, for example, \[1\]) they have proved one of the fundamental results in the recent development of translating solitons theory conjectured by Wang in \[21\]:

**Theorem 1.** \[20, Theorem 1.1\] Let $\Sigma \subset \mathbb{R}^3$ be a complete immersed two-sided translating soliton with nonnegative mean curvature. Then $\Sigma$ is convex.

In this paper we extend the results in \[20\] to mean convex $[\phi, \vec{e}_3]$-minimal surfaces with empty boundary in $\mathbb{R}^3$, $\alpha = \{ p \in \mathbb{R}^3 | \langle p, \vec{e}_3 \rangle > \alpha \}$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying

\begin{equation}
\dot{\phi} > 0, \quad \ddot{\phi} \geq 0 \quad \text{on } [\alpha, \infty].
\end{equation}

More precisely, we see that mean convex examples are stable (Proposition \[4.4\]), and we prove area estimates (Theorem \[4.10\]) when

\begin{equation}
\Gamma := \sup_{[\alpha, +\infty]} (2\ddot{\phi} - \dot{\phi}^2) < +\infty.
\end{equation}

To obtain curvature bounds, we need a good control at infinity of the function $\phi$. To be more precise, we are going to consider that $z \mapsto \frac{\dot{\phi}(z)}{z}$ is analytic at $+\infty$; i.e., $\dot{\phi}$ has the following series expansion at $+\infty$:

\begin{equation}
\dot{\phi}(u) = \Lambda u + \beta + \sum_{i=1}^{\infty} c_i u^i, \quad u \text{ large enough},
\end{equation}

with $\Lambda \geq 0$ and $\beta > 0$ if $\Lambda = 0$.

It is worth to note that condition \(1.7\) implies \(1.6\). Apart of a natural extension of the best known examples, conditions \(1.6\) and \(1.7\) are interesting because under these assumptions it is possible to know explicitly the asymptotic behavior of rotational and translational invariant examples (see \[16\]).

The main results obtained in this paper can be summarized in the following two theorems:

**Theorem A.** Let $\Sigma$ be a properly embedded $[\phi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3$ with non positive mean curvature, locally bounded genus and $\phi : \mathbb{R} \to \mathbb{R}$ satisfying \(1.5\) and \(1.7\). Then $|S|/\dot{\phi}$ is bounded on $\Sigma$. In particular, if $\Lambda = 0$, $|S|$ is bounded and if $\Lambda \neq 0$, $|S|$ may go to infinity but with at most a linear growth in height.

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Theorem B. Let $\Sigma$ be a properly embedded $[\varphi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3_\alpha$ with non positive mean curvature, locally bounded genus and $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying (1.5), (1.7) and $\varphi \leq 0$ on $]\alpha, +\infty[$. Then $\Sigma$ is convex if and only if the function $\Lambda K$ is bounded from below.

The paper is organized as follows. In Sections 2 and 3, we show some facts about the geometry of the Ilmanen’s space, introduced in [13] and give some notations and fundamental equations of $[\varphi, \vec{e}_3]$-minimal surfaces. Following a similar approach as in [20] and using a compactness result of White, [23, Theorem 2.1], we obtain a blow-up theorem for $[\varphi, \vec{e}_3]$-minimal which allow us to prove Theorem A. This is needed for the proof of Theorem B in section 5 which is based on a generalized Omori-Yau’s maximum principle (see [1, Theorem 3.2]).

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2 Geometry of the Ilmanen’s space.

Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a smooth function defined in an open interval $I$ of $\mathbb{R}$. Following [13], consider the Ilmanen’s space $\Omega^\varphi$ as the Riemannian manifold $\Omega = \mathbb{R}^2 \times I$ with the Euclidean conformal metric $\langle \cdot, \cdot \rangle_p$ defined at any point $p = (x_1, x_2, x_3) \in \Omega$ by $\langle \cdot, \cdot \rangle_p = e^{\varphi(x_3)} \langle \cdot, \cdot \rangle_p$.

Denote by $D$ and $R$ (respectively, $D^\varphi$ and $R^\varphi$) the Levi-Civita connection and the curvature tensor of the Euclidean space $\mathbb{R}^3$ (respectively, of the Ilmanen’s space $\Omega^\varphi$). Then, for any orthonormal frame $\{e_i\}_{i=1,2,3}$ of $\mathbb{R}^3$ we obtain

\begin{align*}
D^\varphi_X Y &= D_X Y + \frac{1}{2} \dot{\varphi} (\langle X, e_3 \rangle Y + \langle Y, e_3 \rangle X - \langle X, Y \rangle e_3), \\
R^\varphi(X, Y, Y, X) &= -\frac{\varepsilon^\varphi}{4} |X|^2 \left( (2\dot{\varphi} - \ddot{\varphi}) \langle Y, e_3 \rangle (X, e_3) + |Y|^2 \ddot{\varphi}^2 \right) \\
&\quad + \frac{\varepsilon^\varphi}{4} \langle X, Y \rangle \left( (2\dot{\varphi} - \ddot{\varphi}) \langle Y, e_3 \rangle \langle X, e_3 \rangle + \langle X, Y \rangle \ddot{\varphi}^2 \right) \\
&\quad + \frac{\varepsilon^\varphi}{4} \langle X, Y \rangle \left( (2\dot{\varphi} - \ddot{\varphi}) \langle Y, e_3 \rangle \langle X, e_3 \rangle \right) \\
&\quad - \frac{\varepsilon^\varphi}{4} |Y|^2 \left( (2\dot{\varphi} - \ddot{\varphi}) \langle X, e_3 \rangle \right)
\end{align*}

for any tangent vector fields $X, Y \in T\Omega$ and

From (2.1) and (2.2), we also have the following
Lemma 2.1. Consider the orthonormal frame of $\Omega^\varphi$ given by \(\{e_i^\varphi = e^{-\varphi/2} e_i\}\). Then,

\[
(D_{e_i^\varphi}^\varphi e_j^\varphi, e_k^\varphi)^\varphi = \frac{1}{2} e^{-\varphi/2} \dot{\varphi} (\delta_{ij} \delta_{3k} - \delta_{ik} \delta_{3j})
\]

\[
K^\varphi(e_i^\varphi, e_j^\varphi) = \frac{1}{4} e^{-\varphi/2} (\dot{\varphi}^2 - 2\ddot{\varphi}) \delta_{3j} - \ddot{\varphi}^2) \text{ for } i \neq j.
\]

\[
\nabla^\varphi K^\varphi(e_i^\varphi, e_j^\varphi) = \frac{1}{4} (\dot{\varphi}^3 - (\dot{\varphi}^3 - 2\ddot{\varphi}) \delta_{3j} + 2(\dot{\varphi} \ddot{\varphi} - \dddot{\varphi}) \delta_{3j} - 2\dddot{\varphi}) e_3,
\]

where $\nabla^\varphi$ and $K^\varphi$ are, respectively, the usual gradient operator and the sectional curvature of $\Omega^\varphi$.

Definition 2.2. We say that the Ilmanen’s space $\Omega^\varphi$ has bounded geometry if the sectional curvature $K^\varphi$ is bounded and the injectivity radius is bounded from below.

From Lemma 2.1 and the work of Cheeger, Gromov and Taylor [3], we can prove,

Proposition 2.3. The following statements hold

1. If $\varphi$ is a positive smooth function outside of a compact set, then the Ilmanen’s space is complete.

2. If $\varphi$ is a smooth function such that $e^{-\varphi} \max\{\dot{\varphi}^2, \ddot{\varphi}\}$ is bounded outside a compact set, then the Ilmanen’s space has bounded geometry.

Remark 2.4. Throughout this paper, we will consider $\Sigma$ as a connected and orientable surface with empty boundary in $\Omega \subseteq \mathbb{R}^3$.

Denote by $N$ and $S$ the Gauss map and the second fundamental form of $\Sigma$ in $\mathbb{R}^3$, respectively. Then the shape operators $S^\varphi$ and $S$ of $\Sigma$ in $\Omega^\varphi$ and $\mathbb{R}^3$, respectively, satisfy

\[
-S^\varphi_p u = D_{e_i^\varphi}^\varphi N^\varphi = e^{-\varphi/2} \left( -S_p u + \frac{1}{2} \dot{\varphi} \langle N(p), e_3 \rangle u \right).
\]

for any point $p \in \Sigma$ and any vector $u \in T_p \Sigma$, where $N^\varphi = e^{-\varphi/2} N$ is the Gauss map of $\Sigma$ in the Ilmanen’s space. The above relation gives

Proposition 2.5.

\[
S^\varphi_p (u, v) = e^{\varphi/2} \left( S_p (u, v) + \frac{1}{2} \dot{\varphi} \langle N(p), e_3 \rangle (u, v) \right),
\]

\[
k^\varphi_i (p) = e^{-\varphi/2} \left( k_i(p) + \frac{1}{2} \dot{\varphi} \langle N(p), e_3 \rangle \right)
\]

for any $u, v \in T_p \Sigma$, where $S^\varphi$ and $k^\varphi_i$ (respectively, $S$ and $k_i$) are the second fundamental form and the principal curvatures of $\Sigma$ in the $\Omega^\varphi$ (respectively in $\mathbb{R}^3$).

In particular, the corresponding mean curvatures satisfy

\[
H^\varphi = e^{-\varphi/2} (H + \dot{\varphi} \langle N, e_3 \rangle).
\]
3 Short background of \([\varphi, \vec{e}_3]\)-minimal surfaces.

In what follows, \(\nabla, \Delta\) and \(\nabla^2\) will denote, respectively, the gradient, Laplacian and Hessian operators on \(\Sigma\) associated to the induced metric from \(\mathbb{R}^3\).

**Definition 3.1.** An orientable immersion \(\Sigma\) in \(\mathbb{R}^3\) is called \([\varphi, \vec{e}_3]\)-minimal if and only if the mean curvature \(H\) verifies that \(H = -\langle \nabla \varphi, N \rangle\), where \(\nabla\) is the gradient in \(\mathbb{R}^3\).

A \([\varphi, \vec{e}_3]\)-minimal \(\Sigma\) in \(\mathbb{R}^3\) can be viewed either as a critical point of the weighted area functional

\[
A^\varphi(\Sigma) := \int_{\Sigma} e^\varphi \, dA,
\]

where \(dA\) is the volume element of \(\Sigma\), or as a minimal surface in the Ilmanen’s space \(\Omega^\varphi\). From this property of minimality, a tangency principle can be applied and any two different \(\varphi\)-minimal surfaces cannot “touch” each other at one interior or boundary point (see [7, Theorem 1 and Theorem 1a]).

Let \(\Sigma\) be a \([\varphi, \vec{e}_3]\)-minimal surface and denote by \(\mu := \langle p, \vec{e}_3 \rangle\), \(\eta := \langle \nabla p, \vec{e}_3 \rangle\), \(p \in \Sigma\), their height and angle function, respectively. The following list of fundamental equations that will appear throughout this paper were proved in the Section 2 of [16].

**Lemma 3.2.** Then, the following relations hold,

1. \(\nabla \mu = \vec{e}_3^T\), \(\langle \nabla \eta, \cdot \rangle = S(\nabla \mu, \cdot)\),
2. \(\varphi^2 = \varphi^2 |\nabla \mu|^2 + H^2\),
3. \(\varphi \nabla^2 \mu = HS\),
4. \(\nabla^2 \eta = \nabla \nabla \mu S - \eta S^{[2]}\),
5. \(\Delta \mu = \varphi^2 (1 - |\nabla \mu|^2)\),
6. \(\Delta \varphi + \varphi \nabla \eta + \varphi \eta \nabla \mu + |S|^2 N = 0\),
7. \(\nabla^2 H = -\eta \nabla^2 \varphi - \nabla \nabla \varphi S - HS^{[2]} + B\),
8. \(\Delta S + \nabla \nabla \varphi S + \eta \nabla^2 \varphi + |S|^2 S - B = 0\),

where \(\vec{e}_3^T\) denotes the tangent projection of \(\vec{e}_3\) in \(T \Sigma\), \(\nabla_X\) is the Levi-Civita connection induced by \(D\) and \(S^{[2]}\) and \(B\) are the symmetric 2-tensors given by the following expressions:

\[
S^{[2]}(X, Y) = \sum_k S(X, E_k) S(E_k, Y)
\]

\[
B(X, Y) = \langle \nabla \varphi, X \rangle S(\nabla \mu, Y) + \langle \nabla \varphi, Y \rangle S(\nabla \mu, X)
\]

for any vector fields \(X, Y \in T \Sigma\) and any orthonormal frame \(\{E_i\}\) of \(T \Sigma\).
From the strong maximum principle, the equation 6 in Lemma 3.2 and Definition 3.1 the following result holds:

**Theorem 3.3.** Let \( \varphi : [a, b] \to \mathbb{R} \) be a strictly increasing function satisfying
\[
\ddot{\varphi} + \lambda \dot{\varphi}^2 \geq 0, 
\]
for some \( \lambda > 0 \), and let \( \Sigma \) be a \([\varphi, \vec{e}_3]\)-minimal immersion in \( \mathbb{R}^2 \times [a, b] \) with \( H \leq 0 \). If \( H \) vanishes anywhere, then \( H \) vanishes everywhere and \( \Sigma \) lies in a vertical plane.

Using the Hamilton’s principle (see [18, Section 2]) we also can prove,

**Theorem 3.4.** Let \( \varphi : [a, b] \to \mathbb{R} \) be a strictly increasing function satisfying
\[
\ddot{\varphi} \leq 0
\]
and let \( \Sigma \) be a locally convex \([\varphi, \vec{e}_3]\)-minimal immersion in \( \mathbb{R}^2 \times [a, b] \). If the Gauss curvature \( K \) vanishes anywhere, then \( K \) vanishes everywhere.

### 4 Stability of \([\varphi, \vec{e}_3]\)-minimal surfaces.

In this section, we will study the stability of \([\varphi, \vec{e}_3]\)-minimal surfaces, where stable means stability as minimal surface in the Ilmanen’s space, i.e., its weighted area functional \( A^\varphi \) is locally minimal.

**Proposition 4.1.** (see [1 Appendix]) Let \( X \) be a compactly supported variational vector field on the normal bundle of \( \Sigma \) and \( F_t \) the normal variation associated to \( X \). If \( \Sigma \) is an oriented \([\varphi, \vec{e}_3]\)-minimal surface, then the second derivative of the weighted area functional \( A^\varphi \) is given by,
\[
\frac{d^2}{dt^2} \bigg|_{t=0} A^\varphi(F_t(\Sigma)) = Q_\varphi(u, u), \text{ for any } u \in C_0^\infty(\Sigma).
\]
where \( Q_\varphi \) is the symmetric bilinear
\[
Q_\varphi(f, g) = \int_\Sigma e^\varphi \left( \langle \nabla f, \nabla g \rangle - (|S|^2 - \varphi \eta^2)fg \right) d\Sigma.
\]

**Definition 4.2.** We say that an oriented \([\varphi, \vec{e}_3]\)-minimal surface \( \Sigma \) without boundary is stable if and only if for any compactly supported smooth function \( u \), it holds that
\[
Q_\varphi(u, u) = -\int_\Sigma u\mathcal{L}_\varphi(u) e^\varphi d\Sigma \geq 0,
\]
where \( \mathcal{L}_\varphi \) is a gradient Schrödinger operator defined on \( C^2(\Sigma) \) by
\[
\mathcal{L}_\varphi(\cdot) := \Delta^\varphi(\cdot) + (|S|^2 - \varphi \eta^2)(\cdot)
\]
and \( \Delta^\varphi \) is the drift Laplacian given by \( \Delta^\varphi(\cdot) = \Delta(\cdot) + \langle \nabla \varphi, \nabla(\cdot) \rangle \).

**Remark 4.3.** The existence of stable surfaces it is not guaranteed for any function \( \varphi \). Cheng, Mejia and Zhou [4] proved that if \( \Omega^\varphi \) is complete and \( \ddot{\varphi} \leq -\varepsilon < 0 \) for some positive constant \( \varepsilon \), then there are not stable surfaces without boundary and with finite weighted area.
Proposition 4.4. Let $\varphi : \alpha, +\infty \rightarrow \mathbb{R}$ be a regular function satisfying (1.5) and $\Sigma$ be an oriented $[\varphi, \vec{e}_3]$-minimal immersion in $\mathbb{R}^3$ with $H \leq 0$. Then, $\Sigma$ is stable.

Proof. From Theorem 3.3 we can assume that $H < 0$ everywhere otherwise $\Sigma$ is a vertical plane and as we are going to see in Corollary 4.6 $\Sigma$ will be stable.

Suppose $H < 0$ and consider $w = \log(\eta)$, then by Equation 6 of Lemma 3.2 we get that

\[
\Delta w + \langle \nabla \varphi, \nabla w \rangle = -\frac{|\nabla \eta|^2}{\eta^2} - |S|^2 - \frac{|\varphi| \nabla \mu|^2}.
\]

Now, fix any compact domain $K$ on $\Sigma$ and consider $u$ as an arbitrary function $C^2(\Sigma)$ with compact support inside $K$. Applying the divergence theorem to the expression $\text{div} (\phi u^2 \nabla w)$ we have,

\[
\int_{\Sigma} e^\varphi u^2 (\Delta w + \langle \nabla \varphi, \nabla w \rangle) \, d\Sigma = -2 \int_{\Sigma} e^\varphi u \langle \nabla u, \nabla w \rangle \, d\Sigma.
\]

Now, from (4.4), (4.5) and (4.1) we obtain,

\[
Q(u, u) = \int_{\Sigma} e^\varphi \left( |\nabla u - \frac{u}{\eta} \nabla \eta|^2 + \varphi u^2 \right) \, d\Sigma \geq 0.
\]

which concludes the proof.

Fischer-Colbrie and Schoen [9] gave a condition on the first eigenvalue $\lambda_1(\mathcal{L}_\varphi)$ of $\mathcal{L}_\varphi$ which characterizes the stability of minimal surfaces in 3-manifolds. Using this characterization we have,

Proposition 4.5. Let $\Sigma$ be a complete oriented $[\varphi, \vec{e}_3]$-immersion in $\mathbb{R}^3$. The following statements are equivalent

1. $\Sigma$ is stable.

2. The first eigenvalue $\lambda_1(\mathcal{L}_\varphi)(K) < 0$ on any compact domain $K \subset \Sigma$.

3. There exists a positive function $u \in C^2(\Sigma)$ such that $\mathcal{L}_\varphi(u) = 0$.

As consequence of Proposition 4.4 we have the following corollary:

Corollary 4.6. Let $\Sigma$ be a complete oriented $[\varphi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3$. Then,

1. If $\Sigma$ is a graph with respect to a Killing vector $V$ lying in the orthogonal complement of $\vec{e}_3$, then $\Sigma$ is stable for any smooth function $\varphi$.

2. If $\varphi$ is an increasing convex smooth function and $\Sigma$ is a graph with respect to $\vec{e}_3$, then $\Sigma$ is stable.
Proof. Consider the following smooth function $\nu = \langle V, N \rangle$. By the assumption, $\nu$ is a positive function on $\Sigma$ and from Equation 6 of Lemma 3.2, we get that
\begin{equation}
\Delta \nu = -\dot{\varphi}\langle V, \nabla \eta \rangle - \ddot{\varphi}\eta \langle V, \nabla \mu \rangle - |S|^2 \nu.
\end{equation}
On the other hand, by Equation 1 in Lemma 3.2, the following relations hold,
\begin{align*}
\langle \nabla \varphi, \nabla \nu \rangle &= -\dot{\varphi}\langle S(V, \nabla \mu), N \rangle = \dot{\varphi}S(\nabla \mu, N) = \varphi \langle \nabla \eta, V \rangle, \\
\langle V, \nabla \mu \rangle &= \langle V, e_3 - \eta N \rangle = -\eta \nu.
\end{align*}
From the above expressions and (4.6), we have $L_\varphi(u) = 0$ and the first statement holds. The second assertion is a consequence of Proposition 4.4.

Remark 4.7. Some results about stable $[\varphi, e_3]$-minimal surface with $\ddot{\varphi} < 0$ can be found in [8].

Finally, from Theorem 3 in [4] and Corollary 4.6, we also can prove the following non-existence result:

**Theorem 4.8.** Let $V$ be a Killing vector field in the orthogonal complement of $e_3$. If $\varphi$ a smooth function such that $\ddot{\varphi} \leq -\varepsilon < 0$, for some $\varepsilon > 0$, and the Ilmanen’s space is complete, then there are not $[\varphi, e_3]$-minimal graphs respect to $V$ with finite weighted area.

### 4.1 Intrinsic area estimates

To prove intrinsic area bounds we will follow the same method as in [20].

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying (1.5) and (1.6) and $\Sigma$ be a $[\varphi, e_3]$-immersion in $\mathbb{R}^3$ with $H \leq 0$. Consider $D_\rho(p)$ an intrinsic ball in $\Sigma$ of radius $\rho$ centered at $p$.

**Lemma 4.9.** If $\rho \ddot{\varphi}(\rho + \mu(p)) < \sqrt{2}\pi$, then $D_\rho(p)$ is disjoint from the conjugate locus of $p$ and
\begin{equation}
\int_\Sigma |S|^2 u^2 d\Sigma \leq e^{2\rho \ddot{\varphi}(\rho + \mu(p))} \int_\Sigma (|\nabla u|^2 + \ddot{\varphi}\eta^2 u^2) d\Sigma,
\end{equation}
for any $u \in H^2_0(D_\rho(p))$.

**Proof.** As $|\nabla \mu|^2 \leq 1$, it is clear that for any $q \in D_\rho(p)$, $\mu(p) - \rho \leq \mu(q) \leq \mu(p) + \rho$. Hence
\begin{align*}
\varphi(\mu(p) - \rho) \leq \varphi(\mu(q)) \leq \varphi(\mu(p)), \quad q \in D_\rho(p)
\end{align*}
and we have the following control of the curvature
\begin{equation*}
2K \leq H^2 \leq \ddot{\varphi}^2(\mu(q)) \leq \ddot{\varphi}^2(\rho + \mu(p)) \text{ on } D_\rho(p).
\end{equation*}
Consequently, the first statement follows from the Rauch comparison Theorem. Finally, the inequality (4.7) follows from the above inequalities, Proposition 4.4 and the stability inequality (4.2).
Theorem 4.10 (Boundness of area). Let $\Sigma$ be a $[\varphi, e_3]$- immersion in $\mathbb{R}^3$ with $H \leq 0$ and $\varphi$ satisfying (1.5) and (1.6). If $2 \rho \dot{\varphi}(\rho + \mu(p)) < \log(2)$ and $\sqrt{\Gamma} \rho < 1$, then the geodesic disk $D_\rho(p)$ of radius $\rho$ centered at $p$ is disjoint from the cut locus of $p$ and

$$A(D_\rho(p)) < 4\pi \rho^2,$$

where $A(\cdot)$ is the intrinsic area of $\Sigma$ in $\mathbb{R}^3$.

Proof. First, we prove the inequality (4.8). Since $|S|^2 = H^2 - 2K$, from (1.5), (1.6) and Lemma 4.9, we get that for any $u \in H^2_0(D_\rho(p))$

$$-2 \int_\Sigma K u^2 d\Sigma \leq e^{2\rho \dot{\varphi}(\rho + \mu(p))} \int_\Sigma (|\nabla u|^2 + \varphi^2 u^2) d\Sigma - \int_\Sigma \varphi^2 u^2 d\Sigma$$

$$\leq 2 \int_\Sigma |\nabla u|^2 d\Sigma + \Gamma \int_\Sigma \eta^2 u^2 d\Sigma.$$

Moreover, by Gauss-Bonnet, the variation of the length $l(s)$ of $\partial D_\rho(p)$ is given by,

$$l'(s) = \int_{\partial D_\rho(p)} k_g d\sigma = 2\pi - \int_{\partial D_\rho(p)} K d\Sigma = 2\pi - K(s),$$

where $k_g$ is the geodesic curvature of $\partial D_\rho(p)$. If $u$ is a radial function satisfying $u' \leq 0$ and $u(\rho) = 0$, the coarea formula gives,

$$\int_{D_\rho(p)} K u^2 d\Sigma = \int_0^\rho u^2(s) \int_{\partial D_\rho(p)} K d\sigma ds = \int_0^\rho u^2(s) K'(s) ds,$n

$$\int_{D_\rho(p)} |\nabla u|^2 d\Sigma = \int_0^\rho \int_{\partial D_\rho(p)} |\nabla u|^2 d\sigma ds = \int_0^\rho (u'(s))^2 l(s) ds.$$

In particular, by taking $u(s) = 1 - \frac{s}{\rho}$, applying integration by parts and using (4.10) and the above expressions, we have

$$-4\pi + \frac{4A(D_\rho(p))}{\rho^2} = -\frac{4}{\rho} \int_0^\rho (2\pi - l'(s))(1 - \frac{s}{\rho}) ds$$

$$\leq 2\frac{A(D_\rho(p))}{\rho^2} + \Gamma \int_\Sigma \eta^2 u^2 d\Sigma.$$

If $\Gamma \leq 0$, then the inequality (4.8) trivially holds. If $\Gamma > 0$, using that $\sqrt{\Gamma} \rho < 1$ and (4.11) we get

$$A(D_\rho(p)) \leq \frac{4\pi}{2 - \Gamma \rho^2} \rho^2 < 4\pi \rho^2.$$

Now we will see that $D_\rho(p)$ is disjoint from the cut locus of $p$. Otherwise, there exists $q \in \partial D_{\rho_0}(p)$ that lies in the cut locus of $p$ where $\rho_0 = \text{Inj}(\Sigma)(p) \leq \rho$. Since $\rho \dot{\varphi}(\rho + \mu(p)) < \sqrt{2\pi}$, from Lemma 4.9 and a Klingenberg-type argument
(see for example [14, Chapter 5]), there exist two geodesics from \( p \) to \( q \) which bound a smooth domain \( D \subset \mathcal{D}_{r_0}(p) \) with a possible corner at \( p \). By the Gauss-Bonnet,

\[
2\pi = 2\pi - \int_{\partial D} k_g \, d\sigma = \int_D K \, d\Sigma \leq \frac{1}{2} \varphi^2(r_0 + \mu(p)) \mathcal{A}(D).
\]

Hence,

\[
\mathcal{A}(\mathcal{D}_{r_0}(p)) \geq \mathcal{A}(D) \geq \frac{4\pi}{\varphi^2(r_0 + \mu(p))}. \]

From the area estimate (4.8) for \( \rho = r_0 \) and the fact that \( \rho \varphi(\rho + \mu(p)) < \frac{\log(2)}{2} \), we get that

\[
4\pi > 4\pi r_0^2 \varphi^2(r_0 + \mu(p)) \geq 4\pi,
\]

which is a contradiction. \( \Box \)

### 4.2 Blow-up and curvature estimate.

For later use we will need the following compactness result which is a consequence of Theorem 2.1 in [23]:

**Theorem 4.11.** Let \( \Omega \) be an open subset of \( \mathbb{R}^3 \). Let \( \{ \varphi_n \} \) be a sequence of smooth functions on \( \Omega \) convergeing smoothly to \( \varphi_\infty \). Let \( \Sigma_n \) be a sequence of properly embedded minimal surfaces in the corresponding Ilmanen’s space \( \Omega^I_{\varphi_n} \). Suppose also that the area and the genus of \( \Sigma_n \) are bounded uniformly on compact subsets of \( \Omega \). Then, the total curvatures of \( \Sigma_n \) are also uniformly bounded on compact subsets of \( \Omega \) and after passing to a subsequence, \( \Sigma_n \) converge to a smooth properly embedded minimal \( \Sigma_\infty \) in \( \Omega^I_{\varphi_\infty} \). The convergence is smooth away from a discrete set \( C \) and for each connected component \( \Sigma_\infty^0 \) of \( \Sigma_\infty \) either,

1. the convergence to \( \Sigma_\infty^0 \) is smooth everywhere with multiplicity 1, or
2. the convergence to \( \Sigma_\infty^0 \) is smooth with some multiplicity grater than 1 away from \( \Sigma_\infty \cap C \). In this case, if \( \Sigma_\infty \) is two-sided, the it must be stable.

If the total curvatures of \( \Sigma_n \) are bounded by \( \beta \), the set \( C \) has at most \( \beta/4\pi \) points.

Following the same method as in [22], we prove

**Lemma 4.12** (Monoticity formula). Let \( \Sigma \) be a \([\varphi, \vec{e}_3]\)-minimal immersion in \( \mathbb{R}^3_\alpha \) with \( \varphi \) satisfying (1.5). Fix any point \( q \in \Sigma \) and consider \( B(q, r) \) the Euclidean ball of radius \( r \) centered at \( q \). Denote by \( \Sigma_r = \Sigma \cap B(q, r) \) and by \( \partial^* \Sigma_r = \Sigma \cap \partial B(q, r) \) and define \( A(r) = \mathcal{A}(\Sigma \cap B(q, r)) \) and \( L(r) = \text{length}(\Sigma \cap \partial B(q, r)) \). If there exists \( \varepsilon > 0 \) such that \( 0 \leq \varphi(\varepsilon) < 1 \), then the function

\[
\mathcal{O}_\Sigma(r) = \frac{\varphi(r) A(r)}{4\pi r^2}
\]

is increasing in \( r \) over the interval \([0, \varepsilon] \).
Proof. If we take on $\Sigma$ the vector field $X(p) = p - q$, $p \in \Sigma$, then, from the Divergence Theorem, we get that

$$2A(r) = \int_{\Sigma_r} \text{div}(X) d\Sigma_r = \int_{\partial \Sigma_r} (X, \nu) d\sigma - \int_{\Sigma_r} H(X, N) d\Sigma_r$$

$$= \int_{\partial \Sigma_r} (X, \nu) d\sigma + \int_{\Sigma_r} \phi \eta \langle X, N \rangle d\Sigma_r$$

where $\nu$ is the conormal vector over $\partial \Sigma_r$, $d\Sigma_r$ is the volume element of $\Sigma$ induced by the Euclidean metric and $d\sigma$ is the length element of $\partial \Sigma_r$. From hypothesis, we have that $0 \leq \phi(r) \leq 1$ for any $r < \varepsilon$. Moreover, as in the proof of Theorem 3 in [22], $L(r) \leq A'(r)$ for any $r$ and joining both inequalities to the expression (4.12), we have

$$0 \leq r A'(r) + r \phi(r) A(r) - 2A(r).$$

Finally, multiplying by $r^{-3} \phi(r)$ in (4.13), we get

$$0 \leq r^{-2} \phi(r) A'(r) + r^{-2} \phi(r) \phi(r) A(r) - 2r^{-3} \phi(r) A(r)$$

$$\leq r^{-2} \phi(r) A'(r) + r^{-2} \phi(r) A(r) - 2r^{-3} \phi(r) A(r) = (r^{-2} \phi(r) A(r))'.$$

which concludes the proof.

**Theorem 4.13 (Blow-up).** Let $\Sigma$ be a properly embedded $[\phi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3$ with $H \leq 0$, locally bounded genus and $\phi$ satisfying (1.5) and (1.7). Consider any sequence $\{\lambda_n\} \to +\infty$ and suppose that there exists a sequence $\{p_n\} \in \Sigma$ such that $\{\phi(\mu(p_n))/\lambda_n\} \to C$ for some constant $C \geq 0$. Then, after passing to a subsequence, $\Sigma_n = \lambda_n(\Sigma - p_n)$ converge smoothly to

(i) a plane when $C = 0$,

(ii) one of the following translating soliton when $C > 0$:

(a) vertical plane,

(b) grim reaper surface,

(c) titled grim reaper surface,

(d) bowl soliton,

(e) $\Delta$-Wing translating soliton.

**Proof.** Consider the sequence of properly embedded surfaces $\Sigma_n = \lambda_n(\Sigma - p_n)$ in $\mathbb{R}^3$. Each $\Sigma_n$ is a minimal surface in the Ilmanen’s space $\Omega^{\phi_n}$ where $\Omega = \mathbb{R}^3$ and

$$\phi_n(x_3) = \varphi \left( \frac{x_3}{\lambda_n} + \mu(p_n) \right) - \varphi(\mu(p_n)).$$

It is clear form our assumption that

$$\varphi_n \to \varphi_\infty, \text{ with } \varphi_\infty(x_3) = C x_3.$$

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For any compact set $K$ in $\Omega$, we can consider $r$ large enough such that $K$ is contained in the Euclidean ball $B(0, r)$ of radius $r$ centered at the origin. Then, for any $\epsilon_0 > 0$ and $n$ large enough, it follows from (4.15) that

$$A^\varphi (\Sigma_n \cap K) \leq \int_{\Sigma_n \cap B(0, r)} e^{\varphi} d\Sigma_n = \int_{\Sigma_n \cap B(0, r)} e^{C_{q+\epsilon_0}} d\Sigma_n \leq \lambda_n^2 \int_{\Sigma \cap B(p_n, \frac{r}{\lambda_n})} e^{C_{r+\epsilon_0}} d\Sigma = e^{C_{r+\epsilon_0}} \lambda_n^2 A(\Sigma \cap B(p_n, r/\lambda_n)).$$

As $\varphi$ can be chosen up to a constant, we can assume that there exists $\epsilon > 0$ such that $0 < \varphi(\varepsilon) < 1$. Since $r/\lambda_n \rightarrow 0$, it follows Lemma 4.12 that there must be $n_0$ such that $r/\lambda_n \leq \varepsilon$ and $O_\Sigma(r/\lambda_n) \leq O_\Sigma(\varepsilon)$ for any $n \geq n_0$. Thus,

$$A(\Sigma \cap B(p_n, r/\lambda_n)) \leq \frac{\varphi(\varepsilon)}{\varphi(r/\lambda_n)} \left( \frac{r}{\lambda_n} \right)^2 \frac{A(\Sigma \cap B(p_n, \varepsilon))}{\varepsilon^2}.$$

Joining both inequalities we have that, for $n$ large enough,

$$A^\varphi (\Sigma_n \cap K) \leq \frac{e^{C_{r+\epsilon_0}} \varphi(\varepsilon)}{\varphi(r/\lambda_n)} r^2 \frac{A(\Sigma \cap B(p_n, \varepsilon))}{\varepsilon^2} \leq 4\pi e^{C_{r+\epsilon_0}} \varphi(\varepsilon) r^2.$$

As $\lambda_n \rightarrow +\infty$, there exists a positive constant $\Theta$, depending only of $\varphi$, such that

$$A^\varphi (\Sigma_n \cap K) \leq \Theta \pi e^{C_{r} r^2}.$$

Consequently, $\Sigma_n$ have area uniformly bounded on compact subsets of $\mathbb{R}^3$. From Theorem 4.11, $\Sigma_n$ converge to a properly embedded $[\varphi_\infty, \varphi_3]$-minimal surface $\Sigma_\infty$ in $\mathbb{R}^3$. Since each $\Sigma_n$ is stable, $\Sigma_\infty$ must be a plane in $\mathbb{R}^3$ if $C = 0$ (see [9]). If $C > 0$, then $\Sigma_\infty$ is a mean convex properly embedded translating soliton in $\mathbb{R}^3$ and from the results in [11] and [20], $\Sigma_\infty$ must be either a vertical plane, a grim reaper surface, a titled grim greaper surface, a bowl soliton or a $\Delta$-Wing translating soliton.

Finally, if $p_n \in \Sigma_n$ converge to $p \in \Sigma_\infty$ and the length of the second fundamental form of $\Sigma_n$ at $p_n$ are such that $|S_n(p_n)| \rightarrow +\infty$, then from the stability of $\Sigma_n$ and Theorem 2.2 in [23], if we set $\lambda_n = |S_n(p_n)|$, we conclude that $\Sigma_n = \lambda_n (\Sigma_n - p_n)$ converge smoothly with multiplicity 1 to a plane. But this is a contradiction since the length of the second fundamental of $\Sigma_n$ at the origin satisfy $|S_n'(0)| \rightarrow 1$. In particular the convergence of $\Sigma_n$ is smooth. 

Now, by combining the methods of Rosenberg, Souam and Toubiana [17], and Spruck and Xiao [20], we will prove the Theorem A:

**Proof of Theorem A**

Suppose that there exists a sequence of points $\{p_n\}$ in $\Sigma$ such that

$$\lambda_n = |S_n(p_n)| \rightarrow +\infty, \quad \lim_{n \rightarrow +\infty} \frac{\lambda_n}{\varphi(\mu(p_n))} = +\infty.$$
Then, for a subsequence of \( \{p_n\} \) we have \( \frac{\phi(p_n)}{\lambda_n} \to 0 \) and from Theorem 4.13, the sequence \( \Sigma_n = \lambda_n(\Sigma - p_n) \) converges smoothly to a plane \( \Sigma_\infty \) in \( \mathbb{R}^3 \). Since, \( |S_n(p_n)| = 1 \) for each \( n \) we also have, \( |S_\infty(0)| = 1 \), which is a contradiction. \( \square \)

The following results are consequences of Lemma 2.1 and the results in [17] [22].

**Theorem 4.14.** Let \( \phi \) a smooth function such that
\[
\frac{1}{2} e^{-\phi} \left( |\max\{\phi^2, \tilde{\phi}\}| + |\max\{\phi^3, 2\tilde{\phi}\tilde{\phi}, \tilde{\phi}\} | \right) \geq \rho,
\]
for some constant \( \rho > 0 \) and let \( \Sigma \) be a minimal surface (possible with boundary) in the Ilmanen’s space with total absolute curvature is at most \( \lambda < 4\pi \). Then there exists a constant \( C \) depending of \( \lambda \) such that
\[
|S^\phi| \min\{d_\phi(p, \partial \Sigma), R\} \leq C \text{ for any } p \in \Sigma,
\]
where
\[
R = (\sup|K^\phi| + \sup|\nabla^\phi K^\phi|^{1/2})^{-1}.
\]

**Theorem 4.15.** Let \( \phi \) a smooth function such that the Ilmanen’s space is a complete Riemannian manifold with bounded geometry whose sectional curvature \( |K^\phi| \leq A \) for some constant \( A > 0 \). For any stable minimal immersion \( \Sigma \) in the Ilmanen’s space (with possible boundary), there exists a constant \( C \) such that
\[
|S^\phi| \min\{d_\phi(p, \partial \Sigma), \pi/2\sqrt{A}\} \leq C.
\]

## 5 A Spruck-Xiao’s type Theorem.

Using a delicate maximum principle argument, Spruck and Xiao [20] proved that any complete translating soliton in \( \mathbb{R}^3 \) with \( H \leq 0 \) is convex. A slightly simplified proof of this result is presented by Hoffman, Ilmanen, Martín and White [12]. In this section, we consider the same problem for properly embedded \( [\phi, \bar{\phi}] \)-minimal surfaces in \( \mathbb{R}^3 \) with \( \phi : \mathbb{R} \to \mathbb{R} \) satisfying (1.5), (1.7) and \( \phi \leq 0 \) on \( \alpha, +\infty \).

We start with some results we will use:

**Theorem 5.1.** Generalized Omori-Yau maximum principle for \( \Delta^\psi \) [1] Theorem 3.2] Let \( \Sigma \) be a surface in \( \mathbb{R}^3 \) and \( \Delta^\psi \) the drift laplacian operator associated to \( \psi \in C^2(\Sigma) \). Let \( \gamma \in C^2(\Sigma) \) be such that
\[
\begin{align*}
(5.1) & \quad \gamma(p) \to +\infty \quad \text{as } p \to \infty \\
(5.2) & \quad \Delta^\psi \gamma \leq C \quad \text{outside a compact subset of } \Sigma \\
(5.3) & \quad |\nabla \gamma| \leq C \quad \text{outside a compact subset of } \Sigma
\end{align*}
\]
for some constant $C > 0$. If $\nu \in C^2(\Sigma)$ and $\nu^* = \sup_{\Sigma} \nu < +\infty$, then there exists a sequence of points $\{p_n\} \subset \Sigma$ satisfying

\begin{align}
(5.4) \quad & (i) \, \nu(p_n) > \nu^* - \frac{1}{n}, \\
& (ii) \, \Delta^\nu \nu(p_n) < \frac{1}{n}, \\
& (iii) \, |\nabla \nu(p_n)| < \frac{1}{n},
\end{align}

for each $n \in \mathbb{N}$.

**Lemma 5.2.** Let $k_i$ be the principal curvatures of an immersion $\Sigma$ in $\mathbb{R}^3$ and $U$ the set of totally umbilical points of $\Sigma$. If $\{v_i\}$ is an orthonormal frame of principal directions in $T\Sigma$, then the following statements hold,

1. $\nabla v_i v_i = \alpha_i v_j, \quad \nabla v_j v_i = \alpha_j v_j$ with $\alpha_i = -\alpha_j$.

2. The coefficients $\alpha_i$ are determined by the formula,

$$\alpha_i = \frac{h_{12,i}}{k_1 - k_2} \text{ in } \Sigma - U,$$

where $h_{ij,k} = (\nabla v_i S)(v_j, v_k)$.

**Proof.** The first item is trivially obtained by differentiating $\langle v_i, v_j \rangle = \delta_{ij}$. On the other hand, differentiating $S(v_1, v_2) = 0$ and using the first item we get that

$$0 = (\nabla v_i S)(v_1, v_2) + S(\nabla v_1, v_2) + S(\nabla v_2, v_1) = h_{12,i} + \alpha_i (k_2 - k_1).$$

**Lemma 5.3.** If $\Sigma$ is a $[\varphi, e_3]$-minimal immersion in $\mathbb{R}^3$, then

$$\Delta^\varphi k_i = -|S|^2 k_i - \eta \nabla^2 \varphi(v_i, v_i) + B(v_i, v_i) + 2(-1)^i \frac{Q^2}{k_1 - k_2} \text{ in } \Sigma - U,$$

where $B$ is the bilinear form defined in Lemma 3.2 and

$$Q^2 = h_{12,1}^2 + h_{12,2}^2 = h_{11,2}^2 + h_{22,1}^2.$$

**Proof.** We only prove the formula for the first principal curvature $k_1$ because the reasoning for $k_2$ is the same. Fix any point $p \in \Sigma - U$ and consider a geodesic frame $\{u_1, u_2\}$ of $T_p \Sigma$. Then,

\begin{align}
(5.5) \quad & \Delta k_1 = \sum_{i=1}^2 (\nabla u_i \nabla k_1, u_i) = \sum_{i=1}^2 (\nabla u_i \nabla S(v_1, v_1), u_i).
\end{align}

From item 1. of Lemma 5.2, $S(\nabla u_1, v_1) = 0$ and we have,

\begin{align}
(5.6) \quad & \nabla S(v_1, v_1) = \sum_{i=1}^2 ((\nabla u_i S)(v_1, v_1)) u_i.
\end{align}

By using (5.6) and (5.5), we prove that

\begin{align}
(5.7) \quad & \Delta k_1 = \sum_{i=1}^2 (\nabla u_i (\nabla u_i S)(v_1, v_1) u_i, u_i) = (\Delta S)(v_1, v_1) + 2 \frac{Q^2}{k_1 - k_2}
\end{align}

and the Lemma follows from item 8. of Lemma 3.2. \qed
Lemma 5.4. Let \( \Sigma \) be a \([\varphi, \vec{e}_3]\)-minimal immersion in \( \mathbb{R}^3_\alpha \) with \( k_1 < 0, H = k_1 + k_2 < 0 \). If for any positive smooth function \( \psi : \Sigma \rightarrow ]0, +\infty[ \) take the operator
\[
J^{\psi} := \Delta \varphi + 2 \log \psi,
\]
then on \( \Sigma \setminus \mathcal{U} \) we have
\[
(5.8) \quad J^{\eta - k_1 \eta} = \frac{\eta \Delta \varphi k_2 - k_2 \Delta \varphi \eta}{\eta^2} \quad \text{and} \quad J^{-k_1} = \frac{k_1 \Delta \varphi \eta - \eta \Delta \varphi k_1}{k_1^2}.
\]

In particular, if \( \bar{\varphi} \leq 0 \) on \( \alpha \), \(+\infty[ \) and \( \varphi \) satisfies (1.5), then
\[
(5.10) \quad J^{\eta - k_1 \eta} \geq 0 \quad \text{on} \quad \{ p \in \Sigma : k_2(p) > 0 \}.
\]

Proof. It is not difficult to see that
\[
(5.11) \quad J^{\eta - k_1 \eta} = \frac{\eta \Delta \varphi k_2 - k_2 \Delta \varphi \eta}{\eta^2} \quad \text{and} \quad J^{-k_1} = \frac{k_1 \Delta \varphi \eta - \eta \Delta \varphi k_1}{k_1^2}.
\]
Moreover, from Lemma 3.2 and Lemma 5.3 we get that
\[
(5.12) \quad \eta \Delta \varphi k_1 = -|S|^2 k_1 \eta - \eta^2 (\bar{\varphi} (\nabla \varphi, v_1)^2 - \varphi' k_1) + 2 \varphi k_1 (\nabla \varphi, v_1)^2
\]
\[
- 2(-1)^{1+i+1} \frac{Q^2}{k_1 - k_2},
\]
\[
(5.13) \quad k_2 \Delta \varphi \eta = -\bar{\varphi} k_2 |\nabla \mu|^2 - |S|^2 k_2 \eta,
\]
\[
(5.14) \quad k_1 \Delta \varphi \eta = -\bar{\varphi} k_1 |\nabla \mu|^2 - |S|^2 k_1 \eta,
\]
and we may conclude from (5.11), (5.12), (5.13) and (5.14) by a straightforward computation.

Lemma 5.5. Let \( \Sigma \) be a properly embedded \([\varphi, \vec{e}_3]\)-minimal surface without boundary in \( \mathbb{R}^3_\alpha \) with \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) satisfying (1.5) and (1.7). Then, \( \Sigma \) is complete and the generalized Omori-Yau maximum principle can be applied to \( \Delta \varphi \).

Proof. Consider the function \( \gamma : \Sigma \rightarrow \mathbb{R} \) given by \( \gamma(p) = 2 \log |p| \), then as \( \Sigma \) is properly embedded and \( \varphi \) satisfies (1.5) and (1.7), we have
\[
(5.15) \quad \gamma(p) \rightarrow +\infty \quad \text{as} \quad p \rightarrow \infty
\]
\[
(5.16) \quad |\nabla \gamma(p)| = 2 \frac{|p|}{|p|^2} \leq 2, \quad |p| \gg 0
\]
\[
(5.17) \quad \Delta \varphi \gamma(p) = -4 \frac{|p|}{|p|^2} + \frac{2 \mu(p) \varphi(p) + 4}{|p|^2} \leq 2A + 1, \quad |p| \gg 0.
\]
and from Theorem 5.1, we can apply the generalized Omori-Yau maximum principle to $\Delta \varphi$.

By taking $\gamma$ along any divergent geodesic, it is clear from (5.15) and (5.16) that any properly embedded surface in $\mathbb{R}^3$ is complete.

**Proof of Theorem B**

Let $\Sigma$ be a properly embedded $[\varphi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3$ with $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying (1.5), (1.7) and $\varphi \leq 0$ on $]0, +\infty]$. Then from Theorem 3.3, we can assume that $\eta > 0$ everywhere. Take $k_1 < 0$, $k_1 \leq k_2$, $H = k_1 + k_2 < 0$.

We only need to prove that $\Sigma$ is convex provided $\Lambda K$ is bounded from below since the converse is trivial. For proving that, we will argue by contradiction and suppose there exists a point $p_0 \in \Sigma$ such that $K(p_0) < 0$.

(5.18) $0 < \vartheta := \sup_{\Sigma} \frac{k_2}{\eta} = \sup_{\Omega^+} \frac{k_2}{\eta}$

where $\Omega^+ = \{ p \in \Sigma \mid k_2(p) > 0 \}$.

**Claim 5.1.** The supremum $\vartheta$ is not attained.

**Proof.** Suppose it is attained at a point $p$, then from (5.10) and the strong maximum principle, see [10, Theorem 3.5], $\frac{k_2}{\eta}$ is constant on $\Sigma$ and $Q \equiv 0$. Thus, from Lemma 5.2, $\{v_1, v_2\}$ is parallel and then $k_1k_2 \equiv 0$, getting a contradiction with (5.18). \( \Box \)

**Claim 5.2.** If $\{p_n\} \subset \Omega^+$ is a sequence of points such that $\frac{k_2}{\eta}(p_n) \to \vartheta$, then after passing to a subsequence, $\mu(p_n) \to +\infty$ and

1. if $\Lambda = 0$ and $\frac{\eta}{k_1}(p_n) \to 0$, then $\eta(p_n) \to 0$.

2. if $\Lambda \neq 0$, then $\eta(p_n) \to 0$ and $\frac{\eta}{k_1}(p_n) \to 0$.

**Proof.** From (1.7), the function $2\dot{\varphi} - \varphi^2$ is upper bounded on $]0, +\infty]$ and we can apply the Theorem 4.10, getting that the sequence $\Sigma_n = \Sigma - p_n$ has area uniformly bounded on compact subsets of $\mathbb{R}^3$.

Each $\Sigma_n$ is a $[\varphi_n, \vec{e}_3]$-minimal surface with

(5.19) $\varphi_n(u) = \varphi(u + \mu(p_n)) - \varphi(\mu(p_n))$, for each $n \in \mathbb{N}$.

If $\sup \{\mu(p_n)\} < +\infty$ then, by taking an accumulation point $\mu_\infty$ of $\{\mu(p_n)\}$ and applying the compactness Theorem 4.11, we get that, after passing to a subsequence, $\mu(p_n) \to \mu_\infty \in \mathbb{R}$, $\Sigma_n$ converges to a properly embedded $[\varphi_\infty, \vec{e}_3]$-minimal surface $\Sigma_\infty$ with $H \leq 0$ where $\varphi_\infty(u) := \varphi(u + \mu_\infty) - \varphi(\mu_\infty)$. From
Theorem A, the length of second fundamental form of \( \Sigma_n \) is bounded, therefore the convergence must be smooth at the origin and so the function \( \frac{k_2}{\eta} \) reaches its supremum at the origin. This is a contradiction with Claim 5.1.

If \( \Lambda = 0 \) and \( \frac{\eta}{k_1}(p_n) \to 0 \), we consider \( \Sigma'_n = \lambda_n(\Sigma - p_n) \) where \( \lambda_n = -\frac{k_1}{\eta}(p_n) \) then, from (1.5) and (1.7) and after passing to a subsequence, we get that

\[
\eta(p_n) \to \eta_\infty, \quad \frac{\varphi(\mu(p_n))}{\lambda_n} = 1 + \frac{k_2}{k_1}(p_n) \to 0.
\]

Applying Theorem 4.13, \( \Sigma'_n \) converge smoothly to a plane \( \Sigma_\infty \), with principal curvatures at the origin given by

\[
k_1 = -\eta_\infty \quad \text{and} \quad k_2 = \eta_\infty,
\]

which implies that \( \eta_\infty = 0 \).

If \( \Lambda \neq 0 \), since \( \frac{k_1}{\eta} + \frac{k_2}{\eta} = -\varphi \), we have that \( \frac{\eta}{k_1}(p_n) \to 0 \). Let us suppose by contradiction that \( \eta(p_n) \to \eta_\infty \neq 0 \). Then \( k_1(p_n) \to -\infty \) and \( k_2(p_n) \to \vartheta \), getting to a contradiction with the hypothesis that \( \Lambda K \) is bounded from below.

We will distinguish the case that \( \varphi \) is bounded (\( \Lambda = 0 \)) from the unbounded case (\( \Lambda \neq 0 \)):

- **Case \( \Lambda = 0 \).**

In this case, from (1.7)

\[
0 < \varphi < \sup_{|\alpha, +\infty|} \varphi = \beta.
\]

**Claim 5.3.** The case \( \vartheta = +\infty \) is not possible.

**Proof.** Assume there exists a sequence of points \( \{p_n\} \) such that \( \frac{k_2}{\eta}(p_n) \to +\infty \).

Using that

\[
\left(\frac{k_1}{k_2}\right) + 1 = -\varphi\left(\frac{\eta}{k_2}\right), \quad \frac{k_1 + k_2}{\eta} = -\varphi,
\]

we get \( (k_1/k_2)(p_n) \to -1 \) and \( (\eta/k_1)(p_n) \to 0 \). In particular,

\[
\tau = \sup_{\Sigma} \eta \frac{k_1}{k_2} = 0.
\]

and \( \tau \) is not attained at an interior point. Now we may apply the generalized Omori-Yau maximum principle for \( \Delta^\varphi \) and conclude that there exists a sequence of points, \( \{q_n\} \subset \Sigma, |q_n| \to +\infty \), such that

\[
\frac{\eta}{k_1}(q_n) \to 0, \quad |\nabla \left(\frac{\eta}{k_1}\right) (q_n) | \to 0 \quad \text{and} \quad \Delta^\varphi \left(\frac{\eta}{k_1}\right)(q_n) \leq 0.
\]
Consequently, it follows from (5.21), and (5.23) that \((k_2/\eta)(q_n) \to +\infty\) and so, for \(n\) large enough \(\{q_n\} \subset \Omega^+\). In particular there exists \(n_0 \in \mathbb{N}\) such that (5.8) and (5.9) hold for \(n \geq n_0\). For the rest of the proof of Theorem B, any statement that some quantity tends to a limit refers only to the quantity at the corresponding points.

Now, from Claim 5.2, after passing to subsequence, \(\mu \to +\infty\), \(\eta \to 0\) and \(k_2/k_1 = -\dot{\varphi}\eta - 1 \to -1\). Thus, form Lemma 3.2, we have

\[
\left| \frac{\nabla \eta}{k_1} \right|^2 = (\nabla \mu, v_1)^2 + \left( \frac{k_2}{k_1} \right)^2 (\nabla \mu, v_2)^2 \to 1.
\]

and by (5.23) and (5.24),

\[
\frac{\nabla \eta}{k_1} \to \mathcal{X}, \quad \frac{\eta}{k_1} \frac{\nabla k_1}{k_1} \to \mathcal{X}, \quad \mathcal{X} \neq 0
\]

Since

\[
\frac{\eta}{k_1} \frac{\nabla H}{k_1} = \frac{\eta}{k_1} \frac{\nabla k_1}{k_1} + \frac{\eta}{k_1} \frac{\nabla k_2}{k_1} = -\frac{\eta^2 \nabla \varphi}{k_1} - \frac{\eta \dot{\varphi} \nabla \eta}{k_1},
\]

it follows from Lemma 3.2, (5.25) and (5.26) that

\[
\frac{\eta}{k_1} \frac{h_{11,2}}{k_1} \to < \mathcal{X}, v_2 >, \quad \frac{\eta}{k_1} \frac{h_{22,1}}{k_1} \to -< \mathcal{X}, v_1 >.
\]

In particular,

\[
\frac{\eta^2}{k_1^2} Q^2 \to |\mathcal{X}| = 1
\]

Multiplying by \((\eta/k_1)^2\) in (5.9), we obtain

\[
\left( \frac{\eta}{k_1} \right) \Delta^2 \left( \frac{\eta}{k_1} \right) + 2 \left( \frac{\eta}{k_1} \right) (\nabla \left( \frac{\eta}{k_1} \right), \frac{\nabla k_1}{k_1}) = \dot{\varphi} (\nabla \mu, v_1)^2 \left( \frac{\eta}{k_1} \right)^3
\]

\[
- \varphi \left( \frac{\eta}{k_1} \right)^2 (1 + 2(\nabla \mu, v_1)^2) - 2k_1 \left( \frac{\eta^2}{k_1^2} \right) \frac{Q^2}{k_1 - k_2}.
\]

Using that \(k_2/k_1 \to -1\), (1.5), (1.7), (5.23) and (5.27), we can take limit when \(n \to +\infty\) in the above equality to get:

\[
0 \leq -1,
\]

a contradiction. \(\square\)

**Claim 5.4.** If \(\{p_n\} \subset \Sigma\) is a sequence of points such that \(\frac{k_2}{\eta} \to \vartheta < +\infty\), then after passing to a subsequence,

\[
\mu \to +\infty, \quad \eta \to 0, \quad \frac{k_1}{k_2} \to -\frac{\beta}{\vartheta} - 1.
\]

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Proof. By taking $\Sigma_n = \Sigma - p_n$, we can argue as in the first part of Claim 5.2 to prove that after passing to a subsequence, $\mu \to +\infty$. Then, from (5.19),

$$\varphi_n \to \varphi_\infty, \quad \text{with} \quad \varphi_\infty(u) = \beta u,$$

and using again the compactness Theorem 4.11 after passing to a subsequence, we have that $\Sigma_n$ converges to a properly embedded translating soliton $\Sigma_\infty$ containing the origin with $H \leq 0$. But from Theorem A, the length of the second fundamental form of $\Sigma_n$ is bounded, so the convergence is smooth and we conclude that if $\Sigma_\infty$ is not a vertical plane, $k_2/\eta$ attains its supremum value at the origin of $\Sigma_\infty$ which contradicts Claim 5.1.

Thus, $\eta \to 0$ and

$$\frac{k_1}{k_2} = \frac{H}{k_2} - 1 = -\frac{\varphi}{k_2} - 1 \to -\frac{\vartheta}{\vartheta} - 1.$$  \hfill \Box

Claim 5.5. The case $0 < \vartheta < +\infty$ is not possible.

Proof. If

$$0 < \vartheta = \sup_{\Sigma} \frac{k_2}{\eta} = \sup_{\Omega^+} \frac{k_2}{\eta} < \infty,$$

then from Lemma 5.5, Theorem 5.1 and Claim 5.1, there exists a sequence of points $\{p_n\} \subset \Omega^+$, $|p_n| \to +\infty$ such that

$$\left(\frac{k_2}{\eta}\right) \to \vartheta, \quad \left|\nabla \left(\frac{k_2}{\eta}\right)\right| \to 0, \quad \Delta^\varphi \left(\frac{k_2}{\eta}\right) (p_n) \leq 0.$$  \hfill (5.29)

From Claim 5.4 we get

$$\nabla \mu \to \vec{e}_3, \quad \mu \to +\infty, \quad \frac{k_1}{k_2} \to -\frac{\beta}{\vartheta} - 1,$$

and from Lemma 3.2,

$$\left|\frac{k_2 \nabla \eta}{\eta \eta}\right|^2 = \frac{k_2^2}{\eta^2} \left(\frac{k_2^2}{\eta^2} (\nabla \mu, v_1)^2 + (\nabla \mu, v_2)^2\right) \to C \neq 0,$$

where $C$ is a constant such that $C \in [\vartheta^4, 2\vartheta^4 + \vartheta^2(\beta^2 + 2\beta \vartheta)]$. Then, by (5.29),

$$\nabla k_2 \eta \to \mathcal{Y}, \quad \frac{k_2 \nabla \eta}{\eta \eta} \to \mathcal{Y}, \quad \mathcal{Y} \neq 0.$$  \hfill (5.30)

Arguing as in Claim 5.4, we can prove that

$$\left(\frac{h_{11,2}}{\eta}\right) \to -\vartheta^2 \left(\frac{\beta}{\vartheta} + 1\right) \langle \vec{e}_3, v_2\rangle, \quad \left(\frac{h_{22,1}}{\eta}\right) \to -\vartheta^2 \left(\frac{\beta}{\vartheta} + 1\right) \langle \vec{e}_3, v_1\rangle.$$  \hfill (5.31)
and then,

\begin{equation}
(5.32) \quad \frac{Q^2}{\eta^2} = \frac{h_{1,2}^2 + h_{2,1}^2}{\eta^2} \to \vartheta \left( \frac{\beta}{\vartheta} + 1 \right)^2
\end{equation}

Multiplying by \((k_2/\eta)\) in \((5.8)\), we obtain

\begin{equation}
(5.33) \quad \frac{k_2}{\eta} \Delta \varphi \left( \frac{k_2}{\eta} \right) + 2 \frac{k_2}{\eta} \left( \nabla \left( \frac{k_2}{\eta} \right) \cdot \nabla \eta \right) = -\varphi \frac{k_2}{\eta} \left( \nabla \mu, v_2 \right)^2 + \varphi \left( \frac{k_2}{\eta} \right)^2 \left( 1 + 2 \left( \nabla \mu, v_2 \right)^2 \right) - 2 \left( \frac{Q^2}{\eta^2} \right) \frac{k_2}{k_1 - k_2}.
\end{equation}

Using that \(\frac{k_1}{k_2} \to -\frac{\beta}{\vartheta} - 1\), \((1.5)\), \((1.7)\), \((5.29)\) and \((5.32)\), we can take limit when \(n \to +\infty\) in the above equality to get

\[0 \geq 2 \frac{\vartheta^4 \left( \frac{\beta}{\vartheta} + 1 \right)^2}{\vartheta^2 + 2} > 0\]

which is contradiction.

\[\square\]

\bullet Case \(\Lambda \neq 0\).

As the supremum of \(k_2/\eta\) is not attained on \(\Sigma\), we can take any divergent sequence of points \(\{p_n\} \subset \Omega^+\) such that \(k_2/\eta \to \vartheta\).

\textbf{Claim 5.6.} If \(\Lambda \neq 0\) and \(\{p_n\} \subset \Sigma\) is a sequence of points such that \(\frac{k_2}{\eta} \to \vartheta < +\infty\), then after passing to a subsequence

\[\frac{\eta}{k_1} \to 0, \quad \mu \to +\infty, \quad \eta \to 0, \quad \frac{k_2}{k_1} \to 0.\]

\textit{Proof.} By taking \(\Sigma_n = \Sigma - p_n\), we can argue as in the first part of Claim 5.2 to prove that after passing to a subsequence, \(\mu \to +\infty\). Since \(\frac{k_1 + k_2}{\eta} = -\varphi\), we have that \(\frac{\eta}{k_1} \to 0\) and Claim 5.2 gives that, after passing to subsequence, \(\eta \to 0\). Finally,

\[\frac{k_1}{k_2} = \frac{H}{k_2} - 1 = -\varphi \frac{\eta}{k_2} \to -\infty.\]

\[\square\]

\textbf{Claim 5.7.} The case \(0 < \vartheta < +\infty\) is not possible.

\textit{Proof.} From Theorem 5.1 and Claim 5.1 there exists a sequence of points \(\{q_n\} \subset \Omega^+, \ |q_n| \to +\infty\) such that

\begin{equation}
(5.34) \quad \frac{k_2}{\eta}(q_n) \to \vartheta, \quad \left| \nabla \left( \frac{k_2}{\eta} \right) \right|(q_n) \to 0, \quad \Delta \varphi \left( \frac{k_2}{\eta} \right)(q_n) \leq 0.
\end{equation}
By a straightforward computation we obtain
\[
\eta^2 k_1 k_2 \nabla k_2 = \frac{\eta^3}{k_1 k_2^2} \nabla \left( \frac{k_2}{\eta} \right) + \frac{\eta}{k_1 k_2} \nabla \eta,
\]
\[
\eta^2 k_1 k_2 \nabla k_1 = -\frac{\eta^3}{k_1 k_2^2} \nabla \left( \frac{k_2}{\eta} \right) + \frac{\eta}{k_2} \nabla \eta - \frac{\eta^3 \varphi}{k_1 k_2} \nabla \mu,
\]
and using Claim 5.6 and (5.34),
\[
\eta^2 k_1 k_2 \frac{h_{22,1}}{1} \to \frac{1}{\vartheta} \langle \vec{e}_3, v_1 \rangle, \quad \eta^2 k_1 k_2 \frac{h_{11,2}}{1} \to \frac{1}{\vartheta} \langle \vec{e}_3, v_2 \rangle.
\]
which gives
\[
(5.35) \quad \eta^4 k_1^2 k_2^2 Q^2 = \frac{\eta^4}{k_1^2 k_2^2} (h_{11,2}^2 + h_{22,1}^2) \to \frac{1}{\vartheta^2} > 0.
\]
As the equation (5.38) holds on \( \Omega^+ \), multiplying by \( \frac{\eta^3}{k_1 k_2} \) we get that
\[
(5.36) \quad \frac{\eta^3}{k_1 k_2} \Delta \varphi \left( \frac{k_2}{\eta} \right) + 2 \frac{\eta^3}{k_1 k_2^2} \nabla \left( \frac{k_2}{\eta} \right) \cdot \frac{\nabla \eta}{\eta} = -\varphi \frac{\eta^3}{k_1 k_2} (\nabla \mu, v_2)^2
\]
\[
+ \varphi \frac{\eta^3}{k_1 k_2^2} \left( \frac{k_2}{\eta} \right) (1 + 2(\nabla \mu, v_2)^2) - 2 \frac{\eta^3}{k_1 k_2^2} \frac{1}{\eta k_1 - k_2} Q^2.
\]

\[\Box\]

**Claim 5.8.** The case \( \vartheta = +\infty \) is not possible.

**Proof.** Assume by contradiction that \( \vartheta = +\infty \). Let \( g : \mathbb{R} \to ]-1,1[ \) be a bounded smooth function satisfying:
\[
(5.37) \quad \dot{g} \geq 0, \quad \text{on } \mathbb{R},
\]
\[
(5.38) \quad g(x) = 1 - \frac{1}{x}, \quad \text{on } [1, +\infty[.
\]
Let \( h : \Sigma \to \mathbb{R} \) be the function \( h(p) = g \left( \frac{k_2}{\eta}(p) \right) \). Using (5.38), a straightforward computation provides,
\[
\Delta \varphi h + 2 \langle \nabla h, \frac{\nabla \eta}{\eta} \rangle = \dot{g} \left| \nabla \left( \frac{k_2}{\eta} \right) \right|^2 - \dot{g} \varphi \langle \nabla \mu, v_2 \rangle^2
\]
\[
+ \dot{g} \varphi \left( \frac{k_2}{\eta} \right) (1 + 2(\nabla \mu, v_2)^2) - 2 \frac{\dot{g}}{\eta} \frac{Q^2}{k_1 - k_2}.
\]
Since \( \vartheta = +\infty \), it is clear that
\[
(5.40) \quad \sup_{\Sigma} \{ h \} = 1,
\]

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and it is not attained on $\Sigma$. Now, from Lemma 5.5 we can apply the Theorem 5.1 and there exists a divergent sequence $\{q_n\}$ such that

\[(5.41) \quad h \to 1, \quad |\nabla h| \to 0, \quad \text{and} \quad \Delta^\varphi h(q_n) \leq 0.\]

Thus, $\frac{k_2}{\eta} \to +\infty$, $\frac{\eta}{k_1} \to 0$ and, from Claim 5.2 after passing to a subsequence we have also that $\mu \to +\infty$ and $\eta \to 0$. Now, we can argue as in Claim 5.7 to get that

$$
\frac{\eta}{k_1 k_2} h_{22,1} \to \langle \hat{e}_3, v_1 \rangle, \quad \frac{\eta}{k_1 k_2} h_{11,2} \to \langle \hat{e}_3, v_2 \rangle.
$$

which gives

\[(5.42) \quad \frac{\eta^2}{k_1^2 k_2^2} Q^2 = \frac{\eta^2}{k_1^2 k_2^2} (h_{11,2}^2 + h_{22,1}^2) \to 1 > 0.\]

Using that $\frac{k_2}{k_1} \in ]-1, 0[ \cap \Omega^+$ and that for $n$ large enough,

$$
\dot{g} \left( \frac{k_2}{\eta} (p_n) \right) = \frac{\eta^2}{k_2^2} (p_n), \quad \ddot{g} \left( \frac{k_2}{\eta} (p_n) \right) = -2 \frac{\eta^3}{k_2^2} (p_n).
$$

If we multiply by $\frac{\eta}{k_1}$ in the expression (5.39) take limit when $n \to +\infty$, we get

$$
0 \leq - \frac{2}{1 - C} < 0,
$$

where $\frac{k_2}{k_1} \to C \in [-1, 0]$, which is a contradiction. \hfill \Box

From the above Claims, the only possibility is that $\vartheta \leq 0$, which concludes the proof. \hfill \Box

From Theorem 4.13, Theorem B and arguing as in [1] Corollary 2.3, we may obtain

**Corollary 5.9.** Let $\Sigma$ be as in Theorem B with $\Lambda K$ bounded from below. If $\{p_n\}$ is any divergent sequence in $\Sigma$ and $\{\lambda_n\} \to +\infty$ any sequence such that $\{\dot{\varphi}(\mu(p_n))/\lambda_n\} \to C$ for some constant $C > 0$. Then, $\Sigma_n = \lambda_n (\Sigma - p_n)$ converge smoothly (after passing to a subsequence) to a vertical plane, a grim reaper surface, or a titled grim reaper surface.

Moreover, from Theorem 3.4 and Theorem B, we have

**Corollary 5.10.** Let $\Sigma$ be as in Theorem B with $\Lambda K$ bounded from below. If $K$ vanishes anywhere, then $\Sigma$ has vanishing curvature.
Some interesting questions

We conclude this paper with two questions related to our Theorem B, the first is whether an entire \( [\varphi, e_3] \)-minimal vertical graph in \( \mathbb{R}^3 \) with \( \varphi \) satisfying (1.5) and (1.7) is convex. The second is whether an entire \( [\varphi, e_3] \)-minimal vertical graph in \( \mathbb{R}^3 \) with \( H(p) \rightarrow 0 \) as \( |p| \rightarrow \infty \) and \( \varphi \) satisfying (1.5) and (1.7) is rotationally symmetric. We expect affirmative answers to both questions.

References

[1] L. Alías, P. Mastrolia and M. Rigoli: Maximum Principles and Geometric Applications Springer Monographs in Mathematics. Springer, Cham, 2016.

[2] R. Böme, S. Hildebrant, E. Tausch.: The two-dimensional analogue of the catenary. Pacific J. Math. 88 No. 2 (1980) 247-278.

[3] J. Cheeger, M. Gromov and M. Taylor: Finite propagation speed, kernel estimates for functions of the Laplace operator and the geometry of complete Riemannian manifolds, J. Differential Geom, 17 (1982), pp 15-53.

[4] X. Cheng, T. Mejia and D. Zhou: Stability and compactness for complete \( f \)-minimal surfaces. Trans. Amer. Math. Soc., pages 4041-4059, volume 367, number 6, (2015).

[5] T. H. Colding and W. P. Minicozzi II: Estimates for parametric elliptic integrands, Int. Math.Res.Not. 6 (2002),291-297.

[6] T. H. Colding and W. P. Minicozzi II: A course in minimal surfaces Graduate studies in Mathematics v.121, American Mathematical Society, Providence, R.I., 2011

[7] J. H. Eschenburg: Maximum principle for hypersurfaces. Manuscr. Math., 64 (1989), 55-75.

[8] J. M. Espinar: Gradient Schrödinger operators, manifolds with density and applications. J. Math. Anal. Appl. 455 (2017), no. 2, 1505–1528.

[9] D. Fischer-Colbrie and R. Schoen: The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature, Comm.Pure Appl. Math. 33 (1980), 199-211.

[10] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[11] Hoffman, D.; Ilmanen, T.; Martín, F.; White, B.: Graphical translators for mean curvature flow. Calculus of Variations and PDE’s 58 (2019), art. 117.
[12] D. Hoffman, T. Ilmanen, F. Martín and B. White: Notes on translating solitons for mean curvature flow, to appear in the proceedings of M:IV - Minimal surfaces: Integrable systems and Visualisation workshops, Springer Proceedings in Mathematics and Statistics., arXiv: 1901.09101v1, (2019).

[13] T. Ilmanen: Elliptic regularization and partial regularity for motion by mean curvature. Men. Amer. Math. Soc, 108 No. 520 (1994).

[14] P. Petersen: Riemannian Geometry. Third Edition Springer, (2016).

[15] S.D. Poisson. Sur les surfaces elastique. Men. CL. Sci. Math. Phys. Inst. Frace, deux, 167-225 (1975).

[16] A. Martínez. and A. L. Martínez Triviño: Equilibrium of Surfaces in a Vertical Force Field. Preprint: arXiv: 1910.07795, (2019).

[17] H. Rosenberg, R. Souam and E. Toubiana: General curvature estimates for stable H-surfaces in 3-manifolds and applications. J. Differential Geom, volume 84, Number 3 (2010), 623-648.

[18] A. Savas-Halilaj and K. Smoczyk: Bernstein theorems for length and area decreasing minimal maps, Calc. Var, 50 (2014) 549-577.

[19] R. Schoen: Estimates for stable minimal surfaces in three-dimensional manifolds. Seminar on minimal submanifolds, 111-126, Ann. of Math. Stud., 103, Princeton Univ. Press, Princeton, NJ, (1983).

[20] J. Spruck, and L. Xiao: Complete translating solitons to the mean curvature flow in $\mathbb{R}^3$ with nonnegative mean curvature, Amer. J. Math. 142 (2020), no. 3, 993–1015.

[21] X. Wang: Convex solutions to the mean curvature flow, Annals of Mathematics, 173 (2011), 1185-1239.

[22] B. White: Lectures on minimal surfaces theory prePrint. arXiv: 1308.3325v4, 2016.

[23] B. White: On the compactness theorem for embedded minimal surfaces in 3-manifolds with locally bounded area and genus Comm. Anal. Geom. 26 (2018), no. 3, 659–678.