Bell-like inequality for spin-orbit separability of a classical laser beam

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In analogy with Bell’s inequality for two-qubit quantum states we propose an inequality criterion for the non-separability of the spin-orbit degrees of freedom of a classical laser beam. A definition of separable and non-separable spin-orbit modes is used in consonance with the one presented in Phys. Rev. Lett. 99, 160401 (2007). As the usual Bell’s inequality can be violated for entangled two-qubit quantum states, we show both theoretically and experimentally that the proposed spin-orbit inequality criterion can be violated for non-separable modes. A discussion on the classical-quantum transition is also presented.

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Experiments to show violation of Bell-like inequalities have attracted much attention in the last years due to the possibility of ruling out classical hidden-variables theories which jeopardize the need of a quantum mechanical model to describe nature. The majority of proposed experiments relies on a pair of entangled quantum particles for genuine non-locality tests [1] or not necessarily entangled in the case of non-contextuality tests [2].

Entanglement in single particle degrees of freedom has already been investigated in ref. [3], where a Bell-like inequality was violated by entangling the spin and the beam path of single neutrons in an interferometer. The same kind of single particle scheme has been proposed for photonic setups using the polarization and transverse (spin-orbit) modes [4]. We have proposed a similar setup to investigate the spin-orbit separability of a classical laser beam in [5]. Simulations of Bell-inequalities in classical optics have also been discussed in waveguides [6] and imaging systems [7]. In this work we present our experimental results together with the theoretical background developed for the analogy between the usual quantum mechanical context of Bell inequality and our classical spin-orbit counterpart. Moreover, we discuss the classical-quantum transition by assuming an initial coherent state of the non-separable spin-orbit mode. We show that no genuine entanglement is present in the classical optics implementation as expected. However, entanglement can be reached by post selection of the single photon components of the initial coherent state.

Following ref. [8], we define as separable those spin-orbit modes that can be written in the form \( \tilde{E}_S(\vec{r}) = \psi(\vec{r})\hat{e} \), where \( \psi(\vec{r}) \) is a normalized c-number function of the transverse spatial coordinates (transverse mode) and \( \hat{e} \) is a normalized polarization vector. However, there are modes that cannot be written in this form, which we shall refer to as non-separable. For example, consider the following normalized mode

\[
\tilde{E}_{MNS}(\vec{r}) = \frac{1}{\sqrt{2}} [\psi_V(\vec{r})\hat{e}_V + \psi_H(\vec{r})\hat{e}_H],
\]

where \( \psi_H(\vec{r}) \) and \( \psi_V(\vec{r}) \) are the first order Hermite-Gaussian transverse modes with horizontal (H) and vertical (V) orientations [9], and \( \hat{e}_H \) and \( \hat{e}_V \) are the horizontal (H) and vertical (V) linear polarization unit vectors. This mode cannot be written as product of a spatial part times a polarization vector. In the space of spin-orbit modes of a classical beam, it plays a role analogous to a maximally entangled two-qubit state, and we shall refer to it as a maximally non-separable mode (MNS). While the separable spin-orbit modes have a single polarization state over the beam wavefront, the non-separable modes exhibit a polarization gradient leading to a polarization-vortex behavior [10].

We can define an arbitrary classical spin-orbit mode as:

\[
\tilde{E}(\vec{r}) = A_1\psi_V(\vec{r})\hat{e}_V + A_2\psi_V(\vec{r})\hat{e}_H + A_3\psi_H(\vec{r})\hat{e}_V + A_4\psi_H(\vec{r})\hat{e}_H,
\]

and discuss its separability with the aid of a concurrence-like quantity [8] [11]:

\[
C = 2|A_2A_3 - A_1A_4|
\]

where \( A_i \) (\( i = 1 \ldots 4 \)) are complex numbers satisfying\( \sum_{i=1}^{4}|A_i|^2 = 1 \). It turns out that \( 0 < C \leq 1 \) for non-separable modes. In particular we say that \( C = 1 \) corresponds to a maximally non-separable mode. It can be easily verified that any separable mode of the form:

\[
\tilde{E}_S(\vec{r}) = [B_1\psi_V(\vec{r}) + B_2\psi_H(\vec{r})](B_3\hat{e}_V + B_4\hat{e}_H),
\]

where \( B_i \) are arbitrary complex coefficients, has \( C = 0 \).

To develop the spin-orbit inequality it will be useful to define the following rotated basis of polarization and transverse modes:

\[
\hat{e}_{\alpha+} = \cos(2\alpha)\hat{e}_V + \sin(2\alpha)\hat{e}_H,
\]

\[
\hat{e}_{\alpha-} = \sin(2\alpha)\hat{e}_V - \cos(2\alpha)\hat{e}_H,
\]

\[
\psi_{\beta+}(\vec{r}) = \cos(2\beta)\psi_V(\vec{r}) + \sin(2\beta)\psi_H(\vec{r}),
\]

\[
\psi_{\beta-}(\vec{r}) = \sin(2\beta)\psi_V(\vec{r}) - \cos(2\beta)\psi_H(\vec{r}).
\]
Re-writing the maximally non-separable mode given by Eq. 6 in the rotated basis we get:
\[
\vec{E}_{MNS}(\vec{r}) = A_e(\psi_{\beta+}(\vec{r})\hat{e}_{\alpha+} + \psi_{\beta-}(\vec{r})\hat{e}_{\alpha-}) + \\
A_o(\psi_{\beta-}(\vec{r})\hat{e}_{\alpha+} - \psi_{\beta+}(\vec{r})\hat{e}_{\alpha-}) ,
\]
where
\[
A_e = \cos(2\alpha)\cos(2\beta) + \sin(2\alpha)\sin(2\beta) , \\
A_o = \cos(2\alpha)\sin(2\beta) - \sin(2\alpha)\cos(2\beta) .
\]

Let \( I_{\pm\pm}(\alpha, \beta) \) be the squared amplitude of the \( \psi_{\beta\pm}(\vec{r})\hat{e}_{\alpha\pm} \) component in the expansion of \( \vec{E}_{MNS}(\vec{r}) \) in the rotated basis. They play the same role as the detection probabilities in the quantum mechanical context. Due to the orthonormality of \( \{\psi_{\beta+}, \psi_{\beta-}\} \) it can be easily shown that:
\[
I_{++}(\alpha, \beta) + I_{+-}(\alpha, \beta) + I_{--}(\alpha, \beta) + I_{-+}(\alpha, \beta) = 1 .
\]

Following the analogy with the usual quantum mechanical Bell inequality for spin 1/2 particles, we can define
\[
M(\alpha, \beta) = I_{++}(\alpha, \beta) + I_{--}(\alpha, \beta) - I_{+-}(\alpha, \beta) - I_{-+}(\alpha, \beta) ,
\]
and derive a Bell-type inequality for the quantity
\[
S = M(\alpha_1, \beta_1) + M(\alpha_1, \beta_2) - M(\alpha_2, \beta_1) + M(\alpha_2, \beta_2) .
\]

For any separable mode, \(-2 \leq S \leq 2\), however this condition can be violated for non-separable modes. A maximal violation of the previous inequality, corresponding to \( S = 2\sqrt{2} \), can be obtained for the set:
\[
\alpha_1 = \pi/16 , \quad \alpha_2 = 3\pi/16 , \quad \beta_1 = 0 , \quad \beta_2 = \pi/8 .
\]

The experimental setup to observe the maximum violation of the non-separability inequality is shown in Figure 1 and is composed by two stages: preparation of the maximally non-separable mode and measurement of the intensities \( I_{\pm\pm}(\alpha, \beta) \). The preparation stage consists of a Mach-Zender (MZ) interferometer with a half-wave plate (HWP) oriented at 45° with respect to the horizontal plane in one arm and a Dove prism (DP) also oriented at 45° with respect to the horizontal plane in the other arm. Before the MZ interferometer, a holographic mask positioned in the path of a horizontally polarized TEM\(_{00}\) laser beam produces mode \( \vec{E}(\vec{r}) = \psi_V(\vec{r})\hat{e}_H \) at the first diffraction order. In the MZ interferometer, the half-wave plate converts \( \hat{e}_H \) into \( \hat{e}_V \) and the Dove prism changes \( \psi_V(\vec{r}) \) into \( \psi_H(\vec{r}) \) so the resulting mode at the output BS2 is:
\[
\vec{E}(\vec{r}) = \frac{1}{\sqrt{2}} [\psi_H(\vec{r})\hat{e}_H + e^{i\phi}\psi_V(\vec{r})\hat{e}_V] ,
\]
where \( \phi \) is the phase difference between the two arms of the MZ interferometer. Mirror M1 is mounted on a piezoelectric transducer (PZT) to allow fine control of the phase difference \( \phi \). Our goal is to prepare the initial mode with \( \phi = 0 \), but we carry out this phase in the calculations to show how our results depends on it. The other output of BS2 is used to check the alignment between the two components of the mode prepared.

The measurement stage is composed by a Dove prism oriented at a variable angle \( \beta \) (DP@\( \beta \)), a half-wave plate oriented at a variable angle \( \alpha \) (HWP@\( \alpha \)), a Mach-Zender interferometer with an additional mirror (MZIM) [12], and one polarizing beam splitter (PBS) after each of the MZIM outputs. Four photo-current detectors (D1, D2, D3 and D4) are used to measure the intensities at the PBS outputs. HWP@\( \alpha \) combined with DP@\( \beta \) define in which basis we are going to measure our initial mode.

We want MZIM to work as a parity selector delivering odd modes \( \psi_V(\vec{r})\hat{e}_H \) and \( \psi_H(\vec{r})\hat{e}_V \) in one port and even modes \( \psi_V(\vec{r})\hat{e}_V \) and \( \psi_H(\vec{r})\hat{e}_H \) in the other port. Parity is evaluated according to the eigenvalue of the respective mode under reflection over the horizontal plane. Let \( \chi \) be the optical phase difference between the two arms of the MZIM. Note that proper functioning of the MZIM as a parity selector occurs only when \( \chi = 2n\pi \) (\( n = 0, 1, 2, \ldots \)). For \( \chi = (2n + 1)\pi \), the even and odd outputs are interchanged.

After propagating through DP@\( \beta \) and through HWP@\( \alpha \), the maximally non-separable mode given by
Eq. (12) transforms to:

$$
E'(\vec{r}) = A^{++}(\phi)\psi_V(\vec{r})\hat{e}_V + A^{+-}(\phi)\psi_V(\vec{r})\hat{e}_H
+ A^{-+}(\phi)\psi_H(\vec{r})\hat{e}_V + A^{--}(\phi)\psi_H(\vec{r})\hat{e}_H, \quad (13)
$$

where

$$
A^{++}(\phi) = e^{i\phi}\cos(2\alpha)\cos(2\beta) + \sin(2\alpha)\sin(2\beta),
$$

$$
A^{+-}(\phi) = e^{i\phi}\sin(2\alpha)\cos(2\beta) - \cos(2\alpha)\sin(2\beta),
$$

$$
A^{-+}(\phi) = e^{i\phi}\cos(2\alpha)\sin(2\beta) - \sin(2\alpha)\cos(2\beta),
$$

$$
A^{--}(\phi) = e^{i\phi}\sin(2\alpha)\sin(2\beta) + \cos(2\alpha)\cos(2\beta). \quad (14)
$$

If MZIM phase \( \chi = 0 \) then the four amplitudes above would be the ones measured by the detectors since MZIM interferometer together with PBS1 and PBS2 would separate the modes \( \psi_V(\vec{r})\hat{e}_V, \psi_V(\vec{r})\hat{e}_H, \psi_H(\vec{r})\hat{e}_V \) and \( \psi_H(\vec{r})\hat{e}_H \). But we will still consider the case in which \( \chi \) may differ from zero, and then the corresponding intensities normalized to the total intensity are given by:

\[
\begin{align*}
I_1 &= \sin^2(\chi/2)|A^{--}(\phi)|^2 + \cos^2(\chi/2)|A^{+-}(\phi)|^2, \\
I_2 &= \sin^2(\chi/2)|A^{++}(\phi)|^2 + \cos^2(\chi/2)|A^{-+}(\phi)|^2, \\
I_3 &= \cos^2(\chi/2)|A^{++}(\phi)|^2 + \sin^2(\chi/2)|A^{-+}(\phi)|^2, \\
I_4 &= \cos^2(\chi/2)|A^{+-}(\phi)|^2 + \sin^2(\chi/2)|A^{--}(\phi)|^2. \quad (15)
\end{align*}
\]

We can test the violation of the non-separability inequality by making measurements in the bases \( (\alpha_1, \beta_1), (\alpha_1, \beta_2), (\alpha_2, \beta_1) \) and \( (\alpha_2, \beta_2) \) and obtaining the values of \( M(\alpha, \beta) \) and subsequently \( S \). The value of \( S \) for arbitrary \( \phi \) and \( \chi \) is given by:

\[
S(\chi, \phi) = \sqrt{2}\cos(1 + \cos \phi). \quad (16)
\]

Thus maximal violation of the non-separability inequality is accomplished for \( \phi = \chi = 0 \). This is a key result because it shows that experimental errors in the phases will only diminish the violation, not increase it.

In our experiment the MZIM phase \( \chi \) is continuously varied by applying a voltage ramp to the PZT on M2 while intensities \( I_1 \) through \( I_4 \) are monitored at the oscilloscope. An example of our experimental results is presented in Fig. 2 showing the oscillations caused by the variation of \( \chi \). We know from the intensities dependence on \( \phi \) and \( \chi \) that \( \chi = 0 \) corresponds to the peaks in the graphics, and \( \phi = 0 \) corresponds to a maximal visibility of these oscillations. Since we have a few repetitions of these peaks, we obtain an ensemble of intensities which allows us to calculate the average of \( M(\alpha, \beta) \) and \( S \), whose values are shown in Table I. The standard deviations for the MNS mode results is within 2%. In this table we also show our experimental results for a separable initial mode \( \psi_V(\vec{r})\hat{e}_V \), which is easily obtained by blocking the Dove Prism arm of the preparation MZ interferometer.

\textbf{FIG. 2:} Experimental results for the maximally non-separable initial mode, measured in the \((\alpha_1, \beta_1)\) basis. Time parametrizes the MZIM phase \( \chi \).

| TABLE I: Mean values for \( M \) and \( S \) for maximally non-separable and separable modes. |
|---|---|---|---|---|---|
| mode \((\alpha_1, \beta_1)\) | \((\alpha_1, \beta_2)\) | \((\alpha_2, \beta_1)\) | \((\alpha_2, \beta_2)\) | \( S \) |
| MNS | 0.615 | 0.49 | -0.525 | 0.49 | 2.11 |
| \( S \) | 0.47 | 0.00 | -0.56 | 0.00 | 1.03 |

In Table I we show our best experimental results for the intensities \( I_1 \) through \( I_4 \) which lead to a violation with \( S = 2, 17 \).

Attenuation of the intense laser beam down to the photon count regime brings the setup to the quantum mechanical domain. In order to discuss the classical-quantum transition of this experiment, let us first use a coherent state to represent the intense laser beam prepared in the maximally non-separable mode

\[
|\alpha\rangle_{MNS} = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{|a_{MNS}^n\rangle}{n!} |0\rangle, \quad (17)
\]

where \( a_{MNS}^\dagger \) is the creation operator associated with the MNS mode. Its action on the vacuum state \( |0\rangle \) produces a one-photon Fock state in the MNS mode. Since this mode is decomposed as in eq. (11), its corresponding creation operator can be written as

\[
a_{MNS}^\dagger = (a_{VV}^\dagger + a_{HH}^\dagger)/\sqrt{2},
\]

\textbf{TABLE II:} Best experimental data for maximally non-separable initial mode (Intensities are given in 10mV).

| basis | \( I_1 \) | \( I_2 \) | \( I_3 \) | \( I_4 \) | \( I_{tot} \) | \( M \) |
|---|---|---|---|---|---|---|
| \((\alpha_1, \beta_1)\) | 7.99 | 10.4 | 41.6 | 38.4 | 98.39 | 0.626 |
| \((\alpha_1, \beta_2)\) | 13.6 | 10.4 | 30.8 | 43.6 | 98.4 | 0.512 |
| \((\alpha_2, \beta_1)\) | 36.4 | 40.4 | 10.8 | 12.8 | 100.4 | -0.530 |
| \((\alpha_2, \beta_2)\) | 13.2 | 11.6 | 29.6 | 45.2 | 99.6 | 0.502 |
where the first index corresponds to the transverse mode and the second to the polarization mode, so that

\[
|\alpha\rangle_{MNS} = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \sum_{q=0}^{n} \left( \alpha \over \sqrt{2} \right)^{n-q} (a_{VV}^\dagger)^{n-q} q! \\
\times \left( \alpha \over \sqrt{2} \right)^{n-q} (a_{HH}^\dagger)^{n-q} (n-q)! |0\rangle.
\]  

(18)

Exchanging the order of the summations and defining \( m \equiv n-q \), we obtain

\[
|\alpha\rangle_{MNS} = e^{-|\alpha|^2} \sum_{q=0}^{\infty} \left( \alpha \over \sqrt{2} \right)^{q} (a_{VV}^\dagger)^{q} q! \\
\times e^{-|\alpha|^2} \sum_{m=0}^{\infty} \left( \alpha \over \sqrt{2} \right)^{m} (a_{HH}^\dagger)^{m} m! |0\rangle \\
= |\alpha/\sqrt{2}\rangle_{V V} |\alpha/\sqrt{2}\rangle_{H H},
\]

(19)

which clearly shows that it is a product of coherent states at modes \( HH \) and \( VV \) with complex amplitude \( \alpha/\sqrt{2} \).

However, it is instructive to take a look at the Fock state decomposition of \( |\alpha\rangle_{MNS} \):

\[
|\alpha\rangle_{MNS} = e^{-|\alpha|^2} \left[ |0\rangle + \alpha \over \sqrt{2} (|1_{HH}0_{VV}\rangle + |0_{HH}1_{VV}\rangle) + \ldots \right]
\]

(20)

The single photon component is clearly a maximally entangled state. Therefore, post-selection of single photon states from \( |\alpha\rangle_{MNS} \) followed by the experimental setup used here allows one to investigate the usual Bell inequality from probability measurements.

In conclusion, we have investigated both theoretically and experimentally an inequality criterion, as a sufficient condition, for the spin-orbit non-separability of a classical laser beam. The notion of separable and non-separable spin-orbit modes in classical optics builds an useful analogy with entangled quantum states, allowing for the study of some of their important mathematical properties. This analogy has already been successfully exploited in our group to investigate the topological nature of the phase evolution of an entangled state under local unitary operations [8]. Many quantum computing tasks require entanglement but do not need nonlocality, so that using different degrees of freedom of single particles can be useful. This is the type of entanglement whose properties can be studied in the classical optical regime allowing one to replace time-consuming measurements based on photon count by the much more efficient measurement of photocurrents.

Although helpful, the notion of mode non-separability must not be confused with genuine quantum entanglement. In order to avoid this confusion, we have included a brief discussion of the classical-quantum transition. We show that a coherent quantum state of a maximally non-separable mode can be written as the product of two coherent states in the separable components of the MNS mode. On the other hand, its Fock state decomposition shows that the single photon component exhibits entanglement that can be accessed through post-selection.

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