Nosheen Akbar

Masses, Radii and Regge Trajectories of $\Sigma^-_{\mu}$ State Hybrid Charmonium

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Abstract In this paper, masses, radii and $E_1$ radiative transitions of $\Sigma^-_{\mu}$ state hybrid charmonium mesons for radial and orbital excitations are calculated by numerically solving the Schrödinger equation with non-relativistic potential model. Results for calculated masses of $\Sigma^-_{\mu}$ states charmonium hybrid mesons are found to be close to the results obtained through lattice simulations. Calculated masses are used to construct Regge trajectories. It is found that the trajectories are almost linear and parallel.

1 Introduction

With the discovery of highly excited states such as X(3940), X(4140/4274), Y(4230/4260), Y(4360), Y(4660), X(4500) and X(4700), Z$^{+}$ (4430) at Belle, BaBar, LHCb, BESIII, CDF and CLEO Collaborations, study of charmonium hybrid mesons has become very important. The mesons consisting of charm quark–antiquark pairs bounded with ground state gluonic field are called conventional charmonium mesons. These mesons are described by the quark model. Hybrid charmonium mesons are described by two different approaches. First one is the constituent gluon approach in which hybrid meson is a three-body object made up of quark, antiquark and gluon. The second is Born–Oppenheimer approach in which hybrid meson is a two body object consisting of quar–antiquark pair bounded with excited gluonic field. In Born–Oppenheimer approach, meson is treated like a di-atomic molecule. Slow moving heavy quarks and antiquarks are considered at fixed positions like nuclei. The fast moving gluonic field between quark–antiquark is treated as electron. This gluonic field is considered as a string or flux tube. Slow moving heavy quarks and antiquarks are considered at fixed positions like nuclei. The fast moving gluonic field between quark–antiquark is treated as electron. This gluonic field can fluctuate to excited state and each excited state has an adiabatic potential. Conventional mesons have adiabatic potentials in the lowest lying state while hybrid mesons have adiabatic potentials in higher states due to excitation of gluonic field. These adiabatic potentials are calculated in refs. [1,2] through lattice simulations. These potentials for conventional and hybrid states of quarkonium are plotted in Fig. 3 of ref. [2]. These potential states are characterized by quantum numbers, $J, L, S, \Lambda, \eta, \text{ and } \epsilon$, where $J = L \oplus S$ with $L$ as the orbital angular momentum quantum number and $S$ as the spin angular momentum quantum number. $\Lambda$ is the projection of the total angular momentum of gluons and for $\Lambda = 0, \pm 1, \pm 2, \pm 3, \ldots$, meson states are represented as $\Sigma, \Pi, \Delta$ and so on [1]. $\eta$ is the combination of parity and charge and for $\eta = P \circ C = +, -, states are labelled by sub-script $g, u$ [1]. $\epsilon$ is the eigen value corresponding to the operator $P$ and is equal to $+, -. Parity and charge for hybrid static potentials are defined as [1]

$$P = \epsilon(-1)^{L+\Lambda+1}, C = \epsilon\eta(-1)^{L+\Lambda+S},$$

(1)

The static potential states are labelled as $\Sigma^+_g, \Sigma^-_g, \Sigma^+_u, \Sigma^-_u, \Pi_g, \Pi_u, \Delta_g, \Delta_u$ and so on [1]. $\Sigma^+_g$, the low-lying potential state, represents the conventional mesons and its potential is approximated by a coulomb plus
linear potential. The $\Pi_u$ and $\Sigma_u^-$ represent hybrid meson potential states with the low lying gluonic excited potentials. The potential model for $\Pi_u$ state is suggested in ref. [3] by adding an additional term in ground state (coulomb plus linear) potential. This additional term was found by fitting it with the difference of $\Sigma_u^-$ and $\Pi_u$ lattice data [2]. This extended potential model [3] is tested by finding properties of mesons for a variety of $J^{PC}$ states in refs. [3–6]. The potential model for $\Sigma_u^-$ state is suggested in ref. [7] by adding an ansatz in ground state potential. The additional ansatz is obtained by fitting it with the lattice data [1] for the difference of ground and $\Sigma_u^-$ potentials. The validity of suggested ansatz is tested by calculating the spectrum of $\Sigma_u^-$ state bottomonium mesons [7]. In this paper, $\Sigma_u^-$ state potential model [7] is used to calculate the spectrum, radii and $E_1$ radiative partial widths of $\Sigma_u^-$ charmonium hybrid mesons.

Many theoretical techniques are developed to understand hybrid mesons. Mandula and Horn are the first theorists who consider hybrid mesons with constituent gluons [8]. They used a linear confining potential model and a massless gluon. Later on, the constituent gluon model is used by [9–11] to understand the hybrid charmonium mesons. Lattice quantum chromodynamics (QCD) [1,2,12–14], flux tube model [15–20], QCD string model [21,22], effective field theory [23–26] and QCD sum rules [27–34] are used to investigate the charmonium hybrid mesons. In ref. [35], masses of hybrids has been calculated by solving Schrödinger equation in Born–Oppenheimer approximation. In [3–7,14,24,25], Lattice gauge theory results on hybrid static states bottomonium mesons [7]. In this paper, $\Sigma_u^-$ state potential model [7] is used to calculate the spectrum, radii and $E_1$ radiative partial widths of $\Sigma_u^-$ charmonium hybrid mesons.

This paper is organised as: In the Sect. 2 of this paper, Potential models for $\Sigma_u^+$ and $\Sigma_u^-$ states are defined while methodology of calculation of mass is explained in Sect. 3. The expressions for the calculation of radii and $E_1$ transitions is explained in Sects. 4 and 5. Regge trajectories are constructed in Sect. 6, while the discussion on results and concluding remarks are written in Sect. 7.

2 Potential Model for $\Sigma_u^+$ and $\Sigma_u^-$ States

The potential model for conventional mesons is defined in ref. [46] as:

$$V_{\Sigma_u^+}^{(r)} = \frac{-4\alpha_s}{3r} + br + \frac{32\pi \alpha_s}{9m_c m_\pi} \left( \frac{\sigma}{\sqrt{\pi}} \right)^3 e^{-\sigma^2 r^2} S_c S_{\pi} + \frac{4\alpha_s}{m_c^2 r^3} S_T + \frac{1}{m_c^2} \left( \frac{2\alpha_s}{r^3} - \frac{c}{2r} \right) L.S$$

(2)

where $-4/3$ is the color singlet of quark-antiquark pair with ground state gluonic field. $\alpha_s$ is the coupling constant, $b$ is the string tension, $m_c$ is the mass of charm quark. Coulomb term ($\frac{\alpha_s}{3r}$) dominates at short distance (high energy), while linear potential term ($br$) dominates at large distance (low energy). Constituent mass of charm quark, $m_c$, is taken to be 1.454 GeV [47]. The parameters ($\alpha_s$, $b$, $\sigma$) used in this potential model are taken from [47] and equal to 0.5315, 0.1583 GeV, and 1.105 GeV$^2$ respectively. $S_c S_\pi$ is defined in [4,7,46] as:

$$S_c S_\pi = \frac{S(S+1)}{2} - \frac{3}{4}$$

(3)

$L.S$ describes the spin orbit interactions defined as :

$$L.S = [J(J+1) - L(L+1) - S(S+1)]/2.$$  

(4)
$S_T$ is the tensor operator defined in [46] as:

$$<\hat{L}_J | S_T | \hat{L}_J> = \begin{cases} 
\frac{L}{6(2L+3)}, & J = L + 1 \\
\frac{1}{2}, & J = L \\
-\frac{L+1}{6(2L-1)}, & J = L - 1.
\end{cases} (5)$$

To study the $\Sigma_u^-$ hybrid mesons, the potential model defined in Eq. (2) is modified in ref. [7] by considering the hybrid meson as two body object with excited gluonic field. The potential for gluonic excited state is defined by adding an additional term $A'\exp(-B'r^{P'}) + C'$ in the ground state ($\Sigma_g^+$) potential model. This additional term is obtained by fitting it with the lattice data for the difference of potentials of $\Sigma_u^-$ and $\Sigma_g^+$ state potentials. Extended potential model for $\Sigma_u^-$ charmonium hybrid state is defined in ref. [7] as:

$$V_{\Sigma_u^-}(r) = V_{\Sigma_u^+}(r) + A'\exp(-B'r^{P'}) + C', (6)$$

$V_{\Sigma_u^+}(r)$ is defined in Eq. (2). The parameters ($A', B', P', C'$) are found in ref. [7] by fitting $A'\exp(-B'r^{P'}) + C'$ with the lattice data [2] and values of these fitted parameters are: $A' = 11.5917 \pm 1.1514$ GeV, $B' = 2.9224 \pm 0.0913$, $P' = 0.2810 \pm 0.012$, $C' = 0.9589 \pm 0.006$ GeV.

3 Radial Schrödinger Equation for $\Sigma_g^+$ and $\Sigma_u^-$ States and Mass of $\Sigma_u^-$ States

Radial Schrödinger equation can be written as:

$$U''(r) + 2\mu(E - V_{\text{eff}}(r))U(r) = 0, (7)$$

where

$$V_{\text{eff}}(r) = V_{\bar{Q}Q}(r) + \frac{<\hat{L}_Q^2>}{2\mu r^2}. (8)$$

Here, $V_{\bar{Q}Q}(r)$ is the potential of quark-anti-quark pair. For conventional meson, $V_{\bar{Q}Q}(r) = V_{\Sigma_u^+}(r)$ which is defined above in Eq. (2). For $\Sigma_u^-$ hybrid charmonium state, $V_{\bar{Q}Q}(r) = V_{\Sigma_u^-}(r)$ which is defined in Eq. (6). $U(r) = r R(r)$, where $R(r)$ is the radial wave function and $\mu$ is the reduced mass.

The total angular momentum is defined as:

$$J = L + S, (9)$$

where $S$ is the sum of the spin of quark and anti-quark. $L$ is angular momentum of quark-antiquark pair and defined as

$$L = L_{\bar{Q}Q} + J_g (10)$$

Here $J_g$ is the angular momentum due to gluonic field.

$$<\hat{L}_{\bar{Q}Q}^2> = <\hat{L}^2> - 2 <\hat{L} \cdot \hat{J}_g> + <\hat{J}_g^2> (11)$$

Here, $<\hat{L}^2> = L(L+1)$. $<\hat{L} \cdot \hat{J}_g>$ is evaluated in ref. [2] by expressing $\hat{L}$ and $\hat{J}_g$ in components in the body-fixed frame, and it is found that $<\hat{L} \cdot \hat{J}_g> = \Lambda^2$.

$$<\hat{L}_{\bar{Q}Q}^2> = L(L+1) - 2\Lambda^2 + <\hat{J}_g^2> (12)$$

For $\Sigma$ states, the quantum number $\Lambda$ is equal to zero [1]. For $\Sigma_g^+$ state, $<J_g^2> = 0$; while $<J_g^2> = 2$ for $\Sigma_u^-$ state [1]. Thus, for $\Sigma_g^+$ state, radial Schrödinger equation becomes

$$U''(r) + 2\mu(E - V_{\Sigma_g^+}(r) + \frac{L(L+1)}{2\mu r^2})U(r) = 0. (13)$$
and for $\Sigma_u^-$ state, Schrödinger equation becomes

$$U''_{\Sigma_u^-}(r) + 2\mu(E_{\Sigma_u^-} - V_{\Sigma_u^-}(r)) + \frac{L(L + 1) + 2}{2\mu r^2}U_{\Sigma_u^-}(r) = 0.$$  (14)

Numerical solutions of the Schrödinger equation for $\Sigma_g^+$ and $\Sigma_u^-$ states are found by the shooting method. In this method, the numerical solution of Schrödinger equation is found many times by varying the value of $E$ (or $E_{\Sigma_u^-}$) from $-5$ to $5$ GeV. The numerical solution is obtained with RK method [48]. Only those energy values are considered for which solution is continuous and satisfy the boundary conditions. Normalized solution of the Schrödinger equation are obtained by multiplying the solution with the normalization constant. At high energy ($r \to 0$), the effective potential $V_{\Sigma_g^+}(r)$ becomes $\sim \frac{2\alpha_s}{r^3}(L.S + 2S_T)$. For $J = L, L - 1$ with $S = 1, L.S + 2S_T$ becomes negative. It means that potential becomes attractive at short distance (or high energy). Under this circumstance, wave function becomes unstable at short distance. This problem is solved by applying smearing of position co-ordinates by using the method discussed in ref. [41]. Due to smearing, potential becomes less divergent than $\frac{1}{r^2}$ at short distance. In this way, centrifugal repulsive potential becomes dominant at short distance. Comparison of potentials with and without smearing is shown in Fig. 1 of ref. [6]. The behaviour of wave functions for conventional and hybrid charmonium mesons for $L = 0, 1$ are shown in Figs. 1 and 2.

Masses of $\Sigma_u^-$ state charmonium mesons are calculated by adding the constituent quark masses to the energy $E_{\Sigma_u^-}$, i.e;

$$M_{\Sigma_u^-} = m_c + m_{\tau} + E_{\Sigma_u^-}. \quad (15)$$

Relativistic correction in mass is incorporated by applying perturbation theory. This lowest-order relativistic correction to Hamiltonian is incorporated by adding $-(\frac{1}{4m_c^2})p^4$ in the mass.

4 Radii

To find the root mean square radii of the gluonic excited $\Sigma_u^-$ charmonium states, following relation is used:

$$\sqrt{\langle r^2_{\Sigma_u^-} \rangle} = \sqrt{\int U^*_{\Sigma_u^-} r^2 U_{\Sigma_u^-} dr}. \quad (16)$$

Calculated masses and radii for $\Sigma_u^-$ states are reported in Table 1.
5 Radiative Transitions

Hybrid charmonium can transit from one state to other state through emitting one virtual photon. $E_1$ radiative partial widths for $\Sigma_{u}^{-}$ hybrid charmonium mesons transitions are calculated by using the following expression given in ref. [46] as:

$$\Gamma_{E1}(n^{2S+1}L_J \rightarrow n'^{2S'+1}L'_{J'}, + \gamma) = \frac{4}{3}C_{fj}\delta_{SS'}\Delta e^2\alpha |<\Psi_{\Sigma_{u}^{-}f}|r|\Psi_{\Sigma_{u}^{-}i}>|^2E^3_{y}\frac{E_{\Sigma_{u}^{-}f}}{M_{\Sigma_{u}^{-}i}}.$$ \hspace{2cm} (17)

Here $E_y$, $E_{\Sigma_{u}^{-}f}$, and $M_{\Sigma_{u}^{-}i}$ stand for final photon energy ($E_y = \frac{M_{\Sigma_{u}^{-}f}^2 - M_{\Sigma_{u}^{-}i}^2}{2M_{\Sigma_{u}^{-}i}}$), energy of the final $c\bar{c}$ meson, and mass of initial state of charmonium respectively, and

$$C_{fj} = \max(L, L')(2J' + 1)\left\{\begin{array}{ccc}J' & J & S
\end{array}\right\}^2.$$ \hspace{2cm} (18)

$E_1$ radiative partial widths are calculated using non-relativistic masses.

6 Regge Trajectories

Using the calculated masses of $\Sigma_{u}^{-}$ hybrid charmonium mesons (given in Table 1), Regge trajectories are constructed in $(J, M_{\Sigma_{u}^{-}}^2)$ and $(n, M_{\Sigma_{u}^{-}}^2)$ planes. To construct the Regge trajectories, particle’s states are selected with consecutive values of $J$. In Figs. 3, 4, 5 and 6, Regge trajectories are constructed in $(J, M_{\Sigma_{u}^{-}}^2)$ planes for $\Sigma_{u}^{-}$ state hybrid charmonium mesons by using the following relation [44]

$$M_{\Sigma_{u}^{-}}^2 = \alpha J + \alpha_0.$$ \hspace{2cm} (19)

Slope ($\alpha$) and intercept ($\alpha_0$) are found by fitting the above equation with equation of straight line for particular set of particles that lie on a single trajectory. The fitted slopes and intercepts are given in Table 2.

In Fig. 7, Regge trajectories are constructed in $(n, M_{\Sigma_{u}^{-}}^2)$ plane for $\Sigma_{u}^{-}$ state hybrid charmonium by using the following relation:

$$M_{\Sigma_{u}^{-}}^2 = \beta(n - 1) + \beta_0.$$ \hspace{2cm} (20)

Here, $n$ is the principle quantum number. Slope ($\beta$) and intercept ($\beta_0$) of the above equation are found by fitting it with equation of straight line for particular set of particles that lie on a single trajectory. The fitted slopes and intercepts are given in Table 3.
Table 1 Calculated masses and radii of $\Sigma_c^-$ state hybrid charmonium mesons

| Meson | $J^{PC}$ | NR mass (GeV) | Rel mass (GeV) | Mass [2] (GeV) | Mass [14] (GeV) | Mass [24] (GeV) | Radii (fm) |
|-------|----------|--------------|---------------|---------------|----------------|----------------|------------|
| $\eta_c^0(1^3S_0)$ | 0++ | 4.7086 | 4.6859 | 4.487(5) | 4.407 | 4.486 | 0.7573 |
| $J/\psi^h(1^1S_0)$ | 1++ | 4.7131 | 4.6920 | | | | 0.7648 |
| $\eta_c^0(2^1S_0)$ | 0++ | 5.0973 | 5.0433 | 4.933(9) | 4.822 | 4.920 | 1.1250 |
| $J/\psi^h(2^3S_1)$ | 1++ | 5.1036 | 5.0528 | | | | 1.1342 |
| $\eta_c^0(3^1S_0)$ | 0++ | 5.4373 | 5.3476 | | | | 1.4397 |
| $J/\psi^h(3^3S_1)$ | 1++ | 5.4444 | 5.3592 | | | | 1.4490 |
| $\eta_c^0(4^1S_0)$ | 0++ | 5.7461 | 5.6173 | | | | 1.7231 |
| $J/\psi^h(4^3S_1)$ | 1++ | 5.7536 | 5.6302 | | | | 1.7323 |
| $\eta_c^0(5^1S_0)$ | 0++ | 6.0326 | 5.8592 | | | | 1.9851 |
| $J/\psi^h(5^3S_1)$ | 1++ | 6.0402 | 5.8735 | | | | 1.9939 |
| $\eta_c^0(6^1S_0)$ | 0++ | 6.3021 | 6.0592 | | | | 2.2309 |
| $J/\psi^h(6^3S_1)$ | 1++ | 6.3097 | 6.0729 | | | | 2.2394 |
| $\eta_c^0(1^3P_1)$ | 0+ | 4.8342 | 4.8166 | 4.623(6) | 4.544 | | 0.8705 |
| $\chi_c^0(1^3P_0)$ | 0+ | 4.7934 | 4.7600 | | | | 0.8256 |
| $\chi_c^0(1^3P_1)$ | 1+ | 4.8373 | 4.8098 | | | | 0.8629 |
| $\chi_c^0(2^3P_1)$ | 2+ | 4.8576 | 4.8336 | | | | 0.8885 |
| $h_{c1}^1(2^1P_1)$ | 1++ | 5.2114 | 5.1525 | 5.058(10) | | | 1.2216 |
| $\chi_c^0(2^3P_0)$ | 0+ | 5.1838 | 5.1182 | | | | 1.1981 |
| $\chi_c^0(2^3P_1)$ | 1+ | 5.2102 | 5.1511 | | | | 1.2202 |
| $\chi_c^0(2^3P_2)$ | 2+ | 5.2212 | 5.1658 | | | | 1.2335 |
| $h_{c1}^1(3^1P_1)$ | 1++ | 5.5390 | 5.4436 | | | | 1.5263 |
| $\chi_c^0(3^3P_0)$ | 0+ | 5.5198 | 5.4184 | | | | 1.5112 |
| $\chi_c^0(3^3P_1)$ | 1+ | 5.5398 | 5.4450 | | | | 1.5273 |
| $\chi_c^0(3^3P_2)$ | 2+ | 5.5471 | 5.4561 | | | | 1.5364 |
| $h_{c1}^1(4^1P_1)$ | 1++ | 5.8392 | 5.7039 | | | | 1.8028 |
| $\chi_c^0(4^3P_0)$ | 0+ | 5.8248 | 5.6841 | | | | 1.7920 |
| $\chi_c^0(4^3P_1)$ | 1+ | 5.8412 | 5.7072 | | | | 1.8051 |
| $\chi_c^0(4^3P_2)$ | 2+ | 5.8465 | 5.7162 | | | | 1.8120 |
| $h_{c1}^1(5^1P_1)$ | 1++ | 6.1193 | 5.9365 | | | | 2.0594 |
| $\chi_c^0(5^3P_0)$ | 0+ | 6.1080 | 5.9209 | | | | 2.0514 |
| $\chi_c^0(5^3P_1)$ | 1+ | 6.1220 | 5.9412 | | | | 2.0626 |
| $\chi_c^0(5^3P_2)$ | 2+ | 6.1261 | 5.9487 | | | | 2.0681 |
| $h_{c1}^1(6^1P_1)$ | 1++ | 6.3836 | 6.1130 | | | | 2.3011 |
| $\chi_c^0(6^3P_0)$ | 0+ | 6.3746 | 6.1024 | | | | 2.2949 |
| $\chi_c^0(6^3P_1)$ | 1+ | 6.3869 | 6.1190 | | | | 2.3047 |
| $\chi_c^0(6^3P_2)$ | 2+ | 6.3902 | 6.1244 | | | | 2.3094 |
| $\eta_c^0(1^1D_2)$ | 2++ | 5.0195 | 4.9854 | 4.814(7) | | | 1.0166 |
| $\psi^1(1^1D_2)$ | 1+ | 4.9855 | 4.9429 | | | | 0.9678 |
| $\psi^1(1^3D_2)$ | 2+ | 5.0167 | 4.9814 | | | | 1.0088 |
| $\psi^1(1^3D_3)$ | 3+ | 5.0340 | 5.0037 | | | | 1.0431 |
| $\eta_c^0(2^1D_2)$ | 2++ | 5.3644 | 5.2962 | | | | 1.3490 |
| $\psi^1(2^3D_2)$ | 1+ | 5.3417 | 5.2648 | | | | 1.3218 |
| $\psi^1(2^3D_3)$ | 3+ | 5.3752 | 5.3117 | | | | 1.3651 |
| $\eta_c^0(3^1D_2)$ | 2++ | 5.6771 | 5.5711 | | | | 1.6424 |
| $\psi^1(3^3D_2)$ | 1+ | 5.6596 | 5.5449 | | | | 1.6229 |
| $\psi^1(3^3D_2)$ | 2+ | 5.6766 | 5.5698 | | | | 1.6405 |
| $\psi^1(3^3D_3)$ | 3+ | 5.6859 | 5.5851 | | | | 1.6548 |
| $\eta_c^0(4^1D_2)$ | 2++ | 5.9667 | 5.8191 | | | | 1.9108 |
| $\psi^1(4^3D_2)$ | 1+ | 5.9525 | 5.7965 | | | | 1.8956 |
| $\psi^1(4^3D_3)$ | 2+ | 5.9688 | 5.8187 | | | | 1.9100 |
| $\psi^1(4^3D_3)$ | 3+ | 5.9743 | 5.8322 | | | | 1.9213 |
7 Discussion and Conclusions

In this paper, potential model for lowest lying $\Sigma_u^-$ state hybrids is used to calculate the spectrum and radii of $1S-6S, 1P-6P, 1D-4D$ hybrid charmonium states and results are written in Table 1. Masses calculated by adding relativistic corrections are reported in 4th column of Table 1. Calculated masses are close to the the results given in ref. [13] as shown in Table 1. In ref. [13], spectrum is calculated without including the spin, so the same value of mass is given for $\eta_c^h$ and $J/\psi^h$. However, our proposed potential model gives distinguished results for $S = 0$ and $S = 1$. As observed from Table 1, the lowest calculated mass of the $\Sigma_u^-$ state hybrid charmonium is 4.6859 GeV. In ref. [2], the lowest mass of $\Sigma_u^-$ charmonium state is 4.6 GeV. This shows that my calculated results for mass are close to lattice simulation predictions. However, difference between the mass obtained in [13] and my calculated mass is 0.1989 GeV for the lowest state. This difference is due to incorporation of spin-orbit and color tensor terms which are not incorporated in ref. [13].
Fig. 5 Regge trajectories for $^3S_1$, $^3P_2$, $^3D_3$ hybrid charmonium $\Sigma_u^-$ states. Dashed lines are for Regge trajectories constructed with calculated masses (shown by dots). Solid lines are for the Regge trajectories obtained with $M_{\Sigma_u^-}^2 = \alpha J + \alpha_0$

As potential model ($V_{\Sigma_\mu}(r)$) depends on parameters ($A', B', P', C'$). The % error in the masses due to $\pm 1.1514$ GeV, $\pm 0.0913$, $\pm 0.012$, $\pm 0.006$ GeV change in $A'$, $B'$, $C'$, $P'$ is calculated to be within range $(0.1228–0.4310)$, $(0.1578–0.5619)$, $(0.0939–0.1274)$ and $(0.0953–0.2292)$ respectively.

Charmonium states such as $X(4350)$, $X(4660)$, $X(3940)$, $X(4700)$ and many more give signal of the existence of hybrid mesons. $X(4660)$ state has experimentally measured mass equal to $4.664 \pm 12$ GeV [49] with $J^{PC} = 1^{--}$. By comparing the experimental mass of $X(4660)$ or $Y(4660)$ particles with calculated masses of $1^{--}$ state, $X(4600)$ can be assigned as $h_b^1(2^1P_1)$ state of $\Sigma_u^-$ hybrid charmonium.

From Table 1, it is observed that masses and radii are increased with increase of the orbital quantum number ($L$) or principle quantum number ($n$). It shows that masses and radii increase with radial and orbital excitations. The similar behaviour is observed in ref. [4] in which $\Sigma_u^+$ and $\Pi_u$ states of charmonium mesons are investigated.

The % error in the radii due to $\pm 1.1514$ GeV, $\pm 0.0913$, $\pm 0.012$, $\pm 0.006$ GeV change in $A'$, $B'$, $C'$, $P'$ is calculated to be within range $(0.2072–1.88)$, $(0.2267–1.2872)$, $(0.0048–0.0721)$, $(0.0031–3.041)$ respectively.
Table 2 $S \rightarrow P$ E1 radiative transitions. The masses are taken from above mentioned Table 1.

| Transition | Initial Meson | Final Meson | $\Gamma_{E1}$ for hybrids (keV) |
|------------|---------------|-------------|---------------------------------|
| $2S \rightarrow 1P$ | $^{2}S_{1}$ | $^{1}P_{2}$ | 168.0790 |
| | | $^{1}P_{1}$ | 101.8150 |
| | | $^{1}P_{0}$ | 36.1523 |
| | $^{2}S_{0}$ | $^{1}P_{1}$ | 302.1300 |
| $3S \rightarrow 2P$ | $^{3}S_{1}$ | $^{2}P_{2}$ | 273.0790 |
| | | $^{2}P_{1}$ | 157.8590 |
| | | $^{2}P_{0}$ | 52.4628 |
| | | $^{1}P_{0}$ | 477.267 |
| $3S \rightarrow 1P$ | $^{3}S_{1}$ | $^{1}P_{2}$ | 18.2068 |
| | | $^{1}P_{1}$ | 19.5651 |
| | | $^{1}P_{0}$ | 12.0584 |
| | $^{3}S_{0}$ | $^{1}P_{1}$ | 43.1064 |
| $4S \rightarrow 3P$ | $^{3}S_{1}$ | $^{3}P_{2}$ | 347.2740 |
| | | $^{3}P_{1}$ | 198.4620 |
| | | $^{3}P_{0}$ | 65.4825 |
| | $^{3}S_{0}$ | $^{3}P_{1}$ | 601.9950 |
| $4S \rightarrow 2P$ | $^{4}S_{0}$ | $^{3}P_{2}$ | 38.5213 |
| | | $^{2}P_{1}$ | 32.1291 |
| | | $^{2}P_{0}$ | 15.8919 |
| | $^{4}S_{0}$ | $^{2}P_{1}$ | 77.9832 |
| $4S \rightarrow 1P$ | $^{4}S_{1}$ | $^{1}P_{2}$ | 5.8349 |
| | | $^{1}P_{1}$ | 8.3988 |
| | | $^{1}P_{0}$ | 6.5053 |
| | $^{4}S_{0}$ | $^{1}P_{1}$ | 15.8959 |

Fig. 7 Regge trajectories for $S$, $P$, $D$ hybrid charmonium $\Sigma_{u}$ states. Solid green color lines are for Regge trajectories constructed with $M^{2}_{\Sigma_{u}} = \beta(n - 1) + \beta_{0}$. Blue curves are for Regge trajectories constructed with calculated masses shown by dots. $^{1}S_{0}$ state trajectory is shown by blue solid lines while $^{3}S_{1}$ state trajectory is shown by dashed blue line. $^{1}P_{1}$, $^{3}P_{0}$, $^{3}P_{1}$, $^{3}P_{2}$ are shown by dashed, dotted, dot-dashed and thick lines respectively. $^{1}D_{2}$, $^{3}D_{1}$, $^{3}D_{2}$, $^{3}D_{3}$ are shown by dashed, dotted, dot-dashed and thick lines respectively.
Table 3 1P and 2P E1 radiative transitions

| Transition | Initial Meson | Final Meson | Our calculated $\Gamma_{E1}$ for hybrids (keV) |
|------------|---------------|-------------|-----------------------------------------------|
| $1P \rightarrow 1S$ | $1P_1$ | $1P_1$ | 38.2804 |
| | $1P_2$ | $1P_2$ | 60.2833 |
| | $1P_0$ | $1P_0$ | 10.3259 |
| | $1P_1$ | $1P_1$ | 47.0464 |
| | $2P_2$ | $2P_2$ | 61.0663 |
| | $2P_1$ | $2P_1$ | 47.0464 |
| | $2P_0$ | $2P_0$ | 20.8182 |
| | $2P_1$ | $2P_1$ | 55.6099 |
| $2P \rightarrow 1S$ | $2P_1$ | $2P_1$ | 11.0723 |
| | $2P_2$ | $2P_2$ | 29.1256 |
| | $2P_0$ | $2P_0$ | 65.4230 |
| | $2P_1$ | $2P_1$ | 18.4292 |
| $2P \rightarrow 1D$ | $2P_2$ | $1D_3$ | 111.3130 |
| | $2P_1$ | $1D_2$ | 19.7688 |
| | $2P_0$ | $1D_1$ | 1.3752 |
| | $2P_1$ | $1D_2$ | 99.8065 |
| | $2P_0$ | $1D_1$ | 36.8195 |

Table 4 3P E1 radiative transitions

| Transition | Initial Meson | Final Meson | Our calculated $\Gamma_{E1}$ (keV) |
|------------|---------------|-------------|----------------------------------|
| $3P \rightarrow 3S$ | $\chi_3(3P_2)$ | $J/\psi(3S_1)$ | 63.0179 |
| | $\chi_1(3P_1)$ | | 52.4695 |
| | $\chi_0(3P_0)$ | | 27.1787 |
| | $h_8(3P_1)$ | $\eta_b(3S_0)$ | 61.0943 |
| $3P \rightarrow 2S$ | $\chi_3(3P_2)$ | $\Upsilon(2S_1)$ | 15.7780 |
| | $\chi_1(3P_1)$ | | 35.5119 |
| | $\chi_0(3P_0)$ | | 76.8645 |
| | $h_8(3P_1)$ | $\eta_b(2S_0)$ | 22.8907 |
| $3P \rightarrow 1S$ | $\chi_3(3P_2)$ | $\Upsilon(1S_1)$ | 9.4940 |
| | $\chi_1(3P_1)$ | | 16.5409 |
| | $\chi_0(3P_0)$ | | 25.2498 |
| | $h_8(3P_1)$ | $\eta_b(1S_0)$ | 12.2372 |

It is observed from Table 2 that slope ($\alpha$) decreases and intercept ($\alpha_0$) increases toward higher states. As observed from Table 1 that mass is increased toward higher states, so it can be concluded that slopes decrease (or intercepts increase) with increase of mass. Similar characteristic of the Regge trajectories is observed in [43] for charmed strange mesons. From the Tables 1 and 2, it is concluded that the slopes ($\alpha$) of trajectories presented with respect to angular momentum are found to be greater as compared to the slopes ($\beta$) of trajectories presented with respect to principal quantum number. The similar behaviour is observed in ref. [42] in which slopes of trajectories with the orbital excitations is found to be in average 1.3 times greater than the slope of the trajectories with radial excitation for the light mesons.

In Fig. 5, $1S_0$ and $3S_1$ states trajectories are almost same. Similarly for $P$ states with different spin quantum number and $D$ states with different spin quantum numbers, trajectories are almost same. From Figs. 3, 4, 5, 6 and 7, it is observed that the calculated masses of $\Sigma$ state hybrid charmonium fit to the linear trajectories presented in $(M_{\Sigma}^2, J)$ and $(M_{\Sigma}^2, n)$ planes. It is also observed that the trajectories are almost parallel. Same characteristics for conventional mesons trajectories are observed in [41,42]. These Regge trajectories can be helpful for the identification of higher excited states.
Table 5 Parameters for $\Sigma_u^+$ states of charmonium meson

| $n$ | $\alpha$ | $\alpha_0$ |
|-----|----------|------------|
| 1   | 1.5122   | 22.0954    |
| 2   | 1.3972   | 25.9088    |
| 3   | 1.3326   | 29.4921    |
| 4   | 1.2919   | 32.9466    |

| $n$ | $\alpha$ | $\alpha_0$ |
|-----|----------|------------|
| 1   | 1.8785   | 22.9767    |
| 2   | 1.6620   | 26.8718    |
| 3   | 1.5629   | 30.4682    |
| 4   | 1.5040   | 33.9283    |

| $n$ | $\beta$ | $\beta_0$ |
|-----|---------|-----------|
| $^1S_0$ | 3.4975 | 22.3971 |
| $^3S_1$ | 3.5077 | 22.4477 |
| $^1P_1$ | 33.4499 | 23.64 |
| $^3P_0$ | 3.5161 | 23.2412 |
| $^3P_1$ | 3.4684 | 23.6 |
| $^3P_2$ | 3.4402 | 23.7616 |
| $^1D_2$ | 3.671 | 25.2501 |
| $^3D_1$ | 3.5228 | 24.9288 |
| $^3D_2$ | 3.4767 | 25.2241 |
| $^3D_3$ | 3.449 | 25.3904 |

Radiative transitions of hybrid charmonium provide a way to investigate the experimentally discovered charmonium states. Hadronic decay of hybrid mesons for $\psi \rightarrow \gamma X$ is discussed in detail in ref. [50]. Radiative transitions from charmonium hybrid states ($0^{--}$, $1^{--}$, $2^{--}$, $1^{--}$) to conventional meson are discussed in ref. [51] and radiative transitions from hybrid to hybrid transitions for $\pi$ state charmonium are reported in ref. [4]. In the present paper, $E1$ radiative partial widths for $\Sigma_u^-$ state hybrid to hybrid charmonium transitions are reported in Tables 4, 5 and 6. $3P \rightarrow 2D$, $3P \rightarrow 1D$, $1D \rightarrow 1P$, $2D \rightarrow 2P$, and $2D \rightarrow 1P$ transitions are also possible. These predicted radiative transitions along with radii and masses will be helpful in identifying the experimentally discovered excited charmonium states.

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