On the Tractability of Minimal Model Computation for Some CNF Theories

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Abstract

Designing algorithms capable of efficiently constructing minimal models of CNFs is an important task in AI. This paper provides new results along this research line and presents new algorithms for performing minimal model finding and checking over positive propositional CNFs and model minimization over propositional CNFs. An algorithmic schema, called the Generalized Elimination Algorithm (GEA) is presented, that computes a minimal model of any positive CNF. The schema generalizes the Elimination Algorithm (EA) [5], which computes a minimal model of positive head-cycle-free (HCF) CNF theories. While the EA always runs in polynomial time in the size of the input HCF CNF, the complexity of the GEA depends on the complexity of the specific eliminating operator invoked therein, which may in general turn out to be exponential. Therefore, a specific eliminating operator is defined by which the GEA computes, in polynomial time, a minimal model for a class of CNF that strictly includes head-elementary-set-free (HEF) CNF theories [14], which form, in their turn, a strict superset of HCF theories. Furthermore, in order to deal with the high complexity associated with recognizing HEF theories, an “incomplete” variant of the GEA (called IGEA) is proposed: the resulting schema, once instantiated with an appropriate elimination operator, always constructs a model of the input CNF, which is guaranteed to be minimal if the input theory is HEF. In the light of the above results, the main contribution of this work is the enlargement of the tractability fron-
tier for the minimal model finding and checking and the model minimization
problems.

Key words: CNF theories, minimal model, head-cycle-free CNF theories,
head-elementary-set-free CNF theories, computational complexity.

1. Introduction

Minimal models play a vital role in many systems that are dedicated to
knowledge representation and reasoning. The concept of minimal model is
at the heart of several tasks in Artificial Intelligence including circumscrip-
tion [28, 29, 25], default logic [31], minimal diagnosis [10], planning [20],
and in answering queries posed on logic programs under the stable model
semantics [17, 6] and deductive databases under the generalized closed-world
assumption [30].

On the more formal side, the task of reasoning with minimal models
has been the subject of several studies [8, 7, 23, 12, 9, 3, 4, 21]. Given
a propositional CNF theory $\Pi$, among others, the tasks of Minimal Model
Finding and Minimal Model Checking have been considered. The former task
consists of computing a minimal model of $\Pi$, the latter one is the problem
of checking whether a given set of propositional letter is indeed a minimal
model for $\Pi$.

Findings regarding the complexity of reasoning with minimal models show
that these problems are intractable in the general case. Indeed, it turns out
that even when the theory is positive (that is, it does not contain constraints),
finding a minimal model is $\text{P}^{\text{NP}[O(\log n)]}$-hard [8] (note that positive theories
always have a minimal model)\textsuperscript{1}, and checking whether a model is minimal
for a given theory is co-NP-complete [7].

The above formidable complexities characterizing the two above men-
tioned problems have motivated several researchers to look for heuristics
[27, 3, 4, 1] as long as, due to the complexity results listed above and to the
still unresolved P vs NP conundrum, all exact algorithms for solving these
problems remain exponential in the worst case.

\textsuperscript{1}We recall that $\text{P}^{\text{NP}[O(\log n)]}$ is the class of decision problems that are solved by
polynomial-time bounded deterministic Turing machines making at most a logarithmic
number of calls to an oracle in NP. For a precise characterization of the complexity of
model finding, given in terms of complexity classes of functions, see [9].
One orthogonal direction of research concerns singling out significant fragments of CNF theories for which dealing with minimal models is tractable. The latter approach has also the merit of providing insights that can help improve the efficiency of heuristics for the general case. For instance, algorithms designed for a specific subset of general CNF theories can be incorporated into algorithms for computing minimal models of general CNF theories [3, 32, 18, 19].

Within this scenario, in [5] efficient algorithms are presented for computing and checking minimal models of a restricted subset of positive CNF theories, called Head Cycle Free (HCF) theories [2]. To illustrate, HCF theories are positive CNF theories satisfying the constraint that there is no cyclic dependence involving two positive literals occurring in the same clause. Head-cycle-freeness can also be checked efficiently [2]. These results have been then exploited by other authors to improve model finding algorithms for general theories. For example, the system dlv looks for HCF fragments into general disjunctive logic programs to be processed in order to improve efficiency [24, 22].

The research presented here falls into the groove traced in [5]. The central contribution of this work is a polynomial time algorithm for computing a minimal model for (a superset of) the class of positive HEF (Head Elementary-Set Free) CNF theories, the definition of which we adapt from the homonym one given in [14] for disjunctive logic programs and which form, in their turn, a strict superset of the class of HCF theories studied in [5].

To the best of our knowledge positive HCF theories form the largest class of CNFs for which a polynomial time algorithm solving the Minimal Model Finding problem is known so far. Since HCF theories are a strict subset of HEF ones, our main contribution is the enlargement of the tractability frontier for the minimal model finding problem.

It is worth noting that a relevant difference holds here that while HCF theories are recognizable in polynomial time, for HEF ones the same task is co-NP-complete [13]. Although this undesirable property seems to reduce the applicability of the above result, we will show that our approach leads to techniques to compute a model of any positive CNF theory in polynomial time, while the computed model is guaranteed to be minimal at least for all positive HEF theories. Notice that this latter property holds without the need to recognize whether the input theory is HEF or not.

The rest of the paper is organized as follows. In Section 2, we provide preliminary definitions about CNF theories, present the problems and the
sub-classes of CNF theories of interest here, depict contributions of the work, and discuss application examples. In Section 3, we introduce the Generalized Elimination Algorithm (GEA), that is the basic algorithm presented in this paper, and the concept of eliminating operator that it makes use of. Then, in Section 4, we formally define HEF CNF theories and then construct an eliminating operator that enables GEA to compute a minimal model for a positive HEF CNF theory in polynomial time. In Section 5, we study the behavior the GEA when applied to a general CNF theory and introduce the Incomplete GEA which is able to compute a minimal model for a positive HEF CNF theory in polynomial time without the need to know in advance whether the input theory is HEF or not. Concluding remarks are provided in Section 6. For the sake of presentation, some of the intermediate result proofs are reported in the Appendix.

2. Our problems and application scenarios

In this section, first we define the problems we are dealing with in this paper and then depict some application scenarios.

2.1. Preliminary definitions

In this section we recall or adapt the definitions of propositional CNF theories and their subclasses (head-cycle-free, head-elementary-set-free) which are of interest here.

An atom is a propositional letter (aka, positive literal). A clause (aka, rule – in the following we shall make use of the two terms interchangeably) is an expression of the form \( H \leftarrow B \), where \( H \) and \( B \) are sets of atoms. \( H \) and \( B \) are referred to as, respectively, the head and body of the clause; the atoms in \( H \) are also called head atoms while the atoms in \( B \) are also called body atoms. With a little abuse of terminology, if \( |H| > 1 \), we shall say the clause is disjunctive, otherwise it is a Horn, or non-disjunctive\(^3\). Moreover, if \( |H| = 1 \) the clause is called single-head. A fact is a single-head rule with empty body. A theory \( \Pi \) is a finite set of clauses. If there is some disjunctive rule in \( \Pi \) then \( \Pi \) is called disjunctive, otherwise it is called non-disjunctive. \( \text{atom}(\Pi) \) denotes the set of all the atoms occurring in \( \Pi \). A set \( S \) of atoms

\(^2\)We prefer to adopt the implication-based syntax for clauses in the place of the more usual disjunction-based one to slightly ease the foregoing presentation.

\(^3\)We will use the terms *Horn* and *non-disjunctive* interchangeably.
Symbol | Description
--- | ---
\(\Pi\) | A CNF theory
\(\Pi^{\text{nd}}\) | A non-disjunctive theory obtained from \(\Pi\) by deleting all disjunctive clauses
\(\text{atom}(\Pi)\) | The set of atoms appearing in \(\Pi\)
\(c_X\) | The clause obtained by projecting the clause \(c\) on the set of atoms \(X\): if \(c \equiv H \leftarrow B\) then \(c_X \equiv H_X \leftarrow B_X\) with \(H_X = H \cap X\) and \(B_X = B \cap X\)
\(c_X^{\leftarrow}\) | The clause obtained by projecting the head of the clause \(c\) on the set of atoms \(X\): if \(c \equiv H \leftarrow B\) then \(c_{X^{\leftarrow}} \equiv H_X^{\leftarrow} \leftarrow B\) with \(H_X = H \cap X\)
\(\Pi_X\) | The set of all the non-empty head clauses \(c_X\) with \(c\) in \(\Pi\)
\(\Pi_X^{\leftarrow}\) | The set of all the non-empty head clauses \(c_{X^{\leftarrow}}\) with \(c\) in \(X\)
\(\sigma_M(\Pi)\) | The set of all the non-empty clauses \(c_M\) with \(c\) in \(\Pi\)
\(\Sigma_M(\Pi)\) | A shortcut for \((\sigma_M(\Pi))_M\mid_S\)
\(\mathcal{G}(\Pi)\) | The dependency graph associated with the theory \(\Pi\)
\(\check{\mathcal{G}}(\Pi)\) | The elementary graph associated with the non-disjunctive theory \(\Pi\)

Table 1: Summary of the symbols employed throughout the paper.
is called a disjunctive set for \( \Pi \) if there exists at least one rule \( H \leftarrow B \) in \( \Pi \) such that \(|H \cap S| > 1\). A constraint is an empty-head clause. A theory \( \Pi \) is said to be positive if no constraint occurs in \( \Pi \).

The semantics of CNF theories relies on the concepts of interpretation and model, which are recalled next. An interpretation \( I \) for the theory \( \Pi \) is a set of atoms from \( \Pi \). An atom is true (resp., false) in the interpretation \( I \) if \( a \in I \) (resp., \( a \notin I \)). A rule \( H \leftarrow B \) is true in \( I \) if either at least one atom occurring in \( H \) is true in \( I \) or at least one atom occurring in \( B \) is false in \( I \).

An interpretation \( I \) is a model for a theory \( \Pi \) if all clauses occurring in \( \Pi \) are true in \( I \). A model \( M \) for \( \Pi \) is minimal if no proper subset of \( M \) is a model for \( \Pi \).

A directed graph \( G(\Pi) \), called positive dependency graph, can be associated with a theory \( \Pi \). Specifically, nodes in \( G(\Pi) \) are associated with atoms occurring in \( \Pi \) and, moreover, there is a directed edge \((m, n)\) from a node \( m \) to a node \( n \) in \( G(\Pi) \) if and only if there is a clause \( H \leftarrow B \) of \( \Pi \) such that the atom associated with \( m \) is in \( B \) and the atom associated with \( n \) is in \( H \).

Given a clause \( c \equiv H \leftarrow B \) and a set of atoms \( X \), \( c_X \) denotes the clause \( H \cap X \leftarrow B \), whereas \( c_X \) denotes the clause \( H \cap X \leftarrow B \cap X \). Given a theory \( \Pi \) and a set of atoms \( X \), the theory \( \Pi_{X_{\leftarrow}} \) includes all non-empty head clauses \( c_{X_{\leftarrow}} \), with \( c \) a clause in \( \Pi \). Analogously, the theory \( \Pi_X \) includes all non-empty head clauses \( c_X \), with \( c \) a clause in \( \Pi \). Given a theory \( \Pi \), the theory \( \Pi_{\leftarrow}^nd \subseteq \Pi \) includes all Horn clauses of \( \Pi \). In the following, we assume that the operators \( \cdot_{X_{\leftarrow}} \) and \( \cdot_{X} \) have precedence over the operator \( \cdot_{\leftarrow}^nd \), thus that the expression \( \Pi_{X_{\leftarrow}}^nd \) (\( \Pi_{X}^nd \), resp.) is to be intended equivalent to \( (\Pi_X_{\leftarrow})^nd \) ((\( \Pi_X \))^nd, resp.).

Table 1 summarizes some of the symbols used throughout the paper (some of them are defined in subsequent sections).
2.2. Problems

Table 2 summarizes the problems and the classes of CNF theories of interest here and reports the associated complexities.

As for the classes of CNF theories, other than general one here we consider HEF and HCF theories:

— *Head Cycle Free* (HCF) theories [2] are CNF theories such that no connected component of the associated dependency graph contains two positive literals occurring in the same clause;

— *Head Elementary-Set Free* (HEF) CNF theories, the definition of which we adapt from the homonym one given in [14] for disjunctive logic programs (see Section 4 for the formal definition of HEF theories), form a strict superset of the class of HCF theories.

The problems (listed in the table) are:

— *Recognition Problem* (REC): Given a CNF theory Π and a class C of CNF theories, decide if Π belongs to the class C;

— *Model Finding Problem* (MFP): Given a CNF theory Π, compute a model M for Π;

— *Model Minimization Problem* (MMP): Given a CNF theory Π and a model M for Π, compute a minimal model MM for Π contained in M;

— *Minimal Model Checking Problem* (MMCP): Given a CNF theory Π and a model M for Π, check if M is indeed a minimal model for Π;

— *Minimal Model Finding Problem* (MMFP): Given a CNF theory Π, compute a minimal model M for Π.

The MFP problem is NP-hard unless the theory is positive. Indeed, in the latter case, the set consisting of all the literals occurring in the theory is always a model.

In this work we will focus on the MMP, MMCP, and MMFP problems.

As for MMFP, it turns out that, over positive CNF theories, this is hard to solve. In particular, it is known that on positive theories MMFP is \(P^{NP[O(log n)]}\)-hard [8] (even though positive CNF theories always have a minimal model!).
Given a CNF theory $\Pi$ and a model $\mathcal{M}$ for $\Pi$, it is worth noticing that the theory $\Pi_{\mathcal{M}}$ is always a positive CNF and that the models of $\Pi_{\mathcal{M}}$ are a subset of those of $\Pi$. This explains the fact that the complexity of the MMP and MMCP problems, which have in input a model $\mathcal{M}$ other than the theory $\Pi$, does not depend on positiveness of the theory.

Moreover, we notice that MMFP is not easier than MMP and MMCP since the latter problems can be reduced to the former one as follows:

- As for MMP, return $\text{MMFP}(\Pi_{\mathcal{M}})$;
- As for MMCP, return true if $\text{MMFP}(\Pi_{\mathcal{M}}) = \mathcal{M}$ and false otherwise.

Thus, if for a certain class of theories the MMFP were tractable, then both MMP and MMCP would become tractable as well.

Moreover, if attention is restricted to positive theories, the MMP and MMFP problems coincide (since this time MMFP can be reduced to MMP by setting $\mathcal{M}$ to the set of all the literals occurring in $\Pi$) and, consequently, MMP on general theories is equivalent to MMFP on positive theories.

We notice that, on the other hand, for head-cycle-free CNF theories things are easier than for the general case: indeed it was proved in [5] that the MMFP is solvable in polynomial time if the input theory is HCF.

All that given, the following section details the contributions of the paper.

2.3. Contributions and algorithms road map

In this work we investigate the MMP and MMCP problems on CNF theories and the MMFP on positive CNF theories.

Among the main contributions offered here, we will show that MMP and MMCP are tractable on generic HEF theories, while MMFP is tractable on positive HEF theories.

In order to provide a uniform treatment of these problems, we will concentrate on algorithms for the MMP, which can be considered the most general of them since its input consists of both a CNF theory and a (not necessarily) non-minimal model of the theory. Specifically, we provide a polynomial time algorithm solving the MMP on general HEF CNF theories which, because of the observations made above, can be directly used to solve in polynomial time the following five problems (see also cells of Table 2 reported in bold): (i) MMP on non-positive HEF CNF theories, (ii) MMCP on non-positive HEF CNF theories, (iii) MMP on positive HEF CNF theories, (iv) MMCP on positive HEF CNF theories, and (v) MMFP on positive HEF CNF theories.
Also already noticed, differently from HCF theories, which turn out to be recognizable in polynomial time [2], recognizing HEF theories is an intractable problem [13]. This undesirable property may seem to limit the applicability of the above complexity results. However, as better explained next, we show that our MMP algorithm can be fed with any CNF theory \( \Pi \) and any model \( M \) of \( \Pi \) and it is guaranteed to correctly minimize \( M \) at least in the case that the theory \( \Pi \) is HEF. Notice that this property holds without the need to recognize whether the input theory is HEF or not.

To illustrate, we start by presenting an algorithmic schema, called the *Generalized Elimination Algorithm (GEA)* for model minimization over CNF theories. The GEA invokes a suitable *eliminating operator* in order to converge towards a minimal model of the input theory. Intuitively, an eliminating operator is any function that, given a model as the input, returns a model strictly included therein, if one exists. Therefore, the actual complexity of the GEA depends on the complexity of the specific eliminating operator one decides to employ. Clearly, the trivial eliminating operator may enumerate (in exponential time) all the interpretations contained in the given model and check for satisfiability of the theory, while we shall consider actually interesting only those eliminating operators that accomplish their task in polynomial time.

A specific eliminating operator, denoted by \( \xi_{\text{HEF}} \), is henceforth defined, by which the GEA computes a minimal model of any HEF theory in polynomial time. However, the intractability of the recognition problem for HEF CNF theories may seem to narrow the applicability of the results sketched above and to reduce their significance to a mere theoretical result. This seemingly relevant limitation can fortunately be overcome by suitably readapting the structure of our algorithm: to this end, we introduce the *Incomplete Generalized Elimination Algorithm (IGEA)* that, once instantiated with a suitable operator, outputs a model of the input theory, which is guaranteed to be minimal at least over HEF theories.

The design of IGEA leverages on the notion of *fallible eliminating operator*, which is defined later in this paper. Then, by coupling IGEA with the \( \xi_{\text{HEF}} \) operator, we call this instance of the algorithm IGEA_{\( \xi_{\text{HEF}} \)}, we obtain a polynomial-time algorithm that always minimizes the input model of a HEF CNF theory without the need of knowing in advance whether the input CNF theory is HEF or not. As for non-HEF theories, we show that IGEA_{\( \xi_{\text{HEF}} \)} always returns a model of the input theory which may be minimal or not, depending on the structure of the input theory. This kind of behavior
\[ \Pi = \{ \begin{array}{l}
g \lor j \leftarrow \\
f \lor h \leftarrow \\
  \quad b \leftarrow a \\
  \quad c \leftarrow b \\
  \quad a \leftarrow c \\
  \quad d \leftarrow a, b \\
  \quad c \leftarrow d \\
  \quad e \leftarrow b \\
  \quad h \leftarrow b \\
  \quad f \leftarrow e, i \\
  \quad i \leftarrow e, j \\
  \quad g \leftarrow f \\
  \quad e \leftarrow g \\
  \quad j \leftarrow e \\
  \quad h \leftarrow j \\
  \quad j \leftarrow h \\
  \quad c \leftarrow h, e \\
\end{array} \} \]

Figure 1: A positive CNF theory and the associated dependency graph.

on non-HEF theories is clearly the expected one since, as already noticed, recognizing HEF theories is co-NP-complete. Interestingly, this latter characteristics of IGEA\(\xi_{\text{HEF}}\) further enhances its relevance, since its application is not restricted to the class of HEF CNF theories, but to a even broader class thereof.

2.4. Application scenarios

In this section we consider generic CNF theories without concentrating on the particular class (that is, general, HEF or HCF) they belong to. Later, in Section 4.2, we specialize some of the examples provided next in the context of HEF theories, which is a main focus in our investigation.

As already noticed, the minimal model finding problem is a formidable one and remains intractable even in the case attention is restricted to positive CNFs. The following positive CNF theory will be employed in order to describe the various concepts introduced throughout the paper.

Example 1 (Minimal models of positive CNF theories). Figure 1 reports an example of positive CNF theory \(\Pi\) (on the left) together with the
Figure 2: A logic program $P$, a model $\mathcal{M}$ of $P$, and the reduct $P^\mathcal{M}$.

associated dependency graph $\mathcal{G}(\Pi)$ (on the right). The set $\text{atom}(\Pi)$ is $\{a, b, c, d, e, f, g, h, i, j\}$ and it is the largest model of $\Pi$. This theory has several models, but only a minimal one, which is $\{j, h\}$.

To illustrate a setting in which positive CNFs naturally arise, consider Logic Programming, a central tool in Knowledge Representation and Reasoning. In the field of Logic Programming, the notion of negation by default poses the problem of defining a proper notion of model of the program. Among the several proposed semantics for logic programs with negation, the Stable Models and Answer Sets semantics are nowadays the reference one for closed world scenarios [16]. An interesting application of our techniques concerns stable model (or answer set) checking. To illustrate, stable models exploit the concept of the reduct of the program, as clarified in the following definition.

**Definition 1 (Stable Model [16]).** Given a logic program $P$ and a model $\mathcal{M}$ of $P$, the reduct of $P$ w.r.t $\mathcal{M}$, also denoted by $P^\mathcal{M}$, is the program built from $P$ by (i) removing all rules that contain a negative literal $\text{not } a$ in the body with $a \in \mathcal{M}$, and (ii) removing all negative literals from the remaining rules. A model $\mathcal{M}$ of $P$ is stable if $\mathcal{M}$ is a minimal model of $P^\mathcal{M}$.

**Example 2 (Stable Models of Logic Programs).** Figure 2 shows, on the left, a logic program $P$ and, on the right, the reduct $P^\mathcal{M}$ of $P$ w.r.t. the model $\mathcal{M} = \{a, d\}$. In this case, $\mathcal{M}$ is a minimal model of $P^\mathcal{M}$ and, hence, it is a stable model of $P$.

It is worth noticing that $P^\mathcal{M}$ is a CNF since, by definition of the reduct, negation by default does not occur in any clause of $P^\mathcal{M}$. Moreover, $\mathcal{M}$ is always a model of $P^\mathcal{M}$ and is given in input.
\[ \Pi = \{ \begin{align*} &b \leftarrow a \\ &c \leftarrow a \\ &a \leftarrow b, c \\ &b, c \leftarrow \\ &d \leftarrow \\ &\leftarrow b, d \end{align*} \} \]

\[ \Pi^+ = \{ \begin{align*} &b \leftarrow a \\ &c \leftarrow a \\ &a \leftarrow b, c \\ &b, c \leftarrow \\ &d \leftarrow \\ &\phi \leftarrow b, d \\ &a \leftarrow \phi \\ &b \leftarrow \phi \\ &c \leftarrow \phi \\ &d \leftarrow \phi \end{align*} \} \]

Figure 3: A CNF \( \Pi \) and its positive form \( \Pi^+ \).

Therefore, by setting \( \Pi = P^\mathcal{M} \) the problem of verifying if a given model \( \mathcal{M} \) for the logic program \( P \) is stable fits the minimal model checking problem for positive CNFs and, as such, can be suitably dealt with using the techniques this paper proposes.

In order to analyze a different application scenario, let us assume a positive CNF is given. Next we show that the given theory can be indeed reduced to a positive theory whose models have some clear relationship with the models of the original theory.

Let us first consider the definition of positive form of a CNF.

**Definition 2 (Positive Form of a CNF theory).** The theory \( \Pi^+ \), also said the *positive form* of \( \Pi \), is defined as follows: (1) for each clause \( H \leftarrow B \) of \( \Pi \), if \( H \) is not empty then the clause \( H \leftarrow B \) is in \( \Pi^+ \); (2) for each clause \( \leftarrow B \) of \( \Pi \), the clause \( \phi \leftarrow B \) is in \( \Pi^+ \); (3) for each atom \( a \) occurring in \( \Pi \), the clause \( a \leftarrow \phi \) is in \( \Pi^+ \).

The following result relates models of \( \Pi \) with minimal models of \( \Pi^+ \).

**Proposition 2.1.** *Given a CNF theory \( \Pi \), if \( \phi \) belongs to the (unique) minimal model of \( \Pi^+ \) then \( \Pi \) is inconsistent, otherwise the set of minimal models of \( \Pi \) and \( \Pi^+ \) coincide.*

*Proof.* First of all, we will observe that each model of \( \Pi \) is a model of \( \Pi^+ \) as well and, then, \( \phi \) is not in \( \mathcal{M} \).
Observation 1. Let $M$ be a model for $\Pi$ and consider the theory $\Pi^+$. All the clauses (1) in $\Pi^+$ are also in $\Pi$ and then are true. Since $M$ is a model for $\Pi$ all the empty-head clauses of $\Pi$ don’t have the body fully contained in $M$ and, therefore, $M$ satisfies all the clauses (2) of $\Pi^+$. Finally, since $\phi$ is not in $M$ all the clauses (3) are true.

Now, let $M^+$ be a minimal model of $\Pi^+$ and $\text{atom}(\Pi^+)$ be the set of all atoms occurring in $\Pi^+$.

Note that, because of the presence of the set of clauses (3), two cases are possible, that are: either $M^+$ contains $\phi$ and then all the atoms occurring in $\Pi^+$; or $M^+ \subset \text{atom}(\Pi^+)$ and, in particular, $\phi \notin M^+$.

1. As for the first case, if $\Pi$ had a model $M$ then, due to Observation 1, $M$ would be a model of $\Pi^+$ as well and then $M^+$ would not be minimal. Thus, $\Pi$ is inconsistent.

2. As for the second case, $M^+$ does not contain $\phi$. Consider now the theory $\Pi$. All the non-empty-head clauses in $\Pi$ are also in $\Pi^+$ and, then, are satisfied by $M^+$. Consider, now, the empty-head clauses in $\Pi$. Because of the presence of clauses (2) in $\Pi^+$, and since $M^+$ does not contain $\phi$, it is the case that the body of such clauses is not fully contained in $M^+$. Thus, the correspondent clauses in $\Pi$ are satisfied by $M^+$.

This implies that $M^+$ is a model of $\Pi$ as well. □

To illustrate, consider the following example.

Example 3 (General CNF theories). Consider the CNF reported in Figure 3 on the left. In the same figure, on the right, it is reported the positive form $\Pi^+$ of $\Pi$. $\Pi$ has only one minimal model, namely $\{c, d\}$, which is precisely the unique minimal model of $\Pi^+$. □

3. Generalized Elimination Algorithm

In this section, a generalization of the elimination algorithm proposed in [5], called Generalized Elimination Algorithm, is introduced. We begin by providing some preliminary concepts, notably, those of steady set and eliminating operator.
Intuitively, given a model $\mathcal{M}$ for a theory $\Pi$, the steady set is the subset of $\mathcal{M}$ containing atoms which “cannot” be erased from $\mathcal{M}$, for otherwise $\mathcal{M}$ would no longer be a model for $\Pi$. As proved next, the steady set can be obtained by computing the model of a certain non-disjunctive theory.

**Definition 3 (Steady set).** Given a CNF theory $\Pi$ and a model $\mathcal{M}$ for $\Pi$, the minimal model $S_t \subseteq \mathcal{M}$ of the theory $\Pi_{\mathcal{M} \leftarrow}^{nd}$ is called the steady set of $\mathcal{M}$ for $\Pi$.

Note that the steady set $S_t$ of $\mathcal{M}$ for $\Pi$ always exists and is unique. Indeed, $\Pi_{\mathcal{M} \leftarrow}^{nd}$ is a Horn positive CNF and it is known that these kinds of theories have one and only one minimal model (which can be computed in polynomial time) [11, 26].

**Property 3.1 ($\mathcal{M}\mathcal{M}$-containment).** Given a positive CNF theory $\Pi$, a model $\mathcal{M}$ for $\Pi$ and the steady set $S_t$ of $\mathcal{M}$ for $\Pi$, it holds that each model of $\Pi$ contained in $\mathcal{M}$ contains $S_t$.

**Proof.** First, notice that the models of the positive CNF theory $\Pi$ which are contained in the model $\mathcal{M}$ of $\Pi$ coincide with the models of the positive CNF theory $\Pi_{\mathcal{M} \leftarrow}$.

Since $\Pi_{\mathcal{M} \leftarrow}^{nd}$ is contained in $\Pi_{\mathcal{M} \leftarrow}$, by monotonicity of propositional logic, it follows that all logical consequences of $\Pi_{\mathcal{M} \leftarrow}^{nd}$ are also logical consequences of $\Pi_{\mathcal{M} \leftarrow}$ and, hence, each model of $\Pi_{\mathcal{M} \leftarrow}$ contains the unique minimal model of $\Pi_{\mathcal{M} \leftarrow}^{nd}$, which is the steady set of $\mathcal{M}$ for $\Pi$. $\square$

**Definition 4 (Erasable set).** Let $\mathcal{M}$ be a model of a positive CNF theory $\Pi$. A non-empty subset $\mathcal{E}$ of $\mathcal{M}$ is said to be erasable in $\mathcal{M}$ for $\Pi$ if $\mathcal{M} \setminus \mathcal{E}$ is a model of $\Pi$.

The following result holds.

**Proposition 3.1.** Let $\mathcal{M}$ be a model of a positive CNF theory $\Pi$, let $S_t$ be the steady set of $\mathcal{M}$ for $\Pi$, and let $\mathcal{E}$ be a set erasable in $\mathcal{M}$ for $\Pi$. Then, $\mathcal{E} \subseteq \mathcal{M} \setminus S_t$.

**Proof.** For the sake of contradiction, assume that $\mathcal{E} \cap S_t \neq \emptyset$. Then, $\mathcal{M} \setminus \mathcal{E}$ is a model of $\Pi$ that does not contain $S_t$, which contradicts the fact that $S_t$ has the $\mathcal{M}\mathcal{M}$-containment property in $\mathcal{M}$ for $\Pi$ (See Property 3.1). $\square$
Algorithm 1: Generalized Elimination Algorithm with operator $\xi$, $GEA_\xi(\Pi, M)$

**Input:** A CNF theory $\Pi$ and a model $M$ of $\Pi$

**Output:** A minimal model $M^*$ of $\Pi$ contained in $M$

1: remove all constraints from $\Pi$

2: $stop = false$

3: repeat

4: compute the minimal model $St$ of $\Pi^d_{M^*}$

5: if $St$ is a model of $\Pi$ then

6: \[ M^* = St \text{ stop = true} \]

7: else

8: $E = \xi(\Pi, M)$

9: if ($E = \emptyset$) then

10: \[ M^* = M \text{ stop = true} \]

11: else

12: \[ M = M \setminus E \]

13: until $stop$

14: return $M^*$

Figure 4: Generalized Elimination Algorithm with operator $\xi$, $GEA_\xi(\Pi, M)$

**Definition 5 (Eliminating operator).** Let $M$ be a model of a positive CNF theory $\Pi$. An eliminating operator $\xi$ is a mapping that, given $M$ and $\Pi$ in input, returns an erasable set in $M$ for $\Pi$, if one exists, and an the empty set, otherwise.

It immediately follows that if $\xi(\Pi, M) = \emptyset$ then $M$ is a minimal model of $\Pi$. This is easily shown by observing that $\xi(\Pi, M) = \emptyset$ implies that there is no erasable set in $M$, namely no subset of $M$ is a model for $\Pi$.

We are now ready to present our algorithmic schema, referred to as the Generalized Elimination Algorithm (GEA) throughout the paper, which is summarized in Figure 4. Note that GEA has an operator $\xi$ as its parameter$^4$.

Our first result states that GEA is correct under the condition that the operator parameter $\xi$ is an eliminating operator.

$^4$The term *schema* is used here since actual algorithms are obtained only after instantiating the generic $\xi$ operator invoked in the GEA to a specific operator.
Theorem 3.1 (GEA correctness). Let $\Pi$ be a CNF theory and $\mathcal{M}$ be a model of $\Pi$. If $\xi$ is an eliminating operator, then the set returned by $\text{GEA}_\xi$ on input $\Pi$ and $\mathcal{M}$ is a minimal model for $\Pi$ contained in $\mathcal{M}$.

Proof. First of all, since $\mathcal{M}$ is a model of $\Pi$, by definition of model all the constraints (aka empty-head clauses) of $\Pi$ are true in $\mathcal{M}$ and are also true in any subset of $\mathcal{M}$. Hence, they can be disregarded during the subsequent steps (see line 1).

Moreover, note that, by definition of steady set, it follows that the set $St$ computed at the beginning of each iteration of the algorithm (line 4) is a (not necessarily proper) subset of every minimal model contained in $\mathcal{M}$. Let $n$ be the number of atoms in the model $\mathcal{M}$ computed at line 1 of the GEA.

Three cases are possible, which are discussed next:

1. $St$ is a model of $\Pi$. Since $St$ is the steady set of $\mathcal{M}$ for $\Pi$, if $St$ is a model for $\Pi$, then it is also minimal; so the algorithm stops and returns a correct solution.\footnote{We recall that if the steady set $St$ of $\mathcal{M}$ for $\Pi$ is a model of $\Pi$ then it is the unique minimal model of $\Pi$ contained in $\mathcal{M}$. Hence, the test at line 4 serves the purpose of accelerating the termination of the algorithm. However, operations in lines 3-6 could be safely dropped without affecting the correctness of the algorithm.}

2. $E = \emptyset$. By definition of eliminating operator, if $E$ is empty, then $\mathcal{M}$ is a minimal model; so the algorithm stops and returns a correct solution.

3. $E \neq \emptyset$. In this case, a non-empty set of atoms is deleted from $\mathcal{M}$, letting (by definitions of eliminating operator and erasable set) $\mathcal{M}$ still be a model for $\Pi$. Thus, at the next iteration, the algorithms will work with a smaller (possibly not minimal) model $\mathcal{M}$. Hence, after at most $n$ iterations, either case 1 or case 2 applies.

The next result states the time complexity of the GEA that, clearly, will depend on the complexity $C_\xi$ associated with the evaluation of the eliminating operator $\xi$.

Proposition 3.2. Let $n$ and $m$ denote the number of atoms occurring in the heads of $\Pi$ and, overall, in $\Pi$, respectively. Then, for any model $\mathcal{M}$ of $\Pi$, $\text{GEA}_\xi(\Pi, \mathcal{M})$ runs in time $O(nm + nC_\xi)$.
Proof. Since at each iteration (if the stopping condition is not matched) at least one atom is removed, the total number of iterations is $O(n)$. As for the cost spent at each iteration, the dominant operations are: (i) computing the (unique) minimal model of a non-disjunctive theory (line 4) which can be accomplished in linear time w.r.t. $m$ by the well-known unit propagation procedure [11]; (ii) checking if a set of atoms is a model (line 5) which can be accomplished in linear time in $m$ as well; (iii) applying the eliminating operator (line 8), whose cost is $C_\xi$. This closes the proof. □

In particular, consider the naive operator $\xi_{\text{exp}}$ that enumerates all the $2^n$ non-empty subsets of $\mathcal{M}$ and either returns one of these, call it $\mathcal{E}$, such that $\mathcal{M} \setminus \mathcal{E}$ is a model for $\Pi$, or an empty set if such a set $\mathcal{E}$ does not exist. The resulting algorithm GEA$\xi_{\text{exp}}$ returns a minimal model of $\Pi$ but requires exponential running time.

Conversely, as an example of instance of the GEA algorithm having polynomial time complexity on a specific class of CNF theories, consider the Elimination Algorithm presented in [5]. This algorithm can be obtained from the GEA by having the operator $\xi_{\text{HCF}}$ (described next) as the eliminating operator $\xi$ and the set $\text{atom}(\Pi)$ as the input model $\mathcal{M}$. Indeed, as shown in [5], the Elimination Algorithm computes a minimal model of a positive HCF theory in polynomial time. The definition of $\xi_{\text{HCF}}$ operator follows [5]. Let $\Pi$ be a positive HCF CNF theory and let $\mathcal{M}'$ be the set of the heads of the disjunctive rules in $\Pi_{\mathcal{M}\leftarrow}$ which are false in $\mathcal{M}$. Then, $\xi_{\text{HCF}}(\Pi, \mathcal{M})$ is defined to return a source of $\mathcal{M}'$, where a source of the set of atoms $\mathcal{M}'$ is a connected component in the subgraph of $G(\Pi)$ induced by $\mathcal{M}'$ which does not have incoming arcs.

Before leaving this section, we provide two further results which will be useful when discussing the MMCP and the MMFP.

Lemma 3.1. Given a CNF theory $\Pi$, an eliminating operator $\xi$ and a model $\mathcal{M}$ of $\Pi$, $\mathcal{M}$ is minimal for $\Pi$ if and only if GEA$_{\xi}(\Pi, \mathcal{M})$ outputs $\mathcal{M}$.

Proof. The proof follows by noticing that GEA always outputs a (possibly non-proper) minimal sub-model of the initial model $\mathcal{M}$ as its output. □

Lemma 3.2. Given a positive CNF theory $\Pi$ and an eliminating operator $\xi$, then GEA$_{\xi}(\Pi, \text{atom}(\Pi))$ outputs a minimal model of $\Pi$.

Proof. The proof follows by noticing that $\text{atom}(\mathcal{M})$ is a model of $\mathcal{M}$, being $\Pi$ a positive theory, and by Lemma 3.1. □
4. Model minimization on HEF CNF theories

We have noticed above that the complexity of GEA depends on the complexity characterizing, in its turn, the specific elimination operator it invokes. On the other hand, the MMP being $P^{NP[O(\log n)]}$-hard [8] implies that, unless the polynomial hierarchy collapses, the GEA will generally require exponential time to terminate when called on a generic input CNF theory. Therefore, it is sensible to single out significant subclasses of CNF theories for which it is possible to devise a specific eliminating operator guaranteeing a polynomial running time for the GEA.

In this respect, it is a simple consequence of the results presented in [5] that a model of any head-cycle-free theory can be indeed minimized in polynomial time using the Elimination Algorithm. So, the interesting question remains open of whether we can do better than this. Our answer to this question is affirmative and this section serves the purpose of illustrating this result. In particular, we shall show that by carefully defining the eliminating operator, we can have that the GEA minimizes in polynomial time a model of any HEF CNF theory. In Section 5, we shall moreover show that there also exist CNF theories which are not HEF but for which the algorithm, equipped with a proper eliminating operator, efficiently minimizes a model.

4.1. Head-elementary-set-free theories and super-elementary sets

Next, we recall the definition of head-cycle-free theories [2], adapt that of head-elementary-cycle-free theories [14] to our propositional context and provide a couple of preliminary results which will be useful in the following.

We proceed by introducing the concepts of outbound and elementary set.

**Definition 6 (Outbound Set (adapted from [14])).** Let $\Pi$ be a CNF theory. For any set $Y$ of atoms occurring in $\Pi$, a subset $Z$ of $Y$ is outbound in $Y$ for $\Pi$ if there is a clause $H \leftarrow B$ in $\Pi$ such that: (i) $H \cap Z \neq \emptyset$; (ii) $B \cap (Y \setminus Z) \neq \emptyset$; (iii) $B \cap Z = \emptyset$ and (iv) $H \cap (Y \setminus Z) = \emptyset$.

Intuitively, $Z \subseteq Y$ is outbound in $Y$ for $\Pi$ if there exists a rule $c$ in $\Pi$ such that the partition of $Y$ induced by $Z$ (namely, $\langle Z; Y \setminus Z \rangle$) “separates” head and body atoms of $c$. 
Example 4 (Outbound set). Consider the theory

\[ \Pi = \{ \begin{array}{l} b, c \leftarrow a \\ b \leftarrow c \\ c \leftarrow b \\ a \leftarrow b \\ d \leftarrow b, c \end{array} \} \]

and the set \( E = \{a, b, c\} \). Consider, now, the subset \( O = \{a, b\} \) of \( E \). \( O \) is outbound in \( E \) for \( \Pi \) because of the clause \( b \leftarrow c \), since \( c \in E \setminus O \), \( c \not\in O \), \( b \in O \) and \( b \not\in E \setminus O \). □

Let \( O \) be a non-outbound set in \( X \) for \( \Pi \). \( O \) is minimal non-outbound if any proper subset \( O' \subset O \) is outbound in \( X \) for \( \Pi \).

Definition 7 (Elementary Set (adapted from [14])). Let \( \Pi \) be a CNF theory. For any non-empty set \( Y \subseteq \text{atom}(\Pi) \), \( Y \) is elementary for \( \Pi \) if all non-empty proper subsets of \( Y \) are outbound in \( Y \) for \( \Pi \).

For example, the set \( E_{\text{ex}} \) of Example 4 is elementary for the theory \( \Pi_{\text{ex}} \), since each non-empty proper subset of \( E_{\text{ex}} \) is outbound in \( E_{\text{ex}} \) for \( \Pi_{\text{ex}} \).

Definition 8 (Head-Elementary-Set-Free CNF theory (adapted from [15])).\[ \text{Let } \Pi \text{ be a CNF theory. } \Pi \text{ is head-elementary-set-free (HEF) if for each clause } H \leftarrow B \text{ in } \Pi, \text{ there is no elementary set } E \text{ for } \Pi \text{ such that } |E \cap H| > 1. \]

So, a CNF theory \( \Pi \) is HEF if there is no elementary set containing two or more atoms appearing in the same head of a rule of \( \Pi \). An immediate consequence of Definition 8 is the following property.

Property 4.1. A theory \( \Pi \) is not HEF if and only if there exists a set \( X \) of atoms of \( \Pi \) such that \( X \) is both a disjunctive and an elementary set for \( \Pi \).

For instance, the theory \( \Pi_{\text{ex}} \) of Example 4 is not HEF, since for the rule \( b, c \leftarrow a \) and the elementary set \( E_{\text{ex}} \), we have \( |E_{\text{ex}} \cap \{b, c\}| > 1 \).
4.2. Examples of HEF theories

Now the examples already introduced in Section 2.4 are discussed in the context of HEF CNF theories.

Example 1 (Minimal models of positive CNF theories – continued). Consider the theory reported in Figure 1. This is an HEF CNF theory since no superset of \{g, j\} and no superset of \{f, h\} is an elementary set for \(\Pi\). \(\square\)

Example 2 (Stable Models of Logic Programs – continued). A logic program \(P\) is HEF if the CNF \(\hat{P}\) obtained by removing all the literals of the form \textit{not a} from the body of its rules is HEF [15].

Importantly, it holds that if the logic program \(P\) is HEF and \(M\) is a model of \(P\), then also \(P^M\) is HEF. This follows since, by definition, a logic program \(P\) is HEF if and only if the CNF \(\hat{P}\) is HEF, and by Lemma 4.3 (reported in Section 4.4) any subset of clauses of a HEF CNF is HEF as well, and \(P^M\) is precisely a subset of \(\hat{P}\).

Notably, even if \(P\) is not HEF, it could be anyway the case that \(P^M\) is HEF, and this broadens the range of applicability of the techniques proposed here.

As an example, consider again Figure 2. The program \(P\) there reported is not HEF, since the set \(S = \{a, b, c\}\) is both disjunctive and elementary.

Conversely, \(P^M\) is HEF since the set \(S\) is no longer elementary because the subsets \(\{a, c\}\) and \(\{b, c\}\) of \(S\) are not outbound in \(S\).

Moreover, we notice that the subgraph of \(G(P^M)\) induced by \(S\) is a connected component and then both \(P\) and \(P^M\) are not HCF. \(\square\)

Example 3 (General CNF theories – continued). Given a non-positive CNF \(\Pi\), it holds that if \(\Pi\) is not HEF, then also \(\Pi^+\) is not HEF.

Let \(\Pi'\) be the subset of \(\Pi\) obtained by removing the contraints in \(\Pi\). Notice that \(\Pi'\) can be obtained from \(\Pi^+\) by first removing the clauses of the form \(a \leftarrow \phi\) for each \(a \in \text{atom}(\Pi)\) (see point (3) of Definition 2) and then projecting it on \(\text{atom}(\Pi)\). Since the HEF property does not depend on the contraints, it follows from Lemma 4.3 (reported in Section 4.4) that if \(\Pi\) is not HEF, then \(\Pi^+\) is not HEF as well.

Conversely, if \(\Pi\) is HEF, then \(\Pi^+\) can happen to be either HEF or not.

As an example, consider the theories displayed in Figure 3 of Section 2.4. In this case \(\Pi\) and \(\Pi^+\) are both HEF. Conversely, consider the theories reported in Figure 5. In this case, \(\Pi\) is HEF, whereas \(\Pi^+\) is not. \(\square\)
\( \Pi = \{ \begin{align*} & b \leftarrow a \\ & c \leftarrow a \\ & a \leftarrow b, c \\ & b \leftarrow c \\ & b, c \leftarrow \\ & d \leftarrow \\ & \leftarrow b, d \\ & \leftarrow c, d \end{align*} \} \)

\( \Pi^+ = \{ \begin{align*} & b \leftarrow a \\ & c \leftarrow a \\ & a \leftarrow b, c \\ & b \leftarrow c \\ & b, c \leftarrow \\ & d \leftarrow \\ & \phi \leftarrow b, d \\ & \phi \leftarrow c, d \\ & a \leftarrow \phi \\ & b \leftarrow \phi \\ & c \leftarrow \phi \\ & d \leftarrow \phi \end{align*} \} \)

Figure 5: An HEF CNF \( \Pi \) and its positive form \( \Pi^+ \) which is not HEF.

4.3. Super-elementary sets

We introduce next the definition of simplified theory and of super-elementary set that will play a relevant role in the definition of the eliminating operator for HEF theories.

**Definition 9 (Simplified theory).** Let \( \Pi \) be a CNF theory and \( \mathcal{M} \) be a model of \( \Pi \). Then the simplified theory of \( \Pi \) w.r.t. \( \mathcal{M} \), denoted as \( \Sigma_{\mathcal{M}}(\Pi) \), is the CNF theory \( (\sigma_{\mathcal{M}}(\Pi))_{\mathcal{M}\setminus St} \), where

\[
\sigma_{\mathcal{M}}(\Pi) = \{ H \leftarrow B \in \Pi : H \cap St = \emptyset \text{ and } \mathcal{M} \supseteq B \}
\]

and \( St \) is the steady set of \( \mathcal{M} \) in \( \Pi \).

The clauses in \( \sigma_{\mathcal{M}}(\Pi) \) are those clauses of \( \Pi \) having the body fully contained in \( \mathcal{M} \) and some atoms of the head contained in \( \mathcal{M} \) but not in \( St \). Note that it cannot be the case for the head of any clause in \( \Pi \) to have empty intersection with \( \mathcal{M} \) (or, analogously, the head is empty) since, in such a case, \( \mathcal{M} \) would not be a model for \( \Pi \). Then, intuitively, \( \sigma_{\mathcal{M}}(\Pi) \) contains the subset of the clauses of \( \Pi \) which could be falsified if atoms would be eliminated from the model \( \mathcal{M} \), so that we would have a model for \( \Pi \) no longer. Note that, we do not consider the case that atoms of \( St \) are eliminated from \( \mathcal{M} \) since, by
definition of steady set, if any atom of \( St \) were eliminated we would have no longer models for \( \Pi \) in \( \mathcal{M} \). Simplified theories enjoy two useful properties.

As for the first, we observe that, for any CNF theory \( \Pi \) and model \( \mathcal{M} \) of \( \Pi \), \( \sigma_{\mathcal{M}}(\Pi) \) is positive.

The second one, summarized in the following Lemma, tells that \( \sigma_{\mathcal{M}}(\Pi) \) contains no facts.

**Lemma 4.1.** Let \( \Pi \) be a CNF theory, let \( \mathcal{M} \) be a model of \( \Pi \), and let \( St \) be the steady set of \( \mathcal{M} \) for \( \Pi \). Then no clause of the form \( h \leftarrow \), with \( h \) a single letter, occurs in the theory \( \Sigma_{\mathcal{M}}(\Pi) \).

Next, we introduce the notion of super-elementary set which will be used for defining the eliminating operator for HEF theories.

**Definition 10 (Super-elementary set).** Given a CNF theory \( \Pi \) and a set \( X \subseteq \text{atom}(\Pi) \), \( X \) is super-elementary for \( \Pi \) if \( X \) is both an elementary set for \( \Pi \) and a non-outbound set in \( \text{atom}(\Pi) \) for \( \Pi \).

Intuitively, a super-elementary set \( X \) for \( \Pi \) is a set of atoms such that for no disjunctive clause \( c \) in \( \Pi \), the body of \( c \) is satisfied by atoms not occurring in \( X \) and its head is contained in \( X \) (as will be clear in the proof of Theorem 4.1). Notice that, as a consequence, no clause may become unsatisfied by removing a super-elementary set \( X \) from a model.

**4.4. On the erasability properties of Super-elementary sets**

Next, we are going to show that, given any theory \( \Pi \) and model \( \mathcal{M} \) of \( \Pi \), any super-elementary set is erasable in \( \mathcal{M} \) for \( \Pi \). In order to do that, we shall:

1. demonstrate a one-to-one correspondence between the erasable sets in \( \mathcal{M} \) for \( \Pi \) and the erasable sets in \( \mathcal{M} \setminus St \) for \( \Sigma_{\mathcal{M}}(\Pi) \), where \( St \) is the steady set of \( \mathcal{M} \) for \( \Pi \) (Lemma 4.2),
2. show that the property of a theory \( \Pi \) being HEF is retained by the subsets of \( \Pi \) (Lemma 4.3): this implies that if a theory \( \Pi \) is HEF then, for each model \( \mathcal{M} \) of \( \Pi \), also \( \Sigma_{\mathcal{M}}(\Pi) \) is HEF,
3. prove that, for any HEF theory \( \Pi \) and any model \( \mathcal{M} \) of \( \Pi \), any super-elementary set is erasable in \( \Sigma_{\mathcal{M}}(\Pi) \) (Theorem 4.1), whereby the sought result is obtained.
The following results are conducive to the achievement of the aforementioned objectives. To ease readability, some of the proof are reported in the appendix.

**Lemma 4.2.** Let $\Pi$ be a CNF theory, let $\mathcal{M}$ be a model of $\Pi$ and let $\mathcal{S}t$ be the steady set of $\mathcal{M}$ for $\Pi$. A set of atoms $\mathcal{E}$ is erasable in $\mathcal{M}$ for $\Pi$ if and only if $\mathcal{E}$ is erasable in $(\mathcal{M} \setminus \mathcal{S}t) \supseteq \text{atom}(\Sigma_{\mathcal{M}}(\Pi))$ for $\Sigma_{\mathcal{M}}(\Pi)$.

**Lemma 4.3.** Let $\Pi$ be a HEF CNF theory. For each set of clauses $\Pi' \subseteq \Pi$ and for each set of atoms $X$, the theory $\Pi'_X$ is HEF.

**Theorem 4.1.** Let $\Pi$ be a HEF CNF theory, let $\mathcal{M}$ be a model for $\Pi$ and let $\mathcal{S}t$ be the steady set of $\mathcal{M}$ for $\Pi$. If $\mathcal{E} \subseteq \text{atom}(\Sigma_{\mathcal{M}}(\Pi))$ is super-elementary for $\Sigma_{\mathcal{M}}(\Pi)$ then $\mathcal{E}$ is erasable in $\mathcal{M}$ for $\Pi$.

**Proof.** By Lemma 4.2 it suffices to prove that $\mathcal{E}$ is erasable in $\text{atom}(\Sigma_{\mathcal{M}}(\Pi))$ for $\Sigma_{\mathcal{M}}(\Pi)$, which is accounted for next.

First of all, recall that $\Sigma_{\mathcal{M}}(\Pi)$ is a positive theory. Moreover, by Lemma 4.3, $\Sigma_{\mathcal{M}}(\Pi)$ is HEF, since $\Pi$ is HEF.

Clearly, $\text{atom}(\Sigma_{\mathcal{M}}(\Pi))$ is a model of $\Sigma_{\mathcal{M}}(\Pi)$. It must be proved that each clause of $\Sigma_{\mathcal{M}}(\Pi)$ is true in $\text{atom}(\Sigma_{\mathcal{M}}(\Pi)) \setminus \mathcal{E}$. Let $H \leftarrow B$ be a generic clause of $\Sigma_{\mathcal{M}}(\Pi)$ such that $\mathcal{E}$ contains $H$: this is the only kind of clause that might become false in $\text{atom}(\Sigma_{\mathcal{M}}(\Pi)) \setminus \mathcal{E}$. Next, it is proved that $H \leftarrow B$ is true in $\text{atom}(\Sigma_{\mathcal{M}}(\Pi)) \setminus \mathcal{E}$. First notice that, by definition of $\Sigma_{\mathcal{M}}(\Pi)$, it cannot be the case that $H$ is empty and $|B| > 1$. Thus, the following three cases have to be considered:

1. $B$ is empty and $|H| = 1$. By Lemma 4.1, such a clause cannot exist.
2. $B$ is empty and $|H| > 1$. Notice that, since $\mathcal{E}$ is an elementary set for $\Sigma_{\mathcal{M}}(\Pi)$, it cannot be the case that $|H \cap \mathcal{E}| > 1$ or, in other words, that $\mathcal{E} \supseteq H$, since the theory $\Sigma_{\mathcal{M}}(\Pi)$ is HEF. Hence, the clause $H \leftarrow B$ is true also in $\text{atom}(\Sigma_{\mathcal{M}}(\Pi)) \setminus \mathcal{E}$;
3. $B$ is not empty. By contradiction, assume that $H \leftarrow B$ is false in $\text{atom}(\Sigma_{\mathcal{M}}(\Pi)) \setminus \mathcal{E}$. Then, $H \leftarrow B$ is such that $H \subseteq \mathcal{E}$ and $B \subseteq \text{atom}(\Sigma_{\mathcal{M}}(\Pi)) \setminus \mathcal{E}$, namely, none of the atoms in $B$ occurs in $\mathcal{E}$. But this rule cannot exist, since $\mathcal{E}$ is non-outbound in $\text{atom}(\Sigma_{\mathcal{M}}(\Pi))$ for $\Sigma_{\mathcal{M}}(\Pi)$.

\[\square\]
4.5. On the existence of a super-elementary set in a HEF theory

Next, we are going to show that, under the condition that \( \text{atom}(\Pi) = \text{atom}(\Pi^{nd}) \), any HEF theory \( \Pi \) has a super-elementary set. This result, stated as Theorem 4.2 below, shall be attained by preliminarily proving that:

1. a super-elementary set for the non-disjunctive subset of a positive CNF theory is also super-elementary for the whole theory (Lemma 4.4),
2. each HEF CNF theory \( \Pi \) has a super-elementary set (Lemma 4.5).

Lemma 4.4. Let \( \Pi \) be a HEF CNF theory. If \( O \subseteq \text{atom}(\Pi^{nd}) \) is super-elementary for \( \Pi^{nd} \) then \( O \) is super-elementary for \( \Pi \).

Lemma 4.5. Let \( \Pi \) be a non-disjunctive CNF theory. Each minimal non-outbound set in \( \text{atom}(\Pi) \) for \( \Pi \) is super-elementary for \( \Pi \).

The following result eventually states another key property of super-elementary sets in HEF CNF theories.

Theorem 4.2. Let \( \Pi \) be a disjunctive HEF CNF theory such that \( \text{atom}(\Pi) = \text{atom}(\Pi^{nd}) \). Then, there exists a non-empty set of atoms \( O \subseteq \text{atom}(\Pi) \) such that \( O \) is super-elementary for \( \Pi \).

Proof. Since \( \Pi \) is a disjunctive HEF CNF theory, it cannot be the case that \( \text{atom}(\Pi) = \text{atom}(\Pi^{nd}) \) is elementary for \( \Pi^{nd} \), for otherwise \( \text{atom}(\Pi) \) would be elementary also for \( \Pi \), implying that \( \Pi \) is not HEF. Since \( \text{atom}(\Pi^{nd}) \) is not elementary in \( \Pi^{nd} \), by definition, there exists a set of atoms which is non-outbound in \( \text{atom}(\Pi^{nd}) \) for \( \Pi^{nd} \) and, in particular, there exists a minimal non-outbound set \( O \subseteq \text{atom}(\Pi^{nd}) \) for \( \text{atom}(\Pi^{nd}) \) in \( \Pi^{nd} \). To conclude, \( O \) is super-elementary for \( \Pi^{nd} \) by Lemma 4.5 and, since \( O \subseteq \text{atom}(\Pi^{nd}) = \text{atom}(\Pi) \), \( O \) is super-elementary for \( \Pi \) by Lemma 4.4.

4.6. Computing a super-elementary set

This section is devoted to proving that a super-elementary set of an HEF CNF theory can be, in fact, computed in polynomial time.

The task of computing a super-elementary set is accomplished by the function \texttt{find\_super\_elementary\_set} shown in Figure 8.

At each iteration, the function \texttt{find\_super\_elementary\_set} makes use of the function \texttt{compute\_elementary\_subgraph}, which is detailed in Figure 6. The latter function receives as input a theory \( \Pi \) and a set of atoms \( X \), an returns
Function \texttt{compute\_elementary\_subgraph}

\begin{verbatim}
Input: A non-disjunctive CNF theory \(\Pi\)
\hspace{1em} a set of atoms \(X \subseteq \text{atom}(\Pi)\)

Output: The elementary subgraph \(\widehat{G}(\Pi, X)\) of \(\Pi\)

1: \(i = 0\)
2: \(E_i = \emptyset\)
3: \(\widehat{G}_i = \langle X, E_i \rangle\)
4: repeat
5: \(\text{let } C \text{ be the set of clauses } h \leftarrow B \text{ in } \Pi \text{ s.t. the subgraph of } \widehat{G}_i \text{ induced by } B \text{ is strongly connected}\\)
6: \(E_{i+1} = E_i \cup \{ (b, h) \mid b \in B \text{ and } h \leftarrow B \in C \}\)
7: \(\widehat{G}_{i+1} = \langle X, E_{i+1} \rangle\)
8: remove \(C\) from \(\Pi\)
9: \(i = i + 1\)
10: until \(C = \emptyset\)
11: return \(\widehat{G}_i\)
\end{verbatim}

Figure 6: The compute\_elementary\_subgraph function

a graph, also denoted by \(\widehat{G}(\Pi, X)\), called the elementary subgraph of \(X\) for \(\Pi\) [14]. The function reported in Figure 6 is substantially the same as that described at page 4 of [14]. Specifically, in the pseudo-code, by \(\langle X, E \rangle\) it is denoted a graph where \(X\) is the set of nodes and \(E\) is the set of arcs.

\textbf{Example 5 (Elementary subgraph).} Figure 7 reports an example of computation of an elementary subgraph.

Since \(E_0 = \emptyset\), \(\widehat{G}_0\) is a graph including nodes but no arcs (see Figure 7(c)). The clauses in \(\Pi\) whose body is fully contained in one strongly connected component of \(\widehat{G}_0\) are all the clauses with just one atom in the body, namely \(C = \{c_1, c_2, c_3, c_5\}\). Thus, \(E_1\) consists in set of arcs \(\{(b, c) \mid b \in B\}\) and the clauses \(c_1, c_2, c_3\) and \(c_5\) are removed from \(\Pi\).

The graph \(\widehat{G}_1\) is shown in Figure 7(d). The unique clause left in \(\Pi\) whose body is fully contained in a strongly connected is \(c_4\), then \(C = \{c_4\}\), \(E_2 = \{(a, d), (c, d)\}\), and \(c_4\) is removed from \(\Pi\).

Figure 7(e) reports the graph \(\widehat{G}_2\). Since the body of \(c_6\) does not belong to a strongly connected component of \(\widehat{G}_2\), the procedure stops returning \(\widehat{G}_2\) as the elementary subgraph \(\widehat{G}(\Pi, X)\) of \(X\) for \(\Pi\).

Next, we recall the main result stated in [14], concerning elementary
\[ \Pi = \{ c_1 \equiv b \leftarrow a \\
\hspace{1em} c_2 \equiv a \leftarrow c \\
\hspace{1em} c_3 \equiv c \leftarrow a \\
\hspace{1em} c_4 \equiv d \leftarrow a, c \\
\hspace{1em} c_5 \equiv a \leftarrow d \\
\hspace{1em} c_6 \equiv e \leftarrow a, b \} \]

\[ X = \{ a, b, c, d, e \} \]

(a) The program \( \Pi \) and the set of atoms \( X \)

(b) The dependency graph of \( \Pi \)

(c) \( \hat{G}_0 = \langle X, E_0 \rangle \)

(d) \( \hat{G}_1 = \langle X, E_1 \rangle \)

(e) \( \hat{G}_2 = \langle X, E_2 \rangle \)

Figure 7: An example of elementary subgraph construction

subgraphs.

**Proposition 4.1 (Theorem 2 of [14]).** For any non-disjunctive theory \( \Pi \) and any set \( X \) of atoms occurring in \( \Pi \), \( X \) is an elementary set for \( \Pi \) if and only if the elementary subgraph of \( X \) for \( \Pi \) is strongly connected.

Moreover, as also proved in [14], the following proposition holds.

**Proposition 4.2 ([14]).** The procedure \texttt{compute_elementary_subgraph} terminates in polynomial time.

Indeed, at each iteration, a non-empty set of clauses (for otherwise the algorithm would stop) is taken into account and each clause of the theory is considered at most once. Thus, the number of iterations is at most linear w.r.t. the number of clauses of the theory. As for the cost of a single iteration, we have first to find a clause \( c \) such that the subgraph of \( \hat{G}_i \) induced by the body of \( c \) is strongly connected. This task can be clearly accomplished in polynomial time. Second, we have to build the new graph \( \hat{G}_{i+1} \) by adding new arcs to \( \hat{G}_i \), a task that can be accomplished also in polynomial time.

Let us now resort to the function \texttt{find_super-elementary_set} (see Figure 8).
Function find_super-elementary_set

Input: An HEF CNF theory Π such that \( \text{atom}(\Pi) = \text{atom}(\Pi^{nd}) \)
Output: A super-elementary set in \( \text{atom}(\Pi) \) for Π

1: \( X_0 = \text{atom}(\Pi) \)
2: \( i = 0 \)
3: \( \text{stop} = \text{false} \)
4: repeat
5: compute the elementary subgraph \( \mathcal{G}_i = \tilde{G}(X_i, \Pi^{nd}) \)
6: if \( \mathcal{G}_i \) is strongly connected then
7: \( \text{stop} = \text{true} \)
8: else
9: select a connected component \( C \) in the last level of \( \mathcal{G}_i \)
10: \( X_{i+1} = X_i \setminus C \)
11: \( i = i + 1 \)
12: until \( \text{stop} \)
13: return \( X_i \)

Figure 8: The find_super-elementary_set function.

Assume that the set \( \text{atom}(\Pi^{nd}) \) is not elementary for \( \Pi^{nd} \). Then the elementary graph \( \tilde{G}(\text{atom}(\Pi^{nd}), \Pi^{nd}) \) is not strongly connected (by Proposition 4.1). Therefore, the graph \( \tilde{G}(\text{atom}(\Pi^{nd}), \Pi^{nd}) \) can be partitioned into the sets \( C_1, \ldots, C_k \) of its maximal strongly connected components and organized into \( m \geq 1 \) levels, such that if there is an arc from a node in a connected component \( C_i \) to a node in a connected component \( C_j \), then the level of \( C_i \) precedes the level of \( C_j \). Isolated connected components possibly occurring in the graph are assumed to be part of the last level \( m \).

The following Theorem states the correctness of the function find_super-elementary_set.

Theorem 4.3. Let \( \Pi \) be a disjunctive HEF CNF theory such that \( \text{atom}(\Pi) = \text{atom}(\Pi^{nd}) \). Then, the function find_super-elementary_set(\( \Pi \)) computes a super-elementary set for \( \Pi \).

In order to prove the theorem the following result is useful.
Claim 1. For each \( i \geq 0 \), \( X_i \) is a non-empty non-outbound set in \( \text{atom}(\Pi^{nd}) \) for \( \Pi^{nd} \).

Proof of Claim 1. The proof is by induction.

We start by noticing that the non-empty set \( X_0 = \text{atom}(\Pi) = \text{atom}(\Pi^{nd}) \) is non-outbound in \( \text{atom}(\Pi^{nd}) \) for \( \Pi^{nd} \), by definition of outbound set. Moreover, consider the graph \( G_0 \), namely, the elementary graph associated with the set of atoms \( X_0 = \text{atom}(\Pi^{nd}) \) and the theory \( \Pi^{nd} \). Note that this graph is not strongly connected since \( \Pi \) is, by hypothesis, a disjunctive HE F theory such that \( \text{atom}(\Pi) = \text{atom}(\Pi^{nd}) \) and then \( \text{atom}(\Pi) \) is not elementary for \( \Pi^{nd} \).

Now, for \( i > 1 \), assume by induction hypothesis that \( X_i \) is non-outbound in \( \text{atom}(\Pi^{nd}) \) for \( \Pi^{nd} \) and that the graph \( G_i \) is not strongly connected (for otherwise the algorithm would have stopped). Consider a strongly connected component \( C \) of the last level of \( G_i \) and the set \( X_{i+1} = X_i \setminus C \). Note that \( X_{i+1} \) is non-empty since, by induction hypothesis, \( G_i \) is not strongly connected and note, moreover, that also \( C \) is not empty.

Next, it is shown that \( X_{i+1} \) is non-outbound in \( \text{atom}(\Pi^{nd}) \) for \( \Pi^{nd} \) or, in other words, that there does not exist any clause \( c \equiv h \leftarrow B \) such that \( B \subseteq X_0 \setminus X_{i+1} \) and \( h \in X_{i+1} \), (note that this means that, without loss of generality, we can limit ourselves to focus only on such single-head clauses where the atom in the head belongs to \( X_{i+1} \) and the body is in \( X_0 \setminus X_{i+1} \)).

So, assume by contradiction that one such a clause \( c \) indeed exists. Two cases are possible.

\( B \cap C = \emptyset \). In this case, \( B \subseteq X_0 \setminus X_i \). Therefore, \( c \) cannot exist in \( \Pi^{nd} \) since \( X_i \) is non-outbound in \( \text{atom}(\Pi^{nd}) \) for \( \Pi^{nd} \).

\( B \cap C \neq \emptyset \). Also in this case, the clause \( c \) cannot exist in \( \Pi^{nd} \). Indeed, the clause \( c_{X_i} \) obtained by projecting \( c \) on \( X_i \), has its body contained in \( C \) and its head in \( X_{i+1} \). Since this clause would belong to \( \Pi^{nd}_{X_i} \), then it would be the case that \( C \) would not belong to the last level of \( G_i \).

This concludes the proof of Claim 1.

Using Claim 1, the statement of Theorem 4.3 easily follows, as shown next.
Proof of Theorem 4.3. When the algorithm find\_super-elementary\_set stops, the last set $X_i$ is elementary for $\Pi^\text{nd}$, since the graph $G_i$ is strongly connected. By Claim 1, the set $X_i$ is also non-empty and non-outbound in atom($\Pi$) for $\Pi^\text{nd}$. To conclude, by Lemma 4.4, the set $X_i$ is super-elementary for $\Pi$. $\Box$

Example 1 (Minimal models of positive CNF theories – continued). Consider again the theory $\Pi$ reported in Figure 1 and the function find\_super-elementary\_set($\Pi$). The connected components of the elementary subgraph $G_0$ are shown in Figure 9 on the left. Thus, there is a unique connected component in the last level of $G_0$, which is $C = \{j, h\}$, and $X_1$ is set to $\{a, b, c, d, e, f, g, i\}$. Notice that the connected components of the elementary subgraph $G_1$, which are reported in Figure 9 on the right, are not a subset of those of $G_0$. The set $X_2$ is then $\{a, b, c, d\}$ and it is the super-elementary set returned by the function. $\Box$

The next theorem accounts for the complexity of the function find\_super-elementary\_set.

Theorem 4.4. For any CNF theory $\Pi$, the function find\_super-elementary\_set($\Pi$) terminates in polynomial time in the size of the theory.

Proof. Initially $X_0$ contains all the atoms occurring in the input theory. Then, at each iteration, either the graph $G_i$ is strongly connected and then the function stops and returns $X_i$, or $G_i$ is not strongly connected and in
such case some node is removed from $X_i$. In the latter case, there exist at least two strongly connected components in graph $G_i$. $C$ is one of them and is such that $X_i \supset C \supset \emptyset$. Thus, $X_{i+1}$ is always non-empty. As for the convergence, it is ensured by the fact that the singleton set is strongly connected by definition.

The number of iterations executed by the $\text{find\_super\_elementary\_set}$ function is at most equal to the number of atoms occurring in the input theory, since in the worst case $C$ consists in just one single atom at each iteration. The statement follows by the fact that each iteration can be accomplished in polynomial time. □

4.7. Defining an eliminating operator for HEF CNF theories

In previous sections, we showed that:

- given a HEF CNF theory $\Pi$ and a model $\mathcal{M}$ for $\Pi$, a super-elementary set for $\Sigma_{\mathcal{M}}(\Pi)$ is erasable in $\mathcal{M}$ for $\Pi$ (Theorem 4.1 in Section 4.4),

- given a HEF CNF theory $\Pi$, if the set of atoms of $\Pi$ coincides with that of its non-disjunctive fragment, a super-elementary set always exists (see Theorem 4.2 in Section 4.5) and can be indeed computed in polynomial time (see Theorems 4.3 and 4.4 in Section 4.6).

Putting things together, given an HEF CNF, it can be concluded that if $\text{atom}(\Sigma_{\mathcal{M}}(\Pi))$ coincides with $\text{atom}(\Sigma_{\mathcal{M}}^{\text{nd}}(\Pi))$, an erasable set $\mathcal{E}$ in $\mathcal{M}$ for $\Pi$ can be obtained by computing a super-elementary set for $\Sigma_{\mathcal{M}}(\Pi)$ (as detailed in Section 4.6).

In order to build a suitable eliminating operator for HEF theories, it remains to prove that if $\text{atom}(\Sigma_{\mathcal{M}}(\Pi))$ is a strict superset of $\text{atom}(\Sigma_{\mathcal{M}}^{\text{nd}}(\Pi))$ then it is always possible to find in polynomial time a model $\mathcal{M}' \subseteq \mathcal{M}$ such that $\text{atom}(\Sigma_{\mathcal{M}'}(\Pi))$ coincides with $\text{atom}(\Sigma_{\mathcal{M}'}^{\text{nd}}(\Pi))$.

Proposition 4.3. Given a CNF theory $\Pi$ and a model $\mathcal{M}$ for $\Pi$, a model $\mathcal{M}' \subseteq \mathcal{M}$ such that $\text{atom}(\Sigma_{\mathcal{M}'}(\Pi))$ coincides with $\text{atom}(\Sigma_{\mathcal{M}'}^{\text{nd}}(\Pi))$ can be computed in polynomial time.

The above result, which is valid not only for HEF CNF theories but, rather, for any CNF theory, will make the strategy above depicted generally applicable to any HEF CNF theory.

In order to prove Proposition 4.3, the intermediate results stated in technical Lemmas 4.6 and 4.7 are preliminarily needed.
Lemma 4.6. Let $\Pi$ be a CNF theory, let $\mathcal{M}$ be a model of $\Pi$ and let $St$ be the steady set of $\mathcal{M}$ for $\Pi$. Then, $E = (\mathcal{M} \setminus St) \setminus \text{atom}(\Sigma_M(\Pi))$ is erasable in $\mathcal{M}$ for $\Pi$.

Lemma 4.7. Let $\Pi$ be a CNF theory and let $\mathcal{M}$ be a model of $\Pi$. If there exists an atom $a$ such that $a \in \mathcal{M} \setminus \text{atom}(\Pi^\text{nd}_{\mathcal{M}})$ then $\{a\}$ is erasable in $\mathcal{M}$ for $\Pi$.

We are now in the position of proving Proposition 4.3.

Proof of Proposition 4.3. Let $St$ denote the steady set of $\mathcal{M}$ in $\Pi$. The two following transformations (see points 1-2) can be recursively applied, till the condition $\text{atom}(\Sigma_M(\Pi)) = \text{atom}(\Sigma_M^\text{nd}(\Pi))$ is met:

1. If $\mathcal{M} \setminus St$ is a strict superset of $\text{atom}(\Sigma_M(\Pi))$ then by Lemma 4.6 the atoms in the non-empty set $E = (\mathcal{M} \setminus St) \setminus \text{atom}(\Sigma_M(\Pi))$ are erasable in $\mathcal{M}$ for $\Pi$ and $\mathcal{M}'$ can be set to $\mathcal{M} \setminus E$;
2. Else if $\mathcal{M} \setminus St$ is a strict superset of $\text{atom}(\Sigma_M^\text{nd}(\Pi))$ then any atom $a \in (\mathcal{M} \setminus St) \setminus \text{atom}(\Sigma_M^\text{nd}(\Pi))$ is such that $\{a\}$ is erasable in $\mathcal{M} \setminus St$ for $\Sigma_M(\Pi)$ (by Lemma 4.7, since $\mathcal{M} \setminus St$ is a model for $\Sigma_M^\text{nd}(\Pi)$) and also erasable in $\mathcal{M}$ for $\Pi$ (by Lemma 4.2); hence, let $E = \{a\}$ an arbitrarily chosen atom in $(\mathcal{M} \setminus St) \setminus \text{atom}(\Sigma_M^\text{nd}(\Pi))$, then $\mathcal{M}'$ can be set to $\mathcal{M} \setminus E$;
3. Else it is the case that $\text{atom}(\Sigma_M(\Pi)) = \text{atom}(\Sigma_M^\text{nd}(\Pi))$.

The whole process can be completed polynomial time. \hfill \Box

Before describing the $\xi_{HEF}$ eliminating operator, the following technical result is needed.

Lemma 4.8. Let $\Pi$ be a CNF theory, let $\mathcal{M}$ be a model of $\Pi$, and let $St$ be the steady set of $\mathcal{M}$ for $\Pi$. If the theory $\Sigma_M(\Pi)$ is non-disjunctive, then $\emptyset$ is its minimal model.

Figure 10 shows a realization of the $\xi_{HEF}$ eliminating operator. The following theorem asserts the most relevant result of this section, that is, that a minimal model for an HEF CNF theory can be indeed computed in polynomial time.

Theorem 4.5. Let $\Pi$ be a HEF CNF theory and $\mathcal{M}$ be a model of $\Pi$. Then, $\text{GEA}_{\xi_{HEF}}(\Pi, \mathcal{M})$ computes, in polynomial time, a minimal model of $\Pi$ contained in $\mathcal{M}$.
Function $\xi_{\text{HEF}}$

**Input:** An HEF CNF theory $\Pi$ and a model $\mathcal{M}$ of $\Pi$

**Output:** An erasable set $\mathcal{E}$ in $\mathcal{M}$ for $\Pi$

1. $\mathcal{E}' = \emptyset$
2. repeat
3. $\mathcal{M} = \mathcal{M} \setminus \mathcal{E}'$
4. Compute the steady set $St$ of $\mathcal{M}$ for $\Pi$
5. $\Delta\mathcal{E} = \emptyset$
6. if ($\mathcal{M} \setminus St$) $\supset$ $\text{atom}(\Sigma_{\mathcal{M}}(\Pi))$ then
7. $\Delta\mathcal{E} = (\mathcal{M} \setminus St) \setminus \text{atom}(\Sigma_{\mathcal{M}}(\Pi))$
8. else if ($\mathcal{M} \setminus St$) $\supset$ $\text{atom}(\Sigma_{\mathcal{M}}^{nd}(\Pi))$ then
9. Select an atom $a$ in ($\mathcal{M} \setminus St$) $\setminus$ $\text{atom}(\Sigma_{\mathcal{M}}^{nd}(\Pi))$
10. $\Delta\mathcal{E} = \{a\}$
11. $\mathcal{E}' = \mathcal{E}' \cup \Delta\mathcal{E}$
12. until $\Delta\mathcal{E} = \emptyset$
13. if $\Sigma_{\mathcal{M}}(\Pi)$ is non-disjunctive then
14. $\mathcal{E}'' = \mathcal{M} \setminus St$
15. else
16. $\mathcal{E}'' = \text{find super-elementary set}(\Sigma_{\mathcal{M}}(\Pi))$
17. $\mathcal{E} = \mathcal{E}' \cup \mathcal{E}''$
18. return $\mathcal{E}$

Figure 10: The $\xi_{\text{HEF}}$ eliminating operator.
Proof. Because of Theorem 3.1 and Proposition 3.2, in order to prove the statement, it is sufficient to show that (i) $\xi_{HEF}$ returns an erasable set, if such a set exists, and an empty one otherwise (namely that $\xi_{HEF}$ is, in fact, an eliminating operator) and that (ii) $\xi_{HEF}$ runs in polynomial time.

Let us consider first point (i). Lines 2-12 in Figure 10 serve the purpose of finding a subset $M' \subseteq M$ such that $atom(\Sigma_{M'}(\Pi))$ coincides with $atom(\Sigma_{nd}^{nd}(\Pi))$ according to the strategy depicted in the proof of Proposition 4.3 Notice that, the set $E' = M \setminus M'$ is an erasable set.

We can now assume that $atom(\Sigma_{M}(\Pi))$ coincides with $atom(\Sigma_{nd}^{nd}(\Pi))$. If the theory $\Sigma_{M}(\Pi)$ is non-disjunctive, then by Lemmata 4.8 and 4.2, the set $E'' = M \setminus St$ is an erasable set in $M$ for $\Pi$ and the operator returns $E' \cup E''$ (see lines 13-14).

Otherwise, $\Sigma_{M}(\Pi)$ is disjunctive. Then, by Theorem 4.2 there exists a non-empty set of atoms $E'' \subseteq (M \setminus St)$ such that $E''$ is super-elementary for $\Sigma_{M}(\Pi)$ and, by Theorem 4.1, the set $E''$ is erasable in $M \setminus St$ for $\Sigma_{M}(\Pi)$. In this case, the operator returns the erasable set $E' \cup E''$.

As far as point (ii) is concerned, this is a direct consequence of Theorem 4.4 and this concludes the proof. □

As for minimal model checking, we have the following result.

**Theorem 4.6.** Given a positive HEF CNF theory $\Pi$ and a set of atoms $N \subseteq atom(\Pi)$, checking if $N$ is a minimal model of $\Pi$ can be accomplished in polynomial time.

**Proof.** The proof follows immediately from Theorem 3.1 and Theorem 4.5. □

**Example 1 (Minimal models of positive CNF theories – continued).** Let us consider the execution of $GEA_{\xi_{HEF}}(\Pi, M)$, where $\Pi$ is the HEF theory $\Pi$ reported in Figure 1 and $M = atom(\Pi)$. During the first main iteration, the eliminating operator $\xi_{HEF}$ returns the super-elementary set $\{a, b, c, d\}$, as shown in the example of Section 4.6 and $M$ is set to $\{e, f, g, h, i, j\}$. As for the next iteration, the output of $\xi_{HEF}$ is $\{e, f, g, i\}$ and $M$ becomes $\{j, h\}$. Since now $M$ coincides with the steady set of $\Pi_M = \{j \leftarrow; h \leftarrow; h \leftarrow; j; j \leftarrow; h\}$, the algorithm stops returning $\{j, h\}$ as a minimal model of $\Pi$. □
5. Beyond HEF

In the previous section, we have shown that $\text{GEA}(\xi_{\text{HEF}})$ computes a minimal model of a positive HEF CNF theory in polynomial time. Unfortunately, however, deciding if a given theory is head-elementary-free is a coNP-complete problem [13]6. In other words, while a minimal model for an input HEF CNF theory $\Pi$ can be indeed computed in polynomial time, checking whether $\Pi$ is actually HEF is intractable.

Thus, it is sensible to study the behavior of $\text{GEA}(\xi_{\text{HEF}})$ as applied to a general CNF theory, which is the subject of this section. Recall that, by Theorem 4.4, the $\text{find\_super\_elementary\_set}$ function runs in polynomial time independently of the kind of theory it is applied to.

Next, we will show that there are non-HEF theories for which $\text{GEA}(\xi_{\text{HEF}})$ successfully returns a minimal model and others for which $\text{GEA}(\xi_{\text{HEF}})$ ends failing to construct a correct output7 (recall that, on the basis of the results of the previous section, GEA always returns a correct solution on HEF theories). The following example should help in clarifying this latter issue.

**Example 6 (Behavior on non-HEF theories).** Consider the following two theories:

$$\begin{align*}
P &= \{ & a & \leftarrow \\ & b, c & \leftarrow a \\ & c & \leftarrow b \\ & b & \leftarrow c \} \\
Q &= \{ & a & \leftarrow \\ & b, c, d & \leftarrow a \\ & c & \leftarrow b \\ & b & \leftarrow c \\ & d & \leftarrow c \}
\end{align*}$$

Both theories are not HEF. Indeed, the set $\{b, c\}$ is a disjunctive elementary set, both for $P$ and for $Q$. However, while $\text{GEA}(\xi_{\text{HEF}})$ does not return a minimal model of $P$, it does correctly compute a minimal model of $Q$.

To show that, consider first running $\text{GEA}(\xi_{\text{HEF}})$ on $P$. Let $M$ be $\{a, b, c\}$ (this is the model obtained by taking the union of all the heads). At line 3 of GEA, $St$ is set to $\{a\}$, which is not a model of $P$ and, then, $\xi_{\text{HEF}}$ is invoked. In particular, the $\text{find\_super\_elementary\_set}$ function is executed on the theory $P' = \{b, c \leftarrow; c \leftarrow b; b \leftarrow c\}$. In the execution of the function,

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6We note that the reduction therein presented is still valid for positive HEF CNFs.
7That the algorithm is not always returning the correct answer is indeed the expected behavior due to the intractability of the general problem and since $\xi_{\text{HEF}}$ runs in polynomial time (under the assumption that $P \neq \text{coNP}$).

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$X_0$ is $\{b, c\}$. Since the elementary graph associated with $P^{\text{nd}}_{X_0}$ is strongly connected, the function stops and returns $\{b, c\}$. As a consequence, the set $\mathcal{E}$ is $\{b, c\}$ and the new set $\mathcal{M}$ is $\{a\}$. It turns out that, since the set $\mathcal{M}$ is not a model for $P$ any longer, GEA($\xi_{\text{HEF}}$) is not able to return a minimal model of $P$. Specifically, at the second iteration of GEA($\xi_{\text{HEF}}$), $\xi_{\text{HEF}}$ is invoked on the theory $P$ and on the set $\mathcal{M} = \{a\}$. The steady set $St$ computed at line 1 in Figure 10 is equal to $\{a\}$ (since $\Pi^{\text{nd}}_{\mathcal{M} \leftarrow}$ is the theory $\{a \leftarrow\}$) and the theory $\Pi$ is empty. The set of atoms $\mathcal{R} = \mathcal{M} \setminus St$ computed at line 3 and $\Pi_{\mathcal{R}}$ are empty too. Then the condition at line 9 is true and $\mathcal{R} = \emptyset$ is returned. Thus, at the second iteration of GEA($\xi_{\text{HEF}}$), $St$ is set to $\mathcal{M} = \{a\}$ and returned. Concluding, GEA($\xi_{\text{HEF}}$) on $P$ ends returning $\{a\}$ which is not a minimal model of $\Pi$.

Consider, now, the theory $Q$. Let $\mathcal{M}$ be $\{a, b, c, d\}$, which is the model obtained by taking the union of all the heads. The set $St$ is set to $\{a\}$ which is not a model of $Q$ and, then, $\xi_{\text{HEF}}$ is invoked. In particular, the $\text{find}\_\text{super-elementary}\_\text{set}$ function is executed on the theory $Q' = \{b, c, d \leftarrow; c \leftarrow b; b \leftarrow c; d \leftarrow c\}$. In the execution of the function, $X_0$ is $\{b, c, d\}$. The elementary graph associated with $Q'_{X_0}$ is not strongly connected; actually, it includes the strongly connected component $C_1$ containing $b$ and $c$ and the strongly connected component $C_2$ containing $d$. Moreover, there is an edge from $C_1$ to $C_2$ but not vice versa. Then, $C_2$ belongs to the last level of the graph, and $X_1$ is set to $X_0 \setminus \{d\} = \{b, c\}$. The elementary subgraph associated with $Q'_{X_1}$ is strongly connected; therefore the function stops and returns the set $X_1 = \{b, c\}$. As a consequence the set $\mathcal{E}$ is $\{b, c\}$ and, now, the set $\mathcal{M}$ is $\{a, d\}$ and the theory $Q'_{\mathcal{M} \leftarrow}$ is $\{a \leftarrow; d \leftarrow a\}$ whose minimal model is $St = \{a, d\}$. Since $St$ is a model of $Q$, GEA($\xi_{\text{HEF}}$) stops by returning $St$ as the result, which is indeed a minimal model of $Q$. \hfill $\square$

Summarizing, the algorithm GEA($\xi_{\text{HEF}}$) always runs in polynomial time and correctly returns a minimal model of HEF CNF theories, but its correctness on non-HEF theories is seemingly unpredictable: the rest of this section is devoted to devise a suitable variant of GEA able to tell about the correctness of the result it returns. In order to proceed, some further definitions and results are needed.

**Definition 11 (Fallible eliminating operator).** Let $\mathcal{M}$ be a model of a positive CNF theory $\Pi$. A fallible eliminating operator $\xi_f$ is a polynomial time computable function that returns a subset of $\mathcal{M} \setminus St$, with $St$ the steady...
set of $\Pi$, with the constraint that if $\xi_f$ returns the empty set, then $\mathcal{M}$ is a minimal model of $\Pi$.

**Proposition 5.1.** Let $\Pi$ be a positive CNF theory and $\xi_f$ be a fallible eliminating operator. Checking if the set returned by running $\text{GEA}(\xi_f)$ over $\Pi$ is a minimal model for $\Pi$ is attainable in polynomial time.

**Proof.** By Theorem 3.1, we know that if the set returned by the operator employed in GEA is an erasable set then the algorithm returns a minimal model. Thus, it is sufficient to check if, at each iteration, $\mathcal{E}$ is an erasable set, namely it must be checked if $\mathcal{M} \setminus \mathcal{E}$ is a model for $\Pi$. Since this latter operation can be done in polynomial time, the statement follows. $\square$

As a consequence of our previous results, we are now able to present the modified GEA, called the *Incomplete Generalized Elimination Algorithm* (IGEA, for short), which is reported in Figure 11.

The following Theorem describes the correctness of IGEA as well as its computational complexity.

**Theorem 5.1.** For any fallible eliminating operator $\xi_f$, $\text{IGEA}(\xi_f)$ always terminates (with either success or failure) in polynomial time, returning a model of the input theory. If it succeeds, then the returned model is a minimal one.

**Proof.** If the *if* branch at line 12 is never taken, then $\mathcal{M}$ is, at each iteration, a model for $\Pi$ and $\mathcal{E}$ is an erasable set. In this case, the fallible eliminating operator $\xi_f$ is indeed an eliminating operator, whereby $\text{IGEA}(\xi_f)$ behaves as $\text{GEA}(\xi_f)$ does. This immediately implies that if $\text{IGEA}(\xi_f)$ does not report a “failure” then it returns a minimal model for $\Pi$.

As far as the time complexity of the algorithm is concerned, following the same line of reasoning as before, if the *if* branch at line 12 is never taken, then $\text{IGEA}(\xi_f)$ requires exactly the same number of iterations as $\text{GEA}(\xi_f)$. Conversely, if the *if* branch at line 12 is taken, the algorithm ends. Thus, $\text{IGEA}(\xi_f)$ does not require more iterations than $\text{GEA}(\xi_f)$. As for the cost of a single iteration, $\text{IGEA}(\xi_f)$ has only one operation more than $\text{GEA}(\xi_f)$, consisting in checking if $\mathcal{M} \setminus \mathcal{E}$ is a model for $\Pi$ (line 12). Since this operation is the same as that accomplished at line 4, the asymptotic temporal cost of the algorithm is not affected. Thus, the cost of $\text{IGEA}(\xi_f)$ is exactly that reported in Proposition 3.2 for the GEA, where $C_{\xi_f}$ is polynomial, by definition of fallible eliminating operator. $\square$
Algorithm 2: Incomplete Generalized Elimination Algorithm, IGEA(\(\xi_f\))

**Input:** A positive CNF theory \(\Pi\)

**Output:** A minimal model for \(\Pi\) and an indication of a “success” or a model for \(\Pi\) and an indication of a “failure”

1. \(\mathcal{M} = \{ h \mid H \leftarrow B \in \Pi \text{ and } h \in H \} \)  // \(\mathcal{M}\) is a (possibly non-minimal) model of \(\Pi\)
2. \(\text{stop} = \text{false}\)
3. repeat
4. compute the minimal model \(S_t\) of \(\Pi^{nd}_{\mathcal{M} \leftarrow}\)
5. if \(S_t\) is a model of \(\Pi\) then
6. \(\text{stop} = \text{true}\)
7. else
8. \(\mathcal{E} = \xi_f(\Pi, \mathcal{M})\)
9. if \((\mathcal{E} = \emptyset)\) then
10. \(S_t = \mathcal{M}\)
11. \(\text{stop} = \text{true}\)
12. else
13. if \((\mathcal{M} \setminus \mathcal{E} \text{ is not a model of } \Pi)\) then
14. \(\text{return } \mathcal{M}\text{ and “Failure”}\)
15. \(\mathcal{M} = \mathcal{M} \setminus \mathcal{E}\)
16. until \(\text{stop}\)
17. return \(S_t\) and “Success”

Figure 11: Incomplete Generalized Elimination Algorithm, IGEA(\(\xi_f\))

To conclude this section, we show that \(\xi_{HEF}\) can, in fact, be safely adopted as fallible eliminating operator in IGEA. The following preliminary proposition is useful.

**Proposition 5.2.** Let \(\Pi\) be CNF theory, \(\mathcal{M}\) be a model for \(\Pi\) and \(S_t\) be the steady set of \(\mathcal{M}\) for \(\Pi\). If \(S_t\) is not a model for \(\Pi\) then, on input \(\Pi\) and \(\mathcal{M}\), the operator \(\xi_{HEF}\) returns a non-empty set.

**Proof.** Consider the theory \((\sigma_{\mathcal{M}(\Pi)}),_{\mathcal{M} \setminus S_t}\). Since \(S_t\) is not a model for \(\Pi\), there are rules in \(\Pi\) which are not true in \(S_t\) but are true in \(\mathcal{M}\), thus that \((\sigma_{\mathcal{M}(\Pi)}),_{\mathcal{M} \setminus S_t}\) is not empty. Therefore, it is enough to prove that, whenever the function \(\text{find\_super\_elementary\_set}\) is run over a non-empty theory, it returns a non-empty set.
Consider the function \texttt{find\_super\_elementary\_set} reported in Figure 8. First of all, note that if $\Pi$ is a non-empty theory and $X$ is a non-empty set of atoms occurring in $\Pi$, then the elementary graph $G(\Pi_X \cap d)$ is non-empty as well. Thus, in the function, if $X_i$ is non-empty then $G_i$ is non-empty.

The set $X_0$ at line 1 is non-empty since the function is invoked over a non-empty theory. By induction, assuming that $X_i$ is non-empty, we prove that $X_{i+1}$ is non-empty as well.

Consider the $(i + 1)$-th iteration. Two cases are possible:

(i) $G_i$ is strongly connected and the function ends returning the set $X_i$, which is non-empty by the induction hypothesis.

(ii) $G_i$ is not strongly connected and includes at least two connected components. In such a case, only the atoms of one of the connected components are removed from $X_i$, call it $C$. Then $X_{i+1} = X_i \setminus C$ is not empty.

\[\square\]

**Proposition 5.3.** The operator $\xi_{\text{HEF}}$ is a fallible eliminating operator.

**Proof.** Let $\Pi$ be a general positive CNF theory, $M$ be a model for $\Pi$ and $St$ be the steady set of $M$ for $\Pi$. The proposition is an immediate consequence of the following facts: (i) $\xi_{\text{HEF}}(\Pi, M)$ runs in polynomial time (by Theorem 4.4); (ii) the set returned by the operator $\xi_{\text{HEF}}(\Pi, M)$ is a subset of $M \setminus St$; (iii) the set returned by $\xi_{\text{HEF}}(\Pi, M)$ is not empty (by Proposition 5.2). \[\square\]

Concluding, since $\xi_{\text{HEF}}$ is a fallible eliminating operator, for any CNF theory $\Pi$, IGEA($\xi_{\text{HEF}}$) runs in polynomial time returning a model and, on HEF theories, we are guaranteed that the returned model is minimal. Thus, the successful termination of IGEA($\xi_{\text{HEF}}$) can be also seen as a necessary condition for a theory to be HEF (but it is not a sufficient condition, unless coNP collapses onto P).

The next theorem, finally, summarizes the results of this section.

**Theorem 5.2.** The algorithm IGEA($\xi_{\text{HEF}}$) terminates in polynomial time for any input positive CNF theory. Moreover, if the input theory $\Pi$ is HEF, then IGEA($\xi_{\text{HEF}}$) succeeds returning a minimal model for $\Pi$; otherwise either the algorithm declares success returning a minimal model for $\Pi$ or the algorithm declares failure returning a model for $\Pi$.

**Proof.** The proof immediately follows from Theorem 5.1 and Proposition 5.3. \[\square\]
6. Conclusions

Tasks related with computing with minimal models are relevant to several AI applications.

The focus of this paper has been devising efficient algorithms to deal with minimal models of CNF theories. Particularly, three problems have been mainly considered, that are, minimal model checking, minimal model finding and model minimization. All these problems prove themselves to be intractable for general CNF theories, while it was known that they become tractable for the class of head-cycle-free theories [5] and, in fact, to the best of our knowledge, positive HCF theories form the largest class of CNFs for which polynomial time algorithms solving minimal model finding and minimal model checking are known so far. The research presented here follows the same research target as that of [5] and the main contribution of this work has been that of designing a polynomial time algorithm for computing a minimal model for (a superset of) the class of positive HEF (Head Elementary-Set Free) CNF theories, a strict superset of the class of HCF theories, whose definition naturally stems for the analogous one given in the context of logic programming in [14]. This contribution thus broadens the tractability frontier associated with minimal model computation problems.

In more detail, we have introduced the Generalized Elimination Algorithm (GEA), that computes a minimal model of any positive CNF, whose complexity depends on the complexity of the specific eliminating operator it invokes. Therefore, in order to attain tractability, a specific eliminating operator has been defined which allows for the algorithm to compute in polynomial time a minimal model for a class of CNF that strictly includes HEF theories.

However, it is unfortunately already known that recognizing HEF theories is “per sé” an intractable problem, which may apparently limit the applicability range of our algorithmic schema. In order to overcome such a problem, an “incomplete” variant of the GEA (called IGEA) is proposed: the resulting schema, once instantiated with an appropriate elimination operator, always constructs a model of the input CNF, which is guaranteed to be minimal at least if the input theory is HEF. We note that this latter algorithm is able to “declare” if the returned model is indeed minimal or not.

The research work presented here can be continued along some interesting direction. As a major research direction, since the IGEA is capable to deal also with theories that are not HEF, it would be relevant to define, via a syn-
tactic specification, as those pinpointing HCF and HEF theories, a superset HEF theories coinciding with those on which the IGEA stops returning a success. While it is not at all clear if this can be reasonably attained, we might consider it enough to get close (from below) to this class of theories. Very related to the above line of research, there is the assessment of the practical occurrence of theories having the HEF property or the property of guaranteeing success to the IGEA and also the assessment of the success rate of the IGEA on generic CNF theories. Moreover, enhancing stable models and answer set engines for logic programs with the IGEA appears a potentially fruitful line of investigation.

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Appendix

Proof of Lemma 4.2. In order to prove the theorem, we have to show that there is no clause in \(\Pi\) which is false in \(M\setminus E\) if and only if there is no clause in \(\Sigma_{M}(\Pi)\) which is false in \((M\setminus St)\setminus E\).

The proof is organized in two parts, both proved by contradiction.

\((\Rightarrow)\) First of all note that, for each clause \(c \equiv H \leftarrow B\) in \(\Sigma_{M}(\Pi)\) there is, by definition, a clause \(c' \equiv H' \leftarrow B'\) in \(\sigma_{M}(\Pi)\) such that \(H'_{M\setminus St} = H \neq \emptyset\) and \(B'_{M\setminus St} = B\). Moreover, note that (by construction of \(\sigma_{M}(\Pi)\)) \(c'\) is also in \(\Pi\) and that \(B' \subseteq M\).

For the sake of contradiction, assume that \(c\) is true in \(M\setminus St\) and false in \((M\setminus St)\setminus E\). We aim at proving that if such a rule exists then there exists at least one rule (actually \(c'\)) which is false in \(M\setminus E\).

If \(c\) is false in \((M\setminus St)\setminus E\), then \(B\) is contained in \((M\setminus St)\setminus E\) and \(E\) contains \(H\). Note that since \(B \cap E = \emptyset\) and since \(B = B' \cap (M\setminus St)\), it follows that also \(B' \cap E = \emptyset\) (and, then, \(B' \subseteq M\setminus E\)).

But, since \(H' \cap St = \emptyset\) (by construction of \(\sigma_{M}(\Pi)\)) and since \(H = H' \cap (M\setminus St)\), if \(H \subseteq E\) then also \(H' \subseteq E\).

So, we have proved that \(B' \subseteq (M\setminus E)\) and \(H' \subseteq E\); it follows that \(c'\) is false in \(M\setminus E\), which concludes the first part of the proof.

\((\Leftarrow)\) For the sake of contradiction, assume that there exists a clause \(c' \equiv H \leftarrow B\) in \(\Pi\) which is true in \(M\) and false in \(M\setminus E\). Thus, \(B \subseteq (M\setminus E)\) and \(\emptyset \subset H \subseteq E\) implying that \(H \cap St = \emptyset\) since, by definition of steady set
Consider, now, the clause \( c \equiv H \leftarrow B \leftarrow \sigma \). Since \( H \) is contained in \( \mathcal{E} \) and \( \mathcal{E} \cap \mathcal{S} = \emptyset \), it follows that \( H \) is not empty, then \( c \) is by definition in \( \Sigma_M(\Pi) \). Conversely, since \( B \subseteq (\mathcal{M} \setminus \mathcal{E}) \) then \( \mathcal{M} \setminus \mathcal{S} \cap \mathcal{E} = \emptyset \).

Thus, \( c \) is a clause of \( \Sigma_M(\Pi) \) whose head is included in \( \mathcal{E} \) meaning that \( c \) is false in \( (\mathcal{M} \setminus \mathcal{S}) \setminus \mathcal{E} \). But, since \( \mathcal{M} \setminus \mathcal{S} \) is a model of \( \Sigma_M(\Pi) \), \( c \) is true in \( \mathcal{M} \setminus \mathcal{S} \). This concludes the proof since it contradicts that \( E \) is an erasable set in \( \mathcal{M} \setminus \mathcal{S} \) for \( \Sigma_M(\Pi) \).

\( \square \)

**Proof of Lemma 4.1.** For the sake of contradiction, assume that such a clause \( c \equiv h \leftarrow \) belongs to \( \Sigma_M(\Pi) \). Clearly, in this case there exists at least one clause \( c' \) in \( \sigma_M(\Pi) \subseteq \Pi \) such that \( c = c' \leftarrow \sigma \). By definition of \( \sigma_M(\Pi) \), it is the case that \( h \) is the only atom in \( \mathcal{M} \) (and also in \( \mathcal{M} \setminus \mathcal{S} \)) occurring in the head of the clause \( c' \) and, hence, it can be concluded that \( c' \leftarrow \sigma \) belongs also to the theory \( \Pi^{nd}_{\mathcal{M} \leftarrow} \). The body of \( c' \) is contained in \( \mathcal{M} \) by definition of \( \sigma_M(\Pi) \). Since the body of \( c \) is empty, there are two possibilities:

1. The body of \( c' \) is also empty: but in this case \( h \) should belong to \( \mathcal{S} \) and the clause \( h \leftarrow \) cannot be in \( \Sigma_M(\Pi) \), a contradiction;
2. The body of \( c' \) is contained in \( \mathcal{S} \): but this means that \( c' \leftarrow \sigma \) is a clause of \( \Pi^{nd}_{\mathcal{M} \leftarrow} \) which is false in \( \mathcal{S} \), which contradicts the fact that \( \mathcal{S} \) is the minimal model of \( \Pi^{nd}_{\mathcal{M} \leftarrow} \).

\( \square \)

**Proof of Lemma 4.3.** The proof is given by contraposition: assuming that \( \Pi'_X \) is not HEF it is derived that \( \Pi \) is not HEF. If \( \Pi'_X \) is not HEF then, by Proposition 4.1, there is a set of atoms \( E \subseteq X \) such that \( E \) is both a disjunctive and an elementary set for \( \Pi'_X \). Clear enough, if \( E \) is a disjunctive set for \( \Pi'_X \) then it is a disjunctive set for \( \Pi \) as well. Moreover, if \( E \) is elementary for \( \Pi'_X \) then, for each proper subset \( O \subseteq E \), there is a clause \( c_X \equiv H_X \leftarrow B_X \) in \( \Pi'_X \) such that \( H_X \cap O \neq \emptyset \), \( H_X \cap (E \setminus O) = \emptyset \), \( B_X \cap O = \emptyset \) and \( B_X \cap (E \setminus O) \neq \emptyset \). By definition of \( \Pi'_X \), \( c_X \in \Pi'_X \) implies that there is a clause \( c \equiv H \leftarrow B \in \Pi \) such that \( H_X = H \cap X \) and \( B_X = B \cap X \). Thus, since \( O \subseteq E \subseteq X \), it follows that \( H \cap O = H_X \cap O \neq \emptyset \), \( H \cap (E \setminus O) = H_X \cap (E \setminus O) = \emptyset \), \( B \cap O = B_X \cap O = \emptyset \) and \( B \cap (E \setminus O) = B_X \cap (E \setminus O) \neq \emptyset \). Therefore, \( O \) is outbound in \( E \) also for \( \Pi \). As a consequence, \( E \) is elementary. 
for Π and, since E is also a disjunctive set for Π, it is the case that Π is not HEF.

Proof of Lemma 4.4. Before stating the Lemma, it is needed to recall a result given in [13] asserting that if \( O \subseteq \text{atom}(\Pi^\text{nd}) \) is elementary for \( \Pi^\text{nd} \) then it is elementary also for Π.

Claim 2 (Rephrased from [13]). Let Π be a CNF theory and \( \Pi' \subseteq \Pi \) any CNF consisting of a subset of the clauses of Π. If \( O \subseteq \text{atom}(\Pi') \) is an elementary set for Π', then O is an elementary set for Π as well.

By hypothesis, O is non-outbound in \( \text{atom}(\Pi^\text{nd}) \) for \( \Pi^\text{nd} \). In order to complete the proof, it is enough to prove that, since Π is HEF, O is non-outbound in \( \text{atom}(\Pi) \) for Π, which is shown next.

By contradiction, assume that O is outbound in \( \text{atom}(\Pi) \) for Π. Since \( \text{atom}(\Pi) \) is the set of all the atoms appearing in Π, then there exists a clause \( H \leftarrow B \) in Π such that \( B \subseteq \text{atom}(\Pi) \setminus O \) and \( H \subseteq O \). Since O is non-outbound in \( \text{atom}(\Pi^\text{nd}) \) for \( \Pi^\text{nd} \), then the clause \( H \leftarrow B \) is not in \( \Pi^\text{nd} \) and it holds that \(|H| \geq 2\). As a consequence, O is an elementary set for Π and there is a clause \( H \leftarrow B \) such that \(|H \cap O| \geq 2\). That is to say, Π is not HEF, a contradiction.

Proof of Lemma 4.5. Let O be a minimal non-outbound set in \( \text{atom}(\Pi) \) for Π. Hence, there is no single head clause \( h \leftarrow B \) in Π such that \( B \subseteq \text{atom}(\Pi) \setminus O \) and \( h \in O \).

Consider, now, any non-empty proper subset \( O' \) of O. By hypothesis of minimality of O, the set \( O' \) is outbound in \( \text{atom}(\Pi) \) for Π and, hence, there exists a (single head) clause \( h \leftarrow B \) in Π such that \( B \subseteq \text{atom}(\Pi) \setminus O' \) and \( h \in O' \). Moreover, notice that it is the case that the body of such a clause has non-empty intersection with the set O, for otherwise the set O would be outbound.

Thus, it holds that \( B \cap O \neq \emptyset \) and, since \( B \cap O' = \emptyset \), it also holds that \( B \cap (O \setminus O') \neq \emptyset \).

It can be therefore concluded that \( h \leftarrow B \) is a clause such that \( B \cap O' = \emptyset \), \( B \cap (O \setminus O') \neq \emptyset \) and \( h \in O' \) holds. The existence of such a clause implies that \( O' \) is outbound in O for Π.

Since any proper subset of O is outbound in O for Π, O is also elementary for Π and, hence, super-elementary for Π. □
Proof of Lemma 4.6. The proof is an immediate consequence of Lemma 4.2. □

Proof of Lemma 4.7. Let $a$ be an atom occurring in $\mathcal{M}$ but not in $atom(\Pi_{M\leftarrow}^{ anecdotes})$. Since $a$ does not occur in $atom(\Pi_{M\leftarrow}^{ anecdotes})$, two cases have to be taken into account, that are: (i) $a$ occurs only in the body of some disjunctive rule in $\Pi_{M\leftarrow}$, and (ii) $a$ occurs in the head of some disjunctive rule in $\Pi_{M\leftarrow}$.

In the first case, the set $\mathcal{M}\setminus\{a\}$ is a model of $\Pi$, since the only effect of removing $a$ from $\mathcal{M}$ is to falsify the body of some rule of $\Pi$.

In the second case, the head of no rule $H\leftarrow B$ in $\Pi$ can be falsified since if $\{a\}\subset H$ holds, then $|\mathcal{M}\cap H|\geq2$ holds. Thus, $\mathcal{M}\setminus\{a\}$ is a model for $\Pi$. □

Proof of Lemma 4.8. The proof follows immediately from Lemma 4.1. □