Gauge Parameterization of the $n$-Field

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Received April 24, 2019; revised May 10, 2019; accepted May 10, 2019

Abstract—We propose a gauge parameterization of the three-dimensional $n$-field using an orthogonal $SO(3)$-matrix, which, in turn, is defined by a field taking values in the Lie algebra $so(3)$ (rotation-angle field). The rotation-angle field has an additional degree of freedom, which corresponds to the gauge degree of freedom of rotations around the $n$-field. As a result, we obtain a gauge model with local $SO(2) \simeq U(1)$ symmetry that does not contain a $U(1)$ gauge field.

DOI: 10.1134/S0081543819050122

1. INTRODUCTION

Gauge models are an essential part of modern mathematical physics. The gauge invariance of Yang–Mills models is achieved by introducing gauge fields which are components of the local connection form for the corresponding principal fiber bundle (see, e.g., [6]). It is these models that are usually called gauge models. In the present paper, a gauge model is understood in a wider sense: it is any field model that is invariant under some local transformation group whose parameters can depend in a sufficiently smooth way on a space–time point. In this sense, general relativity is also a gauge model, because the Hilbert–Einstein action is invariant with respect to general coordinate transformations parameterized by four arbitrary functions. In addition, the action depends only on the metric or vierbein, which are not gauge fields in the strict sense.

Thus, the models invariant under local transformations do not always contain gauge fields. In the present paper, we construct a new class of models with local $U(1) \simeq SO(2)$ invariance that does not include a gauge $U(1)$-field. This model arose in the geometric theory of defects [1–5]. Namely, some continuous media possess a spin structure in addition to elastic properties. For instance, the ferromagnetic properties of media are described by the distribution of magnetic moments. In the continuum approximation, such a medium is considered as a three-dimensional manifold $\mathbb{M} \approx \mathbb{R}^3$ with given unit vector field $n(x): \mathbb{M} \to \mathbb{S}^2$ that describes the spin distribution in the medium. If the unit vector field is sufficiently smooth, then we say that the spin structure has no defects and write down some Lagrangian for the $n$-field. However, in nature, the spin structure often contains defects, which are called disclinations. These are any discontinuities and other singularities of the $n$-field, whose supports can be located at points, on lines, or on surfaces. If there are few disclinations, then one can pose a problem for the $n$-field outside the defects with appropriate boundary conditions at the discontinuities of the $n$-field. This approach is applicable to a small number of separate disclinations. However, if there are many disclinations (which is the most common case for real media), the boundary conditions become so complicated that one cannot hope to solve the corresponding boundary value problems. In the limiting case of continuous distribution of disclinations, the $n$-field has discontinuities at every point, which means that it does not exist.

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at all. Therefore, the \( n \)-field is not suitable for describing media with disclinations, and we need a new formalism.

In order to describe single disclinations as well as their continuous distribution, the geometric theory of defects was proposed [1–5]. In this approach, the \( n \)-field is substituted by a new variable, an \( \mathfrak{so}(3) \)-connection, which is nonsingular for a continuous distribution of disclinations. For single disclinations, it may have singularities at points, on lines, or on surfaces. The new variable is introduced as follows. We fix some direction in space and parameterize the \( n \)-field by an orthogonal rotation matrix. In turn, the rotation matrix is parameterized by an element of the Lie algebra \( \mathfrak{so}(3) \); i.e., we have a rotation-angle field \( \omega(x) : \mathbb{M} \to \mathfrak{so}(3) \). If there are no disclinations, then the rotation-angle field \( \omega(x) \) is a smooth function and the partial derivatives \( \partial_\mu \omega \) exist. In the presence of disclinations, the partial derivatives may not exist and we introduce a new variable \( \partial_\mu \omega \mapsto \omega_\mu \), which is a 1-form with values in the Lie algebra \( \mathfrak{so}(3) \) and which is identified with the components of a local \( \mathfrak{so}(3) \)-connection form. In this case, disclinations exist if and only if the curvature tensor for the \( \mathfrak{so}(3) \)-connection is nonzero. On simply connected domains with zero curvature tensor, the \( \mathfrak{so}(3) \)-connection is a pure gauge and one can reconstruct the rotation-angle field \( \omega \) and the \( n \)-field.

In the other cases, the rotation-angle field and \( n \)-field do not exist, as it should be, for example, for a continuous distribution of disclinations.

The change of variables \( n(x) \mapsto \omega(x) \) is a necessary attribute of the geometric theory of defects and thus needs to be carefully analyzed. The problem is that this change of variables is not one-to-one: the \( n \)-field has two degrees of freedom because of the condition \( n^2 = 1 \), and the rotation-angle field \( \omega \) has three degrees of freedom. The additional degree of freedom corresponds to \( \mathfrak{so}(2) \)-rotations around the \( n \)-field and is a gauge one. This question is the subject of the present paper.

2. ANGLE PARAMETERIZATION OF THE \( n \)-FIELD

In the geometric theory of defects, a unit vector field \( n(x) : \mathbb{R}^3 \to \mathbb{S}^2 \), which describes, for example, the distribution of magnetic moments in ferromagnets, is parameterized by a rotation-angle field. To this end, we fix some direction in the Euclidean space \( \mathbb{R}^3 \) by choosing a unit vector \( n_0 \). Then the unit vector field is uniquely represented by an orthogonal matrix:

\[
n^i(x) := n_0^j S_j^i(\omega(x)), \quad S_j^i \in \mathbb{O}(3).
\]  

In turn, the matrix is uniquely parameterized by an element \( \omega(x) = (\omega^i(x)) \) (rotation-angle vector field) of the Lie algebra \( \mathfrak{so}(3) \). The rotation angle \( \omega \) parameterizes the proper rotation subgroup \( \mathbb{SO}(3) \subset \mathbb{O}(3) \) as follows. The direction of the vector \( \omega \) coincides with the rotation axis, and its length is equal to the rotation angle. For definiteness, we assume that the rotation angle varies in the range \( |\omega| \leq \pi \). Then the end of \( \omega \) runs over all points of the closed ball \( \mathbb{B}_\pi^3(0) \subset \mathbb{R}^3 \) of radius \( \pi \) centered at the origin. In addition, the diametrically opposite points of the bounding sphere \( \mathbb{S}_\pi^2(0) = \partial \mathbb{B}_\pi^3(0) \) must be identified, since they correspond to the same rotation.

The change of variables \( n(x) \mapsto \omega(x) \) is not a parameterization in the strict sense of the word. Each value of the rotation-angle field uniquely defines the \( n \)-field by formula (2.1), but the converse statement is not true for two reasons. First, the \( n \)-field does not define the orthogonal matrix \( S \) uniquely, because equality (2.1) does not change if it is multiplied (at every point \( x \)) by an arbitrary orthogonal matrix corresponding to rotations around the vector \( n(x) \) itself. Second, infinitely many elements of the Lie algebra \( \mathfrak{so}(3) \) are mapped to the same element of \( \mathbb{O}(3) \). It is the ambiguity of the “map” \( n(x) \mapsto \omega(x) \) that we study in the present section.

The full rotation group consists topologically of two connected components: \( \mathbb{O}(3) = \mathbb{S}_+ \cup \mathbb{S}_- \), where \( \mathbb{S}_+ \) and \( \mathbb{S}_- \) are the sets of orthogonal matrices with positive and negative determinants, respectively. The component \( \mathbb{S}_+ \) is the Lie subgroup of special orthogonal matrices, \( \mathbb{S}_+ \approx \mathbb{SO}(3) \subset \mathbb{O}(3) \) (the connected component of unity). The component \( \mathbb{S}_- \) is a coset of elements: \( \mathbb{S}_- = \mathbb{S}_+ g \), where
Let \( \psi \) be a smooth field related by the gauge transformation \( n \mapsto k_0 n_0 \psi \). This can be seen even by counting the number of independent variables: the vector field \( n \) has two independent components due to the condition \( n^2 = 1 \), while the rotation-angle field \( \omega \) has three independent components. We will see in what follows that the additional degree of freedom is a gauge one and can be eliminated by a gauge transformation.

The following statement is the main result of the paper.

**Theorem 2.1.** Let \( n_0 \) be a fixed unit vector and \( (k(x), \omega(x)) \) and \( (k'(x), \omega'(x)) \) be two sets of smooth fields related by the gauge transformation

\[
\sin \omega' = \frac{2 \sin(\omega/2) \sin v (\cos(\omega/2) \sin v \cos \alpha - \cos v \sin \alpha)}{1 - (\cos v \cos \alpha + \cos(\omega/2) \sin v \sin \alpha)^2}
\]
or
\[
\cos \omega' = \frac{1 - 2 \sin^2(\omega/2) \sin^2 \nu - (\cos \nu \cos \alpha + \cos(\omega/2) \sin \nu \sin \alpha)^2}{1 - (\cos \nu \cos \alpha + \cos(\omega/2) \sin \nu \sin \alpha)^2}
\] (2.8)
and
\[
k'^i = k^i \cos \alpha + \left(-k^i \cos \frac{\omega}{2} \cos \nu + n^i_0 \cos \frac{\omega}{2} + n^i_0 k^j \varepsilon^i_{jl} \sin \frac{\omega}{2}\right) \sin \frac{\omega}{2} \sin \nu,
\] (2.9)
where the angle \(\nu\) is defined by the equality
\[
\cos \nu := (n_0, k)
\] (2.10)
and \(\alpha(x) \in \mathbb{R}\) is an arbitrary smooth transformation parameter. Then formulas (2.1) and (2.4) define the same field \(n(x)\). Any two sets of fields \((k(x), \omega(x))\) and \((k'(x), \omega'(x))\) that define the same field \(n(x)\) are related by the transformation (2.7)–(2.9) for some parameter \(\alpha(x)\).

**Proof.** To prove the theorem, we need a rather cumbersome but elementary construction, which is illustrated in Fig. 1. Assume that the rotation takes the vector \(n_0\) to a vector \(n \neq n_0\). This rotation does not define the rotation matrix uniquely, because after the rotation the vector \(n\) can be additionally multiplied by a rotation matrix whose rotation axis \(k\) coincides with \(n\) (see (2.6)). This can be done independently at every point \(x \in \mathbb{M}\), which corresponds to the gauge \(\mathbb{U}(1)\) freedom \(\psi(x) \mapsto \psi(x) + \alpha(x)\), where \(\alpha(x)\) is the transformation parameter.

Let us perform calculations. The rotation angle \(\omega_0\) is minimal if and only if the rotation axis \(k_0\) is perpendicular to the plane passing through the vectors \(n_0\) and \(n\). In this case, the unit vector along the rotation axis is given by the vector product:
\[
k_0^i := \frac{\varepsilon^{ijl} n_0 j n_l}{\sin \omega_0}.
\] (2.11)

The corresponding rotation angle is defined by the equality
\[
\cos \omega_0 := (n_0, n) := n_0^i n^j \delta_{ij},
\] (2.12)
where the parentheses denote the ordinary scalar product in \(\mathbb{R}^3\).
The vector \( n \) can be obtained from \( n_0 \) if and only if the rotation is around an axis \( k \) lying in the plane passing through the vectors \( k_0 \) and \( n_0 + n \). Let \( m \) be the unit vector along the sum \( n_0 + n \). Then its components are

\[
m^i := \frac{n_0^i + n^i}{\sqrt{2(1 + (n_0, n))}} = \frac{n_0^i + n^i}{2\cos(\omega/2)},
\]

(2.13)

Any unit vector \( k \) in the plane \( k_0, m \) has the form

\[
k^i = k_0^i \cos \phi + m^i \sin \phi, \quad \phi \in (-\pi, \pi),
\]

(2.14)

for some angle \( \phi \) of rotation in the plane \( k_0, m \).

Assume that the vector \( n_0 \) is fixed and we are given values of the variables \( \omega_0, k_0 \), and \( \phi \) (three independent variables due to the conditions \( k_0^2 = 1 \) and \( (n_0, k_0) = 0 \)). Then we have to find \( \omega \) and \( k \) to define the rotation matrix \( S_\omega^k(\omega, k) \). The vector \( k \) is given by (2.14) with

\[
n^i = n_0^i S_j^i(\omega_0, k_0) = n_0^i \cos \omega_0 + n_0^i k_0^j \varepsilon_{ij} \sin \omega_0.
\]

(2.15)

To find the angle \( \omega \), we make the following construction. Consider the right triangle \( ABC \) lying in the plane perpendicular to the vector \( k \). Let \( v \) be the angle between the vectors \( n_0 \) and \( k \) (see (2.10)). Then

\[AB = \sin v = \sqrt{1 - (n_0, k)^2} = \sqrt{1 - (n_0, m)^2 \sin^2 \phi},\]

where we used equality (2.14). On the other hand, considering the right triangle \( OBC \), we see that

\[BC = \sin \frac{\omega_0}{2}\]

Consequently,

\[
\sin \frac{\omega}{2} = \frac{BC}{AB} = \frac{\sin(\omega_0/2)}{\sqrt{1 - \cos^2(\omega_0/2) \sin^2 \phi}},
\]

(2.16)

since \( (n_0, m) = \cos(\omega_0/2) \).

Straightforward calculations yield the formulas

\[
\sin \omega = \frac{\sin \omega_0 \cos \phi}{1 - \cos^2(\omega_0/2) \sin^2 \phi}, \quad \cos \omega = \frac{\cos \omega_0 - \cos^2(\omega_0/2) \sin^2 \phi}{1 - \cos^2(\omega_0/2) \sin^2 \phi}.
\]

(2.17)

In view of (2.15), we have

\[
k^i = k_0^i \cos \phi + \left( n_0^i \cos \frac{\omega_0}{2} + n_0^j k_0^j \varepsilon_{ij} \sin \frac{\omega_0}{2} \right) \sin \phi.
\]

(2.18)

Thus, formulas (2.17) and (2.18) express \( \omega \) and \( k \) in terms of \( \omega_0, k_0 \), and \( \phi \) for a fixed vector \( n_0 \). Moreover, the vector \( n \) does not depend on \( \phi \):

\[
n^i = n_0^i S_j^i(\omega_0, k_0) = n_0^i S_j^i(\omega, k).
\]

When we construct a model in the framework of the geometric theory of defects, we regard the components of the field \( \omega(x) \) (three variables) or, equivalently, \( \omega(x) \) and \( k(x) \) with the additional condition \( k^2 = 1 \) as independent variables. Thus, the number of variables in \( O(3) \) models increases from two to three, because the \( n \)-field does not depend on the field \( \phi(x) \), which was introduced in (2.14). This field is a gauge parameter of the \( U(1) \) transformation \( (\omega, k) \mapsto (\omega', k') \), because

\[
n^i(x) = n_0^i S_j^i(\omega, k) = n_0^i S_j^i(\omega', k'),
\]

(2.19)
where the primed fields $\omega'$ and $k'$ are built for the field $\phi'(x) := \phi(x) + \alpha(x)$ with the transformation parameter $\alpha$ for the same $\omega_0$ and $k_0$. To find an explicit form of the gauge transformations, which is rather cumbersome, we consider the sequence $(\omega, k) \mapsto (\omega_0, k_0, \phi) \mapsto (\omega', k')$ of one-to-one transformations. We find first the transformation $(\omega, k) \mapsto (\omega_0, k_0, \phi)$ for a given $\phi$. The rotation matrix (2.4) immediately implies an expression for the rotation angle $\omega_0$:

$$
\cos \omega_0 = (n, n_0) = 1 - 2 \sin^2 \frac{\omega}{2} \sin^2 v.
$$

Straightforward calculations yield an expression for the sine:

$$
\sin \omega_0 = 2 \sin \frac{\omega}{2} \sin v \sqrt{1 - \sin^2 v \sin^2 \frac{\omega}{2}}.
$$

In what follows, we will also need half-angle expressions

$$
\sin \frac{\omega}{2} = \sin \frac{\omega}{2} \sin v, \quad \cos \frac{\omega}{2} = \sqrt{1 - \sin^2 v \sin^2 \frac{\omega}{2}}.
$$

To find $k_0$, we first compute $\phi$. Multiplying (2.14) by $n_0$, we get

$$
\sin \phi = \frac{\cos v}{\sqrt{1 - \sin^2 v \sin^2 (\omega/2)}}, \quad \cos \phi = \frac{\cos (\omega/2) \sin v}{\sqrt{1 - \sin^2 v \sin^2 (\omega/2)}}.
$$

Now equality (2.14) implies an expression for the components of $k_0$:

$$
k_0^i = \frac{k^i \cos (\omega/2) - (n_0^i \cos (\omega/2) + n_0^j k^j \varepsilon_{ij} \sin (\omega/2)) \cos v}{\sin v \sqrt{1 - \sin^2 v \sin^2 (\omega/2)}}.
$$

To find an explicit expression for the gauge transformations with parameter $\alpha(x)$, we have to substitute the obtained expressions (2.19)–(2.23) into the formulas $\omega' = \omega'(\omega_0, k_0, \phi')$ and $k' = k'(\omega_0, k_0, \phi')$ and put $\phi' := \phi + \alpha$. Explicit formulas are presented in the statement of the theorem. □

Thus, we have obtained explicit expressions for the gauge $U(1)$ transformations $(\omega, k) \mapsto (\omega', k')$ with parameter $\alpha(x)$. To check the expressions found, we may assume that the initial state coincides with the state in which the rotation angle is minimal. Then it is easy to see that under the substitution $(\omega, k, \alpha) \mapsto (\omega_0, k_0, \phi)$ (in this case $v = \pi/2$) formulas (2.7)–(2.9) transform into (2.17) and (2.18).

For infinitesimal gauge transformations ($\alpha \ll 1$), formulas (2.7)–(2.9) in the linear approximation in $\alpha$ are simplified:

$$
\omega' = \omega + 2\alpha \sin \frac{\omega}{2} \cos \omega \cot v,
$$

$$
k'^i = k^i + \left( -k^i \cos \frac{\omega}{2} \cos v + n_0^i \cos \frac{\omega}{2} + n_0^j k^j \varepsilon_{ij} \sin \frac{\omega}{2} \right) \frac{\alpha}{\sin v}.
$$

Note that the gauge $U(1)$ transformations in the case under consideration are realized without introducing the gauge field. The $n$-field does not change under these transformations. Therefore, after the substitution $n \mapsto \omega$ according to (2.1), any expression for the Lagrangian of the $n$-field will be invariant under local transformations (2.7)–(2.9) with an arbitrary parameter $\alpha(x)$. This is not a very unusual situation. Indeed, we are used to the fact that gauge invariance arises after the introduction of gauge fields (components of a local connection form) in the Yang–Mills theory. However, there exist other models with local invariance. For example, general relativity is invariant under local transformations (general coordinate transformations), with the metric being not a gauge field.

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3. ACTION FOR THE HEISENBERG FERROMAGNET
IN THE GEOMETRIC THEORY OF DEFECTS

In the geometric theory of defects, the n-field is parameterized by the rotation-angle field \( \omega(x) \) and the unit vector field \( k(x) \), \( k^2 = 1 \), which defines the axis of rotation. In addition, one of the three degrees of freedom is gauge, as shown in the previous section. To construct the action, we first consider the simplest case when the rotation axis \( k_0 \) is perpendicular to the vector \( n_0 \), which defines the orientation of the target space in space–time (see Fig. 1). In this case, the n-field is defined by formula (2.15), and the fields \( \omega_0 \) and \( k_0 \) subject to the two conditions \( k_0^2 = 1 \) and \( (k_0, n_0) = 0 \) are independent variables (the gauge freedom is absent).

For definiteness, we choose the vector \( n_0 \) along the \( z \) axis, i.e., set \( n_0 = (0,0,1) \). Then the vector \( k_0 \) lies in the \( x,y \) plane and can be specified in spherical coordinates by one polar angle \( \Psi(x) \):

\[
k_0 = (\cos \Psi(x), \sin \Psi(x), 0).
\]

It follows from (2.15) that the components of the n-field are

\[
\begin{align*}
  n^1 &= S_3^1 = k_0^l \varepsilon_{l3}^1 = \sin \Psi \sin \omega_0, \\
  n^2 &= S_3^2 = k_0^l \varepsilon_{l3}^2 = -\cos \Psi \sin \omega_0, \\
  n^3 &= S_3^3 = \cos \omega_0.
\end{align*}
\]

In this case, the angular parameterization of the n-field is equivalent to the choice of spherical coordinates in the target space, which is given by the simple identification \( \omega_0 = \Theta \) and \( \Psi = \Phi + \pi/2 \). That is, the Lagrangian of the \( \mathbb{O}(3) \) model is

\[
L = \frac{1}{2} (\partial \omega_0^2 + \sin^2 \omega_0 \partial \Psi^2).
\]

Now we consider a gauge model of ferromagnet in general variables \( \omega, k \). The form of the rotation matrices (2.2), (2.4) implies that generally the n-field has the components

\[
\begin{align*}
  n^i(x) &= n_0^i S^{-1}_j(\omega(x), k(x)) = n_0^i \cos \omega + n_0^i k^l \varepsilon_{lj}^i \sin \omega + k^i(n_0, k)(1 - \cos \omega), \\
  n_i(x) &= S^{-1}_j(\omega(x), k(x)) n_{0j} = n_{0i} \cos \omega - k^l \varepsilon_{li}^j n_{0j} \sin \omega + k_i(n_0, k)(1 - \cos \omega).
\end{align*}
\]

Simple straightforward calculations show that the Lagrangian of the Heisenberg ferromagnet in the new variables has the form

\[
L = \frac{1}{2} (\partial \omega_0 n, \partial_0 n)
\]

\[
= \frac{1}{2} \left[ 1 - (n_0, k)^2 \right] (\partial \omega)^2 - 2 \left( n_{0i} \cos \omega + n_0^i k^l \varepsilon_{lj} \sin \omega \right) (n_0, k) \sin \frac{\omega}{2} \partial^\omega \partial_0 k^i
\]

\[
+ 2 \left[ (\delta_{ij} - n_{0i} n_{0j}) \cos^2 \frac{\omega}{2} - n_0^l k^j \varepsilon_{li} n_{0j} \sin \omega + (\delta_{ij}(n_0, k)^2 + n_{0i} n_{0j}) \sin^2 \frac{\omega}{2} \right]
\]

\[
\times \sin \frac{\omega}{2} \partial^\omega k^i \partial_0 k^j.
\]

This Lagrangian depends on the four fields \((\omega, k^i)\) with one condition \(k^2 = 1\). It is invariant with respect to the gauge \( U(1) \) transformations (2.7)–(2.9) with an arbitrary parameter \( \alpha(x) \). The field \( \phi \) from the previous section is transformed in a simple way:

\[
\phi \mapsto \phi' = \phi + \alpha.
\]

By construction, the Lagrangian (3.4) does not depend on \( \alpha \).
As far as we know, the Lagrangian (3.4) is a new kind of a gauge model. The abelian U(1) symmetry is realized nonlinearly, and gauge fields are absent.

Let us rewrite the Lagrangian in terms of the vector $\omega = (\omega^i)$ (an element of the algebra $\mathfrak{so}(3)$). The definition of $k$ implies the equalities

$$ k^i := \frac{\omega^i}{\omega}, \quad \partial_\alpha k^i = \frac{\partial_\alpha \omega^i}{\omega} - \frac{\omega^j \partial_\alpha \omega_j}{\omega}, \quad (\partial_\alpha k, \partial_\beta k) = \frac{(\partial_\alpha \omega, \partial_\beta \omega)}{\omega^2} - \frac{\partial \omega^2}{\omega^2}. $$

The substitution of the obtained expressions into the Lagrangian (3.4) yields a more complicated expression

$$ L = \frac{1}{2} \left[ 1 - \frac{\sin^2 \omega}{\omega^2} - \frac{(n_0, \omega)^2}{\omega^2} \left( 1 - \frac{\sin \omega}{\omega} \right)^2 \right] + \frac{(\partial_\alpha \omega, \partial_\beta \omega)}{2\omega^2} \left[ \sin^2 \omega + \frac{2(n_0, \omega)^2}{\omega^2} \sin^4 \frac{\omega}{2} \right] $$

$$ - \frac{2(n_0, \partial_\alpha \omega)^2}{\omega^2} \sin^2 \frac{\omega}{2} \cos \omega - \frac{\partial^2 \omega(n_0, \partial_\alpha \omega)(n_0, \omega)}{\omega^2} \left( \sin \omega - \frac{4}{\omega} \sin^2 \frac{\omega}{2} \cos \omega \right) $$

$$ - \frac{2\partial^2 \omega \omega^j n_0^{\alpha \beta \gamma} \epsilon_{ijl}}{\omega^3} \sin^2 \frac{\omega}{2} \left[ (n_0, \partial_\alpha \omega) \sin \omega + \partial_\alpha \omega(n_0, \omega) \left( 1 - \frac{\sin \omega}{\omega} \right) \right]. $$

The corresponding action depends only on the three fields $\omega^i$, which are varied without any restrictions.

4. CONCLUSIONS

We have constructed a new gauge parameterization of the Heisenberg ferromagnet $n$-field by the rotation-angle field $\omega$, which is needed in the geometric theory of defects. In this parameterization, we have three independent components of the rotation-angle field $\omega$ instead of the two independent components of the $n$-field. We have shown that this additional degree of freedom is gauge and corresponds to local rotations around the $n$-field. Explicit formulas of gauge transformations are found. In addition, any Lagrangian for the $n$-field leads to a gauge $U(1) \simeq SO(2)$ model in terms of the new variable $\omega$. As an example, we have considered a gauge parameterization of the Heisenberg ferromagnet. These models do not contain a U(1) gauge field but are invariant with respect to local U(1) transformations.

FUNDING

The work was supported in part by the Russian Government Program of Competitive Growth of Kazan Federal University (Russian Academic Excellence Project “5-100”).

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This article was submitted by the author simultaneously in Russian and English.