On the local zeta functions and the \( b \)-functions of certain hyperplane arrangements

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with an appendix by Willem Veys

Abstract
Conjectures of Igusa for \( p \)-adic local zeta functions and of Denef and Loeser for topological local zeta functions assert that (the real part of) the poles of these local zeta functions are roots of the Bernstein–Sato polynomials (that is, the \( b \)-functions). We prove these conjectures for certain hyperplane arrangements, including the case of reduced hyperplane arrangements in three-dimensional affine space.

1. Introduction

Let \( K \) be a \( p \)-adic field, that is, a finite extension of \( \mathbb{Q}_p \), and \( \mathcal{O}_K \) be the ring of integers of \( K \). We have the norm defined by
\[
|x|_K = q^{-v(x)} \text{ for } x \in K^*, \quad \text{where } v(x) \in \mathbb{Z}
\]
is the valuation (or the order) of \( x \in K \) and \( q \) is the cardinality of the residue field \( \mathcal{O}_K / \mathfrak{m}_K \) with \( \mathfrak{m}_K \) the maximal ideal.

For a nonconstant polynomial \( f \in K[x_1, \ldots, x_n] \), Igusa’s \( p \)-adic local zeta function (associated with the characteristic function of \( \mathcal{O}_K^n \subset K^n \) (see [19, 22])) is defined by the meromorphic continuation of the integral
\[
Z^p_f(s) = \int_{\mathcal{O}_K^n} |f(x)|_K^s \, dx \quad \text{for } \Re s > 0.
\]
Here \( dx \) denotes the Haar measure on the compact open subgroup \( \mathcal{O}_K^n \) of \( K^n \), which is the \( p \)-adic analog of the polydisk \( \Delta^n \) in \( \mathbb{C}^n \). Note that \( Z^p_f(s) \) is closely related to the Poincaré series associated with the numbers of solutions of \( f = 0 \) in \( (\mathcal{O}_K / \mathfrak{m}_K)^n \) for \( i > 0 \) in the case \( f \in \mathcal{O}_K[x_1, \ldots, x_n] \).

On the other hand, the Bernstein–Sato polynomial (that is, the \( b \)-function) of a polynomial \( f \in K[x_1, \ldots, x_n] \) is the monic polynomial \( b_f(s) \) of the least degree satisfying the relation
\[
b_f(s)f^s = Pf^{s+1} \quad \text{in } R_f[s]f^s \quad \text{for some } P \in \mathcal{D}_n[s],
\]
where \( R_f \) is the localization of \( R := K[x_1, \ldots, x_n] \) by \( f \), and \( \mathcal{D}_n \) is the Weyl algebra that is generated over \( K \) by \( x_1, \ldots, x_n \) and \( \partial/\partial x_1, \ldots, \partial/\partial x_n \). Here \( K \) may be any field of characteristic 0, and \( b_f(s) \) is invariant by extensions of the field \( K \); see Section 3.1. (There is a shift in the variable \( s \) by 1 if one uses the definition of the Bernstein polynomial in [3] since \( f^s \) is replaced by \( f^{s-1} \) there.) The local \( b \)-function \( b_{f,x}(s) \) is defined by replacing the Weyl algebra \( \mathcal{D}_n \) with \( \mathcal{D}_{X,x} \). Note that, for a homogeneous polynomial \( f \), we have \( b_f(s) = b_{f,0}(s) \).

A conjecture of Igusa [20] asserts the following.

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Conjecture (A)$^p$. The real part of any pole of the $p$-adic local zeta function $Z_f^p(s)$ is a root of $b_f(s)$.

Inspired by this conjecture, Denef and Loeser [12] defined the topological local zeta function $Z_{f,x}^{\text{top}}(s)$ (see (2.1)) for a nonconstant polynomial $f$ and $x \in f^{-1}(0)$ in the case $K = \mathbb{C}$, and conjectured the following.

Conjecture (A)$^{\text{top}}$. Any pole of the topological local zeta function $Z_{f,x}^{\text{top}}(s)$ is a root of $b_{f,x}(s)$.

There is a weaker version of the conjectures, due to Igusa, and Denef and Loeser, respectively, and called the monodromy conjecture, as follows.

Conjecture (B)$^p$. For any pole $\alpha$ of the $p$-adic local zeta function $Z_f(s)$, $e^{2\pi i \text{Re}(\alpha)}$ is an eigenvalue of the Milnor monodromy of $f_C$ at some $x \in f_C^{-1}(0) \subset \mathbb{C}^n$, choosing an embedding $K \hookrightarrow \mathbb{C}$, where $f_C$ is the image of $f$ in $\mathbb{C}[x_1, \ldots, x_n]$.

Conjecture (B)$^{\text{top}}$. For any pole $\alpha$ of the topological local zeta function $Z_{f,x}^{\text{top}}(s)$, $e^{2\pi i \alpha}$ is an eigenvalue of the Milnor monodromy of $f$ at $y \in f^{-1}(0)$ sufficiently near $x$.

In Conjecture (B)$^p$, it is enough to consider an embedding $K^f \hookrightarrow \mathbb{C}$, where $K^f$ is the subfield of $K$ generated by the coefficients of the linear factors of $f$. Originally, Conjectures (A)$^p$ and (B)$^p$ are stated for a polynomial $f \in F[x_1, \ldots, x_n]$ with $F$ a number field and $K$ the completion of $F$ at a prime of $F$ (except possibly for a finite number of primes). In the hyperplane arrangement case, however, this assumption does not seem to be essential since Conjecture (B)$^p$ is already proved by [6] and Conjecture (A)$^p$ is reduced to Conjecture (C).

By Conjecture (A), we mean Conjecture (A)$^p$ or Conjecture (B)$^{\text{top}}$ depending on whether $K$ is the $p$-adic or complex number field, and similarly for Conjecture (B). Note that the eigenvalues of the Milnor monodromies in Conjecture (B) can be defined in a purely algebraic way using the $V$-filtration of Kashiwara [24] and Malgrange [33] on the $D_n[s]$-module $R_f[s]f^*$, and the union of the eigenvalues of the Milnor monodromies for $x \in f_C^{-1}(0) \subset \mathbb{C}^n$ is independent of the choice of the embedding $K^f \hookrightarrow \mathbb{C}$; see Section 3.1. Moreover, we have the following.

Proposition 1.1. Let $K$ be a subfield of $\mathbb{C}$, and $f \in K[x_1, \ldots, x_n]$.

(i) A complex number $\lambda \in \mathbb{C}$ is an eigenvalue of the Milnor monodromy of $f_C$ at some $x \in f_C^{-1}(0) \subset \mathbb{C}^n$ if and only if there is a root $\alpha$ of $b_f(s)$ such that $\lambda = e^{-2\pi i \alpha}$.

(ii) If $K = \mathbb{C}$, then, for any $x \in f^{-1}(0)$, there is an open neighborhood $U$ of $x$ in classical topology such that, for any open neighborhood $U'$ of $x$ in $U$, the following two conditions are equivalent.

(a) The number $\lambda$ is an eigenvalue of the Milnor monodromy of $f$ at some $y \in f^{-1}(0) \cap U'$.

(b) There is a root $\alpha$ of $b_{f,x}(s)$ such that $\lambda = e^{-2\pi i \alpha}$.
This follows from [23, 33]. By Proposition 1, Conjecture (B) can be viewed as the modulo \( \mathbb{Z} \) version of Conjecture (A), and is weaker than the latter. It is known that Conjectures (A) and (B) are rather difficult to prove; see, for example, [1, 2, 11, 12, 21, 22, 25–28, 31, 32, 36, 41, 43–46]. For a generalization to the ideal case, see [18, 42] (using [5]).

In this paper, we prove Conjecture (A) for certain affine hyperplane arrangements \( D \) in \( K^n \). Let \( D_i \) be the irreducible components of \( D \), and \( m_i \) be the multiplicity of \( D \) along \( D_i \). Let \( f \) be a defining equation of \( D \). Set \( d := \deg D = \deg f = \sum_i m_i \). In [6], Conjecture (A) is reduced to the following.

**Conjecture (C).** Let \( D \) be an indecomposable essential central hyperplane arrangement in \( \mathbb{C}^n \) with degree \( d \). Then \( b_f(-n/d) = 0 \).

Here central and essential, respectively, mean that \( 0 \in D_i \) for any \( i \) and \( \dim \cap_i D_i = 0 \). We say that \( D \) is indecomposable if it is not a union of the pullbacks of arrangements by the two projections of some decomposition \( \mathbb{C}^n = \mathbb{C}^{n'} \times \mathbb{C}^{n''} \) as a vector space. Note that the proof of Conjecture (B) in [6] implies that \( -n/d - 1 \) is a root of \( b_f(s) \) in case \( -n/d \) is not, since the roots of \( b_f(s) \) are in \((-2, 0)\); see [37].

As for the reduction of Conjecture (A) to Conjecture (C), we have more precisely the following.

**Theorem 1.2** [6]. For an affine hyperplane arrangement \( D \) in \( K^n \), Conjecture (A) holds if Conjecture (C) for \( (D/L)_C \) holds for every dense edge \( L \) of \( D \).

Here an edge means an intersection of \( D_i \), and \( D/L \) denotes the arrangement in \( K^n/L \) defined by the \( D_i \) containing \( L \) and with the same multiplicity \( m_i \), where we may assume \( 0 \in L \) replaces the origin of \( K^n \) if necessary. We call an edge \( L \neq K^n \) dense if \( D/L \) is indecomposable. If \( K \) is a \( p \)-adic field, then \( (D/L)_C \) denotes the scalar extension of \( D/L \) defined by choosing an embedding \( K^f \hookrightarrow C \) where \( K^f \subset K \) is the smallest subfield such that \( f \) and all the \( D_i \) are defined over \( K^f \). We have \( (D/L)_C = D/L \) in the case \( K = \mathbb{C} \).

Theorem 1.2 is proved by using a resolution of singularities obtained by blowing up only the proper transforms of the dense edges in [39] (together with Igusa’s calculation of candidates for poles of the \( p \)-adic zeta functions [19] in the \( p \)-adic case; see also (2.3)). Because of this very special kind of resolution, all the obtained candidates for poles contribute at least to the monodromy eigenvalues, and Conjecture (B) is proved in [6] for all the candidates for poles using the calculation of the Milnor cohomology of hyperplane arrangements in [8] (or [13, Proposition 6.4.6]) together with a result of [39] on the relation between the indecomposability and nonvanishing of the Euler characteristic of the projective complement. This is contrary to most other cases where lots of cancelations of apparent poles occur; see [11, 31, 43–45] (and Remark 2.1). Recently, Veyes informed us that there are examples of hyperplane arrangements of degree \( d \) in \( \mathbb{C}^n \) such that \( -n/d \) is not a pole of \( Z_{f,0}^\top(s) \) in the case \( n = 3 \) with \( D \) nonreduced or \( n = 5 \) with \( D \) reduced; see Appendix. These examples imply a negative answer to Question (Q) in Section 2.2. There are no such examples if \( n = 2 \) or \( n = 3 \) with \( D \) reduced; see Propositions 2.3 and 2.6.

In this paper, we prove the following.

**Theorem 1.3.** Conjecture (C) holds in the following cases.

(i) The edge \( \{0\} \) is a good dense edge of \( D \);

(ii) The divisor \( D \) is reduced with \( n \leq 3 \);

iii)
(iii) The divisor \( D \) is reduced, \((n, d) = 1\), and \( D_d \) is generic relative to the other \( D_j \) \((j \neq d)\).

Here \( L \) is called a good dense edge if, for any dense edges \( L' \supset L \), we have
\[
n(L)/d(L) \leq n(L')/d(L'),
\]
where \( d(L) = \text{mult}_L D = \sum_{D \supset L} m_i \) and \( n(L) = \text{codim} L \). We say that \( D_d \) is generic relative to the other \( D_j \) \((j \neq d)\) if any nonzero intersection of \( D_j \) \((j \neq d)\) is not contained in \( D_d \); see [17, Example 4.5].

In case (i), Theorem 1.3 follows from Teitler’s refinement [40] of Mustaţă’s formula [34] for multiplier ideals using only dense edges, together with a well-known relation between the jumping coefficients and the roots of \( b_f(s) \); see [14]. In case (ii) or (iii), we use a generalization of Malgrange’s formula for the roots of \( b_f(s) \) in the isolated singularity case (see [37, 38]) reducing the assertion to a certain combinatorial problem which can be solved under condition (ii) or (iii), where we need a result from [17] in case (iii).

Combining Theorems 1.2 and 1.3, we get the following theorem.

**Theorem 1.4.** For an affine hyperplane arrangement \( D \) in \( K^n \), conjecture (A) holds if, for every dense edge \( L \) of \( D \), one of the three conditions in Theorem 1.3 is satisfied for \((D/L)_C\). In particular, Conjecture (A) holds in the following cases.

(i) \( D \) is of moderate type;
(ii) \( D \) is reduced with \( n \leq 3 \);
(iii) \( D \) is reduced with \( n = 4 \), and for each 0-dimensional dense edge \( L \) of \( D \), either condition (ii) or (iii) in Theorem 1.3 is satisfied for \((D/L)_C\).

Here \( D \) is called of moderate type if all the dense edges are good. Note that in the case (iii), condition (ii) in Theorem 1.3 is satisfied for \((D/L)_C\) with \( L \neq 0 \). It seems quite difficult to generalize the arguments in this paper to the nonreduced case, even for \( n = 3 \), or to the four-dimensional case, even for reduced \( D \).

In Section 2, we recall some facts from the theory of local zeta functions. In Section 3, we explain how to calculate the \( b \)-functions of homogeneous polynomials, and prove Theorem 1.3 in cases (i) and (iii). In Section 1.3, we prove Theorem 1.3 in case (ii). In the Appendix by Veys, we describe some examples related to Question (Q) in Section 2.2.

### 2. Local zeta functions

2.1. Let \( K \) be the complex or \( p \)-adic number field. Let \( X \) be a complex manifold of dimension \( n \) with \( f \) a holomorphic function on \( X \) if \( K = \mathbb{C} \), and \( X = K^n \) with \( f \in K[x_1, \ldots, x_n] \) if \( K \) is a \( p \)-adic field. Set \( D = f^{-1}(0) \). Let \( \sigma : (\hat{X}, E) \to (X, D) \) be an embedded resolution with \( E_j \) the irreducible components of \( E := \sigma^* D \). Set
\[
E^0_j = \bigcap_{i \in I_j} E_j \setminus \bigcup_{i \notin I_j} E_j, \quad m_j = \text{mult}_{E_j} \sigma^* D, \quad r_j = \text{mult}_{E_j} \text{det(Jac}(\sigma)).
\]

If \( K = \mathbb{C} \), the topological local zeta function for \( x \in D \) is defined by
\[
Z^\text{top}_{f,x}(s) = \sum_I \chi(E^0_j \cap \sigma^{-1}(x)) \prod_{j \in I} \frac{1}{m_j s + r_j + 1}, \quad (2.1)
\]
which is independent of the choice of the resolution (see [12]). So, we get candidates for poles
\[
\alpha_j := -\frac{r_j + 1}{m_j}, \quad (2.2)
\]
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Note that each \( \alpha_j \) is not necessarily a pole of \( Z_{f,x}^{\text{top}}(s) \) in general. It is not easy to determine exactly false poles since there are cancelations of poles in many cases; see \([11, 31, 43-45]\) (and Remark 2.1). In the hyperplane arrangement case, however, there is a special kind of resolution by Schechtman et al. \([39]\) so that Conjecture (B) is proved for the above candidates for poles although it is still unclear whether they are really poles.

The situation is similar in the \( p \)-adic case where Igusa’s calculation (see, for example, \([22, \text{Theorem 8.2.1}] \) or \([11]\)) implies that the poles of the local zeta function are among the complex numbers

\[
\alpha_{j,k} := -\frac{r_j + 1}{m_j} - \frac{2\pi \sqrt{-1} k}{m_j \log q} \quad (k \in \mathbb{Z}). \tag{2.3}
\]

**Remark 2.1.** It is known that there are remarkable cancelations of poles by the summation in the definition (2.1). So, it is not easy to eliminate false poles, although the curve case is rather well understood; see \([11, 31, 43-45]\). (For a relatively simple proof of Conjecture (B) for \( n = 2 \), see \([36]\).) It is also known that only a few of the roots of \( b_f(s) \) can be detected by the local zeta function.

**Proposition 2.2.** Let \( D \) be a hyperplane arrangement defined by a polynomial \( f \). Then the topological local zeta function \( Z_{f,x}^{\text{top}}(s) \) is a combinatorial invariant.

**Proof.** By the definition of \( Z_{f,x}^{\text{top}}(s) \) in (2.1), we may assume that \( D \) is central, \( x = 0 \). We have to calculate the Euler characteristic of each open stratum of a stratification of \( \sigma^{-1}(0) \) which is induced from the canonical stratification of a divisor with normal crossings. In this case, \( \sigma \) is obtained by taking first the blow-up \( X' \to X = \mathbb{C}^n \) along the origin of \( \mathbb{C}^n \), and then the base change of an embedded resolution of \( (\mathbb{P}^{n-1}, Z) \) by the projection \( X' \to \mathbb{P}^{n-1} \) where \( Z := \mathbb{P}(D) \). The Euler characteristic of an open stratum is calculated from those of the closed strata contained in the closure of the given stratum. So, the assertion follows by induction on \( n \) using \([9]\) together with the embedded resolution of \( (\mathbb{P}^{n-1}, Z) \) obtained by blowing up along the proper transforms of all the edges of \( Z \). Indeed, any intersection of the proper transforms of exceptional divisors can be written as a product of embedded resolutions for certain induced arrangements (see \([9]\) and \([7, \text{Proposition 2.7}]\)) in this case. (If we blow up only the proper transforms of dense edges, then we cannot apply an inductive argument since there is a problem as follows: for two dense edges \( L \subset L' \) of \( Z \subset \mathbb{P}^{n-1}, L \) is not necessarily a dense edge of the induced arrangement in \( L' \).) This concludes the proof of Proposition 2.2.

2.2. Analog of Conjecture (C). The following question arises naturally:

**Question (Q).** Let \( D \) be an indecomposable essential central hyperplane arrangement in \( \mathbb{C}^n \) defined by a polynomial \( f \) of degree \( d \). Then, is \(-n/d\) a pole of \( Z_{f,0}^{\text{top}}(s) \)?

We have a positive answer to this question if \( n = 2 \) or \( n = 3 \) and \( D \) is reduced; see Propositions 2.3 and 2.6. Recently, Veys informed us that the answer is negative in general, more precisely, if \( n = 3 \) with \( D \) nonreduced or \( n = 5 \) with \( D \) reduced; see the Appendix.

Assume, for example, \( n = 2 \) and \( d = \sum_{i=1}^{e} m_i \) with \( m_i = \text{mult}_{D_i} D \). Then

\[
Z_{f,0}^{\text{top}}(s) = \frac{1}{ds + 2} \left( 2 - e + \sum_{i=1}^{e} \frac{1}{m_is + 1} \right). \tag{2.4}
\]
This immediately follows from the definition of the zeta function since the embedded resolution is obtained by one blow-up and \( 2 - e \) is the Euler characteristic of the open stratum in \( \mathbb{P}^1 \). So, \(-2/d\) is a pole of order 2 if and only if \( 2m_i = d \) for some \( i \). If \(-2/d\) is not a pole of order 2, then the coefficient \( C_{-2/d} \) of \( 1/(ds + 2) \) is given by

\[
C_{-2/d} = 2 - e + \sum_{i=1}^{e} \frac{d}{d - 2m_i}.
\]

The next proposition gives a positive answer to Question (Q) in Section 2.2 for \( n = 2 \) where \( D \) may be nonreduced. This is a special case of Veys [45, Proposition 2.8].

**Proposition 2.3 (Veys [45]).** With the above notation, assume that \( n = 2 \). Then \(-2/d\) is a pole of \( Z^{\text{top}}_{f,0}(s) \). More precisely, if \(-2/d\) is not a pole of order 2, then \( C_{-2/d} > 0 \) if \( \{0\} \) is a good dense edge of \( D \), and \( C_{-2/d} < 0 \) otherwise.

**Proof.** For the proof, see [45, Proposition 2.8]. \( \square \)

**Proposition 2.4.** Assume that \( n = 3 \), and that \( D \) is reduced. Let \( \nu_m \) (\( m \geq 2 \)) be the number of points of \( Z := \mathbb{P}(D) \) with multiplicity \( m \). Then

\[
Z^{\text{top}}_{f,0}(s) = \frac{1}{ds + 3} \left( \chi(\mathbb{P}^2 \setminus Z) + \frac{\chi(Z \setminus Z^{\text{sing}})}{s + 1} + \sum_{m} \left( 2 - m + \frac{m}{s + 1} \right) \frac{\nu_m}{ms + 2} \right).
\]

In particular, \(-3/d\) is the only candidate for the pole of order 2 of \( Z^{\text{top}}_{f,0}(s) \), and is really a pole of order 2 if and only if \( d/3 \in \mathbb{Z} \) and \( \nu_{2d/3} \neq 0 \). If \(-3/d\) is not a pole of order 2, the coefficient \( C_{-3/d} \) of \( 1/(ds + 3) \) is given by

\[
C_{-3/d} = \frac{9}{d - 3} \left( d - 1 + \sum_{m \neq 2d/3} \frac{m(m - 1)}{2d - 3m} \nu_m \right).
\]

**Proof.** Since the embedded resolution of \( (\mathbb{P}^2, Z) \) is obtained by blowing up along the singular points of \( Z \), the first assertion follows from the definition of \( Z^{\text{top}}_{f,0}(s) \) using the partition of the summation over \( m \neq 2 \) and \( m = 2 \). This implies the second assertion since the coefficient of the double pole is given up to a nonzero multiplicative constant by

\[
2 - m + \frac{md}{d - 3} = \frac{2a - 1}{a - 1} \neq 0, \quad \text{where} \quad m = 2a \quad \text{with} \quad a := d/3 \in \mathbb{Z}.
\]

For the simple pole case, we have

\[
C_{-3/d} = \chi(\mathbb{P}^2 \setminus Z) + \frac{\chi(Z \setminus Z^{\text{sing}}) d}{d - 3} + \sum_{m \neq 2d/3} \left( 2 - m + \frac{md}{d - 3} \right) \frac{\nu_m d}{2d - 3m}.
\]

Here

\[
\chi(\mathbb{P}^2 \setminus Z) = 3 - 2d + \sum_{m} (m - 1) \nu_m,
\]
\[
\chi(Z \setminus Z^{\text{sing}}) = 2d - \sum_{m} m \nu_m.
\]

Indeed, the first equality is reduced to the calculation of \( \chi(Z) \), which is obtained by using the short exact sequence \( 0 \to \mathbb{Q}_Z \xrightarrow{\iota} \bigoplus_i \mathbb{Q}_{Z_i} \to \text{Coker} \iota \to 0 \), since the cokernel of \( \iota \) is supported on the singular points of \( Z \) and its rank at \( p \) is \( m_p - 1 \) where \( m_p \) is the multiplicity of \( Z \) at \( p \).
Substituting these, we see that $C_{-3/d}$ is given by

$$3 - 2d + \frac{2d^2}{d-3} + \sum_{m \neq 2d/3} \nu_m \left( m - 1 - \frac{md}{d-3} + \frac{(2-m)d}{2d-3m} + \frac{md^2}{(d-3)(2d-3m)} \right).$$

After some calculation, this is transformed to

$$\frac{9}{d-3} \left( d - 1 + \sum_{m \neq 2d/3} \frac{m(m-1)}{2d-3m} \nu_m \right).$$

(The detail is left to the reader.) This completes the proof of Proposition 2.4.

**Remark 2.5.** A strong form of the conjecture in [12] predicts that the multiplicity of each root of the zeta function is at most that of the $b$-function. In general, the multiplicity of the root $-1$ of the $b$-function of a reduced essential central hyperplane arrangement is $n$ (see [38, Theorem 1]), and this settles the problem for the root $-1$. However, the problem is rather difficult for the roots with multiplicity 2 even in the case $n = 3$. In this case, the only such root is $-3/d$ with $d/3 \in \mathbb{N}$ and $\nu_{2d/3} \neq 0$ by Proposition 2.4, but it is not easy to calculate the $b$-function. (Indeed, the multiplicity is calculated only in the case $\nu_m = 0$ for $m > 3$ in [38].)

Using Proposition 2.4, we get the following proposition, which gives a positive answer to Question (Q) in Section 2.2 if $D$ is reduced and $n = 3$. Vey has informed us that he had verified an analog of it for the (finer) motivic or Hodge zeta functions. (Here ‘finer’ means that the nonvanishing of the pole for these does not imply that for $Z_{f,d}^\text{top}(s)$ although the converse is true.)

**Proposition 2.6.** Let $D$ be an indecomposable essential central hyperplane arrangement of degree $d$ in $\mathbb{C}^3$. Assume that $D$ is reduced. Then $-3/d$ is a pole of $Z_{f,d}^\text{top}(s)$. More precisely, if $-3/d$ is not a pole of order 2, then the coefficient $C_{-3/d}$ of $1/(ds + 3)$ is strictly positive if $\{0\}$ is a good dense edge of $D$, that is, if $m < 2d/3$ for any $m$ with $\nu_m \neq 0$, and $C_{-3/d}$ is strictly negative otherwise.

**Proof.** We may assume $n = 3$ since the case $n = 2$ is trivial. We may further assume that $\{0\}$ is not a good dense edge of $D$, since the assertion in the good dense edge case easily follows from Proposition 2.4. We may thus assume $\nu_{m_0} \neq 0$ for some $m_0 := 2a + e$ with $0 < e < a := d/3$ where we do not assume $a \in \mathbb{Z}$. Since the sum of the multiplicities of any two singular points of $Z$ is at most $d + 1$, we have $\nu_{m_0} = 1$ and

$$m \leq 3a - m_0 + 1 = a - e + 1$$

for any $m \neq m_0$ with $\nu_m \neq 0$.

By Proposition 2.4, the assertion $C_{-3/d} < 0$ is equivalent to

$$\frac{(2a+e)(2a+e-1)}{e} > 3(3a-1) + \sum_{m \leq a-e+1} \frac{m(m-1)}{2a-m} \nu_m.$$ 

To show the last inequality, we may replace $\frac{m(m-1)}{2a-m}$ with $\frac{m(m-1)}{a+e-1}$, since $\frac{1}{a+e-1} \geq \frac{1}{2a-m}$ for $m \leq a-e+1$. Using $\binom{m}{2} = \sum_{m} \binom{m}{2} \nu_m$, the assertion is then reduced to

$$\frac{(2a+e)(2a+e-1)}{e} > 3(3a-1) + \frac{3a(3a-1) - (2a+e)(2a+e-1)}{a+e-1},$$

which is clear.
that is,

\[(2a + e)(2a + e - 1)(a + 2e - 1) - 3(3a - 1)(2a + e - 1)e = 2(2a + e - 1)(a - e)(a - e - 1) > 0.\]

Here \(a > e + 1\), that is, \(m_0 = 2a + e < d - 1\) since \(D\) is indecomposable. So, the assertion is proved.

\[\square\]

3. Calculation of \(b\)-functions

3.1. For a nonconstant polynomial \(f \in K[x_1, \ldots, x_n]\) with \(\text{char} K = 0\), the \(b\)-function \(b_f(s)\) can be defined to be the minimal polynomial of the action of \(s\) on

\[\mathcal{D}_n[s]^s / \mathcal{D}_n[s]^s.\]

This implies that \(b_f(s)\) is invariant by extensions of \(K\), and its roots are rational numbers since the last assertion holds for \(K = \mathbb{C}\) by Kashiwara [23].

Let \(i_f : X \hookrightarrow X \times A^n_K\) denote the graph embedding of \(f\), where \(X = A^n_K\). Then, via the global section functor, \(R_f[s]^s\) is identified with the direct image by \(i_f\) of the \(\mathcal{D}_X\)-module \(\mathcal{O}_X[1/f]\) in the notation of the introduction. This is compatible with extensions of \(K\). Moreover, the regular holonomic \(\mathcal{D}_X\)-module \(\text{Gr}_f\) of \((i_j)_!\mathcal{O}_X[1/f]\) corresponds via the global section functor to \(\text{Gr}_f(R_f[s]^s)\), and via the de Rham functor to the \(\lambda\)-eigenspace of Deligne’s nearby cycle sheaf \(\psi_f\mathcal{C}_X\) [10] with \(\lambda = e^{-2\pi i\alpha}\) if \(K = \mathbb{C}\); see [24, 33].

This implies that the union of the eigenvalues of the Milnor cohomology by the action of the monodromy \(T\) is determined by the monodromies around the irreducible components \(\mathcal{L}(k)\) of rank 1 on \(U\) such that

\[H^j(U, L^{(k)}) = H^j(F_f, \mathbb{C})_{\lambda},\]

where \(\lambda = \exp(-2\pi ik/d)\) with \(d = \deg f\). Here the local system \(L^{(k)}\) is defined by the decomposition

\[\pi_* C_{F_f} = \sum_{k=0}^{d-1} L^{(k)},\]

where \(\pi\) is the canonical projection from the affine Milnor fiber \(F_f := f^{-1}(1) \subset \mathbb{C}^n\) onto \(U \subset \mathbb{P}^{n-1}\), and the action of the monodromy is the multiplication by \(\exp(-2\pi ik/d)\) on \(L^{(k)}\) so that (3.2) holds; see [8] or [13, Proposition 6.4.6]. Since \(\mathbb{P}^{n-1}\) is simply connected, the local system \(L^{(k)}\) is determined by the monodromies around the irreducible components \(Z_j\) of \(Z\). These are given by the multiplication by \(\exp(2\pi im_jk/d)\), where \(m_j\) is the multiplicity of the divisor \(Z\) along \(Z_j\).
We can identify locally $\mathcal{L}^{(k)}$ with $\mathcal{O}_Y(*Z)h^{-k/d}$ as a $\mathcal{D}_Y$-module if $h$ defines locally $Z \subset Y := \mathbb{P}^{n-1}$. Then the pole-order filtration $P$ on $\mathcal{L}^{(k)}$ is defined by

$$P_i\mathcal{L}^{(k)} = \mathcal{O}_Y h^{-k/d - i} \quad \text{if } i \geq 0, \text{ and } 0 \text{ otherwise.} \quad (3.3)$$

Note that the residue of the logarithmic connection on $P_i\mathcal{L}^{(k)}$ at a general point of $Z_j$ is the multiplication by

$$\left(\frac{-k}{d} - i\right) m_j. \quad (3.4)$$

The filtration $P_i = P_{-i}$ on $H^{n-1}(U, L^{(k)}) = H^{n-1}(F_j, C)_\lambda$ is induced by $P_{n-1-i}$ on $\mathcal{L}^{(k)}$ using the de Rham complex

$$\mathcal{L}^{(k)} \to \mathcal{L}^{(k)} \otimes_{\mathcal{O}_Y} \Omega_Y^1 \to \cdots \to \mathcal{L}^{(k)} \otimes_{\mathcal{O}_Y} \Omega_Y^{n-1},$$

since the latter has the filtration $P_i = P_{-i}$ defined by

$$P_{n-i}\mathcal{L}^{(k)} \to P_{n-1-i}\mathcal{L}^{(k)} \otimes_{\mathcal{O}_Y} \Omega_Y^1 \to \cdots \to P_{n-1-i}\mathcal{L}^{(k)} \otimes_{\mathcal{O}_Y} \Omega_Y^{n-1}.$$

We have also the Hodge filtration $F$ on $\mathcal{L}^{(k)}$ such that

$$F_i\mathcal{L}^{(k)} \subset P_i\mathcal{L}^{(k)},$$

and the Hodge filtration $F$ on $H^{n-1}(U, L^{(k)}) = H^{n-1}(F_j, C)_\lambda$ is induced by the above formula with $P$ replaced by $F$.

3.3. Calculation of the cohomology of $L^{(k)}$. From now on, assume that $D = f^{-1}(0)$ is a central hyperplane arrangement in $\mathbb{C}^n$. Let $D_i$ ($i = 1, \ldots, e$) be the irreducible components of $D$ with multiplicity $m_i$. Then $Z = \mathbb{P}(D) \subset \mathbb{P}^{n-1}$ and $Z_i = \mathbb{P}(D_i)$. Let $D^{\text{nnn}}$ denote the smallest subset of $D$ such that $D \setminus D^{\text{nnn}}$ is a divisor with normal crossings. Set $Z^{\text{nnn}} = \mathbb{P}(D^{\text{nnn}}) \subset \mathbb{P}^{n-1}$.

Note that $d = \deg f = \sum_{i=1}^e m_i$.

For $k \in \{0, \ldots, d-1\}$ and $I \subset \{1, \ldots, e-1\}$ with $|I| = k - 1$, define

$$\alpha_I^k = \begin{cases} -m_i k/d & \text{if } i \notin I \cup \{e\}, \\ 1 - m_i k/d & \text{if } i \in I \cup \{e\}. \end{cases}$$

$$\alpha_I^L = \sum_{D_i \supset L} \alpha_I^k.$$

$$\Sigma^I = \{p \in Z^{\text{nnn}} \setminus Z_e \mid \alpha_p^I = 0\},$$

where $L$ is an edge of $D$, and we set $\alpha_p^I := \alpha_p^L$ if $\mathbb{P}(L) = \{p\}$. (See Remark 4.4(iii) for another way of defining the $\alpha_p^I$.) Here, it should be noted that, in order to apply the theory in [15] (and also in [39]), we must have a regular singular connection on a trivial line bundle, that is, the following condition should be satisfied:

$$\sum_{i=1}^e \alpha_I^k = 0. \quad (3.6)$$

This is satisfied in this case since $d = \sum_{i=1}^e m_i$. Note also that $\alpha_e^I$ is used in an essential way for (3.8) (that is, the condition of [39]) although it does not appear in the definition of the connection on the affine space $\mathbb{C}^{n-1}$, which is given below.

For $i \in \{1, \ldots, e-1\}$, let $e_i = dg_i/g_i$ with $g_i$ a linear function defining $Z_i \setminus Z_e$ in $\mathbb{P}^{n-1} \setminus Z_e \cong \mathbb{C}^{n-1}$. Set

$$\omega_I := \sum_{i=1}^{e-1} \alpha_I^k e_i.$$

It defines a connection $\nabla^{\omega_I}$ on $\mathcal{O}_U$ (where $U = \mathbb{P}^{n-1} \setminus Z$) such that

$$\nabla^{\omega_I} u = du + u\omega_I \quad \text{for } u \in \mathcal{O}_U.$$
The corresponding local system is isomorphic to $L^{(k)}$ by comparing local monodromies as remarked in Section 3.2. Consider the de Rham cohomology $H^{*}_{\text{DR}}(U, (\mathcal{O}_U, \nabla^i))$, which is calculated by the complex of rational forms $(\Omega^{*}_U(U), \nabla^i)$ since $U$ is affine. Set
\[
\mathcal{A}^p = \sum_{i_1 < \ldots < i_p} \mathbb{C}e_{i_1} \wedge \ldots \wedge e_{i_p}.
\]
Then, we have a natural inclusion of complexes
\[
i^j_I : (\mathcal{A}^{*, \omega_1} \Lambda) \hookrightarrow (\Omega^{*}_U(U), \nabla^i),
\]
where the source is called the Aomoto complex. Note that we have, for $a \in \mathcal{A}^0 = \mathbb{C}$,
\[
i^0_I(a e_{i_1} \wedge \ldots \wedge e_{i_p}) = i^0_I(a) e_{i_1} \wedge \ldots \wedge e_{i_p},
\]
and the image of the injection $i^0_I : \mathcal{A}^0(= \mathbb{C}) \hookrightarrow \Gamma(U, \mathcal{O}_U)$ depends on the choice of $I$. Indeed, $\operatorname{Im} \ i^0_I$ depends on the trivialization of the line bundle $\mathcal{L}^{(k)}$ which is determined by $I$; see the proof of Theorem 3.2 and Remark 3.3(i).

By [15, 39], (3.7) is a quasi-isomorphism if the following condition is satisfied:
\[
a^*_L \notin \mathbb{Z}_{>0} \text{ for any nonzero dense edges } L \subset D.
\]

**Remark 3.1.** Assume that $D$ is reduced (that is, $m_i = 1$) and $(k, d) = 1$. Then condition (3.8) is satisfied for any $I$ with $|I| = k - 1$ since $a^*_L \notin \mathbb{Z}$ for any nonzero edge $L$. Moreover, this assumption implies that $\psi_{f, \lambda} X$, the nearby cycle sheaf with eigenvalue $\lambda := \exp(-2\pi ik/d)$, is supported at the origin. (Indeed, in case the last assertion is not true, there is $d' \in (0, d)$ and $k' \in \mathbb{N}$ such that $k/d = k'/d'$. This follows from the calculation of the Milnor cohomology in Section 3.2 at the point $x \in \mathbb{C} \setminus \{0\}$. Here, the degree $d'$ of the defining equation of $D$ at $x \in D \setminus \{0\}$ becomes strictly smaller. But, this contradicts the assumption $(k, d) = 1$.) The above assertion implies further the vanishing of the lower Milnor cohomology $H^j(F_J, \mathbb{C})$ for $j < n - 1$, since the nearby cycle sheaf $\psi_{f, \lambda} X$ is a perverse sheaf up to the shift of complex by $n - 1$. If, moreover, $D$ is indecomposable, then we get the nonvanishing of the highest Milnor cohomology $H^{n-1}(F_J, \mathbb{C})$ by (3.2), since the indecomposability is equivalent to the nonvanishing of the Euler characteristic $\chi(U)$; see [39].

Note that Theorem 4.2(e) in [38] remains valid in the nonreduced case as follows.

**Theorem 3.2.** Let $V(I)'$ be the subspace of $\mathcal{A}^{n-1}$ generated by $e_{J} := e_{j_1} \wedge \ldots \wedge e_{j_{n-1}}$ for any $J = \{j_1, \ldots, j_{n-1}\} \subset I$ with $j_1 < j_2 < \ldots < j_{n-1}$. Let $V(I)$ be the image of $V(I)'$ in $H^{n-1}(\mathcal{A}^{*, \omega_1})$, where $\omega_1$ and $\alpha^I = (\alpha^I_L)$ are as in Section 3.3. Assume that $V(I) \neq 0$ and that (3.8) holds. Then $b_I(-k/d) = 0$.

**Proof.** By (3.1) it is enough to show that the image of $e_{J}$ by the injection $i^{n-1}_{I}$ in (3.7) is contained $P_{0} \mathcal{L}^{(k)} \otimes \Omega^{n-1}_{V}$ in the notation of Section 3.2. Here $P_{-1} \mathcal{L}^{(k)} = 0$. By definition the image of $a \in \mathcal{A}^0 = \mathbb{C}$ by $i^{0}_{I}$ is a global section $v_{a}$ of a free $\mathcal{O}_{Y}$-submodule $\mathcal{L}_{I}$ of $\mathcal{L}^{(k)}$ such that the residue of the connection at the generic point of $Z_{i}$ is the multiplication by $\alpha^I_{L}$ in (3.5). Set $Z^{I_{\cup} \{e\}} := \bigcup_{k=1}^{n} Z_{j_k}$ with $j_{n} := e$. Then
\[
v_{a} \otimes e_{J} \in \mathcal{L}_{I}(Z^{I_{\cup} \{e\}}) \otimes \Omega^{n-1}_{V},
\]
since $e_{J} \in \Omega^{n-1}(\log Z^{I_{\cup} \{e\}}) = \Omega^{n-1}_{Y}(Z^{I_{\cup} \{e\}})$. Thus, the assertion is reduced to
\[
\mathcal{L}_{I}(Z^{I_{\cup} \{e\}}) \subset P_{0} \mathcal{L}^{(k)},
\]
and this is shown by comparing (3.4) and (3.5). Indeed, the eigenvalue of the residue of the connection on \( \mathcal{L}_I(Z^{I \cup \{e\}}) \) is shifted by \(-1\) at the generic point of \( Z_j \) for \( j \in I \cup \{e\} \), but it is not smaller than \(-m_jk/d\) even after this shift, by (3.5). So, Theorem 3.2 is proved.

**Proof of Theorem 1.3 in cases (i) and (iii).** In case (i), \( n/d \) is a jumping coefficient by Teitler’s refinement \([40]\) of Mustaţă’s formula \([34]\) for multiplier ideals using only dense edges. Hence, it is a root of \( b_I(s) \) up to a sign by Ein et al. \([14]\).

In case (iii), condition (3.8) is satisfied for any \( I \) with \( |I| = n - 1 \) since \( k = n \) and \( (n, d) = 1 \); see Remark 3.1. By Falk and Terao \([17, Example 4.5]\), the highest degree cohomology of the Aomoto complex \( H^{n-1}(\mathcal{A}^*, \omega_{\mathcal{I}}\wedge) \) has a monomial basis (independently of \( I \)) under the genericity condition on \( D_d \). Take a subset

\[ I = \{i_1, \ldots, i_{n-1}\} \subset \{1, \ldots, d-1\}, \]

such that the corresponding form \( e_I = e_{i_1} \wedge \ldots \wedge e_{i_{n-1}} \) is a member of the obtained monomial basis. Since (3.8) is satisfied, the image of \( e_I \) in the cohomology of the local system does not vanish. So, the assertion follows from Theorem 3.2 (that is, \([38, Theorem 4.2(e)]\)).

**Remarks 3.3.** (i) In the above argument, the image of \( e_I \) by \( t_I^{n-1} \) is independent of the choice of \( I \) up to a nonzero constant multiple. Indeed, the injection \( \iota_I^0 \) in (3.7) is defined by using the trivial line bundle \( \mathcal{L}_I \) in the proof of Theorem 3.2 which is determined by the eigenvalues \( \alpha_I^j \) in (3.5). If we take another \( I' \subset \{1, \ldots, d-1\} \) with \( |I'| = n - 1 \) and \( e_{I'} \neq 0 \), then, using the trivialization given by \( \mathcal{L}_I \), a nonzero constant section of \( \mathcal{L}_{I'} \) is identified with the rational function \( cg_{I'}/g_I \) where \( c \in \mathbb{C}^* \) and \( g_I = \prod_{i \in I} g_i \) in the notation of Section 3.3. This gives the difference between \( t_I^j \) and \( t_{I'}^j \) for any \( j \). So, the independence follows since \( g_I e_I = c' g_{I'} e_{I'} \) with \( c' \in \mathbb{C}^* \).

(ii) We can also identify the image of \( e_I \) by \( t_I^{n-1} \) with an element of the Gauss–Manin system of \( f \). The problem is then closely related to the torsion of the Brieskorn lattice.

4. The rank 3 case

In this section, we assume \( n = 3 \) and give two proofs of the case (ii) in Theorem 1.3. Note that the case \( n \leq 2 \) is well known. Indeed, it follows, for instance, from \([14, 34]\).

**Conditions 4.1.** From now on, we assume that

\[ n = k = 3. \]

We write \( p \subset i \) if \( \{p\} \subset Z_i \), and set \( \alpha_p^I = \alpha_I^p \) if \( P(L) = \{p\} \).

In the notation of (3.5), we study the following three conditions:

(a) \( \alpha_p^I \notin \mathbb{Z}_{>0} \) for any \( p \in \mathbb{Z}^{nmc} = P(D^{nmc}) \);

(b) \( \exists p_0 \in (\bigcup_{i \in I} Z_i)^{\text{sing}} \setminus Z_e \);

(c) \( Z \setminus (Z_e \cup \Sigma' \cup \{p_0\}) \) is connected.

**Remarks 4.2.**

(i) In the case \( n = 3 \), condition (a) coincides with condition (3.8), which implies that the inclusion (3.7) is a quasi-isomorphism. Note that we have always inequality of the dimensions; see \([30, Proposition 4.2]\).

(ii) For \( i, j, k \supset p \), there is a well-known relation

\[ e_i \wedge e_j = e_i \wedge e_k - e_j \wedge e_k, \quad (4.1) \]
which is easily checked by setting \( g_1 = x, g_j = y \) and \( g_k = x + y \). This also follows from the relations of the Orlik–Solomon algebra which are given by \( \partial(e_i \wedge e_j \wedge e_k) \) for \( i, j, k \supset p \); see, for example, [35, p. 60]. As in [4, Lemma 1.4], this implies, for \( \eta = \sum_{i=1}^{\varepsilon-1} \beta_i e_i \) and \( p \in Z^{\text{unc}} \setminus Z_e \), that

\[
\text{If } \pi_p(\omega_I \wedge \eta) = 0, \text{ then } \alpha^I_p \beta_i = \beta_p \alpha^I_i \text{ for any } i \supset p. \quad (4.2)
\]

Here \( \beta_p = \sum_{i \supset p} \beta_i \) and \( \pi_p(\omega_I \wedge \eta) \) are the \( p \)-components in the direct sum decomposition in [4, 1.3.2]

\[
H^2(U, Q) = \bigoplus_p L_p,
\]

where \( p \) runs over \( (Z_{\text{red}})^{\text{sing}} \setminus Z_e \), and \( L_p \) is a vector space of rank \( m'_p - 1 \) with \( m'_p \) the multiplicity of \( Z_{\text{red}} \) at \( p \). More precisely, \( L_p \) has a basis consisting of \( e_i \wedge e_k \) with \( i \supset p \) and \( i \neq k \), where \( k \) is any fixed member such that \( k \supset p \). This also follows from the definition of the Orlik–Solomon algebra mentioned after (4.1); see, for example, [35, p. 60]. We also get

\[
\text{If } p \in (Z_i \cap Z_j) \setminus (Z^{\text{unc}} \cup Z_e), \text{ then } \alpha^I_p \beta_j = \alpha^I_j \beta_i. \quad (4.3)
\]

In case \( \alpha^I_p \neq 0 \) (that is, \( m_i \neq d/3 \)) for any \( i \in I \), we have by (4.2–4.3)

\[
\text{If } \pi_p(\omega_I \wedge \eta) = 0 \text{ and } p \notin \Sigma^I, \text{ then } \beta_i/\alpha^I_i \text{ is independent of } i \supset p. \quad (4.4)
\]

(iii) Lemma 1.4 in [4] or (4.2) is essentially known to the specialists; see [30, Lemma 3.1] (and also [16], [29], [47]). Here, the situation is localized at \( p \), that is, the lines not passing through \( p \) are neglected, by using the fact that the relations of the Orlik–Solomon algebra are of the form \( \partial(e_j) \) for certain \( J \) and are compatible with the decomposition by \( p \).

**Proposition 4.3.** With the notation and the assumption of Section 3.3, assume \( n = k = 3 \) and there is \( I \subset \{1, \ldots, e - 1\} \) such that \( |I| = 2 \) and (a), (b) and (c) in Conditions 4.1 are satisfied. Then \( b_f(-3/d) = 0 \), where \( f \) is a defining polynomial of \( D \).

**Proof.** Let \( p_0 \) be as in condition (b) in Conditions 4.1, and assume that the following condition is satisfied:

\[
\pi_p(\omega_I \wedge \eta) = 0 \quad \text{for any } p \neq p_0.
\]

Then, \( \eta \) is a multiple of \( \omega_I \), that is, \( \beta_i/\alpha^I_i \) is independent of \( i \); see Remark 4.2(ii). So, we can apply Theorem 3.2 (that is, [38, Theorem 4.2(e)]), and conclude that \( b_f(-3/d) = 0 \). This completes the proof of Proposition 4.3.

4.1. **One proof of Theorem 1.3(ii).** We may assume that \( \{0\} \) is not a good dense edge, since we can apply the case (i) otherwise. By Proposition 4.3, it is sufficient to show the following assertion.

**Assertion.** There is an irreducible component \( Z_e \) of \( Z \) together with a subset \( I \subset \{1, \ldots, e - 1\} \) such that \( |I| = 2 \) and (a), (b) and (c) in Conditions 4.1 are satisfied, changing the order of \( \{1, \ldots, e\} \) if necessary.

Note first that \( \alpha^I_p \) can be an integer only in the case \( d/3 \in \mathbb{Z} \). (Indeed, we have \( m_p := \sum_{i \supset p} m_i < d \), and hence \( \alpha^I_p \equiv 3m_p/d \not\equiv 0 \mod \mathbb{Z} \) unless \( d/3 \in \mathbb{Z} \).) Then, the above assertion is shown in the case \( d/3 \notin \mathbb{Z} \) as follows.

Since \( \alpha^I_p \notin \mathbb{Z} \) for any \( p \in Z^{\text{unc}} \), condition (a) is trivially satisfied and \( \Sigma^I = \emptyset \) for any choice of \( I \). Assuming \( D \) to be central and indecomposable, there is \( p_0 \in Z^{\text{sing}} \) together with \( Z_e \) and \( I \) satisfying condition (b). As for condition (c), it is not satisfied only in the case there is \( Z_i \).
passing through $p_0$ and such that $Z_i \cap Z_{i'} \subset (Z_i \cap Z_e) \cup \{p_0\}$ for any $i' \notin \{i, e\}$. (Otherwise, for any $Z_i$ passing through $p_0$, there is $Z_{i'}$ such that $Z_i \cap Z_{i'} \notin Z_e \cup \{p_0\}$.) In this case, every $Z_i$ passes through either $p_0$ or $Z_i \cap Z_e$. This implies that $|Z^\text{nnc}| = 2$ since $D$ is indecomposable. Then, replacing $Z_e$ with $Z_i$ containing $Z^\text{nnc}$, we may take $p_0$ to be any point of $Z^\text{nnc} \setminus Z^\text{nnc}$ and $I$ is chosen so that $\{p_0\} = \bigcap_{i \in I} Z_i$. Thus, the assertion is proved in this case.

We may now assume
$$a := d/3 \in \mathbb{Z}.$$ 
Since $\{0\}$ is not a good dense edge, there is $p_1 \in Z^\text{nnc}$ with multiplicity $> 2a$. On the other hand, we may assume that there is $p_2 \in Z^\text{nnc}$ with $\alpha_{p_2}^I \in \mathbb{Z}$, that is, its multiplicity is divisible by $a$, since otherwise the above conditions are easily satisfied. Thus, we may assume that there are $p_1, p_2 \in Z^\text{nnc}$ with multiplicities $2a + 1$ and $a$, respectively, and hence $Z^\text{nnc} = \{p_1, p_2\}$, since $d = 3a$. So, the assertion is proved by the same argument as above.

4.2. Another proof of Theorem 1.3(ii). It is also possible to prove Theorem 1.3(ii) by taking $p_0$ to be the point with multiplicity $m_{p_0} > \frac{2}{3}d$, which exists since we may assume that $\{0\}$ is not a good dense edge as in Section 4.1. In this case, there is a line $Z_{d-1}$ which is different from the line at infinity $Z_d$ and does not contain $p_0$ since $D$ is indecomposable. Moreover, there are at least two lines $Z_1, Z_2$ passing through $p_0$ such that their intersections with $Z_{d-1}$ are ordinary double points of $Z$ and, furthermore, their intersections with $Z_d$ do not have multiplicity $a$ so that conditions (a) and (b) in Conditions 4.1 are satisfied by setting $I = \{1, 2\}$. Indeed, we have $m_{p_0} > \frac{2}{3}d$, $d - m_{p_0} \geq 2$, and hence $d > 6$, and moreover the number of lines $Z_i$ such that $i \supset p_0$ and $Z_i \cap Z_{d-1}$ is an ordinary double point of $Z$ is at least
$$m_{p_0} - 1 - (d - 2 - m_{p_0}) > 1,$$
since $|\bigcup_{i \supset p_0} Z_i \cap Z_{d-1}| \geq m_{p_0} - 1$. So, the condition on the intersection of $Z_1$ and $Z_2$ with $Z_{d-1}$ is satisfied. For the intersection with $Z_d$, we can exclude the case where a point of $Z$ has multiplicity $a$ since this case has a very special structure as explained at the end of Section 4.1 (for example, the singular points of $Z$ other than this point and $p_0$ are ordinary double points) so that we can easily choose $Z_1, Z_2$ satisfying the above conditions in this case.

We can then prove Theorem 1.3(ii) without using Proposition 4.3 but using (4.1). Indeed, by Theorem 3.2 (that is, [38, Theorem 4.2(e)]), it is enough to show that
$$\text{If } \left(\sum_i \alpha_i^I e_i\right) \wedge \left(\sum_j \beta_j e_j\right) = c e_1 \wedge e_2 \text{ for some } c \in \mathbb{Q}, \text{ then } c = 0. \quad (4.5)$$
Under the assumption of (4.5) we get, by using (4.1),
$$\alpha_{p_0}^I \beta_i = \beta_{p_0} \alpha_i^I \text{ if } i \supset p_0 \text{ and } i > 2. \quad (4.6)$$
Here we have $\alpha_{p_0}^I \neq 0$ since $m_{p_0} > \frac{2}{3}d$. So we may assume
$$\beta_i = 0 \text{ if } i \supset p_0 \text{ and } i > 2, \quad (4.7)$$
by replacing $\beta_i$ with $\beta_i - c' \alpha_i^I$ for any $i$ where $c' := \beta_{p_0}/\alpha_{p_0}^I$. (Note that this change of $\beta_i$ does not affect the hypothesis of (4.5).) Since $m_{p_0} > 4$, (4.6) and (4.7) imply
$$\beta_1 + \beta_2 = \beta_{p_0} = 0.$$
On the other hand, by (4.3) applied to the intersections of $Z_1, Z_2$ with $Z_{d-1}$, we get
$$\beta_1/\alpha_1^I = \beta_{d-1}/\alpha_{d-1}^I = \beta_2/\alpha_2^I,$$
where $\alpha_1^I = \alpha_2^I \neq 0$ and $\alpha_{d-1}^I \neq 0$ since $Z$ is reduced. So, $\beta_1 = \beta_2 = 0$, and (4.5) follows.
Remarks 4.4. (i) It does not seem easy to generalize the above arguments to the nonreduced case. If \( p_0 \) is taken to be the point with the highest multiplicity, there is an example as follows: assume \( a > 6 \), and let

\[
f = (xy(x - y))^{a-2} (x + y - z)(x + y - 2z)(x + 2y - 2z)(2x + y - 2z)z^2.
\]

Here \( d = 3a \), and there does not exist \( I \) such that the argument in Section 4.2 can be applied if we set \( p_0 = (0, 0, 1) \). Indeed, let \( Z_i (i = 1, \ldots, 8) \) denote the lines defined by the linear factors of \( f \) respecting the order of the factors, where \( e = 8 \). Here, \( Z_e \) must be the line defined by \( z = 0 \) since (a) and (b) in Conditions 4.1 cannot be satisfied otherwise. Then, the singular points of \( Z \setminus \{p_0\} \cup Z_e \) contained in \( Z_1 \) or \( Z_2 \) all have multiplicity \( a \), and moreover \( Z_{\text{red}} \) has multiplicity 3 at these points. So, the argument in Section 4.2 cannot be applied.

(ii) For a more complicated example, we might consider the following: let \( E \) be an elliptic curve in the dual projective space \( \mathbb{P}^2 \), and \( G \) be the subgroup of torsion points of order 3. This defines a projective hyperplane arrangement in \( \mathbb{P}^2 \) with \( e = |G| = 9 \); see, for example, \([29]\). Let \( G_0 \) be a subgroup of \( G \) with order 3. Assume \( a > 6 \). To the lines corresponding to the elements of \( G_0 \), we give multiplicity \( a - 2 \), while the other lines have multiplicity 1. Then \( d = 3a \), and \( I \cup \{e\} \) should correspond to \( G_0 + p \subset G \) for some \( p \in G \) in order to satisfy (a) in Conditions 4.1. (Indeed, if there are \( g_1, g_2 \in I \cup \{e\} \) such that their images in \( G/G_0 \) are different, then there is \( g_3 \in G \) such that the images of \( g_1, g_2, g_3 \) in \( G/G_0 \) are all different and moreover \( g_1 + g_2 + g_3 = 0 \). The last condition is equivalent to the condition that the three lines corresponding to \( g_1, g_2, g_3 \) intersect at one point. Then condition (a) is not satisfied at this point.) So, \( p_0 \) is contained in \( Z_e \), and hence condition (b) cannot be satisfied. Thus, we cannot prove a generalization of Theorem 2 in this case by using Theorem 3.2 (that is, the generalization of \([38], \text{Theorem 4.2(e)}\), to the nonreduced case).

(iii) In order to apply the theory in \([15, 39]\), we have to choose the residues \( \alpha_i \) of the connection satisfying the two conditions \((3.6)\) and \((3.8)\). In our case, we have \( \alpha_i = n_i - m_i k/d \) with \( n_i \in \mathbb{Z} \) by the monodromy condition, and \( \sum_i n_i = k \) since \( \sum_i m_i = d \). Then, to satisfy \((3.6)\), an easy way is to choose a subset \( J \) of \( \{1, \ldots, e\} \) with \( |J| = k \) and set \( n_i = 1 \) for \( i \in J \) and \( n_i = 0 \) otherwise. Here, there are two possibilities depending on whether \( e \in J \) or \( e \notin J \).

\[
\alpha_i = \begin{cases} 
-m_i k/d & \text{if } i \notin I, \\
1 - m_i k/d & \text{if } i \in I.
\end{cases}
\]

In the latter case, however, it is usually more difficult to satisfy the three conditions in Conditions 4.1.

(iv) If \( n = 3, d \leq 7 \) and \( \text{mult}_p Z = 3 \) for any \( p \in Z^{\text{unc}} \) in the notation of Section 3.3, the \( b \)-function of a reduced hyperplane arrangement is calculated in \([38]\).

Appendix

by Willem Veys

This appendix describes some examples solving Question (Q) in Section 2.2 negatively. I thank the authors of this paper for writing some details.
EXAMPLE A.1. We first explain an example of a nonreduced hyperplane arrangement with $n = 3$, $d = 9$. Let

$$f = xy(x - y)z^2(x - z)^4.$$ 

This gives a negative answer to Question (Q) in Section 2.2. Indeed, we have $\chi(U) = 1$ by using the affine space defined by $x \neq 0$. We have $\chi(Z^0_o) = -1$ except for the line defined by $x = 0$, and the Euler characteristic is 0 for the latter. So, we get

$$Z_{f,0}^{\text{top}}(s) = \frac{1}{2^{s+3}} \left( \frac{1}{s+1} \right) \frac{1}{3s+2} + \left( \frac{1}{2^{s+3}} \right) \frac{1}{7s+2} \left( \frac{1}{2s+1} + \frac{1}{4s+1} \right) \cdot$$

Set $\Phi(s) = (9s^3 + 3)Z_{f,0}^{\text{top}}(s)$. Since $1/(2s + 1) + 1/(4s + 1)$ vanishes by substituting $s = -1/3$, we get

$$\Phi(-1/3) = 1 - 3 \left( \frac{1}{2} \right) - 3 \left( \frac{3}{2} \right) = 0.$$ 

So, the pole at $-1/3$ vanishes. (It vanishes also for the motivic or Hodge zeta function.)

Note that the above example does not give a counterexample to Conjecture (C). This is shown by using Theorem 3.2 and Remark 4.2(ii). Here, $p = (0 : 1 : 0) \in \mathbf{P}^2$, the line at infinity is $(y = 0)$, and $I$ corresponds to the two lines with multiplicities 2 and 4. This assertion is also shown by a calculation using the computer program Asir.

EXAMPLE A.2. There is an example of a reduced hyperplane arrangement with $n = 5$ and $d = 10$, giving a negative answer to Question (Q) in Section 2.2, and which is defined by a polynomial $f$ as below:

$$f = (x - y)(x - 2y)(x - 3y)(x - 4y)(x - 5y)(x + y + z)zuv(u + v + z).$$

In fact, let $Z_1, Z_2, Z_3$ be closed subvarieties of $Y := \mathbf{P}^4$ defined by

$$Z_1 = \{x = y = z = 0\}, \quad Z_2 = \{u = v = z = 0\}, \quad Z_3 = \{x = y = 0\}.$$ 

Let $\rho : Y' \to Y$ be the composition of the blow-up of $Y$ along $Z_1, Z_2$ and the blow-up along the proper transform of $Z_3$. This gives an embedded resolution of $(Y, Z)$ where $Z := \{f = 0\} \subset Y$. We have a partition $\{S_i\}_{i=0,\ldots,3}$ of $Y = \mathbf{P}^4$ defined by

$$S_0 = \{z \neq 0\}, \quad S_i = Z_i \quad (i = 1, 2), \quad S_3 = \{z = 0\} \setminus (Z_1 \cup Z_2).$$

Consider the pullback of the partition

$$S'_i := \rho^{-1}(S_i) \quad (i = 0, \ldots, 3).$$

Let $x', y', u', v'$ be affine coordinates of $S_0$ defined, respectively, by $x/z, y/z, u/z, v/z$. Then

$$S'_0 = \mathbf{P}^2_{x',y'} \times \mathbf{P}^2_{u',v'}, \quad S'_1 = \mathbf{P}^2_{x,y,z} \times \mathbf{P}^1_{u,v}, \quad S'_2 = \mathbf{P}^2_{u,v,z} \times \mathbf{P}^1_{x,y},$$

where $\mathbf{C}^2_{x',y'}$ and $\mathbf{P}^2_{x,y,z}$ are the blow-up of $C^2_{x',y'}$ along $(0,0)$ and the blow-up of $\mathbf{P}^2_{x,y,z}$ along $(0:0:1)$, respectively. Here, the lower indices $x, y$ and so on indicate the coordinates. Note that each $S_i$ is a union of strata of the stratification associated with the divisor with normal crossings $\rho^{-1}(Z)$. So, we get

$$Z_{f,0}^{\text{top}}(s) = \sum_{i=0}^{3} \frac{\Psi_i(s)}{10s + 5},$$

where $\Psi_i(s)$ is the local zeta function at $S'_i$. 

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where $\Psi_i(s)/(10s + 5)$ is the sum of the factors of $Z_{f,0}^{\top}(s)$ associated with the strata contained in $S'_i$. Since the stratification is compatible with the above product structure, we get

$$
\Psi_0(s) = \left(4 - \frac{9}{s + 1} + \frac{5}{(s + 1)^2} + \left(-3 + \frac{5}{s + 1}\right) \frac{1}{5s + 2}\right) \cdot \left(1 - \frac{3}{s + 1} + \frac{3}{(s + 1)^2}\right),
$$

$$
\Psi_1(s) = \frac{1}{7s + 3} \left(4 - \frac{13}{s + 1} + \frac{11}{(s + 1)^2} + \left(-3 + \frac{5}{s + 1}\right) \frac{1}{5s + 2}\right) \cdot \left(-1 + \frac{3}{s + 1}\right),
$$

$$
\Psi_2(s) = \frac{1}{4s + 3} \left(1 - \frac{4}{s + 1} + \frac{6}{(s + 1)^2}\right) \cdot \left(-4 + \frac{6}{s + 1}\right),
$$

$$
\Psi_3(s) = 0.
$$

Indeed, let $Z_0'$ be the divisor on $\mathbb{P}^2_{x,y,z}$ defined by the product of linear factors of $f$ that are linear combinations of $x, y, z$, and similarly for $Z''_0$ with $x, y$ replaced by $u, v$. Then

$$
\chi(\mathbb{P}^2 \setminus Z_0') = 4, \quad \chi(\mathbb{P}^2 \setminus Z''_0) = 1, \quad \chi(Z_0' \setminus \text{Sing} Z_0') = -13, \quad \chi(Z''_0 \setminus \text{Sing} Z''_0) = -4,
$$

and the number of ordinary double points of $Z_0'$ and $Z''_0$ are, respectively, 11 and 6. The calculation for $\mathbb{P}^1_{u,v}$ and $\mathbb{P}^1_{x,y}$ is similar, and we get the formulas for $\Psi_1(s)$ and $\Psi_2(s)$ since the definition of $\Psi_1(s), \Psi_2(s)$ is compatible with the above product structure using the formula $\chi(X_1 \times X_2) = \chi(X_1) \cdot \chi(X_2)$ for topological spaces $X_1, X_2$. As for the first terms, note that the codimensions of the centers $Z_1, Z_2$ are 3, and the multiplicities of $f$ at the generic points of $Z_1$ and $Z_2$ are, respectively, 7 and 4. The term $(-3 + 5/(s + 1))/(5s + 2)$ comes from the exceptional divisor of the blow-up along the proper transform of $Z_3$, where the multiplicity of $f$ at the generic point of $Z_3$ is 5 and $Z_3$ has codimension 2.

The argument is similar for $\Psi_0(s)$. Here, the Euler number of the smooth part and the number of ordinary double points change since the varieties are restricted to (the blow-up of) the affine space $\mathbb{C}^2$. The vanishing of $\Psi_3(s)$ follows from the $\mathbb{C}^*$-action on $S'_3 = S_3$ compatible with the stratification, which is defined by $\lambda(x : y : u : v) = (\lambda x : \lambda y : u : v)$ for $\lambda \in \mathbb{C}^*$.

Substituting $s = -\frac{1}{2}$ in the above formulas, we get

$$
\Psi_0\left(-\frac{1}{2}\right) = -8 \cdot 7, \quad \Psi_1\left(-\frac{1}{2}\right) = -2 \cdot 8 \cdot 5, \quad \Psi_2\left(-\frac{1}{2}\right) = 17 \cdot 8,
$$

and hence the pole of $Z_{f,0}^{\top}(s)$ at $s = -\frac{1}{2}$ vanishes. For the moment, it is not clear whether $-\frac{1}{2}$ is a root of $b_f(s)$.

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